Instabilité de la dynamique en l'absence de décompositions dominées.

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Introduction Générale

1.1 Histoire de la notion de décomposition dominée

Un difféomorphisme $f \in \mathrm{Diff}^r(M)$, où $r \geq 1$, est dit *structurellement stable* si tout difféomorphisme voisin g est conjugué à f par un homéomorphisme h de M. On rappelle qu'un point périodique de période p d'un difféomorphisme f est *hyperbolique* si la p-ème itérée de la différentielle df au dessus de ce point n'a aucune valeur propre de module 1. Lorsque de plus il y a des valeurs propres de modules inférieurs à 1 et d'autres de modules supérieurs à 1, on dit que c'est une *selle*. Dans les années trente, Andronov et Pontriaguine [5] on remarqué une correspondance entre la stabilité structurelle et les comportements hyperboliques. De fait, la stabilité structurelle locale autour d'un point selle découle d'un théorème de Hartmann-Grobmann (voir [31])

Au début des années soixante, D.V. Anosov [6] a prouvé la stabilité structurelle des flots géodésiques des variétés riemanniennes compactes de courbure négative. Ce résultat [7] s'applique en fait à toute une classe de dynamiques : un flot d'une variété compacte riemannienne M est *Anosov* s'il n'a pas de singularité et si le fibré tangent TM se décompose en la somme directe de trois sous-fibrés :

- un fibré *stable*, c'est-à-dire uniformément contracté : il existe un réel $T > 0$ tel que le temps T du flot y contracte strictement tout vecteur non nul.
- un fibré *instable*, c'est-à-dire uniformément dilaté : il existe un réel $T > 0$ tel que le temps T du flot y dilate strictement tout vecteur non nul.
- un fibré unidimensionnel qui n'est ni contracté, ni dilaté.

On définit de même la notion de *difféomorphisme d'Anosov*, en oubliant le troisième fibré.

A la même époque, S. Smale [59] exhibait une nouvelle classe de dynamiques particulièrement simples - les dynamiques *Morse-Smale* - et structurellement stables. Un point x est *non-errant* pour un diffeomorphisme f si pour tout voisinage $\mathcal{U}$ de x, il existe un entier $n \geq 1$ tel que $f^n(\mathcal{U})$ intersecte $\mathcal{U}$. Un diffeomorphisme f est Morse-Smale si l'ensemble $\Omega(f)$ de ses points non-errants est une réunion finie de points périodiques hyperboliques, et si pour tout couple de points dans $\Omega(f)$ la variété stable de l'un et la variété instable de l'autre se rencontrent transversalement en tout point d'intersection.

1.1.1 La théorie hyperbolique

S. Smale a créé une notion générale qui regroupe les difféomorphismes d'Anosov et Morse-Smale. Un compact invariant K d'un difféomorphisme f est *hyperbolique* si et seulement la restriction du fibré tangent à K se décompose en deux sous-fibrés supplémentaires E^s et E^u invariants par la différentielle df, tels qu'une itérée de df contracte E^s uniformément et dilate E^u uniformément. Plus précisément, il existe $n \in \mathbb{N}$ et $0 < \lambda < 1$ tels que pour tous vecteurs unitaires u, v dans E^s, E^u, on a $\|df^n(u)\| < \lambda^{-1}$ et $\|df^n(v)\| > \lambda$.

Un difféomorphisme est dit *hyperbolique* ou *Axiome A* si son ensemble non-errant est hyperbolique et si l'ensemble $\mathrm{Per}(f)$ des points périodiques de f est dense dans l'ensemble non-errant $\Omega(f)$. D'après le théorème de décomposition spectrale de Smale, le non-errant d'un difféomorphisme hyperbolique s'écrit comme la réunion $\Omega(f) = \Lambda_1 \cup ... \cup \Lambda_i$ de *pièces "basiques"* : ce sont des compacts deux à deux disjoints, isolés, invariants par f et transitifs (un ensemble est dit *transitif* lorsqu'il contient une orbite dense). De plus l'α-limite comme l'ω-limite de toute orbite est contenue dans un Λ_i. En particulier, si $\Omega(f) = M$, il n'y a qu'une pièce basique et f est un difféomorphisme d'Anosov.

Un difféomorphisme hyperbolique est dit *sans cycle* si et seulement s'il n'y a aucun cycle de pièces basiques. En d'autres termes, il n'y a pas de suite périodique de pièces basiques telles que l'ensemble instable de chacune intersecte l'ensemble stable de la suivante. Le difféomorphisme satisfait la *condition de transversalité forte* si pour tous x, y dans l'ensemble non-errant, la variété stable de l'un et la variété instable de l'autre sont transverses.

Dans [47], J. Palis et S. Smale ont avancé une *conjecture de stabilité* : Pour tout difféomorphisme $f \in \mathrm{Diff}^r(M)$,

- hyperbolicité et transversalité forte équivaut à stabilité structurelle.
- hyperbolicité et condition *sans cycle* équivaut à Ω-stabilité : f est conjugué à tout difféomorphisme g C^r-voisin, par restriction aux ensembles non-errants respectifs.

Smale [62] a montré qu'hyperbolicité *sans cycle* implique Ω-stabilité. Puis Robbin [53] en topologie C^2, puis Robinson [54] en C^1, ont montré qu'un difféomorphisme hyperbolique satisfaisant la condition de transversalité forte est structurellement stable. A la fin des années 80, Ricardo Mañé [39] a achevé de prouver que la stabilité structurelle implique l'hyperbolicité et la transversalité forte. En s'appuyant sur les techniques de Mañé, Palis [45] a obtenu qu'Ω-stabilité implique hyperbolicité et *sans cycle*, terminant ainsi la preuve en classe C^1 de la conjecture de stabilité, près de 20 ans après l'avoir énoncée. On pensait dans les années 60 que l'ensemble des difféomorphismes hyperboliques d'une variété compacte était dense dans l'espace des difféomorphismes. Mais rapidement sont apparus des exemples d'ouverts de difféomorphismes non structurellement stables [60] et de difféomorphismes non Ω-stables [4].

1.1.2 Hyperbolicité partielle et formes faibles de stabilité

Afin d'étudier l'existence et la dynamique des difféomorpismes loin de l'hyperbolicité, Hirsch, Pugh, Shub [35] et Brin, Pesin [19] ont introduit les sytèmes *partiellement hyperboliques*, i.e. tels que le fibré tangent se décompose en une somme directe $E^s \oplus E^c \oplus E^u$ en trois sous-fibrés invariants, où E^s est uniformément contracté par la dynamique, E^u est uniformément dilaté, et E^c est central, c'est-à-dire moins contracté que E^s et moins dilaté que E^u. On montre aisément que les fibrés d'une décomposition partiellement hyperbolique sont continus, et persistent en variant continûment par perturbations C^1.

Ils ont notamment montré l'intégrabilité des fibrés stables et instables : étant donné un système partiellement hyperbolique de classe C^r, il existe un feuilletage $\mathcal{F}^s$ tangent au fibré stable et un feuilletage $\mathcal{F}^u$ tangent au fibré instable. Le fibré central, quant à lui, n'est en général pas intégrable. On trouvera des indications dans [61, Section 3.1] pour la construction sur le tore de dimension 6 de difféomorphismes robustement non-hyperbolique, et partiellement hyperboliques avec un fibré non-intégrable.

Mais lorsque le feuilletage central existe, on a en général une forme faible de stabilité structurelle. On dit que le feuilletage central de f partiellement hyperbolique est *strucurellement stable* si pour tout difféomorphisme g C^1-voisin, le fibré central de g est intégrable, et s'il existe un homéomorphisme h de M qui envoie le feuilletage central de f sur le feuilletage central de g, de sorte que $h \circ f \circ h^{-1}$ soit isotope à g le long des feuilles centrales. Le résultat principal de [35] est que si un système partiellement hyperbolique a un feuilletage central expansif par plaques (*plaque-expansive*), alors ce feuilletage est structurellement stable.

1.1.3 Décompositions dominées

Une forme encore plus faible d'hyperbolicité est celle de *décomposition dominée*, qui fut introduite indépendamment par Liao [36] et Mañé [38] dans leurs travaux sur la conjecture de stabilité. Une décomposition dominée pour une dynamique fixée est une décomposition du fibré tangent en une somme directe de sous-fibrés invariants, tels que sur chaque sous-fibré la dynamique est moins dilatante ou plus contractante que sur la suivante. On renvoie le lecteur à la section 1.3.1 pour une définition formelle.

De l'existence de champs de cônes stables autour des fibrés, on déduit qu'une décomposition dominée persiste par C^1-perturbation de la dynamique, et que les fibrés sont continus dans le tangent TM et varient continûment (voir [14]) par perturbation. Les fibrés d'une décomposition dominée ne sont généralement pas intégrables, à part les fibrés extrémaux lorsque uniformément contractants ou dilatants.

1.2 Dynamiques robustement non-hyperboliques

Une propriété d'un C^r-difféomorphisme f est dit C^r-robuste si elle persiste par C^r-perturbation de f. Des exemples de difféomorphismes C^1-robustement non hyperboliques on été construits par Abraham et Smale [4] en dimension ≥ 4, puis Simon [58] en dimension ≥ 3.

1.2.1 Bifurcations homoclines

En dimension 2, il n'est pas connu si les difféomorphismes hyperboliques sont denses en topologie C^1, cependant, S. Newhouse a exhibé un ouvert $\mathcal{U}$ de $\mathrm{Diff}^2(M^2)$ tel que tout f dans un résiduel de $\mathcal{U}$ a une infinité de puits ou de sources. La propriété de posséder une infinité de puits ou de source est appelée *phénomène de Newhouse*. Pour construire cet exemple, Newhouse a étudié la dynamique au voisinage des tangences homoclines.

Définition 1.2.1. Une *tangence homocline* est une intersection non transverse des variétés stable et instable d'une selle.

La tangence homocline est très instable et peut-être cassée par une petite perturbation de la dynamique. Cependant, Newhouse [42, 43] a démontré que sur les surfaces en topologie C^2, près de tout difféomorphisme admettant une tangence homocline, il y a un ouvert $\mathcal{U}$ de difféomorphismes et un résiduel $\mathcal{R} \in \mathcal{U}$ tels que tout $g \in \mathcal{R}$ présente un phénomène de Newhouse.

Un autre angle d'étude des dynamiques non-hyperboliques était la notion de transitivité. On dit qu'une dynamique sur M est *transitive* s'il existe une orbite dense dans M. On s'est demandé si les difféomorphismes d'Anosov étaient caractérisés par la transitivité robuste. Plusieurs exemples d'ouverts de difféomorphismes robustement transitifs et non hyperboliques ont été obtenus en dimension ≥ 3.

Comme conséquence de la stabilité structurelle des feuilletages centraux, il est exhibé dans [35] un difféomorphisme robustement transitif avec deux selles d'indices différents, par C^0-perturbation d'un difféomorphisme d'Anosov du tore de dimension 4. S'il existait un difféomorphisme voisin hyperbolique, par transitivité, l'unique pièce basique serait M tout entier. Il ne peut cependant pas y avoir deux selles d'indices différents dans une même pièce basique. C'est donc un difféomorphisme robustement non-hyperbolique.

Par des méthodes différentes, Mañé [37] a trouvé que certaines C^0-perturbations d'un difféomorphisme d'Anosov du tore de dimension 3 sont robustement transitives et admettent deux selles d'indices différents. Cet exemple est loin des tangences homoclines mais est dans l'adhérence des difféomorphismes admettant un cycle hétérodimensionnel :

Définition 1.2.2. Un *cycle hétérodimensionnel* est un couple de selles d'indices différents telles que la variété instable de l'une intersecte la variété stable de l'autre.

Un cycle hétérdimensionnel est également un phénomène instable. Cependant C. Bonatti et L. Diaz [12] ont montré que près d'un cycle hétérodimensionnel de codimension 1 (la différence d'indice entre les deux selles est 1), la fermeture des difféomorphismes présentant un cycle hétérodimensionnel est d'intérieur non vide.

1.2.2 Conjectures de densité de Palis

Palis a proposé de caractériser les dynamiques robustement non hyperboliques par l'accumulation par des tangences homoclines ou des cycles hétérodimensionnels.

Conjecture 1.2.3 (Conjecture de densité C^r de Palis). *La réunion des difféomorphismes hyperboliques et des difféomorphismes admettant une tangence homocline ou un cycle hétérodimensionnel est dense dans* $\mathrm{Diff}^r(M)$.

Une version faible de cette conjecture avance l'existence d'une dichotomie entre difféomorphismes Morse-Smale et fers à cheval.

Conjecture 1.2.4 (Conjecture faible de densité C^r). *L'ensemble des difféomorphismes Morse-Smale et des difféomorphismes admettant une intersection homocline sont deux ouverts dont la réunion est dense dans* $\mathrm{Diff}^r(M)$.

En 1967, Pugh [51] a montré le lemme de fermeture C^1 (C^1-closing lemma), qui dit que si un point $x \in M$ est récurrent pour un difféomorphisme f, alors il existe une C^1-perturbation g arbitrairement faible de f pour laquelle x est un point périodique. Dans les années 90, Hayashi [33] a montré le lemme de connection C^1 (C^1-Connecting Lemma) : sous des hypothèses d'accumulation sur les variétés stables et instables de deux selles d'un difféomorphisme, une perturbation arbitrairement faible crée un cycle entre ces deux selles.

Ces deux lemmes sont fondamentaux dans la création par perturbation de selles périodiques (à partir desquelles, sous certaines conditions, on crée des tangences homoclines), et dans la création de cycles hétérodimensionnels.

Ils sont le point de départ de tous les résultats partiels connus, dans le cadre de la conjecture de densité C^1. A l'heure actuelle, aucun equivalent de ces lemmes n'est démontré en topologie C^r pour $r > 1$, aussi n'a-t-on aucune idée de la façon dont on pourrait répondre aux conjectures de densité C^r.

La conjecture faible de densité C^1 a été montrée par Bonatti, Gan et Wen [15] en dimension 3. Très récemment, S.Crovisier [23], utilisant les travaux de Wen [65] et étudiant de manière originale la dynamique dans la direction centrale, l'a prouvée en toute dimension.

Théorème 1.2.1 (Crovisier). *L'ensemble des difféomorphismes Morse-Smale et l'ensemble des difféomorphismes admettant une intersection homocline sont deux ouverts disjoints dont la réunion est dense dans* $\mathrm{Diff}^1(M)$.

Des avancées récentes laissent espérer qu'une preuve de la conjecture de Palis est accessible en C^1.

En 2000, E. Pujals et M. Sambarino dans [52] l'ont montré pour les surfaces : un difféomorphisme d'une surface compacte peut être approchée en topologie C^1 soit par des difféomorphismes hyperboliques, soit par des difféomorphismes admettant une tangence homocline. Précisons les deux étapes principales de leur preuve. Ils ont d'abord montré (Lemme 2.0.2) qu'en dehors de la fermeture de l'ensemble des difféomorphismes avec tangence homocline, il y a un ouvert dense de difféomorphismes g dont le non-errant $\Omega(g)$ admet une décomposition dominée non-triviale.

Un second résultat (Théorème B.) dit que si f est un difféomorphisme C^2 sur une surface compacte et si $T_{|\Lambda}M = E \oplus F$ est une décomposition dominée au-dessus d'un compact Λ invariant pour f, alors Λ est la réunion disjointe d'un ensemble hyperbolique et d'un nombre fini de courbes fermées simples $\mathcal{C}^i$ normalement hyperboliques, telles que que les restrictions $f^{p_i}|\mathcal{C}^i$ sont conjuguées à une rotation irrationnelle (p_i est la période de la courbe $\mathcal{C}^i$).

On obtient facilement la conjecture de densité C^1 en combinant ces deux résultats. En transposant cette méthode en dimension supérieure, on peut naturellement déjà chercher à montrer que loin des tangences homoclines il y a des décompositions dominées fines sur le non-errant (ou sur les classes homoclines, ou l'ensemble récurrent par chaînes).

Hayashi avait annoncé des avancées dans la preuve de la conjecture de Palis en toute dimension ; cependant rien n'a encore été publié. Dans [65], L. Wen a obtenu des résultats partiels remarquables. A partir de résultats antérieurs [64] et du lemme de sélection de Liao (voir [36]), il montre que génériquement, loin des dynamiques hyperboliques, des tangences homoclines et des cycles hétérodimensionnels, les ensembes minimaux non-hyperboliques admettent une décomposition partiellement hyperbolique avec un ou deux fibrés centraux de dimension 1.

1.3 Définitions élémentaires et notations

Avant d'énoncer les résultats prinipaux de cette thèse, on rappelle les définitions de fibrés, cocycles linéaires et décompositions dominées. On évoque les classes homoclines et la théorie de Conley, outils essentiels en dynamique C^1-générique.

1.3.1 Cocycles linéaires et décompositions dominées

Soit $\mathcal{E} = (E, \Sigma, \pi\colon E \to \Sigma)$ un fibré vectoriel de dimension d au-dessus d'une base Σ, i.e., pour tout $x \in \Sigma$, la fibre $E_x = \pi^{-1}(x)$ est un espace vectoriel de dimension d. En reprenant les notations de [16] et [13], on dit qu'un couple de bijections $\mathcal{A} = (f\colon \Sigma \to \Sigma, A\colon E \to E)$ est un *cocycle linéaire* ou *automorphisme* de $\mathcal{E}$, si et seulement si le diagramme suivant commute :

$$\begin{array}{ccc} E & \xrightarrow{A} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Sigma & \xrightarrow{f} & \Sigma \end{array} \quad . \tag{1.1}$$

Un *fibré euclidien* $(\mathcal{E}, \|.\|)$ est un fibré vectoriel tel que chaque fibre est muni d'une métrique euclidienne $\|.\|$. Pour tout fibré euclidien $\mathcal{E}$ et toute bijection f de sa base Σ, l'ensemble des cocycles linéaires au dessus de la dynamique f est muni de la distance suivante :

$$\mathrm{dist}(\mathcal{A}, \mathcal{B}) = \sup\{\|\mathcal{A}(u) - \mathcal{B}(u)\|, \|\mathcal{A}^{-1}(u) - \mathcal{B}^{-1}(u)\| \quad / \quad u \in \mathcal{E}, \|u\| = 1\}.$$

Un cocycle linéaire est dit *borné par* $C > 0$ si pour tout vecteur unitaire $u \in \mathcal{E}$, on a $\|\mathcal{A}(u)\|, \|\mathcal{A}^{-1}(u)\| < C$.

Définition 1.3.1. Etant donné un cocycle linéaire $\mathcal{A}$ sur un fibré euclidien $\mathcal{E}$, on appelle *décomposition dominée* pour $\mathcal{A}$ toute décomposition $\mathcal{A}$-invariante $\mathcal{E} = E_1 \oplus ... \oplus E_\ell$ en sous-fibrés vectoriels vérifiant la propriété suivante : il existe $C > 0$ et $0 < \lambda < 1$ tels que pour chaque $1 \leq i < \ell$, pour tout couple de vecteurs unitaires $u, v \in E_i, E_{i+1}$ et tout $n \in \mathbb{N}$,

$$\|f^n(u)\| < C\lambda^n \|f^n(v)\|.$$

On dit alors que E_i est dominé par E_{i+1}, et on écrit $E_i \prec E_{i+1}$. De façon évidente, si la décomposition $\mathcal{E} = E_1 \oplus ... \oplus E_\ell$ est dominée, alors pour tout $1 \leq i < j \leq \ell$, on a $E_i \prec E_j$. En construisant des champs de cônes stables, on montre que toute décomposition dominée persiste par perturbation et varie continûment : pour toute famille de voisinages $\mathcal{V}_1, ..., \mathcal{V}_\ell$ des fibrés $E_1, ..., E_\ell$, il existe $\epsilon > 0$ tel que tout cocycle linéaire $\mathcal{B}$ vérifiant $\mathrm{dist}(\mathcal{A}, \mathcal{B}) < \epsilon$ admet une décomposition dominée $F_1 \oplus ... \oplus F_\ell$, où $F_i \in \mathcal{V}_i$, pout tout i.

De plus dans le cas particulier où $\mathcal{E}$ est l'espace tangent d'une variété M restreint à un compact Σ, et où $\mathcal{A}$ est la différentielle df d'un difféomorphisme de M laisant invariant le compact Σ, alors (voir [14, Page 292], par exemple) les sous-fibrés E_i sont continus.

On dit qu'un fibré invariant F est *stable* ou *uniformément contracté* par $\mathcal{A}$ si et seulement s'il existe $C > 0$ et $0 < \lambda < 1$ tels que pour tout vecteur unitaire $u \in F$ et tout $n \in \mathbb{N}$, on a $\|\mathcal{A}^n(u)\| \leq C.\lambda^n$. On dit qu'il est *instable* ou *uniformément dilaté* s'il est uniformément contracté par le cocycle linéaire inverse $\mathcal{A}^{-1}$.

Définition 1.3.2. Etant donné un cocycle linéaire $\mathcal{A}$ sur un fibré euclidien $\mathcal{E}$, on appelle *décompostion partiellement hyperbolique* pour $\mathcal{A}$ toute une décomposition dominée $\mathcal{E} = E^s \oplus E_1 \oplus ... \oplus E_\ell \oplus E^u$ telle que E^s est stable et E^u est instable pour $\mathcal{A}$.

Etant donné un fibré euclidien $\mathcal{E}$ avec une base compacte Σ, et une décomposition dominée $\mathcal{E} = E_1 \oplus ... \oplus E_\ell$ pour un cocycle linéaire $\mathcal{A}$, une *métrique adaptée* à cette décomposition dominée est une métrique $\|.\|_*$ sur chaque fibre de $\mathcal{E}$ telle que pour tout $1 \leq i < \ell$, pour tous vecteurs unitaires $u, v \in E_i, E_{i+1}$, on a $\|\mathcal{A}(u)\|_* < \|\mathcal{A}(v)\|_*$. Si $\mathcal{A}$ admet une décomposition partiellement hyperbolique $\mathcal{E} = E^s \oplus E_1 \oplus ... \oplus E_\ell \oplus E^u$, la métrique $\|.\|_*$ y est *adaptée* si et seulement si elle est adaptée à la décomposition dominée correspondante, et si pour tous vecteurs unitaires $u, v \in E^s, E^u$ on a $\|\mathcal{A}(u)\|_* < 1$ et $\|\mathcal{A}(v)\|_* > 1$.

1.3.2 Force d'une domination

On dit qu'une décomposition $\mathcal{E} = F \oplus G$ invariante pour un cocycle linéaire $\mathcal{A}$ est N*-dominée* si, après N itérations, les vecteurs de F sont deux fois plus dilatés, ou deux fois moins contractés que ceux de E, c'est-à-dire que pour tous vecteurs unitaires $u, v \in F, G$, on a $\|\mathcal{A}^N(u)\| < 1/2\|\mathcal{A}^N(v)\|$. On dira qu'une domination est d'autant plus faible que N est grand.

Remarque 1.3.3. *D'après cette définition, si une décomposition invariante est N-dominée, elle n'est pas forcément M-dominée pour tout $M > N$.*

Cette remarque n'a cependant pas lieu lorsque la métrique $\|.\|$ est adaptée à la décomposition dominée.

Dans [29], la définition de décomposition N-dominée diffère de celle que nous venons de donner ; une décomposition est N-dominée d'après [29] si elle est L-dominée pour un certain $L < N$. On verra que ces deux définitions sont équivalentes, dans une certaine mesure (voir le corollaire 1.4.1).

1.3.3 Notions de récurrence

Classes homoclines

Dans cette section, $r \geq 1$ et M est une variété compacte riemannienne. Soit f un élement de $\mathrm{Diff}^r(M)$. On dit qu'un compact f-invariant Λ est *hyperbolique* si et seulement s'il existe une décomposition invariante $TM_{|\Lambda} = E^s \oplus E^u$ où les fibrés E^s et E^u sont respectivement stable et instable.

Un point périodique P de f est une *selle* si son orbite $\mathrm{Orb}_f(P)$ est hyperbolique si les fibrés stable et instable sont tous les deux non triviaux. L'ensemble stable $W^s(\mathrm{Orb}_f(P)) = \{x \in M / \operatorname{dist}(f^n(P), \mathrm{Orb}_f(P)) \to 0\}$ et l'ensemble instable $W^u(\mathrm{Orb}_f(P)) = \{x \in M / \operatorname{dist}(f^{-n}(P), \mathrm{Orb}_f(P)) \to 0\}$ sont tous les deux des variétés plongées de classe C^r, et sont tangents aux fibrés E^s et E^u.

On dit que deux points selles sont *homocliniquement reliés* si la variété instable de chacun intersecte transversalement la variété stable de l'autre. Ceci définit une relation d'équivalence sur les selles.

Définition 1.3.4. La *classe homocline* $H(Q, f)$ d'un point selle Q d'un difféomorphisme f est la fermeture de l'ensemble des point hyperboliques homocliniquement reliés à Q.

Par définition une classe homocline contient un sous-ensemble dense de points périodiques, et de fait est dans l'ensemble non-errant de f. La classe homocline d'une selle Q est topologiquement transitive ; elle peut être définie de manière équivalente comme la fermeture de l'ensemble des intersections transverses des variétés stables et instables de Q (voir [44]).

Théorie de Conley

Dans ce paragraphe, on décrit brièvement une notion de récurrence introduite par Conley [21]. Une suite (x_n) est appelée *ϵ-pseudo-orbite* pour un $\epsilon > 0$, si et seulement si la distance $\operatorname{dist}(f(x_n), x_{n+1})$ est plus petite que ϵ pour tout n. L'ensemble *récurrent par chaînes* $\mathcal{R}(f)$ est l'ensemble des points x tels que pour tout $\epsilon > 0$ il y a une ϵ-pseudo-orbite non réduite à un seul point, et allant de x à x. On définit une relation d'équivalence $\sim$ sur $\mathcal{R}(f)$ de la façon suivante : $x \sim y$ si et seulement si pour tout $\epsilon > 0$, il existe une ϵ-pseudo-orbite allant de x à y et une autre allant de y à x.

Les *classes de récurrence par chaînes* sont les classes d'équivalence de $\sim$ dans $\mathcal{R}(f)$. Comme M est compacte, l'ensemble récurrent par chaînes est compact et non vide, et chaque classe de récurrence par chaînes est compacte. L'ensemble non errant est clairement contenu dans l'ensemble récurrent par chaînes, et comme toute classe homocline est transitive, toute classe homocline est dans une classe de récurrence par chaînes.

Un *résiduel* d'un espace topologique est une intersection dénombrable d'ouverts denses. Une propriété dynamique $\mathcal{P}$ est dite *C^r-générique* si elle est satisfaite par un résiduel de l'ensemble des dynamiques C^r. On dit également que C^r*-génériquement* la dynamique satisfait la propriété $\mathcal{P}$. Dans [10], C. Bonatti et S. Crovisier ont montré que C^1-génériquement, les classes de récurrence par chaînes non apériodiques et les classes homoclines coïncident :

Théorème 1.3.1 (Bonatti, Crovisier). *Etant donnée une variété compacte M, il y a un résiduel $\mathcal{HCR}$ dans $\mathrm{Diff}^1(M)$ de diffeomorphismes f tels que chaque classe homocline de f est une classe de récurrence par chaînes, et réciproquement, chaque classe de récurrence par chaînes qui n'est pas une classe homocline est apériodique (ne contient pas de point périodique).*

Crovisier [22] a récemment montré que génériquement une classe de récurrence par chaînes est limite de Hausdorff d'une suite d'orbites de points selles. Il est à noter que ces résultats ont été obtenus en travaillant sur la preuve du lemme de fermeture de Pugh, et qu'elles peuvent être vues comme des généralisations de celui-ci. De fait, le lemme de fermeture découle directement du résultat de Crovisier.

1.3.4 Perturbations de difféomorphismes

Soit M une variété compacte riemannienne. On a une distance canonique $\mathrm{dist}\colon TM \times TM \to \mathbb{R}^+$ sur le fibré tangent TM. Un difféomorphisme f est une *ϵ-perturbation de g* en topologie C^1 si et seulement si, pour tout vecteur unitaire $v \in TM$, on a $\mathrm{dist}(df(v), dg(v)) \le \epsilon$ et $\mathrm{dist}(df^{-1}(v), dg^{-1}(v)) \le \epsilon$. Pour un sous-ensemble $S \setminus M$, f est une *pertubation de g sur S* si $f = g$ en-dehors de S.

Un lemme de Franks [26] permet de prolonger une perturbation de la différentielle d'un difféomorphisme le long d'un point périodique en une perturbation locale du difféomorphisme. On le formule ainsi précisément :

Lemme 1.3.5 (Franks). *Soit f un difféomorphisme d'une variété compacte riemannienne M. Pour tout $\delta > 0$, il existe $\epsilon > 0$ tel que :*

si Σ est un ensemble f-invariant fini et si $\mathcal{A}$ est un cocycle linéaire sur $TM_{|\Sigma}$ avec $\mathrm{dist}(\mathcal{A}, df_{|\Sigma}) < \epsilon$, alors on trouve une δ-perturbation g de f sur un voisinage arbitrairement petit de Σ tel que $dg_{|\Sigma} = \mathcal{A}$.

Grâce à ce lemme, on ramènera à chaque fois les problèmes de perturbations de difféomorphismes le long d'orbites périodiques au cas linéaire. On travaillera ainsi sur des cocycles linéaires dans la preuve du Théorème 4.5.2, et sur ce qu'on appelera difféomorphismes selle linéaires (*linear saddle diffeomorphisms*) dans la preuve du Théorème 5.1.3. Remarquons que le lemme de Franks est particulier à la topologie C^1.

1.4 Enoncé des résultats

L'objet de cette thèse est principalement l'étude de la dynamique locale des difféomorphismes le long d'orbites périodiques, afin de déterminer quelles dynamiques peuvent être obtenues par perturbations en l'absence de décomposition dominée. On en déduit, entre autres, des dichotomies génériques entre phénomènes de Newhouse et difféomorphismes dont la dynamique présente une certaine forme de domination, et entre tangences homoclines et dominations stable/instable.

1.4.1 Métriques adaptées

Dans le chapitre 3, on répond à une ancienne question de Hirsch, Pugh, Shub [35, page 5] à propos de l'existence de métriques adaptées aux décompositions dominées. On savait que les décompositions hyperboliques admettent des métriques adaptées. En voici un résumé de la preuve :

Soit K un compact invariant d'un difféomorphisme f, et soit $TM_{|K} = E^s \oplus E^u$ une décomposition hyperbolique pour df. Alors, pour tout vecteur $u \in E^s$, $\|df^n(u)\|$ tend exponentiellement vers 0 lorsque n va à l'infini, où $\|.\|$ est la métrique riemannienne initiale sur M.

On peut alors définir une nouvelle métrique $\|.\|_s$ sur le fibré E^s comme la somme des poussés en avant de la métrique initiale par les itérées positives de df. Clairement, comme $\|u\|_s = \|u\| + \|df(u)\|_s$ on a $\|df(u)\|_s < \|u\|_s$ pour tout vecteur non-nul u. Symétriquement, on définit une métrique $\|.\|_u$ sur E^u telle que $\|df(u)\|_u > \|u\|_u$. On construit alors une métrique $\|.\|_*$ sur M qui coincide avec $\|.\|_s$ sur E_s et $\|.\|_u$ et E^u : elle est adaptée par construction. On obtiendrait de même une métrique adaptée dérivant d'un produit scalaire en prenant la série des poussés en avant par df du produit scalaire initial sur M. Cette métrique peut alors être lissée tout en restant adaptée.

Hirsch, Pugh and Shub on remarqué que lorsqu'il n'y avait que deux fibrés dans une décomposition dominée et que l'un des deux fibrés était de dimension un, une série similaire permettait encore la construction d'une métrique adaptée. Cependant, l'existence d'une telle métrique leur était inconnue dans le cas général pour les décompositions dominées ou partiellement hyperboliques. C'est pourquoi il leur faut distinguer hyperbolicité normale immédiate (*immediate normal hyperbolicity*) et hyperbolicité normale relative (*relative normal hyperbolicity*) [35, page 3]. Le théorème suivant répond complètement à leur question :

Théorème 1.4.1 (G.). *Soit f un difféomorphisme d'une variété riemannienne M. Supposons que f admette une décomposition dominée sur un compact invariant K, c'est-à-dire que le cocycle linéaire df_K défini comme la restriction de df à $TM_{|K}$ admet une décomposition dominée. Alors il existe une métrique riemannienne lisse sur M qui est adaptée à cette métrique. Si* *la décomposition est partiellement hyperbolique, on a encore une métrique riemannienne lisse qui y est adaptée.*

La méthode est la construction de séparateurs entre les fibrés de la décomposition dominée, c'est-à-dire des fonctions de K dans $\mathbb{R}^+$ qui correspondraient à l'ajout à la dynamique de fibrés de dimension 1 entre chaque couple de fibrés consécutifs. Alors sur chaque fibré, grâce à une série de même inspiration que les précédentes, on trouve une métrique qui est adaptée aux dominations relatives aux deux séparateurs qui l'encadrent. Lorsque les séparateurs sont choisis convenablement, on a alors une métrique adaptée à la décomposition dominée.

Par conséquent, pour toute décomposition dominée, il y a une métrique sur le fibré telle que si on a une N-domination entre un couple de fibrés, alors on a une L-domination pour tout $L > N$. Le corollaire suivant peut être aisément obtenu sans métriques adaptées, celles-ci ne facilitent que légèrement la preuve.

Corollaire 1.4.1. *Soit f un difféormorphisme d'une variété compacte M. Alors pour tout $L \in \mathbb{N}$, il existe $N \in \mathbb{N}$ tel que si une décomposition $TM_{|K} = E \oplus F$ au dessus d'un compact K n'est pas N'-dominée pour un certain $N' > N$, alors elle n'est L'-dominée pour aucun $L' \leq L$.*

Preuve : On raisonne par l'absurde et on suppose qu'il existe une suite K_n, une suite L_n d'entiers entre 1 et L, et une suite N_n tendant vers $+\infty$, telles qu'il existe une décomposition L_n-dominée $T_{|K_n} = E_n \oplus F_n$, qui n'est pas N_n-dominée. Quitte à extraire, on peut supposer que les décompositions $T_{|K_n} = E_n \oplus F_n$ ont même indice i et sont L_0-dominées. Alors on a une décomposition L_0-dominée $E \oplus F$ d'indice i sur la fermeture de la réunion $\cup_{n\in\mathbb{N}} K_n$. On a alors une métrique riemannienne $\|.\|_*$ sur M adaptée à cette décomposition dominée. En particulier, pour tout $\epsilon > 0$, lorsque N est plus grand qu'un certain N_ϵ, on a pour tous vecteurs unitaires $u \in E, v \in F$ que $\|df^N(u)\|_* < \epsilon \|df^N(v)\|_*$. Mais la métrique $\|.\|_*$ est équivalente à la métrique initiale $\|.\|$, ce qui contredit le fait que pour tout N_n il existe $(u, v) \in E_n \times F_n$ tels que $\|df^{N_n}(u)\| > 1/2 \|df^{N_n}(v)\|$. □

1.4.2 Phénomènes de Newhouse en toute dimension

Avant que Pujals et Sambarino ne prouvent la conjecture de densité C^1 en dimension 2, R. Mañé [38] a obtenu une dichotomie C^1-générique entre phénomènes de Newhouse et difféomorphismes hyperboliques. Ce résultat a été généralisé par Bonatti, Díaz et Pujals [13] en toute dimension. Ils ont prouvé que génériquement, un difféomorphisme dont une des classes homoclines n'a pas de décomposition dominée a une infinité de puits ou de sources. C'est un corollaire du théorème suivant :

Théorème 1.4.2 (Bonatti, Díaz, Pujals). *Soit P une selle d'un difféomorphisme $f \in \mathrm{Diff}^1(M)$, alors au moins l'une des deux assertions suivantes est vérifiée :*
- *la classe homocline $H(P, f)$ admet une décomposition dominée ;*
- *pour tout voisinage $\mathcal{U}$ de $H(P, f)$, pour tout $k \in \mathbb{N}$, g est arbitrairement C^1-proche de f avec k puits ou sources dont les orbites restent dans $\mathcal{U}$.*

La preuve de Bonatti-Diaz-Pujals s'appuient sur l'existence de ce qu'ils appellent des transitions. Les transitions sont la traduction en termes de cocycles linéaires de la propriété suivante des classes homoclines : étant donnés deux orbites périodiques d'une classe homocline, il y a des points périodiques qui passent arbitrairement près de P, puis arbitrairement près de Q, et ainsi de suite ; on peut de surcroît contrôler le temps que l'orbite va passer près de P puis près de Q, de sorte que la différentielle le long de cette orbite est alternativement très proche de la dérivée le long de $\mathrm{Orb}(P)$, et de la dérivée le long de $\mathrm{Orb}(Q)$ sur des intervalles de temps de longueurs prescrites.

Aussi les résultats de [13] reposent-ils fortement sur le fait qu'ils se placent à l'intérieur de classes homoclines. Dans cette thèse on montre un énoncé similaire en ne travaillant que sur une seule orbite d'un difféomorphisme f. On obtient précisément le résultat suivant :

Théorème 4.5.2 (Bonatti, G., Vivier). *Soit un difféomorphisme $f \in \mathrm{Diff}^1(M)$. Alors pour tout $\epsilon > 0$, il existe $N \in \mathbb{N}$ tel que :*

si Q est un point périodique et si le cocycle $df_{|\,\mathrm{Orb}_f(Q)}$ n'a pas de de décomposition N-dominée, alors il y a une ϵ-perturbation g de f sur un voisinage arbitrairement petit de l'orbite de Q, qui préserve cette orbite, et telle que l'application de premier retour $dg^p_{|\,\mathrm{Orb}(Q)}$ a des valeurs propres réelles de même module différent de 1.

En particulier, Q est un puits ou une source pour g. Le Chapitre 4 est consacré à la preuve du théorème 4.5.2. Avec un théorème de Crovisier [22] qui dit que génériquement toute classe de récurrence par chaînes est limite de Hausdorff d'une suite d'orbites périodiques, Abdenur, Bonatti, Crovisier [2] en ont déduit le résultat générique suivant :

Théorème 1.4.3 (Abdenur, Bonatti, Crovisier). *Il existe un résiduel $\mathcal{R} \in \mathrm{Diff}^1(M)$ de difféomorphismes f tels que pour toute classe de récurrence par chaînes K de f, on est en présence de l'un des deux cas suivants :*
- *il y a une décomposition dominée sur K ;*
- *K est limite de Hausdorff d'une suite de puits ou de sources.*

On en déduit une généralisation de la dichotomie générique de Mañé :

Théorème 1.4.4 (Abdenur, Bonatti, Crovisier). *Il existe un résiduel $\mathcal{R} \in \mathrm{Diff}^1(M)$ de difféomorphismes f tels que l'un des deux points suivants est vérifié :*

– *le non-errant de f admet une décomposition $\Omega(f) = \Lambda_1 \cup ... \cup \Lambda_i$ en compacts f-invariants deux à deux disjoints, qui sont chacun l'union de classes de récurrence par chaînes, et sur chacun desquels on a une décomposition dominée non-triviale ;*
– *f admet une infinité de puits ou de sources.*

1.4.3 Création de tangences homoclines loin des dominations

En dimension 2, Pujals et Sambarino [52] ont montré que s'il n'y a pas de décompostion dominée non-triviale sur l'ensemble non-errant d'un difféomorphisme f, alors par une perturbation arbitrairement petite, on crée une tangence homocline.

Wen [64] a généralisé ce résultat en dimension plus grande. Il a montré que si pour un entier $1 \leq i \leq d$, l'ensemble *i-prépériodique* (l'ensemble des points qui par perturbations arbitrairement faibles peuvent être transformées en des selles d'indice i) n'admet pas de décomposition dominée d'indice i, alors il y a une perturbation arbitrairement petite de f qui crée une tangence homocline. Il se peut cependant que la selle à laquelle est reliée la tangence homocline ait été créée au cours de la perturbation, et l'indice de cette selle ne peut être contrôlée.

La présente thèse répond à ces problèmes, en travaillant encore sur des petits voisinages d'orbites périodiques. On va montrer que si une selle a une longue période, et si la domination entre les fibrés stables et instables est suffisamment faible, alors par une petite perturbation de la dynamique autour de l'orbite on crée une tangence homocline. On dit qu'une selle Q d'un difféomorphisme f est *N-dominée* si la décomposition en les fibrés stable et instable du cocycle $df_{|\,\mathrm{Orb}(Q)}$ est N-dominée. Voici un premier énoncé du résultat annoncé :

Théorème 5.1.3 (G.). *Soit f un difféomophisme d'une variété compacte riemannienne M. Pour tout $\epsilon > 0$, on a deux entiers $N, P > 0$ tels que si Q est une selle de période $p > P$ non N-dominée, alors :*

il existe une ϵ-perturbation de f sur un voisinage arbitrairement petit $\mathcal{U}$ de l'orbite de Q, qui préserve l'orbite de Q et son indice, et qui crée une tangence homocline reliée à Q dans $\mathcal{U}$.

On peut de plus exiger que la perturbation préserve un nombre fini de points dans les variétés stables et instables de la selle périodique Q. Précisément :

Théorème 6.1.1 (G.). *Soit f un difféomorphisme d'une variété compacte riemannienne M. Pour tout $\epsilon > 0$, on a deux entiers $N, P > 0$ tels que si Q est une selle de période $p > P$ non N-dominée,*

alors pour tous ensembles finis Γ^s et Γ^u respectivement dans les variétés stable et instable, il existe une ϵ-perturbation g de f sur un voisinage arbitrairement petit $\mathcal{U}$ de l'orbite de Q, qui préserve l'orbite de Q et son indice, *qui crée une tangence homocline reliée à Q dans $\mathcal{U}$, et telle que Γ^s est dans la variété stable de g et Γ^u est dans l'instable.*

On peut en fait même demander que si $x \in \Gamma^s \cup \Gamma^u$ est dans l'une des variétés stable/instable forte de f, alors la variété stable/instable forte correpondante est également définie pour g et contient x. En particulier une telle tangence homocline peut être crée en préservant une relation homocline entre Q et tout autre point. Bien que l'une soit une conséquence de l'autre, afin d'épargner au lecteur les difficultés techniques de la preuve du théorème 6.1.1 (voir chapitre 6), on donne une preuve indépendante du théorème 5.1.3 au chapitre 5.

On déduira du théorème 6.1.1 que si la classe homocline d'une selle d'indice i n'admet pas de décomposition dominée d'indice i, alors par perturbation on obtient une tangence homocline reliée à cette classe homocline. Plus précisément :

Corollaire 6.6.2 (G.). *Soit Q un point selle de f dont la classe homocline $H(Q, f)$ est non-triviale (non réduite à l'orbite de Q) et n'admet pas de décomposition dominée de même indice que Q. Alors, il existe une perturbation arbitairement petite g de f qui préserve la dynamique sur un voisinage de Q et telle qu'il y a une tangence homocline reliée à Q.*

Abdenur, Bonatti, Crovisier, Díaz et Wen [3] ont montré que pour un difféomorphisme générique, pour toute classe homocline, l'ensemble des indices des points selles dans cette classe homocline est un intervalle. Le Corollaire 6.6.2 et ce résultat donnent une réponse partielle à la [3, Conjecture 1] :

Théorème 6.6.8. *Pour tout difféomorphisme C^1-générique f, soit $H(P, f)$ une classe homocline contenant des selles d'indice α et β, $\alpha < \beta$. Alors on a la dichotomie suivante :*
- *soit il existe une perturbation aritrairement faible g de f admettant une tangence homocline associée à la continuation d'un des points périodiques de $H(P, f)$;*
- *soit on a une décomposition dominée*

$$T_{H(P,f)}M = E \oplus E_1^c \oplus .... \oplus E_{\beta-\alpha}^c \oplus F,$$

où $dim(E) = \alpha$ et chaque E_i^c est de dimension 1 et non hyperbolique.

1.5 Schéma de preuve du théorème 4.5.2

1.5.1 Cocycles à grande périodes

Comme évoqué précédemment, le lemme de Franks 1.3.5 permet de se ramener à l'étude des perturbations de la différentielle le long d'une orbite périodique : en effet, il suffit de trouver une perturbation de la différentielle df de f restreinte à l'orbite d'un point périodique Q qui rend toutes les valeurs propres au premier retour de même module ; puis le lemme de Franks permet de réaliser cette perturbation par une perturbation du difféomorphisme f. Plus précisément, on trouvera une perturbation g de f sur un petit voisinage de l'orbite de Q, qui préserve l'orbite de Q et telle que dg le long de l'orbite de Q est égale à la perturbation de df voulue.

On s'est donc ramené à la perturbation de cocycles linéaires. On va maintenant montrer qu'à borne fixée sur les normes des cocycles, si un cocycle n'a pas de décompostion dominée suffisamment forte, une petite perturbation rend égaux les modules des valeurs propres de l'application de premier retour. Bien que [16] ne le prouve que pour les grandes périodes, il se trouve que cette hypothèse est inutile. Le cas des grandes périodes est traité dans les sections 4.3 et 4.4. On en déduit facilement le cas général, dont la preuve est faite dans l'appendice du même chapitre.

L'idée dans le cas des grandes périodes est de considérer les perturbations qui minimisent le diamètre spectral, c'est-à-dire le diamètre de l'ensemble des exposants de Lyapunov le long de l'orbite.

Un cocycle linéaire $\mathcal{A} = (A, f)$ sur un fibré $\mathcal{E} = (E, \Sigma)$ est appelé *cocycle à grandes périodes* si les trois conditions suivantes sont vérifiées :

- Σ est infini ;
- tout point $x \in \Sigma$ est périodique pour f ;
- pour tout $p \in \mathbb{N}$, l'ensemble des points de période p dans Σ est fini.

Un tel cocycle $\mathcal{A}$ est dit *strictement sans domination* si les seuls ensembles en restriction desquels le cocycle $\mathcal{A}$ admet une décomposition dominée non-triviale sont finis. Les valeurs propres d'un cocycle linéaire sont les valeurs propres de l'application de premier retour le long d'une orbite périodique. On dit que les valeurs propres d'un cocycle ont des *modules distincts* si les valeurs propres de chaque application de premier retour sont de modules deux à deux distincts.

Une *perturbation* d'un cocycle à grandes périodes $\mathcal{A} = (A, f)$ sur un fibré $\mathcal{E} = (E, \Sigma)$ est un cocycle $\mathcal{B} = (B, f)$ sur $\mathcal{E}$ tel que pour tout $\epsilon > 0$, il n'y a qu'un nombre fini de points $x \in \Sigma$ au dessus desquels il y a un vecteur unitaire u vérifiant $\|\mathcal{A}(u) - \mathcal{B}(u)\| \geq \epsilon$. En d'autre termes, $\mathcal{B}$ est ϵ-proche de $\mathcal{A}$ presque partout. Une perturbation d'une perturbation est une perturbation. Cette définition va nous épargner la manipulation de multiples ε et rendra énoncés et preuves plus lisibles et plus agréables.

Enonçons maintenant le résultat principal sur les cocycles :

Théorème 1.5.1. *Soit $\mathcal{A}$ un cocycle borné à grandes périodes. Alors si $\mathcal{A}$ est strictement sans domination, il existe une perturbation $\mathcal{B}$ de $\mathcal{A}$ telle que pour tout $x \in \Sigma$, l'application de premier retour $\mathcal{B}^p$ en x a des valeurs propres réelles de même module.*

1.5.2 Résumé de la preuve du théorème 1.5.1

On constate d'abord qu'il est suffisant de montrer que sous les mêmes hypothèses, il existe un ensemble infini $\Sigma' \subset \Sigma$ sur lequel une perturbation de $\mathcal{A}$ a ses valeurs propres réelles et de même module. Dans ce but on commence par montrer qu'il y a une perturbation $\mathcal{B}$ de $\mathcal{A}$ qui rend toutes les valeurs propres réelles et de modules distincts.

Etant donné un cocycle linéaire, on appelle *diamètre spectral au point* $x \in \Sigma$ la différence entre le plus grand exposant de Lyapunov et le plus petit en ce point. On essaie maintenant de minimiser par perturbations le diamètre spectral sur un sous-ensemble infini de Σ. Précisément, on veut minimiser le *diamètre spectral inférieur* de $\mathcal{B}$, c'est-à-dire la borne inférieure des nombres $\rho > 0$ tels qu'il y ait une infinité de points de Σ en lesquels le diamètre spectral est plus grand que ρ.

Remarquons que si le diamètre spectral inférieur d'une perturbation $\mathcal{B}$ de $\mathcal{A}$ est nul, alors on trouve un sous-ensemble infini $\Sigma' \subset \Sigma$ et une seconde perturbation $\mathcal{C}$ de $\mathcal{B}$ telle que par restriction a Σ', $\mathcal{C}$ a toutes ses valeurs propres réelles et de même module. Enfin $\mathcal{C}$ est évidemment une perturbation de $\mathcal{A}$, ce qui termine la preuve. Il reste donc a trouver un tel $\mathcal{B}$.

On montre en fait qu'il existe une perturbation $\mathcal{B}$ de $\mathcal{A}$ qui a des valeurs propres réelles de modules distincts, et un sous-ensemble infini $\Sigma' \subset \Sigma$ tel que la restriction $\mathcal{B}'$ à Σ' est *incompressible*, c'est-à-dire

– elle admet un *diamètre spectral* $\delta(\mathcal{B}')$: pour tout $\epsilon > 0$ il n'y a qu'un nombre fini de de points $x \in \Sigma'$ auxquels le diamètre spectral est hors de l'intervalle $[\delta(\mathcal{B}') - \epsilon, \delta(\mathcal{B}') + \epsilon]$;
– elle minimise le diamètre spectral inférieur : toute perturbation de $\mathcal{B}'$ a un diamètre spectral inférieur plus grand que que $\delta(\mathcal{B}')$.

On remarquera alors que si $\mathcal{A}$ est strictement sans domination, $\mathcal{B}$ l'est également, ainsi que la restriction $\mathcal{B}'$. Il est maintenant clairement suffisant de montrer que si un cocycle est incompressible et strictement sans domination, alors son diamètre spectral est nul. Dans le paragraphe suivant on en donne une idée de démonstration, concluant ainsi le schéma de preuve du théorème 1.5.1.

Voici de façon résumée comment, par récurrence sur la dimension du fibré, on obtient qu'un cocycle incompressible et strictement sans domination a un diamètre spectral nul. En dimension 2, les techniques de R. Mañé [38] en donnent une preuve. Supposons maintenant par l'absurde qu'un cocycle incompressible $\mathcal{A}$ d'un fibré $\mathcal{E}$ de dimension ≥ 3 n'a strictement pas de domination, et a un diamètre spectral strictement positif. On montre d'abord l'existence d'un sous-ensemble infini $\Sigma' \subset \Sigma$ et d'une décomposition invariante $\mathcal{E}_{|\Sigma'} = F \oplus G$ au-dessus de Σ' en deux sous-fibrés non-triviaux tels que les valeurs propres dans F sont de plus petits modules que dans G, et tels que les restrictions $\mathcal{A}_{|F}$ et $\mathcal{A}_{|G}$ sont strictement sans domination. De là, par incompressibilité, on déduit la nullité des diamètres spectraux de $\mathcal{A}_{|F}$ et $\mathcal{A}_{|G}$.

Comme la dimension est plus grande que 3 et les valeurs propres sont réelles, on peut supposer que $\dim(F) \geq 2$ et trouver un sous-fibré unidimensionnel invariant $H \subset F$. En adaptant un lemme de [13], comme $\mathcal{A}_{|F\oplus G}$ est strictement sans domination, au moins l'une des deux assertions suivantes est vérifiée :

- la décomposition $H \oplus G$ n'est pas dominée pour $\mathcal{A}$;
- la décomposition $F/H \oplus G/H$ n'est pas dominée pour le cocycle quotient $\mathcal{A}/H$.

Par hypothèse de récurrence, on montre enfin que chacun des deux cas autorise une nouvelle perturbation qui décroît strictement le diamètre spectral inférieur de $\mathcal{A}$, ce qui contredit l'incompressibilité. CQFD.

1.6 Schéma de preuve du théorème 5.1.3

Remarque 1.6.1. *On prévient le lecteur que la définition d'N-domination que nous utiliserons dans les preuves des théorèmes 5.1.3 et 6.1.1 est légèrement différente de la définition utilisée jusqu'à présent. Cependant, par le corollaire 1.4.1, le théorème que nous montrerons ainsi implique le théorème annoncé.*

La preuve du théorème 6.1.1 suit approximativement le même chemin ; elle est cependant beaucoup plus technique et la nécessité d'une nouvelle terminologie ne permet pas qu'on la résume ici.

Comme pour le théorème 1.5.1, on travaille sur des petits voisinages d'orbites périodiques ; cependant on doit contrôler plus que la simple dérivée le long de l'orbite périodique. C'est pourquoi l'outil bien pratique de [16] ne s'appliquera pas ici.

On perturbe la dynamique sur des petits voisinages d'orbites selles, tout en préservant ces orbites. Ceci justifie la notion de difféomorphisme selle : un *difféomorphisme selle* est un difféomorphisme f d'un fibré vectoriel euclidien $\mathcal{E}$ au dessus d'une base finie Σ (on voit ce fibré comme une variété riemannienne non connexe), tel que f permute cycliquement les fibres, et la section nulle $0_{\mathcal{E}}$ du fibré est l'orbite d'une selle. Un difféomorphisme selle est *N-dominé* si la décomposition en les directions stable et instable de df au dessus de la section nulle est N-dominée.

On veut montrer que si un difféomorphisme selle a une grande période et n'est pas suffisamment fortement dominé, alors on peut le perturber sur un voisinage arbitrairement petit de la section nulle afin d'obtenir une tangence homocline reliée à la section nulle. De même que dans la preuve du théorème 1.5.1, on voit que si la période est grande, alors une petite perturbation du cocycle $df_{|0_{\mathcal{E}}}$ a toutes ses valeurs propres réelles et de modules deux à deux distincts. Par le lemme de Franks, on déduit que si un difféomorphisme selle f a une période suffisamment grande, alors il existe une petite perturbation de f qui est encore un difféomorphisme selle et qui a des valeurs propres réelles de modules deux à deux distincts.

Par une autre perturbation, on linéarise localement et on se ramène à l'étude d'un difféomorphisme selle linéaire, à valeurs propres réelles et de modules distincts. On montre alors par récurrence sur la dimension du fibré, que si un tel difféomorphisme selle n'est pas suffisamment fortement dominé, alors une petite perturbation crée une tangence homocline.

La dimension 2 a été prouvée par Pujals and Sambarino, sous une forme équivalente. En plus grande dimension, on écrit la décomposition stable/instable $\mathcal{E} = F \oplus G$. Quitte à changer f en f^{-1}, on peut supposer que la direction stable F du difféomorphisme linéaire f est de dimension ≥ 2. Comme les valeurs propres sont réelles, elle contient un sous-fibré unidimensionnel H invariant. Comme la domination entre les directions stable et instable est faible, on applique encore le lemme de [13], de sorte qu'on est dans l'un des deux cas suivants :

- la décompostion $H \oplus G$ est faiblement dominée pour f,
- la décomposition $F/H \oplus G/H$ est faiblement dominée pour le difféomorphisme linéaire quotient f/H.

Dans le premier cas, par hypothèse de récurrence, une petite perturbation de la restriction $f_{|H\oplus G}$ admet une tangence homocline. On montre alors qu'elle peut être étendue en une petite perturbation de f qui présentera clairement une tangence homocline.

Dans le second cas, par hypothèse de récurrence, une petite perturbation du quotient f/H admet une tangence homocline. On la relève en une petite perturbation de f, et on montre qu'une telle perturbation admet également une tangence homocline.

Remarquons que dans les deux cas on n'a a priori aucun contrôle de la taille des voisinages de $0_{\mathcal{E}}$ sur lesquels on perturbe f. Néanmoins, en conjuguant par une homothétie on peut faire en sorte que l'orbite de la tangence homocline reste proche de $0_{\mathcal{E}}$, et grâce à une version plus sophistiquée du lemme de Franks, on construit enfin une perturbation de f sur un petit voisinage de $0_{\mathcal{E}}$, qui admet une tangence homocline. CQFD.

Chapter 2

General Introduction

2.1 Birth of the notion of dominated splitting

A diffeomorphism $f \in \mathrm{Diff}^k(M)$, for some $k \geq 1$, is said to be *structurally stable* if any neighbouring diffeomorphism g is conjugate to f via a homeomorphism h of M. We recall that a periodic point of period p for a diffeomorphism f is *hyperbolic* if the p-th iterate of the derivative df^p has no eigenvalue of modulus one. It is a *saddle* if it has both eigenvalues of moduli greater than one and smaller than one. In the thirties, Andronov and Pontrjagin [5] saw a correspondence between structural stability and hyperbolic behaviours. Indeed, the local structural stability of a dynamics around a saddle point comes from a Hartmann-Grobmann theorem (see [31]).

In the early sixties, D.V. Anosov [6] proved the structural stability of the geodesic flow of Riemannian compact manifolds of negative curvature. He generalized that result [7] to a whole class of flows and diffeomorphisms: an *Anosov flow* on a compact Riemannian manifold M is a flow without singularity for which the tangent bundle TM splits into three supplementary invariant bundles:

- a *stable bundle*, that is, a uniformly contracted one: there is a real number $T > 0$ such that the time-T flow strictly contracts any nonzero vector in it,
- an *unstable bundle*, that is, a uniformly expanded one: there is a real number $T > 0$ such that the time-T flow strictly expands any nonzero vector in it.
- a one-dimensional bundle which is neither contracted nor expanded,

The notion of *Anosov diffeomorphism* is defined similarly, omitting the third bundle.

Contemporarily to it, S. Smale [59] exhibited a new class of very simple dynamical systems now known as *Morse-Smale*, which were proven to be structurally stable. A point x is *non-wandering* for a diffeomorphism f if for any neighbourhood $\mathcal{U}$ of x, there is an integer $n \geq 1$ such that $f^n(\mathcal{U})$ intersects $\mathcal{U}$. A diffeomorphism is called Morse-Smale, if its non-wandering set $\Omega(f)$ (the set of its non-wandering points) is composed of finitely many hyperbolic periodic points, and if for any pair of points in $\Omega(f)$ the stable manifold of the one and the unstable manifold of the other are in general position, that is, the two manifolds meet transversely at any intersection point.

2.1.1 The hyperbolic theory

Smale introduced a general notion which encloses Anosov and Morse-Smale diffeomorphisms. A compact invariant set K for a diffeomorphism f is *hyperbolic* if and only if there is a splitting of the tangent bundle restricted to K into two supplementary bundles E^s and E^u, invariant by the derivative df, such that the one is uniformly contracted and the other uniformly expanded by an iterate of df. Precisely, there exists $n \in \mathbb{N}$ and $0 < \lambda < 1$ such that for any unit vectors u, v in E^s, E^u, $\|df^n(u)\| < \lambda^{-1}$ and $\|df^n(v)\| > \lambda$.

A diffeomorphism is *hyperbolic* or *Axiom A* if its non-wandering set is hyperbolic and if the set $\mathrm{Per}(f)$ of periodic points of f is dense in $\Omega(f)$. Smale's Spectral Decomposition Theorem applies to any Axiom A diffeomorphism and provides a decomposition of the non-wandering set into a union $\Omega(f) = \Lambda_1 \cup ... \cup \Lambda_i$ of a finite number of pairwise disjoint, compact, isolated, f-invariant, transitive sets (a set is transitive, if it contains some dense orbit). Moreover, the α-limit set and ω-limit set of every orbit are respectively contained in some Λ_i. In particular, if $\Omega(f) = M$, there is only one basic set and f is an Anosov diffeomorphism.

An axiom A diffeomorphism satisfies the *no-cycle condition* if and only if there is no cycle of basic pieces, that is, no cyclic sequence of basic pieces, such that the unstable set of each basic piece intersects the stable set of the next. It satisfies the *strong transversality condition* if for any x, y in the non-wandering set, the unstable manifold of the one and the unstable manifold of the other are transversal.

J. Palis and S. Smale conjectured in [47] that

- Axiom A and strong transversality condition is equivalent to structural stability, that is, conjugacy to any neighbouring diffeomorphism.

- a diffeomorphism $f \in \mathrm{Diff}^r(M)$ is Axiom A and *no-cycle condition* if and only if it is Ω-stable, that is, for any neighbouring g, the restrictions $f_{|\Omega(f)}$ and $g_{|\Omega(g)}$ to the respective non-wandering sets are conjugate by a homeomorphism.

This was called the *Stability Conjecture.* Smale [62] showed that Axiom A and *no-cycle condition* implies Ω-stability. Robbin [53] first showed that in

C^2 topology, Axiom A and strong transversality implies structural stability, then Robinson [54] showed it in C^1 topology. In the late eighties, Ricardo Mañé [39] completed the proof that C^1 structural stability implies hyperbolicity and strong transversality. Relying on Mañé's techniques, Palis [45] showed that Ω-stability implies Axiom A and no-cycle.

It was believed in the sixties that the hyperbolic diffeomorphisms of a compact manifold M were C^r dense in the set of diffeomorphisms for some $r \geq 1$. But it quickly appeared that there are open sets of non-structurally stable diffeomorphisms [60] and of non-Ω-stable diffeomorphisms [4].

2.1.2 Partial Hyperbolicity and weaker forms of stability

In order to study the existence and behaviour of diffeomorphisms far from hyperbolicity, Hirsch, Pugh, Shub [35] and Brin, Pesin [19] considered *partially hyperbolic systems*, that is, admitting an invariant splitting of the tangent bundle into a direct sum $E^s \oplus E^c \oplus E^u$ of three subbundles, where E^s is uniformly contracted by the dynamics, E^u uniformly expanded, and E^c is central, i.e. less contracted than E^s and less dilated than E^u. It is easily shown that the bundles of a partially hyperbolic splitting are continuous, persist and vary continuously by C^1 perturbations.

Most notably they showed the integrability of the stable and unstable bundles. Precisely, for a partially hyperbolic C^r system, there exists a smooth stable foliation $\mathcal{F}^s$ that is tangent to the stable bundle E^s, and symmetrically there is a smooth unstable foliation $\mathcal{F}^u$ that is tangent to the unstable bundle E^u. On the other hand, in general, the central bundle is not integrable. Clues are given in [61, Section 3.1] to build diffeomorphisms on the 6-dimensional torus that are robustly non-hyperbolic, partially hyperbolic with a non-integrable central bundle.

But, when a central foliation exists, we have in general a weak form of structural stability. One says that a central foliation of a partially hyperbolic f is structurally stable if for any nearby C^1-diffeomorphism g, the central bundle of g is integrable and if there is a homeomorphism h of M, that sends the central foliation of f on the central one of g, such that $h \circ f \circ h^{-1}$ is isotopic to g along the central leaves. The main result of [35] is that if a partially hyperbolic system has a central foliation that is plaque-expansive, then it is structurally stable.

2.1.3 On dominated splittings

A weaker form of hyperbolicity is the notion of *dominated splitting*. It was independantly introduced by Liao [36] and Mañé [38] in their works on the stability conjecture. A dominated splitting for a given dynamics is a splitting of the tangent manifold into a direct sum of invariant subbundles, such that the vector expansion on each bundle is uniformly less (or the vector contraction is uniformly greater) than on the next bundle. A precise definition is given in section 2.3.1.

From stability of conefields around the bundles, one shows that a dominated splitting persists by C^1-perturbation of the dynamics, and that the bundles are continuous in TM and vary continuously (see [14]). The bundles of a dominated splitting are in general not integrable, except the extremal bundles if uniformly contracting or expanding.

2.2 Persistently non-hyperbolic behaviours

A property on a C^r diffeomorphism f is said to be *(C^r-)robust* if it persists by C^r-perturbations of f. Abraham and Smale [4] in dimension ≥ 4, then Simon [58] in dimension ≥ 3 built examples of C^1-robust non Axiom A diffeomorphisms.

2.2.1 Homoclinic bifurcations

In dimension 2, it is not known if Axiom A diffeomorphisms are dense for the C^1 topology, but S. Newhouse showed the existence of an open subset $\mathcal{U}$ of $\text{Diff}^2(M^2)$ such that any f in a residual subset of $\mathcal{U}$ has an infinite number of sinks or sources. The property of having infinitely many sinks or sources is called *Newhouse phenomenon*. To build this example, Newhouse studied the dynamics in the neighbourhood of homoclinic tangencies.

Definition 2.2.1. A *Homoclinic tangency* is a non-transverse intersection between the stable and unstable manifolds of a saddle orbit.

A homoclinic tangency is very unstable and can be broken by any slight perturbation. However, Newhouse [42, 43] showed that close to any C^2 diffeomorphims f of a surface admitting a homoclinic tangency, there is an open set $\mathcal{U}$ of diffeomorphisms, and a residual subset $\mathcal{R} \in \mathcal{U}$, such that every $g \in \mathcal{R}$ displays a Newhouse phenomenon.

Another viewpoint to study non-hyperbolic behaviors was that of transitivity. One says that a dynamics on M is *transitive* if and only if there is a dense orbit in M. It was asked whether robust transitivity charaterized Anosov diffeomorphisms. Several examples of open sets of robustly transitive, non-hyperbolic diffeomorphisms appeared then in dimensions ≥ 3.

As a consequence of the structural stability of central foliations, [35] obtains, by a C^0-perturbation of an Anosov diffeomorphism on the four-dimensional torus, a diffeomorphism that is robustly transitive and has two saddles of different indices. If a neighbouring diffeomorphism was hyperbolic, by transitivity, the unique basic piece would be M. However there can not be two periodic saddles of different indices in a same basic piece. Hence, this is a robustly non-hyperbolic diffeomorphism.

By different methods, Mañé [37] proves that some C^0-perturbation of an Anosov diffeomorphism on the three-dimensional torus is also robustly transitive and admits two saddle points of different indices. That example is far from homoclinic tangencies but is in the closure of a set of diffeomorphisms that have a heterodimensional cycle:

Definition 2.2.2. A *heterodimensional cycle* is a pair of saddle points of different indices, such that the unstable manifold of one intersects the stable manifold of the other.

A heterodimensional cycle is also an unstable phenomenon, however C. Bonatti and L. Diaz [12] showed that close to a codimension 1 heterodimensional cycle (i.e. the difference of indexes between the two saddles is 1), the closure of the diffeomorphisms that present a heterodimensional cycle has nonempty interior.

2.2.2 Palis' density conjectures

Palis proposed to characterise robustness of non-hyperbolic behaviour by local density of homoclinic tangencies or heterodimensional cycles. This is the purpose of his

Conjecture 2.2.3 (Palis' C^r-density conjecture). *The union of hyperbolic diffeomorphisms, and diffeomorphisms admitting a homoclinic tangency or a heterodimensional cycle is dense in* $\mathrm{Diff}^r(M)$*.*

A weak version of that conjecture asserts that there is a dichotomy between Morse-Smale diffeomorphisms and diffeomorphisms that admit a homoclinic intersection.

Conjecture 2.2.4 (weak C^r-density conjecture). *The set of Morse-Smale diffeomorphisms and the set of diffeomorphisms that admit a homoclinic intersection, are two disjoint open sets whose union is dense in* $\mathrm{Diff}^r(M)$*.*

In 1967, Pugh [51] showed the C^1-closing lemma, that is, if x is a recurrent point for a diffeomorphism f, then there is an arbitrarily small C^1-perturbation g of f for which x is a periodic point. In the 90s Hayashi [33] showed the C^1-Connecting Lemma: under some recurrence-like condition on the stable and unstable manifolds of a pair of saddle points of a diffeomorphism f, an arbitrarily small perturbation of f creates a cycle between these two saddles.

These two lemmas are all-important to perturb and create saddle periodic points, therefore homoclinic tangencies, and heterodimensional cycles. They are the basis of all existing developments towards the C^1-density conjecture. However no C^r-closing nor connecting lemmas are known for $r > 1$: the C^r-density conjectures seem out of reach for the time being.

The weak C^1-density conjecture was shown by Bonatti, Gan and Wen [15] in dimension 3. Very recently, S.Crovisier [23], using the works of Wen [65] and introducing an original way to study the dynamics along the central bundles, proved it in any dimension:

Theorem 2.2.5 (Crovisier). *The set of Morse-Smale diffeomorphisms and the set of diffeomorphisms that admit a homoclinic intersection, are two disjoint open sets whose union is dense in* $\mathrm{Diff}^1(M)$.

Recent advances suggest that a proof of the C^1-density conjecture might be attainable.

In 2000, E. Pujals and M. Sambarino in [52] showed the C^1-density conjecture of Palis for surfaces: a surface diffeomorphism can be C^1 approximated either by Axiom A diffeomorphisms or by diffeomorphisms admitting a homoclinic tangency. Let us give the two main steps of their proof. They first showed (Lemma 2.0.2) that in the complementary set of the closure of the diffeomorphisms with a homoclinic tangency, there is an open and dense set of diffeomorphims g whose non-wandering set $\Omega(g)$ admits a non-trivial dominated splitting.

Then a second result (Theorem B.) says if f is a C^2 diffeomorphism on a compact surface and if $T_{|\Lambda}M = E \oplus F$ is a dominated splitting above a compact invariant set Λ for f, then Λ is the disjoint union of a hyperbolic set and a finite number of periodic simple closed curves $\mathcal{C}^i$ that are normally hyperbolic, and such that the restriction $f^{p_i}{}_{|}\mathcal{C}^i$ is conjugate to an irrational rotation (p_i is the period of the curve $\mathcal{C}^i$).

The C^1-density conjecture in dimension two is easily obtained combining these two results. Paralleling that method, one natural way in higher dimension is first to show that far from homoclinic tangencies, there are thin enough dominated splittings on the non-wandering set (or chain-recurrent set, or homoclinic classes).

Hayashi claimed some breakthroughs for a proof of the C^1 Palis conjecture in higher dimensions, however no paper has yet been released. In [65], L. Wen made some progress towards it, cleverly combining previous results [64] and Liao's selecting lemma [36] to prove that generically, far from hyperbolic dynamics, homoclinic tangencies and heterodimensional cycles, the minimal non hyperbolic sets admit a partially hyperbolic splitting, with one or two one-dimensional central bundles.

2.3 Basic definitions and notations

Before stating the main results of this thesis, we recall elementary definitions on bundles, linear cocycles, and dominated splittings. We also define homoclinic classes and mention Conley's theory, as important tools to study C^1-generic dynamics.

2.3.1 Linear cocycles and dominated splittings

Let $\mathcal{E} = (E, \Sigma, \pi \colon E \to \Sigma)$ be a dimension d linear bundle above a base Σ, that is, for all $x \in \Sigma$, the fibre $E_x = \pi^{-1}(x)$ above x is a dimension d vector space. As similarly defined in [16] and [13], we say that a couple of bijections $\mathcal{A} = (f \colon \Sigma \to \Sigma, A \colon E \to E)$ is a *linear cocycle* or *automorphism* on $\mathcal{E}$ if and only if, for all $x \in \Sigma$, the map A induces by restriction a vector spaces isomorphism from the fibre E_x to $E_{f(x)}$, that is, the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{A} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Sigma & \xrightarrow{f} & \Sigma \end{array} \quad . \tag{2.1}$$

A dimension d *Euclidiean bundle* $(\mathcal{E}, \|.\|)$ is a dimension d linear bundle such that each fibre is endowed with a Euclidean metric $\|.\|$. For any Euclidean bundle $\mathcal{E}$, for any bijection f on its base Σ, we endow the set of linear cocycles above the dynamics f with the following distance:

$$\mathrm{dist}(\mathcal{A}, \mathcal{B}) = \sup\{\|\mathcal{A}(u) - \mathcal{B}(u)\|, \|\mathcal{A}^{-1}(u) - \mathcal{B}^{-1}(u)\| / u \in \mathcal{E}, \|u\| = 1\}.$$

We say that a linear cocycle is *bounded by* $C > 0$ if and only if for any unit vector $u \in \mathcal{E}$, we have $\|\mathcal{A}(u)\|, \|\mathcal{A}^{-1}(u)\| < C$.

Definition 2.3.1. A *dominated splitting* for a linear cocycle $\mathcal{A}$ on a Euclidiean bundle $\mathcal{E}$ is an $\mathcal{A}$-invariant splitting $\mathcal{E} = E_1 \oplus ... \oplus E_l$ such that there is $C > 0$ and $0 < \lambda < 1$ such that for any $1 \le i < l$, for any pair of unit vectors $u, v \in E_i, E_{i+1}$, for all $n \in \mathbb{N}$, we have $\|f^n(u)\| < C\lambda^n \|f^n(v)\|$.

We then say that E_i is dominated by E_{i+1}, and write $E_i \prec E_{i+1}$. Obviously, if $\mathcal{E} = E_1 \oplus ... \oplus E_l$ is dominated, then for all $1 \le i < j \le l$, we have $E_i \prec E_j$. Finding stable cone fields, one shows that a dominated splitting persists by perturbation, and varies continuously: for any family of neighbourhoods $\mathcal{V}_1, ..., \mathcal{V}_l$ of the bundles $E_1, ..., E_l$, there is $\epsilon > 0$ such that any linear cocycle $\mathcal{B}$ so that $\mathrm{dist}(\mathcal{A}, \mathcal{B}) < \epsilon$, admits a dominated splitting $F_1 \oplus ... \oplus F_l$, where $F_i \in \mathcal{V}_i$ for all i.

Moreover, in the particular case where $\mathcal{E}$ is the tangent space of a manifold M restricted to a compact set Σ, and $\mathcal{A}$ is the derivative df of a diffeomorphism of M that leaves the compact set Σ invariant, then (see [14, Page 292], for instance) the subbundles E_i of $TM_{|\Sigma}$ are continuous.

We say that an invariant bundle F is *stable* or *uniformly contracted* by $\mathcal{A}$ if and only if there exist $C > 0$ and $0 < \lambda < 1$ such that, for any unit vector $u \in F$, for all $n \in \mathbb{N}$, we have $\|\mathcal{A}^n(u)\| \le C.\lambda^n$. We say that it is *unstable* or *uniformly expanded* if it is uniformly contracted by the converse linear cocycle $\mathcal{A}^{-1}$.

Definition 2.3.2. A *partially hyperbolic splitting* for a linear cocycle $\mathcal{A}$ on $\mathcal{E}$ is a dominated splitting $\mathcal{E} = E^s \oplus E_1 \oplus ... \oplus E_l \oplus E^u$ such that E^s is stable and E^u is unstable for $\mathcal{A}$.

Given a Euclidean bundle $\mathcal{E}$ with compact base Σ, and a dominated splitting $\mathcal{E} = E_1 \oplus ... \oplus E_l$ for a linear cocycle $\mathcal{A}$, an *adapted metric* for that dominated splitting is a metric $\|.\|_*$ on each fibre of $\mathcal{E}$ such that for all $1 \leq i < l$, for all unit vectors $u, v \in E_i, E_{i+1}$, we have $\|\mathcal{A}(u)\|_* < \|\mathcal{A}(v)\|_*$. If $\mathcal{A}$ admits a partially hyperbolic splitting $\mathcal{E} = E^s \oplus E_1 \oplus ... \oplus E_l \oplus E^u$, the metric $\|.\|_*$ is *adapted* to it if and only if it is adapted to the corresponding dominated splitting, and if for all unit vectors $u, v \in E^s, E^u$ we have $\|\mathcal{A}(u)\|_* < 1$ and $\|\mathcal{A}(v)\|_* > 1$.

2.3.2 Strength of domination

We say that an invariant splitting $\mathcal{E} = F \oplus G$ for a linear cocycle $\mathcal{A}$ is N-*dominated* if, after N iterations, the vectors of F are twice more dilated, or twice less contracted than those of E, that is, for any unit vectors $u, v \in F, G$, we have $\|\mathcal{A}^N(u)\| < 1/2\|\mathcal{A}^N(v)\|$. Therefore, the greater N is, the weaker the domination is.

Remark 2.3.3. *According to that definition, if an invariant splitting is N-dominated, it is not necessarily M-dominated, for all $M > N$.*

However, that remark is addressed if the metric $\|.\|$ is adapted to the dominated splitting. In [29], the definition of an N-dominated splitting differs from the one we gave here; namely, a splitting is N-dominated according to [29] if it is L-dominated for some $L < N$. We will see that these two definitions are equivalent, to some extent (see Corollary 2.4.2).

2.3.3 About recurrence notions

Homoclinic classes

In this section, M is a compact Riemannian manifold and $\text{Diff}^r(M)$ is the set of C^r-diffeomorphisms on M, for some $r \geq 1$. Let f be a diffeomorphism in $\text{Diff}^r(M)$. One says that an f-invariant compact set Λ is *hyperbolic*, if and only if there is an invariant splitting $TM_{|\Lambda} = E^s \oplus E^u$ where E^s is a stable bundle for df and E^u is an unstable bundle.

A periodic point P for f is a *saddle* if its orbit $\text{Orb}_f(P)$ is hyperbolic and the stable and unstable bundles are both non-trivial. The stable set $W^s(\text{Orb}_f(P)) = \{x \in M/\operatorname{dist}(f^n(P), \text{Orb}_f(P)) \to 0\}$ and the unstable set $W^u(\text{Orb}_f(P)) = \{x \in M/\operatorname{dist}(f^{-n}(P), \text{Orb}_f(P)) \to 0\}$ are both C^r embedded manifolds that are tangent to the bundles E^s and E^u, respectively.

We say that two saddle points are *homoclinically related* if and only if the unstable manifold of the orbit of one intersects transversely the stable manifold of the orbit of the other. This defines an equivalence relation on periodic saddle points.

Definition 2.3.4. The *homoclinic class* $H(Q, f)$ of a saddle point Q of a diffeomorphism f is the closure of the set of hyperbolic points that are homoclinically related to Q.

By definition a homoclinic class contains a dense subset of periodic points, therefore it is in the non-wandering set of f. The homoclinic class of Q is topologically transitive and can be equivalently defined as the closure of the transverse intersections of the stable and unstable manifolds of Q (see [44]).

Conley Theory

We briefly describe a recurrence notion introduced by Conley [21]. A sequence (x_n) is an *ϵ-pseudo-orbit* for some $\epsilon > 0$ if and only if the distance $\mathrm{dist}(f(x_n), x_{n+1})$ is less than ϵ for each n. The *chain-recurrent set* $\mathcal{R}(f)$ is the set of points x such that, for all $\epsilon > 0$, there is an ϵ-pseudo-orbit that is not reduced to a single point and that goes from x to x. We define an equivalence relation $\sim$ on $\mathcal{R}(f)$ in the following way: $x \sim y$ if and only if, for all $\epsilon > 0$, there is an ϵ-pseudo-orbit from x to y, and an ϵ-pseudo-orbit from y to x.

The *chain-recurrent classes* are the equivalence classes of $\sim$ in $\mathcal{R}(f)$. For M is compact, the chain recurrent set is compact and not empty, and each chain recurrent class is compact. Clearly, the non-wandering set is contained in the chain-recurrent set, and since a homoclinic class is transitive, each homoclinic class is in a chain-recurrent class.

A *residual subset* of a topological space is a countable intersection of dense open sets. A dynamical property $\mathcal{P}$ is *C^r-generic* if it is satisfied by a residual set of dynamics for the C^r-topology. We also say that *C^r-generically*, the dynamics satisfies property $\mathcal{P}$. In [10], C. Bonatti and S. Crovisier showed that C^1-generically, the non-aperiodic chain recurrent classes and the homoclinic classes coincide:

Theorem 2.3.5 (Bonatti, Crovisier). *Given a compact manifold M, there is a residual subset $\mathcal{HCR}$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that each homoclinic class of f is a chain-recurrent class and conversely, each chain-recurrent class is either aperiodic (does not contain any periodic point) or is a homoclinic class.*

Crovisier [22] recently showed that generically a chain-recurrent class is a Hausdorff limit of saddle orbits. Note that these results were obtained working on Pugh's closing lemma and Hayashi's connecting lemma and can be seen as generalizations of them; indeed, Theorem 2.3.5 implies the C^1-connecting lemma, and Crovisier's clearly implies the C^1-closing lemma.

2.3.4 Perturbations of diffeomorphisms

Let M be a Riemannian compact connected manifold. Then there is a canonical distance $\text{dist}\colon TM \times TM \to \mathbb{R}^+$ on the tangent bundle TM. A diffeomorphism f is an *ϵ-perturbation of g* for the C^1 topology, if and only if, for any unit vector $v \in TM$, we have $\text{dist}(df(v), dg(v)) \leq \epsilon$ and $\text{dist}(df^{-1}(v), dg^{-1}(v)) \leq \epsilon$. Given a subset S of M, f is a *pertubation of g on S* if $f = g$ outside S.

A lemma by Franks [26] allows to extend a perturbation of the derivative of a diffeomorphism along a periodic orbit into a local perturbation of the diffeomorphism on the whole manifold. We restate it:

Lemma 2.3.6 (Franks). *Let f be a diffeomorphism on a compact Riemannian manifold M. For any $\delta > 0$ there is $\epsilon > 0$ such that the following holds:*

if Σ is a finite f-invariant set, and if $\mathcal{A}$ is a cocycle on $TM_{|\Sigma}$ with $\text{dist}(\mathcal{A}, df_{|\Sigma}) < \epsilon$*, then there is a δ-perturbation g of f on an arbitrarily small neighbourhood of Σ such that $dg_{|\Sigma} = \mathcal{A}$.*

With this lemma, we will reduce the perturbation problems to linear perturbation ones. It will allow to work on linear cocycles in the proof of Theorem 4.5.2, and to work on what we will call linear saddle diffeomorphisms in the proof of Theorem 5.1.3. Note that Franks' lemma is specific to the C^1-topology.

2.4 Statement of Results

This thesis essentially studies the local dynamics of diffeomorphisms along periodic orbits, to find what behaviours can be obtained by small perturbations in absence of some dominated splittings along that orbit. The consequences will be, among others, generic dichotomies between Newhouse phenomenons and diffeomorphims admitting some dominated behaviours, dichotomies between homoclinic tangencies and stable/unstable dominated splittings.

2.4.1 Adapted metrics

In Chapter 3, we answer an old question from Hirsch, Pugh, Shub [35, page 5] on existence of adapted metrics for dominated splittings. It was known that a hyperbolic splitting admits an adapted metric. We give an outline of a proof of it:

Let K be a compact invariant set for a diffeomorphism f, and $TM_{|K} = E^s \oplus E^u$ be an invariant hyperbolic splitting for f. Then, for any vector $u \in E^s$, $\|df^n(u)\|$ converges exponentially to zero as n goes to ∞, where $\|.\|$ is the initial Riemannian metric on M.

One can then define a new metric $\|.\|_s$ on the bundle E^s as the sum of the push forwards of the initial metric by the positive iterates of df. Clearly, $\|df(u)\|_s < \|u\|_s$ for any non-zero vector u, since $\|u\|_s = \|u\| + \|df(u)\|_s$. Symmetrically, one defines a metric $\|.\|_u$ on E^u such that $\|df(u)\|_u > \|u\|_u$. One then builds a metric $\|.\|_*$ on M that coincides with $\|.\|_s$ on E_s and $\|.\|_u$ on E^u; that metric is adapted to the hyperbolic splitting, by construction. An adapted Riemannian structure would be obtained taking the serie of the push forwards of the inner product on M by the iterates of df. That metric can then be smoothed and remain adapted.

Hirsch, Pugh and Shub noticed that it worked similarly in the case of a two-bundles dominated splitting, one of the bundles being one-dimensional. However no proof was known for general dominated splittings, or partially hyperbolic splittings. This is why they need to make a difference between 'immediate normal hyperbolicity' and 'relative normal hyperbolicity' [35, page 3]. The next theorem provides a complete answer to their question:

Theorem 2.4.1 (G.). *Let f be a diffeomorphism on a Riemannian manifold M. Suppose that f admits a dominated splitting on a compact invariant set K, that is, the linear cocycle df_K defined as the restriction of df to $TM_{|K}$ admits a dominated splitting. Then there exists an adapted smooth Riemannian metric on M for that dominated splitting. Similarly, if the splitting is partially hyperbolic, there is an adapted smooth Riemannian metric on M.*

The strategy is to build *separators* between the bundles of the dominated splitting, that is, functions from K to $\mathbb{R}^+$ that would correspond to add to the dynamics one-dimensional bundles between each pair of consecutive bundles of the dominated splitting. Then some serie provides on each bundle a metric that is adapted to the domination with respect to the two neighbouring separators (or one-dimensional bundles). When the separators are properly chosen, this provides an adapted metric for the dominated splitting.

Consequently, for any dominated splitting, there is a metric on the bundle such that N-domination between a pair of bundles implies L-domination for any $L > N$. The following corollary can be easily obtained without adapted metrics, knowing that such metrics exist only makes it slightly easier.

Corollary 2.4.2. *Let f be a diffeomorphism on a compact manifold M. Then for any $L \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that if an invariant splitting $TM_{|K} = E \oplus F$ on a compact set K is not N'-dominated for some $N' > N$, then it is not L'-dominated, for all $L' \leq L$.*

Proof : We reason by contradiction. If the statement above is not true, then we can find a sequence K_n, a sequence L_n of integers between 1 and L and a sequence N_n that tends to $+\infty$, such that there is an L_n-dominated splitting $T_{|K_n} = E_n \oplus F_n$, that is not N_n-dominated. We may extract and suppose that the splittings $T_{|K_n} = E_n \oplus F_n$ have constant index i, and are L_0-dominated.

Then there is an L_0-dominated splitting $E \oplus F$ of index i on the closure of the union $\cup K_n$. Let $\|.\|_*$ be a Riemannian adapted metric on M for that dominated splitting. In particular, for all $\epsilon > 0$, if N is greater than some N_ϵ, for any unit vectors $u \in E, v \in F$, we have $\|df^N(u)\|_* < \epsilon \|df^N(v)\|_*$. But the metric $\|.\|_*$ is equivalent to the initial metric $\|.\|$, this contradicts the fact that for all N_n, for some $(u,v) \in E_n \times F_n$, $\|df^{N_n}(u)\| > 1/2 \|df^{N_n}(v)\|$.
□

2.4.2 Newhouse phenomenons in any dimensions

Prior to Pujals and Sambarino's proof of the C^1-density conjecture in dimension 2, R. Mañé [38] obtained a C^1-generic dichotomy between Newhouse phenomenons and Axiom A diffeomorphims. This result was generalized by Bonatti, Díaz and Pujals [13] in greater dimension. They showed that generically, a diffeomorphism such that one of its homoclinic classes has no dominated splitting has infinitely many sinks or sources. This is a corollary of the following:

Theorem 2.4.3 (Bonatti, Díaz, Pujals). *Let P be a saddle of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, then*

- *either the homoclinic class $H(P,f)$ admits a dominated splitting,*
- *or for any neighbourhood $\mathcal{U}$ of $H(P,f)$, for any $k \in \mathbb{N}$, there is g arbitrarily C^1-close to f having k sinks or sources whose orbits are included in $\mathcal{U}$.*

In their proof, Bonatti-Diaz-Pujals need the existence of what they call transitions. Transitions are the translation on linear cocycles of the following property of homoclinic classes: given two periodic orbits P and Q in a homoclinic class, there are periodic points passing arbitrarily close to P, thereafter arbitrarily close to Q, and so on; moreover the successive times the periodic points spend close to P and Q can be chosen, so that the derivative along that periodic orbit is alternatively very close to the derivative on $\mathrm{Orb}(P)$, and very close to the derivative on $\mathrm{Orb}(Q)$ on prescribed segments of time.

Therefore, the [13] results heavily rely on the fact they work inside homoclinic classes. One results of this thesis allows to obtain a corresponding result, working along a single orbit of a diffeomorphism f. Precisely, we get the following:

Theorem 4.5.2 (Bonatti, G., Vivier). *Let f be a diffeomorphism of $\mathrm{Diff}(M)$. For all $\epsilon > 0$, there is $N \in \mathbb{N}$ such that the following stands:*

if Q is a periodic point and the cocycle $df_{|\mathrm{Orb}_f(Q)}$ has no N-dominated splitting, then there is an ϵ-perturbation g of f on an arbitrarily small neighbourhood of the orbit of Q, such that the orbit of Q is preserved, and the *linear isomorphism* $dg^p_{|\mathrm{Orb}(Q)}$ *has real eigenvalues with moduli all equal and different from* 1.

In particular, Q is a sink or a source for g. Chapter 4 deals with the proof of Theorem 4.5.2. From this and a theorem of Crovisier [22] that states that, generically, any chain-recurrent class is the Hausdorff limit of a sequence of periodic orbits, Abdenur, Bonatti, Crovisier [2] got the following generic result:

Theorem 2.4.4 (Abdenur, Bonatti, Crovisier). *There exists a residual subset* $\mathcal{R} \in \mathrm{Diff}^1(M)$ *of diffeomorphisms* f *such that given any chain-recurrent class* K *of* f*,*

- *either there is a dominated splitting on* K*,*
- *or the set* K *is the Hausdorff limit of a sequence of periodic sinks/sources of* f*.*

A consequence of that theorem is the following generalization of Mañé's generic dichotomy:

Theorem 2.4.5 (Abdenur, Bonatti, Crovisier). *There is a residual subset* $\mathcal{R} \in \mathrm{Diff}^1(M)$ *of diffeomorphisms* f *such that*

- *either the non-wandering set of* f *admits a decomposition* $\Omega(f) = \Lambda_1 \cup ... \cup \Lambda_i$ *into pairwise disjoint* f*-invariant compact sets, each of which admits a non trivial dominated splitting and is a union of chain-recurrent classes.*
- *or there are infinitely many periodic sinks/sources.*

2.4.3 Creation of homoclinic tangencies away from domination

In dimension 2, Pujals and Sambarino [52] proved that if there is no non-trivial dominated splitting on the non-wandering set of a diffeomorphism f, then an arbitrarily small perturbation creates a homoclinic tangency.

Wen [64] generalised that result in higher dimensions. He showed that if for some integer $1 \leq i \leq d$, the i-preperiodic set (the set of points that can be turned into a saddle of index i by an arbitrarily small C^1-perturbation) does not admit a dominated splitting of index i, then there is an arbitrarily small perturbation that turns f to admit a homoclinic tangency. However the saddle for which the homoclinic tangency occurs may have been created in the perturbation that he builds. Moreover the index of that saddle cannot a priori be controlled.

We will address these problems, again working on small neighbourhoods of periodic orbits. We will show that if a saddle orbit has a long period and the domination between the stable and unstable bundles is weak enough, then a homoclinic tangency can be obtained by perturbation of the dynamics around the orbit. We say that a saddle periodic point Q for a diffeomorphism f is *N-dominated* if the splitting into the stable and unstable bundles of the cocycle $df_{|\,\mathrm{Orb}(Q)}$ is N-dominated. Here is a first version of our result:

Theorem 5.1.3 (G.). *Let f be a diffeomorphism on a compact Riemannian manifold M. For any $\epsilon > 0$, there are two integers $N, P > 0$ such that, if Q is a saddle point of period $p > P$ and not N-dominated,*

then there exists an ϵ-perturbation of f on an arbitrarily small neighbourhood $\mathcal{U}$ of the orbit of Q, that preserves the orbit of Q and its index, and creates a homoclinic tangency related to Q in $\mathcal{U}$.

Moreover, one can ask the perturbation to preserve a finite number of points in the stable and unstable manifolds of the periodic saddle point Q. Precisely:

Theorem 6.1.1 (G.). *Let f be a diffeomorphism on a compact Riemannian manifold M. For any $\epsilon > 0$, there are two integers $N, P > 0$ such that, if Q is a saddle point of period $p > P$ and not N-dominated,*

then for any finite sets Γ^s in the stable manifold and Γ^u in the unstable one, there exists an ϵ-perturbation g of f on an arbitrarily small neighbourhood $\mathcal{U}$ of the orbit of Q, that preserves the orbit of Q, creates a homoclinic tangency related to Q in $\mathcal{U}$, and such that Γ^s is in the stable manifold of g, and Γ^u is in the unstable one.

Actually, one can even ask that if $x \in \Gamma^s \cup \Gamma^u$ is in some strong stable/unstable manifold for f, the corresponding strong stable/unstable manifold is defined also for g, and contains x. In particular such a homoclinic tangency can be created preserving the homoclinic relationship between Q and any other saddle point. Although one is a consequence of the other, to spare the reader the technical difficulties of the proof of Theorem 6.1.1 (see Chapter 6), we give an independant proof of Theorem 5.1.3 in Chapter 5.

We will deduce from Theorem 6.1.1 that if the homoclinic class of a saddle of index i does not admit a dominated splitting of index i, then by a perturbation, one obtains a homoclinic tangency related to the homoclinic class:

Corollary 6.6.2 (G.). *Let Q be a saddle point for f whose homoclinic class $H(Q, f)$ is non-trivial (not reduced to the orbit of Q) and does not admit a dominated splitting of same index as Q. Then, there is an arbitrarily small perturbation g of f, that preserves the dynamics on a neighbourhood of Q, and such that there is a homoclinic tangency related to Q.*

Abdenur-Bonatti-Crovisier-Díaz-Wen [3] showed that for a generical diffeomorphism, for any homoclinic class, the set of indices of the saddles points in that homoclinic class is an interval. Corollary 6.6.2, and this result give a partial answer to [3, Conjecture 1]:

Theorem 6.6.8. *For every C^1-generic diffeomorphism f, let $H(P,f)$ be a homoclinic class containing saddles of indices α and β, $\alpha < \beta$. Then the following dichotomy holds:*

- *either there is an arbitrarily small perturbation g of f admitting a homoclinic tangency associated to the continuation of some saddle of $H(P,f)$;*
- *or there is a dominated splitting for f on $H(P,f)$*

$$T_{H(P,f)}M = E \oplus E_1^c \oplus .... \oplus E_{\beta-\alpha}^c \oplus F,$$

where $dim(E) = \alpha$ and each E_i^c is 1-dimensional and not hyperbolic.

2.5 Sketch of the proof of Theorem 4.5.2

2.5.1 Large periods cocycles

As we mentioned before, Franks' Lemma 2.3.6 allows to reduce the problem to studying what can be obtained by perturbing the derivative along a periodic orbit: indeed, we first find a perturbation of the derivative df of f along a periodic orbit Q that turns all the eigenvalues at the first return to have same modulus, then Franks' Lemma allows to realize that perturbation by a perturbation of the diffeomorphism f. Precisely, we find a perturbation g of f on a small neighbourhood of the orbit of Q, that preserves the orbit of Q and such that dg on the orbit of Q is equal to the expected perturbation of df.

Hence we are reduced to perturbing linear cocycles. Roughly, it is enough to show that given a bound on the norms of cocycles, if a cocycle has no strong dominated splitting along some orbit, then a small perturbation of the cocycle turns the eigenvalues of the first return linear map on that orbit to have same modulus. In [16] it is only proved for large periods, but it happens that the large period hypothesis is not necessary. The large period case is treated in sections 4.3 and 4.4. The general case is easily deduced and is treated in the appendix of the same chapter.

The idea for large period systems will be to look for the perturbations that minimize the Lyapunov diameter, that is the diameter of the set of Lyapunov exponents, along one orbit.

We define a linear cocycle $\mathcal{A} = (A,f)$ on a bundle $\mathcal{E} = (E,\Sigma)$ to be a *large periods cocycle* if the three next conditions are statisfied:

- Σ is infinite
- any point $x \in \Sigma$ is periodic for f

- for any $p \in \mathbb{N}$, the set of p-periodic points in Σ is finite.

We say that such a cocycle $\mathcal{A}$ is *strictly without domination* if the only sets in restriction to which the cocycle $\mathcal{A}$ admits a non-trivial dominated splitting are finite. The eigenvalues of a linear cocycle are the eigenvalues of the first return map along a periodic orbit. We will say that the eigenvalues of a cocycle *have different moduli* if the eigenvalues of each first return map have different moduli.

A *perturbation* of a large periods cocycle $\mathcal{A} = (A, f)$ on a bundle $\mathcal{E} = (E, \Sigma)$ is a cocycle $\mathcal{B} = (B, f)$ on $\mathcal{E}$ such that for all $\epsilon > 0$, there is only a finite set of points $x \in \Sigma$, for which there is a unit vector u above x satisfying $\|\mathcal{A}(u) - \mathcal{B}(u)\| \geq \epsilon$. That is, $\mathcal{B}$ is ϵ-close to $\mathcal{A}$ almost everywhere. A perturbation of a perturbation is a perturbation. That definition spares us the use and handling of multiple ε-s and makes the statements and proofs much lighter.

We now can state the main result on cocycles:

Theorem 2.5.1. *Let $\mathcal{A}$ be a bounded large periods cocycle. Then if $\mathcal{A}$ is strictly without domination, there exists a perturbation $\mathcal{B}$ of $\mathcal{A}$ such that for any $x \in \Sigma$, the first return map $\mathcal{B}^p$ at x has all eigenvalues real, with same modulus.*

2.5.2 Outline of the proof of Theorem 2.5.1

We first see that it is sufficient to show that with the same hypothesis, there is an infinite subset $\Sigma' \subset \Sigma$ on which some perturbation of $\mathcal{A}$ has a return map with all eigenvalues real, with same modulus. We prove that there is a perturbation $\mathcal{B}$ of $\mathcal{A}$ that turns all the eigenvalues to be real with different moduli.

Given a linear cocycle, we call *Lyapunov diameter at a point $x \in \Sigma$* the difference between the greatest and smallest Lyapunov exponents of the cocycle at that point. We now try minimize the Lyapunov diameter on an infinite subset of Σ. Precisely we minimize the *lower Lyapunov diameter* of $\mathcal{B}$, that is, the infimum of the numbers $\rho > 0$ such that there is an infinite subset of Σ of points at which the Lyapunov diameter of $\mathcal{B}$ is greater than ρ.

We notice that if the lower Lyapunov diameter of a perturbation $\mathcal{B}$ of $\mathcal{A}$ is 0, then we find an infinite subset $\Sigma' \subset \Sigma$ on which some perturbation of $\mathcal{A}$ has a return map with all eigenvalues real, with same modulus, which would conclude the proof. We are left to find such a $\mathcal{B}$.

In fact, we show that there is a perturbation $\mathcal{B}$ of $\mathcal{A}$ that has real eigenvalues with different moduli, and an infinite subset $\Sigma' \subset \Sigma$, such that the restriction $\mathcal{B}'$ to Σ' is *incompressible*, that is

- it admits a *Lyapunov diameter* $\delta(\mathcal{B}')$, i.e. for all $\epsilon > 0$, there is only a finite set of point $x \in \Sigma'$ at which the Lyapunov diameter for $\mathcal{B}'$ is outside the interval $[\delta(\mathcal{B}') - \epsilon, \delta(\mathcal{B}') + \epsilon]$.

- it minimizes the lower Lyapunov diameter: any perturbation of $\mathcal{B}'$ has a lower Lyapunov diameter greater than $\delta(\mathcal{B}')$.

We will notice that $\mathcal{A}$ being strictly without domination, so is the perturbation $\mathcal{B}$, and so is the restriction $\mathcal{B}'$. It is now clearly sufficient to show that if a cocycle is incompressible and is strictly without domination, then its Lyapunov diameter is 0. The next section gives an idea of the way we show it and thus concludes the sketch of the proof of Theorem 2.5.1.

We summarize how, by induction on the dimension of the bundle, we obtain that an incompressible cocycle strictly without domination has Lyapunov diameter equal to zero. In dimension 2, the techniques of R. Mañé [38] provide a proof. Assume by contradiction that an incompressible cocycle $\mathcal{A}$ on a bundle $\mathcal{E}$ of dimension ≥ 3 is strictly without domination, and has Lyapunov diameter > 0. We first show that there is an infinite subset $\Sigma' \subset \Sigma$ and an invariant splitting $\mathcal{E}_{|\Sigma'} = F \oplus G$ above Σ' into two non-trivial subbundles such that the eigenvalues on F are smaller than on G above each point $x \in \Sigma'$, and the restrictions $\mathcal{A}_{|F}$ and $\mathcal{A}_{|G}$ are strictly without domination. From this and by incompressibility, we get that the Lyapunov diameter of both $\mathcal{A}_{|G}$ and $\mathcal{A}_{|G}$ is zero.

Since the dimension is ≥ 3, we may assume that $\dim(F) \geq 2$ and find a one-dimensional invariant subbundle $H \subset F$. Adapting a Lemma from [13], since $\mathcal{A}$ is strictly without domination, we get that

- either the splitting $H \oplus G$ is not dominated for $\mathcal{A}$,

- or the splitting $F/H \oplus G/H$ is not dominated for the quotient cocycle $\mathcal{A}/H$.

We finally show that both case authorise, by the induction hypothesis, a new perturbation that strictly decreases the lower Lyapunov diameter of $\mathcal{A}$, which contradicts incompressibility. QED.

2.6 Sketch of the proof of Theorem 5.1.3

Remark 2.6.1. *As we already warned in section 2.3.2, the definition we take for N-domination in the proofs Theorems 5.1.3 and 5.1.3 is slightly different from the definition we used until now. However, by Corollary 2.4.2, the theorem that we will actually show implies the theorem that we announced.*

The proof of Theorem 5.1.3 roughly follows the same steps, but due to much more technical difficulties and the need of abstruse terminology, we do not summarise it here.

As for Theorem 2.5.1 we will work on small neighbourhoods of periodic orbits, however we need more precise information than the derivative on the periodic orbit. This is why the cocycle tool of [16] will not directly apply here.

We will perturb on small neighbourhoods of periodic saddle orbits, preserving these orbits. This motivates the notion of saddle diffeomorphisms: a *saddle diffeomorphism* is a diffeomorphism f on a linear Euclidean bundle $\mathcal{E}$ over a finite base Σ (we see that bundle as a non-connected Riemannian manifold), such that f does a cyclic permutation of the fibres, and the zero-section $0_{\mathcal{E}}$ is the orbit of a saddle point. A saddle diffeomorphism is *N-dominated* if the splitting into the stable and unstable directions of df above the zero-section is N-dominated.

We want to show that if a saddle diffeomorphism has period great enough, and if it is not dominated enough, then we can perturb it on an arbitrarily small neighbourhood of the zero-section to obtain a homoclinic tangency related to the zero-section. The same way as in the proof of Theorem 2.5.1, we see that if the period is great, then a small perturbation of the cocyle $df_{|0_{\mathcal{E}}}$ above the zero-section has real eigenvalues with pairwise distinct moduli. Again from Franks' Lemma, we deduce that if the period of a saddle diffeomorphism f is great enough then there is a small saddle-perturbation of f that has real eigenvalues with pairwise distinct moduli.

By another perturbation we linearize locally and reduce us to studying a saddle diffeomorphism that is linear, and has real eigenvalues with pairwise distinct moduli. We will prove by induction on the dimension of the bundle, that if such a linear saddle diffeomorphism is not strongly enough dominated, then a small perturbation turns it to admit a homoclinic tangency.

Dimension 2 was proved, albeit not in that form, by Pujals and Sambarino. In greater dimension, let $\mathcal{E} = F \oplus G$ be the stable/unstable splitting. One may assume that the stable direction F of the linear f has dimension ≥ 2. As the eigenvalues are real, it contains a one-dimensional invariant subbundle H. Since the domination between the stable and unstable directions is weak, we use again the lemma from [13], and get that

- either the splitting $H \oplus G$ is weakly dominated for f,
- or the splitting $F/H \oplus G/H$ is weakly dominated for the quotient linear diffeomorphism f/H.

In the first case, the induction hypothesis provides a small perturbation of the restriction $f_{|H\oplus G}$ that admits a homoclinic tangency. We will then show that it can be extended to a small perturbation of f, which will also crearly present a homoclinic tangency.

In the second case, the induction hypothesis provides a small perturbation of the quotient f/H that admits a homoclinic tangency. We will lift it into a small perturbation of f, and show that such a lift also admits a homoclinic tangency.

Notice that in both cases we have a priori no control on the size of the neighbourhood of $0_{\mathcal{E}}$ on which we perturb f. Nevertheless, conjugating by a homothety, we will ensure that the orbit of the homoclinic tangency is close to $0_{\mathcal{E}}$, and thanks to a refined Franks Lemma, we finally find a perturbation of f on a small neighbourhood of $O_{\mathcal{E}}$ that has a homoclinic tangency. QED.

Chapter 3

Adapted metrics

3.1 Indroduction

The best known and simplest examples of chaotic dynamical systems are uniformly hyperbolic systems, like Anosov diffeomorphisms. A diffeomorphism f on a compact Riemannian manifold M is said to be an *Anosov diffeomorphism* if there exists a splitting of the tangent bundle TM into two supplementary, df-invariant subbundles, called the *stable* and the *unstable* bundles that are uniformly contracted and expanded, respectively, by an iterate of f. If the hyperbolic systems are now well understood, many dynamical systems are (robustly) not hyperbolic, so that several authors have tried to weaken the notion of hyperbolicity, in order to recover some of its properties on a larger class of systems.

In this spirit, Brin, Pesin [19] and Hirsch, Pugh, Shub [35] extended the notion of hyperbolic diffeomorphism to that of *partially hyperbolic* diffeomorphism, that is, admitting an invariant splitting $TM = E^s \oplus E^c \oplus E^u$, where the stable bundle E^s is uniformly contracted, the unstable one E^u is uniformly expanded, and the central one E^c is uniformly less contracted (resp. less expanded) than E^s (resp. E^u). Hirsch, Pugh, Shub showed the structural stability of the central bundle of a partially hyperbolic diffeomorphism (under some extra hypothesis, see [35, Theorem 7.1]).

Working on the stability conjecture[1], Liao and Mañé [38] were led to the following general notion : a *dominated splitting* for f is a splitting of TM into two supplementary invariant subbundles such that there exists an iterate of df that uniformly contracts more (or expands less) the first subbundle than the second one. This notion is a key tool for understanding non-hyperbolic systems:

- In dimension 2, Pujals and Sambarino [52] proved that a diffeomor-

[1] The stability conjecture, proved by Mañé in [39] for diffeomorphisms and then by Hayashi for flows in [32], asserts that any C^1-structurally stable system is hyperbolic, i.e. satifies the Axiom A and the strong transversality condition.

phism with a dominated splitting may be C^1-approached by hyperbolic ones, and diffeomorphisms without dominated splitting may be approached by diffeomorphisms exhibiting a homoclinic tangency: as a consequence any diffeomorphism of a compact surface can be C^1-approximated either by hyperbolic (Axiom A) diffeomorphisms, or by diffeomorphisms that exhibit a homoclinic tangency (this was conjectured by Palis).

- In any dimension, Bonatti, Diaz, Pujals showed in [13] that a robustly transitive generic diffeomorphism in $\mathrm{Diff}^1(M)$, admits a non-trivial dominated splitting defined on the whole M.

As recalled above, for a hyperbolic set K of a diffeomorphism f, the vectors in the stable and unstable bundles are uniformly contracted and expanded, respectively, by the derivative df^n, for some $n > 0$. The hyperbolicity of K does not depend on the metric on the manifold, but the smallest time n where the contraction/expansion phenomena are seen depends on the metric; a Riemannian metric is called *adapted* to the hyperbolic set K if one can take $n = 1$. Applying Holmes' theorem (see [35, page 15]), we obtain that any hyperbolic set admits an adapted Riemannian metric. We will adapt this theorem to the case of dominated behaviours, to show Lemma 3.4.2. It was asked in [35, page 5] if there existed an *adapted metric for a dominated splitting*, that is a metric such that df uniformly contracts more (or expands less) the first subbundle than the second one, at the first iteration.

The aim of this paper is to give a complete positive answer to this question, proving that such an adapted metric exists for any dominated splitting:

Theorem . *Let f be a diffeomorphism of a Riemannian manifold M, and K a compact invariant subset of M, such that the restriction of f to K admits a dominated splitting $TM\,|_K = E^1 \oplus E^2 \oplus \cdots \oplus E^d$, where the vectors in E^i are uniformly less expanded than those in E^{i+1} by df^n for some $n > 0$. Then there exists a Riemannian metric $\|.\|$ on M (necessarily equivalent to the first metric) and adapted to the dominated splitting: there exists a constant $0 < \mu < 1$ such that for any $x \in K$, any $i \in \{1, \ldots, d-1\}$, and any unit vectors $u \in E^i_x$, $v \in E^{i+1}_x$, one has $\|df(u)\| < \mu.\|df(v)\|$.*

This result was already known by [35] for a dominated splitting in 2 bundles, $TM|_K = E_1 \oplus E_2$, such that $dim(E_1) = 1$ or $dim(E_2) = 1$. In addition, they showed that any absolutely normally hyperbolic system, admits an adapted metric, but it was not known whether it was true for a relatively normally hyperbolic system, which is, with our definitions, a partially hyperbolic system. We answer by showing (see Theorem 3.4.6) that an adapted metric exists for any partially hyperbolic splitting, that is a metric adapted to the corresponding dominated splitting, and such that the stable/unstable bundles are uniformly expanded/contracted at the first iterate. Finally, we show in Section 3.5 how to transpose these results from diffeomorphisms to flows.

In order to present more clearly the idea of the proof, we will first focus, in Section 3.3, on dominated splittings into two subbundles, over an invariant compact set of a diffeomorphism. Then, we will show that there exists an adapted Finsler for any dominated splitting into d bundles for a finite dimensional Banach bundle automorphism (see Theorem 3.4.1).

3.2 Definition and notations

For a morphism A of normed vector spaces, define the *norm* and the *minimum norm* of A:

$$\|A\| = \sup_{\|u\|=1} \|A(u)\|, \quad \mathfrak{m}(A) = \inf_{\|u\|=1} \|A(u)\|$$

When A is invertible, $\mathfrak{m}(A) = \|A^{-1}\|^{-1}$. For a Banach bundle E, we denote by E_x the fibre of E above a point x of the base. If $\mathcal{A}$ is an automorphism of a Banach bundle E with compact base K, then for any point x of K, we denote by $\mathcal{A}_x$ the restriction of $\mathcal{A}$ to the fibre E_x. We refer the reader to [35] for definitions.

We say that a sequence of functions $g_n(x)\colon K \to \mathbb{R}$ *converges exponentially to zero* if there exists positive constants C and $\mu < 1$ such that for all x and n,

$$|g_n(x)| \leq C\mu^n.$$

Given an automorphism $\mathcal{A}$ of a Banach bundle E with compact base K

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{A}} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow{f} & K \end{array}$$

and a positive continuous function $r\colon K \to \mathbb{R}$, we denote by $R_n(x)$ the product

$$R_n(x) = \prod_{i=0}^{n-1} r[f^i(x)] = r(x)r[f(x)]...r[f^{n-1}(x)]$$

Definition 3.2.1. A positive continuous function $r\colon K \to \mathbb{R}$ *dominates* $\mathcal{A}$, if the sequence of ratios $x \to \|\mathcal{A}_x^n\|/R_n(x)$ converges exponentially to zero as $n \to \infty$, where $\mathcal{A}_x^n = \mathcal{A}^n|_{E_x}$.

In this case we write $df_{|E} \prec r$. Symmetrically, we say that r is *dominated* by $\mathcal{A}$ and we write $r \prec \mathcal{A}$ if and only if the ratio $R_n/\mathfrak{m}(\mathcal{A}^n)$ goes exponentially to zero as $n \to \infty$. Notice that $r \prec \mathcal{A}$ is equivalent to $\mathcal{A}^{-1} \prec 1/r$.

Definition 3.2.2. Let E be a finite dimensional Banach bundle over a compact base K, and $E = E^1 \oplus ... \oplus E^d$ be an invariant splitting for an automorphism $\mathcal{A}$, where the E^i are vector subbundles with constant dimension. Then we say that it is a *dominated splitting* if, for each integer $0 < i < d$, the ratio $\|\mathcal{A}^n_{|E^i}\|/\mathfrak{m}(\mathcal{A}^n_{|E^{i+1}})$ tends exponentially to zero, as n goes to infinity.

We have written $\|\mathcal{A}^n_{|E^i}\|/\mathfrak{m}(\mathcal{A}^n_{|E^{i+1}})$ for the function $x \mapsto \|\mathcal{A}^n_{|E^i_x}\|/\mathfrak{m}(\mathcal{A}^n_{|E^{i+1}_x})$. In this case, we say that $\mathcal{A}_{|E^i}$ is *dominated* by $\mathcal{A}_{|E^{i+1}}$, and we write $\mathcal{A}_{|E^i} \prec \mathcal{A}_{|E^{i+1}}$. We recall that the subbundles E_i are necessarily continuous (see [14, Appendix B] for a proof).

Remark 3.2.3. *Since the bundles and automorphisms are continuous, and the base K is compact, the definitions of domination and dominated splitting are independant of the Finsler. Thus we will be allowed to change to equivalent metrics.*

A Finsler $\|.\|_*$ is *adapted* to the dominated splitting if and only if, for all $i < d$, we have

$$\frac{\|\mathcal{A}_{|E^i}\|_*}{\mathfrak{m}_*(\mathcal{A}_{|E^{i+1}})} < 1$$

where $\|.\|_*$ and $\mathfrak{m}_*$ are the norm and the minimum norm, with respect to the Finsler $\|.\|_*$. Equivalently, by compactness of the base, there exists a real number $0 < C < 1$ such that, for any $x \in K$, for any nonzero unit vectors $u \in E^i_x$, $v \in E^{i+1}_x$, we have $\|\mathcal{A}(u)\|_* < C\|\mathcal{A}(v)\|_*$.

3.3 Two-bundle splittings

Let M be a compact smooth manifold endowed with a Riemannian metric $\|.\|$, let f be a diffeomorphism of M and let K be an invariant compact set in M. We will show the following:

Theorem 3.3.1. *If $T_K M = E \oplus F$ is a dominated splitting for the diffeomorphism f on the compact K, then there exists a smooth Riemannian metric on M that is adapted to that dominated splitting.*

The proof consists in building first a *separator* for the dominated splitting, that is, a positive function $r\colon K \to \mathbb{R}$ such that we have $df_{|E} \prec r \prec df_{|F}$. Then by a dominated version of Holmes' theorem, we will build two metrics $\|.\|_E$ and $\|.\|_F$ on the bundles E and F, such that, for any $x \in K$, for any unit vectors $u \in E_x$ and $v \in F_x$, we have

$$\|df(u)\|_E < r(x) < \|df(v)\|_F.$$

These metrics will induce, up to perturbation, an adapted Riemannian metric on M.

Lemma 3.3.2. *A two-bundle dominated splitting has a separator.*

Proof : In the following, we fix a dominated splitting $T_K M = E \oplus F$ for the diffeomorphism f. For simplicity, call $df_{|E} = \mathcal{A}$ and $df_{|F} = \mathcal{B}$. By hypothesis, the ratio $\|\mathcal{A}^n\|/\mathfrak{m}(\mathcal{B}^n)$ tends exponentially to zero. In particular, for N large enough, the function $x \mapsto \|\mathcal{A}^N_x\|/\mathfrak{m}(\mathcal{B}^N_x)$ is smaller than $1/2$. Therefore, for $a > 1 > b$ close enough to 1, we have for all x in K:

$$a\|\mathcal{A}^N_x\|^{1/N} < b\mathfrak{m}(\mathcal{B}^N_x)^{1/N}.$$

Hence, Lemma 3.3.2 comes from the following claim. □

Claim 1. *Any continuous function $r\colon K \to \mathbb{R}$ such that $a.\|\mathcal{A}^N_x\|^{1/N} \leq r(x) \leq b.\mathfrak{m}(\mathcal{B}^N_x)^{1/N}$ separates the splitting.*

Proof : For any integer $n > N$, and each $0 \leq k \leq N-1$ we can write the iterate $\mathcal{A}^n_x$ as the composition

$$\mathcal{A}^l_{f^{k+mN}(x)} \circ \mathcal{A}^{mN}_{f^k(x)} \circ \mathcal{A}^k_x$$

for some integers $0 \leq l \leq N-1$ and $m \geq 0$. Precisely, take the integer part of $(n-k)/N$ for m, and $l = n - nM - k$. Denote by c the upper bound of the norms of the i-th forward or backward iterates of A, for $i \leq N$:

$$c = \sup_{|i| \leq N, y \in K} \|A^i_y\|.$$

It is finite, as K is compact. We have then

$$\begin{aligned} \|\mathcal{A}^n_x\| &\leq \|\mathcal{A}^l_{f^{k+nm}(x)}\| \left(\textstyle\prod_{i=0}^{m-1} \|\mathcal{A}^N_{f^{k+iN}(x)}\|\right) \|\mathcal{A}^k_x\| \\ \|\mathcal{A}^n_x\| &\leq c^2 \textstyle\prod_{j \in J_k} \|\mathcal{A}^N_{f^j(x)}\| \end{aligned} \qquad (1)_k$$

where J_k is the set of integers $\{k + iN, i = 0...m-1\}$, that is the set of integers of the form $k + iN$ and comprised between 0 and $n - N$. Obviously, the sets J_k for $k = 0, ..., N-1$, are pairwise disjoint and their union is the interval $\{0, ..., n-N\}$. Hence, taking the product of inequalities $(1)_k$, for $k = 0, ..., N-1$ we obtain

$$\|\mathcal{A}^n_x\|^N \leq c^{2N} \prod_{j \in \{0,...,n-N\}} \|\mathcal{A}^N_{f^j(x)}\|.$$

Since $\|\mathcal{A}^N_{f^j(x)}\|^{-1} \leq \|\mathcal{A}^{-N}_{f^{j+N}(x)}\| \leq c$, we get

$$\|\mathcal{A}^n_x\|^N \leq c^{2N} c^N \prod_{j \in \{0,...,n\}} \|\mathcal{A}^N_{f^j(x)}\|.$$

Thus, as $a\|\mathcal{A}^N_{f^j(x)}\|^{1/N} \leq r[f^j(x)]$, we get that, for any $x \in K$,

$$\|\mathcal{A}^n_x\| \leq c^3 a^{-n} R_n(x),$$

which prooves that $\mathcal{A} \prec r$. Notice that $1/b.\|\mathcal{B}^{-N}_x\| < 1/r(x)$, for all x. Thus, we have the same way $\mathcal{B}^{-1} \prec 1/r$ and then $r \prec \mathcal{B}$. This ends the proof of the claim, and that of Lemma 3.3.2. □

We now show the following Lemma (which can actually be seen as a particular case of Lemma 3.4.2 stated below):

Lemma 3.3.3. *Let $r\colon K \to \mathbb{R}$ be a positive function that separates the continuous splitting $E \oplus F$, that is $df_{|E} \prec r \prec df_{|F}$. Then there exists a Riemannian metric $\|.\|_*$ on M that is* adapted *to the domination; namely, for all $x \in K$, for all unit vectors $u \in E_x$, $v \in F_x$ we have:*

$$\|df(u)\|_* < r(x) < \|df(v)\|_*$$

Proof : We define on E a metric $\|.\|_E$ by

$$\|u\|_E^2 = \sum_{n=0}^{\infty} \frac{\|df^n(u)\|^2}{[R_n(x)]^2}$$

for any $u \in E_x$, where $R_n(x) = r(x)...r[f^{n-1}(x)]$ as above. By domination, this is a sum of a normally convergent series of continuous functions; therefore $\|.\|_E$ is well-defined and continuous. As a sum of quadratic forms, $\|.\|_E^2$ is a quadratic form, thus $\|.\|_E$ is a Hilbertian metric (it arises from an inner product). Moreover, we have:

$$\|df(u)\|_E^2 = \sum_{n=0}^{\infty} \frac{\|df^{n+1}(u)\|^2}{[R_n(x)]^2} = \sum_{n=1}^{\infty} \frac{\|df^n(u)\|^2}{[R_{n-1}(x)]^2} = r(x)^2 \sum_{n=1}^{\infty} \frac{\|df^n(u)\|^2}{[R_n(x)]^2}$$

since $R_{n-1}(x) = R_n(x)/r(x)$. We obtained $\|df(u)\|_E^2 = r(x)^2[\|u\|_E^2 - \|u\|^2]$ where $\|u\|^2$ is the first term of the series defining $\|u\|_E^2$. Hence, for any nonzero u, $\|df(u)\|_E < r(x)\|u\|_E$. Up to change f into f^{-1} and r into $1/r$, we find the same way a Hilbertian metric $\|.\|_F$ on F such that, for all nonzero v in F, $r(x)\|v\|_F < \|df(v)\|_F$. Consider now the Hilbertian metric $\|.\|_*$ on $T_K M$ that extends $\|.\|_E$ and $\|.\|_F$ and that makes E and F orthogonal. It is continuous, since $\|.\|_E$ and $\|.\|_F$ are.

The inequality $\|df(u)\|_* < r(x) < \|df(v)\|_*$ holds for all unit vectors $u \in E$, $v \in F$ above each point x of the base K. We extend the metric $\|.\|_*$ to the whole M, and smooth it into a Riemannian metric by a small perturbation, so that, by compactness of K, the inequality is preserved. □

This together with the existence of a separator (Lemma 3.3.2) ends the proof of Theorem 3.3.1.

3.4 Multiple bundles splittings

We will show in this section the most general result of our paper:

Theorem 3.4.1. *Let E be a finite dimensional Banach bundle on a compact base, and let $\mathcal{A}$ be an automorphism of E. If $E = E^1 \oplus ... \oplus E^d$ is a dominated splitting for $\mathcal{A}$, then there is a Finsler $\|.\|_*$ on E adapted to the domination, that is, for each $i = 1...d-1$, for any $x \in K$, we have*

$$\|\mathcal{A}_{|E^i_x}\|_* < \mathfrak{m}_*(\mathcal{A}_{|E^{i+1}_x}).$$

Furthermore, if the original metric on E is Hilbertian, then the adapted metric can be chosen to be also Hilbertian.

Let F be a Banach bundle with compact base K, and $\mathcal{B}$ be an automorphism

$$\begin{array}{ccc} F & \xrightarrow{\mathcal{B}} & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow{f} & K \end{array} .$$

Then we have this dominated version of Holmes' Theorem (see [35, page 15]):

Lemma 3.4.2. *Let $r, s \colon K \to \mathbb{R}$ be two positive continuous functions such that the domination $r \prec \mathcal{B} \prec s$ and the inequality $r < s$ hold on K. Then there is a Finsler $\|.\|_*$ on F that is adapted to the domination, namely, for any $x \in K$, for any $u \in F_x \setminus \{0\}$, we have*

$$r(x)\|u\|_* < \|\mathcal{B}(u)\|_* < s(x)\|u\|_*$$

Proof : For all u in F, we define

$$\|u\|_*^2 = \sum_{n=1}^{\infty} R_n^2[f^{-n}(x)]\|\mathcal{B}^{-n}(u)\|^2 + \sum_{n=0}^{\infty} \frac{\|\mathcal{B}^n(u)\|^2}{[S_n(x)]^2}$$

where $\|.\|$ is the original metric on F, and where $R_n[f^{-n}(y)] = r(f^{-n}(y))...r(f^{-1}(y))$, $S_n(y) = s(y)s(f(y))...s(f^{n-1}(y))$ as before. Still by domination, the series normally converges to a continuous function. Thus $\|.\|_*$ is a well-defined Finsler. We have

$$\begin{aligned} \|\mathcal{B}(u)\|_*^2 &= \sum_{n=1}^{\infty} R_n^2[f^{-n+1}(x)]\|\mathcal{B}^{-n+1}(u)\|^2 + \sum_{n=0}^{\infty} \frac{\|\mathcal{B}^{n+1}(u)\|^2}{[S_n(f(x))]^2} \\ &= \sum_{n=0}^{\infty} R_{n+1}^2[f^{-n}(x)]\|\mathcal{B}^{-n}(u)\|^2 + \sum_{n=1}^{\infty} \frac{\|\mathcal{B}^n(u)\|^2}{[S_{n-1}(f(x))]^2} \end{aligned}$$

For we have $R_{n+1}[f^{-n}(x)] = R_n[f^{-n}(x)].r(x)$ and $S_{n-1}[f(x)] = S_n(x)/s(x)$, we get

$$\|\mathcal{B}(u)\|_*^2 = [r(x)]^2 \sum_{n=0}^{\infty} R_n^2[f^{-n}(x)]\|\mathcal{B}^{-n}(u)\|^2 + [s(x)]^2 \sum_{n=1}^{\infty} \frac{\|\mathcal{B}^n(u)\|^2}{[S_n(x)]^2}$$

On the other hand, we have $R_0(x)\|\mathcal{B}^{-0}(v)\| = \|v\| = \|\mathcal{B}^0(v)\|/S_0(x)$ since R_0 and S_0 are empty products equal to 1. So, in the expression of $\|u\|_*^2$, we can take the first term of the second sum to the first sum:

$$\|u\|_*^2 = \sum_{n=0}^{\infty} R_n^2[f^{-n}(x)]\|\mathcal{B}^{-n}(u)\|^2 + \sum_{n=1}^{\infty} \frac{\|\mathcal{B}^n(u)\|^2}{[S_n(x)]^2}$$

Finally, since $r < s$, we obtain, for any nonzero vector u, the inequality

$$[r(x)]^2\|u\|_*^2 < \|\mathcal{B}(u)\|_*^2 < [s(x)]^2\|u\|_*^2$$

the square root of which concludes the proof. □

Remark 3.4.3. *Having chosen this quadratic construction, if $\|.\|$ is a Hilbertian metric, then $\|.\|_*$ is still a Hilbertian metric.*

Proof of Theorem 3.4.1 : By definition, the ratios $\|\mathcal{A}^n_{|E^i}\|/\mathfrak{m}(\mathcal{A}^n_{|E^{i+1}})$ converge exponentially to zero, for each i, as n goes to infinity. Thus we can find an integer N such that for each i, the ratio $\|\mathcal{A}^N_{|E^i}\|/\mathfrak{m}(\mathcal{A}^N_{|E^{i+1}})$ is smaller than 1/4. The proof of Lemma 3.3.2 still works when $TM_{|K} = E \oplus F$ is replaced by the Banach bundle $E^i \oplus E^{i+1}$, and when $df_{|K}$ is replaced by the automorphism $\mathcal{A}_{|E^i\oplus E^{i+1}}$.

Choose then a family $(r_i)_{0<i<d}$ of continuous functions such that $2^{1/N}\|\mathcal{A}^N_{|E^i}\|^{1/N} < r_i < 2^{-1/N}\mathfrak{m}(\mathcal{A}^N_{|E^{i+1}})^{1/N}$. We have then $r_1(x) < ... < r_{d-1}(x)$, for all $x \in K$; furthermore, by Claim 1, we have

$$E_1 \prec r_1 \prec E_2 \prec ... \prec r_{d-1} \prec E_d.$$

In order to have two-sided dominations for the extremal bundles, we may add two functions $r_0 \prec E_1$ and $E_d \prec r_d$, with $0 < r_0 < r_1$ and $r_{d-1} < r_d$. We now apply Lemma 3.4.2 to find a Finsler $\|.\|_i$ on each E_i that is adapted to the domination $r_i \prec E_{i+1} \prec r_{i+1}$. Define the new metric

$$\|u\|_* = \sqrt{\sum_{i=1..d} \|p_i(u)\|_i^2},$$

for all $u \in E$, where p_i is the projection on E_i along $E_1 \oplus ... \oplus E^{i-1} \oplus E^{i+1} ... \oplus E^d$. It is a Finsler that is clearly adapted to the dominated splitting: for any unit vectors $u \in E^i_x$, $v \in E^{i+1}_x$ we have $\|u\|_* = \|u\|_i < r_i(x) < \|v\|_{i+1} = \|v\|_*$. □

Remark 3.4.4. *Obviously by the previous remark, if the original metric* $\|.\|$ *was a Hilbertian metric, then the metrics* $\|.\|_i$ *are so, and the metric* $\|.\|_*$ *we built is still a Hilbertian metric.*

After smoothing the adapted Hilbertian metric, we obtain the following, which is a reformulation of Theorem :

Corollary 3.4.5. *Let* M *be a Riemannian manifold,* K *a compact invariant set for a diffeomorphism* f*, and* $T_K M = E^1 \oplus ... \oplus E^d$ *a dominated splitting for* f *above* K*. Then there exists a smooth Riemannian metric on* M *that is adapted to it.*

The existence of an adapted metric was shown for *absolute*- and not *relative*-normally hyperbolic systems (see [35] for proofs and definitions). The bases of all Banach bundles are still compact. A dominated splitting $E = E^1 \oplus ... \oplus E^d$ for an automorphism $\mathcal{A}$ is *partially hyperbolic* if and only if, for some $1 \le k < k+1 < l \le d$, the bundles $E^s = E^1 \oplus ... \oplus E^k$ and $E^u = E^l \oplus ... \oplus E^d$ are respectively stable and unstable, that is $\|\mathcal{A}^n_{|E^s}\|$ and $\|\mathcal{A}^{-n}_{|E^u}\|$ converge exponentially to zero as n goes to infinity. We say that a metric $\|.\|_*$ is *adapted* to such partially hyperbolic splitting, if it is adapted to the dominated splitting, and if $\|\mathcal{A}_{|E^s}\| < 1$ and $\mathfrak{m}(\mathcal{A}_{|E^u}) > 1$.

Theorem 3.4.6. *A partially hyperbolic splitting has an adapted metric.*

Proof : We show it in the three-bundle case (it is the same idea for the general case). Consider a partially hyperbolic splitting $E = E^s \oplus E^c \oplus E^u$ with compact base for an automorphism $\mathcal{A}$. Then $\|\mathcal{A}^n_{|E^s}\|$ and $\|\mathcal{A}^{-n}_{|E^u}\|$ tend exponentially to zero as n goes to infinity. From the construction we gave in the proof of Lemma 3.3.2, we can find two functions $0 < r < 1 < s$ such that

$$\mathcal{A}_{|E^s} \prec r \prec \mathcal{A}_{|E^c} \prec s \prec \mathcal{A}_{|E^u}.$$

With respect to this domination, the metric $\|.\|_*$ produced in the proof of Theorem 3.4.1 is adapted to the dominated splitting and satisfies $\|\mathcal{A}_{|E^s}\| < 1 < \mathfrak{m}(\mathcal{A}_{|E^u})$. Hence, it is adapted to the partially hyperbolic splitting. □

3.5 Dominated splittings for flows

In this section, we briefly show that the same results apply for flows. In the following, ϕ is a flow on a compact subset K of a Riemannian manifold M. A Finsler $\|.\|_*$ on M is *adapted* to a dominated splitting $T_K M = E^1 \oplus ... \oplus E^d$ for ϕ, if and only if, for any point $x \in K$, for all unit vectors v, w in any pair E^i_x, E^{i+1}_x, we have

$$\forall t > 0, \quad \|d\phi^t(v)\|_* < \|d\phi^t(w)\|_*,$$

where $d\phi^t$ is the derivative of the time-t map of ϕ. The existence of adapted metrics for flows is not a straightforward consequence of our results on diffeomorphisms. At best, applying the former results would provide, for each $\epsilon > 0$, a metric $\|.\|_*$ such that the inequality above holds for all $t > \epsilon$. To get adapted metrics, we have to transpose the notion of separator to the flow case.

Let E be a subbundle of $T_K M$, invariant by ϕ. Fix two strictly positive, continuous functions $r, s\colon K \to \mathbb{R}$. Then, for any $x \in K$, for all $t \in \mathbb{R}$, define

$$\begin{aligned} R_t(x) &= \exp\left(\int_0^t \ln[r(\phi^u(x))].du\right), \\ S_t(x) &= \exp\left(\int_0^t \ln[s(\phi^u(x))].du\right). \end{aligned}$$

For any fixed x, the functions $t \mapsto R_t(x)$ and $t \mapsto S_t(x)$ are C^1, and for all real numbers t, k,

$$R_{t+k}(x) = R_t(x).R_k[\phi^t(x)], \tag{3.1}$$
$$S_{t+k}(x) = R_t(x).S_k[\phi^t(x)]. \tag{3.2}$$

Assume that $r < s$, and that we have the domination relation $r \prec d_E\phi \prec s$, that is, for all $x \in K$, for any vector $v \in E_x$, the quantities $\|d\phi^t(v)\|/S_t(x)$ and $\|d\phi^t(v)\|/R_t(x)$ go exponentially to zero, respectively, as t goes to $+\infty$, and as t goes to $-\infty$. Then, define the Finsler $\|.\|_*$ on E by

$$\|v\|_*^2 = \int_{-\infty}^0 \frac{\|d\phi^t(v)\|^2}{R_t(x)^2}.dt + \int_0^\infty \frac{\|d\phi^t(v)\|^2}{S_t(x)^2}.dt$$

for any $x \in K$, for all $v \in E_x$.

Claim 2. *For any nonzero vector $v \in E$ above any point $x \in K$, the metric $\|.\|_*$ satisfies*

$$\forall k > 0, \quad R_k(x).\|v\|_* < \|d\phi^k(v)\|_* < S_k(x).\|v\|_*.$$

That is, the metric $\|.\|_*$ is *adapted* to the domination $r \prec d_E\phi \prec s$.

Remark 3.5.1. *This is merely Lemma 3.4.2 for flows.*

Proof : After a change of variable, we get:

$$\begin{aligned} \|d\phi^k(v)\|_*^2 &= \int_{-\infty}^k \frac{\|d\phi^t(v)\|^2}{R_{t-k}^2 \circ \phi^k(x)}.dt + \int_k^{+\infty} \frac{\|d\phi^t(v)\|^2}{S_{t-k}^2 \circ \phi^k(x)}.dt \\ &= R_k^2(x).\int_{-\infty}^k \frac{\|d\phi^t(v)\|^2}{R_t^2(x)}.dt + S_k^2(x).\int_k^{+\infty} \frac{\|d\phi^t(v)\|^2}{S_t^2(x)}.dt, \end{aligned}$$

by formulae (3.1) and (3.2). Let $\theta(k)$ be the quotient $S_k^2(x)/R_k^2(x)$, and define the function:

$$f\colon k \mapsto \frac{\|d\phi^k(v)\|_*^2}{R_k^2(x)} = \int_{-\infty}^{k} \frac{\|d\phi^t(v)\|^2}{R_t^2(x)}.dt + \theta(k).\int_k^{+\infty} \frac{\|d\phi^t(v)\|^2}{S_t^2(x)}.dt,$$

Since $\theta(k) = \exp\left(2.\int_0^k \ln \frac{s}{r}[\phi^u(x)].du\right)$, and $s < r$, the derivative θ' is strictly positive. The function f is obviously C^1, and its derivative, after some calculation, is

$$f'(k) = \theta'(k).\int_k^{+\infty} \frac{\|d\phi^t(v)\|^2}{S_t^2(x)}.dt.$$

Hence f' is strictly positive, and f is strictly increasing. For we have $f(0) = \|v\|_*^2/R_0^2(x) = \|v\|_*^2$, the inequality $f(k) > f(0)$ leads to $R_k(x)^2.\|v\|_*^2 < \|d\phi^k(v)\|_*^2$, for all $k > 0$. The inequality $\|d\phi^k(v)\|_*^2 < S_k^2(x).\|v\|_*^2$ comes the same way, considering this time the function $g : \left(k \mapsto \|d\phi^k(v)\|_*^2/S_k^2(x)\right)$. This concludes the proof of the claim. □

On the other hand, any dominated splitting for a flow has a separator. Let $T_K M = E \oplus F$ be a dominated splitting. For simplicity, Φ will denote the restriction of $d\phi$ to the bundle E. For instance, we will write $\|\Phi_x^t\|$ for the maximum norm of the restriction of $d\phi$ to E_x. Precisely, we assert the following:

Claim 3. *The function $r = \left(x \mapsto a.\|\Phi_x^T\|^{1/T}\right)$ is a separator $E \prec r \prec F$, for some $a > 1$, and some T large enough.*

Remark 3.5.2. *This is Lemma 3.3.2 for flows; the proof is comparable step by step to that of Claim 1.*

Proof : Fix a real $t > T$. Let m be the largest integer such that $T + mT \leq t$. For any real $0 \leq \kappa \leq T$, we decompose Φ^t to obtain, for all $x \in K$,

$$\|\Phi_x^t\| \leq \|\Phi^\lambda_{\phi^{\kappa+mT}(x)}\|.\|\Phi^T_{\phi^{\kappa+(m-1)T}(x)}\|...\|\Phi^T_{\phi^{\kappa+T}(x)}\|.\|\Phi^T_{\phi^\kappa(x)}\|.\|\Phi_x^\kappa\|,$$

where $\lambda > 0$ satisfies $t = \kappa + mT + \lambda$. Denote by c the upper bound $c = \sup_{|\tau|\leq 2T, y\in K} \|\Phi_y^\tau\|$, and take the logarithm of the inequality:

$$\ln\|\Phi_x^t\| \leq 2\ln(c) + \ln\|\Phi^T_{\phi^{\kappa+(m-1)T}(x)}\| + ... + \ln\|\Phi^T_{\phi^\kappa(x)}\|.$$

This stands for all $0 \leq \kappa \leq T$. Therefore we have:

$$\begin{aligned}
\int_0^T \ln\|\Phi_x^t\|.d\kappa &\leq \int_0^T 2.\ln(c).d\kappa + \int_0^T \ln\|\Phi^T_{\phi^{\kappa+(m-1)T}(x)}\|.d\kappa + ... + \int_0^T \ln\|\Phi^T_{\phi^\kappa(x)}\|.d\kappa \\
T.\ln\|\Phi_x^t\| &\leq 2T.\ln(c) + \int_0^{mT} \ln\|\Phi^T_{\phi^u(x)}\|.du \\
T.\ln\|\Phi_x^t\| &\leq 4T.\ln(c) + \int_0^t \ln\|\Phi^T_{\phi^u(x)}\|.du,
\end{aligned}$$

since $0 \leq t - mT \leq 2T$, and $-\ln \|\Phi_y^T\| \leq \ln \|\Phi_y^{-T}\| \leq \ln(c)$. Then, dividing by T, we obtain

$$\ln \|\Phi_x^t\| \quad \leq \quad \ln(c^4) + \int_0^t \ln[a^{-1}.r(\phi^u(x))].du.$$

Writing the exponential form, we get $\|\Phi_x^t\| \leq c^4 a^{-t} R_t(x)$. Which means that $\Phi = d_E\phi \prec r$. We are left to check that $r \prec d_F\phi$ for some $T > 0$ big enough, and some $a > 1$. This is done the same way as in Lemma 3.3.2. □

Clearly, referring the reader to the proof of Theorem 3.4.1, flow versions of Lemmas 3.3.2 and 3.4.2 are all the ingredients we need, to transpose the results of Section 3.4 to flows:

Theorem 3.5.3. *A dominated (resp. partially hyperbolic) splitting for a flow on a compact subset of a Riemannian manifold admits an adapted metric.*

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Chapter 4

Newhouse Phenomenons

Les résultats énoncés dans ce chapitre ont été obtenus en collaboration avec Christian Bonatti et Thérèse Vivier. Nous montrons que si une orbite périodique n'admet pas une décomposition dominée de force prescrite, elle peut être transformée en un puits ou une source par une petite C^1-perturbation le long de l'orbite. A partir de ce résultat [16] montre que le flot de Poincaré linéaire d'un champ de vecteurs C^1 admet une décomposition dominée au-dessus de tout ensemble robustement transitif.

The results stated in this chapter are a common work with Christian Bonatti and Thérèse Vivier. We show that a periodic orbit period of a diffeomorphism or flow, either admits a dominated splitting of a prescribed strength, or can be turned into a sink or a source by a C^1-small perturbation along the orbit. Using that result [16] proves that the linear Poincaré flow of a C^1-vector field admits a dominated splitting over any robustly transitive set.

4.1 Introduction

In [38], R. Mañé proved that any robustly transitive diffeomorphism f of a closed surface S is an Anosov diffeomorphism. Let us give a sketch of his proof.

Mañé first establishes that the tangent bundle of the surface splits into a direct sum $TS = E \oplus F$ of two line bundles, and that this splitting is dominated: the vectors in F are uniformly more expanded by Df than those in E. In order to prove this, he shows that, if there were no dominated splitting, then it would be possible to perturb (in the C^1-topology) the differential of the diffeomorphism along some periodic orbits in order to create a complex eigenvalue, thus creating a sink or a source and breaking the transitivity. This is a purely linear-algebraic argument on periodic sequences of matrices in $GL(2,\mathbb{R})$. It remains to show that the vectors in E are indeed uniformly contracted and the vectors in F uniformly expanded. This requires deeper arguments, in particular the ergodic closing lemma.

The first argument of this proof has been adapted in higher dimension, first in [24] in dimension 3 and then in [13] in any dimension: any robustly transitive diffeomorphisms admits a dominated splitting. However, the arguments in these papers fail to be purely algebraic: both of them use the classical fact that, if P and Q are homoclinically related hyperbolic saddles, there are periodic saddles (with arbitrarily long periods) that remain arbitrarily close to the orbit of P during an arbitrarily long time, then jump into a neighborhood of Q in a bounded time, remain there during an arbitrarily long time, come back in a small neighborhood of P in a bounded time, and so on. This dynamical argument allows in some sense to multiply large positive iterates of the derivative of f corresponding to different periodic orbits. This "semi-group-like" property has been formalized in [13] through the notion of *linear cocycles with transitions*, whose archetype is the cocycle induced by Df, over the set of homoclinically related periodic orbits in some homoclinic class. [13] shows that, if a linear cocycle with transitions does not admit any dominated splitting, then a small perturbation of the diffeomorphism enables to turn the differential along a periodic orbit into a homothety, thus turning the periodic orbit into a sink or a source.

The notion of transition introduced in [13] is a very strong tool, but it turns out to be also heavy and not very flexible. For instance, it is not easy to adapt its definition to continuous time systems. Another problem was that [13] does not give any information on the existence of dominated splittings for sequences of periodic orbits with trivial homoclinic class.

However, transitions are needed to get the linear result in [13]. Given a linear cocycle without transition and without dominated splitting, it is indeed not always possible to turn the matrix at the period into an homothety by a small perturbation of the cocycle [1]. We can nevertheless recover almost the same property: for any linear cocycle over periodic orbits (without transition hypothesis) without dominated splitting, there are arbitrarily small perturbations such that the matrix at the period (corresponding to some periodic orbit of the system) has all its eigenvalues real and with same modulus.

Actually, the article [16] only shows it for long periods systems; we added a complement (see section 4.5) that deals with the general case. We state

[1]Consider for example a sequence of periodic points x_n with periods $p_n \to \infty$ of a diffeomorphism f such that the differential along the orbit can be written (in local coordinates) as: $Df(f^i(x_n)) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \forall i \in \{0, \dots p_n - 2\}$ and $Df(f^{p_n-1}(x_n)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then these orbits admit no dominated splitting, but there is $\epsilon > 0$ such that for any ε-perturbation g of f preserving the orbit of one x_n, the differential $Dg^{p_n}(x_n)$ is not an homothety.

the consequence for the periodic orbits of a diffeomorphism:

Corollary 4.2.21 *Let $f: M \to M$ be a diffeomorphism of a compact Riemannian manifold. Then for any $\varepsilon > 0$ there are integers ℓ, n such that, for any periodic point x of period $p(x) \geq n$:*

- *either f admits an ℓ-dominated splitting along the orbit of x;*
- *or, for any neighborhood U of the orbit of x, there exists an ε-perturbation g of f for the C^1-topology, coinciding with f outside U and on the orbit of x, and such that x is a source or a sink of g for which the differential $Dg^{p(x)}(x)$ has all eigenvalues real and with same modulus.*

One obtains an analogous result for flows (see [16]).

Our results can be connected to a recent result by S. Gan in [27][2]: given any $\varepsilon > 0$, there is $\ell > 0$ such that, for any periodic point x,

- either there is an ℓ-dominated splitting $E \oplus F$ along the orbit of x with $dimE = i$,
- or there is an ε-small C^1-perturbation of f, that makes equal the moduli of the i^{th} and the $i+1^{\text{th}}$ eigenvalues associated to the orbit of x.

Our results on periodic orbits of diffeomorphisms derive from analogous results obtained for linear cocycles (see subsection 4.2.1 for the definition and subsection 4.2.5 for the statement of the result) together with a lemma of Franks which enables to achieve any perturbation of the differential of a diffeomorphism as the differential of a perturbation of the diffeomorphism.

We would like to thank Marie-Claude Arnaud and Sylvain Crovisier for many enlightening discussions, and the referee for pointing out the relation between Gan's result and ours.

[2]It would be tempting to try to deduce our result from Gan's result. If a periodic orbit has no ℓ-dominated splitting, then one may apply Gan's result to any pair of eigenvalues. So, one can hope to make equal, successively, every pair of eigenvalues. This idea relies on the fact that, what is done at a step does not destroy the effect of the previous perturbations; in other words, it needs the robustness of the proximity of some of the Lyapunov exponents, independently of the period of the periodic orbit. Unfortunately, one can easily build sequences of periodic orbits γ_n, whose period tends to infinity, having a pair of equal Lyapunov exponents that are separated by more than a constant (for n large enough) by an arbitrarily small perturbation.

4.2 Statement of the results

4.2.1 Elementary definitions : linear cocycles

Definition 4.2.1. We shall call *linear cocycle of dimension* d any 4-uple $\mathcal{A} = (\Sigma, f, E, A)$ such that:

- Σ is a set and $f : \Sigma \to \Sigma$ is a one-to-one map.
- $\pi : E \to \Sigma$ is a linear bundle of dimension d over Σ, whose fibers are endowed with a Euclidean metric $\| \ \|$. The fiber over the point $x \in \Sigma$ will be denoted by E_x.
- $A : x \in \Sigma \mapsto A_x \in GL(E_x, E_{f(x)})$ is a map.

We shall say that a linear cocycle $\mathcal{A}$ is *bounded* if there exists a constant $K > 0$ such that, for any point $x \in \Sigma$, we have $\|A_x\| < K$ and $\|A_x^{-1}\| < K$. We shall say that $\mathcal{A}$ is a *cocycle of matrices* if the bundle $\pi : E \to \Sigma$ is trivial and if the metric on each fiber is the standard euclidean norm of $\mathbb{R}^d$. We shall then consider A_x as an element of $GL(\mathbb{R}, d)$. Notice that any linear cocycle $\mathcal{A}$ bounded by K is, up to a choice of orthonormal basis over each fiber E_x, conjugated to a cocycle of matrices bounded by the same constant K.

Let $\mathcal{A}$ be a linear cocycle. For any integer n, and any point $x \in \Sigma$, we shall denote by A_x^n the product $A_x^n = A_{f^{n-1}(x)} \circ ... \circ A_{f(x)} \circ A_x$. Moreover, $\mathcal{A}^n = (\Sigma, f^n, E, A^n)$ is a linear cocycle bounded by K^n.

We shall say that a subbundle $F \subset E$ is *invariant* if its fibers F_x have constant dimension for any point $x \in \Sigma$ and if $F_{f(x)} = A_x(F_x)$. We denote by $F^\perp$ the (a priori non invariant) orthogonal subbundle of F, that is, for each point $x \in \Sigma$, $F_x^\perp$ is the orthogonal supplement of F_x in E_x. These subbundles are both naturally endowed with the metric induced by the metric defined on E.

Definition 4.2.2. Let $\mathcal{A}$ be a linear cocycle, Σ' an f-invariant subset of Σ and F an invariant subbundle over Σ.

1. The *restriction* of $\mathcal{A}$ to the subset Σ' is a linear cocycle (Σ', f, E, A) denoted by $\mathcal{A}_{|\Sigma'}$.
2. The linear cocycle $\mathcal{A}_F$ *induced* by A on F is the linear cocycle (Σ, f, F, A) obtained by restricting $\mathcal{A}$ to the subbundle F.
3. The *quotient cocycle* of $\mathcal{A}$ by F, denoted by $\mathcal{A}/F$, is the linear cocycle $(\Sigma, f, F^\perp, A/F)$, where $(A/F)_x \colon F_x^\perp \to F_{f(x)}^\perp$ is the projection on $F_{f(x)}^\perp$, parallel to $F_{f(x)}$, of the restriction $\left(A_{|F^\perp}\right)_x : F_x^\perp \to E_{f(x)}$ of A_x.

Notice that if $\mathcal{A}$ is bounded by K, then $\mathcal{A}_{|\Sigma'}$, $\mathcal{A}_F$ and $\mathcal{A}/F$ are also bounded by K: this is obvious for the first two cocycles. This is true for $\mathcal{A}/F$ because the invariance of F by A implies

$$(\mathcal{A}/F)^{-1} = \mathcal{A}^{-1}/F,$$

and because the use of an orthogonal projection decreases the norm of the application (in fact, for any integer n, $(\mathcal{A}/F)^n = \mathcal{A}^n/F$).

We shall use the natural notions of transverse subbundles, and of direct sum of transverse subbundles.

4.2.2 Dominated decomposition

Definition 4.2.3. Let $\mathcal{A}$ be a linear cocycle bounded by K, and F, G two invariant subbundles. We shall say that G *dominates* F, denoted by $F \prec G$ or $F \prec_\ell G$, if there exists an integer ℓ such that, for any point $x \in \Sigma$ and any pair of vectors $(u, v) \in F_x \times G_x$, the following inequality holds:

$$\frac{\|A_x^\ell(u)\|}{\|u\|} \leq \frac{1}{2}.\frac{\|A_x^\ell(v)\|}{\|v\|}.$$

Remark 4.2.4.

1. *ℓ-domination does not imply $(\ell+1)$-domination;*

2. *For any constant $K > 0$, any integer ℓ, there is an integer L such that, for any linear cocycle bounded by K and any ℓ-dominated decomposition $F \prec_\ell G$, the following assertion holds:*

$$\forall \ell' \geq L,\ F \prec_{\ell'} G.$$

We then use the following notation: $F \prec^L G$. We call characteristic time of domination *the smallest L such that $F \prec^L G$.*

Definition 4.2.5. A bounded linear cocycle $\mathcal{A}$ *admits a dominated splitting* (or *dominated decomposition*) if there exist two transverse invariant subbundles F, G such that $E = F \oplus G$ and $F \prec G$.

More generally, for any $F_1, ..., F_k$ invariant subbundles such that $E = F_1 \oplus \cdots \oplus F_k$, the decomposition is said *dominated* if, for any $i \in \{1, ..., k-1\}$, $F_i \prec F_{i+1}$: indeed, [13] proves that, under these assumptions, the decomposition $E = \left(\bigoplus_1^i F_j\right) \oplus \left(\bigoplus_{i+1}^k F_j\right)$ is dominated for any $i \in \{1, ..., k-1\}$.

We shall also use the following Lemma ([13, Lemma 4.4])

Lemma 4.2.6. *Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle and assume that E admits an invariant decomposition $E = F \oplus G \oplus H$. Consider the* *quotient cocycle $\mathcal{A}/G$. The projection on E/G of the subbundles F and H induces an invariant decomposition denoted by $E/G = F/G \oplus H/G$.*

Assume that we have the dominations $F/G \prec H/G$ and $G \prec H$, then we have $(F \oplus G) \prec H$. Symmetrically, we get:

$$(F \prec G \text{ and } F/G \prec H/G) \Longrightarrow F \prec (G \oplus H).$$

4.2.3 Periodic cocycles

We shall consider linear cocycles over a system (Σ, f) satisfying the following three properties:

(P1) Σ is infinite,

(P2) any point $x \in \Sigma$ is periodic, and we shall denote its period by $p(x)$,

(P3) for any integer $k > 0$, the set of points $x \in \Sigma$ such that $p(x) \leq k$ is finite.

A system verifying $(P2)$ is a *periodic system*. A system (Σ, f) verifying (P1), (P2) and (P3) is a *large periods system*. In particular, for any large periods system (Σ, f) the set Σ is countable, and up to an indexation of the orbits, Σ can be regarded as a sequence of periodic orbits whose periods tend to infinity. The restriction of (Σ, f) to any infinite invariant subset of Σ is another large periods system.

Let (Σ, f) be a periodic system. For any linear cocycle $\mathcal{A} = (\Sigma, f, E, A)$ and any $x \in \Sigma$ we define:

$$M_{x,A} = A_x^{p(x)} \colon E_x \to E_x.$$

We shall use the abridged notation M_x when there is no ambiguity on the considered cocycle.

Definition 4.2.7. Let $\mathcal{A} = (\Sigma, f, E, A)$ be a linear bounded cocycle over an infinite periodic system.

1. We shall say that the cocycle $\mathcal{A}$ is *strictly without dominated decomposition* (or equivalently *strictly without domination*) if the only invariant subsets Σ', in restriction to which the cocycle admits a dominated splitting, are finite.

2. An invariant splitting $E = F \oplus G$ shall be said *strictly not dominated* if the only invariant subsets Σ' in restriction to which the splitting $F \oplus G$ is dominated are finite.

Remark 4.2.8. *If $\mathcal{A}$ is strictly without domination and if $\Sigma' \subset \Sigma$ is an infinite f-invariant subset, then the restriction $\mathcal{A}_{|\Sigma'}$ is also strictly without domination.*

Lemma 4.2.9. *Let $\mathcal{A} = (\Sigma, f, E, A)$ be a linear bounded cocycle over an infinite periodic system.*

1. *Let $E = F \oplus G$ be an invariant splitting, then:*
   - *either there exists an infinite invariant subset $\Sigma' \subset \Sigma$ such that the splitting is strictly not dominated in restriction to Σ';*
   - *or there exists a finite invariant subset $\Sigma_0 \subset \Sigma$ such that the splitting is dominated in restriction to $\Sigma \setminus \Sigma_0$.*
2. *In any case, $\mathcal{A}$ verifies one of the following properties:*
   - *either there exists an infinite invariant subset $\Sigma' \subset \Sigma$ such that the cocycle is strictly without dominated decomposition in restriction to Σ';*
   - *or there exists a partition $\Sigma = \Sigma_0 \cup \Sigma_1 \cdots \cup \Sigma_k$, with $k < d$, such that, for any $i \in \{0, ..., k\}$ the subsets Σ_i are f-invariant, Σ_0 is a finite set and the cocycle admits a dominated decomposition in restriction to each $\Sigma_i,\ i \geq 1$.*

Proof:

1. Let $E = F \oplus G$ be an invariant splitting. For any integer $L \geq 1$, let us consider the set Σ_L of points $x \in \Sigma$ such that the restriction of the cocycle over the orbit of x verifies $F \prec^L G$. If there exists an integer L such that the $\Sigma \setminus \Sigma_L$ is finite, then the second assertion holds. Assume that for any integer L, $\Sigma \setminus \Sigma_L$ is an infinite set, then we can construct an infinite sequence of points x_n with disjoints orbits such that, for any n, x_n belongs to $\Sigma - \Sigma_n$. The union Σ' of the orbits of the points x_n satisfies the first assertion.

2. Assume $\mathcal{A}$ admits no dominated splitting and consider, for any pair of integers $(L \geq 0, i \in \{1, ..., d-1\})$, the set $\Sigma_{(L,i)}$ of points such that the restriction of the cocycle over the orbit of x admits a splitting $E = F \oplus G$ verifying $F \prec^L G$ and $dim(F) = i$. If there exists an integer L such that the set $\Sigma \setminus \bigcup_{i=1}^{d-1} \Sigma_{(L,i)}$ is finite, then the second assertion holds. Assume that for any integer L, $\Sigma \setminus \bigcup_{i=1}^{d-1} \Sigma_{(L,i)}$ is an infinite set, then we can construct an infinite sequence of points x_n with disjoint orbits such that, for any n, x_n belongs to $\Sigma - \bigcup_i \Sigma_{(n,i)}$. The union Σ' of the orbits of the points x_n satisfies the first assertion.

$\square$

4.2.4 Perturbations

Definition 4.2.10.

1. Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle. For any given ε, any bounded linear cocycle $\mathcal{B} = (\Sigma, f, E, B)$ such that $\|A_x - B_x\| \leq \varepsilon$ and $\|A_x^{-1} - B_x^{-1}\| \leq \varepsilon$ for any $x \in \Sigma$ is an *ε-perturbation of $\mathcal{A}$*.

2. Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle over an infinite periodic system. A linear cocycle $\mathcal{B} = (\Sigma, f, E, B)$ is a *perturbation of $\mathcal{A}$* if, for any $\varepsilon > 0$, the set $\{x \in \Sigma,\ \|A_x - B_x\| \geq \varepsilon\}$ is finite.

Remark 4.2.11.

1. *Any linear cocycle $\mathcal{A}$ over a finite (hence periodic) system (Σ, f) admits a dominated decomposition if and only if there exist an integer $i \in \{1, .., d-1\}$ and, for any point $x \in \Sigma$ two spaces F_x and G_x invariant by M_x with dimension i, (respectively $d - i$,) such that the modulus of any eigenvalue of the restriction $M_{x|F_x}$ is strictly smaller than the modulus of any eigenvalue of $M_{x|G_x}$.*

2. *As a consequence, any linear cocycle over a finite system (Σ, f) with dimension greater than 3 admits arbitrarily small perturbations such that there exists a decomposition of the set $\Sigma = \Sigma_1 \cup \Sigma_2$ verifying the two following assertions: the restriction of the linear cocycle to Σ_1 admits a dominated splitting $E = F_1 \oplus G_1$ where $dim(F_1) = 1$, and the restriction of the linear cocycle to Σ_2 admits a dominated splitting $E = F_2 \oplus G_2$ where $dim(F_2) = 2$.*

Both these remarks explain why we shall neglect finite invariant subsets of the system in our further study.

Remark 4.2.12. *Let (Σ, f) be an infinite periodic system, and $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be three bounded linear cocycles of dimension d over (Σ, f). Then :*

- *if $\mathcal{B}$ is a perturbation of $\mathcal{A}$, then $\mathcal{A}$ is a perturbation of $\mathcal{B}$;*
- *if $\mathcal{B}$ is a perturbation of $\mathcal{A}$ and if $\mathcal{C}$ is a perturbation of $\mathcal{B}$, then $\mathcal{C}$ is a perturbation of $\mathcal{A}$;*

In other words, "to be a perturbation of" defines an equivalence relation on the set of bounded cocycles over (Σ, f).

Remark 4.2.13. *Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle over an infinite periodic system and let $\mathcal{B}$ be a perturbation of $\mathcal{A}$. Then there is a scalar function $\alpha \colon \Sigma \to \mathbb{R}$ with the following property:*

- *There is a finite subset $\Sigma_0 \subset \Sigma$ such that, for all $x \in \Sigma \setminus \Sigma_0$ the determinant of the matrix $\alpha(x).B_x$ is equal to the determinant of A_x;*

- *$\alpha(x)$ converges to 1 when $x \to \infty$, that is, for all $\varepsilon > 0$ the set $\{x \in \Sigma \mid |\alpha(x) - 1| > \varepsilon\}$ is finite.*

In particular, the cocycle $\mathcal{C}$ defined by $C_x = \alpha(x)B_x$ is a perturbation of the cocycle $\mathcal{A}$.

Given a bounded linear cocycle admitting a dominated splitting, [13] proves that any small enough perturbation of the cocycle admits a dominated splitting (with same dimension of the subbundles). More precisely, [13] proves that:

Lemma 4.2.14. *Given any dimension d, any positive constant K and any integer $\ell > 0$, there is an $\varepsilon > 0$ such that, for any linear cocycle $\mathcal{A} = (\Sigma, f, E, A)$ bounded by K, of dimension d, admitting an ℓ-dominated splitting $F \prec_\ell G$, one has that:*

any ε-perturbation $\mathcal{B}$ of $\mathcal{A}$ admits a dominated splitting $F' \prec G'$ with $dim(F') = dim(F)$.

Corollary 4.2.15. *Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle over an infinite periodic system. Assume $\mathcal{A}$ is strictly without dominated splitting. Then any perturbation of $\mathcal{A}$ is strictly without dominated splitting.*

Proof: We argue by contradiction: assume that there exists a perturbation $\mathcal{B}$ of $\mathcal{A}$ whose restriction to an infinite invariant subset Σ' admits a dominated splitting $E = F \oplus G$. Since $\mathcal{A}$ is bounded, the linear cocycle $\mathcal{B}$ is bounded by some positive constant K. Fix $\ell > 0$ such that $F \prec_\ell G$. By Lemma 4.2.14, there exists an $\varepsilon > 0$ such that any ε-perturbation of $\mathcal{B}$ admits a dominated splitting. By definition of a perturbation, there exists a finite invariant subset Σ_0 such that the restriction of the linear cocycle $\mathcal{A}$ to the infinite set $\Sigma' \setminus \Sigma_0$ is an ε-perturbation of $\mathcal{B}$, hence admits a dominated splitting. This contradicts the assumption that $\mathcal{A}$ is strictly without dominated splitting. □

4.2.5 Statement of the results for linear cocycles

Theorem 4.2.16. *Any bounded linear cocycle $\mathcal{A}$ over a large periods system admits a perturbation $\mathcal{B}$ such that for any point $x \in \Sigma$, all eigenvalues of $M_{x,B}$ are real, with multiplicity 1 and different moduli.*

Scholium 4.2.17. *For any $x \in \Sigma$, $\Lambda(x, \mathcal{A})$ denotes the d-uple $(\sigma_1, \ldots, \sigma_d)$ of the Lyapunov exponents of x, considered with multiplicity and in increasing order. That is, each σ_i is of the form*

$$\sigma_i = \frac{\log(|\lambda_i|)}{p(x)},$$

where λ_i is an eigenvalue of the matrix $M_{x,A}$.

The proof of Theorem 4.2.16 will show that, in the statement of Theorem 4.2.16, we can require that $\Lambda(x,\mathcal{B})-\Lambda(x,\mathcal{A})\in\mathbb{R}^d$ converges to $0\in\mathbb{R}^d$ when $p(x)$ goes to infinity.

The main result in this section is the following:

Theorem 4.2.18. *Let $\mathcal{A}$ be a bounded linear cocycle over a large periods system. Assume that $\mathcal{A}$ is strictly without dominated decomposition. Then there exists a perturbation $\mathcal{B}$ and an infinite invariant subset Σ' such that, for any point $x\in\Sigma'$, all eigenvalues of $M_{x,B}$ are real, with same modulus.*

This result can be restated in the following stronger version:

Corollary 4.2.19. *Let $\mathcal{A}$ be a bounded linear cocycle over a large periods system. Assume that $\mathcal{A}$ is strictly without dominated decomposition. Then there exists a perturbation $\mathcal{B}$ such that, for any point $x\in\Sigma$, all eigenvalues of $M_{x,B}$ are real, with same modulus.*

Let us prove Corollary 4.2.19 using Theorem 4.2.18:

Proof: For any $\varepsilon>0$, let us denote by Σ_ε the set of points $x\in\Sigma$ such that there exists an ε-perturbation $\mathcal{B}$ over the reduced system $(Orb(x),f)$ verifying: for any integer k, all eigenvalues of $M_{f^k(x),B}$ are real, with same modulus.

Let us first remark that $\Delta_\varepsilon=\Sigma\setminus\Sigma_\varepsilon$ is a finite set. Indeed, if Δ_ε were an infinite set, we could apply Theorem 4.2.18 to the linear cocycle $\mathcal{A}$ restricted to Δ_ε, which is strictly without domination: it contradicts the definition of Δ_ε.

We shall now use the following decomposition

$$\Sigma=\Delta_1\cup\left(\Sigma_1\setminus\Sigma_{1/2}\right)\cup\left(\Sigma_{1/2}\setminus\Sigma_{1/3}\right)\cdots\cup\left(\Sigma_{1/n}\setminus\Sigma_{1/(n+1)}\right)\cdots\cup\left(\cap\Sigma_{1/n}\right).$$

Let us consider the linear cocycle $\mathcal{B}$ defined as follows:

1. Δ_1 being a finite set, the restriction of $\mathcal{B}$ to Δ_1 can be any cocycle verifying $M_{x,B}=Id$ for any point $x\in\Delta_1$;

2. by definition, for any integer k, there exists an $1/k$-perturbation $\mathcal{B}$ of the restriction of $\mathcal{A}$ to the set $\Sigma_{1/k}\setminus\Sigma_{1/(k+1)}$ such that for any point $x\in\Sigma_{1/k}\setminus\Sigma_{1/(k+1)}$, $M_{x,B}$ has all eigenvalues real with same modulus;

3. since $\mathcal{A}$ satisfies the announced assertion over the set $\cap\Sigma_{1/n}$, $\mathcal{B}$ can be taken equal to $\mathcal{A}$ in restriction to the set $\cap\Sigma_{1/n}$.

The set $\Sigma_{1/n}\setminus\Sigma_{1/(n+1)}$ is included in $\Delta_{1/(n+1)}$ which is finite, hence the linear cocycle $\mathcal{B}$ is a perturbation of $\mathcal{A}$. □

Corollary 4.2.20. *Given any dimension d, any positive constant K and any $\varepsilon > 0$, there exists two integers ℓ and n such that, any K-bounded linear cocycle $\mathcal{A}$ with dimension d over a periodic orbit with period greater than n satisfies one of the following two assertions:*

- *either $\mathcal{A}$ admits an ℓ-dominated splitting;*
- *or there exists an ε-perturbation $\mathcal{B}$ of $\mathcal{A}$ such that $M_{x,B}$ has all eigenvalues real with same modulus.*

Proof: Assume, arguing by contradiction, that corollary 4.2.20 is wrong: there is d, K and $\varepsilon > 0$ such that, given any integer n, there is a K-bounded linear cocycle $\mathcal{A}_n$ with dimension d, over a periodic orbit γ_n with period greater than n, verifying the two following properties:

- for every $k \leq n$, $\mathcal{A}_n$ has no k-dominated splitting;
- there is no ϵ-perturbation $\tilde{\mathcal{A}}_n$ of $\mathcal{A}_n$ such that, for $x_n \in \gamma_n$, the eigenvalues of $M_{x_n,\tilde{A}_n}$ are all real with same modulus.

Consider $\Sigma = \bigcup_{n\in\mathbb{N}} \gamma_n$: this set is a large periods system. Let $\mathcal{A}$ be the linear cocycle defined over Σ, such that its restriction to γ_n is $\mathcal{A}_n$. The cocycle $\mathcal{A}$ is a K-bounded linear cocycle which, by construction, is strictly without domination: any invariant set on which $\mathcal{A}$ admits an ℓ-dominated decomposition is included in $\bigcup_{n=0}^{\ell} \gamma_n$ and hence is finite. Furthermore, for any perturbation $\mathcal{B}$ of $\mathcal{A}$, the set of points x for which the matrix $M_{x,B}$ has all eigenvalues real with same modulus is finite: the cocycle $\mathcal{B}$ is (by definition of a perturbation) an ε-perturbation of $\mathcal{A}$ out of a finite set. This contradicts Theorem 4.2.18, and this contradiction concludes the proof. □

4.2.6 Statement of the results for diffeomorphisms

A lemma of Franks in [26] allows to realize any small perturbation of the derivative of a diffeomorphism along a finite set by a C^1-perturbation of the diffeomorphism. This enables us to restate Corollary 4.2.20 for diffeomorphisms.

Corollary 4.2.21. *Let $f\colon M \to M$ be a diffeomorphism of a compact manifold, endowed with a Riemannian metric $\|\cdot\|$. Then for any $\varepsilon > 0$ there are two integers ℓ and n such that, for any periodic point x of period $p(x) \geq n$:*

- *either f admits an ℓ-dominated splitting along the orbit of x;*
- *or, for any neighborhood U of the orbit of x, there exists an ε-perturbation g of f for the C^1-topology, coinciding with f outside U and on the orbit of x, and such that the differential $Dg^{p(x)}(x)$ has all eigenvalues real and with same modulus. This modulus can furthermore be chosen different from 1 so that the orbit of x is a source or a sink of g.*

Remark 4.2.22. *In fact, the pair (ℓ, n) given by the preceding result only depends on ε and on an upper bound of $\|Df\|$. The result henceforth holds with the same (ℓ, n) for a C^1-neighborhood of f.*

Let f be a diffeomorphism on a compact manifold M. Let us recall that a point $x \in M$ is *chain-recurrent* if there exist pseudo-orbits starting and ending at x with arbitrarily small jumps. [10] proves that any chain-recurrent point can be turned into a periodic point by an arbitrarily small C^1-perturbation.

Corollary 4.2.23. *Let f be a diffeomorphism on a compact manifold. Then for any ε, there exist a pair of integers (ℓ, n) such that for any chain-recurrent point x, one of the following two assertions holds:*

1. *either x belongs to an invariant compact set admitting an ℓ-dominated splitting;*
2. *or there is an ε-perturbation g of f (in the C^1-topology) for which x is a periodic sink or source.*

4.3 Proof of Theorem 4.2.16

4.3.1 Perturbations on subbundles and quotient bundles

Throughout this paragraph, we shall denote by $\mathcal{A} = (\Sigma, f, E, A)$ a K-bounded linear cocycle over a infinite periodic system. Let F be an invariant subbundle of E. Let us consider an orthonormal basis of vectors of F and an orthonormal basis of vectors of $F^{\perp}$: this provides an orthonormal basis of vectors in which we can write A in blocks of the form:

$$\begin{pmatrix} A_F & C \\ 0 & A/F \end{pmatrix},$$

where C is bounded. We thus get the following two lemmas:

Lemma 4.3.1. *For any perturbation $\mathcal{B}_F$ of the induced cocycle $\mathcal{A}_F$ there exists a perturbation $\mathcal{B}$ of $\mathcal{A}$ with the following properties:*

- *F is invariant by $\mathcal{B}$;*
- *the induced cocycle obtained by restriction of $\mathcal{B}$ to the subbundle F is $\mathcal{B}_F$;*
- *the quotient cocycle $\mathcal{B}/F$ coincides with $\mathcal{A}/F$. In particular, the eigenvalues of $M_{x,A}$ associated to eigenvectors out of F_x remain unchanged.*

Lemma 4.3.2. *For any perturbation $\mathcal{B}_{F^\perp}$ of the quotient cocycle $\mathcal{A}/F$, there exists a perturbation $\mathcal{B}$ of $\mathcal{A}$ with the following properties:*

- *F is invariant by $\mathcal{B}$;*
- *the quotient cocycle of $\mathcal{B}$ by F coincides with $\mathcal{B}_{F^\perp}$;*
- *the induced cocycle obtained by restriction of $\mathcal{B}$ to F coincides with $\mathcal{A}_F$.*

Definition 4.3.3. Let E and E' be two bundles of same dimension over a system (Σ, f). Any change of basis P from E to E' (that is, for any point $x \in \Sigma$, $P_x \in GL(E_x, E'_x)$) is *bounded* by $K > 0$ if and only if, for any $x \in \Sigma$, $\|P_x\| \leq K$ and $\|P_x^{-1}\| \leq K$.

We then get the following two lemmas:

Lemma 4.3.4. *Let E and E' be two bundles of same dimension over a system (Σ, f), and let P be a bounded change of basis from E to E'. Let $\mathcal{B} = (\Sigma, f, E', B)$ be the bounded linear cocycle defined by $B_x = P_{f(x)} \circ A_x \circ P_x^{-1}$ for any point $x \in \Sigma$. Then the following two statements hold:*

- *if $\mathcal{A}$ admits a dominated splitting, then so does $\mathcal{B}$;*
- *for any perturbation $\tilde{\mathcal{A}}$ of $\mathcal{A}$, the linear cocycle $\tilde{\mathcal{B}}$ defined by $P_{f(x)} \circ \tilde{A}_x \circ P_x^{-1}$ for any point $x \in \Sigma$ is a perturbation of $\mathcal{B}$.*

Lemma 4.3.5. *Let F and G be two invariant subbundles of E with trivial intersection. Assume that, the angle $\widehat{F_x, G_x}$ is bounded from below by a uniform constant for any point $x \in \Sigma$. Then, there exists a bounded change of basis $P : E \to E$ such that the subbundles $P(F)$ and $P(G)$ are orthogonal. (Notice that the subbundles $P(F)$ and $P(G)$ are necessarily invariant by the linear cocycle $\mathcal{B}$ obtained by conjugating $\mathcal{A}$ by P: $B_x = P_{f(x)} \circ A_x \circ P_x^{-1}$ for any point $x \in \Sigma$.)*

As the bundles of a dominated splitting have their angles bounded from below, one deduces from the previous lemma:

Corollary 4.3.6. *Let $\mathcal{A}$ be a bounded linear cocycle admitting a dominated splitting $E = F \oplus G$. Then, up to a change of basis, we can assume the dominated decomposition to be orthogonal, thus we can write A in blocks of the form:*

$$\begin{pmatrix} A_F & 0 \\ 0 & A_G \end{pmatrix}.$$

4.3.2 Cocycle of dimension 2

Proposition 4.3.7. *Let $\mathcal{A}$ be a K-bounded linear cocycle of dimension 2 over a large periods system. There exists a perturbation $\mathcal{B}$ of $\mathcal{A}$ such that, for any point $x \in \Sigma$, $M_{x,B}$ has all eigenvalues real of multiplicity 1, and different modulus.*

Furthermore, the Lyapunov exponents of $\mathcal{B}$ can be chosen arbitrarily close to those of $\mathcal{A}$.

This is a consequence of [10, lemme 6.6] which is restated below:

Lemma 4.3.8. *For any $\varepsilon > 0$, there exists $N(\varepsilon) \geq 1$ such that, for any integer $n \geq N(\varepsilon)$ and any finite sequence $A_0, ..., A_n$ of elements in $SL(2, \mathbb{R})$, there exists a sequence $\alpha_0, ..., \alpha_n$ in $]-\varepsilon, \varepsilon[$ such that the following assertion holds:*

for any $i \in \{0, ..., n\}$ if we denote by $B_i = R_{\alpha_i} \circ A_i$ the composition of A_i with the rotation R_{α_i} of angle α_i, then the matrix $B_n \circ B_{n-1} \circ \cdots \circ B_0$ has real eigenvalues.

Let us deduce the proof of Proposition 4.3.7:

Proof: Let $\mathcal{A}$ be a K-bounded linear cocycle of dimension 2 over a large periods system (Σ, f). First notice that, if a matrix in $GL(2, \mathbb{R})$ has a real eigenvalue, then there is an arbitrarily small perturbation of this matrix that has two real eigenvalues of multiplicity 1 with different moduli. So we just need to build a perturbation of $\mathcal{A}$ such that the matrices $M_{x,B}$ have at least one real eigenvalue (of modulus arbitrarily close to the moduli of the eigenvalues of $M_{x,A}$).

Consider $\Sigma_1 \subset \Sigma$ the set of points x for which the matrix $M_{x,\mathcal{A}}$ has a pair of complex (non-real) conjugated eigenvalues. If Σ_1 is finite, we are done (it suffices to define $\mathcal{B}$ on Σ_1, such that $M_{x,B}$, $x \in \Sigma_1$ is the homothety transformation whose ratio is the modulus of the complex eigenvalue of $M_{x,A}$.

Assume now that Σ_1 is infinite. Fix a sequence ε_n decreasing to 0 and consider the sets $\Gamma_n = \{x \in \Sigma_1 \mid p(x) \geq N(\varepsilon_n)\}$. As Σ is a large periods system, the complement of each Γ_n is finite.

Remark 4.3.9. *There is a sequence δ_n converging to 0 such that, for any $\alpha \in [-\varepsilon_n, \varepsilon_n]$ and any matrix $A \in GL(2, \mathbb{R})$ with $\|A\| < K$ one has $\|A - B\| < \delta_n$ where $B = R_\alpha \circ A$.*

For any $x \in \Gamma_n \setminus \Gamma_{n+1}$, Lemma 4.3.8 gives a sequence (α_i), $i = 0, \ldots, p(x) - 1$, with $|\alpha_i| \leq \varepsilon_n$ such that the matrix $\prod_0^{p(x)-1} R_{\alpha_i} \circ A(f^i(x))$ has a real eigenvalue. Define $\mathcal{B}$ on the orbit of x by

$$B_{f^i(x)} = R_{t(x) \cdot \alpha_i} \circ A(f^i(x))$$

where $t(x)$ is the infimum of the $t \in]0,1]$ such that the matrix $\prod_0^{p(x)-1} R_{t\cdot\alpha_i} \circ A(f^i(x))$ has a real eigenvalue. Then $t(x) \in]0,1]$, and $M_{x,\mathcal{B}}$ has a real eigenvalue of multiplicity 2 and whose modulus coincides with the modulus of the eigenvalue of $M_{x,A}$.

Then Remark 4.3.9 implies that the cocycle $\mathcal{B}$ defined that way on $\bigcup_n (\Gamma_n \setminus \Gamma_{n+1})$ is a perturbation of $\mathcal{A}$; moreover as $(\Gamma_n)_n$ is a decreasing sequence and $\bigcap \Gamma_n = \emptyset$, one gets that $\Gamma_0 = \bigcup_n (\Gamma_n \setminus \Gamma_{n+1})$. Finally $\Sigma_1 \setminus \Gamma_0$ is finite, so that one can complete this perturbation in a perturbation of $\mathcal{A}$ on Σ_1: define $\mathcal{B}$ on the finite set $\Sigma_1 \setminus \Gamma_0$ in the same way as when Σ_1 is finite. We complete this perturbation on Σ by defining $B_x = A_x$ for $x \notin \Sigma_1$, thus obtaining the announced perturbation. □

4.3.3 Proof of Theorem 4.2.16 and Scholium 4.2.17

We proceed by induction on the dimension d of the cocycle. The case $d = 1$ is trivial and $d = 2$ is solved by Proposition 4.3.7.

Assume the result is true for any $d' < d$, and consider a bounded cocycle $\mathcal{A}$ of dimension d over a large periods system (Σ, f). Notice that, for any $d > 1$, any linear isomorphism of $\mathbb{R}^d$ admits an invariant 2-plane. As a direct consequence, any linear cocycle of dimension d over (Σ, f) admits an invariant subbundle F of dimension 2. Applying the induction assumption to the induced cocycle $\mathcal{A}_F$ and to the quotient cocycle $\mathcal{A}/F$, we obtain the perturbations $\mathcal{B}_F$ of $\mathcal{A}_F$ and $\mathcal{B}_{F^\perp}$ of $\mathcal{A}/F$. Lemmas 4.3.1 and 4.3.2 ensure then the existence of a perturbation $\mathcal{B}$ of $\mathcal{A}$ inducing the cocycle $\mathcal{B}_F$ on F and whose quotient $\mathcal{B}/F$ is $\mathcal{B}_{F^\perp}$. Notice that, for any x, all eigenvalues of $M_{x,B}$ are real, and that an arbitrarily small additional perturbation can make their moduli pairwise distinct.

4.4 Proof of Theorem 4.2.18

4.4.1 Lyapunov diameter

Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle over a large periods system. Define the *Lyapunov diameter* of $x \in \Sigma$ by:

$$\delta(x, \mathcal{A}) = \max\left\{\frac{|\log(|\lambda_1|) - \log(|\lambda_2|)|}{p(x)},\ \lambda_i \in \mathrm{Spec}(M_x)\right\},$$

i.e. the difference between the largest and the smallest Lyapunov exponent of x for the cocycle $\mathcal{A}$. We shall use the notation $\delta(x)$ whenever there is no ambiguity on the considered cocycle.

We then denote by $\delta_+(\mathcal{A})$ (the *upper Lyapunov diameter of* $\mathcal{A}$) and $\delta_-(\mathcal{A})$ (the *lower Lyapunov diameter*) the upper and the lower limit, respectively, of the $\delta(x)$ for $x \in \Sigma$.

Lemma 4.4.1. *Let (Σ, f) be a large periods system and $\mathcal{A}$ a bounded linear cocycle such that $\delta_-(\mathcal{A}) = 0$. Then there exists a perturbation $\mathcal{B}$ of $\mathcal{A}$ and an infinite invariant subset Σ' such that, for any point $x \in \Sigma'$, all eigenvalues of $M_{x,B}$ are real, with same modulus.*

Proof: Using Theorem 4.2.16 and Scholium 4.2.17, one builds a perturbation $\mathcal{A}'$ of $\mathcal{A}$ such that $\delta_-(\mathcal{A}') = 0$ and such that, for any $x \in \Sigma$, all eigenvalues of $M_{x,A'}$ are real, with multiplicity 1, and different moduli. We can then choose, for any $x \in \Sigma$, an orthonormal basis b_x of each fiber E_x such that each linear map A'_x has an upper triangular matrix in this basis.

As $\delta_-(\mathcal{A}') = 0$, there is some infinite invariant subset $\Sigma' \subset \Sigma$ such that $\lim_{x\to\infty} \delta(x, \mathcal{A}') = 0$ (that is, for any $\varepsilon > 0$, the set $\{x \in \Sigma' \mid \delta(x, \mathcal{A}') > \varepsilon\}$ is finite). For any $x \in \Sigma'$ we consider $C_x\colon E_x \to E_x$ the linear map whose matrix in the basis b_x is the diagonal matrix (α_i) where $\alpha_i > 0$ verifies:

$$\alpha_i^{p(x)} = \frac{|\det M_{x,A'}|^{\frac{1}{d}}}{|\lambda_i|}$$

where λ_i the i^{th} eigenvalue of $M_{x,A'}$. Since $\delta(x)$ converges to 0 as $p(x)$ goes to ∞ for $x \in \Sigma'$, the matrix C_x converges to the Identity matrix.

Define now $\mathcal{B}$ by:

- $B_x = A_x$ if $x \notin \Sigma'$;
- $B_x = A_x \circ C_x$ if $x \in \Sigma'$.

The linear cocycle $\mathcal{B}$ is a perturbation of $\mathcal{A}$ and, for any $x \in \Sigma'$, all eigenvalues of $M_{x,B}$ are real and have the same modulus. □

Given a bounded cocycle $\mathcal{A}$ over a large periods system (Σ, f), we define the *minimal Lyapunov diameter of* $\mathcal{A}$, denoted by $\delta_{min}(\mathcal{A})$, as the infimum of the $\delta_-(\mathcal{B})$ for all perturbation $\mathcal{B}$ of $\mathcal{A}$.

Remark 4.4.2. *If $\Sigma' \subset \Sigma$ is an infinite invariant subset, then the Lyapunov diameter of the restricted cocycle $\mathcal{A}_{|\Sigma'}$ verifies: $\delta_{min}(\mathcal{A}_{|\Sigma'}) \geq \delta_{min}(\mathcal{A})$.*

Lemma 4.4.3. *There is a perturbation $\mathcal{B}$ of $\mathcal{A}$ such that $\delta_-(\mathcal{B}) = \delta_{min}(\mathcal{A})$. Furthermore, $\mathcal{B}$ can be chosen so that, for any $x \in \Sigma$, all eigenvalues of $M_{x,B}$ are real, with multiplicity* 1 *and with different moduli.*

Proof: For any integer $n > 0$ there is a perturbation $\mathcal{B}_n$ of $\mathcal{A}$ such that $\delta_-(\mathcal{B}_n) < \delta_{min} + \frac{1}{n}$. Then, by definition of $\delta_-(\mathcal{B}_n)$ and of a perturbation of

$\mathcal{A}$, for any $n > 0$ there is an infinite f-invariant subset $\Sigma_n \subset \Sigma$ such that any $x \in \Sigma_n$ verifies:

$$\delta(x) < \delta_{min} + \frac{2}{n} \quad \text{and} \quad |A_n - B_n| < \frac{1}{n}.$$

Choose by iteration an infinite sequence $(x_n)_n \in \Sigma^{\mathbb{N}}$ as follows. Fix $x_1 \in \Sigma_1$. Once $x_1, \dots, x_n$ chosen, choose $x_{n+1} \in \Sigma_{n+1} \setminus \cup_{i=1}^{n} Orb(x_i)$; this is possible because Σ_{n+1} is an infinite set whereas the union of the orbits of x_i, $i \in \{1, \dots, n\}$ is finite.

Define the perturbation $\mathcal{B}$ of $\mathcal{A}$ as follows:

- if $x \in \Sigma$ belongs to the orbit of x_i, for some integer $i \geq 1$, then $B_x = B_{i,x}$;
- otherwise, $B_x = A_x$.

One easily verifies that the so-defined linear cocycle $\mathcal{B}$ is a perturbation of $\mathcal{A}$ verifying $\delta_-(\mathcal{B}) = \delta_{min}(\mathcal{A})$. A new perturbation of $\mathcal{B}$ given by Theorem 4.2.16 and Scholium 4.2.17 allows to turn all eigenvalues into real ones with multiplicity 1, different moduli, without modifying $\delta_-(\mathcal{B})$. □

Remark 4.4.4. *In the proof of Lemma 4.4.3, denote* $\Sigma' = \cup_{i=1}^{n} Orb(x_i)$ *and let* $\mathcal{B}'$ *be the restriction of* $\mathcal{B}$ *to the infinite invariant subset* Σ'*. Then*

$$\delta_+(\mathcal{B}') = \delta_-(\mathcal{B}') = \delta_{min}(\mathcal{B}') = \delta_{min}(\mathcal{B}).$$

This remark motivates the following definition

Definition 4.4.5. Let (Σ, f) be a large periods system. A bounded linear cocycle $\mathcal{A}$ over (Σ, f) is called *incompressible* if it verifies both following assumptions:

1. $\delta_+(\mathcal{A}) = \delta_-(\mathcal{A}) = \delta_{min}(\mathcal{A})$;
2. for any $x \in \Sigma$, all eigenvalues of $M_{x,\mathcal{A}}$ are real, with multiplicity 1 and different moduli.

If $\mathcal{A}$ is incompressible, we denote by $\delta(\mathcal{A})$ the *Lyapunov diameter of* $\mathcal{A}$ defined by $\delta(\mathcal{A}) = \delta_+(\mathcal{A}) = \delta_-(\mathcal{A}) = \delta_{min}(\mathcal{A})$.

Remark 4.4.6. *Let* $\mathcal{A} = (\Sigma, f, E, A)$ *be an incompressible bounded linear cocycle over a large periods system* (Σ, f)*, and let* $\Gamma \subset \Sigma$ *be an invariant infinite subset. Then the restricted cocycle* $\mathcal{A}_{|\Gamma}$ *is incompressible.*

Theorem 4.2.18 is now a corollary of the following result:

Theorem 4.4.7. *Let* (Σ, f) *be a large periods system and* $\mathcal{A}$ *a bounded linear cocycle over* (Σ, f)*. Assume that* $\mathcal{A}$ *is incompressible and strictly without domination. Then* $\delta(\mathcal{A}) = 0$*.*

The proof of Theorem 4.4.7 is the aim of the next sections. Let us first prove Theorem 4.2.18 using Theorem 4.4.7:

Proof: Consider a bounded linear cocycle $\mathcal{A}$ over a large periods system (Σ, f) and assume that $\mathcal{A}$ is strictly without domination. Then Lemma 4.4.3 and Remark 4.4.4 ensure the existence of a perturbation $\mathcal{B}$ of $\mathcal{A}$ and of an infinite f-invariant subset $\Sigma' \subset \Sigma$ such that the restriction $\mathcal{B}' = \mathcal{B}_{|\Sigma'}$ is incompressible. Corollary 4.2.15 and Remark 4.2.8 imply that $\mathcal{B}'$ is strictly without domination.

Then Theorem 4.4.7 asserts that $\delta(\mathcal{B}') = 0$, and Lemma 4.4.1 finally implies the existence of a perturbation $\mathcal{C}'$ of $\mathcal{B}'$ and the existence of an infinite invariant subset $\tilde{\Sigma} \subset \Sigma'$ such that, for any point $x \in \tilde{\Sigma}$, all eigenvalues of $M_{x,C'}$ are real, with same modulus. The cocycle $\mathcal{C}$ on Σ defined by $C_x = A_x$ when $x \notin \tilde{\Sigma}$, and $C_x = C'_x$ when $x \in \tilde{\Sigma}$, is a perturbation of $\mathcal{A}$ satisfying the conclusion of Theorem 4.2.18. □

Remark 4.4.8. *In fact we proved that, if Theorem 4.4.7 holds for d-dimensional cocycles, then Theorem 4.2.18 also holds for d-dimensional cocycles.*

We are left to prove Theorem 4.4.7. We argue by induction on the dimension d of the cocycle. We shall first prove that for any incompressible cocycle $\mathcal{A} = (\Sigma, f, E, A)$ strictly without domination, there exists a splitting into two invariant subbundles $E = F \oplus G$ such that the induced cocycles $\mathcal{A}_{|F}$ and $\mathcal{A}_{|G}$ are both strictly without domination.

4.4.2 Splitting in subbundles strictly without domination

We aim in this section at proving the following:

Proposition 4.4.9. *Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle strictly without domination over a large periods system. Assume that, for any $x \in \Sigma$, all eigenvalues of M_x are real with multiplicity 1 and with different modulus. Then there exists an invariant partition $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_{d-1}$ such that for any $i \in \{1, \ldots, d-1\}$, either the set Σ_i is empty or the set Σ_i is infinite and in this case, there exists an invariant decomposition $E = F_i \oplus G_i$ over Σ_i such that*

- *$dim(F_i) = i$;*
- *the induced cocycles $\mathcal{A}_{F_i}$ and $\mathcal{A}_{G_i}$ over Σ_i are strictly without domination;*
- *for any $x \in \Sigma_i$, the moduli of the eigenvalues of M_x corresponding to F_i are strictly smaller than the ones corresponding to G_i.*

Lemma 4.4.10. *Let $\mathcal{A}$ be a bounded linear cocycle defined over a large periods system (Σ, f). Let $E = F \oplus G$ be an invariant splitting such that*

1. *the cocycles induced by restriction $\mathcal{A}_F$ and $\mathcal{A}_G$ are strictly without domination;*

2. *for any $x \in \Sigma$, the eigenvalues of M_x corresponding to F are strictly smaller than the ones corresponding to G;*

3. *the decomposition $E = F \oplus G$ is strictly not dominated.*

Then the cocycle $\mathcal{A}$ is strictly without domination.

Proof: We proceed by contradiction. Let $F' \oplus G'$ be a dominated splitting on some infinite invariant subset $\Sigma' \subset \Sigma$. Then the moduli of the eigenvalues corresponding to F' are strictly smaller than the ones corresponding to G'. If $dim(F') = dim(F)$ (and hence $dim(G') = dim(G)$) then the second condition of the lemma implies that $F' = F$ and $G' = G$, which is in contradiction with the first condition of the lemma.

Note that

- either $dim(F') > dim(F)$ and $dim(G') < dim(G)$;
- or $dim(F') < dim(F)$ and $dim(G') > dim(G)$.

Both cases are symmetric. We shall consider the first case.

As the moduli of the eigenvalues corresponding to F are strictly smaller than the ones corresponding to G, we get that $G' \subset G$ and $F \subset F'$. We thus deduce that F' is transverse to G and that G admits the following decomposition $G = (F' \cap G) \oplus G'$ over Σ'. This splitting (over the infinite subset Σ') is dominated by assumption, which is in contradiction with the assumption of G being strictly without domination. $\square$

Lemma 4.4.11. *Let $\mathcal{A}$ be a linear cocycle defined over a large periods system (Σ, f), and $E = E_1 \oplus \cdots \oplus E_k$ be an invariant splitting such that:*

1. *for any integer i, the cocycle induced by restriction $\mathcal{A}_{E_i}$ is strictly without domination;*

2. *for any integer i, for any point $x \in \Sigma$, the moduli of the eigenvalues of M_x corresponding to E_i are strictly smaller than the ones corrresponding to E_{i+1};*

3. *the cocycle $\mathcal{A}$ is strictly without domination.*

For any $i \in \{1, \ldots, k-1\}$ define $F_i = E_i \oplus E_{i+1}$. Then there exists an invariant partition $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_{k-1}$ such that, either Σ_i is a finite set, or the induced cocycle $\mathcal{A}_{F_i}$ is strictly without domination over Σ_i.

Proof: For any $x \in \Sigma$, we denote by $L_i(x)$ the characteristic time (cf Remark 4.2.4) of the domination of the splitting $F_i = E_i \oplus E_{i+1}$ over the periodic orbit of x. Let $i_x \in \{1, \dots, k-1\}$ be an integer chosen so that the characteristic time $L_{i_x}(x)$ realizes the maximum of $\{L_i(x),\ i \in \{1, \dots, k-1\}$. Note that i_x can be chosen constant over the periodic orbit of x, hence the function $x \mapsto i_x$ is invariant by f.

Let $\Sigma_i = \{x \in \Sigma,\ i_x = i\}$. Then $\Sigma_1 \cup \dots \cup \Sigma_{k-1}$ is an invariant partition of Σ. Choose i such that Σ_i is an infinite set.

Claim 4. *The decomposition $F_i = E_i \oplus E_{i+1}$ is strictly not dominated over Σ_i.*

Proof: We proceed by contradiction. Assume the decomposition $F_i = E_i \oplus E_{i+1}$ is dominated over an infinite invariant subset $\Sigma_i' \subset \Sigma_i$. There exists an integer L such that $E_i \prec^L E_{i+1}$ over Σ_i'. For any $x \in \Sigma_i'$, L is by definition bigger than $L_i(x)$. As $i_x = i$, we get that $L_j(x) \leq L$ for any $j \in \{1, \dots, k-1\}$. As a conclusion, $E_j \prec^L E_{j+1}$ for any j over Σ_i', hence the decomposition $E = E_1 \oplus \dots \oplus E_k$ is dominated over Σ', which contradicts the assumption of $\mathcal{A}$ being strictly without domination. □

Note that, thanks to the previous claim, the decomposition $F_i = E_i \oplus E_{i+1}$ over Σ_i satisfies all the assumptions of the lemma 4.4.10. We thus get that $\mathcal{A}_{F_i}$ is strictly without domination over Σ_i, which concludes the proof of the lemma. □

We are now ready to prove Proposition 4.4.9.

Proof: We are going to show by a decreasing inductive argument that, for any $k \in \{2, \dots d\}$, there exists an integer i_k and an invariant finite partition $\mathcal{P}_k = \{\Sigma_1^k, \dots, \Sigma_{i_k}^k\}$ of Σ such that for any $j \in \{1, \dots, i_k\}$ either the set Σ_j^k is finite, or the set Σ_j^k is infinite and in this case there is an invariant splitting $E = E_1 \oplus \dots \oplus E_k$ such that

(P1) for any $\ell \in \{1, \dots, k\}$, the induced cocycle $\mathcal{A}_{E_\ell}$ over Σ_j^k is strictly without domination;

(P2) for any $\ell \in \{1, \dots, k-1\}$ and for any $x \in \Sigma_j^k$, the moduli of the eigenvalues of M_x corresponding to E_ℓ are strictly smaller than the ones corresponding to $E_{\ell+1}$;

(P3) the restricted cocycle $\mathcal{A}_{|\Sigma_j^k}$ is strictly without domination.

For $k = d$, denote by $E_1, \dots, E_d$ the one-dimensional subbundles corresponding, for any $x \in \Sigma$, to the eigenspaces of M_x ordered in the increasing way by the moduli of the eigenvalues. The trivial partition $\Sigma = \Sigma$ and this splitting satisfy all the required conditions.

Assume the proposition verified for $k+1$. Let $\Sigma_j^{k+1} \in \mathcal{P}_{k+1}$ be one of the subsets given by the inductive assumption. If this subset is finite, Σ_j^{k+1}

remains in the partition $\mathcal{P}_k$. Assume Σ_j^{k+1} is an infinite set. Denote by $E = E_1 \oplus \cdots \oplus E_{k+1}$ the corresponding invariant splitting. Applying Lemma 4.4.11 to the restricted cocycle $\mathcal{A}_{|\Sigma_j^{k+1}}$, we get a finite invariant partition of the set Σ_j^{k+1} in subsets Σ' verifying the following dichotomy:

- either the subset Σ' is finite;
- or the subset Σ' is infinite and there is an integer i such that the induced cocycle $\mathcal{A}_{E_i \oplus E_{i+1}}$ is strictly without domination. Endow the restricted cocycle to Σ' with the decomposition in k subbundles obtained by gathering E_i together with E_{i+1} in the decomposition (in $k+1$ subbundles) associated to Σ_j^{k+1}. In other words, if $F = E_i \oplus E_{i+1}$, the decomposition associated to Σ' is $E_1 \oplus \cdots \oplus E_{i-1} \oplus F \oplus E_{i+2} \oplus \cdots \oplus E_{k+1}$. This splitting satisfies the required conditions.

Gathering together the partitions built for each Σ_j^{k+1} in $\mathcal{P}_{k+1}$, we get the announced partition $\mathcal{P}_k$, which ends the induction argument.

We thus get a partition $\mathcal{P}_2$. Let us first denote by Σ'_0 the union of all finite subsets of the partition $\mathcal{P}_2$. Over each infinite set of the partition $\mathcal{P}_2$, there is a splitting of E into two subbundles $E = E_1 \oplus E_2$ verifying the conditions (P1), (P2) and (P3). For any integer $i \in \{1, \dots, d-1\}$, denote by Σ'_i the union all infinite subsets of the partition $\mathcal{P}_2$ such that $dim(E_1) = i$.

Choose an integer $i_0 \in \{1, \dots, d-1\}$ such that the set Σ'_{i_0} is not empty. Denote by Σ_{i_0} the set $\Sigma'_0 \cup \Sigma' i_0$ and extend the subbundle E_1 defined over Σ'_{i_0} to Σ'_0 by considering the sum of the first i_0 eigenspaces with smaller moduli of eigenvalues. We then conclude the proof by taking $\Sigma_i = \Sigma'_i$ for any integer $i \neq i_0$. □

4.4.3 Decreasing the Lyapunov spectrum: proof of Theorem 4.4.7

We proceed by induction on the dimension d of the cocycle. When $d = 2$, Mañé ([38]) ensures the existence of perturbations along arbitrarily long periodic orbits, with complex eigenvalues at the period. Applying Proposition 4.3.7, there exists a perturbation whose eigenvalues at the period are real, and of moduli arbitrarily close to the modulus of the complex eigenvalues. This concludes the proof of Theorem 4.4.7 in the 2-dimensional case.

Assume now that Theorem 4.4.7 holds for any cocycle of dimension strictly less than d. We will prove that it also holds for d-dimensional cocycles. Notice that this implies that Theorem 4.2.18 also holds for d-dimensional cocycles (see Remark 4.4.8).

Consider a large periods system (Σ, f) and a bounded d-dimensional linear cocycle $\mathcal{A} = (\Sigma, f, E, A)$ that is incompressible and strictly without domination. We will show, arguing by contradiction, that $\delta(\mathcal{A}) = 0$. Assume (by contradiction) that $\delta(\mathcal{A}) > 0$. Notice that any cocycle $\mathcal{B}$ obtained by restriction of the cocycle $\mathcal{A}$ to an infinite subset of Σ verifies the following:

(H1) $\mathcal{B}$ is incompressible;

(H2) $\mathcal{B}$ is strictly without domination;

(H3) $\delta(\mathcal{B}) = \delta(\mathcal{A}) > 0$.

Proposition 4.4.9 implies the existence of an infinite invariant subset $\Gamma \subset \Sigma$ such that the restricted cocycle $\mathcal{B} = \mathcal{A}_{|\Gamma}$ admits a (not dominated) splitting $E = F \oplus G$ with the following properties:

(H4) the induced cocycle $\mathcal{B}_F$ and $\mathcal{B}_G$ are strictly without domination;

(H5) for any $x \in \Gamma$, the moduli of the eigenvalues of M_x corresponding to F are strictly smaller than the ones corresponding to G.

Lemma 4.4.12. *We have:*

$$\delta_+(\mathcal{B}_F) = \delta_+(\mathcal{B}_G) = 0.$$

Proof: We proceed by contradiction, assuming (for instance) that $\delta_+(\mathcal{B}_F) = \delta > 0$. There exists an infinite invariant subset $\Gamma_0 \subset \Gamma$ such that, for any point $x \in \Gamma_0$, $\delta(x, \mathcal{B}_F) > \delta/2$. The restricted cocycle $(\mathcal{B}_F)_{|\Gamma_0}$ is strictly without domination, and its dimension is strictly less than d. Then, the induction assumption allows to apply Theorem 4.2.18 to the cocycle $(\mathcal{B}_F)_{|\Gamma_0}$. Thus there is a perturbation $\widetilde{(\mathcal{B}_F)_{\Gamma_0}}$ such that $\delta_+\left(\widetilde{(\mathcal{B}_F)_{\Gamma_0}}\right) = 0$. Furthermore, by Remark 4.2.13, we can assume that this perturbation preserves the determinant of the matrix $(B_F)_x$ for any point $x \in \Gamma_0$.

We can now consider a perturbation $\mathcal{C}$ of the cocycle $\mathcal{A}$ such that

- outside Γ_0, $\mathcal{C}$ coincides with $\mathcal{A}$;
- on Γ_0, $\mathcal{C}_F = \widetilde{(\mathcal{B}_F)}$ and $\mathcal{C}/F = \mathcal{A}/F$;

Claim 5. *For any point $x \in \Gamma_0$,*

$$\delta(x, \mathcal{C}) \leq \delta(x, \mathcal{A}) - \frac{\delta}{2d}.$$

Proof: Recall that

- the moduli of the eigenvalues of $M_{x,A}$ corresponding to F are strictly smaller than the ones corresponding to G;
- the eigenvalues of $M_{x,C}$ associated to eigenspaces outside F coincide with the eigenvalues of $M_{x,A}$ associated to G;

- since the considered perturbation of the matrix A_x preserves the determinant of $(A_F)_x$, the modulus of the eigenvalues of $M_{x,C}$ corresponding to F is equal to the geometric average of the moduli of the eigenvalues of $M_{x,A}$.

For any cocycle $\mathcal{D}$, let us denote the smallest and the greatest Lyapunov exponents of a point x by

$$\sigma^+(x,\mathcal{D}) = \max\left\{\frac{|\log(|\lambda|)|}{p(x)},\ \lambda \in Spec\,(M_{x,D})\right\}$$

and

$$\sigma^-(x,\mathcal{D}) = \min\left\{\frac{|\log(|\lambda|)|}{p(x)},\ \lambda \in Spec\,(M_{x,D})\right\}.$$

So, the Lyapunov diameter of x is $\delta(x,\mathcal{D}) = \sigma^+(x,\mathcal{D}) - \sigma^-(x,\mathcal{D})$.

As a consequence of the previous remarks, for any $x \in \Gamma_0$,

$$\sigma^+(x,\mathcal{C}) = \sigma^+(x,\mathcal{A}) \text{ and } \sigma^-(x,\mathcal{C}) = \frac{\log(|\det(M_{x,A_F})|)}{\dim(F).p(x)}.$$

Thus, for any point $x \in \Gamma_0$, $\delta(x,\mathcal{A}) - \delta(x,\mathcal{C}) = \sigma^-(x,\mathcal{C}) - \sigma^-(x,\mathcal{A})$. Denote by $|\lambda_1(x)| < \ldots < |\lambda_{\dim(F)}(x)|$ the moduli of the eigenvalues of $M_{x,A}$ corresponding to F and $\sigma_i(x) = \log(|\lambda_i(x)|)/p(x)$. We then get the following:

- $\sigma^-(x,\mathcal{A}) = \sigma_1(x)$;
- $\sigma^-(x,\mathcal{C}) = \frac{\sum_{k=1}^{\dim(F)} \sigma_k(x)}{\dim(F)}$;
- $\sigma_{\dim(F)}(x) - \sigma_1(x) > \delta/2$ since $x \in \Gamma_0$.

We then get by an easy calculus that $\sigma^-(x,\mathcal{C}) \geq \sigma^-(x,\mathcal{A}) + \frac{\delta}{2.\dim(F)}$, thus $\delta(x,\mathcal{C}) \leq \delta(x,\mathcal{A}) - \frac{\delta}{2d}$. □

Since Γ_0 is an infinite set, we deduce from the preceding claim that $\delta_-(\mathcal{C}) \leq \delta(\mathcal{A}) - \frac{\delta}{2d}$ which is in contradiction with the assumption of incompressibility of $\mathcal{A}$. This concludes the proof of the Lemma. □

As a direct corollary we get:

Corollary 4.4.13. *Let $F' \subset F$ and $G' \subset G$ be invariant subbundles defined over an infinite invariant subset $\Gamma' \subset \Gamma$.*

Then the induced cocycle $(\mathcal{B}_{|\Gamma'})_{F'}$ and $(\mathcal{B}_{|\Gamma'})_{G'}$ are strictly without domination. Furthermore the quotient cocycles $\mathcal{B}/F$ and $\mathcal{B}/G$ verify

$$\delta_+(\mathcal{B}/F) = \delta_+(\mathcal{B}/G) = 0,$$

hence $\mathcal{B}/F$ and $\mathcal{B}/G$ are strictly without domination (and the same holds for any cocycle induced by $\mathcal{B}/F$ or $\mathcal{B}/G$ over an infinite invariant subset).

Proof: Let us start by two easy criteria given by the Lyapunov diameter:

- If a bounded linear cocycle $\mathcal{C}_0$ over a periodic system admits a dominated splitting, then $\delta_-(\mathcal{C}_0) > 0$;
- If a bounded linear cocycle $\mathcal{C}$ over a periodic system is not strictly without domination, then it admits a dominating splitting over an infinite invariant subsystem, hence it verifies $\delta_+(\mathcal{C}) > 0$.

On the one hand, Lemma 4.4.12 implies that, if $F' \subset F$ is an invariant subbundle over an infinite invariant subset $\Gamma' \subset \Gamma$, then $\delta_+((\mathcal{B}_{|\Gamma'})_{F'} \leq \delta_+(\mathcal{B}_F) = 0$, so that $(\mathcal{B}_{|\Gamma'})_{F'}$ is strictly without domination.

On the other hand, for any $x \in \Gamma$, the linear map $M(x, \mathcal{B}/F)$ is conjugated to $M(x, \mathcal{B}_G)$. As a consequence they both have same spectrum, thus $\delta(x, \mathcal{B}/F) = \delta(x, \mathcal{B}_G)$. Similarly, $\delta(x, \mathcal{B}/G) = \delta(x, \mathcal{B}_F)$. Hence $\delta_+(\mathcal{B}/F) = \delta_+(\mathcal{B}_G) = 0$ and $\delta_+(\mathcal{B}/G) = \delta_+(\mathcal{B}_F) = 0$. □

Proposition 4.4.14. *Let $\mathcal{B}$ be a linear cocycle over a large periods system $(\Gamma \subset \Sigma, f)$ satisfying the properties $(H1)$, $(H2)$, $(H3)$, $(H4)$ and $(H5)$. Assume that $H \subset E$ is a proper $\mathcal{B}$-invariant subbundle containing F as a proper subbundle (i.e. $0 \subsetneq F_x \subsetneq H_x \subsetneq E_x$ for any $x \in \Gamma$). Then the splitting $H = F \oplus (G \cap H)$ is a dominated splitting for the cocycle $\mathcal{B}_H$.*

Similarly, if $L \subset E$ is a proper $\mathcal{B}$-invariant subbundle containing G as a proper subbundle, then the splitting $L = (F \cap L) \oplus G$ is a dominated splitting of the cocycle $\mathcal{B}_L$.

Proof: The two statements of the proposition being completely symmetrical, we shall just prove the first one. We proceed by contradiction: consider a proper $\mathcal{B}$-invariant subbundle $H \subset E$ containing F as a proper subbundle, and assume that the splitting $H = F \oplus (G \cap H)$ is not dominated. We shall contradict the assumption $(H1)$.

As the moduli of the eigenvalues of the matrix $M(x, B)$ corresponding to F are strictly smaller than the ones corresponding to G (hence, to $G \cap H$), this splitting is dominated over any finite subsystem. This remark together with the assumption that the splitting $H = F \oplus (G \cap H)$ is not dominated imply, by Lemma 4.2.9, the existence of an infinite invariant subset $\Gamma' \subset \Gamma$ over which the splitting is strictly not dominated. Corollary 4.4.13 then states that the cocycles $(\mathcal{B}_{|\Gamma'})_F$ and $(\mathcal{B}_{|\Gamma'})_{G \cap H}$ are also strictly without domination: we infer by Lemma 4.4.10 that the cocycle $\mathcal{C} = (\mathcal{B}_{|\Gamma'})_H$ is strictly without domination.

As H is a proper subbundle of E, its dimension is strictly smaller than d. The induction assumption hence implies that Theorem 4.2.18 can be applied to $\mathcal{C}$: there is a perturbation $\tilde{\mathcal{C}}$ of $\mathcal{C}$ over an infinite invariant subset $\tilde{\Gamma} \subset \Gamma'$ such that, for any $x \in \tilde{\Gamma}$, all eigenvalues of $M_{x,\tilde{\mathcal{C}}}$ have same modulus. Furthermore, Remark 4.2.13 allows to assume that this perturbation preserves the determinant for any $x \in \tilde{\Gamma}$.

Using Lemmas 4.3.1 and 4.3.2, we build a perturbation $\tilde{B}$ of $\mathcal{B}$ verifying the following properties:

1. $\tilde{\mathcal{B}}$ coincides with $\mathcal{B}$ out of $\tilde{\Gamma}$;
2. $\tilde{\mathcal{B}}$ leaves the subbundle H invariant;
3. the quotient cocycle $(\tilde{\mathcal{B}})/H$ coincides with $\mathcal{B}/H$;
4. the induced cocycle $(\tilde{\mathcal{B}}_{|\tilde{\Gamma}})_H$ is $\tilde{\mathcal{C}}$.

The following lemma (which contradicts the incompressibility of $\mathcal{B}$) will conclude the proof of the proposition:

Lemma 4.4.15. $\delta_-(\tilde{\mathcal{B}}) < \delta(\mathcal{B})$.

Proof: Notice first that $(G \cap H)(x)$, being not reduced to 0, contains eigenvectors of $M_{x,B}$. As $\delta_+(\mathcal{B}_F) = \delta_+(\mathcal{B}_G) = 0$, the set

$$\{x \in \Gamma,\ |\delta(x, \mathcal{B}_H) - \delta(x, \mathcal{B})| \geq \varepsilon\}$$

is finite for any $\varepsilon > 0$. As a direct consequence, $\delta_-(\mathcal{C}) = \delta_+(\mathcal{C}) = \delta(\mathcal{B}) = \delta > 0$.

Consider the variations of the extremities of the Lyapunov spectrum of $x \in \tilde{\Gamma}$, under the perturbations $\mathcal{B} \to \tilde{\mathcal{B}}$. For any $x \in \tilde{\Gamma}$:

- $\sigma^+(x, \tilde{\mathcal{B}}) \leq \sigma^+(x, \mathcal{B})$;
- $\sigma^-(x, \tilde{\mathcal{B}}) = \inf\left\{\sigma^-(x, \tilde{\mathcal{C}}), \sigma^-(x, \mathcal{B}/H)\right\}$.

Notice that $\sigma^+(x, \mathcal{B}) - \sigma^+(x, \mathcal{B}/H)$ converges to 0 when x tends to infinity because this difference corresponds to Lyapunov exponents of $\mathcal{B}$ associated to G and $\delta_+(\mathcal{B}_G) = 0$. In order to prove the lemma, we are left to verify that:

Claim 6.
$$\liminf_{x \to \infty} \left(\sigma^-(x, \tilde{\mathcal{C}}) - \sigma^-(x, \mathcal{C})\right) > 0.$$

The determinant of $M_{x,\tilde{C}}$ and $M_{x,C}$ are equal, and all the eigenvalues of $M_{x,\tilde{C}}$ have the same modulus, so that

$$\sigma^-(x, \tilde{\mathcal{C}}) = \frac{1}{\dim(H)} \frac{\log|\det(M_{x,\mathcal{C}})|}{p(x)}.$$

This determinant is the product of the determinant of M_{x,B_F} by $\dim(H \cap G) = \dim(H) - \dim(F)$ eigenvalues of $M_{x,B}$ corresponding to G. Moreover, the difference of any Lyapunov exponent of M_{x,B_G} and any Lyapunov exponent of M_{x,B_F} converges to δ as $x \to \infty$ (this can be deduced easily from $\delta_+(\mathcal{B}_F) = \delta_+(\mathcal{B}_G) = 0$ and $\delta_+(\mathcal{B}) = \delta_-(\mathcal{B}) = \delta(\mathcal{B})$). Now we get:

$$\lim_{x\to\infty}\left(\sigma^-(x,\tilde{\mathcal{C}})-\sigma^-(x,\mathcal{C})\right)=\frac{(\dim(H)-\dim(F))\,\delta(\mathcal{B})}{\dim(H)}>0 \tag{4.1}$$

This implies the claim and thus concludes the proofs of the lemma and of Proposition 4.4.14. □ □

Proposition 4.4.16. *Let $\mathcal{B}$ be a linear cocycle over a large periods system $(\Gamma\subset\Sigma,f)$ satisfying the properties $(H1)$, $(H2)$, $(H3)$, $(H4)$ and $(H5)$. Assume that $H\subset E$ is a proper $\mathcal{B}$-invariant subbundle containing F as a proper subbundle. The splitting $E=F\oplus G$ induces by projection a natural splitting $E/H=F/H\oplus G/H$ on the quotient cocycle $\mathcal{B}/H$. This splitting is then a dominated splitting for the cocycle $\mathcal{B}_H$.*

Similarly, if $L\subset G$ is a proper $\mathcal{B}$-invariant subbundle, then the splitting $E/L=F/L\oplus G/L$ is a dominated splitting for the cocycle $\mathcal{B}/L$.

The proof of Proposition 4.4.16 follows the same argument as the one of Proposition 4.4.14, so we just give the sketch of the proof.

Proof: We proceed by contradiction: consider a proper $\mathcal{B}$-invariant subbundle $H\subset F$ and assume that the splitting $E/H=F/H\oplus G/H$ is not dominated. Using Lemma 4.4.12, one verifies that $\delta_+((\mathcal{B}_F)/H)=\delta_+((\mathcal{B}_G)/H)=0$ proving that the two cocycles $(\mathcal{B}_F)/H$ and $(\mathcal{B}_G)/H$ are strictly without domination. As the splitting $E/H=F/H\oplus G/H$ is not dominated, there is an infinite invariant subset $\Gamma'\subset\Gamma$ over which the splitting is strictly not dominated. Lemma 4.4.10 implies now that the quotient cocycle $\mathcal{B}/H$ is strictly without domination over Γ'. As the dimension of E/H is strictly less than d the induction hypothesis asserts that we can apply Theorem 4.2.18 (and Remark 4.2.13) to $(\mathcal{B}/H)_{|\Gamma'}$: there is an infinite invariant subset $\tilde\Gamma\subset\Gamma'$ and a perturbation $\tilde{\mathcal{C}}$ of $(\mathcal{B}/H)_{|\tilde\Gamma}$ (preserving the determinant) such that, for any $x\in\tilde\Gamma$, all eigenvalues of $M_{x,\tilde{C}}$ have same modulus. Now, using Lemmas 4.3.1 and 4.3.2 we build a perturbation $\tilde{B}$ of $\mathcal{B}$ verifying the following properties:

1. $\tilde{\mathcal{B}}$ coincides with $\mathcal{B}$ outside $\tilde\Gamma$;
2. $\tilde{\mathcal{B}}$ leaves the subbundle H invariant;
3. the induced cocycle $(\tilde{\mathcal{B}})_H$ coincides with $\mathcal{B}_H$;
4. the quotient cocycle $(\tilde{\mathcal{B}})/H$ coincides with $\tilde{\mathcal{C}}$ over $\tilde\Gamma$.

The following lemma (contradicting the incompressibility of $\mathcal{B}$) hence concludes the proof of the proposition:

Lemma 4.4.17. $\delta_-(\tilde{\mathcal{B}})<\delta(\mathcal{B})$.

The proof of Lemma 4.4.17 follows exactly the proof of Lemma 4.4.15 and we let it to the reader.

□

We conclude the proof of Theorem 4.4.7 (and hence also of Theorem 4.2.18)by proving Lemma 4.4.18 below which contradicts the fact that $\mathcal{B}$ is strictly without domination:

Lemma 4.4.18. *The splitting* $E = F \oplus G$ *is a dominated splitting for* $\mathcal{B}$ *over* Γ.

Proof: As $d \geq 3$, one of the two subbundles F or G has dimension greater than 2. Assume for instance that $dim(F) \geq 2$.

Recall that, for $x \in \Gamma$, all the eigenvalues of $M(x, \mathcal{B})$ have multiplicity one, and pairwise distinct moduli. Let $F_1 \subset F$ be the one-dimensional subbundle corresponding to the eigenvalue of smallest modulus, and $F_2 \subset F$ be the codimension one subbundle directed by the $dim(F) - 1$ other eigendirections. These subbundles are clearly invariant by $\mathcal{B}$, and we get an invariant splitting $F = F_1 \oplus F_2$.

Then, Proposition 4.4.14 applied to $L = F_2 \oplus G$ implies that $F_2 \prec G$. Furthermore, Proposition 4.4.16 applied to $H = F_2$ implies that the splitting $E/F_2 = F/F_2 \oplus G/F_2$ is dominated. Notice that F/F_2 is the projection on E/F_2 of the subbundle F_1, and hence coincides with F_1/F_2 (according to the notations used in Lemma 4.2.6). Now Lemma 4.2.6 asserts that $F_1 \oplus F_2 \prec G$, that is $F \prec G$. □

4.5 Appendix

It happens that in the results we stated previously (article [16]), the large period hypothesis is unnecessary. Actually we have the following on cocycles:

Proposition 4.5.1. *Given any dimension d any period p, any positive constant K and any $\varepsilon > 0$, there exists an integer ℓ_p such that for any K-bounded linear cocycle $\mathcal{A}$ with dimension d over a periodic orbit of period p,*

- *either $\mathcal{A}$ admits an ℓ_p-dominated splitting;*
- *or there exists an ε-perturbation $\mathcal{B}$ of $\mathcal{A}$ such that the first return map has all eigenvalues real with same modulus.*

Then from Franks' Lemma and the large period statement on cocycle (Corollary 4.2.20), we obtain

Theorem 4.5.2. *Let f be a diffeomorphism on a compact Riemannian manifold. Then for all $\epsilon > 0$, there is an integer N such that for any peiodic point x :*

- *either f admits an N-dominated splitting on the orbit* $\mathrm{Orb}_f(x)$ *of x,*
- *or there is an ϵ-perturbation g of f on an arbitrarily small neighbourhood of the orbit* $\mathrm{Orb}_f(x)$*, that preserves the orbit of x, and such that the Lyapunov exponents of g at x are all equal.*

Proof of Theorem 4.5.2 from Proposition 4.5.1 and Corollary 4.2.20 : Let f be a diffeomorphism of $\mathrm{Diff}(M)$. Let $K > 0$ be a bound for the cocycle df on TM, and let $\delta > 0$. Fix $\epsilon > 0$ as in Franks' Lemma 2.3.6 with respect to δ and the bound $A = K$. Choose $n, \ell \in \mathbb{N}$ as in Corollary 4.2.20, with respect to K, ε and the dimension d of M. For all $p \leq n$, choose $\ell_p \in \mathbb{N}$ as in Proposition 4.5.1.

Denote by ℓ_0 the least common multiple of the integers ℓ and ℓ_p, for $p = 1, ..., n$: if x is a periodic point such that the cocycle $df_{|\,\mathrm{Orb}_f(x)}$ has no ℓ_0-dominated splitting, it has neither any ℓ-dominated splitting, nor any ℓ_p-dominated splitting. Then, for any periodic point x, by Proposition 4.5.1 (if x has period less than n) and by Corollary 4.2.20 (if x has period greater than n), there is a linear cocycle $\mathcal{B}$ such that $\mathrm{dist}(df_{|\,\mathrm{Orb}_f(x)}, \mathcal{B}) < \epsilon$ and $\mathcal{B}$ has all Lyapunov exponents equal. Franks' Lemma gives a δ-perturbation g of f on an arbitrarily small neighbourhood of the orbit of x, such that the orbit of x is preserved, and $dg_{|\,\mathrm{Orb}_g(x)} = \mathcal{B}$, therefore $dg^p_{|\,\mathrm{Orb}(x)}$ has real eigenvalues with moduli all equal and different from 1. □

Proof of Proposition 4.5.1 : Let $K > 0$ and $d, p \in \mathbb{N}$. let $\mathcal{A}$ be a cocycle on a bundle $\mathcal{E}$ as in the hypothesis of the proposition. Let $\lambda_1 < ... < \lambda_k$ be the moduli of the eigenvalues of the first return map $\mathcal{A}^p$. One finds a flag $\mathcal{E}_1 \subset \mathcal{E}_1 \subset ... \subset \mathcal{E}_k = \mathcal{E}$ of $\mathcal{A}$ invariant subbundles of $\mathcal{E}$, such that for all i, the moduli of the eigenvalues of $\mathcal{A}^p$ restricted to $\mathcal{E}_i$ are $\lambda_1, ..., \lambda_i$.

Let $\mathcal{F}_1 = \mathcal{E}_1$ and for $1 < i \leq k$ let $\mathcal{F}_i$ be the orthogonal bundle of $\mathcal{E}_{i-1}$ in $\mathcal{E}_i$. We have $\mathcal{E} = \mathcal{F}_1 \oplus .... \oplus \mathcal{F}_k$. Then the linear cocycle $\mathcal{B}$ defined by $\mathcal{B}(v_i) = \lambda_k/\lambda_i.\mathcal{A}(v_i)$, for all $1 < i \leq k$ and all $v_i \in \mathcal{F}_i$, has all eigenvalues with modulus equal to λ_k. Clearly, for any $\epsilon > 0$, there exists a constant $\delta > 1$ depending only on K, d, p (and ϵ) such that, if the ratio between each of the pairs λ_{i+1}, λ_i is less than δ, then $\mathcal{B}$ is ϵ-close to $\mathcal{A}$.

Therefore, we are done if we show that if the ratio between some pair λ_{i+1}, λ_i is greater than δ, then there exists a integer N depending only on K, d, p and δ, such that the cocyle $\mathcal{A}$ admits an N-dominated splitting.

Let $1 \leq i < k$ and suppose that the ratio between λ_i and λ_{i+1} is greater than δ. Write then the invariant splitting $\mathcal{E} = F \oplus G$ such that the restriction $\mathcal{A}_{|F}$ has all eigenvalues with moduli less or equal to λ_i, and $\mathcal{A}_{|G}$ has all eigenvalues with moduli greater or equal to λ_{i+1}.

The map $\mathcal{A}^p_{|F}$ has norm less than K^p, hence finding an orthogonal basis of F on which $\mathcal{A}^p_{|F}$ is upper triangular, we get a splitting $\mathcal{A}^p_{|F} = D + N$, where N is strictly upper triangular and has norm less than K^p, and D is diagonal.

Then for all $n \in \mathbb{N}$, we develop and write $\mathcal{A}^{pn}_{|F} = (D+N)^n$ as the sum of products of the form $T_1....T_n$ where $T_i \in \{D, N\}$, and N appears less than d times. There are less than n^d of these terms, and the norm of such a term $T_1....T_n$ is less than

$$\max(\|N\|^d, 1). \max_{i=n-d,\dots,n} \|D^i\| = K^d.(\lambda_i^{n-d} + \lambda_i^n).$$

Therefore, the norm of $\mathcal{A}^{pn}_{|F}$ is less than $(nK)^d.(\lambda_i^{n-d} + \lambda_i^n)$. Notice that λ_i is between K^{-p} and K^p. Hence, for any $\alpha > 1$, there is an integer n_α depending only on K, d, p and α, such that for any unit vector u in F, we have $\|\mathcal{A}^{p.n_\alpha}(u)\| \leq \alpha.{\lambda_i}^{n_\alpha}$.

Symmetrically, there is an integer m_α depending only on K, d, p and α, such that for any unit vector v in G, we have $\|\mathcal{A}^{p.m_\alpha}(v)\| \geq \alpha^{-1}.{\lambda_{i+1}}^{m_\alpha}$. Take $\alpha < \delta^{1/3}$ and ℓ a multiple of $n_\alpha m_\alpha$ such that $\delta^{\ell/3} > 2$. The integer $N = \ell p$ depends only on K, d, p and δ, and the splitting $\mathcal{E} = F \oplus G$ is N-dominated for $\mathcal{A}$. This, as announced, ends the proof of Proposition 4.5.1. $\square$

Chapter 5

Creating homoclinic tangencies

5.1 Introduction

An invariant set Σ for a diffeomorphism of a compact manifold is *hyperbolic* if and only if the tangent bundle TM_Σ above S splits into two invariant bundles, the first being uniformly contracted by an iterate of the diffeomorphism, and the other uniformly expanded. A hyperbolic diffeomorphism will be a diffeomorphism such that its chain-recurrent set (see [21]), is hyperbolic. This is equivalent to being Axiom A and satisfying the *no cycle condition*. J.Palis and S.Smale conjectured in [47] that

- Axiom A and *no cycle condition* is equivalent to Ω-stability, that is, conjugacy to any neighbouring diffeomorphism, by restriction to respective non-wandering set (which happens here to be the chain recurrent set).

- Axiom A and strong transversality condition is equivalent to structural stability, that is, conjugacy to any neighbouring diffeomorphism.

Smale [62] showed that Axiom A and *no cycle condition* implies Ω-stability. Robbin [53] first showed that in C^2 topology, Axiom A and strong transversality implies structural stability, then Robinson [54] showed it in C^1 topology. In the late eighties, Ricardo Mañé [39] completed the proof that C^1 structural stability implies hyperbolicity and strong transversality. Relying on Mañé's techniques, Palis [45] showed that Ω-stability implies hyperbolicity. It was believed in the sixties that the hyperbolic diffeomorphisms of a compact manifold M were C^r dense in the set of diffeomorphisms for some $r \geq 1$. But soon it appeared that there was open sets of non-structurally stable diffeomorphisms [60] and of non-Ω-stable diffeomorphisms [4]. Such examples now abound, in dimension ≥ 2 for C^2 topology, and in dimension ≥ 3 for C^1 topology. Palis proposed to characterise robustness of non-hyperbolic behaviour by local density of two type of bifurcations:

- homoclinic tangencies, that is, tangency between the stable and unstable manifolds of a periodic saddle point (orbit),
- heterodimensional cycles, that is a pair of saddle points of different index (dimension of the unstable manifold) such that the unstable manifold of one intersects the stable manifold of the other.

This is the purpose of his

Conjecture 5.1.1 (Palis' C^r-density conjecture). *The union of hyperbolic diffeomorphisms, and diffeomorphisms admitting a homoclinic tangency or a heterodimensional cycle is dense in* $\mathrm{Diff}^r(M)$.

In order to study dynamics away from hyperbolic behaviours, a weak form of hyperbolicity was created: let f be a diffeomorphism on a compact manifold, K a compact invariant set, and let $TM_{|K} = E \oplus F$ be an invariant splitting of the tangent bundle restricted to K, for the derivative df. We say that this splitting is *dominated* of index $\dim(F)$ if some iterate of df uniformly contracts more (or expands less) the first bundle than the second one.

The existing results in C^1 topology, and recent developments inspire hope for a proof of the C^1-density conjecture. However no such tools as the Closing Lemma and Franks' Lemma exist in C^r topology for $r > 1$. In the following, we will work exclusively in C^1-topology.

In a groundbreaking paper, E. Pujals and M. Sambarino in [52] showed the C^1 density conjecture of Palis for surfaces: a surface diffeomorphism can be C^1 approximated either by Axiom A diffeomorphisms or by diffeomorphisms admitting a homoclinic tangency. As a first step, they proved that given a diffeomorphism f and $\epsilon > 0$, if the hyperbolic splitting along a periodic orbit of f is not dominated enough then a C^1 ϵ-perturbation of f creates a homoclinic tangency associated to that same orbit. Roughly, they follow a technique of Mañé [38] to create a small angle between the two eigendirections by a pertubation of the derivative, thus "bringing near" both manifolds (see [14, page 132] for a summarizing picture). A second perturbation slightly pushes one of the manifolds to meet the other at some point.

Hayashi claimed some breakthroughs for a proof of the C^1 Palis conjecture in higher dimensions, however no paper has yet been released. In [65], L. Wen made some interesting progress towards it, proving that for a generical diffeomorphism far from hyperbolic dynamics, from homoclinic tangencies and from heterodimensional cycles, the minimal non hyperbolic sets admit a partially hyperbolic splitting, with one or two one-dimensional central bundles.

A weak Palis conjecture says that there is a dense open subset of diffeomorphism that are either Morse-Smale, or admit a homoclinic intersection.

It was shown by Bonatti, Gan and Wen [15] in dimension 3. Very recently, S.Crovisier [23], using the works of Wen [65] and introducing an original way to study the dynamics along the central bundles, proved the weak Palis conjecture in any dimension:

Theorem 5.1.2. *The set of Morse-Smale diffeomorphisms and the set of diffeomorphisms that admit a homoclinic tangency, are two disjoint open sets whose union is dense in* $\mathrm{Diff}^1(M)$*.*

To prove the main results of [65], Wen cleverly conbined Liao's selecting lemma (see [36]) and the main theorem of a previous paper [64, Theorem A], which is a generalization of Pujals and Sambarino's first step of their proof. Precisely, Wen showed that if for some integer $1 \leq i \leq d$, the i-preperiodic set (the set of points that can be turned into a saddle of index i by an arbitrarily small C^1-perturbation) does not admit a dominated splitting of index i, then there is an arbitrarily small perturbation that turns f to admit a homoclinic tangency. However he insists the saddle for which the homoclinic tangency occurs may have not existed before the perturbation. Moreover the index of that saddle cannot a priori be controlled.

One aim of this paper is to address these underlying questions. We will show that if along a saddle point with great period, the splitting between the stable and unstable directions is not strongly enough dominated, then one can C^1-perturb the dynamics on an arbitrarily small neighbourhood of the orbit, preserving the orbit of that saddle, and its index, and creating a homoclinic tangency related to it.

A diffeomorphism f of M is *bounded by A* if the derivatives df and df^{-1} have norms less than A, that is, for any unit vector $v \in TM$, we have $\|df(v)\|, \|df^{-1}(v)\| \leq A$. In section 5.2.3 we endow $\mathrm{Diff}^1(M)$ with a metric canonically associated to the Riemannian structure of M; we say that g is an *ϵ-perturbation* of f if its distance to f is less than ϵ.

Let $N \in \mathbb{N}$. We say that a saddle point Q for a diffeomorphism f is *N-dominated*, if for some $0 \leq n \leq N$, for all unit vectors u and v above x, tangent to the stable and unstable manifolds, respectively, we have $\|df^n(u)\| < 1/2\|df^n(v)\|$. A *homoclinic tangency* related to Q is a point at which the stable and unstable manifolds of Q intersect non-transversely. We now can state our first result, which precises Wen's one:

Theorem 5.1.3. *Fix $A > 0$, $\epsilon > 0$ and an integer $d \geq 2$. There exists two integers $N_d, P > 0$ such that, if f is a diffeomorphism bounded by A on a d-dimensional Riemannian manifold, and Q is a saddle point of period $p > P$, not N_d-dominated,*

then there exists an ϵ-perturbation of f on an arbitrarily small neighbourhood $\mathcal{U}$ of the orbit of Q, that preserves the orbit of Q and its index, and creates a homoclinic tangency related to Q in $\mathcal{U}$.

In particular, if $\mathcal{U}$ is small enough, the tangency will be between the stable and unstable manifolds of a same point of the orbit of Q. Moreover, one can ask that the index of Q is preserved. This is what Pujals and Sambarino showed in dimension $d = 2$; we will show it for $d \geq 3$ in section 5.3 by induction on the dimension. It happens that the technique we use is adaptable to some more constraint such as preserving a finite set in the strong stable/unstable manifolds. Theorem 5.1.3 can be completed to:

Theorem 6.1.1. *Fix $A > 0$, $\epsilon > 0$ and an integer $d \geq 2$. There exists two integers $N_d, P > 0$ such that, if f is a diffeomorphism bounded by A on a d-dimensional Riemannian manifold, and Q is a saddle point of period $p > P$, not N_d-dominated,*

then for any finite set Γ of the manifold, there exists an ϵ-perturbation of f on an arbitrarily small neighbourhood $\mathcal{U}$ of the orbit of Q, that preserves the orbit of Q, creates a homoclinic tangency related to Q in $\mathcal{U}$, and such that if $x \in \Gamma$ was in some strong stable or unstable manifold of Q, then it "still is".

Let us explain the last words of this statement: in section 6.1 we will introduce the notion of flag-configuration for a finite set Γ. The flag-configuration of Γ with respect to a dynamics is the piece of information that tells for each $x \in \Gamma$ what is the dimension of the strongest stable/unstable manifold (if any) that contains x. We will say that the flag-configuration of Γ is preserved by a perturbation if, for any $x \in \Gamma$, the dimension of the strongest stable/unstable manifold (if any) that contains x is preserved or decreased. In other words, if x is in some stong stable manifold, it will stay in it. With this terminology, Theorem 6.1.1 says that we can apply Theorem 5.1.3 preserving the flag-configuration of any finite set Γ of the manifold.

Corollary 5.1.4. *If the homoclinic class $H(P, f)$ of a saddle point P for f does not admit a dominated splitting of same index as P, then there is an arbitrarily small perturbation of f that creates a homoclinic tangency related to P.*

Although the first theorem is a particular case of the second, we will show them separately for the reader's convenience, keeping in mind that the skeleton of the proof of the second is that of the first. Perturbing on a small neighbourhood of Q is equivalent to perturbing *cyclic diffeomorphisms* (see section 5.2.1). We will show (see Reduction Proposition, section 6.4) that it is sufficient to prove the theorems for linear cyclic diffeomorphism, such that the eigenvalues along the orbit are all real and pairwise distinct. This will allow to restrict to an invariant bundle, or to go to the quotient by it, to reduce dimension.

In this chapter, we focus on the proof of Theorem 5.1.3. The proof of theorem 6.1.1, although it is based on that of Theorem 5.1.3, is much more technical and will be done in Chapter 6.

5.2 Notations and preliminaries

5.2.1 Cyclic diffeomorphims and saddle diffeomorphisms on a vector bundle

Let $\mathcal{E} = (E, \Sigma, \pi\colon E \to \Sigma)$ be a dimension d linear bundle above a base Σ, that is, for all $x \in \Sigma$, the fibre $E_x = \pi^{-1}(x)$ above x is a dimension d vector space. As similarly defined in [16] and [13], we say that a couple of bijections $\mathcal{A} = (f\colon \Sigma \to \Sigma, A\colon E \to E)$ is a *linear cocycle* or *automorphism* on $\mathcal{E}$, if and only if, for all $x \in \Sigma$, the map A induces by restriction a vector spaces isomorphism from the fibre E_x to $E_{f(x)}$, that is, the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\ A\ } & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Sigma & \xrightarrow{\ f\ } & \Sigma \end{array} \quad . \tag{5.1}$$

We will consider the particular case of *cyclic automorphisms*, that is, automorphisms $\mathcal{A} = (f, A)$ on bundles whose base Σ is finite and such that f is a cyclic permutation of Σ. In the following, all bundles of the form $\{1, ..., p\} \times \mathbb{R}^d$ will be endowed, fiber by fiber, with the canonical Euclidian metric $\|.\|$, and thus will be viewed both as Riemannian manifolds and vector bundles above the base $\{1, ..., p\}$. We will denote by $0_{\mathcal{E}}$ the trivial subbundle $\{1, ..., p\} \times \{0\}$ of a bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$.

Definition 5.2.1. A diffeomorphism f on a bundle $\mathcal{E} = \{1, .., p\} \times \mathbb{R}^d$ will be said to be a *cyclic diffeomorphism on* $\mathcal{E}$ if for all $0 < i < p$, we have $f(i, 0) = f(i + 1, 0)$ and $f(p, 0) = (1, 0)$.

We say that p is the *period* of such a cyclic diffeomorphism. We say that a cyclic diffeomorphism on a bundle $\mathcal{E}$ is a *linear (cyclic) diffeomorphism* if moreover it is a linear cocycle.

Definition 5.2.2. A cyclic diffeomorphism on a bundle $\mathcal{E} = \{1, .., p\} \times \mathbb{R}^d$ will be said to be a *saddle diffeomorphism on* $\mathcal{E}$ if the orbit $0_{\mathcal{E}}$ of the point $(1, 0)$ is a saddle for f, i.e. the derivative $df^p(1, 0)$ has eigenvalues of moduli smaller and greater than 1, but none equal to 1.

We say that a saddle diffeomorphism on a bundle $\mathcal{E}$ is a *linear saddle* if moreover it is linear. Given a saddle diffeomorphism f on $\mathcal{E}$, we denote by $W^s(f)$ (resp. $W^u(f)$) the stable (resp. unstable) manifold of f on $\mathcal{E}$, that is the set of points $x \in \mathcal{E}$ such that the norm $\|f^n(x)\|$ (resp. $\|f^{-n}(x)\|$) goes to 0 as n goes to $+\infty$.

We say that a real number λ is an eigenvalue of a linear saddle f on $\mathcal{E}$, if and only if there is a point $x \in \mathcal{E}$ such that $f^p(x) = \lambda.x$. We call *eigenbundle associated to* λ the set of points x such that $f^p(x) = \lambda.x$; it is an invariant subbundle of $\mathcal{E}$, with constant dimension. Observe that a linear cyclic diffeomorphism verifies all the diagonalisation and trigonalisation properties of the linear maps.

5.2.2 Restrictions, quotients of cyclic diffeomorpihsms

Given a cyclic diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ and an invariant subbundle $F \subset \mathcal{E}$ for f, we will naturally consider the restriction $f_{|F}$ of f to the bundle F. The bundle F is canonically endowed with the restricted Euclidean metric, and $f_{|F}$ is a cyclic diffeomorphism of the bundle F. Moreover, if f is a saddle diffeomorphism and F intersects both the contracting and the expanding manifolds of f, then $f_{|F}$ is a saddle diffeomorphism.

Write F as the pairwise disjoint union $\bigsqcup_{i=1}^p F_i$ of the fibers $F_i \subset E_i = \{i\} \times \mathbb{R}^d$. The vector spaces F_i have same dimension d'. The quotient spaces $E_i/F_i = \{a + F_i, a \in E_i\}$ are $(d-d')$-dimensional, and canonically endowed with a Euclidean metric $\|.\|_i'$ defined by $\|a + F_i\|_i' = \|\pi_i(a)\|$; where π is the orthogonal projection of a on the orthogonal space of F_i. We will denote by $\mathcal{E}/F$ the quotient vector bundle defined as the disjoint union $\bigsqcup_{i=1}^p E_i/F_i$. As seen before, it is canonically endowed with a Euclidean metric.

If f preserves the directions parallel to F (that is, for any i, for any $a \in E_i$, f induces a bijection from $a + F_i$ on $f(a) + F_{i+1}$, if $i < p$, or on $f(a) + F_1$, if $i = p$) then there exists a unique cyclic diffeomorphism f/F on the quotient bundle such that the following graph commutes:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ f\ } & \mathcal{E} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{E}/F & \xrightarrow{\ f/F\ } & \mathcal{E}/F \end{array} .$$

We say that f is a *lift* of f/F. If F contains neither all the contracting eigendirections, nor all the expanding directions, then f/F is a saddle diffeomorphism of $\mathcal{E}/F$.

5.2.3 Perturbations in C^1 topology

Let M be a Riemannian manifold, not necessarily connected nor compact. Call $\|.\|$ the Riemannian metric on TM, and ∇ the corresponding Levi-Civita connection. We recall that the connection induces a distance on each connected component of TM in the following way: for any points x, y in a same connected component of M, and any vectors $u \in T_xM, v \in T_yM$ we define

$$\mathrm{dist}(u, v) = \inf\{\|v - \nabla_\gamma(u)\| + \ell(\gamma), \gamma \in \mathcal{C}_{xy}\}$$

where $\mathcal{C}_{xy}$ is the set of C^1 curves that go from x to y, $\ell(\gamma)$ is the length of the curve γ, and $\nabla_\gamma(u) \in T_yM$ is the parallel transport of the vector u along γ by the connection ∇. We extend the distance to whole TM, letting $\mathrm{dist}(u,v) = +\infty$ if u and v are not in a same connected component. With this definition of the distance, the following inequality is obvious:

$$\forall A > 1, \mathrm{dist}(A.u, A.v) \leq A. \mathrm{dist}(u,v). \tag{5.2}$$

A diffeomorphism f of M will be said to be *bounded by* $A > 1$, if and only if, for any unit vector $v \in TM$, we have $A^{-1} \leq \|df(v)\| \leq A$. Define

$$|f| = \sup\{\|df(v)\|, \|df^{-1}(v)\|/v \in TM, \|v\| = 1\},$$

that is, $|f|$ is the smallest number such that f is bounded by A. We say that f is simply *bounded* if and only if $|f| < \infty$.

Remark 5.2.3. *If f is bounded by A, so are the restriction $f_{|F}$ and the quotient f/F.*

Definition 5.2.4. A diffeomorphism f is an *ϵ-perturbation of g* for the C^1 topology, if for some $\epsilon_0 < \epsilon$, for any unit vector $v \in TM$, we have $\mathrm{dist}(df(v), dg(v)) \leq \epsilon_0$ and $\mathrm{dist}(df^{-1}(v), dg^{-1}(v)) \leq \epsilon_0$.

Thus f is an ϵ-perturbation of g if and only if f^{-1} is an ϵ-perturbation of g^{-1}. If moreover g coincides with f outside a set $\mathcal{U}$, then we say that g is an *ϵ-perturbation of f on $\mathcal{U}$*. Notice that with the definition we gave, if g is an ϵ-perturbation of f on a compact set, then for some $\epsilon' < \epsilon$, g is an ϵ'-perturbation of f. This, together with the following remark, will allow to perturb an ϵ-perturbation of f into another ϵ-perturbation of f.

Remark 5.2.5. *If h is an η-perturbation of g, which is an ϵ-perturbation of f then h is an $(\eta + \epsilon)$-perturbation of f. For any ϵ-perturbation g of f, there exists $\nu > 0$ such that any ν-perturbation of g is an ϵ-perturbation of f. Moreover, if f is bounded by A then g is bounded by $A + \epsilon$, precisely, $|g| \leq |f| + \epsilon$.*

5.2.4 Perturbations of cyclic/saddle diffeomorphisms

A map g is a *cyclic ϵ-perturbation* of a cyclic diffeomorphism f of the bundle $\mathcal{E} = \{1, ..., p\} = \times \mathbb{R}^d$ if and only if it is both an ϵ-perturbation of f, and a cyclic diffeomorphism on $\mathcal{E}$. We say that it is a *cyclic local ϵ-perturbation* if moreover the perturbation is only local, that is, g coincides with f outside a compact set. We define similarly *saddle ϵ-perturbation*, replacing the word 'cyclic' by 'saddle'.

Notice that if two linear saddles f and g on $\mathcal{E}$ are different then there is no ϵ for which f is an ϵ-perturbation of g. This is why we define two linear saddles f and g to be *linearly ϵ-close* if and only if, for any point $x \in \mathcal{E}$ with norm 1, we have $\|f(x)-g(x)\|, \|f^{-1}(x)-g^{-1}(x)\| < \epsilon$. Eventually, we may forget the adjective *linearly*.

When composing two small C^1-perturbations, a Lipschitz control of the derivatives is needed to make sure that the perturbed composition is still a small perturbation of the initial composition. However simple statements can be made in very particular cases:

Remark 5.2.6. *Let f be a linear (thus necessarily bounded) cyclic diffeomorphism on a bundle $\mathcal{E}$, if Φ is a diffeomorphism of $\mathcal{E}$ that is an ϵ-perturbation of $Id_{\mathcal{E}}$, then $f \circ \Phi$ and $\Phi \circ f$ are $|f|\epsilon$-perturbations of f.*

Moreover, if a cyclic diffeomorphism f is linear on $\mathcal{U}$ and Φ is an ϵ-perturbation of $Id_{\mathcal{E}}$ on $\mathcal{U}$, then $f \circ \Phi$ is an $|f|\epsilon$-perturbation of f on $\mathcal{U}$. If f^{-1} is linear on $\mathcal{U}$, if Φ is an ϵ-perturbation of $Id_{\mathcal{E}}$ on $\mathcal{U}$, then $\Phi \circ f$ is an $|f|\epsilon$-perturbation of f on $f^{-1}(\mathcal{U})$.

Lemma 5.2.7. *Let f be a linear cyclic diffeomorphism on a bundle $\mathcal{E}$, let g be a cylic ϵ-perturbation of f, and ϕ be an η-perturbation of $Id_{\mathcal{E}}$. Then $g \circ \phi$ is an $\epsilon + \eta.(|f| + \epsilon)$-perturbation of f.*

Proof : Let u be a unit vector of the tangent bundle $T\mathcal{E}$. Let $v = d\phi(u)$. We have $\text{dist}(v,u) \leq \eta$, and $\|v\| \leq 1 + \eta$. Since f is linear, we have $\text{dist}(df(v), df(u)) \leq |f|. \text{dist}(u,v)$. Therefore

$$\begin{aligned}
\text{dist}(dg \circ \phi(u), df(u)) &\leq \text{dist}(dg(v), df(v)) + \text{dist}(df(v), df(u)) \\
&\leq \epsilon\|v\| + |f|. \text{dist}(u,v) \\
&\leq \epsilon + \eta.(|f| + \epsilon) \qquad (5.3)
\end{aligned}$$

Conversely, let $v = dg^{-1}(u)$. We have $\|v\| \leq |g| \leq |f| + \epsilon$.

$$\begin{aligned}
\text{dist}(d\phi^{-1} \circ dg^{-1}(u), df^{-1}(u)) &\leq \text{dist}(dg^{-1}(u), df^{-1}(u)) + \text{dist}(d\phi^{-1}(v), v) \\
&\leq \epsilon + \eta.\|v\| \\
&\leq \epsilon + \eta.(|f| + \epsilon)
\end{aligned}$$

Hence $g \circ \phi$ is an $\epsilon + \eta.(|f| + \epsilon)$-perturbation of f. □

We also state without a proof the following useful

Lemma 5.2.8. *Let g_k be a sequence of diffeomorphisms on a manifold M that tends to a diffeomorphim f for the C^1 topology. Let K be a compact and Φ_k be a sequence of cyclic perturbations of Id_M on K, that tends to Id_M for the C^1 topology. Then $g_k \circ \Phi_k$ tends to f for the C^1 topology.*

About conjugating a perturbation of a linear cyclic diffeomorphism by a homothety:

Remark 5.2.9. *Let f be a linear cyclic diffeomorphism on a bundle $\mathcal{E}$, and g a cyclic ϵ-perturbation of f on a set $\mathcal{U}$. Then, for any real number $\lambda > 1$, the diffeomorphism $\lambda^{-1}.g \circ \lambda.Id_{\mathcal{E}}$ is a cyclic ϵ-perturbation of f on $\lambda^{-1}.\mathcal{U}$.*

We define the C^1 norm of a map f from a Riemannian manifold M to the vector bundle $\mathcal{E} = \{1,..,p\} \times \mathbb{R}^d$ by $\|f\| = \sup\{\|f(x)\|, x \in M\} + \sup\{\|df(v)\|, v \in TM, \|v\| = 1\}$. In the particular case where $M = \mathcal{E}$, the tangent bundle $T\mathcal{E}$ can be identified to a vector bundle $\{1, ..., p\} \times \mathbb{R}^d \times \mathbb{R}^d$. This induces on each fibre a distance that coincides with the distance *dist* we defined previously. Let us state without a proof an elementary lemma:

Lemma 5.2.10. *There exists $\alpha > 0$ such that, for any $0 < \epsilon \leq \alpha$, if a map $\tilde{\epsilon} \colon \mathcal{E} \to \mathcal{E}$ sends each fibre of $\mathcal{E}$ into itself, and has C^1 norm less than ϵ, then the map $Id_{\mathcal{E}} + \tilde{\epsilon}$ is a diffeomorphism and is a 2ϵ-perturbation of identity.*

Notice that trivially, we have the following converse statement: if a map is an ϵ-perturbation of identity, we can write it as a sum $Id + \tilde{\epsilon}$, where $\tilde{\epsilon}$ has C^1-norm less than ϵ. Then we have this useful corollary:

Corollary 5.2.11. *Let f be a linear cyclic diffeomorphism on a bundle $\mathcal{E}$. Let $g \colon \mathcal{E} \to \mathcal{E}$ be a map such that for any $x \in \mathcal{E}$, $f(x)$ and $g(x)$ are in the same fibre (that is, the map $f - g$ is defined), and such that the C^1 norm of $f - g$ is less than $\varepsilon = \alpha/|f|$. Then g is diffeomorphism of $\mathcal{E}$ and is a $2|f|^2\varepsilon$-perturbation of f.*

Proof : We put $\tilde{\varepsilon} = f - g$. Then $f^{-1} \circ g = f^{-1} \circ (f + \tilde{\varepsilon}) = Id_{\mathcal{E}} + f^{-1} \circ \tilde{\varepsilon}$ by linearity of f. Besides, the map $f^{-1} \circ \tilde{\varepsilon}$ has C^1 norm smaller than $|f|\varepsilon \leq \alpha$. Thus from Lemma 5.2.10, $f^{-1} \circ g$ is a diffeomorphism and more precisely a $2|f|\varepsilon$-perturbation of $Id_{\mathcal{E}}$. By Remark 5.2.6, $g = f \circ f^{-1} \circ g$ is a $2|f|^2\varepsilon$-perturbation of f. □

5.2.5 Preliminary perturbation and extension lemmas

Franks' Lemma and some more precise perturbation lemmas

Let Σ be a subset of a manifold M. Then given two linear cocycles $\mathcal{A}$ and $\mathcal{B}$ on TM_Σ, we define the distance between them to be ∞ if they do not lift from the same dynamics on Σ, otherwise we define

$$\mathrm{dist}(\mathcal{A}, \mathcal{B}) = \sup_{v \in TM_\Sigma, \|v\|=1, \iota \in \{-1,1\}} (\|\mathcal{A}^\iota(v) - \mathcal{B}^\iota(v)\|).$$

We say that two linear cocycles are *ϵ-close* if the distance from one to another is strictly less than ϵ. We first state without a proof a very classical linearization lemma that perturbs a cyclic diffeomorphism to be *locally* linear, that is, linear on a neighbourhood of the zero-section $0_{\mathcal{E}}$.

Lemma 5.2.12. *Let f be a cyclic diffeormorphism on a bundle $\mathcal{E}$. Then for all $\epsilon > 0$, there is a cyclic ϵ-perturbation of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$ that is locally linear, with same derivative as f above $0_{\mathcal{E}}$, that is $dg_{0_{\mathcal{E}}} = df_{0_{\mathcal{E}}}$.*

In particular if g is a cyclic ϵ-perturbation of f, there is a cyclic ϵ-perturbation h of f that has same derivative as g above $0_{\mathcal{E}}$. We state now a version of Franks' Lemma [26, lemma 1.1] for cyclic diffeomorphisms:

Lemma 5.2.13 (Franks). *Let $A > 0$. Then for all $\delta > 0$, there is $\epsilon > 0$ such that the following holds: if f is a cyclic diffeomorphism on a bundle $\mathcal{E}$, and is bounded by A. If a linear cocycle $\mathcal{B}$ on $T\mathcal{E}_{0_{\mathcal{E}}}$ is ϵ-close to the derivative of df, then there is a cyclic δ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, such that the derivative of g above $0_{\mathcal{E}}$ is $\mathcal{B}$.*

We define the radius of a path $(\mathcal{A}_t)_{t\in[0,1]}$ of cocycles in TM_Σ as

$$\sup_{t\in[0,1]} [\operatorname{dist}(\mathcal{A}_t, \mathcal{A}_0)].$$

In the particular case where some point of Σ is a sink for f and for $\mathcal{A}$, we would like to know when the basin of attraction of that point for g is the same as for f. This is the purpose of the following proposition:

Proposition 5.2.14. *Let f be a cylic diffeomorphism of a bundle $\mathcal{E}$, such that the orbit $0_{\mathcal{E}}$ is a hyperbolic sink for f. Let $\mathcal{A}_t, t \in [0,1]$ be a path of linear cocycles on $T\mathcal{E}_{0_{\mathcal{E}}}$ of radius strictly less than $\varepsilon > 0$, that starts at $\mathcal{A}_0 = df_{0_{\mathcal{E}}}$ and such that for any $t \in [0,1]$, the orbit $0_{\mathcal{E}}$ is a sink for $\mathcal{A}_t$. Then there is a cyclic ε-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$ such that $dg_{0_{\mathcal{E}}} = \mathcal{A}_1$, and such that the basin of attration of $0_{\mathcal{E}}$ for g is the same as for f .*

We will use this proposition in dimension 2 to turn the eigenvalues of a sink to be real, preserving the flag-configuration of its whole basin of attraction (see section 6.4.2).

Lemma 5.2.15. *Let $\mathcal{F}$ be a compact (for the topology of cocycles) family of cyclic linear diffeomorphisms on $\mathcal{E}$ such that any $f \in \mathcal{F}$ is a sink. Then there is $\nu > 0$ such that any cyclic ν-perturbation g of any $f \in \mathcal{F}$ is a hyperbolic sink whose stable manifold is the whole $\mathcal{E}$.*

Proof : We only give a few clues: by compactness, there is an integer $n \in \mathbb{N}$ such that for any $f \in \mathcal{F}$, for any $x \in \mathcal{E}$, $\|f^n(x)\| \leq \|x\|/2$.

Then by compactness of $\mathcal{F}$ and linearity of its elements, the reader can check that for any $n \in \mathbb{N}$, there is $\nu > 0$ such that if g is a cyclic ν-perturbation g of $f \in \mathcal{F}$, then for all $x \in \mathcal{E}$, $\|f^n(x) - g^n(x)\| \leq \|x\|/3$. These two inequalities ensure that g is a hyperbolic sink whose stable manifold is the whole $\mathcal{E}$. □

Proof of Proposition 5.2.14 : Let f be a linear cyclic diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^n$, and let $(\mathcal{A}_t)_{t\in[0,1]}$ be a path of linear cocycles as in the hypothesis of the proposition. Thus, for some $\ell < \varepsilon$, for all $t \in [0,1]$, we have $\operatorname{dist}(\mathcal{A}_0, \mathcal{A}_t) < \ell$. Choose $0 < \delta < \varepsilon - \ell$.

Claim 7. *There is a a cyclic perturbation g of f, such that $dg_{0_\mathcal{E}} = \mathcal{A}_1$, its basin of attraction being the whole $\mathcal{E}$, and such that for all $x \in \mathcal{E}$, the derivatives dg_x and df_x, seen as linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, are $\ell + \delta$-close (that is $\|dg_x - df_x\|, \|dg^{-1}_{g(x)} - df^{-1}_{f(x)}\| < \ell + \delta$).*

There exists $A > 0$ such that for all t, the linear cyclic diffeomorphism f_t that identifies to $\mathcal{A}_t$ is bounded by A. Choose $\epsilon > 0$ as in Lemma 5.2.13 with respect to A and δ. The family of linear cocycles $(\mathcal{A}_t)$ is compact, therefore we can apply Lemma 5.2.15 and find $0 < \nu < \epsilon$ such that any cyclic ν-perturbation g of any f_t is a hyperbolic sink whose stable manifold is the whole $\mathcal{E}$, where f_t is the linear cyclic diffeomorphism that identifies to $\mathcal{A}_t$.

Let $0 = t_0 < t_1 < ... < t_k = 1$ be a sequence such that $\mathrm{dist}(\mathcal{A}_{t_i}, \mathcal{A}_{t_{i+1}}) < \nu$, for all $0 \leq i < k$. Then by Lemma 5.2.13, we find a sequence $(g_i)_{i=1,\dots,k}$ of diffeomorphisms of $0_\mathcal{E}$ such that for all $0 \leq i < k$, g_{i+1} is a cyclic δ-perturbation of f_{t_i} on a bounded neighourhood $\mathcal{U}_{i+1}$ of $0_\mathcal{E}$, with equality of derivatives $dg_{i+1} = df_{t_{i+1}}$ above $0_\mathcal{E}$. We may apply linearization lemma 5.2.12 to suppose moreover that the g_i are locally linear, that is linear on a neighbourhood $\mathcal{V}_i$ of $0_\mathcal{E}$. In other words, g_i coincides locally with f_{t_i}.

Recall that each g_i is a ν-perturbation of a linear cyclic diffeomorphism f_t, for some $t \in [0,1]$, and by Lemma 5.2.15, g_i is a sink whose stable manifold is the whole $\mathcal{E}$. From Remark 5.2.9, we may conjugate each of the g_i by a convenient homothety, and suppose that $\mathcal{U}_{i+1} \subset g_i(\mathcal{V}_i)$. We fit successively these k perturbations together, to obtain a local cyclic perturbation g of f that coincides with f_k on an neighbourhood of $0_\mathcal{E}$, and whose stable manifold is the whole $\mathcal{E}$.

Note finally that for all $x \in \mathcal{E}$, the derivative dg_x is equal to the derivative of some g_i above x, which is ν-close to the derivative of f_{t_i}. Since $\mathrm{dist}(\mathcal{A}_{t_i}, \mathcal{A}_0 \equiv df) \leq \ell$, and $\nu < \delta$, we get that the derivatives dg_x and df_x, seen as linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, are $\ell + \delta < \varepsilon$-close. This ends the proof of the Claim.

We may again conjugate g by a homothety, and suppose that g is a perturbation of f on an arbitrarily small neighbourhood. Applying the mean value theorem to the conclusions of the claim, we obtain that when such a g coincide with f outside a sufficiently small neighbourhood of $0_\mathcal{E}$, it is an ε-perturbation of f. Which ends the proof of Proposition 5.2.14. □

Define the map $(\epsilon, A) \mapsto \eta_{\epsilon,A}$ from $(\mathbb{R}^{+*})^2$ to $\mathbb{R}^{+*}$ by $\eta_{\epsilon,A} = \min\{\alpha/3A, \epsilon/6A^2, \epsilon\}$. Here follows a lemma to turn a perturbation of a saddle diffeomorphism to a local perturbation, preserving the dynamics of the initial perturbation on a neighbourhood of the zero section $0_\mathcal{E}$.

Lemma 5.2.16 (Localization). *Let $\epsilon > 0$, let f be linear cyclic diffeomorphism on a bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, and $\mathcal{U}$ be a neighbourhood of $0_\mathcal{E}$.*

Then there is a bounded neighbourhood $\mathcal{V}$ of $0_\mathcal{E}$ such that if g is a cyclic $\eta_{\epsilon,|f|}$-perturbation of f, there is a cylic ϵ-perturbation h of f that coincides with g on $\mathcal{U}$ and with f, outside $\mathcal{V}$. Precisely h writes as $\phi.g + \psi.f$ where $\phi + \psi$ is a partition unit on $\mathcal{E}$.

We insist that $\mathcal{V}$ only depends on $\mathcal{U}$, ϵ and f, not on g.

Proof : To shorten notation, we put $\eta = \eta_{\epsilon,A}$. Let $\phi + \psi = 1$ be a unit partition on $\mathcal{E}$, such that $\psi = 1$ on $\mathcal{U}$ and $\psi = 0$ outside a bounded neighbourhood $\mathcal{V}$ of $0_\mathcal{E}$. We choose the set $\mathcal{V}$ and the unit partition so that the derivative of ϕ (therefore that of ψ) has derivative less than 1.

Let g be a cyclic $\eta_{\epsilon,|f|}$-perturbation of f. Define the map h on $\mathcal{E}$ by $h(x) = \phi(x).f(x)+\psi(x).g(x)$. For all $x \in \mathcal{E}$, we have $f(x)-h(x) = \psi(x).(f-g)(x) = \psi(x)(f-g)(y)$. Thus the distance $\|f(x) - h(x)\|$ between $f(x)$ and $h(x)$ is less than $\psi(x).\eta \leq \eta$. We calculate the derivative:

$$d(f - h) = d\psi.(f - g) + \psi(df - dg)$$

and get similarly that $\|d_x(f - h)\| < 2.\eta$, since $\|d_x\psi\| \leq 1$. Hence the map $f - h$ has C^1-norm striclty less than 3η. We recall that $\eta = \eta_{\epsilon,|f|} < \alpha/3|f|$. We apply Corollary 5.2.11, and obtain that h is a diffeomorphism on $\mathcal{E}$ and a $6|f|^2\eta$-perturbation of f. Hence for any cyclic $\eta_{\epsilon,|f|}$-perturbation g of f, we found a cyclic ϵ-perturbation h of f on $\mathcal{V}$ that coincides with g on $\mathcal{U}$. Which ends the proof of the lemma. □

Extension and Lifting Lemmas.

In the following, f is a linear cyclic diffeomorphism and H an invariant subbundle. In this section we show to what extent it is possible to extend a perturbation of the restriction $f_{|H}$, or to lift a pertubation of the quotient f/H into a perturbation of f.

Lemma 5.2.17 (Extension). *Let f be a linear cyclic diffeomorphism on a bundle $\mathcal{E} = \{1, ..., p\}$, that admits an invariant bundle H and let $\epsilon < \alpha/|f|$. If g' is a cyclic ϵ-perturbation of the restriction $f_{|H}$, then there exists a cyclic $|f|\epsilon$-perturbation g of f that extends g', that is $g_{|H} = g'$, and such that the quotients g/H and f/H are defined and equal.*

In the following we naturally identify $\mathcal{E}/H$ with the orthogonal bundle D of H in $\mathcal{E}$, thus the canonical projection $\pi\colon \mathcal{E} \to \mathcal{E}/H$ will be the orthogonal projection π_D on D. Besides π_H will be the orthogonal projection on the bundle H.

Proof : Define the map $g = f\circ\pi_D + g'\circ\pi_H$ on $\mathcal{E}$. We have $f = f\circ\pi_D + f_{|H}\circ\pi_H$, therefore $f-g$ is defined and $f-g = (f_{|H}-g')\circ\pi_H$. Since the derivative of π_H has norm 1, and $f_{|H} - g'$ has C^1-norm less than ϵ, $f - g$ has C^1-norm less than ϵ. Since $\epsilon < \alpha/|f|$, from Corollary 5.2.11, g is a diffeomorphism of $\mathcal{E}$ and a $2|f|^2\varepsilon$-perturbation of f. The diffeomorphism g clearly coincides with g' by restriction to H and goes to the quotient by H. Obviously g acts on $\mathcal{E}/H$, that is on D, as f does. In other words $f/H = g/H$. □

Define $\nu_{\epsilon,|f|} = \min\{\alpha/|f|, \frac{\eta_{\epsilon,|f|}}{2|f|^2}\}$. With Lemma 5.2.16, we get the following localized statement:

Corollary 5.2.18. *Let f be a linear cyclic diffeomorphism on a bundle $\mathcal{E} = \{1, ..., p\}$, that admits an invariant bundle H. If g is a cyclic local $\nu_{\epsilon,|f|}$-perturbation of the restriction $f_{|H}$, then there exists a cyclic local ϵ-perturbation g of f that extends g' (i.e. $g_{|H} = g'$), and such that the quotients g/H and f/H are defined and equal.*

Proof : Apply successively the extension lemma and Lemma 5.2.16. □

Lemma 5.2.19 (Lifting). *Let f be a linear cyclic diffeomorphism on a bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ that admits an invariant subbundle H, and let $\epsilon < \alpha/|f|$. If g' is a cyclic ϵ-perturbation of the quotient cyclic diffeomorphism f/H on $\mathcal{E}/H$, then there exists a cyclic $2|f|^2\epsilon$-perturbation g of f that is a lift of g', i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\;g\;} & \mathcal{E} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{E}/H & \xrightarrow{\;g'\;} & \mathcal{E}/H \end{array} \quad . \tag{5.4}$$

Moreover, g can be chosen such that, for all $x \in \mathcal{E}/H$, for $\iota = \pm 1$, if $f/H^{\iota}(x) = g/H^{\iota}(x)$ then $f^{\iota} = g^{\iota}$ on the affine space $x + H$.

Remark 5.2.20. *As a consequence, the diffeomorphism g can be chosen so that if for some x, for all $n \in \iota\mathbb{N}$, we have $f/H^n(x) = g/H^n(x)$, then for all $n \in \iota\mathbb{N}$ we have $f^n = g^n$ on the affine space $x + H$.*

In particular, by restriction to H, $g^n = f^n$ for any integer n.

The last assertion will be useful in preserving homoclinic relations, to verify that a cyclic diffeomorphism is firmly flag-configuration respecting (defined in section 6.2.2).

Proof : We define the map $g \colon \mathcal{E} \to \mathcal{E}$ by $g = g' \circ \pi_D + \pi_H \circ f$. We let the reader notice that

$$f = \pi_D \circ f \circ \pi_D + \pi_H \circ f \tag{5.5}$$

$$= f/H \circ \pi_D + \pi_H \circ f \tag{5.6}$$

Therefore $f - g$ is defined and we have $f - g = (f/H - g') \circ \pi_D$. Since the derivative of π_D has norm 1, and the C^1-norm of $f/H - g'$ is less than ϵ, the C^1-norm of $f-g$ is less than ϵ. Hence, from Corollary 5.2.11, g is a diffeomorphism of $\mathcal{E}$ and is a $2|f|^2\varepsilon$-perturbation of f.

We have $\pi_D \circ g = \pi_D \circ g' \circ \pi_D + \pi_D \circ \pi_H \circ f = g' \circ \pi_D$. Hence, g is a lift of g'.

We are left to verify the last assertion of the lemma: If $x \in \mathcal{E}/H$ satisfies $f/H(x) = g/H(x)$, that is $\pi_D \circ f_{|D} = g'$, then $g_{|x+H} = g'(x) + \pi_H \circ f_{|x+H} = \pi_D \circ f(x) + \pi_H \circ f_{|x+H}$. Since H is invariant by f, which is linear, we have $f(x+H) = f(x) + H$ and $\pi_D \circ f_{|x+H} = \pi_D \circ f(x)$. We finally get $g_{|x+H} = \pi_D \circ f_{|x+H} + \pi_H \circ f_{|x+H} = f_{|x+H}$. If $f/H^{-1}(x) = g/H^{-1}(x) = y$, thus $f/H(y) = g/H(y)$ and $g_{|y+H} = f_{|y+H}$, the image of which is $x+H$. Hence $g^{-1}_{|x+H} = f^{-1}_{|x+H}$. This ends the proof of the lemma. □

5.2.6 Dominated splittings, N-domination and homoclinic tangencies

Let f be a diffeomorphism of a Riemannian manifold M, and K a compact invariant subset of M. We denote by $E^s(f_{|K})$, the *stable bundle* of $f_{|K}$, that is, the set of vectors $v \in TM_{|K}$ of the tangent bundle restricted to K, such that $\|df^n(v)\|$ goes to zero as n tends to infinity. We denote by $E^u(f_{|K})$ the *unstable bundle* of $f_{|K}$, that is, the stable bundle of $f^{-1}_{|K}$. A invariant bundle E is *uniformly contracting* for f if and only if there exists $N > 0$ such that for any $v \in E^s$, $\|df^N(v)\| < 1/2$. It is *uniformly expanding* if and only if it is uniformly contracting for the reverse dynamics.

Definition 5.2.21. We say that an invariant compact set K for f is a *hyperbolic set* if $TM_{|K} = E^s \oplus E^u$, if E^s is uniformly contracted and if E^u is uniformly expanded.

Let us define the notion of dominated splitting, which is a weak form of hyperbolicity.

Definition 5.2.22. An invariant splitting $TM_{|K} = E \oplus F$ of the tangent bundle restricted to K for f, is a *dominated splitting* if there exists N such that, for any point $x \in M$, for any unit vectors $u \in E_x$, $v \in F_x$ above x, we have $df^N(u) < 1/2.df^N(v)$.

We will say that a splitting is *M-dominated* if and only if there exists a positive integer $N \leq M$ that satisfies the above condition. This is a slightly different definition from that of [13] and [16]. A useful consequence of this very definition is that if a splitting is not N-dominated then it is not N'-dominated for any positive integer $N' \leq N$.

Let Q be a periodic saddle point for f, denote by $\mathrm{Orb}_f(Q)$ its orbit, and by $TM_{|\,\mathrm{Orb}_f(Q)} = E^s(Q) \oplus E^u(Q)$ the hyperbolic splitting along its orbit. We call it an *N-dominated saddle* if and only if the splitting $E^s(Q) \oplus E^u(Q)$

is N-dominated. In this article, we denote by $W^s(Q)$ and $W^u(Q)$ the stable and unstable manifolds of the orbit of Q, that is the set of points x such that the distance $\mathrm{dist}(f^n(x), \mathrm{Orb}_f(x))$ goes to 0 as n goes to $+\infty/-\infty$. A *homoclinic intersection* for a saddle Q is an intersection between the stable and unstable manifolds of its orbit elsewhere than at Q.

Definition 5.2.23. A homoclinic intersection x for a saddle Q is called a *homoclinic tangency* if the intersection of the tangent spaces $T_xW^s(Q)$ and $T_xW^u(Q)$ is not reduced to x. By a dimension argument, this is equivalent to saying that the stable and unstable manifolds of $\mathrm{Orb}(Q)$ do not intersect transversely at x.

In this case, we say that x is a homoclinic tangency *related to the saddle* Q. If an open neighbourhood $\mathcal{U}$ of the orbit $\mathrm{Orb}_f(Q)$ of Q contains the orbit of x, then we say that x is a *homoclinic tangency related to* Q *in* $\mathcal{U}$.

In particular, we say that a cyclic diffeomorphism f on a bundle $\mathcal{E} = \{1, .., p\} \times \mathbb{R}^d$ is an *N-dominated saddle on* $\mathcal{E}$ if and only if the periodic point $(0, 0)$, whose orbit is the zero section $0_{\mathcal{E}}$, is N-dominated. We denote by $W^s(f)$ and $W^u(f)$ the stable and unstable manifolds of the orbit $0_{\mathcal{E}}$ for f. We say that the saddle f *admits a homoclinic intersection* (resp. *homoclinic tangency*) if the saddle point $(1, 0)$ admits a homoclinic intersection (resp. homoclinic tangency).

5.3 Proof of Theorem 5.1.3

For the sake of readability, we first present a proof Theorem 5.1.3, although it is a weak version of Theorem 6.1.1. This will allow to introduce clearly the tools we will use for the proof of Theorem 6.1.1.

5.3.1 Reduction of the theorem

Clearly, choosing local charts around each point of the orbit of Q, Theorem 5.1.3 can be equivalently stated in terms of saddle diffeomorphisms:

Theorem 5.3.1. *Fix $A > 0$, $\epsilon > 0$ and an integer $d \leq 2$. There exists two integers $N_d, P > 0$ such that, if f is a saddle diffeomorphism on a bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ and satisfies:*

- *the diffeomorphism f is bounded by A,*
- *the period p is greater than P,*
- *the saddle f is not N_d-dominated,*

then there exists a saddle ϵ-perturbation of f on an arbitrarily small neighbourhood of the zero-section $0_{\mathcal{E}}$ that admits a homoclinic tangency in $\mathcal{U}$.

With the following proposition, the problem can be reduced to show Theorem 5.3.1 in the case where f is locally linear, that is linear on a neighbourhood of $0_{\mathcal{E}}$, and $df^p_{0_{\mathcal{E}}}$ has real eigenvalues with pairwise distinct moduli.

Proposition 5.3.2 (Reduction Proposition). *Fix $\epsilon > 0$ and $A > 0$ and an integer $d > 0$. Then there is an integer $P > 0$ such that, for any $p > P$, the following holds:*

if a saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ is bounded by A, then there exists a locally linear saddle ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, such that $dg^p_{0_{\mathcal{E}}}$ has real eigenvalues with pairwise distinct moduli.

We can restate [16, Theorem 2.1] as follows:

Theorem 5.3.3. *Fix $\epsilon > 0$, $A > 0$ and an integer $d > 0$. Then there is an integer $P > 0$ such that, for any $p > P$, the following holds:*

if a saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ is bounded by A, then there exists an ϵ-perturbation $\mathcal{B}$ of the derivative $df_{0_{\mathcal{E}}}$ such that the p-th iterate $\mathcal{B}^p$ of $\mathcal{B}$ has real eigenvalues with pairwise distinct moduli.

Proof of Proposition 5.3.2 : Apply successively Theorem 5.3.3, Franks' Lemma 5.2.13, and finally linearize locally by Lemma 5.2.12. □

We recall that we defined a diffeomorphism to be a *local* perturbation of another, if they coincide outside a compact set. We will show now that Proposition 5.3.2 reduces us to show the following proposition:

Proposition 5.3.4. *Fix $A > 0$, $\epsilon > 0$ and $d \geq 2$. There exists an integer $N_d > 0$ such that, if f is a linear saddle on a bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, and satisfies:*

- *the diffeomorphism f is bounded by A,*
- *the saddle f is not N_d-dominated,*
- *it has real eigenvalues.*

Then there exists a saddle local ϵ-perturbation of f that admits a homoclinic tangency.

Remark 5.3.5. *A saddle diffeomorphism f is bounded by A and not N_d-dominated if and only if the saddle diffeomorphism f^{-1} is bounded by A and not N_d-dominated. Besides a saddle diffeormophism g is a local ϵ-perturbation of f that admits a homoclinic tangency, if and only if g^{-1} is a local ϵ-perturbation of f^{-1} that admits a homoclinic tangency.*

Thus showing that the conclusions of the proposition hold for f is equivalent to showing that they hold for f^{-1}. This will allow us to change f into f^{-1} in the proof of it.

This proposition is the main part of the proof of Theorem 5.1.3, it will be showed by induction on the dimension d in section 5.3.2. The following remark, which complements Remark 5.2.9, says that once we have a saddle local perturbation that satisfies the conclusions of the proposition, we have a saddle local perturbation on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$ that satisfies the same conclusions.

Remark 5.3.6. *Assume that g is a saddle ϵ-perturbation of the linear saddle f on a bounded set $\mathcal{U}$, that admits a homoclinic tangency in $\mathcal{U}$, that is, the orbit of the tangency is in $\mathcal{U}$. Then, for all $\lambda > 1$, the conjugation by $\lambda.Id$ of g, $g_\lambda = \lambda^{-1}.g \circ \lambda.Id$, is a saddle ϵ-perturbation that admits a homoclinic tangency in $\lambda^{-1}.\mathcal{U}$.*

Hence, f has a saddle ϵ-perturbation on an arbitrarily small neighbourhood $\mathcal{U}$ that admits a homoclinic tangency in it.

We restate [16, Lemma 2.14], which says that any perturbation of a cocycle that admits a dominated splitting still admits a dominated splitting.

Lemma 5.3.7. *For any $A > 0$ and any integers $d \geq 2$, $N > 0$, there is $\nu > 0$ and an integer $M > 0$ such that if $\mathcal{A}$ is a linear cocycle bounded by A on a d-dimensional bundle $\mathcal{E}$, and $\mathcal{E} = F \oplus G$ is an N-dominated splitting for $\mathcal{A}$, then for any ν-perturbation $\mathcal{B}$ of $\mathcal{A}$, there is a (unique) invariant splitting $\mathcal{E} = F' \oplus G'$ for $\mathcal{B}$ that is M-dominated, and such that* $\dim(F') = \dim(F)$.

We have then the straightforward corollary:

Corollary 5.3.8. *For any $A > 0$ and any integers $d \geq 2$, $N > 0$, there is $\nu > 0$ and an integer $M > 0$ such that if g is a saddle diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ that is bounded by A and N-dominated, then any saddle ν-perturbation f of g of same index as g, is M-dominated.*

Remark 5.3.9. *The contrapositive of this corollary is the following: for any $A > 0$ and any integers $d \geq 2$ and $N > 0$, there exists an integer $M > 0$ and a real number $\nu > 0$ such that given a saddle f that is not M-dominated, any saddle ν-perturbation g of f of same index as f is not N-dominated.*

Proof of Theorem 5.3.1 from Propositions 5.3.4 and 5.3.2 : Fix $A > 0$, $\varepsilon > 0$, an integer $d \geq 2$. Apply Proposition 5.3.4 with the bound $A+\varepsilon/2$, with $\epsilon = \varepsilon/2$ and with dimension d to find the corresponding integer N_d. Now choose and fix $M > 0$ and $0 < \nu < \varepsilon/2$ as Remark 5.3.9 allows to do for $N = N_d$. Then apply Proposition 5.3.2 for our chosen ν and A, to find the corresponding integer P_0.

Let f be a d-dimensional saddle bounded by A that is not M-dominated, with period greater than P_0. By Proposition 5.3.2, there exists a locally linear saddle ν-perturbation g of f on an arbitrarily small open neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$ that has pairwise distinct real eigenvalues. By Remark 5.3.9, g is not N_d-dominated. Call $\widehat{g}$ the linear saddle that coincides with g on an open neighbourhood $\mathcal{V}$ of $0_{\mathcal{E}}$. The linear saddle $\widehat{g}$ is bounded by $A+\nu \leq A+\varepsilon/2$, has real eigenvalues and is not N_d-dominated. Therefore, by Proposition 5.3.4, there exists a saddle local $\epsilon/2$-perturbation $\widehat{h}$ of $\widehat{g}$ that admits a homoclinic tangency at a point x.

By Remark 5.3.6, for any neighbourhood $\mathcal{W}$ of the zero section $0_{\mathcal{E}}$, the saddle $\widehat{h}$ can be chosen so that it is a $\varepsilon/2$-perturbation of $\widehat{g}$ on $\mathcal{W}$, and so that $\mathcal{W}$ contains the orbit $\mathrm{Orb}_{\widehat{h}}(x)$ of the tangency x. Choose $\mathcal{W}$ so that its closure is in $\mathcal{V}$.

Then define h to be the saddle that coincides with g outside $\mathcal{W}$, and with $\widehat{h}$ on $\mathcal{W}$ (and in fact on $\mathcal{V}$, since $\widehat{h}$ and g coincide on $\mathcal{V} \setminus \mathcal{W}$). For g is an $\varepsilon/2$-pertubation of f on $\mathcal{U}$, and h is an $\varepsilon/2$ perturbation of g on $\mathcal{W} \subset \mathcal{U}$, we have that h is an ε-perturbation of f on $\mathcal{U}$. Finally, as h and $\widehat{h}$ coincide on $\mathcal{W}$, which contains the orbit of the homoclinic tangency x of $\widehat{h}$, x is also a homoclinic tangency for h in $\mathcal{W}$. This concludes the proof of the theorem. □

5.3.2 Proof of Proposition 5.3.4

In the following, we will only work on bundles/manifolds of the form $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, endowed with the canonical Euclidean metric. We will show Proposition 5.3.4 by induction on the dimension d of the bundle $\mathcal{E}$. We first briefly describe the structure of the demonstration.

Sketch of the proof

We initiate the induction process in dimension $d = 2$, thanks to a smart argument by Pujals and Sambarino in [52]. In dimension $d \geq 3$, we write the hyperbolic splitting: $\mathcal{E} = F \oplus G$, where F is the stable space and G the unstable one. By Remark 5.3.5, we may replace f by f^{-1} and assume that F has dimension ≥ 2.

Fix a non-trivial invariant space $H \subset F$, with $dim(H) = 1$. From Lemma 5.3.10, if the saddle is not strongly enough dominated, i.e. if the splitting $E = F \oplus G$ is not N_d-dominated for N_d large, then one of the following situations occurs:

- The splitting $E' = H \oplus G$ is not strongly enough dominated for the restriction of f to E' so that, by induction hypothesis, we can find a saddle local ϵ-perturbation of $f_{|E'}$ that has a homoclinic tangency.

- The splitting $F/H \oplus G/H$ of the quotient bundle E/H is not dominated enough for the quotient linear saddle f/H so that, by induction hypothesis we can find a saddle local ϵ-perturbation of f/H that admits a homoclinic tangency.

In the first case, by Corollary 5.2.18 we can extend the cycle local small perturbation of $f_{|E'}$ to a cycle local small perturbation g of f. The diffeomorphism g extends a saddle cycle that has a homoclinic tangency; thus it also has a homoclinic tangency.

In the second case, by Lemma 5.2.19 we can lift the small perturbation of f/H into an small pertubation g of f that coincides with f on H. Then we will show in section 5.3.2 that, g/H admitting a homoclinic tangency, so does g. Finally, we choose a bounded open neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$ such that the homoclinic tangency of g can be seen in it, and we find a local small perturbation h of f coinciding with g on $\mathcal{U}$. In these conditions h is a saddle local small perturbation of f with a homoclinic tangency.

Hence, we are done in both cases: we found a cycle local perturbation that turns f to admit a homoclinic tangency. QED.

The proof in details

We restate [13, Lemma 4.4] in terms of saddle diffeomorphisms:

Lemma 5.3.10. *Fix a real number $A > 0$ and an integer $d \geq 2$. For any $N \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that the following holds: let f be a linear saddle on a d-dimensional bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, so that*

- *it is bounded by A,*
- *its hyperbolic splitting $\mathcal{E} = F \oplus G$ is not M-dominated.*

Then for any invariant non-trivial subbundle H of F, either the splitting $H \oplus G$ is not N-dominated for the restriction $f_{|H \oplus F}$, or the splitting $F/H \oplus G/H$ is not N-dominated for the quotient linear automorphism f/H.

Let us notice that, although not explicitly, Proposition 5.3.4 is shown by Pujals and Sambarino [52] in dimension 2 (it suffices to combine Lemmas 2.0.1 and 2.2.1). For a complete proof, see section 6.5.1. We are now ready to prove Proposition 5.3.4 by induction.

Proof of Proposition 5.3.4 : As we just pointed out, it is done for $d = 2$. Fix $d > 2$, and suppose now that the theorem is true for any dimension $2 \leq d' \leq d-1$. Fix $A > 0$ and $\varepsilon > 0$. Let $\epsilon > 0$ be less than $\nu_{\varepsilon,A}$, which is as defined for Corollary 5.2.18, less than $\frac{\eta_{\varepsilon,A}}{2A^2}$ where η is as defined for Lemma 5.2.16, and less than α/A, where α is as defined in Lemma 5.2.10. The induction hypothesis gives us, for each $2 \leq d' \leq d-1$, an integer $N_{d'}$ that verifies the following conditions:

for any linear saddle f of a bundle $\{1, ..., p\} \times \mathbb{R}^{d'}$ such that

- it is bounded by A and not $N_{d'}$-dominated,

- it has real eigenvalues,

there exists a cycle local ϵ-perturbation g of f such that it admits a homoclinic tangency. Let N be the maximum of the integers $N_{d'}$ for $2 \leq d' \leq d-1$. Fix $M \in \mathbb{N}$ as in Lemma 5.3.10, with respect to A, d and N. We will show that the conclusions of Proposition 6.5.2 will hold for $N_d = M$, and a perturbation of size ε.

Let f be a linear saddle on a d-dimensional bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ such that

- it is bounded by A,
- it has real eigenvalues and is not M-dominated.

By Remark 5.3.5, we may replace f by f^{-1} and assume that the stable space F has dimension greater than two. Since f has real pairwise distinct eigenvalues, there is an invariant splitting $F = F_1 \oplus ... \oplus F_k$ of eigendirections, where F_1 is the strongest stable eigendirection. Call H the subbundle $F_2 \oplus ... \oplus F_k$. From Lemma 5.3.10, either the splitting $H \oplus G$ is not N-dominated for the restriction $f_{|H \oplus F}$, or the splitting $F/H \oplus G/H$ is not N-dominated for the quotient linear cocycle f/H. Call d_1 and d_2 the dimensions of $H \oplus F$ and $F/H \oplus G/H$, respectively. By definition, N is greater than N_{d_1} and N_{d_2}; we reformulate the dichotomy:

- either the splitting $E' = H \oplus G$ is not N-dominated, and thus not N_{d_1}-dominated, for the restriction $f_{|H \oplus F}$. Then, by induction hypothesis, we find a saddle local ϵ-perturbation g' of $f_{|E'}$ that admits a homoclinic tangency. The following proposition, shown in section 5.3.2 ends the study of this case:

  Proposition 5.3.11. *For $\epsilon > 0$ smaller than $\nu = \nu_{\epsilon,A}$ as defined for Corollary 5.2.18, if g' is a saddle local ϵ-perturbation of $f_{|E'}$ with a homoclinic tangency, then there is a saddle local ε-perturbation g of f that has a homoclinic tangency.*

- or the splitting $E/H = F/H \oplus G/H$ is not N-dominated, and thus not N_{d_2}-dominated, for the saddle quotient f/H. Then, by induction hypothesis, we find a saddle local ϵ-perturbation g' of f/H that admits a homoclinic tangency. We conclude with the following proposition, shown in section 5.3.2:

  Proposition 5.3.12. *For $\epsilon > 0$ smaller than $\mu = \min(\frac{\eta_{\varepsilon,A}}{2A^2}, \alpha/A)$, if g' is a saddle local ϵ-perturbation of f/H with a homoclinic tangency related to the orbit $0_{\mathcal{E}}$, then there is a saddle local ε-perturbation g of f that admits a homoclinic tangency related to $0_{\mathcal{E}}$.*

We are done in both cases: this ends the proof of Proposition 5.3.4. □

The restriction case: proof of Proposition 5.3.11

Let $\varepsilon > 0$ and $A > 0$. Let $\nu = \nu_{\epsilon,A}$. Let f be a linear saddle on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ that is bounded by A, and let E' be an invariant bundle for f. Let g' be a saddle local ν-perturbation of $f_{|E'}$ that admits a homoclinic tangency. We have $|f| < A$, therefore $\nu = \nu_{\epsilon,A} < \nu_{\epsilon,|f|}$.

Then by Corollary 5.2.18, since g' is a $\nu_{\epsilon,|f|}$-perturbation of $f_{|E'}$, there exists a cyclic local ϵ-perturbation g of f that extends g'. In particular, since $g_{|E'}$ admits a homoclinic tangency, so does g.

The quotient case: proof of Proposition 5.3.12

Let $\varepsilon > 0$ and $A > 0$. Let $\mu = \min(\frac{\eta_{\varepsilon,A}}{2A^2}, \alpha/A)$. Let f be a linear saddle on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ that is bounded by A, and let H be an invariant subbundle of the stable bundle of f. Let g' be a saddle local μ-perturbation of f/H that admits a homoclinic tangency.

We have $\mu \leq \frac{\eta_{\varepsilon,A}}{2A^2} \leq \frac{\eta_{\varepsilon,|f|}}{2|f|^2}$. Then by Lemma 5.2.19, since g' is a $\frac{\eta_{\varepsilon,|f|}}{2|f|^2}$-perturbation of f/H, there is an $\eta_{\varepsilon,|f|}$-perturbation g of f that is a lift of g'. By Remark 5.2.20, g can be chosen such that $g_{|H} = f_{|H}$.

Lemma 5.3.13. *If g/H admits a homoclinic tangency, then g also admits one.*

Claim 8. *The stable manifold of g is the inverse image $\pi^{-1}[W^s(g/H)]$ of the stable manifold $W^s(g/H)$ of g/H by the canonical projection $\pi\colon \mathcal{E} \to \mathcal{E}/H$.*

Proof : The linear saddle $f_{|H} = g_{|H}$ is a sink, thus there an integer $k \in \mathbb{N}$ such that g^k divides by four the norms on H: for any $x \in H$, $\|g^k(x)\| \leq \|x\|/4$. Since g is a diffeomorphism that is C^1-bounded, there is a neighbourhood $\mathcal{U}$ of the zero-section $0_{\mathcal{E}}$ such that for any $x \in \mathcal{U}$, for any $y = x + v$ where v is a vector in H, we have $\|g^k(x) - g^k(y)\| \leq \|x - y\|/3$.

Therefore, since g is C^1, for any $C > 0$ there is a neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}/H}$ such that if $x \in \pi^{-1}(\mathcal{U})$ has norm greater than C, then $\|g^k(x)\| \leq \|x\|/2$.

We recall that $(g/H)^k[\pi(x)] = \pi[g^k(x)]$, by commuting diagram 5.4. Thus for k greater than some $N \in \mathbb{N}$, $g^k(x)$ belongs to $\pi^{-1}(\mathcal{U})$. Therefore for any $k \geq N$, $\|g^{k+1}(x)\| \leq \|g^k(x)\|/2$ if $\|g^k(x)\| \geq C$. This proves that $\limsup_{k\to\infty} \|g^k(x)\| \leq 2C$. This holds of course for all $C > 0$. Therefore, any $x \in \pi^{-1}[W^s(g/H)]$ is in the stable manifold of g.

The converse inclusion is much easier: if x is in the stable manifold of g, then $g^k(x)$ goes to zero, and so does $\pi \circ g^k(x) = (g/H)^k[\pi(x)]$. Therefore $\pi(x) \in W^s(g/H)$. This ends the proof of the claim. □

Claim 9. *The map π induces by restriction a diffeomorphism from the unstable manifold $W^u(g)$ of g to the unstable one $W^u(g/H)$ of g/H.*

Proof : As H is in the stable manifold of g, the tangent bundle to $W^u(g)$ at the zero-section has a trivial intersection with the kernel H of π; thus on a neighbourhood of the zero section, π induces an immersion from $W^u(g)$ to $W^u(g/H)$. By dimension equality, $\pi(W^u(g))$ is a neighbourhood of $0_{\mathcal{E}}$ in $W^u(g/H)$, that is, π induces by restriction a diffeomorphism between a neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$ in $W^u(g)$ and a neighbourhood $\mathcal{V}$ of $0_{\mathcal{E}}$ in $W^u(g/H)$.

Then it clearly induces a diffeomorphism between $g^k(\mathcal{U})$ and $(g/H)^k(\mathcal{V})$. The neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ can be chosen such that the sequences $g^k(\mathcal{U})$ and $(g/H)^k(\mathcal{V})$ are increasing. The limits of these sequences are the unstable manifolds of g and g/H, respectively. Hence π defines a diffeomorphism from $W^u(g)$ to $W^u(g/H)$. □

Proof of Lemma 5.3.13 : Let $x \in \mathcal{E}/H$ be a homoclinic tangency for g/H: the tangent spaces T_1^s and T_1^u at x of $W^s(g/H)$ and $W^u(g/H)$ have non-trivial intersection. Then, by Claim 9, there exists an element $y \in \pi^{-1}(x)$ such that y belongs to $W^u(g)$. By Claim 8, it belongs also to $W^s(g)$. Consider the tangent spaces T_2^s (which contains H) and T_2^u of the stable and unstable manifold of g at y. The map π defines a bijection from T_2^u to T_1^u and, since the stable manifold of g is $\pi^{-1}[W^s(g/H)]$, we have $T_2^s = \pi^{-1}(T_1^s)$. Hence π sends $T_2^u + T_2^s$ in $T_1^u + T_1^s$, and from the rank theorem $\dim(T_2^u + T_2^s) \leq \dim(T_1^u + T_1^s) + \dim[Ker(\pi)]$.

We have $\dim(T_1^u + T_1^s) < \dim(T_1^u) + \dim(T_1^s)$, since they have non-trivial intersection. On the other hand, $\dim(T_2^u) = \dim(T_1^u)$ as they are in bijection by π, and $\dim(T_2^s) = \dim(T_1^s) + \dim(H)$ as $T_2^s = \pi^{-1}(T_1^s)$ and $T_1^s \cap H = \{0\}$. From these equalities and inequalities, we get:

$$\dim(T_2^u + T_2^s) < \dim(T_2^s) + \dim(T_2^u)$$

Hence the two tangent spaces T_2^u and T_2^s have non-trivial intersection: the intersection of the stable and unstable manifolds of g at y is a homoclinic tangency. □

Next claim will end the proof of Proposition 5.3.12:

Proposition 5.3.14. *There exists a saddle local ε-perturbation of f with a homoclinic tangency.*

Proof : Choose an open bounded neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$ that contains the orbit of the homoclinic tangency y by g. Then any saddle diffeomorphism that coincides with g on $\mathcal{U}$ admits a homoclinic tangency at x. As g is an $\eta_{\varepsilon,|f|}$-perturbation of f, Lemma 5.2.16 provides an ε-perturbation h of f that coincides with g on $\mathcal{U}$, and that coindices with f on a neighbourhood of infinity. Thus we found a saddle local ε-perturbation of f that admits a homoclinic tangency. □

Chapter 6

Creating tangencies preserving homoclinic cycles

Here we show that the construction of the previous chapter can be done in order to preserve homolinic cycles. In rough terms, we will show that if the stable/unstable splitting on a long-period saddle point is not dominated enough, then a small perturbation creates a homoclinic tangency related to the saddle point, preserving a fixed finite set in the strong stable/unstable manifolds. To state it precisely, we will introduce the notion of flag-configuration (see below). Among other consequences, we deduce that if a homoclinic class of a saddle P does not admit a dominated splitting of same index as P, then an arbitrarily small perturbation creates a homoclinic tangency related to P.

6.1 Prolegomena and statement of results

Two periodic saddle points R and S of a diffeomorphism f are said to form a *cycle* if and only if the unstable manifold $W^u(R)$ of R intersects the stable manifold $W^s(S)$ of S, and the unstable manifold $W^u(S)$ intersects the stable manifold $W^s(R)$. We will show in this chapter that, with the same hypothesis as in Theorem 5.1.3, it is possible to create a homoclinic tangency at a saddle, preserving a finite number of cycles involving the saddle. Precisely:

Theorem 6.1.1. *Fix $A > 0$, $\epsilon > 0$ and an integer $d \geq 2$. There exists two integers $N_d, P > 0$ such that, if f is a diffeomorphism on a d-dimensional Riemannian manifold, and Q a periodic saddle point, that satisfy:*

- *the diffeomorphism f is stricly bounded by A,*
- *the saddle Q has period p greater than P and is not N_d-dominated,*

then for any finite set Σ of saddle points that form a cycle with Q, there exists an ϵ-perturbation of f on an arbitrarily small neighbourhood $\mathcal{U}$ of the *orbit of Q, that preserves the orbit of Q, that creates a homoclinic tangency related to Q in $\mathcal{U}$, and such that Q still forms a cycle with each of the points of Σ.*

A more precise result is possible; we introduce a few definitions and notations. Let R be a saddle point with period p, for a diffeomorphism f on a d-dimensional Riemannian manifold $\mathcal{M}$. We denote by $\mathrm{Spec}_f(R)$ the spectrum of $df^p(R)$, that is the d-uple $(\lambda_1, ..., \lambda_d)$ of the eigenvalues of $df^p(R)$ counted with multiplicity, where the λ_i are ordered by increasing moduli: for each $1 \leq i < d$, $|\lambda_i| \leq |\lambda_{i+1}|$.

If $|\lambda_i| < \min(|\lambda_{i+1}|, 1)$, we denote by $W_f^{s,i}(R)$ the set of points x of M such that, for any real number $\lambda > |\lambda_i|$, we have $\mathrm{dist}(f^{pn}(x), \mathrm{Orb}_f(R)) = o(\lambda^n)$, as n goes to infinity. This set is an i-dimensional C^1-submanifold of $W_f^s(R)$. We call it the *i-dimensional strong stable manifold* of $\mathrm{Orb}_f(R)$ for f. For convenience, we define the 0-dimensional strong manifold to be the orbit of R. To simplify the statements, if for some integer i, the i-dimensional strong stable manifold of R does not exist, then we let $W_f^{s,i}(R) = \emptyset$. Let the integers $i_1 < ... < i_k$ be the dimensions for which there exists a strong stable manifold. Then we have $W_f^{s,i_1}(R) \subset W_f^{s,i_2}(R) \subset ... \subset W_f^{s,i_k} R = W_f^s(R)$. We call that family of manifolds, the *stable flag* of R for f.

Symmetrically, if it exists, we denote by $W_f^{u,i}(R)$ and call *i-dimensional strong unstable manifold* of $\mathrm{Orb}_f(R)$ for f, the i-dimensional strong stable manifold of $\mathrm{Orb}_f(R)$ for f^{-1}, and we define similarly the *unstable flag*. For any point $x \in M$, we define the *flag-type of* x for the saddle R of f to be the pair (s_x, u_x) defined by

$$s_x = \inf\{i \in \mathbb{N} / x \in W_f^{s,i}(R)\}, \qquad u_x = \inf\{i \in \mathbb{N} / x \in W_f^{u,i}(R)\},$$

with the convention $\inf \emptyset = +\infty$.

Remark 6.1.2. *Let x be of flag-type (s_x, u_x), and i, j be a such that $i \geq s_x$, and $j \geq u_x$. Then if $W_f^{s,i}$ and $W_f^{u,j}$ exist, an arbitrarily small perturbation of f on an arbitrarily small neighourhood of x increases the flag-type of x from (s_x, u_x) to (i, j).*

We will say that a perturbation g of f that preserves the orbit of the saddle R, *respects the flag-configuration* of a finite set Γ for f if and only if, for any $x \in \Gamma$, it preserves or decreases the flag-type of x for R, which means that:

$$x \in W_f^{s,i}(R) \implies x \in W_g^{s,i}(R). \tag{6.1}$$

$$x \in W_f^{u,i}(R) \implies x \in W_g^{u,i}(R). \tag{6.2}$$

In the following, we will ask for more than preserving cycles as in Theorem 6.1.1. For any fixed finite set Γ, we want to find a perturbation that creates a homocinic tangency, and respects the flag-configuration of Γ. We state it precisely:

Theorem 6.1.3. *Fix $A > 0$, $\epsilon > 0$ and an integer $d \geq 2$. There exists two integers $N_d, P > 0$ such that, if Q is a periodic saddle point for a diffeomorphism f on a d-dimensional Riemannian manifold $\mathcal{M}$, that satisfy:*

- *the diffeomorphism f is stricly bounded by A,*
- *the saddle Q has period p greater than P and is not N_d-dominated,*

then for any finite subset Γ of $\mathcal{M}$, there exists an ϵ-perturbation of f on an arbitrarily small neighbourhood $\mathcal{U}$ of the orbit of Q, that preserves the orbit of Q, creates a homoclinic tangency related to Q in $\mathcal{U}$, and respects the flag-configuration of Γ.

The following remark will allow to replace f by f^{-1} in the proofs (see section 6.5.2).

Remark 6.1.4. *g is a saddle ϵ-perturbation of f that respects the flag-configuration of Γ for f, if and only if g^{-1} is a saddle ϵ-perturbation of f^{-1} that respects the flag-configuration of Γ for f^{-1}.*

Now we show that respecting flag-configuration allows to preserve cycles:

Proof of Theorem 6.1.1, from Theorem 6.1.3 : Let Q be a saddle for a diffeomorphism f, and let $\Sigma = \{R_1, ..., R_k\}$ be a finite set of saddles of f that form each a cycle with Q. Suppose first that Q is not in Σ. Denote by s_i (respectively u_i) one of the intersections between $W^s(R_i)$ and $W^u(Q)$ (respectively between $W^u(R_i)$ and $W^s(Q)$). The union $K = \{f^n(s_i), n \in \mathbb{N}, 1 \leq i \leq k\} \cup \{f^{-n}(u_i), n \in \mathbb{N}, 1 \leq i \leq k\} \cup \Sigma$ is clearly a compact set that does not intersect the orbit of Q. Fix a neighbourhood $\mathcal{U}$ of $\mathrm{Orb}_f(Q)$ that does not intersect K. Theorems 6.1.1 and 6.1.3 have the same hypothesis.

Under these hypothesis, by Theorem 6.1.3, we find a perturbation g of f on $\mathcal{U}$ that creates a homoclinic tangency related to Q in $\mathcal{U}$, and respects the flag-configuration of $\Gamma = \{s_i, u_i, 1 \leq i \leq k\}$. For $\mathcal{U}$ does not intersect K, we have that Q and any R_i still form a cycle. Hence that perturbation satisfies the conlusion of Theorem 6.1.1. If Q is in Σ, that is admits a homoclinic intersection at a point x, then add to Γ the point x and apply Theorem 6.1.3 as previously. Then the point x will still be a homoclinic point, and Q will still form a cycle with itself. The conclusions of Theorem 6.1.1 are satisfied. □

If f is a saddle diffeomorphism on a bundle $\mathcal{E}$, then, by an abuse of notation, we call *i-dimensional strong stable manifold of f* and denote by $W^{s,i}(f)$ the i-dimensional strong stable manifold of the orbit $0_{\mathcal{E}}$ of $(0,1)$ for f. Of course we use the symmetrical notations, replacing stable by unstable and s by u. We can restate Theorem 6.1.3 in terms of saddle diffeomorphisms:

Theorem 6.1.5. *Fix $A > 0$, $\epsilon > 0$ and an integer $d \geq 2$. There exist two integers $N_d, P > 0$ such that, if f is a saddle diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ that satisfies:*

- *the diffeomorphism f is bounded by A,*
- *the period p is greater than P,*
- *the saddle f is not N_d-dominated,*

then for any finite subset Γ of $\mathcal{E}$, there exists a saddle ϵ-perturbation of f on an arbitrarily small neighbourhood $\mathcal{U}$ of the zero section $O_{\mathcal{E}}$, that admits a homoclinic tangency in $\mathcal{U}$, and that respects the flag-configuration of Γ.

Given a saddle diffeomorphism f on $\mathcal{E}$, and a finite set Γ in $\mathcal{E}$, we will denote by Γ^s the subset of points of Γ that belong to the stable manifold of f, and Γ^u the subset of points that belong to the unstable one. In our proofs we may have to do successively many flag-configuration-respecting perturbation of a saddle diffeomorphism f. Notice then that Γ^s (resp. Γ^u) is a subset of the stable (resp. unstable) manifold of any saddle perturbation of f that respects the flag-configuration of Γ.

Remark 6.1.6. *Since the points of Γ that are not in the stable or unstable manifolds of f do not appear in the conclusions of the theorem, it will be enough to show it for any finite subset Γ of the union $W^s_f \cup W^u_f$, that is, such that $\Gamma = \Gamma^s \cup \Gamma^u$. Besides, it is clearly sufficient to show it when there is no pair $\{x, y\}$ in Γ such that x is an iterate of y by f. This is what we will suppose in all the following, to ease redaction, even if not mentioned.*

We will again considerably reduce the problem, adapting the lemmas we used in section 5.3. However much more work will be needed here.

6.2 Technical definitions and lemmas

6.2.1 Generalized eigendirections and eigenmanifolds

Let A be a linear isomorphism of a vector space. We say that an invariant space E_λ is a *generalized eigendirection* with eigenvalue $\lambda \in \mathbb{C}$, for A, if and only if one of the two following situations occurs:

- The space E_λ has dimension one, λ is a real number, and A induces a homothety of ratio λ on E_λ.
- The space E_λ has dimension two, λ is not a real number, and A is conjugate to a similitude of ratio λ on that space.

We call *simple isomorphism* any linear isomorphism A of a vector space E that admits an invariant splitting $E = E_{\lambda_1} \oplus ... \oplus E_{\lambda_l}$, where the λ_i are complex numbers with moduli different from 1 such that $|\lambda_1| < |\lambda_2| < ... < |\lambda_l|$, and each E_{λ_i} is a generalized eigenspace for A. We call *good splitting* for a simple isomorphism A the splitting $\mathcal{E} = E_{\lambda_1} \oplus ... \oplus E_{\lambda_l}$ through which we see the simplicity of A.

We say that a cyclic diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ has a *simple derivative* if and only if df^p induces a simple isomorphism on each fibre of $\mathcal{E}$. We call *good splitting* for f the splitting $\mathcal{E} = E_{\lambda_1} \oplus ... \oplus E_{\lambda_l}$ through which we see the simplicity of df^p. Let f be now a linear cyclic diffeomorphism on $\mathcal{E}$. We say that an invariant bundle is an *eigenbundle* or *eigendirection* if and only if there is $\lambda \in \mathbb{R}$ such that, for any point x in the bundle, we have $f^p(x) = \lambda.x$. Then we say that λ is the eigenvalue of that eigendirection.

In the following the adverb *locally* will mean 'on a neighbourhood of $0_{\mathcal{E}}$'. Let f be a cyclic diffeomorphism on $\mathcal{E}$ that has a simple derivative and is locally linear. Then the generalized eigendirections E_{λ_i} of the locally linear f provide local manifolds $W^{loc}_{\lambda_i}$ that are invariant by f if stable, or by f^{-1} if unstable. The union W_{λ_i} of the negative iterates of $W^{loc}_{\lambda_i}$ if $|\lambda_i| < 1$, or positive iterates if $|\lambda_i| > 1$, is an invariant embedded manifold which we will call *generalized eigenmanifold of* f associated to the eigenvalue λ_i. Obviously, the generalized eigenmanifold W_{λ_i} is uniquely defined and any point x in it ends up (by positive iterates if $|\lambda_i| < 1$, or negative else) in the generalized eigendirection E_{λ_i} of the locally linear f.

Remark 6.2.1. *An eigenmanifold has no dynamical meaning and depends on the linearization one chooses. We introduced that notion only for convenience.*

Let f be a linear cyclic diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ with a simple derivative. Fix integers $1 \leq i < l$ such that $0 < |\lambda_{i+1}| < 1$. Call d_1 and d_2 the dimensions of $E_{\lambda_1} \oplus ... \oplus E_{\lambda_i}$ and $E_{\lambda_1} \oplus ... \oplus E_{\lambda_{i+1}}$. We recall the following classical results:

Proposition 6.2.2. *The d_2-dimensional stable manifold of f is defined and admits a dimension d_1 strong stable foliation $\mathcal{F}$, such that for any pair of points x and y in the same leaf, we have* $\mathrm{dist}(f^{pn}(x), f^{pn}(y)) = \mathcal{O}(|\lambda_i|^n)$. *If moreover f is locally linear, then the generalized eigenmanifold $W_{\lambda_{i+1}}$ intersects transversely, exactly once, each of the leaves of $\mathcal{F}$. Finally, for any point $x \in W_{\lambda_{i+1}} \setminus 0_{\mathcal{E}}$, there is a constant C such that for any n great enough,* $\mathrm{dist}(0_{\mathcal{E}}, f^{pn}(x)) = C.|\lambda_{i+1}|^n$.

Remark 6.2.3. *The simple isomorphisms of $\mathbb{R}^d$ are an open dense subset of $GL(\mathbb{R}, d)$. The simple isomorphisms with real eigenvalues form an open dense subset of the set of isomorphisms with real eigenvalues.*

6.2.2 Common multiples and firmly flag-respecting perturbations

Here we introduce a property on the eigenvalues of a linear saddle that will allow localized statements: under some conditions, from a saddle local perturbation that respects the flag-configuration of a set Γ, we will get saddle perturbations on arbitrarily small neighbourhoods of $0_{\mathcal{E}}$ that respect the configuration of Γ and are locally dynamically conjugate to the original. Thus, we will parallel Remark 5.3.6, and complement it to match the flag-configuration requirements.

We say that a finite set of complex numbers $\{\lambda_i, 1 \leq i \leq l\}$ have a *common multiple* λ, if and only if, for any $i \in I$, there is an integer n_i such that $\lambda_i^{n_i} = \lambda$. An arbitrarily small perturbation of any finite set $\{\lambda_i, 1 \leq i \leq l\}$ turns each of the arguments $\theta_i = \arg(\lambda_i)$ to be commensurable to π and each of the logarithms $\log|\lambda_i|$ to be a nonzero rational number. Hence, an arbitrarily small perturbation turns that set to have a real common multiple $\lambda > 1$, which allows the following remark:

Remark 6.2.4. *The isomorphism of $\mathbb{R}^d$ such that their eigenvalues have a common multiple $\lambda > 1$, are dense in $GL(\mathbb{R}, d)$. The isomorphisms that have real eigenvalues with a common multiple $\lambda > 1$ form a dense subset of the set of isomorphisms with real eigenvalues.*

To abridge statements, we may say that a *cycle diffeomorphism f has a common multiple* λ if λ is a common multiple of the eigenvalues of the isomorphism $df^p_{0_{\mathcal{E}}}$. Let f be a linear cyclic diffeomorphism, let $\Gamma \subset \mathcal{E}$ be a finite set. Then we will say that a perturbation g of f *firmly respects the flag-configuration of* Γ if and only if

- it respects the flag-configuration of Γ,
- g^n coincides with f^n on Γ^u, and g^{-n} coincides with f^{-n} on Γ^s, for all $n \in \mathbb{N}$.

Let H be a subbundle of $\mathcal{E}$. In the following, π will denote the canonical projection from $\mathcal{E}$ to $\mathcal{E}/H$, Γ/H will denote the projection $\pi(\Gamma)$ of the set Γ, and $\Gamma_{|H}$ the intersection $\Gamma \cap H$. We may write $\Gamma + H$ for $\pi^{-1}(\Gamma/H)$. We say that a perturbation g of f firmly respects the flag-configuration of Γ *to the quotient by* H if and only if

- g and f go to the quotient by H
- g/H respects the flag-configuration of Γ/H for f/H
- g^n coincides with f^n on $\Gamma^u + H$, and g^{-n} coincides with f^{-n} on $\Gamma^s + H$, for all $n \in \mathbb{N}$.

Of course, saying that g firmly respects the flag-configuration of Γ, corresponds to saying that g firmly respects the flag-configuration of Γ to the quotient by the trivial subbundle $0_{\mathcal{E}}$. The following fundamental remark corresponds to Remark 5.3.6:

Remark 6.2.5. *Let f be a linear cycle with a simple derivative, and a common multiple $\lambda > 1$. Let H be a (maybe trivial) subbundle of $0_{\mathcal{E}}$, and Γ be a finite subset of the generalized eigendirections of f. Let g be a cycle ϵ-perturbation of f on $\mathcal{U}$ that firmly respects the flag-configuration of Γ (to the quotient by H). Then, for any positive integer n, the cycle diffeomorphism $g_{\lambda^k} = \lambda^{-k}.g \circ \lambda^k.Id_{\mathcal{E}}$ is a cycle ϵ-perturbation of f that firmly respects the flag-configuration of Γ (to the quotient by H).*

6.3 Perturbation Propositions: transformations that respect flag-configuration

We will state in this section two technical propositions that will describe general circumstance under which flag-configuration respecting perturbations are possible. More precisely, given a cyclic perturbation, or a family of cyclic perturbations of a cyclic diffeomorphism f on $\mathcal{E}$, we will give some conditions under which there is a small and local cyclic perurbation of f that preserves some dynamical features of the initial cyclic perturbations (such as local behaviour around $0_{\mathcal{E}}$, local linearity, homoclinic tangency, ...) and that respects the flag-configuration of a fixed finite set of points of $\mathcal{E}$.

The first one will be useful to linearize locally or to perturb the derivative of f, respecting the flag-configuration of a finite set (see proofs of Lemmas 6.4.1 and 6.4.2). The proof is postponed until section 6.3.1.

Proposition 6.3.1. *Let f be a cyclic diffeomorphism on a bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$. Let $\mathcal{U}_k$ be a sequence of neighbourhoods of $0_{\mathcal{E}}$ that tends to $0_{\mathcal{E}}$ for the Hausdorff topology, and g_k a sequence of cyclic perturbations of f on $\mathcal{U}_k$ that converges to f for the C^1-topology. Let Γ be a finite subset of $\mathcal{E}$. Then for any $\epsilon > 0$, for any k great enough, there exists an ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that coincides with g_k on a neighbourhood of $0_{\mathcal{E}}$, and that respects the flag-configuration of Γ.*

The second proposition will allow, under some conditions, to preserve the flag-configuration when a perturbation of f is done by induction on the dimension, lifting a perturbation of a quotient f/H of f by an invariant bundle, or extending a perturbation of the restriction $f_{|H}$ of f. For that, H needs to be central:

Definition 6.3.2. We say that a cyclic diffeomorphism f of $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ admits a *central bundle H* if f goes to the quotient by H (in particular H is invariant), and if the eigenvalues of $f_{|H}$ are strictly weaker than the other eigenvalues of f. Precisely, for any eigenvalue λ of the quotient cocycle $df^p/H_{0_\mathcal{E}}$, for any eigenvalue μ of the restricted cocycle $df^p_{|H}$, we have

$$|\lambda|, |\mu| \leq 1 \quad \Rightarrow \quad |\mu| < |\lambda| \tag{6.3}$$

$$|\lambda|, |\mu| \geq 1 \quad \Rightarrow \quad |\mu| > |\lambda| \tag{6.4}$$

Here is first an unformal outline of the proposition: if a small perturbation g of a linear cylic diffeomorphism f - both admitting a central bundle H - firmly respects the flag-configuration of Γ by restriction to H and to the quotient by H, then we can find a small perturbation of f on an arbitrarily small neighbourhood of $0_\mathcal{E}$, that respects the flag-configuration of the whole Γ, and that has locally the same dynamical features as g.

Now we state it precisely, and will prove it in section 6.3.2. We remind that the restriction $\Gamma_{|H}$ is $\Gamma \cap H$ and that $(\epsilon, A) \mapsto \eta_{\epsilon,A}$ was defined in Lemma 5.2.16.

Proposition 6.3.3. *Let $\epsilon > 0$. Let f be a linear cyclic diffeomorphism on $\mathcal{E}$ with a simple derivative and a common multiple $\lambda > 1$, Γ a finite set in the union of the generalized eigendirections of f, and g a cyclic $\eta_{\epsilon,|f|}$-perturbation of f. If H is a central bundle for both f and g such that*

- *$g_{|H}$ firmly respects the flag-configuration of $\Gamma_{|H}$ for $f_{|H}$,*
- *g firmly respects the flag-configuration of $\Gamma \setminus H$ for f, to the quotient by H.*

Then for any bounded set $\mathcal{B}$ of $\mathcal{E}$, there is a cyclic ϵ-perturbation h on an arbitrarily small neighbourhood of $0_\mathcal{E}$, that respects the flag-configuration of Γ, and such that $g_{|\mathcal{B}} = \gamma.h \circ \gamma^{-1}.Id_\mathcal{B}$ for some $\gamma > 0$.

In particular, h can be found to be conjugate to g on a bounded neighbourhood of $0_\mathcal{E}$, therefore to satisfy $dh_{0_\mathcal{E}} = dg_{0_\mathcal{E}}$. Besides, if g has a homoclinic tangency at a point x, if we choose $\mathcal{B}$ to contain the (bounded) orbit of x in its interior, h will also admit a homoclinic tangency.

6.3.1 Proof of Proposition 6.3.1

We recall, without a proof, three folklore lemmas. The two first are equivalent.

Lemma 6.3.4. *Let x_k be a sequence of points in $\mathbb{R}^n$ that tends to x. Then for any neighbourhood $\mathcal{U}$ of x, for any $\epsilon > 0$, for k great enough, there exists an ϵ-perturbation of Id_n on $\mathcal{U}$ that sends x on x_k.*

Lemma 6.3.5. *For any $\epsilon > 0$, there is a ratio r such that, if x and y are two points in an open set $\mathcal{U}$ of $\mathbb{R}^n$ with diameter smaller than 1, if we have*

$$\frac{\text{dist}(x, y)}{\text{dist}(x, \mathcal{E} \setminus \mathcal{U})} < r,$$

then there is an ϵ-perturbation of Id_n on $\mathcal{U}$ that sends x on y.

Lemma 6.3.6. *Let g_k be a sequence of diffeomorphisms that converges to a diffeomorphism f for the C^1-topology. Let P be a periodic saddle point for f. If the i-dimensional strong stable manifold $W^{s,i}(f)$ of f exists, for k great enough, it also exists for g_k, and the sequence $W^{s,i}(g_k)$ C^1-converges to $W^{s,i}(f)$ on the compacts. Moreover, for any point $x \in W^{s,i}(f)$, there is a sequence $x_k \in W^{s,i}(g_k)$ such that the sequence $\sup\{\mathrm{dist}[f^n(x), g_k^n(x_k)], n \in \mathbb{N}\}$ tends to zero as k goes to ∞.*

Of course, the symmetrical statement holds for the strong unstable manifolds.

We need now some definitions and notations. Let f be a cyclic diffeomorphism and g_k be a sequence of cyclic diffeomorphisms. Let $\Delta = \Delta^s \cup \Delta^u$ be a finite set in the union of the stable and unstable manifolds of f such that no point of Δ is an iterate of another by f. To all $k \in \mathbb{N}$, to all $x \in \Delta$, we associate a point x_k of $\mathcal{E}$. let $\Delta_k = \{x_k, x \in \Delta\}$. We say that the sequence (g_k, Δ_k) *approximates the flag-configuration* of (f, Δ) if and only if

- for all $x \in \Delta^s$, the sequence $(g_k^n(x_k))_{n\in\mathbb{N}}$ converges to the sequence $(f^n(x))_n \in \mathbb{N}$ as k goes to $+\infty$, that is
$$\sup\{\mathrm{dist}[f^n(x), g_k^n(x_k)], n \in \mathbb{N}\} \to_{k\to\infty} 0$$
- for all $x \in \Delta^u$, the sequence $\left(g_k^{-n}(x_k)\right)_{n\in\mathbb{N}}$ converges to the sequence $(f^{-n}(x))_n \in \mathbb{N}$ as k goes to $+\infty$, that is
$$\sup\{\mathrm{dist}[f^{-n}(x), g_k^{-n}(x_k)], n \in \mathbb{N}\} \to_{k\to\infty} 0$$
- For all $k \in \mathbb{N}$, the flag-type of x_k for g_k is less or equal to the flag-type of x for f.

We will denote by Δ_k^u (Δ_k^s) the set of such points $x_k \in \Delta_k$ that are in the stable (unstable) manifold of g_k. Notice that Lemma 6.3.6 asserts that if f is a cyclic diffeomorphism and Δ is a finite set in $\Delta^s \cup \Delta^u$, if g_k is a sequence of cyclic diffeomorphisms that converges to f in $\mathrm{Diff}^1(M)$, then there is a sequence Δ_k such that (g_k, Δ_k) approximates the flag-configuration of (f, Δ). We now state the following technical lemma:

Lemma 6.3.7. *If a sequence (g_k, Δ_k) approximates the flag-configuration of (f, Δ), then for any neighbourhood $\mathcal{W}$ of $\Lambda = \Delta^s \cup f^{-1}(\Delta^u)$, there is a sequence Φ_k of perturbations of $Id_{\mathcal{E}}$ on $\mathcal{W}$, that tends to $Id_{\mathcal{E}}$ in $\mathrm{Diff}^1(\mathcal{E})$, and such that for k great enough $g_k \circ \Phi_k$ respects the flag-configuration of Δ for f.*

Proof : With the same notation as previously, we write $\Delta = \{x_k, x \in \Delta\}$. For $g_k \circ \Phi_k$ to respect the flag-configuration of Δ for f, it suffices that

- Φ_k send x on x_k, for all $x \in \Delta^s$ and send $g_k^{-1}(x_k)$ on $g_k^{-1}(x)$, for all $x \in \Delta^u$,
- it be identity by restriction to the positive iterates $\mathcal{I}_k^s = \cup_{n\geq 1} g_k^n(\Delta_k^s)$, the negative iterates $\mathcal{I}_k^u = \cup_{n\geq 2} g_k^{-n}(\Delta_k^u)$, and on a neighbourhood of $0_{\mathcal{E}}$.

Indeed, the positive orbit of any $x \in \Delta^s$ by such $g_k \circ \Phi_k$ coincides after the first iterate with the positive orbit of x_k by g_k. The diffeomorphism $g_k \circ \Phi_k$ coincides with g_k on a neighbourhood of $0_{\mathcal{E}}$, therefore x has same flag-type for $g_k \circ \Phi_k$ as x_k for g_k, which is less or equal to the flag-type of x for f. Thus $g_k \circ \Phi_k$ respects the flag-configuration of Δ^s for f. The unstable case is symmetrical.

Now we have to show that given a neighbourhood $\mathcal{W}$ of Λ, and $\epsilon > 0$, for k great enough, we can find an ϵ-perturbation Φ_k of $Id_{\mathcal{E}}$ on $\mathcal{W}$ that satisfies these conditions. By hypothesis, the sequence $\mathcal{I}_k^s$ converges to $\mathcal{I}^s = \cup_{n\geq 1} f^n(\Delta^s)$, and the sequence $\mathcal{I}_k^u$ converges to $\mathcal{I}^u = \cup_{n\geq 2} f^{-n}(\Delta^u)$, for the Hausdorff topology. Thus the minimun distance from Λ to $\mathcal{I}_k = \mathcal{I}_k^s \cup \mathcal{I}_k^u$ (that is the infimum of the distances $\mathrm{dist}(x,y)$ where $x \in \Lambda$ and $y \in \mathcal{I}_k$) goes to the minimum distance ρ from Λ to $\mathcal{I} \cup 0_{\mathcal{E}}$. We have $\rho > 0$ since we put ourselves under the conditions given by Remark 6.1.6.

Let $\mathcal{W}$ be a union of closed balls $\mathcal{W}_x$ of radius less than $\rho/2$, centered at x if $x \in \Delta^s$ and centered at $f^{-1}(x)$ if $x \in \Delta^u$. It is disjoint from $0_{\mathcal{E}}$, and for k great enough, $\mathcal{W}$ does not intersect $\mathcal{I}_k^s$ not $\mathcal{I}_k^u$. Hence, a perturbation Φ_k on $\mathcal{W}$ satisfies the second item for k great enough.

The distances $\mathrm{dist}(x, x_k)$, $\mathrm{dist}(g_k^{-1}(x), f^{-1}(x))$ and $\mathrm{dist}(g_k^{-1}(x_k), f^{-1}(x))$ go to zero as k goes to ∞. We can apply Lemma 6.3.4 on each $\mathcal{W}_x$ and find, for any k great enough, an ϵ-perturbation of $Id_{\mathcal{E}}$ on $\mathcal{W}$ that sends x on x_k and $g_k^{-1}(x_k)$ on $g_k^{-1}(x)$. This ends the proof of the lemma. □

Proof of Proposition 6.3.1 : Fix f, Γ and the sequences $\mathcal{U}_k$ and g_k as in the hypothesis of the lemma. By Remark 6.1.6, we assume that $\Gamma = \Gamma^s \cup \Gamma^u$, and that there is no point of Γ that is an iterate of another by f. Fix $\epsilon > 0$ and an open neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$. For some $l \in \mathbb{N}$, the set $\Lambda = f^l(\Gamma^s) \cup f^{-l-1}(\Gamma^u)$ is a subset of $\mathcal{U}$. Define $\Delta = f^l(\Gamma^s) \cup f^{-l}(\Gamma^u)$.

Let $\mathcal{W}$ be a compact neighbourhood of Λ in $\mathcal{U}$ that does not intersect $0_{\mathcal{E}}$. By Lemma 6.3.6, there is a sequence Δ_k such that (g_k, Δ_k) approximates the flag-configuration of (f, Δ). We apply Lemma 6.3.7 and find a sequence Φ_k of cyclic pertubations of $Id_{\mathcal{E}}$ on $\mathcal{W}$ that tends to $Id_{\mathcal{E}}$ in $\mathrm{Diff}^1(\mathcal{E})$ and such that, for k great enough, $g_k \circ \Phi_k$ respects the flag-configuration of Δ for f. For any $k \in \mathbb{N}$, $g_k \circ \Phi_k$ is a cyclic diffeomorphism that coincides with g_k outside $\mathcal{W}$, therefore coincides with f outside $\mathcal{W} \cup \mathcal{U}_k$.

From Lemma 5.2.8, the sequence $g_k \circ \Phi_k$ converges to f for the C^1-topology. Therefore, for k great enough, $g_k \circ \Phi_k$ is a cyclic ϵ-perturbation of f on $\mathcal{W} \cup \mathcal{U}_k$ that respects the flag-configuration of Δ for f. Choose $\mathcal{W}$ so that it does not intersect the sets $f^i(\Gamma^s)$ for $i = 0, ..., l-1$, and $f^j(\Gamma^u)$ for $j = -l, ..., 0$ and choose k great enough so that $\mathcal{U}_k$ does not intersect these same sets. Then, for k great enough, $g = g_k \circ \Phi_k$ respects the flag-configuration of Γ for f. We found for k great enough an ϵ-perturbation g of f on $\mathcal{U}$ that respects the flag-configuration of f, and that coincides with g_k on a neighbourhood of $0_{\mathcal{E}}$ (since the compact set $\mathcal{W}$ does not intersect $0_{\mathcal{E}}$). This ends the proof of Proposition 6.3.1. □

6.3.2 Proof of Proposition 6.3.3

Lemma 6.3.8. *Let H be a central bundle for a cyclic diffeomorphism f of $\mathcal{E}$. If the i-dimensional strong stable manifold $W^{s,i}(f/H)$ of f/H exists, it also exists for f and the canonical projection $\pi\colon \mathcal{E} \to \mathcal{E}/H$ induces a diffeomorphism from $W^{s,i}(f)$ on $W^{s,i}(f/H)$. If the i-dimensional strong unstable manifold $W^{u,i}(f/H)$ of f/H exists, then it exists for f and π induces a diffeomorphism from $W^{u,i}(f)$ on $W^{u,i}(f/H)$.*

Proof : Suppose that the i-dimensional strong stable manifold $W^{s,i}(f/H)$ of f/H exists. Then we write the sequence $\lambda_1, ..., \lambda_k$ of the eigenvalues of f/H, counted with multiplicity, and ordered by increasing moduli (of course $k \geq i$). We have $|\lambda_i| < 1$, and if $k > i$, there is a gap between the i-th and the $(i+1)$-th eigenvalue, that is, $|\lambda_i| < |\lambda_{i+1}|$. Since H is central, the eigenvalues of f/H are strictly stronger than those of $f_{|H}$, therefore the i-th eigenvalue of f is also λ_i and there is a gap between it and the next eigenvalue. Hence f has an i-dimensional strong stable manifold.

For any $x \in W^{s,i}(f)$, and for any $\lambda > |\lambda_i|$, we have $\|f^n(x)\| = o(\lambda^n)$. Since π is a projection, $\|f/H^n \circ \pi(x)\| = \|\pi \circ f^n(x)\| = o(\lambda^n)$. Thus $f/H(x)$ is in the i-dimensional strong stable manifold of f/H. The bundle H is locally transverse to $W^{s,i}(f)$ at $0_{\mathcal{E}}$, since the eigenvalues of $f_{|H}$ are strictly weaker than λ_i. Precisely, we find a neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$ in the local $W^{s,i}(f)$, at which the fibres of the form $x+H$ meet $W^{s,i}(f)$ transversely. Since for all $n \in \mathbb{N}$, f^{-n} sends the fibres $x+H$ on others, and $W^{s,i}(f)$ on itself, the fibres $x+H$ meet $W^{s,i}(f)$ transversely at the preimages $f^{-n}(\mathcal{U})$. The union of these is $W^{s,i}(f)$. Hence the projection $\pi\colon \mathcal{E} \to \mathcal{E}/H$ defines an immersion π_i from $W^{s,i}(f)$ on its image.

We are left to show that that immersion is a bijection from $W^{s,i}(f)$ to $W^{s,i}(f/H)$. Since π_i is an immersion and its image contains $0_{\mathcal{E}}$, since $W^{s,i}(f)$ and $W^{s,i}(f/H)$ have same dimension, a neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$ in $W^{s,i}(f)$ is sent bijectively by π_i on a neighbourhood $\mathcal{V}$ of $0_{\mathcal{E}}$ in $W^{s,i}(f/H)$. We may choose $\mathcal{U}$ and $\mathcal{V}$ such that $f(\mathcal{U}) \subset \mathcal{U}$ and $f/H(\mathcal{V}) \subset \mathcal{V}$. Note that, since the following graph commutes

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{E} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{E}/H & \xrightarrow{f/H} & \mathcal{E}/H \end{array} \tag{6.5}$$

we have $\pi_i = f/H^{-n} \circ \pi_i \circ f^n$. Let $\pi_{i,\mathcal{U}}$ be the restriction of π_i to $\mathcal{U}$. It is a bijection. Therefore, $\pi_{i,f^{-n}(\mathcal{U})} = f/H^{-n} \circ \pi_{i,\mathcal{U}} \circ f^n$ is a bijection from $f^{-n}(\mathcal{U})$ on $f/H^{-n}(\mathcal{V})$, and coincides with π_i. As the sequences $f^{-n}(\mathcal{U})$ and $f/H^{-n}(\mathcal{V})$ increase to $W^{s,i}(f)$ and $W^{s,i}(f/H)$, respectively, we get that π_i is a bijection from $W^{s,i}(f)$ on $W^{s,i}(f/H)$. Consequently, π induces by restriction a diffeomorphism from $W^{s,i}(f)$ on $W^{s,i}(f/H)$. The strong unstable case is symmetrical. □

Corollary 6.3.9. *Let H be a central bundle for a linear cyclic diffeomorphism f of $\mathcal{E}$ with a simple derivative. Let $x \in \mathcal{E} \setminus H$ be in a generalized eigendirection of f, and let (s_x, u_x) be its flag-type for f. Then*

- *either x is in the stable manifold of f and $s_x \neq +\infty$. Then f/H has an s_x-dimensional strong stable manifold, and the s_x-dimensional strong stable manifold of f is sent on it by π diffeomorphically.*
- *or x is in the unstable manifold of f and $u_x \neq +\infty$. Then f/H has a u_x-dimensional strong stable manifold, and the u_x-dimensional strong unstable manifold of f is sent on it by π diffeomorphically.*

Proof : Let $x \in \mathcal{E} \setminus H$ be in a generalized eigendirection of f and let $\lambda_x, \bar{\lambda}_x$ be the corresponding eigenvalue(s) (λ might be real). Since f has simple derivative, $|\lambda_x| \neq 1$, therefore x is either in the stable manifold or in the unstable. Suppose it is in the stable (replace f by f^{-1} for the other case). We recall that s_x was defined to be ∞ when x is not in any strong stable manifold. Then $s_x \neq \infty$. Besides, as $x \notin H$, x is not in $0_{\mathcal{E}}$ and $s_x > 0$.

Then we can write the sequence of the s_x first eigenvalues $\lambda_1, ..., \lambda_{s_x}$ of f, counted with multiplicity, and ordered by increasing moduli. Notice that $\lambda_{s_x} = \lambda_x$ or $\bar{\lambda}_x$. Since H is central and does not contain x, the eigenvalues of H are strictly weaker than λ_{s_x}. Therefore, the sequence $\lambda_1, ..., \lambda_{s_x}$ is also the sequence of the s_x first eigenvalues of f/H counted with multiplicity. This shows that f/H admits an s_x-dimensional strong stable manifold. We conclude with Lemma 6.3.8. □

We are now ready to prove Proposition 6.3.3. We put ourselves under its hypothesis and notations, we define $\Delta = \Gamma \setminus H$, $\Delta^s = (\Gamma \setminus H) \cap W^s(f)$, $\Delta^u = (\Gamma \setminus H) \cap W^u(f)$, and $\Lambda = \Delta^s \cup f^{-1}(\Delta^u)$. In the following, we denote $\eta_{\epsilon,|f|}$ simply by η. For all $k \in \mathbb{N}$, we define $g_k = \lambda^{-k}.g \circ \lambda^k.Id_{\mathcal{E}}$.

We will show the proposition in three steps: we will first show that there is a sequence Δ_k of finite sets such that (g_k, Δ_k) approximates the flag-configuration of (f, Δ). By Lemma 6.3.7, for any neighbourhood $\mathcal{W}$ of Λ, this will provide a sequence Φ_k of cyclic pertubations of $Id_{\mathcal{E}}$ on $\mathcal{W}$ that tends to $Id_{\mathcal{E}}$ in $\mathrm{Diff}^1(\mathcal{E})$ and such that, for k great enough, $\widehat{g}_k = g_k \circ \Phi_k$ firmly respects the flag-configuration of Δ for f. Then we will conjugate again $\widehat{g}_k$ by a homothety $\lambda^\ell.Id_{\mathcal{E}}$, to obtain a perturbation h on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$. Choosing k and ℓ great enough, such an h will satisfy the conclusions of the proposition.

Claim 10. *There is a sequence Δ_k of finite sets such that (g_k, Δ_k) approximates the flag-configuration of (f, Δ).*

Proof : Let $x \in \Delta^s$, then it is not in H, and by assumption it is in a generalized eigendirection of f. Let (s_x, u_x) be the flag type of x. Since $x \in W^s(f) \setminus 0_{\mathcal{E}}$, we have $s_x > 0$ and $u_x = 0$. From Corollary 6.3.9, $W^{s,s_x}(f)$ projects diffeomorphically by π on $W^{s,s_x}(f/H)$. The flag type of x/H for f/H is smaller than $(s_x, 0)$. Since g/H firmly respects the flag-configuration of Γ/H for f/H, by Remark 6.2.5, x/H is in the j-dimensional strong stable manifold of $g_k/H = \lambda^{-k}.g/H\lambda^k.Id_{\mathcal{E}/H}$, for some $0 < j \leq s_x$.

By Lemma 6.3.8, there exists a unique point x_0 in the j-dimensional strong stable manifold of $g_0 = g$ such that $x_0/H = x/H$. In particular, the flag-type of x_0 for g_0 is less or equal to the flag-type of x for f. We recall that λ is a multiple of the eigenvalue of x, thus we find n_x such that $f^{-n_x}(x) = \lambda.x$. We let

$$x_k = g_k^{-k.n_x}(\lambda^{-k}.x_0).$$

The flag-type of x_0 for g_0 is the flag type of $\lambda^{-k}.x_0$ for g_k, therefore it is the flag-type of $x_k = g_k^{-k.n_x}(\lambda^{-k}.x_0)$ for g_k.

There is a vector $v \in H$ such that $x_0 = x + v$. Since g firmly respects the flag-configuration of Γ, to the quotient by H, we have $g^{-n}(x_0) = f^{-n}(x_0) = f^{-n}(x) + f^{-n}(v)$. Since x is in a stable eigendirection of f, and v is a central bundle for f, the ratio $\frac{\|f^{-n}(v)\|}{\|f^{-n}(x)\|} = \frac{\|g^{-n}(x_0) - f^{-n}(x)\|}{\|f^{-n}(x)\|}$ goes to zero as n goes to $+\infty$. Since the family $\{g^k(x_0) - f^k(x_0)\}_{k \in \mathbb{N}}$ is bounded, and $\|f^{-n}(x)\|$ goes to ∞, we obtain that

$$\frac{\sup_{l \geq -n} \|g^l(x_0) - f^l(x)\|}{\|f^{-n}(x)\|}$$

goes to zero as n tends to $+\infty$. Then

$$\frac{\sup_{l \geq -k.n_x} \|g^l(x_0) - f^l(x)\|}{\|f^{-k.n_x}(x)\|} = \sup_{l \geq -k.n_x} \|\lambda^{-k}.g^l(x_0) - \lambda^{-k}.f^l(x)\| / \|x\|$$

goes to zero as k goes to $+\infty$. Notice that the set $\{\lambda^{-k}.f^l(x), l \geq -k.n_x\}$ is the set of the positive iterates of x by f. Besides $\lambda^{-k}.g^l(x_0) = g_k^l(\lambda^{-k}.x_0)$, therefore $\{\lambda^{-k}.g^l(x_0), l \geq -k.n_x\}$ is the set of the positive iterates of x_k by g_k. Thus we just have shown that the sequence $(g_k, \{x_k\})$ approximates the flag-configuration of $(f, \{x\})$.

We do this for any $x \in \Delta^s$, and symmetrically for any $x \in \Delta^u$: we found a sequence Δ_k of finite sets such that (g_k, Δ_k) approximates the flag-configuration of (f, Δ). This ends the proof of the claim. $\square$

Claim 11. *For any bounded set $\mathcal{B}$, for any neighbourhood $\mathcal{W}$ of Λ, we can build an $\eta_{\epsilon,|f|}$-perturbation g' of f that firmly respects the flag-configuration of Γ for f, and such that some restriction $g'_{|\mathcal{B}'}$ of g' is conjugate to $g_{|\mathcal{B}}$ by a homothety.*

Proof : Let $\mathcal{B}$ be a bounded set in $\mathcal{E}$. Let $\mathcal{W}$ be a closed neighbourhood of Λ that does not intersect H, nor any $f^{-n}(\Delta^s) + H$, for $n \geq 1$, nor any $f^n(\Delta^u) + H$, for $n \geq 0$.

From the previous claim, we find a sequence Δ_k of sets such that the sequence (g_k, Δ_k) approximates the flag-configuration of (f, Δ). Apply Lemma 6.3.7, and obtain a sequence Φ_k of cyclic perturbations of $Id_{\mathcal{E}}$ on $\mathcal{W}$, that tends to $Id_{\mathcal{E}}$ in $\mathrm{Diff}^1(\mathcal{E})$, and such that for k great enough $g'_k = g_k \circ \Phi_k$ respects the flag-configuration of Δ for f. It coincides with g_k on $\mathcal{B}'$, thus by restriction to $\mathcal{B}'$, g'_k is conjugate to $g_{|\mathcal{B}}$ by a homothety of $\mathcal{E}$. We write that Φ_k is a ν_k-perturbation of $Id_{\mathcal{E}}$, where $\nu_k \to 0$.

We recall that g is an $\eta_{\epsilon,|f|}$-perturbation of f, therefore it is an η'-perturbation of f for some $0 < \eta' < eta_{\epsilon,|f|}$. From Lemma 5.2.7, g'_k is a $[\eta' + \nu_k(|f| + \epsilon)]$-perturbation of f. Thus for k great enough, it is an $\eta_{\epsilon,|f|}$-perturbation of f that respects the flag-configuration of Δ for f. Since $\mathcal{W}$ is closed, and does not intersect $0_{\mathcal{E}}$, for k great enough, $\mathcal{B}' = \lambda^{-k}(\mathcal{B})$ does not intersect $\mathcal{W}$ and $g'_k = g_k$ on $\mathcal{B}'$.

There is an integer $k_1 \in \mathbb{N}$ such that all the properties we just stated are satisfied. That is, g'_{k_1} is an $\eta_{\epsilon,|f|}$-perturbation of f that respects the flag-configuration of Δ for f, and $g'_{k_1|\mathcal{B}'} = g_{k_1|\mathcal{B}'}$ is conjugate to $g_{|\mathcal{B}}$ by the homothety $\lambda^{k_1}.Id_{\mathcal{E}}$.

Since g firmly respects the flag-configuration of Δ for f, to the quotient by H, g_{k_1} also does (Remark 6.2.5), and since $\mathcal{W}$ does not intersect any $f^{-n}(\Delta^s) + H$ for $n \geq 1$, nor any $f^n(\Delta^u) + H$ for $n \geq 0$, g'_{k_1} also firmly respects the flag-configuration of Δ for f. For $\mathcal{W}$ does not intersect H, and g firmly respects the flag-configuration of $\Gamma_{|H}$, g_{k_1} and g'_{k_1} also firmly respects the flag-configuration of $\Gamma_{|H}$ for f. Take $g' = g'_{k_1}$: we proved the claim. $\square$

Proof of Proposition 6.3.3 : Let g' be an $\eta_{\epsilon,|f|}$-perturbation of f as we found in the previous claim. Let $\mathcal{U}$ be an open neighbourhood of $0_{\mathcal{E}}$ that contains the positive orbit of Γ^s, the negative one of Γ^u and $\mathcal{B}'$. Then by Lemma 5.2.16, there is a cylic ϵ-perturbation h of f that coincides with g' on $\mathcal{U}$, and with f outside a bounded set $\mathcal{V}$. Precisely h writes as $\phi.g' + \psi.f$

where $\phi + \psi$ is a partition unit on $\mathcal{E}$. Since $h = g'$ on $\mathcal{U}$, h respects the flag-configuration of Γ for g', thus for f, and the restriction $h_{|\mathcal{B}'} = g'_{|\mathcal{B}'}$ of h is conjugate to $g_{|\mathcal{B}}$ by a homothety.

Since $h = \phi.g' + \psi.f$ and g', f both firmly respect the flag-configuration of Γ for f,the diffeomorphism h also firmly repects the flag-configuration of Γ for f. Thus we can conjugate h by another homothety, and obtain a cyclic ϵ-perturbation on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that respects the flag-configuration of Γ, and such that it is conjugate to $g_{|\mathcal{B}}$ by a homothety of $\mathcal{E}$, by restriction to some bounded set. This ends the proof of the proposition.
□

6.4 Reduction Lemmas and Proposition

In this section, we will show the Reduction Proposition which will allow to reduce the proof of Theorem 6.1.5 to the case of a linear saddle whose eigenvalues are real, have pairwise distinct moduli, a common multiple, and are huge (defined in this section). Moreover, we will be allowed to suppose that the set Γ is in the union of the eigendirections of the linear saddle. The proof of that proposition (6.4.5) will be split into four technical lemmas: Simplification, Linearization, Shifting to Real Eigenvalues, and Getting Huge Eigenvalues. The proofs of the three last lemmas being done in the three next sections. We first state a lemma that allows to reduce the proof to the case of cyclic diffeomorphisms with simple derivatives.

Lemma 6.4.1 (Simplification). *Let f be a cycle diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ and Γ a finite subset, then there exists an arbitrarily small perturbation of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$ that has a simple derivative, with a common multiple $\lambda > 1$, and that respects the flag-configuration of Γ. Moreover, if f has real eigenvalues, then the perturbation can be chosen to have also real eigenvalues.*

Proof : By Remark 6.2.4, there is a sequence $\mathcal{A}_k$ of linear cocycles on $T\mathcal{E}_{0_{\mathcal{E}}}$ that converges to the derivative of f on $0_{\mathcal{E}}$, and such that for all k, $\mathcal{A}_k^p$ is simple and its eigenvalues admit a common multiple λ_k. We write that $\mathcal{A}_k$ is η_k-close to the derivative $df_{0_{\mathcal{E}}}$, where η_k is a sequence that tends to zero. Then, by Franks' Lemma 5.2.13, there is a sequence $\mathcal{U}_k$ of neighbourhoods of $0_{\mathcal{E}}$ that tends to $0_{\mathcal{E}}$, a sequence ϵ_k that tends to zero, and a sequence g_k of saddle diffeomorphisms, such that for any k, g_k is a saddle ϵ_k-perturbation of f on U_k, and its derivative along $0_{\mathcal{E}}$ is $\mathcal{A}_k$. Therefore, all the hypothesis of Proposition 6.3.1 are satisfied.

We apply it: for any $\epsilon > 0$, there is a saddle local ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$ that respects the flag-configuration of Γ and that has coincides with some g_k on a neighbourhood of $0_{\mathcal{E}}$, hence it has a simple derivative. □

We then state a version of linearization Lemma 5.2.12 that turns a cyclic diffeomorphism into a locally linear one, by a perturbation that respects the flag-configuration of a finite set Γ. We remind the reader of Remark 6.1.6, and suppose that $\Gamma = \Gamma^u \cup \Gamma^s$ is made of points of the stable and unstable manifolds. Then it can be required that the generalized eigenmanifolds of the perturbated cyclic diffeomorphism contain Γ. It will be proved in Section 6.4.1.

Lemma 6.4.2 (Linearization). *Let f be a cyclic diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ that has a simple derivative. Then, for all $\epsilon > 0$, for any open neighbourhood $\mathcal{U}$ of the zero section $0_{\mathcal{E}} = \{1, ..., p\} \times \{0\}$, and for any finite set $\Gamma = \Gamma^u \cup \Gamma^s$, there exists a cyclic local ϵ-perturbation g of f on $\mathcal{U}$ that is locally linear and such that:*

- *the derivative of g equals that of f along the zero bundle $0_{\mathcal{E}}$,*
- *it respects the flag-configuration of Γ,*
- *each point of Γ is in a generalized eigenmanifold of g.*

The next lemma is a key to reduce to cyclic diffeomorphisms whose derivative have real eigenvalues. Provided the period is great enough, one finds a perturbation of a linear saddle that turns the eigenvalues to be real, and that respects the flag-configuration of a finite set.

Lemma 6.4.3 (Shifting to Real Eigenvalues). *Fix real numbers $\epsilon > 0$, $A > 0$, and an integer d. There exists an integer P such that, for any integer $p > P$, the following holds:*

- *for any linear cyclic diffeomorphism f on the bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, with simple derivative, bounded by A, and such that its eigenvalues admit a common multiple $\lambda > 1$,*
- *for any finite set Γ of points in the generalized eigenmanifolds of f,*

we can find a cyclic ϵ-perturbation g of f on an arbitrarily small neighbourhood of $O_{\mathcal{E}}$, that has real eigenvalues, that preserves the moduli of the eigenvalues, and respects the flag-configuration of Γ.

When we apply this lemma, we will not use the fact that the perturbation g of f preserves the moduli of the eigenvalues, therefore such a slightly weaker statement would be sufficient. However, we prove it by induction on the dimension d of the bundle $\mathcal{E}$, and moduli preservation is needed as an induction hypothesis. The proof of Lemma 6.4.3 will be tackled in section 6.4.2.

Finally, we state a lemma that allows to restrict our study to saddle diffeomorphisms whose derivatives have *huge* eigenvalues, that is, any eigenvalue in modulus is either stricity less than $1/2$ or strictly greater than 2.

Lemma 6.4.4 (Getting Huge Eigenvalues). *Fix real numbers $\epsilon > 0$ and $A > 0$, and an integer d. There exists an integer P such that, for any integer $p > P$, the following holds:*

- *for any linear saddle diffeomorphism f on the bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, with simple derivative, bounded by A, with real eigenvalues that admit a common multiple $\lambda > 1$,*
- *for any finite set Γ of points in the generalized eigenspaces of f,*

we can find a saddle ϵ-perturbation of f on an arbitrarily small neighbourhood of $O_{\mathcal{E}}$, that has real strong eigenvalues, and respects the flag-configuration of Γ.

We will show it in section 6.4.3. These three lemmas may be synthetized into a single result:

Proposition 6.4.5 (Reduction Proposition). *Fix $\epsilon > 0$ and $A > 0$ and an integer $d > 0$. Then there is an integer $P > 0$ such that, for any $p > P$, the following holds:*

if a saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ is bounded by A, then for any finite set Γ in the union of the stable and unstable manifolds of f, there exists a locally linear saddle ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$,

- *that has real huge eigenvalues with a common multiple $\lambda > 1$ and pairwise distinct moduli,*
- *that respects the flag-configuration of Γ, in such a way that Γ is contained in the generalized eigenmanifolds of g.*

This proposition allows to restrict the proof of Theorem 6.1.5 to very particular cases, namely, linear saddles that match the two items of the conclusion. We do not give a precise proof, but it can be summarized this way:

first apply Lemma 6.4.1 to obtain a simple derivative with a common multiple. Then, linearize with Lemma 6.4.2, and switch to real eigenvalues by Lemma 6.4.3. Use once more Lemma 6.4.2 to linearize locally thus being able to use Lemma 6.4.4 to get real huge eigenvalues. A priori, the derivative is not anymore simple: apply again Lemma 6.4.1 to obtain a simple derivative with a common multiple; thus the eigenvalues are real, with pairwise distinct moduli. This last perturbation can be chosen arbitrarily small, so that the eigenvalues are still huge. Finally we apply Lemma 6.4.2 a third and last time to obtain a locally linear saddle, and to push Γ into the generalized eigendirections.

Of course, all the seven successive perturbations respect the flag-configuration of Γ.

6.4.1 Proof of the Linearization Lemma

We may first show the following partial result:

Lemma 6.4.6. *Let f be a cyclic diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$. Then, for all $\epsilon > 0$, for any open neighbourhood $\mathcal{U}$ of the zero section $0_\mathcal{E} = \{1, ..., p\} \times \{0\}$, and for any finite set $\Gamma = \Gamma^u \cup \Gamma^s$, there exists a cyclic local ϵ-perturbation g of f on $\mathcal{U}$ that is locally linear and such that:*

- *the derivative of g equals that of f along the zero bundle $0_\mathcal{E}$,*
- *it respects the flag-configuration of Γ.*

Proof : By Lemma 5.2.12, we can find a sequence $\epsilon_k > 0$ tending to zero, a sequence $\mathcal{U}_k$ of neighbourhoods of $0_\mathcal{E}$ tending to $0_\mathcal{E}$, and for any k, a saddle local ϵ_k-perturbation g_k of f on $\mathcal{U}_k$ that is locally linear, and whose derivative along $0_\mathcal{E}$ coincides with that of f. Since the strong stable and unstable manifolds vary continuously on compact sets by C^1 perturbation, those of g_k converge to those of f on the compacts. Hence we can apply Proposition 6.3.1 to find, for all $\epsilon > 0$, for all neighbourhood $\mathcal{U}$ of $0_\mathcal{E}$, an ϵ-perturbation g of f on $\mathcal{U}$, that coincides with some g_k on a neighbourhood of $0_\mathcal{E}$, and that respects the flag-configuration of Γ. Hence g satisfies the two items of the conclusion of the lemma. □

We now are ready for the proof of the Linearization Lemma:

Proof of Lemma 6.4.2 : We apply Lemma 6.4.6 to find an arbitrarily small, arbitrarily local, cyclic perturbation g of f that has same derivative, that is locally linear, and respects the flag-configuration of Γ. Let $\mathcal{E} = E_{\lambda_1} \oplus ... \oplus E_{\lambda_l}$ be the good splitting for the locally linear g.

Fix now $x \in \Gamma^s \setminus 0_\mathcal{E}$ and let (u_x, s_x) be its flag-type, that is x is in the s_x-dimensional strong stable manifold, with $s_x > 0$. Since g is locally linear, that strong stable manifold coincides locally with $E_{\lambda_1} \oplus ... \oplus E_{\lambda_i}$, for some $1 \leq i \leq l$. If $i = 1$, it coincides locally with the generalized eigendirection E_{λ_1}, thus with the generalized eigenmanifold W_{λ_1}. Therefore the s_x-dimensional strong stable manifold is W_{λ_1}, by definition, and x is already in a generalized eigenmanifold.

If that manifold is tangent at $0_\mathcal{E}$ to the bundle E_{λ_1}, it is in the generalized eigenmanifold W_{λ_1}. Thus x is in a generalized eigenmanifold. If not, the d_2-dimensional strong stable manifold is tangent at $0_\mathcal{E}$ to $E_{\lambda_1} \oplus ... \oplus E_{\lambda_i} \oplus E_{\lambda_{i+1}}$, for some $i \geq 1$.

We will associate to x a point x' in the following way: Proposition 6.2.2 provides a strong stable foliation $\mathcal{F}$ on the s_x-dimensional strong stable manifold, such that the leaf that goes through x, intersects the generalized eigenmanifold $W_{\lambda_{i+1}}$ at a unique point x'. Notice that x' is not in $0_\mathcal{E}$, otherwise x would be in a stronger stable manifold than the s_x-dimensional, which contradicts that (u_x, s_x) is the flag-type of x. Still from the proposition, we have

$$\mathrm{dist}(g^{pn}(x), g^{pn}(x')) = \mathcal{O}(|\lambda_i|^n) \tag{6.6}$$
$$\mathrm{dist}(0_{\mathcal{E}}, g^{pn}(x')) = C.|\lambda_{i+1}|^n, \tag{6.7}$$

for some constant C, and all n great enough. For all $k \leq 0$, We denote by x_n (resp. x'_n) the n-th iterate of x (resp. x') by g. We may perturb slightly g, preserving the derivative of g along $0_{\mathcal{E}}$ and respecting the flag-configuration of Γ, to get that if $x \neq y$ are two points of Γ, then, for all integers $n, m \geq 0$, we have $x'_n \neq y'_m$.

Let $\Lambda^s = 0_{\mathcal{E}} \cup \{x_n, x \in \Gamma, n \in \mathbb{N}\} \cup \{x'_n, x \in \Gamma, n \in \mathbb{N}\}$. We define symmetrically the set Λ^u, and call Λ the union of these two sets. Then, thanks to equalities (6.6) and (6.7), for any $x \in \Gamma$, the ratio

$$\frac{\mathrm{dist}(x_n, x'_n)}{\mathrm{dist}(x'_n, \Lambda \setminus \{x_n, x'_n\})}$$

tends to zero as n tends to infinity. Define $\Lambda_n = \cup_{x \in \Gamma}\{x_n, x'_n\}$. Then, for n great enough, we can apply Lemma 6.3.5 to each pair $\{x_n, x'_n\}$. Thus we find a small perturbation Φ of $Id_{\mathcal{E}}$ on a small neighbourhood of Λ_n, such that

- for all $x \in \Gamma^s$, for all $m \neq n$, we have $\Phi(x_n) = x'_n$, $\Phi(x_m) = x_m$, $\Phi(x'_m) = x'_m$,
- for all $x \in \Gamma^u$, for all $m \neq n$, we have $\Phi(x'_n) = x_n$, $\Phi(x_m) = x_m$, $\Phi(x'_m) = x'_m$,
- the diffeomorphism Φ is equal to the identity map on a small neighbourhood of $0_{\mathcal{E}}$.

In other words, Φ pushes the points of Γ in the generalized eigenmanifolds of g. The composition $g \circ \Phi$ is a locally linear small perturbation of g that has same derivative as g, the points of Γ are in its generalized eigenmanifolds, and it respects the flag-configuration of g (this last point comes from the fact that x_n and x'_n have same flag-type for g).

Hence $g \circ \Phi$ is a small perturbation of f that satisfies the three items of the conclusions of Lemma 6.4.2. This concludes the proof of the lemma. □

6.4.2 Shifting to real eigenvalues: proof of Lemma 6.4.3

We will prove it by induction on the dimension d of $\mathcal{E}$. We start at dimension $d = 2$. We first reformulate [10, lemme 6.6]. In the following R_{β_i} denotes the rotation of angle $\beta_i > 0$ on the fibre $\{i\} \times \mathbb{R}^2$ of $\mathcal{E}_p = \{1, ..., p\} \times \mathbb{R}^2$.

Lemma 6.4.7 (Bonatti, Crovisier). *For any $\varepsilon > 0$, there is an integer P, such that for any $p \geq P$, and any linear cocycle f of $\mathcal{E}_p$, there is a sequence $\beta = \beta_1, ..., \beta_p$ with $-\varepsilon \leq \beta_i \leq \varepsilon$, that satisfies the following:*

If we denote by Φ_β the diffeomorphism that coincides with R_{β_i} on any fibre $\{i\} \times \mathbb{R}^2$, then the linear cyclic diffeomorphism $f \circ \Phi_\beta$ has real eigenvalues.

In particular, if f has not real eigenvalues, let $t_0 \in [0,1]$ be the smallest such that $f \circ \Phi_{t_0\beta}$ has real eigenvalues, where $t_0\beta$ is the sequence $t_0\beta_1, ..., t_0\beta_p$. Then $f \circ \Phi_{t_0\beta}$ has a real double eigenvalue, and $f \circ \Phi_{t\beta}$, for any $t \in [0, t_0[$ has complex eigenvalues; the moduli of these eigenvalues are equal to the modulus of the eigenvalues of f, by preservation of the determinant. Notice finally that the path of cocycles defined by $(f \circ \Phi_{t\beta})_{t\in[0,t_0]}$ has radius strictly less than $\varepsilon|f| \leq \varepsilon.A$.

Proof of Lemma 6.4.3 in dimension 2 : Let $\varepsilon = \epsilon/A$. In dimension $d = 2$, if the eigenvalues are not real, the linear cyclic diffeomorphism is either a sink or a source. We assume it is a sink, the source case is symmetrical. There is only one generalized eigenmanifold, the 2-dimensional unstable manifold: the flag-type of each point of Γ is 2. By Lemma 6.4.7 and the comment that follows, for A fixed, if p is great enough, we find a path $\mathcal{A}_{t\in[0,1]}$ of length $\varepsilon.A = \epsilon$ in the set of cocycles above $0_\mathcal{E}$ such that $\mathcal{A}_0 = df_{0_\mathcal{E}}$ and $\mathcal{A}_1$ has a real double eigenvalue. Finally, for all $t \in [0,1]$ the eigenvalues of $\mathcal{A}_t$ have same modulus as the eigenvalues of $\mathcal{A}_0$, that is, strictly less than 1.

Thus, the $\mathcal{A}_t$ are all sinks. We can apply Proposition 5.2.14, and obtain a cyclic ϵ-perturbation of f that has a real double eigenvalue, with the same modulus as the initial complex eigenvalues, and such that its stable manifold is the whole $\mathcal{E}$. The flag-type of each point of Γ is either 1 or 2: the flag-configuration is respected. □

Now $d \geq 3$ and we assume that Lemma 6.4.3 is shown for any dimension $d' < d$.

Claim 12. *For all $\epsilon > 0$, $A > 0$, there is an integer P such that, for any integer $p > P$,*

- *for any linear cyclic diffeomorphism f on the bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, with simple derivative, bounded by A, and such that its eigenvalues admit a common multiple $\lambda > 1$,*
- *for any finite set Γ of points in the generalized eigenspaces of f,*

we can find an ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_\mathcal{E}$ that respects the flag-configuration of Γ, that preserves the one (if real) or two (if complex) smallest eigenvalues in modulus, and that turns the other eigenvalues to be real, preserving their moduli.

Proof : Let $\epsilon > 0$ and $A > 0$. Let $\eta = \eta_{\epsilon,A}$ as defined for Lemma 5.2.16 and let $\nu_{\eta,A}$ be as defined for Corollary 5.2.18. From the induction hypothesis, there exists an integer P such that, for any integers $p > P$ and $0 < d' < d$, for any linear cyclic diffeomorphism f' on $\mathcal{E}' = \{1, ..., p\} \times \mathbb{R}^{d'}$ with simple derivative and bounded by A, for any finite set Γ' of points in the generalized eigendirections of f', there is a $\nu_{\eta,A}$-perturbation g' of f' on an arbitrarily small neighbourhood of $0_{\mathcal{E}'}$, that has real eigenvalues, that preserves the moduli of the eigenvalues on the orbit $0_{\mathcal{E}'}$, and that respects the flag-configuration of Γ'.

Let f be a linear cyclic diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, with simple derivative, bounded by A, and such that its eigenvalues admit a common multiple $\lambda > 1$. Let $\mathcal{E} = E_{\lambda_1} \oplus ... \oplus E_{\lambda_l}$ be the good splitting associated to the simple linear automorphism f and let Γ be a finite set in $E_{\lambda_1} \cup ... \cup E_{\lambda_l}$. Let H be the invariant subbundle $E_{\lambda_2} \cup ... \cup E_{\lambda_l}$.

Applying the induction hypothesis, we find a local $\nu_{\eta,A}$-perturbation $\widehat{g}'$ of $f_{|H}$ on an arbitrarily small neighbourhood $\mathcal{V}$ of 0_H, that preserves the moduli of the eigenvalues along the periodic orbit 0_H, that has all eigenvalues real, and that respects the flag-configuration of $\Gamma_{|H}$. Choosing $\mathcal{V}$ small enough, we may suppose that $\widehat{g}'$ firmly respects the flag-configuration of $\Gamma_{|H}$.

By definition, since $|f| \leq A$, we have $\nu_{\eta,A} \leq \nu_{\eta,|f|}$; then by Corollary 5.2.18 there exists a local η-perturbation $\widehat{g}$ of f that extends $\widehat{g}'$, and such that $\widehat{g}/H = f/H$. Thus, the perturbation $\widehat{g}$ of f preserves the moduli of the eigenvalues, and turns all the eigenvalues to be real, but possibly the two smallest in modulus. Since $\widehat{g}$ is a local perturbation of f and $\widehat{g}/H = f/H$ respects the flag-configuration of Γ/H for f/H, for some integer k great enough, $\widehat{g}_k = \lambda^{-k}.\widehat{g} \circ \lambda^k.Id_{\mathcal{E}}$ is an $(\eta = \eta_{A,\epsilon})$-perturbation of f that firmly respects the flag-configuration of $\Gamma \setminus H$ to the quotient by H. We recall that since $|f| \leq A$, we have $\eta_{\epsilon,A} \leq \eta_{\epsilon,|f|} \leq \epsilon$.

To avoid double indexing, let $g = \widehat{g}_k$, and $g' = \widehat{g}_{k|H}$. Although we know that g respects the flag-configuration of $\Gamma \cap H$ for f, it does a priori not respect that of $\Gamma \setminus H$. Two cases may occur:

- Either $|\lambda_1| > 1$. Then any $x \in \mathcal{E} \setminus H$ is in the d-dimensional unstable manifold (and in no other strong unstable manifold), therefore has flag-type $(0, d)$ for f. Its flag-type can only be decreased by $\widehat{g}_k$. Hence $\widehat{g}_k$ respects the flag-configuration of Γ for f, which concludes the study of this case.

- Or $|\lambda_1| < 1$. Then H is a central bundle for f (see Definition 6.3.2) and for g. Trivially, $\widehat{g}_{k|H} = f_{|H}$ firmly respects the flag-configuration of $\Gamma_{|H}$ for $f_{|H}$. Finally, by assumption, Γ is in the union of the generalized eigendirections of f. All the hypothesis of Proposition 6.3.3 are satisfied: for any bounded neighbourhood $\mathcal{B}$ of $0_{\mathcal{E}}$, there is an ϵ-perturbation h of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that respects the flag-configuration of Γ, and is locally conjugate to $\widehat{g}_{k\mathcal{B}}$ by an homothety $\gamma.Id_{\mathcal{E}}$. In particular $dh_{0_{\mathcal{E}}} = d\widehat{g}_{k0_{\mathcal{E}}} = d\widehat{g}_{0_{\mathcal{E}}}$, thus the perturbation h of f preserves the moduli of the eigenvalues counted with multiplicity, and turns all the eigenvalues to be real, but possibly the two smallest in modulus.

Hence we are done for both cases. This ends the proof of the claim. □

Changing f into f^{-1} and g into g^{-1}, we get of course the symmetrical claim:

Claim 13. *For all $\epsilon > 0$ and $A > 0$, there is an integer P such that, for any integer $p > P$,*

- *for any linear cyclic diffeomorphism f on a bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, with simple derivative, bounded by A, and such that its eigenvalues admit a common multiple $\lambda > 1$,*
- *for any finite set Γ of points in the generalized eigenspaces of f,*

we can find a ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$ that respects the flag-configuration of Γ, that preserves the one (if real) or two (if complex) greatest eigenvalues in modulus, and that turns the other eigenvalues to be real preserving their moduli.

We show shortly how, applying these two claims one after the other, we get all eigenvalues real:

Proof of Lemma 6.4.3 : Apply Claim 12 to perturb f into g. Notice that if $\lambda > 1$ is a common multiple for the eigenvalues of f, it is a multiple of the one or two smallest eigenvalues of g (which are preserved). Since g has preserved the moduli of the other eigenvalues, they all are of the form $\pm|\mu|$ where μ is an eigenvalue of f. We have then $\mu^k = \lambda = |\mu|^k$, for some $k \in \mathbb{Z}$, since λ is a real positive number. Thus λ is again a common multiple for the eigenvalues of g. We apply Lemma 6.4.2 to linearize locally and to push Γ in the generalized eigenspaces, while preserving the derivative. This cyclic perturbation coincides locally with a linear cyclic diffeomorphism to which we can apply Claim 13, and turn the possibly remaining two complex eigenvalues into real ones. □

6.4.3 Getting huge eigenvalues: proof of Lemma 6.4.4

Definition 6.4.8. We say that an eigenvalue λ of a saddle cycle f, is a *huge stable* eigenvalue if it has modulus less than $1/2$. We say that it is a *huge unstable* eigenvalue if it has modulus greater than 2.

We naturally can split the proof of Lemma 6.4.4, first obtaining huge stable eigenvalues with a first perturbation, and then huge unstable eigenvalues with a second one.

Lemma 6.4.9. *Fix real numbers $\epsilon > 0$ and $A > 0$, and an integer d. There exists an integer P such that, for any integer $p > P$, the following holds:*

- *for any linear cyclic diffeomorphism f on the bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, bounded by A, with simple derivative and real eigenvalues that admit a common multiple $\lambda > 1$,*
- *for any finite set Γ of points in the eigendirections of f,*

we can find a cyclic ϵ-perturbation of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that has real huge unstable eigenvalues, preserves the stable eigenvalues, and respects the flag-configuration of Γ.

We have of course the symmetrical lemma swapping "stable" and "unstable", which is a straightforward consequence of Lemma 6.4.9, changing f into f^{-1}. Lemma 6.4.4 is easily obtained applying successively these two lemmas.

We are left to prove Proposition 6.4.9: in short terms, we do a perturbation on the quotient f/F of f by its stable bundle F, that respects the flag-configuration of Γ/F and turns the (unstable) eigenvalues to be huge. Then we lift it into a cyclic diffeomorphism of $\mathcal{E}$ that does not *a priori* respect the flag-configuration of Γ. We finally use Proposition 6.3.3 to obtain a cyclic perturbation of f that respects the flag-configuration of Γ, that has the same stable eigenvalues as f and has huge unstable eigenvalues.

Let us first give without a proof a particular case of Franks' Lemma:

Lemma 6.4.10. *For all $\nu > 0$, $A > 0$, there exists $\kappa > 1$ such that, for any bundle $\mathcal{F}$ of the form $\{1, ..., p\} \times \mathbb{R}^d$, for any linear saddle f on $\mathcal{F}$ that is bounded by A, there exists a local ν-perturbation of f that coincides with $f_\kappa = f \circ \kappa.Id_{\mathcal{E}}$ on a neighbourhood of $0_{\mathcal{F}}$ and writes as $\Phi.f + \Psi.f_\kappa$ where $1 = \Phi + \Psi$ is a unit partition.*

Proof of Lemma 6.4.9 : Let $\epsilon > 0$, $A > 0$, and fix an integer $d > 0$. Let $\nu > 0$ such that $2A^2\nu < \min(\eta_{\epsilon,A}, \alpha/A)$. Choose $\kappa > 1$ corresponding to the previous lemma, with respect to ν and A. Choose an integer $P > \frac{\ln(2)}{\ln(\kappa)}$, and let $p > P$. Let f be a linear cylic diffeomorphism on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$, bounded by A, and Γ a finite set that satisfy the assumptions of Lemma 6.4.9. Let $\mathcal{E} = F \oplus G$ be the hyperbolic splitting for f (it exists, as f has a simple derivative). The cyclic diffeomorphism f/F on $\mathcal{E}/F$ has eigenvalues with moduli greater than 1, and is bounded by A.

From Lemma 6.4.10, we get a local ν-perturbation g' of f/F on a bounded set $\mathcal{U}$, that writes as $\Phi.f/F + \Psi.f/F \circ \kappa.Id_{\mathcal{E}/F}$, and that coincides with $f/F \circ \kappa.Id_{\mathcal{E}/F}$ on a neighbourhood of $0_{\mathcal{E}/F}$. Clearly, g' has same strong unstable manifolds as f/F and its derivative has huge eigenvalues, since $\kappa^p > 2$. Thus g' respects the flag-configuration of Γ/G. Thus, for all $\theta > 1$, $g'_\theta = \theta^{-1} g' \circ \theta.Id_{\mathcal{E}/F}$ has same strong unstable manifolds as f/F, and respects the flag-configuration of Γ/F for f/F. Since $\eta^p > 2$, the eigenvalues of g', therefore those of g'_θ, have moduli all greater than 2.

For g' is a local perturbation of f/H, we can choose $\theta > 1$ great, so that for all $n \in \mathbb{N}$ we have $g'^n_\theta = f/H^n$ on Γ^u/H and $g'^{-n}_\theta = f/H^{-n}$ on Γ^s/H. Notice that $\nu < \alpha/A \leq \alpha/|f|$. Then by Lemma 5.2.19, there exists a cyclic $2|f|^2\nu$-perturbation g of f that is a lift of g'_θ. By Remark 5.2.20, g can be chosen such that for all $n \in \mathbb{N}$ we have $g^n = f^n$ by restriction to $\Gamma^u + H$, $g^{-n} = f^{-n}$ by restriction to $\Gamma^s + H$, and $g_{|H} = f_{|H}$. Thus g firmly respects the flag-configuration of Γ for f to the quotient by H, and $g_{|H}$ trivially firmly respects the flag-configuration of $\Gamma_{|H}$ for $f_{|H}$.

We recall that ν was chosen so that $2|f|^2\nu \leq 2A^2\nu < \eta_{\epsilon,|f|}$ and that the bundle F is a central bundle for f. Hence we can apply Proposition 6.3.3 and find an ϵ-perturbation of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that respects the flag-configuration of Γ, and that is locally conjugate to g by a homothety. Such a perturbation has strong unstable eigenvalues, and has same stable eigenvalues as f, since $g_{|F} = f_{|F}$. This ends the proof of the lemma. □

6.5 Proof of Theorem 6.1.5

The first idea of the author to prove Theorem 6.1.5 was to reduce it to a theorem that would be a counterpart of Proposition 5.3.4, with flag-configuration preservation. This would be

Fix $\epsilon > 0$, $A > 0$ and an integer $d > 0$. Then there is an integer $N_d > 0$ such that, for any linear saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$,

- *that is bounded by A,*
- *that has real eigenvalues with pairwise distinct moduli,*
- *that is not N_d-dominated,*

For any finite set Γ of points of the eigendirections of f, there exists a saddle ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that has a homoclinic tangency and that respects the flag-configuration of Γ.

However, in the proof of Proposition 5.3.4, Remark 5.3.6 is fundamental: we need conjugacy by $\lambda.Id$ to respect flag-configuration, which is usually not the case. This works when λ is a common multiple of the eigenvalues of f. Besides, adapting step by step the proof of Proposition 5.3.4, we have to make sure that we can extend, or lift the perturbation (see section 5.3.2), respecting the flag-configuration of Γ. To achieve the retriction case, we can state a useful

Remark 6.5.1. *Let f be a linear saddle on $\mathcal{E} = F \oplus G$, where F is the stable bundle and G the unstable one. Let $F = F_1 \oplus H$ be a dominated splitting for $f_{|F}$. If g' is a saddle ϵ-perturbation of the restriction f' of f to $E' = H \oplus G$ and respects the flag-configuration of a finite set $\Gamma \subset E'$, then any saddle diffeomorphism g that extends g' to $\mathcal{E}$ respects the flag-configuration of Γ for f.*

However, it turned out in the quotient case, that the perturbation made on the quotient f/H, might change the order of the eigenvalues of f in such a way that the flag-type of some points of Γ would be irrecoverably increased. Indeed, our induction proof still relies on Pujals and Sambarino's one on dimension 2; the perturbation they do does not preserve the moduli of the eigenvalues. We will show in section 6.5.1 that the proof of Pujals and Sambarino can be done preserving the moduli of the eigenvalues, and respecting flag-configuration, if we assume that the eigenvalues are strong enough. This is why the counterpart of Proposition 5.3.4 in the flag-configuration respecting case, has to be stated as follows:

Proposition 6.5.2. *Fix $\epsilon > 0$ and $A > 0$ and an integer $d > 0$. Then there is an integer $N_d > 0$ such that, for any linear saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$,*

- *that is bounded by A,*
- *that has huge real eigenvalues with a common multiple $\lambda > 1$ and pairwise distinct moduli,*
- *that is not N_d-dominated,*

For any finite set Γ of points of the eigendirections of f, there exists a saddle ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that has a homoclinic tangency, that respects the flag-configuration of Γ, and that preserves the moduli of the eigenvalues.

This will be showed by induction on the dimension d of $\mathcal{E}$ in section 6.5.2, and for the sake of respecting flag-configuration, we need to enclose preservation of the eigenvalues in the induction process. This motivates the seemingly overly precise conclusion. The proof of that proposition is the core of the paper, and the main part of this section.

Proof of Theorem 6.1.5 from Propositions 6.4.5 and 6.5.2 : This works exactly the same way as the proof of Theorem 5.3.1 from Proposition 5.3.4 and Proposition 5.3.2: apply first the Reduction Proposition 6.4.5 and use Remark 5.3.9 to perturb a saddle diffeomorphism f into g that coincides locally with a linear saddle diffeomorphism $\tilde{g}$ satisfying all the hypothesis of Proposition 6.5.2. □

6.5.1 Proof of Proposition 6.5.2 in dimension 2

Let us reformulate the proposition in the particular case of dimension 2:

Theorem 6.5.3. *Fix $\epsilon > 0$ and $A > 0$. Then are integers $P, N_2 > 0$ such that for any $p > P$, for any linear saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^2$*

- *that is bounded by A,*
- *such that the two huge eigenvalues $|\lambda| < 1/2$ and $|\mu| > 2$ have a common multiple $\lambda > 1$,*
- *that is not N_2-dominated,*

For any finite set Γ of points of the eigendirections of f, there exists a saddle ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and respects the flag-configuration of Γ.

We will split the proof into two steps, as Pujals and Sambarino did, we prove that there exists a perturbation on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that respects the flag-configuration of Γ, preserves the eigenvalues, and creates a small angle between the two eigendirections at $0_{\mathcal{E}}$. We state it precisely:

Lemma 6.5.4. *Fix $\epsilon > 0$, $A > 0$ and $\theta > 0$. Then there are two integers $N_2 > 0$ and $P > 0$ such that, for any linear saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^2$*

- *that is bounded by A,*
- *that is not N_2-dominated,*
- *such that the two eigenvalues have a common multiple $\lambda > 1$,*

for any finite subset Γ of points of the eigendirections of f, there exists a saddle ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that preserves the eigenvalues of the derivative, respects the flag-configuration of Γ, and such that at some point $i \in \{1, ..., p\}$, the two eigendirections of dg at the origin make an angle smaller than θ.

The proof of this lemma is postponed until section 6.5.1. Then we will show in section 6.5.1 how with a small angle we can obtain a homoclinic tangency from a small flag-configuration-preserving lemma:

Lemma 6.5.5. *Fix $\epsilon > 0$, $A > 0$. There exists an angle $\theta > 0$ such that, for any linear saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^2$*

- *that is bounded by A,*

- *that has huge eigenvalues,*
- *such that somewhere the angle between the eigendirections is smaller than θ,*

for any finite subset Γ of the eigendirections, there is an ϵ-perturbation g on an arbitrarily small neighbourhood $\mathcal{V}$ of $0_{\mathcal{E}}$ that preserves the eigenvalues, that respects the flag-configuration of Γ and creates a homoclinic tangency in $\mathcal{V}$.

To prove Proposition 6.5.3, first apply Lemma 6.5.4, linearize locally with Lemma 6.4.2, and apply Lemma 6.5.5. A lengthy proof follows, if needed.

Proof of Lemma 6.5.3 from Lemmas 6.5.4 and 6.5.5 : Fix $\epsilon > 0$ and $A > 0$. Then Lemma 6.5.5 provides $\theta > 0$ with respect to $\epsilon_0 = \epsilon/2$ and $A_0 = A + \epsilon/2$. Lemma 6.5.4 provides two integers $N_2, P > 0$ with respect to A, $\epsilon_1 = \epsilon/2$ and θ. Let f be a linear saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^2$, with $p > P$

- that is bounded by A,
- such that the two huge eigenvalues $|\lambda| < 1/2$ and $|\mu| > 2$ have a common multiple $\lambda > 1$,
- that is not N_2-dominated,

Then by Lemma 6.5.4, for any finite set Γ of points of the eigendirections of f, there exists a saddle $\epsilon/2$-perturbation g of f on an arbitrarily small neighbourhood $\mathcal{W}$ of $0_{\mathcal{E}}$, that preserves the eigenvalues of the derivative, respects the flag-configuration of Γ, and such that at some point $i \in \{1, ..., p\}$, the two eigendirections of dg at the origin make an angle smaller than θ. We may apply Lemma 6.4.2 and assume g is locally linear. Let $\tilde{g}$ be the linear saddle to which g is equal on a neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$. We may change Γ into $\Delta = g^{n_s}(\Gamma^s) \cup g^{-n_u}(\Gamma^u) \subset \mathcal{U}$, where the integers $n_s, n_u > 0$ are such that any positive iterate of $x \in g^{n_s}(\Gamma^s)$ and any negative iterate of $x \in g^{n_s}(\Gamma^s)$ are in $\mathcal{U}$.

Then, apply Lemma 6.5.5 to $\tilde{g}$ (which is bounded by $A + \epsilon/2$ and has stable and unstable eigendirections making an angle smaller than θ somewhere) to find an $\epsilon/2$-perturbation $\tilde{h}$ of $\tilde{g}$ on an arbitrarily small neighbourhood $\mathcal{V}$ that preserves the eigenvalues, creates a homoclinic tangency in $\mathcal{V}$, and respects the flag-configuration of Δ. Choosing $\mathcal{V}$ to be small enough, we finally find a saddle diffeomorphism h that coincides with $\tilde{h}$ on $\mathcal{U}$, and with g outside $\mathcal{U}$ (thus outside $\mathcal{V}$). This will be an ϵ perturbation of f on $\mathcal{W}$ (which we recall was arbitrarily small), that preserves the eigenvalues, that has a homoclinic tangency, and respects the flag-configuration of Γ. This concludes the proof of the two dimensionnal case. □

Obtaining a small angle: proof of Lemma 6.5.4

Naturally, we first show that one can perturb the derivative of f on $0_{\mathcal{E}}$ to obtain somewhere a small angle between the two eigendirections. We recall that in section 5.2.3 we defined two linear saddles f and g to be linearly ϵ-close if they are ϵ-close as linear cocyles. The following proof is strongly inspired from techniques of [38] and [13, Lemma 3.4], for a picture that summarizes the proof, see [14, page 132].

Lemma 6.5.6. *For all $\epsilon > 0$, $A > 0$, $\theta > 0$, there are two integers $N_2 > 0$ and $P > 0$ such that for any linear saddle f on $\{1, ..., p\} \times \mathbb{R}^2$ that is bounded by A, there is a linear saddle g ϵ-close to f, that has the same eigenvalues as f, that preserves one of the eigendirections of f, and such that somewhere the angle between the two eigendirections is smaller than θ.*

Given a linear saddle f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^2$ and the correponding hyperbolic splitting $F \oplus G$. Denote by P_f a diffeomorphism of $\{1, ..., p\} \times \mathbb{R}^2$ that is linear, stabilizes each fibre, and sends isometrically the bundle F on $\{1, ..., p\} \times \mathbb{R} \times \{0\}$, and the bundle G on $\{1, ..., p\} \times \{0\} \times \mathbb{R}$. Then $f^{\perp} = P_f \circ f \circ P_f^{-1}$ is a linear saddle whose stable eigendirection is $\{1, ..., p\} \times \mathbb{R} \times \{0\}$, and unstable eigendirection is $\{1, ..., p\} \times \{0\} \times \mathbb{R}$, and is N-dominated if and only f is. Clearly, if f is bounded by A, then so is $f^{\perp}$, and if the angles between the stable and unstable eigendirections of f is bounded from below by θ, then P_f is bounded by a quantity $B(\theta)$ which is equivalent to θ^{-1}.

Proof of Lemma 6.5.6 : Fix $\epsilon > 0$, $A > 0$, $\theta > 0$. Then, from the bounds A and $B(\theta)$ above, there are $\eta > 0$ and β such that, for any linear saddle f that is bounded by A whose eigendirections make an angle greater than θ, if g is η-close to $P_f \circ f \circ P_f^{-1}$, then $P_f^{-1} \circ g \circ P_f$ is ϵ-close to f; if the angle between the eigendirections of g is smaller than β, then the angle between the eigendirections of f is smaller than θ.

Thus it suffices to show the following:

Claim 14. *Fix $A, \eta, \beta > 0$. There are two integers $P, N > 0$ such that if f is a linear saddle on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^2$ (with $p > P$), that is bounded by A, that is not N-dominated, whose stable bundle is $\{1, ..., p\} \times \mathbb{R} \times \{0\}$, whose unstable bundle is $\{1, ..., p\} \times \{0\} \times \mathbb{R}$,*

then there is a linear saddle g, η-close to f that is equal to f on $\{1, ..., p\} \times \mathbb{R} \times \{0\}$, that has same eigenvalues as f, and such that the angle between the eigendirections of g is somewhere smaller than β.

We may represent f by a sequence of matrices

$$A_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}, i = 1, ..., p$$

where A_i is the matrix in the canonical bases of the induced map $f \colon E_i \to E_{i+1}$.

If f is not N-dominated, then by definition, it is not N' dominated for any $N' < N$, thus we can assume $N < P/2$. We have

$$\prod_{i=1,\dots,p} a_i < \prod_{i=1,\dots,p} b_i, \tag{6.8}$$

and for there is no N-domination, we may reindex the sequence A_i, so that $2\prod_{i=1,\dots,N} a_i \geq \prod_{i=1,\dots,N} b_i$. Where reindexing means to fix an integer j and change each A_i to $A_{(i+j)[p]+1}$, where $(i+j)[p]$ is the rest of the euclidian division of $i+j$ by p.

Define

$$B_1 = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, B_i = \begin{pmatrix} 1 & 0 \\ 0 & 1+\mu \end{pmatrix} \text{ for } i = 2, ..., N,$$

$$B_i = \begin{pmatrix} 1 & 0 \\ 0 & (1+\mu)^{-1} \end{pmatrix} \text{ for } i = N+1, ..., 2N-1.$$

where $\mu > 0$ is the greatest real number such that if a matrix M is bounded by A, then MB_1, B_1M and MB_i, $i = 2, ..., 2N-1$ are η-close to M. The image of the line $\mathbb{R}(0,1)$ (spanned by $(0,1)$) is sent by the matrix B_1 on $\mathbb{R}(\mu, 1)$, then by inequality 6.8 the product $A_p...A2N+1A_{2N}A_{2n-1}B_{2N-1}...A_1$ is hyperbolic with stable direction $\mathbb{R}(1,0)$ and unstable $\mathbb{R}(0,1)$. Hence that product sends the line $\mathbb{R}(\mu, 1)$ on a line $\mathbb{R}(\rho, 1)$ with $|\rho| < \mu$. Finally define

$$B_p = \begin{pmatrix} 1 & -\rho \\ 0 & 1 \end{pmatrix}$$

That matrix sends back $\mathbb{R}(\rho, 1)$ on the line $\mathbb{R}(0,1)$. Let g be the linear cyclic diffeomorphism of $\mathcal{E}$ whose matricial representation is the sequence $A_1B_1, ..., A_{2N-1}B_{2N-1}, A_{2N}, ..., B_pA_p$. The way we chose $0 < \rho < \mu$ ensures that g is η close to f. Clearly, g^p sends the line $\{(1,0)\} \times \mathbb{R}$ on itself (its orbit is still the unstable eigenbundle). Besides, if N is great enough, precisely greater than a constant that depends only on A and β, g^N sends the line $\{1\} \times \{0\} \times \mathbb{R}$ on a line that makes an angle smaller than β with $\{N+1\} \times \mathbb{R} \times \{0\}$. Notice finally that $g = f$ on $\{1, ..., p\} \times \mathbb{R} \times \{0\}$, and the Jacobians of g^p and f^p are equal. Hence the eigenvalues are preserved. This ends the proof of the claim, and that of the lemma. □

Now that found how to perturb the derivative at $0_{\mathcal{E}}$ to obtain small angles, we have to extend that pertubation into a saddle perturbation. This is achieved by the following:

Lemma 6.5.7. *Let f be a linear saddle of $\mathcal{E}$. Let $\epsilon < \alpha/4|f|$ where α is the constant defined in Lemma 5.2.10. Let g be a linear saddle that is linearly ϵ-close to f, and such that $f = g$ on a one-dimensional subbundle H. Then there is a $8|f|^2\epsilon$-perturbation h of f that goes to the quotient by H, with $f/H = g/H$, such that $h = g$ on a neighbourhood of $0_{\mathcal{E}}$ and $h = f$ outside the strip $C_1 = \{x \in \mathcal{E} / \operatorname{dist}(x, H) < 1\}$.*

Proof : Define a map h on $\mathcal{E}$ by $h(x) = \phi(x).f(x) + \psi(x).g(x)$, where $\phi + \psi = 1$ is a unit partition on $\mathcal{E}$ such that $\psi(x)$ and $\phi(x)$ only depend on the projection x/H, $\psi = 1$ on a small neighbourhood of H, and $\psi = 0$ outside C_1. Therefore, the map h goes to the quotient by H, coincides with g on a neighbourhood of H and with f outside C_1. Clearly ψ and ϕ can be chosen to have derivatives with norms less than 2.

If $x \notin C_1$ then $f(x) = h(x)$ and $d_x f = d_x h$. If x is in C_1, then we can write $x = x_H + y$ where $x_H \in H$ and y has norm smaller than 1. We have $f(x) - h(x) = \psi(x).(f-g)(x) = \psi(x)(f-g)(y)$, for $f = g$ on H. Thus $\|f(x) - h(x)\| < 1.\epsilon\|y\| < \epsilon$. We calculate the derivative:

$$d(f-h) = d\psi.(f-g) + \psi(df - dg)$$

and get similarly that $\|d_x(f-h)\| \leq 2\epsilon + 1.\epsilon$. Hence the map $f - h$ has C^1-norm less than $4\epsilon < \alpha/|f|$. Then by corollary 5.2.11, h is a diffeomorphism, precisely a $8|f|^2\epsilon$-perturbation of f. $\square$

Proof of Lemma 6.5.4 : Fix $\epsilon > 0$, $A > 0$ and $\theta > 0$. Let $\varepsilon = \min\left(\alpha/4|f|, \frac{\eta_{\epsilon,|f|}}{8|f|^2}\right)$. Then fix two integers $N_2 > 0$ and $P > 0$ as Lemma 6.5.6 enables us to do. Let f be a linear saddle that is bounded by A, that is not N_2-dominated, and such that the two eigenvalues have a common multiple $\lambda > 1$. Let H be the stable eigendirection of f. Then there is a linear saddle $\widehat{g}$ that is linearly ε-close to f, that coincides with f on F, that has same eigenvalues as f, and such that the angle between the two eigendirections of $\widehat{g}$ is somewhere smaller than θ.

We have $\varepsilon \leq \alpha/4|f|$, hence by Lemma 6.5.7 there is a $8|f|^2\varepsilon$-perturbation, therefore an $\eta_{\epsilon,|f|}$-perturbation of f that goes to the quotient by H, with $f/H = h/H$, such that $h = \widehat{g}$ on a neighbourhood of $0_{\mathcal{E}}$ and $h = f$ outside the strip $C_1 = \{x \in \mathcal{E}/\operatorname{dist}(x,H) < 1\}$. Let Γ be a finite set of points of the eigendirections of f, and let $\gamma > 1$ be such that γ^{-1} is smaller than the distance from H to Γ^u (in other words $\gamma^{-1}.C_1$ does not intersect Γ^u). Then the diffeomorphism $h' = \gamma^{-1}.h \circ \gamma.Id_{\mathcal{E}}$ coincides with f outside C_1. Therefore, it firmly respects the flag-configuration of Γ^u for f to the quotient by H (notice that $f/H = h'/H$).

Obviously, $f_{|H} = h'_{|H}$ by restriction to H, thus $h'_{|H}$ firmly respects the flag-configuration of $\Gamma_{|H}$ for $f_{|H}$. We apply Proposition 6.3.3 and find an ϵ-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that respects the flag-configuration of Γ and that is locally conjugate to h', therefore to $\widehat{g}$. That g has same eigenvalues as f, and the angle between its eigendirections is somewhere smaller than θ. This ends the proof of Lemma 6.5.4 $\square$

Creating a homoclinic tangency from a small angle: proof of Lemma 6.5.5

Once we have a small angle between the two eigendirections, we can adapt the argument of Pujals and Sambarino to preserve a finite set Γ in the stable and unstable manifolds of f.

Under the hypothesis of lemma 6.5.5, let D^s and D^u be two *simple fundamental domains*, respectively, of the stable and unstable eigendirections of f, that is D^s writes as a union $]-x,-|\lambda^s|.x] \cup [|\lambda^s|.x, x[$ of two intervals, where x is a point of the stable direction and λ^s the corresponding eigenvalue, and D^u writes as $]-|\lambda^u|.x,-x] \cup [x,|\lambda^u|.x[$, where x is a point of the unstable direction, and λ^u the corresponding eigenvalue.

We suppose moreover that for any $x \in \Gamma^s$, there is a positive iterate $x' = f^k(x)$ in D^s, and for any $x \in \Gamma^u$, there is a negative iterate $x' = f^{-k}(x)$ in D^u. Call Γ' the union of these x'. It is obviously sufficient to find g that respects the flag-configuration of Γ', provided that g coincides with f on a small enough neighbourhood of $0_{\mathcal{E}}$. From that, we can do a

Remark 6.5.8. *It is sufficient to prove 6.5.5 in the case where Γ^u and Γ^s are simple fundamental domains of the stable and unstable eigendirections of f, respectively.*

Lemma 6.5.9. *Let f be a linear saddle on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^2$ and Γ^s, Γ^u two finite subsets of fundamental intervals D^s and D^u, respectively. Then, for any $\rho > 1$, there is an arbitrarily small saddle perturbation g of f on an arbitralily small $\mathcal{U}$ neighbourhood of $0_{\mathcal{E}}$, that is equal to f on a small neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $0_{\mathcal{E}}$, that preserves the flag-configuration of Γ, and such that any integer k greater than some $K > 0$, we have that $g^k(\Gamma^s)$ and $g^{-k}(\Gamma^u)$ are included in intervals of the form $[-\rho.x, -x] \cup [x, \rho.x]$, where x is in the stable or unstable eigendirections.*

Proof : Under the hypothesis of the lemma, let $\mathcal{U}$ be a neighbourhood of $0_{\mathcal{E}}$. We may change Γ^s into $f^n(\Gamma^s)$ and Γ^u into $f^{-n}(\Gamma^u)$, to suppose that $\Gamma^s \subset D^s \subset \mathcal{U}$ and $\Gamma^u \subset D^u \subset \mathcal{U}$.

We write $D^s =]-y^s, -x^s] \cup [x^s, y^s[$ and $D^u =]-y^u, -x^u] \cup [x^u, y^u[$. We find $z^s \in D^s$ and $z^u \in D^u$ such that $\Gamma^s \subset]-z^s, -x^s] \cup [x^s, z^s[$ and $\Gamma^u \subset]-z^u, -x^u] \cup [x^u, z^u[$. Clearly, there is a compact box B^s in $\mathcal{U}$, that intersects $W^s(f) \cup W^u(f)$ into $[-z^s, -x^s] \cup [x^s, z^s]$, and such that the boxes $|\lambda^s|^k.B^s$, for $k \in \mathbb{N}$, are pairwise disjoint. As a consequence, these boxes do not contain $0_{\mathcal{E}}$.

For all $\eta > 0$, there exists a diffeomorphism Φ^s of B^s that sends the set $]-z^s, -x^s] \cup [x^s, z^s[$ on itself, that can be extended to $\mathcal{E}$ by the identity map into an η-perturbation of $Id_{\mathcal{E}}$, and such that, for the restricted diffeomorphism $\Phi^s_{|]-z^s,-x^s] \cup [x^s,z^s[}$, the points $\pm x^s$ are sinks whose stable manifolds are $[x^s, z^s[$ and $]-z^s, -x^s]$, respectively. The compact set Γ^s is in these manifolds, thus there is an iterate $(\Phi^s)^K(\Gamma^s) \subset [-\rho.x^s, -x^s] \cup [x^s, \rho.x^s]$. Let Ψ^s be the diffeomorphism of $\mathcal{E}$ that coincides with $|\lambda^s|^k.\Phi^s \circ |\lambda^s|^{-k}.Id_{B^s}$ on the boxes $|\lambda^s|^k.B^s$, for all $0 \leq k \leq K$, and with the identity map outside these boxes. The diffeomorphism Ψ^s is an η-perturbation of $Id_{\mathcal{E}}$ on $\mathcal{U}$ that coincides with $Id_{\mathcal{E}}$ on a neighbourhood $\mathcal{V}$ of $0_{\mathcal{E}}$ and on the unstable manifold of f.

By Remark 5.2.6, $h = f \circ \Psi^s$ is a saddle $|f|\eta$-perturbation of f on $\mathcal{U}$. It is equal to f on $\mathcal{V}$ and on the unstable manifold of f, and for $k > K$, the set $h^k(\Gamma^s)$ is in $[-\rho.h^k(x^s), h^k(x^s)] \cup [h^k(x^s), \rho.h^k(x^s)]$. The dynamics on a neighbourhood of the unstable manifold is not affected by that perturbation. Thus, we can do the symmetrical perturbation Ψ^u on the unstable manifold, and obtain finally an arbitrarily small (choose η small enough) saddle perturbation $g = f \circ \Psi^s \circ \Psi^u$ on $\mathcal{U}$, that satisfies all the conclusions of the lemma. This concludes the proof of Lemma 6.5.9. □

To prove Lemma 6.5.5, we are now left to show the following:

Lemma 6.5.10. *Fix $\epsilon > 0$, $A > 0$. Then there is an angle $\theta > 0$, and a constant $\rho > 1$ such that, for any linear saddle diffeomorphism f on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^2$, that is bounded by A, that has two huge eigenvalues, and such that somewhere the angle between the eigendirections is smaller than θ,*

for any finite subset $\Gamma = \Gamma^u \cup \Gamma^s$ such that $\Gamma^s \subset [-\rho.x, -x] \cup [x, \rho.x]$ and $\Gamma^u \subset [-\rho.y, -y] \cup [y, \rho.y]$, where x is the stable direction and y in the unstable direction, there is an ϵ-perturbation g on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$ that preserves the eigenvalues, creates a homoclinic tangency, and respects the flag-configuration of Γ.

This step correspond to Lemma 2.2.1 in [52]. The idea and the proof are roughly the same, thus we only give a short proof. We use without a proof the following statement, slightly more precise than Lemma 6.3.5, as it controls derivatives.

Lemma 6.5.11. *For any $\eta > 0$, there exists a ratio r such that, for any pair of points x and y in an open set $\mathcal{U}$ of $\mathbb{R}^n$ with diameter smaller than 1, for any pair of unit vectors $u \in T_x\mathbb{R}^n$ and $v \in T_y\mathbb{R}^n$, if we have*

$$\frac{\text{dist}(x, y)}{\text{dist}(x, \mathcal{E} \setminus \mathcal{U})} < r,$$

and if the angle between u and v is smaller than r, then there is an η-perturbation of Id_n on $\mathcal{U}$ that sends x on y, and whose derivative sends u on v.

Short proof of Lemma 6.5.5 : Fix $0 < \epsilon < 1$, $A > 0$. And let $0 < \theta$ and $1 < \rho < 2^{1/4}$. We let the reader draw a picture and notice that there is a function $r(\theta) > 0$ that tends to 0 as θ goes to zero that satisfies the following: under the hypothesis of the lemma, for any neighourhood $\mathcal{V}$ of $0_{\mathcal{E}}$, there are a pair of points x and y in the stable and unstable manifolds of f and a compact neighbourhood $\mathcal{U} \subset \mathcal{V}$ of $\{x, y\}$ such that

- $\mathcal{U}$ does not intersect the orbit of Γ, nor $0_{\mathcal{E}}$,
- it intersects each $W^u(f)$ and $W^s(f)$ into an interval that is a subset of a simple fundamental domain of f,
- the ratio $\frac{\operatorname{dist}(x,y)}{\operatorname{dist}(x,\mathcal{E}\setminus\mathcal{U})}$ is smaller than $r(\theta)$.

Let $u \in T_x\mathcal{E}$ and $v \in T_y\mathcal{E}$ be the unit vectors pointing to $0_{\mathcal{E}}$, therefore tangent to the stable and unstable manifold, respectively. They make an angle θ. If θ is small enough, $r(\theta)$ is small enough to apply Lemma 6.5.11 and find an $\epsilon/|f|$-perturbation Ψ of $Id_{\mathcal{E}}$ on $\mathcal{U}$ that sends x^s on x^u, and such that $d\Psi(u) = v$. Then, the diffeomorphism $g = f \circ \Psi$ is a saddle perturbation of f on $\mathcal{U}$ that preserves the orbits of each point of Γ (therefore respects the flag-configuration of Γ), that has the same eigenvalues as f, and such that its stable and unstable manifold go through x^s, tangently to u; in other words it admits a homoclinic tangency. By Remark 5.2.6, g is an ϵ-perturbation of f on $\mathcal{U}$. The neighbourhood $\mathcal{V} \supset \mathcal{U}$ of $0_{\mathcal{E}}$ could be chosen arbitrarily small. Therefore g matches the conclusions of Lemma 6.5.5. □

6.5.2 The induction: proof of Proposition 6.5.2 in dimension $d \geq 3$

Sketch of the proof

This sketch should be compared to that of the proof of Proposition 5.3.4 (see section 5.3.2). We merely adapt it to the constraint of flag-configuration preservation. We initiate the induction process in dimension $d = 2$: it is shown in section 6.5.1 that when the eigenvalues are huge, Pujals and Sambarino's construction can be done preserving the eigenvalues, and adapted to respect the flag-configuration of a finite set Γ.

In dimension $d \geq 3$, we write the hyperbolic splitting: $\mathcal{E} = F \oplus G$, where F is the stable bundle of f and G the unstable one. By Remark 6.1.4, we can assume as in the proof of Proposition 5.3.4 that F has dimension ≥ 2. We recall that f has real eigenvalues with pairwise distinct moduli, therefore F splits into a direct sum $F_1 \oplus ... \oplus F_k$ of eigendirections, where F_1 is the strongest stable eigendirection. Call H the subbundle $F_2 \oplus ... \oplus F_d$. From Lemma 5.3.10, if the saddle is not dominated enough then we have again the same dichotomy:

- The splitting $E' = H \oplus G$ is not dominated enough for the restriction of f to E' so that, by induction hypothesis, we can find a saddle small-perturbation g' of $f_{|E'}$ on an arbitrarily small neighbourhood of $0'_E$, that respects the flag-configuration of $\Gamma' = \Gamma \cap E'$, that preserves the eigenvalues of the derivative, and that admits a homoclinic tangency.

- The splitting $F/H \oplus G/H$ of the quotient bundle $\mathcal{E}/H$ is not dominated enough for the quotient linear saddle f/H so that, by induction hypothesis, we can find a saddle small-perturbation g' of f/H on an arbitrarily small neighbourhood of $0_{\mathcal{E}/H}$, that respects the flag-configuration of $\Gamma/H = \pi(\Gamma)$, that preserves the eigenvalues of the derivative, and that admits a homoclinic tangency.

If we are in the first case, by Corollary 5.2.18 we can extend g' to a cycle local small-perturbation g of f such that the quotients g/E' and f/E' are defined and equal. The diffeomorphism g extends a saddle diffeomorphism that has a homoclinic tangency; thus it also has a homoclinic tangency. The cyclic diffeomorphism g respects the flag-configuration of Γ' for f but *a priori* the flag-configuration of the remainder of Γ (that is $\Gamma \setminus \Gamma = \Gamma \cap F_1$), is not respected. We see that E' is a central bundle for f and g, and choosing $g' = g_{|E'}$ to be a perturbation of $f_{|E'}$ on a sufficiently small neighbourhood of $0_{E'}$, we can apply Proposition 6.3.3.

If we are in the second case, by Lemma 5.2.19, we can lift the perturbation g' of f/H into a small-perturbation g of f that coincides with f on H. We showed in section 5.3 that, g' admitting a homoclinic tangency, so does g. The saddle diffeomorphism g respects the flag-configuration of Γ_H for f, but the flag-configuration of the remainder of Γ is not a priori respected. The bundle H is central for f and g, and choosing g' to be a perturbation of f/H on a sufficiently small neighbourhood $\mathcal{U}'$ of $0_{\mathcal{E}/H}$, we can apply Proposition 6.3.3.

Thus, in both cases we can obtain a cyclic perturbation of f on an arbitrarily small neighbourhood, that respects the flag-configuration of Γ, and that is locally conjugate to g in such a way that it also admits a homoclinic tangency. It is easily checked that the eigenvalues are preserved in both cases. QED.

A detailed proof

Fix $d > 2$, and suppose now that the theorem is true for any dimension $2 \leq d' \leq d-1$. Fix $A > 0$ and $\varepsilon > 0$. Let $\epsilon > 0$ be less than $\nu_{\eta_{\varepsilon,A},A}$, which is as defined for Corollary 5.2.18, less than $\frac{\eta_{\varepsilon,A}}{2A^2}$ where η is as defined for Lemma 5.2.16, and less than α/A, where α is as defined in Lemma 5.2.10. Use Proposition 6.5.2 for each $2 \leq d' \leq d-1$, to find an integer $N_{d'}$ with respect to A and ϵ (this is the induction hypothesis).

Let N be the maximum of the integers $N_{d'}$ for $2 \leq d' \leq d-1$. Fix $M \in \mathbb{N}$ as in Lemma 5.3.10, with respect to A, d and N. We will show that the conclusions of Proposition 6.5.2 will hold for $N_d = M$, and a perturbation of size ε.

Let f be a linear saddle on a d-dimensional bundle $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ such that

- it is bounded by A,
- it has real huge eigenvalues with a common multiple $\lambda > 1$ and pairwise distinct moduli,
- it is not N_d-dominated,

and let Γ be a finite subset in the eigendirections of f. The linear saddle f^{-1} also satisfies the three items and Γ is in the eigendirections of f^{-1}, besides (Remark 6.1.4) the two following statements are equivalent:

- g is a saddle ε-perturbation of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and respects the flag-configuration of Γ.
- g^{-1} is a saddle ε-perturbation of f^{-1} on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and respects the flag-configuration of Γ.

So, we may replace f by f^{-1} and assume that in the invariant hyperbolic splitting $\mathcal{E} = F \oplus G$, the stable space F has dimension greater than two. Since f has real pairwise distinct eigenvalues, there is an invariant splitting $F = F_1 \oplus ... \oplus F_k$ of eigendirections, where F_1 is the strongest stable eigendirection. Call H the subbundle $F_2 \oplus ... \oplus F_d$. From Lemma 5.3.10, either the splitting $H \oplus G$ is not N-dominated for the restriction $f_{|H \oplus F}$, or the splitting $F/H \oplus G/H$ is not N-dominated for the quotient linear cocycle f/H. Call d_1 and d_2 the dimensions of $H \oplus F$ and $F/H \oplus G/H$, respectively. By definition, N is greater than N_{d_1} and N_{d_2}; we reformulate the dichotomy:

- either the splitting $E' = H \oplus G$ is not N-dominated, and thus not N_{d_1}-dominated, for the restriction $f_{|E'}$. Then, by induction hypothesis, we find a saddle ϵ-perturbation g' of $f_{|E'}$ on an arbitrarily small neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$, that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and respects the flag-configuration of $\Gamma' = \Gamma \cap E'$. Choosing $\mathcal{U}$ small enough, one can suppose that the saddle diffeomorphism g' firmly respects the flag-configuration of Γ'.

  The following proposition, shown in section 6.5.2 ends the study of this case:

  Proposition 6.5.12. *Let $\varepsilon > 0$ and $\nu = \nu_{\eta_\varepsilon, A, A}$. If there is a saddle ν-perturbation g' of $f_{|E'}$ that has a homoclinic tangency, that preserves the eigenvalues of the derivative of $f_{|E'}$, and firmly respects the flag-configuration of Γ', then there is a saddle ϵ-perturbation g on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$ of f that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and respects the flag-configuration of Γ.*

- or the splitting $E/H = F/H \oplus G/H$ is not N-dominated, and thus not N_{d_2}-dominated, for the saddle quotient f/H. Then, by induction hypothesis, we find a saddle local ϵ-perturbation g' of f/H on an arbitrarily small neighbourhood $\mathcal{U}$ of $0_{\mathcal{E}}$, that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and respects the flag-configuration of $\Gamma/H = \pi(\Gamma)$. In particular, choosing $\mathcal{U}$ small enough, one can suppose that the saddle diffeomorphism g' firmly respects the flag-configuration of Γ/H. We conclude with the following proposition, shown in section 5.3.2:

  Proposition 6.5.13. *Let $\varepsilon > 0$ and $\mu = \min(\frac{\eta_{\varepsilon,A}}{2A^2}, \alpha/A)$. If there is a saddle local μ-perturbation g' of f/H, that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and firmly respects the flag-configuration of $\Gamma/H = \pi(\Gamma)$, then there is a saddle ε-perturbation g of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and respects the flag-configuration of Γ.*

We recall indeed that ϵ was chosen less than μ and ν defined above. Thus having chosen ϵ sufficiently small we found in both cases a saddle ε-perturbation of f on an arbitrarily small neighbourhood of $0_{\mathcal{E}}$, that has a homoclinic tangency, that preserves the eigenvalues of the derivative, and respects the flag-configuration of Γ. This ends the proof of Proposition 5.3.4.

The restriction case: proof of Proposition 6.5.12

Let $\varepsilon > 0$ and $A > 0$. Let $\eta = \eta_{\varepsilon,A}$. Let f be a linear saddle on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ that is bounded by A, and such that its eigenvalues admit a common multiple $\lambda > 1$. Let $\nu = \nu_{\eta,A}$ be as defined for Corollary 5.2.18. Let E' be an invariant central bundle for f, Γ a finite set in the union of the eigendirections of f, and g' a saddle ν-perturbation of $f_{|E'}$, that has a homoclinic tangency, that preserves the eigenvalues of the derivative of $f_{|E'}$, and firmly respects the flag-configuration of Γ'.

We have $\nu_{\eta,A} \leq \nu_{\eta,|f|}$, therefore by Corollary 5.2.18, g' can be extended to a local η-perturbation g of f, such that $g/E' = f/E'$. Of course it also admits a homoclinic tangency, and its eigenvalues coincide with those of f. By hypothesis, $g_{|E'}$ firmly respects the flag-configuration of $\Gamma_{|E'}$ for $f_{|E'}$, and since $g/E' = f/E'$, g firmly respects the flag-configuration of $\Gamma \setminus E'$ for f, to the quotient by E'. Since $g_{|E'}$ has the same eigenvalues as $f_{|E'}$, and $g/E' = f/E'$, E' is also a central bundle for g. Finally $\eta = \eta_{\varepsilon,A} \leq \eta_{\varepsilon,|f|}$.

Let $\mathcal{B}$ be a bounded neighbourhood of $0_{\mathcal{E}}$ and the orbit of the tangency of g. We can apply Proposition 6.3.3 and find an ε-perturbation h of f on an arbitrarily small neighbourhood, that respects the flag-configuration of Γ for f and such that $\gamma.h \circ \gamma^{-1}.Id_{\mathcal{B}} = g$, for some $\gamma > 0$. Hence h also has a homoclinic tangency and has the same eigenvalues as f. This ends the proof of Proposition 6.5.12.

The quotient case: proof of Proposition 6.5.13

Let $\varepsilon > 0$ and $A > 0$. Let $\mu = \min(\frac{\eta_{\varepsilon,A}}{2A^2}, \alpha/A)$. Let f be a linear saddle on $\mathcal{E} = \{1, ..., p\} \times \mathbb{R}^d$ that is bounded by A, and such that its eigenvalues admit a common multiple $\lambda > 1$. Let H be an invariant central bundle for f, Γ a finite set in the union of the eigendirections of f, and g' a saddle μ-perturbation of f/H that has a homoclinic tangency, that preserves the eigenvalues of the derivative of f/H, and firmly respects the flag-configuration of Γ/H.

As $\alpha/A < \alpha/|f|$, we can apply Lemma 5.2.19 and find a saddle $2A^2\mu$-perturbation g of f that is a lift of g'. By Remark 5.2.20, since g/H firmly respects the flag-configuration of Γ/H for f/H, g can be chosen so that it firmly respects the flag-configuration of Γ for f, to the quotient by H, and so that $g_{|H} = f_{|H}$. Thus trivially $g_{|H}$ firmly respects the flag-configuration of $\Gamma_{|H}$ for $f_{|H}$. As g/H and f/H have same eigenvalues and $g_{|H} = f_{|H}$, g has same eigenvalues as f, and H is also a central bundle for g. Finally we recall that $2A^2\mu \leq \eta_{\varepsilon,A} \leq \eta_{\varepsilon,|f|}$: all the hypothesis of Proposition 6.3.3 are satisfied.

We showed in section 5.3, Lemma 5.3.13, that if g' admits a homoclinic tangency, so does g. Let $\mathcal{B}$ be a neighbourhood of $0_{\mathcal{E}}$ and of the orbit of that homoclinic tangency. We apply Proposition 6.3.3 and find an ε-perturbation h of f on an arbitrarily small neighbourhood, that respects the flag-configuration of Γ for f and such that $\gamma.h \circ \gamma^{-1}.Id_{\mathcal{B}} = g$, for some $\gamma > 0$. Hence h also has a homoclinic tangency and has the same eigenvalues as f. This ends the proof of Proposition 6.5.13.

6.6 Consequences

6.6.1 General results

We say that two periodic saddle points are *homoclinically related* if and only if the unstable manifold of the orbit of one intersects transversely the stable manifold of the orbit of the other. This, by the λ-lemma (see [46, Page 80], for instance), is an equivalence relation.

Definition 6.6.1. Let Q be a saddle point for some diffeomorphism f. The *homoclinic class* $H(Q, f)$ of Q is the closure of the set of hyperbolic points that are homoclinically related to Q.

One of the consequences of Theorem 6.1.1 is

Corollary 6.6.2. *Let Q be a saddle point for f whose homoclinic class $H(Q, f)$ is non-trivial (not reduced to the orbit of Q) and does not admit a* *dominated splitting of same index as Q. Then, there is an arbitrarily small perturbation g of f, that preserves the dynamics on a neighbourhood of Q, and such that there is a homoclinic tangency related to Q.*

We need first a few preliminary results. We show that if two saddles are homoclinically related, then a homoclinic tangency associated to one of them can be turned by a small perturbation into a homoclinic tangency associated to the other.

Proposition 6.6.3. *Let Q and R be saddle points of f that are homoclinically related. If f admits a homoclinic tangency x associated to R, then for any neighbourhood $\mathcal{U}$ of x, for any $\epsilon > 0$, there is an ϵ-perturbation g of f on $\mathcal{U}$, that preserves the saddle Q and such that g admits a homoclinic tangency related to Q.*

We state without a proof a folklore λ-lemma-like:

Lemma 6.6.4. *Let R be a saddle for a diffeomorphism f of index i. Let $x \in W^u(R)$ and $y \in W^s(R)$, and let D be a i-dimensional C^1-disk centered at y and transverse to $W^s(R)$ at y. Then for any $\eta > 0$, there is a point $z \in D$ such that, for some $n_0, n_1 \in \mathbb{N}$, for any $0 \le n \le n_0$, we have*

$$\mathrm{dist}(f^n(y), f^n(z)) < \eta,$$

for any $n_0 \le n \le n_1$, we have

$$\mathrm{dist}(f^n(z), f^{n-n_1}(x)) < \eta,$$

and the tangent space of the disk $f^{n_1}(D)$ at $f^{n_1}(z)$ is η-close to the tangent space of $W^u(R)$ at x.

Proof of Proposition 6.6.3 : Under the hypothesis of the lemma, let a be a transverse intersection between $W^u(Q)$ and $W^s(R)$, and b be a transverse intersection between $W^s(Q)$ and $W^u(R)$. Let δ be the minimum distance between x and

$$\mathrm{Orb}_f(x) \setminus \{x\} \cup \mathrm{Orb}_f(a) \cup \mathrm{Orb}_f(b) \cup \mathrm{Orb}(Q) \cup \mathrm{Orb}(R).$$

Let $\mathcal{U}$ be a neighbourhood of x with diameter less than $\delta/2$.

Let D^s_a be a ball in $W^u(Q)$ centered on a, small enough so that the distance between $f^{-n}(D^s_a)$ and $f^{-n}(a)$ is less than $\delta/2$, for all $n \in \mathbb{N}$. Let $\eta < \delta/2$. Apply the previous lemma, and find $z \in D^s_a$ such that for some $n_0, n_1 \in \mathbb{N}$, for any $0 \le n \le n_0$ we have $\mathrm{dist}(f^n(a), f^n(z)) < \eta$, for any $n_0 \le n \le n_1$, we have $\mathrm{dist}(f^n(z), f^{n-n_1}(x)) < \eta$, and tangent space of the disk $f^{n_1}(D^s_a)$ at $f^{n_1}(a)$ is η-close to the tangent space of $W^u(R)$ at x.

Hence, for all $\eta > 0$, we found a point x^u in $W^u(Q)$ such that $\mathrm{dist}(x, x^u) < \eta$, such that $T_{x^u}W^u(Q)$ and $T_xW^u(R)$ are η-close, and for all

$n \in \mathbb{N}$, $\mathrm{dist}(f^{-n}(x^u), x)$ is greater than $\delta/2$. Symmetrically we find a point x^s in $W^s(Q)$ such that $\mathrm{dist}(x, x^s) < \eta$, such that $T_{x^s}W^s(Q)$ and $T_xW^s(R)$ are η-close, and for all $n \in \mathbb{N}$, $\mathrm{dist}(f^n(x^s), x)$ is greater than $\delta/2$.

Let HT_x be the intersection of $T_xW^u(R)$ and $T_xW^s(R)$. There is a subspace HT^s_x of $T_{x^s}W^s(Q)$ and a subspace HT^u_x of $T_{x^u}W^u(Q)$ such that both are η-close to HT_x. If Φ is a perturbation of Id_M on $\mathcal{U}$ that sends x^u on x^s and HT^u_x on HT^s_x, then $g = f \circ \Phi$ is a perturbation of f that admits a homoclinic tangency at x^u. Taking η small enough, such a Φ can be found to be an arbitrarily small perturbation of Id_M on bounded $\mathcal{U}$. Therefore, by Remark 5.2.8, g can be found to be arbitrarily close to f. □

We recall the following classical

Lemma 6.6.5. *Let f be a diffeomorphism on a compact manifold M with dimension d. Let g_n be a sequence of diffeomorphisms that tends to f in* $\mathrm{Diff}^1(M)$ *and Q_n be a sequence of periodic points for f, such that the sequence of their orbits* $\mathrm{Orb}(Q_n)$ *converges for the Hausdorff topology to a compact set K for f. Then K is invariant for f. Moreover there is a dominated splitting of index i for f on K if and only if, for some integer $N > 0$, for any n great enough, there is an invariant splitting of index i of the tangent space along the orbit of Q_n that is N-dominated for g_n.*

We now are ready for the proof of the corollary.

Proof of Corollary 6.6.2 : We recall that in a homoclinic class $H(P, f)$, there is a sequence P_n of saddle orbits of f, with same index as P, that tends to $H(P, f)$ for the Hausdorff topology. Since $H(P, f)$ is not trivial, it is infinite, therefore the period of the sequences P_n tends to $+\infty$. From Lemma 6.6.5, there is a sequence N_n in $\mathbb{N}$ that tends to $+\infty$ and such that the invariant splitting composed of the stable and unstable bundles of P_n is not N_n-dominated. Therefore we can apply Theorem 6.1.1 to P_n for n great enough, and the cycle it forms with P: for any $\epsilon > 0$, for n great enough, there is an ϵ-perturbation g of f on an arbitrarily small neighbourhood of P_n that creates a homoclinic tangency at P_n and preserves the cycle between P and P_n. Therefore $H(P, g)$ contains P_n which admits a homoclinic tangency related to it. Finally, from Proposition 6.6.3, we perturb again g on an arbitrarily small neighbourhood of the tangency x for P_n, to obtain a perturbation h of f such that the dynamics is preserved on a neighbourhood of $\mathrm{Orb}_f(P)$ and such that there is a homoclinic tangency associated to the saddle P. □

6.6.2 Generic results

Let us recall shortly some definitions from Conley's theory [21]. Let f be a diffeomorphism of a manifold M. A sequence (x_n) is an *ϵ-pseudo-orbit* for some $\epsilon > 0$ if and only if the distance $\mathrm{dist}(f(x_n), x_{n+1})$ is less than ϵ for each

n. The *chain-recurrent set* $\mathcal{R}(f)$ is the set of points x such that for all $\epsilon > 0$ there is an ϵ-pseudo-orbit that is not reduced to a single point and that goes from x to x. We define an equivalence relation $\sim$ on $\mathcal{R}(f)$ in the following way: $x \sim y$ if and only if, for all $\epsilon > 0$, there is an ϵ-pseudo-orbit from x to y, and an ϵ-pseudo-orbit from y to x.

The *chain-recurrent classes* are the equivalence classes of $\sim$ in $\mathcal{R}(f)$. If M is compact, the chain-recurrent set is not empty, and the chain-recurrent classes are compact.

A *residual subset* of a topological space is a countable intersection of dense open sets. A dynamical property $\mathcal{P}$ is C^k*-generic* if it is satisfied by a residual set of dynamics for the C^k-topology. We also say that C^k*-generically*, the dynamics satisfies property $\mathcal{P}$. In [10], C. Bonatti and S. Crovisier showed that C^1-generically, the non-aperiodic chain recurrent classes and the homoclinic classes coincide:

Theorem 6.6.6 (Bonatti, Crovisier). *Given a compact manifold M, there is a residual subset $\mathcal{HCR}$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that each homoclinic class of f is a chain-recurrent class and conversely, each chain-recurrent class is either aperiodic (does not contain any periodic point) or is a homoclinic class.*

Besides, with F. Abdenur, L. Díaz and L. Wen [3], they showed the following result:

Theorem 6.6.7 (Abdenur, Bonatti, Crovisier, Díaz, Wen). *There is a residual subset $\mathcal{I}$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that any homoclinic class $H(p,f)$ containing hyperbolic saddles of indices α and β contains a dense subset of saddles of index τ, for all $\tau \in [\alpha,\beta]$.*

Corollary 6.6.2, and this result give partial answer to [3, Conjecture 1]:

Theorem 6.6.8. *For every C^1-generic diffeomorphism f, let $H(P,f)$ be a homoclinic class containing saddles of indices α and β, $\alpha < \beta$. Then the following dichotomy holds:*

- *either there is an arbitrarily small perturbation g of f admitting a homoclinic tangency associated to the continuation of some saddle of $H(P,f)$;*
- *or there is a dominated splitting for f on $H(P,f)$*

$$T_{H(P,f)}M = E \oplus E_1^c \oplus .... \oplus E_{\beta-\alpha}^c \oplus F,$$

where $dim(E) = \alpha$ and each E_i^c is 1-dimensional and not hyperbolic.

However, we do not know yet whether E and F are uniformly contracted and uniformly expanded when α is minimal and β maximal. In a recent preprint, S. Crovisier [22] showed the following:

Theorem 6.6.9 (Crovisier). *Given a a compact manifold M, there is a residual subset $\mathcal{R}_2$ of $\mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{R}_2$, for each chain recurrent class $\mathcal{C}$ of f, there is a sequence of periodic orbits of f that tends to $\mathcal{C}$ for the Hausdorff topology.*

Still, as explained in [3], for we have no precise information on the indexes of the saddle points, this is not sufficient to generalize the previous dichotomy to all the chain-recurrent classes. Precisely, with Theorem 6.6.6 we have the dichotomy only for chain recurrent-classes that contain a periodic orbit.

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