Sur l'homologie de Khovanov-Rozansky des graphes et des entrelacs.

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par

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Imagination is more important than knowledge. For knowledge is limited to all we now know and understand, while imagination embraces the entire world, and all there ever will be to know and understand.

Albert Einstein (1879-1955)

Remerciements

Mes premières pensées vont naturellement à Marie, ma mère, ma grand-mère et mes proches qui m'ont soutenu, encouragé et parfois subi durant ces années. Merci.

Je veux pour commencer remercier mon directeur, Vladimir TURAEV, qui a dirigé mon travail et mon apprentissage durant ces années de thèse. Il a su me faire profiter de son expérience, de ses connaissances et de ses intuitions. Je le remercie particulièrement de m'avoir dès le début incité à voyager et à rencontrer d'autres chercheurs.

Je suis honoré par l'attention que Christian BLANCHET, Christian KASSEL et Paul TURNER ont portée à mon travail et je leur suis reconnaissant d'avoir accepté de rapporter ce travail. Je remercie Benjamin ENRIQUEZ d'avoir bien voulu faire partie de mon jury.

Je suis très reconnaissant à Gwénaël d'avoir minutieusement relu une première version de ce manuscrit.

Je remercie enfin tous ceux avec qui par leurs discussions, leurs questions et leurs réponses ont alimenté ma curiosité mathématiques. Agnès, Thomas, Anne-Laure, François, Gwénaël, Bertrand et tous les autres. Merci.

Je remercie aussi Hugh MORTON de m'avoir chaleureusement accueilli durant quatre mois à l'Université de Liverpool. Ce déplacement a été financé par une bourse Marie Curie de la Communauté Européenne.

Je remercie également le GDR Tresses et Topologie de Basse Dimension. Les rencontres du GDR ont toujours été des moments privilégiés dans mon travail de recherche.

Cette thèse a été financée par une Allocation du Ministère de l'Education Nationale, de la Recherche et de la Technologie ainsi qu'un monitorat et un poste d'ATER de l'Université Louis Pasteur. Je l'ai effectuée à l'Institut de Recherche Mathématiques Avancées (Strasbourg).

Résumé

Cette thèse est consacrée à la catégorification d'invariants polynomiaux d'entrelacs et de graphes. Pour tout entier strictement positif n, Khovanov et Rozansky ont introduit en 2004 une homologie bigraduée d'entrelacs, ainsi qu'une homologie de graphes planaires. Etant donné n, leur homologie d'entrelacs catégorifie la n-ième spécialisation du polynôme d'entrelacs HOMFLYPT et leur homologie de graphes planaires catégorifie un polynôme de graphes associé.

Dans cette thèse, on étudie ces homologies et on généralise leur construction en introduisant une graduation supplémentaire. Tout d'abord, on généralise une formule de Jaeger pour les polynômes d'entrelacs aux polynômes de graphes planaires, ainsi qu'à l'homologie de graphes planaires; on étend ensuite l'homologie d'entrelacs de Khovanov–Rozansky aux graphes plongés. Puis on construit une homologie trigraduée d'entrelacs. Cette homologie recouvre l'homologie bigraduée d'entrelacs de Khovanov et Rozansky. Enfin, on donne des exemples, des applications et des généralisations de l'homologie trigraduée d'entrelacs. On développe des outils d'algèbre homologique qui permettent de calculer explicitement l'homologie trigraduée d'entrelacs pour des exemples et on considère des déformations de l'homologie trigraduée d'entrelacs.

Contents

Introduction

Cette thèse est consacrée à un développement récent en théorie des nœuds: la catégorification d'invariants. Rappelons que la théorie des nœuds a pour but de classer à isotopie près les nœuds, c'est-à-dire les plongements continus du cercle dans l'espace ambiant $\mathbb{R}^3$. Le problème central de la théorie des nœuds est le suivant : étant donnés deux nœuds, ou plus généralement deux entrelacs, déterminer s'ils sont isotopes ou non (c'est-à-dire s'ils s'obtiennent par déformation continue l'un de l'autre). Pour cela, on construit généralement des invariants algébriques d'isotopie et on les évalue sur les entrelacs à distinguer. Il s'agit donc d'associer à chaque entrelacs un objet algébrique qui peut être un nombre, un polynôme ou encore un groupe. Pour construire de tels invariants, on utilise souvent un diagramme pour présenter l'entrelacs, c'est-à-dire une projection générique dans le plan réel $\mathbb{R}^2$. Afin de vérifier que l'on construit bien un invariant, on vérifie que la quantité associée à l'entrelacs ne dépend pas du diagramme choisi en s'assurant qu'elle est invariante par mouvements de Reidemeister. Parmi ces invariants citons le groupe fondamental du complémentaire de l'entrelacs dans $\mathbb{R}^3$ et le polynôme d'Alexander [**8**].

En 1984 Vaughan Jones [**15**] a construit un tel invariant à valeurs dans l'anneau $\mathbb{Z}[q, q^{-1}]$ des polynômes de Laurent à coefficients entiers. Cette découverte a entraîné d'autres développements en théorie des nœuds et la construction d'autres invariants, appelés "invariants quantiques", en raison de leur lien avec la théorie, également nouvelle, des groupes quantiques. En particulier, l'invariant HOMFLYPT est un polynôme à deux variables P d'entrelacs orientés qui est uniquement déterminé par sa valeur sur le nœud trivial

$$P\left(\bigcirc \right) = \frac{a - a^{-1}}{q - q^{-1}},$$

et la relation d'écheveaux suivante [**12**]:

$$aP\left(\times \right) - a^{-1}P\left(\times \right) = (q - q^{-1})P\left(\smile\frown \right).$$

Pour tout entier strictement positif n, la spécialisation $a = q^n$ produit un invariant P_n à valeurs dans l'anneau $\mathbb{Z}[q, q^{-1}]$ des polynômes de Laurent à coefficients entiers. Le cas $n = 2$ fournit le polynôme de Jones. Pour tout entier n strictement positif, le polynôme de Laurent P_n est intimement lié à la catégorie

des représentations du groupe quantique $\mathcal{U}_q(\mathfrak{sl}_n)$, [**16**].

Avant de passer à la catégorification de ces invariants polynomiaux d'entrelacs, expliquons brièvement la démarche mise en œuvre dans le procédé de catégorification. La catégorification consiste par exemple à voir un nombre entier strictement positif comme la dimension d'un espace vectoriel et un nombre entier en général comme la caractéristique d'Euler d'une homologie. De manière plus élaborée, on exprime un polynôme de Laurent à coefficients positifs comme la dimension graduée d'un espace vectoriel gradué. La graduation sert ici à coder l'information de degré, alors que dans le cas de l'homologie la graduation sert à coder l'information de signe. De manière plus générale, on voit que pour catégorifier un polynôme de Laurent quelconque, on a au moins besoin d'un espace vectoriel bigradué.

En 2000, Mikhail Khovanov [**22**] définit un nouvel invariant $\mathcal{H}(L)$ des entrelacs L sous la forme d'une famille finie de groupes d'homologie à coefficients rationnels $\{\mathcal{H}^{i,j}(L)\}_{(i,j)\in\mathbb{Z}^2}$ ayant la propriété remarquable suivante :

$$\sum_{(i,j)\in\mathbb{Z}^2}(-1)^i q^j \dim_{\mathbb{Q}}\mathcal{H}^{i,j}(L) = P_2(L)(q).$$

Le polynôme de Jones s'exprime comme la caractéristique d'Euler graduée d'une homologie bigraduée $\mathcal{H}$. L'homologie $\mathcal{H}$ est appelée *homologie de Khovanov* et sa classe d'isomorphisme est un invariant d'entrelacs orientés. On dit que l'homologie de Khovanov catégorifie le polynôme de Jones. La construction de $\mathcal{H}$ repose sur le crochet de Kauffman [**18**] et fait jouer un rôle central à l'algèbre de Frobenius $\mathbb{Q}[x]/x^2$.

En 2004, M. Khovanov et L. Rozansky [**24**] généralisent l'approche précédente et construisent pour tout entier n strictement positif et tout entrelacs L une famille finie de groupes d'homologie sur $\mathbb{Q}$, $HKR_n(L) = \{H^iKR_n^j(L)\}_{(i,j)\in\mathbb{Z}^2}$ qui catégorifie la spécialisation $\mathfrak{sl}_n$ du polynôme HOMFLYPT

$$\sum_{(i,j)\in\mathbb{Z}^2}(-1)^i q^j \dim_{\mathbb{Q}}H^iKR_n^j(L) = P_n(L)(q).$$

L'homologie bigraduée HKR_n est appelée *homologie d'entrelacs de Khovanov et Rozansky* et sa classe d'isomorphisme est un invariant d'entrelacs orientés. Cette catégorification repose sur un calcul graphique du polynôme P_n [**20, 36**] et sur l'utilisation de factorisations matricielles.

Rappelons leur approche. On considère des graphes planaires orientés obtenus comme singularisation de diagrammes d'entrelacs orientés (c'est-à-dire des diagrammes d'entrelacs ou chaque croisement est transformé en un sommet 4-valent). De tels graphes sont appelés *graphes planaires réguliers*. Pour tout entier $n > 0$, L. Kauffman et P. Vogel [**20**] associent à chaque graphe planaire régulier Γ un

polynôme $P_n(\Gamma) \in \mathbb{Z}[q, q^{-1}]$ par les formules suivantes:

$$P_n(\ \times\ ) = q^{-1}P_n(\ )(\ ) - q^{-n}P_n(\ \times\ ) = qP_n(\ )(\ ) - q^nP_n(\ \times\ ).$$

Ces polynômes de graphes peuvent aussi être définis de manière intrinsèque grâce à une somme d'état [**36**]. Il apparaît ainsi que ces polynômes n'ont que des coeffficients positifs. Dans un premier temps, Khovanov et Rozansky catégorifient ces polynômes de graphes. Ils associent à chaque graphe planaire régulier Γ un complexe de chaînes graduées $KR_n(\Gamma)$ 2-périodique à coefficients rationnels. Ce complexe est construit de manière locale en utilisant des factorisations matricielles qui généralisent les complexes 2-périodiques. L'homologie

$$HKR_n(\Gamma) = \oplus_{j\in\mathbb{Z}} HKR_n^j(\Gamma)$$

de $KR_n(\Gamma)$ est un $\mathbb{Q}$-espace vectoriel $\mathbb{Z}$-gradué de dimension finie qui catégorifie le polynôme des graphes:

$$\sum_{j\in\mathbb{Z}} q^j \dim_{\mathbb{Q}} HKR_n^j(\Gamma) = P_n(\Gamma)(q).$$

Enfin, ils associent à chaque diagramme D d'entrelacs orienté un complexe $KR_n(D)$ de complexes de chaînes graduées 2-périodiques. L'homologie de ce complexe est notée $HKR_n(D)$ et la classe d'isomorphisme de l'homologie $HKR_n(D)$ ne dépend que de l'entrelacs L représenté par D.

Dans cette thèse, on étudie les homologies introduites par Khovanov–Rozansky, puis on les raffine en introduisant une graduation supplémentaire.

Dans un premier temps, on généralise des résultats sur les invariants polynômiaux aux homologies de Khovanov–Rozansky. On étend une formule de Jaeger [**14**] pour les polynômes d'entrelacs aux polynômes de graphes planaires réguliers ainsi qu'à leur homologie. Cette formule permet, par exemple, d'exprimer $P_k(\Gamma)$ comme une somme de produits $P_n(\Gamma_1)\ P_m(\Gamma_2)$ où Γ_1 et Γ_2 sont certains sous-graphes de Γ et m, $n \geq 1$ sont des entiers tels que $k = m + n$. On étend ensuite la construction de Khovanov et Rozansky à des graphes orientés plongés dans $\mathbb{R}^3$. Les diagrammes de ces graphes plongés correspondent à des singularisations partielles de diagrammes d'entrelacs orientés. Kauffman a défini pour ces diagrammes de graphes plongés des mouvements de Reidemeister généralisés [**19**]. On vérifie que la construction étendue est invariante par ces mouvements de Reidemeister généralisés.

Dans un deuxième temps, on construit pour tout entier $n > 0$ une homologie *trigraduée* d'entrelacs. Cette construction est indépendante de l'étude précédente de l'homologie des graphes. On associe à chaque diagramme D d'entrelacs orienté, un bicomplexe de complexes de chaînes 2-périodique de $\mathbb{Q}$-espaces vectoriels gradués. On note $\mathbb{H}_n(D)$ son homologie. Cette construction repose aussi sur

l'utilisation de factorisations matricielles, mais remplace l'utilisation des graphes planaires réguliers par celle de diagrammes virtuels. On obtient le résultat suivant.

THÉORÈME. *Pour tout diagramme d'entrelacs orienté D, pour tout $(i,j,k) \in \mathbb{Z}^3$, le $\mathbb{Q}$-espace vectoriel $\mathbb{H}_n^{i,j,k}(D)$ est invariant à isomorphisme près par mouvements de Reidemeister.*

En outre, nous montrons que cette homologie triplement graduée recouvre l'homologie bigraduée de Khovanov et Rozansky.

THÉORÈME. *Pour tout diagramme d'entrelacs orienté D, l'homologie $\mathbb{H}_n(D)$ détermine l'homologie d'entrelacs $HKR_n(D)$ de Khovanov–Rozansky. Pour tout $(i,k) \in \mathbb{Z}^2$, on a*

$$H^i KR_n^k(D) \cong \oplus_{j\in\mathbb{Z}}\, \mathbb{H}_n^{i,j,k-j(n+1)}(D).$$

On étudie ensuite l'homologie trigraduée $\mathbb{H}_n$. On précise le rôle joué par l'algèbre de Frobenius $\mathbb{Q}[x]/x^n$ dans cette homologie, ce qui permet de calculer explicitement l'homologie $\mathbb{H}_n$ pour le nœud de trèfle et l'entrelacs de Hopf. On développe ensuite pour cette théorie d'homologie des outils d'algèbre homologique tels que les suites exactes et les suites spectrales. Comme application, nous calculons l'homologie $\mathbb{H}_n$ pour les nœuds toriques à deux brins et le nœud de huit.

Pour tout entrelacs orienté L, on définit le polynôme de Poincaré $P_n^{\mathbb{H}}$ de $\mathbb{H}_n$,

$$P_n^{\mathbb{H}}(L)(u,v,q) = \sum_{(i,j,k)\in\mathbb{Z}^3} u^i v^j q^k \dim_{\mathbb{Q}} \mathbb{H}_n^{i,j,k}(L).$$

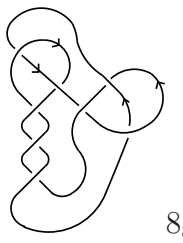
En particulier,

$$Q_n(L)(v,q) = P_n^{\mathbb{H}}(L)(-1,v,q)$$

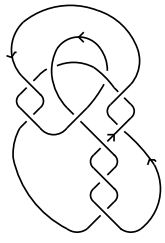
est un invariant polynomial d'entrelacs orientés qui détermine la spécialisation $\mathfrak{sl}_n$ de HOMFLYPT,

$$P_n(L)(q) = Q_n(L)(q^{n+1},q).$$

On utilise cet invariant pour prouver que l'homologie $\mathbb{H}_2$ est un invariant d'entrelacs plus fin que HKR_2. Par exemple, pour les deux nœuds 8_8 et $\overline{10_{129}}$ (i.e., l'image mirroir de 10_{129})



8_8



$\overline{10_{129}}$

on a

$$\mathbb{H}_2(8_8) \ncong \mathbb{H}_2(\overline{10_{129}}),$$

alors que

$$HKR_2(8_8) \cong HKR_2(\overline{10_{129}}).$$

On introduit ensuite des déformations de la précédente construction. Etant donné un diagramme D d'un entrelacs orienté L et un polynôme

$$p(x) = \sum_{k=1}^{n+1} a_{k-1}x^k \in \mathbb{C}[x, x^{-1}],$$

étant fixé, on construit un bicomplexe $C_p(D)$ de complexes de chaînes 2-periodique de $\mathbb{C}$-espaces vectoriels filtrés. L'homologie $\mathbb{H}_p(D)$ de ce bicomplexe est un espace vectoriel de dimension finie bigradué et filtré sur $\mathbb{C}$; sa classe d'isomorphisme est un invariant d'entrelacs orientés. De plus, lorsque $p(x) = x^{n+1} - (n+1)x$, l'homologie $\mathbb{H}_p(D)$ est de dimension n^l où l est le nombre de composantes de L.

Cette thèse est organisée comme suit. Un chapitre préliminaire introduit les objets utilisés dans la suite. En particulier, après des rappels sur les invariants polynomiaux d'entrelacs et de graphes planaires, on introduit les factorisations matricielles. Ces factorisations matricielles sont centrales dans les catégorifications de Khovanov et Rozansky [**24, 26**], ainsi que dans tout le reste de cette thèse. Les catégorifications de Khovanov–Rozansky sont résumées dans le reste du chapitre préliminaire. Le chapitre 2 est consacrée à l'homologie de graphes planaires et à l'homologie de graphes plongés. Dans le chapitre 3, on construit l'homologie trigraduée d'entrelacs $\mathbb{H}_n$, on prouve son invariance par mouvements de Reidemeister et on explique comment en extraire l'homologie bigraduée d'entrelacs de Khovanov et Rozansky. Le chapitre 4 est consacrée à l'étude de l'homologie trigraduée d'entrelacs $\mathbb{H}_n$. Les chapitres 3 et 4 sont indépendants du chapitre 2. Dans une appendice, on démontre que les polynômes de graphes planaires sont uniquement déterminés par un ensemble de relations.

La suite du présent texte est rédigée en anglais.

CHAPTER 1

Link invariants and Khovanov-Rozansky categorification

In this Chapter, we present the general ideas on the procedure of categorification developped by Khovanov and Rozansky [**24**]. We start in Section 1 by recalling various definitions of polynomial invariants of graphs and links which are categorified in Section 3, 4 and 5. In Section 2, we recall definitions and properties of matrix factorizations. Matrix factorizations are the main tools for the various constructions of Section 3, 4, 5, and Chapter 3. In Section 3, we present a categorification by Khovanov and Rozansky of graph polynomials. In Section 4, we describe the categorification by Khovanov and Rozansky of the $\mathfrak{sl}_n$ link invariants. Finally, in Section 5, we give a variant description of the categorification of Section 4 [**26**].

1. Polynomial invariants of links and graphs

We recall some generalities about polynomial link invariants and related polynomial graph invariants, [**29, 17**].

1.1. Link invariants. The two-variable HOMFLY-PT polynomial P of oriented links in $\mathbb{R}^3$ is uniquely determined by its value on the unknot and by the following skein relation , see [**12**]:

$$aP\left(\ \times\ \right) - a^{-1}P\left(\ \times\ \right) = bP\left(\ )(\ \right).$$

The specialization $a = q^n$ and $b = q - q^{-1}$ for a positive integer n gives a Laurent polynomial in one-variable q. We denote this one-variable polynomial by $P_n(L)$, where L is an oriented link, or by $P_n(D)$ if D is a diagram for L; the normalization here is

$$P_n\left(\ \circlearrowright\ \right) = [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

For all positive integer n, the Laurent polynomial P_n is related to the representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}_n)$. This polynomial is defined for any oriented link diagram, invariant under Reidemeister moves and therefore is an invariant of oriented links.

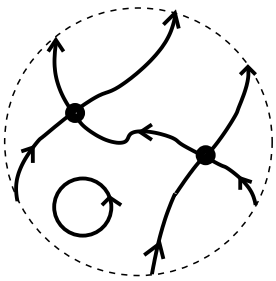


FIGURE 1. An open planar regular graph

1.2. Graph invariants. We consider finite oriented graphs of a particular kind embedded in a disk (for an example see Figure 1). The edges are oriented, and oriented loops (without vertices) are allowed. The vertices are either 4-valent and their neighborhood looks like

 ,

or they are 1-valent and belong to the boundary of the disk. An oriented edge is *internal* if none of its endpoints is on the boundary of the disk. Otherwise, the edge is called *external*. Such graphs are called *open planar regular graphs*. A graph with no boundary points is called a *(closed) planar regular graph*.

Expanding all vertices of a planar regular graph Γ as follows,

$$\begin{aligned} P_n(\ \times\ ) &= qP_n(\ )(\ ) - q^n P_n(\ \times\ ) \\ &= q^{-1}P_n(\ )(\ ) - q^{-n}P_n(\ \times\ ), \end{aligned}$$

Kauffman and Vogel [**20**] defined for any integer $n \geq 1$ a Laurent polynomial $P_n(\Gamma)$ in one variable q. Notice that $P_n(\Gamma)$ is preserved under ambient isotopy of Γ in $\mathbb{R}^2$. The polynomial P_n can also be defined as the only polynomial invariant of planar regular graphs invariant under ambient isotopy of graphs in $\mathbb{R}^2$ and satisfying the relations in Figure 2, see [**20**] and the Appendix. In other words, these relations are sufficient to compute $P_n(\Gamma)$ recursively. Murakami, Ohtsuki and Yamada [**36**] gave a state sum formula for $P_n(\Gamma)$ and deduced that $P_n(\Gamma)$ has only non-negative coefficients for any planar regular graph Γ and any $n \geq 1$.

2. Matrix factorizations

We describe in this section an algebraic object, called matrix factorization which will be useful for the categorifications of Section 3, 4, and 5. They were introduced in the context of knot theory by Khovanov and Rozansky [**24**]. They appeared earlier in the theory of singularities, see [**11, 55**].

(1) $$P_n(\ \bigcirc\ ) = \frac{q^n - q^{-n}}{q - q^{-1}} = [n]_q$$

(2) $$P_n(\ \ ) = [n-1]_q\ P_n(\ \ )$$

(3) $$P_n\left(\ \ \right) = [2]_q\ P_n(\ \ )$$

(4) $$P_n(\ \ ) = P_n(\ \ ) + [n-2]_q\ P_n(\ \ )$$

(5) $$P_n\left(\ \ \right) + P_n\left(\ \ \right) = P_n\left(\ \ \right) + P_n\left(\ \ \right)$$

FIGURE 2. Graph relations

2.1. Generalities. Let k be a positive integer, $\underline{x}$ a set of variables $\{x_1, \ldots, x_k\}$, $R = \mathbb{Q}[\underline{x}] = \mathbb{Q}[x_1, \cdots, x_k]$ the commutative polynomial $\mathbb{Q}$-algebra in the variable $\underline{x}$, and $w \in R$. Consider the ideal I_w generated by the partial derivatives $\partial_{x_i} w$, $i = 1, \ldots, k$. We say that w is a *potential* if R/I_w is a finite-dimensional vector space over $\mathbb{Q}$.

DEFINITION 1.1. *A (R, w)-matrix factorization (briefly mf) C of potential w over the ring R consists of two free R-modules C^0, C^1 and two R-homomorphisms*

$$C^0 \xrightarrow{c^0} C^1 \xrightarrow{c^1} C^0$$

such that $c^{j+1} \circ c^j = w\mathrm{Id}_{C^j}$ for all $j \in \mathbb{Z}/2\mathbb{Z}$.

A matrix factorization C can be seen as a $\mathbb{Z}/2\mathbb{Z}$-graded R-module, i.e. $C = C^0 \oplus C^1$ endowed with an endomorphism $D = \begin{pmatrix} 0 & c^1 \\ c^0 & 0 \end{pmatrix}$ such that $D^2 = w\mathrm{Id}_C$. It explains the term *matrix factorization.*

If $w = 0$ and $\underline{x} = \emptyset$, a matrix factorization is called a 2-periodic chain complex.

DEFINITION 1.2. *A mf C is of* finite rank *if C^0 and C^1 are free R-modules of finite rank.*

If C is a mf of finite rank, then $\mathrm{rank}_R(C^0) = \mathrm{rank}_R(C^1)$

EXAMPLE 1.1. *Consider the following (R, ab) matrix factorization*

$$R \xrightarrow{\times a} R \xrightarrow{\times b} R,$$

with $a, b \in R$. We denote this matrix factorization by $(a, b)_R$.

DEFINITION 1.3. *Given two (R, w)-matrix factorizations C and D, a* homomorphism *$f : C \to D$ is a pair of R-homomorphisms $f^0 : C^0 \to D^0$ and $f^1 : C^1 \to D^1$ such that the following diagram commutes:*

$$\begin{array}{ccccc} C^0 & \xrightarrow{c^0} & C^1 & \xrightarrow{c^1} & C^0 \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} \\ D^0 & \xrightarrow{d^0} & D^1 & \xrightarrow{d^1} & D^0. \end{array}$$

Let MF_w be the category whose objects are matrix factorizations of potential w and morphisms are homomorphisms of matrix factorizations. This category is R-linear and additive. Direct sum of mf's is defined in the obvious way. We denote by $\langle . \rangle$ the shift of the $\mathbb{Z}/2\mathbb{Z}$-grading, i.e. for all $i \in \mathbb{Z}/2\mathbb{Z}$, $C\langle 1 \rangle^i = C^{i+1}$.

DEFINITION 1.4. *A homotopy h between homomorphisms $f, g : (C, c) \to (D, d)$ of mf's is a pair of R-homomorphisms $h^0 : C^0 \to D^1$ and $h^1 : C^1 \to D^0$ such that*

$$f^0 - g^0 = h^1 \circ c^0 + d^1 \circ h^0, \text{ and } f^1 - g^1 = h^0 \circ c^1 + d^0 \circ h^1.$$

Let HMF_w be the category whose objects are matrix factorizations of potential w and morphisms are homomorphisms of matrix factorizations up to homotopy.

DEFINITION 1.5. *A homomorphism $f : C \to D$ of mf's is a homotopy equivalence if there exists a homomorphism $g : D \to C$ such that $f \circ g$ is homotopic to Id_D and $g \circ f$ is homotopic to Id_C.*

REMARK 1.1. *Two mf's C and D are said to be* homotopic *if there exists a homotopy equivalence $f : C \to D$. A mf homotopic to 0 is said to be* contractible.

2.2. Tensor products. We define three types of tensor products of matrix factorizations.

Internal tensor products. Let $(C, c) \in MF_w$ over R and $(D, d) \in MF_{-w}$ over R. The *tensor product* $C \otimes_R D$ is the 2-periodic chain complex

$$(C \otimes_R D)^0 \xrightarrow{\delta^0} (C \otimes_R D)^1 \xrightarrow{\delta^1} (C \otimes_R D)^0,$$

defined by

$$(C \otimes_R D)^j = \oplus_{i \in \mathbb{Z}/2\mathbb{Z}} C^i \otimes_R D^{j-i}, \text{ for all } j \in \mathbb{Z}/2\mathbb{Z}$$

and for all $(j, k) \in (\mathbb{Z}/2\mathbb{Z})^{\times 2}$, $a \in C^k$, $b \in D^{j-k}$,

$$\delta^j(a \otimes b) = (-1)^{j-k} c^k(a) \otimes b + a \otimes d^{j-k}(b).$$

Notice that we use a unusual sign convention.

EXAMPLE 1.2. *Consider two mf's as in Example 1.1, $(a, b)_R$ and $(c, d)_R$, we denote their internal tensor product over R by $\begin{pmatrix} a\ , & b \\ c\ , & d \end{pmatrix}_R$. It is a mf over R of potential $ab + cd$.*

Tensor product for sums of potentials. Let k, r, and s be positive integers. Let $\underline{x} = \{x_1, \ldots, x_k\}$, $\underline{y} = \{y_1, \ldots, y_r\}$, and $\underline{z} = \{z_1, \ldots, z_s\}$ be sets of variables. Define $R = \mathbb{Q}[\underline{x}]$, $R_1 = \mathbb{Q}[\underline{x}, \underline{y}]$, $R_2 = \mathbb{Q}[\underline{x}, \underline{z}]$, and $R_3 = \mathbb{Q}[\underline{x}, \underline{y}, \underline{z}]$. Consider $w \in R$, $w_1 \in \mathbb{Q}[\underline{y}]$, and $w_2 \in \mathbb{Q}[\underline{z}]$. Let $C \in MF_{w_1(\underline{y})-w(\underline{x})}$ over R_1 and $D \in MF_{w(\underline{x})-w_2(\underline{z})}$ over R_2. Define $C \otimes_{\mathbb{Q}[\underline{x}]} D \in MF_{w_1(\underline{y})-w_2(\underline{z})}$ over R_3 by

$$(C \otimes_{\mathbb{Q}[\underline{x}]} D)^i = \oplus_{j \in \mathbb{Z}/2\mathbb{Z}} (C^j \otimes_{\mathbb{Q}[\underline{x}]} D^{i-j}), \text{ for all } j \in \mathbb{Z}/2\mathbb{Z},$$

and for all $(j,k) \in (\mathbb{Z}/2\mathbb{Z})^{\times 2}$, $a \in C^k$, $b \in D^{j-k}$,

$$\delta^j(a \otimes b) = (-1)^{j-k} c^k(a) \otimes b + a \otimes d^{j-k}(b).$$

Here, R_3 is identified with $R_1 \otimes_{\mathbb{Q}[\underline{x}]} R_2$. Notice that the internal tensor product is the particular case when $\underline{y} = \underline{z} = \emptyset$ and $w_1 = w_2 = 0$.

External tensor product. When the set $\underline{x}$ is empty in the previous tensor product, $R = \mathbb{Q}$, and we call it the *external tensor product* $C \otimes_{\mathbb{Q}} D$.

We emphasize that the potential is additive under tensor products: Given a mf C of potential w over R and a mf C' of potential w' over R' , the mf $C \otimes C'$ is a mf of potential $w + w'$, for all kind of tensor products.

Conventions

In order to give explicit descriptions of tensor products of mf's, we fix two conventions. Given two mf's with prefered basis, these two conventions fix a prefered basis for the tensor product of two mf's. First, given two mf's C and D, we fix an ordering on the free R-modules composing $(C \otimes_{\mathbb{Q}} D)^0$ and $(C \otimes_{\mathbb{Q}} D)^1$,

$$(C \otimes_{\mathbb{Q}} D)^0 = C^0 \otimes_{\mathbb{Q}} D^0 \oplus C^1 \otimes_{\mathbb{Q}} D^1,$$
$$(C \otimes_{\mathbb{Q}} D)^1 = C^1 \otimes_{\mathbb{Q}} D^0 \oplus C^0 \otimes_{\mathbb{Q}} D^1.$$

Second, given two free R-modules U and V of finite rank k and l respectively, choose a prefered basis $(u_1, \ldots, u_k)$ for U and $(v_1, \ldots, v_l)$ for V. Then our prefered basis for $U \otimes_{\mathbb{Q}} V$ is

$$(u_1 \otimes v_1, u_2 \otimes v_1, \ldots, u_k \otimes v_1, u_1 \otimes v_2, \ldots, u_k \otimes v_l)$$

We apply the same conventions when tensor products are over other polynomial $\mathbb{Q}$-algebras.

2.3. $\mathbb{Z}$-graded matrix factorizations. Consider the following grading on the ring R: $\deg(x_i)=2$ for all $i = 1, \ldots, k$. Fix a positive integer n.

Definition 1.6. *A $\mathbb{Z}$-graded matrix factorization (briefly, gmf) C of potential w over R is a matrix factorization such that $C^0 = \oplus_{i \in \mathbb{Z}} C^{i,0}$, $C^1 = \oplus_{i \in \mathbb{Z}} C^{i,1}$ are*

graded free R-modules, w is a homogeneous element of R, and $\deg(d^0) = \deg(d^1) = n+1$.

Notice that when $C^0 = \oplus_{i\in I} R^{\otimes k_i}$ and $C^1 = \oplus_{j\in J} R^{\otimes l_j}$ where the k_i's and l_j's are positive integers and I and J are finite sets, then the grading on R induces a grading on C^0 and C^1 in the obvious way. A homomorphism of $\mathbb{Z}$-graded matrix factorizations is required to have degree 0, while a homotopy should have degree $-n-1$. We denote by curly brackets $\{.\}$ the shift of the $\mathbb{Z}$-grading: for $i,k \in \mathbb{Z}$ and $j \in \mathbb{Z}/2\mathbb{Z}$, $C^{i,j}\{k\} = C^{i-k,j}$.We denote $\mathcal{MF}_w$ the category of graded matrix factorizations of homogeneous potential w over R. It is R-linear and additive. We denote by $\mathcal{HMF}_w$ the homotopy category of $\mathbb{Z}$-graded matrix factorizations of homogeneous potential w over R. It is a graded version of HMF_w.

Example 1.3. *The matrix factorization $(a,b)_R$ of Example 1.1 is graded if a and b are homogeneous,*

$$\deg(a) + \deg(b) = 2(n+1),$$

and if we introduce the following shift:

$$R \xrightarrow{\times a} R\{n+1-\deg(a)\} \xrightarrow{\times b} R.$$

Definition 1.7. *A gmf C is indecomposable if C and $D \oplus E$ isomorphic in $\mathcal{HMF}_w$ implies that D is contractible or that E is contractible.*

An additive category is called *Krull–Schmidt* if any object has the unique decomposition property. In other words, if $C \cong \oplus_{i\in I} C_i$ and $C \cong \oplus_{j\in J} D_j$ for some sets I, J and indecomposable C_i, D_j, then there is a bijection $z : I \to J$ such that $C_i \cong D_{z(i)}$.

Proposition 1.1. [**24**, p. 46] *The category $\mathcal{HMF}_w$ is Krull-Schmidt.*

2.4. Homological algebra and graded matrix factorizations.

We consider complexes of gmf's, bicomplexes of gmf's and cones of homomorphisms of gmf's.

Complexes of graded matrix factorizations.

To any additive category $\mathcal{C}$ associate the category $K(\mathcal{C})$ whose objects are bounded complexes of objects of $\mathcal{C}$ and morphisms are homomorphisms of complexes up to homotopy.

Recall that $\mathcal{HMF}_w$ is the category of graded matrix factorizations with potential w, up to homotopy. Let

$$K_w = K(\mathcal{HMF}_w),$$

be the homotopy category of bounded complexes over $\mathcal{HMF}_w$. Thus, an object of K_w is $(\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2)$-graded, where the first grading is the grading on the

complex, the second is the grading on the ring R and the last the grading on the matrix factorizations. Notice that the differential of a complex of gmf's is required to have square equal to zero up to homotopy of the gmf's. Similarly a homomorphism of complex of gmf's should only commute with the differentials up to homotopy of gmf's.

Tensor product of complexes of gmf's.

Consider a complex (A, d_A) of gmf's of potential w over R and a complex (B, d_B) of gmf's of potential w' over R',

$$0 \to A^0 \stackrel{d_A^0}{\to} \dots \stackrel{d_A^{k-1}}{\to} A^k \to 0,$$

$$0 \to B^0 \stackrel{d_B^0}{\to} \dots \stackrel{d_B^{l-1}}{\to} B^l \to 0.$$

Define $A \otimes_{\mathbb{Q}} B$ to be the complex of gmf's of potential $w + w'$ over $R \otimes_{\mathbb{Q}} R'$ by

$$(A \otimes_{\mathbb{Q}} B)^i = \oplus_{j \in \mathbb{Z}} (A^j \otimes_{\mathbb{Q}} B^{i-j}), \text{ for all } i \in \mathbb{Z}$$

and for all $(i, j) \in \mathbb{Z}^2, a \in A^j$, $b \in B^{i-j}$,

$$\Delta^i(a \otimes b) = (-1)^{i-j} d_A^j(a) \otimes b + a \otimes d_B^{i-j}(b).$$

We have similar definitions when the tensor products are performed over other polynomial $\mathbb{Q}$-algebras.

Example 1.4. *As a particular case when $l = 0$ and $k = 1$, denote $A^0 = C$, $A^1 = D$, $B^0 = E$, and $d_A^0 = f$. We obtain*

$$\begin{array}{ccccc} (C \otimes E)^0 & \xrightarrow{P^0} & (C \otimes E)^1 & \xrightarrow{P^1} & (C \otimes E)^0 \\ \downarrow{\scriptstyle F^0} & & \downarrow{\scriptstyle F^1} & & \downarrow{\scriptstyle F^0} \\ (D \otimes E)^0 & \xrightarrow{Q^0} & (D \otimes E)^1 & \xrightarrow{Q^1} & (D \otimes E)^0. \end{array}$$

More precisely,

$$\begin{array}{ccccc} C^0 \otimes E^0 \oplus C^1 \otimes E^1 & \xrightarrow{P^0} & C^1 \otimes E^0 \oplus C^0 \otimes E^1 & \xrightarrow{P^1} & C^0 \otimes E^0 \oplus C^1 \otimes E^1 \\ \downarrow{\scriptstyle F^0} & & \downarrow{\scriptstyle F^1} & & \downarrow{\scriptstyle F^0} \\ D^0 \otimes E^0 \oplus D^1 \otimes E^1 & \xrightarrow{Q^0} & D^1 \otimes E^0 \oplus D^0 \otimes E^1 & \xrightarrow{Q^1} & D^0 \otimes E^0 \oplus D^1 \otimes E^1 \end{array}$$

with

$$P^0 = \begin{pmatrix} c^0 \otimes \mathrm{Id} & \mathrm{Id} \otimes e^1 \\ \mathrm{Id} \otimes e^0 & -c^1 \otimes \mathrm{Id} \end{pmatrix}, \; P^1 = \begin{pmatrix} c^1 \otimes \mathrm{Id} & \mathrm{Id} \otimes e^1 \\ \mathrm{Id} \otimes e^0 & -c^0 \otimes \mathrm{Id} \end{pmatrix},$$

$$F^0 = \begin{pmatrix} f^0 \otimes \mathrm{Id} & 0 \\ 0 & f^1 \otimes \mathrm{Id} \end{pmatrix}, \; F^1 \begin{pmatrix} f^1 \otimes \mathrm{Id} & 0 \\ 0 & f^0 \otimes \mathrm{Id} \end{pmatrix},$$

$$Q^0 = \begin{pmatrix} d^0 \otimes \mathrm{Id} & \mathrm{Id} \otimes e^1 \\ \mathrm{Id} \otimes e^0 & -d^1 \otimes \mathrm{Id} \end{pmatrix}, Q^1 = \begin{pmatrix} d^1 \otimes \mathrm{Id} & \mathrm{Id} \otimes e^1 \\ \mathrm{Id} \otimes e^0 & -d^0 \otimes \mathrm{Id} \end{pmatrix}.$$

This complex of gmf's is denoted by $f \otimes E$. Similarly, we define $E \otimes f$. In this case, the morphism of gmf's is given by the following pair of matrices

$$F'^0 = \begin{pmatrix} \mathrm{Id} \otimes f^0 & 0 \\ 0 & \mathrm{Id} \otimes f^1 \end{pmatrix} \quad \textit{and} \quad F'^1 = \begin{pmatrix} \mathrm{Id} \otimes f^0 & 0 \\ 0 & \mathrm{Id} \otimes f^1 \end{pmatrix}.$$

Bicomplexes of gmf's

Let

$$BK_w = K(K_w)$$

be the homotopy category of bounded complexes over K_w. Thus, an object of BK_w is a bicomplex over $\mathcal{HMF}_w$; It is $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$-graded. where the two first grading are the gradings on the bicomplex, the third is the grading on the ring R adn the last is the grading on the matrix factorization.

Consider a bicomplex $(A, \delta_A = (\delta_{A,h}, \delta_{A,v})$ of gmf's of potential w over R and a bicomplex $(B, \delta_B = (\delta_{B,h}, \delta_{B,v})$ of gmf's of potential w' over R',

$$A^{i,j} \xrightarrow{\delta^{i,j}_{A,h}} A^{i+1,j}, \text{ and } A^{i,j} \xrightarrow{\delta^{i,j}_{A,v}} A^{i,j+1},$$

$$B^{i,j} \xrightarrow{\delta^{i,j}_{B,h}} B^{i+1,j}, \text{ and } B^{i,j} \xrightarrow{\delta^{i,j}_{B,h}} B^{i+1,j},$$

where

$$\delta^{i+1,j}_{A,h} \circ \delta^{i,j}_{A,h} = 0, \quad \delta^{i,j+1}_{A,v} \circ \delta^{i,j}_{A,v} = 0, \quad \delta^{i,j+1}_{A,h} \circ \delta^{i,j}_{A,v} = \delta^{i+1,j}_{A,v} \circ \delta^{i,j}_{A,h},$$
$$\delta^{i+1,j}_{B,h} \circ \delta^{i,j}_{B,h} = 0, \quad \delta^{i,j+1}_{B,v} \circ \delta^{i,j}_{B,v} = 0, \quad \delta^{i,j+1}_{B,h} \circ \delta^{i,j}_{B,v} = \delta^{i+1,j}_{B,v} \circ \delta^{i,j}_{B,h},$$

for all i and j. Define $A \otimes_{\mathbb{Q}} B$ to be the bicomplex of gmf's of potential $w + w'$ over $R \otimes_{\mathbb{Q}} R'$ by

$$(A \otimes_{\mathbb{Q}} B)^{i,j} = \oplus_{k,l}(A^{k,l} \otimes_{\mathbb{Q}} B^{i-k,j-l}), \text{ for all } i, j \in \mathbb{Z}^2,$$

and for all $(i, j, k, l) \in \mathbb{Z}^4, a \in A^{k,l}, b \in B^{i-k,j-l}$,

$$\Delta^{i,j}_h(a \otimes b) = (-1)^{i-k}\delta^{k,l}_{A,h} \otimes b + a \otimes \delta^{i-k,j-l}_{B,h},$$
$$\Delta^{i,j}_v(a \otimes b) = (-1)^{j-l}\delta^{k,l}_{A,v} \otimes b + a \otimes \delta^{i-k,j-l}_{B,v}.$$

We have similar definitions when the tensor products are performed over other polynomial $\mathbb{Q}$-algebras. We state now a general Lemma of homological algebra.

LEMMA 1.1. *Given an additive category $\mathcal{C}$, two chain complex A and A' of $K(\mathcal{C})$ such that*

$$A \cong A',$$

in $K(\mathcal{C})$, then for all $B \in K(\mathcal{C})$,

$$A \otimes B \cong A' \otimes B \text{ in } K(\mathcal{C}).$$

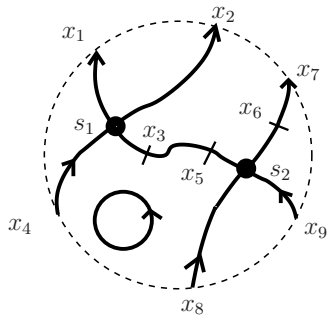


FIGURE 3. An open planar regular graph with marks

PROOF. Straightforward. □

In the rest of this thesis, we will use this Lemma with $\mathcal{C} = K_w$.

Cones

Given a homogeneous map $f : C \to D$ of gmf's of potential w having degree $n+1$, define $\mathrm{Cone}(f)$ to be the following gmf of potential w:

$$\mathrm{Cone}(f)^0 \xrightarrow{\Delta^0} \mathrm{Cone}(f)^1 \xrightarrow{\Delta^1} \mathrm{Cone}(f)^0,$$

with

$$\mathrm{Cone}(f)^0 = C^0 \oplus D^1,\ \mathrm{Cone}(f)^1 = C^1 \oplus D^0,$$
$$\Delta^0 = \begin{pmatrix} c^0 & 0 \\ f^0 & d^1 \end{pmatrix}, \text{ and } \Delta^1 = \begin{pmatrix} c^1 & 0 \\ -f^1 & d^0 \end{pmatrix}.$$

Notice that the map f is required to have degree $n+1$ in order that Δ^0 and Δ^1 are homogeneous of the same degree $n+1$.

3. Graded matrix factorizations for planar graphs

Given a positive integer n, we follow Khovanov and Rozansky [**24**] and associate to any open planar regular graph a gmf and to any closed planar regular graph a graded 2-periodic chain complex. Given a closed planar regular graph, the homology of the associated graded 2-periodic chain complex categorifies the graph polynomial defined in Section 1.

Graded matrix factorization from a planar graph. We decorate an open planar regular graph with some marks as follows: an internal edge has one or more marks, boundary points are treated as marks, additional marks on external edges are allowed, and an oriented loop may or may not have marks.

If Γ is an open planar regular graph with marks, we denote by $m(\Gamma)$ the set of its marks and by $\partial\Gamma \subset m(\Gamma)$ the set of boundary points. If $i \in \partial\Gamma$, the sign $s(i)$

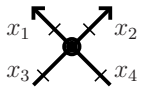


FIGURE 4. Near a 4-valent vertex

of i is 1 if the edge at i is oriented outward, and -1 if the edge is oriented inward. For instance, boundary points marked $1, 2, 7$ in Figure 3 have sign 1, while points $4, 8, 9$ have sign -1.

Consider $\underline{x} = \{x_i\}_{i\in m(\Gamma)}$, $\overline{x} = \{x_i\}_{i\in\partial\Gamma}$, $R = \mathbb{Q}[\underline{x}]$ and $R' = \mathbb{Q}[\overline{x}]$. The ring R' is a subring of R. Consider the grading on the ring R and R': $\deg(x_i)=2$, for all $i \in m(\Gamma)$. We associate to Γ the potential

$$w(\Gamma) = \sum_{i\in\partial\Gamma} s(i)x_i^{n+1} \in R'.$$

To Γ we now associate a $\mathbb{Z}$-graded matrix factorization $KR_n(\Gamma)$ over the ring R' with potential $w(\Gamma)$. First, to a 4-valent vertex s bounded by marks $1, 2, 3, 4$ as in Figure 4 we associate a gmf with potential

$$w_s = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}$$

over $R_s = \mathbb{Q}[x_1, x_2, x_3, x_4]$. Starting with formal variables x, y, we write $x^{n+1}+y^{n+1}$ as a polynomial g in $x + y$ and xy,

$$g(x + y, xy) = x^{n+1} + y^{n+1}.$$

Explicitly,

$$g(s_1, s_2) = s_1^{n+1} + (n + 1) \sum_{1\le i\le\frac{n+1}{2}} \frac{(-1)^i}{i} \binom{n-i}{i-1} s_2^i s_1^{n+1-2i}.$$

The potential w_s can be written as follows

$$\begin{aligned} w_s &= g(x_1 + x_2, x_1x_2) - g(x_3 + x_4, x_3x_4) \\ &= g(x_1 + x_2, x_1x_2) - g(x_3 + x_4, x_1x_2) + g(x_3 + x_4, x_1x_2) - g(x_3 + x_4, x_3x_4) \\ &= \frac{g(x_1 + x_2, x_1x_2) - g(x_3 + x_4, x_1x_2)}{x_1 + x_2 - x_3 - x_4}(x_1 + x_2 - x_3 - x_4) \\ &\quad + \frac{g(x_3 + x_4, x_1x_2) - g(x_3 + x_4, x_3x_4)}{x_1x_2 - x_3x_4}(x_1x_2 - x_3x_4). \end{aligned}$$

Denote

$$\begin{aligned} u_1 &= u_1(x_1, x_2, x_3, x_4) = \frac{g(x_1 + x_2, x_1x_2) - g(x_3 + x_4, x_1x_2)}{x_1 + x_2 - x_3 - x_4}, \\ u_2 &= u_2(x_1, x_2, x_3, x_4) = \frac{g(x_3 + x_4, x_1x_2) - g(x_3 + x_4, x_3x_4)}{x_1x_2 - x_3x_4}. \end{aligned}$$

FIGURE 5. Arc of D

Then $w_s = u_1(x_1 + x_2 - x_3 - x_4) + u_2(x_1x_2 - x_3x_4)$ and s is assigned the gmf

$$KR_s = \begin{pmatrix} u_1\ , & x_1 + x_2 - x_3 - x_4 \\ u_2\ , & x_1x_2 - x_3x_4 \end{pmatrix}_{R_s} \{-1\}.$$

In other words, KR_s is the tensor product over R_s of gmf's

$$R_s \xrightarrow{\times u_1} R_s\{1-n\} \xrightarrow{\times(x_1+x_2-x_3-x_4)} R_s$$

and

$$R_s \xrightarrow{\times u_2} R_s\{3-n\} \xrightarrow{\times} (x_1x_2 - x_3x_4)R_s,$$

with the grading shifted down by 1.

In general, a 4-valent vertex s will be bounded by marks i, j, k, l. Then KR_s is defined as above, with $1, 2, 3, 4$ converted into i, j, k, l.

If α is an arc in an oriented edge (or in an oriented loop) bounded by marks i and j and oriented from j to i, with no additional marks between (see Figure 5) we associate to α the gmf L^i_j

$$R_\alpha \xrightarrow{\times\pi_{ij}} R_\alpha\{1-n\} \xrightarrow{\times(x_i-x_j)} R_\alpha$$

where $R_\alpha = \mathbb{Q}[x_i, x_j]$ and

$$\pi_{ij} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}.$$

Consider the graded $\mathbb{Q}$-vector space $V_n = (\mathbb{Q}[x]/x^n)\{1-n\}$. It can be considered as a 2-periodic chain complex as follows:

$$V_n \longrightarrow 0 \longrightarrow V_n.$$

This 2-periodic chain complex is also denoted V_n. In fact, we just put the graded $\mathbb{Q}$-vector space V_n in $\mathbb{Z}/2\mathbb{Z}$-grading zero and took zero differentials. In general, a graded $\mathbb{Q}$-vector space can always in this way be seen as a graded 2-periodic chain complex.

To an oriented loop without marks we assign the 2-periodic chain complex $V_n\langle 1\rangle$:

$$0 \longrightarrow (\mathbb{Q}[x]/x^n)\{1-n\} \longrightarrow 0.$$

Finally, we define $KR_n(\Gamma)$ as the tensor product of the KR_s, for all 4-valent vertices s, of the L^i_j, for all arcs α, and of the $V_n\langle 1\rangle$, for all markless loops in Γ. The tensor product is formed over suitable intermediate rings so that $KR_n(\Gamma)$ is a free module of finite rank over R.

EXAMPLE 1.5. *For instance, to form $KR_n(\Gamma)$ for Γ in Figure 3, we first tensor KR_{s_1} with L_5^3 over the ring $\mathbb{Q}[x_3]$. Then tensor the result with KR_{s_2} over $\mathbb{Q}[x_5]$. Then we tensor $KR_{s_1} \otimes_{\mathbb{Q}[x_3]} L_5^3 \otimes_{\mathbb{Q}[x_5]} KR_{s_2}$ with L_6^7 over $\mathbb{Q}[x_6]$. Finally, we tensor with $V_n\langle 1\rangle$ over $\mathbb{Q}$.*

We now list some of the main properties of the gmf's associated to open planar graphs.

Properties

Clearly, $KR_n(\Gamma)$ is a gmf of $w(\Gamma)$. We treat it as a gmf over the ring R' of polynomials in boundary variables. Then $w(\Gamma)$ is a potential in this ring. If a graph has at least one internal mark, $KR_n(\Gamma)$ has infinite rank as an R'-module (but is of finite rank over R).

PROPOSITION 1.2. [**24**] *For any open planar regular graph Γ with marks, the gmf $KR_n(\Gamma)$ is homotopic to a finite rank gmf over R'.*

Suppose that Γ' is obtained from Γ by placing a different collection of internal marks on oriented edges and loops of Γ. Then the two graphs have the same potential $w(\Gamma') = w(\Gamma)$ assigned to them, and gmf's $KR_n(\Gamma)$ and $KR_n(\Gamma')$ are objects of the category $\mathcal{HMF}_{w(\Gamma)}$ since the graphs are the same.

PROPOSITION 1.3. [**24**] *There is a canonical isomorphism in $\mathcal{HMF}_{w(\Gamma)}$*

$$KR_n(\Gamma') \cong KR_n(\Gamma).$$

In other words, there exist a homotopy equivalence $KR_n(\Gamma) \to KR_n(\Gamma')$ which is canonical up to homotopy.

PROPOSITION 1.4. [**24**] *Given a pair of open planar regular graphs Γ_1, Γ_2 with potential w_1 and w_2, there is a canonical isomorphism in $\mathcal{HMF}_{w_1+w_2}$*

$$KR_n(\Gamma_1 \sqcup \Gamma_2) \cong KR_n(\Gamma_1) \otimes_{\mathbb{Q}} KR_n(\Gamma_2),$$

where $\Gamma_1 \sqcup \Gamma_2$ is the disjoint union of Γ_1 and Γ_2.

COROLLARY 1.1. [**24**] *If an open planar regular graph Γ_2 is the disjoint union of an open planar regular graph Γ_1 and a circle then*

$$KR_n(\Gamma_2) \cong KR_n(\Gamma_1) \otimes_{\mathbb{Q}} V_n\langle 1\rangle.$$

We recall some relations proved by Khovanov and Rozansky for the gmf's associated to open planar regular graphs. In order to simplify the notations we omit all the marks.

THEOREM 1.1. [**24**]
The following isomorphisms hold in $\mathcal{HMF}_w$:

$$KR_n(\ \bigcirc\ ) \cong V_n\langle 1\rangle,$$

$$KR_n(\ \langle 1\rangle \cong \oplus_{i=0}^{n-2} KR_n(\ )\{2-n+2i\},$$

$$KR_n\left(\ \right) \cong KR_n(\ )\{1\} \oplus KR_n(\ )\{-1\},$$

$$KR_n(\ ) \cong KR_n(\ ) \oplus \left(\oplus_{i=0}^{n-3} KR_n(\ )\langle 1\rangle\{3-n+2i\}\right),$$

$$KR_n\left(\ \right) \oplus KR_n\left(\ \right) \cong KR_n\left(\ \right) \oplus KR_n\left(\ \right).$$

This theorem is central in the construction of Khovanov and Rozansky; it contains the technical results needed for the proof of the invariance of their link homology under Reidemeister moves.

Definition 1.8. *Given a (closed) planar regular graph Γ (i.e. an open planar regular graph without boundary points), $KR_n(\Gamma)$ is a graded 2-periodic chain complex. Denote its homology by $HKR_n(\Gamma) = H^0KR_n(\Gamma) \oplus H^1KR_n(\Gamma)$.*

Given a planar regular graph Γ, perform on each 4-valent vertex of Γ the transformation of Figure 6 to obtain a disjoint union of oriented circles in $\mathbb{R}^2$. Denote $p(\Gamma)$ the number of oriented circles modulo 2.

Corollary 1.2. [**24**]
For any planar regular graph Γ, the homology $HKR_n(\Gamma)$ of $KR_n(\Gamma)$ is concentrated in one of the two $\mathbb{Z}/2\mathbb{Z}$ grading:

$$HKR_n(\Gamma) = H^{p(\Gamma)}KR_n(\Gamma).$$

Furthermore, the 2-periodic chain complex $KR_n(\Gamma)$ is homotopic to

$$0 \longrightarrow H^{p(\Gamma)}KR_n(\Gamma) \longrightarrow 0,$$

where $H^{p(\Gamma)}KR_n(\Gamma)$ is in homological $\mathbb{Z}/2\mathbb{Z}$-grading $p(\Gamma)$.

The $\mathbb{Z}$-grading on R induce a $\mathbb{Z}$-grading on $HKR_n(\Gamma)$,

$$HKR_n(\Gamma) = \oplus_{i\in\mathbb{Z}} HKR_n^i(\Gamma).$$

Corollary 1.3. [**24**] *For any planar regular graph Γ,*

$$P_n(\Gamma)(q) = \sum_{i\in\mathbb{Z}} q^i \dim_{\mathbb{Q}} HKR_n^i(\Gamma).$$

Corollary 1.3 follows from Theorem 1.1 and the graph relations on Figure 2.

Hence, for any planar regular graph Γ, the graded 2-periodic chain complex $KR_n(\Gamma)$ is a categorification of the graph polynomial $P_n(\Gamma)$. Notice that since the graph polynomial $P_n(\Gamma)$ has only positive coefficients [**36**], we only require a $\mathbb{Z}$-graded vector space to categorify it, i.e. $HKR_n(\Gamma) = H^{p(\Gamma)}KR_n(\Gamma)$.

FIGURE 6. Smoothing

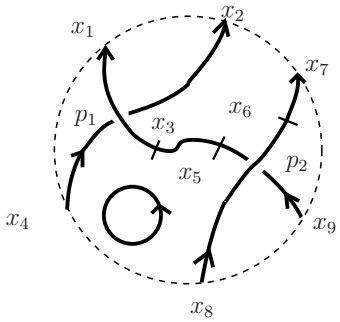


FIGURE 7. A marked diagram of a tangle

4. Link diagrams and complexes of graded matrix factorizations

Given a positive integer n, we associate, following Khovanov and Rozansky [**24**] to any oriented tangle diagram a chain complex of gmf's and to any oriented link diagram a chain complex of graded 2-periodic chain complexes. Given an oriented link diagram, the homology of the chain complex associated categorifies the $\mathfrak{sl}_n$ polynomial link invariant defined in Section 1.

By a *tangle* T we mean a proper embedding of an oriented compact 1-manifold into the ball B^3. We fix a great circle on the boundary 2-sphere of B^3 and require that the boundary points of the embedded 1-manifold lie on this great circle. An isotopy of a tangle should not move its boundary points. A *diagram* D of T is a generic projection of T onto the plane of the great circle. A *marked diagram* (also denoted by D) is a diagram with several marks placed on D so that any segment bounded by crossings has at least one mark (see an example in Figure 7). Boundary points also count as marks.

Let $m(D)$ be the set of marks of D, and let $\partial D \subset m(D)$ be the set of boundary points. Let R be the ring of polynomials in x_i, over $i \in m(D)$, and R' the ring of polynomials in x_i, over $i \in \partial D$.

Each crossing of D has two types of resolution: the smoothing and the graph, see Figure 8. We resolve every crossing in one of these two ways and obtain a resolution Γ of D which is an open planar regular graph in $\mathbb{R}^2$. A resolution Γ of

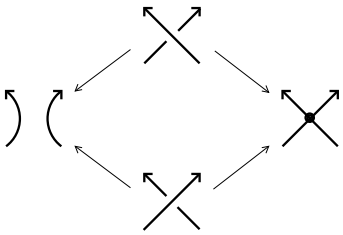
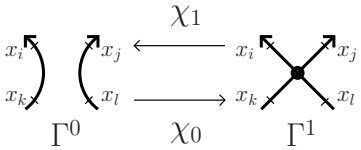


FIGURE 8. Resolutions

D inherits the marks in the obvious way. We associate to Γ a gmf over R as in Section 3.

We define two maps χ_0 and χ_1 of gmf's between elementary gmf's:



We define $\chi_0 : KR_n(\Gamma^0) \to KR_n(\Gamma^1)$ by the pair of matrices

$$U_0 = \begin{pmatrix} x_k - x_j & 0 \\ a & 1 \end{pmatrix}, U_1 = \begin{pmatrix} x_k & -x_j \\ -1 & 1 \end{pmatrix}$$

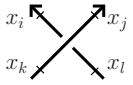
acting on $KR_n^0(\Gamma^0)$ and $KR_n^1(\Gamma^0)$ respectively, where $a = -u_2 + (u_1 + x_i u_2 - \pi_{jl})/(x_i - x_k)$.

The morphism $\chi_1 : KR_n(\Gamma^1) \to KR_n(\Gamma^0)$ is defined by the pair of matrices

$$V_0 = \begin{pmatrix} 1 & 0 \\ -a & x_k - x_j \end{pmatrix}, V_1 = \begin{pmatrix} 1 & x_j \\ 1 & x_k \end{pmatrix}$$

acting on $KR_n^0(\Gamma^1)$ and $KR_n^1(\Gamma^1)$ respectively, where $a = -u_2 + (u_1 + x_i u_2 - \pi_{jl})/(x_i - x_k)$. The maps χ_0 and χ_1 are of degree 1 (for the grading coming from R).

We associate to every crossing of D a short complex of gmf's as follows. To simplify the figures, we omit all the marks. To a positive crossing



,

we associate the complex concentrated in degree 0 and -1,

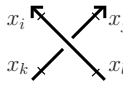
$$KR_n(\ \times\ ) = \left((KR_n^{-1}(\ \times\ ) \xrightarrow{\delta} KR_n^0(\ \times\ ) \right)$$

where

$$KR_n^0(\ \times\ ) = KR_n(\ )(\ )\{n-1\},$$

$$KR_n^{-1}(\ \times\ ) = KR_n(\ \times\ )\{n\} \text{ and } \delta = \chi_1.$$

All other terms of the complex are equal to 0. To a negative crossing



,

we associate the complex concentrated in degree 0 and 1,

$$KR_n(\ \nearrow\!\!\!\!\!\!\searrow\ ) = \Big(KR_n^0(\ \nearrow\!\!\!\!\!\!\searrow\ ) \xrightarrow{\delta} KR_n^1(\ \nearrow\!\!\!\!\!\!\searrow\ )\Big)$$

where

$$KR_n^0(\ \nearrow\!\!\!\!\!\!\searrow\ ) = KR_n(\ )(\ )\{1-n\},$$

$$KR_n^1(\ \nearrow\!\!\!\!\!\!\searrow\ ) = KR_n(\ \times\ )\{-n\} \text{ and } \delta = \chi_0.$$

Again, all other terms of the complex are equal to 0. For simplicity, we denote $KR_n(p)$ for the complex of gmf's associated to a crossing p. To an arc of D

,

we associate the complex concentrated in degree 0,

$$KR_n(\ )\ ) = KR_n^0(\ )\ )$$

where $KR_n^0(\ )\ ) = L_k^i$. Again, all other terms of the complex are equal to 0. For simplicity, we replace the notation $KR_n(\ )\ )$ by L_k^i, thinking of L_k^i as a complex of gmf's and not just a gmf.
To an oriented loop $\mathcal{C}$ without marks we assign the complex of graded 2-periodic chain complexes concentrated in degree 0,

$$KR_n(\mathcal{C}) = KR_n^0(\mathcal{C})$$

where $KR_n^0(\mathcal{C}) = V_n\langle 1\rangle$. Again, all other terms of the chain complex are equal to 0. Similary to the case of arcs, we use the notation $V_n\langle 1\rangle$ instead of $KR_n(\mathcal{C})$.

To D associate the complex of gmf's $KR_n(D)$ which is the tensor product of the $KR_n(p)$ for all crossings, the L_i^j for all arcs and the $V_n\langle 1\rangle$ for all markless circles of D. The tensoring is done over appropriate polynomial rings so that $KR_n(D)$, as an R-module, turns out to be free of finite rank.

Example 1.6. *For instance, to produce $KR_n(D)$, from D in Figure 7, we tensor $KR_n(p_1)$ with $KR_n(p_2)$ over $\mathbb{Q}$ and tensor the result with L_5^3 over $\mathbb{Q}[x_3, x_5]$ and with L_6^7 over $\mathbb{Q}[x_6]$. Finally tensor the result with $V_n\langle 1\rangle$ over $\mathbb{Q}$, so that*

$$KR_n(D) = KR_n(p_1) \otimes_{\mathbb{Q}} KR_n(p_2) \otimes_{\mathbb{Q}[x_3,x_5]} L_5^3 \otimes_{\mathbb{Q}[x_6]} L_6^7 \otimes_{\mathbb{Q}} V_n\langle 1\rangle.$$

For any diagram D, $KR_n(D)$ is a complex of finite graded (R', w)-matrix factorizations, where

$$w = \sum_{i \in \partial D} \pm x_i^{n+1},$$

with signs determined by orientation of D near boundary points. Thus, $KR_n(D)$ is an object of the category K_w over R'.

PROPOSITION 1.5. [**24**] *The complexes $KR_n(D)$ and $KR_n(D')$ are canonically isomorphic in $K_w = K(\mathcal{HMF}_w)$if D' differs from D only by marks.*

THEOREM 1.2. [**24**] *The complexes $KR_n(D)$ and $KR_n(D')$ are isomorphic in K_w if D and D' are two diagrams of the same tangle T.*

Link homology. When D is a diagram of an oriented link L, the ring R' is equal to $\mathbb{Q}$, and $\mathcal{HMF}_0$ is the category of graded 2-periodic chain complexes, up to homotopy. Applying the homology functor to $\mathcal{HMF}_0$, we arrive in the category of finite-dimensional $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Q}$-vector spaces. Note that for any resolution Γ of D the homology $HKR_n(\Gamma)$ is nontrivial only in the $\mathbb{Z}/2\mathbb{Z}$-grading $p(\Gamma)$ corresponding to the number of Seifert circles of D modulo 2. Taking the homology for the 2-periodic chain complexes, we obtain a complex of graded vector spaces and its homology $HKR_n(D)$ is considered as $\mathbb{Z} \oplus \mathbb{Z}$-graded:

$$HKR_n(D) = \oplus_{i,j \in \mathbb{Z}} H^i KR_n^j(D),$$

where i is the grading corresponding to the chain complex structure and j is the grading corresponding to the grading of R. The homology $HKR_n(D)$ categorifies the link polynomial $P_n(D)$ in the sense that,

$$P_n(D)(q) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}} H^i KR_n^j(D).$$

The latter follows from Corollary 1.3 and the relations between the graph polynomials and the $\mathfrak{sl}_n$ link polynomial invariants, see Section 1.

5. Cones and Khovanov-Rozansky construction

In the Appendix of [**26**], Khovanov and Rozansky give another description of the complex of gmf's associated to an oriented tangle diagram D. They describe the gmf associated to a 4-valent vertex as the cone of a map between the gmf associated to the smoothing and a gmf associated to a virtual crossing. This construction is related to the bicomplex construction in Chapter 3.

Define the gmf $KR_n($ $)$ associated to the virtual crossing on Figure 9 to be $(L_l^i \otimes_{\mathbb{Q}} L_k^j)\langle 1 \rangle$.

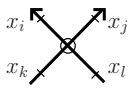


FIGURE 9. Virtual crossing

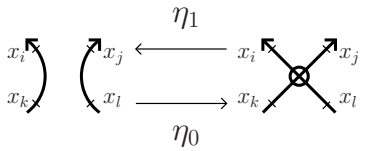


FIGURE 10. Morphisms η_0 and η_1

We define two"saddle" morphisms η_0 and η_1 of the gmf's presented graphically in Figure 10:

$$L_k^i \otimes_{\mathbb{Q}} L_l^j \underset{\eta_1}{\overset{\eta_0}{\rightleftarrows}} L_l^i \otimes_{\mathbb{Q}} L_k^j\langle 1\rangle$$

with η_0 given by the following pair of matrices, acting on KR_n^0 and KR_n^1 respectively,

$$\eta_0^0 = \begin{pmatrix} \pi_{ikl} & 1 \\ -\pi_{jkl} & 1 \end{pmatrix}, \eta_0^1 = \begin{pmatrix} 1 & -1 \\ \pi_{jkl} & \pi_{ikl} \end{pmatrix},$$

and with η_1 given by the following pair of matrices, acting on KR_n^0 and KR_n^1 respectively,

$$\eta_1^0 = \begin{pmatrix} 1 & -1 \\ \pi_{jkl} & \pi_{ikl} \end{pmatrix}, \eta_1^1 = \begin{pmatrix} \pi_{ikl} & 1 \\ -\pi_{jkl} & 1 \end{pmatrix},$$

where

$$\pi_{ijk} = \sum_{\substack{0 \leq a,b,c, \\ a+b+c=n-1,}} x_i^a x_j^b x_k^c.$$

The maps η_0 and η_1 preserve the $\mathbb{Z}/2\mathbb{Z}$-grading and increase the $\mathbb{Z}$-grading by $n-1$. Notice that the maps η_0 and η_1 depend on the variables x_i, x_j, x_k, and x_l. For convenience, we will generally omit these variables from the notation. If necessary these morphisms will be denoted by η_0^{ijkl} and η_1^{ijkl}. Define the following cones:

$$\text{Cone}(\eta_0) = \text{Cone}(L_k^i \otimes_{\mathbb{Q}} L_l^j\{-1\} \xrightarrow{\eta_0} L_l^i \otimes_{\mathbb{Q}} L_k^j\langle 1\rangle\{1\})$$
$$\text{Cone}(\eta_1) = \text{Cone}(L_l^i \otimes_{\mathbb{Q}} L_k^j\langle 1\rangle\{-1\} \xrightarrow{\eta_1} L_k^i \otimes_{\mathbb{Q}} L_l^j\{1\})$$

Notice that the grading shifts make η_0 and η_1 maps of degree $n+1$.

THEOREM 1.3. [**26**] *The cones* $\mathrm{Cone}(\eta_0)$ *and* $\mathrm{Cone}(\eta_1)\langle 1\rangle$ *are homotopic to the gmf* $KR_n($  $)$.

Consider the two natural homomorphisms μ_0 and μ_1,

$$\mu_0 : \mathrm{Cone}(\eta_0) \longrightarrow L_k^i \otimes_{\mathbb{Q}} L_l^j\{-1\},$$
$$\mu_1 : L_k^i \otimes_{\mathbb{Q}} L_l^j\{1\} \to \mathrm{Cone}(\eta_1)\langle 1\rangle.$$

Consider the following complex of gmf's,

$$\overline{KR}_n($$ $$) = \left(\overline{KR}_n^{-1}($$  $$) \xrightarrow{\overline{\delta}} \overline{KR}_n^{0}($$  $$)\right)$$

where

$$\overline{KR}_n^{0}($$ $$) = KR_n($$  $$)\{n-1\},$$
$$\overline{KR}_n^{-1}($$ $$) = \mathrm{Cone}(\eta_0)\{n\}, \text{ and } \overline{\delta} = \mu_0.$$

All other terms of the complex are equal to 0. We also define

$$\overline{KR}_n($$ $$) = \left(\overline{KR}_n^{0}($$  $$) \xrightarrow{\overline{\delta}} \overline{KR}_n^{1}($$  $$)\right)$$

where

$$\overline{KR}_n^{0}($$ $$) = KR_n($$  $$)\{1-n\},$$
$$\overline{KR}_n^{1}($$ $$) = \mathrm{Cone}(\eta_1)\langle 1\rangle\{-n\}, \text{ and } \overline{\delta} = \mu_1.$$

Again, all other terms of the complex are equal to 0.

THEOREM 1.4. [**26**] *The following isomorphisms hold in* K_w,

$$\overline{KR}_n($$  $$) \cong KR_n($$ $$)$$

and

$$\overline{KR}_n($$ $$) \cong KR_n($$ $$).$$

CHAPTER 2

Khovanov-Rozansky graph homology

1. Composition product for planar graphs

In analogy with a recursive formula for the HOMFLY-PT polynomial of links given by Jaeger, we give a recursive formula for the graph polynomial introduced by Kauffman and Vogel. We show how this formula extends to the Khovanov–Rozansky graph homology and derive a direct computational formula for this graph homology.

1.1. Graph polynomials, graph homology and a recursive formula. Jaeger [**14**] introduced a recursive formula for the HOMFLY-PT polynomial. In particular, for any oriented link diagram D and any integers $m, n \geq 1$, this formula allows a computation of $P_{n+m}(D)$ as a sum of products $P_n(D_1)\ P_m(D_2)$ where D_1 and D_2 run over certain subdiagrams of D. Jaeger calls this formula a *composition product*.

In analogy with Jaeger's composition product, we give a formula computing $P_{n+m}(\Gamma)$ as a sum of products $P_n(\Gamma_1)\ P_m(\Gamma_2)$ where Γ_1 and Γ_2 run over certain planar regular subgraphs of a planar regular Γ and $m, n \geq 1$ are integers (see Chapter 1 for a definition of planar regular graph). More precisely, define a *labelling* of Γ to be a mapping f from the set of edges of Γ to the set $\{1, 2\}$ (an oriented circle is treated as an edge without vertices). We denote by $\mathcal{L}(\Gamma)$ the set of labellings of Γ that satisfy the following local condition.

Conservation law: At every vertex v of Γ the number of adjacent edges labelled by 1 (resp. by 2) directed towards v is equal to the number of adjacent edges labelled by 1 (resp. by 2) directed out of v.

Given $f \in \mathcal{L}(\Gamma)$, we can erase all edges labelled by 2 (resp. by 1), smooth all 2-valent vertices (see Figure 1) and obtain thus a regular graph denoted $\Gamma_{f,1}$ (resp. $\Gamma_{f,2}$).

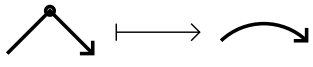


FIGURE 1. Smoothing of a 2-valent vertex

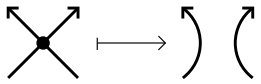


FIGURE 2. Smoothing

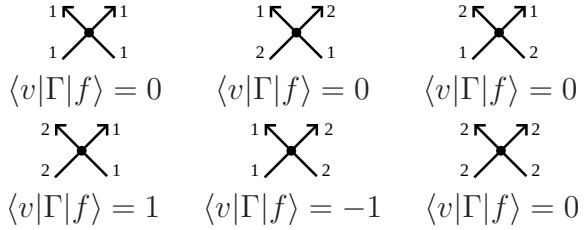


FIGURE 3. Definition of the interaction $\langle v|\Gamma|f\rangle$

We define the *rotation number* of Γ. Smooth all the vertices of Γ as in Figure 2. This gives a disjoint union of oriented circles embedded in $\mathbb{R}^2$; we call these circles the *Seifert circles* of Γ. The rotation number of Γ, denoted by $r(\Gamma)$, is the sum of the signs of these circles where the sign of a Seifert circle is $+1$ if it is oriented counterclockwise and -1 otherwise.

We define the *interaction* $\langle v|\Gamma|f\rangle \in \mathbb{Z}$ of a vertex v of Γ with a labelling f as shown in Figure 3, where 1 and 2 are the values of f on the corresponding edges. Set $\langle \Gamma|f\rangle = \sum_v \langle v|\Gamma|f\rangle$, where v runs over all vertices of Γ. Given integers $m,\ n\ \geq 1$, set

$$\sigma(\Gamma, f) = \sigma_{m,n}(\Gamma, f) = \langle \Gamma, f\rangle + m\, r(\Gamma_{f,1}) - n\, r(\Gamma_{f,2}) \in \mathbb{Z}.$$

LEMMA 2.1. *For all regular graphs $\Gamma \subset \mathbb{R}^2$ and for all integers $m,\ n \geq 1$,*

$$P_{n+m}(\Gamma) = \sum_{f \in \mathcal{L}(\Gamma)} q^{\sigma(\Gamma,f)}\ P_n(\Gamma_{f,1}) P_m(\Gamma_{f,2}).$$

We have seen in Chapter 1 that Khovanov and Rozansky categorified the graph polynomial for all positive integer n, in the sense that they constructed a $\mathbb{Z}$-graded $\mathbb{Q}$-vector space $HKR_n(\Gamma) = \oplus_{i\in\mathbb{Z}} HKR_n^i(\Gamma)$ such that

$$P_n(\Gamma) = \sum_{i\in\mathbb{Z}} \dim_{\mathbb{Q}} HKR_n^i(\Gamma)\ q^i. \tag{6}$$

We state now our main result of this subsection.

(7) $$Q(\ \bigcirc\ ) = \frac{q^{n+m} - q^{-(n+m)}}{q - q^{-1}} = [n+m]_q$$

(8) $$Q(\ \ ) = [n+m-1]_q\ Q(\ \ )$$

(9) $$Q\left(\ \ \right) = [2]_q\ Q(\ \ )$$

(10) $$Q(\ \ ) = Q(\ \ ) + [n+m-2]_q\ Q(\ \ )$$

(11) $$Q\left(\ \ \right) + Q\left(\ \ \right) = Q\left(\ \ \right) + Q\left(\ \ \right)$$

FIGURE 4. Graph relations

THEOREM 2.1. *For all regular graphs $\Gamma \subset \mathbb{R}^2$, for all $m,\ n \geq 1$, $i \in \mathbb{Z}$ and $j \in \mathbb{Z}/2\mathbb{Z}$, we have the folllowing isomorphism of $\mathbb{Q}$-vector spaces:*

$$H^j KR^i_{n+m}(\Gamma) \cong \bigoplus_{\substack{f \in \mathcal{L}(\Gamma) \\ k,l \in \mathbb{Z},\, k+l+\sigma(\Gamma,f) = i \\ r,s \in \mathbb{Z}/2\mathbb{Z},\, r+s=j}} H^r KR^k_n(\Gamma_{f,1}) \otimes_{\mathbb{Q}} H^s KR^l_m(\Gamma_{f,2})\{\sigma(\Gamma,f)\}.$$

This Theorem yields a categorification of Lemma 2.1 and gives a computation of $HKR_{n+m}(\Gamma)$ in terms of $HKR_n(\Gamma_{f,1})$ and $HKR_m(\Gamma_{f,2})$. We also derive from Theorem 2.1 a direct computational formula for $HKR_n(\Gamma)$, see Corollary 2.1.

1.2. Proofs.

1.2.1. *Proof of Lemma 2.1.* Fix $m, n \geq 1$. For any regular graph $\Gamma \subset \mathbb{R}^2$, set

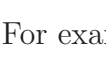
$$Q(\Gamma) = Q_{n+m}(\Gamma) = \sum_{f \in \mathcal{L}(\Gamma)} q^{\sigma(\Gamma,f)} P_n(\Gamma_{f,1}) P_m(\Gamma_{f,2}) \in \mathbb{Z}[q, q^{-1}].$$

In order to prove the lemma, it is enough to check that Q satisfies the five relations (7)-(11) on Figure 4, see [**20**] and the Appendix for a detailed treatment. First we verify (7):

$$\begin{aligned} Q(\ \bigcirc\ ) &= q^{-m} P_n(\ \bigcirc\ ) + q^n P_m(\ \bigcirc\ ) \\ &= q^{-m}[n]_q + q^n [m]_q = [n+m]_q. \end{aligned}$$

To prove the other identities, we need to fix more notation: Given a regular graph Γ, a labelling $f_0 \in \mathcal{L}(\Gamma)$, and a subset E_0 of the set of edges of Γ, set

$$Q(\Gamma_{f_0,E_0}) = \sum_{f\in\mathcal{L}(\Gamma), f|_{E_0}=f_0|_{E_0}} q^{\sigma(\Gamma,f)} P_n(\Gamma_{f,1}) P_m(\Gamma_{f,2}).$$

For example, $Q($  $)$ is given by a sum over all labellings whose values on the pictured edges are 1. We recall that the graph polynomial P_n satisfy the relation 1-5 of Chapter 1, Figure 2. We now check (8). We have

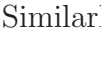
(12) $Q($  $) = Q($  $) + Q($ $) + Q($ $) + Q($ $).$

Applying relation (1) of Chapter 1 to the P_m-terms in $Q($  $)$, we obtain

$$Q($ $) = q^{n-1}[m]_q \; Q($ $),$$

and applying relation (2) to the P_n-terms in $Q($  $)$, we get

$$Q($ $) = q^{-m}[n-1]_q \; Q($  $).$$

Similarly, we can apply relation (2) to the P_m-terms in $Q($  $)$ and relation (1) to the P_n-terms in $Q($ $)$ and we get that the right hand-side of (12) is equal to

$$\begin{aligned}
& q^{-m}[n-1]_q \; Q($  $) + q^{n-1}[m]_q \; Q($ $) \\
& +q^{n}[m-1]_q \; Q($ $) + q^{-m+1}[n]_q \; Q($ $) \\
= \; & \left(q^{-m}[n-1]_q + q^{n-1}[m]_q\right) \; Q($ $) + \left(q^{n}[m-1]_q + q^{-m+1}[n]_q\right) \; Q($ $) \\
= \; & [n+m-1]_q \left(Q($ $) + Q($ $)\right) \\
= \; & [n+m-1]_q \; Q($ $).
\end{aligned}$$

Hence Q satisfies (8).

We check (4):

$$
\begin{aligned}
Q\left(\ \right) \;=\; & Q\left(\ \right)+Q\left(\ \right)+Q\left(\ \right)\\
& +Q\left(\ \right)+\ Q\left(\ \right)+Q\left(\ \right)\\
& +Q\left(\ \right)+Q\left(\ \right)+\ Q\left(\ \right)\\
& +Q\left(\ \right)\\
=\; & Q\left(\ \right)+Q\left(\ \right)+q^{-1}Q\left(\ \right)+qQ\left(\ \right)\\
& +\ qQ\left(\ \right)+q^{-1}Q\left(\ \right)+qQ\left(\ \right)\\
& +q^{-1}Q\left(\ \right)+\ q^{-1}Q\left(\ \right)+qQ\left(\ \right).
\end{aligned}
$$

Using the relation (3) for the P_n-terms and P_m-terms, we easily obtain (4).

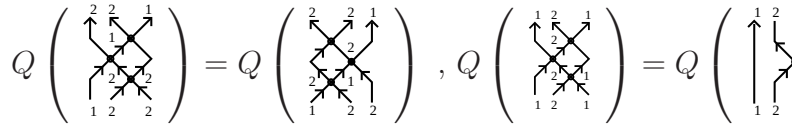
We now check (10).

$$
\begin{aligned}
Q\left(\ \right) \;=\; & Q\left(\ \right)+Q\left(\ \right)\\
& +Q\left(\ \right)+Q\left(\ \right)\\
& +\ Q\left(\ \right)+Q\left(\ \right)\\
& +Q\left(\ \right)+Q\left(\ \right)\\
& +\ Q\left(\ \right)+Q\left(\ \right).
\end{aligned}
$$

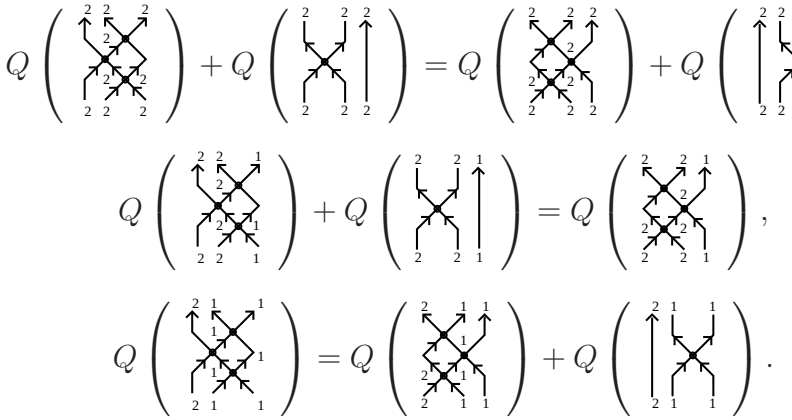
We apply the relations (1), (2), and (4) to the P_n-terms and P_m-terms, so the latter expression is equal to:

$$
\begin{aligned}
& Q\left(\ \right)+q^{-m}[n-2]_q\ Q\left(\ \right)+q^{n-2}[m]_q\ Q\left(\ \right)\\
& +Q\left(\ \right)+q^{n}[m-2]_q\ Q\left(\ \right)+q^{2-m}[n]_q\ Q\left(\ \right)\\
& +q^{1-m}[n-1]_q\ Q\left(\ \right)+q^{n-1}[m-1]_q\ Q\left(\ \right)\\
& +q^{1-n}[m-1]_q\ Q\left(\ \right)+q^{m-1}[n-1]_q\ Q\left(\ \right)\\
& +\ Q\left(\ \right)+Q\left(\ \right)\\
=\; & Q\left(\ \right)+[n+m-2]_q\ Q\left(\ \right).
\end{aligned}
$$

In order to prove the last relation (11), we put in correspondence the labellings occuring on the two sides of (11) as in the following two examples:



There are 28 different labellings possible for  . Among these labellings, 14 are identified with labellings of  as in the first example, and 8 with labellings of  as in the second example. The remaining 6 labellings are involved in the following equalities (three other equalities are obtained by exchanging 1 and 2):



In this way we obtain that Q satisfies (11).
Hence, Q satisfies equations (7-11) and since these equations determine P_{n+m}, we conclude that $Q = P_{n+m}$. Lemma 1 is proved.

1.2.2. *Proof of Theorem 2.1.* Since the graph polynomial has only non negative coefficients, we have directly from Lemma 2.1 and Formula (6):

(13)

$$HKR^i_{n+m}(\Gamma) \cong \bigoplus_{\substack{f \in \mathcal{L}(\Gamma),\\ k,l \in \mathbb{Z}, k+l+\sigma(\Gamma,f)=i}} HKR^k_n(\Gamma_{f,1}) \otimes_{\mathbb{Q}} HKR^l_m(\Gamma_{f,2})\{\sigma(\Gamma,f)\}.$$

We need now to precise the $\mathbb{Z}/2\mathbb{Z}$-grading. It follows from Corollary 1.2 that for any planar regular graph Γ, $n \geq 1$, and $i \in \mathbb{Z}$,

$$H^j KR^i_n(\Gamma) = 0 \quad \text{if } j = r(\Gamma) + 1 \pmod 2.$$

Furthermore, given a planar regular graph Γ and $f \in \mathcal{L}(\Gamma)$, we have

$$r(\Gamma) = r(\Gamma_{f,1}) + r(\Gamma_{f,2}).$$

Theorem 2.1 follows from (13) and the latter formula.

1.3. Consequences. Given a planar regular graph Γ, we denote by $L(\Gamma)$ the subset of labellings of Γ such that $\Gamma_{f,2}$ is a disjoint union of circles. We also define

$$\mathcal{S}(\Gamma) = \{\Gamma_{f,1} | f \in L(\Gamma)\}.$$

Given a planar regular subgraph $\Delta \in \mathcal{S}(\Gamma)$ of Γ, there is a unique labelling $f_\Delta \in L(\Gamma)$ such that $\Delta = \Gamma_{f,1}$. For $\Delta \in \mathcal{S}(\Gamma)$, set $\beta(\Gamma, \Delta) = \langle \Gamma, f_\Delta \rangle$ where $\langle \Gamma, f_\Delta \rangle$ is defined in Subsection 1.1. We need to fix more notations. For all $k, l \in \mathbb{Z}$,

$$\mathbb{Q}\{k\}\langle l \rangle = \oplus_{i \in \mathbb{Z}, j \in \mathbb{Z}/2\mathbb{Z}} \mathbb{Q}\{k\}\langle l \rangle^{i,j},$$

where

$$\mathbb{Q}\{k\}\langle l \rangle^{i,j} = \begin{cases} \mathbb{Q} & \text{if } i = k \text{ and } j = l \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

We state a corollary of Theorem 2.1.

Corollary 2.1. *For all regular graphs $\Gamma \subset \mathbb{R}^2$ and all integers $n \geq 2$,*

$$HKR_n(\Gamma) \cong \bigoplus_{\Delta_1 \in \mathcal{S}(\Gamma), \Delta_2 \in \mathcal{S}(\Delta_1), \ldots, \Delta_{n-1} \in \mathcal{S}(\Delta_{n-2})} \mathbb{Q}\{\delta(\Delta_1, \ldots, \Delta_{n-1})\}\langle r(\Gamma) \rangle$$

where

$$\begin{aligned} \delta(\Delta_1, \ldots, \Delta_{n-1}) &= \sum_{i=0}^{n-2} (\beta(\Delta_i, \Delta_{i+1}) + (n-i)\ r(\Delta_{i+1}) - (n-1-i)\ r(\Delta_i)) \\ &= \sum_{i=0}^{n-2} (\beta(\Delta_i, \Delta_{i+1}) + 2\ r(\Delta_{i+1})) - (n-1)\ r(\Gamma) \end{aligned}$$

with the convention $\Delta_0 = \Gamma$.

Proof. From [**24**, p. 11] we have

$$HKR_1(\Gamma) = \begin{cases} \mathbb{Q} & \text{if } \Gamma \text{ is a union of circles,} \\ 0 & \text{otherwise.} \end{cases}$$

The rest of the argument is just a straightforward induction argument using the case $m = 1$ of Theorem 2.1. □

We illustrate Theorem 2.1 and Corollary 2.1, for $n = 2$ and $m = 1$ in the following example.

EXAMPLE 2.1.

$$HKR_3\left(\ \right) \cong HKR_2\left(\ \right) \oplus HKR_2(\ )\{-3\}\langle 1\rangle$$
$$\oplus HKR_2(\ )\{3\}\langle 1\rangle \oplus HKR_2(\ )\{1\}\langle 1\rangle$$
$$\oplus HKR_2(\ )\{-1\}\langle 1\rangle$$
$$\cong \mathbb{Q}\{-2\} \oplus \mathbb{Q}\{2\} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}\{-2\} \oplus \mathbb{Q}\{-4\}$$
$$\oplus \mathbb{Q}\{4\} \oplus \mathbb{Q}\{2\} \oplus \mathbb{Q}\{2\} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}\{-2\}$$

Remark. Corollary 2.1 suggest an equivalent but direct definition of $HKR_n(\Gamma)$, namely

$$HKR_n(\Gamma) = \bigoplus_{\Delta_1 \in \mathcal{S}(\Gamma), \Delta_2 \in \mathcal{S}(\Delta_1), \ldots, \Delta_n \in \mathcal{S}(\Delta_{n-1})} \mathbb{Q}\{\delta(\Delta_1, \ldots, \Delta_n)\}\langle r(\Gamma)\rangle.$$

Using this expression as a starting definition of $HKR_n(\Gamma)$, it would be interesting to exhibit explicitly the isomorphism of Theorem 2.1.

1.4. Application. Given an oriented link diagram D, define

$$g_n^{\max} = \max\{k \in \mathbb{Z} \mid HKR_n^k(D) = \oplus_{j\in\mathbb{Z}} H^j KR_n^k(D) \neq 0\},$$
$$g_n^{\min} = \min\{k \in \mathbb{Z} \mid HKR_n^k(D) = \oplus_{j\in\mathbb{Z}} H^j KR_n^k(D) \neq 0\}.$$

These are numerical invariants for links. Theorem 2.1 was used by Wu, in order to give a new proof of the following theorem, which is a generalization of the Morton-Frank-Williams inequality on the degree of the link polynomial invariant P_n.

THEOREM 2.2. [**53**] *Let L be a closed braid with m strands, c_+ positive crossings and c_- negative crossings. Then*

$$(n-1)(w-m) - 2c_- \leq g_n^{\min}(L) \leq g_n^{\max}(L) \leq (n-1)(w+m) + 2c_+,$$

where $w = c_+ - c_-$ is the writhe of L.

2. Generalization to embedded graphs

We generalize Khovanov–Rozansky construction to the case of graphs embedded in B^3. More precisely, we prove that a certain complex of gmf's associated to a graph embbeded in B^3 is invariant under the graph moves introduced by Kauffman.

2.1. Open regular graphs and complex of graded matrix factorizations.

We consider finite oriented graphs embedded into a ball $B^3 \subset \mathbb{R}^3$. We fix a great circle on the boundary 2-sphere of B^3 and require that the boundary points of the embedded graph lie on this great circle. These graphs are called *open regular graph*. A *diagram* Γ of an open regular graph is a generic projection of the graph onto the plane of the great circle. An isotopy of such a graph should not move its boundary points. An embedded graph into B^3 without boundary points is called a *(closed) regular graph*.

In Chapter 1, we associated to any open planar regular graph Γ and any positive integer n a gmf $KR_n(\Gamma)$. Moreover, we associated to any tangle diagram D and any positive integer n a complex of gmf's $KR_n(D)$. In particular, we recall that to an arc in an oriented edge (or in an oriented loop) of an open planar regular graph bounded by marks i and j and oriented from j to i, without any other interior mark as on Figure 5 of Chapter 1 we associate the gmf L^i_j. To an arc of a tangle diagram bounded by marks i and j and oriented from j to i, without any other interior mark as on Figure 5 of Chapter 1 we associate the complex of gmf's,

$$KR_n(\ \overset{\curvearrowleft}{\ }\ ) = KR^0_n(\ \overset{\curvearrowleft}{\ }\ )$$

where $KR^0_n(\ \overset{\curvearrowleft}{\ }\ ) = L^i_j$. All other terms of the complex are equal to 0. The difference is just that the gmf L^i_j is now seen as complex of gmf's in the most obvious way. Similarly , in Chapter 1 we associate to a 4-valent vertex s of an open planar regular graph a gmf $KR_n(s)$. Now to a 4-valent vertex in an open regular graph we associate the complex of gmf's

$$KR_n(\ \times\ ) = KR^0_n(\ \times\ )$$

where $KR^0_n(\ \times\ ) = KR_n(s)$. Again, all other terms of the complex are equal to 0. Given an open regular graph Γ, we associate a complex of gmf's in the same way as in the case of tangle diagrams: Thus, we define

$$KR_n(\Gamma) = \left(\bigotimes_a L^i_j\right) \bigotimes \left(\bigotimes_p KR_n(p)\right) \bigotimes \left(\bigotimes_s KR_n(s)\right)$$

where a runs through all arcs in Γ starting and ending in marked points that contain no crossings and no other marked points, p runs through all the positive and negative crossings of Γ and s runs through all 4-valent vertices of Γ .

Notice that when there are no 4-valent vertices, then this construction gives the chain complex of gmf's associated by Khovanov and Rozansky to the link. Furthermore, if there are no crossings, we obtain a trivial chain complex of gmf's

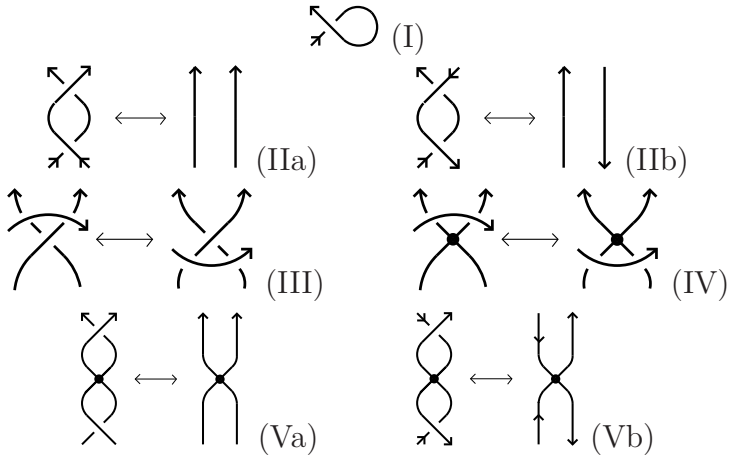


FIGURE 5. Graph moves that generate rigid vertex isotopy

consisting in the gmf associated by Khovanov and Rozansky to the planar regular graph, lying in homological degree zero.

2.2. Reidemeister moves for graphs. We will consider open regular graphs embedded in B^3 as graphs with rigid vertices. As explained in [**19**], a 4-valent graph with rigid vertices can be regarded as an embedding of a graph whose vertices have been replaced by rigid disks. Each disk has four strands attached to it, and the cyclic order of these strands is determined via the rigidity of the disk. An *RV-isotopy or rigid vertex isotopy* of the embedding of such an open regular graph Γ in B^3 consists in affine motions of the disks, coupled with topological ambient isotopies of the strands (corresponding to the edge of Γ). An RV-isotopy of an open regular graph should not move its boundary points. The notion of RV-isotopy is a mixture of mechanical (Euclidian) and topological concepts. It arise naturally in the building of models for graph embeddings, and it also arises naturally in regard to creating invariants of graph embeddings.

In [**19**], Kauffman derived a collection of moves, analogous to Reidemeister moves, that generates RV-isotopy for diagrams of 4-valent graph embeddings. As we will only be interested in 4-valent oriented graph embeddings whose oriented rigid vertex take the basic form

we will present the RV-moves in this case, see Figure 5.

2.3. The main result. We can now state the main result of this section.

THEOREM 2.3. *Let $KR_n(\Gamma_1)$ and $KR_n(\Gamma_2)$ be complexes of gmf's associated to diagrams Γ_1 and Γ_2 of open regular graphs in B^3. If there exist a sequence*

FIGURE 6. Type (IV) move

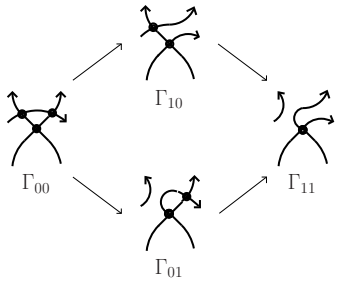


FIGURE 7. Four resolution of Γ in the type (IV) move

of RV-moves such that Γ_2 *is obtained from* Γ_1 *then* $KR_n(\Gamma_1)$ *and* $KR_n(\Gamma_2)$ *are homotopy equivalent.*

As pointed out by Kauffman and Vogel [**20**], link polynomial invariants give rise to graph invariants. Thus, the same is true for Khovanov-Rozansky link homology.

PROOF OF THEOREM 2.3. In [**24**], Khovanov and Rozansky have proved the invariance of $KR_n(\Gamma)$ under type (I), (II) and (III) moves, see Figure 5. We prove the invariance under type (IV) and (V). The invariance under type (IV) follows directly from the proof of invariance under (III). We will use at many level the proofs by Khovanov and Rozansky, see [**24**].

Invariance under (IV)

As pointed out by Wu [**52**], the Khovanov-Rozansky's proof of the invariance under Reidemeister (III) can be simplified by using Bar-Natan's algebraic trick [**5**], i.e by using the fact that the homotopy equivalence used for the proof of the invariance under Reidemeister move (IIa) is a strong deformation retract. If we think of the proof that way, then the proof of invariance under (IV) is contained in the one of (III).

We need to show that $KR_n(\Gamma)$ and $KR_n(\Gamma')$ are isomorphic for Γ, Γ' in Figure 6. The diagram Γ has 4 resolutions, denoted by Γ_{ij} for $i, j \in \{0, 1\}$: See Figure 7. The complex $KR_n(\Gamma)\{-2n\}$ has the form

$$0 \to KR_n(\Gamma_{00}) \overset{\partial^{-2}}{\to} \begin{pmatrix} KR_n(\Gamma_{01})\{-1\} \\ KR_n(\Gamma_{10})\{-1\} \end{pmatrix} \overset{\partial^{-1}}{\to} KR_n(\Gamma_{11})\{-2\} \to 0$$

with $KR_n(\Gamma_{11})\{-2\}$ in cohomological degree 0. This complex is shown in Figure 7.

Khovanov and Rozansky proved [**24**] the following isomorphism:

$$KR_n(\Gamma_{01}) \cong KR_n(\Gamma_{11})\{+1\} \oplus KR_n(\Gamma_{11})\{-1\}. \tag{14}$$

Furthermore, they proved that

$$KR_n(\Gamma_{00}) \cong KR_n(\Gamma_{11}) \oplus \Upsilon, \tag{15}$$

where Υ is defined in [**24**, Prop. 33].

The differential ∂^{-2} is injective on $KR_n(\Gamma_{11}) \subset KR_n(\Gamma_{00})$. In fact, the map to $KR_n(\Gamma_{01})\{-1\}$ is injective (which follows from the inclusion $KR_n(\Gamma_{11}) \subset KR_n(\Gamma_{00})$ and the proof of invariance under (IIa), see [**24**]). The gmf $\partial^{-2}(KR_n(\Gamma_{00}))$ is a direct summand of $KR_n^{-1}(\Gamma)\{-2n\}$. Thus $KR_n(\Gamma)\{-2n\}$ contains a contractible summand

$$0 \to KR_n(\Gamma_{11}) \stackrel{\partial^{-2}}{\to} KR_n(\Gamma_{11}) \to 0. \tag{16}$$

The direct sum decomposition (14) can be chosen so that

$$KR_n(\Gamma_{01})\{-1\} \cong p_{01}\partial^{-2}KR_n(\Gamma_{11}) \oplus KR_n(\Gamma_{11})\{-2\},$$

where p_{01} is the projection of $KR_n^{-1}(\Gamma)\{-2n\}$ onto $KR_n(\Gamma_{01})\{-1\}$. The differential ∂^{-1} is injective on $KR_n(\Gamma_{11})\{-2\} \subset KR_n(\Gamma_{01})\{-1\}$. Furthermore, the image of $KR_n(\Gamma_{11})\{-2\} \subset KR_n(\Gamma_{01})\{-1\}$ under ∂^{-2} is a direct summand of $KR_n^0(\Gamma)\{-2n\}$. Hence the complex $KR_n(\Gamma)\{-2n\}$ contains a contractible direct summand isomorphic to

$$0 \to KR_n(\Gamma_{11})\{-2\} \stackrel{\partial^{-1}}{\to} KR_n(\Gamma_{11})\{-2\} \to 0 \tag{17}$$

After splitting off contractible direct summands (16) and (17), the complex $KR_n(\Gamma)\{-2n\}$ reduces to the complex C defined by

$$0 \to \Upsilon \stackrel{\partial^{-2}}{\to} KR_n(\Gamma_{10})\{-1\} \to 0$$

Since both $KR_n(\Gamma)\{-2n\}$ and $KR_n(\Gamma')\{-2n\}$ contain $KR_n(\Gamma_{10})\{-1\}$, [**24**, Prop. 33] ensures that we can perform exactly the same reduction to $KR_n(\Gamma')\{-2n\}$. Finally, we conclude that

$$KR_n(\Gamma') \cong KR_n(\Gamma').$$

Invariance under (Va)

We need to show that $KR_n(\Gamma)$ and $KR_n(\Gamma')$ are isomorphic for the graphs Γ, Γ' shown in Figure 8. The diagram Γ has 4 resolutions, denoted by Γ_{ij} for $i, j \in \{0, 1\}$ and shown in Figure 9. The complex $KR_n(\Gamma)$ has the form

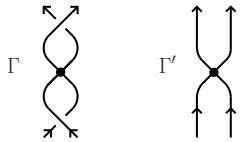


FIGURE 8. Type (Va) move

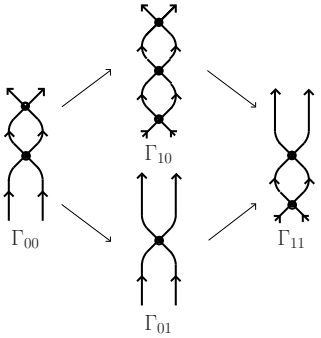


FIGURE 9. Four resolution of Γ in the type (Va) move

$$0 \to KR_n(\Gamma_{00})\{+1\} \stackrel{\partial^{-1}}{\to} \begin{pmatrix} KR_n(\Gamma_{01}) \\ KR_n(\Gamma_{10}) \end{pmatrix} \stackrel{\partial^0}{\to} KR_n(\Gamma_{11})\{-1\} \to 0$$

where $KR_n(\Gamma_{01})$ and $KR_n(\Gamma_{10})$ are in cohomological degree 0. We have depicted this complex in Figure 9. Since Γ_{00} and Γ_{11} are isotopic, $KR_n(\Gamma_{00})$ and $KR_n(\Gamma_{11})$ are isomorphic. Khovanov and Rozansky [**24**] proved that

$$KR_n(\Gamma_{10}) \cong KR_n(\Gamma_{00})\{+1\} \oplus KR_n(\Gamma_{00})\{-1\}. \tag{18}$$

Khovanov-Rozansky's proof of invariance under (IIa) ensures that the differential ∂^{-1} is injective on $KR_n^{-1}(\Gamma_{00})$. Direct sum decomposition (18) can be chosen so that

$$KR_n(\Gamma_{10}) \cong p_{10}\partial^{-1}KR_n(\Gamma_{00})\{+1\} \oplus KR_n(\Gamma_{00})\{-1\}.$$

Thus, $KR_n(\Gamma)$ contains a contractible summand

$$0 \to KR_n(\Gamma_{00})\{+1\} \stackrel{\partial^{-1}}{\to} KR_n(\Gamma_{00})\{+1\} \to 0. \tag{19}$$

Furthermore, we have

$$KR_n(\Gamma_{11})\{-1\} \cong KR_n(\Gamma_{01}) \oplus KR_n(\Gamma_{01})\{-2\}. \tag{20}$$

Differential ∂^0 is surjective onto $KR_n(\Gamma_{01}) \subset KR_n(\Gamma_{11})\{-1\}$. Thus $KR_n(\Gamma)$ contains a contractible summand

$$0 \to KR_n(\Gamma_{01}) \stackrel{\partial^{-1}}{\to} KR_n(\Gamma_{01}) \to 0. \tag{21}$$

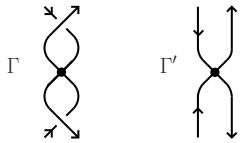


FIGURE 10. Type (Vb) move

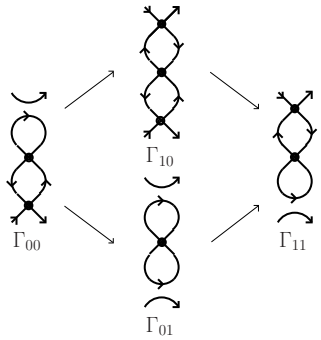


FIGURE 11. Four resolution of Γ in the type (Vb) move

After splitting off the contractible direct summands (19) and (21), the complex $KR_n(\Gamma)$ reduces to the complex C of the form

$$0 \to KR_n(\Gamma_{11})\{-1\} \xrightarrow{\partial^0} KR_n(\Gamma_{01})\{-2\} \to 0$$

The decomposition (20) ensures that C is homotpy equivalent to

$$0 \to KR_n(\Gamma_{01}) \to 0.$$

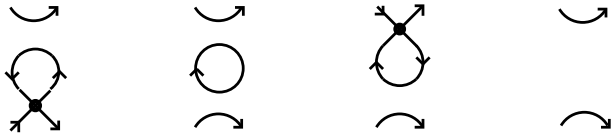
Invariance under (Vb)
We need to show that $KR_n(\Gamma)$ and $KR_n(\Gamma')$ are isomorphic for the graphs Γ, Γ' shown in Figure 10. The diagram Γ has 4 resolutions, denoted by Γ_{ij} for $i, j \in \{0, 1\}$ and shown in Figure 11. The complex $KR_n(\Gamma)$ has the form

$$0 \to KR_n(\Gamma_{00})\{+1\} \xrightarrow{\partial^{-1}} \begin{pmatrix} KR_n(\Gamma_{01}) \\ KR_n(\Gamma_{10}) \end{pmatrix} \xrightarrow{\partial^0} KR_n(\Gamma_{11})\{-1\} \to 0$$

where $KR_n(\Gamma_{01})$ and $KR_n(\Gamma_{10})$ are in cohomological degree 0. We have depicted this complex in Figure 11.

Applying Khovanov-Rozansky's results, we have the following isomorphisms:

$$KR_n(\Gamma_{00}) \cong \bigoplus_{i=0}^{n-2} KR_n(G_{00})\{2 - n + 2i\}\langle 1\rangle, \tag{22}$$



G_{00} G_{01} G_{11} G

FIGURE 12. The graph G_{00}, G_{01}, G_{11}, G

$$KR_n(\Gamma_{01}) \cong \bigoplus_{i=0}^{n-2} KR_n(G_{01})\{2-n+2i\}\langle 1\rangle, \tag{23}$$

$$KR_n(\Gamma_{11}) \cong \bigoplus_{i=0}^{n-2} KR_n(G_{11})\{2-n+2i\}\langle 1\rangle, \tag{24}$$

and

$$KR_n(\Gamma_{00}) \cong \left(\bigoplus_{i=0}^{n-3} KR_n(G_{11})\{3-n+2i\}\langle 1\rangle\right) \oplus KR_n(\Gamma'), \tag{25}$$

where G_{00}, G_{01} and G_{11} are depicted in Figure 12.

We can twist the direct sum decompositions (22), (23) and (24) so that ∂^{-1} and ∂^0 have diagonal form following this decomposition. Furthermore, consider the following short complexes: For all $i \in [\![0, n-2]\!]$

$$0 \to KR_n(G_{00})\{3-n+2i\} \xrightarrow{\partial_i^{-1}} KR_n(G_{01})\{2-n+2i\} \xrightarrow{\partial_i^{0}} KR_n(G_{11})\{1-n+2i\} \to 0$$

The proof of invariance under (I) by Khovanov and Rozansky implies that ∂_i^{-1} is split injective and ∂_i^0 is split surjective. Since the category $\mathcal{HMF}_w$ has splitting idempotents (see [**24**, p. 46]), we can decompose $KR_n^0(\Gamma)$ as the direct sum

$$KR_n^0(\Gamma) \cong \left(\bigoplus_{i=0}^{n-2} Im(\partial_i^{-1})\right) \oplus \left(\bigoplus_{i=0}^{n-2} Y_1^i\right) \oplus Y_2$$

in such a way that ∂_i^0 restrict to an isomorphism from Y_1^i to $KR_n(G_{11})\{1-n+2i\}$ for all $i = 0, \dots, n-2$ and $\partial_i^0(Y_2) = 0$. Therefore, $KR_n(\Gamma)$ is isomorphic to the direct sum of complexes

$$\begin{array}{ccccccc} & & 0 & \to & Y_2 & \to & 0, \\ 0 & \to & KR_n(G_{00})\{3-n+2i\} & \xrightarrow{\cong} & \mathrm{Im}(\partial_i^{-1}) & \to & 0, \\ & & 0 & \to & Y_1^i & \xrightarrow{\cong} & KR_n(G_{11})\{1-n+2i\} \to 0. \end{array}$$

We can decompose further the sum decompositions (22), (23) and (24) so that we obtain

$$KR_n(\Gamma_{00}) \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-2} KR_n(G)\{4-2n+2(i+j)\}, \tag{26}$$

$$KR_n(\Gamma_{01}) \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-1} KR_n(G)\{3-2n+2(i+j)\}, \tag{27}$$

$$KR_n(\Gamma_{11}) \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-2} KR_n(G)\{4-2n+2(i+j)i\}, \tag{28}$$

and

$$KR_n(\Gamma_{00}) \cong \left(\bigoplus_{i=0}^{n-3} \bigoplus_{j=0}^{n-2} KR_n(G)\{5-2n+2(i+j)\} \right) \oplus KR_n(\Gamma'), \tag{29}$$

where G is the right-most graph depicted in Figure 12. From formula (26) to (29) we obtain

$$KR_n^0(\Gamma) \cong KR_n(\Gamma_{01}) \oplus KR_n(\Gamma_{10}) \cong KR_n(\Gamma_{00})\{+1\} \oplus KR_n(\Gamma_{11})\{-1\} \oplus KR_n(\Gamma').$$

From Proposition 1.1, we know that the category $\mathcal{HMF}_w$ is Krull-Schmidt; it implies that $Y_2 \cong KR_n(\Gamma')$. Therefore, the complexes $KR_n(\Gamma)$ and $0 \to KR_n(\Gamma') \to 0$ are isomorphic. This concludes our proof of the invariance under type (Vb) move. Theorem 2.3 follows. □

CHAPTER 3

Triply graded link homology $\mathbb{H}_n$

For each positive integer n, we build a triply graded link homology $\mathbb{H}_n$ from which we recover the Khovanov–Rozansky bigraded link homology. In Section 1, we describe the construction and state the two main theorems. In Section 2, we investigate further the matrix factorizations, in order to prepare the proofs of Theorem 3.1 and Theorem 3.2 in Section 3. This chapter contains the two main theorems of this thesis. We emphasize that this construction is related to the cone description of Khovanov-Rozansky construction, see Chapter 1.5. Rooughly speaking, we ignore the cones in Khovanov–Rozansky construction and associate to any link diagram, a bicomplex of 2-periodic chain complexes over $\mathbb{Q}$.

1. The main construction: from bicomplexes to triple grading

Fix a positive integer n. Let D be an oriented link diagram in $\mathbb{R}^2$. Let k be a positive integer and $\{x_1, \ldots, x_k\}$ be a set of marks on the diagram D such that every arc between two crossings has at least one mark. Each crossing of D has two types of resolution: the smoothing and the virtualization, see Figure 1. We resolve every crossing in one of these two ways and obtain a resolution $\mathcal{D}$ of D, which is a union of oriented circles in $\mathbb{R}^2$ with virtual crossings.

We associate to each such resolution $\mathcal{D}$ a $\mathbb{Z}$-graded 2-periodic chain complex. The resolution $\mathcal{D}$ of D inherits the marks $\{x_1, \ldots, x_k\}$ in the obvious way. We now build a gmf out of the elementary pieces shown on Figure 2. We associate to an arc, as on the left of Figure 2, the gmf L_j^i of potential $x_i^{n+1} - x_j^{n+1} \in \mathbb{Q}[x_i, x_j]$,

$$\mathbb{Q}[x_i, x_j] \xrightarrow{\times \pi_{ij}} \mathbb{Q}[x_i, x_j]\{1-n\} \xrightarrow{\times (x_i - x_j)} \mathbb{Q}[x_i, x_j],$$

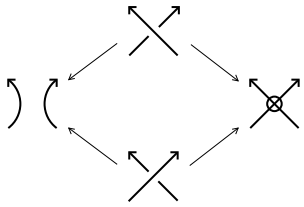


FIGURE 1. Resolutions

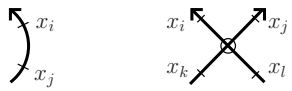


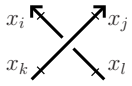
FIGURE 2. Elementary pieces that decompose $\mathcal{D}$

with

$$\pi_{ij} = \sum_{k=0}^{n} x_i^k x_j^{n-k} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}.$$

By definition, the gmf $C_n($  $)$ associated to the virtual crossing as on the right of Figure 2 is equal to $(L_l^i \otimes_{\mathbb{Q}} L_k^j)\langle 1\rangle$.
To the resolution $\mathcal{D}$ of D, we associate the gmf $C_n(\mathcal{D})$ obtained by taking the tensor products over suitable polynomial $\mathbb{Q}$-algebras of the gmf's associated to all pieces of $\mathcal{D}$ (see Chapter 1 for the definitions of the tensor products). The potential of a gmf is additive with respect to these tensor products. As a consequence $C_n(\mathcal{D})$ is a graded 2-periodic chain complex over $R = \mathbb{Q}[x_1, \ldots, x_k]$. Moreover, this 2-periodic chain complex has only homology in one of the two $\mathbb{Z}/2\mathbb{Z}$-grading (it is a consequence of Lemma 3.7 and Lemma 3.8).

Using the saddle morphisms η_0 and η_1 defined in Chapter 1.5, we associate to every crossing of D a bicomplex of gmf's. To simplify the figures, we omit all the marks. To a positive crossing



we associate the bicomplex of gmf's over $\mathbb{Q}[x_i, x_j, x_k, x_l]$

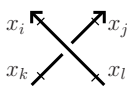
$$C_n(\ \times\ ) = \begin{array}{ccc} C_n^{-1,1}(\ \times\ ) & \longrightarrow & 0 \\ \uparrow{\scriptstyle \delta_v} & & \uparrow \\ C_n^{-1,0}(\ \times\ ) & \xrightarrow{\delta_h} & C_n^{0,0}(\ \times\ ) \end{array}$$

where

$$C_n^{0,0}(\ \times\ ) = C_n^{-1,0}(\ \times\ ) = C_n(\ )\ (\ )\{n-1\},$$

$$C_n^{-1,1}(\ \times\ ) = C_n(\ \otimes\ ),\ \delta_h = \mathrm{Id} \text{ and } \delta_v = \eta_0.$$

All other terms of the bicomplex are equal to 0. To a negative crossing



we associate the bicomplex of gmf's over $\mathbb{Q}[x_i, x_j, x_k, x_l]$

$$\begin{array}{ccc} C_n(\ \times\ ) = C_n^{0,0}(\ \times\ ) & \xrightarrow{\delta_h} & C_n^{1,0}(\ \times\ ) \\ \uparrow & & \uparrow{\scriptstyle \delta_v} \\ 0 & \longrightarrow & C_n^{1,-1}(\ \times\ ) \end{array}$$

where

$$C_n^{0,0}(\ \times\ ) = C_n^{1,0}(\ \times\ ) = C_n(\ )\,(\ )\{1-n\},$$

$$C_n^{1,-1}(\ \times\ ) = C_n(\ \otimes\ ),\ \delta_h = \mathrm{Id}, \text{ and } \delta_v = \eta_1.$$

Again, all other terms of the bicomplex are equal to 0. To an arc of D

we associate the bicomplex

$$C_n(\ )\ ) = C_n^{0,0}(\ )\ )$$

where $C_n^{0,0}(\ )\ ) = L_k^i$. Again, all other terms of the bicomplex are equal to 0. To an oriented circle without marks, we associate the bicomplex

$$C_n(\ \bigcirc\ ) = C_n^{0,0}(\ \bigcirc\ )$$

where $C_n^{0,0}(\ \bigcirc\ ) = V_n\langle 1\rangle$ (see page 22 for a definition of $V_n\langle 1\rangle$). Again, all other terms of the bicomplex are equal to 0.

We finally associate to D a bicomplex $C_n(D) = (C_n^{i,j}(D))_{i,j\in\mathbb{Z}^2}$ of graded 2-periodic chain complexes over $\mathbb{Q}$ by taking the tensor products of the bicomplexes of the gmf's associated to crossings and arcs of D. Set

$$\mathbb{H}_n(D) = H(H(H(C_n(D), d), \delta_v), \delta_h).$$

The symbol d stands for the differential on the 2-periodic chain complexes $C_n^{i,j}(D)$, $(i,j) \in \mathbb{Z}^2$. The symbol δ_v stands for the differential along the columns and δ_h for the differential along the rows. Thus, the $\mathbb{Q}$-vector space $\mathbb{H}_n(D)$ has four gradings: three homological gradings and one additional grading called quantum grading. Two of the homological gradings are $\mathbb{Z}$-gradings and come from the bicomplex structure. The third homological grading is a $\mathbb{Z}/2\mathbb{Z}$-grading and comes from the 2-periodic chain complex structure. We refer to the $\mathbb{Z}$-grading coming from the R-module structure on $C_n(D)$ as the *quantum grading*.

Since $\mathbb{H}_n(D)$ is concentrated in one of the $\mathbb{Z}/2\mathbb{Z}$-grading, we now forget the $\mathbb{Z}/2\mathbb{Z}$-grading and consider $\mathbb{H}_n(D)$ as triply graded:

$$\mathbb{H}_n(D) = \bigoplus_{(i,j,k)\in\mathbb{Z}^3} \mathbb{H}_n^{i,j,k}(D)$$

where i refers to the horizontal bicomplex grading, j to the vertical bicomplex grading and k to the quantum grading.

LEMMA 3.1. *Given an oriented link diagram D, for all $(i,j,k) \in \mathbb{Z}^3$, the $\mathbb{Q}$-vector space $\mathbb{H}_n^{i,j,k}(D)$ is independent of the choice of marks, up to isomorphism.*

PROOF. This lemma is a consequence of Lemma 3.7 and Lemma 3.8 below. □

We now state our two main theorems.

THEOREM 3.1. *Given an oriented link diagram D, for all $(i,j,k) \in \mathbb{Z}^3$, the $\mathbb{Q}$-vector space $\mathbb{H}_n^{i,j,k}(D)$ is invariant under Reidemeister moves, up to isomorphism.*

The next theorem shows that our construction recovers the Khovanov–Rozansky link homology.

THEOREM 3.2. *Given an oriented link diagram D, the bigraded link homology $HKR_n(D)$ is determined by the triply graded link homology $\mathbb{H}_n$. For all $(i,k) \in \mathbb{Z}^2$,*

$$H^iKR_n^k(D) \cong \oplus_{j\in\mathbb{Z}}\ \mathbb{H}_n^{i,j,k-j(n+1)}(D).$$

Given an oriented link diagram D, we consider the Poincaré polynomial associated to $\mathbb{H}_n(D)$:

$$P_n^{\mathbb{H}}(D)(u,v,q) = \sum_{(i,j,k)\in\mathbb{Z}^3} u^iv^jq^k \dim_{\mathbb{Q}}\ \mathbb{H}_n^{i,j,k}(D).$$

COROLLARY 3.1. *The $\mathfrak{sl}_n$-polynomial P_n is determined by $P_n^{\mathbb{H}}$:*

$$P_n(D)(q) = P_n^{\mathbb{H}}(D)(-1, -q^{n+1}, q).$$

2. More on matrix factorizations

We give useful results on matrix factorizations for the proofs of Theorem 3.1 and Theorem 3.2. In a first subsection, we give some general results on the cones and matrix factorizations. In a second subsection, we recall precisely the procedure of removing marks and the Frobenius structure of V_n. We give also some properties of the "saddle" morphisms.

2.1. Cones and matrix factorizations. Consider a complex of matrix factorizations

$$C : C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1}$$

Consider the following procedure which transforms the complex C of matrix factorizations into a matrix factorization denoted $\mathrm{Cone}(C)$. We begin for example, by taking the cone of the differential d^i, we obtain then a complex of matrix factorizations of lengh $k-1$:

$$C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots C^{i-1} \xrightarrow{d^{i-1}} \mathrm{Cone}(d^i) \xrightarrow{d^{i+1}} C^{i+2}\langle 1\rangle \cdots \xrightarrow{d^{k-1}} C^k\langle 1\rangle \xrightarrow{d^k} C^{k+1}\langle 1\rangle$$

We repeat this elementary procedure to all the differentials of the complex. The final result does not depend on the order in which we took the differentials. This follows from the lemma below.

LEMMA 3.2. *Given a short complex of mf's*

$$C : 0 \to E \xrightarrow{d^0} F \xrightarrow{d^1} G \to 0,$$

the following mf's are isomorphic:

$$\mathrm{Cone}(\mathrm{Cone}(d^0) \to G\langle 1\rangle) \cong \mathrm{Cone}(E \to \mathrm{Cone}(d^1))$$

PROOF. The isomorphism is given by the following pair of matrices:

$$Q^0 = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & -\mathrm{Id} \end{pmatrix} \text{ and } Q^1 = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & -\mathrm{Id} \end{pmatrix}.$$

□

Consider mf C, C', D, and D' and two mf's morphisms f and g:

$$f : C \to C' \text{ and } g : D \to D'.$$

LEMMA 3.3. *There is an isomorphism between the matrix factorizations* $Cone(f) \otimes Cone(g)$ *and* $Cone(f \otimes g)$.

We do not make precise tensor products, potentials, and rings, because the result holds in all cases. Furthermore, it is true for gmf's when f and g are of degree $n+1$.

PROOF. We have

$$\mathrm{Cone}(f) = \; C^0 \oplus C'^1 \xrightarrow{\begin{pmatrix} d^0 & 0 \\ f^0 & d'^1 \end{pmatrix}} C^1 \oplus C'^0 \xrightarrow{\begin{pmatrix} d^1 & 0 \\ -f^1 & d'^0 \end{pmatrix}} C^0 \oplus C'^1 \;,$$

and

$$\mathrm{Cone}(g) = \; D^0 \oplus D'^1 \xrightarrow{\begin{pmatrix} \delta^0 & 0 \\ g^0 & \delta'^1 \end{pmatrix}} D^1 \oplus D'^0 \xrightarrow{\begin{pmatrix} \delta^1 & 0 \\ -g^1 & \delta'^0 \end{pmatrix}} D^0 \oplus D'^1 \;.$$

Hence we have

$$\mathrm{Cone}(f)\otimes\mathrm{Cone}(g) = \; E^0 \xrightarrow{P^0} E^1 \xrightarrow{P^1} E^0$$

with

$$\begin{aligned} E^0 &= (C^0\oplus C'^1)\otimes(D^0\oplus D'^1)\oplus(C^1\oplus C'^0)\otimes(D^1\oplus D'^0)\\ &= (C^0\otimes D^0)\oplus(C^0\otimes D'^1)\oplus(C'^1\otimes D^0)\oplus(C'^1\otimes D'^1)\\ &\quad\oplus(C^1\otimes D^1)\oplus(C^1\otimes D'^0)\oplus(C'^0\otimes D^1)\oplus(C'^0\otimes D'^0),\end{aligned}$$

$$\begin{aligned} E^1 &= (C^0\oplus C'^1)\otimes(D^1\oplus D'^0)\oplus(C^1\oplus C'^0)\otimes(D^0\oplus D'^1)\\ &= (C^0\otimes D^1)\oplus(C^0\otimes D'^0)\oplus(C'^1\otimes D^1)\oplus(C'^1\otimes D'^0)\\ &\quad\oplus(C^1\otimes D^0)\oplus(C^1\otimes D'^1)\oplus(C'^0\otimes D^0)\oplus(C'^0\otimes D'^1),\end{aligned}$$

$$P^0 = \begin{pmatrix}
\mathrm{Id}\otimes\delta^0 & 0 & 0 & 0 & d^1\otimes\mathrm{Id} & 0 & 0 & 0\\
\mathrm{Id}\otimes g^0 & \mathrm{Id}\otimes\delta'^1 & 0 & 0 & 0 & d^1\otimes\mathrm{Id} & 0 & 0\\
0 & 0 & \mathrm{Id}\otimes\delta^0 & 0 & -f^1\otimes\mathrm{Id} & 0 & d'^0\otimes\mathrm{Id} & 0\\
0 & 0 & \mathrm{Id}\otimes g^0 & \mathrm{Id}\otimes\delta'^1 & 0 & -f^1\otimes\mathrm{Id} & 0 & d'^0\otimes\mathrm{Id}\\
d^0\otimes\mathrm{Id} & 0 & 0 & 0 & -\mathrm{Id}\otimes\delta^1 & 0 & 0 & 0\\
0 & d^0\otimes\mathrm{Id} & 0 & 0 & \mathrm{Id}\otimes g^1 & -\mathrm{Id}\otimes\delta'^0 & 0 & 0\\
f^0\otimes\mathrm{Id} & 0 & d'^1\otimes\mathrm{Id} & 0 & 0 & 0 & -\mathrm{Id}\otimes\delta^1 & 0\\
0 & f^0\otimes\mathrm{Id} & 0 & d'^1\otimes\mathrm{Id} & 0 & 0 & \mathrm{Id}\otimes g^1 & -\mathrm{Id}\otimes\delta'^0
\end{pmatrix},$$

and

$$P^1 = \begin{pmatrix}
\mathrm{Id}\otimes\delta^1 & 0 & 0 & 0 & d^1\otimes\mathrm{Id} & 0 & 0 & 0\\
-\mathrm{Id}\otimes g^1 & \mathrm{Id}\otimes\delta'^0 & 0 & 0 & 0 & d^1\otimes\mathrm{Id} & 0 & 0\\
0 & 0 & \mathrm{Id}\otimes\delta^1 & 0 & -f^1\otimes\mathrm{Id} & 0 & d'^0\otimes\mathrm{Id} & 0\\
0 & 0 & -\mathrm{Id}\otimes g^1 & \mathrm{Id}\otimes\delta'^0 & 0 & -f^1\otimes\mathrm{Id} & 0 & d'^0\otimes\mathrm{Id}\\
d^0\otimes\mathrm{Id} & 0 & 0 & 0 & -\mathrm{Id}\otimes\delta^0 & 0 & 0 & 0\\
0 & d^0\otimes\mathrm{Id} & 0 & 0 & -\mathrm{Id}\otimes g^0 & -\mathrm{Id}\otimes\delta'^1 & 0 & 0\\
f^0\otimes\mathrm{Id} & 0 & d'^1\otimes\mathrm{Id} & 0 & 0 & 0 & -\mathrm{Id}\otimes\delta^0 & 0\\
0 & f^0\otimes\mathrm{Id} & 0 & d'^1\otimes\mathrm{Id} & 0 & 0 & -\mathrm{Id}\otimes g^0 & -\mathrm{Id}\otimes\delta'^1
\end{pmatrix}.$$

On the other hand, we have

$$f\otimes g := \left(C\otimes D \xrightarrow{\begin{pmatrix}\mathrm{Id}\otimes g\\ f\otimes\mathrm{Id}\end{pmatrix}} C\otimes D'\oplus C'\otimes D \xrightarrow{\begin{pmatrix} f\otimes\mathrm{Id}, & -\mathrm{Id}\otimes g\end{pmatrix}} C'\otimes D \right).$$

and

$$\mathrm{Cone}(f\otimes g) = \; F^0 \xrightarrow{Q^0} F^1 \xrightarrow{Q^1} F^0$$

with

$$\begin{aligned} F^0 &= (C\otimes D)^0\oplus(C\otimes D')^1\oplus(C'\otimes D)^1\oplus(C'\otimes D')^0\\ &= (C^0\otimes D^0)\oplus(C^1\otimes D^1)\oplus(C^0\otimes D'^1)\oplus(C^1\otimes D'^0)\\ &\quad\oplus(C'^0\otimes D^1)\oplus(C'^1\otimes D^0)\oplus(C'^0\otimes D'^0)\oplus(C'^1\otimes D'^1), \end{aligned}$$

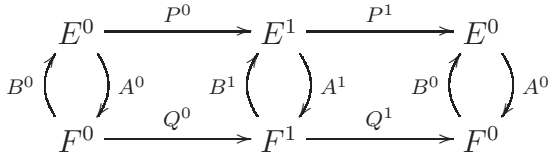
$$\begin{aligned} F^1 &= (C\otimes D)^1\oplus(C\otimes D')^0\oplus(C'\otimes D)^0\oplus(C'\otimes D')^1\\ &= (C^0\otimes D^1)\oplus(C^1\otimes D^0)\oplus(C^0\otimes D'^0)\oplus(C^1\otimes D'^1)\\ &\quad\oplus(C'^0\otimes D^0)\oplus(C'^1\otimes D^1)\oplus(C'^0\otimes D'^1)\oplus(C'^1\otimes D'^0), \end{aligned}$$

$$Q^0=\begin{pmatrix} \mathrm{Id}\otimes\delta^0 & d^1\otimes\mathrm{Id} & 0 & 0 & 0 & 0 & 0 & 0\\ d^0\otimes\mathrm{Id} & -\mathrm{Id}\otimes\delta^1 & 0 & 0 & 0 & 0 & 0 & 0\\ \mathrm{Id}\otimes g^0 & 0 & \mathrm{Id}\otimes\delta'^1 & d^1\otimes\mathrm{Id} & 0 & 0 & 0 & 0\\ 0 & \mathrm{Id}\otimes g^1 & d^0\otimes\mathrm{Id} & -\mathrm{Id}\otimes\delta'^0 & 0 & 0 & 0 & 0\\ f^0\otimes\mathrm{Id} & 0 & 0 & 0 & \mathrm{Id}\otimes\delta^1 & d'^1\otimes\mathrm{Id} & 0 & 0\\ 0 & f^1\otimes\mathrm{Id} & 0 & 0 & d'^0\otimes\mathrm{Id} & -\mathrm{Id}\otimes\delta^0 & 0 & 0\\ 0 & 0 & f^0\otimes\mathrm{Id} & 0 & -\mathrm{Id}\otimes g^1 & 0 & \mathrm{Id}\otimes\delta'^0 & d'^1\otimes\mathrm{Id}\\ 0 & 0 & 0 & f^1\otimes\mathrm{Id} & 0 & -\mathrm{Id}\otimes g^0 & d'^0\otimes\mathrm{Id} & -\mathrm{Id}\otimes\delta'^1 \end{pmatrix},$$

and

$$Q^1=\begin{pmatrix} \mathrm{Id}\otimes\delta^1 & d^1\otimes\mathrm{Id} & 0 & 0 & 0 & 0 & 0 & 0\\ d^0\otimes\mathrm{Id} & -\mathrm{Id}\otimes\delta^0 & 0 & 0 & 0 & 0 & 0 & 0\\ -\mathrm{Id}\otimes g^1 & 0 & \mathrm{Id}\otimes\delta'^0 & d^1\otimes\mathrm{Id} & 0 & 0 & 0 & 0\\ 0 & -\mathrm{Id}\otimes g^0 & d^0\otimes\mathrm{Id} & -\mathrm{Id}\otimes\delta'^1 & 0 & 0 & 0 & 0\\ -f^0\otimes\mathrm{Id} & 0 & 0 & 0 & \mathrm{Id}\otimes\delta^0 & d'^1\otimes\mathrm{Id} & 0 & 0\\ 0 & -f^1\otimes\mathrm{Id} & 0 & 0 & d'^0\otimes\mathrm{Id} & -\mathrm{Id}\otimes\delta^1 & 0 & 0\\ 0 & 0 & -f^0\otimes\mathrm{Id} & 0 & \mathrm{Id}\otimes g^0 & 0 & \mathrm{Id}\otimes\delta'^1 & d'^1\otimes\mathrm{Id}\\ 0 & 0 & 0 & -f^1\otimes\mathrm{Id} & 0 & \mathrm{Id}\otimes g^1 & d'^0\otimes\mathrm{Id} & -\mathrm{Id}\otimes\delta'^0 \end{pmatrix}.$$

We exhibit an isomorphism between $\mathrm{Cone}(f)\otimes\mathrm{Cone}(g)$ and $\mathrm{Cone}(f\otimes g)$:



with

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, A^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$B^0 = (A^0)^{-1}$, and $B^1 = (A^1)^{-1}$. □

Consider two mf's C and D.

LEMMA 3.4. *There exists an isomorphism τ from $C \otimes D$ to $D \otimes C$.*

Similarly this result holds for all choices of tensor products, potentials and rings.

PROOF. The isomorphism is given by the following pair of matrices:

$$\tau^0 = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \text{ and } \tau^1 = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

□

Consider a morphism $f : (C, c) \to (D, d)$ of 2-periodic chain complexes. Consider the 2-periodic chain complex $\mathrm{Cone}(f)$. Suppose there exist a homotopy equivalence $F : (C', d') \to (C, c)$ and $F' : (C, c) \to (C', c')$ such that $F' \circ F = \mathrm{Id}$, $F \circ F' - \mathrm{Id} = h \circ c + c \circ h$, and $h \circ F = 0$. Such a homotopy is called a *strong deformation retract*, see [**5**].

LEMMA 3.5. *The 2-periodic chain complexes* $\mathrm{Cone}(f)$ *and* $Cone(f \circ F)$ *are homotopic.*

PROOF. We give a homotopy equivalence $\overline{F}$, $\overline{F'}$ between $\mathrm{Cone}(f)$ and $\mathrm{Cone}(f \circ F)$:

$$\begin{array}{ccccccccc} & & & \overset{H^0}{\frown} & & & & \overset{H^1}{\frown} & \\ \mathrm{Cone}(f) = & & C^0 \oplus D^1 & \xrightarrow{P^0} & C^1 \oplus D^0 & & \xrightarrow{P^1} & & C^0 \oplus D^1 \\ \overline{F}\big(\quad\big)\overline{F'} & & \overline{F}^0\big(\quad\big)\overline{F'}^0 & & \overline{F}^1\big(\quad\big)\overline{F'}^1 & & & \overline{F}^0\big(\quad\big)\overline{F'}^0 & \\ \mathrm{Cone}(f \circ F) = & & C'^0 \oplus D^1 & \xrightarrow{Q^0} & C'^1 \oplus D^0 & & \xrightarrow{Q^1} & & C'^0 \oplus D^1 \end{array}$$

where

$$P^0 = \begin{pmatrix} c^0 & 0 \\ f^0 & d^1 \end{pmatrix}, P^1 = \begin{pmatrix} c^1 & 0 \\ -f^1 & d^0 \end{pmatrix},$$

$$Q^0 = \begin{pmatrix} c'^0 & 0 \\ f^0 \circ F^0 & d^1 \end{pmatrix}, Q^1 = \begin{pmatrix} c'^1 & 0 \\ -f^1 \circ F^1 & d^0 \end{pmatrix},$$

$$\overline{F'}^0 = \begin{pmatrix} F'^0 & 0 \\ -f^1 \circ h^1 & \mathrm{Id} \end{pmatrix}, \overline{F'}^1 = \begin{pmatrix} F'^1 & 0 \\ f^0 \circ h^0 & \mathrm{Id} \end{pmatrix},$$

$$\overline{F}^0 = \begin{pmatrix} F^0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \overline{F}^1 = \begin{pmatrix} F^1 & 0 \\ 0 & \mathrm{Id} \end{pmatrix},$$

$$H^0 = \begin{pmatrix} h^0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } H^1 = \begin{pmatrix} h^1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The homotopy on the bottom is zero. □

Suppose there exist a strong deformation retract $G : (D, d) \to (D', d')$ and $G' : (D', d') \to (D, d)$ such that $G \circ G' = \mathrm{Id}$, $G' \circ G - \mathrm{Id} = h \circ d + d \circ h$, and $G \circ h = 0$.

Lemma 3.6. *The 2-periodic chain complexes $Cone(f)$ and $Cone(G \circ f)$ are homotopic.*

Lemma 3.5 and Lemma 3.6 hold in the graded case if f is of degree $n + 1$, F, G, F' and G' are of degree 0.

Proof. We give a strong deformation retract $\overline{G}$, $\overline{G'}$ between $Cone(f)$ and $Cone(G \circ f)$:

$$\begin{array}{ccc} \mathrm{Cone}(f) = & C^0 \oplus D^1 \xrightarrow{P^0} C^1 \oplus D^0 \xrightarrow{P^1} C^0 \oplus D^1 \\ \overline{G'}\begin{pmatrix} & \end{pmatrix}\overline{G} & \overline{G'}^0\begin{pmatrix} & \end{pmatrix}\overline{G}^0 \quad \overline{G'}^1\begin{pmatrix} & \end{pmatrix}\overline{G}^1 \quad \overline{G'}^0\begin{pmatrix} & \end{pmatrix}\overline{G}^0 \\ \mathrm{Cone}(G \circ f) = & C^0 \oplus D'^1 \xrightarrow{Q^0} C^1 \oplus D'^0 \xrightarrow{Q^1} C^0 \oplus D'^1 \end{array}$$

where

$$P^0 = \begin{pmatrix} c^0 & 0 \\ f^0 & d^1 \end{pmatrix}, P^1 = \begin{pmatrix} c^1 & 0 \\ -f^1 & d^0 \end{pmatrix},$$

$$Q^0 = \begin{pmatrix} c^0 & 0 \\ G^0 \circ f^0 & d'^1 \end{pmatrix}, Q^1 = \begin{pmatrix} c^1 & 0 \\ -G^1 \circ f^1 & d'^0 \end{pmatrix},$$

$$\overline{G'}^0 = \begin{pmatrix} \mathrm{Id} & 0 \\ h^1 \circ f^0 & G'^1 \end{pmatrix}, \overline{G'}^1 = \begin{pmatrix} \mathrm{Id} & 0 \\ -h^0 \circ f^1 & G'^0 \end{pmatrix},$$

$$\overline{G}^0 = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & G^1 \end{pmatrix}, \overline{G}^1 = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & G^0 \end{pmatrix},$$

$$H^0 = \begin{pmatrix} 0 & 0 \\ 0 & h^1 \end{pmatrix}, \text{ and } H^1 = \begin{pmatrix} 0 & 0 \\ 0 & h^0 \end{pmatrix}.$$

The homotopy on the bottom is zero. □

2.2. Removing marks, "saddle morphisms" and Frobenius algebra. We first recall a lemma of Khovanov and Rozansky about the procedure of removing marks, which is the main tool in the proof of Lemma 3.1 and whose proof will be useful in the proof of Theorem 3.1. Recall that L^i_j is the following gmf

$$\mathbb{Q}[x_i, x_j] \xrightarrow{\times \pi_{ij}} \mathbb{Q}[x_i, x_j]\{1-n\} \xrightarrow{\times (x_i - x_j)} \mathbb{Q}[x_i, x_j].$$

LEMMA 3.7. [**24**, p. 38] *The gmf's $L^k_j \otimes_{\mathbb{Q}[x_k]} L^i_k$ and L^i_j are canonically homotopy equivalent as matrix factorizations of potential $x_i^{n+1} - x_j^{n+1}$ over $R = \mathbb{Q}[x_i, x_j]$.*

PROOF. Denote $R' = \mathbb{Q}[x_i, x_j, x_k]$. We exhibit the homotopy equivalence given in[**24**] between $L^k_j \otimes_{\mathbb{Q}[x_k]} L^i_k$ and L^i_j:

$$\begin{array}{ccccc} R' \oplus R'\{2-2n\} & \xrightarrow{P^0} & R'\{1-n\} \oplus R'\{1-n\} & \xrightarrow{P^1} & R' \oplus R'\{2-2n\} \\ {\scriptstyle G^0}\big(\quad \big){\scriptstyle F^0} & & {\scriptstyle G^1}\big(\quad \big){\scriptstyle F^1} & & {\scriptstyle G^0}\big(\quad \big){\scriptstyle F^0} \\ R & \xrightarrow{\times \pi_{ij}} & R\{1-n\} & \xrightarrow{\times (x_i - x_j)} & R \end{array}$$

where

$$P^0 = \begin{pmatrix} \pi_{jk} & x_i - x_k \\ \pi_{ik} & x_j - x_k \end{pmatrix}, \; P^1 = \begin{pmatrix} x_k - x_j & x_i - x_k \\ \pi_{ik} & -\pi_{jk} \end{pmatrix},$$

$$F^0 = \begin{pmatrix} \phi & 0 \end{pmatrix}, \; F^1 = \begin{pmatrix} \phi & 0 \end{pmatrix},$$

$$G^0 = \begin{pmatrix} 1 \\ \pi_{ijk} \end{pmatrix}, \; G^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$\phi : \mathbb{Q}[x_i, x_j, x_k] \longrightarrow \mathbb{Q}[x_i, x_j]$$

defined by $\phi(x_k) = x_i$, $\phi(x_i) = x_i$, and $\phi(x_j) = x_j$. For a proof that it is in fact a homotopy equivalence, see [**24**, Prop. 9] and [**24**, Prop. 20]. □

REMARK 3.1. *The morphism F that appears in the proof of Lemma 3.7 is homotopic to the following morphism $\overline{F}$, see* [**24**, Prop. 19]*:*

$$\overline{F}^0 = \begin{pmatrix} \psi & 0 \end{pmatrix}, \; \overline{F}^1 = \begin{pmatrix} 0 & \psi \end{pmatrix},$$

with

$$\psi : \mathbb{Q}[x_i, x_j, x_k] \longrightarrow \mathbb{Q}[x_i, x_j]$$

defined by $\psi(x_k) = x_j$, $\psi(x_i) = x_i$, and $\psi(x_j) = x_j$. The morphisms F, $\overline{F}$ and G will be useful in order to reduce the sizes of the matrix factorizations involved in the proof of the invariance under Reidemeister moves. They allow to simplify objects in the category $K(K(\mathcal{HMF}_w))$.

Applying Lemma 3.7 to the case when $j = i$, we get a homotopy equivalence between the graded 2-periodic chain complex associated to a circle with two marks and the graded 2-periodic chain complex associated to a circle with one mark. The next Lemma gives a homotopy equivalence between the graded 2-periodic chain complex L_x^x associated to a circle with one mark and the graded 2-periodic chain complex $V_n\langle 1\rangle$ associated to a markless circle.

LEMMA 3.8. [**24**, p. 40] *The graded 2-periodic chain complexes over* $\mathbb{Q}$, L_x^x *and* $V_n\langle 1\rangle$, *are homotopic.*

PROOF. We give from [**24**, p. 40] a homotopy equivalence between L_x^x and $V_n\langle 1\rangle$:

$$\begin{array}{ccccc} & & \overset{H}{\curvearrowleft} & & \\ \mathbb{Q}[x] & \xrightarrow{\times(n+1)x^n} & \mathbb{Q}[x]\{1-n\} & \xrightarrow{0} & \mathbb{Q}[x] \\ {\scriptstyle G^0}\big(\ \big){\scriptstyle F^0} & & {\scriptstyle G^1}\big(\ \big){\scriptstyle F^1} & & {\scriptstyle G^0}\big(\ \big){\scriptstyle F^0} \\ 0 & \xrightarrow{0} & (\mathbb{Q}[x]/x^n)\{1-n\} & \xrightarrow{0} & 0 \end{array}$$

where $F^0 = G^0 = 0$, the morphism $F^1 : \mathbb{Q}[x]\{1-n\} \to (\mathbb{Q}[x]/x^n)\{1-n\}$ is the natural projection, and the morphism $G^1 : (\mathbb{Q}[x]/x^n)\{1-n\} \to \mathbb{Q}[x]\{1-n\}$ is the injection. The homotopy $H : \mathbb{Q}[x]\{1-n\} \to \mathbb{Q}[x]$ is defined as follows:

$$H(x^k) = 0 \text{ if } k < n \text{ and } H(x^k) = \frac{x^{k-n}}{n+1} \text{ if } k \geq n.$$

Furthermore, we have $F \circ G = \mathrm{Id}$. □

We recall the Frobenius structure [**27**] of $V_n = (\mathbb{Q}[x]/x^n)\{1-n\}$. The multiplication m is given by

$$m : \ V_n \otimes V_n \ \longrightarrow \ V_n, \ \ x^i \otimes x^j \ \longmapsto \ x^{i+j} \ ,$$

and the comultiplication Δ is given by

$$\Delta : \ V_n \ \longrightarrow \ V_n \otimes V_n, \ \ x^i \ \longmapsto \ (n+1)\textstyle\sum_{k=0}^{n-1-i} x^{i+k} \otimes x^{n-1-k} \ .$$

Notice that m and Δ are of degree $n-1$. Define the unit $\iota : \mathbb{Q} \to V_n$, by $\iota(1) = 1$. The counit $\epsilon : V_n \to \mathbb{Q}$ is defined as follows:

$$\epsilon(x^{n-1}) = \frac{1}{n+1} \text{ and } \epsilon(x^i) = 0 \text{ if } i \neq n-1.$$

The unit and the counit are of degree $1-n$. Each of ι and ϵ induces a natural map between the graded 2-periodic chain complex $V_n\langle 1\rangle$ and the following graded 2-periodic chain complex,

$$0 \longrightarrow \mathbb{Q} \longrightarrow 0.$$

We still denote these maps ι and ϵ.

To finish this section we give homotopy equivalent descriptions of the "saddle" morphisms η_0 and η_1. These morphims will appear in the next section. Define two other "saddle" morphisms $\overline{\eta}_0$ and $\overline{\eta}_1$:

$$L_k^i \otimes_{\mathbb{Q}} L_l^j \underset{\overline{\eta}_1}{\overset{\overline{\eta}_0}{\rightleftarrows}} L_l^i \otimes_{\mathbb{Q}} L_k^j\langle 1\rangle$$

with $\overline{\eta}_0$ given by the following pair of matrices, acting on C_n^0 and C_n^1 respectively,

$$\overline{\eta}_0^0 = \begin{pmatrix} \pi_{ijl} & 1 \\ -\pi_{ijk} & 1 \end{pmatrix}, \overline{\eta}_0^1 = \begin{pmatrix} 1 & -1 \\ \pi_{ijl} & \pi_{ijk} \end{pmatrix},$$

and with $\overline{\eta}_1$ given by the following pair of matrices, acting on C_n^0 and C_n^1 respectively,

$$\overline{\eta}_1^0 = \begin{pmatrix} 1 & -1 \\ \pi_{ijk} & \pi_{ijl} \end{pmatrix}, \overline{\eta}_1^1 = \begin{pmatrix} \pi_{ijk} & 1 \\ -\pi_{ijl} & 1 \end{pmatrix},$$

where

$$\pi_{ijk} = \sum_{\substack{0 \le a,b,c, \\ a+b+c=n-1,}} x_i^a x_j^b x_k^c.$$

LEMMA 3.9. *The maps η_i and $\overline{\eta}_i$ are homotopic, $i = 0, 1$.*

PROOF. First for $n = 1$, $\eta_i = \overline{\eta}_i$, for $i = 0, 1$. Suppose now $n > 1$ and denote

$$\pi_{ijkl} = \sum_{\substack{0 \le a,b,c,d, \\ a+b+c+d=n-2,}} x_i^a x_j^b x_k^c x_l^d.$$

Denote $R = \mathbb{Q}[x_i, x_j, x_k, x_l]$. We prove that η_0 and $\overline{\eta}_0$ are homotopic.

$$\begin{array}{ccccccc} R \oplus R\{2-2n\} & \xrightarrow{P^0} & R\{1-n\} \oplus R\{1-n\} & \xrightarrow{P^1} & R \oplus R\{2-2n\} \\ \downarrow{\scriptstyle \eta_0^0 - \overline{\eta}_0^0} & \overset{H^0}{\searrow} & \downarrow{\scriptstyle \eta_0^1 - \overline{\eta}_0^1} & \overset{H^1}{\searrow} & \downarrow{\scriptstyle \eta_0^0 - \overline{\eta}_0^0} \\ R\{1-n\} \oplus R\{1-n\} & \xrightarrow{Q^0} & R \oplus R\{2-2n\} & \xrightarrow{Q^1} & R\{1-n\} \oplus R\{1-n\} \end{array}$$

where

$$P^0 = \begin{pmatrix} \pi_{ik} & x_j - x_l \\ \pi_{jl} & x_k - x_i \end{pmatrix}, P^1 = \begin{pmatrix} x_i - x_k & x_j - x_l \\ \pi_{jl} & -\pi_{ik} \end{pmatrix},$$

$$Q^0 = \begin{pmatrix} x_i - x_l & x_j - x_k \\ \pi_{jk} & -\pi_{il} \end{pmatrix}, Q^1 = \begin{pmatrix} \pi_{il} & x_j - x_k \\ \pi_{jk} & x_l - x_i \end{pmatrix}$$

$$H^0 = \begin{pmatrix} 0 & 0 \\ -\pi_{ijkl} & 0 \end{pmatrix}, \text{ and } H^1 = 0.$$

The proof that η_1 and $\overline{\eta}_1$ are homotopic is similar and involves also a homotopy which contains the term π_{ijkl}. □

3. Proofs of Theorem 3.1 and 3.2

We first prove Theorem 3.2 and see that in fact the vertical grading is hidden in Khovanov-Rozansky construction. Then, we prove Theorem 3.1.

3.1. Proof of Theorem 3.2. In Section 5 of Chapter 1, we recalled a description of the construction of Khovanov and Rozansky [**26**] in terms of cones. We will use this description on order to prove Theorem 3.2. We introduce some new notations. Consider the following chain complexes of gmf's:

$$C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/1.pdf}}\ ) = \left(C_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/2.pdf}}\ )\{-1\} \xrightarrow{\eta_0} C_n^1(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/3.pdf}}\ )\{-n\} \right)$$

with $C_n^1(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/4.pdf}}\ ) = C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/5.pdf}}\ )$ and $C_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/6.pdf}}\ ) = C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/7.pdf}}\ )$, where all other terms are equal to zero, and

$$C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/8.pdf}}\ ) = \left(C_n^{-1}(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/9.pdf}}\ )\{n\} \xrightarrow{\eta_1} C_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/10.pdf}}\ )\{1\} \right)$$

with $C_n^{-1}(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/11.pdf}}\ ) = C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/12.pdf}}\ )$ and $C_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/13.pdf}}\ ) = C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/14.pdf}}\ )$, where again all other terms are equal to zero. Notice that the homological grading here corresponds to the vertical grading in the bicomplexes. The gmf's $C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/15.pdf}}\ )$ and $C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/16.pdf}}\ )$ were defined in Section 1 of Chapter 4. In this setting, the bicomplex associated to a positive crossing can be described as follows:

$$C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/17.pdf}}\ ) = \left(C_n^{-1}(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/18.pdf}}\ )\{n\} \xrightarrow{\delta_h} C_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/19.pdf}}\ )\{n-1\} \right)$$

with $C_n^{-1}(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/20.pdf}}\ ) = C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/21.pdf}}\ )$ and $C_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/22.pdf}}\ ) = C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/23.pdf}}\ )$, where all other terms are equal to zero. The morphism δ_h corresponds to the natural projection to a quotient complex. The bicomplex associated to a negative crossing can be described as follows,

$$C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/24.pdf}}\ ) = \left(C_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/25.pdf}}\ )\{1-n\} \xrightarrow{\delta_h} C_n^1(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/26.pdf}}\ )\{-n\} \right)$$

with $C_n^1(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/27.pdf}}\ ) = C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/28.pdf}}\ )$ and $C_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/29.pdf}}\ ) = C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/30.pdf}}\ )$, where all other terms are equal to zero. The morphism δ_h corresponds to the natural injection of a subcomplex. Consider the following modification that transforms the graded complexes of gmf's $C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/31.pdf}}\ )$ and $C_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/32.pdf}}\ )$ into filtered complexes of gmf's. Here the grading corresponds to the quantum grading. Denote

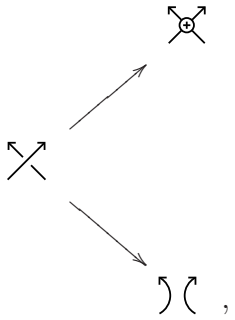
$$\overline{C}_n(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/33.pdf}}\ ) = \left(\overline{C}_n^0(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/34.pdf}}\ )\{-1\} \xrightarrow{\eta_0} \overline{C}_n^1(\ \raisebox{-0.3em}{\includegraphics[height=1.2em]{figures/35.pdf}}\ )\{1\} \right)$$

with $\overline{C}_n^1($ $) = C_n($  $)$ and $\overline{C}_n^0($ $) = C_n($  $)$, where all other terms are equal to zero, and

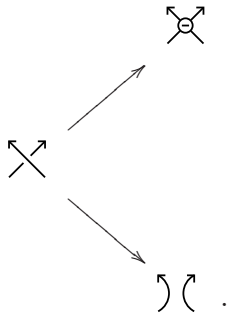
$$\overline{C}_n(\ \text{}\ ) = \left(\overline{C}_n^{-1}(\ \text{}\ )\{-1\} \xrightarrow{\eta_1} \overline{C}_n^0(\ \text{}\ )\{1\} \right)$$

with $\overline{C}_n^{-1}($ $) = C_n($ $)$ and $\overline{C}_n^0($ $) = C_n($ $)$, where again all other terms are equal to zero.

We have shifted the quantum grading such that the morphisms η_0 and η_1 appearing in $\overline{C}_n($ $)$ and $\overline{C}_n($ $)$ are now maps of degree $n+1$. We have now that $\mathrm{Cone}(\overline{C}_n($ $)) = \mathrm{Cone}(\eta_0)$ and $\mathrm{Cone}(\overline{C}_n($ $)) = \mathrm{Cone}(\eta_1)$, see Section 5 of Chapter 1. In terms of resolutions of oriented link diagrams, we resolve every positive crossing as follows



and every negative crossing as follows



Define a *signed closed planar regular graph* to be a closed planar regular graph where we assign to any 4-valent vertex a sign. Hence a resolution of an oriented link diagram gives a signed closed planar regular graph. In the previous construction, we associate to such a graph a complex C of graded 2-periodic chain complexes, whereas in the Khovanov–Rozansky construction we associate a 2-periodic chain complex KR. As we have seen this 2-periodic chain complex KR can be obtained as the cone of a shifted complex $\overline{C}$ of 2-periodic chain complexes. Denote by H the homology of C and by $\overline{H}$ the homology of $\overline{C}$. In order to prove

Theorem 3.2, we need to prove two facts:

Fact 1: The homology$H$ and $\overline{H}$ are related as follows:

$$\overline{H}^{j,l} = H^{j,l-j(n+1)} \text{ for all } (j,l) \in \mathbb{Z}^2.$$

Fact 2: The homology of KR is equal to the homology of $\overline{C}$.

The differential of the complex $\overline{C}$ does not respect the quantum grading, but is still homogeneous of degree $n+1$. From Lemma 3.7 and Lemma 3.8 , we know that the graded 2-periodic chain complexes appearing in $\overline{C}$ and C are homotopic to the graded 2-periodic chain complexes of their homology. Moreover, their homology is concentrated in one of the two $\mathbb{Z}/2\mathbb{Z}$ grading. Hence the complexes C and $\overline{C}$ can be seen as complexes of graded $\mathbb{Q}$-vector spaces. Denote $C = \oplus_{(j,k)\in\mathbb{Z}^2} C^{j,k}$ where j is the homological grading and k is the quantum grading. Notice that

$$\overline{C} = \oplus_{(j,l)\in\mathbb{Z}^2} \overline{C}^{j,l} = \oplus_{(j,k)\in\mathbb{Z}^2} C^{j,k-j(n+1)}.$$

Consider the decreasing filtration associated to the quantum grading on the chains of $\overline{C}$,

$$F^i\overline{C} = \oplus_{j\in\mathbb{Z}, l\geq i} \overline{C}^{j,l}.$$

The differential on $\overline{C}$ respects this filtration. When one has a filtered chain complex, there is a spectral sequence associated, see [**34**]. This spectral sequence converges to the homology of this filtered chain complex which is bigraded. Denote $(\overline{H}^{j,l})_{(j,l)\in\mathbb{Z}^2}$ the homology of $\overline{C}$ and $(H^{j,k})_{(j,k)\in\mathbb{Z}^2}$ the homology of C. In our case, since the differentials of our filtered chain complex $\overline{C}$ are homogeneous of degree $n+1$, the spectral sequence tells us that the relation that holds on the chains still holds on homology. Finally, we have

$$\overline{H}^{j,l} = H^{j,l-j(n+1)} \text{ for all } (j,l) \in \mathbb{Z}^2.$$

This proves Fact 1 and explains the shift in Theorem 3.2.

In order to prove Fact 2, we need to show that the homology of the cone of a complex (see Section 2 of Chapter 3) is equal to the homology of the same complex. This is false in general. Nevertheless, since we are taking the cone with respect to a 2-periodic structure, the result will hold. Let us prove Fact 2. It follows from Lemma 3.3 that we do not need to take the cone of every saddle morphism and then perform the tensor product; We can first take tensor products and obtain $\overline{C}$ and then take the cone. Moreover we know that all graded 2-periodic chain complexes appearing in $\overline{C}$ are homotopic to the 2-periodic chain complexes of their homology. Since these homologies are concentrated in one of the two $\mathbb{Z}/2\mathbb{Z}$-grading, it implies that the homotopies involved satisfy the conditions of Lemma 3.5 and Lemma 3.6. As a consequence, the homotopy type of Cone($\overline{C}$) does not change under these modifications. We still denote it Cone($\overline{C}$). Suppose that the homology of the

2-periodic chain complexes is concentrated in degree 0. The filtered complex $\overline{C}$ has the form

$$\overline{C}^k \overset{d^k}{\to} \overline{C}^{k+1} \cdots \overset{d^{l-1}}{\to} \overline{C}^l$$

where for all $i = k, \ldots, l$, $\overline{C}^i$ is a graded $\mathbb{Q}$-vector space and the differentials d^i are of degree $n+1$. Remember that the $\overline{C}^i$'s are in the zero $\mathbb{Z}/2\mathbb{Z}$-grading. Then Cone$(\overline{C})$ is the following graded 2-periodic chain complex:

$$\oplus_{j\geq 0}\overline{C}^{k+2j} \xrightarrow{D^0} \oplus_{j\geq 0}\overline{C}^{k+1+2j} \xrightarrow{D^1} \oplus_{j\geq 0}\overline{C}^{k+2j},$$

where

$$D^0 = \begin{pmatrix} d^k & 0 & 0 & \cdots & 0 \\ 0 & d^{k+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

and

$$D^1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ d^{k+1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}.$$

We can directly verify that the homology of Cone$(\overline{C})$ is equal to the homology of $\overline{C}$. Hence the homology of Cone$(\overline{C})$ is a graded vector space $\oplus_{l\in\mathbb{Z}}H^l$ and for all $l \in \mathbb{Z}$, $H^l = \oplus_{j\in\mathbb{Z}}\overline{H}^{j,l}$. Notice now that in order to obtain the graded vector space associated to the underlying planar regular graph, we do not just take the cone of the saddle morphisms, but also introduce a shift in the $\mathbb{Z}/2\mathbb{Z}$-grading, see Theorem 1.3. Nevertheless, since in both constructions we forget the $\mathbb{Z}/2\mathbb{Z}$-grading, it does not affect the result. Theorem 3.2 follows.

Along the proof of Theorem 3.2, we introduced some notations useful for computations, see Chapter 4. Moreover, we now state a lemma which will be useful in the proof of Theorem 3.1.

LEMMA 3.10. *Given an oriented link diagram D and a resolution Γ of D into a signed planar regular graph. The complex of graded 2-periodic chain complexes $C_n(\Gamma)$ has only homology in even degrees.*

PROOF. From the proof of Theorem 3.2, we know that the homology of C is equal up to shift in the quantum grading to the homology of $\overline{C}$. Moreover the homology of $\overline{C}$ is equal to the homology of Cone$(\overline{C})$. Consider the case when the homology is concentrated in the zero $\mathbb{Z}/2\mathbb{Z}$-grading as in the end of Theorem 3.2 proof. The homology of Cone$(\overline{C})$ is equal after a suitable shift of the $\mathbb{Z}/2\mathbb{Z}$-grading to the homology of $KR_n(\Gamma)$. The shift corresponds to the parity of the number of negative vertices in Γ. This number is also the lowest homological grading for which the chains of C are non-zero. In the example of Theorem 3.2 proof, this is the number k. Moreover, from Lemma 1.2, $KR_n(\overline{\Gamma})$ has only homology in one of the

two $\mathbb{Z}/2\mathbb{Z}$-gradings ($\overline{\Gamma}$ is the planar regular graph obtained by forgetting the signs in the signed planar regular graph Γ). In this case it is zero (from the choice we made for C). Suppose k is even, then the chain complex can only have homology in the chains $\oplus_{j\geq 0}\overline{C}^{k+2j}$ which corresponds to even grading for the homological grading of C. Similarly if k is odd, we need to shift by 1 the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathrm{Cone}(\overline{C})$ in order to obtain $KR_n(\overline{\Gamma})$ which has still only homology in degree 0. Hence $\mathrm{Cone}(\overline{C})$ has only homology in degree 1, ie. in the chains $\oplus_{j\geq 0}\overline{C}^{k+1+2j}$. These chains still correspond to even grading for the homological grading of C. Similarly we can work out the case when the homology of C and $KR_n(\Gamma)$ are concentrated in the degree 1 of the $\mathbb{Z}/2\mathbb{Z}$-grading. □

3.2. Proof of Theorem 3.1. In order to prove Theorem 3.1, we prove the invariance of $\mathbb{H}_n$ under Reidemeister moves. The invariance of $\mathbb{H}_n$ under Reidemister moves RI, RII will follow from the invariance of the homotopy type of the bicomplexes of gmf's and does not require Khovanov–Rozansky proofs whereas the invariance of $\mathbb{H}_n$ under Reidemeister move RIII will follow from a short exact sequence argument and from Lemma 3.10. Notice that the invariance of $\mathbb{H}_n$ under Reidemeister move RIII requires some properties of the graph homology HKR_n.

Proof of the invariance under RI

We prove that the one-term bicomplex of graded matrix factorizations

$$C_n(\ \ ) = \ C_n^{0,0}(\ \ )$$

with $C_n^{0,0}(\ \ ) = (L_l^i \otimes_{\mathbb{Q}} L_k^j/\langle x_l = x_j\rangle)\langle 1\rangle$, is homotopy equivalent to the bicomplex associated to the diagram on Figure 3.

$$\begin{array}{ccc} C_n(\ \ ) = \ C_n^{0,0}(\ \ ) & \xrightarrow{\delta_h} & C_n^{1,0}(\ \ ) \\ \uparrow & & \uparrow{\scriptstyle \delta_v} \\ 0 & \longrightarrow & C_n^{1,-1}(\ \ ) \end{array}$$

with

$$C_n^{1,-1}(\ \ ) = (L_l^i \otimes_{\mathbb{Q}} L_k^j/\langle x_l = x_j\rangle\rangle\langle 1\rangle,$$

$$C_n^{0,0}(\ \ ) = C_n^{1,0}(\ \ ) = (L_k^i \otimes_{\mathbb{Q}} L_l^j/\langle x_l = x_j\rangle)\{1-n\},$$

$$\delta_h = \mathrm{Id}, \text{ and } \delta_v = (\eta_1)_{x_l = x_j}.$$

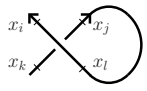


FIGURE 3. Marked diagram for RI

First we will reduce the bicomplex $C_n(\curvearrowright\!\bigcirc)$ by removing marks. It is isomorphic in $K(K(\mathcal{HMF}_w))$ over $\mathbb{Q}[x_i, x_k]$ to the following bicomplex:

$$\begin{array}{ccc} (L_k^i \otimes V_n)\langle 1\rangle\{1-n\} & \xrightarrow{\delta_h'} & (L_k^i \otimes V_n)\langle 1\rangle\{1-n\}\ , \\ \uparrow & & \uparrow{\scriptstyle \delta_v'} \\ 0 & \longrightarrow & L_k^i\langle 1\rangle \end{array}$$

with $\delta_h' = \mathrm{Id}$ and $\delta_v' = -(\pi_{ijk} + \overline{\pi}_{jk})\mathrm{Id}$ where $\overline{\pi}_{jk} = \sum_{s=0}^{n-1}(s+1)x_j^s x_k^{n-1-s}$. The fact that these two bicomplexes are isomorphic follows from the following commutative diagram,

$$\begin{array}{ccc} (L_l^i \otimes_{\mathbb{Q}} L_k^j/\langle x_l = x_j\rangle)\langle 1\rangle & \xrightarrow{\delta_v} & (L_k^i \otimes_{\mathbb{Q}} L_l^j/\langle x_l = x_j\rangle)\{1-n\} \\ {\scriptstyle g_1}\big(\quad\big){\scriptstyle f_1} & & {\scriptstyle g_2}\big(\quad\big){\scriptstyle f_2} \\ L_k^i\langle 1\rangle & \xrightarrow{\delta_v'} & (L_k^i \otimes V_n)\langle 1\rangle\{1-n\}, \end{array}$$

where the homomorphisms f_1, g_1, f_2, and g_2 are given by

$$f_1^0 = \begin{pmatrix} 0 & \phi \end{pmatrix},\ f_1^1 = \begin{pmatrix} \phi & 0 \end{pmatrix},$$
$$g_1^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ g_1^1 = \begin{pmatrix} 1 \\ -\pi_{ijk} \end{pmatrix},$$
$$f_2^0 = \begin{pmatrix} 0 & -1 \end{pmatrix},\ f_2^1 = \begin{pmatrix} 0 & 1 \end{pmatrix},$$
$$g_2^0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ g_2^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $\phi : \mathbb{Q}[x_i, x_j, x_k] \to \mathbb{Q}[x_i, x_k]$ which sends x_j on x_i, x_i on x_i, and x_k on x_k. Notice that f_1,g_1 and f_2, and g_2 are mutual homotopy equivalences of gmf's, see Lemma 3.7 and Lemma 3.8. Hence, the diagram is commutative up to homotopy. We will produce a morphism of bicomplexes of gmf's which will be a homotopy equivalence, see Figure 4. We consider $F = \delta_v'$, $G = -\mathrm{Id} \otimes \epsilon$, $h = \mathrm{Id} \otimes \epsilon$, and $H = -\delta_v' \circ (\mathrm{Id} \otimes \epsilon) - \mathrm{Id} \otimes \mathrm{Id}$, where ϵ is the counit defined in Subsection 2.2

First we check that $G \circ F = \mathrm{Id}$. It is a consequence of the fact that

$$-\epsilon(-\pi_{ijk} - \overline{\pi}_{jk}) = \frac{1}{n+1} + \frac{n}{n+1} = 1.$$

Then $F \circ G - \mathrm{Id} = \delta_v' \circ h + h \circ \delta_v' + H \circ \delta_h' + \delta_h' \circ H$. We need also to check that H, G, and F are well-defined morphisms. For G, there is nothing to check. For H,

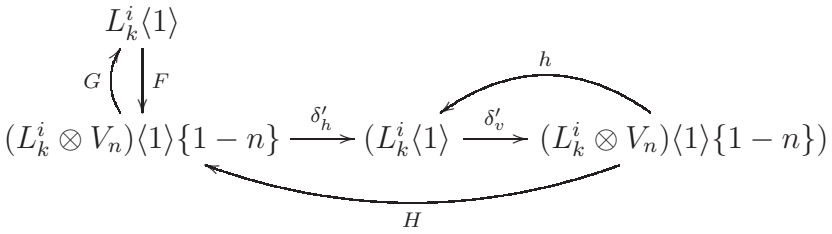
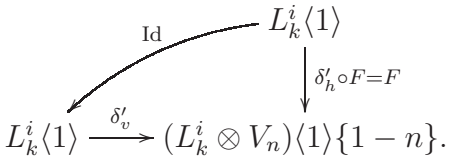


FIGURE 4. Reidemeister RI

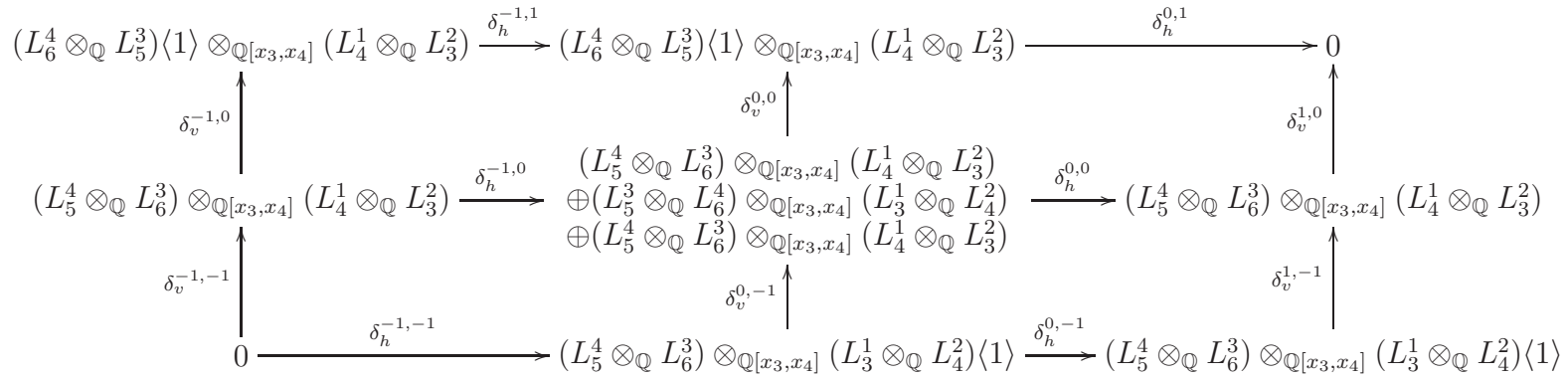
we check that $H \circ \delta_v' = 0$, which can be deduced from the fact that $G \circ F = \mathrm{Id}$. For F, we need that $\delta_h' \circ F = 0$. This is true, up a homotopy on the column, i.e.



The proof of the invariance under the Reidemeister RI move involving a positive crossing is similar.

Proof of the invariance under RIIa

In order to prove the invariance under Reidemeister move RIIa, we first simplify the bicomplex of gmf's, using homotopically equivalent gmf's. We describe graphically in Figure 7 the bicomplex associated to the Figure 6 and algebraically in Figure 5. To simplify the figures, we omit all the marks in the graphical description.



where

$$\delta_h^{-1,1} = \mathrm{Id},\ \delta_h^{0,-1} = -\mathrm{Id},\ \delta_v^{-1,0} = \eta_0^{4356} \otimes \mathrm{Id},\ \delta_v^{1,-1} = \mathrm{Id} \otimes \eta_1^{1243},$$

$$\delta_h^{-1,0} = \begin{pmatrix} \mathrm{Id} \\ 0 \\ \mathrm{Id} \end{pmatrix},\ \delta_h^{0,0} = \begin{pmatrix} -\mathrm{Id} & 0 & \mathrm{Id} \end{pmatrix},\ \delta_v^{0,-1} = \begin{pmatrix} \mathrm{Id} \otimes \eta_1^{1243} \\ -\eta_0^{4356} \otimes \mathrm{Id} \\ 0 \end{pmatrix},\ \text{and}\ \delta_v^{0,0} = \begin{pmatrix} \eta_0^{4356} \otimes \mathrm{Id} & \mathrm{Id} \otimes \eta_1^{1243} & 0 \end{pmatrix}.$$

FIGURE 5. Algebraic bicomplex for RIIa

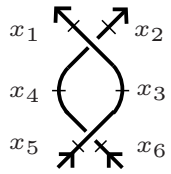


FIGURE 6. Marked diagram for Reidemeister move RIIa

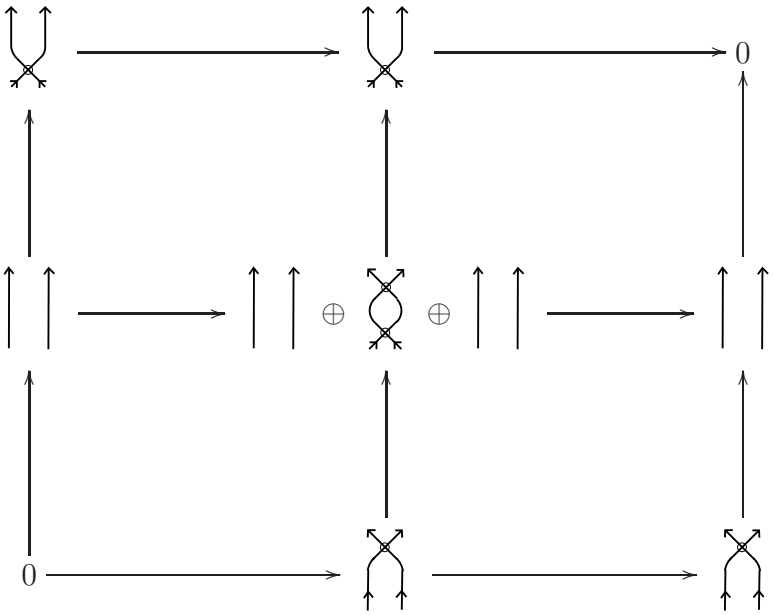


FIGURE 7. Graphical bicomplex for RIIa

We first simplify all the gmf's of the bicomplex, by removing internal marks. We exhibit a homotopy equivalence of bicomplexes. A homotopy equivalence F is described in Figure 8, whereas his converse G is described in Figure 9. Notice that these morphisms of bicomplexes are well defined only up to homotopy of gmf's and that the bicomplexes of gmf's on Figure 8 and 9 are also only well defined, up to homotopy of gmf's (i.e. the square of the differentials is 0 up to homotopy of gmf's).

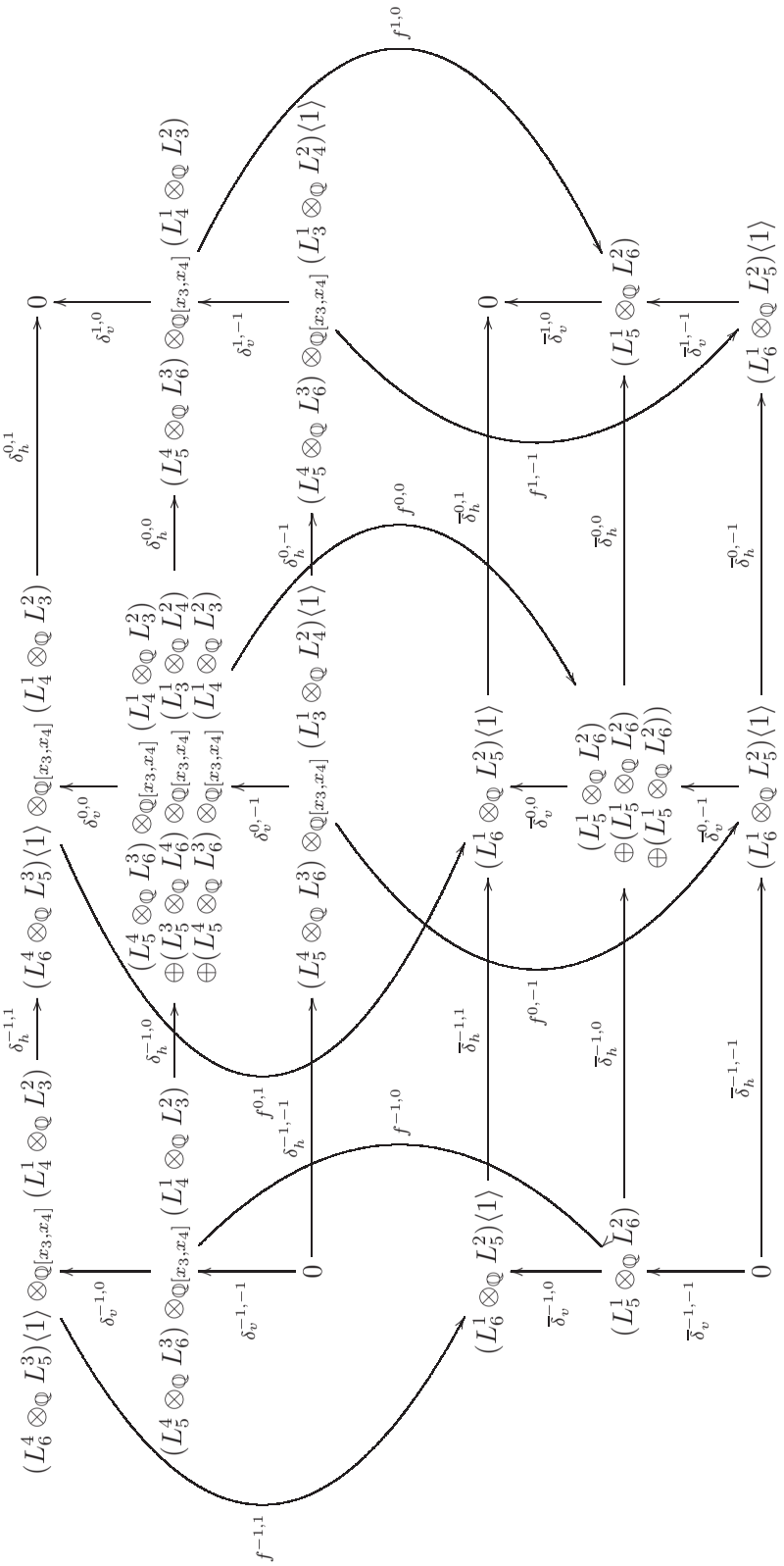


FIGURE 8. Removing marks for RIIa

The morphism of bicomplexes of gmf's F on Figure 8 is given by,

$$f^{0,1} = f^{-1,1} = \begin{cases} (f^{0,1})^0 = (f^{-1,1})^0 = \begin{pmatrix} \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ (f^{0,1})^1 = (f^{-1,1})^1 = \begin{pmatrix} \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{cases}$$

$$f^{-1,0} = f^{1,0} = \begin{cases} (f^{-1,0})^0 = (f^{1,0})^0 = \begin{pmatrix} \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ (f^{-1,0})^1 = (f^{1,0})^1 = \begin{pmatrix} \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{cases}$$

$$f^{0,-1} = f^{1,-1} = \begin{cases} (f^{0,-1})^0 = (f^{1,-1})^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \psi & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi & 0 & 0 & 0 \end{pmatrix}, \\ (f^{0,-1})^1 = (f^{1,-1})^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi & 0 & 0 \end{pmatrix}, \end{cases}$$

$$f^{0,0} = \begin{pmatrix} f_1^{0,0} & 0 & 0 \\ 0 & f_2^{0,0} & 0 \\ 0 & 0 & f_3^{0,0} \end{pmatrix}$$

where $f_1^{0,0} = f^{-1,0}$, $f_3^{0,0} = f^{-1,0}$ and $f_2^{0,0} = f$,

$$f = \begin{cases} f^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi & 0 & 0 \end{pmatrix}, \\ f^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\psi & 0 & 0 \end{pmatrix}, \end{cases}$$

$$\phi : \mathbb{Q}[x_1, \ldots, x_6] \longrightarrow \mathbb{Q}[x_1, \ldots, x_6],$$

defined by $\phi(x_4) = x_1$, $\phi(x_3) = x_2$, and $\phi(x_i) = x_i$ for $i = 1, 2, 5, 6$, and

$$\psi : \mathbb{Q}[x_1, \ldots, x_6] \longrightarrow \mathbb{Q}[x_1, \ldots, x_6],$$

defined by $\psi(x_3) = x_1$, $\psi(x_4) = x_2$, and $\psi(x_i) = x_i$ for $i = 1, 2, 5, 6$.

As for the differentials of the bottom bicomplex of gmf's on Figure 8, they are given by,

$$\overline{\delta}_h^{-1,1} = \mathrm{Id},\ \overline{\delta}_h^{0,-1} = -\mathrm{Id},\ \overline{\delta}_v^{-1,0} = \eta_0^{1256},\ \overline{\delta}_v^{1,-1} = -\overline{\eta}_1^{1256}$$

$$\overline{\delta}_h^{-1,0} = \begin{pmatrix} \mathrm{Id} \\ 0 \\ \mathrm{Id} \end{pmatrix},\ \overline{\delta}_h^{0,0} = \begin{pmatrix} -\mathrm{Id} & 0 & \mathrm{Id} \end{pmatrix},$$

$$\overline{\delta}_v^{0,-1} = \begin{pmatrix} -\overline{\eta}_1^{1256} \\ \eta_1^{1256} \\ 0 \end{pmatrix}, \text{ and } \overline{\delta}_v^{0,0} = \begin{pmatrix} \eta_0^{1256} & \overline{\eta}_0^{1256} & 0 \end{pmatrix}.$$

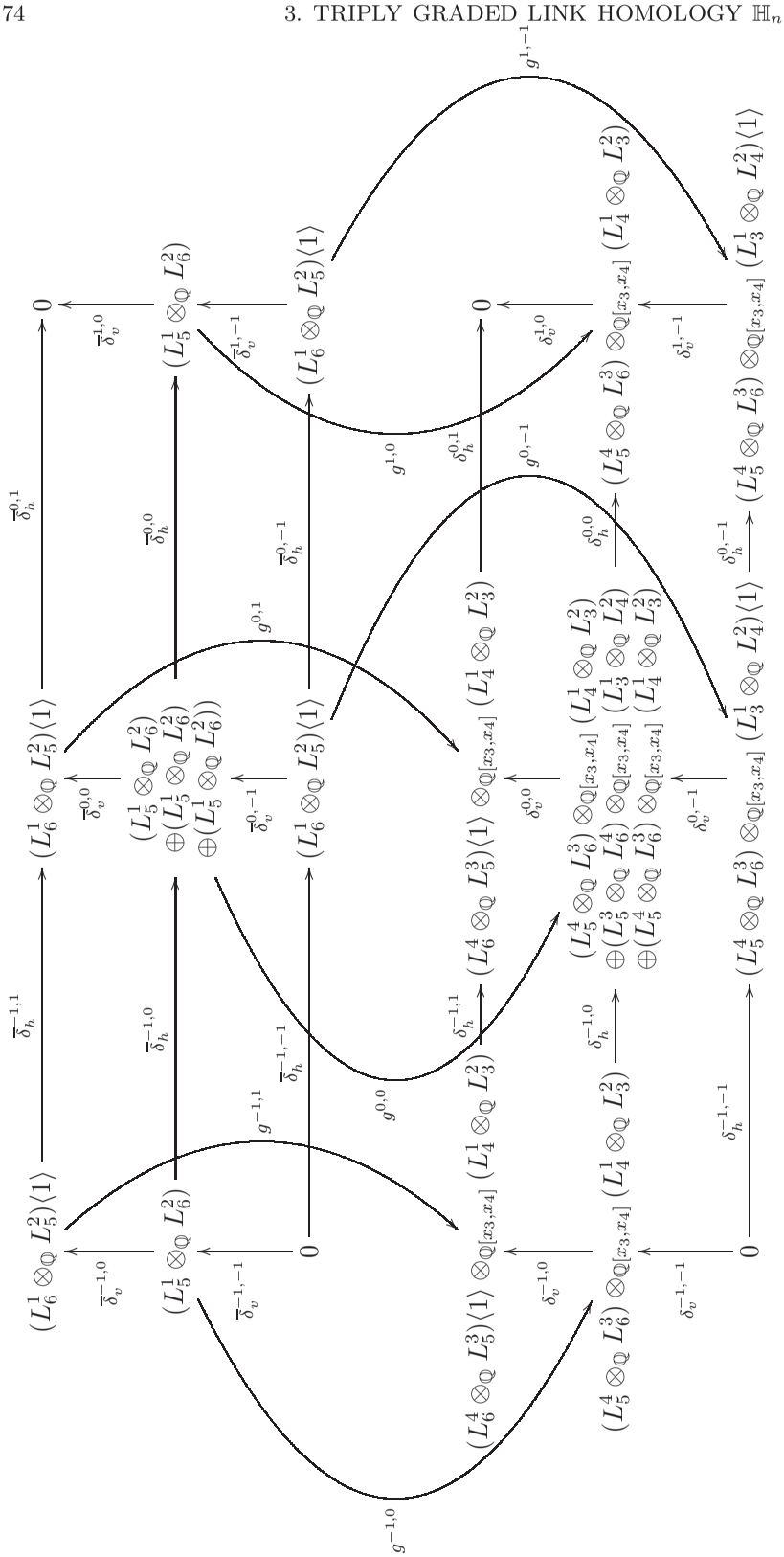


FIGURE 9. Adding marks for RIIa

Let us now describe the converse homotopy equivalence G: See Figure 9 which we now comment.

$$g^{0,1} = g^{-1,1},\ g^{-1,0} = g^{1,0},\ g^{0,-1} = g^{1,-1},$$

$$(g^{0,1})^0 = (g^{-1,1})^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \pi_{146} \\ -\pi_{235} & 0 \\ 1 & 0 \\ 0 & -\pi_{146} \\ 0 & 1 \\ \pi_{235} & 0 \end{pmatrix}, (g^{0,1})^1 = (g^{-1,1})^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ -\pi_{146}\pi_{235} & 0 \\ \pi_{146} & 0 \\ 0 & -1 \\ 0 & 1 \\ \pi_{235} & 0 \end{pmatrix},$$

$$(g^{-1,0})^0 = (g^{1,0})^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ -\pi_{145}\pi_{236} & 0 \\ \pi_{145} & 0 \\ 0 & -1 \\ 0 & 1 \\ \pi_{236} & 0 \end{pmatrix}, (g^{-1,0})^1 = (g^{1,0})^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \pi_{145} \\ -\pi_{236} & 0 \\ 1 & 0 \\ 0 & -\pi_{145} \\ 0 & 1 \\ \pi_{236} & 0 \end{pmatrix},$$

$$(g^{0,-1})^0 = (g^{1,-1})^0 = \begin{pmatrix} -1 & 0 \\ 0 & -\pi_{136} \\ 0 & -1 \\ \pi_{245} & 0 \\ 0 & 1 \\ 1 & 0 \\ -\pi_{245} & 0 \\ 0 & \pi_{136} \end{pmatrix}, (g^{0,-1})^1 = (g^{1,-1})^1 = \begin{pmatrix} 0 & -1 \\ \pi_{136} & 0 \\ \pi_{245} & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -\pi_{245}\pi_{136} & 0 \end{pmatrix},$$

$$g^{0,0} = \begin{pmatrix} g_1^{0,0} & 0 & 0 \\ 0 & g_2^{0,0} & 0 \\ 0 & 0 & g_3^{0,0} \end{pmatrix},$$

where $g_1^{0,0} = g^{-1,0}$, $g_3^{0,0} = g^{-1,0}$ and $g_2^{0,0} = g$ with g,

$$g^0 = \begin{pmatrix} -\pi_{135} & 0 \\ 0 & 1 \\ 0 & -1 \\ -\pi_{246} & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ -\pi_{135}\pi_{246} & 0 \end{pmatrix}, g^1 = \begin{pmatrix} 1 & 0 \\ 0 & -\pi_{135} \\ 0 & 1 \\ \pi_{246} & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & -\pi_{135} \\ \pi_{246} & 0 \end{pmatrix}.$$

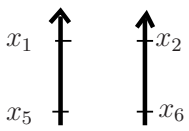


FIGURE 10. Marked smoothing for RIIa

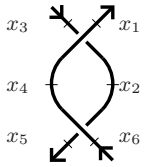


FIGURE 11. Marked diagram for Reidemeister move RIIb

We can now complete the proof of the invariance under RIIa. We have $F \circ G = \mathrm{Id}$ and $G \circ F$ is homotopic to the Id (the homotopies required are homotopies of gmf's). Hence, F and G are inverse isomorphisms in $K(K(\mathcal{HMF}_w))$. We give on Figure 12 a homotopy equivalence between the reduced bicomplex of gmf's obtained by removing marks and the bicomplex associated to the marked diagram on Figure 10. The homotopy h and the homotopy equivalence F, G on Figure 12 are given by the following formulas:

$$h^{0,1} = -\mathrm{Id},\ h^{0,0} = \begin{pmatrix} -\mathrm{Id} & -\mathrm{Id} & 0 \end{pmatrix},\ h^{1,-1} = \mathrm{Id},$$

$$h^{1,0} = \begin{pmatrix} \mathrm{Id} \\ -\mathrm{Id} \\ 0 \end{pmatrix},\ F = \begin{pmatrix} -\mathrm{Id} \\ \mathrm{Id} \\ -\mathrm{Id} \end{pmatrix},\ \text{and } G = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & -\mathrm{Id} \end{pmatrix}.$$

Proof of the invariance under Reidemeister move RIIb

Consider the bicomplex associated to the marked diagram on Figure 11. This bicomplex is described graphically on Figure 13 and algebraically on Figure 14. We first simplify all the gmf's of the bicomplex, by removing internal marks. We exhibit a homotopy equivalence of bicomplexes. A homotopy equivalence F is described in Figure 15, whereas his converse G is described in Figure 16. Notice that these morphisms of bicomplexes are well-defined only up to homotopy of gmf's and that the bottom bicomplex of gmf's on Figure 15 is also only well defined, up to homotopy of gmf's (i.e. the square of the differentials is 0 only up to homotopy of gmf's).

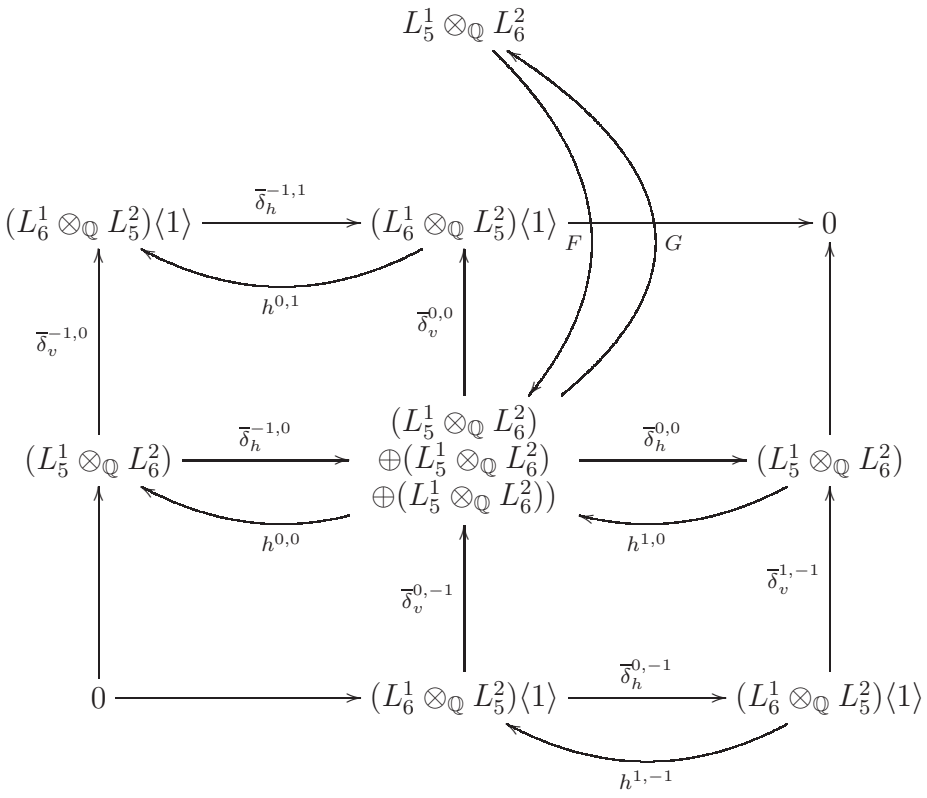


FIGURE 12. Final homotopy equivalence for RIIa

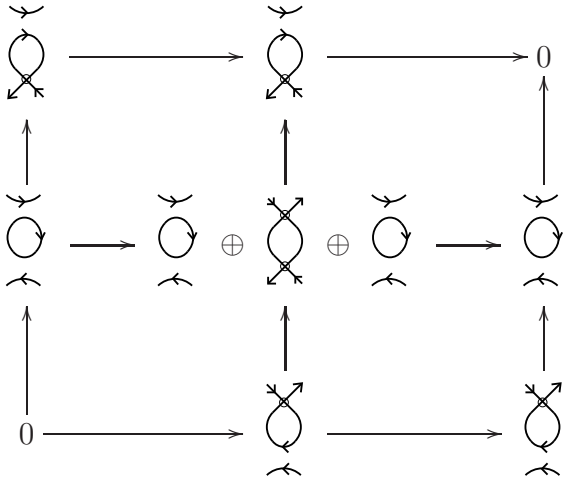
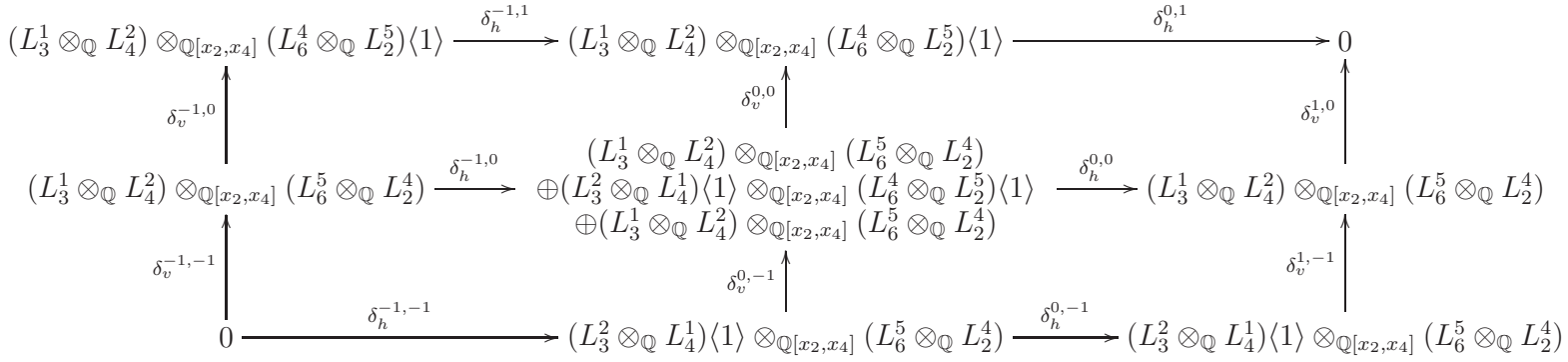


FIGURE 13. Graphical bicomplex for RIIb



where

$$\delta_h^{-1,1} = \mathrm{Id},\ \delta_h^{0,-1} = -\mathrm{Id},\ \delta_v^{-1,0} = \eta_0^{5462} \otimes \mathrm{Id},\ \delta_v^{1,-1} = \mathrm{Id} \otimes \eta_1^{1234},$$

$$\delta_h^{-1,0} = \begin{pmatrix} \mathrm{Id} \\ 0 \\ \mathrm{Id} \end{pmatrix},\ \delta_h^{0,0} = \begin{pmatrix} -\mathrm{Id} & 0 & \mathrm{Id} \end{pmatrix},\ \delta_v^{0,-1} = \begin{pmatrix} \mathrm{Id} \otimes \eta_1^{1234} \\ -\eta_0^{5462} \otimes \mathrm{Id} \\ 0 \end{pmatrix},\ \text{and } \delta_v^{0,0} = \begin{pmatrix} \eta_0^{5462} \otimes \mathrm{Id} & \mathrm{Id} \otimes \eta_1^{1234} & 0 \end{pmatrix}.$$

FIGURE 14. Algebraic bicomplex for RIIb

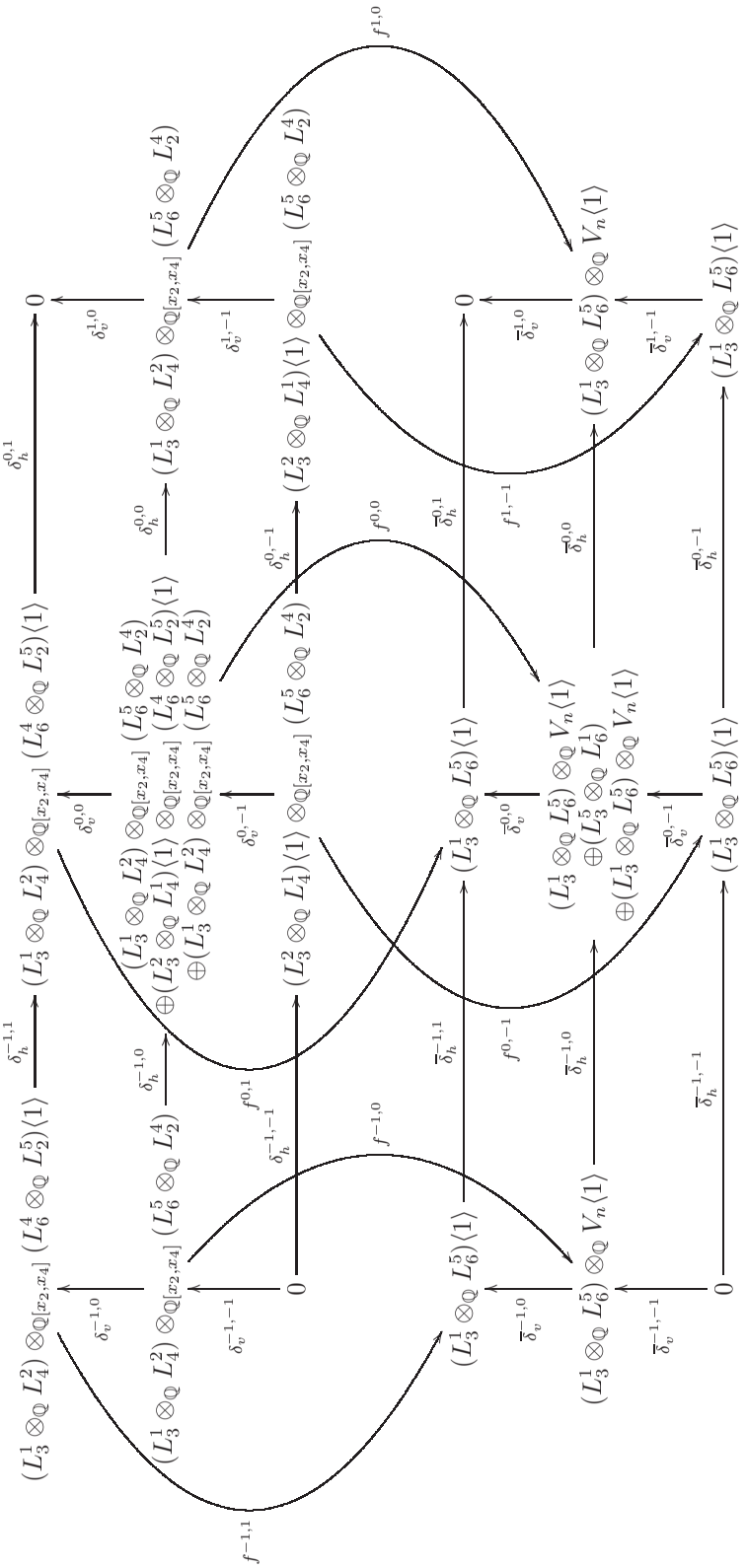


FIGURE 15. Removing marks for RIIb

The morphism of bicomplexes of gmf's F on Figure 15 is given by:

$$f^{0,1} = f^{-1,1} = \begin{cases} (f^{0,1})^0 = (f^{-1,1})^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi & 0 & 0 \end{pmatrix}, \\ (f^{0,1})^1 = (f^{-1,1})^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\phi & 0 & 0 \end{pmatrix}, \end{cases}$$

$$f^{-1,0} = f^{1,0} = \begin{cases} (f^{-1,0})^0 = (f^{1,0})^0 = \begin{pmatrix} 0 & -\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi & 0 & 0 \end{pmatrix}, \\ (f^{-1,0})^1 = (f^{1,0})^1 = \begin{pmatrix} 0 & \psi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\psi & 0 & 0 \end{pmatrix}, \end{cases}$$

$$f^{0,-1} = f^{1,-1} = \begin{cases} (f^{0,-1})^0 = (f^{1,-1})^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \varphi & 0 \\ 0 & 0 & 0 & 0 & \varphi & 0 & 0 & 0 \end{pmatrix}, \\ (f^{0,-1})^1 = (f^{1,-1})^1 = \begin{pmatrix} \varphi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{cases}$$

$$f^{0,0} = \begin{pmatrix} f_1^{0,0} & 0 & 0 \\ 0 & f_2^{0,0} & 0 \\ 0 & 0 & f_3^{0,0} \end{pmatrix},$$

where $f_1^{0,0} = f^{-1,0}$, $f_3^{0,0} = f^{-1,0}$ and $f_2^{0,0} = f$ with

$$f = \begin{cases} f^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \chi & 0 & 0 & 0 \\ -\chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ f^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\chi & 0 & 0 & 0 \\ \chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{cases}$$

with

$$\phi : \mathbb{Q}[x_1, \ldots, x_6] \longrightarrow \mathbb{Q}[x_1, \ldots, x_6],$$

defined by $\phi(x_2) = x_5$, $\phi(x_4) = x_6$, and $\phi(x_i) = x_i$ for $i = 1, 3, 5, 6$,

$$\psi : \mathbb{Q}[x_1, \ldots, x_6] \longrightarrow \mathbb{Q}[x_1, \ldots, x_6],$$

defined by $\psi(x_4) = x_2$, and $\psi(x_i) = x_i$ for $i = 1, 2, 3, 5, 6$,

$$\varphi : \mathbb{Q}[x_1, \ldots, x_6] \longrightarrow \mathbb{Q}[x_1, \ldots, x_6],$$

defined by $\varphi(x_2) = x_3$, $\varphi(x_4) = x_1$, and $\varphi(x_i) = x_i$ for $i = 1, 3, 5, 6$, and

$$\chi : \mathbb{Q}[x_1, \ldots, x_6] \longrightarrow \mathbb{Q}[x_1, \ldots, x_6],$$

defined by $\chi(x_4) = x_1$, $\phi(x_2) = x_5$, and $\phi(x_i) = x_i$ for $i = 1, 3, 5, 6$.

The differentials of the bottom bicomplex of gmf's on Figure 15 are given by:

$\overline{\delta}_h^{-1,1} = \mathrm{Id}$, $\overline{\delta}_h^{0,-1} = -\mathrm{Id}$, $\overline{\delta}_v^{-1,0} = \phi\ \mathrm{Id}$, $\overline{\delta}_v^{1,-1} = (\pi_{123} + \overline{\pi}_{23})\mathrm{Id}$, $\overline{\delta}_h^{0,0} = \begin{pmatrix} -\mathrm{Id} & 0 & \mathrm{Id} \end{pmatrix}$,

$\overline{\delta}_h^{-1,0} = \begin{pmatrix} \mathrm{Id} \\ 0 \\ \mathrm{Id} \end{pmatrix}$, $\overline{\delta}_v^{0,-1} = \begin{pmatrix} (\pi_{123} + \overline{\pi}_{23})\mathrm{Id} \\ \tau \circ \overline{\eta_1}^{1536} \\ 0 \end{pmatrix}$, and $\overline{\delta}_v^{0,0} = \begin{pmatrix} \phi\ \mathrm{Id} & -\eta_0^{1356} \circ \tau & 0 \end{pmatrix}$.

where τ was defined in the proof of Lemma 3.4.

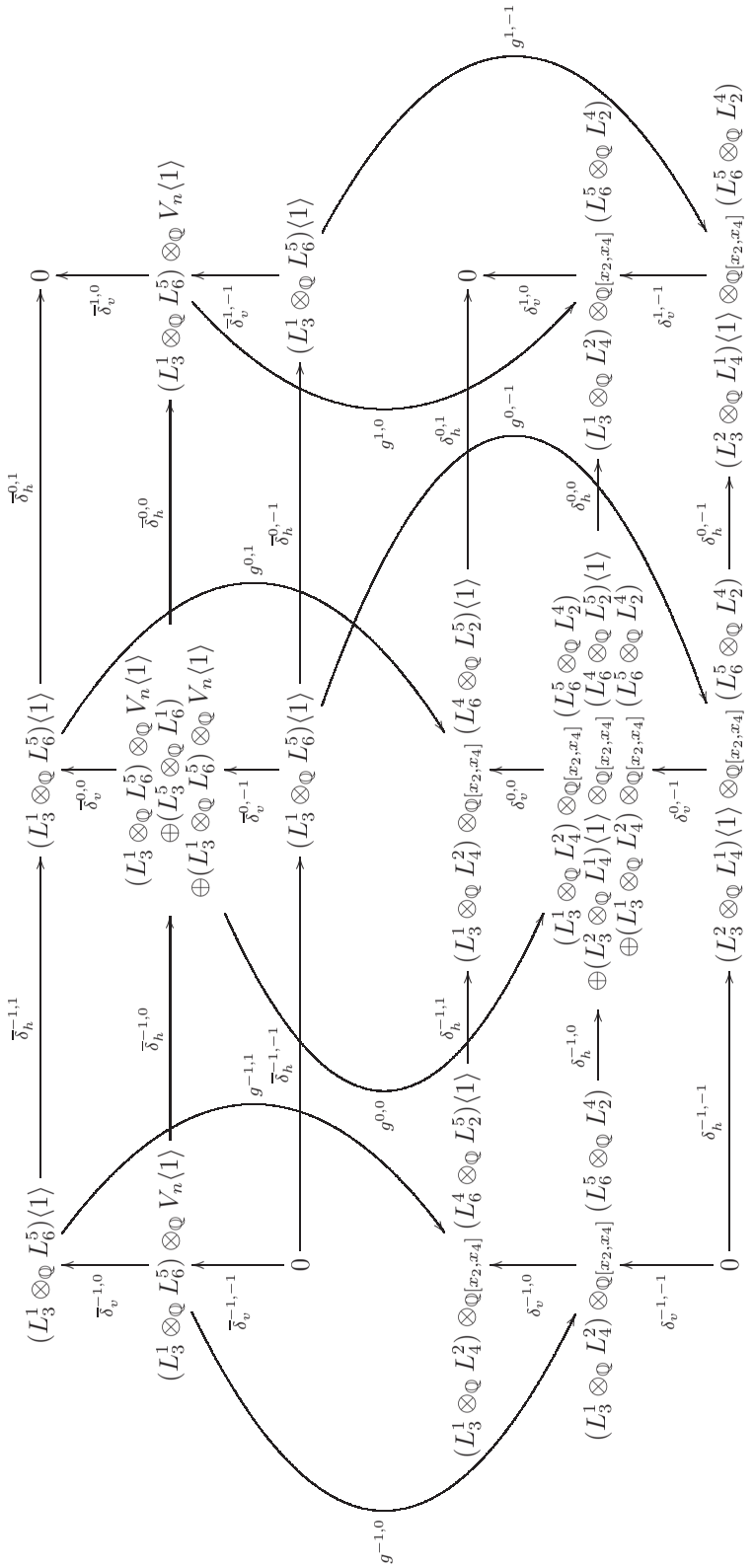


FIGURE 16. Adding marks for RIIb

The morphism of bicomplexes of gmf's G in Figure 16 is given by,

$$g^{0,1} = g^{-1,1},\ g^{-1,0} = g^{1,0},\ g^{0,-1} = g^{1,-1},$$

$$(g^{0,1})^0 = (g^{-1,1})^0 = \begin{pmatrix} 0 & -1 \\ \pi_{456} & 0 \\ 0 & -1 \\ -\pi_{245} & 0 \\ 1 & 0 \\ 0 & 1 \\ \pi_{456} & 0 \\ 0 & -\pi_{245} \end{pmatrix}, (g^{0,1})^1 = (g^{-1,1})^1 = \begin{pmatrix} 0 & 1 \\ -\pi_{456} & 0 \\ 0 & 1 \\ \pi_{245} & 0 \\ -1 & 0 \\ 0 & -1 \\ -\pi_{456} & 0 \\ 0 & \pi_{245} \end{pmatrix},$$

$$(g^{-1,0})^0 = (g^{1,0})^0 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, (g^{-1,0})^1 = (g^{1,0})^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$(g^{0,-1})^0 = (g^{1,-1})^0 = \begin{pmatrix} 0 & -1 \\ \pi_{146} & 0 \\ -\pi_{235} & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -\pi_{235}\pi_{146} & 0 \end{pmatrix}, (g^{0,-1})^1 = (g^{1,-1})^1 = \begin{pmatrix} 0 & 1 \\ -\pi_{146} & 0 \\ 1 & 0 \\ 0 & -\pi_{235} \\ -1 & 0 \\ 0 & -1 \\ 0 & \pi_{235} \\ \pi_{146} & 0 \end{pmatrix},$$

$$g^{0,0} = \begin{pmatrix} g_1^{0,0} & 0 & 0 \\ 0 & g_2^{0,0} & 0 \\ 0 & 0 & g_3^{0,0} \end{pmatrix},$$

where $g_1^{0,0} = g^{-1,0}$, $g_3^{0,0} = g^{-1,0}$, and $g_2^{0,0} = g$, with g given by

$$g^0 = \begin{pmatrix} 0 & -1 \\ \pi_{146} & 0 \\ -\pi_{235} & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -\pi_{235}\pi_{146} & 0 \end{pmatrix}, g^1 = \begin{pmatrix} 0 & 1 \\ -\pi_{146} &) \\ 1 & 0 \\ 0 & -\pi_{235} \\ -1 & 0 \\ 0 & -1 \\ 0 & \pi_{235} \\ \pi_{146} & 0 \end{pmatrix}.$$

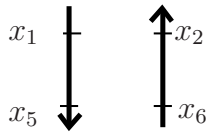


FIGURE 17. Marked smoothing for RIIb

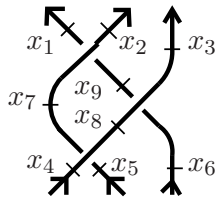


FIGURE 18. Marked diagram for RIII

We now complete the proof of the invariance under Reidemeister move RIIb. We have $F \circ G = \mathrm{Id}$ and $G \circ F$ is homotopic to the identity (the homotopies required are homotopies of gmf's). Hence, F and G are inverse isomorphisms in $K(K(\mathcal{HMF}_w))$. We give on Figure 19 a homotopy equivalence between the reduced bicomplex of gmf's obtained by removing marks and the bicomplex associated to the marked diagram on Figure 17. The homotopy h and the homotopy equivalence F, G on Figure 19 are given by the following formulas:

$$h^{0,1} = -\mathrm{Id},\ h^{0,0} = \begin{pmatrix} -\phi\mathrm{Id} & \eta_0^{1356} \circ \tau & \phi\mathrm{Id} \end{pmatrix},\ h^{1,0} = \begin{pmatrix} \eta_0^{1356} \circ \overline{\eta_1}^{1536} \circ \epsilon \\ \tau \circ \overline{\eta_1}^{1536} \circ \epsilon \\ \eta_0^{1356} \circ \overline{\eta_1}^{1536} \circ \epsilon \end{pmatrix},$$

$$h^{1,-1} = \mathrm{Id},\ F = \begin{pmatrix} \eta_0^{1356} \circ \tau \\ \mathrm{Id} \\ \eta_0^{1356} \circ \tau \end{pmatrix},\ \text{and } G = \begin{pmatrix} \tau \circ \overline{\eta_1}^{1536} \circ \epsilon & \mathrm{Id} & \tau \circ \overline{\eta_1}^{1536} \circ \epsilon \end{pmatrix},$$

where τ was defined in the proof of Lemma 3.4.

Proof of the invariance under Reidemeister move RIII

In order to prove the invariance under Reidemeister move RIII, we first simplify the bicomplex of gmf's using homotopically equivalent gmf's as in the case of Reidemeister II moves. We remove marks. We give directly on Figure 20 the simplified algebraic bicomplex associated to the diagram on Figure 18. For a graphical description of this bicomplex, see also Figure 20.

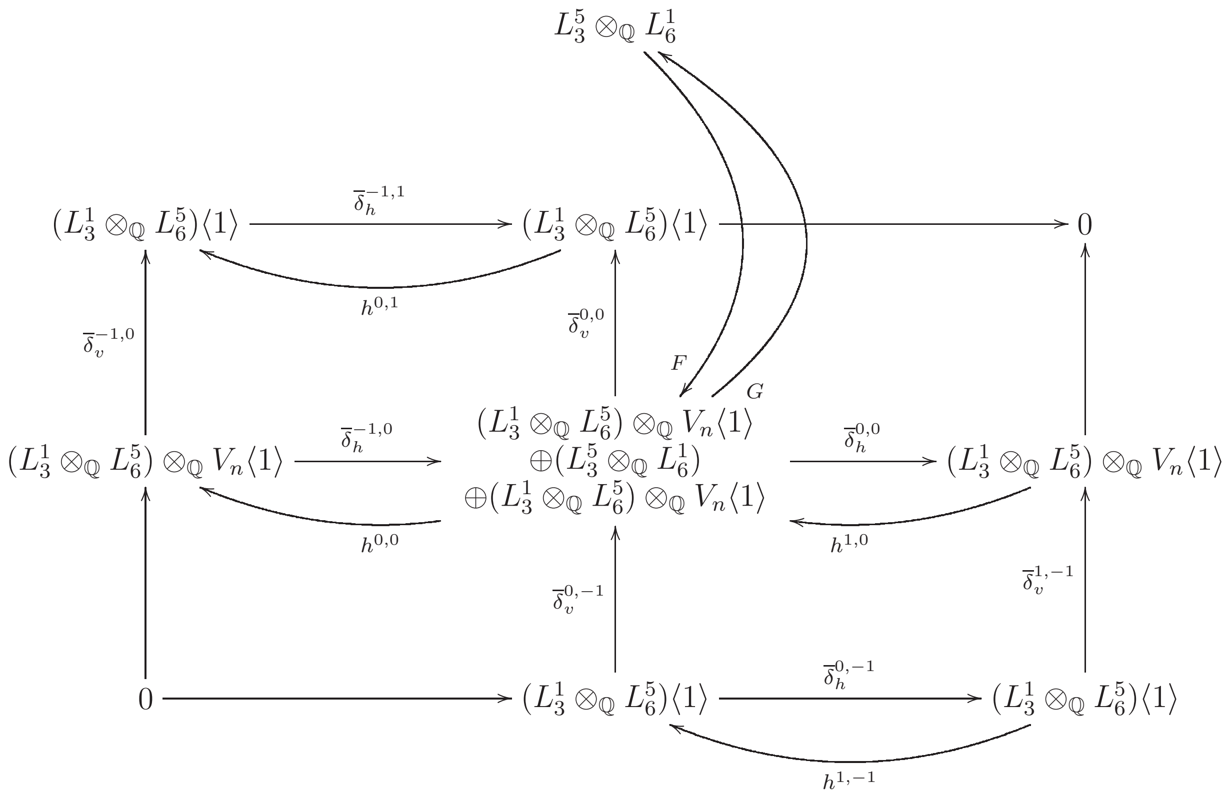


FIGURE 19. Final simplification for RIIb

$$\begin{tikzcd}
(L_6^1 \otimes L_5^2 \otimes L_4^3)\langle 1\rangle & & & \\
(L_4^1 \otimes L_5^2 \otimes L_6^3) \oplus (L_5^1 \otimes L_6^2 \otimes L_4^3) \oplus (L_6^1 \otimes L_4^2 \otimes L_5^3) \arrow[u, "\delta_v^{-3,2}"] \arrow[r, "\delta_h^{-3,2}"] & \begin{array}{c}(L_4^1 \otimes L_5^2 \otimes L_6^3)\\ \oplus(L_5^1 \otimes L_6^2 \otimes L_4^3)\\ \oplus(L_6^1 \otimes L_4^2 \otimes L_5^3)\end{array} & & \\
(L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle \oplus (L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle \oplus (L_4^1 \otimes L_6^2 \otimes L_5^3)\langle 1\rangle \arrow[u, "\delta_v^{-3,1}"] \arrow[r, "\delta_h^{-3,1}"] & \begin{array}{c}(L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle \oplus (L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle\\ \oplus(L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle \oplus (L_4^1 \otimes L_6^2 \otimes L_5^3)\langle 1\rangle\\ \oplus(L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle \oplus (L_4^1 \otimes L_6^2 \otimes L_5^3)\langle 1\rangle\end{array} \arrow[u, "\delta_v^{-2,1}"] \arrow[r, "\delta_h^{-2,1}"] & \begin{array}{c}(L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle\\ \oplus(L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle\\ \oplus(L_4^1 \otimes L_6^2 \otimes L_5^3)\langle 1\rangle\end{array} & \\
(L_4^1 \otimes L_5^2 \otimes L_6^3) \arrow[u, "\delta_v^{-3,0}"] \arrow[r, "\delta_h^{-3,0}"] & \begin{array}{c}(L_4^1 \otimes L_5^2 \otimes L_6^3)\\ \oplus(L_4^1 \otimes L_5^2 \otimes L_6^3)\\ \oplus(L_4^1 \otimes L_5^2 \otimes L_6^3)\end{array} \arrow[u, "\delta_v^{-2,0}"] \arrow[r, "\delta_h^{-2,0}"] & \begin{array}{c}(L_4^1 \otimes L_5^2 \otimes L_6^3)\\ \oplus(L_4^1 \otimes L_5^2 \otimes L_6^3)\\ \oplus(L_4^1 \otimes L_5^2 \otimes L_6^3)\end{array} \arrow[u, "\delta_v^{-1,0}"] \arrow[r, "\delta_h^{-1,0}"] & (L_4^1 \otimes L_5^2 \otimes L_6^3)
\end{tikzcd}$$

Figure 20. Graphical and simplified algebraic bicomplex for Reidemeister III

The differentials of the simplified algebraic bicomplex on Figure 20 are given by:

$$\delta_v^{-3,2} = \begin{pmatrix} -\overline{\eta_0}^{1346} & \overline{\eta_0}^{1256} & \eta_0^{2345} \end{pmatrix}, \delta_h^{-1,0} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \end{pmatrix},$$

$$\delta_v^{-3,1} = \begin{pmatrix} \overline{\eta}_0^{1245} & -\eta_0^{1245} & 0 \\ \overline{\eta_0}^{2346} & 0 & -\overline{\eta_0}^{1345} \\ 0 & -\overline{\eta_0}^{1356} & \overline{\eta_0}^{1246} \end{pmatrix}, \delta_v^{-3,0} = \begin{pmatrix} \eta_0^{1245} \\ \overline{\eta_0}^{1245} \\ \overline{\eta}_0^{2356} \end{pmatrix},$$

$$\delta_v^{-2,0} = \begin{pmatrix} \eta_0^{1245} & 0 & 0 \\ \overline{\eta_0}^{1245} & 0 & 0 \\ 0 & \eta_0^{1245} & 0 \\ 0 & \overline{\eta}_0^{2356} & 0 \\ 0 & 0 & \overline{\eta_0}^{1245} \\ 0 & 0 & \overline{\eta}_0^{2356} \end{pmatrix}, \delta_h^{-3,1} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ 0 & -\mathrm{Id} & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$\delta_v^{-2,1} = \begin{pmatrix} \overline{\eta}_0^{1245} & -\eta_0^{1245} & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{\eta_0}^{2346} & -\eta_0^{1345} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\overline{\eta_0}^{1356} & \overline{\eta_0}^{1246} \end{pmatrix},$$

$$\delta_v^{-1,0} = \begin{pmatrix} \eta_0^{1245} & 0 & 0 \\ 0 & \overline{\eta_0}^{1245} & 0 \\ 0 & 0 & \overline{\eta}_0^{2356} \end{pmatrix}, \delta_h^{-3,2} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$\delta_h^{-3,0} = \begin{pmatrix} -\mathrm{Id} \\ \mathrm{Id} \\ \mathrm{Id} \end{pmatrix}, \delta_h^{-2,1} = \begin{pmatrix} \mathrm{Id} & 0 & \mathrm{Id} & 0 & 0 & 0 \\ 0 & -\mathrm{Id} & 0 & 0 & -\mathrm{Id} & 0 \\ 0 & 0 & 0 & -\mathrm{Id} & 0 & \mathrm{Id} \end{pmatrix},$$

$$\delta_h^{-2,0} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & 0 \\ -\mathrm{Id} & 0 & -\mathrm{Id} \\ 0 & -\mathrm{Id} & \mathrm{Id} \end{pmatrix}.$$

Here, the notations have been simplified as follows: η_0^{ijkl} (resp. $\overline{\eta_0}^{ijkl}$) denotes the tensor product of the morphism η_0^{ijkl} (resp. $\overline{\eta_0}^{ijkl}$) with some identity morphisms. These tensor products are obvious from the gmf's on the source and the target of the differentials. Notice that the algebraic bicomplex on Figure 20 contains differentials involving morphisms of the type η_0 and $\overline{\eta_0}$. Since we are working in the category $K(K(\mathcal{HMF}_w))$, it follows from Lemma 3.9 that we can replace all morphisms of the type $\overline{\eta_0}$ by morphisms of the type η_0.

We simplify now the algebraic bicomplex on Figure 20. We exhibit a homotopy equivalence to a simpler bicomplex, see Figure 21. We give a graphical description of this homotopy equivalence on Figure 22.

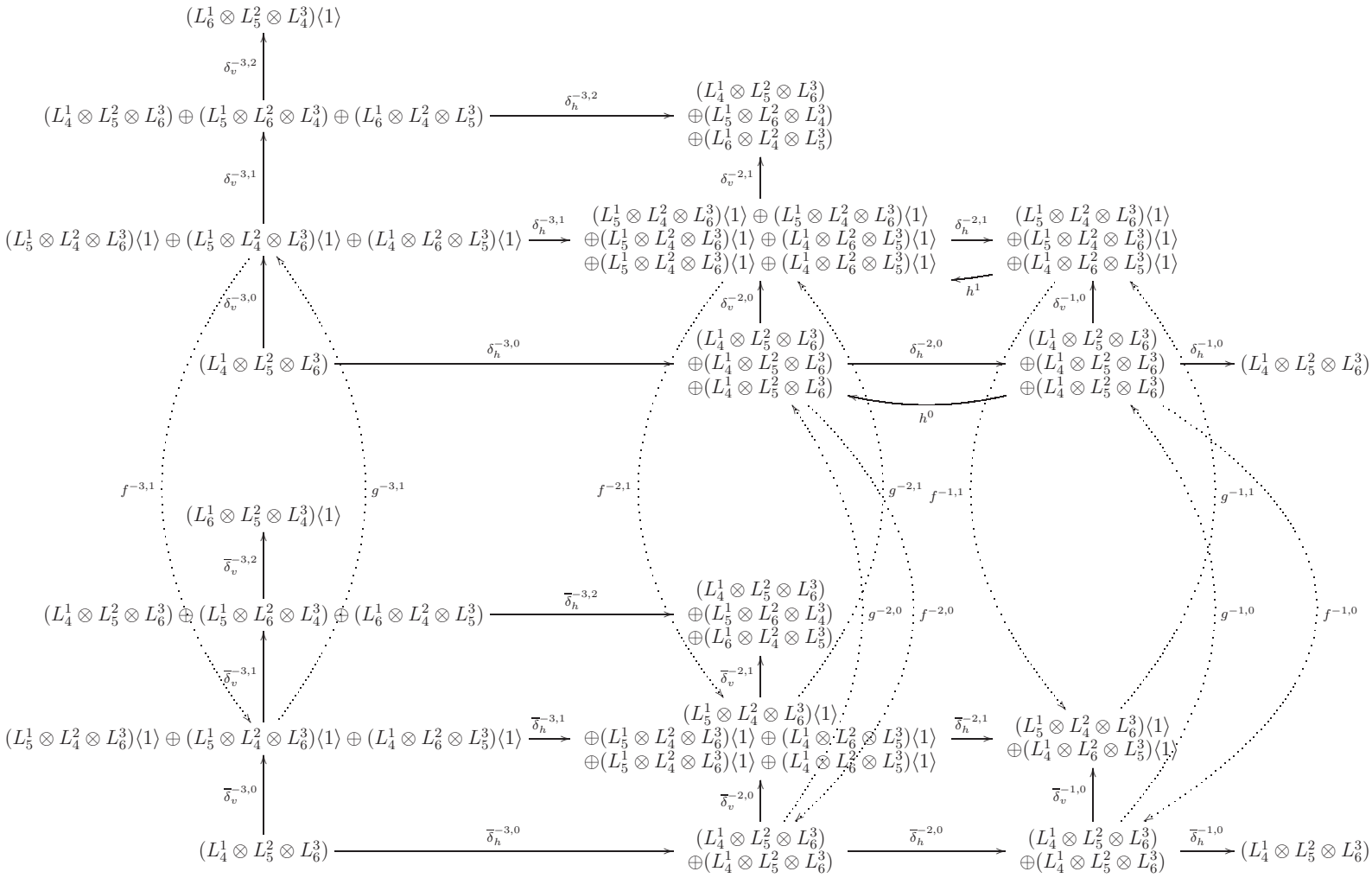


FIGURE 21. A reduced version of the bicomplex for RIII

The homotopy equivalence on Figure 21 or Figure 22 is given by

$$f^{-3,1} = \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \; g^{-3,1} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$f^{-2,1} = \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id} \end{pmatrix}, \; g^{-2,1} = \begin{pmatrix} 0 & -\mathrm{Id} & 0 & 0 & 0 \\ -\mathrm{Id} & -\mathrm{Id} & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$f^{-2,0} = \begin{pmatrix} 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \; f^{-1,1} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \; f^{-1,0} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$g^{-1,0} = \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \; g^{-2,0} = \begin{pmatrix} -\mathrm{Id} & 0 \\ \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \; g^{-1,1} = \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix},$$

$$h^0 = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; h^1 = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ -\mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

All other arrows of the homotopy equivalence are identities. Furthermore, the differentials of the new bicomplex are given by

$$\overline{\delta}_v^{-3,2} = \begin{pmatrix} -\eta_0^{1346} & \eta_0^{1256} & \eta_0^{2345} \end{pmatrix}, \; \overline{\delta}_h^{-1,0} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} \end{pmatrix}$$

$$\overline{\delta}_v^{-3,1} = \begin{pmatrix} \eta_0^{1245} & 0 & 0 \\ \eta_0^{2346} & \eta_0^{2346} & -\eta_0^{1345} \\ 0 & -\eta_0^{1356} & \eta_0^{1246} \end{pmatrix}, \; \overline{\delta}_v^{-3,0} = \begin{pmatrix} 0 \\ \eta_0^{1245} \\ \eta_0^{2356} \end{pmatrix}, \; \overline{\delta}_h^{-3,2} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$\overline{\delta}_v^{-2,0} = \begin{pmatrix} 0 & 0 \\ \eta_0^{1245} & 0 \\ \eta_0^{2356} & 0 \\ 0 & \eta_0^{1245} \\ 0 & \eta_0^{2356} \end{pmatrix}, \; \overline{\delta}_h^{-3,1} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ \mathrm{Id} & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$\overline{\delta}_v^{-2,1} = \begin{pmatrix} \eta_0^{1245} & 0 & 0 & 0 & 0 \\ 0 & \eta_0^{2346} & -\eta_0^{1345} & 0 & 0 \\ 0 & 0 & 0 & -\eta_0^{1356} & \eta_0^{1246} \end{pmatrix}, \; \overline{\delta}_v^{-1,0} = \begin{pmatrix} \eta_0^{1245} & 0 \\ 0 & \eta_0^{2356} \end{pmatrix},$$

$$\overline{\delta}_h^{-3,0} = \begin{pmatrix} \mathrm{Id} \\ \mathrm{Id} \end{pmatrix}, \; \overline{\delta}_h^{-2,1} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & 0 & -\mathrm{Id} & 0 \\ 0 & 0 & -\mathrm{Id} & 0 & \mathrm{Id} \end{pmatrix}, \; \overline{\delta}_h^{-2,0} = \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}.$$

We now exhibit on Figure 23 a short exact sequence of bicomplexes.

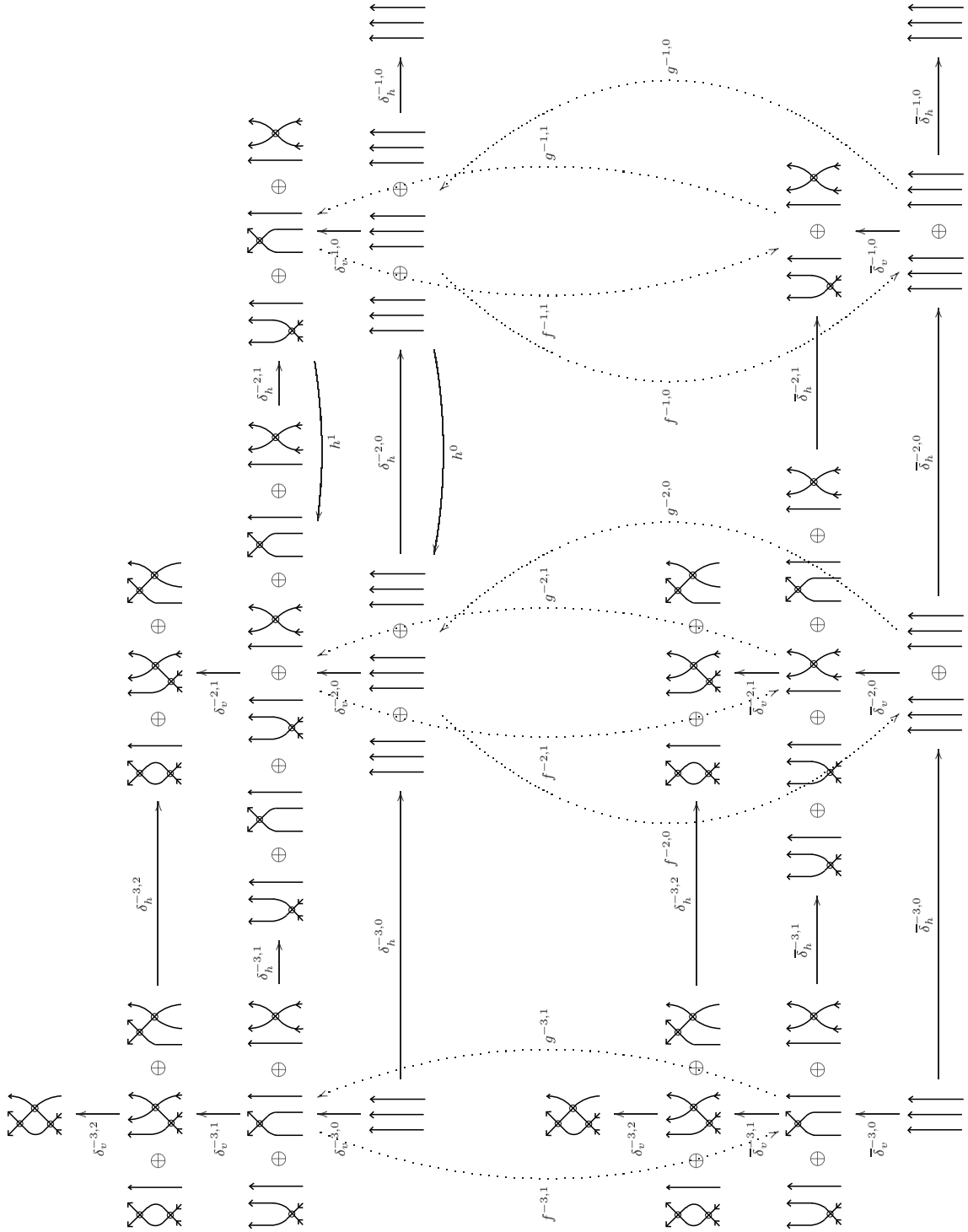


FIGURE 22. Graphical simplification

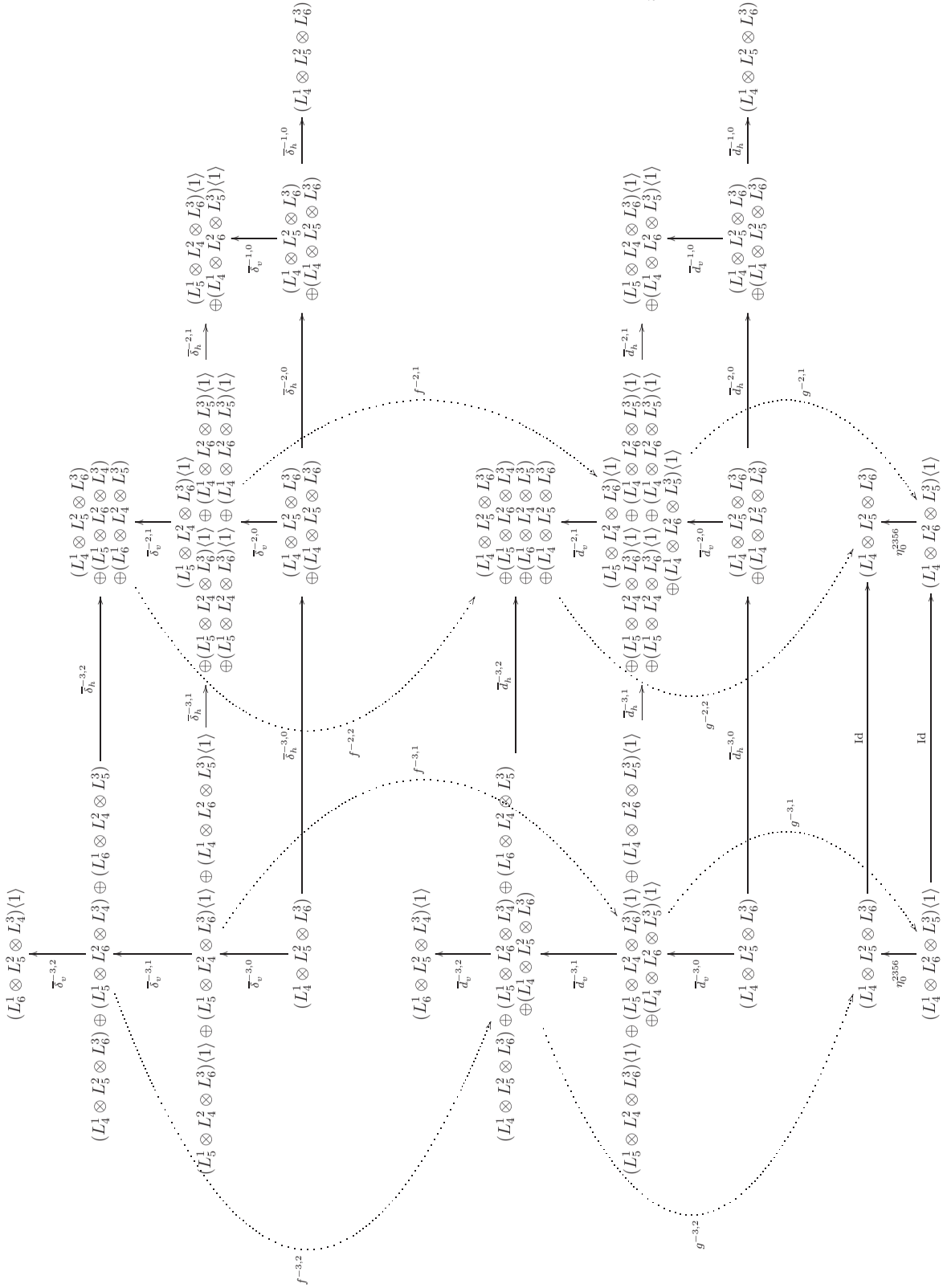


FIGURE 23. A short exact sequence of bicomplexes of gmf's for RIII

We describe the differentials on the middle bicomplex of gmf's on Figure 23:

$$\overline{d}_v^{-3,2} = \begin{pmatrix} -\eta_0^{1346} & \eta_0^{1256} & \eta_0^{2345} & -\eta_0^{1346} \end{pmatrix}, \overline{d}_h^{-1,0} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} \end{pmatrix},$$

$$\overline{d}_v^{-3,1} = \begin{pmatrix} \eta_0^{1245} & 0 & 0 & 0 \\ \eta_0^{2346} & \eta_0^{2346} & -\eta_0^{1345} & 0 \\ 0 & -\eta_0^{1356} & \eta_0^{1246} & \eta_0^{1246} \\ 0 & 0 & 0 & \eta_0^{2356} \end{pmatrix}, \overline{d}_h^{-3,2} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & -\mathrm{Id} \end{pmatrix},$$

$$\overline{d}_v^{-2,0} = \begin{pmatrix} 0 & 0 \\ \eta_0^{1245} & 0 \\ \eta_0^{2356} & 0 \\ 0 & \eta_0^{1245} \\ 0 & \eta_0^{2356} \\ 0 & 0 \end{pmatrix}, \overline{d}_h^{-3,1} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 & 0 \\ \mathrm{Id} & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} & \mathrm{Id} \\ 0 & 0 & 0 & -\mathrm{Id} \end{pmatrix},$$

$$\overline{d}_v^{-3,0} = \begin{pmatrix} 0 \\ \eta_0^{1245} \\ \eta_0^{2356} \\ 0 \end{pmatrix}, \overline{d}_v^{-2,1} = \begin{pmatrix} \eta_0^{1245} & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_0^{2346} & -\eta_0^{1345} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta_0^{1356} & \eta_0^{1246} & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_0^{2356} \end{pmatrix},$$

$$\overline{d}_v^{-1,0} = \begin{pmatrix} \eta_0^{1245} & 0 \\ 0 & \eta_0^{2356} \end{pmatrix},$$

$$\overline{d}_h^{-3,0} = \begin{pmatrix} \mathrm{Id} \\ \mathrm{Id} \end{pmatrix}, \overline{d}_h^{-2,1} = \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & 0 & -\mathrm{Id} & 0 & 0 \\ 0 & 0 & -\mathrm{Id} & 0 & \mathrm{Id} & \mathrm{Id} \end{pmatrix}, \overline{d}_h^{-2,0} = \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}.$$

The morphisms F and G of the short exact sequence on Figure 23 are given by

$$f^{-3,2} = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & 0 & 0 \end{pmatrix}, f^{-3,1} = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & 0 & 0 \end{pmatrix},$$

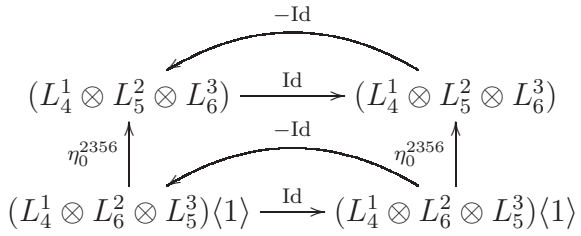
$$g^{-3,2} = \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id} \end{pmatrix}, g^{-3,1} = \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$f^{-2,2} = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & 0 & 0 \end{pmatrix}, f^{-2,1} = \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g^{-2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathrm{Id} \end{pmatrix}, g^{-2,2} = \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

All other arrows of F are identities, whereas all other arrows of G are equal to zero. One can see easily that F and G are morphisms of bicomplexes of gmf's and that the image of F is the kernel of G. The morphism F is injective, whereas the

morphism G is surjective. Remark that the bicomplex on the bottom of Figure 23 is homotopic to zero:



Now consider a marked oriented link diagram D which is locally of the form shown on Figure 18. We associate to such a diagram a bicomplex of 2-periodic chain complexes. It can be obtained by taking a suitable tensor product of the bicomplex of gmf's on the top of Figure 23 with the bicomplex C of gmf's associated to the rest of the diagram. We perform the same tensor product of C with the bicomplexes of gmf's on the middle and the bottom of Figure 23.

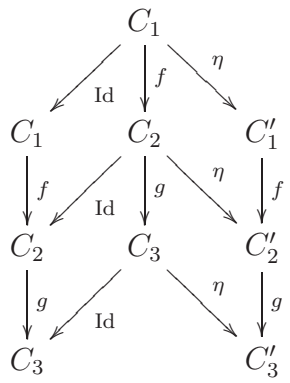
Fact 1: The short exact sequence on Figure 23 is still exact after tensoring all the bicomplexes of gmf's with C.

Fact 1 is a consequence of the fact that the short exact sequence on Figure 23 is just an injection of a subbicomplex and a projection on a quotient bicomplex. All the morphisms are identities or zero. More precisely, the tensor product with C can be decomposed in tensor products with the elementary bicomplexes of gmf's associated to the crossings of the rest of the diagram. We denote C_1, C_2 and C_3 the bicomplexes of gmf's on the top, the middle and the bottom of Figure 23. Since we want to tensor with the bicomplex of gmf's associated to a crossing first we tensor with the gmf associated to a virtual crossing; we obtain three bicomplexes of gmf's C_1', C_2', and C_3'. We still have a short exact sequence,

$$C_1' \longrightarrow C_2' \longrightarrow C_3'.$$

Tensoring the bicomplexes C_1, C_2 and C_3 with the gmf associated to the smoothing, produces bicomplexes homotopic to C_1, C_2 and C_3. Furthermore we have saddle morphisms from the C_i to the C_i', $i = 1, 2, 3$, which commute with morphisms of

the short exact sequences. Hence we obtain the following commutative diagram:



where all the collumns are exact. Now take the cone of the arrows corresponding to the saddle morphisms with respect to the vertical grading of the bicomplex and then the cone of the identities on the left with respect to the horizontal grading of the bicomplex. We obtain the bicomplexes of gmf's obtained by tensoring with the bicomplex associated to a negative crossing. We have still a short exact sequence.

Fact 2: The bicomplex of 2-periodic chain complexes obtained on the bottom of this short exact sequence is homotopic to zero.

The fact 2 follows from the more general fact that: given a chain complex E homotopic to zero then for any chain complex D, $E \otimes D$ is homotopic to zero.

Finally for any marked oriented link diagram which locally is of the form on Figure 18, we have a short exact sequence of bicomplexes of 2-periodic chain complexes,

$$C_1 \longrightarrow C_2 \longrightarrow C_3.$$

First take the homology with respect to the differential of the 2-periodic chain complexes. We have now a short exact sequence of bicomplexes of graded vector spaces over $\mathbb{Q}$. We take then the homology with respect to the differential δ_v. It gives a long exact sequence of complexes of graded vector spaces over $\mathbb{Q}$. Since we know from Lemma 3.10 that taking homology with respect to the differential δ_v only produces homology in the even vertical grading, we have that the long exact sequence of complexes of graded vector spaces over $\mathbb{Q}$ splits into many short exact sequences of complexes of graded vector space over $\mathbb{Q}$. Since C_3 is homotopic to zero, it implies

$$\mathbb{H}_n(C_1) \cong \mathbb{H}_n(C_2).$$

Similarly, the bicomplex of gmf's associated to the marked diagram on Figure 25 can be seen as a subbicomplex of the bicomplex of gmf's C_2, see Figure 24.

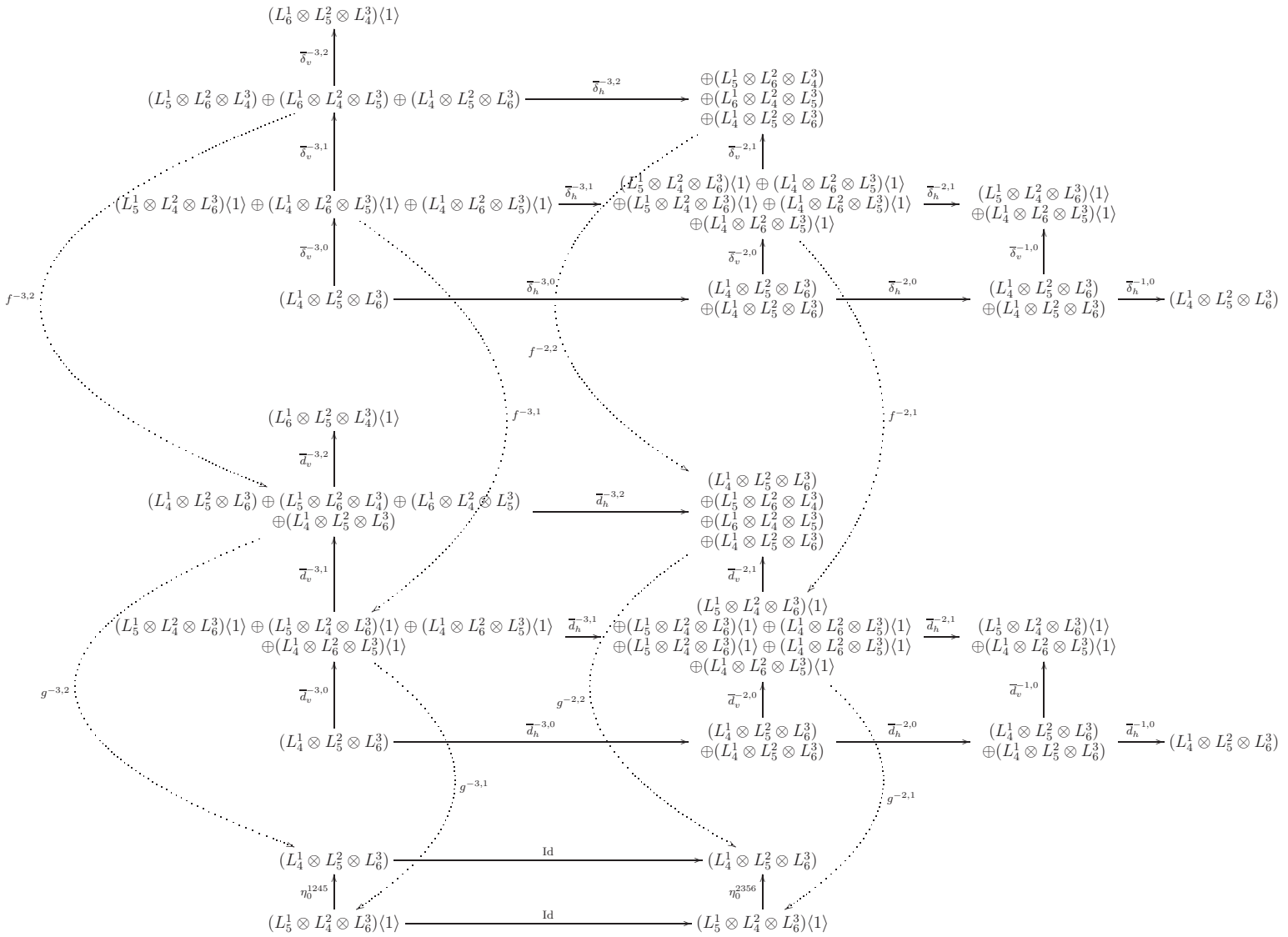


FIGURE 24. Another short exact sequence for RIII

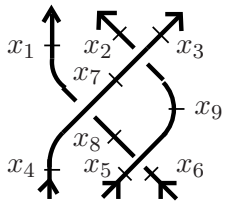


FIGURE 25. Marked diagram for RIII

The morphisms F and G of the short exact sequence on Figure 24 are given by

$$f^{-3,2} = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \; f^{-3,1} = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$g^{-3,2} = \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 \end{pmatrix}, \; g^{-3,1} = \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 \end{pmatrix},$$

$$f^{-2,2} = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \; f^{-2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mathrm{Id} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id} \end{pmatrix},$$

$$g^{-2,1} = \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 & 0 \end{pmatrix}, \; g^{-2,2} = \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 \end{pmatrix}.$$

All other arrows of F are identities, whereas all other arrows of G are equal to zero. One can see easily that F and G are morphisms of bicomplex of gmf's and that the image of F is the kernel of G. The morphism F is injective, whereas the morphism G is surjective. Remark that the bicomplex on the bottom of Figure 24 is homotopic to zero:

$$\begin{array}{ccc} (L_4^1 \otimes L_5^2 \otimes L_6^3) & \xrightarrow{\mathrm{Id}} & (L_4^1 \otimes L_5^2 \otimes L_6^3) \\ {\scriptstyle \eta_0^{1245}}\uparrow & \overset{-\mathrm{Id}}{\swarrow} & \uparrow{\scriptstyle \eta_0^{1245}} \\ (L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle & \xrightarrow{\mathrm{Id}} & (L_5^1 \otimes L_4^2 \otimes L_6^3)\langle 1\rangle \end{array} \qquad \overset{-\mathrm{Id}}{\curvearrowleft}$$

Let $\overline{C}_1$ be the bicomplex associated to the link diagram D' which differs only from D inside a disk; the local picture on Figure 18 is replaced by the local picture on Figure 25. Using the same arguments, we have $\mathbb{H}_n(\overline{C}_1) \cong \mathbb{H}_n(C_2)$ and finally

$$\mathbb{H}_n(C_1) \cong \mathbb{H}_n(\overline{C}_1).$$

This completes the proof of Theorem 3.1.

CHAPTER 4

Properties of $\mathbb{H}_n$

We explain how our additional grading can be used to compute $\mathbb{H}_n$ for some knots. In Section 1, we make explicit the role played by the Frobenius algebra $\mathbb{Q}[x]/x^n$ for the triply graded homology $\mathbb{H}_n$. As an application, we compute $\mathbb{H}_n$ for the trefoil knot and the Hopf link. In Section 2 and 3, we describe short exact sequences and a spectral sequence for the homology $\mathbb{H}_n$; as an application, we compute $\mathbb{H}_n$ for the two strand torus knots and the figure-eight knot. In Section 4, we derive a polynomial link invariant Q_n from the Poincaré polynomial of $\mathbb{H}_n$ and use it to prove that $\mathbb{H}_2$ is a stronger link invariant than HKR_2. Moreover, the polynomial link invariant Q_n satisfies a cubic skein relation. In Section 5, inspired by the work of Lee [**28**], Gornik [**13**] and Wu [**52**] on KR_n, we introduce some deformations of $\mathbb{H}_n$. In Section 6, we generalize $\mathbb{H}_n$ to virtual links.

1. The underlying TQFT

We explain why η_0 and η_1 (defined in Chapter 1.5) should be regarded as "saddle" morphisms. Given an oriented link diagram D, the morphisms η_0 and η_1 go pictorially from a union of virtual oriented circles to another one with one more or one less circle. We would like to see a multiplication or a comultiplication of a Frobenius algebra, see [**27**]. This is the content of the next lemma.

LEMMA 4.1. *The morphisms of gmf's*

$$L_k^i \otimes_{\mathbb{Q}} L_l^j \underset{\eta_1}{\overset{\eta_0}{\rightleftarrows}} L_l^i \otimes_{\mathbb{Q}} L_k^j\langle 1\rangle$$

induce the multiplication m and the comultiplication Δ of the graded Frobenius algebra V_n defined in Chapter 3.2. There exist isomorphisms such that the following square diagrams commute:

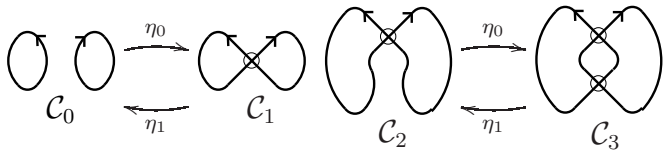
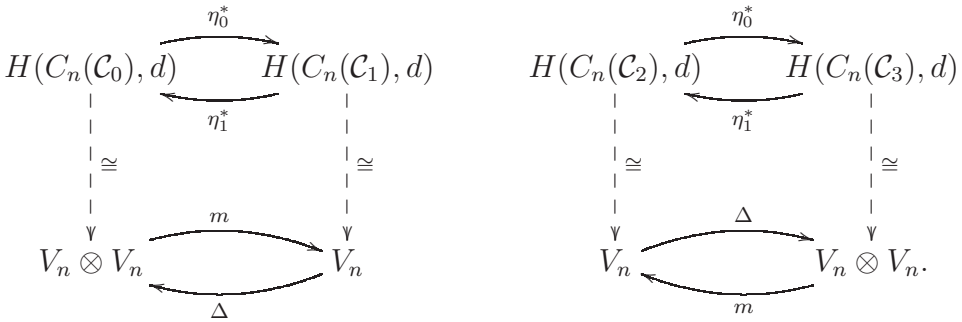


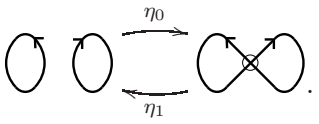
FIGURE 1. Two types of local morphisms (we omit all the marks)



where $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are the diagram resolutions shown on Figure 1 and η_0^* and η_1^* are the morphisms induced respectively by η_0 and η_1 on the homology of the graded 2-periodic chain complexes.

Given an oriented link diagram D, the bicomplex $H(C_n(D), d)$ of graded vector spaces (the differential d stands for the differential on the 2-periodic chain complexes) can be alternatively described without matrix factorizations.

PROOF. We want to see how the morphisms η_0 and η_1 look like for the two kinds of closure of the local picture on Figure 10 of Chapter 1. For the first one, we have



This amounts to considering the morphisms η_0 and η_1 with the additional relations $x_i = x_k$ and $x_j = x_l$

$$L_k^i \otimes_{\mathbb{Q}} L_l^j/(x_i = x_k, x_j = x_l) \underset{\eta_1}{\overset{\eta_0}{\rightleftarrows}} L_l^i \otimes_{\mathbb{Q}} L_k^j/\langle x_i - x_k, x_j - x_l\rangle\langle 1\rangle \ .$$

We have

$$H(L_l^i \otimes_{\mathbb{Q}} L_k^j/(x_i = x_k, x_j = x_l)) \cong \mathbb{Q}[x]/\langle x^n\rangle\{1-n\} \cong V_n$$

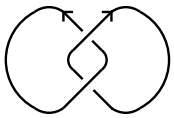
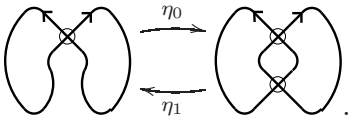


FIGURE 2. Hopf link

and

$$H(L_k^i \otimes_{\mathbb{Q}} L_l^j/\langle x_i = x_k, x_j = x_l\rangle) \cong \mathbb{Q}[x,y]/\langle x^n, y^n\rangle\{2-2n\} \cong V_n \otimes_{\mathbb{Q}} V_n.$$

In this case η_0 is the multiplication and η_1 is the comultiplication of the graded Frobenius algebra V_n. For the second one, we have



This amounts to considering the morphisms η_0 and η_1 with the additional relations $x_i = x_l$ and $x_j = x_k$ and shifting the $\mathbb{Z}/2\mathbb{Z}$-grading by 1. In this case, η_0 is the comultiplication of V_n and η_1 the multiplication of V_n. □

Given an oriented link diagram D, the bicomplex $H(C_n(D), d)$ of graded vector spaces, can be described only by using the Frobenius algebra V_n. Compare to the complex of graded vector spaces constructed by Khovanov and Rozansky, our chains are easier to handle, the differentials are easier to understand since they are just identities, multiplications and comultiplications of the Frobenius algebra, but we increase the number of chains.

As a first application of Lemma 4.1, we compute the homology $\mathbb{H}_n$ for the Hopf link and for the trefoil knot.

EXAMPLE 4.1. *Consider the bicomplex $H(C_n(2_1^2), d)$ of graded vector spaces associated to the Hopf link 2_1^2 on Figure 2 (the differential d stands for the differential on the 2-periodic chain complexes):*

$$\begin{array}{ccccc}
V_n \otimes V_n & & & & \\
\uparrow{\scriptstyle \delta_v^{-2,1}} & & & & \\
(V_n \oplus V_n)\{n-1\} & \xrightarrow{\delta_h^{-2,1}} & V_n\{n-1\} & & \\
 & & \oplus V_n\{n-1\} & & \\
\uparrow{\scriptstyle \delta_v^{-2,0}} & & \uparrow{\scriptstyle \delta_v^{-1,0}} & & \\
(V_n \otimes V_n)\{2n-2\} & \xrightarrow{\delta_h^{-2,0}} & (V_n \otimes V_n)\{2n-2\} & \xrightarrow{\delta_h^{-1,0}} & (V_n \otimes V_n)\{2n-2\} \\
 & & \oplus (V_n \otimes V_n)\{2n-2\} & &
\end{array}$$

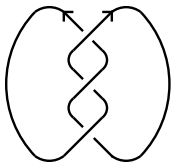


FIGURE 3. Trefoil knot

where

$$\delta_v^{-2,1} = \begin{pmatrix} \Delta & \Delta \end{pmatrix}, \delta_h^{-2,1} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix},$$

$$\delta_v^{-2,0} = \begin{pmatrix} m \\ -m \end{pmatrix}, \delta_h^{-2,0} = \begin{pmatrix} \mathrm{Id} \\ \mathrm{Id} \end{pmatrix}, \delta_h^{-1,0} = \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \end{pmatrix}, \delta_v^{-1,0} = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}.$$

In a nutshell, the bicomplex $H(C_n(2_1^2), d)$ can be summarized as follows:

$$\begin{array}{ccc} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{array}$$

where the upper-left corner is in bidegree $(-2, 2)$ and where the crosss stand for non-zero graded vector spaces.

A direct computation gives the following Poincaré polynomial:

$$P_n^{\mathbb{H}}(2_1^2)(u, v, q) = u^{-2}v^2q^{-1}[n]_q[n-1]_q + q^{n-1}[n]_q.$$

In particular,

$$P_n^{\mathbb{H}}(2_1^2)(u, q^{n+1}, q) = u^{-2}q^{2n+1}[n]_q[n-1]_q + q^{n-1}[n]_q$$

is the Poincaré polynomial for HKR_n, see [**38**] *and*

$$P_n^{\mathbb{H}}(2_1^2)(-1, q^{n+1}, q) = q^{2n+1}[n]_q[n-1]_q + q^{n-1}[n]_q = P_n(2_1^2)(q).$$

EXAMPLE 4.2. *Consider now the bicomplex $H(C_n(3_1), d)$ of graded vector spaces associated to the trefoil knot 3_1 on Figure 3:*
The bicomplex $H(C_n(3_1), d)$ can be summarized as follows:

$$\begin{array}{cccc} \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \end{array}$$

where the upper-left corner is in bidegree $(-3, 3)$.

$$
\begin{array}{ccccccc}
 & & V_n & & & & \\
 & & \uparrow{\scriptstyle \delta_v^{-3,2}} & & & & \\
 & & ((V_n\otimes V_n)\oplus(V_n\otimes V_n)\oplus(V_n\otimes V_n))\{n-1\} & \xrightarrow{\delta_h^{-3,2}} & \begin{array}{c}(V_n\otimes V_n)\{n-1\}\\ \oplus(V_n\otimes V_n)\{n-1\}\\ \oplus(V_n\otimes V_n)\{n-1\}\end{array} & & \\
 & & \uparrow{\scriptstyle \delta_v^{-3,1}} & & \uparrow{\scriptstyle \delta_v^{-2,1}} & & \\
 & & (V_n\oplus V_n\oplus V_n)\{2n-2\} & \xrightarrow{\delta_h^{-3,1}} & \begin{array}{c}(V_n\oplus V_n)\{2n-2\}\\ \oplus(V_n\oplus V_n)\{2n-2\}\\ \oplus(V_n\oplus V_n)\{2n-2\}\end{array} & \xrightarrow{\delta_h^{-2,1}} & \begin{array}{c}V_n\{2n-2\}\\ \oplus V_n\{2n-2\}\\ \oplus V_n\{2n-2\}\end{array} \\
 & & \uparrow{\scriptstyle \delta_v^{-3,0}} & & \uparrow{\scriptstyle \delta_v^{-2,0}} & & \uparrow{\scriptstyle \delta_v^{-1,0}} \\
(V_n\otimes V_n)\{3n-3\} & \xrightarrow{\delta_h^{-3,0}} & \begin{array}{c}(V_n\otimes V_n)\{3n-3\}\\ \oplus(V_n\otimes V_n)\{3n-3\}\\ \oplus(V_n\otimes V_n)\{3n-3\}\end{array} & \xrightarrow{\delta_h^{-2,0}} & \begin{array}{c}(V_n\otimes V_n)\{3n-3\}\\ \oplus(V_n\otimes V_n)\{3n-3\}\\ \oplus(V_n\otimes V_n)\{3n-3\}\end{array} & \xrightarrow{\delta_h^{-1,0}} & (V_n\otimes V_n)\{3n-3\}
\end{array}
$$

where

$$
\delta_v^{-3,2}=\begin{pmatrix} -m & m & m\end{pmatrix},\ \delta_h^{-1,0}=\begin{pmatrix}\mathrm{Id} & \mathrm{Id} & \mathrm{Id}\end{pmatrix},\ \delta_h^{-3,0}=\begin{pmatrix}-\mathrm{Id}\\ \mathrm{Id}\\ \mathrm{Id}\end{pmatrix},\ \delta_h^{-2,1}=\begin{pmatrix}\mathrm{Id}&0&\mathrm{Id}&0&0&0\\ 0&-\mathrm{Id}&0&0&-\mathrm{Id}&0\\ 0&0&0&-\mathrm{Id}&0&\mathrm{Id}\end{pmatrix},
$$

$$
\delta_h^{-2,0}=\begin{pmatrix}\mathrm{Id}&\mathrm{Id}&0\\ -\mathrm{Id}&0&-\mathrm{Id}\\ 0&-\mathrm{Id}&\mathrm{Id}\end{pmatrix},\ \delta_v^{-3,1}=\begin{pmatrix}\Delta&-\Delta&0\\ \Delta&0&-\Delta\\ 0&-\Delta&\Delta\end{pmatrix},\ \delta_v^{-3,0}=\begin{pmatrix}m\\ m\\ m\end{pmatrix},\ \delta_v^{-1,0}=\begin{pmatrix}m&0&0\\ 0&m&0\\ 0&0&m\end{pmatrix},
$$

$$\delta_h^{-3,2} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \delta_v^{-2,0} = \begin{pmatrix} m & 0 & 0 \\ m & 0 & 0 \\ 0 & m & 0 \\ 0 & m & 0 \\ 0 & 0 & m \\ 0 & 0 & m \end{pmatrix},$$

$$\delta_h^{-3,1} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ 0 & -\mathrm{Id} & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \delta_v^{-2,1} = \begin{pmatrix} \Delta & -\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta & -\Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta & \Delta \end{pmatrix}.$$

Again, a direct computation gives the following Poincaré polynomial:

$$P_n^{\mathbb{H}}(3_1)(u,v,q) = u^{-3}v^2q^{2n-1}[n-1]_q + u^{-2}v^2q^{-1}[n-1]_q + q^{2n-2}[n]_q.$$

In particular,

$$P_n^{\mathbb{H}}(3_1)(u,q^{n+1},q) = u^{-3}q^{4n+1}[n-1]_q + u^{-2}q^{2n+1}[n-1]_q + q^{2n-2}[n]_q$$

is the Poincaré polynomial for HKR_n, see [**38**] *and*

$$P_n^{\mathbb{H}}(3_1)(-1,q^{n+1},q) = -q^{4n+1}[n-1]_q + q^{2n+1}[n-1]_q + q^{2n-2}[n]_q = P_n(3_1)(q).$$

2. A canonical spectral sequence

Given an oriented link L with l components, consider the bicomplex of graded vector spaces $H(C_n(L),d)$. The next theorem relates through a spectral sequence the homology $\mathbb{H}_n(L)$ to the homology of the l-component unlink.

THEOREM 4.1. *Suppose L is an oriented link with l components, there is a spectral sequence E_k which has E_2 term $\mathbb{H}_n(L)$ and converges to $V_n^{\otimes l}$.*

PROOF. Since $H(C_n(L),d)$ is a bicomplex of graded vector spaces, there are two spectral sequences E_k and $\overline{E}_k$ converging to the total homology of the bicomplex $H(C_n(D),d)$ and such that

$$\overline{E}_2 = H(H(H(C_n(D),d),\delta_h)\delta_v) \text{ and } E_2 = H(H(H(C_n(D),d),\delta_v)\delta_h),$$

see [**34**]. More precisely, we have in particular

$$E_2^{p,q} = \oplus_{k\in\mathbb{Z}}\mathbb{H}_n^{p,q,k}(D).$$

Consider now $\overline{E}_1 = H(H(C_n(D),d),\delta_h)$. Since the differential δ_h is the identity, it follows that $\overline{E}_1$ is equal to the bicomplex of graded vector space associated to the virtual resolution of D (i.e. the resolution where all crossings are virtualized).

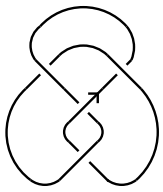


FIGURE 4. Figure-eight knot

By construction, this bicomplex is just a graded vector space isomorphic to $V_n^{\otimes l}$. Hence, $\overline{E}_1$ is equal to the graded vector space $V_n^{\otimes l}$ lying in bidegree $(-w, w)$ where w is the writhe of D. The spectral sequence $\overline{E}_k$ collapses at its first page and the total homology of $H(C_n(D), d)$ is thus isomorphic to $V_n^{\otimes l}$ in degree $0 = w - w$. $\square$

We apply this spectral sequence to the computation of $\mathbb{H}_n$ for the figure-eight knot shown on Figure 4.

EXAMPLE 4.3. *Consider the bicomplex* $(H(C_n(4_1), d)$ *which is of the form*

$$\begin{array}{ccccc} \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \end{array},$$

where the left-upper corner corresponds to bidegree $(-2, 2)$.

It follows from Lemma 3.10 that the bicomplex $H(H(C_n(4_1), d), \delta_v)$ *is of the form*

$$\begin{array}{ccccc} \times & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \times & \times & \times \end{array},$$

where the left-upper corner corresponds to bidegree $(-2, 2)$.

By direct computations one can directly see that the differential $(\delta_h^{-2,0})^*$ *is injective and that the differential* $(\delta_h^{1,0})^*$ *is surjective. Hence the bicomplex* $H(H(H(C_n(4_1), d), \delta_v), \delta_h)$ *is of the form*

$$\begin{array}{ccccc} \times & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \times & \times & \times \end{array},$$

Theorem 4.1 shows that $H(H(H(C_n(4_1),d),\delta_v),\delta_h)$ simplifies to

$$\begin{array}{ccccc} \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times \end{array}.$$

This follows from the fact that the differentials d_k of E_k are of bidegree $(k,1-k)$ and that the infinite page of E_k lies on the diagonal. A direct computation gives the following Poincaré polynomial:

$$P_n^{\mathbb{H}}(4_1)(u,v,q) = u^{-2}v^2q^{-1}[n-1]_q + u^{-1}q[n-1]_q + [n] + uq^{-1}[n-1]_q + u^2v^{-2}q[n-1]_q.$$

In particular,

$$P_n^{\mathbb{H}}(4_1)(u,q^{n+1},q) = u^{-2}q^{2n+1}[n-1]_q + u^{-1}q[n-1]_q + [n] + uq^{-1}[n-1]_q + u^2q^{-2n-1}[n-1]_q$$

is the Poincaré polynomial for HKR_n, see [**38**] *and*

$$P_n^{\mathbb{H}}(4_1)(-1,q^{n+1},q) = q^{2n+1}[n-1]_q - [2]_q[n-1]_q + [n] + q^{-2n-1}[n-1]_q = P_n(4_1)(q).$$

3. Skein exact sequences for $\mathbb{H}_n$

We describe two short exact sequences of bicomplexes that induce long exact sequences for $\mathbb{H}_n$. As an application we compute $\mathbb{H}_n$ for the $(2,l)$-torus knots.

We introduce a few notations. In the proof of Theorem 3.2, we introduced the complexes of gmf's $C_n($  $)$ and $C_n($ $)$. We will consider these complexes as a bicomplexes lying in horizontal homological grading 0.

Lemma 4.2. *The two following short exact sequences*

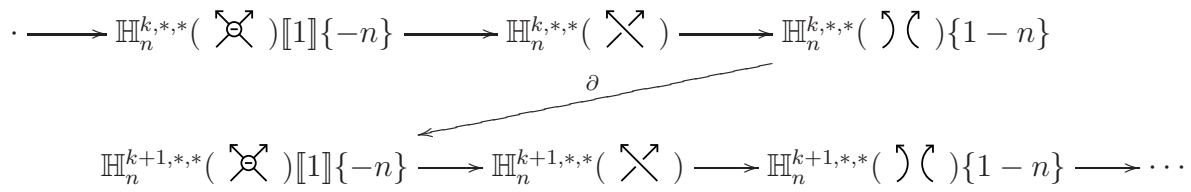
$$0 \longrightarrow C_n(\ )(\ )\{n-1\} \xrightarrow{f^+} C_n(\ \times\ ) \xrightarrow{g^+} C_n(\ \oplus\ )[\![-1]\!]\{n\} \longrightarrow 0,$$

$$0 \longrightarrow C_n(\ \ominus\ )[\![1]\!]\{-n\} \xrightarrow{f^-} C_n(\ \times\ ) \xrightarrow{g^-} C_n(\ )(\ )\{1-n\} \longrightarrow 0,$$

of bicomplexes induce the following long exact sequences

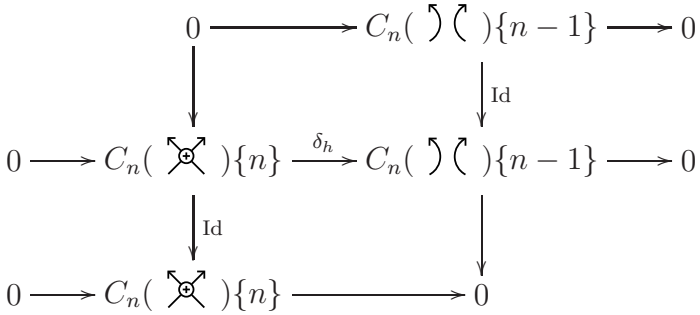
$$\cdot \longrightarrow \mathbb{H}_n^{k,*,*}(\ )(\ )\{n-1\} \longrightarrow \mathbb{H}_n^{k,*,*}(\ \times\ ) \longrightarrow \mathbb{H}_n^{k,*,*}(\ \oplus\ )[\![-1]\!]\{n\}$$
$$\xrightarrow{\partial}$$
$$\mathbb{H}_n^{k+1,*,*}(\ )(\ )\{n-1\} \longrightarrow \mathbb{H}_n^{k+1,*,*}(\ \times\ ) \longrightarrow \mathbb{H}_n^{k+1,*,*}(\ \oplus\ )[\![-1]\!]\{n\} \longrightarrow \cdots$$

and



where the differentials respect the vertical homological grading and the quantum grading and $[\![\cdot]\!]$ is the shift in the horizontal grading.

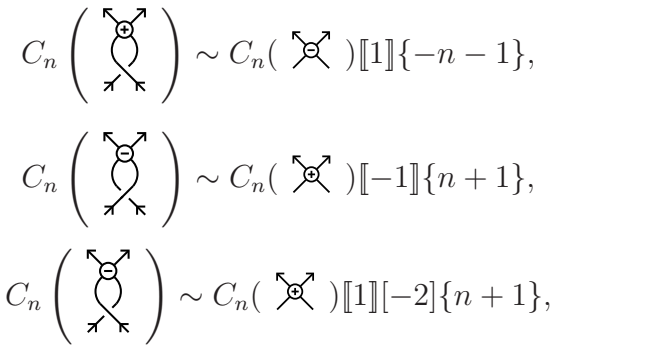
Proof. The first short exact sequence can be described as follows



A similar argument as in the proof of the invariance under Reidemeister RIII move using Lemma 3.10 implies the result. More precisely, taking homology with respect to δ_v produce a family of short exact sequences of complexes of graded vector spaces. This family can be incorporated into a short exact sequence of complexes of bigraded vector spaces where the differentials respect both gradings. The differentials of this complexes are induced by δ_h's. This short exact sequence produces the long exact sequence for $\mathbb{H}_n$. The proof is similar for the other short exact sequence. □

In order to compute $\mathbb{H}_n$ for the $(2, l)$-torus knots, we first consider the following technical lemma.

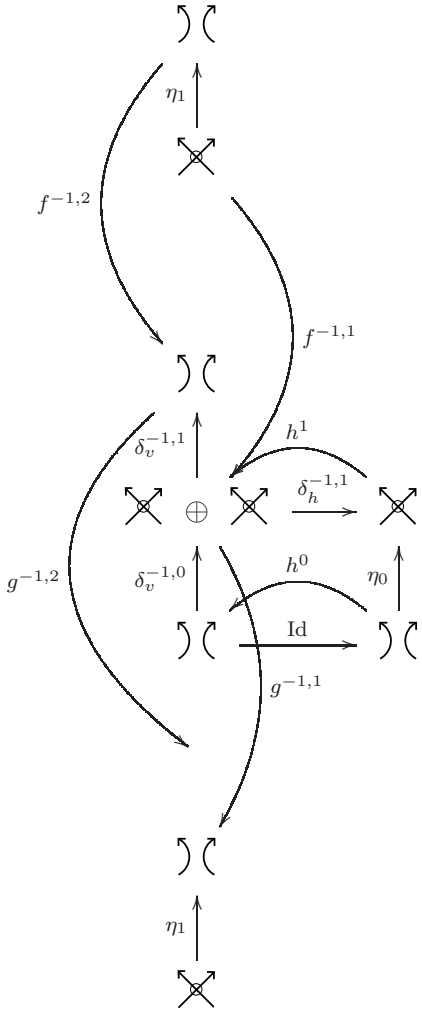
Lemma 4.3. *The following bicomplexes of gmf's are homotopy equivalent:*



$$C_n\left(\ \right) \sim C_n(\ )[\![-1]\!][2]\{-n-1\}.$$

where $[\cdot]$ *is the shift in the vertical grading,* $[\![\cdot]\!]$ *is the shift in the horizontal grading and* $\sim$ *is for homotopy equivalent.*

PROOF. We exhibit a homotopy equivalence between $C_n\left(\ \right)$ and $C_n(\ )[2][\![-1]\!]\{-n-1\}$. As before, we first remove the internal marks and simplify the gmf's, then we exhibit a homotopy equivalence



where

$$\delta_h^{-1,1} = \begin{pmatrix} \mathrm{Id} & 0 \end{pmatrix},\ \delta_v^{-1,1} = \begin{pmatrix} -\eta_1 & \eta_1 \end{pmatrix},\ \delta_v^{-1,0} = \begin{pmatrix} \eta_0 \\ \eta_0 \end{pmatrix},$$

$$f^{-1,2} = g^{-1,2} = \mathrm{Id},\ h^0 = \mathrm{Id},\ h^1 = \begin{pmatrix} \mathrm{Id} \\ \mathrm{Id} \end{pmatrix},\ f^{-1,1} = \begin{pmatrix} 0 \\ \mathrm{Id} \end{pmatrix},\ g^{-1,1} = \begin{pmatrix} -\mathrm{Id} & \mathrm{Id} \end{pmatrix}.$$

We have omitted to precise the shifts in the quantum grading. The proof of the other relations is similar. □

REMARK 4.1. *This lemma can be thought as a first step in the proof of a theorem similar to Theorem 2.3 but in the case of signed regular graphs and associated bicomplexes C_n.*

Hence with the short exact sequence and Lemma 4.3 and since the homology of the Hopf link $T_{2,2}$ and the trefoil knot $T_{2,3}$ have already been computed, we can compute by induction the homology of the $(2,l)$-torus knots $T_{2,l}$ for any integer $l \geq 4$.

PROPOSITION 4.1. *The Poincaré polynomial of $\mathbb{H}_n$ for the torus knot $T_{2,l}$ is equal to*

$$\begin{aligned} P_n^{\mathbb{H}}(T_{2,2k})(u,v,q) &= [n]_q[n-1]_q q^{-1}u^{-2k}v^{2k} + q^{(n-1)(2k-1)}[n]_q \\ &\quad + \left(\sum_{i=1}^{k-1} u^{-2i}v^{2i}q^{(n-1)(2k-2i-1)}[n-1]_q\right) q^{-1}(1+u^{-1}q^{2n}), \end{aligned}$$

for $k \geq 2$ and

$$P_n^{\mathbb{H}}(T_{2,2k+1})(u,v,q) = q^{(n-1)(2k)}[n]_q + \left(\sum_{i=1}^{k} u^{-2i}v^{2i}q^{(n-1)(2k-2i)}[n-1]_q\right) q^{-1}(1+u^{-1}q^{2n}),$$

with $k \geq 1$.

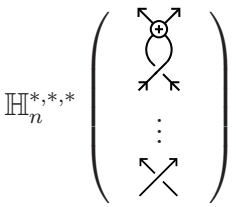
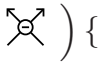
PROOF. Suppose that we can compute $\mathbb{H}_n(T_{2,k})$ for all $k \leq l$. We want to compute $\mathbb{H}_n(T_{2,l+1})$. There are two cases: l odd or l even. First consider the case l odd. Applying the short exact sequence of Lemma 4.2 we get:

$$\mathbb{H}_n^{-l-1,*,*}(T_{2,l+1}) \longrightarrow \mathbb{H}_n^{-l-1,*,*}\left(\ \vdots\ \right)\{n\}$$

$$\downarrow \partial$$

$$\mathbb{H}_n^{-l,*,*}(T_{2,l})\{n-1\} \longrightarrow \mathbb{H}_n^{-l,*,*}(T_{2,l+1}) \longrightarrow \cdots$$

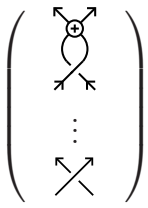
Furthermore, by repeated applications of Lemma 4.3,

$\mathbb{H}_n^{*,*,*}$  $\{n\} \cong \mathbb{H}_n^{-l-1,l+1,*}$  $\{-1\}$.

By the induction hypothesis, we have

$$\mathbb{H}_n^{-l,*,*}(T_{2,l})\{n-1\} \cong \mathbb{H}_n^{-l,l-1,*}(T_{2,l})\{n-1\}.$$

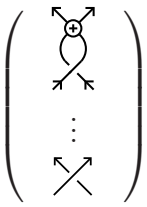
In particular, we obtain $\partial = 0$ since ∂ respects the vertical grading. This implies that we can compute the Poincaré polynomial of the $(2, l+1)$-torus knot.
Suppose now that l is odd. In this case, the short exact sequence is the following one

$\mathbb{H}_n^{-l-1,*,*}(T_{2,l+1}) \longrightarrow \mathbb{H}_n^{-l-1,*,*}$  $\{n\}$

$\downarrow \partial$

$\mathbb{H}_n^{-l,*,*}(T_{2,l})\{n-1\} \longrightarrow \mathbb{H}_n^{-l,*,*}(T_{2,l+1}) \longrightarrow \cdots$

Furthermore, we have by repeated applications of Lemma 4.3

$\mathbb{H}_n^{*,*,*}$  $\{n\} \cong \mathbb{H}_n^{-l-1,l,*}$  $\{n\}$.

By the induction hypothesis, we have

$$\mathbb{H}_n^{-l,*,*}(T_{2,l})\{n-1\} \cong \mathbb{H}_n^{-l,l,*}(T_{2,l})\{n-1\} \cong \mathbb{H}_n^{-l,l,*}$$ $\{n-2\}$.

The connecting morphism ∂ respects the quantum grading and is induced by the identity. The result follows from the following equality on graded dimensions:

$$q^n[n]_q[n-1]_q - q^{n-2}[n]_q[n-1]_q = q^{2n-1}[n-1]_q - q^{-1}[n-1]_q.$$

□

4. A polynomial invariant derived from $\mathbb{H}_n$

We derive from the Poincaré polynomial of $\mathbb{H}_n$ a polynomial invariant Q_n which satisfies a certain cubic skein relation. Furthermore, using this invariant we prove that $\mathbb{H}_2$ is a stronger invariant than HKR_2.

Consider the Laurent polynomial Q_n defined as follows. For any oriented link L, define

$$Q_n(L)(v,q) = P_n^{\mathbb{H}}(L)(-1,v,q).$$

So Q_n is the bigraded Euler characteristic of the complex of bigraded vector spaces $H(H(C_n(D),d),\delta_v)$.

THEOREM 4.2. *The Laurent polynomial Q_n satisfies the following skein relation:*

$$Q_n\left(\ \right) = q^{n-1}Q_n\left(\ \right) + v^2Q_n(\ \ ) - v^2q^{n-1}Q_n(\ \ \ ).$$

This cubic skein relation is not sufficient to evaluate Q_n on all oriented link diagram, see [**9**]. We need another local relation. This relation is said to be cubic because it involves cube of the braid group generators.

PROOF. Since Q_n is the bigraded Euler characteristic of the complex of bigraded vector spaces $H(H(C_n(D),d),\delta_v)$, we have

$$Q_n(\ \ ) = q^{-1}Q_n(\ \ ) - q^{-n}Q_n(\ \ ),$$

and

$$Q_n(\ \ ) = qQ_n(\ \ ) - q^nQ_n(\ \ ).$$

Hence, we have

$$Q_n\left(\ \right) = q^{n-1}Q_n(\ \ ) - q^nQ_n\left(\ \right)$$

By Lemma 4.3, we get

$$Q_n\left(\ \right) = q^{n-1}Q_n(\ \ ) + q^{-1}v^2Q_n(\ \ ).$$

Finally, we have

$$Q_n\left(\ \right) = q^{n-1}Q_n(\ \ ) + v^2Q_n(\ \ ) - q^{n-1}v^2Q_n(\ \ ).$$

Since Q_n is a link invariant, it is in particular invariant under the Reidemeister move RIIa. Then the previous equality is equivalent to the cubic skein relation. □

As an application, we will prove that

$$\mathbb{H}_2(8_8) \ncong \mathbb{H}_2(\overline{10_{129}}).$$

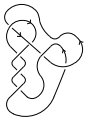
More precisely, we evaluate the difference $Q_n(8_8) - Q_n(\overline{10_{129}})$ in terms of $Q_n(6_1)$ and of an explicit Laurent polynomial in q and v. Suppose then that $\mathbb{H}_2(8_8) \cong \mathbb{H}_2(\overline{10_{129}})$, then $Q_2(8_8) - Q_2(\overline{10_{129}}) = 0$ and we get an explicit expression for $Q_2(6_1)$. After that by computing partially $\mathbb{H}_2(6_1)$, we obtain a contradiction. As a consequence,

$$\mathbb{H}_2(8_8) \ncong \mathbb{H}_2(\overline{10_{129}}),$$

whereas by [**48**] we have

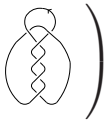
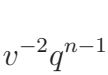
$$HKR_2(8_8) \cong HKR_2(\overline{10_{129}}).$$

Expand $Q_n(8_8)$ using the cubic skein relation:

Q_n  $= q^{n-1}Q_n$  $- q^{n-1}v^2Q_n$  $+ v^2Q_n$ 

$= q^{n-1}\Big[q^{1-n}Q_n$  $- q^{1-n}v^{-2}Q_n$ 

$+v^{-2}Q_n$  $\Big] - q^{n-1}v^2Q_n$  $+ v^2Q_n$ 

$= Q_n$  $- v^{-2}Q_n$  $+ v^{-2}q^{n-1}Q_n$ 

$-q^{n-1}v^2Q_n$  $+ v^2Q_n$  .

Expand $Q_n(6_1)$ in term of $Q_n(\overline{6_1})$:

$$
\begin{aligned}
Q_n\left(\ \right) &= Q_n\left(\ \right) \\
&= v^2 Q_n\left(\ \right) - q^{1-n}v^2 Q_n\left(\ \right) + q^{1-n} Q_n\left(\ \right) \\
&= v^2 Q_n\left(\ \right) - v^2 q^{1-n}[n]_q^2 + q^{1-n} Q_n\left(\ \right) \\
&= v^2 Q_n\left(\ \right) - v^2 q^{1-n}[n]_q^2 + q^{1-n} Q_n\left(\ \right).
\end{aligned}
$$

Hence we obtain

(30)

$$
-v^{-2}q^{1-n}Q_n\left(\ \right) + q^{1-n}[n]_q^2 = Q_n\left(\ \right) - v^{-2}Q_n\left(\ \right).
$$

Expand $Q_n(\overline{10_{129}})$:

$$
\begin{aligned}
Q_n\left(\ \right) &= Q_n\left(\ \right) \\
&= q^{1-n}Q_n\left(\ \right) - q^{1-n}v^{-2}Q_n\left(\ \right) + v^{-2}Q_n\left(\ \right) \\
&= q^{1-n}\Bigg[q^{1-n}Q_n\left(\ \right) - q^{1-n}v^{-2}Q_n\left(\ \right) \\
&\qquad v^{-2}Q_n\left(\ \right)\Bigg] - q^{1-n}v^{-2}Q_n\left(\ \right) + v^{-2}[n]_q
\end{aligned}
$$

$$
\begin{aligned}
&= q^{2(1-n)}[n]_q - v^{-2}q^{2(1-n)}\Bigg[q^{n-1}Q_n\left(\ \right) - q^{n-1}v^2Q_n\left(\ \right)\\
&\quad +v^2Q_n\left(\ \right)\Bigg] + v^{-2}[n]_q\\
&\quad +v^{-2}q^{1-n}Q_n\left(\ \right) - v^{-2}q^{1-n}Q_n\left(\ \right)\\
&= -v^{-2}q^{1-n}Q_n\left(\ \right) + q^{1-n}[n]_q^2 - q^{2(1-n)}Q_n\left(\ \right)\\
&\quad +q^{2(1-n)}[n]_q + v^{-2}[n]_q\\
&= -v^{-2}q^{1-n}Q_n\left(\ \right) + q^{1-n}[n]_q^2 - q^{2(1-n)}Q_n\left(\ \right)\\
&\quad +q^{2(1-n)}[n]_q + v^{-2}[n]_q.
\end{aligned}
$$

Applyinq equality (30), we obtain

$$
\begin{aligned}
Q_n\left(\ \right) &= -q^{2(1-n)}Q_n\left(\ \right) + q^{2(1-n)}[n]_q + v^{-2}[n]_q\\
&\quad Q_n\left(\ \right) - v^{-2}Q_n\left(\ \right).
\end{aligned}
$$

We compute the difference $Q_n(8_8) - Q_n(\overline{10_{129}})$:

$$
\begin{aligned}
&\quad Q_n\left(\ \right) - Q_n\left(\ \right)\\
&= (-q^{2(1-n)} - v^{-2})Q_n\left(\ \right) + q^{2(1-n)}[n]_q + v^{-2}[n]_q\\
&\quad +v^{-2}Q_n\left(\ \right) - v^{-2}q^{n-1}Q_n\left(\ \right) + q^{n-1}v^2Q_n\left(\ \right) - v^2Q_n\left(\ \right)
\end{aligned}
$$

Furthermore, $Q_n(5_2)$ is equal to:

$$\begin{aligned} Q_n\left(\right) &= q^{1-n}Q_n\left(\right) - v^{-2}q^{1-n}Q_n\left(\right) + v^{-2}Q_n\left(\right) \\ &= q^{1-n}\left[v^{-2}Q_n\left(\right) + q^{1-n}Q_n\left(\right) - v^{-2}q^{1-n}Q_n\left(\right)\right] \\ &\quad -v^{-2}q^{1-n}Q_n\left(\right) + v^{-2}[n]_q \\ &= -v^{-2}q^{2(1-n)}Q_n\left(\right) + (q^{2(1-n)} + v^{-2})[n]_q \end{aligned}$$

Hence we obtain

$$Q_n\left(\right) - Q_n\left(\right)$$

$$\begin{aligned} = \quad & (-q^{2(1-n)} - v^{-2})Q_n\left(\right) + q^{2(1-n)}[n]_q + v^{-2}[n]_q - (v^2q^{2(1-n)} + 1)[n]_q \\ & +(v^{-2} + q^{2(1-n)})Q_n\left(\right) - v^{-2}q^{n-1}Q_n\left(\right) + q^{n-1}v^2Q_n\left(\right) \\ = \quad & (-q^{2(1-n)} - v^{-2})Q_n\left(\right) + (v^{-2} + q^{2(1-n)})(1 - v^2)[n]_q - v^{-2}q^{2(n-1)}[n]_q \\ & +(v^{-2} + q^{2(1-n)})Q_n\left(\right) - q^{n-2}[n]_q[n-1]_q + v^2[n]_q + q^n[n]_q[n-1]_q. \end{aligned}$$

Suppose now that $Q_n(8_8) - Q_n(\overline{10_{129}}) = 0$, then

$$Q_n\left(\right) = (1 - v^2)(1 - q^{2(n-1)})[n]_q + Q_n\left(\right).$$

In particular, it follows from Example 4.3 that

$$Q_n(6_1) = Q_n\left(\right) = a_nv^{-2} + b_n + c_nv^2, \tag{31}$$

where a_n, b_n, c_n are elements of $\mathbb{Q}[q,q^{-1}]$. Consider now the bicomplex of graded vector spaces $H(H(C_n(6_1),d),\delta_v)$; it has the following form

$$\begin{array}{ccccccc}
\times & \times & \times & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & \times & \times & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & \times & \times & \times & \times \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \times & \times & \times & \times & \times
\end{array}$$

where the upper-left corner is in bidegree $(-4,4)$. We consider the $\mathbb{Q}$-vector spaces in bidegree $(-4,4)$, $(-3,4)$ and $(-2,4)$. We denote them by $V^{-4,4}$, $V^{-3,4}$ and $V^{-2,4}$. Consider the following morphisms

$$\Delta\otimes \mathrm{Id} : V_n^{\otimes 2} \longrightarrow V_n^{\otimes 3},$$
$$\mathrm{Id}\otimes \Delta : V_n^{\otimes 2} \longrightarrow V_n^{\otimes 3},$$
$$(\mathrm{Id}\otimes\tau)\circ(\Delta\otimes \mathrm{Id}) : V_n^{\otimes 2} \longrightarrow V_n^{\otimes 3}.$$

where τ is the flip. Then by construction, we have

$$V^{-4,4} = \mathrm{Coker}(\Delta\otimes \mathrm{Id}),$$

$$V^{-3,4} = \mathrm{Coker}(\Delta\otimes\mathrm{Id})\cap\mathrm{Coker}(\mathrm{Id}\otimes\Delta)\oplus\mathrm{Coker}(\Delta\otimes\mathrm{Id})\cap\mathrm{Coker}((\mathrm{Id}\otimes\tau)\circ(\Delta\otimes\mathrm{Id})),$$
$$V^{-2,4} = \mathrm{Coker}(\Delta\otimes \mathrm{Id})\cap\mathrm{Coker}(\mathrm{Id}\otimes\Delta)\cap\mathrm{Coker}((\mathrm{Id}\otimes\tau)\circ(\Delta\otimes\mathrm{Id})).$$

We restrict to the case $n=2$ and compute the dimensions:

$$\dim(\mathrm{V}^{-4,4}) = 4,$$
$$\dim(\mathrm{V}^{-3,4}) = 4,$$
$$\dim(\mathrm{V}^{-2,4}) = 1.$$

As a consequence we conclude that $\dim(\mathbb{H}_2^{*,4,*}(6_1)) > 0$ and the Laurent polynomial $Q_2(6_1)$ contains a term of the form av^4 where $a\in\mathbb{Q}[q,q^{-1}]$. We obtain a contradiction with 31. It follows that

$$\mathbb{H}_2(8_8) \ncong \mathbb{H}_2(\overline{10_{129}}).$$

5. Deformations of $\mathbb{H}_n$

Consider a polynomial $p\in\mathbb{C}[x,x^{-1}]$ of the form

$$p(x) = \sum_{k=1}^{n+1} a_{k-1}x^k.$$

Suppose for convenience that $a_n = 1$. We refine a few notations introduced before. Denote

$$\pi_{ij}^l = \frac{x_i^{l+1} - x_j^{l+1}}{x_i - x_j} \in \mathbb{C}[x_i,x_j],$$

and

$$\pi_{ijk}^l = \sum_{\substack{0 \leq a,b,c,\\ a+b+c=l-1,}} x_i^a x_j^b x_k^c \in \mathbb{C}[x_i, x_j, x_k],$$

for any integer $l \geq 1$. Set $\pi_{ij}^0 = 1$ and $\pi_{ijk}^0 = 0$.

We consider now the more general setting of filtered matrix factorizations (briefly, fmf), see [**52**] for a detailed account. Moreover we work over $\mathbb{C}$ instead of $\mathbb{Q}$. We just mention that we consider the increasing filtration associated to the grading on the polynomial algebra. In particular, define ${}^pL_j^i$ to be the following fmf:

$$R \overset{\sum_{k=0}^n a_k \pi_{ij}^k}{\longrightarrow} R\{1-n\} \overset{x_i - x_j}{\longrightarrow} R$$

where $R = \mathbb{Q}[x_i, x_j]$. Notice that in particular, d^0 and $w = p(x_i) - p(x_j)$ are not homogeneous. Moreover, d^0 and d^1 increase the filtration by $n+1$. Furthermore, similarly to the case of gmf's, we can define tensor products of fmf's. Define now two maps ${}^p\eta_0$ and ${}^p\eta_1$:

$${}^pL_k^i \otimes_{\mathbb{Q}}^p L_l^j \underset{{}^p\eta_1}{\overset{{}^p\eta_0}{\rightleftarrows}} {}^pL_l^i \otimes_{\mathbb{Q}}^p L_k^j\langle 1 \rangle$$

with ${}^p\eta_0$ given by the following pair of matrices, acting on C_n^0 and C_n^1 respectively,

$${}^p\eta_0^0 = \begin{pmatrix} \sum_{r=0}^n a_r \pi_{ikl}^r & 1 \\ -\sum_{r=0}^n a_r \pi_{jkl}^r & 1 \end{pmatrix}, \; {}^p\eta_0^1 = \begin{pmatrix} 1 & -1 \\ \sum_{r=0}^n a_r \pi_{jkl}^r & \sum_{r=0}^n a_r \pi_{ikl}^r \end{pmatrix},$$

and with ${}^p\eta_1$ given by the following pair of matrices, acting on C_n^0 and C_n^1 respectively:

$${}^p\eta_1^0 = \begin{pmatrix} 1 & -1 \\ \sum_{r=0}^n a_r \pi_{jkl}^r & \sum_{r=0}^n a_r \pi_{ikl}^r \end{pmatrix}, \; {}^p\eta_1^1 = \begin{pmatrix} \sum_{r=0}^n a_r \pi_{ikl}^r & 1 \\ -\sum_{r=0}^n a_r \pi_{jkl}^r & 1 \end{pmatrix}.$$

Replacing L_j^i by ${}^pL_j^i$, η_0 by ${}^p\eta_0$ and η_1 by ${}^p\eta_1$, we can construct for any oriented link diagram D a bicomplex $C_p(D)$ of fmf's. In particular, notice that $H(C_p(D), d)$ is bicomplex of filtered $\mathbb{C}$-vector spaces and that the differentials respect the filtration. Denote

$$\mathbb{H}_p(D) = H(H(H(C_p(D), d), {}^p\delta_v)^p\delta_h).$$

It is bigraded and filtered $\mathbb{C}$-vector space. The bigrading corresponds to the bigrading of the bicomplex and the filtration to the quantum filtration. Again, we forget the $\mathbb{Z}/2\mathbb{Z}$-grading induced by the mf's. We can now state a result similar to Theorem 3.1.

THEOREM 4.3. *For any positive integer n and any polynomial $p = \sum_{k=1}^{n+1} a_{k-1}x^k \in \mathbb{C}[x, x^{-1}]$, given an oriented link diagram D, the $\mathbb{C}$-vector space $\mathbb{H}_p(D)$ is invariant under Reidemeister moves as a finite dimensional bigraded and filtered $\mathbb{C}$-vector space up to isomorphism.*

PROOF. We point out the main ingredients in the proof of Theorem 3.1 and adapt them in this case. First, we have always simplified the bicomplex of gmf's by removing marks. This can also be done in this setting. In particular, there exist filtered versions of Lemma 3.7 and Lemma 3.8 respectively, just by replacing π_{ijk} by $\sum_{r=0}^{n} a_r \pi_{ijk}^r$ in the proof of Lemma 3.7 and V_n by ${}^pV_n = \mathbb{C}[x]/p'(x)\{1-n\}$ in Lemma 3.8. So we can also perform this simplication for $\mathbb{H}_p$. The end of proof of the invariance under RIIa is exactly the same. For Reidemeister move RI and RII, we used in the homotopy equivalence the underlying Frobenius structure of V_n. Since ${}^pV_n$ is a filtered Frobenius algebra, the proof works similarly. For the proof of invariance under Reidemeister RIII, we used the fact that $H(H(C_n(D), d), \delta_v)$ has only homology in the even homological degree. This result was a consequence of the fact that the Khovanov-Rozansky graph homology is concentrated in one of the two $\mathbb{Z}/2\mathbb{Z}$ grading. Since Wu proved a similar result for the deformations of Khovanov-Rozansky graph homology (see [**52**, Prop. 2.19]), it follows that $H(H(C_p(D), d),{}^p\delta_v)$ has only homology in the even homological degree. The rest of the proof of invariance under RIII is exactly the same. □

We have seen that the bicomplex $H(C_n(D), d)$ can be described without any matrix factorizations just using the Frobenius algebra V_n. The same is true for $H(C_p(D), d)$ using the Frobenius algebra ${}^pV_n = \mathbb{C}[x]/p'(x)\{1-n\}$. The next theorem relates $\mathbb{H}_n$ and $\mathbb{H}_p$ by a spectral sequence

THEOREM 4.4. *For any positive integer n and any polynomial $p = \sum_{k=1}^{n+1} a_{k-1}x^k \in \mathbb{C}[x, x^{-1}]$, given an oriented link diagram D, there exists a spectral sequence E_k converging to $H(H(C_p(D), d),{}^p\delta_v)$ with $E_1 = H(H(C_n(D), d), \delta_v)$.*

We do not know how this spectral sequence fit with the differentials δ_h.

PROOF. We consider the spectral sequence associated to the filtered chain complex $H(H(C_p(D), d),{}^p\delta_v)$, see [**34**] It can easily be checked that the part of the differential ${}^p\delta_v$ which respect the quantum grading is exactly δ_v. The result follows. □

We now restrict our attention to the case $p(x) = x^{n+1} - (n+1)x$ considered by Gornik [**13**]. We denote $\overline{C}_n(D) = H(C_{x^{n+1}-(n+1)x}(D), d)$, $\overline{\mathbb{H}}_n(D) = \mathbb{H}_{x^{n+1}-(n+1)x}(D)$, and $\overline{V}_n = \mathbb{C}[x]/(x^n-1)\{1-n\}$. The next theorem is similar to Gornik's result [**13**].

THEOREM 4.5. *Given a l-component oriented link $L = L_1 \cup \cdots \cup L_s$, the dimension of $\overline{\mathbb{H}}_n(L)$ equals n^s. Moreover,*

$$\overline{\mathbb{H}}_n(L) \cong \oplus_{k\in\mathbb{Z}} \overline{\mathbb{H}}_n^{-k,k}(L)$$

and to each map

$$\psi : \{\textit{components of } L\} \to [\![0, n-1]\!]$$

we assign an element a_ψ *in homological bidegree* $(-k, k)$ *where*

$$k = 2 \sum_{(\epsilon_1,\epsilon_2)\in[\![0,n-1]\!]^2,\epsilon_1\neq\epsilon_2} lk(\psi^{-1}(\epsilon_1), \psi^{-1}(\epsilon_2)).$$

The set of the a_ψ*'s generate* $\overline{\mathbb{H}}_n(L)$*.*

Proof. Consider the basis $\{f_0(x), \dots, f_{n-1}(x)\}$ of $\overline{V}_n$ introduced by Gornik [**13**]:

$$f_k(x) = \sum_{l=0}^{n-1} x^l e^{-2kl\pi i/n} \in \overline{V}_n$$

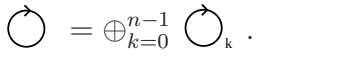
and i is the square root of -1. Moreover, the multiplication and the comultiplication of $\overline{V}_n$ act as follows on this basis [**52**]:

$$m(f_k(x) \otimes f_l(x)) = n\delta_{kl} f_k(x),$$

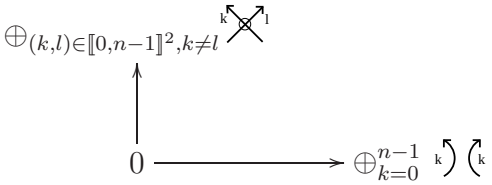
$$\Delta(f_k(x)) = ne^{-\frac{2k\pi}{n}i} f_k(x) \otimes f_k(x),$$

where $(k, l) \in [\![0, n-1]\!]^2$.

The decomposition $\overline{V}_n = \oplus_{k=0}^{n-1} \mathbb{C} f_k(x)$ can be written graphically:



In this setting, we can describe the local bicomplex associated to a positive crossing as follows:



Similarly, we can describe the bicomplex associated to a negative crossing. Notice now that for a knot K, the only resolutions that are allowed are the smoothing. Then we obtain that for a knot K, $\overline{\mathbb{H}}_n(K)$ is of dimension n and lies in bidegree $(0, 0)$. In general, for an oriented link L, any self-crossing of a component L_i has to be resolved as a smoothing, and any crossing between two components L_i and L_j has to be resolved as a smoothing if L_i and L_j are colored by k and to be resolved as a virtual crossing if L_i is colored by k and L_j by l $(k \neq l)$. Hence, the dimension of $\overline{\mathbb{H}}_n(K)$ is n^s and the generators are in one to one correspondance with maps

$$\psi : \{\text{components of } L\} \to [\![0, n-1]\!].$$

Moreover if L_i is colored by k and L_j by l, we need to resolve every crossing between L_i and L_j as a virtual crossing and by construction this add $(-r, r)$ to the homological bidegree where $r = 2\mathrm{lk}(L_i, L_j)$.

□

6. Generalization to virtual links

The construction of the triply graded link homology $\mathbb{H}_n$ generalize in the obvious way to the case of oriented virtual links. Furthermore, the homology obtained is clearly invariant under the virtual Reidemeister moves and the semi-virtual Reidemeister move. More precisely the invariance under these moves follows from Lemma 3.7.

THEOREM 4.6. *If two oriented diagram D and D' represent the same oriented virtual links, then the corresponding homology are isomorphic as triply graded $\mathbb{Q}$-vector space:*

$$\mathbb{H}_n(D) \cong \mathbb{H}_n(D')$$

One of the consequences of this categorification is that we can define now a polynomial invariant of virtual links that generalize the classical $\mathcal{S}l_n$ polynomial invariants of links, for all integer $n \geq 3$. If we still denote $P_n^{\mathbb{H}}(u, v, q)$ the Poincaré polynomial of the homology and $Q_n(v, q) = P_n^{\mathbb{H}}(-1, v, q)$ then Q_n satisfies the following relation:

$$Q_n\left(\ \right) = vq^{n-1}Q_n(\ ) - vQ_n(\ )(\ ) + q^{n-1}Q_n(\ ).$$

Appendix: Graph relations and polynomial invariants of graphs

For any positive integer n, the existence and the uniqueness of a graph polynomial P_n satisfying a set of relations is central in the categorification of Khovanov and Rozansky. It can be seen as a consequence of the existence and unicity of the $\mathfrak{sl}_n$ link polynomial invariants. Nevertheless, we give a direct proof of the fact that the graph polynomial P_n can uniquely be determined by the set of the graph relations shown on Figure 2. This proof uses a variation of Vogel's algorithm for links [**47**].

THEOREM 4.7. *For each integer $n \geq 1$, there exist a unique function*

$$P_n : \{\textit{regular planar graphs}\} \to \mathbb{Z}[q, q^{-1}]$$

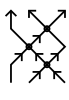
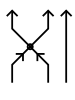
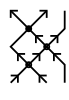
satifying the following five relations:

(32) $P_n($  $) = \dfrac{q^n - q^{-n}}{q - q^{-1}} = [n]_q,$

(33) $P_n($  $) = [n-1]_q \; P_n($  $),$

(34) $P_n\Big($  $\Big) = [2]_q \; P_n($  $),$

(35) $P_n($ $) = P_n($ $) + [n-2]_q \; P_n($ $),$

(36) $P_n\Big($  $\Big) + P_n\Big($  $\Big) = P_n\Big($  $\Big) + P_n\Big($  $\Big).$

We introduce first a few notations and we adapt different lemmas due to Vogel.

DEFINITION 4.1. *Given two oriented circle C_1 and C_2 in S^2, C_1 and C_2 are said to be* compatible *if their orientations come from one of the unique annulus $A \subset S^2$ such that $\partial A = C_1 \cup C_2$. They are said to be* incompatible *otherwise.*

Given an oriented planar regular graph Γ, consider its Seifert circles. Define $r(\Gamma)$ to be the number of Seifert circles and $h(\Gamma)$ to be the number of pairs of incompatibles Seifert circles (seen in S^2). Notice that

$$0 \leq h(\Gamma) \leq \frac{r(\Gamma)(r(\Gamma)-1)}{2}$$

DEFINITION 4.2. *A* face *f of Γ is a connected component of $\mathbb{R}^2 - \Gamma$.*

A face f is adjacent to an edge *e of Γ if $e \subset \overline{f}$. A face f is* adjacent to a Seifert circle *C of Γ if f is adjacent to at least one edge of Γ contained in C.*

A face f is a defect face *if f is adjacent to distinct edges e_1, e_2 of Γ such that the Seifert circles C_1 and C_2 of Γ going along e_1, e_2 are distinct and incompatible.*

In this case, an oriented embedded arc $c \in \mathbb{R}^2$ leading from a point of e_1 to a point of e_2 and lying (except the endpoints) in f is called a reduction arc *$c \in \mathbb{R}^2$ of Γ in f.*

LEMMA 4.4. *Given an oriented planar regular graph Γ and a reduction arc c, let Γ' and Γ''' be the two regular planar graphs obtained by* bending *(see Figure 5), then*

$$\begin{cases} r(\Gamma) = r(\Gamma''), \\ r(\Gamma) = r(\Gamma') + 1, \end{cases} \quad \textit{and} \quad \begin{cases} h(\Gamma) = h(\Gamma'') + 1, \\ h(\Gamma') < h(\Gamma). \end{cases}$$

PROOF. The relations $r(\Gamma) = r(\Gamma'')$ and $h(\Gamma) = h(\Gamma'') + 1$ are similar to the case of links, see [**47**]. Furthermore, the relation $r(\Gamma) = r(\Gamma') + 1$ is obvious and implies clearly $h(\Gamma') < h(\Gamma)$. □

LEMMA 4.5. [**47**] *An oriented regular planar graph Γ has a defect face if and only if $h(\Gamma) \neq 0$.*

PROOF. Vogel's proof works in this framework. □

Define a *regular braid graph* to be an oriented 4-valent graph obtained as the singularization of an oriented braid closure diagram.

LEMMA 4.6. [**47**] *An oriented regular planar graph Γ with $h(\Gamma) = 0$ is isotopic in $\mathbb{R}^2$ to a regular braid graph in S^2.*

PROOF. Vogel's proof works in this framework. □

PROOF OF THEOREM 4.7. The existence of such a polynomial follows from the work of Murakami, Ohtsuki and Yamada [**36**]. Let us prove the uniqueness Given a regular planar graph Γ, if there is a defect face, apply relation

$P_n($ $) = [n-2]_q \; P_n($ $) - P_n($ $).$

Then we have

$$P_n(\Gamma) = P_n(\Gamma') - [n-2]_q P_n(\Gamma''),$$

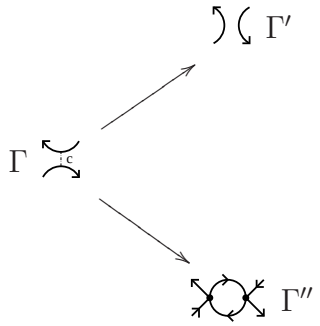


FIGURE 5. Bending

where $h(\Gamma') < h(\Gamma)$ and $h(\Gamma'') < h(\Gamma)$. Repeating this operation permits to express $P_n(\Gamma)$ as follows:

$$P_n(\Gamma) = \sum_{i \in I} q^{\alpha(\Gamma^i)} P_n(\Gamma^i),$$

where I is finite set, $\alpha(\Gamma^i)$'s are integers and Γ^i's are regular braid graph. Moreover it has been proved by Wu [**51**] and Rasmussen [**39**] that the relations (32), (33), (34) and (36) determine $P_n(\Gamma)$ when Γ is a regular braid graph. Theorem 4.7 follows. □

List of Figures

Bibliography

[1] M. Asaeda, J. H. Przytycki, and A. S. Sikora. *Categorification of the Kauffman bracket skein module of I-bundles over surfaces,* Algebr. Geom. Topol. 4 (2004), 1177–1210.

[2] B. Audoux and T. Fiedler. *A Jones polynomial for braid-like isotopies of oriented links and its categorification,* Algebr. Geom. Topol. 5 (2005), 1535–1553.

[3] D. Bar-Natan *Fast Khovanov Homology Computations,* J. Knot Theory Ramifications, 16-3 (2007), 243–255.

[4] D. Bar-Natan. *On Khovanov's categorification of the Jones polynomial,* Algebr. Geom. Topol. 2 (2002), 337–370.

[5] D. Bar-Natan. *Khovanov's homology for tangles and cobordisms,* Geom. Topol. 9 (2005), 1443–1499 (electronic).

[6] D. Bar-Natan and S. Morrison. *The Karoubi envelope and Lee's degeneration of Khovanov homology,* Algebr. Geom. Topol. 6 (2006), 1459–1469 (electronic).

[7] A. Beliakova and S. Wehrli. *Categorification of the colored Jones polynomial and Rasmussen invariant of links,* math,QA/0510382.

[8] R. H. Crowell and R. H. Fox. *Introduction to knot theory.* Reprint of the 1963 original. Graduate Texts in Mathematics, No. 57. Springer-Verlag, New York-Heidelberg, 1977. x+182 pp.

[9] M. Dąbkowski, Mieczysław and J. H. Przytycki. *Burnside obstructions to the Montesinos-Nakanishi 3-move conjecture.* Geom. Topol. 6 (2002), 355–360 (electronic).

[10] N. Dunfield, S. Gukov and J. Rasmussen *The superpolynomial for knot homologies,* Experimental Mathematics 15 (2006), 129–161.

[11] D. Eisenbud. *Homological algebra on a complete intersection, with an application to group representations.* Trans. Amer. Math. Soc. 260 (1980), no. 1, 35–64.

[12] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millet and A. Ocneanu, *A new polynomial invariant of knots and links,* Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 2, 239–246.

[13] B. Gornik. *Note on Khovanov link cohomology,* math.QA/0402266.

[14] F. Jaeger. *Composition product and models for the Homfly polynomial,* Enseign. Math. 35 (1989), 323–361.

[15] V. Jones. *A polynomial invariant for knots via von Neumann algebras,* Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 1, 103–111.

[16] C. Kassel. *Quantum Groups,* Grad. Texts in Math., vol. 155. Springer, Berlin Heidelberg New York, 1995.

[17] L. Kauffman. *On knots,* Princeton UP 1987.

[18] L. Kauffman. *State models and the Jones polynomial,* Topology 26 (1987), 395–407.

[19] L.Kauffman. *Invariants of graphs in three-space,* Trans. Amer. Math. Soc. 311 (1989), no. 2, 697–710.

[20] L. Kauffman and P. Vogel *Link polynomials and a graphical calculus,* J. Knot Theory Ramifications 1 (1992), no. 1, 59-104.

[21] K. Kawamuro. *Khovanov-Rozansky homology and the braid index of a knot,* math.GT/0707.1130.

[22] M. Khovanov. *A categorification of the Jones polynomial,* Duke Math J., 101 (2000),359-426.
[23] M. Khovanov. *sl(3) Link Homology,* Algebr. Geom. Top. 4 (2004), 1045–1081.
[24] M. Khovanov and L. Rozansky. *Matrix factorizations and link homology,* math.QA/0401268.
[25] M. Khovanov and L. Rozansky. *Matrix factorizations and link homology II,* math.QA/0505056.
[26] M. Khovanov and L. Rozansky. *Virtual crossings, convolutions and a categorification of SO(2n) Kauffman polynomial,* math.QA/0701333.
[27] J. Kock. *Frobenius algebras and 2D topological quantum field theories,* London Mathematical Society Student Texts, 59. Cambridge University Press, Cambridge, 2004. xiv+240 pp.
[28] E. S. Lee. *An endomorphism of the Khovanov invariant,* Adv. Math. 197 (2005), no. 2, 554–586.
[29] W. Lickorish. *An introduction to knot theory,* Springer 1997, 200p.
[30] A. Lobb *A slice genus lower bound from SL(n) Khovanov-Rozansky homology,* math.GT/0702393.
[31] M. Mackaay, M. Stosic, and P. Vaz. *Sl(N) link homology using foams and the Kapustin-Li formula,* math.GT/0708.2228.
[32] M. Mackaay and P. Turner. *Bar-Natan's Khovanov homology for coloured links,* Pacific J. Math. 229 (2007), no. 2, 429–446.
[33] M. Mackaay and P. Vaz. *The universal sl_3-link homology,* math.GT/0603307.
[34] J. McCleary *A user's guide to spectral sequences,* 2nd ed., Cambridge Stud. Adv. Math. 58, Cambridge Univ. Press.
[35] V. O. Manturov. *Khovanov homology for virtual links with arbitrary coefficients,* math.GT/0601152.
[36] H. Murakami,T. Ohtsuki, and S. Yamada *Homfly polynomial via an invariant of colored plane graphs,* Enseign. Math. (2) 44 (1998), no. 3-4, 325–360.
[37] J. Rasmussen. *Khovanov homology and the slice genus,* math.GT/0402131.
[38] J. Rasmussen. *Khovanov-Rozansky homology of two-bridge knots and links,* math.GT/0508510.
[39] J. Rasmussen. *Some differentials on Khovanov-Rozansky homology,* math.GT/0607544.
[40] A. Shumakovitch. *Rasmussen invariant, Slice-Bennequin inequality, and sliceness of knots,* math.GT/0411643.
[41] M. Stosíc. *Properties of Khovanov homology for positive braid knots,* math.QA/0511529.
[42] V. Turaev and P. Turner. *Unoriented topological quantum field theory and link homology,* Algebr. Geom. Topol. 6 (2006), 1069–1093 (electronic).
[43] P. Turner. *A spectral sequence for Khovanov homology with an application to (3,q)-torus links,* math.GT/0606369.
[44] P. Turner. *A remark on Rasmussen's invariant of knots,* math.GT/0509692.
[45] P .Turner. *Calculating Bar-Natan's characteristic two Khovanov homology,* J. Knot Theory Ramifications 15 (2006), no. 10, 1335–1356.
[46] O. Viro. *Khovanov homology, its definitions and ramifications,* Fund. Math. Vol 184 (2004), 317–342.
[47] P. Vogel. *Representation of links by braids, a new algorithm,* Comment. Math. Helvetici 65 (1990), 104–113.
[48] L. Watson. *Knots with identical Khovanov homology,* math.GT/0606630.
[49] B. Webster. *Khovanov-Rozansky homology via a canopolis formalism,* Algebr. Geom. Topol., 7 (2007), 673–699.
[50] S. M. Wehrli. *Khovanov Homology and Conway Mutation,* math.GT/0301312.
[51] H. Wu. *Braids, transversal knots and the Khovanov-Rozansky theory,* math.GT/0508064.
[52] H. Wu *On the quantum filtration of the Khovanov-Rozansky cohomology,* math.GT/0612406.

[53] H. Wu *The Khovanov-Rozansky Cohomology and Bennequin Inequalities,* math.GT/0703210.

[54] Y.Yonezawa. *Matrix factorizations and double line in* $\mathfrak{sl}_n$ *quantum link invariant,* math.GT/0703779.

[55] Y. Yoshino. *Cohen-Macaulay modules over Cohen-Macaulay rings,* London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990. viii+177 pp.