– Computational geometry – From theory to practice, From linear objects to curved objects

Monique Teillaud

Overall message

Synergy between

- implementation
- theory (mathematics/algorithmics)

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- implementation
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- no good software is possible
  without a clean theoretical background
- implementation is raising new good theoretical questions

Overall message

Synergy between

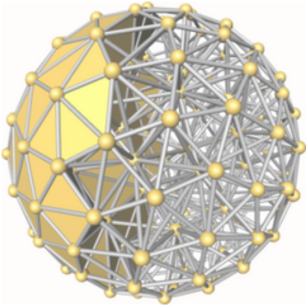
- implementation in CGAL
- theory (mathematics/algorithmics)

- no good software is possible
  without a clean theoretical background
- implementation is raising new good theoretical questions

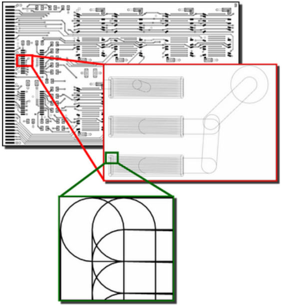
C++ concepts
mathematical concepts

can converge in CGAL

Outline



I - Triangulations



II - Curved objects

Thanks to Pierre Alliez for the beautiful pictures.

III - Current and future work

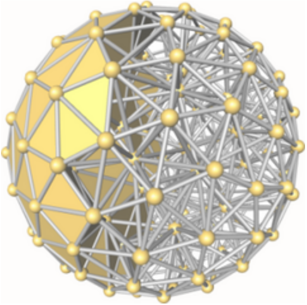
Outline

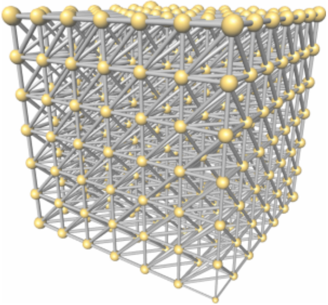
Overview of some of the links of the chain
leading to reliable and largely distributed software:

- mathematical background
- algorithmic and combinatorial study
- representation of objects and structures
- robustness issues
- design choices
- efficient programming
- ...

Part I

Triangulations

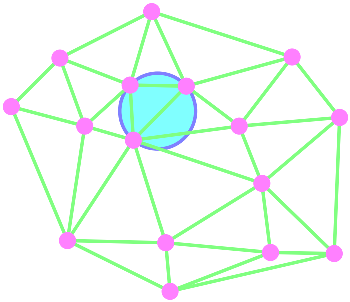




Triangulations

- mathematical background
- algorithmic and combinatorial study
- representation of objects and structures
- robustness issues
- design choices
- efficient programming

Delaunay triangulation in $\mathbb{R}^d$



All balls circumscribing simplices are empty.

Well known properties

Size $O\left(n^{\lceil \frac{d}{2} \rceil}\right)$ (linear in $\mathbb{R}^2$, worst-case quadratic in $\mathbb{R}^3$)

Randomized Incremental Algorithms
General Data Structure: the History graph.

Randomized Incremental Algorithms
General Data Structure: the History graph.

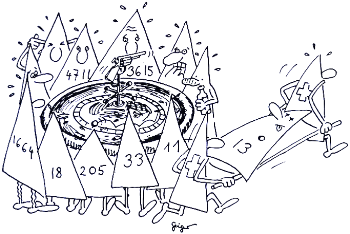
Case of Delaunay triangulations:

Fully dynamic algorithm

Optimal *expected* computation

$O\left(n^{\lfloor \frac{d+1}{2} \rfloor}\right)\ (d \geq 3)$

$O(n \log n)\ (d = 2)$



Bernhard Geiger

Randomized Incremental Algorithms
General Data Structure: the History graph.

Case of Delaunay triangulations:

Fully dynamic algorithm

Optimal *expected* computation

$O\left(n^{\lfloor\frac{d+1}{2}\rfloor}\right)$ $(d \geq 3)$

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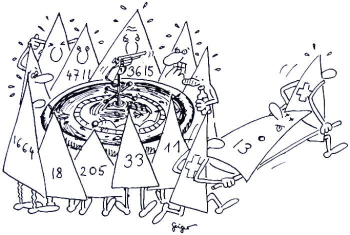


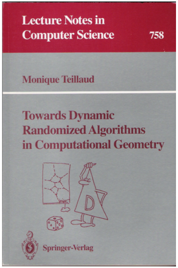
Bernhard Geiger

Practical framework:

- Realistic analysis (no assumption on input data)
- Implemented algorithms (Pascal, then C, . . .)

Randomized Incremental Algorithms
General Data Structure: the History graph.

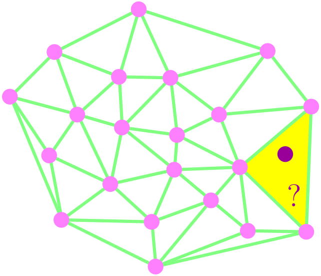
Boissonnat, Devillers,
T., Yvinec $\leq$'93
PhD Thesis '91

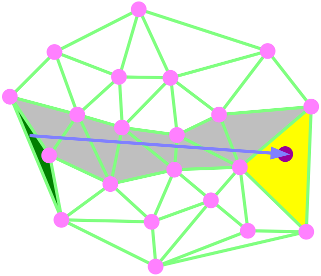


Case of Delaunay triangulations:
Devillers '02 better in practice

Practical framework:

- Realistic analysis (no assumption on input data)
- Implemented algorithms (Pascal, then C, then C++. . .)



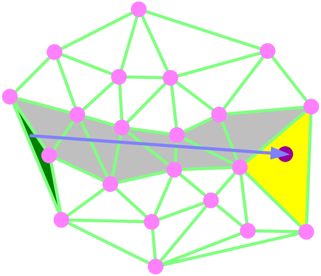


2D:
worst-case $2n$ triangles
Delaunay, random pts $|pq|\sqrt{n}$
`Devroye...`

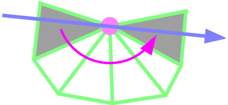
2 orient. tests per triangle

3D:
worst-case $O(n^2)$ tetrahedra
Delaunay, random pts?

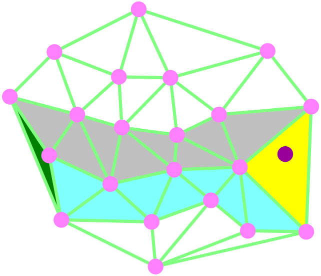
3 orient. tests per tetrahedron



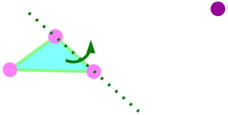
Handling degeneracies:

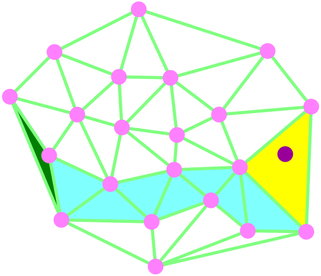


worse in 3D...



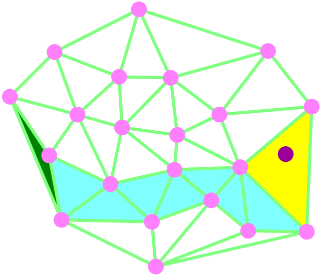
Visibility walk



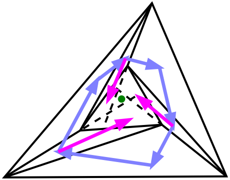


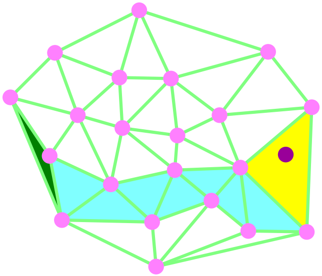
Delaunay 2D:
worst case $2n$
random points?
$\leq$ 1.5 orient. tests per triangle

Delaunay 3D:
worst-case $O(n^2)$ tetrahedra
random points?
$\leq$ 2 orient. tests per tetra.

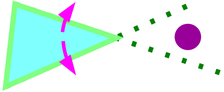


May loop for non Delaunay tr.





Stochastic walk

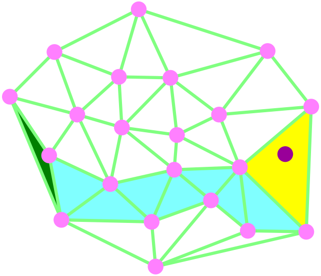


random choice

Always terminates

Average complexity
$\leq$ exponential

Exponential example

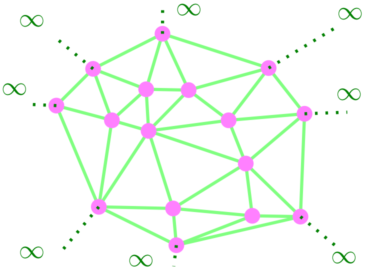


Easy to code, even in 3D
no degeneracies...

Efficient in practice

Devillers, Pion, T. *SoCG'01, Int. J. Found. Comp. Sc.'02*

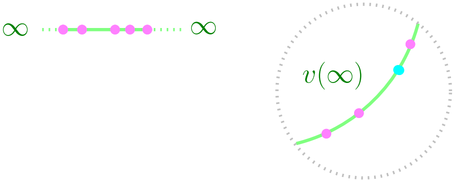
Combinatorial triangulation of the sphere



Devillers, T., Yvinec

Degenerate dimensions

geometric



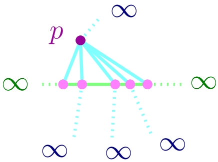
combinatorial

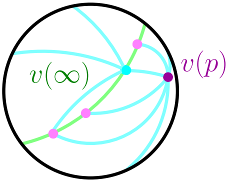
Increasing the dimension

T. *EuroCG'99*

Degenerate dimensions

geometric





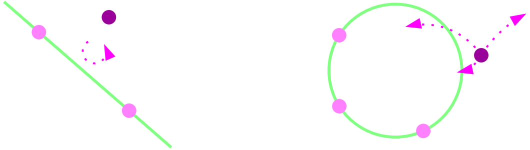
combinatorial

Increasing the dimension

T. *EuroCG'99*

Algorithms rely on

predicates = elementary operations



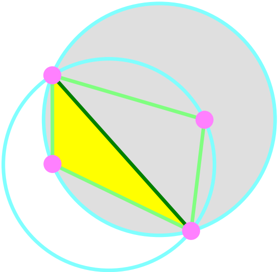
evaluated numerically.

Typically, signs of determinants

(small degree polynomial expressions)

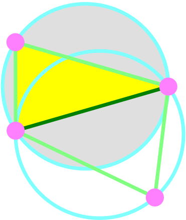
Exact Geometric Computing framework Yap

Filtered exact computations Yap...Mehlhorn...Pion...



Delaunay triangulation depends on the sign of

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ p_x & q_x & r_x & s_x \\ p_y & q_y & r_y & s_y \\ 1+p_x^2+p_y^2 & 1+q_x^2+q_y^2 & 1+r_x^2+r_y^2 & 1+s_x^2+s_y^2 \end{vmatrix}$$



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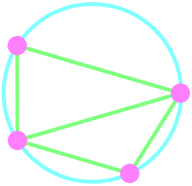
Degenerate configurations handled explicitely.

Cospherical points

Delaunay triangulation
not uniquely defined



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Unique definition?

Required: consistent choice of sign.

Degenerate configurations handled explicitely.

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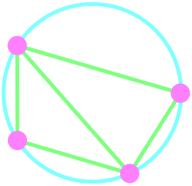


$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ p_x & q_x & r_x & s_x \\ p_y & q_y & r_y & s_y \\ 1+p_x^2+p_y^2 & 1+q_x^2+q_y^2 & 1+r_x^2+r_y^2 & 1+s_x^2+s_y^2 \end{vmatrix} = 0$$

Unique definition?

Required: consistent choice of sign.

Symbolic perturbation of the predicate

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ p_x & q_x & r_x & s_x \\ p_y & q_y & r_y & s_y \\ 1+p_x^2+p_y^2+\varepsilon^i & 1+q_x^2+q_y^2+\varepsilon^j & 1+r_x^2+r_y^2+\varepsilon^k & 1+s_x^2+s_y^2+\varepsilon^l \end{vmatrix} = P(\varepsilon)$$

coefficients =

orientation predicates of the non-perturbed points

Sign = sign of the first non-null coefficient.

Symbolic perturbation of the predicate

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ p_x & q_x & r_x & s_x \\ p_y & q_y & r_y & s_y \\ 1+p_x^2+p_y^2+\varepsilon^i & 1+q_x^2+q_y^2+\varepsilon^j & 1+r_x^2+r_y^2+\varepsilon^k & 1+s_x^2+s_y^2+\varepsilon^l \end{vmatrix} = P(\varepsilon)$$

coefficients =

orientation predicates of the non-perturbed points

Sign = sign of the first non-null coefficient.

Delaunay triangulation uniquely defined
by indexing the points e.g. lexicographic ordering.

This perturbation

- Does not create any flat tetrahedron
- Easy to code
- Allows implementation of vertex removal in 3D
- Generalizes to regular triangulations

`Devillers, T.` *SODA'03*

`Devillers, T.` *RR INRIA'07*

Enforced decoupling between geometry and combinatorics

Kettner *CGAL Polyhedron*

`Triangulation_3 < Traits_3, TrDataStructure_3 >`

- `Triangulation_3`: point location. . .
- `TrDataStructure_3`: insertion. . .

T. *EuroCG'99*

CGAL 3D Triangulation package

- Fully dynamic (insertion, removal)
- Robust
- Efficient ($\sim 30,000$ pts/sec)
- Generic
- Flexible
- Publicly available (QPL)
- Documented

`T.` *CGAL 2.1 - 2.2, '00*

`Pion, T.` *CGAL 2.3 - 3.3, '01-'07*

Participation of C.Delage, O.Devillers, A. Fabri,. . .

2D: `Yvinec` *CGAL 0.9-...* `Pion, Yvinec`

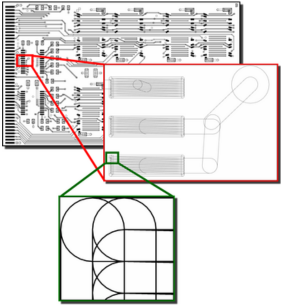
CGAL 3D Triangulation package

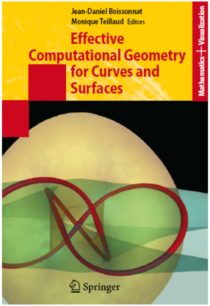
Many users

- CGAL packages (meshing, reconstruction)
  `Rineau, Yvinec`
- academic users
  `Dey, Giesen, Oudot, Chaine, Amenta, Levy, Bernauer, Robbins...`
- commercial users (through GeometryFactory)
  `Midland Valley Exploration, Total, BSAP, British Telecom, France Telecom`

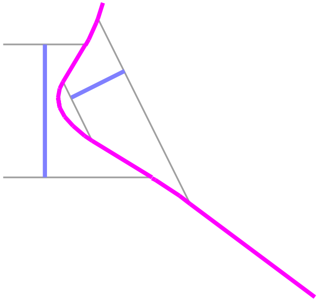
Part II

Curved Objects



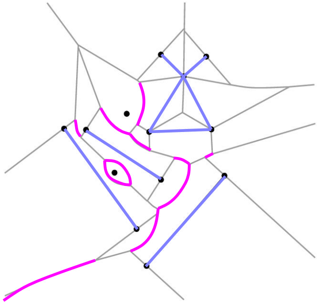


Curves already appear for linear input



Bisecting curve

Curves already appear for linear input

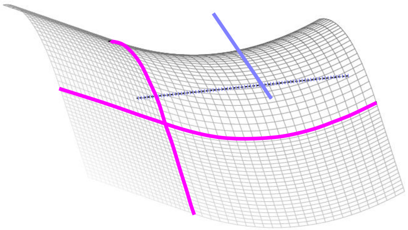


Karavelas, CGAL

Voronoi diagram

2D line segments
arcs of parabolas

Curves already appear for linear input



Voronoi diagram

3D line segments
patches of quadric surfaces

Mostly linear objects handled in

- computational geometry literature
- software, CGAL.

Start with low degree algebraic objects (circles, spheres)

Curved Objects

- mathematical background
- • algorithmic and combinatorial study and
  • robustness issues
- design choices
- representation of objects and structures
- efficient programming

Sweeping plane approach
Computes the so-called vertical decomposition
volumic approach

Comparisons of algebraic numbers of degree 16...
Mourrain, Técourt, T. *CGTA '05*

Complexity $O(n \log^2 n + V \log n)$, $V = O(n^3.2^{\alpha(n)^{16}})$
Halperin et al
Chazelle, Edelsbrunner, Guibas, Sharir

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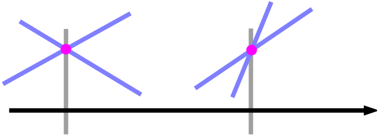
`Halperin et al`
`Chazelle, Edelsbrunner, Guibas, Sharir`

Case of spheres: algebraic numbers of degree 4

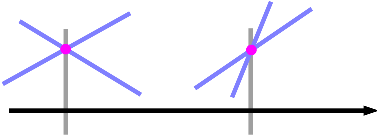
New decomposition: degree 2 only
+ Degenerate cases `Russel, T.` *in progress*
CGAL implementation `Russel` *in progress*
uses CGAL 3D cellular data structure `Bru, T.` *in progress*

Comparing intersection points



signs of
polynomial expressions

Comparing intersection points



signs of
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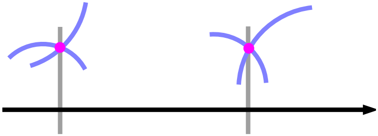

comparison of
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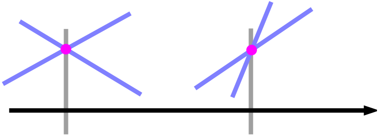
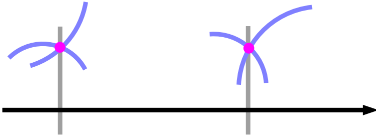




signs of
polynomial expressions

comparison of
algebraic numbers

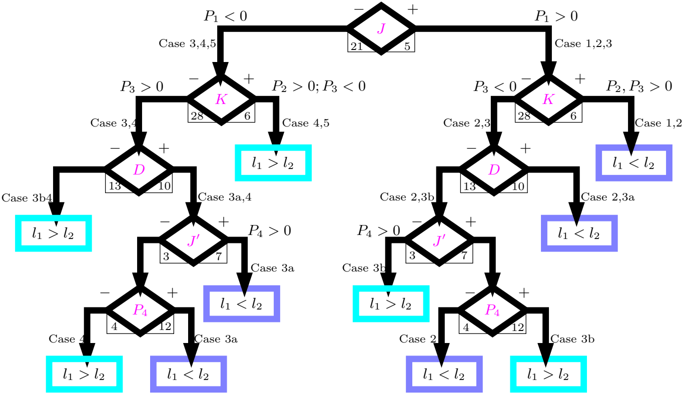
Algebraic tools $\longrightarrow$
signs of
polynomial expressions

Comparing algebraic numbers of degree 2

Measure of complexity

- degree of polynomials
- number of arithmetic operations

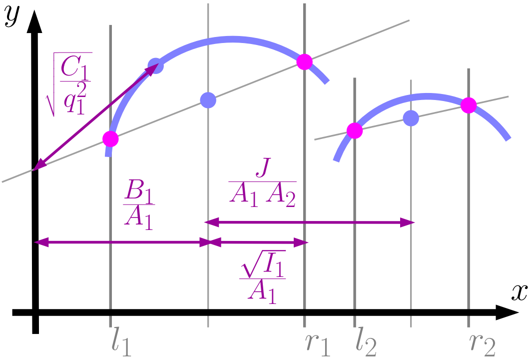
Comparing algebraic numbers of degree 2



Polynomial expressions pre-computed + arithmetic filtering

`Devillers, Fronville, Mourrain, T.` *SoCG '00 CGTA '02*

Polynomial expressions have an intrinsic geometric meaning



$K = 0 \iff l_1, r_1, l_2, r_2$ harmonic division

geometric reasoning?

Enforced decoupling between geometry and algebra

`Curved_kernel < LinearKernel, AlgebraicKernel >`

"Kernel": basic geometric objects and manipulations

allows to

- interchange algebraic kernels
- compare different approaches

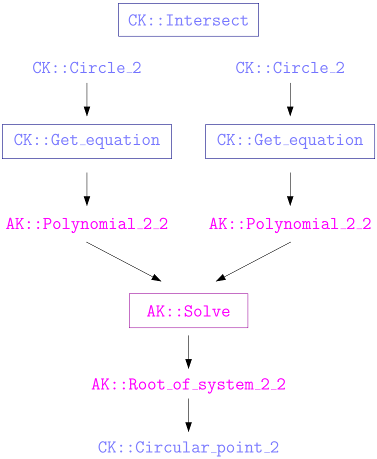
Definition of the `AlgebraicKernel` C++ concept

=

identification of mathematical concepts underlying the computations

Enforced decoupling between geometry and algebra

Careful definition of the interface



Enforced decoupling between geometry and algebra

High-level interface for the algebraic kernel

- `Solve` polynomial system
- `Sign_at` of polynomial at the roots of a system
- ...

Emiris, Kakargias, Pion, Tsigaridas, T. *SoCG '04*

Full specifications, general degree:

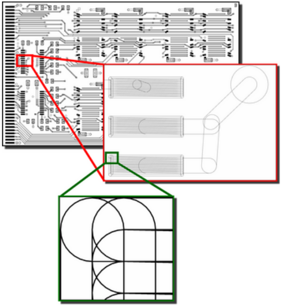
Berberich, Hemmer, Karavelas, T. *CGAL, submitted*

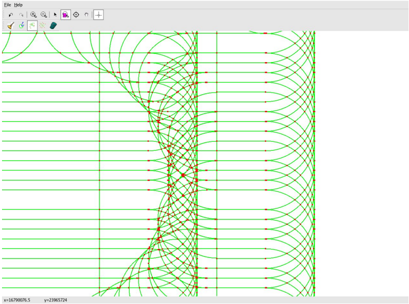
why not: non-algebraic curves...?

Circular arcs in 2D

Benchmarking on industrial VLSI data

with CGAL `Arrangement_2` package





Circular arcs in 2D

Benchmarking on industrial VLSI data
with CGAL `Arrangement_2` package

Efficiency improved:

- caching
- reference counting
- optimization of special cases (rational intersections)
- arithmetic filtering in algebraic numbers
- representation of algebraic numbers
  to reduce length of multi-precision coefficients
- geometric filtering

de Castro, Pion, T. *EuroCG '07*

CGAL packages

- 2D Circular Kernel `Pion, T.` *CGAL 3.2-3.3 '06-'07*
  - Research license `Dassault Systèmes`
- 3D Spherical Kernel `de Castro, T.` *CGAL, Submitted*
  Extension in progress
  `de Castro, Cazals, Loriot, T.` *RR INRIA'07*
  - Used by `Russel, Loriot`

Minimality of the set of predicates
necessary to run an algorithm or compute a structure

Degree measure of precision

Minimality of the set of predicates
necessary to run an algorithm or compute a structure

- number of predicates
- complexity of predicates

Degree measure of precision

Minimality of the set of predicates
necessary to run an algorithm or compute a structure

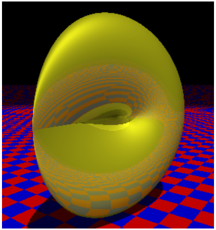
- number of predicates
- complexity of predicates

Degree measure of precision

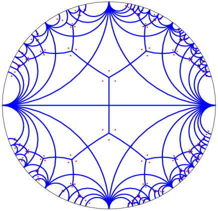
- of a given geometric predicate
- of an algorithm
- of a geometric problem

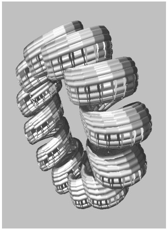
Part III

Future? More Triangulations!



©S. Popescu





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Future? More Triangulations!

Most of the computational geometry literature in $\mathbb{R}^d$

Implementations $\mathbb{R}^2$ or $\mathbb{R}^3$

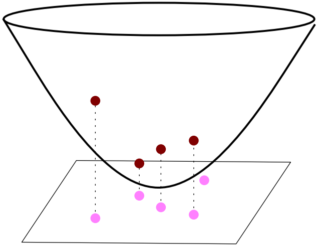
Other geometries?

- hyperbolic
- projective
- periodic
- . . . ?

Future? More Triangulations! Hyperbolic

The space of spheres

$$\begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R}^{d+1} \\ S:(C,r) & \mapsto & s=(C, \|C\|^2 - r^2) \end{array}$$

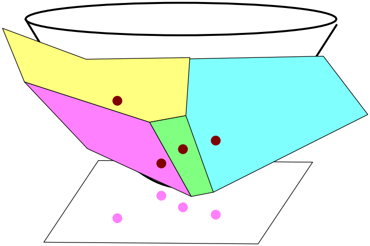


Unified framework for the [mostly known] duality results:
various generalized Voronoi diagrams $\longleftrightarrow$ lower envelopes

Future? More Triangulations! Hyperbolic

The space of spheres

$$
\begin{array}{ccc}
\mathbb{R}^d & \longrightarrow & \mathbb{R}^{d+1} \\
S : (C, r) & \mapsto & s = (C, \|C\|^2 - r^2)
\end{array}
$$



Unified framework for the [mostly known] duality results:
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Future? More Triangulations! Hyperbolic

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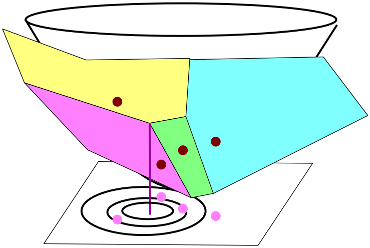

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Future? More Triangulations!

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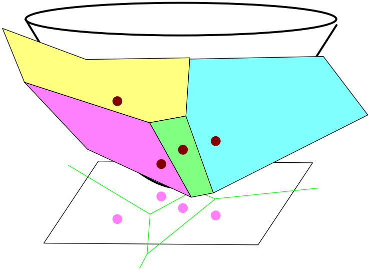
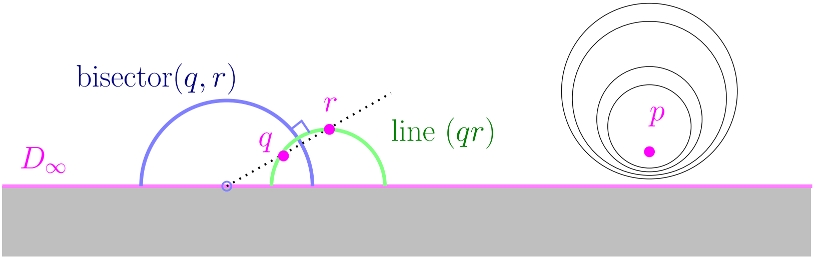


Unified framework for the [mostly known] duality results:
various generalized Voronoi diagrams $\longleftrightarrow$ lower envelopes

Poincaré model of the hyperbolic plane:

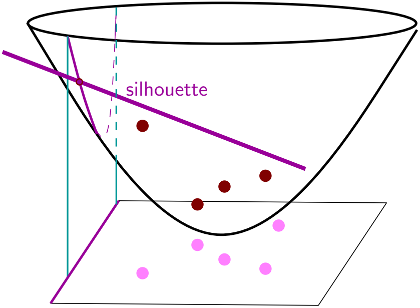


Hyperbolic line
=
Half Euclidean circle orthogonal to D_∞

Hyperbolic circle centered at p
=
Euclidean circle of the pencil with limit point p and radical axis D_∞

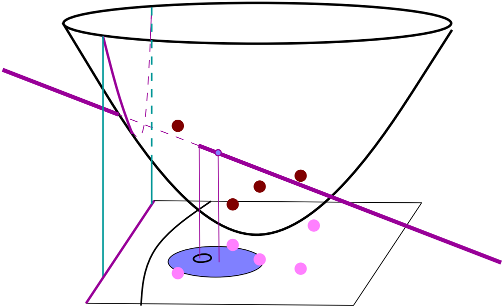
Future? More Triangulations! Hyperbolic

$\mathbb{R}^d$		$\mathbb{R}^{d+1}$
Pencil of circles with given radical axis	$\longleftrightarrow$	Line with given direction

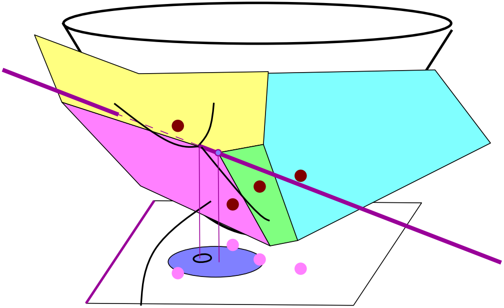


Future? More Triangulations!

$\mathbb{R}^d$		$\mathbb{R}^{d+1}$
Pencil of circles with given radical axis	$\longleftrightarrow$	Line with given direction



$\mathbb{R}^d$		$\mathbb{R}^{d+1}$
Pencil of circles with given radical axis	$\longleftrightarrow$	Line with given direction



Future? More Triangulations! Hyperbolic

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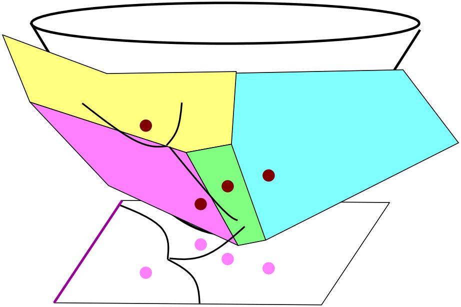
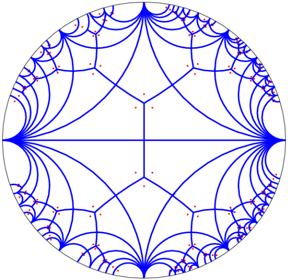


Future? More Triangulations! Hyperbolic



Berger *'77 '87*

Boissonnat, Cérézo, Devillers, T. *CCCG'91 IJCGA'96*

Devillers, Meiser, T. *RR INRIA'92*

Perspectives

CGAL implementation and arising issues

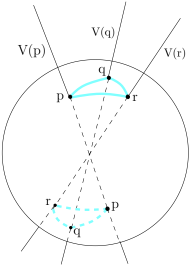
Applications (study of cristalline structures. . . ?)

Triangulation of the projective plane
mostly studied from a graph-theoretic perspective.

In computational geometry:

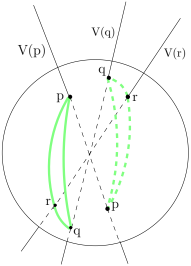
- algorithms in $\mathbb{R}^d$ based on orientation predicates
- oriented projective plane Stolfi

Projective plane non-orientable



$$\mathbb{P}^2 = \mathbb{R}^3 - \{0\} \;\; / \sim$$
$$p \sim p' : p = \lambda p', \lambda \in \mathbb{R}^\star$$

Projective plane non-orientable



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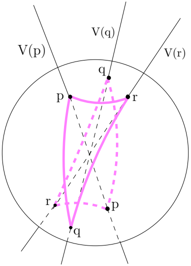

$$\mathbb{P}^2 = \mathbb{R}^3 - \{0\} \ / \sim$$
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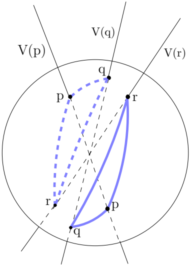
Future? More Triangulations! Projective

Projective plane non-orientable



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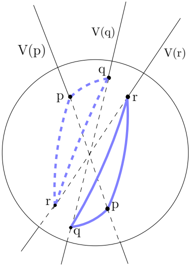
Projective plane non-orientable



Still:
interior of a closed curve well defined.

Incremental algorithm `Aanjaneya, T.` *RR INRIA'07*

Perspectives

Definition of a Delaunay-like triangulation

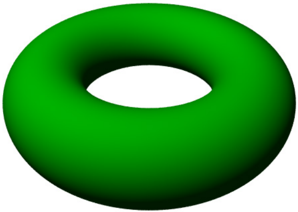
Generalization to 3D (dD)

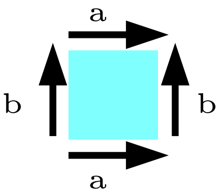
CGAL implementation

Future? More Triangulations!

Periodic triangulations (2D and 3D) widely used for simulations

Parameter space : torus





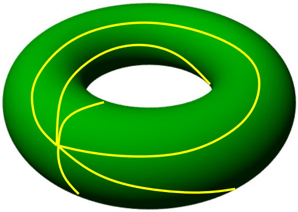
2D:

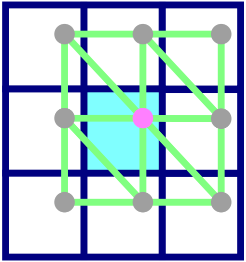
Future? More Triangulations! Periodic

Periodic triangulations (2D and 3D) widely used for simulations

Parameter space : torus

Few points: the “triangulation” is not a simplicial complex.





2D:

Future? More Triangulations!

Method

- Compute first in a 3-sheeted covering (27 copies in 3D),
- Switch to the 1-sheeted covering as soon as simplicial complex.

Implementation

- Redesign the CGAL triangulation package,
  Allow one more level of genericity.

```
Triangulation_3
      < Traits_3, TrDataStructure_3, Space >
```

Kruithof, T.

Caroli, T. *in progress*

Perspectives

Other periodic spaces (cylinders)

Meshes in periodic spaces. . .

Conclusion

Implementing mathematical structures has

- a good future,
- many applications.

> Par ce texte élémentaire, l'auteur espère faire partager sa joie de la découverte d'une théorie mathématique dont la beauté l'émeut, la richesse le ravit, la profondeur l'impressionne, et le pouvoir de description de l'univers le déconcerte.

André Cérézo ’91