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Robust stability and control of switched linear systems

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Contents

Acronymes

BMI - Bilinear Matrix Inegalities

GDDLKF - General Delay Dependent Lyapunov-Krasovskii Function

HDS - Hybrid Dynamical Systems

LMI - Linear Matrix Inegalities

LPV - Linear Parameter-Varying

LTI - Linear Time Invariant

NCS - Networked Control Systems

SPDLF - Switched Parameter Dependent Lyapunov Functions

Notations

- $M > 0$ - square symmetric positive definite matrix,
- $M < 0$ - square symmetric negative definite matrix,
- $M < N$ - the $M - N$ matrix is a square symmetric negative definite matrix,
- $\mathbf{I}$ - identity matrix,
- $det(M)$ - determinant of the square matrix M,
- $\|M\|$ - induced euclidean norm of M,
- $\|x\|$ - induced euclidean norm of a vector x,
- $eig_{max}(M)$ and $eig_{min}(M)$ - the maximum and the minimum eigenvalue of a symmetric matrix M.
- M^{-1} - inverse of a non-singular matrix M,
- M^T - transpose of M,
- $M = \begin{bmatrix} A & B \\ * & D \end{bmatrix}$ - symmetric matrix M where $*$ means B^T
- $diag(a_1, a_2, \ldots, a_n)$ - diagonal matrix with $a_1, a_2, \ldots, a_n$ on the dominant diagonal

- $co(\mathcal{S})$ - convex hull of the set $\mathcal{S}$,

- $\langle , \rangle$ - vector dot product, $\langle x, y \rangle = x^T y$,

General Introduction

Hybrid Dynamical Systems (HDS) are dynamical systems simultaneously containing mixtures of logic and continuous dynamics [94, 114]. The classical example is the case of continuous time processes that are supervised using logical decision-making algorithms. Switched linear systems are an important class of HDS. They represent a set of linear systems and a rule that orchestrates the switching among them.

Goal

This work is concerned with stability analysis and control synthesis of switched linear systems based on Lyapunov stability method and on linear matrix inequalities (LMI). For this class of hybrid systems there are several tools which can be used for stability analysis in the case of an arbitrary switching law. However, in real applications, parameter uncertainties are unavoidable. Until now, few results are concerned with the robust stability in the context of hybrid systems. In this research we address the robust stability and control design problems for switched linear systems.

We are particularly interested in the case of discrete time switched linear systems with parameter uncertainties and uncertain switching law. We model parametric uncertainties as matrix polytopes and we derive robust stability and control synthesis LMI conditions.

The second objective of our research concerns the interaction between digital controllers and continuous time systems. We consider the discrete time representations of LTI and switched systems in digital control loops. In this case timing problems such as sampling jitter or delayed command actuation, induce parametric uncertainties in the discrete time representation. These problems are unavoidable in the context of networked control systems (NCS). We show that an event-based modeling allows us to express the original digital control problem as a stability problem for switched systems with polytopic uncertainties. The methodology proposed for the uncertain switched systems case can be applied for studying the stability/stabilizability of continuous time LTI systems in digital control loops. The results are extended to the case of continuous time switched linear systems.

Structure

This thesis is organized in four chapters that are structured as follows:

Chapter 1

The first chapter is a literature survey. The general stability problem is recalled in order to establish fundamental concepts that are necessary to the comprehension of our work. Then, we present several stability problems and stability criteria that are encountered in the domain of switched linear systems.

Chapter 2

In the second chapter, the robust stability and stabilizability problems are treated in the context of switched linear systems. The chapter is divided in two sections that concern the parametric uncertainties and the uncertainties that are related to the switching law. The goal is to develop less conservative LMI robust stability/stabilizability tools using Lyapunov functions that take into account the uncertain parameters. Furthermore, we show how can we use dwell time stability criteria in the case of switched systems with an uncertain switching law.

Chapter 3

This chapter studies the relation between the stability of discrete time systems with time varying delays and the switched system stability problem. The stability of discrete-time systems with time varying delays can be analyzed by using a discrete-time extension of the classical Lyapunov-Krasovskii approach. In the domain of networked control systems, a similar delay stability problem is treated using a switched system transformation approach. Here we will show that using the switched system transformation is equivalent to using a general delay dependent Lyapunov-Krasovskii function. This function represents the most general form that can be obtained using sums of quadratic terms. Necessary and sufficient LMI conditions for the existence of such functions are presented. These results are used later, in chapter 4.

Chapitre 4

The goal of the last chapter is to present a unique methodology to deal with the main timing problems in the context of digital systems. We present a new event based discrete-time model and we show that the stabilizability of this system can be achieved by finding a control for a switched polytopic system with an additive norm bounded uncertainty. The main problem is to obtain the most representative polytopic representation for the closed-loop system and to reduce its complexity. This allows to treat the stability problem using classical numerical tools. The methodology is extended to the case of switched system.

Personal publications

The research exposed in this thesis can be found in the following publications:

Book chapter

- L. Hetel, J. Daafouz, C. Iung - *About stability analysis for discrete time systems with time varying delays* - Chapter 19 in Taming Heterogeneity and Complexity of Embedded Control - International Scientific and Technical Encyclopedia (ISTE), London, 2006

Journals

- L. Hetel, J. Daafouz, C. Iung - *Stabilization of Arbitrary Switched Linear Systems With Unknown Time-Varying Delays* - IEEE Transactions on Automatic Control, Oct. 2006, Volume: 51, Issue: 10, page(s): 1668- 1674

- R. Bourdais, L. Hetel, J. Daafouz, W. Perruquetti - *Stabilité et stabilisation d'une classe de systèmes dynamiques hybrides* - Journal Européen Systèmes Automatisés, accepted, to appear

- L. Hetel, J. Daafouz, C. Iung - *Equivalence between the Lyapunov-Krasovskii functional approach for discrete delay systems and the stability conditions for switched systems* - Nonlinear Analysis: special issue on Hybrid Systems and Applications - accepted, to appear

- L. Hetel, J. Daafouz, C. Iung - *Analysis and control of LTI and switched systems in digital loops via an event-based modeling* - International Journal of Control - accepted, to appear

International Conferences

- L. Hetel, J. Daafouz, C. Iung - *Robust stability analysis and control design for switched uncertain polytopic systems* - 5th IFAC Workshop on Robust Control (ROCOND 06) - Toulouse, France - 2006

- L. Hetel, J. Daafouz, C. Iung - *Stabilization of switched linear systems with unknown time varying delays* - 2nd IFAC Conference on Analysis and Design of Hybrid Systems - Alghero, Sardinia, Italy - 2006

- L. Hetel, J. Daafouz, C. Iung - *LMI control design for a class of exponential uncertain systems with application to network controlled switched systems* - American Control Conference (ACC) - USA - 2007

- L. Hetel, J. Daafouz, C. Iung -*Equivalence between the Krasovskii-Lyapunov functional approach for discrete delay systems and the stability conditions for switched systems* - IFAC Workshop on Time Delay Systems - Nantes, France - 2007

- L. Hetel, J. Daafouz, C. Iung - *Stability analysis for discrete time switched systems with temporary uncertain switching signal* - IEEE Conference on Decision and Control - New Orleans, USA - 2007

Chapter 1

Basic concepts

In this chapter, we intend to present several basic concepts about switched systems. First, the mathematical definition of a switched system will be given. Next some general concepts of stability will be recalled and the switched system stability and stabilizability problems will be formulated. Some significant results from the literature will be presented. Although the contribution of this thesis is mainly concerned with discrete time switched systems, this first chapter focuses on continuous time switched systems. Actually, from a historical perspective, the problems have been formulated in a continuous time setting. Here we intend to preserve the historical aspect of the results. The most important stability notions will be presented in continuous time and the particularities of the discrete-time case will be indicated when necessary.

1.1 Switched systems - formal definition

Switched systems are a fascinating class of hybrid dynamical systems due to theirs simplicity and to the complexity of phenomena that they can describe. Formally, a switched system in continuous-time is defined by the relation:

$$\dot{x}(t) = f_{\sigma(t)}(t, x(t), u(t)), \tag{1.1}$$

where $\sigma(t)$, $\sigma : \mathbb{R}^+ \to \mathcal{I} = \{1, 2, \dots, N\}$ represents a piecewise constant function, called *switching signal*, which takes values in a set of index $\mathcal{I}$. $x(t) \in \mathbb{R}^n$ represents the system state, $u(t) \in \mathbb{R}^m$ the command, and $f_i(\cdot, \cdot, \cdot)$, $\forall i \in \mathcal{I}$ are vector fields describing the different modes. Similarly, a discrete-time switched system is defined by

$$x(k+1) = f_{\sigma(k)}(k, x(k), u(k)), \tag{1.2}$$

with $\sigma : \mathbb{Z}^+ \to \mathcal{I}$. The switching signal $\sigma(t)$ (or $\sigma(k)$ for the discrete-time case) specifies the active system mode (the active sub-system). Only one sub-system is active at a given instant of time. The choice of the active sub-system can be based on time criteria, on state space regions or on the evolution of some physical parameters.

The models (1.1) and (1.2) are very general. In particular, if the vector fields take the form $A_i x(t)$, $\forall\ i \in \mathcal{I}$, then we obtain a *switched linear system*

$$\dot{x}(t) = A_{\sigma(t)} x(t). \tag{1.3}$$

A taxonomy of switched systems can be defined based on the switching function σ. In this context, one can identify a *controlled* aspect (when the switching function represents a discrete input) and, in opposition, an *autonomous* aspect (for example switching due to state space transitions). A survey of different switched system classes and the associated problems is given in [55, 25, 90] and [86].

1.2 Classical stability concepts

An important problem in the domain of switched system is the research of stability criteria. Before discussing this aspect, some fundamental concepts from the stability theory are recalled. Intuitively, stability is a system property that corresponds to returning to its *equilibrium* position when it is punctually disturbed. Consider a non-linear time invariant autonomous system

$$\dot{x}(t) = f(x(t)) \tag{1.4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. Formally, the equilibrium points x^* represent the real solutions of the equation $f(x) = 0$.

Definition 1 *The equilibrium point of the system (1.4) is*

- *stable if* $\forall\ \epsilon > 0\ \exists \delta = \delta(\epsilon) > 0$ *such that*

$$\|x(0) - x^*\| < \delta \ \Rightarrow\ \|x(t) - x^*\| < \epsilon,\ \forall t \geq 0;$$

- *asymptotically stable if* x^* *is stable and* δ *may be taken such that*

$$\|x(0) - x^*\| < \delta \ \Rightarrow\ \lim_{t\to\infty} x(t) = x^*;$$

- *exponentially stable if there exist three positive real scalars* c, K *and* λ *such that*

$$\|x(t) - x^*\| \leq K \|x(0) - x^*\| e^{-\lambda t},\ \forall\ \|x(0) - x^*\| < c;$$

- *globally asymptotically stable if* x^* *is stable and* $\forall\ x(0) \in \mathcal{R}^n$

$$\lim_{t\to\infty} x(t) = x^*.$$

By translation, the equilibrium point can be moved to the origin ($x^* = 0$), which often simplifies the stability analysis.

The concept of stability leads to the Lyapunov stability theory. This theory establishes the fact that a system whose trajectories are attracted toward an asymptotically stable equilibrium point is progressively loosing its energy, in a monotone fashion. Lyapunov generalized the energy notion by using a function $V(x)$ which depends on the system state. This function is usually a norm.

Theorem 2 *Considering the non-linear system*

$$\dot{x}(t) = f(x(t)) \tag{1.5}$$

with an isolated equilibrium point ($x^* = 0 \in \Omega \subset \mathcal{R}^n$). If there exist a locally Lipschitz function $V : \mathcal{R}^n \to \mathcal{R}$ that has continuous partial derivatives and two $\mathcal{K}$ functions [1] α and β such that

$$\alpha\left(\|x\|\right) \leq V(x) \leq \beta\left(\|x\|\right), \ \forall\ x \in \Omega \subset \mathcal{R}^n,$$

the origin $x = 0$ of system (1.4) is

- *stable if*

$$\frac{dV(x)}{dt} \leq 0, \ \forall x \in \Omega, \ x \neq 0;$$

- *asymptotically stable if there exists a $\mathcal{K}$ function φ such that*

$$\frac{dV(x)}{dt} \leq -\varphi\left(\|x\|\right), \ \forall x \in \Omega, \ x \neq 0;$$

- *exponentially stable if there exist four positive constant scalars $\bar{\alpha}, \bar{\beta}, \gamma, p$ such that*

$$\alpha\left(\|x\|\right) = \bar{\alpha}\left\|x\right\|^p, \ \beta\left(\|x\|\right) = \bar{\beta}\left\|x\right\|^p, \ \varphi\left(\|x\|\right) = \gamma\left\|x\right\|.$$

The extensions of this theorem for the case of non-autonomous systems is given in [49].

[1] A function $\varphi : [0, a) \to [0, \infty)$ is a $\mathcal{K}$ function, if it is strictly decreasing and $\varphi(0) = 0$. It is a $\mathcal{K}_\infty$ function if $a = \infty$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Discrete Time

Theorem 3 *Considering the non-linear discrete time system*

$$x(k+1) = f(x(k))$$

with the origin ($x^* = 0 \in \Omega \subset \mathbb{R}^n$) as equilibrium. If there exist a function $V : \mathcal{R}^n \to \mathcal{R}$ and two $\mathcal{K}$ functions such that

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \; \forall \, x \in \Omega \subset \mathbb{R}^n,$$

The origin of the system is

- *stable if*

$$\Delta V(x(k)) \leq 0, \; \forall \, x \in \Omega, \; x \neq 0$$

  where

$$\begin{aligned} \Delta V(x(k)) &= V(x(k+1)) - V(x(k)) \\ &= V(f(x(k))) - V(x(k)); \end{aligned}$$

- *asymptotically stable if there exists a $\mathcal{K}$ function φ such that*

$$\Delta V(x(k)) \leq -\varphi(\|x(k)\|), \; \forall \, x(k) \in \Omega, \; x(k) \neq 0;$$

- *exponentially stable if there exist four positive constant scalars $\bar{\alpha}, \bar{\beta}, \gamma, p$ such that*

$$\alpha(\|x\|) = \bar{\alpha}\|x\|^p, \; \beta(\|x\|) = \bar{\beta}\|x\|^p, \; \varphi(\|x\|) = \gamma\|x\|, \; \forall \, x \in \Omega, \; x \neq 0.$$

Remarks. These local definitions are globally valid if the given functions are $\mathcal{K}_\infty$ functions.

Definition 4 *The function $V(x)$ that verifies the properties given in the previous theorems is called a Lyapunov function for the system.*

Very often, for simplicity reasons, the term *stable system* is used to describe a system that has a stable equilibrium point.

1.3 Switched systems stability problematics and results

The switched systems stability problem is complex and interesting. The example of asymptotically stable systems that, by switching, lead to an instable trajectory, is well known. The case of unstable systems that can lead to a stable behavior by switching is also notable. For autonomous switched linear systems:

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ \forall\ \sigma(t) \in \mathcal{I}$$

a well known classification of stability problems, has been proposed by Liberzon and Morse [57]:

Problem A Find stability conditions such that the switched system is asymptotically stable for any switching function.

Problem B Given a switching law, determinate if the switched system is asymptotically stable.

Problem C Give the switching signal which makes the system asymptotically stable.

1.3.1 Stability of differential inclusions

Similar stability problems have been discussed in the literature for ordinary differential equation with discontinuous right side member, and more precisely for the case of differential inclusions [4].

Consider the linear differential inclusion described by

$$\dot{x} \in F(x) = \{y : y = Ax, A \in \mathcal{A}\} \tag{1.6}$$

where $\mathcal{A}$ is a compact set. A switched linear system under the form

$$\dot{x}(t) = A_{\sigma(t)}x(t),$$

with $A_{\sigma(t)} \in \{A_1, A_2, \dots, A_N\}$, $\forall\ \sigma(t) \in \mathcal{I}$, can be considered as a differential inclusion (1.6) with $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$.

The linear differential inclusion stability analysis (1.6) is connected to the analysis of his convex hull.

Theorem 5 *[64] The inclusion (1.6) is asymptotically stable if and only if the convex differential inclusion*

$$\dot{x} \in \{y : y = Ax, A \in co\mathcal{A}\} \tag{1.7}$$

is stable.

Molchanov and Pyatnitskiy expressed this stability problem in terms of quasi-quadratic Lyapunov functions:

Theorem 6 *[64] The origin $x = 0$ of the linear differential inclusion (1.6) is asymptotically stable iff there exists a Lyapunov function $V(x)$ strictly convex, homogeneous (of second order) and quasi-quadratic :*

$$V(x) = x^T\mathcal{P}(x)x, \tag{1.8}$$

$$\mathcal{P}(x) = \mathcal{P}^T(x) = \mathcal{P}(\tau x),\ x \neq 0,\ \tau \neq 0 \tag{1.9}$$

whose derivative satisfies the inequality:

$$\dot{V}^* = \sup_{y\in F(x)} \lim_{h\to 0} h^{-1}\left\{V(x+hy) - V(x)\right\} \leq -\gamma \left\|x\right\|^2, \gamma > 0. \tag{1.10}$$

When the set $\mathcal{A}$ is a convex polyhedron, an algebraic criteria can be deduced:

Theorem 7 *For the asymptotic stability of the origin $x = 0$ of the convex linear differential inclusion*

$$\dot{x} \in F(x) = \{y : y = Ax, A \in co\left\{A_1, \ldots, A_M\right\}\} \tag{1.11}$$

it is necessary and sufficient that there exists a number $m > n$, a rank n matrix $\mathcal{L}$ and M row diagonal negatives matrices $(m \times m)$

$$\Gamma_s = \left(\gamma_{ij}^{(s)}\right)_{i,j=1}^{m}, \ \forall\ s = 1, \ldots M,$$

with

$$\gamma_{ii}^{(s)} + \sum_{i\neq j} \left|\gamma_{ij}^{(s)}\right| < 0,\ \forall\ i = 1, \ldots, m,\ \ s = 1, \ldots, M,$$

such that the relation

$$A_s^T \mathcal{L} = \mathcal{L}\ \Gamma_s^T,\ \forall\ s = 1, \ldots, M$$

is verified.

This stability criterium is closely related with the auxiliary stable differential inclusion

$$\dot{z} \in G(z) = \{y : y = \Lambda z, \Lambda \in co\left\{\Lambda_1, \ldots, \Lambda_N\right\}\} \tag{1.12}$$

in the augmented space $\mathbb{R}^m$ whose solutions contain the trajectories of the original inclusion. The $\mathcal{L}$ matrix, with $z = \mathcal{L}^T x$, represents the associated transformation matrix. The proof is based on the existence of a quasi-quadratic Lyapunov function $V(x) = x^T\mathcal{P}(x)x$.

These two theorems can be directly applied to the case of switched linear systems [24, 57, 60, 73]. This means that the stability of a switched system is based on the existence of a common Lyapunov function for all the sub-systems (in the case of the Problem A). However, from a practical point of view, it is very difficult to verify the criteria proposed by the previous theorems. Generally, the numerical / analytical research of a quasi-quadratic Lyapunov function $V(x) = x^T\mathcal{P}(x)x$ or transformation matrix $\mathcal{L}$ is very difficult.

In order to verify the stability of differential inclusions, several authors investigated existence conditions for a common quadratic Lyapunov function $V(x) = x^T P x$. Its existence, a sufficient stability condition, can be expressed as a linear matrix inequality (LMI) [13].

Theorem 8 *Consider system (1.11). If there exists a matrix P, $0 < P = P^T$, solution of the LMIs:*

$$A_i^T P + P A_i < 0, \ \forall \ i = 1, \dots, N \tag{1.13}$$

then the quadratic function $V(x) = x^T P x$ is a Lyapunov function for the system (1.11), i.e. the origin $x = 0$ is globally exponentially stable.

When a common quadratic Lyapunov function exists, we may say that the system is *quadratically stable* and the term *quadratic stability* is used. This implies that there exists a scalar ϵ such that

$$\frac{dV(x)}{dt} < -\epsilon \|x\| .$$

Discrete Time

In discrete time, the concept of *joint spectral radius* [8] gives a necessary and sufficient condition for the stability of difference inclusions.
The joint spectral radius is the maximal growing rate which may be obtained using long products of matrices from a given set. Consider the notation $\mathcal{A} = \{A_1, \dots, A_N\}$. The joint spectral radius of the set $\mathcal{A}$ is formally defined as :

$$\rho(\mathcal{A}) \triangleq \limsup_{p\to\infty} \rho_p(\mathcal{A})$$

where

$$\rho_p(\mathcal{A}) = \sup_{A_{i_1}, A_{i_2} \dots, A_{i_p} \in \mathcal{A}} \left\| A_{i_1} . A_{i_2} \dots A_{i_p} \right\|^{1/p} .$$

The linear difference inclusion

$$x(k+1) \in F(x) = \{y : y = Ax, A \in \mathcal{A}\}$$

is asymptotically stable if and only if the joint spectral radius satisfies the inequality:

$$\rho(\mathcal{A}) < 1.$$

This condition can be directly applied to the case of discrete time switched linear systems

$$x(k+1) = A_{\sigma(k)} x(k), \ A_{\sigma(k)} \in \mathcal{A}.$$

The main difficulty of this approach is the practical computation of the joint spectral radius [93]. An approximation procedure is given in [8]. When ellipsoidal norms are used for computing the approximation, it is possible to find a relation between the the joint spectral radius approach and the existence of a common quadratic Lyapunov function. However this approximation implies some conservatism. Less conservative approximations are given in [8, 75].

1.3.2 Common quadratic Lyapunov Function and algebraical stability criteria

A Lie algebra approach for verifying the stability of a switched linear system has been proposed by Liberzon [55].

Consider the system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad \forall \sigma(t) \in \mathcal{I}. \tag{1.14}$$

The system dynamic is described by the matrix set $\mathcal{A} = \{A_1, A_2, \ldots, A_N\}$. The Lie algebra $\mathfrak{g} = \text{Lie}\{A_i : i \in \mathcal{I}\}$ corresponds to the set of matrices A_i, $\forall\, i \in \mathcal{I}$ and all the iterative commutators obtained using the Lie operator,

$$[A_i, A_j] = A_iA_j - A_jA_i, \ \forall\, i, j \in \mathcal{I}.$$

Several algebraical stability criteria in relation with this Lie algebra have been presented in the literature. If all the state matrices A_i, $\forall\, i \in \mathcal{I}$ are pairwise commutative, i.e. if the Lie operator $[A_i, A_j]$ is zero for all the pairs A_i, A_j, $i, j \in \mathcal{I}$, the switched system (1.14) is asymptotically stable [68, 1]. Gurvirts indicates that if the Lie algebra $\mathfrak{g}$ is nilpotent, then the system is asymptotically stable [37].

Independently of these works, Yoshihiro Mori and Kuroe [66] showed that if the A_i, $\forall\, i \in \mathcal{I}$ matrices simultaneously accept an upper / lower triangulation, then there exists a common quadratic Lyapunov function.

Theorem 9 *[66] Consider the system (1.14). If all the A_i, $i \in \mathcal{I}$ matrices are Hurwirtz stable and if there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that all the matrices*

$$\Lambda_i = T^{-1}A_iT, \ \forall\, i \in \mathcal{I}$$

are upper (or lower) triangular, then there exists a common quadratic Lyapunov function

$$V(x) = x^TPx$$

for the family of systems

$$\{\dot{x} = A_ix, \quad \forall\, i \in \mathcal{I}\}$$

and the switched system (1.14) is asymptotically stable.

Liberzon generalizes the previous results for complex T transformation matrices, $T \in \mathbb{C}^{n\times n}$. He proposes a sufficient condition for a simultaneous upper triangulation of a matrix set in terms of solvable Lie algebra [56]. If

$$\mathfrak{g} = \text{Lie}\{A_i : \ \forall\, i \in \mathcal{I}\}$$

is a solvable Lie algebra, then the system family

$$\{\dot{x} = A_ix, \quad \forall\, i \in \mathcal{I}\}$$

simultaneously accepts an upper/lower triangulation and the switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad \forall\ \sigma(t) \in \mathcal{I}$$

is asymptotically stable.

This result may be locally applied for switched non-linear systems [1, 61] by studying the local linear approximation of the system. The result of Liberzon is interesting because, when all the matrices of the set $\{A_i,\ \forall i \in \mathcal{I}\}$ are pairwise commutative, or they generate a nilpotent Lie algebra $\mathfrak{g}$, they also generate a solvable Lie algebra. However, it represents only a sufficient condition for simultaneous triangulation. The approach is important from a theoretical point of view since it establishes the connection between the Lie algebra based stability conditions and the simultaneous triangulation criteria. However all of these criteria represent sufficient conditions for the existence of a common Lyapunov function which implies some conservatism.

1.3.2.1 Necessary and sufficient criteria for special cases

In order to reduce the conservatism of the previous results, necessary and sufficient conditions for the existence of a common quadratic Lyapunov function have been proposed. Shorten and Narendra [87] developed such a condition for a pair of second order systems.

Consider the convex envelope described by the matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$:

$$co\{A_1, A_2\} \triangleq \{\lambda A_1 + (1-\lambda)A_2 : \lambda \in [0,1]\}.$$

The set of systems

$$\{\dot{x} = A_1x, \dot{x} = A_2x\},\ A_1, A_2 \in \mathbb{R}^{2\times 2}$$

has a common quadratic Lyapunov function if and only if all the matrices of the two convex envelopes $co\{A_1, A_2\}$ and $co\left\{A_1, A_2^{-1}\right\}$ are Hurwitz stables.

Some extensions exist for the case of several second order subsystems [85] and for a pair of third order systems [50], but the extension to the general case is very difficult.

Other algebraic criteria for the existence of a common quadratic Lyapunov function are given in [116], where it is shown that for symmetric matrices

$$A_i = A_i^T,\ \forall\ i \in \mathcal{I}$$

and normal systems

$$A_i.A_i^T = A_i^T.A_i,\ \forall\ i \in \mathcal{I},$$

a necessary and sufficient condition for both the existence of a common quadratic Lyapunov function and the asymptotic stability is that all the subsystems are Hurwitz stable.

1.3.2.2 LMI Conditions for the existence of a common quadratic Lyapunov function

In the literature there exists a purely numerical method for finding a common quadratic Lyapunov function. The method is based on the resolution of the system of linear matrix inequalities (1.13). We can remark that these conditions are necessary and sufficient conditions for the existence of a common quadratic Lyapunov function, for any system order.

On the other hand, it is also useful to verify that a common quadratic Lyapunov function does not exist for a family of systems [13] :

Theorem 10 *If there exist matrices* $R_i = R_i^T, \ \forall i \in \mathcal{I}$ *solutions of the linear matrix inequalities :*

$$R_i > 0, \forall \ i \in \mathcal{I}, \ \sum_{i=1}^{N} A_i^T R_i + R_i A_i > 0.$$

then the family of systems

$$\{\dot{x} = A_i x, \ \forall i \in \mathcal{I}\}.$$

does not have a common quadratic Lyapunov function.

The drawback of using linear matrix inequalities is that in some cases of large dimensional matrices the existing numerical algorithms may not find the solution. Actually, there are cases for which the existence of a common quadratic Lyapunov function can be analytically proved while the usual LMI solvers (MATLAB LMI Toolbox) are not able to find it.

1.3.3 Multiple Lyapunov functions

Several stability criteria based on common quadratic Lyapunov functions exist in the literature. However, the existence of a common quadratic Lyapunov function is only a sufficient stability condition, not a necessary one. In [24], it is analytically shown (for the Problem A) that there exist switched linear systems that are asymptotically stable for which no common quadratic Lyapunov function exists. This means that looking for such a function may be too conservative. This motivates the search of other types of Lyapunov functions. In the literature, we can find several types of Lyapunov functions that may be classified, in a generic framework, under the name of *multiple Lyapunov functions*. The multiple Lyapunov functions describe a family of function of the form

$$V(x) = x^T P(\sigma, x) x$$

where the Lyapunov matrix may depend on the state vector and/or on the switching law. The concatenation of these functions forms one common, non quadratic, Lyapunov function.

1.3.3.1 Piecewise linear Lyapunov functions

Applying the stability criteria proposed by Molchanov and Pyatnitskiy in the context of switched systems, one can notice that it is necessary and sufficient to have a quasi-quadratic Lyapunov function $V(x) = x^T P(x)x$, whose Lyapunov matrix is varying according to the system state (see Theorem 6). Based on these ideas, the scientists, derived the concept of *piecewise linear Lyapunov functions*. This concept denotes a set of functions which placed side by side results in an overall non-quadratic common Lyapunov function. The first approach for deriving such function was to approximate the level functions of the quasi-quadratic Lyapunov function $V(x) = x^T \mathcal{P}(x)x$ (see Theorem 6) by a piecewise linear Lyapunov function [64, 71]:

$$V_m(x) = \max_{1 \le i \le m} |\langle l_i, x \rangle| \,. \tag{1.15}$$

The operator $\langle , \rangle$ denotes the classical scalar product and the elements $l_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, represent constant vectors called generator vectors. For a sufficiently large number m of generator vectors, it is possible to show that the existence of such a function is necessary and sufficient for the stability [65]. Few methods exist for verifying the existence of such functions. The difficulty lies in the a priori specification of generator vectors l_i. The existence of such a piecewise linear Lyapunov function may be verified by analyzing the spectrum of all the matrices in the convex hull generated by A_i, $\forall i \in \mathcal{I}$. For a pair of second order matrices, necessary and sufficient conditions for the existence of such a function with $m = 4$ generators are given in [100].

In the general case, other numerical criteria for the construction of a piecewise linear Lyapunov function are given in [109, 110], for non-linear systems, and in [111] for switched systems.

1.3.3.2 Multiple Lyapunov-like functions

Another approach is presented in the literature. It is based on the concept of Lyapunov-like functions. These are families of piecewise continuous and piecewise differentiable functions that concatenated together produce a single non-traditional Lyapunov function. One of the early results was given by [76]. The discontinuous structure of switched systems suggest the use of discontinuous Lyapunov functions. The authors propose to use a family of *Lyapunov-like functions* $\left\{V_i(x) = x^T P_i x,\ i \in \mathcal{I}\right\}$ such that each vector field $A_i x,\ i \in \mathcal{I}$ has its own Lyapunov function. The particularity of Lyapunov-like functions is that the decay of the function is required only when the system is active. A multiple Lyapunov-like function satisfies the following conditions:

- $V_i(x) = x^T P_i x$ is positive definite $\forall\ x \neq 0$ and $V_i(0) = 0$;
- the derivative of the each $V_i(x) = x^T P_i x$ function satisfies the relation

$$\dot{V}_i(x) = \frac{\partial V_i}{\partial x} A_i x(t) \le 0 \forall\ i \in \mathcal{I} \tag{1.16}$$

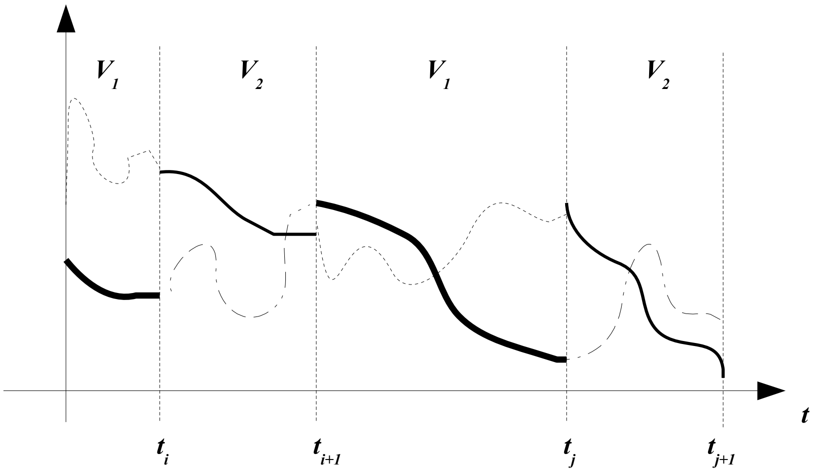

Figure 1.1: Multiple Lyapunov functions

when the ith sub-system is active.

The stability results developed in this context are based on the decay of the Lyapunov function at any successive instants for which a sub-system is switched-in.

Theorem 11 *[76] Consider a family of Lyapunov-like functions V_σ, each associated with a vector field $A_\sigma x$. For $i < j$, let $t_i < t_j$ be the switching times for which $\sigma(t_i) = \sigma(t_j)$. If there exists a $\gamma > 0$ such that*

$$V_{\sigma(t_j)}(x(t_{j+1})) - V_{\sigma(t_i)}(x(t_{i+1})) \leq -\gamma \left\| x(t_{i+1}) \right\|^2$$

then the switched system is asymptotically stable.

Extensions to the non-linear case have been proposed in [14, 15] and [25]. A more general result, assuming a so-called *weak Lyapunov function*, is given by [107]. In this case the condition (1.16) is replaced by

$$V_i(x(t)) \leq \alpha \left(V_i(x(t_j)) \right), \; \forall t \in [t_j, t_{j+1}]$$

where $[t_j, t_{j+1}]$ is the time interval for which a sub-system i is active, t_j, is any switching instant for which the system i is activated and $\alpha : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$ is a continuous function that satisfies $\alpha(0) = 0$. This allows defining Lyapunov functions that may occasionally increase, but their growth is bounded.

The results are difficult to apply to the case of arbitrary switching sequences. We can remark that the theorem requires the system trajectory to be known at least at the switching instants. Another problem is that no analytical or numerical method for the construction of such a Lyapunov-like function is given.

These problems can be solved for particular cases, for example when the switching sequence is a priori known or when the switching law depends on a partition of the state space.

Consider the dynamical system:

$$\dot{x}(t) = A_i x(t) \text{ for all } x(t) \in X_i,\ \forall\ i\ \in \mathcal{I} \tag{1.17}$$

where X_i are bounded sets with disjoint interiors such that $\cup_i X_i = \mathbb{R}^n$. Consider S_{ij} the state space region where the switch from mode i to mode j is allowed if the i^{th} sub-system is currently active, $\forall\ i, j \in \mathcal{I}$.

Consider the family of Lyapunov-like functions $V_i : \mathbb{R}^n \to \mathbb{R}$, each associated with the vector field $A_i x$ and the region :

$$\Omega^i \triangleq \left\{ x \in \mathbb{R}^n : \dot{V}_i(x) = \frac{\partial V}{\partial x} A_i x(t) \leq 0 \right\}, \tag{1.18}$$

$$\Omega^{ij} \triangleq \left\{ x \in \mathbb{R}^n : V_i(x) \geq V_j(x) \right\}. \tag{1.19}$$

If there exists a set of Lyapunov-like functions $V_i, i \in \mathcal{I}$ such that $X_i \subseteq \Omega^i$ and $S_{ij} \subseteq \Omega^{ij}$, then one can show, using an extension of Theorem 11, that the system (1.17) is stable [79, 80]. Moreover, if Lyapunov functions of the form $V_i = x^T P_i x$ are considered and if the state space regions X_i and S_{ij} are conic regions

$$X_i = \left\{ x \in \mathbb{R}^n : x^T Q_i x \leq 0,\ Q_i = Q_i^T \right\}, \tag{1.20}$$

$$S_{ij} = \left\{ x \in \mathbb{R}^n : x^T Q_{ij} x \leq 0,\ Q_{ij} = Q_{ij}^T \right\} \tag{1.21}$$

then, using the $S-$procedure [48, 81], LMI stability criteria are obtained.

Theorem 12 *If there exist matrices $P_i > 0$, scalars $\alpha_i \geq 0$ and $\alpha_{ij} \geq 0\ \forall\ i, j \in \mathcal{I}$ such that :*

$$A_i^T P_i + P_i A_i - \alpha_i Q_i < 0, \tag{1.22}$$

$$(P_i - P_j) - \alpha_{ij} Q_{ij} \leq 0, i \neq j, \tag{1.23}$$

then system (1.17) is asymptotically stable.

This theorem has several practical advantages. Comparing with Theorem 11, the obtained stability conditions can be verified using numerical algorithms, without any knowledge of the system solutions. Moreover, they are very flexible : the Lyapunov-like functions $x^T P_i x$ are constrained locally strictly decreasing along the system solutions, only in X_i, where the vector field $A_i x$ is active. It is possible to generalize the approach using several Lyapunov-like functions for the same vector field or state space partitions different from the ones given for the system [80, 81].

Discrete Time

The discrete time version of the previous theorem is given in [63]. For discrete time switched linear systems

$$x(k+1) = A_{\sigma(k)}x(k) \tag{1.24}$$

under an arbitrary switching law $\sigma(k) \in \mathcal{I}$, the stability analysis and the existence of multiple Lyapunov functions can be expressed in terms of linear matrix inequalities. In this case a *Poly-quadratic Lyapunov function* is obtained. This result, proposed in [21], is based on the extension of stability criteria for polytopic uncertain systems [22] to the case of switched systems .

Theorem 13 *The following statements are equivalent:*
1)There exists a poly-quadratic Lyapunov function

$$V(k, x(k), \sigma(k)) = x^T(k)P_{\sigma(k)}x(k),$$

strictly decreasing along system (1.24) trajectories $\forall\ \sigma \in \mathcal{I}$.

2) There exist N *symmetric matrices* $P_i = P_i^T > 0,\ \forall\ i \in \mathcal{I}$, *satisfying the LMIs:*

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > 0,\ \forall (i,j) \in \mathcal{I} \times \mathcal{I}. \tag{1.25}$$

3) There exist N *symmetric matrices* $S_i = S_i^T > 0$, *and* N *matrices* G_i $\forall\ i \in \mathcal{I}$, *satisfying the LMIs :*

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i^T \\ A_i G_i & S_j \end{bmatrix} > 0,\ \forall (i,j) \in \mathcal{I} \times \mathcal{I}. \tag{1.26}$$

Remarks. The poly-quadratic function approach represents a generalization of the common quadratic approach. With the constraint $P_i = P,\ \forall\ i \in \mathcal{I}$ we obtain the LMIs for the quadratic case. This generalization allows to relax the constraints imposed by the quadratic method and to obtain less conservative stability conditions.

1.4 Stabilization

In this section, two stabilization problems are being discussed: the design of a stabilizing switching sequence and the design of a control law when the switching law is arbitrary.

1.4.1 Design of a stabilizing switching law

The search of a stabilizing switching law is often formalized as follows: what restrictions should we consider for the switching law in order to guarantee the stability of the switched system ? Here we consider two types of restrictions

- state space restrictions, when the switching law is a control signal;

- time domain restrictions, when it is possible to control only the dwell time of each mode, and the switching law is arbitrary.

1.4.1.1 State space dependent constraints for the switching law

The design of a stabilizing switching law may be obtained by dividing the state space in several regions such that the obtained piecewise linear system is asymptotically stable. The problem is trivial when at least one of the A_i matrices is Hurwitz stable : the stable system is always activated. A more delicate stabilization problem occurs when all the $A_i,\ i \in \Gamma$ matrices are unstables. A necessary condition for the stabilization by switching law has been proposed in [91] :

Theorem 14 *If there exists a stabilizing switching sequence, then there exists a sub-system $\dot{x}(t) = A_i x(t),\ i \in \mathcal{I}$ such that at least one of the eigenvalues of $A_i + A_i^T$ is negative .*

Also, the results of Wicks [98] are based on the fact that the trajectory of any convex combination $\{A_1 x, A_2 x\}$ can be approximated by fast switchings among the two sub-systems. The first result is given as follows :

Theorem 15 *[97],[26] Consider a pair of unstables systems with the associated matrices $\{A_1, A_2\}$ $(M = 2)$. If there exists a stable convex combination, i.e. if there exists a scalar $\alpha \in (0, 1)$ such that the matrix $A_{eq} = \alpha A_1 + (1 - \alpha)A_2$ has all of its eigenvalues in the left side of the complex plane, there exists a switching sequence such that the system $\dot{x}(t) = A_{\sigma(t)}x(t),\ \sigma(t) \in \{1, 2\}$ is asymptotically stable.*

One should remark that this theorem gives a sufficient stabilization condition. Necessary and sufficient conditions exist for the particular case of second order systems [106] where the existence of a stabilizing switching law may be verified by analyzing the vectors fields. Several methods for the construction of the matrix A_{eq} are proposed in [97].

As follows, we recall a basic method for deriving a stabilizing state space partition [97, 96] in relation with Theorem 15.

Let $A_{eq} = \alpha A_1 + (1 - \alpha)A_2,\ \alpha \in (0, 1)$ denote the stable convex combination. For this matrix there exist two matrices positive definite P and Q such that :

$$A_{eq}^T P + P A_{eq} = -Q$$

This condition can be expressed as :

$$\alpha(A_1^T P + PA_1) + (1-\alpha)(A_2^T P + PA_2) = -Q$$

which is the same as

$$\alpha \cdot x^T(A_1^T P + PA_1)x + (1-\alpha) \cdot x^T(A_2^T P + PA_2)x = -x^T Q x < 0,$$

$\forall\, x \in \mathbb{R}^n \setminus \{0\}$. The terms $x^T(A_1^T P + PA_1)x$ and $x^T(A_2^T P + PA_2)x$ are weighted by positive coefficients ($\alpha \in (0,1)$). This means that, for any value of the state vector, at least one of the terms should be negative, i.e. $x^T(A_1^T P + PA_1)x < 0$ or $x^T(A_2^T P + PA_2)x < 0$. This implies that the state space is covered by the conic regions:

$$X_i = \left\{x \in \mathbb{R}^n : x^T(A_i^T P + PA_i)x < 0\right\},\ i = 1, 2$$

The function $V(x) := x^T P x$ is strictly decreasing in the region X_1 for any solution of $\dot{x} = A_1 x$ and in the region X_2 for the solutions of $\dot{x} = A_2 x$. Then it is possible to give switching surfaces such that V is strictly decreasing for the solutions of the switched linear system.

The results presented here are based on the existence of a stable convex combination A_{eq}. However, in general, finding a stable convex combination is a NP-hard problem [88, 9]. One should remark that the existing theorems provide sufficient stabilizability criteria. There are classes of systems for which no stable convex combination exists, yet, a stabilizing switching law may be obtained.

1.4.1.2 Time domains constraints

Here we present the concept of *dwell time* and its importance for the switched system stabilization. The dwell time represents the time interval between two instances of switch. The basic idea is very simple. Consider that all the sub-systems $\dot{x} = A_i x$, with $i \in \mathcal{I}$ are asymptotically stable. It is natural to think that the switched system is exponentially stable provided that the dwell time is sufficiently large to allow each sub-system to reach the steady-state. The problem is to compute the minimum time τ_D between two successive switch instants that ensures the stability of the switched system [119, 67]. Let $\Phi_i(t,\tau)$ denote the fundamental matrix of the ith sub-system $\dot{x} = A_i x,\ i \in \mathcal{I}$. Since all the sub-systems are asymptotically stable, one can find two scalars $\mu > 0$ and $\lambda_0 > 0$ such that

$$\|\Phi_i(t,\tau)\| \leq \mu e^{-\lambda_0(t-\tau)},\ t \geq \tau \geq 0, \forall i \in \mathcal{I}.$$

The scalar λ_0 may be interpreted as a common decay rate for the family of sub-systems. Actually the scalars μ and λ_0 may be computed, $\mu \triangleq \max_{i\in\Gamma} \mu_i$ and $\lambda_0 \triangleq \max_{i\in\Gamma} \lambda_i$ where μ_i and λ_i are the constants that define the convergence of each sub-system $\dot{x} = A_i x,\ \forall\, i \in \Gamma$. Consider $t_1, t_2, \ldots, t_k$, the switching instants in the time interval (τ, t), such that $t_i - t_{i-1} \geq \tau_D$. Then the value of the state vector at a given instant of time t is given by

$$x(t) = \Phi_{\sigma(t_k)}(t, t_k)\Phi_{\sigma(t_{k-1})}(t_k, t_{k-1}) \ldots \Phi_{\sigma(t_1)}(t_2, t_1)\Phi_{\sigma(t_1)}(t_1, \tau)x(\tau).$$

The transition matrices between two successive switching times satify the relation:

$$\left\|\Phi_{\sigma(t_{l-1})}(t_l, t_{l-1})\right\| \leq \mu e^{-\lambda_0(t_l-t_{l-1})} \leq \mu e^{-\lambda_0\tau_D},\ \forall\, l \in \{2,3,\ldots,k\}\,.$$

Then the system is asymptotically stable if $\mu e^{-\lambda_0\tau_D} \leq 1$. This condition may be satisfied for

$$\tau_D \geq \frac{\log\mu}{\lambda_0-\lambda} \tag{1.27}$$

$\forall\ \lambda \in (0,\lambda_0)$.

Theorem 16 *[119, 67] Consider the switched linear system $\dot{x} = A_\sigma x$, $\sigma \in \mathcal{S}(\mathcal{I},\tau_D)$ where all the sub-systems $\dot{x} = A_i x$, $\forall i \in \mathcal{I}$ are asymptotically stable with the stability margin λ_0. Here $\mathcal{S}(\mathcal{I},\tau_D)$ denotes the set of switching sequences for which the dwell time is greater then τ_D. For any scalar $\lambda \in (0,\lambda_0)$, the system is asymptotically stable with the stability margin λ if the minimum dwell time τ_D satisfies the condition (1.27).*

A more general result is given in [38] with the concept of *average dwell time*, τ_{moy}. The idea is that the system is asymptotically stable if in average the switching intervals are less than τ_{moy}. This allows, occasionally, to switch "faster" than the rate corresponding to the average dwell time τ_{moy}. Several extensions exist for the case of non-linear switched systems [77, 78] and for switched systems with instable sub-systems [115, 108].

Discrete Time

In the case of discrete time switched linear systems, the dwell time stability problem can be treated using a set of linear matrix inequalities.

Theorem 17 *[35] Consider the system $x(k+1) = A_{\sigma(k)}x(k)$, $\sigma(k) \in \mathcal{I}$. For a given $\tau_D \geq 1$ if there exists a set of positive definite matrices $\{P_1,\ldots P_N\}$ such that*

$$A_i^T P_i A_i - P_i < 0,\ \forall\ i \in \mathcal{I}$$

and

$$A_i^{T\tau_D} P_j A_i^{\tau_D} - P_i < 0,\ \forall\ i,j \in \mathcal{I},\ i \neq j$$

then the system is globally asymptotically stable for a dwell time greater than τ_D.

In this theorem, posing $\tau_D = 1$, we find the LMI stability condition (1.25), proposed in [22].

1.4.2 Control design for arbitrary switching

When the switching signal is uncontrollable, the switched system stabilization problem is closely related to the stabilization of uncertain systems (if the switching signal is unkwnon) or to the gain scheduling problem in the context of linear system with time varying parameters (LPV). In this case, the switching signal must be available in real time.

Consider the following polytopic system :

$$\dot{x}(t) = A(\lambda(t))x(t) + B(\lambda(t))u(t) \tag{1.28}$$

where

$$A(\lambda(t)) = \sum_{i=1}^{N} \lambda_i(t)A_i, \; B(\lambda(t)) = \sum_{i=1}^{N} \lambda_i(t)B_i$$

and

$$\lambda(t) \in \Lambda = \left\{ \lambda = [\lambda_1 \dots \lambda_N] \in \mathbb{R}^N : \lambda_i \geq 0, \; \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

The switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \; \sigma(t) \in \mathcal{I} \tag{1.29}$$

can be expressed as a particular case of a polytopic system with λ_i parameters restricted to two discrete values $\lambda_i \in \{0, 1\}$, such that $\lambda_i = 1$ when $\sigma(t) = i$. There exist a large variety of approaches for the polytopic system stabilization. For example, when the switching law is not known, a design method for a LMI static state feedback has been proposed by Boyd in [13]. The result is based on the existence of a common quadratic Lyapunov function for the closed loop system. It requires the existence of a symmetric positive definite matrix S and of a matrix Y that verify the LMI

$$SA_i^T + A_iS + B_iY + Y^TB_i < 0, \; \forall \; i \; \in \mathcal{I}.$$

In this case, the common quadratic Lyapunov is given with the Lyapunov matrix $P = S^{-1}$. The state feedback is described by

$$u(t) = Kx(t)$$

with $K = YS^{-1}$.

If the switching signal is available in real time then we can look for a *switched state feedback* :

$$u(t) = K_{\sigma(t)}x(t). \tag{1.30}$$

The solution depends on the existence of several matrices Y_i and the previous LMI is replaced by

$$SA_i^T + A_iS + B_iY_i + Y_i^TB_i < 0, \; \forall \; i \; \in \mathcal{I}. \tag{1.31}$$

The state feedback is given by the equation (1.30) with $K_i = Y_iS^{-1}, \; \forall \; i \; \in \mathcal{I}$.

Discrete time

In discrete time, a more general approach is possible. This approach is based on the use of parameter dependent Lyapunov functions

$$V(x) = x^T \sum_{i=1}^{N} \lambda_i P_i x.$$

These functions have been proposed by Daafouz and Bernoussou [21] for polytopic systems. The extension to the case of switched systems is given in [22].

Theorem 18 *The switched system*

$$x(k+1) = A_\sigma x(k+1) + B_\sigma u(k),\ \forall\ \sigma \in \mathcal{I}$$

can be stabilized using the switched state feedback $u = K_\sigma x$ if there exist positive definite matrices S_i and R_i, G_i matrices ,$\forall\ i\ \in \mathcal{I}$, such that the LMIs

$$\begin{bmatrix} G_i + G_i^T - S_i & \left(A_i^T G_i + B_i R_i\right)^T \\ A_i^T G_i + B_i R_i & S_j \end{bmatrix} > 0,\ \forall (i,j) \in \mathcal{I} \times \mathcal{I} \tag{1.32}$$

are satisfied $\forall\ i, j \in \mathcal{I}$. The state feedback is given with $K_i = R_i G_i^{-1}$ and the Lyapuynov matrices are $P_i = S_i^{-1}$.

The use of parameter dependent Lyapunov functions for continuous time switched linear systems is not possible for the moment unless particular constraints are being considered for the switching law. In the case of of an arbitrary switching law one must use a common Lyapunov function. More efficient stabilization criteria can be derived when a priori knowledge about the switching function is used. A notable result in this context is the work of Johanson [47] for switched systems with state space dependent switching laws. The author uses multiple Lyapunov-like functions in order to obtain a switched state feedback for piecewise linear systems. The controller is derived via optimal control methods.

1.5 Conclusion

This chapter gave an overview of some important stability/stabilizability criteria encountered in the context of switched linear systems. In the following chapters we intend to provide robust control methods for stability analysis and control design for switched linear systems.

Chapter 2

Uncertainty in switched systems

In this chapter, stability and control synthesis problems for switched uncertain systems will be discussed. The first section is dedicated to parametric uncertainties. The case of discrete time systems with polytopic uncertainty under an arbitrary switching law will be considered. We assume that the uncertain parameters are time varying.

The second section is a study of switched systems with uncertain switching law. These uncertainties are unavoidable in the context of practical applications, where delays occur between the moment the switching signal is computed and the effective application at actuators level. Such uncertainties also appear when the switching signal is an exogenous event, whose occurrence is not perfectly known. The closed loop stability in the context of such switched systems will be adressed.

2.1 Parametric uncertainties

The stability problem is very complex when parameter uncertainty is considered. In this case, the dynamic of each mode is affected by uncertainty. Until now, few results are concerned with robust stability in the context of switched systems with arbitrary switching law. In the specialized literature [117, 46, 103] the results obtained for the non-uncertain case are directly applied : a simple quadratic Lyapunov function is associated to each uncertain subsystem. The Lyapunov functions is switched similarly to the subsystem but it does take into account the uncertain parameters. The main drawback of such quadratic stability based approaches is the conservatism inherent to the use of a common function over the whole uncertainty.

In the robust control domain, conditions based on parameter dependent Lyapunov functions (PDLF) are proposed in order to reduce the conservatism related to uncertainty problems [23, 26]. Recently, in publication [21], these Lyapunov functions were used to analyze the stability of discrete systems with polytopic uncertainty. The solution is a class of Lyapunov functions that depend in a polytopic way on the uncertain parameter and that can be derived from LMI conditions.

In this section we intend to study the robust stability and control synthesis of discrete time uncertain switching systems under arbitrary switching. We will consider that the uncertainty can be modeled in a polytopic way. We intend to reduce the conservatism related to uncertainty problems using Lyapunov functions that depend on the uncertain parameter and that take into account the structure of the system.

Two approaches will be presented. First, we will prove that stability analysis for switched uncertain systems with polytopic uncertainty can be addressed as the problem of analyzing a unique equivalent matrix polytope for which the parameter dependent Lyapunov function approach presented in publication [21] can be applied. The results are well situated for stability analysis problems. However, the method is not obvious to apply for control synthesis. Second, we will show that using parameter dependent Lyapunov functions that have a structure similar to the system leads to LMI conditions for both stability analysis and control synthesis. A new concept will be introduced : the switched parameter dependent Lyapunov functions, i.e. functions that depend in a polytopic way on the uncertainty of each mode and that are switched following the structure of the system. We will obtain a necessary and sufficient LMI condition for the existence of these functions. A numerical example will illustrate the advantage of these approaches compared to other conditions. We will also show how these functions can be used for control synthesis problems.

2.1.1 Preliminaries

In the case of discrete time switched linear systems, the uncertain model is described by :

$$x(k+1) = \hat{A}_{\sigma(k)}\, x(k) \tag{2.1}$$

where $\hat{A}_i$ represents an uncertain matrix that belongs to the domain $\mathcal{A}_i \subset \mathbb{R}^{n\times n}$, $\forall\ i \in \mathcal{I}$. Several types of uncertain domains exist in the literature [7]. Usually we consider compact $\mathcal{A}_i$ domains, such as

- *norm bounded uncertainties*, with

$$\hat{A}_i \in \mathcal{A}_i = \{A_{i,0} + D_i F_i(k) E_i,\ \|F_i(k)\| \leq 1\}\,, \tag{2.2}$$

  where the $A_{i,0}$ matrices describe a nominal behavior, $F_i(k)$ is an uncertain time varying matrix and D_i,E_i $\forall\ i \in \mathcal{I}$ are known matrices;

- *polytopic uncertainties*, with

$$\hat{A}_i \in \mathcal{A}_i = co\,\{A_{i,1}, A_{i,2}, \ldots, A_{i,n_i}\}\,,\ \forall\ i \in \mathcal{I} \tag{2.3}$$

  where the $A_{i,1}, A_{i,2}, \ldots, A_{i,n_i}$ matrices are called *vertex*.

An important problem is to extend the methods given for the classical switched systems case to the case when the switched models are uncertain.

2.1.1.1 Stability criteria for polytopic systems

Here we recall a stability result based on parameter dependent Lyapunov functions and show how it can help for robust stability analysis in the case of switched uncertain systems with polytopic uncertainty.

An uncertain discrete time polytopic system is described by :

$$x(k+1) = \sum_{i=1}^{n} \alpha_i(k) A_i x(k), \tag{2.4}$$

$$\sum_{i=1}^{n} \alpha_i(k) = 1, \ \alpha_i \geq 0, \ \forall \ i = 1, \ldots, na$$

$\alpha_i \geq 0$ represent the uncertain parameters.

In [21], the stability of such a system is checked using PDLFs of the form :

$$V(k) = x^T(k) \sum_{i=1}^{n} \alpha_i(k) P_i \, x(k), \ P_i = P_i^T > 0. \tag{2.5}$$

These functions depend on the uncertain parameters $\alpha_i, \forall \ i = 1, \ldots, na$.

A system is said *poly-quadratically stable* if there exists a PDLF (2.5) whose difference is negative definite [21].

Necessary and sufficient conditions for the poly-quadratic stability are given by the following theorem:

Theorem 19 *[21] System (2.4) is poly-quadratically stable iff there exists symmetric positive definite matrices S_i, S_j and matrices G_i of appropriate dimension such that :*

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i^T \\ A_i G_i & S_j \end{bmatrix} > 0 \tag{2.6}$$

for all $i = 1, \ldots, N$ and $j = 1, \ldots, N$. In this case, the Lyapunov function is given by (2.5) with $P_i = S_i^{-1}$.

2.1.1.2 Application to the case of uncertain switched systems

Consider the uncertain switched system :

$$x(k+1) = \hat{A}_{\sigma(k)}(k) x(k) \tag{2.7}$$

where $\left\{ \hat{A}_i : i \in I \right\}$ with $I = \{1, 2, .., N\}$, is a family of matrices and $\sigma : Z^+ \to I$ is the switching signal. $\hat{A}_\sigma(k)$ is the uncertain matrix :

$$\hat{A}_\sigma(k) = \sum_{j=1}^{n_\sigma} \alpha_{\sigma j}(k) A_{\sigma j}, \ \sum_{j=1}^{n_\sigma} \alpha_{\sigma j}(k) = 1, \ \alpha_{\sigma j}(k) \geq 0$$

where the coefficients α_{ij} describe the polytopic uncertainty of the i^{th} mode of the systems, $A_{\sigma j}$ denote the extreme points of the polytope $\hat{A}_\sigma$ and n_σ is the number of such points.

Consider $\mathcal{S} = \{A_{11}, \ldots, A_{1n_1}, \ldots, A_{N1}, \ldots, A_{Nn_N}\}$ the set of all vertices defining the dynamic of system (2.7) and $\mathcal{E} = \{E \mid E \in \mathcal{S},\ co\mathcal{S} \neq co(\mathcal{S} - \{E\})\}$, $\mathcal{E} = \{E_1 \ldots E_M\}$, the set of extreme points of $co\mathcal{S}$. Here $co\mathcal{S}$ is the convex hull of $\mathcal{S}$ and M is the number of extreme points of $\mathcal{S}$ [2].

Theorem 20 *[41] System (2.7) is stable if there exists symmetric positive definite matrices S_i, S_j and matrices G_i of appropriate dimension such that :*

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T E_i^T \\ E_i G_i & S_j \end{bmatrix} > 0 \tag{2.8}$$

for all $i = 1, \ldots, M$ and $j = 1, \ldots, M$. The parameter dependent Lyapunov function is constructed with $P_i = S_i^{-1}$.

Proof. The proof is based on the fact that a convex combination of convex polytopes is also a convex polytope, in other words, on the fact that the system (2.7) may be expressed as (2.4), for which we have a stability criterion.

System (2.7) is equivalent to

$$x(k+1) = \sum_{i=1}^{N} \sum_{j=1}^{n_i} \xi_i(k)\alpha_{ij}(k)A_{ij}x(k), \tag{2.9}$$

with

$$\xi_i : \mathbb{Z}^+ \to \{0,1\},\ \sum_{i=1}^{N} \xi_i(k) = 1,\ \forall\, k \in \mathbb{Z}^+.$$

Here, the switching function σ is replaced by the parameters ξ_i; $\xi_i = 1$ when $\sigma = i$ and 0 otherwise. This representation is strictly equivalent with (2.7). Therefore, no additional conservatism is introduced.

Consider the notation :

$$\mathcal{A} = \sum_{i=1}^{N} \xi_i(k)\hat{A}_i = \sum_{i=1}^{N} \sum_{j=1}^{n_i} \xi_i(k)\alpha_{ij}(k)A_{ij}. \tag{2.10}$$

that is, $\mathcal{A}$ is a convex combination of $A_{ij} \in \mathcal{S}$. As E_p are the extreme points of $co\mathcal{S}$, we can write

$$\mathcal{A} = \sum_{p=1}^{M} \Lambda_p(k)E_p,\ \Lambda_p \geq 0,\ \sum_{p=1}^{M} \Lambda_p(k) = 1.$$

[2]Numerical methods for convex hull and extreme points computation are described in [72, 5].

Therefore $\mathcal{A}$ is a polytopic uncertainty similar to that in equation (2.4). From equation (2.9), it can be noticed that any switched uncertain system with polytopic uncertainty (2.7) may be expressed as a simple uncertain system of the form (2.4). By applying Theorem 19, the proof is obvious. □

Similar to publication [21], this approach can be directly applied to the switched state feedback stabilization problem, when the input matrix is not uncertain, it is known and constant for all system modes. However, in the general case, when both the dynamic and the input matrix are switched and uncertain, the construction of a switched state feedback control leads to BMI conditions, that are difficult to solve.

2.1.2 Switched parameter dependent Lyapunov functions

For the control synthesis, a switched parameter dependent Lyapunov function is introduced. It can be used for both proving the asymptotic stability of switched uncertain systems and constructing a switched state feedback when both the dynamic and the input matrix are switched and uncertain.

2.1.2.1 Robust stability analysis

Consider the switched uncertain system (2.7) and its equivalent representation (2.9). Based on a structure similar to the uncertainty description, we propose the following type of Lyapunov function:

$$\begin{aligned} V(k) &= x^T(k)\hat{P}_\sigma(k)x(k), \quad \text{with } \hat{P}_\sigma(k) = \sum_{j=1}^{n_\sigma} \alpha_{\sigma j}(k) P_{\sigma j} \\ &= x^T(k) \sum_{i=1}^{N} \sum_{j=1}^{n_i} \xi_i(k)\alpha_{ij}(k) P_{ij}\, x(k) \end{aligned} \tag{2.11}$$

where $P_{ij} = P_{ij}^T > 0$.

System (2.9) is asymptotically stable if the difference of the Lyapunov function along the solutions of (2.9)

$$\mathcal{L} = V(k+1) - V(k)$$

satisfies :

$$\mathcal{L} = x^T(k)\left(\mathcal{A}^T \mathcal{P}_+ \mathcal{A} - \mathcal{P}\right) x(k) < 0 \tag{2.12}$$

where

$$\begin{aligned}
\mathcal{A} &= \sum_{i=1}^{N}\sum_{j=1}^{n_i} \xi_i(k)\alpha_{ij}(k)A_{ij}, \\
\mathcal{P} &= \sum_{i=1}^{N}\sum_{j=1}^{n_i} \xi_i(k)\alpha_{ij}(k)P_{ij}, \\
\mathcal{P}_+ &= \sum_{i=1}^{N}\sum_{j=1}^{n_i} \xi_i(k+1)\alpha_{ij}(k+1)P_{ij} \\
&= \sum_{m=1}^{N}\sum_{n=1}^{n_m} \xi_m(k)\alpha_{mn}(k)P_{mn},
\end{aligned}$$

$\forall\ k,\ \forall\ x(k),\ \forall\ \xi_i$ and $\forall\ \alpha_{ij}$.

Definition 21 *The functions (2.11) that satisfy (2.12) are called switched parameter dependent Lyapunov functions (SPDLF) for the system (2.7) (also for the equivalent system (2.9)).*

Theorem 22 *[41] A switched parameter dependent Lyapunov function can be constructed iff there exist symmetric positive definite matrices S_{ij}, S_{mn} and matrices G_{ij} of appropriate dimension such that :*

$$\begin{bmatrix} G_{ij}+G_{ij}^T-S_{ij} & G_{ij}^T A_{ij}^T \\ A_{ij}G_{ij} & S_{mn} \end{bmatrix} > 0 \tag{2.13}$$

for all $i=1,\ldots,N$ and $j=1,\ldots,n_i$, $m=1,\ldots,N$ and $n=1,\ldots,n_m$. The switched parameter dependent Lyapunov function is constructed with $P_{ij}=S_{ij}^{-1}$.

Proof. The proof follows similar arguments to the ones in [21].

To prove sufficiency, assume that the condition is feasible. Then

$$G_{ij}+G_{ij}^T-S_{ij} > 0.$$

Therefore G_{ij} is non singular and as S_{ij} is strictly positive definite, we have :

$$(S_{ij}-G_{ij})^T S_{ij}^{-1}(S_{ij}-G_{ij}) \geq 0,$$

which is equivalent to

$$G_{ij}^T S_{ij}^{-1} G_{ij} \geq G_{ij}^T + G_{ij} - S_{ij}.$$

Therefore the relation (2.13) implies

$$\begin{bmatrix} G_{ij}^T S_{ij}^{-1} G_{ij} & G_{ij}^T A_{ij}^T \\ A_{ij}G_{ij} & S_{mn} \end{bmatrix} > 0. \tag{2.14}$$

Pre- and post- multiplying the inequality (2.14) by $diag(G_{ij}^{-T}, S_{mn}^{-1})$ and its transpose gives

$$\begin{bmatrix} S_{ij}^{-1} & A_{ij}^T S_{mn}^{-1} \\ S_{mn}^{-1} A_{ij} & S_{mn}^{-1} \end{bmatrix} > 0 \tag{2.15}$$

Defining $P_{ij} = S_{ij}^{-1}$ the inequality (2.15) becomes

$$\begin{bmatrix} P_{ij} & A_{ij}^T P_{mn} \\ P_{mn} A_{ij} & P_{mn} \end{bmatrix} > 0$$

for all $i = 1, \dots, N$ and $j = 1, \dots, n_i$, $m = 1, \dots, N$ and $n = 1, \dots, n_m$. Repeatedly multiplying by the appropriate coefficients and summing one obtains :

$$\begin{bmatrix} \mathcal{P} & \mathcal{A}^T \mathcal{P}_+ \\ \mathcal{P}_+ \mathcal{A} & \mathcal{P}_+ \end{bmatrix} > 0$$

Using the Schur complement, this is equivalent to

$$\mathcal{A}^T \mathcal{P}_+ \mathcal{A} - \mathcal{P} < 0 \tag{2.16}$$

which implies the existence of the switched parameter dependent Lyapunov function (2.11).

To prove necessity, assume that $\mathcal{L}$ satisfies the relation (2.12). Then the condition

$$\mathcal{A}^T \mathcal{P}_+ \mathcal{A} - \mathcal{P} < 0 \tag{2.17}$$

is true which implies that

$$P_{ij} - A_{ij}^T P_{pq} A_{ij} > 0$$

for all $i, p = 1..N,\ j = 1..n_i,\ q = 1..n_p$. Let $S_{ij} = P_{ij}^{-1}$ and $S_{pq} = P_{pq}^{-1}$. Applying the Schur complement lemma one obtains

$$S_{pq} - A_{ij} S_{ij} A_{ij}^T = T_{ijpq} > 0.$$

For $G_{ij} = S_{ij} + 2g_{ij}I$ with g_{ij} a positive scalar, there a g_{ij} sufficiently small such that

$$g_{ij}^{-2}(S_{ij} + 2g_{ij}I) > A_{ij}^T T_{ijpq}^{-1} A_{ij}$$

Using the Schur complement lemma, the previous relation is equivalent to

$$\begin{bmatrix} S_{ij} + 2g_{ij}\mathbf{I} & -g_{ij} A_{ij}^T \\ -A_{ij} g_{ij} & T_{ijpq} \end{bmatrix} > 0$$

and it can be expressed as

$$\begin{bmatrix} G_{ij} + G_{ij}^T - S_{ij} & S_{ij} A_{ij}^T - G_{ij} A_{ij}^T \\ A_{ij} S_{ij} - A_{ij} G_{ij} & S_{pq} - A_{ij} S_{ij} A_{ij}^T \end{bmatrix} > 0$$

which is equivalent to

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -A_{ij} & \mathbf{I} \end{bmatrix} \begin{bmatrix} G_{ij} + G_{ij}^T - S_{ij} & G_{ij}^T A_{ij}^T \\ A_{ij} G_{ij} & S_{pq} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -A_{ij}^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix} > 0.$$

□

2.1.2.2 Robust Control Synthesis

The switched state feedback control synthesis problem is considered for the following switching uncertain system :

$$x(k+1) = \hat{A}_\sigma(k)x(k) + \hat{B}_\sigma(k)u(k), \tag{2.18}$$

where

$$\hat{A}_\sigma(k) = \sum_{j=1}^{na_\sigma} \alpha_{\sigma j}(k)A_{\sigma j}, \text{ and } \hat{B}_\sigma(k) = \sum_{l=1}^{nb_\sigma} \beta_{\sigma l}(k)B_{\sigma l}, \tag{2.19}$$

$$\sum_{j=1}^{na_\sigma} \alpha_{\sigma j}(k) = 1, \ \alpha_{\sigma j}(k) \geq 0,$$

$$\sum_{l=1}^{nb_\sigma} \beta_{\sigma l}(k) = 1, \ \beta_{\sigma l}(k) \geq 0, \ \forall\, k \in \mathbb{Z}^+$$

represent the uncertainty on the dynamic and input matrix, respectively. The switching signal is given by σ. Here $\alpha_{\sigma j}$ and $\beta_{\sigma l}$ are the uncertain parameters while na_σ and nb_σ represent the number of extreme points in the uncertainty $\hat{A}_\sigma$ and $\hat{B}_\sigma$, respectively.

Assumption 23 *We assume that the switching signal $\sigma(k)$ is not known a priori but is is available in real time for control implementation.*

The problem is to find a switched state feedback

$$u(k) = K_{\sigma(k)}x(k) \tag{2.20}$$

which stabilizes the closed loop switched system.

The solution to this robust control problem is given in the following theorem:

Theorem 24 *[41] There exists a switched state feedback (2.20) stabilizing the system (2.18) if there exists symmetric positive definite matrices S_{ijl}, and G_i, R_i matrices, solutions of the LMIs :*

$$\begin{bmatrix} G_i + G_i^T - S_{ijl} & G_i^T A_{ij}^T + R_i^T B_{il}^T \\ A_{ij}G_i + B_{il}R_i & S_{pwv} \end{bmatrix} > 0, \tag{2.21}$$

$\forall i = 1, \ldots, N,\ j = 1, \ldots, na_i,\ l = 1, \ldots, nb_i,\ p = 1, \ldots, N,\ w = 1, \ldots, na_p$ and $v = 1, \ldots, nb_p$. The switched state feedback is given by the equation (2.23) with

$$K_i = R_i G_i^{-1}.$$

Proof. The system is written as :

$$x(k+1) = \sum_{i=1}^{N} \xi_i(k)\hat{A}_i(k)x(k) + \sum_{i=1}^{N} \xi_i(k)\hat{B}_i(k)u(k), \tag{2.22}$$

where the $\xi_i(k)$ parameters are used instead of the switching law $\sigma(k)$ such that $\xi_i = 1$ when $\sigma(k) = i$ and $\xi_i = 0$ in the opposite case.

The switched state feedback (2.20) may be expressed in the following form

$$u(k) = \sum_{i=1}^{N} \xi_i(k) K_i x(k) \tag{2.23}$$

and the closed loop switched linear system is given by the equation

$$x(k+1) = \sum_{i=1}^{N} \xi_i(k)(\hat{A}_i + \hat{B}_i K_i)x(k).$$

Considering the equation (2.19), the closed loop system becomes

$$x(k+1) = \sum_{i=1}^{N} \xi_i(k) \left(\sum_{j=1}^{na_i} \alpha_{ij}(k) A_{ij} + \sum_{l=1}^{nb_i} \beta_{il}(k) B_{il} K_i \right) x(k)$$

which is equivalent to:

$$x(k+1) = \sum_{i=1}^{N} \sum_{j=1}^{na_i} \sum_{l=1}^{nb_i} \xi_i(k)\alpha_{ij}(k)\beta_{il}(k) H_{ijl} x(k)$$

with $H_{ijl} = A_{ij} + B_{il} K_i$.

Consider the Lyapunov function

$$V(k) = x^T(k)\mathcal{P}x(k)$$

with

$$\mathcal{P} = \sum_{i=1}^{N} \sum_{j=1}^{na_i} \sum_{l=1}^{nb_i} \xi_i(k)\alpha_{ij}(k)\beta_{il}(k) P_{ijl}. \tag{2.24}$$

The difference of the Lyapunov function is given by

$$V(k+1) - V(k) = x(k)(\mathcal{H}^T \mathcal{P}_+ \mathcal{H} - \mathcal{P})x(k),$$

where

$$\mathcal{H} = \sum_{i=1}^{N} \sum_{j=1}^{na_i} \sum_{l=1}^{nb_i} \xi_i(k)\alpha_{ij}(k)\beta_{il}(k) H_{ijl},$$

and

$$\begin{aligned} \mathcal{P}_+ &= \sum_{i=1}^{N} \sum_{j=1}^{na_i} \sum_{l=1}^{nb_i} \xi_i(k+1)\alpha_{ij}(k+1)\beta_{il}(k+1) P_{ijl} \\ &= \sum_{p=1}^{N} \sum_{w=1}^{na_p} \sum_{v=1}^{nb_p} \xi_p(k)\alpha_{pw}(k)\beta_{pv}(k) P_{pwv}. \end{aligned}$$

Assume that the conditions (2.21) are verified. Using

$$K_i = R_i G_i^{-1}$$

in the conditions (2.21), one obtains :

$$\begin{bmatrix} G_i + G_i^T - S_{ijl} & G_i^T A_{ij}^T + G_i^T K_i^T B_{il}^T \\ A_{ij} G_i + B_{il} K_i G_i & S_{pwv} \end{bmatrix} > 0,$$

which is equivalent to

$$\begin{bmatrix} G_i + G_i^T - S_{ijl} & G_i^T H_{ijl}^T \\ H_{ijl} G_i & S_{pwv} \end{bmatrix} > 0. \tag{2.25}$$

Using similar arguments to the ones given for the proof of theorem 22, we can show that (2.25) is equivalent to:

$$\begin{bmatrix} P_{ijl} & H_{ijl}^T P_{pwv} \\ P_{pwv} H_{ijl} & P_{pwv} \end{bmatrix} > 0$$

with $P_{ijl} = S_{ijl}^{-1}$, for all $i = 1, \ldots, N$, $j = 1, \ldots, na_i$, $l = 1, \ldots, nb_i$, $p = 1, \ldots, N$, $w = 1, \ldots, na_p$ and $v = 1, \ldots, nb_p$. By repeatedly multiplying by the appropriate coefficients and summing one gets:

$$\begin{bmatrix} \mathcal{P} & \mathcal{H}^T \mathcal{P}_+ \\ \mathcal{P}_+ \mathcal{H} & \mathcal{P}_+ \end{bmatrix} > 0$$

which implies, using the Schur complement lemma, that the difference of the Lyapunov function (2.24) is strictly decreasing for all the solutions of the system.

Remarks.

- In a similar manner to the case of switched systems without uncertainties [22], it is very easy to show that the proposed stabilization conditions generalize the classical conditions that use Lyapunov functions of the form $x^T P_\sigma x$ or $x^T P x$.

- By duality, the results obviously apply to the state reconstruction problem for uncertain switched systems with uncertain output matrix.

- When the uncertain matrices $\hat{A}_\sigma$ and $\hat{B}_\sigma$ depend on a common uncertain parameter, a particular case of (2.19) with $j = l$, $na_\sigma = nb_\sigma$, and $\alpha_{\sigma j} = \beta_{\sigma l} = \lambda_{\sigma j}$, the LMI conditions (2.21) become :

$$\begin{bmatrix} G_i + G_i^T - S_{ij} & G_i^T A_{ij}^T + R_i^T B_{ij}^T \\ A_{ij} G_i + B_{ij} R_i & S_{uv} \end{bmatrix} > 0, \tag{2.26}$$

  for all $i, u = 1..N$, $j, v = 1..na_\sigma$, where S_{ij} and S_{uv} are symmetric positive definite matrices.

2.1.3 Numerical examples

The efficiency of the proposed stability analysis and control design methods presented in the previous sub-sections, is illustrated with some academical examples.

Example 2.1 To illustrate the LMI stability conditions derived in the previous sections, we will consider a switched uncertain system with affine uncertainty of the form :

$$x(k+1) = \hat{A}_\sigma x(k)$$

where

$$\hat{A}_\sigma(k) = A_{0\sigma} + D_\sigma \rho(k) E_\sigma, \ \rho(k) \in [-1, 1].$$

The interest of using such type of uncertainty is the fact that it can be represented both as a norm bounded uncertainty (2.2) with $\rho(k) = F(k)$, and as a polytopic uncertainty

$$\hat{A}_\sigma(k) = \alpha_{\sigma 1}(k) A_{\sigma 1} + \alpha_{\sigma 2}(k) A_{\sigma 2},$$

with

$$A_{\sigma 1} = A_{0\sigma} + D_\sigma E_\sigma \text{ and } A_{\sigma 2} = A_{0\sigma} - D_\sigma E_\sigma,$$

$$\alpha_{\sigma 1}(k), \alpha_{\sigma 2}(k) > 0, \ \alpha_{\sigma 1}(k) + \alpha_{\sigma 2}(k) = 1, \ \forall\, k \in \mathbb{Z}^+.$$

This allows us to compare our results with other LMI conditions. The conditions that exist in the literature, the only ones available in the case of uncertain switched systems, are based on quadratic stability [46] and on Lyapunov functions that do not depend on uncertain parameters [103], for the norm bounded uncertainty case.

The example that we choose corresponds to

$$A_{01} = \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.1 & -0.5 \\ 0.8 & 0 & -0.1 & -0.3 & 0.3 \\ 0 & -0.3 & -0.4 & 0 & 0 \\ 0 & 0.3 & 0.1 & 0.3 & 0.5 \\ -0.2 & 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$A_{02} = \begin{bmatrix} -0.7 & -0.7 & 0 & 0 & 0.2 \\ 0.5 & 0.3 & 0.3 & -0.3 & 0 \\ 0.3 & 0.4 & 0.3 & 0.6 & 0.3 \\ 0.3 & -0.8 & 0 & 0 & 0 \\ 0.1 & -0.7 & 0.1 & -0.3 & 0.3 \end{bmatrix},$$

$$D_1^T = [0.2\ 0.5\ -0.1\ 0.3\ 0.2],$$

$$D_2^T = [-0.5\ 0.38\ 0.5\ 0.2\ 0.5],$$

$$E_1 = [-0.3\ -0.3\ -0.5\ 0.2\ 0.3]$$

$$E_2 = [-0.2\ 0.1\ -0.1\ -0.05\ 0.7].$$

In this case, the LMI conditions given in [46, 103] are too conservative and do not have any solution.

Yet, the LMI conditions (2.8) and (2.13) here presented have solutions. The existence of these solutions was tested by a numerical LMI solver (SEDUMI). Hence, a parameter dependent and a switched parameter dependent Lyapunov function can be constructed using conditions (2.8) and (2.13).

Example 2.2 We consider the uncertain switched system :

$$x(k+1) = \hat{A}_\sigma(k)x(k) + \hat{B}_\sigma(k)u(k), \tag{2.27}$$

where

$$\hat{A}_\sigma(k) = A_{0\sigma} + D_\sigma \rho(k) E^A_\sigma,$$

and

$$\hat{B}_\sigma(k) = B_{0\sigma} + D_\sigma \rho(k) E^B_\sigma,\ \forall \rho(k) \in [-1, 1],$$

represent norm bounded uncertainties similar to ones given in the previous example.

The uncertain matrices $\hat{A}_\sigma$ and $\hat{B}_\sigma$, depend on a common uncertain parameter $\rho(k)$, similarly to the case treated in [103]. We consider the following matrices:

$$A_{01} = \begin{bmatrix} -0.1 & 0.7 & -0.2 \\ -0.4 & 0.7 & 1 \\ 0.3 & 0.3 & 0 \end{bmatrix},\ A_{02} = \begin{bmatrix} 1 & 0.7 & 0.7 \\ 0.4 & 0.6 & 0.2 \\ 1 & 0.7 & 0 \end{bmatrix},$$

$$B_{01} = [0.1\ 0.8\ 0.8]^T,\ B_{02} = [0.2\ 0.9\ 0.2]^T$$

$$D_1 = [-0.2328\ 0.4340\ -0.4590]^T,$$

$$D_2 = [-0.2645\ 0.2681\ 0.9316]^T,$$

$$E^A_1 = [0.7461\ -0.4767\ 0.1131],$$

$$E^A_2 = [-0.4787\ 0.4671\ 0.4731],$$

$$E^B_1 = 0.8194 \text{ and } E^B_2 = -0.7610.$$

The LMIs proposed in [46, 103] have no solution and do not allow to build a stabilizing switched state feedback.

Using theorem 24, we can construct a switched state feedback (2.23). In fact the LMI (2.26) have a solution and the switched gains are given by

$$K_1 = [0.1956\ -0.8403\ -0.7902]$$

$$K_2 = [-1.1285\ -1.1554\ -0.353].$$

2.2 Uncertain switching law

In addition to the classical parametric uncertainties, in the case of switched systems one should also consider uncertainties that are related to the switching signal. These uncertainties are unavoidable in the context of practical applications, where delays occur between the moment the switching signal is computed and the effective application at actuators level. Such uncertainties also appear when the switching signal is an exogenous event, whose occurrence is not perfectly known.

In the literature, few results are dealing with these robustness aspects. To our knowledge, only the case of piecewise affine systems has been treated (see for example [80]). For these systems, the uncertainties are due to the different switching surfaces modeling problems. In [80], the authors give an LMI solution by modifying the constraints on the linear matrix inequalities (1.22).

In this section, the case of discrete time systems stabilized by a switched state feedback is analyzed. The switching law is assumed to be arbitrary.

First, we will study the closed loop system's stability when the switching law uses a switching law that may be temporary uncertain. The problem is formalized as follows: assuming that the gains are designed such that the closed loop system is stable when the appropriate switching signal is used, find the restrictions that should be imposed for the switching law in order to guarantee that the system is stable in the uncertain case. The stability conditions are given in terms of *dwell time* [119, 67]. We intend to find the relation between the time spent in an uncertain switching signal configuration and the minimum time using the appropriate switching signal such that the system is still asymptotically stable.

Second, we suppose that the switching signal is *partially known*: the exact value is not available, but at each instant of time we know a sub-set of possible values. In this context, a robust switched state feedback will be proposed.

2.2.1 Temporary uncertain switching signal - problem formulation

This subsection is dedicated to the mathematical formalization of the problem under study.

Consider the following discrete time switched system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{2.28}$$

where $x(k) \in \mathbb{R}^n$ represents the system state, $u(k) \in \mathbb{R}^m$ the system input, $\sigma : \mathbb{Z}^+ \to \mathcal{I} = \{1, 2, \ldots, N\}$, the switching function and $A_{\sigma(k)} \in \mathbb{R}^{n\times n}$ and $B_{\sigma(k)} \in \mathbb{R}^{n\times m}$, the state and the input matrices. These matrices are switched according to the switching function $\sigma(k)$. Consider that the system is controlled via a command that takes into account the switching function. For the sake of simplicity we consider here the case of a switched state feedback

$$u(k) = K_{\gamma(k)}x(k) \tag{2.29}$$

where $\gamma : \mathbb{Z} \rightarrow \mathcal{I}$ represents the switching signal used for the control. We assume that the gains $K_{\gamma(k)}$ are designed such that the closed loop system is asymptotically stable when $\gamma(k) = \sigma(k)$.

Definition 25 (Nominal configuration) *When the system uses the appropriate switching signal, i.e. $\gamma(k) = \sigma(k)$, we say that the system is in a nominal configuration.*

Definition 26 (Uncertain configuration) *When the switching signal is uncertain, i.e. when the couple $(\gamma(k), \sigma(k)) \in \mathcal{I} \times \mathcal{I}$, we say that the system is in an uncertain configuration.*

Definition 27 (Mixed mode dynamic) *When $\gamma(k) \neq \sigma(k)$ we say that the closed loop system behaves according to a mixed mode dynamic. These dynamics are described by the following equation:*

$$x(k+1) = (A_i + B_i K_j)\, x(k), \quad i, j \in \mathcal{I}, \; i \neq j.$$

Ideally the switching signal used in the control $\gamma(k)$ is the same as the real one, $\sigma(k)$. In many cases this hypothesis is not realistic. When the switching law is uncertain, it is possible that inappropriate gains are used for control computation. The stabilizability of the system in a *mixed mode* configuration, with $\gamma(k) \neq \sigma(k)$ must be studied. These mixed modes are not necessarily stable. The gains are designed to ensure the stability in a nominal configuration, for $\sigma(k) = \gamma(k)$. They do not guarantee anything when mixed dynamics occur. If these dynamics occurs frequently then the systems performances and even its stability may be affected. The stability criteria should be reconsidered in order to take into account such behaviors.

The closed loop system is equivalent to a system that is switched both on the *real switching signal* $\sigma(k)$ and the one used for control, $\gamma(k)$:

$$x(k+1) \quad = \quad \bar{A}_{(\sigma(k),\gamma(k))}x(k) \tag{2.30}$$

with

$$\bar{A}_{(\sigma(k),\gamma(k))} \quad = \quad A_{\sigma(k)} + B_{\sigma(k)}K_{\gamma(k)}. \tag{2.31}$$

Assumption 28 (Temporary uncertain switching signal) *Here we consider the case when Δ^u samples are spent using an uncertain switching control (in a* *possible mixed mode configuration) each time the system is switched from one mode to another. We assume that this period can be bounded,*

$$\Delta^u_{min} \leq \Delta^u \leq \Delta^u_{max}. \tag{2.32}$$

Moreover, we assume that the system is controlled using the appropriate gains for at least Δ^n samples before another switching occurs.

The problem is to verify if the closed loop system is still stable when the switching signal is temporary uncertain. This implies finding the relation between Δ^n, Δ^u_{max}, Δ^u_{min} and the system parameters such that the closed loop system is asymptotically stable.

2.2.2 Stability conditions for systems with temporary uncertain switching signal

In this sub-section we present stability conditions in terms of dwell time. This conditions are based on the evaluation of the *decay rate* in a nominal configuration

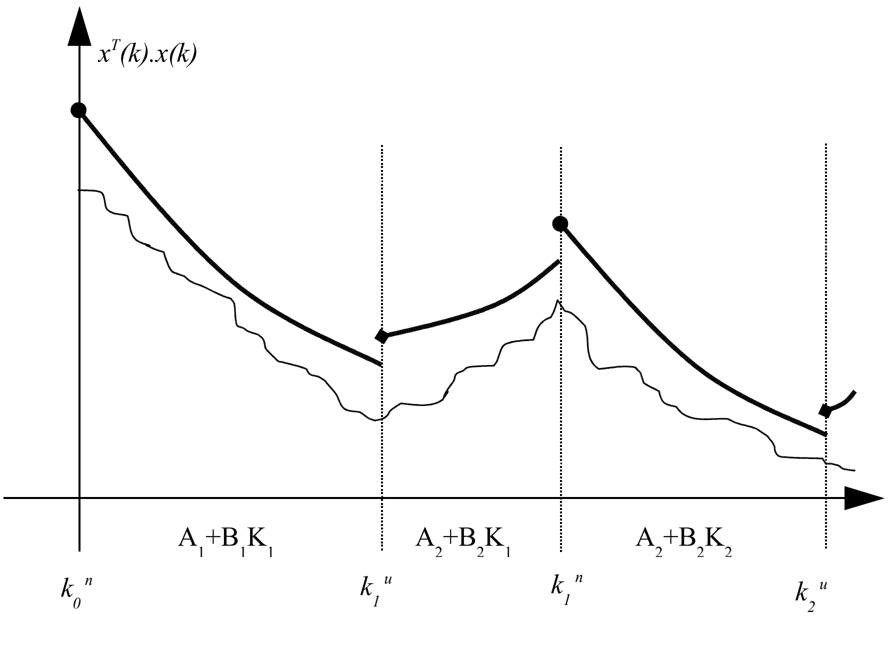

Figure 2.1: The stability problem may be treated by evaluating the upper bounds for the evolution of $\|x(k)\|^2$ in nominal and uncertain configurations . These upper bound, functions of the form $c_n \cdot {\lambda_n}^{k-k^n_j} \left\|x(k^n_j)\right\|^2$, for the nominal configuration and $c_u \cdot {\lambda_u}^{k-k^u_j} \left\|x(k^u_j)\right\|^2$, in the uncertain case, are represented using bold lines in the figure.

and of the *growth rate* in an uncertain configuration. These concepts are defined as follows:

Definition 29 (Decay rate) *Consider an exponentially stable system. Then there exists two scalars $c > 0$ and $0 < \lambda < 1$ that verify the inequality*

$$\|x(k)\|^2 < c \cdot \lambda^{k-k_0} \|x(k_0)\|^2, \ \forall k > k_0.$$

Any constant λ that verifies this inequality is called the decay rate of the system.

Definition 30 (Growth rate) [3] *Consider an unstable system. Then there exist two scalars $c > 0$ and $\lambda > 1$ that verify the inequality*

$$\|x(k)\|^2 < c \cdot \lambda^{k-k_0} \|x(k_0)\|^2, \ \forall k > k_0.$$

Any constant λ that verifies this inequality is called the growth rate of the system.

Consider the scalars $\lambda_n > 0$, $\lambda_u > 0$, and the symmetric positive definite matrices P^n_σ, $P^u_{\sigma,\gamma}$, with $(\sigma, \gamma) \in \mathcal{I} \times \mathcal{I}$ solutions of the following linear matrix inequalities [13]:

$$\bar{A}^T_{(\sigma,\gamma)} P^u_{(\sigma,\gamma)} \bar{A}_{(\sigma,\gamma)} - \lambda_u P^u_{(\sigma,\gamma)} < 0, \ \ \forall \ (\sigma, \gamma) \in \mathcal{I} \times \mathcal{I}, \tag{2.33}$$

$$\bar{A}^T_{(\sigma,\sigma)} P^n_\sigma \bar{A}_{(\sigma,\sigma)} - \lambda_n P^n_\sigma < 0, \ \ \forall \ \sigma \in \mathcal{I}. \tag{2.34}$$

Furthermore, consider the scalars

$$c_n = \frac{\max_{\sigma \in \mathcal{I}} eig_{max} \left(P^n_\sigma\right)}{\min_{\sigma \in \mathcal{I}} eig_{min} \left(P^n_\sigma\right)}, \tag{2.35}$$

$$c_u = \frac{\max_{(\sigma,\gamma) \in \mathcal{I} \times \mathcal{I}} eig_{max} \left(P^u_{(\sigma,\gamma)}\right)}{\min_{(\sigma,\gamma) \in \mathcal{I} \times \mathcal{I}} eig_{min} \left(P^u_{(\sigma,\gamma)}\right)} \tag{2.36}$$

where by $eig_{max}(X)$ and $eig_{min}(X)$ we denote the maximum and the minimum eigenvalues of X, respectively.

Theorem 31 *[43] Let λ^*_n, λ^*_u, be solutions of the following optimization problem $\lambda^*_n = \min \lambda_n$, $\lambda^*_u = \min \lambda_u$ subject to the LMI constraints (2.33) and (2.34). The closed loop system (2.30) is asymptotically stable if*

$$c_n \cdot c_u \cdot (\lambda^*_n)^{\Delta^n} (\lambda^*_u)^{\Delta^u_{max}} < 1, \ \textit{when} \ \lambda^*_u > 1 \tag{2.37}$$

or

$$c_n \cdot c_u \cdot (\lambda^*_n)^{\Delta^n} (\lambda^*_u)^{\Delta^u_{min}} < 1, \ \textit{when} \ \lambda^*_u < 1, \tag{2.38}$$

i.e. when all the mixed modes dynamics are stable.

[3] A brief idea about the way these two rates can be computed for the case of LTI systems is given in the appendix.

Proof. Consider the following functions:

$$V^n(k) = x^T(k)P^n_{\sigma(k)}x(k), \tag{2.39}$$

$$V^u(k) = x^T(k)P^u_{(\sigma(k),\gamma(k))}x(k). \tag{2.40}$$

The function V^n is associated to a nominal configuration. V^u is associated to uncertain dynamics.

From (2.33) and (2.34) one can notice that

$$V^n(k_1) < (\lambda_n^*)^{k-k_0}V(k_0),$$

if the appropriate switching signal is used for the control (if $\sigma(k) = \gamma(k) = \sigma(k_0), \ \ \forall\, k \in [k_0, k_1)$). Also,

$$V^u(k_1) < (\lambda_u^*)^{k-k_0}V(k_0)$$

if the switching signal in the control is not necessarily the same with the real one (the pair $\sigma(k), \gamma(k)$ takes an arbitrary value $\sigma(k_0), \gamma(k_0)$ in $\mathcal{I} \times \mathcal{I} \ \ \forall\, k \in [k_0, k_1)$). We can remark that $\lambda_n < 1$, since the gains K_σ are designed such that the matrix

$$A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma(k)}$$

is Schur stable $\forall\, \sigma \ \in \ \mathcal{I}$ while λ_u may be greater than one. This implies that the function $V^n(k)$ is strictly decreasing and its decay rate is bounded by (λ_n^*). On the other hand, since the gains $K_{\gamma(k)}$ are not designed for stabilizing other combinations than

$$A_{\gamma(k)} + B_{\gamma(k)}K_{\gamma(k)},$$

the function $V^u(k)$ may diverge. However its growth rate can be bounded by (λ_u^*).

The functions $V^u(k)$, $V^n(k)$ can be bounded using the minimum and maximum eigenvalues of P^u_σ, P^n_σ, that is

$$\min_{\sigma\in\mathcal{I}} eig_{min}\left(P^n_\sigma\right)\|x(k)\|^2 < V^n(k) <$$

$$< \max_{\sigma\in\mathcal{I}} eig_{max}\left(P^n_\sigma\right)\|x(k)\|^2,$$

for the nominal configuration and

$$\min_{\sigma\in\mathcal{I}} eig_{min}\left(P^u_{(\sigma,\gamma)}\right)\|x(k)\|^2 < V^u(k) <$$

$$\max_{(\sigma,\gamma)\in\mathcal{I}\times\mathcal{I}} eig_{max}\left(P^u_{(\sigma,\gamma)}\right)\|x(k)\|^2,$$

for the uncertain switching configuration. Therefore we can deduce the state vector norm decay/growth rate in a nominal configuration ($\sigma(k) = \gamma(k)$) and in an uncertain switching signal configuration as follows:

$$\|x(k_1)\|^2 < \begin{cases} c_n \cdot (\lambda_n^*)^{k-k_0} \|x(k_0)\|^2, & \sigma(k) = \gamma(k), \\ c_u \cdot (\lambda_u^*)^{k-k_0} \|x(k_0)\|^2, & (\sigma(k), \gamma(k)) \in \mathcal{I} \times \mathcal{I}, \end{cases} \tag{2.41}$$

$\forall\ k \in [k_0, k_1)$ with c_n and c_u given by (2.35) and (2.36) respectively.

The system behavior is described by the sequential succession of nominal configurations ($\sigma(k) = \gamma(k)$) and uncertain switching signal configurations ($(\sigma(k), \gamma(k)) \in \mathcal{I} \times \mathcal{I}$ with the possibility that $\sigma(k) \neq \gamma(k)$). Let k_j^u describe the instants the closed loop system jumps to an uncertain configuration and k_j^n, the instant the system enters in a nominal configuration, $j = 1, 2, 3 \ldots$. Using our hypothesis on the minimum dwell time in a nominal configuration Δ^n and on the bounds of the uncertain switching signal period, we can say that

$$\Delta^u_{min} \leq k_j^n - k_j^u \leq \Delta^u_{max},\ k_{j+1}^u - k_j^n \geq \Delta^n.$$

Without any lose of generality, we may consider that the system starts with an uncertain configuration, that is $k_j^u < k_j^n$. Consider the evolution of the system for a sequence. We analyze the system behavior from k_j^u to k_{j+1}^u. Since $\sigma(k) \neq \gamma(k),\ \forall k \in \left[k_j^u, k_j^n\right)$ and $\sigma(k) = \gamma(k),\ \forall k \in \left[k_j^n, k_{j+1}^u\right)$, one can notice using (2.41) that the norm of the state at the end of a sequence can be upper bounded such that

$$\left\|x(k_{j+1}^u)\right\|^2 < c_n \cdot c_u \cdot (\lambda_n^*)^{k_{j+1}^u - k_j^n} \cdot (\lambda_u^*)^{k_j^n - k_j^u} \cdot \left\|x(k_j^u)\right\|^2.$$

Using the fact that λ_n^* is less then one and that the minimum dwell time on the nominal configuration is Δ^n we can deduce that the decay rate in nominal configuration satisfies the relation

$$(\lambda_n^*)^{k_{j+1}^u - k_j^n} \leq (\lambda_n^*)^{\Delta^n}.$$

On the other hand, since λ_u^* may be both less than one (when the set of gains K_σ stabilize each all the subsystems) or greater than one (when at least one of the mixed modes $A_{\sigma(k)} + B_{\sigma(k)} K_{\gamma(k)}$ is unstable) we find that

$$(\lambda_u^*)^{k_j^n - k_j^u} \leq \begin{cases} (\lambda_u^*)^{\Delta^u_{max}}, & \lambda_u^* > 1 \\ (\lambda_u^*)^{\Delta^u_{min}}, & \lambda_u^* \leq 1. \end{cases}$$

Using the equation (2.41) we obtain that the system is asymptotically stable if the conditions (2.37) and (2.38) are satisfied. □

Remarks. The previous theorem is based on the evolution of the minimum decay rate on nominal configuration and of the maximum growth rate in an uncertain configuration, which are computed using linear matrix inequalities. Instead of requiring the decay of a Lyapunov function at all instants of time, the theorem simply requires that the euclidean norm of the system state is strictly decreasing between two switching instants. Moreover if there exists a scalar $0 < \lambda < 1$ such that

$$(\lambda_n^*)^{\Delta^n} (\lambda_u^*)^{\Delta^u_{max}} < \lambda^{\Delta^u_{max}+\Delta^n}, \text{ for } \lambda_u^* > 1$$

or

$$(\lambda_n^*)^{\Delta^n} (\lambda_u^*)^{\Delta^u_{min}} < \lambda^{\Delta^u_{min}+\Delta^n}, \text{ for } \lambda_u^* < 1.$$

then the system is exponentially stable with the decay rate λ.

2.2.3 Partially known switching signal - stabilization

In this sub-section we consider the control synthesis problem when the switching law is partially uncertain. Consider the discrete time switched system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{2.42}$$

and the switched state feedback

$$u(k) = K_{\gamma(k)}x(k). \tag{2.43}$$

We assume that the switching law $\sigma(k)$ is *partially known*. Its exact value is not available, but for each of its values $\sigma(k) = i$, $\forall i \in \mathcal{I}$, we know a subset of possible values $\mathcal{E}_i \subset \mathcal{I}$. The switching signal $\gamma(k)$ used in the command takes values in the sub-set $\mathcal{E}_{\sigma(k)}$. In this context we study the design of a robust switched state feedback.

The closed loop switched system is given by

$$x(k+1) = \left(A_{\sigma(k)} + B_{\sigma(k)}K_{\gamma(k)}\right) x(k),$$
$$\forall\ \sigma(k)\ \in\ \mathcal{I} \text{ and } \forall\ \gamma(k)\ \in\ \mathcal{E}_{\sigma(k)}.$$

Theorem 32 *Consider the switched system (2.42) and the state feedback (2.43) with $\sigma(k) \in \mathcal{I}$ and $\gamma(k) \in \mathcal{E}_{\sigma(k)}$. If there exist symmetric positive definite matrices $S_{\alpha,\beta}$ and G_β, R_β matrices for all $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{E}_\alpha$, solutions of the LMIs:*

$$\begin{bmatrix} -S_{\alpha_1,\beta_1} + G_{\beta_1} + G_{\beta_1}^T & G_{\beta_1}^T A_{\alpha_1}^T + R_{\beta_1}^T B_{\alpha_1}^T \\ A_{\alpha_1}G_{\beta_1} + B_{\alpha_1}R_{\beta_1} & S_{\alpha_2,\beta_2} \end{bmatrix} > 0 \tag{2.44}$$

$\forall\ \alpha_1, \alpha_2 \in \mathcal{I},\ \beta_1, \in\ \mathcal{E}_{\alpha_1},\ \beta_2, \in\ \mathcal{E}_{\alpha_2}$, then the closed loop system is asymptotically stable and the gains that stabilize (2.42) are given by

$$K_\gamma = R_\gamma G_\gamma^{-1}, \forall\ \sigma\ \in\ \mathcal{I},\ et\ \forall\ \gamma\ \in\ \mathcal{E}_\sigma.$$

Proof. We use similar arguments to the ones given for the proof of Theorem 22. One can show that when the conditions (2.44) are satisfied, the following relation is true :

$$-S_{\alpha_1,\beta_1} + G_{\beta_1} + G_{\beta_1}^T \leq G_{\beta_1}^T S_{\alpha_1,\beta_1}^{-1} G_{\beta_1}$$

which implies that

$$\begin{bmatrix} G_{\beta_1}^T S_{\alpha_1,\beta_1}^{-1} G_{\beta_1} & G_{\beta_1}^T A_{\alpha_1}^T + R_{\beta_1}^T B_{\alpha_1}^T \\ A_{\alpha_1} G_{\beta_1} + B_{\alpha_1} R_{\beta_1} & S_{\alpha_2,\beta_2} \end{bmatrix} > 0. \tag{2.45}$$

Using the fact that

$$R_{\beta_1} = G_{\beta_1} K_{\beta_1}$$

and multiplying from left and from right by

$$\begin{bmatrix} G_{\beta_1}^{T\ -1} & \mathbf{0} \\ \mathbf{0} & S_{\alpha_2,\beta_2}{}^{-1} \end{bmatrix} \tag{2.46}$$

and its transpose, we obtain

$$\begin{bmatrix} S_{\alpha_1,\beta_1}^{-1} & (A_{\alpha_1} + B_{\alpha_1} K_{\beta_1})^T S_{\alpha_2,\beta_2}^{-1} \\ S_{\alpha_2,\beta_2}^{-1} (A_{\alpha_1} + B_{\alpha_1} K_{\beta_1}) & S_{\alpha_2,\beta_2}^{-1} \end{bmatrix} > 0.$$

With

$$P_{\alpha_1,\beta_1} = S_{\alpha_1,\beta_1}^{-1} \text{ et } P_{\alpha_2,\beta_2} = S_{\alpha_2,\beta_2}^{-1},$$

$\forall\ \alpha_1, \alpha_2 \in \mathcal{I},\ \beta_1 \in \mathcal{E}_{\alpha_1},\ \beta_2 \in \mathcal{E}_{\alpha_2}$, we can show that

$$\begin{bmatrix} P_{\alpha_1,\beta_1} & (A_{\alpha_1} + B_{\alpha_1} K_{\beta_1})^T P_{\alpha_2,\beta_2} \\ P_{\alpha_2,\beta_2} (A_{\alpha_1} + B_{\alpha_1} K_{\beta_1}) & P_{\alpha_2,\beta_2} \end{bmatrix} > 0.$$

Applying the Schur complement lemma, the previous equation implies

$$P_{\alpha_1,\beta_1} - (A_{\alpha_1} + B_{\alpha_1} K_{\beta_1})^T P_{\alpha_2,\beta_2} (A_{\alpha_1} + B_{\alpha_1} K_{\beta_1}) > 0.$$

With $\sigma(k) = \alpha_1$, $\gamma(k) = \beta_1$, $\sigma(k+1) = \alpha_2$, $\gamma(k+1) = \beta_2$ and by multiplying from left and from right by $x^T(k)$ and $x(k)$ we obtain that

$$x^T(k) P_{\sigma(k),\gamma(k)} x(k) -$$

$$x^T(k) \left(A_{\sigma(k)} + B_{\sigma(k)} K_{\gamma(k)}\right)^T P_{\sigma(k+1),\gamma(k+1)} \left(A_{\sigma(k)} + B_{\sigma(k)} K_{\gamma(k)}\right) x(k) > 0$$

which means that the poly-quadraric Lyapunov function

$$V(x) = x^T(k) P_{\sigma(k),\,\gamma(k)}\ x(k),$$

which is switched both on the original switching law $\sigma(k)$ and the switching signal used in the command $\gamma(k)$, has a strictly negative definite difference along the closed loop system solutions. □

2.2.4 Numerical examples

Example 2.3 We consider following state-space model :

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t),\ \sigma \in \{1,2\}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.49 & 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 43.12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.98 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix},\quad B_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0.5 \end{bmatrix}$$

The system is sampled with the sampling period $T = 0.026s$ and it is controlled by the state feedback gains

$$K_1 = [137.2\ 43.53\ 8.9\ 16.68],$$

$$K_2 = [84.23\ 20.77\ 7.09\ 12]$$

with a delay of three samples in the detection of the switching signal ($\Delta^u = 3$). Using the criteria proposed here we obtain that the system is asymptotically stable if at least eight sampling periods are spent in a nominal configuration between two switches ($\Delta^n = 8$).

Example 2.4 Consider the system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)$$

with $\mathcal{I} = \{1,2,3,4\}$,

$$A_1 = \begin{bmatrix} 0.5476 & -0.9500 \\ -0.6465 & 0.0351 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.2989 & -0.9185 \\ -0.9542 & 0.0741 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.9378 & 0.1735 \\ 1.7643 & 0.0995 \end{bmatrix},\ A_4 = \begin{bmatrix} 1.5828 & 0.2617 \\ 0.9066 & -0.0178 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 1.0174 \\ 0.5982 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.8536 \\ 0.3955 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.1609 \\ 0.4881 \end{bmatrix},\ B_4 = \begin{bmatrix} 0.2617 \\ 0.3541 \end{bmatrix}.$$

From the control computation point of view, the switching law is not perfectly known. The uncertainties are defined by the sets

$$\mathcal{E}_1 = \mathcal{E}_2 = \{1,2\} \text{ and } \mathcal{E}_3 = \mathcal{E}_4 = \{3,4\}.$$

Theorem 32 is applied in order to build a robust switched state feedback under the form

$$u(k) = K_{\gamma(k)}x(k)$$

with $\gamma(k) \in \mathcal{E}_{\sigma(k)}$. The LMI conditions (2.44) have a solution and the gains are given by

$$K_1 = K_2 = [-0.2694\ 0.8871],$$

$$K_3 = [-4.1620\ -0.4344],\ K_4 = [-3.4314\ 0.3672].$$

We can remark that for sub-systems 1 and 2 the LMI solver proposes a common gain.

2.3 Conclusion

In this chapter the robust stability analysis and control synthesis problems for discrete time switched systems have been studied. In the first section, dedicated to parametric uncertainties, the case of polytopic uncertainties was considered. The goal of this study was to propose a method for conservatism reduction in the case of parametric uncertainties.

Two approaches have been proposed. First it is shown that the stability of an uncertain switched system may be approached by analyzing one global polytopic system. The stability is checked using the LMI stability conditions from [21]. However, the approach is not suitable for control synthesis. In this case, the approach leads to the resolution of non convex optimization problems (BMI). Second, we propose an LMI approach that can be used for both stability analysis and control design.

A new type of Lyapunov functions has been introduced : the *switched parameter dependent Lyapunov functions* (SPDLF), i.e. functions that are based on a structure similar to the uncertain switched system. Less conservative LMI conditions have been proposed. The approach was extended to the case of control synthesis via a switched state feedback. These results can be easily applied to the dual problem which is state reconstruction for uncertain switched systems with uncertain output matrix.

In the second section we treated the case of discrete time switched systems with uncertain switching signal. We showed how dwell time stability conditions can be applied for stability analysis in the case of switched systems with temporary uncertain switching signal and partially known switching signal. The approach can be extended to the case of switched systems with time varying delays in the input, as we will see in the next chapter.

Chapter 3

Uncertain time delays

In this chapter, robustness is studied with respect to time delays. We consider the case of discrete time switched linear systems and we assume that the closed-loop command may be affected by unknown time varying delays. The stability analysis of systems with unknown time varying delays in the input is a very complex problem, even in the linear time invariant case. Our study will be decomposed into two sub-problems.

First we consider the case of linear time invariant systems. In this context the stability is usually studied using the Lyapunov-Krasovskii functional method. However, faced to practical examples, the obtained stability conditions are very conservative. A different approach, encountered in the domain of networked control systems [83], is possible. It consists in expressing the original problem as a switched system stability problem [70]. A switching law that depends on the delay describes the closed-loop dynamic for different delay values. The first objective of this chapter is to study the relation between the Lyapunov-Krasovkii functional method and the switched system transformation. We will show that applying the "switched system transformation approach" is equivalent to using a general, delay dependent Lyapunov-Krasovskii function. This function represent the most general form that can be obtained using sum of quadratic terms. This result justifies the representation of the delay as a switching parameter.

Second, the case of switched systems will be considered. We consider that a state feedback controller uses gains that are changed according to the switching law. The variations of the switching law are arbitrary. The "switched system transformation approach" for the delay manipulation in the case of LTI systems will be extended to the case of switched systems with time varying delay. An equivalent system, that is switched both on the original switching function and on the delay, is obtained. A problem similar to the case of uncertain switching law is encountered : the delay occurrence in the closed loop implies some particular mixed mode dynamics. We will show the way the stability criteria developed in the previous chapter can be applied in this case.

3.1 Stability of LTI systems with time varying delay

In this section the stability analysis for discrete-time linear systems with time varying delays in the state will be considered. Over the past few years, a lot of attention has been paid to the stability of time delay systems. This problem is very challenging when dealing with time varying delays. For continuous time systems with time varying delays, delay independent and delay dependent sufficient stability criteria can be derived by using Razumikhin-Lyapunov or Krasovskii-Lyapunov functionals (see for example [36, 82] and the references therein). Delay independent stability criteria do not take the size of the delay into account, therefore they suffer of conservatism. The main advantage of the Krasovskii-Lyapunov approach is that it leads to linear matrix inequalities (LMIs) conditions [13]. Also, in [51], a numerical approach based on the resolution of Riccati equations has been proposed. A notable result is the descriptor system transformation approach that has been introduced by Fridman [28].

In the context of discrete time systems with time-varying delays, sufficient LMI stability conditions have been obtained by considering a discrete time equivalent of the Krasovskii-Lyapunov approach [89, 17, 29, 105, 104, 12, 34, 120, 10, 11]. Considering Fridman's work for continuous time systems with time varying delay [28], several authors ([17, 29, 30] and [12]) applied the descriptor system approach in the context of LMI based stability analysis. A popular approach is the result of Gao [33] that has been extended recently for stochastic time delay systems [34]. The authors use *Moon et al.'s* inequality [27] and derive a sufficient LMI condition for the existence of a very complex Krasovskii-Lyapunov function. The conditions depend on the minimum and maximum delay bounds.

Among the open problems encountered when studying delay systems, it is not clear what form should have the most general Lyapunov-Krasovskii function. Which form is *necessary and sufficient* for the stability of the discrete-time linear systems with time varying delays ? This challenging problem is very hard to solve. Here we restrict the discussion to the following questions:

1. *Which is the least conservative Lyapunov-Krasovskii function that involves only quadratic forms ?*

2. *Is it possible to derive necessary and sufficient LMI for the existence of Lyapunov-Krasovskii functions involving quadratic forms ?*
   Until now, only sufficient LMI conditions for the existence of Lyapunov-Krasovskii functions have been obtained. The authors usually propose a Lyapunov-Krasovskii function and an associated LMI. The LMI is not a *necessary* condition for the existence of the function. This means that when the LMI are found infeasible we still do not know anything about the function. It is possible that LMI conditions in the literature are found infeasible

and still there may exist an associated Lyapunov-Krasovskii function.

3. *Can we derive LMI conditions directly from the difference of the function, without bounding its terms ?*
   The most important problem in deriving LMI conditions is the conservatism that stems from bounding the terms in the difference of the Lyapunov-Krasovskii function.

These are fundamental stability problems that are encountered in many research areas. When dealing with digital control implementations, a similar "delay stability" problem is encountered in the domain of networked control systems (NCS). In this context, the control loop is affected by sensing, communication or control computation delays. For several networked control set-ups, the closed-loop system can be transformed into an equivalent discrete time delay system [101]. The stability of discrete time systems with time varying delays is treated indirectly, when analyzing the stability of the networked system (see for example [70, 112, 113, 95]).

In the NCS literature, significant methods for the stability analysis are the state augmentation and the switched system transformation. We cite the work in [59, 70, 69] for the general switched system transformation and [53, 16, 102, 118] for the markovian jump system approach. These approaches reduce the closed loop stability problem to the analysis of a finite dimensional time varying system by augmenting the system model to include delayed variables (past values of plant state, input or output) as additional states. Specific stability criteria for discrete time switched systems [38, 22] or markovian jump systems [20] are applied. In particular, the paper [70] uses *necessary and sufficient* LMI conditions for the existence of multiple (switched) Lyapunov functions [22] in order to analyze the stability of a networked control system.

At first sight, there is no relation between the Krasovskii-Lyapunov approach used in the classical stability theory for delay systems and the switched system transformation, approach used for networked control systems. However, from a practical point of view, the two stability problems are similar.

Here we intend to establish a theoretical link between the Krasovskii-Lyapunov approach and the switched system transformation in the context of discrete time systems with time varying delays. It will be shown that using the multiple Lyapunov functions in the switched systems transformation is equivalent to using a new, delay dependent, Krasovskii-Lyapunov function. This function generalizes the classical functions from the discrete delay systems literature [89, 17, 29, 105, 104, 12, 34, 120, 10, 11]. Using this approach, there is no need of bounding the terms in the difference of the Krasovskii-Lyapunov function. The method avoids the conservatism problem that is usually encountered. Based on an adaptation of the results from [22] and [70], *necessary and sufficient* LMIs conditions for the *existence* of such functions will be given.

3.1.1 Context

Consider the discrete time system with time delay

$$x(k+1) = Ax(k) + A_d x(k - \tau(k)),\ \forall k \in \mathbb{Z}^+, \tag{3.1}$$
$$x(k) = \phi(k)\,,\ \forall k \in [-m, 0]\,,$$

where $x(k) \in \Re^n$ represents the system state vector at time $k \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, $\tau(k)$ is a positive integer representing the time varying delay and A, A_d are known constant $\Re^{n\times n}$ matrices. Here $\phi(k)$, is a real-valued initial function on $[-m, 0]$ where $m > 0$. We consider that the time varying delay $\tau(k)$ is upper-bounded, that is $0 < \tau(k) \leq m$. For the sake of simplicity the lower delay bound is considered to be zero, the results can be easily generalized to a non zero lower bound in a similar manner to [33]. We are interested in the stability analysis for this class of discrete time systems with delays.

In the specialized literature, when dealing with continuous-time LTI delay systems, sufficient LMI conditions are obtained by using Lyapunov-Krasovskii functionals [52, 82]. In the case of discrete-time systems with time varying delay, discrete-time equivalents of such functionals are considered. Here, a very simple example of discrete Krasovkii-Lyapunov functions, presented in [89], is recalled:

$$V_S(k) = x^T(k)Px(k) + \sum_{i=1}^{m} \sum_{j=k-i}^{k-1} x^T(j)Qx(j). \tag{3.2}$$

In the literature, the authors derive sufficient LMI conditions for the existence of such functions. For the case of $V_S(k)$, the stability conditions are given as follows:

Lemma 33 *[89] Suppose that there exist positive definite matrices P and Q such that the following linear matrix inequality (LMI) holds:*

$$\begin{bmatrix} P - A^T PA - mQ & A^T PA_d \\ A_d^T PA & Q - A_d^T PA_d \end{bmatrix} > 0 \tag{3.3}$$

Then the difference of the Lyapunov-Krasovskii function (3.2) is strictly negative along system (3.1) solutions and the system is uniformly asymptotically stable.

Remark. One should notice that in this case the stability is analyzed using a single LMI. Such LMI conditions are obtained using the evaluation of an upper bound of the difference of the Lyapunov function, leading to conservatism. For example in [89], the term $x^T(k-\tau)Qx(k-\tau)$ was enlarged to $\sum_{i=1}^{m} x^T(k-i)Qx(k-i)$ (see equation (2.14) in [89]). Similar maximizations can be found in most of the publications that deal with discrete time Lyapunov-Krasovskii functions. This means that a LMI such as (3.3) may be found infeasible even when the system is asymptotically stable. Moreover, since the LMI (3.3) is only a sufficient condition for the existence of the Lyapunov-Krasovskii function (3.2), it is possible to find a Lyapunov-Krasovskii function under the form (3.2) for which the LMI condition (3.3) is not satisfied.

3.1.2 General delay dependent Lyapunov-Krasovskii function

Many Lyapunov-Krasovskii functions have been presented in the literature. It can be noticed that all of them are sums of quadratic forms that depend on the delayed states. Analyzing these functions, we can imagine other functions in which the Lyapunov matrices are not necessarily related to a unique matrix Q and in which all combinations of delayed states are used in the quadratic forms. These ideas lead to considering the most *general form* that can be obtained using the sum of all the possible combinations of quadratic forms. This is expressed as the following *delay dependent Lyapunov-Krasovskii function*:

$$V(k) = V(k, x(k) \dots x(k-m), \tau(k)) = \sum_{i=0}^{m} \sum_{j=0}^{m} x^T(k-i) P_{\tau(k)}^{i,j} x(k-j). \quad (3.4)$$

We notice that the function involves all the combinations of delayed states

$$x^T(k-i) P_{\tau(k)}^{i,j} x(k-j), \; i,j \in \{1, \dots, m\}.$$

Each quadratic form presents different Lyapunov matrices $P_{\tau(k)}^{i,j}$ which change according to the delay in a set

$$\mathcal{P}^{i,j} = \left\{ P_1^{i,j}, P_2^{i,j}, \dots, P_m^{i,j} \right\}.$$

The matrices $P_\tau^{i,j}$, $i,j \in \{0, \dots, m\}$, $\tau \in \{1, \dots, m\}$, are defined such that the function is always positive definite, i.e.

$$V(k, x(k) \dots x(k-m), \tau(k)) > 0$$

which is the same as

$$\Phi(\tau) = \begin{bmatrix} P_\tau^{0,0} & P_\tau^{0,1} & \dots & P_\tau^{0,m} \\ P_\tau^{0,1} & P_\tau^{1,1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ P_\tau^{0,m} & \dots & \dots & P_\tau^{m,m} \end{bmatrix} > 0, \quad (3.5)$$

for any delay τ in the set $\{1, \dots, m\}$. Without any loss of generality we may consider that $P_\tau^{i,j} = P_\tau^{j,i^T}$.

Remarks. Notice that the Lyapunov matrices $P_{\tau(k)}^{i,j}$ are not necessarily positive definite. Only the block matrix $\Phi(\tau)$ is required to be a symmetric positive definite matrix. The proposed Lyapunov function generalizes the classical functions [89, 17, 29, 105, 104, 12, 34, 32, 120, 10, 11].

By using traditional methods, the construction of LMI stability conditions from the difference of the general delay dependent Lyapunov-Krasovskii function

(3.4) seems a very difficult task. As follows we give necessary and sufficient LMI conditions for the existence of this function. These LMI conditions use augmented state methods and are inspired from the switched systems theory.

Consider the notation :

$$\Lambda(\tau) = \begin{bmatrix} A & \Xi_1(\tau) & \Xi_2(\tau) & \dots & \Xi_m(\tau) \\ \mathbf{I} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{3.6}$$

with

$$\Xi_i(\tau) = \begin{cases} A_d, & i = \tau \\ \mathbf{0}, & i \neq \tau \end{cases} \quad \forall i = 1, \dots, m \tag{3.7}$$

for all $\tau \in \{1, \dots, m\}$.

Theorem 34 *[40] The following statements are equivalent:*

- *There exist $m \cdot (m+1)^2$ matrices $P_\tau^{i,j}$, $\tau = 1, \dots, m$, $i, j = 0, \dots, m$, such that the block matrices $\Phi(\tau), \tau = 1, \dots, m$ defined by (3.5) satisfy the LMIs*

$$\begin{bmatrix} \Phi(\tau_1) & \Lambda^T(\tau_1)\Phi(\tau_2) \\ \Phi(\tau_2)\Lambda(\tau_1) & \Phi(\tau_2) \end{bmatrix} > 0, \tag{3.8}$$

  for all $\tau_1, \tau_2 \in \{1, \dots, m\}$.

- *There exist a delay dependent Lyapunov-Krasovskii function (3.4) whose difference is strictly negative definite along system (3.1) solutions.*

Proof. Consider the difference of the Lyapunov-Krasovskii function (3.4)

$$V(k) - V(k+1) > 0, \tag{3.9}$$

which is the same as

$$\sum_{i=0}^{m} \sum_{j=0}^{m} x^T(k-i) P_{\tau(k)}^{i,j} x(k-j) - \sum_{i=0}^{m} \sum_{j=0}^{m} x^T(k+1-i) P_{\tau(k+1)}^{i,j} x(k+1-j) > 0.$$

Deriving LMI stability conditions directly from the previous expression is possible only by assuming particular forms for the $P_{\tau(k)}^{i,j}$ matrices and by bounding the obtained expression. Here a different approach is applied. Using the augmented vector

$$\bar{x}(k) = \left[x^T(k) \dots x^T(k-m)\right]^T$$

and the notation (3.5), the equation (3.9) can be expressed as

$$\bar{x}^T(k)\Phi(\tau(k))\,\bar{x}(k) - \bar{x}^T(k+1)\Phi(\tau(k+1))\,\bar{x}(k+1) > 0. \tag{3.10}$$

Using the notations (3.6) we can notice that

$$\bar{x}(k+1) = \Lambda\left(\tau(k)\right)\bar{x}(k),\ \forall\ \tau(k) \in \{1,\dots,m\}\,. \tag{3.11}$$

With $\tau_1 = \tau(k)$ and $\tau_2 = \tau(k+1)$ the equation (3.10) becomes

$$\bar{x}^T(k)\Phi(\tau_1)\bar{x}(k) - \bar{x}^T(k)\Lambda^T(\tau_1)\Phi(\tau_2)\Lambda(\tau_1)\bar{x}(k) > 0. \tag{3.12}$$

Equations similar to (3.11) and (3.12) are classical in the domain of discrete time varying system [21].

It is easy to see that the equation (3.12) is equivalent to

$$\Phi(\tau_1) - \Lambda^T(\tau_1)\Phi(\tau_2)\Lambda(\tau_1) > 0. \tag{3.13}$$

Applying the Schur complement, the previous equation is strictly the same as the LMIs (3.8), which means that the difference (3.9) of the general Lyapunov-Krasovskii function (3.4) and (3.8) are equivalent. □

Remark. Until now, only sufficient LMI conditions have been presented for the existence of such functions. This is due to the use of an upper bound for the difference of the Lyapunov-Krasovskii functions. In the proof of the previous theorem, there is no need of bounding the terms in the difference of the Lyapunov-Krasovskii function. In this case, the LMI conditions (3.8) are necessary and sufficient conditions for the existence of the general delay dependent Lyapunov-Krasovskii function (3.4). This means that when the LMIs (3.8) are found infeasible it is impossible to find a Lyapunov-Krasovskii function of the form (3.4). Furthermore, since (3.4) is the most general function that uses quadratic forms, this means that it is impossible to find a Lyapunov-Krasovskii function under the forms proposed in the literature, or any other function that involves sums of quadratic forms.

The LMIs (3.8) can be easily extended to the case of discrete time systems with multiple time varying delays such as

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} A_i x(k - \tau_i(k)),$$

where all the delays $\tau_i(k)$ are varying $\tau_i(k) \in \{1,2,\dots,m\}$, $\forall i = 1,\dots,m$. This generalization is obtained by reconsidering the notation (3.7) with $\Xi_i(\tau_1,\dots,\tau_m) = \sum_{j,\ \tau_j(k)=i} A_j$, $\forall i = 1,\dots,m$ and considering as many matrices $\Lambda(\tau_1,\dots,\tau_m)$ and Lyapunov matrices $\Phi(\tau_1,\dots,\tau_m)$ as possible delay combinations.

3.1.3 Equivalence between the switched Lyapunov function and the general delay dependent Krasovkii-Lyapunov function

In this section we show that using the multiple Lyapunov functions is equivalent to using the general, delay dependent, Lyapunov-Krasovskii function.

Considering the augmented state vector

$$z(k) = \left[x^T(k) \dots x^T(k-m)\right]^T, \tag{3.14}$$

the dynamics of the delay system (3.1) can be represented as the following switched system

$$z(k+1) = \bar{A}_{\sigma(k)} z(k). \tag{3.15}$$

The state matrix $\bar{A}_{\sigma(k)}$ switches in the set of possible matrices $\{\bar{A}_1, \dots, \bar{A}_m\}$ according to the parameter $\sigma(k)$. We denote by I the set of values the switching function $\sigma(k)$ may take, $\sigma : \mathbb{Z}^+ \to I$. In the case of time delay systems, the matrices $\bar{A}_i,\ i \in I = \{1, \dots, m\}$ are given by

$$\bar{A}_i = \begin{bmatrix} A & \mathbf{0} & \dots & \mathbf{0} & A_d & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \dots & \dots & \dots & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \dots & \dots & \dots & \mathbf{0} \\ \vdots & & & & & & & \vdots \\ \mathbf{0} & \dots & \dots & \dots & \dots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{3.16}$$

where the block A_d changes its positions according to i, such as it is found on the $(i+1)^{th}$ column of the first line of $\bar{A}_i$. Moreover, the switching function $\sigma : \mathbb{Z}^+ \to I = \{1, 2, .., m\}$ changes according to the delay, that is $\sigma(k) = i$ if $\tau(k) = i$. The asymptotic stability of a switched system can be checked by using multiple switched Lyapunov functions [22]

$$V(k) = z^T(k) \bar{P}_{\sigma(k)} z(k) \tag{3.17}$$

where $\bar{P}_1, \dots, \bar{P}_m$ are symmetric positive definite matrices. In this case the asymptotic stability is ensured if the matrices $\bar{P}_i,\ \forall\ i = 1 \dots, m$ are solutions of the following LMIs:

$$\begin{bmatrix} \bar{P}_i & \bar{A}_i^T \bar{P}_j \\ \bar{P}_j \bar{A}_i & \bar{P}_j \end{bmatrix} > 0,\ \forall (i,j) \in I \times I. \tag{3.18}$$

From a theoretical perspective such a result is rather surprising, since most of the existing stability analysis approaches of time delay systems are based on using the Lyapunov-Krasovskii method and several publications the claim that the state augmentation is useless in time-varying delay case [29, 33]. In order to analyze the relation between the two approaches, it seems interesting to construct an equivalent Lyapunov-Krasovskii function. Since $\sigma(k) = i$ if $\tau(k) = i$, the switched Lyapunov function (3.17) can be expressed as a function that explicitly depends on the delayed variables $x(k), \dots, x(k-m)$ and the delay τ:

$$V(k, \bar{x}(k), \tau(k))) = \bar{x}^T(k) \bar{P}_{\tau(k)} \bar{x}(k),$$

for all $\tau(k) \in \{1, \dots, m\}$. Let $\bar{P}_{\tau(k)}$ be described by a block matrix similar to the matrix $\Phi(\tau(k))$ given in (3.5), for the Lyapunov-Krasovskii approach. We notice that the switched Lyapunov function (3.17) is strictly equivalent to *the general delay dependent Lyapunov-Krasovskii function* (3.4). Moreover, posing $\Lambda(\tau) = \bar{A}_\tau$ for all $\tau = 1, \ldots, m$, the LMIs (3.8) are strictly identical to (3.18). Since the delay dependent Lyapunov-Krasovskii function (3.4) is the most general Lyapunov-Krasovskii function, this actually justifies the switched system transformation.

3.2 Closed loop switched systems with time varying delay

In this section, we present a generalization of the time varying delay stability problem in the case of closed loop switched systems with time varying delay. We consider a state feedback controller that uses gains that are changed according to the switching law. The variations of the switching law are arbitrary. The "switched system transformation approach" for the delay manipulation in the case of LTI systems will be extended to the case of switched systems with time varying delay.

Consider the switched delay system:

$$\begin{aligned} x(k+1) &= A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k-\tau(k)), \\ u(k) &= K_{\sigma(k)}x(k) \\ x(k) &= \phi(k), \ \forall k \in \left[-\overline{\theta}, 0\right], \\ \sigma(k) &= \epsilon(k), \ \forall k \in \left[-\overline{\theta}, 0\right], \end{aligned} \tag{3.19}$$

where $\sigma : \mathbb{Z}^+ \to \mathcal{I} = \{1, 2, .., N\}$ represents the switching function, and $\tau(k) \in \mathcal{T} = \left\{\underline{\theta}, \ldots, \overline{\theta}\right\}$ the time varying delay [4]. Here $\phi(k)$, $\epsilon(k)$ represent initial value functions on $\left[-\overline{\theta}, 0\right]$ where $\overline{\theta} > 0$ represents the maximum delay. Similarly, we consider a minimum delay $\underline{\theta}$. We assume here that $0 \leq \underline{\theta} \leq \tau(k) \leq \overline{\theta}$. Consider Δ the minimum dwell time between two switchings. The time that can be spent using a wrong switching command depends on the delay τ, i.e. $\Delta^u_{min} = \underline{\theta}$ and $\Delta^u_{max} = \overline{\theta}$. We can conclude that the minimum time spent using the "appropriate" switching signal is $\Delta^n = \Delta - \overline{\theta}$.

The closed loop system is given by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}K_{\gamma(k)}x(k-\tau(k)).$$

where γ represents the switching function used by the gains $\gamma : \mathbb{Z} \to \mathcal{I}, \ \gamma(k) = \sigma(k - \tau(k))$.

Consider the augmented state vector

$$z(k) = \left[x^T(k) \ldots x^T(k-\overline{\theta})\right]^T.$$

[4] The set $\mathcal{T}$ does not necessarily include all the values between $\underline{\theta}$ and $\overline{\theta}$.

The closed-loop system can be expressed as

$$z(k+1) = \left(\bar{A}_{\sigma(k)} + \bar{B}_{\sigma(k)}\bar{K}_{(\gamma(k),\tau(k))}\right) z(k) \tag{3.20}$$

where

$$\bar{A}_{\sigma(k)} = \begin{bmatrix} A_{\sigma(k)} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \bar{B}_{\sigma(k)} = \begin{bmatrix} B_{\sigma(k)} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

$$\bar{K}_{(\gamma(k),\tau(k))} = \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & K_{\gamma(k)} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

such that the term $K_{\gamma(k)}$ is found on the $(\tau+1)^{th}$ column of $\bar{K}_{(\gamma(k),\tau(k))}$. This represents a system that is switched on the original switching function $\sigma(k)$ and two supplementary switching parameters $\gamma(k)$ and $\tau(k)$. Moreover it can be written as

$$z(k+1) = H_{(\sigma(k),\gamma(k),\tau(k))} z(k)$$

with

$$H_{(\sigma(k),\gamma(k),\tau(k))} = \bar{A}_{\sigma(k)} + \bar{B}_{\sigma(k)}\bar{K}_{(\gamma(k),\tau(k))}.$$

If one has no hypothesis on the dwell time, then the stability of the closed-loop system can be verified by checking if there exists positive definite matrices $P_{(\sigma,\gamma,\tau)} > 0, \ \forall \ (\sigma,\gamma,\tau) \in \mathcal{D} = \mathcal{I} \times \mathcal{I} \times \mathcal{T}$, such that

$$H_i^T P_j H_i - P_i < 0, \ \forall \ i,j \in \mathcal{D} \times \mathcal{D}.$$

The condition is also the unique method of checking the stability if the minimum time spent using the good gain is less than one sample, $\Delta^n < 1$ (this means that it is possible that we always use the wrong switching signal in the command). The condition may be found too restrictive since the feedback gains $K_{\gamma(k)}$ have to ensure the decay of the Lyapunov function $V(k) = z^T(k)P_{(\sigma,\gamma,\tau)}z(k)$ for all delay $\tau(k)$ and system mode $\sigma(k)$ combinations.

The stability conditions may be relaxed using Theorem 31 from the previous chapter. In this case the decay/growth rates are computing using $\lambda_n^* = \min \lambda_n$, $\lambda_u^* = \min \lambda_u$ solutions of

$$H_{(\sigma,\gamma,\tau_1)}^T P_{(\sigma,\gamma,\tau_2)}^u H_{(\sigma,\gamma,\tau_1)} - \lambda_u P_{(\sigma,\gamma,\tau_1)}^u < 0$$

$$H_{(\sigma,\sigma,\tau_1)}^T P_{(\sigma,\tau_2)}^n H_{(\sigma,\gamma,\tau_1)} - \lambda_n P_{(\sigma,\tau_1)}^n < 0$$

$\forall \, (\sigma,\gamma) \in \mathcal{I} \times \mathcal{I}$, and $(\tau_1,\tau_2) \in \mathcal{T} \times \mathcal{T}$.

Considering the scalars:

$$c_n = \frac{\max_{(\sigma,\tau)\in\mathcal{I}\times\mathcal{T}} eig_{max}\left(P^n_{\sigma,\tau}\right)}{\min_{(\sigma,\tau)\in\mathcal{I}\times\mathcal{T}} eig_{min}\left(P^n_{\sigma,\tau}\right)}, \tag{3.21}$$

$$c_u = \frac{\max_{(\sigma,\gamma,\tau)\in\mathcal{D}} eig_{max}\left(P^u_{(\sigma,\gamma,\tau)}\right)}{\min_{(\sigma,\gamma,\tau)\in\mathcal{D}} eig_{min}\left(P^u_{(\sigma,\gamma,\tau)}\right)} \tag{3.22}$$

the system is stable if

$$c_n \cdot c_u \cdot (\lambda_n^*)^{\Delta-\overline{\theta}} (\lambda_u^*)^{\overline{\theta}} < 1, \text{ when } \lambda_u^* > 1$$

or

$$c_n \cdot c_u \cdot (\lambda_n^*)^{\Delta-\overline{\theta}} (\lambda_u^*)^{\underline{\theta}} < 1, \text{ when } \lambda_u^* < 1.$$

The approach can be easily extended for control synthesis by imposing the system decay rates λ_n^* and λ_u^* as constraints and using a constrainted state feedback with the form $\bar{K}_\gamma$ for the augmented system.

3.3 Numerical Examples

The proposed delay dependent Lyapunov-Krasovskii function and the associated LMI stability conditions depend on the complete range of possible delayed state. For large delay values this may lead to a large number of variables in the LMI (3.8) or (3.18). However, feasibility is illustrated using some examples.

Example 3.1 Consider the system (3.1) with

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix} \text{ and } A_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}.$$

Gao [33] and Fridman [30] show that the system is asymptotically stable for $3 \le \tau(k) \le 5$ and $3 \le \tau(k) \le 10$, respectively. Using the delay dependent Krasovskii-Lyapunov function or the switched Lyapunov function (3.17), the LMIs (3.8) and (3.18) show that the system (3.1) is asymptotically stable for $3 \le \tau(k) \le 13$. Moreover, using this approach we can consider discrete delays. For example, it is possible to show that the system is stable for a delay $\tau \in \{1, 6, 11, 16\}$.

Example 3.2 Consider the LTI state-space model of an inverted pendulum on a cart presented in publication [62]

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)\cdot g}{M\cdot l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m\cdot g}{M} & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \theta \\ \omega \\ x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{M\cdot l} \\ 0 \\ \frac{1}{M} \end{bmatrix} u(t),\ y(t) = [1\ 0\ 0\ 0] \cdot \begin{bmatrix} \theta \\ \omega \\ x \\ v \end{bmatrix}$$

where $M = 2kg$, $m = 0.1kg$, $l = 0.5m$ and $g = 9.81m/s^2$. The system is sampled with the sampling period $T = 20ms$ and it is stabilized by a digital state feedback control that is affected by delays:

$$u(t) = Kx(k - \tau), \ \forall t \in [kT, (k+1)T]$$

where

$$K = [56 \ 12 \ 0.45 \ 1.4] \, .$$

In order to check the stability of the digital setup, the discrete time model of the system is considered. The LMIs (3.8) and (3.18) show that the system is asymptotically stable for a $1 \leq \tau(k) \leq 4$. In fact, for a constant delay $\tau = 5$ the system is unstable (this can be verified by checking the eigenvalues of the augmented state matrix). This means that using the proposed LMIs we are able to verify the stability of the system for the maximum delay range for which it is actually stable.

Remarks. The previous numerical examples clearly illustrate the practical applicability of the LMIs (3.8),(3.18) and the conservatism reduction. An important aspect should be considered: when dealing with systems that are stable independently of the delay, such as the case treated in [70], it is possible that one reaches the limit of possible variables of the numerical solver and the LMIs may be found infeasible. This may be misinterpreted as a reduced delay range for which the system is stable. In such cases, it may be useful to apply delay independent stability criteria, such as the one in [11].

3.4 Conclusion

In this chapter, robust stability analysis methods for systems with time varying delay have been presented. In the case of LTI systems we have established the equivalence between the switched system transformation approach and the classical Lyapunov-Krasovskii method. It is shown that applying the switched systems transformation is equivalent to using a general delay dependent, Lyapunov-Krasovskii function. This function represents the most general form that can be obtained using sums of quadratic terms. It generalizes the classical functions from the literature. Necessary and sufficient LMI conditions for the existence of such functions are presented. The method has been extended to the case of switched linear systems with time varying delay in the input. Since mixed dynamics occur, the stability criteria proposed in the previous chapter are applied.
In the future, other stability criteria from switched systems such as the dwell or the average dwell time stability criteria, should be introduced in the domain of time delay systems. This aspect could lead to less conservative stability criteria which use Lyapunov Functions that may occasionally increase (see [38, 54]).

The obtained stability conditions may be extended to the case of systems that are affected both by parametric uncertainties and time varying delay in the input.

This aspect will be considered in the next chapter, in the context of continuous time systems in digital control loops.

Chapter 4

Application to digital control systems

This chapter is dedicated to the modeling of LTI continuous time systems in digital control loops. We consider the digital control problem on non-uniform sampling periods. Moreover we assume that time varying delays that may have a variation range larger than a sampling period affect the closed-loop. Our goal is to present a unique model that is able to include these problems simultaneously and that can be handled by classical control synthesis tools. We present a new event based discrete-time model (an exponential uncertain system with delay) and we show that the stabilizability of this system can be achieved by finding a control for a switched polytopic system with an additive norm bounded uncertainty. The methodology is extended to the case of switched system.

4.1 Context

In digital control applications *timing problems* are unavoidable. Ideally all the components (sensors, computer and actuators) are synchronous, but in reality sensing delays, computer load, communication errors, etc. interfere in the control loop [99]. They are responsible for sampling jitter and for significantly large time-varying delays. An important challenge in the context of digital control systems [3] and in particular for networked control systems [101, 83] is the modeling of such timing problems. When ignored, they compromise the system performances and even its stability [92]. A realistic discrete-time model of the control loop that can be used for stability analysis and control design is required. The difficulty is mainly due to *exponential uncertainties* [39] that are introduced in the discrete model by the timing problems. Loosely speaking, exponential uncertainties represent terms like $e^{M\tau}$ or $\int_0^\tau e^{Ms}ds$ that depend on an unknown, time varying parameter τ. The uncertain nature and the particular structure of the exponential terms make the stability analysis or the control synthesis very difficult tasks.

In the literature there are several attempts for dealing with the timing problems [3, 101, 58, 70, 69, 2, 31, 84, 18, 19]. Usually, the obtained models treat one problem at a time. For sampling jitter or, more generally, for the digital control on non-uniform time domains, models are given in [6, 45]. In publication ([6]) the exponential uncertainties were expressed as a norm bounded uncertainty. A similar approach is applied when dealing with networked induced delay in [74]. This approach leads to using a common Lyapunov function for all the values of the uncertain matrices, which may give very conservative results. Generally, when considering time varying delays, the delay variation is limited to a sampling period. In fact the models are obtained by considering the discrete representation of the system over a sampling period. Applying this methodology in the general case, with sampling jitter and delay variations larger than the sampling period, leads to very complex models that are very difficult to analyze. This is due to the fact that several actuations may occur during a sampling period. This is the case for systems with the delay variation range greater than m sampling periods. The discrete representation over a sampling period has then $m+1$ time-varying exponential uncertain terms corresponding to each command that may arrive during this sampling period. The unique alternative given in the literature consists in taking supplementary hypothesis on the delay. By assuming that the delay is known, a compensation method can be used [101]. Delays that belong to a finite set of discrete nominal values are treated in [58, 70, 69, 2] where a switched system model of the closed loop system is used. The problem now is to deduce the behavior of the system when the delay belongs in the different intervals between the nominal values [5].

Here we intend to present a unique general model for the interaction between digital control and continuous time systems. We will present a new discrete-time model that can simultaneously treat non-uniform sampling and time varying delays. Instead of considering the discrete system representative over a sampling period, an *event-based representation* is proposed. We consider the different asynchronous events that occur in the closed loop system (sampling, actuations) and we give the discrete-time representation between two such events. The obtained model represents then a discrete-time system with exponential uncertainty and time varying delay. The polytopic embedding methodology that we proposed in [42] is generalized. We show that, even for the LTI systems, the obtained event-based model can be treated as a switched polytopic model with an additive norm bounded uncertainty: the delay is treated as a supplementary switching parameter while the exponential uncertainty is bounded in a convex polytope with a reduced number of vertices and an additive norm bounded term. This representation considerably reduces the computational complexity and allows to use Parameter Dependent Lyapunov functions for efficient LMI based control synthesis [13]. The modeling approach is extended to the digital control of switched systems.

[5]Extending the previous approach to this case implies solving an infinite number of linear matrix inequalities.

4.2 Discrete time models

We consider a continuous time LTI system that is controlled by a digital controller

$$\frac{dx_c(t)}{dt} = Mx_c(t) + Nu_c(t) \tag{4.1}$$

where $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times m}$ represent the state and the input matrices, respectively, $x_c(t) \in \mathbb{R}^n$ represents the system state and $u_c(t) \in \mathbb{R}^m$ the input. The values of the continuous time signals $x_c(t)$ are available by sampling. The sampling may be non uniform. By $x(k)$, we denote the digital version of the system state. This signal is sent to a computer that produces the digital control $u(k)$. A zero order holder transforms $u(k)$ into a piecewise constant continuous time signal $u_c(t) = u(k)$ until a new value is available.

Several timing problems can be considered. Firstly, the command $u_c(t)$ may be affected by delays. These delays are time-varying, possibly unknown, they may be greater than the sampling period and they may have a very large variation. Such delays make the system very difficult to model. The usual discrete time representation of the digital control loop considers the system representation over a sampling period (see [99, 101]). This representation is not appropriate for treating delay variations larger than a sampling period. In this case the system input $u_c(t)$ may change its value several times during the same sampling period. We illustrate such a model as follows. Consider that during the k^{th} sampling cycle the command may take values in the set $\{u(k), u(k-1), \ldots, u(k-k_{max})\}$. For a constant sampling period T, let $\tau^j(k)$, with $kT \leq \tau^j(k) \leq (k+1)T, j = 0, \ldots, k_{max}$, be the instant of time the command $u(k-j)$ occurred during the k^{th} sampling period. The classical discrete time representation over a sampling period is given by

$$\begin{aligned} x(k+1) = e^{MT}x(k) + \int_{kT}^{\tau^{k_{max}-1}(k)} e^{M[(k+1)T-s]}dsNu(k-k_{max}) + \ldots \\ \ldots + \int_{\tau^1(k)}^{\tau^0(k)} e^{M[(k+1)T-s]}dsNu(k-1) + \int_{\tau^0(k)}^{(k+1)T} e^{M[(k+1)T-s]}dsNu(k). \end{aligned} \tag{4.2}$$

This model is very complex: it has $k_{max}+1$ non-linear *uncertain exponential matrices*, each one depending on different unknown time-varying parameters $\tau^j(k)$ (see for example the model given in [19]). Using traditional methods it is very difficult to analyze the system. The analysis is even more complex when the sampling period is time varying. This problem is relevant to many practical fields [6].

4.2.1 Event-based discrete model

Here we propose a new *event-based discrete-time representation*. This model is able to represent both the system with time varying delays and non-uniform sampling. Consider the different events that occur in the control loop: sampling event, representing the fact that new system data are taken for control computation, actuation event, reflecting the different input changes. When delays occur in the control loop, sampling and actuation are distinct events that are succeeding at non uniform time intervals in an arbitrary way. It is possible that several actuations occur between two sampling events or no actuation at all. Let $t_i,\ i \in \mathbb{N}$ be the instants of time when such events occur. It may correspond to the $\tau^j(k)$ parameter in the model (4.2). The system evolution between two events is described by :

$$x_c(t_{i+1}) = e^{M(t_{i+1}-t_i)}x_c(t_i) + \int_0^{t_{i+1}-t_i} e^{Ms}dsNu_c(t_i). \tag{4.3}$$

This model decomposes the behavior of the systems (4.1) and (4.2) on time intervals for which the command is constant. Obviously, the command $u_c(t_i)$ changes only when t_i corresponds to an actuation event. Similarly, only the state values of $x_c(t_i)$ with t_i corresponding to a sampling event ($t_i = kT,\ k \in \mathbb{Z}^+$) are available for control computation.

The command $u_c(t_i)$ may depend on a sample $x_c(t_j)$ with $t_i \geq t_j$. Let ρ_i denote the time interval between two successive events. ρ_i is an unknown, bounded, time varying parameter, i.e. :

$$0 = \underline{\rho} < \rho_i = t_{i+1} - t_i \leq \overline{\rho}.$$

Furthermore, we denote by

$$\eta_i \ = \ x_c(t_i), \text{ and } u_i = u_c(t_i). \tag{4.4}$$

the state and input vectors, respectively, at the moment an event occurs. Using these notations, system (4.3) can be expressed as a discrete time uncertain system with *exponential uncertainty*

$$\eta_{i+1} = A(\rho_i)\eta_i + B(\rho_i)u_i \tag{4.5}$$

where by *exponential uncertainty* we denote the terms

$$A(\rho_i) = e^{M\rho_i},\ B(\rho_i) = \int_0^{\rho_i} e^{Ms}dsN.$$

Stabilizing a system affected by timing problems is the same as finding a control for system (4.5). As an example, consider the case where the control signal $u_c(t_i)$ is a function of past values of the digital system state,

$$u_c(t_i) = Kx_c(t_j) = Kx_c(t_{i-\theta_i}),\ \theta_i \in \mathcal{T} = \left\{\theta \in \mathbb{Z}^+ :\ \underline{\theta} \leq \theta \leq \overline{\theta}\right\}$$

where $\theta_i = i - j$ represents the distance between sampling and actuation in the space of event indexes. Since a finite number of actuations may occur during a finite time interval, the positive integer θ_i can be bounded. We denote by $\underline{\theta}$ and

$\overline{\theta}$ the lower and upper bounds, respectively, $\underline{\theta} \leq \theta_i \leq \overline{\theta}$. The system (4.5) input vector u_i is equivalent to

$$u_i = K\eta_{i-\theta_i}$$

for which the closed loop event-based model becomes:

$$\eta_{i+1} = A(\rho_i)\eta_i + B(\rho_i)K\eta_{i-\theta_i}.$$

The closed loop system is an uncertain system with exponential uncertainty and time varying delay in the state. Now finding a gain K that stabilizes this model ensures the stabilizability of the original continuous time system (4.1).

Remarks. The first problem is the exponential uncertainty, that is finding the most appropriate representation of such uncertainty. The second problem is the time varying delay. These problems are discussed in the following parts.

4.3 Convex polytopic model of a LTI system in a digital control loop

In this section we provide a method for expressing the event based model (4.5) as an uncertain polytopic model. This representation is important since it allows the application of several classical stability/stabilizability results from the domain of robust control. For convex polytopes the Parameter Dependent Lyapunov Function approach [21] can be used to derive efficient stability and stabilizability LMI criteria. Using such Lyapunov functions is less conservative than using a common Lyapunov function for all the values of the uncertain parameter ρ_i. The goal is to provide the most appropriate representation of the exponential uncertainties in system (4.5) for LMI control synthesis.

Consider the exponential uncertain matrices:

$$A(\rho_i) = e^{M\rho_i}, \; B(\rho_i) = \int_0^{\rho_i} e^{Ms} ds N. \tag{4.6}$$

For particular cases, a very simple and efficient approach can be applied. Consider the case when the state matrix M is diagonalizable, i.e. there exist invertible matrices T and diagonal matrices D such that $M = TDT^{-1}$, where $D = diag(\lambda_1, \dots, \lambda_n)$ and where $\lambda_j, j = 1 \dots n$ are the eigenvalues of M. For the sake of simplicity we present here the case of real eigenvalues λ_j. The approach can be easily extended to the complex eigenvalues case. For real eigenvalues, the exponential uncertainties can be expressed as :

$$A(\rho_i) = T diag\left(a_1(\rho_i), \dots, a_n(\rho_i)\right) T^{-1} \tag{4.7}$$

and

$$B(\rho_i) = T diag\left(b_1(\rho_i), \dots, b_n(\rho_i)\right) T^{-1} N \tag{4.8}$$

where

$$a_j(\rho_i) = e^{\lambda_j \rho_i}$$

and

$$b_j(\rho_i) = \int_0^{\rho_i} e^{\lambda_j s} ds$$

for all $j = 1, \ldots, n$. This means that each exponential uncertainty depends on n unknown time varying parameters. The bounds of the parameters a_j and b_j are given by :

$$\overline{a}_j = \max_{\underline{\rho}<\rho_i<\overline{\rho}} a_j(\rho_i),\ \underline{a}_j = \min_{\underline{\rho}<\rho_i<\overline{\rho}} a_j(\rho_i), \tag{4.9}$$

$$\overline{b}_j = \max_{\underline{\rho}<\rho_i<\overline{\rho}} b_j(\rho_i) \text{ and } \underline{b}_j = \min_{\underline{\rho}<\rho_i<\overline{\rho}} b_j(\rho_i). \tag{4.10}$$

As $A(\rho_i)$ and $B(\rho_i)$ depend on n unknown parameters, each of them included in a bounded interval, these uncertain exponential matrices are bounded by a convex polytope with 2^n vertices defined by the possible combinations of the $\overline{a}_j$, $\underline{a}_j$ $\overline{b}_j$ and $\underline{b}_j$ parameters or, equivalently, that the matrix $[A(\rho_i)\ \ B(\rho_i)]$ belongs to a convex polytope with 2^{2n} vertices:

$$[A(\rho_i)\ \ B(\rho_i)] \in \sum_{j=1}^{2^{2n}} \mu_j \left[U_j^A\ \ U_j^B\right],\ \sum_{j=1}^{2^{2n}} \mu_j = 1,\ \mu_j > 0\ \forall j = 1, \ldots, 2^{2n}$$

where

$$U_j^A = TE_j^a T^{-1},\ U_j^B = TE_j^b T^{-1} N,$$

such that $\left[E_j^a\ \ E_j^b\right]$, $j = 1, \ldots, 2^{2n}$ are the vertices of the convex polytope

$$\mathcal{E} = \left\{ E = [diag(a_1, \ldots, a_n)\ \ diag(b_1, \ldots, b_n)] : \underline{a}_j \leq a_j \leq \overline{a}_j, \underline{b}_j \leq b_j \leq \overline{b}_j, j = 1, \ldots, n \right\}.$$

4.3.1 Reformulation of the exponential uncertainty as a polytopic uncertainty with an additive norm bounded term

The previous approach can be extended to the general case by using Jordan canonical forms [19]. One may also try the direct convex embedding [18]. However, in these cases it is possible to obtain convex polytopes with a large number of vertices, which leads to a considerable computational complexity. For example, for an n-order system the direct convex embedding leads to a hyper-rectangle with $2^{n \times n}$ vertices. Moreover it is possible that such methods provide a too large approximation of the uncertainty.

In order to reduce the computational complexity we reformulate the exponential uncertainties as convex polytopes with an additive norm bounded term. This formulation can be derived from a Taylor series approximation of each exponential uncertainty. The polytopic uncertainty is derived from the h-order truncation of the Taylor series (a h-degree matrix polynomial) while the norm bounded term is derived from the remainder of the Taylor series approximation. In practical applications, the idea is to convexify the uncertainty such that the remainder of the Taylor approximation is less than the numerical precision.

4.3.1.1 h-order Taylor series expansion

Using a h-order Taylor series expansion, the exponential uncertainties $A(\rho_i)$ and $B(\rho_i)$ can be expressed as matrix polynomials with additive terms that can be bounded.

Fact 35 *The uncertain exponential matrices :*

$$A(\rho_i) = e^{M\rho_i},\ B(\rho_i) = \int_0^{\rho_i} e^{Ms} ds N.$$

can be expressed as

$$A(\rho_i) = A^h(\rho_i) + \Delta A^h(\rho_i) \text{ and } B(\rho_i) = B^h(\rho_i) + \Delta B^h(\rho_i) \tag{4.11}$$

where

$$A^h(\rho_i) = \sum_{j=0}^{h} \frac{M^j}{j!}\rho_i^j \text{ and } B^h(\rho_i) = \sum_{j=1}^{h} \frac{M^{j-1}}{j!}\rho_i^j N,$$

are polynomials that represent the h-order Taylor series approximation of the uncertain matrices and

$$\begin{aligned} \Delta A^h(\rho_i) &= e^{M\rho_i} - \sum_{j=0}^{h} \frac{M^j}{j!}\rho_i^j \text{ and} \\ \Delta B^h(\rho_i) &= \int_0^{\rho_i} e^{Ms} ds N - \sum_{j=1}^{h} \frac{M^{j-1}}{j!}\rho_i^j N, \end{aligned} \tag{4.12}$$

are the remainders of the approximation.

4.3.1.2 Vertices number reduction and polytopic transformation of the h-order Taylor series approximation

Here we show how polynomial matrices with positive parameters can be bounded by convex polytopes. In general, any h-degree polynomial

$$L(\rho_i) = L_0 + \rho_i L_1 + \rho_i^2 L_2 + \ldots + \rho_i^h L_h, \tag{4.13}$$

with a ρ_i parameter bounded in a closed interval can be bounded by a convex polytope with 2^h vertices, a hyper-rectangle. In our case we deal with a ρ_i parameter bounded and positive. For this case it is possible to find a convex polytope with only $h+1$ vertices inside the hyper-rectangle that gives a tighter approximation of the Taylor series polynomial.

Lemma 36 *[44] Consider the uncertain parameter dependent polynomial matrix (4.13) such that the uncertain parameter ρ_i is bounded and positive : $0 < \underline{\rho} < \rho_i < \overline{\rho}$. Then one can find a convex polytope with $h+1$ vertices that envelopes the polynomial matrix $L(\rho_i)$, i.e. there exists parameters $\mu_j(\rho_i)$,*

$$\sum_{j=1}^{h+1} \mu_j(\rho_i) = 1,\ \ \mu_j > 0\ \forall\ j = 1, \ldots, h+1$$

such that

$$L(\rho_i) = \sum_{j=1}^{h+1} \mu_j(\rho_i) U_j \tag{4.14}$$

where U_j represent the polytope vertices given as follows :

$$\begin{aligned} U_1 &= L_h \underline{\rho}^{\,h} + L_{h-1}\underline{\rho}^{\,h-1} + \ldots + \underline{\rho}^{\,2} L_2 + \underline{\rho} L_1 + L_0, \\ U_2 &= L_h \underline{\rho}^{\,h} + L_{h-1}\underline{\rho}^{\,h-1} + \ldots + \underline{\rho}^{\,2} L_2 + \overline{\rho} L_1 + L_0, \\ U_3 &= L_h \underline{\rho}^{\,h} + L_{h-1}\underline{\rho}^{\,h-1} + \ldots + \overline{\rho}^{\,2} L_2 + \overline{\rho} L_1 + L_0, \\ &\vdots \\ U_{h+1} &= L_h \overline{\rho}^{\,h} + L_{h-1}\overline{\rho}^{\,h-1} + \ldots + \overline{\rho}^{\,2} L_2 + \overline{\rho} L_1 + L_0. \end{aligned} \tag{4.15}$$

The relations between the uncertain parameters ρ_i and μ are given by:

$$\mu_1 = 1 - \frac{\rho - \underline{\rho}}{\overline{\rho} - \underline{\rho}}, \mu_j = \frac{\rho^{\,j-1} - \underline{\rho}^{\,j-1}}{\overline{\rho}^{\,j-1} - \underline{\rho}^{\,j-1}} - \frac{\rho^{\,j} - \underline{\rho}^{\,j}}{\overline{\rho}^{\,j} - \underline{\rho}^{\,j}},\ \ j = 2..h+1. \tag{4.16}$$

Proof.

From (4.13), (4.14) and (4.15), one can notice that proving lemma 36 is the same as showing that there exists a set of uncertain parameters $\mu_j(\rho)$ that represent barycentric coordinates, solution of the linear system :

$$\begin{bmatrix} 1 & \ldots & \ldots & \ldots & 1 \\ \underline{\rho} & \overline{\rho} & \ldots & \ldots & \overline{\rho} \\ \underline{\rho}^{\,2} & \underline{\rho}^{\,2} & \overline{\rho}^{\,2} & \ldots & \overline{\rho}^{\,2} \\ \vdots & & & \ddots & \vdots \\ \underline{\rho}^{\,h} & \ldots & \ldots & \underline{\rho}^{\,h} & \overline{\rho}^{\,h} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{h+1} \end{bmatrix} = \begin{bmatrix} 1 \\ \rho \\ \rho^2 \\ \vdots \\ \rho^h \end{bmatrix}$$

The existence of a unique solution $[\mu_1\ \ldots\ \mu_{h+1}]$ is guaranteed by the fact that the system determinant is not null as the columns of the system matrix are linearly independent (the parameters in the columns are chosen between the vertices of a hyper-rectangles, see equation (4.15)).

Using the classical Gauss method, the solutions can be computed by the recursive formula :

$$\mu_1 = 1 - \frac{\rho - \underline{\rho}}{\overline{\rho} - \underline{\rho}}, \tag{4.17}$$

$$\mu_j = \frac{\rho^{\,j-1} - \underline{\rho}^{\,j-1}}{\overline{\rho}^{\,j-1} - \underline{\rho}^{\,j-1}} - \sum_{l=j+1}^{h+1} \mu_l,\ \ j = 2..h. \tag{4.18}$$

One can prove that

$$\mu_j = \frac{\rho^{j-1} - \underline{\rho}^{j-1}}{\overline{\rho}^{j-1} - \underline{\rho}^{j-1}} - \frac{\rho^{j} - \underline{\rho}^{j}}{\overline{\rho}^{j} - \underline{\rho}^{j}},\ j = 2..h+1. \tag{4.19}$$

which is strictly positive since the function $f : \Re \rightarrow \Re$,

$$f(x) = \frac{\rho^x - \underline{\rho}^x}{\overline{\rho}^x - \underline{\rho}^x}$$

is monotone decreasing for $x \in (0, \infty)$. Therefore U_i define a convex polytope that includes $L(\rho)$. $\Box$

Remark. The previous lemma gives not only a method for reducing the number of vertices of the polytope to which the polynomial matrix belongs, but also indicates a tighter approximation of the polynomial uncertainty: instead of taking the complete 2^h hyper-rectangle, we consider only the $h+1$ convex polytope to which the uncertainty effectively belongs.

4.3.1.3 Exponential uncertainty modeling as a polytope with additive norm bounded uncertainty

Consider the notation :

$$\phi_1 = \begin{bmatrix} \underline{\rho}^h\mathbf{I} \\ \underline{\rho}^{h-1}\mathbf{I} \\ \vdots \\ \underline{\rho}^2\mathbf{I} \\ \underline{\rho}\mathbf{I} \\ \mathbf{I} \end{bmatrix}, \phi_2 = \begin{bmatrix} \underline{\rho}^h\mathbf{I} \\ \underline{\rho}^{h-1}\mathbf{I} \\ \vdots \\ \underline{\rho}^2\mathbf{I} \\ \overline{\rho}\mathbf{I} \\ \mathbf{I} \end{bmatrix}, \ldots, \phi_{h+1} = \begin{bmatrix} \overline{\rho}^h\mathbf{I} \\ \overline{\rho}^{h-1}\mathbf{I} \\ \vdots \\ \overline{\rho}^2\mathbf{I} \\ \overline{\rho}\mathbf{I} \\ \mathbf{I} \end{bmatrix}. \tag{4.20}$$

Using the previous lemma (equations (4.14) and (4.15)) we establish a new representation of the exponential uncertainty.

Fact 37 *The exponential uncertainties (4.6) can be expressed as :*

$$\begin{array}{l} A(\rho_i) = \sum_{j=1}^{h+1} \mu_j(\rho_i) U_j^{Ah} + \Delta A^h(\rho_i) \\ B(\rho_i) = \sum_{j=1}^{h+1} \mu_j(\rho_i) U_j^{Bh} + \Delta B^h(\rho_i) \end{array} \tag{4.21}$$

with

$$\begin{array}{rcl} U_j^{Ah} & = & \left[\frac{M^h}{h!} \cdots \frac{M^2}{2!} \ \ M \ \ \boldsymbol{I}\right] \phi_j, \\ U_j^{Bh} & = & \left[\frac{M^{h-1}}{h!} \cdots \frac{M}{2!} \ \ \boldsymbol{I} \ \ \boldsymbol{0}\right] \phi_j N, \end{array}$$

where ϕ_j are given by (4.20). The norm bounded terms $\Delta A^h(\rho_i), \Delta B^h(\rho_i)$ are found in (4.12). The relation between the uncertain parameter ρ and the coordinates $\mu_j(\rho)$ is given by the equations (4.16).

In the robust control literature there are several techniques for dealing with convex polytopes. The obtained convex polytopic model of the exponential uncertainties simplifies the analysis of the event based model (4.5).

4.4 Control synthesis

In this section we present LMI control synthesis criteria for the event based model (4.5). Consider the event based model (4.5) from Section 4.2 :

$$\eta_{i+1} = A(\rho_i)\eta_i + B(\rho_i)u_i \tag{4.22}$$

with

$$A(\rho_i) = e^{M\rho_i},\ B(\rho_i) = \int_0^{\rho_i} e^{Ms} ds N.$$

We assume that the command u_i is a state feedback. As we have seen in Section 4.2, when a time varying delay occurs in control loop, the command u_i can be expressed under the form

$$u_i = K\eta_{i-\theta_i} \tag{4.23}$$

with $\theta_i \in \mathcal{T} = \left\{\theta \in \mathbb{Z}^+ :\ \underline{\theta} \leq \theta \leq \overline{\theta}\right\}$. We show first the way the delay can be treated using an augmented model of the event based system. Second we include the polytopic modeling of the exponential uncertainty in the augmented event-based system and we give LMI criteria.

4.4.1 Including the delay as an arbitrary switching parameter

Here we transform system (4.22) into a non delayed switched system by including the delay as a switching parameter. Consider the augmented vector

$$z_i = \left[\eta_i^T\ \eta_{i-1}^T\ \ldots \eta_{i-\overline{\theta}}^T\right]^T$$

and the matrices

$$\bar{A}(\rho_i) = \begin{bmatrix} A(\rho_i) & \mathbf{0} & \ldots & \ldots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \ldots & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix},\ \bar{B}(\rho_i) = \begin{bmatrix} B(\rho_i) \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

The feedback (4.23) can be expressed as a function of the augmented state vector z_i:

$$u_i = \bar{K}_{(\theta_i)} z_i, \tag{4.24}$$

where the gains $\bar{K}_{(\theta_i)}$ are given by

$$\bar{K}_{(\theta_i)} = [\mathbf{0}\ \ \ldots\ \ \mathbf{0}\ \ K\ \ \mathbf{0}\ \ \ldots\ \ \mathbf{0}]$$

with K on the $(\theta_i + 1)^{th}$ column. Now the system (4.22) with the state feedback (4.23) can be expressed as the closed-loop augmented system:

$$z_{i+1} = \left(\bar{A}(\rho_i) + \bar{B}(\rho_i)\bar{K}_{(\theta_i)}\right) z_i \tag{4.25}$$

which is a switched system without delay. The delay is included as a switching parameter in the gain matrix $\bar{K}_{(\theta_i)}$.

4.4.2 Reformulation of the exponential uncertainty for the event-based representation

Consider the system (4.22), its augmented version (4.25) and the following notations:

$$\bar{A}_j^h = \begin{bmatrix} U_j^{Ah} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \; \bar{B}_j^h = \begin{bmatrix} U_j^{Bh} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

$$\Delta\bar{A}^h(\rho_i) = \begin{bmatrix} \Delta A^h(\rho_i) & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \; \Delta\bar{B}^h(\rho_i) = \begin{bmatrix} \Delta B^h(\rho_i) \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

Using the results from the previous section (Fact 37), one can remark that the solutions of system (4.25) can be expressed as solutions of the polytopic system with a normed bounded uncertainty and a switched control

$$z_{i+1} = \left(\sum_{j=1}^{h+1} \mu_j(\rho_i)\bar{A}_j^h + \Delta\bar{A}^h(\rho_i)\right) z_i + \left(\sum_{j=1}^{h+1} \mu_j(\rho_i)\bar{B}_j^h + \Delta\bar{B}^h(\rho_i)\right) \bar{K}_{(\theta_i)} z_i, \tag{4.26}$$

where

$$\sum_{j=1}^{h+1} \mu_j(\rho_i) = 1, \; \mu_j(\rho_i) > 0 \; \forall \; j = 1 \dots, h+1, \forall \; i \in \mathbb{Z}^+$$

and where the terms $\Delta\bar{A}^h(\rho_i)$ and $\Delta\bar{B}^h(\rho_i)$ are bounded while $\underline{\rho} < \rho_i < \overline{\rho}$ (see the equation (4.12)). We denote their bound by

$$\epsilon_A^{1/2} = \sup_{\underline{\rho}<\rho<\overline{\rho}} \left\|\Delta\bar{A}^h(\rho)\right\|$$

and

$$\epsilon_B^{1/2} = \sup_{\underline{\rho}<\rho<\overline{\rho}} \left\|\Delta\bar{B}^h(\rho)\right\|,$$

respectively.

4.4.3 LMI state feedback synthesis

Here we present LMI control synthesis methods for the switched polytopic uncertain model with additive norm bounded uncertainty (4.26) obtained in the previous subsection. It is clear that finding a control for the system (4.26) ensures the stabilizability of the event based model (4.5).

Theorem 38 *[44] Consider the closed-loop switched polytopic system with norm bounded uncertainty (4.26) and the matrices*

$$G_\theta = \begin{bmatrix} & G_\theta^1 & \\ G_\theta^2 & G^3 & G_\theta^4 \\ & G_\theta^5 & \end{bmatrix}, \tag{4.27}$$

$G_\theta^1 \in \mathbb{R}^{n\cdot\theta\times n\cdot(\bar{\theta}+1)}$, $G_\theta^2 \in \mathbb{R}^{n\times n\cdot\theta}$, $G^3 \in \mathbb{R}^{n\times n}$, $G^4 \in \mathbb{R}^{n\times n\cdot(\bar{\theta}-\theta)}$, $G^5 \in \mathbb{R}^{n\cdot(\bar{\theta}-\theta)\times n\cdot(\bar{\theta}+1)}$

$$R_\theta = \begin{bmatrix} R_\theta^2 & R^3 & R_\theta^4 \end{bmatrix}, \tag{4.28}$$

$R^2 \in \mathbb{R}^{m\times n\cdot\theta}$, $R^3 \in \mathbb{R}^{m\times n}$, $R^4 \in \mathbb{R}^{m\times n\cdot(\bar{\theta}-\theta)}$, *such that*

$$G_\theta^2 = G^3\Omega_\theta^2,\ G_\theta^4 = G^3\Omega_\theta^4,\ R^2 = R^3\Omega_\theta^2,\ R^4 = R^3\Omega_\theta^4 \tag{4.29}$$

where $\Omega_\theta^2, \Omega_\theta^4$ *are arbitrary matrices of appropriate dimension for all* $\theta \in \mathcal{T}$. *If there exist two scalars* $\lambda_1 > 0$, $\lambda_2 > 0$, *matrices* G_θ, R_θ *and symmetric positive definite matrices* $S_{j,\theta}$, *such that*

$$\begin{bmatrix} \lambda_2 \boldsymbol{I} & \boldsymbol{0} & R_{\theta_1} & \boldsymbol{0} \\ * & \lambda_1 \boldsymbol{I} & G_{\theta_1} & \boldsymbol{0} \\ * & * & -S_{j_1,\theta_1} + G_{\theta_1} + G_{\theta_1}^T & G_{\theta_1}^T A_{j_1}^T + R_{\theta_1}^T B_{j_1}^T \\ * & * & * & S_{j_2,\theta_2} - \lambda_2\epsilon_B^2 \boldsymbol{I} - \lambda_1\epsilon_A^2 \boldsymbol{I} \end{bmatrix} > 0 \tag{4.30}$$

for all $\theta_1, \theta_2 \in \mathcal{T}$, *then the closed loop system (4.26) is stabilized using the gains*

$$\bar{K}_\theta = R_\theta G_\theta^{-1},\ \forall\, \theta \in \mathcal{T}.$$

Proof. Consider the switched system obtained using the vertices of the polytopic system (4.26).

$$z_{i+1} = \left(\bar{A}_j^h + \Delta\bar{A}^h\right) z_i + \left(\bar{B}_j^h + \Delta\bar{B}^h\right) \bar{K}_{(\theta_i)} z_i$$

with $\theta \in \mathcal{T}$. Applying Proposition 43 from the Appendix with $\alpha = (j)$ and $\beta = (\theta)$ we obtain that the system can be stabilized using the gains $\bar{K}_{(\theta_i)}$ if the LMI (4.30) has a solution. In this case there exists a switched Lyapunov function

$$V(i) = z_i^T P_{j,\theta} z_i \ \text{ with } P_{j,\theta} = S_{j,\theta}^{-1}$$

strictly decreasing. This implies that

$$\begin{bmatrix} P_{j_1,\theta_1} & \left(\bar{A}_{j_1}^h + \Delta\bar{A}^h + \left(\bar{B}_{j_1}^h + \Delta\bar{B}^h\right)\bar{K}_{\theta_1}\right)^T \cdot P_{j_2,\theta_2} \\ * & P_{j_2,\theta_2} \end{bmatrix} > 0$$

$\forall j_1, j_2 = 1, \ldots, h+1$. Multiply by μ_{j_1} and sum for $j_1 = 1, \ldots, h+1$. Then repeat the procedure with μ_{j_2}, j_2 and apply the Schur complement. We obtain that

$$\sum_{j_1=1}^{h+1} \mu_{j_1} P_{j_1,\theta_1} - \sum_{j_1=1}^{h+1} \mu_{j_1} \left(\bar{A}_{j_1}^h + \Delta\bar{A}^h + \left(\bar{B}_{j_1}^h + \Delta\bar{B}^h\right)\bar{K}_{\theta_1}\right)^T \times$$

$$\times \sum_{j_2=1}^{h+1} \mu_{j_2} P_{j_2,\theta_2} \sum_{j_1=1}^{h+1} \mu_{j_1} \left(\bar{A}_{j_1}^h + \Delta\bar{A}^h + \left(\bar{B}_{j_1}^h + \Delta\bar{B}^h\right)\bar{K}_{\theta_1}\right) > 0, \forall\theta_1, \theta_2 \in \mathcal{T}$$

which implies that

$$V(i) = z_i^T \sum_{j=1}^{h+1} \mu_j P_{j,\theta} z_i$$

is a Lyapunov function for system (4.26). □

Remarks. The previous theorem provides a constant digital state feedback of the form

$$u(k) = Kx(k)$$

for the original continuous time system (4.1). The gain matrix K is obtained by finding a constrained state feedback for the augmented polytopic system (4.26). Using the particular form of the G_θ matrices, with constant terms G^3, the previous theorem gives feedback gains $\bar{K}_\theta$ of the form

$$\bar{K}_\theta = \left[\mathbf{0}_{m\times\theta}\ K\ \mathbf{0}_{m\times\left(\bar{\theta}-\theta\right)}\right]$$

which depend on the constant gain K. The event-based system is stabilized by $K = R^3(G^3)^{-1}$. The arbitrary matrices Ω_θ^2 and Ω_θ^4 in (4.29) ensure that the system of equations

$$\begin{cases} KG_\theta^2 &= R_\theta^2 \\ KG^3 &= R^3 \\ KG_\theta^4 &= R_\theta^4 \end{cases}$$

has a solution, K. In the equation (4.30), only G_θ, R_θ and $S_{j,\theta}$ are LMI variables. Ω_θ^2 and Ω_θ^4 are arbitrary matrices not LMI variables. For simple cases Ω_θ^2 and Ω_θ^4 can be taken as null matrices. Ω_θ^2 and Ω_θ^4 matrices different from zero are useful for relaxing the LMI constrains and giving more freedom in the choice of the G_θ matrix. Notice that introducing this matrices as optimization variables leads to $BMIs$. The obtained gains K ensure the stabilizability of the discrete event based representation (4.5).

There are cases where using a priori information could lead to an approximative knowledge of the delay. For example, in the case of networked control systems, it is possible to indicate in which sampling period the control will arrive. This can be done using time-stamped messages [101]. In this case, θ_i may be known. It could be interesting to compute a delay-dependent feedback controller, that is to find feedback gains that depend on θ_i. The previous theorem can be adapted for giving a delay-dependent state feedback by changing the matrices G_θ and R_θ such that G^3 and R^3 are no longer constant, they depend on θ.

When networked protocols that discard the old data are used, the given LMI constraints can be relaxed. Since the actuator always receives fresh data, not older than the ones used in the previous commands, the delay θ_i is increased by one (up to the maximum value) or it takes the minimum value $\underline{\theta}$. In this context, not all the combinations of θ_1, θ_2 are possible. Therefore, one should reconsider the Theorem 38 (i.e. the LMI (4.30)) with $\theta_1 \in \mathcal{T}$, $\theta_2 \in \min\left\{\theta_1 + 1, \overline{\theta}\right\} \cup \{\underline{\theta}\}$.

Using similar arguments to the ones in [42], the system behavior can be analyzed between the events and it can be shown that the obtained control ensures the stabilizability of the original continuous time system.

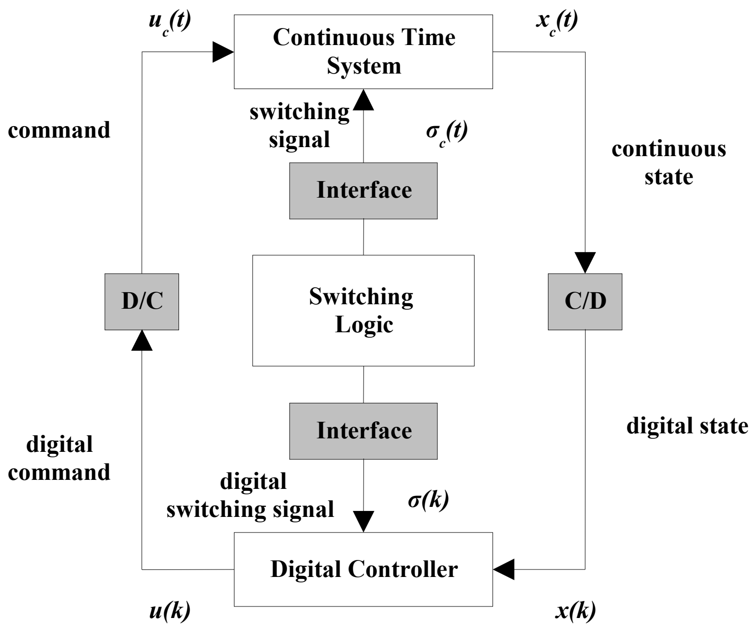

Figure 4.1: System architecture

4.5 Switched systems in digital control loops

In this section we consider the case of continuous time switched system described by :

$$\frac{dx_c(t)}{dt} = M_{\sigma_c(t)}x_c(t) + N_{\sigma_c(t)}u_c(t). \tag{4.31}$$

Here $\{M_i \in \mathbb{R}^{n\times n} : i \in P\}$ and $\{N_i \in \mathbb{R}^{n\times m} : i \in P\}$ are two families of matrices. Each pair (M_i, N_i), $i \in P = \{1, 2, .., N\}$ describes a continuous time model representing the different regimes of system behavior. We denote by $\sigma_c(t)$ the continuous time switching signal. Formally $\sigma_c(t) : \mathbb{R}^+ \to P$ is a piecewise constant signal that gives a particular index i indicating the active system regime. For the sake of generality we do not consider any hypothesis on the nature of the switching mechanism. It may represent a random process, possibly uncontrollable, or a very complex digital control device, too difficult to model.

According to the analogical or digital nature of the switching mechanism, interface blocks are used between the switching mechanism and the other elements. They may be simple continuous to discrete or discrete to continuous converters.

Here we assume that the switching function is arbitrary, unknown a priori but available in real time. We assume that the values of the continuous time signals $x_c(t)$ and $\sigma_c(t)$ are available by sampling via an analogical to digital converter (the C/D block in Figure 4.1) and they are used for control computation.

By assuming that an infinite number of switching cannot occur in a finite period of time and considering the switching phenomena as an event, we can generalize the *event-based representation* (4.5) for the case of switched linear systems:

$$\eta_{i+1} = A_{\varsigma_i}(\rho_i)\eta_i + B_{\varsigma_i}(\rho_i)u_i \tag{4.32}$$

where

$$\begin{aligned}
\eta_i &= x_c(t_i), && (4.33)\\
\varsigma_i &= \sigma_c(t_i), && (4.34)\\
u_i &= x_c(t_i) && (4.35)\\
A_{\varsigma_i}(\rho_i) &= e^{M_{\varsigma_i}\rho_i},\ B_{\varsigma_i}(\rho_i) = \int_0^{\rho_i} e^{M_{\varsigma_i}s}ds N_{\varsigma_i} && (4.36)\\
0 &= \underline{\rho} < \rho_i \le \overline{\rho}. && (4.37)
\end{aligned}$$

For this model we can apply the convex polytopic modeling approach. The proposed methodology is applied for the case of a switched state feedback:

$$u_c(t_i) = K_{\sigma_c(t_j)}x_c(t_j),$$

where t_i represents an arbitrary event, t_j with $t_i \geq t_j$ corresponds to some past sampling event. Let $\theta_i = j - i$ represent the number of events that passed between the sampling (at the moment t_j) and the actuation of the command $K_{\sigma_c(t_j)} x_c(t_j)$ (at the moment t_i). Since only a finite number of switchings can occur in a finite interval of time, the parameter θ can be bounded similar to the LTI case, $\theta_i \in \mathcal{T} = \left\{\theta \in \mathbb{Z}^+ : \underline{\theta} \leq \theta \leq \overline{\theta}\right\}$. For system (4.32) the switched state feedback can be expressed as

$$u_i = K_{\gamma_i} \eta_{i-\theta_i}, \tag{4.38}$$

where $\gamma : \mathbb{Z}^+ \to P$ is a function that represents the switching signal in the input at the i^{th} event, $\gamma_i = \varsigma_{i-\theta_i}$.

Consider the augmented vector

$$z_i = \left[\eta_i^T \ \eta_{i-1}^T \ \ldots \eta_{i-\overline{\theta}}^T\right]^T$$

and the matrices

$$\bar{A}^h_{\varsigma_i,j} = \begin{bmatrix} U^{Ah}_{\varsigma_i,j} & \mathbf{0} & \ldots & \ldots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \ldots & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \bar{B}^h_{\varsigma_i,j} = \begin{bmatrix} U^{Bh}_{\varsigma_i,j} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

$$\Delta\bar{A}^h_{\varsigma_i}(\rho_i) = \begin{bmatrix} \Delta A^h_{\varsigma_i}(\rho_i) & \mathbf{0} & \ldots & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ldots & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \Delta\bar{B}^h_{\varsigma_i}(\rho_i) = \begin{bmatrix} \Delta B^h_{\varsigma_i}(\rho_i) \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \tag{4.39}$$

$$\bar{K}_{(\gamma_i,\theta_i)} = [\mathbf{0} \ \ \ldots \ \ \mathbf{0} \ \ K_{\gamma_i} \ \ \mathbf{0} \ \ \ldots \ \ \mathbf{0}]$$

with K_{γ_i} on the $(\theta_i + 1)^{th}$ column. The terms

$$\begin{aligned} U^{Ah}_{\varsigma,j} &= \left[\frac{M^h_\varsigma}{h!} \ldots \frac{M^2_\varsigma}{2!} \ \ M_\varsigma \ \ \mathbf{I}\right] \phi_j, \\ U^{Bh}_{\varsigma,j} &= \left[\frac{M^{h-1}_\varsigma}{h!} \ldots \frac{M_\varsigma}{2!} \ \ \mathbf{I} \ \ \mathbf{0}\right] \phi_j N_\varsigma, \end{aligned}$$

(where ϕ_j are given by (4.20)) and the norm bounded matrices $\Delta A^h(\rho_i), \Delta B^h(\rho_i)$

$$\begin{aligned} \Delta A^h_\varsigma(\rho_i) &= e^{M_\varsigma \rho_i} - \sum_{j=0}^{h} \frac{M^j_\varsigma}{j!} \rho^j \text{ and} \qquad (4.40) \\ \Delta B^h_\varsigma(\rho_i) &= \int_0^{\rho_i} e^{M_\varsigma s} ds N_\varsigma - \sum_{j=1}^{h} \frac{M^{j-1}_\varsigma}{j!} \rho^j N_\varsigma, \end{aligned}$$

are obtained by applying the Fact 37 to the case of the uncertainties (4.36).

The *convex polytopic representation* of the switched event-based model (4.32) is given by

$$z_{i+1} = \left(\sum_{j=1}^{h+1} \mu_j(\rho_i) \bar{A}^h_{\varsigma_i,j} + \Delta \bar{A}^h_{\varsigma_i}(\rho_i) \right) z_i + \left(\sum_{j=1}^{h+1} \mu_j(\rho_i) \bar{B}^h_{\varsigma_i,j} + \Delta \bar{B}^h_{\varsigma_i}(\rho_i) \right) \bar{K}_{(\gamma_i,\theta_i)} z_i, \tag{4.41}$$

with

$$\epsilon_A^{1/2} = \sup_{\underline{\rho}<\rho<\overline{\rho}} \left\| \Delta \bar{A}^h_{\varsigma_i}(\rho) \right\| \text{ and } \epsilon_B^{1/2} = \sup_{\underline{\rho}<\rho<\overline{\rho}} \left\| \Delta \bar{B}^h_{\varsigma_i}(\rho) \right\|.$$

For this system the LMI control synthesis is expressed as follows.

Theorem 39 *Consider the closed-loop switched polytopic system with norm bounded uncertainty (4.41) and the matrices*

$$G_{\gamma,\theta} = \left[\begin{array}{ccc} & G^1_{\gamma,\theta} & \\ G^2_{\gamma,\theta} & G^3_{\gamma} & G^4_{\gamma,\theta} \\ & G^5_{\gamma,\theta} & \end{array} \right], \tag{4.42}$$

$G^1_{\gamma,\theta} \in \mathbb{R}^{n\cdot\theta\times n\cdot(\overline{\theta}+1)}$, $G^3_\gamma \in \mathbb{R}^{n\times n}$, $G^2_{\gamma,\theta} \in \mathbb{R}^{n\times n\cdot\theta}$,

$$R_{\gamma,\theta} = \left[\begin{array}{ccc} R^2_{\gamma,\theta} & R^3_\gamma & R^4_{\gamma,\theta} \end{array} \right], \tag{4.43}$$

$R^2_\gamma \in \mathbb{R}^{m\times n\cdot\theta}$, $R^3_\gamma \in \mathbb{R}^{m\times n}$, $R^3_\gamma \in \mathbb{R}^{m\times n\cdot(\overline{\theta}-\theta)}$, *such that*

$$G^2_{\gamma,\theta} = G^3_\gamma \Omega^2_{\gamma,\theta},\ G^4_{\gamma,\theta} = G^3_\gamma \Omega^4_{\gamma,\theta},\ R^2_\gamma = R^3_\gamma \Omega^2_{\gamma,\theta},\ R^4_\gamma = R^3_\gamma \Omega^4_{\gamma,\theta} \tag{4.44}$$

where $\Omega^2_{\gamma,\theta}, \Omega^4_{\gamma,\theta}$ *are arbitrary matrices of appropriate dimension with* $\varsigma \in P$, $j = 1,\ldots,h+1$, $\gamma \in P$ *and* $\theta \in \mathcal{T}$. *If there exist two scalars* $\lambda_1 > 0$, $\lambda_2 > 0$, *matrices* $G_{\gamma,\theta}$, $R_{\gamma,\theta}$ *and symmetric positive definite matrices* $S_{\varsigma,j,\gamma,\theta}$ *such that*

$$\left[\begin{array}{cccc} \lambda_2 \boldsymbol{I} & \boldsymbol{0} & R_{\gamma_1,\theta_1} & \boldsymbol{0} \\ * & \lambda_1 \boldsymbol{I} & G_{\gamma_1,\theta_1} & \boldsymbol{0} \\ * & * & -S_{\varsigma_1,j_1,\gamma_1,\theta_1} + G_{\gamma_1,\theta_1} + G^T_{\gamma_1,\theta_1} & G^T_{\gamma_1,\theta_1} {A^h}^T_{\varsigma_1,j_1} + R^T_{\gamma_1,\theta_1} {B^h}^T_{\varsigma_1,j_1} \\ * & * & * & S_{\varsigma_2,j_2,\gamma_2,\theta_2} - \lambda_2 \epsilon_B^2 \boldsymbol{I} - \lambda_1 \epsilon_A^2 \boldsymbol{I} \end{array} \right] > 0 \tag{4.45}$$

for all $\varsigma_1, \varsigma_2 \in P$, $j_1, j_2 = 1,\ldots,h+1$, $\gamma_1,\gamma_2 \in P$ *and* $\theta_1,\theta_2 \in \mathcal{T}$, *then the closed loop system (4.41) is stabilized using the gains*

$$\bar{K}_{\gamma,\theta} = R_{\gamma,\theta} G^{-1}_{\gamma,\theta},\ \forall\ (\gamma,\theta) \in P \times \mathcal{T}.$$

Proof. Consider the switched system obtained using the vertices of the polytopic system (4.41). The LMIs (4.45) are obtained by applying Proposition 43 from the Appendix with $\alpha = (\varsigma, j)$ and $\beta = (\gamma,\theta)$ and using similar arguments to the ones in the proof of Theorem 38. □

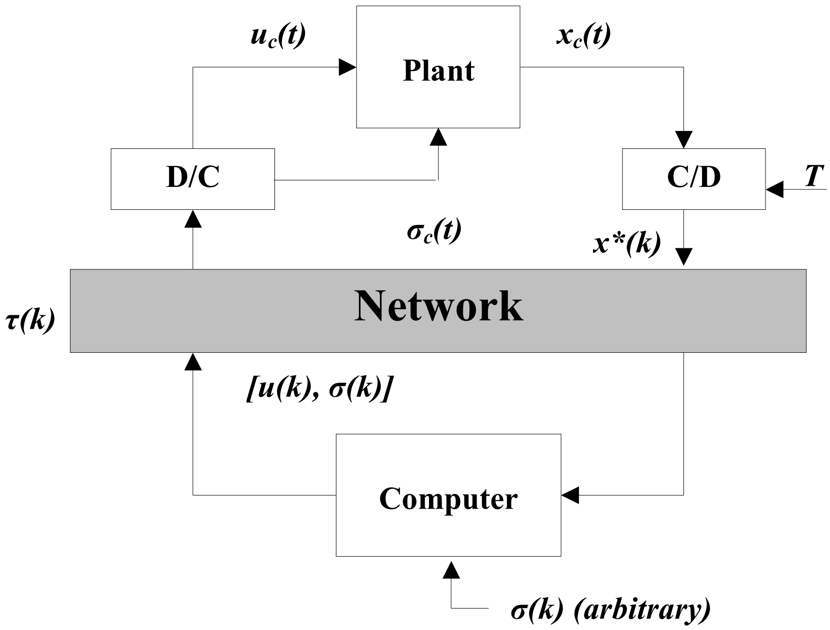

Figure 4.2: Remote control of a switched system

4.6 Applications

In this section we present an application of our modeling approach and some numerical examples.

Event-based modeling for the remote control of switched systems

As follows we illustrate the construction of event-based models and the switched polytopic modeling for a simple case of switched system.

We consider a continuous time switched system (4.1) that is controlled through a network (see Figure 4.2). The switching signal represents a command from an upper level supervisor. We consider that its value is arbitrary and that it can be available in real time. Assuming an arbitrary switching can be very useful in many practical applications such as the case when $\sigma(k)$ is computed via complex algorithms by a higher level supervisor or when it is generated by a human operator (for example the switch of gears in a car). The discrete state of the system, $x(k)$, is sent to a computer that produces the digital version of the control $u(k)$ and synchronizes it to the switching signal $\sigma(k)$. We assume that both $u(k)$ and $\sigma(k)$ can be transmitted in a single packet through the network. Event-driven actuator (the D/C blocks in the figure) transform them into piecewise constant continuous time signals $u_c(t)$ and $\sigma_c(t)$ as soon as the signals are available. Since

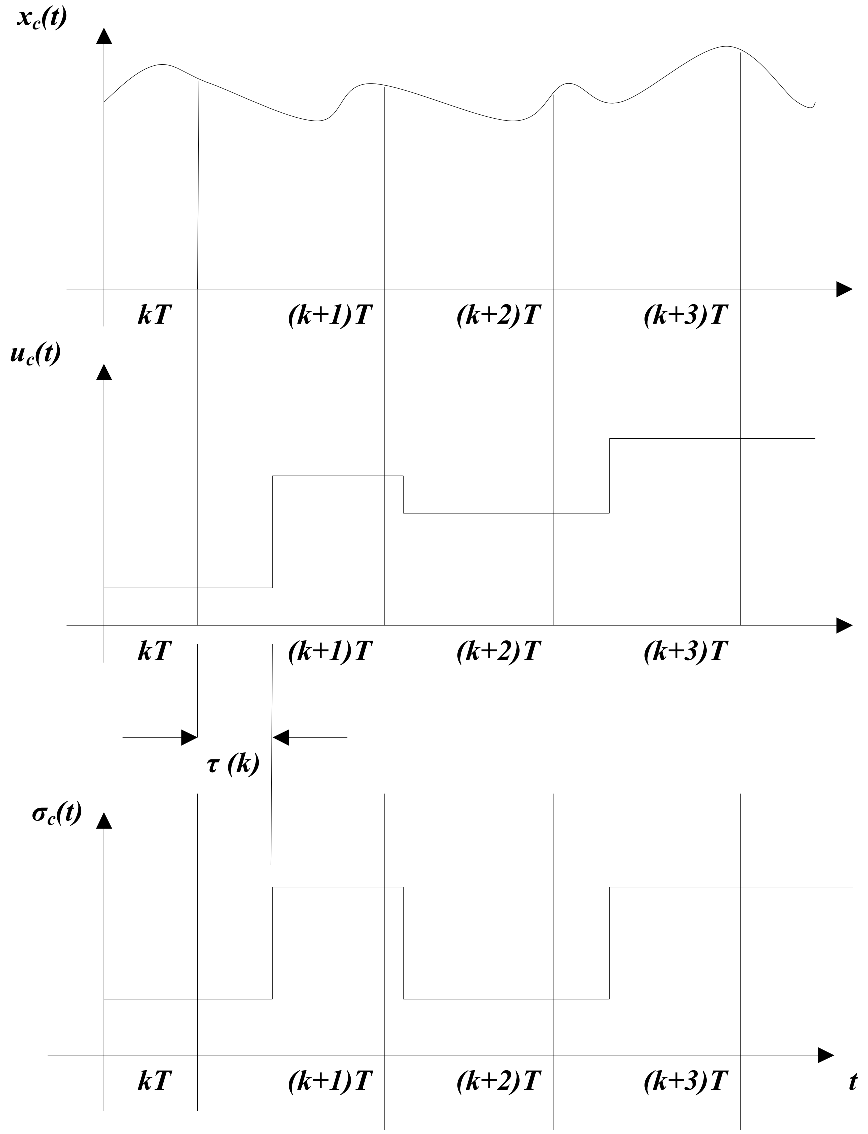

Figure 4.3: The control and the switching signals

the value of the switching signal is known and synchronized with command $u(k)$ we can be sure to use the *real* value of the switching signal in the command.

The information transfer through the network introduces time varying delays. We consider here the case of constant sampling period T and total delay $\tau_{min} < \tau(k) < \tau_{max}$ smaller than T. The timing of signals is presented in Figure 4.3. The continuous-time signals affected by the delay are given by :

$$\sigma_c(t) = \begin{cases} \sigma(k-1), t \in [kT, kT + \tau(k)) \\ \sigma(k), t \in [kT + \tau(k), (k+1)T) \end{cases} \tag{4.46}$$

$$\text{and } u_c(t) = \begin{cases} u(k-1), t \in [kT, kT + \tau(k)) \\ u(k), t \in [kT + \tau(k), (k+1)T) \end{cases} \tag{4.47}$$

Here we give the construction of an event based model of the form (4.32) for the remote switched system. The system evolves between two successive events: the sampling ($t_i = kT$) and the actuation/switching event ($t_i = kT + \tau(k)$). The state of the event based model is

$$\eta_i = \begin{cases} x_c(kT), & \text{sampling event }, \\ x_c(kT + \tau(k)), & \text{actuation/switching event }, \end{cases}$$

but the value of the state η_i is known only for $t_i = kT$.

The uncertain parameter ρ_i successively takes the values $\tau(k)$ and $T - \tau(k)$ and its bounds are $\underline{\rho} = \min(\tau_{min}, T - \tau_{max})$, $\overline{\rho} = \max(\tau_{max}, T - \tau_{min})$. Since η_i is known only for $t_i = kT$ it is clear that we will never have a command that depends on the value of the continuous system state at the moment of the actuation. The system evolution is described by:

$$\eta_{i+1} = A_{\varsigma_i}(\rho_i)\eta_i + B_{\varsigma_i}(\rho_i)u_i.$$

Consider that the command is a switched state feedback:

$$u(k) = K_{\sigma(k)}x(k).$$

In the space of event indexes

$$u_i = \begin{cases} K_{\gamma_i}\eta_{i-1}, & \text{if } i \text{ is an actuation event and} \\ \\ K_{\gamma_i}\eta_{i-2}, & \text{if } i \text{ corresponds to a sampling event.} \end{cases}$$

We obtain a system of the form (4.32) with $\theta_i \in \{1, 2\}$. Since the next value of switching signal $\sigma(k)$ is known at the moment of the control computation we may use it for the control, i.e. $\gamma_i = \varsigma_i$. Notice that the value of the switching signal changes only at $t_i = kT + \tau(k)$, which means that $\varsigma_i = \varsigma_{i-1}$ if $\theta_i = 2$. Therefore, the command is given by

$$u_i = K_{\varsigma_i}\eta_{i-\theta_i}$$

and the closed loop system has the form

$$\eta_{i+1} = A_{\varsigma_i}(\rho_i)\eta_i + B_{\varsigma_i}K_{\varsigma_i}\eta_{i-\theta_i}.$$

We use an augmented state vector $z_i = \left[\eta_i^T \ \eta_{i-1}^T \ \eta_{i-2}^T\right]^T$ for which we obtain an augmented system (without delay) of the form

$$z_{i+1} = \bar{A}_{\varsigma_i}(\rho_i)z_i + \bar{B}_{\varsigma_i}(\rho_i)\bar{K}_{(\varsigma_i,\theta_i)}z_i$$

with

$$\bar{A}_{\varsigma_i}(\rho_i) = \begin{bmatrix} A_{\varsigma_i}(\rho_i) & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \bar{B}_{\varsigma_i}(\rho_i) = \begin{bmatrix} B_{\varsigma_i}(\rho_i) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

The delay is included as a supplementary switching parameter in the control gains $\bar{K}_{(\varsigma_i,\theta_i)}$:

$$\bar{K}_{(\varsigma_i,1)} = [\mathbf{0} \ \ K_{\varsigma_i} \ \ \mathbf{0}], \ \ \bar{K}_{(\varsigma_i,2)} = [\mathbf{0} \ \ \mathbf{0} \ \ K_{\varsigma_i}].$$

For this system we use the polytopic model of the uncertainty in order to design a state feedback control. Considering a first order approximation, the exponential uncertainties $A_{\varsigma_i}(\rho_i)$ and $A_{\varsigma_i}(\rho_i)$ can be expressed as

$$A_{\varsigma_i}(\rho) = \mathbf{I} + M_{\varsigma_i} \cdot \rho + \Delta A_{\varsigma_i}, \ B_{\varsigma_i}(\rho) = N_{\varsigma_i} \cdot \rho + \Delta B_{\varsigma_i}.$$

We obtain switched polytopic uncertain system with two vertices per polytope:

$$\bar{A}_{\varsigma_i,1} = \begin{bmatrix} \mathbf{I} + M_{\varsigma_i} \cdot \underline{\rho} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \bar{B}_{\varsigma_i,1} = \begin{bmatrix} N_{\varsigma_i} \cdot \underline{\rho} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

$$\bar{A}_{\varsigma_i,2} = \begin{bmatrix} \mathbf{I} + M_{\varsigma_i} \cdot \overline{\rho} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \bar{B}_{\varsigma_i,2} = \begin{bmatrix} N_{\varsigma_i} \cdot \overline{\rho} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

The additive norm bounded uncertainties $\Delta\bar{A}_{\varsigma_i}$ and $\Delta\bar{B}_{\varsigma_i}$ can be adapted from (4.39). For this model we can apply Theorem 39 with $\gamma_i = \varsigma_i$ in order to obtain a stabilizing feedback.

As follows the obtained LMI stabilizability criteria are illustrated by some numerical examples.

Control synthesis for LTI systems with sampling jitter and delays

Consider the LTI model of a servo-motor positioning system given by the transfer function

$$G_p(s) = \frac{1000}{s\,(0.25s + 1)}. \tag{4.48}$$

The system is sampled every $T = 0.015s$ with a sampling jitter $\delta T \in [0, 0.005s]$. The feedback loop is affected by a delay τ that is varying between $\tau_{min} = 0.001s$

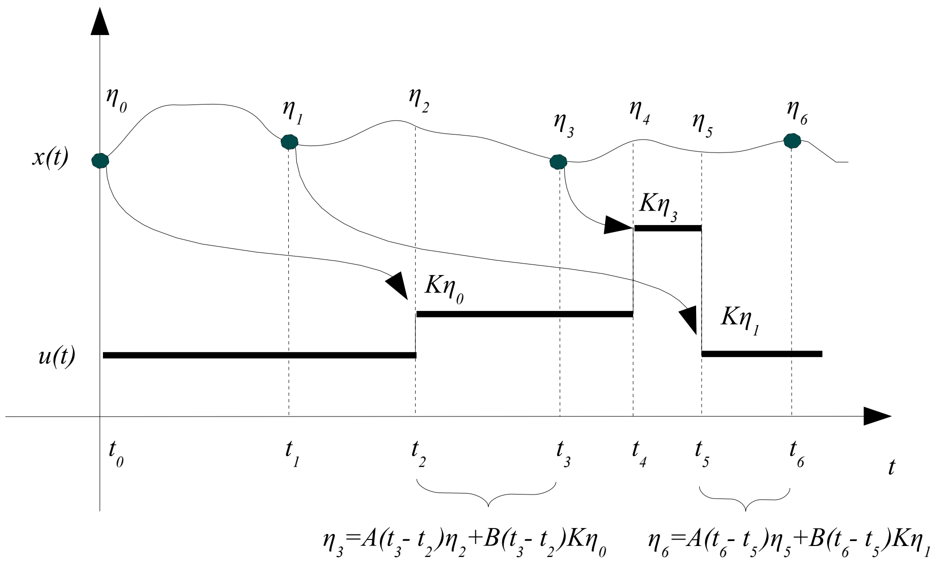

Figure 4.4: LTI system with delay. The values of the state vector that are obtained by sampling are marked by a black dot. t_i specifies the moment the ith event occurs and η_i is the value of the state vector $x(t)$ for $t = t_i$.

and $\tau_{max} = 0.018s$. This implies that two actuations may occur during a sampling period.

We obtain an event-based model of the form (4.5) with an uncertain parameter $\rho \in [0, 0.02s]$ while the new discrete delay θ takes values in the finite set $\{1, 2, 3, 4\}$ (see Figure 4.4). As we see, the price of reducing the number of nonlinear terms is paid by increasing the number of delays in the new model.

We consider now the Taylor development of the uncertain terms. For this example, using a first order approximation of the exponential uncertainty ($h = 1$) the norms of the remainders are less than 1.3×10^{-4}. We obtain a polytopic model of the uncertainty with two vertices. By using the state augmentation and including the delay as a switching parameter we obtain a 10^{th} order system that switches among 4 polytopes (each one with 2 vertices). We consider that the delay is completely unknown and we look for a constant state feedback. Using Theorem 38, we obtain a feedback gain $K = [-0.9942 \;\; -0.5431]$.

Control of switched system on non-uniform time domains

Here we consider a switched system on uncertain time domains. The system is characterized by a nominal behavior (described by the equation (4.48)) and a reduced input behavior, where the input is multiplied by a 50 percent factor. We assume that the sampling period T is varying between $T_{min} = 0.007s$ and

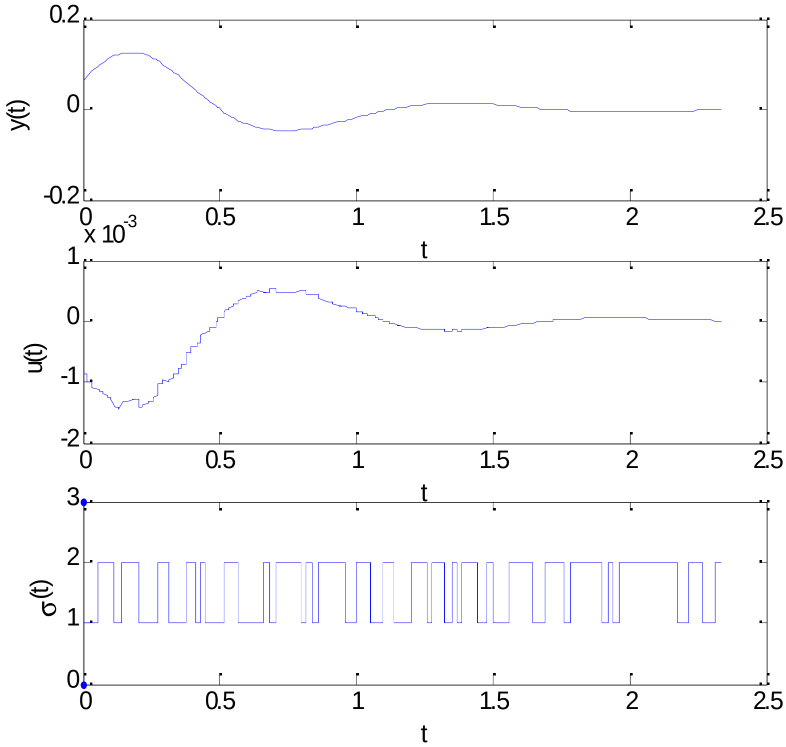

Figure 4.5: Control on non-uniform time intervals.

$T_{max} = 0.018s$. The switching between these two modes is randomly given by an upper level supervisor, and it is synchronized with the instant of sampling. We aim giving a switched control that stabilizes the system. Using an event based representation with $\rho \in [0, 0.018s]$, $\theta \in \{1, 2\}$ and a first order approximation of the exponential uncertainty we obtain the following feedback gains:

$$K_1 = [-1.1582 - 0.0344]\,,\ K_2 = [-2.3165 - 0.0689]\,,$$

where K_1 ensures the stability of the nominal system while K_2 compensates the input reduction. An example of system evolution from an arbitrary initial condition is given in Figure 4.5.

4.7 Conclusion

The goal of this chapter was to provide a unique methodology for dealing with timing problems in the digital control of LTI and switched systems. It is shown that using state augmentation and polytopic modeling of exponential uncertainty we can obtain a switched polytopic model with additive norm bounded uncertainty for which the control synthesis problem can be solved using LMI stabilizability conditions. The obtained model is able to treat simultaneously all the timing problems (sampling jitter, feedback delays) even if the delay variation is larger than a sampling period. The methodology was extended to the case of switched systems. Numerical examples illustrate the approach. The paper treats the most general case that we could imagine. The control synthesis results can certainly be improved when a priori knowledge about the switching function or about the implementation protocols are available. .

General Conclusion

This PhD thesis is dedicated to the robust stability analysis and control design for switched linear systems. The research is principally focused on discrete time switched linear systems with polytopic uncertainties and uncertainties on the switching law. The stability problems are treated in the framework of the Lyapunov Theory and it is based on the resolution of linear matrix inequalities. In this context we showed how it is possible to take into account the uncertain parameters in order to design efficient control synthesis laws: the uses of Switched Parameter Dependent Lyapunov Functions allowed us to derive less conservative stability and control design criteria.

We showed that the interaction between digital controllers and continuous time systems leads to switched uncertain systems with polytopic uncertainties. We consider the discrete time model of a LTI continuous time system in digital control loops. Uncertainties due to the timing problems, i.e. the inaccuracy of sampling and control actuation and the delay occurrence in the control loop, have been considered. We showed that an event-based model allows as to formulate the digital stabilization problem as a stability problem for switched uncertain systems with polytopic uncertainties.

The results are based on the contribution of chapter 3 where it is shown that modeling a discrete delay as a switching parameter is equivalent to treating the delay stability problem using the most general quadratic Lyapunov-Krasovskii functional. The control design methodology proposed for switched uncertain system is applied for the stabilization of LTI continuous time systems with timing problems. Finally, we presented an extension of these results to the case of continuous time switched linear systems in digital control loops. The given modeling approach may seem very complex in the most general case : switched systems with delay larger than the sampling period, arbitrary switching law and mixed dynamics. In practical application the control synthesis results can certainly be improved since a priori knowledge about the switching function or about the implementation protocols are often available.

In the future, a first point to develop is the design of delay dependent switching controllers. There are cases where using a priori information could lead to an approximative knowledge of the delay. For example, in the case of networked control systems, it is possible to indicate in which sampling period the control will arrive (this can be done using time-stamped messages). In this case, the delay may be known. It would be interesting in this case to define clusters in the delay space and to use adaptive gains that are switched according to the instantaneous value of the delay.

A second point which could be studied is the relation between the sampling frequency and the polytopic approximation of exponential uncertainties, $e^{M\tau}$. In this case the Shanon theorem may give more informations on the uncertain parameter variation domain. This informations may be used for selecting the Taylor series development order, which can help in reducing the computational complexity.

In this thesis, we studied the stabilization using a state feedback. For practical reasons, the obtained results should be extended other stabilizing laws such as the output feedback or dynamic output feedback. At last, performance constraints should also be considered for control synthesis.

Appendix

Lemma 40 (Schur Complement lemma) *[13] Consider the matrix*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with D, a non-singular (invertible) matrix. The matrix

$$S := A - BD^{-1}C$$

is called the Schur complement of D in M. If M is a symmetric definite matrix then $M > 0$ is equivalent to

$$D > 0 \text{ and } S = A - BD^{-1}C > 0.$$

Lemma 41 (Norm bounded real) *[13] Consider a constant scalar $\lambda > 0$ and the M, N and Y matrices of appropriate dimension, the following relation holds:*

$$MYN + N^T Y^T M^T \leq \lambda M M^T + \lambda^{-1} N^T N$$

where $Y^T Y \leq I$.

Lemma 42 *(adapted from [13] for the discrete time case) Consider a LTI discrete time system*

$$x(k+1) = Ax(k),$$

the constant scalar $\lambda > 0$ and a positive definite symmetric matrix P verifying the following matrix inequality

$$A^T PA - \lambda P < 0. \tag{1}$$

Then the norm of the state vector satisfies the relation

$$\|x(k)\|^2 < c \cdot \lambda^{k-k_0} \|x(k_0)\|^2, \ \forall k > k_0$$

with

$$c = \frac{eig_{max}(P)}{eig_{min}(P)}.$$

Proposition 43 *Consider a closed-loop switched system with norm bounded uncertainty*

$$x(k+1) = \left(A_{\alpha(k)} + \Delta A_{\alpha(k)}\right) x(k) + \left(B_{\alpha(k)} + \Delta B_{\alpha(k)}\right) K_{\beta(k)} x(k). \qquad (2)$$

that is switched according to two parameters $\alpha(k)$, $\beta(k)$ *that take values in the sets* $\mathcal{I}_\alpha$ *and* $\mathcal{I}_\beta$, *respectively. For this system* $\alpha(k) \in \mathcal{I}_\alpha$, *represents the switching functions for the state matrix and* $\beta(k) \in \mathcal{I}_\beta$ *indicates the active gain matrix; the uncertainties bounds are given as follows:* $\Delta A^T_{\alpha(k)} \Delta A_{\alpha(k)} < \epsilon_\alpha \boldsymbol{I}$, $\Delta B^T_{\beta(k)} \Delta B_{\beta(k)} < \epsilon_\beta \boldsymbol{I}$, $\forall \alpha(k) \in \mathcal{I}_\alpha$, $\beta(k) \in \mathcal{I}_\beta$.

If there exist symmetric positive definite matrices $S_{\alpha,\beta}$, *matrices* G_β, R_β *and scalars* $\lambda_\alpha > 0$, $\lambda_\beta > 0$, *with* $\alpha \in \mathcal{I}_\alpha$, $\beta \in \mathcal{I}_\beta$ *solutions of the linear matrix inequalities*

$$\begin{bmatrix} \lambda_\beta \boldsymbol{I} & \boldsymbol{0} & R_{\beta_1} & \boldsymbol{0} \\ * & \lambda_\alpha \boldsymbol{I} & G_{\beta_1} & \boldsymbol{0} \\ * & * & -S_{\alpha_1,\beta_1} + G_{\beta_1} + G^T_{\beta_1} & G^T_{\beta_1} A^T_{\alpha_1} + R^T_{\beta_1} B^T_{\alpha_1} \\ * & * & * & S_{\alpha_2,\beta_2} - \lambda_\beta \epsilon^2_\beta \boldsymbol{I} - \lambda_\alpha \epsilon^2_\alpha \boldsymbol{I} \end{bmatrix} > 0 \qquad (3)$$

$\forall \alpha_1, \alpha_2 \in \mathcal{I}_\alpha,\ \beta_1, \beta_2 \in \mathcal{I}_\beta$ *then the closed loop system (2) is asymptotically stable using the feedback gains*

$$K_\beta = R_\beta G_\beta^{-1},\ \forall\ \beta \in \mathcal{I}_\beta,$$

that is there exists a strictly decreasing switched Lyapunov function

$$V(x) = x^T(k) P_{\alpha,\beta} x(k)$$

with $P_{\alpha,\beta} = S^{-1}_{\alpha,\beta}$ *for the closed loop system.*

Proof. Using similar arguments to the ones in proof of Theorem 1 from ([21]) one can show that when the condition (3) is satisfied the following relation holds

$$\begin{bmatrix} \lambda_\beta \mathbf{I} & \mathbf{0} & K_{\beta_1} & \mathbf{0} \\ * & \lambda_\alpha \mathbf{I} & \mathbf{I} & \mathbf{0} \\ * & * & P_{\alpha_1,\beta_1} & \left(A_{\alpha_1} + B_{\alpha_1} K_{\beta_1}\right)^T P_{\alpha_2,\beta_2} \\ * & * & * & P_{\alpha_2,\beta_2} - P_{\alpha_2,\beta_2}\left(\lambda_\beta \epsilon^2_\beta \mathbf{I} - \lambda_\alpha \epsilon^2_\alpha \mathbf{I}\right) P_{\alpha_2,\beta_2} \end{bmatrix} > 0$$

where $P_{\alpha,\beta} = S^{-1}_{\alpha,\beta}, \forall \alpha \in \mathcal{I}_\alpha,\ \beta \in \mathcal{I}_\beta$. Using the Schur complement lemma the previous equation is the same as

$$\begin{bmatrix} \lambda_\alpha \mathbf{I} & \mathbf{I} & \mathbf{0} \\ * & P_{\alpha_1,\beta_1} & \left(A_{\alpha_1} + B_{\alpha_1} K_{\beta_1}\right)^T P_{\alpha_2,\beta_2} \\ * & * & P_{\alpha_2,\beta_2} - P_{\alpha_2,\beta_2}\left(\lambda_\alpha \epsilon^2_\alpha \mathbf{I}\right) P_{\alpha_2,\beta_2} \end{bmatrix} - \lambda_\beta^{-1} \begin{bmatrix} \mathbf{0} \\ -K^T_{\beta_1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -K_{\beta_1} & \mathbf{0} \end{bmatrix} -$$

$$-\lambda_\beta \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \epsilon_\beta P^T_{\alpha_2,\beta_2} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \epsilon_\beta P^T_{\alpha_2,\beta_2} \end{bmatrix} > 0$$

From Lemma 41 we obtain that

$$\begin{bmatrix} \lambda_\alpha \mathbf{I} & \mathbf{I} & \mathbf{0} \\ * & P_{\alpha_1,\beta_1} & \left(A_{\alpha_1} + B_{\alpha_1} K_{\beta_1}\right)^T P_{\alpha_2,\beta_2} \\ * & * & P_{\alpha_2,\beta_2} - P_{\alpha_2,\beta_2}\left(\lambda_\alpha \epsilon_\alpha^2 \mathbf{I}\right) P_{\alpha_2,\beta_2} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ -K_{\beta_1}^T \\ \mathbf{0} \end{bmatrix} \Delta B_{\alpha_1}^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & P_{\alpha_2,\beta_2}^T \end{bmatrix} -$$

$$- \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ P_{\alpha_2,\beta_2}^T \end{bmatrix} \Delta B_{\alpha_1} \begin{bmatrix} \mathbf{0} & -K_{\beta_1} & \mathbf{0} \end{bmatrix} > 0$$

which is the same as

$$\begin{bmatrix} \lambda_\alpha \mathbf{I} & \mathbf{I} & \mathbf{0} \\ * & P_{\alpha_1,\beta_1} & \left(A_{\alpha_1} + \left(B_{\alpha_1} + \Delta B_{\alpha_1}\right) K_{\beta_1}\right)^T P_{\alpha_2,\beta_2} \\ * & * & P_{\alpha_2,\beta_2} - P_{\alpha_2,\beta_2}\left(\lambda_\alpha \epsilon_\alpha^2 \mathbf{I}\right) P_{\alpha_2,\beta_2} \end{bmatrix} > 0.$$

By repeating the same procedure (Schur complement and Lemma 41 with ΔA_{α_1}) we obtain

$$\begin{bmatrix} P_{\alpha_1,\beta_1} & \left(A_{\alpha_1} + \Delta A_{\alpha_1} + \left(B_{\alpha_1} + \Delta B_{\alpha_1}\right) K_{\beta_1}\right)^T P_{\alpha_2,\beta_2} \\ * & P_{\alpha_2,\beta_2} \end{bmatrix} > 0$$

which implies that

$$V(x) = x^T(k) P_{\alpha,\beta} x(k)$$

is a switched Lyapunov function for the closed-loop system. □

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Résumé

Les travaux de cette thèse portent sur l'analyse de stabilité et la synthèse de commandes robustes pour les systèmes linéaires à commutation en temps discret avec des incertitudes polytopiques et des incertitudes sur la loi de commutation. On considère des lois de commutations arbitraires et on montre que l'utilisation des fonctions de Lyapunov commutées dépendant de paramètres permet de déterminer des critères de stabilité et de stabilisation robuste moins conservatifs. Ensuite, des conditions de stabilité robuste pour les systèmes en temps discret avec une loi de commutation incertaine sont présentées en termes de temps minimum de séjour. Les résultats obtenus s'avèrent utiles dans le contexte de la commande numérique des systèmes continus en présence d'imprécisions sur les instants d'échantillonnage et d'application des commandes. Nous montrons comment une modélisation à base d'évènements permet de ramener le problème original à un problème spécifique aux systèmes à commutation avec des incertitudes polytopiques. Les résultats sont étendus au cas des systèmes à commutation continus commandés par des correcteurs numériques.

Mots-clés: systèmes à commutation, robustesse, incertitudes paramétriques, retards variables inconnus, contrôle numérique.

Abstract

This PhD thesis is dedicated to the study of robust stability analysis and control synthesis for discrete time uncertain switching systems under arbitrary switching. Polytopic uncertainties are considered. We show that Lyapunov functions that depend on the uncertain parameter and that take into account the structure of the system may be used in order to reduce the conservatism related to uncertainty problems. Next, we consider the case of discrete time switched systems that are stabilized by a switched state feedback for which the switching signal may be temporary uncertain. Dwell time conditions for stability analysis of such systems are given. These results are usefull in the context of continuous time are stabilized via a computer when uncertainties occur on the sampling and actuation events. We present a new event based discrete-time model and we show that the stabilizability of this system can be achieved by finding a control for a switched polytopic system. The methodology is extended to the case of switched system.

Keywords: switched systems, robustness, parametric uncertainty, time varing delays, digital control.