Analyse locale dans les variétés presque complexes

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Contents

Introduction

Nous savons depuis les travaux de A.Newlander et L.Nirenberg [57] qu'il n'existe génériquement pas de coordonnées pseudoholomorphes sur une variété presque complexe, rendant problématique l'étude locale d'une telle variété (absence de noyau de Bergman, de théorie L^2, ...). Cependant toute structure presque complexe est localement une petite déformation de la structure standard ; c'est ce principe qui permit à A.Nijenhuis et W.Woolf [58] de montrer l'existence locale de disques pseudoholomorphes pour des structures à faible régularité, en considérant l'équation satisfaite par de telles applications comme une perturbation elliptique de l'équation standard de Cauchy-Riemann. L'importance des courbes pseudoholomorphes est connue aussi bien en analyse et géométrie complexes qu'en géométrie symplectique, grâce entre autres aux travaux de M.Gromov [40], d'H.Alexander [2], [1], ou de H.Hofer [43] (voir aussi [13], [18] ou [56] pour des références plus complètes sur le sujet). Dans le cadre des variétés presque complexes, les disques pseudoholomorphes permettent l'analyse géométrique locale des sous-variétés à courbure, dont nous abordons certains aspects dans cette thèse.

Il est naturel de transposer l'étude des domaines à courbure (feuilletage par des courbes pseudoholomorphes, prolongement au bord de difféomorphismes, ...) à l'étude de sous-fibrés totalement réels du fibré cotangent de la variété. Plusieurs structures presque complexes ont été construites sur le fibré cotangent, par I.Sato [63] ou K.Yano et S.Ishihara [44]. Nous unifions ces constructions et les caractérisons par le choix d'une connexion linéaire dans la première partie de cette thèse.

Les fonctions plurisousharmoniques jouent un rôle fondamental en géométrie presque complexe. Néanmoins, il n'existe que très peu d'exemples intéressants de telles fonctions. Nous devons à E.Chirka [19] l'existence de fonctions antipic plurisousharmoniques et plus généralement à J.-P.Rosay [61] la pluripolarité des disques pseudoholomorphes. Enfin K.Diederich et A.Sukhov [29] montrent que les domaines pseudoconvexes relativement compacts admettent une fonction bornée d'exhaustion strictement plurisousharmonique. Notons aussi les travaux de N.Pali [59] sur la caractérisation de la plurisousharmonicité en termes de courants. Dans la seconde partie de cette thèse, nous avons construit des fonctions locales pic plurisousharmoniques pour des domaines de type de D'Angelo fini dans une variété presque complexe de dimension réelle quatre, généralisant les travaux de J.E.Fornaess et N.Sibony [31].

Un aspect récurrent dans l'étude que nous menons est le comportement asymptotique de la pseudométrique de Kobayashi dans les domaines pseudoconvexes. Son comportement au voisinage du bord est relié à certaines questions fascinantes d'analyse locale dans les variétés comme les phénomènes de prolongement au bord des difféomorphismes ou encore la classification des domaines, et fournit des informations intéressantes sur les propriétés géométriques et dynamiques de la variété. Dans le but de montrer que tout point d'une variété presque complexe possède une base de voisinages hyperboliques complets au sens de Kobayashi (résultat dû à R.Debalme et S.Ivashkovich [28] dans le cas de la dimension réelle quatre), H.Gaussier et A.Sukhov [35], et indépendamment S.Ivashkovich et J.-P.Rosay [45], ont donné des estimées locales de la pseudométrique de Kobayashi dans les domaines strictement pseudoconvexes. Dans cette thèse nous nous intéressons à cette question pour des domaines pseudoconvexes de type de D'Angelo fini. Par ailleurs, en raffinant les estimées de [35] nous nous sommes intéressés, dans la troisième partie de cette thèse, à la notion d'hyperbolicité au sens de Gromov. Introduite dans les années 1980 (voir [41], [16] et [38]), l'hyperbolicité au sens de Gromov a largement contribué au développement de la théorie géométrique des groupes. Il est naturel de s'intéresser au lien entre l'hyperbolicité au sens de Gromov et au sens de Kobayashi. Cette question a été initiée par Z.M.Balogh et M.Bonk [3] qui montrent que tout domaine relativement compact strictement pseudoconvexe de l'espace Euclidien complexe est hyperbolique au sens de Gromov. Cette notion d'hyperbolicité étant purement métrique, sa définition ne nécessite aucun argument d'analyse ou de géométrie complexe. Aussi, nous avons généralisé les résultats de Z.M.Balogh et M.Bonk au cadre non intégrable en éliminant les arguments holomorphes utilisés dans [3]. Nous montrons, par exemple, que tout point d'une variété presque complexe de dimension réelle quatre admet une base de voisinages Gromov hyperboliques.

Nous allons maintenant présenter les chapitres 2, 3 et 4 de cette thèse, le premier chapitre rassemblant quelques rappels de géométrique presque complexe.

Structures presque complexes sur le fibré cotangent

Dans le second chapitre de cette thèse, nous étudions les structures presque complexes sur le fibré cotangent. Il existe un lien étroit entre l'analyse locale sur les variétés complexes (et presque complexes) et les fibrés canoniques. Par exemple, le fibré cotangent est profondément relié à l'extension au bord des biholomorphismes (voir [23]) et à l'étude des disques stationnaires introduits par L.Lempert [51] (voir aussi [68] et [67]). Le but de ce chapitre est d'introduire un relevé de structure presque complexe au fibré cotangent, appelé le relevé général horizontal, qui permet d'unifier et de caractériser les relevés complets définis par I.Sato [63] et horizontaux construits par S.Ishihara-K.Yano [44].

Soit M une variété réelle lisse de dimension paire n munie d'une structure presque complexe J, ie d'un champ de tenseurs de type $(1,1)$ qui vérifie $J^2 = -Id$. Considérons des systèmes de coordonnées locales $(x_1, \cdots, x_n)$ sur M et $(x_1, \cdots, x_n, p_1, \cdots, p_n)$ sur le fibré cotangent T^*M. Nous notons $\Gamma(TM)$ (resp. $\Gamma(T^*M)$) les sections du fibré tangent (resp. cotangent), autrement dit les champs de vecteurs (resp. formes).

Définissons dans un premier temps le relevé complet formel J^c introduit par I.Sato

[63]. Soit θ la forme de Liouville définie sur le fibré cotangent T^*M et d'expression locale $\theta = p_i dx^i$. La différentiation de θ munit T^*M d'une forme symplectique canonique $\omega_{st} := d\theta$. Nous introduisons une 1-forme $\theta(J)$ sur le fibré cotangent T^*M qui contracte la forme de Liouville θ et la structure presque complexe $J = J_l^k dx^l \otimes \partial x_k$ de la manière suivante $\theta(J) = p_k J_l^k dx^l$. Puisque la forme symplectique canonique ω_{st} du fibré cotangent induit un isomorphisme entre les 2-formes et les tenseurs de type $(1,1)$, le *relevé complet formel* J^c est défini par $d(\theta(J)) = \omega_{st}(J^c.,.)$. Néanmoins le tenseur de type $(1,1)$ J^c n'est génériquement pas une structure presque complexe sur le fibré cotangent T^*M. Plus précisément, S.Ishihara et K.Yano [44] montrent que J^c est une structure presque complexe si et seulement si J est une structure intégrable sur M, c'est-à-dire si et seulement si M est une variété complexe. En introduisant un terme correctif induit par la non intégrabilité de J mesurée par le tenseur de Nijenhuis $N_J(X,Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$, (pour $X, Y \in \Gamma(TM)$), I.Sato a obtenu une structure presque complexe $\widetilde{J}$, appelée le *relevé complet* et définie par

$$\widetilde{J} := J^c - \frac{1}{2}\gamma(JN_J),$$

où γ contracte les tenseurs de type $(2,1)$ en des tenseurs de type $(1,1)$ de la manière suivante : pour un tenseur R de type $(2,1)$ de coordonnées $R_{i,j}^k$, nous définissons le tenseur $\gamma(R)$ de type $(1,1)$ dont la représentation matricielle est :

$$\gamma(R) = \begin{pmatrix} 0 & 0 \\ p_k R_{j,i}^k & 0 \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}).$$

Nous rappelons à présent la définition du relevé horizontal d'une structure presque complexe construit par S.Ishihara et K.Yano dans [44]. Nous munissons M d'une connexion (linéaire) $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ sur M, ie d'une loi de dérivation sur les champs de vecteurs. Notons T sa torsion définie par $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$, pour tout champ $X, Y \in \Gamma(TM)$ et introduisons la connexion symétrique (ie de torsion nulle) suivante $\widetilde{\nabla} := \nabla - \frac{1}{2}T$. Le *relevé horizontal* de J est défini par

$$J^{H,\nabla} := J^c + \gamma([\widetilde{\nabla} J]),$$

où le tenseur de type $(2,1)$ $[\widetilde{\nabla} J]$ est défini par

$$[\widetilde{\nabla} J](X,Y) := -\widetilde{\nabla}_X JY + J\widetilde{\nabla}_X Y + \widetilde{\nabla}_Y JX - J\widetilde{\nabla}_Y X,$$

pour tout $X, Y \in \Gamma(TM)$. Nous savons depuis S.Ishihara et K.Yano que le relevé $J^{H,\nabla}$ est une structure presque complexe sur le fibré cotangent T^*M. Cependant, contrairement au relevé complet, construire le relevé horizontal nécessite la donnée d'une connexion ∇ sur M, et que par suite sa définition et ses propriétés en sont fortement dépendantes. Par ailleurs, il est fondamental de symétriser la connexion ∇ pour assurer que le relevé horizontal d'une structure presque complexe reste une structure presque complexe. Par la suite nous souhaitons nous affranchir de cette étape.

L'une de nos principales motivations de ce chapitre est d'appréhender les relevés précédents d'un point de vue plus canonique et plus géométrique. Nous introduisons dans ce but le relevé général horizontal d'une structure presque complexe au fibré cotangent. Notre approche est basée sur la remarque suivante inspirée par la construction d'une structure presque complexe sur l'espace des jets d'application pseudoholomorphes de P.Gauduchon [33]. Soit $x \in M$ et soit $\xi \in T^*_xM$. Considérons une distribution H sur le fibré cotangent satisfaisant la décomposition locale $T_\xi T^*M = H_\xi \oplus T^*_xM$. Il est naturel de définir un relevé de structure presque complexe en respectant la décomposition de H, ie $J \oplus {}^tJ$ sur $H_\xi \oplus T^*_xM$. Plus précisément, soit ∇ une connexion sur M et considérons la distribution horizontale H^∇ définie par $H^\nabla_\xi := \{d_xs(X),\ X \in T_xM, s \in \Gamma(T^*M), s(x) = \xi, \nabla_X s = 0\} \subseteq T_\xi T^*M$. Nous avons une décomposition en somme directe $T_\xi T^*M = H^\nabla_\xi \oplus T^*_xM$, en fibres horizontale et verticale-cotangente. L'isomorphisme induit par la projection $\pi : T^*M \to M$ entre une fibre horizontale H^∇_ξ et l'espace tangent T_xM permet de définir, pour un vecteur $Y = (X, v^\nabla(Y)) \in T_\xi T^*M = H^\nabla_\xi \oplus T^*_xM$, le *relevé général horizontal* associé à la connexion ∇ par :

$$J^{G,\nabla}(Y) := (JX, {}^tJ(v^\nabla(Y))),$$

où $v^\nabla : T_\xi T^*M \longrightarrow T^*_xM$ est la projection verticale sur T^*_xM parallèlement à H^∇_ξ.

Les descriptions locale (matricielle) et tensorielle du relevé général horizontal $J^{G,\nabla}$ permettent de considérer ce dernier comme une correction du relevé complet formel. Ainsi nous montrons que

$$J^{G,\nabla} = J^c + \gamma(S),$$

où $S(X,Y) := -(\nabla J)(X,Y) + (\nabla J)(Y,X) + T(JX,Y) - JT(X,Y)$. Nous remarquons que les trois relevés de structures presque complexes introduits précédemment apparaissent comme des corrections du relevé complet formel J^c, montrant alors que pour un certain choix de connexion, le relevé général horizontal $J^{G,\nabla}$ coïncide avec les relevés complets $\widetilde{J}$ et horizontaux $J^{H,\nabla}$. Ce résultat constitue le théorème suivant et met en lumière la nature géométrique de $\widetilde{J}$.

Théorème 1.

1. *Le relevé général horizontal $J^{G,\nabla}$ coïncide avec le relevé complet $\widetilde{J}$ si et seulement si $S = -\frac{1}{2}JN_J$.*

2. *$J^{G,\nabla}$ coïncide avec le relevé horizontal $J^{H,\nabla}$ si et seulement si $T(J.,.) = T(.,J.)$.*

3. *Pour toute connexion presque complexe (ie $\nabla_X(JY) = J\nabla_XY$ pour tout $X, Y \in \Gamma(TM)$) et minimale (ie $T = \frac{1}{4}N_J$), les trois structures relevées coïncident.*

Le troisième point de ce théorème s'appuie sur l'existence de connexions presque complexes et minimales établie par A.Lichnerowicz [53] et montre finalement que cette famille de connexions est la plus canonique possible sur une variété presque complexe.

Afin de caractériser le relevé complet de I.Sato, nous étudions certaines propriétés géométriques des relevés de structures. Naturellement, du fait de la construction des différents relevés au fibré cotangent, la projection sur la base $\pi : T^*M \to (M, J)$ et la section nulle $s : (M, J) \to T^*M$ sont pseudoholomorphes. Considérons un difféomorphisme $f : (M_1, J_1) \to (M_2, J_2,)$, (J_1, J_2)-holomorphe entre deux variétés presque complexes et soit $\widetilde{f} := (f, {}^t(df)^{-1}) : T^*M_1 \to T^*M_2$ son relevé au fibré cotangent. Une question naturelle est de savoir sous quelles conditions $\widetilde{f}$ est pseudoholomorphe pour les relevés de structures. Notons que cette question a été étudiée afin d'obtenir la généralisation au cadre presque complexe du théorème d'extension de Fefferman (voir [23]). Nous obtenons alors le résultat suivant, établi dans un premier temps pour le relevé général horizontal et étendu ensuite aux relevés complets et horizontaux par l'intermédiaire du Théorème 1 :

Proposition 2.

1. *Le relevé d'un difféomorphisme $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ au fibré cotangent est $(J_1^{G,\nabla_1}, J_2^{G,\nabla_2})$-holomorphe si et seulement si f est une application (J_1, J_2)-holomorphe satisfaisant $f_*S_1 = S_2$.*

2. *Le relevé d'un difféomorphisme $f : (M_1, J_1) \longrightarrow (M_2, J_2)$ au fibré cotangent est $(\widetilde{J}_1, \widetilde{J}_2)$-holomorphe si et seulement si f est (J_1, J_2)-holomorphe.*

3. *Le relevé d'un difféomorphisme $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ au fibré cotangent est $(J_1^{H,\nabla_1}, J_2^{H,\nabla_2})$-holomorphe si et seulement si f est (J_1, J_2)-holomorphe et vérifié $f_*[\widetilde{\nabla_1} J_1] = [\widetilde{\nabla_2} J_2]$.*

Nous nous intéressons par ailleurs aux conditions géométriques sous lesquelles la multiplication fibre à fibre $Z : T^*M \longrightarrow T^*M$ par un nombre $a + ib \in \mathbb{C}$ (avec $b \neq 0$), définie localement par $Z(x, p) := (x, (a + b^tJ(x))p)$, est pseudoholomorphe. Cette question a été étudiée dans un premier temps pour les relevés de structure presque complexe au fibré tangent TM (voir [49], [52]). Cette propriété est motivée par le souhait de munir le fibré canonique considéré d'une structure de fibré vectoriels presque holomorphe, où tous les objets associés sont pseudoholomorphes. Néanmoins dans le cadre du fibré tangent la multiplication fibre à fibre n'est génériquement pas pseudoholomorphe (voir [49], [52]). Plus précisément la pseudoholomorphie est établie uniquement lorsque la structure sur la variété base est intégrable. Dans le cas du fibré cotangent T^*M, et contrairement au fibré tangent, nous obtenons des conditions ne faisant pas intervenir l'intégrabilité de la structure J sur M mais seulement reliées au choix d'une connexion. Nous montrons ainsi que la multiplication fibre à fibre est pseudoholomorphe pour le relevé complet de I.Sato.

Proposition 3.

1. *La multiplication sur une fibre $Z : T^*M \longrightarrow T^*M$ d'un nombre $a + ib \in \mathbb{C}$, est $(J^{G,\nabla}, J^{G,\nabla})$-holomorphe si et seulement si $(\nabla J)(J., .) = (\nabla J)(., J.)$.*

2. *La multiplication Z est $(\widetilde{J}, \widetilde{J})$-holomorphe et,*

3. *Z est $(J^{H,\nabla}, J^{H,\nabla})$-holomorphe si et seulement si $(\widetilde{\nabla} J)(J., .) = (\widetilde{\nabla} J)(., J.)$.*

La forme de Liouville θ induit sur le fibré cotangent T^*M une structure canonique de variété symplectique (T^*M, ω_{st}). Par ailleurs, l'isomorphisme entre les 2-formes et les tenseurs de type $(1,1)$ induit par la forme ω_{st} est à la base de la construction du relevé complet formel J^c. Aussi nous semble-t-il naturel d'étudier la compatibilité des relevés de structures avec la forme symplectique canonique ω_{st}. Nous montrons que le couple formé par le relevé complet et par ω_{st} sur le fibré cotangent est dans un certain sens déterminé par les propriétés des fibrés conormaux des hypersurfaces strictement pseudoconvexes. Pour un point $x \in M$, la fibre conormale d'une hypersurface $\Gamma \subset M$ est définie par :

$$N_x^*(\Gamma) := \{p_x \in T_x^*M, (p_x)_{|T_x\Gamma} = 0\},$$

et le fibré conormal de $\Gamma \subset M$ comme l'union disjointe

$$N^*(\Gamma) := \bigcup_{x \in \Gamma} N_x^*(\Gamma).$$

Nous savons depuis les travaux de S.Webster [69] (voir aussi [35], [66]) que le fibré conormal d'une hypersurface strictement pseudoconvexe Γ dans une variété (presque) complexe (M, J) est une sous-variété totalement réelle maximale (ie $TN^*(\Gamma) \cap \widetilde{J}(TN^*(\Gamma)) = \{0\}$) du fibré cotangent T^*M muni du relevé complet $\widetilde{J} = J^c$. Néanmoins les preuves de ce résultat sont purement complexes bien que la définition du relevé complet $\widetilde{J}$ fasse intervenir la structure symplectique canonique ω_{st}. La proposition suivante explique la raison pour laquelle cette approche a été privilégiée au détriment d'une preuve symplectique.

Proposition 4. *Soit (M, J, ∇) une variété presque complexe munie d'une connexion ∇. Soit ω une forme symplectique sur T^*M compatible avec le relevé généralisé $J^{G,\nabla}$, (resp. le relevé complet $\widetilde{J}$ ou le relevé horrizontal $J^{H,\nabla}$). Alors, il n'existe pas d'hypersurface strictement J-pseudoconvexe dans M telle que le fibré conormal soit Lagrangien pour ω.*

Régions pseudoconvexes de type fini au sens de D'Angelo dans les variétés presque complexes de dimension quatre

Dans le troisième chapitre de cette thèse, nous menons une étude locale des régions pseudoconvexes de type de D'Angelo fini dans les variétés presque complexes de dimension quatre. Plus précisément nous nous intéressons au comportement asymptotique de la pseudométrique de Kobayashi.

Le type apparaît naturellement dans les variétés complexes et est relié au comportement au voisinage du bord du $\overline{\partial}$, au noyau de Bergman, ou encore aux métriques invariantes (voir [25],[24],[47],[15]). La motivation sous-jacente au type est de mesurer les singularités de la forme de Levi aux points où elle dégénère. Aussi, plusieurs notions du type ont été définies et coïncident dans les variétés complexes de dimension deux. En outre nous savons depuis les travaux de J.D'Angelo [25], [24] que la condition géométrique pour obtenir de la régularité pour l'unique solution du problème du $\overline{\partial}$-Neumann en dimension quelconque s'exprime en terme de type de D'Angelo.

Définissons le *type de D'Angelo* d'un point p contenu dans le bord d'un domaine D d'une variété presque complexe (M, J) :

$$\Delta^1(\partial D, p) := \sup\left\{\frac{\delta_p(\partial D, u)}{\delta(u)},\ u : \Delta \to \left(\mathbb{R}^4, J\right)\ J\text{-holomorphe non constant},\ u(0) = p\right\},$$

où $\delta_p(\partial D, u)$ est l'ordre de contact de u avec ∂D en p (ie, le degré du premier terme non nul dans le développement de Taylor de $\rho \circ u$) et où $\delta(u)$ est la multiplicité de u en $0 \in \mathbb{C}$. Ainsi défini, le type de D'Angelo mesure l'obstruction à l'existence d'un germe en p d'une courbe J-holomorphe non constante dans l'hypersurface réelle ∂D. Similairement au cas des variétés complexes de dimension deux, nous montrons dans un premier temps que le type de D'Angelo coïncide avec le type régulier, permettant alors de ne considérer que des disques pseudoholomorphes réguliers. Ainsi $\Delta^1(\partial D, p) = \sup\{\delta_p(\partial D, u),\ u : \Delta \to (\mathbb{R}^4, J)\ J\text{-holomorphe}, u(0) = p, d_0 u \neq 0\}$.

Rappelons qu'une *région J-pseudoconvexe* dans une variété presque complexe (M, J) est un domaine $D = \{\rho < 0\}$ où ρ est une fonction définissante pour D, de classe $\mathcal{C}^2$ et J-plurisousharmonique sur un voisinage du bord $\overline{D}$. La description locale des régions J-pseudoconvexes de type de D'Angelo fini permet d'établir un système de coordonnées normales (x_1, y_1, x_2, y_2) dans lequel la structure presque complexe est diagonale et coïncide le long d'un disque J-holomorphe plat d'ordre de contact maximal avec la structure standard J_{st} et tel que la fonction définissante ρ s'écrive :

$$\rho = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + \widetilde{H}(z_1, z_2) + O\left(|z_1|^{2m+1} + |z_2||z_1|^m + |z_2|^2\right),$$

où H_{2m} est un polynôme homogène de degré $2m$, sousharmonique admettant une partie non harmonique, notée H_{2m}^* et où $\widetilde{H}(z_1, z_2) = \Re e \sum_{k=1}^{m-1} \alpha_k z_1^k z_2$. Dans les écritures précédentes, nous notons $z_k = x_k + i y_k$, pour $k = 1, 2$. Afin d'obtenir de telles coordonnées, nous considérons un disque $u : \Delta \to \mathbb{R}^4$ J-holomorphe régulier d'ordre de contact maximal $2m$. Nous choisissons des coordonnées telles que u est donné par $u(\zeta) = (\zeta, 0)$, et telles que $J(z_1, 0) = J_{st}$. Par ailleurs nous pouvons supposer que l'espace tangent complexe $T_0\partial D \cap J(0) T_0 \partial D$ est égal à $\{z_2 = 0\}$. Nous considérons ensuite deux feuilletages transversaux par des disques J-holomorphes que l'on redresse en droites par un difféomorphisme local (voir Figure 1).

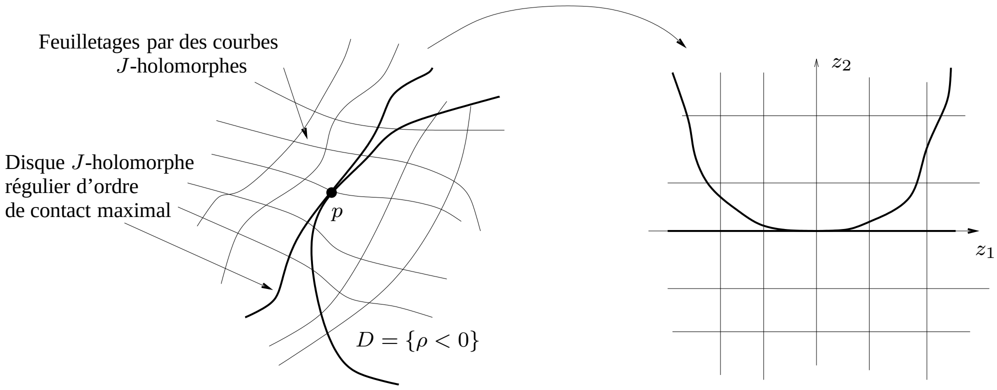

Figure 1. Coordonnées normales pour une région J-pseudoconvexe en $dim_{\mathbb{R}} = 4$.

L'analyse locale des domaines pseudoconvexes de type de D'Angelo fini se base de manière essentielle sur l'existence de fonctions locales pic J-plurisousharmoniques en un point donné du bord. Rappelons que pour un point $p \in \partial D$, une telle fonction φ doit notamment vérifier $\varphi(p) = 0$ et $\varphi < 0$ sur $\overline{D} \cap U \backslash \{p\}$, où U est un voisinage de p. Nous devons à J.E.Fornaess et N.Sibony [31] la construction d'une fonction locale pic plurisousharmonique pour des domaines pseudoconvexes de type de D'Angelo fini dans des variétés complexes de dimension deux. Aussi, la généralisation de cette construction au cadre non intégrable est une question fondamentale. Nous montrons alors :

Théorème 5. *Soit $D = \{\rho < 0\}$ une région J-pseudoconvexe de type de D'Angelo fini dans une variété presque complexe (M, J) de dimension quatre. Il existe une fonction φ locale pic J-plurisousharmonique en tout point du bord.*

La difficulté principale de la démonstration réside dans le fait que la J-plurisousharmonicité réagit très mal aux perturbations, aussi petites soient-elles. Notre preuve s'articule de la manière suivante. Plaçons-nous dans un système de coordonnées normales (x_1, y_1, x_2, y_2). Dans un premier temps, nous souhaitons contrôler les directions d'annulation de la forme de Levi du polynôme $H_{2m}(z_1, \overline{z_1})$ en un vecteur donné $(v_1, 0)$. Nous savons depuis J.E.Fornaess et N.Sibony (Lemme 2.4 dans [31]), qu'il existe une fonction $g \ : \mathbb{R} \to \mathbb{R}$ 2π-périodique négative bornée et qui vérifie :

$$\Delta \left(H_{2m} + \delta \|H_{2m}^*\| g\left(\theta\right) |z_1|^{2m} \right) > \delta^2 \|H_{2m}^*\| |z_1|^{2(m-1)},$$

pour une certaine constante $\delta > 0$. Plus précisément g redresse les directions d'annulations du Laplacien de H_{2m} en la direction normale $\{z_1 = 0\}$ et diminue de manière contrôlée le Laplacien de H_{2m} dans les directions strictement sousharmoniques. Nous montrons alors qu'il existe deux constantes positives L et C telles que la fonction

$$\begin{aligned}\varphi \ := \ & \Re e z_2 + 2L \left(\Re e z_2\right)^2 - L \left(\Im m z_2\right)^2 + H_{2m}(z_1, \overline{z_1}) + \delta \|H_{2m}^*\| g\left(\theta\right) |z_1|^{2m} + \\ & \widetilde{H}(z_1, z_2) + C |z_1|^2 |z_2|^2\end{aligned}$$

est locale pic J-plurisousharmonique en l'origine. Expliquons les grandes lignes de ce résultat. En rajoutant le terme $2L \left(\Re e z_2\right)^2 - L \left(\Im m z_2\right)^2$, nous assurons la stricte positivité de la forme de Levi de φ dans les directions tangentes ; par ailleurs la description locale du domaine D assure que $2L\,(\Re e z_2)^2$ est contrôlé par un $O((|z_1|^{2m} + |\Im m z_2|^2)\|z\|)$. Parallèlement, de par sa construction, $\delta\|H^*_{2m}\|g\,(\theta)\,|z_1|^{2m}$ contrôle les directions d'annulations de la forme de Levi de H_{2m} et, g étant négative, joue un rôle crucial dans le caractère pic de φ. En rajoutant le terme $C|z_1|^2|z_2|^2$, Nous garantissons finalement la stricte positivité de la forme de Levi de φ dans les directions normales, tout en s'assurant qu'il ne perturbe pas le fait que φ soit pic.

La construction d'une telle famille de fonctions permet d'établir des propriétés d'attraction des disques pseudolomorphes. Plus précisément, nous montrons pour une région $D = \{\rho < 0\}$ J-pseudoconvexe de type de D'Angelo fini dans une variété presque complexe (M, J) la propriété d'attraction suivante. Soit $p \in \bar{D}$ et soit U un voisinage de p dans M. Il existe une constante $s > 0$, et un voisinage $V \subset U$ de p dans M, tels que pour tout disque J-holomorphe $u\ : \Delta \to D \cap U$ dont le centre $u(0) \in D \cap V$ on ait :

$$u(\Delta_s) \subset D \cap U,$$

ou de manière équivalente pour tout $q \in D \cap V$ et tout $v \in T_qM$ on ait :

$$K_{(D,J)}\,(q, v) \geq sK_{(D\cap U,J)}\,(q, v)\,.$$

La démonstration de ce principe de localisation est une légère modification de la preuve du Théorème 3 de N.Sibony [64] (voir aussi [7] et [35]). Elle repose essentiellement sur l'existence de fonctions pic J-plurisousharmoniques que nous avons établies et sur la construction de fonctions antipic J-plurisousharmoniques établie par E.Chirka [19] (voir par exemple [45] ou [35] pour une preuve).

En outre nous obtenons des estimées locales de la pseudométrique de Kobayashi impliquant notamment l'hyperbolicité locale au sens de Kobayashi et plus généralement :

Proposition 6. *Soit $D = \{\rho < 0\}$ une région J-pseudoconvexe (de classe $\mathcal{C}^2$) relativement compacte de type de d'Angelo fini dans une variété presque complexe (M, J) de dimension quatre. Supposons en outre qu'il existe une fonction globalement strictement J-plurisousharmonique sur (M, J). Alors D est un domaine taut.*

Notons que K.Diederich et A.Sukhov [29] ont obtenu ce résultat pour des domaines J-pseudoconvexes à bord de classe $\mathcal{C}^3$ dans des variétés presque complexes de dimensions $2n$, en construisant une fonction bornée d'exhaustion J-plurisousharmonique.

L'existence d'un prolongement Hölder continu au bord est une question centrale pour l'étude des difféomorphismes pseudoholomorphes entre domaines contenus dans des variétés presque complexes, mettant en jeu les propriétés géométriques du bord. Plus précisément, ce phénomène est entièrement gouverné par les propriétés au voisinage du bord de la pseudométrique de Kobayashi. Nous raffinons les estimées obtenues en nous apuyant fortement sur le comportement au bord des fonctions pic J-plurisousharmoniques construites, pour établir que la pseudométrique de Kobayashi en un point $p \in D$ est de l'ordre de $1/\mathrm{dist}(p, \partial D)^{2m}$ au voisinage d'un point du bord de type de D'Angelo $2m$, entraînant finalement :

Proposition 7. *Soit $D = \{\rho < 0\}$ et $D' = \{\rho' < 0\}$ deux régions pseudoconvexes relativement compactes de type de D'Angelo $2m$ dans deux variétés presque complexes (M, J) et (M', J') de dimension quatre. Soit $f : D \to D'$ un difféomorphisme (J, J')-holomorphe. Alors f se prolonge en un homéomorphisme Hölder d'exposant $1/2m$ entre $\overline{D}$ et $\overline{D'}$.*

La méthode de démonstration est classique et repose sur les estimées raffinées de la pseudométrique de Kobayashi que nous avons obtenues et sur la version presque complexe du lemme de Hopf établi par B.Coupet, H.Gaussier et A.Sukhov [23] impliquant une propriété de conservation des distances par les difféomorphismes pseudoholomorphes.

Nous désirons à présent fournir des estimées précises et optimales de la pseudométrique de Kobayashi. En privilégiant une approche basée sur des minorations de la métrique de Carathéodory et les estimées L^2 de Hörmander, D.Catlin [17] fut le premier à obtenir de telles estimées dans les domaines de type de D'Angelo fini dans $(\mathbb{C}^2, J_{st})$. Néanmoins sa preuve ne peut se transposer au cadre presque complexe. Nous suivons alors une preuve donnée par de F.Berteloot [8] fondée sur un principe de Bloch. Encore une fois, l'existence de fonctions locales pic J-plurisousharmoniques est primordiale puisqu'elle réduit l'obtention d'estimées optimales à un problème purement local. L'aspect technique de notre preuve réside dans l'élaboration d'une méthode de dilatations adaptée au cadre presque complexe. En fait, la difficulté est d'obtenir un domaine limite et une structure limite $(\tilde{D}, \tilde{J})$ tels que $\tilde{D}$ ne contienne pas de courbes $\tilde{J}$-complexes entières. Comme nous le montrons, cela est possible en supposant que le domaine considéré est de type de D'Angelo (au plus) quatre. Pour les domaines de type de D'Angelo strictement plus grand que quatre, construire une méthode de changement d'échelle adaptée est une question ouverte et constitue une perspective intéressante. Comme nous le montrons dans le premier appendice de ce chapitre, génériquement, la suite de structures presque complexes induite par une méthode de dilatations polynomiale ne converge pas (génériquement) vers la structure standard. Notons que ce n'est pas la première fois qu'apparaît une différence entre les types inférieurs ou égal à quatre et strictement plus grands que quatre. En effet selon J.D'Angelo [26], pour une hypersurface H réelle dans $\mathbb{C}^n$, si le type régulier de $p \in H$ est plus petit que quatre alors les types régulier et de D'Angelo coïncident.

Evoquons à présent les points clés de notre méthode de dilatations. Plaçons-nous dans un système de coordonnées normales (x_1, y_1, x_2, y_2) dans lequel la structure J est diagonale et satisfait $J = J_{st} + O(|z_2|)$ et tel que la fonction définissante de D s'écrive :

$$\rho = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

où H_{2m} est un polynôme homogène de degré $2m$ sousharmonique et admettant une partie non harmonique. Considérons une suite de points p_ν de $D \cap U$ convergeant vers l'origine. Pour ν suffisament grand, nous notons $p_\nu^* \in \partial D \cap U$ la projection de p_ν sur le bord $p_\nu^* = p_\nu + (0, \delta_\nu)$, où $\delta_\nu > 0$. Remarquons que d'après J.-F.Barraud et E.Mazzilli [4] le type régulier est semi-continu supérieurement dans les variétés presque complexes de dimension quatre. Ainsi le type de D'Angelo de p_ν^* est nécessairement plus petit que $2m$. En considérant alors un système de coordonnées normales centré en p_ν^*, nous trouvons un difféomorphisme local Φ^ν satisfaisant les propriétés suivantes :

1. $\Phi^\nu\left(p_\nu^*\right)=0$ et $\Phi^\nu\left(p_\nu\right)=(0,-\delta_\nu)$,

2. Φ^ν converge vers $Id\ :\mathbb{R}^4\to\mathbb{R}^4$ sur les sous ensembles compacts de $\mathbb{R}^4$ pour la topologie $\mathcal{C}^2$,

3. la fonction définissante $\rho^\nu := \rho\circ\left(\Phi^\nu\right)^{-1}$ du domaine image $D^\nu := \Phi^\nu\left(D\cap U\right)$ s'écrit :

$$\rho^\nu\left(z_1,z_2\right)=\Re e z_2+\sum_{k=2l_\nu}^{2m} P_k\left(z_1,\overline{z_1},p_\nu^*\right)+O\left(|z_1|^{2m+1}+|z_2|\|z\|\right),$$

où le polynôme homogène $P_{2l_\nu}\neq 0$ de degré $2l_\nu$ admet une partie non harmonique. Par ailleurs $J^\nu := \left(\Phi^\nu\right)_* J$ est diagonale et vérifie $J=J_{st}+O(|z_2|)$.

A l'aide d'un biholomorphisme (pour la structure standard) polynomial de $\mathbb{C}^2$, nous enlevons ensuite les termes harmoniques du polynôme $\sum_{k=2l_\nu}^{2m-1} P_k\left(z_1,\overline{z_1},p_\nu^*\right)$. Cela permet de construire alors un difféomorphisme local Φ_ν satisfaisant les points 1 et 2 précédents et vérifiant maintenant:

3'. la fonction définissante $\rho_\nu := \rho\circ\left(\Phi_\nu\right)^{-1}$ du domaine $D_\nu := \Phi_\nu\left(D\cap U\right)$ s'écrit localement :

$$\rho_\nu\left(z_1,z_2\right)=\Re e z_2+\sum_{k=2l_\nu}^{2m-1} P_k^*\left(z_1,\overline{z_1},p_\nu^*\right)+P_{2m}\left(z_1,\overline{z_1},p_\nu^*\right)+O\left(|z_1|^{2m+1}+|z_2|\|z\|\right),$$

où le polynôme $\sum_{k=2l_\nu}^{2m-1} P_k^*\left(z_1,\overline{z_1},p_\nu^*\right)$ ne contient pas de termes harmoniques et où $P_{2l_\nu}^*\neq 0$. Enfin la structure image $J_\nu := \left(\Phi_\nu\right)_* J$ n'est génériquement plus diagonale.

Fixons une norme $\|.\|$ sur l'espace vectoriel des polynômes de degré au plus $2m$ en, $z_1,\overline{z_1}$ et introduisons, pour ν suffisament grand, le réel

$$\tau\left(p_\nu^*,\delta_\nu\right) := \min_{k=2l_\nu,\cdots,2m}\left(\frac{\delta_\nu}{\|P_k^*\left(.,p_\nu^*\right)\|}\right)^{\frac{1}{k}}.$$

Nous définissons ainsi une dilatation anisotrope Λ_ν de $\mathbb{C}^2$ par :

$$\Lambda_\nu(z_1,z_2) := \left(\tau\left(p_\nu^*,\delta_\nu\right)\right)^{-1}z_1,\delta_\nu^{-1}z_2).$$

Nous montrons que le domaine $\tilde{D_\nu} := \Lambda_\nu\left(D_\nu\right)$ converge (au sens de la convergence de Hausdorff locale pour les ensembles) vers un domaine pseudoconvexe $\tilde{D}=\{\Re e z_2+P\left(z_1,\overline{z_1}\right)<0\}$, où P est polynôme non nul sousharmonique de degré plus petit que $2m$ admettant une partie non harmonique. De plus, **lorsque l'origine est de type de D'Angelo inférieur ou égal à quatre pour** D, la suite de structures presque complexes

$\tilde{J}_\nu := (\Lambda_\nu)_* (J_\nu)$ converge vers J_{st} pour la topologie $\mathcal{C}^2$ sur les compacts de $\mathbb{R}^4$. Cependant lorsque le type de D'Angelo de l'origine pour D est plus strictement plus grand que quatre, le fait de ne pas contrôler les termes harmoniques contenus dans $\displaystyle\sum_{k=2l_\nu}^{2m-1} P_k\left(z_1, \overline{z_1}, p_\nu^*\right)$, implique que $\tilde{J}_\nu$ diverge génériquement.

Les estimées de Catlin restent valides dans le cas d'une région J-pseudoconvexe de type de D'Angelo inférieur ou égal à quatre :

Théorème 8. *Soit $D = \{\rho < 0\}$ une région J-pseudoconvexe relativement compacte de type de D'Angelo inférieur ou égal à quatre dans une variété presque complexe (M, J) de dimension quatre. Alors il existe une constante $C > 0$ satisfaisant la propriété suivante : pour tout $p \in D$ et $v \in T_pM$ il existe un difféomorphisme Φ_{p^*} dans un voisinage U de p tel que :*

$$K_{(D,J)}(p, v) \geq C \left(\frac{|\left(d_p\Phi_{p^*}v\right)_1|}{|\rho(p)|^{\frac{1}{4}}} + \frac{|\left(d_p\Phi_{p^*}v\right)_2|}{|\rho(p)|} \right).$$

Notre preuve se décompose ainsi. Pour $p \in D \cap U$ suffisament proche du bord ∂D, nous notons $p^* \in \partial D \cap U$ l'unique point tel que $p^* = p + (0, \delta)$, avec $\delta > 0$. Nous remarquons que δ est équivalent à $\mathrm{dist}(p, \partial D)$. Définissons une pseudométrique infinitésimale N sur $D \cap U \subseteq \mathbb{R}^4$ pour $p \in D \cap U$ et $v \in T_p\mathbb{R}^4$:

$$N(p, v) := \frac{|\left(d_p\Phi_{p^*}v\right)_1|}{\tau\left(p^*, |\rho(p)|\right)} + \frac{|\left(d_p\Phi_{p^*}v\right)_2|}{|\rho(p)|},$$

où Φ_{p^*} est obtenu en considérant un système de coordonnées normales centrées en $p^* \in \partial D \cap U$.

Pour prouver l'estimée du Théorème 8, il nous suffit de trouver une constante $C > 0$ telle que pour tout disque $u : \Delta \to D \cap U$, J-holomorphe, l'on ait $N\left(u(0), d_0u\left(\partial/\partial_x\right)\right) \leq C$. Nous raisonnons alors par l'absurde. Il existe ainsi une suite de disque J-holomorphes $u_\nu : \Delta \to D \cap U$ tels que $N\left(u_\nu(0), d_0u_\nu\left(\partial/\partial_x\right)\right) \geq \nu^2$. Un procédé de renormalisation de type Zalcman, permet de construire à partir des disques u_ν des disques J-holomorphes $g_\nu : \Delta_\nu \to D \cap U$ tels que $g_\nu(0)$ converge vers l'origine et dont les dérivées en l'origine, mesurées avec la pseudométrique N, sont uniformément minorées. Nous appliquons la méthode de dilatation à la suite de points $g_\nu(0)$, obtenant alors une suite de disques $\tilde{J}_\nu$-holomorphes

$$\tilde{g_\nu} := \Lambda_\nu \circ \Phi_\nu \circ g_\nu : \Delta_\nu \to \tilde{D}_\nu.$$

Afin d'extraire à partir $\tilde{g_\nu}$ une suite de disques qui converge vers une droite J_{st}-holomorphe entière contenu dans $\tilde{D} = \{\Re e z_2 + P(z_1, \overline{z_1}) < 0\}$ nous remarquons qu'il existe une constante $r_0 > 0$ telle que :

1. il existe $C_1 > 0$ telle que

$$\tilde{g_\nu}\left(r_0\Delta_\nu\right) \subset \Delta_{C_1} \times \Delta_{C_1},$$

2. pour une constante $C_2 > 0$ et ν suffisament grand, nous avons :

$$\|d\tilde{g_\nu}\|_{\mathcal{C}^0(r_0\Delta_\nu)} \leq C_2.$$

Le premier point découle d'une localisation des disques $\Phi_\nu \circ g_\nu$ dans des polydisques du type $Q\left(0,\delta_\nu\right) := \{z \in \mathbb{C}^2 \ : |z_1| \leq \tau\left(p_\nu^*, \delta_\nu\right), |z_2| \leq \delta_\nu\}$. La seconde partie résulte de la convergence de $\|\tilde{J_\nu} - J_{st}\|_{\mathcal{C}^1(\overline{\Delta_{C_1} \times \Delta_{C_1}})}$ vers zéro et des estimées elliptiques des courbes pseudoholomorphes obtenues par J.-C.Sikorav [65].

Ainsi, par un procédé d'extraction diagonal, nous construisons une sous-suite de $\tilde{g_\nu}$ qui converge pour la topologie $\mathcal{C}^1$ vers une droite J_{st}-holomorphe

$$\tilde{g} \ : \mathbb{C} \rightarrow \left(\{Rez_2 + P\left(z_1, \overline{z_1}\right) < 0\}, J_{st}\right).$$

Le polynôme P étant sousharmonique et admettant une partie non harmonique, une droite J_{st}-homorphe contenue dans le domaine limite $\left(\{Rez_2 + P\left(z_1, \overline{z_1}\right) < 0\}, J_{st}\right)$ est nécessairement constante, contredisant finalement la minoration uniforme des dérivées en l'origine mesurées avec la pseudométrique N des disques g_ν.

Remarquons que la méthode de démonstration redonne les estimées précises obtenues par H.Gaussier et A.Sukhov [35] pour les domaines strictement J-pseudoconvexes dans les variétés presque complexes de dimension $2n$; nous le montrons dans le second appendice du chapitre 3.

Le théoreme de Wong-Rosay met en lumière le lien entre la géométrie au voisinage du bord et la géométrie globale d'un domaine ; il établit qu'un domaine (de classe $\mathcal{C}^2$) dans $(\mathbb{C}^n, J_{st})$, admettant un automorphisme dont une orbite s'accumule en un point de stricte pseudoconvexité du bord, est biholomorphe à la boule unité $\mathbb{B} \subset \mathbb{C}^n$ (voir [34], [60], [70]). Remarquons que pour un domaine D borné de $\mathbb{C}^n$, il est équivalent de supposer que D admette un automorphisme dont une orbite s'accumule en un point du bord et de supposer la non compacité du groupe d'automorphismes de D. La généralisation au cadre presque complexe du théorème de Wong-Rosay est due à H.Gaussier and A.Sukhov [35] pour des variétés de dimension quatre et à K.H.Lee [50] en dimension (paire) quelconque. Notons que contrairement aux variétés complexes, le demi plan de Siegel $\mathbb{H} = \{\Re ez_n + |z_1|^2 + \cdots + |z_{n-1}|^2 < 0\}$, pour $n > 2$, peut être muni d'une infinité de structures presque complexes $(J_t)_{t\in\mathbb{R}}$ (non intégrables) et telles que $(\mathbb{H}, J_t)$ n'est pas biholomorphe à $(\mathbb{H}, J_{t'})$ pour $t \neq t'$. Ainsi, K.H.Lee montre que la version non intégrable du théorème de Wong-Rosay met en jeu des structures (limites) dites modèles, introduites par H.Gaussier et A.Sukhov dans l'article [35].

A l'image des domaines strictement pseudoconvexes (dont le type de D'Angelo est égal à deux), classifier les domaines de type de d'Angelo fini est une question fondamentale, étudiée notamment par E.Bedford et S.I.Pinchuk [5] et F.Berteloot et G.Coeuré [9]. Aussi nous intéressons nous à une caractérisation dans les variétés presque complexes des domaines ayant un automorphisme s'accumulant en un point (du bord) de type de D'Angelo quatre.

Corollaire 9. *Soit $D = \{\rho < 0\}$ une région J-pseudoconvexe relativement compacte de type de D'Angelo type inférieur ou égale à quatre dans une variété presque com-*

plexe (M,J) de dimension quatre. Supposons qu'il existe un automorphisme de D admettant une orbite s'accumulant en un point du bord. Alors il existe un polynôme P de degré au plus quatre, sans termes harmoniques tel que (D,J) est biholomorphe à $(\{\Re e z_2 + P(z_1,\overline{z_1}) < 0\}, J_{st})$.

Exposons maintenant la méthode de démonstration. Supposons que pour un point $p_0 \in D$, il existe une suite f_ν d'automorphismes de (D,J) tels que $p_\nu := f_\nu(p_0)$ converge vers $0 \in \partial D$. Nous appliquons la méthode de dilatation des coordonnées à la suite p_ν. Les difféomorphismes $(J,\tilde{J}_\nu)$-holomorphes

$$F_\nu := \Lambda_\nu \circ \Phi_\nu \circ f_\nu \ : f_\nu^{-1}(D \cap U) \to \tilde{D}_\nu$$

satisfont les trois propriétés suivantes :

1. les domaines $\left(f_\nu^{-1}(D\cap U)\right)_\nu$ converge au sens de la convergence de Hausdorff locale pour les ensembles vers le domaine D. Ce point résulte des estimées précises obtenues dans le Théorème 8.
2. $\tilde{D}_\nu$ converge vers le domaine J_{st}-pseudoconvexe $\tilde{D} = \{Rez_2 + P(z_1,\overline{z_1}) < 0\}$, où P est un polynôme non nul sousharmonique de dégrée ≤ 4, ne contenant pas de termes harmoniques purs.
3. Pour chaque compact $K \subset D$, la suite $\left(\|F_\nu\|_{\mathcal{C}^1(K)}\right)_\nu$ est bornée.

Ainsi, nous obtenons une sous suite de $(F_\nu)_\nu$ convergeant, sur les compacts de D pour la topologie $\mathcal{C}^\infty$, vers une application $F \ : D \longrightarrow \bar{\tilde{D}}$, (J,J_{st})-holomorphique. Finalement nous montrons que F est un (J,J_{st})-biholomorphisme de D vers $\tilde{D}$.

Afin d'obtenir des estimées de la pseudométrique de Kobayashi au voisinage d'un point de type de D'Angelo arbitraire, nous privilégions une approche non tangentielle, en s'inspirant de la démarche de I.Graham [39], qui fut l'un des premiers à obtenir des estimées de la pseudométrique de Kobayashi dans les variétés complexes.

Théorème 10. *Soit $D = \{\rho < 0\}$ une région J-pseudoconvexe relativement compacte dans une variété presque complexe (M,J) de dimension quatre. Soit $q \in \partial D$ un point de type de D'Angelo $2m$ et soit $\Lambda \subset D$ le cône de sommet q et d'axe l'axe (réel) normal. Alors il existe une constante $C > 0$ telle que pour tout $p \in D \cap \Lambda$ et $v = v_n + v_t \in T_pM$:*

$$K_{(D,J)}(p,v) \geq C\left(\frac{|v_n|}{|\rho(p)|^{\frac{1}{2m}}} + \frac{|v_t|}{|\rho(p)|}\right),$$

où v_n et v_t sont les composantes normale et tangentielle du vecteur v en q.

La preuve s'articule essentiellement comme celle du Théorème 8. Néanmoins, en privilégiant une approche non tangentielle, il n'est plus nécessaire de recentrer la suite de points convergeant vers l'origine par le difféomorphisme Φ_ν. Ainsi les dilatations de $\mathbb{C}^2$ que nous considérons sont définies par :

$$\Lambda_\nu \ : (z_1,z_2) \mapsto \left(\delta_\nu^{\frac{-1}{2m}} z_1, \delta_\nu^{-1} z_2\right),$$

et nous assurent finalement la convergence des domaines et structures dilatés vers un domaine Brody hyperbolique.

Estimées fines de la pseudométrique de Kobayashi et hyperbolicité au sens de Gromov

Il est important de noter que plusieurs notions d'hyperbolicité ont été introduites, basées sur différentes propriétés géométriques des variétés. Il est naturel de s'intéresser aux liens qui les unissent : par exemple, le lien entre l'hyperbolicité complexe (au sens de Kobayashi ou de Brody) et l'hyperbolicité symplectique a été étudié par A.-L.Biolley [13]. Dans le quatrième chapitre de cette thèse, nous nous intéressons au lien entre l'hyperbolicité au sens de Kobayashi (complexe) et l'hyperbolicité au sens de Gromov (métrique). Plus précisément, nous montrons l'hyperbolicité au sens de Gromov des domaines strictement J-pseudoconvexes d'une varitété presque complexe (M, J) de dimension quatre. Notre approche s'appuie sur les travaux de Z.M.Balogh et M.Bonk [3] et de D.Ma [54].

Dans un espace métrique (X, d) géodésique (ie tel que deux points quelconques peuvent être reliés par une géodésique) l'hyperbolicité au sens de Gromov est définie en terme de finesse des triangles géodésiques. Plus généralement, pour un espace métrique (X, d) quelconque, la Gromov hyperbolicité est quantifiée à l'aide de l'inégalité suivante :

$$(1) \qquad d(x, y) + d(z, \omega) \leq \max(d(x, z) + d(y, \omega), d(x, \omega) + d(y, z)) + 2\delta,$$

pour $x, y, z, \omega \in X$ et une constante positive uniforme δ.

En considérant des estimées fines de la métrique de Kobayashi obtenues par D.Ma [54], Z.M.Balogh et M.Bonk [3] ont montré la Gromov hyperbolicité des domaines D bornés strictement pseudoconvexes de l'espace Euclidien complexe. Leur preuve est basée sur une description du comportement au voisinage du bord ∂D de la distance de Kobayashi d_D permettant de la comparer à une application de $D \times D$ vers $[0, +\infty)$ satisfaisant la condition (1) de Gromov hyperbolicité. Cette description est purement métrique et ne s'appuie sur aucun argument d'analyse complexe. Il résulte de ce fait la motivation d'obtenir des estimées fines de la pseudométrique de Kobayashi dans des domaines relativement compacts strictement J-pseudoconvexes pour des structures non intégrables. Nous montrons alors :

Théorème 11. *Soit $D = \{\rho < 0\}$ un domaine lisse relativement compact dans une variété presque complexe (M, J) de dimension quatre. Nous supposons que ρ est une fonction J-plurisousharmonique au voisinage de $\overline{D}$ et strictement J-plurisousharmonique sur un voisinage de ∂D. Alors il existe des constantes $C > 0$ et $s > 0$ telles que pour tout $p \in D$*

suffisamment proche du bord et $v = v_n + v_t \in T_pM$ *on ait :*

$$e^{-C\delta(p)^s}\left(\frac{|v_n|^2}{4\delta(p)^2}+\frac{\mathcal{L}_J\rho(p^*,v_t)}{2\delta(p)}\right)^{\frac{1}{2}} \leq K_{(D,J)}(p,v)$$

$$\leq e^{C\delta(p)^s}\left(\frac{|v_n|^2}{4\delta(p)^2}+\frac{\mathcal{L}_J\rho(p^*,v_t)}{2\delta(p)}\right)^{\frac{1}{2}}, \tag{2}$$

où p^* *désigne l'unique point de* ∂D *tel que* $\delta(p) := \mathrm{dist}(p, \partial D) = \|p - p^*\|$.

Si notre preuve suit les grandes lignes de celle de D.Ma [54] pour des domaines de $(\mathbb{C}^n, J_{st})$, il est nécessaire d'éliminer tous les arguments complexes contenus dans [54] comme l'introduction de fonctions holomorphes pic. Évoquons en les points clés.

Dans un premier temps, similairement à F.Forstneric et J.-P.Rosay [32], nous obtenons un principe fin de localisation de la pseudométrique de Kobayashi au voisinage d'un point $p^* \in \partial D$ de stricte J-pseudoconvexité. Néanmoins la preuve donnée dans [32] s'appuie sur l'existence de fonctions holomorphes pic et ne peut se généraliser tel quel au cadre presque complexe. Nous contournons cet obstacle en mesurant précisément la longueur au sens de Kobabayashi des chemins s'éloignant de p^* à l'aide d'estimées de la pseudométrique de Kobayashi obtenues par K.H.Lee [50] (voir aussi l'article de S.Ivashkovich et J.-P.Rosay [45]). Le principe de localisation ainsi obtenu dépend du voisinage de p^*, mais cette dépendance sera quantifiée en considérant des polydisques anisotropes de taille entièrement contrôlée.

Nous considérons un point $p \in D = \{\rho < 0\}$ suffisamment proche de ∂D, nous notons $p^* \in \partial D$ son projeté sur le bord et $\delta = \delta(p) = \|p - p^*\|$. Afin d'estimer $K_{(D,J)}(p, v)$, nous travaillons localement, supposant alors :

1. $p^* = 0$ et $p = (\delta, 0)$,
2. $D \cap U \subset \mathbb{R}^4$,
3. la structure J est triangulaire supérieure et coïncide avec J_{st} le long de l'espace tangent complexe $\{z_1 = 0\}$. Par ailleurs notons que les composantes $dz_2 \otimes \dfrac{\partial}{\partial z_1}$ et $dz_2 \otimes \dfrac{\partial}{\partial \overline{z_1}}$ de J s'expriment en $O(|z_1||z_2| + |z_2|^3)$,
4. la fonction définissante ρ s'écrit :

$$\rho(z) = -2\Re e z_1 + 2\Re e \sum \rho_{j,k} z_j z_k + \sum \rho_{j,\overline{k}} z_j \overline{z_k} + O(\|z\|^3),$$

   où $\rho_{j,k}$ et $\rho_{j,\overline{k}}$ sont des constantes telles que $\rho_{j,k} = \rho_{k,j}$, avec $\rho_{2,2} = 0$ et $\rho_{j,\overline{k}} = \overline{\rho}_{k,\overline{j}}$; de plus la stricte J-pseudoconvexité de D permet de supposer $\rho_{2,\overline{2}} = 1$ (voir [23], [35]).

Nous considérons maintenant les polydisques suivants :

$$Q_{(\delta,\alpha)} := \{z \in \mathbb{C}^2, |z_1| < \delta^{1-\alpha}, |z_2| < c\delta^{\frac{1-\alpha}{2}}\},$$

où α est une constante suffisamment petite à fixer et où la constante c, indépendante de p du fait de la stricte pseudoconvexité de D, est choisie telle que :

$$D \cap U \cap \partial Q_{(\delta,\alpha)} \subset \{z \in \mathbb{C}^2, |z_1| = \delta^{1-\alpha}\}.$$

Posons $\Omega := D \cap U \cap Q_{(\delta,\alpha)}$. Le principe de localisation précédemment obtenu s'écrit alors :

$$\left(1 - 2\delta^{\beta}\right) K_{(\Omega,J)}(p,v) \leq K_{(D\cap U,J)}(p,v) \leq K_{(\Omega,J)}(p,v).$$

pour une constante β indépendante de $p = (\delta, 0)$. Par ailleurs, à l'aide d'une fonction plateau, nous faisons l'hypothèse que la structure J est globalement définie sur $\mathbb{R}^4$ et coïncide avec J_{st} en dehors de Ω.

Considérons la dilatation Ψ_δ de $\mathbb{C}^2$:

$$\Psi_\delta(z_1, z_2) := \left(\frac{z_1 - \delta}{z_1 + \delta}, \frac{\sqrt{2\delta}z_2}{z_1 + \delta}\right).$$

Une telle application présente l'avantage suivant : dilater anisotropiquement les coordonnées de $\mathbb{C}^2$, puis se ramemer à la version bornée de la boule unité de $\mathbb{C}^2$. Cela permet, similairement à [54], de localiser le domaine image $\Psi_\delta(\Omega)$ entre deux boules pour lesquelles la métrique de Kobayashi peut être estimée plus aisément :

$$\mathbb{B}\left(0, e^{-C\delta^{\alpha'}}\right) \subset \Psi_\delta(\Omega) \subset \mathbb{B}\left(0, e^{C\delta^{\alpha'}}\right), \tag{3}$$

pour une constante $C > 0$. Ainsi il résulte de l'invariance par biholomorphismes de la métrique de Kobayashi et de (3) :

$$K_{\left(\mathbb{B}(0,e^{C\delta^{\alpha'}}),\widetilde{J^\delta}\right)}(0, d_p\Psi_\delta(v)) \leq K_{(\Omega,J)}(p,v) \leq K_{\left(\mathbb{B}(0,e^{-C\delta^{\alpha'}}),\widetilde{J^\delta}\right)}(0, d_p\Psi_\delta(v)), \tag{4}$$

où $d_p\Psi_\delta(v) = v_1/2\delta + v_2/\sqrt{2\delta}$, et où $\widetilde{J^\delta}$ désigne l'image de la structure J par Ψ_δ. Par ailleurs nous montrons l'inégalité importante suivante :

$$\left\|\widetilde{J^\delta} - J_{st}\right\|_{\mathcal{C}^1(\overline{\mathbb{B}(0,2)})} \leq c\delta^s, \tag{5}$$

pour des constantes $c > 0$ et $s > 0$.C'est précisément dans le but d'obtenir un tel contrôle de l'ordre d'une puissance de δ que nous avons introduit les polydisques $Q_{(\delta,\alpha)}$ plutôt que les boules de taille fixe qu'utilise D.Ma.

quiation

Nous montrons l'estimée inférieure de (2) à l'aide de (4) en considérant (nous inspirant encore une fois de [64]) une fonction plurisousharmonique construite à l'aide de la fonction antipic plurisousharmonique $\log\|z\|^2 + A_\delta\|z\|$ introduite par E.Chirka [19], où la constante A_δ est calculée explicitement compte tenu de (5).

Enfin pour établir l'estimée supérieure souhaitée, il suffit de construire un disque J-holomorphe centré en l'origine et dont la dérivée en l'origine vaut $rv/\|v\|$, avec $r = 1 - c'\delta^{s'}$ pour des constantes $c' > 0$ et $s' > 0$. Nous considérons pour cela un disque J_{st}-holomorphe dont la dérivée en l'origine sera fixée. Nous construisons à l'aide d'un théorème des fonctions implicites quantitatif un disque J-holomorphe dont la dérivée en l'origine est une petite déformation de celle du disque standard. Cette perturbation étant encore une fois explicitement contrôlée du fait de (5), une nouvelle application du théorème des fonctions implicites fournit le disque souhaité.

Finalement, la Gromov hyperbolicité des domaines relativement compacts strictement J-pseudoconvexes résulte du Théorème 1.1 de [3] et du Théorème 11 :

Théorème 12. *Soit $D = \{\rho < 0\}$ un domaine lisse relativement compact dans une variété presque complexe (M, J) de dimension quatre. Nous supposons que ρ est une fonction J-plurisousharmonique au voisinage de $\overline{D}$ et strictement J-plurisousharmonique sur un voisinage de ∂D. Alors D muni de la distance de Kobayashi $d_{(D,J)}$ est hyperbolique au sens de Gromov.*

Les "petites" boules d'une variété presque complexe (M, J) vérifiant les hypothèses du Théorème 12, nous obtenons :

Corollaire 13. *Soit (M, J) une variété presque complexe de dimension quatre. Alors tout point $p \in M$ admet une base de voisinages hyperboliques au sens de Gromov.*

Chapter 1

Preliminaries

In this chapter, we give some properties of almost complex geometry. Let TM and T^*M be the tangent and cotangent bundles over a manifold M. We denote by Δ the unit disc of $\mathbb{C}$ and by Δ_r the disc of $\mathbb{C}$ centered at the origin of radius $r > 0$. We denote by $\mathbb{B}$ the unit ball of $\mathbb{R}^{2n}$, for every n.

1.1 Almost complex structures

An *almost complex structure* J on a real smooth manifold M is a smooth field of endomorphisms of the tangent bundle TM which satisfies $J^2 = -Id$. The pair (M, J) is called an *almost complex manifold*. An almost complex structure J defines a complex structure on each fiber of TM, by

$$(a + ib)v = av + bJ(p)v,$$

where $a, b \in \mathbb{R}$, $p \in M$ and $v \in T_pM$.

The basic example is the complex space $\mathbb{C}^n$ endowed with the standard complex structure $J_{st}^{(2n)}$. Identifying $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ by $z_k = x_k + iy_k$, for any $k = 1, \cdots, n$, $J_{st}^{(2n)}$ is defined by

$$J_{st}^{(2n)}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \quad \text{and} \quad J_{st}^{(2n)}\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},$$

for any $k = 1, \cdots, n$. The matricial interpretation of $J_{st}^{(2n)}$ is given by

$$J_{st}^{(2n)} = \begin{pmatrix} J_{st}^{(2)} & & & & & \\ & J_{st}^{(2)} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & J_{st}^{(2)} \end{pmatrix},$$

where $J_{st}^{(2)}$ of $\mathbb{R}^2$ is the following matrix:

$$J_{st}^{(2)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By an abuse of notation, we simply denote by J_{st} the standard complex structure on $\mathbb{R}^{2n}$, for every n.

The following lemma (see [35]) states that locally any almost complex manifold can be seen as the unit ball of $\mathbb{C}^n$ endowed with a small smooth perturbation of the standard integrable structure J_{st}.

Lemma 1.1.1. *Let (M, J) be an almost complex manifold, with J of class $\mathcal{C}^k$, $k \geq 0$. Then for every point $p \in M$ and every $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinate diffeomorphism $z : U \rightarrow \mathbb{B}$ centered a p (ie $z(p) = 0$) such that the direct image of J satisfies $z_* J\,(0) = J_{st}$ and $||z_*\,(J) - J_{st}||_{\mathcal{C}^k\left(\bar{\mathbb{B}}\right)} \leq \lambda_0$.*

This is simply done by considering a local chart $z : U \rightarrow \mathbb{B}$ centered a p (ie $z(p) = 0$), composing it with a linear diffeomorphism to insure $z_* J\,(0) = J_{st}$ and dilating coordinates.

1.1.1 Vectors fields and differentiable forms

Let (M, J) be an almost complex manifold. The complex tangent bundle $T_{\mathbb{C}}M$ of (M, J) is a bundle such that each fibre is the complexification $\mathbb{C} \otimes T_pM$ of T_pM. Recall that $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ where

$$T^{(1,0)}M := \{X \in T_{\mathbb{C}}M \ : JX = iX\} = \{v - iJv, v \in TM\},$$

and

$$T^{(0,1)}M := \{X \in T_{\mathbb{C}}M \ : JX = -iX\} = \{v + iJv, v \in TM\}.$$

We point out that $T^{(1,0)}M$ (resp. $T^{(0,1)}M$) is the eigenspace corresponding to the eigenvalue i (resp. $-i$) of the endomorphism J. Identifying $\mathbb{C} \otimes T^*M$ with $T^*_{\mathbb{C}}M := Hom(T_{\mathbb{C}}M, \mathbb{C})$, we define the set of complex $(1, 0)$-forms on M by :

$$T^*_{(1,0)}M = \{\omega \in T^*_{\mathbb{C}}M \ : \omega(X) = 0, \forall X \in T^{(0,1)}M\}$$

and the set of complex $(0, 1)$-forms on M by :

$$T^*_{(0,1)}M = \{\omega \in T^*_{\mathbb{C}}M \ : \omega(X) = 0, \forall X \in T^{(1,0)}M\}.$$

Then $T^*_{\mathbb{C}}M = T_{(1,0)}M \oplus T_{(0,1)}M$.

1.1.2 Integrability

A complex manifold is a smooth real manifold M of dimension $2n$ equipped with holomorphic charts with values in $\mathbb{C}^n$; this means that the transition maps are holomorphic. One may define an almost complex structure J on M by pulling back the standard complex structure J_{st}. The structure defined in this way coincides with J_{st} on a neighborhood of each point of M. Thus it is natural to ask under what conditions an almost complex manifold is a complex manifold. This was studied by A.Newlander and L.Nirenberg in [57].

Let N_J be the Nijenhuis tensor with respect to the almost complex structure J defined by:

$$N_J(X,Y) := [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y],$$

for any $X,Y \in \Gamma(TM)$. Then $N_J \equiv 0$ if and only the bundle $T^{0,1}M$ is integrable, that is closed under Lie brackets. The following theorem due to A.Newlander and L.Nirenberg [57] proves that, generically, an almost complex manifold is not a complex manifold.

Theorem 1.1.2. *An almost complex manifold (M,J) is a complex manifold if and only if the bundle $T^{0,1}M$ is integrable.*

In other words, the Nijenhuis tensor measures the lack of complex coordinates of almost complex manifolds. A structure J on M is said to be *integrable* if $N_J \equiv 0$ on $TM \times TM$.

Remark 1.1.3. *Since the Nijenhuis tensor on a real manifold of dimension two is identically zero, any almost complex structure on a Riemann surface is integrable.*

1.2 Pseudoholomorphic discs

A differentiable map $f : (M', J') \longrightarrow (M,J)$ between two almost complex manifolds is said to be (J',J)-*holomorphic* if:

$$J(f(p)) \circ d_p f = d_p f \circ J'(p),$$

for every $p \in M$. A (J',J)-holomorphic map f is called a (J',J)-*biholomorphism* if f is a diffeomorphism.

In case $f : (M,J) \longrightarrow M'$ is a diffeomorphism, we define an almost complex structure, denoted by f_*J, on M' as the direct image of J by f:

$$f_*J\,(q) := d_{f^{-1}(q)} f \circ J\left(f^{-1}\,(q)\right) \circ d_q f^{-1},$$

for every $q \in M'$.

In case $M' = \Delta \subset \mathbb{C}$ and $J' = i$, a (i,J)-holomorphic map is called a *pseudoholomorphic disc*. The J-holomorphy equation for a pseudoholomorphic disc $u : \Delta \to U \subseteq \mathbb{R}^{2n}$ is given by

$$\frac{\partial u}{\partial y} - J\,(u)\,\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

or equivalently by

$$(J(u) + J_{st})\frac{\partial u}{\partial \overline{\zeta}} = (J\,(u) - J_{st})\frac{\partial u}{\partial \zeta},$$

Since, according to Lemma 1.1.1, $J + J_{st}$ is locally invertible, the pseudoholomorphic disc u satisfies the following local J-holomorphy equation:

$$\frac{\partial u}{\partial \overline{\zeta}} + Q_J(u)\frac{\partial u}{\partial \zeta} = 0,$$

where the endomorphism $Q_J(u)$ is defined by

$$Q_J(u) := -(J(u) + J_{st})^{-1}(J(u) - J_{st}).$$

A.Nijenhuis and W.Woolf [58], proved the local existence of pseudoholomorphic curves with prescribed one-jets. The generalization for prescribed k-jets for arbitrary positive $k \in \mathbb{N}$ is due to S.Ivashkovich and J.-P.Rosay [45] and is stated as follows:

Proposition 1.2.1. *Let $k \in \mathbb{N}$, $k \geq 1$, and $0 < \alpha < 1$. Let J be a $\mathcal{C}^{k-1,\alpha}$ almost complex structure defined near the origin in R^{2n}. For any $p \in \mathbb{R}^{2n}$ sufficiently close the origin, and every $V = (v_1, \ldots, v_k) \in (\mathbb{R}^{2n})^k$ small enough, there is a $\mathcal{C}^{k,\alpha}$ J-holomorphic disc $u_{p,V} : \Delta \to R^{2n}$ such that*

$$u_{p,V}(0) = p, \quad \textit{and} \quad \frac{\partial^j u_{p,V}}{\partial x^j}(0) = v_j,$$

for any $1 \leq j \leq k$. If the structure J is of class $\mathcal{C}^{k,\alpha}$, then $u_{p,V}$ may be chosen with $\mathcal{C}^1$ dependence (in $\mathcal{C}^{k,\alpha}$) on the parameters (p, V) in $\mathbb{R}^{2n} \times (\mathbb{R}^{2n})^k$.

The proof they gave, assuming the $\mathcal{C}^{k,\alpha}$ regularity of the structure J is a consequence of the implicit function theorem. As they noticed, in case the structure is only supposed to be $\mathcal{C}^{k-1,\alpha}$, the continuous dependence on parameters probably fails.

1.2.1 First order estimate for pseudohomorphic discs

In this subsection we present a theorem stated by J.-C. Sikorav in [65] which provides a generalization of the Cauchy estimates for pseudoholomorphic discs.

Let $k \in \mathbb{N}$, $k \geq 1$, $0 < \alpha < 1$, and let us consider the following elliptic Beltrami PDE:

$$\frac{\partial u}{\partial \overline{\zeta}} + q(u)\frac{\partial u}{\partial \zeta} = 0, \tag{1.2}$$

where $q : \mathbb{B} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ is an endomorphism with regularity $\mathcal{C}^{k,\alpha}$, and u is a differentiable map from Δ to $\mathbb{B}$.

Theorem 1.2.2. *Let $0 < r < 1$. Let D be a relatively compact domain in $\mathbb{R}^{2n}$. Then there are positive constants ε and C such that if $\|q\|_{\mathcal{C}^{k,\alpha}} \leq \varepsilon$, then any map u satisfying (1.2) is of class $\mathcal{C}^{k+1,\alpha}$ on Δ_{1-r} and verify:*

$$\|u\|_{\mathcal{C}^{k+1,\alpha}(\Delta_{1-r})} \leq C\|u\|_{L^\infty}.$$

As a direct consequence, if we suppose that J is of class $\mathcal{C}^{1,r}$, then the set of J-holomorphic disc is closed for the topology of the uniform convergence over compact subsets.

Remark 1.2.3. *L.Blanc-Centi obtained in [14] explicit estimates for pseudoholomorphic disc attached to a maximal totally real submanifold E, involving the curvature of E.*

1.2.2 Normal coordinates

As noticed by J.-C.Sikorav in [65], a corollary of Proposition 1.2.1 is the existence of normal coordinates on a four dimensional almost complex manifold (M, J), where J is smooth enough.

Lemma 1.2.4. *Let (M, J) be an almost complex manifold where J is of class $\mathcal{C}^{1,r}$ at least. Then near each point p there are $\mathcal{C}^{2,\alpha}$ coordinates $z \in \mathbb{C}^2$ centered at p such that J satisfies $J(p) = J_{st}$ and admits a block diagonal matrix representation:*

$$J(z) = \begin{pmatrix} J_1(z) & 0 \\ 0 & J_2(z) \end{pmatrix}.$$

As illustrated by Figure 2, this is done by considering the family of vectors $(1, 0)$ at base points $(0, t)$ for $t \neq 0$ small enough. Due to the (local) existence of pseudoholomorphic discs (see Proposition 1.2.1), we obtain a family of J holomorphic discs u_t such that $u_t(0) = (0, t)$ and $d_0 u_t(\partial/\partial_x) = (0, 1)$ and according to the parameters dependence, we may straighten these discs into the complex lines $\{z_2 = t\}$. We then consider a transversal foliation by J-holomorphic discs and straighten these lines into $\{z_1 = c\}$.

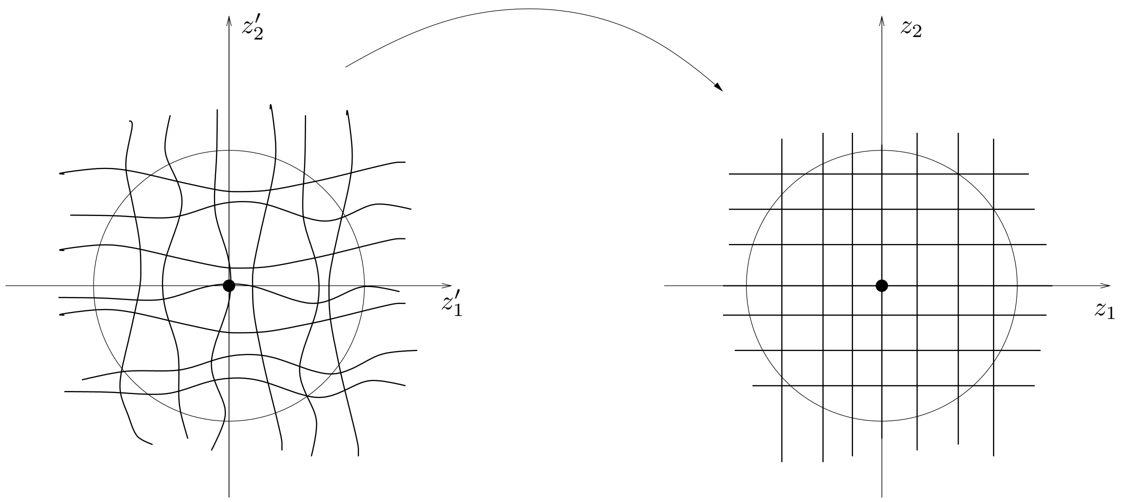

Figure 2. Normal coordinates in a four dimensional almost complex manifold.

We point out that a J-holomorphic disc u satisfies the following diagonal J-holomorphy equation:

$$\frac{\partial u_k}{\partial y} = J_k(u) \frac{\partial u_k}{\partial x},$$

for $k = 1, 2$.

Remark 1.2.5. *Generically and in higher dimension, there is no coordinate such that J is diagonal (ie such that it admits a block diagonal matrix representation). Indeed for an almost complex manifold (M, J) of dimension $2n$, there is no, generically, submanifold of real dimension greater than two (and of real codimension greater than one) in (M, J) closed under J.*

There is a normal form for an almost complex structure along a regular pseudoholomorphic disc, illustrated by Figure 3. Let $(t_1, t_2, \cdots, t_{2n})$ be coordinates of $\mathbb{R}^{2n}$. Let $J = J_l^k dt^l \otimes \partial t_k$ be structure in $\mathbb{R}^{2n}$ and consider a (regular) J-holomorphic disc u in $(\mathbb{R}^{2n}, J)$. After a change of variables, u may be expressed by the flat pseudoholomorphic disc $u(\zeta) = (\zeta, 0, \cdots, 0)$. Moreover let us consider the linear diffeomorphism Φ of $\mathbb{R}^{2n}$ defined by:

$$\begin{aligned}\Phi^{-1}(z) \quad := \quad & \Big(x_1 + \sum_{k=1}^{n} J_{2k-1}^{1}(u(z_1))y_k, \sum_{k=1}^{n} J_{2k-1}^{2}(u(z_1))y_k, \cdots \\ & \cdots, x_n + \sum_{k=1}^{n} J_{2k-1}^{2n-1}(u(z_1))y_k, \sum_{k=1}^{n} J_{2k-1}^{2n}(u(z_1))y_k\Big).\end{aligned}$$

In that change of variables the structure J is transformed into an almost complex structure that coincides with J_{st} along $\mathbb{C} \times \{0\} \subset \mathbb{C}^n$

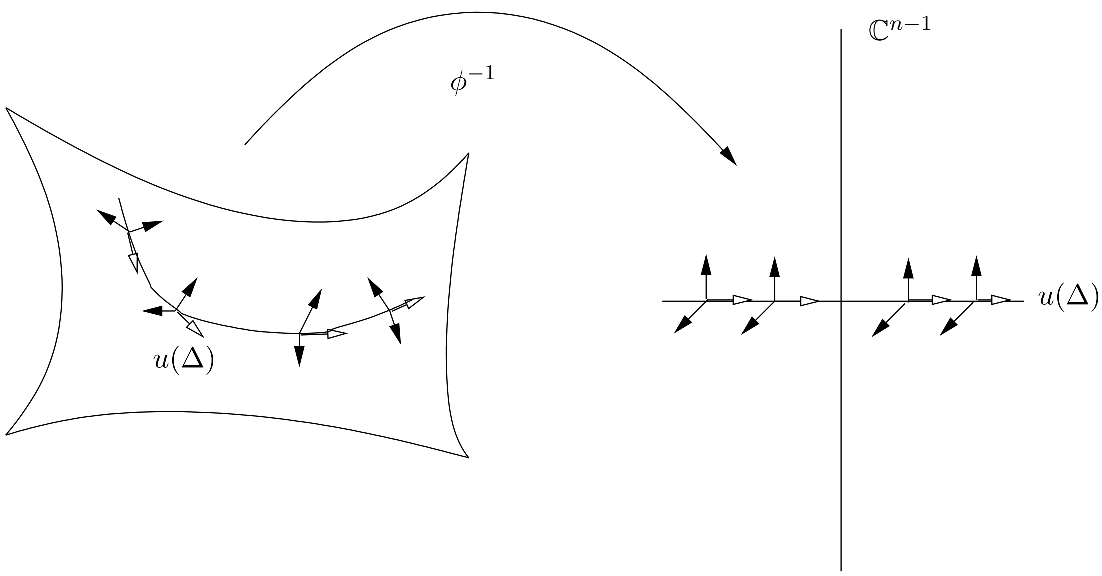

Figure 3.

1.3 Levi geometry

Let ρ be a $\mathcal{C}^2$ real valued function on a smooth almost complex manifold (M, J) .

1.3.1 The Levi form

We denote by $d^c_J\rho$ the differential form defined by

$$d^c_J\rho\,(v) := -d\rho\,(Jv)\,, \tag{1.3}$$

where v is a section of TM. The *Levi form* of ρ at a point $p \in M$ and a vector $v \in T_pM$ is defined by

$$\mathcal{L}_J\rho\,(p,v) := d\,(d^c_J\rho)\,(p)\,(v, J(p)v) = dd^c_J\rho(p)\,(v, J(p)v)\,.$$

In case $(M,J) = (\mathbb{C}^n, J_{st})$, then $\mathcal{L}_{J_{st}}\rho$ is, up to a positive multiplicative constant, the usual standard Levi form:

$$\mathcal{L}_{J_{st}}\rho(p,v) = 4\sum \frac{\partial^2\rho}{\partial z_j\partial\overline{z_k}}v_j\overline{v_k}.$$

We investigate now how close is the Levi form with respect to J from the standard Levi form. For $p \in M$ and $v \in T_pM$, we easily get:

$$\mathcal{L}_J\rho\,(p,v) = \mathcal{L}_{J_{st}}\rho(p,v) + d(d^c_J - d^c_{J_{st}})\rho(p)(v, J(p)v) + dd^c_{J_{st}}\rho(p)(v, J(p) - J_{st})v). \tag{1.4}$$

In local coordinates $(t_1, t_2, \cdots, t_{2n})$ of $\mathbb{R}^{2n}$, (1.4) may be written as follows

$$\begin{aligned}\mathcal{L}_J\rho\,(p,v) &= \mathcal{L}_{J_{st}}\rho(p,v) + {}^t v(A - {}^tA)J(p)v + {}^t(J(p) - J_{st})vDJ_{st}v + \\ &\quad {}^t(J(p) - J_{st})vD(J(p) - J_{st})v\end{aligned} \tag{1.5}$$

where

$$A := \left(\sum_l \frac{\partial u}{\partial t_l}\frac{\partial J^l_j}{\partial t_k}\right)_{1\leq j,k\leq 2n} \quad \text{and} \quad D := \left(\frac{\partial^2 u}{\partial t_j\partial t_k}\right)_{1\leq j,k\leq 2n}.$$

Let f be a (J', J)-biholomorphism from (M', J') to (M, J). Then for every $p \in M$ and every $v \in T_pM$:

$$\mathcal{L}_{J'}\rho\,(p,v) = \mathcal{L}_J\rho \circ f^{-1}\,(f\,(p)\,, d_pf\,(v))\,.$$

This expresses the invariance of the Levi form under diffeomorphisms.

The next proposition is useful in order to compute the Levi form (see [27] and [45]).

Proposition 1.3.1. *Let $p \in M$ and $v \in T_pM$. Then*

$$\mathcal{L}_J\rho\,(p,v) = \Delta\,(\rho\circ u)\,(0)\,,$$

where $u\;:\Delta \to (M,J)$ is any J-holomorphic disc satisfying $u\,(0) = p$ and $d_0u\,(\partial/\partial_x) = v$.

1.3.2 J-plurisubharmonic functions

Proposition 1.3.1 leads to the following proposition-definition:

Proposition 1.3.2. *The two statements are equivalent:*

1. *$\rho \circ u$ is subharmonic for any J-holomorphic disc $u : \Delta \to M$.*

2. *$\mathcal{L}_J\rho(p, v) \geq 0$ for every $p \in M$ and every $v \in T_pM$.*

If one of the previous statements is satisfied we say that ρ is *J-plurisubharmonic*. We say that ρ is *strictly J-plurisubharmonic* if $\mathcal{L}_J\rho(p, v)$ is positive for any $p \in M$ and any $v \in T_pM \setminus \{0\}$. Plurisubharmonic functions play a very important role in almost complex geometry: they give attraction and localization properties for pseudoholomorphic discs. For this reason the construction of J-plurisubharmonic functions is crucial.

The basic example of a J-plurisubharmonic function on (M, J) is:

Example 1. *For every point $p \in (M, J)$ there exists a neighborhood U of p and a diffeomorphism $z : U \to \mathbb{B}$ centered at p such that the function $|z|^2$ is J-plurisubharmonic on U.*

The next example less trivial is due to E.Chirka [19] (see [35] or [45] for a proof). We will give a quantitative version of this lemma in Chapter 4 (see Lemma 4.1.1).

Lemma 1.3.3. *Let p be a point in an almost complex manifold (M, J). There exist a neighborhood U of p in M, a diffeomorphism $z : U \to \mathbb{B}$ centered at p and a positive constant A, such that the function $log|z| + A|z|$ is J-plurisubharmonic on U. Such a function is called a local antipeak J-plurisubharmonic function at p.*

Consequently any point p in a smooth almost complex manifold (M, J) is a polar set; and more generally, J.-P.Rosay [61] proved that J-holomorphic discs are polar sets. As suggested by J.-P.Rosay, a very interesting open problem is the construction of a J-plurisubharmonic function whose polar set is a pseudoholomorphic disc with a cusp.

1.3.3 J-pseudoconvexity

Similarly to the integrable case, one may define the notion of pseudoconvexity in almost complex manifolds. Let D be a domain in (M, J). We denote by $T^J\partial D := T\partial D \cap JT\partial D$ the J-invariant subbundle of $T\partial D$.

Definition 1.3.4.

1. *The domain D is J-pseudoconvex (resp. it strictly J-pseudoconvex) if $\mathcal{L}_J\rho(p, v) \geq 0$ (resp. > 0) for any $p \in \partial D$ and $v \in T^J_p\partial D$ (resp. $v \in T^J_p\partial D \setminus \{0\}$).*

2. *A J-pseudoconvex region is a domain $D = \{\rho < 0\}$ where ρ is a $\mathcal{C}^2$ defining function, J-plurisubharmonic on a neighborhood of $\overline{D}$.*

We recall that a defining function for D satisfies $d\rho \neq 0$ on ∂D.

1.4 Kobayashi hyperbolicity

1.4.1 The Kobayashi pseudometric

The existence of local pseudoholomorphic discs (see Proposition 1.2.1) allows to define the *Kobayashi-Royden pseudometric*, abusively called the *Kobayashi pseudometric*, $K_{(M,J)}$ for $p \in M$ and $v \in T_pM$:

$$\begin{aligned} K_{(M,J)}(p,v) &:= \inf\left\{\frac{1}{r} > 0, u : \Delta \to (M,J) \ J\text{-holomorphic}, u(0) = p, \right. \\ &\qquad\qquad \left. d_0u(\partial/\partial x) = rv\right\}. \\ &= \inf\left\{\frac{1}{r} > 0, u : \Delta_r \to (M,J), \ J\text{-holomorphic}, u(0) = p, \right. \\ &\qquad\qquad \left. d_0u(\partial/\partial x) = v\right\}. \end{aligned}$$

Since the composition of pseudoholomorphic maps is still pseudoholomorphic, the Kobayashi pseudometric satisfies the decreasing property:

Proposition 1.4.1. *Let $f : (M', J') \to (M, J)$ be a (J', J)-holomorphic map. Then for any $p \in M'$ and $v \in T_pM'$ we have*

$$K_{(M,J)}(f(p), d_pf(v)) \leq K_{(M',J')}(p,v).$$

For a complex manifold M, the Kobayashi pseudometric is an upper semicontinuous function on the tangent bundle TM. According to B.Kruglikov [48], the same is true in almost complex manifolds whenever the structure is smooth enough. More precisely S.Ivashkovich and J.-P.Rosay [45] proved that it is upper semicontinuous if the structure is of class $\mathcal{C}^{1,r}$ at least. In [46], there is an example of an almost complex structure J of class $\mathcal{C}^{2/3}$ on the bidisc $\Delta \times \Delta$ such that $K_{(\Delta\times\Delta,J)}$ is not upper semicontinuous on the tangent bundle. We do not know what happen in case the structure is $\mathcal{C}^\alpha$ with $2/3 < \alpha \leq 1$.

Since the structures we consider are smooth enough, we may define the integrated pseudodistance $d_{(M,J)}$ of $K_{(M,J)}$:

$$d_{(M,J)}(p,q) := \inf\left\{\int_0^1 K_{(M,J)}(\gamma(t), \dot{\gamma}(t))\,dt, \ \gamma : [0,1] \to M, \ \gamma(0) = p, \gamma(1) = q\right\}.$$

Similarly to the standard integrable case, B.Kruglikov [48] proved that the integrated pseudodistance of the Kobayashi pseudometric coincides with the Kobayashi pseudodistance defined by chains of pseudholomorphic discs.

We now define the Kobayashi hyperbolicity:

Definition 1.4.2.

1. *The manifold (M, J) is Kobayashi hyperbolic if the Kobayashi pseudodistance $d_{(M,J)}$ is a distance.*

2. *The manifold (M, J) is local Kobayashi hyperbolic at $p \in M$ if there exist a neighborhood U of p and a positive constant C such that*

$$K_{(M,J)}(q, v) \geq C\|v\|$$

for every $q \in U$ and every $v \in T_qM$.

3. *A Kobayashi hyperbolic manifold (M, J) is complete hyperbolic if it is complete for the distance $d_{(M,J)}$.*

Another way to say that (M, J) is complete hyperbolic is to say that the Kobayashi ball

$$\mathbb{B}_{(M,J)}(p, r) := \left\{q \in M, d_{(M,J)}(p, q) < r\right\}$$

are relatively compact in M for any $p \in M$ and any real positive r.

Remark 1.4.3. *In case the manifold (M, J) is Kobayashi hyperbolic then the topology induced by the Kobayashi distance coincides with the usual topology on M.*

Remark 1.4.4. *The manifold (M, J) is Kobayashi hyperbolic if and only if it is local Kobayashi hyperbolic at each point of M. This statement is due to H.L.Royden [62] in the integrable case and its proof is identical in the almost complex setting.*

1.4.2 Tautness

In this subsection, we give the definition of the tautness and its link with the Kobayashi hyperbolicity.

Definition 1.4.5. *A domain D in an almost complex manifold (M, J) is taut if for every $(u_\nu)_\nu$ sequence of J-holomorphic discs, either*

1. *there is a subsequence which converges to J-holomorphic disc in D, or*
2. *is compactly divergent, that is, for all compact subsets $K \subset \Delta$ and $K' \subset D$, there is an integer ν_0 such that $u_\nu(K) \cap K' = \emptyset$ for $\nu \geq \nu_0$.*

We point out that a relatively compact Kobayashi hyperbolic domain D is taut if and only if for every sequence $(u_\nu)_\nu$ of J-holomorphic discs in D converging to a J-holomorphic disc u in $\overline{D}$, the limit satisfies either $u(\Delta) \subset D$, or $u(\Delta) \subset \partial D$.

The link between the tautness and the Kobayashi hyperbolicity of a domain is given by:

Proposition 1.4.6. *Let D be a domain in an almost complex manifold (M, J). Then*

$$D \text{ is complete hyperbolic } \Rightarrow D \text{ is taut } \Rightarrow D \text{ is hyperbolic.}$$

Chapter 2

Almost complex structures on the cotangent bundle

This chapter follows the paper published in Complex Variables and Elliptic Equations 52 (2007), 741-754. [10].

Résumé

Nous construisons un relevé de structure presque complexe sur le fibré cotangent, que l'on nomme *relevé horizontal généralisé*, en utilisant une connexion sur la variété base. Cette construction unifie le *relevé complet* de I.Sato et le *relevé horizontal* défini par S.Ishihara et K.Yano. Nous étudions certaines propriétés géométriques de ce relevé, permettant ainsi de caractériser génériquement le relevé complet, et étudions sa compatibilité avec les formes symplectiques sur le fibré cotangent.

Abstract

We construct some lift of an almost complex structure to the cotangent bundle, using a connexion on the base manifold. This unifies the complete lift defined by I.Sat and the horizontal lift introduced by S.Ishihara and K.Yano. We study some geometric properties of this lift and its compatibility with symplectic forms on the cotangent bundle.

Introduction

There is a natural and deep connection between local analysis on complex and almost complex manifolds and canonical bundles. For instance, the cotangent bundle is tightly related to extension of biholomorphisms and to the study of stationary discs. Moreover, it is well known that the cotangent bundle plays a very important role in symplectic geometry and its applications, since this carries a canonical symplectic structure induced by the Liouville form.

Several lifts of an almost complex structure on a base manifold are constructed on the cotangent bundle. These are essentially due to I.Sato in [63] and S.Ishihara-K.Yano in

[44]. I.Sato defined a lift of the ambient structure as a correction of the formal complete lift; S.Ishihara-K.Yano introduced the horizontal lift obtained via a symmetric connection. The aim of the present chapter is to unify and to generalize these lifts by introducing a more natural almost complex lift called the generalized horizontal lift.

It turns out that our construction depends on the introduction of some connection: we study the dependence of the lift on it. Our main result states that the structure defined by I.Sato and the horizontal lift are special cases of our general construction, obtained by particular choices of connections (Theorem 2.2.4). We establish some geometric properties of this general lift (Theorems 2.3.1 and 2.3.3). Then we characterize generically the structure constructed by I.Sato by the holomorphy of the lift of a given diffeomorphism on the bases and by the holomorphy of the complex fiberwise multiplication (Corollary 2.3.2 and Corollary 2.3.4).

Finally, we study the compatibility between lifted almost complex structures and symplectic forms on the cotangent bundle. The conormal bundle of a strictly pseudoconvex hypersurface is a totally real maximal submanifold in the cotangent bundle endowed with the structure defined by I.Sato. This was proved by S.Webster [69] for the standard complex structure, and by A.Spiro [66], and independently by H.Gaussier-A.Sukhov [36], for the almost complex case. One can search for a symplectic proof of this, since every Lagrangian submanifold in a symplectic manifold is totally real for almost complex structures compatible with the symplectic form. We prove that for every almost complex manifold and every symplectic form on T^*M compatible with the generalized horizontal lift, the conormal bundle of a strictly pseudoconvex hypersurface is not Lagrangian (Proposition 2.4.2). This illustrates the singular fact that to study local complex (or almost complex) geometry, we naturally use structures which are not compatible with the canonical symplectic form of the cotangent bundle.

2.1 Preliminaries

Let (M, J) be an almost complex manifold of even dimension n. We denote by TM and T^*M the tangent and cotangent bundles over M, by $\Gamma(TM)$ and $\Gamma(T^*M)$ the sets of sections of these bundles and by $\pi : T^*M \longrightarrow M$ the fiberwise projection. We consider local coordinates systems $(x_1, \cdots, x_n)$ in M and $(x_1, \cdots, x_n, p_1, \cdots, p_n)$ in T^*M. We suppose that, in local coordinates, the structure J is given by $J = J_l^k dx^l \otimes \partial x_k$. We do not write any sum symbol; we use Einstein summation convention.

2.1.1 Tensors and contractions

Let θ be *the Liouville form* on T^*M. This one-form is locally given by

$$\theta = p_i dx^i.$$

The two-form $\omega_{st} := d\theta$ is the *canonical symplectic form* on the cotangent bundle, with local expression

$$\omega_{st} = -dx^k \wedge dp^k.$$

We stress out that these forms do not depend on the choice of coordinates on T^*M.

We denote by $T_q^r M$ the space of q covariant and r contravariant tensors on M. For positive q, we consider the contraction map $\gamma : T_q^1 M \to T_{q-1}^1(T^*M)$ defined by:

$$\gamma(R) := p_k R^k_{i_1,\cdots,i_q} dx^{i_1} \otimes \cdots \otimes dx^{i_{q-1}} \otimes \partial p_{i_q} \tag{2.1}$$

for $R = R^k_{i_1,\cdots,i_q} dx^{i_1} \otimes \cdots \otimes dx^{i_q} \otimes \partial x_k$.

We also define a q-form on T^*M by

$$\theta(R) := p_k R^k_{i_1,\cdots,i_q} dx^{i_1} \otimes \cdots \otimes dx^{i_q} \tag{2.2}$$

for a tensor $R \in T_q^1 M$ on M and we notice that:

$$\theta(R)(X_1, \cdots, X_q) = \theta(R(d\pi(X_1), \cdots, d\pi(X_q)))$$

for $X_1, \cdots, X_q \in \Gamma(T^*M)$.

Since the canonical symplectic form ω_{st} establishes a correspondence between q-forms and $T_{q-1}^1 M$, one may define the contraction map γ using the Liouville form θ and ω_{st} by setting, for $X_1, \cdots, X_q \in \Gamma(T^*M)$:

$${}^t(\theta(R))(X_1, \cdots, X_q) = -\omega_{st}(X_1, \gamma(R)(X_2, \cdots, X_q)), \tag{2.3}$$

where

$${}^t(\theta(R))(X_1, \cdots, X_q) = \theta(R)(X_2, \cdots, X_q, X_1).$$

For a tensor $R \in T_2^1 M$, we have a matricial interpretation of the contraction γ; if $R^k_{i,j}$ are the coordinates of R then $\gamma(R)$ is given by:

$$\gamma(R) = \begin{pmatrix} 0 & 0 \\ p_k R^k_{j,i} & 0 \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}).$$

2.1.2 Connections

Let (E, π, M) be a vector bundle on a real (smooth) manifold M and denote by $\Gamma(E)$ the set of sections of this bundle. A *connection* ∇ on (E, π, M) is a $\mathbb{R}$-bilinear map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ satisfying:

$$\begin{cases} \nabla_{fX}\sigma & = & f\nabla_X \sigma \\ \nabla_X(f\sigma) & = & (X.f)\sigma + f\nabla_X \sigma \end{cases}$$

for every $X \in \Gamma(M)$, $\sigma \in \Gamma(E)$ and smooth function f. In what follows connections will be taken on the tangent bundle except when it is clearly announced.

Let ∇ be a connection on an almost complex manifold (M, J). We denote by $\Gamma^k_{i,j}$ its Christoffel symbols defined by

$$\nabla_{\partial x_i} \partial x_j = \Gamma^k_{i,j} \partial x_k.$$

Let also $\Gamma_{i,j}$ defined in local coordinates $(x_1 \cdots, x_n, p_1, \cdots, p_n)$ on the cotangent bundle T^*M by the equality

$$p_k \Gamma^k_{i,j} = \Gamma_{i,j}.$$

The *torsion* T of ∇ is defined by:

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$

for every $X, Y \in \Gamma(TM)$. There are natural families of connections on an almost complex manifold.

Definition 2.1.1. *A connection ∇ on M is called:*

1. *almost complex when $\nabla_X(JY) = J\nabla_X Y$ for every $X, Y \in \Gamma(TM)$,*
2. *minimal when its torsion T is equal to $\frac{1}{4}N_J$,*
3. *symmetric when its torsion T is identically zero.*

A.Lichnerowicz proved in [53] that for any almost complex manifold, the set of almost complex and minimal connections is nonempty. This fact is crucial in the following.

We introduce a tensor $\nabla J \in T^1_2 M$ which measures the lack of almost complexity of the connection ∇:

$$(\nabla J)(X,Y) := \nabla_X JY - J\nabla_X Y \tag{2.4}$$

for every $X, Y \in \Gamma(TM)$. Locally we have

$$(\nabla J)^k_{i,j} = \partial x_i J^k_j - J^k_l \Gamma^l_{i,j} + J^l_j \Gamma^k_{i,l}. \tag{2.5}$$

To the connection ∇ we associate three other connections:

1. $\overline{\nabla} := \nabla - T$. The Christoffel symbols $\overline{\Gamma}^k_{i,j}$ of $\overline{\nabla}$ are given by
$$\overline{\Gamma}^k_{i,j} = \Gamma^k_{j,i}.$$
2. $\widetilde{\nabla} := \nabla - \frac{1}{2}T$. The connection $\widetilde{\nabla}$ is a symmetric connection and its Christoffel symbols $\widetilde{\Gamma}^k_{i,j}$ are given by:
$$\widetilde{\Gamma}^k_{i,j} = \frac{1}{2}(\Gamma^k_{i,j} + \Gamma^k_{j,i}).$$
3. a connection on the cotangent bundle (T^*M, π, M), still denoted by ∇, and defined by:
$$(\nabla_X s)(Y) := X.s(Y) - s\nabla_X Y, \tag{2.6}$$
for every $X, Y \in \Gamma(TM)$ and every $s \in \Gamma(T^*M)$.

Let $x \in M$ and let $\xi \in T^*M$ be such that $\pi(\xi) = x$. The *horizontal distribution* H^∇ of ∇ is defined by:

$$H^\nabla_\xi := \{d_x s(X),\ X \in T_x M, s \in \Gamma(T^*M), s(x) = \xi, \nabla_X s = 0\} \subseteq T_\xi T^*M.$$

We recall that $d_\xi \pi$ induces an isomorphism between H^∇_ξ and $T_x M$. Moreover we have the following decomposition:

$$T_\xi T^*M = H^\nabla_\xi \oplus T^*_x M.$$

So an element $Y \in T_\xi T^*M$ decomposes as $Y = (X, v^\nabla(Y))$, where

$$v^\nabla : T_\xi T^*M \longrightarrow T^*_x M$$

is the projection on the vertical space $T^*_x M$ parallel to H^∇_ξ.

2.2 Generalized horizontal lift on the cotangent bundle

Let (M, J) be an almost complex manifold. We first recall the definitions of the structures constructed by I.Sato and S.Ishihara-K.Yano. Then we introduce a new almost complex lift of J to the cotangent bundle T^*M over M and we prove that this unifies the complete lift and the horizontal lift.

2.2.1 Complete lift

We consider the formal complete lift denoted by J^c and defined by I.Sato in [63] as follows: let $\theta(J)$ be the one-form on T^*M with local expression

$$\theta(J) = p_k J_l^k dx^l.$$

Since the canonical symplectic form ω_{st} gives a correspondence between two-forms and tensors of type $(1, 1)$, one may define J^c by the identity

$$d(\theta(J)) = \omega_{st}(J^c., .).$$

Then J^c is locally given by:

(2.7)
$$J^c = \begin{pmatrix} J_j^i & 0 \\ p_k(\partial x_j J_i^k - \partial x_i J_j^k) & J_i^j \end{pmatrix}.$$

The formal complete lift J^c is an almost complex structure on T^*M if and only if J is an integrable structure on M, that is if and only if M is a complex manifold. Introducing a correction term which involves the non integrability of J, I.Sato [63] obtained an almost complex structure $\widetilde{J}$ on the cotangent bundle and called the *complete lift* of J; this is given by:

(2.8)
$$\widetilde{J} := J^c - \frac{1}{2}\gamma(JN_J).$$

The coordinates of JN_J are given by:

$$JN_J(\partial x_i, \partial x_j) = \left[-\partial x_j J_i^k + \partial x_i J_j^k + J_s^k J_i^q \partial x_q J_j^s - J_s^k J_j^q \partial x_q J_i^s\right] dx^k.$$

Thus we have the following local expression of $\widetilde{J}$:

(2.9)
$$\widetilde{J} = \begin{pmatrix} J_j^i & 0 \\ B_j^i & J_i^j \end{pmatrix},$$

with

$$B_j^i = \frac{p_k}{2}\left[\partial x_j J_i^k - \partial x_i J_j^k + J_s^k J_i^q \partial x_q J_j^s - J_s^k J_j^q \partial x_q J_i^s\right].$$

2.2.2 Horizontal lift

We now recall the definition of the horizontal lift of an almost complex structure on the cotangent bundle T^*M. Let ∇ be a connection on M and let $\widetilde{\nabla} := \nabla - \frac{1}{2}T$ be its symmetrized associated connection. The *horizontal lift* of J is defined in [44] by:

$$J^{H,\nabla} := J^c + \gamma([\widetilde{\nabla} J]), \tag{2.10}$$

where the tensor $[\widetilde{\nabla} J] \in T^1_2 M$ is given by:

$$[\widetilde{\nabla} J](X,Y) := -(\widetilde{\nabla} J)(X,Y) + (\widetilde{\nabla} J)(Y,X),$$

for every $X, Y \in \Gamma(TM)$ (with $\widetilde{\nabla} J$ defined as in (2.4)).

S.Ishihara and K.Yano [44] proved that $J^{H,\nabla}$ is an almost complex structure on T^*M. But it is important to notice that without symmetrizing ∇, the horizontal lift of J is not an almost complex structure. The structure $J^{G,\nabla}$ is locally given by:

$$J^{H,\nabla} = \begin{pmatrix} J^i_j & 0 \\ \widetilde{\Gamma}_{i,l} J^l_j - \widetilde{\Gamma}_{j,l} J^l_i & J^j_i \end{pmatrix}. \tag{2.11}$$

The complete and the horizontal lifts are both a correction of the formal complete lift J^c. Our aim is to unify and to characterize these two almost complex structures.

2.2.3 Construction of the generalized horizontal lift

Let $x \in M$ and let $\xi \in T^*M$ be such that $\pi(\xi) = x$. Assume that H is a distribution satisfying the local decomposition $T_\xi T^*M = H_\xi \oplus T^*_x M$. From an algebraic point of view it is natural to lift the almost complex structure J as a product structure, that is $J \oplus {}^tJ$ with respect to $H_\xi \oplus T^*_x M$. Since any such distribution determines and is determined by a unique connection one may define a lifted almost complex structure using a connection (this point of view is inspired by P.Gauduchon in [33]).

Let ∇ be a connection on M. We consider the connection induced by ∇ on (M, T^*M), defined by (2.6). As illustrated by Figure 4, we define, for a vector $Y = (X, v^\nabla(Y)) \in T_\xi T^*M = H^\nabla_\xi \oplus T^*_x M$:

$$J^{G,\nabla}(Y) := (JX, {}^tJ(v^\nabla(Y))),$$

where

$$JX = (d_\xi \pi_{|H^\nabla_\xi})^{-1}(J(x) d_\xi \pi(X)).$$

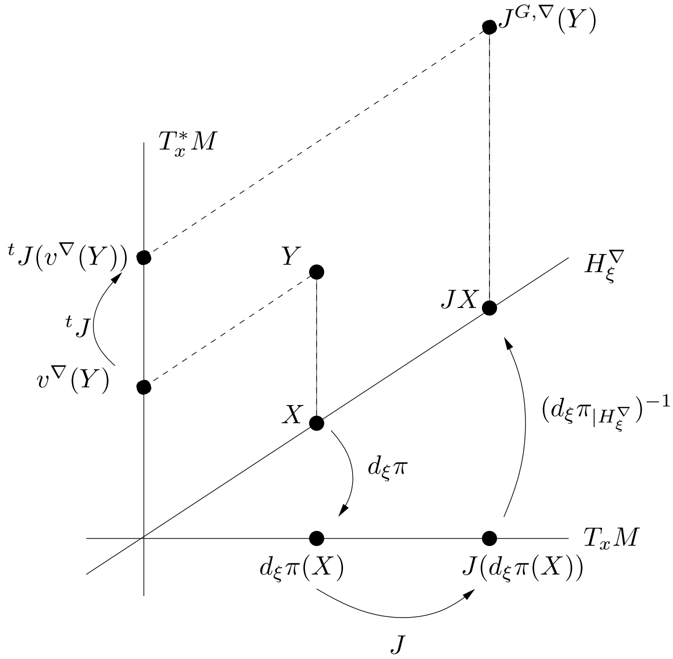

Figure 4. Construction on the generalized horizontal lift $J^{G,\nabla}$.

Definition 2.2.1. *The almost complex structure $J^{G,\nabla}$ is called the generalized horizontal lift of J associated to the connection ∇.*

We first study the dependence of $J^{G,\nabla}$ on the connection ∇.

Proposition 2.2.2. *Assume that ∇ and ∇' are two connections on (M,J). Then $J^{G,\nabla} = J^{G,\nabla'}$ if and only if the tensor $L := \nabla' - \nabla$ satisfies $L(J.,.) = L(.,J.)$.*

Proof. Let ∇ and ∇' be two connections on (M,J) and let $L \in T_2^1(M)$ be the tensor defined by $L := \nabla' - \nabla$. We notice that, considering the induced connections on (M, T^*M), we have:

$$\nabla'_X s = \nabla_X s - s(L(X,.)).$$

Moreover :

$$v^{\nabla'}(Y) = v^\nabla(Y) - \xi(L(d_\xi\pi(X),.)),$$

where $Y = (X, v^\nabla(Y)) \in T_\xi T^*M$.

A vector $Y \in T_\xi T^*M$ can be written

$$Y = (X, v^\nabla(Y))$$

in the decomposition $H_\xi^\nabla \oplus T_x^*M$ of $T_\xi T^*M$ and

$$Y = (X', v^{\nabla'}(Y))$$

in $H_\xi^{\nabla'} \oplus T_x^*M$, with $d_\xi\pi(X) = d_\xi\pi(X')$. By construction we have

$$d_\xi\pi(JX) = d_\xi\pi(JX').$$

Thus $J^{G,\nabla'} = J^{G,\nabla}$ if and only if $v^{\nabla}(J^{G,\nabla'}Y) = v^{\nabla}(J^{G,\nabla}Y)$ for every $\xi \in T^*M$ and every $Y \in T_\xi T^*M$. Let us compute $v^{\nabla}(J^{G,\nabla'}Y)$:

$$\begin{aligned} v^{\nabla}(J^{G,\nabla'}Y) &= v^{\nabla'}(J^{G,\nabla'}Y)) + \xi(L(Jd_\xi\pi(X),.)) \\ &= {}^tJ(v^{\nabla'}(Y)) + \xi(L(Jd_\xi\pi(X),.)) \\ &= {}^tJ(v^{\nabla}(Y)) - {}^tJ\xi(L(d_\xi\pi(X),.)) + \xi(L(Jd_\xi\pi(X),.)) \\ &= v^{\nabla}(J^{G,\nabla}Y) - \xi(L(d_\xi\pi(X),J.)) + \xi(L(Jd_\xi\pi(X),.)). \end{aligned}$$

So $J^{G,\nabla'} = J^{G,\nabla}$ if and only if $L(d_\xi\pi(X),J.) = L(Jd_\xi\pi(X),.)$. Since $d_\xi\pi_{|H_\xi^{\nabla}}$ is a bijection between H_ξ^{∇} and T_xM, we obtain the desired result. □

As a direct consequence of Proposition 2.2.2 we get the following Corollary:

Corollary 2.2.3. *Let ∇ and ∇' be two minimal almost complex connections. One has* $J^{G,\nabla'} = J^{G,\nabla}$.

Proof. Since ∇ and ∇' have the same torsion, the tensor $L := \nabla - \nabla'$ is symmetric. Moreover, the almost complexity of connections ∇ and ∇' leads to:

$$\begin{aligned} L(X,JY) &= \nabla_X(JY) - \nabla'_X(JY) \\ &= J(\nabla_XY - \nabla'_XY) \\ &= JL(X,Y), \end{aligned}$$

for every $X, Y \in \Gamma(TM)$. Finally we have $L(J.,.) = JL(.,.) = L(.,J.)$ providing $J^{G,\nabla'} = J^{G,\nabla}$. □

We see from Corollary 2.2.3 that minimal almost complex connections are natural connections in an almost complex manifold to construct generalized horizontal lifts of structures.

The links between the generalized horizontal lift $J^{G,\nabla}$, the complete lift $\widetilde{J}$, and the horizontal lift $J^{H,\nabla}$ are given by the following Theorem:

Theorem 2.2.4. *We have:*

1. $J^{G,\nabla} = \widetilde{J}$ *if and only if* $S = -\frac{1}{2}JN_J$, *where*

$$S(X,Y) := -(\nabla J)(X,Y) + (\nabla J)(Y,X) + T(JX,Y) - JT(X,Y), \tag{2.12}$$

2. $J^{G,\nabla} = J^{H,\nabla}$ *if and only if* $T(J.,.) = T(.,J.)$ *and,*

3. *For every almost complex and minimal connection, we have* $J^{G,\nabla} = \widetilde{J} = J^{H,\nabla}$.

2.2.4 Proof of Theorem 2.2.4

The main idea of the proof is to find a tensorial expression of the generalized horizontal structure $J^{G,\nabla}$, involving the formal complete lift J^c. In that way, we first describe locally the horizontal distribution H^∇ :

Lemma 2.2.5. *Let $x \in M$ and let $\xi \in T^*M$ be such that $\pi(\xi) = x$. We have*

$$H_\xi^\nabla = \left\{ \begin{pmatrix} X \\ \Gamma_{j,k} X^j \end{pmatrix}, X \in T_x M \right\}.$$

Proof. Let us prove that

$$H_\xi^\nabla \subseteq \left\{ \begin{pmatrix} X \\ \Gamma_{j,k} X^j \end{pmatrix}, X \in T_x M \right\}.$$

Let $Y \in H_\xi^\nabla$; Y is equal to $d_x s(X)$ where $X \in T_x M$ and s is a section of the cotangent bundle such that $\nabla_X s = 0$. Locally we set

$$\begin{cases} s &= s_i dx^i \\ X &= X^i \partial x_i, \end{cases}$$

and so:

$$Y = \begin{pmatrix} X \\ X^j \partial x_j s_i \end{pmatrix}.$$

Since $\nabla_X s = 0$ we obtain:

$$\begin{aligned} 0 &= X^j \nabla_{\partial x_j}(s_i dx^i) \\ &= X^j s_i \nabla_{\partial x_j} dx^i + X^j \partial x_j s_i dx^i \\ &= -X^j s_i \Gamma_{j,k}^i dx^k + X^j \partial x_j s_k dx^k. \end{aligned}$$

Therefore

$$X^j \partial x_j s_k = X^j s_i \Gamma_{j,k}^i = X^j \Gamma_{j,k}.$$

This proves the desired inclusion.

Moreover the following decomposition insures the equality:

$$T_\xi T^*M = \left\{ \begin{pmatrix} X \\ \Gamma_{j,k} X^j \end{pmatrix}, X \in T_x M \right\} \oplus T_x^* M.$$

□

The following proposition gives the local expression of the generalized horizontal lift which is necessary to obtain the desired tensorial expression stated in part 2.

Proposition 2.2.6.

1. *With respect to the local coordinates system* $(x_1, \cdots, x_n, p_1, \cdots, p_n)$ *on the cotangent bundle* T^*M*, the structure* $J^{G,\nabla}$ *is given by:*

$$J^{G,\nabla} = \begin{pmatrix} J^i_j & 0 \\ \Gamma_{l,i}J^l_j - \Gamma_{j,l}J^l_i & J^j_i \end{pmatrix}.$$

2. *We have*

$$J^{G,\nabla} = J^c + \gamma(S), \tag{2.13}$$

where S *is defined by (2.12).*

Proof. We first prove part 1. We denote by δ^i_j the Kronecker symbol. With respect to the local coordinates system $(x_1, \cdots, x_n, p_1, \cdots, p_n)$, the structure $J^{G,\nabla}$ is locally given by:

$$J^{G,\nabla} = \begin{pmatrix} J^i_j & 0 \\ a^i_j & J^j_i \end{pmatrix},$$

for some a^i_j we have to determine. Since $\begin{pmatrix} \delta^j_i \\ \Gamma_{i,j} \end{pmatrix} \in H^\nabla_\xi$, it follows from Lemma 2.2.5, that for every $i \in \{1, \cdots, n\}$:

$$J^{G,\nabla} \begin{pmatrix} \delta^j_i \\ \Gamma_{i,j} \end{pmatrix} = \begin{pmatrix} J^j_i \\ \Gamma_{k,j}J^k_i \end{pmatrix}.$$

Hence we have:

$$a^i_j = \Gamma_{l,i}J^l_j - \Gamma_{j,l}J^l_i.$$

This concludes the proof of part 1.

Then we prove part 2. Using the local expression of the formal complete lift J^c (see (2.7)), we get:

$$J^{G,\nabla} = J^c + \begin{pmatrix} 0 & 0 \\ -p_k \partial x_j J^k_i + p_k \partial x_i J^k_j + \Gamma_{l,i}J^l_j - \Gamma_{j,l}J^l_i & 0 \end{pmatrix}.$$

Since

$$\nabla_{\partial x_i}(J\partial x_j) = \partial x_i J^k_j \partial x_k + \Gamma^k_{i,l} J^l_j \partial x_k,$$

it follows that:

$$-p_k \partial x_j J^k_i + p_k \partial x_i J^k_j + \Gamma_{l,i}J^l_j - \Gamma_{j,l}J^l_i = p_k dx^k[-\nabla_{\partial x_j}(J\partial x_i) + \overline{\nabla}_{\partial x_i}(J\partial x_j)].$$

We define

$$\begin{aligned} S'(X,Y) &:= -\nabla_X(JY) + \overline{\nabla}_Y(JX) \\ &= -\nabla_X(JY) + \nabla_Y JX + T(JX, Y) \end{aligned}$$

and we notice that

$$S'(\partial x_i, \partial x_j) = -\nabla_{\partial x_i}(J\partial x_j) + \overline{\nabla}_{\partial x_j}(J\partial x_i).$$

We point out that S' is not a tensor. However introducing a correction term, we obtain a tensor S of type $(2,1)$:

$$\begin{aligned} S(X,Y) &:= S'(X,Y) + J[X,Y] \\ &= -\nabla_X(JY) + \nabla_Y(JX) + T(JX,Y) + J\nabla_X Y - J\nabla_Y X - JT(X,Y) \\ &= -(\nabla J)(X,Y) + (\nabla J)(Y,X) + T(JX,Y) - JT(X,Y). \end{aligned}$$

By construction of S we have:

$$S(\partial x_i, \partial x_j) = S'(\partial x_i, \partial x_j).$$

This leads to (2.13). □

Hence we may compare the three lifted structures via their intrinsic expressions given by:

$$\begin{cases} \widetilde{J} &= J^c - \frac{1}{2}\gamma(JN_J) \quad \text{(see (2.8))}, \\ J^{H,\nabla} &= J^c + \gamma([\widetilde{\nabla} J]) \quad \text{(see (2.10))}, \\ J^{G,\nabla} &= J^c + \gamma(S) \quad \text{(see (2.13))}. \end{cases}$$

The lecture of the two first expressions gives part 1 of Theorem 2.2.4.

To prove the second part of Theorem 2.2.4, we notice that:

$$\begin{aligned} [\widetilde{\nabla} J](X,Y) &= -(\widetilde{\nabla} J)(X,Y) + (\widetilde{\nabla} J)(Y,X) \\ &= -(\nabla J)(X,Y) + (\nabla J)(Y,X) + \tfrac{1}{2}T(X,JY) + \tfrac{1}{2}T(JX,Y) \\ &\quad -JT(X,Y). \end{aligned}$$

Let us prove the third part of Theorem 2.2.4. The equality $J^{G,\nabla} = \widetilde{J}$ follows from the fact that $\nabla J = 0$ because the connection ∇ is almost complex and from the fact that

$$-T(J.,.) + JT(.,) = \frac{1}{4}JN_J + \frac{1}{4}JN_J.$$

Since $T = \frac{1}{4}N_J$ and $N_J(J.,.) = N_J(.,J.)$ we finally obtain

$$J^{G,\nabla} = J^{H,\nabla}.$$

The proof of Theorem 2.2.4 is now achieved. □

We end this section with the following corollary:

Corollary 2.2.7. *We have* $J^{H,\nabla} = J^{G,\widetilde{\nabla}}$.

Proof. This is a direct consequence of Theorem 2.2.4 since $J^{H,\nabla} = J^{H,\widetilde{\nabla}}$ and $J^{G,\widetilde{\nabla}} = J^{H,\widetilde{\nabla}}$ by part 2. □

We point out that Corollary 2.2.7 may also be proved using Lemma 2.2.5 and the distribution D of horizontal lifted vectors defined by S.Ishihara and K.Yano [44] as follows: let $x \in M$ and $\xi \in T^*M$ such that $\pi(\xi) = x$. Assume $X^{H,\nabla}$ is the horizontal lift of a tangent vector $X \in T_xM$ on the cotangent bundle defined by:

$$X^{H,\nabla} = \begin{pmatrix} X \\ \widetilde{\Gamma}_{j,k} X^j \end{pmatrix} \in T_\xi T^*M.$$

Then the distribution D of horizontal lifted vectors is defined by

$$D_\xi = \left\{ X^{H,\nabla}, X \in T_xM \right\}.$$

S.Ishihara and K.Yano proved that $J^{H,\nabla} = J \oplus {}^tJ$ in the decomposition $T_\xi T^*M = D_\xi \oplus T_x^*M$. From Lemma 2.2.5 we have $D = H^{\widetilde{\nabla}}$ and finally

$$J^{H,\nabla} = J \oplus {}^tJ = J^{G,\widetilde{\nabla}}$$

with respect to the decomposition $T_\xi T^*M = D_\xi \oplus T_x^*M = H_\xi^{\widetilde{\nabla}} \oplus T_x^*M$.

2.3 Geometric properties of the generalized horizontal lift

2.3.1 Lift properties

In Theorem 2.3.1 we state the lift properties of the generalized horizontal lift of an almost complex structure.

Theorem 2.3.1.

1. *The projection* $\pi : T^*M \longrightarrow M$ *is* $(J^{G,\nabla}, J)$*-holomorphic.*
2. *The zero section* $s : M \longrightarrow T^*M$ *is* $(J, J^{G,\nabla})$*-holomorphic.*
3. *The lift of a diffeomorphism* $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ *to the cotangent bundle is* $(J_1^{G,\nabla_1}, J_2^{G,\nabla_2})$*-holomorphic if and only if* f *is a* (J_1, J_2)*-holomorphic map satisfying* $f_* S_1 = S_2$.

We recall that the lift $\widetilde{f}$ of a diffeomorphism $f : M_1 \longrightarrow M_2$ to the cotangent bundle is defined by

$$\widetilde{f} := (f, {}^t(df)^{-1}).$$

Its differential $d\widetilde{f}$ is locally given by:

$$d\widetilde{f} = \begin{pmatrix} df & 0 \\ (*) & {}^t(df)^{-1} \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}), \tag{2.14}$$

where $(*)$ denotes a $(n \times n)$ block of derivatives of f with respect to $(x_1, \cdots, x_n)$.

Proof of Theorem 2.3.1. Parts 1 and 2 are consequences of the first part of Proposition 2.2.6.

Let us prove the third part. Assume that $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ is a (J_1, J_2)-holomorphic diffeomorphism satisfying $\widetilde{f}_* S_1 = S_2$, where S_i is defined by (2.12) for $i = 1, 2$, and denote by $\widetilde{f}$ its lift to the cotangent bundle T^*M. According to Proposition 2.2.6, we have

$$J^{G,\nabla_i} = J^c + \gamma(S_i)$$

for $i = 1, 2$. We denote by θ_i and $\omega_{i,st}$ the Liouville form and the canonical symplectic form of T^*M_i. The invariance by lifted diffeomorphisms of these forms insures that

$$\begin{cases} \widetilde{f}_*\theta_1 = \theta_2 \\ \\ \widetilde{f}_*\omega_{1,st} = \omega_{2,st}. \end{cases}$$

We also recall that

$${}^t(\theta_i(S_i)) = -\omega_{i,st}(.,\gamma(S_i).).$$

Let us establish that

$$\widetilde{f}_*(J_1^{G,\nabla_1}) = J_2^{G,\nabla_2}.$$

The first step consists in proving that the direct image of J_1^c by $\widetilde{f}$ is J_2^c. By the nondegeneracy of $\omega_{2,st}$, it is equivalent to obtain

$$\omega_{2,st}(\widetilde{f}_* J_1^c.,.) = \omega_{2,st}(J_2^c.,.).$$

We compute

$$\begin{aligned} \omega_{2,st}(\widetilde{f}_* J_1^c.,.) &= \omega_{2,st}(d\widetilde{f} \circ J_1^c \circ (d\widetilde{f})^{-1}.,.) \\ \\ &= \omega_{1,st}(J_1^c \circ (d\widetilde{f})^{-1}.,(d\widetilde{f})^{-1}) \\ \\ &= \widetilde{f}_*(\omega_{1,st}(J_1^c.,.)) \\ \\ &= \widetilde{f}_* d(\theta_1(J_1)) \end{aligned}$$

and

$$\omega_{2,st}(J_2^c.,.) = d(\theta_2(J_2)).$$

Thus we need to prove that the pull-back of $\theta_2(J_2)$ by $\widetilde{f}$ is equal to $\theta_1(J_1)$. According to the local expression of $d\widetilde{f}$ (see (2.14)), we have

$$\widetilde{f}^*(\theta_2(J_2)) = \theta_2(J_2 \circ df)$$

and so:

$$\widetilde{f}^*(\theta_2(J_2)) = \theta_2(df \circ J_1) = (\widetilde{f}^*\theta_2)(J_1) = \theta_1(J_1).$$

Thus we obtain

$$\widetilde{f}_* d(\theta_1(J_1)) = d(\theta_2(J_2)),$$

that is

$$\widetilde{f}_* J_1^c = J_2^c.$$

To show the desired result, it remains to prove that the direct image of $\gamma(S_1)$ by $\widetilde{f}$ is $\gamma(S_2)$. We prove more generally that $f_*(S_1) = S_2$ if and only if $\widetilde{f}_*(\gamma(S_1)) = \gamma(S_2)$ which is equivalent to prove that $f_*(S_1) = S_2$ if and only if $\omega_{2,st}(.,\widetilde{f}_*(\gamma(S_1)).) = \omega_{2,st}(.,\gamma(S_2).)$. We have:

$$\begin{aligned}
\omega_{2,st}(.,\widetilde{f}_*\gamma(S_1).) &= \omega_{2,st}(.,d\widetilde{f}\circ\gamma(S_1)\circ(d\widetilde{f})^{-1}.) \\
&= \omega_{1,st}((d\widetilde{f})^{-1}.,\gamma(S_1)\circ(d\widetilde{f})^{-1}.,) \\
&= \widetilde{f}_*(\omega_{1,st}(.,\gamma(S_1).)).
\end{aligned}$$

Due to (2.3), this leads to

$$\omega_{2,st}(.,\widetilde{f}_*\gamma(S_1).) = -\widetilde{f}_*({}^t\theta_1(S_1)).$$

Let us check that $f_*(S_1) = S_2$ if and only if $\widetilde{f}_*{}^t(\theta_1(S_1)) = {}^t(\theta_2(S_2))$. We have:

$$\begin{cases}
\widetilde{f}^*(\theta_2(S_2)) = \theta_2(S_2(df,df)) \\
\theta_1(S_1) = (\widetilde{f}^*\theta_2)(S_1) = \theta_2(df\circ S_1).
\end{cases}$$

According to this fact and to (2.2), it follows that $f_*S_1 = S_2$ if and only if $\theta_2(S_2(df,df)) = \theta_2(df\circ S_1)$. So $f_*(S_1) = S_2$ if and only if $\widetilde{f}_*(\gamma(S_1)) = \gamma(S_2)$. Finally we have proved that if $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ is a (J_1, J_2)-holomorphic diffeomorphism satisfying $f_*S_1 = S_2$ then $\widetilde{f}$ is $(J_1^{G,\nabla_1}, J_2^{G,\nabla_2})$-holomorphic.

Reciprocally if $\widetilde{f}$ is $(J_1^{G,\nabla_1}, J_2^{G,\nabla_2})$-holomorphic then f is (J_1, J_2)-holomorphic. Indeed the zero section $s_1 : M_1 \longrightarrow T^*M_1$ is (J_1, J_1^{G,∇_1})-holomorphic by part 2 of Theorem 2.3.1, the projection $\pi_2 : T^*M_2 \longrightarrow M_2$ is (J_2^{G,∇_2}, J_2)-holomorphic by part 1 of Theorem 2.3.1 and we have $f = \pi_2\circ\widetilde{f}\circ s_1$. Since f is (J_1, J_2)-holomorphic we have

$$\widetilde{f}_* J_1^c = J_2^c.$$

Then the $(J_1^{G,\nabla_1}, J_2^{G,\nabla_2})$-holomorphy of $\widetilde{f}$ implies the equality

$$\widetilde{f}_*(\gamma(S_1)) = \gamma(S_2),$$

that is

$$f_*S_1 = S_2.$$

□

As a corollary, we obtain the lift properties of the complete and the horizontal lifts by considering special connections. We point out that Theorem 2.3.1 and Corollary 2.3.2 characterize the complete lift via the lift of diffeomorphisms.

Corollary 2.3.2.

1. *The lift of a diffeomorphism $f : (M_1, J_1) \longrightarrow (M_2, J_2)$ to the cotangent bundle is $(\widetilde{J}_1, \widetilde{J}_2)$-holomorphic if and only if f is (J_1, J_2)-holomorphic.*
2. *The lift of a diffeomorphism $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ to the cotangent bundle is $(J_1^{H,\nabla_1}, J_2^{H,\nabla_2})$-holomorphic if and only if f is a (J_1, J_2)-holomorphic map satisfying $f_*[\widetilde{\nabla_1 J_1}] = [\widetilde{\nabla_2 J_2}]$.*

Proof. To prove part 1, we consider almost complex and minimal connections ∇_1 and ∇_2 on M_1 and M_2. Hence

$$\widetilde{J}_k = J^{G,\nabla_k} = J_k^c + \gamma(S_k)$$

for $k = 1, 2$. Moreover we have

$$S_k = -\frac{1}{2} J_k N_{J_k}$$

for $k = 1, 2$. We notice that if $f : (M_1, J_1) \longrightarrow (M_2, J_2)$ is a (J_1, J_2)-holomorphic diffeomorphism then

$$f_* N_{J_1} = N_{J_2}$$

and so

$$f_* J_1 N_{J_1} = J_2 N_{J_2}.$$

According to Theorem 2.3.1, the lift of a diffeomorphism f to the cotangent bundle is $(\widetilde{J}_1, \widetilde{J}_2)$-holomorphic if and only if f is (J_1, J_2)-holomorphic.

Finally, part 2 follows from the equality $J^{G,\widetilde{\nabla}} = J^{H,\nabla}$ obtained in Corollary 2.2.7 and from Theorem 2.3.1.

□

We point out that the projection (resp. the zero section) is (J', J)-holomorphic (resp (J, J')-holomorphic) for $J' = \widetilde{J}, J^{H,\nabla}$ due to local expressions of the complete lift and of the horizontal lift (see (2.9) and (2.11)).

2.3.2 Fiberwise multiplication

We consider the multiplication map $Z : T^*M \longrightarrow T^*M$ by a complex number $a + ib$ with $b \neq 0$ on the cotangent bundle. This is locally defined by

$$Z(x, p) := (x, (a + b\,{}^tJ(x))p).$$

For $(x, p) \in T^*M$ we have

$$d_{(x,p)}Z = \begin{pmatrix} Id & 0 \\ C & aId + b\,{}^tJ \end{pmatrix},$$

where

$$C_j^i = bp_k \partial x_j J_i^k.$$

Theorem 2.3.3. *The multiplication map Z is $J^{G,\nabla}$-holomorphic if and only if $(\nabla J)(J.,.) = (\nabla J)(.,J.)$.*

Proof. Let us evaluate

$$d_{(x,p)}Z \circ J^{G,\nabla}(x,p) - J^{G,\nabla}(x, ap + b^t Jp) \circ d_{(x,p)Z}.$$

This is equal to:

$$\begin{pmatrix} 0 & 0 \\ CJ + (aId + b^t J)B(x,p) - B(x, ap + {}^t Jp) - {}^t JC & 0 \end{pmatrix},$$

where

$$B_j^i(x,p) = p_k(\Gamma_{l,i}^k J_j^l - \Gamma_{j,l}^k J_i^l).$$

We first notice that

$$aB_j^i(x,p) - B_j^i(x, ap + b^t Jp) = -bp_k J_s^k(\Gamma_{l,i}^s J_j^l - \Gamma_{j,l}^s J_i^l).$$

Let us compute

$$D := CJ + (aId + b^t J)B(x,p) - B(x, ap + {}^t Jp) - {}^t JC.$$

We have:

$$D_j^i = bp_k[\underbrace{J_j^l \partial x_l J_i^k}_{(1)} + \underbrace{J_i^l \Gamma_{s,l}^k J_j^s}_{(2)} - \underbrace{J_i^l \Gamma_{j,s}^k J_l^s}_{(2)'} - \underbrace{J_s^k \Gamma_{l,i}^s J_j^l}_{(3)} + \underbrace{J_s^k \Gamma_{j,l}^s J_i^l}_{(3)'} - \underbrace{J_i^l \partial x_j J_l^k}_{(1)'}].$$

We obtain

$$\begin{cases} (1) + (2) + (3) & = & J_j^l(\partial x_l J_i^k + J_i^s \Gamma_{l,s}^k - J_s^k \Gamma_{l,i}^s), \\ (1)' + (2)' + (3)' & = & J_i^l(\partial x_j J_l^k + J_l^s \Gamma_{j,s}^k - J_s^k \Gamma_{j,l}^s). \end{cases}$$

We recognize the coordinates of the tensor ∇J (see (2.5)):

$$\begin{cases} \partial x_l J_i^k - J_s^k \Gamma_{l,i}^s + J_i^s \Gamma_{l,s}^k & = & (\nabla J)_{l,i}^k, \\ \partial x_j J_l^k - J_s^k \Gamma_{j,l}^s + J_l^s \Gamma_{j,s}^k & = & (\nabla J)_{j,l}^k. \end{cases}$$

Finally we obtain

$$D_j^i = bp_k[J_j^l(\nabla J)_{l,i}^k - J_i^l(\nabla J)_{j,l}^k].$$

Then Z is $J^{H,\nabla}$-holomorphic if and only if $J_j^l(\nabla J)_{l,i}^k = (\nabla J)_{j,l}^k J_i^l$. Since

$$(\nabla J)_{j,l}^k J_i^l \partial x_k = (\nabla J)(\partial x_j, J\partial x_i)$$

and since

$$J_j^l(\nabla J)_{l,i}^k \partial x_k = (\nabla J)(J\partial x_j, \partial x_i),$$

this concludes the proof of Theorem 2.3.3. ∎

In particular, the almost complex lift $\widetilde{J}$ may be characterized generically by the holomorphy of Z; more precisely we have:

Corollary 2.3.4.

1. *The multiplication map Z is $\widetilde{J}$-holomorphic and,*
2. *Z is $J^{H,\nabla}$-holomorphic if and only if $(\widetilde{\nabla}J)(J.,.) = (\widetilde{\nabla}J)(.,J.)$.*

Proof. Let us prove part 1. Assume ∇ is an almost complex minimal connection on M. We have $\widetilde{J} = J^{G,\nabla}$ and by almost complexity of ∇, ∇J is identically equal to zero. Theorem 2.3.3 implies the $\widetilde{J}$-holomorphy of Z.

Part 2 follows from Theorem 2.3.3 and from the equality $J^{H,\nabla} = J^{G,\widetilde{\nabla}}$ stated in Corollary 2.2.7.

□

Remark 2.3.5. *In the case of the tangent bundle TM, the fiberwise multiplication is holomorphic for the complete lift of J if and only if J is integrable. More precisely, the lack of pseudoholomorphy of this map is measured by the Nijenhuis tensor (see [49] and [52]).*

2.4 Compatible lifted structures and symplectic forms

Assume (M, J) is an almost complex manifold. Let $\Gamma = \{\rho = 0\}$ be a real smooth hypersurface of M, where $\rho : M \to \mathbb{R}$ is a defining function of Γ.

Definition 2.4.1.

1. *A submanifold N of a symplectic manifold (M', ω') is called Lagrangian for ω' if $\omega'(X, Y) = 0$ for every $X, Y \in \Gamma(TN)$.*
2. *A submanifold N of an almost complex manifold (M', J') is totally real if $TN \cap J(TN) = \{0\}$.*

For $x \in \Gamma$ we define the *conormal space*

$$N_x^*(\Gamma) := \{p_x \in T_x^*M, (p_x)_{|T_x\Gamma} = 0\}$$

The *conormal bundle* over Γ, defined by the disjoint union

$$N^*(\Gamma) := \bigcup_{x\in\Gamma} N_x^*(\Gamma),$$

is a totally real submanifold of T^*M endowed with the complete lift (see [69], [35] and [66]). In order to look for a symplectic proof of this fact, we search for a symplectic form, ω', compatible with the complete lift for which $N^*(\Gamma)$ is Lagrangian. More generally we are interested in the compatibility with the generalized horizontal lift. Proposition 2.4.2 states that one cannot find such a form.

Proposition 2.4.2. *Assume (M, J, ∇) is an almost complex manifold equipped with a connection. Let ω be a symplectic form on T^*M compatible with the generalized horizontal lift $J^{G,\nabla}$. There is no strictly pseudoconvex hypersurface in M whose conormal bundle is Lagrangian with respect to ω.*

Proof. Let Γ be a strictly pseudoconvex hypersurface in M and let $x \in \Gamma$. Since the problem is purely local we can suppose that $M = \mathbb{R}^{2m}$, $J = J_{st} + O|(x_1, \cdots, x_{2m})|$ and $x = 0$. Since Γ is strictly pseudoconvex we can also suppose that

$$T_0\Gamma = \{X \in \mathbb{R}^{2m}, X_1 = 0\}.$$

The two-form ω is given by

$$\omega = \alpha_{i,j} dx^i \wedge dx^j + \beta_{i,j} dp^i \wedge dp^j + \gamma_{i,j} dx^i \wedge dp^j.$$

Assume that $\omega(X, Y) = 0$ for every $X, Y \in TN^*(\Gamma)$. We have

$$\begin{aligned} N_0^*(\Gamma) &= \{p_0 \in T_0^*\mathbb{R}^{2m}, (p_0)_{|T_0\Gamma} = 0\} \\ &= \{(P_1, 0, \cdots, 0), P_1 \in \mathbb{R}\}. \end{aligned}$$

Then a vector $Y \in T_0N^*(\Gamma)$ can be written

$$Y = X_2 \partial x_2 + \cdots + X_{2m} \partial x_{2m} + P_1 \partial p_1.$$

So we have for $2 \leq i < j \leq 2m$:

$$\omega(0)(\partial x_i, \partial x_j) = \alpha_{i,j} = 0.$$

Then $\omega(0)$ is given by

$$\omega(0) = \alpha_{1,j} dx^1 \wedge dx^j + \beta_{i,j} dp^i \wedge dp^j + \gamma_{i,j} dx^i \wedge dp^j.$$

Since r

$$J^{G,\nabla}(0) = \begin{pmatrix} J_{st} & 0 \\ 0 & J_{st} \end{pmatrix}$$

we have

$$J^{G,\nabla}(0)Y' = \partial x_{2m}$$

for $Y' = \partial x_{2m-1} \neq 0 \in T_0(T^*\Gamma)$. Thus

$$\omega(0)(Y', J^{G,\nabla}(0)Y') = 0$$

and so ω is not compatible with $J^{G,\nabla}$.

□

Proposition 2.4.2 is also established for complete and horizontal lifts because $J^{G,\nabla}(0) = \widetilde{J}(0) = J^{H,\nabla}(0)$.

Remark 2.4.3. *Since the conormal bundle of a (strictly pseudoconvex) hypersurface is Lagrangian for the symplectic form ω_{st} on T^*M, Proposition 2.4.2 shows that ω_{st} and $J^{G,\nabla}$ are not compatible.*

Chapter 3

Pseudoconvex regions of finite D'Angelo type in four dimensional almost complex manifolds

This chapter follows [11].

Résumé Soit D une région J-pseudoconvexe dans une variété presque complexe (M,J) de dimension quatre. Nous construisons une fonction locale pic J-plurisubharmonique en tout point $p \in bD$ de type de D'Angelo fini. Nous montrons ensuite des estimées de la pseudométrique de Kobayashi, impliquant l'hyperbolicité locale au sens de Kobayashi du domaine D en p. Lorsque le point $p \in \partial D$ est de type de D'Angelo inférieur ou égal à quatre, ou lorsque nous privilégions une approche non tangentielle, nous donnons des estimées précises de la pseudométrique de Kobayashi.

Abstract Let D be a J-pseudoconvex region in a smooth almost complex manifold (M,J) of real dimension four. We construct a local peak J plurisubharmonic function at every point $p \in bD$ of finite D'Angelo type. As applications we give local estimates of the Kobayashi pseudometric, implying the local Kobayashi hyperbolicity of D at p. In case the point p is of D'Angelo type less than or equal to four, or the approach is nontangential, we provide sharp estimates of the Kobayashi pseudometric.

Introduction

In the present chapter we study the behaviour of the Kobayashi pseudometric of a J-pseudoconvex region of finite D'Angelo type in an almost complex manifold (M,J) of dimension four. Finite D'Angelo type appeared naturally in complex manifolds when considering the boundary behaviour of the $\overline{\partial}$ operator (see [25],[24],[47],[15]). Moreover on complex manifolds of dimension two, the D'Angelo type unifies many type conditions as the finite regular type. Finite regular type was recently characterized intrinsically by J.-F.Barrault-E.Mazzilli [4] by means of Lie brackets, which generalizes in the non integrable case, a result of T.Bloom-I.Graham [15].

Our main result is the construction of a local peak J-plurisubharmonic function on pseudoconvex regions provided by theorem A3 (see also Theorem 3.1.7):

Theorem A3. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in an almost complex manifold (M, J) of dimension four. We suppose that ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Let $p \in \partial D$ be a boundary point. Then there exists a local peak J-plurisubharmonic function at p.*

Theorem A3 allows to localize pseudoholomorphic discs and to obtain lower estimates of the Kobayashi pseudometric which provide the local Kobayashi hyperbolicity of J-pseudoconvex regions of D'Angelo type $2m$ (Proposition 3.2.2 and Proposition 3.2.8). As an application we prove the $1/2m$-Hölder extension of biholomorphisms up to the boundary (Proposition 3.2.7). In order to obtain sharp lower estimates of the Kobayashi pseudometric similar to those given in complex manifolds by D.Catlin [17] (see also [8]), we consider a natural scaling method. However this reveals the fact that for a domain of finite D'Angelo type greater than four, the sequence of almost complex structures obtained by any polynomial scaling process does not converge generically to the standard structure; this is presented in the Appendix. This may be related to the fact that finite D'Angelo type is based on purely complex considerations, as the boundary behaviour of the Cauchy-Riemann equations. Hence we provide sharp lower estimates of the Kobayashi pseudometric for a region of finite D'Angelo type four (see also Theorem 3.3.1):

Theorem B3. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type less than or equal to four in an almost complex manifold (M, J) of dimension four, where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Then there is a positive constant C with the following property: for every $p \in D$ and every $v \in T_pM$ there exists a diffeomorphism Φ_{p^*} in a neighborhood U of p, such that:*

$$K_{(D,J)}(p, v) \geq C \left(\frac{|\,(d_p\Phi_{p^*}v)_1\,|}{|\rho(p)\,|^{\frac{1}{4}}} + \frac{|\,(d_p\Phi_{p^*}v)_2\,|}{|\rho(p)\,|} \right).$$

We point out that the approach we use, based on some renormalization principle of pseudoholomorphic discs, gives also a different proof of precise lower estimates obtained by H.Gaussier-A.Sukhov in [35] for strictly J-pseudoconvex domains in arbitrary dimension. As an application of Theorem B3, we obtain the (local) complete hyperbolicity of J-pseudoconvex regions of D'Angelo type less than or equal to four (Corollary 3.3.5) and we give a Wong-Rosay theorem for regions with noncompact automorphisms group (Corollary 3.3.6).

Finally, in order to obtain precise estimates near a point of arbitrary finite D'Angelo type, we are interested in the nontangential behaviour of the Kobayashi pseudometric (see also Theorem 3.3.9):

Theorem C3. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in an almost complex manifold (M, J) of dimension four, where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Let $q \in \partial D$ be a boundary point of D'Angelo type $2m$ and let $\Lambda \subset D$ be a cone with vertex at q and axis the inward normal axis. Then there exists a positive constant C such that for every $p \in D \cap \Lambda$ and every $v = v_n + v_t \in T_pM$:*

$$K_{(D,J)}(p,v) \geq C\left(\frac{|v_n|}{|\rho(p)|^{\frac{1}{2m}}} + \frac{|v_t|}{|\rho(p)|}\right),$$

where v_n and v_t are the normal and the tangential parts of v with respect to q.

3.1 Construction of a local peak plurisubharmonic function

This section is devoted to the proof of Theorem A3 (see Theorem 3.1.7).

3.1.1 Pseudoconvex regions of finite D'Angelo type

In this subsection we describe a pseudonconvex region on a neighborhood of a boundary point of finite D'Angelo type. We point out that all our considerations are purely local. Assume that $D = \{\rho < 0\}$ is a J-pseudoconvex region in $\mathbb{C}^2$ and that the structure J is defined on a fixed neighborhood U of $\overline{D}$. We suppose that the origin is a boundary point of D.

Definition 3.1.1. *Let $u : (\Delta, 0) \to (\mathbb{R}^4, 0, J)$ be a J-holomorphic disc satisfying $u(0) = 0$. The order of contact $\delta_0(\partial D, u)$ with ∂D at the origin is the degree of the first term in the Taylor expansion of $\rho \circ u$. We denote by $\delta(u)$ the multiplicity of u at the origin.*

We now define the D'Angelo type and the regular type of the real hypersurface ∂D at the origin.

Definition 3.1.2.

1. *The D'Angelo type of ∂D at the origin is defined by:*

$$\Delta^1(\partial D, p) := \sup\left\{\frac{\delta_p(\partial D, u)}{\delta(u)},\ u : \Delta \to \left(\mathbb{R}^4, J\right) \text{ } J\text{-holomorphic nonconstant}, \quad u(0) = p\right\},$$

2. *The regular type of ∂D at origin is defined by:*

$$\Delta^1_{\text{reg}}(\partial D, 0) := \sup\{\delta_0(\partial D, u),\ u : \Delta \to \left(\mathbb{R}^4, J\right) \text{ } J\text{-holomorphic}, \quad u(0) = 0, d_0u \neq 0\}.$$

Since the regular type of ∂D at the origin consists in considering only regular discs we have:

$$\Delta^1_{\text{reg}}(\partial D, 0) \leq \Delta^1(\partial D, 0). \tag{3.1}$$

The type condition as defined in part 1 of Definition 3.1.2 was introduced by J.-P.D'Angelo [25], [24] who proved that this coincides with the regular type in complex manifolds of dimension two. After Proposition 3.1.3, we will also prove that the D'Angelo type and the regular type coincide in four dimensional almost complex manifolds (see Proposition 3.1.5).

We suppose that the origin is a point of finite regular type. Then let $u : \Delta \to \mathbb{R}^4$ be a regular J-holomorphic disc of maximal contact order $2m$. We choose coordinates such that u is given by $u(\zeta) = (\zeta, 0)$, $J(z_1, 0) = J_{st}$ and such that the complex tangent space $T_0\partial D \cap J(0)T_0\partial D$ is equal to $\{z_2 = 0\}$. Then by considering the family of vectors $(1,0)$ at base points $(0,t)$ for $t \neq 0$ small enough, we obtain a family of J holomorphic discs u_t such that $u_t(0) = (0,t)$ and $d_0u_t(\partial/\partial_x) = (0,1)$. Due to the parameters dependence of the solution to the J-holomorphy equation (1.1), we straighten these discs into the complex lines $\{z_2 = t\}$. We then consider a transversal foliation by J-holomorphic discs and straighten these lines into $\{z_1 = c\}$. In these new coordinates still denoted by z, the matricial representation of J is diagonal:

$$J = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & -a_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & -a_2 \end{pmatrix}. \tag{3.2}$$

Since $J(z_1, 0) = J_{st}$ we have

$$J = J_{st} + O(|z_2|). \tag{3.3}$$

In the next fundamental proposition we describe precisely the local expression of the defining function ρ.

Proposition 3.1.3. *The J-plurisubharmonic defining function for the domain D has the following local expression:*

$$\rho = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + \widetilde{H}(z_1, z_2) + O\left(|z_1|^{2m+1} + |z_2||z_1|^m + |z_2|^2\right)$$

where H_{2m} is a homogeneous polynomial of degree $2m$, subharmonic which is not harmonic and

$$\widetilde{H}(z_1, z_2) = \Re e \sum_{k=1}^{m-1} \rho_k z_1^k z_2.$$

Before proving Proposition 3.1.3, we establish the following lemma.

Lemma 3.1.4. *Assume that J is a diagonal almost complex structure on $\mathbb{R}^4$ that coincides with the standard structure J on $\mathbb{C} \times \{0\}$. To fix notations we suppose that J satisfies (3.2). Then the Levi form of some smooth real valued function f at a point (z_1, z_2) and $v = (1,0,0,0)$ is equal to*

$$\mathcal{L}_J f\,(z, v) = -c_1 \Delta_1 f + O\left(|z_2|\right)).$$

where $\Delta_1 f := \dfrac{\partial^2 f}{\partial x_1 \partial x_1} + \dfrac{\partial^2 f}{\partial y_1 \partial y_1}$.

Proof. Let us compute the Levi form of some smooth real valued function f at a point (z_1, z_2) and $v = (1,0,0,0)$:

$$\begin{aligned} c_1^{-1} \mathcal{L}_J f\,(z, v) &= -\Delta_1 f + \left[-2 \frac{\partial^2 f}{\partial x_1 \partial y_1} a_1 + \frac{\partial^2 f}{\partial x_1 \partial x_1} (1 + b_1) + \frac{\partial^2 f}{\partial y_1 \partial y_1} (c_1 - 1) \right] + \\ & \quad \frac{\partial f}{\partial x_1} \left[\frac{\partial b_1}{\partial x_1} - \frac{\partial a_1}{\partial y_1} \right] + \frac{\partial f}{\partial y_1} \left[\frac{\partial a_1}{\partial x_1} + \frac{\partial c_1}{\partial y_1} \right] \\ &= -\Delta_1 f + \left[-2 \frac{\partial^2 f}{\partial x_1 \partial y_1} O\left(|z_2|\right) + \frac{\partial^2 f}{\partial x_1 \partial x_1} O\left(|z_2|\right) + \frac{\partial^2 f}{\partial y_1 \partial y_1} O\left(|z_2|\right) \right] + \\ & \quad \frac{\partial f}{\partial x_1} O\left(|z_2|\right) + \frac{\partial f}{\partial y_1} O\left(|z_2|\right) \\ &= -\Delta_1 f + O\left(|z_2|\right). \end{aligned}$$

□

Proof of Proposition 3.1.3. Since $T_0 \partial D \cap J(0) T_0 \partial D = \{z_2 = 0\}$, we have

$$\rho = \Re e z_2 + O(\|z\|^2).$$

Moreover the disc $\zeta \mapsto (\zeta, 0)$ being a regular J-holomorphic disc of maximal contact order $2m$, the defining function ρ has the following local expression:

$$\rho = \Re e z_2 + H_{2m}\left(z_1, \overline{z_1}\right) + O\left(|z_1|^{2m+1} + |z_2| \|z\|\right),$$

where H_{2m} is a homogeneous polynomial of degree $2m$.

We prove that the polynomial H_{2m} is subharmonic using a standard dilation argument. Consider the non-isotropic dilation of $\mathbb{C}^2$

$$\Lambda_\delta\left(z_1, z_2\right) := \left(\delta^{-\frac{1}{2m}} z_1, \delta^{-1} z_2\right).$$

Due to Proposition 1.3.1, the domain

$$\Lambda_\delta(D) = \{\delta^{-1}\left(\rho \circ \Lambda_\delta^{-1}\left(z_1, z_2\right)\right) < 0\}$$

is $(\Lambda_\delta)_*(J)$-pseudoconvex. Moreover $\Lambda_\delta(D)$ converges in the sense of local Hausdorff set convergence to

$$\tilde{D} := \{Re(z_2) + H_{2m}(z_1, \overline{z_1}) < 0\},$$

as δ tends to zero and the sequence of structures $(\Lambda_\delta)_* J$ converges to the standard structure J_{st}. It follows that the limit domain $\tilde{D}$ is J_{st}-pseudoconvex implying that H_{2m} is subharmonic.

Now we prove that H_{2m} contains a nonharmonic part. By contradiction, we assume that H_{2m} is harmonic. Then H_{2m} can be written $\Re e z_1^{2m}$. According to Proposition 1.1 of [45], and since the structure J is smooth there exists, for a sufficiently small $\lambda > 0$, a pseudoholomorphic disc $u : \Delta \to (\mathbb{R}^4, J)$ such that:

$$\begin{cases} u(0) &= 0 \\ \dfrac{\partial u}{\partial x}(0) &= \left(\lambda^{\frac{1}{2m}}, 0, 0, 0\right) \\ \dfrac{\partial^k u}{\partial x^k}(0) &= (0,0,0,0)\,, \text{ for } 1 < k < 2m \\ \dfrac{\partial^{2m} u}{\partial x^{2m}}(0) &= (0, 0, -\lambda(2m)!, 0)\,. \end{cases}$$

We prove that the contact order of such a regular disc u is greater than $2m$ which contradicts the fact that D is of regular type $2m$. We denote by $[\rho \circ u]_{2m}$ the homogeneous part of degree $2m$ in the Taylor expansion of $\rho \circ u$ at the origin:

$$[\rho \circ u]_{2m}(x, y) = \sum_{k=0}^{2m} a_k x^k y^{2m-k}.$$

Let us prove that $a_k = \dfrac{\partial^k}{\partial x^k}\dfrac{\partial^{2m-k}}{\partial y^{2m-k}}\rho \circ u(0)$ is equal to zero for each $0 \leq k \leq 2m$.

For a_{2m}, we have:

$$\begin{aligned} \frac{\partial^{2m}}{\partial x^{2m}}\rho \circ u(0) &= \Re e\frac{\partial^{2m}}{\partial x^{2m}}u_2(0) + \Re e\frac{\partial^{2m}}{\partial x^{2m}}u_1^{2m}(0) \\ &= -\lambda(2m)! + \Re e\frac{\partial^{2m}}{\partial x^{2m}}u_1^{2m}(0)\,. \end{aligned}$$

Since $u_1(0) = 0$, it follows that the only non vanishing term in $\Re e\dfrac{\partial^{2m}}{\partial x^{2m}}u_1^{2m}(0)$ is

$$(2m)!\Re e\left(\frac{\partial u_1}{\partial x}(0)\right)^{2m} = \lambda(2m)!.$$

This proves that $a_{2m} = 0$.

Then let $0 \leq k < 2m$:

$$\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}\rho \circ u\,(0) \quad = \quad \Re e\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}u_2\,(0) + \Re e\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}u_1^{2m}\,(0)\,.$$

For the same reason as previously, the only term to consider in $\Re e\dfrac{\partial^k}{\partial x^k}\dfrac{\partial^{2m-k}}{\partial y^{2m-k}}u_1^{2m}\,(0)$ is

$$(2m)!\Re e\left(\frac{\partial}{\partial x}u_1\,(0)\right)^k\left(\frac{\partial}{\partial y}u_1\,(0)\right)^{2m-k} = \lambda^{\frac{k}{2m}}\,(2m)!\Re e\left(\frac{\partial u_1}{\partial y}\,(0)\right)^{2m-k}.$$

Then, since u is J-holomorphic, it satisfies the diagonal J-holomorphy equation:

$$\frac{\partial u_l}{\partial y} = J_l\,(u)\,\frac{\partial u_l}{\partial x},$$

for $l = 1, 2$, where

$$J_l = \begin{pmatrix} a_l & b_l \\ c_l & -a_l \end{pmatrix} \text{ (see (3.2) for notations).}$$

It follows that

$$\begin{aligned}\lambda^{\frac{k}{2m}}\,(2m)!\Re e\left(\frac{\partial u_1}{\partial y}\,(0)\right)^{2m-k} \quad &= \quad \lambda^{\frac{k}{2m}}\,(2m)!\Re e\left(J_1\,(u\,(0))\,\frac{\partial u_1}{\partial x}\,(0)\right)^{2m-k} \\ &= \quad \lambda\,(2m)!\Re e\,(i)^{2m-k}\,.\end{aligned}$$

Moreover due to the condition $\dfrac{\partial^k u_2}{\partial x^k}\,(0) = (0,0)$, for $1 \leq k < 2m$, it follows that the only part we need to consider in $\dfrac{\partial^{2m-k}}{\partial y^{2m-k}}u_2\,(0)$ is $J_2\,(u)\,\dfrac{\partial}{\partial x}\dfrac{\partial^{2m-k-1}}{\partial y^{2m-k-1}}u_2\,(0)$ and by induction $(J_2\,(u))^{2m-k}\,\dfrac{\partial^{2m-k}}{\partial x^{2m-k}}u_2\,(0)$. Finally

$$\begin{aligned}\Re e\frac{\partial^k}{\partial x^k}\frac{\partial^{2m-k}}{\partial y^{2m-k}}u_2\,(0) \quad &= \quad \Re e\,(J_2\,(u\,(0)))^{2m-k}\,\frac{\partial^{2m}u_2}{\partial x^{2m}}\,(0) \\ &= \quad -\lambda\,(2m)!\Re e\left(J_2\,(u\,(0))^{2m-k}\,(1,0)\right) \\ &= \quad -\lambda\,(2m)!\Re e\,(i)^{2m-k}\,.\end{aligned}$$

This proves that the homogeneous part $[\rho \circ u]_{2m}$ is equal to zero.

For smaller order terms it is a direct consequence of $u(0) = 0$ and $\frac{\partial^k u}{\partial x^k}(0) = (0,0,0,0)$, for $1 < k < 2m$.

It remains to prove there are no term $\Re e \rho_{\overline{k}} z_1^k \overline{z_2}$ with $k < m$ in the defining function ρ. This is done by contradiction and by computing the Levi form of ρ at a point $z_0 = (z_1, 0)$ and at a vector $v = (X_1, 0, X_2, 0)$. Assume that

$$\rho = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + \widetilde{H}(z_1, z_2) + \Re e \rho_{\overline{k}} z_1^k \overline{z_2} + O\left(|z_1|^{2m+1} + |z_2||z_1|^{k+1} + |z_2|^2\right),$$

with $k < m$. Replacing z_1 by $(\rho_{\overline{k}})^{\frac{1}{k}} z_1$ if necessary, we suppose $\rho_{\overline{k}} = 1$.

The Levi form of $\Re e z_2$ at a point $z_0 = (z_1, 0)$ and at a vector $v = (X_1, 0, X_2, 0)$ is equal to

$$\begin{aligned}\mathcal{L}_J \Re e z_2(z_0, v) &= \left[(a_1 - a_2)(z_0)\frac{\partial a_2}{\partial x_1}(z_0) + c_1(z_0)\frac{\partial a_2}{\partial y_1}(z_0) - c_2(z_0)\frac{\partial b_2}{\partial x_1}(z_0)\right] X_1 X_2 + \\ &\quad c_2(z_0)\left[\frac{\partial a_2}{\partial y_2}(z_0) - \frac{\partial b_2}{\partial x_2}(z_0)\right] X_2^2.\end{aligned}$$

Due to (3.3) we have

$$\begin{cases} a_1(z_0) = a_2(z_0) = 0, \\ c_2(z_0) = 1, \\ \frac{\partial a_2}{\partial y_1}(z_0) = \frac{\partial b_2}{\partial x_1}(z_0) = 0. \end{cases}$$

So the Levi form of $\Re e z_2$ at $z_0 = (x_1, 0, 0, 0)$ and at a vector $v = (X_1, 0, X_2, 0)$ is

$$\mathcal{L}_J \Re e z_2(z_0, v) = \left[\frac{\partial a_2}{\partial y_2}(z_0) - \frac{\partial b_2}{\partial x_2}(z_0)\right] X_2^2.$$

According to Lemma 3.1.4, the Levi form of $H_{2m} + O(|z_1|^{2m+1})$ at z_0 and $v_1 = (X_1, 0, X_2, 0)$ is equal to

$$\mathcal{L}_J(H_{2m} + O(|z_1|^{2m+1}))(z_0, v) = \Delta\left(H_{2m} + O(|z_1|^{2m+1})\right) X_1^2 + O(|z_1|^{2m-1}) X_1 X_2.$$

According to the fact that the Levi form for the standard structure of $\widetilde{H}(z_1, z_2)$ is identically equal to zero, and due to (1.5) and to (3.3), it follows that the Levi form of $\widetilde{H}(z_1, z_2)$ at z_0 is equal to

$$\mathcal{L}_J \widetilde{H}(z_0, v) = O(|z_1|) X_2^2.$$

Now the Levi form of $O(|z_2|^2)$ is equal to

$$\mathcal{L}_J O(|z_2|^2)(z_0, v) = O(1) X_2^2.$$

And the Levi form of $\Re e z_1^k \overline{z_2}$ is equal

$$\mathcal{L}_J \Re e z_1^k \overline{z_2}\,(z_0, v) \;=\; (k \Re e z_1^{k-1}) X_1 X_2 + O(|z_1|^k) X_2^2.$$

Finally the Levi form of the defining function ρ at a point $z_0 = (z1, 0)$ and at a vector $v = (X_1, 0, X_2, 0)$ is equal to:

$$\begin{aligned}\mathcal{L}_J \rho\,(z_0, v) \;&=\; O\left(|z_1|^{2m-2}\right) X_1^2 + \left[4k \Re e z_1^{k-1} + O(|z_1|^{2m-1})\right] X_1 X_2 \\ &\quad + \left[\frac{\partial a_2}{\partial y_2}(z_0) - \frac{\partial b_2}{\partial x_2}(0) + O(1) + O\left(|z_1|\right)\right] X_2^2.\end{aligned}$$

It follows that since $k < m$ there are z_1, X_1 and X_2 such that $\mathcal{L}_J \rho\,(z_0, v)$ is negative, providing a contradiction.

□

Now we prove that the D'Angelo type coincides with the regular type in the non integrable case.

Proposition 3.1.5. *We have*

$$\Delta^1_{\text{reg}}\left(\partial D, 0\right) = \Delta^1\left(\partial D, 0\right).$$

Proof. We suppose that the origin is a point of finite D'Angelo type. According to (3.1) we may write:

$$\Delta^1_{\text{reg}}\left(\partial D, 0\right) = 2m < +\infty.$$

So we may assume that $u\,(\zeta) = (\zeta, 0)$ is a regular J-holomorphic disc of maximal contact order $2m$, and that the structure J satisfies (3.2) and (3.3). Moreover the defining function ρ has the following local expression:

$$\rho = \Re e z_2 + H_{2m}\left(z_1, \overline{z_1}\right) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right).$$

Now consider a J-holomorphic disc $v = (f_1, g_1, f_2, g_2) : (\Delta, 0) \to (\mathbb{R}^4, 0, J)$ of finite contact order satisfying $v\,(0) = 0$ and such that $\delta\,(v) \geq 2$ (see definition 3.1.1 for notations).

We set $v_1 := f_1 + i g_1$ and $v_2 := f_2 + i g_2$. The J-holomorphy equation for the disc v is given by:

$$\left\{\begin{aligned} a_k\,(v)\,\frac{\partial f_k}{\partial x} + b_k\,(v)\,\frac{\partial g_k}{\partial x} \;&=\; \frac{\partial f_k}{\partial y}, \\ c_k\,(v)\,\frac{\partial f_k}{\partial x} - a_k\,(v)\,\frac{\partial g_k}{\partial x} \;&=\; \frac{\partial g_k}{\partial y}, \end{aligned}\right.$$

for $k = 1, 2$. Since $J\,(v) = J_{st} + O\left(|v_2|\right)$ and $\delta\,(v) \geq 2$, it follows that:

$$\left\{\begin{aligned} \delta\,(v_1) \;&=\; \delta\,(f_1) \;=\; \delta\,(g_1)\,, \\ \delta\,(v_2) \;&=\; \delta\,(f_2) \;=\; \delta\,(g_2)\,. \end{aligned}\right. \tag{3.4}$$

Then consider

$$\rho \circ v(\zeta) = f_2(\zeta) + H_{2m}\left(v_1(\zeta), \overline{v_1(\zeta)}\right) + O\left(|v_1(\zeta)|^{2m+1} + |v_2(\zeta)|\|v(\zeta)\|\right). \tag{3.5}$$

Equation (3.4) implies that the term $O(|v_2|\|v\|)$ in (3.5) vanishes to order larger than f_2.

Case 1: $\delta(f_2) > \delta(H_{2m}(v_1, \overline{v_1}))$. In that case

$$\delta_0(\partial D, u) = \delta(H_{2m}(v_1, \overline{v_1})) = 2m\delta(v_1).$$

Thus we get:

$$\frac{\delta_0(\partial D, v)}{\delta(v)} = \frac{2m\delta(v_1)}{\delta(v_1)} = 2m.$$

Case 2: $\delta(f_2) \leq \delta(H_{2m}(v_1, \overline{v_1}))$. We have two subcases.

Subcase 2.1: $f_2 + H_{2m}(v_1, \overline{v_1}) \not\equiv 0$. Thus

$$\delta_0(\partial D, u) = \delta(\Re e v_2) = \delta(v_2),$$

and so

$$\frac{\delta_0(\partial D, v)}{\delta(v)} = \frac{\delta(v_2)}{\delta(v)} \leq \frac{\delta(H_{2m}(v_1, \overline{v_1}))}{\delta(v)} = \frac{2m\delta(v_1)}{\delta(v)}.$$

This means that:

$$\frac{\delta_0(\partial D, v)}{\delta(v)} = 1 \text{ if } \delta(v) = \delta(v_2)$$

or

$$\frac{\delta_0(\partial D, v)}{\delta(v)} \leq 2m \text{ if } \delta(v) = \delta(v_1).$$

Subcase 2.2: $f_2 + H_{2m}(v_1, \overline{v_1}) \equiv 0$. Let $w : \Delta \to (\mathbb{R}^4, J_{st})$ be a standard holomorphic disc satisfying $w(0) = 0$ and:

$$\frac{\partial^k w}{\partial x^k}(0) = \frac{\partial^k v}{\partial x^k}(0),$$

for $k = 1, \cdots, 2m\delta(v)$. Since $\delta(v_2) = 2m\delta(v_1) = 2m\delta(v) < +\infty$ and since $J(v) = J_{st} + O(|v_2|)$, any differentiation of $J(v)$, of order smaller than $2m\delta(v)$, is equal to zero. Combining this with the J-holomorphy equation (1.1) of v we obtain:

$$\frac{\partial^{k+l} w}{\partial x^k \partial y^l}(0) = \frac{\partial^{k+l} v}{\partial x^k \partial y^l}(0),$$

for $k + l = 1, \cdots, 2m\delta(v)$. Since $\rho \circ v$ vanishes to an order greater than $2m\delta(v)$ at 0 and since it involves only the $2m\delta(v)$-jet of v, it follows that $\rho \circ w$ vanishes to an order greater than $2m\delta(v)$ at 0. Finally we have constructed a standard holomorphic disc w such that

$$\begin{cases} \delta(w) & = \delta(v), \\ \delta_0(\partial D, w) & > 2m\delta(w), \end{cases}$$

which is not possible since, according Proposition 3.1.3, the type for the standard structure of ∂D at the origin is equal to $2m$. □

3.1.2 Construction of a local peak plurisubharmonic function

We first give the definition of a local peak J-plurisubharmonic function for a domain D.

Definition 3.1.6. *Let D be a domain in an almost complex manifold (M, J). A function φ is called a local peak J-plurisubharmonic function at a boundary point $p \in \partial D$ if there exists a neighborhood U of p such that φ is continuous up to $\overline{D} \cap U$ and satisfies:*

1. *φ is J-plurisubharmonic on $D \cap U$,*
2. *$\varphi(p) = 0$,*
3. *$\varphi < 0$ on $\overline{D} \cap U \backslash \{p\}$.*

The existence of local peak J_{st}-plurisubharmonic functions was first proved by E.Fornaess and N.Sibony in [31]. For almost complex manifolds the existence was proved by S.Ivashkovich and J.-P.Rosay in [45] whenever the domain is strictly J-pseudoconvex. In the next Proposition we state the existence for J-pseudoconvex regions of finite D'Angelo type. As mentioned earlier our considerations are purely local. In particular, the assumptions of J-plurisubharmonicity and of finite D'Angelo type may be restricted to a neighborhood of a boundary point. For convenience of writing, we state them globally.

Theorem 3.1.7. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in a four dimensional almost complex manifold (M, J). We suppose that ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Let $p \in \partial D$ be a boundary point. Then there exists a local peak J-plurisubharmonic function at p.*

Proof. Since the existence of a local peak function near a boundary point of type 2 was proved in [45], we assume that p is a boundary point of D'Angelo type $2m > 2$. The problem being purely local we assume that $D \subset \mathbb{C}^2$ and that $p = 0$. According to Proposition 3.1.3 the defining function ρ has the following local expression on a neighborhood U of the origin:

$$\rho = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + \widetilde{H}(z_1, z_2) + O\left(|z_1|^{2m+1} + |z_2||z_1|^m + |z_2|^2\right)$$

where H_{2m} is a subharmonic polynomial containing a nonharmonic part, denoted by H_{2m}^*, and

$$\widetilde{H}(z_1, z_2) = \Re e \sum_{k=1}^{m-1} \rho_k z_1^k z_2.$$

According to [31] (see Lemma 2.4), the polynomial H_{2m} satisfies the following Lemma:

Lemma 3.1.8. *There exist a positive $\delta > 0$ and a smooth function $g : \mathbb{R} \to \mathbb{R}$ with period 2π with the following properties:*

1. *$-2 < g(\theta) < -1$,*

2. $\|g\| < 1/\delta$,
3. $\max\left(\Delta H_{2m}, \Delta\left(\|H^*_{2m}\| g(\theta)|z_1|^{2m}\right)\right) > \delta\|H^*_{2m}\||z_1|^{2(m-1)}$, *for* $z_1 = |z_1|e^{i\theta} \neq 0$ *and,*
4. $\Delta\left(H_{2m} + \delta\|H^*_{2m}\| g(\theta)|z_1|^{2m}\right) > \delta^2\|H^*_{2m}\||z_1|^{2(m-1)}$.

We denote by P the function defined by

$$P(z_1, \overline{z_1}) := H_{2m}(z_1, \overline{z_1}) + \delta\|H^*_{2m}\| g(\theta)|z_1|^{2m}.$$

Theorem 3.1.7 will be proved by establishing the following claim.

Claim. There are positive constants L and C such that the function

$$\varphi := \Re e z_2 + 2L(\Re e z_2)^2 - L(\Im m z_2)^2 + P(z_1, \overline{z_1}) + \widetilde{H}(z_1, z_2) + C|z_1|^2|z_2|^2$$

is a local peak J-plurisubharmonic function at the origin.

Proof of the claim. We first prove that the function φ is J-plurisubharmonic. We set:

$$dd^c_J\varphi = \alpha_1 dx_1\wedge dy_1 + \alpha_2 dx_2\wedge dy_2 + \alpha_3 dx_1\wedge dx_2 + \alpha_4 dx_1\wedge dy_2 + \alpha_5 dy_1\wedge dx_2 + \alpha_6 dy_1\wedge dy_2,$$

where α_k, for $k = 1, \cdots, 6$, are real valued function. According to the matricial representation of J (see (3.2)), the Levi form of φ at a point $z \in D\cap U$ and at a vector $v = (X_1, Y_1, X_2, Y_2) \in T_z\mathbb{R}^4$ can be written

$$\begin{aligned}\mathcal{L}_J\varphi(z, v) &= c_1\alpha_1 X_1^2 - 2a_1\alpha_1 X_1Y_1 - b_1\alpha_1 Y_1^2 + \beta_3 X_1X_2 + \beta_4 X_1Y_2 + \\ &\quad \beta_5 Y_1X_2 + \beta_6 Y_1Y_2 + c_2\alpha_2 X_2^2 - 2a_2\alpha_2 X_2Y_2 - b_2\alpha_2 Y_2^2,\end{aligned}$$

with

$$\begin{cases}\beta_3 &:= \alpha_3(a_2 - a_1) + \alpha_4 c_2 - \alpha_5 c_1\\ \beta_4 &:= -\alpha_4(a_1 + a_2) + \alpha_3 b_2 - \alpha_6 c_1\\ \beta_5 &:= \alpha_5(a_1 + a_2) - \alpha_3 b_1 + \alpha_6 c_2\\ \beta_6 &:= \alpha_6(a_1 - a_2) - \alpha_4 b_1 + \alpha_5 b_2.\end{cases}$$

Moreover due to (3.3) we have for $k = 1, 2$

$$\begin{cases}a_k = O(|z_2|)\\ b_k = -1 + O(|z_2|)\\ c_k = 1 + O(|z_2|).\end{cases}$$

This implies that for $k = 1, 2$:

$$c_k \alpha_k X_k^2 - 2a_k \alpha_k X_k Y_k - b_k \alpha_k Y_k^2 \geq \frac{\alpha_k}{2}\left(X_k^2 + Y_k^2\right).$$

Thus we obtain

$$\begin{aligned}\mathcal{L}_J \varphi(z, v) \geq\ & \frac{\alpha_1}{4} X_1^2 + \beta_3 X_1 X_2 + \frac{\alpha_2}{4} X_2^2 + \frac{\alpha_1}{4} Y_1^2 + \beta_5 Y_1 X_2 + \frac{\alpha_2}{4} X_2^2 + \\ & \frac{\alpha_1}{4} X_1^2 + \beta_4 X_1 Y_2 + \frac{\alpha_2}{4} Y_2^2 + \frac{\alpha_1}{4} Y_1^2 + \beta_6 Y_1 Y_2 + \frac{\alpha_2}{4} Y_2^2.\end{aligned}$$

In order to prove that φ is J-plurisubharmonic, we need to see that:

1. $\alpha_k \geq 0$, for $k = 1, 2$,

2. $4\beta_j^2 \leq \alpha_1 \alpha_2$, for $j = 3, \cdots, 6$.

The coefficient α_2 is obtained by the differentiation of $\Re e z_2$, $2L\left(\Re e z_2\right)^2 - L\left(\Im m z_2\right)^2$, $\widetilde{H}(z_1, z_2)$ and $C|z_1|^2|z_2|^2$. Hence we have for z sufficiently close to the origin

$$\alpha_2 \geq L > 0.$$

The coefficient α_1 is obtained by differentiating P, $\widetilde{H}(z_1, z_2)$ and $C|z_1|^2|z_2|^2$. This is equal to

$$\begin{aligned}\alpha_1 &= \Delta P + O(|z_1|^{2m-2}|z_2|) + O(|z_2|^2) + C|z_2|^2 + O(|z_2|^3) \\ &\geq \frac{\delta^2 \|H_{2m}^*\|}{2}|z_1|^{2m-2} + \frac{C}{2}|z_2|^2,\end{aligned}$$

for z sufficiently small and $C > 0$ large enough. Hence α_1 is nonnegative.

Finally it sufficient to prove that

$$4\beta_j^2 \leq L\left(\frac{\delta^2 \|H_{2m}^*\|}{2}|z_1|^{2m-2} + \frac{C}{2}|z_2|^2\right),$$

to insure the J-plurisubharmonicity of φ. The coefficient $|\beta_j|$ is equal to

$$\begin{aligned}|\beta_j| &= O(|z_2|) + LO(|z_2|^2) + O(|z_1|^{2m-1}) + CO(|z_1||z_2|) \\ &\leq C'(|z_2| + |z_1|^{2m-1}),\end{aligned}$$

for a positive constant C' (not depending on L and C). It follows that φ is J-plurisubharmonic on a neighborhood of the origin.

We prove now that φ is local peak at the origin, that is there exists $r > 0$ such that $\overline{D} \cap \{0 < \|z\| \leq r\} \subset \{\varphi < 0\}$. Assuming that $z \in \{\rho = 0\} \cap \{0 < \|z\| \leq r\}$ we have:

$$\begin{aligned}\varphi(z) =\ & \delta \|H_{2m}^*\| g(\theta) |z_1|^{2m} + 2L\left(\Re e z_2\right)^2 - L\left(\Im m z_2\right)^2 + C|z_1|^2|z_2|^2 + \\ & O\left(|z_1|^{2m+1}\right) + O\left(|z_2||z_1|^m\right) + O\left(|z_2|^2\right).\end{aligned}$$

Since $g < -1$ and increasing L if necessary we have

$$O\left(|\Im m z_2||z_1|^m\right) \leq -\frac{1}{2}\delta \|H^*_{2m}\| g(\theta) |z_1|^{2m} + \frac{1}{2} L \left(\Im m z_2\right)^2,$$

whenever z is sufficiently close to the origin. Thus

$$\begin{aligned}
\varphi(z) &\leq -\frac{1}{2}\delta\|H^*_{2m}\||z_1|^{2m} + \left(2L + C|z_1|^2\right)\left(\Re e z_2\right)^2 - \frac{1}{2} L \left(\Im m z_2\right)^2 + C|z_1|^2 (\Im m z_2)^2 + \\
&\quad O\left(|z_1|^{2m+1}\right) + O\left(|\Re e z_2||z_1|^m\right) + O\left(|z_2|^2\right) \\
&\leq -\frac{1}{4}\delta\|H^*_{2m}\||z_1|^{2m} + \left(2L + C|z_1|^2\right)\left(\Re e z_2\right)^2 - \frac{1}{4} L \left(\Im m z_2\right)^2 + O\left(|\Re e z_2||z_1|^m\right) + \\
&\quad O\left(|z_2|^2\right).
\end{aligned}$$

There is a positive constant C'' such that

$$O\left(|z_2|^2\right) \leq C''|\Re e z_2|^2 + C''|\Im m z_2|^2.$$

Thus increasing L if necessary:

$$\begin{aligned}
\varphi(z) &\leq -\frac{1}{4}\delta\|H^*_{2m}\||z_1|^{2m} + \left(2L + C|z_1|^2\right)\left(\Re e z_2\right)^2 + O(|\Re e z_2|^2) \\
&\quad - \left(\frac{1}{4}L - C''\right)\left(\Im m z_2\right)^2 + O\left(|\Re e z_2||z_1|^m\right) + O(|\Im m z_2|^2\|z\|). \\
&\leq -\frac{1}{4}\delta\|H^*_{2m}\||z_1|^{2m} + \left(2L + C|z_1|^2\right)\left(\Re e z_2\right)^2 + O(|\Re e z_2|^2) + O\left(|\Re e z_2||z_1|^m\right) \\
&\quad -\frac{1}{2}\left(\frac{1}{4}L - C''\right)\left(\Im m z_2\right)^2.
\end{aligned}$$

Since

$$-\Re e z_2 (1 + O(|z|)) = H_{2m}\left(z_1, \overline{z_1}\right) + O\left(|z_1|^{2m+1} + |\Im m z_2||z_1| + |\Im m z_2|^2\right),$$

we have

$$(\Re e z_2)^2 (1 + O(|z|)) = O\left(|z_1|^{4m} + |\Im m z_2||z_1|^{2m+1} + |\Im m z_2|^2\|z\|\right).$$

We finally obtain for z small enough

$$\varphi(z) \leq -\frac{1}{8}\delta\|H^*_{2m}\||z_1|^{2m} - \frac{1}{4}\left(\frac{1}{4}L - C''\right)\left(\Im m z_2\right)^2.$$

Thus φ is negative for $z \in \{\rho = 0\} \cap \{0 < \|z\| \leq r\}$, with r small enough. It follows that, reducing r if necessary,

$$\overline{D} \cap \{0 < \|z\| \leq r\} \subset \{\varphi < 0\},$$

which achieves the proof of the claim and of Theorem 3.1.7. □

We notice that in case $\mathcal{L}_J \Re e z_2 \equiv 0$, we may give a simpler expression for a local peak J-plurisubharmonic function.

Proposition 3.1.9. *If $\mathcal{L}_J \Re e z_2 \equiv 0$, then there exists a real positive number L such that the function*

$$\varphi := \Re e z_2 + 2L\left(\Re e z_2\right)^2 - L\left(z_2\right)^2 + P\left(z_1, \overline{z_1}\right)$$

is local peak J-plurisubharmonic at the origin.

3.2 Estimates of the Kobayashi pseudometric

In this section we prove standard estimates of the Kobayashi pseudometric on J-pseudoconvex regions of finite D'Angelo type in an almost complex manifold.

3.2.1 Hyperbolicity of pseudoconvex regions of finite D'Angelo type

In order to localize pseudoholomorphic discs, we need the following technical Lemma (see [35] for a proof).

Lemma 3.2.1. *Let $0 < r < 1$ and let θ_r be a smooth nondecreasing function on $\mathbb{R}^+$ such that $\theta_r(s) = s$ for $s \leq r/3$ and $\theta_r(s) = 1$ for $s \geq 2r/3$. Let (M, J) be an almost complex manifold, and let p be a point of M. Then there exist a neighborhood U of p, positive constants $A = A(r) \geq 1$, $B = B(r)$, and a diffeomorphism $z : U \to \mathbb{B}$ such that $z(p) = 0$, $z_* J(p) = J_{st}$ and the function $\log\left(\theta_r\left(|z|^2\right)\right) + \theta_r\left(A|z|\right) + B|z|^2$ is J-plurisubharmonic on U.*

In the next Proposition we give a priori estimates and a localization principle of the Kobayashi pseudometric. This proves the local Kobayashi hyperbolicity of J-pseudoconvex $\mathcal{C}^2$ regions of finite D'Angelo type. If (M, J) admits a global J-plurisubharmonic function, then K.Diederich and A.Sukhov proved in [29] the (global) Kobayashi hyperbolicity of a relatively compact J-pseudoconvex domain (with $\mathcal{C}^3$ boundary) by constructing a bounded strictly J-plurisubharmonic exhaustion function. We notice that, in our case, if the manifold (M, J) admits a global J-plurisubharmonic function then J-pseudoconvex $\mathcal{C}^2$ relatively compact regions of finite D'Angelo type are also (globally) Kobayashi hyperbolic.

Proposition 3.2.2. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in an almost complex manifold (M, J), where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic in a neighborhood of $\overline{D}$. Let $p \in \overline{D}$ and let U be a neighborhood of p in M. Then there exist positive constants C and s, and a neighborhood $V \subset U$ of p in M, such that for each $q \in D \cap V$ and each $v \in T_q M$:*

$$K_{(D,J)}(q, v) \geq C\|v\|, \tag{3.6}$$

$$K_{(D,J)}(q, v) \geq s K_{(D\cap U,J)}(q, v). \tag{3.7}$$

This Proposition is a classical application of Lemma 3.2.1. This is due to N.Sibony [64] (see also [7] and [35] for a proof). For convenience we give the proof.

Proof. According to Theorem 3.1.7, there exists a local peak J-plurisubharmonic function φ at p for D. We can choose constants $0 < \alpha < \alpha' < \beta' < \beta$ and $N > 0$ such that $\varphi \geq -\beta^2/N$ on $\{\|z\| < \alpha\}$ and $\varphi \leq -2\beta^2/N$ on $\overline{D} \cap \{\alpha' \leq \|z\| \leq \beta'\}$.

We define $\tilde{\varphi}$ by:

$$\tilde{\varphi} := \begin{cases} \max\left(N\varphi + \|z\|^2 - \beta^2, -2\beta^2\right) & \text{if } z \in D \cap \{\|z\| \leq \beta'\}, \\ -2\beta^2 & \text{on } D \backslash \{\|z\| \leq \beta'\}. \end{cases}$$

The function $\|z\|^2$ is J-plurisubharmonic on $\{q \in U \ : |z(q)| < 1\}$ if $\|z_* J - J_{st}\|_{\mathcal{C}^2(\mathbb{B})}$ is sufficiently small. Then it follows that $\tilde{\varphi}$ is J-plurisubharmonic on D. We may also suppose that $\tilde{\varphi}$ is negative on D. Moreover the function $\tilde{\varphi} - \|z\|^2$ is J-plurisubharmonic on $D \cap \{q \in U \ : |z(q)| \leq \alpha\}$.

Let θ_{α^2} be a smooth non decreasing function on $\mathbb{R}^+$ such that $\theta_{\alpha^2}(s) = s$ for $s \leq \alpha^2/3$ and $\theta_{\alpha^2}(s) = 1$ for $s \geq 2\alpha^2/3$. Set $V = \{q \in U \ : |z(q)| \leq \alpha^2\}$. According to Lemma 3.2.1, there are uniform positive constants $A \geq 1$ and B such that the function

$$\log\left(\theta_{\alpha^2}\left(|z - z(q)|^2\right)\right) + \theta_{\alpha^2}\left(A|z - z(q)|\right) + B\|z\|^2$$

is J-plurisubharmonic on U for every $q \in D \cap V$.

We define for each $q \in D \cap V$ the function:

$$\Psi_q := \begin{cases} \theta_{\alpha^2}\left(|z - z(q)|^2\right)\exp\left(\theta_{\alpha^2}\left(A|z - z(q)|\right)\right)\exp\left(B\tilde{\varphi}(z)\right) & \text{on } D \cap \{\|z\| < \alpha\}, \\ \exp\left(1 + B\tilde{\varphi}\right) & \text{on } D \setminus \{\|z\| < \alpha\}. \end{cases}$$

The function $\log\Psi_q$ is J-plurisubharmonic on $D \cap \{\|z\| < \alpha\}$ and, on $D \setminus \{\|z\| < \alpha\}$, it coincides with $1 + B\tilde{\varphi}$ which is J-plurisubharmonic. Finally $\log\Psi_q$ is J-plurisubharmonic on the whole domain D.

Let $q \in V$ and let $v \in T_q M$ and consider a J-holomorphic disc $u \ : \Delta \to D$ such that $u(0) = q$ and $d_0 u(\partial/\partial x) = rv$ where $r > 0$. For ζ sufficiently close to 0 we have

$$u(\zeta) = q + d_0 u(\zeta) + \mathcal{O}\left(|\zeta|^2\right).$$

We define the following function

$$\phi(\zeta) := \frac{\Psi_q(u(\zeta))}{|\zeta|^2}$$

which is subharmonic on $\Delta\backslash\{0\}$ since $\log\phi$ is subharmonic. If ζ close to 0, then

$$\phi(\zeta) = \frac{|u(\zeta) - q|^2}{|\zeta|^2}\exp\left(A|u(\zeta) - q|\right)\exp\left(B\tilde{\varphi}(u(\zeta))\right). \tag{3.8}$$

Setting $\zeta = \zeta_1 + i\zeta_2$ and using the J-holomorphy condition $d_0u \circ J_{st} = J \circ d_0u$, we may write :

$$d_0u\,(\zeta) = \zeta_1 d_0u\,(\partial/\partial x) + \zeta_2 J\,(d_0u\,(\partial/\partial x))\,.$$

(3.9) $$|d_0u\,(\zeta)\,| \leq |\zeta|\,(\|I + J\|\,\|d_0u\,(\partial/\partial x)\,\|)$$

According to (3.8) and to (3.9), we obtain that $\limsup_{\zeta\to 0}\phi\,(\zeta)$ is finite. Moreover setting $\zeta_2 = 0$ we have

$$\limsup_{\zeta\to 0}\phi\,(\zeta) \geq \|d_0u\,(\partial/\partial x)\,\|^2 \exp\,(B\tilde{\varphi}\,(q))\,.$$

Applying the maximum principle to a subharmonic extension of ϕ on Δ we obtain the inequality

$$\|d_0u\,(\partial/\partial x)\,\|^2 \leq \exp\,(1 - B\tilde{\varphi}\,(q))\,.$$

Hence, by definition of the Kobayashi pseudometric, we obtain for every $q \in D \cap V$ and every $v \in T_qM$:

$$K_{(D,J)}\,(q, v) \geq (\exp\,(-1 + B\tilde{\varphi}\,(q)))^{\frac{1}{2}}\,\|v\|.$$

This gives estimate (3.6).

Now in order to obtain estimate (3.7), we prove that there is a neighborhood $V \subset U$ and a positive constant s such that for any J-holomorphic disc $u\ : \Delta \to D$ with $u\,(0) \in V$ then $u\,(\Delta_s) \subset D \cap U$. Suppose this is not the case. We obtain a sequence ζ_ν of Δ and a sequence of J-holomorphic discs u_ν such that ζ_ν converges to 0, $u_\nu\,(0)$ converges to p and $\|u_\nu\,(\zeta_\nu)\,\| \notin D \cap U$ for every ν. According to the estimate (3.6), we obtain for a positive constant $c > 0$:

$$c \leq d_{(D,J)}\,(u_\nu\,(0)\,, u_\nu\,(\zeta_\nu)) \leq d_\Delta\,(\zeta_\nu, 0)\,.$$

This contradicts the fact that ζ_ν converges to 0. □

The (global) Kobayashi hyperbolicity is provided if we suppose that there is a global strictly J-plurisubharmonic function on (M, J).

Corollary 3.2.3. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type in an almost complex manifold (M, J) of dimension four, ρ being a defining function of D, J-plurisubharmonic in a neighborhood of $\overline{D}$. Assume that (M, J) admits a global strictly J-plurisubharmonic function. Then (D, J) is Kobayashi hyperbolic.*

As an application of the a priori estimate (3.6) of Proposition 3.2.2, we prove the tautness of D.

Corollary 3.2.4. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type in an almost complex manifold (M, J) of dimension two. Assume that ρ is J-plurisubharmonic in a neighborhood of $\overline{D}$. Moreover suppose that (M, J) admits a global strictly J-plurisubharmonic function. Then D is taut.*

Proof. Let $(u_\nu)_\nu$ be a sequence of J-holomorphic discs in D. According to Corollary 3.2.3 the domain D is hyperbolic. Thus the sequence $(u_\nu)_\nu$ is equicontinuous, and then by Ascoli Theorem, we can extract from this sequence a subsequence still denoted $(u_\nu)_\nu$ which converges to a map $u \ : \Delta \rightarrow \overline{D}$. Passing to the limit the equation of J-holomorphy of each u_ν, it follows that u is a J-holomorphic disc. Since ρ is J-plurisubharmonic defining function for D, we have, by applying the maximum principle to $\rho \circ u$, the alternative: either $u(\Delta) \subset D$ or $u(\Delta) \subset \partial D$. □

We point out that the tautness of the domain D was proved, using a different method, by K.Diederich-A.Sukhov in [29].

3.2.2 Uniform estimates of the Kobayashi pseudometric

In order to obtain more precise estimates, we need to uniform estimates (3.6) of the Kobayashi pseudometric for a sequence of domains.

Proposition 3.2.5. *Assume that $D = \{\Re e z_2 + P(z_1, \overline{z_1}) < 0\}$ is a J_{st}-pseudoconvex region of $\mathbb{R}^4$, where P is a homogeneous polynomial of degree $2k \leq 2m$ admitting a nonharmonic part. Let D_ν be a sequence of J_ν-pseudoconvex region of $\mathbb{R}^4$ such that $0 \in \partial D_\nu$ is a boundary point of finite D'Angelo type $2l_\nu \leq 2m$. Suppose that D_ν converges in the sense of local Hausdorff set convergence to D when ν tends to $+\infty$ and that J_ν converges to J_{st} in the $\mathcal{C}^2$ topology when ν tends to $+\infty$. Then there exist a positive constant C and a neighborhood $V \subset U$ of the origin in $\mathbb{R}^4$, such that for large ν and for every $q \in D_\nu \cap V$ and every $v \in T_q\mathbb{R}^4$*

$$K_{(D_\nu, J)}(q, v) \geq C\|v\|.$$

Proof. Under the conditions of Proposition 3.2.5 we have the following Lemma:

Lemma 3.2.6. *For every large ν, there exists a diffeomorphism $\Phi_\nu \ : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ with the following property:*

1. *The map $\zeta \mapsto (\zeta, 0)$ is a $(\Phi_\nu)_* J_\nu$-holomorphic disc of maximal contact order $2l_\nu$.*

2. *The almost complex structure $(\Phi_\nu)_* J_\nu$ satisfies conditions (3.2) and (3.3).*

3. *$\Phi_\nu(D_\nu) = \{\rho_\nu < 0\}$ with*

$$\rho_\nu = \Re e z_2 + \sum_{j=2l_\nu}^{2m} P_{j,\nu}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right) < 0,$$

   where $P_{j,\nu}$ are homogeneous polynomials of degree j and $P_{2l_\nu,\nu}$ contains a nonharmonic part denoted by $P^*_{2l_\nu,\nu} \neq 0$.

4. *we have $inf_\nu\{\|P_{2l_\nu,\nu}\|\} > 0$.*

Moreover the sequence of diffeomorphisms Φ_ν converges to the identity on any compact subsets of $\mathbb{R}^4$ in the $\mathcal{C}^2$ topology.

The crucial fact used to prove Proposition 3.2.5 is the point (4), which is a direct consequence of the convergence of $\Phi_\nu(D_\nu)$ to D. Hence the proof of Proposition 3.2.5 is similar to Theorem 3.1.7 and Theorem 3.2.2, where all the constants are uniform.

□

3.2.3 Hölder extension of diffeomorphisms

This subsection is devoted to the boundary continuity of diffeomorphisms. This is stated as follows:

Proposition 3.2.7. *Let $D = \{\rho < 0\}$ and $D' = \{\rho' < 0\}$ be two relatively compact domains of finite D'Angelo type $2m$ in four dimensional almost complex manifolds (M, J) and (M', J'). We suppose that ρ (resp. ρ') is a J(resp J')-plurisubharmonic defining function on a neighborhood of D (resp. $\overline{D'}$). Let $f : D \to D'$ be a (J, J')-biholomorphism. Then f extends as a Hölder homeomorphism with exponent $1/2m$ between $\overline{D}$ and $\overline{D'}$.*

Estimates of the Kobayashi pseudometric obtained by H.Gaussier and A.Sukhov in [35] provide the Hölder extension with exponent $1/2$ up to the boundary of a biholomorphism between two strictly pseudoconvex domains (see Proposition 3.3 of [23]). Similarly, in order to obtain Proposition 3.2.7, we begin by establishing a more precise estimate than (3.6) of Proposition 3.2.2.

Proposition 3.2.8. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in a four dimensional almost complex manifold (M, J), where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic in a neighborhood of $\overline{D}$. Let $p \in \partial D$ and let U be a neighborhood of p in M. Then there are positive constant C and a neighborhood $V \subset U$ of p in M, such that for every $q \in D \cap V$ and every $v \in T_qM$:*

$$K_{(D,J)}(q, v) \geq C \frac{\|v\|}{\operatorname{dist}(q, \partial D)^{1/2m}}. \tag{3.10}$$

Proof of Proposition 3.2.8. Let $p \in \partial D$. We may suppose that $D \subset \mathbb{R}^4$, $p = 0$ and that J satisfies (3.2) and (3.3). Let q' be a boundary point in a neighborhood of the origin and let $\varphi_{q'}$ be the local peak J-plurisubharmonic function at q' given by Theorem 3.1.7. There are positive constants C_1 and C_2 such that

$$-C_1\|z - q'\| \leq \varphi_{q'}(z) \leq -C_2\Psi_{q'}(z), \tag{3.11}$$

where

$$\Psi_{q'}(z) := |z_1 - q'_1|^{2m} + |z_2 - q'_2|^2 + |z_1 - q'_1|^2|z_2 - q'_2|^2$$

is a J-plurisubharmonic function on a neighborhood U of the origin.

Now consider a J-holomorphic disc $u : \Delta \to D$, such that $u(0)$ is sufficiently close to the origin and then, according to Proposition 3.2.2, we have $u(\Delta_s) \subset D \cap U$, for some $0 < s < 1$ depending only on $u(0)$. We assume that q' is such that $\operatorname{dist}(u(0), \partial D) = \|u(0) - q'\|$. According to the J-plurisubharmonicity of $\Psi_{q'}$, we have for $|\zeta| \leq s$:

$$\Psi_{q'}(u(\zeta)) \leq \frac{C_3}{2\pi} \int_0^{2\pi} \Psi_{q'}\left(u\left(re^{i\theta}\right)\right) d\theta,$$

for some positive constant C_3. Hence using (3.11) and the J-plurisubharmonicity of $\varphi_{q'}$ we obtain:

$$\Psi_{q'}\left(u\left(\zeta\right)\right) \leq -\frac{C_3}{2\pi C_2}\int_0^{2\pi}\varphi_{q'}\left(u\left(re^{i\theta}\right)\right)d\theta \leq -\frac{C_3}{C_2}\varphi_{q'}\left(u\left(0\right)\right).$$

Since there is a positive constant C_4 such that

$$\|u\left(\zeta\right)-q'\|^{2m} \leq C_4\Psi_{q'}\left(u\left(\zeta\right)\right)$$

and using (3.11), we finally obtain:

$$\|u\left(\zeta\right)-q'\|^{2m} \leq \frac{C_1C_3C_4}{C_2}\mathrm{dist}\left(u\left(0\right),\partial D\right).$$

Hence there exists a positive constant C_5 such that:

$$\mathrm{dist}\left(u\left(\zeta\right),\partial D\right) \leq C_5\mathrm{dist}\left(u\left(0\right),\partial D\right)^{1/2m},$$

whenever $\zeta \leq s$.

According to Lemma 1.5 of [45] there is a positive constant C_6 such that:

$$\|\nabla u\left(0\right)\| \leq C_6\sup_{|\zeta|<s}\|u\left(\zeta\right)-u\left(0\right)\| \leq C_5C_6\mathrm{dist}\left(u\left(0\right),\partial D\right)^{1/2m},$$

wich provides the desired estimate. □

We also need the two next lemmas provided by [23]:

Lemma 3.2.9. *Let D be a domain in an almost complex manifold (M,J). Then there is a positive constant C such that for any $p \in D$ and any $v \in T_pM$:*

$$K_{(D,J)}\left(p,v\right) \leq C\frac{\|v\|}{\mathrm{dist}\left(p,\partial D\right)}. \tag{3.12}$$

Lemma 3.2.10. *(Hopf lemma) Let D be a relatively compact domain with a $\mathcal{C}^2$ boundary on an almost complex manifold (M,J). Then for any negative J-plurisubharmonic function ρ on D there exists a constant $C>0$ such that for any $p \in D$:*

$$|\rho(p)| \geq C\mathrm{dist}(p,\partial D).$$

Now we can go on the proof of Proposition 3.2.7.

Proof of Proposition 3.2.7. Let $f : D \to D'$ be a (J,J')-biholomorphism. According to Proposition 3.2.8 and to the decreasing property of the Kobayashi pseudometric there is a positive constant C such that for every $p \in D$ sufficiently close to the boundary and every $v \in T_pM$

$$C\frac{\|d_pf\left(v\right)\|}{\mathrm{dist}\left(f\left(p\right),\partial D'\right)^{\frac{1}{2m}}} \leq K_{(D',J')}\left(f\left(p\right),d_pf\left(v\right)\right) = K_{(D,J)}\left(p,v\right).$$

Due to Lemma 3.2.9 there exists a positive constant C_1 such that:

$$K_{(D,J)}(p,v) \leq C_1 \frac{\|v\|}{\text{dist}\,(p,\partial D)}.$$

This leads to:

$$\|d_p f(v)\| \leq \frac{C_1}{C} \frac{\text{dist}\,(f(p),\partial D')^{\frac{1}{2m}}}{\text{dist}\,(p,\partial D)} \|v\|.$$

Moreover the Hopf lemma 3.2.10 for almost complex manifolds applied to $\rho' \circ f$ and $\rho \circ f^{-1}$ and the fact that ρ and ρ' are defining functions, provides the following boundary distance preserving property:

$$\frac{1}{C_2}\text{dist}\,(p,\partial D) \leq \text{dist}\,(f(p),\partial D') \leq C_2 \text{dist}\,(p,\partial D),$$

for some positive constant C_2. Finally this implies:

$$\|d_p f(v)\| \leq \frac{C_1 C_2}{C} \frac{\|v\|}{\text{dist}\,(p,\partial D)^{\frac{2m-1}{2m}}}.$$

This gives the desired statement. □

3.3 Sharp estimates of the Kobayashi pseudometric

In this section we give sharp lower estimates of the Kobayashi pseudometric in a pseudoconvex region near a boundary point of finite D'Angelo type less than or equal to four. This condition will appear necessary, in our proof, as explained in the appendix. Moreover in order to give sharp estimates near a point of arbitrary finite D'Angelo type, we are also interested in the nontangential behaviour of the Kobayashi pseudometric.

The main result of this section is the following theorem (see also Theorem B3):

Theorem 3.3.1. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type less than or equal to four in an almost complex manifold (M,J) of dimension four, where ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. Then there exists a positive constant C with the following property: for every $p \in D$ and every $v \in T_pM$ there is a diffeomorphism, Φ_{p^*}, in a neighborhood U of p, such that:*

$$K_{(D,J)}(p,v) \geq C\left(\frac{|(d_p\Phi_{p^*}v)_1|}{\tau(p^*,|\rho(p)|)} + \frac{|(d_p\Phi_{p^*}v)_2|}{|\rho(p)|}\right), \tag{3.13}$$

where $\tau(p^*,|\rho(p)|)$ is defined by (3.15).

As a direct consequence we have:

$$K_{(D,J)}(p,v) \geq C'\left(\frac{|(d_p\Phi_{p^*}v)_1|}{|\rho(p)|^{\frac{1}{4}}} + \frac{|(d_p\Phi_{p^*}v)_2|}{|\rho(p)|}\right), \tag{3.14}$$

for a positive constant C'.

In complex manifolds, D.Catlin [17] first obtained such an estimate, based on lower estimates of the Carathéodory pseudometric. F.Berteloot [8] gave a different proof based on a Bloch principle. Our proof which is inspired by the proof of F.Berteloot is based on some scaling method.

3.3.1 The scaling method

We consider here a pseudoconvex region $D = \{\rho < 0\}$ of finite D'Angelo type $2m$ in $\mathbb{R}^4$, where ρ has the following expression on a neighborhood U of the origin:

$$\rho(z_1, z_2) = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right).$$

where H_{2m} is a homogeneous subharmonic polynomial of degree $2m$ admitting a nonharmonic part.

Assume that p_ν is a sequence of points in $D \cap U$ converging to the origin. For each p_ν sufficiently close to ∂D, there exists a unique point $p_\nu^* \in \partial D \cap U$ such that

$$p_\nu^* = p_\nu + (0, \delta_\nu),$$

with $\delta_\nu > 0$. Notice that for large ν, the quantity δ_ν is equivalent to $\operatorname{dist}(p_\nu, \partial D \cap U)$ and to $|\rho(p_\nu)|$.

We consider a diffeomorphism $\Phi^\nu : \mathbb{R}^4 \to \mathbb{R}^4$ satisfying:

1. $\Phi^\nu(p_\nu^*) = 0$ and $\Phi^\nu(p_\nu) = (0, -\delta_\nu)$.
2. Φ^ν converges to $Id : \mathbb{R}^4 \to \mathbb{R}^4$ on any compact subset of $\mathbb{R}^4$ in the $\mathcal{C}^2$ sense.
3. When we denote by $D^\nu := \Phi^\nu(D \cap U)$ which admits the defining function is $\rho^\nu := \rho \circ (\Phi^\nu)^{-1}$ and by $J^\nu := (\Phi^\nu)_* J$, then ρ^ν is given by:

$$\rho^\nu(z_1, z_2) = \Re e z_2 + \sum_{k=2l_\nu}^{2m} P_k(z_1, \overline{z_1}, p_\nu^*) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

   where the polynomial P_{2l_ν} contains a nonharmonic part. Moreover J^ν satisfies (3.2) and (3.3).

This is done by considering first the translation T^ν of $\mathbb{R}^4$ given by $z \mapsto z - p_\nu^*$. According to J.-F.Barraud and E.Mazzilli [4] that the D'Angelo type is an upper semicontinuous function in a four dimensional almost complex manifold. Thus the D'Angelo type of points in a small enough neighborhood can only be smaller than at the point itself. Then we consider a $(T^\nu)_* J$-holomorphic disc u of maximal contact order $2l_\nu$, where $2l_\nu \leq 2m$ is the D'Angelo type of p_ν^*. We choose coordinates such that u is given by $u(\zeta) = (\zeta, 0)$, and such that $(T^\nu)_* J(z_1, 0) = J_{st}$ and $T_0(\partial T^\nu(D)) \cap J(0) T_0(\partial T^\nu(D)) = \{z_2 = 0\}$. Then by considering the family of vectors $(1, 0)$ at base points $(0, t)$ for $t \neq 0$ small enough, we obtain a family of pseudoholomorphic discs u_t such that $u_t(0) = (0, t)$ and $d_0 u_t(\partial/\partial_x) = (0, 1)$. Due to the parameters dependence of the solution to the J^ν-holomorphy equation, we straighten these discs into the lines $\{z_2 = t\}$. Next we consider a transversal foliation by pseudo-holomorphic discs passing through $(t, 0)$ and $(t, -\delta_\nu)$ for t small enough and we straighten these lines into $\{z_1 = c\}$. This leads to the desired diffeomorphism Φ^ν of $\mathbb{R}^4$.

Now, we need to remove harmonic terms from the polynomial

$$\sum_{k=2l_\nu}^{2m-1} P_k\left(z_1, \overline{z_1}, p_\nu^*\right).$$

So we consider a biholomorphism (for the standard structure) of $\mathbb{C}^2$ with the following form:

$$\varphi_\nu\left(z_1, z_2\right) := \left(z_1, z_2 + \sum_{k=2l_\nu}^{2m-1} \Re e\left(c_{k,\nu} z_1^k\right)\right),$$

where $c_{k,\nu}$ are well chosen complex numbers. Then the diffeomorphism $\Phi_\nu := \varphi_\nu \circ \Phi^\nu$ satisfies:

1. $\Phi_\nu\left(p_\nu^*\right) = 0$ and $\Phi_\nu\left(p_\nu\right) = (0, -\delta_\nu)$.
2. Φ_ν converges to $Id : \mathbb{R}^4 \to \mathbb{R}^4$ on any compact subset of $\mathbb{R}^4$ in the $\mathcal{C}^2$ sense.
3. If we denote by $D_\nu := \Phi_\nu\left(D \cap U\right)$ the domain with the defining function $\rho_\nu := \rho \circ \left(\Phi_\nu\right)^{-1}$, then ρ_ν is given by:

   $$\rho_\nu\left(z_1, z_2\right) = \Re e z_2 + \sum_{k=2l_\nu}^{2m-1} P_k^*\left(z_1, \overline{z_1}, p_\nu^*\right) + P_{2m}\left(z_1, \overline{z_1}, p_\nu^*\right) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

   where the polynomial

   $$\sum_{k=2l_\nu}^{2m-1} P_k^*\left(z_1, \overline{z_1}, p_\nu^*\right)$$

   does not contain any harmonic terms. Moreover the polynomial $P_{2l_\nu}^*$ is not idencally zero. Moreover, generically, $J_\nu := \left(\Phi_\nu\right)_* J$ is no more diagonal.

Since the origin is a boundary point of D'Angelo type $2m$ for D, it follows that, denoting by P_{2m}^* the nonharmonic part of P_{2m}, we have $P_{2m}^*\left(., 0\right) = H_{2m}^* \neq 0$, where H_{2m}^* is the nonharmonic part of H_{2m}. This allows to define for large ν:

$$\tau\left(p_\nu^*, \delta_\nu\right) := \min_{k=2l_\nu,\cdots,2m} \left(\frac{\delta_\nu}{\|P_k^*\left(., p_\nu^*\right)\|}\right)^{\frac{1}{k}}. \tag{3.15}$$

Moreover the following inequalities hold:

$$\frac{1}{C}\delta_\nu^{\frac{1}{2}} \leq \tau\left(p_\nu^*, \delta_\nu\right) \leq C\delta_\nu^{\frac{1}{2m}}, \tag{3.16}$$

where C is a positive constant. The right inequality comes from the fact that $\| P_{2m}^{*}\left(.,p_{\nu}^{*}\right) \| \geq C_1 > 0$ for large ν. And the left one comes the fact that there exists a positive constant C_2 such that for every $2l_\nu \leq k \leq 2m$ we have $\| P_k^{*}\left(.,p_{\nu}^{*}\right) \| \leq C_2$.

Now we consider the nonisotropic dilation Λ_ν of $\mathbb{C}^2$:

$$\Lambda_\nu \ : (z_1, z_2) \mapsto \left(\tau\left(p_\nu^*, \delta_\nu\right)^{-1} z_1, \delta_\nu^{-1} z_2\right).$$

We set $\tilde{D}_\nu := \Lambda_\nu\left(D_\nu\right)$ the domain admitting the defining function $\tilde{\rho_\nu} := \delta_\nu^{-1}\rho_\nu \circ \Lambda_\nu^{-1}$ and $\tilde{J}_\nu := \left(\Lambda_\nu\right)_*\left(J_\nu\right)$ the direct image of J_ν under Λ_ν.

The next lemma is devoted to describe $(\tilde{D}_\nu, \tilde{J}_\nu)$ when passing at the limit.

Lemma 3.3.2.

1. *The domain $\tilde{D}^\nu$ converges in the sense of local Hausdorff set convergence to a (standard) pseudoconvex domain $\tilde{D} = \{\tilde{\rho} < 0\}$, with*

$$\tilde{\rho}\left(z\right) = \Re e z_2 + P\left(z_1, \overline{z_1}\right),$$

where P is a nonzero subharmonic polynomial of degree smaller than or equal to $2m$ which admits a nonharmonic part.

2. ***In case the origin is of D'Angelo type four for*** *D, the sequence of almost complex structures $\tilde{J}_\nu$ converges on any compact subsets of $\mathbb{C}^2$ in the $\mathcal{C}^2$ sense to J_{st}.*

Proof. We first prove part 1. Due to inequalities (3.16), the defining function of $\tilde{D}_\nu$ satisfies:

$$\tilde{\rho_\nu} = \Re e z_2 + \sum_{k=2l_\nu}^{2m} \delta_\nu^{-1}\tau\left(p_\nu^*, \delta_\nu\right)^k P_k^*\left(z_1, \overline{z_1}, p_\nu^*\right) + \delta_\nu^{-1}\tau\left(p_\nu^*, \delta_\nu\right)^{2m} P_{2m}\left(z_1, \overline{z_1}, p_\nu^*\right) + O\left(\tau\left(\delta_\nu\right)\right).$$

Passing to a subsequence, we may assume that the polynomial

$$\sum_{k=2l_\nu}^{2m} \delta_\nu^{-1}\tau\left(p_\nu^*, \delta_\nu\right)^k P_k^*\left(z_1, \overline{z_1}, p_\nu^*\right) + \delta_\nu^{-1}\tau\left(p_\nu^*, \delta_\nu\right)^{2m} P_{2m}\left(z_1, \overline{z_1}, p_\nu^*\right)$$

converges uniformly on compact subsets of $\mathbb{C}^2$ to a nonzero polynomial P of degree $\leq 2m$ admitting a nonharmonic part. Since the pseudoconvexity is invariant under diffeomorphisms, it follows that the domains $\tilde{D}^\nu$ are $\tilde{J}_\nu$-pseudoconvex, and then passing to the limit, the domain $\tilde{D}$ is J_{st}-pseudoconvex. Thus the polynomial P is subharmonic.

We next prove part 2. The complexification of the almost complex structure J_ν is given by

$$\begin{aligned}(J_\nu)_{\mathbb{C}} &= \sum_{l=1}^{2}\Big(A_{l,l}(z)\,dz_l\otimes\frac{\partial}{\partial z_l}+B_{l,l}(z)\,dz_l\otimes\frac{\partial}{\partial\overline{z_l}}+\overline{B_{l,l}}(z)\,d\overline{z_l}\otimes\frac{\partial}{\partial z_l}+\\&\overline{A_{l,l}}(z)\,d\overline{z_l}\otimes\frac{\partial}{\partial\overline{z_l}}\Big)+A_{1,2}(z)\,dz_1\otimes\frac{\partial}{\partial z_2}+B_{1,2}(z)\,dz_1\otimes\frac{\partial}{\partial\overline{z_2}}+\\&\overline{B_{1,2}}(z)\,d\overline{z_1}\otimes\frac{\partial}{\partial z_2}+\overline{A_{1,2}}(z)\,d\overline{z_1}\otimes\frac{\partial}{\partial\overline{z_2}},\end{aligned}$$

where

$$\begin{cases}A_{l,l}(z) &= i+O\left(\left|z_2+\sum_{k=2}^{3}c_{k,\nu}z_1^k\right|^2\right)\ \text{for } l=1,2,\\ B_{l,l}(z) &= O\left(\left|z_2+\sum_{k=2}^{3}c_{k,\nu}z_1^k\right|\right)\ \text{for } l=1,2,\\ A_{1,2}(z) &= \sum_{k=2}^{3}kc_{k,\nu}z_1^{k-1}O\left(\left|z_2+\sum_{k=2}^{3}c_{k,\nu}z_1^k\right|^2\right),\\ B_{1,2}(z) &= \sum_{k=2}^{3}k\left(c_{k,\nu}z_1^{k-1}-\overline{c_{k,\nu}z_1^{k-1}}\right)O\left(\left|z_2+\sum_{k=2}^{3}c_{k,\nu}z_1^k\right|\right).\end{cases}$$

By a direct computation, the complexification of $\tilde{J}_\nu$ is equal to:

$$\begin{aligned}\left(\tilde{J}_\nu\right)_{\mathbb{C}} &= \sum_{l=1}^{2}(A_{l,l}(\Lambda_\nu^{-1}(z))dz_l\otimes\frac{\partial}{\partial z_l}+B_{l,l}(\Lambda_\nu^{-1}(z))dz_l\otimes\frac{\partial}{\partial\overline{z_l}}+\\&\overline{B_{l,l}}(\Lambda_\nu^{-1}(z))d\overline{z_l}\otimes\frac{\partial}{\partial z_l}+\overline{A_{l,l}}(\Lambda_\nu^{-1}(z))d\overline{z_l}\otimes\frac{\partial}{\partial\overline{z_l}})+\\&\tau(p_\nu^*,\delta_\nu)\delta_\nu^{-1}A_{1,2}(\Lambda_\nu^{-1}(z))dz_1\otimes\frac{\partial}{\partial z_2}+\tau(p_\nu^*,\delta_\nu)\delta_\nu^{-1}B_{1,2}(\Lambda_\nu^{-1}(z))dz_1\otimes\frac{\partial}{\partial\overline{z_2}}+\\&\tau(p_\nu^*,\delta_\nu)\delta_\nu^{-1}\overline{B_{1,2}}(\Lambda_\nu^{-1}(z))d\overline{z_1}\otimes\frac{\partial}{\partial z_2}+\tau(p_\nu^*,\delta_\nu)\delta_\nu^{-1}\overline{A_{1,2}}(\Lambda_\nu^{-1}(z))d\overline{z_1}\otimes\frac{\partial}{\partial\overline{z_2}}.\end{aligned}$$

According to (3.16) and since $c_{k,\nu}$ converges to zero when ν tends to $+\infty$ for $k=2,3$, it follows that $\tilde{J}_\nu$ converges to J_{st}. This proves part (2). □

3.3.2 Complete hyperbolicity in D'Angelo type four condition

In this subsection we prove Theorem 3.3.1. Keeping notations of the previous subsection; we start by establishing the following lemma which gives a precise localization of pseudoholomorphic discs in boxes.

Lemma 3.3.3. *Assume the origin $\in \partial D$ is a point of D'Angelo type four. There are positive constants C_0, δ_0 and r_0 such that for any $0 < \delta < \delta_0$, for any large ν and for any J_ν-holomorphic disc $g_\nu : \Delta \to D_\nu$ we have :*

$$g_\nu(0) = (0, -\delta_\nu) \Rightarrow g_\nu(r_0\Delta) \subset Q(0, C_0\delta_\nu),$$

where $Q(0,\delta_\nu) := \{z \in \mathbb{C}^2 : |z_1| \le \tau(p_\nu^*, \delta_\nu), |z_2| \le \delta_\nu\}$.

Proof. Proof of Lemma 3.3.3. Assume by contradiction that there are a sequence $(C_\nu)_\nu$ that tends to $+\infty$ as ζ_ν converges to 0 in Δ, and J_ν-holomorphic discs $g_\nu : \Delta \to D_\nu$ such that $g_\nu(0) = (0, -\delta_\nu)$ and $g_\nu(\zeta_\nu) \notin Q(0, C_\nu\delta_\nu)$. We consider the nonisotropic dilations of $\mathbb{C}^2$:

$$\Lambda_\nu^r \ : (z_1, z_2) \mapsto \left(r^{\frac{1}{4}}\tau(p_\nu^*, \delta_\nu)^{-1} z_1, r\delta_\nu^{-1} z_2\right),$$

where r is a positive constant to be fixed. We set $h_\nu := \Lambda_\nu^r \circ g_\nu$, $\tilde{\rho_\nu^r} := r\delta_\nu^{-1}\rho_\nu \circ (\Lambda_\nu^r)^{-1}$ and $\tilde{J_\nu^r} := (\Lambda_\nu^r)_* J_\nu$. It follows from Lemma 3.3.2 that $\tilde{\rho_\nu^r}$ converges to

$$\tilde{\rho} = Re(z_2) + P(z_1, \overline{z_1})$$

uniformly on any compact subset of $\mathbb{C}^2$ and $\tilde{J_\nu^r}$ converges to J_{st}, uniformly on any compact subset of $\mathbb{C}^2$. According to the stability of the Kobayashi pseudometric stated in Proposition 3.2.5, there exist a positive constant C and a neighborhood V of the origin in $\mathbb{R}^4$, such that for every large ν, for every $q \in \tilde{D_\nu} \cap V$ and every $v \in T_q\mathbb{R}^4$:

$$K_{\left(\tilde{D_\nu}, \tilde{J_\nu}\right)}(q, v) \ge C\|v\|.$$

Therefore, there exists a constant $C' > 0$ such that

$$\| \, dh_\nu(\zeta) \, \| \le C'$$

for any $\zeta \in (1/2)\Delta$ satisfying $h_\nu(\zeta) \in \tilde{D_\nu} \cap V'$, with $V' \subset V$. Now we choose the constant r such that $h_\nu(0) = (0, -r) \in Int(V')$. On the other hand, the sequence $|h_\nu(\zeta_\nu)|$ tends to $+\infty$. Denote by $[0, \zeta_\nu]$ the segment (in $\mathbb{C}$) joining the origin and ζ_ν and let $\zeta'_\nu = r_\nu e^{i\theta_\nu} \in [0, \zeta_\nu]$ be the point closest to the origin such that $h_\nu([0, \zeta'_\nu]) \subset \tilde{D_\nu} \cap V$ and $h_\nu(\zeta'_\nu) \in \partial V$. Since $h_\nu(0) \in Int(V')$, we have

$$\|h_\nu(0) - h_\nu(\zeta'_\nu)\| \ge C''$$

for some constant $C'' > 0$. It follows that:

$$\|h_\nu(0) - h_\nu(\zeta'_\nu)\| \le \int_0^{r_\nu} \left\|dh_\nu\left(te^{i\theta_\nu}\right)\right\| dt \le C' r_\nu \longrightarrow 0.$$

This contradiction proves Lemma 3.3.3. □

Now we go on the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. Due to the localization of the Kobayashi pseudometric established in Proposition 3.2.2, it suffices to prove Theorem 3.3.1 in a neighborhood U of $q \in \partial D$. Choosing local coordinates $z : U \to \mathbb{B} \subset \mathbb{R}^4$ centered at q, we may assume that $D \cap U = \{\rho < 0\}$ is a J-pseudonconvex region of $(\mathbb{R}^4, J)$, that $q = 0 \in \partial D$ and that J satisfies (3.2) and (3.3). We also suppose that the complex tangent space $T_0\partial D \cap J(0)T_0\partial D$ at 0 of ∂D is given by $\{z_2 = 0\}$. Moreover the defining function ρ is expressed by:

$$\rho(z) = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right)$$

For $p \in D \cap U$ be sufficiently close to the boundary ∂D, there exists a unique point $p^* \in \partial D \cap U$ such that

$$p^* = p + (0, \delta),$$

with $\delta > 0$. We define an infinitesimal pseudometric N on $D \cap U \subseteq \mathbb{R}^4$ by:

$$N(p,v) := \frac{|(d_p\Phi_{p^*}v)_1|}{\tau(p^*, |\rho(p)|)} + \frac{|(d_p\Phi_{p^*}v)_2|}{|\rho(p)|}, \tag{3.17}$$

for every $p \in D \cap U$ and every $v \in T_p\mathbb{R}^4$, where Φ_{p^*} is defined as diffeomorphisms Φ^ν (of previous subsection) for p^* instead of p^*_ν.

To prove estimate (3.13) of Theorem 3.3.1, it suffices to find a positive constant C such that for any J-holomorphic disc $u : \Delta \to D \cap U$, we have:

$$N(u(0), d_0u(\partial/\partial_x)) \leq C. \tag{3.18}$$

Indeed, for a J-holomorphic disc u such that $u(0) = p$ and $d_0u(\partial/\partial_x) = rv$, (3.18) leads to

$$\frac{1}{r} = \frac{N(p,v)}{N(u(0), d_0u(\partial/\partial_x))} \geq \frac{N(p,v)}{C}.$$

Suppose by contradiction that (3.18) is not true, that is, there is a sequence of J-holomorphic discs $u_\nu : \Delta \to D \cap U$ such that $N(u_\nu(0), d_0u_\nu(\partial/\partial_x)) \geq \nu^2$. Then we consider a sequence $(y_\nu)_\nu$ of points in $\overline{\Delta_{1/2}}$ such that:

1. $|y_\nu| \leq \dfrac{2\nu}{N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x))}$,
2. $N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x)) \geq \nu^2$, and
3. $y_\nu + \Delta_{\nu/N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x))} \subseteq \overline{\Delta_{1/2}}$ for sufficiently large ν.

This allows to define a sequence of J-holomorphic discs $g_\nu : \Delta_\nu \to D \cap U$ by

$$g_\nu(\zeta) := u_\nu\left(y_\nu + \frac{\zeta}{2N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x))}\right).$$

Consider the sequence $g_\nu = u_\nu(y_\nu)$ in $D \cap U$. Since $|y_\nu| \leq 2/\nu$ and since the $\mathcal{C}^1$ norm of any J-holomorphic disc u_ν is uniformly bounded it follows that $g_\nu(0)$ converges to the origin.

We apply the scaling method to the sequence $g_\nu(0)$. We denote by $g_\nu(0)^*$ the boundary point given by $g_\nu(0)^* := g_\nu(0) + (0, \delta_\nu)$. We set the scaled disc $\tilde{g_\nu} := \Lambda_\nu \circ \Phi_\nu \circ g_\nu$, where diffeomorphisms Λ_ν and Φ_ν are define in the subsection about the scaling method. In order to extract from $\tilde{g_\nu}$ a subsequence which converges to a Brody curve, we need the following Lemma.

Lemma 3.3.4. *There is a positive constant r_0 such that:*

1. *There exists a positive constant C_1 such that*

$$\tilde{g_\nu}(r_0\Delta_\nu) \subset \Delta_{C_1} \times \Delta_{C_1}. \tag{3.19}$$

2. *There exists a positive constant C_2 such that for every large ν we have :*

$$\|d\tilde{g_\nu}\|_{\mathcal{C}^0(r_0\Delta_\nu)} \leq C_2. \tag{3.20}$$

Proof. We prove the first part. We define a J_ν-holomorphic disc $h_\nu(\zeta) := \Phi_\nu \circ g_\nu(\nu\zeta)$ from the unit disc Δ to D_ν. According to Lemma 3.3.3, since $h_\nu(0) = \Phi_\nu \circ g_\nu(0) = (0, -\delta_\nu)$, we have

$$h_\nu(r_0\Delta) \subseteq Q(0, C_0\delta_\nu)$$

for some positive constants r_0 and C_0. Hence

$$\Phi_\nu \circ g_\nu(r_0\Delta_\nu) \subseteq Q(0, C_0\delta_\nu).$$

After dilations, this leads to (3.19).

Then we prove the second part. According to Lemma 3.3.2, the sequence of almost complex structures $\tilde{J}_\nu$ converges on any compact subsets of $\mathbb{C}^2$ in the $\mathcal{C}^2$ sense to J_{st}. Then for sufficiently large ν, the norm $\|\tilde{J}_\nu - J_{st}\|_{\mathcal{C}^1(\overline{\Delta_{C_1} \times \Delta_{C_1}})}$ is as small as necessary. So for large ν, and due to Proposition $2.3.6$ of J.-C.Sikorav in [65] there exists $C_2 > 0$ such that (3.20) holds. □

Hence according to Lemmas 3.3.2 and 3.3.4 we may extract from $\tilde{g_\nu}$ a subsequence, still denoted by $\tilde{g_\nu}$ which converges in $\mathcal{C}^1$ topology to a standard complex line

$$\tilde{g} : \mathbb{C} \to (\{Rez_2 + P(z_1, \overline{z_1}) < 0\}, J_{st}).$$

The polynomial P is subharmonic and contains a nonharmonic part; this implies that the domain $(\{Rez_2 + P(z_1, \overline{z_1}) < 0\}, J_{st})$ is Brody hyperbolic and so the complex line $\tilde{g}$ is constant. To obtain a contradiction, we prove that the derivative of $\tilde{g}$ at the origin is nonzero:

$$\frac{1}{2} = N(g_\nu(0), d_0 g_\nu(\partial/\partial_x)) = \frac{|(d_0(\Phi_\nu \circ g_\nu)(\partial/\partial_x))_1|}{\tau(g_\nu(0)^*, |\rho(g_\nu(0))|)} + \frac{|(d_0(\Phi_\nu \circ g_\nu)(\partial/\partial_x))_2|}{|\rho(g_\nu(0))|}.$$

Since $|\rho(g_\nu(0))|$ is equivalent to δ_ν, it follows that for some positive constant C_3 and for large ν, we have:

$$\frac{1}{2} \leq C_3 \left(\frac{|(d_0(\Phi_\nu \circ g_\nu)(\partial/\partial_x))_1|}{\tau(g_\nu(0)^*, \delta_\nu)} + \frac{|(d_0(\Phi_\nu \circ g_\nu)(\partial/\partial_x))_2|}{\delta_\nu} \right) = C_3 \|d_0 \tilde{g_\nu}(\partial/\partial_x)\|_1.$$

Since $\tilde{g_\nu}$ converges to $\tilde{g}$ in the $\mathcal{C}^1$ sense, it follows that $d_0\tilde{g}(\partial/\partial_x) \neq 0$, providing a contradiction. This achieves the proof of Theorem 3.3.1. □

Estimate (3.14) of the Kobayashi pseudometric allows to study the completeness of the Kobayashi pseudodistance D.

Corollary 3.3.5. *Let $D = \{\rho < 0\}$ be a relatively compact domain of finite D'Angelo type less than or equal to four in an almost complex manifold (M, J) of dimension four, where ρ is a defining function of D, J-plurisubharmonic in a neighborhood of $\overline{D}$. Assume that (M, J) admits a global strictly J-plurisubharmonic function. Then (D, J) is complete hyperbolic.*

Proof. The fact that (M, J) admits a global strictly J-plurisubharmonic function and estimate (3.6) of Proposition 3.2.2 leads to the Kobayashi hyperbolicity of D. Then estimate (3.14) of the Kobayashi pseudometric stated in Theorem 3.3.1 gives the completeness of the metric space $\left(D, d_{(D,J)}\right)$ by a classical integration argument. □

3.3.3 Regions with noncompact automorphisms group

The next corollary is devoted to regions with noncompact automorphisms group.

Corollary 3.3.6. *Let $D = \{\rho < 0\}$ be a relatively compact domain in a four dimensional almost complex manifold (M, J) of finite D'Angelo type less than or equal to four. Assume that ρ is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. If there is an automorphism of D with orbit accumulating at a boundary point then there exists a polynomial P of degree at most four, without harmonic part such that (D, J) is biholomorphic to $(\{\Re e z_2 + P(z_1, \overline{z_1}) < 0\}, J_{st})$.*

If the domain D is a relatively compact strictly J-pseudoconvex domain with noncompact automorphisms group then (D, J) is biholomorphic to a model domain. This was proved by H.Gaussier and A.Sukhov in [35] in the four dimensional case and by K.H.Lee in [50] in arbitrary (even) dimension. Although this theorem is new until now, its proof is quite similar to the proof of the equivalent theorem for strictly J-pseudoconvex domains given by K.H.Lee in [50]. Indeed the proof is mainly based on the explosion of the Kobayashi pseudodistance near the boundary ∂D, which is new in type four condition.

Proof. We suppose that for some point $p_0 \in D$, there is a sequence φ_ν of automorphisms of (D, J) such that $p_\nu := \varphi_\nu(p_0)$ converges to $0 \in \partial D$. We apply the scaling method to the sequence p_ν. Still keeping notations of previous subsections, we set

$$F_\nu := \Lambda_\nu \circ \Phi_\nu \circ \varphi_\nu \ : \varphi_\nu^{-1}(D \cap U) \to \tilde{D}_\nu.$$

In order to extract from $(F_\nu)_\nu$ a subsequence converging to map F, and to describe the limit F we need the two next lemmas.

Lemma 3.3.7. *Let K be a compact in D such that $p_0 \in K$. Then for large ν*

$$\varphi_\nu(K) \subset D \cap U. \tag{3.21}$$

Proof. There exists a constant C_K such that

$$d_{(D,J)}(p_0, q) \leq C_K,$$

for every $q \in K$. Since the kobayashi pseudodistance in invariant under biholomorphims, it follows that

$$d_{(D,J)}(p_\nu, \varphi_\nu(q)) \leq C_K.$$

Moreover according to Corollary 3.3.5, the distance $d_{(D,J)}(p_\nu, D \cap \partial U)$ tends to $+\infty$ as ν tends to $+\infty$. This finally implies that (3.21) is satisfied for large ν. □

Lemma 3.3.8. *For any compact subset $K \subset D$,*

1. *the sequence $\left(\|F_\nu\|_{\mathcal{C}^0(K)}\right)_\nu$ is bounded.*
2. *there is a positive constant C''_K such that*

   $$\|d_q F_\nu(v)\| \leq C''_K \|v\|, \tag{3.22}$$

   for every $q \in K$ and $v \in T_q M$.

Proof. We proof the first part. We consider a finite covering $U_{q_j}, j = 0, \cdots, N$ of K, with $q_0 = p_0$, where U_{q_j} is a neighborhood of $q_j \in K$ such that there is a family $\mathcal{F}_j$ of J-holomorphic discs passing trough q_j and satisfying

$$U_{q_j} \subset \bigcup_{u \in \mathcal{F}_j} u(\Delta_{r(q_j)}),$$

with $r(q_j) < r_0$ (see [27], [45], [48]), where r_0 is given in Lemma 3.3.3. We may assume that $U_{q_j} \cap U_{q_{j+1}} \neq \emptyset$. We set

$$r := \max r(q_j) < r_0.$$

According to Lemma 3.3.3, since $\Phi_\nu \circ \varphi_\nu(p_0) \in Q(0, \delta_\nu)$ it follows that

$$\Phi_\nu \circ \varphi_\nu \circ u(\Delta_r) \subset Q(0, C_{r_0}\delta_\nu)$$

for any $u \in \mathcal{F}_0$. Hence we have

$$\Phi_\nu \circ \varphi_\nu(U_{q_0}) \subset Q(0, C_{r_0}\delta_\nu).$$

There is a disc $u \in \mathcal{F}_{q_1}$ and a point $\xi_1 \in \Delta_r$ such that $u(\xi_1) \in U_{q_0} \cap U_{q_1}$. Then consider the following J-holomorphic disc

$$g(\xi) := u\left(\frac{\xi + \xi_1}{1 + \overline{\xi_1}\xi}\right)$$

satisfying

$$\begin{cases} g(0) = u(\xi_1) & \in & Q(0, C_{r_0}\delta_\nu), \\ g(\xi_1) & = & u(0). \end{cases}$$

It follows that:

$$\Phi_\nu \circ \varphi_\nu(q_1) \in Q(0, C_{r_0}^2\delta_\nu),$$

and then

$$\Phi_\nu \circ \varphi_\nu(U_{q_1}) \subset Q(0, C_{r_0}^3\delta_\nu)$$

for any $u \in \mathcal{F}_1$. Continuing this process, we obtain

$$\Phi_\nu \circ \varphi_\nu(U_{q_j}) \subset Q(0, C_{r_0}^{2j+1}\delta_\nu).$$

Finally there is a positive constant C'_K such that

$$\Phi_\nu \circ \varphi_\nu(K) \subset Q(0, C'_K\delta_\nu).$$

It follows that the sequence $(\|F_\nu\|_{\mathcal{C}^0(K)})_\nu$ is bounded.

Let us prove part 2. It is sufficient to prove (3.22) for small v. Let $q \in K$ and $v \in T_qD$ such that $\|v\|$ is sufficiently small. Then consider a J-holomorphic disc $u : \Delta \rightarrow D$ passing trough q with $d_0u(\partial/\partial_x) = v$. Since the restriction of F_ν on the disc $u_{|\Delta_r}$ is uniformly bounded in the $\mathcal{C}^0$ norm, it follows from Proposition 2.3.6 of [65] that there exists a positive constant C''_K such that

$$\|d_qF_\nu(v)\| = \|d_0(F_\nu \circ u)(\partial/\partial_x)\| \leq C''_K.$$

This ends the proof of Lemma 3.3.8. □

We know from Lemma 3.3.2 that the domain $\tilde{D}_\nu$ converges in the sense of local Hausdorff set convergence to a pseudoconvex domain $\tilde{D} = \{Rez_2 + P(z_1, \overline{z_1}) < 0\}$, where P is a nonzero subharmonic polynomial of degree ≤ 4 which contains a non harmonic part. Changing $\tilde{D}$ by applying a standard biholomorphism if necessary, we may suppose that $P(z_1, \overline{z_1})$ is without harmonic terms.

According to Lemma 3.3.7 and Lemma 3.3.8, we may extract from $(F_\nu)_\nu$ a subsequence converging, uniformly on compact subsets of D, to a (J, J_{st})-holomorphic map

$$F : D \longrightarrow \bar{\tilde{D}}.$$

It remains to proof that F is a biholomorphism from D to $\tilde{D}$. We denote by $G_\nu : \tilde{D}_\nu \rightarrow \varphi_\nu^{-1}(D \cap U)$ the $(\tilde{J}_\nu, J)$-biholomorphism satisfying $G_\nu = (F_\nu)^{-1}$. According to Lemma 3.3.2, for any relatively compact neighborhood V of $(0, -1)$ in $\tilde{D}$, we have $V \subset \tilde{D}_\nu$ for large ν. Moreover the domain $(\tilde{D}, J_{st})$ is also complete hyperbolic. Since the $\{G_\nu(0, -1),\ \nu\} = \{p_0\}$ is relativily compact, it follows from Proposition 2.3 of [50], that for any relatively compact neighborhood V of $(0, -1)$ in $\tilde{D}$, the sequence $((G_\nu)_{|V})_\nu$ admits a subsequence converging to a (J_{st}, J)-holomorphic map $G : V \rightarrow D$. Then there is a (J_{st}, J)-holomorphic map, still denoted $G : \tilde{D} \to D$ which is subsequential limit of $(G_\nu)_\nu$ on every compact exhaustion of $\tilde{D}$. By passing at the limit, we obtain:

$$\begin{cases} F \circ G & = & Id_{\tilde{D}} \\ & & \\ G \circ F_{|F^{-1}(\tilde{D})} & = & Id_{F^{-1}(\tilde{D})}. \end{cases}$$

Let us proove that $F^{-1}(\tilde{D}) = D$ by contradiction. Let $q \in D \cap \partial F^{-1}(\tilde{D}) = F^{-1}(\partial \tilde{D})$, and let $(q_j)_j$ be a sequence of $F^{-1}(\tilde{D})$ converging to q. Since the domain $(\tilde{D}, J_{st})$ is complete hyperbolic, the distance $d_{(\tilde{D},J_{st})}((0,-1), F(q_j))$ tends to $+\infty$ as ν tends to $+\infty$. This contradicts the following:

$$d_{(\tilde{D},J_{st})}((0,-1), F(q_j)) \leq d_{(D,J)}(p_0, q_j) \to d_{(D,J)}(p_0, q) < +\infty.$$

Finally F is a (J, J_{st})-biholomorphism from D to $\tilde{D}$.

□

3.3.4 Nontangential approach in the general setting

In this subsection, refering to I.Graham [39], we give a sharp estimate of the Kobayashi pseudometric of a pseudoconvex region in a cone with vertex at a boundary point of arbitrary finite D'Angelo type. We denote by $\Lambda := \{-\Re e z_2 > k\|z\|\}$, where $0 < k < 1$, the cone with vertex at the origin and axis the negative real z_2 axis.

Theorem 3.3.9. *Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in $(\mathbb{R}^4, J)$, where*

$$\rho(z_1, z_2) = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

is a $\mathcal{C}^2$ defining function of D, J-plurisubharmonic on a neighborhood of $\overline{D}$. We suppose that H_{2m} is a homogeneous subharmonic polynomial of degree $2m$ admitting a nonharmonic part. Then there exists a positive constant C such that for every $p \in D \cap \Lambda$ and every $v \in T_pM$:

$$K_{(D,J)}(p, v) \geq C\left(\frac{|v_1|}{|\rho(p)|^{\frac{1}{2m}}} + \frac{|v_2|}{|\rho(p)|}\right).$$

Before proving Theorem 3.3.9 we need the following crucial lemma.

Lemma 3.3.10. *There exist a neighborhood U of the origin and a positive constant C such that if $p \in D \cap U \cap \Lambda$ then*

$$p \in \left\{z \in \mathbb{C}^2 \ : |z_1| < C_1 \mathrm{dist}(p, \partial D)^{\frac{1}{2m}}, |z_2| < C_1 \mathrm{dist}(p, \partial D)\right\}.$$

Proof. According to the fact that $\mathrm{dist}(z, \partial D)$ is equivalent to $|\rho(z)| = -\Re e z_2 + O(\|z\|^2)$ and to the definition of the cone Λ, we have:

$$\lim_{z \to 0, z \in D \cap \Lambda} \frac{-\Re e z_2}{\mathrm{dist}(z, \partial D)} = 1.$$

This implies the existence of a positive constant C_1 such that

$$\|p\| < -\frac{1}{k}\Re e p_2 \leq C_1 \mathrm{dist}\,(p, \partial D)\,,$$

whenever $p \in D \cap \Lambda$ is sufficiently close to the origin. Thus

$$p \in \left\{ z \in \mathbb{C}^2 \ : |z_1| < C_1 \mathrm{dist}\,(p, \partial D)^{\frac{1}{2m}}\,, |z_2| < C_1 \mathrm{dist}\,(p, \partial D) \right\},$$

for $p \in D \cap \Lambda$ sufficiently close to the origin.

□

The proof of Theorem 3.3.9 is similar and easier than proof of Theorem 3.3.1. For convenience, we write it.

Proof of Theorem 3.3.9. Let U be a neighborhood of the origin. We define an infinitesimal pseudometric N on $D \cap U \subseteq \mathbb{R}^4$ by:

$$N\,(p, v) := \frac{|v_1|}{|\rho\,(p)\,|^{\frac{1}{2m}}} + \frac{|v_2|}{|\rho\,(p)\,|},$$

for every $p \in D \cap U$ and every $v \in T_p\mathbb{C}^2$.

We have to find a positive constant C such that for every J-holomorphic disc $u \ : \Delta \rightarrow D \cap U$, such that if $u\,(0) \in \Lambda$ then:

$$N\,(u\,(0)\,, d_0 u\,(\partial/\partial_x)) \leq C.$$

Suppose by contradiction that this inequality is not true, that is, there exists a sequence of J-holomorphic discs $u_\nu \ : \Delta \rightarrow D \cap U$ such that

$$u_\nu\,(0) \in \Lambda \quad \text{and} \quad N\,(u_\nu\,(0)\,, d_0 u_\nu\,(\partial/\partial_x)) \geq \nu^2.$$

Then consider a sequence $(y_\nu)_\nu$ of points in $\overline{\Delta_{1/2}}$ such that

1. $|y_\nu| \leq \dfrac{2\nu}{N\,(u_\nu\,(y_\nu)\,, d_{y_\nu} u_\nu\,(\partial/\partial x))}$,

2. $N\,(u_\nu\,(y_\nu)\,, d_{y_\nu} u_\nu\,(\partial/\partial x)) \geq \nu^2$, and

3. $y_\nu + \Delta_{\nu/N(u_\nu(y_\nu), d_{y_\nu} u_\nu(\partial/\partial x))} \subseteq \overline{\Delta_{1/2}}$ for sufficiently large ν.

Then we define a sequence of J-holomorphic discs $g_\nu \ : \Delta_\nu \rightarrow D \cap U$ by

$$g_\nu\,(\zeta) := u_\nu \left(y_\nu + \frac{\zeta}{2N\,(u_\nu\,(y_\nu)\,, d_{y_\nu} u_\nu\,(\partial/\partial x))} \right).$$

For large ν, we have $g_\nu(0) = u_\nu(y_\nu)$ in $D \cap U \cap \Lambda$ and $g_\nu(0)$ converges to the origin. Set

$$\delta_\nu := \operatorname{dist}(g_\nu(0), \partial D),$$

and consider the following dilations of $\mathbb{C}^2$:

$$\Lambda_\nu \ : (z_1, z_2) \mapsto \left(\delta_\nu^{\frac{-1}{2m}} z_1, \delta_\nu^{-1} z_2\right).$$

In order to extract from $\Lambda_\nu \circ g_\nu$ a subsequence which converges to a Brody curve, we need the following Lemma.

Lemma 3.3.11. *There exists a positive constant r_0 such that:*

1. *there exists a positive constant C_1 such that:*

$$\Lambda_\nu \circ g_\nu(r_0 \Delta_\nu) \subset \Delta_{C_1} \times \Delta_{C_1}, \tag{3.23}$$

2. *there is a positive constant C_2 such that for every large ν we have :*

$$\|d(\Lambda_\nu \circ g_\nu)\|_{\mathcal{C}^0(r_0\Delta_\nu)} \le C_2. \tag{3.24}$$

Proof. We first prove (3.23). We define a new J-holomorphic disc $h_\nu(\zeta) := g_\nu(\nu\zeta)$ from the unit disc Δ to D_ν. According to Lemma 3.3.10, we have

$$h_\nu(0) = g_\nu(0) \in \{z \in \mathbb{C}^2 : |z_1| \le C_1 \delta_\nu^{\frac{1}{2m}}, |z_2| \le C_1 \delta_\nu\}.$$

This implies:

$$h_\nu(r_0\Delta) \subseteq \{z \in \mathbb{C}^2 \ : |z_1| \le C_0 \delta_\nu^{\frac{1}{2m}}, |z_2| < C_0 \delta_\nu\},$$

for positive constants r_0 and C_0, since Lemma 3.3.3 is true if we replace $\tau(p_\nu^*, \delta_\nu)$ by $\delta_\nu^{\frac{1}{2m}}$. Hence

$$g_\nu(r_0\Delta_\nu) \subseteq \{z \in \mathbb{C}^2 \ : |z_1| < C_0 \delta_\nu^{\frac{1}{2m}}, |z_2| \le C_0 \delta_\nu\}.$$

After dilations, this leads to (3.23).

The proof of (3.24) is similar to (3.20) of Lemma 3.3.4, since the sequence of structures $(\Lambda_\nu)_* J$ converges on any compact subset of $\mathbb{C}^2$ in the $\mathcal{C}^1$ sense to J_{st} because J is diagonal. □

Hence according to Lemma 3.3.11 we may extract from $\Lambda_\nu \circ g_\nu$ a subsequence, still denoted by $\Lambda_\nu \circ g_\nu$ which converges in the $\mathcal{C}^1$ sense to a standard complex line $\tilde{g} \ : \mathbb{C} \to (\{Rez_2 + H_{2m}(z_1, \overline{z_1}) < 0\}, J_{st})$, where the domain $(\{Rez_2 + P(z_1, \overline{z_1}) < 0\}, J_{st})$ is Brody hyperbolic since $H_{2m}(z_1, \overline{z_1})$ contains a nonharmonic part. Then the standard complex line $\tilde{g}$ is constant. To obtain a contradiction, we prove that the derivative of $\tilde{g}$ is nonzero:

$$\frac{1}{2} = N(g_\nu(0), d_0 g_\nu(\partial/\partial_x)) = \frac{|(d_0 g_\nu(\partial/\partial_x))_1|}{|\rho(g_\nu(0))|^{\frac{1}{2m}}} + \frac{|(d_0 g_\nu(\partial/\partial_x))_2|}{|\rho(g_\nu(0))|}.$$

Since $|\rho(g_\nu(0))|$ is equivalent to δ_ν, it follows that for some positive constant C_3 we have for large ν:

$$\frac{1}{2} \leq C_3 \left(\frac{|(d_0(g_\nu)(\partial/\partial_x))_1|}{\delta_\nu^{\frac{1}{2m}}} + \frac{|(d_0(g_\nu)(\partial/\partial_x))_2|}{\delta_\nu} \right) = C_3 \|d_0(\Lambda_\nu \circ g_\nu)(\partial/\partial_x)\|_1.$$

This provide a contradiction. □

3.4 Appendix 1: Convergence of the structures involved by the scaling method.

In this appendix, we prove that, generically, the convergence of the sequence of structures involved by the scaling method to the standard structure J_{st} occurs only on a neighborhood of boundary points of D'Angelo type less than or equal to four.

Let $D = \{\rho < 0\}$ be a pseudoconvex region of finite D'Angelo type $2m$ in $\mathbb{R}^4$, where ρ has the following expression on a neighborhood U of the origin:

$$\rho(z_1, z_2) = \Re e z_2 + H_{2m}(z_1, \overline{z_1}) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right),$$

where H_{2m} is a homogeneous subharmonic polynomial of degree $2m$ admitting a nonharmonic part. Assume that p_ν is a sequence of points in $D \cap U$ converging to the origin, and, for large ν, consider the sequence of diffeomorphisms $\Phi^\nu : \mathbb{R}^4 \to \mathbb{R}^4$ given in the scaling method. We suppose that the function $\rho^\nu = \rho \circ (\Phi^\nu)^{-1}$ is given by:

$$\rho^\nu(z_1, z_2) = \Re e z_2 + \Re e\left(\alpha_\nu z_1^2\right) + \beta\nu|z_1|^2 + \sum_{k=3}^{2m} P_k(z_1, \overline{z_1}, p_\nu^*) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right).$$

Moreover the structure $J^\nu := (\Phi^\nu)_* J$ satisfies (3.2) and (3.3). To fix notations, we set:

$$J^\nu = \begin{pmatrix} a_1^\nu & b_1^\nu & 0 & 0 \\ c_1^\nu & -a_1^\nu & 0 & 0 \\ 0 & 0 & a_2^\nu & b_2^\nu \\ 0 & 0 & c_2^\nu & -a_2^\nu \end{pmatrix}.$$

Now, consider the following diffeomorphism of $\mathbb{R}^4$ defined by:

$$\Psi_\nu^{-1}(x_1, y_1, x_2, y_2) = (x_1 + R_{1,\nu}, y_1 + S_{1,\nu}, x_2 + R_{2,\nu}, y_2 + S_{2,\nu}) \tag{3.25}$$

converging to the identity and such that $d_0\Psi_\nu^{-1} = Id$. We suppose that $R_{k,\nu}$ and $S_{k,\nu}$, for $k = 1, 2$ are real functions depending smoothly on x_1, y_1 and y_2 and that $R_{2,\nu}$ and $S_{2,\nu}$ are given by:

$$\begin{cases} R_{2,\nu} = -\alpha_\nu x_1^2 + \alpha_\nu y_1^2 + O\left(|z_1|^3 + y_2^2 + |y_2|\|z\|\right), \\ S_{2,\nu} = -2\alpha_\nu x_1 y_1 + O\left(|z_1|^3 + y_2^2 + |y_2|\|z\|\right). \end{cases} \tag{3.26}$$

We write:

$$(3.27)\quad \begin{cases} R_{1,\nu} &= r_{5,\nu}x_1^2 + r_{6,\nu}x_1y_1 + r_{7,\nu}y_1^2 + r_{1,\nu}x_1^3 + r_{2,\nu}x_1^2y_1 + r_{3,\nu}x_1y_1^2 + \\ & \quad r_{4,\nu}y_1^3 + O\left(|z_1|^4 + y_2^2 + |y_2|\|z\|\right) \\ S_{1,\nu} &= s_{5,\nu}x_1^2 + s_{6,\nu}x_1y_1 + s_{7,\nu}y_1^2 + s_{1,\nu}x_1^3 + s_{2,\nu}x_1^2y_1 + s_{3,\nu}x_1y_1^2 + \\ & \quad s_{4,\nu}y_1^3 + O\left(|z_1|^4 + y_2^2 + |y_2|\|z\|\right). \end{cases}$$

It follows that:

$$\rho_\nu \circ \Psi_\nu^{-1}(z_1, z_2) = \Re e z_2 + \beta_\nu |z_1^2| + \sum_{k=3}^{2m} P'_k(z_1, \overline{z_1}, \nu) + O\left(|z_1|^{2m+1} + |z_2|\|z\|\right).$$

Then we define

$$\tau_\nu := \min\left(\left(\frac{\delta_\nu}{|\beta_\nu|}\right)^{\frac{1}{2}}, \min_{k=3,\cdots,2m-1}\left(\frac{\delta_\nu}{\|P'_k(.,\nu)\|}\right)^{\frac{1}{k}}, \delta_\nu^{\frac{1}{2m}}\right).$$

And we consider the following anisotropic dilations of $\mathbb{C}^2$:

$$\Lambda_\nu(z_1, z_2) := \left(\tau_\nu^{-1} z_1, \delta_\nu^{-1} z_2\right).$$

If we write $J_\nu := (\Psi_\nu)_* J^\nu$ as:

$$J_\nu = \begin{pmatrix} J_{1,\nu} & B_{1,\nu} \\ C_{1,\nu} & J_{2,\nu} \end{pmatrix} \text{ with } C_{1,\nu} := \begin{pmatrix} (J_\nu)_1^3 & (J_\nu)_2^3 \\ (J_\nu)_1^4 & (J_\nu)_2^4 \end{pmatrix},$$

then we have:

$$(\Lambda_\nu)_* J_\nu(z) = \begin{pmatrix} J_{1,\nu}(\tau_\nu z_1, \delta_\nu z_2) & \tau_\nu^{-1}\delta_\nu B_{1,\nu}(\tau_\nu z_1, \delta_\nu z_2) \\ \tau_\nu \delta_\nu^{-1} C_{1,\nu}(\tau_\nu z_1, \delta_\nu z_2) & J_{2,\nu}(\tau_\nu z_1, \delta_\nu z_2) \end{pmatrix}.$$

We have generically the following situation:

Proposition 3.4.1. *The sequence of structures $(\Lambda_\nu)_* J_\nu$ converges to the standard structure J_{st} if and only if the D'Angelo type of the origin is less than or equal to four.*

Proof. We notice that $(\Lambda_\nu)_* J_\nu$ converges to J_{st} if and only if $C_{1,\nu} = O\left(|z_1|^{2m-1}\right) + O\left(|z_2|\right)$. Indeed if $C_{1,\nu} = O\left(|z_1|^{2m-1}\right) + O\left(|z_2|\right)$ then

$$\tau_\nu \delta_\nu^{-1} C_{1,\nu}(\tau_\nu z_1, \delta_\nu z_2) = \tau_\nu^{2m}\delta_\nu^{-1} O|z_1|^{2m} + \tau_\nu^{2m} O|z_1|^{2m},$$

which converges to the zero 2 by 2 matrix since $\tau_\nu \leq \delta_\nu^{\frac{1}{2m}}$ and since $C_{1,\nu}$ tends to the zero 2 by 2 matrix. Conversely if $C_{1,\nu} = O\left(|z_1|^k\right) + O\left(|z_2|\right)$, with $k < 2m-1$, then $(\Lambda_\nu)_* J_\nu$ converges to a polynomial integrable structure $\tilde{J} = J_{st} + O|z_1|^2$ which is generically different from J_{st}.

We have proved in Lemma 3.3.2 that when the origin is a point of D'Angelo type four, then $C_{1,\nu} = O\left(|z_1|^3\right) + O\left(|z_2|\right)$ and so $(\Lambda_\nu)_* J_\nu = \left(\Lambda_\nu \circ \Psi_\nu\right)_* J^\nu$ converges to J_{st} when ν tends to $+\infty$, with:

$$\begin{cases} R_{1,\nu} &= S_{1,\nu} = 0, \\ R_{2,\nu} &= -\alpha_\nu x_1^2 + \alpha_\nu y_1^2, \\ S_{2,\nu} &= -2\alpha_\nu x_1 y_1. \end{cases}$$

In case the D'Angelo type of the origin is greater than four, we cannot guarantee the convergence of $\tau_\nu \delta_\nu^{-1} C_1^\nu \left(\tau_\nu z_1, \delta_\nu z_2\right)$ when we only remove harmonic terms. So we need to find a more general sequence of diffeomorphisms Ψ_ν defined by (3.25), (3.26) and (3.27) and such that $C_{1,\nu} = O\left(|z_1|^{2m-1}\right) + O\left(|z_2|\right)$.

Claim. There are no polynomial $R_{1,\nu}, S_{1,\nu}, R_{2,\nu}$ and $S_{2,\nu}$ such that $C_{1,\nu}$ does not contain any order three terms in x_1 and y_1.

A direct computation leads to:

$$\begin{aligned} \alpha_\nu^{-1} \left(J_\nu\right)_1^3(z) \quad = \quad & \left(a_2^\nu - a_1^\nu\right)\left(\Psi_\nu^{-1}(z)\right) x_1 - \left(c_1^\nu + b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right) y_1 - y_1 \frac{\partial R_{1,\nu}}{\partial x_1} \\ & - x_1 \frac{\partial R_{1,\nu}}{\partial y_1} - x_1 \frac{\partial S_{1,\nu}}{\partial x_1} + y_1 \frac{\partial S_{1,\nu}}{\partial y_1} + x_1 \frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial S_{1,\nu}}{\partial x_1} + y_1 \frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial S_{1,\nu}}{\partial y_1} \\ & - y_1 \frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial S_{1,\nu}}{\partial x_1} + x_1 \frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial S_{1,\nu}}{\partial y_1} - y_1 \left(\frac{\partial S_{1,\nu}}{\partial x_1}\right)^2 - y_1 \left(\frac{\partial S_{1,\nu}}{\partial y_1}\right)^2 \\ & - x_1 \frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial R_{2,\nu}}{\partial y_2} + x_1 \frac{\partial S_{1,\nu}}{\partial y_1}\frac{\partial R_{2,\nu}}{\partial y_2} + y_1 \frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial R_{2,\nu}}{\partial y_2} + \\ & y_1 \frac{\partial S_{1,\nu}}{\partial x_1}\frac{\partial R_{2,\nu}}{\partial y_2} + O\left(|z_1|^4 + |z_2|\|z\|\right) \end{aligned}$$

and to

$$\begin{aligned}\alpha_\nu^{-1}\left(J_\nu\right)_2^3(z) &= \left(b_1^\nu - b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1 + \left(a_1^\nu + a_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1 + x_1\frac{\partial R_{1,\nu}}{\partial x_1}\\ &-y_1\frac{\partial R_{1,\nu}}{\partial y_1} - y_1\frac{\partial S_{1,\nu}}{\partial x_1} - x_1\frac{\partial S_{1,\nu}}{\partial y_1} - x_1\left(\frac{\partial R_{1,\nu}}{\partial x_1}\right)^2 - x_1\left(\frac{\partial R_{1,\nu}}{\partial y_1}\right)^2 +\\ &y_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial S_{1,\nu}}{\partial x_1} + x_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial S_{1,\nu}}{\partial y_1} - x_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial S_{1,\nu}}{\partial x_1} + y_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial S_{1,\nu}}{\partial y_1}\\ &-x_1\frac{\partial R_{1,\nu}}{\partial y_1}\frac{\partial R_{2,\nu}}{\partial y_2} - x_1\frac{\partial S_{1,\nu}}{\partial x_1}\frac{\partial R_{2,\nu}}{\partial y_2} - y_1\frac{\partial R_{1,\nu}}{\partial x_1}\frac{\partial R_{2,\nu}}{\partial y_2} + y_1\frac{\partial S_{1,\nu}}{\partial y_1}\frac{\partial R_{2,\nu}}{\partial y_2} +\\ &O\left(|z_1|^4 + |z_2|\|z\|\right).\end{aligned}$$

The only order two terms in x_1 and y_1 of $\alpha_\nu^{-1}\left(J^\nu\right)_1^3(z)$ and of $\alpha_\nu^{-1}\left(J^\nu\right)_2^3(z)$ are those contained, respectively, in

$$-y_1\frac{\partial R_{1,\nu}}{\partial x_1} - x_1\frac{\partial R_{1,\nu}}{\partial y_1} - x_1\frac{\partial S_{1,\nu}}{\partial x_1} + y_1\frac{\partial S_{1,\nu}}{\partial y_1}$$

and

$$x_1\frac{\partial R_{1,\nu}}{\partial x_1} - y_1\frac{\partial R_{1,\nu}}{\partial y_1} - y_1\frac{\partial S_{1,\nu}}{\partial x_1} - x_1\frac{\partial S_{1,\nu}}{\partial y_1}.$$

Vanishing these order two terms leads to:

$$\begin{cases} R_{1,\nu} = r_{5,\nu}x_1^2 - 2s_{5,\nu}x_1y_1 - r_{5,\nu}y_1^2 + r_{1,\nu}x_1^3 + r_{2,\nu}x_1^2y_1 + r_{3,\nu}x_1y_1^2 + r_{4,\nu}y_1^3 +\\ \qquad O\left(|z_1|^4 + y_2^2 + |y_2|\|z\|\right)\\ S_{1,\nu} = s_{5,\nu}x_1^2 + 2s_{5,\nu}x_1y_1 - s_{5,\nu}y_1^2 + s_{1,\nu}x_1^3 + s_{2,\nu}x_1^2y_1 + s_{3,\nu}x_1y_1^2 + s_{4,\nu}y_1^3 +\\ \qquad O\left(|z_1|^3 + y_2^2 + |y_2|\|z\|\right).\end{cases}$$

Then it follows that:

$$\begin{aligned}\alpha_\nu^{-1}\left(J_\nu\right)_1^3(z) &= \left(a_2^\nu - a_1^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1 - \left(c_1^\nu + b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1 - y_1\frac{\partial R_{1,\nu}}{\partial x_1}\\ &-x_1\frac{\partial R_{1,\nu}}{\partial y_1} - x_1\frac{\partial S_{1,\nu}}{\partial x_1} + y_1\frac{\partial S_{1,\nu}}{\partial y_1} + O\left(|z_1|^4 + |z_2|\|z\|\right),\end{aligned}$$

and that

$$\begin{aligned}\alpha_\nu^{-1}\left(J_\nu\right)_2^3(z) &= \left(b_1^\nu - b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1 + \left(a_1^\nu + a_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1 + x_1\frac{\partial R_{1,\nu}}{\partial x_1}\\ &-y_1\frac{\partial R_{1,\nu}}{\partial y_1} - y_1\frac{\partial S_{1,\nu}}{\partial x_1} - x_1\frac{\partial S_{1,\nu}}{\partial y_1} + O\left(|z_1|^4 + |z_2|\|z\|\right).\end{aligned}$$

Since J^ν satisfies (3.3), we have:

$$\begin{cases} \left(a_2^\nu - a_1^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1 - \left(c_1^\nu + b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1 &= H_{3,\nu}(x_1,y_1) + \\ & O\left(|z_1|^4 + |z_2|\|z\|\right) \\ \left(b_1^\nu - b_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)x_1 + \left(a_1^\nu + a_2^\nu\right)\left(\Psi_\nu^{-1}(z)\right)y_1 &= H'_{3,\nu}(x_1,y_1) + \\ & O\left(|z_1|^4 + |z_2|\|z\|\right), \end{cases}$$

where $H_{3,\nu}(x_1,y_1)$ and $H'_{3,\nu}(x_1,y_1)$ are real homogeneous polynomials of degree three in x_1 and y_1 which are generically non identically zero. Since we cannot insure the convergence of

$$\alpha_\nu\tau_\nu\delta_\nu^{-1}H_{3,\nu}(\tau_\nu x_1,\tau_\nu y_1) = \alpha_\nu\tau_\nu^4\delta_\nu^{-1}H_{3,\nu}(x_1,y_1)$$

and

$$\alpha_\nu\tau_\nu\delta_\nu^{-1}H'_{3,\nu}(\tau_\nu x_1,\tau_\nu y_1) = \alpha_\nu\tau_\nu^4\delta_\nu^{-1}H'_{3,\nu}(x_1,y_1),$$

we want to cancel polynomials $H_{3,\nu}(x_1,y_1)$ and $H'_{3,\nu}(x_1,y_1)$ by order three terms in x_1 and y_1 contained in

$$-y_1\frac{\partial R_{1,\nu}}{\partial x_1} - x_1\frac{\partial R_{1,\nu}}{\partial y_1} - x_1\frac{\partial S_{1,\nu}}{\partial x_1} + y_1\frac{\partial S_{1,\nu}}{\partial y_1}$$

and

$$x_1\frac{\partial R_{1,\nu}}{\partial x_1} - y_1\frac{\partial R_{1,\nu}}{\partial y_1} - y_1\frac{\partial S_{1,\nu}}{\partial x_1} - x_1\frac{\partial S_{1,\nu}}{\partial y_1}.$$

Finally, vanishing order three terms in x_1 and y_1 of $\alpha_\nu^{-1}\left(J^\nu\right)_1^3(z)$ and of $\alpha_\nu^{-1}\left(J^\nu\right)_2^3(z)$ involve the following system of linear equations:

$$\begin{pmatrix} 3 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} r_{1,\nu} \\ r_{2,\nu} \\ r_{3,\nu} \\ r_{4,\nu} \\ s_{1,\nu} \\ s_{2,\nu} \\ s_{3,\nu} \\ s_{4,\nu} \end{pmatrix} = Y$$

Since this 8×8 system of linear equations is not a Cramer system, it follows that there does not exist, generically, polynomials $R_{1,\nu}$ and $S_{1,\nu}$ such that there are no order three term in x_1 and y_1 in $\left(J^\nu\right)_1^3(z)$ and $\left(J^\nu\right)_2^3(z)$. $\square$

3.5 Appendix 2: Estimates of the Kobayashi metric on strictly pseudoconvex domains

In the recent paper [35], H.Gaussier and A.Sukhov obtained precise lower estimates of the Kobayashi pseudometric of a strictly pseudoconvex domain in an almost complex manifold. In this section we obtain these estimates by a different approach based on some renormalization principle of pseudoholomorphic discs inspired by F.Berteloot [8]. The theorem we want to give a proof may be stated as follows:

Theorem 3.5.1. *Let $D = \{\rho < 0\}$ be a relatively compact domain in (M, J). We assume that ρ is a $\mathcal{C}^2$ defining function of D, strictly J-plurisubharmonic on a neighborhood of $\bar{D}$. Then there is a positive constant C with the following property: for every $p \in D$ and every $v \in T_pM$ there exists a diffeomorphism, Φ_{p^*}, in a neighborhood U of p, such that:*

$$K_{(D,J)}(p, v) \geq C \left[\frac{\| (d_p\Phi_{p^*}v)' \|}{|\rho(p)|^2} + \frac{| (d_p\Phi_{p^*}v)_n |^2}{|\rho(p)|^2} \right]^{1/2}, \tag{3.28}$$

for every $p \in D$ and every $v \in T_pM$.

In the above theorem we use the standard notations $(z_1, \cdots, z_{n-1}, z_n) = (z', z_n)$. Let $D = \{\rho < 0\}$ be a relatively compact domain in (M, J). We assume that ρ is a $\mathcal{C}^2$ defining function of D, strictly J-plurisubharmonic on a neighborhood of $\bar{D}$. Let $q \in \partial D$ be a boundary point. Due to the localization of the Kobayashi pseudometric (see Proposition 3 in [35] or Lemma 2.1 in [45]), it suffices to prove Theorem 3.5.1 on a neighborhood U of $q \in \partial D$. Choosing a coordinate system $\Phi\ : U\ \to \Phi(U) \subseteq \mathbb{R}^{2n}$ such that $\Phi(q) = 0$, we may identify $0 = q$, $\Phi(U) = U$, $\rho \circ \Phi^{-1} = \rho$ and $\Phi_* J = J$. Moreover we may suppose that:

1. the complex tangent space $T_0(\partial D) \cap J(0)\, T_0(\partial D)$ at 0 of ∂D is given by $\{z_n = 0\}$,

2. the defining function ρ can be expressed locally by:
$$\rho(z) = \Re e z_n + 2\Re e \sum \rho_{j,k} z_j z_k + \sum \rho_{j,\overline{k}} z_j \overline{z_k} + O(|z|^3),$$
where $\rho_{j,k}$ and $\rho_{j,\overline{k}}$ are constants satisfying $\rho_{j,k} = \rho_{k,j}$ and $\rho_{j,\overline{k}} = \overline{\rho}_{k,\overline{j}}$,

3. the structure J satisfies $J(0) = J_{st}$.

3.5.1 The scaling method

Assume that p_ν is a sequence of points in $D \cap U$ converging to the origin. For each p_ν sufficiently close to ∂D, there exists a unique point $p_\nu^* \in \partial D \cap U$ such that

$$\delta_\nu := d(p_\nu, \partial D) = \|p_\nu - p_\nu^*\|.$$

Notice that for large ν, the quantity δ_ν is equivalent to $|\rho(p_\nu)|$.

We consider a diffeomorphism $\Phi^\nu : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying:

1. $\Phi^\nu(p_\nu^*) = 0$ and $\Phi^\nu(p_\nu) = (0', -\delta_\nu)$,

2. Φ^ν converges to $Id : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ on any compact subset of $\mathbb{R}^{2n}$ in the $\mathcal{C}^2$ sense,

3. when we set $D^\nu := \Phi^\nu(D \cap U)$ and $J^\nu := (\Phi^\nu)_* J$, then the complex tangent space at 0 of ∂D^ν is equal to $\{z_n = 0\}$ and $J^\nu(0) = J_{st}$.

Moreover the sequence of defining functions $\rho^\nu := \rho \circ (\Phi^\nu)^{-1}$ converges to ρ in the $\mathcal{C}^2$ sense and J^ν converges to the structure J in the $\mathcal{C}^1$ sense. To fix the notations, we set:

$$\rho^\nu(z) = \Re e z_n + 2\Re e \sum \rho^\nu_{j,k} z_j z_k + \sum \rho^\nu_{j,\overline{k}} z_j \overline{z_k} + O(|z|^3).$$

We consider now the nonisotropic dilations of $\mathbb{C}^n$:

$$\Lambda_\nu : (z', z_n) \mapsto \left(\delta_\nu^{-\frac{1}{2}} z', \delta_\nu^{-1} z_n\right).$$

We set $\tilde{D^\nu} := \Lambda_\nu(D^\nu) = \{\tilde{\rho_\nu} = \delta_\nu^{-1}\rho_\nu \circ \Lambda_\nu^{-1} < 0\}$, and $\tilde{J_\nu} := (\Lambda_\nu)_* (J_\nu)$.

The following lemma (see [23], [35] or [50] for a proof) states that passing to the limit, we obtain a model domain:

Lemma 3.5.2.

1. *The sequence of almost complex structures $\tilde{J_\nu}$ converges on any compact subsets of $\mathbb{C}^n$ in the $\mathcal{C}^1$ sense to $\tilde{J}(z', z_n) = J_{st} + L(z', 0)$, where $L(z', 0) = (L_{k,j}(z', 0))_{k,j}$ denotes a matrix with $L_{k,j} = 0$ for $k = 1 \cdots, n-1$, $j = 1, \cdots, n$, $L_{n,n} = 0$, and $L_{n,j}(z', 0)$, $j = 1, \cdots n-1$ being real linear forms in z'.*

2. *The domain $\tilde{D^\nu}$ converges in the sense of local Hausdorff set convergence to $\tilde{D} = \{\tilde{\rho} < 0\}$, with*

$$\tilde{\rho}(z) = Re z_n + Re \sum_{j,k=1}^{n-1} \rho_{j,k} z_j z_k + \sum_{j,k=1}^{n-1} \rho_{j,\overline{k}} z_j \overline{z_k}.$$

The next lemma is crucial for the proof of Theorem 3.5.1. This implies that the domain $\tilde{D}$ does not contain any nonconstant $\tilde{J}$-complex line.

Lemma 3.5.3. *The domain $\tilde{D}$ is strictly $\tilde{J}$-pseudoconvex and admits a global $\tilde{J}$-plurisubharmonic defining function.*

Proof. By the invariance of the Levi form we have:

$$\mathcal{L}_J(\rho)(0, \Lambda_\nu^{-1}(v)) = \mathcal{L}_{\tilde{J}_\nu}(\rho \circ \Lambda_\nu^{-1})(0, v).$$

Since ρ is strictly J-plurisubharmonic, multiplying by δ^{-1} and passing to the limit at the right side as $\delta \longrightarrow 0$, we obtain:

$$\mathcal{L}_{\tilde{J}}(\tilde{\rho})(0,v) \geq 0$$

for any v. Now let $v = ('v, 0)$. Then $\Lambda_\nu^{-1}(v) = \delta^{1/2}v$ and so

$$\mathcal{L}_J(\rho)(0,v) = \mathcal{L}_{\tilde{J}_\nu}(\tilde{\rho_\nu})(0,v).$$

Passing to the limit as δ tends to zero, we obtain

$$\mathcal{L}^{J_0}(\tilde{\rho})(0,v) > 0$$

for any $v = ('v, 0)$ with $'v \neq 0$. This proves $\tilde{D}$ is strictly $\tilde{J}$-pseudoconvex.

Moreover since the Levi form is invariant under diffeomorphisms, we obtain, passing to the limit, that the defining function $\tilde{\rho}$ is J-plurisubharmonic. □

3.5.2 Proof of Theorem 3.5.1

Proof of Theorem 3.5.1. We define an infinitesimal pseudometric N on $D \cap U \subseteq \mathbb{R}^{2n}$ by:

$$N(p,v) := \frac{|\,(d_p\Phi_{p^*}v)'\,|}{|\rho(p)|^{\frac{1}{2}}} + \frac{|\,(d_p\Phi_{p^*}v)_n\,|}{|\rho(p)|},$$

for every $p \in D \cap U$ and every $v \in T_p\mathbb{R}^{2n}$, where the diffeomorphism Φ_{p^*} is defined as diffeomorphisms Φ^ν for p^* instead of p^*_ν, p^* being the unique boundary point such that $\|p - p^*\| = dist(p, \partial D)$.

To prove (3.28), it suffices to find a positive constant C such that for every J-holomorphic disc $u\ : \Delta \rightarrow D \cap U$, we have:

$$N\left(u\left(0\right), d_0u\left(\partial/\partial_x\right)\right) \leq C. \tag{3.29}$$

Suppose by contradiction that (3.29) is not true; there is a sequence of J-holomorphic discs $u_\nu : \Delta \rightarrow D \cap U$ such that

$$N\left(f_\nu\left(0\right), d_0f_\nu\left(\partial/\partial_x\right)\right) \geq \nu^2.$$

Then consider a sequence $(y_\nu)_\nu$ of points in $\overline{\Delta_{1/2}}$ such that

1. $|y_\nu| \leq \dfrac{2\nu}{N\left(u_\nu\left(y_\nu\right), d_{y_\nu}u_\nu\left(\partial/\partial x\right)\right)}$,

2. $N\left(u_\nu\left(y_\nu\right), d_{y_\nu}u_\nu\left(\partial/\partial x\right)\right) \geq \nu^2$,

3. $y_\nu + \Delta_{\nu/N(u_\nu(y_\nu), d_{y_\nu}u_\nu(\partial/\partial x))} \subseteq \overline{\Delta_{1/2}}$, for sufficiently large ν.

Then we define a sequence of J-holomorphic discs $g_\nu : \Delta_\nu \to D \cap U$ by

$$g_\nu(\zeta) := u_\nu \left(y_\nu + \frac{\zeta}{2N(u_\nu(y_\nu), d_{y_\nu} u_\nu(\partial/\partial x))} \right).$$

Consider the sequence $p_\nu := g_\nu(0) = f_\nu(y_\nu)$ in $D \cap U$, which converges to the origin. We apply the scaling method to the sequence p_ν. We define the scaled pseudoholomorphic disc $\tilde{g_\nu} := \Lambda_\nu \circ \Phi_\nu \circ g_\nu$, where diffeomorphisms Λ_ν and Φ_ν are described in previous subsection.

Due to the following lemma (see [37]), we may localize J-holomorphic discs $\Phi_\nu \circ g_\nu$.

Lemma 3.5.4. *There exist $C_0 > 0$, $\delta_0 > 0$ and $r_0 > 0$ such that for every $0 < \delta < \delta_0$, for every $\nu >> 1$ and for every J_ν-holomorphic disc $h_\nu : \Delta \to D^\nu$ we have:*

$$h_\nu(0) \in Q(0,\delta) \Rightarrow h_\nu(\Delta_{r_0}) \subset Q(0, C_0\delta),$$

where $Q(0,\delta) := \{z = (z', z_n) \in \mathbb{C}^n : |z'| \leq \delta^{\frac{1}{2}}, |z_n| \leq \delta\}$.

We apply this lemma to the J_ν-holomorphic discs $h_\nu(\zeta) := \Phi_\nu \circ g_\nu(\nu\zeta)$. Since

$$h_\nu(0) = (0', -\delta_\nu) \in Q(0, \delta_\nu),$$

we obtain

$$h_\nu(\Delta_{r_0}) \subseteq Q(0, C_0\delta_\nu)$$

for some positive constants r_0 and C_0 and finally:

$$\Phi_\nu \circ g_\nu(\Delta_{r_0\nu}) \subseteq Q(0, C_0\delta_\nu). \tag{3.30}$$

It follows from (3.30) that:

$$\tilde{g_\nu}(r_0\Delta_\nu) \subseteq Q(0, C_0). \tag{3.31}$$

According to Lemma 3.5.2, the sequence of almost complex structures $\tilde{J}_\nu$ converges on any compact subsets of $\mathbb{R}^{2n}$ in the $\mathcal{C}^1$ sense to a model structure $\tilde{J}$. Moreover, taking J as close as J_{st}, we may suppose that, due to the expression of the model structure $\tilde{J}$, involving only order one terms of J, $\tilde{J}$ is sufficiently close to J_{st} in $\mathcal{C}^1(\overline{Q(0,C_0)})$. Finally for large ν, and due to Proposition $2.3.6$ of J.-C.Sikorav in [65] there exists $C_2 > 0$ such that

$$\|d\tilde{g_\nu}\|_{\mathcal{C}^0(r_0\Delta_\nu)} \leq C_2. \tag{3.32}$$

Hence according to (3.31) and (3.32) we may extract from $\tilde{g_\nu}$ a subsequence, still denoted by $\tilde{g_\nu}$ which converges in $\mathcal{C}^1$ topology to a $\tilde{J}$-holomorphic line $\tilde{g} : \mathbb{C} \to \tilde{D}$. Due to Lemma 3.5.3 the $\tilde{J}$-complex line $\tilde{g}$ is constant. The contradiction is obtained by showing that the derivative of $\tilde{g}$ at the origin is nonzero:

$$\frac{1}{2} = N(g_\nu(0), d_0 g_\nu(\partial/\partial_x)) = \frac{|(d_0(\Phi_\nu \circ g_\nu)(\partial/\partial_x))'|}{|\rho(p_\nu)|^{\frac{1}{2}}} + \frac{|(d_0(\Phi_\nu \circ g_\nu)(\partial/\partial_x))_n|}{|\rho(g_\nu(0))|}.$$

Since $|\rho(p_\nu)|$ is equivalent to δ_ν, it follows that for some positive constant C_3 we have for large ν and for some positive constant C_3:

$$\frac{1}{2} \leq C_3 \left(\frac{|\left(d_0\left(\Phi_\nu \circ g_\nu\right)\left(\partial/\partial_x\right)\right)'|}{\delta_\nu^{\frac{1}{2}}} + \frac{|\left(d_0\left(\Phi_\nu \circ g_\nu\right)\left(\partial/\partial_x\right)\right)_n|}{\delta_\nu} \right) = C_3 \| d_0 \tilde{g_\nu}\left(\partial/\partial_x\right)\|_1.$$

This provides a contradiction. □

3.5.3 Remark on the previous proof

K.H.Lee [50] proved a localization result for pseudoholomorphic discs and their derivatives. Keeping notation of previous subsections we have:

Lemma 3.5.5. *Let r be a sufficiently small real positive number. There are positive constants C_r and δ_r such that for every $0 < \delta < \delta_r$ and every J_ν-holomorphic discs $h_\nu\ :\Delta \to D^\nu$ with $h(0) \in Q(0,\delta)$, we have:*

$$\begin{cases} h_\nu(\Delta_r) \subset Q(0, C_r\delta) \\ \|h_\nu'\|_{\mathcal{C}^1(\Delta_r)} \leq \sqrt{C_r\delta} \\ \|(h_\nu)_n\|_{\mathcal{C}^1(\Delta_r)} \leq C_r\delta \end{cases}$$

where $Q(0,\delta) := \{z = (z', z_n) \in \mathbb{C}^n\ : |z'| \leq \delta^{\frac{1}{2}}, |z_n| \leq \delta\}$.

If we apply this lemma to the J_ν-holomorphic discs $h_\nu(\zeta) := \Phi_\nu \circ g_\nu(\nu\zeta)$, since

$$h_\nu(0) = (0', -\delta_\nu) \in Q(0, \delta_\nu),$$

we obtain

$$\tilde{g_\nu}(r\Delta_\nu) \subseteq Q(0, C_r), \tag{3.33}$$

and

$$\begin{cases} \|(d(\Phi_\nu \circ g_\nu))'\|_{\mathcal{C}^0(\Delta_{r\nu})} & \leq \frac{(C_1\delta_\nu)^{\frac{1}{2}}}{\nu} \\ \|(d(\Phi_\nu \circ g_\nu))_n\|_{\mathcal{C}^0(\Delta_{r\nu})} & \leq \frac{C_1\delta_\nu}{\nu}. \end{cases}$$

This finally gives:

$$\|d\tilde{g_\nu}\|_{\mathcal{C}^0(\Delta_{r\nu})} \leq \frac{C}{\nu}, \tag{3.34}$$

for a positive constant C.

Then (3.33) and (3.34) implies directly that we may extract from $\tilde{g_\nu}$ a subsequence which converges in $\mathcal{C}^1$ topology to a $\tilde{J}$-holomorphic line $\tilde{g}\ :\mathbb{C} \to \tilde{D}$. And according to (3.34) it follows that the $\tilde{J}$-complex line $\tilde{g}$ is constant.

This could be seen as an alternative way to end the proof of Theorem 3.5.1 instead of using Lemma 3.5.4. But in a first hand the localization lemma 3.5.5 established by K.H.Lee is very technical. In a second hand once the pseudoholomorphic discs and their derivatives are controlled as in Lemma 3.5.5, it is rather simple to give the desired precise lower estimates, without using any scaling method: actually, Lemma 3.5.5 may be seen as an alternative (but equivalent) way to state Theorem 3.5.1.

Chapter 4

Sharp estimates of the Kobayashi pseudometric and Gromov hyperbolicity

The present chapter follows [12].

Résumé Soit $D = \{\rho < 0\}$ un domaine lisse relativement compact dans une variété presque complexe (M, J) de dimension quatre, où ρ est une fonction J-plurisousharmonique au voisinage de $\overline{D}$ et strictement J-plurisousharmonique sur un voisinage de ∂D. Nous donnons des estimées fines de la pseudométrique de Kobayashi $K_{D,J}$ en nous appuyant sur une description locale quantitative du domaine D et de la structure presque complexe J au voisinage d'un point du bord. Grâce aux résultats de Z.M.Balogh et M.Bonk [3], ces estimées fines montrent l'hyperbolicité au sens de Gromov du domaine D.

Abstract Let $D = \{\rho < 0\}$ be a smooth relatively compact domain in a four dimensional almost complex manifold (M, J), where ρ is a J-plurisubharmonic function on a neighborhood of $\overline{D}$ and strictly J-plurisubharmonic on a neighborhood of ∂D. We give sharp estimates of the Kobayashi pseudometric $K_{D,J}$. Our approach is based on a local quantitative description of the domain D and of the almost complex structure J near a boundary point. Following Z.M.Balogh and M.Bonk [3], these sharp estimates provide the Gromov hyperbolicity of the domain D.

Introduction

In this chapter, we give sharp estimates of the Kobayashi pseudometric on stricly pseudoconvex domains in four almost complex manifolds:

Theorem A4. *Let D be a relatively compact strictly J-pseudoconvex smooth domain in a four dimensional almost complex manifold (M, J). Then for every $\varepsilon > 0$, there exists $0 < \varepsilon_0 < \varepsilon$ and positive constants C and s such that for every $p \in D \cap N_{\varepsilon_0}(\partial D)$ and every*

$v = v_n + v_t \in T_pM$ *we have*

$$e^{-C\delta(p)^s}\left(\frac{|v_n|^2}{4\delta(p)^2}+\frac{\mathcal{L}_J\rho(\pi(p),v_t)}{2\delta(p)}\right)^{\frac{1}{2}} \leq K_{(D,J)}(p,v)$$

$$\leq e^{C\delta(p)^s}\left(\frac{|v_n|^2}{4\delta(p)^2}+\frac{\mathcal{L}_J\rho(\pi(p),v_t)}{2\delta(p)}\right)^{\frac{1}{2}}. \tag{4.1}$$

In the above theorem, $\delta(p) := \mathrm{dist}(p, \partial D)$, where dist is taken with respect to a Riemannian metric. For p sufficiently close to the boundary the point $\pi(p)$ denotes the unique boundary point such that $\delta(p) = \|p - \pi(p)\|$. Moreover $N_{\varepsilon_0}(\partial D) := \{q \in M, \delta(q) < \varepsilon_0\}$. We point out that the splitting $v = v_n + v_t \in T_pM$ in tangent and normal components in (4.1) is understood to be taken at $\pi(p)$.

Our proof is inspired by a result by D.Ma [54]. However the proof he gives is based on some purely complex analysis argument as the local existence of peak holomorphic functions. Since such functions do not exist generically in almost complex manifolds, we consider a quantitative approach using a well chosen family of polydiscs. Notice that this also gives a different way to obtain estimates in [54] in complex manifolds without using any complex analysis tools.

In the complex Euclidean space, Z.M.Balogh and M.Bonk [3] proved the Gromov hyperbolicity of strictly pseudoconvex domains. Their proof is based on sharp estimates of the Kobayashi pseudometric obtained by D.Ma [54] similar to the ones provided by (4.1), and on some sub-Riemannian geometry. This gives as a corollary of Theorem A4:

Theorem B4.*Let D be a relatively compact strictly J-pseudoconvex smooth domain in an almost complex manifold (M, J) of dimension four. Then the metric space $(D, d_{(D,J)})$ is Gromov hyperbolic.*

4.1 Preliminaries

4.1.1 Splitting of the tangent space

Assume that J is a diagonal almost complex structure defined in a neighborhood of the origin in $\mathbb{R}^4$ and such that $J(0) = J_{st}$. Consider a basis (ω_1, ω_2) of $(1,0)$ differential forms for the structure J in a neighborhood of the origin. Since J is diagonal, we may choose

$$\omega_j = dz^j - B_j(z)d\bar{z}^j,\ j = 1, 2.$$

Denote by (Y_1, Y_2) the corresponding dual basis of $(1,0)$ vector fields. Then

$$Y_j = \frac{\partial}{\partial z^j} - \beta_j(z)\frac{\partial}{\partial \overline{z^j}},\ j = 1, 2.$$

Moreover $B_j(0) = \beta_j(0) = 0$ for $j = 1, 2$. The basis $(Y_1(0), Y_2(0))$ simply coincides with the canonical (1,0) basis of $\mathbb{C}^2$. In particular $Y_1(0)$ is a basis vector of the complex tangent space $T_0^J(\partial D)$ and $Y_2(0)$ is normal to ∂D. Consider now for $t \geq 0$ the translation $\partial D - t$ of the boundary of D near the origin. Consider, in a neighborhood of the origin, a $(1,0)$ vector field X_1 (for J) such that $X_1(0) = Y_1(0)$ and $X_1(z)$ generates the J-invariant tangent space $T_z^J(\partial D - t)$ at every point $z \in \partial D - t$, $0 \leq t << 1$. Setting $X_2 = Y_2$, we obtain a basis of vector fields (X_1, X_2) on D (restricting D if necessary). Any complex tangent vector $v \in T_z^{(1,0)}(D,J)$ at point $z \in D$ admits the unique decomposition $v = v_t + v_n$ where $v_t = \alpha_1 X_1(z)$ is the tangent component and $v_n = \alpha_2 X_2(z)$ is the normal component. Identifying $T_z^{(1,0)}(D,J)$ with $T_z D$ we may consider the decomposition $v = v_t + v_n$ for each $v \in T_z(D)$. Finally we consider this decomposition for points z in a neighborhood of the boundary.

4.1.2 A few remarks on Levi geometry

We need the following lemma due to E.Chirka [19] (see also Lemma 1.3.3).

Lemma 4.1.1. *Let J be an almost complex structure of class $\mathcal{C}^1$ defined in the unit ball $\mathbb{B}$ of $\mathbb{R}^{2n}$ satisfying $J(0) = J_{st}$. Then there exist positive constants ε and $A_\varepsilon = O(\varepsilon)$ such that the function $\log\|z\|^2 + A_\varepsilon\|z\|$ is J-plurisubharmonic on $\mathbb{B}$ whenever $\|J - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})} \leq \varepsilon$.*

Proof. This is due to the fact that for $p \in \mathbb{B}$ and $\|J - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})}$ sufficiently small, we have:

$$\begin{aligned}
\mathcal{L}_J A\|z\|(p,v) &\geq A\Big(\frac{1}{\|p\|} - \frac{2}{\|p\|}\|J(p) - J_{st}\| \\
&\quad -2(1 + \|J(p) - J_{st}\|)\|J - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})}\Big)\|v\|^2 \\
&\geq \frac{A}{2\|p\|}\|v\|^2
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_J \ln\|z\|(p,v) &\geq \Big(-\frac{2}{\|p\|^2}\|J(p) - J_{st}\| - \frac{1}{\|p\|^2}\|J(p) - J_{st}\|^2 - \frac{2}{\|p\|}\|J - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})} \\
&\quad -\frac{2}{\|p\|}\|J(p) - J_{st}\|\|J - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})}\Big)\|v\|^2 \\
&\geq -\frac{6}{\|p\|}\|J - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})}\|v\|^2.
\end{aligned}$$

So taking $A = 24\|J - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})}$ the Chirka's lemma follows. ☐

The strict J-pseudoconvexity of a relatively compact domain D implies that there is a constant $C \geq 1$ such that:

$$\frac{1}{C}\|v\|^2 \leq \mathcal{L}_J\rho(p,v) \leq C\|v\|^2, \tag{4.2}$$

for $p \in \partial D$ and $v \in T_p^J(\partial D)$.

Let ρ be a defining function for D, J-plurisubharmonic on a neighborhood of $\overline{D}$ and strictly J-plurisubharmonic on a neighborhood of the boundary ∂D. Consider the one-form $d_J^c\rho$ defined by (1.3) and let α be its restriction on the tangent bundle $T\partial D$. It follows that $T^J\partial D = \mathrm{Ker}\alpha$. Due to the strict J-pseudoconvexity of ρ, the two-form $\omega := dd_J^c\rho$ is a symplectic form (ie nondegenerate and closed) on a neighborhood of ∂D, that tames J. This implies that

$$g_R := \frac{1}{2}(\omega(.,J.) + \omega(J.,.)) \tag{4.3}$$

defines a Riemannian metric. We say that $T^J\partial D$ is a *contact structure* and α is *contact form* for $T^J\partial D$. Consequently vector fields in $T^J\partial D$ span the whole tangent bundle $T\partial D$. Indeed if $v \in T^J\partial D$, it follows that $\omega(v, Jv) = \alpha([v, Jv]) > 0$ and thus $[v, Jv] \in T\partial D \setminus T^J\partial D$. We point out that in case $v \in T^J\partial D$, the vector fields v and Jv are orthogonal with respect to the Riemannian metric g_R.

4.2 Gromov hyperbolicity

In this section we give some backgrounds about Gromov hyperbolic spaces. Furthermore according to Z.M.Balogh and M.Bonk [3], proving that a domain D with some curvature is Gromov hyperbolic reduces to providing sharp estimates for the Kobayashi pseudometric $K_{(D,J)}$ near the boundary of D.

4.2.1 Gromov hyperbolic spaces

Let (X, d) be a metric space.

Definition 4.2.1. *The Gromov product of two points $x, y \in X$ with respect to the basepoint $\omega \in X$ is defined by*

$$(x|y)_\omega := \frac{1}{2}(d(x,\omega) - d(y,\omega) - d(x,y)).$$

The Gromov product measures the failure of the triangle inequality to be an equality and is always nonnegative. Figure 5 provides a geometric interpretation of the Gromov product of x, y with respect to ω in the Euclidean plane. The Gromov product of x, y with respect to ω satisfies $(x|y)_\omega = \|x' - \omega\| = \|y' - \omega\|$.

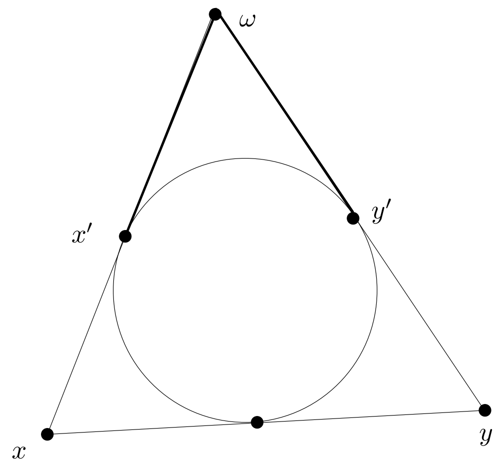

Figure 5.

Definition 4.2.2. *The metric space X is Gromov hyperbolic if there is a nonnegative constant δ such that for any $x, y, z, \omega \in X$ one has:*

$$(x|y)_\omega \geq \min((x|z)_\omega, (z|y)_\omega) - \delta. \tag{4.4}$$

We point out that (4.4) can also be written as follows:

$$d(x, y) + d(z, \omega) \leq \max(d(x, z) + d(y, \omega), d(x, \omega) + d(y, z)) + 2\delta, \tag{4.5}$$

for $x, y, z, \omega \in X$.

There is a family of metric spaces for which Gromov hyperbolicity may be defined by means of geodesic triangles. A metric space (X,d) is said to be *geodesic space* if any two points $x, y \in X$ can be joined by a *geodesic segment*, that is the image of an isometry $g : [0, d(x, y)] \to X$ with $g(0) = x$ and $g(d(x, y)) = y$. Such a segment is denoted by $[x, y]$. A *geodesic triangle* in X is the subset $[x, y] \cup [y, z] \cup [z, x]$, where $x, y, z \in X$. For a geodesic space (X, d), one may define equivalently (see [38]) the Gromov hyperbolicity as follows:

Definition 4.2.3. *The geodesic space X is Gromov hyperbolic if there is a nonnegative constant δ such that for any geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ and any $\omega \in [x, y]$ one has*

$$d(\omega, [y, z] \cup [z, x]) \leq \delta.$$

4.2.2 Gromov hyperbolicity of strictly pseudoconvex domains in almost complex manifolds of dimension four

Let $D = \{\rho < 0\}$ be a relatively compact J-strictly pseudoconvex smooth domain in an almost complex manifolds (M, J) of dimension four. Although the boundary of a compact complex manifold with pseudoconvex boundary is always connected, this is not the case in almost complex setting. Indeed D.McDuff obtained in [55] a compact almost complex manifold (M, J) of dimension four, with a disconnected J-pseudoconvex boundary. Since D is globally defined by a smooth function, J-plurisubharmonic on a neighborhood of $\overline{D}$ and strictly J-plurisubharmonic on a neighborhood of the boundary ∂D, it follows that the boundary ∂D of D is connected. Moreover this also implies that there are no J-complex line contained in D and so that $(D, d_{D,J})$ is a metric space.

A $\mathcal{C}^1$ curve $\alpha \ : [0,1] \rightarrow \partial D$ is *horizontal* if $\dot{\alpha}(s) \in T^J_{\alpha(s)}\partial D$ for every $s \in [0,1]$. This is equivalent to $\dot{\alpha}_n \equiv 0$. Thus we define the *Levi length* of a horizontal curve by

$$\mathcal{L}_J\rho - \mathrm{length}(\alpha) := \int_0^1 \mathcal{L}_J\rho(\alpha(s), \dot{\alpha}(s))^{\frac{1}{2}} ds.$$

We point out that, due to (4.3),

$$\mathcal{L}_J\rho - \mathrm{length}(\alpha) = \int_0^1 g_R(\alpha(s), \dot{\alpha}(s))^{\frac{1}{2}} ds.$$

Since $T^J\partial D$ is a *contact structure*, a theorem due to Chow [21] states that any two points in ∂D may be connected by a $\mathcal{C}^1$ horizontal curve. This allows to define the *Carnot-Carathéodory metric* as follows:

$$d_H(p,q) := \{\mathcal{L}_J\rho - \mathrm{length}(\alpha), \alpha \ : [0,1] \rightarrow \partial D \ \text{ horizontal } , \alpha(0) = p, \alpha(1) = q\}.$$

Equivalently, we may define locally the *Carnot-Carathéodory metric* by means of vector fields as follows. Consider two g_R-orthogonal vector fields $v, Jv \in T^J\partial D$ and the *sub-Riemannian metric* associated to v, Jv:

$$g_{SR}(p,w) := \inf\left\{a_1^2 + a_2^2, \ a_1 v(p) + a_2(Jv)(p) = w\right\}.$$

For a horizontal curve α, we set

$$g_{SR} - \mathrm{length}(\alpha) := \int_0^1 g_{SR}(\alpha(s), \dot{\alpha}(s))^{\frac{1}{2}} ds.$$

Thus we define:

$$d_H(p,q) := \{g_{SR} - \mathrm{length}(\alpha), \alpha \ : [0,1] \rightarrow \partial D \ \text{ horizontal } , \alpha(0) = p, \alpha(1) = q\}.$$

We point out that for a small horizontal curve α, we have

$$\dot{\alpha}(s) = a_1(s)v(\alpha(s)) + a_2(s)J(\alpha(s))v(\alpha(s)).$$

Consequently

$$g_R(\alpha(s), \dot{\alpha}(s)) = \left[a_1^2(s) + a_2^2(s)\right] g_R(\alpha(s), v(\alpha(s))).$$

Although the role of the bundle $T^J\partial D$ is crucial, it is not essential to define the Carnot-Carathéodory metric with g_{SR} instead of g_R. Actually, two Carnot-Carathéodory metrics defined with different Riemannian metrics are bi-Lipschitz equivalent (see [42]).

According to A.Bellaiche [6] and M.Gromov [42] and since $T\partial D$ is spanned by vector fields of $T^J\partial D$ and Lie Brackets of vector fields of $T^J\partial D$, balls with respect to the Carnot-Carathéodory metric may be anisotropically approximated. More precisely

Proposition 4.2.4. *There exists a positive constant C such that for ε small enough and $p \in \partial D$:*

$$\text{Box}\left(p, \frac{\varepsilon}{C}\right) \subseteq \mathbb{B}_H(p,\varepsilon) \subseteq \text{Box}(p, C\varepsilon), \tag{4.6}$$

where $\mathbb{B}_H(p,\varepsilon) := \{q \in \partial D, d_H(p,q) < \varepsilon\}$ and $\text{Box}(p,\varepsilon) := \{p + v \in \partial D, |v_t| < \varepsilon, |v_n| < \varepsilon^2\}$.

The splitting $v = v_t + v_n$ is taken at p. We point out that choosing local coordinates such that $p = 0$, $J(0) = J_{st}$ and $T_0^J\partial D = \{z_1 = 0\}$, then $\text{Box}(p,\varepsilon) = \partial D \cap Q(0,\epsilon)$, where $Q(0,\epsilon)$ is the classical polydisc $Q(0,\epsilon) := \{z \in \mathbb{C}^2, |z_1| < \varepsilon^2, |z_2| < \varepsilon\}$.

As proved by Z.M.Balogh and M.Bonk [3], (4.6) allows to approximate the Carnot-Carathéodory metric by a Riemannian anisotropic metric:

Lemma 4.2.5. *There exists a positive constant C such that for any positive κ*

$$\frac{1}{C} d_\kappa(p,q) \leq d_H(p,q) \leq C d_\kappa(p,q),$$

whenever $d_H(p,q) \geq 1/\kappa$ for $p, q \in \partial D$. Here, the distance $d_\kappa(p,q)$ is taken with respect to the Riemannian metric g_κ defined by:

$$g_\kappa(p,v) := \mathcal{L}_J\rho(p, v_h) + \kappa^2 |v_n|^2,$$

for $p \in \partial D$ and $v = v_t + v_n \in T_p\partial D$.

The crucial idea of Z.M.Balogh and M.Bonk [3] to prove the Gromov hyperbolicity of D is to introduce a function on $D \times D$, using the Carnot-Carathéodory metric, which satisfies (4.4) and which is roughly similar to the Kobayashi distance.

For $p \in D$ we define a boundary projection map $\pi : D \to \partial D$ by

$$\delta(p) = \|p - \pi(p)\| = \text{dist}(p, \partial D).$$

We notice that $\pi(p)$ is uniquely determined only if $p \in D$ is sufficiently close to the boundary. We set

$$h(p) := \delta(p)^{\frac{1}{2}}.$$

Then we define a map $g : D \times D \to [0, +\infty)$ by:

$$g(p,q) := 2\log\left(\frac{d_H(\pi(p),\pi(q)) + \max\{h(p), h(q)\}}{\sqrt{h(p)h(q)}}\right),$$

for $p, q \in D$. The map π is uniquely determined only near the boundary. But an other choice of π gives a function g that coincides up to a bounded additive constant that will not disturb our results. The motivation of introducing the map g is related with the Gromov hyperbolic space $\mathrm{Con}(Z)$ defined by M.Bonk and O.Schramm in [16] (see also [41]) as follows. Let (Z,d) be a bounded metric space which does not consist of a single point and set

$$\mathrm{Con}(Z) := Z \times (0, diam(Z)].$$

Let us define a map $\widetilde{g} : \mathrm{Con}(Z) \times \mathrm{Con}(Z) \to [0,+\infty)$ by

$$\widetilde{g}\left((z,h),(z',h')\right) := 2\log\left(\frac{d(z,z') + \max\{h,h'\}}{\sqrt{hh'}}\right).$$

M.Bonk and O.Schramm in [16] proved that $(\mathrm{Con}(Z), \widetilde{g})$ is a Gromov hyperbolic (metric) space.

In our case the map g is not a metric on D since two different points $p \neq q \in D$ may have the same projection; nevertheless

Lemma 4.2.6. *The function g satisfies (4.5) (or equivalently (4.4)) on D.*

Proof. Let r_{ij} be real nonnegative numbers such that

$$r_{ij} = r_{ji} \text{ and } r_{ij} \leq r_{ik} + r_{kj},$$

for $i, j, k = 1, \cdots, 4$. Then

$$r_{12}r_{34} \leq 4\max(r_{13}r_{24}, r_{14}r_{23}). \tag{4.7}$$

Consider now four points $p_i \in D$, $i = 1, \cdots, 4$. We set $h_i = \delta(p_i)^{\frac{1}{2}}$ and $d_{i,j} = d_{(H,J)}(\pi(p_i), \pi(p_j))$. Then applying (4.7) to $r_{ij} = d_{i,j} + \min(h_i, h_j)$, we obtain:

$$(d_{1,2} + \min(h_1,h_2))(d_{3,4} + \max(h_3,h_4))$$
$$\leq 4\max((d_{1,3} + \max(h_1,h_3))(d_{2,4} + \min(h_2,h_4), (d_{1,4} + \min(h_1,h_4))(d_{2,3} + \max(h_2,h_3)).$$

Then:

$$g(p_1,p_2) + g(p_3,p_4) \leq \max(g(p_1,p_3) + g(p_2,p_4), g(p_1,p_4) + g(p_2,p_3)) + 2\log 4,$$

which proves the desired statement. □

As a direct corollary, if a metric d on D is roughly similar to g, then the metric space (D,d) is Gromov hyperbolic:

Corollary 4.2.7. *Let d be a metric on D verifying*

$$-C + g(p,q) \leq d(p,q) \leq g(p,q) + C \tag{4.8}$$

for some positive constant C, and every $p, q \in D$. Then d satisfies (4.5) and so the metric space (D,d) is Gromov hyperbolic.

Z.M.Balogh and M.Bonk [3] proved that if the Kobayashi pseudometric (with respect to J_{st}) of a bounded strictly pseudoconvex domain satisfies (4.1), then the Kobayashi distance is rough similar to the function g. Their proof is purely metric and does not use complex geometry or complex analysis. We point out that the strict pseudoconvexity is only needed to obtain (4.2) or the fact that $T\partial D$ is spanned by vector fields of $T^{J_{st}}\partial D$ and Lie Brackets of vector fields of $T^{J_{st}}\partial D$. In particular their proof remains valid in the almost complex setting and, consequently, Theorem A4 implies:

Theorem 4.2.8. *Let D be a relatively compact strictly J-pseudoconvex smooth domain in an almost complex manifold (M,J) of dimension four. There is a nonnegative constant C such that for any $p,q \in D$*

$$g(p,q) - C \leq d_{(D,J)}(p,q) \leq g(p,q) + C.$$

According to Corollary 4.2.7 we finally obtain the following theorem (see also Theorem B4):

Theorem 4.2.9. *Let D be a relatively compact strictly J-pseudoconvex smooth domain in an almost complex manifolds (M,J) of dimension four. Then the metric space $(D, d_{(D,J)})$ is Gromov hyperbolic.*

Example 2. *There exist a neighborhood U of p and a diffeomorphism $z : U \to \mathbb{B} \subseteq \mathbb{R}^4$, centered at p, such that the function $\|z\|^2$ is strictly J-plurisubharmonic on U and $\|z_*(J) - J_{st}\|_{\mathcal{C}^2(U)} \leq \lambda_0$. Hence the unit ball $\mathbb{B}$ equipped with the metric $d_{(\mathbb{B}(0,1),z_*J)}$ is Gromov hyperbolic.*

As a direct corollary of Example 2 we have:

Corollary 4.2.10. *Let (M,J) be a four dimensional almost complex manifold. Then every point $p \in M$ has a basis of Gromov hyperbolic neighborhoods.*

4.3 Sharp estimates of the Kobayashi pseudometric

In this section we give a precise localization principle for the Kobayashi pseudometric and we prove Theorem A4.

Let $D = \{\rho < 0\}$ be a domain in an almost complex manifold (M,J), where ρ is a $\mathcal{C}^3$ defining strictly J-plurisubharmonic function. For a point $p \in D$ we define

(4.9) $$\delta(p) := \operatorname{dist}(p, \partial D),$$

and for p sufficiently close to ∂D, we define $\pi(p) \in \partial D$ as the unique boundary point such that:

(4.10) $$\delta(p) = \|p - \pi(p)\|.$$

For $\varepsilon > 0$, we introduce

(4.11) $$N_\varepsilon := \{p \in D, \delta(p) < \varepsilon\}.$$

4.3.1 Sharp localization principle

F.Forstneric and J.-P.Rosay [32] obtained a sharp localization principle of the Kobayashi pseudometric near a strictly J_{st}-pseudoconvex boundary point of a domain $D \subset \mathbb{C}^n$. However their approach is based on the existence of some holomorphic peak function at such a point; this is purely complex and cannot be generalized in the nonintegrable case. The sharp localization principle we give is based on some estimates of the Kobayashi length of a path near the boundary.

Proposition 4.3.1. *There exists a positive constant r such that for every $p \in D$ sufficiently close to the boundary and for every sufficiently small neighborhood U of $\pi(p)$ there is a positive constant c such that for every $v \in T_pM$:*

$$K_{(D\cap U,J)}(p,v) \geq (1 - c\delta(p)^r)K_{(D\cap U,J)}(p,v). \tag{4.12}$$

We will give later a more precise version of Proposition 4.3.1, where the constants c and r are given explicitly (see Lemma 4.3.4).

Proof. We consider a local diffeomorphism z centered at $\pi(p)$ from a sufficiently small neighborhood U of $\pi(p)$ to $z(U)$ such that

1. $z(p) = (\delta(p), 0)$,
2. the structure z_*J satisfies $z_*J(0) = J_{st}$ and is diagonal,
3. the defining function $\rho \circ z^{-1}$ is locally expressed by:

$$\rho \circ z^{-1}(z) = -2\Re e z_1 + 2\Re e \sum \rho_{j,k} z_j z_k + \sum \rho_{j,\overline{k}} z_j \overline{z_k} + O(\|z\|^3),$$

   where $\rho_{j,k}$ and $\rho_{j,\overline{k}}$ are constants satisfying $\rho_{j,k} = \rho_{k,j}$ and $\rho_{j,\overline{k}} = \overline{\rho}_{k,\overline{j}}$.

According to Lemma 4.8 in [50], there exists a positive constant c_1 ($C_{1/4}$ in the notations of [50]), independent of p, such that, shrinking U if necessary, for any $q \in D \cap U$ and any $v \in T_q\mathbb{R}^4$:

$$K_{(D,J)}(q,v) \geq c_1 \frac{\|d_q\chi(v)\|}{\chi(q)},$$

where $\chi(q) := |z_1(q)|^2 + |z_2(q)|^4$.

Let $u : \Delta \to D$ be a J-holomorphic discs satisfying $u(0) = p \in D$. Assume that $u(\Delta) \not\subset D \cap U$ and let $\zeta \in \Delta$ such that $u(\zeta) \in D \cap \partial U$. We consider a $\mathcal{C}^\infty$ path $\gamma : [0;1] \to D$ from $u(\zeta)$ to the point p; so $\gamma(0) = u(\zeta)$ and $\gamma(1) = p$. Without loss of generality we may suppose that $\gamma([0,1[) \subseteq D \cap U$. From this we get that the Kobayashi length of γ satisfies:

$$\begin{aligned} L_{(D,J)}(\gamma) &:= \int_0^1 K_{(D,J)}(\gamma(t),\dot\gamma(t))\mathrm{d}t \\ &\geq c_1 \int_0^1 \frac{\|d_{\gamma(t)}\chi(\dot\gamma(t))\|}{\chi(\gamma(t))} dt. \end{aligned}$$

This leads to:

$$L_{(D,J)}(\gamma) \quad \geq \quad c_1 \int_{\chi(p)}^{\chi(u(s\zeta))} \frac{dt}{t} = c_1 \left| \log \frac{\chi(u(s\zeta))}{\chi(p)} \right| = c_1 \log \frac{\chi(u(s\zeta))}{\chi(p)},$$

for p sufficiently small. Since there exists a positive constant $c_2(U)$ such that for all $z \in D \cap \partial U$:

$$\chi(z) \geq c_2(U),$$

and since $\chi(p) = \delta(p)^2$ it follows that

$$L_{(D,J)}(\gamma) \geq c_1 \log \frac{c_2(U)}{\delta(p)^2}, \tag{4.13}$$

We set $c_3(U) = c_1 \log(c_2(U))$.

According to the decreasing property of the Kobayashi distance, we have:

$$d_{(D,J)}(p, u(\zeta)) \leq d_{(\Delta, J_{st})}(0, \zeta) = \log \frac{1 + |\zeta|}{1 - |\zeta|}. \tag{4.14}$$

Due to (4.13) and (4.14) we have:

$$\frac{e^{c_3(U)} - \delta(p)^{2c_1}}{e^{c_3(U)} + \delta(p)^{2c_1}} \leq |\zeta|,$$

and so for p sufficiently close to its projection point $\pi(p)$:

$$1 - 2e^{-c_3(U)} \delta(p)^{2c_1} \leq |\zeta|,$$

This finally proves that

$$u(\Delta_s) \subset D \cap U$$

with $s := 1 - 2e^{-c_3(U)} \delta(p)^{2c_1}$. □

4.3.2 Sharp estimates of the Kobayashi metric

In this subsection we give the proof of Theorem A4.

Proof. Let $p \in D \cap N_{\varepsilon_0}$ and set $\delta := \delta(p)$. Considering a local diffeomorphism $z : U \to z(U) \subset \mathbb{R}^4$ such that Proposition 4.3.1 holds, me may assume that:

1. $\pi(p) = 0$ and $p = (\delta, 0)$.
2. $D \cap U \subset \mathbb{R}^4$,
3. The structure J is diagonal and coincides with J_{st} on the complex tangent space $\{z_1 = 0\}$:

$$J_{\mathbb{C}} = \begin{pmatrix} a_1 & \overline{b_1} & 0 & 0 \\ b_1 & \overline{a_1} & 0 & 0 \\ 0 & 0 & a_2 & \overline{b_2} \\ 0 & 0 & a_2 & \overline{a_2} \end{pmatrix}, \tag{4.15}$$

with

$$\begin{cases} a_l &= i + O(\|z_1\|^2), \\ b_l &= O(\|z_1\|), \end{cases}$$

for $l = 1, 2$.

4. The defining function ρ is expressed by:

$$\rho(z) = -2\Re e z_1 + 2\Re e \sum \rho_{j,k} z_j z_k + \sum \rho_{j,\overline{k}} z_j \overline{z_k} + O(\|z\|^3),$$

where $\rho_{j,k}$ and $\rho_{j,\overline{k}}$ are constants satisfying $\rho_{j,k} = \rho_{k,j}$ and $\rho_{j,\overline{k}} = \overline{\rho}_{k,\overline{j}}$.

Since the structure J is diagonal, the Levi form of ρ at the origin with respect to the structure J coincides with the Levi form of ρ at the origin with respect to the structure J_{st} on the complex tangent space. It follows essentially from [23] (see also [35]).

Lemma 4.3.2. *Let $v_2 = (0, v_2) \in \mathbb{R}^4$ be a tangent vector to ∂D at the origin. We have:*

$$\rho_{2,\overline{2}} |v_2|^2 = \mathcal{L}_{J_{st}} \rho(0, v_2) = \mathcal{L}_J \rho(0, v_2). \tag{4.16}$$

Proof of Lemma 4.3.2. Let $u : \Delta \to \mathbb{C}^2$ be a J-holomorphic disc such that $u(0) = 0$ and tangent to v_2,

$$u(\zeta) = \zeta v_2 + \mathcal{O}(|\zeta|^2).$$

Since J is a diagonal structure, the J-holomorphy equation leads to:

$$\frac{\partial u_1}{\partial \overline{\zeta}} = q_1(u) \overline{\frac{\partial u_1}{\partial \zeta}}, \tag{4.17}$$

where $q_1(z) = O(\|z\|)$. Moreover, since $d_0 u_1 = 0$, (4.17) gives:

$$\frac{\partial^2 u_1}{\partial \zeta \partial \overline{\zeta}}(0) = 0.$$

This implies that

$$\frac{\partial^2 \rho \circ u}{\partial \zeta \partial \overline{\zeta}}(0) = \rho_{2,\overline{2}} |v_t|^2.$$

Thus, the Levi form with respect to J coincides with the Levi form with respect to J_{st} on the complex tangent space of ∂D^δ at the origin. □

Remark 4.3.3. *More generally, even if $J(0) = J_{st}$, the Levi form of a function ρ with respect to J at the origin does not coincide with the Levi form of ρ with respect to J_{st}. According to Lemma 4.3.2 if the structure is diagonal then they are equal at the origin on the complex tangent space; but in real dimension greater than four, the structure can not be (genericaly) diagonal. K.Diederich and A.Sukhov [29] proved that if the structure J satisfies $J(0) = J_{st}$ and $d_z J = 0$ (which is always possible by a local diffeomorphism in arbitrary dimensions), then the Levi forms coincide at the origin (for all the directions).*

Lemma 4.3.2 implies that since the domain D is strictly J pseudoconvex at $\pi(p) = 0$, we may assume that $\rho_{2,\overline{2}} = 1$.

Consider the following biholomorphism Φ (for the standard structure J_{st}) that removes the harmonic term $2\Re e(\rho_{2,2} z_2^2)$:

$$\Phi(z_1, z_2) := (z_1 - \rho_{2,2} z_2^2, z_2). \tag{4.18}$$

The complexification of the structure $\Phi_* J$ admits the following matricial representation:

$$(\Phi_* J)_{\mathbb{C}} = \begin{pmatrix} a_1(\Phi^{-1}(z)) & \overline{b_1(\Phi^{-1}(z))} & c_1(z) & \overline{c_2(z)} \\ b_1(\Phi^{-1}(z)) & \overline{a_1(\Phi^{-1}(z))} & c_2(z) & \overline{c_1(z)} \\ 0 & 0 & a_2(\Phi^{-1}(z)) & \overline{b_2(\Phi^{-1}(z))} \\ 0 & 0 & b_2(\Phi^{-1}(z)) & \overline{a_2(\Phi^{-1}(z))} \end{pmatrix}, \tag{4.19}$$

where

$$\begin{cases} c_1(z) & := & 2\rho_{2,2} z_2 \left(a_1(\Phi^{-1}(z)) - a_2(\Phi^{-1}(z)\right) \\ \\ c_2(z) & := & 2\rho_{2,2} z_2 b_1(\Phi^{-1}(z)) - \overline{\rho_{2,2} z_2} b_2(\Phi^{-1}(z)). \end{cases}$$

In what follows, we need a quantitative version of Proposition 4.3.1. So we consider the following polydisc $Q_{(\delta,\alpha)} := \{z \in \mathbb{C}^2, |z_1| < \delta^{1-\alpha}, |z_2| < c\delta^{\frac{1-\alpha}{2}}\}$ centered at the origin, where c is chosen such that

$$\Phi(D \cap U) \cap \partial Q_{(\delta,\alpha)} \subset \{z \in \mathbb{C}^2, |z_1| = \delta^{1-\alpha}\}. \tag{4.20}$$

Lemma 4.3.4. *Let $0 < \alpha < 1$ be a positive number. There is a positive constant β such that for every sufficiently small δ we have:*

$$K_{(D\cap U,J)}(p,v) = K_{(\Phi(D\cap U),\Phi_* J)}(p,v) \geq \left(1 - 2\delta^{\beta}\right) K_{(\phi(D\cap U)\cap Q_{(\delta,\alpha)},\Phi_* J)}(p,v), \tag{4.21}$$

for $p = (\delta, 0)$ and every $v \in T_p\mathbb{R}^4$.

Proof. The proof is a quantitative repetition of the proof of Proposition 4.3.1; we only notice that according to (4.20) we have $c_2 = \delta^{1-\alpha}$, implying $\beta = 2\alpha c_1$. □

Let $0 < \alpha < \alpha' < 1$ to be fixed later, independently of δ. For every sufficiently small δ, we consider a smooth cut off function $\chi : \mathbb{R}^4 \to \mathbb{R}$:

$$\begin{cases} \chi & \equiv & 1 \text{ on } Q_{(\delta,\alpha)}, \\ \\ \chi & \equiv & 0 \text{ on } \mathbb{R}^4 \setminus Q_{(\delta,\alpha')}, \end{cases}$$

with $\alpha' < \alpha$. We point out that χ may be chosen such that

$$\|d_z\chi\| \leq \frac{c}{\delta^{1-\alpha'}}, \tag{4.22}$$

for some positive constant c independent of δ. We consider now the following endomorphism of $\mathbb{R}^4$:

$$q'(z) := \chi(z)q(z),$$

for $z \in Q_{(\delta,\alpha')}$, where

$$q(z) := (\Phi_* J(z) + J_{st})^{-1}(\Phi_* J(z) - J_{st}).$$

According to the fact that $q(z) = O(|z_1 + \rho_{2,2}z_2^2|)$ (see (4.19)) and according to (4.22), the differential of q' is upper bounded on $Q_{(\delta,\alpha')}$, independently of δ. Moreover the $dz_2 \otimes \dfrac{\partial}{\partial z_1}$ and the $dz_2 \otimes \dfrac{\partial}{\partial \overline{z_1}}$ components of the structure $\Phi_* J$ are $O(|z_1 + \rho_{2,2}z_2^2||z_2|)$ by (4.19); this is also the case for the endomorphism q'. We define an almost complex structure on the whole space $\mathbb{R}^4$ by:

$$J'(z) = J_{st}(Id + q'(z))(Id - q'(z))^{-1},$$

which is well defined since $\|q'(z)\| < 1$. It follows that the structure J' is identically equal to $\Phi_* J$ in $Q_{(\delta,\alpha)}$ and coincides with J_{st} on $\mathbb{R}^4 \setminus Q_{(\delta,\alpha')}$ (see Figure 6). Notice also that since $\chi \equiv d\chi \equiv 0$ on $\partial Q_{(\delta,\alpha')}$, J' coincides with J_{st} at first order on $\partial Q_{(\delta,\alpha')}$. Finally the structure J' satisfies:

$$J' = J_{st} + O(|z_1 + \rho_{2,2}z_2^2|)$$

on $Q_{(\delta,\alpha')}$. To fix the notations, the almost complex structure J' admits the following matricial interpretation:

$$J'_{\mathbb{C}} = \begin{pmatrix} a'_1 & \overline{b'_1} & c'_1 & \overline{c'_2} \\ b'_1 & \overline{a'_1} & c'_2 & \overline{c'_1} \\ 0 & 0 & a'_2 & \overline{b'_2} \\ 0 & 0 & b'_2 & \overline{a'_2} \end{pmatrix}. \tag{4.23}$$

with

$$\begin{cases} a'_l & = & i + O(\|z\|^2), \\ b'_l & = & O(\|z\|), \\ c'_l & = & O(|z_2|\|z\|), \end{cases}$$

for $l = 1, 2$.

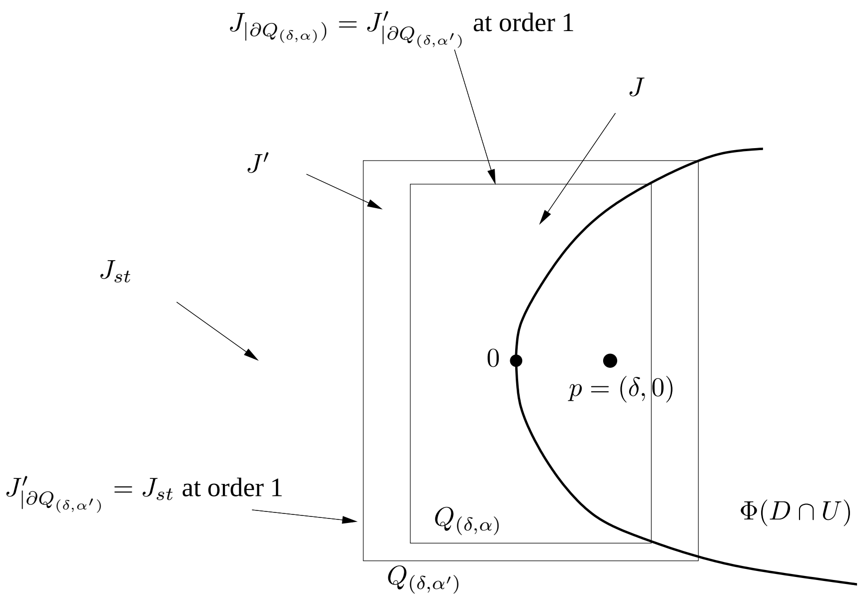

Figure 6. Extension of the almost complex structure J.

Furthermore, according to the decreasing property of the Kobayashi pseudometric we have for $p = (\delta, 0)$:

(4.24) $K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha)},\Phi_* J)}(p, v) = K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha)},J')}(p, v) \geq K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha')},J')}(p, v).$

Finally, (4.21) and (4.24) lead to:

(4.25) $$K_{(D\cap U,J)}(p, v) \geq (1 - 2\delta^{\beta})K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha')},J')}(p, v).$$

This implies that in order to obtain the lower estimate of Theorem A4 it is sufficient to prove lower estimates for $K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha')},J')}(p, v)$.

We set $\Omega := \Phi(D \cap U) \cap Q_{(\delta,\alpha')}$. Let T_δ be the translation of $\mathbb{C}^2$ defined by

$$T_\delta(z_1, z_2) := (z_1 - \delta, z_2),$$

and let φ_δ be a linear diffeomorphism of $\mathbb{R}^4$ such that the direct image of J' by $\varphi_\delta \circ T_\delta \circ \Phi$, denoted by J'^{δ}, satisfies:

(4.26) $$J'^{\delta}(0) = J_{st}.$$

To do this we consider a linear diffeomorphism such that its differential at the origin transforms the basis $(e_1, (T_\delta \circ \Phi)_* J'(0)(e_1), e_3, (T_\delta \circ \Phi)_* J'(0)e_3)$ into the canonical basis (e_1, e_2, e_3, e_4) of $\mathbb{R}^4$. According to (4.18) and (4.19), we have

$$(T_\delta \circ \Phi)_* J'(0) = \Phi_* J'(\delta, 0) = J'(\delta, 0).$$

This means that the endomorphism $(T_\delta \circ \Phi)_* J'(0)$ is block diagonal. This and the fact that $J'(\delta, 0) = J'_{st} + O(\delta)$ imply that the desired diffeomorphism is expressed by:

(4.27) $$\varphi_\delta(z) := (z_1 + O(\delta|z_1|), z_2 + O(\delta|z_2|)),$$

for $z \in T_\delta(\Omega)$, and that:

$$(J'^\delta)_{\mathbb{C}}(z) = \begin{pmatrix} a'_{1,\delta}(z) & \overline{b'_{1,\delta}(z)} & c'_{1,\delta}(z) & \overline{c'_{2,\delta}(z)} \\ b'_{1,\delta}(z) & \overline{a'_{1,\delta}(z)} & c'_{2,\delta}(z) & \overline{c'_{1,\delta}(z)} \\ 0 & 0 & a'_{2,\delta}(z) & \overline{b'_{2,\delta}(z)} \\ 0 & 0 & b'_{2,\delta}(z) & \overline{a'_{2,\delta}(z)} \end{pmatrix}, \tag{4.28}$$

where

$$\begin{cases} a'_{k,\delta}(z) & := & a'_k(\Phi^{-1} \circ T_\delta^{-1} \circ \varphi_\delta^{-1}(z)) + O(\delta) \\ b'_{k,\delta}(z) & := & b'_k(\Phi^{-1} \circ T_\delta^{-1} \circ \varphi_\delta^{-1}(z)) + O(\delta) \\ c'_{k,\delta}(z) & := & c'_k(T_\delta^{-1} \circ \varphi_\delta^{-1}(z)) + O(\delta) \end{cases}$$

for $k = 1, 2$. Furthermore we notice that the structure J'^δ is constant and equal to $J_{st} + O(\delta)$ on $\mathbb{R}^4 \setminus (\varphi_\delta \circ T_\delta \circ (\Omega))$,

We consider now the following anisotropic dilation Λ_δ of $\mathbb{C}^2$:

$$\Lambda_\delta(z_1, z_2) := \left(\frac{z_1}{z_1 + 2\delta}, \frac{\sqrt{2\delta} z_2}{z_1 + 2\delta} \right).$$

Its inverse is given by:

$$\Lambda_\delta^{-1}(z) = \left(2\delta \frac{z_1}{1 - z_1}, \sqrt{2\delta} \frac{z_2}{1 - z_1} \right). \tag{4.29}$$

Let

$$\Psi_\delta := \Lambda_\delta \circ \varphi_\delta \circ T_\delta.$$

We have the following matricial representation for the complexification of the structure $\widetilde{J^\delta} := (\Lambda_\delta)_* J^\delta$:

$$\begin{pmatrix} A'_{1,\delta}(z) & \overline{B'_{1,\delta}(z)} & C'_{1,\delta}(z) & \overline{C'_{2,\delta}(z)} \\ B'_{1,\delta}(z) & \overline{A'_{1,\delta}((z)} & C'_{2,\delta}(z) & \overline{C'_{1,\delta}(z)} \\ D'_{1,\delta}(z) & \overline{D'_{2,\delta}(z)} & A'_{2,\delta}(z) & \overline{B'_{2,\delta}(z)} \\ D'_{2,\delta}(z) & \overline{D'_{1,\delta}(z)} & B'_{2,\delta}(z) & \overline{A'_{2,\delta}(z)} \end{pmatrix}, \tag{4.30}$$

with

$$\begin{cases} A'_{1,\delta}(z) & := & a'_{1,\delta}(\Lambda_\delta^{-1}(z)) + \dfrac{1}{\sqrt{2\delta}} z_2 c'_{1,\delta}(\Lambda_\delta^{-1}(z)) \\ A'_{2,\delta}(z) & := & a'_{2,\delta}(\Lambda_\delta^{-1}(z)) - \dfrac{1}{\sqrt{2\delta}} z_2 c'_{1,\delta}(\Lambda_\delta^{-1}(z)) \\ B'_{1,\delta}(z) & := & \dfrac{(1-\overline{z_1})^2}{(1-z_1)^2} b'_{1,\delta}(\Lambda_\delta^{-1}(z)) + \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})^2 z_2}{(1-z_1)^2} c'_{2,\delta}(\Lambda_\delta^{-1}(z)) \\ B'_{2,\delta}(z) & := & \dfrac{1-\overline{z_1}}{1-z_1} b'_{2,\delta}(\Lambda_\delta^{-1}(z)) - \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})\overline{z_2}}{1-z_1} c'_{2,\delta}(\Lambda_\delta^{-1}(z)) \\ C'_{1,\delta}(z) & := & \dfrac{1}{\sqrt{2\delta}} (1-z_1) c'_{1,\delta}(\Lambda_\delta^{-1}(z)) \\ C'_{2,\delta}(z) & := & \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})^2}{1-z_1} c'_{2,\delta}(\Lambda_\delta^{-1}(z)) \\ D'_{1,\delta}(z) & := & \dfrac{z_2}{1-z_1} (a'_{2,\delta}(\Lambda_\delta^{-1}(z)) - a'_{1,\delta}(\Lambda_\delta^{-1}(z))) - \dfrac{1}{\sqrt{2\delta}} \dfrac{z_2^2}{1-z_1} c'_{1,\delta}(\Lambda_\delta^{-1}(z)) \\ D'_{2,\delta}(z) & := & \dfrac{1-\overline{z_1}}{(1-z_1)^2} (z_2 b'_{2,\delta}(\Lambda_\delta^{-1}(z)) - \overline{z_2} b'_{1,\delta}(\Lambda_\delta^{-1}(z))) \\ & & -\frac{1}{\sqrt{2\delta}} \frac{(1-\overline{z_1})|z_2|^2}{(1-z_1)^2} c'_{2,\delta}(\Lambda_\delta^{-1}(z)). \end{cases}$$

Direct computations lead to:

$$\begin{cases} A'_{1,\delta}(z) & = & a'_1(\tilde{z}_1 + \rho_{2,2}\tilde{z}_2{}^2, \tilde{z}_2) + \dfrac{1}{\sqrt{2\delta}} z_2 O(|\tilde{z}_2||\tilde{z}_1 + \rho_{2,2}\tilde{z}_2{}^2|) + O(\sqrt{\delta}) \\ B'_{1,\delta}(z) & = & \dfrac{(1-\overline{z_1})^2}{(1-z_1)^2} b'_1(\tilde{z}_1 + \rho_{2,2}\tilde{z}_2{}^2, \tilde{z}_2) + \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})^2}{1-z_1^2} z_2 O(|\tilde{z}_2||\tilde{z}_1 + \rho_{2,2}\tilde{z}_2{}^2|) \\ & & +O(\sqrt{\delta}) \\ C'_{1,\delta}(z) & = & \dfrac{1}{\sqrt{2\delta}} (1-z_1) O(|\tilde{z}_2||\tilde{z}_1 + \rho_{2,2}\tilde{z}_2{}^2|) + O(\sqrt{\delta}) \\ D'_{1,\delta}(z) & = & \dfrac{z_2}{1-z_1} [(a'_2 - a'_1)(\tilde{z}_1 + \rho_{2,2}\tilde{z}_2{}^2, \tilde{z}_2)] + \dfrac{1}{\sqrt{2\delta}} \dfrac{z_2^2}{1-z_1} O(|\tilde{z}_2||\tilde{z}_1 + \rho_{2,2}\tilde{z}_2{}^2|) \\ & & +O(\sqrt{\delta}). \end{cases}$$

where

$$\begin{cases} \tilde{z_1} & := & 2\delta\dfrac{z_1}{1-z_1} + \delta + O\left(\delta^2\left|\dfrac{z_1}{1-z_1}\right|\right) \\ \tilde{z_2} & := & \sqrt{2\delta}\dfrac{z_2}{1-z_1} + O\left(\delta^{3/2}\left|\dfrac{z_2}{1-z_1}\right|\right). \end{cases}$$

Notice that:

$$\begin{cases} \dfrac{\partial}{\partial z_1}\tilde{z_1} & := & 2\delta\dfrac{1}{(1-z_1)^2} + \dfrac{\partial}{\partial z_1}O\left(\delta^2\left|\dfrac{z_1}{1-z_1}\right|\right) \\ \dfrac{\partial}{\partial z_1}\tilde{z_2} & := & -\sqrt{2\delta}\dfrac{z_2}{(1-z_1)^2} + \dfrac{\partial}{\partial z_1}O\left(\delta^{3/2}\left|\dfrac{z_2}{1-z_1}\right|\right). \end{cases}$$

The crucial step is to control $\|\widetilde{J'^\delta} - J_{st}\|_{\mathcal{C}^1(\overline{\Psi_\delta(\Omega)})}$ by some positive power of δ. Working on a small neighborhood of the unit ball $\mathbb{B}$ (see next Lemma 4.3.5), it is sufficient to prove that the differential of $\widetilde{J'^\delta}$ is controlled by some positive constant of δ. We first need to determine the behaviour of a point $z = (z_1, z_2) \in \Psi_\delta(\Omega)$ near the infinite point $(1, 0)$. Let $\omega = (\omega_1, \omega_2) \in \Omega$ be such that $\Psi_\delta(\omega) = z$; then:

$$z_1 = \frac{\omega_1 - \delta + O(\delta|\omega_1 - \delta|)}{\omega_1 + \delta + O(\delta|\omega_1 - \delta|)},$$

where the two terms $O(\delta|\omega_1 - \delta|)$ are equal, and so

$$\left|\frac{1}{1-z_1}\right| = \left|\frac{\omega_1 + \delta + O(\delta|\omega_1 - \delta|)}{2\delta}\right| \leq c_1\delta^{-\alpha'}. \tag{4.31}$$

for some positive constant c_1 independent of z. Moreover there is a positive constant c_2 such that

$$|z_2| = \sqrt{2\delta}\left|\frac{\omega_2 + O(\delta|\omega_2|)}{\omega_1 + \delta + O(\delta|\omega_1 - \delta|)}\right| \leq c_2\delta^{\alpha'/2}. \tag{4.32}$$

All the behaviours being equivalent, we focus for instance on the derivative $\frac{\partial}{\partial z_1}D'_{1,\delta}(z)$. In this computation we focus only on terms that play a crucial role:

$$\begin{aligned}
\frac{\partial}{\partial z_1} D'_{1,\delta}(z) &= -\frac{z_2}{(1-z_1)^2}[(a'_2 - a'_1)(\tilde{z_1} + \rho_{2,2}\tilde{z_2}^2, \tilde{z_2})] + \\
& \frac{z_2}{(1-z_1)}\left[\frac{\partial}{\partial z_1}(a'_2 - a'_1).\left(2\delta\frac{1}{(1-z_1)^2} - 4\rho_{2,2}\delta\frac{z_2^2}{(1-z_1)^3}\right)\right] + \\
& \frac{z_2}{(1-z_1)}\left[\frac{\partial}{\partial z_2}(a'_2 - a'_1).\sqrt{2\delta}\frac{z_2}{(1-z_1)^2}\right] + \\
& \frac{-1}{\sqrt{2\delta}}\frac{z_2^2}{(1-z_1)^2} O(|\tilde{z_2}||\tilde{z_1} + \rho_{2,2}\tilde{z_2}^2|) \\
& + \frac{1}{\sqrt{2\delta}}\frac{z_2^2}{1-z_1}\frac{\partial}{\partial z_1} O(|\tilde{z_2}||\tilde{z_1} + \rho_{2,2}\tilde{z_2}^2|) + R(z).
\end{aligned}$$

According to (4.31), to (4.32) and to the fact that $(a'_2 - a'_1)(z) = O|z|$, it follows that for α' small enough

$$\left|\frac{\partial}{\partial z_1} D'_{1,\delta}(z)\right| \leq c\delta^s$$

for positive constants c and s. By similar arguments on other derivatives, it follows that there are positive constants, still denoted by c and s such that

$$\|d\widetilde{J'^\delta}\|_{\mathcal{C}^0(\overline{\Psi_\delta(\Omega)})} \leq c\delta^s.$$

In view of the next Lemma 4.3.5, since $\Psi_\delta(\Omega)$ is bounded, this also proves that

(4.33) $$\|\widetilde{J'^\delta} - J_{st}\|_{\mathcal{C}^1(\overline{\Psi_\delta(\Omega)})} \leq c\delta^s.$$

Moreover on $\mathbb{B}(0,2) \setminus \Psi_\delta(\Omega)$, by similar and easier computations we see that $\|\widetilde{J'^\delta} - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}(0,2)\setminus\Psi_\delta(\Omega)})}$ is also controlled by some positive constant of δ. This finally implies the crucial control :

(4.34) $$\begin{cases} \widetilde{J'^\delta}(0) &= J_{st}, \\ \\ \|\widetilde{J'^\delta} - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}(0,2)})} &\leq c\delta^s. \end{cases}$$

In order to obtain estimates of the Kobayashi pseudometric, we need to localize the domain $\Psi_\delta(\Omega) = \Psi_\delta(\Phi(D \cap U) \cap \Phi(Q_{(\delta,\alpha')}))$ between two balls (see Figure 7). This technical result is essentially due to D.Ma [54].

Lemma 4.3.5. *There exists a positive constant C such that:*

$$\mathbb{B}\left(0, e^{-C\delta^{\alpha'}}\right) \subset \Psi_\delta(\Omega) \subset \mathbb{B}\left(0, e^{C\delta^{\alpha'}}\right).$$

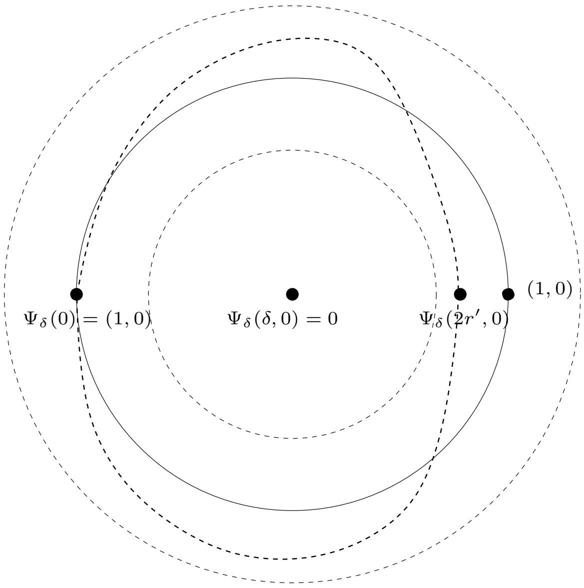

Figure 7. Approximation of $\Psi_\delta(\Omega)$.

Proof of Lemma 4.3.5. We have:

$$\Psi_\delta(z) = \left(\frac{z_1 - \delta + O(\delta|z_1-\delta|)}{z_1+\delta+O(\delta|z_1-\delta|)}, \sqrt{2\delta}\frac{z_2+O(\delta|z_2|)}{z_1+\delta+O(\delta|z_1-\delta|)}\right). \tag{4.35}$$

Consider the following expression:

$$\begin{aligned} L(z) &:= |z_1+\delta+O(\delta|z_1-\delta|)|^2(\|\Psi_\delta(z)\|^2-1) \\ &= |z_1-\delta+O(\delta|z_1-\delta|)|^2 + 2\delta|z_2+O(\delta|z_2|)|^2 \\ &\quad -|z_1+\delta+O(\delta|z_1-\delta|)|^2. \end{aligned}$$

Since $O(\delta|z_1-\delta|)$ in the first and last terms of the right hand side of the previous equality are equal, this leads to

$$L(z) = 2\delta M(z) + \delta^2 O(|z_1|) + \delta^2 O(|z_2|^2),$$

where

$$M(z) := -2\Re e z_1 + |z_2|^2.$$

Let $z \in \Omega = \Phi(D\cap U)\cap Q_{(\delta,\alpha')}$. For δ small enough, we have:

$$\begin{aligned} |z_1+\delta+O(\delta|z_1-\delta|)|^2 &\geq |z_1|^2+\delta^2+\delta^2 O(|z_1|+\delta)+\delta O(|z_1|^2+\delta|z_1|)+ \\ &\quad \delta^2 O(|z_1|+\delta)^2 + 2\delta\Re e z_1 \\ &\geq |z_1|^2+\delta^2+\delta O(|z_1|^2)+\delta^2 O(|z_1|)+O(\delta^3)+2\delta\Re e z_1 \\ &\geq \frac{3}{4}(|z_1|^2+\delta^2)+2\delta\Re e z_1. \end{aligned} \tag{4.36}$$

Moreover

$$2\Re e z_1 > 2\Re e \rho_{1,1} z_1^2 + 2\Re e \rho_{1,2} z_1 z_2 + \sum \rho_{j,\overline{k}} z_j \overline{z_k} + O(\|z\|^3).$$

Since the defining function ρ is strictly J-plurisubharmonic, we know that, for z small enough, $\sum \rho_{j,\overline{k}} z_j \overline{z_k} + O(\|z\|^3)$ is nonnegative. Hence :

$$2\Re e z_1 \geq 2\Re e \rho_{1,1} z_1^2 + 2\Re e \rho_{1,2} z_1 z_2$$

for z sufficiently small and so there is a positive constant C_1 such that:

(4.37) $$2\Re e z_1 \geq -C_1 |z_1| \|z\|.$$

Finally, (4.36) and (4.37) lead to:

$$|z_1 + \delta + O(\delta |z_1 - \delta|)|^2 \geq \frac{1}{2}(|z_1|^2 + \delta^2)$$

for z small enough. Hence we have:
(4.38)
$$|\|\Psi_\delta(z)\|^2 - 1| = \frac{|L(z)|}{|z_1 + \delta + O(\delta|z_1 - \delta|)|^2} \leq \frac{4\delta|M(z)| + \delta^2 O(|z_1|) + \delta^2 O(|z_2|^2)}{|z_1|^2 + \delta^2}.$$

The boundary of Ω is equal to $V_1 \cup V_2$ (see Figure 8), where:

$$\begin{cases} V_1 & := & \Phi(\overline{D \cap U}) \cap \partial Q_{(\delta,\alpha')}, \\ V_2 & := & \Phi(\partial(D \cap U)) \cap Q_{(\delta,\alpha')}. \end{cases}$$

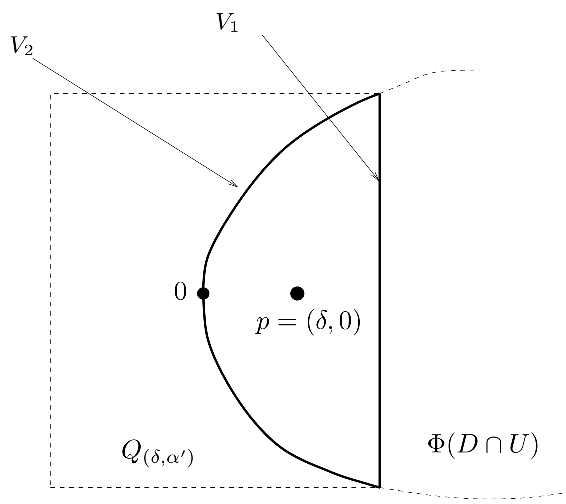

Figure 8. Boundary of Ω.

Let $z \in V_1$. According (4.38) we have:

$$
\begin{aligned}
|\|\Psi_\delta(z)\|^2 - 1| &\leq \frac{4\delta|M(z)| + \delta^2 O(|z_1|) + \delta^2 O(|z_2|^2)}{|z_1|^2 + \delta^2} \\
&\leq \frac{4\delta|z_1| + 4\delta|z_2|^2 + C_2\delta^{3-\alpha'}}{\delta^{2-2\alpha'} + \delta^2} \\
&\leq \frac{C_3\delta^{2-\alpha'}}{\delta^{2-2\alpha'} + \delta^2} \\
&\leq C_4\delta^{\alpha'}
\end{aligned}
$$

for some positive constants C_1, C_2, C_3 and C_4, and for α' small enough.

If $z \in V_2$, then

$$M(z) = -2\Re e z_1 + |z_2|^2 = O(|z_2|^3 + |z_1|\|z\|)$$

and so there is a positive constant C_5 such that:

$$M(z) \leq C_5 \delta^{\frac{3}{2}(1-\alpha')}. \tag{4.39}$$

We finally obtain from (4.38) and (4.39):

$$
\begin{aligned}
|\|\Psi_\delta(z)\|^2 - 1| &\leq 2C_5 \frac{\delta^{\frac{5-3\alpha'}{2}}}{|z_1|^2 + \delta^2} + C_2 \frac{\delta^{3-\alpha'}}{|z_1|^2 + \delta^2} \\
&\leq 2C_5\delta^{\frac{1-3\alpha'}{2}} + C_2\delta^{1-\alpha'} \\
&\leq (2C_5 + C_2)\delta^{\frac{1-3\alpha'}{2}}.
\end{aligned}
$$

This proves that:

$$\mathbb{B}\left(0, 1 - C\delta^{\alpha'}\right) \subset \Psi_\delta(\Omega) \subset \mathbb{B}\left(0, 1 + C\delta^{\alpha'}\right),$$

for some positive constant C. □

Lemma 4.3.5 provides for every $v \in T_0\mathbb{C}^2$:

$$K_{\left(\mathbb{B}(0,e^{C\delta^{\alpha'}}),\widetilde{J'^\delta}\right)}(0, v) \leq K_{\left(\Psi_\delta(\Omega),\widetilde{J'^\delta}\right)}(0, v) \leq K_{\left(\mathbb{B}(0,e^{-C\delta^{\alpha'}}),\widetilde{J'^\delta}\right)}(0, v). \tag{4.40}$$

Lower estimate

In order to give a lower estimate of $K_{\left(\mathbb{B}(0,e^{C\delta\alpha'}),\widetilde{J'^{\delta}}\right)}(0,v)$ we need the following proposition:

Proposition 4.3.6. *Let $\widetilde{J}$ be an almost complex structure defined on $\mathbb{B} \subseteq \mathbb{C}^2$ such that $\widetilde{J}(0) = J_{st}$. There exist positive constants ε and $A_\varepsilon = O(\varepsilon)$ such that if $\|\widetilde{J} - J_{st}\|_{\mathcal{C}^1(\mathbb{B})} \leq \varepsilon$ then we have:*

$$K_{(\mathbb{B},\widetilde{J})}(0,v) \geq \exp\left(-\frac{A_\varepsilon}{2}\right)\|v\|. \tag{4.41}$$

Proof of Proposition 4.3.6. Due to Lemma 4.1.1, there exist positive constants ε and $A_\varepsilon = O(\varepsilon)$ such that the function $\log\|z\|^2 + A_\varepsilon\|z\|$ is $\widetilde{J}$-plurisubharmonic on $\mathbb{B}$ if $\|\widetilde{J} - J_{st}\|_{\mathcal{C}^1(\mathbb{B})} \leq \varepsilon$. Consider the function Ψ defined by:

$$\Psi := \|z\|^2 e^{A_\varepsilon\|z\|}.$$

Let $u : \Delta \to \mathbb{B}$ be a $\widetilde{J}$-holomorphic disc such that $u(0) = 0$ and $d_0u(\partial/\partial x) = rv$ where $v \in T_q\mathbb{C}^2$ and $r > 0$. For ζ sufficiently close to 0 we have

$$u(\zeta) = q + d_0u(\zeta) + \mathcal{O}(|\zeta|^2).$$

Setting $\zeta = \zeta_1 + i\zeta_2$ and using the $\widetilde{J}$-holomorphy condition $d_0u \circ J_{st} = \widetilde{J} \circ d_0u$, we may write:

$$d_0u(\zeta) = \zeta_1 d_0u\left(\frac{\partial}{\partial x}\right) + \zeta_2\widetilde{J}\left(d_0u\left(\frac{\partial}{\partial x}\right)\right).$$

This implies

$$|d_0u(\zeta)| \leq |\zeta|\|I + \widetilde{J}\|\left\|d_0u\left(\frac{\partial}{\partial x}\right)\right\|. \tag{4.42}$$

We now consider the following function

$$\phi(\zeta) := \frac{\Psi(u(\zeta))}{|\zeta|^2} = \frac{\|u(\zeta)\|^2}{|\zeta|^2}\exp(A_\varepsilon|u(\zeta)|),$$

which is subharmonic on $\Delta\backslash\{0\}$ since $\log\phi$ is subharmonic. According to (4.42) $\limsup_{\zeta\to 0}\phi(\zeta)$ is finite. Moreover setting $\zeta_2 = 0$ we have:

$$\limsup_{\zeta\to 0}\phi(\zeta) \geq \left\|d_0u\left(\frac{\partial}{\partial x}\right)\right\|^2.$$

Applying the maximum principle to a subharmonic extension of ϕ on Δ we obtain the inequality:

$$\left\|d_0u\left(\frac{\partial}{\partial x}\right)\right\|^2 \leq \exp A_\varepsilon.$$

Hence, by definition of the Kobayashi infinitesimal pseudometric, we obtain for every $q \in D \cap V$, $v \in T_q M$:

$$K_{(D,\widetilde{J})}(q,v) \geq \exp\left(-\frac{A_\varepsilon}{2}\right)\|v\|. \tag{4.43}$$

This gives the desired estimate (4.41). □

In order to apply Proposition 4.3.6 to the structure $\widetilde{J'^\delta}$, it is necessary to dilate isotropically the ball $\mathbb{B}(0, e^{C\delta^{\alpha'}})$ to the unit ball $\mathbb{B}$. So consider the dilation of $\mathbb{C}^2$:

$$\Gamma(z) = e^{-C\delta^{\alpha'}} z.$$

$$K_{\left(\mathbb{B}(0,e^{C\delta^{\alpha'}}),\widetilde{J'^\delta}\right)}(0,v) = e^{-C\delta^{\alpha'}} K_{\left(\mathbb{B},\Gamma_*\widetilde{J'^\delta}\right)}(0,v). \tag{4.44}$$

According to (4.40) we obtain:

$$e^{-C\delta^{\alpha'}} K_{\left(\mathbb{B},\Gamma_*\widetilde{J'^\delta}\right)}(0,v) \leq K_{\left(\Psi_\delta(\Omega),\widetilde{J'^\delta}\right)}(0,v). \tag{4.45}$$

Then applying Proposition 4.3.6 to the structure $\Gamma_*\widetilde{J'^\delta} = \widetilde{J'^\delta}(e^{C\delta^{\alpha'}}.)$ and to $\varepsilon = c\delta^s$ (see (4.34)) provides the existence of a positive constant C_1 such that:

$$K_{\left(\mathbb{B},\Gamma_*\widetilde{J'^\delta}\right)}(0,v) \geq e^{-C_1\delta^s}\|v\|. \tag{4.46}$$

Moreover

$$K_{(\Omega,J')}((\delta,0),v) = K_{\left(\Psi_\delta(\Omega),\widetilde{J'^\delta}\right)}(0, d_{(\delta,0)}\Psi_\delta(v)), \tag{4.47}$$

where

$$\begin{aligned} d_{(\delta,0)}\Psi_\delta(v) &= d_0\Lambda_\delta \circ d_0\varphi_\delta \circ d_{(\delta,0)}T_\delta(v) \\ &= \left(\frac{1}{2\delta}(v_1 + O(\delta)v_1), \frac{1}{\sqrt{2\delta}}(v_2 + O(\delta)v_2)\right). \end{aligned}$$

According to (4.25), (4.46), (4.45) and (4.47), we finally obtain:

$$K_{(D,J)}(p,v) \geq e^{-C_2\delta^{\beta''}} \left(\frac{|v_1|^2}{4\delta^2} + \frac{|v_2|^2}{2\delta}\right)^{\frac{1}{2}}, \tag{4.48}$$

for some positive constant C_2 and β''.

Upper estimate

Now, we want to prove the existence of a positive constant C_3 such that

$$K_{(D,J)}(p,v) \leq e^{C_3\delta^{\alpha'}} \left(\frac{|v_1|^2}{4\delta^2} + \frac{|v_2|^2}{2\delta} \right)^{\frac{1}{2}}.$$

According to the decreasing property of the Kobayashi metric it is sufficient to give an upper estimate for $K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha)},J)}(p,v)$. Moreover, due to (4.40) and (4.47) it is sufficient to prove:

(4.49)
$$K_{\left(\mathbb{B}(0,e^{-C\delta^{\alpha'}}),\widetilde{J^\delta}\right)}(0,v) \leq e^{C_4\delta^{\alpha'}} \|v\|.$$

In that purpose we need to deform quantitatively a standard holomorphic disc contained in the ball $\mathbb{B}(0,e^{-C\delta^{\alpha'}})$ into a $\widetilde{J^\delta}$-holomorphic disc, controlling the size of the new disc, and consequently its derivative at the origin. As previously by dilating isotropically the ball $\mathbb{B}(0,e^{-C\delta^{\alpha'}})$ into the unit ball $\mathbb{B}$, we may suppose that we work on the unit ball endowed with $\widetilde{J^\delta}$ satisfying (4.34).

We define for a map g with values in a complex vector space, continuous on $\overline{\Delta}$, and for $z \in \Delta$ the *Cauchy-Green operator* by:

$$T_{CG}(g)(z) := \frac{1}{\pi} \int_\Delta \frac{g(\zeta)}{z-\zeta} dxdy.$$

We consider now the operator $\Phi_{\widetilde{J^\delta}}$ from $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B}(0,2))$ into $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{R}^4)$ by:

$$\Phi_{\widetilde{J^\delta}}(u) := \left(Id - T_{CG}\, q_{\widetilde{J^\delta}}(u) \frac{\partial}{\partial z} \right) u,$$

which is well defined since $\widetilde{J^\delta}$ satisfying (4.34). Let $u\ : \Delta \to \mathbb{B}$ be a $\widetilde{J^\delta}$-holomorphic disc in $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B})$. According to the continuity of the Cauchy-Green operator from $\mathcal{C}^r(\overline{\Delta},\mathbb{R}^4)$ into $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{R}^4)$ and since $\widetilde{J^\delta}$ satisfies (4.34), we get:

$$\begin{aligned}
\left\| T_{CG}\, q_{\widetilde{J^\delta}}(u) \frac{\partial}{\partial z} u \right\|_{\mathcal{C}^{1,r}(\overline{\Delta})} &\leq c \left\| q_{\widetilde{J^\delta}}(u) \frac{\partial}{\partial z} u \right\|_{\mathcal{C}^{r}(\overline{\Delta})} \\
&\leq c \left\| q_{\widetilde{J^\delta}} \right\|_{\mathcal{C}^1(\overline{\mathbb{B}})} \|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \\
&\leq c' \left\| \widetilde{J^\delta} - J_{st} \right\|_{\mathcal{C}^1(\overline{\mathbb{B}})} \|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \\
&\leq c''\delta^s \|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})}
\end{aligned}$$

for some positive constants c, c' and c''. Hence

(4.50)
$$(1-c''\delta^s)\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \leq \left\| \Phi_{\widetilde{J^\delta}}(u) \right\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \leq (1+c''\delta^s)\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})}$$

for any $\widetilde{J^\delta}$-holomorphic disc $u\ :\Delta\to\mathbb{B}$. This implies that the map $\Phi_{\widetilde{J^\delta}}$ is a $\mathcal{C}^1$ diffeomorphism from $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B})$ onto $\Phi_{\widetilde{J^\delta}}(\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B}))$. Furthermore the following property is classical: the disc u is $\widetilde{J^\delta}$-holomorphic if and only if $\Phi_{\widetilde{J^\delta}}(u)$ is J_{st}-holomorphic. According to (4.50), there exists a positive constant c_3 such that for $w\in\mathbb{R}^4$ with $\|w\|=1-c_3\delta^s$, the map $h_w\ :\Delta\to\mathbb{B}(0,1-c_3\delta^s)$ defined by $h_w(\zeta)=\zeta w$ belongs to $\Phi_{\widetilde{J^\delta}}(\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B}))$. In particular, the map $\Phi^{-1}_{\widetilde{J^\delta}}(h_w)$ is a $\widetilde{J^\delta}$-holomorphic disc from Δ to the unit ball $\mathbb{B}$.

Consider now $w\in\mathbb{R}^4$ such that $\|w\|=1-c_3\delta^s$, and h_w the associated standard holomorphic disc. Let us estimate the derivative of the $\widetilde{J^\delta}$-holomorphic disc $u:=\Phi^{-1}_{\widetilde{J^\delta}}(h_w)$ at the origin:

$$\begin{aligned} w &= \frac{\partial h}{\partial x}(0)\\ &= \frac{\partial}{\partial x}\left(\Phi_{\widetilde{J^\delta}}(u)\right)(0)\\ &= \frac{\partial}{\partial x}u(0)+\frac{\partial}{\partial x}T_{CG}q_{\widetilde{J^\delta}}(u)\frac{\partial u}{\partial z}\\ &= \frac{\partial}{\partial x}u(0)+T_{CZ}\left(q_{\widetilde{J^\delta}}(u)\frac{\partial u}{\partial z}\right)(0) \end{aligned} \tag{4.51}$$

where T_{CZ} denotes the *Calderon-Zygmund* operator. This is defined by:

$$T_{CZ}(g)(z):=\frac{1}{\pi}\int_\Delta\frac{g(\zeta)}{(z-\zeta)^2}dxdy,$$

for a map g with values in a complex vector space, continuous on $\overline{\Delta}$ and for $z\in\Delta$, with the integral in the sense of principal value. Since T_{CZ} is a continuous operator from $\mathcal{C}^r(\overline{\Delta},\mathbb{R}^4)$ into $\mathcal{C}^r(\overline{\Delta},\mathbb{R}^4)$, we have:

$$\left\|T_{CZ}\left(q_{\widetilde{J^\delta}}(u)\frac{\partial u}{\partial z}\right)(0)\right\|\leq c\left\|q_{\widetilde{J^\delta}}(u)\frac{\partial}{\partial z}u\right\|_{\mathcal{C}^r(\overline{\Delta})}\leq c'''\delta^s\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \tag{4.52}$$

for some positive constant c and c'''. Moreover, according to (4.50) we have:

$$\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})}=\left\|\Phi^{-1}_{\widetilde{J^\delta}}(h_w)\right\|_{\mathcal{C}^{1,r}(\overline{\Delta})}\leq(1+c''\delta^s)\|h_w\|_{\mathcal{C}^{1,r}(\overline{\Delta})}\leq 2\|w\|. \tag{4.53}$$

Finally (4.51), (4.52) and (4.53) lead to:

$$(1-2c'''\delta^s)\|w\|\leq\left\|\frac{\partial}{\partial x}\left(\Phi^{-1}_{\widetilde{J^\delta}}(h_w)\right)(0)\right\|\leq(1+2c'''\delta^s)\|w\|. \tag{4.54}$$

This implies that the map $w\mapsto\dfrac{\partial}{\partial x}\left(\Phi^{-1}_{\widetilde{J^\delta}}h_w\right)(0)$ is a small continuously differentiable perturbation of the identity. More precisely, using (4.54), there exists a positive constant c_4 such that for every vector $v\in\mathbb{R}^4\setminus\{0\}$ and for $r=1-c_4\delta^s$, there is a vector $w\in\mathbb{R}^4$ satisfying $\|w\|\leq 1+c_3\delta^s$ and such that $\frac{\partial}{\partial x}\left(\Phi^{-1}_{\widetilde{J^\delta}}h_w\right)(0)=rv/\|v\|$ (see Figure 9).

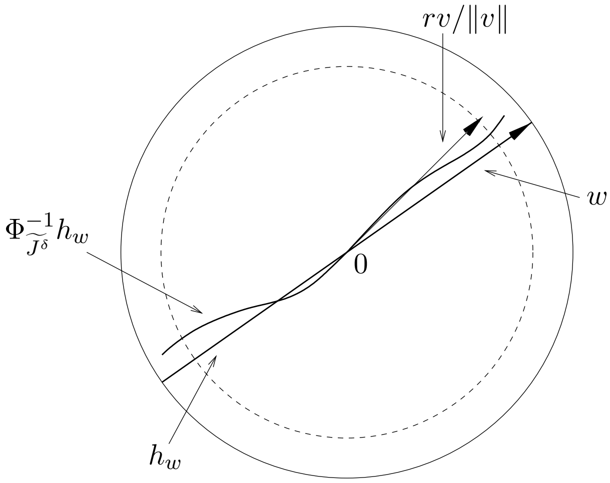

Figure 9. Deformation of a standard holomorphic disc.

Hence the $\widetilde{J^\delta}$-holomorphic disc $\Phi^{-1}_{\widetilde{J^\delta}} h_w : \Delta \to \mathbb{B}$ satisfies

$$\begin{cases} \Phi^{-1}_{\widetilde{J^\delta}} h_w(0) & = 0, \\ \frac{\partial}{\partial x} \Phi^{-1}_{\widetilde{J^\delta}} h_w(0) & = r \frac{v}{\|v\|}. \end{cases}$$

This proves estimate (4.49), giving the upper estimate of Theorem A4.

The lower estimate (4.48) and the upper estimate (4.49) imply estimate (4.1) of Theorem A4.

□

Conclusion et perspectives

Dans le second chapitre de cette thèse, nous avons introduit un relevé de structure presque complexe au fibré cotangent induit par une connexion. Nous avons montré que cette construction généralise et unifie les relévés complets et horizontaux pour des choix canoniques de connexions. Nous avons étudié certaines propriétés géométriques de ce nouveau relevé comme la pseudoholomorphie des relevés de difféomorphismes et la multiplication sur une fibre, et qui permettent de caractériser le relevé complet. Nous nous sommes aussi intéressés à la compatibilité entre les relevés de structures presque complexes et les formes symplectiques sur le fibré cotangent. Plus précisément, nous avons montré qu'étant données une variété presque complexe (M, J) et une forme symplectique sur le fibré cotangent T^*M compatible avec le relevé de structure que nous avons construit, le fibré conormal d'une hypersurface strictement J-pseudoconvexe n'est pas Lagrangien.

Le troisième chapitre a été dédié aux régions pseudoconvexes de type de D'Angelo fini dans le cadre presque complexe. L'étude analytique locale de tels domaines est une question importante et est reliée au comportement au bord de l'équation de Cauchy-Riemann. Dans un premier temps, nous avons construit une fonction pic plurisousharmonique au voisinage de tout point du bord de type fini. En fournissant des propriétés d'attraction des disques pseudoholomorphes, les fonctions plurisousharmoniques constituent un outil fondamental dans le cadre presque complexe et leur construction fait l'objet de nombreux travaux actuels. Dans notre cas, l'existence de fonctions pic plurisousharmoniques nous a permis de prouver l'hyperbolicité locale d'une région pseudoconvexe de type fini et le prolongement Hölderien des difféomorphismes pseudoholomorphes. Nous avons ensuite établi des estimées précises de la pseudométrique de Kobayashi au voisinage d'un point de type au plus quatre en développant une méthode de changement d'échelle adaptée au cadre presque complexe. Ces estimées nous ont permis de caractériser les domaines pseudoconvexes possédant un difféomorphisme pseudoholomorphe dont une orbite s'accumule en un point du bord de type au plus quatre. Afin de fournir des estimées précises dans le cas de type arbitraire nous nous sommes aussi intéressés à une approche non tangentielle.

Dans le quatrième chapitre, nous nous sommes intéressés au lien qui unissait une hyperbolicité métrique et une hyperbolicité (presque) complexe. Plus précisément, nous avons prouvé l'hyperbolicité au sens de Gromov des domaines strictement J-pseudoconvexes d'une variété presque complexe (M, J) de dimension réelle quatre. Notre démonstration suit dans les grandes lignes celle donnée par D.Ma [54] pour l'espace Euclidien complexe. Néanmoins, outre l'élimination des arguments d'analyse complexe utilisés par D.Ma (comme l'usage des fonctions pics holomorphes), notre preuve repose sur l'introduction d'une famille de polydisques qui permettent un contrôle quantitatif des structures provenant d'un changement d'échelle. Le lien entre l'hyperbolicité au sens de Gromov et la Kobayashi hyperbolicité permet, comme le soulignent Z.M.Balogh et M.Bonk [3], d'obtenir une nouvelle approche des domaines strictement pseudoconvexes. Par exemple, notre résultat redonne le prolongement continu au bord des applications pseudoholomorphes propres.

Présentons à présent quelques perspectives ; trois grands axes se dégagent.

Gromov hyperbolicité dans les variétés presque complexes

Il semble naturel de généraliser les liens entre l'hyperbolicité de Gromov et l'hyperbolicité au sens de Kobayashi au cas de la dimension quelconque. Notre démonstration s'appuie sur une normalisation propre à la dimension quatre et qui permet de contrôler les structures induites par un changement d'échelle par rapport à la structure standard. Dans le cas de la dimension quelconque, nous n'obtenons un tel contrôle que par rapport à une structure modèle, ce qui constitue une différence fondamentale. Une des idées pour résoudre ce problème, est de calculer explicitement les géodesiques pour la pseudométrique de Kobayashi des domaines modèles.

Nous souhaitons aussi étudier l'hyperbolicité au sens de Gromov des régions relativement compactes pseudoconvexes de type fini dans une variété presque complexe de dimension réelle quatre. Similairement au cas des domaines strictement pseudoconvexes, cette question est reliée à une description fine du comportement de la pseudométrique de Kobayashi. Un premier pas dans cette direction serait alors d'obtenir des estimées précises au voisinage d'un point du bord de type fini strictement plus grand que quatre, en élaborant une méthode polynomiale de changement d'échelle. La difficulté majeure est d'obtenir la Brody hyperbolicité du domaine limite.

Pseudométrique de Kobayashi dans les variétés presque complexes

Récemment, R.Debalme et S.Ivashkovich [28] ont étudié l'hyperbolicité complète au sens de Kobayashi du complément d'une courbe presque complexe dans un voisinage hyperbolique d'une variété presque complexe de dimension réelle quatre (citons aussi S.Ivashkovich et J.-P.Rosay [45] dans le cas plus général du complément d'une hypersurface en dimension quelconque). Ils ont prouvé que tout point d'une courbe lisse C contenue dans un voisinage hyperbolique D est à distance infinie du complémentaire $D \setminus C$. Le cas d'une courbe pseudoholomorphe singulière reste un problème ouvert. Cette considération est motivée par le théorème de compacité de M.Gromov [40], grâce auquel les courbes pseudoholomorphes avec des singularités de type cusp apparaissent naturellement en géométrie presque complexe comme limites de courbes pseudoholomorphes lisses. Le résultat que nous envisageons de montrer s'énonce sous la forme suivante :

Conjecture. *Soit J une structure presque complexe de classe $\mathcal{C}^2$ définie au voisinage de* *l'origine dans $\mathbb{R}^4$ et soit C une courbe J-holomorphe, singulière en l'origine. Alors pour tout voisinage hyperbolique complet D de l'origine, l'ensemble $(D \backslash C, J)$ est hyperbolique complet.*

Une idée est de transposer ce problème à l'eclaté $\widetilde{D}$ de D en l'origine. Nous savons depuis les travaux de J.Duval [30], qu'il est possible de relever la structure presque complexe J à $\widetilde{D}$ en une structure $\widetilde{J}$ avec une perte de régularité. Le problème se réduit alors à montrer que $(\widetilde{D} \backslash (E \cup \widetilde{C}), \widetilde{J})$ est hyperbolique complet, où E est le diviseur exceptionnel et $\widetilde{C}$ est l'éclaté de la courbe C. Cependant une difficulté pour obtenir un tel résultat provient de la non hyperbolicité de $\widetilde{D}$. En effet, localement, nous arrivons facilement à montrer l'hyperbolicité complète de $(\widetilde{D} \setminus (E \cup \widetilde{C}), \widetilde{J})$; du fait de la non hyperbolicité de $\widetilde{D}$, cela n'apporte aucune information sur l'hyperbolicité complète (globale). Cette obstruction est relativement déroutante puisque nous ne savons pas montrer, en raisonnant uniquement sur l'éclaté, que $(\widetilde{D} \setminus (E \cup \widetilde{C}), \widetilde{J})$, avec $C = \{z_2 = 0\}$ (régulière !), est hyperbolique complet. Une autre idée pour montrer cette conjecture est de prouver qu'une courbe pseudoholomorphe se désingularise par un nombre fini d'éclatements, ce qui constitue en soi un résultat remarquable.

Une autre direction de travail dans cette thématique concerne la semi-continuité supérieure de la pseudométrique de Kobayashi. S.Ivashkovich et J.-P.Rosay [45] ont prouvé la semi-continuité supérieure de la pseudométrique de Kobayashi pour toute structure Hölderienne $\mathcal{C}^{1,\alpha}$ avec $\alpha > 0$. Dans l'article [46], S.Ivashkovich, S.Pinchuk et J.-P.Rosay ont donné un exemple d'une structure presque complexe de classe $\mathcal{C}^{\frac{2}{3}}$ sur le bidisque $\Delta \times \Delta \subseteq \mathbb{R}^4$ pour laquelle la pseudométrique de Kobayashi n'est pas semi-continue supérieurement. Il peut être intéressant de comprendre le comportement de la pseudométrique de Kobayashi pour des structures $\mathcal{C}^{\alpha}$ avec $2/3 < \alpha \leq 1$. En particulier, quelle est la borne inférieure pour α pour obtenir la semi-continuité supérieure ?

Théorie du pluripotentiel

La théorie du pluripotentiel joue un rôle important en géométrie (presque complexe) en fournissant des informations dynamiques sur les variétés. Nous savons depuis les travaux de E.Chirka que tout point d'une variété presque complexe lisse est un ensemble pluripolaire, et plus généralement, J.-P.Rosay [61] a montré que toute courbe pseudoholomorphe est un ensemble pluripolaire. Un problème naturel est de prouver que les courbes pseudoholomorphes singulières sont pluripolaires.

La notion de disque stationnaire a été introduite par L.Lempert [50]. Il a prouvé, pour un domaine strictement convexe, que les disques stationnaires coïncident avec les disques extrémaux pour la pseudométrique de Kobayashi, et a introduit un analogue multi dimensionnel de l'application de Riemann. L'importance de cet objet provient notamment de son lien avec la théorie du pluripotentiel. La question suivante est étudiée en collaboration avec H.Gaussier et J.-C. Joo. Soit $\mathbb{B} = \{\rho := -1 + \|z\|^2 < 0\}$ la boule unité de $\mathbb{R}^{2n}$ et soit $\{J_t, t \in [0,1]\}$ une famille de structures presque complexes vérifiant $J_0 = J_{st}$. Nous envisageons de comprendre les conditions symplectique sur le couple (ρ, J_t) impliquant un feuilletage de $\mathbb{B}$ par des disques stationnaires J_t-holomorphes pour tout $t \in [0,1]$. Notons que dans le cas où J_t est une petite perturbation de la structure standard J_{st}, ce résultat provient des travaux de B.Coupet, H.Gaussier et A.Sukhov [22].

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Résumé.

Dans cette thèse, nous abordons certains aspects de l'analyse locale dans les variétés presque complexes. Dans un premier temps, nous étudions le fibré cotangent qui est un outil important pour l'analyse et la géométrie complexe. Nous construisons un relevé de structure presque complexe, à l'aide d'une connexion, qui unifie les relevés complets de I.Sato et horizontaux de S.Ishihara et K.Yano. Par ailleurs, nous dégageons les principales propriétés analytiques et symplectiques du relevé ainsi construit. Dans les deux études qui suivent, nous nous intéressons aux propriétés locales des domaines pseudoconvexes de type de D'Angelo fini d'une variété presque complexe de dimension réelle quatre. Nous construisons des fonctions locales pic plurisousharmoniques, généralisant des travaux de J.E.Fornaess et N.Sibony. La construction d'une telle famille de fonctions permet d'établir des propriétés d'attraction et de localisation des disques pseudoholomorphes. En particulier, elle réduit l'étude de la pseudométrique de Kobayashi à un problème purement local. Le comportement asymptotique de cette pseudométrique est relié à certaines questions fascinantes d'analyse locale dans les variétés comme les phénomènes de prolongement au bord des difféomorphismes ou encore la classification des domaines, et fournit des informations intéressantes sur les propriétés géométriques et dynamiques de la variété. Nous donnons alors des estimées locales de cette pseudométrique au voisinage du bord. De plus, dans le cas de stricte pseudoconvexité, nous obtenons des estimées très fines nous permettant d'étudier les liens entre l'hyperbolicité au sens de Kobayashi et l'hyperbolicité au sens de Gromov ; nous généralisons ainsi, au cadre presque complexe, un résultat dû à Z.M.Balogh et M.Bonk.

Abstract.

In this thesis, we study some aspects of local analysis in almost complex manifolds. We first study the cotangent bundle which is a fundamental tool for complex analysis and geometry. We construct a lifted almost complex structure, using a connection on the base manifold; this unifies the complete lift defined by I.Sato and the horizontal lift introduced by S.Ishihara and K.Yano. Moreover, we study some geometric properties of this lift and its compatibility with symplectic forms on the cotangent bundle. In the next chapters, we are interested in local analysis of pseudoconvex domains of finite D'Angelo type in a four dimensional almost complex manifold. We construct local peak plurisubharmonic functions, generalizing a result of J.E.Fornaess and N.Sibony. Such plurisubharmonic functions give attraction and localization properties for pseudoholomorphic discs. In particular, this reduces the study of the Kobayashi pseudometric to a purely local problem. The Kobayashi pseudometric is an important tool for the study of pseudoholomorphic maps and for the classification of domains, and gives informations on the geometric and dynamic properties of the manifold. We give local estimates of this pseudometric on a neighborhood of the boundary, and, for a strictly pseudoconvex domain, we obtain sharp estimates. As an application we study the links between the Kobayashi hyperbolicity and the Gromov hyperbolicity; we generalize, in the almost complex setting, a result of Z.M.Balogh and M.Bonk.