On several problems of differential geometry related to the theory of elasticity

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Introduction et Bibliographie

1. Introduction

Cette thèse vise à approfondir les liens entre la géométrie différentielle et la théorie de l'élasticité, linéaire ou nonlinéaire. En s'appuyant sur cette analogie, notre objectif est de déduire des résultats nouveaux tant en élasticité (chapitres 1 et 2), qu'en géométrie différentielle (chapitres 3 et 4). Dans un appendice à cette thèse, nous présentons quelques résultats d'analyse utilisés dans la thèse et dont la démonstration n'est pas toujours facile à trouver dans la littérature.

La théorie de l'élasticité étudie les déformations des solides élastiques sous l'effet d'efforts extérieurs. Transposé en géométrie différentielle, ce problème revient à étudier les immersions (l'équivalent d'une déformation) dans l'espace euclidien tridimensionnel d'une variété différentielle (l'équivalent du solide dans une configuration donnée). Les liens entre les différents points matériels du solide dans une configuration donnée sont modélisés par une métrique définie sur la variété différentielle correspondante. Ces liens changent lorsque le solide subit une déformation, ce qui correspond à un changement de métrique définie sur la variété différentielle.

Le problème statique en élasticité nonlinéaire consiste à déterminer la déformation du solide minimisant le changement des liens entre les points matériels du solide (mesuré pour une certaine norme) effectué pour répondre à un effort extérieur. En géométrie différentielle, ce problème revient à déterminer l'immersion d'une variété différentielle dans l'espace euclidien tridimensionnel minimisant le changement de métrique (pour une certaine norme) entre l'image de la variété par l'immersion et la variété elle-même. En élasticité linéarisée, ce changement de métrique est remplacé par sa partie linéaire par rapport au champ de déplacements, c'est-à-dire le champ de vecteurs reliant les points de la variété différentielle à leurs images par l'immersion.

En élasticité tridimensionnelle (respectivement bidimensionnelle), la géométrie du solide est modélisée par une variété différentielle tridimensionnelle (respectivement bidimensionnelle) plongée dans l'espace euclidien tridimensionnel identifié à $\mathbb{R}^3$. La possibilité de passer d'une variété bidimensionnelle à une variété tridimensionnelle en rajoutant une troisième dimension le long des normales à la variété bidimensionnelle constitue l'idée centrale de notre preuve de l'inégalité de Korn sur une surface présentée dans le premier chapitre.

Le caractère tensoriel des termes apparaissant dans cette inégalité nous conduit dans le second chapitre à une généralisation de cette inégalité à des surfaces définies par plusieurs cartes locales, dont les surfaces sans bord constituent l'exemple typique.

L'équivalence des approches extrinsèque et intrinsèque en géométrie différentielle (voir par exemple [17, 36, 42, 43, 44, 59, 63, 65, 66, 68]) permet de considérer la métrique comme l'inconnue du problème de l'élasticité tridimensionnelle en lieu et place de l'immersion réalisant cette métrique (immersion qui correspond au champ de déformations du solide). En effet, l'on peut recouvrir une telle immersion à partir d'une métrique donnée, pourvu que cette dernière soit suffisamment régulière et satisfasse les équations de Riemann-Christoffel. Comme la solution du problème de l'élasticité statique minimise une fonctionnelle définie par l'intégrale d'une certaine densité, il convient de la chercher dans des espaces de fonctions de faible régularité, en particulier dans des espaces de type Sobolev. Ceci nous a amené à considérer dans le troisième chapitre le problème du recouvrement d'une immersion isométrique pour les variétés de Riemann peu régulières.

Enfin, le problème analogue en élasticité bidimensionnelle nous a conduit à étudier dans le quatrième chapitre le problème du recouvrement d'une surface à partir de ses première et deuxième formes fondamentales sous des hypothèses de régularité faibles de ces formes.

Nous présentons maintenant d'une manière générale les travaux de cette thèse, puis nous préciserons l'énoncé de nos résultats chapitre par chapitre.

L'inégalité de Korn sur une surface joue un rôle essentiel dans la théorie de coques linéairement élastiques puisqu'elle permet d'établir l'existence d'une solution pour le modèle linéaire de Koiter d'une part (voir e.g. Bernadou, Ciarlet et Miara [7], Blouza et Le Dret [8], Ciarlet [24]), et intervient dans l'analyse asymptotique de coques élastiques d'autre part (voir e.g. Ciarlet, Lods et Miara [28], Ciarlet [24]). Elle affirme que l'on peut contrôler le champ de déplacements d'une surface (mesuré en une norme appropriée de type Sobolev) par les tenseurs linéarisés de changement de métrique et de courbure de la surface, mesurés en norme L^2. Cette inégalité a été établie par Bernadou et Ciarlet [6]. Une preuve plus simple a été donnée dans Ciarlet et Miara [31]. Enfin, Blouza et Le Dret [8] l'ont établie sous des hypothèses de régularité plus faibles sur la surface.

L'objet du premier chapitre, qui est un travail en commun avec Philippe G. Ciarlet, est d'établir que cette inégalité est une conséquence de l'inégalité de Korn tridimensionnelle en coordonnées curvilignes, elle-même pouvant être déduite de l'inégalité de Korn classique (posée en coordonnées cartésiennes), à savoir

$$\|\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\|_{\boldsymbol{L}^2(\Omega)} \geq C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$$

pour tout $\boldsymbol{u} \in H^1(\Omega; \mathbb{R}^3)$ s'annulant sur une partie de mesure non-nulle du bord de Ω, où Ω est un ouvert borné et connexe de $\mathbb{R}^3$ et $C > 0$ est une constante. Nous souhaitons indiquer ici qu'un résultat analogue a été établi par Akian [2].

L'inégalité de Korn sur une surface établie au premier chapitre s'applique uniquement à des surfaces définies par une seule carte. Ce travail pose naturellement la question de savoir si cette restriction est essentielle.

Nous répondons à cette question dans le deuxième chapitre, en établissant une inégalité de type Korn sur une surface compacte sans bord. Cette inégalité est en fait valable pour toutes les surfaces pouvant être définies par un nombre fini de cartes locales, comme c'est le cas des surfaces compactes.

Considérons maintenant un espace riemannien (Ω, g), où Ω est un ouvert connexe et simplement connexe de $\mathbb{R}^d$ et g est une métrique riemanniene sur Ω, et supposons que le tenseur de courbure de Riemann associé à la métrique g s'annule. Si la métrique est de classe C^2, alors un résultat classique en géométrie différentielle (voir e.g. Blume [9], Choquet-Bruhat, Dewitt-Morette et Dillard-Bleick [20], Spivak [63] pour le résultat local ou Ciarlet et Larsonneur [26] pour le résultat global) montre que l'espace riemannien peut être plongé dans l'espace euclidien d-dimensionnel par une immersion isométrique. Autrement dit, il existe une application $\Theta : \Omega \to \mathbb{R}^d$ de classe C^3 réalisant cette métrique, i.e.

$$(\nabla\Theta)^T \nabla\Theta = g \text{ dans } \Omega.$$

De plus, l'application Θ est unique aux isométries de $\mathbb{R}^d$ près. Ce résultat a été étendu récemment dans C. Mardare [49] pour des métriques de classe C^1.

L'objet du troisième chapitre est d'affaiblir davantage les hypothèses de régularité sur la métrique sous lesquelles l'espace riemannien (Ω, g) est plongé dans $\mathbb{R}^d$. Plus précisément, nous montrons que si la métrique est lipschitzienne localement, alors il existe une immersion isométrique Θ, unique aux isométries de $\mathbb{R}^d$ près, réalisant cette métrique (l'application et ses dérivées partielles du premier ordre étant lipschitziennes localement).

La principale difficulté dans ce chapitre est de résoudre un système de Pfaff dont les coefficients sont seulement de classe L^∞_{loc}, alors que dans le cas classique les coefficients de ce système sont de classe C^1.

Considérons enfin un champ de matrices symétriques définies positives d'ordre deux $(a_{\alpha\beta})$ et un champ de matrices symétriques d'ordre deux $(b_{\alpha\beta})$ et supposons qu'ils vérifient ensemble les équations de Gauss et de Codazzi-Mainardi dans un ouvert ω de $\mathbb{R}^2$. Si le premier champ de matrices est de classe C^2 et le deuxième champ de matrices est de classe C^1, alors le théorème fondamental de la théorie des surfaces (voir, e.g., do Carmo [15] et Klingenberg [44]) affirme qu'il existe localement une surface S dont les champs de matrices $(a_{\alpha\beta})$ et $(b_{\alpha\beta})$ sont respectivement les composantes covariantes des première et deuxième formes fondamentales.

Les hypothèses de ce théorème ont été affaiblies dans Hartmann et Wintner [41] : l'existence de la surface S est assurée dans l'espace $C^2(\omega; \mathbb{R}^3)$ si $a_{\alpha\beta}$ et $b_{\alpha\beta}$ sont respectivement de classe $C^1(\omega)$ et $C^0(\omega)$ et satisfont ensemble les équations de Gauss et de Codazzi-Mainardi sous une forme intégrale le long des courbes de Jordan de classe C^1 contenues dans ω.

L'objet du quatrième chapitre est de montrer que ce résultat reste vrai sous l'hypothèse que le champ de matrices $(a_{\alpha\beta})$ est lipschitzien localement et le champ de matrices $(b_{\alpha\beta})$ est mesurable et essentiellement borné localement. Nous revenons aux équations classiques de Gauss et de Codazzi-Mainardi (et non pas sous une forme intégrale comme dans Hartmann et Wintner [41]) et supposons qu'elles sont satisfaites au sens des distributions. Lorsque l'ouvert ω est simplement connexe, on montre l'existence globale de la surface immergée S (c'est-à-dire que la carte définissant S est définie sur tout ω), ainsi que l'unicité aux isométries de $\mathbb{R}^3$ près.

Nous précisons maintenant l'énoncé de nos résultats. Nous renvoyons le lecteur aux différents chapitres de cette thèse pour les preuves et la bibliographie complète.

Chapitre I : **Sur les inégalités de Korn en coordonnées curvilignes**

Le travail de ce premier chapitre a été fait en collaboration avec Philippe G. Ciarlet.

Dans ce chapitre, nous considérons une surface définie par une seule carte. Dans ce cas, l'*inégalité de Korn sur une surface* s'énonce ainsi : Soit $S := \boldsymbol{\theta}(\overline{\omega})$ une surface plongée dans l'espace euclidien tridimensionnel identifié à $\mathbb{R}^3$, où $\omega \subset \mathbb{R}^2$ est un ouvert borné, connexe, de frontière lipschitzienne et $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ est une immersion injective de classe C^3 dans $\overline{\omega}$. Alors il existe une constante $C > 0$ telle que

$$\|\gamma(\boldsymbol{\eta})\|^2_{\boldsymbol{L}^2(\omega)} + \|\rho(\boldsymbol{\eta})\|^2_{\boldsymbol{L}^2(\omega)} \geq C\left(\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{H}^1(\omega)} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^2(\omega)}\right) \tag{1}$$

pour tout $\boldsymbol{\eta} \in \boldsymbol{V}_K(\omega)$. Dans cette inégalité, $\gamma(\boldsymbol{\eta})$ est le tenseur linéarisé de changement de métrique, $\rho(\boldsymbol{\eta})$ est le tenseur linéarisé de changement de courbure entre la surface $\boldsymbol{\theta}(\omega)$ et la surface déformée $(\boldsymbol{\theta} + \boldsymbol{\eta})(\omega)$, et

$$\boldsymbol{V}_K(\omega) := \{\boldsymbol{\eta} = \boldsymbol{\eta}_\tau + \boldsymbol{\eta}_\nu;\ \boldsymbol{\eta}_\tau \in H^1_{\gamma_0}(\omega;\mathbb{R}^3), \boldsymbol{\eta}_\nu \in H^2_{\gamma_0}(\omega;\mathbb{R}^3)\}.$$

Les champs de vecteurs $\boldsymbol{\eta}_\tau$ et $\boldsymbol{\eta}_\nu$ désignent respectivement les composantes tangentielle et normale (à la surface S) du champ de vecteurs $\boldsymbol{\eta}$, γ_0 désigne une partie de mesure strictement positive de la frontière de ω, $H^1_{\gamma_0}(\omega;\mathbb{R}^3)$ désigne l'espace des champs de vecteurs de classe H^1 dans Ω s'annulant sur γ_0, et $H^2_{\gamma_0}(\omega;\mathbb{R}^3)$ désigne l'espace des champs de vecteurs $\boldsymbol{\eta} \in H^2(\omega;\mathbb{R}^3)$ tels que $\boldsymbol{\eta}$ et sa dérivée normale s'annulent sur γ_0.

L'inégalité de Korn sur une surface (1) a été établie par Bernadou et Ciarlet [6]. D'autres preuves se trouvent dans [7, 8, 24, 31].

Cette inégalité est l'analogue en élasticité bidimensionnelle de l'*inégalité de Korn tridimensionnelle en coordonnées curvilignes* : Soit $\Theta : \overline{\Omega} \to \Theta(\overline{\Omega}) \subset \mathbb{R}^3$ un difféomorphisme de classe C^2, où Ω est un ouvert connexe et borné de $\mathbb{R}^3$ de frontière Γ lipschitzienne. Alors il existe une constante $C > 0$ telle que

$$\|e(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)} \geq C\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \text{ pour tout } \boldsymbol{v} \in \boldsymbol{V}(\Omega), \tag{2}$$

où $e(\boldsymbol{v})$ est le tenseur linéarisé de changement de métrique entre la variété tridimensionnelle $\Theta(\Omega)$ et la variété déformée $(\Theta + \boldsymbol{v})(\Omega)$ et

$$\boldsymbol{V}(\Omega) := \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^3); \boldsymbol{v} = 0 \text{ sur } \Gamma_0\},$$

Γ_0 étant une partie de mesure strictement positive de Γ.

L'inégalité de Korn tridimensionnelle en coordonnées curvilignes peut être établie comme une conséquence de l'inégalité de Korn classique (en coordonnées cartesiennes) ou par une preuve directe comme dans Ciarlet [21, 24]. Pour l'énoncé et la démonstration de l'inégalité de Korn classique nous renvoyons le lecteur à e.g. Ciarlet [22], Duvaut et Lions [35] ou Taylor [67] .

L'objet de ce chapitre est d'établir que les inégalités (1) et (2) ci-dessus sont en fait reliées :

Théorème 1. *L'inégalité de Korn sur une surface est une conséquence de l'inégalité de Korn tridimensionnelle en coordonnées curvilignes.*

Autrement dit, l'inégalité (1) est une conséquence de l'inégalité (2).

La démonstration repose sur le relèvement de la surface $S = \boldsymbol{\theta}(\overline{\omega})$ à une variété tridimensionnelle avec bord $\Theta(\overline{\Omega}_0)$ défini par

$$\Theta(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ pour tout } (y, x_3) \in \Omega_0,$$

où $\boldsymbol{a}_3(y)$ est un vecteur unitaire normal à la surface S au point $\boldsymbol{\theta}(y)$ et $\Omega_0 := \omega \times]-\varepsilon_0, \varepsilon_0[$, avec ε_0 suffisamment petit (tel que Θ soit une immersion injective).

Les éléments de l'espace $\boldsymbol{V}_K(\omega)$ (i.e. les déplacements de la surface S) sont aussi relevés à des éléments de l'espace $\boldsymbol{V}(\Omega_0)$ (i.e. les déplacements de la variété tridimensionnelle $\Theta(\overline{\Omega}_0)$) par l'application linéaire

$$F : \boldsymbol{\eta} \in \boldsymbol{V}_K(\omega) \to \boldsymbol{v} \in \boldsymbol{V}(\Omega_0)$$

définie par $\boldsymbol{v}(\cdot, x_3) = \boldsymbol{\eta} - x_3(\partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha$, où $(\boldsymbol{a}^1(y), \boldsymbol{a}^2(y))$ désigne la base contravariante du plan tangent à la surface S au point $\boldsymbol{\theta}(y)$.

Le théorème ci-dessus est alors obtenu comme une conséquence du lemme suivant :

Lemme 1. *Il existe $\varepsilon \in]0, \varepsilon_0[$ tel que l'application $F : \boldsymbol{V}_K(\omega) \to \boldsymbol{V}(\Omega)$, où $\Omega := \omega \times] - \varepsilon, \varepsilon[$, soit un isomorphisme entre les espaces de Hilbert $\boldsymbol{V}_K(\omega)$ et*

$$\boldsymbol{V}_{KL}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{V}(\Omega);\ \partial_i \boldsymbol{v} \cdot \partial_3 \Theta + \partial_3 \boldsymbol{v} \cdot \partial_i \Theta = 0\}.$$

Il est à remarquer que le lemme ci-dessus donne en particulier une caractérisation des champs de déplacement de Kirchhoff-Love linéarisés à l'intérieur d'un corps ayant la géométrie de la variété $\Theta(\Omega)$.

La preuve de ce lemme se fait en plusieurs étapes. D'abord, nous montrons que l'application F est continue, injective, et que son image est l'espace $\boldsymbol{V}_{KL}(\Omega)$ défini ci-dessus. A partir de là, le théorème du graphe fermé entraîne que l'application inverse de F est aussi continue, ce qui conclut la preuve du lemme.

L'inégalité de Korn sur une surface (voir (1)) est ensuite obtenue en combinant l'inégalité inverse donnée par le lemme ci-dessus avec l'inégalité de Korn tridimensionnelle en coordonnées curvilignes (voir (2)) appliquée aux champs de vecteurs $\boldsymbol{v}$ appartenant au sous-espace $\boldsymbol{V}_{KL}(\Omega)$ de $\boldsymbol{V}(\Omega)$.

Chapitre II : **Inégalité de type Korn sur une surface compacte**

Dans ce chapitre, une surface régulière de classe C^k est un sous-ensemble connexe S de $\mathbb{R}^3$ qui est localement l'image d'un ouvert de $\mathbb{R}^2$ par une immersion injective de classe C^k, l'image étant un ouvert relatif dans S (voir e.g. do Carmo [15] ou Klingenberg [44]).

Le premier pas vers l'obtention d'une inégalité de type Korn sur une surface compacte (donc sans bord, vu la définition ci-dessus) est d'établir qu'une telle surface peut être définie à l'aide d'un nombre fini de cartes locales définies sur des ouverts connexes, bornés et de frontière lipschitzienne.

Le modèle de Koiter, défini usuellement en coordonnées locales sur la surface, est récrit ensuite sous une forme intrinsèque, à savoir

$$\begin{aligned}
&\boldsymbol{\zeta} \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S),\\
&\int_S A(\boldsymbol{\zeta}, \boldsymbol{\eta})\, ds = \int_S \boldsymbol{f} \cdot \boldsymbol{\eta}\, ds \text{ pour tout } \boldsymbol{\eta} \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S),
\end{aligned}$$

ce qui nous permet de le transposer *verbatim* dans le cas d'une surface compacte. Dans ce modèle, l'application bilinéaire $A(\boldsymbol{\zeta}, \boldsymbol{\eta})$ est définie en fonction des tenseurs de l'élasticité bidimensionnelle et de changement linéarisé de métrique et de courbure. L'expression exacte n'est pas essentielle à ce stade, il suffit de savoir qu'elle vérifie l'inégalité

$$A(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq C \left(\|\gamma(\boldsymbol{\eta})\|_{\boldsymbol{L}^2(S)} + \|\rho(\boldsymbol{\eta})\|_{\boldsymbol{L}^2(S)} \right)$$

pout tout $\boldsymbol{\eta} \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$, où $C > 0$ est une constante. La notation $\boldsymbol{\eta} \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$ signifie que les parties tangentielle et normale (à la surface S) de $\boldsymbol{\eta}$ sont respectivement dans $\boldsymbol{H}^1(S)$ et $\boldsymbol{H}^2(S)$.

Avec toutes ces notations, le résultat principal du deuxième chapitre s'énonce ainsi :

Théorème 2. *Soit S une surface régulière compacte de classe C^3. Alors il existe une constante $c > 0$ telle que*

$$\left(\|\gamma(\boldsymbol{\eta})\|_{\boldsymbol{L}^2(S)} + \|\rho(\boldsymbol{\eta})\|_{\boldsymbol{L}^2(S)} \right) \geq c \|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(S)} \text{ pour tout } \dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(S),$$

où $\boldsymbol{\eta}$ est un représentant arbitraire de $\dot{\boldsymbol{\eta}}$, $\dot{\boldsymbol{V}}(S)$ est l'espace quotient défini par

$$(\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S))/\ker B,$$

et $\ker B := \{\boldsymbol{\eta} \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S);\ \gamma(\boldsymbol{\eta}) = \rho(\boldsymbol{\eta}) = 0\}$.

Une conséquence immédiate de ce théorème est que le modèle de Koiter défini ci-dessus est bien posé sur l'espace quotient $\dot{\boldsymbol{V}}(S)$.

Il est à remarquer que cet espace quotient est isomorphe à un sous-espace de codimension 6 de l'espace $\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$ et que l'espace $\ker B$ peut être caractérisé explicitement, à savoir

$$\ker B = \{S \ni p \mapsto \boldsymbol{c} + \boldsymbol{d} \wedge \overrightarrow{Op};\ \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3\}.$$

Ce résultat est obtenu à partir du lemme des déplacements rigides infinitésimaux pour des surfaces définies par une seule carte.

L'idée de la preuve du théorème ci-dessus est d'appliquer l'inégalité de Korn sans conditions au bord (le premier membre de l'inégalité de Korn usuelle (1) contient alors le terme supplémentaire $\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\omega)}$) pour chaque surface définie par une carte locale. Ces inégalités seront ensuite récrites dans un cadre fonctionnel plus général permettant d'utiliser des méthodes d'analyse fonctionnelle concernant les espaces quotient. L'inégalité de Korn sur une surface compacte est alors une conséquence du résultat d'analyse fonctionnelle suivant :

Lemme 2. *Soit* $(E, \|\cdot\|)$ *un espace de Banach, soit* $|\cdot|$ *une seminorme définie sur* E *et soit* $(\tilde{E}, \|\cdot\|_0)$ *un espace normé tel que*

(i) $E \subset \tilde{E}$,

(ii) l'inclusion $(E, \|\cdot\|) \hookrightarrow (\tilde{E}, \|\cdot\|_0)$ *est compacte,*

(iii) il existe deux constantes $c, c_0 > 0$ *telles que*

$$c|x| \leq \|x\| \leq c_0(\|x\|_0 + |x|) \text{ pour tout } x \in E.$$

Alors il existe une constante $C > 0$ *telle que*

$$\|\hat{x}\|_{E/F} \leq C|\hat{x}|_{E/F} \text{ pour tout } \hat{x} \in E/F,$$

où $F := \{x \in E\,;\ |x| = 0\}$, $\|\hat{x}\|_{E/F} := \inf_{x \in \hat{x}} \|x\|$ *et* $|\hat{x}|_{E/F} := \inf_{x \in \hat{x}} |x|$.

La preuve de ce lemme se trouve dans le corps de la thèse. Nous souhaitons indiquer ici qu'un résultat voisin, mais non équivalent, d'analyse fonctionnelle lié à la théorie de l'élasticité tridimensionnelle a été établi dans Duvaut et Lions [35].

Chapitre III : **Sur les immersions isométriques d'un espace de Riemann**

Dans ce chapitre, ainsi que dans le suivant, on utilise la convention suivante pour les classes de fonctions modulo la relation d'égalité presque partout : si $\dot{f} \in L^\infty_{\text{loc}}(\Omega)$, nous utilisons toujours le représentant f donné par

$$f(x) := \liminf_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} \widetilde{f}(y)dy,$$

où $\widetilde{f}$ est un représentant quelconque de la classe $\dot{f} \in L^\infty_{\text{loc}}(\Omega)$ (cette définition est clairement indépendante du choix du représentant $\widetilde{f}$). De même, pour $\dot{f} \in W^{1,\infty}_{\text{loc}}(\Omega)$, nous utilisons le représentant continu f de $\dot{f}$. Par souci de simplicité, la même notation est utilisée pour la classe et son représentant, la distinction devant être claire d'après le contexte.

Nous considérons dans le troisième chapitre une métrique riemannienne définie dans un ouvert $\Omega \subset \mathbb{R}^d$ par un champ de matrices symétriques définies positives (g_{ij}) d'ordre d et l'on suppose que son tenseur de courbure de Riemann s'annule, i.e., que

$$\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^l_{ik}\Gamma^p_{jl} - \Gamma^l_{ij}\Gamma^p_{kl} = 0 \text{ dans } \Omega$$

pour tout $i, j, k, p \in \{1, 2, ..., d\}$, où

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right)$$

sont les symboles de Christoffel associés à la métrique (g_{ij}).

L'objet de ce chapitre est d'établir, sous des hypothèses faibles de régularité sur la métrique, que l'espace riemannien $(\Omega; (g_{ij}))$ peut être plongé localement dans l'espace euclidien d-dimensionnel identifié à $\mathbb{R}^d$ par une immersion isométrique. Autrement dit, pour tout point $x \in \Omega$, il existe un voisinage V de x et une application $\Theta : V \to \mathbb{R}^d$ telle que

$$g_{ij}(x) = \frac{\partial \Theta(x)}{\partial x_i} \cdot \frac{\partial \Theta(x)}{\partial x_j}$$

pour tout $x = (x_1, x_2, ..., x_d) \in V$. Lorsque l'ouvert Ω est connexe et simplement connexe, il existe une immersion isométrique définie sur Ω tout entier, c'est-à-dire que $V = \Omega$ dans la définition ci-dessus. De plus, une telle immersion isométrique Θ est unique aux isométries de $\mathbb{R}^d$ près.

Dans le cas où la métrique est de classe C^2, ce résultat est classique. Nous renvoyons le lecteur à Spivak [63] pour une preuve du résultat local et à Ciarlet et Larsonneur [26] pour une preuve du résultat global. Voir aussi [9, 18, 20, 65]. Dans le cas où la métrique est seulement de classe C^1, une preuve du résultat global se trouve dans C. Mardare [49].

Le but de ce chapitre est d'établir l'existence, avec unicité aux isométries de $\mathbb{R}^d$ près, d'une immersion isométrique Θ lorsque la métrique est lipschitzienne localement. Comme le résultat local est évidemment une conséquence du résultat global, seulement ce dernier est énoncé :

Théorème 3. *Soit Ω un ouvert connexe et simplement connexe de $\mathbb{R}^d$. On considère une métrique définie sur Ω par un champ de matrices $(g_{ij}) \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{S}^d_>)$ dont le tenseur de courbure de Riemann s'annule au sens des distributions.*

(i) Alors il existe une application $\Theta \in W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^d)$, unique aux isométries de $\mathbb{R}^d$ près, telle que

$$\partial_i \Theta \cdot \partial_j \Theta = g_{ij} \text{ in } \Omega.$$

(ii) Si $(g_{ij}) \in W^{1,\infty}(\Omega; \mathbb{S}^d_>)$, $(g_{ij})^{-1} \in L^\infty(\Omega; \mathbb{M}^d)$, et le diamètre géodésique de l'ouvert Ω est fini, alors $\Theta \in W^{2,\infty}(\Omega; \mathbb{R}^d)$.

Notons que le diamètre géodésique d'un ouvert borné, connexe, simplement connexe et qui satisfait la proprieté du cône est toujours fini.

Le point central de la preuve du théorème ci-dessus est la résolution du système matriciel

$$\partial_i F = F\Gamma_i \text{ dans } \Omega,$$

où les coefficients Γ_i sont des champs de matrices définies par les symboles de Christoffel associés à la métrique (g_{ij}), i.e.,

$$\Gamma_i = \left(\Gamma_{ij}^k\right) : \Omega \to \mathbb{M}^d$$

On a affaire ici à un système de Pfaff dont les coefficients sont de classe L^∞_{loc} et satisfont les équations de compatibilité

$$\partial_i\Gamma_j - \partial_j\Gamma_i + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i = 0$$

au sens des distributions. En effet, l'on observe que l'ensemble de ces conditions est équivalent à l'annulation du tenseur de courbure de Riemann associé à la métrique définie par le champ de matrices (g_{ij}).

La résolution d'un tel système est standard lorsque les coefficients sont de classe C^1 (ou en raisonnant comme dans [49] lorsqu'ils sont de classe C^0) et est basée sur l'intégration de ces équations le long des courbes dans Ω (voir Cartan [18], Malliavin [48], Thomas [69]). Ces méthodes ne peuvent s'appliquer dans le cas où les coefficients Γ_i sont de classe L^∞, en particulier parce qu'on ne peut définir leurs restrictions à toutes les courbes de Ω. Il est à remarquer également qu'une méthode d'approximation basée sur la régularisation par convolution des coefficients Γ_i ne peut aboutir car les équations de compatibilité ci-dessus sont non linéaires.

La nouveauté par rapport à la preuve classique est résumée dans le résultat d'existence et unicité suivant pour un système de Pfaff à coefficients peu réguliers :

Lemme 3. *Soit Ω un ouvert connexe et simplement connexe de $\mathbb{R}^d$, soit x^0 un point de Ω, et soit Y^0 une matrice de $\mathbb{M}^{q,l}$. On se donne des champs de matrices $A_\alpha \in L^\infty_{\text{loc}}(\Omega; \mathbb{M}^l)$ et $B_\alpha \in L^\infty_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ tels que*

$$\partial_\alpha A_\beta + A_\alpha A_\beta = \partial_\beta A_\alpha + A_\beta A_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^l),$$
$$\partial_\alpha B_\beta + B_\alpha A_\beta = \partial_\beta B_\alpha + B_\beta A_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l})$$

pour tout $\alpha, \beta \in \{1, 2, ..., d\}$.

(i) Alors il existe une solution unique $Y \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ du système

$$\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}),$$
$$Y(x^0) = Y^0.$$

(ii) Si de plus le diamètre géodésique de l'ensemble Ω est fini et les champs de matrices A_α et B_α appartiennent respectivement aux espaces $L^\infty(\Omega; \mathbb{M}^l)$ et $L^\infty(\Omega; \mathbb{M}^{q,l})$, alors la solution Y du système ci-dessus appartient à l'espace $W^{1,\infty}(\Omega; \mathbb{M}^{q,l})$.

La démonstration de ce lemme se fait en deux étapes : D'abord, on résout localement ce système dans tout parallélépipède inclus dans Ω en intégrant succesivement les équations du système le long de certaines "lignes d'intégration admissibles". Ensuite, on recolle ces solutions locales grâce à un résultat d'unicité démontré au préalable d'une part, et au fait que l'ouvert Ω est simplement connexe d'autre part. Nous souhaitons indiquer ici que cette méthode permettant de passer du résultat local au résultat global ne demande aucune régularité sur la frontière de Ω et peut s'appliquer dans d'autres situations, en particulier pour montrer que l'ont peut déduire le résultat global classique, c'est-à-dire pour une métrique régulière, à partir du résultat local. Notons enfin que l'existence des "lignes d'intégration admissibles" mentionnées ci-dessus est assurée par un théorème de type Lebesgue-Besicovitch, énoncé ici sous la forme :

Lemme 4. *Soit $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Alors il existe un ensemble $X_d \subset \mathbb{R}$ de mesure nulle tel que*

$$f(\cdot, \overline{x}_d) \in L^1_{\mathrm{loc}}(\mathbb{R}^{d-1}) \text{ et } \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\overline{x}_d-\varepsilon}^{\overline{x}_d+\varepsilon} \int_{\omega'} \left| f(x', x_d) - f(x', \overline{x}_d) \right| dx = 0$$

pout tout ouvert borné $\omega' \subset \mathbb{R}^{d-1}$ et tout $\overline{x}_d \in \mathbb{R} \setminus X_d$.

L'idée de la preuve, utilisée dans Evans et Gariepy [37] dans un contexte différent, consiste à utiliser la séparabilité de l'espace $L^1(\mathbb{R}^{d-1})$ pour approcher la fonction $f(\cdot, \overline{x}_d)$ par une suite de fonctions q_n, puis d'appliquer le théorème de différentiation de Lebesgue-Besicovitch (voir [55, 71]) aux fonctions

$$x_d \in \mathbb{R} \mapsto \int_{\omega'} \left| f(x', x_d) - q_n(x') \right| dx'.$$

Chapitre IV : **Sur le théorème fondamental de la théorie des surfaces**

On considère un champ $(a_{\alpha\beta})$ de matrices symétriques définies positives d'ordre deux et un champ $(b_{\alpha\beta})$ de matrices symétriques d'ordre deux définis dans un ouvert connexe et simplement connexe ω de $\mathbb{R}^2$. On suppose que les fonctions $a_{\alpha\beta}$ et $b_{\alpha\beta}$ sont respectivement de classe C^2 et C^1 dans ω et qu'elles satisfont ensemble les équations de Gauss et de Codazzi-Mainardi,

$$\begin{aligned} &\partial_\gamma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\sigma\beta} = b_{\alpha\beta} b^\tau_\gamma - b_{\alpha\gamma} b^\tau_\beta, \\ &\partial_\gamma b_{\alpha\beta} - \partial_\beta b_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta} b_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma} b_{\sigma\beta} = 0. \end{aligned}$$

pour tout $\alpha, \beta, \gamma, \tau \in \{1, 2\}$. Les symboles de Christoffel $\Gamma^\tau_{\alpha\beta}$ associés à la métrique $(a_{\alpha\beta})$, ainsi que d'autre notations, sont définis dans le corps de la thèse.

Sous ces hypothèses, le théorème fondamental des surfaces (voir, e.g., Ciarlet et Larsonneur [26], Jacobowitz [42], Klingenberg [44]) affirme qu'il existe une application $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ de classe C^3 telle que les composantes covariantes des première et deuxième formes fondamentales de la surface immergée $S = \boldsymbol{\theta}(\omega)$ (ici, comme dans toute la thèse, les composantes covariantes sont associées à la carte $\boldsymbol{\theta}$) sont respectivement donnés par les champs de matrices $(a_{\alpha\beta})$ et $(b_{\alpha\beta})$, i.e.,

$$\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ et } \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} = b_{\alpha\beta} \text{ dans } \omega.$$

Ce résultat contient un “paradoxe de différentiabilité” (voir Hartmann et Wintner [40]) dans le sens que les première et deuxième formes fondamentales peuvent être définies pour toute surface S de classe C^2 (elles sont alors respectivement de classe C^1 et C^0), alors que pour reconstruire une surface à partir des première et deuxième formes fondamentales données, ces dernières doivent être respectivement de classe C^2 et C^1. Néanmoins, Hartmann et Wintner ont montré dans [41] qu'il est possible de reconstruire une surface à partir des première et deuxième formes fondamentales respectivement de classe C^1 et C^0 à condition qu'elles satisfassent ensemble les équations de Gauss et de Codazzi-Mainardi sous une forme intégrale, c'est-à-dire

$$\begin{aligned}
&\int_J (\Gamma^\tau_{\alpha 1} dy_1 + \Gamma^\tau_{\alpha 2} dy_2) = \int_{\mathrm{dom} J} (\Gamma^\sigma_{\alpha 1}\Gamma^\tau_{\sigma 2} - \Gamma^\sigma_{\alpha 2}\Gamma^\tau_{\sigma 1} - b_{\alpha 1} b^\tau_2 + b_{\alpha 2} b^\tau_1)\, dy,\\
&\int_J (b_{\alpha 1} dy_1 + b_{\alpha 2} dy_2) = \int_{\mathrm{dom} J} (\Gamma^\sigma_{\alpha 1} b_{\sigma 2} - \Gamma^\sigma_{\alpha 2} b_{\sigma 1})\, dy
\end{aligned}$$

pour tout $\alpha, \tau \in \{1, 2\}$ et toute courbe Jordan $J \subset \omega$ de classe C^1, où $\mathrm{dom} J$ désigne l'ouvert borné de $\mathbb{R}^2$ dont J est la frontière.

Le résultat principal de ce chapitre est d'établir que le théorème fondamental des surfaces reste vrai sous des hypothèses de régularité encore plus faibles. Il s'énonce ainsi :

Théorème 4. *Si les champs de matrices* $(a_{\alpha\beta}) \in W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ *et* $(b_{\alpha\beta}) \in L^\infty_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ *satisfont ensemble les équations de Gauss et de Codazzi-Mainardi, i.e., pour tout* $\alpha, \beta, \gamma, \tau \in \{1, 2\}$,

$$\begin{aligned}
&\partial_\gamma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\sigma\beta} = b_{\alpha\beta} b^\tau_\gamma - b_{\alpha\gamma} b^\tau_\beta,\\
&\partial_\gamma b_{\alpha\beta} - \partial_\beta b_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta} b_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma} b_{\sigma\beta} = 0,
\end{aligned}$$

au sens des distributions, alors il existe une application $\boldsymbol{\theta} \in W^{2,\infty}_{\mathrm{loc}}(\omega, \mathbb{R}^3)$, *unique aux isométries propres de* $\mathbb{R}^3$ *près, telle que les composantes covariantes des première et deuxième formes fondamentales de la surface immergée* $S = \boldsymbol{\theta}(\omega)$ *soient respectivement donnés par les champs de matrices* $(a_{\alpha\beta})$ *et* $(b_{\alpha\beta})$.

De plus, si le diamètre géodésique de l'ouvert ω *est fini,* $a_{\alpha\beta} \in W^{1,\infty}(\omega)$, $b_{\alpha\beta} \in L^\infty(\omega)$, *et* $(a_{\alpha\beta})^{-1} \in L^\infty(\omega, \mathbb{M}^2)$, *alors l'application* $\boldsymbol{\theta}$ *appartient à l'espace* $W^{2,\infty}(\omega, \mathbb{R}^3)$.

La démonstration de ce résultat repose d'abord sur l'observation que les équations de Gauss et de Codazzi-Mainardi sont satisfaites si et seulement si l'équation matricielle suivante est satisfaite au sens des distributions pour tout $\alpha, \beta \in \{1,2\}$:

$$\partial_\alpha \Gamma_\beta + \Gamma_\alpha \Gamma_\beta = \partial_\beta \Gamma_\alpha + \Gamma_\beta \Gamma_\alpha.$$

Les champs de matrices $\Gamma_\alpha : \omega \to \mathbb{M}^3$ sont définies par

$$\Gamma_\alpha := \begin{pmatrix} \Gamma^1_{\alpha 1} & \Gamma^1_{\alpha 2} & -b^1_\alpha \\ \Gamma^2_{\alpha 1} & \Gamma^2_{\alpha 2} & -b^2_\alpha \\ b_{\alpha 1} & b_{\alpha 2} & 0 \end{pmatrix}$$

et sont de classe $L^\infty_{\rm loc}$. Il est à remarquer que les équations de Gauss et de Codazzi-Mainardi se ramènent ainsi à un système d'équations matricielles ayant exactement la même forme que le système d'équations matricielles correspondant à l'annulation du tenseur de courbure de Riemann associé à une métrique (g_{ij}) mentionné au troisième chapitre.

Ainsi, grâce au Lemme 3 (page 15) démontrée au troisième chapitre, il existe un champ de matrices $F \in W^{1,\infty}_{\rm loc}(\omega; \mathbb{M}^3)$ satisfaisant le système de Pfaff

$$\partial_\alpha F = F\,\Gamma_\alpha.$$

Nous montrons ensuite que les deux premières colonnes du champ de matrices F sont les dérivées de l'application recherchée $\boldsymbol{\theta}$, dont l'existence est donnée par un théorème de type Poincaré. Le reste de la preuve consiste à démontrer que les composantes covariantes des première et deuxième formes fondamentales de la surface immergée $S = \boldsymbol{\theta}(\omega)$ sont effectivement les composantes des champs de matrices $(a_{\alpha\beta})$ et $(b_{\alpha\beta})$, puis que cette application $\boldsymbol{\theta}$ est unique aux isométries de $\mathbb{R}^3$ près.

Appendice

Dans cette dernière partie de la thèse, on fournit d'une part les preuves complètes de certains résultats d'analyse qui ont été utilisés dans les chapitres 3 et 4 et l'on établit d'autre part une version plus générale du Lemme 3 ci-dessus (page 15).

On établit en particulier une formule de changement de variables pour des fonctions de classe $W^{1,\infty}$ dans un ouvert de $\mathbb{R}^d$ sous des hypothèses plus faibles que celles trouvées habituellement dans la littérature (voir par exemple Brezis [14], Proposition IX.6). Ce résultat s'énonce comme suit :

Théorème 5. *Soit Ω et $\widetilde{\Omega}$ deux ouverts de $\mathbb{R}^d$, soit $f \in W^{1,\infty}(\Omega)$, et soit $G \in C^1(\widetilde{\Omega}; \mathbb{R}^d)$ une application telle que*

$$G(\widetilde{\Omega}) \subset \Omega,\ \nabla G \in L^\infty(\widetilde{\Omega}; \mathbb{M}^d) \text{ et } \mathcal{L}^d(\{x \in \widetilde{\Omega}\ ;\ J_G(x) = 0\}) = 0,$$

où les notations J_G et $\mathcal{L}^d$ désignent respectivement le Jacobien de l'application G et la mesure de Lebesgue dans $\mathbb{R}^d$. Alors $f \circ G \in W^{1,\infty}(\widetilde{\Omega})$ et

$$\partial_i(f \circ G) = \sum_{j=1}^{d} (\partial_j f \circ G)\partial_i G_j \text{ pour tout } i \in \{1, \dots, d\}\,.$$

Il est à remarquer que l'application G peut être ni injective, ni surjective. Dans la démonstration de ce résultat, on utilise le lemme suivant qui est essentiellement la réciproque du théorème de Sard.

Lemme 5. *Soit D un ouvert de $\mathbb{R}^d$ et soit $G \in C^1(D;\mathbb{R}^d)$ une application telle que la $\mathcal{L}^d$-mesure de l'ensemble $\{x \in D; J_G(x) = 0\}$ s'annule. Alors $\mathcal{L}^d(G^{-1}(A)) = 0$ pour tout ensemble $A \subset \mathbb{R}^d$ tel que $\mathcal{L}^d(A) = 0$.*

Il est à remarquer que ce résultat est implicitement utilisé lorsque l'on définit la composition $f \circ G$ entre une classe de fonctions f modulo l'égalité presque partout et une application G correspondant à un changement de variable.

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CHAPITRE 1

Sur les inégalités de Korn en coordonnées curvilignes

Ce travail a été fait en collaboration avec Philippe G. Ciarlet. Il a fait l'objet des publications suivantes :

Ciarlet P.G., Mardare S., *On Korn inequalities in curvilinear coordinates*, Math. Models Methods in Appl Sci 11, no. 8, 2001, 1379-1391.

Ciarlet P.G., Mardare S., *Sur les inégalités de Korn en coordonnées curvilignes*, C.R. Acad. Sci. Paris, sér. I, 331, 2000, 337-343.

On Korn's inequalities in curvilinear coordinates

Joint work with Philippe G. CIARLET.

1. The three-dimensional Korn inequality in curvilinear coordinates

The Euclidean inner product and the exterior product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are denoted $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ and the Euclidean norm of $\mathbf{a} \in \mathbb{R}^3$ is denoted $|\mathbf{a}|$. Latin indices and exponents vary in the set $\{1, 2, 3\}$. A *domain* in $\mathbb{R}^3$ is a bounded, open and connected subset of $\mathbb{R}^3$, with a Lipschitz-continuous boundary Γ, the set Ω being locally on one side of Γ. The norms in $L^2(\Omega)$ and $H^1(\Omega)$ are denoted $|\cdot|_{0,\Omega}$ and $\|\cdot\|_{1,\Omega}$. If $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega) = (H^1(\Omega))^3$, then $\|\mathbf{v}\|_{1,\Omega} := \{\sum_i \|v_i\|_{1,\Omega}^2\}^{1/2}$.

It is well known that the *three-dimensional Korn inequality* plays a fundamental rôle in three-dimensional linearized elasticity. While it is usually established in *Cartesian coordinates* (see e.g. Duvaut and Lions [18]), it can also be directly established in *curvilinear coordinates*, as shown by Ciarlet [9] (see also Theorem 1.7-4 of Ciarlet [11]). It takes then the following form:

Theorem 1.1. *Let Ω be a domain in $\mathbb{R}^3$ with boundary Γ, and let Γ_0 be a measurable subset of Γ, with* $\operatorname{area}\Gamma_0 > 0$*. Define the space*

$$\mathbf{V}(\Omega) := \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}.$$

Let $\mathbf{\Theta} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3)$ *be a* $\mathcal{C}^2$*-diffeomorphism of* $\overline{\Omega}$ *onto its image* $\mathbf{\Theta}(\overline{\Omega})$*. Then there exists a constant* $C = C(\Omega, \Gamma_0, \mathbf{\Theta})$ *such that*

$$\|\mathbf{v}\|_{1,\Omega} \leq C \left\{ \sum_{i,j} |e_{i\|j}(\mathbf{v})|^2_{0,\Omega} \right\}^{1/2}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$*, where*

$$e_{i\|j}(\mathbf{v}) := \frac{1}{2}(v_{i\|j} + v_{j\|i}).$$

A generic point in $\overline{\Omega}$ being denoted $x = (x_i)$, we let $\partial_i := \partial/\partial x_i$. We recall that the *covariant derivatives* $v_{i\|j}$ and the *Christoffel symbols* Γ^p_{ij} associated with the mapping $\mathbf{\Theta} : \overline{\Omega} \to \mathbb{R}^3$ are defined by

$$v_{i\|j} := \partial_j v_i - \Gamma^p_{ij} v_p = \partial_j(v_k \mathbf{g}^k) \cdot \mathbf{g}_i \ \text{ and } \ \Gamma_{ij} := \mathbf{g}^p \cdot \partial_i \mathbf{g}_j,$$

where

$$\mathbf{g}_i := \partial_i \mathbf{\Theta}$$

and the vectors $\mathbf{g}^j$ are defined by the relations

$$\mathbf{g}^j \cdot \mathbf{g}_i = \delta^j_i.$$

For details about these classical notations, see e.g. Sec. 1.4. of Ciarlet [11].

The *three-dimensional Korn inequality "in curvilinear coordinates"* of Theorem 1.1 thus displays *one* tensor, the *linearized change of metric tensor* associated with a displacement field $v_i \mathbf{g}^i$ of the set $\mathbf{\Theta}(\overline{\Omega})$. This tensor is represented here by means of its *covariant components* $e_{i\|j}(\mathbf{v})$ expressed as functions of the *curvilinear coordinates* of the set $\mathbf{\Theta}(\overline{\Omega})$, i.e. the coordinates x_i of the points $x \in \overline{\Omega}$. The interpretation of this tensor, as the linearized part "around $\mathbf{v} = \mathbf{0}$" of the "full" change of metric tensor associated with the field $v_i \mathbf{g}^i$, is given in Theorem 2.4-1 of Ciarlet [11].

This inequality can be established either from the "classical" three-dimensional inequality *in Cartesian coordinates* (see Exercise 1.10 of Ciarlet [11]), or directly in *curvilinear coordinates.*

Let us briefly sketch the latter approach (for details, see Sec. 1.7 of Ciarlet [11]), which is essentially an extension of the proof of Duvaut and Lions [18] in Cartesian coordinates.

First, the following Korn *inequality "without boundary conditions"*, i.e. valid for *all* vector fields $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$, is established: *There exists a constant* $C_0 = C_0(\Omega, \mathbf{\Theta})$ *such that*

$$\|\mathbf{v}\|_{1,\Omega} \leq C_0 \left\{ \sum_i |v_i|^2_{0,\Omega} + \sum_{i,j} |e_{i\|j}(\mathbf{v})|^2_{0,\Omega} \right\}^{1/2}$$

for all $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$*.* This inequality is essentially a consequence of a fundamental *lemma of J.L. Lions*, which states that, if a distribution $v \in H^{-1}(\Omega)$ is such that all its first order partial derivatives are also in

$H^{-1}(\Omega)$, then v is in $L^2(\Omega)$ (that this implication in due to J.L. Lions was first mentioned in Magenes and Stampacchia [23] p. 320, Note (27); its first published proof is in Duvaut and Lions [18] for domains with smooth boundaries; various extensions to domains with Lipschitz-continuous boundaries have then been given in Amrouche and Girault [1], Bolley and Camus [5], Borchers and Sohr [6], and Geymonat and Suquet [19]).

Second, a *linearized rigid displacement lemma in curvilinear coordinates* is established, to the effect that the *semi-norm*

$$\mathbf{v} \mapsto \left\{ \sum_{i,j} |e_{i\|j}(\mathbf{v})|^2_{0,\Omega} \right\}^{1/2}$$

becomes a *norm* on the space $\mathbf{V}(\Omega)$.

Third, the Korn inequality of Theorem 1.1 is established by contradiction, as a consequence of the above Korn inequality "without boundary conditions" and linearized rigid displacement lemma.

2. The inequality of Korn's type on a surface

Greek indices (except ν in ∂_ν) and exponents take theirs values in the set $\{1,2\}$. The *inequality of Korn's type on a surface* was first established in [2], then given a simpler proof (briefly sketched below) by Ciarlet and Miara [15]; see also Bernadou, Ciarlet and Miara [3]. This inequality, which plays a fundamental rôle in the mathematical analysis of the *linear Koiter shell equations*, so named after Koiter [21], and more generally in the *asymptotic analysis of linearly elastic shells* (see Chaps. 4-7 of Ciarlet [11], takes the following form ($|\cdot|_{0,\omega}$, $\|\cdot\|_{1,\omega}$, and $\|\cdot\|_{2,\omega}$ designate the norms of the spaces $L^2(\omega)$, $H^1(\omega)$, and $H^2(\omega)$):

Theorem 1.2. *Let ω be a domain in $\mathbb{R}^2$, i.e. a bounded, connected, open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, the set ω being locally on one side of γ, and let γ_0 be a measurable subset of γ, with* length$\gamma_0 > 0$. *Define the space*

$$\mathbf{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \ \text{on } \gamma_0\}.$$

Let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points in $\overline{\omega}$. Then there exists a constant $c = c(\omega, \gamma_0, \boldsymbol{\theta})$ such that

$$\left\{ \sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + |\eta_3\|^2_{2,\omega} \right\}^{1/2} \le c \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} \right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$, where

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3,$$
$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := \eta_{3|\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha \eta_{\sigma|\beta} + b^\tau_\beta \eta_{\tau|\alpha} + b^\tau_\beta|_\alpha \eta_\tau,$$

and $b_{\alpha\beta}$ and b_α^σ respectively designate the covariant and mixed components of the second fundamental form of the surface $\boldsymbol{\theta}(\overline{\omega})$.

A generic point in $\overline{\omega}$ being denoted $y = (y_\alpha)$, we let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$. We recall that the *covariant derivatives* $\eta_{\alpha|\beta}$, $\eta_{3|\alpha\beta}$, $b_\beta^\tau|_\alpha$, the *Christoffel symbols* $C_{\alpha\beta}^\sigma$, and the functions $b_{\alpha\beta}$ and b_α^σ, associated with the mapping $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ are given by

$$\eta_{\alpha|\beta} := \partial_\beta \eta_\alpha - C_{\alpha\beta}^\sigma \eta_\sigma, \quad \eta_{3|\alpha\beta} := \partial_{\alpha\beta}\eta_3 - C_{\alpha\beta}^\sigma \partial_\sigma \eta_3,$$
$$b_\beta^\tau|_\alpha := \partial_\alpha b_\beta^\tau + C_{\alpha\sigma}^\tau b_\beta^\sigma - C_{\alpha\beta}^\sigma b_\sigma^\tau, \quad C_{\alpha\beta}^\sigma := \mathbf{a}^\sigma \cdot \partial_\alpha \mathbf{a}_\beta,$$
$$b_{\alpha\beta} := \mathbf{a}^3 \cdot \partial_\alpha \mathbf{a}_\beta \quad \text{and} \quad b_\alpha^\sigma := a^{\beta\sigma} b_{\alpha\beta},$$

where

$$\mathbf{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \quad \text{and} \quad \mathbf{a}_3 := \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|},$$

and the vectors $\mathbf{a}^i$ are defined by the relations

$$\mathbf{a}^i \cdot \mathbf{a}_j = \delta_j^i.$$

For more details on the differential geometry of surfaces, see e.g. do Carmo [7], Klingenberg [20] or Chap. 2 of Ciarlet [11].

The *inequality of Korn's type on a surface* of Theorem 1.2 thus displays *two* tensors, the *linearized change of metric tensor* and the *linearized change of curvature tensor*, both associated with a displacement field $\eta_i \mathbf{a}^i$ of the surface $\boldsymbol{\theta}(\overline{\omega})$. These tensors are represented here by means of their *covariant components* $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ expressed as functions of the *curvilinear coordinates* of the surface $\boldsymbol{\theta}(\overline{\omega})$, i.e. the coordinates y_α of the points $y \in \overline{\omega}$. The interpretation of these tensors, as the linearized part "around $\boldsymbol{\eta} = \mathbf{0}$" of the "full" change of metric and change of curvature tensors associated with the field $\eta_i \mathbf{a}^i$, is given in Theorems 2.4-1 and 2.5-1 of Ciarlet [11].

As shown by Ciarlet and Miara [15] and Bernadou, Ciarlet and Miara [3] (see also Sec. 2.6 of Ciarlet [11]), this inequality can be established by a proof (only briefly sketched here) that bears *strong resemblance* in its *principle* to that of Theorem 1.1:

First, the following Korn *inequality of Korn's type "without boundary conditions" on a surface* is established, again as a consequence of the same *lemma of J.L. Lions* as in Sec. 1: There exists a constant $c_0 = c_0(\omega, \boldsymbol{\theta})$ such that

$$\left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2 \right\}^{1/2}$$
$$\leq c_0 \left\{ \sum_\alpha |\eta_\alpha|_{0,\omega}^2 + \|\eta_3\|_{1,\omega}^2 + \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

Second, a *linearized rigid displacement lemma on a surface* is established, to the effect that the *semi-norm*

$$\boldsymbol{\eta} = (\eta_i) \mapsto \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} \right\}^{1/2}$$

becomes a *norm* on the space $\mathbf{V}_K(\omega)$.

Third, the inequality of Korn's type of Theorem 1.2 is established by contradiction, as a consequence of the above inequality of Korn's type "without boundary conditions" and linearized rigid displacement lemma.

Remark 1.1. (1) Blouza and Le Dret [4] have shown how the *regularity* assumption made in Theorem 1.2 on the mapping $\boldsymbol{\theta}$ can be weakened to $\boldsymbol{\theta} \in W^{2,\infty}(\omega; \mathbb{R}^3)$, provided the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are replaced by *ad hoc* extensions. They obtain in this fashion an inequality of Korn's type *on a surface "with little regularity"*, while that considered here is an inequality of Korn's type *on a "smooth" surface* $\boldsymbol{\theta}(\overline{\omega})$, in that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$.

(2) The inequality of Korn's type of Theorem 1.2 holds on a surface $S = \boldsymbol{\theta}(\overline{\omega})$ having an "arbitrary geometry", i.e. for any mapping $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ satisfying the assumption of this theorem. If the surface S is "elliptic", i.e. the Gaussian curvature is everywhere > 0, *another* inequality of Korn's type also holds; cf. Ciarlet and Lods [13] and Ciarlet and Sanchez-Palencia [16].

3. The inequality of Korn's type on a surface as a consequence of the three-dimensional Korn inequality in curvilinear coordinates

The objective of this paper is to show how the strong analogies observed between the three-dimensional inequality and the inequality of Korn's type on a surface, and between their proof, can be rigorously substantiated. To this end, we establish the following result, which has been announced in Ciarlet and Mardare [14]:

Theorem 1.3. *The inequality of Korn's type on a surface (Theorem 1.2) may be established as a consequence of the three-dimensional Korn inequality in curvilinear coordinates (Theorem 1.1) for ad hoc choices of the set* Ω*, mapping* $\boldsymbol{\Theta}$*, and fields* $\boldsymbol{v}$*.*

In other words, Theorem 1.2 is a corollary of Theorem 1.1.

For clarity, the proof of Theorem 1.3 is established by means of a series of six lemmas. Lemmas 1.1 and 1.2 and their proofs can be found in Theorems 3.1-1 and 2.6-2 of Ciarlet [11]; they are nevertheless reproduced here for the sake of completeness.

The summation convention with respect to repeated indices or exponents (Latin or Greek) is systematically used.

We are thus given once and for all a domain $\omega \subset \mathbb{R}^2$, a subset γ_0 of γ with *length*$\gamma_0 > 0$, and an injective mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ such that the two vectors $\mathbf{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points in $\overline{\omega}$.

Lemma 1.1. *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\mathbf{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$, and let $\mathbf{a}_3 = (\mathbf{a}_1 \wedge \mathbf{a}_2)/|\mathbf{a}_1 \wedge \mathbf{a}_2|$. Then there exists $\varepsilon_0 > 0$ such that the mapping $\boldsymbol{\Theta} : \overline{\Omega}_0 \to \mathbb{R}^3$ defined by*

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \mathbf{a}_3(y) \quad \text{for all } (y, x_3) \in \overline{\Omega}_0,$$

where $\Omega_0 := \omega \times]-\varepsilon_0, \varepsilon_0[$, is a $\mathcal{C}^2$-diffeomorphism from $\overline{\Omega}_0$ onto $\boldsymbol{\Theta}(\overline{\Omega}_0)$ and $\det(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) > 0$ in $\overline{\Omega}_0$, where $\mathbf{g}_i := \partial_i \boldsymbol{\Theta}$ (we let $\partial_i = \partial/\partial x_i$, where $x = (x_i)$ with $x_\alpha = y_\alpha$ denotes a generic point in the set $\overline{\Omega}_0$).

Proof. The assumed regularity on $\boldsymbol{\theta}$ implies that $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\omega} \times [-\varepsilon, \varepsilon];\mathbb{R}^3)$ for any $\varepsilon > 0$. The relations

$$\mathbf{g}_\alpha = \partial_\alpha \boldsymbol{\Theta} = \mathbf{a}_\alpha + x_3 \partial_\alpha \mathbf{a}_3 \quad \text{and} \quad \mathbf{g}_3 = \partial_3 \boldsymbol{\Theta} = \mathbf{a}_3$$

imply that

$$\det(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)|_{x_3=0} = \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) > 0 \text{ in } \overline{\omega}.$$

Hence it is clear that $\det(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) > 0$ on $\overline{\omega} \times [-\varepsilon, \varepsilon]$ if $\varepsilon > 0$ is small enough.

Therefore, the *implicit function theorem* can be applied: It shows that, locally, the mapping $\boldsymbol{\Theta}$ is a $\mathcal{C}^2$-diffeomorphism: Given any $y \in \overline{\omega}$, there exist a neighborhood $U(y)$ of y in $\overline{\omega}$ and $\varepsilon(y) > 0$ such that $\boldsymbol{\Theta}$ is a $\mathcal{C}^2$-diffeomorphism from the set $U(y) \times [-\varepsilon(y), \varepsilon(y)]$ onto $\boldsymbol{\Theta}(U(y) \times [-\varepsilon(y), \varepsilon(y)])$. See e.g. Chap 3, Sec. 8 of Schwartz [24]; the proof of the implicit function theorem, which is almost invariably given for functions defined over open sets, can be easily extended to functions defined over *closures of domains*, such as the sets $\overline{\omega} \times [-\varepsilon, \varepsilon]$ (the definition of differentiable functions over the closure of a domain poses no difficulty; see e.g. Stein [25]).

To establish that the mapping $\boldsymbol{\Theta} : \overline{\omega} \times [-\varepsilon, \varepsilon] \to \mathbb{R}^3$ is injective provided $\varepsilon > 0$ is small enough, we proceed by contradiction: If this property is false, there exist $\varepsilon_n > 0$, (y^n, x_3^n), and $(\widetilde{y}^n, \widetilde{x}_3^n)$, $n \in \mathbb{N}$ such that

$$\varepsilon^n \to 0 \text{ as } n \to \infty, \quad y^n \in \overline{\omega}, \quad \widetilde{y}^n \in \overline{\omega}, \quad |x_3^n| \leq \varepsilon_n, \quad |\widetilde{x}_3^n| \leq \varepsilon_n,$$

$$(y^n, x_3^n) \neq (\widetilde{y}^n, \widetilde{x}_3^n) \quad \text{and} \quad \boldsymbol{\Theta}(y^n, x_3^n) = \boldsymbol{\Theta}(\widetilde{y}^n, \widetilde{x}_3^n).$$

Since the set $\overline{\omega}$ is compact, there exist $y \in \overline{\omega}$, $\widetilde{y} \in \overline{\omega}$, and a subsequence, still indexed by n for convenience, such that

$$y^n \to y, \quad \widetilde{y}^n \to \widetilde{y}, \quad , x_3^n \to 0, \quad , \widetilde{x}_3^n \to 0$$

as $n \to \infty$. Hence

$$\boldsymbol{\theta}(y) = \lim_{n\to\infty} \boldsymbol{\Theta}(y^n, x_3^n) = \lim_{n\to\infty} \boldsymbol{\Theta}(\widetilde{y}^n, \widetilde{x}_3^n) = \boldsymbol{\theta}(\widetilde{y}),$$

by the continuity of the mapping $\boldsymbol{\Theta}$ and thus $y = \widetilde{y}$ since the mapping $\boldsymbol{\theta}$ is injective by assumption. But these properties contradict the local injectivity (established *supra*) of the mapping $\boldsymbol{\Theta}$. Hence there exists $\varepsilon_0 > 0$ such that $\boldsymbol{\Theta}$ is injective on the set $\overline{\Omega}_0 = \overline{\omega} \times [-\varepsilon_0, \varepsilon_0]$. $\square$

The following result is due to Chapelle [8]. It shows that a vector field $\eta_i \mathbf{a}^i$ on a surface may be "canonically" extended to a three-dimensional vector field $v_i \mathbf{g}^i$ in such a way that all the components $e_{i\|j}(\mathbf{v})$ of the associated "three-dimensional" linearized change of metric tensor have remarkable expressions in terms of the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ of the linearized change of metric and curvature tensors of the surface vector field:

Lemma 1.2. *Let the assumptions on the mapping $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$, the open set $\Omega_0 = \omega \times]-\varepsilon_0, \varepsilon_0[$, and the $\mathcal{C}^2$-diffeomorphism $\boldsymbol{\Theta} : \overline{\Omega}_0 \to \mathbb{R}^3$, be as in Lemma 1.1. In particular then, the three vectors $\mathbf{g}_i := \partial_i \boldsymbol{\Theta}$ are linearly independent at all points of $\overline{\Omega}_0$. With any vector field $\boldsymbol{\eta} = (\eta_i)$ in the space*

$$\mathbf{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

let there be associated the vector field $\mathbf{v} = (v_i) : \overline{\Omega}_0 \to \mathbb{R}^3$ defined on $\overline{\Omega}_0$ by

$$v_i(y, x_3)\mathbf{g}^i(y, x_3) = \eta_i(y)\mathbf{a}^i(y) - x_3(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma)\mathbf{a}^\alpha(y)$$

for all $(y, x_3) \in \overline{\Omega}_0$, where the vectors $\mathbf{g}^i$ form the contravariant basis associated with the vectors $\mathbf{g}_i$ (i.e. $\mathbf{g}^i \cdot \mathbf{g}_j = \delta_j^i$). Then

$$\mathbf{v} \in \mathbf{V}(\Omega_0) := \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega_0);\ \mathbf{v} = \mathbf{0} \text{ on } \gamma_0 \times [-\varepsilon_0, \varepsilon_0]\},$$

so that the above relation defines a linear mapping

$$\mathbf{F} : \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega) \to \mathbf{F}(\boldsymbol{\eta}) := \mathbf{v} \in \mathbf{V}(\Omega_0).$$

The covariant components $e_{i\|j}(\mathbf{v}) := (v_{i\|j} + v_{j\|i})/2$ of the associated "three-dimensional" linearized change of metric tensor (Sec. 1) *are then given by*

$$\begin{aligned} e_{\alpha\|\beta}(\mathbf{v}) =& \gamma_{\alpha\beta}(\boldsymbol{\eta}) - x_3 \rho_{\alpha\beta}(\boldsymbol{\eta}) \\ &+ \frac{x_3^2}{2}\{b_\alpha^\sigma \rho_{\beta\sigma}(\boldsymbol{\eta}) + b_\beta^\tau \rho_{\alpha\tau}(\boldsymbol{\eta}) - 2 b_\alpha^\sigma b_\beta^\tau \gamma_{\sigma\tau}(\boldsymbol{\eta})\}, \\ e_{i\|3}(\mathbf{v}) =& 0. \end{aligned}$$

Proof. As in the above expressions of the functions $e_{\alpha\|\beta}(\mathbf{v})$, the dependence on x_3 is explicit, but the dependence with respect to $y \in \overline{\omega}$ is omitted, throughout the proof.

(i) Given functions $\eta_\alpha, \chi_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$, let the vector field $v_i \mathbf{g}^i$ be defined on $\overline{\Omega}_0$ by

$$v_i \mathbf{g}^i = \eta_i \mathbf{a}^i + x_3 \chi_\alpha \mathbf{a}^\alpha.$$

Then the functions v_i are in $H^1(\overline{\Omega}_0)$ and the covariant components $e_{i\|j}(\mathbf{v})$ of the linearized change of metric tensor associated with the field $v_i \mathbf{g}^i$ are given by

$$\begin{aligned} e_{\alpha\|\beta}(\mathbf{v}) =& \frac{1}{2}(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3 \\ &+ \frac{x_3}{2}\{\chi_{\alpha|\beta} + \chi_{\beta|\alpha} - b_\alpha^\sigma(\eta_{\sigma|\beta} - b_{\beta\sigma}\eta_3) - b_\beta^\tau(\eta_{\tau|\alpha} - b_{\alpha\tau}\eta_3)\} \\ &+ \frac{x_3^2}{2}\{-b_\alpha^\sigma\chi_{\sigma|\beta} - b_\beta^\tau\chi_{\tau|\alpha}\}, \\ e_{\alpha\|3}(\mathbf{v}) =& \frac{1}{2}(\chi_\alpha + \partial_\alpha\eta_3 + b_\alpha^\sigma\eta_\sigma), \\ e_{3\|3}(\mathbf{v}) =& 0, \end{aligned}$$

where $\eta_{\alpha|\beta} = \partial_\beta\eta_\alpha - C_{\alpha\beta}^\sigma\eta_\sigma$ and $\chi_{\alpha|\beta} = \partial_\beta\chi_\alpha - C_{\alpha\beta}^\sigma\chi_\sigma$.

Since

$$\partial_\alpha\mathbf{a}_3 = -b_\alpha^\sigma\mathbf{a}_\sigma$$

by the second formula of Weingarten, the vectors of the covariant basis associated with the mapping $\boldsymbol{\Theta} = \boldsymbol{\theta} + x_3\mathbf{a}_3$ are given by

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha - x_3 b_\alpha^\sigma\mathbf{a}_\sigma \quad \text{and} \quad \mathbf{g}_3 = \mathbf{a}_3.$$

Since

$$v_i = (v_j\mathbf{g}^j)\cdot\mathbf{g}_i = (\eta_j\mathbf{a}^j + x_3\chi_\alpha\mathbf{a}^\alpha)\cdot\mathbf{g}_i,$$

the assumed regularities of the functions η_i and χ_α imply that $v_i \in H^1(\Omega_0)$, since $\mathbf{g}_i \in \mathbf{C}^2(\overline{\Omega}_0)$. The announced expressions for the functions $e_{i\|j}(\mathbf{v})$ are obtained by simple computations, based on the relations

$$v_{i\|j} = \{\partial_j(v_k\mathbf{g}^k)\}\cdot\mathbf{g}_i \quad \text{and} \quad e_{i\|j}(\mathbf{v}) = \frac{1}{2}(v_{i\|j} + v_{j\|i}).$$

(ii) When $\chi_\alpha = -(\partial_\alpha\eta_3 + b_\alpha^\sigma\eta_\sigma)$, the functions $e_{i\|j}(\mathbf{v})$ in (i) take the expressions announced in the statement of the theorem.

We first note that $\chi_\alpha \in H^1(\omega)$ (since $b_\alpha^\sigma \in \mathcal{C}^1(\overline{\omega})$) and that $e_{\alpha\|3}(\mathbf{v}) = 0$ in this case. It thus remains to find the explicit forms of the functions $e_{\alpha\|\beta}(\mathbf{v})$. Replacing the functions χ_α by their expressions and using the *Codazzi-Mainardi identities* $b_\alpha^\sigma|_\beta = b_\beta^\sigma|_\alpha$, we find that

$$\begin{aligned} &\frac{1}{2}\{\chi_{\alpha|\beta} + \chi_{\beta|\alpha} - b_\alpha^\sigma(\eta_{\sigma|\beta} - b_{\beta\sigma}\eta_3) - b_\beta^\tau(\eta_{\tau|\alpha} - b_{\alpha\tau}\eta_3)\} \\ &\quad = -\eta_{3|\alpha\beta} - b_\alpha^\sigma\eta_{\sigma|\beta} - b_\beta^\tau\eta_{\tau|\alpha} - b_\beta^\tau|_\alpha\eta_\tau + b_\alpha^\sigma b_{\sigma\beta}\eta_3. \end{aligned}$$

Hence the factor of x_3 in $e_{\alpha\|\beta}(\mathbf{v})$ is equal to $-\rho_{\alpha\beta}(\boldsymbol{\eta})$. Finally,

$$\begin{aligned} &- b_\alpha^\sigma\chi_{\sigma|\beta} - b_\beta^\tau\chi_{\tau|\alpha} \\ &\quad = b_\alpha^\sigma(\eta_{3|\beta\sigma} + b_\sigma^\tau|_\beta\eta_\tau + b_\sigma^\tau\beta\eta_{\tau|\beta}) + b_\beta^\tau(\eta_{3|\alpha\tau} + b_\tau^\sigma|_\alpha\eta_\sigma + b_\tau^\sigma\beta\eta_{\sigma|\alpha}) \\ &\quad = b_\alpha^\sigma(\rho_{\beta\sigma}(\boldsymbol{\eta}) - b_\beta^\tau\eta_{\tau|\sigma} + b_\beta^\tau b_{\tau\sigma}\eta_3) + b_\beta^\tau(\rho_{\alpha\tau}(\boldsymbol{\eta}) - b_\alpha^\sigma\eta_{\sigma|\tau} + b_\alpha^\sigma b_{\sigma\tau}\eta_3) \\ &\quad = b_\alpha^\sigma\rho_{\beta\sigma}(\boldsymbol{\eta}) + b_\beta^\tau\rho_{\alpha\tau}(\boldsymbol{\eta}) - 2b_\alpha^\sigma b_\beta^\tau\gamma_{\sigma\tau}(\boldsymbol{\eta}), \end{aligned}$$

i.e. the factor of $x_3^2/2$ in $e_{\alpha\|\beta}(\mathbf{v})$ is that announced in the theorem.

(iii) If $\eta_i = \partial_\nu \eta_3 = 0$ on γ_0, the functions $\chi_\alpha = -(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma)$ vanish on γ_0, since $\eta_3 = \partial_\nu \eta_3 = 0$ on γ_0 implies $\partial_\alpha \eta_3 = 0$ on γ_0; consequently,

$$v_i = (v_j \mathbf{g}^j) \cdot \mathbf{g}_i = (\eta_j \mathbf{a}^j + x_3 \chi_\alpha \mathbf{a}^\alpha) \cdot \mathbf{g}_i = 0 \text{ on } \gamma_0 \times [-\varepsilon_0, \varepsilon_0].$$

□

The rest of the proof essentially consists of studying the properties of the mapping $\mathbf{F}$ defined in Lemma 1.2. As the proof of this lemma, the explicit dependence on $y \in \overline{\omega}$ is henceforth omitted whenever no confusion should arise.

Lemma 1.3. *Let the spaces* $\mathbf{V}_K(\omega)$ *and* $\mathbf{V}(\Omega_0)$ *be respectively equipped with their product norms, viz.,*

$$\boldsymbol{\eta} = (\eta_i) \to \left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2 \right\}^{1/2} \text{ and } \mathbf{v} = (v_i) \to \left\{ \sum_i \|v_i\|_{1,\Omega_0}^2 \right\}^{1/2}.$$

Then the linear mapping $\mathbf{F} : \mathbf{V}_K(\omega) \to \mathbf{V}(\Omega_0)$ *defined in Lemma 1.2 is continuous.*

Proof. Given $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$, let $(v_i) = \mathbf{F}(\boldsymbol{\eta})$. A simple computation, based on the relations

$$v_i = (v_j \mathbf{g}^j) \cdot \mathbf{g}_i, \quad \mathbf{g}_\alpha = \mathbf{a}_\alpha - x_3 b_\alpha^\sigma \mathbf{a}_\sigma, \quad \mathbf{g}_3 = \mathbf{a}_3,$$

shows that

$$v_\alpha = \eta_\alpha - x_3(\partial_\alpha \eta_3 + 2 b_\alpha^\sigma \eta_\sigma) + x_3^2 (b_\alpha^\beta \partial_\beta \eta_3 + b_\alpha^\sigma b_\sigma^\beta \eta_\beta) \text{ and } v_3 = \eta_3.$$

The continuity of the mapping $\mathbf{F}$ easily follows from these relations (note that $b_\alpha^\beta \in \mathcal{C}^1(\overline{\omega})$). □

Lemma 1.4. *There exists* ε_1 *satisfying* $0 < \varepsilon_1 \le \varepsilon_0$ *such that the mapping*

$$\mathbf{F} : \mathbf{V}_K(\omega) \to \mathbf{V}(\Omega) := \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \gamma_0 \times [-\varepsilon_1, \varepsilon_1]\},$$

where

$$\Omega := \omega \times] - \varepsilon_1, \varepsilon_1[,$$

is injective.

Proof. Let $\boldsymbol{\eta} = (\eta_i)$ be such that $\mathbf{F}(\boldsymbol{\eta}) = \mathbf{0}$. Then the computation from the preceding proof shows that

$$\eta_3 = 0 \quad \text{and} \quad \eta_\alpha - (2 x_3 b_\alpha^\beta - x_3^2 b_\alpha^\sigma b_\sigma^\beta) \eta_\beta = 0.$$

Clearly, there exists ε_1 such that the matrix (of order two) with elements $(\delta_\alpha^\beta - 2 x_3 b_\alpha^\beta + x_3^2 b_\alpha^\sigma b_\sigma^\beta)$ is invertible for all $|x_3| \le \varepsilon_1$, in which case $\eta_\alpha = 0$. Evidently, there is no loss of generality in assuming that $\varepsilon_1 \le \varepsilon_0$. □

Lemma 1.5. *The image* $\operatorname{Im} \mathbf{F}$ *of the mapping* $\mathbf{F} : \mathbf{V}_K(\omega) \to \mathbf{V}(\Omega)$ *is the space*

$$\mathbf{V}_{KL}(\Omega) := \{\mathbf{v} \in \mathbf{V}(\Omega); \quad e_{i\|3}(\mathbf{v}) = 0 \text{ in } \Omega\}.$$

Proof. The computations used in this proof rely on the following easily proved observations:

(i) If $\theta \in H^{-1}(\Omega)$ and $g \in \mathcal{C}^1(\overline{\Omega})$, the product θg is well-defined as a distribution in $H^{-1}(\Omega)$.

(ii) If $w \in L^2(\Omega)$ and $g \in \mathcal{C}^1(\overline{\Omega})$, then $\partial_i(wg) = (\partial_i w)g + w(\partial_i g)$, as an equality in $H^{-1}(\Omega)$.

(iii) Let $\mathbf{e}^k$ denote the vectors of the Cartesian basis in $\mathbb{R}^3$ and let $\mathbf{a} = [\mathbf{a}]_k \mathbf{e}^k$ denote the expansion of a vector $\mathbf{a} \in \mathbb{R}^3$ over this basis. Any distribution $\mathbf{z} = [\mathbf{z}]_k \mathbf{e}^k \in (H^{-1}(\Omega))^3$ can be expanded as $\mathbf{z} = z_j \mathbf{g}^j$ over the vectors $\mathbf{g}^j$ of the contravariant bases, the components $z_j \in H^{-1}(\Omega)$ being given by $z_j = \mathbf{z} \cdot \mathbf{g}_j$; equivalently, $[\mathbf{z}]_k = z_j [\mathbf{g}^j]_k$, where $z_j = [\mathbf{z}]_l [\mathbf{g}_j]_l$ (these equalities make sense by (i)).

(iv) Let $\mathbf{z} \in (H^{-1}(\Omega))^3$ be such that $\mathbf{z} \cdot \mathbf{g}_i = 0$ for all i. Then $\mathbf{z} = \mathbf{0}$.

Returning to the proof of the lemma *per se*, we observe that it suffices to establish the inclusion $\mathbf{V}_{KL} \subset \operatorname{Im} \mathbf{F}$, since the other inclusion holds by Lemma 1.2. Let then $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$ satisfy $e_{i\|3}(\mathbf{v}) = 0$ in Ω and $\mathbf{v} = \mathbf{0}$ on Γ_0.

Noticing that $\mathbf{g}_3 = \mathbf{a}_3$ is independent of x_3 and letting $\widetilde{\mathbf{v}} := v_i \mathbf{g}^i$, we obtain

$$e_{3\|3}(\mathbf{v}) = (\partial_3 \widetilde{\mathbf{v}}) \cdot \mathbf{g}_3 = \partial_3(\widetilde{\mathbf{v}} \cdot \mathbf{g}_3) = 0.$$

Hence $v_3 = (\widetilde{\mathbf{v}} \cdot \mathbf{g}_3)$ is independent of x_3, and

$$(\partial_{33} \widetilde{\mathbf{v}}) \cdot \mathbf{g}_3 = \partial_3(\partial_3(\widetilde{\mathbf{v}} \cdot \mathbf{g}_3)) = 0,$$

as an equality in $H^{-1}(\Omega)$. Next, the relations

$$2e_{\alpha\|3}(\mathbf{v}) = (\partial_\alpha \widetilde{\mathbf{v}}) \cdot \mathbf{g}_3 + (\partial_3 \widetilde{\mathbf{v}}) \cdot \mathbf{g}_\alpha = 0$$

imply that ($\partial_3 \mathbf{g}_\alpha = -b_\alpha^\sigma \mathbf{a}_\sigma$ since $\mathbf{g}_\alpha = \mathbf{a}_\alpha - x_3 b_\alpha^\sigma \mathbf{a}_\sigma$):

$$2\partial_3 e_{\alpha\|3}(\mathbf{v}) = (\partial_{\alpha 3} \widetilde{\mathbf{v}}) \cdot \mathbf{a}_3 + (\partial_{33} \widetilde{\mathbf{v}}) \cdot \mathbf{g}_\alpha - b_\alpha^\sigma (\partial_3 \widetilde{\mathbf{v}}) \cdot \mathbf{a}_\sigma = 0,$$

hence that (by the second formula of Weingarten, $-\partial_\alpha \mathbf{a}_3 = b_\alpha^\sigma \mathbf{a}_\sigma$)

$$(\partial_{33} \widetilde{\mathbf{v}}) \cdot \mathbf{g}_\alpha = -(\partial_{\alpha 3} \widetilde{\mathbf{v}}) \cdot \mathbf{a}_3 - (\partial_3 \widetilde{\mathbf{v}}) \cdot \partial_\alpha \mathbf{a}_3 = -\partial_\alpha(\partial_3 \widetilde{\mathbf{v}} \cdot \mathbf{a}_3) = 0.$$

The three relations $(\partial_{33} \widetilde{\mathbf{v}}) \cdot \mathbf{g}_i = 0$ thus imply that the field $\widetilde{\mathbf{v}} \in \mathbf{H}^1(\Omega)$ satisfies

$$\partial_{33} \widetilde{\mathbf{v}} = \mathbf{0} \text{ in } \Omega,$$

in the sense of distribution theory. There thus exist fields $\widetilde{\boldsymbol{\eta}} \in \mathbf{H}^1(\omega)$ and $\widetilde{\boldsymbol{\eta}}^1 \in \mathbf{H}^1(\omega)$ such that (for a proof, see e.g. Lemmas 4.1 and 4.2 of Le Dret [22]):

$$\widetilde{\mathbf{v}}(y, x_3) = \widetilde{\boldsymbol{\eta}}(y) + x_3 \widetilde{\boldsymbol{\eta}}^1(y) \text{ for almost all } (y, x_3) \in \Omega.$$

Furthermore, $\widetilde{\boldsymbol{\eta}} = \widetilde{\boldsymbol{\eta}}^1 = \mathbf{0}$ on γ_0 since $\widetilde{\mathbf{v}} = v_i \mathbf{g}^i$ vanish on $\Gamma_0 = \gamma_0 \times [-\varepsilon_1, \varepsilon_1]$.

Expanding the fields $\widetilde{\boldsymbol{\eta}}$ and $\widetilde{\boldsymbol{\eta}}^1$ over the vectors $\mathbf{a}^i$ of the contravariant bases, we infer from the last relations that there exist fields $\boldsymbol{\eta} = (\eta_i) \in \mathbf{H}^1(\omega)$ and $\boldsymbol{\eta}^1 = (\eta_i^1) \in \mathbf{H}^1(\omega)$ vanishing on γ_0 such that

$$\widetilde{\mathbf{v}}(y, x_3) = v_i(y, x_3)\mathbf{g}^i(y, x_3) = \eta_i(y)\mathbf{a}^i(y) + x_3\eta_i^1(y)\mathbf{a}^i(y)$$

for almost all $(y, x_3) \in \Omega$. Since $v_3 = \widetilde{\mathbf{v}} \cdot \mathbf{g}_3 = \eta_3 + x_3\eta_3^1$ is independent of x_3, we conclude that $\eta_3^1 = 0$. Hence the field $\widetilde{\mathbf{v}} = v_i\mathbf{g}^i$ is of the form

$$v_i\mathbf{g}^i = (\eta_\beta + x_3\eta_\beta^1)\mathbf{a}^\beta + \eta_3\mathbf{a}^3.$$

Using this expression in the relations $e_{\alpha\|3}(\mathbf{v}) = 0$, together with the formulas of Gauss and Weingarten, viz.,

$$\partial_\alpha\mathbf{a}^\beta = -C^\beta_{\alpha\sigma}\mathbf{a}^\sigma + b^\beta_\alpha\mathbf{a}^3 \quad \text{and} \quad \partial_\alpha\mathbf{a}^3 = -b^\sigma_\alpha\mathbf{a}_\sigma,$$

we then obtain

$$0 = (\partial_\alpha\widetilde{\mathbf{v}}) \cdot \mathbf{a}_3 + (\partial_3\widetilde{\mathbf{v}}) \cdot (\mathbf{a}_\alpha - x_3b^\sigma_\alpha\mathbf{a}_\sigma) = b^\beta_\alpha\eta_\beta + \eta^1_\alpha + \partial_\alpha\eta_3.$$

The relations $\partial_\alpha\eta_3 = -b^\beta_\alpha\eta_\beta - \eta^1_\alpha$, together with the previously established relations $\eta_3 \in H^1(\omega)$ and $\eta_3 = 0$ on γ_0, then show that $\eta_3 \in H^2(\omega)$ and that $\partial_\nu\eta_3 = 0$ on γ_0. We have thus found $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$ such that $\mathbf{F}(\boldsymbol{\eta}) = \mathbf{v}$, as was desired. □

Lemma 1.6. *The inequality of Korn's type of Theorem 1.2 is a consequence of the three-dimensional Korn inequality of Theorem 1.1 applied with the mapping* $\boldsymbol{\Theta}$ *defined in Lemma 1.1 on the open set* Ω *defined in Lemma 1.4, and of Lemmas 1.2, 1.3 and 1.5.*

Proof. The linear mapping $\mathbf{F} : \mathbf{V}_K(\omega) \to \mathbf{V}_{KL}(\Omega)$ is continuous (Lemma 1.3), injective (Lemma 1.4), and surjective (Lemma 1.5). The spaces $\mathbf{V}_K(\omega)$ and $\mathbf{V}_{KL}(\Omega)$ being Hilbert spaces ($\mathbf{V}_{KL}(\Omega)$ is a closed subspace of $\mathbf{V}(\Omega)$), the *closed graph theorem* shows that the inverse mapping $\mathbf{F}^{-1} : \mathbf{V}_{KL}(\Omega) \to \mathbf{V}_K(\omega)$ is also continuous, i.e. that there exists a constant C_2 such that

$$\left\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\right\}^{1/2} \leq C_2\|\mathbf{F}(\boldsymbol{\eta})\|_{1,\Omega} \quad \text{for all } \boldsymbol{\eta} \in \mathbf{V}_K(\omega).$$

Next, there exists by Theorem 1.1 a constant C such that

$$\|\mathbf{F}(\boldsymbol{\eta})\|_{1,\Omega} \leq C\left\{\sum_{i,j} |e_{i\|j}(\mathbf{F}(\boldsymbol{\eta}))|^2_{0,\Omega}\right\}^{1/2} \quad \text{for all } \boldsymbol{\eta} \in \mathbf{V}_K(\omega).$$

Finally, the expressions of the functions $e_{i\|j}(\mathbf{F}(\boldsymbol{\eta}))$ given in Lemma 1.2 show that there exists a constant C_1 such that

$$\left\{\sum_{i,j}|e_{i\|j}(\mathbf{F}(\boldsymbol{\eta}))|^2_{0,\Omega}\right\}^{1/2} = \left\{\sum_{\alpha,\beta}|e_{\alpha\|\beta}(\mathbf{F}(\boldsymbol{\eta}))|^2_{0,\Omega}\right\}^{1/2}$$
$$\leq C_1\left\{\sum_{\alpha,\beta}|\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}+\sum_{\alpha,\beta}|\rho_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\right\}^{1/2}$$
$$\text{for all } \boldsymbol{\eta}\in\mathbf{V}_K(\omega).$$

Hence the inequality of Korn's type of Theorem 1.2 holds, with $c = CC_1C_2$. The proof of Theorem 1.3 is thus complete. □

4. Commentary

The above proof thus shows that the formal analogies observed between the statements of Theorems 1.1 and 1.2 reflect in fact the existence of an isomorphism, the mapping $\mathbf{F}$ introduced in Lemma 1.2, between the spaces $\mathbf{V}_{KL}(\Omega)$ and $\mathbf{V}_K(\omega)$. Thanks to this isomorphism, the inequality of Korn's type on a surface becomes a consequence of the application of the three-dimensional Korn inequality to the particular fields $\mathbf{F}(\boldsymbol{\eta})\in\mathbf{V}_{KL}(\Omega)$, the fields $\boldsymbol{\eta}$ varying in the space $\mathbf{V}_K(\omega)$. The existence of this isomorphism was first established by Destuynder [17], albeit by a different proof and for different purposes.

The identification of the image $\operatorname{Im}\mathbf{F}$ as the space $\mathbf{V}_{KL}(\Omega)$ has an interest *per se* in *linearized shell theory.* This result shows that, inside a *shell*, i.e. an elastic body whose reference configuration is of the form $\boldsymbol{\Theta}(\overline{\Omega})$, where $\overline{\Omega} := \overline{\omega}\times[-\varepsilon,\varepsilon]$ and the mapping $\boldsymbol{\Theta}:\overline{\Omega}\to\mathbb{R}^3$ is of the form given in Lemma 1.1, the *linearized Kirchhoff-Love displacement fields*, i.e. those displacement fields $v_i\mathbf{g}^i$ that satisfy the relations $e_{i\|3}(\mathbf{v})=0$ in Ω, are of the form

$$v_i\mathbf{g}^i=\eta_i\mathbf{a}^i-x_3(\partial_\alpha\eta_3+b^\sigma_\alpha\eta_\sigma)\mathbf{a}^\alpha \quad \text{with } \eta_\alpha\in H^1(\omega),\ \eta_3\in H^2(\omega),$$

and *vice versa.* This identification thus constitutes an extension of the well-known identification of *Kirchhoff-Love displacement fields* inside an *elastic plate* (cf. Ciarlet and Destuynder [12] and also Theorem 1.4-4 of Ciarlet [10]).

Note Added in Proof

After this work was completed, it was brought to the authors' attention that another derivation of the inequality of Korn's type on a surface from the three-dimensional Korn inequality had been simultaneously carried out by means of a related, but nevertheless different, method in an unpublished ONERA technical report by Jean-Luc Akian. This work will also appear in journal form.

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CHAPITRE 2

Inegalité de type Korn sur une surface compacte

Ce travail a fait l'objet de la publication suivante :

MARDARE S., *Inequality of Korn's type on compact surfaces without boundary.* Chin. Ann. Math., vol. 24, no. 2., Ser. B, 2003, 191-204.

Inequality of Korn's type on compact surfaces without boundary

"Compact surfaces without boundary" were already considered by Şlicaru [12] in his doctoral dissertation, where he studied the asymptotic behaviour of thin elastic shells with such middle surfaces.

For defining a surface with boundary, one mapping is usually enough. In fact, this is the way we define the surface, as the image of that map. It is easily seen that this is no longer possible in the case of surfaces without boundary. It suffices to consider the case of the sphere: it is well known that a sphere is not homeomorphic with a part of a plane. So the study we want to make here seems to be more complicated. However, this is compensated by the fact that the functional analysis which is behind is simplier than in the case of surfaces with boundary. That is why, when possible, we will recast our problems in a general functional analysis setting about quotient spaces. In this setting, we establish some general theorems which will give in particular the desired inequality of Korn's type.

We start with some definitions and results about surfaces and Sobolev spaces on surfaces. These are needed for justifying the inequality of Korn's type in the case of compact surfaces without boundary.

1. Some Elements of Surface Theory

Let there be given a three-dimensional vector space, in which we fix an origin O and a basis; in this way the three-dimensional vector space is identified with $\mathbb{R}^3$.

In this paper we use the classical definition of a regular surface as found, for instance, in Do Carmo [4] or Klingenberg [11].

Definition 2.1. A connected subset $S \subset \mathbb{R}^3$ is a regular surface of class C^k if, for each point $p \in S$, there exists a neighborhood V of p in $\mathbb{R}^3$ and a map $\theta : U \to V \cap S$ of the open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

(1) θ is of class C^k.

(2) θ is a homeomorphism (the topology of S is the induced topology of the usual topology on $\mathbb{R}^3$).

(3) For each $q \in U$, the differential $d\theta_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The third condition is equivalent with the fact that the vectors $\partial_\alpha \theta$ ($\alpha \in \{1, 2\}$) are linearly independent at all points $q \in U$. This means that $\theta : U \to \mathbb{R}^3$ is an immersion.

Another observation is that we can replace the neighborhood V appearing in Definition 2.1 with an open neighborhood $V' \subset \mathbb{R}^3$. Indeed, since V is a neighborhood of p, there exists an open neighborhood $V' \subset V$ of p. Now, since $\theta : U \to V \cap S$ is continuous, $U' := \theta^{-1}(V' \cap S)$ is an open set in U, hence in $\mathbb{R}^2$ (because U is open in $\mathbb{R}^2$). Now we can see that the map $\theta : U' \to V' \cap S$ also satisfies the conditions of Definition 2.1.

Conversely, if we have a collection of maps $(\theta_t)_{t\in A}$ (where A is an arbitrary set of indices) and a family $(\omega_t)_{t\in A}$ of open sets of $\mathbb{R}^2$ such that:

(1) $\theta_t : \omega_t \to \mathbb{R}^3$ is of class C^k,

(2) $\theta_t : \omega_t \to \theta_t(\omega_t)$ is an homeomorphism,

(3) For each $q \in \omega_t$, the differential $d\theta_{t_{|q}}$ is one-to-one,

(4) $\theta_t(\omega_t)$ is open in $\cup_{t\in A}\theta_t(\omega_t)$,

(5) $\cup_{t\in A}\theta_t(\omega_t)$ is connected,

then $S := \cup_{t\in A}\theta_t(\omega_t)$ is a regular surface of classe C^k. In particular, this shows that Definition 2.1 is equivalent with the definition of an embedded surface as found in Klingenberg (Definition 6.1.1 in [11]).

The second viewpoint is the analog in the general case of a surface defined as the image of a single map (obviously, the fourth condition is satisfied in the case of a single map).

Before passing to the case of compact surfaces, we recall some important results about general surfaces:

1. *Change of parameters.* If $\theta : U \to S$ and $\psi : V \to S$ are two parameterizations of a regular surface of class C^k such that $\theta(U) \cap \psi(V) = W \neq \emptyset$, then the "change of coordinates" $\theta^{-1} \circ \psi : \psi^{-1}(W) \to \theta^{-1}(W)$ is of class C^k. For a proof, see for instance Do Carmo [4], § 2.3, Proposition 1.

This shows that a regular surface of class C^k is, in particular, a *differentiable manifold of class* C^k. So we can use all the results on differentiable manifolds and especially those concerning tensor fields.

2. Using the same technique as in the proof of Proposition 1 of [4], we can show that conditions (1) and (3) (here, we suppose that $k \geq 1$), together with the fact that θ_t is one-to-one, imply that $\theta_t : \omega_t \to \theta_t(\omega_t)$ is a homeomorphism.

In what follows, a regular surface of class C^k will be defined as a connected subset of $\mathbb{R}^3$ that can be writen as $S := \bigcup_{t\in A} \theta_t(\omega_t)$, where

$$(2.1)\quad \begin{array}{l}(1)\ \omega_t \subset \mathbb{R}^2 \text{ are open sets,}\\ (2)\ \theta_t : \omega_t \to \mathbb{R}^3 \text{ are injective immersions of class } C^k \text{ over } \omega_t,\\ (3)\ \theta_t(\omega_t) \text{ are open in } S.\end{array}$$

If S is in addition compact, then we can assume that A is a finite set. More specifically, we will use the following property of compact surfaces, which allow to apply on those parts of the surface that are images of a single map the results already known for surfaces defined by a single map.

Theorem 2.1. *Let S be a compact regular surface of class C^k. Then there exists a finite number of maps $(\theta_t, \omega_t)_{t=1}^N$ such that $S = \cup_{t=1}^N \theta_t(\omega_t) = \cup_{t=1}^N \theta_t(\overline{\omega}_t)$, where*

(1) $\omega_t \subset \mathbb{R}^2$ are open, bounded and connected sets with Lipschitz-continuous boundary,

(2) $\theta_t : \overline{\omega}_t \to \mathbb{R}^3$ are injective, C^k-differentiable immersions,

(3) $\theta_t(\omega_t)$ are open in S.

Here, we consider that a function is of class C^k over an arbitrary nonempty set, if it is the restriction to that set of a function of class C^k on a larger open set.

Proof. Since S is a compact regular surface of class C^k, there exist $(\theta_t, D_t)_{t=1}^N$, where $\theta_t : D_t \to \mathbb{R}^3$, such that $S = \cup_{t=1}^N \theta_t(D_t)$, where

(1) $D_t \subset \mathbb{R}^2$ are open sets,

(2) $\theta_t : D_t \to \mathbb{R}^3$ are injective, of class C^k over D_t, and $\partial_\alpha \theta_t$ are linearly independent at all points of D_t,

(3) $\theta_t(D_t)$ are open in S.

Now we will use the following theorem due to Lebesgue:

Let K be a compact metric space and let $K = \cup_{t=1}^N V_t$, where $V_t \subset K$ are open sets. Then there exists an $\varepsilon > 0$ such that for all $x \in K$, there exists $t_x \in \{1, \dots, N\}$ such that $B(x, \varepsilon) \subset V_{t_x}$ (here, $B(x, \varepsilon)$ denotes the open ball centered at x with radius ε).

We use this theorem with $K = S$ and $V_t = \theta_t(D_t)$. We consider that S is endowed with the metric induced by the Euclidean metric on $\mathbb{R}^3$. With ε given by Lebesgue's theorem, define $V_t' := \{x \in V_t\, ;\ d(x, K \setminus V_t) > \frac{\varepsilon}{2}\}$. Obviously, V_t' is open in K. We will show that $K = \cup_{t=1}^N V_t'$.

Let $x \in K$. Then, by Lebesgue's Theorem, the ball $B(x, \varepsilon)$ is included in some V_{t_x}, which implies that $d(x, K \setminus V_{t_x}) \geq \varepsilon > \frac{\varepsilon}{2}$. Consequently, $x \in V_{t_x}'$, so that $x \in \cup_{t=1}^N V_t'$.

Since $\overline{V_t'} = \{x \in V_t\, ;\ d(x, K \setminus V_t) \geq \frac{\varepsilon}{2}\}$, we have $V_t' \subset \overline{V_t'} \subset V_t$. Define $\omega_t' := \theta_t^{-1}(V_t')$ and note that ω_t' is open. Since θ_t is a homeomorphism (as observed above), we have $V_t' = \theta_t(\omega_t')$ and $\overline{\omega}_t' = \theta_t^{-1}(\overline{V_t'}) \subset \theta_t^{-1}(V_t) = D_t$. We also see that the sets ω_t' are bounded (indeed, θ_t is a homeomorphism and $\overline{V_t'}$ is a compact set, so $\overline{\omega}_t'$ is compact and therefore bounded). Since $\overline{\omega}_t' \subset D_t$, there exists a bounded open set ω_t with Lipschitz-continuous boundary, such that $\overline{\omega}_t' \subset \omega_t \subset \overline{\omega}_t \subset D_t$. So $\theta_t(\overline{\omega}_t) \subset S$ and $\theta_t(\omega_t)$ is open in S, because $\theta_t(\omega_t) \subset \theta_t(D_t)$ and $\theta_t : D_t \to \theta_t(D_t)$ is a homeomorphism.

Now it is clear that θ_t satisfies the regularity conditions on $\overline{\omega}_t$ (since it satisfies these conditions on D_t). Finally, we have

$$S = \cup_{t=1}^{N} V_t' = \cup_{t=1}^{N} \theta_t(\omega_t') \subset \cup_{t=1}^{N} \theta_t(\omega_t) \subset \cup_{t=1}^{N} \theta_t(\overline{\omega}_t) \subset S$$

To conclude the proof, we observe that we can assume that the sets ω_t are connected: otherwise, we take the connected components of ω_t and use the compactness of S to again obtain a finite number of maps.

□

Before passing to the next section, let us introduce some classical elements of a surface, which will be used in the present paper. Here and in the sequel, Greek indices and exponents (except ε and ν) take their values in the set $\{1,2\}$, Latin indices and exponents (except t) take their values in the set $\{1,2,3\}$, and the summation convention with respect to repeted indices and exponents is used. The Euclidean scalar and vector products are denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$, respectively. We consider a regular surface of class C^k ($k \geq 2$) denoted by S. Let $p \in S$ be a point of the surface and let (θ, ω) be a local map at p with $\theta(x) = p$. We define the following elements of either p or x by the same symbol (we write this dependence explicitly only for the first definition; in applications, we will consider functions of either p or x, depending on whether we work on the surface or on the set ω defining the surface through the map θ:

$\boldsymbol{a}_\alpha(p) = \boldsymbol{a}_\alpha(x) := \partial_\alpha \theta(x)$ are the vectors of the *covariant basis* of T_pS (the tangent space to S at p) associated with the map θ,

$\boldsymbol{a}^\alpha$ are the vectors of the *contravariant basis* of T_pS associated with the map θ, and they are defined by the relations $\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta_\beta^\alpha$, where δ_β^α designates the Kronecker's delta,

$\boldsymbol{a}_3 = \boldsymbol{a}^3 := \dfrac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$ is the unit normal vector to S at p,

$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ are the covariant components of the metric tensor,

$a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$ are the contravariant components of the metric tensor,

$a := \det(a_{\alpha\beta})$ is the square of the surface element,

$\Gamma_{\alpha\beta}^\sigma := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha = \boldsymbol{a}^\sigma \cdot \partial_{\alpha\beta}\theta$ are the Christoffel symbols,

$b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha = \boldsymbol{a}^3 \cdot \partial_{\alpha\beta}\theta$ are the covariant components of the curvature tensor,

$b_\alpha^\beta := a^{\beta\sigma} b_{\sigma\alpha}$ are the mixed components of the curvature tensor,

$ds := \sqrt{a}\, dx$ is the area element on S.

2. Sobolev Spaces on Surfaces

Let S be a regular surface of class C^k and let $f : S \to \mathbb{R}$ be a real valued function on S. We say that $f \in H^m(S)$, $(m \le k)$ if, for every $p \in S$ and for a map (θ, ω) such that $p = \theta(x) \in \theta(\omega)$, the derivatives of $f \circ \theta$ of order $\le m$ are in $L^1_{loc}(\omega)$ and the following expression is finite:

$$\|f\|_{H^m(S)} := \left(\int_S f^2 \, ds + \sum_{l=1}^{m} \int_S a^{\alpha_1\beta_1} \ldots a^{\alpha_l\beta_l} f_{|\alpha_1\ldots\alpha_l} f_{|\beta_1\ldots\beta_l} \, ds \right)^{\frac{1}{2}}, \tag{2.2}$$

where $f_{|\alpha_1\ldots\alpha_l}$ are the l-covariant derivatives of f.

It is easily seen that such a property of weak-differentiability does not depend on the map we choose; this comes from the fact that S is a C^k-differentiable manifold (see section 1). As regards the expression in (2.2), we know from tensor theory that it is intrinsic, which means that it does not depend on the maps we choose to calculate it.

In the case of a compact surface, we can replace these norms by the following equivalent norms which are simplier: for a fixed collection of maps $(\theta_t, \omega_t)_{t=1}^{N}$ as those in Theorem 2.1, define

$$\left(\sum_{t=1}^{N} \|f \circ \theta_t\|^2_{H^m(\omega_t)} \right)^{\frac{1}{2}} .$$

These norms are no more intrinsic, but this is not inconvenient for our analysis, since they are equivalent (for a fixed collection of maps) with the intrinsic norms defined in (2.2).

Let us show this assertion for $m = 1, 2$ (in this paper, we will use only the $H^1(S)$ and $H^2(S)$-norms). First of all, we have

$$\|f\|_{H^1(S)} := \left(\int_S f^2 \, ds + \int_S a^{\alpha\beta} f_{|\alpha} f_{|\beta} \, ds \right)^{\frac{1}{2}}, \tag{2.3}$$

$$\|f\|_{H^2(S)} := \left(\int_S f^2 \, ds + \int_S a^{\alpha\beta} f_{|\alpha} f_{|\beta} \, ds + \int_S a^{\alpha\beta} a^{\sigma\tau} f_{|\alpha\sigma} f_{|\beta\tau} \, ds \right)^{\frac{1}{2}}, \tag{2.4}$$

where

$$f_{|\alpha}(p) = f_{|\alpha}(x) := \frac{\partial(f \circ \theta)}{\partial x_\alpha}(x) \text{ and}$$
$$f_{|\alpha\beta}(p) = f_{|\alpha\beta}(x) := \frac{\partial^2(f \circ \theta)}{\partial x_\alpha \partial x_\beta}(x) - \Gamma^\sigma_{\alpha\beta}(x) f_{|\sigma}(x),$$

with $p = \theta(x)$, are the first and second covariant derivatives of the function f in p.

For the $H^1(S)$-norm, thanks to the positive definitness of $(a^{\alpha\beta})$, we have $\tilde{c} \sum_\alpha (f_{|\alpha})^2 \le a^{\alpha\beta} f_{|\alpha} f_{|\beta} \le \tilde{C} \sum_\alpha (f_{|\alpha})^2$ for some positive constants $\tilde{c}$ and $\tilde{C}$. We multiply this relation by $\sqrt{a}$, add $(f \circ \theta_t)^2 \sqrt{a}$, then integrate over ω_t.

Using in addition the fact that a is a stictly positive function on $\bar{\omega}_t$, we obtain that there exist two constants $c > 0$ and $C > 0$ such that

$$c\|f \circ \theta_t\|^2_{H^1(\omega_t)} \leq \int_{\theta_t(\omega_t)} (f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta})\, ds \leq C\|f \circ \theta_t\|^2_{H^1(\omega_t)} . \tag{2.5}$$

Since the function $f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta}$ is positive on S, we have the following inequalities

$$\int_{\theta_t(\omega_t)} (f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta})\, ds \leq \int_S (f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta})\, ds$$
$$\leq \sum_{t=1}^{N} \int_{\theta_t(\omega_t)} (f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta})\, ds .$$

Taking the sum with respect to t in (2.5) and using the last inequalities, we obtain

$$\frac{c}{N} \sum_{t=1}^{N} \|f \circ \theta_t\|^2_{H^1(\omega_t)} \leq \|f\|^2_{H^1(S)} \leq C \sum_{t=1}^{N} \|f \circ \theta_t\|^2_{H^1(\omega_t)} ,$$

which is the sought equivalence for $m = 1$.

Now, let us prove the equivalence for the $H^2(S)$-norm. It suffices to find two constants $c > 0$ and $C > 0$ such that

$$c \sum_{t=1}^{N} \|f \circ \theta_t\|^2_{H^2(\omega_t)} \leq \|f\|^2_{H^2(S)} \leq C \sum_{t=1}^{N} \|f \circ \theta_t\|^2_{H^2(\omega_t)} . \tag{2.6}$$

Using in particular Theorem 3.3-2 of [6], which states that $(a^{\alpha\beta} a^{\sigma\tau})$ is uniformly positive definite, we obtain on the one hand that

$$\tilde{c} \sum_{\alpha,\beta} (f_{|\alpha\beta})^2 \leq a^{\alpha\beta} a^{\sigma\tau} f_{|\alpha\sigma} f_{|\beta\tau} \leq \tilde{C} \sum_{\alpha,\beta} (f_{|\alpha\beta})^2 \tag{2.7}$$

for some constants $\tilde{c} > 0$ and $\tilde{C} > 0$. On the other hand, since the functions $\Gamma^\sigma_{\alpha\beta}$ are bounded on ω_t, we deduce from the definition of $f_{|\alpha\beta}$ that there exist two constants $c_1 > 0$ and $C_1 > 0$ such that

$$c_1 \Big\{ \big(\partial_{\alpha\beta}(f \circ \theta_t)\big)^2 + \sum_\sigma \big(\partial_\sigma (f \circ \theta_t)\big)^2 \Big\} \leq (f_{|\alpha\beta})^2 + \sum_\sigma (f_{|\sigma})^2$$
$$\leq C_1 \Big\{ \big(\partial_{\alpha\beta}(f \circ \theta_t)\big)^2 + \sum_\sigma \big(\partial_\sigma (f \circ \theta_t)\big)^2 \Big\} .$$

We introduce this inequality in (2.7), then multiply the result by $\sqrt{a}$ (which is a strictly positive and bounded function on $\bar{\omega}_t$) and finally integrate on

ω_t. This gives

$$c_2\left(\sum_{\alpha,\beta}\|\partial_{\alpha\beta}(f\circ\theta_t)\|^2_{L^2(\omega_t)}+\sum_{\sigma}\|\partial_{\sigma}(f\circ\theta_t)\|^2_{L^2(\omega_t)}\right)$$
$$\leq\int_{\theta_t(\omega_t)}\left(a^{\alpha\beta}a^{\sigma\tau}f_{|\alpha\sigma}f_{|\beta\tau}+a^{\alpha\beta}f_{|\alpha}f_{|\beta}\right)ds$$
$$\leq C_2\left(\sum_{\alpha,\beta}\|\partial_{\alpha\beta}(f\circ\theta_t)\|^2_{L^2(\omega_t)}+\sum_{\sigma}\|\partial_{\sigma}(f\circ\theta_t)\|^2_{L^2(\omega_t)}\right),$$

for some positive constants c_2 and C_2. Since the function $a^{\alpha\beta}a^{\sigma\tau}f_{|\alpha\sigma}f_{|\beta\tau}+a^{\alpha\beta}f_{|\alpha}f_{|\beta}$ is positive on S, we can use the same method than that used for the $H^1(S)$-norm in order to obtain (2.6).

We say that a *spatial vector field* $\boldsymbol{\eta}$ (which means that with each $p\in S$, we associate a vector $\boldsymbol{\eta}(p)$ in space, not necessarily in the tangent space T_pS) is in the space $\boldsymbol{H}^m(S)$ if all its components in a fixed basis of $\mathbb{R}^3$ belong to $H^m(S)$.

Now, let us consider the tangential and the normal components of $\boldsymbol{\eta}$. More specifically, let $\boldsymbol{\eta}=\boldsymbol{\eta}_\tau+\boldsymbol{\eta}_\nu$, where $\boldsymbol{\eta}_\tau(p)\in T_pS$ and $\boldsymbol{\eta}_\nu(p)$ is parallel to the normal to S at the point p. We say that $\boldsymbol{\eta}\in\boldsymbol{H}^m_\tau(S)\oplus\boldsymbol{H}^n_\nu(S)$ if $\boldsymbol{\eta}_\tau\in\boldsymbol{H}^m(S)$ and $\boldsymbol{\eta}_\nu\in\boldsymbol{H}^n(S)$.

3. Koiter's Model for a Linearly Elastic Shell

Throughout this paragraph, S is a compact regular surface of class C^3. Throughout the sequel, the points $p\in S$ and $x\in\omega$ (or ω_t) are related by the relation $p=\theta(x)$. Our aim is to establish an inequality, thereafter called inequality of Korn's type, which eventually will imply the existence (and uniqueness) of a solution to Koiter's model for a linearly elastic shell with a compact regular middle surface. To establish such an inequality, we make an analogy with the case of *one-mapping surfaces* (surfaces which are parameterized by a single map) and we retain from this case only the intrinsic quantities. To begin with, let us define the two-dimensional Koiter equations for a linearly elastic shell. We consider a shell with middle surface S and thickness 2ε, subjected to applied body forces. In Koiter's model, the unknown is the displacement field $\boldsymbol{\zeta}^\varepsilon_K:S\to\mathbb{R}^3$ of the middle surface of the shell. In the case where the surface is defined by a single map (θ,ω) satisfying properties (1) and (2) (with $k=3$) of Theorem 2.1, the problem under consideration is the following:

$$(2.8)\quad \left\{\begin{aligned} &\text{Find } \tilde{\boldsymbol{\zeta}}^{\varepsilon}_K \in \boldsymbol{V}_K(\omega) \text{ such that}\\ &\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\tilde{\boldsymbol{\zeta}}^{\varepsilon}_K)\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(\tilde{\boldsymbol{\zeta}}^{\varepsilon}_K)\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\Big\}\sqrt{a}\,dx\\ &\qquad\qquad = \int_\omega \tilde{\boldsymbol{f}}^{\varepsilon}\cdot\tilde{\boldsymbol{\eta}}\sqrt{a}\,dx \text{ for all } \tilde{\boldsymbol{\eta}} = (\tilde{\eta}_i) \in \boldsymbol{V}_K(\omega), \end{aligned}\right.$$

where $V_K(\omega)$ is a closed subspace of the space $H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ considered with the usual norm, $\tilde{\boldsymbol{f}}^{\varepsilon} = (\tilde{f}^{\varepsilon,i}) \in L^2(\omega)\times L^2(\omega)\times L^2(\omega)$ (where $\boldsymbol{f}^{\varepsilon} = \tilde{f}^{\varepsilon,i}\boldsymbol{a}_i$ account for the applied body forces) and

$$a^{\alpha\beta\sigma\tau,\varepsilon}(p) = a^{\alpha\beta\sigma\tau,\varepsilon}(x) := \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon+2\mu^\varepsilon}a^{\alpha\beta}a^{\sigma\tau} + 2\mu^\varepsilon(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}),$$

$$\begin{aligned}\gamma_{\alpha\beta}(\boldsymbol{\eta})(p) = \gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})(x) :&= \frac{1}{2}(\partial_\beta\boldsymbol{\eta}\cdot\boldsymbol{a}_\alpha + \partial_\alpha\boldsymbol{\eta}\cdot\boldsymbol{a}_\beta)\\ &= \frac{1}{2}(\partial_\beta\tilde{\eta}_\alpha + \partial_\alpha\tilde{\eta}_\beta) - \Gamma^\sigma_{\alpha\beta}\tilde{\eta}_\sigma - b_{\alpha\beta}\tilde{\eta}_3\,,\end{aligned}$$

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta})(p) = \rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})(x) :&= (\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\boldsymbol{\eta})\cdot\boldsymbol{a}_3\\ &= \partial_{\alpha\beta}\tilde{\eta}_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\tilde{\eta}_3 - b^\sigma_\alpha b_{\sigma\beta}\tilde{\eta}_3\\ &\quad + b^\sigma_\alpha(\partial_\beta\tilde{\eta}_\sigma - \Gamma^\tau_{\beta\sigma}\partial_\sigma\tilde{\eta}_\tau) + b^\tau_\beta(\partial_\alpha\tilde{\eta}_\tau - \Gamma^\sigma_{\alpha\tau}\partial_\sigma\tilde{\eta}_\sigma)\\ &\quad + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\tilde{\eta}_\tau\,,\end{aligned}$$

where $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$ are the Lamé constants of the elastic material constituting the shell, and $\boldsymbol{\eta}(p) = \tilde{\eta}_i(x)\boldsymbol{a}^i(p)$. Then, in Koiter's model, the displacement field of the middle surface of the shell is given by $\boldsymbol{\zeta}^\varepsilon_K = \zeta^\varepsilon_{K,i}\boldsymbol{a}^i$. Throughout this section, we denote by the same symbol a function of p (defined on the surface) or of x (provided that $\theta(x) = p$). We also make the convention that the two functions are equal. Recall that we have already used this convention in section 2. For further details about Koiter's model, see [6].

Let us notice that matrices $\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})(p)\big)$ and $\big(\rho_{\alpha\beta}(\boldsymbol{\eta})(p)\big)$ are symmetric and that the functions $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ and $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ belong to $L^2(\omega)$, since $\tilde{\boldsymbol{\eta}} \in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$. As long as we are only interested in the existence and uniqueness of a solution to problem (2.8), the coefficients ε and $\frac{\varepsilon^3}{3}$ are not really relevant, so we make the convention that $\varepsilon = 1$ and $\frac{\varepsilon^3}{3} = 1$ in (2.8). Accordingly, the expression appearing in (2.8) became:

$$A(\boldsymbol{\zeta},\boldsymbol{\eta}) := a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta}),$$

where $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ are two spatial vector fields. We already know that $(a^{\alpha\beta\sigma\tau})$ are the contravariant components of a tensor field of rank 4 (the two-dimensional elasticity tensor), that $\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})\big)$ are the covariant components of a tensor field of rank 2 (the linearized change of metric tensor), and that $\big(\rho_{\alpha\beta}(\boldsymbol{\zeta})(\rho_{\sigma\tau}(\boldsymbol{\eta})\big)$ are the covariant components of a tensor field of rank 4.

Finally, by inner multiplication, we see that the expression $A(\boldsymbol{\zeta},\boldsymbol{\eta})$ is a tensor field of rank 0, i. e., a function. This means that this expression does not depend on the choice of maps, but only on the vector fields $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ defining the tensor fields $\big(\gamma_{\alpha\beta}(\boldsymbol{\zeta})\big)$, $\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})\big)$ and $\big(\rho_{\alpha\beta}(\boldsymbol{\zeta})(\rho_{\sigma\tau}(\boldsymbol{\eta})\big)$.

As we shall see later in this paper, the fact that $\tilde{\boldsymbol{\eta}} \in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ is equivalent with $\boldsymbol{\eta}\in \boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S)$. Now, problem (2.8) (with the simplifying convention that $\varepsilon = 1$ and $\frac{\varepsilon^3}{3} = 1$) takes the folowing intrinsic form:

$$\text{(2.9)}\qquad \begin{aligned}&\boldsymbol{\zeta}\in \boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S),\\ &\int_S A(\boldsymbol{\zeta},\boldsymbol{\eta})\,ds = \int_S \boldsymbol{f}\cdot\boldsymbol{\eta}\,ds \text{ for all } \eta\in \boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S).\end{aligned}$$

Note that this form of Koiter's model can be transposed *verbatim* in the case where the surface S is compact.

Let us consider the bilinear form

$$\text{(2.10)}\qquad B(\boldsymbol{\zeta},\boldsymbol{\eta}) := \int_S A(\boldsymbol{\zeta},\boldsymbol{\eta})\,ds$$

defined over the space $\boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S)$, and the linear form

$$L(\boldsymbol{\eta}) := \int_S \boldsymbol{f}\cdot\boldsymbol{\eta}\,ds$$

defined over the same space. Then we rewrite problem (2.9) in a functional form, that is:

$$\text{(2.11)}\qquad \begin{aligned}&\text{Find } \boldsymbol{\zeta}\in E := \boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S) \text{ such that}\\ &B(\boldsymbol{\zeta},\boldsymbol{\eta}) = L(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta}\in E.\end{aligned}$$

We are interested in proving the existence and uniqueness of the solution to this problem. To this end, we will use the Lax-Milgram theorem; naturally, proving the ellipticity of the bilinear form is the only difficulty. The object of the following section is to establish an inequality which allow to prove the ellipticity of the bilinear form B.

4. Korn Inequality for Compact Surfaces

Let us begin by making some observations on the bilinear form B. We can verify that $\boldsymbol{\eta}\mapsto\sqrt{B(\boldsymbol{\eta},\boldsymbol{\eta})}$ is a seminorm on E. This come from the fact that B is a symmetric bilinear form which satisfies $B(\boldsymbol{\eta},\boldsymbol{\eta})\geq 0$ for all $\boldsymbol{\eta}\in E$. To prove this last property of B, we use the fact that $a^{\alpha\beta\sigma\tau}t_{\sigma\tau}t_{\alpha\beta}\geq c\sum_{\alpha,\beta}|t_{\alpha\beta}|^2$ for all symmetric matrices $(t_{\alpha\beta})$ (for a proof, see [6], Theorem 3.3-2). We shall see later that $\boldsymbol{\eta}\mapsto\sqrt{B(\boldsymbol{\eta},\boldsymbol{\eta})}$ is not a norm on E.

We consider the following framework, which is well suited to our problem: let $(E, \|\cdot\|)$ be a Banach space, let $L\in E'$ be a linear form over E, and let B be a symmetric bilinear form over E which satisfies $B(x,x) \geq 0$ for all $x \in E$. Consider the problem

$$\text{Find } x \in E \text{ such that } B(x,y) = L(y) \text{ for all } y \in E. \tag{2.12}$$

Define the seminorm $x \mapsto |x| := \sqrt{B(x,x)}$ and the set $F := \{x \in E\,;\, |x| = 0\}$. We can easily verify the following properties:

1. F is a vector space.

2. We must have $L(y) = 0$ for all $y \in F$ if we wish that problem (2.12) have solutions. Indeed, since $B(x,x) \geq 0$ for all $x \in E$, we have the Cauchy-Schwarz inequality: $|B(x,y)| \leq \sqrt{B(x,x)}\sqrt{B(y,y)} = |x||y|$. So, if $x \in E$ and $y \in F$, we have $|B(x,y)| \leq |x||y| = 0$. In other words,

$$B(x,y) = 0 \text{ for all } x \in E,\ y \in F. \tag{2.13}$$

Therefore, if x is a solution of (2.12) and $y \in F$, then $B(x,y) = 0 = L(y)$.

Remark 2.1. Property (2.13) sais that $F = \ker B$, where

$$\ker B := \{y \in E\,;\ B(x,y) = 0 \text{ for all } x \in E\}.$$

3. If x is a solution of (2.12) and $\tilde{x} \in F$ then $x + \tilde{x}$ is also a solution of (2.12).

So, if we wish that problem (2.12) be well posed (in the sense that it has one and only one solution), we have to impose the condition

$$L_{|F} = 0 \tag{2.14}$$

and try to solve the problem over the quotient space E/F, not over E. Note that L is well defined on E/F, thanks to the compatibility condition (2.14). The same remark holds for B, thanks to relation (2.13). Now, the problem we want to solve reads:

$$\text{Find } \hat{x} \in E/F \text{ such that } B(\hat{x},\hat{y}) = L(\hat{y}) \text{ for all } \hat{y} \in E/F. \tag{2.15}$$

The following abstract result gives the ellipticity of B on E/F under some additional assumptions. This is equivalent with saying that the seminorm induced by B is a norm equivalent with the norm of E/F. Applying the following theorem to our case give us the Korn inequality for compact surfaces. Note that some ideas of the proof are close to those used by Duvaut and Lions in [9], chapter 3, where they have studied the three-dimensional elasticity problem without boundary conditions.

Theorem 2.2. *Let $(E, \|\cdot\|)$ be a Banach space, let $|\cdot|$ be a seminorm on E and let $(\tilde{E}, \|\cdot\|_0)$ be a larger normed space ($E \subset \tilde{E}$) such that*

(i) There exists $c > 0$ such that $|x| \leq c\|x\|$ for all $x \in E$,

(ii) The inclusion $(E, \|\cdot\|) \hookrightarrow (\tilde{E}, \|\cdot\|_0)$ is compact,

(iii) There exists $c_0 > 0$ such that $\|x\| \leq c_0(\|x\|_0 + |x|)$ for all $x \in E$.

Then there exists $C > 0$ such that $\|\hat{x}\|_{E/F} \leq C|\hat{x}|_{E/F}$ for all $\hat{x} \in E/F$, where $F := \{x \in E\,;\, |x| = 0\}$, $\|\hat{x}\|_{E/F} := \inf\{\|x\|\,;\, x \in \hat{x}\}$ and $|\hat{x}|_{E/F} := \inf\{|x|\,;\, x \in \hat{x}\}$.

In the case of problem (2.11), (i) means the continuity of the quadratic form $\boldsymbol{\eta} \mapsto B(\boldsymbol{\eta}, \boldsymbol{\eta})$ and will be given by the continuity of the bilinear form B, (ii) will be the compact inclusion of $\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$ in $\boldsymbol{L}^2_\tau(S) \oplus \boldsymbol{H}^1_\nu(S)$ and (iii) will be an inequality of Korn's type without boundary conditions.

Proof. We argue by contradiction. If the announced inequality is false, then there exists a sequence $(\hat{x}_n)_{n\in\mathbb{N}}$ in E such that $\|\hat{x}_n\|_{E/F} = 1$ and $|\hat{x}_n|_{E/F} \to 0$ when $n \to +\infty$.

For each $\hat{x}_n$ we choose a representative x_n such that $\|x_n\| \le 2$. The inclusion $(E, \|\cdot\|) \hookrightarrow (\tilde{E}, \|\cdot\|_0)$ is compact, so there exists a subsequence (also denoted by (x_n) for simplicity of notations) such that (x_n) converges in the norm $\|\cdot\|_0$ to some element of E. In particular, (x_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_0$. By using (iii), we obtain

$$\|x_n - x_m\| \le c_0(\|x_n - x_m\|_0 + |x_n - x_m|) \le c_0(\|x_n - x_m\|_0 + |x_n| + |x_m|).$$

Using the fact that $|x_n| = |\hat{x}_n|_{E/F}$ for all representative x_n of $\hat{x}_n$ (indeed, if x_n and x'_n are two representatives of $\hat{x}_n$, then

$$x'_n - x_n \in F \text{ and } |x'_n| = |x_n + x'_n - x_n| \le |x_n| + |x'_n - x_n| = |x_n|;$$

in the same way we get $|x_n| \le |x'_n|$, so $|x'_n| = |x_n|$) and the fact that $|\hat{x}_n|_{E/F} \to 0$, we obtain that (x_n) is a Cauchy sequence with respect to the norm $\|\cdot\|$.

Since $(E, \|\cdot\|)$ is a Banach space, there exists $x \in E$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

But $\|\hat{x}_n - \hat{x}\|_{E/F} = \|\widehat{x_n - x}\|_{E/F} \le \|x_n - x\|$, so $\lim_{n\to\infty} \|\hat{x}_n - \hat{x}\|_{E/F} = 0$. Consequently, $\|\hat{x}_n\|_{E/F} \to \|\hat{x}\|_{E/F}$, so we have $\|\hat{x}\|_{E/F} = 1$.

On the other hand, we have

$$\big||\hat{x}_n|_{E/F} - |\hat{x}|_{E/F}\big| \le |\hat{x}_n - \hat{x}|_{E/F} = |x_n - x| \le c\|x_n - x\| \overset{n\to\infty}{\longrightarrow} 0. \tag{2.16}$$

But $|\cdot|_{E/F}$ is a norm on E/F. Indeed, if $|\hat{x}|_{E/F} = 0$, then there exists a sequence $(x + y_n)$ of representatives of $\hat{x}$, $x + y_n$, (where x is a fixed representative of $\hat{x}$ and $y_n \in F$) such that $|x + y_n| < \frac{1}{n}$, for all $n \in \mathbb{N}^*$. We have

$$|x| \le |x + y_n| + |y_n| = |x + y_n| < \frac{1}{n} \text{ for all } n \in \mathbb{N}^*.$$

So $|x| = 0$, which implies that $x \in F$ and finally $\hat{x} = \hat{0}$.

Now, using (2.16) and the fact that $|\hat{x}_n|_{E/F} \overset{n\to\infty}{\longrightarrow} 0$, we obtain $|\hat{x}|_{E/F} = 0$, so $\hat{x} = \hat{0}$. But this is in contradiction with the fact that $\|\hat{x}\|_{E/F} = 1$ and the proof is complete. □

Now we can solve problem (2.15) by the following corollary:

Corollary 2.1. *Let $(E, \|\cdot\|)$ be a Banach space and let $B : E \times E \to \mathbb{R}$ be a symmetric continuous bilinear form (there exists $c > 0$ such that $|B(x, y)| \le c\|x\|\|y\|$ for all $x, y \in E$) which satisfies $B(x, x) \ge 0$ for all $x \in E$. Let $F := \{x \in E\,;\, B(x, x) = 0\}$ and let $L \in E'$ be a linear form on E which* *satisfies* $L_{|F} = 0$. *Assume that there exists a larger space* $(\tilde{E}, \|\cdot\|_0)$ *such that*

(i) The inclusion of $(E, \|\cdot\|)$ *in* $(\tilde{E}, \|\cdot\|_0)$ *is compact,*

(ii) There exists $c_0 > 0$ *such that* $\|x\| \leq c_0\big(\|x\|_0 + \sqrt{B(x,x)}\big)$ *for all* $x \in E$.

Then there exists one and only one solution of the variational problem

$$\textit{Find } \hat{x} \in E/F \textit{ such that } B(\hat{x}, \hat{y}) = L(\hat{y}) \textit{ for all } \hat{y} \in E/F.$$

Proof. We consider the seminorm $x \in E \mapsto |x| := \sqrt{B(x,x)}$ and the induced norm on E/F. We have seen in the proof of Theorem 2.2 that $|\cdot|_{E/F}$ is a norm on E/F.

The continuity of B imply that $B(x,x) \leq c\|x\|^2$ for all $x \in E$, so

$$|x| \leq \sqrt{c}\|x\| \text{ for all } x \in E. \tag{2.17}$$

Taking the *inf* with respect to $x \in \hat{x}$ in (2.17), we obtain that $|\hat{x}|_{E/F} \leq \sqrt{c}\|\hat{x}\|_{E/F}$ for all $\hat{x} \in E/F$. In addition, inequality (2.17) shows that assumption (i) of Theorem 2.2 is satisfied. The other two hypotheses are given in the statement of the corollary, so we can apply Theorem 2.2. Consequently, there exists $C > 0$ such that

$$\|\hat{x}\|_{E/F} \leq C|\hat{x}|_{E/F} \text{ for all } \hat{x} \in E/F.$$

Therefore, the norms $\|\cdot\|_{E/F}$ and $|\cdot|_{E/F}$ are equivalent.

We also see that $L \in (E/F, \|\cdot\|_{E/F})'$. Indeed, if $\|\hat{y}_n - \hat{y}\|_{E/F} \to 0$, then we can choose y_n and y as representatives for $\hat{y}_n$, respectively $\hat{y}$, such that $\|y_n - y\| \to 0$ (using the same technique as in Theorem 2.2). Since L is a linear form over E, we have $L(y_n) \to L(y)$. Since $L(y_n) = L(\hat{y}_n)$ and $L(y) = L(\hat{y})$, we obtain

$$L(\hat{y}_n) \to L(\hat{y}).$$

We know from the general theory that $(E/F, \|\cdot\|_{E/F})$ is a Banach space. Therefore, since the norms $\|\cdot\|_{E/F}$ and $|\cdot|_{E/F}$ are equivalent, we have that $(E/F, |\cdot|_{E/F})$ is a Banach space too. Moreover, the last one is a Hilbert space, because

$$(\hat{x}, \hat{y}) \mapsto B(\hat{x}, \hat{y})$$

is a scalar product. From the equivalence of the norms, we also deduce the equality $(E/F, |\cdot|_{E/F})' = (E/F, \|\cdot\|_{E/F})'$. So $L \in (E/F, |\cdot|_{E/F})'$. Now, we can conclude by applying Riesz's Theorem to the Hilbert space $\big(E/F, B(\cdot,\cdot)\big)$ and to the linear form L. $\square$

Let us come back to our particular case, where

$$E = \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S), \text{ which is a Banach space,}$$
$$\tilde{E} = \boldsymbol{L}^2_\tau(S) \oplus \boldsymbol{H}^1_\nu(S),$$
$$B : E \times E \to \mathbb{R}\ ,\ B(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \int_S A(\boldsymbol{\zeta}, \boldsymbol{\eta})\, ds,$$
$$L : E \to \mathbb{R}\ ,\ L(\boldsymbol{\eta}) = \int_S \boldsymbol{f} \cdot \boldsymbol{\eta}\, ds,$$

where $\boldsymbol{f}$ is a vector field in $\boldsymbol{L}^2(S)$ and where

$$A(\boldsymbol{\zeta}, \boldsymbol{\eta})(p) := a^{\alpha\beta\sigma\tau}(p)\gamma_{\sigma\tau}(\boldsymbol{\zeta})(p)\gamma_{\alpha\beta}(\boldsymbol{\eta})(p) + a^{\alpha\beta\sigma\tau}(p)\rho_{\sigma\tau}(\boldsymbol{\zeta})(p)\rho_{\alpha\beta}(\boldsymbol{\eta})(p),$$

the expression in the right hand side being taken for a map (θ, ω) such that $p \in \theta(\omega)$. We have already seen that the value of this expression does not depend on the chosen map.

It is a classical result that $E \hookrightarrow \tilde{E}$ is a compact inclusion and it is not difficult to verify that L is a linear form over E and that B is a symmetric continuous bilinear form over the same space. We know from [6], Theorem 3.3-2, that $A(\boldsymbol{\eta}, \boldsymbol{\eta})(p) \geq 0$ on S, because $\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})(p)\big)$ and $\big(\rho_{\alpha\beta}(\boldsymbol{\eta})(p)\big)$ are symmetric matrices. So $B(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq 0$ for all $\boldsymbol{\eta} \in E$.

In order to apply Theorem 2.2, it remains to verify hypothesis (*iii*), which can be viewed in this setting as a “weak” inequality of Korn’s type in the entire space E. To this end, we shall use the inequality of Korn’s type without boundary conditions for one-mapping surfaces that is proved in [6].

Since S is a compact regular surface of class C^3, we can apply Theorem 2.1 and find N maps (θ_t, ω_t) of class C^3, satisfying conditions $(1) - (3)$ of this theorem such that $S = \cup_{t=1}^N \theta_t(\omega_t)$. We apply the inequality of Korn’s type without boundary conditions for each map. Accordingly, if $\boldsymbol{\eta}(p) = \tilde{\eta}_i(x)\boldsymbol{a}^i(p)$, we have for all $t \in \{1, \ldots, N\}$ that

$$\sum_\alpha \|\tilde{\eta}_\alpha\|^2_{H^1(\omega_t)} + \|\tilde{\eta}_3\|^2_{H^2(\omega_t)} \leq c_t \Big\{ \sum_\alpha \|\tilde{\eta}_\alpha\|^2_{L^2(\omega_t)} + \|\tilde{\eta}_3\|^2_{H^1(\omega_t)} + \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\|^2_{L^2(\omega_t)} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\|^2_{L^2(\omega_t)} \Big\} \tag{2.18}$$

for all $\tilde{\boldsymbol{\eta}} = (\tilde{\eta}_i) \in H^1(\omega_t) \times H^1(\omega_t) \times H^2(\omega_t)$. Here, we have applied Theorem 2.6-1 of [6]. The hypotheses of this theorem are satisfied, since $\boldsymbol{\eta} \in \boldsymbol{H}^m_\tau(S) \oplus \boldsymbol{H}^n_\nu(S)$ (with $m, n \leq 2$) implies that $\boldsymbol{\eta} \in \boldsymbol{H}^m_\tau(\theta_t(\omega_t)) \oplus \boldsymbol{H}^n_\nu(\theta_t(\omega_t))$ for all t, which is equivalent with $\tilde{\eta}_\alpha \in H^m(\omega_t)$, $\tilde{\eta}_3 \in H^n(\omega_t)$. Moreover, the norms $\|\boldsymbol{\eta}_\tau\|_{\boldsymbol{H}^m(\theta_t(\omega_t))}$ and $\|\boldsymbol{\eta}_\nu\|_{\boldsymbol{H}^n(\theta_t(\omega_t))}$ are equivalent with the norms $\|(\tilde{\eta}_\alpha)\|_{H^m(\omega_t) \times H^m(\omega_t)}$ and respectively $\|\tilde{\eta}_3\|_{H^n(\omega_t)}$.

Indeed, we have $\tilde{\eta}_i(x) = (\boldsymbol{\eta} \cdot \boldsymbol{a}^i)(p) = (\boldsymbol{\eta}_\tau + \boldsymbol{\eta}_\nu)(p) \cdot \boldsymbol{a}^i(p)$, so

$$\tilde{\eta}_\alpha(x) = (\boldsymbol{\eta}_\tau \cdot \boldsymbol{a}^\alpha)(p) \text{ and } \tilde{\eta}_3(x) = (\boldsymbol{\eta}_\nu \cdot \boldsymbol{a}^3)(p).$$

Conversely, we have $\boldsymbol{\eta}_\tau(p) = (\tilde{\eta}_\alpha \boldsymbol{a}^\alpha)(x)$ and $\boldsymbol{\eta}_\nu(p) = (\tilde{\eta}_3 \boldsymbol{a}^3)(x)$. The desired equivalences come from the fact that $\boldsymbol{a}_i$ and $\boldsymbol{a}^i$ are $\boldsymbol{C}^2(\overline{\omega}_t)$-vector fields.

By using Theorem 3.3-2 of [6], we deduce the existence of a constant $\tilde{c}_t > 0$ such that

$$A(\boldsymbol{\eta},\boldsymbol{\eta})(p) \geq \tilde{c}_t\Big(\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})(x)|^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})(x)|^2\Big). \tag{2.19}$$

Therefore, we infer from (2.18) that

$$\begin{aligned} &\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{H}^1(\theta_t(\omega_t))} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^2(\theta_t(\omega_t))} \\ &\qquad \leq C_t \left\{ \|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{L}^2(\theta_t(\omega_t))} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^1(\theta_t(\omega_t))} + \int_{\theta_t(\omega_t)} A(\boldsymbol{\eta},\boldsymbol{\eta})\, ds \right\} \\ &\qquad\qquad \text{for all } \boldsymbol{\eta} \in \boldsymbol{H}^1_\tau(\theta_t(\omega_t)) \oplus \boldsymbol{H}^2_\nu(\theta_t(\omega_t)). \end{aligned}$$

Since $A(\boldsymbol{\eta},\boldsymbol{\eta}) \geq 0$ over S, we also have

$$\begin{aligned} &\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{H}^1(\theta_t(\omega_t))} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^2(\theta_t(\omega_t))} \\ &\qquad \leq C_t \left\{ \|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{L}^2(\theta_t(\omega_t))} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^1(\theta_t(\omega_t))} + \int_S A(\boldsymbol{\eta},\boldsymbol{\eta})\, ds \right\} \end{aligned}$$

for all $\boldsymbol{\eta} \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$. Taking the sum with respect to t, we obtain

$$\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{H}^1(S)} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^2(S)} \leq c \left\{ \|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{L}^2(S)} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^1(S)} + N \int_S A(\boldsymbol{\eta},\boldsymbol{\eta})\, ds \right\}.$$

Consequently, there exists a constant $c_0 > 0$ such that

$$\|\boldsymbol{\eta}\|_E \leq c_0(\|\boldsymbol{\eta}\|_{\tilde{E}} + \sqrt{B(\boldsymbol{\eta},\boldsymbol{\eta})})$$

for all $\boldsymbol{\eta} \in E$, which is exactly hypothesis (iii) of Theorem 2.2, where we have denoted $|\boldsymbol{\eta}| := \sqrt{B(\boldsymbol{\eta},\boldsymbol{\eta})}$.

For simplicity, let us denote $E/\ker B$ by $\dot{\boldsymbol{V}}(S)$. We recall that the space F that appears in Theorem 2.2 is in fact $\ker B$ (see remark 2.1). Applying Theorem 2.2 to our case, we obtain the following theorem which establishes the coercivity of the bilinear form appearing in the Koiter's model:

Theorem 2.3. *Let S be a regular compact surface of class C^3. Then there exists a constant $c > 0$ such that*

$$\|\hat{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(S)} \leq c\Big(\int_S A(\boldsymbol{\eta},\boldsymbol{\eta})\, ds\Big)^{\frac{1}{2}} \text{ for all } \hat{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(S),$$

where $\boldsymbol{\eta}$ is an arbitrary representative of $\hat{\boldsymbol{\eta}}$.

Of course, the same result still holds true if one replace the bilinear form A with the following bilinear form

$$A^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta}) := \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta}).$$

Indeed, with the notation $B^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta}) := \int_S A^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta})\,ds$, it is obvious that A^ε, respectively B^ε, satisfies the properties of A, respectively B, that we have used in our analysis. Moreover, we have $\ker B^\varepsilon = \ker B$.

Now, if the bilinear form is defined by

$$B(\boldsymbol{\zeta},\boldsymbol{\eta}) := \int_S \left(a^{\alpha\sigma}a^{\beta\tau}\{\gamma_{\alpha\beta}(\boldsymbol{\zeta})\gamma_{\sigma\tau}(\boldsymbol{\eta}) + \rho_{\alpha\beta}(\boldsymbol{\zeta})\rho_{\sigma\tau}(\boldsymbol{\eta})\}\right)(p)\,ds,$$

we obtain the following theorem which gives the desired inequality of Korn's type on compact surfaces without boundary:

Theorem 2.4. *Let S be a regular compact surface of class C^3. Then there exists a constant $c > 0$ such that*

$$\|\hat{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(S)} \le c(\|\gamma(\boldsymbol{\eta})\|_{\boldsymbol{L}^2(S)} + \|\rho(\boldsymbol{\eta})\|_{\boldsymbol{L}^2(S)}) \text{ for all } \hat{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(S),$$

where $\boldsymbol{\eta}$ is an arbitrary representative of $\hat{\boldsymbol{\eta}}$ and

$$\|\gamma(\boldsymbol{\eta})\|^2_{\boldsymbol{L}^2(S)} := \int_S \left(a^{\alpha\sigma}a^{\beta\tau}\gamma_{\alpha\beta}(\boldsymbol{\eta})\gamma_{\sigma\tau}(\boldsymbol{\eta})\right)(p)\,ds,$$

$$\|\rho(\boldsymbol{\eta})\|^2_{\boldsymbol{L}^2(S)} := \int_S \left(a^{\alpha\sigma}a^{\beta\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\rho_{\sigma\tau}(\boldsymbol{\eta})\right)(p)\,ds.$$

The compatibility condition on L^ε (where $L^\varepsilon := \int_S \boldsymbol{f}^\varepsilon \cdot \boldsymbol{\eta}\,ds$) become in our particular case

$$\int_S \boldsymbol{f}^\varepsilon \cdot \boldsymbol{\eta}\,ds = 0 \text{ for all } \boldsymbol{\eta} \in \ker B.$$

We recall that, if $\hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(s)$ and $(\boldsymbol{\zeta},\boldsymbol{\zeta'})$, $(\boldsymbol{\eta},\boldsymbol{\eta'})$ are two pairs of representatives for $\hat{\boldsymbol{\zeta}}$ and respectively $\hat{\boldsymbol{\eta}}$, then $B^\varepsilon(\boldsymbol{\zeta'},\boldsymbol{\eta'}) = B^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta})$ (thanks to (2.13)) and $L(\boldsymbol{\eta'}) = L(\boldsymbol{\eta})$ (thanks to the compatibility condition). By applying corollary 2.1 to the spaces E, $\tilde{E}$, to the bilinear form B^ε and to the linear form L^ε appearing in our particular case, we establish the existence of a solution to the Koiter's model (2.20) for a linearly elastic shell with a compact middle surface (note that problem (2.20) is the analogue of the problem (2.8) in the case of compact regular surfaces):

Theorem 2.5. *Let S be a regular compact surface of class C^3 and let $\boldsymbol{f}^\varepsilon \in \boldsymbol{L}^2(S)$ be a vector field on S such that $\int_S \boldsymbol{f}^\varepsilon \cdot \boldsymbol{\eta}\,ds = 0$ for all $\boldsymbol{\eta} \in \ker B^\varepsilon$. Then there exists one and only one solution to the problem*

$$\text{(2.20)} \qquad \left\{\begin{array}{l} \textit{Find } \hat{\boldsymbol{\zeta}}^{\boldsymbol{\varepsilon}} \in \dot{\boldsymbol{V}}(S) \textit{ such that} \\ \displaystyle\int_S A^\varepsilon(\boldsymbol{\zeta}^{\boldsymbol{\varepsilon}},\boldsymbol{\eta})\,ds = \int_S \boldsymbol{f}^\varepsilon \cdot \boldsymbol{\eta}\,ds \textit{ for all } \hat{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(S), \end{array}\right.$$

where $\boldsymbol{\zeta}^{\boldsymbol{\varepsilon}}$, $\boldsymbol{\eta}$ are arbitrary representatives of $\hat{\boldsymbol{\zeta}}^{\boldsymbol{\varepsilon}}$ and respectively $\hat{\boldsymbol{\eta}}$.

Now, we would like to better describe the space $\dot{\boldsymbol{V}}(S)$. For exemple, the first natural question is to find out if $\dot{\boldsymbol{V}}(S)$ is a proper quotient space of $\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$, i. e., to find out if $\ker B = \{\boldsymbol{0}\}$ or not. The answer is given by the following theorem, which describes the space $\ker B$ in the case of general regular surfaces (defined by (2.1)).

Theorem 2.6 (infinitesimal rigid displacement lemma on a general regular surface)**.** *Let S be a regular surface of class C^3 and let $\boldsymbol{\eta}$ be a vector field in $\ker B$. Then there exists two vectors $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$ such that*

$$\boldsymbol{\eta}(p) = \boldsymbol{c} + \boldsymbol{d} \wedge \boldsymbol{p}, \text{ for all } p \in S,$$

where $\boldsymbol{p} := \overrightarrow{Op}$ is the position vector of p.

Proof. Let $p_0 \in S$ and let (θ, ω) be a local map at p_0 (i.e., $p_0 \in \theta(\omega) \subset S$) such that $\omega \subset \mathbb{R}^2$ is connected, $\theta : \omega \to \mathbb{R}^3$ is an injective application of class C^3, $\theta(\omega)$ is open in S, and the vectors $(\partial_\alpha \theta)$ are linearly independent in all points of ω. Notice that such a map exists, by the definition of a regular surface of class C^3 (see section 1).

Since $\boldsymbol{\eta} \in \ker B$, we have in particular $B(\boldsymbol{\eta}, \boldsymbol{\eta}) = 0$, which is equivalent with $A(\boldsymbol{\eta}, \boldsymbol{\eta}) = 0$. Consider the surface $S' := \theta(\omega)$ and notice that $A(\boldsymbol{\eta}_{|S'}, \boldsymbol{\eta}_{|S'}) = A(\boldsymbol{\eta}, \boldsymbol{\eta})_{|S'} = 0$ on S'. Therefore, it follows from (2.19) that $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ vanish on S'. Applying Theorem 2.6-3 of [6] to the surface S' and to the vector field $\boldsymbol{\eta}_{S'}$ thus gives the existence of two vectors $\boldsymbol{c}(\theta)$ and $\boldsymbol{d}(\theta)$ such that $\boldsymbol{\eta}(p) = \boldsymbol{c}(\theta) + \boldsymbol{d}(\theta) \wedge \theta(x)$ for all $x \in \omega$, or equivalently, that

$$\boldsymbol{\eta}(p) = \boldsymbol{c}(\theta) + \boldsymbol{d}(\theta) \wedge \boldsymbol{p} \text{ for all } p \in S'.$$

So "locally", the theorem is true. To prove the global result, it suffices to show that $\boldsymbol{c}(\theta)$ and $\boldsymbol{d}(\theta)$, which apparently depend on the local map, are in fact constants over the entire surface S. To this end, the decisive argument is the connectedness of S.

Since S is a regular surface of class C^3, there exists a collection of maps $(\theta_t, \omega_t)_{t \in A}$ such that ω_t are connected, $S = \cup_{t \in A} \theta_t(\omega_t) = \cup_{t \in A} S_t$, and θ_t, ω_t satisfy conditions $(1) - (3)$ of (2.1). Notice that $S_t := \theta_t(\omega_t)$ are also regular surfaces of class C^3.

We have seen in the first part of the proof that, for all $t \in A$, there exist $\boldsymbol{c}_t, \boldsymbol{d}_t \in \mathbb{R}^3$ such that $\boldsymbol{\eta}(p) = \boldsymbol{c}_t + \boldsymbol{d}_t \wedge \boldsymbol{p}$, for all $p \in S_t$. Now, fix $t_0 \in A$ and define the sets

$$A_0 := \{t \in A\, ;\, \boldsymbol{c}_t = \boldsymbol{c}_{t_0} \text{ and } \boldsymbol{d}_t = \boldsymbol{d}_{t_0}\} \text{ and } A_1 = A \setminus A_0.$$

Then $S_0 := \cup_{t \in A_0} S_t$ and $S_1 := \cup_{t \in A_1} S_t$ are open sets in S, since each S_t is open in S. Obviously, $S = S_0 \cup S_1$ and $t_0 \in A_0$, so $S_{t_0} \subset S_0$, which proves that $S_0 \neq \emptyset$.

Now, let us prove that $S_1 = \emptyset$. We argue be contradiction. Suppose that $S_1 \neq \emptyset$, which is equivalent with $A_1 \neq \emptyset$. Then $S_0 \cap S_1 \neq \emptyset$, because S is connected. So there exists $t_1 \in A_1$ such that $S_{t_1} \cap S_0 \neq \emptyset$. For all $p \in S_{t_1} \cap S_0$, we have

$$\boldsymbol{\eta}(p) = \boldsymbol{c}_{t_1} + \boldsymbol{d}_{t_1} \wedge \boldsymbol{p} = \boldsymbol{c}_{t_0} + \boldsymbol{d}_{t_0} \wedge \boldsymbol{p}. \tag{2.21}$$

Since $S_{t_1} \cap S_0$ is open in S, there exist three non-colinear points $p, q, r \in S_{t_1} \cap S_0$ (i.e. $\boldsymbol{p} - \boldsymbol{q} \neq \lambda(\boldsymbol{p} - \boldsymbol{r})$ for all $\lambda \in \mathbb{R}$). Otherwise, for any local map that describes a portion of $S_{t_1} \cap S_0$, the vectors $\boldsymbol{a}_\alpha$ cannot be linearly independent. We write (2.21) for $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$:

$$\begin{aligned}\boldsymbol{c}_{t_1} + \boldsymbol{d}_{t_1} \wedge \boldsymbol{p} &= \boldsymbol{c}_{t_0} + \boldsymbol{d}_{t_0} \wedge \boldsymbol{p}\\ \boldsymbol{c}_{t_1} + \boldsymbol{d}_{t_1} \wedge \boldsymbol{q} &= \boldsymbol{c}_{t_0} + \boldsymbol{d}_{t_0} \wedge \boldsymbol{q}\\ \boldsymbol{c}_{t_1} + \boldsymbol{d}_{t_1} \wedge \boldsymbol{r} &= \boldsymbol{c}_{t_0} + \boldsymbol{d}_{t_0} \wedge \boldsymbol{r}.\end{aligned}$$

Substracting the second equation from the first one, we obtain $\boldsymbol{d}_{t_1} \wedge (\boldsymbol{p} - \boldsymbol{q}) = \boldsymbol{d}_{t_0} \wedge (\boldsymbol{p} - \boldsymbol{q})$, so that $(\boldsymbol{d}_{t_1} - \boldsymbol{d}_{t_0}) \wedge (\boldsymbol{p} - \boldsymbol{q}) = 0$. Therefore $\boldsymbol{d}_{t_1} - \boldsymbol{d}_{t_0}$ is colinear with $\boldsymbol{p} - \boldsymbol{q}$. We make the same operations with the first and the last equation and we obtain that $\boldsymbol{d}_{t_1} - \boldsymbol{d}_{t_0}$ is also colinear with $\boldsymbol{p} - \boldsymbol{r}$. That is, there exists $\lambda, \mu \in \mathbb{R}$ such that

$$\boldsymbol{d}_{t_1} - \boldsymbol{d}_{t_0} = \lambda(\boldsymbol{p} - \boldsymbol{q}) = \mu(\boldsymbol{p} - \boldsymbol{r}).$$

But $\boldsymbol{p} - \boldsymbol{q}$ and $\boldsymbol{p} - \boldsymbol{r}$ are non-colinear, so we must have $\lambda = \mu = 0$. Consequently, $\boldsymbol{d}_{t_1} = \boldsymbol{d}_{t_0}$. Moreover, $\boldsymbol{c}_{t_1} = \boldsymbol{c}_{t_0}$, thanks to (2.21). But this proves that $t_1 \in A_0$, which contradicts the fact that $t_1 \in A_1$ (because A_0 and A_1 are disjoint sets, by definition).

So $S_1 = \emptyset$ and $S = S_0$. To conclude the proof, we take $\boldsymbol{c} = \boldsymbol{c}_{t_0}$ and $\boldsymbol{d} = \boldsymbol{d}_{t_0}$. □

Remarks 2.2. 1. Using the same arguments as in the previous proof, where we have shown that $S_1 = \emptyset$, we can prove that $\boldsymbol{c}$ and $\boldsymbol{d}$ of Theorem 2.6 are unique for a given vector field $\boldsymbol{\eta} \in \ker B$.

2. Theorem 2.6 shows not only that $\dot{\boldsymbol{V}}(S) \neq \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$, but more precisely, that $\dot{\boldsymbol{V}}(S)$ is isomorphic with a subspace of $\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$ of codimension 6.

3. Theorems 2.3, 2.4 and 2.5 also holds true for a larger class of surfaces, more specifically, for surfaces of class C^3 with Lipschitz boundary such that the set $S \cup \partial S$ is compact (here, ∂S is the boundary of S as a surface, not as a subset of $\mathbb{R}^3$). Indeed, we have used the compactedness of the surface only in the proof of Theorem 2.1, or this theorem also holds true for the surfaces described above.

4. Let S be a general bounded surface with boundary and let Γ be a measurable subset of the boundary of S whose relative measure is > 0. If we require that the solution of Koiter's problem (2.9) satisfy in addition some boundary conditions on Γ, then the quotient space $\dot{\boldsymbol{V}}(S) = E/\ker B$ appearing in Theorem 2.3 coincides with the entire space E. More specifically, problem (2.9) is posed in this case over the space $E = V_K(S)$, where

$$\boldsymbol{V}_K(S) := \left\{\boldsymbol{\eta} \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)\,;\, \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta}_\nu = 0 \text{ on } \Gamma\right\},$$

where the normal derivative of any vector field $\boldsymbol{\xi} = (\xi_i) \in \boldsymbol{H}^2(S)$ is defined by $\partial_\nu \boldsymbol{\xi} := (\nabla \xi_i \cdot \boldsymbol{\nu})$ over the boundary of S. Here $\nabla \xi_i := [\xi_i]_{|\alpha} \boldsymbol{a}^\alpha$, where $[\xi_i]_{|\alpha}$ are the covariant derivatives of the function ξ_i. The equality $E/\ker B = E$

comes from the fact that $\ker B = \{0\}$ over $V_K(S)$. One can prove this by using theorem 2.6 together with a connectedness argument. If, in addition, S is a surface with Lipschitz boundary such that the set $S \cup \partial S$ is a compact set (see the remark above), then we obtain the analogue of theorems 2.4 and 2.5 with $\dot{\boldsymbol{V}}(S)$ replaced by $\boldsymbol{V}_K(S)$. Note that in the case of one-mapping surfaces, these theorems have already been proved in [6] (see Theorem 2.6-4 and the begining of chapter 7).

5. Even in the case of compact surfaces without boundary, we can avoid considering quotient spaces. It suffices to consider the space $\boldsymbol{V}_\perp(S) := \big\{\boldsymbol{\eta} \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)\,;\, \int_S \boldsymbol{\eta} \cdot \boldsymbol{\zeta}\, ds = 0 \text{ for all } \boldsymbol{\zeta} \in \ker B\big\}$ instead of the space $\dot{\boldsymbol{V}}(S)$. Note that the space $\boldsymbol{V}_\perp(S)$ is in fact the subspace $(\ker B)^\perp$ of $\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$, where the orthogonal of $\ker B$ is taken with respect to the scalar product $(\boldsymbol{\eta}, \boldsymbol{\zeta}) \mapsto \int_S \boldsymbol{\eta} \cdot \boldsymbol{\zeta}\, ds$. This idea has already been used by Şlicaru in his doctoral dissertation [12]. However, fixing in this manner a representative of $\hat{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(S)$ (since this is what we do eventually) does not correspond to any physical requirement or principle. This is why we have preferred to solve the problem over the quotient space $\dot{\boldsymbol{V}}(S)$.

6. The regularity of the surface S can be weakened by using (for each local map) the results of Blouza and Le Dret [3] and of Anicic, Le Dret and Raoult [1] on shells with little regularity. For instance, Theorem 2.6 still holds true if the surface S is of class $W^{2,\infty}$ and

$$\boldsymbol{\eta} \in \ker B := \{\boldsymbol{\eta} \in \boldsymbol{H}^1(S);\ B(\boldsymbol{\eta}, \boldsymbol{\zeta}) = 0 \text{ for all } \boldsymbol{\zeta} \in \boldsymbol{H}^1(S)\}.$$

In this case, the covariant components of the linearized change of metric and curvature tensors are defined by

$$\begin{aligned} \gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta \boldsymbol{\eta} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{a}_\beta) \\ \rho_{\alpha\beta}(\boldsymbol{\eta}) &:= (\partial_{\alpha\beta} \boldsymbol{\eta} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \boldsymbol{\eta}) \cdot \boldsymbol{a}_3. \end{aligned}$$

For further details see Blouza and Le Dret [3].

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CHAPITRE 3

Sur les immersions isométriques d'un espace de Riemann

Ce travail a fait l'objet des publications suivantes :

MARDARE S., *On isometric immersions of a Riemannian space with little regularity*, accepté dans Analysis and Applications.

MARDARE S., *On isometric immersions of a Riemannian space under weak regularity assumptions*, C.R. Acad. Sci. Paris, Ser. I 337, 2003, 785-790.

On isometric immersions of a Riemannian space with little regularity

1. Introduction

Let a Riemannian metric in an open subset Ω of $\mathbb{R}^d$ be given by a symmetric positive definite matrix field (g_{ij}) and assume that its Riemann curvature tensor vanishes. This means that

$$\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^l_{ik} \Gamma^p_{jl} - \Gamma^l_{ij} \Gamma^p_{kl} = 0 \text{ in } \Omega$$

for all $i, j, k, p \in \{1, 2, ..., d\}$, where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} \right)$$

are the Christoffel symbols associated with the Riemannian metric.

If the components g_{ij} of the metric tensor are of class C^2, then a classical theorem in differential geometry asserts that the Riemannian space $(\Omega, (g_{ij}))$ is locally isometrically immersed in the d-dimensional Euclidean space, that is, for any given point of Ω, there exist a neighborhood V of the given point and an application Θ of class C^3 from V into the d-dimensional Euclidean space such that $g_{ij}(x) = \frac{\partial \Theta(x)}{\partial x_i} \cdot \frac{\partial \Theta(x)}{\partial x_j}$ for all $x = (x_1, x_2, ..., x_d) \in V$ (see [11] for instance). If in addition Ω is simply-connected, then there exists a global isometric immersion as shown in [3]. Besides, such an isometric immersion is unique up to isometries in the d-dimensional Euclidean space.

This classical result has been recently improved: if the components g_{ij} of the metric tensor are of class C^1, then it has been proved in [7] that the

Riemannian space is locally isometrically immersed in the d-dimensional Euclidean space by a mapping Θ of class C^2.

The purpose of this paper is to show that the continuity of the derivatives of the metric (g_{ij}) can be further relaxed in order that the Riemannian space be locally isometrically immersed in the d-dimensional Euclidean space. More specifically, we assume that the metric is of class $W^{1,\infty}_{\rm loc}$ in Ω and that its Riemann curvature tensor vanishes in a distributional sense. Then we show that the Riemannian space is locally isometrically immersed in the d-dimensional Euclidean space by a mapping Θ of class $W^{2,\infty}_{\rm loc}$ in its domain of definition. If in addition Ω is simply-connected, then we show that the Riemannian space $(\Omega, (g_{ij}))$ is isometrically immersed in the d-dimensional Euclidean space, that is, the isometric immersion Θ is defined over Ω. Since the local result is an immediate consequence of the global result, only the latter is presented in this paper.

2. Preliminaries

All functions and fields appearing in this paper are real valued and the summation convention with respect to repeated indices and exponents is used. Boldface letters designate vectors or vector fields and capital letters (except Θ and Φ, which designate vector fields) designate matrices or matrix fields.

The d-dimensional Euclidean space will be identified with $\mathbb{R}^d$. Let $\boldsymbol{u}\cdot\boldsymbol{v}$ designates the Euclidean inner product for $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d$ and let $|\boldsymbol{u}| = \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}$ denote the Euclidean norm of $\boldsymbol{u}\in\mathbb{R}^d$.

Let $\mathbb{M}^d$ designate the set of all square matrices of order d, let $\mathbb{S}^d_>$ designate its subset of all symmetric, positive definite matrices of order d, and let $\mathbb{O}^d$ designate the set of all orthogonal matrices of order d. An isometry in $\mathbb{R}^d$ is an element of the set

$$\{\boldsymbol{u}\in\mathbb{R}^d \mapsto \boldsymbol{a} + Q\boldsymbol{u}; \boldsymbol{a}\in\mathbb{R}^d, Q\in\mathbb{O}^d\}.$$

The notation $\mathbb{M}^{q,l}$ designates the space of all matrices with q rows and l columns. The space $\mathbb{M}^{q,l}$ is endowed with the operator norm $|\cdot|$ defined by

$$|A| := \sup_{x\in\mathbb{R}^l\setminus\{0\}} \frac{|Ax|}{|x|}.$$

It is well known that $|A|$ is also given by the square root of the largest eigenvalue of the matrix A^TA (A^T denotes the transpose of the matrix A). Notations such as (a_{ij}), or (a^i_j), designate the matrix whose entries are the elements a_{ij}, or a^i_j, which may be either real numbers or real functions. The first, or upper, index is the row index and the second, or lower, index is the column index.

The open ball of radius $R>0$ centered at $x\in\mathbb{R}^d$ is denoted $B(x,R)$ and the distance between two subsets A and B of $\mathbb{R}^d$ is defined by

$$\operatorname{dist}(A,B) = \inf_{x\in A, y\in B} |x-y|.$$

Let Ω be an open subset of $\mathbb{R}^d$. Given two points $x, y \in \Omega$, a *path joining x to y* is any mapping $\boldsymbol{\gamma} \in C^0([0,1];\mathbb{R}^d)$ that satisfies $\boldsymbol{\gamma}(t) \in \Omega$ for all $t \in [0,1]$ and $\boldsymbol{\gamma}(0) = x$ and $\boldsymbol{\gamma}(1) = y$. The image by $\boldsymbol{\gamma}$ of the interval $[0,1]$ is denoted $\mathrm{Im}\boldsymbol{\gamma} := \{\boldsymbol{\gamma}(t); t \in [0,1]\}$. Such an open set Ω is connected if and only if, for all $x, y \in \Omega$, there exists a path joining x to y. The set Ω is simply-connected if and only if, for all $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1 \in C^0([0,1];\Omega)$ such that $\boldsymbol{\gamma}_0(0) = \boldsymbol{\gamma}_1(0)$ and $\boldsymbol{\gamma}_0(1) = \boldsymbol{\gamma}_1(1)$, there exists a homotopy joining $\boldsymbol{\gamma}_0$ to $\boldsymbol{\gamma}_1$, an application $\boldsymbol{\varphi} \in C^0([0,1] \times [0,1];\Omega)$ such that

$$\begin{aligned}
&\boldsymbol{\varphi}(t,0) = \boldsymbol{\gamma}_0(t) \text{ and } \boldsymbol{\varphi}(t,1) = \boldsymbol{\gamma}_1(t) \text{ for all } t \in [0,1],\\
&\boldsymbol{\varphi}(0,s) = \boldsymbol{\gamma}_0(0) \text{ and } \boldsymbol{\varphi}(1,s) = \boldsymbol{\gamma}_0(1) \text{ for all } s \in [0,1].
\end{aligned}$$

A subset Ω of $\mathbb{R}^d$ satisfies the *cone property* if each point of Ω is the vertex of a cone contained in Ω along with its closure, the cone being defined by the inequalities $y_1^2 + y_2^2 + ... + y_{d-1}^2 < by_d^2$ and $0 < y_d < a$ in some Cartesian system, where a, b are constants. Notice that a bounded open set with a Lipschitz boundary satisfies the cone property.

The geodesic distance between two points $x, y \in \Omega$ is the infimum of the length, denoted $L(\boldsymbol{\gamma})$, of all paths joining x to y and the geodesic diameter of Ω is the number $D_\Omega \in [0,\infty]$ defined by

$$D_\Omega := \sup_{x,y\in\Omega} \inf_{\boldsymbol{\gamma}} \{L(\boldsymbol{\gamma});\ \boldsymbol{\gamma} \text{ is a path joining } x \text{ to } y\}.$$

For any open subset Ω of $\mathbb{R}^d$, $d \geq 1$, the space of indefinitely derivable real functions with compact support included in Ω is denoted $\mathcal{D}(\Omega)$. The support of a continuous function $\varphi : \Omega \to \mathbb{R}$ is defined as

$$\mathrm{supp}\varphi = \overline{\{x \in \Omega, \varphi(x) \neq 0\}}.$$

The space of distributions over Ω is denoted $\mathcal{D}'(\Omega)$. The usual Sobolev space being denoted $W^{m,p}(\Omega;\mathbb{M}^{q,l})$, we let

$$W^{m,p}_{loc}(\Omega;\mathbb{M}^{q,l}) := \{Y \in \mathcal{D}'(\Omega;\mathbb{M}^{q,l}); Y \in W^{m,p}(U;\mathbb{M}^{q,l}) \text{ for all open set } U \Subset \Omega\},$$

where the notation $U \Subset \Omega$ means that the closure of U in $\mathbb{R}^d$ is a compact subset of Ω. For real-valued function spaces, we shall use the notation $W^{m,p}(\Omega)$ instead of $W^{m,p}(\Omega;\mathbb{R})$, $\mathcal{D}(\Omega)$ instead of $\mathcal{D}(\Omega;\mathbb{R})$, etc.

Let $x = (x_1, x_2, ..., x_d)$, $x' = (x_1, x_2, ..., x_{d-1})$ and $x'' = (x_1, x_2, ..., x_{d-2})$ respectively denote a generic point in $\mathbb{R}^d$, $\mathbb{R}^{d-1}$ and $\mathbb{R}^{d-2}$. The Lebesgue measure in $\mathbb{R}^d$ is denoted $\mathcal{L}^d$, or dx when it is used in a Lebesgue integral. A subset in $\mathbb{R}^k$, $k \geq 1$, is said to have zero measure if its $\mathcal{L}^k$-measure is zero. Finally, let

$$\partial_i := \frac{\partial}{\partial x_i} \text{ and } \partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_d^{\alpha_d}},$$

where $\alpha := (\alpha_1, \alpha_2, ..., \alpha_d)$ is a multi-index and $|\alpha| := \alpha_1 + \alpha_2 + ... + \alpha_d$.

We also make the following convention for classes of functions with respect to the equality almost everywhere, which is an equivalence relation: if $\dot{f} \in L^\infty_{\rm loc}(\Omega)$, we will always use the representative f of $\dot{f}$ given by

$$f(x) := \liminf_{\varepsilon\to 0} \frac{1}{\mathcal{L}^d(B(x,\varepsilon))} \int_{B(x,\varepsilon)} \widetilde{f}(y)dy,$$

where $\widetilde{f}$ is any representative of the class $\dot{f} \in L^\infty_{\rm loc}(\Omega)$ (this definition is clearly independent of the choice of the representative $\widetilde{f}$). This choice of the representative insures that

$$\|\dot{f}\|_{L^\infty(U)} = \sup_{x\in U} |f(x)| \text{ for all open subset } U \Subset \Omega.$$

Also, for any $\dot{f} \in W^{1,\infty}_{\rm loc}(\Omega)$, we will choose the continuous representative f of $\dot{f}$.

The notation $(g_{ij}) \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{S}^d_>)$ means that each component of the matrix belongs to the space $W^{1,\infty}_{\rm loc}(\Omega)$ and that $(g_{ij}(x)) \in \mathbb{S}^d_>$ *for all* $x \in \Omega$. For simplicity, we will use the same notation for a class of functions and its representative chosen as before (i.e., f will denote either the class of f in $L^\infty_{\rm loc}(\Omega)$ or its representative chosen as before), the distiction between them being made according to the context.

3. Preliminary lemmas

We gather here two lemmas which are needed in the proof of the main result (see Theorem 3.1) of the next section. The first lemma is a stronger form of the Lebesgue-Besicovitch theorem (see [8, 12]). In its statement below, $\mathcal{L}^1_{loc}(\mathbb{R}^d)$ designate the space of all measurable functions (*not* classes of functions) whose modulus is locally integrable.

Lemma 3.1. *Let* $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$*. Then there exists a set* $X_d \subset \mathbb{R}$ *with zero measure such that*

$$f(\cdot, \overline{x}_d) \in \mathcal{L}^1_{loc}(\mathbb{R}^{d-1})$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{\overline{x}_d-\varepsilon}^{\overline{x}_d+\varepsilon} \int_{\omega'} \left|f(x', x_d) - f(x', \overline{x}_d)\right| dx = 0 \tag{3.1}$$

for all bounded open sets $\omega' \subset \mathbb{R}^{d-1}$ *and all numbers* $\overline{x}_d \in \mathbb{R} \setminus X_d$*.*

Proof. Since any bounded subset of $\mathbb{R}^{d-1}$ is contained in a ball $B_n := B(0, n)$, where $n \in \mathbb{N}$, it suffices to prove (3.1) for all $\omega' = B_n$, $n \in \mathbb{N}$. Note that f belongs to $\mathcal{L}^1\big(B_n \times (-n, n)\big)$ for all $n \in \mathbb{N}$. Then Fubini's theorem shows that there exists a set $\widetilde{X}_n \subset (-n, n)$, with zero measure, such that $f(\cdot, \overline{x}_d) \in \mathcal{L}^1(B_n)$ for all $\overline{x}_d \in (-n, n) \setminus \widetilde{X}_n$. Let $\widetilde{X} = \cup_{n\in\mathbb{N}} \widetilde{X}_n$. It is easily seen that the measure of $\widetilde{X}$ is zero and that $f(\cdot, \overline{x}_d) \in \mathcal{L}^1(B_n)$ for all $\overline{x}_d \in \mathbb{R} \setminus \widetilde{X}$ and all $n \in \mathbb{N}$. Therefore $f(\cdot, \overline{x}_d) \in \mathcal{L}^1_{\rm loc}(\mathbb{R}^{d-1})$ for all $\overline{x}_d \in \mathbb{R} \setminus \widetilde{X}$.

Note that if q is a function of $\mathcal{L}^1(\mathbb{R}^{d-1})$, then the function

$$x_d \in \mathbb{R} \longmapsto \int_{B_n} |f(x', x_d) - q(x')|\, dx' \in \mathbb{R}$$

belongs to $\mathcal{L}^1_{\text{loc}}(\mathbb{R})$. Let Q be a countable and dense subset of $\mathcal{L}^1(\mathbb{R}^{d-1})$. By Lebesgue-Besicovitch theorem, for each $q \in Q$, there exists a set $X(q,n) \subset \mathbb{R}$, with zero measure, such that

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{\overline{x}_d-\varepsilon}^{\overline{x}_d+\varepsilon} \left\{ \int_{B_n} \left| f(x', x_d) - q(x') \right| dx' \right\} dx_d$$
$$= \int_{B_n} \left| f(x', \overline{x}_d) - q(x') \right| dx'$$

for all $\overline{x}_d \in \mathbb{R} \setminus X(q,n)$. Let

$$X_d = \cup_{\substack{q\in Q\\ n\in\mathbb{N}}} X(q,n) \cup \widetilde{X}$$

and note that its measure vanishes. We are now able to prove (3.1) for $\omega' := B_n$.

Let $\overline{x}_d \in \mathbb{R} \setminus X_d$ and let a sequence $(q_m) \in Q$ be such that

$$\lim_{m\to\infty} \int_{B_n} \left| f(x', \overline{x}_d) - q_m(x') \right| dx' = 0.$$

Then

$$\limsup_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{\overline{x}_d-\varepsilon}^{\overline{x}_d+\varepsilon} \int_{B_n} \left| f(x', x_d) - f(x', \overline{x}_d) \right| dx$$
$$\leq \limsup_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{\overline{x}_d-\varepsilon}^{\overline{x}_d+\varepsilon} \int_{B_n} \left| f(x', x_d) - q_m(x') \right| dx$$
$$+ \int_{B_n} \left| f(x', \overline{x}_d) - q_m(x') \right| dx'$$
$$= 2 \int_{B_n} \left| f(x', \overline{x}_d) - q_m(x') \right| dx'$$

for all $m \in \mathbb{N}$. Since the right hand side goes to zero as m goes to ∞, relation (3.1) is proved for $\omega' = B_n$. This implies that relation (3.1) holds for all bounded open subsets ω' of $\mathbb{R}^{d-1}$. □

Remark 3.1. If in addition $f \in \mathcal{L}^\infty(\mathbb{R}^d)$, then the set X_d can be chosen in such a way that $f(\cdot, \overline{x}_d) \in \mathcal{L}^\infty(\mathbb{R}^{d-1})$ for all $\overline{x}_d \in \mathbb{R} \setminus X_d$.

Lemma 3.2. *Let $\omega = \omega' \times (\overline{x}_d - \varepsilon_d, \overline{x}_d + \varepsilon_d)$ be an open parallelepiped in $\mathbb{R}^d$, where $\omega' = \prod_{i=1}^{d-1}(\overline{x}_i - \varepsilon_i, \overline{x}_i + \varepsilon_i)$ and $d \geq 1$, and let $A, B, C \in L^\infty(\omega; \mathbb{M}^l)$, $l \geq 1$. Let $M > 0$ be a constant such that*

$$\max\left\{ \|A\|_{L^\infty(\omega;\mathbb{M}^l)}, \|B\|_{L^\infty(\omega;\mathbb{M}^l)}, \|C\|_{L^\infty(\omega;\mathbb{M}^l)} \right\} \leq M,$$

and let U_0, V_0 *be given in* $L^\infty(\omega; \mathbb{M}^{q,l})$. *For almost all* $x' \in \omega'$ *and all* $x_d \in (\overline{x}_d - \varepsilon_d, \overline{x}_d + \varepsilon_d)$, *define* $U_n(x', x_d), V_n(x', x_d) \in \mathbb{M}^{q,l}$ *by*

$$\begin{aligned} U_{n+1}(x', x_d) &= \int_{\overline{x}_d}^{x_d} (U_n A)(x', t_d) dt_d, \\ V_{n+1}(x', x_d) &= \int_{\overline{x}_d}^{x_d} (V_n B)(x', t_d) dt_d + \int_{\overline{x}_d}^{x_d} (U_n C)(x', t_d) dt_d \end{aligned}$$

for all $n \in \mathbb{N}$. *Then* $U_n, V_n \in L^\infty(\omega; \mathbb{M}^{q,l})$ *and*

$$\begin{aligned} \|U_n\|_{L^\infty(\omega;\mathbb{M}^{q,l})} &\leq \frac{M^n \varepsilon_d^n}{n!} \|U_0\|_{L^\infty(\omega;\mathbb{M}^{q,l})}, \\ \|V_n\|_{L^\infty(\omega;\mathbb{M}^{q,l})} &\leq \frac{M^n \varepsilon_d^n}{n!} \Big(\|V_0\|_{L^\infty(\omega;\mathbb{M}^{q,l})} + n \|U_0\|_{L^\infty(\omega;\mathbb{M}^{q,l})} \Big). \end{aligned}$$

Proof. By Fubini's theorem, the integrals appearing in the right-hand-sides of the relations defining U_{n+1} and V_{n+1} are well defined for almost all $x' \in \omega'$. Furthermore, U_{n+1} and V_{n+1} are mesurable. For, if f is a given element of $L^1(\omega)$ and $g : \omega \to \mathbb{R}$ is defined by $g(x', x_d) = \int_{\overline{x}_d}^{x_d} f(x', t_d)\, dt_d$, then the function g is measurable. Indeed, with $\widetilde{f} : \omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{f}(x', x_d, t_d) := \begin{cases} f(x', t_d) & \text{if } x_d > \overline{x}_d \text{ and } t_d \in (\overline{x}_d, x_d), \\ -f(x', t_d) & \text{if } x_d < \overline{x}_d \text{ and } t_d \in (x_d, \overline{x}_d), \\ 0 & \text{otherwise,} \end{cases}$$

we have $g(x', x_d) = \int_{\mathbb{R}} \widetilde{f}(x', x_d, t_d)\, dt_d$ and the mesurability of g is given by Fubini's theorem. The boundedness of U_{n+1} and V_{n+1} follows immediately.

Now, let us prove the inequalities announced in Lemma 3.2. We will first prove the following inequalities

$$\begin{aligned} |U_n(x', x_d)| &\leq \frac{M^n |x_d - \overline{x}_d|^n}{n!} \|U_0\|_{L^\infty(\omega;\mathbb{M}^{q,l})}, \\ |V_n(x', x_d)| &\leq \frac{M^n |x_d - \overline{x}_d|^n}{n!} \Big(\|V_0\|_{L^\infty(\omega;\mathbb{M}^{q,l})} + n \|U_0\|_{L^\infty(\omega;\mathbb{M}^{q,l})} \Big) \end{aligned}$$

for almost all $x = (x', x_d) \in \omega$ and all $n \in \mathbb{N}$. Since $|x_d - \overline{x}_d| \leq \varepsilon_d$, the desired inequalities follow.

We proceed by a recursion argument. It is obvious that the inequalities above are verified for $n = 0$. Assume that they are verified for a given n.

Then, for all $(x', x_d) \in \omega$, we have

$$\begin{aligned}
&|U_{n+1}(x', x_d)| \\
&\quad \leq \left| \int_{\overline{x}_d}^{x_d} \big|U_n(x', t_d)\big| \; \big|A(x', t_d)\big| \, dt_d \right| \\
&\quad \leq M \left| \int_{\overline{x}_d}^{x_d} \big|U_n(x', t_d)\big| \, dt_d \right| \\
&\quad \leq M \left| \int_{\overline{x}_d}^{x_d} \frac{M^n |t_d - \overline{x}_d|^n}{n!} \|U_0\|_{L^\infty(\omega;\mathbb{M}^{q,l})} \, dt_d \right| \\
&\qquad\qquad\qquad = \frac{M^{n+1}|x_d - \overline{x}_d|^{n+1}}{(n+1)!} \|U_0\|_{L^\infty(\omega;\mathbb{M}^{q,l})}.
\end{aligned}$$

The second inequality is obtained in the same manner. □

4. A key preliminary result

The following result constitutes a key step towards establishing the main result of this paper (Theorem 3.3).

Theorem 3.1. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^d$ and let a point $x^0 \in \Omega$ and a matrix $Y^0 \in \mathbb{M}^{q,l}$ be fixed. Let matrix fields $A_\alpha \in L^\infty_{\text{loc}}(\Omega; \mathbb{M}^l)$ and $B_\alpha \in L^\infty_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ be given that satisfy*

$$\begin{aligned}
&\partial_\alpha A_\beta + A_\alpha A_\beta = \partial_\beta A_\alpha + A_\beta A_\alpha \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^l), \\
&\partial_\alpha B_\beta + B_\alpha A_\beta = \partial_\beta B_\alpha + B_\beta A_\alpha \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}).
\end{aligned} \tag{3.2}$$

Then the Cauchy problem

$$\begin{aligned}
&\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}), \\
&Y(x^0) = Y^0
\end{aligned}$$

has a unique solution in $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$.

Proof. We wish to solve the Cauchy problem

$$\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}), \tag{3.3}$$

$$Y(x^0) = Y^0. \tag{3.4}$$

This cannot be done by classical methods since the coefficients A_α and B_α appearing in this system are only of class L^∞ locally.

The outline of the proof is as follows. We first solve the system (3.3) locally by integrating the equations appearing in this system along a "special" set of straight lines, defined with the help of Lemma 3.1. Because of the lack of regularity of the coefficients A_α and B_α, we cannot integrate these equations directly, but only through a sequence of approximating solutions. Then, the local solutions to the system (3.3) will be glued together by using sequences of local solutions along curves starting from a given point x^0. We show that this definition is unambiguous thanks to the simple-connectedness of the domain Ω and to a uniqueness result for the local solution to the system (3.3). The proof is broken into six steps, numbered (i) to (vi).

(i) *Conventions and notations.* Throughout the proof, the integers $d, q, l \geq 1$ (which appear in $\mathbb{R}^d$, $\mathbb{M}^{q,l}$, or $\mathbb{M}^l$ for instance) are fixed once and for all and the integer n will be used exclusively for indexing sequences. Greek indices vary in the set $\{1, 2, ..., d\}$ and Latin indices vary either in the set $\{1, 2, ..., q\}$ or in the set $\{1, 2, ..., l\}$ according to the context. For instance, i varies in the set $\{1, 2, ..., l\}$ when used for indexing the rows of a matrix $(A^i_{\alpha j}) \in \mathbb{M}^l$, while i varies in the set $\{1, 2, ..., q\}$ when used for indexing the rows of a matrix $(B^i_{\alpha j}) \in \mathbb{M}^{q,l}$.

If $x := (x_1, x_2, ..., x_d)$ is a point in $\mathbb{R}^d$ and $\alpha, \beta \in \{1, 2, ..., d\}$ with $\alpha < \beta$ are two given indices, then we let

$$\begin{aligned} x_{\alpha...\beta} &:= (x_\alpha, x_{\alpha+1}, ..., x_\beta), \\ (x_{...}, x_\beta, \overline{x}_{...}) &:= (x_1, ..., x_{\beta-1}, x_\beta, \overline{x}_{\beta+1}, ..., \overline{x}_d), \\ (x_{...}, t_\beta) &:= (x_1, ..., x_{\beta-1}, t_\beta), \\ dx_{\alpha...\beta} &:= dx_\alpha dx_{\alpha+1} ... dx_\beta, \text{ etc.} \end{aligned}$$

Let $A^i_{\alpha j}$ and $B^i_{\alpha j}$ respectively denote the elements of the matrices A_α and B_α, where i is the row index and j is the column index. Recall that, according to the conventions made in section 2, the functions $A^i_{\alpha j}$ and $B^i_{\alpha j}$ satisfy

$$\sup_{x \in U} |A^i_{\alpha j}(x)| = \|A^i_{\alpha j}\|_{L^\infty(U)} \text{ and } \sup_{x \in U} |B^i_{\alpha j}(x)| = \|B^i_{\alpha j}\|_{L^\infty(U)}$$

for all open subset $U \Subset \Omega$.

(ii) *Definition of a subset $\mathcal{S}$ of $\mathbb{R}^d$.* For each integer $n \geq 1$, define the open set $\Omega_n := \{x \in \Omega;\ |x| < n \text{ and } \text{dist}(x, \Omega^c) > 1/n\}$, where $\Omega^c := \mathbb{R}^d \setminus \Omega$, and define the functions $\widehat{A}^i_{\alpha j}(n), \widehat{B}^i_{\alpha j}(n) : \mathbb{R}^d \to \mathbb{R}$ by

$$\widehat{A}^i_{\alpha j}(n)(x) = \begin{cases} A^i_{\alpha j}(x) & \text{if } x \in \Omega_n, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega_n \end{cases}$$

and

$$\widehat{B}^i_{\alpha j}(n)(x) = \begin{cases} B^i_{\alpha j}(x) & \text{if } x \in \Omega_n, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega_n. \end{cases}$$

Then $\Omega = \cup_n \Omega_n$, $\Omega_n \Subset \Omega_{n+1}$ and $\widehat{A}^i_{\alpha j}(n), \widehat{B}^i_{\alpha j}(n) \in L^\infty(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$.

Therefore, Lemma 3.1 (and Remark 3.1) shows that there exists a set $X_d(\alpha, i, j, n)$ with zero measure such that

$$\widehat{A}^i_{\alpha j}(n)(\cdot, \overline{x}_d) \in L^\infty(\mathbb{R}^{d-1}) \text{ and } \widehat{B}^i_{\alpha j}(n)(\cdot, \overline{x}_d) \in L^\infty(\mathbb{R}^{d-1})$$

and that the following two relations hold

$$(3.5)\qquad \begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\overline{x}_d-\varepsilon}^{\overline{x}_d+\varepsilon}\int_{\omega'}\left|\widehat{A}^i_{\alpha j}(n)(x',x_d)-\widehat{A}^i_{\alpha j}(n)(x',\overline{x}_d)\right|dx=0,\\ &\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\overline{x}_d-\varepsilon}^{\overline{x}_d+\varepsilon}\int_{\omega'}\left|\widehat{B}^i_{\alpha j}(n)(x',x_d)-\widehat{B}^i_{\alpha j}(n)(x',\overline{x}_d)\right|dx=0\end{aligned}$$

for all $\overline{x}_d \in \mathbb{R}\setminus X_d(\alpha,i,j,n)$ and all open bounded subset $\omega'\subset\mathbb{R}^{d-1}$. Let $X_d=\cup_{\alpha,i,j,n}X_d(\alpha,i,j,n)$ so that X_d has zero measure. Therefore, if $\overline{x}_d\in\mathbb{R}\setminus X_d$, relations (3.5) hold for every α,i,j,n and every bounded open subset $\omega'\subset\mathbb{R}^{d-1}$.

For each $\overline{x}_d\in\mathbb{R}\setminus X_d$, we apply Lemma 3.1 to the functions $\widehat{A}^i_{\alpha j}(\cdot,\overline{x}_d)$ and $\widehat{B}^i_{\alpha j}(\cdot,\overline{x}_d)$. Accordingly, there exists a subset $X_{d-1}(\overline{x}_d,\alpha,i,j,n)$ of $\mathbb{R}$ with zero measure such that

$$\widehat{A}^i_{\alpha j}(n)(\cdot,\overline{x}_{d-1},\overline{x}_d),\widehat{B}^i_{\alpha j}(n)(\cdot,\overline{x}_{d-1},\overline{x}_d)\in L^\infty(\mathbb{R}^{d-2})$$

and the following two relations hold:

$$(3.6)\qquad \begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\overline{x}_{d-1}-\varepsilon}^{\overline{x}_{d-1}+\varepsilon}\int_{\omega''}\left|\widehat{A}^i_{\alpha j}(n)(x'',x_{d-1},\overline{x}_d)-\widehat{A}^i_{\alpha j}(n)(x'',\overline{x}_{d-1},\overline{x}_d)\right|dx'=0,\\ &\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\overline{x}_{d-1}-\varepsilon}^{\overline{x}_{d-1}+\varepsilon}\int_{\omega''}\left|\widehat{B}^i_{\alpha j}(n)(x'',x_{d-1},\overline{x}_d)-\widehat{B}^i_{\alpha j}(n)(x'',\overline{x}_{d-1},\overline{x}_d)\right|dx'=0\end{aligned}$$

for all $\overline{x}_{d-1}\in\mathbb{R}\setminus X_{d-1}(\overline{x}_d,\alpha,i,j,n)$ and all bounded open subset $\omega''\subset\mathbb{R}^{d-2}$. Let $X_{d-1}(\overline{x}_d)=\cup_{\alpha,i,j,n}X_{d-1}(\overline{x}_d,\alpha,i,j,n)$ so that $X_{d-1}(\overline{x}_d)$ has zero measure. Therefore, if $\overline{x}_d\in\mathbb{R}\setminus X_d$ and $\overline{x}_{d-1}\in\mathbb{R}\setminus X_{d-1}(\overline{x}_d)$, relations (3.6) hold for every α,i,j,n and every bounded open subset $\omega''\subset\mathbb{R}^{d-2}$.

In a similar way, we define the subset $X_{d-2}(\overline{x}_{d-1},\overline{x}_d)$ of $\mathbb{R}$ for every numbers $\overline{x}_d\in X_d$ and $\overline{x}_{d-1}\in X_{d-1}(\overline{x}_d)$, and, after $(d-1)$ steps, the set $X_2(\overline{x}_3,...,\overline{x}_d)$ for every numbers $\overline{x}_d\in X_d,\overline{x}_{d-1}\in X_{d-1}(\overline{x}_d),...,\overline{x}_3\in X_3(\overline{x}_4,...,\overline{x}_d)$. Since the sets X_d, $X_{d-1}(\overline{x}_d)$, ..., and $X_2(\overline{x}_3,...,\overline{x}_d)$ have zero measures, the sets $\mathbb{R}\setminus X_d$, $\mathbb{R}\setminus X_{d-1}(\overline{x}_d)$, ..., and $\mathbb{R}\setminus X_2(\overline{x}_3,...,\overline{x}_d)$ are dense in $\mathbb{R}$.

We next define the set of points of $\mathbb{R}^d$,

$$(3.7)\qquad \begin{aligned}\mathcal{S}=\big\{\overline{x}:=(\overline{x}_1,\overline{x}_2,...,\overline{x}_d)\in\mathbb{R}^d\,;\,\overline{x}_d\in\mathbb{R}\setminus X_d,\\ \overline{x}_{d-1}\in\mathbb{R}\setminus X_{d-1}(\overline{x}_d),...,\overline{x}_2\in\mathbb{R}\setminus X_2(\overline{x}_3,...,\overline{x}_d),\,\overline{x}_1\in\mathbb{R}\big\}.\end{aligned}$$

Clearly, the set $\mathcal{S}$ is dense in $\mathbb{R}^d$ and the set $\mathcal{S}\cap\Omega$ is dense in Ω.

In the next step, we will show that the system (3.3) possesses a solution in any open subset of $\mathbb{R}^d$ of the form

$$\omega=\prod_{i=1}^d(\overline{x}_i-\varepsilon_i,\overline{x}_i+\varepsilon_i),$$

where $\overline{x} \in \mathcal{S} \cap \Omega$ and $\varepsilon_i > 0$ are such that $\overline{\omega} \subset \Omega$. We also denote

$$\omega_\alpha = \prod_{i=1}^{\alpha} (\overline{x}_i - \varepsilon_i, \overline{x}_i + \varepsilon_i),$$

for all $\alpha \in \{1, 2, ..., d\}$.

Since $\Omega = \cup_n \Omega_n$ and $\Omega_n \subset \Omega_{n+1}$ for all n, there exists an integer $n_0 \in \mathbb{N}$ such that $\omega \subset \Omega_{n_0}$. Therefore, $A^i_{\alpha j}(x) = \widehat{A}^i_{\alpha j}(n_0)(x)$ and $B^i_{\alpha j}(x) = \widehat{B}^i_{\alpha j}(n_0)(x)$ for all $x \in \omega$. This implies that, for all points $\overline{x}$ in the set $\mathcal{S}$ and for all $\tau \in \{2, 3, ..., d\}$, the functions $A^i_{\alpha j}(\cdot, \overline{x}_\tau, ..., \overline{x}_d)$ and $B^i_{\alpha j}(\cdot, \overline{x}_\tau, ..., \overline{x}_d)$ belong to $L^\infty(\omega_{\tau-1})$ and also that

$$(3.8) \quad \begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\overline{x}_\tau - \varepsilon}^{\overline{x}_\tau + \varepsilon} \int_{\omega_{\tau-1}} \left| A^i_{\alpha j}(x_{...}, x_\tau, \overline{x}_{...}) - A^i_{\alpha j}(x_{...}, \overline{x}_\tau, \overline{x}_{...}) \right| dx_{1...\tau} = 0, \\ &\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\overline{x}_\tau - \varepsilon}^{\overline{x}_\tau + \varepsilon} \int_{\omega_{\tau-1}} \left| B^i_{\alpha j}(x_{...}, x_\tau, \overline{x}_{...}) - B^i_{\alpha j}(x_{...}, \overline{x}_\tau, \overline{x}_{...}) \right| dx_{1...\tau} = 0. \end{aligned}$$

(iii) *Let two constants $c_1 \geq 0$ and $c_2 \geq 0$ satisfy*

$$\max_\alpha \|A_\alpha\|_{L^\infty(\omega;\mathbb{M}^l)} \leq c_1 \text{ and } \max_\alpha \|B_\alpha\|_{L^\infty(\omega;\mathbb{M}^{q,l})} \leq c_2,$$

and let there be given a matrix $\overline{Y} \in \mathbb{M}^{q,l}$. Then there exists a solution $Y \in W^{1,\infty}(\omega; \mathbb{M}^{q,l})$ to the system

$$(3.9) \quad \begin{aligned} &\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ in } \mathcal{D}'(\omega; \mathbb{M}^{q,l}), \text{ for all } \alpha \in \{1, 2, ..., d\} \\ &Y(\overline{x}) = \overline{Y}. \end{aligned}$$

In addition, this solution satisfies the inequality:

$$(3.10) \quad \|Y\|_{L^\infty(\omega;\mathbb{M}^{q,l})} \leq e^{c_1(\varepsilon_1 + ... + \varepsilon_d)} \left\{ |\overline{Y}| + c_2(\varepsilon_1 + ... + \varepsilon_d) \right\}.$$

In what follows, the partial differential equations are understood in a distributional sense. We construct a solution to the system (3.9) recursively. First, we define a solution $Y_1 = Y_1(\cdot, \overline{x}_{2...d}) \in W^{1,\infty}(\omega_1; \mathbb{M}^{q,l})$ to the system

$$\begin{aligned} &\partial_1 Y_1 = Y_1 A_1(\cdot, \overline{x}_{2...d}) + B_1(\cdot, \overline{x}_{2...d}), \\ &Y_1(\overline{x}_1) = \overline{Y}. \end{aligned}$$

Then we use Y_1 to define a solution $Y_2 = Y_2(\cdot, \overline{x}_{3...d}) \in W^{1,\infty}(\omega_2; \mathbb{M}^{q,l})$ to the system

$$\begin{aligned} &\partial_\alpha Y_2 = Y_2 A_\alpha(\cdot, \overline{x}_{3...d}) + B_\alpha(\cdot, \overline{x}_{3...d}), \alpha \in \{1, 2\}, \\ &Y_2(\overline{x}_1, \overline{x}_2) = \overline{Y}, \end{aligned}$$

and, after d steps, we use Y_{d-1} to define a solution $Y_d \in W^{1,\infty}(\omega_d; \mathbb{M}^{q,l})$ to the system

$$\begin{aligned} &\partial_\alpha Y_d = Y_d A_\alpha + B_\alpha, \alpha \in \{1, 2, ..., d\}, \\ &Y_d(\overline{x}) = \overline{Y}. \end{aligned}$$

Finally, the solution to the system (3.9) will be given by $Y = Y_d$. In order to prove the existence of $Y_1, Y_2, ..., Y_d$, it suffices to prove the existence of Y_β for a fixed $\beta \in \{1, 2, ..., d\}$ under the assumption that $Y_1, Y_2, ..., Y_{\beta-1}$ exist. We make the convention that $Y_0 := \overline{Y}$, so that the existence of Y_1 reduces to the general case. From now on until the end of this step of the proof, β is kept fixed, α vary in the set $\{1, 2, ..., \beta - 1\}$, $\widetilde{A}_\alpha(x_1, ..., x_\beta) := A_\alpha(x_1, ..., x_\beta, \overline{x}_{\beta+1}, ..., \overline{x}_d)$ and $\widetilde{B}_\alpha(x_1, ..., x_\beta) := B_\alpha(x_1, ..., x_\beta, \overline{x}_{\beta+1}, ..., \overline{x}_d)$.

We wish to prove the existence of a solution $Y_\beta \in W^{1,\infty}(\omega_\beta; \mathbb{M}^{q,l})$ to the system

$$(3.11) \qquad \begin{aligned} &\partial_\tau Y_\beta = Y_\beta \widetilde{A}_\tau + \widetilde{B}_\tau \text{ in } \mathcal{D}'(\omega_\beta; \mathbb{M}^{q,l}), \tau \in \{1, 2, ..., \beta\}, \\ &Y_\beta(\overline{x}_{1...\beta}) = \overline{Y}. \end{aligned}$$

To this end, we define a sequence of "approximating solutions" $(Y_\beta^n)_{n\geq 0}$ that will eventually converge to a solution to (3.11). For almost all $(x_{1...\beta-1}) \in \omega_{\beta-1}$ and all $x_\beta \in (\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta + \varepsilon_\beta)$, we define the sequence $\Big(Y_\beta^n(x_{1...\beta})\Big)_{n\in\mathbb{N}}$ by letting

$$(3.12) \qquad \begin{aligned} &Y_\beta^0(x_{1...\beta}) := 0, \\ &Y_\beta^{n+1}(x_{1...\beta}) := Y_{\beta-1}(x_{1...\beta-1}) + \int_{\overline{x}_\beta}^{x_\beta} (Y_\beta^n \widetilde{A}_\beta + \widetilde{B}_\beta)(x_{...}, t_\beta) dt_\beta. \end{aligned}$$

Let us first prove that the fields Y_β^n belong to the space $W^{1,\infty}(\omega_\beta; \mathbb{M}^{q,l})$ for all $n \geq 0$. We proceed by a recursion argument, at the end of which the fields Y_β^0 and Y_β^1 will be shown to belong to this space. We now prove that if the fields Y_β^{n-1} and Y_β^n, $n \geq 1$, belong to the space $W^{1,\infty}(\omega_\beta; \mathbb{M}^{q,l})$, so does Y_β^{n+1}.

From (3.12), we deduce that $Y_\beta^{n+1} \in L^\infty(\omega_\beta; \mathbb{M}^{q,l})$ and that

$$(3.13) \qquad \partial_\beta Y_\beta^{n+1} = Y_\beta^n \widetilde{A}_\beta + \widetilde{B}_\beta \text{ in } \mathcal{D}'(\omega_\beta; \mathbb{M}^{q,l}).$$

Hence $\partial_\beta Y_\beta^{n+1} \in L^\infty(\omega_\beta; \mathbb{M}^{q,l})$. It remains to prove that the partial derivatives $\partial_\alpha Y_\beta^{n+1}$ belong to the space $L^\infty(\omega_\beta; \mathbb{M}^{q,l})$.

Let $\varphi \in \mathcal{D}(\omega_\beta)$ be a fixed, but otherwise arbitrary, function. Then

$$(3.14) \qquad \begin{aligned} \int_{\omega_\beta} (Y_\beta^{n+1} \partial_\alpha \varphi)(x_{1...\beta}) dx_{1...\beta} &= \int_{\omega_\beta} Y_{\beta-1}(x_{1...\beta-1}) \partial_\alpha \varphi(x_{1...\beta}) dx_{1...\beta} \\ &+ \int_{\omega_\beta} \left\{ \int_{\overline{x}_\beta}^{x_\beta} (Y_\beta^n \widetilde{A}_\beta + \widetilde{B}_\beta)(x_{...}, t_\beta) dt_\beta \right\} \partial_\alpha \varphi(x_{1...\beta}) dx_{1...\beta}, \end{aligned}$$

which implies on the one hand that

$$(3.15)\quad \int_{\omega_\beta} (Y_\beta^{n+1}\partial_\alpha\varphi)(x_{1\dots\beta})dx_{1\dots\beta} = -\int_{\omega_\beta} \partial_\alpha Y_{\beta-1}(x_{1\dots\beta-1})\varphi(x_{1\dots\beta})dx_{1\dots\beta}$$
$$-\int_{\omega_\beta}\int_{\overline{x}_\beta}^{x_\beta} \partial_\alpha Y_\beta^n(x_{\dots},t_\beta)\varphi(x_{1\dots\beta})\widetilde{A}_\beta(x_{\dots},t_\beta)dt_\beta dx_{1\dots\beta}$$
$$+\int_{\omega_\beta}\int_{\overline{x}_\beta}^{x_\beta} \partial_\alpha\left(Y_\beta^n(x_{\dots},t_\beta)\varphi(x_{1\dots\beta})\right)\widetilde{A}_\beta(x_{\dots},t_\beta)dt_\beta dx_{1\dots\beta}$$
$$+\int_{\omega_\beta}\int_{\overline{x}_\beta}^{x_\beta} \widetilde{B}_\beta(x_{\dots},t_\beta)\partial_\alpha\varphi(x_{1\dots\beta})dt_\beta dx_{1\dots\beta}.$$

On the other hand, a lengthy calculation, which uses in particular the "compatibility conditions" (3.2) and the definition of the set $\mathcal{S}$ (see the Appendix at the end of the paper for the complete proof of the following two relations), shows that the last two terms of the relation above are given by

$$(3.16)\quad \int_{\omega_\beta}\int_{\overline{x}_\beta}^{x_\beta} \partial_\alpha\left(Y_\beta^n(x_{\dots},t_\beta)\varphi(x_{\dots},x_\beta)\right)\widetilde{A}_\beta(x_{\dots},t_\beta)dt_\beta dx_{1\dots\beta}$$
$$=\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{x_\beta}(\partial_\beta Y_\beta^n\widetilde{A}_\alpha)(x_{\dots},t_\beta)dt_\beta\right\}\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}$$
$$+\int_{\omega_\beta}(Y_\beta^n\widetilde{A}_\alpha)(x_{\dots},\overline{x}_\beta)\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}-\int_{\omega_\beta}(Y_\beta^n\widetilde{A}_\alpha\varphi)(x_{\dots},x_\beta)dx_{1\dots\beta}$$
$$+\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{x_\beta}(Y_\beta^n(\widetilde{A}_\alpha\widetilde{A}_\beta-\widetilde{A}_\beta\widetilde{A}_\alpha))(x_{\dots},t_\beta)dt_\beta\right\}\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}$$

and

$$(3.17)\quad \int_{\omega_\beta}\int_{\overline{x}_\beta}^{x_\beta} \partial_\alpha\varphi(x_{\dots},x_\beta)\widetilde{B}_\beta(x_{\dots},t_\beta)dt_\beta dx_{1\dots\beta}$$
$$=\int_{\omega_\beta}\widetilde{B}_\alpha(x_{\dots},\overline{x}_\beta)\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}-\int_{\omega_\beta}(\widetilde{B}_\alpha\varphi)(x_{\dots},x_\beta)dx_{1\dots\beta}$$
$$+\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{x_\beta}(\widetilde{B}_\alpha\widetilde{A}_\beta-\widetilde{B}_\beta\widetilde{A}_\alpha)(x_{\dots},t_\beta)dt_\beta\right\}\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}.$$

Using these last two relations in (3.15) yields

$$
\begin{aligned}
(3.18)\quad -\int_{\omega_\beta}(Y_\beta^{n+1}\partial_\alpha\varphi)(x_{\dots},x_\beta)dx_{1\dots\beta}
&= \int_{\omega_\beta}(Y_\beta^n\widetilde{A}_\alpha+\widetilde{B}_\alpha)(x_{\dots},x_\beta)\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}\\
&+\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{x_\beta}\Big((\partial_\alpha Y_\beta^n - Y_\beta^{n-1}\widetilde{A}_\alpha-\widetilde{B}_\alpha)\widetilde{A}_\beta\Big)(x_{\dots},t_\beta)dt_\beta\right\}\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}\\
&-\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{x_\beta}\Big((Y_\beta^n - Y_\beta^{n-1})(\widetilde{A}_\alpha\widetilde{A}_\beta-\widetilde{A}_\beta\widetilde{A}_\alpha)\Big)(x_{\dots},t_\beta)dt_\beta\right\}\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}\\
&-\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{x_\beta}\Big((\partial_\beta Y_\beta^n - Y_\beta^{n-1}\widetilde{A}_\beta-\widetilde{B}_\beta)\widetilde{A}_\alpha\Big)(x_{\dots},t_\beta)dt_\beta\right\}\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}\\
&+\int_{\omega_\beta}\left\{\partial_\alpha Y_{\beta-1}(x_1,\dots,x_{\beta-1})-\Big(Y_\beta^n\widetilde{A}_\alpha+\widetilde{B}_\alpha\Big)(x_{\dots},\overline{x}_\beta)\right\}\varphi(x_{\dots},x_\beta)dx_{1\dots\beta}.
\end{aligned}
$$

But the last two integrals vanish since

$$\partial_\beta Y_\beta^n = (Y_\beta^{n-1}\widetilde{A}_\beta+\widetilde{B}_\beta) \text{ a.e. in } \omega_\beta$$

and

$$
\begin{aligned}
\partial_\alpha Y_{\beta-1} &= Y_{\beta-1}\widetilde{A}_\alpha(\cdot,\overline{x}_\beta)+\widetilde{B}_\alpha(\cdot,\overline{x}_\beta)\\
&= (Y_\beta^n\widetilde{A}_\alpha+\widetilde{B}_\alpha)(\cdot,\overline{x}_\beta) \text{ a.e. in } \omega_{\beta-1},
\end{aligned}
$$

thanks to relation (3.12) with n replaced by $(n-1)$. Note also that in the second relation we have used the fact that the continuous representative of the class of Y_β^n in $W^{1,\infty}$ takes the value $Y_{\beta-1}(x_{1\dots\beta-1})$ at $(x_{1\dots\beta-1},\overline{x}_\beta)$ since the representative defined in (3.12) (with n replaced by $(n-1)$) is continuous at $(x_{1\dots\beta-1},\overline{x}_\beta)$ and takes the same value (i.e. $Y_{\beta-1}(x_{1\dots\beta-1})$) at this point. Therefore, relation (3.18) becomes

$$
\begin{aligned}
(3.19)\quad \partial_\alpha Y_\beta^{n+1}(x_{\dots},x_\beta) &= (Y_\beta^n\widetilde{A}_\alpha+\widetilde{B}_\alpha)(x_{\dots},x_\beta)\\
&+\int_{\overline{x}_\beta}^{x_\beta}\left\{(\partial_\alpha Y_\beta^n - Y_\beta^{n-1}\widetilde{A}_\alpha-\widetilde{B}_\alpha)\widetilde{A}_\beta\right\}(x_{\dots},t_\beta)dt_\beta\\
&-\int_{\overline{x}_\beta}^{x_\beta}\left\{(Y_\beta^n - Y_\beta^{n-1})(\widetilde{A}_\alpha\widetilde{A}_\beta-\widetilde{A}_\beta\widetilde{A}_\alpha)\right\}(x_{\dots},t_\beta)dt_\beta
\end{aligned}
$$

in $\mathcal{D}'(\omega_\beta;\mathbb{M}^3)$. Since the right-hand-side above belongs to $L^\infty(\omega_\beta;\mathbb{M}^{q,l})$, the derivative $\partial_\alpha Y_\beta^{n+1}$ belongs to the same space. Since we have already proved that Y_β^{n+1} and $\partial_\beta Y_\beta^{n+1}$ belong to $L^\infty(\omega_\beta;\mathbb{M}^{q,l})$, the field Y_β^{n+1} belongs to the space $W^{1,\infty}(\omega_\beta;\mathbb{M}^{q,l})$.

In order to complete the recursion argument, we have to prove that Y_β^0 and Y_β^1 belong to the space $W^{1,\infty}(\omega_\beta;\mathbb{M}^{q,l})$. It is clear from the definition

(3.12) of the sequence (Y^n_β) that $Y^0_\beta \in W^{1,\infty}(\omega_\beta; \mathbb{M}^{q,l})$, $Y^1_\beta \in L^\infty(\omega_\beta; \mathbb{M}^{q,l})$, and

$$\partial_\beta Y^1_\beta = \widetilde{B}_\beta \text{ in } \mathcal{D}'(\omega_\beta; \mathbb{M}^{q,l}).$$

Since relations (3.14) and (3.17) hold for $n = 0$, combining them shows that

$$\partial_\alpha Y^1_\beta(x_{...}, x_\beta) = \partial_\alpha Y_{\beta-1}(x_1, ..., x_{\beta-1}) - \widetilde{B}_\alpha(x_{...}, \overline{x}_\beta) + \widetilde{B}_\alpha(x_{...}, x_\beta) - \int_{\overline{x}_\beta}^{x_\beta} (\widetilde{B}_\alpha \widetilde{A}_\beta - \widetilde{B}_\beta \widetilde{A}_\alpha)(x_{...}, t_\beta) dt_\beta$$

in $\mathcal{D}'(\omega_\beta; \mathbb{M}^{q,l})$. Since the right-hand-sides of the two relations above belong to the space $L^\infty(\omega_\beta; \mathbb{M}^{q,l})$, the field Y^1_β belongs to the space $W^{1,\infty}(\omega_\beta; \mathbb{M}^{q,l})$. This completes the recursion argument. Therefore, $Y^n_\beta \in W^{1,\infty}(\omega_\beta; \mathbb{M}^{q,l})$ for all $n \geq 0$. Moreover, relations (3.13) and (3.19) show that the derivatives of the fields Y^{n+1}_β, $n \geq 1$, are given by (recall that $\alpha \in \{1, 2, ..., \beta - 1\}$):

$$\begin{aligned} \partial_\alpha Y^{n+1}_\beta(x_{1...\beta}) &= (Y^n_\beta \widetilde{A}_\alpha + \widetilde{B}_\alpha)(x_{1...\beta}), \\ &+ \int_{\overline{x}_\beta}^{x_\beta} \left\{ (\partial_\alpha Y^n_\beta - Y^{n-1}_\beta \widetilde{A}_\alpha - \widetilde{B}_\alpha) \widetilde{A}_\beta \right\} (x_{...}, t_\beta) dt_\beta \\ &+ \int_{\overline{x}_\beta}^{x_\beta} \left\{ (Y^n_\beta - Y^{n-1}_\beta) \left(\widetilde{A}_\beta \widetilde{A}_\alpha - \widetilde{A}_\alpha \widetilde{A}_\beta \right) \right\} (x_{...}, t_\beta) dt_\beta, \\ \partial_\beta Y^{n+1}_\beta(x_{1...\beta}) &= (Y^n_\beta \widetilde{A}_\beta + \widetilde{B}_\beta)(x_{1...\beta}). \end{aligned} \tag{3.20}$$

We now prove that the sequence (Y^n_β) converges in $L^\infty(\omega_\beta; \mathbb{M}^{q,l})$ and that its limit satisfies the system (3.11). Let

$$M = \max_\tau \left\{ \|A_\tau\|_{L^\infty(\omega; \mathbb{M}^p)}, 2\|A_\tau\|^2_{L^\infty(\omega; \mathbb{M}^p)} \right\}.$$

From (3.12), we infer that

$$(Y^{n+1}_\beta - Y^n_\beta)(x_{...}, x_\beta) = \int_{\overline{x}_\beta}^{x_\beta} ((Y^n_\beta - Y^{n-1}_\beta) \widetilde{A}_\beta)(x_{...}, t_\beta) dt_\beta$$

By applying Lemma 3.2 to this relation with $U_n := (Y^{n+1}_\beta - Y^n_\beta)$ and to (3.20) with $V_n := (\partial_\alpha Y^{n+1}_\beta - Y^n_\beta \widetilde{A}_\alpha - \widetilde{B}_\beta)$, we deduce that

$$\|Y^{n+1}_\beta - Y^n_\beta\|_{L^\infty(\omega_\beta; \mathbb{M}^{q,l})} \leq \frac{M^n \varepsilon^n_\beta}{n!} \|Y^1_\beta\|_{L^\infty(\omega_\beta; \mathbb{M}^{q,l})},$$

and

$$\|\partial_\alpha Y^{n+1}_\beta - Y^n_\beta \widetilde{A}_\alpha - \widetilde{B}_\alpha\|_{L^\infty(\omega_\beta; \mathbb{M}^{q,l})} \leq \frac{M^n \varepsilon^n_\beta}{n!} \left(\|\partial_\alpha Y^1_\beta - \widetilde{B}_\alpha\|_{L^\infty(\omega_\beta; \mathbb{M}^{q,l})} + n \|Y^1_\beta\|_{L^\infty(\omega_\beta; \mathbb{M}^{q,l})} \right).$$

This implies that (Y^n_β) is a Cauchy sequence in $L^\infty(\omega_\beta; \mathbb{M}^{q,l})$ and that

$$(\partial_\alpha Y^{n+1}_\beta - Y^n_\beta \widetilde{A}_\alpha - \widetilde{B}_\alpha) \to 0 \tag{3.21}$$

in $L^\infty(\omega_\beta;\mathbb{M}^{q,l})$. Consequently, there exists $Y_\beta \in L^\infty(\omega_\beta;\mathbb{M}^{q,l})$ such that $Y_\beta^n \to Y_\beta$ in $L^\infty(\omega_\beta;\mathbb{M}^{q,l})$. This implies that

$$(\partial_\alpha Y_\beta^{n+1} - Y_\beta^n \widetilde{A}_\alpha - \widetilde{B}_\alpha) \to (\partial_\alpha Y_\beta - Y_\beta \widetilde{A}_\alpha - \widetilde{B}_\alpha) \text{ and}$$
$$(\partial_\beta Y_\beta^{n+1} - Y_\beta^n \widetilde{A}_\beta - \widetilde{B}_\beta) \to (\partial_\beta Y_\beta - Y_\beta \widetilde{A}_\beta - \widetilde{B}_\beta)$$

in $\mathcal{D}'(\omega_\beta;\mathbb{M}^{q,l})$. Using then relations (3.13) and (3.21), we find that $\partial_\alpha Y_\beta = Y_\beta \widetilde{A}_\alpha + \widetilde{B}_\alpha$ and $\partial_\beta Y_\beta = Y_\beta \widetilde{A}_\beta + \widetilde{B}_\beta$ in $\mathcal{D}'(\omega_\beta;\mathbb{M}^{q,l})$. Therefore $Y_\beta \in W^{1,\infty}(\omega_\beta;\mathbb{M}^{q,l})$ and

$$\partial_\tau Y_\beta = Y_\beta \widetilde{A}_\tau + \widetilde{B}_\tau \text{ for all } \tau \in \{1,2,...,\beta\},$$
$$Y_\beta(\overline{x}_{1...\beta}) = \lim_{n\to\infty} Y_\beta^n(\overline{x}_{1...\beta}) = Y_{\beta-1}(\overline{x}_{1...\beta-1}) = \overline{Y}.$$

Note that in the second relation, we have used the fact that the continuous representative of the class of Y_β^n in $W^{1,\infty}$ takes the value $\overline{Y}$ at $\overline{x}_{1...\beta}$ since the representative defined in (3.12) (with n replaced by $(n-1)$) is continuous at $\overline{x}_{1...\beta}$ and takes the same value (i.e. $\overline{Y}$) at this point.

We now establish inequality (3.10). Letting $n \to \infty$ in (3.12) shows that

$$\text{(3.22)} \qquad Y_\beta(x_{1...\beta-1}, x_\beta) = Y_{\beta-1}(x_{1...\beta-1}) + \int_{\overline{x}_\beta}^{x_\beta} (Y_\beta \widetilde{A}_\beta + \widetilde{B}_\beta)(x_{...}, t_\beta) dt_\beta$$

for almost all $(x_1, ..., x_\beta) \in \omega_\beta$. Let $f_\beta(x_\beta) := \|Y_\beta(\cdot, x_\beta)\|_{L^\infty(\omega_{\beta-1},\mathbb{M}^{q,l})}$ (with no summation on β). Then we deduce from the previous relation that

$$f_\beta(x_\beta) \le \|Y_{\beta-1}\|_{L^\infty(\omega_{\beta-1};\mathbb{M}^{q,l})} + c_2\varepsilon_\beta + c_1 \int_{\overline{x}_\beta}^{x_\beta} f_\beta(t_\beta) dt_\beta,$$

from which we deduce by Gronwall's inequality that

$$\|Y_\beta\|_{L^\infty(\omega_\beta;\mathbb{M}^{q,l})} = \sup_{x_\beta \in (\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta + \varepsilon_\beta)} |f_\beta(x_\beta)|$$
$$\le e^{c_1\varepsilon_\beta} \left(\|Y_{\beta-1}\|_{L^\infty(\omega_{\beta-1};\mathbb{M}^{q,l})} + c_2\varepsilon_\beta \right).$$

By iterating this inequality, we finally find that

$$\|Y_\beta\|_{L^\infty(\omega_\beta,\mathbb{M}^{q,l})} \le e^{c_1(\varepsilon_1+...+\varepsilon_\beta)} \left\{ |\overline{Y}| + c_2(\varepsilon_1 + ... + \varepsilon_\beta) \right\}.$$

Thus inequality (3.10) is established.

In what follows, our aim is to glue together the local solutions of the system (3.3) found in the previous step. These local solutions are defined over parallelepipeds centered at points $\overline{x}$ belonging to $\Omega \cap \mathcal{S}$, where the set $\mathcal{S}$ is defined by (3.7). This restriction on $\overline{x}$ will be removed in the following step. Besides, since distances between parallelepipeds are not easy to estimate, the local solutions will be instead defined over open balls centered at arbitrary points of Ω.

(iv) *Let* $y \in \Omega$ *and let* $r > 0$ *such that* $r\sqrt{d} < \operatorname{dist}(y, \Omega^c)$ *where* $\Omega^c := \mathbb{R}^d \setminus \Omega$. *Let* Y_* *be a fixed, but otherwise arbitrary, matrix in* $\mathbb{M}^{q,l}$. *Then there exists a solution* $Y \in W^{1,\infty}(B(y,r); \mathbb{M}^{q,l})$ *to the system*

$$\begin{aligned} &\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B(y,r); \mathbb{M}^{q,l}), \alpha = 1, 2, ..., d \\ &Y(y) = Y_*. \end{aligned} \tag{3.23}$$

Let any $R > r$ be fixed such that the cube

$$\omega_R := \prod_{i=1}^d (y_i - R, y_i + R).$$

satisfies $B(y,r) \subset \omega_R \Subset \Omega$, let two constants $c_1 \geq 0$ and $c_2 \geq 0$ satisfy

$$\max_\alpha \|A_\alpha\|_{L^\infty(\omega_R; \mathbb{M}^l)} \leq c_1 \text{ and } \max_\alpha \|B_\alpha\|_{L^\infty(\omega_R; \mathbb{M}^{q,l})} \leq c_2,$$

and finally, let a sequence of cubes

$$\omega^n = \prod_{i=1}^d (\overline{x}_i^n - R^n, \overline{x}_i^n + R^n),$$

where $\overline{x}^n \in \Omega \cap \mathcal{S}$ and $0 < R^n < R$, be such that $\overline{x}^n \to y$ as $n \to \infty$ and $B(y,r) \subset \omega^{n+1} \subset \omega^n \subset \omega_R$.

By the previous step, there exists a solution $Y(n) \in W^{1,\infty}(\omega^n; \mathbb{M}^{q,l})$ to the system

$$\begin{aligned} &\partial_\alpha Y(n) = Y(n) A_\alpha + B_\alpha \text{ in } \mathcal{D}'(\omega^n; \mathbb{M}^{q,l}), \alpha = 1, 2, ..., d, \\ &Y(n)(\overline{x}^n) = Y_*, \end{aligned} \tag{3.24}$$

and this solution satisfies the following inequality:

$$\|Y(n)\|_{L^\infty(\omega^n; \mathbb{M}^{q,l})} \leq e^{c_1 d R} \{\|Y_*\| + c_2 d R\}. \tag{3.25}$$

Let c_3 denote the right-hand-side of this inequality and let any $n_1, n_2 \in \mathbb{N}$, $n_1 < n_2$, be fixed. We infer from (3.22) that

$$Y_\beta(n_1)(\widetilde{x}^{n_2}_{1...\beta}) - Y_{\beta-1}(n_1)(\widetilde{x}^{n_2}_{1...\beta-1}) = \int_{\overline{x}^{n_1}_\beta}^{\widetilde{x}^{n_2}_\beta} (Y_\beta(n_1) A_\beta + B_\beta)(\widetilde{x}^{n_2}_{...}, t_\beta, \overline{x}^{n_1}_{...}) dt_\beta$$

for almost all $\widetilde{x}^{n_2}_{1...\beta} \in \omega_\beta$, which next gives (since $Y_\beta(n_1) = Y(n_1)(x_{...}, x_\beta, \overline{x}_{...})$)

$$|Y_\beta(n_1)(\widetilde{x}^{n_2}_{1...\beta}) - Y_{\beta-1}(n_1)(\widetilde{x}^{n_2}_{1...\beta-1})| \leq (c_3 c_1 + c_2)|\widetilde{x}^{n_2}_\beta - \overline{x}^{n_1}_\beta|.$$

Since this inequality holds true for the continuous représentants of the classes Y_β and $Y_{\beta-1}$, letting $\widetilde{x}^{n_2}_{1...\beta}$ be as close to $\overline{x}^{n_2}_{1...\beta}$ as we wish gives

$$|Y_\beta(n_1)(\overline{x}^{n_2}_{1...\beta}) - Y_{\beta-1}(n_1)(\overline{x}^{n_2}_{1...\beta-1})| \leq (c_3 c_1 + c_2)|\overline{x}^{n_2}_\beta - \overline{x}^{n_1}_\beta|.$$

which in turn gives

$$(3.26) \quad |Y(n_1)(\overline{x}^{n_2}) - Y_*| \le (c_3c_1 + c_2)\sum_{\beta=1}^{d} |\overline{x}^{n_2}_\beta - \overline{x}^{n_1}_\beta| \le (c_3c_1 + c_2)\sqrt{d}|\overline{x}^{n_2} - \overline{x}^{n_1}|.$$

Let $H := Y(n_1) - Y(n_2)$, which is defined over ω^{n_2} since $\omega^{n_2} \subset \omega^{n_1}$. Then $H \in W^{1,\infty}(\omega^{n_2}; \mathbb{M}^{q,l})$ and satisfies

$$\partial_\alpha H = HA_\alpha \text{ in } \mathcal{D}'(\omega^{n_2}; \mathbb{M}^{q,l}), \alpha = 1, 2, ..., d,$$
$$H(\overline{x}^{n_2}) = Y(n_1)(\overline{x}^{n_2}) - Y_*.$$

Applying inequality (3.10) to this problem gives

$$\|H\|_{L^\infty(\omega^{n_2};\mathbb{M}^{q,l})} \le e^{c_1 dR^{n_2}}|Y(n_1)(\overline{x}^{n_2}) - Y_*| \le e^{c_1 dR}|Y(n_1)(\overline{x}^{n_2}) - Y_*|.$$

By combining this inequality with (3.26), one can see that there exists a constant $c_4 > 0$ such that

$$\|Y(n_1) - Y(n_2)\|_{L^\infty(B(y,r);\mathbb{M}^{q,l})} \le c_4|\overline{x}^{n_2} - \overline{x}^{n_1}|.$$

This means that $(Y(n))$ is a Cauchy sequence in the space $L^\infty(B(y, r); \mathbb{M}^{q,l})$. Hence there exists $Y \in L^\infty(B(y, r); \mathbb{M}^{q,l})$ such that

$$Y(n) \to Y \text{ in } L^\infty(B(y, r); \mathbb{M}^{q,l}) \text{ as } n \to \infty.$$

Then we infer from the system (3.24) that Y satisfies

$$\partial_\alpha Y = YA_\alpha + B_\alpha \text{ in } \mathcal{D}'((B(y, r); \mathbb{M}^{q,l}),$$

which next implies that $Y \in W^{1,\infty}(B(y, r); \mathbb{M}^{q,l})$. On the other hand, letting first $n_2 \to \infty$, then $n_1 \to \infty$, in inequality (3.26) shows that $Y(y) = Y_*$. Consequently, Y is a solution to the system (3.23).

In order to define a global solution, i.e. defined over Ω, to the system (3.3)–(3.4) by glueing together the local solutions found in the previous step, one needs to prove the uniqueness of the solution to problem (3.23). In fact, we will prove a more general result (which will also be used in establishing the global uniqueness result in Theorem 3.1).

(v) *Let U be a connected open subset of Ω and let $Y, \widetilde{Y} \in W^{1,\infty}_{loc}(U; \mathbb{M}^{q,l})$ be such that*

$$\partial_\alpha Y = YA_\alpha + B_\alpha, \text{ in } \mathcal{D}'(U; \mathbb{M}^{q,l}),$$
$$\partial_\alpha \widetilde{Y} = \widetilde{Y}A_\alpha + B_\alpha, \text{ in } \mathcal{D}'(U; \mathbb{M}^{q,l}),$$

for all $\alpha \in \{1, 2, ..., d\}$. Assume that there exists a point $x^* \in U$ such that $Y(x^*) = \widetilde{Y}(x^*)$. Then $Y(x) = \widetilde{Y}(x)$ for all $x \in U$.

Let $H(x) = Y(x) - \widetilde{Y}(x)$. Then $H \in W^{1,\infty}_{\text{loc}}(U; \mathbb{M}^{q,l})$ satisfies the following equations

$$\begin{aligned} &\partial_\alpha H = H A_\alpha \quad \text{in } \mathcal{D}'(U; \mathbb{M}^{q,l}) \text{ and} \\ &H(x^*) = 0 \in \mathbb{M}^{q,l}. \end{aligned} \tag{3.27}$$

We wish to prove that $H = 0$ in $W^{1,\infty}_{\text{loc}}(U; \mathbb{M}^{q,l})$. To this end, we shall use a connectedness argument. We first infer from the Sobolev imbedding theorem that $H \in C^0(U; \mathbb{M}^{q,l})$. Then we define the following subset of U:

$$\mathcal{A} := \{x \in U; H(x) = 0 \in \mathbb{M}^{q,l}\}.$$

Since $x^* \in \mathcal{A}$ by (3.27), the set $\mathcal{A}$ is non-empty. The continuity of the application $H : U \to \mathbb{M}^{q,l}$ next implies that $\mathcal{A}$ is closed in U. We now prove that $\mathcal{A}$ is also open in U.

Let $x \in \mathcal{A}$ and let $B(x, \widetilde{\varepsilon})$ be an open ball such that $B(x, \widetilde{\varepsilon}) \Subset U$. Let $\varepsilon \in]0, \widetilde{\varepsilon}[$ be such that $\varepsilon < \frac{1}{2dM}$, where

$$M := \max_{\alpha \in \{1,2,...,d\}} \|A_\alpha\|_{L^\infty(B(x,\widetilde{\varepsilon}); \mathbb{M}^{q,l})}.$$

Then we infer from (3.27) that

$$\|\partial_\alpha H\|_{L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})} \le M \|H\|_{L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})} \tag{3.28}$$

for all $\alpha \in \{1, 2, ..., d\}$.

Next, define the space

$$W^{1,\infty}_x(B(x,\varepsilon); \mathbb{M}^{q,l}) := \{Y \in W^{1,\infty}(B(x,\varepsilon); \mathbb{M}^{q,l}); Y(x) = 0 \text{ in } \mathbb{M}^{q,l}\},$$

and let Y be an element of this space. By a result about Sobolev spaces (see Corollary A.1, page 134), we know that for almost all $z \in \partial B(x, \varepsilon)$, the restriction of Y to the segment $]x, z[$ belongs to $W^{1,\infty}(]x, z[; \mathbb{M}^{q,l})$,

$$\|\nabla Y\|_{L^\infty(]x,z[; \mathbb{M}^{q,l})} \le \|\nabla Y\|_{L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})}$$

and

$$Y(y) = \int_0^1 \frac{d}{dt} Y(x + t(y-x)) dt = \int_0^1 \sum_{\alpha=1}^d \partial_\alpha Y(x + t(y-x))(y_\alpha - x_\alpha) dt$$

for all $y \in]x, z[:= \{x + t(z-x); t \in]0,1[\}$. This gives the following inequality of Poincaré's type:

$$\|Y\|_{L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})} \le d\varepsilon \max_{\alpha \in \{1,2,...,d\}} \|\partial_\alpha Y\|_{L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})} \tag{3.29}$$

for all $Y \in W^{1,\infty}_x(B(x,\varepsilon); \mathbb{M}^{q,l})$.

By combining this inequality with $Y = H$ with (3.28), we obtain that

$$\|H\|_{L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})} \le d\varepsilon M \|H\|_{L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})} \le \frac{1}{2} \|H\|_{L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})}$$

thanks to the choice of ε. Hence $H = 0$ in $L^\infty(B(x,\varepsilon); \mathbb{M}^{q,l})$, and thus $H = 0$ also in the space $C^0(B(x,\varepsilon); \mathbb{M}^{q,l})$ since H is continuous. This implies that the entire ball $B(x, \varepsilon)$ is included in $\mathcal{A}$. Therefore, the set $\mathcal{A}$ is open in U.

Since the set $\mathcal{A}$ is non-empty, closed, and open in U, we deduce from the conectedness of U that $\mathcal{A} = U$. Therefore $Y(x) - \widetilde{Y}(x) = H(x) = 0$ for all $x \in U$.

In the following step of the proof, we define a solution Y to the system (3.3)–(3.4) over the entire set Ω by glueing together some sequences of local solutions defined in step (iv) along curves starting from the given point x^0. We shall prove that this definition is unambiguous thanks to the uniqueness result proved in steps (v) and to the fact that the set Ω is simply-connected.

(vi) *There exists a unique solution* $Y \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ *to the system:*

$$\begin{aligned} &\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}), \\ &Y(x^0) = Y^0. \end{aligned} \tag{3.30}$$

In order to define a solution of problem (3.30), let x be a fixed, but otherwise arbitrary, point of Ω. Let $\boldsymbol{\gamma}$ be a path joining x^0 to x, let $R > 0$ be such that $R\sqrt{d} < \text{dist}(\text{Im}\boldsymbol{\gamma}, \Omega^c)$, and let $\Delta = \{t_0, t_1, t_2, ..., t_N\}$ be an R-admissible division for the path $\boldsymbol{\gamma}$, in the sense that $0 = t_0 < t_1 < t_2 < ... < t_N = 1$ and, for all $i \in \{0, 1, ..., N\}$,

$$\boldsymbol{\gamma}(t) \in B(x^i, R) \text{ for all } t \in [t_{i-1}, t_{i+1}],$$

where $x^i := \boldsymbol{\gamma}(t_i)$, $t_{-1} := 0$ and $t_{N+1} := 1$. Note that an R-admissible division for the path $\boldsymbol{\gamma}$ always exists, since $\boldsymbol{\gamma}$ is uniformly continuous over the set $[0, 1]$. A triple $(\boldsymbol{\gamma}, R, \Delta)$ will be called x-admissible if $\boldsymbol{\gamma}$ is a path joining x^0 to x, $0 < R\sqrt{d} < \text{dist}(\text{Im}\boldsymbol{\gamma}, \Omega^c)$ and Δ is an R-admissible division for the path $\boldsymbol{\gamma}$.

Let $B_i := B(x^i, R)$. For $i = 0, 1, 2, ..., N$, we successively define $Y^i := Y^{[t_i]} \in W^{1,\infty}(B_i; \mathbb{M}^{q,l})$ to be the solutions to the systems

$$\begin{aligned} &\partial_\alpha Y^i = Y^i A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B_i; \mathbb{M}^{q,l}), \\ &Y^i(x^i) = Y^{i-1}(x^i), \end{aligned} \tag{3.31}$$

with the convention that $Y^{-1}(x^0) := Y^0$. Since $R\sqrt{d} < \text{dist}(x^i, \Omega^c)$, this system has a unique solution thanks to steps (iv) and (v). We are now able to define the solution to the system (3.30) by letting

$$Y(x) := Y^N(x). \tag{3.32}$$

In this way, we have associated a value for $Y(x)$ to each x-admissible triple $(\boldsymbol{\gamma}, R, \Delta)$, . In what follows, we shall prove that this definition is unambiguous, that is, it does not depend on the choice of the triple $(\boldsymbol{\gamma}, R, \Delta)$. We first prove that this definition does not depend on Δ, then on R, and finally on $\boldsymbol{\gamma}$.

Let an x-admissible triple $(\boldsymbol{\gamma}, R, \Delta)$ be given and let $t^* \in]t_k, t_{k+1}[$. Then one can see that the triple $(\boldsymbol{\gamma}, R, \Delta^*)$, where

$$\Delta^* = \{t_0, t_1, ..., t_k, t^*, t_{k+1}, ..., t_N\},$$

is x-admissible. With the triple $(\boldsymbol{\gamma}, R, \Delta)$, we have associated the functions Y^i, solutions to the systems (3.31). In the same way, with the triple $(\boldsymbol{\gamma}, R, \Delta^*)$, we associate the functions Y_*^0, Y_*^1,..., Y_*^k, Y_*, Y_*^{k+1},...,Y_*^N. We wish to show that $Y^N(x) = Y_*^N(x)$, so that definition (3.32) does not depend on the division Δ. By the uniqueness of the solution to the system (3.31) with $i = 0, 1, 2, ..., k$, we see that $Y_*^i = Y^i$ in B_i for all $i = 0, 1, 2, ..., k$. Let $x^* := \boldsymbol{\gamma}(t^*)$ and $B_* := B(x^*, R)$. Since Y_* and Y^k satisfy

$$\begin{aligned}
&\partial_\alpha Y_* = Y_* A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B_*; \mathbb{M}^{q,l}),\\
&\partial_\alpha Y^k = Y^k A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B_k; \mathbb{M}^{q,l}),\\
&Y_*(x^*) = Y_*^k(x^*) = Y^k(x^*),
\end{aligned}$$

we infer from step (v) that $Y_* = Y^k$ in $B_* \cap B_k$. In particular, $Y_*(x^{k+1}) = Y^k(x^{k+1})$, which next implies that $Y_*^{k+1}(x^{k+1}) = Y^{k+1}(x^{k+1})$. Since Y_*^{k+1} and Y^{k+1} satisfy in addition the relations

$$\begin{aligned}
&\partial_\alpha Y_*^{k+1} = Y_*^{k+1} A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B_{k+1}; \mathbb{M}^{q,l}),\\
&\partial_\alpha Y^{k+1} = Y^{k+1} A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B_{k+1}; \mathbb{M}^{q,l}),
\end{aligned}$$

we infer from step (v) that $Y_*^{k+1} = Y^{k+1}$ in B_{k+1}. By the uniqueness of the solution to the system (3.31) with $i = k+2, k+3, ..., N$, we finally obtain that $Y_*^i = Y^i$ for all $i = k+2, k+3, ..., N$. In particular, $Y_*^N(x) = Y^N(x)$.

Now, let $\Delta = \{t_0, t_1, t_2, ..., t_N\}$ and $\Delta' = \{t'_0, t'_1, t'_2, ..., t'_M\}$ be two R-admissible divisions for the path $\boldsymbol{\gamma}$. Let $\Delta \cup \Delta' = \{s_0, s_1, s_2, ..., s_P\}$. Beginning with Δ and "refining" it to $\Delta \cup \Delta'$, using the previous argument a finite number of times shows that $Y^{[s_P]} = Y^{[t_N]}$ in $B(x, R)$. In the same way, but "refining" Δ' to $\Delta \cup \Delta'$, we also find that $Y^{[s_P]} = Y^{[t'_M]}$ in $B(x, R)$. Therefore, $Y^{[t_N]} = Y^{[t'_M]}$ in $B(x, R)$. This implies that the definition (3.32) of $Y(x)$ does not depend on the division Δ.

We now prove that the definition (3.32) of $Y(x)$ does not depend on the number R. Let $\boldsymbol{\gamma}$ be a path joining x^0 to x and let numbers R_1, R_2 be such that

$$0 < R_1 < R_2 < \frac{1}{\sqrt{d}} \text{dist}(\text{Im}\boldsymbol{\gamma}, \Omega^c).$$

Let a division $\Delta = \{t_0, t_1, t_2, ..., t_N\}$ be R_1-admissible and R_2-admissible for the path $\boldsymbol{\gamma}$. Let $x^i := \boldsymbol{\gamma}(t_i)$. For $i \in \{0, 1, 2, ..., N\}$, let Y_1^i belonging to the space $W^{1,\infty}(B(x^i, R_1); \mathbb{M}^{q,l})$ be the solution to the system

$$\begin{aligned}
&\partial_\alpha Y_1^i = Y_1^i A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B(x^i, R_1); \mathbb{M}^{q,l}),\\
&Y_1^i(x^i) = Y_1^{i-1}(x^i),
\end{aligned}$$

and let $Y_2^i \in W^{1,\infty}(B(x^i, R_2); \mathbb{M}^{q,l})$ be the solution to the system

$$\begin{aligned}
&\partial_\alpha Y_2^i = Y_2^i A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B(x^i, R_2); \mathbb{M}^{q,l}),\\
&Y_2^i(x^i) = Y_2^{i-1}(x^i),
\end{aligned}$$

where $Y_1^{-1}(x^0) = Y_2^{-1}(x^0) := Y^0$. Then we prove by a recursion argument that $Y_1^i = Y_2^i$ in $B(x^i, R_1)$. For $i = 0$ this is a consequence of step (v). Assume now that $Y_1^k = Y_2^k$ in $B(x^k, R_1)$ for a fixed $k \in \{0, 1, 2, ..., N-1\}$. Since $x^{k+1} \in B(x^k, R_1) \subset B(x^k, R_2)$, it follows that $Y_1^k(x^{k+1}) = Y_2^k(x^{k+1})$. By the uniqueness of the solution to the first system above with $i := k + 1$, this implies that $Y_1^{k+1} = Y_2^{k+1}$ in $B(x^{k+1}, R_1)$. After a finite number of iterations, we obtain that $Y_1^N = Y_2^N$ in $B(x, R_1)$, which means that definition (3.32) of $Y(x)$ does not depend on the number R.

We now prove that the definition (3.32) of $Y(x)$ does not depend on the path $\boldsymbol{\gamma}$. Let there be given two paths $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$ joining x^0 to x. Since Ω is simply connected, there exists a homotopy $\boldsymbol{\varphi} \in C^0([0,1] \times [0,1]; \Omega)$ such that

$$\boldsymbol{\varphi}(t, 0) = \boldsymbol{\gamma}(t), \boldsymbol{\varphi}(t, 1) = \widetilde{\boldsymbol{\gamma}}(t),$$
$$\boldsymbol{\varphi}(0, s) = x^0, \boldsymbol{\varphi}(1, s) = x.$$

Let a number $r > 0$ be fixed such that $2r\sqrt{d} < \text{dist}(\boldsymbol{\varphi}([0,1] \times [0,1]), \Omega^c)$ (such a number r exists since $\boldsymbol{\varphi}([0,1] \times [0,1]) \Subset \Omega$). Since $\boldsymbol{\varphi}$ is uniformly continuous over the compact set $[0,1] \times [0,1]$, there exists an integer $N > 0$ such that

$$|\boldsymbol{\varphi}(t, s) - \boldsymbol{\varphi}(t', s')| < r \tag{3.33}$$

for all $(t, s), (t', s') \in [0,1] \times [0,1]$ such that $\{|t - t'|^2 + |s - s'|^2\}^{1/2} \leq 1/N$. Let $t_k = s_k := k/N$ for all $k = 0, 1, 2, ..., N$. Let $\boldsymbol{\gamma}^k(t) := \boldsymbol{\varphi}(t, s_k)$ and note that $\boldsymbol{\gamma}^k$ are paths joining x^0 to x that satisfy

$$|\boldsymbol{\gamma}^k(t) - \boldsymbol{\gamma}^{k-1}(t)| < r \text{ for all } k \in \{1, 2, ..., N\} \text{ and all } t \in [0,1].$$

Let also $\Delta := \{t_0, t_1, t_2, ..., t_N\}$ and note that the triples $(\boldsymbol{\gamma}^k, 2r, \Delta)$ are x-admissible for all $k = 0, 1, 2..., N$ and the division Δ is r-admissible for the path $\boldsymbol{\gamma}^k$.

In order to prove that the definition (3.32) of $Y(x)$ does not depend on the choice of the path joining x^0 to x, it suffices to prove that, for a given $k \in \{1, 2, ..., N\}$, the definition (3.32) based on the triple $(\boldsymbol{\gamma}^{k-1}, 2r, \Delta)$ coincides with the definition (3.32) based on the triple $(\boldsymbol{\gamma}^k, 2r, \Delta)$.

Let $x^{k,i} := \boldsymbol{\gamma}^k(t_i)$ and $B_{k,i} := B(x^{k,i}, 2r)$. For each $i \in \{0, 1, 2, ..., N\}$, let $Y^{k,i} \in W^{1,\infty}(B_{k,i}; \mathbb{M}^{q,l})$ be the solution to the system (see (3.31))

$$\begin{aligned} &\partial_\alpha Y^{k,i} = Y^{k,i} A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B_{k,i}; \mathbb{M}^{q,l}), \\ &Y^{k,i}(x^{k,i}) = Y^{k,i-1}(x^{k,i}), \end{aligned} \tag{3.34}$$

where $Y^{k,-1}(x^{k,0}) := Y^0$. We wish to prove that $Y^{k-1,N}(x) = Y^{k,N}(x)$. In fact, we will prove that $Y^{k-1,N} = Y^{k,N}$ in the open ball $B(x, 2r)$.

First, notice that $Y^{k-1,0} = Y^{k,0}$ in $B_{k-1,0} \cap B_{k,0}$ thanks to the uniqueness of the solution to the system (3.34) with $i := 0$ (see step (v)). Assume that for a fixed $i \in \{0, 1, 2, ..., N-1\}$ we have $Y^{k-1,i} = Y^{k,i}$ in $B_{k-1,i} \cap B_{k,i}$. In particular, $Y^{k-1,i}(x^{k-1,i+1}) = Y^{k,i}(x^{k-1,i+1})$, which implies on the one hand that

$$Y^{k-1,i+1}(x^{k-1,i+1}) = Y^{k,i}(x^{k-1,i+1}).$$

On the other hand, step (v) implies that $Y^{k,i+1} = Y^{k,i}$ in the set $B_{k,i+1} \cap B_{k,i}$. Since $x^{k-1,i+1}$ belongs to this set, we obtain in particular that

$$Y^{k,i+1}(x^{k-1,i+1}) = Y^{k,i}(x^{k-1,i+1}).$$

Combining the two relations above gives that

$$Y^{k-1,i+1}(x^{k-1,i+1}) = Y^{k,i+1}(x^{k-1,i+1}).$$

We then infer from step (v) that $Y^{k-1,i+1} = Y^{k,i+1}$ in $B_{k-1,i+1} \cap B_{k,i+1}$. After N iterations, we eventually find that $Y^{k-1,N} = Y^{k,N}$ in $B_{k-1,N} \cap B_{k,N} = B(x, 2r)$. Therefore, the definition (3.32) of $Y(x)$ does not depend on the choice of the path $\boldsymbol{\gamma}$ joining x^0 to x.

It remains to prove that the matrix field Y belongs to $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ and that it satisfies the system (3.30). Let x be a fixed, but otherwise arbitrary, point in Ω and let $(\boldsymbol{\gamma}, R, \Delta)$ be an x-admissible triple with $2R\sqrt{d} < \text{dist}(\text{Im}\boldsymbol{\gamma}, \Omega^c)$. Let $x^i := \boldsymbol{\gamma}(t_i)$. Then

$$Y(x) := Y^N(x), \tag{3.35}$$

where $Y^i \in W^{1,\infty}(B(x^i, R); \mathbb{M}^{q,l})$, $i = 0, 1, 2, ..., N$, is the solution to the system

$$\begin{aligned} &\partial_\alpha Y^i = Y^i A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B(x^i, R); \mathbb{M}^{q,l}), \\ &Y^i(x^i) = Y^{i-1}(x^i), \end{aligned} \tag{3.36}$$

with the convention that $Y^{-1}(x^0) := Y^0$ (see the begining of step (vi)).

We now show that

$$Y = Y^N \text{ in the open ball } B(x, R). \tag{3.37}$$

To this end, let $\widetilde{x}$ be a fixed, but otherwise arbitrary, point in $B(x, R)$ and let $\widetilde{\boldsymbol{\gamma}}$ be the path obtained by joining $\text{Im}\boldsymbol{\gamma}$ with the segment $[x, \widetilde{x}]$. For instance, $\widetilde{\boldsymbol{\gamma}}$ is given by

$$\widetilde{\boldsymbol{\gamma}}(\widetilde{t}) = \begin{cases} \boldsymbol{\gamma}(2\widetilde{t}) & \text{for all } \widetilde{t} \in [0, 1/2]\,, \\ (2 - 2\widetilde{t})x + (2\widetilde{t} - 1)\widetilde{x} & \text{for all } \widetilde{t} \in (1/2, 1]\,. \end{cases}$$

Notice that $\widetilde{\boldsymbol{\gamma}}$ is a path joining x^0 to $\widetilde{x}$. Let $\widetilde{\Delta} := \{\widetilde{t}_0, \widetilde{t}_1, \ldots, \widetilde{t}_N, \widetilde{t}_{N+1}\}$, where $\widetilde{t}_i = t_i/2$ for all $i \in \{0, 1, 2, \ldots, N\}$ and $\widetilde{t}_{N+1} = 1$. Then one can see that the triple $(\widetilde{\boldsymbol{\gamma}}, R, \widetilde{\Delta})$ is $\widetilde{x}$-admissible. To this end, one has to prove in particular that $\text{dist}(\text{Im}\widetilde{\boldsymbol{\gamma}}, \Omega^c) > R\sqrt{d}$. This can be shown in the following way. If $\widetilde{t} \in [0, \widetilde{t}_N]$, then

$$\text{dist}(\widetilde{\boldsymbol{\gamma}}(\widetilde{t}), \Omega^c) = \text{dist}(\boldsymbol{\gamma}(t), \Omega^c) > 2R\sqrt{d} > R\sqrt{d}.$$

If $\widetilde{t} \in [\widetilde{t}_N, 1]$, then we have $\text{dist}(x, \Omega^c) \leq |x - \widetilde{\boldsymbol{\gamma}}(\widetilde{t})| + \text{dist}(\widetilde{\boldsymbol{\gamma}}(\widetilde{t}), \Omega^c)$, so that

$$\text{dist}(\widetilde{\boldsymbol{\gamma}}(\widetilde{t}), \Omega^c) \geq \text{dist}(x, \Omega^c) - |x - \widetilde{\boldsymbol{\gamma}}(\widetilde{t})| > 2R\sqrt{d} - R > R\sqrt{d}.$$

Therefore we can define $Y(\widetilde{x})$ by means of $\widetilde{\gamma}$, R and $\widetilde{\Delta}$ in the same way that $Y(x)$ was defined by means of γ, R and Δ (see (3.35)). More specifically, let $\widetilde{x}^i := \widetilde{\gamma}(\widetilde{t}_i)$. Then

$$Y(\widetilde{x}) := \widetilde{Y}^{N+1}(\widetilde{x}),$$

where, for all $i = 0, 1, 2, ..., N+1$, $\widetilde{Y}^i \in W^{1,\infty}(B(\widetilde{x}^i, R); \mathbb{M}^{q,l})$ is the solution to the system

$$\begin{aligned} &\partial_\alpha \widetilde{Y}^i = \widetilde{Y}^i A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B(\widetilde{x}^i, R); \mathbb{M}^{q,l}), \\ &\widetilde{Y}^i(\widetilde{x}^i) = \widetilde{Y}^{i-1}(\widetilde{x}^i), \end{aligned} \tag{3.38}$$

where $\widetilde{Y}^{-1}(\widetilde{x}^0) := Y^0$.

Since $x^i = \widetilde{x}^i$ for all $i \in \{0, 1, 2, ..., N\}$, we infer from step (v) that $\widetilde{Y}^i = Y^i$ in $B(x^i, R)$. Hence $\widetilde{Y}^N = Y^N$ in $B(x^N, R)$, which implies in particular that $\widetilde{Y}^N(\widetilde{x}) = Y^N(\widetilde{x})$. On the other hand, since $\widetilde{x} = \widetilde{x}^{N+1}$, we have $\widetilde{Y}^{N+1}(\widetilde{x}) = \widetilde{Y}^N(\widetilde{x})$ by (3.38). By combining these relations, we finally obtain

$$Y(\widetilde{x}) = \widetilde{Y}^{N+1}(\widetilde{x}) = \widetilde{Y}^N(\widetilde{x}) = Y^N(\widetilde{x}).$$

This completes the proof of relation (3.37).

So far, we have defined a matrix field $Y : \Omega \to \mathbb{M}^{q,l}$ and showed that $Y = Y^N$ in the open ball $B(x, R)$. Since Y^N belongs to the space $W^{1,\infty}(B(x, R); \mathbb{M}^{q,l})$ and satisfies the system (see (3.36))

$$\partial_\alpha Y^N = Y^N A_\alpha + B_\alpha \text{ in } \mathcal{D}'(B(x, R); \mathbb{M}^{q,l}),$$

letting x vary in the set Ω shows that Y belongs to the space $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ and satisfies the system

$$\begin{aligned} &\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}), \\ &Y(x^0) = Y^0. \end{aligned}$$

The uniqueness of the solution to (3.30) is given by step (v). □

Under an additional assumption on the set Ω, if the coefficients A_α and B_α are of class L^∞ in Ω, then the solution Y to the system appearing in Theorem 3.1 belong to $W^{1,\infty}$ in Ω:

Corollary 3.1. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^d$ whose geodesic diameter is finite. Let there be given a point $x^0 \in \Omega$, a matrix $Y^0 \in \mathbb{M}^{q,l}$, and some matrix fields $A_\alpha \in L^\infty(\Omega; \mathbb{M}^l)$, $B_\alpha \in L^\infty(\Omega; \mathbb{M}^{q,l})$ such that*

$$\begin{aligned} &\partial_\alpha A_\beta + A_\alpha A_\beta = \partial_\beta A_\alpha + A_\beta A_\alpha \textit{ in } \mathcal{D}'(\Omega; \mathbb{M}^l), \\ &\partial_\alpha B_\beta + B_\alpha A_\beta = \partial_\beta B_\alpha + B_\beta A_\alpha \textit{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}). \end{aligned}$$

Then the system

$$\begin{aligned} &\partial_\alpha Y = Y A_\alpha + B_\alpha \textit{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}), \\ &Y(x^0) = Y^0 \end{aligned}$$

has a unique solution in $W^{1,\infty}(\Omega;\mathbb{M}^{q,l})$.

Proof. Theorem 3.1 shows that the system above has a unique solution in $W^{1,\infty}_{\text{loc}}(\Omega;\mathbb{M}^{q,l})$. It remains to prove that this solution belongs to $W^{1,\infty}(\Omega;\mathbb{M}^{q,l})$.

Let $c_1, c_2 \geq 0$ be two constants such that

$$\|A_\alpha\|_{L^\infty(\Omega;\mathbb{M}^l)} \leq c_1 \text{ and } \|B_\alpha\|_{L^\infty(\Omega;\mathbb{M}^{q,l})} \leq c_2.$$

Let $x \in \Omega$ be a fixed, but otherwise arbitrary, point in Ω and let $\varepsilon > 0$. Then the definition of the geodesic diameter D_Ω shows that there exists a path $\widetilde{\boldsymbol{\gamma}}$ joining x^0 to x such that

$$L(\widetilde{\boldsymbol{\gamma}}) \leq D_\Omega + \varepsilon.$$

Let $r := \frac{1}{4}\text{dist}(\text{Im}\widetilde{\boldsymbol{\gamma}}, \Omega^c)$ and note that $r > 0$. Since $\widetilde{\boldsymbol{\gamma}}$ is uniformly continuous over $[0,1]$, there exist numbers $t_0, t_1, t_2, ..., t_N \in [0,1]$ such that $0 = t_0 < t_1 < t_2 < ... < t_N = 1$ and

$$|\widetilde{\boldsymbol{\gamma}}(t_i) - \widetilde{\boldsymbol{\gamma}}(t_{i-1})| < r \text{ for all } i \in \{1,2,...,N\}.$$

Let $\delta := \min(r, \frac{\varepsilon}{2N})$ and let the points $x^1, x^2, ..., x^N \in \Omega$ be such that $x^i \in B(\widetilde{\boldsymbol{\gamma}}(t_i), \delta)$ and such that the field $Y \circ \boldsymbol{\gamma}^i : [0,1] \to \mathbb{M}^{q,l}$ satisfies the following relations

$$\begin{aligned} Y(\boldsymbol{\gamma}^i(t)) &= Y(\boldsymbol{\gamma}^i(0)) + \int_0^t (Y \circ \boldsymbol{\gamma}^i)'(s)ds \\ &= Y(\boldsymbol{\gamma}^i(0)) + \int_0^t \partial_\alpha Y(\boldsymbol{\gamma}^i(s))(\gamma^i_\alpha)'(s)ds \\ &= Y(\boldsymbol{\gamma}^i(0)) + \int_0^t \left\{(YA_\alpha)(\boldsymbol{\gamma}^i(s)) + B_\alpha(\boldsymbol{\gamma}^i(s))\right\}(\gamma^i_\alpha)'(s)ds \end{aligned}$$

for all $t \in [0,1]$ and all $i \in \{0,1,...,N-1\}$, where $\boldsymbol{\gamma}^i : t \mapsto \left\{x^i + t(x^{i+1} - x^i)\right\}$ maps the interval $[0,1]$ onto the segment joining x^i to x^{i+1}. This can be done successively, by choosing first the segment $[x^0, x^1]$ such that $x^1 \in B(\widetilde{\boldsymbol{\gamma}}(t_1), \delta)$ and such that the restriction of the field $Y \in W^{1,\infty}(B(x^0, 3r);\mathbb{M}^{q,l})$ to the open segment (x^0, x^1) is of class $W^{1,\infty}$ and its derivative satisfies the relations above with $i = 0$ (see Corollary A.1, page 134, for further details), then by choosing the segment $[x^1, x^2]$ such that $x^2 \in B(\widetilde{\boldsymbol{\gamma}}(t_2), \delta)$ and the restriction of the field $Y \in W^{1,\infty}(B(x^1, 3r);\mathbb{M}^{q,l})$ to the open segment (x^1, x^2) is of class $W^{1,\infty}$ and its derivative satisfies the relations above with $i = 1$, etc. Note that the segment $[x^i, x^{i+1}]$ is contained in the ball $B(x^i, 3r)$ and that this ball is contained in the set Ω.

Let $\boldsymbol{\gamma} = (\gamma_\alpha) : [0,N] \to \Omega$ be the union of the paths $\boldsymbol{\gamma}^i$ defined by $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}^i(t - i)$ for all $t \in [i, i+1]$ and all $i \in \{0,1,...,N-1\}$. Then the previous relations together imply that

$$Y(\boldsymbol{\gamma}(t)) = Y(\boldsymbol{\gamma}(0)) + \int_0^t \{(YA_\alpha)(\boldsymbol{\gamma}(s)) + B_\alpha(\boldsymbol{\gamma}(s))\}\gamma'_\alpha(s)ds.$$

for all $t \in [0, N]$. Consequently,

$$|Y(\boldsymbol{\gamma}(t))| \leq |Y^0| + \int_0^t (|c_1 Y(\boldsymbol{\gamma}(s))| + c_2) \sum_\alpha |\gamma'_\alpha(s)| ds$$
$$\leq |Y^0| + c_2\sqrt{d}\int_0^t |\boldsymbol{\gamma}'(s)|ds + c_1\sqrt{d}\int_0^t |Y(\boldsymbol{\gamma}(s))||\boldsymbol{\gamma}'(s)|ds,$$

which next implies that the function $f : [0, N] \to \mathbb{R}$ defined by $f(t) := |Y(\boldsymbol{\gamma}(t))|$ satisfies the inequality

$$f(t) \leq (|Y^0| + c_2\sqrt{d}L(\boldsymbol{\gamma})) + c_1\sqrt{d}\int_0^t |\boldsymbol{\gamma}'(s)|f(s)ds.$$

We then infer from Gronwall inequality that

$$f(t) \leq (|Y^0| + c_2\sqrt{d}L(\boldsymbol{\gamma}))e^{c_1\sqrt{d}\int_0^t |\boldsymbol{\gamma}'(s)|ds},$$

which implies in particular that

$$|Y(x^N)| \leq (|Y^0| + c_2\sqrt{d}L(\boldsymbol{\gamma}))e^{c_1\sqrt{d}L(\boldsymbol{\gamma})}.$$

But the length of the path $\boldsymbol{\gamma}$ satisfies

$$L(\boldsymbol{\gamma}) = \sum_{i=0}^{N-1} |x^{i+1} - x^i| \leq \sum_{i=0}^{N-1} (|x^{i+1} - \widetilde{\boldsymbol{\gamma}}(t_{i+1})| + |\widetilde{\boldsymbol{\gamma}}(t_{i+1}) - \widetilde{\boldsymbol{\gamma}}(t_i)| + |\widetilde{\boldsymbol{\gamma}}(t_i) - x^i|)$$
$$\leq L(\widetilde{\boldsymbol{\gamma}}) + \sum_{i=0}^{N-1} \frac{\varepsilon}{N} \leq D_\Omega + 2\varepsilon.$$

Consequently,

$$|Y(x^N)| \leq (|Y^0| + c_2\sqrt{d}(D_\Omega + 2\varepsilon))e^{c_1\sqrt{d}(D_\Omega + 2\varepsilon)}.$$

Since x^N goes to x when ε goes to zero, letting ε go to zero in the inequality above gives

$$|Y(x)| \leq (|Y^0| + c_2\sqrt{d}D_\Omega)e^{c_1\sqrt{d}D_\Omega}. \tag{3.39}$$

This implies in particular that the matrix field Y belongs to the space $L^\infty(\Omega, \mathbb{M}^{q,l})$. Since the derivatives of the field Y satisfy the relations $\partial_\alpha Y = YA_\alpha + B_\alpha$, the field Y belongs to the space $W^{1,\infty}(\Omega, \mathbb{M}^{q,l})$. □

Remark 3.2. Inequality (3.39) furnishes another proof to the uniqueness result announced in Corollary 3.1.

With a proof similar to that of Theorem 3.1, the following more general result can be established:

Theorem 3.2. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^d$. Let the matrix fields $A_\alpha \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^l)$, $B_\alpha \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$ and $C_\alpha \in$*

$L^\infty_{\rm loc}(\Omega;\mathbb{M}^q)$ such that

$$\begin{aligned}&\partial_\alpha A_\beta + A_\alpha A_\beta = \partial_\beta A_\alpha + A_\beta A_\alpha,\\ &\partial_\alpha C_\beta + C_\beta C_\alpha = \partial_\beta C_\alpha + C_\alpha C_\beta\\ &\partial_\alpha B_\beta + B_\alpha A_\beta + C_\beta B_\alpha = \partial_\beta B_\alpha + B_\beta A_\alpha + C_\alpha B_\beta.\end{aligned}$$

Let a point $x^0 \in \Omega$ and a matrix $Y^0 \in \mathbb{M}^{q,l}$ be fixed. Then the Cauchy problem

$$\begin{aligned}&\partial_\alpha Y = Y A_\alpha + C_\alpha Y + B_\alpha \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^{q,l}),\\ &Y(x^0) = Y^0\end{aligned}$$

has a unique solution in $W^{1,\infty}_{\rm loc}(\Omega;\mathbb{M}^{q,l})$.

Remark 3.3. The above theorem can also be established as a corollary to Theorem 3.1 (see Theorem A.4, page 137, for a proof).

5. Existence of an isometric immersion

Let Ω be a connected and simply-connected open subset of $\mathbb{R}^d$ and let $(\Omega,(g_{ij}))$ be a Riemannian space whose metric is given by a matrix field $(g_{ij}) \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^d_>)$. We recall that, according to the conventions made in section 2, the field (g_{ij}) is the continuous representative of the class, still denoted by, $(g_{ij}) \in W^{1,\infty}_{\rm loc}(\Omega;\mathbb{S}^d_>)$. Therefore, $(g_{ij}(x)) \in \mathbb{S}^d_>$ *for all* $x \in \Omega$. Define the Christoffel symbols

$$\Gamma^k_{ij}(x) = \frac{1}{2} g^{kl}(x)\left(\partial_i g_{jl}(x) + \partial_j g_{li}(x) - \partial_l g_{ij}(x)\right)$$

for almost all $x \in \Omega$, where $(g^{kl}(x))$ is the inverse of the matrix $(g_{ij}(x))$. Since $(g_{ij}) \in C^0(\Omega;\mathbb{S}^d_>)$ by the Sobolev imbeddings, the inverse matrix $(g^{kl}) \in C^0(\Omega;\mathbb{S}^d_>)$. This implies that the Christoffel symbols satisfy $\Gamma^k_{ij} \in L^\infty_{\rm loc}(\Omega)$.

We assume that the corresponding Riemann curvature tensor vanishes in $\mathcal{D}'(\Omega)$, that is,

$$\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^l_{ik}\Gamma^p_{jl} - \Gamma^l_{ij}\Gamma^p_{kl} = 0$$

for all $i,j,k,p \in \{1,2,...,d\}$; this means that

$$\int_\Omega \Gamma^p_{ij}\partial_k\varphi dx - \int_\Omega \Gamma^p_{ik}\partial_j\varphi dx + \int_\Omega \left(\Gamma^l_{ik}\Gamma^p_{jl} - \Gamma^l_{ij}\Gamma^p_{kl}\right)\varphi dx = 0,$$

for all $\varphi \in \mathcal{D}(\Omega)$.

The aim of this section is to find a mapping $\Theta \in W^{2,\infty}_{\rm loc}(\Omega;\mathbb{R}^d)$ such that the restriction of the d-dimensional Euclidean metric to $\Theta(\Omega)$ is given by the matrix field (g_{ij}), i.e., such that

$$g_{ij} = \partial_i\Theta \cdot \partial_j\Theta \text{ in } \Omega.$$

Thus the mapping Θ is an isometric immersion of the Riemannian space $(\Omega,(g_{ij}))$ in the d-dimensional Euclidean space. In other words, from a given metric in $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{S}^d_>)$ whose Riemann curvature tensor vanishes in a distributional sense, one can recover a manifold in the d-dimensional Euclidean space whose metric is the given one. This result is established in Theorem 3.3.

We first need to prove the following lemma:

Lemma 3.3. *Let* $(g_{ij}) \in \mathbb{S}^d_>$. *Then there exist* d *vectors* $\boldsymbol{g}_i \in \mathbb{R}^d$, $i \in \{1, 2, ..., d\}$, *such that* $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij}$.

Proof. Since (g_{ij}) is symmetric, there exists a diagonal matrix D and an orthogonal matrix P such that $(g_{ij}) = P^T DP$. Since (g_{ij}) is positive definite, the elements of the diagonal of the matrix D are > 0. Let $D^{1/2}$ be the unique positive definite square root of the diagonal matrix D. Then, $(g_{ij}) = (D^{1/2}P)^T(D^{1/2}P)$. Let $\boldsymbol{g}_i \in \mathbb{R}^d$ be the i-th column of the matrix $D^{1/2}P$. Clearly, $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij}$ and the proof is complete. □

We are now in a position to prove our main result:

Theorem 3.3. *Let* Ω *be a connected and simply connected open subset of* $\mathbb{R}^d$ *and let a metric be given in* Ω *by the means of a matrix field* $(g_{ij}) \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{S}^d_>)$. *Assume that the corresponding Riemann curvature tensor vanishes, that is,*

$$\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^l_{ik}\Gamma^p_{jl} - \Gamma^l_{ij}\Gamma^p_{kl} = 0 \text{ in } \mathcal{D}'(\Omega).$$

(i) Then there exists a mapping $\Theta \in W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^d)$, *unique up to isometries in* $\mathbb{R}^d$, *such that*

$$\partial_i \Theta \cdot \partial_j \Theta = g_{ij} \text{ in } \Omega.$$

(ii) Let there be given $x^0 \in \Omega$ *and* $\Theta^0, \boldsymbol{g}^0_i \in \mathbb{R}^d$ *such that* $\boldsymbol{g}^0_i \cdot \boldsymbol{g}^0_j = g_{ij}(x^0)$ *(such vectors exist thanks to Lemma 3.3). Then there exists one and only one mapping* $\Theta \in W^{2,\infty}_{\text{loc}}(\Omega; \mathbb{R}^d)$ *such that* $\partial_i\Theta \cdot \partial_j\Theta = g_{ij}$ *in* Ω *and*

$$\Theta(x^0) = \Theta^0 \text{ and } \partial_i \Theta(x^0) = \boldsymbol{g}^0_i.$$

Proof. The outline of the proof is as follows. Let the matrix field $\Gamma_i : \Omega \to \mathbb{M}^d$ be given by $\Gamma_i(x) = \left(\Gamma^k_{ij}(x)\right) \in \mathbb{M}^d$, where j is the column index and k is the row index of the matrix.

We begin by finding a matrix field $F \in W^{1,\infty}(\Omega; \mathbb{M}^d)$ such that

$$\begin{aligned} &\partial_i F = F\Gamma_i \text{ in } \Omega, \\ &F(x^0) = F^0, \end{aligned} \tag{3.40}$$

where $F^0 \in \mathbb{M}^d$ is the matrix whose i-th column is $\boldsymbol{g}^0_i \in \mathbb{R}^d$. The existence of such a field is insured by Theorem 3.1.

Then the columns of the matrix $F(x)$, denoted by $\boldsymbol{g}_i(x) \in \mathbb{R}^d$, will turn out to be the derivatives of the sought mapping Θ, whose existence will again be given by Theorem 3.1 (which in that case is a generalized form of Poincaré's theorem).

The proof is broken into five steps, numbered (i) to (v).

(i) *There exists a solution* $F \in W^{1,\infty}_{\text{loc}}(\Omega;\mathbb{M}^d)$ *to the system*

$$\begin{aligned} &\partial_i F = F\Gamma_i \text{ a.e. in } \Omega,\\ &F(x^0) = F^0, \end{aligned} \tag{3.41}$$

where F^0 *is the matrix whose i-th column is* $\boldsymbol{g}_i^0$.

Since $(g_{ij}) \in W^{1,\infty}_{\text{loc}}(\Omega;\mathbb{S}^d_>) \subset C^0(\Omega;\mathbb{S}^d_>)$, we have $\det(g_{ij}(x)) > 0$ for all $x \in \Omega$. This implies that the coefficients g^{kl} of the inverse of the matrix (g_{ij}) belong to the space $C^0(\Omega)$. Then the definition of the Christoffel symbols shows that $\Gamma^k_{ij} \in L^\infty_{\text{loc}}(\Omega)$, so that $\Gamma_i \in L^\infty_{\text{loc}}(\Omega;\mathbb{M}^d)$.

Now, one can see that the Riemann compatibility conditions (3.49) are satisfied if and only if

$$\partial_i\Gamma_j - \partial_j\Gamma_i + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i = 0 \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^d),$$

that is, the following relations hold

$$\int_\Omega \{-\Gamma_j\partial_i\varphi + \Gamma_i\partial_j\varphi + \Gamma_i\Gamma_j\varphi - \Gamma_j\Gamma_i\varphi\}(x)dx = 0 \tag{3.42}$$

for all $\varphi \in \mathcal{D}(\Omega)$. The assumptions of Theorem 3.1 applied to the system (3.41) being satisfied, the existence of a solution F to this system follows.

(ii) *Let* $\boldsymbol{g}_i(x)$ *denote the i-th column of the matrix* $F(x)$, $x \in \Omega$. *Then there exists a solution* $\Theta \in W^{2,\infty}_{\text{loc}}(\Omega;\mathbb{R}^d)$ *to the system*

$$\begin{aligned} &\partial_i\Theta = \boldsymbol{g}_i \text{ in } \Omega, i = 1,2,...,d,\\ &\Theta(x^0) = \Theta^0. \end{aligned} \tag{3.43}$$

Thanks to step (i), we have that $\boldsymbol{g}_i \in W^{1,\infty}_{\text{loc}}(\Omega;\mathbb{R}^d)$ and

$$\partial_j\boldsymbol{g}_i = \Gamma^k_{ji}\boldsymbol{g}_k = \Gamma^k_{ij}\boldsymbol{g}_k = \partial_i\boldsymbol{g}_j. \tag{3.44}$$

These relations, together with the assumption that Ω is simply-connected, allow to apply Theorem 3.1 to the system (3.43). This shows that there exists a unique solution $\Theta \in W^{1,\infty}_{\text{loc}}(\Omega;\mathbb{R}^d)$ to this system. Since $\partial_i\Theta = \boldsymbol{g}_i \in W^{1,\infty}_{\text{loc}}(\Omega;\mathbb{R}^d)$, the mapping Θ belongs in fact to the space $W^{2,\infty}_{\text{loc}}(\Omega;\mathbb{R}^d)$.

(iii) *The mapping* Θ *satisfies the relation*

$$\partial_i\Theta(x)\cdot\partial_j\Theta(x) = g_{ij}(x) \text{ for all } x \in \Omega \text{ and all } i,j. \tag{3.45}$$

Since the fields $\boldsymbol{g}_i := \partial_i\Theta$, $i = 1,2,...,d$, are the columns of the matrix field F that satisfies the system (3.41), we obtain on the one hand that

$$\begin{aligned} \partial_k(\boldsymbol{g}_i(x)\cdot\boldsymbol{g}_j(x)) &= \partial_k\boldsymbol{g}_i(x)\cdot\boldsymbol{g}_j(x) + \boldsymbol{g}_i(x)\cdot\partial_k\boldsymbol{g}_j(x)\\ &= \Gamma^p_{ki}(x)\boldsymbol{g}_p(x)\cdot\boldsymbol{g}_j(x) + \Gamma^p_{kj}\boldsymbol{g}_i(x)\cdot\boldsymbol{g}_p(x) \end{aligned}$$

for almost all $x \in \Omega$ and all $i, j, k \in \{1, 2, ..., d\}$, and also that

$$\boldsymbol{g}_i(x^0) \cdot \boldsymbol{g}_j(x^0) = \boldsymbol{g}_i^0 \cdot \boldsymbol{g}_j^0 = g_{ij}(x^0).$$

On the other hand, thanks to the definition of the Christoffel symbols, we have

$$g_{ip}\Gamma^p_{kj} = \frac{1}{2}\left\{\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{kj}\right\},$$
$$g_{jp}\Gamma^p_{ki} = \frac{1}{2}\left\{\partial_k g_{ji} + \partial_i g_{jk} - \partial_j g_{ki}\right\}$$

for almost all $x \in \Omega$ and all $i, j, k \in \{1, 2, ..., d\}$. Adding these two relations gives

$$\partial_k g_{ij} = \Gamma^p_{ki} g_{pj} + \Gamma^p_{kj} g_{ip},$$

since $g_{ij} = g_{ji}$.

Let $X_{ij}(x) := \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) - g_{ij}(x)$. Then the functions X_{ij} belong to $W^{1,\infty}_{\text{loc}}(\Omega)$ and satisfy the relations

$$\partial_k X_{ij} = \Gamma^p_{ki} X_{pj} + \Gamma^p_{kj} X_{ip} \text{ in } L^\infty(\Omega),$$
$$X_{ij}(x^0) = 0$$

for all $i, j, k \in \{1, 2, ..., d\}$. Let the matrix field $X : \Omega \to \mathbb{M}^d$ be given by $X(x) := (X_{ij}(x)) \in \mathbb{M}^d$, where i is the row index and j is the column index of the matrix. Then $X \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^d)$ and

$$\begin{aligned} &\partial_k X = X\Gamma_k + \Gamma_k^T X \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^d), \\ &X(x^0) = 0 \text{ in } \mathbb{M}^d. \end{aligned} \tag{3.46}$$

We wish to prove that X vanish in Ω. To this end, we shall use a connectedness argument. First, note that $X \in C^0(\Omega; \mathbb{M}^d)$ by the Sobolev imbeddings. Then define the following subset of Ω,

$$S := \{x \in \Omega; X(x) = 0 \in \mathbb{M}^d\}.$$

The set S is non-empty (see (3.46)) and closed in Ω (since the application $X : \Omega \to \mathbb{M}^d$ is continuous). Now, let us prove that S is also open in Ω.

Let $y \in S$, let $B_0 := B(y, \varepsilon_0)$ be an open ball such that $B_0 \Subset \Omega$ and let $M := \max_{k\in\{1,2,...,d\}} \|\Gamma_k\|_{L^\infty(B_0;\mathbb{M}^d)}$. Let also $B := B(y, \varepsilon)$, where

$$\varepsilon = \min\left\{\frac{1}{2d(1+\sqrt{d})M}, \varepsilon_0\right\}.$$

Using the inequality

$$|A^T| \le \sqrt{\text{tr}(AA^T)} = \sqrt{\text{tr}(A^TA)} \le \sqrt{d|A|^2} = \sqrt{d}|A|$$

valid for all matrices $A \in \mathbb{M}^d$, we deduce from (3.46) that

$$\|\partial_k X\|_{L^\infty(B;\mathbb{M}^d)} \le (1+\sqrt{d})M\|X\|_{L^\infty(B;\mathbb{M}^d)} \tag{3.47}$$

for all $k \in \{1, 2, ..., d\}$.

Now, define the space

$$W_y^{1,\infty}(B;\mathbb{M}^d) := \{Y \in W^{1,\infty}(B;\mathbb{M}^d); Y(y) = 0 \text{ in } \mathbb{M}^d\}.$$

Let Y be an element of this space. Then a result about Sobolev spaces (see Corollary A.1, page 134, for a proof) shows that, for almost all $z \in \partial B$ (with respect to the surface measure), the restriction of Y to the open segment $]y,z[$ belongs to $W^{1,\infty}(]y,z[;\mathbb{M}^d)$, $\|\nabla Y\|_{L^\infty(]y,z[;\mathbb{M}^d)} \leq \|\nabla Y\|_{L^\infty(B;\mathbb{M}^d)}$ and

$$Y(x) = \int_0^1 \frac{d}{dt} Y(y+t(x-y))dt = \int_0^1 \sum_{k=1}^d \partial_k Y(y+t(x-y))(x_k - y_k)dt$$

for all $x \in]y,z[:= \{y + t(z-y); t \in]0,1[\}$. This gives the following inequality of Poincaré's type:

$$\|Y\|_{L^\infty(B;\mathbb{M}^d)} \leq d\varepsilon \max_{k\in\{1,2,\ldots,d\}} \|\partial_k Y\|_{L^\infty(B;\mathbb{M}^d)}$$

for all $Y \in W_y^{1,\infty}(B;\mathbb{M}^d)$. By combining this inequality with $Y = X$ with (3.47), we obtain that

$$\|X\|_{L^\infty(B;\mathbb{M}^d)} \leq d\varepsilon(1+\sqrt{d})M\|X\|_{L^\infty(B;\mathbb{M}^d)} \leq \frac{1}{2}\|X\|_{L^\infty(B;\mathbb{M}^d)}$$

thanks to the choice of ε. Hence $X = 0$ in $L^\infty(B;\mathbb{M}^d)$, and therefore in $C^0(B;\mathbb{M}^d)$. This implies that the entire ball $B = B(y,\varepsilon)$ is included in S. Therefore, the set S is open in Ω.

Since the set S is non-empty, closed, and open in Ω, the conectedness of Ω implies that $S = \Omega$. This completes the proof of step (iii).

(iv) *There exists at most one solution in* $W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ *to the system*

$$\begin{aligned} &\partial_i\Theta(x)\cdot\partial_j\Theta(x) = g_{ij}(x) \textit{ for all } x \in \Omega, \\ &\Theta(x^0) = \Theta^0, \partial_i\Theta(x^0) = \boldsymbol{g}_i^0. \end{aligned} \tag{3.48}$$

Let $\Theta, \widetilde{\Theta} \in W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ be solutions to the system above. Let $F(x)$ denote the matrix whose i-th column is $\boldsymbol{g}_i(x) := \partial_i\Theta(x)$ and let $\widetilde{F}(x)$ denote the matrix whose i-th column is $\widetilde{\boldsymbol{g}}_i(x) := \partial_i\widetilde{\Theta}(x)$. Then one can see that $F, \widetilde{F} \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{M}^d)$ both satisfy the system:

$$\begin{aligned} &\partial_i F = \Gamma_i F \text{ in } \Omega, \\ &F(x^0) = F^0. \end{aligned}$$

The uniqueness result established in Theorem 3.1 then shows that $F = \widetilde{F}$ in Ω. Since Θ and $\widetilde{\Theta}$ both satisfy the system

$$\begin{aligned} &\partial_i\Theta = \boldsymbol{g}_i \text{ in } \Omega, \\ &\Theta(x^0) = \Theta^0, \end{aligned}$$

a classical result about distributions (see e.g. [10]), or Theorem 3.1, shows that $\Theta = \widetilde{\Theta}$ in Ω.

(v) *Let* $\Theta, \widetilde{\Theta} \in W^{2,\infty}_{\rm loc}(\Omega;\mathbb{R}^d)$ *satisfy*

$$\partial_i\Theta \cdot \partial_j\Theta = g_{ij} \textit{ and } \partial_i\widetilde{\Theta} \cdot \partial_j\widetilde{\Theta} = g_{ij} \textit{ in } \Omega.$$

Then there exist a vector $\boldsymbol{a} \in \mathbb{R}^d$ *and an orthogonal matrix* $Q \in \mathbb{M}^d$ *such that*

$$\widetilde{\Theta}(x) = \boldsymbol{a} + Q\Theta(x) \textit{ for all } x \in \Omega.$$

Let $F : \Omega \to \mathbb{M}^d$ be the matrix field whose i-th column is $\partial_i\Theta$ and let $\widetilde{F} : \Omega \to \mathbb{M}^d$ be the matrix field whose i-th column is $\partial_i\widetilde{\Theta}$. Let a point $x^0 \in \Omega$ be fixed and define

$$Q := \widetilde{F}(x^0)F(x^0)^{-1} \text{ and } \boldsymbol{a} := \widetilde{\Theta}(x_0) - Q\Theta(x_0),$$

the matrix $F(x^0)$ being invertible since $F(x^0)^T F(x^0) = (g_{ij}(x^0)) \in \mathbb{S}^d_>$. Let

$$\Phi(x) := \boldsymbol{a} + Q\Theta(x) \text{ for all } x \in \Omega.$$

Then one can see that the matrix Q is orthogonal, that $\Phi(x^0) = \widetilde{\Theta}(x^0)$ and $\partial_i\Phi(x^0) = \partial_i\widetilde{\Theta}(x^0)$, and that

$$\partial_i\Phi \cdot \partial_j\Phi = g_{ij} \text{ in } \Omega.$$

Consequently, the uniqueness result proved in step (iv) shows that $\Phi = \widetilde{\Theta}$ in Ω.

The proof is now complete. □

Remark 3.4. In order to prove the local existence of Θ in step (ii), we can also proceed as in [1]. Let $B := B(\overline{x}, r) \Subset \Omega$ be an open ball and let $\overline{\Theta}$ be a given point in $\mathbb{R}^d$. First, we construct a sequence $(\boldsymbol{g}_i^\varepsilon)$ of regular enough vector fields satisfying relations (3.44) such that $\boldsymbol{g}_i^\varepsilon \to \boldsymbol{g}_i$ in $W^{1,p}(B;\mathbb{R}^d)$, $1 \le p < \infty$, as ε goes to 0 (such vector fields $\boldsymbol{g}_i^\varepsilon$ can be defined by taking the convolution of $\boldsymbol{g}_i$ with a sequence of mollifiers). Then we apply the classical Poincaré theorem to the problem of finding Θ^ε such that

$$\begin{aligned} \partial_i\Theta^\varepsilon &= \boldsymbol{g}_i^\varepsilon \text{ in } B,\\ \Theta^\varepsilon(\overline{x}) &= \overline{\Theta}. \end{aligned}$$

In this way, we will find a solution Θ^ε of class C^∞ to the system above. Then Θ will be found as the limit of Θ^ε in $W^{2,p}(B;\mathbb{R}^d)$ as ε goes to zero. Since $\partial_i\Theta = \boldsymbol{g}_i \in W^{1,\infty}(B;\mathbb{R}^d)$, the mapping Θ belongs in fact to the space $W^{2,\infty}(B;\mathbb{R}^d)$.

Under the additional assumptions that the geodesic diameter of Ω is finite and $(g_{ij}) \in W^{1,\infty}(\Omega;\mathbb{S}^d_>)$ and $(g_{ij})^{-1} \in L^\infty(\Omega;\mathbb{M}^d)$, we can show that the isometric immersion Θ given by the previous theorem belongs to the space $W^{2,\infty}(\Omega;\mathbb{R}^d)$:

Theorem 3.4. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^d$ such that its geodesic diameter is finite. Let a matrix field $(g_{ij}) \in W^{1,\infty}(\Omega; \mathbb{S}^d_>)$ be given such that $(g_{ij})^{-1} \in L^\infty(\Omega; \mathbb{M}^d)$. Assume that the Riemann curvature tensor associated with the metric (g_{ij}) vanishes, that is,*

$$\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^l_{ik}\Gamma^p_{jl} - \Gamma^l_{ij}\Gamma^p_{kl} = 0 \text{ in } \mathcal{D}'(\Omega). \tag{3.49}$$

(i) Then there exists a mapping $\Theta \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{R}^d)$, unique up to isometries in $\mathbb{R}^d$, such that

$$\partial_i \Theta \cdot \partial_j \Theta = g_{ij} \text{ in } \Omega.$$

(ii) Let there be given $x^0 \in \Omega$, $\Theta^0 \in \mathbb{R}^d$ and $\boldsymbol{g}^0_i \in \mathbb{R}^d$ such that $\boldsymbol{g}^0_i \cdot \boldsymbol{g}^0_j = g_{ij}(x^0)$ (such vectors exist thanks to Lemma 3.3). Then there exists one and only one mapping $\Theta \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{R}^d)$ such that $\partial_i\Theta \cdot \partial_j \Theta = g_{ij}$ in Ω and

$$\Theta(x^0) = \Theta^0 \text{ and } \partial_i \Theta(x^0) = \boldsymbol{g}^0_i.$$

Proof. By Theorem 3.3, there exists a mapping $\Theta \in W^{2,\infty}_{\text{loc}}(\Omega, \mathbb{R}^d)$, unique up to proper isometries in $\mathbb{R}^d$, such that $\partial_i\Theta \cdot \partial_j\Theta = g_{ij}$ in Ω. Then the definition of the Christoffel symbols shows that

$$\partial_i \boldsymbol{g}_j = \Gamma^k_{ij} \boldsymbol{g}_k \text{ in } \mathcal{D}'(\Omega; \mathbb{R}^d),$$

where $\boldsymbol{g}_j := \partial_j \Theta$. These equations (for all i, j) are equivalent with the matriceal equation $\partial_i F = F\,\Gamma_i$, where F is the matrix whose i-th column is $\boldsymbol{g}_i$ and $\Gamma_i := (\Gamma^k_{ij})$. The regularity assumptions on the matrix field (g_{ij}) implies that the coefficients Γ_i belong to $L^\infty(\Omega; \mathbb{M}^d)$. Since in addition the geodesic diameter of Ω is finite, we can apply Corollary 3.1 to the system

$$\begin{aligned} &\partial_i Y = Y\,\Gamma_i \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^d), \\ &Y(x^0) = F(x^0), \end{aligned}$$

where x^0 is a fixed point in Ω. This implies that the unique solution F to this system belongs to the space $W^{1,\infty}(\Omega; \mathbb{M}^d)$. Then the columns $\boldsymbol{g}_i$ of the matrix field F belong to $L^\infty(\Omega; \mathbb{R}^d)$ and thus we can apply Corollary 3.1 to the system

$$\begin{aligned} &\partial_i \Phi = \boldsymbol{g}_i \text{ in } \mathcal{D}'(\Omega; \mathbb{R}^d), \\ &\Phi(x^0) = \Theta(x^0). \end{aligned}$$

This implies that the unique solution Θ to this problem belongs to the space $W^{1,\infty}(\Omega; \mathbb{R}^d)$. Since we have already seen that $\partial_i \Theta \in W^{1,\infty}(\Omega; \mathbb{R}^d)$, the mapping Θ belongs to the space $W^{2,\infty}(\Omega; \mathbb{R}^d)$. □

Remark 3.5. The geodesic diameter of a bounded open $\Omega \in \mathbb{R}^d$ that satisfies the cone property is finite.

6. Appendix

Proof of identities (3.16) and (3.17) used in the proof of theorem 3.1. The assumptions of theorem 3.1 imply that

$$\begin{aligned}\partial_\beta A_\alpha - \partial_\alpha A_\beta - A_\alpha A_\beta + A_\beta A_\alpha = 0,\\ \partial_\beta B_\alpha - \partial_\alpha B_\beta - B_\alpha A_\beta + B_\beta A_\alpha = 0\end{aligned} \tag{3.50}$$

in $\mathcal{D}'(\Omega)$. Hence

$$\begin{aligned}\partial_\beta \widetilde{A}_\alpha - \partial_\alpha \widetilde{A}_\beta - \widetilde{A}_\alpha \widetilde{A}_\beta + \widetilde{A}_\beta \widetilde{A}_\alpha = 0 \text{ and}\\ \partial_\beta \widetilde{B}_\alpha - \partial_\alpha \widetilde{B}_\beta - \widetilde{B}_\alpha \widetilde{A}_\beta + \widetilde{B}_\beta \widetilde{A}_\alpha = 0\end{aligned} \tag{3.51}$$

in $\mathcal{D}'(\omega_\beta)$. To see this, let $\rho \in \mathcal{D}(\mathbb{R})$ be given such that $\operatorname{supp}\rho \subset [-1,1]$ and $\int_{\mathbb{R}} \rho(t)dt = 1$. Then, given any $\psi \in \mathcal{D}(\omega_\beta)$, let

$$\psi_\delta(x) = \psi(x_{1\ldots\beta}) \left(\frac{1}{\delta_{\beta+1}}\rho(\frac{x_{\beta+1} - \overline{x}_{\beta+1}}{\delta_{\beta+1}})\right) \ldots \left(\frac{1}{\delta_d}\rho(\frac{x_d - \overline{x}_d}{\delta_d})\right),$$

where $\delta := (\delta_{\beta+1}, \ldots, \delta_d)$. Since $\psi_\delta \in \mathcal{D}(\Omega)$ for δ_i small enough, the first relation of (3.50) implies that

$$\int_\Omega (A_\beta \partial_\alpha \psi_\delta - A_\alpha \partial_\beta \psi_\delta)dx = \int_\Omega (A_\alpha A_\beta - A_\beta A_\alpha)\,\psi_\delta dx.$$

Letting $\delta_d \to 0, \ldots, \delta_{\beta+1} \to 0$ in this relation and using the first relation of (3.8) yields

$$\begin{aligned}\int_{\omega_\beta} (\partial_\alpha \psi(x_{1\ldots\beta}) \widetilde{A}_\beta(x_{1\ldots\beta}) - \partial_\beta \psi(x_{1\ldots\beta}) \widetilde{A}_\alpha(x_{1\ldots\beta}))dx_{1\ldots\beta}\\ = \int_{\omega_\beta} \psi(x_{1\ldots\beta}) \left(\widetilde{A}_\alpha \widetilde{A}_\beta - \widetilde{A}_\beta \widetilde{A}_\alpha\right)(x_{1\ldots\beta})dx_{1\ldots\beta}.\end{aligned} \tag{3.52}$$

Hence the first relation of (3.51) is established. In a similar way, we have also

$$\begin{aligned}\int_{\omega_\beta} (\partial_\alpha \psi(x_{1\ldots\beta}) \widetilde{B}_\beta(x_{1\ldots\beta}) - \partial_\beta \psi(x_{1\ldots\beta}) \widetilde{B}_\alpha(x_{1\ldots\beta}))dx_{1\ldots\beta}\\ = \int_{\omega_\beta} \psi(x_{1\ldots\beta}) \left(\widetilde{B}_\alpha \widetilde{A}_\beta - \widetilde{B}_\beta \widetilde{A}_\alpha\right)(x_{1\ldots\beta})dx_{1\ldots\beta}\end{aligned} \tag{3.53}$$

and the second relation of (3.51) is established.

By a denseness argument, these last two relations hold for all $\psi \in W_0^{1,1}(\omega_\beta)$, the closure of $\mathcal{D}(\omega_\beta)$ in $W^{1,1}(\omega_\beta)$. Moreover, relation (3.52) still holds true for any $\Psi \in W_0^{1,1}(\omega_\beta; \mathbb{M}^{q,l})$, the multiplication by the scalar ψ being replaced by the multiplication by the matrix Ψ (one can see this by writing this relation componentwise).

Now, let $t_\beta \in (\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta + \varepsilon_\beta)$ be a fixed, but otherwise arbitrary, number. For every $\varepsilon > 0$, define the function $\chi_\varepsilon \in C^0(\mathbb{R})$, which also depends on t_β, in the following way: for $t_\beta \in (\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta]$, let

$$\chi_\varepsilon(x_\beta) = \begin{cases} -1 & \text{if } x_\beta \in [t_\beta, \overline{x}_\beta], \\ 0 & \text{if } x_\beta \notin [t_\beta - \varepsilon, \overline{x}_\beta + \varepsilon], \\ \text{affine} & \text{otherwise,} \end{cases}$$

and, for $t_\beta \in (\overline{x}_\beta, \overline{x}_\beta + \varepsilon_\beta)$, let

$$\chi_\varepsilon(x_\beta) = \begin{cases} 1 & \text{if } x_\beta \in [\overline{x}_\beta, t_\beta], \\ 0 & \text{if } x_\beta \notin [\overline{x}_\beta - \varepsilon, t_\beta + \varepsilon], \\ \text{affine} & \text{otherwise.} \end{cases}$$

Note that $\text{supp}\chi_\varepsilon \subset (\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta + \varepsilon_\beta)$ for ε small enough. Let a matrix field $\Psi : \omega_\beta \to \mathbb{M}^{q,l}$ be defined by $\Psi(x_{1\ldots\beta}) = Y_\beta^n(x_{1\ldots\beta})\varphi(x_{\ldots}, t_\beta)\chi_\varepsilon(x_\beta)$. Since $\Psi \in W_0^{1,1}(\omega_\beta; \mathbb{M}^{q,l})$, we can use it as a test function in (3.52). Letting then $\varepsilon \to 0$ gives

$$\begin{aligned} (3.54) \quad & \int_{\omega_{\beta-1}} \int_{\overline{x}_\beta}^{t_\beta} \partial_\alpha \left(Y_\beta^n(x_{\ldots}, x_\beta)\varphi(x_{\ldots}, t_\beta) \right) \widetilde{A}_\beta(x_{\ldots}, x_\beta) dx_\beta dx_{1\ldots\beta-1} \\ &= \int_{\omega_{\beta-1}} \int_{\overline{x}_\beta}^{t_\beta} (\partial_\beta Y_\beta^n \widetilde{A}_\alpha)(x_{\ldots}, x_\beta)\varphi(x_{\ldots}, t_\beta) dx_\beta dx_{1\ldots\beta-1} \\ & + \int_{\omega_{\beta-1}} \int_{\overline{x}_\beta}^{t_\beta} \left(Y_\beta^n \left(\widetilde{A}_\alpha \widetilde{A}_\beta - \widetilde{A}_\beta \widetilde{A}_\alpha \right) \right) (x_{\ldots}, x_\beta)\varphi(x_{\ldots}, t_\beta) dx_\beta dx_{1\ldots\beta-1} \\ & + \lim_{\varepsilon \to 0} \int_{\omega_\beta} (Y_\beta^n \widetilde{A}_\alpha)(x_{\ldots}, x_\beta)\varphi(x_{\ldots}, t_\beta)\partial_\beta \chi_\varepsilon(x_\beta) dx_\beta dx_{1\ldots\beta-1}. \end{aligned}$$

Now, for all $t_\beta \in (\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta + \varepsilon_\beta) \setminus X_\beta(\overline{x}_{\beta+1}, ..., \overline{x}_d)$ (recall that $\overline{x}_\beta$ also belongs to the set $\mathbb{R} \setminus X_\beta(\overline{x}_{\beta+1}, ..., \overline{x}_d)$), we have

$$\begin{aligned}
&\Big| \int_{\omega_\beta} (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, x_\beta)\varphi(x_{...}, t_\beta)\partial_\beta \chi_\varepsilon(x_\beta) dx_\beta dx_{1...\beta-1} \\
&\qquad - \int_{\omega_{\beta-1}} (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, \overline{x}_\beta)\varphi(x_{...}, t_\beta) dx_{1...\beta-1} \\
&\qquad + \int_{\omega_{\beta-1}} (Y_\beta^n \widetilde{A}_\alpha(x_{...}, t_\beta)\varphi(x_{...}, t_\beta) dx_{1...\beta-1} \Big| \\
&\leq \frac{1}{\varepsilon^*}\int_{t_\beta-\varepsilon^*}^{t_\beta}\int_{\omega_{\beta-1}} \Big| \Big\{ (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, t_\beta) - (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, x_\beta) \Big\} \varphi(x_{...}, t_\beta) \Big| dx_{1...\beta} \\
&+ \frac{1}{\varepsilon^*}\int_{\overline{x}_\beta}^{\overline{x}_\beta+\varepsilon^*}\int_{\omega_{\beta-1}} \Big| \Big\{ (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, x_\beta) - (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, \overline{x}_\beta) \Big\} \varphi(x_{...}, t_\beta) \Big| dx_{1...\beta} \\
&\leq \frac{1}{\varepsilon^*}\int_{t_\beta-\varepsilon^*}^{t_\beta}\int_{\omega_{\beta-1}} \big| \{ Y_\beta^n(x_{...}, t_\beta) - Y_\beta^n(x_{...}, x_\beta) \} \varphi(x_{...}, t_\beta) \big| \Big| \widetilde{A}_\alpha(x_{...}, t_\beta) \Big| dx_{1...\beta} \\
&+ \frac{1}{\varepsilon^*}\int_{t_\beta-\varepsilon^*}^{t_\beta}\int_{\omega_{\beta-1}} \big| Y_\beta^n(x_{...}, x_\beta)\varphi(x_{...}, t_\beta) \big| \Big| \widetilde{A}_\alpha(x_{...}, t_\beta) - \widetilde{A}_\alpha(x_{...}, x_\beta) \Big| dx_{1...\beta} \\
&+ \frac{1}{\varepsilon^*}\int_{\overline{x}_\beta}^{\overline{x}_\beta+\varepsilon^*}\int_{\omega_{\beta-1}} \big| Y_\beta^n(x_{...}, x_\beta)\varphi(x_{...}, t_\beta) \big| \Big| \widetilde{A}_\alpha(x_{...}, x_\beta) - \widetilde{A}_\alpha(x_{...}, \overline{x}_\beta) \Big| dx_{1...\beta} \\
&+ \frac{1}{\varepsilon^*}\int_{\overline{x}_\beta}^{\overline{x}_\beta+\varepsilon^*}\int_{\omega_{\beta-1}} \big| \{ Y_\beta^n(x_{...}, x_\beta) - Y_\beta^n(x_{...}, \overline{x}_\beta) \} \varphi(x_{...}, t_\beta) \big| \Big| \widetilde{A}_\alpha(x_{...}, \overline{x}_\beta) \Big| dx_{1...\beta},
\end{aligned}$$

where $\varepsilon^* = \varepsilon$ if $t_\beta \leq \overline{x}_\beta$ and $\varepsilon^* = -\varepsilon$ if $t_\beta > \overline{x}_\beta$. Noting that $Y_\beta^n \in W^{1,\infty}(\omega_\beta; \mathbb{M}^{q,l}) \subset C^0(\omega_\beta; \mathbb{M}^{q,l})$, we infer from the definition of the set $X_\beta(\overline{x}_{\beta+1}, ..., \overline{x}_d)$ that the right-hand-side above goes to zero as $\varepsilon \to 0$. Consequently,

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{\omega_\beta} (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, x_\beta)\varphi(x_{...}, t_\beta)\partial_\beta \chi_\varepsilon(x_\beta) dx_\beta dx_{1...\beta-1} \\
&\qquad = \int_{\omega_{\beta-1}} (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, \overline{x}_\beta)\varphi(x_{...}, t_\beta) dx_{1...\beta-1} \\
&\qquad\qquad - \int_{\omega_{\beta-1}} (Y_\beta^n \widetilde{A}_\alpha)(x_{...}, t_\beta)\varphi(x_{...}, t_\beta) dx_{1...\beta-1}.
\end{aligned}$$

Using this relation in (3.54) gives

$$\int_{\omega_{\beta-1}}\int_{\overline{x}_\beta}^{t_\beta}\partial_\alpha\left(Y^n_\beta(x_{\dots},x_\beta)\varphi(x_{\dots},t_\beta)\right)\widetilde{A}_\beta(x_{\dots},x_\beta)dx_\beta dx_{1\dots\beta-1}$$
$$=\int_{\omega_{\beta-1}}\int_{\overline{x}_\beta}^{t_\beta}(\partial_\beta Y^n_\beta\widetilde{A}_\alpha)(x_{\dots},x_\beta)\varphi(x_{\dots},t_\beta)dx_\beta dx_{1\dots\beta-1}$$
$$+\int_{\omega_{\beta-1}}(Y^n_\beta\widetilde{A}_\alpha)(x_{\dots},\overline{x}_\beta)\varphi(x_{\dots},t_\beta)dx_{1\dots\beta-1}$$
$$-\int_{\omega_{\beta-1}}(Y^n_\beta\widetilde{A}_\alpha)(x_{\dots},t_\beta)\varphi(x_{\dots},t_\beta)dx_{1\dots\beta-1}$$
$$+\int_{\omega_{\beta-1}}\int_{\overline{x}_\beta}^{t_\beta}(Y^n_\beta(\widetilde{A}_\alpha\widetilde{A}_\beta-\widetilde{A}_\beta\widetilde{A}_\alpha))(x_{\dots},x_\beta)\varphi(x_{\dots},t_\beta)dx_\beta dx_{1\dots\beta-1}.$$

for all $t_\beta\in(\overline{x}_\beta-\varepsilon_\beta,\overline{x}_\beta+\varepsilon_\beta)\setminus X_\beta(\overline{x}_{\beta+1},...,\overline{x}_d)$. By integrating this relation with respect to t_β over $(\overline{x}_\beta-\varepsilon_\beta,\overline{x}_\beta+\varepsilon_\beta)$ (recall that the subset $X_\beta(\overline{x}_{\beta+1},...,\overline{x}_d)$ of $\mathbb{R}$ has zero measure), we obtain that

$$\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{t_\beta}\partial_\alpha\left(Y^n_\beta(x_{\dots},x_\beta)\varphi(x_{\dots},t_\beta)\right)\widetilde{A}_\beta(x_{\dots},x_\beta)dx_\beta\right\}dx_{1\dots\beta-1}dt_\beta$$
$$=\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{t_\beta}(\partial_\beta Y^n_\beta\widetilde{A}_\alpha)(x_{\dots},x_\beta)dx_\beta\right\}\varphi(x_{\dots},t_\beta)dx_{1\dots\beta-1}dt_\beta$$
$$+\int_{\omega_\beta}(Y^n_\beta\widetilde{A}_\alpha)(x_{\dots},\overline{x}_\beta)\varphi(x_{\dots},t_\beta)dx_{1\dots\beta-1}dt_\beta$$
$$-\int_{\omega_\beta}(Y^n_\beta\widetilde{A}_\alpha)(x_{\dots},t_\beta)\varphi(x_{\dots},t_\beta)dx_{1\dots\beta-1}dt_\beta$$
$$+\int_{\omega_\beta}\left\{\int_{\overline{x}_\beta}^{t_\beta}(Y^n_\beta(\widetilde{A}_\alpha\widetilde{A}_\beta-\widetilde{A}_\beta\widetilde{A}_\alpha))(x_{\dots},x_\beta)dx_\beta\right\}\varphi(x_{\dots},t_\beta)dx_{1\dots\beta-1}dt_\beta.$$

This completes the proof of relation (3.16).

We will now establish relation (3.17). Let $\psi:\omega_\beta\to\mathbb{R}$ be defined by $\psi(x_{1\dots\beta})=\varphi(x_{\dots},t_\beta)\chi_\varepsilon(x_\beta)$, where the function χ_ε is that defined before. Using this function as a test function in (3.53) and letting $\varepsilon\to0$ gives

$$\int_{\omega_{\beta-1}}\int_{\overline{x}_\beta}^{t_\beta}\partial_\alpha\varphi(x_{\dots},t_\beta)\widetilde{B}_\beta(x_{\dots},x_\beta)dx_\beta dx_{1\dots\beta-1}\tag{3.55}$$
$$=\int_{\omega_{\beta-1}}\int_{\overline{x}_\beta}^{t_\beta}(\widetilde{B}_\alpha\widetilde{A}_\beta-\widetilde{B}_\beta\widetilde{A}_\alpha)(x_{\dots},x_\beta)\varphi(x_{\dots},t_\beta)dx_\beta dx_{1\dots\beta-1}$$
$$+\lim_{\varepsilon\to0}\int_{\omega_\beta}\widetilde{B}_\alpha(x_{\dots},x_\beta)\varphi(x_{\dots},t_\beta)\partial_\beta\chi_\varepsilon(x_\beta)dx_\beta dx_{1\dots\beta-1}.$$

Now, for all $t_\beta \in (\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta + \varepsilon_\beta) \setminus X_\beta(\overline{x}_{\beta+1}, ..., \overline{x}_d)$ (recall that $\overline{x}_\beta$ also belongs to $\mathbb{R} \setminus X_\beta(\overline{x}_{\beta+1}, ..., \overline{x}_d)$), we have

$$\begin{aligned}
&\left| \int_{\omega_\beta} \widetilde{B}_\alpha(x_{...}, x_\beta)\varphi(x_{...}, t_\beta)\partial_\beta\chi_\varepsilon(x_\beta)dx_\beta dx_{1...\beta-1} \right. \\
&\qquad \left. - \int_{\omega_{\beta-1}} \left(\widetilde{B}_\alpha(x_{...}, \overline{x}_\beta) - \widetilde{B}_\alpha(x_{...}, t_\beta)\right)\varphi(x_{...}, t_\beta)dx_{1...\beta-1} \right| \\
&\quad \leq \frac{1}{\varepsilon^*}\int_{t_\beta-\varepsilon^*}^{t_\beta}\int_{\omega_{\beta-1}} \left| \left\{\widetilde{B}_\alpha(x_{...}, t_\beta) - \widetilde{B}_\alpha(x_{...}, x_\beta)\right\}\varphi(x_{...}, t_\beta)\right| dx_{1...\beta} \\
&\quad + \frac{1}{\varepsilon^*}\int_{\overline{x}_\beta}^{\overline{x}_\beta+\varepsilon^*}\int_{\omega_{\beta-1}} \left| \left\{\widetilde{B}_\alpha(x_{...}, x_\beta) - \widetilde{B}_\alpha(x_{...}, \overline{x}_\beta)\right\}\varphi(x_{...}, t_\beta)\right| dx_{1...\beta},
\end{aligned}$$

where $\varepsilon^* = \varepsilon$ if $t_\beta \leq \overline{x}_\beta$ and $\varepsilon^* = -\varepsilon$ if $t_\beta > \overline{x}_\beta$. We then infer from (3.8) that the right-hand-side above goes to zero as $\varepsilon \to 0$. Consequently,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\omega_\beta} &\widetilde{B}_\alpha(x_{...}, x_\beta)\varphi(x_{...}, t_\beta)\partial_\beta\chi_\varepsilon(x_\beta)dx_\beta dx_{1...\beta-1} \\
&= \int_{\omega_{\beta-1}} \widetilde{B}_\alpha(x_{...}, \overline{x}_\beta)\varphi(x_{...}, t_\beta)dx_{1...\beta-1} \\
&\qquad - \int_{\omega_{\beta-1}} \widetilde{B}_\alpha(x_{...}, t_\beta)\varphi(x_{...}, t_\beta)dx_{1...\beta-1}.
\end{aligned}$$

Using this relation in (3.55) gives

$$\begin{aligned}
\int_{\omega_{\beta-1}}\int_{\overline{x}_\beta}^{t_\beta} &\partial_\alpha\varphi(x_{...}, t_\beta)\widetilde{B}_\beta(x_{...}, x_\beta)dx_\beta dx_{1...\beta-1} \\
&= \int_{\omega_{\beta-1}}\int_{\overline{x}_\beta}^{t_\beta}(\widetilde{B}_\alpha\widetilde{A}_\beta - \widetilde{B}_\beta\widetilde{A}_\alpha)(x_{...}, x_\beta)\varphi(x_{...}, t_\beta)dx_\beta dx_{1...\beta-1} \\
&\qquad + \int_{\omega_{\beta-1}} \widetilde{B}_\alpha(x_{...}, \overline{x}_\beta)\varphi(x_{...}, t_\beta)dx_{1...\beta-1} \\
&\qquad\qquad - \int_{\omega_{\beta-1}} \widetilde{B}_\alpha(x_{...}, t_\beta)\varphi(x_{...}, t_\beta)dx_{1...\beta-1}.
\end{aligned}$$

for all $t_\beta \in (\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta + \varepsilon_\beta) \setminus X_\beta(\overline{x}_{\beta+1}, ..., \overline{x}_d)$. By integrating this relation with respect to t_β over $(\overline{x}_\beta - \varepsilon_\beta, \overline{x}_\beta + \varepsilon_\beta)$ (recall that the subset

$X_\beta(\overline{x}_{\beta+1}, ..., \overline{x}_d)$ of $\mathbb{R}$ has zero measure), we obtain that

$$\int_{\omega_\beta} \left\{ \int_{\overline{x}_\beta}^{t_\beta} \partial_\alpha \varphi(x_{...}, t_\beta) \widetilde{B}_\beta(x_{...}, x_\beta) dx_\beta \right\} dx_{1...\beta-1} dt_\beta$$

$$= \int_{\omega_\beta} \widetilde{B}_\alpha(x_{...}, \overline{x}_\beta) \varphi(x_{...}, t_\beta) dx_{1...\beta-1} dt_\beta$$

$$- \int_{\omega_\beta} \widetilde{B}_\alpha(x_{...}, t_\beta) \varphi(x_{...}, t_\beta) dx_{1...\beta-1} dt_\beta$$

$$+ \int_{\omega_\beta} \left\{ \int_{\overline{x}_\beta}^{t_\beta} (\widetilde{B}_\alpha \widetilde{A}_\beta - \widetilde{B}_\beta \widetilde{A}_\alpha)(x_{...}, x_\beta) dx_\beta \right\} \varphi(x_{...}, t_\beta) dx_{1...\beta-1} dt_\beta.$$

This completes the proof of relation (3.17).

□

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CHAPITRE 4

Sur le théorème fondamental de la théorie des surfaces

Ce travail a fait l'objet des publications suivantes :

MARDARE S., *The fundamental theorem of surface theory for surfaces with little regularity*, accepté dans Journal of Elasticity.

MARDARE S., *On the fundamental theorem of surface theory under weak regularity assumptions*, C.R. Acad. Sci. Paris, Ser. I 338, 2004, 71-76.

The fundamental theorem of surface theory for surfaces with little regularity

1. Introduction

Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ and let there be given a symmetric positive definite matrix field $(a_{\alpha\beta})$ of order two and a symmetric matrix field $(b_{\alpha\beta})$ of order two, both defined over the set ω. Assume that the functions $a_{\alpha\beta}$ are of class C^2 over ω, that the functions $b_{\alpha\beta}$ are of class C^1 over ω and that they satisfy the Gauss equations,

$$\partial_\gamma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\sigma\beta} = b_{\alpha\beta} b^\tau_\gamma - b_{\alpha\gamma} b^\tau_\beta \text{ in } \omega, \tag{4.1}$$

and the Codazzi-Mainardi equations,

$$\partial_\gamma b_{\alpha\beta} - \partial_\beta b_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta} b_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma} b_{\sigma\beta} = 0 \text{ in } \omega. \tag{4.2}$$

The Christoffel symbols $\Gamma^\tau_{\alpha\beta}$ (associated with $(a_{\alpha\beta})$) together with further notations and conventions are defined in section 3.

Under these assumptions, the fundamental theorem of surface theory asserts that there exists a mapping $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ of class C^3 such that the first and second fundamental forms of the immersed surface $S = \boldsymbol{\theta}(\omega)$ are given by the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ (see [5, 9, 11] for the global result and [3, 16, 17, 18, 19] for the local result).

This means that

$$\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ and } \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} = b_{\alpha\beta}$$

over ω. The mapping $\boldsymbol{\theta}$ then defines an isometric immersion of the Riemannian space $(\omega, (a_{\alpha\beta}))$ in the three-dimensional Euclidean space.

This result contain a "paradox of differentiability", as pointed out in [7, 8]: in order to define the first and second fundamental forms $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ associated with a given isometric immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$, one usually assumes that it belongs only to $C^2(\omega; \mathbb{R}^3)$; this implies that $a_{\alpha\beta} \in C^1(\omega)$ and $b_{\alpha\beta} \in C^0(\omega)$. But in order to recover the mapping $\boldsymbol{\theta}$ from two matrix fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$, one has to assume that $a_{\alpha\beta} \in C^2(\omega)$ and $b_{\alpha\beta} \in C^1(\omega)$, the mapping $\boldsymbol{\theta}$ being then in $C^3(\omega; \mathbb{R}^3)$. This extra regularity assumption on $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$, which is needed so as the derivatives appearing in the Gauss and Codazzi-Mainardi equations be well defined.

However, Hartman and Wintner [8] get rid of this extra regularity assumption by requiring that $a_{\alpha\beta}$ and $b_{\alpha\beta}$, this time assumed to be in $C^1(\omega)$ and $C^0(\omega)$ respectively, satisfy the Gauss and Codazzi-Mainardi equations in an "integrated" form, namely

$$\int_J (\Gamma^\tau_{\alpha 1} dy_1 + \Gamma^\tau_{\alpha 2} dy_2) = \int_{\mathrm{dom} J} (\Gamma^\sigma_{\alpha 1}\Gamma^\tau_{\sigma 2} - \Gamma^\sigma_{\alpha 2}\Gamma^\tau_{\sigma 1} - b_{\alpha 1} b^\tau_2 + b_{\alpha 2} b^\tau_1)\, dy \tag{4.3}$$

and

$$\int_J (b_{\alpha 1} dy_1 + b_{\alpha 2} dy_2) = \int_{\mathrm{dom} J} (\Gamma^\sigma_{\alpha 1} b_{\sigma 2} - \Gamma^\sigma_{\alpha 2} b_{\sigma 1})\, dy \tag{4.4}$$

for all Jordan curve of class C^1 contained in ω, where dom$J$ designates the bounded open set whose boundary is the Jordan curve J.

In this paper, we reconsider the Gauss and Codazzi-Mainardi equations (as given by (4.1) and (4.2)) and assume that they are satisfied in a distributional sense by a matrix field $(a_{\alpha\beta})$ of class $W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ and a matrix field $(b_{\alpha\beta})$ of class $L^\infty_{\mathrm{loc}}(\omega; \mathbb{S}^2)$. Then we establish the existence of a mapping $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ of class $W^{2,\infty}_{\mathrm{loc}}(\omega)$ such that $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are the first and second fundamental forms of the immersed surface $S = \boldsymbol{\theta}(\omega)$ (see Theorem 4.2).

The result of Hartman and Wintner [8] can be obtained as a consequence (see Corollary 4.1) of our more general result since, for first and second fundamental forms respectively of class C^1 and C^0 over a connected and simply-connected open subset of $\mathbb{R}^2$, the Gauss and Codazzi-Mainardi equations (understood in a distributional sense) are satisfied if and only if relations (4.3) and (4.4) hold for all regular Jordan curves of class C^2 included in ω. Therefore, the assumption of Hartman and Wintner [8] are stronger than our assumptions (see Corollary 4.1). The proof of this assertion is not given in this paper.

2. Preliminaries

An immersed surface is a connected subset S of $\mathbb{R}^3$ that can be written as $S = \cup_{t\in A}\theta_t(\omega_t)$, where ω_t are open subsets of $\mathbb{R}^2$ and $\theta_t : \omega_t \to \mathbb{R}^3$ are immersions of class C^1 over ω_t. Since in this paper we consider only immersed surfaces, an immersed surface will be simply called "surface".

All functions and fields appearing in this paper are real-valued and the summation convention with respect to repeated indices and exponents is used. Matrix-valued, vector valued, and scalar-valued functions (except when they are components of a matrix field, as e.g. the components X_{ij} of matrix field X) are respectively denoted by capital, boldface, and lower case letters. The Kronecker symbol is defined by

$$\delta_i^j := \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

For any integer $d \geq 2$, the d-dimensional Euclidean space will be identified with $\mathbb{R}^d$. Let $\boldsymbol{u} \cdot \boldsymbol{v}$ denote the Euclidean inner product for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$ and let $|\boldsymbol{v}| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ denote the Euclidean norm of $\boldsymbol{v} \in \mathbb{R}^d$.

The notations $\mathbb{M}^d, \mathbb{S}^d, \mathbb{S}^d_>$ and $\mathbb{O}^d$ respectively designate the set of all square matrices, of all symmetric matrices, of all positive definite symmetric matrices, and of all orthogonal matrices, of order d. The set of all proper orthogonal matrices is defined by

$$\mathbb{O}^d_+ := \{Q \in \mathbb{O}^d;\ \det Q = 1\}.$$

The notation $\mathbb{M}^{q,l}$ designates the space of all matrices with q rows and l columns. The space $\mathbb{M}^{q,l}$ is endowed with the operator norm $|\cdot|$ defined by

$$|A| := \sup_{x \in \mathbb{R}^l \setminus \{0\}} \frac{|Ax|}{|x|}.$$

It is well known that $|A|$ is also given by the square root of the largest eigenvalue of the matrix $A^T A$. Notations such as (a_{ij}), or (a^i_j), designate the matrix whose entries are the elements a_{ij}, or a^i_j, which may be either real numbers or real functions. The first, or upper, index is the row index and the second, or lower, index is the column index.

An isometry in $\mathbb{R}^d$ is an element of the set

$$\{\boldsymbol{u} \in \mathbb{R}^d \mapsto \boldsymbol{a} + Q\boldsymbol{u};\ \boldsymbol{a} \in \mathbb{R}^d, Q \in \mathbb{O}^d\},$$

and a proper isometry in $\mathbb{R}^d$ is an element of the set

$$\{\boldsymbol{u} \in \mathbb{R}^d \mapsto \boldsymbol{a} + Q\boldsymbol{u};\ \boldsymbol{a} \in \mathbb{R}^d, Q \in \mathbb{O}^d_+\}.$$

Let Ω be an open subset of $\mathbb{R}^d$. Given two points $x, y \in \Omega$, a *path joining x to y* is any mapping $\boldsymbol{\gamma} \in C^0([0,1];\mathbb{R}^d)$ that satisfies $\boldsymbol{\gamma}(t) \in \Omega$ for all $t \in [0,1]$ and $\boldsymbol{\gamma}(0) = x$ and $\boldsymbol{\gamma}(1) = y$. Such an open set Ω is connected if and only if, for all $x, y \in \Omega$, there exists a path joining x to y. The set Ω is simply-connected if, for all $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1 \in C^0([0,1];\Omega)$ such that $\boldsymbol{\gamma}_0(0) = \boldsymbol{\gamma}_1(0)$ and $\boldsymbol{\gamma}_0(1) = \boldsymbol{\gamma}_1(1)$, there exists a homotopy joining $\boldsymbol{\gamma}_0$ to $\boldsymbol{\gamma}_1$, i.e., an application $\boldsymbol{\varphi} \in C^0([0,1] \times [0,1];\Omega)$ such that

$$\begin{aligned} &\boldsymbol{\varphi}(t,0) = \boldsymbol{\gamma}_0(t), \boldsymbol{\varphi}(t,1) = \boldsymbol{\gamma}_1(t), \\ &\boldsymbol{\varphi}(0,s) = \boldsymbol{\gamma}_0(0), \boldsymbol{\varphi}(1,s) = \boldsymbol{\gamma}_0(1). \end{aligned}$$

The geodesic distance between two points $x, y \in \Omega$ is the infimum of the length, denoted $L(\gamma)$, of all paths joining x to y and the geodesic diameter of Ω is the number $D_\Omega \in [0, \infty]$ defined by

$$D_\Omega := \sup_{x,y\in\Omega} \inf_\gamma \{L(\gamma);\ \gamma \text{ is a path joining } x \text{ to } y\}.$$

For any open subset Ω of $\mathbb{R}^d$, $d \geq 1$, the space of indefinitely derivable real functions with compact support included in Ω is denoted $\mathcal{D}(\Omega)$. The support of a continuous function $\varphi : \Omega \to \mathbb{R}$ is defined as

$$\text{supp}\varphi = \overline{\{x \in \Omega, \varphi(x) \neq 0\}}.$$

The space of distributions over Ω is denoted $\mathcal{D}'(\Omega)$. The usual Sobolev spaces being denoted $W^{m,p}(\Omega; \mathbb{M}^{q,l})$, we let

$$W^{m,p}_{loc}(\Omega; \mathbb{M}^{q,l}) := \{Y \in \mathcal{D}'(\Omega; \mathbb{M}^{q,l}); Y \in W^{m,p}(U; \mathbb{M}^{q,l}) \text{ for all open set } U \Subset \Omega\},$$

where the notation $U \Subset \Omega$ means that the closure of U in $\mathbb{R}^d$ is a compact subset of Ω. For real-valued function spaces, we shall use the notation $W^{m,p}(\Omega)$ instead of $W^{m,p}(\Omega; \mathbb{R})$, $\mathcal{D}(\Omega)$ instead of $\mathcal{D}(\Omega; \mathbb{R})$, etc.

Let $y = (y_1, y_2, ..., y_d)$ denote a generic point in $\mathbb{R}^d$ and let dy denote the Lebesgue measure in $\mathbb{R}^d$. A subset in $\mathbb{R}^d$ is said to have zero measure if its $\mathbb{R}^d$-Lebesgue measure is zero. Finally, let

$$\partial_\alpha := \frac{\partial}{\partial x_\alpha} \text{ and } \partial_{\alpha\beta} := \frac{\partial^2}{\partial x_\alpha \partial x_\beta}.$$

We also make the following convention for classes of functions with respect to the equality almost everywhere, which is an equivalence relation: if $\dot{f} \in L^\infty_{\text{loc}}(\Omega)$, we will always use the representative f of $\dot{f}$ given by

$$f(x) := \liminf_{\varepsilon\to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} \widetilde{f}(y)dy$$

where $\widetilde{f}$ is any representative of the class $\dot{f} \in L^\infty_{\text{loc}}(\Omega)$ (this definition is clearly independent of the choice of the representative $\widetilde{f}$). This choice of the representative insures that

$$\|\dot{f}\|_{L^\infty(U)} = \sup_{x\in U} |f(x)| \text{ for all open subset } U \Subset \Omega.$$

Also, for any $\dot{f} \in W^{1,\infty}_{\text{loc}}(\Omega)$, we will choose the continuous representative f of $\dot{f}$.

The notation $(a_{\alpha\beta}) \in W^{1,\infty}_{\text{loc}}(\omega; \mathbb{S}^2_>)$ means that each component of the matrix belongs to the space $W^{1,\infty}_{\text{loc}}(\omega)$ and that $(a_{\alpha\beta}(y)) \in \mathbb{S}^2_>$ *for all* $y \in \omega$. For simplicity, we will use the same notation for a class of functions and its representative chosen as before (i.e., f will denote either the class of f in $L^\infty_{\text{loc}}(\Omega)$ or its representative chosen as before), the distiction between them being made according to the context.

The following result, established in Theorem 3.1 and Corollary 3.1 of Chapter 3, is crucial in the proof of our main result (Theorem 4.2) of the next section.

Theorem 4.1. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^d$ and let a point $x^0 \in \Omega$ and a matrix $Y^0 \in \mathbb{M}^{q,l}$ be fixed. Let matrix fields $A_\alpha \in L^\infty_{\rm loc}(\Omega;\mathbb{M}^l)$ and $B_\alpha \in L^\infty_{\rm loc}(\Omega;\mathbb{M}^{q,l})$ be given that satisfy*

$$\begin{aligned}\partial_\alpha A_\beta + A_\alpha A_\beta &= \partial_\beta A_\alpha + A_\beta A_\alpha \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^l),\\ \partial_\alpha B_\beta + B_\alpha A_\beta &= \partial_\beta B_\alpha + B_\beta A_\alpha \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^{q,l}).\end{aligned} \tag{4.5}$$

Then the system

$$\begin{aligned}&\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ in } \mathcal{D}'(\Omega;\mathbb{M}^{q,l}),\\ &Y(x^0) = Y^0\end{aligned}$$

has a unique solution in $W^{1,\infty}_{\rm loc}(\Omega;\mathbb{M}^{q,l})$.

If the geodesic diameter of Ω is finite and the matrix fields A_α and B_α belong respectively to the spaces $L^\infty(\Omega;\mathbb{M}^l)$ and $L^\infty(\Omega;\mathbb{M}^{q,l})$, then the solution to the above system belongs to the space $W^{1,\infty}(\Omega;\mathbb{M}^{q,l})$.

3. The fundamental theorem of surface theory revisited

Throughout this section, Greek indices vary in the set $\{1,2\}$, Latin indices vary in the set $\{1,2,3\}$ and the summation convention with respect to repeted indices is used in conjunction with these rules.

Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ and let $y = (y_\alpha)$ denote a generic point in ω. Let there be given two matrix fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$ with $a_{\alpha\beta} \in W^{1,\infty}_{\rm loc}(\omega)$ and $b_{\alpha\beta} \in L^\infty_{\rm loc}(\omega)$ for all $\alpha,\beta \in \{1,2\}$. Define the Christoffel symbols associated with $(a_{\alpha\beta})$ by letting

$$\Gamma^\tau_{\alpha\beta} := \frac{1}{2} a^{\tau\sigma}\left(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\sigma\alpha} - \partial_\sigma a_{\alpha\beta}\right),$$

where $(a^{\tau\sigma}(y))$ is the inverse of the matrix $(a_{\alpha\beta}(y))$ at each $y \in \omega$. Let $b^\beta_\alpha := a^{\beta\sigma} b_{\alpha\sigma}$. Since $a_{\alpha\beta} \in W^{1,\infty}_{\rm loc}(\omega) \subset C^0(\omega)$ and $\det(a_{\alpha\beta}(y)) > 0$ for all $y \in \omega$, the coefficients $a^{\tau\sigma}$ belong to $L^\infty_{\rm loc}(\omega)$. Hence $\Gamma^\tau_{\alpha\beta}$ and b^β_α belong to the space $L^\infty_{\rm loc}(\omega)$.

Assume that the Gauss and Codazzi-Mainardi equations,

$$\begin{aligned}&\partial_\gamma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\sigma\beta} = b_{\alpha\beta} b^\tau_\gamma - b_{\alpha\gamma} b^\tau_\beta\\ &\partial_\gamma b_{\alpha\beta} - \partial_\beta b_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta} b_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma} b_{\sigma\beta} = 0\end{aligned} \tag{4.6}$$

are satisfied *in* $\mathcal{D}'(\omega)$ for all $\alpha, \beta, \gamma, \tau \in \{1,2\}$. Note that these equations make sense in $\mathcal{D}'(\omega)$ since $\Gamma^\tau_{\alpha\beta}$, $b_{\alpha\beta}$ and $a^{\sigma\tau}$ belong to $L^\infty_{\text{loc}}(\omega)$. More specifically, relations (4.6) mean that the relations

$$\int_\omega \left(\Gamma^\tau_{\alpha\gamma}\partial_\beta\varphi - \Gamma^\tau_{\alpha\beta}\partial_\gamma\varphi + \Gamma^\sigma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma}\varphi - \Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\sigma\beta}\varphi\right) dx = \int_\omega \left(b_{\alpha\beta}b^\tau_\gamma - b_{\alpha\gamma}b^\tau_\beta\right)\varphi dx$$

and

$$\int_\omega \left(b_{\alpha\gamma}\partial_\beta\varphi - b_{\alpha\beta}\partial_\gamma\varphi + \Gamma^\sigma_{\alpha\beta}b_{\sigma\gamma}\varphi - \Gamma^\sigma_{\alpha\gamma}b_{\sigma\beta}\varphi\right) dx = 0,$$

are satisfied for all $\varphi \in \mathcal{D}(\omega)$. Our aim is to prove the existence of an immersed surface in the three-dimensional Euclidean space whose (covariant components of the) first and second fundamental forms are given by the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$, respectively. More specifically, we wish to find a mapping $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ such that the restriction of the three-dimensional Euclidean metric to the surface $S := \boldsymbol{\theta}(\omega)$ be given by the matrix field $(a_{\alpha\beta})$, i.e.,

$$a_{\alpha\beta}(y) = \partial_\alpha\boldsymbol{\theta}(y) \cdot \partial_\beta\boldsymbol{\theta}(y)$$

for all $y \in \omega$, and that the second fundamental form of the surface S be given by the matrix field $(b_{\alpha\beta})$, i.e.,

$$b_{\alpha\beta}(y) = \partial_{\alpha\beta}\boldsymbol{\theta}(y) \cdot \frac{\partial_\alpha\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)|}$$

for almost all $y \in \omega$ (note that $|\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)| = \sqrt{\det(a_{\alpha\beta}(y))} \neq 0$). Our main result is the following:

Theorem 4.2. *Assume that ω is a connected and simply-connected open subset of $\mathbb{R}^2$ and that the matrix fields $(a_{\alpha\beta}) \in W^{1,\infty}_{\text{loc}}(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^\infty_{\text{loc}}(\omega; \mathbb{S}^2)$ satisfy the Gauss and Codazzi-Mainardi equations in $\mathcal{D}'(\omega)$. Then there exists a mapping $\boldsymbol{\theta} \in W^{2,\infty}_{\text{loc}}(\omega, \mathbb{R}^3)$ such that*

$$a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta},$$
$$b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}$$

a.e. in ω. Moreover, the mapping $\boldsymbol{\theta}$ is unique in $W^{2,\infty}_{\text{loc}}(\omega, \mathbb{R}^3)$ up to proper isometries in $\mathbb{R}^3$.

Proof. The proof is broken into six steps, numbered (i) to (vi).

(i) *Preliminaries.* Throughout the proof, we fix a point $y^0 \in \omega$, a vector $\boldsymbol{\theta}^0 \in \mathbb{R}^3$, and two vectors $\boldsymbol{a}^0_\alpha \in \mathbb{R}^3$ such that $\boldsymbol{a}^0_\alpha \cdot \boldsymbol{a}^0_\beta = a_{\alpha\beta}(y^0)$. We also define a unit vector normal to $\boldsymbol{a}^0_\alpha$ by letting

$$\boldsymbol{a}^0_3 := \frac{\boldsymbol{a}^0_1 \wedge \boldsymbol{a}^0_2}{|\boldsymbol{a}^0_1 \wedge \boldsymbol{a}^0_2|}.$$

The vectors $\boldsymbol{a}_\alpha^0$ can be chosen for instance in the following manner. Since $(a_{\alpha\beta}(y^0))$ is symmetric, there exists a diagonal matrix D and an orthogonal matrix P such that $(a_{\alpha\beta}(y^0)) = P^T D P$. Since $(a_{\alpha\beta}(y^0))$ is positive definite, the elements of the diagonal of the matrix D are > 0. Let $D^{1/2}$ be the unique positive definite square root of the diagonal matrix D. Then, $(a_{\alpha\beta}(y^0)) = (D^{1/2}P)^T(D^{1/2}P)$. Let $\widetilde{\boldsymbol{a}}_\alpha^0 := (a_{\alpha 1}^0, a_{\alpha 2}^0)^T \in \mathbb{R}^2$ be the α-th column of the matrix $D^{1/2}P$. Clearly, $\widetilde{\boldsymbol{a}}_\alpha^0 \cdot \widetilde{\boldsymbol{a}}_\beta^0 = a_{\alpha\beta}(y^0)$. Then we can choose $\boldsymbol{a}_\alpha^0 := (a_{\alpha 1}^0, a_{\alpha 2}^0, 0)^T \in \mathbb{R}^3$.

Finally, we define the matrix fields $\Gamma_\alpha : \omega \to \mathbb{M}^3$ by letting

$$\Gamma_\alpha := \begin{pmatrix} \Gamma_{\alpha 1}^1 & \Gamma_{\alpha 2}^1 & -b_\alpha^1 \\ \Gamma_{\alpha 1}^2 & \Gamma_{\alpha 2}^2 & -b_\alpha^2 \\ b_{\alpha 1} & b_{\alpha 2} & 0 \end{pmatrix}. \tag{4.7}$$

The outline of the proof is as follows. We begin by finding a matrix field $F \in W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{M}^3)$ such that

$$\partial_\alpha F = F\,\Gamma_\alpha \text{ in } \mathcal{D}'(\omega; \mathbb{M}^3),$$
$$F(y^0) = F^0,$$

where $F^0 \in \mathbb{M}^3$ is the matrix whose i-th column is $\boldsymbol{a}_i^0 \in \mathbb{R}^3$. The existence of such a matrix field F is given by Theorem 3.1. Then the first two columns of the matrix $F(x)$, denoted by $\boldsymbol{a}_\alpha(x) \in \mathbb{R}^3$, will turn out to be the derivatives of the sought mapping $\boldsymbol{\theta}$, whose existence will be given by Theorem 3.1.

(ii) *The Gauss and Codazzi-Mainardi equations are satisfied if and only if the following matrix equation is satisfied*

$$\partial_\alpha \Gamma_\beta + \Gamma_\alpha \Gamma_\beta = \partial_\beta \Gamma_\alpha + \Gamma_\beta \Gamma_\alpha \textit{ in } \mathcal{D}'(\omega; \mathbb{M}^3), \tag{4.8}$$

i.e., if and only if the following equality holds for all $\varphi \in \mathcal{D}(\omega)$*:*

$$\int_\omega (\Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha)\varphi dy = \int_\omega (\Gamma_\beta \partial_\alpha \varphi - \Gamma_\alpha \partial_\beta \varphi) dy.$$

Since $\det(a_{\alpha\beta}(y)) > 0$ for all $y \in \omega$ and since $a_{\alpha\beta} \in W^{1,\infty}_{\mathrm{loc}}(\omega) \subset C^0(\omega)$, the coefficients of the matrix field $(a^{\alpha\beta})$ belong to $L^\infty_{\mathrm{loc}}(\omega)$. This implies that $\Gamma_{\alpha\beta}^\tau \in L^\infty_{\mathrm{loc}}(\omega)$ and that $b_\alpha^\tau := a^{\tau\beta} b_{\alpha\beta} \in L^\infty_{\mathrm{loc}}(\omega)$. Hence $\Gamma_\alpha \in L^\infty_{\mathrm{loc}}(\omega; \mathbb{M}^3) \subset \mathcal{D}'(\omega; \mathbb{M}^3)$, so that the equations (4.8) make sense. Written componentwise, equation (4.8) reads

$$\partial_\alpha \Gamma_{\beta\sigma}^\tau + \Gamma_{\alpha\gamma}^\tau \Gamma_{\beta\sigma}^\gamma - b_\alpha^\tau b_{\beta\sigma} = \partial_\beta \Gamma_{\alpha\sigma}^\tau + \Gamma_{\beta\gamma}^\tau \Gamma_{\alpha\sigma}^\gamma - b_\beta^\tau b_{\alpha\sigma}, \tag{4.9}$$

$$\partial_\alpha b_{\beta\sigma} + \Gamma_{\beta\sigma}^\gamma b_{\alpha\gamma} = \partial_\beta b_{\alpha\sigma} + \Gamma_{\alpha\sigma}^\gamma b_{\beta\gamma}, \tag{4.10}$$

$$\partial_\alpha b_\beta^\tau + \Gamma_{\alpha\gamma}^\tau b_\beta^\gamma = \partial_\beta b_\alpha^\tau + \Gamma_{\beta\gamma}^\tau b_\alpha^\gamma, \tag{4.11}$$

$$b_{\alpha\gamma} b_\beta^\gamma = b_{\beta\gamma} b_\alpha^\gamma. \tag{4.12}$$

Relations (4.9) are equivalent to the Gauss equation, relations (4.10) are the Codazzi-Mainardi equations, and relations (4.12) are trivial identities since $b_{\alpha\gamma}b_\beta^\gamma = b_{\alpha\gamma}b_{\beta\sigma}a^{\gamma\sigma} = b_{\beta\gamma}b_\alpha^\gamma$. Therefore it suffices to prove that the Gauss and Codazzi-Mainardi equations implies the remaining equation (4.11).

On the one hand, the definition of the Christoffel symbols gives

$$2\Gamma_{\alpha\tau}^\gamma a_{\sigma\gamma} = \partial_\alpha a_{\tau\sigma} + \partial_\tau a_{\alpha\sigma} - \partial_\sigma a_{\alpha\tau} \text{ and}$$
$$2\Gamma_{\alpha\sigma}^\gamma a_{\tau\gamma} = \partial_\alpha a_{\sigma\tau} + \partial_\sigma a_{\alpha\tau} - \partial_\tau a_{\alpha\sigma},$$

which next implies that

$$\partial_\alpha a_{\sigma\tau} = \Gamma_{\alpha\sigma}^\gamma a_{\tau\gamma} + \Gamma_{\alpha\tau}^\gamma a_{\sigma\gamma}, \tag{4.13}$$

since the matrix $(a_{\alpha\beta})$ is symmetric. On the oher hand, we infer from the Codazzi-Mainardi equations that

$$\partial_\alpha(b_\beta^\tau a_{\sigma\tau}) + \Gamma_{\beta\sigma}^\gamma b_{\alpha\gamma} = \partial_\beta(b_\alpha^\tau a_{\sigma\tau}) + \Gamma_{\alpha\sigma}^\gamma b_{\beta\gamma},$$

which next gives

$$a_{\sigma\tau}\partial_\alpha b_\beta^\tau + b_\beta^\tau\partial_\alpha a_{\sigma\tau} + \Gamma_{\beta\sigma}^\gamma b_{\alpha\gamma} = a_{\sigma\tau}\partial_\beta b_\alpha^\tau + b_\alpha^\tau\partial_\beta a_{\sigma\tau} + \Gamma_{\alpha\sigma}^\gamma b_{\beta\gamma}. \tag{4.14}$$

The products $a_{\sigma\tau}\partial_\alpha b_\beta^\tau$ and $a_{\sigma\tau}\partial_\beta b_\alpha^\tau$ appearing in the equation above are well defined since $a_{\sigma\tau} \in H_{loc}^1(\omega)$ and $\partial_\alpha b_\beta^\tau, \partial_\beta b_\alpha^\tau \in H_{loc}^{-1}(\omega)$ (the space of all distributions $f \in \mathcal{D}'(\omega)$ whose restrictions to $\mathcal{D}(\widetilde{\omega})$ belong to $H^{-1}(\widetilde{\omega})$ for all open subset $\widetilde{\omega} \Subset \omega$). For, if $f \in H_{loc}^{-1}(\omega)$ and $g \in H_{loc}^1(\omega)$, then the product fg is the distribution defined by

$$\varphi \in \mathcal{D}(\omega) \longmapsto < fg, \varphi > := < f, g\varphi >_{(H^{-1}(\widetilde{\omega}), H_0^1(\widetilde{\omega}))},$$

where $\widetilde{\omega}$ is an open set such that $\operatorname{supp}\varphi \subset \widetilde{\omega} \Subset \omega$. If in addition $g \in W_{loc}^{1,\infty}(\omega)$, then $fg \in H_{loc}^{-1}(\omega)$. Relation (4.14) was obtained by using the Leibniz formula, which holds true for products of elements of $W_{loc}^{1,\infty}(\omega)$ with elements of $L_{loc}^\infty(\omega)$ (the derivatives of elements in $L_{loc}^\infty(\omega)$ belong to $H_{loc}^{-1}(\omega)$ since $L_{loc}^\infty(\omega) \subset L_{loc}^2(\omega)$).

Using now relation (4.13) and the identity $b_\alpha^\tau a_{\tau\gamma} = b_{\alpha\gamma}$ in relation (4.14) gives

$$a_{\sigma\tau}\partial_\alpha b_\beta^\tau + b_{\beta\gamma}\Gamma_{\alpha\sigma}^\gamma + b_\beta^\tau\Gamma_{\alpha\tau}^\gamma a_{\sigma\gamma} + \Gamma_{\beta\sigma}^\gamma b_{\alpha\gamma}$$
$$= a_{\sigma\tau}\partial_\beta b_\alpha^\tau + b_{\alpha\gamma}\Gamma_{\beta\sigma}^\gamma + b_\alpha^\tau\Gamma_{\beta\tau}^\gamma a_{\sigma\gamma} + \Gamma_{\alpha\sigma}^\gamma b_{\beta\gamma}.$$

Hence

$$a_{\sigma\tau}\partial_\alpha b_\beta^\tau + b_\beta^\tau\Gamma_{\alpha\tau}^\gamma a_{\sigma\gamma} = a_{\sigma\tau}\partial_\beta b_\alpha^\tau + b_\alpha^\tau\Gamma_{\beta\tau}^\gamma a_{\sigma\gamma}.$$

Since $a_{\sigma\tau}a^{\sigma\psi} = \delta_\tau^\psi$, multiplying the previous relation with $a^{\sigma\psi}$ gives the desired relation (4.11). Consequently, the result announced in step (ii) is established.

(iii) *There exists a matrix field* $F \in W^{1,\infty}_{\mathrm{loc}}(\omega, \mathbb{M}^3)$ *such that*

$$\begin{aligned} &\partial_\alpha F = F\Gamma_\alpha \textit{ in } \mathcal{D}'(\omega; \mathbb{M}^3), \\ &F(y^0) = F^0, \end{aligned} \tag{4.15}$$

where F^0 *is the matrix whose* i*-th column is* $\boldsymbol{a}_i^0$*.*

To see this, it suffices to apply Theorem 3.1 to the Cauchy problem (4.15), the assumptions of this theorem being satisfied thanks to the previous step.

(iv) *Let* $\boldsymbol{a}_i(y)$ *denote the* i*-th column of the matrix* $F(y)$*. Then there exists a solution* $\boldsymbol{\theta} \in W^{2,\infty}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ *to the system*

$$\begin{aligned} &\partial_\alpha \boldsymbol{\theta}(y) = \boldsymbol{a}_\alpha(y) \textit{ for all } y \in \omega, \\ &\boldsymbol{\theta}(y^0) = \boldsymbol{\theta}^0. \end{aligned} \tag{4.16}$$

By writing the first equation of (4.15) using the columns of the matrix field F, one can easily show that the vector fields $\boldsymbol{a}_i$ satisfy the equations

$$\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3 \text{ in } \mathcal{D}'(\omega; \mathbb{R}^3).$$

Hence they also satisfy

$$\partial_\beta \boldsymbol{a}_\alpha = \Gamma^\sigma_{\beta\alpha} \boldsymbol{a}_\sigma + b_{\beta\alpha} \boldsymbol{a}_3 \text{ in } \mathcal{D}'(\omega; \mathbb{R}^3).$$

Since $\Gamma^\sigma_{\beta\alpha} = \Gamma^\sigma_{\alpha\beta}$ and $b_{\beta\alpha} = b_{\alpha\beta}$, the two previous relations yield

$$\partial_\alpha \boldsymbol{a}_\beta = \partial_\beta \boldsymbol{a}_\alpha.$$

Since in addition ω is simply connected and $\boldsymbol{a}_\alpha$ belong to $L^\infty_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ (in particular), we can apply Theorem 3.1 to problem (4.16). This shows that there exists a unique solution $\boldsymbol{\theta} \in W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ to problem (4.16). Since $\partial_\alpha \boldsymbol{\theta} = \boldsymbol{a}_\alpha \in W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$, the mapping $\boldsymbol{\theta}$ belongs in fact to $W^{2,\infty}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$.

(v) *The mapping* θ *satisfies the relations*

$$\begin{aligned} &a_{\alpha\beta}(y) = \partial_\alpha \boldsymbol{\theta}(y) \cdot \partial_\beta \boldsymbol{\theta}(y), \\ &b_{\alpha\beta}(y) = \partial_{\alpha\beta} \boldsymbol{\theta}(y) \cdot \frac{\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)}{|\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)|} \end{aligned} \tag{4.17}$$

for almost all $y \in \omega$*.*

Define $a_{\alpha 3}(y) = a_{3\alpha}(y) := 0$ and $a_{33}(y) = 1$ for all $y \in \omega$. We first prove that $\boldsymbol{a}_i(y) \cdot \boldsymbol{a}_j(y) = a_{ij}(y)$ for all $y \in \omega$, which will give in particular the first relations of (4.17). Let $\Gamma^p_{\alpha i}$ denote the coefficients of the matrix Γ_α, i.e., $\Gamma^3_{\alpha\beta} := b_{\alpha\beta}$, $\Gamma^\tau_{\alpha 3} := -b^\tau_\alpha$, $\Gamma^3_{\alpha 3} := 0$ and $\Gamma^\tau_{\alpha\beta} := \frac{1}{2} a^{\tau\sigma}(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\beta\sigma} - \partial_\sigma a_{\alpha\beta})$.

Then one can see that these functions satisfy the relations

$$\partial_\alpha a_{ij} = \Gamma^p_{\alpha i} a_{pj} + \Gamma^p_{\alpha j} a_{ip}.$$

Now, let the functions $h_{ij} : \omega \to \mathbb{R}$ be defined by $h_{ij}(y) := \boldsymbol{a}_i(y) \cdot \boldsymbol{a}_j(y)$ for all $y \in \omega$. Then

$$\partial_\alpha h_{ij} = \partial_\alpha \boldsymbol{a}_i \cdot \boldsymbol{a}_j + \boldsymbol{a}_i \cdot \partial_\alpha \boldsymbol{a}_j = \Gamma^p_{\alpha i} h_{pj} + \Gamma^p_{\alpha j} h_{ip}$$

and $h_{ij}(y^0) = \boldsymbol{a}_i(y^0) \cdot \boldsymbol{a}_j(y^0) = \boldsymbol{a}^0_i \cdot \boldsymbol{a}^0_j = a_{ij}(y^0)$ (see (4.15)).

Let $X_{ij}(y) := h_{ij}(y) - a_{ij}(y)$. Then the functions X_{ij} belong to $W^{1,\infty}_{\mathrm{loc}}(\omega)$ and satisfy the following equations

$$\begin{aligned} &\partial_\alpha X_{ij} = \Gamma^p_{\alpha i} X_{pj} + \Gamma^p_{\alpha j} X_{ip} \text{ in } L^\infty(\omega), \\ &X_{ij}(y^0) = 0 \end{aligned}$$

for all i, j, α. Let the matrix field $X : \omega \to \mathbb{M}^3$ be given by $X(y) := (X_{ij}(y)) \in \mathbb{M}^3$, where i is the row index and j is the column index of the matrix. Then

$$X \in W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{M}^3)$$

and

$$\begin{aligned} &\partial_\alpha X = \Gamma_\alpha X + X \Gamma^T_\alpha \text{ in } L^\infty_{loc}(\omega; \mathbb{M}^3), \\ &X(y^0) = 0 \text{ in } \mathbb{M}^3. \end{aligned} \tag{4.18}$$

We wish to prove that the field X vanishes in ω. To this end, we shall use a connectedness argument. First, note that $X \in C^0(\omega; \mathbb{M}^3)$ by the Sobolev imbedding theorem. Then the following subset of ω,

$$\mathcal{A} := \{y \in \omega; X(y) = 0 \in \mathbb{M}^3\},$$

is non-empty (since $y^0 \in \mathcal{A}$) and closed in ω (since the application $X : \omega \to \mathbb{M}^3$ is continuous). We now prove that the set $\mathcal{A}$ is also open in ω.

Let $y \in \mathcal{A}$, let $B_0 := B(y, \varepsilon_0)$ be an open ball such that $B_0 \Subset \omega$ and let $C := \max_{\alpha \in \{1,2\}} \|\Gamma_\alpha\|_{L^\infty(B_0; \mathbb{M}^3)}$. Let also $B := B(y, \varepsilon)$, where $\varepsilon = \min\left\{\frac{1}{12C}, \varepsilon_0\right\}$. Using the inequality

$$\|A^T\| \leq \sqrt{\mathrm{tr}(AA^T)} = \sqrt{\mathrm{tr}(A^T A)} \leq \sqrt{3\|A\|^2} = \sqrt{3}\|A\| \leq 2\|A\|$$

valid for all matrices $A \in \mathbb{M}^3$, we deduce from (4.18) that

$$\|\partial_\alpha X\|_{L^\infty(B; \mathbb{M}^3)} \leq 3C \|X\|_{L^\infty(B; \mathbb{M}^3)} \tag{4.19}$$

for all $\alpha \in \{1, 2\}$.

Now, define the space

$$W^{1,\infty}_y(B; \mathbb{M}^3) := \{Y \in W^{1,\infty}(B; \mathbb{M}^3); Y(y) = 0 \text{ in } \mathbb{M}^3\}.$$

Then a result about Sobolev spaces (see Corollary A.1, page 134, for a proof) shows that for almost all $z \in \partial B$, the restriction of Y to the segment $]y, z[$ belongs to $W^{1,\infty}(]y, z[; \mathbb{M}^3)$, $\|\nabla Y\|_{L^\infty(]y,z[; \mathbb{M}^3)} \leq \|\nabla Y\|_{L^\infty(B; \mathbb{M}^3)}$ and

$$Y(x) = \int_0^1 \frac{d}{dt} Y(y + t(x - y)) dt = \int_0^1 \sum_{\alpha=1}^2 \partial_\alpha Y(y + t(x - y))(x_\alpha - y_\alpha) dt.$$

for all $x \in]y, z[:= \{y + t(z - y); t \in]0, 1[\}$. This gives the following inequality of Poincaré's type:

$$\|Y\|_{L^\infty(B;\mathbb{M}^3)} \leq 2\varepsilon \max_{\alpha\in\{1,2\}} \|\partial_\alpha Y\|_{L^\infty(B;\mathbb{M}^3)} \tag{4.20}$$

for all $Y \in W^{1,\infty}_y(B; \mathbb{M}^3)$. By combining this inequality with $Y = X$ with (4.19), we obtain that

$$\|X\|_{L^\infty(B;\mathbb{M}^3)} \leq 6C\varepsilon\|X\|_{L^\infty(B;\mathbb{M}^3)} \leq \frac{1}{2}\|X\|_{L^\infty(B;\mathbb{M}^3)}$$

thanks to the choice of ε. Hence $X = 0$ in $L^\infty(B; \mathbb{M}^3)$, and thus $X = 0$ also in $C^0(B; \mathbb{M}^3)$ since the field X is continuous over B. This implies that the entire ball $B = B(y, \varepsilon)$ is included in $\mathcal{A}$. Therefore, the set $\mathcal{A}$ is open in ω.

Since the set $\mathcal{A}$ is non-empty, closed, and open in ω, the connectedness of ω implies that $\mathcal{A} = \omega$. Hence we have

$$\begin{aligned}
&\boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y) = a_{\alpha\beta}(y),\\
&\boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_3(y) = 0,\\
&\boldsymbol{a}_3(y) \cdot \boldsymbol{a}_3(y) = 1
\end{aligned}$$

for all $y \in \omega$. These relations show that the first relation announced in step (v) holds. They also show that

$$\text{either } \boldsymbol{a}_3(y) = \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|} \text{ or } \boldsymbol{a}_3(y) = -\frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|},\ y \in \omega,$$

and that $F^T F = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in ω (recall that F is the matrix whose i-th column is $\boldsymbol{a}_i$). Therefore, $(\det F(y))^2 > 0$, which implies in particular that $\det F(y) \neq 0$ for all $y \in \omega$. On the other hand,

$$\boldsymbol{a}_3(y^0) = \frac{\boldsymbol{a}^0_1 \wedge \boldsymbol{a}^0_2}{|\boldsymbol{a}^0_1 \wedge \boldsymbol{a}^0_2|} = \frac{\boldsymbol{a}_1(y^0) \wedge \boldsymbol{a}_2(y^0)}{|\boldsymbol{a}_1(y^0) \wedge \boldsymbol{a}_2(y^0)|},$$

which implies that $\det F(y^0) > 0$. Noting that $\det F$ is a continuous function over the connected set ω, we conclude that $\det F > 0$ over ω. Hence we must have

$$\boldsymbol{a}_3(y) = \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$$

for all $y \in \omega$. On the other hand, step (iii) shows that

$$\partial_\alpha \boldsymbol{a}_\beta(y) = \Gamma^\sigma_{\alpha\beta}(y)\boldsymbol{a}_\sigma(y) + b_{\alpha\beta}(y)\boldsymbol{a}_3(y)$$

for almost all $y \in \omega$. By combining these last two relations, we finally get

$$b_{\alpha\beta}(y) = \partial_\alpha \boldsymbol{a}_\beta(y) \cdot \boldsymbol{a}_3(y) = \partial_{\alpha\beta}\boldsymbol{\theta}(y) \cdot \frac{\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)|},$$

which is the second relation announced in step (v).

(vi) *The mapping $\boldsymbol{\theta}$ is unique up to proper isometries of* $\mathbb{R}^3$*.* Let another mapping $\boldsymbol{\phi}$ satisfy the conditions of the theorem. Let $F_0 \in \mathbb{M}^3$ be the matrix whose i-th column is $\boldsymbol{a}_i^0$ and let $E_0 \in \mathbb{M}^3$ be the matrix whose first, second and third column are respectively $\partial_1\boldsymbol{\phi}(y^0)$, $\partial_2\boldsymbol{\phi}(y^0)$ and $(\partial_1\boldsymbol{\phi}(y^0) \wedge \partial_2\boldsymbol{\phi}(y^0))/|\partial_1\boldsymbol{\phi}(y^0) \wedge \partial_2\boldsymbol{\phi}(y^0)|$. Define the mapping $\widehat{\boldsymbol{\theta}} : \omega \to \mathbb{R}^3$ by letting

$$\widehat{\boldsymbol{\theta}}(y) = \boldsymbol{\theta}^0 + Q(\boldsymbol{\phi}(y) - \boldsymbol{\phi}(y^0)) \text{ for all } y \in \omega, \tag{4.21}$$

where $Q := F_0 E_0^{-1}$. The matrix Q is proper ortogonal since $\det Q > 0$ and $Q^TQ = I$, the latter relation being an immediate consequence of the following relation

$$F_0^T F_0 = E_0^T E_0 = \begin{pmatrix} a_{11}(y^0) & a_{12}(y^0) & 0 \\ a_{21}(y^0) & a_{22}(y^0) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then a simple calculation shows that the mapping $\widehat{\boldsymbol{\theta}}$ satisfies the conditions of the theorem. This implies that the Gauss and Weingarten equations are satisfied, i.e.,

$$\partial_\alpha \widehat{\boldsymbol{a}}_\beta = \Gamma_{\alpha\beta}^\sigma \widehat{\boldsymbol{a}}_\sigma + b_{\alpha\beta}\widehat{\boldsymbol{a}}_3 \text{ in } \mathcal{D}'(\omega;\mathbb{R}^3),$$
$$\partial_\alpha \widehat{\boldsymbol{a}}_3 = -b_\alpha^\sigma \widehat{\boldsymbol{a}}_\sigma \text{ in } \mathcal{D}'(\omega;\mathbb{R}^3),$$

where $\widehat{\boldsymbol{a}}_\alpha(y) := \partial_\alpha \widehat{\boldsymbol{\theta}}(y)$ and

$$\widehat{\boldsymbol{a}}_3(y) := \frac{\partial_1\widehat{\boldsymbol{\theta}}(y) \wedge \partial_2\widehat{\boldsymbol{\theta}}(y)}{|\partial_1\widehat{\boldsymbol{\theta}}(y) \wedge \partial_2\widehat{\boldsymbol{\theta}}(y)|}$$

for all $y \in \omega$. These equations are equivalent with the matrix equation $\partial_\alpha \widehat{F} = \widehat{F}\,\Gamma_\alpha$, where $\widehat{F}$ is the matrix whose i-th column is $\widehat{\boldsymbol{a}}_i$. In the same manner, $\partial_\alpha F = F\,\Gamma_\alpha$, where F is the matrix whose i-th column is $\boldsymbol{a}_i$.

On the other hand, since $\widehat{\boldsymbol{a}}_i(y^0) = \boldsymbol{a}_i^0$, we also have $F(y^0) = \widehat{F}(y^0) = F_0$. Then Theorem 3.1 implies that $F = \widehat{F}$ in Ω. Therefore $\partial_\alpha\boldsymbol{\theta} = \partial_\alpha\widehat{\boldsymbol{\theta}}$ for $\alpha \in \{1, 2\}$, which next implies that $\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}$ is a constant function. Since $\boldsymbol{\theta}(y^0) = \widehat{\boldsymbol{\theta}}(y^0) = \boldsymbol{\theta}^0$ and ω is connected, we finally obtain that $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$. Then relation (4.21) shows that the mapping $\boldsymbol{\theta}$ satisfying the conditions of the theorem is unique up to proper isometries of $\mathbb{R}^3$.

The proof is now complete. □

Remark 4.1. The equalities $h_{ij} = a_{ij}$ appearing in step (v) of the previous proof can also be established by applying Theorem 3.2 of Chapter 3 to the system (4.18).

The proof of Theorem 4.2 also shows that the mapping $\boldsymbol{\theta}$ is unique if it is required to satisfy some additional conditions. More specifically, it shows that the following result holds.

Theorem 4.3. *Let ω be a connected and simply connected open subset of $\mathbb{R}^2$ and let matrix fields $(a_{\alpha\beta}) \in W^{1,\infty}_{\mathrm{loc}}(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^\infty_{\mathrm{loc}}(\omega;\mathbb{S}^2)$ be given* *that satisfy together the Gauss and Codazzi-Mainardi equations in $\mathcal{D}'(\omega)$. Let there be given $y^0 \in \omega$, $\boldsymbol{\theta}^0 \in \mathbb{R}^3$ and $\boldsymbol{a}_\alpha^0 \in \mathbb{R}^3$ such that $\boldsymbol{a}_\alpha^0 \cdot \boldsymbol{a}_\beta^0 = a_{\alpha\beta}(y^0)$. Then there exists one and only one map $\boldsymbol{\theta} \in W^{2,\infty}_{\rm loc}(\omega;\mathbb{R}^3)$ such that $\boldsymbol{\theta}(y^0) = \boldsymbol{\theta}^0$, $\partial_\alpha \boldsymbol{\theta}(y^0) = \boldsymbol{a}_\alpha^0$ and the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ are given respectively by the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$.*

The result of Hartman and Wintner [8] can now be obtained as a corollary to Theorem 4.2. More specifically, we have the following:

Corollary 4.1. *Assume that ω is a connected and simply-connected open subset of $\mathbb{R}^2$. Let the matrix fields $(a_{\alpha\beta}) \in C^1(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in C^0(\omega;\mathbb{S}^2)$ be such that they satisfy the Gauss and Codazzi-Mainardi equations,*

$$\partial_\gamma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\sigma\beta} = b_{\alpha\beta}b^\tau_\gamma - b_{\alpha\gamma}b^\tau_\beta,$$
$$\partial_\gamma b_{\alpha\beta} - \partial_\beta b_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta} b_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma} b_{\sigma\beta} = 0,$$

in $\mathcal{D}'(\omega)$. Then there exists a mapping $\boldsymbol{\theta} \in C^2(\omega,\mathbb{R}^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and}$$
$$b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \tag{4.22}$$

in ω. Moreover, the mapping $\boldsymbol{\theta}$ is unique in $C^2(\omega,\mathbb{R}^3)$ up to proper isometries in $\mathbb{R}^3$.

Proof. By Theorem 4.2, there exists a mapping $\boldsymbol{\theta} \in W^{2,\infty}_{\rm loc}(\omega,\mathbb{R}^3)$, unique up to proper isometries in $\mathbb{R}^3$, such that relations (4.22) hold true for almost all $y \in \omega$. It remains to prove that $\boldsymbol{\theta} \in C^2(\omega,\mathbb{R}^3)$.

First, we have $\boldsymbol{\theta} \in C^1(\omega,\mathbb{R}^3)$ by the Sobolev imbedding $W^{2,\infty}_{\rm loc}(\omega,\mathbb{R}^3) \subset C^1(\omega,\mathbb{R}^3)$. Then the Gauss and Weingarten equations show that

$$\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\tau_{\alpha\beta}\boldsymbol{a}_\tau + b_{\alpha\beta}\boldsymbol{a}_3,$$
$$\partial_\alpha \boldsymbol{a}_3 = -b^\tau_\alpha \boldsymbol{a}_\tau,$$

where $\boldsymbol{a}_\beta(y) := \partial_\beta \boldsymbol{\theta}(y)$ and

$$\boldsymbol{a}_3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$$

for all $y \in \omega$. Since $\boldsymbol{a}_\tau, \boldsymbol{a}_3 \in W^{1,\infty}_{\rm loc}(\omega;\mathbb{R}^3) \subset C^0(\omega;\mathbb{R}^3)$, $b^\tau_\alpha = b_{\alpha\sigma}a^{\sigma\tau} \in C^0(\omega)$, and

$$\Gamma^\tau_{\alpha\beta} = \frac{1}{2}a^{\tau\sigma}\left(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\sigma\alpha} - \partial_\sigma a_{\alpha\beta}\right) \in C^0(\omega),$$

we deduce from the Gauss and Weingarten equations that $\boldsymbol{a}_\beta, \boldsymbol{a}_3 \in C^1(\omega;\mathbb{R}^3)$. Hence $\partial_\beta \boldsymbol{\theta} \in C^1(\omega;\mathbb{R}^3)$. Since we have already seen that $\boldsymbol{\theta} \in C^1(\omega,\mathbb{R}^3)$, we finally deduce that $\boldsymbol{\theta} \in C^2(\omega,\mathbb{R}^3)$. □

Another consequence of Theorem 4.2 combined with Theorem 4.1 is that, if the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are respectively of class $W^{1,\infty}$ and L^∞ over ω and the matrix field $(a^{\sigma\tau})$ is of class L^∞ over ω, then the surface is given by a mapping of class $W^{2,\infty}$ over ω, provided that ω has a finite geodesic diameter. More specifically, we have the following

Corollary 4.2. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ with a finite geodesic diameter. Let there be given two matrix fields $(a_{\alpha\beta}) \in W^{1,\infty}(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^\infty(\omega;\mathbb{S}^2)$ such that $(a_{\alpha\beta})^{-1} \in L^\infty(\omega,\mathbb{M}^2)$. If these fields satisfy the Gauss and Codazzi-Mainardi equations in $\mathcal{D}'(\omega)$, then there exists a mapping $\boldsymbol{\theta} \in W^{2,\infty}(\omega,\mathbb{R}^3)$ such that*

$$
\begin{aligned}
&a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \text{ and} \\
&b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \text{ a.e. in } \omega.
\end{aligned} \tag{4.23}
$$

Moreover, the mapping $\boldsymbol{\theta}$ is unique in $W^{2,\infty}(\omega,\mathbb{R}^3)$ up to proper isometries in $\mathbb{R}^3$.

Proof. By Theorem 4.2, there exists a mapping $\boldsymbol{\theta} \in W^{2,\infty}_{\text{loc}}(\omega,\mathbb{R}^3)$, unique up to proper isometries in $\mathbb{R}^3$, such that relations (4.23) hold true for almost all $y \in \omega$. This implies that the Gauss and Weingarten equations are satisfied, i.e., that

$$
\begin{aligned}
&\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta} \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3 \text{ in } \mathcal{D}'(\omega;\mathbb{R}^3), \\
&\partial_\alpha \boldsymbol{a}_3 = -b^\sigma_\alpha \boldsymbol{a}_\sigma \text{ in } \mathcal{D}'(\omega;\mathbb{R}^3),
\end{aligned}
$$

where

$$
\boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3 := \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}.
$$

These equations are equivalent with the matrix equation $\partial_\alpha F = F\,\Gamma_\alpha$, where F is the matrix whose i-th column is $\boldsymbol{a}_i$ and Γ_α is the matrix defined by (4.7).

The regularity assumptions on the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ imply that the coefficients Γ_α belong to $L^\infty(\omega;\mathbb{M}^3)$. Since in addition the geodesic diameter of ω is finite, we can apply Theorem 4.1 to the system

$$
\begin{aligned}
&\partial_\alpha Y = Y\,\Gamma_\alpha \text{ in } \mathcal{D}'(\omega;\mathbb{M}^3), \\
&Y(y^0) = F(y^0),
\end{aligned}
$$

where y^0 is a fixed point in ω. This implies that the unique solution F to this system belongs to $W^{1,\infty}(\omega;\mathbb{M}^3)$. Then the first two columns $\boldsymbol{a}_\alpha$ of the matrix field F belong to $L^\infty(\omega;\mathbb{R}^3)$ and thus we can apply Theorem 4.1 to the system

$$
\begin{aligned}
&\partial_\alpha \boldsymbol{\phi} = \boldsymbol{a}_\alpha \text{ in } \mathcal{D}'(\omega;\mathbb{R}^3), \\
&\boldsymbol{\phi}(y^0) = \boldsymbol{\theta}(y^0).
\end{aligned}
$$

This implies that the unique solution $\boldsymbol{\theta}$ to this problem belongs to $W^{1,\infty}(\omega;\mathbb{R}^3)$. Since we have already seen that $\partial_\alpha\boldsymbol{\theta} \in W^{1,\infty}(\omega;\mathbb{R}^3)$, we conclude that the mapping $\boldsymbol{\theta}$ belongs to the space $W^{2,\infty}(\omega;\mathbb{R}^3)$ and the corollary is proved. □

A proof similar to that of Theorem 4.2 yields the following more general result for hypersurfaces immersed in $\mathbb{R}^d$, $d \geq 2$. All definitions and notations (for the Christoffel symbols in particular) that have been used for surfaces immersed in $\mathbb{R}^3$ are translated *verbatim* to hypersurfaces immersed in $\mathbb{R}^d$ with the new convention that Greek indices vary in the set $\{1, 2, ..., d-1\}$, Latin indices vary in the set $\{1, 2, ..., d\}$ and the summation convention with respect to repeted indices is used in conjunction with these rules. The normal vector to the hypersurface $\boldsymbol{\theta}(\omega)$, where $\omega \subset \mathbb{R}^{d-1}$ and $\boldsymbol{\theta} : \omega \to \mathbb{R}^d$, is defined by

$$\boldsymbol{a}_d(y) := \frac{\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y) \wedge \cdots \wedge \partial_{d-1}\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y) \wedge \cdots \wedge \partial_{d-1}\boldsymbol{\theta}(y)|}.$$

The exterior product $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_{d-1}$ of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, ...\boldsymbol{a}_{d-1} \in \mathbb{R}^d$ is defined as the unique vector $\boldsymbol{a}$ (obtained by Riesz Theorem) that satisfies

$$\det_{\mathcal{B}}(\boldsymbol{a}_1, \boldsymbol{a}_2, ...\boldsymbol{a}_{d-1}, \boldsymbol{v}) = \boldsymbol{a} \cdot \boldsymbol{v} \text{ for all } \boldsymbol{v} \in \mathbb{R}^d,$$

where $\mathcal{B}$ is the canonical basis of $\mathbb{R}^d$.

Theorem 4.4. *Assume that ω is a connected and simply-connected open subset of $\mathbb{R}^{d-1}$ and that the matrix fields $(a_{\alpha\beta}) \in W^{1,\infty}_{\rm loc}(\omega;\mathbb{S}^{d-1}_>)$ and $(b_{\alpha\beta}) \in L^\infty_{\rm loc}(\omega;\mathbb{S}^{d-1})$ satisfy the equations (4.6) in $\mathcal{D}'(\omega)$ (where the Greek indices vary in the set $\{1, 2, \ldots, d-1\}$). Then there exists a mapping $\boldsymbol{\theta} \in W^{2,\infty}_{\rm loc}(\omega, \mathbb{R}^d)$ such that*

$$a_{\alpha\beta}(y) = \partial_\alpha\boldsymbol{\theta}(y) \cdot \partial_\beta\boldsymbol{\theta}(y),$$
$$b_{\alpha\beta}(y) = \partial_{\alpha\beta}\boldsymbol{\theta}(y) \cdot \frac{\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y) \wedge \cdots \wedge \partial_{d-1}\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y) \wedge \cdots \wedge \partial_{d-1}\boldsymbol{\theta}(y)|}$$

for almost all $y \in \omega$. Moreover, the mapping $\boldsymbol{\theta}$ is unique in $W^{2,\infty}_{\rm loc}(\omega, \mathbb{R}^d)$ up to proper isometries in $\mathbb{R}^d$.

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Appendice

Appendice

1. Introduction

Dans cette dernière partie de la thèse, on fournit d'une part les preuves complètes de certains résultats d'analyse qui ont été utilisés dans les chapitres 3 et 4 et, d'autre part, on établit que le Théorème 3.2 énoncé au troisième chapitre est une conséquence du Théorème 3.1 énoncé et démontré dans ce même chapitre.

Dans la première section de cet appendice, on établit en particulier le résultat suivant, qui a été utilisé à plusieurs reprises dans les deux derniers chapitres de la thèse : Soit $\Omega := B(x^0, R)$ une boule ouverte dans l'espace $\mathbb{R}^d$ et soit $Y \in W^{1,\infty}(\Omega; \mathbb{M}^{q,l})$. Alors pour presque tout $z \in \partial\Omega$ (par rapport à la mesure surfacique), la restriction de l'application Y au segment $]x^0, z[$ appartient à l'espace $W^{1,\infty}(]x^0, z[; \mathbb{M}^{q,l})$, $\|\nabla Y\|_{L^\infty(]x^0,z[;\mathbb{M}^{q,l})} \leq \|\nabla Y\|_{L^\infty(B(x,R);\mathbb{M}^{q,l})}$ et

$$Y(x) = \int_0^1 \sum_{\alpha=1}^d \partial_\alpha Y(x^0 + t(x - x^0))(x_\alpha - x_\alpha^0) dt$$

pour tout $x \in]x^0, z[:= \{x^0 + t(z - x^0); t \in]0, 1[\}$. On rappelle que cette relation, ou l'une de ses variantes, intervient d'une manière essentielle dans la démonstration du Corollaire 3.1 du troisième chapitre et est utilisée dans l'étape (v) de la démonstration du Théorème 3.1 du troisième chapitre ou du Théorème 4.2 du quatrième chapitre.

Ce résultat est établi dans le Corollaire A.1, qui est obtenu comme une conséquence des Théorèmes A.1 et A.2. Le Théorème A.1, obtenu à l'aide des Lemmes A.1 et A.2, est un résultat classique concernant les espaces de Sobolev dont la démonstration est reproduite ici pour rendre l'appendice autocontenu. Le Théorème A.2, établi à l'aide des Lemmes A.3 et A.4, fournit une formule de changement de variables pour les fonctions de classe $W^{1,\infty}$ dans un ouvert de $\mathbb{R}^d$ sous des hypothèses plus faibles que celles trouvées habituellement dans la littérature (voir par exemple Brezis [3], Proposition IX.6).

Le Lemme A.3 ci-dessous est essentiellement la réciproque du théorème de Sard. Il est implicitement utilisé lorsque l'on définit la composition $f \circ G$ entre une classe de fonctions f modulo l'égalité presque partout et une application G correspondant à un changement de variables (par exemple pour passer des coordonnées cartésiennes aux coordonnées sphériques).

Notons également que le Lemme A.4 ci-dessous permet de donner une autre preuve des inégalités (3.29) du troisième chapitre et (4.20) du quatrième chapitre.

Dans la deuxième section de cet appendice, on établit essentiellement que la résolution du système matriciel

$$\partial_\alpha Y = Y A_\alpha + C_\alpha Y + B_\alpha$$

se ramène à la resolution des systèmes plus simples du type

$$\partial_\alpha X = X A_\alpha \text{ (ou } \partial_\alpha Z = C_\alpha Z)$$

et

$$\partial_\alpha X = \widetilde{B}_\alpha,$$

pour un choix approprié du champ de matrices $\widetilde{B}_\alpha$.

2. Quelques résultats d'analyse

Dans cette section, toutes les fonctions sont à valeurs réelles et la règle de la sommation par rapport aux indices répétés est utilisée.

Pour tout entier $d \geq 1$, l'espace euclidien d-dimensionnel est identifié à $\mathbb{R}^d$. L'adhérence d'un sous-ensemble Ω de $\mathbb{R}^d$ est notée $\overline{\Omega}$ et la frontière de Ω est notée $\partial\Omega$. Un point courant de $\mathbb{R}^d$ est noté $x = (x', x_d) = (x_1, x_2, ..., x_d)$, où $x' = (x_1, x_2, ..., x_{d-1})$, et les dérivées partielles sont notées $\partial_i := \partial/\partial x_i$. La notation $B(x, R)$ désigne la boule ouverte de centre $x \in \mathbb{R}^d$ et de rayon $R > 0$, et

$$\operatorname{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|$$

désigne la distance entre les sous-ensembles A et B de $\mathbb{R}^d$. Le support d'une fonction continue $\varphi : \Omega \to \mathbb{R}$ est l'adhérence de l'ensemble des points où f est non nulle, i.e.,

$$\operatorname{supp} \varphi = \overline{\{x \in \Omega, \varphi(x) \neq 0\}}.$$

Si Ω est un sous-ensemble de $\mathbb{R}^d$, la notation $\omega \Subset \Omega$ signifie que ω est un sous-ensemble de Ω tel que son adhérence $\overline{\omega}$ est un ensemble compact contenu dans Ω.

Si Ω est un ouvert de $\mathbb{R}^d$ et $G = (G_1, G_2, ..., G_d) : \Omega \to \mathbb{R}^d$ est une fonction différentiable en $x \in \Omega$, le gradient de G en x est la matrice $\nabla G(x) := (\partial_i G_j(x))$, où i est l'indice de colonne et j est l'indice de ligne, et le Jacobien de G en x est le nombre $J_G(x) := \det(\nabla G(x))$ (le déterminant de la matrice $(\nabla G(x))$. L'image réciproque par l'application G d'un ensemble $A \subset \mathbb{R}^k$ est définie par

$$G^{-1}(A) := \{x \in \Omega;\ G(x) \in A\}.$$

La mesure de Lebesque dans $\mathbb{R}^d$ est notée $\mathcal{L}^d$ ou dx lorsqu'elle est utilisée dans une intégrale, qui est alors comprise au sens de Lebesgue. Un sous-ensemble de $\mathbb{R}^d$ est dit mesurable s'il est mesurable par rapport à la mesure de Lebesgue $\mathcal{L}^d$ et une fonction définie dans un sous-ensemble mesurable de $\mathbb{R}^d$ est dite mesurable si elle est mesurable par rapport à la mesure de Lebesgue $\mathcal{L}^d$.

Soit Ω un ouvert de $\mathbb{R}^d$. Les notations $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, $C^k(\Omega)$, et $W^{m,p}(\Omega)$ désignent respectivement l'espace des fonctions indéfiniment dérivables à support compact dans Ω, l'espace des distributions définies dans Ω, l'espace de fonctions continues dans Ω telles que leurs dérivées partielles jusqu'à l'ordre k sont également continues dans Ω, et l'espace de Sobolev usuel. On utilise également la notation

$$W^{m,p}_{loc}(\Omega;\mathbb{R}^d) := \{u \in \mathcal{D}'(\Omega;\mathbb{R}^d);\\ u \in W^{m,p}(U;\mathbb{R}^d) \text{ pour tout ouvert } U \Subset \Omega\}.$$

Les espaces des fonctions à valeurs dans un espace vectoriel X sont notés $\mathcal{D}(\Omega;X)$, $\mathcal{D}'(\Omega;X)$, $C^k(\Omega;X)$, $W^{m,p}(\Omega;X)$, etc.

Pour toute classe de fonctions $\dot{f} \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, on choisit le représentant continu, noté f, de $\dot{f}$. Par souci de simplicité, on utilise dans ce qui suit la même notation pour une classe et son représentant continu, la distinction devant être claire d'après le contexte. Ainsi, la notation $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega;X)$ signifie que f et toutes ses dérivées partielles sont essentiellement bornées et que $f(x)$ est bien défini et appartient à l'ensemble X en tout point x de Ω.

On commence par un résultat classique concernant les fonctions d'une seule variable (voir par exemple Brezis [3]) :

Lemme A.1. *Soit $f \in W^{1,1}_{loc}(]a,b[)$, où $a,b \in [-\infty,+\infty]$, et soit $x_0 \in]a,b[$. Alors f admet un représentant continu et pour ce représentant (noté aussi f), on a*

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt\ ,$$

pour tout $x \in]a,b[$, où f' est un représentant arbitraire de la derivée de f au sens des distributions dans $\mathcal{D}'(]a,b[)$.

Démonstration. Soit la fonction $\tilde{f} :]a,b[\to \mathbb{R}$ définie par $\tilde{f}(x) = \int_{x_0}^{x} f'(t)\,dt$ et notons que $\tilde{f}$ est continue et satisfait la relation $\tilde{f}(x) = \tilde{f}(y) + \int_{y}^{x} f'(t)\,dt$ pour tout $x, y \in]a,b[$.

Soit $\varphi \in \mathcal{D}(]a,b[)$. Il existe $a' > a$ et $b' < b$ tels que $\operatorname{supp}\varphi \subset [a',b'] \subset]a,b[$. Alors on a

$$\begin{aligned}\int_a^b \tilde{f}(t)\varphi'(t)\,dt = \int_{a'}^{b'} \tilde{f}(t)\varphi'(t)\,dt &= \int_{a'}^{b'} \Big(\tilde{f}(a') + \int_{a'}^{t} f'(s)\,ds\Big)\varphi'(t)\,dt \\ &= \tilde{f}(a')\int_{a'}^{b'} \varphi'(t)\,dt + \int_{a'}^{b'} \Big(\int_{a'}^{t} f'(s)\,ds\Big)\varphi'(t)\,dt \\ &= \int_{a'}^{b'}\int_{s}^{b'} f'(s)\varphi'(t)\,dt\,ds = \int_{a'}^{b'} f'(s)\Big(\int_{s}^{b'} \varphi'(t)\,dt\Big)\,ds \\ &= -\int_{a'}^{b'} f'(s)\varphi(s)\,ds = -\int_a^b f'(t)\varphi(t)\,dt,\end{aligned}$$

où pour la quatrième égalité ci-dessus, nous avons utilisé le Théorème de Fubini pour la fonction

$$(s,t) \in [a',b']^2 \mapsto f'(s)\varphi'(t)\mathbf{1}_{[a',t]}(s),$$

qui est évidemment intégrable sur $[a',b']^2$. La notation $\mathbf{1}_{[a',t]}$ désigne la fonction indicatrice de l'intervalle $[a',t]$, c'est-à-dire que

$$\mathbf{1}_{[a',t]}(s) = \begin{cases} 1 \text{ si } s \in [a',t], \\ 0 \text{ sinon }. \end{cases}$$

On obtient donc $\tilde{f}' = f'$, les dérivées étant prises au sens des distributions. Par conséquent, il existe une constante $c \in \mathbb{R}$ telle que $f = \tilde{f} + c$ dans $\mathcal{D}'(]a,b[)$ (voir par exemple Brezis [3] ou Schwartz [10]), ce qui est équivalent à $f(x) = \tilde{f}(x) + c$ pour presque tous $x \in]a,b[$. Pour conclure, il suffit d'observer que la fonction $f := \tilde{f} + c$ satisfait les propriétés demandées dans l'énoncé du Théorème. □

Remarque A.1. Le lemme précédent donne en particulier le résultat classique suivant : Si $f \in W^{1,1}_{loc}(]a,b[)$, alors la fonction f est absolument continue sur tout ensemble compact inclus dans l'intervalle $]a,b[$.

Le théorème suivant est aussi classique, il peut être trouvé sous différentes formes dans e.g. [7, 9]. Avant de l'énoncer, on fait les conventions de notation suivantes : si $\Omega \subset \mathbb{R}^d$ est un ouvert et $x' \in \mathbb{R}^{d-1}$ alors $\Omega_{x'} \subset \mathbb{R}$ est l'ensemble défini par

$$\Omega_{x'} := \{x_d \ ; \ (x', x_d) \in \Omega\}.$$

Notons que $\Omega_{x'}$ est un sous-ensemble ouvert de $\mathbb{R}$. Si $f : \Omega \to \mathbb{R}^d$ est une fonction, on note $f(x',\cdot)$ la restriction de f à $\Omega_{x'}$ et l'on note $f'(x',\cdot)$ la dérivée de $f(x',\cdot)$ dans $\mathcal{D}'(\Omega_{x'})$ (lorsque cette dérivée existe). La notation $\partial_d f := \frac{\partial f}{\partial x_d}$ désigne la dérivée au sens des distributions dans $\mathcal{D}'(\Omega)$. Lorsque $f \in W^{1,1}_{loc}(\Omega)$, la notation $\partial_d f(x',\cdot)$ désigne la restriction d'un représentant de $\partial_d f$ à $\Omega_{x'}$.

Dans l'énoncé suivant, on se fixe un représentant pour f et un représentant pour $\partial_d f$.

Lemme A.2. *Soit Ω un ouvert de $\mathbb{R}^d$ et $f \in W^{1,1}_{loc}(\Omega)$. Alors il existe un ensemble $E \subset \mathbb{R}^{d-1}$ de $\mathcal{L}^{d-1}$-mesure nulle tel que*

$$f(x',\cdot) \in W^{1,1}_{loc}(\Omega_{x'}) \text{ et } f'(x',\cdot) = \frac{\partial f}{\partial x_d}(x',\cdot)$$

pour tout $x' \in \mathbb{R}^{d-1} \setminus E$.

Démonstration. On considère une suite d'ouverts U_m, $m \in \mathbb{N}$, tels que $U_m \Subset \Omega$, $U_m \subset U_{m+1}$, et $\Omega = \cup_m U_m$. Ceci entraîne que $\Omega_{x'} = \cup_m (U_m)_{x'}$ et $(U_m)_{x'} \subset (U_{m+1})_{x'}$ pour tout $x' \in \mathbb{R}^{d-1}$. Il suffit donc de montrer que pour tout ouvert $U \Subset \Omega$, il existe un ensemble $E_U \subset \mathbb{R}^{d-1}$ de $\mathcal{L}^{d-1}$-mesure nulle tel que

$$f(x',\cdot) \in W^{1,1}(U_{x'}) \text{ et } f'(x',\cdot) = \frac{\partial f}{\partial x_d}(x',\cdot) \text{ pour tout } x' \in \mathbb{R}^{d-1} \setminus E_U.$$

L'ensemble E apparaissant dans l'énoncé du lemme sera ensuite donné par

$$E := \cup_m E_{U_m}.$$

Soit $f \in W^{1,1}_{loc}(\Omega)$ et un ouvert $U \Subset \Omega$. Le théorème de Meyers-Serrin [8] (voir aussi [1, 3, 6]) affirme qu'il existe une suite $f_n \in C^1(U) \cap W^{1,1}(U)$ telle que f_n converge vers f dans $W^{1,1}(U)$ lorsque $n \to \infty$. En particulier,

$$\int_U (|f_n(x) - f(x)| + |\partial_d f_n(x) - \partial_d f(x)|)dx$$
$$= \int_{\mathbb{R}^{d-1}} \int_{U_{x'}} (|f_n(x) - f(x)| + |\partial_d f_n(x) - \partial_d f(x)|)dx_d dx' \to 0$$

lorsque $n \to \infty$. Quitte à extraire une sous-suite, il existe alors un ensemble $E_1 \subset \mathbb{R}^{d-1}$ de $\mathcal{L}^{d-1}$-mesure nulle tel que

$$\text{(A.1)} \quad \int_{U_{x'}} (|f_n(x',x_d) - f(x',x_d)| + |\partial_d f_n(x',x_d) - \partial_d f(x',x_d)|)dx_d \to 0$$

pour tout $x' \in \mathbb{R}^{d-1} \setminus E_1$ (par convention, cette dernière intégrale est réduite à zéro lorsque l'ensemble $U_{x'}$ est vide).

D'autre part, pour tout n, on a

$$\int_U (|f_n(x)| + |\partial_d f_n(x)|)dx = \int_{\mathbb{R}^{d-1}} \int_{U_{x'}} (|f_n(x)| + |\partial_d f_n(x)|)dx_d dx' < \infty.$$

Par conséquent, pour tout n, il existe un ensemble $E_2^n \subset \mathbb{R}^{d-1}$ de $\mathcal{L}^{d-1}$-mesure nulle tel que

$$\int_{U_{x'}} (|f_n(x',x_d)| + |\partial_d f_n(x',x_d)|)dx_d < \infty$$

pour tout $x' \in \mathbb{R}^{d-1} \setminus E_2^n$. Ceci entraîne que

$$\text{(A.2)} \quad f_n(x',\cdot) \in W^{1,1}(U_{x'}) \text{ pour tout } n \text{ et pour tout } x' \in \mathbb{R}^{d-1} \setminus E_2,$$

où $E_2 = \cup_n E_2^n$.

L'ensemble $E_U := E_1 \cup E_2$ est de $\mathcal{L}^{d-1}$-mesure nulle. De plus, les relations (A.1) et (A.2) ensemble impliquent que $f(x',\cdot)$ et $\partial_d f(x',\cdot)$ appartiennent à l'espace $L^1(U_{x'})$ et que l'on a les convergences

$$f_n(x',\cdot) \to f(x',\cdot) \text{ dans } L^1(U_{x'}) \text{ et} \tag{A.3}$$

$$\partial_d f_n(x',\cdot) \to \partial_d f(x',\cdot) \text{ dans } L^1(U_{x'}) \tag{A.4}$$

pour tout $x' \in E_U$. L'inégalité (A.1) et l'inégalité triangulaire montrent que la suite $(f_n(x',\cdot))$ est de Cauchy dans $W^{1,1}(U_{x'})$. Par conséquent, sa limite dans $L^1(U_{x'})$ appartient à l'espace $W^{1,1}(U_{x'})$, c'est-à-dire que $f(x',\cdot) \in W^{1,1}(U_{x'})$. Comme

$$f_n'(x',\cdot) \to f'(x',\cdot) \text{ dans } \mathcal{D}'(U_{x'}) \text{ lorsque } n \to \infty$$

(grâce à la relation (A.3)) et comme $f_n'(x',\cdot) = \partial_d f_n(x',\cdot)$, la relation (A.4) entraîne que $f'(x',\cdot) = \partial_d f(x',\cdot)$, c'est-à-dire que la dérivée au sens des distributions dans $\mathcal{D}'(U_{x'})$ de $f(x',\cdot)$ est donnée par $\partial_d f(x',\cdot)$.

□

Remarque A.2. En raisonnant comme dans l'Appendix du troisième chapitre (page 97), on peut démontrer le Lemme A.2 à l'aide du théorème de type Lebesgue-Besicovitch suivant : *Si $f \in L^1_{loc}(\mathbb{R}^d)$, alors il existe un ensemble $E \subset \mathbb{R}^{d-1}$ de $\mathcal{L}^{d-1}$-mesure nulle tel que*

$$f(\overline{x}',\cdot) \in L^1_{loc}(\mathbb{R}) \text{ et}$$
$$\lim_{\varepsilon\to 0} \frac{1}{\mathcal{L}^{d-1}(B(\overline{x}',\varepsilon))} \int_{B(\overline{x}',\varepsilon)} \int_U |f'(x',x_d) - f'(\overline{x}',x_d)|\,dx_d dx' = 0$$

pour tout $\overline{x}' \in \mathbb{R}^{d-1} \setminus E$ et pour tout ouvert borné $U \subset \mathbb{R}$. La preuve de ce résultat est analogue à celle du Lemme 3.1, page 68.

En combinant les deux lemmes précédentes, on obtient le résultat suivant :

Théorème A.1. *Soit ω un ouvert de $\mathbb{R}^{d-1}$, soit I un intervalle ouvert de $\mathbb{R}$ et soit $a \in I$. On se donne $f \in W^{1,\infty}(\omega \times I)$. Alors il existe un ensemble $E \subset \omega$ de $\mathcal{L}^{d-1}$-mesure nulle tel que*

$$f(x',\cdot) \in W^{1,\infty}(I) \text{ et}$$
$$f(x',x_d) = f(x',a) + \int_a^{x_d} \partial_d f(x',t)\,dt$$

pour tout $(x',x_d) \in (\omega \setminus E) \times I$, où f est le représentant continu de $f \in W^{1,\infty}(\omega \times I)$, et $\partial_d f$ est un représentant de la classe (notée aussi) $\partial_d f \in L^\infty(\omega \times I)$.

Les deux lemmes suivants sont nécessaires pour établir une formule de changement de variables (voir Théorème A.2) sous des hypothèses plus faibles que celles usuellement rencontrées dans la littérature (voir par exemple Brezis [3]). On rappelle que le Jacobien d'une application G est noté J_G.

Lemme A.3. *Soit D un ouvert de $\mathbb{R}^d$ et soit $G \in C^1(D;\mathbb{R}^d)$ une application telle que la $\mathcal{L}^d$-mesure de l'ensemble $\{x \in D; J_G(x) = 0\}$ s'annule. Alors $\mathcal{L}^d(G^{-1}(A)) = 0$ pour tout ensemble $A \subset \mathbb{R}^d$ tel que $\mathcal{L}^d(A) = 0$.*

Démonstration. La fonction $J_G : D \to \mathbb{R}$ définie ci-dessus étant continue (puisque $G \in C^1(D)$), l'ensemble

$$E := \{x \in D; J_G(x) \neq 0\}$$

est ouvert. Par conséquent, pour tout $x \in E$, il existe un voisinage ouvert V_x de x inclus dans E tel que l'application $G : V_x \to G(V_x)$ est un difféomorphisme de classe C^1 (grâce au théorème d'inversion locale, voir par exemple [4]). Grâce à la propriété de Lindelöf (voir par exemple Bourbaki [2]), on peut recouvrir l'ensemble E par un ensemble dénombrable de tels voisinages, c'est-à-dire qu'il existe les voisinages $V_n := V_{x_n}$, $n \in \mathbb{N}$, tels que $E = \cup_n V_n$.

Soit $A \subset \mathbb{R}^d$ un ensemble de $\mathcal{L}^d$-mesure nulle. Il existe alors un ensemble borelien B de $\mathcal{L}^d$-mesure nulle tel que $A \subset B$. Comme l'application G est continue, l'ensemble $G^{-1}(B)$ est aussi un ensemble borelien. Avec la notation

$$B_n := V_n \cap G^{-1}(B),$$

l'on a $G(B_n) \subset G(G^{-1}(B)) \subset B$, d'où $\mathcal{L}^d(G(B_n)) = 0$. D'autre part, comme le sous-ensemble B_n de V_n est mesurable et $G : V_n \to G(V_n)$ est un difféomorphisme de classe C^1, la formule classique de changement de variable sous l'intégrale montre que

$$\mathcal{L}^d(G(B_n)) = \int_{B_n} |J_G(x)| dx.$$

Comme $|J_G(x)| \neq 0$ pour tout $x \in B_n$, la relation précédente entraîne que $\mathcal{L}^d(B_n) = 0$.

Par conséquent,

$$\begin{aligned} G^{-1}(A) &= \left(E \cap G^{-1}(A)\right) \cup \left(E^c \cap G^{-1}(A)\right) \\ &= (\cup_n\{V_n \cap G^{-1}(A)\}) \cup \left(E^c \cap G^{-1}(A)\right) \\ &\subset (\cup_n B_n) \cup \left(E^c \cap G^{-1}(A)\right), \end{aligned}$$

où $E^c := D \setminus E$. Comme $\mathcal{L}^d(B_n) = 0$ et $\mathcal{L}^d(E^c) = \mathcal{L}^d(\{x \in D; J_G(x) = 0\}) = 0$ (voir les hypothèses du théorème), on en déduit que $\mathcal{L}^d(G^{-1}(A)) = 0$. □

Remarques A.3. 1. Le Lemme précédent peut également être obtenu comme une conséquence de la formule de l'aire établie dans Evans et Gariepy [5] (voir le Théorème 1, p. 96). Cette formule permet d'obtenir le Lemme précédent sous l'hypothèse affaiblie que l'application G est seulement lipschitzienne. Dans ce cas, le Jacobien de G est bien défini presque partout dans D grâce au théorème de Rademacher (voir [5], Théorème 2, p. 81).

2. Une conséquence du lemme précédent est que l'image réciproque par G d'un ensemble mesurable est un ensemble mesurable, c'est-à-dire que

$G^{-1}(E)$ est $\mathcal{L}^d$-mesurable pour tout sous-ensemble $\mathcal{L}^d$-mesurable $E \subset \mathbb{R}^d$. Ainsi, si $f : \mathbb{R}^d \to \mathbb{R}$ est une fonction $\mathcal{L}^d$-mesurable, alors la fonction $f \circ G : D \to \mathbb{R}$ est $\mathcal{L}^d$-mesurable.

Lemme A.4. *Soit Ω un ouvert de $\mathbb{R}^d$ et soit $f \in W^{1,\infty}(\Omega)$. Alors pour tout ouvert $\omega \Subset \Omega$, il existe une suite de fonctions $(\overline{f}_n) \subset \mathcal{D}(\mathbb{R}^d)$, $n \in \mathbb{N}$, telle que leurs restrictions à ω, notées $f_n := \overline{f}_n|_\omega$, satisfont les trois conditions suivantes :*

$$f_n \text{ converge vers } f \text{ dans } W^{1,p}(\omega) \text{ pour tout } 1 \leq p < \infty,$$
$$\|f_n\|_{L^\infty(\omega)} \leq \|f\|_{L^\infty(\Omega)}, \text{ pour tout } n \in \mathbb{N}$$
$$\|\partial_i f_n\|_{L^\infty(\omega)} \leq \|\partial_i f\|_{L^\infty(\Omega)}, \text{ pour tout } n \in \mathbb{N} \text{ et } i \in \{1, 2, ..., d\}.$$

Démonstration. Soit $\omega \Subset \Omega$ un ouvert, soit $n_0 \in \mathbb{N}$ tel que

$$\frac{1}{n_0} < d(\omega, \Omega^c),$$

et soit

$$\omega_0 = \omega + B(0, \frac{1}{n_0}) := \left\{ x + y;\ x \in \omega,\ y \in B(0, \frac{1}{n_0}) \right\}.$$

L'ensemble ω_0 est ouvert et satisfait $\omega_0 \Subset \Omega$.

On définit ensuite la fonction $\overline{f} : \mathbb{R}^d \to \mathbb{R}$ par

$$\overline{f}(x) := \begin{cases} f(x) \text{ si } x \in \omega_0, \\ 0 \text{ si } x \in \mathbb{R}^d \setminus \omega_0, \end{cases}$$

et les fonctions $\overline{\partial_i f} : \mathbb{R}^d \to \mathbb{R}$, $i = 1, 2, ..., d$, par

$$\overline{\partial_i f}(x) := \begin{cases} \partial_i f(x) \text{ si } x \in \omega_0, \\ 0 \text{ si } x \in \mathbb{R}^d \setminus \omega_0. \end{cases}$$

Clairement, les fonctions $\overline{f}$ et $\overline{\partial_i f}$, $i = 1, 2, ..., d$, appartiennent à l'espace $L^p(\mathbb{R}^d)$ pour tout $p \in [1, \infty]$.

On se donne également une suite de fonctions régularisantes $\rho_n \in \mathcal{D}(\mathbb{R}^d)$, où $n \in \mathbb{N}$ et $n \geq n_0$, telle que

$$\rho_n \geq 0 \text{ dans } \mathbb{R}^d, \quad \operatorname{supp} \rho_n \subset B(0, \frac{1}{n}) \text{ et } \int_{\mathbb{R}^d} \rho_n(x) dx = 1.$$

Pour tout $n \geq n_0$, on définit les fonctions $\overline{f}_n := \rho_n * \overline{f} : \mathbb{R}^d \to \mathbb{R}$ et $(\overline{\partial_i f})_n := \rho_n * \overline{\partial_i f} : \mathbb{R}^d \to \mathbb{R}$ par

$$\overline{f}_n(x) = \int_{\mathbb{R}^d} \overline{f}(y) \rho_n(x - y) dy \text{ et } (\overline{\partial_i f})_n(x) = \int_{\mathbb{R}^d} \overline{\partial_i f}(y) \rho_n(x - y) dy$$

pour tout $x \in \mathbb{R}^d$. Les fonctions $\overline{f}_n$ et $(\overline{\partial_i f})_n$ appartiennent alors à l'espace $\mathcal{D}(\mathbb{R}^d)$ et satisfont (voir par exemple [3])

$$\text{(A.5)} \qquad \begin{aligned} &\overline{f}_n \to \overline{f} \text{ dans } L^p(\mathbb{R}^d) \text{ lorsque } n \to \infty \\ &(\overline{\partial_i f})_n \to \overline{\partial_i f} \text{ dans } L^p(\mathbb{R}^d) \text{ lorsque } n \to \infty \end{aligned}$$

pour tout $1 \le p < \infty$, ainsi que

(A.6)

$$|\overline{f}_n(x)| \le \|\overline{f}\|_{L^\infty(\mathbb{R}^d)}\|\rho_n\|_{L^1(\mathbb{R}^d)} = \|\overline{f}\|_{L^\infty(\mathbb{R}^d)} = \|f\|_{L^\infty(\omega_0)} \le \|f\|_{L^\infty(\Omega)},$$
$$|(\overline{\partial_i f})_n(x)| \le \|\overline{\partial_i f}\|_{L^\infty(\mathbb{R}^d)}\|\rho_n\|_{L^1(\mathbb{R}^d)} = \|\partial_i f\|_{L^\infty(\omega_0)} \le \|\partial_i f\|_{L^\infty(\Omega)}$$

pour tout $x \in \mathbb{R}^d$ et tout $n \ge n_0$.

Montrons maintenant que les restrictions de $\overline{f}_n$ à ω, notées $f_n := \overline{f}_n|_\omega$, satisfont les propriétés requises par le théorème. On commence par montrer que la dérivée au sens des distributions de f_n est donnée par $\partial_i f_n = (\overline{\partial_i f})_n|_\omega$.

Soit $\varphi \in \mathcal{D}(\omega)$. Alors

$$\begin{aligned}
\int_\omega f_n(x)\partial_i\varphi(x)dx &= \int_{\mathbb{R}^d} \overline{f}_n(x)\partial_i\varphi(x)dx \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \overline{f}(y)\rho_n(x-y)dy\right)\partial_i\varphi(x)dx \\
&= \int_{\mathbb{R}^d} \overline{f}(y)\left(\int_{\mathbb{R}^d} \partial_i\varphi(x)\rho_n(x-y)dx\right)dy \\
&= \int_{\mathbb{R}^d} \overline{f}(y)\partial_i(\check{\rho}_n * \varphi)(y)dy,
\end{aligned}$$

où $\check{\rho}_n(t) = \rho_n(-t)$ pour tout $t \in \mathbb{R}^d$. Comme

$$\operatorname{supp}(\check{\rho}_n * \varphi) \subset \operatorname{supp}\varphi + \operatorname{supp}\check{\rho}_n \subset \operatorname{supp}\varphi + B(0, \frac{1}{n_0}) \subset \omega_0$$

pour tout $n \ge n_0$, la relation précédente entraîne que (en appliquant le théorème de Fubini comme ci-dessus)

$$\begin{aligned}
\int_\omega f_n\partial_i\varphi dx &= \int_{\omega_0} \overline{f}\partial_i(\check{\rho}_n * \varphi)dx = \int_{\omega_0} f\partial_i(\check{\rho}_n * \varphi)dx \\
&= \int_\Omega f\partial_i(\check{\rho}_n * \varphi)dx = -\int_\Omega \partial_i f(\check{\rho}_n * \varphi)dx \\
&= -\int_{\omega_0} \partial_i f(\check{\rho}_n * \varphi)dx = -\int_{\mathbb{R}^d} \overline{\partial_i f}(\check{\rho}_n * \varphi)dx \\
&= -\int_{\mathbb{R}^d} \rho_n * \overline{\partial_i f}\varphi dx.
\end{aligned}$$

Par conséquent,

$$\int_\omega f_n\partial_i\varphi dx = -\int_\omega \left(\overline{\partial_i f}\right)_n \varphi dx$$

pour tout $\varphi \in D(\omega)$, ce qui entraîne que

$$\partial_i f_n = (\overline{\partial_i f})_n|_\omega \text{ dans } \mathcal{D}'(\omega).$$

Notons que $f_n \in W^{1,p}(\omega)$ car $\overline{f}_n \in \mathcal{D}(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$. Ensuite, les relations (A.5) montrent que

$$\|f_n - f\|_{L^p(\omega)} \leq \|\overline{f}_n - \overline{f}\|_{L^p(\mathbb{R}^d)} \to 0 \text{ lorsque } n \to \infty,$$
$$\|\partial_i f_n - \partial_i f\|_{L^p(\omega)} \leq \|(\overline{\partial_i f})_n - \overline{\partial_i f}\|_{L^p(\mathbb{R}^d)} \to 0 \text{ lorsque } n \to \infty$$

pour tout $1 \leq p < \infty$, et les inégalités (A.6) montrent que

$$\|f_n\|_{L^\infty(\omega)} \leq \|f\|_{L^\infty(\Omega)},$$
$$\|\partial_i f_n\|_{L^\infty(\omega)}| \leq \|\partial_i f\|_{L^\infty(\Omega)}$$

pour tout $n \geq n_0$. La preuve est complète. □

Remarque A.4. Comme la fonction $\overline{f}$ est continue sur ω_0, on obtient aussi la convergence $f_n \to f$ dans $L^\infty(\omega)$ lorsque $n \to \infty$.

Le résultat suivant fournit une formule de changement de variables sous des hypothèses plus faibles que celles trouvées habituellement dans la littérature (voir par exemple Brezis [3], Proposition IX.6). On rappelle que le Jacobien d'une application G est noté J_G.

Dans l'énoncé suivant, il est à remarquer que l'application G peut être ni injective, ni surjective.

Théorème A.2. *Soit Ω et $\widetilde{\Omega}$ deux ouverts de $\mathbb{R}^d$, soit $f \in W^{1,\infty}(\Omega)$, et soit $G \in C^1(\widetilde{\Omega}; \mathbb{R}^d)$ une application telle que*

$$G(\widetilde{\Omega}) \subset \Omega, \ \nabla G \in L^\infty(\widetilde{\Omega}; \mathbb{M}^d) \ \text{et} \ \mathcal{L}^d(\{x \in \widetilde{\Omega} \ ; \ J_G(x) = 0\}) = 0.$$

Alors $f \circ G \in W^{1,\infty}(\widetilde{\Omega})$ et

$$\partial_i(f \circ G) = \sum_{j=1}^{d} (\partial_j f \circ G) \partial_i G_j \ \text{pour tout} \ i \in \{1, \ldots, d\}.$$

Démonstration. Le Lemme A.3 montre que $\mathcal{L}^d(G^{-1}(A)) = 0$ pour tout $A \subset \Omega$ de mesure nulle. Ceci entraîne que les fonctions $(f \circ G)$ et $((\partial_j f \circ G)\partial_i G_j)$ sont bien définies dans $\widetilde{\Omega}$ (c'est-à-dire qu'elle sont indépendantes, modulo la relation d'égalité presque partout, du choix des représentants de classes de f et de $\partial_i f$), mesurables, et appartiennent à l'espace $L^\infty(\widetilde{\Omega})$. Il reste à montrer que les dérivées au sens des distributions de $f \circ G$ sont données par

$$\partial_i(f \circ G) = \sum_{j=1}^{d} (\partial_j f \circ G) \partial_i G_j \text{ pour tout } i \in \{1, \ldots, d\}.$$

Soit $\varphi \in \mathcal{D}(\widetilde{\Omega})$. Comme l'application G est continue et l'ensemble $\operatorname{supp} \varphi$ est compact, l'image $G(\operatorname{supp} \varphi)$ est un compact inclus dans Ω. Il existe alors un ouvert ω de $\mathbb{R}^d$ tel que $G(\operatorname{supp} \varphi) \subset \omega \Subset \Omega$. On a ainsi la relation $\operatorname{supp} \varphi \subset G^{-1}(G(\operatorname{supp} \varphi)) \subset G^{-1}(\omega)$, donc le support de φ est inclus dans l'ensemble $\widetilde{\omega} := G^{-1}(\omega)$. Notons que l'ensemble $\widetilde{\omega}$ est ouvert et que $\widetilde{\omega} \subset \widetilde{\Omega}$.

Comme $\omega \Subset \Omega$, le Lemme A.4 montre qu'il existe une suite de fonctions $\overline{f}_n \in \mathcal{D}(\mathbb{R}^d)$, $n \in \mathbb{N}$, telle que les restrictions $f_n := \overline{f}_n|_\omega$ appartiennent à l'espace $W^{1,1}(\omega)$ et satisfont les relations :

$$f_n \to f \text{ dans } W^{1,1}(\omega) \text{ lorsque } n \to \infty,$$
$$\|f_n\|_{L^\infty(\omega)} \leq \|f\|_{L^\infty(\Omega)} \text{ pour tout } n \in \mathbb{N},$$
$$\|\partial_j f_n\|_{L^\infty(\omega)} \leq \|\partial_j f\|_{L^\infty(\Omega)} \text{ pour tout } n \in \mathbb{N} \text{ et } j \in \{1, 2, ..., d\}.$$

Les fonctions composées $f_n \circ G, (\partial_j f_n) \circ G : \widetilde{\omega} \to \mathbb{R}$ sont clairement de classe C^1 dans $\widetilde{\omega}$. De plus, les deux inégalités ci-dessus montrent que ces fonctions satisfont

$$\|f_n \circ G\|_{L^\infty(\widetilde{\omega})} \leq \|f\|_{L^\infty(\Omega)} \text{ et}$$
$$\|\partial_j f_n \circ G\|_{L^\infty(\widetilde{\omega})} \leq \|\partial_j f\|_{L^\infty(\Omega)}, j \in \{1, 2, ..., d\},$$

pour tout $n \in \mathbb{N}$. Comme $f_n \to f$ dans $W^{1,1}(\omega)$, le Lemme A.3 et le théorème de la convergence dominée montrent que, quitte à extraire une sous-suite, l'on a

$$\begin{aligned} &f_n \circ G \to f \circ G \text{ dans } L^1(\widetilde{\omega}), \\ &\partial_j f_n \circ G \to \partial_j f \circ G \text{ dans } L^1(\widetilde{\omega}), j = 1, ..., d \end{aligned} \tag{A.7}$$

lorsque $n \to \infty$.

Avec l'ouvert $\widetilde{\omega}$ et les fonctions f_n construits ci-dessus, l'on a (on utilise la règle de la sommation par rapport aux indices répétés)

$$\int_{\widetilde{\omega}} (f_n \circ G)\partial_i \varphi dx = -\int_{\widetilde{\omega}} ((\partial_j f_n \circ G)\partial_i G_j)\, \varphi dx. \tag{A.8}$$

Par passage à la limite $n \to \infty$, l'on obtient alors grâce aux relations (A.7) que

$$\int_{\widetilde{\omega}} (f \circ G)\partial_i \varphi dx = -\int_{\widetilde{\omega}} ((\partial_j f \circ G)\partial_i G_j) \varphi dx.$$

Comme le support de φ est inclus dans $\widetilde{\omega}$, il s'ensuit que

$$\int_{\widetilde{\Omega}} (f \circ G)\partial_i \varphi dx = -\int_{\widetilde{\Omega}} ((\partial_j f \circ G)\partial_i G_j)\, \varphi dx,$$

donc que $\partial_i (f \circ G) = (\partial_j f \circ G)\partial_i G_j$. □

Remarques A.5. 1. Le théorème précédent reste vrai si l'on remplace l'hypothèse que $G \in C^1(\widetilde{\Omega}; \mathbb{R}^d)$ par l'hypothèse affablie que l'application G est lipschitzienne sur $\widetilde{\Omega}$. Pour le montrer, il suffit d'utiliser la Remarque A.3 (donc d'appliquer le Lemme A.3 pour une application G lipschitzienne) et la relation (A.8) ci-dessus lorsque G est seulement lipschitzien. Cette relation peut être établie par un passage à la limite en utilisant l'analogue du Lemme A.4 pour une fonction à valeurs vectorielles.

2. L'hypothèse que $f \in W^{1,\infty}(\Omega)$ ne peut être remplacée dans l'énoncé du Théorème précédent par l'hypothèse plus faible que $f \in W^{1,p}(\Omega)$ avec

$1 \leq p < +\infty$, même lorsque G est un C^1-difféomorphisme. Pour le voir, il suffit de considérer l'exemple suivant :

$$\Omega := B(0,1) \setminus \{(x,0);\ x \geq 0\} \subset \mathbb{R}^2,$$
$$\widetilde{\Omega} :=]0, 2\pi[\times]0,1[,$$

et l'application $G : \widetilde{\Omega} \to \Omega$ est définie par

$$G(\varphi, r) := (r\cos\varphi, r\sin\varphi) \text{ pour tout } (\varphi, r) \in \widetilde{\Omega}.$$

Notons que cette application est un C^1-difféomorphisme tel que

$$\nabla G \in L^\infty(\widetilde{\Omega}; \mathbb{M}^2) \text{ et } \{x \in \widetilde{\Omega}\ ;\ J_G(x) = 0\} = \emptyset.$$

Ensuite, si $1 < p < \infty$, on définit la fonction $f : \Omega \to \mathbb{R}$ par

$$f(x) := |x|^{1-\frac{1}{p}},$$

et, si $p = 1$, on définit la fonction $f : \Omega \to \mathbb{R}$ par

$$f(x) := |x|^{-\frac{1}{2}}.$$

Un calcul élémentaire montre alors que $f \in W^{1,p}(\Omega)$ et que $f \circ G \notin W^{1,p}(\widetilde{\Omega})$.

Le Théorème précédent permet d'établir un résultat analogue à celui du Théorème A.1 en coordonnées sphériques. Plus précisement, l'on a :

Corollaire A.1. *Soit $\Omega := B(x^0, R)$ une boule ouverte de $\mathbb{R}^d$, où $x^0 \in \mathbb{R}^d$ et R est un réel > 0. Pour tout $z \in \partial\Omega$, on définit une paramétrisation du segment $[x^0, z]$ par l'application (dépendant de z)*

$$\boldsymbol{\gamma}^z = (\gamma_1^z, \gamma_2^z, ..., \gamma_d^z) : [0,1] \to \mathbb{R}^d, \text{ où } \boldsymbol{\gamma}^z(t) := x^0 + t(z - x^0).$$

Soit $f \in W^{1,\infty}(\Omega)$. Alors pour presque tout $z \in \partial\Omega$ (par rapport à la mesure surfacique sur $\partial\Omega$), l'on a

$$f \circ \boldsymbol{\gamma}^z \in W^{1,\infty}(]0,1[),$$
$$\|(\partial_j f) \circ \boldsymbol{\gamma}^z\|_{L^\infty(]0,1[)} \leq \|\partial_j f\|_{L^\infty(\Omega)} \text{ pour tout } j \in \{1, 2, ..., d\}, \text{ et}$$
$$f(\boldsymbol{\gamma}^z(t)) = f(\boldsymbol{\gamma}^z(0)) + \int_0^t \partial_j f(\boldsymbol{\gamma}^z(s))(\gamma_j^z)'(s)ds \tag{A.9}$$

pour tout $t \in [0,1]$, où f est le représentant continu de $f \in W^{1,\infty}(\Omega)$ et $\partial_j f$, $j = 1, 2, ..., d$, est un représentant de $\partial_j f \in L^\infty(\Omega)$.

Démonstration. On définit l'application $G = (G_1, G_2, ..., G_d) : \mathbb{R}^d \to \mathbb{R}^d$ par

$$G(\varphi_1, ..., \varphi_{d-1}, r) = (x_1, x_2, ..., x_d),$$

où

$$\begin{aligned}
x_1 &= x_1^0 + r\cos\varphi_1 ... \cos\varphi_{d-3}\cos\varphi_{d-2}\cos\varphi_{d-1},\\
x_2 &= x_2^0 + r\cos\varphi_1 ... \cos\varphi_{d-3}\cos\varphi_{d-2}\sin\varphi_{d-1},\\
x_3 &= x_3^0 + r\cos\varphi_1 ... \cos\varphi_{d-3}\sin\varphi_{d-2},\\
&\vdots\\
x_d &= x_d^0 + r\sin\varphi_1.
\end{aligned}$$

Soit $\widetilde{\Omega} := \omega\times]-R,R[\subset \mathbb{R}^d$, où $\omega := \mathbb{R}^{d-1}$. Un point courant de $\widetilde{\Omega}$ est noté (φ, r), où $r\in]-R,R[$ et $\varphi = (\varphi_1,\varphi_2,...,\varphi_{d-1})\in\omega$, et un point courant de Ω est noté $x = (x_1, x_2, ..., x_d)$.

Clairement, $G\in C^1(\widetilde{\Omega};\mathbb{R}^d)$, $G(\widetilde{\Omega})\subset\Omega$, et $\nabla G\in L^\infty(\widetilde{\Omega};\mathbb{M}^d)$. De plus,

$$J_G(\varphi,r) = (-1)^{d(d-1)/2} r^{d-1}(\cos\varphi_1)^{d-2}(\cos\varphi_2)^{d-3}...(\cos\varphi_{d-2}),$$

ce qui montre que la $\mathcal{L}^d$-mesure de l'ensemble $\{x\in\widetilde{\Omega}\ ;\ J_G(x)=0\}$ s'annule.

Soit $f\in W^{1,\infty}(\Omega)$. Le Théorème A.2 montre que $f\circ G\in W^{1,\infty}(\widetilde{\Omega})$ et que $\partial_i(f\circ G) = (\partial_j f\circ G)\partial_i G_j$. Le Théorème A.1 appliqué à la fonction $f\circ G\in W^{1,\infty}(\omega\times]-R,R[)$ montre qu'il existe un ensemble $E\subset\omega$ de $\mathcal{L}^{d-1}$-mesure nulle tel que

$$(f\circ G)(\varphi,\cdot)\in W^{1,\infty}(]-R,R[) \text{ et} \tag{A.10}$$

$$(f\circ G)(\varphi,r) = (f\circ G)(\varphi,0) + \int_0^r (\partial_j f\circ G)(\frac{\partial G_j}{\partial r})(\varphi,t)dt \tag{A.11}$$

pour tout $(\varphi,r)\in(\omega\setminus E)\times]-R,R[$.

Avec les notations $x := G(\varphi,r)$ et $z := G(\varphi,R)$, l'on a

$$G(\varphi,t) = x^0 + t\left(\frac{z-x^0}{R}\right).$$

Alors la relation (A.11) montre que

$$f(x) = f(x^0) + \int_0^r \partial_j f(x^0 + t\frac{z-x^0}{R})\frac{z_j - x_j^0}{R}dt \tag{A.12}$$

pour tout $z\in G((\omega\setminus E)\times\{R\})$ et $r\in]-R,R[$. Comme

$$\begin{aligned}\partial\Omega\setminus G(E\times\{R\}) &= G(\omega\times\{R\})\setminus G(E\times\{R\})\\ &\subset G((\omega\setminus E)\times\{R\}),\end{aligned} \tag{A.13}$$

la formule (A.12) est valable pour tout $z\in\partial\Omega\setminus F$ et $r\in]-R,R[$, où

$$F := G(E\times\{R\}).$$

Par conséquent,

$$f(x) = f(x^0) + \int_0^{r/R}\partial_j f(x^0 + s(z-x^0))(z_j - x_j^0)ds$$

pour tout $(z, r/R)\in(\partial\Omega\setminus F)\times]-1,1[$

Comme $x^0+s(z-x^0)=\boldsymbol{\gamma}^z(s)$ pour tout $s\in[0,1]$, la relation précédente combinée avec (A.10) donne en particulier

$$\begin{aligned}&f\circ\boldsymbol{\gamma}^z\in W^{1,\infty}(]0,1[)\text{ et}\\ &f(\boldsymbol{\gamma}^z(t))=f(\boldsymbol{\gamma}^z(0))+\int_0^t\partial_jf(\boldsymbol{\gamma}^z(s))(\gamma_j^z)'(s)ds\end{aligned}\tag{A.14}$$

pour tout $(z,t)\in(\partial\Omega\setminus F)\times[0,1]$ (l'égalité en $t=1$ est obtenue grâce à la continuité de la fonction f sur $\overline{\Omega}$). De plus, la mesure surfacique de l'ensemble F s'annule (car la $\mathcal{L}^{d-1}$-mesure de l'ensemble E s'annule).

D'autre part, comme l'application $(\partial_jf)\circ G$ appartient à l'espace $L^\infty(\widetilde{\Omega})$, il existe un ensemble E_j de $\mathcal{L}^{d-1}$-mesure nulle tel que

$$\|((\partial_jf)\circ G)(\varphi,\cdot)\|_{L^\infty(]-R,R[)}\le\|(\partial_jf)\circ G\|_{L^\infty(\widetilde{\Omega})}\tag{A.15}$$

pour tout $\varphi\in\omega\setminus E_j$. Comme $G(\varphi,t)=\boldsymbol{\gamma}^z(t/R)$, cette relation entraîne que

$$\|(\partial_jf)\circ\boldsymbol{\gamma}^z\|_{L^\infty(]0,1[)}\le\|\partial_jf\|_{L^\infty(\Omega)}\tag{A.16}$$

pour tout $z\in G((\omega\setminus E_j)\times\{R\})$. Grâce à la relation A.13, la relation précédente a lieu a fortiori pour tout $z\in\partial\Omega\setminus F_j$, où

$$F_j:=G(E_j\times\{R\}).$$

Notons que la mesure surfacique de l'ensemble F_j s'annule car la $\mathcal{L}^{d-1}$-mesure de l'ensemble E_j s'annule.

Ainsi, les relations A.14 et A.16 sont satisfaites pour tout $t\in[0,1]$, pour tout $z\in\partial\Omega\setminus\widetilde{F}$, et pour tout j, où $\widetilde{F}:=F\cup(\cup_jF_j)$. La preuve est complète. □

Remarques A.6. 1. Sous les hypothèses du corollaire précédent, l'on a également l'égalité

$$f(x)=f(x^0)+\int_0^1\partial_jf(x^0+s(x-x^0))(x_j-x_j^0)ds\tag{A.17}$$

pour tout $z\in\partial\Omega\setminus\widetilde{F}$ (donc pour presque tout $z\in\partial\Omega$) et pour tout x appartenant au segment $[x^0,z]$. Pour le montrer, il suffit de faire le changement de variable $t=rs$ dans l'intégrale apparaissant dans la formule (A.12) et d'utiliser la relation

$$\frac{z-x^0}{R}=\frac{x-x^0}{r}.$$

2. En appliquant les formules (A.9) et (A.17) pour chaque composante, on observe que celles-ci restent valables pour des fonctions f à valeurs matricielles. C'est sous cette forme matricielle que ces formules sont utilisées dans les troisième et quatrième chapitres.

3. Le Théorème 3.2 est une conséquence du Théorème 3.1

Les notations et les conventions sont celles du troisième chapitre. Rappellons d'abord l'énoncé du Théorème 3.1 du troisième chapitre (voir le Théorème A.3 ci-dessous) et du Théorème 3.2 du troisième chapitre (voir le Théorème A.4 ci-dessous) :

Théorème A.3. *Soit Ω un ouvert connexe et simplement connexe de $\mathbb{R}^d$. On se donne deux champs de matrices $A_\alpha \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^l)$ et $B_\alpha \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$ tels que*

$$\partial_\alpha A_\beta + A_\alpha A_\beta = \partial_\beta A_\alpha + A_\beta A_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^l),$$
$$\partial_\alpha B_\beta + B_\alpha A_\beta = \partial_\beta B_\alpha + B_\beta A_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}).$$

Soient donnés un point $x^0 \in \Omega$ et une matrice $Y^0 \in \mathbb{M}^{q,l}$. Alors le système

$$\partial_\alpha Y = Y A_\alpha + B_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}),$$
$$Y(x^0) = Y^0$$

admet une solution unique dans $W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$.

Théorème A.4. *Soit Ω un ouvert connexe et simplement connexe de $\mathbb{R}^d$. On se donne trois champs de matrices $A_\alpha \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^l)$, $B_\alpha \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$ et $C_\alpha \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^q)$ tels que*

$$\partial_\alpha A_\beta + A_\alpha A_\beta = \partial_\beta A_\alpha + A_\beta A_\alpha, \tag{A.18}$$
$$\partial_\alpha C_\beta + C_\beta C_\alpha = \partial_\beta C_\alpha + C_\alpha C_\beta \tag{A.19}$$
$$\partial_\alpha B_\beta + B_\alpha A_\beta + C_\beta B_\alpha = \partial_\beta B_\alpha + B_\beta A_\alpha + C_\alpha B_\beta. \tag{A.20}$$

Soient donnés un point $x^0 \in \Omega$ et une matrice $Y^0 \in \mathbb{M}^{q,l}$. Alors le système

$$\left.\begin{aligned} &\partial_\alpha Y = Y A_\alpha + C_\alpha Y + B_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}), \\ &Y(x^0) = Y^0 \end{aligned}\right. \tag{A.21}$$

admet une solution unique dans $W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$.

Le Théorème A.3 a été démontré au troisième chapitre. Le Théorème A.4 peut être démontré par une méthode analogue, mais cette démonstration n'a pas été détaillée. Le but de cette section est d'établir le Théorème A.4 comme une conséquence du Théorème A.3. Pour ce faire, on a besoin des deux lemmes suivants.

Lemme A.5. *Soit Ω un ouvert de $\mathbb{R}^d$. On se donne un champ de matrices $X \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^l)$ tel que la matrice $X(x)$ soit inversible pour tout $x \in \Omega$. Alors le champ de matrices $X^{-1} : \Omega \to \mathbb{M}^l$, défini par $X^{-1}(x) := (X(x))^{-1}$ pour tout $x \in \Omega$, appartient à l'ensemble $W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^l)$ et satisfait la relation*

$$\partial_\alpha(X^{-1}) = -X^{-1}(\partial_\alpha X)X^{-1}.$$

Démonstration. Comme $X \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^l)$, il admet un représentant continu $X \in C^0(\Omega; \mathbb{M}^l)$ par les inclusions de Sobolev. D'autre part, la matrice $X(x)$ est inversible pour tout $x \in \Omega$. Ceci entraîne que, pour tout ouvert ω tel que $\omega \Subset \Omega$, il existe une constante $\varepsilon := \varepsilon(\omega) > 0$ telle que

$$\det X(x) \geq \varepsilon \text{ pour tout } x \in \overline{\omega}.$$

Comme $X \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^l)$, la formule donnant la matrice inverse montre alors que le champ de matrices X^{-1} appartient à l'espace $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^l)$.

Comme les champs X et X^{-1} appartiennent à l'espace $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^l)$, l'on peut dériver l'équation $X(X^{-1}) = I_l$ au sens des distributions ; l'on obtient ainsi que

$$(\partial_\alpha X)X^{-1} + X\partial_\alpha(X^{-1}) = 0 \in \mathbb{M}^l,$$

donc que

$$\partial_\alpha(X^{-1}) = -X^{-1}(\partial_\alpha X)X^{-1}.$$

□

Lemme A.6. *Soit Ω un ouvert connexe de $\mathbb{R}^d$ et soit $X \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^l)$ une solution du système*

$$\partial_\alpha X = XA_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^l), \alpha = 1, 2, ..., d. \tag{A.22}$$

On suppose qu'il existe un point $x^0 \in \Omega$ tel que la matrice $X(x^0)$ soit inversible. Alors la matrice $X(x)$ est inversible en tout point $x \in \Omega$.

Démonstration. Supposons par l'absurde qu'il existe un point $x^1 \in \Omega$ tel que la matrice $X(x^1)$ ne soit pas inversible. Par conséquent, la matrice transposée $X(x^1)^T$ admet une valeur propre nulle, ce qui entraîne l'existence d'un vecteur non nul $\mathbf{v} \in \mathbb{R}^l$ tel que

$$X(x^1)^T\mathbf{v} = \mathbf{0} \in \mathbb{R}^l.$$

Comme le champ X satisfait le système (A.22), le champ de matrices $(\mathbf{v}^T X) : \Omega \to \mathbb{M}^{1,l}$, défini par $(\mathbf{v}^T X)(x) := \mathbf{v}^T X(x)$, satisfait le système

$$\partial_\alpha(\mathbf{v}^T X) = (\mathbf{v}^T X)A_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{1,l}),$$
$$(\mathbf{v}^T X)(x^1) = 0 \in \mathbb{M}^{1,l}.$$

Or le champ de matrices identiquement nul est aussi solution de ce système. Le résultat d'unicité fournit par le Théorème A.3 appliqué au système ci-dessus montre alors que le champ de matrices $(\mathbf{v}^T X)$ est identiquement nul dans Ω. En particulier, $(\mathbf{v}^T X)(x^0) = 0$. Or la matrice $X(x^0)$ est inversible par l'hypothèse, ce qui entraîne que $\mathbf{v}^T = 0$. Or cette égalité contredit le fait que $\mathbf{v}$ est un vecteur propre associé à la matrice $X^T(x^1)$. On conclut alors que la matrice $X(x)$ est inversible pour tout $x \in \Omega$. □

On est maintenant en mesure de montrer que le Théorème A.4 est une conséquence du Théorème A.3. La preuve en est comme suit :

Démonstration. On construit une solution du système (A.21) de la manière suivante : On résout d'abord les systèmes

$$\begin{aligned} &\partial_\alpha X = XA_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^l), \\ &X(x^0) = I_l, \end{aligned} \tag{A.23}$$

où I_l est la matrice identité dans $\mathbb{M}^l$, et

$$\begin{aligned} &\partial_\alpha Z = C_\alpha Z \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^q), \\ &Z(x^0) = I_q, \end{aligned} \tag{A.24}$$

où I_q est la matrice identité dans $\mathbb{M}^q$.

On résout ensuite le système

$$\begin{aligned} &\partial_\alpha V = Z^{-1} B_\alpha X^{-1} \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}), \\ &V(x^0) = Y^0, \end{aligned} \tag{A.25}$$

où X et Z sont les solutions des deux systèmes ci-dessus, qui sont inversibles en tout point de Ω grâce au Lemme A.6.

Finalement, l'on définit le champ de matrices

$$Y := ZVX \tag{A.26}$$

et l'on montre qu'il est l'unique solution du système (A.21).

La preuve est en quatre étapes.

(i) *Le système (A.23) admet une solution unique $X \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^l)$ et le système (A.24) admet une solution unique $Z \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^q)$. De plus, les matrices $X(x)$ et $Z(x)$ sont inversibles en tout point x de Ω.*

Le Théorème A.3 et la relation (A.18) assurent l'existence d'une solution unique X du système (A.23). De plus, la matrice $X(x)$ est inversible pour tout $x \in \Omega$ grâce au Lemme A.6.

On observe que le champ Z est solution du système (A.24) si et seulement si le champ des matrices transposées Z^T est solution du système

$$\begin{aligned} &\partial_\alpha Z^T = Z^T C_\alpha^T \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^q), \\ &Z^T(x^0) = I_q. \end{aligned}$$

Or ce dernier système admet une solution unique grâce au Théorème A.3 et la relation

$$\partial_\alpha C_\beta^T + C_\alpha^T C_\beta^T = \partial_\beta C_\alpha^T + C_\beta^T C_\alpha^T,$$

qui est déduite de la relation (A.19). De plus, la matrice $Z(x)^T$ est inversible pour tout $x \in \Omega$ grâce au Lemme A.6. Donc la matrice $Z(x)$ est également inversible pour tout $x \in \Omega$.

(ii) *Le système (A.25) admet une solution unique $V \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$.*

Montrons que l'on peut appliquer le Théorème A.3 au système (A.25). Pour ce faire, il suffit de montrer que les champs de matrices $\widetilde{B}_\alpha := Z^{-1}B_\alpha X^{-1}$ appartiennent à l'espace $L^\infty_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ et qu'ils satisfont la relation

$$\partial_\beta \widetilde{B}_\alpha = \partial_\alpha \widetilde{B}_\beta \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}) \text{ pour tout } \alpha, \beta.$$

Premièrement, l'étape précédente et le Lemme A.5 montrent que les champs de matrices X^{-1} et Z^{-1} appartiennent respectivement aux espaces $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^l)$ et $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^q)$ et qu'ils satisfont les relations

$$\begin{aligned} \partial_\alpha(X^{-1}) &= -X^{-1}(\partial_\alpha X)Z^{-1}, \\ \partial_\alpha(Z^{-1}) &= -Z^{-1}(\partial_\alpha Z)Z^{-1}. \end{aligned}$$

Comme $B_\alpha \in L^\infty_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ par l'hypothèse, l'on obtient que $\widetilde{B}_\alpha \in L^\infty_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$.

Deuxièmement, les champs $\widetilde{B}_\alpha$ (qui appartiennent en particulier à l'espace de distributions $\mathcal{D}'(\Omega; \mathbb{M}^{q,l})$) satisfont la relation

$$\partial_\beta \widetilde{B}_\alpha = \partial_\alpha \widetilde{B}_\beta \text{ pour tout } \alpha, \beta,$$

au sens des distributions. Pour le voir, calculons d'abord la dérivée

$$\begin{aligned} \partial_\beta \widetilde{B}_\alpha &= \partial_\beta(Z^{-1}B_\alpha X^{-1}) \\ &= (\partial_\beta Z^{-1})B_\alpha X^{-1} + Z^{-1}(\partial_\beta B_\alpha)X^{-1} + Z^{-1}B_\alpha(\partial_\beta X^{-1}) \\ &= -Z^{-1}(\partial_\beta Z)Z^{-1}B_\alpha X^{-1} + Z^{-1}(\partial_\beta B_\alpha)X^{-1} - Z^{-1}B_\alpha X^{-1}(\partial_\beta X)X^{-1} \\ &= -Z^{-1}\left((C_\beta Z)Z^{-1}B_\alpha - \partial_\beta B_\alpha + B_\alpha X^{-1}(XA_\beta)\right)X^{-1}, \end{aligned}$$

ce qui donne finalement

$$\text{(A.27)} \qquad \partial_\beta \widetilde{B}_\alpha = -Z^{-1}\left(C_\beta B_\alpha - \partial_\beta B_\alpha + B_\alpha A_\beta\right)X^{-1} \text{ dans } D'(\Omega; \mathbb{M}^{q,l}).$$

Notons que dans les relations précédentes, le produit entre la dérivée au sens des distributions d'un champ de matrices appartenant à l'espace $L^\infty_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ et un champ de matrices appartenant à l'espace $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$ est bien défini au sens des distributions ; par exemple, le produit $(M\partial_\beta B_\alpha)$, où $M := (M_{ik}) \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^q)$, est donné par ses composantes $(M\partial_\beta B_\alpha)_{ij}$ définies par

$$\begin{aligned} < (M\partial_\beta B_\alpha)_{ij}, \varphi >= &\sum_{k=1}^q < M_{ik}(\partial_\beta B_\alpha)_{kj}, \varphi > \\ &:= \sum_{k=1}^q < \partial_\beta (B_\alpha)_{kj}, M_{ik}\varphi >= -\sum_{k=1}^q \int_\Omega (B_\alpha)_{kj}\partial_\beta(M_{ik}\varphi)\,dx \end{aligned}$$

pour tout $\varphi \in D(\Omega)$ (voir aussi l'étape (ii) de la preuve du Théorème 4.1 du quatrième chapitre pour une définition similaire).

De la même manière, l'on obtient

$$\text{(A.28)} \qquad \partial_\alpha \widetilde{B}_\beta = -Z^{-1}\left(C_\alpha B_\beta - \partial_\alpha B_\beta + B_\beta A_\alpha\right)X^{-1} \text{ dans } D'(\Omega; \mathbb{M}^{q,l}).$$

L'hypothèse (A.20) montre alors que les membres de droite des relations (A.27) et (A.28) sont égaux, ce qui entraîne que

$$\partial_\beta \widetilde{B}_\alpha = \partial_\alpha \widetilde{B}_\beta \text{ dans } D'(\Omega; \mathbb{M}^{q,l}).$$

Donc les hypothèses du Théorème A.3 appliqué au système (A.25) sont satisfaites, ce qui entraîne l'existence et l'unicité de la solution $V \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$ de ce système.

(iii) *Le champ de matrices $Y := ZVX$ est une solution du système (A.21).*

En effet, l'on déduit de tout ce qui precède que le champ de matrices Y appartient à l'espace $W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$ et qu'il satisfait l'équation (au sens des distributions)

$$\begin{aligned}
\partial_\alpha Y &= (\partial_\alpha Z)VX + Z(\partial_\alpha V)X + ZV(\partial_\alpha X)\\
&= (C_\alpha Z)VX + Z(Z^{-1}B_\alpha X^{-1})X + ZV(XA_\alpha)\\
&= C_\alpha Y + B_\alpha + YA_\alpha,
\end{aligned}$$

et l'équation

$$Y(x^0) = I_q Y^0 I_l = Y^0.$$

(iv) *Le système (A.21) admet une solution unique.*

Soit $Y, \widetilde{Y} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$ deux solutions du système (A.21). L'étape (i) et le Lemme A.5 montrent que le système

$$\begin{aligned}
&\partial_\alpha Z = C_\alpha Z \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^q),\\
&Z(x^0) = I_q,
\end{aligned}$$

admet une solution unique $Z \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^q)$, que la matrice $Z(x)$ est inversible pour tout $x \in \Omega$, et que $Z^{-1} \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^q)$.

Alors les champs de matrices $T := Z^{-1}Y$ et $\widetilde{T} := Z^{-1}\widetilde{Y}$ appartiennent à l'espace $W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l})$ et satisfont le système

$$\begin{aligned}
&\partial_\alpha T = TA_\alpha + Z^{-1}B_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}),\\
&T(x^0) = Y^0.
\end{aligned}$$

En effet,

$$\begin{aligned}
\partial_\alpha T &= \partial_\alpha(Z^{-1})Y + Z^{-1}(\partial_\alpha Y)\\
&= -Z^{-1}(\partial_\alpha Z)Z^{-1}Y + Z^{-1}(YA_\alpha + C_\alpha Y + B_\alpha)\\
&= -Z^{-1}(C_\alpha Z)Z^{-1}Y + Z^{-1}(YA_\alpha + C_\alpha Y + B_\alpha)\\
&= Z^{-1}YA_\alpha + Z^{-1}B_\alpha.
\end{aligned}$$

Le Théorème A.3 montre alors que $T = \widetilde{T}$, donc que $Y = \widetilde{Y}$ dans Ω. La preuve est complète. □

Remarques A.7. 1. La preuve précédente montre en fait que la resolution des systèmes tels qu'ils apparaissent dans les Théorèmes A.3 et A.4 se ramène à la resolution de deux systèmes plus simples, à savoir

$$\text{(A.29)} \qquad \begin{aligned} &\partial_\alpha X = X A_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^l), \\ &X(x^0) = I_l, \end{aligned}$$

et

$$\text{(A.30)} \qquad \begin{aligned} &\partial_\alpha V = F_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}), \\ &V(x^0) = V^0, \end{aligned}$$

où $A_\alpha \in L^\infty_{\text{loc}}(\Omega; \mathbb{M}^l)$, $F_\alpha \in L^\infty_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$, $V^0 \in \mathbb{M}^{q,l}$ sont tels que

$$\begin{aligned} &\partial_\alpha A_\beta + A_\alpha A_\beta = \partial_\beta A_\alpha + A_\beta A_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^l), \\ &\partial_\alpha F_\beta = \partial_\beta F_\alpha \text{ dans } \mathcal{D}'(\Omega; \mathbb{M}^{q,l}). \end{aligned}$$

Les solutions X et V ci-dessus appartiennent alors respectivement aux espaces $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^l)$ et $W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{M}^{q,l})$.

2. L'existence et unicité de la solution du système (A.29) est obtenue en suivant la preuve du Théorème A.3 donnée au troisième chapitre. Cette preuve, qui comporte six étapes numerotées (i)-(vi), peut être légèrement simplifiée en établissant le résultat de l'étape (iv) à l'aide du Lemme A.6. Il s'agit de montrer que le système

$$\begin{aligned} &\partial_\alpha Y = Y A_\alpha \text{ dans } \mathcal{D}'(B(y,r); \mathbb{M}^l), \\ &Y(y) = Y_* \in \mathbb{M}^l, \end{aligned}$$

admet une solution $Y \in W^{1,\infty}_{\text{loc}}(B(y,r); \mathbb{M}^l)$, sachant que le système

$$\begin{aligned} &\partial_\alpha X = X A_\alpha \text{ dans } \mathcal{D}'(\omega; \mathbb{M}^l), \\ &X(\overline{x}) = I_l \in \mathbb{M}^l, \end{aligned}$$

admet une solution $X \in W^{1,\infty}_{\text{loc}}(\omega; \mathbb{M}^l)$, où ω est un ouvert connexe contenant la boule $B(y,r)$ et $\overline{x}$ est un point de ω. Comme la matrice $X(x)$ est inversible pour tout $x \in \omega$ grâce au Lemme A.6, le champ Y donné par

$$Y(x) := Y_*(X(y))^{-1} X(x) \text{ pour tout } x \in B(y,r),$$

est bien défini et satisfait le système ci-dessus.

3. La preuve du résultat d'existence et unicité de la solution du système (A.30) est beaucoup plus simple. Elle est obtenue en suivant, tout en simplifiant, la preuve du Théorème A.3. Il est facile de voir que l'hypothèse de regularité des F_α, $\alpha = 1, 2, ..., d$, peut être affablie puisqu'il suffit de prendre $F_\alpha \in L^p_{loc}(\Omega; \mathbb{M}^{q,l})$, $1 \leq p \leq \infty$. Dans le cas où Ω est l'espace $\mathbb{R}^d$ tout entier, il a été établi dans Schwartz [10] (voir le Théorème VI, p. 59) que le système $\partial_\alpha V = F_\alpha$, $\alpha = 1, 2, ..., d$, admet une solution unique modulo les constantes lorsque $F_\alpha \in \mathcal{D}'(\mathbb{R}^d)$.

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