Intégrabilité, renormalisation et fractions continues

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Intégrabilité, renormalisation et fractions continues

(Systèmes Dynamiques)

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Chapter 1

Introduction et Résumé

Ce texte a pour cadre l'étude des systèmes dynamiques continus et discrets. Il est divisé en quatre chapitres et contient les résultats principaux obtenus après ma thèse.

Dans le Chapitre 2 nous nous intéressons au problème d'intégrabilité du problème plan des trois corps (voir [T1], [T2]).

Un système Hamiltonien est dit intégrable s'il a suffisamment de lois de conservation, i.e d'intégrales premières comme l'énergie, le moment cinétique, etc. Depuis cinquante ans, il existe une grande variété de résultats concernant les systèmes intégrables. En particulier, on observe qu'ils sont localement "tous semblables" dans le sens où le mouvement d'un tel système est (en dehors des singularités) un mouvement quasi périodique sur des tores. Par contre, les systèmes non-intégrables sont "tous différents" et la nature de leur non–intégrabilité est spécifique dans chaque cas particulier. Ce thème a rencontré récemment un vif regain d'intérêt, à travers les travaux de Ziglin, Morales et Ramis (voir l'exposé de M. Audin au séminaire Bourbaki [2] pour un panorama des travaux récents, incluant les notres). A ce jour, la théorie d'intégrabilité moderne compte parmi ses méthodes principales :

La méthode directe : on cherche des intégrales premières sous une forme particulièrement simple, par exemple comme des fonctions algébriques. Cette méthode est très souvent utilisée dans l'analyse des équations de la mécanique mais nécessite souvent de très gros calculs.

Les méthodes perturbatives, comme la théorie KAM conduisant à la notion d'intégrabilité sur un feuilletage cantorien, la méthode de Poincaré-Melnikov fournissant des résultats de non-integrabilité, et les idées développées récemment dans le cadre de la diffusion d'Arnold. Ces méthodes s'appliquent au cas d'un système hamiltonien suffisamment proche d'un système intégrable, néanmoins elles constituent une part très élaborée de la théorie d'intégrabilité réelle.

Les techniques de type complexe : utilisées depuis les années quatre-vingt (travaux de Morales, Ramis et Ziglin) elles sont beaucoup plus puissantes que les techniques réelles.

La mécanique céleste fournit des problèmes d'intégrabilité les plus profonds, avec des implications fondamentales pour l'astrophysique, la théorie de notre système solaire et les missions spatiales.

Dans ses *Principia*, après avoir résolu le problème képlérien de deux corps (qui décrivent toujours des trajectoires coniques), Isaac Newton considérait un système constitué par le Soleil, la Terre et la Lune. Le problème des trois corps ainsi posé, où l'on considère trois masses ponctuelles en interaction gravitationnelle, ne fait intervenir qu'un corps de plus par rapport au problème képlérien, mais il s'est avéré infiniment plus complexe. Du point de vue analytique, le problème des trois corps dans le plan est un système de six équations différentielles dont les solutions possèdent une extrême sensibilité aux conditions initiales.

A la suite des travaux de Poincaré, Moser, Kolmogorov et Arnold, on appelle maintenant ce phénomène le *chaos déterministe*. Cette sensibilité est responsable de l'imprévisibilité à très long terme du mouvement de la Lune, Mercure, Venus, la Terre [60]. Cette étude fait partie de la théorie la plus générale, celle de l'intégrabilité où l'on étudie des transcendantes nouvelles qui apparaissent comme des solutions et des intégrales premières (i.e lois de conservation) des équations différentielles. C'est pour cela que, en suivant le chemin indiqué par Kovalevskaya et Painlevé, il faut sortir du domaine réel et étudier des fonctions complexes multiformes. C'est ce phénomène de multiformité, “invisible” d'un point de vue réel, qui nous intéresse et qui nous guidera dans notre tentative de comprendre l'origine de la complexité des solutions réelles.

Parmi les techniques dont on dispose dans ce domaine je mentionne d'abord l'approche de Ziglin [96], basée sur l'étude du groupe de monodromie des équations linéarisées du flot et le théorème de Morales et Ramis [67], [68] qui permet de démontrer que certains systèmes Hamiltoniens ne sont pas intégrables en utilisant un groupe de Galois différentiel.

Dans ma thèse [T7], (voir aussi [T8], [T6]), j'ai montré que, au voisinage de la solution particulière de Lagrange, le problème plan des trois corps n'admet pas *deux* intégrales premières méromorphes supplémentaires indépendantes.

Le même résultat a été obtenu par Boucher [14] à l'aide du théorème de non-intégrabilité de Morales et Ramis avec l'hypothèse supplémentaire selon laquelle toutes les intégrales sont en involution. Dans la Section 2.2 du texte nous montrons comment une synthèse fructueuse de la méthode géométrique de Ziglin et de l'approche algébrique infinitésimale de Morales et Ramis nous permet aboutir au résultat beaucoup plus général : l'absence, sauf dans quelques cas exceptionnels, d'une seule intégrale première méromorphe supplémentaire.

Notre preuve est basée sur l'étude du groupe de monodromie de l'équation aux variations normales le long des solutions paraboliques de Lagrange. Il arrive que ce groupe ait une structure assez particulière, en particulier il possède deux générateurs unipotents et un centralisateur non trivial. Grâce à cette propriété on peut passer directement à l'étude de son algèbre de Lie et réduire encore, sauf en certains cas exceptionnels, le nombre des intégrales premières possibles.

Posons

$$\sigma = \frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{(m_1 + m_2 + m_3)^2}.$$

Voici les résultats principaux du Chapitre 2 publiés dans [T1], [T2].

Theorem 1. *Soit*

$$\sigma \notin \left\{ \frac{1}{3}, \frac{2}{9}, \frac{2^3}{3^3}, \frac{7}{48}, \frac{5}{24} \right\},$$

alors, au voisinage de la solution de Lagrange Γ*, le problème plan des trois corps n'admet pas d'intégrale première méromorphe supplémentaire et fonctionnellement indépendante du Hamiltonien* H*.*

Theorem 2. *Dans les cas* $\sigma = 1/3$ *et* $\sigma = 2^3/3^3$*, l'équation aux variations normales du système le long de la solution* Γ *a une nouvelle intégrale première méromorphe.*

Le Théorème 1 généralise le résultat de Bruns ([12], p. 358), qui a montré en 1887 que le problème général des trois corps n'admet pas d'intégrale première algébrique, autre que les intégrales déjà connues.

Voici l'esquisse de la démonstration du Théorème 1. Considérons trois corps P_1, P_2, P_3 dans le plan avec des masses $m_1 > 0$, $m_2 > 0$, $m_3 > 0$ qui s'attirent conformément à la loi de Newton. En supposant que le centre de gravité est fixe et en utilisant l'intégrale des aires, on obtient les équations du mouvement sous la forme suivante ([93], p. 353) :

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r}, \quad (r = 1, 2, 3). \tag{1.1}$$

Ici

$$H=\frac{M_1}{2}\left\{p_1^2+\frac{1}{q_1^2}P^2\right\}+\frac{M_2}{2}(p_2^2+p_3^2)+\frac{1}{m_3}\left\{p_1p_2-\frac{p_3}{q_1}P\right\}-$$
$$-\frac{m_1m_3}{r_1}-\frac{m_3m_2}{r_2}-\frac{m_1m_2}{r_3},$$
$$P=p_3q_2-p_2q_3-k,$$

et

$$r_1=q_1,\quad r_2=\sqrt{q_2^2+q_3^2},\quad r_3=\sqrt{(q_1-q_2)^2+q_3^2}$$

sont les distances mutuelles des corps et k est la constante du moment cinétique.

Supposons que $k\neq 0$; alors il existe une solution particulière Γ dans laquelle les trois corps forment un triangle équilatéral et décrivent chacun une parabole ([59]).

Nous avons trouvé dans [T6] l'équation aux variations normales du système (1.1) le long de la solution Γ :

$$\frac{dx}{d\tau}=\left(\frac{A}{\tau-\tau_0}+\frac{B}{\tau-\tau_1}+\frac{C}{\tau-\tau_2}\right)x,\quad x\in\mathbb{C}^4. \tag{1.2}$$

C'est un système linéaire avec quatre singularités régulières τ_0, τ_1, τ_2, ∞ et A, B, C sont les 4×4 matrices constantes qui dépendent des masses m_1, m_2, m_3.

Soit $G\subset Sp(4,\mathbb{C})$ le groupe de monodromie de l'èquation (1.2). Alors, d'après le lemme 3.1 de [T6], le groupe G est engendré par deux matrices unipotentes $T_i=I+d_i$, $i=1,2$ où $d_i^2=0$ et où I est la matrice unité. Ici T_1 et T_2 sont les généateurs de G correspondants respectivement aux groupes de monodromie locaux autour les singularités τ_1 et τ_2.

Plaçons la singularité $\tau=\infty$ de l'équation (1.2) en $\tau=0$ à l'aide du changement $\tau=1/z$. Nous obtenons alors le système suivant :

$$\frac{dx}{dz}=\left(\frac{A}{z-\tau_0^{-1}}+\frac{B}{z-\tau_1^{-1}}+\frac{C}{z-\tau_2^{-1}}+\frac{A_\infty}{z}\right)x,\quad x\in\mathbb{C}^4, \tag{1.3}$$

où $A_\infty=-(A+B+C)$.

Par le calcul effectué dans [T6] :

$$\text{Spectre}(A_\infty)=\{\lambda_1,\lambda_2,3-\lambda_1,3-\lambda_2\},$$

où

$$\lambda_1=\frac{3}{2}+\frac{1}{2}\sqrt{13+\sqrt{\theta}},\quad \lambda_2=\frac{3}{2}+\frac{1}{2}\sqrt{13-\sqrt{\theta}},\quad \theta=144\,(1-3\sigma)\,. \tag{1.4}$$

Soit $T_1T_2 = T_\infty^{-1}$ où T_∞ est l'élement de G correspondant au groupe de monodromie local autor le point $\tau = \infty$ (autour le point $z = 0$ pour l'équation (1.3)). Posons

$$E = \left\{ \frac{1}{3}, \frac{2}{9}, \frac{2^3}{3^3}, \frac{7}{48}, \frac{5}{2^4} \right\}.$$

Lemma 1 ([T2]). *Supposons que $\sigma \notin E$. Alors la matrice T_∞ a des valeurs propres simples.*

Voici l'idée de la démonstration. On peut montrer par un calcul direct à l'aide des formules (1.4) que si $\sigma \notin E$ alors la matrice A_∞ n'a pas de valeurs caractéristiques distinctes différant entre elles par un entier. Nous pouvons donc établir (voir par exemple [4]) que la solution $x(z)$ du système (1.3) au voisinage de point $z = 0$ est de la forme suivante :

$$x(z) = a(z)z^{A_\infty},$$

où $a(z)$ est une matrice analytique au voisinage de $z = 0$. Dans ce cas, la matrice de monodromie T_∞ est conjuguée à $\exp(2\pi i A_\infty)$ qui a des valeurs propres simples.

On va montrer maintenant que si $\sigma \notin E$ alors la représentation du groupe G est réductible.

Lemma 2 ([T2]). *Soit $\sigma \notin E$. Alors, relativement à une base convenable, les matrices T_1 et T_2 sont de la forme :*

$$T_1 = \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}, \tag{1.5}$$

où $A_{1,2}$, $B_{1,2}$ sont des matrices unipotentes.

Le point–clé de la démonstration du théorème 1 est le lemme suivant, dû à Morales et Ramis (voir [68]) et à Ziglin [96].

Lemma 3. *Si, dans un voisinage de la solution particulière de Lagrange Γ, le système* (1.1) *possède une intégrale première méromorphe et fonctionnellement indépendante de l'énergie H, alors le groupe de Galois (le groupe de monodromie) de l'équation* (1.2) *a un invariant rationnel.*

Theorem 3 ([T2]). *Supposons qu'on ait $\sigma \notin E$. Alors le groupe de monodromie G ne possède pas d'invariant rationnel.*

L'idée de la démonstration est d'étudier l'invariant rationnel $I(x_1, x_2, x_3, x_4)$ du groupe $G =< T_1, T_2 >$ où les matrices T_1, T_2 sont données par (1.5).

Aux générateurs T_1, T_2 on associe deux opérateurs l'inéaires différentiels :

$$\delta = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3},$$

$$\Delta = (ax_1 + a_1 x_2)\frac{\partial}{\partial x_1} + (a_2 x_1 - a x_2)\frac{\partial}{\partial x_2} + (b x_3 + b_1 x_4)\frac{\partial}{\partial x_3} + (b_2 x_3 - b x_4)\frac{\partial}{\partial x_4}.$$

Soient $\Delta_1 = [\delta, \Delta]$, $\Delta_2 = [\delta, \Delta_1]$ leur commutateurs. Alors, d'après le lemme 4.3 du [T6] qui utilise le caractère unipotent de $T_{1,2}$, nous avons que :

$$\delta I = \Delta I = \Delta_1 I = \Delta_2 I = 0.$$

En étudiant les conditions de la compatibilité de ce système et en utilisant la condition $\sigma \notin E$, on peut démontrer [T2] qu'il ne possède pas de solution rationnelle $I(x_1, x_2, x_3, x_4)$. Ceci achève la démonstration du théorème 3. Le théorème 1 est alors une conséquence du lemme 3. Le cas $\sigma = 1/3$ est équivalent au cas $m_1 = m_2 = m_3$ des masses égales. On peut noter que, dans les cas $\sigma = 1/3, 2^3/3^3$, le groupe de monodromie G a un invariant polynomial qui correspond à une intégrale première univalente de l'équation aux variations normales du système (1.3).

Il est évident que la plupart des systèmes d'équations différentielles sont non-intégrables. Néanmoins, il y a des exceptions très remarquables. Voici un exemple contenu dans notre travail [T13].

Considérons le système de six équations différentielles dérivé des équations d'Euler sur l'algèbre de Lie so(4) [32] :

$$\dot{M} = [M, A\,M], \quad \dot{L} = [L, B\,M], \quad M, L \in \mathbb{R}^3, \tag{1.6}$$

où $A = \mathrm{diag}(a_1, a_2, a_3)$, $B = \mathrm{diag}(b_1, b_2, b_3)$ et $[\,\cdot\,]$ est le produit vectoriel dans $\mathbb{R}^3$. D'après l'interprétation de Veselov, ces équations décrivent le mouvement d'une toupie contenant à l'intérieure une cavité elliptique remplie de liquide.

Les intégrales premières connues sont : $I_1 = (M, AM)$, $I_2 = (M, M)$, $I_3 = (\gamma, \gamma)$, où $(\,\cdot\,)$ est le produit scalaire dans $\mathbb{R}^3$.

En 1988, Fomenko [32] a suggéré, en appuyant sur l'analyse de Kovalevskaya, que le système (1.6) a une nouvelle intégrale première supplémentaire I_4, si les matrices diagonales A et B sont liées par la relation $B = k\,A$ avec k un nombre entier impair. Dans notre travail [T13] pour chaque $k = 1, 3, 5, ...$ nous avons trouvé I_4 sous la forme d'un polynôme homogène de M et γ de degré $k + 1$:

$$I_4 = (P(M), D\gamma), \quad P(M) = \mathcal{K}(M)P_0, \quad P_0 \in \mathbb{R}^3,$$

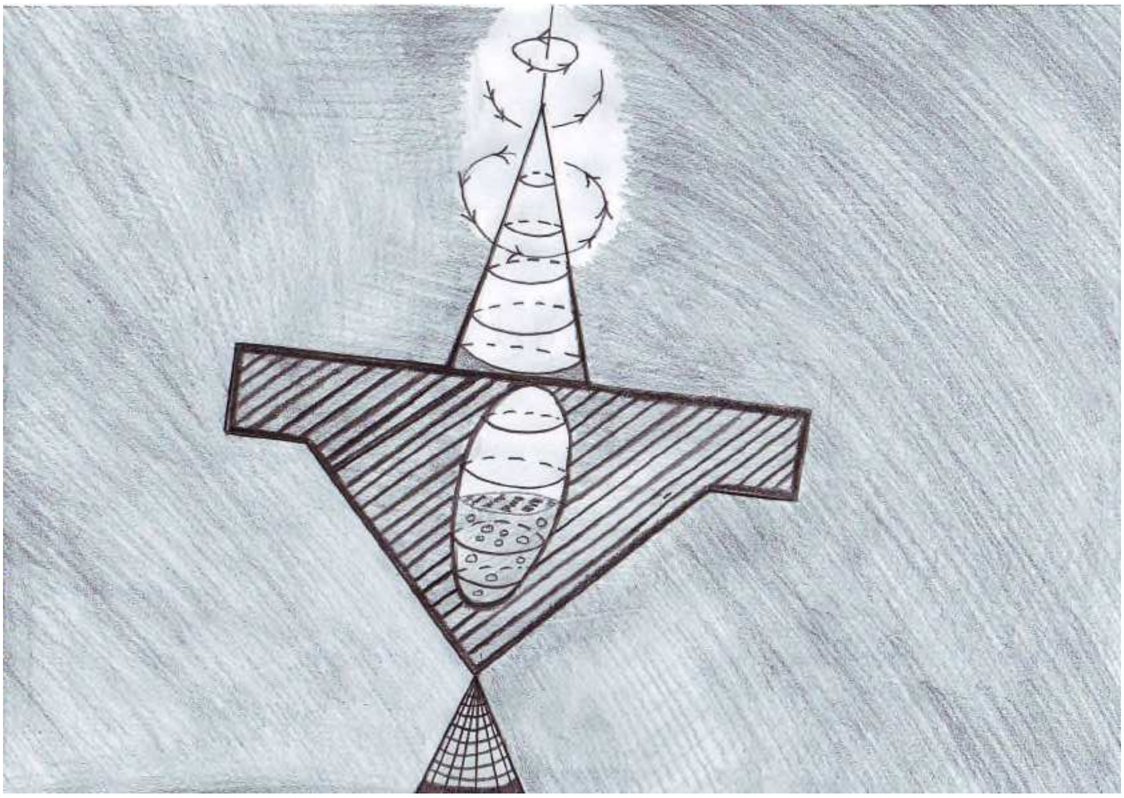

Figure 1.1: La toupie de Fomenko–Veselov.

où $\mathcal{K}(M)$ est une matrice qui depend de $M = (m_1, m_2, m_3)$:

$$\mathcal{K}(M) = \operatorname{diag}(m_1, m_2, m_3) \prod_{n=0}^{\frac{k-3}{2}} \left(J_n \operatorname{diag}(m_1^2, m_2^2, m_3^2)\right),$$

et $J_n \in GL(3, \mathbb{C})$ sont des matrices constantes connues.

A ce jour, l'intégration du système (1.6) en quadratures reste toujours un problème ouvert.

Finalement, dans la Section 2.3, nous exposons nos résultats récents dans le problème d'intégrabilité de la pierre celte (rattleback). C'est une toupie amusante dont la stabilité des mouvements dépend du sens de la rotation (Figure 2.8, page 57).

Le Chapitre 3 du texte est consacré à mes travaux sur la théorie de renormalisation (voir [TMO1], [TMO2], [T4]).

En 1978, Feigenbaum (regarde aussi [17]), pour expliquer les propriétés universelles obtenues numériquement de certaines familles d'applications unimodales dépendantes d'un paramètre, proposent une méthode de démonstration basée sur l'analyse du comportement d'itération d'un certain opérateur, dit de *renormalisation*. Le point fixe $g(x)$ de cet opérateur satisfait l'équation fonctionnelle de Feigenbaum :

$$g(x) = -\lambda^{-1}\, g(g(-\lambda\, x)), \quad g(0) = 1, \quad \lambda = -g(1)\,, \tag{1.7}$$

et joue un rôle fondamental dans la théorie.

En 1986, H. Epstein a montré que si $u(x)$ est une fonction réciproque de $g(x)$, alors $u(x)$ est une fonction anti-Herglotzienne.

Dans le travail [T4], en utilisant la théorie analytique des fractions continues, nous prouvons que $u(x)$ s'écrit toujours sous la forme suivante :

$$u(x) = \cfrac{1}{1 + \cfrac{g_1\, f_\lambda(x)}{1 + \cfrac{(1-g_1)g_2\, f_\lambda(x)}{1 + \cfrac{(1-g_2)g_3\, f_\lambda(x)}{1 + \cdots}}}},$$

où $f_\lambda(x)$ est une fonction rationnelle connue de x et λ, et les $g_i \in (0, 1)$.

Par conséquent, on en déduit des nouvelles caractéristiques de la solution $g(x)$ de l'équation de Feigenbaum (1.7). Ces résultats sont exposés dans la section 3.2.

Dans [TMO2] nous avons étudié, dans le cadre de la théorie de la renormalisation, des familles d'applications unimodales asymétriques [64]. Notre résultat principal est la caractérisation de l'hyperbolicité du point fixe de l'opérateur de renormalisation correspondant au cas asymétrique. C'est une généralisation naturelle des résultats classiques qu'on trouve dans la situation symétrique (le cas de Feigenbaum) obtenus par Epstein et Eckmann dans [22] en 1990. La méthode qu'on utilise est basée sur l'étude des espaces fonctionnels herglotziens auxquels appartiennent les points fixes des opérateurs de renormalisation.

Voici un résumé bref des résultats exposés dans le Chapitre 3.100.

En suivant Arneodo *et al* [6] nous considérons une application unimodale asymétrique :

$$f = (f_L, f_R) \,:\, [-1, 1] \to [-1, 1]\,,$$

de degré $d > 1$. L'exemple plus simple d'une application de ce type est donné par :

$$f(x) = \begin{cases} f_L(x) = 1 - a_1|x|^d & \text{si}\, x \leq 0\,; \\ f_R(x) = 1 - a_2|x|^d & \text{si}\, x \geq 0\,. \end{cases}$$

où $d > 1$ et a_1, $a_2 \in \mathbb{R}^+$.
On rappelle que pour une application unimodale $f : [-1,1] \to [-1,1]$, $f(0) = 1$, l'opérateur de renormalisation est défini par :

$$R(f) = -\lambda^{-1} f(f(-\lambda x)), \quad \lambda = -f(1)\,. \tag{1.8}$$

Dans le cas asymétrique, on étudie les points fixes de R de période 2, ce que revient à écrire un système de quatre équations fonctionnelles (voir [64]) :

$$\begin{aligned}
\tilde{f}_L(x) &= -\lambda^{-1} f_R f_R(-\lambda x)\,,\\
\tilde{f}_R(x) &= -\lambda^{-1} f_R f_L(-\lambda x)\,,\\
f_L(x) &= -\tilde{\lambda}^{-1} \tilde{f}_R \tilde{f}_R(-\tilde{\lambda} x)\,,\\
f_R(x) &= -\tilde{\lambda}^{-1} \tilde{f}_R \tilde{f}_L(-\tilde{\lambda} x)\,,
\end{aligned}$$

avec la normalisation $f_L(0) = f_R(0) = \tilde{f}_L(0) = \tilde{f}_R(0) = 1$ et $\lambda = -f_R(1) > 0$, $\tilde{\lambda} = -\tilde{f}_R(1) > 0$.
La solution de ce système formée par $(f_L, f_R, \tilde{f}_L, \tilde{f}_R)$ dépend du *module d'asymétrie* :

$$\mu = \frac{f_L^{(d)}(0-)}{f_R^{(d)}(0+)}\,.$$

et de degré $d > 1$.
Le cas $\mu = 1$ correspond au cas symétrique de Feigenbaum.

Theorem 4 ([65])**.** *Pour tout $\mu > 0$ et tout $d > 1$, il existe une solution de $R^2 f = f$ avec $f = (f_L, f_R)$ analytique au voisinage de l'intervalle $[-1,1]$.*

On vérifie que l'action de R sur f définie par (1.8) préserve le degré d et inverse le module μ. Cela explique notre passage à l'étude de points fixes de R^2 dans le cas asymétrique.
Il est préférable de travailler avec l'opérateur R_p donné par $R_p(f, \tilde{f}) = (R(\tilde{f}), R(f))$, $\tilde{f} = R(f)$. Dans ce cas, le point fixe de R_p correspond à une orbite périodique de période 2 de R. Le spectre de dR^2 est lié au spectre de R_p : chaque valeur propre ρ^2 de $dR^2(f)$ correspond à une paire $\pm\rho \in \mathrm{Spectre}(dR_p(f))$.
Pour des applications unimodales asymétriques de la forme (1.8) on observe le même phénomène d'universalité que dans le cas symétrique étudié en 1975 par Feigenbaum pour la famille d'applications quadratiques $f_\mu : [-1,1] \to [-1,1]$:

$$f_\mu(x) = 1 - \mu x^2, \quad \mu \in (0,2]\,. \tag{1.10}$$

La dynamique de f_μ dépend de façon très sensible des variations de paramètre μ. Pour $\mu < 3/4$, on a un point fixe stable de f_μ donné par $x_\mu = \frac{\sqrt{1+4\mu}-1}{2\mu}$ Si l'on fait croître lentement μ, de très intéressants phénomènes apparaissent : pour des valeurs successives de μ : $\mu_1 = 3/4$, μ_2, μ_3 etc... se produit ce qu'on appelle la *bifurcation du doublement de période* – une orbite de période 2^i dévient instable en donnant naissance à une orbite stable de période $2 \cdot 2^i = 2^{i+1}$ (la *cascade* de doublements de période).

Soit μ_n la valeur de μ pour laquelle apparaît une cycle de longueur 2^n. On observe alors que la suite $\{\mu_n\}_{n=1}^{n=\infty}$ est croissante et $\lim\limits_{n\to\infty} \mu_n = \mu_\infty < 1$.

Les *constantes universelles* de Feigenbaum δ et α sont définies par :

$$\delta = \lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} = 4.6692016\ldots, \tag{1.11}$$

et

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = -2.502907875\ldots,$$

où d_n est la distance entre l'orbite de la période 2^n et $x = 0$.

Le mot "universalité" vient du fait qu'en remplaçant la famille d'applications (1.10) par une autre famille d'applications unimodales quadratiques *arbitraire*, nous arrivons aux mêmes nombres de Feigenbaum. Par exemple, nous pouvons bien étudier la famille d'applications quadratiques transcendantes : $f_\mu(x) = 1 - \mu \sin(x)^2$ et obtenir ainsi les mêmes valeurs de α et δ.

Voici comment on explique le scénario du doublage de période et son universalité. Imaginons l'action de l'opérateur R défini par (1.8) dans l'espace fonctionnel $\mathcal{F}$ formé par toutes les applications analytiques réelles $f : [-1, 1] \to [-1, 1]$, $f(0) = 1$. L'idée de Feigenbaum était d'admettre que R possède dans $\mathcal{F}$ *un point fixe hyperbolique* $g(x)$ qui est à son tour la solution de l'équation fonctionnelle (1.7). La variété instable W^u de $g(x)$ de dimension 1 correspond à l'unique valeur propre instable de $dR(g)$ donnée par $\delta > 1$.

Soit Γ_1 l'ensemble de codimension 1 de toutes les applications dans $\mathcal{F}$ pour lesquelles le point fixe attractif de R est en train de bifurquer en une orbite stable de période 2. Alors, Γ_1 intersecte transversalement la variété instable W^u de g. L'image réciproque $\Gamma_n = R^{-n}(R(\Gamma_0))$ est exactement l'ensemble de toutes les application dans $\mathcal{F}$ dont l'orbite stable de période 2^{n-1} bifurque en une orbite stable de période 2^n. Les ensembles Γ_n de codimension 1 ainsi obtenus s'accumulent vers la variété stable W^s de g qui est de codimension 1. Soit f_μ une famille d'applications unimodales intersectant W^s transversalement et μ_n sa suite de bifurcations du doublement de période. On a $f_{\mu_n} \in \Gamma_{\mu_n}$ et $f_{\mu_\infty} \in W^s$.

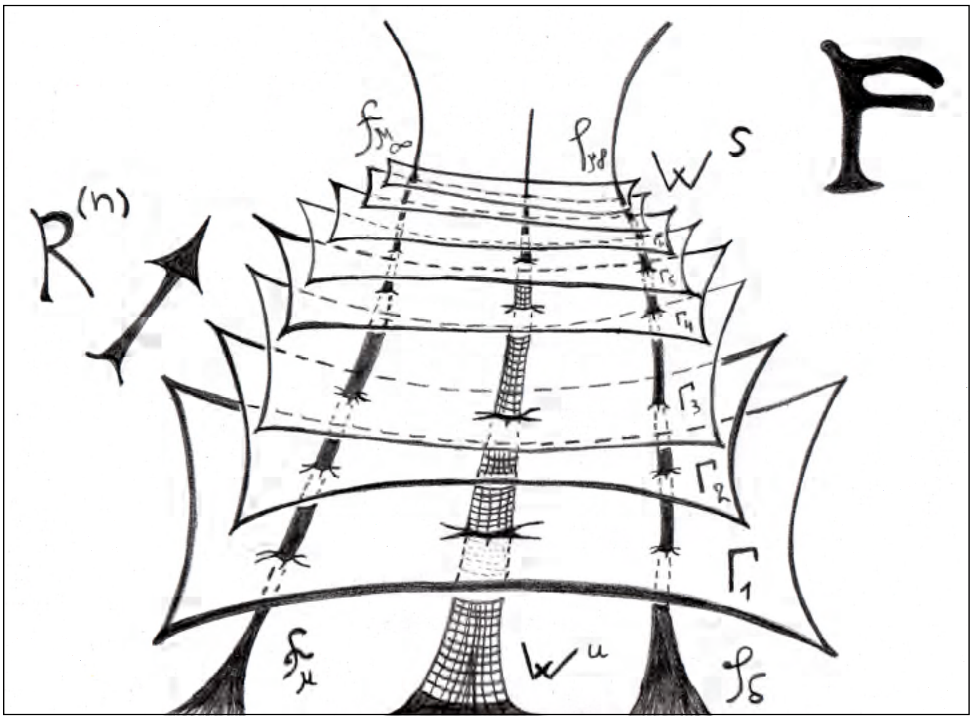

Figure 1.2: L'action de R.

Le fait que g est un point fixe hyperbolique implique que μ_n s'accumulent vers μ_∞ de la façon indiquée par (1.11) (voir [58] pour la démonstration). Le même scénario se produit dans le cas d'applications unimodales de degré quelconque $d > 1$. Il faut souligner que les valeurs de δ et α (pour d fixé) ne dépendent ni de la famille transverse ni de son paramétrage. Donc l'analyse de l'opérateur R nous fournit les valeurs de ces exposantes et prouve leur universalité.

Pour une famille d'applications asymétriques (1.8) (où μ et d sont fixés), l'universalité est toujours reflétée par la dynamique de R_p dans un espace de Banach fonctionnel bien choisi de paires (f_L, f_R). En regardant le point fixe f de R_p, nous allons maintenant étudier le spectre de $dR_p(f)$.

En supposant que λ et $\tilde{\lambda}$ sont constantes (ce que va introduire tout simplement deux valeurs propres dans le Spectre(dR_p)), on écrit la dérivée de R_p en point fixe $(f, \tilde{f})$ comme :

$$dR_P(f,\tilde{f})\begin{pmatrix} \delta f \\ \delta\tilde{f} \end{pmatrix} = \begin{pmatrix} -\tilde{\lambda}^{-1}\delta\tilde{f}(\tilde{f}(-\tilde{\lambda}x)) - \tilde{\lambda}^{-1}\tilde{f}'(\tilde{f}(-\tilde{\lambda}x))\delta\tilde{f}(-\tilde{\lambda}x) \\ -\lambda^{-1}\delta f(f(-\lambda x)) - \lambda^{-1}f'(f(-\lambda x))\delta f(-\lambda x)) \end{pmatrix}.$$

Par définition des applications unimodales, il existe des fonctions $F, \tilde{F}$ telles que $f(x) = F(|x|^d)$ et $\tilde{f}(x) = \tilde{F}(|x|^d)$. Nous posons $(v(x), \tilde{v}(x)) = (\frac{\delta F}{F'}, \frac{\delta\tilde{F}}{\tilde{F}'})$. En suivant [22], nous définissons l'application $q : \mathbb{R} \to \mathbb{R}$ par :

$$q(x) = \mathrm{sign}(x)|x|^d.$$

On définit également L et $\tilde{L}$ par :

$$L(x) = q(F(x)), \quad \tilde{L}(x) = q(\tilde{F}(x)), \quad x \in [0,1].$$

L'opérateur dR_p s'écrit alors sous la forme plus compacte :

$$T\begin{pmatrix} v(x) \\ \tilde{v}(x) \end{pmatrix} = \begin{pmatrix} \tilde{t}^{-1}(\tilde{v}(\tilde{t}x) + \tilde{v}(\tilde{L}(\tilde{t}x))\tilde{L}'(\tilde{t}x)^{-1}) \\ t^{-1}(v(tx) + v(L(tx))L'(tx)^{-1}) \end{pmatrix}.$$

où $t = \mu\lambda^d$, $\tilde{t} = \mu^{-1}\tilde{\lambda}^d$.

On définit dans [TMO2] un espace B de paires des fonctions $(v(x), \tilde{v}(x))$ qui est invariant par l'action de T, avec la propriété que T possède sur B une unique valeur propre de valeur absolue maximale $\delta > 0$. L'espace propre correspondant est engendré par un élément de B qui est dans l'intérieur d'un cône $\Gamma \subset B$, lui aussi invariant par T. Le théorème de Krein et Rutman [48] nous permet alors d'estimer la valeur de δ. Notre résultat principal (en collaboration avec B Mestel and A Osbaldestin) est donné par le théorème suivant.

Theorem 5 ([TMO2])**.** *Il existe un espace fonctionnel B invariant par T sur lequel T est compact et possède un valeur propre $\delta > 0$ tel que :*

$$1 < \frac{1}{(\lambda\tilde{\lambda})^{(d-1)}(1+\sqrt{\lambda\tilde{\lambda}})^2} < \delta^2 < \frac{1}{(\lambda\tilde{\lambda})^d}.$$

Finalement, dans la Section 3.3, nous parlons de nos recherches récentes dans le problème de la renormalization des applications non-commutatives du cercle.

Dans le Chapitre 4 nous nous intéressons à la question de convergence de fractions continues ∞–périodiques :

$$\frac{1}{1-}\ \frac{a_1}{1-}\ \frac{a_2}{1-}\ \frac{a_3}{1-\cdots}, \quad \lim_{i\to\infty} a_i = a, \quad a_i \in \mathbb{R}. \tag{1.12}$$

Pour $a < 1/4$ la fraction ci-dessus converge d'après le résultat de Van Vleck [90]. Dans le cas $a = 1/4$ elle est convergente ou divergente selon la vitesse de la convergence de a_i vers a (Gill, [38]).

Dans ses notes [1], Ramanujan a conjecturé que la fraction continue (1.12) est toujours divergente sous la condition $a > 1/4$.

Dans [T5] nous construisons un exemple explicite d'une fraction continue ∞-périodique convergente :

$$\frac{1}{1-}\ \frac{1}{1-}\ \frac{3/2}{1-}\ \frac{1}{1-}\ \frac{1/3}{1-}\ \frac{5/3}{1-}\ \frac{1}{1-}\ \frac{3/5}{1-}\ \frac{7/5}{1-}\ \frac{1}{1-}\ \frac{5/7}{1-}\ \frac{9/7}{1-}\ \frac{1}{1-}\ \frac{7/9}{1-}\ \frac{11/9}{1-}, \tag{1.13}$$

pour laquelle on a $\lim\limits_{i\to\infty} a_i = a = 1 > 1/4$.

Pour expliquer nos idées, on considère l'ensemble $\mathcal{W}$ de fractions continues :

$$g(z) = \frac{1}{1-}\ \frac{g_1 z}{1-}\ \frac{g_2(1-g_1)z}{1-}\ \frac{g_3(1-g_2)z}{1-\cdots}, \quad z \in \mathbb{C}, \quad g_i \in (0,1)\,. \tag{1.14}$$

On peut démontrer [92] que $g(z)$ est toujours une fonction analytique dans $\mathbb{C}_+ \cup \mathbb{C}_- \cup (-\infty, 1)$.

Soit $h_i(z;t) = \dfrac{a_i}{1+t}$, $i \geq 0$ une suite infinie de transformations de Möbius de la variable t avec $a_0 = 1$, $a_1 = -zg_1$, $a_i = -g_i(1-g_{i-1})z$, $i \geq 2$ et soit $H_n(z;t) = h_0 \circ h_1 \circ \cdots \circ h_n(z;t)$.

La fraction continue (1.14) est convergente (dans le sens normal) pour $z \in \mathbb{C}$ si les deux limites

$$\lim_{n\to\infty} H_n(z;0) = \lim_{n\to\infty} H_n(z;\infty)\,,$$

existent dans $\bar{\mathbb{C}}$.

On dit que (1.14) *converge généralement* vers $\alpha \in \bar{\mathbb{C}}$ pour $z \in \bar{\mathbb{C}}$ s'il existe deux suites u_n et v_n dans $\bar{\mathbb{C}}$ telles que :

$$\lim_{n\to\infty} H_n(z;u_n) = \lim_{n\to\infty} H_n(z;v_n) = \alpha, \quad \liminf_{n\to\infty} \sigma(u_n, v_n) > 0\,,$$

où $\sigma(x,y)$ est la distance sphérique entre $x, y \in \bar{\mathbb{C}}$.

En particulière, chaque fraction convergente converge aussi généralement. Il y a des exemples [13] de fractions continues convergentes généralement mais pas normalement.

Dans [79] Runckel a étudié une classe particulière $\mathcal{R}$ d'endomorphismes holomorphes du disque unité $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| = 1\}$ représentés à l'aide des fractions continues de Schur [84] convergentes uniformément à l'intérieur de

$\mathbb{D}$. On peut avoir sur la frontière $\partial\mathbb{D}$ des points (dites de *Runckel*) ou une fraction continue $w(z) \in \mathcal{R}$ converge également. Dans le Chapitre 4 nous établissons la relation entre les deux classes de fractions continues $\mathcal{W}$ et $\mathcal{R}$ de telle manière que si $e(z) \in \mathcal{W}$ converge en un point de Runckel $z \in \partial\mathbb{D}$, alors la g-fractions continue $g_e(z)$ (1.14), correspondante à $e(z)$, converge généralement en un point $\tilde{z} > 1$ (voir le Théorème 40). Remarquons qu'en général, la demi-droite $(1, +\infty)$ est une ligne *singulière* pour $g_e(z)$, où la fraction continue diverge en général.

Dans des cas exceptionnels, la g-fraction $g_e(\tilde{z})$ peut converger dans le sens normal, ce qui se produit, par exemple, pour l'endomorphisme ϕ_p de $\mathbb{D}$ donné par $\phi_p(w) = \dfrac{1 + w^p}{2}$, $p = 2n + 1$, $n \in \mathbb{N}$. Notre fraction continue (1.13) correspond alors au cas $p = 3$.

Dans la Section 4.4, nous discutons comment les g-fractions continues peuvent être utiles dans le problème d'approximation des solutions des équations différentielles ordinaires.

Chapter 2

On the absence of an additional meromorphic first integral in the three-body problem

2.1 General facts about integrability

Let X_H be a Hamiltonian vector field defined on a real symplectic maniflold M of dimension $2n$. According to the theorem of Darboux, in a small neighborhood of any point of M there are local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that the symplectic structure written in these variables takes the form

$$\Omega = \sum_{i=1}^{n} dy_i \wedge dx_i \, .$$

For any two functions f, g we define the Poisson bracket according to

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} \, .$$

By definition, Hamiltonian system X_H is called *completely integrable* or *integrable in the sense of Arnold-Liouville* if there are n functions $f_1 = H$, $f_2, \ldots, f_n$ satisfying the following conditions:

(A) $f_1, \ldots, f_n$ are functionally independent i.e the 1 - forms df_i, $i = 1, 2, \ldots, n$ are linearly independent on some open dense subset $U \subset M$.

(B) $f_1, \ldots, f_n$ form a set in *involution*, i.e $\{f_i, f_j\} = 0$, $i, j = 1, 2, \ldots, n$.

As follows from (B), the functions $f_1, \ldots, f_n$ are first integrals of the Hamiltonian systemX_H.
The simplest completely integrable Hamiltonian system is given by

$$H = H_1(x_1, y_1) + \cdots + H_n(x_n, y_n)\,. \tag{2.1}$$

Here $H_i(x_i, y_i)$ are functionally independent differentiable functions of two variables x_i and y_i. It is easy to see that $\{H_i, H_j\} = 0$, $\forall\, i, j = 1, \ldots, n$ and hence H_i form a complete set of first integrals in involution of (2.1).
Let us take a completely integrable Hamiltonian system X_H together with the corresponding set of first integrals in involution $f_1, \ldots, f_n$. We assume that $M_a = \{z \in M : f_i(z) = a_i, i = 1, \ldots, n\}$, $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is a non singular level surface of $f_1, \ldots, f_n$ (in other words $\mathrm{rank}(df_1, \ldots, df_n) = n$ on M_a). One has the following fundamental result (the theorem of Arnold-Liouville):

(a) M_a is a X_H-invariant manifold. In the case when M_a is compact and connected, it is diffeomorphic to n–dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

(b) In a neighborhood of T^n there exists a local system of coordinates $(I, \phi) = (I_1, \ldots, I_n, \phi_1, \ldots, \phi_n)$, $\phi_i \pmod{2\pi}$, called the *action–angle* variables, such that, written in these variables the Hamiltonian equations become

$$\dot{I}_i = 0, \quad \dot{\phi}_i = \omega_i(I), \quad i = 1, 2, \ldots, n\,.$$

The natural question whether a given Hamiltonian system is integrable or not appears to be very a complicated one. We are going to describe now the known obstructions of different nature to integrability of Hamiltonian systems.

2.1.1 Obstructions of topological nature.

Strangely enough, most results in this domain have been obtained only starting from the end of the nineteenth century. The reason is probably that for mathematicians of XVIII-XIX centuries the word "integrability" in the context of systems of differential equations meant exclusively the solution by means of quadratures. The global behavior of trajectories in the phase space was thus completely ignored leaving place to a local approach.

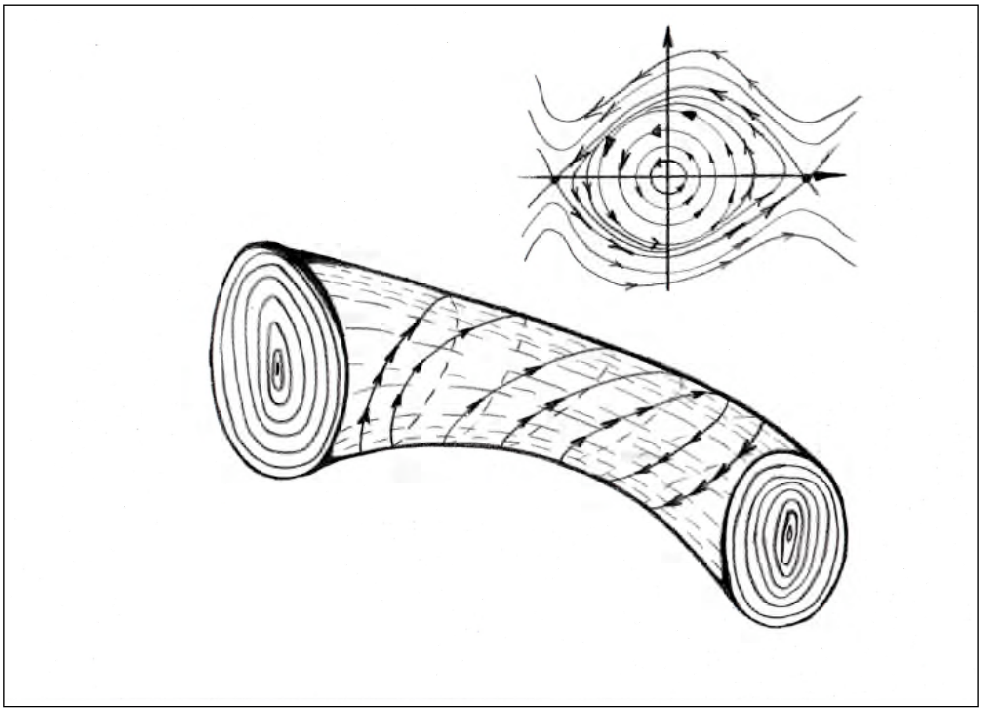

Figure 2.1: The Arnold-Liouville's theorem.

Let us consider a Hamiltonian system with two degrees of freedom

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \tag{2.2}$$

whose configuration space is a two-dimensional compact analytic surface M. One has the following result:

Theorem 6 (**[50]**). *Let the genus g of the surface M be distinct from $0, 1$. Then the Hamiltonian system (2.2) has no additional first integrals analytic on T^*M independent from the Hamiltonian function H.*

The simplest examples of integrable systems corresponding to $g = 0, 1$ are the problems of free motion of a material point on spherical and toroidal surfaces. For $n > 1$ the above result has been generalized by Taimanov:

Theorem 7 ([87], [88]). *Let us assume that the configuration space M^n of a Hamiltonian system with n degrees of freedom is a connected analytic manifold and that the Hamiltonian H is analytic on the whole phase space. If the*

Hamiltonian system has n independent analytic first integrals in involution then the following conditions are fulfilled

$$b_k(M^n) \leq C_k^n, \quad k = 1, \ldots, n\,,$$

where $b_k(M^n)$ is the k^{th}–Betti number of the manifold M.

Let us consider a geodetic flow on a surface M homeomorphic to the two-dimentional sphere S^2. According to the Poincaré theorem, on S^2 there are at least three geodesic curves γ_i. The following theorem belongs to S. Bolotin:

Theorem 8. *Let us assume that the geodesics γ_1, γ_2, γ_3 have no intersections and each of them can be deformed into a point without crossing the others. Then the equations of the geodesic flow on S^2 have no additional analytic first integrals.*

2.1.2 Integrability obstructions related to the intersection of asymptotic surfaces.

In 1888 Poincaré, while studying the three-body problem, established a fundamental relation between the complicated topological behaviour of trajectories and the existence of a completely new qualitative phenomenon: the transversal intersection of asymptotic surfaces.

Let us take a smooth vector field v without singular points defined on a three-dimensional analytic manifold M. We assume that it admits two hyperbolic periodic trajectories γ_1 and γ_2.

Define by Λ_1^+ (resp. Λ_2^-) the stable (unstable) analytic manifold of γ_1 (resp. γ_2).

Theorem 9. *Let us suppose that Λ_1^+ and Λ_2^- intersect transversally. Then the system*

$$\dot{x} = v(x), \quad x \in M\,,$$

does not admit a first integral analytic on M.

This theorem was proved in works of Kozlov [51] and Cushman [20]. The basic idea of the proof is to find a subset of M in a neighborhood of the solutions γ_1, γ_2 on which the possible analytic first integrals must be constant functions. A natural generalization of these ideas in the case of Hamiltonian systems close to integrable ones has been found by Bolotin [10].

We consider a Hamiltonian system given by the analytical Hamiltonian

$$H = H_0(z) + \epsilon H_1(z,t) + o(\epsilon), \quad z = (x,y) \in \mathbb{R}^{2n}\,.$$

such that the non perturbed system corresponding to H_0 has two hyperbolic fixed points $z_\pm$ connected by the doubly asymptotic solution $z_0(t)$, $t \in \mathbb{R}$.
One has the following theorem

Theorem 10 ([10]). *Let us assume that*

1) $\int\limits_{-\infty}^{+\infty} \{H_0, \{H_0, H_1\}\}(z_0(t), t) dt \neq 0,$

2) For sufficiently small values of $\epsilon > 0$ the system has the doubly asymptotic solution $z_\epsilon(t)$ close to $z_0(t)$.
Then for any small fixed value of $\epsilon \neq 0$ the perturbed Hamiltonian system is not completely integrable in a neighborhood of the orbit $z_\epsilon(t)$.

The method of splitting of separatrices has been applied successfully by Kozlov in the problem of rotation of an asymmetric rigid body [52] (the Euler-Poisson system, see the next section). Studying the perturbation of Euler's integrable case it is possible to prove the presence of transversal splitting of asymptotic surfaces and hence to prove the non-integrability. This can be done in all cases except for the Hess-Appelrot's one in which actually there is an additional partial first integral.
The non-integrability of the symmetric case of the Euler-Poisson problem was proved in [53].

2.1.3 Integrability obstructions related to branching of solutions in the complex time plane.

In 1889 S. Kovalevskaya obtained a prize of the French Academy of Sciences for a remarkable discovery concerning the Euler-Poisson problem:

$$\begin{array}{l} A\dot{p} = (B - C)qr + \mu g(y_0\gamma'' - z_0\gamma') \\ B\dot{q} = (C - A)rp + \mu g(z_0\gamma - x_0\gamma'') \\ C\dot{r} = (A - B)pq + \mu g(x_0\gamma' - y_0\gamma) \\ \\ \dot{\gamma} = r\gamma' - q\gamma'' \\ \dot{\gamma}' = p\gamma'' - r\gamma \\ \dot{\gamma}'' = q\gamma - p\gamma' \end{array} \tag{2.3}$$

describing a rotation of a heavy rigid body around a fixed point P.
Here $\gamma = (\gamma, \gamma', \gamma'')$ are the components of the vertical unit vector, $M = (p, q, r)$ is the angular velocity vector , (A, B, C) are body moments of inertia,

$R = (x_0, y_0, z_0)$ is position vector of center of mass, μ is the weight of the body. All vectors are defined in the body fixed axes (with origin in P) coinciding with principal inertia axes.

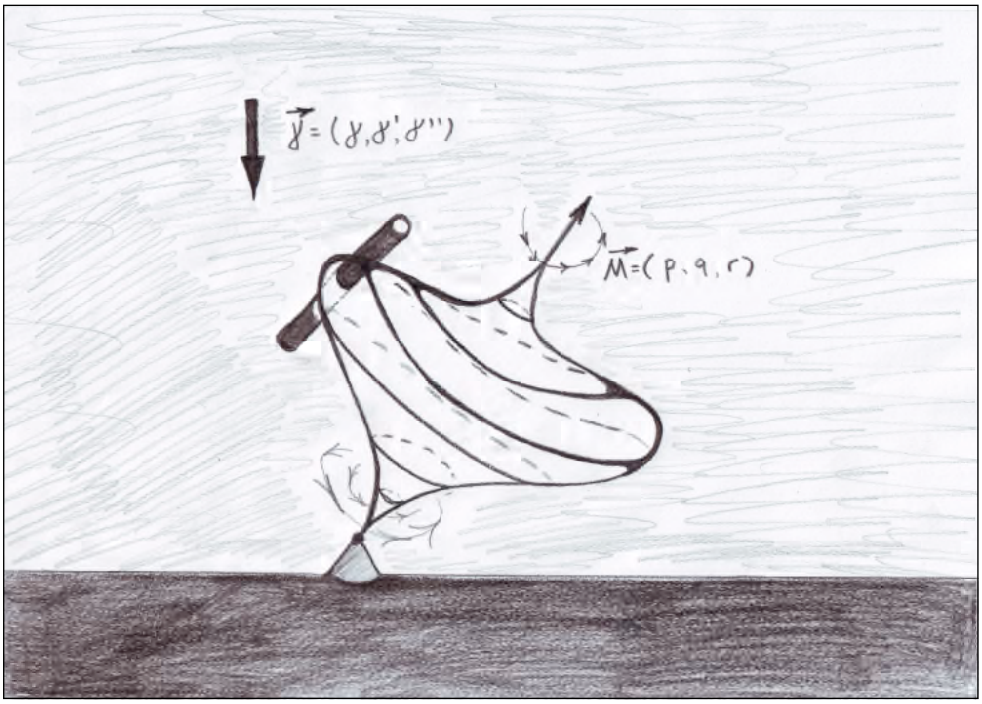

Figure 2.2: The Kovalevskaya Top.

According to the Jacobi's last multiplier theorem we can integrate the equations (2.3) in quadratures once there exists a fourth additional first integral functionally independent from already existing ones.

Up to 1889 the following cases of integrability of the Euler-Poisson system were known: the Euler case corresponding to $R = (0, 0, 0)$ and the Lagrange's one corresponding to $A = B, x_0 = y_0 = 0$.

In these two cases the general solution of the problem can be expressed in theta-functions and hence is a meromorphic function in the complex time plane. This remark played a key role for Sonya Kovalevskaya. She decided to determine all cases in which the general solution of the Euler-Poisson system is meromorphic.

Being extremely skillful in complex analytic function theory (we remind that her teacher was the founder of the modern analysis K. Weierstrass) she succeeded to solve this difficult problem having shown that the general meromorphic solution of the equations (2.3) exists only in the above mentioned cases of Euler and Lagrange and in a new one corresponding to $A = B = 2C$, $z_0 = 0$ (the Kovalevskaya's case).

Let us explain the essence of the Kovalevskaya method (for some historical reasons this method is often called the *Painlevé test*).

Assuming that the general solution of (2.3) is meromorphic we can write it locally as a formal power series:

$$\begin{aligned}&M = t^{-n}(M_0 + tM_1 + \cdots)\\ &\gamma = t^{-m}(\gamma_0 + t\gamma_1 + \cdots), \quad n, m = 1, 2, \ldots .\end{aligned}$$

The main problem consists then in finding conditions which would guarantee that the coefficients M_i, γ_i will contain five arbitrary complex constants $c_1, \ldots, c_5$. This problem can be solved by means of elementary methods and this is how Kovalevskaya found all three above mentioned cases of integrability . The important remark is that in the cases of Euler, Lagrange and Kovalevskaya the corresponding additional first integral is a polynomial function.

Considering the example of the Euler-Poisson system we can ask the following question: whether there is connection between branching of the general solution and the existence of single-valued first integrals ?

Namely, one can ask whether the following statement is true: the property of a general solution to be meromorphic implies the existence of a sufficiently many single-valued first integrals. The answer to this conjecture is negative in general though in the majority of integrable cases this property actually holds. One of the first rigorous results in this direction was obtained by Kozlov for Hamiltonian system on a torus [63].

Let $T^n = \{x_1, \ldots, x_n \mid \bmod 2\pi\}$ be the configuration space of a Hamiltonian system with n degrees of freedom

$$\sum_{j=1}^{n} a_{kj}\ddot{x}_j = F_k(x_1, \ldots, x_n), \quad 1 \le k \le n\,. \tag{2.4}$$

We shall admit that the components of the force field F_i are holomorphic functions on T^n which can be continued meromorphically through the whole space $\mathbb{C}^n$.

Definition. We say that a first integral of (2.4) which is polynomial with respect to $\dot{x}_i$ is *single-valued* if its coefficients satisfy the following conditions:

1) they are 2π-periodic functions of time t,

2) they are holomorphic in the domain $\mathbb{C}^n/P$ where P is the set of poles of the meromorphic functions $F_1, \ldots, F_n$.

Let us consider the meromorphic vector field $f : \mathbb{C} \to \mathbb{C}^n$

$$f(z) = \begin{pmatrix} F_1(az+b) \\ \ldots \\ F_n(az+b) \end{pmatrix},$$

defined by restrictions of $F_1, \ldots, F_n$ to the straight line $az+b, a, b \in \mathbb{C}$.
The following results establishes a relation between branching of solutions and the existence of single-valued first integrals.

Theorem 11 ([54]). *Let us assume that for some $a, b \in \mathbb{C}$ the function $f(z)$ has $m > 0$ poles with non zero residues. Then*

a) The general solution of (2.4) *is a multivalued function of the complex time.*

b) The number k of independent single-valued polynomial first integrals of the equations (2.4) *satisfies the inequality*

$$m + k \leq n\,.$$

An essential role in searching for new integrable cases is played by the theory of *Kovalevskaya exponents* developed in 1983 by Yoshida. His approach establishes the important link between branching of solutions and the existence of algebraic first integrals.
We call a function $A(x)$, $x \in \mathbb{C}^n$ a *quasi-homogeneous* of degree $d \in \mathbb{R}$ with exponents $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ if

$$A(t^{w_1}x_1, \ldots, t^{w_n}x_n) = t^d A(x_1, \ldots, x_n), \quad \forall\, t \in \mathbb{C}^*\,.$$

Let us consider a polynomial system of differential equations

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{C}^n, \quad f \in \mathbb{C}^n[x]\,, \tag{2.5}$$

where all functions f_i are quasi-homogeneous of degrees $w_i - 1$ with exponents $w = (w_1, \ldots, w_n)$.

That implies in particular that the above system is invariant under the transformation

$$t \to \epsilon t, \quad x_i \to \epsilon w_i \, x_i, \quad i = 1, \ldots, n \,. \tag{2.6}$$

Let $\alpha \neq 0$ be a non zero solution of the following algebraic system of n equations:

$$\alpha_i w_i = f_i(\alpha), \quad \alpha \in \mathbb{C}^n \,, \tag{2.7}$$

Then, as follows from (2.6), (2.7), the initial system (2.5) has a particular solution

$$x(t) = \alpha \; t^w \,.$$

The following matrix can be defined

$$K = Df(\alpha) - \operatorname{diag}(w) \,,$$

called the *Kovalevskaya matrix.*

The eigenvalues $\rho_1 = -1, \ldots, \rho_n$ of K are called the *Kovalevskaya exponents.* With the help of ρ_i we can write the local development of a general solution of the system (2.5) near $t = t_0$ as follows

$$x = \alpha (t - t_0)^w P(c_1 \gamma_1 (t - t_0)^{\rho_1}, \ldots, c_m \gamma_n (t - t_0)^{\rho_n}) \,,$$

where γ_i is the eigenvector vector of K corresponding to the eigenvalue ρ_i, P is a power series with coefficients polynomial on $\ln(t - t_0)$.

The following theorem holds:

Theorem 12 ([95])**.** *We assume that the system of differential equations* (2.5) *has an algebraic quasi-homogeneous first integral $I(x)$ of degree d. Let $\alpha \neq 0$ be a solution of the algebraic system* (2.7) *such that* grad $I(\alpha) \neq 0$. *Then d is one of the Kovalevskaya exponents.*

The next result strengthens greatly the above theorem since it does not impose any restrictions on grad $I(\alpha)$:

Theorem 13 ([40])**.** *We assume that the quasi-homogeneous system of differential equations* (2.5) *possesses k independent rational first integrals $I_1, \ldots, I_k$ with homogeneous degrees equal respectively to $d_1, \ldots, d_l$. Let $\rho_1 = -1, \rho_2, \ldots, \rho_n$ be the corresponding Kovalevskaya exponents. Then there exists a $k \times (n-1)$ matrix $N_{i,j}$ with positive integer entries such that*

$$\sum_{j=2}^{n} N_{ij} \rho_j = d_i, \quad i = 1, \ldots, k \,.$$

Corollary 1. *If the Kovalevskaya exponents of the system* (2.5) *are linearly* $\mathbb{Z}$ *-independent (*$\mathbb{N}$ *-independent) then there is no rational (polynomial) first integral.*

Similar conditions can be formulated for the existence of algebraic symmetry fields [55]. As an elementary application of the Kovalevskaya exponents method we consider the Halphen system

$$\begin{aligned} \dot{x}_1 &= x_3x_2 - x_1x_3 - x_1x_2, \\ \dot{x}_2 &= x_1x_3 - x_2x_1 - x_2x_3, \\ \dot{x}_3 &= x_2x_1 - x_3x_2 - x_3x_1 \,. \end{aligned} \tag{2.8}$$

The Kovalevskaya exponents corresponding to a solution $\alpha = (-1, -1, -1)$ of (2.7) are $\rho_1 = \rho_2 = \rho_3 = -1$.

Thus, as follows from the Theorem 13, these equations have no polynomial first integrals. Actually, for the same system the non existence of rational integrals can be proved [83].

The great advantage of the Yoshida method is its simplicity. It provides also *a priori* possible degrees of unknown first integrals (under the assumption $\operatorname{grad} I(\alpha) \neq 0$). When the prescribed degree is small enough one can easily prove (or disprove) the existence of a polynomial first integral.

Among well known examples of quasi-homogeneous systems we mention the Lotka–Volterra equations and the Euler-Poisson system discussed previously.

An important particular subset of quasi-homogeneous systems is given by the class of quadratic homogeneous vector fields.

For these systems the theory of Kovalevskaya exponents can be generalized once we restrict ourselves to the case of polynomial first integrals. In our work [T9] we propose an effective method to calculate polynomial first integrals or to show that they do not exist once the set of Kovalevskay exponents has been found.

This algorithm has been used in [29], where a new integrable case of the generalized Toda equations was discovered (whose existence was conjectured by Kozlov and Treshev in [57]).

In our papers [T11], [T12] the Kovalevskaya exponents were evaluated for some equations from classical mechanics, in particular for the Euler equations on the Lie algebra $so(4)$. Using the Kovalevskaya exponents method we found in [T13] a new family of algebraic first integrals in one important limit case of this problem (see also p. 11).

Our paper [T14] contains calculation of Kovalevskaya exponents for generalized Toda systems in the Minkowski space. We show that in most cases these exponents are non real numbers. In particular, that means that in a generic case these systems are not algebraically integrable.

2.1.4 Ziglin and Morales-Ramis approaches.

We consider a holomorphic vector field X_H defined on a complex symplectic manifold M. Let $\Gamma \subset M$ be an integral curve of X_H different from an equilibrium point. Historically, the idea to relate the branching of solutions of variational equations of X_H along Γ and the absence of single-valued first integrals of X_H goes back to works of Kovalevskaya and Lyapunov. The branching can be measured by the complexity (or more precisely by the *non-abelianity*) of the corresponding monodromy and differential Galois groups. Below we present a short historical review of results which led to the notion of the monodromy group (see also [41] for the complete story).

In 1769 in his "Institutions Calculi Integralis" Euler studies the differential equation

$$x(x-1)\frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{dy}{dx} - \alpha\beta\gamma y = 0, \quad (\alpha,\beta,\gamma) \in \mathbb{C}^3\,, \quad (2.9)$$

and writes its solution by means of the power series:

$$y = 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^3 + \cdots\,. \quad (2.10)$$

convergent for $|x| < 1$.

Following Wallis (1655), Euler calls (2.9) the *hypergeometric* equation. In 1799, Gauss, while studying the iteration of arithmetic and geometric means

$$\begin{aligned} a_{n+1} &= (a_n + b_n)/2, \\ b_{n+1} &= \sqrt{a_n b_n} \end{aligned} \quad , n = 0, 1, \ldots\,,$$

defines two sequences of numbers $\{a_n\}$, $\{b_n\}$ converging to the same limit $M(a_0, b_0)$.

After some heavy calculations Gauss arrives to the function

$$y = M(1+x, 1-x)^{-1} = 1 + \left(\frac{1}{2}\right)^2 x^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^4 + \cdots\,,$$

which satisfies the following linear differential equation

$$(x^3 - x)\frac{d^2y}{dx^2} + (3x^2 - 1)\frac{dy}{dx} + xy = 0\,. \quad (2.11)$$

Later he finds a second linearly independent solution of the same equation given by $y(x) = M(1,x)^{-1}$.

Making in (2.11) the substitution $z = x^2$ we transform it to the hypergeometric form

$$z(1-z)\frac{d^2y}{dz^2} + (1-2z)\frac{dy}{dz} - \frac{1}{4}y = 0\,,$$

called also the Legendre's equation.

In 1812 Gauss published his results concerning properties of the hypergeometric series $F(\alpha,\beta,\gamma,x)$. In particular, for the hypergeometric equation (2.9), in addition to the solution (2.10), previously found by Euler, he discovers a second linearly independent one which allows him to write the general solution of (2.9) as follows

$$y = c_1F(\alpha,\beta,\gamma,x) + c_2F(\alpha,\beta,\alpha+\beta+1-\gamma,1-x)\,.$$

Considering $x = a+ib$ as a complex variable Gauss makes a remarkable observation (cf. [36]): there is a clear distinction between the function $F(\alpha,\beta,\gamma,x)$ defined as the series (2.10) and the function $F(\alpha,\beta,\gamma,x)$ defined as a solution of the linear differential equation (2.9). Indeed, the series (2.10) converges only for $\mid x \mid < 1$ while the equation (2.9) defines $F(\alpha,\beta,\gamma,x)$ for all $x \in \mathbb{C}$ except for three points 0, 1 and ∞.

These ideas were developed later by Riemann [80] whose approach was purely geometric. The key role in his approach is played by the so called P–function

$$P\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \tilde\alpha & \tilde\beta & \tilde\gamma \end{pmatrix}(z)\,,$$

where $(a,b,c) \in \mathbb{C}$ denote the branching points of P. The parameters $(\alpha,\tilde\alpha)$, $(\beta,\tilde\beta)$, $(\gamma,\tilde\gamma)$ correspond to loops around a,b,c and are defined as follows. In a complex neighborhood U_a of the singular point a the function $P(z)$ is defined locally by

$$P(z) = c_1P_1(z) + c_2P_2(z)\,,$$

where $(z-a)^{-\alpha}P_1$, $(z-a)^{\tilde\alpha}P_2$ are holomorphic in U_a.

Parameters $(\beta,\tilde\beta)$, $(\gamma,\tilde\gamma)$ are defined in a similar way. Riemann shows that his function $P(z)$ satisfies the hypergeometric equation

$$\frac{d^2P}{dz^2} + \sum_{(a,b,c)} \frac{1-\alpha-\tilde\alpha}{z-a}\frac{dP}{dz} + \sum_{(a,b,c)} \frac{\alpha\tilde\alpha(a-b)(a-c)}{z-a}\,\frac{P}{(z-a)(z-b)(z-c)} = 0\,. \tag{2.12}$$

The relation between the P–function and $F(\alpha,\beta,\gamma,z)$ is given by

$$P\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \end{pmatrix} = z^{\alpha}(1-z)F(\beta+\alpha+\gamma, \tilde{\beta}+\alpha+\gamma, \alpha-\tilde{\alpha}+1, z)\,.$$

Thus, according to Riemann, in order to know the global behavior of $P(z)$ it is sufficient to know it locally around the branching points a, b, c. Indeed, the functions P_1, P_2 continued analytically in a positive direction along the closed curve surrounding the singular point a, take the new values

$$\tilde{P}_1 = a_1 P_1 + a_2 P_2,$$
$$\tilde{P}_2 = a_3 P_1 + a_4 P_2,$$

with some constants a_i. As a consequence, the matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

determines completely the branching of $P(z)$ in this point.
In a similar way, we construct the matrices B, C corresponding to the singular points b, c. Moreover, one has the identity

$$CBA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since any closed curve can be represented as a product of loops around a, b, c, Riemann concludes: "... the coefficients of matrices A, B, C completely define the branching of the function $P(z)$... "
The matrix group generated by the matrices A, B is called the *monodromy group* of the hypergeometric equation (2.12).
In 1865 Fuchs [33] studies the general linear differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_n y = 0\,, \tag{2.13}$$

where $p_i(x)$ are meromorphic functions.
A point $x = x_0$ is called *singular* for the equation (2.13) if it is a pole of at least one of the functions p_i. Fuchs proves that in a neighborhood of a non singular point the general solution of (2.13) is holomorphic.
He also studies a remarkable class of linear differential equations

$$\frac{d^n y}{dx^n} + \frac{F_{\rho-1}(x)}{\psi}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \frac{F_{n(\rho-1)}}{\psi^n}(x) = 0\,, \tag{2.14}$$

where $\psi(x) = (x - a_1) \cdots (x - a_\rho)$ and $F_s(x)$ is a polynomial of degree s.

To determine the local Taylor expansion of the general solutions of the above equation near $x = a_1$ one starts by finding roots $w_1, \ldots, w_n$ of the characteristic polynomial $P_{a_1}(w) = 0$ easy to find.

After that, the fundamental system of solutions can be written locally as

$$y_i(x) = (x - a_1)^{r_i} \phi_i(x, \ln(x - a_1)) \,,$$

where $w_i = e^{2\pi i r_i}$ and ϕ_i are holomorphic in x and polynomials in $ln(x - a_1)$ functions.

An equation of the form (2.14) is called *Fuchsian* with regular singular points $a_1, \ldots, a_\rho$.

The idea to use group theory in studying of systems of ordinary differential equations goes back to Poincaré [73]. Let us consider a certain Fuchsian group $G \in SL(2, \mathbb{C})$ with elements

$$g_i = \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} \,,$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$ and $\alpha_i \delta_i - \beta_i \gamma_i = 1$.

Following Poincaré, we define the so called *theta-fuchsian* function $\theta(z)$ of weight m according to

$$\theta(z) = \sum_{i=1}^{\infty} H\left(\frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}\right) \frac{1}{(\gamma_i z + \delta_i)^{2m}} \,,$$

where $H(z)$ is any rational function.

The ratio of two theta-fuchsian functions of the same weight defines an *automorphic* function $F(z)$ invariant under the action of G:

$$F\left(\frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}\right) = F(z), \quad \forall\, i \,.$$

Thus, any Fuchsian group possesses an infinite number of invariants.

According to the fundamental result of Poincaré, any two automorphic functions F_1, F_2 defined by the same Fuchsian group, always satisfy the equation

$$P(F_1, F_2) = 0 \,,$$

where P is a certain algebraic function.

Poincaré discovered also the following remarkable relation between automorphic functions and solutions of algebraic differential equations of the first order: if $x = F(z)$ is a Fuchsian function, then $v_1 = \sqrt{F'(z)}$ and $v_2 = z\sqrt{F'(z)}$

are independent solutions of the linear differential equation $y'' = y\phi(x, y)$, where ϕ is algebraic in y.

The study of dynamical properties of complex monodromy groups was initiated by Arnold and Krylov who considered in [7] some ergodic properties of action of finitely generated groups on compact varieties. Let us take a linear differential equation

$$\frac{dx}{dz} = A(z)x\,, \tag{2.15}$$

where z is a complex variable, $x \in \mathbb{C}^n$ and $A(z)$ is a matrix whose entries are holomorphic functions on $Z = \mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$.

The phase space M of (2.15) has real dimension $2n + 2$ and is a direct product of Z and $\mathbb{C}^n$. Its solutions $x = x(z), z \in Z$ define a foliation of M by 2-dimensional surfaces. To every point $z_0 \in Z$ and to every vector $x_0 \in \mathbb{C}^n$ corresponds thus the unique solution $x(z)$ with the initial condition $x(z_0) = x_0$. Thus we get a family of linear maps $\mathbb{C}^n \to \mathbb{C}^n$.

In particular, each closed curve $\gamma \in Z$ defines a linear transformation A_γ of $\mathbb{C}^n$ into itself. Obviously, this transformation A_γ depends only on a homotopy class of $\gamma \subset Z$. That defines a representation of the fundamental group of Z with image in $GL(n, \mathbb{C})$. The group, generated by all transformations $\{A_\gamma\}$ is called the monodromy group of the system (2.15). The following two results demonstrate how its structure reflects the properties of corresponding single-valued first integrals.

Lemma 4 ([7])**.** *If the monodromy group of* (2.15) *is bounded, then it possesses an unique first integral of the form*

$$(B(z)x, \overline{x}) = \text{const}\,,$$

where $B(z)$ is a positive unitary matrix defined for all $z \in Z$.

Theorem 14 ([7])**.** *The hypergeometric equation*

$$z(z-1)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)z]\frac{dy}{dx} - \alpha\beta x = 0, \alpha, \beta, \gamma \in \mathbb{R}\,,$$

always admits the single-valued first integral

$$b_{11}x\overline{x} + b_{12}\overline{y}x + b_{21}y\overline{x} + b_{22}y\overline{y} = \text{const}\,,$$

where $y = dx/dz$ and $b_{ij}(z)$ are functions single-valued for $z \neq 0, 1, \infty$.

In 1831 Evariste Galois showed that the solution in radicals of an algebraic equation was connected to the structure of a group of permutations related to the equation. The link between complete integrability of an analytic Hamiltonian system and properties of the monodromy group of the normal variational equations (NVE) along a non-equilibrium solution was established in 1982 by Ziglin [96]. This result was improved by Morales and Ramis [67], [68], [4] who formulated their criteria of non-integrability in terms of the differential Galois group G of NVE. Namely, in the integrable cases, the identity component of G, under Zariski's topology, must be abelian. A crucial role in both approaches is played by the fact that the meromorphic first integrals of the Hamiltonian system give rise to rational homogeneous invariants of the monodromy (differential Galois) group of NVE. We outline now briefly the Ziglin and Morales-Ramis approaches.

In his work [96] Ziglin considers an analytic Hamiltonian system defined over a symplectic manifold M of complex dimension $2n$

$$\dot{z} = JH'_z, \quad z \in M, \tag{2.16}$$

where $H : M \to \mathbb{C}$ is analytic and J is defined by

$$\begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}.$$

Assuming that (2.16) has a particular solution $z_0(t)$ single-valued on the Riemann surface Γ (which we identify to $z_0(t)$) one writes the variational equations along the integral curve Γ as follows

$$\frac{d\zeta}{dt} = JH_{zz}(\Gamma)\zeta, \quad \zeta \in T_\Gamma M \tag{2.17}$$

where H_{zz} is the Hessian matrix of Hamiltonian H at Γ

These equations always admit the linear first integral $F = (\zeta, H_z(\Gamma))$, where $H_z = grad(H)$ and can be reduced on the normal $2n-1$-dimensional bundle $G = T_\Gamma M/T\Gamma$ of Γ . After the restriction of (2.17) on the surface $F = 0$ we obtain the *normal variational equations* [96] which are the system of $2n-2$ equations

$$\frac{d\eta}{dt} = \tilde{A}(\Gamma)\eta, \quad \eta \in \mathbb{C}^{2n-2}. \tag{2.18}$$

Let $\Sigma(t)$ be the fundamental matrix solution of (2.18).

It can be continued along a closed path $\gamma \subset \Gamma$ with end points at $\tau \in \Gamma$. We obtain thus the function $\tilde{\Sigma}_\gamma(t)$ which also satisfies (2.18). By linearity, it follows that there exists a complex matrix T_γ such that $\tilde{\Sigma}_\gamma(t) = \Sigma(t)T_\gamma$. The set of matrices $G = \{T_\gamma\}$ corresponding to all closed curves in Γ with end points at $\tau \in \Gamma$ clearly forms a subgroup of $Sp(2n-2, \mathbb{C})$. This group is called the *monodromy group* of the normal variational equations (2.18).

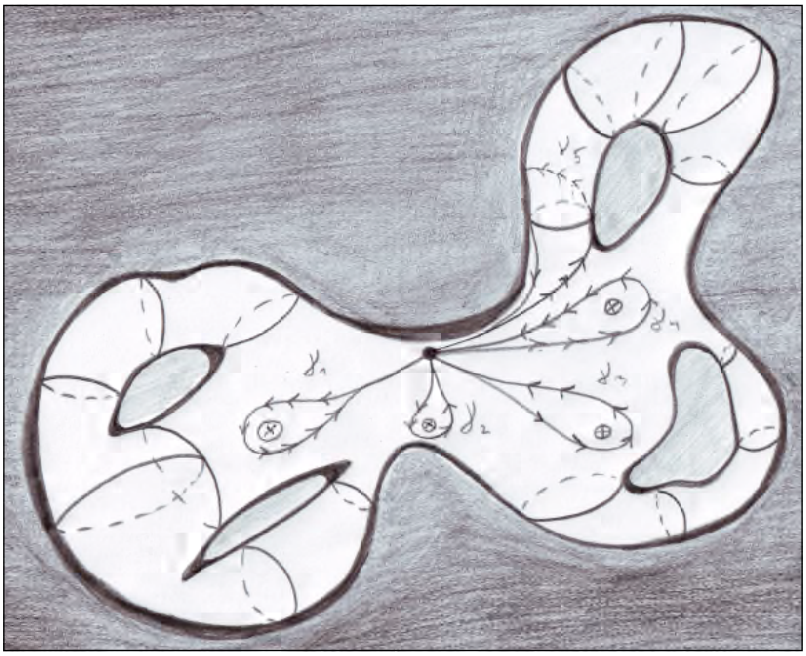

Figure 2.3: The monodromy group.

Definition ([96])**.** We say that the transformation $g \in G \subset Sp(2n, \mathbb{C})$, $\mathrm{Spectr}(g) = \{\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}\}$ is *non-resonant* if the condition

$$\lambda_1^{m_1} \cdots \lambda_n^{m_n} = 1, \quad m_i \in \mathbb{Z}\,,$$

implies $m_k = 0$, $\forall\, k$.

Remark 1. In the case $n = 2$, the resonance means that λ_1 is a root of unity.

Theorem 15 ([96])**.** *Suppose that the monodromy group $G \subset Sp(2n-2, \mathbb{C})$ of the normal variational equations contains a non-resonant transformation g. If the system* (2.16) *has n independent meromorphic first integrals in a connected neighborhood of $X = \{z = z_0(t), t \in \Gamma\}$ (not necessary in involution !) then each $\tilde{g} \in G$, $\tilde{g} \neq g$ permutes the eigendirections of g. If, in addition, no set of eigenvalues of $\tilde{g}$ forms a regular polygon with $n \geq 2$ vertices centered at* 0 *and having at least two vertices (if $n = 2$ it means that the eigenvalues are not $\pm i$), then $[\tilde{g}, g] = 0$.*

As an application of the above theorem we consider the linearized equation of oscillations of a pendulum with a vibrating suspension point:

$$\ddot{z} + (\omega^2 + \epsilon f(t))z = 0, \tag{2.19}$$

where $\omega, \epsilon \in \mathbb{R}$ and f is a doubly-periodic function with periods 2π and $2\pi i$ having an unique pole $t = 0$ of the second order in its parallelogram of the periods.

Firstly we determine the eigenvalues of the monodromy generator around the singular point $t = 0$. One verifies (see [56], p. 331), that the Laurent expansion of $f(t)$ in a neighborhood of $t = 0$ has the form

$$\frac{\alpha}{t^2} + \sum_{n \geq 0} f_n t^n, \quad \alpha \neq 0\,.$$

We are looking for particular solutions of (2.19) of the form

$$z(t) = t^\rho \sum_{n \geq 0} c_n t^n, \quad \rho \in \mathbb{C}, \quad c_0 \neq 0\,.$$

After substitution in (2.19) one obtains

$$\rho(\rho - 1) + \epsilon\alpha = 0\,,$$

that gives two values ρ_1, ρ_2 corresponding to two linearly independent solutions $z_1(t), z_2(t)$.

After going around the singularity at $t = 0$, each of these solutions is multiplied accordingly by $e^{2\pi i \rho_1}$ and $e^{2\pi i \rho_2}$. The corresponding monodromy is trivial provided ρ_1 and ρ_2 are integers, in particular $\epsilon\alpha$ must be an integer. Let g_1, g_2 be the elements of the monodromy group corresponding to the periods 2π and $2\pi i$ of $f(t)$ respectively. Then obviously $g = [g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$.

Let us first consider the non perturbed case $\epsilon = 0$. Then the eigenvalues of the monodromy transformations of g_1 and g_2 are equal respectively to $\lambda_{1,2} = e^{\pm 2\pi\omega i}$ and $\mu_{1,2} = e^{\pm 2\pi\omega}$. It is easy to see that $\mid \mu_{1,2} \mid \neq 1$ if $\omega \neq 0$ and $\lambda_{1,2} \neq \pm i$ for $\omega \neq 1/4 + k\pi, k \in \mathbb{Z}$. By continuity we conclude that if $\omega \neq 1/4 + k\pi$ then for $\epsilon \neq 0$ sufficiently small the eigenvalues $\mu_{1,2}$ are not roots of unity and $\lambda_{1,2} \neq \pm i$. Hence, in these cases, by Theorem 15, the equation (2.19) has no meromorphic first integrals.

The theory of Ziglin was greatly generalized by Morales and Ramis who considered the algebraic approach based on the differential Galois theory.

Let K be the differential field of meromorphic functions on the Riemann surface $\bar{\Gamma}$ where $\tilde{\Gamma}$ is obtained from Γ by adding equilibrium points, singularities of the Hamiltonian vector field (2.16) and points at infinity. Let

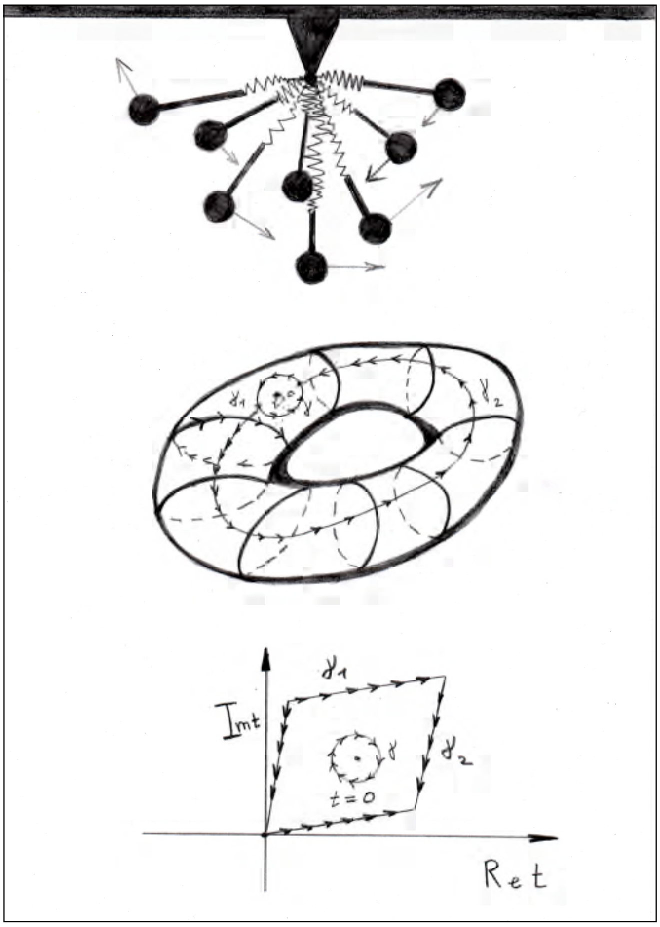

Figure 2.4: The pendulum problem.

$\Sigma(t) = (\Sigma_{ij}(t))$ be the fundamental matrix solution of the normal variational equations (2.18). The differential field extension $K(\Sigma_{ij}(t))$ is called the *Picard-Vessiot extension* V of (2.18). The Galois group G_{Gal} of (2.18) is then the group of all differential automorphisms of V such that any element of K is left invariant.

One of the principal results of the Picard-Vessiot theory says that a linear differential equation is solvable by quadratures if and only if its Galois group is solvable. That is, the integrability problem can be viewed from a purely algebraic point of view. The integrability by quadratures means that the general solution can be obtained by a combination of integrals, exponentials of integrals and algebraic functions of elements of the base differential field. The next theorem by Morales and Ramis shows that the solvability property, in the context of completely integrable Hamiltonian systems, must be replaced by a stronger property of commutativity reflecting the involutivity of first integrals.

Theorem 16 ([67]). *Suppose that the Hamiltonian system* (2.16) *has n functionally independent first integrals in involution which are meromorphic in a neighborhood of Γ and not necessarily independent on Γ itself, then the Galois group G_{Gal} of the normal variational equations* (2.18) *has the following properties*

1) The identity component G^0_{Gal} of G_{Gal} is commutative.

2) The Lie algebra L_{Gal} of G_{Gal} is commutative.

3) G_{Gal} has a commutative invariant subgroup H such that the factor group G/H is finite (i.e., it is virtually commutative).

This theorem contains the essence of the Morales-Ramis method which is a very powerful tool for studying the non-integrability of Hamiltonian systems. In the case of two degrees of freedom, the differential Galois group can be evaluated using the Kovacic algorithm.

We close this section by outlining an important intermediate result fundamental for both Ziglin and Morales-Ramis approaches: the so called *Ziglin's lemma* (already known to Poincaré in the linear case).

Let M be a complex manifold of dimension $2n$. Let $\Phi(z)$, $z \in M$ be an arbitrary function analytic in a neighborhood of the point $z = a \in M$. Suppose that all partial derivatives of order $\leq k-1$ of $\Phi(z)$ vanish at $z = a$, while one of the k-th order derivatives is different from zero.

Definition. Let $\mathbb{F}$ be the homogeneous form of degree k on the tangent space T_aM whose value on the vector $\zeta \in T_aM$ is the k-th derivative of Φ at $z = a$ in the direction ζ (i.e the first non vanishing k–jet). Then $[\Phi]_a = \mathbb{F}/k!$.

Let F be a function meromorphic in a neighborhood of $z = a$. Then $F = P/Q$ is a ratio of two analytic functions and by definition: $[F]_a = [P]_a/[Q]_a$.

We assume that $\Phi(z)$ is a first integral of the Hamiltonian system (2.16) meromorphic in a neighborhood of the particular solution Γ. We define the function $[\Phi] : T_\Gamma M \to \mathbb{C}$ whose restriction to $T_{z_0(t)}M$ equals $[\Phi]_{z_0(t)}$. It is easy to see that $I(t, \zeta) = [\Phi]$ is a meromorphic first integral of the variational equations (2.17). After symplectic reduction, we obtain the meromorphic first integral $\tilde{I}(t, \zeta)$ of the normal variational equations (2.18) which, once $t = t_0 \in \Gamma$ is fixed, becomes a rational invariant of the monodromy group G. The same statement holds in the case of the differential Galois group of (2.16): any meromorphic first integral of (2.16) generates a rational invariant of G_{Gal}.

Lemma 5 ([96])**.** *Suppose that the Hamiltonian system*(2.16) *has, in a neighborhood of the curve* Γ*,* m *meromorphic first integrals, functionally independent together with* H*. Then* G *and* G_{Gal} *have* m *rational, functionally independent invariants.*

In the next chapter the above lemma will be our principal tool in the study of existence of additional first integrals of the planar three-body problem meromorphic in a neighborhood of Lagrangian orbits. Our method will be based essentially on the principal idea of the Morales-Ramis approach: to replace the monodromy group consideration by the infinitesimal analysis. That can be done by introducing a Lie algebra of linear differential operators naturally emerging from the unipotent structure of the corresponding monodromy group.

2.2 On the non-existence of additional meromorphic first integrals in the three-body problem

2.2.1 Introduction and known results

The planar three-body problem is a mechanical system which consists of three mass points m_1, m_2, m_3 in the plane which attract each other according to the Newtonian law [70].

The corresponding equations of motion can be represented in a Hamiltonian form

$$\frac{dx_r}{dt} = \frac{\partial H_1}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial H_1}{\partial x_r}, \quad (r = 1, 2, \ldots, 6),$$

and have 6 degrees of freedom.

Using the classical first integrals one can decrease the number of degrees of freedom up to three. That was done in classical works of Jacoby, Lagrange and Poincaré.

The practical importance of this problem arises from its applications to celestial mechanics: the bodies which constitute the solar system attract each other according to the Newtonian law, and the stability of this system on a long period of time is a fundamental question. Although Sundman [71] gave a power series solution to the three-body problem in 1913 (under the assumption that the total angular momentum is not zero), it was not useful in determining the growth of the system for long intervals of time. Chazy [19] proposed in 1922 the first general classification of motion as $t \to \infty$. In view of the modern analysis [56], this stability problem leads to the problem of integrability of a Hamiltonian system i.e. the existence of a full set of analytic first integrals in involution. Poincaré [75] considered Hamiltonian functions $H(z, \mu)$ which in addition to $z_1, \ldots, z_{2n}$ also depended analytically on a parameter μ near $\mu = 0$. His theorem states that under certain assumptions about $H(z, 0)$, which are in general satisfied, the Hamiltonian system corresponding to $H(z, \mu)$ can have no integrals represented as convergent series in $2n + 1$ variables $z_1, \ldots, z_{2n}$ and μ, other than the convergent series in H, μ. Based on this result he proved in 1889 the non-integrability of the restricted three-body problem [74]. However, this theorem does not assert anything about a fixed parameter value μ.

Bruns [12] showed in 1882 that the classical integrals are the only independent algebraic integrals of the problem of three bodies. His theorem has been extended by Painlevé [76], who has shown that every integral of the problem of n bodies which involves the velocities algebraically (whether the coordinates are involved algebraically or not) is a combination of the classical integrals.

Nevertheless, as was mentioned later by Wintner [94], these elegant negative results by Bruns and Painlevé have no great importance in dynamics, since they do not take into account the peculiarities of the behavior of phase trajectories. Indeed, as far as first integrals are concerned, locally, in a neighborhood of a non-singular point, a complete set of independent integrals always exists. Therefore, the integrability problem makes sense only when it is considered in the whole phase space or in a neighborhood of an invariant set (for example, an equilibrium position or a periodic trajectory).

The algebraic non–integrability of the $n + 1$ bodies problem in $\mathbb{R}^p$ under the conditions $n \geq 2$ and $1 \leq p \leq n+1$ was proved by E. Julliard-Tosel (a student of A. Chenciner, see [43], [44]). At the same time, the four body problem on the line, with masses $(1, m, m)$, was shown by her to be meromorphically non-integrable [45], [46].

2.2.2 The reduction of Whittaker

Following Whittaker [93] let (x_1, x_2) be the coordinates of m_1, (x_3, x_4) the coordinates of m_2, and (x_5, x_6) the coordinates of m_3. Let $y_r = m_k \dfrac{dx_r}{dt}$, where k denotes the greatest integer in $\dfrac{1}{2}(r+1)$. The equations of the motion are

$$\frac{dx_r}{dt} = \frac{\partial H_1}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial H_1}{\partial x_r}, \quad (r = 1, 2, \ldots, 6), \tag{2.20}$$

where

$$H_1 = \frac{1}{2m_1}(y_1^2 + y_2^2) + \frac{1}{2m_2}(y_3^2 + y_4^2) + \frac{1}{2m_3}(y_5^2 + y_6^2) - m_3 m_2\{(x_3 - x_5)^2 + \\ +(x_4 - x_6)^2\}^{-1/2} - m_3 m_1\{(x_5 - x_1)^2 + (x_6 - x_2)^2\}^{-1/2} - m_1 m_2\{(x_1 - x_3)^2 + \\ +(x_2 - x_4)^2\}^{-1/2}.$$

This is a Hamiltonian system with 6 degree of freedom which admits 4 first integrals:
$T_1 = H_1$ – the energy,
$T_2 = y_1 + y_3 + y_5$, $T_3 = y_2 + y_4 + y_6$ – the components of the impulse of the system,
$T_4 = y_1 x_2 + y_3 x_4 + y_5 x_6 - x_1 y_2 - x_3 y_4 - x_5 y_6$ – the integral of angular momentum of the system.
The system (2.20) can be transformed to a system with 4 degrees of freedom by the following canonical change (Poincaré, 1896)

$$x_r = \frac{\partial W_1}{\partial y_r}, \quad g_r = \frac{\partial W_1}{\partial l_r}, \quad (r = 1, 2, \ldots, 6),$$

where

$$W_1 = y_1 l_1 + y_2 l_2 + y_3 l_3 + y_4 l_4 + (y_1 + y_3 + y_5) l_5 + (y_2 + y_4 + y_6) l_6\,. \tag{2.21}$$

Here (l_1, l_2) are the coordinates of m_1 relative to axes through m_3 parallel to the fixed axes, (l_3, l_4) are the coordinates of m_2 relative to the same axes, (l_5, l_6) are the coordinates of m_3 relative to the original axes, (g_1, g_2) are the components of impulse of m_1, (g_3, g_4) are the components of impulse of m_2, and (g_5, g_6) are the components of impulse of the system. It can be shown that in the system of center of gravity the corresponding equations for l_5, l_6, g_5, g_6 disappear from the system and the reduced system takes the following form

$$\frac{dl_r}{dt} = \frac{\partial H_2}{\partial g_r}, \quad \frac{dg_r}{dt} = -\frac{\partial H_2}{\partial l_r}, \quad (r = 1, 2, 3, 4), \tag{2.22}$$

with the Hamiltonian

$$H_2 = \frac{M_1}{2}(g_1^2 + g_2^2) + \frac{M_2}{2}(g_3^2 + g_4^2) + \frac{1}{m_3}(g_1 g_3 + g_2 g_4) - \\ - \frac{m_3 m_2}{\rho_1} - \frac{m_1 m_3}{\rho_2} + \frac{m_1 m_2}{\rho_3},$$

where

$$\rho_1 = \sqrt{l_3^2 + l_4^2}, \quad \rho_2 = \sqrt{l_1^2 + l_2^2}, \quad \rho_3 = \sqrt{(l_1 - l_3)^2 + (l_2 - l_4)^2},$$

are the mutual distances of the bodies and $M_1 = m_1^{-1} + m_3^{-1}$, $M_2 = m_2^{-1} + m_3^{-1}$.

This system admits two first integrals in involution

$K_1 = H_2$ – the energy,

$K_2 = g_2 l_1 + g_4 l_3 + g_6 l_5 - g_1 l_2 - g_3 l_4 - g_5 l_6 = k$ – the integral of angular momentum.

Let us suppose that the Hamiltonian system (2.22) possesses a first integral K different from $K_{1,2}$.

Definition. The first integral K of the system (2.22) is called *meromorphic* if it is representable as a ratio

$$K = \frac{R(l, g, \rho)}{Q(l, g, \rho)},$$

where R, Q are analytic functions of the variables l_i, g_i, $1 \le i \le 4$ and the mutual distances ρ_j, $1 \le j \le 3$.

It can be shown [93] that the system (2.22) possesses an ignorable coordinate which will make possible a further reduction.

Let us make the following canonical transformation

$$l_r = \frac{\partial W_2}{\partial g_r}, \quad p_r = \frac{\partial W_2}{\partial q_r}, \quad (r = 1, 2, 3, 4), \tag{2.23}$$

where

$$W_2 = g_1 q_1 cos q_4 + g_2 q_1 sin q_4 + g_3 (q_2 cos q_4 - q_3 sin q_4) + g_4 (q_2 sin q_4 + q_3 cos q_4).$$

Here q_1 is the distance $m_1 m_3$; q_2 and q_3 are the projections of $m_2 m_3$ on, and perpendicular to $m_1 m_3$; p_1 is the component of momentum of m_1 along $m_3 m_1$; p_2 and p_3 are the components of momentum of m_2 parallel and perpendicular to $m_3 m_1$ (see Fig. 2.5).

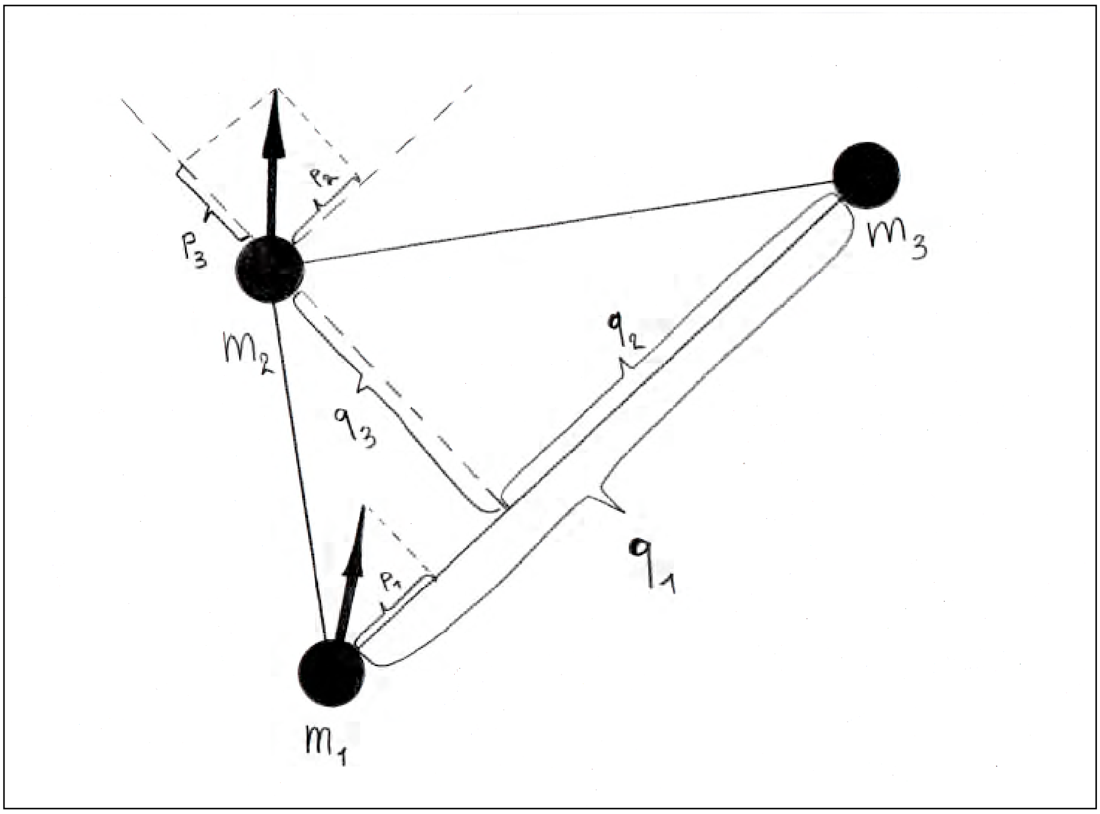

Figure 2.5: Whittaker's variables.

One can write the new equations as follows

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r}, \quad (r = 1, 2, 3)\,, \tag{2.24}$$

and

$$\frac{dq_4}{dt} = \frac{\partial H}{\partial p_4}, \quad \frac{dp_4}{dt} = 0\,, \tag{2.25}$$

with the Hamiltonian

$$H = \frac{M_1}{2}\left\{p_1^2 + \frac{1}{q_1^2}P^2\right\} + \frac{M_2}{2}(p_2^2 + p_3^2) + \frac{1}{m_3}\left\{p_1p_2 - \frac{p_3}{q_1}P\right\} -$$
$$-\frac{m_1m_3}{r_1} - \frac{m_3m_2}{r_2} - \frac{m_1m_2}{r_3},$$
$$P = p_3q_2 - p_2q_3 - p_4,$$

where

$$r_1 = q_1, \quad r_2 = \sqrt{q_2^2 + q_3^2}, \quad r_3 = \sqrt{(q_1 - q_2)^2 + q_3^2}\,,$$

are the mutual distances of the bodies.

Since $p_4 = k = const$ the system (2.24) is a closed Hamiltonian system with 3 degrees of freedom. If this system is integrated then q_4 can be found by a quadrature from (2.25).

2.2.3 The monodromy group around the Lagrangian triangular solution

Let D be the domain in the 6-dimensional complexified phase space $\mathbb{C}^6$ with coordinates (q_i, p_i), $i = 1, 2, 3$ defined by inequalities

$$|q_2| < |q_3|, \quad |q_1 - q_2| < |q_3|\,. \tag{2.26}$$

It is easy to check that the Hamiltonian equations (2.24) are analytic in D.

In the 1870's Lagrange [59] discovered particular solutions of the three-body problem in which the triangle formed by bodies is equilateral and the trajectories of the bodies are similar conics with one focus at the common barycenter. For the equations (2.24) this geometric condition gives

$$\begin{aligned} q_1 &= q, \quad q_2 = \frac{q}{2}, \quad q_3 = \frac{\sqrt{3}q}{2} \\ p_1 &= p, \quad p_2 = Ap + \frac{B}{q}, \quad p_3 = Cp + \frac{D}{q}, \end{aligned} \tag{2.27}$$

where

$$A = \frac{m_2(m_3 - m_1)}{m_1 S_3}, \quad B = -\frac{\sqrt{3}kS_1 m_2 m_3}{S_2 S_3}, \quad C = \frac{\sqrt{3}m_2(m_1 + m_3)}{m_1 S_3},$$

$$D = -\frac{km_2(S_2 + m_1 m_2 - m_3^2)}{S_2 S_3}, \quad S_1 = m_1 + m_2 + m_3,$$

$$S_2 = m_1 m_2 + m_2 m_3 + m_3 m_1, \quad S_3 = m_2 + 2m_3.$$

and the functions $q(t), p(t)$ satisfy

$$\begin{aligned} ap^2 + \frac{bp}{q} + \frac{c}{q} + \frac{d}{q^2} &= h, \\ \frac{dq}{dt} &= \left(M_1 + \frac{A}{m_3}\right) p + \frac{B}{m_3 q}\,, \end{aligned} \tag{2.28}$$

where h is the constant of energy and

$$a = \frac{2S_1S_2}{m_1^2S_3^2},\, b = -\frac{2\sqrt{3}km_2S_1}{m_1S_3^2},\, c = -S_2,\, d = \frac{2k^2S_1(m_2^2+m_2m_3+m_3^2)}{S_3^2S_2}.$$

In the case $h = 0, k \neq 0$ formulas (2.27), (2.28) define a solution Γ in which each of the bodies describes a parabola and which we call the "parabolic" Lagrangian solution. With help of (2.27) we can parametrize q, p as follows

$$\begin{aligned} q = P(w), \quad p = \frac{w}{P(w)}, \\ P(w) = -(aw^2+bw+d)/c, \quad w \in \mathbb{C}\,. \end{aligned} \tag{2.29}$$

One checks that $\Gamma \subset D$. According to (2.29) Γ is a meromorphic function on the Riemann surface $\tilde{\Gamma} = \mathbb{CP}^1 \setminus \{\infty, w_1, w_2\}$ where w_1, w_2 are zeros of $P(w)$ given by formulas

$$w_1 = \frac{(\sqrt{3}m_2 + iS_3)km_1}{2S_2}, \quad w_2 = \frac{(\sqrt{3}m_2 - iS_3)km_1}{2S_2}.$$

The NVE of Hamiltonian equations (2.24) (see for definition [96], p. 183]) along the particular solution Γ were obtained in [T6] and are of Fuchsian type [T1]

$$\frac{dx}{dz} = \left(\frac{A_0}{z-z_0} + \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \frac{A_\infty}{z-z_3}\right)x, \quad x \in \mathbb{C}^4\,, \tag{2.30}$$

where $z_i^{-1} = w_i/k$, $i = 1, 2$, $z_0^{-1} = \sqrt{3}m_1m_2/2S_2$, $z_3 = 0$, $A_0, A_1, A_2, A_\infty = -(A_0 + A_1 + A_2)$ are constant 4×4 matrices depending on m_1, m_2, m_3 as calculated in [T6]. We note that $z = \infty$ is a regular point for (2.30).

Theorem 17 ([T6])**.** *The monodromy group M of* (2.30) *is generated by two unipotent symplectic matrices $T_1, T_2 \in \mathrm{Sp}(\mathbb{C}, 4)$ having the same Jordan form*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The proof is based on the detailed study of local expansions of the general solution of (2.30) in the neighbourhoods of its singular points.

As it was pointed in [T6], the generators T_1 and T_2 correspond to local monodromy groups around the singularities $z = z_1$ and $z = z_2$ respectively. The generator of the monodromy group around the singularity $z = z_0$ turns to be trivial i.e. the general solution of (2.30) is a meromorphic function of z in the neighbourhood of this point.

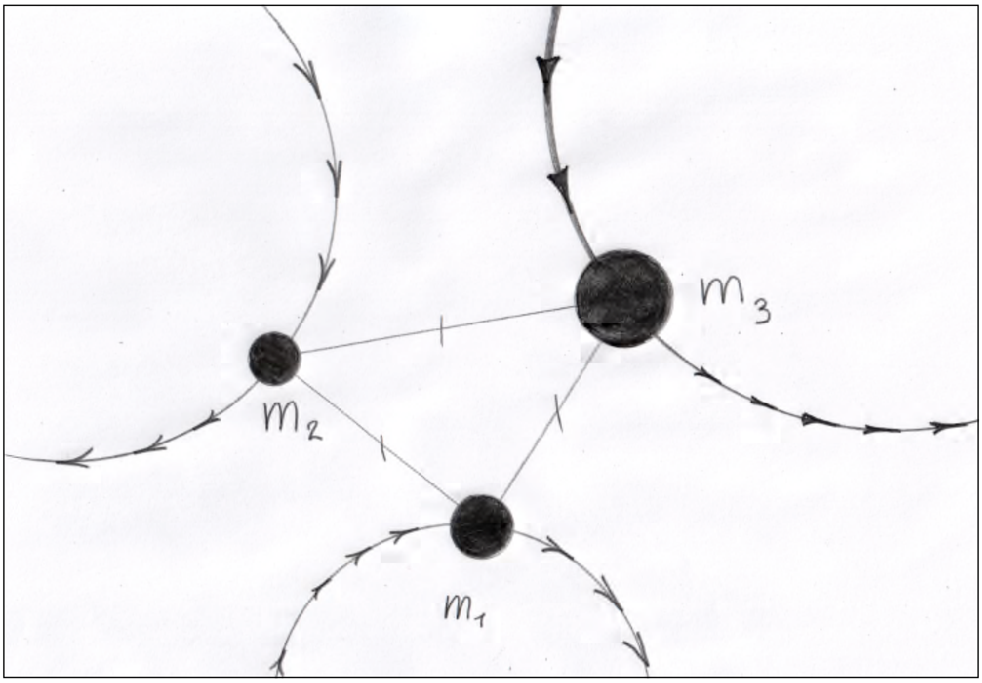

Figure 2.6: The triangular solution of Lagrange.

Definition. Let A be a square matrix with eigenvalues $\lambda_1, ..., \lambda_n$. We call A *non-resonant* if $\lambda_i - \lambda_j \notin \mathbb{Z}$ for all $i \neq j$.

As shown in [T6], the matrix A_∞ has eigenvalues

$$\text{Spectr}(A_\infty) = \{\lambda_1, \lambda_2, 3 - \lambda_1, 3 - \lambda_2\},$$

where

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}\sqrt{13 + x}, \quad \lambda_1 = \frac{3}{2} + \frac{1}{2}\sqrt{13 - x}, \quad x = 12\sqrt{1 - 3\sigma}, \tag{2.31}$$

and

$$\sigma = \frac{m_1 m_2 + m_3 m_2 + m_3 m_1}{(m_1 + m_2 + m_3)^2}.$$

One verifies that for arbitrary $m_i > 0$, $i = 1, 2, 3$ we have $\sigma \in (0, 1/3]$ and $x \in [0, 12)$.

The following result holds [T2].

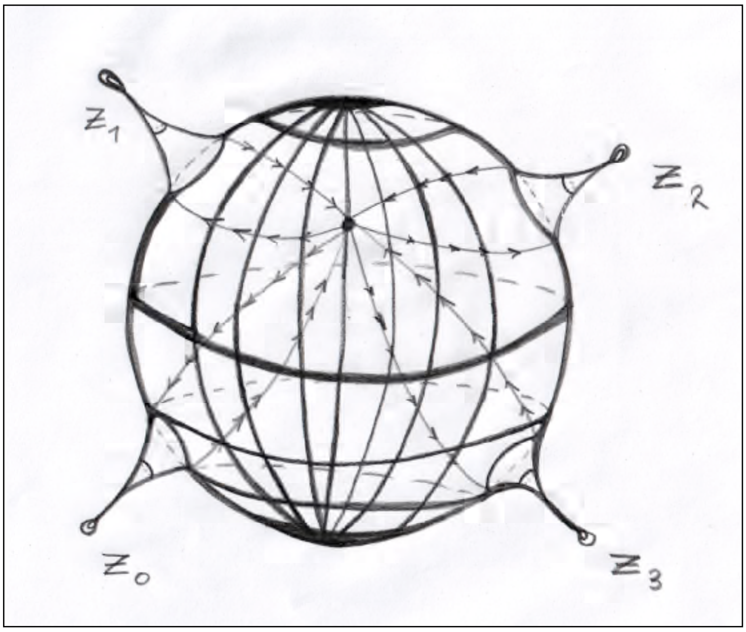

Figure 2.7: The monodromy group of the equation (2.30).

Lemma 6. *The matrix A_∞ is non-resonant if and only if $\sigma \notin E$ where*

$$E = \left\{ \frac{1}{3}, \frac{2^3}{3^3}, \frac{2}{9}, \frac{7}{48}, \frac{5}{24} \right\} .$$

Below we write explicitly the eigenvalues of A_∞ for each $\sigma \in E$:

$$\sigma = \frac{1}{3}, \quad \mathrm{Spectr}(A_\infty) = \left\{ \frac{3}{2} + \frac{\sqrt{13}}{2}, \frac{3}{2} + \frac{\sqrt{13}}{2}, \frac{3}{2} - \frac{\sqrt{13}}{2}, \frac{3}{2} - \frac{\sqrt{13}}{2} \right\},$$

$$\sigma = \tfrac{2^3}{3^3}, \quad \mathrm{Spectr}(A_\infty) = \left\{ \tfrac{3}{2} + \tfrac{\sqrt{17}}{2}, 3, \tfrac{3}{2} - \tfrac{\sqrt{17}}{2}, 0 \right\},$$

$$\sigma = \tfrac{2}{9}, \quad \mathrm{Spectr}(A_\infty) = \left\{ 2 + \sqrt{3}, 1 + \sqrt{3}, 1 - \sqrt{3}, 2 - \sqrt{3} \right\},$$

$$\sigma = \tfrac{7}{48}, \quad \mathrm{Spectr}(A_\infty) = \left\{ \tfrac{3}{2} + \tfrac{\sqrt{22}}{2}, \tfrac{5}{2}, \tfrac{3}{2} - \tfrac{\sqrt{22}}{2}, \tfrac{1}{2} \right\},$$

$$\sigma = \tfrac{5}{2^4}, \quad \mathrm{Spectr}(A_\infty) = \left\{ \tfrac{7}{2}, \tfrac{3}{2} + \tfrac{\sqrt{10}}{2}, -\tfrac{1}{2}, \tfrac{3}{2} - \tfrac{\sqrt{10}}{2} \right\}.$$

Corollary 2. *The matrix A_∞ is diagonalizable for all $m_i > 0$, $i = 1, 2, 3$.*

Indeed, the eigenvalues of A_∞ are always different except in the case $\sigma = 1/3$ where the straightforward calculation shows that A_∞ is diagonalizable. If $\sigma \notin E$ then A_∞ is non-resonant and the fundamental solution of (2.30) in the neighbourhood of $z = 0$ may be written in the form $x(z) = a(z)z^{A_\infty}$ where $a(z)$ is a square 4×4 matrix with elements analytic in the neighbourhood of $z = 0$ (see f.e. [34]). Let T_∞ be the monodromy matrix corresponding to the singular point $z = 0$, then

$$T_\infty^{-1} = T_1 T_2 \,. \tag{2.32}$$

As T_∞ is conjugate to $e^{2\pi i A_\infty}$ one gets

$$\mathrm{Spectr}(T_\infty) = \left\{ e^{2\pi i \lambda_1}, e^{2\pi i \lambda_2}, e^{-2\pi i \lambda_1}, e^{-2\pi i \lambda_2} \right\} .$$

We divide E in two parts as follows

$$E = E_1 \cup E_2, \quad E_1 = \left\{ \frac{1}{3}, \frac{2^3}{3^3} \right\}, \quad E_2 = E \setminus E_1 \,.$$

Using a general theory of linear equations with regular singular points one shows the following

Theorem 18 ([T2])**.** *If $\sigma \in E_1$ then T_∞ is diagonalizable.*

Of course, this result raises the obvious question, namely does T_∞ have non-trivial Jordan blocks for $\sigma \in E_2$? One can show that for each $\sigma \in E_2$ the answer here is "yes" at least for some particular values of masses m_1, m_2, m_3.

2.2.4 Invariants of the monodromy group

Let $x = (x_1, x_2, x_3, x_4)^T$ be coordinates in the linear space $\mathbb{C}^4$ where the monodromy group M, which a subgroup of $Sp(\mathbb{C}, 4)$, acts by multiplication. The function $J(x)$, which always can be supposed to be rational and homogeneous with respect to x, is called an *invariant* of M if $J(m\,x) = J(x)$ for all $m \in M$.

Our aim is to prove the following theorem.

Theorem 19. *If $\sigma \notin E$ then the monodromy group M does not have a rational invariant. For $\sigma \in E_1$ M has a unique rational invariant.*

According to Theorem 17 we can write the generators T_1, T_2 of M in the form $T_i = I + d_i$, $i = 1, 2$ where d_1, d_2 are nilpotent matrices and I is the identity matrix.

We recall that a matrix commuting with all elements of the group M is called a *centralizer* of M. It is not necessary for a centralizer to be in M.

Lemma 7. *The monodromy group M has a centralizer given by $T = d_1 d_2 + d_2 d_1 = T_\infty + T_\infty^{-1} - 2\,I$.*

The proof consist in a simple verification of $[T, T_i] = 0$, $i = 1, 2$ using $d_1^2 = d_2^2 = 0$.

The eigenvalues of T are given by formulas

$$\mathrm{Spectr}(T) = \{\sigma_1, \sigma_1, \sigma_2, \sigma_2\}, \quad \sigma_i = 2(\cos(2\pi\lambda_i) - 1), \quad i = 1, 2\,. \tag{2.33}$$

If $\sigma \notin E$ then $T \neq \alpha\, I$, $\alpha = const$ and hence there exists a linear basis in which the generators T_1, T_2 of M take the form

$$T_i = I + d_i, \quad i = 1, 2, \quad d_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} A & O \\ O & B \end{pmatrix}, \tag{2.34}$$

where A, B are some unknown nilpotent 2×2 matrices which we parametrize as follows

$$A = \begin{pmatrix} a & a_1 \\ a_2 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & b_1 \\ b_2 & -b \end{pmatrix},$$

with arbitrary parameters $a, b, a_i, b_i \in \mathbb{C}$ satisfying relations $a^2 + a_1 a_2 = 0$, $b^2 + b_1 b_2 = 0$.

Let us introduce two linear differential operators associated with generators T_1 and T_2

$$D_1 = (d_1 x, \nabla), \quad D_2 = (d_2 x, \nabla), \quad \nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4})^T,$$

where $(,)$ is a scalar product.
Next, we suppose that the monodromy group M has a rational invariant $J(x_1, x_2, x_3, x_4)$.

Lemma 8. *The function J is a solution of equations*

$$D_1 J = 0, \quad D_2 J = 0 .$$

The elementary proof of this fact can be found in [T6]. From the point of view of the Morales-Ramis theory (see [67], [68]) Lemma 8 means that J is an invariant of the differential Galois group G of (2.30). Then D_1, D_2 are two generators of the corresponding Lie algebra of G.
The operators D_1, D_2 may be written in the explicit form

$$D_1 = x_2\,\partial_{x_1} + x_4\,\partial_{x_3},$$

$$D_2 = (ax_1 + a_1x_2)\,\partial_{x_1} + (a_2x_1 - ax_2)\,\partial_{x_2} + (bx_3 + b_1x_4)\,\partial_{x_3} +$$

$$+(b_2x_3 - bx_4)\,\partial_{x_4} .$$

Their commutators

$$D_3 = [D_1, D_2] = (-a_2x_1 + 2ax_2)\,\partial_{x_1} + a_2x_2\,\partial_{x_2} + (2bx_4 - b_2x_3)\,\partial_{x_3} +$$

$$+b_2x_4\,\partial_{x_4},$$

$$D_4 = -\tfrac{1}{2}\,[D_1, D_2] = a_2x_2\,\partial_{x_1} + b_2x_4\,\partial_{x_3} ,$$

provide us with two additional equations: $D_3\,J = 0$, $D_4\,J = 0$. The idea of the proof of Theorem 19 is to study conditions of compatibility of the system $D_i\,J = 0$, $i = 1, ..., 4$. Equations $D_1\,J = 0$, $D_4\,J = 0$ or equivalently

$$\begin{cases} x_2\,\partial_{x_1} J + x_4\,\partial_{x_3} J = 0 \\ a_2x_2\,\partial_{x_1} J + b_2x_4\,\partial_{x_3} J = 0, \end{cases}$$

yield $a_2 = b_2 = \alpha$ or $\partial_{x_1} J = \partial_{x_3} J = 0$. In the first case, after some calculations, one gets $T = d_1d_2 + d_2d_1 = \alpha\, I$ which is in contrast to our assumption $T \neq \alpha I$ following from $\sigma \notin E$. The second case deserves the particular attention. Conditions $\partial_{x_1} J = \partial_{x_3} J = 0$ imply that J depends only on x_2, x_4. Since we consider non-trivial invariants of M, one must have either $\partial_{x_2} J \neq 0$ or $\partial_{x_4} J \neq 0$. In this situation, the compatibility of equations $D_2 J = 0$, $D_3 J = 0$ is equivalent to the identity

$$\det \begin{pmatrix} a_2x_1 - ax_2 & b_2x_3 - bx_4 \\ a_2x_2 & b_2x_4 \end{pmatrix} = b_2a_2x_1x_4 + (a_2b - b_2a)x_2x_4 - a_2b_2x_2x_3 \equiv 0\,,$$

which splits into three cases:

I. $b_2 = a_2 = 0$,

$$\mathrm{Spectr}(T_\infty) = \{1, 1, 1, 1\}\,.$$

II. $b_2 = b = 0$,

$$\mathrm{Spectr}(T_\infty) = \{1, 1, 1 + \tfrac{a_2}{2} + \tfrac{1}{2}\sqrt{4a_2 + a_2^2 + 4a_1a_2 + 4a^2},$$

$$1 + \tfrac{a_2}{2} - \tfrac{1}{2}\sqrt{4a_2 + a_2^2 + 4a_1a_2 + 4a^2}\}\,.$$

III. $a_2 = a = 0$,

$$\mathrm{Spectr}(T_\infty) = \{1, 1, 1 + \tfrac{b_2}{2} + \tfrac{1}{2}\sqrt{4b_2 + b_2^2 + 4b_1b_2 + 4b^2},$$

$$1 + \tfrac{b_2}{2} - \tfrac{1}{2}\sqrt{4b_2 + b_2^2 + 4b_1b_2 + 4b^2}\}\,.$$

We observe that in all these cases T_∞ has at least two equal eigenvalues which we found with help of (2.32) as eigenvalues of the matrix $(T_1T_2)^{-1}$. This yields that if $\rho_1, ..., \rho_4$ are eigenvalues of A_∞ then $\rho_i - \rho_j \in \mathbb{Z}$ for at least one pair of indexes $i \neq j$ i.e. A_∞ is resonant according to our Definition 2.2.3. But this contradicts Lemma 6. Thus, the first part of the proof of Theorem 19 is done.

We will now show that if $\sigma \in E_1$ then M has an unique invariant. Using (2.33) one can show that in the case $\sigma \in E_1$ all eigenvalues of the matrix T are equal i.e. $\sigma_1 = \sigma_2$. Then, applying Theorem 18 together with $T = T_\infty + T_\infty^{-1} - 2\,I$ one shows that $T = \alpha\, I$, $\alpha \neq 0$. Taking d_1 as given in (2.34) and writing d_2 in its general form

$$d_2 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}\,,$$

with unknown coefficients $a_i, b_i, c_i, d_i \in \mathbb{C}$, we calculate $T = d_1 d_2 + d_2 d_1$

$$T = \begin{pmatrix} b_1 & b_2 + a_1 & b_3 & b_4 + a_3 \\ 0 & b_1 & 0 & b_3 \\ d_1 & d_2 + c_1 & d_3 & d_4 + c_3 \\ 0 & d_1 & 0 & d_3 \end{pmatrix}.$$

One can satisfy conditions $T = \alpha I$, $\alpha \neq 0$ and $d_2^2 = 0$ parametrizing a_i, b_i, c_i, d_i as follows

$$a_1 = \delta_1, \quad a_2 = \tfrac{\delta_2 \delta_3 - \delta_1^2}{\alpha}, \quad a_3 = \delta_2, \quad a_4 = -\tfrac{\delta_1 \delta_2 + \delta_2 \delta_4}{\alpha},$$

$$b_1 = \alpha, \quad b_2 = -\delta_1, \quad b_3 = 0, \quad b_4 = -\delta_2,$$

$$c_1 = -\delta_3, \quad c_2 = \tfrac{\delta_4 \delta_3 + \delta_3 \delta_1}{\alpha}, \quad c_3 = \delta_4, \quad c_4 = \tfrac{\delta_2 \delta_3 - \delta_4^2}{\alpha},$$

$$d_1 = 0, \quad d_2 = \delta_3, \quad d_3 = \alpha, \quad d_4 = -\delta_4\,.$$

where $\delta_{1,2,3,4}$ are arbitrary complex numbers.

As above, we define two linear differential operators corresponding to generators T_1, T_2

$$D_1 = x_2\, \partial_{x_1} + x_4\, \partial_{x_3}$$

$$D_2 = \left(\textstyle\sum_i a_i x_i\right) \partial_{x_1} + \left(\sum_i b_i x_i\right) \partial_{x_2} + \left(\sum_i c_i x_i\right) \partial_{x_3} + \left(\sum_i d_i x_i\right) \partial_{x_4}\,.$$

Due to Lemma 8 the rational invariant J of M is a solution of the system of two equations: $D_1\, J = 0$, $D_2\, J = 0$. The first one can be easy solved in the explicit form and gives $J = J(x_1 x_4 - x_3 x_2, x_2, x_4)$. Introducing the new variables $(y_1, y_2, y_3, y_4) = (x_1 x_4 - x_3 x_2, x_2, x_4, x_3)$ one writes D_2 in the form

$$D_2 = \Delta_0 + y_4\, \Delta_1\,,$$

where

$$\alpha\, y_3\, \Delta_0 = (\alpha\, (\delta_1 - \delta_4)\, y_1\, y_3 + (\delta_4^2 - \delta_1^2)\, {y_3}^2\, y_2 + 2\, \alpha\, \delta_3\, y_2\, y_1 -$$

$$- \delta_3\, (\delta_1 + \delta_4)\, y_3\, {y_2}^2 - \delta_2 (\delta_1 + \delta_4)\, {y_3}^3)\, \partial_{y_1} + \alpha\, (\alpha\, y_1 - \delta_1\, y_3\, y_2 - \delta_2\, {y_3}^2)\, \partial_{y_2} +$$

$$+ \alpha y_3 (\delta_3\, y_2 - \delta_4\, y_3)\, \partial_{y_3} + (\cdots)\, \partial_{y_4},$$

$$y_3\, \Delta_1 = (2\, (\delta_1 - \delta_4)\, y_2\, y_3 + 2\, \delta_2\, y_3^2 + 2\, \delta_3\, y_2^2)\, \partial_{y_1} + \alpha\, y_2\, \partial_{y_2} + \alpha\, y_3\, \partial_{y_3} +$$

$$+ (\cdots)\, \partial_{y_4}\,.$$

In new variables $y_1, ..., y_4$ we have obviously $J = J(y_1, y_2, y_3)$. Thus, $D_2\, J = 0$ is equivalent to two equations: $\Delta_0\, J = 0$, $\Delta_1\, J = 0$. Solving the second one we get $J = J(K_1, K_2)$ with $K_1 = y_3/y_2$, $K_2 = p_1 y_2^2 + p_2 y_3^2 + p_3 y_2 y_3 - \alpha y_1$ where $p_1 = \delta_3$, $p_2 = \delta_2$, $p_3 = \delta_1 - \delta_4$. The following change of variables $(z_1, z_2, z_3) = (K_1, K_2, y_3)$ simplifies Δ_0

$$\Delta_0 = \frac{\alpha z_1^2 z_2}{z_3}\, \partial_{z_1}\, .$$

In variables (z_1, z_2, z_3) J depends on z_1, z_2 only and therefore $\Delta_0\, J = 0$ is equivalent to $J = J(z_2)$. Thus, returning to initial variables $x_1, ..., x_4$, we conclude that for $\sigma \in E_1$ the monodromy group M has an unique invariant given by

$$J = \delta_3 x_2^2 + \delta_2 x_4^2 + (\delta_1 - \delta_4) x_2 x_4 - \alpha(x_1 x_4 - x_3 x_2)\, . \qquad (2.35)$$

The proof of Theorem 19 is now completed.

2.2.5 Non-existence of additional meromorphic first integrals

Let D be defined by (2.26) and $U_\Gamma \subset D$ be the connected neighbourhood of the Lagrangian parabolic solution Γ. Suppose that the Hamiltonian equations of the planar three-body problem (2.24) admit a first integral meromorphic in U_Γ with respect to positions q_i, mutual distances r_i, momenta p_i and which is functionally independent with H. According to Ziglin ([96], p. 183), in this case the monodromy group M of NVE (2.30) has a rational invariant and so, due to Theorem 19, the next result follows immediately

Theorem 20. *Let* $\sigma = \dfrac{m_1 m_2 + m_3 m_2 + m_3 m_1}{(m_1 + m_2 + m_3)^2}$ *and* $\sigma \notin \left\{ \dfrac{1}{3}, \dfrac{2^3}{3^3}, \dfrac{2}{9}, \dfrac{7}{48}, \dfrac{5}{24} \right\}$. *Then the planar three-body problem* (2.24) *has no additional first integrals meromorphic with respect to positions, mutual distances and momenta.*

Let $Y(z)$ be the fundamental solution of (2.30) satisfying $Y(e) = I$ for $e \in \mathbb{C}$ different from $0, z_0, z_1, z_2$. Then, after the analytic continuation of $Y(z)$ starting from e and going along the closed path around z_i, one obtains $\tilde{Y}(z) = Y(z) M_i$, $i = 1, 2$. Consider the function $I(x, z) = J(Y^{-1}(z)x)$ where $J(x)$ is the invariant of M calculated in (2.35). It is easy to see that $I(x, z)$ is a first integral of (2.30) and that it is an analytic function of z on the Riemann surface $R = \mathbb{CP}^1 \setminus \{0, z_0, z_1, z_2\}$ i.e. is a rational function of z. We can give now our final theorem

Theorem 21. *If* $\sigma \in \left\{\frac{1}{3}, \frac{2^3}{3^3}\right\}$ *then the normal variational equations* (2.30) *along the Lagrangian parabolic solution* Γ *of the three-body problem* (2.15) *admit a first integral* $I(x, z)$ *which is a polynomial with respect to* x *and which is a rational function with respect to* z.

2.2.6 The relationship to the result of Bruns

In this section we will comment on the relationship between Theorem 20 and the result of Bruns [12], see also [93], [43], [44].

After the center of mass reduction (Radau, 1868, [77], see also Whittaker [93]) the three-body problem in $\mathbb{R}^3$ can be written as a Hamiltonian system with 6 degrees of freedom

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r}, \quad (r = 1, 2, \ldots, 6)\,, \tag{2.36}$$

where

$$\begin{aligned}
&H = T - U, \\
&T = \tfrac{1}{2\mu}(p_1^2 + p_2^2 + p_3^2) + \tfrac{1}{2\mu'}(p_4^2 + p_5^2 + p_6^2), \\
&U = m_1 m_2 (q_1^2 + q_2^2 + q_3^2)^{-\frac{1}{2}} + m_1 m_2 \{q_4^2 + q_5^2 + q_6^2 + \\
&+ \tfrac{2m_2}{m_1+m_2}(q_1 q_4 + q_2 q_5 + q_3 q_6) + \left(\tfrac{m_2}{m_1+m_2}\right)^2 (q_1^2 + q_2^2 + q_3^2)\}^{-\frac{1}{2}} + \\
&+ m_2 m_3 \{q_4^2 + q_5^2 + q_6^2 - \tfrac{2m_2}{m_1+m_2}(q_1 q_4 + q_2 q_5 + q_3 q_6) + \\
&+ \left(\tfrac{m_2}{m_1+m_2}\right)^2 (q_1^2 + q_2^2 + q_3^2)\}^{-\frac{1}{2}}, \\
&\mu = \tfrac{m_1 m_2}{m_1+m_2},\ \mu' = \tfrac{m_3(m_1+m_2)}{m_1+m_2+m_3}\,.
\end{aligned} \tag{2.37}$$

Theorem 22 (Bruns, 1868)**.** *The classical integrals of the three-body problem* (2.36) *(three for the angular momentum and one for the energy) are the only independent algebraic first integrals (with respect to variables* q_i *and* p_i, $i = 1, \ldots, 6$*).*

Let s denotes the sum of three mutual distances between bodies. We put $q = (q_1, \ldots, q_6)$, $p = (p_1, \ldots, p_6)$.

In his original proof, Bruns derives the previous theorem as a consequence of the following result

Theorem 23. *The only independent first integrals of* (2.36) *of the form*

$$f(q,p,s) = \text{const} \quad \text{with} \quad f \in \mathbb{C}(q,p,s)\,, \tag{2.38}$$

are the classical ones.

One verifies that all mutual distances are certain rational functions of s and q. Thus, one can reformulate Theorem 23 as follows

Theorem 24. *The only independent first integrals of the three-body problem* (2.36) *which are rational functions of q, p and of the mutual distances are the classical ones.*

Hence, it may be argued that Theorem 20 generalizes the result of Bruns given by Theorem 24 in the case of the planar three-body problem written in the variables of Whittaker.

2.3 Current research. The non–integrability of the nonholonomic Rattleback problem

In this section I will quote my current research in collaboration with H. Dullin on the non–integrability of the ellipsoidal Rattleback model (**added after the thesis defense: the material of this chapter was published later in [T5bis]**)

2.3.1 Introduction

The rattleback's amazing mechanical behaviour can be described as follows: when spun on a flat horizontal surface in the clockwise direction this top continues to spin in the same direction until it consumes the initial spin energy; when, however, spun in the counterclockwise direction, the spinning soon ceases, the body briefly oscillates, and then reverses its spin direction and thus spins in the clockwise direction until the energy is consumed. The first mathematical model of this phenomena belongs to Walker (1896) who studied the linearized equations of motion and concluded that the completely stable motion is possible in only one (say clockwise) spin direction. This classical explanation of the rattleback's behavior is quite unsatisfactory since it does not reflect the global dynamical effects explaining the transfer of trajectories from the vicinity of the unstable solution to the stable one. To analyze thoroughly this question we propose to study the nonholonomic equations of the rattleback in the complex domain and particularly to study *the existence of additional analytic first integrals*. It has been observed in many mechanical systems that non existence of analytic first integrals is usually associated to complicated chaotic behavior of trajectories of the system. We mention that actually only numerical evidence for chaos of rattleback systems have been observed [11].

Figure 2.8: One of the Rattleback models.

In our case the rattleback represents a full ellipsoidal body whose center of mass P coincides with its geometric one. All vectors are defined in the body fixed axes (with origin in P) coinciding with principal geometric axes of the ellipsoid

$$E(r) = \frac{r_1^2}{b_1^2} + \frac{r_2^2}{b_2^2} + \frac{r_3^2}{b_3^2} = 1\,. \tag{2.39}$$

Let $<,>$ denotes the euclidean scalar product in $\mathbb{R}^3$ and let $||\cdot||$ be the corresponding norm. In addition, let $[\,,\,]$ denote the cross product in $\mathbb{R}^3$.

As shown in [35], the nonholonomic equations of motion can be written in the following form

$$\Theta \frac{d\omega}{dt} + m\,[r,[\frac{d\omega}{dt},r]] = -[\omega,\Theta\omega] - m\,[r,[\omega,[\omega,r]]] + mg\,[r,\gamma] - m\,[r,[\omega,\frac{dr}{dt}]]\,, \tag{2.40}$$

$$\frac{d\gamma}{dt} = [\gamma,\omega]\,, \tag{2.41}$$

where

$$\Theta = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & 0 \\ \Sigma_{12} & \Sigma_{22} & 0 \\ 0 & 0 & \Sigma_{33} \end{bmatrix}, \tag{2.42}$$

is the inertia matrix; m – mass of the body; g – gravitational constant; $\omega = (\omega_1,\omega_2,\omega_3)^T$ – angular velocity; $\gamma = (\gamma_1,\gamma_2,\gamma_3)^T$ – unit vector normal to the body's surface at the point of contact; $r = (r_1,r_2,r_3)^T$– position of contact point.

To observe the rattleback behavior, one has to impose the following condition on Θ (see f.e. [11]) and b_i

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} I_1\cos^2\delta + I_2\sin^2\delta & (I_1-I_2)\cos\delta\sin\delta \\ (I_1-I_2)\cos\delta\sin\delta & I_1\sin^2\delta + I_2\cos^2\delta \end{bmatrix}, \tag{2.43}$$

$$\Sigma_{33} = I_3, \quad b_1 \neq b_2\,,$$

where I_1, I_2, I_3 are the principal components of the inertia tensor whose principal horizontal axes are rotated by the angle $\delta > 0$ with respect to (r_1,r_2) axes of the ellipsoid.

Below we will always assume that

$$\Sigma_{ij} > 0, \quad \Sigma_{3,3} < \Sigma_{1,1} + \Sigma_{2,2}, \quad b_1 > b_2 > 0, \quad b_3 > 0, \quad m > 0, \quad g > 0\,. \tag{2.44}$$

Using (2.39), the vector $r = R(\gamma)$ is defined by solving $\gamma = -\nabla E/||\nabla E||$ for r, which gives

$$r_i = R_i(\gamma) = \frac{-b_i^2\gamma_i}{\sqrt{b_1^2\gamma_1^2 + b_2^2\gamma_2^2 + b_3^2\gamma_3^2}}, \quad i = 1,2,3\,. \tag{2.45}$$

Excluding $\dot{\omega}$ from (2.40) and using (2.45) we transform the nonholonomic rattleback equations to the standard form

$$\begin{aligned} \tfrac{d\omega}{dt} &= F(\omega,\gamma), \\ \tfrac{d\gamma}{dt} &= [\gamma,\omega]\,, \end{aligned} \tag{2.46}$$

where $F=(F_1,F_2,F_3)^T$ is a vector field rational with respect to variables ω, γ and $s=\sqrt{b_1^2\gamma_1^2+b_2^2\gamma_2^2+b_3^2\gamma_3^2}$.

Let $(\omega,\gamma,s)\in\mathbb{C}^7$. Then the vector field (2.46), as a function of (ω,γ,s), is analytic in the domain $\mathcal{D}=\mathbb{C}^7\setminus(\mathcal{S}\cup\mathcal{L})$ where $\mathcal{L}=\{(\omega,\gamma,s)\in\mathbb{C}^7\,:\,s=0\}$ and $\mathcal{S}\subset\mathbb{C}^7$ is the surface on which the determinant of the matrix U given below is zero

$$U=\Theta-m[r,[r,\cdot]]=\Theta+m\big(<r,r>)\mathrm{Id}-r\otimes r\big)\,. \tag{2.47}$$

This matrix appears when we solve (2.40) to find $\dot\omega$. For an arbitrary non-zero vector $u\in\mathbb{R}^3$

$$<u,Uu>=<u,\Theta u>+m<u,u><r,r>-m<u,r>^2=<u,\Theta u>+$$

$$+m||u\times r||^2>0\,,$$

so that U is positive definite and hence $\det(U)=0$ never occurs in the mechanical case.

The system (2.46) always possesses two first integrals (see [35]):

$$H=\frac{m}{2}||[\omega,R(\gamma)]||^2+\frac{1}{2}<\omega,\Theta\omega>-mg<R(\gamma),\gamma>=h,\quad h\in\mathbb{R}\,, \tag{2.48}$$

– the energy;

$$G=<\gamma,\gamma>=\gamma_1^2+\gamma_2^2+\gamma_3^2=l,\quad l\in\mathbb{R}_+\,, \tag{2.49}$$

– the geometric integral.

We see that $H(\omega,\gamma,s)$ (after introduction of s with help of (2.45)) and $G(\gamma)$ are analytic functions of ω, γ, s in $\mathcal{D}$. The natural question arises whether the equations (2.46) can have a third first integral analytic (meromorphic) with respect to $(\omega,\gamma,s)\in\mathbb{C}^7$ and functionally independent together with H and G. We remark that the absence of first integrals in the meromorphic case is generally harder to prove than in the analytic one.

The paper is organized as follows. In Chapter 2.3.2 we describe one particular solution of the Rattleback problem. Chapter 2.3.3 discusses some properties of the normal variational equations and the monodromy group generators.

Chapter 2.3.4 examines the case in which the normal variations equations have logarithmic branching points. Under these conditions we show the non-existence of additional meromorphic first integrals (Theorem 26). Finally, Chapter 2.3.5 contains the proof of our main theorem about the analytic non-integrability

Theorem 25. *Let us assume that the conditions* (2.44) *and* (2.96) *hold. Then the energy H and the geometric first integral G are the only analytic, with respect to* (ω, γ, s)*, first integrals of the rattleback problem* (2.46).

Rigid body limiting case

The rattleback equations (2.40)-(2.41) formally contain the equations of the heavy rigid body in the singular limiting case $m \to 0$, $mg \not\to 0$. Moreover the functions $R_i(\gamma)$ are constants denoted by r_i again, now designating the position of the center of mass in the body frame. By a rotation the tensor of inertia Θ can be diagonalised so that Σ_{12} can be set to zero in this case, thus violating (2.44). Under these assumptions the system is Hamiltonian with another well known integral

$$L =< \gamma, \Theta\omega >= \gamma_1\omega_1 I_1 + \gamma_2\omega_2 I_2 + \gamma_3\omega_3 I_3 \,. \tag{2.50}$$

2.3.2 The invariant manifold

We consider the vector field (2.46) as a function of six variables (ω, γ). Let $l > 0$ (in the mechanical case we can set $l = 1$). For $(\omega, \gamma) \in \mathbb{R}^6$ the square root in s is assumed to be always positive. It is straightforward to check that the system (2.46) has the invariant manifold $M = \{(\omega, \gamma) \in \mathbb{R}^6 : \omega_1 = \omega_2 = 0, \gamma_3 = 0\}$. The mechanical sense of the motion on M is quite clear: it corresponds to rolling of the body on the line of intersection of its surface with $r_3 = 0$ plane. We note that this invariant manifold exists because of the assumption that the r_3 ellipsoidal axis coincides with one of the principal inertia body axes.

Complexifying (1.7) we denote $U_{l,h} \subset M \subset \mathbb{C}^6$ its orbit corresponding to $(l, h) \in \mathbb{C}^2$ defined as the intersection of two algebraic surfaces

$${\omega_3}^2 \left(\frac{m\,(b_1^4\gamma_1^2 + b_2^4\gamma_2^2)}{b_1^2\gamma_1^2 + b_2^2\gamma_2^2} + \Sigma_{33} \right) + 2\,mg\,\sqrt{b_1^2\gamma_1^2 + b_2^2\gamma_2^2} = 2\,h \in \mathbb{R}, \tag{2.51a}$$

$$\gamma_1^2 + \gamma_2^2 = l \in \mathbb{R}\,. \tag{2.51b}$$

We assume now that $h \neq 0$ and $l = 0$. Obviously the solution obtained in this way cannot be real; it is non-mechanical.

Let $\Gamma \subset \mathbb{C}^6 = (\omega, \gamma)$ be the complex curve defined by

$$\omega_3 = p, \quad \gamma_1 = \frac{p^2 - \alpha^2}{\beta}, \, \gamma_2 = i\frac{p^2 - \alpha^2}{\beta}, \, \omega_1 = \omega_2 = \gamma_3 = 0, \, p \in \mathbb{C}\backslash\{-\alpha, +\alpha\}\,, \tag{2.52}$$

where

$$\alpha^2 = \frac{2h}{m(b_1^2 + b_2^2) + \Sigma_{3,3}}, \quad \beta = -\frac{2mg\sqrt{b_1^2 - b_2^2}}{m(b_1^2 + b_2^2) + \Sigma_{3,3}}, \quad h \in \mathbb{C}\,, \tag{2.53}$$

(the parametrization of $U_{0,h}$).

Let $P_0 = (\omega_0, \gamma_0) \in \Gamma$ such that $\gamma_0 \neq 0$.

The function $s = \sqrt{b_1^2\gamma_1^2 + b_2^2\gamma_2^2 + b_3^2\gamma_3^2}$, once the square-root branching is fixed, is analytic in a small neighborhood $U_{P_0} \subset \mathbb{C}^6$ of P_0. We can analytically continue s and all its derivatives along Γ with help of the parametrization above: $s((\omega, \gamma) \in \Gamma) = s(p) = \sigma\frac{p^2-\alpha^2}{\beta}$ where $\sigma = \sqrt{b_1^2 - b_2^2} > 0$. In particular, it shows that s is single-valued on Γ and hence is analytic in a small neighborhood $B_\Gamma \subset \mathbb{C}^6$ of it. Thus, the vector field (2.46) restricted to B_Γ is an analytic function of complex variables (ω, γ) and has the invariant curve Γ.

We use the reparametrisation $b_i^2 = \rho_i(\rho_1 - \rho_2)$, with $\rho_1 > \rho_2$, $\rho_i = \text{const} > 0$, which makes $\sigma = \rho_1 - \rho_2$ polynomial on the solutions we are considering. In particular for this solution $r = (-\rho_1, -\mathrm{i}\rho_2, 0)^T$, which is a constant.

Since in the *rigid body limiting case* r is not proportional to γ the invariant manifold M only exists under the additional assumption that $r_3 = 0$.

2.3.3 The variational equations and its monodromy group

The relation between the parameters p and t can be easily found by substitution of the parametrization (2.52) in one of the equations (2.46). This gives

$$dp = i\frac{p^2 - \alpha^2}{2}dt\,. \tag{2.54}$$

Remark 2. The change of time (2.54) has infinitely many sheets. Nevertheless, one can always replays the vector field $F(X)$, $X = (\omega, \gamma)^T$ defined in (2.46) by $\tilde{F} = (F/(2^{-1}i(p^2 - \alpha^2)), 1)^T$ i.e consider the new autonomous differential system

$$\frac{dX}{d\tau} = \frac{\tilde{F}(X)}{2^{-1}i(p^2 - \alpha^2)}, \quad \frac{dp}{d\tau} = 1\,, \tag{2.55}$$

which will have the same particular solution Γ defined by (2.52) where p is replaced by τ.

Obviously, an autonomous analytic first integral of (2.46) gives a first integral of the same type for the vector field (2.55). One sees also that $\tilde{F}$ is analytic in the neighborhood of Γ. The further analysis based on variational equations of (2.46) or (2.55) will be essentially the same.

Let $\tilde{\omega}_i$, $\tilde{\gamma}_i$, $i = 1, 2, 3$ be the variations of variables ω_i, γ_i respectively. We use the following notation for coordinates of the variation vector $V = (\tilde{X}, \tilde{Y})^T = ((\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\gamma}_3), (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\omega}_3))^T$. In these variables, the variational equations of (2.46) along Γ, after the substitution (2.54), have the block diagonal form

$$\frac{dX}{dp} = \begin{bmatrix} M & O \\ O & N \end{bmatrix} X, \quad X \in T_\Gamma B_\Gamma \,. \tag{2.56}$$

where $M(p)$, $N(p) \in GL(3, \mathbb{C}(p))$ and O is the zero 3×3 matrix.

One can show that the derivatives of the first integrals H and G with respect to variables ω_1, ω_2, γ_3 vanish along Γ. So, the linear first integrals $\tilde{H} =< dH(\Gamma), \tilde{Y} >$ and $\tilde{G} =< dG, \tilde{Y} >$ are not useful in solving of the first block system

$$\frac{d\tilde{X}}{dp} = M\tilde{X} \,, \tag{2.57}$$

whereas the second subsystem $\frac{d\tilde{Y}}{dp} = N\tilde{Y}$ can be completely solved in radicals with help of $\tilde{H}$ and $\tilde{G}$.

As seen from the parametrization (2.52), the equations (2.57) are the normal variational equations (see [96] for definition) of the system (2.46) along the orbit Γ. The following lemma holds.

Lemma 9 ([67], [96]). *Let us suppose that the system* (2.46) *has a third first integral $H_3(\omega, \gamma)$ which is analytic (meromorphic) in the neighborhood B_Γ and functionally independent together with H and G. Then the monodromy and the differential Galois groups of the normal variational equations* (2.57) *have a non trivial polynomial (rational) invariant.*

We can put the linear system (2.57) into the Fuchsian form with help of the rational transformation

$$\tilde{X} = \mathrm{diag}(p, p, p^2 - \alpha^2)x \,, \tag{2.58}$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{C}^3$. The new system takes the following form

$$\frac{dx}{dp} = T(p)x = \left(\frac{A}{p-\alpha} + \frac{A}{p+\alpha} + \frac{B}{p}\right) x, \quad p \in \mathcal{R} = \bar{\mathbb{C}} \setminus \{-\alpha, +\alpha, 0, \infty\} \,. \tag{2.59}$$

Clearly, the monodromy groups of the systems (2.57) and (2.59) are equivalent. The constant 3×3 matrices A, B are given in the the Appendix A.

Remark 3. One sees that the above linear equations are invariant under the transformation $p \mapsto -p$. That in particular allows to reduce the number of finite singularities up to two via introducing the new time $p = \tau^2$. Nevertheless, we prefer to keep this symmetry inside the equations and to use it later in the study of the monodromy group of (2.59).

Proposition 1. *The general solution of* (2.59) *is meromorphic in the neighborhood of* $p = 0$.

The proof follows from the fact that the matrix $M(p)$ in (2.57) is holomorphic in the neighborhood of $p = 0$ as seen from (2.52), (2.40). In particualr, we have

$$\mathrm{Spectr}(B) = \{0, -1, -1\}\,. \tag{2.60}$$

That can be verified directly with help of formulas for B given in Appendix A.

We will need some technical results. The following proposition will play a crucial role for our non-integrability results below.

Proposition 2. *The characteristic polynomials $P(x)$, $P_\infty(x)$ of the residue matrices A at $p = \pm\alpha$ and $A_\infty = -2A - B$ at $p = \infty$ always have non-real roots.*

Proof. A direct calculation shows that $P(x)$ is of the form

$$P(x) = x^3 + x^2 + \theta_1 x + \theta_0\,, \tag{2.61}$$

where with the definitions

$$\Psi_n = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}, \quad \Psi_d = \Psi_n - (\Sigma_{11} + \Sigma_{22} - \Sigma_{33})\mathrm{Id}, \quad v = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -\rho_1 \\ -\mathrm{i}\rho_2 \end{pmatrix}, \tag{2.62}$$

the coefficients of P are given by

$$\theta_d = \det \Psi_d + m < v, \Psi_d v > \tag{2.63a}$$

$$\theta_d \theta_1 = \det \Psi_n + m\{< v, \Psi_n v > + \rho_3(\Sigma_{11}\rho_1 - \Sigma_{22}\rho_2 + \mathrm{i}\Sigma_{12}(\rho_1 + \rho_2))\} \tag{2.63b}$$

$$\theta_d \theta_0 = \theta_d \theta_1 - m\rho_3(\rho_1 - \rho_2)(\Sigma_{11} + \Sigma_{22} - \Sigma_{33})\,. \tag{2.63c}$$

Therefore

$$\operatorname{Im}\frac{1}{\theta_0-\theta_1}=\frac{2\Sigma_{12}\rho_1\rho_2}{\rho_3(\rho_1-\rho_2)(\Sigma_{11}+\Sigma_{22}-\Sigma_{33})}\,, \tag{2.64}$$

which is obviously non-zero according to (2.44). A direct calculation shows that $P_\infty(x)$ is of the form

$$P_\infty(x)=x^3-4x^2+\chi_1 x+\chi_0 \tag{2.65}$$

In the notation of the previous proposition and the appendix one coefficient is given by

$$\chi_1=5+4a_1^2+4a_2a_3+4(c_3-\mathrm{i}c_1)/\beta \tag{2.66}$$

The coefficients of $P(x)$ and $P_\infty(x)$ are related by

$$12(\theta_0-\theta_1)+\chi_0+3\chi_1=-9\,. \tag{2.67}$$

Therefore

$$\operatorname{Im}\frac{1}{\chi_0+3\chi_1+9}=-\frac{\Sigma_{12}\rho_1\rho_2}{6\rho_3(\rho_1-\rho_2)(\Sigma_{11}+\Sigma_{22}-\Sigma_{33})} \tag{2.68}$$

□

In the rigid body case the matrices A and B become

$$A=\begin{pmatrix}0 & \mathrm{i}(\Sigma_{33}-\Sigma_{22})/\Sigma_{11} & 0\ \mathrm{i}(\Sigma_{11}-\Sigma_{33})/\Sigma_{22}\ \ 0\ \ 0\\ -1/\beta & -\mathrm{i}/\beta & -1\end{pmatrix}$$

$$B=\begin{pmatrix}-1 & 0 & -2\mathrm{i}mgr_2/\Sigma_{11}\\ 0 & -1 & 2\mathrm{i}mgr_1/\Sigma_{22}\\ 0 & 0 & 0\end{pmatrix}.$$

Therefore

$$\operatorname{Spectr}(A)=\left\{-1,-\sqrt{\frac{(\Sigma_{11}-\Sigma_{33})(\Sigma_{22}-\Sigma_{33})}{\Sigma_{11}\Sigma_{22}}},\sqrt{\frac{(\Sigma_{11}-\Sigma_{33})(\Sigma_{22}-\Sigma_{33})}{\Sigma_{11}\Sigma_{22}}}\right\}.$$

For the Kovalesvkaya case $\Sigma_{11}=\Sigma_{22}$, $\Sigma_{33}=2\Sigma_{11}$ the eigenvalues become $\{-1,1,1\}$ and $\{-2,3,3\}$ at infinity with a non-trivial Jordan block.

The following proposition shows that the system (2.59) is not solvable in the Lappo-Danilevsky sense (see [28]).

Proposition 3. *The residue matrices A and B do not have common eigenvectors. In particular, they do not commute.*

Proof. The proof is a straightforward, but since the proposition will play an important role later, we indicate some basic steps of it. A direct computation gives for the eigenvectors $\mathcal{V}_i$ and the corresponding eigenvalues λ_i of B:

$$\begin{array}{ll} \mathcal{V}_1 = (1, v_1, v_2)^T, & \lambda_1 = 0, \\ \mathcal{V}_2 = (1, 0, 0)^T, & \lambda_1 = -1, \\ \mathcal{V}_3 = (0, 1, 0)^T, & \lambda_1 = -1\,, \end{array}$$

where v_i are some known expressions.

One verifies that the matrix $R = (v_1, v_2, v_3)$ is not singular once the condition (2.44) holds. Under the same conditions it is easy to show that $\mathcal{V}_2$ and $\mathcal{V}_3$ are not eigenvectors of A. Here is one method to prove the same statement for $\mathcal{V}_1$. We consider the conjugation $\tilde{A} = R^{-1}AR$ of A by R. It is sufficient to show then that $\tilde{A}_{1,2} \neq 0$ or $\tilde{A}_{1,3} \neq 0$. That can be done quite easily once the condition (2.44) is fixed. Otherwise, if $C = [A, B]$, then the non-commutativity property follows directly from $C_{3,1} = 1/\beta \neq 0$.

□

Fixing a basepoint $e \in \mathcal{R}$, one defines the monodromy group G of the system (2.59) as the image of the fundamental group $\pi(\mathcal{R}, e)$ given by the analytical continuations of the fundamental matrix solution $\Sigma(p)$, $\Sigma(e) = \mathrm{Id}$ of (2.59) along all closed paths $\Gamma \in \pi(\mathcal{R}, e)$ starting from e. One verifies that $\mathrm{tr}(A), \mathrm{tr}(B) \in \mathbb{Z}$. Hence, $G \subset SL(3, \mathbb{C})$. According to Proposition 1, the group $G = \langle M_+, M_- \rangle$ is generated by the local monodromy transformations M_+, M_- around the singularities $p = \alpha$ and $p = -\alpha$ respectively.

2.3.4 On the non-existence of additional meromorphic first integrals in the logarithmic branching case

The next proposition will show that the logarithmic branching of solutions of the normal variational equations (2.59) is not compatible with existence of a new *meromorphic* first integral of the rattleback problem (2.46).

Theorem 26. *Let us assume that the general solution of the variational equations* (2.59) *has logarithmic branching around one of the singularities* $p = \pm\alpha$. *Then the rattleback equations does not have any new meromorphic first integrals.*

Proof. By a suitable linear transormation of the general solution of (2.59), we can always reduce one of the monodromy matrices around $p = \alpha$ or $p = -\alpha$ to its Jordan form. In the case then the general solution has logarithmic branching points, at least one of the monodromy matrices M_+, M_- has a non-trivial Jordan block.

Since $p = \pm\alpha$ enter in (2.59) in the symmetric way, it is sufficient to consider the case of $p = \alpha$. Firstly, we assume that M_+ is of the form

$$M_+ = \begin{bmatrix} q & 1 & 0 \\ 0 & q & 1 \\ 0 & 0 & q \end{bmatrix} . \tag{2.69}$$

Since $M_+ \in SL(3, \mathbb{C})$, one has $q^3 = 1$.

Let

$$\mathrm{Spectr}(A) = (\lambda_1, \lambda_2, \lambda_3) . \tag{2.70}$$

As follows from (2.59) (see f.e. [34])

$$\mathrm{Spectr}(M_+) = \mathrm{Spectr}(M_-) = \{e^{2\pi i\lambda_1}, e^{2\pi i\lambda_2}, e^{2\pi i\lambda_3}\} . \tag{2.71}$$

Thus, in the case (2.69) all eigenvalues of A must be real (even rational) numbers. This is impossible according to Proposition 2.

We suppose now that M_+ is of the form

$$M_+ = \begin{bmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & u^{-2} \end{bmatrix} , \quad u \in \mathbb{C}^* . \tag{2.72}$$

In this case we prove the following lemma.

Lemma 10. *Let $R(x)$, $x = (x_1, x_2, x_3)^T$ be a rational invariant of G. Then $R = R(x_2, x_3)$.*

Proof. We write $R(x)$, $\deg R = m$ as follows

$$R = \frac{\sum\limits_{i=0}^{l} x_3^{l-i}\, P_i(x_1, x_2)}{\sum\limits_{j=0}^{k} x_3^{k-j} Q_j(x_1, x_2)}, \quad l, k \in \{0, 1, 2, \dots\}, \quad m = l - k , \tag{2.73}$$

where $P_i(x_1, x_2)$, $Q_j(x_1, x_2)$ are homogeneous polynomials of degrees i and j respectively. Let us assume that at least one of the polynomials P_i or Q_j

depends on x_1. So, for example, we will find $0 \le \rho, s \le l$ such that the term $x_3^{\rho} P_s(x_1, x_2)$, $\partial P/\partial x_1 \neq 0$, $\deg(P_s) = s$ is a semi-invariant of M_+ i.e satisfies

$$\tilde{M}_+^n P_s(x_1, x_2) = c^n P_s(x_1, x_2), \quad c \in \mathbb{C}^*, \quad \forall n \in \mathbb{N}, \quad \tilde{M}_+ = \begin{bmatrix} u & 1 \\ 0 & u \end{bmatrix}. \tag{2.74}$$

Since $P_s(x_1, x_2)$ is homogeneous of degree s, we can put it in the form

$$P_s(x_1, x_2) = \alpha_0 x_1^s + \alpha_1 x_1^{s-1} x_2 + \cdots + \alpha_s x_2^s. \tag{2.75}$$

We observe that $\tilde{M}_+^n (x_1, x_2)^T = (u^n x_1 + n u^{n-1} x_2, u^n x_2)^T$. Consequently, the polynomial $\tilde{M}_+^n P_s(x_1, x_2)$ will have its coefficient before x_2^s equal to $\theta_n = \sum_{r=0}^{s} \alpha_r n^{s-r} u^{s(n-1)+r}$. Let q be the smallest index such that $\alpha_q \neq 0$. Then the coefficient before $x_1^q x_2^{s-q}$ in $\tilde{M}_+^n P_s(x_1, x_2)$ is $\eta_n = u^{nq} \alpha_q$. Clearly, as $n \to \infty$, the asymptotic behaviors of θ_n and η_n are different. So, the equality (2.74) cannot be true. That finishes the proof of Lemma 10. □

Lemma 11. *All rational homogeneous invariants of M_+ are functions of the invariant $x_2^2 x_3$.*

The proof follows easily from the condition that Spectr(A) is non-real.

Lemma 12. *Let us assume that the monodromy group $G = \langle M_-, M_+ \rangle$ has a rational homogeneous invariant $R(x)$. Then $x_2^2 x_3$ is the invariant of G and $R = (x_2^2 x_3)^m$ for a certain $m \in \mathbb{Z}^*$.*

Proof. Let $R(x_1, x_2, x_3)$ be a rational homogeneous invariant of G written as

$$R = \frac{\sum\limits_{i_1+i_2+i_3=M} a_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}}{\sum\limits_{j_1+j_2+j_3=N} b_{j_1,j_2,j_3} x_1^{j_1} x_2^{j_2} x_3^{j_3}}, \quad i_k, j_p \in \{0, 1, 2, \dots\}, \tag{2.76}$$

where M, N are non-negative integer numbers and all coefficients a_i, b_i are different from zero.
Firstly, we assume that R reduces to the single term

$$R = x_1^{l_1} x_2^{l_2} x_3^{l_3}, \quad l_i \in \mathbb{Z}, \quad (l_1, l_2, l_3) \neq (0, 0, 0). \tag{2.77}$$

Then $l_1 = 0$ according to Lemma 10, and, as follows from Lemma 11, there exists $m \in \mathbb{Z}^*$ such that

$$R = (x_2^2 x_3)^m, \tag{2.78}$$

that in turn proves the result.

If R is not of the form (2.77), it can be written as follows

$$R = \frac{a x_1^{r_1} x_2^{r_2} x_3^{r_3} + b x_1^{q_1} x_2^{q_2} x_3^{q_3} + \cdots}{c x_1^{p_1} x_2^{p_2} x_3^{p_3} + \cdots}, \quad r_i, q_j, p_k \in \{0, 1, 2, \dots\}, \tag{2.79}$$

where $a, b, c \neq 0$ and $(r_1, r_2, r_3) \neq (q_1, q_2, q_3)$, $\sum r_i = \sum q_i = M$.

We note that, as seen from (2.79), the terms $x_1^{r_1} x_2^{r_2} x_3^{r_3}$ and $x_1^{q_1} x_2^{q_2} x_3^{q_3}$ are multiplied by the same constant under the action of M_+. Therefore, the division by $x_1^{q_1} x_2^{q_2} x_3^{q_3}$ shows that $r(x) = x_1^{k_1} x_2^{k_2} x_3^{k_3}$, $k_i = r_i - q_i$, $(k_1, k_2, k_3) \neq (0, 0, 0)$, $\sum k_i = 0$ is the invariant of M_+. We have $k_1 = 0$ according to Lemma 11 and hence $r(x) = (x_2/x_3)^{k_2}$, $k_2 \neq 0$. This is impossible according to Lemma 11.

Thus, we can assume R to have the form (2.78). This immediately fixes the monodromy transformation M_-, also preserving R, as follows

$$M_- = \begin{bmatrix} u_1 & n & k \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix}, \quad u_1 u_2 u_3 = 1, \quad u_i, n, k \in \mathbb{C}. \tag{2.80}$$

We will show that $u_1 = u_2 = u$ and $u_3 = u^{-2}$.

Indeed, since $\mathrm{Spectr}(M_+) = \mathrm{Spectr}(M_-) = \{u, u, u^{-2}\}$, we have necessarily $u_1 = u$ or $u_2 = u$. Let $u_1 = u$, $u_2 = u^{-2}$, $u_3 = u$. With help of R we obtain $u^{-3m} = 1$ and so u is the root of unity. Let now $u_1 = u^{-2}$, $u_2 = u$, $u_3 = u$. Then $u^{3m} = 1$ and we conclude as above. Hence, the monodromy transformation M_- takes the form

$$M_- = \begin{bmatrix} u & n & k \\ 0 & u & 0 \\ 0 & 0 & u^{-2} \end{bmatrix}. \tag{2.81}$$

The rational function $R = x_2^2 x_3$ is clearly invariant under the action of M_- and M_+. □

We shall follow the approach close to the Tannakian one (see f.e. [67]).

Let $\Sigma(p)$ be the fundamental matrix solution of the equations (2.59) and $\Sigma^{-1}(p) = (\Sigma_1, \Sigma_2, \Sigma_3)^T$ where Σ_i are linearly independent vector functions. It is known (see f.e [T1], p. 246) that if $R(x) = x_2^2 x_3$ is a polynomial invariant of the monodromy group G then

$$I(p, x) = R(\Sigma(p)^{-1} x) =< \Sigma_2(p), x >^2 < \Sigma_3(p), x >, \tag{2.82}$$

will be a first integral of (2.59) invariant under the action of G i.e. single-valued as a function of $p \in \mathcal{R}$. One can express this fact by stating that all coefficients $a_{i,j,k}(p)$ in the expression for I:

$$I = \sum_{i+j+k=3} a_{i,j,k}(p)\, x_1^i x_2^j x_3^k\,, \tag{2.83}$$

are rational functions of p.

Since the polynomial I given by (2.83) can be factorized as (2.82), there exist two 3-vector functions $\mathcal{A}(p)$, $\mathcal{B}(p)$, algebraically dependent on p, such that I writes as

$$I(p,x) =< \mathcal{A}(p), x >^2 < \mathcal{B}(p), x > \ . \tag{2.84}$$

The structure of the monodromy group generated by the transformations (2.72), (2.81) suggests that the system (2.59) has two linearly independent particular solutions $X_{1,2}(p)$ of the following form

$$X_1(p) = (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1}\, R_1(p), \quad e^{2\pi i \lambda_1} = u, \quad \lambda_1 \in \mathrm{Spectr}(A)\,, \tag{2.85}$$

$$X_2(p) = (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1} R_2(p) + u^{-1} X_1(p) \left(\frac{\log(p-\alpha)}{2\pi i} + n \frac{\log(p+\alpha)}{2\pi i} \right)\,, \tag{2.86}$$

where $R_{1,2}(p)$ are vector functions rationally dependent on p.

Lemma 13. *We have* $< \mathcal{A}, R_1 >=< \mathcal{B}, R_1 >= 0$.

Proof. Plugging the solution (2.85) into the integral (2.84) we obtain

$$I(p, X_1) = (p-\alpha_1)^{3\lambda_1}(p+\alpha)^{3\lambda_1} < \mathcal{A}, R_1 >^2 < \mathcal{B}, R_1 >= c = \mathrm{const}\,.$$

In the case $c \neq 0$, since $\mathcal{A}, \mathcal{B}, R_1$ are algebraic with respect to p, the last equality implies $\lambda_1 \in \mathbb{Q}$. According to (2.72)

$$\mathrm{Spectr}(A) = \{\lambda, \lambda+k, -2\lambda-k-1\}\,, \tag{2.87}$$

for certain $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$.

So, it is clear that if $\lambda_1 \in \mathbb{R}$, then $\mathrm{Spectr}(A)$ has to be real. According to Proposition 2 we therefore get $< \mathcal{A}, R_1 >^2 < \mathcal{B}, R_1 >= 0$ for all p.

We shall consider the case $< \mathcal{A}, R_1 >= 0$ and $< \mathcal{B}, R_1 > \neq 0$. Obviously, the shift $\tilde{I}(p,x) = I(p, x + X_1(p))$ of the first integral I is again a first integral of the system (2.59) and

$$\tilde{I} = I + (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1} < \mathcal{B}, R_1 >< \mathcal{A}, x >^2 = \mathrm{const}\,. \tag{2.88}$$

Since the equations (2.59) are homogeneous, with help of (2.88), (2.84), we derive the first integral

$$J_1 = \frac{< \mathcal{B}, x >}{(p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1}(\mathcal{B}, R_1)} = (p-\alpha)^{-\lambda_1}(p+\alpha)^{-\lambda_1} < \tilde{B}, x > , \quad (2.89)$$

where $\tilde{B}$ is algebraic on p.

Finally, combining J_1 with I, one gets

$$\begin{aligned} J_2 &= \sqrt{< \mathcal{B}, R_1 > (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1}} < \mathcal{A}, x >= \\ &= (p-\alpha)^{\lambda_1/2}(p+\alpha)^{\lambda_1/2} < \tilde{A}, x > , \end{aligned} \quad (2.90)$$

– the first integral of (2.59) with $\tilde{A}(p)$ algebraic on p.

Obviously, J_1 and J_2 are functionally independent. Indeed, the vectors $\mathcal{A}(p)$, $\mathcal{B}(p)$ are independent as being proportional to the lines $\Sigma_{2,3}(p)$ of the matrix $\Sigma^{-1}(p)$ whose determinant is not identically zero. Finally, in order to find the third linear first integral, we apply the Liouville theorem to the fundamental matrix solution of (2.59) formed by the columns $X_1(p)$, $X_2(p)$ and the arbitrary solution $x(p)$:

$$\begin{aligned} &\Sigma(p) = (X_1(p), X_2(p), x), \\ &\det(\Sigma) = (p-\alpha)^{2\lambda_1}(p+\alpha)^{2\lambda_1}\det(R_1, R_2, x) = \mathrm{const}\cdot\, a(p) , \end{aligned} \quad (2.91)$$

where $a(p)$ is a rational function of p in view of $\mathrm{tr}(A), \mathrm{tr}(B) \in \mathbb{Z}$.

One derives from (2.91) the following first integral of (2.59)

$$J_3 = (p-\alpha)^{-2\lambda_1}(p+\alpha)^{-2\lambda_1} < \mathcal{C}, x > , \quad (2.92)$$

with $\mathcal{C}(p)$ algebraic on p.

In the case the vectors $\tilde{A}, \tilde{B}, \mathcal{C}$ are linearly independent, finding x from the linear system $J_1 = c_1, J_2 = c_2, J_3 = c_3$, $c_{1,2,3} \in \mathbb{C}$, we see that the general solution of (2.59) does not contain logarithmic branching and so we get a contradiction. Let us assume that $\tilde{A}, \tilde{B}, \mathcal{C}$ are linearly dependent. Then $\mathcal{C} = l_1\tilde{A} + l_2\tilde{B}$ where l_1, l_2 are certain algebraic functions of p and the following relation holds in view of (2.89), (2.90), (2.92)

$$(p-\alpha)^{-\frac{5}{2}\lambda_1}(p+\alpha)^{-\frac{5}{2}\lambda_1}c_1 l_1 + (p-\alpha)^{-\lambda_1}(p+\alpha)^{-\lambda_1}c_2 l_2 = c_3 . \quad (2.93)$$

Since J_1, J_2 are functionally independent, the last expression shows that $\lambda_1 \in \mathbb{Q}$ and the proof is finished with help of (2.87). The case $< \mathcal{A}, R_1 > \neq 0$, $< \mathcal{B}, R_1 >= 0$ is treated in the analogous way. □

With help of the previous lemma one obtains $I(p, x + X_2) = I(p, x + (p - \alpha)^{\lambda_1}(p + \alpha)^{\lambda_1} R_2)$ which is a first integral of (2.59). As before we show $<\mathcal{A}, R_2> = <\mathcal{B}, R_2> = 0$. Since $\mathcal{A}$, $\mathcal{B}$ are linearly independent, it follows from (2.85), (2.86) that $X_2 = \theta X_1$ for a certain function $\theta = \theta(p) \neq \text{const}$. This situation can be ruled out by the substitution of X_1, X_2 into the system (2.59) that gives a contradiction. According to Lemma 9 the proof of Proposition 26 is finished.

□

2.3.5 Non-existence of additional analytic first integrals

Our aim now is to prove the non-existence of new analytic first integrals of the rattleback problem (2.46) under one of the following hyperbolicity conditions

$$\text{Spectr}(M_\pm) = \{s_{1,2,3} \in \mathbb{C} \,:\, 0 < |s_1| < 1, |s_2| > 1, |s_3| > 1, s_2 \neq s_3\}\,, \quad (2.94)$$

or

$$\text{Spectr}(M_\pm) = \{s_{1,2,3} \in \mathbb{C} \,:\, |s_1| > 1, 0 < |s_2| < 1, 0 < |s_3| < 1, s_2 \neq s_3\}\,, \quad (2.95)$$

which represent certain restrictions on the eigenvalues of the characteristic polynomial (2.61). Reminding that $\sum_i \lambda_i = -1$ and $s_1 s_2 s_3 = 1$ we deduce from the relations $e^{2\pi i \lambda_i} = s_i$ that at least one of the conditions (2.94) or (2.95) is satisfied once

$$\lambda_1, \lambda_2, \lambda_3 \notin \mathbb{R} \quad \text{and} \quad \operatorname{Im}\lambda_i - \operatorname{Im}\lambda_j \neq 0, \quad \forall\, i \neq j\,, \quad (2.96)$$

where λ_i, $i = 1, 2, 3$ are three roots of the cubic algebraic equation

$$x^3 + x^2 + \theta_1 x + \theta_0 = 0\,, \quad (2.97)$$

whose coefficients depend on ($\Sigma_{i,j}$, b_i, m) and are defined by (2.63).

Up from now, we will assume that the condition (2.96) is fulfilled and that the case (2.94) takes place.

Let $\mathcal{L}$ be the space of linear forms $l = <L, x>$, $L \in \mathbb{C}^n$ dual to $\mathbb{C}^n$. To each $M \in GL(n, \mathbb{C})$ we associate the linear automorphism $M : \mathcal{L} \to \mathcal{L}$ according to $M\,.\,l = <M^T L, x>$. The next results shows how the hyperbolicity of G implies the reducibility of its polynomial invariants.

Proposition 4. *Let the monodromy group $G = \langle M_+, M_- \rangle$ has a polynomial homogeneous invariant $P(x_1, x_{,2}, x_3)$. Then one of the following situations holds:*

a) $P = x_1^\rho x_2^l L(x_1, x_2, x_3)$, $\rho, l \in \mathbb{N}$,
b) $P = x_1^\mu x_3^k M(x_1, x_2, x_3)$, $\mu, k \in \mathbb{N}$,
c) $P = x_1^\eta N(x_1, x_2, x_3)$, $\eta \in \mathbb{N}$ *and* $M_\pm \,.\, x_1 = s_1 x_1$,
where $L, M, N \in \mathbb{C}[x_1, x_2, x_3]$.

Proof. Since P is invariant under the action of M_+ and (2.94) holds, all different monomials entering in P contain a positive degree of x_1. Therefore, P factorizes as follows

$$P = x_1^\rho \, P_1(x_1, x_2, x_3), \quad \rho \in \mathbb{N}\,, \tag{2.98}$$

where P_1 is a homogeneous polynomial non divisible by x_1.
Let $\mathcal{E}$ be the set of all pairwise non-colliniear forms dividing P. In particular we have already $x_1 \in \mathcal{E}$. We denote $\mathcal{E}_0 \subset \mathbb{C}P^2$ the set of directions of all elements from $\mathcal{E}$ equipped with the naturally defined action of G. The G-invariance of P implies clearly this of $\mathcal{E}_0$.
Exchanging the roles of M_+ and M_-, one finds $e \in \mathcal{E}$ – the eigenform of M_-, $M_- \,.\, e = s_1 e$. Let $e = e_1 x_1 + e_2 x_2 + e_3 x_3$. Then, considering the orbit $M_+^n \,.\, e$, $n \geq 1$ and using (2.94) together with $\operatorname{card} \mathcal{E}_0 < \infty$, one sees that if $e_1 \neq 0$ then $e_2 = e_3 = 0$ i.e. $e = x_1$ (case c). If $e_1 = 0$, then either $e_2 e_3 = 0$ and so $e \in \{x_2, x_3\}$ (cases a-b) or $e_2 e_3 \neq 0$ so that $\exists\, m \in \mathbb{N}$ such that $s_2^m = s_3^m$. The last equation implies $\lambda_2 - \lambda_3 \in \mathbb{R}$ which is impossible according to (2.96). The proof is done.

□

Proposition 5. *The monodromy transformations M_+ and M_-, taken in any basis, are permutationally conjugated i.e.* $\exists\, C \in GL(3, \mathbb{C})$ *such that*

$$C M_+ C^{-1} = M_-, \quad C M_- C^{-1} = M_+\,. \tag{2.99}$$

Proof. We take a basepoint $e \in \mathcal{R}$ on the positive imaginary axis $\operatorname{Im} p > 0$. Let $\Sigma(p)$, $\Sigma(e) = \mathrm{Id}$ be the normalised fundamental matrix solution of (2.59) and let G be the corresponding monodromy group. Since the equations (2.59) are invariant under the change of time $p \to -p$, $\tilde{\Sigma}(p) = \Sigma(-p)$ is again a fundamental matrix solutions of (2.59). Let Γ_1, Γ_2 be two loops starting from e and going around the singularities $p = \alpha$ and $p = -\alpha$ respectively. We define the loops $\tilde{\Gamma}_i = -\Gamma_i$, $i = 1, 2$ (symmetric to $\Gamma_{1,2}$ with respect to origin), starting from the point $\tilde{e} = -e$ and having the same orientation as $\Gamma_{1,2}$. Obviously, $\tilde{\Sigma}(\tilde{e}) = \mathrm{Id}$ and we can define the monodromy group $\tilde{G}$ using $\tilde{\Sigma}$ in the usual way. One sees that $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ define now the monodromy transformations around $p = -\alpha$ and $p = \alpha$ respectively. The result follows then from the fact that G and $\tilde{G}$ are always conjugated. □

Remark 4. It follows from the previous proposition that if $C^2 \neq \mathrm{Id}$ then it is a centraliser of G in $GL(3, \mathbb{C})$.

Proposition 6. *Let us assume that the monodromy group G of the normal variational equations* (2.59) *has a polynomial homogeneous invariant and that the conditions* (2.44), (2.96) *hold. Then G is diagonalizable.*

Proof. One first assumes that a)-b) from Proposition 4 hold.
Then we have three possible cases: $\mathcal{E} = \{x_1, x_2\}$ (A), $\mathcal{E} = \{x_1, x_3\}$ (B) or $\mathcal{E} = \{x_1, x_2, x_3\}$ (C).
The group G acts on $\mathcal{E}_0$ by permutations. Hence, exchanging if necessary x_2 and x_3, in A-B we can put

$$M_+ = \mathrm{diag}(s_1, s_2, s_3),\ M_- = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ a & b & \sigma_3 \end{bmatrix},\ \sigma_i \in \mathrm{Spectr}(M_+),\ a, b \in \mathbb{C}\,. \tag{2.100}$$

Indeed, $\mathcal{E}$ always contains an eigenform of M_- corresponding to the stable eigenvalue $s_1 \in \mathrm{Spectr}(M_+) = \mathrm{Spectr}(M_-)$. In particular $\sigma_1 = s_1$ or $\sigma_2 = s_1$.
Using (2.100) one can calculate the unique (modulo multiplication by a constant) nonsingular matrix T such that $T^{-1} M_- T = \mathrm{diag}(s_1, s_2, s_3)$. Then, as follows from Proposition 5, $T^{-1} M_+ T = M_-$. The last equation, together with (2.96) and some elementary calculations implies $a = b = 0$ that proves the result.
If the case C holds, then the matrix $M_- \in SL(3, \mathbb{C})$ is either diagonal, so the proposition is proved, or is a permutation of the eigendirections of M_+ fixing one of them. In this last case, $\mathrm{Spectr}(M_-)$ will contain necessarily a pair of eigenvalues with equal absolute values. This contradicts to (2.96).
We consider now the case c) from Proposition 4. One puts P into the form

$$P = x_1^{\rho}(x_1 P_1(x_1, x_2, x_3) + P_2(x_2, x_3))\,, \tag{2.101}$$

where P_1, P_2 are homogeneous polynomials and $P_2 \neq \mathrm{const}$ since $|s_1| < 1$.
Let $\tilde{M}_+ = \mathrm{diag}(s_2, s_3)$ and let $\tilde{M}_- = (m_{ij})_{2 \leq i,j \leq 3}$ denotes the restriction of the linear operator M_- to the x_2, x_3-plane.
It is clear from (2.101) that P_2 is a polynomial semi-invariant for both $\tilde{M}_+$ and $\tilde{M}_+$. If P_2 contains two different monomials $x_2^n x_3^m$ and $x_2^p x_3^q$, $n + m = p + q = \deg P - \rho$ then $s_2^r = s_3^r$ for $r = n - p = q - m \neq 0$ that contradicts to (2.96).
If x_2^N (resp. x_3^N) is the only monomial entering in P_2, then x_2 (resp. x_3) is the eigenform of $\tilde{M}_-$.

Thus, exchanging if necessary x_2 and x_3, it is sufficient to consider the case

$$M_+ = \operatorname{diag}(s_1, s_2, s_3), \quad M_- = \begin{bmatrix} s_1 & 0 & 0 \\ a & m_{22} & 0 \\ b & m_{32} & m_{33} \end{bmatrix}. \tag{2.102}$$

With help of (2.99), (2.96) and some elementary algebraic computations one shows that $m_{22} = s_2$, $m_{33} = s_3$.

We introduce the matrices $U = M_+^{-1} M_-$ and $K = U - \mathrm{Id}$. One verifies that $\mathrm{Spectr}(U) = \{1, 1, 1\}$ and that if $\mathrm{rang}(K) = 2$ then M_- does not have any polynomial invariants. That can be done by transforming U to its Jordan form. If $\mathrm{rang}(K) = 0$ then $M_- = M_+$ and the proposition is proved. The condition $\mathrm{rang}(I - \mathrm{Id}) = 1$ implies in turn: $a = 0$ (i), $b = m_{32} = 0$ (ii) or $m_{32} = 0$ (iii). In the cases (i), (ii), M_- has the eigenform equal to x_2 or x_3 that corresponds to the case considered before. If (iii) holds, it is sufficient to apply again the conjugacy conditions (2.99) to obtain a contradiction with (2.96).

Finally, we consider the case then $P_2 = c\, x_2^t x_3^l$, $t, l \in N$, $c \in \mathbb{C}^*$. Then $\tilde{M}_-$ either preserves or permutes x_2, x_3-eigendirections of $\tilde{M}_+$. In the first case the proceed as above. In the second one, it is sufficient to verify by a direct computation that $\mathrm{Spectr}(M_-)$ will contain in this case a pair of eigenvalues $\pm s$, $s \in \mathbb{C}^*$ so that the condition (2.96) is violated. The proof of Proposition 6 is finished.

□

The next proposition shows that in our case the Fuchsian system (2.59) never has a diagonal monodromy group.

Proposition 7. *Under the conditions* (2.44) *and* (2.96) *the monodromy group G of the normal variational equations* (2.59) *is not diagonalizable.*

Proof. Let us assume that G is diagonalizable: $M_+ = \operatorname{diag}(s_1, s_2, s_3)$, $M_- = \operatorname{diag}(s_{i_1}, s_{i_2}, s_{i_1})$, $s_k = e^{2\pi i \lambda_k}$, $k = 1, 2, 3$ where (i_1, i_2, i_3) is a certain permutation of $(1, 2, 3)$.

That implies existence of three independent solutions of (2.59) of the form

$$X_k(p) = (p - \alpha)^{\lambda_k} (p + \alpha)^{\lambda_{i_k}} N_k(p), \quad k = 1, 2, 3\,, \tag{2.103}$$

with vector functions N_k rational on p.

Let Y_i, $i = 1, 2, 3$ be three linearly independent eigenvectors of A with corresponding eigenvalues λ_i (we remind that λ_i are pairwise different in view of (2.96)). One deduces from (2.103) the following formulas containing the rational vector functions R_k

$$X_k = (p-\alpha)^{\lambda_{g_k}} (p+\alpha)^{\lambda_{u_k}} R_k(p), \quad k = 1,2,3 \tag{2.104}$$

with $g_k\ u_k \in \{1,2,3\}$ and where now $R_k = Y_{g_k} + Y^+_{g_k}(p-\alpha) + \cdots$ in the neighborhood of $p=\alpha$ and $R_k = Y_{u_k} + Y^-_{g_k}(p+\alpha) + \cdots$ in the neighborhood of $p=-\alpha$, $(k = 1,2,3)$. Since the system (2.59) is invariant under the change $p \to -p$, $X_k(-p)$ is also its solution. It yields, together with condition (2.96), $g_k = u_k$ for $k = 1,2,3$.

Under the same condition (2.96), as a simple argument shows, Y_{g_1}, Y_{g_2}, Y_{g_3} must be pairwise different. Since $M_+ = M_-$ and $M_+M_-M_\infty = \mathrm{Id}$, the similar property holds for the point $p=\infty$.

Thus, one can represent X_k as below

$$X_k = (p-\alpha)^{\lambda_k} (p+\alpha)^{\lambda_k} p^{n_k} P_k(p), \quad k = 1,2,3\,, \tag{2.105}$$

where $n_k \in \mathrm{Spectr}(B) = \{0,-1,-1\}$; P_k are polynomial vector functions such that $P_k(\alpha) = P_k(-\alpha) = Y_k$ for $k = 1,2,3$ and $P_k(0)$ are eigenvectors of B.

Let D_k be the order of X_k at infinity. One finds from (2.105): $D_k = -2\lambda_k - n_k - d_k$ so that $\sum D_k = -2\sum\lambda_k - \sum n_k - \sum d_k \geq -2\cdot(-1) - (0) - \sum d_k$ since $\sum n_k \leq 0$. Otherwise, we know that $\sum D_k = 4$ (see Proposition 2) and hence $\sum d_k \leq 2$. Thus, at least one of the vectors P_k is constant and, as easy follows from substitution of (2.105) into (2.59), is a common eigenvector of A and B. This is impossible according to Proposition 3 and the proof is achieved.

□

The results above can be now summarised by the main Theorem 25 in view of Lemma 9.

2.3.6 Conclusion

The difficulty of the rattleback problem is due to its non-hamiltonian nature. That explains the non-trivial structure of the monodromy group studied in the previous sections. The technical problem in applying our non-integrability Theorem 25 are the hyperbolicity conditions (2.96). Of course, one can write these restrictions directly using Cardano formulas that will lead to quite complicated expressions. It may be interesting to consider a concrete example. Let $I_1 = 0.5$, $I_2 = 0.6$, $I_3 = 0.8$, $\delta = 1.3$, $m = 1$, $g = 1$,

$b_1 = 1$, $b_2 = 2$, $b_3 = 3$. Then the eigenvalues of the characteristic polynomial (2.61) are

$$\lambda_1 = -0.365 - 0.858i, \quad \lambda_2 = -0.435 + 0.963i, \quad \lambda_3 = -0.200 - 0.106i\,,$$

and the conditions (2.96) are obviously satisfied.

We are sure that our non-integrability conditions can be strengthened. Thus, the further detailed analysis of the characteristic polynomial (2.61) is needed.

We note that in the heavy rigid body case, the characteristic polynomial (2.61) always has a real root as seen from (2.3.3). Indeed, it corresponds to existence of the fourth polynomial first integral (2.50) of the Euler-Poisson equations. In the rattleback case, the interesting remaining problem is the existence of new meromorphic first integrals. Our Theorem 26 answers this question only then the variational equations (2.59) have logarithmic singularities. Our intention to avoid the study of the Zariski closure of the monodromy group G was twofold. Firstly, that makes the proofs self-contained and quite elementary. Secondly, we would like to underline the importance of the symmetry conditions (2.99), coming from the mechanical context of the problem and which simplify greatly the non-integrability analysis.

2.3.7 Appendix

The matrix N is given by

$$N = \begin{pmatrix} 0 & * & * \\ \mathrm{i}\frac{p^2-\alpha}{\beta} & 0 & p \\ -\frac{p^2-\alpha}{\beta} & -p & 0 \end{pmatrix}. \tag{2.106}$$

The matrix M is given by

$$M = \begin{pmatrix} a_1 p & a_2 p & c_1 + c_2\frac{\alpha}{p^2-\alpha} \\ a_3 p & -a_1 p & c_3 + c_4\frac{\alpha}{p^2-\alpha} \\ -\mathrm{i}\frac{p^2-\alpha}{\beta} & \frac{p^2-\alpha}{\beta} & 0 \end{pmatrix}. \tag{2.107}$$

The coefficients a_i, c_i are related by one quadratic equation, which is equivalent to (2.67). They are given by

$$\theta_d a_1 = 2\frac{mg}{\beta}(\rho_1-\rho_2)(\Sigma_{12}-\mathrm{i}m\rho_1\rho_2)+(\Sigma_{12}-\mathrm{i}m\rho_1\rho_2)(\Sigma_{11}+\Sigma_{22}+m(\rho_1^2-\rho_2^2)) \tag{2.108a}$$

$$\theta_d a_2 = 2\frac{mg}{\beta}(\rho_1-\rho_2)(\Sigma_{22}+m\rho_1^2) \tag{2.108b}$$

$$+(\Sigma_{12}-\mathrm{i}\Sigma_{22}-\mathrm{i}m\rho_1(\rho_1-\rho_2)(\Sigma_{12}+\mathrm{i}\Sigma_{22}+\mathrm{i}m\rho_1(\rho_1-\rho_2) \tag{2.108c}$$

$$\theta_d a_3 = 2\frac{mg}{\beta}(\rho_1-\rho_2)(\Sigma_{11}-m\rho_2^2)- \tag{2.108d}$$

$$-(\Sigma_{11}-\mathrm{i}\Sigma_{12}-m\rho_2(\rho_1-\rho_2))(\Sigma_{11}+\mathrm{i}\Sigma_{12}+m\rho_2(\rho_1-\rho_2)) \tag{2.108e}$$

$$\theta_d c_2 = -m\beta\rho_3(\Sigma_{12}\rho_1+\mathrm{i}\Sigma_{22}\rho_2) \tag{2.108f}$$

$$\theta_d c_4 = m\beta\rho_3(\Sigma_{11}\rho_1+\mathrm{i}\Sigma_{12}\rho_2) \tag{2.108g}$$

$$2\theta_d c_1 = 2mg(\Sigma_{12}(\rho_1-\rho_3)+\mathrm{i}\Sigma_{22}(\rho_2-\rho_3)+\mathrm{i}m\rho_1\rho_3(\rho_1-\rho_2)) \tag{2.108h}$$

$$-m\beta(\Sigma_{12}(2\rho_1-\rho_2)-\mathrm{i}\Sigma_{22}(\rho_1-\rho_2))+\mathrm{i}m^2\beta\rho_1\rho_3(\rho_1^2-\rho_2^2) \tag{2.108i}$$

$$2\theta_d c_3 = 2mg(\Sigma_{11}(\rho_1-\rho_3)+\mathrm{i}\Sigma_{12}(\rho_2-\rho_3)-m\rho_2\rho_3(\rho_1-\rho_2)) \tag{2.108j}$$

$$-m\beta(\Sigma_{11}(2\rho_1-\rho_2)-\mathrm{i}\Sigma_{12}(\rho_1-\rho_2))-m^2\beta\rho_2\rho_3(\rho_1^2-\rho_2^2) \tag{2.108k}$$

The matrix B is given by

$$B = \begin{pmatrix} -1 & 0 & -2\mathrm{i}(c_1-c_2) \\ 0 & -1 & -2\mathrm{i}(c_3-c_4) \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix A is given by

$$A = \begin{pmatrix} -\mathrm{i}a_1 & -\mathrm{i}a_2 & -\mathrm{i}c_2 \\ -\mathrm{i}a_3 & \mathrm{i}a_1 & -\mathrm{i}c_4 \\ -1/\beta & -\mathrm{i}/\beta & -1 \end{pmatrix}.$$

Rigid body case:

$$r_1=-\rho_1, r_2=-\mathrm{i}\rho_2, r_3=\rho_3=0, \quad m=0, mg\neq 0, \quad \Sigma_{12}=\delta=0.$$

2.4 Open problems and perspectives

As was shown in previous sections, the complex analytic methods are quite powerful in detecting the complex non–integrability. Particularly, one can show that a given Hamiltonian system does not have a full set of first integrals meromorphic in a *complex* neighborhood of a particular orbit. The *real* integrability i.e the existence of real meromorphic first integrals is hence ruled out. Nevertheless, there are some mechanical systems which are not complex analytically integrable but admit real analytic first integrals, for example the pendulum problem considered on p. 37. So, some refined methods are needed to deal with first integrals represented as a ratio of two real analytic functions. Some partial results here were obtained by Ziglin [97] who proved that if one has a whole family of particular solutions, so that the singularities on Riemann surfaces accumulate near the real axis, the real non–integrability can be proved. The similar question can be asked about existence of $\mathcal{C}^n$–first integrals for $n \geq 1$ fixed. The strongest result in this direction has to provide restrictions on coefficients of a given vector field which would guarantee the existence of $\mathcal{C}^1$–first integrals. Another interesting question can be asked about the global complex analytic structure of analytic first integrals obtained by analytic continuation of local ones given by the Cauchy existence theorem. We note that according to the Arnold-Liouville theorem, every completely integrable Hamiltonian system with n degrees of freedom always possesses $2n - 1$ first integrals which can be calculated by quadratures. First n of them are just meromorphic functions while the remaining $n - 1$ ones are integrals of known 1-forms. Thus, one can start here by checking the elementary examples from classical mechanics such as the Kepler or Jacoby two fixed centers integrable problems. It is of great interest to determine the Riemann surface structure of the remarkable figure-eight solution of Chenciner and Mongomery [15] in the planar three-body problem.

The main obstacle to applying Ziglin or Morales–Ramis methods is usually the non–existence of explicit particular solutions with sufficiently reach fundamental group. One elementary example here is the quadratic homogeneous systems studied on p. 27. They always have straight line particular solutions of the form $x(t) = c/t$ so that the corresponding monodromy group of the linearized flow is automatically abelian i.e it does not provide any useful information about possible meromorphic first integrals. One possible break–through in this direction is to study the normal variational equations of higher orders, the approach recently proposed by Morales, Ramis and Simo. One hopes that a full solution to the complex integrability problem will be given with help of the non–linear differential Galois theory (developed by B. Malgrange [61]) so that we can have a classification of all meromorphically integrable analytic Hamiltonian vector fields.

Finally, the exceptional cases given by Theorem 21 in the planar three-body problem have to be investigated and the conjecture on the non–integrability of the Rattleback problem (see p. 60) has to be proved.

Chapter 3

The renormalization theory for period doubling

3.1 Bounds on the unstable eigenvalue for the asymmetric renormalization operator for period doubling

3.1.1 Introduction

The remarkable universality of the scalings witnessed in the period-doubling route to chaos now has a well-established, mathematically rigorous, basis. Soon after discovery by Feigenbaum [30] (see also Coullet-Tresser [17]), the first (computer-assisted) proof was given by Lanford [58], closely followed by the analytic proofs of Epstein [25] and coworkers. More recently the rigorous analysis has reached new levels of sophistication in the works of Sullivan [85] and McMullen [62].

Contemporaneously, Arneodo *et al* [6] initiated the investigation of *asymmetric* unimodal maps. In a recent series of articles [64, 65, 69], we have given a rigorous renormalization analysis of period doubling in degree-d asymmetric unimodal maps. These are unimodal maps possessing a degree-d maximum, but with differing left and right dth derivatives. The maps we have in mind take the form

$$f(x) = \begin{cases} f_L(x) = 1 - a_1|x|^d & \text{if } x \leq 0\,; \\ f_R(x) = 1 - a_2|x|^d & \text{if } x \geq 0\,. \end{cases} \tag{3.1}$$

(The case of differing left- and right-hand degrees appears to be somewhat different in nature. See, e.g., [47].) In brief, for each $d > 1$, the standard Feigenbaum period-doubling renormalization operator has been shown to possess a family of period-two orbits, parametrized by an invariant asymmetry modulus, μ, measuring the ratio of the left and right dth derivatives at the maximum. The period-two orbit is then given by a quartet of functions $(f_L, f_R, \tilde{f}_L, \tilde{f}_R)$ satisfying the functional equations

$$\tilde{f}_L(x) = -\lambda^{-1} f_R f_R(-\lambda x)\,, \tag{3.2a}$$

$$\tilde{f}_R(x) = -\lambda^{-1} f_R f_L(-\lambda x)\,, \tag{3.2b}$$

$$f_L(x) = -\tilde{\lambda}^{-1} \tilde{f}_R \tilde{f}_R(-\tilde{\lambda} x)\,, \tag{3.2c}$$

$$f_R(x) = -\tilde{\lambda}^{-1} \tilde{f}_R \tilde{f}_L(-\tilde{\lambda} x)\,, \tag{3.2d}$$

with the normalizations $f_L(0) = f_R(0) = \tilde{f}_L(0) = \tilde{f}_R(0) = 1$ so that $\lambda = -f_R(1) > 0$ and $\tilde{\lambda} = -\tilde{f}_R(1) > 0$.

The solutions of (3.2) depend on two parameters, viz., the degree d of the critical point and the modulus μ, which (for the case when d is an even integer) is the ratio

$$\mu = \frac{f_L^{(d)}(0-)}{f_R^{(d)}(0+)}\,. \tag{3.3}$$

The case $\mu = 1$ is the standard Feigenbaum scenario in which case the period-two orbit is in fact a fixed point.

Let us denote by R the period-doubling renormalization operator acting on a unimodal map f with $f(0) = 1$, so that

$$R(f)(x) = -\lambda^{-1} f(f(-\lambda x)), \qquad \lambda = -f(1)\,. \tag{3.4}$$

Then R acts on both symmetric and asymmetric unimodal maps, preserving the degree d and inverting the asymmetry modulus μ. The scaling of the parameters in (3.1) undergoing a period-doubling cascade is determined by the expanding eigenvalue of the derivative of R^2 at the period-2 point f. This derivative $dR^2(f)$ is compact on a suitable Banach space of tangent functions δf and numerical results suggest that it is hyperbolic with a single expanding eigenvalue δ^2. It is this expanding eigenvalue which we investigate in this work. More precisely, we study an associated operator T (defined below), which has a positive expanding eigenvalue δ. We give a brief description of the relationship between T and $dR^2(f)$ in Section 3.1.4.

The operator T is defined on a pair of functions $(v, \tilde{v})$ and is given by:

$$T\begin{pmatrix} v(x) \\ \tilde{v}(x) \end{pmatrix} = \begin{pmatrix} \tilde{t}^{-1}(\tilde{v}(\tilde{t}x) + \tilde{v}(\tilde{L}(\tilde{t}x))\tilde{L}'(\tilde{t}x)^{-1}) \\ t^{-1}(v(tx) + v(L(tx))L'(tx)^{-1}) \end{pmatrix}, \tag{3.5}$$

where $t = \mu\lambda^d$, $\tilde{t} = \mu^{-1}\tilde{\lambda}^d$, and $L(x) = F(x)^d$, $\tilde{L}(x) = \tilde{F}(x)^d$. In this article we analyze the positive unstable eigenvalue of T, and, in particular, we shall establish the following theorem. Our work mirrors closely the analysis of Eckmann and Epstein [22] on the expanding eigenvalue of the symmetric Feigenbaum fixed-point. We shall establish the following result,

Theorem 27. *There exists a Banach space of function pairs on which the operator T is well defined, compact and has an eigenvalue $\delta > 0$ satisfying*

$$1 < \frac{1}{(\lambda\tilde{\lambda})^{(d-1)/2}(1+\sqrt{\lambda\tilde{\lambda}})} < \delta < \frac{1}{(\lambda\tilde{\lambda})^{d/2}}\,. \tag{3.6}$$

Several remarks are appropriate for this theorem. Firstly, the theorem establishes the existence of an expanding eigenvalue but does not prove the hyperbolicity of the operator dR_f^2. Secondly, the lower bound for δ, whilst greater than 1, is suboptimal and, indeed, is worse than the bounds $1/\lambda^d - 1/\lambda$ obtained in [22] for the symmetric period-doubling case. Unfortunately, some of the estimates in that paper do not readily generalize to the asymmetric case and our results are accordingly weaker, although they do apply to all degree d and modulus μ.

3.1.2 Notation and background material

In this section we establish our notation and give a brief summary of previous results from [64, 65, 69] that we shall use in this work.

The Herglotz function approach [25] has been an extremely fruitful technique in the analysis of the accumulation of period-doubling. It was used in [65] to prove the existence of a solution of the equations (3.2) for all real $\mu > 0$ and $d > 1$. We recall here how equations (3.2) may be recast as an anti-Herglotz function problem.

Firstly we build the singularity into our functions by defining

$$f_R(x) = F_R(|x|^d)\,, \qquad \tilde{f}_R(x) = \tilde{F}_R(|x|^d)\,. \tag{3.7}$$

The left-hand functions are given in terms of the right-hand ones by

$$f_L(x) = F_R(\mu|x|^d)\,, \qquad \tilde{f}_L(x) = \tilde{F}_R(\mu^{-1}|x|^d)\,. \tag{3.8}$$

We then consider the inverses of these functions by defining

$$F_R(x) = U^{-1}(x)\,, \qquad \tilde{F}_R(x) = \tilde{U}^{-1}(x)\,. \tag{3.9}$$

The functions U and $\tilde{U}$ satisfy the conditions $U(1) = 0$, $U(-\lambda) = 1$, $\tilde{U}(1) = 0$, $\tilde{U}(-\tilde{\lambda}) = 1$. We may further normalize by setting $U(x) = k\psi(x)$, $\tilde{U}(x) = \tilde{k}\tilde{\psi}(x)$, where $k = U(0)$, $\tilde{k} = \tilde{U}(0)$, so that the functions ψ and $\tilde{\psi}$ satisfy $\psi(1) = 0$, $\psi(0) = 1$, $\tilde{\psi}(1) = 0$, $\tilde{\psi}(0) = 1$. We then have $U(x) = {z_1}^d\psi(x)$, $\tilde{U}(x) = \tilde{z}_1^d\tilde{\psi}(x)$, where $z_1 = \psi(-\lambda)^{-1/d}$, $\tilde{z}_1 = \tilde{\psi}(-\tilde{\lambda})^{-1/d}$.

In this new setting our equations become

$$\psi(x) = \tilde{\tau}^{-1}\tilde{\psi}(\tilde{\phi}(x)), \qquad \tilde{\psi}(x) = \tau^{-1}\psi(\phi(x)), \tag{3.10}$$

where $\phi(x) = z_1\psi(-\lambda x)^{1/d} = U(-\lambda x)^{1/d}$, $\tilde{\phi}(x) = \tilde{z}_1\tilde{\psi}(-\tilde{\lambda}x)^{1/d} = \tilde{U}(-\tilde{\lambda}x)^{1/d}$, and $\tau = \psi(z_1)$, $\tilde{\tau} = \tilde{\psi}(\tilde{z}_1)$ satisfy

$$\lambda^d = \frac{\tau z_1^d}{\mu\tilde{z}_1^d}, \qquad \tilde{\lambda}^d = \frac{\mu\tilde{\tau}\tilde{z}_1^d}{z_1^d}. \tag{3.11}$$

Note that $\tau\tilde{\tau} = (\lambda\tilde{\lambda})^d$. The method of the existence proof is now to show that (3.10) has a solution in a space of anti-Herglotz functions.

Let $\mathbb{C}_+$, $\mathbb{C}_-$ denote the upper and lower half planes in $\mathbb{C}$. Recall that a complex analytic function on $\mathbb{C}_+ \cup \mathbb{C}_-$ is said to be Herglotz (resp. anti-Herglotz) if $f(\mathbb{C}_+) \subset \bar{\mathbb{C}}_+$ and $f(\mathbb{C}_-) \subset \bar{\mathbb{C}}_-$ (resp. $f(\mathbb{C}_+) \subset \bar{\mathbb{C}}_-$ and $f(\mathbb{C}_-) \subset \bar{\mathbb{C}}_+$).

For $A < B \in \mathbb{R}$, we let $\Omega(A, B)$ denote $\mathbb{C}_+ \cup \mathbb{C}_- \cup (A, B)$. We denote by $\mathrm{H}(A, B)$ and $\mathrm{AH}(A, B)$ (respectively) the space of Herglotz and anti-Herglotz functions (respectively) analytic on the interval (A, B). Furthermore, if $[0, 1] \subset (A, B)$, let $\mathrm{E}(A, B)$ denote the space of anti-Herglotz functions $\psi \in \mathrm{AH}(A, B)$ which satisfy the normalizations $\psi(0) = 1$, $\psi(1) = 0$. As is normal, we equip $\mathrm{H}(A, B)$, $\mathrm{AH}(A, B)$ and $\mathrm{E}(A, B)$ with the topology of uniform convergence on compact subsets of $\Omega(A, B)$.

In [65] the following existence theorem was proved:

Theorem. *For each $\mu > 0$ and for each $d > 1$, there exists a solution pair $(\psi, \tilde{\psi})$ for (3.10) with $\psi \in E(-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1})$ and $\tilde{\psi} \in E(-\lambda^{-1}, (\lambda\tilde{\lambda})^{-1})$.*

From this it is straightforward to reverse the transformation above to show that (3.2) has a solution. See [65].

One crucial feature of the Herglotz and anti-Herglotz functions is that they satisfy the so-called *a priori* bounds. (See [25, 26, 65].) For the solution pair $(\psi, \tilde{\psi})$ these bounds are, for $x < 0$ and $x > 1$:

$$\frac{1-x}{1-\lambda\tilde{\lambda}x} \le \psi(x) \le \frac{1-x}{1+\tilde{\lambda}x}, \qquad \frac{1-x}{1-\lambda\tilde{\lambda}x} \le \tilde{\psi}(x) \le \frac{1-x}{1+\lambda x}; \tag{3.12}$$

and for $0 < x < 1$:

$$\frac{1-x}{1+\tilde{\lambda}x} \le \psi(x) \le \frac{1-x}{1-\lambda\tilde{\lambda}x}\,, \qquad \frac{1-x}{1+\lambda x} \le \tilde{\psi}(x) \le \frac{1-x}{1-\lambda\tilde{\lambda}x}\,. \tag{3.13}$$

In addition, as in [25], it is straightforward to derive *a priori* bounds on the first and second derivatives:

$$\frac{-2\tilde{\lambda}}{(1+\tilde{\lambda}x)} \le \frac{\psi''(x)}{\psi'(x)} = \frac{U''(x)}{U'(x)} \le \frac{2\lambda\tilde{\lambda}}{(1-\lambda\tilde{\lambda}x)}\,, \qquad x \in (-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1})\,, \tag{3.14a}$$

$$\frac{-2\lambda}{(1+\lambda x)} \le \frac{\tilde{\psi}''(x)}{\tilde{\psi}'(x)} = \frac{\tilde{U}''(x)}{\tilde{U}'(x)} \le \frac{2\lambda\tilde{\lambda}}{(1-\lambda\tilde{\lambda}x)}\,, \qquad x \in (-\lambda^{-1}, (\lambda\tilde{\lambda})^{-1})\,. \tag{3.14b}$$

Let us define $t = \mu\lambda^d$, $\tilde{t} = \mu^{-1}\tilde{\lambda}^d$. Then we have the following properties which are a consequence of the definitions and the results of [65].

1. $t,\ \tilde{t},\ z_1,\ \tilde{z}_1,\ \tau,\ \tilde{\tau} \in (0,1)$;
2. $t < z_1^d$, and $\tilde{t} < \tilde{z}_1^d$.

Following [25], we define the functions $V = \tau^{-1}\psi(z_1 x^{1/d})$, $\tilde{V} = \tilde{\tau}^{-1}\tilde{\psi}(\tilde{z}_1 x^{1/d})$. Let us further define $\alpha = \psi(-\lambda)$, $\tilde{\alpha} = \tilde{\psi}(-\tilde{\lambda})$. Then $V \in \mathrm{AH}(0, \alpha(\lambda\tilde{\lambda})^{-d})$ and $\tilde{V} \in \mathrm{AH}(0, \tilde{\alpha}(\lambda\tilde{\lambda})^{-d})$ and, in view of equations (3.10), we have

$$\psi(x) = \tilde{V}(\tilde{\psi}(-\tilde{\lambda}x))\,, \qquad \tilde{\psi}(x) = V(\psi(-\lambda x)\,. \tag{3.15}$$

Note that $V(1) = \tilde{V}(1) = 1$ and $V(\alpha) = \tilde{V}(\tilde{\alpha}) = 0$. Differentiating (3.15), and evaluating at 0, gives

$$V'(1) = \frac{-\tilde{\psi}'(0)}{\lambda\psi'(0)}\,, \qquad \tilde{V}'(1) = \frac{-\psi'(0)}{\tilde{\lambda}\tilde{\psi}'(0)}\,, \qquad V'(1)\tilde{V}'(1) = \frac{1}{\lambda\tilde{\lambda}}\,. \tag{3.16}$$

Lemma 14. *The functions $U(x)$ and $\tilde{U}(x)$ are injective respectively in domains $\Omega = \mathbb{C}_+ \cup \mathbb{C}_- \cup (-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1})$ and $\tilde{\Omega} = \mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, (\lambda\tilde{\lambda})^{-1})$.*

The proof (see [TMO2]) uses the ideas of Epstein exposed in [26],[27].

3.1.3 Lower bound for d

In this section we shall prove the inequality $d > (1+\lambda\tilde{\lambda})/(1-\lambda\tilde{\lambda})$ which is important in the proof of convexity of the functions L, $\tilde{L}$, V and $\tilde{V}$. The inequality is analogous to Epstein's result for the symmetric Feigenbaum function, viz., $d > (1+\lambda^2)/(1-\lambda^2)$. However, our proof differs somewhat from that in [25].

Lemma 15.

$$d > (1+\lambda\tilde{\lambda})/(1-\lambda\tilde{\lambda})\,. \tag{3.17}$$

The proof of this lemma is based on the estimates contained in the paper [69] and the *a priori* bounds (3.13).

3.1.4 The operator T

In this section we first of all discuss informally the relationship between $dR^2(f)$ and the operator T given in Section 3.1.1.

When analyzing asymmetric maps it is often convenient to work (as in [65]) with a map of pairs, rather than the doubling operator R. Let R_P denote the map $R_P(f,\tilde{f}) = (R(\tilde{f}), R(f))$. Then a fixed point of R_P, with $f \neq \tilde{f} = R(f)$, corresponds to a period-2 point of R and vice versa. The spectra of the derivatives $dR^2(f)$ and $dR_P(f,\tilde{f})$ are related: an eigenvalue ρ^2 of $dR^2(f)$ corresponds to a pair of eigenvalues $\pm\rho$ of $dR_P(f,\tilde{f})$. Indeed, if $\rho^2 \in \mathbb{C}$ is an eigenvalue of $dR^2(f)$ with eigenvector δf, then the pair $(\delta f, \pm\rho^{-1}\,\delta\tilde{f})$, where $\delta\tilde{f} = dR_f\delta f$, is eigenvector of $dR_P(f,\tilde{f})$ with eigenvalue $\pm\rho$, and vice versa. We may therefore study the spectrum of $dR_P(f,\tilde{f})$ in lieu of $dR^2(f)$.

As in [22], a further simplification can be made by studying the operator $\bar{R}_P$ given by R_P with the parameters λ and $\tilde{\lambda}$ held constant at their values at the fixed-point pair $(f,\tilde{f})$. This introduces eigenvalues ± 1 into the spectrum of $d\bar{R}_P(f,\tilde{f})$ but otherwise leaves the spectrum undisturbed. Acting on pairs of tangent functions $(\delta f(x), \delta\tilde{f}(x))$, the operator $d\bar{R}_P(f,\tilde{f})$ is given by:

$$d\bar{R}_P(f,\tilde{f})\begin{pmatrix}\delta f\\ \delta\tilde{f}\end{pmatrix} = \begin{pmatrix}-\tilde{\lambda}^{-1}\delta\tilde{f}(\tilde{f}(-\tilde{\lambda}x)) - \tilde{\lambda}^{-1}\tilde{f}'(\tilde{f}(-\tilde{\lambda}x))\delta\tilde{f}(-\tilde{\lambda}x)\\ -\lambda^{-1}\delta f(f(-\lambda x)) - \lambda^{-1}f'(f(-\lambda x))\delta f(-\lambda x))\end{pmatrix}. \tag{3.18}$$

Furthermore, it is convenient to build in the degree of criticality d by writing $f(x) = F(|x|^d)$, $\tilde{f}(x) = \tilde{F}(|x|^d)$ leading to an induced map $\bar{R}_P$ on pairs $(F,\tilde{F})$ and derivative $d\bar{R}_P(F,\tilde{F})$. Following [22], as a final simplification, we consider tangent vector pairs $(v,\tilde{v}) = (\delta F/F', \delta\tilde{F}/\tilde{F}')$. Following [22] we define a map from $\mathbb{R}$ to $\mathbb{R}$ given by

$$q(x) = \mathrm{sign}(x)|x|^d\,. \tag{3.19}$$

We then define L, $\tilde{L}$ by

$$L(x) = q(F(x)), \quad \tilde{L}(x) = q(\tilde{F}(x)), \quad x \in [0,1]\,, \tag{3.20}$$

and use also the notation

$$L(x) = \begin{cases} L_+(x) = F(x)^d\,, & x \in [0, z_1^d] \\ -L_-(x) = -|F(x)|^d\,, & x \in [z_1^d, 1]\,, \end{cases} \tag{3.21a}$$

$$\tilde{L}(x) = \begin{cases} \tilde{L}_+(x) = \tilde{F}(x)^d\,, & x \in [0, \tilde{z}_1^d] \\ -\tilde{L}_-(x) = -|\tilde{F}(x)|^d\,, & x \in [\tilde{z}_1^d, 1]\,. \end{cases} \tag{3.21b}$$

These functions satisfy the identities

$$L(x) = -\frac{1}{\tilde{\lambda}^d}\tilde{L}(\tilde{L}(\tilde{t}x)), \quad \tilde{L}(x) = -\frac{1}{\lambda^d}L(L(tx))\,, \tag{3.22}$$

or, equivalently,

$$L_+(x) = \frac{1}{\tilde{\lambda}^d}\tilde{L}_-(\tilde{L}_+(\tilde{t}x)),\ \forall\ x \in [0, z_1^d]\,,\ \tilde{L}_+(x) = \frac{1}{\lambda^d}L_-(L_+(tx)),\ \forall x \in [0, \tilde{z}_1^d]$$

$$L_-(x) = \frac{1}{\tilde{\lambda}^d}\tilde{L}_+(\tilde{L}_+(\tilde{t}x)),\ \forall\ x \in [z_1^d, 1]\,,\ \tilde{L}_-(x) = \frac{1}{\lambda^d}L_+(L_+(tx)),\ \forall x \in [\tilde{z}_1^d, 1]\,.$$

The linear operator induced on $(v, \tilde{v})$ by $d\bar{R}_P(F, \tilde{F})$ is the operator T described in the introduction:

$$T\begin{pmatrix} v(x) \\ \tilde{v}(x) \end{pmatrix} = \begin{pmatrix} v_1(x) \\ \tilde{v}_1(x) \end{pmatrix} = \begin{pmatrix} \tilde{t}^{-1}(\tilde{v}(\tilde{t}x) + \tilde{v}(\tilde{L}(\tilde{t}x))\tilde{L}'(\tilde{t}x)^{-1}) \\ t^{-1}(v(tx) + v(L(tx))L'(tx)^{-1}) \end{pmatrix}. \tag{3.23}$$

According to Lemma 14 The functions $F(x)$, $v(x)$ are analytic in the domain $\Delta = U(\Omega)$ and $\tilde{F}(x)$, $\tilde{v}(x)$ are analytic in $\tilde{\Delta} = \tilde{U}(\tilde{\Omega})$, where $\Omega = \mathbb{C}_+ \cup \mathbb{C}_- \cup (-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1})$, $\tilde{\Omega} = \mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, (\lambda\tilde{\lambda})^{-1})$.
We recall that $U(x)$, $\tilde{U}(x)$ satisfy the following functional equations

$$\tilde{t}U(x) = \tilde{U}(\tilde{u}(-\tilde{\lambda}\,x)), \quad \tilde{u}(x) = \tilde{U}(x)^{1/d}, \quad x \in \Omega\,, \tag{3.24a}$$

$$t\tilde{U}(x) = U(u(-\lambda\,x)), \quad u(x) = U(x)^{1/d}, \quad x \in \tilde{\Omega}\,. \tag{3.24b}$$

The following equations are a direct consequence of (3.24):

$$L(t\tilde{U}(x)) = U(-\lambda x), \quad x \in \tilde{\Omega}\,, \qquad \tilde{L}(\tilde{t}U(x)) = \tilde{U}(-\tilde{\lambda}x), \quad x \in \Omega\,, \tag{3.25}$$

which provide (using injectivity of U and $\tilde{U}$) a holomorphic extension of the restriction $L|(0, z_1^d)$ (resp. $\tilde{L}|(0, \tilde{z}_1^d)$) to the complex domain $t\tilde{\Delta}$ (resp. $\tilde{t}\Delta$).
We now consider rigorously the properties of the operator T. Our first task is to show that T is well defined on function pairs $(v, \tilde{v})$ on suitable domains. We first of all show that the domains Δ and $\tilde{\Delta}$ map nicely.

Lemma 16. *The domains Ω, $\tilde{\Omega}$, Δ, $\tilde{\Delta}$ satisfy:*

1. $-\tilde{\lambda}\Omega \subset \tilde{\Omega}, \quad -\lambda\tilde{\Omega} \subset \Omega\,.$
2. $\tilde{t}\,\Delta \subset \tilde{\Delta}, \quad t\,\tilde{\Delta} \subset \Delta.$
3. $L(t\tilde{\Delta}) \subset \Delta, \quad \tilde{L}(\tilde{t}\Delta) \subset \tilde{\Delta}.$

The proof can be found in [TMO2].
The domains Δ, $\tilde{\Delta}$ and $t\tilde{\Delta}$, $\tilde{t}\Delta$ are natural domains on which to define F, $\tilde{F}$ and L, $\tilde{L}$. However, to ensure that T is well defined and compact, we must obtain smaller domains on which T is bounded and analyticity improving. This we do in the next section.

3.1.5 Analyticity-improving domains

Let a, $b \in \mathbb{R}$, $a < b$ and $D(a,b)$ be an open disc in $\mathbb{C}$ with diameter (a,b). We introduce the domains $\Delta_1 = U(D(\alpha_1,\beta_1))$, $\Delta_0 = U(D(\alpha_0,\beta_0))$, $\tilde{\Delta}_1 = \tilde{U}(D(\tilde{\alpha}_1,\tilde{\beta}_1))$, $\tilde{\Delta}_0 = \tilde{U}(D(\tilde{\alpha}_0,\tilde{\beta}_0))$, where

$$\alpha_1 = -\tilde{\lambda}^{-1}\,, \quad \tilde{\alpha}_1 = -\lambda^{-1}\,, \quad \beta_0 = u(-\tilde{\lambda}^{-1})\,, \quad \tilde{\beta}_0 = \tilde{u}(-\lambda^{-1})\,, \tag{3.26}$$

$$\alpha_0 = -3\tilde{\lambda}^{-1}/4 - \lambda\tilde{u}(-\lambda^{-1})/4\,, \quad \tilde{\alpha}_0 = -3\lambda^{-1}/4 - \tilde{\lambda}u(-\tilde{\lambda}^{-1})/4\,, \tag{3.27}$$

$$\beta_1 = (\lambda\tilde{\lambda})^{-1}/2 + u(-\tilde{\lambda}^{-1})/2\,, \quad \tilde{\beta}_1 = (\lambda\tilde{\lambda})^{-1}/2 + \tilde{u}(-\lambda^{-1})/2\,. \tag{3.28}$$

The following inequalities will be important in what follows:

$$1 < u(-\tilde{\lambda}^{-1}) < (\lambda\tilde{\lambda})^{-1}, \quad 1 < \tilde{u}(-\lambda^{-1}) < (\lambda\tilde{\lambda})^{-1}\,. \tag{3.29}$$

We shall now prove (3.29). To show that $u(-\tilde{\lambda}^{-1}) > 1$ we use the fact that $u(x)$ is an anti-Herglotz function which is decreasing in $(-\tilde{\lambda}^{-1}, 1)$ and satisfies the condition $u(-\lambda) = 1$, $\lambda \leq \tilde{\lambda}^{-1}$. This gives $u(-\tilde{\lambda}^{-1}) = \lim_{x\to -\tilde{\lambda}^{-1}} u(x) > 1$. Next, $u(-\tilde{\lambda}^{-1}) = \tilde{z}_1/\tilde{t}^{1/d} = \tilde{z}_1\mu^{1/d}\tilde{\lambda}^{-1}$, so that $u(-\tilde{\lambda}^{-1}) < (\lambda\tilde{\lambda})^{-1}$ if and only if $\tilde{z}_1\mu^{1/d}\lambda < 1$ which follows from the inequalities $\tilde{z}_1 < 1$, $\mu^{1/d}\lambda < 1$. The other inequalities follow similarly.
From the inequalities (3.29), it straightforward to check the following:

$$-\tilde{\lambda}^{-1} = \alpha_1 < \alpha_0 < -\lambda < 0 < 1 < \beta_0 < \beta_1 < (\lambda\tilde{\lambda})^{-1} \tag{3.30a}$$

$$-\lambda^{-1} = \tilde{\alpha}_1 < \tilde{\alpha}_0 < -\tilde{\lambda} < 0 < 1 < \tilde{\beta}_0 < \tilde{\beta}_1 < (\lambda\tilde{\lambda})^{-1}\,, \tag{3.30b}$$

and from these it is easy to check that $\Delta_0 \Subset \Delta_1 \subset \Delta$ and $\tilde{\Delta}_0 \Subset \tilde{\Delta}_1 \subset \tilde{\Delta}$.
We have the following lemma concerning the domains Δ_0, Δ_1, $\tilde{\Delta}_1$, $\tilde{\Delta}_0$.

Lemma 17 ([TMO2])**.** *The domains* Δ_0*,* Δ_1*,* $\tilde{\Delta}_0$*,* $\tilde{\Delta}_1$ *satisfy:*

1. $\tilde{L}(\tilde{t}\,\Delta_1) \subset \tilde{\Delta}_0, \quad L(t\,\tilde{\Delta}_1) \subset \Delta_0\,.$
2. $\tilde{t}\,\Delta_1 \subset \tilde{\Delta}_0, \quad t\,\tilde{\Delta}_1 \subset \Delta_0\,.$
3. $[0,1] \subset \Delta_0, \quad [0,1] \subset \tilde{\Delta}_0\,.$

From Lemma 17 we have that if $(v,\tilde{v})$ are analytic on $\Delta_0 \times \tilde{\Delta}_0$ then $\tilde{v}(\tilde{t}x)$, $\tilde{v}(\tilde{L}(\tilde{t}x))$ are analytic on Δ_1 and $v(tx)$, $v(L(tx))$ are analytic on $\tilde{\Delta}_1$. Furthermore, differentiating the first equation of (3.25) gives $L'(t\tilde{U}(x)) \neq 0$ for all $x \in \tilde{\Omega}$, since U is single-valued on $-\lambda\tilde{\Omega} \subset \Omega$. We deduce that $L'(tx) \neq 0$ for all $x \in \tilde{\Delta} = \tilde{U}(\tilde{\Omega})$. Similarly $\tilde{L}'(tx) \neq 0$ for all $x \in \Delta$.
Since $\Delta_1 \subset \Delta$ and $\tilde{\Delta}_1 \subset \tilde{\Delta}$ we conclude that if $(v,\tilde{v})$ are analytic on $\Delta_0 \times \tilde{\Delta}_0$ then then $T(v,\tilde{v})$ is defined and analytic on $\Delta_1 \times \tilde{\Delta}_1$.
We note that the derivative $L'(tx)$ (resp. $\tilde{L}'(\tilde{t}x)$) vanishes at $x = z_1^d/t \in \partial\tilde{\Delta}_1$ (resp. $x = \tilde{z}_1^d/\tilde{t} \in \partial\Delta_1$). But its reciprocal $1/L'(tx)$ (resp. $1/\tilde{L}'(\tilde{t}x)$) is bounded in any $\tilde{\Delta}' \Subset \tilde{\Delta}_1$ (resp. $\Delta' \Subset \Delta_1$). Hence $T(v,\tilde{v})$ is well defined and bounded on any domain $\Delta_1' \times \tilde{\Delta}_1'$ with $\Delta_1' \Subset \Delta_1$, $\tilde{\Delta}_1' \Subset \tilde{\Delta}_1$.
From this we immediately have the following lemma, which shows that T is analyticity-improving.

Lemma 18. *If* $(v(x),\tilde{v}(x))$ *is a pair of real functions on* $[0,1]$ *which extend to a holomorphic functions on* $\Delta_0 \times \tilde{\Delta}_0 \subset \mathbb{C}^2$ *then the pair* $(v_1(x),\tilde{v}_1(x))$*, defined in* (3.23)*, extend to holomorphic functions on* $\Delta_1 \times \tilde{\Delta}_1$*.*

We now define the Banach space in which we shall work.

Definition. Let B denote the Banach space of pairs of functions $(v(x),\tilde{v}(x))$ holomorphic and bounded on $\Delta_0 \times \tilde{\Delta}_0 \subset \mathbb{C}^2$ which are real on $[0,1]$. We equip B with the norm

$$||(v,\tilde{v})|| = \max\left(\sup_{x\in\Delta_0}|v(x)|, \sup_{x\in\tilde{\Delta}_0}|\tilde{v}(x)|\right)\,. \tag{3.31}$$

The results of this section enable us to conclude that T is compact. Indeed, we have the following, which is a direct consequence of Lemma 16 and Lemmas 18.

Corollary 3. *T is a compact operator on B and $TB \subset B$. Moreover, for* *every* $\Delta' \Subset \Delta_1$, $\tilde{\Delta}' \Subset \tilde{\Delta}_1$ *we have*

$$\sup_{x\in\Delta'} |v_1(x)| \le \tilde{t}^{-1} \left(1 + \sup_{x\in\Delta'} \left|\frac{1}{\tilde{L}'(\tilde{t}x)}\right|\right) \sup_{y\in\tilde{\Delta}_0} |\tilde{v}(y)| \,, \tag{3.32a}$$

$$\sup_{x\in\tilde{\Delta}'} |\tilde{v}_1(x)| \le t^{-1} \left(1 + \sup_{x\in\tilde{\Delta}'} \left|\frac{1}{L'(tx)}\right|\right) \sup_{y\in\Delta_0} |v(y)| \,. \tag{3.32b}$$

3.1.6 Properties of the functions $L(x)$ and $\tilde{L}(x)$

In this section, we prove several properties of the functions L and $\tilde{L}$; in particular we show that they are convex.

Lemma 19 ([TMO2])**.** *The function* L_+ *is convex on* $[0, z_1^d]$*, and the function* L_- *is convex on* $[z_1^d, 1]$*. The function* $\tilde{L}_+$ *is convex on* $[0, \tilde{z}_1^d]$*, and the function* $\tilde{L}_-$ *is convex on* $[\tilde{z}_1^d, 1]$*.*

The next two lemmas give important estimates on L, $\tilde{L}$ and their derivatives.

Lemma 20 ([TMO2])**.** *For all* $x \in [0,1]$*, we have*

$$L_+(tx) > tx, \quad \tilde{L}_+(\tilde{t}x) > \tilde{t}x \,. \tag{3.33}$$

Lemma 21 ([TMO2])**.** *For all* $x \in [0,1]$*, we have*

$$L'(tx) < -1, \quad \tilde{L}'(\tilde{t}x) < -1 \,. \tag{3.34}$$

3.1.7 Invariant cone for T

Generalizing the result obtained in [22], now we can derive the existence of an invariant cone for the operator T in the space of functions $(v(x), \tilde{v}(x))$. We shall then be able to apply the Krein-Rutman theorem.

Definition. Define Γ_1 to be the set of pairs $(v(x), \tilde{v}(x))$ of real smooth functions on $[0,1]$ which, for $x \in [0,1]$, satisfy (i) $v(x) \ge 0$, $\tilde{v}(x) \ge 0$, and (ii) $v'(x) \le 0$, $\tilde{v}'(x) \le 0$.

The following lemma is a generalization of Lemma 3.4 of [22].

Lemma 22 ([TMO2])**.** *Let* $\Gamma = B \cap \Gamma_1$*. Then* T *maps* Γ_1 *into itself and* T^2 *maps any non-zero vector in* Γ *into the interior of* Γ*.*

From the theorem of Krein and Rutman [48] we thus have the following result.

Theorem 28. *The operator T, acting on B, has an eigenvalue of largest modulus $\delta > 0$. The spectral subspace corresponding to δ is one-dimensional and is generated by an element from the interior of Γ which is the only eigenvector of T in Γ.*

In the next section we give some bounds on this eigenvalue δ.

3.1.8 Bounds on the expanding eigenvalue

Let $(v, \tilde{v})$ be an eigenvector with eigenvalue δ in the cone $v \geq 0$, $\tilde{v} \geq 0$, $v' \leq 0$, $\tilde{v}' \leq 0$. We have further that $v(0)$, $\tilde{v}(0) > 0$, since $(v, \tilde{v})$ is in the interior of the cone.
The eigenvector equations are

$$\delta\tilde{v}(x) = t^{-1}\left(v(tx) + \frac{v(L(tx))}{L'(tx)}\right), \qquad \delta v(x) = \tilde{t}^{-1}\left(\tilde{v}(\tilde{t}x) + \frac{\tilde{v}(\tilde{L}(\tilde{t}x))}{\tilde{L}'(\tilde{t}x)}\right). \tag{3.35}$$

Evaluating these at 0 we obtain

$$\delta\tilde{v}(0) = t^{-1}\left(v(0) + \frac{v(1)}{L'(0)}\right), \qquad \delta v(x) = \tilde{t}^{-1}\left(\tilde{v}(0) + \frac{\tilde{v}(1)}{\tilde{L}'(0)}\right). \tag{3.36}$$

Now we have $L'(0)$, $\tilde{L}'(0) < -1$ and $v(1)$, $\tilde{v}(1) > 0$ so that, neglecting the second term on the right hand sides of these equations, and multiplying, we immediately obtain the bound $\delta^2 v(0)\tilde{v}(0) < (t\tilde{t})^{-1}v(0)\tilde{v}(0)$ so that $\delta^2 < (t\tilde{t})^{-1} = (\lambda\tilde{\lambda})^{-d}$, which is the upper bound in Theorem 27.
To obtain the lower bound, we use the convexity of L and $\tilde{L}$. Since v', $\tilde{v}' \leq 0$, we have that $v(1) \leq v(0)$ and $\tilde{v}(1) \leq \tilde{v}(0)$ so that, multiplying the eigenvector equations (3.36), we have

$$\delta^2 v(0)\tilde{v}(0) \geq (t\tilde{t})^{-1}v(0)\tilde{v}(0)\left(1 + \frac{1}{L'(0)}\right)\left(1 + \frac{1}{\tilde{L}'(0)}\right). \tag{3.37}$$

From the convexity of L and $\tilde{L}$ we have $L'(0) < -1/z_1^d < -1$, and $\tilde{L}'(0) < -1/\tilde{z}_1^d < -1$ so that $1 - z_1^d < 1 + 1/L'(0)$ and $1 - \tilde{z}_1^d < 1 + 1/\tilde{L}'(0)$, and, hence,

$$\delta^2 > \frac{1}{t\tilde{t}}(1 - z_1^d)(1 - \tilde{z}_1^d). \tag{3.38}$$

Recall that we have $V'(1)\tilde{V}'(1) = (\lambda\tilde{\lambda})^{-1}$. Then both V and $\tilde{V}$ are convex since they are scaled versions of S_+ and $\tilde{S}_+$ respectively. We also have $V(1) = 1$, $V(\alpha) = 0$, where $\alpha = z_1^{-d} > 1$.

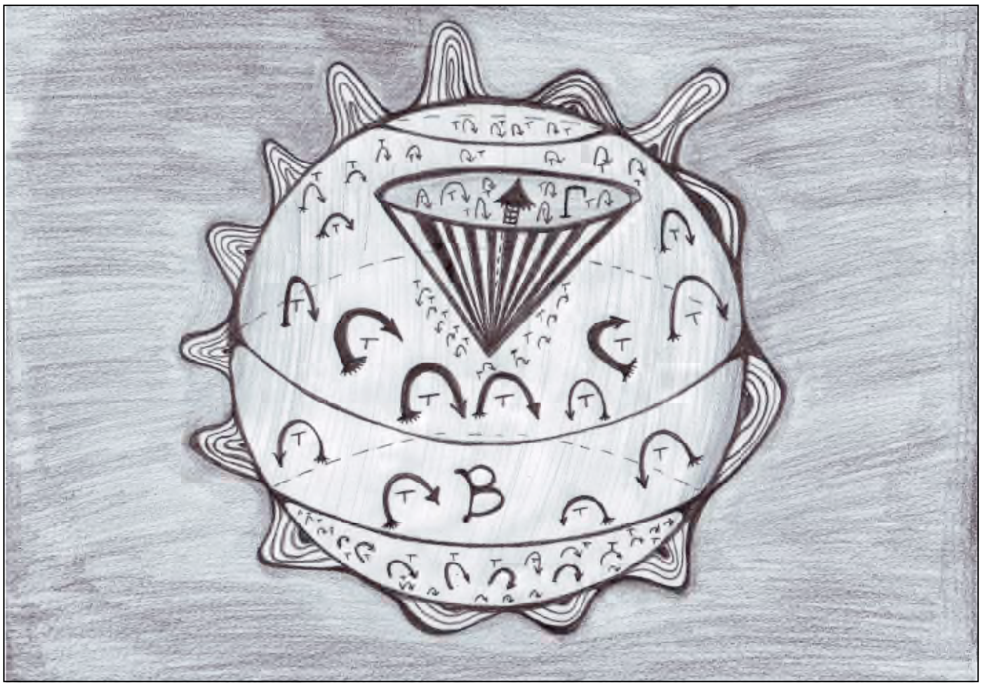

Figure 3.1: The T-invariant cone.

From the convexity of V we have $V'(1) \leq -1/(\alpha - 1)$ so that $\alpha - 1 \geq -1/V'(1)$. Similarly, we have $\tilde{\alpha}-1 \geq -1/\tilde{V}'(1)$, and thus $(\alpha-1)(\tilde{\alpha}-1) \geq \lambda\tilde{\lambda}$. Now if $x, y > 1$ and we have $(x-1)(y-1) \geq C > 0$, then a straightforward application of Lagrange multipliers shows that $(1-x^{-1})(1-y^{-1}) \geq C/(1+\sqrt{C})^2$. We conclude that

$$\delta^2 \geq \frac{1}{t\tilde{t}}(1 - z_1^d)(1 - \tilde{z}_1^d) = \frac{1}{t\tilde{t}}(1-\alpha^{-1})(1-\tilde{\alpha}^{-1}) \geq \frac{1}{(\lambda\tilde{\lambda})^{(d-1)}(1+\sqrt{\lambda\tilde{\lambda}})^2}\,. \tag{3.39}$$

One shows (see [TMO2]) that for $g = \sqrt{\lambda\tilde{\lambda}}$ we have $g^{(d-1)} \leq (1+g+g^3)^{-1} < (1+g)^{-1}$. It follows that $(\lambda\tilde{\lambda})^{(d-1)}(1+\sqrt{\lambda\tilde{\lambda}})^2 = g^{2(d-1)}(1+g)^2 < 1$ and thus

$$1 < \frac{1}{(\lambda\tilde{\lambda})^{(d-1)}(1+\sqrt{\lambda\tilde{\lambda}})^2} < \delta^2 < \frac{1}{(\lambda\tilde{\lambda})^d}\,, \tag{3.40}$$

so, in particular, $\delta > 1$. This completes the proof of Theorem 27.

3.2 A priori bounds for anti-Herglotz functions with some applications to the functionall Feigenbaum-Cvitanović equation

3.2.1 Introduction

Let $D = \mathbb{C}_+ \cup \mathbb{C}_- \cup (-1, +\infty)$. Consider a function $f(z)$ which is analytic in D and real for $z \in (-1, +\infty)$. If conditions $f(\mathbb{C}_+) \subset \mathbb{C}_-, f(\mathbb{C}_-) \subset \mathbb{C}_+$ are satisfied when $f(z)$ is called an *anti-Herglotz* function. In particular, it is well known that in this case $f(z)$ is decreasing and has a positive Schwarzian derivative [21], [89]. We assume that $f^2(z)$ is again an anti-Herglotz function i.e. $-\pi/2 < \arg(f(z)) < 0$ for $z \in \mathbb{C}_+$. Let $c_0 = f(0), c_n = \frac{d^{(n)}}{d^n z} f(0), n = 1, 2, ...$ be the derivatives of $f(z)$ calculated at $z = 0$. The following problem can be stated: find a sequence of pairs of functions $A_k(c_0, c_1, ..., c_k, z), B_k(c_0, c_1, ..., c_k, z)$, analytic in D, real for real values of z and such that

$$A_k(z) \leq f(z) \leq B_k(z), \quad k \geq 0,$$

for all $z \in (-1, +\infty)$.

In addition, we ask that as $k \to \infty$ the functions A_k, B_k converge uniformly on compact subsets of D to $f(z)$.

In this section we give a simple solution of this problem using the analytic theory of continued fractions whose development may be found in the book of H. Wall [92]. The foundation of this theory is based on the natural correspondence between the continued fractions, analytic functions and their integral representations given by a moment problem. For concreteness, let us consider the Stieltjes integral of the form

$$f(z) = \int_0^1 \frac{d\theta(u)}{1 + uz}, \tag{3.41}$$

where $\theta(u)$ is a bounded nondecreasing function in the interval $[0, 1]$.

Then it has the following continued fraction representation [92]

$$f(z) = \frac{\mu_0}{1 + \frac{g_1 z}{1 + \frac{(1-g_1)g_2 z}{1 + \cdots}}},$$

where the numbers $g_p \in [0, 1]$, $p = 1, 2, \ldots$ are certain functions of the *moments* μ_p of $\theta(u)$ defined by

$$\mu_p = \int_0^1 u^p d\theta(u), \quad p = 0, 1, 2, \ldots$$

This continued fraction is called a *g-fraction*.

In Section 3.2.2 we show that an anti-Herglotz function which is analytic in D and have a positive real part is a g-fraction (Theorem 30). Section 3.2.3 is devoted to approximations of g-fractions by rational functions. As an application, in Sections 3.2.4-3.2.5 we consider the Feigenbaum-Cvitanović functional equation

$$g(x) = -\lambda^{-1} g(g(-\lambda x)), \quad g(0) = 1, \tag{3.42}$$

where $g(x)$ is a map of the interval $[-1, 1]$ into itself. We only consider solutions $g(x)$ such that, in $[0, 1)$, $g(x) = F(|x|^d)$, $d > 1$, with $F(x)$ analytic, decreasing, and without critical points on $[0, 1)$. It was shown in [25], [26] that such a solution exists for all $d > 1$. For fixed d, from (3.42) it follows that

$$g(1) = -\lambda, \quad g'(1) = -\lambda^{-d+1}, \quad \lambda \in (0, 1). \tag{3.43}$$

Let $U(x) = F^{(-1)}(x)$ be the inverse function, then $U(x)$ will satisfy

$$U(U(-\lambda x)^{1/d}) = \lambda U(x). \tag{3.44}$$

Theorem 29 ([25], [26]). *$U(x)$ extends to a function holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, \lambda^{-2})$ which is injective there and $U(\mathbb{C}_+) \subset \mathbb{C}_-$, $U(\mathbb{C}_-) \subset \mathbb{C}_+$.*

See also [24], [16] for more details.

Our study focuses mainly on two important characteristics of $U(x)$ given by parameters λ and d. The known results (essentially based on the computer-assisted methods) give evidence that λ increases from 0 to 1 when d increases from 1 to $+\infty$. For some particular values of d the corresponding values of λ were calculated numerically by many authors and are known with a high precision [see for instance [9]. It was noted that as d approaches infinity, we have $\lambda^d \to 0.003338\ldots$ which makes obvious the non trivial nature of the dependence between d and λ. The natural question for a preliminary investigation here is to obtain the lower and upper bounds on d considered as a function of λ. Our main result here is the bound (3.62) on $d(\lambda)$ which implies in particular the uniform bound $\lambda^d < c$, where $c = 0.26308...$ is an algebraic number as was announced in [TMO2].

3.2.2 Continued fraction expansions for anti-Herglotz functions with positive real parts

Our aim is to prove the following theorem

Theorem 30. *A necessary condition for a function $G(z)$ to be anti-Herglotz and have a positive real part in the domain D, and be real for real z, is that it have a continued fraction expansion of the form*

$$G(z) = \frac{\mu_0}{1 + \frac{g_1 z}{1 + \frac{(1-g_1)g_2 z}{1+\cdots}}}, \tag{3.45}$$

where $\mu_0 > 0$, $g_p \in [0, 1]$, $p = 1, 2, \ldots$ and the continued fraction converges uniformly over every finite closed domain in D. We shall agree that in case some partial numerator of the continued fraction vanishes identically, then the continued fraction shall terminate with the first identically vanishing partial quotient. With this agreement, the continued fraction expansions (3.45) *are unique.*

We use the following result due to Wall [92].

Theorem 31. *A necessary and sufficient condition for a function $W(z)$ to be analytic and have a positive real part in the domain D, and be real for real z, is that it have a continued fraction expansion of the form*

$$W(z) = \frac{\mu_0 \sqrt{1+z}}{1 + \frac{g_1 z}{1 + \frac{(1-g_1)g_2 z}{1+\cdots}}}, \tag{3.46}$$

where $\mu_0 > 0$, $g_p \in [0, 1]$, $p = 1, 2, \ldots$ and the continued fraction converges uniformly over every finite closed domain in D.

Definition. We will call a function $W(z)$ with the properties stated in Theorem 31 a *Wall* function.

Proof of Theorem 30. Let $G(z)$ be an anti-Herglotz function which is analytic in the domain D, real for real values of z and having a positive real part. Consider the function $f(z) = \sqrt{1+z}$ where the square root is positive on the axis $(-1, +\infty)$. The function so obtained is a Herglotz function since the argument of $f(z)$ is one half the argument of z. Moreover, it is clear that $f(z)$ is a Wall function. We will check that the product $\Phi = fG$ is again a Wall function. Let $f = U + iP$, $G = Y + iK$ where U, Y and P, K are the corresponding real and imaginary parts of the functions f and G. Then $\mathrm{Re}(\Phi) = UY - PK > 0$ in D. Indeed, $UY > 0$ since both f and G are Wall functions and $-PK > 0$ since the terms P and K always have a different sign in D. Thus, by the Theorem 31 the function $\Phi = \sqrt{1+z}\, G(z)$ can be written in the form (3.46). Cancelling of $\sqrt{1+z}$ achieves the proof of Theorem 30. □

Definition. The function $G(z)$ with the properties stated in Theorem 30 will be called a *Wall-Herglotz* function.

3.2.3 Approximation of Wall-Herglotz functions by rational functions

We need two preparatory lemmas which follow directly from Theorem 1.11, and Theorems 14.2-14.3 of [92].

Lemma 23. *The continued fraction*

$$\phi(z) = \cfrac{r_1}{1 + \cfrac{(1-r_1)r_2 z}{1 + \cfrac{(1-r_2)r_3 z}{1+\cdots}}}, \tag{3.47}$$

where $r_p \in [0,1]$, $p = 1, 2, \ldots$ *converges uniformly for* $z \in [-1, 0]$ *and satisfies*

$$\left|\phi(z) - \frac{1}{2-r_1}\right| \le \frac{1-r_1}{2-r_1}.$$

In particular

$$0 < \phi(z) \le 1. \tag{3.48}$$

Lemma 24. *The continued fraction*

$$l(z) = \cfrac{1}{1 + \cfrac{(1-r_1)r_2 z}{1 + \cfrac{(1-r_2)r_3 z}{1+\cdots}}}, \tag{3.49}$$

where $r_p \in [0,1]$, $p = 1, 2, \ldots$ *converges uniformly for* $z \in [0, \infty)$ *and satisfies*

$$0 \le l(z) < 1. \tag{3.50}$$

We denote by $\mathcal{W}$ the class of all Wall-Herglotz functions with $\mu_0 = 1$. To each $f \in \mathcal{W}$ corresponds an unique sequence of real numbers $\{g_p\}_{p=\infty}^{p=1}$, $g_p \in [0,1]$. The following theorem gives the simplest lower and upper rational bounds for an arbitrary function from $\mathcal{W}$.

Theorem 32. *Let* $f(z) \in \mathcal{W}$ *with corresponding parameters* $\{g_p\}_{p=\infty}^{p=1}$. *Then we have*

$$A_1(z) \le f(z) \le B_1(z) \quad -1 < z < +\infty,$$

where

$$A_1(z) = \frac{1}{1+g_1 z}, \quad B_1(z) = \frac{1+(1-g_1)z}{1+z}.$$

Proof. First consider z from interval $(-1, 0)$. The function $f(z)$ can be represented as follows

$$f(z) = \cfrac{1}{1 + \cfrac{g_1 z}{1+(1-g_1)z\phi(z)}},$$

where $\phi(z)$ has the form (3.47) with $r_i = g_{i+1}$, $i \geq 1$. The inequality (3.48) gives us

$$1 + (1 - g_1)z\phi(z) \in [1 + (1 - g_1)z, 1]$$

Further,

$$1 + \frac{g_1 z}{1 + (1 - g_1)z\phi(z)} \in \left[1 + \frac{g_1 z}{1 + (1 - g_1)z}, 1 + g_1 z\right]$$

and finally we get

$$\frac{1}{1 + \frac{g_1 z}{1+(1-g_1)z\phi(z)}} \in \left[\frac{1}{1 + g_1 z}, \frac{1}{1 + \frac{g_1 z}{1+(1-g_1)z}}\right] = [A_1(z), B_1(z)].$$

Now consider the case $0 \leq z < +\infty$. We write $f(z)$ in the form

$$f(z) = \frac{1}{1 + \frac{g_1 z}{1+(1-g_1)g_2 z l(z)}},$$

where $l(z)$ is given by (3.49) with $r_i = g_{i+1}$, $i \geq 1$. The inequality (3.50) implies

$$1 + (1 - g_1)g_2 z l(z) \in [1, 1 + (1 - g_1)z],$$

where we have used the fact that $g_2 \in [0, 1]$. One can verify that

$$1 + \frac{g_1 z}{1 + (1 - g_1)g_2 z l(z)} \in \left[1 + \frac{g_1 z}{1 + (1 - g_1)z}, 1 + g_1 z\right]$$

and further

$$\frac{1}{1 + \frac{g_1 z}{1+(1-g_1)g_2 z l(z)}} \in [A_1(z), B_1(z)].$$

The proof is done. □

The following theorem is a simple generalization of the above result.

Theorem 33. *a) Let $f \in \mathcal{W}$, $k = 2n + 1$, $n = 0, 1, 2, \ldots$, then*

$$A_k(z) \leq f(z) \leq B_k(z), \quad -1 < z < +\infty,$$

where

$$A_k = \frac{1}{1+\frac{g_1 z}{1+\frac{(1-g_1)g_2 z}{1+\frac{\cdots}{1+(1-g_{k-1})g_k z}}}}, \quad B_k = \frac{1}{1+\frac{g_1 z}{1+\frac{(1-g_1)g_2 z}{1+\frac{\cdots}{1+\frac{(1-g_{k-1})g_k z}{1+(1-g_k)z}}}}}$$

b) Let $f \in \mathcal{W}$, $k = 2n$, $n = 1, 2, \dots,$ then

$$A_k^+ \le f(z) \le B_k^+, \quad 0 \le z < +\infty,$$

$$A_k^- \le f(z) \le B_k^-, \quad -1 < z < 0$$

where
$$A_k^+ = \frac{1}{1+\frac{g_1 z}{1+\frac{(1-g_1)g_2 z}{1+\frac{\cdots}{1+\frac{(1-g_{k-1})g_k z}{1+(1-g_k)z}}}}}, \quad B_k^+ = \frac{1}{1+\frac{g_1 z}{1+\frac{(1-g_1)g_2 z}{1+\frac{\cdots}{1+(1-g_{k-1})g_k z}}}}$$

and $A_k^- = B_k^+$, $B_k^- = A_k^+$.

As it was mentioned in Introduction, each g-fraction (3.45) has an integral representation

$$f(z) = \int_0^1 \frac{d\theta(u)}{1+uz}$$

where $\theta(u)$ is a bounded nondecreasing function. We have hence

$$\mathrm{Im}(f) = -\int_0^1 \frac{y d\theta(u)}{(1+ux)^2 + u^2 y^2}$$

which is always negative for $y > 0$ and thus f is an anti-Herglotz function. In the case then one of the numbers g_p is 0 or 1, f becomes a rational function and the measure $\theta(u)$ is concentrated at poles of f. From this remark we have

Proposition 8. *The functions A_k, B_k, A_k^+, B_k^+ are anti-Herglotz functions for all $k \ge 1$.*

Below we write the explicit formulas for rational bounds corresponding to $k = 2, 3$.

Case $k = 2$.

$$A_2^+ = \frac{(1-g_1 g_2)z+1}{(1+z)(g_1(1-g_2)z+1)}, \quad B_2^+ = \frac{g_2(1-g_1)z+1}{(g_1 - g_1 g_2 + g_2)z+1}$$

$$A_2^- = B_2^+, \quad B_2^- = A_2^+.$$

Case $k = 3$.

$$A_3 = \frac{(g_3+g_2-g_3g_2-g_2g_1)z+1}{g_1g_3(1-g_2)z^2+(g_3+g_2+g_1-g_3g_2-g_1g_2)z+1},$$

$$B_3 = \frac{g_2(1-g_3)(1-g_1)z^2+(1+g_2-g_3g_2-g_1g_2)z+1}{(1+z)((g_1+g_2-g_3g_2-g_1g_2)z+1)}$$

Remark 5. It is easy to see that the coefficient g_p in the expression (3.45) for the g-fraction is uniquely defined by derivatives $\frac{d^{(n)}}{d^n z} f(0), n = 0, 2, ...$ Thus, Theorem 33 solves the problem stated in Introduction.

xxx

3.2.4 Application to the the Feigenbaum-Cvitanović functional equation

The equation (3.44) written for the function $u(z) = U^{1/d}(z)$ has the form

$$u(u(-\lambda z)) = \lambda u(z), \quad \lambda \in (0,1), \tag{3.51}$$

where $u(z)$ is an analytic function positive and decreasing in the interval $(-\lambda^{-1}, 1)$ which satisfies the following conditions

$$u(1) = 0, \quad u(-\lambda) = 1, \quad u'(-\lambda) = -\lambda^{d-1}, \quad d > 1. \tag{3.52}$$

From Theorem 29 we derive

Theorem 34. *For all $d > 1$ there exists a solution $u(z)$ of the equation* (3.51) *witch satisfies* (3.52) *and which is an anti-Herglotz function analytic in the domain $\mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, 1)$.*

Let $u(z)$ be a solution of the equation (3.51) given by Corollary 4.1 and corresponding to an arbitrary $d \geq 2$. Then, $u(z)$ is an anti-Herglotz function which has a positive real part in the domain $\mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, 1)$. To see this let us write $u(z)$ as $u(z) = U(z)^{1/d}$ where one takes a restriction of $U(z)$ on the interval $(-\lambda^{-1}, 1)$. Taking into account the positiveness of $U(z)$ in this interval the desired properties of $u(z)$ are obvious.
Let $\lambda \in (0,1)$ and consider the conformal mapping $f_\lambda : \mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, 1) \mapsto D$ given by

$$f_\lambda(z) = \frac{1}{1-\lambda} \frac{\lambda + z}{1 - z}$$

which sends the real interval $(-\lambda^{-1}, 1)$ into $(-1, +\infty)$ bijectively and satisfies

$$f_\lambda(-\lambda) = 0, \quad f_\lambda(-\lambda^{-1}) = -1. \tag{3.53}$$

We note, that both $f_\lambda(z)$ and its inverse function $f_\lambda^{(-1)}(z)$ are Herglotz functions which implies that the function $u(f_\lambda^{(-1)}(z))$ satisfies all properties imposed by Theorem 30 and hence is a Wall-Herglotz function. Moreover, in view of (3.52), (3.53) we have $u(f_\lambda^{-1}(0)) = \mu_0 = 1$ and hence $u(f_\lambda^{-1}(z)) \in \mathcal{W}$. As an consequence of the Theorem 34 we obtain

Theorem 35. *For $d \geq 2$ the solution of the functional equation* (3.51) *has the form*

$$u(z) = \cfrac{1}{1 + \cfrac{g_1 f_\lambda(z)}{1 + \cfrac{(1-g_1)g_2 f_\lambda(z)}{1+\cdots}}} \tag{3.54}$$

for some $g_p \in (0,1)$, $p \geq 1$ which are defined uniquely.

The restriction $d \geq 2$ can be easy deleted by the simple remark, that the function $u(f_\lambda^{(-1)}(z))^{d/\alpha}$, $\alpha \geq 2/d$ is a Wall-Herglotz function, so we have

Theorem 36. *Let $\alpha \geq 2/d$ be an arbitrary real number. Then for all values of $d > 1$ the solution of the functional equation* (3.51), *existence of which is stated by Theorem 34 can be written in the form*

$$u(z) = \frac{1}{\left(1 + \cfrac{g_1 f_\lambda(z)}{1 + \cfrac{(1-g_1)g_2 f_\lambda(z)}{1+\cdots}}\right)^\alpha}, \tag{3.55}$$

where parameters $g_p \in (0,1)$, $p \geq 1$ are defined uniquely for each α. In particular, on can always put $\alpha = 2$.

Remark 6. In Theorem 35 we have replaced the condition $g_p \in [0,1]$ which appears in Theorem 30 by $g_p \in (0,1)$ since in the case $g_p = 0, 1$ the corresponding function $u(z)$ reduces to a rational function. But it can be shown that the equation (3.51) has no rational solutions corresponding to real λ (see [18]).

Applying Theorem 33 we obtain the following rational bounds on $u(z)$ given by

Corollary 4. *Let $u(z)$ be the solution of the equation* (3.51) *corresponding to $d \geq 2$. Then*

$$A_k(f_\lambda(z)) \leq u(z) \leq B_k(f_\lambda(z)), \quad -\lambda^{-1} < z < 1$$

for odd k and

$$A_k^+(f_\lambda(z)) \leq u(z) \leq B_k^+(f_\lambda(z)), \quad -\lambda \leq z < 1,$$

$$A_k^-(f_\lambda(z)) \leq u(z) \leq B_k^-(f_\lambda(z)), \quad -\lambda^{-1} < z < -\lambda$$

for even k.

The similar bounds hold in the case (3.55).

3.2.5 A lower bound on $d(\lambda)$

In this section we derive a new lower bound on the function $d(\lambda)$ which differs from the bound given in [25]

$$d > \sigma(\lambda) \quad \text{where} \quad \sigma(\lambda) = 1 + \frac{2\lambda^2}{1-\lambda^2}, \quad 0 < \lambda < 1. \tag{3.56}$$

Let $u(z)$ be the solution of (3.51) corresponding to $\lambda \in (0,1)$.
From Theorem 32 and Corollary 4 it follows that

$$a(z) \leq u(z) \leq b(z), \quad -\lambda^{-1} < z < 1,$$

where $a_1(z) = A_1(f_\lambda(z))$, $b_1(z) = B_1(f_\lambda(z))$.
The functions a, b have the following elementary properties easy to check

- They are both strictly decreasing in the interval $(-\lambda^{-1}, 1)$
- $a(-\lambda^{-1}) = (1-g_1)^{-1} > 1$, $a(-\lambda) = 1$, $a(1) = 0$,
- $b(-\lambda^{-1}) = +\infty$, $b(-\lambda) = 1$, $b(1) = 1 - g_1 < 1$

One has

$$\lambda u(z) \in [\lambda a(z), \lambda b(z)] \tag{3.57}$$

and $u(-\lambda z) \in [a(-\lambda z), b(-\lambda z)]$.
The function $u(z)$ is decreasing in the interval $(-\lambda^{-1}, 1)$ and hence

$$u(u(-\lambda z)) \in [a(b(-\lambda z)), b(a(-\lambda z))], \quad z \in (-\lambda^{-1}, 1). \tag{3.58}$$

Taking into account (3.57), (3.58) and the equation (3.51) we arrive to the inequality

$$b(a(-\lambda z)) \geq \lambda a(z) \tag{3.59}$$

for all z from the interval $(-\lambda^{-1}, 1)$.
In particular, for $z = -\lambda^{-1}$, it gives

$$b(0) \geq \lambda a(-\lambda^{-1})$$

which is equivalent to

$$1 - \lambda + (1-g_1)\lambda \geq \frac{\lambda}{1-g_1}$$

The simple analysis shows that it is equivalent to

$$0 < g_1 \leq \theta(\lambda), \quad \text{where} \quad \theta(\lambda) = \frac{1}{2}\,\frac{1 + \lambda - \sqrt{1 - 2\,\lambda + 5\,\lambda^2}}{\lambda}. \tag{3.60}$$

We have the following lemma, whose proof consists of a simple differentiation of $u(z)$.

Lemma 25. *Let $u(z)$ have the form* (3.54) *and $u'(-\lambda) = -\lambda^{d-1}$. Then*

$$g_1 = \lambda^{d-1}(1 - \lambda^2). \tag{3.61}$$

After substitution of (3.61) into (3.60) we obtain

$$d \geq D(\lambda), \quad D(\lambda) = \log\left(\frac{\lambda\theta(\lambda)}{1-\lambda^2}\right)/\log(\lambda). \tag{3.62}$$

The inequality (3.60) can be written in the form $\lambda^d \leq \zeta(\lambda)$, $\zeta(\lambda) = \lambda\theta(\lambda)/(1-\lambda^2)$. It is easy to show that $\zeta'(\lambda)$ vanishes at a unique point λ_c in $(0,1)$ and $\zeta''(\lambda_c) < 0$. Hence in this interval ζ has just one maximum. From this we derive the following uniform bound

$$\lambda^d < c, \quad c = 0.26308..., \quad \text{for} \quad d \geq 2,$$

where we omit the explicit expression for c (which is an algebraic number) due to its complexity. Furthermore, comparing (3.56) and (3.62) we have $D(\lambda) > \sigma(\lambda)$, $\forall\lambda \in (0,1)$ and $D(\lambda)/\sigma(\lambda) \to 3\ \log 2 > 1$ if $\lambda \to 1$.

3.3 Current research. Renormalization for Non-Commuting Critical Circle Maps

In this section I will describe the research in progress in collaboration with B. Mestel and A. Osbaldestion.
My original contribution to this piece of work is contained in the sections 3.3.3-3.3.4 where the equations (3.108) for the fixed point of the renormalization operator for non-commuting critical circle maps are derived.

3.3.1 Introduction. Golden-mean circle maps

Since the 1980s there has been a great deal of study of critical circle maps, not least because they have played an important role in the dynamical systems approach to the onset of turbulence. In particular, the appearance of a single critical point in a circle diffeomorphism with irrational rotation number is an idealized model of the break-up of an invariant torus in phase space on which the flow is quasiperiodic. For a review of this theory we refer the reader to [78].

Let $f : S^1 \to S^1$ be a circle homeomorphism. Its rotation (or winding) number, $\rho(f)$ is defined to be

$$\rho(f) = \lim_{n\to\infty} \frac{1}{n}\hat{f}^n(0) \pmod 1 \tag{3.63}$$

where $\hat{f} : \mathbb{R} \to \mathbb{R}$ is the lift of f to the real line, satisfying

$$\hat{f}(x+1) = \hat{f}(x) + 1, \quad 0 \le \hat{f}(0) < 1 \tag{3.64}$$

Of particular interest is the case of *golden mean* rotation number. The golden mean $\sigma = (\sqrt{5}-1)/2$ is distinguished as the number with all entries equal to one in its continued fraction expansion: $\sigma = [1, 1, 1, 1, \dots]$. The rational convergents to σ are ratios of successive Fibonacci numbers

$$\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \sigma \tag{3.65}$$

where

$$F_0 = 1, F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \ge 2\,. \tag{3.66}$$

In what follows we restrict attention to the case with $\rho = \sigma$.

Following [72] we write a circle map f as a pair (ξ, η) where

$$\xi(x) = \hat{f}(x), \quad \hat{f}(0) - 1 \le x < 0\,, \tag{3.67a}$$

$$\eta(x) = \hat{f}(x) - 1, \quad 0 \le x < \hat{f}(0)\,. \tag{3.67b}$$

The pair (ξ, η) has normalization $\xi(0) - \eta(0) = 1$, corresponding to a circle normalized to have length 1. We note that

$$\xi(\eta(x)) = \eta(\xi(x))\,, \tag{3.68}$$

i.e., the pair commutes.

In general we may consider pairs (ξ, η) of increasing maps satisfying the following conditions:

1. $\eta(0) < 0 < \xi(0)$

2. ξ defined on $[\eta(0), 0]$

3. η defined on $[0, \xi(0)]$

4. $\xi(\eta(0)) = \eta(\xi(0))$

This last condition guarantees that one may 'glue' together the pair (ξ, η) to make a circle homeomorphism. The rotation number of a circle map pair $\rho(\xi, \eta)$ is defined to be the rotation number of the associated circle homeomorphism.

In [72], the golden mean renormalization transformation T is defined by $T(\xi, \eta) = (\tilde{\xi}, \tilde{\eta})$ where:

$$\tilde{\xi}(x) = \beta^{-1}\eta(\beta x), \quad \tilde{\eta}(x) = \beta^{-1}\eta(\xi(\beta x)), \tag{3.69}$$

where $\beta = \eta(0) - \eta(\xi(0))$. The scaling behaviour for golden mean rotation number circle maps ([81]) may be summarized as follows.

For parametrized families of diffeomorphisms f_ω, where ω governs the rotation number, we have so-called trivial scaling:

$$f^{F_n}(0) - F_{n-1} \sim \beta^n, \quad \beta = -\sigma \tag{3.70a}$$

$$\beta^{-n} f^{F_n}(\beta^n x) - p_n \longrightarrow R_\sigma(x) \tag{3.70b}$$

$$\omega_n \sim \delta^{-n}, \quad \delta = -1/\sigma^2\,, \tag{3.70c}$$

where $R_\sigma(x) = x + \sigma$ is a rotation through angle σ, whilst for families of cubic critical circle maps we have

$$f^{F_n}(0) - F_{n-1} \sim \beta^n, \quad \beta < 0 \tag{3.71a}$$

$$\beta^{-n} f^{F_n}(\beta^n x) - F_{n-1} \longrightarrow \phi(x) \tag{3.71b}$$

$$\omega_n \sim \delta^{-n} \tag{3.71c}$$

where β, δ, ϕ depend on the degree of the critical point d. For $d = 3$, the generic case of cubic-critical maps, we have:

$$\beta = -0.776... \tag{3.72a}$$

$$\delta = -2.833... \tag{3.72b}$$

and ϕ is an analytic function of x^3.

In [72] and [31] this scaling behaviour (observed in [81]) has been explained in terms of two fixed-points of the renormalization transformation T:- a linear, so-called trivial fixed-point:

$$(\xi_L(x), \eta_L(x)) = (x + \sigma, x - \sigma^2) \tag{3.73}$$

and a non-trivial, critical fixed-point (ξ_C, η_C) that depends on the order of the critical point d.

The existence of a critical fixed-point has been established first by computer-assisted means by de la Llave in the case of cubic maps, and then analytically for arbitrary degree [23]. Indeed, if the critical point is of order $d > 1$, then there is a non-trivial fixed-point pair (ξ, η) with

$$\xi(x) = E(x^{(d)}), \quad \eta(x) = F(x^{(d)}), \tag{3.74}$$

where E and F are analytic functions in a neighbourhood of 0 and $x^{(d)} = x|x|^{d-1}$. We note that, for d an odd integer, ξ and η are both themselves analytic functions.

One important remark is that both the trivial and critical fixed-points commute in a neighbourhood of 0. Moreover, restricted to the space of commuting pairs (ξ, η), the transformation T is hyperbolic at both fixed-points, each with a single essential eigenvalue $\delta > 1$. However, once commutativity is no longer imposed, hyperbolicity is lost and, as observed in [72], an eigenvalue -1 appears. This eigenvalue indicates the existence of a line of period-two points of T, corresponding to non-commuting circle maps pairs.

The non-critical non-commuting circle map pairs were studied in [66], in the context of understanding the scaling behaviour of implicit complex maps on the boundary of a golden-mean Siegel disc. It was observed that a line of period-two points did indeed exist through the trivial fixed-point (ξ_L, η_L). The period-two points were given by fractional-linear maps and were parametrized by an invariant 'modulus' μ given for a pair (ξ, η) by:

$$\mu = \frac{(\eta\xi)'(0)}{(\xi\eta)'(0)} \tag{3.75}$$

See also the work of Khanin and Vul [49]

Our idea is to extend the work in [66] to consider the case of critical non-commuting circle maps. The formula (3.75) has a natural generalization to degree-d maps where d is an odd integer:

$$\mu = \frac{(\eta\xi)^{(d)}(0)}{(\xi\eta)^{(d)}(0)} \tag{3.76}$$

We shall see below that this is indeed a invariant modulus for degree-d critical non-commuting circle-map pairs.

3.3.2 Renormalization analysis

Let $d > 1$ be fixed, not necessarily integral. We consider pairs (ξ, η) satisfying 1-4 above and which may be written as functions of $x^{(d)}$, as in (3.74). The functions E, F are increasing functions with normalization $E(0) - F(0) = 1$ and satisfying the conditions:

1. $F(0) < 0 < E(0)$
2. E defined on $[F(0)^{(d)}, 0]$
3. F defined on $[0, E(0)^{(d)}]$
4. $E(F(0)^{(d)}) = F(E(0)^{(d)})$

The renormalization transformation T given by (3.69) preserves the space of pairs of this form, and induces a transformation on the pairs (E, F), which we also denote by T. The transformation $T(E, F) = (\tilde{E}, \tilde{F})$ is given by

$$\tilde{E}(x) = -\beta^{-1} E(-|\beta|^d x), \quad \tilde{F}(x) = \beta^{-1} F(E(-|\beta|^d x)^{(d)}) \,. \tag{3.77}$$

In terms of E and F the compositions $\eta\xi$ and $\xi\eta$ are given by

$$\eta(\xi(x)) = H_1(x^{(d)}), \quad H_1(x) = F(E(x)^{(d)}) \,, \tag{3.78a}$$
$$\xi(\eta(x)) = H_2(x^{(d)}), \quad H_2(x) = E(F(x)^{(d)}) \,. \tag{3.78b}$$

We now define the modulus $s = s(\xi, \eta) = s(E, F)$ to be

$$s(E, F) = \frac{H_1'(0)}{H_2'(0)} \tag{3.79}$$

which, in view of the normalization $E(0) = 1$, can be written explicitly in terms of E and F as

$$s(E, F) = \frac{F'(E(0)^{(d)})|E(0)|^{d-1} E'(0)}{E'(F(0)^{(d)})|F(0)|^{d-1} F'(0)} \,. \tag{3.80}$$

We note that, for d an odd integer, this definition agrees with (3.76).

Proposition 9. *$s(T(E, F)) = s(E, F)^{-1}$.*

Proof.

$$\begin{aligned}\tilde{H}_1(x) &= \tilde{F}(\tilde{E}(x)^{(d)}) \\ &= \beta^{-1} F(E(F(-|\beta|^d x)^{(d)})^{(d)}) \\ &= \beta^{-1} F(H_2(-|\beta|^d x)^{(d)}) \,,\end{aligned} \tag{3.81}$$

and

$$
\begin{aligned}
\tilde{H}_2(x) &= \tilde{E}(\tilde{F}(x)^{(d)}) \\
&= \beta^{-1} F(F(E(-|\beta|^d x)^{(d)})^{(d)}) \\
&= \beta^{-1} F(H_1(-|\beta|^d x)^{(d)})\,.
\end{aligned} \tag{3.82}
$$

Differentiating at 0, we obtain

$$
s(\tilde{E},\tilde{F}) = \frac{\tilde{H}_1'(0)}{\tilde{H}_2'(0)} = \frac{H_2'(0)}{H_1'(0)} = s(E,F)^{-1} \tag{3.83}
$$

□

The fact that $s(T(E,F)) = s(E,F)^{-1}$ means that a fixed-point pair of T necessarily has $s(E,F) = 1$. For $s(E,F) \neq 1$, we will have $s(T^2(E,F)) = S(E,F)$, which is certainly consistent with a period-two point of T.

We make the following conjecture.

Conjecture.

1. *(Existence) For all $d > 1$ and all $\mu > 0$ there exists a solution pair (ξ,η) to the equation $T^2(\xi,\eta) = (\xi,\eta)$, with $\xi(x) = E(x^{(d)})$, $\eta(x) = F(x^{(d)})$, with $E'(0), F'(0) \neq 0$ and with $s(\xi,\eta) = s(E,F) = \mu$. Furthermore E and F are analytic on a neighbourhood of 0.*

2. *(Hyperbolicity) Restricted to the space of pairs (ξ,η) with $s(\xi,\eta) = \mu$, the period-two orbit is hyperbolic with a single unstable direction, i.e., the spectrum of $dT^2(\xi,\eta)$ consists of a single eigenvalue Δ, with $|\Delta| > 1$, and all other eigenvalues lie strictly within the unit circle.*

This conjecture is in line with the results for commuting circle maps, and we believe that it may be possible to use the Herglotz-function techniques of Epstein, and Lanford's computer-assisted-proof techniques to prove this conjecture.

A period-two point (ξ,η) satisfies the equations

$$
\xi = \tilde{\beta}^{-1}\tilde{\eta}\tilde{\beta}\,, \tag{3.84a}
$$

$$
\tilde{\xi} = \beta^{-1}\eta\beta\,, \tag{3.84b}
$$

$$
\eta = \tilde{\beta}^{-1}\tilde{\eta}\tilde{\xi}\tilde{\beta}\,, \tag{3.84c}
$$

$$
\tilde{\eta} = \beta^{-1}\eta\xi\beta\,, \tag{3.84d}
$$

with normalizations $\xi(0) - \eta(0) = \tilde{\xi}(0) - \tilde{\eta}(0) = 1$. Writing $\gamma = \beta\tilde{\beta}$, we have

$$\xi = \gamma^{-1}\eta\xi\gamma, \quad \eta = \gamma^{-1}\eta\xi\eta\gamma\,. \tag{3.85}$$

We now assume that the period-two point exists with $s(E, F) = \mu$. We have the following:

Proposition 10.

$$H_1(x) = H_2(\mu x), \quad \text{i.e.,} \quad \eta(\xi(x)) = \xi(\eta(\mu^{1/d}x))\,. \tag{3.86}$$

Proof. Re-writing (3.85) in terms of E and F, we have

$$H_1(x) = G(H_1(\gamma^d x))\,, \quad H_2(x) = G(H_2(\gamma^d x))\,, \tag{3.87}$$

where

$$G(x) = \gamma^{-1}\eta\xi\eta = \gamma^{-1}F(E(F(x^{(d)})^{(d)})^{(d)}) \tag{3.88}$$

so that both H_1 and H_2 conjugate the function G to the linear functions $\gamma^{-d}x$. Since $\mu \neq 0, \infty$, $H_1'(0)$, $H_2'(0) \neq 0$, so that we may invert H_2 about 0 to define $K = H_2^{-1}H_1$. Then K satisfies the equation

$$K(x) = \gamma^{-d}K(\gamma^d x) \tag{3.89}$$

around 0. Since H_1, H_2 are analytic around 0, so is K. It is a simple exercise to obtain that $K(x) = wx$, for some constant w. Using (3.79) and $s(E, F) = \mu$, we obtain $w = \mu$. It follows that $H_1(x) = H_2(\mu x)$ and that $\eta(\xi(x)) = \xi(\eta(\mu^{1/d}x)$. □

The following corollary follows from Propositions 9, 10 and the fact that the equations (3.84) are unchanged while replacing $(\tilde{\xi}, \tilde{\eta})$ by (ξ, η) and $\tilde{\beta}$ by β.

Corollary 5.

$$\tilde{\eta}(\tilde{\xi}(x)) = \tilde{\xi}(\tilde{\eta}(\mu^{-1/d}x))\,. \tag{3.90}$$

3.3.3 Extended domains of definition for ξ, η, $\tilde{\xi}$, $\tilde{\eta}$

Lemma 26. *$\beta, \tilde{\beta} < 0$ and $\eta(\xi(0)) > 0$, $\tilde{\eta}(\tilde{\xi}(0)) > 0$.*

Proof. Puting $x = 0$ in (3.84a) and (3.84b) we obtain $\xi(0) = \tilde{\beta}^{-1}\tilde{\eta}(0)$, $\tilde{\xi}(0) = \beta^{-1}\eta(0)$. But $\xi(0) > 0$, $\tilde{\eta}(0) < 0$ and $\tilde{\xi}(0) > 0$, $\eta(0) < 0$ by definition. It gives $\beta < 0$ and $\tilde{\beta} < 0$. Substituting $x = 0$ in (3.84c) and (3.84d) we obtain $\eta(0) = \tilde{\beta}^{-1}\tilde{\eta}(\tilde{\xi}(0))$, $\tilde{\eta}(0) = \beta^{-1}\eta(\xi(0))$. Thus, using the fact that $\eta(0), \tilde{\eta}(0) < 0$ and $\beta, \tilde{\beta} < 0$, we have $\tilde{\eta}(\tilde{\xi}(0)) > 0$ and $\eta(\xi(0)) > 0$. □

We know that $\xi(x)$ is defined for $x \in (\eta(0), 0)$ and $\tilde{\xi}(x)$ is defined for $x \in (\tilde{\eta}(0), 0)$. Furthermore, there exist $\eta_0 \in (0, \xi(0))$ and $\tilde{\eta}_0 \in (0, \tilde{\xi}(0))$ such that $\eta(\eta_0) = 0$ and $\tilde{\eta}(\tilde{\eta}_0) = 0$. Indeed, $\eta(x), \tilde{\eta}(x)$ are defined in domains $(0, \xi(0))$ and $(0, \tilde{\xi}(0))$ respectively, $\eta(0), \tilde{\eta}(0) < 0$ and $\tilde{\eta}(\tilde{\xi}(0)) > 0$, $\eta(\xi(0)) > 0$ by lemma 26.

Substituting in the hand raight side of (3.84a) the values of x in the order $x \to 0 \to \dfrac{\tilde{\eta}_0}{\tilde{\beta}} \to \dfrac{\tilde{\xi}(0)}{\tilde{\beta}}$ we see that $\xi(x)$ can be extended to the function defined in the interval $(\tilde{\xi}(0)/\tilde{\beta}, 0) \supset (\eta(0), 0)$ and that $\xi(\tilde{\eta}_0/\tilde{\beta}) = 0$. Repeating the same arguments for the function $\tilde{\xi}(x)$ using (3.84b) we obtain the following lemma

Lemma 27. *$\xi(x)$ can be extended to the interval $(\tilde{\xi}(0)/\tilde{\beta}, 0)$ and $\xi(\xi_0) = 0$ where $\xi_0 = \tilde{\eta}_0/\tilde{\beta}$.*
$\tilde{\xi}(x)$ can be extended to the interval $(\xi(0)/\beta, 0)$ and $\tilde{\xi}(\tilde{\xi}_0) = 0$ where $\tilde{\xi}_0 = \eta_0/\beta$.

We consider now the equation (3.84c)

$$\eta(x) = \tilde{\beta}^{-1}\tilde{\eta}(\tilde{\xi}(\tilde{\beta}x)), \quad x \in (0, \xi(0)), \tag{3.91}$$

with $\tilde{\eta}(x)$ being defined by definition only for $0 < x < \tilde{\xi}(0)$. Thus, we must have $\tilde{\beta}x \in (\tilde{\xi}_0, 0)$ in order to have $\tilde{\xi}(\tilde{\beta}x) > 0$. So, $\eta(x)$ is well defined for $x \in (0, \tilde{\xi}_0/\tilde{\beta})$. This proves, together with the similar argument applayed to $\tilde{\eta}(x)$, the following lemma

Lemma 28. *$\eta(x)$ can be extended to the interval $(0, \tilde{\xi}_0/\tilde{\beta}) \supset (0, \xi(0))$.*
$\tilde{\eta}(x)$ can be extended to the interval $(0, \xi_0/\beta) \supset (0, \tilde{\xi}(0))$

Below we summarize the known facts obtained during the proof of lemmas 27-28.

$$\eta_0 < \frac{\tilde{\xi}_0}{\tilde{\beta}}, \quad \tilde{\eta}_0 < \frac{\xi_0}{\beta}, \tag{3.92a}$$

$$\frac{\tilde{\xi}(0)}{\tilde{\beta}} < \eta(0), \quad \frac{\xi(0)}{\beta} < \tilde{\eta}(0), \tag{3.92b}$$

$$\xi_0 = \frac{\tilde{\eta}_0}{\tilde{\beta}}, \quad \tilde{\xi}_0 = \frac{\eta_0}{\beta}, \tag{3.92c}$$

$$\frac{\tilde{\xi}_0}{\tilde{\beta}} < \xi(0), \quad \frac{\xi_0}{\beta} < \tilde{\xi}(0). \tag{3.92d}$$

$$\tag{3.92e}$$

Using again the equations (3.84a), (3.84b) and the lemma 28 we prove the following

Lemma 29. *$\xi(x)$ can be extended to the interval $(\xi_0/\beta\tilde{\beta}, 0)$.*
$\tilde{\xi}(x)$ can be extended to the interval $(\tilde{\xi}_0/\beta\tilde{\beta}, 0)$

Theorem 37. *The intervals of analyticity of functions $\xi(x), \tilde{\xi}(x), \eta(x), \tilde{\eta}(x)$ given by lemmas 28, 29 are the maximal open intervals on which these functions are strictly monotonous.*

Proof. Differentiating the both sides of the equations (3.84) we obtain

$$\xi'(x) = \tilde{\eta}'(\tilde{\beta}x), \tag{3.93a}$$

$$\tilde{\xi}'(x) = \eta'(\beta x), \tag{3.93b}$$

$$\eta'(x) = \tilde{\xi}'(\tilde{\beta}x)\tilde{\eta}'(\tilde{\xi}(\tilde{\beta}x)), \tag{3.93c}$$

$$\tilde{\eta}'(x) = \xi'(\beta x)\eta'(\xi(\beta x)). \tag{3.93d}$$

Substituting $x = \tilde{\xi}_0/\tilde{\beta}$ in (3.93c), $x = \xi_0/\beta$ in (3.93d) we obtain that $\eta'(\tilde{\xi}_0/\tilde{\beta}) = 0$ and $\tilde{\eta}'(\xi_0/\beta) = 0$ since $\xi'(0) = \tilde{\xi}'(0) = 0$ by definition. Equations (3.93a), (3.93b) shows then that $\xi'(\xi_0/\beta\tilde{\beta}) = \tilde{\xi}'(\xi_0/\beta\tilde{\beta}) = 0$. One shows also, with help of (3.93), that the derivatives of $\xi(x), \tilde{\xi}(x), \eta(x), \tilde{\eta}(x)$ are positive in these intervals. □

Lemma 30. $0 < \beta\tilde{\beta} < 1$

Proof. According to (3.92a) we have $\eta_0 < \tilde{\xi}_0/\tilde{\beta} = \eta_0/\beta\tilde{\beta}$ which implies $1 < 1/\beta\tilde{\beta}$. Using lemma 26 we obtain $0 < \beta\tilde{\beta} < 1$. □

We recall that

$$\beta = \eta(0) - \eta(\xi(0)), \quad \xi(0) - \eta(0) = 1, \tag{3.94a}$$

$$\tilde{\beta} = \tilde{\eta}(0) - \tilde{\eta}(\tilde{\xi}(0)), \quad \tilde{\xi}(0) - \tilde{\eta}(0) = 1 \tag{3.94b}$$

or

$$\beta = \xi(0) - 1 - \xi(\xi(0) - 1), \quad \tilde{\beta} = \tilde{\xi}(0) - 1 - \tilde{\xi}(\tilde{\xi}(0) - 1). \tag{3.95a}$$

Using the equations (3.84a), (3.84b) at $x = 0$ we find

$$\xi(0) = \tilde{\beta}^{-1}\tilde{\eta}(0) = \tilde{\beta}^{-1}(\tilde{\xi}(0) - 1), \quad \tilde{\xi}(0) = \beta^{-1}\eta(0) = \beta^{-1}(\xi(0) - 1)\,, \tag{3.96a}$$

which gives

$$\beta = \frac{\xi(0) - 1}{\tilde{\xi}(0)}, \quad \tilde{\beta} = \frac{\tilde{\xi}(0) - 1}{\xi(0)}, \tag{3.97a}$$

$$\xi(0) = \frac{1 + \beta}{1 - \beta\tilde{\beta}}, \quad \tilde{\xi}(0) = \frac{1 + \tilde{\beta}}{1 - \beta\tilde{\beta}}. \tag{3.97b}$$

From these formulas and lemma 30 the following is easy to prove

Lemma 31. $-1 < \beta < 0$, $-1 < \tilde{\beta} < 0$.

3.3.4 Reformulation of the problem

In this section we shall prove that the equations (3.84) which imply in turn (3.86), (3.90) are equivalent to the following system

$$\xi = \tilde{\beta}^{-1}\tilde{\eta}\tilde{\beta}\,, \tag{3.98a}$$

$$\tilde{\xi} = \beta^{-1}\eta\beta\,, \tag{3.98b}$$

$$\eta = \tilde{\beta}^{-1}\tilde{\xi}\tilde{\eta}\tilde{\beta}\mu^{-1/d}\,, \tag{3.98c}$$

$$\tilde{\eta} = \beta^{-1}\xi\eta\beta\mu^{1/d}\,, \tag{3.98d}$$

with normalizations $\xi(0) - \eta(0) = \tilde{\xi}(0) - \tilde{\eta}(0) = 1$. First, we note that the equations (3.98a), (3.98b) and (3.84a), (3.84b) are the same. The equations

(3.98c), (3.98d) follow from (3.86) and (3.90). Thus, the system (3.98) follows from (3.84).

Substituting (3.98d), (3.98b) into (3.98c) we obtain

$$\eta = \gamma^{-1}\eta\xi\eta\gamma\,, \tag{3.99}$$

where $\gamma = \beta\tilde{\beta}$. Expressing $\tilde{\eta}$ from (3.98a) and substituting into (3.98d) we get

$$\xi = \gamma^{-1}\xi\eta\gamma\mu^{1/d}\,. \tag{3.100}$$

The equation (3.99) gives: $\eta\xi = \gamma^{-1}\eta\xi\eta\gamma\xi$. Which can be written as follows

$$V_1(x) = \gamma^{-1}V_1(G_1(x)))\,, \tag{3.101}$$

where $V_1 = \eta\xi$, $G_1 = \eta\gamma\xi$. From the equation (3.100) we obtain: $\xi\eta\mu^{1/d} = \gamma^{-1}\xi\eta\gamma\mu^{1/d}\eta\mu^{1/d}$. Which can be written in the form

$$V_2(x) = \gamma^{-1}V_2(G_2(x))\,, \tag{3.102}$$

where $V_2 = \xi\eta\mu^{1/d}$, $G_2 = \gamma\eta\mu^{1/d}$. The following equalities can be easy established with help of (3.99), (3.100)

$$G_1 = G_2 = \eta\xi\eta\gamma\mu^{1/d}\,. \tag{3.103}$$

Then, it follows from (3.101) and (3.102) that $V_1(x) = cV_2(x)$ for a certain constant $c \neq 0$. The condition $\xi\eta(0) = \eta\xi(0)$ implies $c = 1$ i.e. $V_1(x) = V_2(x)$ or $\eta\xi = \xi\eta\mu^{1/d}$. Thus, we have proved that the commutative relations (3.86) (and hence (3.90)) follow directly from the system (3.98). Applying them to the equations (3.98c) and (3.98d) we derive the equations (3.84c) and (3.84d). This finishes the proof that the systems (3.84) and (3.98) are equivalent.

It will be useful for us to introduce the new functions $e = \alpha^{-1}\xi\alpha$, $\tilde{e} = \alpha^{-1}\tilde{\xi}\tilde{\alpha}$, $h = \alpha^{-1}\eta\alpha$, $\tilde{h} = \alpha^{-1}\tilde{\eta}\tilde{\alpha}$ where $\alpha = \xi(0)^{-1}$, $\tilde{\alpha} = \tilde{\xi}(0)^{-1}$ with the new scaling

$$e(0) = \tilde{e}(0) = 1. \tag{3.104}$$

The equations (3.98) take the form

$$e = -\tilde{\lambda}^{-1}\tilde{h}(-\tilde{\lambda}x)\,, \tag{3.105a}$$

$$\tilde{e} = -\lambda^{-1}h(-\lambda x)\,, \tag{3.105b}$$

$$h = -\tilde{\lambda}^{-1}\tilde{e}(\tilde{h}(-\tilde{\lambda}\mu^{-1/d}x))\,, \tag{3.105c}$$

$$\tilde{h} = -\lambda^{-1}e(h(-\lambda\mu^{1/d}x))\,, \tag{3.105d}$$

where $\lambda = -\tilde{\alpha}\alpha^{-1}\beta > 0$, $\tilde{\lambda} = -\alpha\tilde{\alpha}^{-1}\tilde{\beta} > 0$. Using (3.105a) (3.105b) to express $h, \tilde{h}$ in terms of $e, \tilde{e}$, we obtain from (3.105c), (3.105d) the following equations which contain $e, \tilde{e}$ only

$$e(x) = (\lambda\tilde{\lambda})^{-1} e(-\lambda\tilde{e}(-\tilde{\lambda}\mu^{1/d}x)), \tag{3.106a}$$

$$\tilde{e}(x) = (\lambda\tilde{\lambda})^{-1} \tilde{e}(-\tilde{\lambda}e(-\lambda\mu^{-1/d}x)). \tag{3.106b}$$

With help of equations (3.84) we obtain

$$\xi(\xi_0/\beta\tilde{\beta}) = \tilde{\xi}(0)/\tilde{\beta}, \quad \tilde{\xi}(\tilde{\xi}_0/\beta\tilde{\beta}) = \xi(0)/\beta. \tag{3.107}$$

The function $e(x) = \alpha^{-1}\xi(\alpha x) = \xi(0)^{-1}\xi(\xi(0)x)$ is monotonically increasing in the interval $A = \xi_0/(\beta\tilde{\beta}\xi(0)) < x < 0)$ such that $e(A) = \tilde{\xi}(0)/(\tilde{\beta}\xi(0)) = -\tilde{\lambda}^{-1} < 0$ and $e(0) = 1$. Analogously, the function $\tilde{e}(x) = \tilde{\alpha}^{-1}\tilde{\xi}(\tilde{\alpha}x) = \tilde{\xi}(0)^{-1}\tilde{\xi}(\tilde{\xi}(0)x)$ is monotonically increasing in the interval $\tilde{A} = \tilde{\xi}_0/(\beta\tilde{\beta}\tilde{\xi}(0)) < x < 0)$ such that $\tilde{e}(\tilde{A}) = \xi(0)/(\beta\tilde{\xi}(0)) = -\lambda^{-1} < 0$ and $\tilde{e}(0) = 1$.

Let $e(x) = K(x^{(d)})$, $\tilde{e}(x) = \tilde{F}(x^{(d)})$. Then $e^{-1}(x) = (K^{-1}(x))^{(1/d)}$, $\tilde{e}^{-1}(x) = (\tilde{K}^{-1}(x))^{(1/d)}$. Let $U(x) = -K^{-1}(x)$ and $\tilde{U}(x) = -\tilde{K}^{-1}(x)$, then the equations (3.106) can be written in the form

$$U(x) = -\mu^{-1}\tilde{\lambda}^{-d}\tilde{U}(\lambda^{-1}U^{1/d}(\lambda\tilde{\lambda}x)), \quad x \in (-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1}), \tag{3.108a}$$

$$\tilde{U}(x) = -\mu\lambda^{-d}U(\tilde{\lambda}^{-1}\tilde{U}^{1/d}(\lambda\tilde{\lambda}x)), \quad x \in (-\lambda^{-1}, (\lambda\tilde{\lambda})^{-1}). \tag{3.108b}$$

3.4 Open problems and perspectives

Firstly, as already mentioned on p. 82, the lower bound (3.6) for δ has to be improved so that in the limit $\mu \to 1$ it gives $1/\lambda^d - 1/\lambda$ as obtained in [22] for the symmetric period-doubling case. That can be done only by improving the *a priori* bounds (3.12)–(3.13) for anti-Herglotz functions. As shown in the previous chapter (see [T4]) the more precise bounds can be derived via continued fractions technique. That would be interesting, using the same anti-Herglotz approach, to obtain more information on λ, $\tilde{\lambda}$ then $\mu \to 1$ or $d \to \infty$. This work was initiated in [69].

The numerical simulations make clear the fact that the fixed point operator T (see [65] for details) written for (3.10) is actually a contraction in the functional space of pairs $(\psi, \tilde{\psi})$. No rigorous statements are available in this direction yet. Our conjecture is that, by a natural analogy with dimension one holomorphic endomorphisms of the unit disk, this contraction can be explained by introducing the Kobayashi distace D on a suitable domain of $(\psi, \tilde{\psi})$-space so that D is shrinking by T. On this way one can proof the uniqueness of the fixed point of the renormalization operator and at the same time the continuity of the universal constants α and δ on μ and d.

One can start here by looking at more simple symmetric case, where the truncated form of (3.54) provides a good rational approximation for the fixed point $u(x)$ and can be parametrized by coefficients $g_1, \ldots, g_N$ for $N \in \mathbb{N}$. One then applies the usual Newton method to solve (3.51). We have numerical suggestions that for any $d > 1$ and for quite small N, this method gives excellent approximations for the fixed point function $u(x)$ and the universal parameter λ.

Finally, for the functional equations (3.108), the existence of solutions has to be proved using the Epstein method based on the anti–Herglotz functions technique.

Chapter 4

Convergence of g-fractions fractions at Runckel's points and the Ramanujan's conjecture

4.1 Introduction

Let $\mathcal{E}$ be the class of analytic self maps of the unit disc $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$. For $e \in \mathcal{E}$ we introduce, following Schur [84], the parameters $\{t_i\}_{i=0}^{\infty}$, $|t_i| \leq 1$ of $e(w)$ as follows

$$e_0(w) = e(w), \quad t_0 = e_0(0), \quad e_{n+1}(w) = \frac{1}{w}\frac{e_n(w) - t_n}{1 - \bar{t}_n e_n(w)}, \quad t_{n+1} = e_{n+1}(0)\,.$$

The recursively defined functions

$$[w; t_i] = t_i, \quad [w; t_l, \ldots, t_k] = \frac{t_l + w[w; t_{l+1}, \ldots, t_k]}{1 + \bar{t}_l w[w; t_{l+1}, \ldots, t_k]}, \quad 0 \leq l < k\,,$$

provide then representation of $e(w)$

$$e(w) = \lim_{n\to\infty} [w; t_0, \ldots, t_n]\,,$$

convergent uniformly over every compact subset of $\mathbb{D}$.
Let

$$s_i(w; t) = t_i + \frac{(1 - |t_i|^2)w}{\bar{t}_i w + t^{-1}}, \quad i = 0, 1, \ldots\,,$$

be an infinite sequence of Möbius transformations of the variable t. We define $S_p(w;t) = s_0 \circ s_1 \circ \cdots \circ s_p(w;t)$. The following formula can be easy derived: $[w; t_0, \ldots, t_n] = S_{n-1}(w; t_n)$, $n \geq 1$ and thus $e(w) = \lim_{n\to\infty} S_{n-1}(w; t_n)$.

Conversely with each sequence of complex numbers t_i with $|t_i| < 1$, $i \geq 0$ one can associate an analytic function $e(w)$ in $\mathbb{D}$ with $\sup_{|w|<1} |e(w)| \leq 1$ such that t_i are just as above.

We denote by $\mathcal{W}$ the set of continued g-fractions

$$g(z) = \frac{1}{1-}\,\frac{g_1 z}{1-}\,\frac{g_2(1-g_1)z}{1-}\,\frac{g_3(1-g_2)z}{1-\cdots}, \quad z \in \mathbb{C}, \quad g_i \in (0,1)\,. \tag{4.1}$$

Definition. Let $h_i(z;t) = \dfrac{a_i}{1+t}$, $i \geq 0$ be an infinite sequence of Möbius transformations of the variable t with $a_0 = 1$, $a_1 = -zg_1$, $a_i = -g_i(1-g_{i-1})z$, $i \geq 2$ and let $H_n(z;t) = h_0 \circ h_1 \circ \cdots \circ h_n(z;t)$. Then the continued fraction (4.1) is called convergent at the point $z \in \mathbb{C}$ if the limits

$$\lim_{n\to\infty} H_n(z;0) = \lim_{n\to\infty} H_n(z;\infty)\,,$$

exist in the extended complex plane $\bar{\mathbb{C}}$.

As shown in [92, p. 279], the continued fraction (4.1) converges for all z from the domain $C = \mathbb{C}_- \cup \mathbb{C}_+ \cup (-\infty, 1)$ to an analytic function $g(z)$ with the property $\mathrm{Re}(\sqrt{1-z}\, g(z)) > 0$, $\forall\, z \in C$.

The Definition 4.1 of convergence is a classical one and is rather unsatisfactory as one can imagine situations then $H_n(z;t)$ may converge at many t, but perhaps not when t is 0 or ∞ (see [13, p. 565]). The following refined definition of convergence is due to Lisa Jacobsen [42, p. 480].

Definition. The continued fraction (4.1) converges generally to value $\alpha \in \bar{\mathbb{C}}$ at $z \in \bar{\mathbb{C}}$ if there exist sequences u_n and v_n in $\bar{\mathbb{C}}$ such that

$$\lim_{n\to\infty} H_n(z;u_n) = \lim_{n\to\infty} H_n(z;v_n) = \alpha, \quad \liminf_{n\to\infty} \sigma(u_n, v_n) > 0\,,$$

where $\sigma(x,y)$ is a chordal distance between $x, y \in \bar{\mathbb{C}}$.

One can see arbitrary Möbius transformations $l_n(z)$ as isometries of the hyperbolic space $\mathbb{H}^3$ [13, p. 559]. Then the above definition is fully justified by the following geometric result due to Beardon

Theorem 38 ([13])**.** *A sequence l_n of Möbius maps converges generally to $\alpha \in \bar{\mathbb{C}}$ iff $l_n \to \alpha$ pointwise on $\mathbb{H}^3$.*

A continued fraction convergent in the classical sense always converges generally to the same value (one puts $u_n = 0$, $v_n = \infty$, $n \geq 0$ in Definition 4.1). The next theorem describes the correspondence between functions from the classes $\mathcal{E}$ and $\mathcal{W}$

Theorem 39 ([92])**.** *Let*

$$w = -1 + 2(1 - \sqrt{1-z})/z\,, \tag{4.2}$$

be the conformal mapping of C onto $\mathbb{D}$ with a positive square root branch for $z < 1$. To every function $e(w) \in \mathcal{E}$ corresponds a function $g(z) \in \mathcal{W}$ according to

$$\frac{1+w}{1-w}\frac{1-we(w)}{1+we(w)} = g(z)\,,$$

where the coefficients t_i and g_i are related by $t_{k-1} = 1 - 2g_k$, $k \geq 1$.

4.2 The Runckel's points

Let t_i, $i = 0, 1, \dots$ be a sequence of real numbers with $|t_i| < 1$. We assume that

$$\sum_{i=0}^{\infty} t_i^2 < \infty\,, \tag{4.3}$$

and therefore

$$\lim_{i\to\infty} t_i = 0\,. \tag{4.4}$$

Let $E \subset \mathcal{E}$ be the subset of functions $e(w)$ whose parameters t_i satisfy the above conditions.

Definition. A point $r \neq \pm 1$, $|r| = 1$ is a Runckel's point for $e(w) \in E$ if the limit $e(r) = \lim\limits_{n\to\infty} [r; t_0, \dots, t_n]$ exists and is equal to 1.

We note that, according to Runckel [79, p. 98], if in addition to (4.3), (4.4) there exists a natural p such that $\sum\limits_{i=0}^{\infty} |t_{i+p} - t_i| < +\infty$, then, as $k \to \infty$, $[w; t_0, \dots, t_k]$ converges uniformly over every compact subset of $|w| \leq 1$, $w^p \neq 1$ to $e(w)$ (analytic in $\mathbb{D}$) continuous and $|e(w)| < 1$ for all w in $|w| \leq 1$, $w^p \neq 1$. Thus, in this particular case, every Runckel's point r satisfies $r^p = 1$. We will see some examples of these functions in the next Section.

The following result concerns the general convergence of $g(z)$ at Runckel's points.

Theorem 40 ([T5])**.** *For $e(w) \in E$ let r be its Runckel's point and $g(z) \in \mathcal{W}$ is the corresponding g-fraction given by Theorem 39. Then $g(z)$ converges generally at the point*

$$z_r = 2(1 + \mathrm{Re}(r))^{-1} > 1\,, \tag{4.5}$$

called also a Runckel's point of $g(z)$.

This proof is based on the relation between the partial approximants of the fractions $e(z)$ and $g(z)$ (see Theorem 78.1, [92])

The sequence $H_n(z_r; 0)$, $n \geq 1$ can be still divergent in $\bar{\mathbb{C}}$ i.e $g(z)$ being divergent at $z = z_r$ in the classical sense. Nevertheless we have the following result.

Lemma 32. *One has*

$$\min\{|H_n(z_r; 0) - 1|, |H_{n+1}(z_r; 0) - 1|\} \to 0, \quad \text{as} \quad n \to \infty\,. \tag{4.6}$$

In our proof we follow the Beardon's geometric method described in [13]. Let $\gamma_{0,\infty}$ be the vertical geodesic in $\mathbb{H}^3$ with endpoints 0 and ∞, let γ_{u_n,v_n} be the geodesic with endpoints u_n and v_n. One checks that the hyperbolic distance d between $\gamma_{0,\infty}$ and γ_{u_n,v_n} always satisfies $d > 0$. Since $H_n(z_r; t)$ are isometries of $\mathbb{H}^3$, the distance between $H_n(z_r; \gamma_{0,\infty})$ and $H_n(z_r; \gamma_{u_n,v_n})$ is also d.

As shown in the proof of Theorem 40 (see [T5] for details) there exist points u_n, v_n with the property

$$\lim_{n\to\infty} H_n(z_r; u_n) = \lim_{n\to\infty} H_n(z_r; v_n) = 1\,,$$

as $n \to \infty$.

Thus, the geodesics $H_n(z_r; \gamma_{u_n,v_n})$ shrink to the point $1 \in \bar{\mathbb{C}}$ and hence $\min\{|H_n(z_r; 0) - 1|, |H_n(z_r; \infty) - 1|\} \to 0$ as $n \to \infty$ that gives (4.6).

We denote $\mathrm{dist}(X, Y)$ the Euclidean distance between two subsets $X, Y \subset \bar{\mathbb{C}}$.

The following theorem describes the possible limit behavior of the sequence $H_n(z_r; 0)$ as $n \to \infty$.

Theorem 41 ([T5])**.** $\lim_{n\to\infty} \mathrm{dist}(H_n(z_r; 0), \{1, \left(\frac{1+r}{1-r}\right)^2\}) = 0$

4.3 The classical convergence case

It is interesting to have an example of function $e(w) \in E$ with the classical convergence at its Runckel's points.

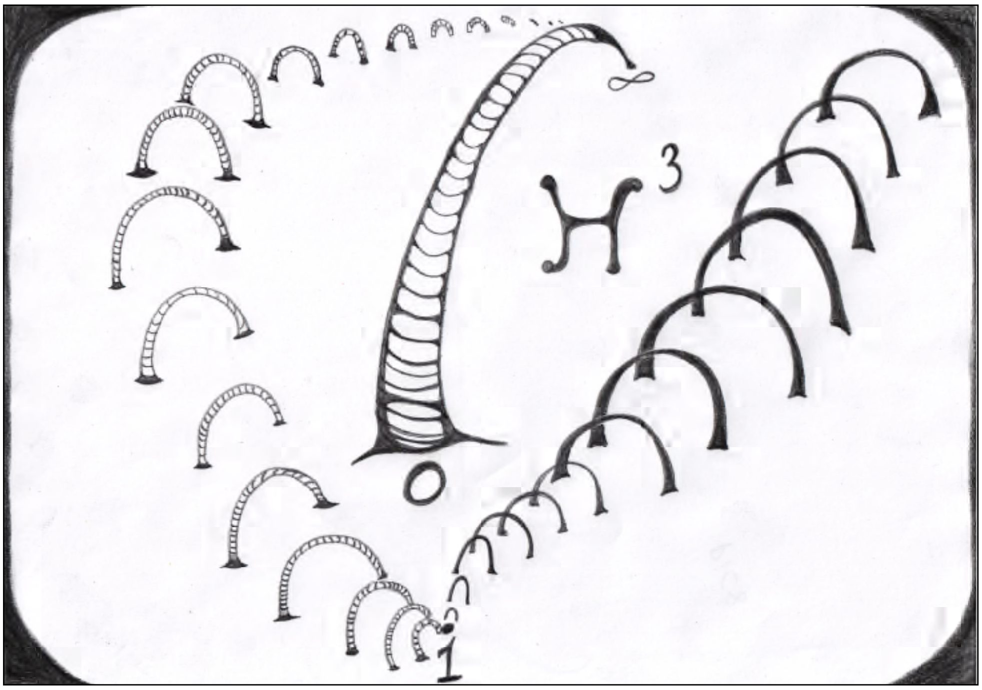

Figure 4.1: The geodesics in the proof of Lemma 32.

We define $e_p(w) = (1 + w^p)/2$, $p \in \mathbb{N}$. Then, as shown in [84, p. 142] (see also [79, p. 106]),

$$\begin{aligned} &e_p(w) = [w; t_0, 0, \ldots, 0, t_1, \ldots, 0, t_2, 0, \ldots], \\ &t_0 = 1/2, \quad t_n = 2/(2n+1), \quad n \geq 1\,, \end{aligned} \tag{4.7}$$

where $p - 1$ zeros are added between t_n and t_{n+1}.
According to Schur, the function $e_p(w) \in E$ given by (4.7) is continuous in $|w| \leq 1$ (with $\lim\limits_{n\to\infty} [w; t_0, \ldots, t_n]$ existing in $\mathbb{C}$) with Runckel's points w given by the roots of $w^p = 1$, $w \neq \pm 1$. The coefficients g_i of the corresponding to $e_p(w)$ continued fraction $g_p(z)$ are defined by relations $g_k = (1 - t_{k-1})/2$ and

$$g_1 = \frac{1}{4}, \quad g_{1+l} = \frac{1}{2}, \quad l \not\equiv 0 \bmod (p), \quad g_{1+pk} = \frac{2k-1}{2(2k+1)}, \quad k \geq 1\,.$$

Introducing the parameters $b_1 = g_1$, $b_i = g_i(1 - g_{i-1})$, $i \geq 2$ one obtains

$$b_1 = \frac{1}{4}, \quad b_2 = \frac{3}{8}, \quad b_{1+l} = \frac{1}{4}, \quad l \not\equiv 0, 1 \bmod (p), \quad b_{1+pk} = \frac{1}{4}\frac{2k-1}{2k+1},$$

$$b_{2+pk} = \frac{1}{4}\frac{2k+3}{2k+1}, \quad k \geq 1\,.$$

In particular $\lim\limits_{i\to\infty} b_i = 1/4$.

The continued fractions $g_p(z)$ has the Runckel's point $z_p = \cos^{-2}(\pi/p) > 1$ corresponding to $r = \exp(2\pi i/p)$ – the Runckel's point of $e_p(w)$.

Repeating the arguments used in the proof of Theorem 41 it is easy to show (see [T5] for details) that for every odd $p > 1$ the limit periodic continued fractions

$$g_p(z_p) = \frac{1}{1-}\,\frac{b_1 z_p}{1-}\,\frac{b_2 z_p}{1-}\,\frac{b_3 z_p}{1-\cdots}, \quad \lim_{i\to\infty} b_i z_p = z_p/4 > 1/4\,, \tag{4.8}$$

converges to 1 and is divergent by oscillations if p is even.

In the simplest case $p = 3$ the above continued fraction takes the form

$$\frac{1}{1-}\,\frac{1}{1-}\,\frac{3/2}{1-}\,\frac{1}{1-}\,\frac{1/3}{1-}\,\frac{5/3}{1-}\,\frac{1}{1-}\,\frac{3/5}{1-}\,\frac{7/5}{1-}\,\frac{1}{1-}\,\frac{5/7}{1-}\,\frac{9/7}{1-}\,\frac{1}{1-}\,\frac{7/9}{1-}, \tag{4.9}$$

and converges to 1.

For a general limit periodic continued fraction

$$\frac{1}{1-}\,\frac{a_1}{1-}\,\frac{a_2}{1-}\,\frac{a_3}{1-\cdots}, \quad \lim_{i\to\infty} a_i = a\,, \tag{4.10}$$

Ramanujan [1, pp. 38-39] stated without proof that it is convergent or not according as $a <$ or $> 1/4$. The convergence in the case $a < 1/4$ was proved in 1904 by Van Vleck [90]. If $a > 1/4$ then (4.10) diverges if a_i tends to a "fast enough" as was shown by Gill in 1973 who also proved [38] that (4.10) may converge if one allows complex a_i. Our continued fraction (4.9) provides an explicit example of convergence with real elements $a_i > 0$ and $\lim\limits_{i\to\infty} a_i = a > 1/4$. Existence of other examples, coming from a different approach, was reported by A. A. Glutsyuk [39].

4.4 Current research. Continued g –fractions and the Sundman power series solution to the 3-body problem

The problem stated by Weierstrass in 1880's asks for a method of constructing of power series solution of the Newtonian 3-body problem converging on the entire t-axis. Karl Sundman showed in 1912 that it is always possible for solutions of non-zero angular momentum but the convergence of the corresponding power series was so slow that they are really useless in practice. We try to suggest an alternative representation of these solutions within the framework of the analytic theory of continued fractions. Based primarily on rigidity properties of bounded holomorphic maps, this approach may provide a better capture of the global dynamics.

By the *Stieltjes summability* one means a general procedure of assigning to a given power series (which can be divergent) a certain convergent continued fraction. Stieltjes worked out a large number of examples in this domain which the interested reader can find in his *Ouevres*, vol. 2, pp. 184-200, 378-391, (see also [92] pp. 362-376). The idea of the Stieltjes summability in the particular case of g-fractions works particulary well. Expanding the function $\phi(z) = g(-z)$, with $g(z)$ given by (4.1), in a Taylor series convergent in the disk $|z| < 1$, we get

$$\phi(z) = 1 - A_1(g_1)\, z + A_2(g_1, g_2)\, z^2 - \cdots + (-1)^n A_n(g_1, g_2, \ldots, g_n)\, z^n + \cdots \,, \tag{4.11}$$

where the coefficients A_k, $k \geq 1$ are recursively defined polynomials on g_i, $1 \leq i \leq k$.

At the same time, expanding the Stieltjes integral (3.41), which represents $\phi(z)$ in the integral form, about the same point $z = 0$, we obtain

$$\phi(z) = 1 - \mu_1\, z + \mu_2\, z^2 - \cdots + (-1)^n \mu_n\, z^n + \cdots \,, \tag{4.12}$$

where $\mu_n \in [0, 1]$ (moments of μ) are defined by $\mu_n = \int_0^1 u^n d\mu(u)$, $n \geq 1$. We note that in general the series (4.12) diverges for $|z| > 1$.

Knowing the derivatives μ_i of all orders of $\phi(z)$ at $z = 0$ one calculates the coefficients g_i obtaining thus the analytic continuation of $\phi(z)$ through the whole domain $\mathbb{H}$ given by the continued fraction (4.1). In particular

$$g_1 = \mu_1, \quad g_2 = \frac{\mu_2 - \mu_1^2}{\mu_1(1 - \mu_1)}, \quad g_3 = \frac{(1 - \mu_1)(\mu_2^2 - \mu_3\mu_1)}{(\mu_2 - \mu_1)(\mu_2 - \mu_1^2)} \,. \tag{4.13}$$

The coefficients g_i, $i \geq 1$ of all orders can be found with help of the Stieltjes recursive formulas (see [92], p. 203).

Our aim now is to obtain the representations for g-fractions up to some conformal changes of the argument z. In this concern, let us consider the holomorphic automorphisms of $\mathbb{H} = \mathbb{C}_+ \cup \mathbb{C}_- \cup (-1, +\infty)$ given by the following Möbius transformations

$$f_\alpha(z) = \alpha z + \alpha - 1, \quad f_\beta(z) = \frac{\beta - z}{1 + z},$$

where $\alpha > 0$, $\beta > -1$. Let $f_\alpha^{-1}(z)$ and $f_\beta^{-1}(z)$ be their inverse functions and $f(z)$ be a Wall function (see Definition 3.2.2). Then the compositions $f_1 = f(f_\alpha^{-1}(z))$ and $f_2 = f(f_\beta^{-1}(z))$ are again Wall functions. Using the known properties of g–fractions (see [92]) one proves that

$$\{g_1, g_2, \ldots |z\} = C_1 \{p_1, p_2, \ldots |\alpha z + \alpha - 1\}, \quad C_1 > 0, \quad \alpha > 0,\ g_i, p_i \in [0,1]\,, \tag{4.14}$$

and

$$\{g_1, g_2, \ldots |z\} = \frac{C_2}{1+z} \{q_1, q_2, \ldots |\frac{\beta - z}{1+z}\}, \quad C_2 > 0,\ \beta > -1,\ g_i, q_i \in [0,1]\,. \tag{4.15}$$

The next result gives a simple characterization of functions bounded in the complex strip with help of g–fractions.

Proposition 11. *Let $f(z)$ be a function satisfying the following conditions:*

1. *$f(z)$ is holomorphic in the strip $S_B = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < B\}$,*

2. *$f(\mathbb{R}) \subset \mathbb{R}$,*

3. *$|f(z)| < M$ for all $z \in S_B$.*

Then for some $\mu_0 > 0$, $g_k \in [0,1]$, $k \geq 1$ one has

$$\begin{aligned} f(z) &= M \tanh\left(\Omega(z) - \tfrac{\pi z}{4B}\right)\,, \\ \Omega(z) &= \log\sqrt{\mu_0 \{g_1, g_2, \ldots \big|\exp(-\tfrac{\pi z}{B}) - 1\}}, \quad z \in S_B\,. \end{aligned} \tag{4.16}$$

Proof. We define the following domains

$$\mathbb{D}_M = \{z \in \mathbb{C} : |z| < M\}, \quad \mathbb{H}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}\,, \tag{4.17}$$

and consider the conformal maps $\theta : \mathbb{D}_M \to \mathbb{H}$, $\phi : \mathbb{H} \to S_B$ given by

$$\theta(z) = \frac{M + z}{M - z}, \quad \phi(z) = \frac{B}{\pi} \log(1 + z)\,. \tag{4.18}$$

One verifies that the composition $F = \theta \circ f \circ \phi$ is a function holomorphic in $\mathbb{H}$ such that $F(\mathbb{H}) \subset \mathbb{H}_+$ and $F(z)$ is real for $z > -1$. $F(z)$ can be written in the form

$$F(z) = \mu_0 \sqrt{1+z} \int_0^1 \frac{d\mu(u)}{1+zu} \,. \tag{4.19}$$

for a certain nondecreasing bounded function $\mu(u)$ with support $\mathrm{supp}\mu = [0,1]$.

$$\mu(u) = \begin{cases} 0 & \text{for } u \le 0\,, \\ \frac{\mu(u-0)+\mu(u+0)}{2} & \text{for } 0 < u < 1\,, \\ 1 & \text{for } u > 1\,. \end{cases}$$

We have for $f = \theta^{-1} \circ F \circ \phi^{-1}$

$$f(z) = M \left(1 - \frac{2}{\exp(\frac{\pi z}{2B})\mu_0 \int_0^1 \frac{d\mu(u)}{1+(\exp(\frac{\pi z}{B})-1)u} + 1} \right) \,. \tag{4.20}$$

One writes the integral in the last formula as g-fraction

$$\int_0^1 \frac{d\mu(u)}{1+uz} = \{\tilde{g}_1, \tilde{g}_2, \ldots | z\} \,, \tag{4.21}$$

for some $\tilde{g}_i \in [0,1]$, $i \ge 1$.
Taking the transformation (4.15) with $\beta = 0$, $C_2 = 1$ (see also [92], p. 281)

$$\{\tilde{g}_1, \tilde{g}_2, \ldots | z\} = \frac{1}{1+z} \left\{ g_1, g_2, \ldots \middle| \frac{-z}{1+z} \right\}, \quad g_i \in [0,1], \ge 1\,, \tag{4.22}$$

we obtain

$$\begin{aligned} &\exp\left(\tfrac{\pi z}{2B}\right) \int_0^1 \frac{d\mu(u)}{1+(\exp\left(\frac{\pi z}{B}\right)-1)u} = \\ &= \mu_0 \exp\left(-\tfrac{\pi z}{2B}\right) \{g_1, g_2, \ldots | \exp\left(-\tfrac{\pi z}{B}\right) - 1\} \end{aligned} \tag{4.23}$$

that together with (4.20) implies (4.16). The proof is complete.

□

With help of (4.13) we can find a connection between $\mu_0 > 0$, $g_1, g_2 \in [0,1]$ and the derivatives $X = f(0)$, $Y = f'(0)$, $Z = f''(0)$

$$X = M\tfrac{\mu_0-1}{\mu_0+1},\ |X| < M\,,$$

$$Y = \frac{M\,\mu_0\,\pi\,(2\,g_1-1)}{(\mu_0+1)^2\,B}\,,$$

$$Z = -\frac{1}{2}\,\frac{M\,\mu_0\,\pi^2\,(-8\,{g_1}^2+8\,g_1+8\,g_2\,{g_1}^2\,\mu_0+8\,g_2\,{g_1}^2-8\,g_2\,g_1\,\mu_0-8\,g_2\,g_1+\mu_0-1)}{(\mu_0+1)^3\,B^2}\,. \tag{4.24}$$

The conditions $0 \le g_1 \le 1$ and $0 \le g_2 \le 1$ are equivalent to

$$|Y| \le \frac{\pi}{4MB}(M^2-X^2)\,, \tag{4.25}$$

and

$$\left|Z + \frac{2Y^2X}{M^2-X^2}\right| \le \Lambda = 2M\frac{\left(\frac{\pi}{4MB}\right)^2(X^2-M^2)^2-Y^2}{M^2-X^2}\,. \tag{4.26}$$

We will apply the above results to the three-body problem, whose solutions in many situations are analytic functions in the strip along the real axis of the complex time plane.

We consider three mass points P_1, P_2, P_3 in $\mathbb{R}^3$ which attract each other according to the Newtonian law with finite positive masses m_1, m_2, m_3. Let $R_i = (x_i, y_i, z_i)$ be the position vector of P_i and r_{ij} the distance between it and mass j. One writes equations of motion as follows:

$$\begin{aligned} m_i\tfrac{dR_i'}{dt} &= -\sum_{j\neq i} m_i m_j \tfrac{(R_i-R_j)}{r_{ij}^3}\,, \\ R_i' &= \tfrac{dR_i}{dt} = (x_i', y_i', z_i'), \quad i = 1,2,3\,. \end{aligned} \tag{4.27}$$

which have the integral of energy:

$$\begin{aligned} T+U &= h = -\tfrac{m_1m_2m_3}{2\Gamma}K\,, \\ T &= \sum_{i=1}^{3}\tfrac{m_i({x_i'}^2+{y_i'}^2+{z_i'}^2)}{2} \\ U &= -\tfrac{m_3m_2}{r_{32}} - \tfrac{m_1m_3}{r_{13}} - \tfrac{m_2m_1}{r_{21}}\,, \\ \Gamma &= m_1+m_2+m_3\,, \end{aligned} \tag{4.28}$$

the first integrals of the impulse of the system:

$$\sum_{i=1}^{3} m_i x_i = 0, \quad \sum_{i=1}^{3} m_i y_i = 0, \quad \sum_{i=1}^{3} m_i z_i = 0\,,$$
$$\sum_{i=1}^{3} m_i x_i' = 0, \quad \sum_{i=1}^{3} m_i y_i' = 0, \quad \sum_{i=1}^{3} m_i z_i' = 0\,, \tag{4.29}$$

and the first integrals of the angular momentum:

$$\begin{cases} \sum_{i=1}^{3} m_i(x_i y_i' - y_i x_i') = c_1,\ \sum_{i=1}^{3} m_i(y_i z_i' - z_i y_i') = c_2,\ \sum_{i=1}^{3} m_i(z_i x_i' - x_i z_i') = c_3\,, \\ c_1, c_2, c_3 = consts\,. \end{cases} \tag{4.30}$$

We shall need the following Theorem due to Sundman [86]

Theorem 42. *Given $\chi > 0$ let $x_i(t)$, $y_i(t)$, $z_i(t)$ be a solution of the 3-body problem (4.27) which satisfy for all $t \in (-\infty, +\infty)$ the inequalities*

$$r_{32}(t) > 14\chi, \quad r_{13}(t) > 14\chi, \quad r_{21}(t) > 14\chi\,. \tag{4.31}$$

Then $R_i(t)$, $1 \le i \le 3$ are holomorphic functions in the strip of the complex t-plane

$$S_{B_\chi}\ :\ |\mathrm{Im}(t)| < B_\chi = \frac{\chi}{\sqrt{\frac{4}{21}\frac{\Gamma^2}{m\chi} + \Gamma|K|}}, \quad m = \min\{m_1, m_2, m_3\}\,, \tag{4.32}$$

and

$$|x_i(t) - x_i(\tilde{t})| < \chi,\ |y_i(t) - y_i(\tilde{t})| < \chi,\ |z_i(t) - z_i(\tilde{t})| < \chi,\ \tilde{t} \in \mathbb{R},\ |t - \tilde{t}| < B_\chi\,. \tag{4.33}$$

Let $x_i(t)$, $y_i(t)$, $z_i(t)$ be a solution of the 3-body problem (4.27) satisfying the conditions of Theorem 42 for a certain $\chi > 0$ and such that $\exists\, \lambda > 0$: $|x_i(t)| < \lambda$, $|y_i(t)| < \lambda$, $|z_i(t)| < \lambda$, $\forall\, t \in (-\infty, \infty)$. We call it a *$\lambda, \chi$-solution.* As follows from (4.33)

$$|x_i(t)| < M_\chi, \quad |y_i(t)| < M_\chi, \quad |z_i(t)| < M_\chi, \quad \text{for} \quad M_{\chi,\lambda} = \lambda + \chi, \quad t \in S_{B_\chi}\,. \tag{4.34}$$

Thus, *to every λ, χ-solution correspond functions $x_i(t)$, $y_i(t)$, $z_i(t)$ satisfying M, B–conditions with $B = B_\chi$, $M = M_{\chi,\lambda}$ given by* (4.32)–(4.34). Our aim now is to describe the regions of the phase space of the 3-body problem swept by λ, χ–solutions.

It is well known that bounded solutions (4.34) can exist only for zero or negative values of the energy h. Together with (4.28) and (4.31) this property gives the following conditions

$$-\frac{1}{14\chi}\sum_{i<j} m_i m_j < h \leq 0\,, \tag{4.35}$$

which express basically the fact that the kinetic energy T is positive, and

$$T < h + \frac{1}{14\chi}\sum m_i m_j\,. \tag{4.36}$$

Applying inequalities (4.25)–(4.26) together with (4.27) we obtain the following system of differential inequalities:

$$|x_i'| \leq \frac{\pi}{4M_{\chi,\lambda}B_\chi}(M^2_{\chi,\lambda} - x_i^2)\,, \tag{4.37}$$

and

$$\left|\frac{\partial U}{\partial x_i} - \frac{2m_i {x_i'}^2 x_i}{M^2_{\chi,\lambda} - {x_i}^2}\right| \leq 2m_i M_{\chi,\lambda}\frac{\left(\frac{\pi}{4M_{\chi,\lambda}B_\chi}\right)^2 ({x_i}^2 - M^2_{\chi,\lambda})^2 - {x_i'}^2}{M^2_{\chi,\lambda} - {x_i}^2}\,, \tag{4.38}$$

with similar inequalities involving y_i, and z_i, $1 \leq i \leq 3$.

It is an interesting problem to analyze the topological structure of the domains in the phase space of the three-body problem (4.27) defined by (4.37) and (4.38).

4.5 Open problems and perspectives

Besides the pioneering work of Beardon [13], very little is known about relation between continued fractions and the hyperbolic geometry of Möbius maps in $\mathbb{H}^3$. The main question here is to establish a basic dictionary between classical results of continued fractions theory and analogous results in complex dynamics and discrete group theory. Little attention has been paid to constructing of explicit examples of divergent continued fractions convergent generally (see Definition 4.1, p. 115). Speaking about g–fractions, there has been as yet no systematic examination of their convergence behavior in connection with the limit behavior of parameters $\{g_i\}_{i=1}^{\infty}$.

That may be useful to construct explicit examples of limit periodic continued fractions with $\lim a_i > 1/4$ different from the family (4.8). What is the optimal convergence speed of $a_i \to a \geq 1/4$ in the convergent case ? The only known result here corresponds to the case $a = 1/4$ (Jacobsen).

Finally, there is work to be done in the theory of continued fractions and generally convergent sequences of Möbius maps in higher dimensions.

Liste des travaux

Travaux après la thèse

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[TMO2] Mestel, B. D.; Osbaldestin, A. H.; Tsygvintsev, A. V. Bounds on the unstable eigenvalue for the asymmetric renormalization operator for period doubling. Comm. Math. Phys. 250 (2004), no. 2, 241–257.

[T4] Tsygvintsev, A. On the connection between g-fractions and solutions of the Feigenbaum-Cvitanović equation. Commun. Anal. Theory Contin. Fract. 11 (2003), 103–112.

[T5] Tsygvintsev, A. On the convergence of continued fractions at Runckel's points. The Ramanujan Journal, à paraître.

[T5bis] Dullin, H., Tsygvintsev, A. On the analytic non-integrability of the Rattleback problem, (with H. Dullin), Annales de la faculté des sciences de Toulouse, à paraître (**cette publication a été ajoutée dans le document après la soutenance !)**

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Travaux avant et durant la thèse

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