SUR LES SYSTEMES ELLIPTIQUES QUASI-LINEAIRES ET ANISOTROPIQUES AVEC EXPOSANTS CRITIQUES DE SOBOLEV.

Khalid Adriouch

UNIVERSITY OF LA ROCHELLE - FRANCE

Lab. de Maths & Applications

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Khalid ADRIOUCH [1]

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Supervisor :	**Abdallah El Hamidi**	Associate professor (HDR), Univ. de La Rochelle
External examinators :	**Olivier Goubet**	Professor, University of Amiens (France)
	Vicentiu Radulescu	Professor, University of Craiova (Romania)
Members of the jury :	**Samir Adly**	Professor, University of Limoges (France)
	Claude-Michel Brauner	Professor, University of Bordeaux I (France
	Mokhtar Kirane	Professor, University of La Rochelle (France)
President of the jury :	**Jean-Michel Rakotoson**	Professor, University of Poitiers (France)

[1]For further remarks or enquiries please contact me on:
kadriouc@univ-lr.fr or khalid.adriouch@gmail.com

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Contents

Introduction

The main aim of this thesis is to present some recent results concerning the existence and the multiplicity of positive solutions of systems of nonlinear elliptic differential equations involving the $(p, q)-$Laplacian operator of the following form:

$$\begin{cases} \Delta_p u &= f(x,u,v), \quad \text{in} \quad \Omega \\ \Delta_q v &= g(x,u,v), \quad \text{in} \quad \Omega \end{cases} \tag{0.1}$$

and systems of anisotropic differential equations of the form:

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) &= f(x,u,v), \quad \text{in} \quad \Omega \\ \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{q_i-2}\frac{\partial v}{\partial x_i}\right) &= g(x,u,v), \quad \text{in} \quad \Omega \end{cases} \tag{0.2}$$

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ and $\Delta_q v = \nabla \cdot (|\nabla v|^{p-2}\nabla v)$ with $p > 1$, $q > 1$, p_i, $q_i > 1$ and Ω an open nonempty subset of $\mathbb{R}^N$. Here, both f and g are functions of Caratheodory type submitted to certain growth conditions in order to guarantee that the Euler-Lagrange functional associated is well defined in a specific space.
Systems of nonlinear elliptic and anisotropic differential equations present some new and interesting phenomenons that are not present in study of a scalar equation. In general, the systems are coupled, or even strongly coupled, in the dependent variables. So, the notions of superlinearity or sublinearity, and that of criticality have to take into consideration such a coupling.
Notice that $p-$Laplacian operator appears in pure mathematics such as problems of curves as well as in applied mathematics. Indeed, it intervenes in numerous fields in experimental sciences: nonlinear reaction-diffusion problems, dynamics of populations, non-Newtonian fluids flows, flows through porous medias, nonlinear elasticity and petroleum extraction [23], torsional creep problems etc.
In literature, there exists numerous papers dedicated to the study of such equations and systems. In fact, the study of scalar equations had really started in the middle of 80s by M. Ôtani [50] in one dimension then in dimension N by F. de Thélin [58] who obtained the first results on the equation of the form: $-\Delta_p u = \lambda u^{\gamma-1}$. The last author and W. M. Ni. Serrin [57] have showed independently the existence and

the uniqueness of radial solutions in $\mathbb{R}^N$. Lately, this result has been generalized by M. Ôtani [49] to any arbitrary open subset of $\mathbb{R}^N$. In 1987, F. de Thélin [59] has extended these results to the equation of the type $\Delta_p u = g(x,u)$ where g is a function controlled by polynomial functions in u. Among the forerunners of the analysis of eigenvalue problems, one can cite G. Barles [10], S. Sakaguchi [56] and A. Anane [8], who studied the equations of the type:

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in a bounded domain } \Omega.$$

Lately in 1990, P. Lindqvist [43] established numerous results on this type of equation which follow the article of A. Anane [8]. Furthermore, there are other results on the uniqueness were stated by J. I. Dìaz and J. E. Saa [24] in 1987 for the equation $-\Delta_p u = f(x,u)$ under the condition that the function $r \mapsto \frac{f(x,u)}{r^{p-1}}$ is decreasing for every $x \in \Omega$. The bifurcation problem from the first eigenvalue has been discussed by R. F. Manásevich and M. A. del Pino [46], while the non-resonance problems involving the $p-$Laplacian was studied by A. Anane and J. P. Gossez [9]. Later, the unbounded case of these equations was studied by P. Drábek [25], P. Drábek and Y. X. Huang [26] and A. Bechah, K. Chaïb and F. de Thélin [11], where the questions of existence and uniqueness were solved for eigenvalue problem as well as for nonlinear problems.

The case of systems presents a new challenge and leads to tremendous complications related to the coupling. The variational systems could be treated by using the theory of critical points as the solutions of these systems are precisely critical points of the functional from where are originate (for more details about critical point theory, see the introduction of this thesis and the book of O. Kavian [42]). The spaces where these functionals are studied depend on boundary conditions that these solutions must satisfy. This method called *direct method of the calculus of variations* whose origins go back to Gauss and Thomson in the middle of the 19^{th} century and had been used Dirichlet and Riemann to solve the Dirichlet problem for the Laplace equation. However, there were gaps in the proof, mathematical rigour needed, as pointed out by Weierstrass in the 1870's. So this procedure had to wait until the begining of the 20th century, when D. Hilbert revived the method and put in the right tracks what was since called the *Dirichlet principle*. Today, the same sort of ideas is still used to other boundary problems for more general elliptic equations and systems. In the simpler case of Dirichlet problem for the Laplace equation, the critical point is a minimum of the associated functional. As a consequence, some new critical point theory had to be developed. Already in the 1930's, Ljusternik and Schnierelman developed a theory of critical points of the min-max type for functionals presenting a Z_2 symmetry. In the 1970's A. Ambrosetti and P. H. Rabinowitz [7] established several results on critical points of the min-max type for functionals without symmetry. The class of variational systems can be split into two classes: the class of *gradient type* if there exists a function $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

of class C^1 such that:

$$\frac{\partial F}{\partial u} = f \quad \text{et} \quad \frac{\partial F}{\partial v} = g, \tag{0.3}$$

and the class of *Hamiltonian type* if there exists a function $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ of class C^1 such that:

$$\frac{\partial H}{\partial v} = f \quad \text{and} \quad \frac{\partial H}{\partial u} = g, \tag{0.4}$$

where the functions f and g are defined as in (0.1) and (0.2).

Concerning the gradient system (0.1)-(0.3), we look for the critical points of the Euler-Lagrange functional associated

$$I(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla v|^q dx - \int_\Omega F(x,u,v)dx$$

which are the weak solutions of the system (0.1). The functional I is defined in the product space $W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We require that the functional derivatives F satisfies the following growth conditions

$$|F_u(x,u,v)| \leq C(|u|^\gamma + |u|^\alpha |v|^{\beta+1}) \quad \text{a.e. in } \Omega \tag{0.5}$$
$$|F_v(x,u,v)| \leq C(|v|^\delta + |u|^{\alpha+1} |v|^{\beta}) \quad \text{a.e. in } \Omega \tag{0.6}$$

where $p^* = \frac{Np}{N-p}$, $q^* = \frac{Nq}{N-q}$ and $1 < p, q < N$ the critical exponents of Sobolev's embeddings $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and $W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega)$. The exponents γ and δ satisfy $1 < \gamma < p^* - 1$ and $1 < \delta < q^* - 1$. Remark that when we impose that F be of class C^1 and satisfy the conditions (0.5) and (0.6) imply that I also is of class C^1.

In 1990, it was F. de Thélin [60] who first instigated the studies on systems involving the $p-$Laplacian operator where he proved a result of existence and uniqueness of the first eigenvalue to the following system

$$\begin{cases} -\Delta_p u &= \lambda |u|^{\alpha-1} u |v|^{\beta+1} \quad \text{in} \quad \Omega \\ -\Delta_q v &= \lambda |u|^{\alpha+1} |v|^{\beta-1} v \quad \text{in} \quad \Omega \end{cases}$$

under the condition $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$. The case of a variational system was discussed by P. Felmer, R. F. Manásevich and F. de Thélin [34] where they studied the existence and uniqueness of positive solutions of a variational system of the type:

$$\begin{cases} -\Delta_p u &= \dfrac{\partial H}{\partial u}(x,u,v) \quad \text{dans} \quad \Omega \\ \\ -\Delta_q v &= \dfrac{\partial H}{\partial v}(x,u,v) \quad \text{dans} \quad \Omega, \end{cases}$$

generalizing therefore the results obtained in the scalar case by J. I. Dìaz and J. E. Saa [24]. Later, these results were extended to a system deriving from a potential

by F. de Thélin and J. Vélin [62], J. Chabrowski [19] and L. Boccardo and D. G. de Figueiredo [14] and started an approach of the non-variational case by imposing growth conditions on the nonlinearities.
In the preceding, authors studied only systems on bounded domains of $\mathbb{R}^N$ while the unbounded case was discussed by J. Fleckinger, R. F. Manásevich, N. M. Stavrakakis and F. de Thélin [34] and A. Bechah, K. Chaïb and F. de Thélin [11]. Notice that the study of the systems involving the $p-$Laplacian operator in $\mathbb{R}^N$ was inspired by the general study made by M. F. Bidaut-Véron [13] on the classical Laplacians and by P. Clément, J. Fleckinger, R. F. Manásevich and F. de Thélin [21] and P. Clément, R. Manásevich and E. Mitidieri [22] who studied the existence question of the solutions for the systems $(p, q)-$Laplacien purely non-variational of the type (0.1).

Currently, several recent research papers are studying systems, particularly the four chapters of the present thesis [1, 2, 3, 4]. Notice that the historic I have drawn up here is far to be exhaustive.

One of the motivations of this thesis is the fact that certain results concerning elliptic systems involving the $p-$Laplacian either in bounded or unbounded domains deserve to be completed and so far in some papers we seemingly impose "nonnatural" conditions on the exponents to guarantee the existence or the nonexistence of the solutions.
Our second work was motivated by the difficulty to prove multiplicity results of positive solutions in the critical case in unbounded domain. The third motivation is to generalize some results obtained by T. Aubin et H. Brézis & L. Nirenberg concerning the critical level where we can affirm the compactness of the minimizing sequences of Palais-Smale type to more general scalar equations and then to elliptic systems involving the $(p, q)-$Laplacian operator. Our last work was motivated by a important recent result due to A. El Hamidi & J. M. Rakotoson [32] where they generalized the famous concentration-compactness principle of P. L. Lions to the case of anisotropic operators.

In the sequel of this concise introduction, we briefly describe the main papers figuring in this thesis.

Chapter 2: In their article [6], A. Ambrosetti, H. Brézis and G. Cerami studied the existence and multiplicity of the following equation:

$$\left\{ \begin{array}{rcll} -\Delta u & = & f_\lambda(u), & x \in \Omega \\ u & = & 0 & x \in \partial\Omega, \end{array} \right.$$

where f_λ is a function presenting a concave-convex nonlinearity for example $f_\lambda(u) = \lambda|u|^{p-1}u + |u|^{q-1}u$ under the assumption $1 < p < 2 < q < 2^*$. They showed, by using an argument of sub and super solutions, the existence of a positive solution corresponding to a small value of $0 < \lambda < \lambda^*$. Moreover, they proved the existence of infinity of solutions if the function f_λ is odd. These result was generalized and improved by A. El Hamidi [29] to the problem involving $p-$Laplacian with Dirichlet or mixed conditions on the boundary $\partial\Omega$:

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-1}u + |u|^{r-1}u, & x \in \Omega \\ \varepsilon |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + a(x)|u|^{p-2}u = 0, & x \in \partial\Omega, \end{cases} \tag{0.7}$$

with $1 < q < p < r < p^*$ and $\varepsilon \in \{0, 1\}$. Using the fibering method, introduced and developed by S. I. Pohozaev [27], A. El Hamidi [29] studied the modified functional energy $\widetilde{E}_\lambda$ defined on $\mathbb{R} \times W^{1,p}_\Gamma(\Omega)$ by:

$$\widetilde{E}_\lambda(t, u) := E_\lambda(tu),$$

where $E_\lambda(u)$ is the Euler-Lagrange functional associated with Problem (0.7). He studied the restriction E^1_λ and E^2_λ of E_λ on the Nehari manifold $\mathcal{N}_\lambda$, which is split into two disjoint subsets $\mathcal{N}^1_\lambda$ and $\mathcal{N}^2_\lambda$ as $0 < \lambda < \widehat{\lambda}$. E^1_λ and E^2_λ are both bounded from below on $\mathbb{S}$, the unit sphere of $W^{1,p}_\Gamma(\Omega)$, and then one can apply the Ekeland variational principle [28] and prove the existence of two different positive Palais-Smale sequences of $W^{1,p}_\Gamma(\Omega)$. These sequences converge to different solutions such that the first one has a negative energy however the second changes the sign of its energy at $\lambda_0 \in (0, \widehat{\lambda})$.

In the article [1], we have studied the following subcritical variational elliptic system:

$$\begin{cases} -\Delta_p u &=& \lambda |u|^{p_1-2}u + (\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1} \quad \text{dans} \quad \Omega, \\ -\Delta_q v &=& \mu |v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v \quad \text{dans} \quad \Omega, \end{cases} \tag{0.8}$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}|_{\Sigma_1} = 0, \\ v|_{\Gamma_2} = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0, \end{cases} \tag{0.9}$$

with $1 < p_1 < p < N$, $1 < \beta + 1 < q < N$, $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$ and $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$. The argument adopted here is of A. El Hamidi in his article [29]. As developed by Y. Bozhkov and E. Mitidieri [16], we introduce the modified energy functional defined by:

$$\widetilde{I}_{\lambda,\mu}(s, u, t, v) = I_{\lambda,\mu}(su, tv)$$

where $I_{\lambda,\mu}$ is Euler-Lagrange functional associated with System (0.8). Exploring the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ associated with $I_{\lambda,\mu}$ defined by all couples $(su, tv) \neq (0, 0)$ satifying $\partial\widetilde{I}_{\lambda,\mu}(s, u, t, v)/\partial s = \partial\widetilde{I}_{\lambda,\mu}(s, u, t, v)/\partial t = 0$, we proved that this manifold is also composed of two disjoint subsets $\mathcal{N}^1_{\lambda,\mu}$ and $\mathcal{N}^2_{\lambda,\mu}$ since the couple (λ,μ) belongs to a subset $\mathcal{D}$ of $\mathbb{R}^2$. The study of the restriction of the functional $I_{\lambda,\mu}$ on $\mathcal{N}^1_{\lambda,\mu}$ and $\mathcal{N}^2_{\lambda,\mu}$ showed that $I_{\lambda,\mu}$ is bounded from below et then one can prove the existence of two positive Palais-Smale belonging to Nehari manifold and converge to two different solutions of System (0.8). Furthermore, we prove that the first solution has a negative energy however there exists a continuous function $\mu \mapsto \lambda_0(\mu)$, whose graph is included in $\mathcal{D}$, such that the energy of the second solution of System (0.8) changes its signs according to the function $\lambda_0(\mu)$.

<u>**Chapter 3:**</u> In this part, we deal with System (**??**), in the unbounded case and under the condition $\frac{\alpha+1}{p^*}+\frac{\beta+1}{q^*}=1$. This step is perilous because we lose the compactness of the injections $W^{1,p}(\Omega)\subset L^{p^*}(\Omega)$ and $W^{1,q}(\Omega)\subset L^{q^*}(\Omega)$. Notice that in this case, the compactness-concentration principle of P. L. Lions [45] is not valid any longer in the unbounded domains. In the literature, this principle and the concentration-compactness principle of G. Bianchi et al. [12] are widely used to overcome the difficulty due to the loss of the compactness of the Palais-Smale sequences. Notice that Struwe decomposition [61] is useful as well in the bounded domains and compact manifolds. In the article [62], J. Vélin and F. de Thélin studied the following problem:

$$\begin{cases} -\Delta_p = u|u|^{\alpha-1}|v|^{\beta+1} & \text{in} \quad \Omega \\ \\ -\Delta_q = |u|^{\alpha+1}|v|^{\beta-1}v & \text{in} \quad \Omega. \end{cases}$$

They proved an existence result of solutions under the hypothesis $\frac{\alpha+1}{p^*}+\frac{\beta+1}{q^*}<1$ and a nonexistence result in the case $\frac{\alpha+1}{p^*}+\frac{\beta+1}{q^*}=1$ and Ω is strictly star-shaped by using Pohozaev identity [53]. Next, the following system

$$\begin{cases} -\Delta_p = u|u|^{\alpha-1}|v|^{\beta+1}+f, & \text{in} \quad \Omega \\ \\ -\Delta_q = |u|^{\alpha+1}|v|^{\beta-1}v+g, & \text{in} \quad \Omega. \\ \\ u=v=0, & \text{on} \quad \partial\Omega \end{cases}$$

with $\frac{\alpha+1}{p^*}+\frac{\beta+1}{q^*}=1$, was studied by J. Chabrowski [19] in the case when $p=q$ and lately by J. Vélin [63] if $p\neq q$. They proved the existence of at least one positive solution under the hypothesis $f\in W^{-1,p'}(\Omega)\setminus\{0\}$ and $g\in W^{-1,q'}(\Omega)\setminus\{0\}$ and $\|f\|_{-1,p'},\ \|g\|_{-1,q'}<k$. In our article [3], we were interested in the following system

$$\begin{cases} -\Delta_p u = a(x)|u|^{p_1-2}u+u|u|^{\alpha-1}|v|^{\beta+1}, & \text{in} \quad \mathbb{R}^N \\ \\ -\Delta_q v = b(x)|v|^{q-2}v+|u|^{\alpha+1}|v|^{\beta-1}v, & \text{in} \quad \mathbb{R}^N \\ \\ \lim\limits_{\|x\|\to+\infty} u(x) = \lim\limits_{\|x\|\to+\infty} v(x)=0, \end{cases}$$

where $\frac{\alpha+1}{p^*}+\frac{\beta+1}{q^*}=1$. We proved the existence of at least one solution but in the contrary of our article [1], where we have got two solutions, in this case we have lost the second solution and succeeded to recover it only for $0 \leq \mu < \mu_1$ (μ_1 is the first eigenvalue of $-\Delta_p$ in $W^{1,p}(\mathbb{R}^N)$) and λ is sufficiently close to 0. These results was obtained by using a gradient convergence theorem of minimizing sequences of Palais-Smale developed by A. El Hamidi and J. M. Rakotoson [31].

Chapter 4: One of the major difficulties in the analysis of nonlinear elliptic variational problems involving critical nonlinearities is the recovering of the compactness of Palais-Smale sequences of the Euler-Lagrange associated. This problem was discussed by H. Brézis and L. Nirenberg in their pioneering article [17]. The compactness by concentration principle due to P. L. Lions is widely used to overcome this type of problems. In the literature, there exists a lot of methods trying to recover the compactness based on the almost everywhere convergence of the gradient of the Palais-Smale sequences. One can cite for example the article of L. Boccardo and F. Murat [15], J. M. Rakotoson [55] for bounded domains and A. El Hamidi and J. M. Rakotoson [31] for arbitrary domains.

The authors in [17], studied the eigenvalue problem with a critical perturbation:

$$\begin{cases} -\Delta u &= \lambda u + u^{2^*-1}, \quad \text{in} \quad \Omega, \\ u &> 0, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{cases}$$

with Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 3$ with a smooth boundary $\partial\Omega$, $2^* = \frac{2N}{N-2}$ is the critical exponent of Sobolev injection $W^{1,2}(\Omega) \subset L^p(\Omega)$ and λ is positive parameter. The authors also introduced a critical level corresponding to the energy of the limit of the Palais-Smale sequences guaranteeing their relative compactness. Indeed, let (u_n) be a Palais-Smale sequence for corresponding the Euler-Lagrange functional:

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |\nabla u|^{2^*}.$$

They proved that if a sequence (u_n) of $(PS)_c$ satisfying:

$$c := \lim_{n\to+\infty} I_\lambda(u_n) < \frac{1}{N} S^{\frac{N}{2}}, \tag{0.10}$$

then (u_n) is relatively compact. In this case, this implies the existence of nontrivial critical points of I_λ. Here, S is the best Sobolev constant of the injection $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$. In this chapter, we give a generalization of Condition (0.10) for the following equation involving the $p-$Laplacian:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u &= \lambda f(x,u) + |u|^{p^*-2}u, \quad \text{in} \quad \Omega, \\ \\ u|_\Gamma &= 0, \quad \text{and} \quad \frac{\partial u}{\partial \nu}|_\Sigma = 0, \end{cases} \tag{0.11}$$

where Ω is a bounded domain of R^N, $N \geq 3$ with smooth boundary such that $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}$ et Γ Σ are smooth submanifolds of $\partial\Omega$ of dimension $(N-1)$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Here, $\frac{\partial}{\partial \nu}$ is the outward normal vector to Ω and f is a subcritical perturbation of $|u|^{p^*-1}$. We proved a similar condition, that if (u_n) is a $(PS)_c$ such that

$$c < c^* \equiv \inf_{u \in \mathcal{N}_0} I_0(u) + \inf_{v \in \mathcal{N}_\lambda \cup \{0\}} I_\lambda(v), \tag{0.12}$$

then (u_n) is relatively compact. The level c^* is critical because we succeeded in constructing a $(PS)_{c^*}$ but not relatively compact. $\mathcal{N}_\lambda$ and $\mathcal{N}_0$ denote respectively the Nehari manifolds relative to Problems (P_λ) and (P_0).

In the second part of this chapter, we deal with the following critical system

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\Delta_p = \lambda f(x,u) + u|u|^{\alpha-1}|v|^{\beta+1}, & \text{in } \Omega \\ \\ -\Delta_q = \mu g(x,v) + |u|^{\alpha+1}|v|^{\beta-1}v, & \text{in } \Omega \end{cases}$$

where $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$ and boundary conditions are of Dirichlet or mixted, with f and g are respectively subcritical perturbations of $|u|^{p^*-1}$ and $|v|^{q^*-1}$. We denote by $p^* = \frac{Np}{N-p}$ and $q^* = \frac{Nq}{N-q}$ the critical exponents relative to the injections de Sobolev $W^{1,p}(\Omega) \subset L^r(\Omega)$ and $W^{1,q}(\Omega) \subset L^r(\Omega)$ respectively. Our approach provides a general condition based on the properties of the Nehari manifold, which can be extended to a more general class of nonlinear elliptic systems. The optimality of our condition was established only for the special case $p = q$ and obtained by the construction of a Palais-Smale condition non relatively compact relative to the level c^*. However, the question for $p \neq q$ is still open because we ignore the maximal functions relative to Problem $(P_{0,0})$ for $p \neq q$. For more general system classes, one can refer to D. G. de Figueiredo [35], D. G. de Figueiredo and P. Felmer [36] and L. Boccardo and D. G. de Figueiredo [14].

<u>**Chapter 6:**</u> This chapter is motivated by the recent results of I. Fragalà et al. [38], C. O. Alves and A. El Hamidi [5] and A. El Hamidi and J. M. Rakotoson [30, 31] concerning the study of scalar anisotropic equation. In fact, the authors in [38] considered the scalar problem and established some existence and regularity results in the subcritical case as well as a nonexistence result in star-shaped domains.

This chapter is devoted to the study of a critical nonlinear anisotropic system of the following form:

$$
\begin{cases}
\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) & = \quad \lambda a(x)|u|^{p-2}u + u|u|^{\alpha-1}|v|^{\beta+1} \quad \text{in} \quad \Omega \\
\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{\partial v}{\partial x_i}\right) & = \quad \mu b(x)|v|^{q-2}v + |u|^{\alpha+1}|v|^{\beta-1}v \quad \text{in} \quad \Omega, \\
u = v = 0 \quad \text{on} \quad \partial\Omega
\end{cases}
\tag{0.13}
$$

with

$$
1 < p < p_i,\ 1 < q < q_i,\ \sum_{i=1}^{N} \frac{1}{p_i} > 1,\ \sum_{i=1}^{N} \frac{1}{q_i} > 1 \quad \text{et} \quad \frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1
$$

where

$$
p^* = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i} - 1}, \quad \text{et} \quad q^* = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{q_i} - 1}.
$$

Here, p^* and q^* designate respectively the effective critical exponents associated with the operators

$$
\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}\right|^{p_i-2} \frac{\partial}{\partial x_i}\right) \quad \text{and} \quad \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}\right|^{q_i-2} \frac{\partial}{\partial x_i}\right).
$$

The authors in [31, 32] generalized the concentration-compactness principle of P. L. Lions [45] in the anisotropic case. Using this generalized principle, they proved that the best Sobolev constant in the critical case is attained. In the article [4], we generalize the existence and regularity of positive solutions results obtained by C. O. Alves and A. El Hamidi [5]. By using this generalized principle, They showed that Sobolev's best constant in a certain critical case critique is attained. In our paper [4], we generalize the existence and regularity results of positive solutions obtained by C. O. Alves and A. El Hamidi [5] to the case of System (0.13) in a bounded domain of $\mathbb{R}^N$, using mini-max methods.

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Chapter 1

The Nehari manifold for systems of nonlinear elliptic equations

Abstract

This paper deals with existence and multiplicity results of nonlocal positive solutions to the following system

$$\begin{cases} -\Delta_p u &=& \lambda|u|^{p_1-2}u + (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v &=& \mu|v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v, \end{cases}$$

together with Dirichlet or mixed boundary conditions, under some hypotheses on the parameters p, p_1, α, β and q. More precisely, the system considered corresponds to a perturbed eigenvalue equation combined with a second equation having concave and convex nonlinearities. The study is based on the extraction of Palais-Smale sequences in the Nehari manifold. The behaviour of the energy corresponding to these positive solutions, with respect to the real parameters λ and μ, is established.

1.1 Introduction

In this work, we consider the system of quasilinear elliptic equations

$$\begin{cases} -\Delta_p u &=& \lambda|u|^{p_1-2}u + (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v &=& \mu|v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v, \end{cases} \tag{1.1}$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}|_{\Sigma_1} = 0, \\ v|_{\Gamma_2} = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0, \end{cases} \tag{1.2}$$

where, Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega = \overline{\Gamma}_i \cap \overline{\Sigma}_i$, where Γ_i are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that

$\Gamma_i \cap \Sigma_i = \emptyset$, $i \in \{1,2\}$. Δ_p is the p-Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. It is clear that when $\Gamma_1 = \Gamma_2 = \partial\Omega$, one deals with homogeneous Dirichlet boundary conditions.
Our aim here is to establish nonlocal existence and multiplicity results, with respect to the real parameters λ and μ, for Problem (1.1). Along this work, the following assumptions will hold

$$1 < p_1 < p < N, \quad q > 1, \quad \alpha > 1, \quad \beta > 1, \tag{1.3}$$

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1, \tag{1.4}$$

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1 \text{ and } \frac{\beta+1}{q} < 1, \tag{1.5}$$

where

$$p^* = \frac{Np}{N-p}, \quad q^* = \frac{Nq}{N-q}$$

are the critical exponents for the p-Laplacian and q-Laplacian respectively. These assumptions mean that we are concerned with a subcritical and super-homogeneous system where the first equation is *concave-convex* and the second equation is only a perturbation of an eigenvalue equation. Also, the following assumptions concerning the real parameters λ and μ will hold

$$\lambda > 0, \quad \mu < \mu_1,$$

where μ_1 is the first eigenvalue of $-\Delta_q$ in Ω.
Problem (1.1), together with (1.2), is posed in the framework of the Sobolev space $W = W^{1,p}_{\Gamma_1}(\Omega) \times W^{1,q}_{\Gamma_2}(\Omega)$, where

$$W^{1,p}_{\Gamma_1}(\Omega) = \{u \in W^{1,p}(\Omega) \;:\; u|_{\Gamma_1} = 0\}, \;\; W^{1,q}_{\Gamma_2}(\Omega) = \{u \in W^{1,q}(\Omega) \;:\; u|_{\Gamma_2} = 0\},$$

are respectively the closure of $C^1_0(\Omega \cap \Gamma_1, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$ and $C^1_0(\Omega \cap \Gamma_2, \mathbb{R})$ with respect to the norm of $W^{1,q}(\Omega)$. We can refer the reader to [9] for a complete description of this space in the case $p = 2$. Notice that $meas(\Gamma_i) > 0$, $i = 1, 2$, imply that the Poincaré inequality is still available in $W^{1,p}_{\Gamma_1}(\Omega)$ and $W^{1,q}_{\Gamma_2}(\Omega)$, so W can be endowed with the norm

$$||(u,v)|| = ||\nabla u||_p + ||\nabla v||_q$$

and $(W, ||\,.\,||)$ is a reflexive and separable Banach space.
Semilinear and quasilinear scalar elliptic equations with concave and convex nonlinearities are widely studied, we can refer the reader to [1, 4, 10, 18] and to the survey article [5]. For the nonlinear elliptic systems, we refer to [2, 3, 6, 8, 11, 14, 20, 21] and to the survey article [13]. In [15], the authors studied the existence of positive solutions to a perturbed eigenvalue problem involving the p-Laplacian operator. In

[6], the authors have generalized the results of [15] to a perturbed eigenvalue system involving p and q-Laplacian operators. Recently, in [10] the first author has considered a semilinear elliptic equation with concave and convex nolinearities, and showed *nonlocal* existence and multiplicity results with respect to the parameter via the extraction of Palais-Smale sequences in the Nehari manifold.
In this paper, we extend this method to the system (1.1) where one equation contains concave and convex nonlinearities and the other one is simply a perturbation of an eigenvalue equation. We show that Problem (1.1) has at least two positive solutions when the pair of parameters (λ, μ) belongs to a subset of $\mathbb{R}^2$ which will be specified below.
For solutions of (1.1) we understand critical points of the Euler-Lagrange functional $I \in C^1(W, \mathbb{R})$ given by

$$I(u, v) = \frac{1}{p} P(u) - \frac{\lambda}{p_1} P_1(u) + \frac{1}{q} (Q(v) - \mu Q_1(v)) - R(u, v),$$

where $P(u) = ||\nabla u||_p^p$, $P_1(u) = ||u||_{p_1}^{p_1}$, $Q(v) = ||\nabla v||_q^q$, $Q_1(v) = ||v||_q^q$ and $R(u, v) = \int_\Omega |u|^{\alpha+1} |v|^{\beta+1} dx$.
Consider the "Nehari" manifold [16] associated to Problem (1.1) given by

$$\mathcal{N} = \{(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) \;/\; D_1 I(u, v)(u) = D_2 I(u, v)(v) = 0\},$$

where $D_1 I$ and $D_2 I$ are the derivatives of I with respect to the first variable and the second variable respectively.
An interesting and useful characterization of $\mathcal{N}$, [15, 18, 22, 10, 7] is the following

$$\mathcal{N} = \{(su, tv) \;/\; (s, u, t, v) \in \mathcal{Z}^* \text{ and } \partial_s I(su, tv) = \partial_t I(su, tv) = 0\},$$

where $\mathcal{Z}^* = (\mathbb{R} \setminus \{0\}) \times (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$ and I is considered as a functional of four variables (s, u, t, v) in $\mathcal{Z} := \mathbb{R} \times W_{\Gamma_1}^{1,p}(\Omega) \times \mathbb{R} \times W_{\Gamma_2}^{1,q}(\Omega)$. For this reason, we introduce the modified Euler-Lagrange functional $\widetilde{I}$ defined on $\mathcal{Z}$ by

$$\widetilde{I}(s, u, t, v) := I(su, tv).$$

1.2 Preliminary results

In this work, we are interested by nontrivial positive solutions $u \neq 0$ and $v \neq 0$ to Problem (1.1). Since the functional $\widetilde{I}$ is even in s and t, we limit our study for $s > 0$, $t > 0$ and for $(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$.

Lemma 1.2.1. *For every $(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$ there exists a unique $\lambda(u, v) > 0$ such that the real-valued function $(s, t) \in (0, +\infty)^2 \mapsto \widetilde{I}(s, u, t, v)$ has exactly two critical points (resp. one critical point) for $0 < \lambda < \lambda(u, v)$ (resp. $\lambda = \lambda(u, v)$). This functional has no critical point for $\lambda > \lambda(u, v)$.*

Proof. Let (u,v) be an arbitrary element in $(W^{1,p}_{\Gamma_1}(\Omega)\setminus\{0\})\times(W^{1,q}_{\Gamma_2}(\Omega)\setminus\{0\})$. Then

$$\widetilde{I}(s,u,t,v)=\frac{s^p}{p}P(u)-\frac{\lambda}{p_1}s^{p_1}P_1(u)+\frac{t^q}{q}(Q(v)-\mu Q_1(v))-s^{\alpha+1}t^{\beta+1}R(u,v).$$

A direct computation gives $\partial_t\widetilde{I}(s,u,t,v)=0$ if and only if

$$t=t(s)=\left[(\beta+1)\frac{R(u,v)}{Q(v)-\mu Q_1(v)}\right]^{\frac{1}{q-(\beta+1)}}s^{\frac{\alpha+1}{q-(\beta+1)}}, \tag{2.6}$$

and

$$\widetilde{I}(s,u,t(s),v)=\frac{s^p}{p}P(u)-\frac{\lambda}{p_1}s^{p_1}P_1(u)-\frac{s^r}{r}A(u,v),$$

where

$$A(u,v)=(\alpha+1)(\beta+1)^{\frac{\alpha+1}{q-(\beta+1)}}\frac{R(u,v)^{\frac{q}{q-(\beta+1)}}}{(Q(v)-\mu Q_1(v))^{\frac{\beta+1}{q-(\beta+1)}}}$$

and $r=\frac{(\alpha+1)q}{q-(\beta+1)}$. It is easy to verify that $r>p$. Now consider the function $s\in(0,+\infty)\mapsto\widetilde{I}(s,u,t(s),v)$ and let us write

$$\partial_s\widetilde{I}(s,u,t(s),v):=s^{p_1-1}F_{\lambda,\mu}(s,u,v).$$

where $F_{\lambda,\mu}(s,u,v):=P(u)s^{p-p_1}-\lambda P_1(u)-A(u,v)s^{r-p_1}$. The function $s\in(0,+\infty)\mapsto F_{\lambda,\mu}(s,u,v)$ is increasing on $(0,\overline{s}_\mu(u,v))$, decreasing on $(\overline{s}_\mu(u,v),+\infty)$ and attains its unique maximum for $s=\overline{s}_\mu(u,v)$, where

$$\overline{s}_\mu(u,v)=\left[\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}\right]^{\frac{1}{r-p}}. \tag{2.7}$$

So, the function $s\in(0,+\infty)\mapsto F_{\lambda,\mu}(s,u,v)$ has two positive zeros (resp. one positive zero) if $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v)>0$ (resp. $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v)=0$) and has no zero if $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v)<0$. On the other hand, a direct computation leads to

$$F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v)=\frac{r-p}{r-p_1}\left[\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}\right]^{\frac{p-p_1}{r-p_1}}P(u)-\lambda P_1(u).$$

Then, $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v)>0$ (resp. $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v)<0$) if $\lambda<\lambda(u,v)$ (resp. $\lambda>\lambda(u,v)$) and $F_{\lambda(u,v),\mu}(\overline{s}_\mu(u,v),u,v)=0$, where

$$\lambda(u,v)=\widehat{c}\frac{P(u)^{\frac{r-p_1}{r-p}}}{P_1(u)A(u,v)^{\frac{p-p_1}{r-p}}}\quad\text{and}\quad\widehat{c}=\frac{r-p}{r-p_1}\left[\frac{p-p_1}{r-p_1}\right]^{\frac{p-p_1}{r-p}}. \tag{2.8}$$

Therefore, if $\lambda\in(0,\lambda(u,v))$, the function $s\in(0,+\infty)\mapsto\partial_s\widetilde{I}(s,u,t(s),v)$ has two positive zeros denoted by $s_1(u,v,\lambda,\mu)$ and $s_2(u,v,\lambda,\mu)$ verifying $0<s_1(u,v,\lambda,\mu)<$

$\overline{s}_\mu(u,v) < s_2(u,v,\lambda,\mu)$. Since $F_{\lambda,\mu}(s_1(u,v,\lambda,\mu),u,v) = F_{\lambda,\mu}(s_2(u,v,\lambda,\mu),u,v) = 0$, $\partial_s F_{\lambda,\mu}(s,u,v) > 0$ for $0 < s < \overline{s}_\mu(u,v)$ and $\partial_s F_{\lambda,\mu}(s,u,v) < 0$ for $s > \overline{s}_\mu(u,v)$ it follows that

$$\partial_{ss}\widetilde{I}(s_1(u,v,\lambda,\mu),u,t(s_1(u,v,\lambda,\mu)),v) \;>\; 0, \tag{2.9}$$
$$\partial_{ss}\widetilde{I}(s_2(u,v,\lambda,\mu),u,t(s_2(u,v,\lambda,\mu)),v) \;<\; 0. \tag{2.10}$$

This implies that the real-valued function $s \in (0,+\infty) \mapsto \widetilde{I}(s,u,t(s),v)$ achieves its unique local minimum at $s = s_1(u,v,\lambda,\mu)$ and its unique local maximum at $s = s_2(u,v,\lambda,\mu)$, which ends the proof. □

Hereafter, we will denote $t_i(u,v,\lambda,\mu) := t(s_i(u,v,\lambda,\mu))$, $i = 1, 2$. At this stage, we introduce the characteristic value

$$\widehat{\lambda}(\mu) := \inf\{\,\lambda(u,v),\ (u,v) \in (W^{1,p}_{\Gamma_1}(\Omega)\setminus\{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega)\setminus\{0\})\,\}.$$

We claim that $\widehat{\lambda}(\mu)$ is great than a positive constant which depends only on μ, p, p_1, q, α, β and Ω. Indeed, using the Hölder inequality, we get

$$R(u,v) \le |\Omega|^{\delta}||u||_{p^*}^{\alpha+1}||v||_{q^*}^{\beta+1},$$

where $\delta > 1$ is such that $\frac{1}{p^*}+\frac{1}{q^*}+\frac{1}{\delta} = 1$. Using the continuous embedding $W^{1,q}_{\Gamma_2}(\Omega) \subset L^{q^*}(\Omega)$ we get

$$A(u,v) \le c_1 \frac{P_*(u)^{\frac{r}{p^*}}}{(\mu_1-\mu)^{\frac{\beta+1}{q-(\beta+1)}}},$$

where $P_*(u) = ||u||_{p^*}^{p^*}$ and $c_1 = c_1(p,p_1,q,\alpha,\beta,\Omega)$. Using again the continuous embeddings $W^{1,p}_{\Gamma_1}(\Omega) \subset L^{p_1}(\Omega)$ and $W^{1,p}_{\Gamma_1}(\Omega) \subset L^{p^*}(\Omega)$ we obtain

$$\lambda(u,v) \ge c_2(\mu_1-\mu)^{\frac{\beta+1}{q-(\beta+1)}\frac{p-p_1}{r-p}},$$

where $c_2 = c_2(p,p_1,q,\alpha,\beta,\Omega)$ and then

$$\widehat{\lambda}(\mu) \ge c_2(\mu_1-\mu)^{\frac{\beta+1}{q-(\beta+1)}\frac{p-p_1}{r-p}},$$

which achieves the claim. Now let us introduce

$$\mathcal{D} := \{(\lambda,\mu) \in (0,+\infty)\times(-\infty,\mu_1)\ :\ \lambda < \widehat{\lambda}(\mu)\}.$$

For every $(\lambda,\mu) \in \mathcal{D}$, the functionals $(u,v) \in (W^{1,p}_{\Gamma_1}(\Omega)\setminus\{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega)\setminus\{0\}) \mapsto \widetilde{I}(s_i(u,v,\lambda,\mu),u,t_i(u,v,\lambda,\mu),v)$ $i = 1, 2$, are well defined and one can show easily that they are bounded below. Hence, for every $(\lambda,\mu) \in \mathcal{D}$, we define

$$\alpha_1(\lambda,\mu) \;:=\; \inf\{\widetilde{I}(s_1(u,v,\lambda,\mu),u,t_1(u,v,\lambda,\mu),v),\ (u,v) \in \widetilde{W}\} \tag{2.11}$$
$$\alpha_2(\lambda,\mu) \;:=\; \inf\{\widetilde{I}(s_2(u,v,\lambda,\mu),u,t_2(u,v,\lambda,\mu),v),\ (u,v) \in \widetilde{W}\} \tag{2.12}$$

where

$$\widetilde{W} := (W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times (W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\}).$$

Our aim in the sequel is to show that $\alpha_1(\lambda, \mu)$ and $\alpha_2(\lambda, \mu)$ are in fact critical values of the Euler-Lagrange functional I for every $(\lambda, \mu) \in \mathcal{D}$. We start with the following

Lemma 1.2.2. *Let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11) (resp. of (2.12)) and let $(U_n^1, V_n^1) := (s_1(u_n, v_n, \lambda, \mu)u_n, t_1(u_n, v_n, \lambda, \mu)v_n)$ (resp. $(U_n^2, V_n^2) := (s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$). Then it holds:*

$$\text{(i)} \quad \limsup_{n\to+\infty} ||(U_n^1, V_n^1)|| < \infty \quad (\textit{resp.} \limsup_{n\to+\infty} ||(U_n^2, V_n^2)|| < \infty).$$

$$\text{(ii)} \quad \liminf_{n\to+\infty} ||(U_n^1, V_n^1)|| > 0 \quad (\textit{resp.} \liminf_{n\to+\infty} ||(U_n^2, V_n^2)|| > 0).$$

Proof. We show the assertion (i), let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11). Since $\partial_s \widetilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$ and $\partial_t \widetilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$, it follows that

$$P(U_n^1) - \lambda P_1(U_n^1) - (\alpha+1)R(U_n^1, V_n^1) = 0, \tag{2.13}$$

$$Q(V_n^1) - \mu Q_1(V_n^1) - (\beta+1)R(U_n^1, V_n^1) = 0. \tag{2.14}$$

Suppose that there is a subsequence, still denoted by (U_n^1, V_n^1), such that $\lim_{n\to+\infty} ||(U_n^1, V_n^1)|| = \infty$. We will distinguish three cases:

Case a) $\lim_{n\to+\infty} ||\nabla U_n^1||_p = \infty$ and $||\nabla V_n^1||_q$ is bounded. By (2.14) we get that $R(U_n^1, V_n^1)$ is bounded. On the other hand, using the continuous embedding $W^{1,p}_{\Gamma_1}(\Omega) \subset L^{p_1}(\Omega)$, we have $P_1(U_n^1) = o_n\left(P(U_n^1)\right)$, as n goes to $+\infty$. By (2.13) we get $R(U_n^1, V_n^1) = \frac{1}{\alpha+1}(1 + o_n(1))P(U_n^1)$ as n goes to $+\infty$ and hence $\lim_{n\to+\infty} R(U_n^1, V_n^1) = +\infty$, which cannot hold true.

Case b) $\lim_{n\to+\infty} ||\nabla V_n^1||_q = \infty$ and $||\nabla U_n^1||_p$ is bounded. By (2.13) we get $R(U_n^1, V_n^1)$ bounded. If $0 < \mu < \mu_1$, using the Sobolev and Young inequalities, for every $\varepsilon \in (0, 1)$, there is a positive constant C_ε such that

$$||V_n^1||_q^q \leq \frac{\varepsilon}{\mu} ||\nabla V_n^1||_q^q + C_\varepsilon,$$

which gives $(\beta+1)R(U_n^1, V_n^1) + \mu C_\varepsilon \geq (1-\varepsilon)Q(V_n^1)$. Then $\lim_{n\to+\infty} R(U_n^1, V_n^1) = +\infty$, which is impossible. If $\mu < 0$, then $Q(V_n^1) - \mu Q_1(V_n^1) = (\beta+1)R(U_n^1, V_n^1) \geq Q(V_n^1)$ so $\lim_{n\to+\infty} R(U_n^1, V_n^1) = +\infty$, which is also impossible.

Case c) $\lim_{n\to+\infty} ||\nabla U_n^1||_p = \lim_{n\to+\infty} ||\nabla V_n^1||_q = \infty$. As in the first case, we have

$$R(U_n^1, V_n^1) = \frac{1}{\alpha+1}(1 + o_n(1))P(U_n^1), \quad \text{as } n \text{ goes to } +\infty.$$

Then $I(U_n^1, V_n^1) = \frac{1}{\alpha+1}\left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1 + o_n(1)\right)P(U_n^1)$ as n goes to $+\infty$. Hence, using the hypothese (1.5), $\lim_{n\to+\infty} I(U_n^1, V_n^1) = +\infty$, which is impossible. Consequently, $\limsup_{n\to+\infty} ||(U_n^1, V_n^1)|| < \infty$. We show in the same way that $\limsup_{n\to+\infty}$ $||(U_n^2, V_n^2)|| < \infty$.

Now, we show the assertion (ii), let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11). Suppose that there is a subsequence, still denoted by (U_n^1, V_n^1), such that $\lim_{n\to+\infty} ||(U_n^1, V_n^1)|| = 0$. By (2.13) we get $\lim_{n\to+\infty} I(U_n^1, V_n^1) = 0$ and this can not hold true because $I(U_n^1, V_n^1) < 0$ for every n.
Similarly, let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.12). Suppose that there is a subsequence, still denoted by (U_n^2, V_n^2), such that
$\lim_{n\to+\infty} ||(U_n^2, V_n^2)|| = 0$. If $p > \alpha + 1$, by (2.10) , we have

$$\partial_{ss} I(U_n^2, V_n^2) = (p-1)P(U_n^2) - \lambda(p_1 - 1)P_1(U_n^2) - \alpha(\alpha+1)R(U_n^2, V_n^2) < 0$$

Then $(p-1)P(U_n^2) - \lambda(p-1)P_1(U_n^2) - \alpha p R(U_n^2, V_n^2) < 0$, which implies that $(p-(\alpha+1))R(U_n^2, V_n^2) < 0$ and this is impossible. Finally, if $p \le \alpha+1$, then $(p-p_1)P(U_n^2) < (\alpha+1)^2 R(U_n^2, V_n^2)$. Since $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$, then there exist $\tilde{p}$ and $\tilde{q}$ satisfying $p < \tilde{p} < p^*$, $q < \tilde{q} < q^*$ and

$$\frac{\alpha+1}{\tilde{p}} + \frac{\beta+1}{\tilde{q}} = 1. \tag{2.15}$$

Therefore,

$$\begin{aligned} R(U_n^2, V_n^2) &\le c(\Omega, p, q)||U_n^2||_{\tilde{p}}^{\alpha+1}||V_n^2||_{\tilde{q}}^{\beta+1} \\ &\le c'(\Omega, p, q)||\nabla U_n^2||_p^{\alpha+1}||\nabla V_n^2||_q^{\beta+1} \end{aligned}$$

and consequently, $(p - p_1) \le c'(\Omega, p, q)(\alpha+1)^2||\nabla U_n^2||_p^{\alpha+1-p}||\nabla V_n^2||_q^{\beta+1}$ which converges to 0 as n goes to $+\infty$. This contradicts the fact $p > p_1$, which ends the proof.
□

1.3 Palais-Smale sequences in the Nehari Manifold

It is interesting to notice that for every $\gamma > 0$, $\delta > 0$, it holds

$$\begin{aligned} \widetilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \widetilde{I}(s, u, t, v), \\ \partial_t \widetilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\delta}\partial_t \widetilde{I}(s, u, t, v), \\ \partial_s \widetilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\gamma}\partial_s \widetilde{I}(s, u, t, v), \\ \partial_{ss} \widetilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\gamma^2}\partial_{ss} \widetilde{I}(s, u, t, v). \end{aligned}$$

This implies that

$$\begin{aligned}
s_1(u,v,\lambda,\mu) &= \frac{1}{\gamma}s_1\left(\frac{u}{\gamma},\frac{v}{\delta},\lambda,\mu\right), \quad \forall\,\delta>0, && (3.16)\\
s_2(u,v,\lambda,\mu) &= \frac{1}{\gamma}s_2\left(\frac{u}{\gamma},\frac{v}{\delta},\lambda,\mu\right), \quad \forall\,\delta>0, && (3.17)\\
t_1(u,v,\lambda,\mu) &= \frac{1}{\delta}t_1\left(\frac{u}{\gamma},\frac{v}{\delta},\lambda,\mu\right), \quad \forall\,\gamma>0, && (3.18)\\
t_2(u,v,\lambda,\mu) &= \frac{1}{\delta}t_2\left(\frac{u}{\gamma},\frac{v}{\delta},\lambda,\mu\right), \quad \forall\,\gamma>0. && (3.19)
\end{aligned}$$

It follows that

$$\begin{aligned}
\alpha_1(\lambda,\mu) &= \inf_{(u,v)\in\mathbb{S}_p\times\mathbb{S}_q}\{\widetilde{I}(s_1(u,v,\lambda,\mu),u,t_1(u,v,\lambda,\mu),v)\}, && (3.20)\\
\alpha_2(\lambda,\mu) &= \inf_{(u,v)\in\mathbb{S}_p\times\mathbb{S}_q}\{\widetilde{I}(s_2(u,v,\lambda,\mu),u,t_2(u,v,\lambda,\mu),v)\}, && (3.21)
\end{aligned}$$

where $\mathbb{S}_p$ and $\mathbb{S}_q$ are the unit spheres of $W^{1,p}_{\Gamma_1}(\Omega)$ and $W^{1,q}_{\Gamma_2}(\Omega)$ respectively. Make precise that $\mathbb{S}_p\times\mathbb{S}_q$ is a 2-codimensional and complete submanifold of W, we will denote it in the sequel by $\mathbb{S}$.

Lemma 1.3.1. *Let $(\lambda,\mu)\in\mathcal{D}$ and let $(u_n,v_n)\in\mathbb{S}$ be a minimizing sequence of (3.20) (resp. of (3.21)). Then $(s_1(u_n,v_n,\lambda,\mu)u_n, t_1(u_n,v_n,\lambda,\mu)v_n)$,*
(resp. $(s_2(u_n,v_n,\lambda,\mu)u_n, t_2(u_n,v_n,\lambda,\mu)v_n)$) is a Palais-Smale sequence for the functional I.

Proof. Let $(\lambda,\mu)\in\mathcal{D}$ and consider a minimizing sequence $(u_n,v_n)\in\mathbb{S}$ of (3.20). Let us set

$$\begin{aligned}
U_n &= s_1(u_n,v_n,\lambda,\mu)u_n,\\
V_n &= t_1(u_n,v_n,\lambda,\mu)v_n.
\end{aligned}$$

The sequence (U_n,V_n) is clearly bounded in W. On the other hand, the gradient (resp. the Hessian determinant) of $\widetilde{I}$ with respect to s and t at $(s,t)=(s_1(u_n,v_n,\lambda,\mu),$
$t_1(u_n,v_n,\lambda,\mu))$ is equal to zero (resp. is strictly negative). So, the implicit function theorem implies that that $s_1(u_n,v_n,\lambda,\mu)$ and $t_1(u_n,v_n,\lambda,\mu)$ are C^1 with respect to (u,v), since $\widetilde{I}$ is.
We introduce now the functional $\mathcal{I}$ defined on $\mathbb{S}$ by

$$\mathcal{I}(u,v)=\widetilde{I}(s_1(u,v,\lambda,\mu),u,t_1(u,v,\lambda,\mu),v),$$

then

$$\alpha_1(\lambda,\mu)=\inf_{(u,v)\in\mathbb{S}}\mathcal{I}(u,v)=\lim_{n\to+\infty}\mathcal{I}(u_n,v_n).$$

Applying the Ekeland variational principle [12, 17, 19, 22] on the complete manifold $(\mathbb{S}, ||\,.\,||)$ to the functional $\mathcal{I}$ we get

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) \leq \frac{1}{n}||(\varphi_n, \psi_n)||, \;\; \forall(\varphi_n, \psi_n) \in T_{(u_n,v_n)}\mathbb{S},$$

where $T_{(u_n,v_n)}\mathbb{S}$ denotes the tangent space to $\mathbb{S}$ at the point (u_n, v_n). Recall that $T_{(u_n,v_n)}\mathbb{S} = T_{u_n}\mathbb{S}_p \times T_{v_n}\mathbb{S}_q$, where $T_{u_n}\mathbb{S}_p$ (resp. $T_{v_n}\mathbb{S}_q$) is the tangent space to $\mathbb{S}_p$ (resp. $\mathbb{S}_q$) at the point u_n (resp. v_n).
Set

$$A_n := (u_n, v_n, \lambda, \mu), \text{ and } B_n := (s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n).$$

For every $(\varphi_n, \psi_n) \in T_{u_n}\mathbb{S}_p \times T_{v_n}\mathbb{S}_q$, one has

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) = D_1\widetilde{I}(B_n)(\varphi_n) + D_2\widetilde{I}(B_n)(\psi_n)$$

where

$$\begin{aligned} D_1\widetilde{I}(B_n)(\varphi_n) &= \partial_u s_1(A_n)(\varphi_n)\partial_s\widetilde{I}(B_n) + \partial_u\widetilde{I}(B_n)(\varphi_n) + \partial_u t_1(A_n)(\varphi_n)\partial_t\widetilde{I}(B_n) \\ &= \partial_u\widetilde{I}(B_n)(\varphi_n). \end{aligned}$$

Similarly, one has

$$D_2\widetilde{I}(B_n)(\psi_n) = \partial_v\widetilde{I}(B_n)(\psi_n).$$

Furthermore, consider the "fiber" maps

$$\begin{aligned} \pi \;:\; W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\} &\longrightarrow \mathbb{R} \times \mathbb{S}_p \\ u &\longmapsto \left(||\nabla u||_p, \tfrac{u}{||\nabla u||_p}\right) := (\pi_1(u), \pi_2(u)), \\ \widetilde{\pi} \;:\; W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\} &\longrightarrow \mathbb{R} \times \mathbb{S}_q \\ v &\longmapsto \left(||\nabla v||_q, \tfrac{v}{||\nabla v||_q}\right) := (\widetilde{\pi}_1(v), \widetilde{\pi}_2(v)). \end{aligned}$$

Applying the Hölder inequality we get, for every $(u, \varphi) \in (W^{1,p}_{\Gamma_1}(\Omega) \setminus \{0\}) \times W^{1,p}_{\Gamma_1}(\Omega)$ and $(v, \psi) \in (W^{1,q}_{\Gamma_2}(\Omega) \setminus \{0\}) \times W^{1,q}_{\Gamma_2}(\Omega)$, the following estimates

$$\begin{aligned} |\pi_1'(u)(\varphi)| &\leq ||\nabla\varphi||_p, \;\; |\pi_2'(u)(\varphi)| \leq 2\frac{||\nabla\varphi||_p}{||\nabla u||_p}, \\ |\widetilde{\pi}_1'(v)(\psi)| &\leq ||\nabla\psi||_q, \;\; |\widetilde{\pi}_2'(v)(\psi)| \leq 2\frac{||\nabla\psi||_q}{||\nabla v||_q}. \end{aligned}$$

On one hand, from Lemma (1.2.2), there is a positive constant K such that $s_1(A_n) \geq K$ and $t_1(A_n) \geq K$, for every integer n. On the other hand, for every $(\varphi, \psi) \in W$,

$$\begin{aligned} D_1I(U_n, V_n)(\varphi) &= \varphi_n^1\partial_s\widetilde{I}(B_n) + \partial_u\widetilde{I}(B_n)(\varphi_n^2) + \varphi_n^1\partial_t\widetilde{I}(B_n) \\ &= \partial_u\widetilde{I}(B_n)(\varphi_n^2). \end{aligned}$$

where $\varphi_n^1 = \pi_1'(u_n)(\varphi)$ and $\varphi_n^2 = \pi_2'(u_n)(\varphi)$. Then the following estimates hold true: $|\varphi_n^1| \leq ||\nabla\varphi||_p$ and $||\nabla\varphi_n^2||_p \leq \frac{2}{K}||\nabla\varphi||_p$. In the same manner, we get

$$\begin{aligned} D_2I(U_n,V_n)(\psi) &= \psi_n^1\partial_s\widetilde{I}(B_n) + \partial_v\widetilde{I}(B_n)(\psi_n^2) + \psi_n^1\partial_t\widetilde{I}(B_n) \\ &= \partial_v\widetilde{I}(B_n)(\psi_n^2). \end{aligned}$$

where $\psi_n^1 = \widetilde{\pi}_1'(v_n)(\psi)$ and $\psi_n^2 = \widetilde{\pi}_2'(v_n)(\psi)$, with the estimates $|\psi_n^1| \leq ||\nabla\psi||_q$ and $||\nabla\psi_n^2||_q \leq \frac{2}{K}||\nabla\psi||_q$. Therefore

$$\begin{aligned} |D_1I(U_n,V_n)(\varphi)| &\leq \frac{1}{n}||\nabla\varphi_n^2||_p \\ &\leq \frac{2}{nK}||\nabla\varphi||_p \end{aligned}$$

and

$$\begin{aligned} |D_2I(U_n,V_n)(\psi)| &\leq \frac{1}{n}||\nabla\psi_n^2||_q \\ &\leq \frac{2}{nK}||\nabla\psi||_q. \end{aligned}$$

We conclude easily that

$$\lim_{n\to+\infty} ||I'(U_n,V_n)||_* = 0,$$

where $I'(U_n,V_n)(\varphi,\psi) = D_1I(U_n,V_n)(\varphi) + D_2I(U_n,V_n)(\psi)$ and $||\ ||_*$ is the norm on the dual space of W.
The arguments are similar if $(u_n,v_n) \in \mathbb{S}$ is a minimizing sequence of (3.21). Hence, the lemma is proved. □

Remark. For every $(u,v) \in \widetilde{W}$ and $(\lambda,\mu) \in \mathcal{D}$, one has $\widetilde{I}(s,u,t,v) = \widetilde{I}(s,|u|,t,|v|)$, $s_i(|u|,|v|,\lambda,\mu) = s_i(u,v,\lambda,\mu)$, $i \in \{1,2\}$ and consequently $t_i(|u|,|v|,\lambda,\mu) = t_i(u,v,\lambda,\mu)$, $i \in \{1,2\}$. Therefore, every minimizing sequence $(u_n,v_n) \in \mathbb{S}_p\times\mathbb{S}_q$ of (3.20) or (3.21) can be considered as a sequence satisfying $u_n \geq 0$ and $v_n \geq 0$ in Ω.

1.4 Positive solutions and the behaviour of their energy

Theorem 1.4.1. *Let $(\lambda,\mu) \in \mathcal{D}$. Then Problem (1.1) has at least two nontrivial solutions (U^i,V^i), $i \in \{1,2\}$, such that $U^i \geq 0$ and $V^i \geq 0$ in Ω and $U^i \neq 0$, $V^i \neq 0$, for $i \in \{1,2\}$.*

Proof. We will use the notations of the previous lemmas. Let $(\lambda,\mu) \in \mathcal{D}$ and consider a nonnegative minimizing sequence $(u_n,v_n) \in \mathbb{S}$ of (3.20). It is known from Lemma (1.3.1) that

$$\begin{aligned} \lim_{n\to+\infty} I(U_n,V_n) &= \alpha_1(\lambda,\mu), \\ \lim_{n\to+\infty} ||I'(U_n,V_n)||_* &= 0 \end{aligned}$$

and that (U_n, V_n) is bounded in W. Passing if necessary to a subsequence, there are $U^1 \in W^{1,p}_{\Gamma_1}(\Omega)$ and $V^1 \in W^{1,q}_{\Gamma_2}(\Omega)$ such that

$$\begin{array}{rcl} U_n & \rightharpoonup & U^1 \ \text{ in } \ W^{1,p}_{\Gamma_1}(\Omega), \\ U_n & \to & U^1 \ \text{ in } \ L^{p_1}(\Omega) \ \text{ and } \ L^{\tilde{p}}(\Omega), \\ V_n & \rightharpoonup & V^1 \ \text{ in } \ W^{1,q}_{\Gamma_2}(\Omega), \\ V_n & \to & V^1 \ \text{ in } \ L^{q_1}(\Omega) \ \text{ and } \ L^{\tilde{q}}(\Omega), \end{array}$$

where $\tilde{p}$ and $\tilde{q}$ are specified in (2.15). At this stage, we use the well known inequalities: $\forall (x, y) \in \mathbb{R}^N$

$$\begin{array}{rcl} |x-y|^\gamma & \leq & C\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right) \cdot (x-y), \ \text{ if } \ \gamma \geq 2, \\ |x-y|^2 & \leq & C(|x|-|y|)^{2-\gamma}\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right) \cdot (x-y), \ \text{ if } \ \gamma < 2. \end{array}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^N$.
In the case $p \geq 2$, we obtain

$$\begin{array}{rcl} P(U_n - U^1) & \leq & C \displaystyle\int_\Omega \left(|\nabla U_n|^{p-2}\nabla U_n - |\nabla U^1|^{p-2}\nabla U^1\right) \cdot (\nabla U_n - \nabla U^1) \\ & = & C(D_1 I(U_n, V_n)(U_n - U^1) - D_1 I(U^1, V^1)(U_n - U^1) \ + \\ & & C\lambda \displaystyle\int_\Omega \left(|U_n|^{p_1-2}U_n - |U^1|^{p_1-2}U\right)(U_n - U^1) \ + \\ & & C(\alpha+1) \displaystyle\int_\Omega \left(U_n|U_n|^{\alpha-1}|V_n|^{\beta+1} - U^1|U^1|^{\alpha-1}|V^1|^{\beta+1}\right)(U_n - U^1). \end{array}$$

Since $\lim_{n\to+\infty} ||I'(U_n, V_n)||_* = 0$, (V_n) is bounded, and using the fact that $U_n \to U^1$ in $L^{p_1}(\Omega)$ and in $L^{\tilde{p}}(\Omega)$, $V_n \to V^1$ in in $L^{\tilde{q}}(\Omega)$, we conclude, by the Hölder inequality, that $P(U_n - U^1) \to 0$, as n goes to $+\infty$, which means that

$$U_n \longrightarrow U^1 \ \text{ in } \ W^{1,p}_{\Gamma_1}(\Omega).$$

In the case $p < 2$, a direct computation gives

$$\begin{array}{rcl} ||\nabla U_n - \nabla U^1||_p^2 & \leq & C\left(||\nabla U_n||_p^{2-p} + ||\nabla U^1||_p^{2-p}\right) \times \\ & & \displaystyle\int_\Omega \left(|\nabla U_n|^{p-2}\nabla U_n - |\nabla U^1|^{p-2}\nabla U^1\right) \cdot (\nabla U_n - \nabla U^1). \end{array}$$

Since $||\nabla U_n - \nabla U^1||_p$ is bounded, the same arguments used above show that $U_n \to U^1$ in $W^{1,p}_{\Gamma_1}(\Omega)$, as n goes to $+\infty$. In a similar way we get $V_n \to V^1$ in $W^{1,q}_{\Gamma_2}(\Omega)$, as n goes to $+\infty$.
Moreover, it is clear that (U^1, V^1) is a nontrivial solution of Problem (1.1) verifying $U^1 \geq 0$ and $V^1 \geq 0$ in Ω and $U^1 \neq 0$, $V^1 \neq 0$. On the other hand, there is a subsequence of (u_n, v_n), still denoted by (u_n, v_n) such that

$$\begin{array}{c} U_n := s_1(u_n, v_n, \lambda, \mu)u_n \longrightarrow U^1 \ \text{ in } \ W^{1,p}_{\Gamma_1}(\Omega), \\ V_n := t_1(u_n, v_n, \lambda, \mu)v_n \longrightarrow V^1 \ \text{ in } \ W^{1,q}_{\Gamma_2}(\Omega). \end{array}$$

According to Lemma (1.2.2), let $(s_1, t_1) \in (0, +\infty)^2$ such that

$$\begin{cases} s_1(u_n, v_n, \lambda, \mu) & \longrightarrow \quad s_1 \text{ in } \mathbb{R}, \\ t_1(u_n, v_n, \lambda, \mu) & \longrightarrow \quad t_1 \text{ in } \mathbb{R}, \\ u_n & \longrightarrow \quad u^1 = \frac{U^1}{s_1} \text{ in } W^{1,p}_{\Gamma_1}(\Omega), \\ v_n & \longrightarrow \quad v^1 = \frac{V^1}{t_1} \text{ in } W^{1,q}_{\Gamma_2}(\Omega), \end{cases}$$

with $u^1 = \frac{U^1}{s^1} \in \mathbb{S}_p$, $v^1 = \frac{V^1}{t^1} \in \mathbb{S}_q$, $s_1 = s_1(u^1, v^1, \lambda, \mu)$ and $t_1 = t_1(u^1, v^1, \lambda, \mu)$. Therefore, $\partial_{ss}\widetilde{I}(s_1(u^1, v^1, \lambda, \mu), u^1, t_1(u^1, v^1, \lambda, \mu), v^1) > 0$.
Proceeding in the same manner with a nonnegative minimizing sequence $(\tilde{u}_n, \tilde{v}_n) \in \mathbb{S}$ of (3.21), we obtain a second nontrivial solution (U^2, V^2) of (1.1) verifying $U^2 \geq 0$ and $V^2 \geq 0$ in Ω and $U^2 \neq 0$, $V^2 \neq 0$.
Now, we have to show that $(U^1, V^1) \neq (U^2, V^2)$. Let $(s_2, t_2) \in (0, +\infty)^2$ such that

$$\begin{cases} s_2(\tilde{u}_n, \tilde{v}_n, \lambda, \mu) & \longrightarrow \quad s_2 \text{ in } \mathbb{R}, \\ t_2(\tilde{u}_n, \tilde{v}_n, \lambda, \mu) & \longrightarrow \quad t_2 \text{ in } \mathbb{R}, \\ \tilde{u}_n & \longrightarrow \quad u^2 = \frac{U^2}{s_2} \text{ in } W^{1,p}_{\Gamma_1}(\Omega), \\ \tilde{v}_n & \longrightarrow \quad v^2 = \frac{V^2}{t_2} \text{ in } W^{1,q}_{\Gamma_2}(\Omega), \end{cases}$$

with $u^2 = \frac{U^2}{s^2} \in \mathbb{S}_p$, $v^2 = \frac{V^2}{t^2} \in \mathbb{S}_q$, $s_2 = s_2(u^2, v^2, \lambda, \mu)$ and $t_2 = t_2(u^2, v^2, \lambda, \mu)$. Therefore, $\partial_{ss}\widetilde{I}(s_2(u^2, v^2, \lambda, \mu), u^2, t_2(u^2, v^2, \lambda, \mu), v^2) < 0$. Hence $(U^1, V^1) \neq (U^2, V^2)$, which ends the proof. □

In the sequel, for every $(\lambda, \mu) \in \mathcal{D}$, the functions (u^1, v^1) and (u^2, v^2) will be denoted by $(u^1(\lambda, \mu), v^1(\lambda, \mu))$ and $(u^2(\lambda, \mu), v^2(\lambda, \mu))$ respectively. Similarly, the solutions (U^i, V^i), $i \in \{1, 2\}$, will be denoted by $(U^i(\lambda, \mu), V^i(\lambda, \mu))$, $i \in \{1, 2\}$.

Theorem 1.4.2. *Let* $(\lambda, \mu) \in \mathcal{D}$. *Then*

$$\text{(i)} \quad I(U^1, V^1) < 0 \ \text{ for } \ \lambda \in]0, \widehat{\lambda}(\mu)[,$$

$$\text{(ii)} \quad \begin{cases} I(U^2, V^2) & > 0 \quad \text{for } \lambda \in]0, \lambda_0(\mu)[, \\ I(U^2, V^2) & < 0 \quad \text{for } \lambda \in]\lambda_0(\mu), \widehat{\lambda}(\mu)[, \end{cases}$$

where

$$\lambda_0(\mu) := \frac{p_1}{r} \left(\frac{r}{p}\right)^{\frac{r-p_1}{r-p}} \widehat{\lambda}(\mu).$$

Proof. In this proof, μ will be fixed in $(-\infty, \mu_1)$, so we will omit the dependence on μ in the expressions which will follow. However, the dependece on λ will be specified. In particular, the Euler-Lagrange functional I will be denoted by I_λ.
(ii) Let (u, v) be an arbitrary element of $\widetilde{W}$. We denote

$$\widetilde{I}_\lambda(s, u, t(s), v) = \frac{s^p}{p} P(u) - \frac{\lambda}{p_1} s^{p_1} P_1(u) - \frac{s^r}{r} A(u, v),$$

and write

$$\widetilde{I}_\lambda(s, u, t(s), v) = s^{p_1} \widetilde{G}_\lambda(s, u, v),$$

where

$$\widetilde{G}_\lambda(s, u, v) = s^{p-p_1} \frac{P(u)}{p} - \lambda \frac{P_1(u)}{p_1} - s^{r-p_1} \frac{A(u, v)}{r}.$$

It follows that

$$\partial_s \widetilde{I}_\lambda(s, u, t(s), v) = p_1 s^{p_1 - 1} \widetilde{G}_\lambda(s, u, v) + s^{p_1} \partial_s \widetilde{G}_\lambda(s, u, v),$$

with

$$\partial_s \widetilde{G}_\lambda(s, u, v) = s^{p-p_1-1} \left\{ \frac{p - p_1}{p} P(u) - \frac{r - p_1}{r} s^{r-p} A(u, v) \right\}.$$

The real valued function $s \longmapsto \widetilde{G}_\lambda(s, u, v)$ is increasing on $]0, s_0(u, v)[$, decreasing on $]s_0(u, v), +\infty[$ and attains its unique maximum for $s = s_0(u, v)$, where

$$s_0(u, v) = \left(\frac{r}{p} \right)^{\frac{1}{r-p}} \overline{s}_\mu(u, v), \tag{4.22}$$

and $\overline{s}_\mu(u, v)$ is defined in (2.7). On the other hand, a direct computation gives

$$\widetilde{G}_\lambda(s_0(u, v), u, v) = \left(\frac{p - p_1}{r - p_1} \frac{P(u)}{A(u, v)} \right)^{\frac{r-p_1}{r-p}} R(u, v) - \lambda P_1(u).$$

Similarly, $\widetilde{G}_\lambda(s_0(u, v), u, v) > 0$ (resp. $\widetilde{G}_\lambda(s_0(u, v), u, v) < 0$) if $\lambda < \lambda_0(u, v)$ (resp. $\lambda > \lambda_0(u, v)$) and $\widetilde{G}_{\lambda_0(u,v)}(s_0(u, v), u, v) = 0$, where

$$\lambda_0(u, v) = \frac{p_1}{r} \left(\frac{r}{p} \right)^{\frac{r-p_1}{r-p}} \lambda(u, v), \tag{4.23}$$

with $\lambda(u, v)$ given by (2.8). Thus, we get

$$\begin{cases} \widetilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) > 0 & \text{if} \quad \lambda < \lambda_0(u, v), \\ \widetilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) = 0 & \text{if} \quad \lambda = \lambda_0(u, v), \\ \widetilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) < 0 & \text{if} \quad \lambda > \lambda_0(u, v). \end{cases} \tag{4.24}$$

First, since the function

$$\begin{array}{ccc}]0,1[& \longrightarrow & \mathbb{R} \\ t & \longmapsto & \frac{\ln t}{1-t} \end{array}$$

is increasing, then for every real numbers x, y such that $0 < x < y < 1$, one has

$$\ln\left[\frac{1}{x}\right] > \frac{1-x}{1-y}\ln\left[\frac{1}{y}\right] = \ln\left[\left(\frac{1}{y}\right)^{\frac{1-x}{1-y}}\right],$$

and consequently

$$0 < x\left(\frac{1}{y}\right)^{\frac{1-x}{1-y}} < 1.$$

In the particular case $x = p_1/r$ and $y = p/r$ we get

$$0 < \frac{p_1}{r}\left(\frac{r}{p}\right)^{\frac{r-p_1}{r-p}} < 1,$$

and therfore $0 < \lambda_0(u,v) < \lambda(u,v)$.
Moreover, for every $(u,v) \in \widetilde{W}$, one has $\widetilde{G}_{\lambda_0(u,v)}(s,u,v) < 0$ for $s \in]0,+\infty[\backslash\{s_0(u,v)\}$ and $\widetilde{G}_{\lambda_0(u,v)}(s_0(u,v),u,v) = 0$. Hence, the real valued function $s \longmapsto \widetilde{I}_{\lambda_0(u,v)}(s,u,t(s),v)$, $(s > 0)$, attains its unique maximum at $s = s_0(u,v)$ and we obtain the following interesting identity

$$s_2(u,v,\lambda_0(u,v),\mu) = s_0(u,v). \tag{4.25}$$

We will set

$$t_0(u,v) := t_2(u,v,\lambda_0(u,v),\mu).$$

On the other hand, it is clear that the functional $\lambda_0(u,v)$ is weakly lower semi-continuous on $\widetilde{W}$. Thus, the value

$$\widehat{\lambda}_0 := \inf_{(u,v)\in\widetilde{W}} \lambda_0(u,v) \tag{4.26}$$

is achieved on $\widetilde{W}$. Since $\lambda_0(u,v)$ is 0-homogeneous in u and v, we can assume that there is some $(u^*,v^*) \in \mathbb{S}_p \times \mathbb{S}_q$ such that $\widehat{\lambda}_0 = \lambda_0(u^*,v^*)$.
Now, let λ be such that $0 < \lambda < \widehat{\lambda}_0$. Then, for every $(u,v) \in \widetilde{W}$ one has $0 < \lambda < \lambda_0(u,v)$ and consequently $\widetilde{I}_\lambda(s_0(u,v),u,t(s_0(u,v)),v) > 0$ holds from (4.24). But, $s \longmapsto \widetilde{I}_\lambda(s,u,t(s),v)$, $(s > 0)$ attains its unique maximum for $s = s_2(u,v,\lambda)$, hence $\widetilde{I}_\lambda(s_2(u,v,\lambda),u,t_2(u,v,\lambda),v) > 0$, for every $(u,v) \in \widetilde{W}$. In particular, we have

$$\widetilde{I}_\lambda(s_2(u^2(\lambda),v^2(\lambda),\lambda),u^2(\lambda),t_2(u^2(\lambda),v^2(\lambda),\lambda),v^2(\lambda)) > 0,$$

i.e. $I_\lambda(U^2(\lambda),V^2(\lambda)) > 0$.

If $\lambda = \widehat{\lambda}_0$, then

$$\begin{aligned} I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) &= \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0)) \\ &= \inf_{(u,v)\in \mathbb{S}_p\times\mathbb{S}_q} \widetilde{I}_{\widehat{\lambda}_0}(s_2(u,v,\widehat{\lambda}_0), u, t_2(u,v,\widehat{\lambda}_0), v) \\ &\leq \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*,v^*), u^*, t_2(u^*,v^*), v^*) \\ &= \widetilde{I}_{\lambda_0(u^*,v^*)}(s_0(u^*,v^*), u^*, t_0(u^*,v^*), v^*) \\ &= 0 \end{aligned}$$

which implies that $I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) \leq 0$. In addition, it is known from (4.24) that

$$\begin{aligned} \widetilde{I}_{\widehat{\lambda}_0}(s_0(u,v), u, t_0(u,v), v) &\geq 0, \\ \widetilde{I}_{\widehat{\lambda}_0}(s_1(u,v,\widehat{\lambda}_0), u, t_1(u,v,\widehat{\lambda}_0), v) &< 0, \end{aligned}$$

for every $(u,v) \in \widetilde{W}$. Then

$$s_0(u,v) > s_1(u,v,\widehat{\lambda}_0), \quad \forall (u,v) \in \widetilde{W}.$$

It follows that

$$\begin{aligned} &\widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0)) \geq \\ &\quad \widetilde{I}_{\widehat{\lambda}_0}(s_0(u^1(\widehat{\lambda}_0), v^1(\widehat{\lambda}_0)), u^1(\widehat{\lambda}_0), t_0(u^1(\widehat{\lambda}_0), v^1(\widehat{\lambda}_0)), v^1(\widehat{\lambda}_0)) \geq 0. \end{aligned}$$

Hence,

$$\begin{aligned} I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) &= \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0)) \\ &= 0. \end{aligned}$$

Finally, assume that $\widehat{\lambda}_0 < \lambda < \widehat{\lambda}$. Since, for every $s \in]0, +\infty[$ and $(u,v) \in \widetilde{W}$, the real valued function $\lambda \longmapsto \widetilde{I}_\lambda(s,u,t(s),v)$ is decreasing, it follows that

$$\widetilde{I}_\lambda(s,u,t(s),v) < \widetilde{I}_{\widehat{\lambda}_0}(s,u,t(s),v), \quad \text{for every } s > 0 \text{ and } (u,v) \in \widetilde{W}. \tag{4.27}$$

In addition, we have

$$\begin{aligned} \widetilde{I}_\lambda(s_2(u^2(\lambda), v^2(\lambda), \lambda), u^2(\lambda), t_2(u^2(\lambda), v^2(\lambda), \lambda), v^2(\lambda)) &= \\ \inf_{(u,v)\in \mathbb{S}_p\times\mathbb{S}_q} \widetilde{I}_\lambda(s_2(u,v,\lambda), u, t_2(u,v,\lambda), v) &\leq \\ \widetilde{I}_\lambda(s_2(u^*,v^*,\lambda), u^*, t_2(u^*,v^*,\lambda), v^*) &< \\ \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*,v^*,\lambda), u^*, t_2(u^*,v^*,\lambda), v^*) & \end{aligned}$$

where the last inequality follows from (4.27). Moreover, the real valued function $s \longmapsto \widetilde{I}_{\widehat{\lambda}_0}(s, u^*, t(s), v^*)$, $(s > 0)$, achieves its unique maximum at $s = s_0(u^*, v^*)$. Thus,

$$\begin{aligned}
\widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*, v^*, \lambda), u^*, t_2(u^*, v^*, \lambda), v^*) &\leq \widetilde{I}_{\widehat{\lambda}_0}(s_0(u^*, v^*), u^*, t_0(u^*, v^*), v^*)\\
&= \widetilde{I}_{\lambda_0(u^*,v^*)}(s_0(u^*, v^*), u^*, t_0(u^*, v^*), v^*)\\
&= 0.
\end{aligned}$$

Hence $\widetilde{I}_{\lambda}(s_2(u^2(\lambda), v^2(\lambda), \lambda), u^2(\lambda), t_2(u^2(\lambda), v^2(\lambda), \lambda), v^2(\lambda)) < 0$, which ends the proof. □

The following result shows the subtle link existing between the characteristic value $\widehat{\lambda}_0$ defined by (4.26) and Problem (1.1).

Theorem 1.4.3. *If (u, v) is a solution of (4.26) then $(s_0(u, v)u, t_0(u, v)v)$ is a solution of the system (1.1) when $\lambda = \widehat{\lambda}_0$.*

Proof. Let (u, v) be a solution of (4.26). In order to simplify the notations, we set $U := s_0(u, v)u$ and $V := t_0(u, v)v$. Thus, for $\lambda = \widehat{\lambda}_0 = \lambda_0(u, v)$ we have:

$$I_{\widehat{\lambda}_0,\mu}(U, V) = \frac{s_0(u, v)^p}{p}P(u) - \widehat{\lambda}_0\frac{s_0(u, v)^{p_1}}{p_1}P_1(u) - \frac{s_0(u, v)^r}{r}A(u, v)$$

and for every $\varphi \in W_0^{1,p}(\Omega)$:

$$D_1I_{\widehat{\lambda}_0,\mu}(U, V)(\varphi) = \frac{1}{p}P'(U)(\varphi) - \frac{\widehat{\lambda}_0}{p_1}P_1'(U)(\varphi) - \frac{1}{r}D_1A(U, V)(\varphi),$$

where

$$\begin{cases}
P'(U)(\varphi) &= s_0(u, v)^{p-1}P'(u)(\varphi),\\
P_1'(U)(\varphi) &= s_0(u, v)^{p_1-1}P_1'(u)(\varphi),\\
D_1A(U, V)(\varphi) &= s_0(u, v)^{r-1}D_1A(u, v)(\varphi).
\end{cases}$$

We calculate now,

$$\begin{aligned}
\widehat{\lambda}_0P_1'(U)(\varphi) &= \lambda_0(u, v)s_0(u, v)^{p_1-1}P_1'(u)(\varphi)\\
&= \frac{p_1}{r}\left(\frac{r}{p}\right)^{\frac{r-p_1}{r-p}}\left(\frac{p-p_1}{r-p_1}\right)^{\frac{p-p_1}{r-p}}\frac{P(u)}{P_1(u)}\left(\frac{P(u)}{A(u,v)}\right)^{\frac{p-1}{r-p}}\\
&\times\ \frac{r-p}{r-p_1}\left(\frac{r}{p}\right)^{\frac{p_1-1}{r-p}}\left(\frac{p-p_1}{r-p_1}\frac{P(u)}{A((u,v)}\right)^{\frac{p_1-1}{r-p}}P_1'(u)(\varphi)\\
&= \frac{r-p}{r-p_1}\frac{p_1}{r}\frac{r}{p}\left(\frac{p-p_1}{r-p_1}\right)^{\frac{p-1}{r-p}}\left(\frac{r}{p}\right)^{\frac{p-1}{r-p}}\frac{P(u)}{P_1(u)}\frac{P(u)}{A(u,v)}^{\frac{p-1}{r-p}}P_1'(u)(\varphi)\\
&= \frac{r-p}{r-p_1}\frac{p_1}{p}\frac{P(u)}{P_1(u)}\left(\left(\frac{r}{p}\right)^{\frac{1}{r-p}}\left(\frac{p-p_1}{r-p_1}\right)^{\frac{1}{r-p}}\frac{P(u)}{A(u,v)}\right)^{p-1}P_1'(u)(\varphi)\\
&= \frac{p_1}{p}\frac{r-p}{r-p_1}P(u)s_0(u, v)^{p-1}\frac{P_1'(u)(\varphi)}{P_1(u)}.
\end{aligned}$$

In addition, one has

$$\begin{aligned} D_1A(U,V)(\varphi) &= s_0(u,v)^{r-1}D_1(u,v)(\varphi) \\ &= \left(\frac{r}{p}\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}\right)^{\frac{p-1}{r-p}}\frac{r}{p}\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}D_1A(u,v)(\varphi) \\ &= \frac{r}{p}\frac{p-p_1}{r-p_1}P(u)s_0(u,v)^{p-1}\frac{D_1A(u,v)(\varphi)}{A(u,v)}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} D_1I_{\widehat{\lambda}_0,\mu}(U,V)(\varphi) &= \left[\frac{P'(u)(\varphi)}{P(u)} - \frac{r-p}{r-p_1}\frac{P_1'(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p_1}\frac{D_1A(u,v)(\varphi)}{A(u,v)}\right] \\ &\times \frac{P(u)s_0(u,v)^{p-1}}{p} \\ &= K\left(\frac{r-p_1}{r-p}\frac{P'(u)(\varphi)}{P(u)} - \frac{P_1'(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p}\frac{D_1A(u,v)(\varphi)}{A(u,v)}\right), \end{aligned}$$

where $K := \frac{r-p}{r-p_1}\frac{P(u)}{p}s_0(u,v)^{p-1}$. On the other hand, a direct computation gives:

$$D_1\lambda_0(u,v)(\varphi) = \widehat{\lambda}_0\left(\frac{r-p_1}{r-p}\frac{P'(u)(\varphi)}{P(u)} - \frac{P_1'(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p}\frac{D_1A(u,v)(\varphi)}{A(u,v)}\right),$$

which is equal to zero by assumption. Hence $D_1I_{\widehat{\lambda}_0,\mu}(U,V)(\varphi) = 0$ since it is proportional to $D_1\lambda_0(u,v)(\varphi)$.
Moreover, for every $\psi \in W_0^{1,q}(\Omega)$, we get

$$D_2\lambda_0(u,v)(\psi) = -\frac{p-p_1}{r-p_1}\lambda_0(u,v)\frac{D_2A(u,v)(\psi)}{A(u,v)},$$

which is also equal to zero by assumption. This implies that $D_2A(u,v)(\psi) = 0$, since $\lambda_0(u,v) = \widehat{\lambda}_0 \neq 0$. Then

$$D_2I_{\widehat{\lambda}_0,\mu}(U,V)(\psi) = -\frac{s_0(u,v)^r}{r}D_2A(u,v)(\psi) = 0.$$

which implies that $(s_0(u,v)u, t_0(u,v)v)$ is well a solution of the problem (1.1) with $\lambda = \widehat{\lambda}_0$. □

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Chapter 2

Nehari manifold for a critical system in $\mathbb{R}^N$

Abstract

In this paper, we are interested in existence and multiplicity results of non local solutions to the following critical system:

$$\begin{cases} -\Delta_p u &= \lambda a(x)|u|^{p_1-2}u + u|u|^{\alpha-1}|v|^{\beta+1} \ \textit{in} \ \mathbb{R}^N, \\ -\Delta_q v &= \mu b(x)|v|^{q-2}v + |u|^{\alpha+1}|v|^{\beta-1}v \ \ \textit{in} \ \ \mathbb{R}^N, \end{cases}$$

under some conditions for the parameters a, b, p, p_1, α, β, q, λ *and* μ *in the critical case:* $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$. *We show these results by developing variational tools. The study consists in the extraction of Palais-Smale sequences in the Nehari manifold. A compactness principle due to A. El Hamidi and J-M. Rokotoson allows us to obtain convergence results for the gradients in our unbounded case.*

2.1 Introduction

We consider the system of quasilinear elliptic equations:

$$\begin{cases} -\Delta_p u &= \lambda a(x)|u|^{p_1-2}u + u|u|^{\alpha-1}|v|^{\beta+1} \ \text{in} \ \mathbb{R}^N, \\ -\Delta_q v &= \mu b(x)|v|^{q-2}v + |u|^{\alpha+1}|v|^{\beta-1}v \ \ \text{in} \ \mathbb{R}^N, \end{cases} \tag{1.1}$$

We are interested in establishing nonlocal existence and multiplicity results for Problem (1.1). Of course *nonlocal* solutions means *with respect to the real parameters* λ *and* μ. Throughout this paper, the following assumptions will hold:

$$a \geq 0,\ a \not\equiv 0,\ a \in L^{\frac{p^*}{p^*-p_1}}(\mathbb{R}^N) \text{ and } b \geq 0,\ b \not\equiv 0,\ b \in L^{\frac{N}{q}}(\mathbb{R}^N), \tag{1.2}$$

$$1 < p_1 < p < N,\ \ 1 < q < N,\ \ \alpha > -1,\ \ \beta > -1, \tag{1.3}$$

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1, \tag{1.4}$$

$$q > \beta + 1, \tag{1.5}$$

where

$$p^* = \frac{Np}{N-p}, \quad q^* = \frac{Nq}{N-q},$$

are the critical exponents for the p-Laplacian and q-Laplacian respectively. These assumptions mean that we are concerned with a critical system where the first equation is *concave-convex* and the second equation is only a critical perturbation of an eigenvalue equation. Also, the following assumptions concerning the real parameters λ and μ will hold

$$\lambda > 0, \quad \mu < \mu_1,$$

where μ_1 is the first eigenvalue of the equation

$$-\Delta_q v = \mu b(x) v|v|^{q-2} \text{ in } \mathbb{R}^N.$$

Thus

$$\mu_1 = \inf_{\psi \in \mathcal{D}^{1,q}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla \psi|^q \, dx}{\int_{\mathbb{R}^N} b(x)|\psi|^q \, dx},$$

where the space $\mathcal{D}^{1,q}(\mathbb{R}^N)$ is the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,q}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^q dx \right)^{\frac{1}{q}}.$$

One can prove that $\mu_1 > 0$ and μ_1 is achieved. Indeed, on one hand, by integrability of b, we claim that the functional

$$\begin{array}{rcccl} Q_b & : & \mathcal{D}^{1,q}(\mathbb{R}^N) & \longrightarrow & \mathbb{R} \\ & & v & \longmapsto & \displaystyle\int_{\mathbb{R}^N} b(x)|v|^q \, dx \cdot \end{array}$$

is weakly continuous. It is clear that the functional Q_b is well defined since q^*/q and N/q are conjugate exponents. Now, let $u_n \rightharpoonup u$ in $\mathcal{D}^{1,q}(\mathbb{R}^N)$ weakly. We are going to prove that $|u_n|^q \rightharpoonup |u|^q$ in $L^{q^*/q}(\mathbb{R}^N)$. Since $\| \, |u_n|^q \, \|_{q^*/q} = \|u_n\|_{q^*}^q$ is bounded we can assume, up to a subsequence, that $|u_n|^q \rightharpoonup v$ in $L^{q^*/q}(\mathbb{R}^N)$. The claim is complete if we show that $v = |u|^q$ because then the limit does not depend of the subsequence. Choose any increasing sequence $(K_n)_{n\in\mathbb{N}}$ of open relatively subsets, with regular boundaries, of $\mathbb{R}^N$ covering $\mathbb{R}^N$: $\mathbb{R}^N = \cup_{n=0}^{\infty} K_n$. By using the compact/continuous embeddings

$$\mathcal{D}^{1,q}(K_n) \hookrightarrow L^q(K_n) \subset L^1(K_n)$$

$$u_n \rightharpoonup u \Longrightarrow u_n \longrightarrow u \Longrightarrow |u_n|^q \longrightarrow |u|^q$$

and

$$L^{q^*/q}(K_n) \subset L^1(K_n)$$

$$|u_n|^q \rightharpoonup v \Longrightarrow |u_n|^q \rightharpoonup v.$$

Thus, $v = |u|^q$ *a.e.* on each K_n. Using the diagonal process of Cantor, we conclude that $v = |u|^q$ *a.e.* in $\mathbb{R}^N$ and the claim is achieved.
On the other hand, let (ψ_n) be a minimizing nonnegative sequence of μ_1, (with $Q_b(\psi_n) = 1$, which is possible by homogeneity arguments), there is a nonnegative function $\psi \in \mathcal{D}^{1,q}(\mathbb{R}^N)$ such that, up to a subsequence,

$$\psi_n \rightharpoonup \psi \text{ in } \mathcal{D}^{1,q}(\mathbb{R}^N) \quad \text{weakly.}$$

Using the claim proved above, we get as $n \to +\infty$

$$Q_b(\psi_n) \to Q_b(\psi) = 1.$$

But,

$$\int_{\mathbb{R}^N} |\nabla \psi|^q \, dx \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla \psi_n|^q \, dx = \mu_1$$

then μ_1 is acicheied by ψ. Finally, suppose that $\mu_1 = 0$, then $\int_{\mathbb{R}} |\nabla \psi|^q \, dx = 0$ wich implies that ψ is a constant function which is positive since $Q_b(\psi) = 1$. But positive constant functions do not belong to $\mathcal{D}^{1,q}(\mathbb{R}^N)$. Notice that ψ satisfies, in the weak sense, the equation

$$-\Delta_q \psi = \mu_1 b(x) \psi |\psi|^{q-2} \text{ in } \mathbb{R}^N.$$

We denote by S_p (resp. S_q) the best Sobolev's constant for the continuous embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ (resp. $\mathcal{D}^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$).
Problem (1.1) is well posed in the framework of the space $W := \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathcal{D}^{1,q}(\mathbb{R}^N)$, where

$$\begin{aligned} \mathcal{D}^{1,p}(\mathbb{R}^N) &= \{u \in L^{p^*}(\mathbb{R}^N) \ : \ |\nabla u| \in L^p(\mathbb{R}^N)\}, \\ \mathcal{D}^{1,q}(\mathbb{R}^N) &= \{v \in L^{q^*}(\mathbb{R}^N) \ : \ |\nabla v| \in L^q(\mathbb{R}^N)\}, \end{aligned}$$

which are, as mentioned above, respectively the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norms of

$$\begin{aligned} \|u\|_{1,p} : &= \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{1}{p}}, \\ \|v\|_{1,q} : &= \left(\int_{\mathbb{R}^N} |\nabla u|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

The space W is endowed by the following norm:

$$\|(u, v)\| = \|u\|_{1,p} + \|v\|_{1,q}$$

which gives to $(W, \|.\|)$ Banach space properties, reflexivity and separability.

For solutions of (1.1) we mean critical points of the Euler-Lagrange functional $I \in C^1(W,\mathbb{R})$ given by

$$I(u,v) := (\alpha+1)\left(\frac{1}{p}P(u) - \frac{\lambda}{p_1}P_{1,a}(u)\right) + \frac{\beta+1}{q}\{Q(v) - \mu Q_b(v)\} - R(u,v),$$

where

$$\begin{aligned} P(u) &= \|u\|_p^p, \quad P_{1,a}(u) = \int_{\mathbb{R}^N} a(x)|u|^{p_1}dx, \\ Q(v) &= \|v\|_q^q, \quad Q_b(v) = \int_{\mathbb{R}^N} b(x)|v|^q dx, \\ R(u,v) &= \int_{\mathbb{R}^N} |u|^{\alpha+1}|v|^{\beta+1}dx. \end{aligned}$$

Remark that the functional I is bounded neither above nor below on W. For this reason we introduce the Nehari manifold corresponding to I, which contains all critical points of I and on which I is bounded below, as we will see in the sequel. For each $(u,v) \in (\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\})\times(\mathcal{D}^{1,q}(\mathbb{R}^N)\setminus\{0\})$, the Nehari manifold associated to the functional I is defined by

$$\mathcal{N}_{\lambda,\mu} := \{(u,v) \in (\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}) \times (\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}) : I'(u,v)(u,v) = 0\}.$$

This manifold can be characterized as follows

$$\mathcal{N}_{\lambda,\mu} = \{(s,u,t,v) \in \mathcal{Z}^* : \ \partial_s I(su,tv) = 0 \ \text{ and } \ \partial_t I(su,tv) = 0\},$$

where $\mathcal{Z}^* = (\mathbb{R}\setminus\{0\}) \times \left(\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}\right) \times (\mathbb{R}\setminus\{0\}) \times \left(\mathcal{D}^{1,q}(\mathbb{R}^N)\setminus\{0\}\right)$ and I is considered as a functional of four variables $(s,u,t,v) \in \mathcal{Z} := \mathbb{R}\times\mathcal{D}^{1,p}(\mathbb{R}^N)\times\mathbb{R}\times\mathcal{D}^{1,q}(\mathbb{R}^N)$. This suggests the introduction of the modified Euler-Lagrange functional $\widetilde{I}$ defined on $\mathcal{Z}$ by

$$\widetilde{I}(s,u,t,v) := I(su,tv).$$

2.2 Some properties of minimizing sequences

Troughout this paper, we will be interested by positive solutions; $u > 0$ and $v > 0$ to the problem (1.1). As the functional $\widetilde{I}$ is even in s and t we can limit our study for $s > 0$, $t > 0$ and $(u,v) \in \widetilde{W} := (\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N)\setminus\{0\})$. In the first lemma, we establish some properties of the modified functional $(s,t) \mapsto \widetilde{I}(s,.,t,.)$:

Lemma 2.2.1. *For every $(u,v) \in \widetilde{W}$, and for all $\mu < \mu_1$ there exists a unique $\lambda(u,v,\mu) > 0$ such that the real-valued function $(s,t) \in (0,+\infty)^2 \mapsto \widetilde{I}(s,u,t,v)$ has exactly two critical points (resp. one critical point) for $0 < \lambda < \lambda(u,v,\mu)$ (resp. $\lambda = \lambda(u,v,\mu)$). This functional has no critical point for $\lambda > \lambda(u,v,\mu)$.*

Proof. Let $(u,v) \in \widetilde{W}$, $(s,t) \in (0,+\infty)^2$ and $\mu < \mu_1$ be arbitrary elements. We have

$$\begin{aligned} \widetilde{I}(s,u,t,v) &= (\alpha+1)\left(\frac{s^p}{p}P(u) - \frac{\lambda}{p_1}s^{p_1}P_{1,a}(u)\right) - s^{\alpha+1}t^{\beta+1}R(u,v) \\ &+ (\beta+1)\frac{t^q}{q}\left(Q(v) - \mu Q_b(v)\right) \end{aligned}$$

Then $\partial_t \widetilde{I}(s,u,t,v) = 0$ if and only if

$$t = t(s) = \left[\frac{R(u,v)}{Q(v) - \mu Q_b(v)}\right]^{\frac{1}{q-(\beta+1)}} s^{\frac{\alpha+1}{q-(\beta+1)}}, \tag{2.6}$$

and consequently

$$\widetilde{I}(s,u,t(s),v) = (\alpha+1)\left(\frac{s^p}{p}P(u) - \frac{\lambda}{p_1}s^{p_1}P_{1,a}(u) - \frac{s^r}{r}A(u,v)\right),$$

where

$$A(u,v) = \frac{R(u,v)^{\frac{q}{q-(\beta+1)}}}{(Q(v) - \mu Q_b(v))^{\frac{\beta+1}{q-(\beta+1)}}}$$

and $r = \frac{(\alpha+1)q}{q-(\beta+1)} > p$. Let us write $\partial_s \widetilde{I}(s,u,t(s),v) = s^{p_1-1}F(s,u,v)$, where $F(s,u,v) := P(u)s^{p-p_1} - \lambda P_{1,a}(u) - A(u,v)s^{r-p_1}$. The function $s \in (0,+\infty) \mapsto F_{\lambda,\mu}(s,u,v)$ is increasing on $(\overline{s}_\mu(u,v),+\infty)$ and attains its unique maximum for $s = \overline{s}_\mu(u,v)$, where

$$\overline{s}_\mu(u,v) = \left[\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}\right]^{\frac{1}{r-p}}. \tag{2.7}$$

Thus, the function $s \in (0,+\infty) \mapsto F_{\lambda,\mu}(s,u,v)$ has two positive zeros (resp. one positive zero) if $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v) > 0$ (resp. $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v) = 0$) and has no zero if $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v) < 0$. On the other hand, a direct computation leads to

$$F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v) = \frac{r-p}{r-p_1}\left[\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u,v)}\right]^{\frac{p-p_1}{r-p_1}} P(u) - \lambda P_{1,a}(u).$$

Then, $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v) > 0$ (resp. $F_{\lambda,\mu}(\overline{s}_\mu(u,v),u,v) < 0$) if $\lambda < \lambda(u,v)$ (resp. $\lambda > \lambda(u,v)$) and $F_{\lambda(u,v),\mu}(\overline{s}_\mu(u,v),u,v) = 0$, where

$$\lambda(u,v) = \widehat{c}\frac{P(u)^{\frac{r-p_1}{r-p}}}{P_{1,a}(u)A(u,v)^{\frac{p-p_1}{r-p}}} \quad \text{and} \quad \widehat{c} = \frac{r-p}{r-p_1}\left[\frac{p-p_1}{r-p_1}\right]^{\frac{p-p_1}{r-p}}. \tag{2.8}$$

Therefore, if $\lambda \in (0,\lambda(u,v))$, the function $s \in (0,+\infty) \mapsto \partial_s \widetilde{I}(s,u,t(s),v)$ has two positive zeros denoted by $s_1(u,v,\lambda,\mu)$ and $s_2(u,v,\lambda,\mu)$ verifying $0 < s_1(u,v,\lambda,\mu) <$

$\overline{s}_\mu(u,v) < s_2(u,v,\lambda,\mu)$. Since $F_{\lambda,\mu}(s_1(u,v,\lambda,\mu),u,v) = F_{\lambda,\mu}(s_2(u,v,\lambda,\mu),u,v) = 0$, $\partial_s F_{\lambda,\mu}(s,u,v) > 0$ for $0 < s < \overline{s}_\mu(u,v)$ and $\partial_s F_{\lambda,\mu}(s,u,v) < 0$ for $s > \overline{s}_\mu(u,v)$ it follows that

$$\begin{aligned} \partial_{ss}\widetilde{I}(s_1(u,v,\lambda,\mu)u, t_1(u,v,\lambda,\mu), v) &> 0, && (2.9)\\ \partial_{ss}\widetilde{I}(s_2(u,v,\lambda,\mu), u, t_2(u,v,\lambda,\mu), v) &< 0. && (2.10)\end{aligned}$$

This implies that the real-valued function $s \in (0,+\infty) \mapsto \widetilde{I}(s,u,t(s),v)$ achieves its unique local minimum at $s = s_1(u,v,\lambda,\mu)$ and its unique local maximum at $s = s_2(u,v,\lambda,\mu)$, which ends the proof. □

Hereafter, we will denote $t_i(u,v,\lambda,\mu) = t(s_i(u,v,\lambda,\mu))$, $i = 1,2$. Notice that for every $(u,v) \in \widetilde{W}$, $\mu < \mu_1$ and $\lambda \in (0, \lambda(u,v,\mu))$, the points $(s_1(u,v,\lambda,\mu), u, t_1(u,v,\lambda,\mu), v)$ and $(s_2(u,v,\lambda,\mu), u, t_2(u,v,\lambda,\mu), v)$ belong to the Nehari manifold $\mathcal{N}_{\lambda,\mu}$.

At this stage, we introduce the characteristic value, for all $\mu < \mu_1$,

$$\widehat{\lambda}(\mu) := \inf_{\substack{(u,v)\,\in\, W\\ u\neq 0,\, v\neq 0}} \lambda(u,v,\mu).$$

We consider the spaces

$$\begin{aligned} L^{p_1}_a(\mathbb{R}^N) &:= \left\{u : \mathbb{R}^N \longrightarrow \mathbb{R}, \text{ measurable } : \textstyle\int_{\mathbb{R}^N} a(x)|u(x)|^{p_1}dx < +\infty\right\},\\ L^q_b(\mathbb{R}^N) &:= \left\{u : \mathbb{R}^N \longrightarrow \mathbb{R}, \text{ measurable } : \textstyle\int_{\mathbb{R}^N} b(x)|u(x)|^{q}dx < +\infty\right\},\end{aligned}$$

endowed with their seminorms

$$\begin{aligned} \|u\|_{p_1,a} &:= \left(\textstyle\int_{\mathbb{R}^N} a(x)|u(x)|^{p_1}dx\right)^{1/p_1}\\ \|u\|_{q,b} &:= \left(\textstyle\int_{\mathbb{R}^N} b(x)|u(x)|^{q}dx\right)^{1/q}.\end{aligned}$$

It is clear that, under these notations, the embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p_1}_a(\mathbb{R}^N)$ and $\mathcal{D}^{1,q}(\mathbb{R}^N) \subset L^q_b(\mathbb{R}^N)$ are continuous.

Remark 2.2.1. *Let $(u_n)_n$ be a sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, then we have the assertion:*

$$u_n \rightharpoonup u \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N) \text{ weakly } \Longrightarrow \lim_{n\to+\infty} \|u_n - u\|_{p_1,a} = 0.$$

We will write this property by

$$u_n \rightharpoonup u \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N) \text{ weakly } \Longrightarrow u_n \;\to\; u \text{ in } L^{p_1}_a(\mathbb{R}^N) \text{ strongly},$$

and that the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p_1}_a(\mathbb{R}^N)$ is compact.
Similarly,

$$v_n \rightharpoonup v \text{ in } \mathcal{D}^{1,q}(\mathbb{R}^N) \text{ weakly } \Longrightarrow v_n \;\to\; v \text{ in } L^{q}_b(\mathbb{R}^N) \text{ strongly},$$

that is, the embedding $\mathcal{D}^{1,q}(\mathbb{R}^N) \hookrightarrow L^q_b(\mathbb{R}^N)$ is compact.

Indeed, fix $(u_n)_n \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ *and a real number* $\varepsilon > 0$*. It is clear that there is a constant* $C > 0$ *such that* $\|u_n - u\|_{p^*} \leq C$*, for every* $n \in \mathbb{N}$*. Since* $a \in L^{\frac{p^*}{p^*-p_1}}(\mathbb{R}^N)$*, there is a compact* $K(a, \varepsilon, N) \subset \mathbb{R}^N$ *such that*

$$\|a\|_{L^{\frac{p^*}{p^*-p_1}}(\mathbb{R}^N \setminus K)} \leq (\varepsilon/C)^{p_1}.$$

On the other hand, by standard compact Sobolev embeddings, there is $n_1 \in \mathbb{N}$ *such that*

$$\int_K a(x)|u_n - u|^{p_1} dx \leq \varepsilon^{p_1}, \ \ \forall n \geq n_1.$$

Therefore, using the Hölder inequality, we get

$$\int_{\mathbb{R}^N} a(x)|u_n - u|^{p_1} dx \leq 2\varepsilon^{p_1}, \ \ \forall n \geq n_1,$$

which ends the claim. The argumentation is the same for the sequence $(v_n)_n \subset \mathcal{D}^{1,q}(\mathbb{R}^N)$*.*

We prove that $\widehat{\lambda}(\mu)$ is greater than a nonnegative constant which depends only on μ, p, p_1, q, α and β. Indeed, using the Hölder inequality, we get

$$R(u, v) \leq \|u\|_{p^*}^{\alpha+1} \|v\|_{q^*}^{\beta+1}.$$

On the other hand we have, also by the Hölder inequality

$$P_{1,a}(u) := \int_{\mathbb{R}^N} a(x)|u|^{p_1} dx \leq \|a\|_{\frac{p^*}{p^*-p_1}} \|u\|_{p^*}^{p_1}.$$

Using the continuous embedding $\mathcal{D}^{1,q}(\mathbb{R}^N) \subset L^{q^*}(\mathbb{R}^N)$, we obtain

$$A(u, v) \leq c_1 \frac{P_*(u)^{\frac{r}{p^*}}}{(\mu - \mu_1)^{\frac{\beta+1}{q-(\beta+1)}}},$$

where $P_*(u) = \|u\|_{p^*}^{p^*}$ and $c_1 = c_1(b, p, p_1, q, \alpha, \beta)$. Then, using the continuous embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L_a^{p_1}(\mathbb{R}^N)$ and $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$ we obtain

$$\lambda(u, v, \mu) \geq c_2 (\mu - \mu_1)^{\frac{\beta+1}{q-(\beta+1)} \frac{p-p_1}{r-p}},$$

where $c_2 = c(a, b, p, p_1, q, \alpha, \beta)$ and $L_a^{p_1}(\mathbb{R}^N) := \{u \text{ measurable} \,/\, a|u|^{p_1} \in L^1(\mathbb{R}^N)\}$. Consequently

$$\widehat{\lambda}(\mu) \geq c_2 (\mu - \mu_1)^{\frac{\beta+1}{q-(\beta+1)} \frac{p-p_1}{r-p}},$$

which achieves the claim. We now introduce the subset of $\mathbb{R}^2$ defined by

$$\mathcal{D} := \{(\lambda, \mu) \in (0, +\infty) \times (-\infty, \mu_1) : \ \lambda < \widehat{\lambda}(\mu)\}.$$

For every $(\lambda, \mu) \in \mathcal{D}$ and $(u, v) \in (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\})$, we have $\partial_s \widetilde{I}(s_1(u,v), u, t_1(u,v), v) = 0$ and (resp. $\partial_s \widetilde{I}(s_2(u,v), u, t_2(u,v), v) = 0$) , it follows that the functional $(u,v) \mapsto \widetilde{I}(s_1(u,v), u, t_1(u,v), v)$ (resp. $(u,v) \mapsto \widetilde{I}(s_2(u,v), u, t_2(u,v), v)$) is bounded below on $(u, v) \in (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\})$. Thus, for every $(\lambda, \mu) \in \mathcal{D}$, we define

$$\alpha_1(\lambda, \mu) = \inf\{\widetilde{I}(s_1(u,v), u, t_1(u,v), v) \ : \ (u,v) \in \widetilde{W}\}, \tag{2.11}$$

$$\alpha_2(\lambda, \mu) = \inf\{\widetilde{I}(s_2(u,v), u, t_2(u,v), v) \ : \ (u,v) \in \widetilde{W}\}. \tag{2.12}$$

Remark 2.2.2. *It is interesting to notice that for every $\gamma > 0$, and $\delta > 0$, it holds that*

$$\begin{aligned}
\widetilde{I}(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}) &= \widetilde{I}(s, u, t, v),\\
\partial_t \widetilde{I}(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}) &= \frac{1}{\delta} \partial_t \widetilde{I}(s, u, t, v),\\
\partial_s \widetilde{I}(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}) &= \frac{1}{\gamma} \partial_s \widetilde{I}(s, u, t, v) \ \textit{and}\\
\partial_{ss} \widetilde{I}(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}) &= \frac{1}{\gamma^2} \partial_{ss} \widetilde{I}(s, u, t, v).
\end{aligned}$$

It follows that

$$s_1(u, v, \lambda, \mu) = \frac{1}{\gamma} s_1\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall\, \delta > 0, \tag{2.13}$$

$$s_2(u, v, \lambda, \mu) = \frac{1}{\gamma} s_2\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall\, \delta > 0, \tag{2.14}$$

$$t_1(u, v, \lambda, \mu) = \frac{1}{\delta} t_1\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall\, \gamma > 0, \tag{2.15}$$

$$t_2(u, v, \lambda, \mu) = \frac{1}{\delta} t_2\left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu\right), \quad \forall\, \gamma > 0. \tag{2.16}$$

Therefore,

$$\alpha_1(\lambda, \mu) = \inf_{(u,\,v)\,\in\,\mathbb{S}_p\,\times\,\mathbb{S}_q} \widetilde{I}(s_1(u,v), u, t_1(u,v), v), \tag{2.17}$$

$$\alpha_2(\lambda, \mu) = \inf_{(u,\,v)\,\in\,\mathbb{S}_p\,\times\,\mathbb{S}_q} \widetilde{I}(s_2(u,v), u, t_2(u,v), v), \tag{2.18}$$

where $\mathbb{S}_p$ and $\mathbb{S}_q$ are the unit spheres of $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $\mathcal{D}^{1,q}(\mathbb{R}^N)$ respectively. Specify that $\mathbb{S}_p \times \mathbb{S}_q$ is a 2-codimensional and complete submanifold of W, and we will denote it by $\mathbb{S}$.

Our aim in this part is to show that $\alpha_1(\lambda, \mu)$ is in fact a critical value of the Euler-Lagrange functional I for every $(\lambda, \mu) \in \mathcal{D}$. We start with characterizing the minimizing sequences of $\alpha_1(\lambda, \mu)$ and $\alpha_2(\lambda, \mu)$, for every $(\lambda, \mu) \in \mathcal{D}$.

Lemma 2.2.2. *Let $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (2.17) (resp. of (2.18)) and let $(U_n^1, V_n^1) := (s_1(u_n, v_n, \lambda, \mu)u_n, t_1(u_n, v_n, \lambda, \mu)v_n)$ (resp. $(U_n^2, V_n^2) := (s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$). Then it holds:*

$$\begin{array}{ll}\text{(i)} & \limsup_{n\to+\infty} ||(U_n^1, V_n^1)|| < \infty \quad (\textit{resp.} \limsup_{n\to+\infty} ||(U_n^2, V_n^2)|| < \infty). \\ \text{(ii)} & \liminf_{n\to+\infty} ||(U_n^1, V_n^1)|| > 0 \quad (\textit{resp.} \liminf_{n\to+\infty} ||(U_n^2, V_n^2)|| > 0).\end{array}$$

Proof. We start by checking the point (i), let $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (2.17). Since $\partial_s \widetilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$ and $\partial_t \widetilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$, it follows that

$$\begin{aligned} P(U_n^1) - \lambda P_{1,a}(U_n^1) - R(U_n^1, V_n^1) &= 0, \\ Q(V_n^1) - \mu Q_b(V_n^1) - R(U_n^1, V_n^1) &= 0. \end{aligned}$$

Suppose that there exists a subsequence of (U_n^1, V_n^1), still denoted by (U_n^1, V_n^1), such that

$$\lim_{n\to+\infty} \|(U_n^1, V_n^1)\| = \infty.$$

We can distinguish three cases:

<u>*case a)*</u> $\lim_{n\to+\infty} \|\nabla U_n^1\|_p = \infty$ and $\|\nabla V_n^1\|_q$ is bounded. By (2.19) we get $R(U_n^1, V_n^1)$ is bounded. On the other hand, the continuous embedding of $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L_a^{p_1}(\mathbb{R}^N)$ enables us to have $P_{1,a}(U_n^1) = o(P(U_n^1))$, as n goes to $+\infty$. By (2.19) we get $R(U_n^1, V_n^1) = (1+o(1))P(U_n^1)$, as n goes to $+\infty$ and consequently $\lim_{n\to+\infty} R(U_n^1, V_n^1) = +\infty$, which leads to a contradiction.

<u>*case b)*</u> $\lim_{n\to+\infty} \|\nabla V_n^1\|_q = \infty$ and $\|\nabla U_n^1\|_p$ is bounded. We obtain by (2.19) the fact $R(U_n^1, V_n^1)$ is bounded. On the other hand, if $0 < \mu < \mu_1$, by Sobolev and Young's inequalities, for $\varepsilon \in (0, 1)$, there is a constant $C_\varepsilon > 0$ such that

$$\|V_n^1\|_q^q \le \frac{\varepsilon}{\mu} \|\nabla V_n^1\|_q^q + C_\varepsilon$$

which gives $R(U_n^1, V_n^1) + \mu C_\varepsilon \ge (1 - \varepsilon)Q(V_n^1)$. Then $\lim_{n\to+\infty} R(U_n^1, V_n^1) = +\infty$, which is impossible. If $\mu \le 0$, then $Q(V_n^1) - \mu Q_b(V_n^1) = R(U_n^1, V_n^1) \ge Q(V_n^1)$ so $\lim_{n\to+\infty} R(U_n^1, V_n^1) = +\infty$, which can not hold not.

<u>*case c)*</u> $\lim_{n\to+\infty} \|\nabla U_n^1\|_p = \lim_{n\to+\infty} \|\nabla V_n^1\|_q = \infty$. As in the first case, we have

$$R(U_n^1, V_n^1) = (1 + o(1))P(U_n^1), \qquad \text{as } n \text{ goes to} + \infty.$$

Then $I(U_n^1, V_n^1) = (\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1 + o_n(1))P(U_n^1)$ as n goes to $+\infty$. Hence, $\lim_{n\to+\infty} I(U_n^1, V_n^1) = +\infty$, which contradicts the hypothesis $I(U_n^1, V_n^1) \le 0$ for every $n \in \mathbb{N}$. The first assertion for (2.18) follows by the same arguments.

Now, let us show the second assertion of the lemma. Let $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (2.17). Suppose that there is a a subsequence, still denoted by (U_n^1, V_n^1),

such that $\lim_{n\to+\infty}\|(U_n^1,V_n^1)\|=0$. By (2.19), we get $\lim_{n\to+\infty} I(U_n^1,V_n^1)=0$ and this can not hold true because $I(U_n^1,V_n^1)\leq 0$ for every $n\in\mathbb{N}$.
Similarly, let $(u_n,v_n)\in\mathbb{S}$ be a minimizing sequence of (2.18). Suppose that there is a subsequence, still denoted by (U_n^2,V_n^2), such that $\lim_{n\to+\infty}\|(U_n^2,V_n^2)\|=0$. If $p>\alpha+1$, by (2.10), we have

$$\partial_{ss}\widetilde{I}(U_n^2,V_n^2)=(\alpha+1)((p-1)P(U_n^2)-\lambda(p_1-1)P_{1,a}(U_n^2)-\alpha R(U_n^2,V_n^2))<0.$$

Then $(p-1)P(U_n^2)-\lambda(p_1-1)P_{1,a}(U_n^2)-\alpha pR(U_n^2,V_n^2)<0$, which implies that $(p-(\alpha+1))R(U_n^2,V_n^2)<0$ and this is impossible. Finally, if $p\leq\alpha+1$, then $(p-p_1)P(U_n^2)<(\alpha+1+p)R(U_n^2,V_n^2)$. Since $\frac{\alpha+1}{p^*}+\frac{\beta+1}{q^*}=1$, then

$$\begin{array}{rcl} R(U_n^2,V_n^2) & \leq & \|U_n^2\|_{p^*}^{\alpha+1}\|V_n^2\|_{q^*}^{\beta+1}\\ & \leq & c(p,q)\|\nabla U_n^2\|_p^{\alpha+1}\|\nabla V_n^2\|_q^{\beta+1}, \end{array}$$

and consequently, $(p-p_1)\leq c(\alpha,p,q)\|\nabla U_n^2\|_p^{\alpha+1-p}\|\nabla V_n^2\|_q^{\beta+1}$, which converges to 0 s n goes to $+\infty$. This contradicts the fact $p>p_1$, which ends the proof. □

Lemma 2.2.3. *Let $(u_n,v_n)\subset\mathbb{S}$ be a minimizing sequence of $\alpha_1(\lambda,\mu)$ (resp. for $\alpha_2(\lambda,\mu)$), then the sequences (U_n^1,V_n^1) and (resp. (U_n^2,V_n^2)) is a Palais-Smale for the functional I, where (U_n^1,V_n^1) and (U_n^2,V_n^2) are defined bellow.*

Proof. Let $(\lambda,\mu)\in\mathcal{D}$ and $(u_n,v_n)\in\mathbb{S}$ be a minimizing sequence of (2.17). Let us set

$$\begin{array}{rcl} U_n & = & s_1(u_n,v_n,\lambda,\mu)u_n,\\ V_n & = & t_1(u_n,v_n,\lambda,\mu)v_n. \end{array}$$

By the previous lemma, it is clear that the sequence (U_n,V_n) is bounded in W. On the other hand, the functional $(s,t)\mapsto\widetilde{I}(.,s,.,t)$ has null gradient and its Hessian determinant is strictly positive. So, the implicit functions theorem allows to confirm that the functions $(u,v)\mapsto s_1(u,v,\lambda,\mu)$ and $(u,v)\mapsto t_1(u,v,\lambda,\mu)$ are $\mathcal{C}^1(W,\mathbb{R})$, since $(u,v)\mapsto\widetilde{I}(.,u,.,v)$ is.
We introduce now the functional $\mathcal{I}$ defined on $\mathbb{S}$ by

$$\mathcal{I}(u,v)=\widetilde{I}(s_1(u,v,\lambda,\mu),u,t_1(u,v,\lambda,\mu),v),$$

Then

$$\alpha_1(\lambda,\mu)=\inf_{(u,v)\in\mathbb{S}}\mathcal{I}(u,v)=\lim_{n\to+\infty}\mathcal{I}(u_n,v_n).$$

By the Ekland variational principle on the complete manifold $(\mathbb{S},\|.\|)$ to the functional $\mathcal{I}$ we get

$$\mathcal{I}'(u_n,v_n)(\varphi_n,\psi_n)\leq\frac{1}{n}\|(\varphi_n,\psi_n)\|,\ \forall(\varphi_n,\psi_n)\in T_{(u_n,v_n)}\mathbb{S},$$

where $T_{(u_n,v_n)}\mathbb{S}$ designs the tangent space to $\mathbb{S}$ at the point (u_n, v_n). We know that the space $T_{(u_n,v_n)}\mathbb{S} = T_{u_n}\mathbb{S}_p \times T_{v_n}\mathbb{S}_q$ where $T_{u_n}\mathbb{S}_p$ and $T_{v_n}\mathbb{S}_q$ are respectively the tangent spaces to $\mathbb{S}_p$ and $\mathbb{S}_q$ at the respective points u_n and v_n.
In order to reduce the notations, we set

$$A_n := (u_n, v_n, \lambda, \mu), \quad \text{and} \quad B_n := (s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n).$$

For every $(\varphi_n, \psi_n) \in T_{u_n}\mathbb{S}_p \times T_{v_n}\mathbb{S}_q$, one has

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) = D_1\widetilde{I}(B_n)(\varphi_n) + D_2\widetilde{I}(B_n)(\psi_n)$$

where

$$\begin{aligned} D_1\widetilde{I}(B_n)(\varphi_n) &= \partial_s s_1(A_n)(\varphi_n)\partial_s\widetilde{I}(B_n) + \partial_u\widetilde{I}(B_n)(\varphi_n) + \partial_u t_1(A_n)(\varphi_n)\partial_t\widetilde{I}(B_n) \\ &= \partial_u\widetilde{I}(B_n)(\varphi_n). \end{aligned}$$

With the same manner, one has

$$D_2\widetilde{I}(B_n)(\psi_n) = \partial_v\widetilde{I}(B_n)(\psi_n).$$

Furthermore, consider the following "fiber" maps

$$\begin{aligned} \pi : \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\} &\to \mathbb{R} \times \mathbb{S}_p \\ u &\mapsto \left(\|u\|_p, \frac{u}{\|u\|_p}\right) := (\pi_1(u), \pi_2(u)), \\ \widetilde{\pi} : \mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\} &\to \mathbb{R} \times \mathbb{S}_q \\ v &\mapsto \left(\|v\|_q, \frac{v}{\|v\|_q}\right) := (\widetilde{\pi}_1(v), \widetilde{\pi}_2(v)). \end{aligned}$$

Applying the Hölder inequality we get

$$|\pi_1'(u)(\varphi)| \leq \|\varphi\|_p, \quad |\pi_2'(u)(\varphi)| \leq 2\frac{\|\varphi\|_p}{\|u\|_p}, \quad \forall (u, \varphi) \in \left(\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}\right) \times \mathcal{D}^{1,p}(\mathbb{R}^N)$$

and

$$|\widetilde{\pi}_1'(v)(\psi)| \leq \|\psi\|_q, \quad |\widetilde{\pi}_2'(v)(\psi)| \leq 2\frac{\|\psi\|_q}{\|v\|_q}, \quad \forall (v, \psi) \in \left(\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\}\right) \times \mathcal{D}^{1,q}(\mathbb{R}^N).$$

From Lemma 2.2.2, there exists a nonnegative constant K such that $s_1(A_n) \geq K$ and $t_1(A_n) \geq K$ for any integer n. On the other hand, for every $(\varphi, \psi) \in W$

$$\begin{aligned} D_1 I(U_n, V_n)(\varphi) &= \varphi_n^1 \partial_s\widetilde{I}(B_n) + \partial_u\widetilde{I}(B_n)(\varphi_n^2) + \varphi_n^1\partial_t\widetilde{I}(B_n) \\ &= \partial_u\widetilde{I}(B_n)(\varphi_n^2), \end{aligned}$$

where $\varphi_n^1 = \pi_1'(u_n)(\varphi)$ and $\varphi_n^2 = \pi_2'(u_n)(\varphi)$. Then we have the following estimates $|\varphi_n^1| \leq \|\varphi\|_p$ and $\|\varphi_n^2\|_p \leq \frac{2}{K}\|\varphi\|_p$. In the same way, we get

$$\begin{aligned} D_2 I(U_n, V_n)(\psi) &= \psi_n^1 \partial_s \widetilde{I}(B_n) + \partial_v \widetilde{I}(B_n)(\psi_n^2) + \psi_n^1 \partial_t \widetilde{I}(B_n) \\ &= \partial_v \widetilde{I}(B_n)(\psi_n^2), \end{aligned}$$

where $\psi_n^1 = \widetilde{\pi}_2'(v_n)(\psi)$ and $\psi_n^2 = \widetilde{\pi}_1'(v_n)(\psi)$. Then we have the following estimates $|\psi_n^1| \leq \|\psi\|_p$ and $\|\psi_n^2\|_p \leq \frac{2}{K}\|\psi\|_q$. Therefore

$$\begin{aligned} |D_1 I(U_n, V_n)(\varphi)| &\leq \frac{1}{n}\|\varphi_n^2\|_p \\ &\leq \frac{2}{nK}\|\varphi\|_p \end{aligned}$$

and

$$\begin{aligned} |D_2 I(U_n, V_n)(\psi)| &\leq \frac{1}{n}\|\psi_n^2\|_p \\ &\leq \frac{2}{nK}\|\psi\|_q \end{aligned}$$

Finally, we have

$$\lim_{n\to+\infty} \|I'(U_n, V_n)\|_{W^*} = 0,$$

where $I'(U_n, V_n)(\varphi, \psi) = D_1 I(U_n, V_n)(\varphi) + D_2 I(U_n, V_n)(\psi)$ and $\| \|_{W^*}$ designs the norm of the dual space of W, which achieves the first claim. The second one follows with similar arguments. □

2.3 Existence and multiplicity results of solutions to the problem

In this section, we will show that there is at least one solution to the system (1.1) and two solutions in the case $p = q$ obtained by considering minimizing sequences of (2.17) and (2.18) under some supplementary conditions on (λ, μ) which belongs to $\mathcal{D}$. We begin by stating the following lemma, due to A. El Hamidi and J.M. Rakotoson [8]

Lemma 2.3.1. *[8] Let $\widehat{\Psi}$ be a Caratheodory function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}^N$ satisfying the usual Leray-Lions growth and monotonicity conditions. Let (u_n) be a bounded sequence of $W^{1,p}_{\text{loc}}(\mathbb{R}^N) = \Big\{ v \in L^p_{\text{loc}}(\mathbb{R}^N),\ |\nabla v| \in L^p_{\text{loc}}(\mathbb{R}^N) \Big\}$, with $1 < p < +\infty$, (f_n) be a bounded sequence of $L^1_{\text{loc}}(\mathbb{R}^N)$ and (g_n) be a sequence of $W^{-1,p'}_{\text{loc}}(\mathbb{R}^N)$ tending strongly to zero.*
Assume that (u_n) satisfies:

$$(H1) \quad \int_{\mathbb{R}^N} \widehat{\Psi}\big(x, u_n(x), \nabla u_n(x)\big) \cdot \nabla\varphi \, dx = \int_{\mathbb{R}^N} f_n \varphi \, dx + < g_n, \varphi >,$$

$$\forall \varphi \in W^{1,p}_{comp}(\mathbb{R}^N) = \Big\{ v \in W^{1,p}(\mathbb{R}^N),\ \textit{with compact support} \Big\},\ \varphi \textit{ bounded.}$$

Then

1. *there exists a function* u *such that* $u_n(x) \to u(x)$ a.e. *in* $\mathbb{R}^N$,
2. $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$,
3. *there exists a subsequence, still denoted* (u_n), *such that*

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e. } \textit{in } \mathbb{R}^N.$$

At this stage, we state and show the following

Lemma 2.3.2. *Let* $(\lambda, \mu) \in \mathcal{D}$ *and* $(u_n, v_n)_n \subset W$ *be a* $(P.S)_c$ *sequences of* I *such that*

$$c := \lim_{n \in +\infty} I(u_n, v_n) < \inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u,v) + \alpha_1(\lambda, \mu). \tag{3.19}$$

Then the sequence (u_n, v_n) *is relatively compact.*

Proof. Let $(\lambda, \mu) \in \mathcal{D}$ and $(u_n, v_n)_n \subset W$ be a $(P.S)_c$ sequence of I satisfying the condition (3.19).
On one hand, We claim that (u_n, v_n) is bounded in W. Since (u_n, v_n) is Palais-Smale sequence of I, then we have

$$\begin{aligned} I(u_n, v_n) &= c + o_n(1) && (3.20)\\ P(u_n) - \lambda P_{1,a}(u_n) &= R(u_n, v_n) + o(\|u_n\|_{p^*}) && (3.21)\\ Q(v_n) - \mu Q_b(v_n) &= R(u_n, v_n) + o(\|v_n\|_{q^*}) && (3.22) \end{aligned}$$

Then we can apply the result of the lemma 2.2.2 to prove that (u_n, v_n) is bounded in W. At this stage, we can assume, up to a subsequence, that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N),\\ v_n &\rightharpoonup v \text{ in } \mathcal{D}^{1,q}(\mathbb{R}^N),\\ u_n &\to u \text{ a.e. in } \mathbb{R}^N,\\ v_n &\to v \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

It is clear that $(u, v) \in \mathcal{N}_{\lambda,\mu} \cup \{(0,0)\}$.
Notice that $I'(u_n, v_n)(\varphi, 0) \longrightarrow 0$ and $I'(u_n, v_n)(0, \psi) \longrightarrow 0$ for every $(\varphi, \psi) \in \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathcal{D}^{1,q}(\mathbb{R}^N)$ imply the hypothesis (H_1) for the sequences $(u_n)_n$ and $(v_n)_n$, in the special case where $\widehat{\Psi}$ corresponds to the $p-$Laplacian or the $q-$Laplacian respectively. Consequently, Lemma 2.3.1 show that, up to subsequences:

$$\begin{aligned} \nabla u_n &\longrightarrow \nabla u \text{ a.e. in } \mathbb{R}^N,\\ \nabla v_n &\longrightarrow \nabla v \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

Let us set $x_n := u_n - u$ and $y_n := v_n - v$. Using the Brézis-Lieb lemma [4], so we obtain the following decompositions

$$\begin{aligned} P(x_n) &= P(u_n) - P(u) + o_n(1), \\ Q(y_n) &= Q(v_n) - Q(v) + o_n(1), \\ P_{1,a}(x_n) &= P_{1,a}(u_n) - P_{1,a}(u) + o_n(1), \\ Q_b(y_n) &= Q_b(v_n) - Q_b(v) + o_n(1), \\ R(x_n, y_n) &= R(u_n, v_n) - R(u, v) + o_n(1). \end{aligned}$$

Using the compactness of the embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L_a^{p_1}(\mathbb{R}^N)$, and $\mathcal{D}^{1,q}(\mathbb{R}^N) \hookrightarrow L_b^q(\mathbb{R}^N)$ we get $P_{1,a}(x_n) = o_n(1)$ and $Q_b(y_n) = o_n(1)$.
It follows that

$$\begin{aligned} P(x_n) &= R(x_n, y_n) + o_n(1), \\ Q(y_n) &= R(x_n, y_n) + o_n(1), \\ I_{0,0}(x_n, y_n) &= c - I(u, v) + o_n(1). \end{aligned}$$

Notice that the Nehari Manifold associated to $I_{0,0}$ is given by

$$\mathcal{N}_{0,0} = \{(s_0(u,v)u, t_0(u,v)v);\ (u,v) \in W,\ u \neq 0,\ v \neq 0\},$$

where

$$s_0(u,v) = \left[\frac{P(u) Q(v)^{\frac{r(\beta+1)}{q(\alpha+1)}}}{R(u,v)^{\frac{r}{\alpha+1}}} \right]^{\frac{1}{r-p}} \text{ and } t_0(u,v) = t(s_0(u,v)),$$

and $s \to t(s)$ is defined by (2.6). Let l be the common limit of $P(x_n)$, $Q_(y_n)$ and $R(x_n, y_n)$. Suppose that $l \neq 0$, we get then

$$\begin{aligned} I_{0,0}(s_0(x_n,y_n)x_n, t_0(x_n,y_n)y_n) &= (\alpha+1)\left(\frac{1}{p} - \frac{1}{r}\right) K(x_n, y_n) && (3.23) \\ &\geq \inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v), && (3.24) \end{aligned}$$

where

$$K(x_n, y_n) = \left[\frac{P(x_n)^{(\alpha+1)} Q(y_n)^{(\beta+1)\frac{p}{q}}}{R(x_n,y_n)^p} \right]^{\frac{r}{(\alpha+1)(r-p)}},$$

which tends to l as n tends to $+\infty$.
Therefore

$$\lim_{n\to+\infty} I_{0,0}(s_0(x_n,y_n)x_n, t_0(x_n,y_n)y_n) = l(\alpha+1)\left(\frac{1}{p} - \frac{1}{r}\right).$$

On the other hand,

$$\begin{aligned} \lim_{n\to+\infty} I_{\lambda,\mu}(x_n, y_n) &= l\left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) \\ &= l(\alpha+1)\left(\frac{1}{p} - \frac{1}{r}\right). \end{aligned}$$

Hence, we obtain

$$l(\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right)=c-I(u,v),$$

and consequently

$$\begin{aligned} c &\geq \inf_{(u,v)\in\mathcal{N}_{0,0}} I(u,v)+I(u,v)\\ &\geq \inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v)+\alpha_1(\lambda,\mu), \end{aligned}$$

which contradicts the hypothesis (3.19), then $l=0$, which achieves the proof. $\square$

Theorem 2.3.1. *The system (1.1) has at least one solution, for every $(\lambda,\mu)\in\mathcal{D}$.*

Proof. Using the Hölder inequality in $R(u,v)$, we get

$$\begin{aligned} \inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) &= \inf_{(u,v)\in\mathbb{S}}(\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right)\left[\frac{P(u)^{(\alpha+1)}Q(v)^{(\beta+1)\frac{p}{q}}}{R(u,v)^p}\right]^{\frac{r}{(\alpha+1)(r-p)}}\\ &\geq (\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right)\left[S_pS_q^{\frac{p(\beta+1)}{q(\alpha+1)}}\right]^{\frac{r}{r-p}}>0, \end{aligned} \tag{3.25}$$

where S_p and S_q are the best Sobolev constants in the embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N)\subset L^{p^*}(\mathbb{R}^N)$ and $\mathcal{D}^{1,q}(\mathbb{R}^N)\subset L^{q^*}(\mathbb{R}^N)$ respectively. Consequently,

$$\forall\ (\lambda,\mu)\in\mathcal{D},\quad \alpha_1(\lambda,\mu)<\inf_{(u,v)\in\mathcal{N}_{0,0}} I(u,v)+\alpha_1(\lambda,\mu)$$

We set $U_n^1:=s_1(u_n,v_n,\lambda,\mu)u_n$ and $V_n^1:=t_1(u_n,v_n,\lambda,\mu)v_n$, where (u_n,v_n) is a minimizing sequence of (2.11). By Lemma 2.2.3 the sequence (U_n^1,V_n^1) is of Palais-smale type whose level is $\alpha_1(\lambda,\mu)$). Then, according to Lemma 2.3.2 there is a subsequence, still denoted (U_n^1,V_n^1), and (U^1,V^1) such that

$$(U_n^1,V_n^1)\longrightarrow(U^1,V^1)\ \text{ strongly in } W.$$

Now, since $\lim_{n\to+\infty}\|I'(U_n^1,V_n^1)\|_{W^*}=0$, we have for every $(\varphi,\psi)\in W$

$$\left\{\begin{aligned} \int_{\mathbb{R}^N}|\nabla U_n^1|^{p-2}\nabla U_n^1.\nabla\varphi\,dx &= \lambda\int_{\mathbb{R}^N}A_n\varphi\,dx+\int_{\mathbb{R}^N}X_n^1\varphi\,dx+o_n(1),\\ \int_{\mathbb{R}^N}|\nabla V_n^1|^{q-2}\nabla V_n^1.\nabla\psi\,dx &= \mu\int_{\mathbb{R}^N}B_n\psi\,dx+\int_{\mathbb{R}^N}Y_n^1\psi\,dx+o_n(1), \end{aligned}\right. \tag{3.26}$$

where $A_n:=a\,|U_n^1|^{p_1-2}U_n^1$, $B_n:=b\,|V_n^1|^{q-2}V_n^1$, $X_n^1:=|U_n^1|^{\alpha-1}U_n^1|V_n^1|^{\beta+1}$ and $Y_n^1:=|U_n^1|^{\alpha+1}|V_n^1|^{\beta-1}V_n^1$. On one hand, the continuity of the embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N)\subset L_a^{p_1}(\mathbb{R}^N)$ and $\mathcal{D}^{1,q}(\mathbb{R}^N)\subset L_b^q(\mathbb{R}^N)$ implies that $\int_{\mathbb{R}^N}A_n\varphi\,dx\longrightarrow\int_{\mathbb{R}^N}a|U^1|^{p_1-2}U^1\varphi\,dx$ and $\int_{\mathbb{R}^N}B_n\psi\,dx\longrightarrow\int_{\mathbb{R}^N}b|V^1|^{q-2}V^1\psi\,dx$ as n tends to $+\infty$. On the other hand, since $A_n\varphi\longrightarrow a|U^1|^{p_1-2}U^1\varphi$ and $B_n\psi\longrightarrow b|V^1|^{q-2}V^1\psi$ a.e. in $\mathbb{R}^N$ as n goes to $+\infty$, the inequalities $\int_{\mathbb{R}^N}A_n\varphi\,dx\leq\|u_n\|_{p^*}^{\alpha}\|v_n\|_{q^*}^{\beta+1}\|\varphi\|_{p^*}$, $\int_{\mathbb{R}^N}B_n\psi\,dx\leq\|u_n\|_{p^*}^{\alpha+1}\|v_n\|_{q^*}^{\beta}\|\psi\|_{q^*}$

and the Lebesgue theorem imply that we can pass to the limit under integral sign in (3.26) to obtain for all $(\varphi,\psi)\in W$

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla U^1|^{p-2}\nabla U^1.\nabla\varphi\,dx &= \lambda\int_{\mathbb{R}^N} a|U^1|^{p_1-2}U^1\varphi\,dx + \int_{\mathbb{R}^N} X^1\varphi\,dx, \\ \int_{\mathbb{R}^N} b|\nabla V^1|^{q-2}\nabla V^1.\nabla\psi\,dx &= \mu\int_{\mathbb{R}^N} b|V^1|^{q-2}V^1\psi\,dx + \int_{\mathbb{R}^N} Y^1\psi\,dx, \end{cases}$$

where $X^1 := |U^1|^{\alpha-1}U^1|V^1|^{\beta+1}$ and $Y^1 := |U^1|^{\alpha+1}|V^1|^{\beta-1}V^1$. Hence (U^1,V^1) is a weak solution to the problem (1.1). □

Remark 2.3.1. *In the scalar case, we obtain the analogous of Theorem 2.3.2 with the same arguments. We note here in this special case, direct computations give*

$$\inf_{u\in\mathcal{N}_0} I_0(u) = \frac{1}{N}S_p^{\frac{N}{p}} \quad \textit{and} \quad \inf_{u\in\mathcal{N}_\lambda\cup\{0\}} I_\lambda(u) = 0,$$

which generalize the famous Brézis-Nirenberg condition for the entire space.

Proposition 2.3.1. *Let $p=q>1$ and (λ,μ) be in $\mathcal{D}$. Then,*

$$\inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) = \frac{p}{N-p}S_p^{\frac{N}{p}}.$$

Proof. Assume that $p=q>1$, then

$$p^* = \alpha+\beta+2 \quad\text{and}\quad (\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right) = \frac{p}{N-p}.$$

By the inequality (3.25), we conclude that

$$\inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) \geq \frac{p}{N-p}S_p^{\frac{N}{p}}.$$

On the other hand, let $(u_n)\subset\mathcal{D}^{1,p}(\mathbb{R}^N)$ be a minimizing sequence of S_p. Then using the identity (3.23), we get

$$\begin{aligned} \inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) &\leq I_{0,0}(s_0(u_n,u_n)u_n, t_0(u_n,u_n)u_n) \\ &= \frac{p}{N-p}\left[\frac{P(u_n)}{P_*(u_n)^{\frac{p}{p^*}}}\right]^{\frac{rp^*}{(\alpha+1)(r-p)}} \\ &= \frac{p}{N-p}\left[\frac{P(u_n)}{P_*(u_n)^{\frac{p}{p^*}}}\right]^{\frac{N}{p}}, \end{aligned}$$

making tend n to $+\infty$ the right hand of the last quantity goes to $\frac{p}{N-p}S_p^{\frac{N}{p}}$, which achieves the proof. □

Theorem 2.3.2. *If $p = q > 1$, the system (1.1) has another nontrivial nonnegative solution different from the solution established in Theorem 2.3.1.*

We start by stating and showing the following

Lemma 2.3.3. *Let $p > 1$, $q > 1$, $(\lambda, \mu) \in \mathcal{D}$ and (u_n, v_n) in W be a Palais-Smale sequence for $I_{\lambda,\mu}$ such that $(u_n, v_n) \rightharpoonup (u, v)$. Then there exists a constant $K > 0$ depending on p, p_1, a and N such that*

$$I(u, v) \geq -K\lambda^{\frac{p}{p-p_1}}.$$

Proof. Let $(u_n, v_n) \in W$ be a Palais-Smale sequence for I converging weakly to (u, v) in W. It is clear that $(u, v) \in \mathcal{N}_{\lambda,\mu} \cup \{(0, 0)\}$ and if $u = 0$ or $v = 0$ then $(u, v) = (0, 0)$. If $(u, v) = (0, 0)$, the proof is achieved. We assume now that $(u, v) \in \mathcal{N}_{\lambda,\mu}$.
Since (u_n, v_n) is a Palais-Smale sequence, then

$$\begin{cases} P(u_n) - \lambda P_{1,a}(u_n) &= R(u_n, v_n) + o_n(1), \\ Q(v_n) - \mu Q_b(v_n) &= R(u_n, v_n) + o_n(1). \end{cases}$$

It follows that

$$I(u_n, v_n) = (\alpha + 1)\left(\left(\frac{1}{p} - \frac{1}{r}\right) P(u_n) - \lambda\left(\frac{1}{p_1} - \frac{1}{r}\right) P_{1,a}(u_n)\right) + o_n(1).$$

We introduce the following function

$$f(t, u) := (\alpha + 1)\left(t^p\left(\frac{1}{p} - \frac{1}{r}\right) P(u) - t^{p_1}\lambda\left(\frac{1}{p_1} - \frac{1}{r}\right) P_{1,a}(u)\right).$$

Then

$$\frac{\partial f}{\partial t}(t, u) = 0 \Longleftrightarrow t = t(u) := \left[\frac{\lambda p_1(\left(\frac{1}{p_1} - \frac{1}{r}\right)}{p(\left(\frac{1}{p} - \frac{1}{r}\right)} \frac{P_{1,a}(u)}{P(u)}\right]^{\frac{1}{p-p_1}},$$

and

$$f(t(u), u) = -\lambda^{\frac{p}{p-p_1}}\left(1 - \frac{p_1}{p}\right)\left(\frac{p_1}{p}\right)^{\frac{p_1}{p-p_1}} \frac{\left(\frac{1}{p_1} - \frac{1}{r}\right)^{\frac{p_1}{p-p_1}}}{\left(\frac{1}{p} - \frac{1}{r}\right)^{\frac{p}{p-p_1}}} \frac{P_{1,a}(u)^{\frac{p}{p-p_1}}}{P(u)^{\frac{p_1}{p-p_1}}}.$$

If S_{p_1} denotes the best constant of the continuous embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p_1}_a(\mathbb{R}^N)$, we get

$$\forall\, u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}, \;\; \frac{P_{1,a}(u)^{\frac{p}{p-p_1}}}{P(u)^{\frac{p_1}{p-p_1}}} \leq \|a\|^{\frac{p}{p-p_1}}_{L^{\frac{p^*}{p^*-p_1}}} S_p^{\frac{pp_1}{p-p_1}}.$$

So there exists a constant $K > 0$ such that

$$\forall\, u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\},\ \ f(t,u) \geq f(t(u),u) \geq -K\lambda^{\frac{p}{p-p_1}}.$$

Therefore, for every $(u,v) \in \mathcal{N}_{\lambda,\mu}$, one has

$$I(u,v) = f(t(u),u) \geq -K\lambda^{\frac{p}{p-p_1}} + o_n(1),$$

which ends the proof. □

Lemma 2.3.4. *If $p = q > 1$ and $(\lambda,\mu) \in \mathcal{D}$, then the functional I satisfies the Palais-Smale condition on the interval $\left(-\infty, \frac{N}{p^*} S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}\right)$.*

Proof. Let (u_n, v_n) be a Palais-Smale sequence such that

$$\lim_{n\to\infty} I(u_n,v_n) = c < \frac{N}{p^*} S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}.$$

By Standard arguments one can prove that (u_n, v_n) is bounded in W, so one can extract a subsequence of (u_n, v_n), still denoted (u_n, v_n), such that

$$\begin{array}{rcl} (u_n,v_n) & \rightharpoonup & (u,v) \text{ in } W, \\ u_n & \to & u \text{ in } L_a^{p_1}(\mathbb{R}^N), \\ v_n & \to & v \text{ in } L_b^{q}(\mathbb{R}^N), \\ u_n(x) & \to & u(x) \text{ a.e } x \text{ in } \mathbb{R}^N \\ v_n(x) & \to & v(x) \text{ a.e } x \text{ in } \mathbb{R}^N. \end{array}$$

Let $x_n := u_n - u$ and $y_n =: v_n - v$, applying again Lemma 2.3.1, we get $\nabla x_n \to 0$ and $\nabla y_n \to 0$ almost everywhere in $\mathbb{R}^N$. So by Brézis-Lieb lemma, it follows

$$\begin{array}{rcl} P(x_n) & = & P(u_n) - P(u) + o_n(1), \\ Q(y_n) & = & Q(v_n) - Q(v) + o_n(1), \\ R(x_n,y_n) & = & R(u_n,v_n) - R(u,v) + o_n(1). \end{array}$$

So

$$\begin{array}{rcl} P(x_n) & = & R(x_n,y_n) + o_n(1), \\ Q(y_n) & = & R(x_n,y_n) + o_n(1), \\ I_{0,0}(x_n,y_n) & = & c - I(u,v) + o_n(1). \end{array}$$

Let l be the common limit of $P(x_n)$, $P(y_n)$ and $R(x_n,y_n)$. If $l \neq 0$, we get

$$\begin{array}{rcl} I_{0,0}(s_0(x_n,y_n)x_n, t_0(x_n,y_n)y_n) & = & \dfrac{N}{p^*} K(x_n,y_n), \\ & \geq & \displaystyle\inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w), \end{array}$$

where

$$K(x_n, y_n) = \left[\frac{P(x_n)^{\frac{\alpha+1}{p}} P(y_n)^{\frac{\beta+1}{p}}}{R(x_n, y_n)} \right]^{\frac{N}{p}}.$$

Direct computations show that

$$K(x_n, y_n) \to l,$$

so

$$\lim_{n\to\infty} I_{0,0}(s_0(x_n, y_n)x_n, t_0(x_n, y_n)y_n) = \frac{N}{p^*} l.$$

On the other hand we have

$$\begin{aligned} \lim_{n\to+\infty} I_{0,0}(x_n, y_n) &= l\left(\frac{\alpha+1}{p} + \frac{\beta+1}{p} - 1\right), \\ &= \frac{N}{p^*} l. \end{aligned}$$

Hence, we obtain

$$\frac{N}{p^*} l = c - I(u, v).$$

Using the lemma 2.3.3 , we have

$$\begin{aligned} c &= \frac{N}{p^*} l + I(u, v), \\ &\geq \frac{N}{p^*} l - K\lambda^{\frac{p}{p-p_1}}, \end{aligned}$$

which cannot hold true, and $l = 0$. □

Lemma 2.3.5. *Let $p = q > 1$. There exists $v \in W^{1,p}(\mathbb{R}^N)$ and $\lambda^* > 0$ such that for $(\lambda, \mu) \in (0, \lambda^*) \times (0, +\infty)$, we have*

$$\sup_{s\geq 0} I_{\lambda,\mu}(sv, sv) < \frac{N}{p^*} l - K\lambda^{\frac{p}{p-p_1}}.$$

In particular,

$$\alpha_2(\lambda, \mu) < \frac{N}{p^*} l - K\lambda^{\frac{p}{p-p_1}}.$$

Proof. Let's consider the following family of functions given by

$$w_\varepsilon = C_N \varepsilon^{(N-p)/p^2} (\varepsilon + |x|^{p'})^{(p-N)/p}$$

which attains the best constant S_p of the Sobolev embedding $W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$. Let $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ in a neighborhood of the origin. We define $u_\varepsilon(x) = \phi(x)w_\varepsilon(x)$. Taking $v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{p^*}}$ and using the following estimates

$$P(v_\varepsilon) = \begin{cases} \frac{S_p}{2^{p/N}} - C\varepsilon^{(p-1)/p} + o(\varepsilon^{(p-1)/p}) + O(\varepsilon^{(N-p)/p}) \text{ if } N \geq p^2, \\ \\ \frac{S_p}{2^{p/N}} - C\varepsilon^{(p-1)/p} f(\varepsilon) + O(\varepsilon^{(N-p)/p}) \text{ if } N < p^2, \end{cases}$$

where C is a positive constant and $\lim_{\varepsilon\to 0} f(\varepsilon) = +\infty$. Let $\delta_2 > 0$ be such that

$$\frac{N}{p^*} S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}} > 0, \ \forall \lambda \in (0, \delta_2).$$

$$\begin{aligned} I_{\lambda,\mu}(sv_\varepsilon, sv_\varepsilon) &= \frac{s^p}{p}\left(p^* P(v_\varepsilon) - (\beta+1)\mu Q_b(v_\varepsilon)\right) - \frac{\alpha+1}{p_1}\lambda s^{p_1} P_{1,a}(v_\varepsilon) - s^{p^*} P_*(v_\varepsilon), \\ &\leq s^p \frac{p^*}{p} P(v_\varepsilon) - \frac{\alpha+1}{p_1}\lambda s^{p_1} P_{1,a}(v_\varepsilon) - s^{p^*} P_*(v_\varepsilon), \\ &\leq s^p \frac{p^*}{p} P(v_\varepsilon) - \frac{\alpha+1}{p_1}\lambda s^{p_1} P_{1,a}(v_\varepsilon) - s^{p^*} \equiv J(s, v_\varepsilon). \end{aligned}$$

As the function $s \mapsto J(s, v_\varepsilon)$ is continuous, $\lim_{s\to+\infty} J(s, v_\varepsilon) = -\infty$, and

$$\sup_{s\geq 0} \left\{ s^p \frac{p^*}{p} P(v_\varepsilon) - \frac{\alpha+1}{p_1}\lambda s^{p_1} P_{1,a}(v_\varepsilon) - s^{p^*} \right\} > 0,$$

then there exists $s_0 \in (0, +\infty)$ such that:

$$\sup_{0\leq s\leq s_0} J(s, v_\varepsilon) < \frac{N}{p^*} S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}, \ \forall \lambda \in (0, \delta_2).$$

If $N \geq p^2$, we have

$$\begin{aligned} J(s, v_\varepsilon) &\leq P(v_\varepsilon)s^p - s^{p^*} - \frac{\alpha+1}{p_1}\lambda s_0^{p_1} P_{1,a}(v_\varepsilon), \\ &\leq \frac{S_p}{2^{\frac{p}{N}}} s^p - s^{p^*} - C\varepsilon^{(p-1)/p} + o(\varepsilon^{(p-1)/p}) + O(\varepsilon^{(N-p)/p}) \\ &- \frac{\alpha+1}{p_1}\lambda s_0^{p_1} P_{1,a}(v_\varepsilon). \end{aligned}$$

Therefore, for all $\lambda \in (0, \delta_3)$

$$\sup_{s\geq s_0} J(s, v_\varepsilon) \leq \frac{N}{2p^*} S_p^{\frac{N}{p}} - C\varepsilon^{(p-1)/p} + o(\varepsilon^{(p-1)/p}) + O(\varepsilon^{(N-p)/p}) - K\lambda^{\frac{p}{p-p_1}},$$

where $\delta_3 = \left(\frac{(\alpha+1)s_0^{p_1}P_{1,a}(v_\varepsilon)}{2Kp_1}\right)^{\frac{p-p_1}{p_1}}$.
As

$$\frac{N-p}{p} - \frac{p-1}{p} \geq \frac{(p-1)^2}{p} > 0,$$

one can fix $\varepsilon > 0$ such that

$$-C\varepsilon^{(p-1)/p} + o(\varepsilon^{(p-1)/p}) + O(\varepsilon^{(N-p)/p}) < 0.$$

If we set $\lambda^* = \min\{\delta_2, \delta_3\}$, we obtain

$$\sup_{s\geq 0} I_{\lambda,\mu}(sv_\varepsilon, sv_\varepsilon) \leq \sup_{s\geq 0} J(s, v_\varepsilon) < \frac{N}{p^*}S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}, \ \forall\, \lambda \in (0, \lambda^*),$$

and finally

$$\alpha_2(\lambda,\mu) < \frac{N}{p^*}S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}, \ \forall\, \lambda \in (0, \lambda^*).$$

The case $N < p^2$ can be proved by following the same steps. □

Theorem 2.3.3. *If $p = q > 1$ and $(\lambda, \mu) \in \mathcal{D}_+ \equiv \mathcal{D} \cap ((0, \lambda^*) \times (0, +\infty))$. Then Problem (1.1) has at least two nonnegative solutions.*

Proof. The first solution (U^1, V^1) corresponding to the level $\alpha_1(\lambda, \mu)$ has been proved in the above. Now, to obtain the second solution, we take the minimizing sequence $(U_n^2, V_n^2) \equiv (s_2(u_n, v_n, \lambda, \mu))u_n, t_2(u_n, v_n, \lambda, \mu))v_n)$ such that

$$I(U_n^2, V_n^2) \to \alpha_2(\lambda, \mu), \ \|I'(U_n^2, V_n^2)\|_* \to 0 \text{ as } n \to +\infty.$$

Notice that If $(\lambda, \mu) \in \mathcal{D}_+$, one has

$$\alpha_2(\lambda,\mu) < \frac{N}{p^*}S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}.$$

Then, we can extract a subsequence of (U_n^2, V_n^2), still denoted (U_n^2, V_n^2), and two nonnegative and nontrivial functions belonging to W such that

$$\begin{array}{rcl} U_n^2 & \to & U^2 \text{ in } W^{1,p}(\Omega), \text{ as } n \to +\infty \\ V_n^2 & \to & V^2 \text{ in } W^{1,q}(\Omega), \text{ as } n \to +\infty. \end{array}$$

So (U^2, V^2) is a solution of Problem (1.1) satisfying

$$\partial_{ss}\widetilde{I}(U^2, V^2) < 0 \text{ and } \partial_{ss}\widetilde{I}(U^1, V^1) > 0,$$

which imply $(U^1, V^1) \neq (U^2, V^2)$. □

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Chapter 3

On local compactness in quasilinear elliptic problems

Abstract

One of the major difficulties in nonlinear elliptic problems involving critical nonlinearities is the compactness of Palais-Smale sequences. In their celebrated work [7], Brézis and Nirenberg introduced the notion of critical level for these sequences in the case of a critical perturbation of the Laplacian homogeneous eigenvalue problem. In this paper, we give a natural and general formula of the critical level for a large class of nonlinear elliptic critical problems. The sharpness of our formula is established by the construction of suitable Palais-Smale sequences which are not relatively compact.

3.1 Introduction

In nonlinear elliptic variational problems involving critical nonlinearities, one of the major difficulties is to recover the compactness of Palais-Smale sequences of the associated Euler-Lagrange functional. Such questions were first studied, in our knowledge, by Brézis and Nirenberg in their well-known work [7]. The concentration-compactness principle due to Lions [12] is widely used to overcome these difficulties. Other methods, based on the convergence almost everywhere of the gradients of Palais-Smale sequences, can be also used to recover the compactness. We refer the reader to the papers by Boccardo and Murat [5] and by J. M. Rakotoson [14] for bounded domains. For arbitray domains, we refer to the recent work by A. El Hamidi and J. M. Rakotoson [9].

In [7], the authors studied the critical perturbation of the eigenvalue problem:

$$\left\{ \begin{array}{rcl} -\Delta u & = & \lambda u + u^{2^*-1} \text{ in } \Omega, \\ u & > & 0 \text{ in } \Omega, \\ u & = & 0 \text{ on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary, $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent of the embedding $W^{1,2}(\Omega) \subset L^p(\Omega)$, and λ is a positive parameter. The authors introduced an important condition on the level corresponding to the energy of Palais-Smale sequences which guarantees their relative compactness. Indeed, let (u_n) be a Palais-Smale sequence for the Euler-Lagrange functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}.$$

More precisely, the authors showed that if

$$\lim_{n\to+\infty} I_\lambda(u_n) < \frac{1}{N} S^{\frac{N}{2}} \tag{1.2}$$

then (u_n) est relatively compact, which implies the existence of nontrivial critical points of I_λ. Here, S denotes the best Sobolev constant in the embedding $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$. In this work, we begin by giving the generalization of condition (1.2) for the quasilinear equation

$$\begin{aligned} &-\Delta_p u = \lambda f(x,u) + |u|^{p^*-2}u \ \text{ in } \ \Omega, \\ &u|_\Gamma = 0 \ \text{ and } \ \tfrac{\partial u}{\partial \nu}|_\Sigma = 0, \end{aligned} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where Γ and Σ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Δ_p is the p-Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. Here, f is a subcritical perturbation of $|u|^{p^*-1}$.

The sharpness of our result is estabished by the construction of suitable Palais-Smale sequences (corresponding to the critical level) which are not relatively compact.

Then we give the analogous condition to (1.2) for a general system with critical exponents

$$\begin{cases} -\Delta_p u &= \lambda f(x,u) + u|u|^{\alpha-1}|v|^{\beta+1} \ \text{ in } \ \Omega \\ \\ -\Delta_q v &= \mu g(x,v) + |u|^{\alpha+1}|v|^{\beta-1}v \ \text{ in } \ \Omega \end{cases}$$

together with Dirichlet or mixed boundary conditions, where f and g are subcritical perturbations of $|u|^{p^*-1}$ and $|v|^{q^*-1}$ respectively, $p^* = \frac{Np}{N-p}$ (resp. $q^* = \frac{Nq}{N-q}$) is the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \subset L^r(\Omega)$ (resp. $W^{1,q}(\Omega) \subset L^r(\Omega)$). Our approach provides a general condition based on the Nehari manifold, which can be extended to a large class of critical nonlinear problems. In this work, we confine ourselves to systems involving $(p,q)-$Laplacian operators and critical nonlinearities. The sharpness of our result is estabished, in the special case $p = q$, by the construction of suitable Palais-Smale sequences which are not relatively compact. The question of sharpness corresponding to the case $p \neq q$ is still open.

For a more complete description of nonlinear elliptic systems, we refer the reader to the papers by De Figueiredo [10] and by De Figueiredo & Felmer [11] and the references therein.

3.2 A general local compactness result

For the reader's convenience, we start with the scalar case and to render the paper selfcontained we will recall or show some well-known facts.

3.2.1 The scalar case

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary $\partial\Omega$. Let $f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function which is measurable in x, continuous in u and satisfying the growth condition at infinity

$$|f(x,u)| = o(u^{p^*-1}) \ \text{ as } \ u \to +\infty, \ \text{ uniformly in } \ x. \tag{2.4}$$

This situation occurs, for example, in the special cases $f(x,u) = u$ or $f(x,u) = u^{q-1}$, $1 < q < p^*$.
Consider the problem

$$\begin{aligned} &-\Delta_p u = \lambda f(x,u) + |u|^{p^*-2}u \ \text{ in } \ \Omega, \\ &u|_\Gamma = 0 \ \text{ and } \ \tfrac{\partial u}{\partial \nu}|_\Sigma = 0, \end{aligned} \tag{2.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where Γ and Σ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Problem (2.5) is posed in the framework of the Sobolev space

$$W^{1,p}_\Gamma(\Omega) = \{u \in W^{1,p}(\Omega) \ : \ u|_\Gamma = 0\},$$

which is the closure of $C^1_0(\Omega \cap \Gamma, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$. Notice that $meas(\Gamma) > 0$ implies that the Poincaré inequality is still available in $W^{1,p}_\Gamma(\Omega)$, so it can be endowed with the norm

$$||u|| = ||\nabla u||_p$$

and $(W^{1,p}_\Gamma(\Omega), ||\,.\,||)$ is a reflexive and separable Banach space. The associated Euler-Lagrange functional is given by

$$J_\lambda(u) := \frac{1}{p}||\nabla u||_p^p - \frac{1}{p^*}||u||_{p^*}^{p^*} - \lambda \int_\Omega F(x,u(x))\,dx$$

the corresponding Euler-Lagrange functional, where $F(x,u) := \int_0^u f(x,s)\,ds$.
We recall here that the Nahari manifold associated to the functional J_λ is given by:

$$\mathcal{N}_{J_\lambda} = \{u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\} \ : \ J'_\lambda(u)(u) = 0\},$$

and it is clear that $\mathcal{N}_{J_\lambda}$ contains all nontrivial critical points of J_λ. This manifold can be characterized more explicitely by the following

$$\mathcal{N}_{J_\lambda} = \left\{ tu, \ (t,u) \in (\mathbb{R} \setminus \{0\}) \times (W^{1,p}_\Gamma(\Omega) \setminus \{0\}) \ : \ \frac{d}{dt} J_\lambda(tu) = 0 \right\},$$

where $t \mapsto J_\lambda(tu)$ is a function defined from $\mathbb{R}$ to itself, for every u given in $W^{1,p}_\Gamma(\Omega)\setminus\{0\}$. We define the critical level associated to Problem (2.5) by:

$$c^*(\lambda) := \inf_{w\in\mathcal{N}_{J_0}} J_0(w) + \inf_{w\in\mathcal{N}_{J_\lambda}\cup\{0\}} J_\lambda(w). \tag{2.6}$$

At this stage, we can state and show our first result

Theorem 3.2.1. *Let $\lambda \in \mathbb{R}$ and (u_n) be a Palais-Smale sequence of J_λ such that*

$$\lim_{n\to+\infty} J_\lambda(u_n) < c^*(\lambda). \tag{2.7}$$

Then (u_n) is relatively compact.

Proof. Let $\lambda \in \mathbb{R}$ and (u_n) be a Palais-Smale sequence for J_λ of level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ for short) satisfying the condition (2.7). We claim that (u_n) is bounded in $W^{1,p}_\Gamma(\Omega)$. Indeed, on has one hand

$$\frac{1}{p}||\nabla u_n||_p^p - \frac{1}{p^*}||u_n||_{p^*}^{p^*} - \lambda\int_\Omega F(x,u_n)\,dx = c + o_n(1), \tag{2.8}$$

and

$$||\nabla u_n||_p^p - ||u_n||_{p^*}^{p^*} - \lambda\int_\Omega f(x,u_n)u_n\,dx = o_n(||\nabla u_n||_p). \tag{2.9}$$

Then,

$$\left(\frac{1}{p}-\frac{1}{p^*}\right)||u_n||_{p^*}^{p^*} + \frac{\lambda}{p}\int_\Omega f(x,u_n)u_n\,dx - \lambda\int_\Omega F(x,u_n)\,dx = c + o_n(1) + o_n(||\nabla u_n||_p).$$

Now, let $\varepsilon > 0$, using the growth condition (2.4), there exists $c_1(\varepsilon) > 0$ such that

$$|f(x,u)| \le \varepsilon|u|^{p^*-1}+c_1 \ \text{ and } \ |F(x,u)| \le \frac{\varepsilon}{p^*}|u|^{p^*}+c_1, \ \text{ a.e. } x\in\Omega \ \text{ and for every } \ u\in\mathbb{R}.$$

Applying the Hölder and the Young inequalities to the last relations, it follows

$$||u_n||_{p^*}^{p^*} \le \varepsilon||\nabla u_n||_p + c_2(|\Omega|,\lambda,\varepsilon). \tag{2.10}$$

Combining (2.10) and (2.8), we deduce that (u_n) is in fact bounded in $W^{1,p}_\Gamma(\Omega)$. So passing, if necessary to a subsequence, we can consider that

$$\begin{array}{rcl} u_n & \rightharpoonup & u \text{ in } W^{1,p}_\Gamma(\Omega), \\ u_n & \to & u \text{ a.e. in } \Omega. \end{array}$$

On the other hand, the growth condition (2.4) implies also that, for almost every $x \in \Omega$, the functions $s \mapsto F(x,s)$ and $s \mapsto sf(x,s)$ satisfy the conditions of the Brézis-Lieb Lemma (see Theorem 2 in [6]). Thus, we get the identities

$$\begin{array}{rcl} \displaystyle\int_\Omega F(x,v_n)\,dx & = & \displaystyle\int_\Omega F(x,u_n) - \int_\Omega F(x,u) + o_n(1), \\ \displaystyle\int_\Omega f(x,v_n)v_n\,dx & = & \displaystyle\int_\Omega f(x,u_n)u_n - \int_\Omega f(x,u)u + o_n(1). \end{array}$$

Moreover, let $\varepsilon > 0$, there is $c_1(\varepsilon) > 0$ such that

$$\left|\int_\Omega f(x, v_n)v_n\,dx\right| \le \varepsilon||v_n||_{p^*}^{p^*} + c_1||v_n||_1.$$

Let $C > 0$ (which is independent of n and ε), such that $||v_n||_{p^*}^{p^*} \le C$. Since (v_n) converges strongly to 0 in $L^1(\Omega)$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $||v_n||_1 \le \varepsilon/c_1$, for every $n \ge n_0(\varepsilon)$, and consequently

$$|\int_\Omega f(x, v_n)v_n\,dx| \le \varepsilon(1 + C), \quad \forall n \ge n_0(\varepsilon).$$

In the same way, rewriting $F(x, v_n) = \int_0^{v_n} f(x, s)\,ds$ and using the same arguments as above, we deduce that

$$\int_\Omega F(x, v_n)\,dx = o_n(1) \tag{2.11}$$

$$\int_\Omega f(x, v_n)v_n\,dx = o_n(1). \tag{2.12}$$

Applying once again the Brézis-Lieb Lemma, we conclude that $u \in \mathcal{N}_{J_\lambda} \cup \{0\}$ and

$$||v_n||^p - ||v_n||_{p^*}^{p^*} = o_n(1), \tag{2.13}$$

$$J_0(v_n) := \frac{1}{p}||v_n||^p - \frac{1}{p^*}||v_n||_{p^*}^{p^*} = c - J_\lambda(u) + o_n(1). \tag{2.14}$$

A direct computation gives

$$\mathcal{N}_{J_0} = \left\{t_0(u)u \;:\; u \in W_\Gamma^{1,p}(\Omega) \setminus \{0\}\right\},$$

where

$$t_0(u) := \left(\frac{||u||^p}{||u||_{p^*}^{p^*}}\right)^{\frac{1}{p^*-p}}.$$

Now, let b be the common limit of $||v_n||^p$ and $||v_n||_{p^*}^{p^*}$. Suppose that $b \neq 0$. On one hand we have

$$\begin{aligned} J_0(t_0(v_n)v_n) &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{||v_n||^p}{||v_n||_{p^*}^{p}}\right)^{\frac{p^*}{p^*-p}} \\ &\ge \inf_{w \in \mathcal{N}_{J_0}} J_0(w). \end{aligned}$$

Then

$$\lim_{n\to+\infty} J_0(t_0(v_n)v_n) = \frac{b}{N} \ge \inf_{w \in \mathcal{N}_{J_0}} J_0(w).$$

On the other hand, the identity (2.14) leads to

$$\frac{b}{N} = c - J_\lambda(u).$$

It follows then

$$\begin{aligned} c &\geq \inf_{w\in\mathcal{N}_{J_0}} J_0(w) + J_\lambda(u) \\ &\geq \inf_{w\in\mathcal{N}_{J_0}} J_0(w) + \inf_{w\in\mathcal{N}_{J_\lambda}\cup\{0\}} J_\lambda(w), \end{aligned}$$

which contradicts the condition (2.7). This achives the proof. □

3.2.2 Sharpness of the critical level formula in the scalar case

To show the sharpness of the critical level formula (2.7), it suffices to carry out a Palais-Smale sequence for J_λ of level $c^*(\lambda)$ which contains no convergent subsequence.
Consider, for a given $\varepsilon > 0$, the extremal function

$$\Phi_\varepsilon(x) = C_N \varepsilon^{\frac{N-p}{p^2}} \left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}} \quad \text{with} \quad C_N := \left(N\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{(N-p)/p^2}$$

which attains the best constant S of the Sobolev embedding

$$D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N).$$

Without loss of generality, we can consider that $0 \in \Sigma$. Moreover, the set $\partial\Omega$ satisfies the following property (see more details in Adimurthi, Pacella and Yadava [1]): There exist $\delta > 0$, an open neighborhood $\mathcal{V}$ of 0 and a diffeomorphism $\Psi : B_\delta(0) \longrightarrow \mathcal{V}$ which has a jacobian determinant equal to one at 0, with $\Psi(B_\delta^+) = \mathcal{V} \cap \Omega$, where $B_\delta^+ = B_\delta(0) \cap \{x \in \mathbb{R}^N : x_N > 0\}$.
Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in a neighborhood of the origin.
We define the sequence defined by

$$\psi_n(x) := \varphi(x)\Phi_{1/n}(x), \quad \text{for} \quad n \in \mathbb{N}^*. \tag{2.15}$$

It is well known that the sequence $(\psi_n) \subset W_\Gamma^{1,p}(\Omega)$ is a Palais-Smale sequence for J_0 of level $\inf_{w\in\mathcal{N}_{J_0}} J_0(w)$, which satisfies

$$\begin{aligned} \psi_n &\rightarrow 0 \text{ a.e. in } \Omega, \\ \nabla\psi_n &\rightarrow 0 \text{ a.e. in } \Omega, \\ ||\psi_n||_{p^*}^{p^*} &\longrightarrow \left[N \inf_{w\in\mathcal{N}_{J_0}} J_0(w)\right]^{p/N} := \ell \text{ as } n \longrightarrow +\infty, \\ ||\nabla\psi_n||_p^p &\longrightarrow \left[N \inf_{w\in\mathcal{N}_{J_0}} J_0(w)\right]^{p/N} := \ell \text{ as } n \longrightarrow +\infty. \end{aligned}$$

Now, let (u_n) be a Palais-Smale sequence of J_λ of level $\inf_{w\in\mathcal{N}_{J_\lambda}\cup\{0\}} J_\lambda(w)$. We will not go into further details concerning which subcritical terms $f(u)$ allow the existence of such sequences, but in the litterature, this occurs for various classes of subcritical terms. Applying Theorem 3.2.1, there exists a subsequence, still denoted by (u_n), which converges to some $u \in W^{1,p}_\Gamma(\Omega)$. Then

$$\begin{array}{rcl} ||u_n+\psi_n||_{p^*} & \leq & C, \\ u_n+\psi_n & \to & u \text{ a.e. in } \Omega, \\ ||\nabla u_n+\nabla\psi_n||_p & \leq & C, \\ \nabla u_n+\nabla\psi_n & \to & \nabla u \text{ a.e. in } \Omega. \end{array}$$

where C a positive constant independent of n. We apply the Brézis-Lieb Lemma to the sequence $(u_n+\psi_n)$ and get

$$||u_n+\psi_n||_{p^*}^{p^*} = ||(u_n-u)+\psi_n||_{p^*}^{p^*} + ||u||_{p^*}^{p^*} + \mathrm{o}_n(1).$$

Moreover, one has

$$-||u_n-u||_{p^*}+||\psi_n||_{p^*}-\ell^{1/p^*} \leq ||(u_n-u)+\psi_n||_{p^*}-\ell^{1/p^*} \leq ||u_n-u||_{p^*}+||\psi_n||_{p^*}-\ell^{1/p^*}$$

which implies that

$$||(u_n-u)+\psi_n||_{p^*}-\ell^{1/p^*} = \mathrm{o}_n(1).$$

Therefore, we conclude that

$$||u_n+\psi_n||_{p^*}^{p^*} = ||u||_{p^*}^{p^*} + \ell + \mathrm{o}_n(1).$$

The same argumets applied to the sequence $(\nabla u_n+\nabla\psi_n)$ give

$$||\nabla u_n+\nabla\psi_n||_p^p = ||\nabla u||_p^p + \ell + \mathrm{o}_n(1).$$

Finally, using the fact that

$$|\psi_n|^{p^*} \overset{*}{\rightharpoonup} \ell\,\delta_0 \quad \text{weakly} * \text{in } \mathcal{M}^+(\Omega) \tag{2.16}$$

$$|\nabla\psi_n|^{p} \overset{*}{\rightharpoonup} \ell\,\delta_0 \quad \text{weakly} * \text{in } \mathcal{M}^+(\Omega) \tag{2.17}$$

where δ_0 is the Dirac measure concentrated at the origin and $\mathcal{M}^+(\Omega)$ is the space of positive finite measures [20]), we get that the sequence $(u_n+\psi_n)$ is a Palais-Smale sequence of J_λ of level $c^*(\lambda)$.
We hence constructed a Palais-Smale sequence $(u_n+\psi_n)$ of J_λ of level $c^*(\lambda)$ which can not be relatively compact in $W^{1,p}_\Gamma(\Omega)$. This justifies the sharpness of the critical level formula (2.7).

Remark 3.2.1. *If we are interested by the homogeneous Dirichlet conditions, i.e. if $\Sigma=\emptyset$, the same arguments developed above are still valid, it suffices to assume that the origin $0\in\Omega$ and consider $\varphi\in C_0^\infty(\Omega)$ such that $\varphi\equiv 1$ in a neighborhood of the origin.*

3.2.3 The system case

Now, consider the system

$$\begin{cases} -\Delta_p u &= \lambda f(x,u) + u|u|^{\alpha-1}|v|^{\beta+1}, \\ \\ -\Delta_q v &= \mu g(x,v) + |u|^{\alpha+1}|v|^{\beta-1}v, \end{cases} \tag{2.18}$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}|_{\Sigma_1} = 0, \\ \\ v|_{\Gamma_2} = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0, \end{cases} \tag{2.19}$$

where, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma}_i \cup \overline{\Sigma}_i$, where Γ_i and Σ_i are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma_i \cap \Sigma_i = \emptyset$, $i \in \{1,2\}$. Δ_p is the p-Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. Also, it is clear that when $\Gamma_1 = \Gamma_2 = \partial\Omega$, one deals with homogeneous Dirichlet boundary conditions. We assume here that

$$1 < p < N, \;\; 1 < q < N, \tag{2.20}$$

and the critical condition

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1. \tag{2.21}$$

Indeed, this condition represents the maximal growth such that the integrability of the product term $|u|^{\alpha+1}|v|^{\beta+1}$ (which will appear in the Euler-Lagrange functional) can be guaranteed by suitable Hölder estimates.
The functions f and g are two caratheodory functions which satisfy the growth conditions

$$|f(x,u)| = o(u^{p^*-1}) \text{ as } u \to +\infty, \text{ uniformly in } x, \tag{2.22}$$
$$|g(x,v)| = o(v^{q^*-1}) \text{ as } v \to +\infty, \text{ uniformly in } x. \tag{2.23}$$

Problem (2.18), together with (2.19), is posed in the framework of the Sobolev space $W = W^{1,p}_{\Gamma_1}(\Omega) \times W^{1,q}_{\Gamma_2}(\Omega)$, where

$$W^{1,p}_{\Gamma_1}(\Omega) = \{u \in W^{1,p}(\Omega) \;:\; u|_{\Gamma_1} = 0\}, \;\; W^{1,q}_{\Gamma_2}(\Omega) = \{u \in W^{1,q}(\Omega) \;:\; u|_{\Gamma_2} = 0\},$$

which are respectively the closure of $C^1_0(\Omega \cap \Gamma_1, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$ and $C^1_0(\Omega \cap \Gamma_2, \mathbb{R})$ with respect to the norm of $W^{1,q}(\Omega)$. Notice that $meas(\Gamma_i) > 0$, $i = 1, 2$, imply that the Poincaré inequality is still available in $W^{1,p}_{\Gamma_1}(\Omega)$ and $W^{1,q}_{\Gamma_2}(\Omega)$, so W can be endowed with the norm

$$||(u,v)|| = ||\nabla u||_p + ||\nabla v||_q$$

and $(W, ||\,.\,||)$ is a reflexive and separable Banach space. The associated Euler-Lagrange functional $I_{\lambda,\mu} \in C^1(W, \mathbb{R})$ is given by

$$I_{\lambda,\mu}(u,v) = (\alpha+1)\left(\frac{P(u)}{p} - \lambda\int_\Omega F(x,u)\right)+(\beta+1)\left(\frac{Q(v)}{q} - \mu\int_\Omega G(x,v)\right)-R(u,v),$$

where $P(u) = ||\nabla u||_p^p$, $Q(v) = ||\nabla v||_q^q$, $F(x,u) = \int_0^u f(x,s)\,ds$, $G(x,v) = \int_0^v g(x,t)\,dt$, and $R(u,v) = \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}dx$. Notice that $R(u,v) \le ||u||_{p^*}^{\alpha+1}||v||_{q^*}^{\beta+1} < +\infty$.
Consider the Nehari manifold associated to Problem (2.18) given by

$$\mathcal{N}_{\lambda,\mu} = \{(u,v) \in W \setminus \{(0,0)\} \;/\; D_1 I_{\lambda,\mu}(u,v)(u) = D_2 I_{\lambda,\mu}(u,v)(v) = 0\},$$

where $D_1 I_{\lambda,\mu}$ and $D_2 I_{\lambda,\mu}$ are the derivative of $I_{\lambda,\mu}$ with respect to the first variable and the second variable respectively.
An interesting and useful characterization of $\mathcal{N}_{\lambda,\mu}$ is the following

$$\mathcal{N}_{\lambda,\mu} = \{(su,tv) \;/\; (s,u,t,v) \in \mathcal{Z}^* \text{ and } \partial_s I_{\lambda,\mu}(su,tv) = \partial_t I_{\lambda,\mu}(su,tv) = 0\},$$

where

$$\mathcal{Z}^* = \{(s,u,t,v);\ (s,t) \in \mathbb{R}^2,\ (u,v) \in W^{1,p}_{\Gamma_1}(\Omega) \times W^{1,q}_{\Gamma_2}(\Omega), (su,tv) \neq (0,0)\}$$

and $I_{\lambda,\mu}$ is considered as a functional of four variables (s,u,t,v) in $\mathcal{Z} := \mathbb{R} \times W^{1,p}_{\Gamma_1}(\Omega) \times \mathbb{R} \times W^{1,q}_{\Gamma_2}(\Omega)$.

Definition 3.2.1. *Let λ and μ be two real parameters. A sequence $(u_n, v_n) \in W$ is a Palais-Smale sequence of the functional $I_{\lambda,\mu}$ if*

- *there exists $c \in \mathbb{R}$ such that* $\lim_{n\to+\infty} I_{\lambda,\mu}(u_n,v_n) = c$ (2.24)
- *$DI_{\lambda,\mu}(u_n,v_n)$ converges strongly in the dual W' of W* (2.25)

where $DI_{\lambda,\mu}(u_n,v_n)$ denotes the Gâteaux derivative of $I_{\lambda,\mu}$.

The last condition (2.25) implies that

$$\begin{aligned} D_1 I_{\lambda,\mu}(u_n,v_n)(u_n) &= \mathrm{o}\,(||u_n||_{p^*}) && (2.26)\\ D_2 I_{\lambda,\mu}(u_n,v_n)(v_n) &= \mathrm{o}\,(||v_n||_{q^*}). && (2.27)\end{aligned}$$

where $D_1 I_{\lambda,\mu}(u_n,v_n)$ (resp. $D_2 I_{\lambda,\mu}(u_n,v_n)$) denotes the Gâteaux derivative of $I_{\lambda,\mu}$ with respect to its first (resp. second) variable.

We introduce the critical level corresponding to Problem (2.18) by

$$c^*(\lambda,\mu) := \inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w) + \inf_{w\in\mathcal{N}_{\lambda,\mu}\cup\{(0,0)\}} I_{\lambda,\mu}(w). \tag{2.28}$$

Then we have the following

Theorem 3.2.2. *Let λ and μ be two real parameters and (u_n, v_n) be a Palais-Smale sequence of $I_{\lambda,\mu}$ such that*

$$c := \lim_{n\to+\infty} I_{\lambda,\mu}(u_n, v_n) < c^*(\lambda, \mu). \tag{2.29}$$

Then (u_n, v_n) relatively compact.

Proof. Let λ and μ be two real parameters and (u_n, v_n) be a Palais-Smale sequence of $I_{\lambda,\mu}$ satisfying the condition (2.29). We claim that (u_n, v_n) is bounded in W. Indeed, on one hand conditions (2.24), (2.26) and (2.27) can be rewritten as the following

$$\begin{aligned} I_{\lambda,\mu}(u_n, v_n) &= c + o_n(1) && (2.30)\\ P(u_n) - \lambda \int_\Omega f(x, u_n)u_n\, dx &= R(u_n, v_n) + \mathrm{o}\,(||u_n||_{p^*}) && (2.31)\\ Q(v_n) - \mu \int_\Omega f(x, v_n)v_n\, dx &= R(u_n, v_n) + \mathrm{o}\,(||v_n||_{q^*}). && (2.32) \end{aligned}$$

Using (2.21), one gets

$$\begin{aligned} R(u_n, v_n) &= \frac{\alpha+1}{p^*}\left(P(u_n) - \lambda \int_\Omega f(x, u_n)u_n\right) + \mathrm{o}\,(||u_n||_{p^*})\\ &+ \frac{\beta+1}{q^*}\left(Q(v_n) - \mu \int_\Omega g(x, v_n)v_n\right) + \mathrm{o}\,(||v_n||_{q^*}). \end{aligned} \tag{2.33}$$

Suppose that there is a subsequence, still denoted by (u_n, v_n) in W which is unbounded, *i.e.* $||\nabla u_n||_p + ||\nabla v_n||_q$ tends to $+\infty$ as n goes to $+\infty$.
If

$$\lim_{n\to+\infty} ||\nabla u_n||_p = +\infty,$$

then using (2.22) one has

$$\begin{aligned} \int_\Omega |f(x, u_n)u_n| &= \mathrm{o}\,(P(u_n)),\\ \int_\Omega |F(x, u_n)| &= \mathrm{o}\,(P(u_n)), \end{aligned}$$

since (2.22) implies that for every $\varepsilon > 0$, there exists $c_1(\varepsilon) > 0$ such that

$$|f(x, s)| \le \varepsilon |s|^{p^*-1} + c_1 \text{ and } |F(x, s)| \le \frac{\varepsilon}{p^*}|s|^{p^*} + c_1, \text{ a.e. } x \in \Omega, \ \forall s \in \mathbb{R}.$$

Similarly, if

$$\lim_{n\to+\infty} ||\nabla v_n||_q = +\infty,$$

then using (2.23) it follows

$$\begin{aligned}\int_\Omega |g(x,v_n)v_n| &= \mathrm{o}\,(Q(v_n)),\\ \int_\Omega |G(x,v_n)| &= \mathrm{o}\,(Q(v_n)).\end{aligned}$$

On one hand, suppose that

$$\lim_{n\to+\infty} ||\nabla u_n||_p = \lim_{n\to+\infty} ||\nabla v_n||_q = +\infty.$$

Substituting (2.33) in (2.30), we obtain

$$\begin{aligned} c+\mathrm{o}_n(1) &= (\alpha+1)\left(\frac{1}{p}-\frac{1}{p^*}+\mathrm{o}\,(P(u_n))^{\frac{p^*-p}{p}}\right)P(u_n)\\ &+ (\beta+1)\left(\frac{1}{q}-\frac{1}{q^*}+\mathrm{o}\,(Q(v_n))^{\frac{q^*-q}{q}}\right)Q(v_n) \longrightarrow_{n\to+\infty} +\infty\end{aligned}$$

which can not hold true. On the other hand, suppose that

$$\lim_{n\to+\infty} ||\nabla u_n||_p = +\infty \text{ and the sequence } ||\nabla v_n||_q \text{ is bounded,}$$

then (2.31) implies that $R(u_n,v_n)$ is unbounded while (2.32) implies, on the contrary, that $R(u_n,v_n)$ is bounded. The case

$$\lim_{n\to+\infty} ||\nabla v_n||_q = +\infty \text{ and the sequence } ||\nabla u_n||_p \text{ is bounded,}$$

leads to a contradiction with the same argument, which achieves the claim.
At this stage, we can assume, up to a subsequence, that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W^{1,p}_{\Gamma_1}(\Omega),\\ v_n &\rightharpoonup v \text{ in } W^{1,q}_{\Gamma_2}(\Omega),\\ u_n &\to u \text{ a.e. in } \Omega,\\ v_n &\to v \text{ a.e. in } \Omega.\end{aligned}$$

It is clear that

$$(u,v)\in\mathcal{N}_{\lambda,\mu}\cup\{(0,0)\}.$$

Let us set

$$X_n = u_n - u \text{ and } Y_n = v_n - v.$$

Using again the growth conditions (2.22) and (2.23), we show easily that the functions, which are defined on $\Omega\times\mathbb{R}$: $(x,s)\mapsto sf(x,s)$, $(x,s)\mapsto sg(x,s)$, $(x,s)\mapsto$

$F(x,s)$ and $(x,s) \mapsto G(x,s)$ satisfy the conditions of the Brézis-Lieb lemma [6]. Then, we have the decompositions

$$\begin{aligned}
\int_\Omega F(x,X_n) &= \int_\Omega F(x,u_n) - \int_\Omega F(x,u) + o_n(1),\\
\int_\Omega f(x,X_n)X_n &= \int_\Omega f(x,u_n)u_n - \int_\Omega f(x,u)u + o_n(1),\\
\int_\Omega G(x,Y_n) &= \int_\Omega G(x,v_n) - \int_\Omega G(x,v) + o_n(1),\\
\int_\Omega g(x,Y_n)Y_n &= \int_\Omega g(x,v_n)v_n - \int_\Omega g(x,v)v + o_n(1).
\end{aligned}$$

Moreover, let $\varepsilon > 0$, then there is $c_1(\varepsilon) > 0$ such that

$$\left| \int_\Omega f(x,X_n)X_n \, dx \right| \leq \varepsilon ||X_n||_{p^*}^{p^*} + c_1 ||X_n||_1.$$

Let C be a positive constant such that $||X_n||_{p^*}^{p^*} \leq C$. Since X_n converges to 0 in $L^1(\Omega)$, there exists $n_0(\varepsilon) \in \mathbb{N}$ verifying $||X_n||_1 \leq \varepsilon/c_1$, for every $n \geq n_0(\varepsilon)$, thus

$$\left| \int_\Omega f(x,X_n)X_n \, dx \right| \leq \varepsilon(1+C), \quad \forall n \geq n_0(\varepsilon).$$

In the same manner, writing $F(x,X_n) = \int_0^{X_n} f(x,s)\,ds$ and using the same arguments as above, we get

$$\int_\Omega F(x,X_n) = o_n(1) \ \text{ and } \ \int_\Omega f(x,X_n)X_n = o_n(1).$$

Similarly, it follows that

$$\int_\Omega G(x,Y_n) = o_n(1) \ \text{ and } \ \int_\Omega g(x,Y_n)Y_n = o_n(1).$$

Applying a slightly modified version of the Brézis-Lieb lemma [13], one has

$$R(X_n,Y_n) = R(u_n,v_n) - R(u,v) + o_n(1).$$

It follows that

$$\begin{aligned}
P(X_n) - R(X_n,Y_n) &= o_n(1),\\
Q(Y_n) - R(X_n,Y_n) &= o_n(1),\\
I_{0,0}(X_n,Y_n) &= c - I_{\lambda,\mu}(u,v) + o_n(1).
\end{aligned}$$

Notice that the Nehari manifold associated to $I_{0,0}$ is given by

$$\mathcal{N}_{0,0} = \left\{ (s_0(u,v)u, t_0(u,v)v); \ (u,v) \in W^{1,p}_{\Gamma_1}(\Omega) \times W^{1,q}_{\Gamma_2}(\Omega), \ u \not\equiv 0, \ v \not\equiv 0 \right\},$$

where

$$s_0(u,v) = \left[\frac{P(u)Q(v)^{\frac{r(\beta+1)}{q(\alpha+1)}}}{R(u,v)^{\frac{r}{\alpha+1}}}\right]^{\frac{1}{r-p}}, \quad t_0(u,v) = t(s_0(u,v)),$$

and

$$r = \frac{(\alpha+1)q}{q-(\beta+1)} > p, \quad t(s) = \left[\frac{R(u,v)}{Q(v)}\right]^{\frac{r}{q(\alpha+1)}} s^{\frac{r}{q}}.$$

Let ℓ be the common limit of $P(X_n)$, $Q(Y_n)$ and $R(X_n,Y_n)$. We claim that $\ell = 0$. By contradiction, suppose that $\ell \neq 0$, then on one hand we get

$$\begin{aligned} I_{0,0}(s_0(X_n,Y_n)X_n, t_0(X_n,Y_n)Y_n) &= (\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right)K(X_n,Y_n), \\ &\geq \inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w), \end{aligned} \tag{2.34}$$

where

$$K(X_n,Y_n) = \left[\frac{P(X_n)^{(\alpha+1)}Q(Y_n)^{(\beta+1)\frac{p}{q}}}{R(X_n,Y_n)^p}\right]^{\frac{r}{(\alpha+1)(r-p)}}.$$

A direct computation shows that

$$\lim_{n\to+\infty} K(X_n,Y_n) = \ell,$$

therefore

$$\lim_{n\to+\infty} I_{0,0}(s_0(X_n,Y_n)X_n, t_0(X_n,Y_n)Y_n) = \ell(\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right).$$

On the other hand,

$$\begin{aligned} \lim_{n\to+\infty} I_{0,0}(X_n,Y_n) &= \ell\left(\frac{\alpha+1}{p}+\frac{\beta+1}{q}-1\right) \\ &= \ell(\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right). \end{aligned}$$

Hence, we obtain

$$\ell(\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right) = c - I_{\lambda,\mu}(u,v),$$

and consequently

$$\begin{aligned} c &\geq \inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w) + I_{\lambda,\mu}(u,v) \\ &\geq \inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w) + \inf_{w\in\mathcal{N}_{\lambda,\mu}\cup\{(0,0)\}} I_{\lambda,\mu}(w). \end{aligned}$$

This leads to a contradiction with (2.29), then $\ell = 0$, which achieves the proof. □

Remark 3.2.2. *1) In the scalar case, we obtain the analogous of Theorem 3.2.2, the proof follows easily with the same arguments. We note here that if we consider the special case (1.1), direct computations show that*

$$\inf_{w\in\mathcal{N}_0} I_0(w) = \frac{1}{N} S^{\frac{N}{2}} \quad \textit{and} \quad \inf_{w\in\mathcal{N}_\lambda\cup\{0\}} I_\lambda(w) = 0,$$

which recovers the famous Brézis-Nirenberg condition (1.2).
2) It is clear that our condition (2.7) or (2.29) can be extended to a large class of quasilinear or semilinear differential operators: Leray-Lions type operators, fourth-order operators.
3) Using the Hölder inequality in the denominator $R(u,v)$, we get

$$\inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) \geq (\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right)\left[S_p S_q^{\frac{p(\beta+1)}{q(\alpha+1)}}\right]^{\frac{r}{r-p}}, \tag{2.35}$$

where S_p (resp. S_q) denotes the best Sobolev constant in the embedding $W^{1,p}_{\Gamma_1}(\Omega) \subset L^{p^*}(\Omega)$ (resp. $W^{1,q}_{\Gamma_2}(\Omega) \subset L^{q^*}(\Omega)$).

We end this note by the following interesting relation arising in the special case $p = q$ and $\Gamma_1 = \Gamma_2$.

Proposition 3.2.1. *Assume that $p = q > 1$. Then,*

$$\inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) = \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

Proof. In the special case $p = q$, direct computations give

$$p^* = \alpha+\beta+2 \quad \text{and} \quad (\alpha+1)\left(\frac{1}{p}-\frac{1}{r}\right) = \frac{p}{N-p}.$$

Then, using (2.35), we conclude that

$$\inf_{(u,v)\in\mathcal{N}_{0,0}} I_{0,0}(u,v) \geq \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

On the other hand, let $(u_n) \subset W^{1,p}_{\Gamma_1}(\Omega)$ be a minimizing sequence of S_p. Then using the identity (2.34), we get

$$\begin{aligned}\inf_{w\in\mathcal{N}_{0,0}} I_{0,0}(w) \leq I_{0,0}(s_0(u_n,u_n)u_n, t_0(u_n,u_n)u_n) &= \frac{p}{N-p}\left[\frac{||\nabla u_n||_p^p}{||u_n||_{p^*}^p}\right]^{\frac{rp^*}{(\alpha+1)(r-p)}} \\ &= \frac{p}{N-p}\left[\frac{||\nabla u_n||_p^p}{||u_n||_{p^*}^p}\right]^{\frac{N}{p}}.\end{aligned}$$

It is clear that the last quantity goes to $\dfrac{p}{N-p} S_p^{\frac{N}{p}}$ as $n + \infty$, which achieves the proof. □

Remark 3.2.3. *For the sharpness of the critical level (2.29), we define the sequence* $\psi_n(x) := \varphi(x)\Phi_{1/n}(x)$ *as in (2.15). We consider then a Palais-Smale sequence* (u_n, v_n) *for* $J_{\lambda,\mu}$ *of level* $\inf_{w\in\mathcal{N}_{\lambda,\mu}\cup\{(0,0)\}} I_{\lambda,\mu}(w)$. *Following the same argumets developed in the scalar case and using Proposition 3.2.1, we prove that the sequence* $(u_n+\psi_n, v_n+\psi_n)$ *is a Palais-Smale sequence for* $J_{\lambda,\mu}$ *of level* $c^*(\lambda,\mu)$ *and which can not be relatively compact in* W. *This implies the sharpness of the critical level formula (2.29).*

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Chapter 4

Existence and Regularity Results for an anisotropic system involving critical exponents

Abstract

In this paper, we establish some existence and regularity results of positive solutions of a critical anisotropic system by using variational methods

$$(P_{\lambda,\mu}) \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) & = & \lambda a(x)|u|^{p-2}u + u|u|^{\alpha-1}|v|^{\beta+1} \ in \ \Omega, \\ -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{q_i-2}\frac{\partial v}{\partial x_i}\right) & = & \mu b(x)|v|^{q-2}v + |u|^{\alpha+1}v|v|^{\beta-1} \ \ in \ \ \Omega, \\ \\ u \geq 0 \quad and \quad v \geq 0 \quad in \ \ \Omega, \\ u = 0 \quad and \quad v = 0 \quad on \quad \partial\Omega \end{cases}$$

where Ω is a bounded open domain of $\mathbb{R}^N$, λ and μ are positive parameters, p^* and q^* are respectively the critical exponents for these classes of problem. The functions a and b belong to spaces which will be specified later.

4.1 Introduction

In this paper, we are interested in existence results of nonlocal solutions to the following critical system:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) & = \lambda a(x)|u|^{p-2}u + u|u|^{\alpha-1}|v|^{\beta+1} \text{ in } \Omega, \\ -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{q_i-2}\frac{\partial v}{\partial x_i}\right) & = \mu b(x)|v|^{q-2}v + |u|^{\alpha+1}v|v|^{\beta-1} \text{ in } \Omega, \\ u \geq 0 \quad \text{and} \quad v \geq 0 \text{ in } \Omega, \\ u = 0 \quad \text{and} \quad v = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda \geq 0$, $\mu \geq 0$ are real parameters and the exponents p_i, α, q_i, β satisfy the following conditions

$$p_i > 1,\ q_i > 1, \sum_{i=1}^{N}\frac{1}{p_i} > 1,\ \sum_{i=1}^{N}\frac{1}{q_i} > 1$$

and

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1,$$

where p^* and q^* are defined by

$$p^* := \frac{N}{\displaystyle\sum_{i=1}^{N}\frac{1}{p_i} - 1} \text{ and } q^* := \frac{N}{\displaystyle\sum_{i=1}^{N}\frac{1}{q_i} - 1}.$$

We assume in the sequel that

$$\max\{p_1, p_2, \cdots, p_N\} < p^* \quad \text{and} \quad \max\{q_1, q_2, \cdots, q_N\} < q^*,$$

p^* and q^* are the effective critical exponents associated to the operators

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}\right|^{p_i-2}\frac{\partial}{\partial x_i}\right) \quad \text{and} \quad \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}\right|^{q_i-2}\frac{\partial}{\partial x_i}\right),$$

respectively [8, 7].

The functions a and b are assumed to be nontrivial, nonnegative , $a \in L^{\frac{p^*}{p^*-p}}(\Omega)$ and $b \in L^{\frac{q^*}{q^*-q}}(\Omega)$.

In this work, we deal with the nonlocal existence, with respect to λ and μ, of nonnegative, nontrivial solutions to Problem (1.1). Consider the Euler-Lagrange functional associated to Problem (1.1) defined by

$$I(u,v) := (\alpha+1)\left(\sum_{i=1}^{N}\frac{P_i(u)}{p_i} - \frac{\lambda}{p}P_a(u)\right) + (\beta+1)\left(\sum_{i=1}^{N}\frac{Q_i(v)}{q_i} - \frac{\mu}{q}Q_b(v)\right) - R(u,v),$$

where $P_i(u) := \displaystyle\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx$, $P_a(u) := \displaystyle\int_\Omega a(x)|u|^p dx$, $Q_i(v) := \displaystyle\int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{q_i} dx$, $Q_b(v) := \displaystyle\int_\Omega b(x)|v|^q dx$ and $R(u,v) := \displaystyle\int_\Omega |u|^{\alpha+1}|v|^{\beta+1} dx$.
The functional I is of class $C^1(W;\mathbb{R})$, where $W := W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$, $W_0^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{q}}(\Omega)$ are respectively the completions of the space $\mathcal{D}(\Omega)$ with respect of the norms :

$$\|u\|_{\vec{p}} := \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i} \quad \text{and} \quad \|v\|_{\vec{q}} := \sum_{i=1}^{N} \left\|\frac{\partial v}{\partial x_i}\right\|_{q_i}.$$

The spaces $W_0^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{q}}(\Omega)$ can also be seen as

$$W_0^{1,\vec{p}}(\Omega) = \left\{ u \in L^{p_+}(\Omega) \quad : \quad \left|\frac{\partial u}{\partial x_i}\right| \in L^{p_i}(\Omega), \quad i = 1, \cdots, N, \quad u_{|\partial\Omega} = 0, \right\},$$

$$W_0^{1,\vec{q}}(\Omega) = \left\{ v \in L^{q_+}(\Omega) \quad : \quad \left|\frac{\partial v}{\partial x_i}\right| \in L^{q_i}(\Omega), \quad i = 1, \cdots, N, \quad v_{|\partial\Omega} = 0 \right\},$$

where $\vec{p} := (p_1, p_2, \cdots, p_N)$ and $\vec{q} := (q_1, q_2, \cdots, q_N)$, the space W is endowed with norm

$$\|(u,v)\| := \|u\|_{\vec{p}} + \|v\|_{\vec{q}}.$$

We introduce the modified Euler-Lagrange functional $\widetilde{I}$ defined on $\mathcal{Z} := \mathbb{R} \times W_0^{1,\vec{p}}(\Omega) \times \mathbb{R} \times W_0^{1,\vec{q}}(\Omega)$ by

$$\widetilde{I}(s,u,t,v) := I(su, tv).$$

In the sequel, we set $p_- = \min\{p_1, p_2, \cdots, p_N\} = p_{i_0}$, $p_+ = \max\{p_1, p_2, \cdots, p_N\} = p_{i_1}$, $q_- = \min\{q_1, q_2, \cdots, q_N\} = q_{j_0}$, $q_+ = \max\{q_1, q_2, \cdots, q_N\} = q_{j_1}$, $P_-(u) = P_{i_0}(u)$, $P_+(u) = P_{i_1}(u)$, $Q_-(v) = Q_{j_0}(v)$ and $Q_+(v) = Q_{j_1}(v)$.

4.2 Preliminary results

Under the following assumptions

$$\begin{cases} p < p_- \leq p_+ < \alpha + 1, \\ q < q_- \leq q_+ < \beta + 1, \\ \lambda > 0, \\ \mu > 0. \end{cases} \tag{2.2}$$

We have the following lemmas

Lemma 4.2.1. *There exist $\lambda^* > 0$, $\mu^* > 0$ and $r > 0$, $\rho > 0$ such that*

$$I(u,v) \geq r, \;\; \forall (\lambda,\mu) \in (0,\lambda^*) \times (0,\mu^*), \;\; \forall (u,v) \in W, \text{ such that } \|(u,v)\| = \rho.$$

Proof. Let $(u, v) \in W$ such that

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i} \leq 1 \quad \text{and} \quad \left\|\frac{\partial v}{\partial x_i}\right\|_{q_i} \leq 1, \quad \forall\, i \in \{1, \cdots, N\}.$$

Let $\varepsilon > 0$ fixed, by Young's inequality, there exists a positive constant C_ε such that

$$R(u, v) \leq (\alpha + 1)\varepsilon \|u\|_{p^*}^{\alpha+1} + (\beta + 1)C_\varepsilon \|v\|_{q^*}^{\beta+1}.$$

Since

$$I(u, v) = (\alpha+1)\left(\sum_{i=1}^{N} \frac{P_i(u)}{p_i} - \lambda \frac{P_a(u)}{p}\right) + (\beta+1)\left(\sum_{i=1}^{N} \frac{Q_i(v)}{q_i} - \mu \frac{Q_b(v)}{q}\right) - R(u, v)$$

it follows that

$$\begin{aligned} I(u, v) \geq (\alpha + 1) &\left(\left(\frac{1}{p_1} + \cdots \frac{1}{p_N}\right) \sum_{i=1}^{N} P_i(u)^{p_+/p_i} - \lambda \frac{P_a(u)}{p} - \varepsilon \|u\|_{p^*}^{\alpha+1}\right) \\ +(\beta + 1) &\left(\left(\frac{1}{q_1} + \cdots \frac{1}{q_N}\right) \sum_{i=1}^{N} Q_i(v)^{q_+/q_i} - \mu \frac{Q_b(v)}{q} - C_\varepsilon \|v\|_{q^*}^{\beta+1}\right). \end{aligned}$$

Now, we calculate, for a fixed $j \in \{1, \cdots, N\}$

$$\begin{aligned} \frac{1}{p^*} - \frac{1}{p_j^*} &= \frac{1}{N}\left(\sum_{i=1}^{N} \frac{1}{p_i} - 1\right) - \left(\frac{1}{p_j} - \frac{1}{N}\right) \\ &= \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{1}{p_i} > 0, \end{aligned}$$

then $p^* < p_j^*$ for all $j \in \{1, \cdots, N\}$, where $p_j^* := p_j N/(N - p_j)$ is the critical exponent of the compact embedding $W^{1,p_j}(\Omega) \hookrightarrow L^r(\Omega)$. As $p^* < p_j^*$, we use then Sobolev inequalities;

$$\|u\|_{p^*} \leq c_{1,j} \|\nabla u\|_{p_j}$$

and then

$$\|u\|_{p^*} \leq c_1 \|u\|_{\vec{p}}.$$

By the same way

$$\|v\|_{q^*} \leq c_2 \|v\|_{\vec{q}}.$$

Consequently, there exist positive constants h_1, h_2, h_3, k_1, k_2 and k_3 such that

$$I(u, v) \geq \left(h_1 \|u\|_{\vec{p}}^{p_+} - h_2 \lambda \|u\|_{\vec{p}}^{p} - h_3 \|u\|_{\vec{p}}^{\alpha+1}\right) + \left(k_1 \|v\|_{\vec{q}}^{q_+} - k_2 \mu \|v\|_{\vec{q}}^{q} - k_3 \|v\|_{\vec{q}}^{\alpha+1}\right).$$

Since $1 < p < p^+ < \alpha + 1$, there exist $\lambda^* > 0$, $r_1 > 0$ and $\rho_1 > 0$ and such that

$$h_1 \|u\|_{\vec{p}}^{p_+} - h_2 \lambda \|u\|_{\vec{p}}^{p} - h_3 \|u\|_{\vec{p}}^{\alpha+1} \geq r_1, \;\; \forall\, u \in W_0^{1,\vec{p}}(\Omega) \; : \; \|u\|_{\vec{p}} = \rho_1, \; \forall \lambda \in (0, \lambda^*).$$

Similarly, Since $1 < q < q^+ < \beta + 1$, there exist $\mu^* > 0$, $r_2 > 0$ and $\rho_2 > 0$ and such that

$$k_1\|v\|_{\vec{q}}^{q+} - k_2\mu\|v\|_{\vec{q}}^{q} - k_3\|v\|_{\vec{q}}^{\beta+1} \geq r_2, \ \ \forall\, v \in W_0^{1,\vec{q}}(\Omega)\ :\ \|v\|_{\vec{q}} = \rho_2,\ \forall \mu \in (0, \mu^*).$$

Therefore, for all $(u, v) \in W$ such that $\|(u, v)\| = \rho := \rho_1 + \rho_2$ and for all $(\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*)$, one has

$$I(u, v) \geq r := r_1 + r_2 > 0,$$

which achieves the proof. □

Lemma 4.2.2. *The functional I is bounded from below in*

$$\overline{B}_\rho(0) = \{(u, v) \in W;\ \|(u, v)\| \leq \rho\}.$$

Moreover,

$$\inf_{(u,v)\in B_\rho(0)} I(u, v) < 0, \quad \forall(\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*). \tag{2.3}$$

Proof. It is easy to check that I is bounded from below in $\overline{B}_\rho(0)$. To prove (2.3), fix $(\phi, \psi) \in \widetilde{W}$ and let $s > 0$ and $t > 0$. Then

$$I(s\phi, t\psi) \leq (\alpha+1)\left(\sum_{i=1}^{N} s^{p_i}\frac{P_i(\phi)}{p_i} - \lambda s^p\frac{P_a(\phi)}{p}\right)+(\beta+1)\left(\sum_{i=1}^{N} t^{q_i}\frac{Q_i(\psi)}{q_i} - \mu t^p\frac{Q_b(\psi)}{q}\right).$$

As $p < p_-$ and $q < q_-$, the last inequality implies for s_0 and t_0 sufficiently small

$$I(s_0\phi, t_0\psi) < 0 \quad \text{and} \quad (s_0\phi, t_0\psi) \in \bar{B}_\rho(0)$$

from where follows the lemma. □

For every $(\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*)$, we introduce

$$\alpha(\lambda, \mu) := \inf_{(u,v)\in\overline{B}_\rho(0)} I(u, v).$$

Applying Ekeland's principle to the functional I on the metric space $(\overline{B}_\rho(0), d)$ endowed with the metric d given by

$$d\left((u_1, v_1), (u_2, v_2)\right) = \|(u_1 - u_2, v_1 - v_2)\| = \|u_1 - u_2\|_{\vec{p}} + \|v_1 - v_2\|_{\vec{q}},$$

there exists a sequence $(u_n, v_n) \subset \overline{B}_\rho(0)$ such that

$$I(u_n, v_n) \longrightarrow \alpha(\lambda, \mu) \quad \text{as} \quad n \to +\infty \tag{2.4}$$

and

$$I(u, v) - I(u_n, v_n) \leq \frac{1}{n}\|(u - u_n, v - v_n)\|, \quad \text{for all} \quad (u, v) \neq (u_n, v_n). \tag{2.5}$$

Using the differentiability of I over W, from the previous inequality it follows that

$$I'(u_n, v_n) \longrightarrow 0, \quad \text{as} \quad n \to +\infty. \tag{2.6}$$

From (2.5) and (2.6)

$$I(u_n, v_n) \longrightarrow \alpha(\lambda, \mu) \quad \text{and} \quad I'(u_n, v_n) \longrightarrow 0, \quad \text{as} \quad n \to +\infty, \tag{2.7}$$

then (u_n, v_n) is a bounded $(PS)_{\alpha(\lambda,\mu)}$ sequence to I. Hereafter, we will denote by $(u^*, v^*) \in W$ the weak limit of (u_n, v_n), up to a subsequence. Moreover, by definition of I we can assume that the sequence (u_n, v_n) is a sequence of nonnegative functions.

Theorem 4.2.1. *The weak limit* (u^*, v^*) *of* (u_n, v_n) *satisfies*

$$I'(u^*, v^*) = 0 \quad \textit{and} \quad (u^*, v^*) \in \widetilde{W}, \quad \textit{for all} \quad (\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*).$$

Proof. Using the properties of (u_n, v_n), one can easily show that

$$I(u_n, v_n) - \frac{1}{p^*} D_1 I(u_n, v_n) u_n - \frac{1}{q^*} D_1 I(u_n, v_n) v_n = \alpha(\lambda, \mu) + o_n(1),$$

where $D_1 I(u_n, v_n)$ and $D_2 I(u_n, v_n)$ are respectively the first and second partial Gâteaux derivatives in u and v of the functional I. Thus

$$\begin{aligned} \alpha(\lambda, \mu) + o_n(1) = (\alpha + 1) & \left(\sum_{i=1}^{N} \left(\frac{1}{p_i} - \frac{1}{p^*} \right) P_i(u_n) - \lambda \left(\frac{1}{p} - \frac{1}{p^*} \right) P_a(u_n) \right) \\ + (\beta + 1) & \left(\sum_{i=1}^{N} \left(\frac{1}{q_i} - \frac{1}{q^*} \right) Q_i(v_n) - \mu \left(\frac{1}{q} - \frac{1}{q^*} \right) Q_b(v_n) \right). \end{aligned} \tag{2.8}$$

As it is said befeore, one has

$$\begin{array}{cccl} u_n & \rightharpoonup & u^* & \text{in} \quad W_0^{1,\vec{p}}(\Omega), \\ u_n & \longrightarrow & u^* & \text{in} \quad L^p(\Omega), \\ v_n & \rightharpoonup & v^* & \text{in} \quad W_0^{1,\vec{q}}(\Omega), \\ v_n & \longrightarrow & v^* & \text{in} \quad L^q(\Omega). \end{array}$$

By the relation (2.8) and using the compact embeddings of $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^p(\Omega)$ and $W_0^{1,\vec{q}}(\Omega) \hookrightarrow L^q(\Omega)$, we obtain

$$\alpha(\lambda, \mu) + o_n(1) \geq -K_1 \|u_n\|_p^p - K_2 \|v_n\|_q^q$$

where K_1 and K_2 are two positive constants. Then we get, as n goes to $+\infty$:

$$0 > \alpha(\lambda, \mu) > -K_1 \|u^*\|_p^p - K_2 \|v^*\|_q^q,$$

and consequently $u^* \neq 0$ and $v^* \neq 0$. Finally, we use the result due to El Hamidi and Rakotoson [6] and obtain

$$\begin{array}{rclcl} \nabla u_n(x) & \longrightarrow & \nabla u^*(x) & \text{a.e. in} & \Omega, \\ \nabla v_n(x) & \longrightarrow & \nabla v^*(x) & \text{a.e. in} & \Omega. \end{array}$$

Therefore, the weak limit (u^*, v^*) is a nonnegative and nontrivial solution to (1.1). □

Under the following assumptions

$$\left\{ \begin{array}{rcl} p & < & \alpha + 1 \\ q & < & \beta + 1 \\ \max\{p_+, q_+\} & < & \min\{p, q\} \\ \lambda & \geq & 0 \\ \mu & \geq & 0 \end{array} \right. \tag{2.9}$$

We have the following lemmas

Lemma 4.2.3. *For every $\lambda > 0$ and $\mu > 0$, the functional I satisfies the following properties:*

a) There exist $r > 0$ and $\rho > 0$ such that

$$I(u, v) \geq r, \quad \forall\, (u, v) \in W, \quad \textit{such that} \quad \|(u, v)\| = \rho.$$

b) There exists $(e_{\lambda,\mu}, f_{\lambda,\mu}) \in W$ such that $\|(e_{\lambda,\mu}, f_{\lambda,\mu})\| \geq \rho$ such that

$$I(e_{\lambda,\mu}, f_{\lambda,\mu}) < 0.$$

Proof.

a) Using the same arguments in the proof of Lemma 4.2.1, we have for every $(u, v) \in W$ and $\|(u, v)\| \leq 1$, there exist positive constants h_1, h_2, h_3, k_1, k_2 and k_3 such that

$$I(u, v) \geq \left(h_1 \|u\|_{\vec{p}}^{p_+} - h_2 \lambda \|u\|_{\vec{p}}^{p} - h_3 \|u\|_{\vec{p}}^{\alpha+1}\right) + \left(k_1 \|v\|_{\vec{q}}^{q_+} - k_2 \mu \|v\|_{\vec{q}}^{q} - k_3 \|v\|_{\vec{q}}^{\alpha+1}\right).$$

As $p_+ < p < \alpha + 1$ and $q_+ < q < \beta + 1$, there exist r_1, r_2, $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$h_1 \|u\|_{\vec{p}}^{p_+} - h_2 \lambda \|u\|_{\vec{p}}^{p} - h_3 \|u\|_{\vec{p}}^{\alpha+1} \geq r_1, \qquad \text{for all} \quad \|u\|_{\vec{p}} \leq \rho_1$$

and

$$k_1 \|v\|_{\vec{q}}^{q_+} - k_2 \mu \|v\|_{\vec{q}}^{q} - k_3 \|v\|_{\vec{q}}^{\alpha+1} \geq r_2, \qquad \text{for all} \quad \|v\|_{\vec{q}} \leq \rho_2.$$

Let us introduce $\rho := \min\{\rho_1, \rho_2\}$, so if $\|(u, v)\| = \rho$ then we have either $\|u\|_{\vec{p}} \leq \rho \leq \rho_1$ and $\|v\|_{\vec{q}} \leq \rho \leq \rho_2$. Consequently, if $\|(u, v)\| = \rho$ we have $I(u, v) \geq r := r_1 + r_2$.

b) Let $(\varphi, \psi) \in W$ such that $\varphi \neq 0$, $\psi \neq 0$ and $R(\varphi, \psi) \neq 0$ then

$$\begin{aligned} I(s\varphi, s\psi) &= (\alpha+1)\left(\sum_{i=1}^{N} \frac{s^{p_i}}{p_i} P_i(\varphi) - \lambda \frac{s^p}{p} P_a(\varphi)\right) - s^{\alpha+\beta+2} R(\varphi, \psi) \\ &+ (\beta+1)\left(\sum_{i=1}^{N} \frac{s^{q_i}}{q_i} Q_i(\psi) - \mu \frac{s^q}{q} Q_b(\psi)\right). \end{aligned}$$

We know that $\alpha + \beta + 2 > \max\{p_+, q_+\}$, then for s sufficiently large, we have

$$I(s\varphi, s\psi) < 0.$$

By a lemma in [6], the equation $\frac{\partial I}{\partial s}(su, sv) = 0$ has only one solution $s_{\lambda,\mu} > 0$ such that $\frac{\partial I}{\partial s}(s_{\lambda,\mu} u, s_{\lambda,\mu} v) = 0$. and $s_{\lambda,\mu}$ satisfies the condition

$$I(s_{\lambda,\mu}\varphi, s_{\lambda,\mu}\psi) = \max_{s \geq 0} I(s\varphi, s\psi).$$

By considering $(u_{\lambda,\mu}, v_{\lambda,\mu}) := (s_{\lambda,\mu}\varphi, s_{\lambda,\mu}\psi)$ we have the following results:

$$\begin{aligned} &(i) \quad I(u_{\lambda,\mu}, v_{\lambda,\mu}) = \max_{s \geq 0} I(su_{\lambda,\mu}, sv_{\lambda,\mu}) \\ &(ii) \quad I(u_{\lambda,\mu}, v_{\lambda,\mu}) \longrightarrow 0 \ \text{ as } \lambda \to +\infty \ \text{ or } \ \mu \to +\infty. \end{aligned}$$

Let us show (i). The function $I(su_{\lambda,\mu}, sv_{\lambda,\mu})$ has a unique maximum value at some point $s_0 > 0$. Since $u_{\lambda,\mu} \neq 0$ and $v_{\lambda,\mu} \neq 0$, we have

$$\frac{(\alpha+1)\lambda s_0^p P_a(u_{\lambda,\mu}) + (\beta+1)\mu s_0^q Q_b(v_{\lambda,\mu}) + (\alpha+\beta+2) s_0^{\alpha+\beta+2} R(u_{\lambda,\mu}, v_{\lambda,\mu})}{\sum_{i=1}^{N} [(\alpha+1) P_i(u_{\lambda,\mu}) s_0^{p_i} + (\beta+1) Q_i(v_{\lambda,\mu}) s_0^{q_i}]} = 1.$$

On the other hand $(u_{\lambda,\mu}, v_{\lambda,\mu})$ satisfies

$$\frac{(\alpha+1)\lambda P_a(u_{\lambda,\mu}) + (\beta+1)\mu Q_b(v_{\lambda,\mu}) + (\alpha+\beta+2) R(u_{\lambda,\mu}, v_{\lambda,\mu})}{\sum_{i=1}^{N} [(\alpha+1) P_i(u_{\lambda,\mu}) + (\beta+1) Q_i(v_{\lambda,\mu})]} = 1.$$

We study the variation of the function

$$\theta \ : \ s \mapsto \frac{(\alpha+1)\lambda s^p P_a(u_{\lambda,\mu}) + (\beta+1)\mu s^q Q_b(v_{\lambda,\mu}) + (\alpha+\beta+2) s^{\alpha+\beta+2} R(u_{\lambda,\mu}, v_{\lambda,\mu})}{\sum_{i=1}^{N} [(\alpha+1) P_i(u_{\lambda,\mu}) s^{p_i} + (\beta+1) Q_i(v_{\lambda,\mu}) s^{q_i}]}$$

The function θ is of the form $\theta(s) = \frac{s^a}{Bs^b+Cs^c}$ with $B > 0$, $C > 0$, $a > b > 0$ and $a > c > 0$. For every $s > 0$ we have

$$\theta'(s) = \frac{B(a-b)s^{a+b-1} + C(a-c)s^{a+c-1}}{(Bs^b + Cs^c)^2} > 0,$$

then the function θ is increasing on $(0, +\infty)$. Therefore it follows that $s_0 = 1$.

We show the claim (ii), notice that

$$\frac{s^p_{\lambda,\mu} P_a(\varphi)}{\displaystyle\sum_{i=1}^{N} [(\alpha+1)P_i(\varphi)s^{p_i}_{\lambda,\mu} + (\beta+1)Q_i(\psi)s^{q_i}_{\lambda,\mu}]} \leq \frac{1}{\lambda},$$

and

$$\frac{s^q_{\lambda,\mu} Q_b(\psi)}{\displaystyle\sum_{i=1}^{N} [(\alpha+1)P_i(\varphi)s^{p_i}_{\lambda,\mu} + (\beta+1)Q_i(\psi)s^{q_i}_{\lambda,\mu}]} \leq \frac{1}{\mu}.$$

Thus,

$$\frac{s^p_{\lambda,\mu} P_a(\varphi)}{\displaystyle\sum_{i=1}^{N} [(\alpha+1)P_i(\varphi)s^{p_i}_{\lambda,\mu} + (\beta+1)Q_i(\psi)s^{q_i}_{\lambda,\mu}]} \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty$$

and

$$\frac{s^q_{\lambda,\mu} Q_b(\psi)}{\displaystyle\sum_{i=1}^{N} [(\alpha+1)P_i(\varphi)s^{p_i}_{\lambda,\mu} + (\beta+1)Q_i(\psi)s^{q_i}_{\lambda,\mu}]} \longrightarrow 0 \quad \text{as} \quad \mu \longrightarrow +\infty,$$

as the function θ satisfies: $\theta(0) = 0$, θ is continuous on $\mathbb{R}^+$ and increasing then $\theta(s) \to 0 \Longleftrightarrow s \to 0$. Thus, we have either

$$s_{\lambda,\mu} \longrightarrow 0 \quad \text{as} \quad \lambda \to +\infty \quad \text{or} \quad \mu \to +\infty.$$

Recalling that

$$0 \leq I(u_{\lambda,\mu}, v_{\lambda,\mu}) \leq (\alpha+1)\sum_{i=1}^{N} \frac{s^{p_i}_{\lambda,\mu}}{p_i} P_i(\varphi) + (\beta+1)\frac{s^{q_i}_{\lambda,\mu}}{q_i} Q_i(\psi) - s^{\alpha+\beta+2}_{\lambda,\mu} R(\varphi,\psi)$$

it follows that

$$I(u_{\lambda,\mu}, v_{\lambda,\mu}) \longrightarrow 0 \quad \text{as} \quad \lambda \to +\infty \quad \text{or} \quad \mu \to +\infty,$$

which proves the claim. □

Hereafter, we fix $(e_{\lambda,\mu}, f_{\lambda,\mu})$ with $e_{\lambda,\mu} = s^*_{\lambda,\mu} u_{\lambda,\mu}$ and $f_{\lambda,\mu} = s^*_{\lambda,\mu} v_{\lambda,\mu}$ such that

$$\|(e_{\lambda,\mu}, f_{\lambda,\mu})\| \geq r \quad \text{and} \quad I(e_{\lambda,\mu}, f_{\lambda,\mu}) < 0.$$

Lemma 4.2.4. *If $c_{\lambda,\mu}$ is the minimax value obtained by the Mountain Pass Theorem applied to the functional I, then we get*

$$c_{\lambda,\mu} \longrightarrow 0 \quad \textit{as} \quad \lambda \longrightarrow +\infty \quad \textit{or} \quad \mu \longrightarrow +\infty.$$

Proof. The maximum value $c_{\lambda,\mu}$ is given by

$$c_{\lambda,\mu} = \inf_{(\gamma,\xi)\in\Gamma} \max_{s\in[0,1]} I(\gamma(s), \xi(s)),$$

where the set of all paths linking $(0,0)$ and $(e_{\lambda,\mu}, f_{\lambda,\mu})$ is defined by

$$\Gamma = \Big\{(\gamma,\xi) \in \mathcal{C}(W;\mathbb{R}), \quad (\gamma(0),\xi(0)) = (0,0) \quad \text{and} \quad (\gamma(1),\xi(1)) = (e_{\lambda,\mu}, f_{\lambda,\mu})\Big\}.$$

Let's consider $\gamma(s) = s e_{\lambda,\mu}$ and $\xi(s) = s f_{\lambda,\mu}$, so then $(\gamma,\xi) \in \Gamma$ and

$$\max_{s\in[0,1]} I(\gamma(s), \xi(s)) = \max_{s\geq 0} I(s u_{\lambda,\mu}, s v_{\lambda,\mu}) = I(u_{\lambda,\mu}, v_{\lambda,\mu})$$

then

$$0 \leq c_{\lambda,\mu} \leq I(u_{\lambda,\mu}, v_{\lambda,\mu}),$$

from where

$$c_{\lambda,\mu} \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty \quad \text{or} \quad \mu \longrightarrow +\infty,$$

which ends the proof. □

Hereafter, we shall denote by $S_{\vec{p}} > 0$ and $S_{\vec{q}} > 0$ the positive constants, see [7]:

$$S_{\vec{p}} = \inf_{u\in\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N), \|u\|_{p^*}=1} \left\{ \sum_{i=1}^N \frac{1}{p_i} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} \right\}$$

and

$$S_{\vec{q}} = \inf_{v\in\mathcal{D}^{1,\vec{q}}(\mathbb{R}^N), \|v\|_{q^*}=1} \left\{ \sum_{i=1}^N \frac{1}{q_i} \left\| \frac{\partial v}{\partial x_i} \right\|_{q_i}^{q_i} \right\}.$$

The following lemma is an immediate consequence of the last one, that is,

Lemma 4.2.5. *There exists* $\lambda^* > 0$ *and* $\mu^* > 0$ *such that for all* $(\lambda, \mu) \in [\lambda^*, +\infty[\times[\mu^*, +\infty[$ *we have*

$$0 < c_{\lambda,\mu} < d_1 \min \left\{ \left(S_{\vec{p}}^{\frac{\alpha+1}{p_+}} S_{\vec{q}}^{\frac{\beta+1}{q_+}} \right)^{\frac{1}{d_1}}, \left(S_{\vec{p}}^{\frac{\alpha+1}{p_+}} S_{\vec{q}}^{\frac{\beta+1}{q_-}} \right)^{\frac{1}{d_2}}, \left(S_{\vec{p}}^{\frac{\alpha+1}{p_-}} S_{\vec{q}}^{\frac{\beta+1}{q_+}} \right)^{\frac{1}{d_3}}, \left(S_{\vec{p}}^{\frac{\alpha+1}{p_-}} S_{\vec{q}}^{\frac{\beta+1}{q_-}} \right)^{\frac{1}{d_4}} \right\}$$

where $d_1 = \frac{\alpha+1}{p_+}+\frac{\beta+1}{q_+}-1$, $d_2 = \frac{\alpha+1}{p_+}+\frac{\beta+1}{q_-}-1$, $d_3 = \frac{\alpha+1}{p_-}+\frac{\beta+1}{q_+}-1$ *and* $d_4 = \frac{\alpha+1}{p_-}+\frac{\beta+1}{q_-}-1$.

Related to the Mountain Pass level $c_{\lambda,\mu}$, there exists a sequence $(u_n, v_n) \subset W$ satisfying

$$I(u_n, v_n) \longrightarrow c_{\lambda,\mu} \quad \text{and} \quad I'(u_n, v_n) \longrightarrow 0 \quad \text{in} \quad W^*.$$

Using standard arguments, we have that (u_n, v_n) is bounded in W, hence we can assume that there exists $(u, v) \in W$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\vec{p}}(\Omega)$$

and

$$v_n \rightharpoonup v \quad \text{in} \quad W_0^{1,\vec{q}}(\Omega).$$

Lemma 4.2.6. *The weak limit* (u, v) *is such that* $u \neq 0$ *and* $v \neq 0$.

Proof. By applying the result of A. El Hamidi and Rakotoson in [5], we prove that we can extract a subsequence of (u_n, v_n), still denoted (u_n, v_n), such that

$$\nabla u_n(x) \longrightarrow \nabla u(x) \quad \text{a.e. in} \quad \Omega,$$

and

$$\nabla v_n(x) \longrightarrow \nabla v(x) \quad \text{a.e. in} \quad \Omega,$$

then $I'(u, v) = 0$.
To prove now that $u \neq 0$ and $v \neq 0$, we assume by contradiction that $u = 0$ and we set $x_n := u_n$ and $y_n := v_n - v$. Using similar arguments as above, we get

$$\begin{cases} \sum_{i=1}^{N} P_i(u_n) - \lambda P_a(u_n) = R(u_n, v_n) + o_n(1) \\ \sum_{i=1}^{N} Q_i(v_n) - \mu Q_b(v_n) = R(u_n, v_n) + o_n(1). \end{cases} \tag{2.10}$$

Using Brézis-Lieb Lemma [3], we obtain

$$\sum_{i=1}^{N} P_i(x_n) = R(x_n, y_n) + o_n(1), \tag{2.11}$$

$$\sum_{i=1}^{N} Q_i(y_n) = R(x_n, y_n) + o_n(1), \tag{2.12}$$

$$I_{0,0}(x_n, y_n) = c_{\lambda,\mu} - I(0, v) + o_n(1). \tag{2.13}$$

On the other hand, we have

$$I(0,v)=(\beta+1)\left(\sum_{i=1}^{N}\frac{1}{q_i}Q_i(v)-\mu\frac{1}{q}Q_b(v)\right),$$

with

$$\sum_{i=1}^{N}Q_i(v)-\mu Q_b(v)=0.$$

Therefore

$$I(0,v)=(\beta+1)\sum_{i=1}^{N}\left(\frac{1}{q_i}-\frac{1}{q}\right)Q_i(v),$$

and consequently $I(0,v)\geq 0$, since $q_+<q$. Now, if $\lim_{n\to+\infty}R(x_n,y_n)=L$ then it follows by (2.11) and (2.12) that

$$\sum_{i=1}^{N}P_i(x_n)\longrightarrow L \quad \text{and} \quad \sum_{i=1}^{N}Q_i(y_n)\longrightarrow L.$$

Then

$$I_{0,0}(x_n,y_n)=(\alpha+1)\sum_{i=1}^{N}\left(\frac{1}{p_i}-\frac{1}{p^*}\right)P_i(x_n)+(\beta+1)\sum_{i=1}^{N}\left(\frac{1}{q_i}-\frac{1}{q^*}\right)Q_i(y_n)+o_n(1),$$

and

$$\lim_{n\to+\infty}I_{0,0}(x_n,y_n)\geq\left[(\alpha+1)\left(\frac{1}{p_+}-\frac{1}{p^*}\right)+(\beta+1)\left(\frac{1}{q_+}-\frac{1}{q^*}\right)\right]L.$$

By the relation (2.13) we get

$$\begin{aligned}
c_{\lambda,\mu} &= \lim_{n\to+\infty}I_{0,0}(x_n,y_n)+I(0,v)\\
&\geq \lim_{n\to+\infty}I_{0,0}(x_n,y_n)\\
&\geq \left[(\alpha+1)\left(\frac{1}{p_+}-\frac{1}{p^*}\right)+(\beta+1)\left(\frac{1}{q_+}-\frac{1}{q^*}\right)\right]L\\
&= \left(\frac{\alpha+1}{p_+}+\frac{\beta+1}{q_+}-1\right)L.
\end{aligned}$$

We use the same arguments as in [2] and [7], to prove that we have either

$$S_{\vec{p}}\|x_n\|_{p^*}^{p_+}\leq\sum_{i=1}^{N}\left\|\frac{\partial x_n}{\partial x_i}\right\|_{p_i}^{p_i} \quad \text{if} \quad \|x_n\|_{p^*}\leq 1 \tag{2.14}$$

$$S_{\vec{q}}\|y_n\|_{q^*}^{q_+}\leq\sum_{i=1}^{N}\left\|\frac{\partial y_n}{\partial x_i}\right\|_{q_i}^{q_i} \quad \text{if} \quad \|y_n\|_{p^*}\leq 1 \tag{2.15}$$

or

$$S_{\vec{p}}\|x_n\|_{p^*}^{p_-} \leq \sum_{i=1}^{N} \left\|\frac{\partial x_n}{\partial x_i}\right\|_{p_i}^{p_i} \quad \text{if} \quad \|x_n\|_{p^*} \geq 1$$

$$S_{\vec{q}}\|y_n\|_{q^*}^{q_-} \leq \sum_{i=1}^{N} \left\|\frac{\partial y_n}{\partial x_i}\right\|_{q_i}^{q_i} \quad \text{if} \quad \|y_n\|_{p^*} \geq 1.$$

From the previous inequalities, if $\|x_n\|_{p^*} \leq 1$ and $\|y_n\|_{p^*} \leq 1$ we obtain

$$\begin{aligned} R(x_n, y_n) &\leq \|x_n\|_{p^*}^{\alpha+1}\|y_n\|_{q^*}^{\beta+1} \\ &\leq \left(\frac{1}{S_{\vec{p}}}\sum_{i=1}^{N} P_i(x_n)\right)^{\frac{\alpha+1}{p_+}} \left(\frac{1}{S_{\vec{q}}}\sum_{i=1}^{N} Q_i(y_n)\right)^{\frac{\beta+1}{q_+}} \end{aligned}$$

as $n \to +\infty$ we have

$$L \geq \left(S_{\vec{p}}^{\frac{\alpha+1}{p_+}} S_{\vec{q}}^{\frac{\beta+1}{q_+}}\right)^{\frac{1}{\frac{\alpha+1}{p_+}+\frac{\beta+1}{q_+}-1}}$$

and

$$c_{\lambda,\mu} \geq (\frac{\alpha+1}{p_+} + \frac{\beta+1}{q_+} - 1)\left(S_{\vec{p}}^{\frac{\alpha+1}{p_+}} S_{\vec{q}}^{\frac{\beta+1}{q_+}}\right)^{\frac{1}{\frac{\alpha+1}{p_+}+\frac{\beta+1}{q_+}-1}}.$$

Then

$$c_{\lambda,\mu} \geq d_1 \min\left\{\left(S_{\vec{p}}^{\frac{\alpha+1}{p_+}} S_{\vec{q}}^{\frac{\beta+1}{q_+}}\right)^{\frac{1}{d_1}}, \left(S_{\vec{p}}^{\frac{\alpha+1}{p_+}} S_{\vec{q}}^{\frac{\beta+1}{q_-}}\right)^{\frac{1}{d_2}}, \left(S_{\vec{p}}^{\frac{\alpha+1}{p_-}} S_{\vec{q}}^{\frac{\beta+1}{q_+}}\right)^{\frac{1}{d_3}}, \left(S_{\vec{p}}^{\frac{\alpha+1}{p_-}} S_{\vec{q}}^{\frac{\beta+1}{q_-}}\right)^{\frac{1}{d_4}}\right\},$$

which cannot hold true. Thus $L = 0$ and $c_{\lambda,\mu} = I(0, v) > 0$, in this case $\lim_{\lambda\to+\infty} c_{\lambda,\mu} = I(0, v) > 0$ which leads to a contradiction with the fact that $\lim_{\lambda\to+\infty} c_{\lambda,\mu} = 0$. Then $u \neq 0$ and $v \neq 0$. $\square$

4.3 Regularity of Weak Solutions

In this section, we show that every weak solution $(u, v) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$ of the following problem

$$(P) \qquad \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = f(x, u, v) & \text{in } \Omega, \\ -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{\partial v}{\partial x_i}\right) = g(x, u, v) & \text{in } \Omega, \\ u \geq 0 \quad \text{and} \quad v \geq 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{and} \quad v = 0 \quad \text{on } \partial\Omega \end{cases}$$

is a *a strong solution*, under some hypothesis on the functions f and g.

Lemma 4.3.1. *Suppose that the functions f and g satisfy the following growth conditions:*
(h_1) There exist $c_1,\ c_2 \geq 0$ and $p \in (1, p^*)$ such that

$$|f(x,u,v)| \leq c_1 u^{p-1} + c_2 u^{\alpha} v^{\beta+1}, \quad \forall u \geq 0 \quad and \quad v \geq 0,$$

(h_2) There exist $c_1',\ c_2' \geq 0$ and $q \in (1, p^*)$ such that

$$|g(x,u,v)| \leq c_1' v^{q-1} + c_2' u^{\alpha+1} v^{\beta}, \quad \forall u \geq 0 \quad and \quad v \geq 0.$$

Then every weak solution $(u,v) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$ of $(P_{\lambda,\mu})$ belongs to $L^r(\Omega) \times L^s(\Omega)$ for all $1 \leq r < +\infty$ and $1 \leq s < +\infty$.

Proof. We will use similar arguments developed by Fragala, Gazzola and Kawohl [8].
Let (u,v) be a weak solution to (P). The assertion that $(u,v) \in L^r(\Omega) \times L^s(\Omega)$ for all $1 \leq r < +\infty$ and $1 \leq s < +\infty$ may be equivalently reformulated as

$$(u,v) \in L^{(1+a)p^*}(\Omega) \times L^{(1+b)q^*}(\Omega) \quad \text{for all } (a,b) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+. \tag{3.16}$$

To prove (3.16) it is enough to show that $(u^{a+1}, v^{b+1}) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$, which is equivalent to

$$\lim_{L \to +\infty} \sum_{i=1}^{N} \left(\int_\Omega |\partial_{x_i}(u \min[u^a, L]|^{p_i} dx \right)^{\frac{1}{p_i}} < +\infty, \tag{3.17}$$

$$\lim_{L \to +\infty} \sum_{i=1}^{N} \left(\int_\Omega |\partial_{x_i}(v \min[v^b, L]|^{q_i} dx \right)^{\frac{1}{q_i}} < +\infty. \tag{3.18}$$

For each L there exist indexes j and k such that

$$\sum_{i=1}^{N} \left(\int_\Omega |\partial_i(u \cdot \min[u^a, L]|^{p_i}\, dx \right)^{\frac{1}{p_i}} \leq C \left(\int_\Omega |\partial_j(u \cdot \min[u^a, L]|^{p_j} dx \right)^{\frac{1}{p_j}}, \tag{3.19}$$

$$\sum_{i=1}^{N} \left(\int_\Omega |\partial_i(v \cdot \min[v^b, L]|^{q_i} dx \right)^{\frac{1}{q_i}} \leq C' \left(\int_\Omega |\partial_k(v \cdot \min[v^b, L]|^{q_k} dx \right)^{\frac{1}{q_k}} \tag{3.20}$$

where C and C' are positive constants independent of L.
For these indexes j and k, and for every $L > 0$, set $\phi_L := \min[u^{ap_j}, L^{p_j}]$ and $\psi_L := v \min[v^{bq_k}, L^{q_k}]$ such that $(\phi_L, \psi_L) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$. Note that for every $1 \leq i \leq N$ and for almost every $x \in \Omega$,

$$|\partial_i u|^{p_i - 2} \partial_i u \partial_i \phi_L \geq \min[u^{ap_j}, L^{p_j}] |\partial_i u|^{p_i},$$

$$|\partial_i v|^{q_i - 2} \partial_i v \partial_i \psi_L \geq \min[v^{bq_k}, L^{q_k}] |\partial_i v|^{q_i},$$

$$|\partial_i(u\cdot\min[u^a,L])|^{p_i}\le (a+1)\min[u^{ap_i},L^{p_i}]|\partial_i u|^{p_i}, \tag{3.21}$$

and

$$|\partial_i(v\cdot\min[v^b,L])|^{q_i}\le (b+1)\min[v^{bq_i},L^{q_i}]|\partial_i v|^{q_i}. \tag{3.22}$$

As (u,v) is a weak solution to (P), we rewrite, for every $k\ge 1$, $\Omega\equiv\Omega_{1,k}\cup\Omega_{2,k}\cup\Omega_{3,k}\cup\Omega_{4,k}$, where

$$\begin{aligned}
\Omega_{1,k} &:= \{x\in\Omega \quad \text{such that} \quad u<k \quad \text{and } v<k\},\\
\Omega_{2,k} &:= \{x\in\Omega \quad \text{such that} \quad u<k \quad \text{and } v\ge k\},\\
\Omega_{3,k} &:= \{x\in\Omega \quad \text{such that} \quad u\ge k \quad \text{and } v<k\},\\
\Omega_{4,k} &:= \{x\in\Omega \quad \text{such that} \quad u\ge k \quad \text{and } v\ge k\}.
\end{aligned}$$

Then we have

$$\sum_{i=1}^N\int_\Omega \min[u^{ap_j},L^{p_j}]|\partial_i u|^{p_i}dx\le\sum_{i=1}^N\int_\Omega|\partial_i u|^{p_i-2}\partial_i u\partial_i\phi_L dx=\int_\Omega f(x,u,v)\phi_L dx,$$

and

$$\sum_{i=1}^N\int_\Omega \min[v^{bq_k},L^{q_k}]|\partial_i v|^{q_i}dx\le\sum_{i=1}^N\int_\Omega|\partial_i v|^{q_i-2}\partial_i v\partial_i\psi_L dx=\int_\Omega g(x,u,v)\psi_L dx.$$

Using $(h1)$, for $L\ge k^a\ge 1$, it follows that

$$|f(x,u,v)|\phi_L\le\tilde C_{k,1} \quad \text{in} \quad \Omega_1 \tag{3.23}$$

$$|g(x,u,v)|\psi_L\le\tilde C_{k,2} \quad \text{in} \quad \Omega_1 \tag{3.24}$$

hence

$$\int_{\Omega_1} f(x,u,v)\phi_L dx\le C_{k,1}=\tilde C_{k,1}|\Omega|,$$

$$\begin{aligned}
\int_{\Omega_{2,k}} f(x,u,v)\phi_L dx &\le \int_{\Omega_2}\left[c_1u^p+c_2u^{\alpha+1}v^{\beta+1}\right]\min[u^{ap_j},L^{p_j}]dx,\\
&\le C_{2,k}+c_2\int_{\Omega_2}u^{\alpha+1}v^{\beta+1}\min[u^{ap_j},L^{p_j}]dx,\\
&\le C_{2,k}+c_2\left(\int_{\Omega_{2,k}}(u^{\alpha+1}\min[u^{ap_j},L^{p_j}])^{p^*/(\alpha+1)}dx\right)^{\frac{\alpha+1}{p^*}}\times\\
&\quad\times\left(\int_{\Omega_{2,k}}v^{q^*}dx\right)^{\frac{\beta+1}{q^*}}\\
&\le C_{2,k}+\varepsilon_kC'_{2,k}
\end{aligned}$$

where ε_k tends to 0 as k tends to $+\infty$.

$$\begin{aligned}
\int_{\Omega_{3,k}} f(x,u,v)\phi_L dx &\leq c_1 \int_{\Omega_{3,k}} u^{p^*} \min[u^{ap_j}, L^{p_j}]dx + \int_{\Omega_{3,k}} u^{\alpha+1} v^{\beta+1} \min[u^{ap_j}, L^{p_j}]dx \\
&\leq c_1 \left(\int_{\Omega_{3,k}} u^{p^*} dx\right)^{\frac{p^*-p_j}{p^*}} \cdot \int_{\Omega_{3,k}} (u^{p_j} \cdot \min[u^{ap_j}, L^{p_j}])^{p^*/p_j} dx \\
&+ c_2 \int_{\Omega_{3,k}} u^{p^*} \min[u^{ap_j}, L^{p_j}] v^{\beta+1} dx \\
&\leq \varepsilon'_k \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx\right)^{p_j/p^*} \\
&+ c_2 \left(\int_{\Omega_{3,k}} u^{p^*-p_j} u^{p_j} \cdot (\min[u^{ap_j}, L^{p_j}])^{p^*/(\alpha+1)} dx\right)^{\frac{\alpha+1}{p^*}} \\
&\times \left(\int_{\Omega_{3,k}} v^{q^*} dx\right)^{\frac{\beta+1}{q^*}} \\
&\leq \varepsilon'_k \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx\right)^{p_j/p^*} + c_2 \|u\|_{p^*,\Omega_{3,k}}^{(\alpha+1)(p^*-p_j)/p^*} \|v\|_{q^*,\Omega_{3,k}}^{\beta+1} \times \\
&\times \left(\int_{\Omega_{3,k}} \left(u^{p_j} \cdot (\min[u^{ap_j}, L^{p_j}])^{p^*/(\alpha+1)}\right)^{\frac{p^*}{p_j}} dx\right)^{\frac{(\alpha+1)p_j}{p^{*2}}} \\
&\leq \varepsilon'_k \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx\right)^{p_j/p^*} + \varepsilon''_k \|v\|_{q^*}^{\beta+1} \times \\
&\times \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx\right)^{\frac{(\alpha+1)p_j}{p^{*2}}} \\
&\leq \varepsilon'_k \left[\sum_{i=1}^{N} \left(\int_{\Omega} |\partial_i (u \cdot \min[u^a, L])|^{p_i} dx\right)^{1/p_i}\right]^{p_j} \\
&+ \varepsilon''_k \left[\sum_{i=1}^{N} \left(\int_{\Omega} |\partial_i \left(u \cdot \min[u^a, L]\right)|^{p_i} dx\right)^{\frac{1}{p_i}}\right]^{(\alpha+1)p_i/p^*}
\end{aligned}$$

where ε'_k and ε''_k tend to 0 as k goes to $+\infty$.

$$\begin{aligned}
\int_{\Omega_{4,k}} f(x,u,v)\phi_L dx &\le \int_{\Omega_{4,k}} \left[c_1 u^p + c_2 u^{\alpha+1} v^{\beta+1}\right] \min[u^{ap_j}, L^{p_j}]dx \\
&\le \varepsilon'_k \left(\int_\Omega (u\cdot\min[u^a,L])^{p^*}dx\right)^{p_j/p^*} + c_2\|u\|^{(\alpha+1)(p^*-p_j)/p^*}_{p^*,\Omega_{4,k}} \times \\
&\times \|v\|^{\beta+1}_{q^*,\Omega_{4,k}} \left(\int_{\Omega_{3,k}} \left(u^{p_j}\cdot(\min[u^{ap_j},L^{p_j}])^{p^*/(\alpha+1)}\right)^{\frac{p^*}{p_j}} dx\right)^{\frac{(\alpha+1)p_j}{p^{*2}}} \\
&\le \varepsilon^1_k \left(\int_\Omega (u\cdot\min[u^a,L])^{p^*}dx\right)^{p_j/p^*} + \\
&+ \varepsilon^2_k \left(\int_\Omega (u\cdot\min[u^a,L])^{p^*}dx\right)^{\frac{(\alpha+1)p_j}{p^{*2}}} \\
&\le \varepsilon^1_k \left[\sum_{i=1}^N \left(\int_\Omega |\partial_i(u\cdot\min[u^a,L])|^{p_i}dx\right)^{1/p_i}\right]^{p_j} + \\
&+ \varepsilon^2_k \left[\sum_{i=1}^N \left(\int_\Omega |\partial_i\left(u\cdot\min[u^a,L]\right)|^{p_i}dx\right)^{\frac{1}{p_i}}\right]^{(\alpha+1)p_i/p^*}
\end{aligned}$$

where ε^1_k and ε^2_k tend to 0 as k goes to $+\infty$.
Inserting (3.19) and (3.20) in the last inequalities, we then obtain,

$$\begin{aligned}
\int_\Omega |\partial_j(u\cdot\min[u^a,L])|^{p_j}dx &\le C'_k + \varepsilon_{k,1}\int_\Omega |\partial_j(u\cdot\min[u^a,L])|^{p_j}dx \\
&+ \varepsilon_{k,2}\left[\int_\Omega |\partial_i\left(u\cdot\min[u^a,L]\right)|^{p_i}dx\right]^{(\alpha+1)/p^*}.
\end{aligned}$$

Choosing k sufficiently large, that are $\varepsilon_{k,1}$ and $\varepsilon_{k,2}$ sufficiently small, the last inequality ensures that the integral $\int_\Omega |\partial_j(u\cdot\min[u^a,L])|^{p_j}dx$ is bounded for L large enough, from where follows (3.17). By the same way we can prove (3.18), and we can conclude then that every weak solution $(u,v)\in W_0^{1,\vec{p}}(\Omega)\times W_0^{1,\vec{q}}(\Omega)$ of (P) belongs to $L^r(\Omega)\times L^s(\Omega)$, for all $r\ge 1$ and $s\ge 1$. □

Proposition 4.3.1. *Under the conditions (h_1) and (h_2), every nonnegative solution $(u,v)\in W_0^{1,\vec{p}}(\Omega)\times W_0^{1,\vec{q}}(\Omega)$ of (P) belongs to $L^\infty(\Omega)\times L^\infty(\Omega)$.*

Proof. For $u\ge 0$ and $v\ge 0$ such that (u,v) is a solution of (P), we set $A_\tau = \{x\in\Omega,\quad u(x)\ge\tau\}$ and $B_\tau=\{x\in\Omega,\quad v(x)\ge\tau\}$, $|A_\tau|$ and $|B_\tau|$ their Lebesgue measures. Recall that Cavalieri principle, based on Fubini theorem, gives:

$$\int_k^{+\infty}|A_\tau|d\tau = \int_\Omega (u-k)_+dx, \quad \text{for all} \quad k\ge 0$$

and
$$\int_k^{+\infty} |B_\tau| d\tau = \int_\Omega (v-k)_+ dx, \quad \text{for all} \quad k \geq 0.$$
Let $\varphi_k = (u-k)_+$, for $k > 0$ fixed. Choosing this function as a test function, combining the Cavalieri principle and Hölder inequality, one gets

$$\begin{aligned}
\sum_{i=1}^N P_i(\varphi_k) &= \int_\Omega (\lambda a u^{p-1} + u^\alpha v^{\beta+1}) \varphi_k dx \\
&\leq c_1 \lambda \int_\Omega |u|^{p-1} \varphi_k dx + c_2 \int_\Omega |u|^\alpha |v|^{\beta+1} \varphi_k dx \\
&\leq c \left(\int_{[u \leq 1]} \varphi_k dx + \int_{[u \leq 1]} |v|^{\beta+1} \varphi_k dx + \int_{[u \geq 1]} |u|^\alpha |v|^{\beta+1} \varphi_k dx \right), \\
&\leq c \left(|A_k|^{1-\frac{1}{p^*}} + |A_k|^{\left(1-\frac{\alpha+1}{p^*}\right)\left(1-\frac{1}{p^*}\right)} \right) \|\varphi_k\|_{p^*}. \qquad (3.25)
\end{aligned}$$

Since $\lim\limits_{n \to +\infty} \|\varphi_k\|_{p^*} = 0$, then for $k \geq k_0 > 0$, $\|\varphi_k\|_{p^*} \leq 1$. Relations (2.14) and (3.25) give

$$\begin{aligned}
S_{\vec{p}} \|\varphi_k\|_{p^*}^{p^+} &\leq \sum_{i=1}^N P_i(\varphi_k) \\
&\leq c \left(|A_k|^{1-\frac{1}{p^*}} + |A_k|^{\left(1-\frac{\alpha+1}{p^*}\right)\left(1-\frac{1}{p^*}\right)} \right) \|\varphi_k\|_{p^*}.
\end{aligned}$$

Thus, for every $k \leq k_0$, we have:

$$\begin{aligned}
\|\varphi_k\|_{p^*} &\leq c \left(|A_k|^{1-\frac{1}{p^*}} + |A_k|^{\left(1-\frac{\alpha+1}{p^*}\right)\left(1-\frac{1}{p^*}\right)} \right)^{\frac{1}{p^+-1}} \\
&\leq c \left(|A_k|^{\left(1-\frac{1}{p^*}\right)\frac{1}{p^+-1}} + |A_k|^{\left(1-\frac{\alpha+1}{p^*}\right)\left(1-\frac{1}{p^*}\right)\frac{1}{p^+-1}} \right). \qquad (3.26)
\end{aligned}$$

Using Cavalieri's principle, Hölder inequality and Relation (3.26), one has for all $k \geq k_0$:

$$\begin{aligned}
\int_k^{+\infty} |A_\tau| d\tau &= \int_\Omega (u-k)_+ dx \\
&\leq |A_k|^{1-\frac{1}{p^*}} \|\varphi_k\|_{p^*} \\
&\leq c \left(|A_k|^{1+\frac{1}{p^*}\frac{p^*-1}{p^+-1}} + |A_k|^{1+\theta} \right), \qquad (3.27)
\end{aligned}$$

where $\theta := -\dfrac{1}{p^*} + \left(1 - \dfrac{\alpha+1}{p^*}\right)\left(1 - \dfrac{1}{p^*}\right)\dfrac{1}{p^+-1} > 0.$
Since
$$\gamma := \frac{1}{p^*}\frac{p^*-1}{p^+-1} \geq \theta,$$

then

$$\int_k^{+\infty} |A_\tau| d\tau \leq c + |A_k|^{1+\gamma}.$$

This inequality is of Gronwall type, which shows that there exists $c_\lambda > 0$ such that

$$\|u\|_\infty \leq c_\lambda.$$

One can prove by exactly the same way the fact that there exists $c_\mu > 0$ such that $\|v\|_\infty \leq c_\mu$. $\square$

4.4 On the weak sub and supersolutions

In this section, we will use some classical tools concerning sub and super solutions for a class of systems involving the anisotropic operators considered above. Notice that the standard laplacian operator and the anisotropic operator were studied respectively in [12] and [2].

Let us consider the following problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) & = & f(x,u,v) \quad \text{in} \quad \Omega \\ -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{q_i-2}\frac{\partial v}{\partial x_i}\right) & = & g(x,u,v) \quad \text{in} \quad \Omega, \\ & & \\ u \geq 0 \quad \text{and} \quad v \geq 0 \quad \text{in} \quad \Omega, & & \\ u = 0 \quad \text{and} \quad v = 0 \quad \text{on} \quad \partial\Omega & & \end{cases} \tag{4.28}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, both f, $g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory satisfying the following property that for each fixed $A > 0$, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$(H) \qquad \begin{cases} |f(x,s,t)| & \leq & C_1, \quad \forall\,(x,s,t) \in \Omega \times [-A,A] \times [-A,A] \\ |g(x,s,t)| & \leq & C_2, \quad \forall\,(x,s,t) \in \Omega \times [-A,A] \times [-A,A]. \end{cases}$$

Definition 4.4.1. *The couple $(u,v) \in W$ is a (weak) sub-solution to (4.28) if $u \leq 0$ and $v \leq 0$ on $\partial\Omega$ and for all $(\varphi,\psi) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ with $\varphi(x) \geq 0$ and $\psi(x) \geq 0$ for all $x \in \Omega$, we have simultaneously*

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx - \int_\Omega f(x,u,v)\varphi dx & \leq & 0 \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{p_i-2} \frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i} dx - \int_\Omega g(x,u,v)\psi dx & \leq & 0. \end{cases}$$

Similarly, $(u,v) \in W$ is a (weak) super-solution to (4.28) if in the above the reverse inequalities hold.

Theorem 4.4.1. *Suppose (H) holds, $(\underline{u}, \underline{v}) \in W$ is sub-solution while $(\bar{u}, \bar{v}) \in W$ is a super-solution to Problem (4.28) and assume that there exist $\underline{c}, \bar{c}, \underline{c}', \bar{c}' \in \mathbb{R}$ there holds $\underline{c} \leq \underline{u} \leq \bar{u} \leq \bar{c}$ and $\underline{c}' \leq \underline{v} \leq \bar{v} \leq \bar{c}'$ almost everywhere in Ω. Then, there exists a weak solution $(u, v) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$ of (4.28), satisfying the condition $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$ almost everywhere in Ω.*

Proof. Let $F(x, u, v) = \int_0^u f(x, s, v)ds$ and $G(x, u, v) = \int_0^v g(x, u, t)dt$ denote respectively primitives of f and g. Let us define $J : W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega) \to \mathbb{R}$ the Euler-Lagrange functional associated to (4.28) given by

$$J(u, v) := \sum_{i=1}^{N} \int_\Omega \left(\frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} + \frac{1}{q_i} \left| \frac{\partial v}{\partial x_i} \right|^{q_i} \right) dx - \int_\Omega (F(x, u, v) + G(x, u, v)dx$$

We introduce the closed and convex subset $\mathcal{M}$ of $W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$ defined by

$$\mathcal{M} = \left\{ (u, v) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega) \; : \; \underline{u} \leq u \leq \bar{u} \text{ and } \underline{v} \leq v \leq \bar{v} \text{ a.e. in } \Omega \right\}.$$

Since $\underline{u}$, $\underline{v}$, $\bar{u}$ and $\bar{v} \in L^\infty$ by assumption, also $\mathcal{M} \in L^\infty \times L^\infty$ and consequently there exists $c > 0$ and $c > 0$ such that $|F(x, u(x), v(x)| \leq c$ and $|G(x, u(x), v(x)) \leq c|$ for all $(u, v) \in \mathcal{M}$ and for almost all $x \in \Omega$. Consequently

$$J(u, v) \geq \sum_{i=1}^{N} \int_\Omega \left(\frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} + \frac{1}{q_i} \left| \frac{\partial v}{\partial x_i} \right|^{q_i} \right) dx - 2c\, meas(\Omega)$$

on $\mathcal{M}$, which implies that J is coercive on $\mathcal{M}$. We claim now that the functional J is weakly lower semi-continuous on $\mathcal{M}$. Indeed, let (u_n, v_n), $(u, v) \subset \mathcal{M}$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ and $v_n \rightharpoonup W_0^{1,\vec{q}}(\Omega)$. We may assume that, up to a subsequence, $u_n \to u$ and $v_n \to v$ pointwise almost everywhere; moreover $|F(x, u_n(x) v_n(x)| \leq c$ and$|G(x, u_n(x) v_n(x)| \leq c$ uniformly. Hence we may appeal to Lebesgue's theorem on dominated convergence which implies that

$$\begin{aligned} \int_\Omega F(x, u_n, v_n)dx \quad &\longrightarrow \quad \int_\Omega F(x, u, v)dx \text{ and} \\ \int_\Omega G(x, u_n, v_n)dx \quad &\longrightarrow \quad \int_\Omega G(x, u, v)dx, \quad \text{as } n \text{ tends to } +\infty, \end{aligned}$$

these end the claim, since the functionals

$$u \in W_0^{1,\vec{p}}(\Omega) \mapsto \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx$$

and

$$v \in W_0^{1,\vec{q}}(\Omega) \mapsto \sum_{i=1}^{N} \int_\Omega \frac{1}{q_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx$$

are clearly weakly lower semi-continuous on the whole space. As the space W is reflexive then there exists $(u,v) \in \mathcal{M}$ such that $J(u,v) = \inf_{(w,s)\in\mathcal{M}} J(w,s)$. We claim that (u,v) solves weakly Problem (4.28), that is $J'(u,v) = 0$. Indeed, fix $(\varphi,\psi) \in \mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ and $\varepsilon > 0$ and consider the couple $(u_\varepsilon, v_\varepsilon) \in \mathcal{M}$ defined on Ω by:

$$u_\varepsilon(x) = \begin{cases} \bar{u}(x) & \text{if} \quad u(x)+\varepsilon\varphi(x) \geq \bar{u}(x), \\ u(x)+\varepsilon\varphi(x) & \text{if} \quad \underline{u}(x) \leq u(x)+\varepsilon\varphi(x) \leq \bar{u}(x), \\ \underline{u}(x) & \text{if} \quad u(x)+\varepsilon\varphi(x) \leq \underline{u}(x). \end{cases}$$

and

$$v_\varepsilon(x) = \begin{cases} \bar{v}(x) & \text{if} \quad v(x)+\varepsilon\psi(x) \geq \bar{v}(x), \\ v(x)+\varepsilon\psi(x) & \text{if} \quad \underline{v}(x) \leq v(x)+\varepsilon\psi(x) \leq \bar{v}(x), \\ \underline{v}(x) & \text{if} \quad v(x)+\varepsilon\psi(x) \leq \underline{v}(x). \end{cases}$$

The functions u_ε and v_ε can be characterised by $u_\varepsilon = (u+\varepsilon\varphi) - (\bar{\varphi}_\varepsilon - \underline{\varphi}_\varepsilon)$ and $v_\varepsilon = (v+\varepsilon\psi) - (\bar{\psi}_\varepsilon - \underline{\psi}_\varepsilon)$, where $\bar{\varphi}_\varepsilon = \max\{0, u+\varepsilon\varphi-\bar{u}\} \geq 0$, $\underline{\varphi}_\varepsilon = -\min\{0, u+\varepsilon\varphi-\underline{u}\} \geq 0$, $\bar{\psi}_\varepsilon = \max\{0, v+\varepsilon\psi-\bar{v}\} \geq 0$ and $\underline{\psi}_\varepsilon = -\min\{0, v+\varepsilon\psi-\underline{v}\} \geq 0$. Note that $\bar{\varphi}_\varepsilon$, $\underline{\varphi}_\varepsilon$, $\bar{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon \in W \cap L^\infty(\Omega)\times L^\infty(\Omega)$. Since (u,v) minimizes J on $\mathcal{M}$ and J is differentiable, then

$$0 \leq D_1(u,v)(u_\varepsilon - u) = \varepsilon D_1J(u,v)(\varphi) + D_1J(u,v)(\underline{\varphi}_\varepsilon) - D_1J(u,v)(\bar{\varphi}_\varepsilon),$$

where $D_1J(u,v)$ denotes the first derivative in u of J, so that

$$D_1J(u,v)(\varphi) \leq \frac{1}{\varepsilon}\left(D_1J(u,v)(\bar{\varphi}_\varepsilon) - D_1J(u,v)(\underline{\varphi}_\varepsilon)\right). \tag{4.29}$$

Using the fact that $(\bar{u},\bar{v})$ is a super-solution to (4.28), we get

$$\begin{aligned} D_1J(u,v)(\bar{\varphi}_\varepsilon) &= D_1J(\bar{u},\bar{v})(\bar{\varphi}_\varepsilon) + \left[D_1J(u,v) - D_1J(\bar{u},\bar{v})\right](\bar{\varphi}_\varepsilon) \\ &\geq \left[D_1J(u,v) - D_1J(\bar{u},\bar{v})\right](\bar{\varphi}_\varepsilon) \\ &= \sum_{i=1}^N \int_{\Omega_\varepsilon} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \bar{u}}{\partial x_i}\right)\frac{\partial}{\partial x_i}(u-\bar{u}+\varepsilon\varphi)dx - \\ &- \int_{\Omega_\varepsilon}\left[f(x,u,v) - f(x,\bar{u},\bar{v})\right](u-\bar{u}+\varepsilon\varphi)dx \\ &\geq \varepsilon\sum_{i=1}^N \int_{\Omega_\varepsilon} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i} - \left|\frac{\partial \bar{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \bar{u}}{\partial x_i}\right)\frac{\partial}{\partial x_i}(\varphi)dx - \\ &- \varepsilon\int_{\Omega_\varepsilon}\left|f(x,u,v) - f(x,\bar{u},\bar{v})\right||\varphi|dx \end{aligned}$$

where $\Omega_\varepsilon = \{x\in\Omega : u(x)+\varepsilon\varphi(x) \geq \bar{u}(x) \quad \text{and} \quad v(x)+\varepsilon\psi(x) \geq \bar{v}(x)\}$. Notice that $meas(\Omega_\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0$. Thus,

$$D_1J(u,v)(\bar{\varphi}_\varepsilon) \geq o(\varepsilon),$$

where $o(\varepsilon)/\varepsilon \longrightarrow 0$ as $\varepsilon \to 0$. Similarly, we conclude that

$$D_1 J(u,v)(\underline{\varphi}_\varepsilon) \leq o(\varepsilon),$$

and consequently, with (4.29), we get

$$D_1 J(u,v)(\varphi) \geq 0$$

for every $\varphi \in \mathcal{D}(\Omega)$. This implies, by reversing the sign of φ, that $D_1 J(u,v)(\varphi) = 0$ for every $\varphi \in \mathcal{D}(\Omega)$. Using the density of $\mathcal{D}(\Omega)$ in $W_0^{1,\vec{p}}(\Omega)$. The proof of $D_2 J(u,v)(\psi) = 0$ where $D_2 J(u,v)$ is the seconde derivative of J in the seconde variable, follows the same steps as in the above. $\square$

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Chapter 5

Appendix

In this part, we will recall some definitions, theorems and essential tools we used in the precedent chapters.

5.1 Palais-Smale Condition

In order to express the compactness of certain minimizing sequences, or in general of sequences converging to a point that it would probably be a critical point, we usually have recourse to the *Palais-Smale condition* (for short (PS)).

Definition 5.1.1. *Let W be a Banach space and $J : W \to \mathbb{R}$ functional of class $\mathcal{C}^1$. If $c \in \mathbb{R}$, we say that J satisfies the Palais-Smale condition (at the level c) and we denote $(PS)_c$, if every sequence $(u_n)_n$ of W such that*

$$J(u_n) \to c \quad in \quad \mathbb{R} \quad and \quad J'(u_n) \to 0 \quad in \quad W'$$

contains a convergent subsequence $(u_{n_k})_k$.

Intuitively, what we require is the compactness of sequences which could realize a critical value. However, even if a functional is bounded from below and $c := \inf J$, it is not always obvious that when $J(u_n) \to c$ then $J'(u_n) \to 0$. We generally have two important theorems permitting the construction of Palais-Smale sequences: *Mountain Pass theorem* due to Ambrosetti-Rabinowitz and *Ekeland's Variational principle.*

5.2 Mountain-Pass Geometry and Mountain-Pass Theorem

The simplest geometric situation for the construction of "almost critical" points by the min-max process is the *mountain pass theorem.* If the value of φ at 0 and at sme point e (valleys) are strictly smaller that the infimum of the values of φ on the

sphere of centre φ and radius $r < \|e\|$ (range of mountains), one can expect that, by taking the infinmum over all the paths joining 0 to e of the supremum of φ over such a path, one will obtain the value of φ at some critical point of mountain pass type. This is not always true, but gives points almost critique. Suitable Palais-Smale conditions over φ lead then to the existence of a critical point having the mointain pass critical value.
The following theorem is due to Ambrosetti-Rabinowitz:

Theorem 5.2.1. *Let W be a Banach space and $I \in \mathcal{C}^1(W, \mathbb{R})$. Assume that there exist $u_0 \in W$, $u_1 \in W$ and an open neighborhood Ω of u_0 such that $u_1 \notin \bar{\Omega}$ and*

$$c_0 := \max\{I(u_0), I(u_1)\} < c_1 := \max_{\partial\Omega} I(v).$$

Let

$$H = \{h \in \mathcal{C}([0,1], W) \quad : \quad h(0) = u_0,\ h(1) = u_1\},$$

$$c := \inf_{h \in H} \sup_{s \in [0,1]} I(h(s)).$$

Then,

$$\inf_{\partial\Omega} I(v) \leq c < +\infty,$$

and for all $\varepsilon \in (0, c_1 - c_0]$ and for each $h \in H$ such that

$$\max_{s \in [0,1]} I(h(s)) \leq c + \varepsilon,$$

there exists $v \in W$ satisfying:

i) $c - \varepsilon \leq I(v) \leq c + \varepsilon$

ii) $dist(v, h([0,1]) \leq \sqrt{\varepsilon}$

iii) $\|I'(v)\| \leq \sqrt{\varepsilon}$.

Another minimax theorem of geometrical type showed by P. H. Rabinowitz.

Theorem 5.2.2. (Geometrical Saddle Point)
Suppose that $W = U \oplus V$ is a Banach space and U and V two closed subspaces, such that $\dim U < +\infty$. For all $\rho > 0$, one consideres the following sets:

$$M := \{u \in U : \ \|u\| \leq \rho\}, \quad M_0 := \{u \in U : \ \|u\| = \rho\}.$$

We define the following space

$$H := \{h \in \mathcal{C}(M, W) \ : \ h|_{M_0} = id\},$$

and $I \in \mathcal{C}^1(W, \mathbb{R})$ *such that*

$$c_0 := \max_{u \in M_0} I(u) < c_1 := \inf_{v \in V} I(v).$$

Let

$$c := \inf_{h \in H} \max_{s \in M} I(h(s)).$$

If $c_1 \leq c$, *then for all* $\varepsilon \in]0, c_1 - c_0]$ *and for all* $h \in H$ *such that*

$$\max_{s \in M} I(h(s)) \leq c + \varepsilon,$$

there exists $u \in W$ *such that*

(i) $c - \varepsilon \leq I(u) \leq \max_{s \in M} I(h(s))$

(ii) $dist\big(u, h(M)\big) \leq \sqrt{\varepsilon}$

(iii) $\|I'(u)\| \leq \sqrt{\varepsilon}$.

Ekeland's Variational Principle

In general, it is not clear that a bounded and lower semi-continuous functional J attains its infimum. For instance, the analytic function $f(x) = \arctan x$ does not attain its minimum nor its maximum over the real line.
The advantage of this principle is the existence Palais-Smale sequences is insured even if the functional is not bounded from below over the whole space but only over closed subset of it.

Theorem 5.2.3. *Let* (M, d) *a complete metric space with* d *its metric and let* $J : M \to \mathbb{R} \cup +\infty$ *lower semi-continuous, bounded from below and* $\not\equiv +\infty$. *Then for each* ε, $\delta > 0$ *and every* $u \in M$ *with*

$$J(u) \leq \inf_M + \varepsilon,$$

there exists an element $v \in M$ *minimizing strictly the functional*

$$J_v(w) \equiv J(w) + \frac{\varepsilon}{\delta} d(v, w).$$

Moreover, one has

$$J(v) \leq J(u) \qquad \textit{whenever} \quad d(u, v) \leq \delta.$$

Proposition 5.2.1. *Let* W *be a Banach space and* $J \in \mathcal{C}^1(W)$ *bounded from below, then there exists a minimizing sequence* (v_n) *for* J *on* W *such that*

$$J(v_n) \longrightarrow \inf_W J, \qquad DJ(v_n) \longrightarrow 0 \quad \textit{in} \quad W' \quad \textit{in} \quad n \to \infty.$$

Critical Level

In order to solve Yamabe's problem in Riemannian geometry, Thierry Aubin introduced the notion of critical level associated to Yamabe's problem , noticed c^*, showed that if a minimizing sequence of the corresponding energy has a level lower than the critical level then the Yamabe's problem possesses solutions.
In a paper of Brézis and Nirenberg, the authors showed that if $(u_n) \subset W$ is Palais-Smale sequence of the functional J such that $J(u_n) \longrightarrow c < c^*$ as n tends to $+\infty$, therefore (u_n) is relatively compact. The level c^* is called critical in the sense where one can construct a Palais-Smale sequence $(w_n) \subset W$ such that $J(w_n) \to c^*$ but we could not extract any convergent subsequence in W.

5.3 Concentration-Compactness Theorem

This method introduced by P. L. Lions is the more general method for treating problems of minimization which they intervene in more varied domains (PDEs, calculus of variations, harmonic analysis, etc...)

Definition 5.3.1. *Let Ω be an open subset of $\mathbb{R}^N$, we define the following sets:*

$$\begin{aligned} \mathcal{K}(\Omega) &:= \{u \in \mathcal{C}(\Omega) : \quad u \text{ of compact support } \Omega\} \\ \mathcal{BC}(\Omega) &:= \left\{ u \in \mathcal{C}(\Omega) \, \|u\|_\infty := \sup_{x\in\Omega} |u(x)| < \infty \right\}. \end{aligned}$$

The space $\mathcal{C}_0(\Omega)$ is the cloture de $\mathcal{K}(\Omega)$ with respect to the uniform norm. A finite measure defined on Ω is a linear continuous application on $\mathcal{C}_0(\Omega)$. The norm of a finite measure μ is defined by

$$\|\mu\| := \sup_{\substack{u \in \mathcal{C}_0(\Omega) \\ \|u\|_\infty = 1}} | < \mu, u > |.$$

We denote by $\mathcal{M}(\Omega)$ (resp. $\mathcal{M}^+(\Omega)$) the space of finite measures (resp. positive finite measures) on Ω. A sequence of finite measures (μ_n) converges weakly to μ in $\mathcal{M}(\Omega)$, we write in this case

$$\mu_n \rightharpoonup \mu,$$

if one has

$$< \mu_n, u > \longrightarrow < \mu, u >, \quad \forall\, u \in \mathcal{C}_0(\Omega).$$

Theorem 5.3.1. *(Concentration Compactness lemma)*
Let $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ a sequence such that

$$\begin{aligned} u_n &\rightharpoonup u \quad in \quad \mathcal{D}^{1,2}(\mathbb{R}^N), \\ |\nabla(u_n - u)|^2 &\rightharpoonup \mu \quad in \quad \mathcal{M}(\mathbb{R}^N), \\ |u_n - u|^{2^*} &\rightharpoonup \nu \quad in \quad \mathcal{M}(\mathbb{R}^N), \\ u_n &\to u \quad a.e.\ in \quad \mathbb{R}^N \end{aligned}$$

et on définit

$$\mu_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\geq R} |\nabla u_n|^2, \quad \nu_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} |u_n|^{2^*}.$$

On a alors

- $\|\nu\|^{2/2^*} \leq S^{-1}\|\mu\|$,
- $\nu_\infty^{2/2^*} \leq S^{-1}\mu_\infty$,
- $\overline{\lim_{n\to\infty}} \|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\mu\| + \mu_\infty$
- $\overline{\lim_{n\to\infty}} \|u_n\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \|\nu\| + \nu_\infty$.

Furthermore, if $u = 0$ and $\|\nu\|^{2/2^*} = S^{-1}\|\mu\|$, then μ and ν are concentrated at a single point.

5.4 Brézis-Lieb Lemma

Théorème 5.1. *(Brézis-Lieb Lemma) Let $j : \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function such that $j(0) = 0$ and satisfy the following condition:*
For every $\varepsilon > 0$ small sufficiently, there exist two continuous and positive functions φ_ε and ψ_ε such that

$$(P) \qquad \forall\, a,\, b \in \mathbb{C}, \quad |j(a+b) - j(a)| \leq \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b).$$

Suppose that j satisfies the above hypothesis and let $f_n = f + g_n$ be a sequence of measurable functions from Ω into $\mathbb{C}$ such that:

(i) $g_n \to 0$ *a.e. in* Ω,

(ii) $j(f) \in L^1$,

(iii) there exist C independent from ε and n such that $\int \varphi_\varepsilon(g_n(x))d\mu(x) \leq C < \infty$,

(iv) $\int \psi_\varepsilon(f(x))\, d\mu(x) < \infty$ for all $\varepsilon > 0$, where μ is the Lebesgue's measure in $\mathbb{R}^N$.

Then as $n \to \infty$,

$$\int |j(f + g_n) - j(g_n) - j(f)|\, d\mu \to 0.$$

Proposition 5.4.1. *The functional considered in precedent chapters § **??**, 1, 2 and 3 of the form $s \mapsto sf(x,s)$ and $s \mapsto F(x,s) = \int_0^s f(x,t)dt$ satisfy the conditions of Brézis-Lieb lemma where f is a fonctional of Carathéodory type and satisfy for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ satisfying*

$$|f(x,u)| \leq \varepsilon|u|^{p^*-1} + C_\varepsilon,$$

uniformly in $x \in \mathbb{R}^N$.

Preuve. Let $\varepsilon > 0$, then

$$\begin{aligned}
|(a+b)f(x,a+b) - af(x,a)| &\leq |(a+b)f(x,a+b)| + |af(x,a)| \\
&< \varepsilon|a+b|^{p^*} + C_\varepsilon|a+b| + \varepsilon|a|^{p^*} + C_\varepsilon|a| \\
&< \varepsilon c(p)\left[|a|^{p^*} + |b|^{p^*}\right] + \varepsilon|a|^{p^*} + 2C_\varepsilon|a| + C_\varepsilon|b| \\
&< \varepsilon\left[(c(p)+1)|a|^{p^*} + (2C_\varepsilon/\varepsilon)|a|\right] + \varepsilon|b|^{p^*} + C_\varepsilon|b|.
\end{aligned}$$

With the same notions as in the Brézis-Lieb lemma , we take

$$\begin{aligned}
\varphi_\varepsilon(a) &= (c(p)+1)|a|^{p^*} + (2C_\varepsilon/\varepsilon)|a| \\
\psi_\varepsilon(b) &= \varepsilon|b|^{p^*} + C_\varepsilon|b|,
\end{aligned}$$

Frme where the relation (P) Brézis-Lieb lemma. In the other hand, for every $\varepsilon > 0$

$$\begin{aligned}
|F(x,a+b) - F(x,a)| &= \left|\int_a^{a+b} f(x,t)dt\right| \\
&\leq \int_a^{a+b} \left[\varepsilon|t|^{p^*-1} + C_\varepsilon\right] dt \\
&< \frac{\varepsilon}{p^*}\left||a+b|^{p^*} - |a|^{p^*}\right| + C_\varepsilon|b| \\
&< \frac{\varepsilon}{p^*}(c(p)+1)|a|^{p^*} + \frac{\varepsilon c(p)}{p^*}|b|^{p^*} + C_\varepsilon|b|.
\end{aligned}$$

Thus

$$\varphi_\varepsilon(a) = \frac{1}{p^*}(c(p)+1)|a|^{p^*}$$

and $$\psi_\varepsilon(b) = \frac{\varepsilon c(p)}{p^*}|b|^{p^*} + C_\varepsilon|b|.$$