Systemes de particules multicolores

Nicolas Lanchier

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THÈSE

en vue de l'obtention du titre de

Docteur de l'Université de Rouen

présentée par

Nicolas Lanchier

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Systèmes de particules multicolores

Soutenue le 22 septembre 2005 devant le jury composé de

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Présentation des résultats

Les processus dont il est question dans cette thèse, appelés *systèmes de particules multicolores*, sont des processus de Markov $\xi_t : \mathbb{Z}^d \longrightarrow F$ où $\mathbb{Z}^d$ désigne l'espace d-dimensionnel à coordonnées entières, et F un ensemble fini appelé *ensemble des couleurs*. Pour décrire l'évolution de nos processus, nous commençons par introduire un *voisinage d'interaction*

$$\mathcal{N} \;=\; \{z_0,\, z_1,\, \ldots,\, z_N\} \;\subset\; \mathbb{Z}^d \qquad \text{avec} \quad z_0 \;=\; 0.$$

Le système étant dans une configuration $\xi \in F^{\mathbb{Z}^d}$, la couleur de $x \in \mathbb{Z}^d$ passe de i à j au taux

$$c_{ij}\,(x, \xi) \;=\; h_{ij}\,(x,\, \xi(x + z_0),\, \xi(x + z_1),\, \ldots,\, \xi(x + z_N)).$$

En d'autres termes, les interactions ont une *portée finie*, i.e. le taux de transition en x dépend uniquement de la position du site x et de la couleur d'un nombre fini de ses voisins. Dans la plupart de nos exemples, les processus seront supposés *invariants par translation*, autrement dit les règles appliquées en x ne sont qu'une translation de celles appliquées en 0, ce qui signifie encore que la fonction h_{ij} ne dépend plus de la position de x. Pour tenter de comprendre le comportement d'un système de particules, notre principal objectif sera d'exhiber ses *mesures stationnaires*, i.e. les mesures μ telles que $P_\mu(\xi_t \in A) = \mu(A)$ pour tous $t \geq 0$ et $A \in \mathcal{F}$, où P_μ désigne la loi du processus issu de μ, et $\mathcal{F}$ la tribu usuelle sur l'ensemble des fonctions $\xi : \mathbb{Z}^d \longrightarrow F$.

1. Le modèle successionnel

Notre premier modèle spatial, appelé modèle successionnel, est un processus de Markov en temps continu dont l'état à l'instant t est une fonction $\xi_t : \mathbb{Z}^d \longrightarrow \{0, 1, 2\}$. Un site $x \in \mathbb{Z}^d$ est dit vide si $\xi(x) = 0$, et occupé par une particule de type 1 (resp. 2) si $\xi(x) = 1$ (resp. 2). L'évolution au site x est décrite par les taux de transition

$$\begin{array}{rclcrcl} c_{01}\,(x, \xi) & = & \lambda_1 \displaystyle\sum_{0 < \|x - z\| \leq R} \mathbb{1}_{\{\xi(z) = 1\}} & \qquad & c_{12}\,(x, \xi) & = & 1 \\ c_{02}\,(x, \xi) & = & \lambda_2 \displaystyle\sum_{0 < \|x - z\| \leq R} \mathbb{1}_{\{\xi(z) = 2\}} & \qquad & c_{20}\,(x, \xi) & = & 1. \end{array}$$

Nous avons baptisé notre processus modèle successionnel en référence aux phénomènes de successions écologiques dont il décrit l'évolution. Dans notre contexte écologique, les 1 sont les individus d'une epèce dite *pionnière* capable d'envahir un milieu hostile. L'apparition de cette première espèce favorise ensuite le developpement d'une seconde espèce, ici représentée par les particules de type 2, la transition spontanée $1 \rightarrow 2$ s'expliquant, dans le cadre des communautés de plantes, par la présence dans le sol d'une banque de graines.

Tout d'abord, si seuls les 2 sont présents, le système est réduit à un processus de contact de paramètre λ_2. Dans ce cas, il existe une valeur critique $\lambda_c \in (0, \infty)$ telle que les assertions suivantes soient satisfaites. Si $\lambda_2 \leq \lambda_c$ alors le processus converge en distribution vers δ_0, la configuration ne contenant que des 0. Sinon, il existe une mesure stationnaire μ_2 concentrée sur les configurations contenant une infinité de 2 (Liggett, 1999). Dans le cas $\lambda_1 \leq \lambda_c$ et $\lambda_2 \leq \lambda_c$, il est en particulier facile de voir qu'il y a extinction, i.e. ξ_t converge vers δ_0.

Afin d'éviter les trivialités, nous supposons désormais que ξ_0 contient une infinité de particules de type 1 et 2. Dans le cas $\lambda_1 \geq \lambda_2 > \lambda_c$ un argument de couplage standard (Harris, 1972) mettant en jeu notre modèle et le processus de contact implique que les 2 survivent. Si $\lambda_2 > \lambda_1 > \lambda_c$ un nouveau couplage assorti des résultats de Durrett et Neuhauser (1997), Section 3, nous permet de montrer que les particules de type 2 l'emportent. En conclusion, nous obtenons le théorème suivant, où $\Rightarrow$ fait référence à la convergence faible.

Théorème 1 *Fixons $\lambda_2 > \lambda_c$. Si $\lambda_1 \geq \lambda_2$ alors les 2 survivent, i.e. $P\,(\forall\, t \geq 0,\, \exists\, x : \xi_t(x) = 2) = 1$. Si $\lambda_1 < \lambda_2$ alors les 2 l'emportent, i.e. $\xi_t \Rightarrow \mu_2$.*

Examinons maintenant le cas $\lambda_2 > 0$ proche de 0. En supprimant la transition $0 \to 2$, c'est-à-dire en fixant $\lambda_2 = 0$, le processus devient un modèle d'incendie de forêt avec 0 = vivant, 1 = en feu, et 2 = brûlé. En utilisant un argument de renormalisation, Durrett et Neuhauser (1991) ont prouvé que, dans le cas d'une dynamique à plus proches voisins en dimension 2, il existe une valeur critique $\alpha_c \in (0, \infty)$ telle que si $\lambda_1 > \alpha_c$ alors le processus admet une mesure stationnaire non triviale ν concentrée sur les configurations contenant une infinité de 1 et de 2. Le théorème 2 nous dit que les 1 et les 2 coexistent également si $\lambda_2 > 0$ est suffisamment proche de 0, i.e. le processus admet une mesure stationnaire $\nu = \nu_{\lambda_1,\lambda_2}$ telle que $\nu(\xi(x) = 1) \neq 0$.

Théorème 2 *Posons $d = 2$ et $\lambda_1 > \alpha_c$. Il existe $\beta_c^{Th2} \in (0, \infty)$ tel que si $\lambda_2 \leq \beta_c^{Th2}$ alors les particules de type 1 et 2 coexistent, i.e. $\xi_t \Rightarrow \nu$ avec $\nu(\xi(x) = 1) \neq 0$.*

Nous arrivons maintenant à notre principal résultat, à savoir que les particules de type 2 l'emportent également si $\lambda_1 = \lambda_2$. Bien que le résultat ne semble pas surprenant, la preuve repose sur une propriété quelque peu intéressante du processus dual.

Théorème 3 *Si $\lambda_1 = \lambda_2 > \lambda_c$ et ξ_0 est invariant par translation alors $\xi_t \Rightarrow \mu_2$.*

Pour décrire le processus dual, nous commençons par construire le système de particules à partir d'une collection de processus de Poisson indépendants. Pour $x,\ z \in \mathbb{Z}^d$ avec $\|x - z\| \leq R$, nous désignons par $\{T_n^{x,z} : n \geq 1\}$ les temps d'arrivée de processus de Poisson indépendants d'intensité λ_1. Aux instants $T_n^{x,z}$, nous traçons une flèche du site x au site z pour indiquer que, si x est occupé et z est vide, alors la particule en x donne naissance à une particule du même type en z. Pour prendre en compte les transitions $1 \to 2$ et $2 \to 0$, nous introduisons deux collections supplémentaires de processus de Poisson indépendants $\{U_n^x : n \geq 1\}$ et $\{V_n^x : n \geq 1\}$, $x \in \mathbb{Z}^d$, d'intensité 1. Enfin, nous plaçons une $\times$ en (x, U_n^x) pour indiquer qu'une particule de type 2 est tuée, et un $\bullet$ en (x, V_n^x) pour indiquer qu'une particule de type 1 change de couleur.

Le lecteur pourra observer que, comme les $\times$ ne tuent qu'un type de particule, la construction d'un processus dual à partir de la représentation graphique décrite ci-dessus produira un objet quelque peu complexe. Le principal ingrédient pour simplifier le processus dual consiste à observer que, comme les 1 (resp. 2) ne voient pas les $\times$ (resp. $\bullet$), nous pouvons superposer les deux symboles. En d'autres termes, nous retirons les $\bullet$ de notre représentation graphique, et prétendons que l'effet des $\times$ est maintenant de tuer les 2, et de changer la couleur des 1.

Nous dirons que (z, s) et (x, t), $0 \leq s \leq t$, sont *fortement connectés* s'il existe une suite d'instants $s_0 = s < s_1 < \cdots < s_{n+1} = t$ et une suite de sites $x_0 = z,\ x_1,\ \ldots,\ x_n = x$ tels que

1. Pour $i = 1,\ 2,\ \ldots,\ n$, il existe une flèche de x_{i-1} à x_i au temps s_i et
2. Pour $i = 0,\ 1,\ \ldots,\ n$, le segment vertical $\{x_i\} \times (s_i, s_{i+1})$ ne contient aucune $\times$.

Si au lieu de la condition 2 ci-dessus

3. L'ensemble $\displaystyle\bigcup_{i=0}^{n} \{x_i\} \times (s_i, s_{i+1})$ contient exactement une $\times$,

nous dirons que (z, s) et (x, t) sont *faiblement connectés*. Nous dirons enfin qu'il existe un *chemin* de (z, s) à (x, t) si les deux points sont soit fortement soit faiblement connectés. Compte tenu de l'effet des $\times$, s'il existe un chemin de (z, s) à (x, t), une particule de type 1 en (z, s) peut certainement être l'*ancêtre* d'une particule de type 2 en (x, t) si elle traverse une $\times$ lors de son ascension vers (x, t). En conclusion, le *processus dual issu de* (x, t) sera défini en posant

$$\hat{\xi}_s^{(x,t)} \;=\; \{z \in \mathbb{Z}^d : \text{il existe un chemin dual de } (x,t) \text{ à } (z, t - s)\}$$

où un *chemin dual* de (x, t) à $(z, t - s)$ indique l'existence d'un chemin de $(z, t - s)$ à (x, t). Notons maintenant que $\{(\hat{\xi}_s^{(x,t)}, s) : 0 \leq s \leq t\}$ exhibe une *structure d'arbre* divisée en deux couches selon le nombre de × (0 ou 1) qu'un ancêtre traverse pour atteindre (x, t). Un site $z \in \mathbb{Z}^d$ sera appelé *sur-ancêtre* (resp. *sous-ancêtre*) au temps s s'il existe un chemin de (z, s) à (x, t) contenant exactement 0 (resp. 1) ×. De même que pour le processus de contact multitype (Neuhauser, 1992), la structure d'arbre du processus dual induit une *hiérarchie*, les ancêtres pouvant être rangés selon l'ordre dans lequel ils déterminent la couleur de (x, t). Le premier ancêtre au sens de cette relation d'ordre sera appelé *particule déterminante*, et le premier sur-ancêtre *particule distinguée.*

Maintenant le processus dual construit, nous pouvons donner l'idée de la preuve. La première étape consiste à prouver que $P(\Theta_1 < \infty) = 1$, où Θ_1 désigne le dernier instant auquel la particule déterminante est un sur-ancêtre. La suite de la preuve repose sur le fait qu'une fois que le premier ancêtre est un sous-ancêtre (après l'instant Θ_1), il ne peut peindre (x, t) qu'avec la couleur 2 puisque l'effet des × est de changer la couleur des 1. En dimension $d \leq 2$, nous attendons jusqu'au temps Θ_1 et prouvons que la particule distinguée et la particule déterminante fusionnent presque sûrement de sorte qu'elles atterrissent, pour t suffisamment grand, sur le même site. Si ce site est initialement occupé par un 1, la particule déterminante peindra (x, t) avec la couleur 2, tandis que s'il est occupé par un 2, la particule distinguée peindra (x, t) avec la couleur 2 à moins qu'un sous-ancêtre ne le fasse avant. En dimension $d \geq 3$, nous exploitons la finitude de Θ_1 pour construire une suite $\zeta_s(n)$ de sous-ancêtres prioritaires sur la particule distinguée dans la hiérarchie des ancêtres, et extrayons une sous-suite de candidats $\zeta_s(n_k)$ ne fusionnant pas entre-eux pour conclure qu'au moins l'un d'eux atterrit sur un 1 au temps 0 et peint (x, t) avec la couleur 2.

2. Le processus de contact multitype avec blocage dynamique

Notre second système de particules est un processus de Markov dont l'état à l'instant t est une fonction $\xi_t : \mathbb{Z}^d \longrightarrow \{0, 1, 2, 3\}$. Comme précédemment, un site $x \in \mathbb{Z}^d$ est dit occupé par une particule de type 1 (resp. 2) si $\xi(x) = 1$ (resp. 2), et vide sinon. Nous distinguons maintenant deux types de sites vides. Plus précisément, un site $x \in \mathbb{Z}^d$ est appelé *site libre* si $\xi(x) = 0$, et appelé *site gelé* si $\xi(x) = 3$. L'évolution au site x est décrite par les taux de transition

$$\begin{array}{rclcrcl} c_{01}(x,\xi) & = & \lambda_1 \displaystyle\sum_{0<\|x-z\|\leq R} \mathbb{1}_{\{\xi(z)=1\}} & \qquad & c_{13}(x,\xi) & = & 1 \\[2ex] c_{31}(x,\xi) & = & \lambda_1 \displaystyle\sum_{0<\|x-z\|\leq R} \mathbb{1}_{\{\xi(z)=1\}} & \qquad & c_{30}(x,\xi) & = & \gamma \\[2ex] c_{02}(x,\xi) & = & \lambda_2 \displaystyle\sum_{0<\|x-z\|\leq R} \mathbb{1}_{\{\xi(z)=2\}} & \qquad & c_{20}(x,\xi) & = & 1. \end{array}$$

En particulier, notre processus est une généralisation du processus de contact multitype (Neuhauser, 1992) où les 1 inhibent la propagation des 2 en gelant les sites qu'ils viennent d'occuper. Réciproquement, le processus de contact multitype n'est autre que le cas limite $\gamma = \infty$. L'interprétation que nous avons à l'esprit est celle d'un modèle spatial d'*allélopathie.* Dans la littérature biologique, l'allélopathie est définie comme un processus impliquant des métabolites secondaires produites par des plantes, des micro-organismes, des virus ou des champignons ayant une influence sur la croissance et le développement d'un système biologique. Dans notre cas, les 1 sont les individus d'une espèce *inhibitrice* et les 2 ceux d'une espèce *sensible.*

Pour étudier notre modèle, nous commençons par observer que, si seuls les 2 sont présents, le processus est réduit comme précédemment à un processus de contact de paramètre λ_2. En particulier, si $\lambda_2 \leq \lambda_c$ alors $\xi_t \Rightarrow \delta_0$, tandis que si $\lambda_2 > \lambda_c$ alors $\xi_t \Rightarrow \mu_2$ (voir Section 1). Si seuls les 1 sont présents, nous obtenons un résultat similaire. Plus précisément, si $\lambda_1 \leq \lambda_c$ alors le processus converge en distribution vers δ_0, tandis que si $\lambda_1 > \lambda_c$ il existe une mesure stationnaire non triviale ν_1 concentrée sur les configurations contenant une infinité de 1 et de 3. En particulier, dans le cas $\lambda_1 \leq \lambda_c$ et $\lambda_2 \leq \lambda_c$, il y a extinction des deux espèces.

Pour éviter les trivialités, nous supposons maintenant que $\lambda_1 > \lambda_c$, $\lambda_2 > \lambda_c$ et ξ_0 contient une infinité de 1 et de 2. Tout d'abord, un argument de couplage standard implique que la probabilité de survie des 1 est croissante en λ_1 et décroissante en λ_2 et γ. Le résultat analogue pour les 2 est également vérifié. En conclusion, nous obtenons le

Théorème 4 *Soit* $\Theta_t^i = \{x \in \mathbb{Z}^d : \xi_t(x) = i\}$ *l'ensemble des sites occupés à l'instant t par une particule de type i. Alors les probabilités de survie* $P\,(\Theta_t^i \neq \varnothing$ *pour tout* $t \geq 0)$, $i = 1, 2$, *sont monotones par rapport à chacun des paramètres* λ_1, λ_2 *et* γ.

En observant en particulier que le modèle de paramètre $\gamma = \infty$ correspond au processus de contact multitype, le théorème 4 ci-dessus ainsi que le théorème 1 de Neuhauser (1992) impliquent que

Théorème 5 *Si* $\lambda_1 > \lambda_2 > \lambda_c$ *et* $\gamma \in (0, \infty)$ *alors* $\xi_t \Rightarrow \nu_1$.

Examinons maintenant le cas $\lambda_1 = \lambda_2$. Comme les lois d'évolution favorisent les 1, nous nous attendons à ce que les processus avec et sans blocage dynamique exhibent des comportements différents. D'après le théorème 6 ci-dessous, les 1 l'emportent en $d \geq 3$ si $\gamma < \infty$, tandis que les deux types de particules coexistent si $\gamma = \infty$. Voir le théorème 3 de Neuhauser (1992). Nous conjecturons que les 1 l'emportent en toute dimension mais notre preuve repose de façon essentielle sur la transience des marches aléatoires symétriques en dimension $d \geq 3$.

Théorème 6 *Si* ξ_0 *est invariant par translation,* $\lambda_1 = \lambda_2 > \lambda_c$ *et* $d \geq 3$ *alors* $\xi_t \Rightarrow \nu_1$.

La preuve du théorème 6 repose en partie sur les techniques de dualité introduites à la Section 1. Pour tenter de comprendre les propriétés du processus dual, nous commençons par construire le système de particules dans le cas $\lambda_1 = \lambda_2$. Pour x, $z \in \mathbb{Z}^d$ avec $\|x - z\| \leq R$, désignons par $\{T_n^{x,z} : n \geq 1\}$, $\{U_n^x : n \geq 1\}$ et $\{V_n^x : n \geq 1\}$ les temps d'arrivée de processus de Poisson indépendants d'intensité λ_1, 1 et γ respectivement. Aux temps $T_n^{x,z}$, nous traçons une flèche de x à z pour indiquer que si x est occupé par un 1 et z vide alors z devient occupé par un 1, tandis que si x est occupé par un 2 et z libre alors z devient occupé par un 2. Enfin, nous plaçons une $\times$ en (x, U_n^x) pour indiquer qu'une mort survient, et un $\bullet$ en (x, V_n^x) pour signaler qu'un site gelé devient libre.

Nous dirons qu'il existe un *chemin* de (z, s) à (x, t), $0 \leq s \leq t$, si (z, s) et (x, t) sont fortement connectés (voir 1 et 2 page 2), et qu'il existe un *chemin dual* de (x, t) à $(z, t - s)$ s'il existe un chemin de $(z, t - s)$ à (x, t). Le *processus dual issu de* (x, t) est alors défini comme pour le processus de contact multitype classique, c'est-à-dire en posant

$$\hat{\xi}_s^{(x,t)} \;=\; \{z \in \mathbb{Z}^d : \text{il existe un chemin dual de } (x,t) \text{ à } (z, t-s)\}.$$

Comme précédemment, l'ensemble $\{(\hat{\xi}_s^{(x,t)}, s) : 0 \leq s \leq t\}$ exhibe une *structure d'arbre* induisant une *hiérarchie*, les ancêtres pouvant être rangés selon l'ordre dans lequel ils déterminent la couleur de (x, t). Les éléments de $\hat{\xi}_s^{(x,t)}$ sont appelés les *sur-ancêtres*, et le premier sur-ancêtre au sens de la relation d'ordre définie ci-dessus la *particule distinguée.*

La principale différence avec le processus de contact multitype est que les 1 produisent maintenant des 3 qui bloquent les 2. En particulier, la couleur de (x, t) ne dépend plus seulement de la couleur des sur-ancêtres au temps 0. L'idée est de prouver que le nombre de sites gelés visités par la particule distinguée lors de son ascension vers (x, t) tend vers l'infini quand $t \to \infty$, ce qui a pour effet de bloquer les 2 pour la détermination de la couleur de (x, t). Plus précisément, nous nous inspirons de la construction introduite à la Section 1. Nous dirons que $z \in \mathbb{Z}^d$ est un *sous-ancêtre* au temps s si (z, s) et (x, t) sont faiblement connectés. Au vu de l'effet des $\times$, si les points (z, s) et (x, t) sont faiblement connectés, une particule de type 1 en (z, s) pourra *geler* la trajectoire de la particule distinguée en certains points. L'objectif est de construire une collection de sous-ancêtres $\zeta_s(n)$ qui soient de bons candidats pour réaliser cet évènement. Si tous ces ancêtres fusionnent, ce qui se réalise presque sûrement en $d \leq 2$, et atterrissent au temps 0 sur un 2, nous ne pouvons conclure. Si, au contraire, $d \geq 3$, ce qu'il faut de transience nous permet de construire une sous-suite infinie $\zeta_s(n_k)$ de sous-ancêtres atterrissant sur un 1 au temps 0, et réalisant notre bon évènement.

En supposant enfin que $\lambda_1 < \lambda_2$, les résultats de Durrett et Neuhauser (1997), Section 3, impliquent que les 2 l'emportent en $d = 2$ pourvu que $\gamma = \infty$. En se basant sur un argument de perturbation (voir Section 5), nous pouvons montrer que le résultat est également satisfait pour γ suffisamment grand. Plus précisément,

Théorème 7 *Supposons que $d = 2$ et $\lambda_2 > \lambda_1 > \lambda_c$. Il existe une valeur critique $\gamma_c \in (0, \infty)$ telle que $\xi_t \Rightarrow \mu_2$ pour tout $\gamma > \gamma_c$.*

Pour synthétiser nos résultats, posons $\beta_c\,(\gamma, \lambda_1) = \inf\{\lambda_2 \geq 0 \ : \ \text{les 1 meurent}\}$ pour des paramètres $\gamma > 0$ et $\lambda_1 > \lambda_c$ donnés, avec la convention habituelle $\inf \varnothing = \infty$. Une application directe des théorèmes 4-7 implique que $\beta_c\,(\gamma, \lambda_1) \downarrow \lambda_1$ quand $\gamma \uparrow \infty$. En particulier, notre processus avec blocage dynamique exhibe un comportement proche de celui du processus de contact multitype quand γ est grand, ce qui peut être vu comme un résultat de continuité.

3. Interactions hôtes-pathogènes et hôtes-mutualistes

Notre troisième processus est destiné à modéliser les interactions locales au sein d'une population d'hôtes et de symbiotes dans le but de comprendre le rôle des symbiotes dans la diversité et la structure des communautés de plantes. De façon générale, un symbiote est un organisme vivant en association avec un hôte. Il peut avoir un effet néfaste, auquel cas nous l'appellerons *pathogène*, ou un effet bénéfique, auquel cas nous l'appellerons *mutualiste*. Une composante essentielle des modèles multi-hôtes, multi-symbiotes est le degré avec lequel différents hôtes et symbiotes sont capables de s'associer entre-eux, ce que l'on nommera *spécificité*. Un symbiote *spécialiste* ne s'associe qu'à un petit nombre d'hôtes tandis qu'un symbiote *généraliste* s'associe à un grand nombre d'hôtes. La facilité avec laquelle un symbiote s'associe à un hôte, appelée *transmissibilité*, est un autre facteur important des interactions hôtes-symbiotes.

Pour décrire la dynamique des hôtes, nous employons le plus simple des modèles interspécifiques, à savoir le modèle des votants (Holley et Liggett, 1975, Clifford et Sudbury, 1973). Au sein de cette population, nous introduisons des symbiotes avec différents degrés de spécificité et de transmissibilité. Plus précisément, notre modèle est un processus de Markov dont l'état au temps t est une fonction $\xi_t : \mathbb{Z}^d \longrightarrow \{1, 2, \ldots, \kappa\} \times \{0, 1, \ldots, \kappa\}$, l'entier κ désignant à la fois le nombre d'hôtes et de symbiotes en interaction. Un site $x \in \mathbb{Z}^d$ est dit occupé par un hôte *non associé* de type i si $\xi(x) = (i, 0)$, et par un hôte de type i *associé* à un symbiote de type j si $\xi(x) = (i, j)$. Dans la suite, nous noterons $\xi_t(x) = (\xi_t^1(x), \, \xi_t^2(x))$, où $\xi_t^1(x)$ et $\xi_t^2(x)$ désignent respectivement les types de l'hôte et du symbiote présents au site x au temps t, la condition $\xi_t^2(x) = 0$ traduisant l'absence de symbiote. L'évolution au site x est décrite par les taux de transition

$$\begin{aligned} c_{(i,j)(k,0)}\,(x, \xi) \quad &= \quad \lambda \sum_{0 < \|x - z\| \leq R_1} \left\{ \ \mathbb{1}_{\{\xi(z) = (k,0)\}} \ + \ g \ \sum_{\ell = 1}^{\kappa} \ \mathbb{1}_{\{\xi(z) = (k,\ell)\}} \ \right\} \\ c_{(i,0)(i,j)}\,(x, \xi) \quad &= \quad c_{ij} \sum_{0 < \|x - z\| \leq R_2} \ \sum_{\ell = 1}^{\kappa} \ \mathbb{1}_{\{\xi(z) = (\ell,j)\}}. \end{aligned}$$

Le taux de naissance des hôtes non associés est égal à λ. Le paramètre g indique la variation du taux de naissance des hôtes associés à un symbiote. En particulier, si $0 \leq g < 1$, le symbiote est un pathogène, si $g = 1$, le symbiote n'a aucun effet sur le taux de naissance des hôtes, et si $g > 1$, le symbiote est un mutualiste. Le paramètre c_{ij} désigne le taux auquel le symbiote j s'associe à l'hôte i, permettant ainsi de modéliser le degré de spécificité des symbiotes. Enfin, les naissances et infections surviennent à l'intérieur d'un voisinage d'interaction, R_1 désignant la portée des naissances pour les hôtes, et R_2 la portée des infections pour les symbiotes. Le cardinal de chacun de ces voisinages sera noté ν_{R_1} et ν_{R_2} respectivement.

Avant de décrire le comportement du modèle spatial, nous examinons le modèle du champ moyen (Durrett et Levin, 1994). En d'autres termes, nous supposons que tous les sites sont indépendants et que le système est homogène. Il en résulte un système d'équations différentielles ordinaires pour les densités d'hôtes associés et non associés. Notons u_i la densité d'hôtes non associés de type i, et v_{ij} la densité d'hôtes de type i associés à un symbiote de type j. Nous supposons en outre que $c_{ii} = \beta$ et $c_{ij} = \alpha$ pour $i \neq j$, avec $0 \leq \alpha \leq \beta$. Posons

$$u_\cdot \ = \ \sum_{i=1}^{\kappa} u_i, \qquad v_{\cdot j} \ = \ \sum_{i=1}^{\kappa} v_{ij}, \qquad v_{i\cdot} \ = \ \sum_{j=1}^{\kappa} v_{ij} \qquad \text{et} \qquad v_{\cdot\cdot} \ = \ \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} v_{ij}.$$

Une façon d'obtenir le modèle du champ moyen est de fixer les portées R_1 et R_2 égales à R et de faire tendre R vers l'infini. Pour obtenir une limite cohérente, nous renormalisons les paramètres λ, α et β par la taille de leur voisinage respectif ν_R en posant $\lambda = \frac{1}{\nu_R}$ (ce qui fixe également l'échelle temporelle),

$$\alpha \ = \ \frac{a}{\nu_R} \qquad \text{et} \qquad \beta \ = \ \frac{b}{\nu_R}.$$

Quand R tend vers l'infini, les sites deviennent indépendants. Si de plus la configuration initiale est invariante par translation, l'évolution des densités pour $i \neq j$ est alors décrite par le système d'équations suivant.

$$\begin{aligned} u_i' \ &= \ (1 - u_i)\,(u_i + g\,v_{i\cdot}) \ - \ u_i \sum_{j \neq i} (u_j + g\,v_{j\cdot}) \ - \ b\,u_i\,v_{\cdot i} \ - \ a \sum_{j \neq i} u_i\,v_{\cdot j} \\ v_{ii}' \ &= \ b\,u_i\,v_{\cdot i} \ - \ v_{ii}\,(u_\cdot + g\,v_{\cdot\cdot}) \\ v_{ij}' \ &= \ a\,u_i\,v_{\cdot j} \ - \ v_{ij}\,(u_\cdot + g\,v_{\cdot\cdot}). \end{aligned}$$

Quand $a = 0$, les symbiotes sont spécialistes. Quand a croît vers b, les symbiotes deviennent généralistes. Si $g = 1$, le système admet des quantités conservatives, à savoir les densités initiales d'hôtes $h_i = u_i + v_{i\cdot}$, $i = 1, 2, \ldots, \kappa$. Si $(\kappa - 1)\,a + b > \kappa$, alors pour $g \neq 1$, il existe un équilibre non trivial avec $u_1 = u_2 = \cdots = u_\kappa \geq 0$ et $v_{1\cdot} = v_{2\cdot} = \cdots = v_{\kappa\cdot} > 0$. Les simulations indiquent que cet équilibre est localement stable pour $a < b$ quand $g < 1$, mais pas quand $g > 1$. Si initialement seuls deux hôtes et un symbiote, disons le symbiote 1, sont présents alors, pour $0 < g < 1$, le pathogène 1 disparaît et les deux hôtes peuvent coexister. Nous conjecturons un comportement similaire pour le modèle spatial, excepté dans le cas d'une dynamique unidimensionnelle à plus proches voisins (voir le théorème 11 et la discussion qui suit le théorème). Quand $g > 1$, l'hôte 2 disparaît pourvu que la densité initiale de mutualistes de type 1 soit suffisamment grande. Le résultat analogue pour le modèle spatial est donné par le théorème 10.

Revenons maintenant au modèle spatial de paramètres λ, α et β. Pour fixer l'échelle temporelle, posons $\lambda = 1$. Nous discutons les cas généraliste $\alpha = \beta$ et spécialiste $\alpha = 0$, et donnons des comparaisons avec le modèle du champ moyen.

Interactions généralisées

Nous considérons le cas généraliste $\alpha = \beta$ pour le modèle spatial. Pour le modèle du champ moyen correspondant, la coexistence des hôtes et symbiotes est possible quand $b > 1$. En revanche, pour $b \leq 1$, les symbiotes sont incapables de survivre. Il est facile de voir que pour $b > 1$ l'équilibre non trivial $(u_\cdot, v_{\cdot\cdot})$ est localement stable et le point frontière $v_{\cdot\cdot} = 0$ instable. Le comportement du modèle spatial est plus complexe et peut dépendre, comme nous le verrons par la suite, de la dimension de l'espace.

Si $g = 1$, les symbiotes n'ont aucun effet sur les hôtes, de sorte que ξ_t^1 et ξ_t^2 sont stochastiquement indépendants. Par ailleurs, il est facile de voir que ξ_t^1 est un modèle des votants, et ξ_t^2 un processus de contact multitype de taux de naissance $\beta\,\nu_{R_2}$ et de taux de mort ν_{R_1}. Voir respectivement Holley et Liggett (1975) et Neuhauser (1992) pour une étude de ces deux processus. Il en résulte l'existence d'une valeur critique $\beta_c(R_1, R_2) \in (0, \infty)$ dépendant de ν_{R_1} et ν_{R_2} telle que les symbiotes peuvent survivre si et seulement si $\beta > \beta_c(R_1, R_2)$. Si l'on ignore le type des hôtes et symbiotes en examinant de préférence les interactions entre hôtes associés et non associés, alors pour $\beta > \beta_c(R_1, R_2)$ et quelque soit la dimension, il existe une mesure stationnaire pour laquelle les hôtes associés et non associés coexistent. Si de plus $d \geq 3$, il existe une mesure stationnaire pour laquelle tous les hôtes et symbiotes coexistent.

Pour analyser le cas $g \neq 1$, nous définissons le processus *daltonien* pour lequel un site est dans l'état 0 s'il est occupé par un hôte non associé, et dans l'état 1 sinon. Nous obtenons ainsi un système de particules $\zeta_t : \mathbb{Z}^d \longrightarrow \{0,\, 1\}$ de taux de transition

$$\begin{aligned} c_{01}\,(x,\zeta) \ &= \ \beta \sum_{0<\|x-z\|\leq R_2} \mathbb{1}_{\{\zeta(z)=1\}} \\ c_{10}\,(x,\zeta) \ &= \ \sum_{0<\|x-z\|\leq R_1} \big\{\mathbb{1}_{\{\zeta(z)=0\}} \ + \ g\ \mathbb{1}_{\{\zeta(z)=1\}}\big\}\,. \end{aligned}$$

Quand $g = 1$, le processus est réduit à un processus de contact de taux de naissance $\beta\,\nu_{R_2}$ et de taux de mort ν_{R_1}. Désignons comme précédemment par $\beta_c(R_1, R_2)$ la valeur critique de ce processus de contact. Un argument de couplage standard nous permet de comparer les processus de paramètres $g \neq 1$ et $g = 1$, et d'en déduire que si $g \leq 1$ et $\beta > \beta_c(R_1, R_2)$ alors ζ_t admet une mesure stationnaire non triviale, tandis que si $g \geq 1$ et $\beta \leq \beta_c(R_1, R_2)$ alors $\zeta_t \Rightarrow \delta_0$. Pour couvrir les cas restants, nous introduisons le processus de contact η_t de taux de naissance $\beta\,\nu_{R_2}$ et de taux de mort $g\,\nu_{R_1}$. Le processus η_t admet une mesure stationnaire non triviale si et seulement si $\beta > g\,\beta_c(R_1, R_2)$ ce qui, avec un nouveau couplage, implique que si $g \leq 1$ et $\beta \leq g\,\beta_c(R_1, R_2)$, alors les pathogènes disparaissent, tandis que si $g \geq 1$ et $\beta > g\,\beta_c(R_1, R_2)$, alors ζ_t admet une mesure stationnaire non triviale. Tous ces résultats sont résumés dans le théorème suivant.

Théorème 8 *Posons $\alpha = \beta$ et supposons que $P\,(\zeta_0(x) = 1) > 0$.*

1. *Si $g \leq 1$, alors $\zeta_t \Rightarrow \delta_0$ si $\beta \leq g\,\beta_c$ et il existe un équilibre non trivial si $\beta > \beta_c$.*
2. *Si $g \geq 1$, alors $\zeta_t \Rightarrow \delta_0$ si $\beta \leq \beta_c$ et il existe un équilibre non trivial si $\beta > g\,\beta_c$.*

Examinons maintenant le cas $\beta > 1$ et $g > 0$ proche de 0. Notons tout d'abord que si $g = 0$ et $R_1 = R_2$ le processus ζ_t n'est autre que le modèle des votants biaisé de paramètres β et 1. En particulier, $P\,(\zeta_t(x) = 0) \to 1$ si $\beta < 1$ alors que $P\,(\zeta_t(x) = 1) \to 1$ si $\beta > 1$. De plus, dans le dernier cas, le processus ξ_t exhibe une *fixation* puisque les hôtes associés à un pathogène sont maintenant stériles. En se basant sur un argument de perturbation (voir Section 5), nous pouvons montrer que les pathogènes survivent également si $\beta > 1$ et $g > 0$ est suffisamment proche de 0.

Théorème 9 *Posons $\alpha = \beta$ et $R_1 = R_2$. Si $\beta > 1$ alors il existe une valeur critique $g_c > 0$ telle que si $g \leq g_c$ alors $\zeta_t \Rightarrow \mu$ avec $\mu\,(\zeta(x) = 1) \neq 0$.*

Interactions spécialisées

Dans le cas d'interactions spécialisées $\alpha = 0$ et $\beta > 0$, l'étude est plus délicate, l'évolution de chaque symbiote dépendant fortement de la structure de la population d'hôtes. En d'autres termes, il n'existe pas de système de particules $\zeta_t : \mathbb{Z}^d \longrightarrow \{0,\, 1\}$ décrivant l'évolution globale des symbiotes ne se souciant pas de leur type. Puisque pour tout $i = 1,\, 2,\, \ldots,\, \kappa$ le symbiote i ne peut vivre qu'en association avec les hôtes du même type, il est toutefois facile de montrer, par un argument de couplage, que si $g \leq 1$ et $\beta \leq \max\,(\beta_c\, g, 1)$, les pathogènes disparaissent.

La prochaine étape est d'étendre les résultats du modèle du champ moyen au modèle spatial correspondant. Nous commençons par considérer une population de deux hôtes et un seul symbiote, disons le symbiote 1, et supposons qu'au temps 0 tous les hôtes de type 1 sont associés. Dans le cas limite $\beta = \infty$, la transition $(1, 0) \to (1, 1)$ est instantanée pourvu que $R_1 \leq R_2$ afin d'éviter les problèmes d'hôtes *isolés* non associés ne pouvant être atteint par aucun symbiote. Il en résulte que le processus ξ_t^1 est un modèle des votants biaisé pour lequel les hôtes de type 1 se reproduisent au taux $g\,\nu_{R_1}$ et ceux de type 2 au taux ν_{R_1}. En particulier, si $g < 1$, alors $\xi_t^1 \Rightarrow \delta_2$, tandis que si $g > 1$, alors $\xi_t^1 \Rightarrow \delta_1$. Le théorème 10 nous dit qu'en toute dimension le résultat est également vérifié si $g > 1$ et $\beta \in (0, \infty)$ est suffisamment grand.

Théorème 10 *Fixons $\alpha = 0$, $g > 1$ et $R_1 \leq R_2$. Au temps 0, $\xi_0(x) = (1,1)$ ou $(2,0)$ pour tout site $x \in \mathbb{Z}^d$. Il existe une valeur critique $\beta_c^{Th10} \in (0,\infty)$ telle que si $\beta > \beta_c^{Th10}$ alors $\xi_t^1 \Rightarrow \delta_1$.*

Les arguments de notre preuve n'impliquent toutefois pas le résultat analogue pour $g < 1$. Nous pensons en fait qu'à l'exception du cas particulier d'une dynamique unidimensionnelle à plus proches voisins, $P\,(\xi_t^1(x) = 2) \not\to 1$. Décrivons, avant de justifier notre conjecture, ce cas particulier. Supposons que $\xi_0^1(x) = 1$ pour $x > 0$, et $\xi_0^1(x) = 2$ pour $x \leq 0$, le système étant initialement occupé par une infinité de pathogènes de type 1 mais dépourvu de pathogènes de type 2. Notons $r_t^2 = \sup\{x \in \mathbb{Z} : \xi_t^1(x) = 2\}$ la position au temps t de l'hôte de type 2 le plus à droite. En particulier, $r_0^2 = 0$. Le résultat suivant implique que pour β suffisamment grand, les 2 se propagent vers la droite en éliminant les 1 et leurs pathogènes.

Théorème 11 *Posons $d = 1$ et $R_1 = 1$. Si $\alpha = 0$ et $g < 1$, il existe $\beta_c^{Th11} \in (0,\infty)$ tel que pour tout $\beta > \beta_c^{Th11}$, le processus $r_t^2 \to \infty$ quand $t \to \infty$ avec probabilité 1.*

Nous pensons que le résultat n'est valable que dans le cas d'une dynamique unidimensionnelle à plus proches voisins. Notons tout d'abord qu'à l'exception de ce cas particulier la dynamique produit des hôtes *isolés*, c'est-à-dire des hôtes de type 1 (resp. 2) encerclés d'hôtes de type 2 (resp. 1). Comme suggéré par le théorème 10, quand $g > 1$, un hôte isolé de type 2 est rapidement *avalé* par les hôtes de type 1 avoisinants. Quand en revanche $g < 1$, un hôte 1 situé au coeur d'un regroupement de 2 en expansion ne peut être infecté par aucun pathogène dès lors que ce regroupement excède une certaine taille critique. Dans cette zone en expansion, le processus se comporte alors comme un modèle des votants (non biaisé) de sorte que les 1 peuvent maintenant rivaliser avec les 2, et survivre.

Nous étudions pour finir la coexistence des symbiotes dans le cas neutre $g = 1$. Notons d'abord que, pour $g = 1$, le processus ξ_t^1 est un modèle des votants de paramètre 1. En particulier, il n'existe pas, en dimension $d \leq 2$, de mesure stationnaire μ telle que $\mu\,(\xi^1(x) = i) \neq 0$ pour toute couleur $i \in \{1, 2, \ldots, \kappa\}$ (voir Holley et Liggett, 1975). Puisque, dans le cas d'interactions spécialisées, $\xi_t^2(x) = i$ implique que $\xi_t^1(x) = i$, la coexistence des symbiotes ne peut être possible. Bien qu'en dimension $d \geq 3$, les hôtes coexistent, étant donnée la formation de regroupements, le problème de la coexistence des symbiotes reste délicat. En d'autres termes, le modèle des votants exhibe des regroupements dont le diamètre peut excéder une longueur critique, empêchant ainsi les symbiotes de se propager. Pour contourner cette difficulté, nous introduisons un nouveau processus, noté $\hat{\xi}_t$, pour lequel les symbiotes évoluent comme précédemment et les hôtes réalisent un modèle des votants de seuil θ. Précisément, les taux de transition sont donnés par

$$\begin{aligned} c_{(i,i)(k,0)}\,(x,\xi) \ &= \ \begin{cases} 1 & \text{si} \ \ \operatorname{card}\{z \in \mathbb{Z}^d : 0 < \|x - z\| \leq R_1 \ \text{et} \ \hat{\xi}^1(z) = k\} \ \geq \ \theta \\ 0 & \text{sinon} \end{cases} \\ c_{(i,0)(i,i)}\,(x,\xi) \ &= \ \ \beta \ \times \ \operatorname{card}\{z \in \mathbb{Z}^d : 0 < \|x - z\| \leq R_2 \ \text{et} \ \hat{\xi}^2(z) = i\}. \end{aligned}$$

L'introduction de ce système de particules est motivée par le théorème 1 de Durrett (1992) selon lequel le modèle des votants de seuil θ admet une mesure stationnaire non triviale suffisamment proche d'une mesure produit pour entraîner notre dernier résultat.

Théorème 12 *Fixons $g = 1$ et $\theta < \nu_{R_1}/\kappa$. Si R_1 et R_2 sont suffisamment grands, alors il existe une valeur critique $\beta_c^{Th12} \in (0,\infty)$, dépendant de κ, telle que si $\beta > \beta_c^{Th12}$, alors tous les hôtes et symbiotes coexistent, tandis que si $\beta < \beta_c^{Th12}$, la coexistence des symbiotes est impossible.*

Comparaison du modèle spatial et du modèle du champ moyen

Les simulations numériques du modèle du champ moyen indiquent que la coexistence des symbiotes n'est possible que pour $g < 1$. Les simulations du modèle spatial révèlent un comportement similaire. Quand $g < 1$ et $\alpha < \beta$, tous les hôtes et symbiotes coexistent. Dans ce cas, nous observons que la taille des regroupements est limitée par la présence de pathogènes. Les regroupements d'hôtes non associés s'étendent jusqu'à la frontière de regroupements d'hôtes associés du même type. Après invasion du symbiote privilégié (celui de taux d'infection β), les regroupements rétrécissent de nouveau. Le cas $g > 1$ et $\alpha < \beta$ est relativement différent. Les regroupements d'hôtes associés à leur mutualiste privilégié se forment et continuent de s'étendre, de même que pour le modèle des votants. Les mutualistes non privilégiés (ceux de taux d'infection α), semblent incapables de résister face au mutualiste privilégié, de même que pour le modèle du champ moyen avec un hôte et deux symbiotes de taux d'infection a et b respectivement ($a < b$). En résumé, les pathogènes ont la capacité de modifier la structure spatiale de leurs hôtes en favorisant la diversité locale, tandis que les mutualistes ne modifient pas cette structure. Cette différence est par ailleurs d'autant plus prononcée avec la spécificité des symbiotes.

4. Modèle des votants et modèle des votants biaisé en milieu hétérogène

Notre dernier processus stochastique est destiné à modéliser le flux de gènes d'une culture transgénique vers une population naturelle par pollinisation. Comme les plantes cultivées sont les descendantes de plantes naturelles, l'existence d'un flux de gènes des unes aux autres n'est pas surprenante. De l'insertion de nouveaux gènes dans le génôme des plantes cultivées, induisant l'expression de protéines ayant des fonctions biologiques spécifiques, naît la préoccupation que ces nouveaux gènes s'échappent dans la nature et confèrent un accroissement de la fitness de certaines espèces, et que les plantes naturelles deviennent des envahisseurs aggressifs. Pour comprendre comment la configuration spatiale de cultures permanentes d'organismes génétiquement modifiés affecte le flux de gènes et l'invasion de transgènes au sein des populations naturelles adjacentes, nous introduisons un modèle stochastique spatial en milieu hétérogène. Le milieu représentera les cultures permanentes d'organismes génétiquement modifiés inscrites dans une matrice de plantes naturelles. Ces cultures étant fixées, le flux de gènes aura lieu uniquement des plantes transgéniques vers les plantes naturelles.

Notre modèle s'inspire du modèle de génétique des populations le plus simple, le modèle haploïde de Wright-Fisher avec sélection. La version spatiale de ce modèle est le modèle des votants biaisé. Nous considérons le modèle des votants en milieu hétérogène avec la présence d'un flux de gènes récurrent depuis les cultures transgéniques vers le milieu naturel. Plus précisément, notre modèle est un processus de Markov $\xi_t : \mathbb{Z}^d \longrightarrow \{0,1\}$, où les 0 représentent les individus porteurs d'un gène naturel, et les 1 les individus porteurs d'un transgène. Un site $x \in \mathbb{Z}^d$ est dit occupé par un 0 (resp. 1) si $\xi(x) = 1$ (resp. 0). Nous utiliserons également la notation $x \in \xi_t$ si et seulement si $\xi_t(x) = 1$. Le sous-ensemble $\Delta \subset \mathbb{Z}^d$ représente la partie de l'habitat occupé par les plantes transgéniques. Nous gelons l'évolution sur Δ en prétendant que si $x \in \Delta$ alors, à tout instant $t \geq 0$, $\xi_t(x) = 1$. Pour décrire la dynamique, nous introduisons une probabilité de transition $p\,(x,z)$ sur $\mathbb{Z}^d$ qui soit invariante par translation, i.e. $p\,(x,z) = q\,(z-x)$, symétrique, et telle que $q\,(z)$ ait une variance $\sigma^2 < \infty$. Un site $x \in \mathbb{Z}^d - \Delta$ occupé par un 0 choisit, avec un taux β et une probabilité $p\,(x,z)$, un site $z \in \mathbb{Z}^d$ dont il prend la couleur, alors que si le site x est occupé par un 1, le choix se fait au taux 1. En d'autres termes,

$$c_{01}\,(x,\xi) \;=\; \beta \sum_z p\,(x,z)\; \mathbb{1}_{\{\xi(z)=1\}} \qquad \text{et} \qquad c_{10}\,(x,\xi) \;=\; \sum_z p\,(x,z)\; \mathbb{1}_{\{\xi(z)=0,\ x\notin\Delta\}}.$$

Excepté dans le cas $\Delta = \varnothing$, nous supposons désormais que $\xi_0(x) = 1$ si et seulement si $x \in \Delta$. En particulier, il résulte de la dynamique que $\xi_t(x) = 1$ pour tous $t \geq 0$ et $x \in \Delta$.

Notons tout d'abord que si $\Delta = \varnothing$ et $\beta = 1$ alors ξ_t est le *modèle des votants*. Dans ce cas, les résultats de Holley et Liggett (1975) révèlent un changement de comportement contrasté selon la dimension. Si $d \leq 2$ alors $\xi_t \Rightarrow \alpha\,\delta_0 + (1-\alpha)\,\delta_1$, pour un $\alpha \in [0,1]$, où δ_i désigne la masse de Dirac en $\xi \equiv i$, tandis que si $d \geq 3$, la coexistence est possible, i.e. il existe une mesure stationnaire μ telle que $\mu\,(\xi(x) = 0) \neq 0$ et $\mu\,(\xi(x) = 1) \neq 0$. Choisir $\Delta \neq \varnothing$ peut modifier drastiquement le comportement limite du processus. En prenant par exemple $\Delta = \{0\}$ et $d \leq 2$, la particule située à l'origine produira un amas de particules qui envahira l'espace dans sa totalité.

Si l'on pose $\Delta = \varnothing$ et $\beta > 1$ alors ξ_t est le *modèle des votants biaisé*. Les résultats de Bramson et Griffeath (1980, 1981) montrent que si $\Omega_\infty = \{\xi_t \neq \varnothing \text{ pour tout } t \geq 0\}$ et qu'à $t = 0$ une seule particule est présente à l'origine, il existe un convexe A tel que sur Ω_∞ et pour tout $\varepsilon > 0$

$$(1-\varepsilon)\, t\, A \ \cap\ \mathbb{Z}^d \ \subset\ \xi_t \ \subset\ (1+\varepsilon)\, t\, A \ \cap\ \mathbb{Z}^d \quad \text{pour tout } t \text{ suffisamment grand.}$$

De plus, sur Ω_∞^c, le processus converge exponentiellement vite vers δ_0.

Fixons maintenant $d = 1$. La discussion ci-dessus implique que si $\Delta \neq \varnothing$ alors $\xi_t \Rightarrow \delta_1$. La première question à laquelle nous souhaiterions répondre est la suivante. Quel est l'effet de la géométrie de Δ sur la vitesse de convergence vers δ_1 ? La première étape est d'étudier le processus de configuration initiale $\xi_0(x) = \mathbb{1}_{\{x \leq 0\}}$. Dans le cas particulier $\Delta = \varnothing$, le comportement du processus à l'interface a été étudié par Cox et Durrett (1995). Pour formuler leur résultat, nous introduisons le *zéro le plus à gauche* et le *un le plus à droite*

$$\ell_t \ = \ \inf\{x \in \mathbb{Z} : \xi_t(x) = 0\} \qquad \text{et} \qquad r_t \ = \ \sup\{x \in \mathbb{Z} : \xi_t(x) = 1\}.$$

Pour des raisons techniques, nous supposons de plus que p est irréductible et que q admet des moments d'ordre 3. Alors $\{r_t - \ell_t : t \geq 0\}$ est stochastiquement compact, i.e. pour tout $\varepsilon_0 > 0$ il existe $M < \infty$ tel que $P\,(r_t - \ell_t > M) \leq \varepsilon_0$. Voir Cox et Durrett (1995), Section 4. En notant par ailleurs $\Phi(x)$ la densité de la loi normale centrée réduite,

$$\lim_{t\to\infty} P\,(r_t/\sigma\sqrt{t} \leq x) \ = \ \lim_{t\to\infty} P\,(\ell_t/\sigma\sqrt{t} \leq x) \ = \ \Phi(x).$$

Dans le cas $\Delta = \mathbb{Z}^-$, l'ensemble des entiers négatifs ou nuls, nous pouvons de nouveau montrer la compacité stochastique de $\{r_t - \ell_t : t \geq 0\}$. L'équation précédente, en revanche, n'est plus satisfaite puisque les processus r_t et ℓ_t sont maintenant condamnés à vivre à droite de zéro. Plus précisément, nous avons le

Théorème 13 *Fixons $\beta = 1$. Si $x \geq 0$ et $x_t = x\,\sigma\sqrt{t}$ alors*

$$\lim_{t\to\infty} P\,(r_t \geq x_t) \ = \ \lim_{t\to\infty} P\,(\ell_t \geq x_t) \ = \ \sqrt{\frac{2}{\pi}} \int_x^\infty \exp\left[-\frac{y^2}{2}\right] dy.$$

En d'autres termes, les particules diffusent vers la droite, révélant une invasion relativement lente. Au vu du théorème 13, notre intuition est que la vitesse d'invasion du transgène est d'autant plus grande que Δ est *fragmenté*. Pour préciser notre argument, considérons le processus sur le tore $\varepsilon\mathbb{Z}/N\mathbb{Z}$ où N est un entier positif et ε un réel proche de 0. Nous accélérons le temps d'un facteur ε^{-2} et introduisons la probabilité de transition $p_\varepsilon\,(x, y) = q\,(\varepsilon^{-1}(y - x) \mod N\varepsilon^{-1})$ pour formuler la dynamique. Fixons $L > 0$ et $K = N/L$ de sorte que K et $L\,\varepsilon^{-1}$ soient des entiers, avec K pair. Pour tout site $z \in \{0,\, 1,\, \ldots,\, K-1\}$, considérons enfin les sous-ensembles

$$B_z \ = \ [\,z\,L,\, (z+1)\,L) \ \cap\ \varepsilon\mathbb{Z} \qquad \text{et} \qquad \Delta \ = \ \bigcup_{z \text{ pair}} B_z.$$

Le lecteur notera que la condition $\xi_0(x) = 1$ si et seulement si $x \in \Delta$ implique que, pour tout choix de L, exactement la moitié des sites est initialement occupée. Désignons par T_{inv} le temps d'arrêt $T_{\text{inv}} = \inf\{t \geq 0 : \xi_t \equiv 1\}$, quantité que l'on nommera *temps de complète invasion.*

Théorème 14 *Fixons $\beta = 1$. Si $p\,(x, y)$ est à support compact alors*

$$\lim_{\varepsilon\to 0} P\,(T_{\text{inv}} \leq t) \ = \ \left\{1 - \int_0^L (\Psi_t * \Psi_t)(x)\, dx\right\}^{N/2L}$$

où

$$\Psi_t(x) \ = \ -\frac{\partial}{\partial x} \sqrt{\frac{2}{\pi}} \int_{x/\sigma\sqrt{t}}^\infty \exp\left[-\frac{y^2}{2}\right] dy.$$

Le théorème 14 nous donne un indice des effets de Δ sur le temps de complète invasion. Les études numériques révèlent que, pour $t \geq 0$ fixé, la fonction $L \mapsto \lim_{\varepsilon \to 0} P\,(T_{\mathrm{inv}} \geq t)$ est croissante. Il en résulte que le temps de complète invasion sera très lent dans le cas extrême $L = N/2$, et de plus en plus rapide au fur et à mesure que le nombre de composantes de Δ augmente.

Nous considérons maintenant le processus sur $\mathbb{Z}$ où les particules donnent naissance selon la probabilité de transition $p\,(x, y)$. Posons $\Delta = \mathbb{Z}^-$ et $\xi_0(x) = 1$ si et seulement si $x \in \Delta$. Supposons de plus que $p\,(x, y)$ est à support compact. La prochaine étape est d'étudier les propriétés du un le plus à droite r_t. Le comportement de r_t quand $\beta = 1$ et $t \to \infty$ découle du théorème 13. En travaillant un peu plus, nous pouvons montrer que le processus revient infiniment souvent à sa configuration initiale.

Théorème 15 *Si* $\beta = 1$ *alors* $P\,(r_t = 0 \text{ i.s.}) = 1$.

Pour étudier le cas $\beta \neq 1$, considérons une dynamique à plus proches voisins, i.e. $p\,(x, y) = 1/2$ si et seulement si $|x - y| = 1$. Si $\beta < 1$, le théorème 15 assorti d'un argument de couplage implique que $r_t = 0$ i.s.. Le processus n'exhibe toutefois pas le comportement décrit par le théorème 13.

Théorème 16 *Si* $\beta < 1$ *alors* $P\,(r_t \geq x) \leq \beta^x$ *à tout instant* $t \geq 0$.

Enfin, si $\beta > 1$, le processus se comporte comme le modèle correspond en milieu homogène, i.e. le processus croît linéairement en temps avec une vitesse de l'ordre de $\beta - 1$.

Théorème 17 *Si* $\beta > 1$, *il existe* $C < \infty$ *et* $\gamma > 0$ *tels que pour tout* $t > 0$ *et tout* $\varepsilon > 0$

$$P\,(|\,r_t - (\beta - 1)\,t\,| \ \geq \ \varepsilon\,t) \ \leq \ C\,e^{-\gamma\varepsilon^2 t}.$$

En conclusion, les théorèmes 13 et 16-17 impliquent que si $\beta \geq 1$ alors, pour tout entier $x \geq 0$ et tout $\varepsilon > 0$, il existe $t_0 \geq 0$ tel que $P\,(r_t \geq x) \geq 1 - \varepsilon$ pour tout $t \geq t_0$, tandis que si $\beta < 1$ alors $P\,(r_t \geq x) \leq \beta^x$ pour tout $t \geq 0$. Les théorèmes 15-17 nous donnent par ailleurs la loi du 0-1

$$P\,(r_t = 0 \ \text{ i.s.}) \ = \ \begin{cases} 0 & \text{si} \quad \beta > 1 \\ 1 & \text{si} \quad \beta \leq 1. \end{cases}$$

En particulier, si $\beta = 1$ et $p\,(x, y)$ est à support compact, le un le plus à droite r_t converge en probabilité vers l'infini quand $t \to \infty$ mais pas presque sûrement.

En conclusion, nos résultats confirment la nécessité d'un contrôle des flux récurrents de gènes depuis les cultures génétiquement modifiées vers le milieu naturel. La sélection et la structure spatiale jouent toutes deux un rôle essentiel dans la vitesse et l'étendue de l'invasion par le transgène. Pour prendre totalement possession de la population naturelle, le transgène a besoin d'un avantage sélectif. Néanmoins, même si le transgène n'a pas d'avantage selectif, il continuera d'être présent dans le mileu naturel, la superficie qu'il pourra envahir dépendant fortement de la portée de dispersion. En considérant que le pollen peut se disperser à travers de longues distances, la pollution génétique des populations naturelles reste donc une préoccupation sérieuse.

5. Résultat de continuité pour les systèmes de particules multicolores

Comme suggéré aux Sections 1-3, étant donné un système de particules $\xi_t : \mathbb{Z}^d \longrightarrow F$, une méthode classique pour prouver l'existence de mesures stationnaires non triviales pour le processus stochastique ξ_t consiste à appliquer un *argument de renormalisation*. L'idée de base est de comparer une certaine collection de bons évènements pour le processus vu sous une échelle spatio-temporelle adéquate à un modèle de percolation orientée. Pour plus de détails sur la percolation orientée, nous renvoyons le lecteur à Durrett (1984). L'argument de renormalisation a été inventé par Bramson et Durrett (1988), et revu par Durrett (1991, 1995).

Pour formuler notre *résultat de continuité*, nous considérons un ouvert Λ et supposons que notre processus dépend d'un paramètre $\lambda \in \Lambda$, i.e. la couleur de $x \in \mathbb{Z}^d$ passe de i à j au taux

$$c_{ij}\,(\lambda, x, \xi) \;=\; h_{ij}\,(\lambda,\, \xi(x+z_1),\, \xi(x+z_2),\, \ldots,\, \xi(x+z_N)).$$

Nous faisons de plus l'hypothèse de continuité

(HC) Pour tout $\delta > 0$, il existe un voisinage $\mathcal{V}$ de λ_0 dans Λ tel que

$$|c_{ij}\,(\lambda, 0, \xi) \,-\, c_{ij}\,(\lambda_0, 0, \xi)| \;<\; \delta \qquad \forall\, \lambda \in \mathcal{V}, \quad \forall\, i,\, j \in F, \quad \forall\, \xi \in F^{\mathbb{Z}^d}.$$

Notre dernier théorème nous dit que, si l'on parvient à montrer, grâce à l'argument de renormalisation, l'existence d'une mesure stationnaire pour le processus de paramètre $\lambda_0 \in \Lambda$, alors il existe un voisinage $\mathcal{V}$ de λ_0 dans Λ tel que les processus de paramètre $\lambda \in \mathcal{V}$ continuent d'exhiber, en un sens qui sera précisé plus loin, le même comportement. Le résultat a déjà été établi dans de nombreux cas particuliers. Voir, e.g., Durrett et Neuhauser (1997), Schinazi (2001, 2002) ou les théorèmes 2 et 9 ci-dessus.

Pour faire la connnexion entre systèmes de particules et percolation orientée, l'idée consiste à transformer l'espace-temps du processus en *mur en briques*, chaque brique étant associée à un bon évènement. Notons $\bar{\xi}_t : \mathbb{Z}^d \longrightarrow F$ le système de particules de paramètre $\lambda_0 \in \Lambda$ construit à partir d'une collection de processus de Poisson indépendants (Harris, 1972), et posons

$$\mathcal{L} \;=\; \{(z,n) \in \mathbb{Z}^2 : z+n \text{ est pair et } n \geq 0\}.$$

Soient L, T et Γ des entiers tels que $T = \Gamma L$, et $\mathcal{H}$ une collection de configurations déterminées par la valeur de $\bar{\xi}$ sur $[-L, L\,]^d$. Notons e_1 le premier vecteur unité de $\mathbb{Z}^d$, et pour $z \in \mathbb{Z}$, désignons par $\mathcal{H}_z$ la collection $\mathcal{H}$ translatée du vecteur Lze_1. Nous dirons alors qu'un site $(z,n) \in \mathcal{L}$ est *occupé* si $\xi_{nT} \in \mathcal{H}_z$. Fixons enfin $k_0,\, j_0 \in \mathbb{N}^*$ et $M = \max\{j_0, k_0\}$, et considérons la boîte

$$B_{z,n} \;=\; (Lze_1,\, nT) \;+\; \{[-k_0\,L, k_0\,L\,]^d \,\times\, [\,0,\, j_0\,T\,]\}.$$

Chaque $(z,n) \in \mathcal{L}$ est associé à un bon évènement $E_{z,n}$ mesurable par rapport à la représentation graphique du processus restreinte à $B_{z,n}$. Nous supposons enfin que pour tout $\varepsilon > 0$ l'hypothèse de comparaison est satisfaite pour le processus de paramètre λ_0, i.e.

($P_{\lambda_0}^{\varepsilon}$) Les paramètres L et Γ peuvent être choisis suffisamment grands de sorte que

1. $P\,(E_{z,n}) \geq 1 - \varepsilon$ et
2. Si (z,n) est occupé et $E_{z,n}$ se réalise, alors $(z-1, n+1)$ et $(z+1, n+1)$ sont occupés.

Ici, (z,n) occupé signifie que $\bar{\xi}_{nT} \in \mathcal{H}_z$. En d'autres termes, dans la définition de $P_{\lambda_0}^{\varepsilon}$, *occupé* s'applique au processus de paramètre λ_0. Désignons par P_{λ_0} la propriété que pour tout $\varepsilon > 0$ l'hypothèse de comparaison $P_{\lambda_0}^{\varepsilon}$ est satisfaite. Nous avons alors le théorème suivant.

Théorème 18 *Supposons HC et P_{λ_0}. Pour tout $\varepsilon > 0$, il existe un voisinage $\mathcal{V}_\varepsilon$ de λ_0 dans Λ tel que pour tout $\lambda \in \mathcal{V}_\varepsilon$, l'hypothèse de comparaison $P_{\lambda}^{\varepsilon}$ est satisfaite.*

Comme suggéré plus haut, le théorème 18 implique que si λ est proche de λ_0 alors les processus de paramètres λ et λ_0 exhibent le même comportement au sens où chacun des deux processus admet une mesure stationnaire ν telle que, pour tout $z \in \mathbb{Z}$, $\nu(\xi \in \mathcal{H}_z) > 0$. Au-delà d'un simple résultat de continuité, le théorème 18 nous donne un indice sur la stratégie à suivre pour étudier le comportement d'un système de particules. Plus précisément, nous avons le

Corollaire 19 *Supposons que ξ_t exhibe une transition de phase en $\lambda_0 \in \Lambda$. Alors, le comportement du processus quand $\lambda = \lambda_0$ ne peut être étudié par le biais du théorème de comparaison.*

La preuve du corollaire 19 est directe. Supposons que l'on réussisse à construire une mesure invariante ν pour le processus de paramètre λ_0 par le biais du théorème de comparaison, autrement dit que l'on réussisse à prouver que l'hypothèse P_{λ_0} ci-dessus est satisfaite. Le théorème 18 nous assure alors de l'existence d'un voisinage $\mathcal{V}$ de λ_0 dans Λ tel que, pour tout $\lambda \in \mathcal{V}$, le processus de paramètre λ continue d'exhiber le même comportement, ce qui contredit l'existence d'une transition de phase en λ_0 et prouve le corollaire 19.

Nous pensons enfin que le théorème 18 n'est pas satisfait pour $\lambda_0 = \infty$ et $\lambda > 0$ grand (voir la discussion qui suit le théorème 11 ci-dessus). L'argument de continuité pour $\lambda_0 = \infty$ et λ grand est toutefois vérifié dans certains cas. Voir, e.g., les théorèmes 7 et 10 pour des illustrations.

6. Simulations numériques

Les possesseurs d'une version récente de Linux intéressés par les simulations peuvent se rendre à l'adresse www.univ-rouen.fr/LMRS/Persopage/Lanchier/index.html où la plupart des modèles spatiaux présentés dans cette thèse peuvent être téléchargés.

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Presentation of the results

The processes we will be interested in all along this thesis, referred to as *multicolor particle systems*, are Markov processes $\xi_t : \mathbb{Z}^d \longrightarrow F$ where $\mathbb{Z}^d$ denotes the d-dimensional space with integer coordinates, and F a finite set called *set of colors*. To describe the evolution of our processes, we first introduce an *interaction neighborhood*

$$\mathcal{N} \ = \ \{z_0, \, z_1, \, \ldots, \, z_N\} \ \subset \ \mathbb{Z}^d \qquad \text{with} \quad z_0 \ = \ 0.$$

If the system is in some configuration $\xi \in F^{\mathbb{Z}^d}$, the color of $x \in \mathbb{Z}^d$ flips from i to j at rate

$$c_{ij}\,(x, \xi) \ = \ h_{ij}\,(x, \, \xi(x + z_0), \, \xi(x + z_1), \, \ldots, \, \xi(x + z_N)).$$

In other words, our interaction is *finite range*, i.e., the flip rates at x depend only on the location of site x and of the color of a finite number of its neighbors. In most of our examples, the processes will be assumed to be *translation invariant*, i.e., the rules applied at x are just a translation of those applied at 0, which means that we will drop the dependence on the location of x in the function h_{ij}. To figure out the behavior of a multicolor particle system, the main objective will be to investigate its *stationary measures*, i.e., the measures μ such that $P_\mu(\xi_t \in A) = \mu(A)$ at any time $t \geq 0$ and for any $A \in \mathcal{F}$, where P_μ denotes the law of the process starting from the distribution μ, and $\mathcal{F}$ the usual σ-algebra on the set of functions $\xi : \mathbb{Z}^d \longrightarrow F$.

1. The successional model

The first one of our spatial models, called successional model, is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \longrightarrow \{0, 1, 2\}$. A site $x \in \mathbb{Z}^d$ is said to be empty if $\xi(x) = 0$, and occupied by a particle of type 1 (resp. 2) if $\xi(x) = 1$ (resp. 2). The evolution at site x is described by the transition rates

$$\begin{array}{rclcrcl} c_{01}\,(x, \xi) & = & \lambda_1 \displaystyle\sum_{0 < \|x - z\| \leq R} \mathbb{1}_{\{\xi(z) = 1\}} & \qquad & c_{12}\,(x, \xi) & = & 1 \\[2ex] c_{02}\,(x, \xi) & = & \lambda_2 \displaystyle\sum_{0 < \|x - z\| \leq R} \mathbb{1}_{\{\xi(z) = 2\}} & \qquad & c_{20}\,(x, \xi) & = & 1. \end{array}$$

We called our process successional model to refer to the ecological succession phenomenon it describes the evolution. In our ecological context, 1's are the individuals of a so-called *pioneer* species that can invade a hostile environment. The appearance of this first species then helps the development of a second species, here the particles of type 2. In plant communities, the spontaneous transition 1 → 2 is explained by the presence of a seed bank in the soil.

First of all, if only 2's are present, the system reduces to the contact process with parameter λ_2. In this case, there is a critical value $\lambda_c \in (0, \infty)$ such that the following holds: If $\lambda_2 \leq \lambda_c$ then the process converges in distribution δ_0, the "all 0" configuration. Otherwise, there exists a stationary measure μ_2 that concentrates on configurations with infinitely many 2's (Liggett, 1999). In particular, in the case $\lambda_1 \leq \lambda_c$ and $\lambda_2 \leq \lambda_c$, it is easy to see that there is extinction, i.e., ξ_t converges to δ_0.

To avoid trivialities, we assume from now on that ξ_0 contains infinitely many 1's and 2's. To begin with, if $\lambda_1 \geq \lambda_2 > \lambda_c$ a standard coupling argument (Harris, 1972) involving our model and the contact process implies that 2's survive. If $\lambda_2 > \lambda_1 > \lambda_c$ a new coupling argument together with the results of Durrett and Neuhauser (1997), Section 3, allows to prove that 2's win. In conclusion, we get the following theorem, where ⇒ denotes weak convergence.

Theorem 1 *We set $\lambda_2 > \lambda_c$. If $\lambda_1 \geq \lambda_2$ then 2's survive, i.e. $P\,(\forall\ t \geq 0, \exists\ x : \xi_t(x) = 2) = 1$. If $\lambda_1 < \lambda_2$ then 2's win, i.e. $\xi_t \Rightarrow \mu_2$, the upper invariant measure of the contact process.*

We now focus on the case $\lambda_2 > 0$ close to 0. If we suppress the transition $0 \to 2$ ($\lambda_2 = 0$), the process becomes a forest fire model in which 0 = alive, 1 = on fire, and 2 = burnt. By using a rescaling argument, Durrett and Neuhauser (1991) proved that, in the 2-dimensional nearest neighbor case, there is a critical value $\alpha_c \in (0, \infty)$ such that if $\lambda_1 > \alpha_c$ then there exists a nontrivial stationary measure ν that concentrates on configurations with infinitely many 1's and 2's. Theorem 2 tells us that 1's and 2's still coexist for $\lambda_2 > 0$ sufficiently small, i.e., the process has a stationary distribution $\nu = \nu_{\lambda_1,\lambda_2}$ such as $\nu(\xi(x) = 1) \neq 0$.

Theorem 2 *We set $d = 2$ and $\lambda_1 > \alpha_c$. There exists $\beta_c^{Th2} \in (0, \infty)$ such that if $\lambda_2 \leq \beta_c^{Th2}$ then 1's and 2's coexist, i.e. $\xi_t \Rightarrow \nu$ with $\nu(\xi(x) = 1) \neq 0$.*

We now come to our main result, that is to prove that 2's still win if $\lambda_1 = \lambda_2$. Although the result is not surprising, its proof relies on a somewhat interesting property of the dual process.

Theorem 3 *If $\lambda_1 = \lambda_2 > \lambda_c$ and ξ_0 is translation invariant then $\xi_t \Rightarrow \mu_2$.*

To describe the dual process, we start by constructing the particle system from a collection of independent Poisson processes. For x, $z \in \mathbb{Z}^d$ with $\|x - z\| \leq R$, let $\{T_n^{x,z} : n \geq 1\}$ be the arrival times of independent Poisson processes with rate λ_1. At times $T_n^{x,z}$, we draw an arrow from site x to site z to indicate that, if x is occupied and z is empty, then the particle at x gives birth to a particle of the same type at z. To take into account the transitions $1 \to 2$ and $2 \to 0$, we introduce two further collections of independent Poisson processes $\{U_n^x : n \geq 1\}$ and $\{V_n^x : n \geq 1\}$, $x \in \mathbb{Z}^d$, each of them has parameter 1. We put a $\times$ at (x, U_n^x) to indicate that a particle of type 2 is killed, and a $\bullet$ at (x, V_n^x) to indicate that a particle of type 1 changes its color.

At this point, the reader should see that, since the $\times$'s do not kill both types of particles, the construction of a dual process from the graphical representation we have just introduced will produce a somewhat complicated object. The main ingredient to simplify the dual process is to observe that, since 1's (resp. 2's) do not see the $\times$'s (resp. the $\bullet$'s), we can superimpose the $\times$'s and the $\bullet$'s. In other words, we remove the $\bullet$'s from our graphical representation, and pretend that the $\times$'s effect is now to kill 2's, and paint 1's the color 2.

We will say that (z, s) and (x, t), $0 \leq s \leq t$, are *strongly connected* if there is a sequence of times $s_0 = s < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 = z, x_1, \ldots, x_n = x$ so that

1. For $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0, 1, \ldots, n$, the vertical segment $\{x_i\} \times (s_i, s_{i+1})$ does not contain any $\times$'s.

If instead of the condition 2 above

3. The set $\displaystyle\bigcup_{i=0}^{n} \{x_i\} \times (s_i, s_{i+1})$ contains exactly one $\times$,

we will say that (z, s) and (x, t) are *weakly connected*. Finally, we will say that there is a *path* from (z, s) to (x, t) if both points are either strongly or weakly connected. In view of the $\times$'s effect, if there is a path from (z, s) to (x, t), a particle of type 1 at (z, s) can certainly be the *ancestor* of a particle of type 2 at (x, t) if it goes through one $\times$ on its way up to (x, t). In conclusion, the *dual process starting at* (x, t) will be defined by letting

$$\hat{\xi}_s^{(x,t)} = \{z \in \mathbb{Z}^d : \text{there is a dual path from } (x, t) \text{ to } (z, t - s)\}$$

where a *dual path* from (x, t) to $(z, t-s)$, $0 \leq s \leq t$, indicates the existence of a path from $(z, t-s)$ to (x, t). We now observe that $\{(\hat{\xi}_s^{(x,t)}, s) : 0 \leq s \leq t\}$ exhibits a *tree structure* divided into two stages depending on the number of $\times$ (0 or 1) an ancestor has to cross to reach (x, t). A site $z \in \mathbb{Z}^d$ is called an *upper ancestor* (resp. a *lower ancestor*) at time s if there is a path from (z, s) to (x, t) containing 0 (resp. 1) $\times$. As in the mutitype contact process (Neuhauser, 1992), the tree structure of the dual process allows us to define an ancestor *hierarchy* in which the members are arranged according to the order they determine the color of (x, t). The first ancestor in the hierarchy will be called the *determining particle*, and the first upper ancestor the *distinguished particle*.

With this construction in mind, we can now sketch the proof of Theorem 3. The first step is to prove that $P(\Theta_1 < \infty) = 1$, where Θ_1 is the last time the determining particle is an upper ancestor. The rest of the proof then relies on the observation that, once the determining particle is a lower ancestor (after time Θ_1), it can only paint (x, t) the color 2 since the ×'s effect is to change the color of 1's. In $d \leq 2$, we wait until time Θ_1 and then prove that the distinguished particle and the determining particle coalesce with probability 1 so that they will land, for t sufficiently large, on the same site. If this site is initially occupied by a 1, the determining particle will paint (x, t) the color 2 while if it is occupied by a 2, the distinguished particle will paint (x, t) the color 2 unless a lower ancestor does it earlier. In $d \geq 3$, we rely on the fact that $\Theta_1 < \infty$ to construct a sequence of lower ancestors $\zeta_s(n)$ that come before the distinguished particle in the ancestor hierarchy, and extract a subsequence of candidates $\zeta_s(n_k)$ that do not coalesce together to conclude that at least one lands on a 1 and paints (x, t) the color 2.

2. The multitype contact process with frozen states

Our second spatially explicit stochastic model is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \longrightarrow \{0, 1, 2, 3\}$. As previously, a site $x \in \mathbb{Z}^d$ is said to be occupied by a particle of type 1 (resp. 2) if $\xi(x) = 1$ (resp. 2), and empty otherwise. We now distinguish two types of empty site, namely a site $x \in \mathbb{Z}^d$ will be a *free site* if $\xi(x) = 0$ and will be a *frozen site* if $\xi(x) = 3$. The evolution at site x is described by the transition rates

$$\begin{array}{rclcrcl}
c_{01}(x, \xi) & = & \lambda_1 \displaystyle\sum_{0 < \|x - z\| \leq R} \mathbb{1}_{\{\xi(z) = 1\}} & \qquad & c_{13}(x, \xi) & = & 1 \\[2em]
c_{31}(x, \xi) & = & \lambda_1 \displaystyle\sum_{0 < \|x - z\| \leq R} \mathbb{1}_{\{\xi(z) = 1\}} & \qquad & c_{30}(x, \xi) & = & \gamma \\[2em]
c_{02}(x, \xi) & = & \lambda_2 \displaystyle\sum_{0 < \|x - z\| \leq R} \mathbb{1}_{\{\xi(z) = 2\}} & \qquad & c_{20}(x, \xi) & = & 1.
\end{array}$$

In particular, the process is a generalization of the multitype contact process (Neuhauser, 1992) in which 1's inhibit the spread of 2's by freezing the sites they have just occupied. Reciprocally, the multitype contact process is just the extreme case $\gamma = \infty$. The interpretation we have in mind is that of a spatial model of *allelopathy*. In biology literature, allelopathy is defined as a process involving secondary metabolites produced by plants, micro-organisms, viruses and fungi that influence growth and development of biological systems. In our case, 1's are the individuals of an *inhibitory* species and 2's the individuals of a *susceptible* species.

To investigate our model, we first observe that if only 2's are present, the process reduces as before to the contact process with parameter λ_2. In particular, if $\lambda_2 \leq \lambda_c$ then the process converges in distribution to δ_0 while if $\lambda_2 > \lambda_c$ then $\xi_t \Rightarrow \mu_2$ (see Section 1). If only 1's are present, we have almost the same result. Namely, if $\lambda_1 \leq \lambda_c$ then the process converges in distribution to δ_0 while if $\lambda_1 > \lambda_c$ there exists a nontrivial stationary measure ν_1 that concentrates on configurations with infinitely many 1's and 3's. In particular, in the case $\lambda_1 \leq \lambda_c$ and $\lambda_2 \leq \lambda_c$, there is extinction of both species.

To avoid trivialities, we assume from now on that both parameters λ_1 and λ_2 are $> \lambda_c$ and that ξ_0 contains infinitely many 1's and 2's. First of all, a standard coupling argument implies that the survival probability of 1's is nondecreasing with respect to λ_1 and nonincreasing with respect to λ_2 and γ. The analogous result holds for 2's. In conclusion, we get the following

Theorem 4 *Let $\Theta_t^i = \{x \in \mathbb{Z}^d : \xi_t(x) = i\}$ be the set of sites occupied at time t by a particle of type i. Then the survival probabilities $P(\Theta_t^i \neq \varnothing$ for all $t \geq 0)$, $i = 1, 2$, are monotonous with respect to each of the parameters λ_1, λ_2 and γ.*

In particular, by observing that the model with parameter $\gamma = \infty$ corresponds to the multitype contact process, Theorem 4 above and Theorem 1 in Neuhauser (1992) imply that

Theorem 5 *If* $\lambda_1 > \lambda_2 > \lambda_c$ *and* $\gamma \in (0, \infty)$ *then* $\xi_t \Rightarrow \nu_1$.

We now focus on the case $\lambda_1 = \lambda_2$. Since the evolution rules help 1's, we expect that the processes with or without frozen states exhibit different behaviors. Theorem 6 tells us that 1's still win in $d \geq 3$ if $\gamma < \infty$ while 1's and 2's coexist if $\gamma = \infty$. See Theorem 3 in Neuhauser (1992). We conjecture that 1's win in any dimension but our proof heavily relies on transience of symmetrical random walks in $d \geq 3$.

Theorem 6 *If* ξ_0 *is translation invariant,* $\lambda_1 = \lambda_2 > \lambda_c$ *and* $d \geq 3$ *then* $\xi_t \Rightarrow \nu_1$.

The proof of Theorem 6 partly relies on the duality techniques introduced in Section 1. To figure out the properties of the dual process, we start by constructing the process from collections of Poisson processes in the case $\lambda_1 = \lambda_2$. For x, $z \in \mathbb{Z}^d$, $\|x - z\| \leq R$, let $\{T_n^{x,z} : n \geq 1\}$, $\{U_n^x : n \geq 1\}$ and $\{V_n^x : n \geq 1\}$ be the arrival times of Poisson processes with rates λ_1, 1 and γ respectively. At times $T_n^{x,z}$, we draw an arrow from x to z to indicate that if x is occupied by a 1 and z is empty then z becomes occupied by a 1 while if x is occupied by a 2 and z is free then z becomes occupied by a 2. At times U_n^x, we put a $\times$ at x to indicate that a death occurs, and at times V_n^x, we put a $\bullet$ at x to indicate that a frozen site becomes free.

We now say that there is a *path* from (z, s) to (x, t), $0 \leq s \leq t$, if (z, s) and (x, t) are strongly connected (see 1 and 2 page 16), and that there is a *dual path* from (x, t) to $(z, t - s)$ if there is a path from $(z, t - s)$ to (x, t). Finally, the *dual process starting at* (x, t) will be defined as for the basic multitype contact process, that is by letting

$$\hat{\xi}_s^{(x,t)} = \{z \in \mathbb{Z}^d : \text{there is a dual path from } (x,t) \text{ to } (z, t-s)\}.$$

As previously, the set $\{(\hat{\xi}_s^{(x,t)}, s) : 0 \leq s \leq t\}$ exhibits a *tree structure* that allows to define an ancestor *hierarchy* in which the members are arranged according to the order they determine the color of (x, t). The elements of $\hat{\xi}_s^{(x,t)}$ are called the *upper ancestors*, and the first upper ancestor in the hierarchy the *distinguished particle*.

The main difference with the multitype contact process is that 1's now produce 3's that are forbidden for 2's. In particular, the color of (x, t) will not depend only on the state of the upper ancestors at time 0. The key idea is to prove that the number of frozen sites visited by the distinguished particle on its way up to (x, t) tends to infinity as $t \to \infty$, which blocks 2's from determining the color of (x, t). To do this, we rely on the idea introduced in Section 1. First of all, we call $z \in \mathbb{Z}^d$ a *lower ancestor* at time s if (z, s) and (x, t) are weakly connected. In view of the $\times$'s effect, if the points (z, s) and (x, t) are weakly connected, a particle of type 1 at (z, s) can *freeze* the path of the distinguished particle at some particular point. The aim is to construct a collection of lower ancestors $\zeta_s(n)$ that are good candidates to realize this event. If all these ancestors coalesce, which occurs with probability 1 in $d \leq 2$, and land at time 0 on a 2, we cannot conclude. If, on the contrary, $d \geq 3$, enough transience allows us to construct an infinite subsequence $\zeta_s(n_k)$ of lower ancestors that land on sites occupied at time 0 by a particle of type 1, and realize our good event.

If we now focus on the case $\lambda_1 < \lambda_2$, the results of Durrett and Neuhauser (1997), Section 3, imply that 2's win in $d = 2$ provided that $\gamma = \infty$. By relying on a perturbation argument (see Section 5), we can prove that the result still holds when γ is sufficiently large. More precisely,

Theorem 7 *We assume that* $d = 2$ *and* $\lambda_2 > \lambda_1 > \lambda_c$. *Then, there exists a critical value* $\gamma_c \in (0, \infty)$ *such that* $\xi_t \Rightarrow \mu_2$ *for any* $\gamma > \gamma_c$.

To merge our results together, we now set $\beta_c(\gamma, \lambda_1) = \inf\{\lambda_2 \geq 0 : \text{1's die out}\}$ for given parameters $\gamma > 0$ and $\lambda_1 > \lambda_c$, with the usual convention $\inf \varnothing = \infty$. A fairly straightforward application of Theorems 4-7 then implies that $\beta_c(\gamma, \lambda_1) \downarrow \lambda_1$ as $\gamma \uparrow \infty$. This tells us in particular that our particle system with frozen states behaves nearly like the basic multitype contact process when γ is large, which can be seen as a continuity result.

3. Host-pathogen and host-mutualist interactions

Our third stochastic model is intended to mimic the local interactions within a population of hosts and symbionts, and figure out the role of symbionts on diversity and structure of plant communities. In a general way, a symbiont is an organism that lives in close association with a host. It can either have a harmful effect, in which case we call it a *pathogen*, or a beneficial effect, in which case we call it a *mutualist*. An important component of multi-host, multi-symbiont models is the degree to which different symbionts and hosts can associate with each other. This is referred to as *specificity*. A *specialist* symbiont associates with a very small number of hosts; a *generalist* symbiont associates with many hosts. The ease of transmission of a symbiont to a host, referred to as *transmissibility*, is another important factor in host-symbiont interactions.

To describe the host dynamics, we employ the simplest of all multi-species models, namely the voter model (Holley and Liggett, 1975, Clifford and Sudbury, 1973). Into this population, we introduce symbionts with varying degrees of specificity and transmissibility. More precisely, our model is a continuous-time Markov process $\xi_t : \mathbb{Z}^d \longrightarrow \{1, 2, \ldots, \kappa\} \times \{0, 1, \ldots, \kappa\}$ where the integer κ denotes both the number of hosts and the number of symbionts involved in the interaction. A site $x \in \mathbb{Z}^d$ is said to be occupied by an *unassociated* host of type i if $\xi(x) = (i, 0)$, and by a host of type i *associated* with a symbiont of type j if $\xi(x) = (i, j)$. Later on, we will write $\xi_t(x) = (\xi_t^1(x), \xi_t^2(x))$, where $\xi_t^1(x)$ (resp. $\xi_t^2(x)$) denotes the type of the host (resp. the symbiont) present at x at time t, with $\xi_t^2(x) = 0$ denoting the absence of a symbiont. The evolution at site x is described by the transition rates

$$\begin{aligned} c_{(i,j)(k,0)}(x, \xi) \ &= \ \lambda \sum_{0 < \|x - z\| \leq R_1} \left\{ \mathbb{1}_{\{\xi(z) = (k,0)\}} \ + \ g \sum_{\ell = 1}^{\kappa} \mathbb{1}_{\{\xi(z) = (k,\ell)\}} \right\} \\ c_{(i,0)(i,j)}(x, \xi) \ &= \ c_{ij} \sum_{0 < \|x - z\| \leq R_2} \sum_{\ell = 1}^{\kappa} \mathbb{1}_{\{\xi(z) = (\ell,j)\}}. \end{aligned}$$

The birth rate of unassociated hosts is equal to λ. The parameter g indicates the variation of the birth rate of hosts associated with a symbiont. If $0 \leq g < 1$, the symbiont is a pathogen; if $g = 1$, the symbiont has no effect on the birth rate of the host; if $g > 1$, the symbiont is a mutualist. The parameters c_{ij} denote the rate at which symbiont j infects host i. This parameter will allow us to mimic specialist and generalist symbionts. Finally, births and infections occur within a local neighborhood, with R_1 denoting the birth range of hosts, and R_2 the infection range of symbionts. The cardinality of each of these neighborhoods will be denoted by ν_{R_1} and ν_{R_2} respectively.

Before we describe the behavior of the spatially explicit stochastic model, we look at the mean-field model (Durrett and Levin, 1994), that is, we pretend that all sites are independent and that the system is spatially homogeneous. This then results in a system of differential equations for the densities of unassociated and associated hosts. We let u_i denote the density of unassociated hosts of type i, and v_{ij} denote the density of host i associated with symbiont j. Furthermore, we assume that $c_{ii} = \beta$ and $c_{ij} = \alpha$ for $i \neq j$, with $0 \leq \alpha \leq \beta$. We define

$$u_{\cdot} \ = \ \sum_{i=1}^{\kappa} u_i, \qquad v_{\cdot j} \ = \ \sum_{i=1}^{\kappa} v_{ij}, \qquad v_{i \cdot} \ = \ \sum_{j=1}^{\kappa} v_{ij} \qquad \text{and} \qquad v_{\cdot\cdot} \ = \ \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} v_{ij}.$$

One way to obtain the mean-field limit is to set the neighborhood ranges, R_1 and R_2 equal to R and then let R go to infinity. To obtain a meaningful limit, we also need to rescale the parameters λ, α, and β by the neighborhood size ν_R, that is, we set $\lambda = \frac{1}{\nu_R}$ (this also sets the time scale), and define

$$\alpha \ = \ \frac{a}{\nu_R} \qquad \text{and} \qquad \beta \ = \ \frac{b}{\nu_R}.$$

In the limit, $R \to \infty$, sites become independent. If, in addition, the initial configuration is translation invariant, the dynamics of the densities for $i \neq j$ is then described by the following system of differential equations, called mean-field equations.

$$\begin{aligned}
u_i' &= (1-u_i)\,(u_i + g\,v_{i\cdot}) \;-\; u_i \sum_{j\neq i} (u_j + g\,v_{j\cdot}) \;-\; b\,u_i\,v_{\cdot i} \;-\; a \sum_{j\neq i} u_i\,v_{\cdot j} \\
v_{ii}' &= b\,u_i\,v_{\cdot i} \;-\; v_{ii}\,(u_{\cdot} + g\,v_{\cdot\cdot}) \\
v_{ij}' &= a\,u_i\,v_{\cdot j} \;-\; v_{ij}\,(u_{\cdot} + g\,v_{\cdot\cdot}).
\end{aligned}$$

When $a = 0$, the symbionts are specialists. As a increases to b, the association turns into a generalist relationship. When $g = 1$, the system has a conserved quantity, namely the initial host densities $h_i = u_i + v_{i\cdot}$, $i = 1, 2, \ldots, \kappa$. If $(\kappa - 1)\,a + b > \kappa$, then for $g \neq 1$, there exists a nontrivial equilibrium with $u_1 = u_2 = \cdots = u_\kappa \geq 0$ and $v_{1\cdot} = v_{2\cdot} = \cdots = v_{\kappa\cdot} > 0$. Numerical simulations indicate that this equilibrium is locally stable for $a < b$ when $g < 1$, but not for $g > 1$. If initially only two hosts and one symbiont, say symbiont 1, are present then for $0 < g < 1$, pathogen 1 will go extinct and both hosts may coexist. We conjecture similar behavior for the spatial model, except in the 1-dimensional, nearest neighbor case (see Theorem 11 and discussion following the theorem). When $g > 1$, host 2 goes extinct provided the initial density of mutualists of type 1 is sufficiently large. The spatial analogue of this result is proved in Theorem 10.

We now return to the spatially explicit model with parameters λ, α, and β. To define the time scale, we set $\lambda = 1$. We discuss both the generalist case $\alpha = \beta$ and the specialist case $\alpha = 0$, and provide comparisons with the mean-field model.

Generalist interactions

We consider the generalist case $\alpha = \beta$ of the spatially explicit stochastic model. In the corresponding mean-field model, coexistence of hosts and symbionts is possible when $b > 1$. On the other hand, for $b \leq 1$, symbionts are unable to persist. It is not hard to see that the nontrivial equilibrium of unassociated hosts, $(u_{\cdot}, v_{\cdot\cdot})$, is locally stable and that the boundary point $v_{\cdot\cdot} = 0$ is unstable for $b > 1$. The behavior of the spatially explicit model is more complicated and may depend on the spatial dimension, as we will see in the following.

If $g = 1$, then the symbionts have no effect on the hosts, which means that the spatially explicit processes ξ_t^1 and ξ_t^2 are stochastically independent. Moreover, it is easy to see that ξ_t^1 is a voter model run at rate 1, and that ξ_t^2 is a multitype contact process in which particles give birth at rate $\beta\,\nu_{R_2}$ and die at rate ν_{R_1}. See respectively Holley and Liggett (1975) and Neuhauser (1992) for a study of these two processes. It follows that there exists a critical value $\beta_c(R_1, R_2) \in (0, \infty)$ that depends on ν_{R_1} and ν_{R_2} such that the symbionts can survive if and only if $\beta > \beta_c(R_1, R_2)$. If we ignore host and symbiont types but rather focus on associated versus unassociated hosts, then for $\beta > \beta_c(R_1, R_2)$, regardless of the spatial dimension, there exists a nontrivial stationary distribution of associated and unassociated hosts. Moreover, if $d \geq 3$, there exists a stationary distribution in which all hosts and symbionts coexist.

To analyze the case $g \neq 1$, we define the *color-blind* process where a site is in state 0 if it is occupied by an unassociated host, and in state 1 if it is occupied by an associated host. We obtain a particle system $\zeta_t : \mathbb{Z}^d \longrightarrow \{0, 1\}$ with transition rates

$$\begin{aligned}
c_{01}\,(x, \zeta) &= \beta \sum_{0 < \|x - z\| \leq R_2} \mathbb{1}_{\{\zeta(z) = 1\}} \\
c_{10}\,(x, \zeta) &= \sum_{0 < \|x - z\| \leq R_1} \left\{ \mathbb{1}_{\{\zeta(z) = 0\}} \;+\; g\;\mathbb{1}_{\{\zeta(z) = 1\}} \right\}.
\end{aligned}$$

When $g = 1$, the process reduces to a contact process with birth rate $\beta\,\nu_{R_2}$ and death rate ν_{R_1}. We denote the critical value of this contact process by $\beta_c(R_1, R_2)$ as above. A coupling argument allows us to compare the processes with $g \neq 1$ and $g = 1$, and to deduce that if $g \leq 1$ and $\beta > \beta_c(R_1, R_2)$ then ζ_t has a nontrivial stationary measure, while if $g \geq 1$ and $\beta \leq \beta_c(R_1, R_2)$ then $\zeta_t \Rightarrow \delta_0$, the “all 0” configuration. To cover the remaining cases, we introduce the contact process η_t in which particles give birth at rate $\beta\,\nu_{R_2}$ and die at rate $g\,\nu_{R_1}$. Then η_t has a nontrivial stationary measure if and only if $\beta > g\,\beta_c(R_1, R_2)$ which, with a new coupling argument, implies that if $g \le 1$ and $\beta \le g\,\beta_c(R_1, R_2)$, then the pathogens die out, while if $g \ge 1$ and $\beta > g\,\beta_c(R_1, R_2)$, then ζ_t has a nontrivial stationary measure. These results are summarized in the following theorem.

Theorem 8 *Assume that $\alpha = \beta$ and that ζ_0 is translation invariant with $P(\zeta_0(x) = 1) > 0$.*

(a) If $g \le 1$, then $\zeta_t \Rightarrow \delta_0$ if $\beta \le g\,\beta_c$, and a nontrivial equilibrium exists if $\beta > \beta_c$.

(b) If $g \ge 1$, then $\zeta_t \Rightarrow \delta_0$ if $\beta \le \beta_c$, and a nontrivial equilibrium exists if $\beta > g\,\beta_c$.

We now focus on the case $\beta > 1$ and $g > 0$ close to 0. First of all, we observe that if $g = 0$ and $R_1 = R_2 = R$ then the process ζ_t is the biased voter model with parameters β and 1. In particular, $P(\zeta_t(x) = 0) \to 1$ if $\beta < 1$ while $P(\zeta_t(x) = 1) \to 1$ if $\beta > 1$. Moreover, in the latter case, *fixation* occurs for the process ξ_t since hosts associated with pathogens are now sterile. By using a perturbation argument (see Section 5), we can show that if $\beta > 1$ and $g > 0$ is sufficiently close to 0 then the pathogens still survive.

Theorem 9 *Assume $\alpha = \beta$ and $R_1 = R_2$. If $\beta > 1$ there exists a critical value $g_c > 0$ such that if $g \le g_c$ then $\zeta_t \Rightarrow \mu$ with $\mu(\zeta(x) = 1) \ne 0$.*

Specialist interactions

In the specialist case $\alpha = 0$ and $\beta > 0$, the process is more difficult to investigate since the evolution of each symbiont strongly depends on the configuration of the host population. That is, there is no particle system $\zeta_t : \mathbb{Z}^d \longrightarrow \{0, 1\}$ which allows us to describe the global evolution of the symbionts regardless of their type. Since for any $i = 1, 2, \ldots, \kappa$ the symbiont i can live only through hosts of type i, it is, however, easy to deduce from a coupling argument that if $g \le 1$ and $\beta \le \max(\beta_c\,g, 1)$, then the pathogens die out.

The next step is to extend the results of the mean-field model to the corresponding spatial model. To do this, we consider a population of two hosts with only one type of symbiont, say symbiont of type 1, and start the evolution with all the hosts of type 1 associated with a symbiont. Then, in the limiting case $\beta = \infty$, the transition $(1, 0) \to (1, 1)$ is instantaneous provided that $R_1 \le R_2$ to avoid the problem of *isolated* unassociated host that cannot be reached by any symbiont. This implies that the process ξ_t^1 is a biased voter model in which hosts of type 1 give birth at rate $g\,\nu_{R_1}$ and hosts of type 2 at rate ν_{R_1}. In particular, if $g < 1$, then $\xi_t^1 \Rightarrow \delta_2$, the "all 2" configuration, while if $g > 1$, then $\xi_t^1 \Rightarrow \delta_1$, the "all 1" configuration. Theorem 10 tells us that, in any dimension, the result still holds if $g > 1$ and $\beta \in (0, \infty)$ is sufficiently large.

Theorem 10 *Assume that $\alpha = 0$, $g > 1$, and $R_1 \le R_2$. At time 0, $\xi_0(x) = (1, 1)$ or $(2, 0)$ for all $x \in \mathbb{Z}^d$. Then, there is a critical value $\beta_c^{Th10} \in (0, \infty)$ such that if $\beta > \beta_c^{Th10}$ then $\xi_t^1 \Rightarrow \delta_1$.*

The arguments in our proof, however, do not imply the analogous result for $g < 1$. We think actually that except in the 1-dimensional nearest neighbor case, $P(\xi_t^1(x) = 2) \not\to 1$. Before explaining our intuition, we describe the behavior of the 1-dimensional process with nearest neighbor interactions for the hosts and short range interactions for the pathogens. Assume that at time 0, $\xi_0^1(x) = 1$ for $x > 0$ with infinitely many pathogens of type 1, and $\xi_0^1(x) = 2$ for $x \le 0$ with no associated pathogens. Denote by $r_t^2 = \sup\{x \in \mathbb{Z} : \xi_t^1(x) = 2\}$ the rightmost host of type 2. Then $r_0^2 = 0$. The following result implies that for β large enough, 2's will spread to the right and eliminate 1's together with their associated pathogens.

Theorem 11 *Assume $d = 1$ and $R_1 = 1$. If $\alpha = 0$ and $g < 1$, there exists $\beta_c^{Th11} \in (0, \infty)$ such that if $\beta > \beta_c^{Th11}$, then $r_t^2 \to \infty$ as $t \to \infty$ with probability 1.*

We conjecture that this result should only be true in the 1-dimensional nearest neighbor case. We first observe that except in the 1-dimensional nearest neighbor case, the dynamics produce *isolated* hosts, that is hosts of type 1 (resp. 2) surrounded by a cluster of hosts of type 2 (resp. 1). As suggested by Theorem 10, when $g > 1$, isolated 2's are *swallowed* very quickly by surrounding 1's. On the other hand, when $g < 1$, an isolated 1 located in a linearly growing cluster of 2's cannot be invaded anymore by any pathogen as soon as the cluster has reached some critical size. In this expanding region, the process then behaves like an unbiased voter model in which 1's can now compete with 2's, and survive.

Lastly, we investigate the coexistence of symbionts in the neutral case $g = 1$. We observe that, in this case, the first coordinate process ξ_t^1 performs a voter model run at rate 1. In particular, in $d \leq 2$, there does not exist any stationary distribution μ such that $\mu(\xi^1(x) = i) \neq 0$ for any $i \in \{1, 2, \ldots, \kappa\}$ (see Holley and Liggett, 1975). Since in the specialist case, $\xi_t^2(x) = i$ implies that $\xi_t^1(x) = i$, the same conclusion holds for the symbionts. In $d \geq 3$, coexistence occurs for the process ξ_t^1, i.e., there is a stationary measure μ which satisfies the condition above. However, due to the formation of clusters, the problem of coexistence of the symbionts remains a difficult question. Namely, the voter model ξ_t^1 exhibits clusters whose diameter can exceed some critical size, which prevents the symbionts from spreading out. To get around this difficulty, we introduce a modification of the particle system, denoted by $\hat{\xi}_t$, in which the symbionts evolve as previously but where the hosts now perform a threshold θ voter model. More precisely, the process $\hat{\xi}_t$ evolves according to the following transition rates

$$
\begin{aligned}
c_{(i,i)(k,0)}(x,\xi) &= \begin{cases} 1 & \text{if} \quad \text{card}\,\{z \in \mathbb{Z}^d : 0 < \|x - z\| \leq R_1 \text{ and } \hat{\xi}^1(z) = k\} \ \geq \ \theta \\ 0 & \text{otherwise} \end{cases} \\
c_{(i,0)(i,i)}(x,\xi) &= \qquad \beta \ \times \ \text{card}\,\{z \in \mathbb{Z}^d : 0 < \|x - z\| \leq R_2 \text{ and } \hat{\xi}^2(z) = i\}.
\end{aligned}
$$

The introduction of this particle system is motivated by Theorem 1 of Durrett (1992) which implies that the threshold θ voter model has a nontrivial stationary distribution which is close enough to a product measure to produce our last result.

Theorem 12 *Let $g = 1$ and $\theta < \nu_{R_1}/\kappa$. If R_1 and R_2 are sufficiently large, then there is a critical value $\beta_c^{Th12} \in (0,\infty)$, depending on κ, such that if $\beta > \beta_c^{Th12}$, then all the hosts and symbionts coexist, while if $\beta < \beta_c^{Th12}$, then coexistence is not possible.*

Comparison of the spatially explicit and the mean-field model

Numerical simulations of the mean-field model indicate that coexistence is only possible when $g < 1$. Simulations of the spatially explicit model show similar behavior. When $g < 1$ and $\alpha < \beta$, then coexistence of hosts and pathogens is possible. We observed that in this case, cluster size is limited by the presence of pathogens: In the absence of pathogens, clusters grow at the expense of neighboring clusters that contain symbionts. Upon invasion by the preferred symbionts (those with infection rate β), the clusters appear to shrink again. The case $g > 1$ and $\alpha < \beta$ is quite different. Clusters of hosts with their preferred mutualists form and appear to continue to grow, just as in the voter model case. Less preferred mutualists (those with infection rate α) do not seem to be able to persist with preferred mutualists, just as is the mean-field case of one host and two symbionts with infection rates a and b, respectively ($a < b$). In summary, pathogens have the ability to alter the spatial structure of their hosts by promoting local diversity, whereas mutualists do not alter the spatial structure of their hosts. This difference in behavior is more pronounced the more host-specific the symbionts are.

4. Voter model and biased voter model in heterogeneous environment

Our last stochastic process is intended to model the gene flow from transgenic crop into natural populations of close relatives through pollination. Since crop plants are descended from wild plants, the existence of a gene flow between each of them is no surprise. With novel genes being inserted into the genomes of crop plants to express proteins for specific biological functions, there is increased concern that these novel genes would escape into the wild and confer increased fitness to some species and that the wild plants could become aggressive invaders. To understand how the spatial configuration of permanent plots of genetically modified crop plants affect gene flow and invasion of transgenes into adjacent natural populations of close relatives, we propose a spatially explicit, stochastic model in a heterogeneous environment. The environment will reflect the permanent plots of genetically modified organisms embedded in a matrix of wild plants. Since we think of the genetically modified crop plants as planted, gene flow will only occur within the wild plants and from the genetically modified plants to the wild plants.

Our model is based on the simplest population genetics model, the haploid Wright-Fisher model with selection. The spatial analog of the Wright-Fisher model is the (biased) voter model. We define the voter model in a heterogeneous environment with gene flow in such a way to address the problem of recurrent gene flow from genetically modified crop plants to their wild relatives. More precisely, our spatial model is a continuous-time Markov process $\xi_t : \mathbb{Z}^d \longrightarrow \{0,1\}$. We think of 1's as representing individuals carrying the transgene, and 0's as the ones carrying the wild type gene. A site $x \in \mathbb{Z}^d$ is said to be occupied by a 1 (resp. 0) if $\xi(x) = 1$ (resp. 0). We will also use the notation $x \in \xi_t$ if and only if $\xi_t(x) = 1$. The subset $\Delta \subset \mathbb{Z}^d$ represents the part of the habitat that is occupied by transgenic crop plants. We freeze the evolution on Δ by pretending that if $x \in \Delta$ then, at any time $t \geq 0$, $\xi_t(x) = 1$. To describe the dynamics, we introduce a transition probability $p\,(x,z)$ on $\mathbb{Z}^d$ that is translation invariant, i.e., $p\,(x,z) = q\,(z-x)$, symmetric, and such that $q\,(z)$ has variance $\sigma^2 < \infty$. If a site $x \in \mathbb{Z}^d - \Delta$ is occupied by a 1, then, at rate 1, it picks a site $z \in \mathbb{Z}^d$ with probability $p\,(x,z)$ and changes to the state of the individual at z. If the site x is occupied by a 0, it chooses a site $z \in \mathbb{Z}^d$ at rate β according to $p\,(x,z)$ and changes to the state of the site z. That is, the evolution is described by the transition rates

$$c_{01}\,(x,\xi) \;=\; \beta \sum_z p\,(x,z)\; \mathbb{1}_{\{\xi(z)=1\}} \qquad \text{and} \qquad c_{10}\,(x,\xi) \;=\; \sum_z p\,(x,z)\; \mathbb{1}_{\{\xi(z)=0,\ x \notin \Delta\}}.$$

Except in the case $\Delta = \varnothing$, we assume from now on that $\xi_0(x) = 1$ if and only if $x \in \Delta$. In particular, it follows from the dynamics that for all $t \geq 0$, $\xi_t(x) = 1$ for $x \in \Delta$.

We first observe that if $\Delta = \varnothing$ and $\beta = 1$ then ξ_t is the d-dimensional *voter model*. In this case, the results of Holley and Liggett (1975) reveal a sharp change in behavior depending on the spatial dimension: If $d \leq 2$ then $\xi_t \Rightarrow \alpha\,\delta_0 + (1-\alpha)\,\delta_1$, for some $\alpha \in [0,1]$, where δ_i denotes the point mass on $\xi \equiv i$, while if $d \geq 3$, coexistence is possible, i.e., there exists a stationary distribution μ such as $\mu\,(\xi(x) = 0) \neq 0$ and $\mu\,(\xi(x) = 1) \neq 0$. Choosing $\Delta \neq \varnothing$ can drastically change the limiting behavior of the process. For instance, if we set $\Delta = \{0\}$ and $d \leq 2$, then the particle located at the origin will produce a cluster that will invade the whole space.

If we set $\Delta = \varnothing$ and $\beta > 1$ then ξ_t is the *biased voter model*. The results of Bramson and Griffeath (1980, 1981) show that if we let $\Omega_\infty = \{\xi_t \neq \varnothing \text{ for all } t \geq 0\}$ and start with a single 1 at the origin, there exists a convex set A so that on Ω_∞ we have for any $\varepsilon > 0$

$$(1-\varepsilon)\,t\,A \;\cap\; \mathbb{Z}^d \;\subset\; \xi_t \;\subset\; (1+\varepsilon)\,t\,A \;\cap\; \mathbb{Z}^d \quad \text{for all } t \text{ sufficiently large.}$$

Moreover, on Ω_∞^c, the process converges to the "all 0" configuration exponentially fast.

We now fix $d = 1$. The discussion above implies that if $\Delta \neq \varnothing$ then $\xi_t \Rightarrow \delta_1$. The first question we would like to answer is: What is the effect of the geometry of Δ on the speed of convergence to the "all 1" configuration? The first step is to study the process with $\xi_0(x) = \mathbb{1}_{\{x \leq 0\}}$. In the special case $\Delta = \varnothing$, the behavior of the process at the interface has been investigated by Cox and Durrett (1995). We consider the *leftmost-zero* and *rightmost-one* processes

$$\ell_t \;=\; \inf\{x \in \mathbb{Z} : \xi_t(x) = 0\} \qquad \text{and} \qquad r_t \;=\; \sup\{x \in \mathbb{Z} : \xi_t(x) = 1\}.$$

Moreover, we assume, for technical reasons, that p is irreducible and that q has finite third moments. Then $\{r_t - \ell_t : t \geq 0\}$ is stochastically compact, that is for any $\varepsilon_0 > 0$ there exists a constant $M < \infty$ such that $P\,(r_t - \ell_t > M) \leq \varepsilon_0$. See Cox and Durrett (1995), Section 4. Moreover, if we denote by $\Phi(x)$ the standard normal distribution function, then

$$\lim_{t\to\infty} P\,(r_t/\sigma\sqrt{t} \leq x) \;=\; \lim_{t\to\infty} P\,(\ell_t/\sigma\sqrt{t} \leq x) \;=\; \Phi(x).$$

In the case $\Delta = \mathbb{Z}^-$, the set of nonpositive integers, we can prove that $\{r_t - \ell_t : t \geq 0\}$ is still stochastically compact. The previous equation, however, becomes false since r_t and ℓ_t are now forced to live on the right side of zero. More precisely, we have the following

Theorem 13 *Let $\beta = 1$. If $x \geq 0$ and $x_t = x\,\sigma\sqrt{t}$ then*

$$\lim_{t\to\infty} P\,(r_t \geq x_t) \ = \ \lim_{t\to\infty} P\,(\ell_t \geq x_t) \ = \ \sqrt{\frac{2}{\pi}} \int_x^\infty \exp\left[-\frac{y^2}{2}\right] dy.$$

In words, the particles diffuse to the right, which reveals a very slow invasion. In view of Theorem 13, our guess is that the more Δ is *scattered*, the faster the transgene will invade the wild population. To make this argument precise, we consider the process on the torus $\varepsilon\mathbb{Z}/N\mathbb{Z}$ where N is a positive integer and ε is close to 0. Moreover, we speed up time by ε^{-2}, and introduce the transition probability $p_\varepsilon\,(x,y) = q\,(\varepsilon^{-1}(y-x) \mod N\varepsilon^{-1})$ to formulate the dynamics. We let $L > 0$ and $K = N/L$ such that both K and $L\,\varepsilon^{-1}$ are integers, with K even. For any site $z \in \{0, 1, \ldots, K-1\}$, we define the subsets

$$B_z \ = \ [\,z\,L,\,(z+1)\,L) \ \cap \ \varepsilon\,\mathbb{Z} \qquad \text{and} \qquad \Delta \ = \ \bigcup_{z \text{ even}} B_z.$$

The reader will note that since we start the process with $\xi_0(x) = 1$ if and only if $x \in \Delta$, for any choice of L, half of the sites are initially occupied, and half of the sites are initially vacant. Finally, we denote by T_{inv} the stopping time $T_{\text{inv}} = \inf\{t \geq 0 : \xi_t \equiv 1\}$. We refer to T_{inv} as the *time to complete invasion.*

Theorem 14 *Let $\beta = 1$. If $p\,(x,y)$ has compact support then*

$$\lim_{\varepsilon\to 0} P\,(T_{\text{inv}} \leq t) \ = \ \left\{1 - \int_0^L (\Psi_t * \Psi_t)(x)\,dx\right\}^{N/2L}$$

where

$$\Psi_t(x) \ = \ -\frac{\partial}{\partial x}\sqrt{\frac{2}{\pi}} \int_{x/\sigma\sqrt{t}}^\infty \exp\left[-\frac{y^2}{2}\right] dy.$$

Theorem 14 gives us insights into the effects of the spatial configuration of Δ on the time to complete invasion. Numerical investigations suggest that, for fixed $t \geq 0$, the function $L \mapsto \lim_{\varepsilon\to 0} P\,(T_{\text{inv}} \geq t)$ is increasing. This implies that for N fixed, the time to complete invasion will be very slow in the extreme case $L = N/2$, and faster and faster while increasing the number of components of Δ.

We now consider the process on $\mathbb{Z}$ where the particles give birth according to the transition probability $p\,(x,y)$. Let $\Delta = \mathbb{Z}^-$ and start the process with $\xi_0(x) = 1$ if and only if $x \in \Delta$. Moreover, we assume that $p\,(x,y)$ has compact support. The next step is to investigate the properties of the rightmost-one process r_t with birth rate β. The behavior of r_t as $t \to \infty$ follows from Theorem 13. By working a little bit more, we can prove that the process comes back to its initial configuration infinitely often.

Theorem 15 *If $\beta = 1$, then $P\,(r_t = 0$ i.o.$) = 1$.*

To investigate the case $\beta \neq 1$, we consider the process with nearest neighbor interactions, i.e., we let $p\,(x,y) = 1/2$ if $|x-y| = 1$. If $\beta < 1$, Theorem 15 accompanied by a coupling argument implies that $r_t = 0$ i.o.; the process however does not exhibit the behavior described in Theorem 13.

Theorem 16 *If $\beta < 1$, then $P\,(r_t \geq x) \leq \beta^x$ at any time $t \geq 0$.*

Finally, if $\beta > 1$, the process behaves like the corresponding biased voter model in homogeneous environment, namely the process grows linearly in time with a wave speed of order $\beta - 1$.

Theorem 17 *If $\beta > 1$, there exist $C < \infty$ and $\gamma > 0$ such that for any $t > 0$ and any $\varepsilon > 0$*

$$P\,(|\,r_t - (\beta - 1)\,t\,| \ \geq \ \varepsilon\,t) \ \leq \ C\,e^{-\gamma\varepsilon^2 t}.$$

In conclusion, Theorems 13 and 16-17 imply that if $\beta \geq 1$ then, for any integer $x \geq 0$ and $\varepsilon > 0$, there is $t_0 \geq 0$ such that $P\,(r_t \geq x) \geq 1 - \varepsilon$ for any $t \geq t_0$, while if $\beta < 1$ then $P\,(r_t \geq x) \leq \beta^x$ for any $t \geq 0$. Moreover, Theorems 15-17 exhibit the 0-1 law:

$$P\,(r_t = 0 \ \text{ i.o.}) \ = \ \begin{cases} 0 & \text{if} \quad \beta > 1 \\ 1 & \text{if} \quad \beta \leq 1. \end{cases}$$

In particular, if $\beta = 1$ and $p\,(x, y)$ has compact support, the rightmost-one process r_t converges in probability to infinity as $t \to \infty$ but not almost surely.

In conclusion, our results confirm the need to monitor recurrent gene flow from genetically modified crops to their wild relatives. Both selection and the spatially explicit structure of our model greatly affect the rate and extent of invasion of transgenes into wild populations. To completely take over a wild population, the transgene needs to be favored by selection. However, even if the transgene is not favored, it will continue to be present in the wild population and the extend to which it can penetrate into the wild population depends strongly on the dispersal range. Considering that pollen can disperse over very large spatial distances, genetic pollution of wild population remains a serious concern, even if the transgene is not favored by selection.

5. Continuity result for multicolor particle systems

As suggested in Sections 1-3, given a particle system $\xi_t : \mathbb{Z}^d \longrightarrow F$, a well-known method for proving the existence of nontrivial stationary distributions for the process ξ_t is to apply a *rescaling argument*. The basic idea is to compare a certain collection of good events for the process viewed on suitable space and time scale with an oriented site percolation process. For more details about oriented percolation, we refer the reader to Durrett (1984). The rescaling argument has been invented by Bramson and Durrett (1988) and is reviewed in Durrett (1991, 1995).

To formulate our *continuity result*, we introduce an open set Λ and assume that our particle systems now depend on a parameter $\lambda \in \Lambda$, i.e., the color of $x \in \mathbb{Z}^d$ flips from i to j at rate

$$c_{ij}\,(\lambda, x, \xi) \ = \ h_{ij}\,(\lambda, \, \xi(x + z_1), \, \xi(x + z_2), \, \ldots, \, \xi(x + z_N)).$$

Moreover, we make the continuity assumption

(CA) For any $\delta > 0$, there exists a neighborhood $\mathcal{V}$ of λ_0 in Λ such that

$$|c_{ij}\,(\lambda, 0, \xi) \ - \ c_{ij}\,(\lambda_0, 0, \xi)| \ < \ \delta \qquad \forall\ \lambda \in \mathcal{V}, \quad \forall\ i, j \in F, \quad \forall\ \xi \in F^{\mathbb{Z}^d}.$$

Our last theorem tells us that, if one succeeds in showing, through the rescaling argument, the existence of a stationary distribution for the process with parameter $\lambda_0 \in \Lambda$, then there is a neighborhood $\mathcal{V}$ of λ_0 in Λ such that the processes with parameter $\lambda \in \mathcal{V}$ still exhibit, in some sense that will be precised later, the same behavior. The result has already been proved in many special cases. See, e.g., Durrett and Neuhauser (1997), Schinazi (2001, 2002) or Theorems 2 and 9 above.

To make the connection between particle systems and oriented percolation, the basic idea is to turn the space-time of the process into a *brick wall*, each brick is associated with a certain good event. Let $\bar{\xi}_t : \mathbb{Z}^d \longrightarrow F$ define the particle system with parameter $\lambda_0 \in \Lambda$ that is constructed from a collection of independent Poisson processes (Harris, 1972), and set

$$\mathcal{L} \ = \ \{(z, n) \in \mathbb{Z}^2 : z + n \text{ is even and } n \geq 0\}.$$

Let L, T and Γ be integers with $T = \Gamma L$, and $\mathcal{H}$ be a collection of configurations determined by the value of $\bar{\xi}$ on $[-L, L]^d$. For any $z \in \mathbb{Z}$, we denote by $\mathcal{H}_z$ the collection $\mathcal{H}$ translated by the vector Lze_1, and say that a site $(z, n) \in \mathcal{L}$ is *occupied* if $\xi_{nT} \in \mathcal{H}_z$. Here, e_1 denotes the first unit vector of the d-dimensional lattice. Finally, we let k_0, $j_0 \in \mathbb{N}^*$, set $M = \max\{j_0, k_0\}$, and introduce the space-time region

$$B_{z,n} = (Lze_1, nT) + \{[-k_0 L, k_0 L]^d \times [0, j_0 T]\}.$$

Each $(z, n) \in \mathcal{L}$ is associated with a certain good event $E_{z,n}$ measurable with respect to the graphical representation of the process in $B_{z,n}$. Finally, we suppose that, for any $\varepsilon > 0$, the comparison assumption holds for the process with parameter λ_0, i.e.,

$(P^{\varepsilon}_{\lambda_0})$ The parameters L and Γ can be chosen sufficiently large so that

1. $P(E_{z,n}) \geq 1 - \varepsilon$ and
2. If (z, n) is occupied and $E_{z,n}$ occurs, then $(z - 1, n + 1)$ and $(z + 1, n + 1)$ are occupied.

Here, (z, n) occupied means that $\bar{\xi}_{nT} \in \mathcal{H}_z$. That is, in the definition of $P^{\varepsilon}_{\lambda_0}$, *occupied* applies to the particle system with parameter λ_0. Let P_{λ_0} be the property that for any $\varepsilon > 0$ the comparison assumption $P^{\varepsilon}_{\lambda_0}$ is satisfied. Then, we get the following theorem.

Theorem 18 *Assume CA and P_{λ_0}. Then, for any $\varepsilon > 0$, there is a neighborhood $\mathcal{V}_\varepsilon$ of λ_0 in Λ such that the following holds: For any $\lambda \in \mathcal{V}_\varepsilon$, the comparison assumption $P^{\varepsilon}_{\lambda}$ is satisfied.*

As previously explained, Theorem 18 tells us that if λ is close to λ_0 then the processes with parameters λ and λ_0 exhibit the same behavior in the sense that each of both processes has a stationary distribution ν such that, for any $z \in \mathbb{Z}$, $\nu(\xi \in \mathcal{H}_z) > 0$. But beyond a simple continuity result, Theorem 18 gives us insights into the strategy to study the behavior of a particle system. More precisely, we get the following

Corollary 19 *Assume that the process ξ_t exhibits a phase transition at $\lambda_0 \in \Lambda$. Then, the behavior of the process when $\lambda = \lambda_0$ cannot be investigated through the comparison result.*

The proof of Corollary 19 is straightforward. Assume that one succeeds in constructing a stationary measure ν for the process with parameter λ_0 through the comparison result, that is proving that the assumption P_{λ_0} holds. Then, Theorem 18 tells us that there is a neighborhood $\mathcal{V}$ of λ_0 in Λ such that, for any $\lambda \in \mathcal{V}$, the process with parameter λ still exhibits the same behavior, which is not consistent with the existence of a phase transition at λ_0 and proves Corollary 19.

Finally, we think that Theorem 18 does not hold in the very useful case $\lambda_0 = \infty$ and $\lambda > 0$ large (see the discussion following Theorem 11 above). The continuity argument when $\lambda_0 = \infty$ and λ is large, however, holds in some special cases. See, e.g., Theorems 7 and 10 for illustrations.

6. Numerical simulations

Possessors of a recent version of Linux who are interested in numerical simulations can go to the address www.univ-rouen.fr/LMRS/Persopage/Lanchier/index.html where most of the spatial models introduced in this thesis can be downloaded.

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Multicolor particle systems and spatial structures

1. Construction of particle systems

In a general way, we will call *particle system* any Markov process ξ_t which evolves on a spatial structure, typically a graph such as $\mathbb{Z}^d$, and with values in a finite set $F = \{0, 1, \ldots, \kappa - 1\}$ called *set of colors*. The value of $\xi_t(x)$ indicates the color of site $x \in \mathbb{Z}^d$ at time $t \geq 0$. To describe the evolution of the processes we are interested in, we consider an *interaction neighborhood*

$$\mathcal{N} \ = \ \{z_0, z_1, \ldots, z_N\} \ \subset \ \mathbb{Z}^d \qquad \text{with} \quad z_0 \ = \ 0.$$

If the system is in some configuration $\xi : \mathbb{Z}^d \longrightarrow F$, the color of $x \in \mathbb{Z}^d$ flips to $i \in F$ at rate

$$c_i\,(x, \xi) \ = \ h_i\,(\xi(x + z_0), \xi(x + z_1), \ldots, \xi(x + z_N)).$$

In other words, the flip rates at x only depend on the colors of site x and of a finite number of neighboring sites. Moreover, our particle systems are *translation invariant*, i.e., if τ_x denotes the translation operator $\tau_x\,\xi(z) = \xi(x + z)$, then

$$c_i\,(x, \xi) \ = \ c_i\,(0, \tau_x\,\xi) \qquad \text{for any} \quad x \in \mathbb{Z}^d.$$

To figure out the behavior of a particle system, the main objective is to investigate its *stationary measures*, i.e., the measures μ such that $P_\mu(\xi_t \in A) = \mu(A)$ for any $t \geq 0$ and $A \in \mathcal{F}$. Here, P_μ denotes the law of the process starting from the distribution μ, and $\mathcal{F}$ the usual σ-algebra on the set of the functions $\xi : \mathbb{Z}^d \longrightarrow F$. To justify the use of a plural, we observe that the transition rates above indicate that the particle systems we are interested in are not irreducible and have, a priori, more than one stationary measure. Finally, if the process has an absorbing state $\bar\xi$, the measure that concentrates on the configuration $\bar\xi$ is a stationary measure, that we will call, for obvious reasons, *trivial stationary measure.*

We now observe that, since the spatial structure on which our particles evolve has infinitely many sites, the jumping times of the process cannot be ordered. In particular, it is not clear that the transition rates $c_i\,(x, \xi)$ specify a unique Markov process. An idea of Harris (1972), however, assures us of the existence and the uniqueness of the particle system provided that

$$c \ = \ \sup \left\{ c_i\,(x, \xi) : i \in F, \ x \in \mathbb{Z}^d, \ \xi \in F^{\mathbb{Z}^d} \right\} \ < \ +\infty.$$

Since the set of colors and the interaction neighborhood are finite, the reader will note that the previous condition holds systematically for our translation invariant processes. The basic idea is to construct the particle system from a collection of independent Poisson processes $\{T_n^{x,i} : n \geq 1\}$, $x \in \mathbb{Z}^d$, $i \in F$, each of them has parameter c. At time $t = T_n^{x,i}$, the color of x flips to i with probability $c_i\,(x, \xi_{t-})/c$, which, in view of the well-known properties of the Poisson processes, will produce the desired flip rate. Now, we draw an unoriented arc between x and y at time $T_n^{x,i}$ if $y - x \in \mathcal{N}$ to indicate that both sites may interact. We fix $t_0 > 0$, and say that there is a *path* between x and y if there exists a sequence of sites $x_0, x_1, \ldots, x_n$ such that $x_0 = x$, $x_n = y$, and for $i = 1, 2, \ldots, n$, the sites x_{i-1} and x_i are connected by time t_0 by an arc. This construction defines an equivalence relation on the set of sites, and so a partition of $\mathbb{Z}^d$. Then, it is easy to prove that, for $t_0 > 0$ sufficiently small, each of these equivalence classes are a.s. finite, so that the color of site $x \in \mathbb{Z}^d$ at time t_0 results from a finite number of interactions. This also implies that the configuration of the process at any time $t \geq 0$ can be computed by induction.

The rest of this paper is devoted to what seems to be the main leitmotiv of the particle systems literature: The connection between particle systems and spatial structures. To investigate such processes, the basic idea is to introduce beforehand a *simplified* version of the model, obtained either by removing the spatial structure, or by turning the discrete lattice into a continuous spatial structure, the effect is to make the initial *stochastic* model a *deterministic* model. To motivate this strategy, the reader will note all along this part that the three versions of a given particle system – *no* spatial structure, *continuous* spatial structure, and *discrete* spatial structure – usually exhibit, in some sense to be precised, the same properties. By increasing in particular the size of the interaction neighborhood or the dimension of the space, which reduces the interactions between adjacent sites, the stochastic model will tend to behave like its deterministic versions.

2. Mean-field model: Removal of the spatial structure

If we assume that all the sites in $\mathbb{Z}^d$ are independent and that the system is spatially homogeneous, then all the concepts related to the spatial structure lose their consistency, the only objects still visible under this assumption are the particle densities u_i for $i \in F$. The new model resulting from this transformation, or *mean-field model* (Durrett and Levin, 1994), is then described by a coupled system of ordinary differential equations

$$u_i'(t) \;=\; g_i\,(u_0\,(t),\, u_1\,(t),\, \ldots,\, u_{\kappa-1}\,(t)), \qquad i \in F$$

called *mean-field equations*. More precisely,

$$u_i' \;=\; \langle\, c_i\,(0,\xi)\; \mathbb{1}_{\{\xi(0) \neq i\}}\,\rangle_u \;-\; \sum_{j \neq i} \langle\, c_j\,(0,\xi)\; \mathbb{1}_{\{\xi(0) = i\}}\,\rangle_u$$

where $\langle\,\phi(\xi)\,\rangle_u$ denotes the expected value of $\phi(\xi)$ under the product measure $\pi(\xi(x) = j) = u_j$. The previous equation, though not very welcoming, is easy to understand. The first member on the right-hand side is just the probability that the color of 0 flips to i when 0 is not already occupied by a particle of type i, which contributes to increase the density u_i. The second member, on the contrary, is the probability that the color of 0 flips from i to some other color $j \neq i$, which contributes to decrease u_i.

As we will see further, the analysis of the mean-field model is essential for a better understanding of the particle system, in the sense that the existence of locally stable fixed points for the nonspatial model is often symptomatic of the existence of stationary measures for the original spatial model. To specify this argument, we consider a particle system $\xi : \mathbb{Z}^d \longrightarrow F$ whose transition rates at site $x \in \mathbb{Z}^d$ depend on some parameter $\lambda \in D$, with D connex, i.e.,

$$c_i\,(\lambda, x, \xi) \;=\; h_i\,(\lambda,\, \xi(x+z_0),\, \xi(x+z_1),\, \ldots,\, \xi(x+z_N)).$$

We will say that $i \in F$ and $\lambda \in D$ are *strongly compatible* if the spatial model with parameter λ has a stationary distribution μ such that $\mu\,(\xi(x) = i) \neq 0$, and will denote by D_i the set of parameters $\lambda \in D$ that are strongly compatible with the color $i \in F$. Now, if there is a color $i \in F$ with $D_i \notin \{\varnothing, D\}$, i.e., there exist parameters $\lambda_1,\ \lambda_2 \in D$ such that the color i is strongly compatible with λ_1 but not with λ_2, we will say that the process exhibits a *phase transition*. In this case, the elements of ∂D_i, the topological boundary of D_i in D, will be called *critical values*. By analogy, we will say that $i \in F$ and $\lambda \in D$ are *weakly compatible* if the nonspatial model with parameter λ has a locally stable fixed point u such that $u_i \neq 0$, and will denote by D_i' the set of parameters $\lambda \in D$ that are weakly compatible with the color $i \in F$. As explained above, the existence of locally stable fixed points for the nonspatial model is often symptomatic of the existence of stationary measures for the spatial model, which we will express by the existence of a homeomorphism $\psi : D \longrightarrow D$ such that $\psi(D_i) = D_i'$ for any color $i \in F$. The existence of such a ψ, however, is not systematic. As we will see further through concrete examples, the introduction of a spatial structure in the form of local interactions can modify the behavior of the system depending on spatial parameters such as the dimension or the range of the interactions.

3. Long range limits and rapid stirring limits: Continuous spatial structure

3.1. Long range processes

We have just seen that if the sites in $\mathbb{Z}^d$ are independent, then our model becomes devoid of spatial structure and, in this way, considerably simplified. The next step is to see that if we increase the range of the interactions, we simplify the original particle system as well, by producing a deterministic spatial model. To formulate the dynamics of our long range processes, it is convenient to assume that the interactions between neighboring sites are described by a transition probability $p\,(x,z)$, i.e., the flip rates $c_i\,(x,\xi)$ now depend on the transition $p\,(x,z)$ and the colors of the sites z such that $p\,(x,z) \neq 0$. For more convenience, we will construct the long range version on $\varepsilon\mathbb{Z}^d$, where the scale parameter $\varepsilon > 0$ is intended to go to zero. Let $q\,(x)$ be a continuous symmetric kernel with compact support, and, for $x \in \varepsilon\mathbb{Z}^d$, let

$$q_\varepsilon\,(x) \;=\; \varepsilon \int_{x+B_\varepsilon} q\,(z)\; dz_1 \cdots dz_d \qquad \text{where} \quad B_\varepsilon \;=\; [-\,\varepsilon/\,2,\,\varepsilon/\,2)^d.$$

Then, the long range version $\xi^\varepsilon : \varepsilon\mathbb{Z}^d \longrightarrow F$ is derived from the original particle system by replacing $p\,(x,z)$ with the transition probability $p_\varepsilon\,(x,z) = q_\varepsilon\,(z-x)$. Intuitively, taking $\varepsilon > 0$ small increases the size of the interaction neighborhood and keeps the process close to a product measure. The rate of change of the densities can then be computed by assuming that adjacent sites are independent.

To make this argument precise and figure out what happens through an example, we now introduce the 1-dimensional long range sexual reproduction process $\xi^\varepsilon : \varepsilon\mathbb{Z}^d \longrightarrow \{0,\,1\}$. The transition rates at site $x \in \mathbb{Z}^d$ are given by

$$0 \;\to\; 1 \quad \text{at rate} \quad \lambda \sum_z \; p_\varepsilon\,(x,z)\; \mathbb{1}_{\{\xi^\varepsilon(z)=1,\; \xi^\varepsilon(z+\varepsilon)=1\}} \qquad\qquad 1 \;\to\; 0 \quad \text{at rate} \quad 1.$$

In other words, if we think of sites in state 0 as empty, and sites in state 1 as occupied, the evolution can be interpreted as follows. Each particle dies at rate 1. A pair of particles located at adjacent sites z and $z+\varepsilon$ produces at rate λ an offspring which is then sent to x with probability $p_\varepsilon\,(x,z)$. If x is empty the birth occurs, otherwise it is suppressed. The behavior of the process when $\varepsilon \to 0$ and $\varepsilon > 0$ is small has been investigated by Neuhauser (1994). Her first result is the

Theorem 1 *Let $u_\varepsilon(t,x) = P\,(\xi_t^\varepsilon(x) = 1)$ and assume that $\xi_0^\varepsilon(x)$, $x \in \varepsilon\mathbb{Z}$, are independent and that $u_\varepsilon(0,x) = \Phi(x)$ is continuous. Then, as $\varepsilon \to 0$, $u_\varepsilon(t,x) \to u(t,x)$ the bounded solution of*

$$\partial_t\, u(t,x) \;=\; -\,u(t,x) \;+\; \lambda\,(1-u(t,x)) \int q\,(x-z)\,u^2\,(t,z)\,dz \quad \textit{with} \quad u(0,x) \;=\; \Phi(x).$$

In particular, as mentioned above, the model defined on a continuous spatial structure, obtained by taking the limit $\varepsilon \to 0$, exhibits a deterministic behavior described by a certain partial differential equation. Although we focus on a particular example, the proof of results such as Theorem 1 is always a three-step process whose first step is to define

The influence set $I_\varepsilon^{x,t}\,(s)$

Let a site $x \in \varepsilon\mathbb{Z}$ and a time $t \geq 0$. In a general way, the *influence set* $I_\varepsilon^{x,t}\,(s)$ contains all the sites which may *influence* the color of x at time t. To define $I_\varepsilon^{x,t}\,(s)$, we need first of all to construct the process from a graphical representation. For any $x \in \varepsilon\mathbb{Z}$, let $\{U_n^x : n \geq 1\}$ be independent Poisson processes with rate 1 and, for any $x,\, z \in \varepsilon\mathbb{Z}$ such that $p_\varepsilon\,(x,z) > 0$, let $\{T_n^{x,z} : n \geq 1\}$ be independent Poisson processes with rate $\lambda\, p_\varepsilon\,(x,z)$. The interpretation is as follows. At times U_n^x, we kill the particle at x if it is present, while at times $T_n^{x,z}$, a new particle is born at x if z and $z+\varepsilon$ are occupied and x is empty. The set $I_\varepsilon^{x,t}\,(s)$ is defined by going backwards in time. Let $I_\varepsilon^{x,t}\,(0) = \{x\}$. If $z \in I_\varepsilon^{x,t}\,(s)$ and $T_n^{z,z'} = t-s$, we add z' and $z'+\varepsilon$ to $I_\varepsilon^{x,t}\,(s)$. The Poisson processes $\{U_n^x : n \geq 1\}$ have no effect for the moment.

We now label $I_\varepsilon^{x,t}(s)$ by letting $X_\varepsilon^k(s)$ denote the location at time $t-s$ of the k-th particle added to the influence set. To describe the family structure of $I_\varepsilon^{x,t}(s)$, we define a collection of random variables μ_ε^k. More precisely, we set $\mu_\varepsilon^{2k} = \mu_\varepsilon^{2k+1} = n$ if the particle X_ε^n is the parent of X_ε^{2k} and X_ε^{2k+1}. The reader will note that, in view of the inversion of time, an offspring in the original process becomes a parent in the influence set. If a death occurs at $X_\varepsilon^k(s)$, we will set $X_\varepsilon^k = \Delta$ and say that the k-th particle is *virtual.* By convention, if a particle is virtual, the pairs of particles generated by this particle will be virtual as well. In the same way, if a particle is not born yet, we will say that this particle is *virtual* and assign the value Υ to its location. To conclude, we will note that the color of x at time t can be computed from knowing $\{X_\varepsilon^k(s), k \geq 1\}$ and $\{\mu_\varepsilon^k, k \geq 1\}$, and the colors $\xi_{t-s}^\varepsilon(z)$ of sites $z \in I_\varepsilon^{x,t}(s)$ at time $t-s$.

Convergence of $u_\varepsilon(t,x)$

The second step is to prove the convergence of $u_\varepsilon(t,x) = P(\xi_t^\varepsilon(x) = 1)$ as $\varepsilon \to 0$. Let a *collision* define the event that there exist $i \neq j$ such that $X_\varepsilon^i(s) = X_\varepsilon^j(s)$ at some time $s \leq t$. Then, for any fixed time t, the probability of any collision by time t tends to 0 as $\varepsilon \to 0$. To prove this point, we first denote by $\mathcal{K}_t$ the number of particles that are born before time t in the influence set. Then, $\mathbb{E}\,\mathcal{K}_t \leq e^{2\lambda t}$. Moreover, if $x \in \varepsilon\mathbb{Z}$ and $M = \sup_{z\in\mathbb{R}} q(z)$, then the probability that an offspring is sent to x is at most εM, whatever the location of the parents. This implies that the probability of any collision by time t can be bounded by

$$P(\text{collisions by time } t \,|\, \mathcal{K}_t \leq K) \;+\; P(\mathcal{K}_t > K) \;\leq\; K^2 \varepsilon M \;+\; K^{-1} e^{2\lambda t}.$$

Finally, taking $K = \varepsilon^{-0.2}$, the right-hand side of the previous equation tends to 0 as $\varepsilon \to 0$. From this result, it is not difficult to deduce that our labeled influence set converges in distribution to a labeled *branching random walk*:

$$\big(\{X_\varepsilon^k(s), s \geq 0\}, \mu_\varepsilon^k\big)_{k\geq 1} \quad \Rightarrow \quad \big(\{Z_s^k, s \geq 0\}, \mu^k\big)_{k \geq 1} \qquad \text{as} \quad \varepsilon \to 0$$

where Z_s^k is the location at time s of the k-th particle in a branching random walk on $\mathbb{R}$ starting at $Z_0^1 = x$ and where particles die at rate 1, and pairs of particles are born at rate λ. More precisely, if the parent is located at x, then the pair is sent to a site z chosen at random according to the kernel $q(x-z)$.

We now assume that $\xi_0^\varepsilon(x)$, $x \in \varepsilon\mathbb{Z}$, are independent and that $u_\varepsilon(0,x) = \Phi(x)$ is continuous. If the particle at Z_t^k is virtual at time 0, we set $\zeta_0(k) = 0$. Otherwise, we flip a coin with success probability $\Phi(Z_t^k)$, and set $\zeta_0(k) = 1$ if there is a success, and $\zeta_0(k) = 0$ otherwise. Now, if there is no collisions, then the influence set $I_\varepsilon^{x,t}(s)$ and the branching random walk $\{Z_s^k, s \geq 0\}$ exhibit the same family structure. In particular, starting from the same configuration, i.e., the sites $X_\varepsilon^k(t)$ and Z_t^k have the same color, then $\xi_t^\varepsilon(x) = \zeta_t(1)$. Since $I_\varepsilon^{x,t}(s)$ is finite and $\Phi(x)$ continuous, it follows from the continuous mapping theorem that $u_\varepsilon(t,x) \to P(\zeta_t(1) = 1)$ as $\varepsilon \to 0$.

The limit $u(t,x)$ satisfies the integro-differential equation

To conclude, we now prove that the function $u(t,x) = P(\zeta_t(1) = 1)$ satisfies the partial differential equation given in Theorem 1. Let x, $z \in \varepsilon\mathbb{Z}$ with $x \neq z$. Then, the result above also implies that the probability of a collision between the particles of the influence sets $I_\varepsilon^{x,t}(s)$ and $I_\varepsilon^{z,t}(s)$ tends to 0 as $\varepsilon \to 0$. This tells us that both sites x and z are asymptotically independent. In other respects, a simple generator calculation leads to

$$\partial_t\, u_\varepsilon(t,x) \;=\; -\,u_\varepsilon(t,x) \;+\; \lambda \sum_z \, p_\varepsilon(x,z)\, P(\xi_t^\varepsilon(x) = 0,\, \xi_t^\varepsilon(z) = \xi_t^\varepsilon(z+\varepsilon) = 1).$$

This and the fact that the sites x, z and $z+\varepsilon$ become independent when $\varepsilon \to 0$ imply that

$$\partial_t\, u(t,x) \;=\; \lim_{\varepsilon\to 0} \partial_t\, u_\varepsilon(t,x) \;=\; -\,u(t,x) \;+\; \lambda\,(1-u(t,x)) \int q(x-z)\, u^2(t,z)\, dz$$

which completes the proof of Theorem 1. For a rigorous proof of the previous equation, we refer one more time the reader to Neuhauser (1994), Section 2.

Even if we do not give the details of the proofs, it is interesting for our purpose to know the end of the story, that is the behavior of the stochastic model when $\varepsilon > 0$ is small. We first observe that any spatially homogeneous solution $v(t) = u(t,x)$ of the integro-differential equation given above is also a solution of the mean-field equation $v' = -v + \lambda\,(1-v)\,v^2$. It is easy to see that if $\lambda > 4$ the previous equation has two nontrivial fixed points $0 < \rho_u < \rho_s$ with ρ_s stable and ρ_u unstable in the sense that if $v(0) < \rho_u$ then $v(t) \to 0$ while if $v(0) > \rho_u$ then $v(t) \to \rho_s$. If we now start from a spatially inhomogeneous density $\Phi(x)$, a result of Weinberger (1982) implies that the system exhibits traveling wave solutions $u(t,x) = U(x - c(\lambda)\,t)$ where the wave speed $c(\lambda)$ is nondecreasing with respect to λ. Moreover, if $c(\lambda) > 0$ and $\Phi(x)$ exceeds ρ_u in a sufficiently large interval, then there is a larger interval in which the density will be close to ρ_s at some later time. Relying on this result and a rescaling argument, Neuhauser (1994) proved that

Theorem 2 *If $c(\lambda) > 0$ and $\varepsilon > 0$ is small, then ξ_t^ε has a nontrivial stationary distribution in which the density of particles is close to ρ_s.*

In other words, the stochastic spatial model exhibits the behavior predicted by its nonspatial deterministic version, provided that the range of the interactions is sufficiently large.

3.2. Rapid stirring

The second way to simplify our stochastic models is to scale space by ε and *stir* the particles at rate ε^{-2}. Taking $\varepsilon > 0$ small will produce the same conclusion as for the long range processes. The stirring mechanism operates at fast rate, and keeps the particle system close to a product measure, so that the rate of change of the densities can be computed assuming that adjacent sites are independent. Making $\varepsilon \to 0$ will produce, as previously, a deterministic spatial model. The evolution of this system, however, will be described by a so called reaction-diffusion equation rather than an integro-differential equation. More precisely, we let $\varepsilon > 0$ and consider the translation invariant particle system $\xi^\varepsilon : \varepsilon\mathbb{Z}^d \longrightarrow F$ which evolves as follows.

1. The state of site $x \in \mathbb{Z}^d$ flips to $i \in F$ at rate

$$c_i\,(x, \xi^\varepsilon) \;=\; h_i\,(\xi^\varepsilon(x),\, \xi^\varepsilon(x + \varepsilon\, z_1),\, \ldots,\, \xi^\varepsilon(x + \varepsilon\, z_N)).$$

2. For any $x,\, z \in \varepsilon\mathbb{Z}^d$ with $\|x - z\|_1 = \varepsilon$, we exchange the values at x and z at rate ε^{-2}.

To construct the process from Harris' graphical representation, we first introduce a collection of independent Poisson processes $\{T_n^{x,i} : n \geq 1\}$, $x \in \mathbb{Z}^d$, $i \in F$, with parameter

$$c \;=\; \sup\left\{c_i\,(x,\xi) : i \in F,\ \xi \in F^{\mathbb{Z}^d}\right\}$$

as well as a sequence $\{U_n^{x,i} : n \geq 1\}$ of independent random variables that are uniform on $[0,1]$. At time $t = T_n^{x,i}$, the state of site x will flip to i if $U_n^{x,i} \leq c_i\,(x, \xi_{t-}^\varepsilon)/c$. To stir the particles, we consider, for any $x,\, z \in \varepsilon\mathbb{Z}^d$ with $\|x - z\|_1 = \varepsilon$, a further collection of independent Poisson processes $\{V_n^{x,z} : n \geq 1\}$ with parameter ε^{-2}. At time $V_n^{x,z}$, we exchange the values at x and z. The behavior of the particle system when $\varepsilon \to 0$ is described by the following *mean-field limit theorem* of De Masi, Ferrari and Lebowitz (1986).

Theorem 3 *Let $u_i^\varepsilon(t,x) = P\,(\xi_t^\varepsilon(x) = i)$ and assume that $\xi_0^\varepsilon(x)$, $x \in \varepsilon\mathbb{Z}$, are independent and that $u_i^\varepsilon(0,x) = \Phi_i(x)$ is continuous. Then, as $\varepsilon \to 0$, $u_i^\varepsilon(t,x) \to u_i(t,x)$ the bounded solution of*

$$\partial_t\, u_i(t,x) \;=\; \Delta\, u_i \;+\; g_i\,(u) \qquad \textit{and} \qquad u_i(0,x) \;=\; \Phi_i(x).$$

We remind the reader that the functions $g_i(u)$, $i \in F$, have been defined in Section 2 by

$$g_i(u) = \langle c_i(0,\xi) \, \mathbb{1}_{\{\xi(0) \neq i\}} \rangle_u - \sum_{j \neq i} \langle c_j(0,\xi) \, \mathbb{1}_{\{\xi(0)=i\}} \rangle_u.$$

This tells us that, as for the long range limits, the spatially homogeneous solutions of the reaction-diffusion equation satisfy the mean-field equation as well. In particular, we expect a one-to-one correspondence between the stationary measures of the stochastic model with $\varepsilon > 0$ small, and the locally stable fixed points of the nonspatial model. The rest of this section is devoted to the proof of Theorem 3. The strategy is the same as previously and we start by the construction of

The influence set $I_\varepsilon^{x,t}(s)$

The role of $I_\varepsilon^{x,t}(s)$ is the same as previously. It allows us to keep track of the ancestors and to deduce the color of site x at time t from the configuration of the particle system at earlier time. First of all, we set $I_\varepsilon^{x,t}(0) = \{x\}$. Then, $I_\varepsilon^{x,t}$ makes transitions as follows.

1. If $z \in I_\varepsilon^{x,t}(s)$ and $T_n^{z,i} = t - s$, then we add the points of $z + \varepsilon \mathcal{N}$ to $I_\varepsilon^{x,t}(s)$.
2. If $z \in I_\varepsilon^{x,t}(s)$ and $V_n^{z,z'} = t - s$, then we move the particle at z to z'.

To label our influence set, we define a sequence $X_\varepsilon^k(s)$, $k \geq 1$, by induction. Let $X_\varepsilon^0(0) = x$, and move $X_\varepsilon^0(s)$ as dictated by the stirring mechanism. If $mN+1$ particles have been created at some arrival time $T_n^{z,i} = t - s$ with $z = X_\varepsilon^k(s)$ for some $k \leq mN+1$, then we set

$$X_\varepsilon^{mN+j+1}(s) = \varepsilon z_j + X_\varepsilon^k(s) \qquad \text{for} \quad j = 1, 2, \ldots, N,$$

and $\mu_\varepsilon^{m+1} = k$ to indicate that $X_\varepsilon^k(s)$ is the mother of $X_\varepsilon^{mN+2}(s), \cdots, X_\varepsilon^{(m+1)N+1}(s)$. After being created, the N particles move according to the stirring described by the rule 2 above. Finally, we set $X_\varepsilon^k(s) = \Upsilon$ before the k-th particle is born. Then, the state of x at time t can be computed from knowing the labeled influence set at time s as well as the states $\xi_{t-s}^\varepsilon(z)$ of sites $z \in I_\varepsilon^{x,t}(s)$.

Convergence of $u_i^\varepsilon(t,x)$

The basic idea is the same as for the long range processes. If a new particle $X_\varepsilon^k(s)$ is created at the location of an existing particle, we call this event a *collision*. It should be intuitively clear that if t is fixed, then the probability that a collision occurs by time t tends to 0 as $\varepsilon \to 0$. In other respects, the stirring mechanism described in the rule 2 above tells us that the process $X_\varepsilon^k(s)$ approaches a Brownian motion as $\varepsilon \to 0$. The last two results imply that our labeled influence set converges in distribution to a labeled *branching Brownian motion*:

$$\left(\{X_\varepsilon^k(s), s \geq 0\}, \mu_\varepsilon^k\right)_{k \geq 1} \quad \Rightarrow \quad \left(\{Z_s^k, s \geq 0\}, \mu^k\right)_{k \geq 1} \qquad \text{as} \quad \varepsilon \to 0$$

where Z_s^k denotes the location at time s of the k-th particle in a d-dimensional branching Brownian motion starting at $Z_0^1 = x$ and where particles give birth to N new particles at rate $c\kappa$. See Durrett and Neuhauser (1994), Section 2, for the details of the proof.

The conclusion is rigorously the same as for the long range processes. The limit $u_i(t,x)$ will be computed from the branching Brownian motion Z_s and independent coin flips. Let $\mathcal{K}_t$ be the number of particles in Z_t and, for $0 \leq k \leq \mathcal{K}_t$, let $\zeta_0(k)$ be independent and equal to i with probability $\Phi_i(Z_t^k)$. Since $I_\varepsilon^{x,t}(s)$ and Z_s exhibit the same family structure and that Φ_i is continuous, the continuous mapping theorem implies that $u_i^\varepsilon(t,x) \to P(\zeta_t(0) = i)$ as $\varepsilon \to 0$.

The limit $u(t,x)$ satisfies the reaction-diffusion equation

This follows from a generator calculation and the fact that the rate of change of the densities can be computed assuming that adjacent sites are independent. Let $\psi_{z,i}$ be the function defined on the set of the configurations by $\psi_{z,i}(\xi) = 1$ if $\xi(z) = i$, and $= 0$ otherwise. If L denotes the generator of the particle system with no stirring, then

$$L\,\psi_{z,i} \;=\; \sum_j \,\Big\{-\; h_{j_0}\,(i,\,j_1,\,\dots,\,j_N)\;\psi_{z,i} \;+\; h_i\,(j_0,\,j_1,\,\dots,\,j_N)\;\psi_{z,j_0}\Big\}\;\prod_{\ell=1}^{N}\;\psi_{z+\varepsilon z_\ell,j_\ell}$$

where the sums are over $j_0,\,j_1,\,\dots,\,j_N \in F$. It follows, for the system with stirring, that

$$\begin{aligned} P\,(\xi_t^\varepsilon(x) = i) \;=\; & \sum_z \; p_t^\varepsilon\,(x,z)\;\Phi_i(z) \\ & + \int_0^t \;\sum_z\; p_s^\varepsilon\,(x,z)\;\mathbb{E}\,\Big\{-\;\sum_j\; h_{j_0}\,(i,\,j_1,\,\dots,\,j_N)\;\psi_{z,i}\,(\xi_{t-s}^\varepsilon)\;\prod_{\ell=1}^{N}\;\psi_{z+\varepsilon z_\ell,j_\ell}\,(\xi_{t-s}^\varepsilon) \\ & \qquad + \sum_j\; h_i\,(j_0,\,j_1,\,\dots,\,j_N)\;\psi_{z,j_0}\,(\xi_{t-s}^\varepsilon)\;\prod_{\ell=1}^{N}\;\psi_{z+\varepsilon z_\ell,j_\ell}\,(\xi_{t-s}^\varepsilon)\;\Big\}\;ds \end{aligned}$$

where $p_s^\varepsilon\,(x,z)$ denotes the transition probability of a random walk that jumps from site x to site z at rate ε^{-2} if $\|x - z\|_1 = \varepsilon$. To deduce an expression of $u_i(t,x)$, we now observe that, since the influence sets $I_\varepsilon^{z,t}(s)$ and $I_\varepsilon^{z',t}(s)$, $z \neq z'$, are asymptotically independent,

$$\lim_{\varepsilon\to 0}\;\mathbb{E}\;\Big\{\;\psi_{z,j_0}\,(\xi_{t-s}^\varepsilon)\;\prod_{\ell=1}^{N}\;\psi_{z+\varepsilon z_\ell,j_\ell}\,(\xi_{t-s}^\varepsilon)\;\Big\}\;=\;\prod_{\ell=0}^{N}\;u_{j_\ell}\,(t-s,z).$$

In particular, if we denote by $p_t(x,z)$ the transition probability of a Brownian motion run at rate 2, and take the limit $\varepsilon \to 0$, then

$$\begin{aligned} u_i(t,x) \;\;=\;\; & \int\; p_t(x,z)\;\Phi_i(z)\;dz \\ & + \int_0^t\!\!\int\; p_s(x,z)\;\Big\{-\;\sum_j\; h_{j_0}\,(i,\,j_1,\,\dots,\,j_N)\;u_i(t-s,z)\;\prod_{\ell=1}^{N}\;u_{j_\ell}(t-s,z) \\ & \qquad + \sum_j\; h_i\,(j_0,\,j_1,\,\dots,\,j_N)\;u_{j_0}(t-s,z)\;\prod_{\ell=1}^{N}\;u_{j_\ell}(t-s,z)\;\Big\}\;dz\;ds \\ \;\;=\;\; & \int\; p_t(x,z)\;\Phi_i(z)\;dz\;+\;\int_0^t\!\!\int\;p_s(x,z)\;g_i\,(u\,(t-s,z))\;dz\;ds. \end{aligned}$$

Differentiating with respect to the time then yields Theorem 3.

4. The stochastic model: Return to the discrete spatial structure

We have just seen that if the range of the interactions is multiplied by a factor ε^{-1}, or that a stirring at rate ε^{-2} is introduced, then, when t is fixed, ξ_t^ε converges, as $\varepsilon \to 0$, to a deterministic limit. This limit is the bounded solution of a certain partial differential equation. In this section, we now focus on the behavior of the particle systems just as they have been introduced in Section 1, that is evolving on a discrete spatial structure. This can be considered as an *inversion of the roles played by space and time*, in the sense that, in the previous section, we fixed a time t and looked at what happened at that time when the spatial structure approaches a continuum, while now, the spatial structure is fixed and we look at what happens when the time goes up to infinity. The reader will note that the main consequence of this return to a discrete spatial structure is to make our spatial model a stochastic model. In particular, our next step is to investigate the existence of nontrivial stationary distributions for the process.

4.1. Oriented percolation

The main method for proving the existence of nontrivial stationary measures, invented by Bramson and Durrett (1988), is based on the comparison of the particle system viewed on suitable length and time scales with oriented percolation. To prepare the comparison, we first introduce oriented percolation and its basic properties. Let

$$\mathcal{L} \ = \ \{(z,n) \in \mathbb{Z}^2 : z+n \text{ is even and } n \geq 0\}.$$

We define random variables $\omega(z,n)$, $(z,n) \in \mathcal{L}$, to be 1 if (z,n) is *open* and to be 0 if (z,n) is *closed.* We will say that there is a *path* from (z,m) to (z',n), or that (z',n) can be reached from (z,m), and write $(z,m) \to (z',n)$, if there is a sequence $z = z_m, \ldots, z_n = z'$ such that

1. For $k = m, \ldots, n-1$, $|z_k - z_{k+1}| = 1$.
2. For $k = m, \ldots, n$, the site (z_k,k) is open, i.e., $\omega(z_k,k) = 1$.

From now on, we assume that the $\omega(z,n)$ are *M-dependent with density* p, i.e.,

$$P\left(\omega(z_i,n_i) = 0 \text{ for } 1 \leq i \leq m\right) \ = \ (1-p)^m$$

for any (z_i,n_i), $1 \leq i \leq m$, such that $\|(z_i,n_i) - (z_j,n_j)\|_\infty > M$ if $i \neq j$. That is, sites that are sufficiently far apart are independent of each other. Let

$$C_0 \ = \ \{(z,n) \in \mathcal{L} : (0,0) \to (z,n)\}$$

be the cluster containing the origin. See Figure 1 for a picture of this cluster when $p = 0.7$ and $p = 0.8$. For our purpose, the event of interest is $\Omega_\infty = \{|C_0| = \infty\}$. If Ω_∞ occurs, we say that percolation occurs. To prepare the proof of the following results, we now let W_n^A be the set of *wet* sites at level n when starting from the initial condition $A \subset 2\mathbb{Z}$, i.e.,

$$W_n^A \ = \ \{z : (x,0) \to (z,n) \text{ for some } x \in A\}.$$

Finally, we let $\tau^A = \inf\{n : W_n^A = \varnothing\}$, and use the short $W_n = W_n^A$ when $A = 2\mathbb{Z}$.

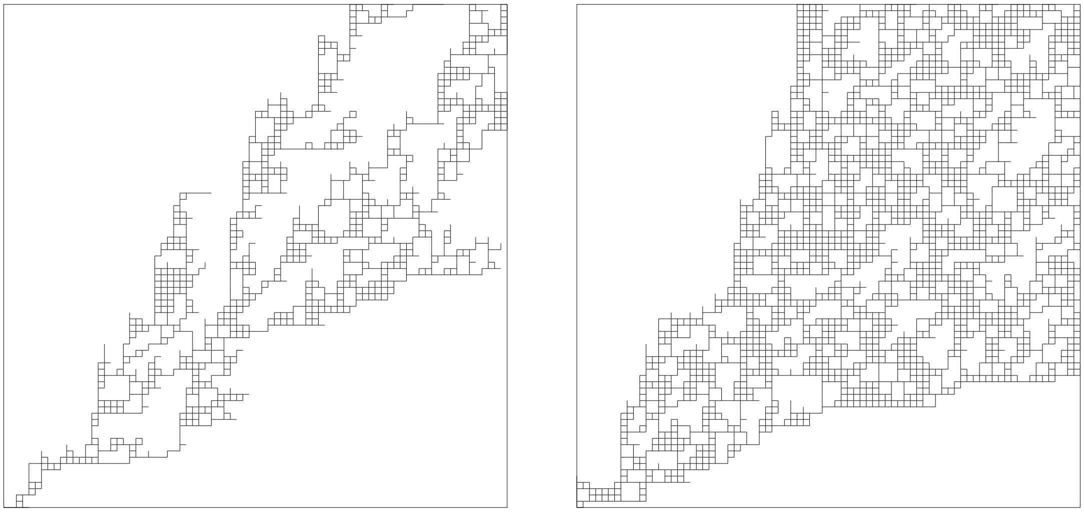

Figure 1. *Pictures of the cluster C_0. The origin $(0,0)$ is at the bottom left corner of the picture. The density of open sites is $p = 0.7$ and $p = 0.8$, respectively.*

Lemma 4.1 *If $p > 1 - 3^{-4(2M+1)^2}$ then $P(\tau^0 = \infty) > 0$.*

Proof The proof is based on a contour argument. To begin with, we prove the result when the sites are assumed to be independent. Let C_N denote the set of wet sites when starting from the initial condition $[-2N, 0]$, and set

$$D \ = \ \{z \in \mathbb{R}^2 : |z_1| + |z_2| \leq 1\} \qquad \text{and} \qquad D_N \ = \ \bigcup_{(z,n) \, \in \, C_N} ((z,n) + D).$$

If the cluster C_N is finite, the contour associated with C_N, denoted by Γ, will be the boundary of the unbounded component of $(\mathbb{R} \times (-1, \infty)) - D_N$. Then, we orient the boundary in such a way that the segment from $(0,-1)$ to $(1,0)$ is oriented in the direction indicated. Now that our contour is defined, the idea of the proof is to estimate

$$P\,(|C_0| < \infty) \ = \ P\,(\Gamma \text{ exists}) \ \leq \ \mathbb{E}\,(\text{number of contours}).$$

We observe that the first segment of the contour is always $(0,-1) \to (1,0)$. Then, there are at most 3 choices at each step. This implies that there are at most 3^{m-1} contours of length m. In other respects, a geometrical argument allows to see that, for a contour of length m to exist, there must be at least $m/4$ closed sites. See Durrett (1984), Section 10, for the details. Finally, since the shortest possible contour has length $2N + 4$, we can conclude that if $p > 1 - 3^{-4}$ then

$$P\,(\tau^{[-2N,0]} < \infty) \ \leq \ \sum_{m=2N+4}^{\infty} 3^{m-1}\,(1-p)^{m/4} \ = \ C\,(3\,(1-p)^{1/4})^{2N}.$$

It is easy to see that the lemma follows for independent site percolation. To deal with the M-dependent case, we observe that, given $(z_0, n_0) \in \mathcal{L}$, there are at most $(2M+1)^2$ sites $(z,n) \in \mathcal{L}$ such that $\|(z_0, n_0) - (z,n)\|_\infty \leq M$. Therefore, for each contour of length m, there is a set of at least $m/4\,(2M+1)^2$ sites which are separated by more than M and which must be closed for the contour to exist. This completes the proof. $\square$

To study the existence of stationary measures, we also need to prove that if we start the process from a *good configuration*, then we will have a positive density of wet sites. To be precise, we introduce $\bar{W}_0^\theta \subset 2\mathbb{Z}$ such that $\{z \in \bar{W}_0^\theta\}$, $z \in 2\mathbb{Z}$, are independent and have probability θ. Finally, let $\bar{W}_n^\theta$ be the set of wet sites at level n when starting from $\bar{W}_0^\theta$.

Lemma 4.2 *If* $p > 1 - 3^{-4(2M+1)^2}$ *and* $\theta > 0$ *then* $\liminf_{n \to \infty} P\,(0 \in \bar{W}_{2n}^\theta) > 0$.

Proof Let A, $B \subset 2\mathbb{Z}$. If we map $\mathcal{L}$ into itself by sending $(z,m) \mapsto (z, 2n-m)$ and reverse the orientation of the graph, the distribution of the process is unchanged, so that

$$P\,(W_{2n}^A \cap B \neq \varnothing) \ = \ P\,(W_{2n}^B \cap A \neq \varnothing).$$

In particular, by choosing $A = \bar{W}_0^\theta$ and $B = \{0\}$, the previous duality equation leads to

$$\begin{aligned} P\,(0 \in \bar{W}_{2n}^\theta) \ &= \ P\,(W_{2n}^0 \cap \bar{W}_0^\theta \neq \varnothing) \ = \ 1 \ - \ \mathbb{E}\,\Big\{(1-\theta)^{|W_{2n}^0|}\Big\} \\ &\geq \ \theta \times P\,(W_{2n}^0 \neq \varnothing) \ \geq \ \theta \times P\,(\tau^0 = \infty). \end{aligned}$$

The result then follows from Lemma 4.1. This completes the proof. $\square$

4.2. The comparison theorem

To make the connection between particle systems and oriented percolation, the first step is to turn the space-time into a brick wall, each brick is associated with a certain *good event.* We consider a translation invariant finite range process $\xi_t : \mathbb{Z}^d \longrightarrow F$ that is constructed from the

Harris' graphical representation given in Section 1, integers L, T and Γ with $T = \Gamma L$, and a collection $\mathcal{H}$ of configurations determined by the value of ξ on $[-L, L]^d$. For any $z \in \mathbb{Z}$, we denote by $\mathcal{H}_z$ the collection $\mathcal{H}$ translated by the vector Lze_1 and say that a site $(z, n) \in \mathcal{L}$ is *occupied* if $\xi_{nT} \in \mathcal{H}_z$. Here, e_1 denotes the first unit vector of the d-dimensional lattice. Finally, we let k_0 and j_0 be two integers, set $M = \max\{j_0, k_0\}$, and introduce the space-time region

$$B_{z,n} \ = \ (Lz\, e_1,\, nT) \ + \ \{[-k_0\, L, k_0\, L]^d \times [0, j_0\, T]\}.$$

Each $(z, n) \in \mathcal{L}$ is associated with a certain good event $E_{z,n}$ measurable with respect to the graphical representation of the process in $B_{z,n}$. See Figure 2 for a picture. Finally, we assume that there exists an $\varepsilon > 0$ small such that

1. $P(E_{z,n}) \geq 1 - \varepsilon$.
2. If (z, n) is occupied and $E_{z,n}$ occurs, then $(z-1, n+1)$ and $(z+1, n+1)$ are occupied.

This holds for any $(z, n) \in \mathcal{L}$. The conditions 1 and 2 above are usually referred to as the *comparison assumption*. To formulate the comparison theorem, we let

$$X_n \ = \ \{z : (z, n) \in \mathcal{L} \text{ with } (z, n) \text{ occupied}\}$$

be the set of occupied sites at level n.

Theorem 4 *If the comparison assumption holds, then the random variables $\omega(z, n)$ can be defined in such a way that X_n dominates an M-dependent oriented percolation process with initial condition $W_0 = X_0$ and density $1 - \varepsilon$, i.e., $W_n \subset X_n$ for any $n \geq 0$.*

Proof The first step is to construct the random variables $\omega(z, n)$ by induction, through a sequence of subsets V_n, $n \geq 0$. To define V_n, we set $V_0 = X_0$ and, for $n \geq 1$, pretend that $z \in V_n$ if there is $x \in V_0$ such that either $(z-1, n-1)$ or $(z+1, n-1)$ can be reached from $(x, 0)$. In particular, $W_n \subset V_n$ so we just need to prove that $V_n \subset X_n$. We assume that $V_n \subset X_n$ at some level $n \geq 0$, and define the $\omega(z, n)$, and hence V_{n+1}, as follows.

1. If $z \in V_n \subset X_n$, we set $\omega(z, n) = 1$ if the good event $E_{z,n}$ occurs, and $= 0$ otherwise.
2. If $z \notin V_n$, the value of $\omega(z, n)$ is unimportant, so we set $\omega(z, n)$ equal to an independent random variable that is 1 with probability $1 - \varepsilon$, and 0 with probability ε.

To prove the inclusion $V_{n+1} \subset X_{n+1}$ at level $n+1$, we first observe that if $z' \in V_{n+1}$ then either $z' - 1 \in V_n$ and $\omega(z'-1, n) = 1$, or $z' + 1 \in V_n$ and $\omega(z'+1, n) = 1$. The point 1 above then implies that either $(z'-1, n)$ is occupied and $E_{z'-1}$ occurs, or $(z'+1, n)$ is occupied and $E_{z'+1}$ occurs. In either case, $z' \in X_{n+1}$. The last thing to check is the M-dependence

$$P\,(\omega(z_i, n_i) = 0 \text{ for } 1 \leq i \leq m) \ \leq \ \varepsilon^{\,m}$$

for any (z_i, n_i), $1 \leq i \leq m$, such that $\|(z_i, n_i) - (z_j, n_j)\|_\infty > M$ if $i \neq j$. To prove this point, we observe that $\omega(z_i, n_i)$ is determined either by independent flip coin, or by the event E_{z_i, n_i}. In the first case, $\omega(z_i, n_i)$ is clearly independent of the $\omega(z_j, n_j)$ for $j \neq i$. In the second case, we observe that, since $M = \max\{j_0, k_0\}$, the boxes B_{z_j, n_j}, $1 \leq j \leq m$, are disjoint, which implies that the events E_{z_j, n_j} are independent. This completes the proof. □

The last step is to deduce from the comparison theorem the existence of a nontrivial stationary distribution for our particle system. First of all, we assume that the comparison assumption holds for some $\varepsilon < 3^{-4(2M+1)^2}$ where $M = \max\{j_0, k_0\}$. We start the process ξ_t from a translation invariant configuration such that the events $\{z \in X_0\}$, $z \in 2\mathbb{Z}$, are independent and have probability $\theta > 0$. Then, run the process to time S, take the Cesaro average of the distribution at times $0 \leq s \leq S$, and extract a convergent subsequence. By Liggett (1985), Proposition 1.8, the limit μ is known to be a stationary measure. We now set $W_0 = X_0$. Then, Lemma 4.2 implies that $\liminf_{n \to \infty} P\,(0 \in W_{2n}) > 0$. Since ξ_0 is translation invariant, Theorem 4 allows us to deduce that, for any $z \in \mathbb{Z}$, $\mu\,(\xi \in \mathcal{H}_z) > 0$, so that μ has the desired property.

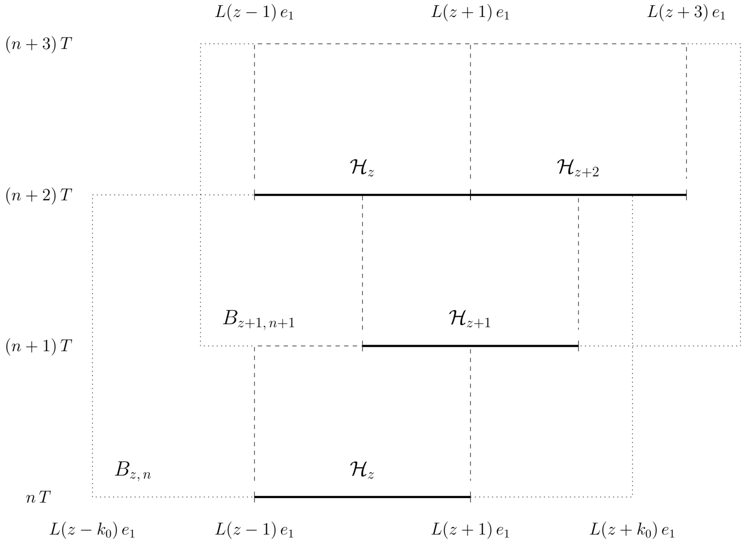

Figure 2. *Picture of the rescaling argument.*

4.3. A continuity result

We now consider a system $\xi_t : \mathbb{Z}^d \longrightarrow F$ whose transition rates $c_i\,(x,\xi) = c_i\,(\lambda, x, \xi)$ depend on a parameter $\lambda \in D$, where D is an open set, and assume that one succeeds in proving the existence of a stationary distribution for the process with parameter $\lambda_0 \in D$, i.e., the evolution is described by the transition rates $c_i\,(\lambda_0, x, \xi)$, through the rescaling argument introduced in § 4.2. The objective of this paragraph is to prove that, under some continuity assumptions, there is a neighborhood $\mathcal{V}$ of λ_0 in D such that the comparison with supercritical oriented site percolation still holds for the set of processes with parameter $\lambda \in \mathcal{V}$ (see Lanchier, 2005c). This tells us that our particle system still exhibits the same behavior after a slight perturbation of the evolution. From now on, this result will be referred to as *continuity result*. The argument is very useful and has been proved in many special cases. See for instance Durrett and Neuhauser (1997), Schinazi (2001, 2002) and Lanchier (2005a). In this paragraph, we prove the result in the general case.

To formulate our result, we consider a particle system $\xi_t : \mathbb{Z}^d \longrightarrow F$ whose transition rates continuously depend on $\lambda \in D$. That is, for any $\lambda_0 \in D$, we have

(CA) For any $\delta > 0$, there exists a neighborhood $\mathcal{V}$ of λ_0 in D such that

$$|c_i\,(\lambda, 0, \xi)\; - \; c_i\,(\lambda_0, 0, \xi)| \; < \; \delta \qquad \forall\; \lambda \in \mathcal{V}, \quad \forall\; i \in F, \quad \forall\; \xi \in F^{\mathbb{Z}^d}.$$

From now on, we denote by $\bar{\xi}_t$ the process with parameter λ_0. Finally, given all the ingredients introduced in § 4.2, we suppose that, for any $\varepsilon > 0$, the comparison assumption holds for the process with parameter λ_0. More precisely,

($P_{\lambda_0}^{\varepsilon}$) The parameters L and Γ can be chosen sufficiently large so that

1. $P\,(E_{z,n}) \geq 1 - \varepsilon$ and
2. If (z,n) is occupied and $E_{z,n}$ occurs, then $(z-1, n+1)$ and $(z+1, n+1)$ are occupied.

Here, (z,n) occupied means that $\bar{\xi}_{nT} \in \mathcal{H}_z$. That is, the assumption $P_{\lambda_0}^{\varepsilon}$ applies to the particle system with parameter λ_0. Let P_{λ_0} be the property that $P_{\lambda_0}^{\varepsilon}$ is satisfied for any $\varepsilon > 0$.

Theorem 5 *Assume CA and P_{λ_0}. Then, for any $\varepsilon > 0$, there is a neighborhood $\mathcal{V}_{\varepsilon}$ of λ_0 in D such that the following holds: For any $\lambda \in \mathcal{V}_{\varepsilon}$, the comparison assumption $P_{\lambda}^{\varepsilon}$ is satisfied.*

As previously explained, Theorem 5 tells us that if λ is close to λ_0 then the processes with parameters λ and λ_0 exhibit the same behavior in the sense that each of both processes has a stationary distribution ν such that, for any $z \in \mathbb{Z}$, $\nu(\xi \in \mathcal{H}_z) > 0$. But beyond a simple continuity result, Theorem 5 gives us insights into the strategy to study the behavior of a particle system. More precisely, we get the following

Corollary 6 *Assume that the process ξ_t exhibits a phase transition at $\lambda_0 \in D$. Then, the behavior of the process when $\lambda = \lambda_0$ cannot be investigated through the comparison result.*

The proof of Corollary 6 is straightforward. Assume that one succeeds in constructing a stationary measure ν for the process with parameter λ_0 through the comparison result, that is proving that the assumption P_{λ_0} holds. Then, Theorem 5 tells us that there is a neighborhood $\mathcal{V}$ of λ_0 in D such that, for any $\lambda \in \mathcal{V}$, the processes with parameter λ and λ_0 exhibit the same behavior, which is not consistent with the existence of a phase transition at λ_0 and proves Corollary 6.

Before going into the details of the proof, we start by observing that, typically, the transition rates $c_i(\lambda, x, \xi)$, $i \in F$, are linear functions of the parameter λ. See, e.g., Durrett (1995). In particular, even though essential, the continuity assumption CA is not very restrictive so that our results can be plentifully applied. Finally, we think that Theorem 5 does not hold in the very useful case $\lambda_0 = \infty$ and $\lambda > 0$ large (see for instance the discussion following Theorem 3 of Lanchier and Neuhauser (2005)). The continuity result when $\lambda_0 = \infty$ and λ is large, however, holds in some special cases, and the strategy of our proof is general enough to be easily adapted to such cases provided that one has an explicit description of the process to be investigated. See for instance Lanchier (2005b) or Lanchier and Neuhauser (2005) for illustrations. The rest of the paragraph is devoted to the proof of Theorem 5.

To begin with, we prove that, with probability close to 1, the dual process $\bar{\xi}_s^{(x,T)}$ starting at (x, T) does not escape from a spatial box to be fixed later. The objective is to fix the appropriate space and time scale so that the comparison assumption holds.

The first step is to construct the process $\bar{\xi}_t$ from a graphical representation and to define its dual process. Let $\{T_n^{x,i} : n \geq 1\}$, $x \in \mathbb{Z}^d$, $i \in F$, be a collection of independent Poisson processes with parameter

$$c_i(\lambda_0) \; = \; \sup \left\{ c_i(\lambda_0, 0, \xi) : \xi \in F^{\mathbb{Z}^d} \right\}$$

and define the evolution of the process $\bar{\xi}_t$ from these exponential clocks as explained in Section 1. The dual process $\bar{\xi}_s^{(x,T)}$ is then defined by going backwards in time. In a general way, it allows us to deduce the color of site x at time T from the configuration at earlier time by keeping track of the ancestors. The dual process starts at $\bar{\xi}_0^{(x,T)} = \{x\}$ and makes transitions as follows.

If $z \in \bar{\xi}_s^{(x,T)}$ and $T_n^{z,i} = t - s$, then we add the points of $z + \mathcal{N}$ to $\bar{\xi}_s^{(x,T)}$

where $\mathcal{N}$ denotes the interaction neighborhood. From the previous definition, it is easy to see that we can determine the color of (x, T) by knowing the colors of sites $z \in \bar{\xi}_s^{(x,T)}$ at time $T - s$. The dual process is naturally defined only for $0 \leq s \leq T$ but it is convenient to assume that the Poisson processes in the construction are defined for negative times and $\bar{\xi}_s^{(x,T)}$ for all $s \geq 0$.

Lemma 4.3 *There exists a convex set $A \subset \mathbb{R}^d$ such that for any $\delta > 0$*

$$P\left((1-\delta)\, t\, A \cap \mathbb{Z}^d \subset \bar{\xi}_t^{(0,T)} \subset (1+\delta)\, t\, A \cap \mathbb{Z}^d\right) \to 1 \qquad \text{as} \quad t \to \infty.$$

Proof The basic idea is that $\bar{\xi}_s^{(0,T)}$ is almost a Richardson's model that is linearly growing in time. Unfortunately, since our dual process increases by "blocks", we did not find any coupling argument to prove the domination by a Richardson's model. The proof of Durrett (1988), however, can be easily extended to our model, so we just give the key ideas and refer the reader to the last reference for the details. The first step is to introduce, for any x, $y \in \mathbb{Z}^d$, the stopping times

$$t(x) \; = \; \inf \left\{ s \geq 0 : x \in \bar{\xi}_s^{(0,T)} \right\} \qquad \text{and} \qquad t(x,y) \; = \; \inf \left\{ s \geq 0 : y \in \bar{\xi}_{s+t(x)}^{(x,T-t(x))} \right\}.$$

For a given site $x \in \mathbb{Z}^d$ and positive integers m and n, let $X_{m,n} = t(mx, nx)$ be the amount of time it takes the dual process starting at site mx to reach nx. Then, $\{X_{m,n} : m \leq n\}$ satisfies the hypothesis of Liggett (1985), Theorem 2.6, Chapter 6, which implies that

$$\frac{X_{0,n}}{n} \ = \ \frac{t(nx)}{n} \ \longrightarrow \ \mu(x) \quad \text{a.s.} \qquad \text{as} \quad n \to \infty.$$

This tells us that $\bar{\xi}_s^{(0,T)}$ grows linearly in each direction and proves the result in $d = 1$. To prove the result in $d \geq 2$, we first turn $\bar{\xi}_s^{(0,T)}$ into a solid blob by letting

$$\tilde{\xi}_s^{(0,T)} \ = \ \Big\{ x + y : x \in \bar{\xi}_s^{(0,T)} \ \text{and} \ y \in [-1/2, 1/2]^d \Big\}.$$

This allows us to extend the definitions of $t(x)$ and of $\mu(x)$ to $x \in \mathbb{R}^d$, the d-dimensional space with real coordinates. This new definition makes μ a norm on $\mathbb{R}^d$. The shape result then follows by taking $A = \{x : \mu(x) \leq 1\}$, the unit ball in that norm. $\square$

Lemma 4.4 *Let $T = \Gamma L$. There exists an integer ℓ_0 such that*

$$P\left(\{\bar{\xi}_s^{(x,T)} : 0 \leq s \leq T\} \subset [-\ell_0 L, \ell_0 L]^d \ \textit{ for any } \ x \in [-2L, 2L]^d\right) \ \to \ 1 \qquad \textit{as} \quad L \to \infty.$$

Proof This is a straightforward consequence of Lemma 4.3 supplemented with a coupling argument. We fix $t_0 > 0$ such that $[-4, 4]^d \subset t_0 A$. Then, the shape result implies that

$$\begin{aligned} & P\left(\{\bar{\xi}_s^{(x,T)} : 0 \leq s \leq T\} \not\subset 2\,(T + t_0 L)\,A \ \text{ for some } \ x \in [-2L, 2L]^d\right) \\ & \quad \leq \ P\left([-2L, 2L]^d \not\subset \bar{\xi}_{t_0 L}^{(0, T + t_0 L)}\right) \ + \ P\left(\bar{\xi}_{T + t_0 L}^{(0, T + t_0 L)} \not\subset 2\,(T + t_0 L)\,A\right) \ \to \ 0 \end{aligned}$$

as $L \to \infty$. The result then follows by taking ℓ_0 such that $2\,(\Gamma + t_0)\,A \subset [-\ell_0, \ell_0]^d$. $\square$

We are now ready to prove Theorem 5. In what follows, we will denote by $\bar{\xi}_t$ the process with fixed parameter $\lambda_0 \in D$, i.e., the process with transition rates $c_i\,(\lambda_0, x, \xi)$, and will denote by ξ_t the process with parameter $\lambda \in D$ close to λ_0. For more convenience, the objects associated with $\bar{\xi}_t$ will be written as their analogous for ξ_t with a bar in addition.

Given an integer $\bar{M} \geq 1$, a collection $\mathcal{H}$ of configurations determined by the value of ξ on the spatial box $[-L, L]^d$, and a collection $\bar{E}_{z,n}$, $(z, n) \in \mathcal{L}$, of good events that are measurable with respect to the graphical representation of the process $\bar{\xi}_t$ in the space-time region

$$\bar{B}_{z,n} \ = \ (Lze_1, \, nT) \ + \ \{[-k_0 L, k_0 L]^d \ \times \ [0, \, j_0 T]\}$$

with $\max\{j_0, k_0\} = \bar{M}$, we assume that

(P_{λ_0}) For any $\varepsilon > 0$, the parameters L and Γ can be chosen sufficiently large so that

1. $P\,(\bar{E}_{z,n}) \geq 1 - \varepsilon$ and
2. If (z, n) is occupied and $\bar{E}_{z,n}$ occurs, then $(z - 1, n + 1)$ and $(z + 1, n + 1)$ are occupied.

Here, (z, n) occupied means that $\bar{\xi}_{nT} \in \mathcal{H}_z$. The proof of Theorem 5 is a three-step process. The basic idea is to define, for $(z, n) \in \mathcal{L}$, a new good event $E_{z,n}$ to be the intersection of three good events $E_{z,n}^j$, $j = 1, 2, 3$. This event will have to insure us that the assumption P_λ^ε holds for the process with parameter λ close to λ_0.

The first good event will be $E_{z,n}^1 = \bar{E}_{z,n}$. Now, to make sure that $E_{z,n}$ is measurable with respect to the graphical representation in some box $B_{z,n}$, we first let $E_{z,n}^2$ be the event that

$$\{\bar{\xi}_s^{(x,(n+1)T)} : 0 \leq s \leq T\} \subset [(z - \ell_0)\,L, (z + \ell_0)\,L]^d \qquad \text{for any} \quad x \in [(z-2)\,L, (z+2)\,L]^d.$$

The event $E_{z,n}^2$ seems to be the bad one since it does not say anything about the process with parameter λ. The third good event, however, will allow us to solve the problem. To define $E_{z,n}^3$, the first step is to construct both processes $\bar{\xi}_t$ and ξ_t by using the same Harris' graphical representation. More precisely, for any color $i \in F$, we let

$$c_i\,(\lambda, \lambda_0) \;=\; \max\,(c_i\,(\lambda), c_i\,(\lambda_0)) \;=\; \sup\,\Big\{\max\,(c_i\,(\lambda, 0, \xi), c_i\,(\lambda_0, 0, \xi)) : \xi \in F^{\mathbb{Z}^d}\Big\}$$

and let $\{T_n^{x,i} : n \geq 1\}$, $x \in \mathbb{Z}^d$, be a collection of Poisson processes with parameter $c_i\,(\lambda, \lambda_0)$, all of them are independent. To complete the construction, for any $x \in \mathbb{Z}^d$ and $i \in F$, we toss a coin with success probability

$$p_i\,(\lambda, \lambda_0) \;=\; \frac{|c_i\,(\lambda) - c_i\,(\lambda_0)|}{c_i\,(\lambda, \lambda_0)} \;=\; \frac{|c_i\,(\lambda) - c_i\,(\lambda_0)|}{\max\,(c_i\,(\lambda), c_i\,(\lambda_0))}.$$

If there is a success, we put a ω_0 at point $(x, T_n^{x,i})$ if $c_i\,(\lambda) < c_i\,(\lambda_0)$, and a ω at point $(x, T_n^{x,i})$ if on the contrary $c_i\,(\lambda) > c_i\,(\lambda_0)$. Then, the process $\bar{\xi}_t$ (resp. ξ_t) can be constructed as explained in Section 1 by using the exponential clock devoid of ω (resp. ω_0). In other words, the process with parameter λ_0 does not see the ω's while the process with parameter λ does not see the ω_0's. Finally, we let m_0 denote an integer to be fixed later and set

$$B_{z,n} \;=\; (Lze_1,\, n\Gamma L) \;+\; \{[-m_0\, L, m_0\, L\,]^d \,\times\, [\,0,\, j_0\,\Gamma L\,]\}.$$

Then, $E_{z,n}^3$ will be the event that none of the exponential clocks in $B_{z,n}$ is labeled, i.e., there is neither ω nor $\bar{\omega}$ in the space-time region $B_{z,n}$.

Now that our three good events $E_{z,n}^j$, $j = 1$, 2, 3, are defined, the next step is to prove that our choice of $E_{z,n}$ implies Theorem 5. To begin with, we let $\varepsilon > 0$ and apply the comparison assumption P_{λ_0} to pick L and Γ so that $P\,(E_{z,n}^1) = P\,(\bar{E}_{z,n}) \geq 1 - \varepsilon/\,3$. In other respects, Lemma 4.4 implies the existence of an integer ℓ_0 independent of L such that, for L sufficiently large, $P\,(E_{z,n}^2) \geq 1 - \varepsilon/\,3$. This last observation allows us to fix the size of $B_{z,n}$ by setting $m_0 = \max\,\{k_0, \ell_0\}$. Now that L and Γ are fixed so that $E_{z,n}^1$ and $E_{z,n}^2$ occur with probability close to 1, we are ready to estimate our third event. A straightforward calculation shows that

$$P\,(E_{z,n}^3) \;\geq\; 1 \;-\; (2\,m_0\,L + 1)^d \,\sum_{i=0}^{\kappa-1}\, \big\{\,1 - \exp\,\big(-|c_i\,(\lambda) - c_i\,(\lambda_0)|\; j_0\,\Gamma L\,\big)\big\}\,.$$

This together with the continuity assumption CA implies the existence of a neighborhood $\mathcal{V}_\varepsilon$ of λ_0 in D such that $P\,(E_{z,n}^3) \geq 1 - \varepsilon/\,3$, for any $\lambda \in \mathcal{V}_\varepsilon$. In conclusion, we have proved that there exist parameters L and Γ, and a neighborhood $\mathcal{V}_\varepsilon$ of λ_0 such that $P\,(E_{z,n}) \geq 1 - \varepsilon$, for any $\lambda \in \mathcal{V}_\varepsilon$.

We are now ready for the final denouement. First of all, we observe that the event $E_{z,n}$ is measurable with respect to the graphical representation in $B_{z,n}$. That is, the configuration of the process at time $(n+1)\,T$ in the box $B_{z-1} \cup B_{z+1}$ only depends on what happens in the space-time region $B_{z,n}$. In particular, if $M = \max\,\{j_0, m_0\}$ then, whenever $\|(z_i, n_i) - (z_j, n_j)\|_\infty > M$, the events E_{z_i,n_i} and E_{z_j,n_j} are independent. To prove Theorem 5, the last thing we have to check is that if (z, n) is occupied and $E_{z,n}$ occurs, then $(z-1, n+1)$ and $(z+1, n+1)$ will be occupied as well, where "occupied" now applies to the process with parameter λ. This results from a coupling argument. More precisely, we assume that $\xi_{nT} \in \mathcal{H}_z$, and set $\bar{\xi}_{nT} = \xi_{nT}$. If both processes are constructed from the same graphical representation, it is easy to see that on $E_{z,n}^2 \cap E_{z,n}^3$

$$\bar{\xi}_s^{(x,(n+1)T)} \;=\; \xi_s^{(x,(n+1)T)} \quad \text{for any} \quad x \,\in\, [(z-2)\,L, (z+2)\,L\,]^d \quad \text{and} \quad 0 \,\leq\, t \,\leq\, T.$$

The properties of the dual process (see above), together with $\bar{\xi}_{nT} = \xi_{nT}$ then implies that

$$\bar{\xi}_{(n+1)T}(x) \;=\; \xi_{(n+1)T}(x) \quad \text{for any} \quad x \,\in\, [(z-2)\,L, (z+2)\,L\,]^d.$$

Finally, if our good event $E_{z,n}$ occurs, the event $E_{z,n}^1$ tells us that $\bar{\xi}_{(n+1)T} \in \mathcal{H}_{z-1} \cap \mathcal{H}_{z+1}$. In conclusion, $\xi_{(n+1)T} \in \mathcal{H}_{z-1} \cap \mathcal{H}_{z+1}$ and Theorem 5 follows.

4.4. Host-symbiont interactions

To conclude this section and figure out how the comparison theorem introduced in § 4.2 works through a concrete instance, we now investigate a particle system modeling the local interactions between hosts and symbionts. An important component of multi-host, multi-symbiont models is the degree to which different symbionts and hosts can associate with each other. This will be referred to as *specificity.* A *specialist* symbiont associates with a very small number of hosts; a *generalist* symbiont associates with many hosts. The ease of transmission of a symbiont to a host, referred to as *transmissibility,* is another important factor in host-symbiont interactions.

The spatially explicit model

To describe the host dynamics, we employ the simplest of all multi-species models, namely the voter model (Holley and Liggett, 1975, Clifford and Sudbury, 1973). Into this population, we introduce symbionts with varying degrees of specificity and transmissibility. More precisely, our model is a continuous-time Markov process $\xi_t : \mathbb{Z}^d \longrightarrow \{1, 2, \ldots, \kappa\} \times \{0, 1, \ldots, \kappa\}$ where the integer κ denotes both the number of hosts and the number of symbionts involved in the interaction. A site $x \in \mathbb{Z}^d$ is said to be occupied by an *unassociated* host of type i if $\xi(x) = (i, 0)$, and by a host of type i *associated* with a symbiont of type j if $\xi(x) = (i, j)$. Later on, we will write $\xi_t(x) = (\xi_t^1(x), \xi_t^2(x))$, where $\xi_t^1(x)$ (resp. $\xi_t^2(x)$) denotes the type of the host (resp. the symbiont) present at x at time t, with $\xi_t^2(x) = 0$ denoting the absence of a symbiont. The evolution at site $x \in \mathbb{Z}^d$ is described by the transition rates

$$(i,j) \ \to \ (k,0) \quad \text{at rate} \quad \lambda \sum_{0<\|x-z\|\leq R_1} \left\{ \mathbb{1}_{\{\xi(z)=(k,0)\}} \ + \ g \sum_{\ell=1}^{\kappa} \mathbb{1}_{\{\xi(z)=(k,\ell)\}} \right\}$$

$$(i,0) \ \to \ (i,j) \quad \text{at rate} \quad c_{ij} \sum_{0<\|x-z\|\leq R_2} \sum_{\ell=1}^{\kappa} \mathbb{1}_{\{\xi(z)=(\ell,j)\}}.$$

The birth rate of unassociated hosts is equal to λ. The parameter g indicates the variation of the birth rate of hosts associated with a symbiont. If $0 \leq g < 1$, the symbiont is a *pathogen*; if $g = 1$, the symbiont has no effect on the birth rate of the host; if $g > 1$, the symbiont is a *mutualist.* The parameters c_{ij} denote the rate at which symbiont j infects host i. Finally, births and infections occur within a local neighborhood, with R_1 denoting the birth range of hosts, and R_2 the infection range of symbionts. The neighborhoods of a site $x \in \mathbb{Z}^d$ will be denoted by $\mathcal{N}_x^1$ and $\mathcal{N}_x^2$, and their cardinality by ν_{R_1} and ν_{R_2} respectively.

To define the time scale, we set $\lambda = 1$. We will discuss the specialist case by setting $c_{ii} = \beta > 0$ for $i = 1, 2, \ldots, \kappa$, and $c_{ij} = 0$ for $i, j = 1, 2, \ldots, \kappa$ with $i \neq j$, and provide comparisons with the corresponding mean-field model. To begin with, we consider a population of two hosts with only one symbiont, say symbiont 1, and start the evolution with all the hosts of type 1 associated with a symbiont. Then, in the limiting case $\beta = \infty$, the transition $(1, 0) \to (1, 1)$ is instantaneous provided that $R_1 \leq R_2$ to avoid the problem of *isolated* unassociated host that cannot be reached by any symbiont. This implies that the process ξ_t^1 is a biased voter model in which hosts of type 1 give birth at rate g and hosts of type 2 at rate 1. In particular, if $g < 1$, then $\xi_t^1 \Rightarrow \delta_2$, the "all 2" configuration, while if $g > 1$, then $\xi_t^1 \Rightarrow \delta_1$, the "all 1" configuration. Theorem 7 tells us that, in any dimension, the result still holds if $g > 1$ and $\beta \in (0, \infty)$ is sufficiently large.

Theorem 7 *Assume that $g > 1$ and $R_1 \leq R_2$. At time 0, $\xi_0(x) = (1, 1)$ or $(2, 0)$ for all $x \in \mathbb{Z}^d$. Then, there is a critical value $\beta_c \in (0, \infty)$ such that if $\beta > \beta_c$ then $\xi_t^1 \Rightarrow \delta_1$.*

The proof of Theorem 7 will give us an illustration of the techniques introduced in § 4.2 and § 4.3. As explained above, the first coordinate process ξ_t^1 performs a biased voter model in the limiting case $\beta = \infty$. Hence, we will start by proving a general result about the biased voter model, and then apply a perturbation argument to extend this result to the region $\beta < \infty$.

Let $\beta_1,\,\beta_2 \in (0,\infty)$, and $\eta_t : \mathbb{Z}^d \longrightarrow \{1,\,2\}$ be the biased voter model with parameters β_1 and β_2, i.e., the process whose state at site x changes as follows.

$$i \ \rightarrow\ j \qquad \text{at rate} \qquad \beta_j \sum_{0<\|x-z\|\leq R} \mathbb{1}_{\{\eta(z)=j\}} \qquad \text{with} \quad R \ =\ R_1.$$

It is a well-known fact that if $\beta_1 > \beta_2$ then $P\,(\eta_t(x) = 1) \to 1$ as $t \to \infty$, provided that at time 0, the process has infinitely many 1's (see, e.g., Durrett (1988), Chapter 3).

To prove Theorem 7, we will follow the strategy described in Durrett and Neuhauser (1997), Section 3. We begin with a rescaling argument to estimate the rate of convergence of $P\,(\eta_t(x) = 1)$. This estimate will have to be good enough so that a perturbation argument can be applied. As explained in § 4.2, the basic idea is to show that for given $\varepsilon > 0$, members of the family of processes under consideration, when viewed on suitable length and time scales, dominate an M-dependent oriented percolation process in which sites are open with probability $1-\varepsilon$ (Durrett, 1995). To compare the process with a percolation process, we consider a positive integer L to be fixed later, and scale space by setting

$$B \ =\ [-L,\,L\,]^d, \qquad \Phi(z) \ =\ L\,z, \qquad \text{and} \qquad B_z \ =\ \Phi(z) \ +\ B$$

for any $z \in \mathbb{Z}^d$. Let Γ be a positive integer, and say that (z,n) is *occupied* if all sites in B_z are occupied by 1's at time $n\,\Gamma L$. The first step in proving Theorem 7 is the following

Proposition 4.5 *Let $\varepsilon > 0$ and $\beta_1 > \beta_2$. Then L and Γ can be chosen in such a way that the set of occupied sites dominates the set of open sites in a 3-dependent oriented site percolation process where sites are open with probability $p = 1 - 2\,\varepsilon/\,3$.*

The key to the proof is duality (Durrett 1988, Chapter 3). To define the dual process of the biased voter model, we consider two collections of independent Poisson processes $\{T_n^{x,z} : n \geq 1\}$ and $\{U_n^{x,z} : n \geq 1\}$ where $0 < \|x-z\| \leq R$, with parameter β_2 and $\beta_1 - \beta_2$ respectively. At times $T_n^{x,z}$ we draw an arrow from z to x and put a δ at site x, while at times $U_n^{x,z}$ we draw an arrow from z to x without putting a δ at the tip. The process is then obtained from the graphical representation as follows. At time $T_n^{x,z}$, the particle at x imitates the one at z. At time $U_n^{x,z}$, the site x becomes occupied by a particle of type 1 if z is. We say that there is a *path* from $(x,0)$ to (z,t) if there is a sequence of times $s_0 = 0 < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 \ =\ x,\, x_1,\, \ldots,\, x_n \ =\ z$ such that the following two conditions hold.

1. For $i = 1,\,2,\,\ldots,\,n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0,\,1,\,\ldots,\,n$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any δ's.

Finally, we say that there exists a *dual path* from (x,t) to $(z,t-s)$, $0 \leq s \leq t$, if there is a path from $(z,t-s)$ to (x,t), and define the *dual process starting at* (x,t) by setting

$$\hat\eta_s^{(x,t)} \ =\ \{z \in \mathbb{Z}^d : \text{there is a dual path from } (x,t) \text{ to } (z,t-s)\}$$

for any $0 \leq s \leq t$. The reason why we introduce the dual process is that it allows us to deduce the state of site x at time t from the configuration at earlier times. More precisely,

$$\eta_t(x) \ =\ 1 \qquad \text{if and only if} \qquad \eta_{t-s}(z) \ =\ 1 \quad \text{for some} \quad z \ \in\ \hat\eta_s^{(x,t)}.$$

See Durrett (1988), Chapter 3. The strategy to proving Proposition 4.5 can then be summarized as follows. Let $T = \Gamma\,L$ and $x \in B_z$ with $\|\,z\,\| = 1$. Then, we will prove that, with probability arbitrarily close to 1, there exists a dual path A_s starting at (x,T) and landing in the target set B. More precisely, we will prove that A_s hits the set $J = [-R,\,R\,]^d$ by time T where $R = R_1 < L$, and then stays inside B until time T. In particular, if B is void of 2's at time 0 then, with probability close to 1, B_z will be void of 2's as well T units of time later. To define the dual path A_s, we start the process at $A_0 = (x,T)$ and go down the graphical representation. If A_s comes across a δ at some time $s = T - T_n^{x,z}$ with $x = A_s$ then move A_s to z. If A_s meets the tip of an arrow that is without a δ at some time $s = T - U_n^{x,z}$ then move A_s to z only if it takes it closer to 0. Intuitively, this should cause A_s to drift towards the set B. We now make this argument precise.

Lemma 4.6 *Assume that* $x \in B_z$, $\| z \| = 1$, *and* $\beta_1 > \beta_2$. *There exist* $C_1, \gamma_1 \in (0, \infty)$ *such that*

$$\sup_{x \in B_z} \ P_x \, (A_s \notin J \textit{ for all } s \leq T) \ \leq \ C_1 \, \exp\,(-\gamma_1 \, L)$$

for L *and* Γ *sufficiently large. Here, the subscript* x *indicates the starting point.*

Proof Let σ_k denote the k-th time A_s encounters the tip of an arrow (with or without a δ). At time σ_k, the arrow does not have a δ at its tip with probability $(\beta_1 - \beta_2)/\beta_1 > 0$. Moreover, if $A_{\sigma_k} \notin J$ and the arrow does not have a δ at its tip, then with probability at least $1/2d > 0$, A_s moves closer to 0. In particular, if $N = \inf\,\{k \geq 1 : A_{\sigma_k} \in J\}$ then there is $c > 0$ such that

$$P\,(N \geq c\,L) \ \leq \ C_2 \, \exp\,(-\gamma_2 \, L)$$

for suitable $C_2, \gamma_2 \in (0, \infty)$. Since $P\,(\sigma_k - \sigma_{k-1} > t) = \exp\,(-\beta_1 \, t)$, the result follows. □

Lemma 4.7 *Assume that* $\beta_1 > \beta_2$. *For any* $y \in J$ *there exist* $C_3 < \infty$ *and* $\gamma_3 > 0$ *such that*

$$\sup_{y \in J} P_y \, (A_s \notin B \textit{ for some } s \leq T) \ \leq \ C_3 \, \exp\,(-\gamma_3 \, L)$$

for L *sufficiently large.*

Proof We let $s_0 = 0$ and, for $k \geq 1$, define the following stopping times

$$\begin{array}{rcl} t_k & = & \inf\,\left\{t > s_{k-1} : A_t \notin (-L/2, \, L/2)^d\right\} \\ s_k & = & \inf\,\{t > t_k : A_t \in J\} \quad \text{and} \quad \tau \ = \ \inf\,\{t > 0 : A_t \notin B\}. \end{array}$$

Moreover, we denote by $M(t) = \sup\,\{k \geq 1 : \sigma_k < t\}$ the number of tips of arrows encountered by A_s by time t. Then for any site $y \in J$

$$\begin{array}{rcl} P_y \, (A_s \notin B \text{ for some } s \leq T) & = & P_y \, (A_{\sigma_k} \notin B \text{ for some } k \leq M(T)) \\ & \leq & P_y \, (A_{\sigma_k} \notin B \text{ for some } k \leq 2\,\beta_1 \, T) \ + \ P\,(M(T) > 2\,\beta_1 \, T) \\ & \leq & P_y \, (s_k > \tau \text{ for some } k \leq 2\,\beta_1 \, T) \ + \ P\,(M(T) > 2\,\beta_1 \, T) \\ & \leq & 2\,\beta_1 \, T \ \sup_{z \in J} P_z \, (s_1 > \tau) \ + \ P\,(M(T) > 2\,\beta_1 \, T). \end{array}$$

Since A_s has a drift towards J and that the time between consecutive jumps has exponential bound, $P\,(s_1 > \tau) \leq C_4 \, \exp\,(-\gamma_4 \, L)$ for appropriate $C_4 < \infty$ and $\gamma_4 > 0$ (see the proof of Lemma 3.2). Furthermore, since $\mathbb{E}\, M(T) = \beta_1 \, T$, large deviation estimates imply that there are $C_5 < \infty$ and $\gamma_5 > 0$ such that $P\,(M(T) > 2\,\beta_1 \, T) \leq C_5 \, \exp\,(-\gamma_5 \, T)$. □

Lemma 4.8 *Assume that* $x \in B_z$, $\| z \| = 1$, *and* $\beta_1 > \beta_2$. *There exist* $C_6, \gamma_6 \in (0, \infty)$ *such that*

$$\sup_{y \in B_z} \ P_x \, (A_T \notin B) \ \leq \ C_6 \, \exp\,(-\gamma_6 \, L)$$

for Γ *and* L *sufficiently large.*

Proof By decomposing according to whether $A_s \in J$ for some $s \leq T$ or not, we obtain

$$P_x \, (A_T \notin B) \ \leq \ P_x \, (A_s \notin J \text{ for all } s \leq T) \ + \ P\,(A_T \notin B \ ; \ A_s \in J \text{ for some } s \leq T).$$

The first term on the right-hand side can be bounded using Lemma 4.6. For the second term, we first observe that

$$P\,(A_T \notin B \ ; \ A_s \in J \text{ for some } s \leq T) \ \leq \ \sup_{y \in J} P_y \, (A_s \notin B \text{ for some } s \leq T)$$

and then apply Lemma 4.7. This completes the proof. $\square$

Since there are at most $(2L+1)^d$ sites in B_z, it follows from Lemma 4.8 and duality that there is a constant $C_7 < \infty$ independent of L such that for Γ and L sufficiently large

$$\begin{aligned} P\,(\eta_T(x) = 2 \text{ for some } x \in B_z) \ &\leq \ \sum_{x \in B_z} P_x\,(A_T \not\in B) \\ &\leq \ (2L+1)^d\, C_6 \,\exp\,(-\gamma_6\, L) \ \leq \ C_7\, L^{-1} \ \leq \ \varepsilon\,/\,3. \end{aligned}$$

Moreover, since each of the dual paths has a drift toward J, we can fix $M > 0$, say $M = 3$, so that for any $\varepsilon > 0$

$$\begin{aligned} &P\,(\text{any of the selected paths is not contained in} \\ &\qquad\qquad [-ML,\, ML]^d \text{ at some time } s \leq T) \ \leq \ \varepsilon\,/\,3 \end{aligned}$$

by choosing L sufficiently large. This shows that boxes that are sufficiently far apart are independent of each other with high probability and completes the proof of Proposition 4.5.

To deduce Theorem 7 from Proposition 4.5, we now rely on a perturbation argument. More precisely, we will extend the result to the region $\beta > 0$ large by proving that if hosts of type 1 become occupied by their associated mutualists quickly enough, then ξ_t^1 will evolve like a biased voter model in the space-time box $B \times [0, \Gamma L]$ with probability close to 1. We first define ξ_t on the same space as the biased voter model η_t introduced above with $\beta_1 = g$ and $\beta_2 = 1$. At time $T_n^{x,w}$, the host present at site w gives birth to an unassociated host of the same type which is then sent to x. At time $U_n^{x,w}$, the birth from w to x occurs only if the host at w is associated with a mutualist. To describe the evolution of the mutualists, we consider one more collection of independent Poisson processes, $\{V_n^{x,w} : n \geq 1\}$, $0 < \|x - w\| \leq R_2$, with parameter β. At time $V_n^{x,w}$, we draw an arrow labeled with a 1 from w to x to indicate that a mutualist (of type 1) present at site w gives birth to a mutualist at site x if this site is already occupied by a host of type 1. We will prove that there exists $\beta_c \in (0, \infty)$ such that if $\beta > \beta_c$ and $\xi_0^1 = \eta_0$ on B, then $\xi_t^1 = \eta_t$ on B_z with $\|\,z\,\| = 1$ at time $t = \Gamma L$ with probability $\geq 1 - \varepsilon\,/\,3$. Since boxes that are sufficiently far apart are independent of each other with probability close to 1, we can focus on $[-ML, ML]^d \times [0, \Gamma L]$, $M = 3$, to estimate this event. Let $x \in [-ML, ML]^d$ and follow the line $\{x\} \times [0, \Gamma L]$ by going forward in time. Each time a host at w attempts to give birth at site x, we require that the next 1-arrow from w to x appears before the host at w is replaced or the host at x gives birth. An easy calculation shows that this event occurs with probability

$$\begin{aligned} &P\,\big(V_1^{x,w} < \min\,(T_1^{y,x}, U_1^{y,x}) \text{ for any } y \in \mathcal{N}_x^1 \quad \text{and} \\ &\qquad\qquad V_1^{x,w} < \min\,(T_1^{w,y}, U_1^{w,y}) \text{ for any } y \in \mathcal{N}_w^1\,\big) \ = \ \beta\,(\beta + 2m)^{-1} \end{aligned}$$

where $m = g\,\nu_{R_1}$. We now denote by $K(x,t)$ the number of unlabeled arrows and δ-arrows that point at site x by time $t > 0$, and set $I_M = [-ML, ML]^d$. Then, by observing that $\mathbb{E}\,K(x,T) = m\,T$, and by decomposing the event to be estimate according to whether $K(x,T) > 2m\,T$ or not, we finally obtain

$$\begin{aligned} &P\,(\xi_t^1 \neq \eta_t \text{ on } B_z \text{ at time } t = \Gamma L) \\ &\qquad \leq \ \sum_{x \,\in\, I_M} P\,(K(x,T) > 2\,m\,T) \ + \ 2\,m\,T \sum_{x \,\in\, I_M} \frac{2\,m}{\beta + 2\,m} \\ &\qquad\qquad \leq \ (2ML)^d \ \times \ \big\{C_8 \,\exp\,(-\gamma_8\, T) \ + \ 4\,m^2\, T\,(\beta + 2m)^{-1}\,\big\} \end{aligned}$$

for appropriate $C_8 < \infty$ and $\gamma_8 > 0$. Taking L and then β sufficiently large so that

$$P\,(\xi_t^1 \neq \eta_t \text{ on } B_z \text{ at time } t = \Gamma L) \ \leq \ \varepsilon\,/\,3,$$

and applying Proposition 4.5 imply that the set of occupied sites dominates the set of open sites in an oriented percolation process with parameter $p = 1 - \varepsilon$. Here (z, n) occupied means that all sites in B_z are occupied by associated hosts of type 1 at time nT. This almost produces Theorem 7. Our last problem is that oriented site percolation has a positive density of unoccupied sites. To prove that there is an in-all-directions expanding region which is void of hosts of type 2, we apply a result from Durrett (1992) which shows that unoccupied sites do not percolate when ε is close enough to 0. Since hosts of either type cannot appear spontaneously, once a region is void of one type, this type can only reappear in the region through invasion from the outside. This then implies that our process has the desired property and completes the proof of Theorem 7.

The mean-field model

The next step is to compare the spatially explicit and the mean-field models in order to provide a new illustration of the purpose introduced in Section 2. To define the mean-field model, we pretend that all sites are independent and that the system is spatially homogeneous. This then results in a system of differential equations for the densities of unassociated and associated hosts. To deal with the case of specialist interactions, we let u_i denote the density of unassociated hosts of type i and v_{ii} denote the density of hosts of type i associated with a symbiont (of the same type). We define

$$u_{\cdot} \;=\; \sum_{i=1}^{\kappa} u_i \qquad \text{and} \qquad v_{\cdot\cdot} \;=\; \sum_{i=1}^{\kappa} v_{ii}.$$

The mean-field limit is obtained by letting both neighborhood ranges, R_1, R_2, go to infinity. To obtain a meaningful limit, we need to rescale the parameters λ and β by their respective neighborhood sizes, that is, we set $\lambda = \frac{1}{\nu_{R_1}}$ (this also sets the time scale), and define $\beta = \frac{b}{\nu_{R_2}}$. The following system of differential equations then describes the mean-field behavior.

$$\begin{aligned} u_i' &= (1-u_i)\,(u_i + g\,v_{ii}) \;-\; u_i \sum_{j \neq i} (u_j + g\,v_{jj}) \;-\; b\,u_i\,v_{ii} \\ v_{ii}' &= b\,u_i\,v_{ii} \;-\; v_{ii}\,(u_{\cdot} + g\,v_{\cdot\cdot}). \end{aligned}$$

When $g = 1$, the system has a conserved quantity, namely the initial host densities $h_i = u_i + v_{ii}$, $i = 1, 2, \ldots, \kappa$. If $b > \kappa$ then for $g \neq 1$, there exists a nontrivial equilibrium with $u_1 = u_2 = \cdots = u_\kappa \geq 0$ and $v_{11} = v_{22} = \cdots = v_{\kappa\kappa} > 0$. Numerical simulations indicate that the nontrivial equilibrium is locally stable when $g < 1$, but not when $g > 1$.

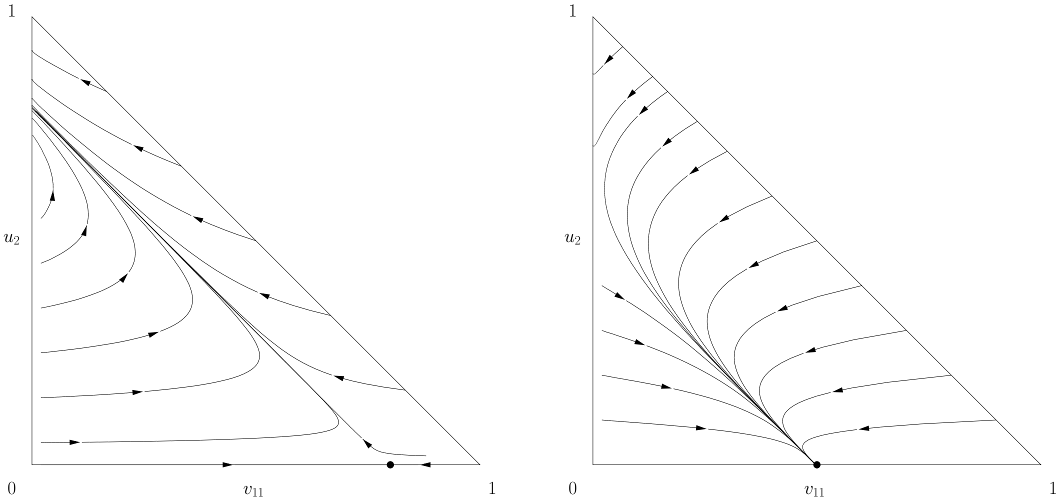

FIGURE 3. *Solution curves of the mean-field model with* $b = 3$. *The parameter* g *is equal to 0.5 and 2, respectively.*

If initially only two hosts and one symbiont, say symbiont 1, are present then for $g < 1$, pathogen of type 1 will go extinct and both hosts may coexist. See Figure 3, picture on left, for an illustration. We will conjecture similar behavior for the spatial model, except in the 1-dimensional, nearest neighbor case. When $g > 1$, host of type 2 goes extinct provided the initial density of mutualists of type 1 is sufficiently large. See Figure 3, picture on right, for an illustration. The spatial analogue of this result is given by Theorem 7 above.

Comparison of the spatially explicit and the mean-field models

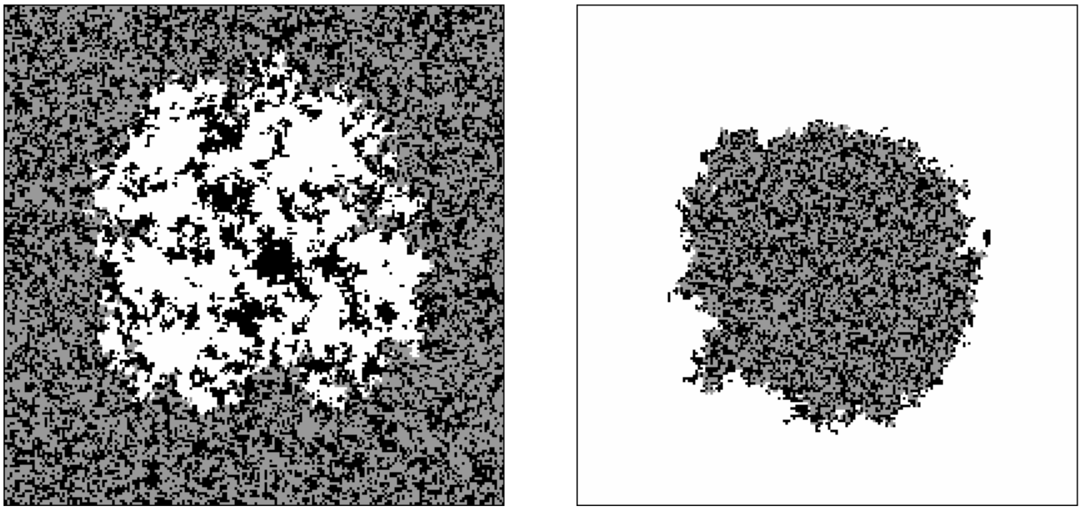

Figure 4. *Process with nearest neighbor interactions on the* 200×200 *torus at time 250. Picture on left:* $\beta = 2$ *and* $g = 0.5$. *The process starts with unassociated white hosts in* $J = (90, 110)^2$ *and black hosts associated with a pathogen outside* J. *Picture on right:* $\beta = 4$ *and* $g = 2$. *The process starts with black hosts associated with a mutualist in* J *and unassociated white hosts outside* J. *In both pictures, gray sites refer to black hosts associated with a symbiont.*

In conclusion, numerical simulations of the mean-field model and Theorem 7 indicate similar behaviors for the spatial and nonspatial models with only two hosts and one symbiont when $g > 1$. In other respects, a careful reading of the proof of Theorem 7 shows that the analogous result holds when $g < 1$, that is $P(\xi_t^1(x) = 2) \to 1$ for any $x \in \mathbb{Z}^d$, but only in the 1-dimensional nearest neighbor case. Numerical simulations of the mean-field model reveal a quite different behavior, namely pathogen will go extinct and both hosts coexist. See Figure 3, picture on left. We think, however, that except in the 1-dimensional nearest neighbor case, $P(\xi_t^1(x) = 2) \not\to 1$. Here is our intuition. We first observe that, if $d \geq 2$ or $R_1 \geq 2$, then the dynamics produce *isolated* hosts, that is hosts of type 1 (resp. 2) surrounded by a cluster of hosts of type 2 (resp. 1). As suggested by Theorem 7, when $g > 1$, isolated 2's are *swallowed* very quickly by surrounding 1's. On the other hand, when $g < 1$, an isolated 1 located in a linearly growing cluster of 2's cannot be invaded anymore by any pathogen as soon as the cluster has reached some critical size. In this expanding region, the process then behaves like an unbiased voter model in which 1's can now compete with 2's, and survive. See Figure 4 for simulations in $d = 2$. In particular, we conjecture similar behaviors for the mean-field model and the spatially explicit model in $d \geq 3$, in the sense described in Section 2. That is, in both spatial and nonspatial models pathogen will go extinct and hosts coexist (see Holley and Liggett, 1975). The result is consistent with the observations of Section 2, namely there is a critical dimension d_0 such that the spatial and nonspatial models exhibit the same behavior if and only if $d \geq d_0$. Moreover, the behavior of the particle system when $d = 1$ and $R_1 = 1$, that is when the spatial interactions are the most emphasized, contrasts with the mean-field model.

5. Geometry of particle systems and duality

We have seen in Section 4 that the construction of stationary measures through the comparison theorem requires that we scale the space-time of the particle system by turning it into a brick wall. Moreover, to make sure in practice that the comparison assumption holds for ε small, the scale parameters L and T have to be chosen sufficiently large. In particular, even if the comparison theorem is a powerful tool to investigate the *macroscopic* behavior, or stationary measures, of a given particle system, it does not tell us anything about its *microscopic* behavior, or the geometry of the configurations. A typical phenomenon we think about in particular is called *clustering*. Rather than describing with thousands of words what we mean by clustering, we prefer to refer the reader to Figures 5 and 6 for pictures of the two processes for which this property has been proved. To have a mathematical definition, let's consider a particle system $\xi_t : \mathbb{Z}^d \longrightarrow F$. We will say that clustering occurs if, for colors $i,\ j \in F$ with $i \neq j$, we have

$$P\,(\xi_t(x) = i,\ \xi_t(y) = j) \ \to \ 0 \qquad \text{as} \ \ t \ \to \ \infty.$$

To study the patterns drawn by a particle system, the appropriate tool is called *duality*. The basic idea is to pick a site $x \in \mathbb{Z}^d$, a time $t \geq 0$, and to work backwards in time, starting from the point (x,t), in order to reconstruct the family tree of the particle present at site x at time t. The resulting process, denoted by $\hat{\xi}_s^{(x,t)}$, is called the *dual process starting at* (x,t). Such a process has already been introduced above in the case of the biased voter model. See the proof of Theorem 7, in § 4.4. In a general way, the consideration of this process leads to a *duality relationship* which allows us to deduce the color of site x at time t from the configuration at earlier time. For this reason, we will call the elements of $\hat{\xi}_s^{(x,t)}$, the *ancestors* of (x,t) at time $t-s$.

With the dual process in hands, we are now ready to describe a general method to prove that clustering occurs. Even if the proof can become a real brain twister, the basic idea is easy to understand. To begin with, we pick two sites $x,\ y \in \mathbb{Z}^d$, and a time $t \geq 0$, and consider the dual processes $\hat{\xi}_s^{(x,t)}$ and $\hat{\xi}_s^{(y,t)}$ starting at (x,t) and (y,t) respectively. Typically, both processes exhibit some *tree structure*, this topological structure allows to define an *ancestor hierarchy* in which the members are arranged according to the order they determine the color of the starting point. We denote respectively by $X_s^{(x,t)}$ and $X_s^{(y,t)}$ the spatial locations of the first ancestor of (x,t) and (y,t) at time $t-s$. Since both $X_s^{(x,t)}$ and $X_s^{(y,t)}$ are constructed from the same exponential clocks, the process $X_s^{(x,t)} - X_s^{(y,t)}$ is stopped when it hits 0. For this reason, we will call them *coalescing processes*. In particular, if one can prove that, by looking backwards in time, both ancestors will eventually collide with probability 1, then

$$P\,(X_t^{(x,t)} \, = \, X_t^{(y,t)}) \ \to \ 1 \qquad \text{as} \ \ t \ \to \ \infty,$$

which, by duality, is equivalent to clustering. On the contrary, if the convergence above does not hold, then duality implies that, at least starting from a good configuration, i.e., a translation invariant product measure in which all the colors are present, coexistence occurs. We now make this argument precise through two concrete examples of particle systems.

5.1. A warming up example: The voter model

The voter model, investigated in details in Holley and Liggett (1975), is the first example of particle system for which clustering has been proved. Before stating their results, we remind the reader that the basic voter model is the translation invariant process $\xi_t : \mathbb{Z}^d \longrightarrow \{0,\,1\}$, whose state at site x flips according to the following transition rates

$$0 \ \to \ 1 \quad \text{at rate} \sum_{0<\|x-z\|\leq R} \mathbb{1}_{\{\xi(z)=1\}} \qquad\qquad 1 \ \to \ 0 \quad \text{at rate} \sum_{0<\|x-z\|\leq R} \mathbb{1}_{\{\xi(z)=0\}}.$$

That is, at some constant rate, the particle at site x picks at random a site $z \in \mathbb{Z}^d$ with $\|x-z\| \leq R$, and mimics the particle present at site z. See Figure 5 for realizations of the process. As previously explained, to study the geometry of a spatial model, the first step is to define the dual process. In a general way, the dual process allows to keep track of the ancestors of a given particle by working backwards in time. To begin with, we construct the particle system from a graphical representation. For any x, $z \in \mathbb{Z}^d$, let $\{T_n^{x,z} : n \geq 1\}$ be the arrival times of a Poisson process with parameter 1. At times $T_n^{x,z}$, we draw an arrow from z to x and put a $\times$ at site x to indicate that the particle at x mimics the particle at z. The dual process is then constructed from the graphical representation as in § 4.4 for the biased voter model. We will say that there exists a *path* from $(x, 0)$ to (z, t) if there is a sequence of times $s_0 = 0 < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 = x, x_1, \ldots, x_n = z$ such that the following two conditions hold.

1. For $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0, 1, \ldots, n$, the vertical segment $\{x_i\} \times (s_i, s_{i+1})$ does not contain any $\times$'s.

Finally, we will say that there exists a *dual path* from (x, t) to $(z, t - s)$, $0 \leq s \leq t$, if there is a path from $(z, t - s)$ to (x, t), and define the *dual process starting at* (x, t) by setting

$$\hat{\xi}_s^{(x,t)} \ = \ \{z \in \mathbb{Z}^d : \text{there is a dual path from } (x,t) \text{ to } (z, t-s)\}$$

for any $0 \leq s \leq t$. In the case of the voter model, however, since all the arrows are now associated with a $\times$, the process $\hat{\xi}_s^{(x,t)}$ is reduced to a single site which has the interpretation that the particle present at site x at time t has the same color as the particle present at site $\hat{\xi}_s^{(x,t)}$ at time $t - s$. Moreover, it is easy to see from the graphical representation that $\hat{\xi}_s^{(x,t)}$ performs a continuous-time random walk which jumps from z to $w \in z + \mathcal{N}$ at rate 1. With the ingredients we have just introduced, we are now ready to prove the

Theorem 8 *If $d \leq 2$, clustering occurs, i.e., for any initial configuration ξ_0, we have*

$$\lim_{t \to \infty} P\,(\xi_t(x) \ \neq \ \xi_t(y)) \ = \ 0 \qquad \forall\ x,\, y \ \in \ \mathbb{Z}^d.$$

Proof First of all, the duality described above implies that if $\hat{\xi}_t^{(x,t)} = \hat{\xi}_t^{(y,t)}$, then the particles at x and y will have the same ancestor at time t, and so the same color. It follows that

$$P\,(\xi_t(x) \ \neq \ \xi_t(y)) \ \leq \ P\,(\hat{\xi}_t^{(x,t)} \ \neq \ \hat{\xi}_t^{(y,t)}).$$

Now, since $\hat{\xi}_s^{(x,t)}$ and $\hat{\xi}_s^{(y,t)}$ are constructed from the same collection of independent Poisson processes, the dual process performs coalescing random walks, i.e., $\hat{\xi}_s^{(x,t)} - \hat{\xi}_s^{(y,t)}$ is stopped when it hits 0. Finally, since our random walks also have mean 0 and finite variance, they are recurrent in $d \leq 2$. In particular, $P\,(\hat{\xi}_t^{(x,t)} \neq \hat{\xi}_t^{(y,t)}) \to 0$ as $t \to \infty$. This completes the proof. □

In higher dimensions, the argument of the proof of Theorem 8 does not hold anymore, since in this case we loose the recurrence property. In fact, the spatial model exhibits a sharp change of behavior when increasing the dimension, namely clustering in $d \leq 2$ and coexistence in $d \geq 3$. More precisely, we have the following result.

Theorem 9 *Assume that $d \geq 3$. If the events $\{\xi_0(x) = 1\}$, $x \in \mathbb{Z}^d$, are independent and have probability θ, then $\xi_t \Rightarrow \mu$, a translation invariant stationary measure with $\mu\,(\xi\,(x) = 1) = \theta$.*

Proof Let $B \subset \mathbb{Z}^d$, and $\hat{\xi}_s^{(B,t)} = \{\hat{\xi}_s^{(x,t)} : x \in B\}$. If we think of ξ_t as a set valued process in which the state at time t is the set of sites occupied by 1's, then by duality

$$P\,(\xi_t \cap B \ = \ \varnothing) \ = \ \mathbb{E}\left\{(1-\theta)^{|\hat{\xi}_t^{(B,t)}|}\right\}.$$

Now, since as explained above $\hat{\xi}_t^{(B,t)}$ is a coalescing random walk, the function $t \mapsto |\hat{\xi}_t^{(B,t)}|$ is nonincreasing, and so has a limit when $t \to \infty$. Moreover, $0 \leq (1-\theta)^{|\hat{\xi}_t^{(B,t)}|} \leq 1$, so it follows from the bounded convergence theorem that

$$\lim_{t \to \infty} \ \mathbb{E}\left\{(1-\theta)^{|\hat{\xi}_t^{(B,t)}|}\right\} \quad \text{exists.}$$

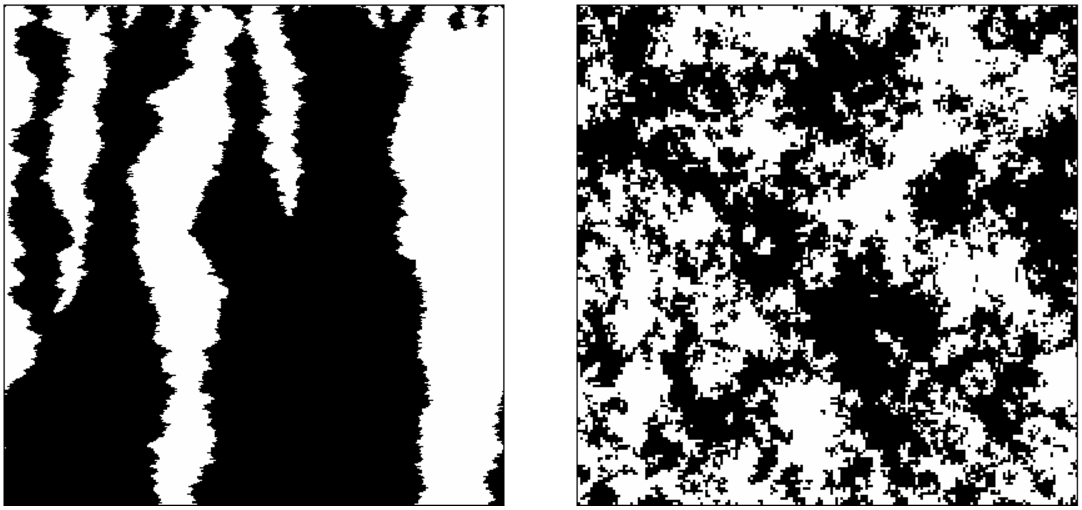

Figure 5. *Realizations of the voter model. The duality between the particle system and coalescing random walks implies a clustering in dimensions 1 and 2.*

This implies that ξ_t converges to a stationary measure μ as $t \to \infty$. We now prove that μ satisfies the desired properties. First of all, since the process is constructed from a translation invariant collection of Poisson processes, μ will be translation invariant as well. In other respects,

$$P(x \in \xi_t) \ = \ P(\hat{\xi}_t^{(x,t)} \in \xi_0) \ = \ \theta.$$

Finally, taking the limit as $t \to \infty$ in the previous equation implies the result. $\square$

To figure out one more time the connection between spatial and nonspatial models, we observe that if all the sites are assumed to be independent, then the densities of particles are preserved by the dynamics, i.e., with the usual notations, $u_0' = u_1' = 0$. This implies that the mean-field model exhibits a one-parameter family of locally stable fixed points. The behavior predicted by the deterministic model only occurs in $d \geq 3$ for the spatial model. As indicated by the proofs of Theorems 8 and 9, this comes from the intrinsic properties of symmetric random walks. However, even without being aware of the proofs, the result is easy to understand: To increase the dimension of the space reduces the dependence between adjacent sites. In particular, the existence of a critical dimension d_0 such that spatial and nonspatial models exhibit the same behavior if and only if $d \geq d_0$ is not so surprising.

5.2. The multitype contact process

The second model that exhibits such a dichotomy, introduced by Neuhauser (1992), is the multitype contact process, i.e., the continuous-time Markov process whose state at time t is a function $\xi_t : \mathbb{Z}^d \longrightarrow \{0,1,2\}$, and whose transition rates at site $x \in \mathbb{Z}^d$ are given by

$$0 \ \to \ 1 \quad \text{at rate} \quad \lambda_1 \sum_{0<\|x-z\|\leq R} \mathbb{1}_{\{\xi(z)=1\}} \qquad\qquad 1 \ \to \ 0 \quad \text{at rate} \quad 1$$

$$0 \ \to \ 2 \quad \text{at rate} \quad \lambda_2 \sum_{0<\|x-z\|\leq R} \mathbb{1}_{\{\xi(z)=2\}} \qquad\qquad 2 \ \to \ 0 \quad \text{at rate} \quad 1.$$

In words, each 1 (resp. 2) tries to give birth onto each of its neighbors at rate λ_1 (resp. λ_2). If the offspring is sent to an empty site, that is a site in state 0, the birth occurs, otherwise, it is suppressed. Moreover, each particle dies at rate 1. In the following, we will investigate the process when $\lambda_1 = \lambda_2$, and will denote by λ the common value of both parameters.

To prove that clustering occurs in $d \leq 2$, the strategy is the same as for the voter model. We will pick two sites x, $y \in \mathbb{Z}^d$, a time $t \geq 0$, and work backwards in time through the dual process to follow the ancestors of the particles present at (x, t) and (y, t). Then, clustering will occur if both ancestors coalesce almost surely together. The main difficulty is that the dual process starting at a single site is now much more complicated than a simple random walk.

To introduce the dual process, the first step is to construct our particle system from a graphical representation. For any x, $z \in \mathbb{Z}^d$ with $\|x - z\| \leq R$, let $\{T_n^{x,z} : n \geq 1\}$ and $\{U_n^x : n \geq 1\}$ be the arrival times of independent Poisson processes with rate λ and 1 respectively. The graphical representation is obtained by drawing an arrow from site x to site z at times $T_n^{x,z}$ to indicate that, if the site x is occupied and the site z is empty, then the particle at x gives birth to a particle of the same type at z. At times U_n^x, we put a × at site x to indicate that any particle present at x is killed. Paths and dual paths are defined as in § 5.1 (see the conditions 1 and 2, page 50), and the dual process starting at (x, t) is given by

$$\hat{\xi}_s^{(x,t)} \ = \ \{z \in \mathbb{Z}^d : \text{there is a dual path from } (x,t) \text{ to } (z, t-s)\}.$$

From the previous definition, it is easy to see that the state of (x, t) can be determined by knowing the states of sites $z \in \hat{\xi}_s^{(x,t)}$ at time $t - s$. The dual process is naturally defined for $0 \leq s \leq t$ only but it is convenient to assume that the Poisson processes in the construction are defined for negative times and $\hat{\xi}_s^{(x,t)}$ for all $s \geq 0$. Unlike the voter model, $\{(\hat{\xi}_s^{(x,t)}, s) : s \geq 0\}$ exhibits a *tree structure*. This tree structure allows to define an *ancestor hierarchy* in which the members are arranged according to the order they determine the color of (x, t). Later on, the first ancestor in this hierarchy will be called the *distinguished particle*, and its location at time $t - s$ denoted by $\hat{\xi}_s^{(x,t)}(1)$. See Figure 7, picture on left, for an example of ancestor hierarchy and an illustration of the path of the distinguished particle.

Even if the dual process seems to be a complicated object, it has a nice property, namely the path of the distinguished particle can be broken at certain points into independent and identically distributed pieces. These points are called *renewal points*. This, together with estimates on how the tree behaves between the renewal points, allows us to trace the history of the tree by looking at the location of the renewal points.

The renewal points of the dual process starting at (x, t) are well defined only if (x, t) *lives forever*, that is $\hat{\xi}_s^{(x,t)} \neq \varnothing$ for any $s \geq 0$, an event with positive probability if $\lambda > \lambda_c$, the critical value of the basic contact process. See, e.g., Liggett (1999). We start the dual process at (x, t) and follow the path of the distinguished particle. Whenever the particle jumps to a site that lives forever, we will call this point a renewal point. Let the spatial displacement between consecutive renewal points be X_i and the corresponding temporal displacement be τ_i so that

$$S_n \ = \ x \ + \ \sum_{i=1}^n \ X_i \qquad \text{and} \qquad T_n \ = \ \sum_{i=1}^n \ \tau_i$$

will be respectively the spatial and temporal locations of the nth renewal point (see Figure 7, picture on right). Then, we have the following

Proposition 5.1 *If the starting point (x, t) lives forever, $\{(X_i, \tau_i)\}_{i \geq 1}$ form an i.i.d. family of random vectors on $\mathbb{Z}^d \times \mathbb{R}^+$. Moreover, we have the exponential bounds*

$$P\,(\|X_i\| > t) \ \leq \ C\,e^{-\beta t} \qquad \textit{and} \qquad P\,(\tau_i > t) \ \leq \ C\,e^{-\beta t}$$

for appropriate $C < \infty$ and $\beta > 0$.

See Neuhauser (1992), Section 2, for a proof. As previously explained, the process $\hat{\xi}_s^{(x,t)}(1)$ is more complicated than a simple random walk. Proposition 5.1, however, tells us that the renewal points define an *embedded random walk* for the distinguished particle. This, together with the exponential

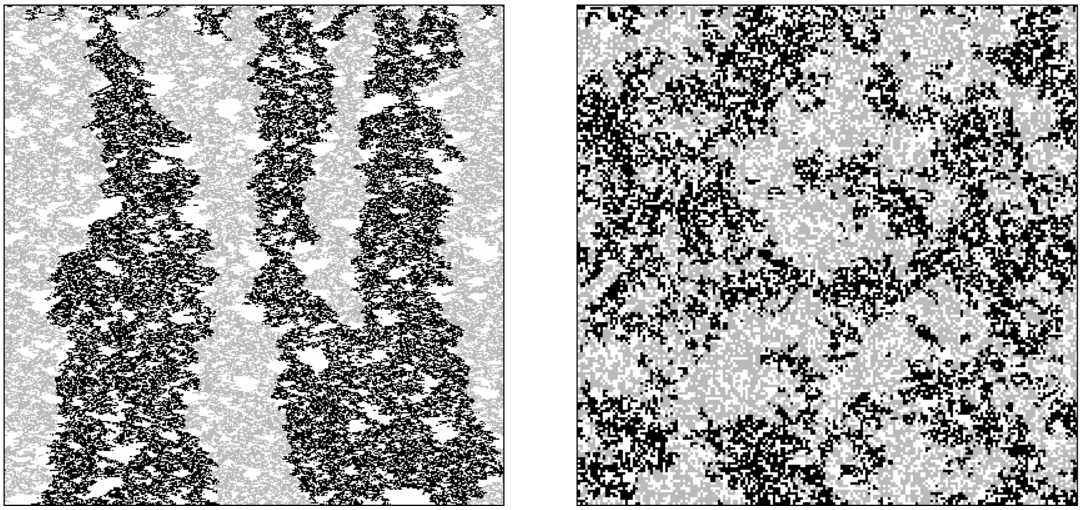

Figure 6. *Realizations of the 1 and 2-dimensional multitype contact process. The particle system exhibits the same behavior as the voter model, that is clustering in $d \le 2$ and coexistence in $d \ge 3$.*

bounds on the spatial and temporal displacement, implies that our random walk is recurrent in $d \le 2$, and transient in $d \ge 3$. Moreover, since the tree structure is linearly growing in time, the distinguished particle will stay within a set linearly growing in time which we will call *triangle*. Whenever a renewal occurs, the next triangle starts at the bottom of the preceding one. Hence, we obtain the picture of a chain of connected triangles where we can find our distinguished particle with probability close to 1.

To prove that clustering occurs in $d \le 2$, we pick two sites x, $y \in \mathbb{Z}^d$, and keep track of the distinguished particles starting at (x,t) and (y,t), respectively. By Proposition 5.1, there exists a finite distance K such that, with probability close to 1, both ancestors move independently to each other without collision of their triangles as long as they are separated by more than K. Recurrence implies that, in $d \le 2$, we can bring the distinguished particles within a finite distance K, this occurs infinitely often. Finally, as soon as both particles are within K, there is a positive probability that they coalesce. In conclusion, we get the

Lemma 5.2 *Let* x, $y \in \mathbb{Z}^d$. *If* $d \le 2$ *then* $P\,(\hat{\xi}_t^{(x,t)}(1) \neq \hat{\xi}_t^{(y,t)}(1)) \to 0$ *as* $t \to \infty$.

The behavior when $d \ge 3$ is quite different. Transience of random walks, together with the exponential bounds given in Proposition 5.1, implies that there is a positive probability that the triangles of both distinguished particles do not collide. More precisely, we have the following

Lemma 5.3 *Let* $\|x - y\| \ge K$. *If* $d \ge 3$ *then there exists* $C < \infty$ *such that*

$$P\,(\|\,\hat{\xi}_t^{(x,t)}(1) - \hat{\xi}_t^{(y,t)}(1)\| \;\ge\; t^{1/8} \text{ for all } t \ge 0) \;\ge\; 1 - C\,K^{-1/10} - \;2\,C\,K^{-3/32}.$$

By working a little bit more, and relying on the same arguments as in the proofs of Theorems 8 and 9, we can deduce from Lemmas 5.2 and 5.3 that

Theorem 10 *If* $\lambda_1 = \lambda_2$ *and* $d \le 2$, *clustering occurs, i.e., for any initial configuration* ξ_0

$$\lim_{t \to \infty} P\,(\xi_t(x) \,\neq\, \xi_t(y)) \;=\; 0 \qquad \forall\; x,\, y \,\in\, \mathbb{Z}^d.$$

Theorem 11 *If* $\lambda_1 = \lambda_2$ *and* $d \ge 3$, *coexistence occurs, i.e., there is a translation invariant stationary distribution* ν *such that* $\nu\,(\xi(x) = 1) \neq 0$ *and* $\nu\,(\xi(x) = 2) \neq 0$.

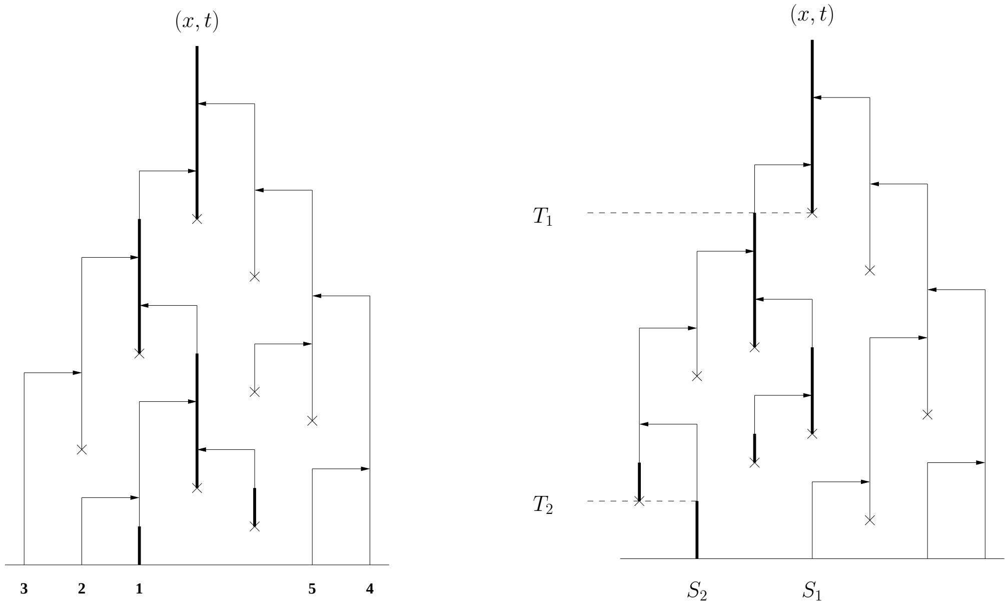

FIGURE 7. *Pictures of the dual process.*

5.3. More duality: The successional model

The aim of this last paragraph is to investigate in detail the duality properties of the successional model introduced in Lanchier (2005a), and to give, through this example, a new application of the notion of renewal points described previously. Our spatial model is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \longrightarrow \{0,1,2\}$. A site $x \in \mathbb{Z}^d$ is said to be empty if $\xi(x) = 0$, and occupied by a particle of type 1 (resp. 2) if $\xi(x) = 1$ (resp. 2). The evolution of our successional model is given by the following transitions at site x.

$$0 \;\to\; 1 \quad \text{at rate} \quad \lambda_1 \sum_{0<\|x-z\|\le R} \mathbb{1}_{\{\xi(z)=1\}} \qquad\qquad 1 \;\to\; 2 \quad \text{at rate} \quad 1$$

$$0 \;\to\; 2 \quad \text{at rate} \quad \lambda_2 \sum_{0<\|x-z\|\le R} \mathbb{1}_{\{\xi(z)=2\}} \qquad\qquad 2 \;\to\; 0 \quad \text{at rate} \quad 1.$$

In words, each 1 (resp. 2) tries to give birth onto each of its neighbors at rate λ_1 (resp. λ_2). If the offspring is sent to an empty site, the birth occurs, otherwise, it is suppressed. Each 1 becomes 2 at rate 1. Finally, each 2 dies, that is becomes 0, at rate 1 as well.

We called our process successional model to refer to the ecological succession phenomenon it describes the evolution. In our ecological context, 1's are the individuals of a so-called *pioneer* species that can invade a hostile environment. The appearance of this first species then helps the development of a second species, here the particles of type 2. In plant communities, the spontaneous transition 1 → 2 is explained by the presence of a seed bank in the soil.

We first observe that, if only 2's are present, the system reduces to the basic contact process with parameter λ_2. In this case, there exists a critical value $\lambda_c \in (0,\infty)$ such that the following holds: If $\lambda_2 \le \lambda_c$ then the process converges in distribution to the all empty state. Otherwise, there exists a stationary measure μ_2 with $\mu_2(\xi(x) = 2) > 0$. See, e.g., Liggett (1999). To avoid trivialities, we now start the evolution with infinitely many 1's and 2's, and focus on the behavior of the particle system when both λ_1 and λ_2 are supercritical. The case $\lambda_1 < \lambda_2$ is easy to deal with. We denote by ξ_t^1 the process with parameters λ_1 and λ_2, and by ξ_t^2 the process obtained from ξ_t^1 by replacing the transition 1 → 2 by 1 → 0. This makes ξ_t^2 a multitype contact process with parameters λ_1 and λ_2. Then, a standard coupling argument allows to define both processes on the same space, starting from the same configuration, in such a way that the successional model ξ_t^1 has more 2's and fewer 1's than the multitype contact process ξ_t^2. Theorem 1 of Neuhauser (1992) then implies that both processes exhibit the same behavior, i.e.,

Theorem 12 *If $\lambda_2 > \lambda_1 > \lambda_c$ then $\xi_t \Rightarrow \mu_2$.*

However, since the spontaneous transition $1 \to 2$ prevents the 1's from surviving without the 2's, the behavior of our successional model when $\lambda_1 \geq \lambda_2$ is quite different of that of the multitype contact process. The next step is to prove that 2's still win if $\lambda_1 = \lambda_2$. Although Theorem 13 is not surprising, the proof relies on a somewhat interesting property of the dual process.

Theorem 13 *If $\lambda_1 = \lambda_2 > \lambda_c$ and ξ_0 is translation invariant then $\xi_t \Rightarrow \mu_2$.*

The rest of this section is devoted to the proof of Theorem 13, which is based on duality. The first step will be to figure out the geometry of the dual process. In the case of the successional model, the dual process will exhibit a tree structure divided into two stages that we will call *upper* and *lower* trees. As for the multitype contact process introduced above, the path of the ancestors which will determine the color of (x,t) can be broken into independent and identically distributed pieces. In particular, in view of the sharp change of behavior of symmetric random walks depending on the dimension, the strategy will be different in $d \leq 2$ and $d \geq 3$. The reader will note, however, that the process now exhibits the same behavior in any dimension.

Geometrical properties of the dual process

To describe the dual process, we start by constructing the particle system from a collection of independent Poisson processes. First of all, we fix $\lambda_1 = \lambda_2$, denote by λ their common value, and, for x, $z \in \mathbb{Z}^d$ with $\|x - z\| \leq R$, let $\{T_n^{x,z} : n \geq 1\}$ be the arrival times of independent Poisson processes with rate λ. At times $T_n^{x,z}$, we draw an arrow from site x to site z to indicate that, if the site x is occupied and the site z is empty, then the particle at x gives birth to a particle of the same type at z. To take into account the transitions $1 \to 2$ and $2 \to 0$, we introduce two further collections of independent Poisson processes $\{U_n^x : n \geq 1\}$ and $\{V_n^x : n \geq 1\}$, $x \in \mathbb{Z}^d$, each of them has parameter 1. We put a × at (x, U_n^x) to indicate that a particle of type 2 is killed, and a $\bullet$ at (x, V_n^x) to indicate that a particle of type 1 changes its color.

At this point, the reader should see that, since the ×'s do not kill both types of particles, the construction of a dual process from the graphical representation we have just introduced will produce a somewhat complicated object. The main ingredient to simplify the dual process is to observe that, since the particles of type 1 (resp. 2) do not see the ×'s (resp. the $\bullet$'s), we can superimpose the ×'s and the $\bullet$'s. In other words, we remove the $\bullet$'s from our graphical representation, and pretend that the ×'s effect is now to kill the particles of type 2, and paint the particles of type 1 the color 2. In view of this new interpretation, it is natural to extend the definition of *path* introduced above for the multitype contact process by allowing paths to contain at most one ×. We now give the details of our construction. We will say that the points (x,s) and (z,t) are *strongly connected*, and write $(x,s) \to (z,t)$, if there is a sequence of times $s_0 = s < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 = x, x_1, \ldots, x_n = z$ so that

1. For $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0, 1, \ldots, n$, the vertical segment $\{x_i\} \times (s_i, s_{i+1})$ does not contain any ×'s.

If instead of the condition 2 above we have the condition

3. The set $\bigcup_{i=0}^{n} \{x_i\} \times (s_i, s_{i+1})$ contains exactly one ×,

we will say that (x,s) and (z,t) are *weakly connected*, and write $(x,s) \rightharpoonup (z,t)$. Finally, we will say that there is a *path* from (x,s) to (z,t) if both points are either strongly or weakly connected. In view of the ×'s effect, if there is a *generalized* path from (x,s) to (z,t), a particle of type 1 at (x,s) can certainly be the *ancestor* of a particle of type 2 at (z,t) if it goes through one × on its way up to (z,t). In conclusion, we will define the *dual process starting at* (x,t) by setting

$$\hat{\xi}_s^{(x,t)} \ = \ \{z \in \mathbb{Z}^d : \text{there is a dual path from } (x,t) \text{ to } (z,t-s)\}$$

where, as usual, a *dual path* from (x,t) to $(z,t-s)$, $0 \leq s \leq t$, indicates the existence of a path from $(z,t-s)$ to (x,t). The next step in proving Theorem 13 is to figure out the geometry of the dual process. To begin with, we observe that $\{(\hat{\xi}_s^{(x,t)}, s) : 0 \leq s \leq t\}$ exhibits a tree structure divided into two *stages* depending on the number of × (0 or 1) an ancestor has to cross to reach (x,t). We denote by Γ the set of points that are strongly connected with (x,t), i.e.,

$$\Gamma \ = \ \{(z,t-s) \in \mathbb{Z}^d \times \mathbb{R}^+ : (z,t-s) \to (x,t)\}.$$

The tree Γ is broken at some points by a × at which a new tree forms. The picture of the dual process we then obtain is that of an expanding cone of trees connected with Γ by some ×'s. In the following, Γ will be called the *upper tree starting at* (x,t), and the trees starting at a × will be called the *lower trees*. For an illustration, see Figure 8 where Γ is drawn in solid lines and the lower trees in dotted lines. We now denote by $\hat{\xi}_s^{x,1}$ and $\hat{\xi}_s^{x,2}$ the dual subsets given by

$$\hat{\xi}_s^{x,1} \ = \ \{z \in \mathbb{Z}^d : (z,t-s) \to (x,t)\} \qquad \text{and} \qquad \hat{\xi}_s^{x,2} \ = \ \{z \in \mathbb{Z}^d : (z,t-s) \rightharpoonup (x,t)\}.$$

By analogy with our partition in upper and lower trees, the elements of $\hat{\xi}_s^{x,1}$ and $\hat{\xi}_s^{x,2}$ will be called *upper ancestors* and *lower ancestors* respectively. As for the multitype contact process, the tree structure of the dual process allows to define an ancestor hierarchy in which the members are arranged according to the order they determine the color of (x,t). Here, the geometry of the dual also plays an important role since the color of (x,t) strongly depends on the type of the ancestors. To specify this idea, we now explain in greater detail how to deduce the color of (x,t) from knowing the ancestor hierarchy and the type (upper or lower) of each ancestor.

From now on, we will call *determining particle* the first ancestor in the hierarchy, and *distinguished particle* the first upper ancestor. For a picture of the path of the distinguished particle, see Figure 8. To determine the color of (x,t), we first look at the determining particle to know which of the following four events occurs.

1. The ancestor is an upper ancestor that lands on a 1.
2. The ancestor is an upper ancestor that lands on a 2 or a lower ancestor that lands on a 1.
3. The ancestor is a lower ancestor that lands on a 2.
4. The ancestor lands on an empty site.

In the case 1 (resp. 2), the determining particle will paint (x,t) the color 1 (resp. 2) and the algorithm is done. In the cases 3 and 4, the determining particle cannot paint (x,t) any color. In the case 4, we repeat the same reasoning with the second ancestor of the hierarchy (instead of the determining particle). In the case 3, the particle of type 2 can block some other ancestors from deciding the color of (x,t). Since these ancestors cannot determine the color of (x,t) anymore, we need to remove them from the hierarchy. To do this, we follow the determining particle on its way up to (x,t) until the first × we encounter, remove all the ancestors of the dual process starting at this × from the hierarchy, and repeat the same reasoning with the first ancestor of (x,t) that is left. If after the second trial none of the ancestors painted (x,t) any color, that is one of both cases 3 or 4 occurs one more time, we start again with the next ancestor, and so on.

Proof of Theorem 13 in dimension $d \leq 2$

The strategy we will follow to prove Theorem 13 is quite different depending on the dimension. In this paragraph, we will deal with the case $d \leq 2$ relying on the recurrence of 1 and 2-dimensional random walks.

For more convenience, we will work backwards in time by sending $s \mapsto t - s$. We first observe that if the upper tree Γ does not live forever then, for t sufficiently large, all the ancestors are

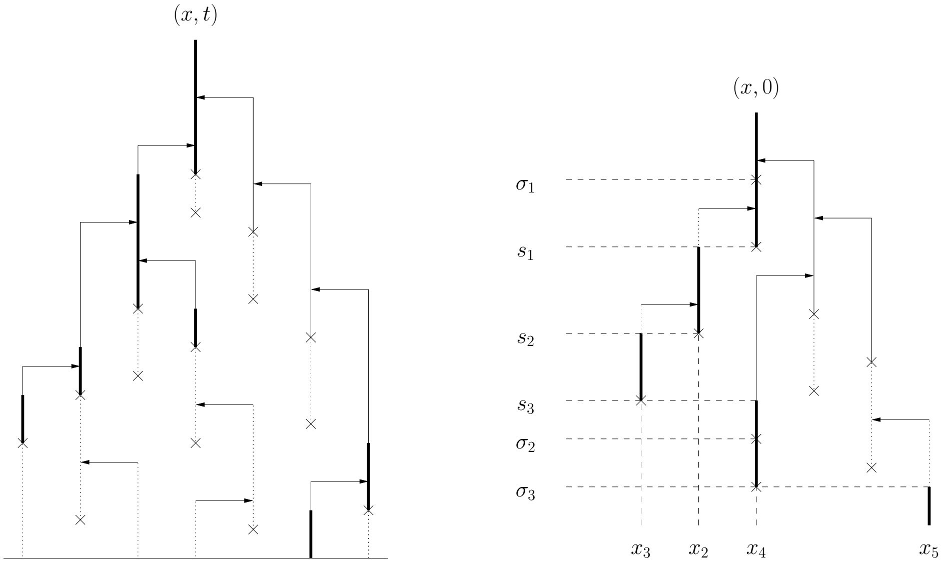

Figure 8. *Pictures of the dual process.*

lower ancestors, i.e., $\hat{\xi}_t^{(x,0)} = \hat{\xi}_t^{x,2}$, which implies that none of the ancestors can paint $(x,0)$ the color 1. To avoid trivialities, we assume from now on that Γ lives forever. We denote by Θ_1 the first time the determining particle penetrates in a lower tree Γ_1 that lives forever. The algorithm of the first ancestor implies that, after time Θ_1, the determining particle stays trapped inside Γ_1. This implies that after time Θ_1 (1) The paths of the determining and distinguished particles have the same law, and (2) The determining particle can only paint $(x,0)$ the color 2. Motivated by the properties (1) and (2) above, we first prove that Θ_1 is a.s. finite.

Lemma 5.4 $P(\Theta_1 = \infty) = 0$.

Proof To begin with, let $\{s_k\}_{k\geq 1}$ be the jumping times of the determining particle, and let x_k be its location between times s_{k-1} and s_k, with the convention $s_0 = 0$. We denote by σ_1 the first time the ancestor meets a ×, and by Ω_1 the lower tree starting at (x,σ_1), that is the first lower tree the determining particle visits. See Figure 8, picture on right. If Ω_1 lives forever then $\Theta_1 = \sigma_1$ and the proof is done. Otherwise, we define a sequence of stopping times $\{\sigma_n\}_{n\geq 1}$ and a sequence of lower trees $\{\Omega_n\}_{n\geq 1}$ as follows. If Ω_{n-1} does not live forever, we let σ_n be the first time the determining particle visits a new lower tree after Ω_{n-1} dies, and by Ω_n this lower tree. On the other hand, if Ω_{n-1} lives forever, we set $\sigma_n = \infty$ and $\Omega_n = \varnothing$. The reader will note that, for any $k \geq 1$, the path the determining particle follows on its way up from (x_k, s_k) to $(x,0)$ contains exactly one × so that if Ω_{n-1} does not live forever, then σ_n is finite with probability 1 and Ω_n is well defined. Let $N = \inf\{n \geq 1 : \sigma_n = \infty\}$, and A_n be the event that Ω_n lives forever. If the event A_{n-1} does not occur, the tree Ω_n is well defined and the event A_n determined by parts of the graph that are after Ω_{n-1} dies so A_{n-1} and A_n are independent. More generally, since the trees Ω_1, Ω_2, …, Ω_n are disjoint, A_1, A_2, …, A_n are independent. Moreover, the probability that A_n occurs is equal to $p(\lambda)$, the survival probability of the contact process with parameter λ starting from one infected site. This implies that

$$P(N > n) \;=\; P(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n^c) \;=\; \prod_{i=1}^{n} \big\{1 - P(A_i)\big\} \;=\; (1 - p(\lambda))^n.$$

The condition $\lambda > \lambda_c$ then leads to $p(\lambda) > 0$ and $P(N = \infty) = 0$. Finally, by observing that $\Theta_1 = \sigma_N$, the lemma follows. □

By Lemma 5.4, the determining particle is trapped, at some time Θ_1 a.s. finite, inside a lower tree that lives forever, which implies that the determining particle can only paint $(x, 0)$ the color 2. At this point, the worst scenario we have in mind is that the determining particle lands on a 2 and the distinguished particle on a 1. In such a case, the determining particle, which is a lower ancestor, cannot paint $(x, 0)$ any color whereas the distinguished particle can bring a particle of type 1 to $(x, 0)$. To see that this *bad event* has probability 0, we prove that the determining and distinguished particles coalesce with probability 1. We first observe that, after time Θ_1, the paths of both ancestors have the same law. Proposition 5.1 then tells us that the path of the determining particle can be broken into independent and identically distributed pieces at some renewal points. The renewal points of the determining particle are defined as in § 5.2 by replacing the upper tree Γ by the lower tree Γ_1. Lemma 5.2 then implies that the determining and distinguished particles coalesce with probability 1. In particular, we can suppose, by taking t sufficiently large, that both ancestors land at time t on the same site. If the site both ancestors land on is occupied by a particle of type 1, the determining particle, that is a lower ancestor by Lemma 5.4, will paint $(x, 0)$ the color 2. If the site is occupied by a 2, the distinguished particle will paint $(x, 0)$ the color 2 unless a lower ancestor succeeds earlier. Finally, if the target site is empty, we start over again with the second ancestor, and so on. Since the tree starting at the point where both particles coalesce is linearly growing in time and the initial configuration is translation invariant, we eventually find, by Lemma 9.14 in Harris (1976), an ancestor landing on an occupied site that will bring a 2 to $(x, 0)$. This concludes the proof of Theorem 13 in $d \leq 2$.

Proof of Theorem 13 in dimension $d \geq 3$

The strategy to deal with the case $d \geq 3$ is quite different. To begin with, we will construct by induction an ordered set of ancestors $\{\eta_t(k)\}_{k \geq 1}$ that are good candidates for painting $(x, 0)$ the color 2. By relying on Lemma 5.4, we will prove that, for $k \geq 1$ and t sufficiently large, $\eta_t(k)$ is a lower ancestor that comes before the distinguished particle in the ancestor hierarchy. Then, relying on the transience of d-dimensional random walks for $d \geq 3$, we will extract a subsequence of ancestors $\{\eta_t(k_i)\}_{i \geq 1}$ that never coalesce together. In particular, the number of sites occupied by these candidates can be made arbitrarily large so that we will eventually find one, landing on a particle of type 1, that will paint $(x, 0)$ the color 2.

We start by constructing inductively the ordered ancestor set $\{\eta_t(k)\}_{k \geq 1}$. The first member of the sequence $\eta_t(1)$ is the determining particle. Before defining the second member, we wait until time Θ_1 that the determining particle jumps in Γ_1. Then, we look at the ancestor hierarchy at that time, and discard all the ancestors that land on $S_0(1)$ or a site that does not live forever. Here, $S_0(1)$ is the spatial location of the determining particle at time Θ_1. The second member of the sequence will be the first ancestor that is left after discarding. The reader will note that the existence of such an ancestor follows from the fact that Γ lives forever. Then, by using the arguments of Lemma 5.4, one can prove that $\eta_t(2)$ also penetrates, at some time Θ_2 a.s. finite, in a lower tree Γ_2 that lives forever. We then repeat the same procedure to define the third candidate, and so on. For any $k \geq 1$, let Γ_k be the infinite lower tree visited by the k-th member, Θ_k the first time the ancestor penetrates in Γ_k, and $S_0(k)$ its spatial location at time Θ_k. By Proposition 5.1, the path of the k-th member can be broken, from time Θ_k, into i.i.d. pieces. We denote by $(S_n(k), T_n(k))$ the n-th renewal point of the k-th member, by $X_i(k)$ the spatial displacement between consecutive renewals, and by $\tau_i(k)$ the corresponding temporal displacement, i.e.,

$$S_n(k) \ = \ S_0(k) \ + \ \sum_{i=1}^n \ X_i(k) \qquad \text{and} \qquad T_n(k) \ = \ \Theta_k \ + \ \sum_{i=1}^n \ \tau_i(k).$$

The strategy to prove that, with probability 1, there exists a subsequence $\eta_t(k_i)$ of lower ancestors that never coalesce together can be formulated as follows. Let $n \in \mathbb{N}^*$, and assume that there exist n lower ancestors $\eta_t(k_1)$, $\eta_t(k_2)$, …, $\eta_t(k_n)$ that never coalesce together. Then, given an integer $m > k_n$, we will prove that, with positive probability, we can

$G_{m,K}$ = Bring $\eta_t(m)$ at least K units apart from each of the $\eta_t(k_i)$ and trap the $n + 1$ ancestors inside large disjoint cubes without collision of their triangles.

Here, G is for good event. Then, relying on Lemma 5.3, we will prove that if the event $G_{m,K}$ occurs for K sufficiently large, then

$$\lim_{t\to\infty} \|\,\eta_t(k_i) - \eta_t(m)\| \;=\; \infty \qquad \text{for} \quad i = 1,\,2,\,\ldots,\,n$$

with probability close to 1. We will then conclude by proving that with probability 1 the good events $G_{m,K}$ occur for infinitely many $m > k_n$. In what follows, we give the proofs for the determining particle only but the same holds for the other members of the sequence.

Lemma 5.5 *Let $K \in \mathbb{N}^*$ and $D_K = [-K, K]^d$. For any time $t \geq \Theta_1$, the event B that the determining particle leaves the box D_K in less than one unit of time, and then survives without giving birth until time $t + K$ has positive probability.*

Proof It is trivial if we think of the process as being generated by the Harris' graphical representation introduced above. □

Lemma 5.6 *Let $H_{t,K}$ be the event that the triangles of the first ancestor are inside a box $z + D_K$ from time t to time $t + K$. For any $\varepsilon_1 > 0$, there exists K such that $P\,(H_{t,K}) \geq 1 - \varepsilon_1$.*

Proof The basic idea is to prove that the events (1) The renewal points of the determining particle belong to $z + [-K/2, K/2\,]^d$ between times t and $t + K$, and (2) The triangles of the determining particle are smaller than $K/2$ between times t and $t + K$, both occur with probability close to 1 for K large. This is a straightforward consequence of Proposition 5.1. □

We are now ready to prove that, with probability 1, there is an integer $m > k_n$ such that $\eta_t(m)$ does not coalesce with the first n members of the subsequence $\eta_t(k_i)$. Let $G_{m,K}$ be the event that, for $i = 1,\,2,\,\ldots,\,n$, there is a time $t_i \in [\Theta_m, \Theta_m + K\,]$ such that $\eta_t(k_i)$ and $\eta_t(m)$ are good and at least K units apart from each other at time $t = t_i$.

Lemma 5.7 *For any $\varepsilon_2 > 0$, there exists K sufficiently large such that on the event $G_{m,K}$*

$$P\,\Big(\lim_{t\to\infty}\,\inf_{1\leq i\leq n}\,\|\,\eta_t(k_i) - \eta_t(m)\| \;=\; +\infty\Big) \;\geq\; 1 - \varepsilon_2.$$

Proof Let $1 \leq i \leq n$, and assume that the ancestors $\eta_t(k_i)$ and $\eta_t(m)$ are good and K units apart at some time $t \geq \Theta_m$. Then, Lemma 5.3 implies that there exists $C > 0$ such that

$$\begin{aligned} &P\,\big(\text{the particles } \eta_t(k_i) \text{ and } \eta_t(m) \text{ do never coalesce}\big) \\ &\qquad =\; P\,\Big(\lim_{t\to\infty}\,\|\,\eta_t(k_i) - \eta_t(m)\| \;=\; +\infty\Big) \;\geq\; 1 \;-\; C\,K^{-1/10} \;-\; 2\,C\,K^{-3/32}. \end{aligned}$$

In particular, for K sufficiently large,

$$\begin{aligned} &P\,\big(\eta_t(k_i) \text{ and } \eta_t(m) \text{ coalesce for some } 1 \leq i \leq n\,\big) \\ &\qquad \leq\; \sum_{i=1}^{n}\, P\,\big(\eta_t(k_i) \text{ and } \eta_t(m) \text{ coalesce}\big) \;\leq\; n\,C\,K^{-1/10} \;+\; 2\,n\,C\,K^{-3/32} \;\leq\; \varepsilon_2. \end{aligned}$$

This proves the lemma. □

Lemma 5.8 *There exists K sufficiently large such that $P\,(\limsup_{m\to\infty} G_{m,K}) = 1$.*

Proof Since the n lower ancestors $\eta_t(k_i)$ do not coalesce and that $\Theta_m \to \infty$, Lemma 5.7 implies that there exists an integer m such that the distances between the ancestors at time Θ_m are bigger than $4\sqrt{d}K$. For $i = 1,\,2,\,\ldots, n$, we denote by H_i the event that the triangles of $\eta_t(k_i)$

are contained, between time Θ_m and time $\Theta_m + K$, in some box $\Omega_i = z_i + D_K$. The reader will observe that m has been chosen so that $\Omega_i \cap \Omega_j = \varnothing$ as soon as $i \neq j$. In particular, the good events H_i, $i = 1, 2, \ldots, n$, are determined by disjoint parts of the graph, and then are independent. This together with Lemma 5.6 implies that, for K sufficiently large,

$$P\,(H_1 \,\cap\, H_2 \,\cap \cdots \cap\, H_n) \;=\; \prod_{i=1}^{n} P\,(H_i) \;\geq\; 1 - n\,\varepsilon_1.$$

Now that the n lower ancestors $\eta_t(k_i)$, $1 \leq i \leq n$, are trapped inside large disjoint cubes, we require each of them to be good at least once between time Θ_m and time $\Theta_m + K$. Since this occurs if each of the n ancestors has at least one renewal in this interval of time, the probability that this event occurs can be bounded from below by

$$\prod_{i=1}^{n} P\,(\tau_1(k_i) < K) \;\geq\; \left[1 - C\,e^{-\beta\,K}\right]^n$$

for appropriate $C < \infty$ and $\beta > 0$. To conclude, the last thing we require is that $\eta_t(m)$ is good and at least K units apart from each other ancestor between time $\Theta_m + 1$ and time $\Theta_m + K$. Since the cubes Ω_i are at least $2K$ units apart from each other, this occurs, by Lemma 5.5, with positive probability. Putting things together, it follows that there exists $\varepsilon_3 > 0$ such that $P(G_{m,K}) \geq \varepsilon_3$ for m sufficiently large. Finally, by observing that the events $G_{m_1,K}$ and $G_{m_2,K}$ are independent as soon as $|\Theta_{m_1} - \Theta_{m_2}| > K$, we can conclude, by the Borel-Cantelli Lemma, that, for infinitely many $m \geq 1$, the good event $G_{m,K}$ occurs. $\square$

To conclude the proof of Theorem 13, we now denote by $B_s^{(x,0)}$ the set of sites $x \in \mathbb{Z}^d$ occupied at time s by a particle of type 1, and set

$$\eta_t \;=\; \{\eta_t(k) : k \geq 1 \text{ with } \Theta_k \leq t\}.$$

First of all, Lemma 5.7 implies that if the event $G_{m,K}$ occurs, then the ancestor $\eta_t(m)$ coalesces with one of the $\eta_t(k_i)$, $1 \leq i \leq n$, with probability $< \varepsilon_2$. In other respects, Lemma 5.8 tells us that the good event $G_{m,K}$ occurs for infinitely many $m \geq 1$. This implies that, by choosing t sufficiently large, the cardinality of η_t can be made arbitrarily large. More precisely, for any $\varepsilon > 0$ and $M > 0$, there exists a time $t_0 \geq 0$ such that $P\,(\text{card } \eta_t < M) \leq \varepsilon$ for $t \geq t_0$. Lemma 9.14 of Harris (1976) and the translation invariance of the initial configuration then imply that

$$\lim_{t \to \infty} \; P\,(\eta_t \,\cap\, B_t^{(x,0)} \;=\; \varnothing) \;=\; 0.$$

In particular, if t is large, there will be a lower ancestor $\eta_t(n)$ occupied by a particle of type 1. We now look at the ancestors that come before $\eta_t(2)$ in the hierarchy until we find one that lands on an occupied site. If a 1 stands at this site, the ancestor will paint $(x, 0)$ the color 2 and the proof is done. If a 2 stands at this site, $S_0(1)$ will be empty at time Θ_1^- whatever the color of the sites the next ancestors land on. So, we look at $\eta_t(2)$, and so on. If none of the ancestors that come before $\eta_t(n)$ in the hierarchy succeeds in painting $(x, 0)$ the color 2, $\eta_t(n)$ will do it. This completes the proof of Theorem 13.

For another illustration of the duality techniques we have just introduced, we refer the reader to Lanchier (2005b) where we investigate the multitype contact process with frozen states.

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Phase transitions and duality properties of a successional model

N. Lanchier

Abstract The first purpose of this article is to study the phase transitions of a new interacting particle system. We have two types of particles. Each type gives birth to particles of the same type as the parent. The second type can die while the first one mutates into the second one. Specifically, we prove that the three possible evolutions of the process, that is extinction, survival of 2's, or coexistence, may occur depending on the selected parameters. The main objective, however, is to investigate the duality properties of the process; the corresponding dual process exhibits a structure somewhat different from that of well-known particle systems.

1. Introduction

The spatial model we introduce in this paper is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \longrightarrow \{0, 1, 2\}$. A site $x \in \mathbb{Z}^d$ is said to be empty if $\xi(x) = 0$, and is said to be occupied by a particle of type 1 (resp. 2) if $\xi(x) = 1$ (resp. 2). The evolution rules can be formulated as follows:

1. Each 1 (resp. 2) tries to give birth onto each of its neighbors at rate λ_1 (resp. λ_2). Here, the neighbors of a site $x \in \mathbb{Z}^d$ is the set of $y \in \mathbb{Z}^d$ such that $\|x - y\| \leq R$ where $\|\cdot\|$ is a norm and where R is a positive constant.
2. If the offspring is sent to an empty site, the birth occurs. Otherwise, it is suppressed.
3. Each 1 becomes 2 at rate 1.
4. Each 2 dies, that is becomes 0, at rate 1.

We called our process successional model to refer to the ecological succession phenomenon it describes the evolution. In our ecological context, 1's are the individuals of a so-called *pioneer* species that can invade a hostile environment. The appearance of this first species then helps the development of a second species, here the particles of type 2. In plant communities, the spontaneous transition 1 → 2 is explained by the presence of a seed bank in the soil.

First of all, if only 2's are present, the system reduces to the contact process with parameter λ_2. In this case, there exists a critical value $\lambda_c \in (0, \infty)$ such that the following holds: If $\lambda_2 \leq \lambda_c$ then the process converges in distribution to the all empty state. Otherwise, there exists a stationary measure μ_2 that concentrates on configurations with infinitely many 2's. See, e.g., Liggett (1999). To avoid trivialities, we assume from now on that ξ_0 contains infinitely many 1's and 2's.

We first suppose that $\lambda_1 \geq \lambda_2 > \lambda_c$ and consider two initial configurations ξ_0^1 and ξ_0^2 such that $\xi_0^2(x) = 0$ if $\xi_0^1(x) = 0$, and $\xi_0^2(x) = 2$ otherwise, i.e., ξ_0^2 can be deduced from ξ_0^1 by replacing each 1 by a 2. This makes ξ_t^2 a contact process with parameter $\lambda_2 > \lambda_c$. Then, relying on a standard argument from Harris (1972) we may run both processes on the same probability space in such a way that if $\xi_t^2(x) = 2$ then $\xi_t^1(x) \neq 0$. This implies, in particular, that 2's survive, i.e., $P(\forall\, t \geq 0, \exists\, x : \xi_t^1(x) = 2) = 1$. We now suppose that $\lambda_2 > \lambda_1 > \lambda_c$. In this case, due to a lack of monotonicity, the previous coupling fails. We then take $\xi_0^1 = \xi_0^2$ and replace the transition 1 → 2 in ξ_t^1 by the transition 1 → 0 in ξ_t^2. This makes ξ_t^2 a multitype contact process with parameters λ_1 and λ_2 (see Neuhauser, 1992). By running both processes on the same space as before, one can prove that ξ_t^1 has more 2's and fewer 1's than ξ_t^2. Then, the results of Section 3 in Durrett and Neuhauser (1997) implies that $\xi_t^1 \Rightarrow \mu_2$. Here, $\Rightarrow$ denotes weak convergence. These results are summarized in the following theorem.

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Theorem 1 *We set $\lambda_2 > \lambda_c$. If $\lambda_1 \geq \lambda_2$ then 2's survive, i.e., $P(\forall\, t \geq 0, \exists\, x : \xi_t(x) = 2) = 1$. If $\lambda_1 < \lambda_2$ then 2's win, i.e., $\xi_t \Rightarrow \mu_2$, the upper invariant measure of the contact process.*

We now focus on λ_2 close to 0. If we suppress the transition $0 \to 2$, namely if $\lambda_2 = 0$, we can view the process as a forest fire model. In words, 0 = alive, 1 = on fire, and 2 = burnt. By using a rescaling argument, Durrett and Neuhauser (1991) proved that, in the 2-dimensional nearest neighbor case, the process exhibits a phase transition. More precisely, there is a critical value $\alpha_c \in (0, \infty)$ such that if $\lambda_1 > \alpha_c$ then there exists a nontrivial stationary measure ν that concentrates on configurations with infinitely many 1's and 2's. Theorem 2 tells us that 1's and 2's still coexist for sufficiently small $\lambda_2 > 0$.

Theorem 2 *Assume that ξ_0 contains infinitely many 1's. If $d = 2$ and $\lambda_1 > \alpha_c$ there is $\beta_c \in (0, \infty)$ such that if $\lambda_2 \leq \beta_c$ then 1's and 2's coexist, i.e., $\xi_t \Rightarrow \nu$ with $\nu(\xi(x) = 1) \neq 0$.*

We now come to the main result of this paper: To improve Theorem 1 by proving that 2's still win if $\lambda_1 = \lambda_2$. Although Theorem 3 is not surprising, the proof relies on a somewhat interesting property of the dual process.

Theorem 3 *Assume that the configuration ξ_0 contains infinitely many 2's and is translation invariant. If $\lambda_1 = \lambda_2 > \lambda_c$ then 2's win, i.e., $\xi_t \Rightarrow \mu_2$.*

The main ingredient of the proof is duality. To figure out the structure of the dual process, we will start by focusing on the multitype contact process that we denote by $\eta_t : \mathbb{Z}^d \longrightarrow \{0, 1, 2\}$. Then we will see how to construct both processes ξ_t and η_t using the same graphical representation and what makes the dual process of ξ_t different from that of the multitype contact process. First of all, we fix $\lambda_1 = \lambda_2$, denote by λ their common value, and, for $x, y \in \mathbb{Z}^d$, $\|x - y\| \leq R$, let $\{T_n^{x,y} : n \geq 1\}$ and $\{U_n^x : n \geq 1\}$ be the arrival times of Poisson processes with rates λ and 1 respectively. At times $T_n^{x,y}$, we draw an arrow from site x to site y while at times U_n^x, we put a cross × at site x. This may be interpreted as follows. If at time $T_n^{x,y}$, the site x is occupied and the site y empty then the particle at x gives birth to a particle of the same type at y. At time U_n^x, we remove the particle at x if it is present. These evolution rules make η_t the *multitype contact process* with parameters $\lambda_1 = \lambda_2 = \lambda$. (See Neuhauser (1992) for a complete study of this process). Finally, we say that there is a *path* from $(y, 0)$ to (x, t) if there is a sequence of times $s_0 = 0 < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 = y, x_1, \ldots, x_n = x$ such that the following two conditions hold.

1. For $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0, 1, \ldots, n$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any ×'s.

Since the ×'s kill both types of particles, the *ancestor* of a particle at x at time t can be located at y at time 0 only if there is a path from $(y, 0)$ to (x, t). In conclusion, the dual process starting at (x, t), that is the set of the potential ancestors of site x at time t, will be defined by reversing the arrows and letting

$$\tilde{\eta}_s^{(x,t)} \;=\; \{y \in \mathbb{Z}^d : \text{there is a path from } (x, \tilde{0}) \text{ to } (y, \tilde{s})\}$$

where $\tilde{s} = t - s$. From a topological point of view, it is known that the set $\{(\tilde{\eta}_s^{(x,t)}, s) : 0 \leq s \leq t\}$ exhibits a *tree structure*. See for instance Durrett (1995), Liggett (1999), or Neuhauser (1992).

A natural way to construct our successional model would be to introduce a new collection of Poisson processes $\{V_n^x : n \geq 1\}$, $x \in \mathbb{Z}^d$, with rate 1, put a dot • at site x at time $\{V_n^x : n \geq 1\}$, and pretend that the ×'s effect is now to kill the particles of type 2 only while the •'s effect is to paint the particles of type 1 the color 2. The first ingredient in proving Theorem 3 is to observe that, since the particles of type 1 (resp. 2) do not see the ×'s (resp. the •'s), we can superimpose the ×'s and the •'s. In other words, we use the same collections of Poisson processes as in the multitype contact process, namely we remove the •'s, and say that if at time U_n^x the site x is occupied by a particle of type 1 (resp. of type 2) then we paint this particle the color 2 (resp. we kill this particle). Relying on this graphical representation, it is natural to extend the definition of *path* by replacing the condition 2 above with

3. The set $\bigcup_{i=0}^{n} \{x_i\} \times (s_i, s_{i+1})$ contains at most one ×.

That is, a path is now allowed to contain at most one ×. In view of the ×'s effect, if there is a path from $(y, 0)$ to (x, t) then a particle of type 1 at y at time 0 can certainly be the *ancestor* of a particle of type 2 located at x at time t if it goes through one × on its way up to (x, t). If we define the dual process starting at (x, t) by letting

$$\tilde{\xi}_s^{(x,t)} \;=\; \{y \in \mathbb{Z}^d : \text{there is a path from } (x, \tilde{0}) \text{ to } (y, \tilde{s})\},$$

the set $\{(\tilde{\xi}_s^{(x,t)}, s) : 0 \leq s \leq t\}$ now exhibits a tree structure divided into two *stages* depending on the number of × (0 or 1) an ancestor has to cross to reach (x, t). Then, a site $y \in \mathbb{Z}^d$ will be called an *upper ancestor* (resp. a *lower ancestor*) at time s if there is a path from (y, s) to (x, t) containing 0 (resp. 1) ×. We will use the terminology *upper stage* (resp. *lower stage*) to refer to the set of the upper (resp. lower) ancestors. Here, as in the mutitype contact process, the tree structure of the dual process allows us to define an ancestor *hierarchy* in which the members are arranged according to the order they determine the color of (x, t). The first ancestor in the hierarchy will be called the *determining particle*, and the first upper ancestor the *distinguished particle*.

With this construction in mind, we can sketch the proof of Theorem 3 as follows. The first step is to prove that the last time Θ_1 at which the determining particle belongs to the upper stage of the tree structure is a.s. finite, i.e., $P(\Theta_1 = \infty) = 0$. The rest of the proof then relies on the observation that, once trapped inside the lower stage, the determining particle can only paint (x, t) the color 2 since the ×'s effect is to change the color of the 1's. In $d \leq 2$, we will wait until time Θ_1 and then prove that the distinguished particle and the determining particle coalesce with probability 1 so that they will land for t sufficiently large on the same site. If this site is initially occupied by a particle of type 1, the determining particle will paint (x, t) the color 2. If the site is occupied by a particle of type 2, the distinguished particle will paint (x, t) the color 2 unless a lower ancestor does it earlier. In $d \geq 3$, we will rely on the fact that $\Theta_1 < \infty$ to construct inductively a sequence of lower ancestors $\zeta_s(k)$ that come before the distinguished particle in the ancestor hierarchy. We will then extract a subsequence of candidates $\zeta_s(k_i)$ that do not coalesce together, and we will conclude that at least one lands on a 1 and paints (x, t) the color 2.

In conclusion, the phase diagram looks like Figure 1 where our theorems are summarized. For the same reasons as Durrett and Neuhauser (1991), we do not know how to prove that the probability that coexistence occurs is increasing with respect to λ_1 but simulations in dimension 2 exhibit such a picture. We also conjecture the existence of a critical interval I_c such that for any $\lambda_1 \in I_c$ the three possible evolutions can occur. In particular, if $\lambda_1 \in I_c$ and $\lambda_2 \leq \lambda_c$ we can notice that the probability that the 2's survive is paradoxically decreasing with respect to λ_2.

Except Section 2 that focuses on Theorem 2, the rest of the article is devoted to the proof of Theorem 3. In Section 3 we investigate in greater details the duality properties of the process. Relying on the construction given in Section 3, we then prove Theorem 3 in two steps. In Section 4, we start by dealing with the case $d \leq 2$, and then conclude in Section 5 with the case $d \geq 3$.

2. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. In particular, we will prove that, in dimension 2, coexistence occurs for an open set of values (λ_1, λ_2) in $\mathbb{R}^2$. We conjecture that such a property holds in any dimension but our proof heavily relies on Lemma 1.1 of Durrett and Neuhauser (1991) that has been proved in $d = 2$ only.

If $\lambda_2 = 0$, we recall that 0 can be interpreted as a living tree, 1 as a burning tree and 2 as a burnt site. We let $B = (-L, L)^2$ and, for any $m \in \mathbb{Z}$, $B_m = mLe_1 + B$, where $e_1 = (1, 0)$ is the first unit vector of $\mathbb{Z}^2$. For $(m, n) \in \mathbb{Z}^2$ with m and n both even or m and n both odd, we say that (m, n) is *occupied* if the following two conditions are satisfied.

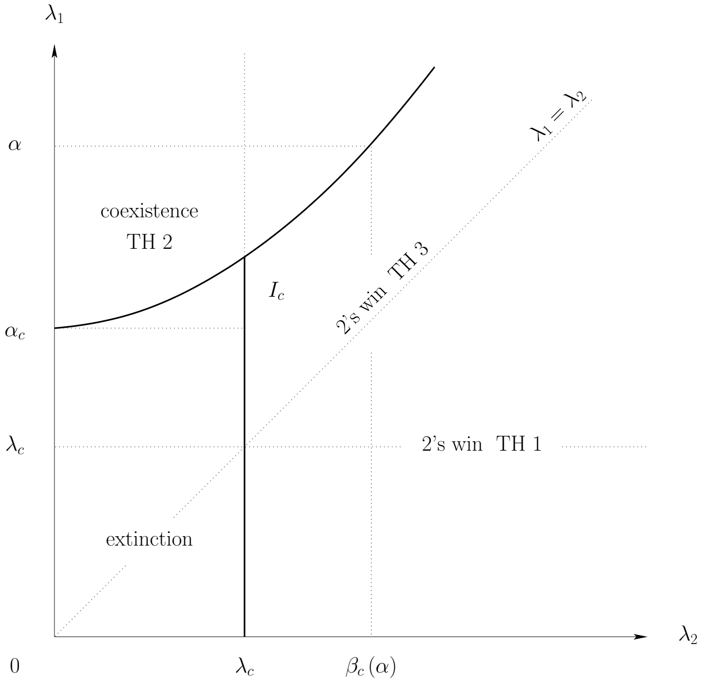

Figure 1. *Phase diagram.*

1. There are more than $\sqrt{L}$ burning trees in B_m at some time $t \in [\,n\,\Gamma\,L, (n+1)\,\Gamma\,L\,]$.
2. There is at least one burning tree in B_m at all times $t \in [(n+1)\,\Gamma\,L, (n+2)\,\Gamma\,L\,]$.

Here Γ is a positive constant that will be fixed later. The following lemma implies that there exists a critical value $\alpha_c \in (0, \infty)$ such that coexistence occurs for $\lambda_1 > \alpha_c$ and $\lambda_2 = 0$.

Lemma 2.1 (Durrett and Neuhauser) *There exists $\alpha_c \in (0, \infty)$ such that if $\lambda_1 > \alpha_c$ and $\lambda_2 = 0$ then Γ and L can be chosen so that the set of occupied sites dominates the set of wet sites in a one-dependent oriented percolation process with parameter $p = 1 - 6^{\,-36}$.*

See Durrett and Neuhauser (1991), Lemma 1.1. Now, given Γ and L, it is clear that there exists $\beta_c \in (0, \infty)$ such that if $\lambda_2 \leq \beta_c$ the probability that the 2's give birth in the space-time box $B_m \times [\,n\,\Gamma\,L, (n+2)\,\Gamma\,L\,]$ can be bounded by $6^{\,-36}$. This implies that Lemma 2.1 holds for $\lambda_2 \leq \beta_c$ and $p = 1 - 2 \times 6^{\,-36}$. In particular, there exists, for $\lambda_1 > \alpha_c$ and $\lambda_2 \leq \beta_c$, an infinite cluster of occupied sites. See Durrett (1984), Section 10. In conclusion, there is a stationary distribution that concentrates on configurations with infinitely many 1's and 2's. See, e.g., Durrett (1995), Section 4. This completes the proof of Theorem 2.

3. Construction and properties of the dual process

From now on and until the end of this paper, we suppose that $\lambda_1 = \lambda_2$ and denote by λ their common value. We start by constructing the process from a collection of Poisson processes in the following way. For x, $y \in \mathbb{Z}^d$, $\|x - y\| \leq R$, we let $\{T_n^{x,y} : n \geq 1\}$ and $\{U_n^x : n \geq 1\}$ be the arrival times of Poisson processes with rates λ and 1 respectively. At times $T_n^{x,y}$, we draw an arrow from x to y to indicate that a birth may occur. More precisely, if x is occupied and y is vacant then the particle present at site x gives birth in y to a particle of the same type. At times U_n^x, we put a cross $\times$ at x to indicate that a particle of type 1 present at x becomes 2 while a particle of type 2 is killed. A result of Harris (1972) implies that such a graphical representation can be used to construct the process starting from any initial configuration $\xi_0 : \mathbb{Z}^d \longrightarrow \{0, 1, 2\}$. See Figure 2 for a picture of the graphical representation.

After constructing the graphical representation, we can now define the dual process. We say that $(x, 0)$ and (y, t) are *strongly connected*, and write $(x, 0) \to (y, t)$, if there is a sequence of times $s_0 = 0 < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 = x, x_1, \ldots, x_n = y$ so that

1. For $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0, 1, \ldots, n$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any ×'s.

If instead of 2 we have the condition

3. The set $\bigcup_{i=0}^{n} \{x_i\} \times (s_i, s_{i+1})$ contains exactly one ×,

we say that $(x, 0)$ and (y, t) are *weakly connected*, and write $(x, 0) \rightharpoonup (y, t)$. Since the ×'s do not kill the 1's but change them into 2's, we must take into account the paths that contain one × in the construction of the dual. So, to define the dual process, we reverse the arrows and time by mapping $\tilde{s} = t - s$, and let

$$\tilde{\xi}_s^{(x,t)} \ = \ \{y \in \mathbb{Z}^d : (x, \tilde{0}) \to (y, \tilde{s}) \text{ or } (x, \tilde{0}) \rightharpoonup (y, \tilde{s})\}.$$

Since it will be easier to work with a forward process than a backward process, we also introduce the dual $\hat{\xi}_s^{(x,0)}$ that is constructed from the graphical representation that has an arrow from x to y at time $T_n^{x,y}$ and that is defined by

$$\hat{\xi}_s^{(x,0)} \ = \ \{y \in \mathbb{Z}^d : (x, 0) \to (y, s) \text{ or } (x, 0) \rightharpoonup (y, s)\}.$$

This can be done because the processes $\tilde{\xi}_s^{(x,t)}$ and $\hat{\xi}_s^{(x,0)}$ have the same law. Before going into the proof of Theorem 3, we now investigate the geometry of the dual process. First of all, we can observe that $\{(\hat{\xi}_s^{(x,0)}, s) : s \geq 0\}$ has a tree structure constituted of the points that are either strongly or weakly connected. Let Γ be the set of points strongly connected with $(x, 0)$, i.e.,

$$\Gamma \ = \ \{(y, t) \in \mathbb{Z}^d \times \mathbb{R}^+ : (x, 0) \to (y, t)\}.$$

It is clear that the tree Γ is broken at some points by a × at which a new tree forms. So the picture we obtain is that of an expanding cone of trees connected with Γ by some ×'s. In the following, Γ will be called the *upper tree starting at* $(x, 0)$ and the trees starting at a × the *lower trees*. For an illustration, see Figure 3 where Γ is drawn in solid lines and the lower trees in dotted lines. Although the dual process seems to be a complicated object, in view of the translation invariance of the graphical representation, one can break it up into identically distributed pieces. More precisely, the upper tree and each of the lower trees have the same law as the tree structure of the contact process with parameter λ. See e.g., Durrett (1995) or Liggett (1999) for a construction of the contact process. We now denote by $\hat{\xi}_s^{x,1}$ and $\hat{\xi}_s^{x,2}$ the dual subsets given by

$$\hat{\xi}_s^{x,1} \ = \ \{y \in \mathbb{Z}^d : (x, 0) \to (y, s)\} \qquad \text{and} \qquad \hat{\xi}_s^{x,2} \ = \ \{y \in \mathbb{Z}^d : (x, 0) \rightharpoonup (y, s)\}.$$

By analogy with the cutting of the tree structure in upper tree and lower trees, the elements of $\hat{\xi}_s^{x,1}$ and $\hat{\xi}_s^{x,2}$ will be called respectively *upper ancestors* and *lower ancestors*. As for the multitype contact process, the tree structure of the dual process allows to define an ancestor hierarchy in which the members are arranged according to the order they determine the color of $(x, 0)$. Here, the geometry of the dual plays an important part since the color of $(x, 0)$ strongly depends on the type of the ancestors. To specify this idea, we now explain in greater detail how to deduce the color of $(x, 0)$ from the ancestor hierarchy and the type of each of them.

First of all, we denote by $\hat{\xi}_s^{(x,0)}(n)$ the n-th member of the ordered ancestor set and let $\hat{\xi}_s^{x,1}(k) = \hat{\xi}_s^{(x,0)}(n_k)$ be the k-th upper ancestor. Later on, $\hat{\xi}_s^{(x,0)}(1)$ and $\hat{\xi}_s^{x,1}(1)$ will be called the *determining particle* and the *distinguished particle* respectively. For a picture of the path of the distinguished particle, see Figure 3. We now give an algorithm to determine the color of $(x, 0)$ depending on the type of each ancestor and the initial configuration. First of all, we look at the determining particle to know which of the following four events occurs.

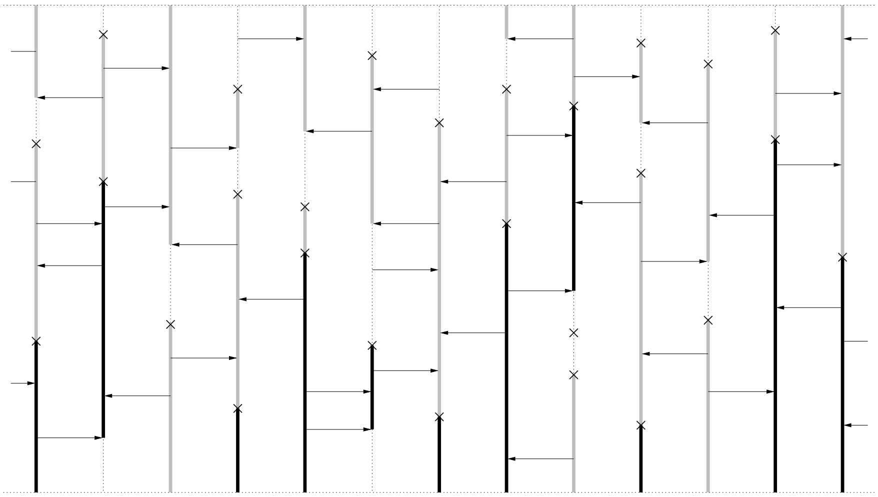

Figure 2. *Harris' graphical representation. The black lines refer to the 1-particles, the gray ones to the 2-particles, and the dotted ones to empty sites.*

1. The ancestor is an upper ancestor that lands on a 1.
2. The ancestor is an upper ancestor that lands on a 2 or a lower ancestor that lands on a 1.
3. The ancestor is a lower ancestor that lands on a 2.
4. The ancestor lands on an empty site.

In the case 1 (resp. 2), the determining particle will paint $(x, 0)$ the color 1 (resp. 2) and the algorithm is done. In the cases 3 and 4, the determining particle cannot paint $(x, 0)$ any color. In the case 4, we repeat the same reasoning with the second ancestor of the hierarchy (instead of the determining particle). In the case 3, the particle of type 2 can block some other ancestors from determining the color of $(x, 0)$. Since these ancestors cannot determine the color of $(x, 0)$ anymore, we need to remove them from the hierarchy. To do this, we follow the determining particle on its way up to $(x, 0)$ until the first × we encounter, remove all the ancestors of the dual process starting at this × from the hierarchy and repeat the same reasoning with the first ancestor of $(x, 0)$ that is left. If after the second trial no ancestor can paint $(x, 0)$ any color, that is one of both cases 3 or 4 occurs one more time, we start again with the next ancestor, and so on.

We now refer to the right side of Figure 3 for an example of application. The determining particle is a lower ancestor that lands on a 2 (case 3) so it cannot paint $(x, 0)$ any color. The first × the determining particle encounters on its way up is located at site x at the top of the picture. The ancestors of this × are the determining particle and the second ancestor so we try again with the third ancestor. The third ancestor lands on an empty site (case 4) so we look at the fourth ancestor. The fourth ancestor is an upper ancestor that lands on a 1 (case 1) so it paints $(x, 0)$ the color 1 and the algorithm is done.

As we will see further, the state of $(x, 0)$ strongly depends on the spatial location of the first ancestor and of the distinguished particle. Fortunately, by using an idea of Kuczek (1989), one can easily manage the path of the distinguished particle, which is crucial to the proof of Theorem 3. To be precise, its path can be broken into i.i.d. pieces at certain points called *renewal points*. To define these points, we follow the path of the distinguished particle starting at $(x, 0)$ and, each time it jumps to a site that lives forever, call this site a renewal point. Let (S_n, T_n) be the location of the n-th renewal. We denote by X_i the spatial displacement between consecutive renewal points, and by τ_i the corresponding temporal displacement so that

$$S_n \;=\; x \;+\; \sum_{i=1}^{n} X_i \qquad \text{and} \qquad T_n \;=\; \sum_{i=1}^{n} \tau_i.$$

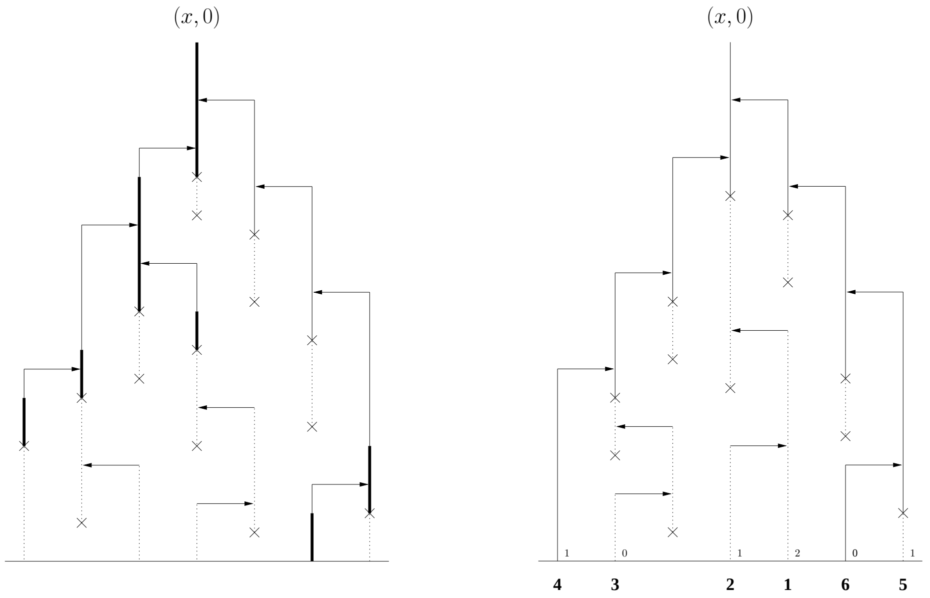

Figure 3. *Pictures of the dual process.*

One of the main ingredients we need to prove Theorem 3 is then given by the following proposition. For the details of the proof, see Neuhauser (1992), Section 2.

Proposition 3.1 (Neuhauser) *If the upper tree lives forever,* $\{(X_i, \tau_i)\}_{i \geq 1}$ *form an i.i.d. family of random vectors on* $\mathbb{Z}^d \times \mathbb{R}^+$. *Moreover, we have the exponential bounds*

$$P(\|X_i\| > t) \ \leq \ C\, e^{-\beta t} \qquad \text{and} \qquad P(\tau_i > t) \ \leq \ C\, e^{-\beta t}$$

for appropriate $C < \infty$ *and* $\beta > 0$.

Sketch of the proof First of all, denote by σ_0 the first jumping time of the distinguished particle, that is $\sigma_0 = \inf\{t > 0 : \hat{\xi}_t^{x,1}(1) \text{ hits a } \times\}$, by x_1 its spatial location after σ_0, and by β_1 the branch of Γ starting at (x_1, σ_0). See Figure 4, picture on right. If β_1 lives forever then (x_1, σ_0) is the first renewal point. Else, we define the sequences $\{(x_k, \sigma_{k-1})\}_{k \geq 1}$ and $(\beta_k)_{k \geq 1}$ as follows. If $\sigma_{k-1} < \infty$, let x_k be the location of the particle after σ_{k-1} and β_k the branch starting at (x_k, σ_{k-1}). Note that such a branch always exists since we have supposed that Γ lives forever. Then denote by σ_k the time when β_k dies out. The sequences are defined until σ_k is equal to infinity. If $\sigma_k = \infty$ then β_k lives forever and (x_k, σ_{k-1}) is the first renewal point of the distinguished particle. To determine the next one, we start over again the whole procedure replacing $(x, 0)$ by (x_k, σ_{k-1}), and so on. Now, it is clear that from time σ_{k-1} the particle stays forever inside β_k so that its path only depends on β_k. In particular, what happens before and after σ_{k-1} is determined by disjoint parts of Γ. This implies that the random variables (X_i, τ_i) are independent. Moreover, since the graphical gadget is translation invariant, the vectors (X_i, τ_i) are also identically distributed. For a proof of the exponential bounds, see Neuhauser (1992), Section 2. □

In conclusion, Proposition 3.1 gives us control over the location of the distinguished particle at the renewal points. Moreover, the contact process grows at most linearly in space (see e.g., Durrett (1988), Section 1) so between consecutive renewals the particle stays within a set linearly growing which we will call *triangle* in the next sections.

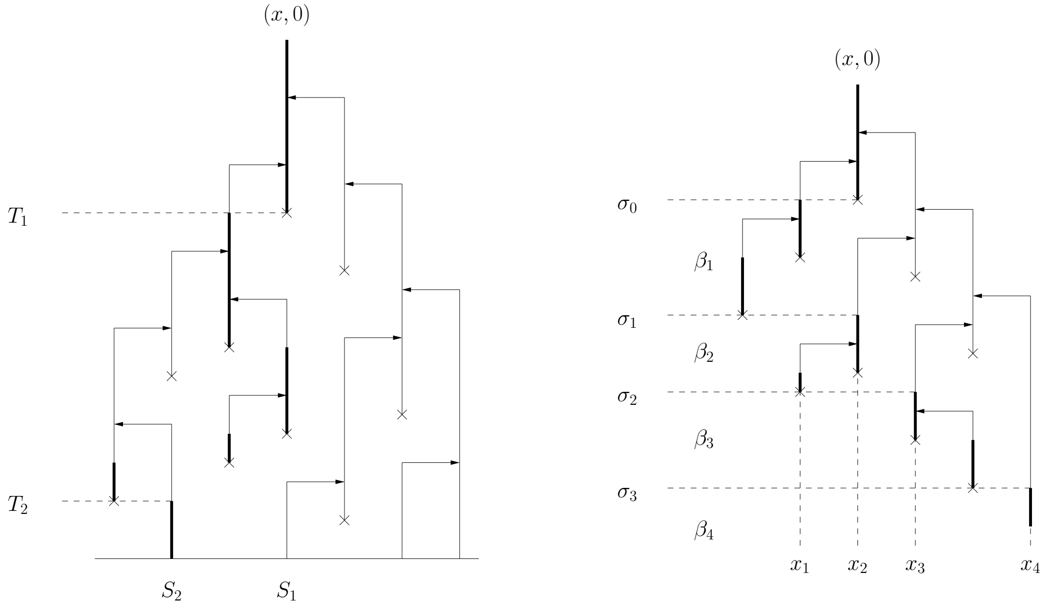

Figure 4. *Pictures of the dual process.*

4. Proof of Theorem 3 in dimension ≤ 2

The techniques and tools we will make use to prove Theorem 3 are quite different depending on the dimension of the state space. In this section, we deal with the case $d \leq 2$ relying on the recurrence of 1 and 2-dimensional random walks. To begin with, we will show that the determining particle is trapped with probability 1 inside a lower tree that lives forever so that, for t large enough, the determining particle is a lower ancestor. At this point, the worst scenario we have in mind is that the distinguished particle lands on a 1 and the determining particle on a 2. In such a case the determining particle, which is a lower ancestor, cannot paint $(x,0)$ any color whereas the distinguished particle can possibly bring a 1 to $(x,0)$. To conclude, we will then prove that this bad event is negligible showing that, with probability 1, we can make coalesce the distinguished particle and the determining particle together and so make them land on the same site. If this site is occupied by a particle of type 1, the determining particle will paint $(x,0)$ the color 2. If the site is occupied by a particle of type 2, the distinguished particle will paint $(x,0)$ the color 2 unless a lower ancestor does it earlier.

First of all, we can observe that if the upper tree starting at $(x,0)$ does not live forever then, for t large enough, $\hat{\xi}_t^{(x,0)} = \hat{\xi}_t^{x,2}$. In particular, since each of the lower ancestors meets one $\times$, the point $(x,0)$ cannot be reached in this case by a 1 (see the description of the ancestor hierarchy in Section 3). So, to avoid trivialities, we suppose that the upper tree Γ lives forever. Note that the probability of such an event is equal to the survival probability of the contact process with parameter λ starting from one infected site, that is positive since $\lambda > \lambda_c$. The first step is to prove that the determining particle is a.s. trapped inside a lower tree after a finite time. More precisely, if we let Θ_1 be the last time the determining particle is strongly connected with $(x,0)$ then

Lemma 4.1 $P(\Theta_1 = \infty) = 0$.

Proof To begin with, let s_k, $k \geq 1$, be the jumping times of the determining particle and x_k its location before s_k. Denote by σ_1 the first time the particle meets a $\times$, i.e.,

$$\sigma_1 \; = \; \inf\{t \geq 0 : \hat{\xi}_t^{(x,0)}(1) \text{ is a lower ancestor}\},$$

and by Ω_1 the lower tree starting at (x_1, σ_1), that is the first lower tree the particle visits. See Figure 5, picture on left. In view of the duality properties, once the particle penetrates in Ω_1, it remains trapped inside (as long as the lower tree is alive). Hence, if Ω_1 lives forever, $\hat{\xi}_t^{(x,0)}(1)$ is a lower ancestor for any $t \geq \sigma_1$, and the proof is done. Otherwise, we denote by σ_2 the first time the determining particle visits a new lower tree after Ω_1 dies and by Ω_2 this lower tree. Note that for all $k \geq 1$ the path the particle takes to climb from (x_k, s_k^-) to $(x, 0)$ contains one $\times$ so σ_2 is a.s. finite and Ω_2 is well defined. While the particle is not trapped in a lower tree that lives forever, we thus construct by induction a sequence of trees Ω_k visited by the first ancestor. Now, denote by B_n the event that the first n trees $\Omega_1, \Omega_2, \ldots, \Omega_n$ are bounded and, for any $k \geq 1$, by A_k the event that the k-th tree lives forever. If A_k does not occur then Ω_{k+1} is well defined and the event A_{k+1} is determined by parts of the graph that are after Ω_k dies so A_k and A_{k+1} are independent. More generally, since the trees $\Omega_1, \Omega_2, \ldots, \Omega_{k+1}$ are disjoint, $A_1, A_2, \ldots, A_{k+1}$ are independent. Moreover, since Ω_k has the same distribution as the tree structure of the contact process, the probability that A_k occurs is given by $p(\lambda)$, the survival probability of the contact process with parameter λ starting from one infected site. This implies that

$$P(B_n) \; = \; P(A_1^c \cap \ldots \cap A_{n-1}^c \cap A_n^c) \; = \; \prod_{k=1}^n P(A_k^c) \; = \; (1 - p(\lambda))^n.$$

Finally, since λ is supercritical, the survival probability of the contact process $p(\lambda)$ is strictly positive so that $\lim_{n \to \infty} P(B_n) = 0$. This completes the proof of the lemma. □

The next step is to prove that the determining and distinguished particles coalesce with probability 1. To do this, we first note that after penetrating in a lower tree Γ_1 that never dies, the determining particle is weakly connected with $(x, 0)$ and so jumps to a new branch each time it meets a $\times$. In particular, from time Θ_1, the path of the determining particle can be broken up, as that of the distinguished particle, into i.i.d. pieces. We define the renewal points of the determining particle as before replacing the upper tree Γ by the lower tree Γ_1 starting at $(S_0(1), \Theta_1)$. Here, $S_0(1)$ is the site where the determining particle jumps at time Θ_1. We denote by $(S_n(1), T_n(1))$ the location of the n-th renewal after Θ_1, and by $X_i(1)$ and $\tau_i(1)$ the spatial and temporal displacements between two consecutive renewals, so

$$S_n(1) \; = \; S_0(1) \; + \; \sum_{i=1}^n X_i(1) \qquad \text{and} \qquad T_n(1) \; = \; \Theta_1 \; + \; \sum_{i=1}^n \tau_i(1).$$

By translation invariance of the graphical representation (see the description of the tree structure in Section 3), the families $\{(X_i, \tau_i)\}_{i \geq 1}$ and $\{(X_i(1), \tau_i(1))\}_{i \geq 1}$ are identically distributed so Proposition 3.1 holds again for the random vectors $(X_i(1), \tau_i(1))$. In particular, as long as their triangles do not collide, both particles behave nearly like independent random walks. This constitutes the main ingredient to prove coalescence.

The first idea to establish coalescence is to extend the notion of renewals for both particles, that is to break up the set of both paths into i.i.d. pieces. To do this, we say that an ancestor is *good* at time t if it did not meet any arrow since its last renewal. Observe that if both particles are good at the same time, what happens before and after that time uses disjoint parts of the graph and then is independent. We now prove that both particles are good i.o. at the same time.

Lemma 4.2 P (*the determining and distinguished particles are good at the same time i.o.*) $= 1$.

Proof To begin with, we construct by induction two sequences of subscripts $(n_k)_{k \geq 0}$ and $(m_k)_{k \geq 1}$ as follows. We let $n_0 = 1$, and for any $k \geq 1$

$$m_k \; = \; \min\{m \geq 1 : T_m > T_{n_{k-1}}(1)\} \qquad \text{and} \qquad n_k \; = \; \min\{n \geq 1 : T_n(1) > T_{m_k}\}.$$

See Figure 5, picture on right, for an illustration. Then, denote by E_k the event that the determining particle lives without giving birth between time $T_{n_{k-1}}(1)$ and time T_{m_k}. Note that if E_k

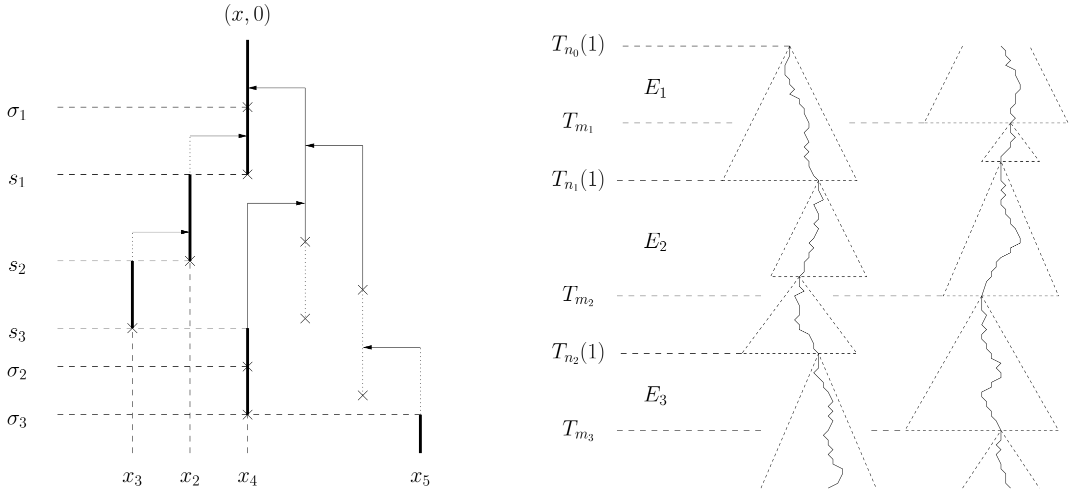

Figure 5. *Pictures of the dual process.*

occurs then both particles are obviously good at time T_{m_k}. Moreover, in view of the exponential bound given by Proposition 3.1, for any time $T > 0$

$$P\,(T_{m_k} - T_{n_{k-1}}(1) > T) \;\leq\; P\,(T_{m_k} - T_{m_k-1} > T) \;=\; P\,(\tau_1 > T) \;\leq\; C\,e^{-\beta T}$$

so that $P\,(E_k) \;\geq\; (1 - C\,e^{-\beta T})\,e^{-(1+2d\lambda)T}$. Here, $e^{-(1+2d\lambda)T}$ is the probability that the determining particle lives without giving birth for T units of time. Since this holds for all $T > 0$, there exists a constant $\varepsilon_0 > 0$ such that $P(E_k) \geq \varepsilon_0$. In other respects, the events E_k are determined by disjoint parts of the graphical gadget so they are independent. Hence, by the Borel-Cantelli Lemma, we can conclude that

$$P\,(\text{both particles are good at the same time i.o.}) \;\geq\; P\,(\limsup_{k\to\infty} E_k) \;=\; 1.$$

This proves the lemma. $\square$

To make coalesce both particles together, we now proceed in two steps. First of all, relying on the recurrence of 1 and 2-dimensional random walks, Neuhauser (1992) proved that with positive probability we can bring both particles within a finite distance K without collision of their triangles. Then, as soon as the particles are close enough to each other, we try to make them coalesce. More precisely, we have the following lemma.

Lemma 4.3 *If the determining and distinguished particles are within a finite distance K at some time $t \geq \Theta_1$ then the event A that they coalesce has positive probability.*

Proof To find a lower bound for $P(A)$, we consider the following particular event. We require the distinguished particle to keep still for $3dK$ units of time and the determining particle to jump toward the distinguished particle until they coalesce. To estimate this event, we observe that since both ancestors are within a distance K it takes the determining particle at most dK steps by increasing or decreasing each of its coordinates to reach the distinguished particle. Moreover, the probability of having neither birth nor death between times 0 and 1, a good oriented arrow between times 1 and 2, and a death between times 2 and 3 is given by

$$e^{-2}\,(1 - e^{-1})\,e^{-(4d-1)\lambda}\,(1 - e^{-\lambda}).$$

Since it takes the determining particle at most $3dK$ units of time to reach the distinguished particle in the manner just described, we can conclude that

$$P\,(A) \;\geq\; e^{-3dK(1+2d\lambda)}\,\Big[e^{-2}\,(1 - e^{-1})\,e^{-(4d-1)\lambda}\,(1 - e^{-\lambda})\Big]^{dK} \;>\; 0$$

where $e^{-3dK(1+2d\lambda)}$ is the probability that the distinguished particle survives without giving birth for $3dK$ units of time. This completes the proof. $\square$

If we do not succeed in gluing the particles together, we use the restart argument given by Lemma 4.2, i.e., we wait until both particles are good at the same time and then start over again the whole procedure. Since the set of both paths is broken into i.i.d. pieces, we can apply the Borel-Cantelli Lemma to conclude that coalescence eventually occurs with probability 1.

To complete the proof of Theorem 3 in $d \leq 2$ we now use the dual process $\tilde{\xi}_s^{(x,t)}$, $0 \leq s \leq t$, starting at (x,t) and determine the ancestor hierarchy after t units of time by going backwards in time. First of all, since both particles coalesce a.s., we can suppose by taking t large enough that they land at time 0 on the same site. If this site is occupied by a 1, the determining particle, that is a lower ancestor by Lemma 4.1, will paint (x,t) the color 2. On the other hand, if both land on a 2, it is the distinguished particle that will paint (x,t) the color 2 unless a lower ancestor succeeds earlier. Finally, if the target site is empty, we start over again with the second ancestor, and so on. Since the tree starting at the point where both particles coalesce is linearly growing in time and ξ_0 is translation invariant, we eventually find, by Lemma 9.14 in Harris (1976), an ancestor landing on an occupied site that will bring a 2-particle to (x,t). This concludes the proof.

5. Proof of Theorem 3 in dimension ≥ 3

The strategy of the proof to deal with the case $d \geq 3$ is quite different. To begin with, we will construct by induction an ordered set of ancestors $\zeta_t(k)$, $k \geq 1$, that are candidates for painting $(x,0)$ the color 2. Using Lemma 4.1, we will prove that for any $k \geq 1$, and for t sufficiently large, $\zeta_t(k)$ is a lower ancestor that comes before the distinguished particle in the ancestor hierarchy. Then, relying on the transience of d-dimensional random walks for $d \geq 3$, we will extract a subsequence of ancestors, denoted by $\zeta_t(k_i)$, $i \geq 1$, that never coalesce together. In particular, the number of sites occupied by these candidates can be made arbitrarily large so that we find one landing on a 1 that will paint $(x,0)$ the color 2.

We start by constructing inductively the ordered ancestor set $\zeta_t(k)$, $k \geq 1$. The first member of the sequence $\zeta_t(1)$ is the determining particle. Before defining the second member, we wait until the first one penetrates in a lower tree Γ_1 that lives forever. By Lemma 4.1, the time Θ_1 when this event occurs is a.s. finite, so Γ_1 is well defined. Then, we look at the ancestor hierarchy at time Θ_1 and discard all the ancestors that land on $S_0(1)$ or a site that does not live forever. Here, $S_0(1)$ is the spatial location of the determining particle at time Θ_1. The second member of the sequence is then the first ancestor that is left. Observe that such an ancestor exists since we assumed that Γ lives forever. Moreover, by using the arguments of Lemma 4.1, one can prove that $\zeta_t(2)$ also penetrates with probability 1 in a lower tree Γ_2 that lives forever. We then repeat the same procedure to define the third candidate, and so on.

For any $k \geq 1$, we let Γ_k be the infinite lower tree visited by the k-th member of the sequence, and denote by Θ_k the first time the ancestor penetrates in Γ_k and by $S_0(k)$ its spatial location at time Θ_k. As previously, one can break up from time Θ_k the path of the k-th member into i.i.d. pieces. We denote by $(S_n(k), T_n(k))$ the n-th renewal point, by $X_i(k)$ the spatial displacement between consecutive renewals, and by $\tau_i(k)$ the corresponding temporal displacement, i.e.,

$$S_n(k) \ = \ S_0(k) \ + \ \sum_{i=1}^{n} X_i(k) \qquad \text{and} \qquad T_n(k) \ = \ \Theta_k \ + \ \sum_{i=1}^{n} \tau_i(k).$$

The strategy to prove that with probability 1 there exists a subsequence $\zeta_t(k_i)$ of lower ancestors that never coalesce together is the following. First of all, we let $n \in \mathbb{N}^*$, the set of positive integers, and assume that there exist n lower ancestors $\zeta_t(k_1)$, $\zeta_t(k_2)$, $\ldots$, $\zeta_t(k_n)$ that never coalesce together. Then, given $m > k_n$, we will prove that with positive probability we can

$G_{m,K}$ = Bring $\zeta_t(m)$ at least K units apart from each of the $\zeta_t(k_i)$ and trap the $n+1$ ancestors inside large disjoint cubes without collision of their triangles.

Here, G is for good event. Then, relying on Lemma 5.5 of Neuhauser (1992), we will prove that on the event $G_{m,K}$ and for K sufficiently large

$$\lim_{t\to\infty} \inf_{1\le i\le n} \ \|\zeta_t(k_i) - \zeta_t(m)\| \ = \ +\infty$$

with probability close to 1. We will then conclude by proving that with probability 1 the good events $G_{m,K}$ occur for infinitely many $m > k_n$. In what follows, we give the proofs for the determining particle only but the same holds for the other members of the sequence.

Lemma 5.1 *Let $K \in \mathbb{N}^*$ and $D_K = [-K, K]^d$. For any $t \ge \Theta_1$ the event B that the determining particle leaves D_K in less than one unit of time and then survives without giving birth until time $t + K$ has positive probability.*

Proof If the determining particle stands out of D_K at time t, then $P(B)$ can be bounded from below by $e^{-(1+2d\lambda)K}$, that is the probability of living without giving birth for K units of time. Otherwise, we require the determining particle to increase or decrease its first spatial coordinate until leaving D_K. A straightforward calculation shows that

$$P\,(B) \ \ge \ e^{-(1+2d\lambda)K} \ \Big[e^{-1/K} \ (1 - e^{-1/2K}) \ e^{-d\lambda/K} \ (1 - e^{-d\lambda/K}) \Big]^K \ > \ 0$$

since it takes at most K steps for the ancestor to reach a face. This concludes the proof. □

Lemma 5.2 *Let $H_{t,K}$ be the event that the triangles of the first ancestor are inside a box $z + D_K$ from time t to time $t + K$. For any $\varepsilon_1 > 0$, there exists K such that $P\,(H_{t,K}) \ge 1 - \varepsilon_1$.*

Proof To make the notations easier, we will omit, all along the proof, the number 1 that refers to the determining particle. Moreover, since the graphical representation is translation invariant, we can suppose that the first renewal point after time t stands on the origin and let $z = 0$. Observe that in this case, $H_{t,K}$ occurs if the spatial locations of the particle at the renewals do not leave the box $\frac{1}{2}\,D_K$ and if each triangle is smaller than $\frac{1}{2}\,K$. See Figure 6 for a picture. Denoting by $\hat{\gamma}_s$ the location of the random walk S_n at time s and by $N_{t,K}$ the number of triangles between times t and $t + K$, we then obtain

$$\begin{aligned} P\,(H_{t,K} \text{ does not occur}) \ &\le \ P\left(N_{t,K} > \frac{2}{m}\,K \right) \\ &+ \ P\left(\text{the largest triangle is } > \frac{K}{2} \ ; \ N_{t,K} \le \frac{2}{m}\,K \right) \\ &+ \ P\left(\|\hat{\gamma}_s\|_\infty > \frac{K}{2} \text{ for some } \ t \le s \le t + K \ ; \ N_{t,K} \le \frac{2}{m}\,K \right) \end{aligned}$$

where $m = \mathbb{E}\,\tau_1$. To begin with, we can bound the first term on the right-hand side by using the large deviation estimate

$$P\left(N_{t,K} > \frac{2}{m}\,K \right) \ \le \ C\,e^{-\beta K}$$

for appropriate $C < \infty$ and $\beta > 0$. Moreover, since the random vectors X_i, $i \ge 1$, have the same distribution, the second term can be bounded as follows

$$\begin{aligned} &P\left(\text{the largest triangle is } > \frac{K}{2} \ ; \ N_{t,K} \le \frac{2}{m}\,K \right) \\ &\qquad \le \ P\left(\max_{1\le i\le 2m^{-1}K} \|X_i\|_\infty > \frac{K}{2} \right) \ \le \ \frac{2}{m}\,K \ P\left(\|X_1\|_\infty > \frac{K}{2} \right) \ \le \ \frac{2}{m}\,K\,C\,e^{-\beta K/2}. \end{aligned}$$

Observe now that if the random walk S_n leaves the box $\frac{1}{2}\,D_K$ then at least one of its coordinates is bigger than $\frac{1}{2}\,K$ so the third term can be bounded by

$$\sum_{i=1}^{d}\;P\left(|\hat{\gamma}_s^{\,(i)}| > \frac{K}{2}\ \text{for some}\ t \le s \le t+K\ ;\ N_{t,K} \le \frac{2}{m}\,K\right)$$

$$=\;d\;P\left(|\hat{\gamma}_s^{\,(1)}| > \frac{K}{2}\ \text{for some}\ t \le s \le t+K\ ;\ N_{t,K} \le \frac{2}{m}\,K\right)$$

where the superscript (i) refers to the i-th coordinate. We can use the reflection principle to bound this last term by

$$2\,d\;P\left(|S^{(1)}_{2m^{-1}K} - S^{(1)}_0| > \frac{K}{2}\right).$$

In other respects, Chebyshev's inequality gives for any $\theta > 0$

$$P\left(|S^{(1)}_{2m^{-1}K} - S^{(1)}_0| > \frac{K}{2}\right)\;\le\;e^{-\theta K/2}\prod_{i=1}^{2m^{-1}K}\mathbb{E}\,e^{\,\theta X_i^{(1)}}\;=\;\exp\left(-\frac{\theta K}{2} + \frac{2K}{m}\,\log\phi(\theta)\right)$$

where $\phi(\theta)$ is the moment generating function of $X_1^{(1)}$. Since $\mathbb{E}\,X_1^{(1)} = 0$ and Var $X_1^{(1)} < \infty$ we can state that $\log\phi(\theta) \le C\,\theta^2$ for some $C > 0$ and for θ small enough. In particular, taking $\theta = \dfrac{1}{\sqrt{K}}$ in the last expression we conclude that

$$P\left(\|\hat{\gamma}_s\|_\infty > \frac{K}{2}\ \text{for some}\ \ t \le s \le t+K\ ;\ N_{t,K} \le \frac{2}{m}\,K\right)\;\le\;2\,d\,e^{-\sqrt{K}/4}$$

for K large enough. Putting things together, we can finally maintain that

$$P\,(H_{t,K})\;\ge\;1\;-\;C\,e^{-\beta K}\;-\;\frac{2}{m}\,K\,C\,e^{-\beta K/2}\;-\;2\,d\;e^{-\sqrt{K}/4}.$$

This completes the proof of the lemma. □

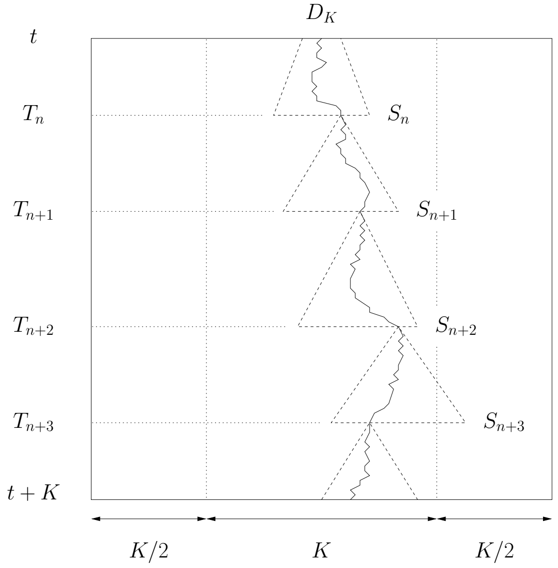

Figure 6. *Picture of the event* $H_{t,K}$.

We are now ready to prove that with probability 1 there is an integer $m > k_n$ such that $\zeta_t(m)$ does not coalesce with the first n members of the subsequence $\zeta_t(k_i)$. We let $G_{m,K}$ be the event that for any $1 \leq i \leq n$ there is a time $t_i \in [\Theta_m, \Theta_m + K\,]$ such that $\zeta_t(k_i)$ and $\zeta_t(m)$ are good and at least K units apart from each other at time $t = t_i$.

Lemma 5.3 *For any $\varepsilon_2 > 0$, there exists K sufficiently large such that on the event $G_{m,K}$*

$$P\Big(\lim_{t\to\infty}\inf_{1\leq i\leq n}\|\zeta_t(k_i) - \zeta_t(m)\| \,=\, +\infty\Big) \;\geq\; 1 - \varepsilon_2.$$

Proof We let $1 \leq i \leq n$ and assume that the ancestors $\zeta_t(k_i)$ and $\zeta_t(m)$ are good and K units apart at some time $t \geq \Theta_m$. Then, Lemma 5.5 of Neuhauser (1992) implies that there exists a constant $C > 0$ such that

$$\begin{aligned}&P\Big(\text{the particles } \zeta_t(k_i) \text{ and } \zeta_t(m) \text{ do never coalesce}\Big)\\ &\qquad = \; P\Big(\lim_{t\to\infty}\|\zeta_t(k_i) - \zeta_t(m)\| \,=\, +\infty\Big) \;\geq\; 1 \;-\; C\,K^{-1/10} \;-\; 2\,C\,K^{-3/32}.\end{aligned}$$

In particular, for K large enough,

$$\begin{aligned}&P\Big(\zeta_t(k_i) \text{ and } \zeta_t(m) \text{ coalesce for some } 1 \leq i \leq n\Big)\\ &\qquad \leq \; \sum_{i=1}^{n} P\Big(\zeta_t(k_i) \text{ and } \zeta_t(m) \text{ coalesce}\Big) \;\leq\; n\,C\,K^{-1/10} \;+\; 2\,n\,C\,K^{-3/32} \;\leq\; \varepsilon_2.\end{aligned}$$

This proves the lemma. □

Lemma 5.4 *There exists K sufficiently large such that $P\left(\limsup_{m\to\infty} G_{m,K}\right) = 1$.*

Proof Since the n lower ancestors $\zeta_t(k_i)$ do not coalesce and that $\Theta_m \to \infty$, we can find by Lemma 5.3 a large enough m such that the distances between the ancestors at time Θ_m are bigger than $4\sqrt{d}K$. We denote by H_i the event that the triangles of $\zeta_t(k_i)$ are contained, between time Θ_m and time $\Theta_m + K$, in some box $\Omega_i = z_i + D_K$. Observe that m has been chosen so that $\Omega_i \cap \Omega_j = \varnothing$ as soon as $i \neq j$. Hence the events H_i are determined by disjoint parts of the graph, and then are independent. This together with Lemma 5.2 implies that for K sufficiently large

$$P\,(H_1 \,\cap\, \cdots \,\cap\, H_n) \;=\; \prod_{i=1}^{n} P\,(H_i) \;\geq\; 1 - n\,\varepsilon_1.$$

Now that the n lower ancestors $\zeta_t(k_i)$ are trapped inside large disjoint cubes, we require each of them to be good at least once between time Θ_m and time $\Theta_m + K$. Since this occurs if each of the n ancestors has at least one renewal in this interval of time, the probability that this event occurs can be bounded from below by

$$\prod_{i=1}^{n} P\,(\tau_1(k_i) < K) \;\geq\; \Big[1 - C\,e^{-\beta\,K}\Big]^n$$

for appropriate $C < \infty$ and $\beta > 0$. The last thing we need to conclude is that $\zeta_t(m)$ is good and at least K units apart from each other ancestor between time $\Theta_m + 1$ and time $\Theta_m + K$. Since the cubes Ω_i are at least $2K$ units apart from each other, this occurs, by Lemma 5.1, with positive probability. Putting things together we can state that there exists an $\varepsilon_3 > 0$ such that $P(G_{m,K}) \geq \varepsilon_3$ for m sufficiently large. By observing that the events $G_{m_1,K}$ and $G_{m_2,K}$ are independent as soon as $|\Theta_{m_1} - \Theta_{m_2}| > K$, we can conclude by the Borel-Cantelli Lemma that with probability 1 the events $G_{m,K}$ occur for infinitely many $m \geq 1$. □

To conclude the proof of Theorem 3, we now use, as in Section 4, the dual process $\tilde{\xi}_s^{(x,t)}$ starting at (x,t) and determine the ancestor hierarchy by going backwards in time. We denote by $B_s^{(x,t)}$

the set of sites occupied at time s by a 1 and let $\zeta_t = \{\zeta_t(k) : k \geq 1 \text{ with } \Theta_k \leq t\}$. Firstly, Lemma 5.3 implies that on the event $G_{m,K}$ the lower ancestor $\zeta_t(m)$ coalesces with one of the $\zeta_t(k_i)$ with probability $< \varepsilon_2$. Moreover, by Lemma 5.4, $G_{m,K}$ occurs for infinitely many $m \geq 1$. This implies in particular that the cardinality of ζ_t can be made arbitrarily large by choosing t large enough, i.e., given $\varepsilon_4 > 0$ and $M > 0$ there exists a time $t_0 \geq 0$ so that $P\,(\operatorname{card}\zeta_t < M) \leq \varepsilon_4$ for any $t \geq t_0$. By Lemma 9.14 of Harris (1976), the translation invariance of ξ_0 then implies that

$$\lim_{t\to\infty} P\,(\zeta_t \cap B_0^{(x,t)} = \varnothing) \;=\; 0.$$

So, for t sufficiently large, there is a lower ancestor $\zeta_t(n)$ that lands at time 0 on a 1. Now, we look at the ancestors that come before $\zeta_t(2)$ in the hierarchy until we find one that lands on an occupied site. If a 1 stands at this site, the ancestor will paint (x,t) the color 2 and the proof is done. On the other hand, if the site is occupied by a 2, $S_0(1)$ will be empty at time $(t-\Theta_1)^+$ whatever the color of the next ancestors. Then, we look at $\zeta_t(2)$, and so on. Finally, if none of the ancestors that come before $\zeta_t(n)$ in the hierarchy succeeds in painting (x,t) the color 2, this last one will do it. This completes the proof of Theorem 3.

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Multitype contact process with frozen states: a spatial model of allelopathy

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Abstract In this paper, we introduce a generalization of the two colors multitype contact process intended to mimic a biological process called allelopathy. Precisely, we have two types of particles. Each type gives birth to particles of the same type and dies at rate 1. When a particle of type 1 dies, it gives way to a frozen site that blocks particles of type 2 for an exponentially distributed amount of time. Specifically, we investigate in details the phase transitions as well as the duality properties of the interacting particle system.

1. Introduction

The model we introduce in this paper is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \longrightarrow \{0,1,2,3\}$. A site $x \in \mathbb{Z}^d$ is said to be occupied by a particle of type 1 (resp. 2) if $\xi(x) = 1$ (resp. 2), and is said to be empty otherwise. We distinguish two types of empty site. Namely, a site $x \in \mathbb{Z}^d$ will be called a *free site* if $\xi(x) = 0$ and a *frozen site* if $\xi(x) = 3$. The evolution rules are defined as follows:

1. Each 1 (resp. 2) tries to give birth onto each of its neighbors at rate λ_1 (resp. λ_2). Here, the neighbors of a site $x \in \mathbb{Z}^d$ is the set of $y \in \mathbb{Z}^d$ such that $\|x - y\| \leq R$ where $\| \cdot \|$ is a norm and where R is a positive constant.
2. If the offspring of a 1 (resp. of a 2) is sent to a site in state 0 or 3 (resp. in state 0), the birth occurs. Otherwise, it is suppressed.
3. Both types die at rate 1. Type 1 (resp. 2) gives way to a frozen site (resp. to a free site).
4. Frozen sites (3) become free (0) at rate $\gamma > 0$.

In particular, the process is a generalization of the multitype contact process (Neuhauser, 1992) in which 1's inhibit the spread of 2's by freezing the sites they have just occupied. Reciprocally, the multitype contact process is just the extreme case $\gamma = \infty$ in which the transition $3 \to 0$ is instantaneous. The interpretation we have in mind is that of a spatial model of *allelopathy.* In biology literature, allelopathy is defined as a process involving secondary metabolites produced by plants, micro-organisms, viruses and fungi that influence growth and development of biological systems. In our case, 1's are the individuals of an *inhibitory* species and 2's the individuals of a *susceptible* species. The reader especially interested in this biological process can refer to Durrett and Levin (1997) as well. Their stochastic spatial model also is a generalization of the multitype contact process but with only three states: 0 = empty, 1 = inhibitory species, and 2 = susceptible species. Particles of type 1 die at rate 1 while there is a constant $c > 0$ such that particles of type 2 die at rate $1 + c \times$ the number of neighbors in state 1. That is, the particles of type 1 increase the death rate of the neighboring particles of type 2. Precisely, their stochastic spatial process models the competition of the colicin-producing *Escherichia coli* and colicin-sensitive bacteria. The particle system we introduce in this paper, on the contrary, is more appropriate to investigate plant competitions involving inhibitory species such as *Hieracium pilosella.* In this case, the inhibitory species produces toxic substances that prevent susceptible species from setting up for a certain amount of time.

To investigate our model, we first observe that if only 2's are present, the process reduces to the basic contact process with parameter λ_2. In such a case, there exists a critical value $\lambda_c \in (0, \infty)$

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such that if $\lambda_2 \le \lambda_c$ then the process converges in distribution to the all empty state while if $\lambda_2 > \lambda_c$ there exists a stationary measure μ_2 that concentrates on configurations with infinitely many 2's. See, e.g., Liggett (1999). If only 1's are present, we have almost the same result: If $\lambda_1 \le \lambda_c$ then the process converges in distribution to the all empty state while if $\lambda_1 > \lambda_c$ there exists a nontrivial stationary measure ν_1 that concentrates on configurations with infinitely many 1's and 3's. To construct this measure, we start the process from a configuration with infinitely many 1's, take the Cesaro average of the distributions from time 0 to time T and extract a convergent subsequence. Then, by Proposition 1.8 of Liggett (1985), the limit ν_1 is known to be an invariant measure. Moreover, since the 1's do not see the 3's, we obtain $\nu_1(\xi(x) = 1) = \mu_2(\xi(x) = 2)$ provided that $\lambda_1 = \lambda_2$. To avoid trivialities, we assume from now on that both λ_1 and λ_2 are $> \lambda_c$ and that ξ_0 contains infinitely many 1's and 2's.

We first set $\gamma_a < \gamma_b$ and denote by ξ_t^i the process with parameters λ_1, λ_2 and γ_i. Then, if we think of the processes as being generated by Harris' graphical representation, we may run ξ_t^a and ξ_t^b simultaneously, starting from the same initial configuration, in such a way that ξ_t^a has more 1's and fewer 2's than ξ_t^b, i.e., for any $x \in \mathbb{Z}^d$, if $\xi_t^a(x) = 2$ then $\xi_t^b(x) = 2$, and if $\xi_t^b(x) = 1$ then $\xi_t^a(x) = 1$. The same coupling argument implies that the process is also monotonous with respect to each of the parameters λ_1 and λ_2. These results are summarized in the following theorem.

Theorem 1 *Let $\Theta_t^i = \{x \in \mathbb{Z}^d : \xi_t(x) = i\}$ be the set of sites occupied at time t by a particle of type i. Then the survival probabilities $P\,(\Theta_t^i \neq \varnothing$ for all $t \ge 0)$, $i = 1, 2$, are monotonous with respect to each of the parameters λ_1, λ_2 and γ.*

In particular, if we set $\gamma_a \in (0, \infty)$ and $\gamma_b = \infty$ then the process ξ_t^a will have more 1's and fewer 2's than ξ_t^b. Now, as explained above, ξ_t^b is the multitype contact process with parameters λ_1 and λ_2. Theorem 1 in Neuhauser (1992) implies that if $\lambda_1 > \lambda_2$ and we start with infinitely many 1's, then $\xi_t^b \Rightarrow \mu_1$, the upper invariant measure of the basic contact process. Here, $\Rightarrow$ denotes weak convergence. It follows that

Theorem 2 *Assume that ξ_0 contains infinitely many 1's. If $\lambda_1 > \lambda_2$ and $\gamma \in (0, \infty)$ then $\xi_t \Rightarrow \nu_1$.*

We now focus on the case $\lambda_1 = \lambda_2$. Since the evolution rules help the 1's, we expect in this case that the processes with or without frozen states exhibit different behaviors. Theorem 3 tells us that if $\lambda_1 = \lambda_2$ and $\gamma < \infty$ the 1's still win in $d \ge 3$ while 1's and 2's coexist if $\gamma = \infty$. See Theorem 3 in Neuhauser (1992). We conjecture that the 1's win in any dimension but our proof heavily relies on transience of symmetrical random walks in $d \ge 3$.

Theorem 3 *Assume that ξ_0 contains infinitely many 1's and is translation invariant. If $\lambda_1 = \lambda_2$ and $d \ge 3$ then $\xi_t \Rightarrow \nu_1$.*

The proof of Theorem 3 partly relies on the duality techniques introduced in Neuhauser (1992) and Lanchier (2005). To figure out the properties of the dual process, we start by constructing the process from collections of Poisson processes in the case $\lambda_1 = \lambda_2$. For x, $y \in \mathbb{Z}^d$, $\|x - y\| \le R$, let $\{T_n^{x,y} : n \ge 1\}$, $\{U_n^x : n \ge 1\}$ and $\{V_n^x : n \ge 1\}$ be the arrival times of Poisson processes with rates λ_1, 1 and γ respectively. At times $T_n^{x,y}$, we draw an arrow from x to y to indicate that if x is occupied by a 1 and y is empty then y becomes occupied by a 1 while if x is occupied by a 2 and y is free then y becomes occupied by a 2. At times U_n^x, we put a cross × at x to indicate that a death occurs, and at times V_n^x, we put a dot • at x to indicate that a frozen site becomes free. Finally, we say that there is a *path* from (y, s) to (x, t) if there exists a sequence of times $s_0 = s < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 = y,\ x_1,\ \cdots\ x_n = x$ such that

1. For $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0, 1, \ldots, n$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any ×'s.

If there is a path from $(y, t - s)$ to (x, t), we say that there is a *dual path* from (x, t) to $(y, t - s)$ and define the *dual process starting at* (x, t) by

$$\tilde{\xi}_s^{(x,t)} \;=\; \{y \in \mathbb{Z}^d : \text{there is a dual path from } (x, t) \text{ to } (y, t - s)\}.$$

Then $\{(\tilde{\xi}_s^{(x,t)}, s) : 0 \le s \le t\}$ exhibits a tree structure that allows us to equip the dual process with an ordered relation in which the members are arranged according to the order they determine the color of (x, t). See Neuhauser (1992), Section 2. The tree $\{(\tilde{\xi}_s^{(x,t)}, s) : 0 \le s \le t\}$ will be called the *upper tree starting at* (x, t) and the elements of $\tilde{\xi}_s^{(x,t)}$ the *upper ancestors*. The first upper ancestor will be called the *distinguished particle*.

The main difference with the multitype contact process is that now 1's produce 3's that are forbidden for the 2's. In particular, the state of (x, t) will not depend only on the state of the upper ancestors at time 0. The key idea is to prove that the number of frozen sites visited by the distinguished particle on its way up to (x, t) tends to infinity as $t \to \infty$, which blocks 2's from determining the color of (x, t). If instead of the condition 2 above we have the condition

3. The set $\bigcup_{i=0}^{n} \{x_i\} \times (s_i, s_{i+1})$ contains exactly one $\times$,

we will say that (y, s) and (x, t) are *weakly connected* and that y is a *lower ancestor*. In view of the $\times$'s effect, if both points $(y, 0)$ and (x, t) are weakly connected, then a particle of type 1 at y at time 0 can *freeze* the path of the distinguished particle at some particular point. The aim will be to construct a collection of lower ancestors $\zeta_t(n)$ that are good candidates to realize this event. If all these ancestors coalesce, which occurs with probability 1 in $d \le 2$ by the results of Section 3 in Neuhauser (1992), and land at time 0 on a 2, we will not be able to conclude. If, on the contrary, $d \ge 3$, the lower ancestors $\zeta_t(n)$ will behave nearly like transient random walks so that it will be possible to extract a subsequence of ancestors $\zeta_t(n_k)$ that will land on sites occupied at time 0 by a particle of type 1.

Finally, if we focus on the case $\lambda_1 < \lambda_2$, it is not clear that the 2's win. Theorem 4 tells us that, in $d = 2$, the particles of type 2 win provided that γ is sufficiently large.

Theorem 4 *Assume that ξ_0 contains infinitely many 2's. If $d = 2$ and $\lambda_1 < \lambda_2$ then there exists a critical value $\gamma_c \in (0, \infty)$ such that $\xi_t \Rightarrow \mu_2$ for any $\gamma > \gamma_c$.*

To figure out the implications of our results, we fix $\lambda_1 > \lambda_c$ and $\gamma > 0$, and denote by $\beta_c(\gamma, \lambda_1)$ the infimum of $\lambda_2 \ge 0$ such that the 1's die out, with the convention $\inf \varnothing = \infty$. A fairly straightforward application of Theorems 1-4 then implies that $\lambda_1 \mapsto \beta_c(\gamma, \lambda_1)$ is nondecreasing and $\beta_c(\gamma, \lambda_1) \downarrow \lambda_1$ as $\gamma \uparrow \infty$. In conclusion, the phase diagram we obtain is given by Figure 1 where our results are summarized.

Unfortunately, we do not know what is the outcome of the competition when particles evolve in a spatial structure and (λ_1, λ_2) is such that $\lambda_1 < \lambda_2 < \beta_c(\gamma, \lambda_1)$. To deal with this case, we look at the mean-field model (Durrett and Levin, 1994), that is we pretend that all the sites are independent and that the system is spatially homogeneous. Then, the evolution can be formulated thanks to the following ordinary differential equations.

$$\begin{aligned} u_1' &= \lambda_1 u_0 u_1 + \lambda_1 u_3 u_1 - u_1 \\ u_2' &= \lambda_2 u_0 u_2 - u_2 \\ u_3' &= u_1 - \lambda_1 u_1 u_3 - \gamma u_3 \end{aligned}$$

where u_i denotes the density of sites in state i. Let $\Omega = \{u : u_i \ge 0,\ u_0 + u_1 + u_2 + u_3 = 1\}$ be the collection of values we are interested in and, for fixed $\gamma > 0$, set

$$\begin{aligned} D_1 &= \{(\lambda_1, \lambda_2) : \lambda_1 > 1 \text{ and } (\lambda_2 - \lambda_1)\gamma < (\lambda_1 - 1)\lambda_1\} \qquad \text{and} \\ D_2 &= \{(\lambda_1, \lambda_2) : \lambda_2 > 1 \text{ and } \lambda_2 > \lambda_1\}. \end{aligned}$$

First of all, a straightforward calculation shows that the ODE has a nontrivial fixed point u on the boundary $u_2 = 0$ if and only if $\lambda_1 > 1$, where nontrivial means $\neq (1, 0, 0, 0)$. Moreover, by studying the eigenvalues of the linearization at point u of the ODE, one can prove that the equilibrium u is stable if $(\lambda_1, \lambda_2) \in D_1$, and unstable otherwise, that is the linearization has an unstable direction that points into int Ω, the interior of Ω. In the same way, if $\lambda_2 > 1$ there is a nontrivial equilibrium

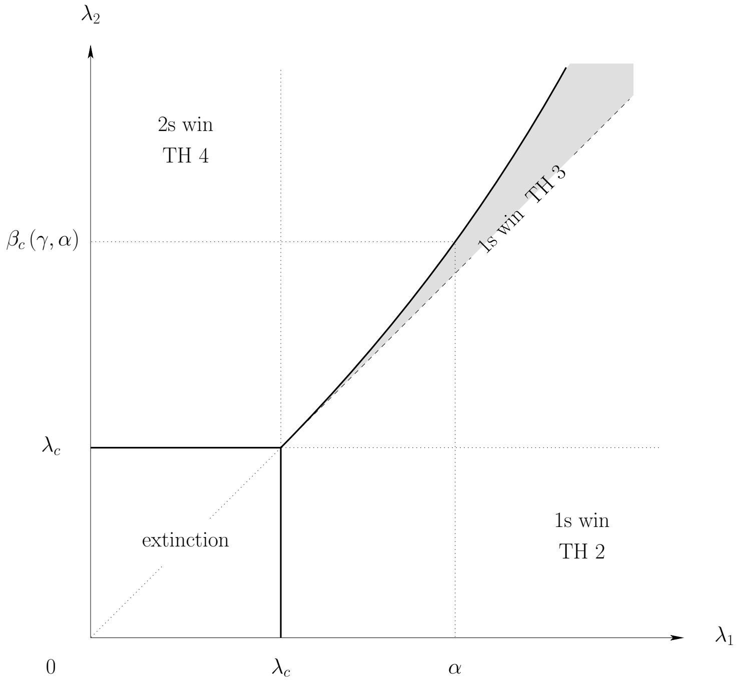

Figure 1. *Phase diagram.*

u on the boundary $u_1 = u_3 = 0$ that is stable if $(\lambda_1, \lambda_2) \in D_2$, and unstable otherwise. Finally, the ODE has a fixed point belonging to int Ω if and only if

$$\lambda_2 \ > \ \lambda_1 \ > \ 1 \qquad \text{and} \qquad \gamma \ < \ \frac{\lambda_1 - 1}{\lambda_2 - \lambda_1}\, \lambda_1$$

that is $(\lambda_1, \lambda_2) \in D_1 \cap D_2$. Our mean-field model, however, exhibits the same property as the mean-field model introduced in Durrett and Levin (1997). Namely, the boundary fixed points only are locally stable. See Figure 2 for a picture of the solution curves when $\lambda_1 = 2$, $\lambda_2 = 3$, and $\gamma = 1$ and 1.5 respectively. In words, if $(\lambda_1, \lambda_2) \in D_1 \cap D_2$ then none of both species can invade the other one in its equilibrium. If the density of particles of type 1 (resp. 2) is close to the equilibrium value, and particles of type 2 (resp. 1) are introduced with a low density, then the density of 2's (resp. 1's) shrinks to 0. In a homogeneously mixing population, the outcome of the competition then depends on the initial densities. Relying on the instability of the interior fixed point, we think that, for the particle system, given a set of parameters $\lambda_1, \lambda_2 > \lambda_c$ and $\gamma > 0$, there is a stronger type that will win the competition provided that ξ_0 contains infinitely many 1's and 2's. In conclusion, we summarize and complete Theorems 2-4 with the

Conjecture 5 *For any $\lambda_1 > \lambda_c$ and $\lambda_2 > \lambda_c$, there is a critical value γ_c such that the following holds: If $\gamma < \gamma_c$ then $\xi_t \Rightarrow \nu_1$ while if $\gamma > \gamma_c$ then $\xi_t \Rightarrow \mu_2$.*

The rest of this paper is devoted to proofs. In Section 2, we will investigate in greater details the duality properties of the process. Relying on the results of Section 2, we will then prove Theorem 3 in Section 3. Finally, the proof of Theorem 4 will be carried out in Section 4.

2. Graphical representation and duality

The first step in proving Theorem 3 and Theorem 4 is to construct the process from a collection of Poisson processes in the case $\lambda_1 \leq \lambda_2$. For x, $y \in \mathbb{Z}^d$, $\|x - y\| \leq R$, let $\{T_n^{x,y} : n \geq 1\}$,

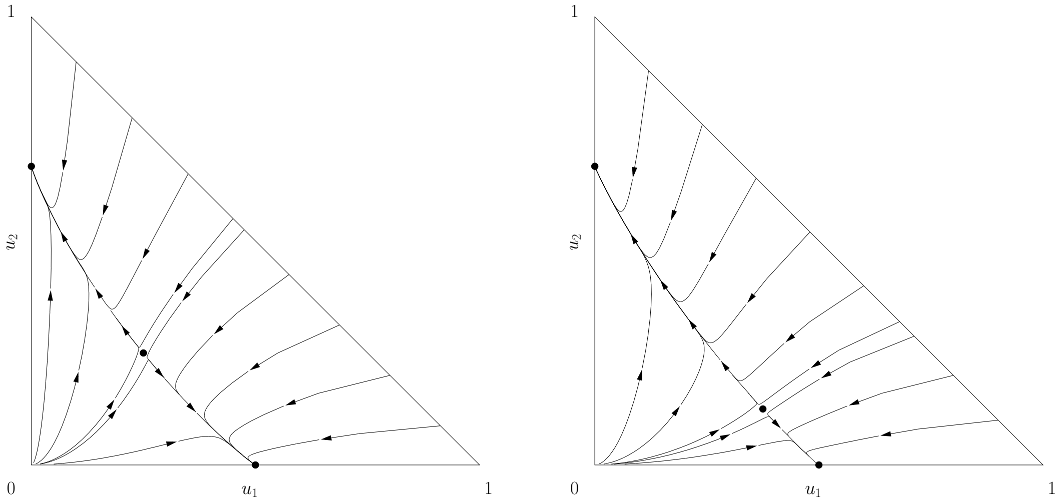

FIGURE 2. *Projection of the solution curves for the mean-field model. Picture on left:* $\lambda_1 = 2$, $\lambda_2 = 3$ *and* $\gamma = 1$. *Picture on right:* $\lambda_1 = 2$, $\lambda_2 = 3$ *and* $\gamma = 1.5$.

$\{U_n^x : n \geq 1\}$ and $\{V_n^x : n \geq 1\}$ be the arrival times of Poisson processes with rates λ_2, 1 and γ respectively. At times $T_n^{x,y}$, we draw an arrow from x to y, toss a coin with success probability $(\lambda_2 - \lambda_1)/\lambda_2$, and if there is a success label the arrow with a 2. If at time $T_n^{x,y}$ the site x is occupied by a 1, the site y is empty, that is free or frozen, and the arrow is unlabeled then y becomes occupied by a 1 while if x is occupied by a 2 and y is free then the site y becomes occupied by a 2. At times U_n^x, we put a cross × at x to indicate that a death occurs, i.e., a 1 gives way to a frozen site and a 2 to a free site. Finally, at times V_n^x, we put a dot • at x to indicate that a frozen site becomes free. A result of Harris (1972) implies that such a graphical representation can be used to construct the process starting from any initial configuration $\xi_0 : \mathbb{Z}^d \longrightarrow \{0, 1, 2, 3\}$. See Figure 3 for a picture. After constructing the graphical representation, we now define the dual process. We say that two points (x, s) and (y, t) in $\mathbb{Z}^d \times \mathbb{R}^+$ are *connected* or that there is a *path* from (x, s) to (y, t) if there exists a sequence of times $s_0 = s < s_1 < s_2 < \cdots < s_n < s_{n+1} = t$ and spatial locations $x_0 = x,\ x_1,\ x_2,\ \cdots\ x_n = y$ such that

1. For $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0, 1, \ldots, n$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any ×'s.

If instead of 2 we have the condition

3. The set $\displaystyle\bigcup_{i=0}^{n} \{x_i\} \times (s_i, s_{i+1})$ contains exactly one ×,

we say that (x, s) and (y, t) are *weakly connected*. Finally, we say that there is a *dual path* from (x, t) to $(y, t - s)$, $0 \leq s \leq t$, if there is a path from $(y, t - s)$ to (x, t). In other words, dual paths move against the direction of time and arrows. We then define the *dual process* by setting

$$\tilde{\xi}_s^{(x,t)} \ = \ \{y \in \mathbb{Z}^d : \text{there is a dual path from } (x, t) \text{ to } (y, t - s)\}$$

for any $0 \leq s \leq t$. Since it will be easier to work with a forward process than a backward process, we also introduce the dual $\hat{\xi}_s^{(x,0)}$ that is defined by

$$\hat{\xi}_s^{(x,0)} \ = \ \{y \in \mathbb{Z}^d : \text{there is a path from } (x, 0) \text{ to } (y, s)\}.$$

The reader will note that both processes $\tilde{\xi}_s^{(x,t)}$ and $\hat{\xi}_s^{(x,0)}$ have the same law. First of all, we observe that $\{(\tilde{\xi}_s^{(x,t)}, s) : 0 \leq s \leq t\}$ exhibits a tree structure. As for the mutitype contact process, such a structure allows us to equip the dual process $\tilde{\xi}_s^{(x,t)}$ with an ordered relation in which the members are arranged according to the order they determine the color of (x,t). See e.g., Neuhauser (1992), Section 2, for a complete description of this hierarchy. From now on, the tree

$$\Gamma \; = \; \{(\tilde{\xi}_s^{(x,t)}, s) : 0 \leq s \leq t\}$$

will be called the *upper tree starting at* (x,t) and the elements of $\tilde{\xi}_s^{(x,t)}$ the *upper ancestors*. We denote by $\tilde{\xi}_s^{(x,t)}(n)$ the n-th member of the ordered ancestor set, and call *distinguished particle* the first upper ancestor. For an example of ancestor hierarchy, see Figure 4, picture on left. The algorithm for deducing the location of the distinguished particle can be described as follows. First of all, we start at (x,t), and go down the graphical representation until the first time a × is encountered. Then, we go back up until the first time the tip of an arrow is encountered. We follow this arrow against its direction to the branch the arrow is attached to, and repeat the above procedure until we reach time 0. If (y,s') and (x,t) are weakly connected, the tree starting at (y,s') will be called a *lower tree* and the elements of $\tilde{\xi}_s^{(y,s')}$ the *lower ancestors*. We observe that, contrary to the multitype contact process, the state of some sites (free or frozen) strongly depends on the lower ancestors. Namely, a lower ancestor can bring a 1 up to a × and then freeze a site belonging to the upper tree.

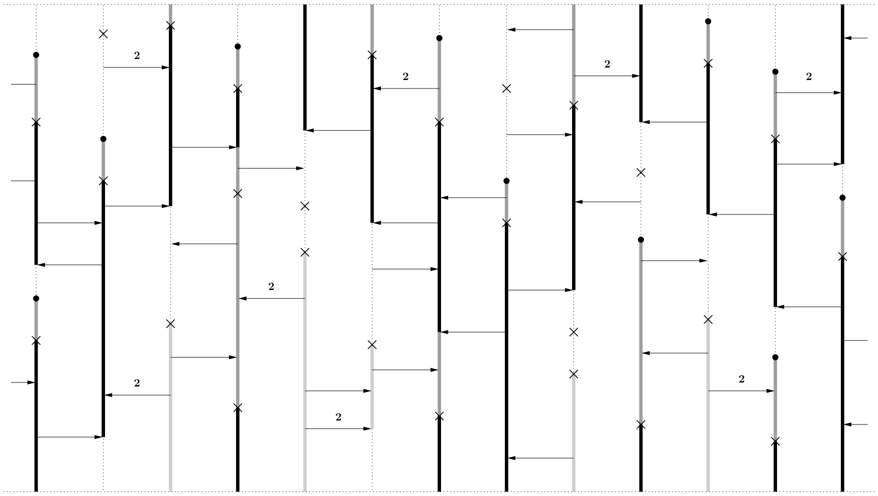

FIGURE 3. *Harris' graphical representation. The black lines refer to 1's, the pale gray ones to 2's, the dark gray ones to frozen sites, and the dotted ones to free sites.*

To conclude this section, we now describe an algorithm to determine the color of (x,t) in the case $\lambda_1 \leq \lambda_2$. We say that an arrow from x to y is *bad* for the 2's if its target site y is frozen. First of all, we determine which of the following three events occurs: The site the distinguished particle lands on at time 0 is (1) in state 1, (2) in state 2, or (3) in state 0 or 3. In the case 1, the distinguished particle will paint (x,t) the color 1 if it does not cross any 2-arrow. Otherwise, we follow the path of the distinguished particle on its way up to (x,t) until the first 2-arrow we encounter, look backwards in time starting from the location where this arrow is attached, and discard all the ancestors of this point. The reason why we need to discard these ancestors is that they are now blocked on their way up to (x,t) by a particle of type 1. In the case 2, the distinguished particle will paint (x,t) the color 2 if it does not cross any arrow bad for the 2's. Otherwise, we discard as previously all the ancestors of the point where the first bad arrow is attached since these ancestors are now blocked by a particle of type 2. Finally, in the case 3, the distinguished particle cannot paint (x,t) any color. If after the first trial, the distinguished particle did not paint (x,t) any color, we repeat the same procedure with the first upper ancestor that is left after discarding, and so on. We now refer to Figure 4, picture on left, for an illustration of this algorithm. The distinguished particle lands on a 2 (case 2) but crosses the arrow bad for the 2's that points from $x-2$ to $x-3$ before reaching (x,t) so we discard all the ancestors of this arrow. Since the only one is the distinguished particle, we now focus on the second ancestor of the hierarchy. The second ancestor lands on a 1 (case 1) but crosses the 2-arrow that points from x to $x-1$. Since the ancestors of this 2-arrow are the second and third ancestors, we look at the fourth ancestor. The fourth ancestor lands on a 2 (case 2) and does not cross any arrow bad for the 2's so (x,t) will be of type 2. The reader will note that, due to the $\bullet$ under its tip, the 2-arrow on the right side of the picture is not bad for the 2's.

3. Proof of Theorem 3

To establish Theorem 3, the strategy consists in proving that if the upper tree Γ lives forever then, with probability 1, the distinguished particle will jump infinitely often on a frozen site. To do this, we will focus on the structure of the lower trees and show that the number of lower ancestors that freeze the sites visited by the distinguished particle tends to infinity as $t\to\infty$. We will finally conclude by exhibiting an upper ancestor that will bring a 1 to (x,t).

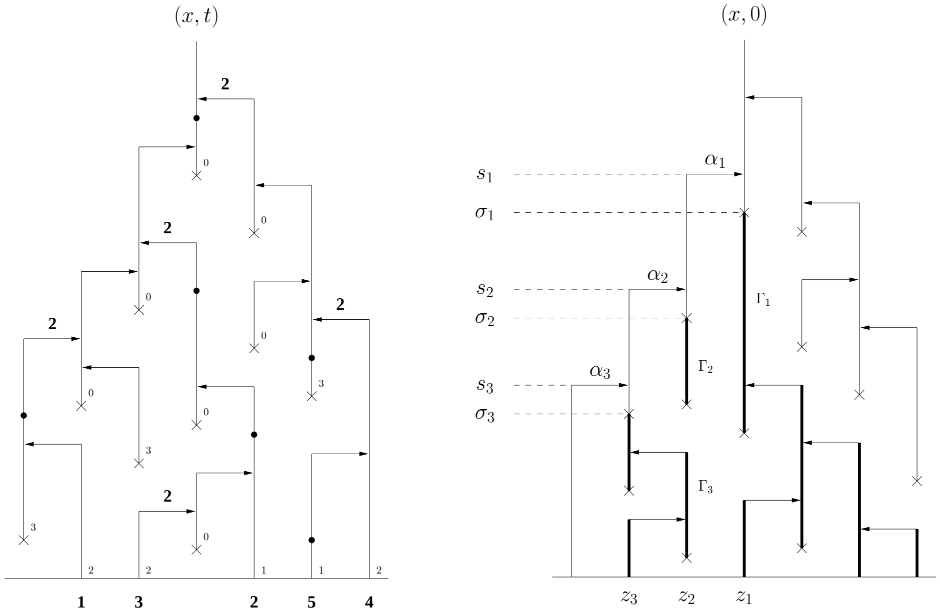

Figure 4. *Dual process.*

From now on, we denote by λ the common value of λ_1 and λ_2 and suppose that Γ lives forever. The reader will observe that such an event occurs with positive probability since $\lambda>\lambda_c$. For more convenience, we use in this section the dual process $\hat{\xi}_s^{(x,0)}$ starting at $(x,0)$. The main objective is to prove that the number of frozen sites visited by the distinguished particle tends to infinity as $t\to\infty$. First of all, we follow the path of the distinguished particle starting from $(x,0)$ and, for $n\geq 1$, denote by α_n the n-th arrow we cross. See Figure 4, picture on right. We let z_n and s_n be respectively the arrival site and the temporal location of the arrow α_n and denote by N_t the number of arrows α_n that point by time t to a frozen site, i.e.,

$$N_t \;=\; \operatorname{card}\{n\geq 1:\xi_{s_n}(z_n)=3 \text{ and } s_n\leq t\}.$$

By construction, N_t also denotes the number of frozen sites visited by time t by the distinguished particle. In particular, the main result we have to prove is the

Proposition 3.1 *If $d\geq 3$ then* $\lim_{t\to\infty} N_t=\infty$ *a.s.*

The intuitive idea of the proof is that the lower ancestors provide enough 1's to freeze the path of the distinguished particle at infinitely many points. First of all, we denote by σ_n the arrival time of the first × located under the tip of α_n, i.e.,

$$\sigma_n \;=\; \min\{U_k^{z_n} : U_k^{z_n} \geq s_n\}$$

and let Γ_n be the lower tree starting at (z_n, σ_n), that is

$$\Gamma_n \;=\; \{(y,s) \in \mathbb{Z}^d \times [\,\sigma_n, \infty) : \text{there is a path from } (z_n,\sigma_n) \text{ to } (y,s)\}.$$

See Figure 4 for a picture. We say that Γ_n is *good* if the following two conditions are satisfied.

1. Γ_n lives forever and
2. The vertical segment $\{z_n\} \times (s_n, \sigma_n)$ does not contain any dots.

As we will see further, the properties 1 and 2 will give us a good opportunity to freeze the site z_n at time s_n. Let G_n be the event that the n-th lower tree is good.

Lemma 3.2 $P\,(\limsup_{n\to\infty} G_n) = 1$.

Proof To begin with, denote by A_n the event that Γ_n lives forever, and by B_n the event that $\{z_n\} \times (s_n, \sigma_n)$ does not contain any dots. The first step is to prove that, for any $n \geq 1$, there exists a.s. an integer $m \geq n$ such that A_m occurs. To do this, we set $\Gamma_{n_1} = \Gamma_n$ and, while Γ_{n_k} is bounded, we denote by $\Gamma_{n_{k+1}}$ the first lower tree that is born after Γ_{n_k} dies. Note that if A_{n_k} does not occur, then $\Gamma_{n_{k+1}}$ is well defined and the event $A_{n_{k+1}}$ is determined by parts of the graph that are after Γ_{n_k} dies so that A_{n_k} and $A_{n_{k+1}}$ are independent. More generally, since the trees $\Gamma_{n_1}, \Gamma_{n_2}, \cdots, \Gamma_{n_{k+1}}$ are disjoint, the events $A_{n_1}, A_{n_2}, \cdots, A_{n_{k+1}}$ are independent. Moreover, the probability that A_{n_k} occurs is given by the survival probability $p\,(\lambda)$ of the basic contact process with parameter λ starting from one infected site so

$$\begin{aligned} P\,(A_n^c \,\cap\, A_{n+1}^c \,\cap\, \cdots\,) \;&\leq\; \lim_{k\to\infty}\; P\,(A_{n_1}^c \,\cap\, A_{n_2}^c \,\cap\, \cdots \,\cap\, A_{n_k}^c) \\ &\leq\; \prod_{k=1}^{\infty} P\,(A_{n_k}^c) \;=\; \lim_{k\to\infty}\; (1 - p\,(\lambda))^k \;=\; 0 \end{aligned}$$

as soon as $\lambda > \lambda_c$. In particular,

$$P\,(\limsup_{n\to\infty} A_n) \;=\; \lim_{n\to\infty}\; P\,(A_n \,\cup\, A_{n+1} \,\cup\, \cdots\,) \;=\; 1.$$

This proves that, with probability 1, there exist infinitely many lower trees Γ_n that live forever. In other respects, since $\sigma_n - s_n$ is exponentially distributed with parameter λ, one can state that

$$P\,(B_n) \;=\; P\,(\sigma_n - s_n \leq V_1^{z_n}) \;=\; \lambda\,\gamma^{-1}\,(\lambda + \gamma)^{-1} \;>\; 0.$$

By independence, we can finally conclude that $P\,(\limsup_{n\to\infty} A_n \,\cap\, B_n) = 1$. □

To complete the proof of Proposition 3.1, we now consider, for any $n \geq 1$ and $s \geq \sigma_n$, the time-translation dual process

$$\hat{\xi}_s^{(z_n,\sigma_n)} \;=\; \{y \in \mathbb{Z}^d : \text{there is a path from } (z_n,\sigma_n) \text{ to } (y,s)\},$$

and denote by $\zeta_s(n)$ the associated distinguished particle, that is the first ancestor of (z_n, σ_n). First of all, we can observe that if the lower tree Γ_n lives forever then $\zeta_s(n)$ is well defined for any $s \geq \sigma_n$. Moreover, if we suppose that Γ_n is good and that $\zeta_s(n)$ lands on a 1 then, in view of the condition 2 above, the site z_n will be frozen at time s_n. In particular, if Γ_{n_k} is a subsequence of good trees given by Lemma 3.2, the proof of Proposition 3.1 can be completed with the

Lemma 3.3 *Let* $\Omega_s = \{\zeta_s(n_k) : \sigma_{n_k} \leq s\}$ *and* Θ_s^1 *be the set of sites occupied at time* s *by a 1. If* ξ_0 *is translation invariant and* $d \geq 3$ *then, starting from infinitely many 1's*

$$\lim_{t \to \infty} \operatorname{card}(\Omega_t \cap \Theta_0^1) = \infty \quad a.s.$$

Proof By Proposition 2.1 of Neuhauser (1992), the path of $\zeta_s(n_k)$ can be broken into independent and identically distributed pieces so that the process $\zeta_s(n_k)$ is transient in $d \geq 3$. See Neuhauser (1992), Sections 4 and 5, for a proof. This, together with Lemmas 5.3 and 5.4 in Lanchier (2005) implies that card $(\Omega_t) \to \infty$. Finally, since ξ_0 is translation invariant, Lemma 9.14 of Harris (1976) tells us that card $(\Omega_t \cap \Theta_0^1) \to \infty$ as $t \to \infty$ with probability 1. □

To conclude the proof of Theorem 3, we now use the dual process $\tilde{\xi}_t^{(x,t)}$ and construct a sequence of upper ancestors $\eta_t^{(x,t)}(k)$, $k \geq 0$, that are candidates to paint (x,t) the color 1. The first member of the sequence will be the distinguished particle. Next, we renumber the sequence of frozen points (z_k, s_k), $k \geq 1$, visited by the distinguished particle by going forward in time, and denote by n_t the number of frozen points encountered. For each $1 \leq k \leq n_t$, we look backwards in time starting from the location where the arrow α_k is attached and discard all the ancestors of this particular point. We then define $\eta_t^{(x,t)}(k)$ as the first upper ancestor that is left after discarding. Let $\eta_t = \{\eta_t^{(x,t)}(k) : 0 \leq k \leq n_t\}$. Proposition 3.1 tells us that $\lim_{t\to\infty} n_t = \infty$ with probability 1. This, together with Lemmas 5.3 and 5.4 in Lanchier (2005), implies that the cardinality of η_t can be made arbitrarily large by choosing t sufficiently large. In particular, a new application of Lemma 9.14 in Harris (1976) gives us that

$$\lim_{t \to \infty} P(\eta_t \cap \Theta_0^1 = \varnothing) = 0.$$

Hence, there exists at least one candidate that lands on a 1. We denote by $\eta_t^{(x,t)}(k_0)$ the first one in the hierarchy. Since the arrow α_{k_0} is bad for the 2's, the upper ancestor $\eta_t^{(x,t)}(k_0)$ will finally paint (x,t) the color 1. This completes the proof of Theorem 3.

4. Proof of Theorem 4

In this section, we assume that $d = 2$, set $\lambda_1 < \lambda_2$ and prove that there is $\gamma_c < \infty$ such that for any $\gamma > \gamma_c$ the 2's win. In view of the evolution rules, the survival of 2's is not clear and tools as coupling and duality fail in proving Theorem 4. We will first rely on the rescaling argument described in Durrett and Neuhauser (1997), Section 3, valid in the case $\gamma = \infty$, and then, prove that taking $\gamma > 0$ sufficiently large does not affect too much the process. We start by introducing the suitable space and time scales. We let L be a positive integer and, for $z = (z_1, z_2)$ in $\mathbb{Z}^2$, set

$$\Phi(z) = (Lz_1, Lz_2), \qquad B = [-L, L]^2, \qquad B(z) = \Phi(z) + B.$$

Moreover, we tile $B(z)$ with $L^{0.1} \times L^{0.1}$ squares by setting

$$\pi(w) = (L^{0.1} w_1, L^{0.1} w_2), \qquad D = (-L^{0.1}/2, L^{0.1}/2]^2,$$

$$D(w) = \pi(w) + D, \qquad I_z = \{w \in \mathbb{Z}^2 : D(w) \subseteq B(z)\}.$$

We say that $B(z)$ is *good* if $B(z)$ is void of 1's and has at least one particle of type 2 in each of the squares $D(w)$ for $w \in I_z$. For $z = (z_1, z_2) \in \mathbb{Z}^2$ with z_1 and z_2 both even for even k, and z_1 and z_2 both odd for odd k, we say that (z, k) is *occupied* if $B(z)$ is good at time kT, where T is an integer to be picked later. Moreover, we require this event to occur for the process restricted to the region $\Phi(z) + [-ML, ML]^2$. We start by assuming that $\gamma = \infty$.

Proposition 4.1 (Durrett and Neuhauser) *If* $\lambda_2 > \lambda_1$ *and* $T = L^2$ *then for any* $\varepsilon > 0$*, the parameters* L *and* M *can be chosen so that the set of occupied sites dominates the set of open sites in an* M*-dependent oriented percolation process with parameter* $1 - \varepsilon$*.*

See Durrett and Neuhauser (1997), Proposition 3.1 and Lemma 3.7. Given $z \in \mathbb{Z}^2$, $\| z \| = 1$, and $x \in B(z)$, the strategy to prove Proposition 4.1 is to construct a dual path A_t starting at $A_0 = (x, T)$ that lands on B at time 0 and blocks 1's from determining the color of (x, T). To make the proof more explicit, we divide it into three steps.

Construction of the selected path

It essentially relies on a procedure called *repositioning algorithm.* We start by breaking up the path of the first ancestor at some points called *renewal points.* To define these points, we follow the dual path of the distinguished particle starting at (x, T) and, whenever it jumps to a branch that lives forever, call its target site a renewal point. See Neuhauser (1992), Section 2. We then call a renewal point *associated with a 2-arrow* if the first arrow a particle crosses on its way up the designated path of the first ancestor starting at the renewal point is a 2-arrow. We set $\sigma_0 = 0$ and, for $i \geq 1$, denote by σ_i the times when A_t jumps to a renewal point associated with a 2-arrow. Between times σ_i, the process A_t follows the path of the first ancestor while its position at times σ_i is determined by the repositioning algorithm. If we pretend that a 1 was able to come up all the way to a renewal point associated with a 2-arrow, this 1 would now be blocked by this arrow and some other site, if it exists, would be the next candidate for a path determining the color of (x, T). We call this site B_{σ_i}. To describe the algorithm, we now embed into the selected path a jump process S_t which stays put except at times σ_i. Denoting by $J = [-L/10, L/10\,]^2$ and by $\mathrm{dist}(z, J)$ the Euclidean distance between a point z and the set J, the process S_t can be defined as follows. If $\mathrm{dist}(B_{\sigma_i}, J) < \mathrm{dist}(A_{\sigma_i}, J)$ and B_{σ_i} lives forever, we set $S_{\sigma_i} = B_{\sigma_i}$, otherwise $S_{\sigma_i} = A_{\sigma_i}$. In either case, A_t continues starting at S_{σ_i} and uses the path determined by the algorithm of the first ancestor until time σ_{i+1} where the repositioning algorithm is applied again. If both sites A_{σ_i} and B_{σ_i} are contained in J, we toss a fair coin to determine which path to continue with.

First step

The objective is now to prove that, with probability close to 1, the selected path starting at (x, T) will reach the set J before time σ_K where K is defined as

$$K \ = \ \min \{k \geq 1 : \sigma_k \geq T - 2L^{0.5}\}.$$

The main ingredient of the proof is provided by Proposition 2.1 of Neuhauser (1992) that allows to break up the path of the first ancestor into i.i.d. pieces. More precisely, if we denote by X_i the spatial displacement between consecutive renewal points, and by τ_i the corresponding temporal displacement, one can prove the following result.

Proposition 4.2 (Neuhauser) *If the upper tree starting at (x, T) lives forever, $\{(X_i, \tau_i)\}_{i \geq 1}$ form an i.i.d. family of random vectors on $\mathbb{Z}^d \times \mathbb{R}^+$. Moreover, we have the exponential bounds*

$$P\,(\|X_i\| > t) \ \leq \ C_1\, e^{-\gamma_1\, t} \qquad \text{and} \qquad P\,(\tau_i > t) \ \leq \ C_1\, e^{-\gamma_1\, t}$$

for appropriate constants $C_1 < \infty$ and $\gamma_1 > 0$.

In particular, the location of the distinguished particle at the renewal points can be controlled using the following large deviations principle.

Lemma 4.3 *Let $\mathcal{H}_n$ be a renewal process whose interarrival times $\mathcal{L}_i = \mathcal{H}_i - \mathcal{H}_{i-1}$ are i.i.d. with mean λ and such that $P(\mathcal{L}_i > t) \leq C_2\, e^{-\gamma_2\, t}$. Then for any $\varepsilon > 0$ there exist constants $C_3 < \infty$ and $\gamma_3 > 0$ such that*

$$P\,(\mathcal{H}_n \geq (1 + \varepsilon)\, n\lambda) \ \leq \ C_3\, e^{-\gamma_3\, n\lambda} \quad \text{and} \quad P\,(\mathcal{H}_n \leq (1 - \varepsilon)\, n\lambda) \ \leq \ C_3\, e^{-\gamma_3\, n\lambda}.$$

In other respects, the spatial displacement X_i having mean 0 due to the translation invariance of the graphical representation, the renewal points define an embedded symmetrical random walk for the distinguished particle. So, if we now apply the repositioning algorithm at times σ_i and select, with positive probability, B_{σ_i} instead of A_{σ_i} the spatial displacement $\|S_{\sigma_{i+1}}\| - \|S_{\sigma_i}\|$ will have negative mean as long as S_t is outside of J. Intuitively, this should cause S_t to drift towards the set J. We now make this argument precise.

Lemma 4.4 *If $\lambda_1 < \lambda_2$ there exist $C_4 < \infty$ and $\gamma_4 > 0$ so that*

$$P\,(\sigma_1 > t) \ \leq \ C_4\, e^{-\gamma_4\, t}.$$

Proof Since the arrows are labeled independently of each other and with probability $(\lambda_2 - \lambda_1)/\lambda_2 > 0$, a geometric number of trials suffices to find a renewal point associated with a 2-arrow. The lemma then follows from the second exponential bound in Proposition 4.2. □

To prove that the selected path reaches the set J by time σ_K we still need an estimate on the maximum size of the spatial displacement $Y_i = S_{\sigma_i} - S_{\sigma_{i-1}}$ of the embedded jump process. We let $M(t)$ be the number of times the repositioning algorithm has been applied by time t.

Lemma 4.5 *We set $T = L^2$. If $\lambda_1 < \lambda_2$ then for any $\gamma_5 > 0$ there exist $C_5 < \infty$ and $\gamma_6 > 0$ so that for L sufficiently large*

$$P\left(\max_{1 \leq k \leq M(T)} \|Y_k\| > \frac{\gamma_6}{\gamma_4}\,\log L\right) \ \leq \ C_5\, L^{-\gamma_5}.$$

Proof We first set $m = \mathbb{E}\, M(t)/t$ and decompose the event according to whether $M(T) > 2\,mT$ or $M(T) \leq 2\,mT$. Then for any $\gamma_6 > 0$

$$P\left(\max_{1 \leq k \leq M(T)} \|Y_k\| > \frac{\gamma_6}{\gamma_4}\,\log L\right) \ \leq \ P\,(M(T) > 2\,mT) \ + \ 2\,mT\,P\left(\|Y_1\| > \frac{\gamma_6}{\gamma_4}\,\log L\right).$$

The first large deviations estimate in Lemma 4.3 takes care of the first term on the right-hand side. In other respects, since between times σ_{k-1} and σ_k the process spreads out at most linearly, the spatial displacement $\|Y_k\|$ is of the same order as the corresponding temporal displacement $\sigma_k - \sigma_{k-1}$. In particular, by Lemma 4.4

$$P\,(\|Y_k\| > t) \ \leq \ C_6\, e^{-\gamma_4\, t}$$

for all $t \geq 0$ and appropriate $C_6 < \infty$. Putting things together, we finally obtain

$$P\left(\max_{1 \leq k \leq M(T)} \|Y_k\| > \frac{\gamma_6}{\gamma_4}\,\log L\right) \ \leq \ C_3\, \exp(-\gamma_3\, mT) \ + \ 2\,mT\, C_6\, L^{-\gamma_6}.$$

Now, for given $\gamma_5 > 0$, we can choose $\gamma_6 > 0$ so that the wanted inequality holds for appropriate $C_5 < \infty$ and L sufficiently large. This completes the proof. □

Lemma 4.6 *Let $x \in B(z)$ with $\|\, z\,\| = 1$. If $\lambda_1 < \lambda_2$ then for any $\gamma_7 > 0$*

$$P_x\,(S_{\sigma_k} \notin J \text{ for all } k \leq K) \ \leq \ C_7\, L^{-\gamma_7}$$

for some $C_7 < \infty$ and L large enough. Here the subscript on P indicates the starting site.

Proof We first observe that

$$\begin{aligned} P_x\,(S_{\sigma_k} \notin J \text{ for all } k \leq K) \ &\leq \ P_x\,(S_{\sigma_k} \notin J \text{ for all } k \leq 2\,mT) \ + \ P\,(K > 2\,mT) \\ &\leq \ P_x\left(S_{\sigma_k} \notin J \text{ for all } k \leq 2\,mT\,; \max_{1 \leq k \leq 2\,mT} \|Y_k\| \leq \frac{\gamma_6}{\gamma_4}\,\log L\right) \\ &\quad + \ P\left(\max_{1 \leq k \leq 2\,mT} \|Y_k\| > \frac{\gamma_6}{\gamma_4}\,\log L\right) \ + \ P\,(K > 2\,mT) \end{aligned}$$

We restrict the jump size $\|Y_k\|$ to ensure that S_{σ_k} will not miss the set J. The last two terms on the right-hand side can be bounded by using Lemma 4.5 and the first large deviations estimate in Lemma 4.3 respectively. For the first term, note that there exists $m_1 > 0$ so that on the set $\{S_{\sigma_{k-1}} \notin J\}$, $\mathbb{E}\,(\|S_{\sigma_k}\| - \|S_{\sigma_{k-1}}\| \,|\, \mathcal{F}_{\sigma_{k-1}}) \leq -\, m_1$ where $\mathcal{F}_{\sigma_{k-1}} = \sigma(S_{\sigma_0}, \, \ldots, \, S_{\sigma_{k-1}})$. It follows that for γ_6 and L sufficiently large there exists m_2 with $0 < m_2 \leq m_1$ so that

$$\mathbb{E}\,\big(\{\|S_{\sigma_k}\| - \|S_{\sigma_{k-1}}\|\}\ \mathbb{1}\{\|Y_k\| \ \leq\ (\gamma_6/\gamma_4)\log L\} \mid \mathcal{F}_{\sigma_{k-1}}\big) \ \leq\ -\, m_2$$

on the set $\{S_{\sigma_{k-1}} \notin J\}$. This implies that

$$P_x\left(S_{\sigma_k} \notin J \text{ for all } k \leq 2\,mT\,;\ \max_{1 \leq k \leq 2\,mT} \|Y_k\| \leq \frac{\gamma_6}{\gamma_4}\,\log L\right) \ \leq\ C_8\, e^{-\gamma_8\, L}$$

for appropriate $C_8 < \infty$ and $\gamma_8 > 0$, which concludes the proof. $\square$

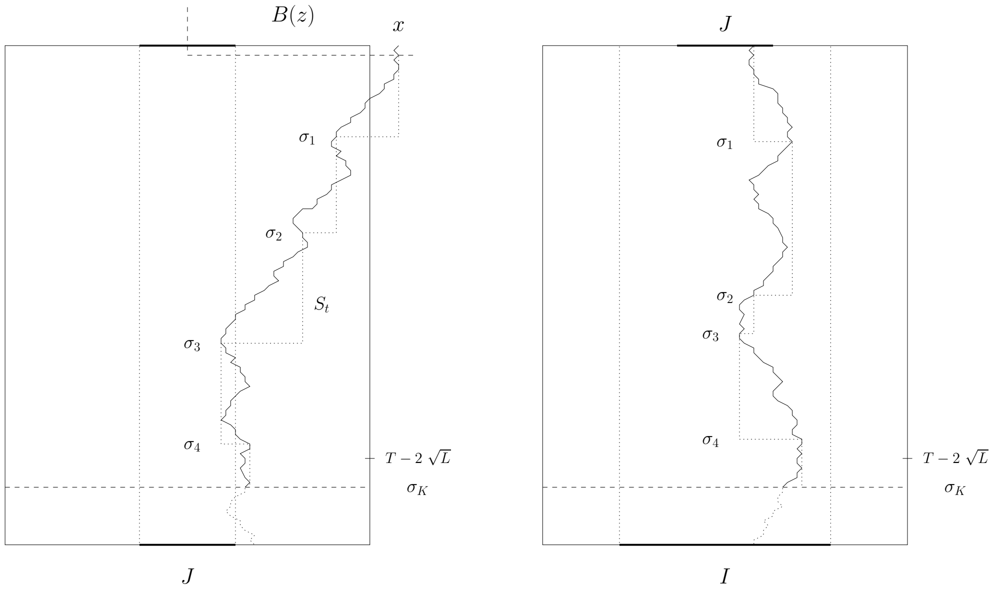

FIGURE 5. *Pictures of the selected path. First step: With probability close to 1, A_t reaches J before time σ_K. Second step: With probability close to 1, A_t belongs to I at time σ_K, that is $S_{\sigma_K} \in I$.*

Second step

We now let $I = [-2L/5, 2L/5\,]^2$ and prove that with probability close to 1 the selected path A_t belongs to I at time σ_K, that is $S_{\sigma_K} \in I$. In view of Lemma 4.6, we just need to show that starting from any site $y \in J$, the process S_t stays inside of I for T units of time.

Lemma 4.7 *If $\lambda_1 < \lambda_2$ then for any $y \in J$ and any $\gamma_9 > 0$*

$$P_y\,(S_{\sigma_k} \notin I \textit{ for some } k \leq M(T)) \ \leq\ C_9\, L^{-\gamma_9}$$

for appropriate $C_9 < \infty$ and L sufficiently large.

Proof We let $s_0 = 0$ and, for $k \geq 1$, define the following stopping times

$$\begin{aligned} t_k &= \inf\,\big\{t > s_{k-1} : S_t \notin [-L\,/\,5,\, L\,/\,5\,]^2\,\big\} \\ s_k &= \inf\,\{t > t_k : S_t \in J\} \quad \text{and} \quad R \ =\ \inf\,\{t > 0 : S_t \notin I\}. \end{aligned}$$

Then for any site $y \in J$

$$\begin{aligned} & P_y\,(S_{\sigma_k} \notin I \text{ for some } k \leq M(T)) \\ & \quad \leq\; P\,\bigg(s_k > R \text{ for some } k \leq 2\,mT\,;\ \max_{1 \leq k \leq 2\,mT} \|Y_k\| \leq \frac{\gamma_6}{\gamma_4}\,\log L\bigg) \\ & \quad +\; P\,\bigg(\max_{1 \leq k \leq 2\,mT} \|Y_k\| > \frac{\gamma_6}{\gamma_4}\,\log L\bigg) \;+\; P\,(M(T) > 2\,mT). \end{aligned}$$

To estimate the first term on the right-hand side, we only need to estimate the probability that S_t leaves I before returning to J. As in Lemma 4.6, we use that $\|S_{\sigma_k}\| - \|S_{\sigma_{k-1}}\|$ has negative drift and bounded increments on $\{\max_{1 \leq k \leq 2\,mT} \|Y_k\| \leq (\gamma_6/\gamma_4)\,\log L\}$. Hence

$$\begin{aligned} & P\,\bigg(s_k > R \text{ for some } k \leq 2\,mT\,;\ \max_{1 \leq k \leq 2\,mT} \|Y_k\| \leq \frac{\gamma_6}{\gamma_4}\,\log L\bigg) \\ & \quad \leq\; 2\,mT\;P\,\bigg(s_1 > R\,;\ \max_{1 \leq k \leq 2\,mT} \|Y_k\| \leq \frac{\gamma_6}{\gamma_4}\,\log L\bigg) \;\leq\; C_{10}\,e^{-\gamma_{10}\,L} \end{aligned}$$

for appropriate $C_{10} < \infty$ and $\gamma_{10} > 0$. Finally, the last two terms can be bounded using Lemma 4.5 and the first large deviations estimate in Lemma 4.3. $\square$

The proof of the second step is now straightforward combining Lemma 4.6 and Lemma 4.7.

Lemma 4.8 *Assume $x \in B(z)$ for some $z \in \mathbb{Z}^2$ with $\|\,z\,\| = 1$. If $\lambda_1 < \lambda_2$ then for any $\gamma_{11} > 0$*

$$P_x\,(S_{\sigma_K} \notin I) \;\leq\; C_{11}\,L^{-\gamma_{11}}$$

for appropriate $C_{11} < \infty$ and L sufficiently large.

Proof By decomposing according to whether $S_{\sigma_k} \in J$ for some $k \leq K$ or not, we obtain

$$P_x\,(S_{\sigma_K} \notin I) \;\leq\; P_x\,(S_{\sigma_k} \notin J \text{ for all } k \leq K) \;+\; P\,(S_{\sigma_K} \notin I \,|\, S_{\sigma_k} \in J \text{ for some } k \leq K).$$

The first term on the right-hand side can be bounded using Lemma 4.6. For the second term, we first observe that

$$P\,(S_{\sigma_K} \notin I \,|\, S_{\sigma_k} \in J \text{ for some } k \leq K) \;\leq\; \sup_{y \in J} P_y\,(S_{\sigma_k} \notin I \text{ for some } k \leq M(T))$$

and then apply Lemma 4.7. This completes the proof. $\square$

Third step

To complete the comparison with oriented percolation process, we still need to prove that provided B is good at time 0, (S_{σ_K}, σ_K) will be occupied, with probability close to 1, by a 2. Since the selected path has been constructed to block 1's, this 2 will determine the type of (x, T) unless another 2 succeeds earlier. This will prove that the 2's in B invade the neighboring boxes $B(z)$ with probability close to 1.

Lemma 4.9 *Assume B is good at time 0. If $S_{\sigma_K} \in I$ there exist $C_{12} < \infty$ and $\gamma_{12} > 0$ so that*

$$P\,((S_{\sigma_K}, \sigma_K) \text{ is not occupied by a } 2) \;\leq\; C_{12}\,\exp(-\gamma_{12}\,L^{0.1}).$$

Proof To begin with, observe that except for exponentially small probability, $\sigma_K \leq T - L^{0.5}$. More precisely, by Lemma 4.4,

$$P\,(\sigma_K > T - L^{0.5}) \;\leq\; P\,(\sigma_1 > L^{0.5}) \;\leq\; C_4\,\exp(-\gamma_4\,L^{0.5}).$$

Now, if $\sigma_K \leq T - L^{0.5}$, we run the dual process starting at (S_{σ_K}, σ_K) for another $L^{0.5} - L^{0.2}$ units of time. Then, it follows from the properties of the dual of the contact process that, except for a probability $\leq C_{13} \exp(-\gamma_{13} L^{0.1})$, we can select $L^{0.1}$ sites at about time $L^{0.2}$ which are contained in the dual process starting at (S_{σ_K}, σ_K) such that (i) all these sites are contained in $[-3L/5, 3L/5]^2$, (ii) they are at least $L^{0.3}$ units apart from each other, and (iii) none of the duals starting at these sites interferes with any of the other duals for the remaining $L^{0.2}$ units of time. Since each of the duals has positive probability of surviving and B is good at time 0, each of these $L^{0.1}$ sites has probability $\eta > 0$ of being occupied by a 2. In conclusion,

$$\begin{aligned} & P\,((S_{\sigma_K}, \sigma_K) \text{ is not occupied by a 2}) \\ & \qquad \leq\; C_4 \exp(-\gamma_4 L^{0.5}) \;+\; C_{13} \exp(-\gamma_{13} L^{0.1}) \;+\; (1-\eta)^{L^{0.1}} \;\leq\; C_{12} \exp(-\gamma_{12} L^{0.1}) \end{aligned}$$

for appropriate $C_{12} < \infty$ and $\gamma_{12} > 0$. $\square$

Conclusion

We are now ready to conclude the proof of Proposition 4.1. First of all, combining Lemma 4.8 and Lemma 4.9 implies that if B is good at time 0 then

$$P\,(\xi_T(x) = 1) \;\leq\; C_{11} L^{-\gamma_{11}} \;+\; C_{12} \exp(-\gamma_{12} L^{0.1}) \;\leq\; C_{14} T^{-4}$$

for appropriate $C_{14} < \infty$ and any $x \in B(z)$ with $\| z \| = 1$. In particular, since there are at most $(2L+1)^2$ sites in $[-L, L]^2$, it follows that

$$P\,(\xi_T(x) = 1 \text{ for some } x \in B(z)) \;\leq\; C_{15} L^{-6} \;\leq\; \varepsilon / 3$$

for L sufficiently large. In other respects, the process dominates a one-color contact process with parameter $\lambda_1 > \lambda_c$ so the probability that there exists an $L^{0.1} \times L^{0.1}$ square $D(w) \subseteq B(z)$ that is empty can be bounded by $C_{16} L^{1,8} \exp(-\gamma_{16} L^{0.2}) \leq \varepsilon / 3$. Finally, since the selected paths have a drift toward J, we can find $M > 0$ so that

$$\begin{aligned} & P\,(\text{any of the selected paths is not contained in} \\ & \qquad\qquad [-ML/3, ML/3]^2 \text{ for some } t \leq T) \;\leq\; \varepsilon / 3. \end{aligned}$$

This shows that boxes that are sufficiently far apart are independent of each other with high probability. Proposition 4.1 then follows.

To generalize the comparison to $\gamma > 0$ sufficiently large, we just need to prove that, with probability close to 1, the process behaves like the multitype contact process (i.e., none of the 2's is blocked by a frozen site) inside the space-time box

$$J(z) \;\times\; [0, T] \qquad \text{where} \qquad J(z) \;=\; \Phi(z) \;+\; [-ML/3, ML/3]^2.$$

Now, the event we are interested in occurs if and only if each time an arrow points at a site $x \in J(z)$ by time T this site is not in state 3. So, to make sure that this occurs, we will follow the line $\{x\} \times [0, T]$ by going forward in time and each time we will encounter a × we will put a $\bullet$ at x before meeting the next tip of arrow. By letting $K(x, t)$ be the number of arrows that point at site x by time t and decomposing according to whether $K(x, T) > 2\lambda_2 T$ or not, we obtain

$$\begin{aligned} P\,(\text{any of the 2's is blocked}) \;&\leq\; \sum_{x \in J(z)} P\,(K(x,T) > 2\lambda_2 T) \;+\; 2\lambda_2 T \sum_{x \in J(z)} P\,(U_1^{x} < V_1^{x}) \\ &\leq\; (2ML/3)^2 \left(C_{17}\, e^{-\gamma_{17} T} \;+\; 2\lambda_2 T \gamma^{-1} (\gamma+1)^{-1}\right) \;\leq\; \varepsilon/3 \end{aligned}$$

for T and γ sufficiently large and appropriate $C_{17} < \infty$ and $\gamma_{17} > 0$. At this point, we have proved that if $\lambda_1 < \lambda_2$ and γ is sufficiently large then there exist L and M so that the set of occupied sites dominates the set of open sites in an M-dependent oriented percolation process with parameter $1 - \varepsilon$. This almost produces Theorem 4. Our last problem is that oriented site percolation has a positive density of unoccupied sites. To prove that there is an in-all-directions expanding region which is void of 1's, we apply a result from Durrett (1992) which shows that unoccupied sites do not percolate when ε is close enough to 0. Since particles of either type cannot appear spontaneously, once a region is void of one type, this type can only reappear in the region through invasion from the outside. This then implies that our process has the desired property and completes the proof of Theorem 4.

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Stochastic spatial models of host-pathogen and host-mutualist interactions

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Abstract Mutualists and pathogens, collectively called symbionts, are ubiquitous in plant communities. While some symbionts are highly host-specific, others associate with multiple hosts. The outcome of multispecies host-symbionts interactions with different degrees of specificity are difficult to predict at this point due to a lack of a general conceptual framework. Complicating our predictive power is the fact that plant populations are spatially explicit and we know from past research that explicit space can profoundly alter plant-plant interactions. We introduce a spatially explicit, stochastic model to investigate the role of explicit space and host-specificity in multispecies host-symbiont interactions. We find that in our model, pathogens can significantly alter the spatial structure of plant communities, promoting coexistence, whereas mutualists appear to have only a limited effect. Effects are more pronounced the more host-specific symbionts are.

1. Introduction

The diversity and structure of plant communities is largely determined by nutrient availability, competition among plants, herbivory, and associations between plants and their symbionts. The first three have been the focus of much ecological research, both empirical and theoretical. The role of symbionts on diversity and structure has received less attention (Aerts, 2002) and will be the topic of this paper.

A symbiont is an organism that lives in close association with a host. It can either have a harmful effect, in which case we call it a pathogen, or a beneficial effect, in which case we call it a mutualist. Symbionts are ubiquitous. For instance, more than 90% of terrestrial plants associate with mycorrhizal fungi (Trappe, 1987), a beneficial association that supplies nutrients to the plant, and, in return, carbon to the fungal partner. Fecundity and viability of virtually all plants are affected by pathogens, sometimes with devastating effects, such as Dutch elm disease or chestnut blight that point to the important role of pathogens in structuring plant communities. Mathematical models play an important role in elucidating the roles of symbionts in community dynamics.

Modeling of disease dynamics has had a long tradition, starting with the model by Kermack and McKendrick (1927), which describes the course of a disease outbreak caused by a single disease infecting a single host. This model and its extensions have yielded enormously valuable insights into disease dynamics and potential control strategies. Although originally developed to describe epidemics in human populations, it can equally well be applied to plant diseases. A key concept of disease dynamics is the basic reproductive rate R_0, which is defined as the expected number of secondary infections caused by an infected individual when introduced into a population of susceptible individuals (Anderson, 1981). The condition for a disease outbreak is given in the biological literature as $R_0 > 1$. This condition is based on a single-host, single-disease model in a non-spatial population. Including spatial structure in the form of local interactions has shown that for a disease to spread, R_0 needs to exceed a threshold that is greater than that for a non-spatial population. The reason for this is the lack of sufficient numbers of susceptible individuals near the location of a disease outbreak once the disease starts spreading. One of the first models where this has been demonstrated mathematically is the contact process (Harris, 1974, Mollison, 1977).

Much of the theoretical work in the epidemiological literature has focused on single-host, single-disease dynamics. A rapidly increasing empirical body of work on multispecies host-disease dy-

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namics necessitates the development of a theoretical framework. This has only begun recently (see Holt et al., 2003 and references therein). An attempt for a broad classification was made by Holt et al. (2003) using a graphical isocline framework that allows for generalizations that are applicable to a wide range of host-pathogen models. A recent model by Dobson (2004) investigates persistence of a pathogen that can infect multiple hosts.

Modeling of host-mutualist interactions has primarily focused on evolutionary questions, such as the evolution of cheaters (i.e., symbionts that receive benefits but do not confer them). Almost no modeling has been done on the effects of local (spatial) interactions on host-mutualist dynamics. Similarly, no theoretical framework has been developed for multispecies host-mutualist interactions.

An important component of multi-host, multi-symbiont models is the degree to which different symbionts and hosts can associate with each other. This is referred to as *specificity.* A *specialist* symbiont associates with a very small number of hosts; a *generalist* symbiont associates with many hosts. The ease of transmission of a symbiont to a host, referred to as *transmissibility*, is another important factor in host-symbiont interactions.

In this paper, we investigate the role of spatial structure caused by local interactions, such as symbiont transmission and host dispersal, on persistence of host-symbiont associations for both generalists and specialists in multi-host, multi-symbiont models. We employ the simplest of all multi-species models to describe the host dynamics, the voter model (Holley and Liggett, 1975, Clifford and Sudbury, 1973). The voter model is defined on the d-dimensional integer lattice, where each lattice site is occupied by an individual characterized by one of a finite number of types. Individuals give birth to offspring of their own kind at a constant rate, and their offspring displace randomly chosen individuals within their dispersal neighborhood. The dynamics imply that all sites remain occupied at all times. Into this population, we introduce symbionts with varying degrees of specificity and transmissibility. More precisely, our spatial model is a continuous-time Markov process $\xi_t : \mathbb{Z}^d \longrightarrow \{1, 2, \ldots, \kappa\} \times \{0, 1, \ldots, \kappa\}$ where the integer κ denotes both the number of hosts and the number of symbionts involved in the interaction. A site $x \in \mathbb{Z}^d$ is said to be occupied by an *unassociated* host of type i, $i = 1, 2, \ldots, \kappa$, if $\xi(x) = (i, 0)$, and by a host of type i, $i = 1, 2, \ldots, \kappa$, *associated* with a symbiont of type j, $j = 1, 2, \ldots, \kappa$, if $\xi(x) = (i, j)$. We will write $\xi_t(x) = (\xi_t^1(x), \xi_t^2(x))$, where $\xi_t^1(x)$ denotes the type of the host present at x at time t and $\xi_t^2(x)$ the type of the symbiont present at x at time t, with $\xi_t^2(x) = 0$ denoting the absence of a symbiont. We set $\| x \| = \sup_{i=1,2,\ldots,d} |x_i|$. The evolution at site $x \in \mathbb{Z}^d$ is described by the transition rates

$$\begin{array}{llll} (i,j) \ \to \ (k,0) & \text{at rate} & \lambda \displaystyle\sum_{0<\|x-z\|\le R_1} & \left\{ \mathbb{1}_{\{\xi(z)=(k,0)\}} \ + \ g \displaystyle\sum_{\ell=1}^{\kappa} \mathbb{1}_{\{\xi(z)=(k,\ell)\}} \right\} \\ (i,0) \ \to \ (i,j) & \text{at rate} & c_{ij} \displaystyle\sum_{0<\|x-z\|\le R_2} & \displaystyle\sum_{\ell=1}^{\kappa} \mathbb{1}_{\{\xi(z)=(\ell,j)\}}. \end{array}$$

The transition $(i, j) \to (k, 0)$ is the birth of an unassociated host at x by either unassociated or associated neighboring hosts. The birth rate of unassociated hosts is equal to λ. The parameter g indicates the variation of the birth rate of hosts associated with a symbiont. If $0 \le g < 1$, the symbiont is a pathogen; if $g = 1$, the symbiont has no effect on the birth rate of the host and we refer to this as the neutral case; if $g > 1$, the symbiont is a mutualist. The transition $(i, 0) \to (i, j)$ is the transmission of a neighboring symbiont j to an unassociated host of type i at x. The parameters c_{ij} denote the rate at which symbiont j infects host i. This parameter will allow us to mimic specialist and generalist symbionts. Births and infections occur within a local neighborhood, with R_1 denoting the birth range of hosts, and R_2 the infection range of symbionts. Neighborhoods are punctured boxes with side $2R_i + 1$, $i = 1, 2$, centered at site x, that is $\mathcal{N}_x^i = \{z \in \mathbb{Z}^d : 0 < \|x - z\| \le R_i\}$. The cardinality of this set is denoted by $\nu_{R_i} = |\mathcal{N}_x^i|$.

Before we describe the behavior of the spatially explicit, stochastic model, we will look at the mean-field model (Durrett and Levin, 1994). The mean-field model is described by a system of differential equations for the densities of unassociated and associated hosts. To define it, we let u_i

denote the density of unassociated hosts of type i, $i = 1, 2, \ldots, \kappa$, and v_{ij} denote the density of host i associated with symbiont j, $i, j = 1, 2, \ldots, \kappa$. It follows from the dynamics of the spatially explicit, stochastic model that at all times

$$\sum_i u_i \;+\; \sum_{i,j} v_{ij} \;=\; 1.$$

Furthermore, we assume that for $i = 1, 2, \ldots, \kappa$, $c_{ii} = \beta$, and for $i, j = 1, 2, \ldots, \kappa$ with $i \neq j$, $c_{ij} = \alpha$ with $0 \leq \alpha \leq \beta$. We define

$$u_{\cdot} \;=\; \sum_{i=1}^{\kappa} u_i, \qquad v_{\cdot j} \;=\; \sum_{i=1}^{\kappa} v_{ij}, \qquad v_{i\cdot} \;=\; \sum_{j=1}^{\kappa} v_{ij} \qquad \text{and} \qquad v_{\cdot\cdot} \;=\; \sum_{i=1}^{\kappa}\sum_{j=1}^{\kappa} v_{ij}.$$

One way to obtain the mean-field limit is to set the neighborhood ranges, R_1 and R_2 equal to R and then let R go to infinity. To obtain a meaningful limit, we also need to rescale the parameters λ, α, and β by the neighborhood size ν_R, that is, we set $\lambda = \frac{1}{\nu_R}$ (this also sets the time scale), and define

$$\alpha \;=\; \frac{a}{\nu_R} \qquad \text{and} \qquad \beta \;=\; \frac{b}{\nu_R}.$$

In the limit, $R \to \infty$, sites become independent. If, in addition, the initial configuration is translation invariant, the dynamics of the densities for $i \neq j$ is then described by the following system of differential equations, called mean-field equations.

$$\begin{aligned}
\frac{du_i}{dt} &= (1-u_i)\,(u_i + g\,v_{i\cdot}) \;-\; u_i \sum_{j\neq i} (u_j + g\,v_{j\cdot}) \;-\; b\,u_i\,v_{\cdot i} \;-\; a \sum_{j\neq i} u_i\,v_{\cdot j}\\
\frac{dv_{ii}}{dt} &= b\,u_i\,v_{\cdot i} \;-\; v_{ii}\,(u_{\cdot} + g\,v_{\cdot\cdot})\\
\frac{dv_{ij}}{dt} &= a\,u_i\,v_{\cdot j} \;-\; v_{ij}\,(u_{\cdot} + g\,v_{\cdot\cdot}).
\end{aligned}$$

When $a = 0$, the symbionts are specialists. As a increases to b, the association turns into a generalist relationship. The following results are proved in Section 2. When $g = 1$, the system has a conserved quantity, namely the initial host densities $h_i = u_i + v_{i\cdot}$, $i = 1, 2, \ldots, \kappa$. If $(\kappa - 1)\,a + b > \kappa$, then for $g \neq 1$, there exists a nontrivial equilibrium with $u_1 = u_2 = \cdots = u_\kappa \geq 0$ and $v_{1\cdot} = v_{2\cdot} = \cdots = v_{\kappa\cdot} > 0$ such that for $i = 1, 2, \ldots, \kappa$,

$$u_i \;=\; \frac{g}{(\kappa-1)\,a + b - \kappa\,(1-g)} \qquad \text{and} \qquad h_i \;=\; \frac{1}{\kappa}.$$

Furthermore, for $i \neq j$

$$\frac{v_{ij}}{v_{ii}} \;=\; \frac{a}{b}.$$

Numerical simulations indicate that the nontrivial equilibrium is locally stable for $a < b$ when $g < 1$, but not for $g > 1$. In addition, if $g = 0$ and $(\kappa - 1)\,a + b > \kappa$, all hosts will be associated at equilibrium. If initially only two hosts and one symbiont, say symbiont 1, are present then for $0 < g < 1$, simulations indicate that pathogen 1 will go extinct and both host 1 and host 2 may coexist. We will conjecture similar behavior for the spatial model, except in the 1-dimensional, nearest neighbor case (see Theorem 3 and discussion following the theorem). When $g > 1$, host 2 goes extinct provided the initial density of mutualists of type 1 is sufficiently large. The spatial analogue of this result is proved in Theorem 2. Both results are illustrated in Figure 1.

We now return to the spatially explicit model with parameters λ, α, and β. To define the time scale, we set $\lambda = 1$. We will discuss both the generalist case $\alpha = \beta$ and the specialist case $\alpha = 0$, and provide comparisons with the mean-field model.

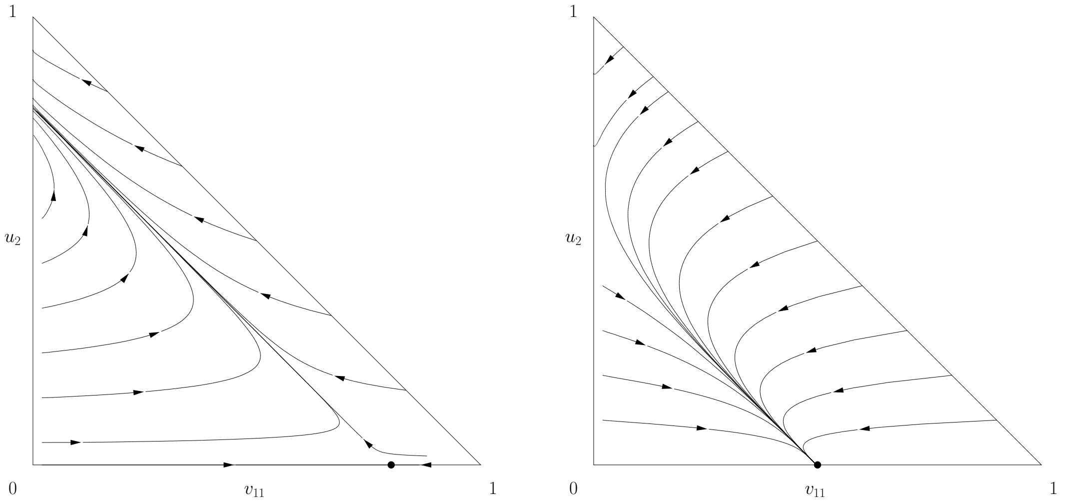

Figure 1. *Solution curves of the mean-field model with $a = 0$ and $b = 3$. The parameter g is equal to 0.5 and 2, respectively.*

Generalist interactions

We consider the generalist case $\alpha = \beta$ of the spatially explicit, stochastic model. In the corresponding mean-field model, coexistence of hosts and symbionts is possible when $b > 1$. For $b \leq 1$, symbionts are unable to persist. When $a = b$, symbionts are no longer host-specific, and the mean-field model can be reduced to one with u_i and $v_{i\cdot}$. It is not hard to see then that the nontrivial equilibrium of unassociated hosts, $(u_{\cdot}, v_{\cdot\cdot})$, is locally stable since in this case

$$\frac{dv_{\cdot\cdot}}{dt} \;=\; v_{\cdot\cdot}\,[(b-1)\,u_{\cdot} - g\,v_{\cdot\cdot}].$$

With $u_{\cdot} + v_{\cdot\cdot} = 1$, it follows that, the boundary point $v_{\cdot\cdot} = 0$ is unstable for $b > 1$ and that

$$(u_{\cdot}, v_{\cdot\cdot}) \;=\; \left(\frac{g}{b-1+g}, \frac{b-1}{b-1+g}\right)$$

is locally stable. Furthermore, any vector $(h_1, h_2, \ldots, h_\kappa)$ with $h_i \geq 0$ and $\sum_i h_i = 1$ gives rise to an equilibrium if we set

$$u_i \;=\; \frac{g}{b-1+g}\,h_i \qquad \text{and} \qquad v_{i\cdot} \;=\; \frac{b-1}{g}\,u_i.$$

The behavior of the spatially explicit model is more complicated and may depend on the spatial dimension, as we will see in the following.

If $g = 1$, then the symbionts have no effect on the hosts, which means that the spatially explicit processes ξ_t^1 and ξ_t^2 are stochastically independent. Moreover, by looking at the transition rates, it is easy to see that ξ_t^1 is a multitype voter model run at rate 1, and that ξ_t^2 is a multitype contact process in which particles give birth at rate $\beta\,\nu_{R_2}$ and die at rate ν_{R_1}. See respectively Holley and Liggett (1975) and Neuhauser (1992) for a study of these two processes. It follows that there exists a critical value $\beta_c(R_1, R_2) \in (0, \infty)$ that depends on the neighborhood sizes ν_{R_1} and ν_{R_2} such that the symbionts can survive if and only if $\beta > \beta_c(R_1, R_2)$. If we ignore host and symbiont types but rather focus on associated versus unassociated hosts, then for $\beta > \beta_c(R_1, R_2)$, regardless of the spatial dimension, there exists a nontrivial stationary distribution of associated and unassociated hosts. Moreover, if $d \geq 3$, there exists a stationary distribution in which all hosts and symbionts coexist.

Unfortunately, we cannot say much about coexistence when $g \neq 1$. To analyze this case, we define the “color-blind” process where a site is in state 0 if it is occupied by an unassociated host, and in state 1 if it is occupied by an associated host. We obtain a particle system $\zeta_t : \mathbb{Z}^d \longrightarrow \{0,\, 1\}$ with transitions at $x \in \mathbb{Z}^d$

$$\begin{array}{lll} 0 \ \to \ 1 & \text{at rate} & \displaystyle \beta \sum_{0 < \|x-z\| \leq R_2} \mathbb{1}_{\{\zeta(z)=1\}} \\ 1 \ \to \ 0 & \text{at rate} & \displaystyle \sum_{0 < \|x-z\| \leq R_1} \big\{ \mathbb{1}_{\{\zeta(z)=0\}} \ + \ g\, \mathbb{1}_{\{\zeta(z)=1\}} \big\}. \end{array}$$

When $g = 0$, the process reduces to a biased voter model. When $g = 1$, it reduces to a contact process with birth rate $\beta\, \nu_{R_2}$ and death rate ν_{R_1}. We denote the critical value of this contact process by $\beta_c(R_1, R_2)$ as above. A standard coupling argument allows us to compare the processes with $g \neq 1$ and $g = 1$, and to deduce that if $g \leq 1$ and $\beta > \beta_c(R_1, R_2)$ then ζ_t has a nontrivial stationary measure, while if $g \geq 1$ and $\beta \leq \beta_c(R_1, R_2)$ then the mutualists die out, i.e., $\zeta_t \Rightarrow \delta_0$, the "all 0" configuration (i.e., $\zeta(x) \equiv 0$). To cover the remaining cases, we introduce the contact process η_t in which particles give birth at rate $\beta\, \nu_{R_2}$ and die at rate $g\, \nu_{R_1}$. Then η_t has a nontrivial stationary measure if and only if $\beta > g\, \beta_c(R_1, R_2)$ which, with a new coupling argument, implies that if $g \leq 1$ and $\beta \leq g\, \beta_c(R_1, R_2)$, then the pathogens die out, while if $g \geq 1$ and $\beta > g\, \beta_c(R_1, R_2)$, then ζ_t has a nontrivial stationary measure.

We now focus on the case $\beta > 1$ and $g > 0$ close to 0. First of all, we observe that if $g = 0$ and $R_1 = R_2$ then the process ζ_t is the biased voter model with parameters β and 1. In particular, $P\,(\zeta_t(x) = 0) \to 1$ if $\beta < 1$ while $P\,(\zeta_t(x) = 1) \to 1$ if $\beta > 1$ provided we start with infinitely many 0's and 1's at time 0. Moreover, in the latter case, fixation occurs for the process ξ_t since hosts associated with pathogens are now sterile. The behavior is identical to that of the mean-field model. We will use a perturbation argument in Section 4 to show that if $\beta > 1$ and $g > 0$ is sufficiently close to 0 then the pathogens still survive. The results are summarized in Figure 2 and in the following theorem where "$\Rightarrow$" denotes weak convergence and δ_0 is the distribution that concentrates on the "all 0" configuration.

Theorem 1 *Assume that $\alpha = \beta$ and that ζ_0 is translation invariant with $P\,(\zeta_0(x) = 1) > 0$.*

(a) If $g \leq 1$, then $\zeta_t \Rightarrow \delta_0$ if $\beta \leq g\, \beta_c$, and a nontrivial equilibrium exists if $\beta > \beta_c$. If $g \geq 1$, then $\zeta_t \Rightarrow \delta_0$ if $\beta \leq \beta_c$, and a nontrivial equilibrium exists if $\beta > g\, \beta_c$.

(b) If $\beta > 1$ there exists $g_c > 0$ such that if $g \leq g_c$ then $\zeta_t \Rightarrow \mu$ with $\mu\,(\zeta(x) = 1) \neq 0$.

Part (b) of this theorem will be proved in Section 4.

Specialist interactions

In the specialist case $\alpha = 0$ and $\beta > 0$, the process is more difficult to investigate since the evolution of each symbiont strongly depends on the configuration of the host population. That is, there is no particle system $\zeta_t : \mathbb{Z}^d \longrightarrow \{0,\, 1\}$ which allows us to describe the global evolution of the symbionts regardless of their type. Since for any $i = 1,\, 2,\, \ldots,\, \kappa$ the symbiont i can live only through hosts of type i, it is, however, easy to deduce from a coupling argument that if $g \leq 1$ then the processes with $\alpha = 0$ and $\alpha = \beta$ can be defined on the same space so that, starting from the same configuration, the process with $\alpha = 0$ has fewer pathogens. In words, the survival of the pathogens is harder to obtain with specialist interactions. In particular, if $g \leq 1$ and $\beta \leq \max\,(\beta_c\, g, 1)$, then the pathogens die out.

The next step is to extend the results of the mean-field model summarized in Figure 1 to the corresponding spatial model with short range interactions. To do this, we consider a population of two hosts with only one type of symbiont, say symbiont of type 1, and start the evolution with all the hosts of type 1 associated with a symbiont. Then, in the limiting case $\beta = \infty$, the transition $(1, 0) \to (1, 1)$ is instantaneous, i.e., unassociated hosts of type 1 become instantaneously associated with a symbiont, provided that $R_1 \leq R_2$ to avoid the problem of *isolated* unassociated host that cannot be reached by any symbiont. This implies that the process ξ_t^1 is a biased voter model in which hosts of type 1 give birth at rate $g\, \nu_{R_1}$ and hosts of type 2 at rate ν_{R_1}. In

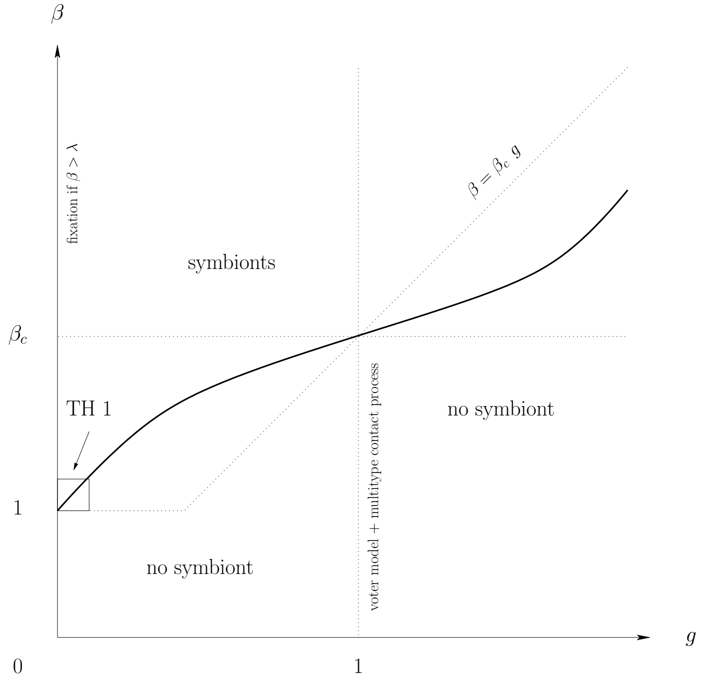

FIGURE 2. *Phase diagram of the spatial model with generalist interactions.*

particular, if $g < 1$, then $\xi_t^1 \Rightarrow \delta_2$, the "all 2" configuration, while if $g > 1$, then $\xi_t^1 \Rightarrow \delta_1$, the "all 1" configuration. Theorem 2 tells us that, in any dimension, the result still holds if $g > 1$ and $\beta \in (0, \infty)$ is sufficiently large.

Theorem 2 *Assume that $\alpha = 0$, $g > 1$, and $R_1 \le R_2$. At time 0, $\xi_0(x) = (1,1)$ or $(2,0)$ for all $x \in \mathbb{Z}^d$. Then, there is a critical value $\beta_{cr}^{Th2} \in (0,\infty)$ such that if $\beta > \beta_{cr}^{Th2}$ then $\xi_t^1 \Rightarrow \delta_1$.*

The arguments in our proof, however, do not imply the analogous result for $g < 1$. We think actually that except in the 1-dimensional nearest neighbor case, $P(\xi_t^1(x) = 2) \not\to 1$. Before explaining our intuition, we describe the behavior of the 1-dimensional process with nearest neighbor interactions for the hosts and short range interactions for the pathogens. Assume that at time 0, $\xi_0^1(x) = 1$ for $x > 0$ with infinitely many pathogens of type 1, and $\xi_0^1(x) = 2$ for $x \le 0$ with no associated pathogens. Denote by $r_t^2 = \sup\{x \in \mathbb{Z} : \xi_t^1(x) = 2\}$ the rightmost host of type 2. Then $r_0^2 = 0$. The following result implies that for β large enough, 2's will spread to the right and eliminate 1's together with their associated pathogens.

Theorem 3 *Assume $d = 1$ and $R_1 = R_2 = 1$. If $\alpha = 0$ and $g < 1$, there exists $\beta_{cr}^{Th3} \in (0,\infty)$ such that if $\beta > \beta_{cr}^{Th3}$, then $r_t^2 \to \infty$ as $t \to \infty$ with probability 1.*

We conjecture that this result should only be true in the 1-dimensional nearest neighbor case. Here is the intuition. We first observe that except in the 1-dimensional nearest neighbor case, the dynamics produce *isolated* hosts, that is hosts of type 1 (resp. 2) surrounded by a cluster of hosts of type 2 (resp. 1). As suggested by Theorem 2, when $g > 1$, isolated 2's are *swallowed* very quickly by surrounding 1's. On the other hand, when $g < 1$, an isolated 1 located in a linearly growing cluster of 2's cannot be invaded anymore by any pathogen as soon as the cluster has reached some critical size. In this expanding region, the process then behaves like an unbiased voter model in which 1's can now compete with 2's, and survive. See Figure 3 in Section 5 for simulations in $d = 2$.

Lastly, we investigate the coexistence of symbionts in the neutral case $g = 1$. We observe that, in this case, the first coordinate process ξ_t^1 performs a voter model run at rate 1. In particular, in $d \leq 2$, there does not exist any stationary distribution μ such that $\mu\,(\xi^1(x) = i) \neq 0$ for any $i \in \{1, 2, \ldots, \kappa\}$ (see Holley and Liggett, 1975). Since in the specialist case, $\xi_t^2(x) = i$ implies that $\xi_t^1(x) = i$, the same conclusion holds for the symbionts. In $d \geq 3$, coexistence occurs for the process ξ_t^1, i.e., there is a stationary measure μ which satisfies the condition above. However, due to the formation of clusters, the problem of coexistence of the symbionts remains a difficult question. Namely, the voter model ξ_t^1 exhibits clusters whose diameter can exceed some critical size, which prevents the symbionts from spreading out. To get around this difficulty, we introduce a modification of the particle system, denoted by $\hat{\xi}_t$, in which the symbionts evolve as previously but where the hosts now perform a threshold θ voter model. More precisely, the process $\hat{\xi}_t$ evolves according to the following transitions at $x \in \mathbb{Z}^d$.

$$(i,j) \ \to \ (k,0) \quad \text{at rate} \quad \begin{cases} 1 & \text{if} \quad \operatorname{card}\{z \in \mathbb{Z}^d : 0 < \|x - z\| \leq R_1 \text{ and } \hat{\xi}^1(z) = k\} \ \geq \ \theta \\ 0 & \text{otherwise} \end{cases}$$

$$(i,0) \ \to \ (i,i) \quad \text{at rate} \qquad \beta \ \operatorname{card}\{z \in \mathbb{Z}^d : 0 < \|x - z\| \leq R_2 \text{ and } \hat{\xi}^2(z) = i\}.$$

The introduction of this particle system is motivated by Theorem 1 of Durrett (1992) which implies that the threshold θ voter model has a nontrivial stationary distribution which is close enough to a product measure to produce our next result. We will prove the following result.

Theorem 4 *Let $g = 1$ and $\theta < \nu_{R_1}/\kappa$. If R_1 and R_2 are sufficiently large, then there is a critical value $\beta_{cr}^{Th4} \in (0, \infty)$, depending on κ, such that the following holds: If $\beta > \beta_{cr}^{Th4}$, then coexistence occurs, and if $\beta < \beta_{cr}^{Th4}$, then coexistence is not possible.*

Here coexistence means that there is a stationary measure μ such that $\mu\,(\hat{\xi}^2(x) = i) \neq 0$ for any type $i \in \{1, 2, \ldots, \kappa\}$. Unfortunately, we do not know how to prove something better than *coexistence is not possible* when $\beta < \beta_{cr}^{Th4}$. However, we conjecture that starting from product measure in which each host is associated with a symbiont and has density $1/\kappa$, all the symbionts die out. To justify our conjecture, we observe that the processes η_t^i defined by $\eta_t^i(x) = 1$ if $\hat{\xi}_t^2(x) = i$ and $\eta_t^i(x) = 0$ otherwise, do not interact since they are confined to their associated hosts. Since Theorem 1 of Durrett (1992) tells us that the hosts coexist with density of each type close to $1/\kappa$, each symbiont should remain subcritical. Finally, since the symbionts can only spread out through their host, we conjecture that β_{cr}^{Th4} is increasing with respect to κ.

Comparison of the spatially explicit and the mean-field model

Numerical simulations of the mean-field model indicate that coexistence is only possible when $g < 1$. Simulations of the spatially explicit model show similar behavior. When $g < 1$ and $\alpha < \beta$, then coexistence of hosts and pathogens is possible. We observed that in this case, cluster size is limited by the presence of pathogens: In the absence of pathogens, clusters grow at the expense of neighboring clusters that contain symbionts. Upon invasion by the preferred symbionts (those with infection rate β), the clusters appear to shrink again. The case $g > 1$ and $\alpha < \beta$ is quite different. Clusters of hosts with their preferred mutualists form and appear to continue to grow, just as in the voter model case. Less preferred mutualists (those with infection rate α) do not seem to be able to persist with preferred mutualists, just as is the mean-field case of one host and two symbionts with infection rates a and b, respectively ($a < b$). In summary, pathogens have the ability to alter the spatial structure of their hosts by promoting local diversity, whereas mutualists do not alter the spatial structure of their hosts. This difference in behavior is more pronounced the more host-specific the symbionts are.

The rest of this paper is devoted to proofs. In Section 2, we will investigate the mean-field model. In Section 3, we will prove a preliminary result about the biased voter model to prepare the proofs of Theorems 1 and 2, which will be carried out in Sections 4 and 5, respectively. Section 6 will be devoted to the proof of Theorem 3. Finally, we will investigate the coexistence of symbionts and prove Theorem 4 in Section 7.

2. The mean-field model

The mean-field model was introduced in Section 1. Our first claim was that the host density $h_i = u_i + v_{i\cdot}$ is a conserved quantity when $g = 1$. A straightforward calculation shows that if $g = 1$,

$$\frac{d}{dt}\,(u_i + v_{i\cdot}) \;=\; 0.$$

from which our claim follows.

We summarize the behavior of the mean-field model in the following proposition.

Proposition 2.1 *For* $(\kappa - 1)\,a + b > \kappa$ *and* $g \geq 0$*, there exists a nontrivial equilibrium with* $u_1 = u_2 = \cdots = u_\kappa \geq 0$ *and* $v_{1\cdot} = v_{2\cdot} = \cdots = v_{\kappa\cdot} > 0$ *such that for* $i = 1, 2, \ldots, \kappa$,

$$u_i \;=\; \frac{g}{(\kappa - 1)\,a + b - \kappa\,(1 - g)} \qquad \text{and} \qquad h_i \;=\; \frac{1}{\kappa}.$$

Furthermore, for $i \neq j$

$$\frac{v_{ij}}{v_{ii}} \;=\; \frac{a}{b}.$$

Proof If we denote by $h_i = u_i + v_{i\cdot}$ the density of host i (both associated and unassociated), then

$$\frac{dh_i}{dt} \;=\; u_i + g\,v_{i\cdot} - h_i\,(u_\cdot + g\,v_{\cdot\cdot}).$$

By setting the right-hand side equal to 0, we obtain

$$h_i \;=\; \frac{u_i + g\,v_{i\cdot}}{u_\cdot + g\,v_{\cdot\cdot}}.$$

It follows that

$$\frac{h_i}{h_j} \;=\; \frac{u_i\,(1 - g) + g\,h_i}{u_j\,(1 - g) + g\,h_j}$$

from which we conclude that

$$\frac{h_i}{h_j} \;=\; \frac{u_i}{u_j} \;=\; \frac{v_{i\cdot}}{v_{j\cdot}}.$$

In the symmetric case, $h_1 = h_2 = \cdots = h_\kappa = \frac{1}{\kappa}$, we find $u_1 = u_2 = \cdots = u_\kappa$ and $v_{1\cdot} = v_{2\cdot} = \cdots = v_{\kappa\cdot}$. The nontrivial equilibrium can then be computed explicitly. We find

$$u_i \;=\; \frac{g}{(\kappa - 1)\,a + b - \kappa\,(1 - g)} \qquad \text{and} \qquad h_i \;=\; \frac{1}{\kappa}.$$

Specifically, when $g = 0$, $u_i = 0$ and consequently all hosts will be associated at equilibrium. The condition for the existence of a nontrivial point equilibrium, namely $(\kappa - 1)\,a + b > \kappa$, follows directly from requiring that $u_i < 1/\kappa$ and $h_i = 1/\kappa$. Furthermore, it follows from

$$b\,u_i\,v_{\cdot i} \;=\; v_{ii}\,(u_\cdot + g\,v_{\cdot\cdot}) \qquad \text{and} \qquad a\,u_i\,v_{\cdot j} \;=\; v_{ij}\,(u_\cdot + g\,v_{\cdot\cdot})$$

that

$$\frac{a}{b} \;=\; \frac{v_{\cdot i}}{v_{\cdot j}}\,\frac{v_{ij}}{v_{ii}}.$$

Since $v_{\cdot i} = v_{\cdot j}$ by symmetry, the last claim follows as well. $\square$

3. Preliminary results about the biased voter model

As explained in the introduction, if $\alpha = \beta$ and $R_1 = R_2$, then the "color-blind" process ζ_t performs a biased voter model when $g = 0$. If we set $\alpha = 0$ and consider a population of two host types with only one symbiont type, then the process ξ_t^1, which describes the evolution of both host types, performs a biased voter model in the limiting case $\beta = \infty$. So, to prove Theorems 1 and 2, we will start by proving a general result about the biased voter model, and then apply a perturbation argument to extend this result to the region $g > 0$ small in the first case, and to the region $\beta < \infty$ large in the second case. Let $\beta_1,\, \beta_2 \in (0, \infty)$, and $\eta_t : \mathbb{Z}^d \longrightarrow \{1,\, 2\}$ be the biased voter model with parameters β_1 and β_2, i.e., the process whose state at site x changes as follows:

$$i \;\to\; j \qquad \text{at rate} \qquad \beta_j \sum_{0 < \|x - z\| \le R_1} \mathbb{1}_{\{\eta(z) = j\}}.$$

It is a well-known fact that if $\beta_1 > \beta_2$ then $P\,(\eta_t(x) = 1) \to 1$ as $t \to \infty$, provided that at time 0, the process has infinitely many 1's and 2's (see, e.g., Durrett (1988), Chapter 3).

To prove Theorems 1 and 2, we will follow the strategy described in Durrett and Neuhauser (1997), Section 3. We begin with a rescaling argument to estimate the rate of convergence of $P\,(\eta_t(x) = 1)$. This estimate will have to be good enough so that a perturbation argument can be applied. The basic idea is to show that for given $\varepsilon > 0$, members of the family of processes under consideration, when viewed on suitable length and time scales, dominate an M-dependent oriented percolation process in which sites are open with probability $1 - \varepsilon$ (Durrett 1995, Section 4). To compare the process with a percolation process, we consider a positive integer L to be fixed later, and scale space by setting

$$B \;=\; [-L,\, L]^d, \qquad \Phi(z) \;=\; L\, z, \qquad \text{and} \qquad B_z \;=\; \Phi(z) \;+\; B$$

for any $z \in \mathbb{Z}^d$. Let Γ be a positive integer, and say that (z, n) is *occupied* if all sites in B_z are occupied by 1's at time $n\,\Gamma L$. The first step in proving Theorems 1 and 2 is the following

Proposition 3.1 *Let $\varepsilon > 0$ and $\beta_1 > \beta_2$. Then M, L and Γ can be chosen in such a way that the set of occupied sites dominates the set of open sites in an M-dependent oriented site percolation process where sites are open with probability $p = 1 - 2\,\varepsilon/\,3$.*

The key to the proof is duality (Durrett 1988, Chapter 3). To define the dual process of the biased voter model, we consider two collections of independent Poisson processes $\{T_n^{x,z} : n \ge 1\}$ and $\{U_n^{x,z} : n \ge 1\}$ where $0 < \|x - z\| \le R_1$, with parameter β_2 and $\beta_1 - \beta_2$ respectively. At times $T_n^{x,z}$ we draw an arrow from z to x and put a δ at site x, while at times $U_n^{x,z}$ we draw an arrow from z to x without putting a δ at the tip. The process is then obtained from the graphical representation as follows: At time $T_n^{x,z}$, the particle at x imitates the one at z. At time $U_n^{x,z}$, the site x becomes occupied by a particle of type 1 if z is. We say that there is a *path* from $(x, 0)$ to (z, t) if there is a sequence of times $s_0 = 0 < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 \;=\; x,\, x_1,\, \ldots,\, x_n \;=\; z$ such that the following two conditions hold:

1. For $i = 1,\, 2,\, \ldots,\, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0,\, 1,\, \ldots,\, n$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any δ's.

Finally, we say that there exists a *dual path* from (x, t) to $(z, t - s)$, $0 \le s \le t$, if there is a path from $(z, t - s)$ to (x, t), and define the *dual process starting at* (x, t) by setting

$$\hat{\eta}_s^{(x,t)} \;=\; \{z \in \mathbb{Z}^d : \text{there is a dual path from } (x,t) \text{ to } (z, t-s)\}$$

for any $0 \le s \le t$. The reason why we introduce the dual process is that it allows us to deduce the state of site x at time t from the configuration at earlier times. More precisely,

$$\eta_t(x) \;=\; 1 \qquad \text{if and only if} \qquad \eta_{t-s}(z) \;=\; 1 \quad \text{for some} \quad z \;\in\; \hat{\eta}_s^{(x,t)}.$$

See Durrett (1988), Chapter 3. The strategy to proving Proposition 3.1 can then be summarized as follows: Let $T = \Gamma L$ and $x \in B_z$ with $\| z \| = 1$. Then, we will prove that, with probability arbitrarily close to 1, there exists a dual path A_s starting at (x, T) and landing in the target set B. More precisely, we will prove that A_s hits the set $J = [-R_1, R_1]^d$ by time T where $R_1 < L/2$, and then stays inside B until time T. Recall that R_1 denotes the range of the interactions. In particular, if B is void of 2's at time 0 then, with probability close to 1, B_z will be void of 2's as well T units of time later. To define the dual path A_s, we start the process at $A_0 = (x, T)$ and go down the graphical representation. If A_s comes across a δ at some time $s = T - T_n^{x,z}$ with $x = A_s$ then move A_s to z. If A_s meets the tip of an arrow that is without a δ at some time $s = T - U_n^{x,z}$ then move A_s to z only if it takes it closer to 0. Intuitively, this should cause A_s to drift towards the set B. We now make this argument precise in a series of lemmas.

Lemma 3.2 *Assume that $x \in B_z$, $\| z \| = 1$, and $\beta_1 > \beta_2$. There exist $C_1, \gamma_1 \in (0, \infty)$ such that*

$$\sup_{x \in B_z} P_x (A_s \notin J \textit{ for all } s \leq T) \ \leq \ C_1 \exp(-\gamma_1 L)$$

for L and Γ sufficiently large. Here, the subscript x indicates the starting point.

Proof Let σ_k denote the k-th time A_s encounters the tip of an arrow (with or without a δ). At time σ_k, the arrow does not have a δ at its tip with probability $(\beta_1 - \beta_2)/\beta_1 > 0$. Moreover, if $A_{\sigma_k} \notin J$ and the arrow does not have a δ at its tip, then with probability at least $1/2d > 0$, A_s moves closer to 0. In particular, if $N = \inf \{k \geq 1 : A_{\sigma_k} \in J\}$ then there is $c > 0$ such that

$$P(N \geq c L) \ \leq \ C_2 \exp(-\gamma_2 L)$$

for suitable $C_2, \gamma_2 \in (0, \infty)$. Since $P(\sigma_k - \sigma_{k-1} > t) = \exp(-\beta_1 t)$, the result follows. □

Lemma 3.3 *Assume that $\beta_1 > \beta_2$. For any $y \in J$ there exist $C_3 < \infty$ and $\gamma_3 > 0$ such that*

$$\sup_{y \in J} P_y (A_s \notin B \textit{ for some } s \leq T) \ \leq \ C_3 \exp(-\gamma_3 L)$$

for L sufficiently large.

Proof We let $s_0 = 0$ and, for $k \geq 1$, define the following stopping times

$$\begin{aligned} t_k &= \inf \left\{ t > s_{k-1} : A_t \notin (-L/2, L/2)^d \right\} \\ s_k &= \inf \{t > t_k : A_t \in J\} \quad \text{and} \quad \tau \ = \ \inf\{t > 0 : A_t \notin B\}. \end{aligned}$$

Moreover, we denote by $M(t) = \sup \{k \geq 1 : \sigma_k < t\}$ the number of tips of arrows encountered by A_s by time t. Then for any site $y \in J$

$$\begin{aligned} P_y (A_s \notin B \text{ for some } s \leq T) &= P_y (A_{\sigma_k} \notin B \text{ for some } k \leq M(T)) \\ &\leq P_y (A_{\sigma_k} \notin B \text{ for some } k \leq 2 \beta_1 T) \ + \ P(M(T) > 2 \beta_1 T) \\ &\leq P_y (s_k > \tau \text{ for some } k \leq 2 \beta_1 T) \ + \ P(M(T) > 2 \beta_1 T) \\ &\leq 2 \beta_1 T \ \sup_{z \in J} P_z (s_1 > \tau) \ + \ P(M(T) > 2 \beta_1 T). \end{aligned}$$

Since A_s has a drift towards J and that the time between consecutive jumps has exponential bound, $P(s_1 > \tau) \leq C_4 \exp(-\gamma_4 L)$ for appropriate $C_4 < \infty$ and $\gamma_4 > 0$ (see the proof of Lemma 3.2). Furthermore, since $\mathbb{E} M(T) = \beta_1 T$, large deviation estimates imply that there are $C_5 < \infty$ and $\gamma_5 > 0$ such that $P(M(T) > 2 \beta_1 T) \leq C_5 \exp(-\gamma_5 T)$. □

Lemma 3.4 *Assume that* $x \in B_z$, $\| z \| = 1$, *and* $\beta_1 > \beta_2$. *There exist* $C_6, \gamma_6 \in (0, \infty)$ *such that*

$$\sup_{y \in B_z} P_x (A_T \notin B) \ \leq \ C_6 \exp(-\gamma_6 L)$$

for Γ *and* L *sufficiently large.*

Proof By decomposing according to whether $A_s \in J$ for some $s \leq T$ or not, we obtain

$$P_x (A_T \notin B) \ \leq \ P_x (A_s \notin J \text{ for all } s \leq T) \ + \ P (A_T \notin B \ ; \ A_s \in J \text{ for some } s \leq T).$$

The first term on the right-hand side can be bounded using Lemma 3.2. For the second term, we first observe that

$$P (A_T \notin B \ ; \ A_s \in J \text{ for some } s \leq T) \ \leq \ \sup_{y \in J} P_y (A_s \notin B \text{ for some } s \leq T)$$

and then apply Lemma 3.3. This completes the proof. □

Since there are $(2L+1)^d$ sites in B_z, it follows from Lemma 3.4 and duality that there is a constant $C_7 < \infty$ independent of L such that for Γ and L sufficiently large

$$\begin{aligned} P (\eta_T(x) = 2 \text{ for some } x \in B_z) \ &\leq \ \sum_{x \in B_z} P_x (A_T \notin B) \\ &\leq \ (2L+1)^d \, C_6 \exp(-\gamma_6 L) \ \leq \ C_7 \, L^{-1} \ \leq \ \varepsilon / 3. \end{aligned}$$

Moreover, since each of the dual paths has a drift toward J, we can fix $M > 0$, say $M = 3$, so that for any $\varepsilon > 0$

$$\begin{aligned} &P (\text{any of the selected paths is not contained in} \\ &\qquad [-ML, ML]^d \text{ at some time } s \leq T) \ \leq \ \varepsilon / 3 \end{aligned}$$

by choosing L sufficiently large. This shows that boxes that are sufficiently far apart are independent of each other with high probability and completes the proof of Proposition 3.1.

4. Proof of Theorem 1

This section is devoted to the proof of part (b) of Theorem 1 which describes the behavior of the process for $\beta > 1$ and g close to 0 in the generalist case $\alpha = \beta$. As already explained in the introduction, if $g = 0$ then the process $\zeta_t : \mathbb{Z}^d \longrightarrow \{0, 1\}$ reduces to the biased voter model with parameters 1 and β so that if $\beta > 1$ then $P(\zeta_t(x) = 1) \to 1$ as $t \to \infty$. To prove that the pathogens still survive when $g > 0$ is sufficiently small, we show that for $M = 3$ and any $\varepsilon > 0$, we can choose L and Γ such that Proposition 3.1 holds for $\beta_1 = \beta$ and $\beta_2 = 1$. Now that Γ and L are fixed and $M = 3$, we can assert that there exists $g_c > 0$ small so that

$$\begin{aligned} &P (\text{some infected host in } [-ML, ML]^d \text{ gives birth to a healthy host sent to } B \\ &\qquad \text{between time 0 and time } T) \ \leq \ (2ML+1)^d \, (1 - \exp(-g_c T)) \ \leq \ \varepsilon / 3. \end{aligned}$$

This tells us that if $g < g_c$, then the set of occupied sites dominates the set of open sites in an oriented percolation process with parameter $p = 1 - \varepsilon$. Here (z, n) occupied means that all sites in B_z are occupied by pathogens at time nT. Finally, by taking $\varepsilon > 0$ sufficiently small so that percolation occurs with positive probability, Theorem 1 follows.

5. Proof of Theorem 2

The proof of Theorem 2 also relies on a perturbation argument. In the case $\beta = \infty$ and $R_1 \leq R_2$, the transition $(1,0) \to (1,1)$ is instantaneous, i.e., unassociated hosts of type 1 become instantaneously associated with a mutualist, provided that all the hosts of type 1 are initially associated with a mutualist. The assumption $R_1 \leq R_2$ is to avoid the problem of births of isolated, unassociated hosts of the same type that are not accessible to mutualists. Under these assumptions, the process ξ_t^1 performs a biased voter model with parameters $\beta_1 = g$ and $\beta_2 = 1$. In particular, well-known results about the biased voter model imply that if $g > 1$, then $P(\xi_t^1(x) = 1) \to 1$ as $t \to \infty$.

To extend the result to the region $\beta > 0$ large, we prove that if hosts of type 1 become occupied by their associated mutualists quickly enough, then ξ_t^1 will evolve like a biased voter model in the space-time box $B \times [0, \Gamma L]$ with probability close to 1. We first define ξ_t on the same space as the biased voter model η_t introduced in Section 3 with $\beta_1 = g$ and $\beta_2 = 1$. At time $T_n^{x,w}$, the host present at site w gives birth to an unassociated host of the same type which is then sent to x. At time $U_n^{x,w}$, the birth from w to x occurs only if the host at w is associated with a mutualist. To describe the evolution of the mutualists, we consider one more collection of independent Poisson processes, $\{V_n^{x,w} : n \geq 1\}$, $0 < \|x - w\| \leq R_2$, with parameter β. At time $V_n^{x,w}$, we draw an arrow labeled with a 1 from w to x to indicate that a mutualist (of type 1) present at site w gives birth to a mutualist at site x if this site is already occupied by a host of type 1. We will prove that there exists $\beta_{cr}^{Th2} \in (0, \infty)$ such that if $\beta > \beta_{cr}^{Th2}$ and $\xi_0^1 = \eta_0$ on B, then $\xi_T^1 = \eta_T$ on B_z with $\|z\| = 1$ at time $T = \Gamma L$ with probability $\geq 1 - \varepsilon/3$. Since boxes that are sufficiently far apart are independent of each other with probability close to 1, we can focus on $[-ML, ML]^d \times [0, \Gamma L]$, $M = 3$, to estimate this event. Let $x \in [-ML, ML]^d$ and follow the line $\{x\} \times [0, \Gamma L]$ by going forward in time. Each time a host at w attempts to give birth at site x, we require that the next 1-arrow from w to x appears before the host at w is replaced or the host at x gives birth. A straightforward calculation shows that this event occurs with probability

$$\begin{aligned} P\big(V_1^{x,w} < \min(T_1^{y,x}, U_1^{y,x}) \text{ for any } y \in \mathcal{N}_x^1 \quad \text{and} \\ V_1^{x,w} < \min(T_1^{w,y}, U_1^{w,y}) \text{ for any } y \in \mathcal{N}_w^1\big) \ = \ \beta\,(\beta + 2m)^{-1} \end{aligned}$$

where $m = g\,\nu_{R_1}$. Let's now denote by $K(x,T)$ the number of unlabeled arrows and δ-arrows that point at site x by time T, and set $I_M = [-ML, ML]^d$. Then, by observing that $\mathbb{E}\,K(x,T) = m\,T$, and by decomposing the event to be estimated according to whether $K(x,T) > 2m\,T$ or not, we finally obtain

$$\begin{aligned} P(\xi_T^1 \neq \eta_T \text{ on } B_z) \ &\leq \ \sum_{x \in I_M} P(K(x,T) > 2\,m\,T) \ + \ 2\,m\,T \sum_{x \in I_M} \frac{2\,m}{\beta + 2\,m} \\ &\leq \ (2ML)^d \ \times \ \big\{C_8 \exp(-\gamma_8\,T) \ + \ 4\,m^2\,T\,(\beta + 2m)^{-1}\big\} \end{aligned}$$

for appropriate $C_8 < \infty$ and $\gamma_8 > 0$. Taking L and then β sufficiently large so that

$$P(\xi_T^1 \neq \eta_T \text{ on } B_z) \ \leq \ \varepsilon/3,$$

and applying Proposition 3.1 imply that the set of occupied sites dominates the set of open sites in an oriented percolation process with parameter $p = 1 - \varepsilon$. Here (z, n) occupied means that all sites in B_z are occupied by associated hosts of type 1 at time $n\,T$. This almost produces Theorem 2. Our last problem is that oriented site percolation has a positive density of unoccupied sites. To prove that there is an in-all-directions expanding region which is void of hosts of type 2, we apply a result from Durrett (1992) which shows that unoccupied sites do not percolate when ε is close enough to 0. Since hosts of either type cannot appear spontaneously, once a region is void of one type, this type can only reappear in the region through invasion from the outside. This implies that $P(\xi_t^1(x) = 2) \to 0$ as $t \to \infty$ for any $x \in \mathbb{Z}^d$ and completes the proof of Theorem 2.

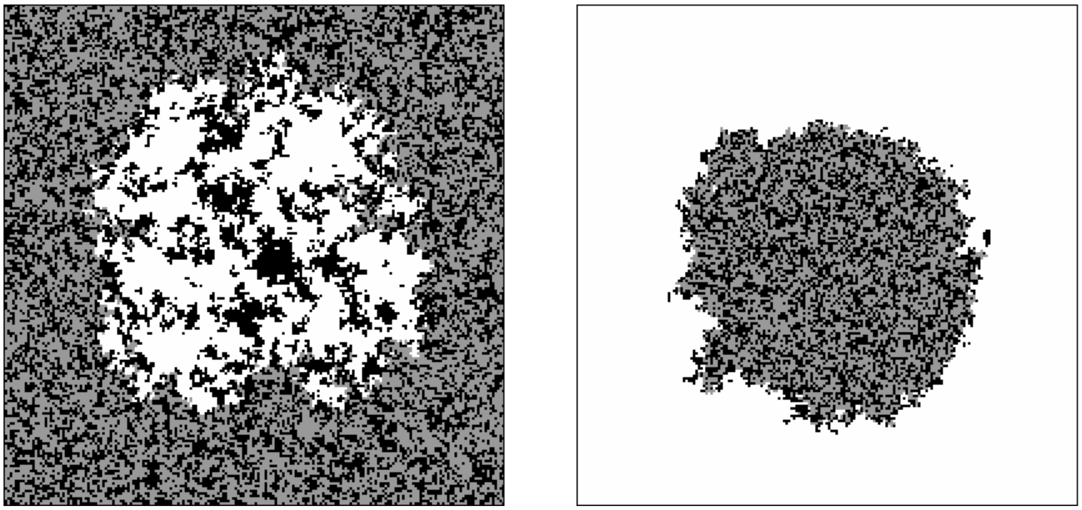

Figure 3. *Process with nearest neighbor interactions on the* 200×200 *torus at time 250. Picture on left:* $\beta = 2$ *and* $g = 0.5$*. The process starts with unassociated white hosts in* $J = (90, 110)^2$ *and black hosts associated with a pathogen outside* J*. Picture on right:* $\beta = 4$ *and* $g = 2$*. The process starts with black hosts associated with a mutualist in* J *and unassociated white hosts outside* J*. In both pictures, gray sites refer to black hosts associated with a symbiont.*

6. Proof of Theorem 3

This section is devoted to the proof of Theorem 3. The method of the proof can also be applied to give a more explicit proof of Theorem 2 without too much more work. For the proof, we will assume that the birth rate λ is not set equal to 1. In fact, we will prove Theorem 3 first for λ small and then change the time scale so that it holds for $\lambda = 1$ as well. We start by introducing the rightmost host 2 process r_t^2 and the leftmost symbiont 1 process ℓ_t^1, i.e.,

$$r_t^2 \ = \ \sup\{x \in \mathbb{Z} : \xi_t^1(x) = 2\} \qquad \text{and} \qquad \ell_t^1 \ = \ \inf\{x \in \mathbb{Z} : \xi_t^2(x) = 1\}.$$

We observe that because of nearest neighbor interactions, $\ell_t^1 - r_t^2 \geq 1$ at any time $t \geq 0$. Moreover, if $G_t = \ell_t^1 - r_t^2 - 1$ denotes the number of sites between both processes, then r_t^2 changes as follows:

$$\text{for } G_t = 0 : \ r_t^2 \ \to \left\{ \begin{array}{ll} r_t^2 + 1 & \text{at rate } \lambda \\ r_t^2 - 1 & \text{at rate } \lambda g \end{array} \right. \qquad \text{and} \qquad \text{for } G_t > 0 : r_t^2 \ \to \left\{ \begin{array}{ll} r_t^2 + 1 & \text{at rate } \lambda \\ r_t^2 - 1 & \text{at rate } \lambda \end{array} \right.$$

which suggests that r_t^2 drifts to the right if $g < 1$ (and to the left if $g > 1$). To make this argument precise, we will prove the following lemma.

Lemma 6.1 *For* $g \neq 1$*, there exists* $\varepsilon_0 > 0$ *such that:* $\displaystyle\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{G_t = 0\}} \, dt \ \geq \ \varepsilon_0.$

To deduce Theorem 3 from Lemma 6.1, we first observe that $\mathbb{E}\, r_t^2 \ \geq \ \lambda(1-g)\, \varepsilon_0\, t$ for $g < 1$. A large deviation estimate then implies that

$$P\,(r_t^2 \ \geq \ (1-\varepsilon)\, \sigma\, t) \ \geq \ 1 - C_9\, e^{-\gamma_9 t}$$

where $\sigma = \lambda(1-g)\, \varepsilon_0$. This implies that if $g < 1$, then $r_t^2 \to \infty$ with probability 1. If $g > 1$, then $\mathbb{E}\, r_t^2 \ \leq \ -\lambda(g-1)\, \varepsilon_0\, t$, and a similar argument implies that $r_t^2 \to -\infty$ with probability 1. The proof of Lemma 6.1 is based on a rescaling argument. The main objective is to prove that the fraction of time the host present at site $r_t^2 + 1$ is associated with a symbiont is greater than some positive constant. To be able to compare the particle system with oriented percolation process, we will artificially freeze r_t^2 by introducing the process seen from the interface, namely $\iota_t(x) = \xi_t(x + r_t^2 + 1)$. From this new point of view, the symbiont evolves on the half-line $\mathbb{Z}^+$. To do this comparison, we need to extend some results on oriented percolation to oriented percolation *in half-space*, i.e., the process in which all sites to the left of 0 are closed.

Oriented percolation in half-space

As explained above, our first objective is to investigate 1-dependent site percolation process in the half-space. Let

$$\Omega \ = \ \{(x,n) : x + n \text{ is even and } n \geq 0\}.$$

For any $(x,n) \in \Omega$, let $\omega(x,n)$ define a random variable with the following property: If $x < 0$ then $\omega(x,n) = 0$, while if $x \geq 0$, then $\omega(x,n) = 1$ with probability p and $\omega(x,n) = 0$ with probability $1-p$. The site (x,n) is said to be open (resp. closed) if $\omega(x,n) = 1$ (resp. 0). Finally, 1-dependent with parameter p means that whenever (x_i, n_i), $1 \leq i \leq m$, is a sequence with $x_i \geq 0$ for any $1 \leq i \leq m$ where $\|(x_i, n_i) - (x_j, n_j)\|_\infty > 1$ for $i \neq j$, then

$$P\,(\omega(x_i, n_i) = 0 \text{ for } 1 \leq i \leq m) \ = \ (1-p)^m.$$

We say that (y,n) can be reached from (x,m) and write $(x,m) \to (y,n)$ if there exists a sequence of points $x = x_m, x_{m+1}, \ldots, x_n = y$ such that $|x_{i+1} - x_i| = 1$ for $m \leq i \leq n-1$, and $\omega(x_i, i) = 1$ for $m \leq i \leq n$. To formulate the next result, we let

$$W_n^A \ = \ \{y : (x,0) \to (y,n) \text{ for some } x \in A\} \qquad \text{and} \qquad \tau^A \ = \ \inf\{n : W_n^A = \varnothing\}$$

for any $A \subset 2\mathbb{Z}^+$. Finally, if $A = 2\mathbb{Z}^+$, we will write $W_n = W_n^A$ and $\tau = \tau^A$.

Lemma 6.2 *If $p > 1 - 3^{-72}$ then there exists $\varepsilon_1 > 0$ such that $P\,(0 \in W_{2n}) \geq \varepsilon_1$ for any $n \geq 0$.*

Proof The proof closely follows Durrett (1984), Section 10, so we will try to be as brief as possible, and refer the reader to this reference for details. The first step is a slight modification of the contour argument applied to independent site percolation process. Let $A = \{0, 2, \ldots, 2N\}$ and $C = \{(y,n) : \text{there exists } x \in A \text{ with } (x,0) \to (y,n)\}$. We set

$$D \ = \ \{(a,b) \in \mathbb{R}^2 : |\,a\,| + |\,b\,| \leq 1\} \qquad \text{and} \qquad W \ = \ \bigcup_{z \in C} \,(z + D).$$

If the set C is finite, we denote by Γ the boundary of the unbounded component of $(\mathbb{R} \times (-1,\infty)) - W$ and orient the boundary in such a way that the segment from $(2N,-1)$ to $(2N+1,0)$ is oriented in the direction indicated. The boundary is a contour line, if it exists, starting at $(2N,-1)$ and ending at $(0,-1)$. There are at most 3^{m-1} contours of length m. Moreover, for a contour of length m to exist, there must be at least $m/8$ closed sites to the right of zero. To prove this point, we call a segment a line segment of the form $x + F$ where $x \in C$ and F is one of the sides of D. The site closest to the right of the segment is the site associated with the segment. Here, right and left are defined according to the orientation introduced above. We call segments of Γ which look like ↖ ↙ ↘ and ↗ segments of types 1, 2, 3 and 4 respectively. By construction, a site associated with a segment of type 1 or 2 must be closed. Let m_i and $\bar{m}_i$ be respectively the number of segments of type i and the number of segments of type i located on the left of zero. Since the contour starts at $(2N,-1)$ and ends at $(0,-1)$, $m_1 + m_2 = m_3 + m_4 + 2N$, so if the contour has length m then $m_1 + m_2 \geq m/2$. The same reasoning leads to $m_1 + m_4 = m_2 + m_3$ so that $\bar{m}_2 + \bar{m}_3 \leq m/2$. Now, since the sites located on the left of zero are closed with probability 1, we obtain $\bar{m}_1 = 0$ and $\bar{m}_2 = \bar{m}_3 \leq m/4$, which implies $(m_1 + m_2) - (\bar{m}_1 + \bar{m}_2) \geq m/4$. Finally, since a site in W^c can be associated with at most two segments of type 1 and 2, it follows that the number of sites on the right of zero that must be closed is $\geq m/8$. Noticing that the shortest possible contour has length $2N + 4$, one can conclude that

$$P\,(\tau^{[0,2N]} < \infty) \ \leq \ \sum_{m=2N+4}^{\infty} 3^m\,(1-p)^{m/8} \ = \ C_{10}\,(3\,(1-p)^{1/8})^{2N}$$

if $p > 1 - 3^{-8}$ and the variables $\omega(x,n)$ are independent. To deal with the 1-dependent case, we observe that there are 9 sites in Ω with $(|\,m\,| + |\,n\,|)/2 \leq 1$ so for each Γ of length m there is a set of $m/72$ sites which are separated by more than one and which must be closed for the contour to exist. In conclusion, if $p > 1 - 3^{-72}$ then $P\,(\tau^0 = \infty) > 0$. Finally, if we map Ω into itself by sending $(x,m) \mapsto (x, n-m)$ and reverse the orientation of the graph then

$$P\,(W_n^A \cap B \neq \varnothing) \;=\; P\,(W_n^B \cap A \neq \varnothing).$$

Taking $A = 2\mathbb{Z}^+$ and $B = \{0\}$, it then follows that $P\,(0 \in W_{2n}) \downarrow P\,(\tau^0 = \infty) > 0$. In conclusion, the lemma holds by setting $\varepsilon_1 = P\,(\tau^0 = \infty)$. $\square$

The process seen from the interface

To prove Lemma 6.1 for $g < 1$, we introduce the nearest neighbor contact process ξ_t^- in which a particle at x dies at rate 2λ, gives birth at rate $\beta\nu_{R_2}$, and sends its offspring to one of the neighbors at random in $\mathcal{N}_x^2$. The process is modified so that particles located in the interval $(-\infty,\, r_t^2]$ are removed. More precisely, each time a particle in ξ_t^- tries to give birth to a particle which is sent to a site in $(-\infty,\, r_t^2]$, the birth is suppressed. Furthermore, if a particle is present at site $r_t^2 + 1$ when the process r_t^2 moves to the right, then this particle is removed. (The proof is similar in the case $g > 1$, provided one replaces ξ^- by ξ^+ where particles give birth at rate $\beta\nu_{R_2}$ but now die at rate $2\lambda g$.) The first step is to prove that ξ_t dominates the process ξ_t^-.

Lemma 6.3 *If $g < 1$, the processes ξ_t and ξ_t^- can be defined on the same probability space in such a way that if the inclusion $\xi_0^- \subset \xi_0^2$ holds at time 0 then $\xi_t^- \subset \xi_t^2$ at any later time.*

Proof Let's start by observing that if the rightmost process r_t^2 jumps to the right, a particle located at $r_t^2 + 1$, if it is present, is removed from both processes ξ_t^- and ξ_t^2 and that, if it jumps to the left, both configurations stay unchanged. Therefore, it suffices to prove that the inclusion holds when r_t^2 is constant, say $r_t^2 = 0$ at any time. This follows from a standard coupling argument so we just need to define a graphical representation that preserves the inclusion. We consider for any $x,\, z \geq 0$ with $|x - z| = 1$ the independent Poisson processes $\{S_n^{x,z} \,:\, n \geq 1\}$ and $\{T_n^{x,z} \,:\, n \geq 1\}$ with rate λg and $\lambda(1-g)$, respectively. For any $x,\, z \geq 0$, with $0 < \|z - x\| \leq R_2$, we also introduce the Poisson process $\{U_n^{x,z} \,:\, n \geq 1\}$ with rate β. The evolution of ξ_t is as follows: At time $S_n^{x,z}$, the host present at site x gives birth to a healthy host of the same type, which is then sent to z. At time $T_n^{x,z}$, the birth occurs only if the host at x is healthy. At time $U_n^{x,z}$, a pathogen (of type 1) at site x infects a host of type 1 at z if it is present. Finally, the process ξ_t^- evolves according to the following rules: At time $U_n^{x,z}$, a particle at site x gives birth to a new particle which is then sent to z. If the site is empty, the birth occurs. Otherwise, it is suppressed. At times $T_n^{x,z}$ and $S_n^{x,z}$, a particle present at z is removed. Such a coupling leads to the desired result. $\square$

We now introduce the process seen from the interface: $\eta_t(x) = \xi_t^-\,(x + r_t^2 + 1)$. To describe this process, we define the translation operators τ_1 and τ_{-1} by setting

$$[\tau_1\,\eta]\,(x) \;=\; \eta\,(x+1) \qquad \text{and} \qquad [\tau_{-1}\,\eta]\,(x) \;=\; \begin{cases} \eta\,(x-1) & \text{if } \; x \geq 1 \\ \quad 0 & \text{if } \; x \leq 0. \end{cases}$$

Then η_t evolves according to the following rules:

1. A particle at x gives birth at rate $\beta\nu_{R_2}$ to a new particle which is then sent to a neighbor within the neighborhood $\mathcal{N}_x^2$. If the target site is empty the birth occurs, otherwise it is suppressed. Moreover, no births are allowed to the left of 0.
2. Each particle dies at rate 2λ.
3. Depending on whether $\xi_t^2\,(r_t^2 + 1) = 0$ or 1, η_t respectively shifts as follows:

$$\eta_t \;\to\; \begin{cases} \tau_1\,\eta_t & \text{at rate } \lambda \\ \tau_{-1}\,\eta_t & \text{at rate } \lambda \end{cases} \qquad \text{and} \qquad \eta_t \;\to\; \begin{cases} \tau_1\,\eta_t & \text{at rate } \lambda \\ \tau_{-1}\,\eta_t & \text{at rate } \lambda g. \end{cases}$$

To compare η_t with an oriented percolation process in half-space, we let Γ and L be two large integers to be fixed later, and scale space by setting

$$B \ = \ [-L, L], \qquad \Phi(z) \ = \ L\,z \qquad \text{and} \qquad B_z \ = \ \Phi(z) \ + \ B.$$

Let $J_z = \Phi(z) + (-L/5, L/5)$, and say that the site $(z, n) \in \Omega$ is *occupied* if there is at least one particle in the interval J_z at time $n\,\Gamma L$. Let's fix $z \geq 0$ even, and start the process η_t with one particle in J_z. The first step is to investigate the process with no deaths inside B_z and modified so that any particle outside B_z is killed. We denote by $\bar{\eta}_t$ this new process.

Lemma 6.4 *Let $\varepsilon_2 = 6^{-72}$ and $\beta > \lambda$. Then Γ and L can be chosen so that*

$$P\,(\bar{\eta}_{\Gamma L} \ \cap \ J_{z+1} \ \neq \ \varnothing) \ \geq \ 1 - 2\,\varepsilon_2/3.$$

Proof A standard coupling argument implies that $\bar{\eta}_t$ has more particles if we increase the range of the interactions. So, we just need to prove the result when the offspring is sent to one of the two nearest neighbors. The idea is to prove that the rightmost particle in $\bar{\eta}_t$ reaches the right edge of B_z by time ΓL, and then stays inside $J_{z+1} \cap B_z$ until time ΓL. Let

$$r_t \ = \ \sup\,\{x \in B_z : \bar{\eta}_t(x) = 1\} \qquad \text{and} \qquad \tau \ = \ \inf\,\{t \geq 0 : r_t = \Phi(z+1)\}.$$

Then, on the set $\{\tau > t\}$, we have $\mathbb{E}\,(r_t - r_0) \geq (\beta - \lambda)\,t$. In particular, since $\beta > \lambda$, the parameters Γ and L can be chosen such that $P\,(\tau > \Gamma L) \leq \varepsilon_2/3$. This implies that the rightmost particle will reach J_{z+1} by time ΓL with high probability. To prove that the rightmost particle does not leave J_{z+1} until time ΓL, we observe that

$$r_t \ \to \ r_t - 1 \ \text{ at rate} \leq \lambda \qquad \text{and} \qquad r_t \ \to \ r_t + 1 \ \text{ at rate } \left\{ \begin{array}{ll} \geq \beta & \text{if } \ r_t < \Phi(z+1) \\ \ \ 0 & \text{if } \ r_t = \Phi(z+1). \end{array} \right.$$

Then well-known estimates about random-walks imply that

$$P\left(\exists\ t \in [\tau,\, \Gamma L] : r_t \notin J_{z+1}\right) \ \leq \ \Gamma L \left(\frac{\lambda}{\beta}\right)^{L/5} \ \leq \ \varepsilon_2/3$$

for L sufficiently large. This completes the proof. $\square$

We now fix Γ and L such that Lemma 6.4 holds. To extend the result to the process η_t, we just need to choose $\lambda > 0$ sufficiently small so that the probability a death occurs in the space-time region $B_z \times [0, \Gamma L\,]$ is smaller than $\varepsilon_2/3$. In other respects, since the result holds for the process modified so that any particle outside B_z is killed, it follows that: If $\lambda > 0$ is small then the set of occupied sites dominates the set of wet sites in an oriented percolation process in the half-space with parameter $1 - 6^{-72}$. Lemma 6.2 then implies that

$$P\,(\text{there is at least one particle in } J_z \text{ at time } 2n\,\Gamma L) \ \geq \ \varepsilon_1 \ > \ 0$$

for any integer $n \geq 0$, provided that η_0 contains infinitely many particles. Now, it is easy to see that there exists a constant $\varepsilon_3 > 0$ independent of n such that: If there is at least one particle in the interval J_z at time $2n\,\Gamma L$ then the probability that $\xi_t^-\,(r_t^2 + 1) = \eta_t(0) = 1$ for at least one unit of time between times $2n\,\Gamma L$ and $2\,(n+1)\,\Gamma L$ is greater than ε_3. This tells us that

$$\lim_{T \to \infty} \ \frac{1}{T} \int_0^T \mathbb{1}_{\{\eta_t(0) \ = \ 1\}}\, dt \ \geq \ \varepsilon_0$$

for some appropriate constant $\varepsilon_0 > 0$. Since Lemma 6.3 implies that $\{\eta_t(0) = 1\} \subset \{G_t = 0\}$, Lemma 6.1 follows from the previous inequality. A time change now allows us to set $\lambda = 1$, which then completes the proof of Theorem 3.

7. Proof of Theorem 4

This section is devoted to the proof of Theorem 4 which addresses coexistence of the symbionts in the neutral case $g = 1$. To remind the reader, we assume that the symbionts evolve as previously but the hosts perform a threshold θ voter model according to the following rate at x:

$$i \ \to \ j \quad \text{at rate} \quad \left\{ \begin{array}{ll} 1 & \text{if} \quad \text{card}\,\{z \in \mathbb{Z}^d : 0 < \|x - z\| \leq R_1 \text{ and } \hat\xi_t^{\,1}(z) = j\} \ \geq \ \theta \\ 0 & \text{otherwise.} \end{array} \right.$$

It is easy to see that the critical value for the infection rate, $\beta_{cr}^{\,Th4}(\kappa)$, is strictly bounded away from 0. Namely, if $\kappa = 1$, then the symbionts perform a basic contact process with death rate 1, provided $\theta \leq \nu_{R_1}$, and birth rate $\beta \nu_{R_2}$. Furthermore, since the contact process is monotone, $\beta_{cr}^{\,Th4}(\kappa) \geq \beta_{cr}^{\,Th4}(1)$ for $\kappa \geq 1$, from which our claim follows.

To prove Theorem 4, we will compare the particle system viewed on suitable length and time scales with a 1-dependent oriented percolation process in two dimensions. The properties of the process in the absence of symbionts was described in Durrett (1992). To apply his results, we introduce, for any $x \in \mathbb{Z}^d$, the *house*

$$H_x \ = \ [\,x_1\, L,\, (x_1 + 1)\, L) \ \times \ \cdots \ \times \ [\,x_d\, L,\, (x_d + 1)\, L),$$

where L is an integer to be fixed later and x_i denotes the i-th coordinate of the vector x. We fix $\sigma < 1/\kappa$ such that $\theta < \sigma\, \nu_{R_1}$, and say that H_x is *good* if it contains at least $\sigma\, L^d$ hosts of each type. For $x \in \mathbb{Z}^d$, we define $\|\, x\,\|_2 = (|x_1|^2 + \cdots + |x_d|^2)^{1/2}$ and set $B_2(x, r) = \{y : \|y - x\|_2 \leq r\}$. We say that $B_2(0, r)$ is *good* if for any $x \in B_2(0, r)$ the house H_x is good. For z even for even n or z odd for odd n, we will say that (z, n) is *occupied* if the following two conditions hold:

1. For any $x \in B_2(zKe_1, K)$, the house H_x is good at time $n\,\Gamma L$.
2. For any $i = 1, 2, \ldots, \kappa$, $B_2(zKLe_1, KL)$ has at least one symbiont of type i at time $n\,\Gamma L$.

Here, e_1 denotes the first unit vector, and K and Γ are large integers that will be fixed later. Note that the set $B_2(zKe_1, K)$ is defined on the rescaled lattice, whereas $B_2(zKLe_1, KL)$ is defined on the original lattice. We will prove the following result.

Proposition 7.1 *Let $\varepsilon > 0$ and $\theta < \nu_{R_1}/\kappa$. There exists $\beta_{cr}^{\,Th4} \in (0, \infty)$ such that if $\beta > \beta_{cr}^{\,Th4}$ then K, L and Γ can be chosen in such a way that the set of occupied sites dominates the set of open sites in a 1-dependent oriented percolation process with parameter $1 - \varepsilon$.*

The first step in proving Proposition 7.1 is to summarize the results of Durrett (1992), Section 2, which describe the behavior of the process in the absence of symbionts. To formulate the result we are interested in, we set $R_1 = L(M_1 + 1)$ where L and M_1 are large integers.

Lemma 7.2 (Durrett) *Let $\varepsilon > 0$ and $\theta < \nu_{R_1}/\kappa$. There exist R_0, M_0 and Γ such that the following holds: If $M_1 \geq M_0$ and $B_2(0, R_0 M_1)$ is good at time 0 then, for L large, $B_2(0, R_0 M_1)$ is good until time ΓL and $B_2(0, 2R_0 M_1)$ is good at time ΓL with probability at least $1 - \varepsilon/\,3$.*

The sets described in Lemma 7.2 provide an environment favorable to the survival of symbionts. To explain this, we introduce, for any type $i = 1, 2, \ldots, \kappa$, the processes $\eta_t^{\,i}$ defined by $\eta_t^{\,i}(x) = 1$ if $\hat\xi_t^{\,2}(x) = i$ and $\eta_t^{\,i}(x) = 0$ otherwise. Since $\alpha = 0$, it is easy to see that, for $i = 1, 2, \ldots, \kappa$, the processes $\eta_t^{\,i}$ do not interact. We fix a type $i \in \{1, 2, \ldots, \kappa\}$, and focus on the process $\eta_t^{\,i}$. The evolution of $\eta_t^{\,i}$ is as follows:

1. Each particle dies at rate at most κ and gives birth at rate $\beta \nu_{R_2}$.
2. A particle born at site x is sent to a site z chosen at random from $\mathcal{N}_x^2$.
3. If the target site z is occupied by an unassociated host of type i then the birth occurs. Otherwise, it is suppressed.

The proof of Theorem 4 relies, like Theorems 1 and 2, on a perturbation argument. More precisely, we first prove Proposition 7.1 in the extreme case $\beta = \infty$, and then extend the result to the region $\beta > 0$ large. We denote by $\bar\eta_t^i$ the process η_t^i modified so that no births are allowed outside $B_2(0, KL)$.

Lemma 7.3 *Assume that $B_2(0, K)$ is good until time ΓL and that at time 0 there exists $x \in B_2(0, KL)$ with $\eta_0^i(x) = 1$. If $R_2 \geq 4\,dL$ and $\beta = \infty$ then*

$$\{x \in \mathbb{Z}^d : \hat\xi_t^1(x) = i\} \cap B_2(0, KL) \ = \ \{x \in \mathbb{Z}^d : \bar\eta_t^i(x) = 1\} \qquad \text{for all } t \leq \Gamma L.$$

Proof This is elementary geometry. To begin with, we cover the set $B_2(0, KL)$ with a finite number of Euclidean balls B_j, $j \in I$, each of them has radius $r = \sqrt{d}L$. Then, it is easy to see that, for any $j \in I$, B_j contains at least one house. In particular, as long as $B_2(0, K)$ is good, B_j contains at least one host of type i provided that $B_j \subset B_2(0, KL)$. At any time $0 \leq t \leq \Gamma L$, let's pick one at random and denote by $X_j(t)$ its spatial location. Now, since $R_2 \geq 4\,r\sqrt{d}$, we have

$$\min \ \big\{\|X_j(t) - X_k(t)\| : k \neq j\big\} \ \leq \ R_2 \qquad \text{for all } j \in I \text{ and for all } t \leq \Gamma L.$$

This implies that, for any x, $z \in B_2(0, KL)$ occupied by a host of type i, there exists a chain of sites $x_0 = x$, $x_1, \ldots, x_n = z$ such that the following two conditions hold:

1. For $k = 1, 2, \ldots, n$, $\|\, x_{k-1} - x_k\| \leq R_2$ and
2. For $k = 0, 1, \ldots, n$, the site x_k is occupied by a host of type i.

In particular, since $\bar\eta_t^i$ starts with at least one particle in $B_2(0, KL)$ and $B_2(0, KL)$ is finite, all the hosts of type i are instantaneously invaded by a symbiont at time 0. It is easy to prove by induction that this holds until time ΓL. If a host of type i gives birth to an unassociated host which is sent to a site $x \in B_2(0, KL)$ at time t, pick $X_j(t)$ such that $\|\, x - X_j(t)\| \leq R_2$. Since $X_j(t)$ is occupied by a symbiont of type i, the host at x will be instantaneously invaded. □

To extend the result to $\beta > 0$ large, it is convenient to construct the process $\hat\xi_t$ from a graphical representation. For any type $i \in \{1, 2, \ldots, \kappa\}$ and $x \in \mathbb{Z}^d$, let $\{T_n^{i,x} : n \geq 1\}$ be independent Poisson processes with rate 1. At time $T_n^{i,x}$ the state of x flips to $(i, 0)$ if the set $\mathcal{N}_x^1$ has at least θ hosts of type i. For $x \in \mathbb{Z}^d$, let $\{U_n^x : n \geq 1\}$ be independent Poisson processes with rate β. At time U_n^x, we choose at random a site z from $\mathcal{N}_x^2$. If a host of a certain type is present at site x, and a symbiont of the same type is present at site z, then the host at site x becomes associated if it is not already. So that Lemma 7.3 holds for $\beta < \infty$ large, we now require the following two good events, denoted by G_1 and G_2, respectively: Firstly, we need a *quick invasion* of the ball $B_2(0, KL)$ by the symbionts. More precisely, G_1 will be the event that if at time 0 there exists $x \in B_2(0, KL)$ with $\eta_0^i(x) = 1$, then for all $z \in B_2(0, KL)$, with z occupied by host i, the host present at site z becomes associated before another host attempts to give birth in $B_2(0, KL)$. To estimate $P(G_1)$, we observe that, for any $z \in B_2(0, KL)$, the host at site z can be reached in at most $4KL/R_2$ steps by a symbiont, i.e., if the host at z is of type i then there is a chain of sites x_0, $x_1, \ldots, x_n = z$ with $n \leq 4KL/R_2$, satisfying the conditions 1 and 2 above and such that x_0 is occupied by a symbiont of type i at time 0. We denote by ν_{KL} the number of sites in the ball $B_2(0, KL)$. Then since the transition $i \to j$ occurs at rate at most 1 and there are κ hosts and ν_{KL} sites in $B_2(0, KL)$, new hosts are born at rate at most $\kappa\,\nu_{KL}$. We set n equal to the integer part of $4KL/R_2$. Then

$$\begin{aligned} P(G_1) \ &\geq \ 1 \ - \ \nu_{KL}\, n \, \frac{\kappa\,\nu_{KL}}{\kappa\,\nu_{KL} + \beta/n} \\ &\geq \ 1 \ - \ \nu_{KL}\, \frac{4KL}{R_2} \, \frac{\kappa\,\nu_{KL}}{\kappa\,\nu_{KL} + R_2\,\beta/4KL}. \end{aligned}$$

Now that $B_2(0, KL)$ has been invaded, we secondly require it to remain *fully occupied* until time ΓL. In other words, G_2 will be the event that given that at time 0 all hosts are associated, each time a host is born at some site $x \in B_2(0, KL)$, it becomes associated before another host is born in the ball $B_2(0, KL)$, this occurs from time 0 to time ΓL. Let N denote the number of times a host is born in $B_2(0, KL)$ from time 0 to time ΓL. Since $\mathbb{E}\, N \leq \kappa\, \nu_{KL} \Gamma L$, we find that

$$P\,(N > 2\,\kappa\,\nu_{KL}\,\Gamma L) \ \leq \ C_{11}\, \exp\,(-\gamma_{11}\,\Gamma L)$$

for appropriate $C_{11} < \infty$ and $\gamma_{11} > 0$. If only one host in $B_2(0, KL)$ is unassociated, it becomes associated at rate at least β. Births of hosts in $B_2(0, KL)$ occur at rate at most $\kappa\,\nu_{KL}$. Let X be a random variable with exponential distribution with parameter β and Y be a random variable with exponential distribution with parameter $\kappa\,\nu_{KL}$. Then

$$\begin{aligned} P\,(G_2^c) \ &\leq \ P\,(N > 2\,\kappa\,\nu_{KL}\,\Gamma L) \ + \ P\,(G_2^c\ ;\ N \leq 2\,\kappa\,\nu_{KL}\,\Gamma L) \\ &\leq \ C_{11}\, \exp\,(-\gamma_{11}\,\Gamma L) \ + \ 2\,\kappa\,\nu_{KL}\,\Gamma L\, P\,(Y \leq X) \\ &\leq \ C_{11}\, \exp\,(-\gamma_{11}\,\Gamma L) \ + \ 2\,\kappa\,\nu_{KL}\,\Gamma L\, \frac{\kappa\,\nu_{KL}}{\beta + \kappa\,\nu_{KL}}. \end{aligned}$$

The proof of Proposition 7.1 is now straightforward. Let $\varepsilon > 0$ and assume that $B_2(0, K)$ is good and that $B_2(0, KL)$ has at least one symbiont of each type at time 0. Fix $R_1 = L(M_1 + 1)$ and $K = R_0 M_1$, then apply Lemma 7.2 and choose L sufficiently large so that $B_2(0, K)$ is good from time 0 to time ΓL and $B_2(0, 2K)$ is good at time ΓL with probability at least $1 - \varepsilon/\,3$. Now, increase L and then choose β sufficiently large so that both probabilities $P\,(G_1)$ and $P\,(G_2\,|\,G_1)$ are greater than $1 - \varepsilon/\,3$. To see that this produces the desired result, we observe that if $B_2(0, K)$ is good from times 0 to ΓL then Lemma 7.3 implies that, on $G_1 \cap G_2$, the balls $B_2(-KLe_1, KL)$ and $B_2(KLe_1, KL)$ contain at least one symbiont of each type. This completes the proof.

To deduce the existence of a nontrivial stationary measure μ from Proposition 7.1, we start the process $\hat{\xi}_t$ from a product measure in which each host is associated with a symbiont and has density $1/\kappa$. Then, we take the Cesaro average of the distributions from time 0 to time T and extract a convergent subsequence. By Proposition 1.8 of Liggett (1985), the limit μ is known to be an invariant measure. To see that μ has the desired property, we observe that if L is large then the law of large numbers implies that $(z, 0)$, z even, is occupied with probability close to 1. Moreover, if $\varepsilon > 0$ is small, well-known percolation results imply that, at any level n, the density of occupied sites is positive, which implies that $\mu\,(\hat{\xi}^{\,2}(x) = i) \neq 0$ for any $i \in \{1,\, 2,\, \ldots,\, \kappa\}$. At this point, we have proved that there is a critical value $\beta_{cr}^{Th4} \in (0, \infty)$ such that if $\beta > \beta_{cr}^{Th4}$ then coexistence occurs.

To see that β_{cr}^{Th4} can be chosen so that if $\beta < \beta_{cr}^{Th4}$ then coexistence does not occur, we rely on a standard coupling argument. If we think of the process as being generated by the Poisson processes introduced above, it is easy to see that if $\beta_1 < \beta_2$, then the processes with parameters β_1 and β_2 can be defined on the same space, starting from the same initial configuration, in such a way that the process with parameter β_1 has fewer symbionts of type i for any $i \in \{1,\, 2,\, \ldots,\, \kappa\}$. This completes the proof of Theorem 4.

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Voter model and biased voter model in heterogeneous environments

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Abstract With the rapid adoption of transgenic crops, gene flow from transgenic crops to wild relatives through pollen dispersal is of significant concern and warrants both empirical and theoretical studies to assess the risk of introduction of transgenes into wild populations. We propose to use the (biased) voter model in a heterogeneous environment to investigate the effects of recurrent gene flow from transgenic crop to wild relatives. Our main findings suggest that unless transgenes confer increased fitness in wild relatives, introgression of transgenes into populations of wild plants is slow and maybe even reversible without intervention. Our study also addresses the effects of different spatial planting patterns of transgenic crops on the rate of introgression.

1. Introduction

The technology of inserting genes into the genomes of organisms for commercial applications was developed in the early 1980s. Insulin was the first commercial product that was produced by a genetically engineered organism. It was approved by the U.S. Federal Drug Administration in 1982. Shortly thereafter, in 1983, the first genetically modified tobacco plant was engineered to express an antibiotic gene from a bacterium. Since then, genetic engineering has become a standard tool of gene technology for drug and agricultural product design. Genetically modified crop plants (transgenic crops) are now being developed to resist insect herbivory, herbicides, or viral pathogens. A well-known example is *Bt* maize that is engineered to express a gene from the soil bacterium *Bacillus thuringiensis* (*Bt*). This gene codes for a toxin that is lethal to the larval stage of maize's main insect pest, the European corn borer *Ostrinia nubilalis* (Hübner) [Lepidoptera: Crambidae]. *Bt* maize has been commercially available since the mid 1990s and was planted on about 32 percent of corn acreage in the U.S. in 2004 (www.ers.usda.gov/data/biotechcrops). Other widely used genetically engineered crops include cotton and soybeans, which have both been engineered to be herbicide tolerant and were planted on 60 percent and 85 percent of U.S. acreage in 2004, respectively (www.ers.usda.gov/data/biotechcrops).

Environmental safety of transgenic crops is a major concern, particularly the potential of gene transfer from transgenic crops into natural populations of close relatives through pollination. That this concern is not just a theoretical possibility was demonstrated by Watrud et al. (2004) in a recent study of gene flow among *Agrostis stolonifera* and some of its close relatives. *A. stolonifera* is a wind-pollinated perennial grass in the genus *Agrostis* that is estimated to contain over 200 species worldwide (Soreng and Peterson 2003). *A. stolonifera* is commonly used on golf courses and as a forage crop. It has been genetically modified to express resistance to glyphosate, the active ingredient of RoundUp herbicide (Monsanto, St. Louis, MO). Watrud et al. (2004) set up a field experiment to measure the gene flow among *A. stolonifera* and some of its close relatives. This experiment showed that gene flow typically occurs within 2 km, but long distance dispersal events of up to 21 km were also observed.

Crop plants are descended from wild plants and have close relatives among them. It is therefore no surprise that gene flow between crop plants and wild plants has occurred in the past (Ellstrand et al. 1999). However, with novel genes being inserted into the genomes of crop plants to express proteins for specific biological functions, there is increased concern that these novel genes would

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escape into the wild and confer increased fitness to some species and that the wild plants could become aggressive invaders.

Few mathematical models have explored the potential of gene flow from genetically engineered crops to wild relatives (Huxel 1999, Wolf et al. 2001, Haygood et al. 2003). They found that even if selection disfavors the transgene, invasion into the wild population is possible. Their models point to the need for closer monitoring of gene flow from genetically engineered crops into wild populations.

None of the models mentioned above is spatially explicit. Since gene flow is an inherently spatial process, the lack of a spatial component is potentially a serious shortcoming. To begin to understand how the spatial configuration of permanent plots of genetically modified crop plants affect gene flow and invasion of transgenes into adjacent natural populations of close relatives, we propose a spatially explicit, stochastic model in a heterogeneous environment. The environment will reflect the permanent plots of genetically modified organisms embedded in a matrix of wild plants. Since we think of the genetically modified crop plants as planted, gene flow will only occur within the wild plants and from the genetically modified plants to the wild plants. The model is based on the simplest population genetics model, the haploid Wright-Fisher model with selection. The spatial analog of the Wright-Fisher model is the (biased) voter model. We will define the voter model in a heterogeneous environment with gene flow in such a way to address the problem of recurrent gene flow from genetically modified crop plants to their wild relatives.

The heterogeneous voter model is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \longrightarrow \{0,1\}$. We think of 1's as representing individuals carrying the transgene, and 0's as the ones carrying the wild type gene. A site $x \in \mathbb{Z}^d$ is said to be occupied by a 1 (resp. 0) if $\xi(x) = 1$ (resp. 0). We will also use the notation $x \in \xi_t$ if and only if $\xi_t(x) = 1$. The subset $\Delta \subset \mathbb{Z}^d$ represents the part of the habitat that is occupied by transgenic crop plants. We freeze the evolution on Δ by pretending that if $x \in \Delta$ then, at any time $t \geq 0$, $\xi_t(x) = 1$. The dynamics are defined as follows. Let $p\,(x,y)$ be a transition probability on $\mathbb{Z}^d$ that is translation invariant, i.e., $p\,(x,y) = q\,(y-x)$, symmetric, i.e., $q\,(z) = q\,(-z)$, and such that $q\,(z)$ has variance $\sigma^2 < \infty$. If a site $x \in \mathbb{Z}^d - \Delta$ is occupied by a 1, then, at rate 1, it picks a site $y \in \mathbb{Z}^d$ with probability $p\,(x,y)$ and changes to the state of the individual at y. If the site x is occupied by a 0, it chooses a site $y \in \mathbb{Z}^d$ at rate β according to $p\,(x,y)$ and changes to the state of the site y. Except in the homogeneous case $\Delta = \varnothing$, we will assume from now on that $\xi_0(x) = 1$ if and only if $x \in \Delta$. It follows from the dynamics that for all $t > 0$, $\xi_t(x) = 1$ for $x \in \Delta$.

We first observe that if $\Delta = \varnothing$ and $\beta = 1$ then ξ_t is the d-dimensional *voter model*. In this case, the results of Holley and Liggett (1975) reveal a sharp change in behavior depending on the spatial dimension: If $d \leq 2$ then $\xi_t \Rightarrow \alpha\,\delta_0 + (1-\alpha)\,\delta_1$, for some $\alpha \in [0,1]$, where δ_i denotes the point mass on $\xi \equiv i$. Furthermore, *clustering* occurs, that is for any x, $y \in \mathbb{Z}^d$,

$$P\,(\xi_t(x) = 0,\, \xi_t(y) = 1) \;\to\; 0 \qquad \text{as} \;\; t \to \infty.$$

If $d \geq 3$, and ξ_0^θ denotes the initial distribution in which the coordinates $\xi_0^\theta(x)$ are independent and equal to 1 with probability θ, then $\xi_t^\theta \Rightarrow \xi_\infty^\theta$, a stationary measure in which $P\,(\xi_\infty^\theta = 1) = \theta$. Choosing $\Delta \neq \varnothing$ can drastically change the limiting behavior of the process. For instance, if we set $\Delta = \{0\}$ and $d \leq 2$, then the particle located at the origin will produce a cluster that will invade the whole space.

If $\Delta = \varnothing$ and $\beta > 1$, then ξ_t is the *biased voter model*. The results of Bramson and Griffeath (1980 and 1981) show that, in any dimension, if we let $\Omega_\infty = \{\xi_t \neq \varnothing \text{ for all } t \geq 0\}$, then starting from a configuration where there is a single 1 at the origin, there is a convex set A so that on the event Ω_∞ we have for any $\varepsilon > 0$

$$(1-\varepsilon)\,t\,A \;\cap\; \mathbb{Z}^d \;\subset\; \xi_t \;\subset\; (1+\varepsilon)\,t\,A \;\cap\; \mathbb{Z}^d \quad \text{for all } t \text{ sufficiently large.}$$

Moreover, on Ω_∞^c, the process converges to the "all 0" configuration exponentially fast.

We now fix $d = 1$. The discussion above implies that if $\Delta \neq \varnothing$ then $\xi_t \Rightarrow \delta_1$. The first question we would like to answer is: What is the effect of the geometry of Δ on the speed of convergence to the all occupied configuration? The first step to answer this question is to investigate the process starting from $\xi_0(x) = \mathbb{1}_{\{x \leq 0\}}$. In the special case $\Delta = \varnothing$, the behavior of the process at the interface has been studied by Cox and Durrett (1995). To introduce their results, let's consider the *leftmost-zero* and the *rightmost-one* processes

$$\ell_t \;=\; \inf\{x \in \mathbb{Z} : \xi_t(x) = 0\} \qquad \text{and} \qquad r_t \;=\; \sup\{x \in \mathbb{Z} : \xi_t(x) = 1\}.$$

Moreover, we assume, for technical reasons, that

(i) p is irreducible, i.e., it is possible to get from 0 to any $x \in \mathbb{Z}$ in a finite number of steps.

(ii) q has finite third moments, i.e., $\sum_z |z|^3\, q\,(z) < \infty$.

Then $\{r_t - \ell_t : t \geq 0\}$ is stochastically compact, that is for any $\varepsilon_0 > 0$ there exists a constant $M < \infty$ such that $P\,(r_t - \ell_t \geq M) \leq \varepsilon_0$. See Cox and Durrett (1995), Section 4. Moreover, if we denote by $\Phi(x)$ the standard normal distribution function, then

$$\lim_{t\to\infty} P\,(r_t/\sigma\sqrt{t} \leq x) \;=\; \lim_{t\to\infty} P\,(\ell_t/\sigma\sqrt{t} \leq x) \;=\; \Phi(x).$$

In the case $\Delta = \mathbb{Z}^-$, the set of nonpositive integers, we will prove that the family $\{r_t - \ell_t : t \geq 0\}$ is still stochastically compact. The previous equation however becomes false since r_t and ℓ_t are now forced to live on the right side of zero. More precisely, we will prove that

Theorem 1 *Let $\beta = 1$. If $x \geq 0$ and $x_t = x\,\sigma\sqrt{t}$ then*

$$\lim_{t\to\infty} P\,(r_t \geq x_t) \;=\; \lim_{t\to\infty} P\,(\ell_t \geq x_t) \;=\; \sqrt{\frac{2}{\pi}} \int_x^\infty \exp\left[-\frac{y^2}{2}\right] dy.$$

In words, the particles diffuse to the right, which reveals a very slow invasion. In view of Theorem 1, our guess is that the more Δ is *scattered*, the faster the transgene will invade the wild population.

To make this argument precise, we consider the process on the torus $\varepsilon\mathbb{Z}/N\mathbb{Z}$ where N is a positive integer and ε is close to 0. Moreover, we speed up time by ε^{-2} and introduce the transition probability $p_\varepsilon\,(x,y) = q\,(\varepsilon^{-1}(y-x) \mod N\varepsilon^{-1})$ to formulate the dynamics. We let $L > 0$ and $K = N/L$ such that both K and $L\,\varepsilon^{-1}$ are integers, with K even. For any $z \in \{0, 1, \ldots, K-1\}$, we define the subsets

$$B_z \;=\; [\,z\,L,\,(z+1)\,L)\;\cap\;\varepsilon\,\mathbb{Z} \qquad \text{and} \qquad \Delta \;=\; \bigcup_{z \text{ even}} B_z.$$

The reader will note that since we start the process with $\xi_0(x) = 1$ if and only if $x \in \Delta$, for any choice of L, half of the sites are initially occupied, and half of the sites are initially vacant. Finally, we denote by $T_{\rm inv}$ the stopping time $T_{\rm inv} = \inf\{t \geq 0 : \xi_t \equiv 1\}$. We refer to $T_{\rm inv}$ as the *time to complete invasion.*

Theorem 2 *Let $\beta = 1$. If $p\,(x,y)$ has compact support then*

$$\lim_{\varepsilon\to 0} P\,(T_{\rm inv} \leq t) \;=\; \left\{1 - \int_0^L (\Psi_t * \Psi_t)(x)\,dx\right\}^{N/2L}$$

where

$$\Psi_t(x) \;=\; -\frac{\partial}{\partial x}\sqrt{\frac{2}{\pi}} \int_{x/\sigma\sqrt{t}}^\infty \exp\left[-\frac{y^2}{2}\right] dy.$$

Theorem 2 gives us insights into the effects of the spatial configuration of Δ on the time to complete invasion. Numerical investigations suggest that, for fixed $t \geq 0$, the function $L \mapsto \lim_{\varepsilon\to 0} P\,(T_{\rm inv} \geq t)$ is increasing. This implies that for N fixed, the time to complete invasion will be slowest in the extreme case $L = N/2$, and faster as the number of components of Δ increases.

Unfortunately, we do not know how to extend Theorem 2 to higher dimensions. To be convinced however that the analogous result holds in $d = 2$, we simulated the process on a 200×200 lattice with periodic boundary conditions and nearest neighbor interactions, tiled into 20×20 squares $B_{i,j}$, $i, j = 1, 2, \ldots, 10$. Figure 1 shows the time to complete invasion in three different environments. In the *chess-board* model, Δ is the set of $B_{i,j}$ with $i+j$ even. In the *striped* model, Δ is the set of $B_{i,j}$ with i even. Finally, in the *four-patch* model, Δ is the set of $B_{i,j}$ with $i, j = 1, 2, \ldots, 5$ or $i, j = 6, 7, \ldots, 10$. As suggested by Figure 1, the simulations reveal that T_{inv} is a decreasing function of the number of components of Δ.

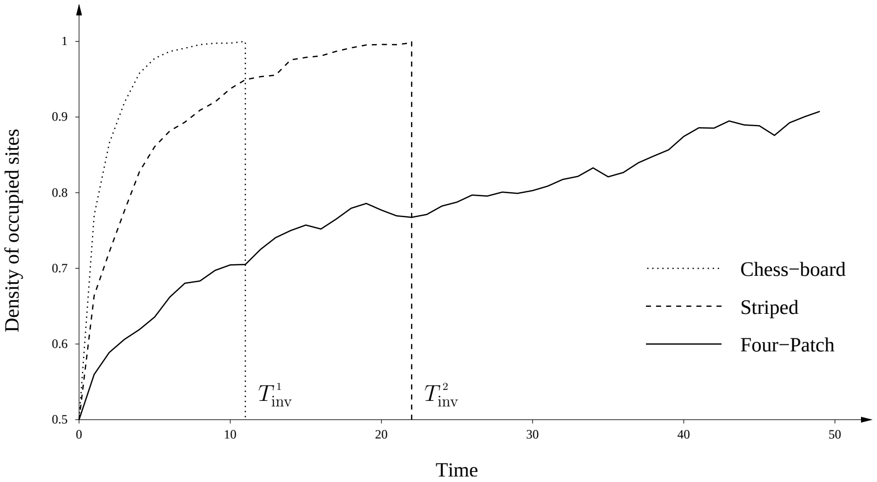

Figure 1. *Evolution to resistance.*

We now consider the process on $\mathbb{Z}$ where the particles give birth according to the transition probability $p(x, y)$. Let $\Delta = \mathbb{Z}^-$ and start the process with $\xi_0(x) = 1$ if and only if $x \in \Delta$. Moreover, we assume that $p(x, y)$ has compact support, i.e., there exists $R > 0$ such that if $|x - y| > R$ then $p(x, y) = 0$. The next step is to investigate the properties of the rightmost-one process r_t with birth rate β. The limiting behavior of r_t as $t \to \infty$ follows from Theorem 2. By working a little bit more, we can prove that the process comes back to its initial configuration infinitely often. See Figure 4 for a picture.

Theorem 3 *Assume that $p(x, y)$ has compact support. If $\beta = 1$, then $P(r_t = 0 \text{ i.o.}) = 1$.*

To investigate the case $\beta \neq 1$, we consider the process with nearest neighbor interactions, i.e., we let $p(x, y) = 1/2$ if $|x - y| = 1$. If $\beta < 1$, Theorem 3 accompanied by a coupling argument implies that $r_t = 0$ i.o.; the process however does not exhibit the behavior described in Theorem 1.

Theorem 4 *Assume nearest neighbor dispersal. If $\beta < 1$ then $P(r_t \geq x) \leq \beta^x$ at any time $t \geq 0$.*

Finally, if $\beta > 1$, the process behaves like the corresponding biased voter model in homogeneous environment, namely the process grows linearly in time with a wave speed of order $\beta - 1$. See Figure 5 for a picture.

Theorem 5 *Assume nearest neighbor dispersal. If $\beta > 1$, there exist $C < \infty$ and $\gamma > 0$ such that for any $t > 0$ and any $\varepsilon > 0$*

$$P(|\, r_t - (\beta - 1)\, t\, | \;\geq\; \varepsilon\, t) \;\leq\; C\, e^{-\gamma \varepsilon^2 t}.$$

In conclusion, Theorems 1 and 4-5 imply that if $\beta \geq 1$ then, for any integer $x \geq 0$ and $\varepsilon > 0$, there is $t_0 \geq 0$ such that $P(r_t \geq x) \geq 1 - \varepsilon$ for any $t \geq t_0$, while if $\beta < 1$ then $P(r_t \geq x) \leq \beta^x$ for any $t \geq 0$. Moreover, Theorems 3-5 exhibit the 0-1 law:

$$P(r_t = 0 \ \text{ i.o.}) \ = \ \begin{cases} 0 & \text{if} \quad \beta > 1 \\ 1 & \text{if} \quad \beta \leq 1. \end{cases}$$

In particular, if $\beta = 1$ and $p(x, y)$ has compact support, the rightmost-one process r_t converges in probability to infinity as $t \to \infty$ but not almost surely.

Our results confirm the need to monitor recurrent gene flow from genetically modified crops to their wild relatives but our results differ in an important way from the results of non-spatial models analyzed by other researchers (Huxel 1999, Wolf et al. 2001, Haygood et al. 2003) who consistently found that the transgene would eventually go to fixation over a wide range of parameters even if not favored by selection. Both selection and the spatially explicit structure of our model greatly affect the rate and extent of invasion of transgenes into wild populations. If the transgene is favored by selection, it will completely take over a wild population. In the neutral case, the transgene can penetrate a wild population arbitrarily far, which can result in fixation when the wild population is only of limited spatial extent, but the invasion is slow. If the transgene is not favored, it will continue to be present in the wild population and the extent to which it can penetrate into the the wild population depends strongly on the strength of selection. Simulations strongly indicate that the dispersal distance is a key factor in how quickly and to what extent transgenes can invade natural populations. Considering that pollen can disperse over very large spatial distances, genetic pollution of wild population remains a serious concern, even if the transgene is not favored by selection.

Modeling at this conceptual level can only point out the various factors that affect transgene introgression. It does by no means replace necessary field work and system specific simulation models that are parametrized by field data. However, our work clearly shows that spatially explicit and stochastic models can contribute to a deeper understanding of the problem of recurrent gene flow by identifying key parameters that need to be measured to assess risk.

The paper is organized as follows. Section 2 provides some preliminary results. Section 3 is devoted to the proofs of Theorems 1 and 2, Section 4 to the proof of Theorem 3, and Section 5 to the proofs of Theorems 4 and 5. Figures 4 and 5 provide some visualizations of the effects of dispersal range and selection strength.

2. Duality. Preliminary results

We start by constructing the process from a collection of Poisson processes in the case $\beta \geq 1$. For each $x \in \mathbb{Z}^d - \Delta$, we let $\{S_n^x : n \geq 1\}$ and $\{T_n^x : n \geq 1\}$ be the arrival times of independent Poisson processes with rate 1 and $\beta - 1$ respectively, and we let $\{U_n^x : n \geq 1\}$ and $\{V_n^x : n \geq 1\}$ be i.i.d. sequences with $P(U_n^x = y) = P(V_n^x = y) = p(x, y)$. At times S_n^x we draw an arrow from U_n^x to x and put a δ at x while at times T_n^x we just draw an arrow from V_n^x to x. Then the process is obtained from the graphical representation as follows: At time S_n^x, the site x imitates the site U_n^x, i.e., becomes occupied by a 1 (resp. 0) if U_n^x is occupied by a 1 (resp. 0). At time T_n^x, the site x becomes occupied by a 1 if V_n^x is occupied by a 1. An idea of Harris (1972) implies that such a graphical representation can be used to construct the process starting from any initial configuration $\xi_0 : \mathbb{Z}^d \longrightarrow \{0, 1\}$. After constructing the graphical representation, we can now define the dual process. We say that there is a *path* from $(z, 0)$ to (x, t) if there is a sequence of times $s_0 = 0 < s_1 < \cdots < s_{n+1} = t$ and spatial locations $x_0 = z, x_1, \ldots, x_n = x$ such that the following two conditions hold:

1. For $i = 1, 2, \ldots, n$, there is an arrow from x_{i-1} to x_i at time s_i and
2. For $i = 0, 1, \ldots, n$, the vertical segment $\{x_i\} \times (s_i, s_{i+1})$ do not contain any δ's.

We say that there exists a *dual path* from (x, t) to $(z, t - s)$, $0 \leq s \leq t$, if there is a path from $(z, t - s)$ to (x, t), and we define the *dual process starting at* (x, t) by setting

$$\hat{\xi}_s^{(x,t)} \ = \ \{z \in \mathbb{Z}^d : \text{there is a dual path from } (x, t) \text{ to } (z, t - s)\}$$

for any $0 \le s \le t$. The dual process allows us to deduce the state of site x at time t from the configuration at earlier times. Recall that $x \in \xi_t$ if and only if $\xi_t(x) = 1$. Then,

$$x \ \in \ \xi_t \qquad \Longleftrightarrow \qquad \hat{\xi}_s^{(x,t)} \ \cap \ \xi_{t-s} \ \neq \ \varnothing.$$

See, e.g., Durrett (1988), Chapter 3. In the homogeneous case $\Delta = \varnothing$, the dual process is a well-known object. If $\beta = 1$, the process $\hat{\xi}_s^{(x,t)}$ performs a continuous-time random walk on $\mathbb{Z}^d$ run at rate 1. Starting the dual process from two sites x and y amounts to running two individual dual processes $\hat{\xi}_s^{(x,t)}$ and $\hat{\xi}_s^{(y,t)}$ independently until their $\mathbb{Z}^d$ coordinates are identical, at which point they coalesce, i.e., $\hat{\xi}_s^{(x,t)} - \hat{\xi}_s^{(y,t)}$ has 0 as an absorbing state. After coalescing, the two dual processes move together according to the rules of the dual process of a single random walk. This implies a duality between the voter model and *coalescing random walks* when $\beta = 1$. If $\beta > 1$, the dual process branches when it encounters the tip of a single arrow. More precisely, we start with a single particle at x at time $s = 0$. If a particle in $\hat{\xi}_s^{(x,t)}$ meets a δ at some time $s = t - S_n^z$ then this particle moves to U_n^z. If a particle in $\hat{\xi}_s^{(x,t)}$ meets the tip of a single arrow at time $s = t - T_n^z$ then this particle gives birth to a new particle which is then sent to V_n^z. In conclusion, for $\beta > 1$, there is a duality between the biased voter model and *coalescing branching random walks.*

In the heterogeneous case $\Delta \neq \varnothing$, we observe that if $z \in \Delta$ then no arrows point at z. If $\beta = 1$, the process $\hat{\xi}_s^{(x,t)}$ then evolves like a continuous-time random walk until the first time it visits Δ where it becomes frozen. It is convenient to construct the dual process in the heterogeneous environment from that in a homogeneous environment: If we denote by ζ_t the process constructed from the graphical representation in homogeneous environment, then ξ_t can be constructed from the graphical representation obtained by removing all the arrows that point at a site $z \in \Delta$. Since both processes have the same graphical representation on $\mathbb{Z}^d - \Delta$, $\hat{\xi}_s^{(x,t)}$ and $\hat{\zeta}_s^{(x,t)}$ move together until they hit Δ when $\hat{\xi}_s^{(x,t)}$ is frozen. In the same way, if $\beta > 1$, the particles in each dual move and give birth together as long as they belong to $\mathbb{Z}^d - \Delta$. When a particle in $\hat{\xi}_s^{(x,t)}$ hits the set Δ it is frozen, whereas the corresponding particle in $\hat{\zeta}_s^{(x,t)}$ continues to move and give birth.

The rest of this section is devoted to the proof of a preliminary result, which is the main ingredient to establish Theorems 1-3. From now on, we set $d = 1$, let $\Delta = \mathbb{Z}^-$ be the set of nonpositive integers and assume that $\xi_0(x) = 1$ if and only if $x \in \Delta$. We define the *leftmost-zero* and the *rightmost-one* processes by

$$\ell_t \ = \ \inf \{x \in \mathbb{Z} : \xi_t(x) = 0\} \qquad \text{and} \qquad r_t \ = \ \sup \{x \in \mathbb{Z} : \xi_t(x) = 1\}.$$

Proposition 2.1 *Let $\Delta = \mathbb{Z}^-$ and $\beta = 1$. Then the family $\{r_t - \ell_t : t \ge 0\}$ is stochastically compact. That is, for any $\varepsilon_0 > 0$ there is $M > 0$ so that $P(r_t - \ell_t \ge M) \le \varepsilon_0$ at any time $t \ge 0$.*

In the case $\Delta = \varnothing$, Proposition 2.1 was proved by Cox and Durrett (1995), Section 4. Their proof relies on a strong analysis of the dual process and random walks estimates. They introduced the following concept, which is key to proving Proposition 2.1: The event that there are x, $y \in \mathbb{Z}$ such that $x < y$ and $\hat{\xi}_t^{(y,t)} \le 0 < \hat{\xi}_t^{(x,t)}$ will be called an *inversion.* If we let B_t be the number of inversions that occur by time t, then Cox and Durrett (1995) showed that

$$-1 \ \le \ r_t \ - \ \ell_t \ \le \ B_t.$$

The proof of Proposition 2.1 can then be reduced to an analysis of the dual process. See Cox and Durrett (1995), Section 4. To deduce Proposition 2.1 from the result in the homogeneous case, we will prove that there exists a constant $C < \infty$ such that $\mathbb{E}(B_t) \le C \times \mathbb{E}(B_t')$, where B_t' denotes the number of inversions by time t for the process ζ_t defined above. To compare the dual processes, we introduce the following notation: For any x, $y \in \mathbb{Z}$, we consider the stopping times

$$\begin{aligned} \sigma_{x,y} \ &= \ \inf \big\{ s \ge 0 : \hat{\xi}_s^{(x,t)} = \hat{\xi}_s^{(y,t)} \text{ or } \hat{\xi}_s^{(x,t)}, \hat{\xi}_s^{(y,t)} \le 0 \big\}, \\ \tau_{x,y} \ &= \ \inf \big\{ s \ge 0 : \hat{\zeta}_s^{(x,t)} = \hat{\zeta}_s^{(y,t)} \big\} \qquad \text{and} \qquad s_x \ = \ \inf \big\{ s \ge 0 : \hat{\xi}_s^{(x,t)} \le 0 \big\}. \end{aligned}$$

Moreover, we let $\varrho_{x,y} = \min(\sigma_{x,y}, \tau_{x,y})$ and $s_{x,y} = \min(s_x, s_y)$. We are now ready to prove Proposition 2.1. For a better understanding of Lemmas 2.2-2.5, we refer the reader to Figure 2, which gives a picture of the stopping times we have just introduced. Finally, we would like to remind the reader that our construction implies that $\hat{\xi}_s^{(x,t)}$ and $\hat{\zeta}_s^{(x,t)}$ move together as long as their position is to the right of zero.

Lemma 2.2 *For any $z \in \mathbb{Z}^-$ and $w \in \mathbb{Z}$, $P(\sigma_{z,w} > 2t) \le P(\tau_{z,w} > t)$.*

Proof If $w \le 0$, the inequality is trivial since in this case $\sigma_{z,w} = 0$. To deal with the case $w > 0$, we observe that $\sigma_{z,w}$ is the first time a rate-one random walk starting at w hits $\mathbb{Z}^-$ while $\tau_{z,w}$ is the first time a rate-two random walk starting at $w - z \ge w$ hits 0. □

Lemma 2.3 *For any $x, y \in \mathbb{Z}$, $P(\sigma_{x,y} > 2t \; ; \; s_{x,y} < \varrho_{x,y}) \le P(\tau_{x,y} > t \; ; \; s_{x,y} < \varrho_{x,y})$.*

Proof We first observe that

$$\begin{aligned} P(\sigma_{x,y} > t \; ; \; s_{x,y} < \varrho_{x,y}) &= P(\sigma_{x,y} > t \; ; \; s_{x,y} < \varrho_{x,y} \; ; \; s_x < s_y) \\ &\quad + P(\sigma_{x,y} > t \; ; \; s_{x,y} < \varrho_{x,y} \; ; \; s_x > s_y). \end{aligned}$$

We will prove the assertion for $s_{x,y} = s_x$. By the Markov property

$$\begin{aligned} P(\sigma_{x,y} > 2t \; ; \; s_x < \varrho_{x,y} \; ; \; s_x < s_y) &= \int_0^\infty P(\sigma_{x,y} > 2t \; ; \; \varrho_{x,y} > s \; ; \; s_y > s) \, P^x(s_x \in ds) \\ &= \sum_{z \le 0} \sum_{w > 0} \int_0^\infty P(\sigma_{x,y} > 2t \; ; \; \varrho_{x,y} > s \; ; \; s_y > s \; ; \; \hat{\xi}_s^{(x,t)} = z \; ; \; \hat{\xi}_s^{(y,t)} = w) \, P^x(s_x \in ds) \\ &= \sum_{z \le 0} \sum_{w > 0} \int_0^\infty P(\sigma_{z,w} > 2t - s) \\ &\quad \times P(\varrho_{x,y} > s \; ; \; s_y > s \; ; \; \hat{\xi}_s^{(x,t)} = z \; ; \; \hat{\xi}_s^{(y,t)} = w) \, P^x(s_x \in ds). \end{aligned}$$

Similarly,

$$\begin{aligned} P(\tau_{x,y} > t \; ; \; s_x < \varrho_{x,y} \; ; \; s_x < s_y) &= \sum_{z \le 0} \sum_{w > 0} \int_0^\infty P(\tau_{z,w} > t - s) \\ &\quad \times P(\varrho_{x,y} > s \; ; \; s_y > s \; ; \; \hat{\zeta}_s^{(x,t)} = z \; ; \; \hat{\zeta}_s^{(y,t)} = w) \, P^x(s_x \in ds). \end{aligned}$$

Now, since $\hat{\xi}_s^{(x,t)} = \hat{\zeta}_s^{(x,t)}$ and $\hat{\xi}_s^{(y,t)} = \hat{\zeta}_s^{(y,t)}$ until time $s = s_{x,y}$ it follows that

$$\begin{aligned} & P(\varrho_{x,y} > s \; ; \; s_y > s \; ; \; \hat{\xi}_s^{(x,t)} = z \; ; \; \hat{\xi}_s^{(y,t)} = w \; ; \; s_x = s) \\ &\qquad = P(\varrho_{x,y} > s \; ; \; s_y > s \; ; \; \hat{\zeta}_s^{(x,t)} = z \; ; \; \hat{\zeta}_s^{(y,t)} = w \; ; \; s_x = s). \end{aligned}$$

Furthermore, Lemma 2.2 implies that for $z \le 0$

$$P(\sigma_{z,w} > 2t - s) \le P(\sigma_{z,w} > 2(t - s)) \le P(\tau_{z,w} > t - s).$$

This completes the proof. □

Lemma 2.4 *For any $x, y \in \mathbb{Z}$, $P(\sigma_{x,y} > 2t) \le P(\tau_{x,y} > t)$.*

Proof We observe that if $s_{x,y} \ge \varrho_{x,y}$ then $\sigma_{x,y} = \tau_{x,y}$ since $\hat{\xi}_s^{(x,t)} = \hat{\zeta}_s^{(x,t)}$ and $\hat{\xi}_s^{(y,t)} = \hat{\zeta}_s^{(y,t)}$ until time $s_{x,y}$. In particular,

$$P(\sigma_{x,y} > 2t \; ; \; s_{x,y} \ge \varrho_{x,y}) \le P(\sigma_{x,y} > t \; ; \; s_{x,y} \ge \varrho_{x,y}) = P(\tau_{x,y} > t \; ; \; s_{x,y} \ge \varrho_{x,y}).$$

The conclusion then follows from Lemma 2.3. □

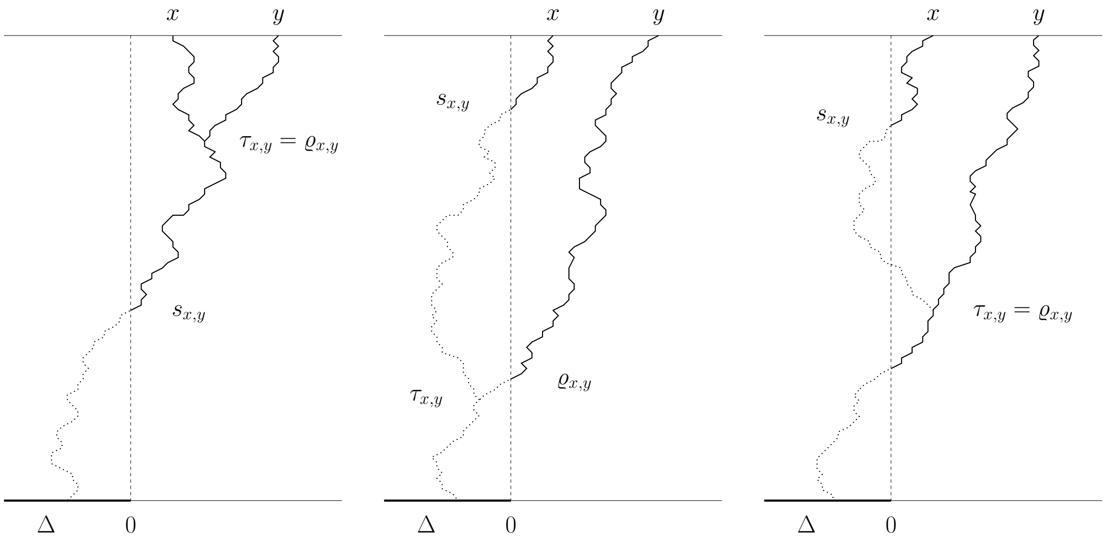

Figure 2. *Pictures of the dual processes. The solid lines refer to the path of the dual processes moving together. The dotted lines refer to the path of* $\hat\zeta_s^{(\cdot,t)}$ *after that* $\hat\xi_s^{(\cdot,t)}$ *has been frozen.*

Lemma 2.5 *For any integer* $y > 0$,

$$\sum_x P\,(\hat\xi_t^{(x+y,t)} \le 0 < \hat\xi_t^{(x,t)} \mid \sigma_{x,x+y} > t) \;\le\; \sum_x P\,(\hat\zeta_t^{(x+y,t)} \le 0 < \hat\zeta_t^{(x,t)} \mid \tau_{x,x+y} > t).$$

Proof We first observe that $P\,(\hat\zeta_t^{(x+y,t)} \le 0 < \hat\zeta_t^{(x,t)}) = P\,(\hat\zeta_t^{(-x,t)} \le 0 < \hat\zeta_t^{(-x-y,t)})$ so that

$$\sum_{x>0} P\,(\hat\zeta_t^{(x+y,t)} \le 0 < \hat\zeta_t^{(x,t)}) \;\le\; \frac{1}{2} \sum_x P\,(\hat\zeta_t^{(x+y,t)} \le 0 < \hat\zeta_t^{(x,t)}).$$

Since $P\,(\hat\xi_t^{(x,t)} > 0) = 0$ for any $x \le 0$, it suffices to prove that

$$P\,(\hat\xi_t^{(x+y,t)} \le 0 < \hat\xi_t^{(x,t)} \mid \sigma_{x,x+y} > t) \;\le\; 2\,P\,(\hat\zeta_t^{(x+y,t)} \le 0 < \hat\zeta_t^{(x,t)} \mid \tau_{x,x+y} > t)$$

for any $x > 0$ and $y > 0$. Now, if an inversion occurs before time t then $s_{x+y} \le t$, and hence

$$\begin{aligned}
P\,(\hat\xi_t^{(x+y,t)} \le 0 < \hat\xi_t^{(x,t)} \mid \sigma_{x,x+y} > t) \;&\le\; P\,(\hat\xi_t^{(x,t)} > 0 \;;\; s_{x+y} \le t \mid \sigma_{x,x+y} > t) \\
&= \int_0^t P\,(\hat\xi_t^{(x,t)} > 0 \;;\; s_{x+y} = s \mid \sigma_{x,x+y} > t)\; P^{x+y}(s_{x+y} \in ds) \\
&\le \int_0^t P\,(\hat\zeta_t^{(x,t)} > 0 \;;\; s_{x+y} = s \;;\; \hat\zeta_t^{(x+y,t)} \ge 0 \mid \tau_{x,x+y} > t)\; P^{x+y}(s_{x+y} \in ds) \\
&\quad + \int_0^t P\,(\hat\zeta_t^{(x,t)} > 0 \;;\; s_{x+y} = s \;;\; \hat\zeta_t^{(x+y,t)} \le 0 \mid \tau_{x,x+y} > t)\; P^{x+y}(s_{x+y} \in ds) \\
&= 2\;P\,(\hat\zeta_t^{(x+y,t)} \le 0 < \hat\zeta_t^{(x,t)} \mid \tau_{x,x+y} > t).
\end{aligned}$$

The last step follows from the fact that if $\hat\xi_s^{(x+y,t)}$ becomes frozen, $\hat\zeta_s^{(x+y,t)}$ can be thought of as starting from a site to the left of 0 and running until time t. This together with the fact that the dispersal probability kernel is symmetric then implies the result. This completes the proof. □

By Cox and Durrett (1995), Section 3, there exists a constant α that depends only on $y - x$ such that $P\,(\tau_{x,y} > t) \sim \alpha/\sqrt{t}$ as $t \to \infty$. Lemma 2.4 then implies that there is $C < \infty$ such that, for t sufficiently large, $P(\sigma_{x,y} > t) \le C \times P(\tau_{x,y} > t)$. This together with Lemma 2.5 leads to

$$\begin{aligned}\mathbb{E}(B_t) &= \sum_{y>0}\sum_{x} P(\hat{\xi}_t^{(x+y,t)} \le 0 < \hat{\xi}_t^{(x,t)}) \\ &= \sum_{y>0}\sum_{x} P(\hat{\xi}_t^{(x+y,t)} \le 0 < \hat{\xi}_t^{(x,t)} \mid \sigma_{x,x+y} > t)\, P(\sigma_{x,x+y} > t) \\ &\le C \sum_{y>0}\sum_{x} P(\hat{\zeta}_t^{(x+y,t)} \le 0 < \hat{\zeta}_t^{(x,t)} \mid \tau_{x,x+y} > t)\, P(\tau_{x,x+y} > t) \;=\; C \times \mathbb{E}(B_t').\end{aligned}$$

Proposition 2.1 then follows from Proposition 1 in Cox and Durrett (1995).

3. Proofs of Theorems 1 and 2

Proof of Theorem 1

By symmetry, we only need to prove the result for r_t. Since $\xi_t(x_t) = 1$ implies that $r_t \ge x_t$, it follows from the duality properties introduced in Section 2 that

$$P(r_t \ge x_t) \;\ge\; P(\xi_t(x_t) = 1) \;=\; P(s_{x_t} \le t).$$

Donsker's Theorem implies that $\zeta_s^{(x_t,t)}/\sigma\sqrt{t}$ converges in distribution to W_s^x, a standard Brownian motion starting at x. We denote by τ_0 the first time W_s^x hits 0. Then

$$\lim_{t\to\infty} P(r_t \ge x_t) \;\ge\; \lim_{t\to\infty} P(\sigma_{x_t} \le t) \;=\; P(\tau_0 \le 1) \;=\; \sqrt{\frac{2}{\pi}} \int_x^\infty \exp\left[-\frac{y^2}{2}\right] dy.$$

For a proof of the last step, see Karlin and Taylor (1975), Section 7.3. To get the opposite inequality, let $\varepsilon_0 > 0$ and refer to Proposition 2.1 to choose $M > 0$ such that $P(r_t - \ell_t \ge M) \le \varepsilon_0$. Since $\xi_t(x_t - M) = 0$ and $r_t \ge x_t$ imply $r_t - \ell_t \ge M$ we obtain

$$P(r_t \ge x_t) \;\le\; P(\xi_t(x_t - M) = 1) \;+\; \varepsilon_0.$$

Finally, since $\lim_{t\to\infty} P(\xi_t(x_t) = 1) = \lim_{t\to\infty} P(\xi_t(x_t - M) = 1)$, the result follows. □

Proof of Theorem 2

For any $z \in \{0, 1, \dots, K-1\}$, let T_z define the stopping time

$$T_z \;=\; \inf\{t \ge 0 : \xi_t \equiv 1 \text{ on } B_z\}.$$

We observe that if z is even then $B_z \subset \Delta$, which implies that $T_z = 0$. Furthermore, if $x \in B_z$ and $y \in B_{z+2}$ then $p_\varepsilon(x,y) = 0$ for $\varepsilon > 0$ sufficiently small so that $T_{\text{inv}} = \max_z T_z$. It follows that for ε sufficiently small, the stopping times T_z are independent. We assume for the rest of the proof that ε is sufficiently small so that both assertions above hold. To estimate T_z for odd z, we now consider the edge processes

$$\begin{aligned}\ell_t(z) &= \inf\{x \in B_z : \hat{\xi}_t^{(x,t)} \in B_z\} \\ r_t(z+1) &= \sup\{x \in B_z : \hat{\xi}_t^{(x,t)} \in B_z\}.\end{aligned}$$

Since $\xi_t \not\equiv 1$ on B_z if and only if $\ell_t(z) < r_t(z+1) + \varepsilon$, we obtain

$$P(T_z > t) \;=\; P(\ell_t(z) - r_t(z+1) < \varepsilon).$$

Unfortunately, in view of the properties of the dual process, $\ell_t(z)$ and $r_t(z+1)$ are not a priori independent. To deal with this problem, we set

$$\begin{aligned}r_t(z) &= \sup\{x \in B_z : \hat{\xi}_t^{(x,t)} \in B_{z-1}\} \\ \ell_t(z+1) &= \inf\{x \in B_z : \hat{\xi}_t^{(x,t)} \in B_{z+1}\}\end{aligned}$$

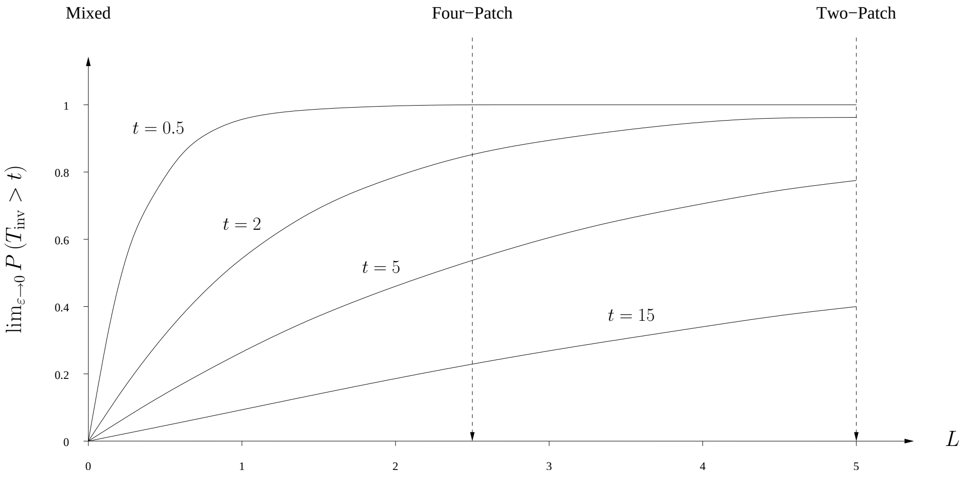

FIGURE 3. $\lim_{\varepsilon\to 0} P\,(T_{\rm inv} > t)$ *with respect to* L *with* $\sigma^2 = 1$ *and* $N = 10$.

and observe that $r_t(z)$ and $\ell_t(z+1)$ move independently as long as $r_t(z) < \ell_t(z+1)$. Moreover, Proposition 2.1 implies that for any $z \in \{0,\, 1,\, \dots,\, K-1\}$,

$$\lim_{\varepsilon\to\, 0} P\,(\ell_t(z) \geq x) \ = \ \lim_{\varepsilon\to\, 0} P\,(r_t(z) \geq x).$$

Hence, for odd z,

$$\lim_{\varepsilon\to\, 0} P\,(T_z > t) \ = \ \lim_{\varepsilon\to\, 0} P\,(r_t(z) - \ell_t(z+1) \leq 0).$$

To investigate the process $r_t(z) - \ell_t(z+1)$, we first observe that $r_t(z) - z\,L$ and $(z+1)\,L - \ell_t(z+1)$ are identically distributed. Furthermore, by Theorem 1,

$$\lim_{\varepsilon\to\, 0} P\,(r_t(z) - z\,L \geq x) \ = \ \sqrt{\frac{2}{\pi}} \int_{x/\sigma\sqrt{t}}^{\infty} \exp\left[-\,\frac{y^2}{2}\right] dy.$$

In particular, $r_t(z) - z\,L$ and $(z+1)\,L - \ell_t(z+1)$ have density

$$\Psi_t(x) \ = \ -\frac{\partial}{\partial x}\sqrt{\frac{2}{\pi}} \int_{x/\sigma\sqrt{t}}^{\infty} \exp\left[-\,\frac{y^2}{2}\right] dy \ = \ \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma\sqrt{t}} \exp\left[-\,\frac{1}{2t}\,\frac{x^2}{\sigma^2}\right].$$

It follows that for odd z,

$$\begin{aligned}\lim_{\varepsilon\to\, 0} P\,(T_z > t) \ &= \ \lim_{\varepsilon\to\, 0} P\,(r_t(z) - \ell_t(z+1) \leq 0) \\ &= \ \lim_{\varepsilon\to\, 0} P\,(r_t(z) - z\,L + (z+1)\,L - \ell_t(z+1) \leq L) \ = \ \int_0^L (\Psi_t * \Psi_t)(x)\,dx.\end{aligned}$$

Since the stopping times T_z are independent, $T_{\rm inv}$ can be estimated

$$\begin{aligned}\lim_{\varepsilon\to\, 0} P\,(T_{\rm inv} \leq t) \ &= \ \lim_{\varepsilon\to\, 0} P\,(\max_z\, T_z \leq t) \\ &= \ \lim_{\varepsilon\to\, 0} \prod_{z \text{ odd}} P\,(T_z \leq t) \ = \ \left\{1 - \int_0^L (\Psi_t * \Psi_t)(x)\,dx\right\}^{N/2L}\end{aligned}$$

which completes the proof of Theorem 2. $\square$

By working a little bit more, one can prove that

$$\begin{aligned}(\Psi_t * \Psi_t)(x) &= \int_{-\frac{x}{2}}^{+\frac{x}{2}} \Psi_t\left(\frac{x}{2}+u\right)\,\Psi_t\left(\frac{x}{2}-u\right)\,du \\ &= \frac{4}{\pi}\,\frac{1}{\sigma^2 t}\,\exp\left[-\frac{x^2}{4\sigma^2 t}\right]\int_0^{\frac{x}{2}}\exp\left[-\frac{u^2}{\sigma^2 t}\right]du.\end{aligned}$$

This expression is difficult to analyze. We therefore resort to numerical simulations of the function $L \mapsto \lim_{\varepsilon\to 0} P\,(T_{\text{inv}} \geq t)$ to convince the reader that $\lim_{\varepsilon\to 0} P\,(T_{\text{inv}} \geq t)$ is increasing with respect to L (see Figure 3).

4. Proof of Theorem 3

Recall that the transition probability $p\,(x,y)$ has compact support, and let $R > 0$ such that if $|x-y| > R$ then $p\,(x,y) = 0$. We start by introducing the stopping times

$$\begin{aligned}\tau_s^{1} &= \inf\{t \geq s : \ell_t = r_t + 1\}, \\ \tau_s^{2} &= \inf\{t \geq s : \ell_t \leq R\}, \qquad \text{and} \qquad \tau_s^{3} = \inf\{t \geq s : r_t = 0\},\end{aligned}$$

and, for $x \geq 0$, the *interface process* $\iota_t(x) = \xi_t(x+\ell_t-1)$. We observe that, since $r_t = 0$ if and only if $\xi_t(x) = \mathbb{1}_{\{x\leq 0\}}$, it suffices to prove that $P\,(\tau_s^{3} < \infty) = 1$. To do this, we will prove that each of the stopping times introduced above is finite with probability 1. Let $\varepsilon_0 > 0$. Then Proposition 2.1 provides a constant $M > 0$ such that

$$P\,(r_s - \ell_s \geq M) \;\leq\; \varepsilon_0 \qquad \forall\; s \geq 0.$$

Furthermore, if $r_s - \ell_s < M$, then the number of possible configurations for the process ι_s is at most $2^M < \infty$. In particular, there is $\varepsilon_1 > 0$ such that the probability that from time s the process $r_t - \ell_t$ decreases to -1 is $\geq \varepsilon_1$. Since $r_s - \ell_s < M$ infinitely often, $P\,(\tau_s^{1} < \infty) = 1$. Let $t_1 = \tau_s^{1}$ and denote by ζ_t the voter model starting at time t_1 from $\zeta_{t_1}(x) = \xi_{t_1}(x) = \mathbb{1}_{\{x\leq r_{t_1}\}}$. Since $p\,(x,y) = 0$ if $|x-y| > R$, it is easy to check that ξ_t and ζ_t can be defined on the same space in such a way that $\xi_t = \zeta_t$ until the first time $t_2 = \tau_{t_1}^{2}$ at which the process ℓ_t visits $[1,R]$. Moreover, Theorem 5 of Cox and Durrett (1995) implies that the leftmost-zero $\ell'_t = \inf\{x : \zeta_t(x) = 0\}$ is recurrent, so $P\,(\tau_{t_1}^{2} < \infty) = 1$ and

$$P\,(\ell_t \text{ visits } [1,R] \text{ i.o.}) \;=\; 1.$$

To conclude, we use the argument described above: If $r_{t_2} - \ell_{t_2} < M$ then $r_{t_2} < R + M$ and the number of possible configurations ξ_{t_2} is at most 2^{R+M}. In particular, there is $\varepsilon_2 > 0$ such that the probability that from time t_2 the process r_t decreases to 0 is $\geq \varepsilon_2$. In conclusion, each time ℓ_t visits the set $[1,R]$, the process r_t returns to zero with probability at least $\varepsilon_2\,(1-\varepsilon_0) > 0$. The Borel-Cantelli Lemma then implies the result.

5. Proofs of Theorems 4 and 5

Proof of Theorem 4

The key of the proof is duality. Recall that if $\beta < 1$, the dual process performs coalescing branching random walks in which each particle gives birth to a new particle at rate $1-\beta$. Let's denote by $X_s^{(x,t)} = \max \hat{\xi}_s^{(x,t)}$ the *right edge* of the dual process. If we start the process from the initial configuration $\xi_0(x) = \mathbb{1}_{\{x\leq 0\}}$, the duality property

$$\xi_t(x) \;=\; 0 \qquad \Longleftrightarrow \qquad \xi_0(z) \;=\; 0 \quad \text{for some} \quad z \;\in\; \hat{\xi}_t^{(x,t)}$$

implies that $\xi_t(x) = 1$ if and only if $X_t^{(x,t)} > 0$. Moreover, in the nearest neighbor case, the right edge process $X_s^{(x,t)}$ is easy to describe.

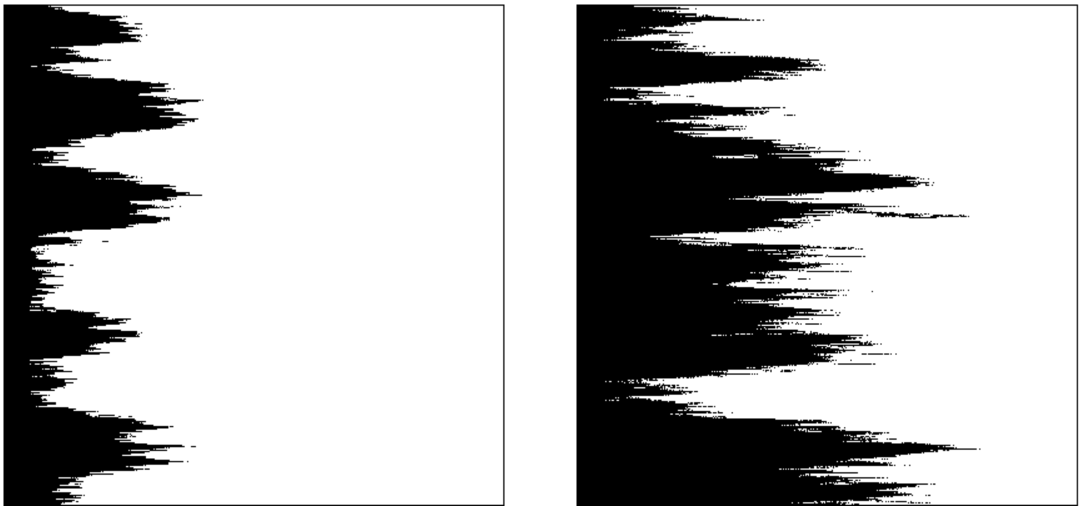

FIGURE 4. *Voter model with $\Delta = \mathbb{Z}^-$ in the interval [-20, 380] from time 0 at the top to time 20 000 at the bottom. Picture on left: The range of the interactions = 2 and the number of returns to zero = 91. Picture on right: The range = 4 and the number of returns to zero = 6.*

1. If $X_s^{(x,t)}$ meets a δ, then it is forced to move along the corresponding δ-arrow.
2. If a particle at $X_s^{(x,t)}$ branches, then $X_s^{(x,t)}$ moves only if it takes it further from 0.

In conclusion, if $x > 0$ then $X_s^{(x,t)}$ is a continuous-time random walk which makes transitions

$$X_s^{(x,t)} \ \to \ \begin{cases} X_s^{(x,t)} + 1 & \text{at rate } 1 \\ X_s^{(x,t)} - 1 & \text{at rate } \beta \end{cases}$$

and with 0 as absorbing state. Let u_x be the probability that $X_s^{(x,t)} = 0$ at some time $s \geq 0$. By decomposing according to whether $X_s^{(x,t)}$ first jumps on the left or on the right gives

$$u_x \ = \ \frac{\beta}{\beta+1}\, u_{x-1} \ + \ \frac{1}{\beta+1}\, u_{x+1} \qquad \forall\ x \geq 1.$$

This implies that $u_{x+1} - u_x$ is a geometrical sequence with parameter β so that

$$u_x \ - \ u_0 \ = \ \sum_{k=0}^{x-1} \beta^k\,(u_1 \ - \ u_0) \ = \ \frac{1-\beta^x}{1-\beta}\,(u_1 \ - \ u_0).$$

Now, since $X_s^{(x,t)}$ drifts to the right, $\lim_{x\to\infty} u_x = 0$. Moreover, $u_0 = 1$ so it follows from the previous equation that $u_1 = \beta$. In conclusion, $u_x = \beta^x$ and

$$P\,(r_t \geq x) \ = \ P\,(X_t^{(x,t)} = 0) \ \leq \ P\,(X_s^{(x,t)} = 0 \text{ at some time } s \geq 0) \ = \ \beta^x.$$

This completes the proof. $\square$

Proof of Theorem 5

We denote by τ the last time the rightmost-one process r_t returns to zero. Since $\beta > 1$, there exist $C < \infty$ and $\gamma > 0$ such that $P\,(\tau > \varepsilon\, t) \leq C\,\exp\,(-\gamma\,\varepsilon\, t)$. In particular, it suffices to prove the result for the continuous-time random walk Y_t starting at $Y_0 = 0$ which makes transitions

$$Y_t \ \to \ \begin{cases} Y_t + 1 & \text{at rate } \beta \\ Y_t - 1 & \text{at rate } 1. \end{cases}$$

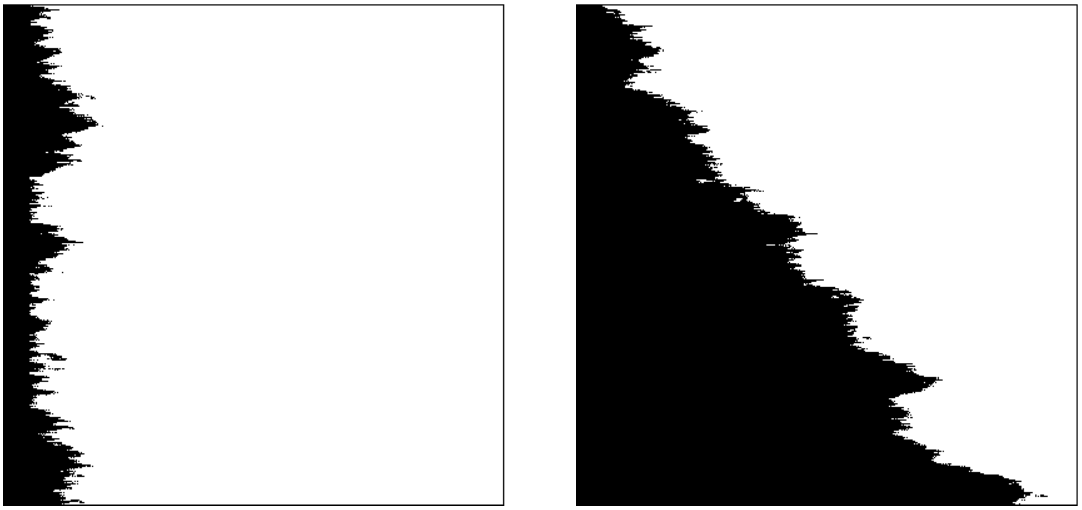

Figure 5. *Range 2 voter model with $\Delta = \mathbb{Z}^-$ in the interval [-20, 380] from time 0 at the top to time 4 000 at the bottom. Picture on left: $\beta = 0.95$. Picture on right: $\beta = 1.05$.*

Let Z_n be the discrete-time version of Y_t, namely the Bernoulli random walk which jumps on the right with probability $\beta/(1+\beta)$ and on the left with probability $1/(1+\beta)$. Then, Proposition 3, Section 1.5, in Spitzer (1976) implies that

$$P\left\{\left|\frac{Z_n}{n} - \frac{\beta-1}{\beta+1}\right| \geq \varepsilon\right\} \leq 2\,e^{-\gamma\varepsilon^2 n}$$

for any $n \geq 1$ and $\varepsilon > 0$. To deduce the analogous result for the continuous-time version Y_t, we observe that large-deviations results for the Poisson distribution imply that the probability of more than $(\beta + 1 + \varepsilon)\,t$ or less than $(\beta + 1 - \varepsilon)\,t$ jumps by time t is smaller than $C\,e^{-\gamma t}$ for some C, $\gamma \in (0, \infty)$. It follows that

$$P\left(|\,Y_t - (\beta-1)\,t\,| \geq \varepsilon\,t\right) \leq C\,e^{-\gamma\varepsilon^2 t}$$

for appropriate $C < \infty$ and $\gamma > 0$. □

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Continuity result for multicolor particle systems

N. Lanchier

Abstract The aim of this paper is to prove a continuity result for particle systems. Let ξ_t be a particle system whose transition rates depend continuously on a parameter $\lambda \in \Lambda$. We give a sufficient condition so that if λ is close enough to a fixed $\lambda_0 \in \Lambda$ then the processes with parameters λ and λ_0 exhibit, in some sense to be precised, the same behavior. This result, referred to as continuity result, has already been proved in many special cases. In this article, we prove the result for a large class of multicolor particle systems. Moreover, we will see that our result gives us insights into the strategy to study the behavior of a particle system.

1. Introduction

The processes we will be interested in all along this paper, that we will refer to as *multicolor particle systems*, are Markov processes $\xi_t : \mathbb{Z}^d \longrightarrow F$ where $\mathbb{Z}^d$ denotes the d-dimensional space with integer coordinates, and F a finite set called *set of colors*. To describe the evolution of our processes, we consider an *interaction neighborhood*

$$\mathcal{N} \ = \ \{z_0, z_1, \ldots, z_N\} \ \subset \ \mathbb{Z}^d \qquad \text{with} \quad z_0 \ = \ 0.$$

If the system is in some configuration $\xi : \mathbb{Z}^d \longrightarrow F$, the color of $x \in \mathbb{Z}^d$ flips to $i \in F$ at rate

$$c_i\,(\lambda, x, \xi) \ = \ h_i\,(\lambda,\, \xi(x + z_0),\, \xi(x + z_1),\, \ldots,\, \xi(x + z_N)).$$

In other words, our interaction is finite range, i.e., the flip rates depend only on the color of x and of a finite number of neighbors, and translation invariant, i.e., the rules applied at x are just a translation of those applied at 0. Moreover, we assume that the flip rates $c_i\,(\lambda, x, \xi)$ depend on a parameter $\lambda \in \Lambda$, where Λ is an open set.

Given a multicolor particle system $\xi_t : \mathbb{Z}^d \longrightarrow F$, a well-known method for proving the existence of nontrivial stationary distributions for the process ξ_t is to apply a *rescaling argument*. The basic idea is to compare a certain collection of good events for the process viewed on suitable space and time scale with an oriented site percolation process. The rescaling argument has been invented by Bramson and Durrett (1988) and is reviewed in Durrett (1991, 1995). Let's now assume that one succeeds in proving the existence of a stationary distribution for the process with parameter $\lambda_0 \in \Lambda$, i.e., the evolution is described by the transition rates $c_i\,(\lambda_0, x, \xi)$, through the rescaling argument. The aim of this paper is to prove that, under some continuity assumptions, there is a neighborhood $\mathcal{V}$ of λ_0 in Λ such that the comparison with supercritical oriented site percolation still holds for the set of processes with parameter $\lambda \in \mathcal{V}$. This tells us in particular that our particle system still exhibits the same behavior after a slight perturbation of the evolution. From now on, this result will be referred to as *continuity result*. The argument is very useful and has been proved in many special cases. See, e.g., Durrett and Neuhauser (1997), Schinazi (2001, 2002) and Lanchier (2005a). In this paper, we prove the result in the general case.

To formulate rigorously our result and prepare its proof, we first remind the strategy of the rescaling argument introduced in Bramson and Durrett (1988). The first step is to define oriented percolation. For more details, we refer the reader to Durrett (1984). To begin with, let

$$\mathcal{L} \ = \ \{(z, n) \in \mathbb{Z}^2 : z + n \text{ is even and } n \geq 0\}.$$

From each $(z, n) \in \mathcal{L}$, we draw an oriented arc to $(z - 1, n + 1)$ and $(z + 1, n + 1)$, and define a random variable $\omega(z, n)$ to be 1 if (z, n) is *open* and to be 0 if (z, n) is *closed*. We will say that there is a *path* from (z, m) to (z', n) if there exists a sequence $z \ = \ z_m, \ldots, z_n \ = \ z'$ such that

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1. For any $k = m, \ldots, n-1$, $|z_k - z_{k+1}| = 1$ and
2. For any $k = m, \ldots, n$, the site (z_k, k) is open, i.e., $\omega(z_k, k) = 1$.

Finally, given an initial condition $W_0 \subset 2\mathbb{Z}$, we let

$$W_n \ = \ \{z : \text{there is a path from } (x,0) \text{ to } (z,n) \text{ for some } x \in W_0\}$$

be the set of *wet* sites at level n. Later on, the random variables $\omega(z,n)$, $(z,n) \in \mathcal{L}$, will be assumed to be M-dependent with density p which means that

$$P\left(\omega(z_i, n_i) = 0 \text{ for } 1 \leq i \leq m\right) \ = \ (1-p)^m$$

for any (z_i, n_i), $1 \leq i \leq m$, such that $\|(z_i,n_i) - (z_j,n_j)\|_\infty > M$ if $i \neq j$. To make the connection between particle systems and oriented percolation, the basic idea is to turn the space-time of the process into a brick wall, each brick is associated with a certain good event. To be precise, we consider a particle system $\xi_t : \mathbb{Z}^d \longrightarrow F$ that is constructed from a collection of independent Poisson processes (Harris, 1972), integers L, T and Γ with $T = \Gamma L$, and a collection $\mathcal{H}$ of configurations determined by the value of ξ on $[-L, L]^d$. For any $z \in \mathbb{Z}$, we denote by $\mathcal{H}_z$ the collection $\mathcal{H}$ translated by the vector Lze_1, and say that a site $(z,n) \in \mathcal{L}$ is *occupied* if $\xi_{nT} \in \mathcal{H}_z$. Here, e_1 denotes the first unit vector of the d-dimensional lattice. Finally, we let k_0 and j_0 be two integers, set $M = \max\{j_0, k_0\}$, and introduce the space-time region

$$B_{z,n} \ = \ (Lze_1, \, nT) \ + \ \{[-k_0\, L, k_0\, L]^d \, \times \, [0, \, j_0\, T]\}.$$

Each $(z,n) \in \mathcal{L}$ is associated with a certain good event $E_{z,n}$ measurable with respect to the graphical representation of the process in $B_{z,n}$. See Figure 1 for a picture. Finally, we assume that there exists an $\varepsilon > 0$ small such that

1. $P(E_{z,n}) \geq 1 - \varepsilon$ and
2. If (z,n) is occupied and $E_{z,n}$ occurs, then $(z-1, n+1)$ and $(z+1, n+1)$ are occupied.

This holds for any $(z,n) \in \mathcal{L}$. The conditions 1 and 2 above are usually referred to as the *comparison assumption.* To formulate the comparison theorem, we let

$$X_n \ = \ \{z : (z,n) \in \mathcal{L} \text{ with } (z,n) \text{ occupied}\}$$

be the set of occupied sites at level n.

Theorem 1 *If the comparison assumption holds, then the random variables $\omega(z,n)$ can be defined in such a way that X_n dominates an M-dependent oriented percolation process with initial condition $W_0 = X_0$ and density $1-\varepsilon$, i.e., $W_n \subset X_n$ for any $n \geq 0$.*

See Durrett (1995), Section 4. Well-known results about oriented percolation (Durrett, 1984) tell us that if $\varepsilon > 0$ is sufficiently small then there is a positive probability that percolation occurs, i.e., $\liminf_{n \to \infty} P(W_n \neq \varnothing) > 0$. Theorem 1 and Liggett (1985), Proposition 1.8, then imply the existence of a stationary distribution ν such that, for any $z \in \mathbb{Z}$, $\nu(\xi \in \mathcal{H}_z) > 0$. In other words, the rescaling argument allows to prove the existence of nontrivial stationary distributions.

To formulate our result, we now consider a multicolor particle system $\xi_t : \mathbb{Z}^d \longrightarrow F$ whose evolution is described by the transition rates $c_i(\lambda, x, \xi)$, $i \in F$, $\lambda \in \Lambda$, introduced above. Moreover, we suppose that following continuity assumption holds.

(CA) For any $\delta > 0$, there exists a neighborhood $\mathcal{V}$ of λ_0 in Λ such that

$$|c_i(\lambda, 0, \xi) \, - \, c_i(\lambda_0, 0, \xi)| \ < \ \delta \qquad \forall\, \lambda \in \mathcal{V}, \quad \forall\, i \in F, \quad \forall\, \xi \in F^{\mathbb{Z}^d}.$$

Let $\lambda_0 \in \Lambda$ fixed, and denote by $\bar{\xi}_t$ the process with parameter λ_0. Finally, given all the ingredients introduced above, we suppose that, for any $\varepsilon > 0$, the comparison assumption holds for the process with parameter λ_0. More precisely,

$(P_{\lambda_0}^{\varepsilon})$ The parameters L and Γ can be chosen sufficiently large so that

1. $P(E_{z,n}) \geq 1 - \varepsilon$ and
2. If (z, n) is occupied and $E_{z,n}$ occurs, then $(z-1, n+1)$ and $(z+1, n+1)$ are occupied.

Here, (z, n) occupied means that $\bar{\xi}_{nT} \in \mathcal{H}_z$. That is, in the definition of $P_{\lambda_0}^{\varepsilon}$, "occupied" applies to the particle system with parameter λ_0. From now on, we let P_{λ_0} be the property that the comparison assumption $P_{\lambda_0}^{\varepsilon}$ is satisfied for any $\varepsilon > 0$.

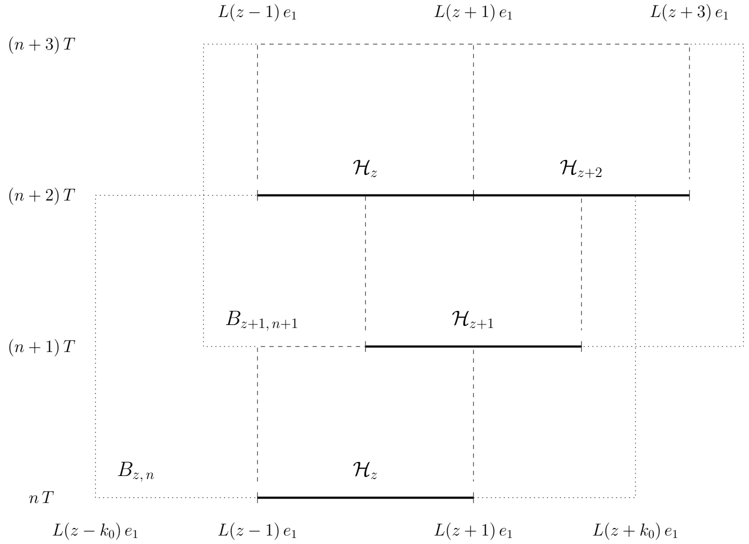

Figure 1. *Picture of the rescaling argument.*

Theorem 2 *Assume CA and P_{λ_0}. Then, for any $\varepsilon > 0$, there is a neighborhood $\mathcal{V}_\varepsilon$ of λ_0 in Λ such that the following holds: For any $\lambda \in \mathcal{V}_\varepsilon$, the comparison assumption P_λ^ε is satisfied.*

As previously explained, Theorem 2 tells us that if λ is close to λ_0 then the processes with parameters λ and λ_0 exhibit the same behavior in the sense that each of both processes has a stationary distribution ν such that, for any $z \in \mathbb{Z}$, $\nu(\xi \in \mathcal{H}_z) > 0$. But beyond a simple continuity result, Theorem 2 gives us insights into the strategy to study the behavior of a particle system. More precisely, we get the following

Corollary 3 *Assume that the process ξ_t exhibits a phase transition at $\lambda_0 \in \Lambda$. Then, the behavior of the process when $\lambda = \lambda_0$ cannot be investigated through the comparison result.*

The proof of Corollary 3 is straightforward. Assume that one succeeds in constructing a stationary measure ν for the process with parameter λ_0 through the comparison result, that is proving that the assumption P_{λ_0} holds. Then, Theorem 2 tells us that there is a neighborhood $\mathcal{V}$ of λ_0 in Λ such that, for any $\lambda \in \mathcal{V}$, the processes with parameters λ and λ_0 exhibit the same behavior, which is not consistent with the existence of a phase transition at λ_0 and proves Corollary 3.

Before going into the details of the proof, we start by observing that, typically, the transition rates $c_i(\lambda, x, \xi)$, $i \in F$, are linear functions of the parameter λ. See, e.g., Durrett (1995). In particular, even though essential, the continuity assumption CA is not very restrictive so that our results can be plentifully applied. Finally, we think that Theorem 2 does not hold in the very useful case $\lambda_0 = \infty$ and $\lambda > 0$ large (see for instance the discussion following Theorem 3 of Lanchier and Neuhauser (2005)). The continuity result when $\lambda_0 = \infty$ and λ is large, however, holds in some special cases, and the strategy of our proof is general enough to be easily adapted to such cases provided that one has an explicit description of the process to be investigated. See for instance Lanchier (2005b) or Lanchier and Neuhauser (2005) for illustrations.

2. Preliminary results about the dual process

In this section, we let $\bar{\xi}_t : \mathbb{Z}^d \longrightarrow F$ be the process with parameter λ_0, i.e., the evolution is described by the transition rates $c_i(\lambda_0, x, \xi)$, and prove that, with probability close to 1, the dual process $\bar{\xi}_s^{(x,T)}$ starting at (x, T) does not escape from a spatial box to be fixed later. The objective is to fix the appropriate space and time scale so that the comparison assumption holds.

The first step is to construct the process $\bar{\xi}_t$ from a graphical representation and to define its dual process. To begin with, we let $\{T_n^{x,i} : n \geq 1\}$, $x \in \mathbb{Z}^d$, $i \in F$, be a collection of independent Poisson processes with parameter

$$c_i(\lambda_0) \ = \ \sup \left\{ c_i(\lambda_0, 0, \xi) : \xi \in F^{\mathbb{Z}^d} \right\}.$$

The reader will note that, since $c_i(\lambda_0, 0, \xi)$ only depends on the color of 0 and of a finite number of neighboring sites, $c_i(\lambda_0) < \infty$. At time $t = T_n^{x,i}$, the color of site $x \in \mathbb{Z}^d$ flips to $i \in F$ with probability $c_i(\lambda_0, x, \bar{\xi}_{t-})/c_i(\lambda_0)$. An argument from Harris (1972) implies that such a graphical representation can be used to construct the process starting from any initial configuration. The dual process $\bar{\xi}_s^{(x,T)}$ is then defined by going backwards in time. In a general way, it allows us to deduce the color of site x at time T from the configuration at earlier time by keeping track of the ancestors. The dual process starts at $\bar{\xi}_0^{(x,T)} = \{x\}$ and makes transitions as follows.

If $z \in \bar{\xi}_s^{(x,T)}$ and $T_n^{z,i} = t - s$, then we add the points of $z + \mathcal{N}$ to $\bar{\xi}_s^{(x,T)}$

where $\mathcal{N}$ denotes the interaction neighborhood. From the previous definition, it is easy to see that we can determine the color of (x, T) by knowing the colors of sites $z \in \bar{\xi}_s^{(x,T)}$ at time $T - s$. The dual process is naturally defined only for $0 \leq s \leq T$ but it is convenient to assume that the Poisson processes in the construction are defined for negative times and $\bar{\xi}_s^{(x,T)}$ for all $s \geq 0$.

Lemma 2.1 *There exists a convex set $A \subset \mathbb{R}^d$ such that for any $\delta > 0$*

$$P\left((1-\delta)\, t\, A \cap \mathbb{Z}^d \ \subset \ \bar{\xi}_t^{(0,T)} \ \subset \ (1+\delta)\, t\, A \cap \mathbb{Z}^d\right) \ \to \ 1 \qquad \text{as} \quad t \to \infty.$$

Proof The basic idea is that $\bar{\xi}_s^{(0,T)}$ is almost a Richardson's model that is linearly growing in time. Unfortunately, since our dual process increases by "blocks", we did not find any coupling argument to prove the domination by a Richardson's model. The proof of Durrett (1988), however, can be easily extended to our model, so we just give the key ideas and refer the reader to the last reference for the details. The first step is to introduce, for any x, $y \in \mathbb{Z}^d$, the stopping times

$$t(x) \ = \ \inf \left\{ s \geq 0 : x \ \in \ \bar{\xi}_s^{(0,T)} \right\} \qquad \text{and} \qquad t(x,y) \ = \ \inf \left\{ s \geq 0 : y \ \in \ \bar{\xi}_{s+t(x)}^{(x,T-t(x))} \right\}.$$

For a given site $x \in \mathbb{Z}^d$ and positive integers m and n, let $X_{m,n} = t(mx, nx)$ be the amount of time it takes the dual process starting at site mx to reach nx. Then, $\{X_{m,n} : m \leq n\}$ satisfies the hypothesis of Liggett (1985), Theorem 2.6, Chapter VI, which implies that

$$\frac{X_{0,n}}{n} \ = \ \frac{t(nx)}{n} \ \longrightarrow \ \mu(x) \quad \text{a.s.} \qquad \text{as} \quad n \to \infty.$$

This tells us that $\bar{\xi}_s^{(0,T)}$ grows linearly in each direction and proves the result in $d = 1$. To prove the result in $d \geq 2$, we first turn $\bar{\xi}_s^{(0,T)}$ into a solid blob by letting

$$\tilde{\xi}_s^{(0,T)} \ = \ \left\{ x + y : x \ \in \ \bar{\xi}_s^{(0,T)} \text{ and } y \ \in \ [-1/2, 1/2]^d \right\}.$$

This allows us to extend the definitions of $t(x)$ and of $\mu(x)$ to $x \in \mathbb{R}^d$, the d-dimensional space with real coordinates. This new definition makes μ a norm on $\mathbb{R}^d$. The shape result then follows by taking $A = \{x : \mu(x) \leq 1\}$, the unit ball in that norm. $\square$

Lemma 2.2 *Let $T = \Gamma L$. There exists an integer ℓ_0 such that*

$$P\left(\{\bar\xi_s^{(x,T)} : 0 \le s \le T\} \subset [-\ell_0 L, \ell_0 L]^d \ \text{ for any } \ x \in [-2L, 2L]^d\right) \ \to\ 1 \qquad \text{as} \quad L \to \infty.$$

Proof This is a straightforward consequence of Lemma 2.1 supplemented with a coupling argument. We fix $t_0 > 0$ such that $[-4,\, 4]^d \subset t_0 A$. Then, the shape result implies that

$$\begin{aligned} & P\left(\{\bar\xi_s^{(x,T)} : 0 \le s \le T\} \not\subset 2\,(T + t_0 L)\, A \ \text{ for some } \ x \in [-2L, 2L]^d\right) \\ & \quad \le\ P\left([-2L, 2L]^d \not\subset \bar\xi_{t_0 L}^{(0, T + t_0 L)}\right) \ +\ P\left(\bar\xi_{T + t_0 L}^{(0, T + t_0 L)} \not\subset 2\,(T + t_0 L)\, A\right) \ \to\ 0 \end{aligned}$$

as $L \to \infty$. The result then follows by taking ℓ_0 such that $2\,(\Gamma + t_0)\, A \subset [-\ell_0,\, \ell_0\,]^d$. □

3. Proof of the continuity result

This section is devoted to the proof of Theorem 2. In what follows, we will denote by $\bar\xi_t$ the process with fixed parameter $\lambda_0 \in \Lambda$, i.e., the process with transition rates $c_i\,(\lambda_0, x, \xi)$, and will denote by ξ_t the process with parameter $\lambda \in \Lambda$ close to λ_0. For more convenience, the objects associated with $\bar\xi_t$ will be written as their analogous for ξ_t with a bar in addition.

Given an integer $\bar M \ge 1$, a collection $\mathcal{H}$ of configurations determined by the value of ξ on the spatial box $[-L, L]^d$, and a collection $\bar E_{z,n}$, $(z, n) \in \mathcal{L}$, of good events that are measurable with respect to the graphical representation of the process $\bar\xi_t$ in the space-time region

$$\bar B_{z,n} \ =\ (Lze_1,\, nT) \ +\ \{[-k_0 L, k_0 L]^d \times [0,\, j_0 T]\}$$

with $\max\{j_0, k_0\} = \bar M$, we assume that

(P_{λ_0}) For any $\varepsilon > 0$, the parameters L and Γ can be chosen sufficiently large so that

1. $P\,(\bar E_{z,n}) \ge 1 - \varepsilon$ and
2. If (z, n) is occupied and $\bar E_{z,n}$ occurs, then $(z - 1, n + 1)$ and $(z + 1, n + 1)$ are occupied.

Here, (z, n) occupied means that $\bar\xi_{nT} \in \mathcal{H}_z$. The proof of Theorem 2 is a three-step process. The basic idea is to define, for $(z, n) \in \mathcal{L}$, a new good event $E_{z,n}$ to be the intersection of three good events $E_{z,n}^{\,j}$, $j = 1, 2, 3$. This event will have to insure us that the assumption P_λ^ε holds for the process with parameter λ close to λ_0.

The first good event will be $E_{z,n}^{\,1} = \bar E_{z,n}$. Now, to make sure that $E_{z,n}$ is measurable with respect to the graphical representation in some box $B_{z,n}$, we first let $E_{z,n}^{\,2}$ be the event that

$$\{\bar\xi_s^{(x,(n+1)T)} : 0 \le s \le T\} \subset [(z - \ell_0)\, L, (z + \ell_0)\, L]^d \qquad \text{for any} \quad x \in [(z - 2)\, L, (z + 2)\, L]^d.$$

The event $E_{z,n}^{\,2}$ seems to be the bad one since it does not say anything about the process with parameter λ. The third good event, however, will allow us to solve the problem. To define the event $E_{z,n}^{\,3}$, the first step is to construct both processes $\bar\xi_t$ and ξ_t by using the same Harris' graphical representation. More precisely, for any color $i \in F$, we let

$$c_i\,(\lambda, \lambda_0) \ =\ \max\,(c_i\,(\lambda), c_i\,(\lambda_0)) \ =\ \sup\left\{\max\,(c_i\,(\lambda, 0, \xi), c_i\,(\lambda_0, 0, \xi)) : \xi \in F^{\mathbb{Z}^d}\right\}$$

and let $\{T_n^{x,i} : n \ge 1\}$, $x \in \mathbb{Z}^d$, be a collection of Poisson processes with parameter $c_i\,(\lambda, \lambda_0)$, all of them are independent. To complete the construction, for any $x \in \mathbb{Z}^d$ and $i \in F$, we toss a coin with success probability

$$p_i\,(\lambda, \lambda_0) \ =\ \frac{|c_i\,(\lambda) - c_i\,(\lambda_0)|}{c_i\,(\lambda, \lambda_0)} \ =\ \frac{|c_i\,(\lambda) - c_i\,(\lambda_0)|}{\max\,(c_i\,(\lambda), c_i\,(\lambda_0))}.$$

If there is a success, we put a ω_0 at point $(x, T_n^{x,i})$ if $c_i\,(\lambda) < c_i\,(\lambda_0)$, and a ω at point $(x, T_n^{x,i})$ if on the contrary $c_i\,(\lambda) > c_i\,(\lambda_0)$. Then, the process $\bar\xi_t$ (resp. ξ_t) can be constructed as explained in

Section 2 by using the exponential clock devoid of ω (resp. ω_0). In other words, the process with parameter λ_0 does not see the ω's while the process with parameter λ does not see the ω_0's. Finally, we let m_0 denote an integer to be fixed later and set

$$B_{z,n} \ = \ (Lze_1,\, n\Gamma L) \ + \ \{[-m_0\, L, m_0\, L\,]^d \, \times \, [\,0,\, j_0\, \Gamma L\,]\}.$$

Then, $E^3_{z,n}$ will be the event that none of the exponential clocks in $B_{z,n}$ is labeled, i.e., there is neither ω nor ω_0 in the space-time region $B_{z,n}$.

Now that our three good events $E^j_{z,n}$, $j = 1, 2, 3$, are defined, the next step is to prove that our choice of $E_{z,n}$ implies Theorem 2. To begin with, we let $\varepsilon > 0$ and apply the comparison assumption P_{λ_0} to pick L and Γ so that $P\,(E^1_{z,n}) = P\,(\bar{E}_{z,n}) \geq 1 - \varepsilon/\,3$. In other respects, Lemma 2.2 implies the existence of an integer ℓ_0 independent of L such that, for L sufficiently large, $P\,(E^2_{z,n}) \geq 1 - \varepsilon/\,3$. This last observation allows us to fix the size of $B_{z,n}$ by setting $m_0 = \max\{k_0, \ell_0\}$. Now that L and Γ are fixed so that $E^1_{z,n}$ and $E^2_{z,n}$ occur with probability close to 1, we are ready to estimate our third event. A straightforward calculation shows that

$$P\,(E^3_{z,n}) \ \geq \ 1 \ - \ (2\, m_0\, L + 1)^d \ \sum_{i=0}^{\kappa-1} \ \big\{\, 1 - \exp\, \big(\, - |c_i\,(\lambda) - c_i\,(\lambda_0)|\; j_0\, \Gamma L \,\big) \big\}\,.$$

This together with the continuity assumption CA implies the existence of a neighborhood $\mathcal{V}_\varepsilon$ of λ_0 in Λ such that $P\,(E^3_{z,n}) \geq 1 - \varepsilon/\,3$, for any $\lambda \in \mathcal{V}_\varepsilon$. In conclusion, we have proved that there exist parameters L and Γ, and a neighborhood $\mathcal{V}_\varepsilon$ of λ_0 such that $P\,(E_{z,n}) \geq 1 - \varepsilon$, for any $\lambda \in \mathcal{V}_\varepsilon$.

We are now ready for the final denouement. First of all, we observe that the event $E_{z,n}$ is measurable with respect to the graphical representation in $B_{z,n}$. That is, the configuration of the process at time $(n+1)\,T$ in the box $B_{z-1} \cup B_{z+1}$ only depends on what happens in the space-time region $B_{z,n}$. In particular, if $M = \max\{j_0, m_0\}$ then, whenever $\|(z_i, n_i) - (z_j, n_j)\|_\infty > M$, the events E_{z_i, n_i} and E_{z_j, n_j} are independent. To prove Theorem 2, the last thing we have to check is that if (z, n) is occupied and $E_{z,n}$ occurs, then $(z-1, n+1)$ and $(z+1, n+1)$ will be occupied as well, where "occupied" now applies to the process with parameter λ. This results from a coupling argument. More precisely, we assume that $\xi_{nT} \in \mathcal{H}_z$, and set $\bar{\xi}_{nT} = \xi_{nT}$. If both processes are constructed from the same graphical representation, it is easy to see that on $E^2_{z,n} \cap E^3_{z,n}$

$$\bar{\xi}_s^{(x,(n+1)T)} \ = \ \xi_s^{(x,(n+1)T)} \quad \text{for any} \quad x \, \in \, [(z-2)\, L, (z+2)\, L\,]^d \quad \text{and} \quad 0 \, \leq \, t \, \leq \, T.$$

The properties of the dual process (see Section 2), together with $\bar{\xi}_{nT} = \xi_{nT}$ then implies that

$$\bar{\xi}_{(n+1)T}(x) \ = \ \xi_{(n+1)T}(x) \quad \text{for any} \quad x \, \in \, [(z-2)\, L, (z+2)\, L\,]^d.$$

Finally, if our good event $E_{z,n}$ occurs, the event $E^1_{z,n}$ tells us that $\bar{\xi}_{(n+1)T} \, \in \, \mathcal{H}_{z-1} \cap \mathcal{H}_{z+1}$. In conclusion, $\xi_{(n+1)T} \in \mathcal{H}_{z-1} \cap \mathcal{H}_{z+1}$ and Theorem 2 follows.

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Résumé : La plupart des modèles mathématiques introduits dans la littérature biologique décrivant des phénomènes spatiaux de populations en interaction consistent en des systèmes d'équations différentielles ordinaires obtenues sous des hypothèses de dispersion globale, excluant par conséquent toute structure spatiale. Les systèmes de particules, au contraire, sont des processus de Markov d'espace d'états F^S où F est un ensemble fini de couleurs et S est une structure spatiale, typiquement $\mathbb{Z}^d$. Ils sont en ce sens parfaitement adaptés à l'étude des conséquences de l'inclusion d'une structure spatiale sous forme d'interactions locales. Nous étudions les propriétés mathématiques (mesures stationnaires, géométrie des configurations, transitions de phases) de différents systèmes de particules multicolores définis sur $\mathbb{Z}^d$. Chacun de ces systèmes est déstiné à modéliser les interactions locales au sein d'une communauté de populations structurée spatialement. Plus précisément, les processus biologiques étudiés sont la succession écologique, l'allélopathie ou compétition entre une espèce inhibitrice et une espèce sensible, les interactions multispécifiques hôtes-symbiontes, et les migrations continues de gènes des cultures transgéniques par pollinisation en milieu hétérogène. Les techniques mathématiques sont purement probabilistes, incluant le couplage, la dualité, les arguments multi-échelle, la percolation orientée, les propriétés asymptôtiques des marches aléatoires, ou encore les estimations de grandes déviations.

Mots-clefs : Processus de Markov, systèmes de particules en interaction, modèle des votants, processus de contact multitype, percolation orientée, argument multi-échelle, dualité, marches aléatoires, coalescence, modèles de compétition, modèles d'épidémie, génétique des populations.

Abstract: Most mathematical models in the biological literature that describe inherently spatial phenomena of interacting populations consist of systems of ordinary differential equations obtained under global dispersion assumptions, thus leaving out any spatial structure. Interacting particle systems are Markov processes with state space F^S where F is a finite set of colors and where S is a spatial structure, typically $\mathbb{Z}^d$. They are ideally suited to study the consequences of the inclusion of a spatial structure in the form of local interactions. We investigate the mathematical properties (stationary distribution, geometry of the configurations, phase transitions) of various multicolor particle systems defined on $\mathbb{Z}^d$. Each of these systems is intended to model local interactions within a spatially structured community of populations. More precisely, the biological processes we investigate are ecological succession, allelopathy or competition between an inhibitory species and a susceptible species, multi-species host-symbiont interactions, and persistent gene flow from transgenic crops to wild relatives through pollination in a heterogeneous environment. The mathematical techniques are probabilistic, including coupling, duality, multiscale arguments, oriented percolation, asymptotic properties of random walks, and large deviations estimates.

Keywords: Markov processes, interacting particle systems, voter model, multitype contact process, oriented percolation, multiscale argument, duality, random walks, coalescence, competition models, epidemics models, population genetics.