Comportement asymptotique des processus de Markov auto-similaires positifs et forêts de Lévy stables conditionnées.

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COMPORTEMENT ASYMPTOTIQUE DES PROCESSUS DE MARKOV AUTO-SIMILAIRES POSITIFS ET FORÊTS DE LÉVY STABLES CONDITIONNÉES

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Contents

Introduction

Cette thèse est compose de sept chapitres qui correspondent à quatre articles qui ont été publiés ou soumis pour publication. Il s'agit de :

- *The lower envelope of positive self-similar Markov processes*, écrit en collaboration avec Loïc Chaumont. Paru à *Electronic Journal of Probability*, **11**, 2006, pp. 1321-1341.
- *On the future infimum of positive self-similar Markov processes.* Paru à *Stochastics and stochastics reports*, **78**, n. 3, 2006, pp. 123-155.
- *The upper envelope of positive self-similar Markov processes.*
- *On the genealogy of conditioned stable Lévy forest,* écrit en collaboration avec Loïc Chaumont.

Comme le titre le suggère, cette thèse est organisée en deux parties indépendantes. La première partie est consacrée à l'étude de l'enveloppe inférieure et supérieure des processus de Markov auto-similaires positifs et la seconde à l'étude des forêts de Lévy stables d'une taille donnée et conditionnées par leur masse. Cette introduction a pour but de décrire les principaux résultats contenus dans cette thèse.

Comportement asymptotique des processus de Markov auto-similaires positifs.

Un processus de Markov $X^{(x)}$ à valeurs dans $\mathbb{R}$, issu de x et dont les trajectoires sont continues à droite avec des limites à gauche (càdlàg) est dit auto-similaire d'indice $\alpha > 0$ si pour tout $k > 0$,

$$\left(kX^{(x)}_{k^{-\alpha}t},\, t \geq 0\right) \overset{(d)}{=} \left(X^{(kx)}_t,\, t \geq 0\right). \tag{0.1}$$

Les processus de Markov auto-similaires apparaissent souvent dans diverses parties de la théorie des probabilités comme limites de processus normalisés . Leurs propriétés ont été etudiées au début des années soixantes à travers les travaux de John Lamperti [**Lamp62, Lamp72**]. La propriété de Markov ajoutée à l'auto-similarité (ou scaling) fournit des propriétés très intéressantes comme l'avait remarqué Lamperti dans [**Lamp72**] où le cas particulier des processus de Markov auto-similaires positifs est étudié. Les processus de Markov auto-similaires apparaissent dans certains domaines de la théorie des probabilités. Mentionnons par exemple la théorie des processus de branchement et arbres aléatoires, la théorie de fragmentation et les fonctionnelles exponentielles des processus de Lévy.
Dans cette première partie, on considérera des processus de Markov auto-similaires positifs, on y fera référence par l'abréviation *pssMp*. Quelques exemples particulièrement bien connus sont des processus de Bessel, les subordinateurs stables ou plus généralement, les processus de Lévy stables conditionnés à rester positifs.
Notre but est de décrire l'enveloppe inférieure et supérieure en 0 et en $+\infty$ au moyen de tests intégraux et de lois du logarithme itéré pour une classe suffisamment grande de pssMp et quelques processus associés, comme le minimum futur et le pssMp réflechi en son minimum futur. Un point crucial dans nos arguments est la célèbre représentation de Lamperti

des pssMp. Cette représentation nous permet de construire les trajectoires d'un pssMp issu de $x > 0$, noté $X^{(x)}$, à partir de celles d'un processus de Lévy. Plus précisément, [**Lamp72**] Lamperti a montré la représentation suivante :

$$X_t^{(x)} = x \exp\Big\{\xi_{\tau(tx^{-\alpha})}\Big\}, \quad 0 \leq t \leq x^\alpha I(\xi)\,, \tag{0.2}$$

sous la loi du processus $X^{(x)}$, noté $\mathbb{P}_x$, où

$$\tau_t = \inf\Big\{s : I_s(\xi) \geq t\Big\}\,, \quad I_s(\xi) = \int_0^s \exp\Big\{\alpha\xi_u\Big\}\,\mathrm{d}u\,, \quad I(\xi) = \lim_{t\to+\infty} I_t(\xi)\,,$$

et ξ est un processus de Lévy réel éventuellement tué en un temps exponentiel indépendent. On remarque que pour $t < I(\xi)$, on a

$$\tau_t = \int_0^{x^\alpha t} \Big(X_s^{(x)}\Big)^{-\alpha} ds,$$

ce qu'implique que (0.2) est inversible et définit une bijection entre l'ensemble des processus de Lévy de temps de vie éventuellement fini et les pMasp jusqu'en leur premier temps d'atteinte de 0.
Ici, on considère des pMasp qui dérivent vers $+\infty$, c'est-à-dire

$$\lim_{t\to+\infty} X_t^{(x)} = +\infty, \qquad \text{presque sûrement,}$$

et qui satisfont la propriété de Feller sur $[0,\infty)$ de sorte que on peut définir la loi d'un pssMp, que on note $X^{(0)}$, issu de 0 et avec le même semi-groupe de transition que $X^{(x)}$, $x > 0$. Bertoin et Caballero [**BeCa02**], et Bertoin et Yor [**BeYo02**] ont montré qu'une condition suffisante pour la convergence de la famille des processus $X^{(x)}$, quand $x \downarrow 0$, au sens des distributions fini-dimensionelle vers un processus non dégénéré (qu'on va désigner par $X^{(0)}$) est que le processus de Lévy associé ξ par la représentation de Lamperti satisfasse la condition suivante

$$(\mathbf{H}) \qquad \xi \quad \text{n'est pas arithmétique} \quad \text{et} \quad 0 < m \overset{\text{(def)}}{=} \mathbb{E}(\xi_1) \leq \mathbb{E}(|\xi_1|) < +\infty\,.$$

Caballero et Chaumont [**CaCh06**] ont montré que cette dernière condition est aussi une condition nécessaire et suffisante pour la convergence de la famille des proccessus $X^{(x)}$, $x > 0$, dans l'espace de Skohorod des trajectoires càdlàg. Dans le même article, les auteurs ont également fourni une construction du processus $X^{(0)}$ que l'on considérera au début du premier chapitre. La loi d'entrée du processus $X^{(0)}$ a été décrite dans [**BeCa02**] et [**BeYo02**] de la manière suivante : pour tout $t > 0$ et toute fonction mesurable $f : \mathbb{R}_+ \to \mathbb{R}_+$, on a

$$\mathbb{E}\left(f\left(X_t^{(0)}\right)\right) = \frac{1}{m}\mathbb{E}\Big(I(-\alpha\xi)^{-1} f\big(tI(-\alpha\xi)^{-1}\big)\Big)\,. \tag{0.3}$$

où

$$I(-\alpha\xi) = \int_0^s \exp\Big\{-\alpha\xi_u\Big\}\,\mathrm{d}u.$$

Remarque: *De la propriété de scaling, on peut facilement vérifier que le processus* $(X^\alpha, \mathbb{P}_x)$, $x > 0$ *est un pssMp dont le coefficient de scaling est égal à* 1. *D'autre part, la fonction* $x \mapsto x^\alpha$ *est une fonctionnelle continue dans l'espace des trajectoires càdlàg, alors sans perte de généralité on peut supposer que* α *est égal à 1 dans la suite.*

Dans le premier chapitre, on présente des propriétés trajectorielles des pssMp qui seront d'une grande utilité pour l'étude de leur comportement asymptotique. En fait, le but de ce premier chapitre est de décomposer les trajectoires du processus limite $X^{(0)}$ en ses premiers et derniers temps de passage près de 0 et de $+\infty$, et aussi de déterminer, quand il sera possible, les lois des premiers et derniers temps de passage en termes du processus de Lévy associé par la représentation de Lamperti.
Les trajectoires du processus $X^{(0)}$ peuvent être décomposées au premier temps de passage de manière naturelle en utilisant la propriété de Markov mais cette décomposition ne nous permet pas de déterminer la loi du premier temps de passage en termes du processus de Lévy associé. La construction de Caballero et Chaumont [**CaCh06**] fournit une décomposition qui satisfait ces conditions. Soit (x_n) une suite décroissante de réels strictement positifs qui converge vers 0, de manière informelle la construction de Caballero et Chaumont est une concaténation d'une suite $(Y^{(n)}, n \geq 1)$ de pMasp, où chaque processus $Y^{(n)}$ est issu de la valeur du processus $Y^{(n+1)}$ pris en son premier temps de passage au-dessus de x_n et tué au premier temps de passage de $Y^{(n)}$ au-dessus de x_{n-1}. On peut trouver cette construction de manière détaillée au début du premier chapitre.
Pour la deuxième décomposition, on va d'abord étudier la loi du processus $X^{(0)}$ retourné en son dernier temps de passage. Soit

$$U_y = \sup\left\{t : X_t^{(0)} \leq y\right\}, \quad y \leq 0,$$

le dernier temps de passage du processus $X^{(0)}$ dessous de y. On définit une famille de pssMp dont la représentation de Lamperti est donnée par

$$\hat{X}^{(x)} = \left(x \exp\left\{\hat{\xi}_{\hat{\tau}(t/x)}\right\}, 0 \leq t \leq xI(\hat{\xi})\right), \quad x > 0, \tag{0.4}$$

où

$$\hat{\xi} = -\xi, \qquad \hat{\tau}_t = \inf\left\{s : \int_0^s \exp\left\{\hat{\xi}_u\right\} \mathrm{d}u \geq t\right\}, \quad \text{et} \quad I(\hat{\xi}) = \int_0^\infty \exp\left\{\hat{\xi}_s\right\} \mathrm{d}s.$$

Par hypothèse, il est clair que le processus $\hat{X}^{(x)}$ atteint 0 de manière continu en un un temps aléatoire presque sûrement fini qui est égal à $xI(\hat{\xi})$.
Pour simplifier la notation, on fixe $\Gamma = X^{(0)}_{U_x-}$ et on note par K, le support de la loi de Γ.

PROPOSITION 1. *La loi du processus $\hat{X}^{(x)}$ est une version régulière de la loi du processus*

$$\hat{X} \overset{(def)}{=} \left(X^{(0)}_{(U_x-t)-}, 0 \leq t \leq U_x\right),$$

conditionnellement á $\{\Gamma = x\}$, $x \in K$.

Grâce à cette proposition, on obtient la décomposition suivante du processus $\hat{X}$. Soit (x_n) une suite décroissante de réels strictement positifs qui converge vers 0. Pour $y > 0$, on définit

$$\hat{S}_y = \inf\left\{t : \hat{X}_t \leq y\right\}.$$

Entre les temps de passage $\hat{S}_{x_n}$ et $\hat{S}_{x_{n+1}}$, le processus peut être décrit de la manière suivante :

$$\left(\hat{X}_{\hat{S}_{x_n}}, 0 \leq t \leq \hat{S}_{x_{n+1}} - \hat{S}_{x_n}\right) = \left(\Gamma_n \exp\left\{\hat{\xi}^{(n)}_{\hat{\tau}^{(n)}(t/\Gamma_n)}\right\}, 0 \leq t \leq H_n\right), \qquad n \geq 1,$$

où les processus $\hat{\xi}^{(n)}$, $n \geq 1$ sont indépendants et ont tous la même loi que $\hat{\xi}$. La suite $(\hat{\xi}^{(n)})$ est indépendante de Γ et

$$\hat{\tau}_t^{(n)} = \inf\left\{s : \int_0^s \exp\left\{\hat{\xi}_u^{(n)}\right\} du \geq t\right\}$$
$$H_n = \Gamma_n \int_0^{\hat{T}^{(n)}(\log(x_{n+1}/\Gamma_n))} \exp\left\{\hat{\xi}_s^{(n)}\right\} ds$$
$$\Gamma_{n+1} = \Gamma_n \exp\left\{\hat{\xi}^{(n)}_{\hat{T}^{(n)}(\log x_{n+1}/\Gamma_n)}\right\}, \quad n \geq 1, \qquad \Gamma_1 = \Gamma,$$
$$\hat{T}_z^{(n)} = \inf\left\{t : \hat{\xi}_t^{(n)} \leq z\right\}.$$

Pour chaque n, Γ_n est indépendant de $\xi^{(n)}$ et

(0.5) $$x_n^{-1}\Gamma_n \overset{(d)}{=} x_1^{-1}\Gamma\,.$$

En particulier, le temps U_{x_n} peut être décomposé en la somme

$$U_{x_n} = \sum_{k\geq n} \Gamma_k \int_0^{\hat{T}^{(k)}_{\log(x_{k+1}/\Gamma_k)}} \exp\left\{\hat{\xi}_s^{(k)}\right\} ds\,, \quad p.s.$$

Comme conséquence de ces résultats, on a l'identité en loi suivante :

$$U_x \overset{(d)}{=} \frac{x}{x_1} \Gamma I(\hat{\xi})\,.$$

Il est important de remarquer qu'on a les mêmes résultats pour x assez grand.

La dernière partie du premier chapitre est consacrée au pMasp sans saut positif qui est une classe remarquable des processus de Markov auto-similaires. Des propriétés trajectorielles de tels processus peuvent être développées sous une forme simple et complète. En particulier, on obtient une nouvelle construction du processus $X^{(0)}$ faisant intervenir ses derniers temps de passage.
Soient (x_n) une suite décroissante de réels strictement positifs qui converge vers 0 et $(\xi^{(n)})$ une suite de processus de Lévy indépendants et ayant tous la même loi que ξ. On définit,

$$Y_t^{(x_n)} = x_n \exp\left\{\bar{\xi}^{(n)}_{\bar{\tau}^{(n)}(t/x_n)}\right\} \quad \text{for } t \geq 0, \quad n \geq 1,$$

où $\bar{\xi}^{(n)} = \left(\xi^{(n)}_{\gamma_0^{(n)}+t}, t \geq 0\right)$, $\gamma_z^{(n)} = \sup\left\{t \geq 0 : \xi_t^{(n)} \leq z\right\}$ et

$$\bar{\tau}^{(n)}(t/x_n) = \inf\left\{s \geq 0 : \bar{A}_s^{(n)} > t/x_n\right\} \quad \text{et} \quad \bar{A}_s^{(n)} = \int_0^s \exp\left\{\bar{\xi}_u^{(n)}\right\} du,$$

ainsi que

$$\sigma^{(n)} = \sup\left\{t \geq 0 : Y_t^{(x_n)} \leq x_{n-1}\right\}.$$

PROPOSITION 2. *Soit $\Sigma'_n = \sum_{k\geq n} \sigma^{(k)}$, alors pour chaque n, $0 < \Sigma'_n < \infty$ p.s. En plus, le processus*

$$Y_t^{(0)} = \begin{cases} Y^{(x_1)}_{t-\Sigma'_2} & \textit{si } t \in [\Sigma'_2, \infty[, \\ Y^{(x_2)}_{t-\Sigma'_3} & \textit{si } t \in [\Sigma'_3, \Sigma'_2[, \\ \vdots & \\ Y^{(x_n)}_{t-\Sigma'_{n+1}} & \textit{si } t \in [\Sigma'_{n+1}, \Sigma'_n[, \\ \vdots & \end{cases}, \quad Y_0^{(0)} = 0, \tag{0.6}$$

est bien défini et est continu à droite sur $[0, \infty)$ et l'on a les propriétés suivantes :

i) *Le processus $Y^{(0)}$ admet des limites à gauche sur $(0, \infty)$, $\lim_{t\to\infty} Y_t^{(0)} = +\infty$, p.s. et $Y^{(0)} > 0$, p.s. pour tout $t \geq 0$.*
ii) *La loi du processus $Y^{(0)}$ ne dépend pas de la suite (x_n).*
iii) *La famille des mesures de probabilités $(\mathbb{Q}_x, x > 0)$ converge faiblement sur $\mathcal{D}$ vers la loi du processus $Y^{(0)}$, quand x tend vers 0.*

Le résultat suivant nous donne une égalité en loi entre le processus $Y^{(0)}$ et le processus limite $X^{(0)}$ défini par la construction de Caballero et Chaumont.

THÉORÈME 1. *Les processus $Y^{(0)}$ et $X^{(0)}$, définis par la construction de Caballero et Chaumont, ont la même loi. De plus :*

i) *Le processus $Y^{(0)}$ satisfait la propriété de scaling, i.e. pour $k > 0$,*
$$\left(kY^{(0)}_{k^{-1}t}, t \geq 0\right) \quad \textit{a la même loi que} \quad Y^{(0)}.$$
ii) *Le processus $Y^{(0)}$ est fortement markovien et a le même semi-groupe de transition que $(X, \mathbb{P}_x)$ pour $x > 0$.*

Une importante application de ces deux constructions est qu'on peut déterminer la loi du processus $X^{(0)}$ retourné au dernier et premier temps de passage sans utiliser la théorie du retournement de temps de Nagasawa. Rappelons que dans la Proposition 1, on a déterminé la loi du processus $X^{(0)}$ retourné au dernier temps de passage dans le cas général. Il reste seulement à étudier, en utilisant la construction de Caballero et Chaumont dans notre cas particulier, la loi du processus $X^{(0)}$ retourné au premier temps de passage. Définissons d'abord, pour chaque $y > 0$, le processus

$$\tilde{X}_t^{(y)} = y \exp\left\{\tilde{\xi}_{\tilde{\tau}(t/y)}\right\} \qquad t \geq 0,$$

où

$$\tilde{\tau}_t = \inf\left\{s \geq 0 : I_s\big(\tilde{\xi}\big) > t\right\}, \qquad I_s\big(\tilde{\xi}\big) = \int_0^s \exp\left\{\tilde{\xi}_u\right\} \mathrm{d}u,$$

et $\tilde{\xi} = (-\xi_{\gamma_0+t}, t \geq 0)$.
Par hypothèse, on peut déduire que le processus $\tilde{X}^{(y)}$ atteint 0 de manière continue en un temps aléatoire presque sûrement fini, noté par $\tilde{\rho}^{(y)} = \inf\{t \geq 0, \tilde{X}_t^{(y)} = 0\}$.

PROPOSITION 3. *Les processus $\left(X^{(0)}_{(S_x-t)-}, 0 \leq t \leq S_x\right)$ et $\left(\tilde{X}_t^{(x)}, 0 \leq t \leq \tilde{\rho}^{(x)}\right)$ ont la même loi.*

Grâce à ce dernier résultat et à la Proposition 1, on obtient les égalités en loi suivantes

$$S_x \stackrel{(\mathcal{L})}{=} x \int_0^\infty \exp\Big\{-\xi_{\gamma(0)+s}\Big\} \mathrm{d}s \quad \text{et} \quad U_x \stackrel{(\mathcal{L})}{=} x \int_0^\infty \exp\Big\{-\xi_s\Big\} \mathrm{d}s.$$

On déduit des constructions de Caballero et Chaumont (cas sans saut positif) et de la Proposition 2 que les processus $(S_x, x \geq 0)$ et $(U_x, x \geq 0)$ sont auto-similaires, croissants et ses accroissements sont indépendants.

Le deuxième chapitre est consacré à l'étude du comportement asymptotique des pssMp et de leur infimum futur en 0 et en $+\infty$. Plusieurs résultats partiels ont été établis sur ce sujet, en particulier pour les processus de Bessel, subordinateurs stables, pMasp croissants et processus de Lévy stables conditionnés à rester positifs.
L'enveloppe inférieure pour les processus de Bessel a été étudiée par Dvoretzky et Erdős [**DvEr51**]. Selon ces auteurs l'enveloppe inférieure pour un processus de Bessel de dimension $\delta > 2$ et issu de 0, désigné par $X^{(0)}$, satisfait le test intégral suivant : soit f une fonction positive et croissante qui diverge quand t tend vers $+\infty$, alors

$$\mathbb{P}\Big(X_t^{(0)} < f(t), \text{ infiniment souvent lorsque } t \to 0\Big) = 0 \text{ ou } 1,$$

suivant que l'intégrale

$$\int_{0^+} \left(\frac{f(t)}{t}\right)^{\frac{\delta-2}{4}} \frac{\mathrm{d}t}{t} \qquad \text{est finie ou infinie.}$$

La propriété d'inversion de temps des processus de Bessel induit le même test intégral pour le comportement en $+\infty$ de $X^{(x)}$, $x \geq 0$.

Le cas du subordinateur stable a été d'abord étudié par Fristed [**Fris64**] et récemment généralisé au cas de pssMp croissants par Rivero [**Rive03**] qui a prouvé la loi du logarithme itéré suivante : soit ξ un subordinateur dont l'exposant de Laplace ϕ est à variation régulière en $+\infty$ avec indice $\beta \in (0,1)$. On définit la fonction

$$f(t) = \frac{\phi(\log|\log t|)}{\log|\log t|}, \qquad t \neq e, \quad t > 0,$$

alors le pMasp $X^{(x)}$ associé au subordinateur ξ par la représentation de Lamperti, satisfait pour tout $x \geq 0$,

$$\liminf_{t\to+\infty} \frac{X_t^{(x)}}{tf(t)} = (1-\beta)^{(1-\beta)} \qquad \text{presque sûrement,}$$

et

$$\liminf_{t\to 0} \frac{X_t^{(0)}}{tf(t)} = (1-\beta)^{(1-\beta)} \qquad \text{presque sûrement.}$$

Le résultat suivant étend le test intégral pour le processus de Bessel et la loi du logarithme itéré pour le pssMp croissants. Pour simplifier les notations, posons

$$I \stackrel{\text{(def)}}{=} \int_0^\infty \exp\Big\{-\xi_s\Big\}\mathrm{d}s \quad \text{ et } \quad I_q \stackrel{\text{(def)}}{=} \int_0^{\hat{T}_{-q}} \exp\Big\{-\xi_s\Big\}\mathrm{d}s, \quad q > 0,$$

où $\hat{T}_x = \inf\{t : \hat{\xi}_t \leq x\}$, pour $x \leq 0$, ainsi que

$$F(t) \stackrel{\text{(def)}}{=} \mathbb{P}(I > t) \qquad \text{et} \qquad F_q(t) \stackrel{\text{(def)}}{=} \mathbb{P}(I_q > t)\,.$$

THÉORÈME 2. *L'enveloppe inférieure du processus $X^{(0)}$ en 0 se décrit de la manière suivante :*
Soit f une fonction croissante.

(i) *Si*

$$\int_{0+} F\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} < \infty\,,$$

alors pour tout $\varepsilon > 0$*,*

$$\mathbb{P}\Big(X_t^{(0)} < (1-\varepsilon)f(t),\ \textit{infiniment souvent lorsque } t \to 0\Big) = 0\,.$$

(ii) *Si pour tout* $q > 0$*,*

$$\int_{0+} F_q\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} = \infty\,,$$

alors pour tout $\varepsilon > 0$*,*

$$\mathbb{P}\Big(X_t^{(0)} < (1+\varepsilon)f(t),\ \textit{infiniment souvent lorsque } t \to 0\Big) = 1\,.$$

(iii) *Supposons que* $t \mapsto f(t)/t$ *est croissante. S'il existe* $\gamma > 1$ *tel que,*

$$\limsup_{t\to+\infty}\frac{\mathbb{P}(I > \gamma t)}{\mathbb{P}(I > t)} < 1 \quad \textit{et si} \quad \int_{0+} F\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} = \infty\,,$$

alors pour tout $\varepsilon > 0$*,*

$$\mathbb{P}\Big(X_t^{(0)} < (1+\varepsilon)f(t),\ \textit{infiniment souvent lorsque } t \to 0\Big) = 1\,.$$

Le même résultat est satisfait en $+\infty$ pour le processus $X^{(x)}$, $x \geq 0$. Dans la suite de cette introduction, nous énoncerons seulement les tests intégraux en 0 car le mêmes résultats sont satisfaits en $+\infty$, pour tout point de départ $x \geq 0$.

Remarque: *Rappelons que on a supposé que* $\alpha = 1$*. Pour obtenir les test intégraux pour tout index strictement positive* α*, il suffit de considérer le processus* $(X^{(0)})^{1/\alpha}$ *dans le résultat précédent. La même remarque vaut pour les résultats suivants.*

Maintenant, on introduit l'infimum futur du processus $X^{(x)}$ qui est défini par

$$J_t^{(x)} = \inf_{s\geq t} X_s^{(x)}, \qquad t \geq 0.$$

Notons que le processus de l'infimum futur $J^{(x)} = (J_t^{(x)}, t \geq 0)$ est un processus auto-similaire croissant. Grâce à l'hypothèse (**H**), on a que le processus $J^{(x)}$ diverge quand t tend vers $+\infty$ pour $x \geq 0$.
La preuve du Théorème 2 dépend de la décomposition de la trajectoire du processus $X^{(0)}$ présentée après la Proposition 1 et du comportement asymptotique des derniers temps de passage. Comme le processus de l'infimum futur peut être vu comme l'inverse généralisé des derniers temps de passage de $X^{(0)}$, sans perdre de généralité, on peut remplacer $X^{(0)}$ par son infimum futur dans le Théorème 2, ainsi que pour la version de ce résultat en $+\infty$.

Dans la deuxième partie de ce chapitre, on va étudier l'enveloppe supérieure des pssMp et celle de ses infimums futurs. D'après Dvoretsky et Erdős [**DvEr51**], l'enveloppe supérieure des processus de Bessel est décrite de la manière suivante : soient $X^{(0)}$ un processus de Bessel de dimension $\delta > 2$ et f une fonction positive croissante qui diverge quand t tend vers $+\infty$, alors

$$\mathbb{P}\Big(X_t^{(0)} > f(t),\ \text{infiniment suivant lorsque } t \to 0\Big) = 0 \text{ ou } 1,$$

suivant que l'intégrale

$$\int_{0+} \left(\frac{f(t)}{t}\right)^{\delta} \exp\Big\{-f^2(t)/2\Big\}\frac{\mathrm{d}t}{t} \qquad \text{est finie ou infinie.}$$

Ce test intégral est connu comme le test intégral de Kolmogorov-Dvoretzky-Erdős. Le comportement en $+\infty$, comme dans le cas de l'enveloppe inférieure, se déduit de la propriété de l'inversion de temps du processus de Bessel. On peut aussi déduire de ce test intégral la loi du logarithm itéré suivante :

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\sqrt{2t\log\log t}} = 1, \quad \text{et} \quad \limsup_{t\to +\infty} \frac{X_t^{(0)}}{\sqrt{2t\log\log t}} = 1, \qquad \text{presque sûrement.}$$

Le comportement asymptotique de $J^{(0)}$, l'infimum futur des processus de Bessel, a été étudié par Khoshnevisan, Lewis et Li [**Khal94**] ainsi que le comportement asymptotique du processus de Bessel réflechi en son infimum futur. Les auteurs ont obtenu dans [**Khal94**] les lois du logarithme itéré suivantes :

$$\limsup_{t\to +\infty} \frac{J_t^{(0)}}{\sqrt{2t\log\log t}} = 1, \qquad \text{et} \qquad \limsup_{t\to +\infty} \frac{X_t^{(0)} - J_t^{(0)}}{\sqrt{2t\log\log t}} = 1, \quad \text{presque sûrement.}$$

Dans [**Khal94**], Khoshnevisan et al. ont donné un test intégral qui décrit la classe des fonctions qui sont plus grandes que l'infimum futur. Plus précisement, soit $f(t) = \sqrt{t}h(t)$ une fonction croissante, qui diverge quand t tends vers $+\infty$, la condition

$$\int_1^{+\infty} \big(f(t)\big)^{\delta-2} \exp\Big\{-f^2(t)/2\Big\}\frac{\mathrm{d}t}{t} < \infty,$$

implique que

$$\mathbb{P}\Big(J_t^{(0)} > f(t), \text{ infiniment souvent lorsque } t \to +\infty\Big) = 0.$$

L'enveloppe supérieure des subordinateurs stables a été études par Khinchin [**Khin38**] où il a obtenu le test intégral suivant : si $X^{(0)}$ désigne un subordinateur stable d'indice $\alpha \in (0,1)$ et f une fonction positive croissante telle que $t \mapsto f(t)/t$ est aussi croissante, alors

$$\mathbb{P}\Big(X_t^{(0)} > h(t), \text{ infiniment souvent lorsque } t \to 0\Big) = 0 \text{ ou } 1,$$

suivant que l'intégrale

$$\int_{0+} \big(h(t)\big)^{-\alpha}\mathrm{d}t \qquad \text{est finie ou infinie.}$$

Il existe une loi du logarithme itéré pour les processus de Lévy stable sans sauts positifs conditionnés à rester positifs. En fait, cette loi a été montrée par Bertoin [**Bert95**] pour tout processus de Lévy sans sauts positifs conditionné à rester positif. Plus précisement, soit $X^{(0)}$ un processus de Lévy stable sans sauts positifs conditionné à rester positif d'indice $\alpha \in (1,2]$, alors il existe une constante $c > 0$ telle que

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{t^{1/\alpha}(\log|\log t|)^{1-1/\alpha}} = c, \qquad \text{presque sûrement.}$$

Les test intégraux qu'on va présenter maintenant généralisent les résultats décrits ci-dessous. Tout d'abord nous allons étudier l'enveloppe supérieure pour l'infimum futur du processus $X^{(0)}$. On définit

$$\bar{F}_\nu(t) \stackrel{(\text{def})}{=} \mathbb{P}\Big(\nu I < t\Big) \qquad \text{et} \qquad \bar{F}(t) \stackrel{(\text{def})}{=} \mathbb{P}\Big(I < t\Big),$$

où ν est indépendante de I et a même loi que $x_1^{-1}\Gamma$. Notons par $\mathcal{H}_0$ la famille des fonctions $h(t)$ positives croissantes qui satisfont

i) $h(0) = 0$, et

ii) il existe $\beta \in (0,1)$ tel que $\sup_{t<\beta} \dfrac{t}{h(t)} < \infty$.

THÉORÈME 3. *Soit* $h \in \mathcal{H}_0$.

i) *Si*

$$\int_{0+} \bar{F}_\nu\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}_0\Big(J_t > (1+\epsilon)h(t), \text{ infiniment souvent lorsque } t \to 0\Big) = 0.$$

ii) *Si*

$$\int_{0+} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}_0\Big(J_t > (1-\epsilon)h(t), \text{ infiniment souvent lorsque } t \to 0\Big) = 1.$$

Notons que dans le cas où le pssMp $X^{(0)}$ est croissant, le processus $X^{(0)}$ et son infimum futur $J^{(0)}$ coïncident ainsi que ses premiers et derniers temps de passage. Alors on déduit que S_1, le premier temps de passage de $X^{(0)}$ au-dessous de 1, a la même loi que νI et que son enveloppe supérieure est décrite par le Théorème 3.
En général l'enveloppe supérieure des pssMp dépend du premier temps de passage. Dans le cas ou il n'y a pas de sauts positifs, on sait que le processus du premier temps de passage a des accroissements indépendants. Cette propriété nous permet d'obtenir le résultat suivant. Définissons

$$\tilde{F}(t) \stackrel{\text{(def)}}{=} \mathbb{P}\Big(I(\tilde{\xi}) < t\Big) \quad \text{et} \quad E \stackrel{\text{(def)}}{=} \mathbb{E}\Big(\log^+ I(\tilde{\xi})^{-1}\Big).$$

THÉORÈME 4. *Soit* $h \in \mathcal{H}_0$.

i) *Si*

$$\int_{0+} \tilde{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t), \text{ infiniment souvent lorsque } t \to 0\Big) = 0.$$

ii) *Supposons que* E *soit fini. Si*

$$\int_{0+} \tilde{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1-\epsilon)h(t), \text{ infiniment souvent lorsque } t \to 0\Big) = 1.$$

Dans le cas général ne semble pas facile de déterminer la loi du premier temps de passage en termes du processus de Lévy associé. En plus le processus du premier temps de passage n'a plus des accroissements indépendants, alors utiliser le même argument que dans les résultats précédents ne paraît pas une bonne idée. En fait le plus simple est de comparer l'enveloppe supérieure du processus $X^{(0)}$ avec l'enveloppe supérieure de son infimum futur. Tout d'abord, on définit

$$G(t) \overset{\text{(def)}}{=} \mathbb{P}\big(S_1 < t\big).$$

PROPOSITION 4. *Soit $h \in \mathcal{H}_0$.*

i) *Si*

$$\int_{0+} G\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}\Big(X_t^{(0)} > (1+\epsilon)h(t),\ \textit{infiniment souvent lorsque } t \to 0\Big) = 0.$$

ii) *Si*

$$\int_{0+} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}\Big(X_t^{(0)} < (1-\epsilon)h(t),\ \textit{infiniment souvent lorsque } t \to 0\Big) = 1.$$

Les chapitres suivants concernent les applications des test intégraux qu'on vient d'établir. Dans le chapitre 3, on étudie le cas où les queues des probabilités $F(t)$, $\bar{F}(t)$ et $\bar{F}_\nu(t)$ sont à variations régulières. Sous l' hypothèse que

$$F(t) \sim \lambda t^{-\gamma} L(t), \qquad t \to +\infty,$$

où $\lambda, \gamma > 0$ et L est une fonction à variations lentes en $+\infty$, on obtient que le test intégral pour l'enveloppe inférieure (Théroème 2) ne dépend plus de ϵ. En plus, on a que pour tout $q > 0$

$$(1 - e^{\gamma q})F(t) \le F_q(t) \le F(t), \qquad t \to +\infty,$$

ce qui implique que l'enveloppe inférieure dépend seulement du comportement de $F(t)$. Maintenant, on suppose que

$$ct^\alpha L(t) \le \bar{F}(t) \le \bar{F}_\nu(t) \le Ct^\alpha L(t), \qquad t \to 0, \tag{0.7}$$

où $\alpha > 0$, c et C sont deux constantes positives telles que $c \le C$ et $L(t)$ est une fonction à variations lentes en 0. Comme dans le cas de l'enveloppe inférieure, on obtient que le test intégral pour l'enveloppe supérieure de l'infimum futur (Théorème 3) ne dépend plus de ϵ. Par (0.7), il est clair Théorème 3 dépend seulement du comportement de $\bar{F}(t)$. Une chose très importante à remarquer est que la queue de la loi du premier temps de passage satisfait que

$$ct^\alpha L(t) \le G(t) \le Kt^\alpha L(t), \qquad t \to 0,$$

où $K \ge C$. Ansi, grâce à la Proposition 4, l'enveloppe supérieure de $X^{(0)}$ et celle de son infimum futur sont les mêmes.

Le cas où $-\log F(t)$, $-\log \bar{F}(t)$ et $-\log \bar{F}_\nu(t)$ sont à variation régulières est étudié dans le chapitre 4. En particulier, sous ce type de comportement, on obtient des lois du logarithme itéré. Le résultat obtenu pour l'enveloppe inférieure généralise le résultat de Rivero [**Rive03**] pour les pMasp croissants. On suppose que

$$-\log F(t) \sim \lambda t^\beta L(t)\,, \quad \text{quand } t \to \infty,$$

où $\lambda > 0$, $\beta > 0$ et L est une fonction à variations lente en $+\infty$. Définissons la fonction Φ par

$$\Phi(t) \stackrel{\text{(def)}}{=} \frac{t}{\inf\{s : 1/F(s) > |\log t|\}}\,, \quad t > 0\,,\ t \neq 1\,.$$

Alors l'enveloppe inférieure de $X^{(0)}$ est décrite par

(i)

$$\liminf_{t\to 0} \frac{X_t^{(0)}}{\Phi(t)} = 1\,, \quad \text{presque sûrement.}$$

(ii) Pour tout $x \geq 0$,

$$\liminf_{t\to +\infty} \frac{X_t^{(x)}}{\Phi(t)} = 1\,, \quad \text{presque sûrement.}$$

Maintentant on suppose que

$$-\log \bar{F}_\nu(1/t) \sim -\log \bar{F}(1/t) \sim \lambda t^\beta L(t), \qquad \text{quand } t \to +\infty,$$

où $\lambda, \beta > 0$ et L est une fonction à variations lente en $+\infty$. Sous cette hypothèse, on a que

$$-\log G(1/t) \sim \lambda t^\beta L(t) \qquad \text{quand} \quad t \to +\infty,$$

ce qu'implique que l'enveloppe supérieure de $X^{(0)}$ et l'enveloppe supérieure de son infimum futur satisfont la même loi de logarithme itéré mais elles ne satisfont pas nécessairement le même test intégral. Définissons la fonction

$$\bar{\Psi}(t) \stackrel{\text{(def)}}{=} t \inf\left\{s : 1/\bar{F}(1/s) > |\log t|\right\}, \quad t > 0, \quad t \neq 1,$$

alors l'enveloppe supérieure de $X^{(0)}$ et son infimum futur $J^{(0)}$ satisfont :

i)

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\bar{\Psi}(t)} = 1 \quad \text{et} \quad \limsup_{t\to 0} \frac{J_t^{(0)}}{\bar{\Psi}(t)} = 1, \quad \text{presque sûrement.}$$

ii) Pour tout $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{X_t^{(x)}}{\bar{\Psi}(t)} = 1 \quad \text{et} \quad \limsup_{t\to 0} \frac{J_t^{(x)}}{\bar{\Psi}(t)} = 1, \quad \text{presque sûrement.}$$

En plus sous l'hypothèse d'absence de saut positifs, le processus $X^{(0)}$ réflechi en son infimum futur satisfait la même loi du logarithme itéré, i.e. pour tout $x \geq 0$,

$$\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{\bar{\Psi}(t)} = 1 \quad \text{et} \quad \limsup_{t\to \infty} \frac{X_t^{(x)} - J_t^{(x)}}{\bar{\Psi}(t)} = 1, \quad \text{presque sûrement.}$$

Finalement, dans le chapitre 5, on traite le cas où $X^{(0)}$ est un processus de Bessel transient. En particulier, on obtient un nouveau test intégral pour l'enveloppe supérieure de l'infimum futur de $X^{(0)}$ qui étend le test intégral obtenu par Khoshnevisan et al. [**Khal94**].

THÉORÈME 5. *Soit $h \in \mathcal{H}_0$, alors :*

i) *Si*

$$\int_{0^+} \Big(h(t)/2t\Big)^{\frac{\delta-4}{2}} \exp\Big\{-h(t)/2t\Big\} \frac{\mathrm{d}t}{t} < \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}\Big(J_t^{(0)} > (1+\epsilon)h(t),\ \textit{infiniment souvent lorsque } t \to 0\Big) = 0.$$

ii) *Si*

$$\int_{0+} \Big(h(t)/2t\Big)^{\frac{\delta-4}{2}} \exp\Big\{-h(t)/2t\Big\}\frac{\mathrm{d}t}{t} = \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}\Big(J_t^{(0)} > (1-\epsilon)h(t), \text{ infiniment souvent lorsque } t \to 0\Big) = 1.$$

Dans le même chapitre, on obtient aussi un nouveau test intégral pour l'enveloppe supérieure de $X^{(0)}$.

THÉORÈME 6. *Soit* $h \in \mathcal{H}_0$,

i) *Si*

$$\int_{0+} \left(\frac{h(t)}{t}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{h(t)}{2t}\right\}\frac{\mathrm{d}t}{t} < \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}\Big(X_t^{(0)} > (1+\epsilon)h(t), \text{ infiniment souvent lorsque } t \to 0\Big) = 0.$$

ii) *Si*

$$\int_{0+} \left(\frac{h(t)}{t}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{h(t)}{2t}\right\}\frac{\mathrm{d}t}{t} = \infty,$$

alors pour tout $\epsilon > 0$

$$\mathbb{P}\Big(X_t^{(0)} > (1-\epsilon)h(t), \text{ infiniment souvent lorsque} t \to 0\Big) = 1.$$

Forêts de Lévy stables conditionnées.

Le but de cette deuxième partie est d'étudier des forêts de Lévy stables d'une taille donnée et conditionnées par leur masse et d'établir un principe d'invariance pour ces forêts conditionnées.

L'objet de base est l'arbre de Galton-Watson de loi de reproduction μ. Dans toute la suite, un élément u de $(\mathbb{N}^*)^n$, où $\mathbb{N}^* = \{1, 2, \ldots\}$, s'écrit $u = (u_1, \ldots u_n)$ et on fixe $|u| = n$. Soit

$$\mathbb{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

avec la convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. La concaténation de deux éléments de $\mathbb{U}$, par exemple $u = (u_1, \ldots u_n)$ et $v = (v_1, \ldots, v_m)$ est désignée par $uv = (u_1, \ldots u_n, v_1, \ldots, v_m)$. Un arbre discret enraciné est un élément τ de $\mathbb{U}$ qui satisfait:

(i) $\emptyset \in \tau$,
(ii) Si $v \in \tau$ et $v = uj$ pour $j \in \mathbb{N}^*$, alors $u \in \tau$,
(iii) Pour tout $u \in \tau$, il existe $k_u(\tau) \geq 0$, tel que $uj \in \tau$ si et seulement si $1 \leq j \leq k_u(\tau)$.

Dans cette définition, $k_u(\tau)$ représente le nombre d'enfants du sommet u. Dénotons par $\mathbb{T}$ l'ensemble des arbres discrets ordonnés enracinés. Le cardinal d'un élément $\tau \in \mathbb{T}$ sera désigné par $\zeta(\tau)$. Si $\tau \in \mathbb{T}$ et $u \in \tau$, alors on définit l'arbre discret issu de u dans τ par

$$\theta_u(\tau) = \{v \in \mathbb{U} : uv \in \tau\}.$$

Soit $u \in \tau$, on dit que u est une feuille de τ si et seulement si $k_u(\tau) = 0$.
Considérons maintenant une mesure de probabilité μ sur $\mathbb{Z}_+$, telle que

$$\sum_{k=0}^{\infty} k\mu(k) \leq 1 \quad \text{and} \quad \mu(1) < 1\,.$$

La loi d'un arbre de Galton-Watson de loi de reproduction μ est l'unique mesure de probabilité $\mathbb{Q}_\mu$ sur $\mathbb{T}$ telle que :

(i) $\mathbb{Q}_\mu(k_\emptyset(\tau) = j) = \mu(j)$, $j \in \mathbb{Z}_+$.

(ii) Pour tout $j \geq 1$, avec $\mu(j) > 0$, les arbres translatés $\theta_1(\tau), \dots, \theta_j(\tau)$ sont indépendants sous la loi $\mathbb{Q}_\mu(\,\cdot \mid k_\emptyset = j)$ et leur loi conditionnelle est $\mathbb{Q}_\mu$.

Une forêt de Galton-Watson de loi de reproduction μ est une suite finie ou infinie d'arbres indépendants de Galton-Watson de loi de reproduction μ, qu'on désignera par $\mathcal{F} = (\tau_k)$.
Il est bien connu qu'un processus de Galton-Watson associé à un arbre de Galton-Watson ne code pas entièrement sa généalogie. Par contre le *processus d'exploration*, le *processus de contour*, le *processus de hauteur* et la marche aléatoire de codage contiennent toute l'information contenue dans l'arbre ou la forêt associée.

Désignons par $u_\tau(0) = \emptyset$, $u_\tau(1) = 1, \dots,$ $u_\tau(\zeta - 1)$ les sommets d'un arbre τ qui sont ordonnés dans l'ordre lexicographique.

(1) La fonction des hauteurs de τ est définie par

$$n \mapsto H_n(\tau) = |u(n)|,\ 0 \leq n \leq \zeta(\tau) - 1\,.$$

(2) La fonction des hauteurs de la forêt $\mathcal{F} = (\tau_k)_{k\geq 1}$ est définie par

$$n \mapsto H_n(\mathcal{F}) = H_{n-(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1}))}(\tau_k),$$
$$\text{si } \ \zeta(\tau_0) + \dots + \zeta(\tau_{k-1}) \leq n \leq \zeta(\tau_0) + \dots + \zeta(\tau_k) - 1,$$

pour $k \geq 1$, et avec la convention $\zeta(\tau_0) = 0$.

Une deuxième façon de coder l'arbre de Galton-Watson est d'en dessiner le contour : imaginons que l'arbre soit injecté dans le demi-plan orienté dans les sens direct et que les arêtes de l'arbre injecté sont des segments de longueur 1. Supposons qu'une particule parcourt l'arbre de gauche à droite, en partant de la racine, à vitesse unité. Chaque arête est parcourue deux fois : une fois en montant et une fois en descendant, si bien que la particule met un temps égal à deux fois le nombre total d'arêtes de l'arbre pour revenir à la racine. Le processus de contour et le processus des hauteurs sont proches l'un de l'autre et le processus de contour peut être obtenu à partir du processus des hauteurs comme suit : posons $K_n = 2n - H_n(\tau)$, alors

$$C_t(\tau) = \begin{cases} (H_n(\tau) - (t - K_n))^+ & \text{si } t \in [K_n, K_{n+1} - 1), \\ (t - K_{n+1} + H_{n+1}(\tau))^+, & \text{si } t \in [K_{n+1} - 1, K_{n+1}), \end{cases} \tag{0.8}$$

Le processus de contour pour une forêt $\mathcal{F} = (\tau_k)$ est défini par

$$C_t(\mathcal{F}) = C_{t-2(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1}))}(\tau_k),\ \text{ if } \ 2(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1})) \leq t \leq 2(\zeta(\tau_0)+\dots+\zeta(\tau_k)).$$

Signalons qu'en général ni le processus des hauteurs ni le processus de contour n'ont une loi facile à décrire. En particulier ne sont pas des processus de Markov.

On peut également coder un arbre de Galton-Watson par un processus dont la loi peut facilement être décrite, la pluspart des auteurs l'appellent la marche associée $S(\tau)$ qui est définie comme suit :

$$S_0 = 0\,, \quad S_{n+1}(\tau) - S_n(\tau) = k_{u(n)}(\tau) - 1, \quad 0 \leq n \leq \zeta(\tau) - 1\,.$$

Ici nous l'appellerons la marche aléatoir de codage. Clairement il est possible de reconstruire τ à partir de $S(\tau)$. Pour chaque n, $S_n(\tau)$ est la somme de frères plus jeunes de chaque ancêtre $u(n)$ en incluant $u(n)$ lui-même. Pour une forêt $\mathcal{F} = (\tau_k)$, le processus $S(\mathcal{F})$ est la concaténation de $S(\tau_1), \dots, S(\tau_n), \dots$:

$$S_n(\mathcal{F}) = S_{n-(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1}))}(\tau_k) - k + 1,$$
$$\text{si } \zeta(\tau_0) + \dots + \zeta(\tau_{k-1}) \le n \le \zeta(\tau_0) + \dots + \zeta(\tau_k).$$

On remarque que les sauts de $S(\tau_1)$ sont plus grands ou égaux à -1. De plus, $S(\tau_1)_n \ge 0$ pour tout $n \in \{0, \dots, \zeta(\tau_1) - 1\}$ et $S(\tau_1)_{\zeta(\tau_1)} = -1$.
Rappelons l'égalité

$$H_n = \text{card}\left\{0 \le k \le n : S_k = \inf_{k \le j \le n} S_j\right\}$$

qui est établie dans [**DuLG02, LGLJ98**]. On peut interpréter cette égalité, pour chaque n, comme le temps que passe la marche aléatoire S en son minimum futur avant n.
Désignons par $\mathcal{F}^{k,n}$ la forêt de Galton-Watson avec k arbres conditionnée à avoir n sommets, c'est à dire la forêt avec la même loi que $\mathcal{F} = (\tau_1, \dots, \tau_k)$ sous la loi $\mathbb{Q}_\mu(\,\cdot \mid \zeta(\tau_1) + \dots + \zeta(\tau_k) = n)$. Le point de départ de cette deuxième partie est le fait que $\mathcal{F}^{k,n}$ peut être codée par une marche aléatoire conditionnée à passer en $-k$ pour la première fois au temps n. Une interprétation de ce résultat se trouve dans [**Pitm02**], Lemme 6.3.

PROPOSITION 5. *Soient $\mathcal{F} = (\tau_j)$ une forêt de Galton-Watson avec loi de reproduction μ et S et H sa marche aléatoire de codage et son processus des hauteurs. Soit W une marche aléatoire définie dans une espace de probabilité $(\Omega, \mathcal{F}, P)$ ayant la même loi que S. On définit $T_i^W = \inf\{j : W_j = -i\}$, pour $i \ge 1$. On choisit k et n tels que $P(T_k^W = n) > 0$. Alors sous la loi $\mathbb{Q}_\mu(\,\cdot \mid \zeta(\tau_1) + \dots + \zeta(\tau_k) = n)$,*

(1) *le processus $(S_j,\, 0 \le j \le \zeta(\tau_1) + \dots + \zeta(\tau_k))$ a la même loi que $(W_j,\, 0 \le j \le T_k^W)$.*

Définissons les processus $H_n^W = \text{card}\left\{k \in \{0, \dots, n-1\} : W_k = \inf_{k \le j \le n} W_j\right\}$ et C^W en utilisant H^W comme dans (0.8), alors

(2) *le processus $(H_j,\, 0 \le j \le \zeta(\tau_1) + \dots + \zeta(\tau_k))$ a même loi que $(H_j^W,\, 0 \le j \le T_k^W)$'*

(3) *le processus $(C_t,\, 0 \le 0 \le t \le 2(\zeta(\tau_1) + \dots + \zeta(\tau_k) - k))$ a même loi que $(C_t^W,\, 0 \le t \le 2(T_k^W - k))$.*

Introduisons les objets "continus" correspondant aux objets discrets évoqués précédemment. Tout d'abord les processus de Lévy sans sauts négatifs sont les analogues des marches aléatoires codant les arbres de Galton-Watson.
Soit X un processus de Lévy sans sauts négatifs dont l'exposant de Laplace ψ, défini par $\mathbb{E}(e^{-\lambda X_t}) = e^{t\psi(\lambda)}$ pour $\lambda \in \mathbb{R}_+$, satisfait la condition suivante :

$$\int_1^\infty \frac{du}{\psi(u)} < \infty\,. \tag{0.9}$$

Le processus des hauteurs, noté $\bar{H} = (\bar{H}_t, t \ge 0)$ associé à X est défini pour chaque $t \ge 0$, comme la "mesure" de l'ensemble :

$$\Big\{s \le t : X_s = \inf_{s \le r \le t} X_r\Big\}.$$

Une signification rigoureuse à cette mesure est donnée par le résultat suivant dû à Le Jan and Le Gall [**LGLJ98**] : Il existe une suite des réels positifs (ε_k) qui converge vers 0, telle que pour chaque t,

$$\bar{H}_t \overset{\text{(def)}}{=} \lim_{k\to+\infty} \frac{1}{\varepsilon_k} \int_0^t \mathbb{1}_{\{X_s - I_t^s < \varepsilon_k\}}\, ds \qquad \text{existe p.s.,}$$

où $I_t^s = \inf_{s\le u\le t} X_u$. Les auteurs dans [**LGLJ98**] ont aussi montré que sous la condition (0.9), $\bar{H}$ est un processus continu.
De la définition précédente, on peut déduire que $\bar{H}$ est une fonctionnelle du processus de Lévy réflechi en son minimum, c'est-à-dire $X - I$ où $I = I^0$. Il est bien connu que le processus réflechi $X - I$ est un processus de Markov et que $-I$ est son temps local en 0. On va désigner par N la mesure des excursions en dehors de 0.
Dans [**DuLG02**], Duquesne et Le Gall ont montré l'existence d'un processus croissant continu $(L_t^a,\, t \ge 0)$ qui est défini par:

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left(\sup_{0\le s\le t} \left| \frac{1}{\varepsilon} \int_0^s du\, \mathbb{1}_{\{a<\bar{H}_u\le a+\varepsilon\}} - L_s^a \right| \right) = 0\,.$$

Le support de la mesure dL_t^a est contenu dans $\{t \ge 0 : \bar{H}_t = a\}$ et en plus $L^0 = -I$. Ceci permet de définir un processus de Poisson ponctuel à partir des excursions du processus $\bar{H}$ en dehors de 0.
Pour définir la version "continue" des arbres de Galton-Watson, on doit d'abord introduire la notion d'arbre réel.

DÉFINITION 1. *On dit qu'un espace métrique $(\mathcal{T}, d)$ est un arbre réel si pour tous $\sigma_1, \sigma_2 \in \mathcal{T}$,*

1. *Il existe une isométrie f_{σ_1,σ_2} de $[0, d(\sigma_1,\sigma_2)]$ vers $\mathcal{T}$ telle que $f_{\sigma_1,\sigma_2}(0) = \sigma_1$ et $f_{\sigma_1,\sigma_2}(d(\sigma_1,\sigma_2)) = \sigma_2$.*
2. *Si g est une application continue injective de $[0,1]$ dans $\mathcal{T}$ telle que $g(0) = \sigma_1$ et $g(1) = \sigma_2$,*
$$g([0,1]) = f_{\sigma_1,\sigma_2}([0, d(\sigma_1,\sigma_2)])\,.$$

Un arbre réel enraciné est un arbre réel $(\mathcal{T}, d)$ avec un point distingué $\rho = \rho(\mathcal{T})$ appelé la racine. Une forêt réelle est une famille d'arbres réels enracinés $\mathcal{F} = \{(\mathcal{T}_i, d_i),\, i \in \mathcal{I}\}$.

Une construction de quelques cas particuliers de ces espaces métriques a été donnée par Aldous [**Aldo91**] et récemment par Duquesne et Le Gall [**DuLG05**] dans un cadre plus général. Soit $f : [0,\infty) \to [0,\infty)$ une fonction continu à support compact, telle que $f(0) = 0$. Pour $0 \le s \le t$, on définit

$$d_f(s,t) = f(s) + f(t) - 2 \inf_{u\in[s,t]} f(u) \tag{0.10}$$

et la relation d'équivalence

$$s \sim t \quad \text{si et seulement si} \quad d_f(s,t) = 0\,.$$

(Notons que $d_f(s,t) = 0$ si et seulement si $f(s) = f(t) = \inf_{u\in[s,t]} f(u)$). On vérifie que la projection de d_f dans l'espace quotient

$$\mathcal{T}_f = [0,\infty)/\sim$$

définit une distance qu'on note encore d_f; $(\mathcal{T}_f, d_f)$ est alors un arbre réel compact. Notons $p_f : [0,\infty) \to \mathcal{T}_f$ la projection canonique, le point $\rho = p_f(0)$ est choisi comme la racine de l'arbre $\mathcal{T}_f$.

Soit $T_s = \inf\{t : -I_t \geq s\}$ l'inverse généralisé du temps local en 0 du processus $X - I$. Posons $T_0- = 0$ et pour $u \geq 0$

$$e_u(v) = \begin{cases} \bar{H}_{T_{u-}+v}\,, & \text{if } 0 \leq v \leq T_u - T_{u-} \\ 0\,, & \text{if } v > T_u - T_{u-} \end{cases} .$$

Pour chaque $s \geq 0$, on définit l'arbre réel $(\mathcal{T}_{e_s}, d_{e_s})$ sous $\mathbb{P}$ comme dans la construction précédente et on déduit que sous la mesure de probabilité $\mathbb{P}$, les processus $(e_s, s \geq 0)$ et $\{(\mathcal{T}_{e_s}, d_{e_s}), s \geq 0\}$ sont des processus de Poisson ponctuels. La mesure caractéristique du processus $(e_s, s \geq 0)$ est la loi de $\bar{H}$ sous la mesure des excursions N tandis que la mesure caractéristique de $\{(\mathcal{T}_{e_s}, d_{e_s}), s \geq 0\}$ est la loi de l'arbre réel $(\mathcal{T}_{\bar{H}}, d_{\bar{H}})$ sous N.

DÉFINITION 2. *L'arbre de Lévy est l'arbre réel $(\mathcal{T}_{\bar{H}}, d_{\bar{H}})$ codé par la fonction $u \mapsto \bar{H}_u$ sous la mesure N. Notons $\Theta(d\mathcal{T})$ la mesure σ-finie sur $\mathbb{T}_c$, l'espace des arbres réels, qui est la loi de l'arbre de Lévy $\mathcal{T}_{\bar{H}}$ sous N. La forêt de Lévy $\mathcal{F}_{\bar{H}}$ est le processus de Poisson ponctuel*

$$(\mathcal{F}_{\bar{H}}(u),\, u \geq 0) \overset{(def)}{=} \{(\mathcal{T}_{e_u}, d_{e_u}), u \geq 0\}$$

avec mesure caractéristique $\Theta(d\mathcal{T})$ sous $\mathbb{P}$. Pour $s \geq 0$, le processus

$$\mathcal{F}^s_{\bar{H}} \overset{(def)}{=} \{(\mathcal{T}_{e_u}, d_{e_u}), 0 \leq u \leq s\} \quad \textit{sous} \quad \mathbb{P},$$

s'appellera la forêt de Lévy de taille s.

Duquesne et Le Gall [**DuLG05**] ont introduit la mesure $\ell^{a,u}$ qui représente la mesure du temps local au niveau $a \geq 0$ sur l'arbre de Lévy $\mathcal{T}_{e_u}$. Pour tout $a > 0$ et toute fonction φ continue, bornée sur $\mathcal{T}_{e_u}$ cette mesure est définie par :

$$\langle \ell^{a,u}, \varphi \rangle = \int_0^{T_u - T_{u-}} dL^a_{T_u+v}\varphi(p_{e_u}(v))\,, \tag{0.11}$$

où L^a est le temps local au niveau a de $\bar{H}$. Par conséquent, la mesure de la masse de l'arbre de Lévy $\mathcal{T}$ est donnée par

$$\mathbf{m}_{\mathcal{T}_{e_u}} = \int_0^\infty da\, l^{a,u} \tag{0.12}$$

et la masse totale de l'arbre se définit naturellement comme $\mathbf{m}_{\mathcal{T}_{e_u}}(\mathcal{T}_{e_u})$. On désigne la masse totale de l'arbre $\mathcal{T}_{e_u}$ par $\mathbf{m}_u$, i.e. $\mathbf{m}_u \overset{\text{(def)}}{=} \mathbf{m}_{\mathcal{T}_{e_u}}(\mathcal{T}_{e_u})$. La masse totale de la forêt de taille s, $\mathcal{F}^s_{\bar{H}}$ est alors

$$\mathbf{M}_s = \sum_{0 \leq u \leq s} \mathbf{m}_u\,.$$

PROPOSITION 6. *$T_s = \mathbf{M}_s$, $\mathbb{P}$-presque sûrement*

Maintenant, on peut construire les processus qui codent la généalogie de la forêt de taille s conditionnée avoir une masse egale à t. Informellement, on définit

$$X^{br} \overset{\text{(def)}}{=} [(X_u,\, 0 \leq u \leq T_s) \,|\, T_s = t]$$
$$\overline{H}^{br} \overset{\text{(def)}}{=} [(\overline{H}_u,\, 0 \leq u \leq T_s) \,|\, T_s = t].$$

Si X est le mouvement Brownien, le processus X^{br} est appelé le premier pont de passage, (voir [**BeCP03**]). Afin de donner une définition appropriée dans le cas général, nous avons besoin de l'hypothèse suivante:

Le semigroupe de $(X, \mathbb{P})$ est absolument continu par raport à la mesure de Lebesgue.

Notons par $p_t(\cdot)$ la densité du semigroupe de X et $\hat{p}_t(x) = p_t(-x)$.

LEMME 1. *La mesure de probabilité définie sur* $\mathcal{G}_t^X \overset{(def)}{=} \sigma\{X_u, u \le t\}$ *par*

$$\mathbb{P}(X^{br} \in \Lambda_u) = \mathbb{E}\left(\mathbb{I}_{\{X\in\Lambda_u, u<T_s\}} \frac{t(s+X_u)}{s(t-u)} \frac{\hat{p}_{t-u}(s+X_u)}{\hat{p}_t(s)} \right), \quad u < t \quad \Lambda_u \in \mathcal{F}_u^X .$$

est une version régulière de la loi du processus $(X_u,\, 0 \le u \le T_s)$ *en sachant* $T_s = t$, *dans le sens : pour tout* $u > 0$, *pour* $s > 0$ $\lambda-$*p.t.p. et* $t > u$,

$$\mathbb{P}(X^{br} \in \Lambda_u) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(X \in \Lambda_u \,|\, |T_s - t| < \varepsilon),$$

où λ *est la mesure de Lebesgue.*

Maintenant, nous pouvons construire le processus des hauteurs $\bar{H}^{br}$ de la trajectoire du premier pont de passage X^{br} de la même manière comme $\bar{H}$ est construit à partir de X ou dans la Définition 1.2.1 dans [**DuLG02**]. La loi du processus $\bar{H}^{br}$ est une version régulière de la loi du processus $(\bar{H}_u, o \le u \le T_s)$ en sachant $T_s = t$. Notons $(e_v^{s,t},\, 0 \le v \le s)$ le processus des excursions de $\bar{H}^{br}$, i.e.

$$(e_v^{s,t},\, 0 \le v \le s) \text{ a la même loi } (e_v,\, 0 \le v \le s) \text{ en sachant que } T_s = t\,.$$

PROPOSITION 7. *La loi du processus* $\{\mathcal{T}_{e_v^{s,t}}, d_{e_v^{s,t}}),\, 0 \le v \le s\}$ *est une version régulière de la loi de la forêt de taille* s, $\mathcal{F}_{\bar{H}}^s$ *en sachant* $\mathbf{M}_s = t$.

Notons $(\mathcal{F}_{\bar{H}}^{s,t}(u),\, 0 \le u \le s)$ pour le processus à valeurs dans $\mathbb{T}_c$ dont la loi sous $\mathbb{P}$ est celle de la forêt de Lévy de taille s conditionné par $\mathbf{M}_s = t$, i.e. conditionné a avoir une masse égale à t.
Supposons maintenant que X soit un processus de Lévy stable de indice α. La condition (0.9) est satisfaite si et seulement si $\alpha \in (1,2)$. Le processus des hauteurs correspondant $\bar{H}$ est aussi auto-similaire d'indice $\alpha/(\alpha-1)$, i.e.:

$$(\overline{H}_t,\, t \ge 0) \overset{(d)}{=} (k^{1/\alpha - 1}\overline{H}_{kt},\, t \ge 0), \quad \text{pour tout } k > 0.$$

L'arbre de Lévy $(\mathcal{T}_{\bar{H}}, d_{\bar{H}})$ associé au mécanisme stable est appelé l'arbre de Lévy stable d'indice α.
Le résultat suivant donne une construction trajectorielle du processus $(X^{br}, \overline{H}^{br})$ à partir de la trajectoire du processus $(X, \overline{H})$.

THÉORÈME 7. *Soit* $g = \sup\{u \le 1 : T_{u^{1/\alpha}} = s \cdot u\}$,

(1) $\mathbb{P}$-*presque sûrement,*

$$0 < g < 1\,.$$

(2) *Sous* $\mathbb{P}$, *le processus*

$$(g^{(1-\alpha)/\alpha}\bar{H}(gu),\, 0 \le u \le 1) \tag{0.13}$$

a la même loi que $\bar{H}^{br}$ *et en plus, il est indépendant de* g.

(3) *La forêt* $\mathcal{F}_{\bar{H}}^{s,1}$ *de taille* s *et masse* 1 *peut être construite à partir du processus* (0.13), *i.e. soit* $u \mapsto \epsilon_u \overset{(def)}{=} (g^{(1-\alpha)/\alpha}e_u(gv), v \ge 0)$ *son processus des excursions en dehors de* 0, *alors sous* $\mathbb{P}$, $\mathcal{F}_{\bar{H}}^{s,1} \overset{(d)}{=} \{(\mathcal{T}_{\epsilon_u}, d_{\epsilon_u}),\, 0 \le u \le s\}$.

D'après Lamperti [**Lamp67, Lam67b**], on sait que une suite de processus de Galton-Watson normalisés converge vers le processus de branchement à espace d'états continu. Une question assez naturelle est : quand peut-on dire que la généalogie d'un arbre ou une forêt de Galton-Watson converge? En particulier, les processus des hauteurs et de contour et la marche aléatoire de codage normalisés convergent-ils? Ces questions ont été déjà étudiées par Duquesne and Le Gall [**DuLG02**]. Maintenant, on se pose les même

questions pour les arbres ou forêt de Lévy conditionnés par leur taille et leur masse. Dans [**Duqu03**], Duquesne a montré que quand la loi ν est dans le domaine d'attraction d'une loi stable, le processus de hauteur, le processus de contour et la marche aléatoire de codage associés á un arbre conditionné par sa masse converge en loi dans l'espace de Skorohod des trajectoires càdlàg. Ce résultat généralise le résultat de Aldous [**Aldo91**] qui a étudie le cas brownien. Notre but est de montrer dans le cas stable un principe d'invariance pour la forêt de Galton-Watson conditionnée par leur taille et leur masse (le cas étudié par Duquesne [**Duqu03**] devient alors le cas particulier ou la taille est égal à 0).
On suppose d'abord que:

$$\textbf{(HA)} \quad \begin{cases} \mu \text{ est apériodique et que il existe une suite croissante } (a_n)_{n\geq 0} \\ \text{telle que } a_n \to +\infty \text{ et } S_n/a_n \text{ converge en loi quand } n \to +\infty \\ \text{vers la loi d'un v.a. non dégénérée } \theta. \end{cases}$$

Notons qu'on est dans le cas critique, i.e. $\sum_k k\mu(k) = 1$, et que la loi de θ est une loi stable. En plus, grâce à que $\nu(-\infty, -1) = 0$, le support de la mesure de Lévy de θ est $[0, \infty)$ et son indice α est tel que $1 < \alpha \leq 2$. La suite (a_n) est une suite à variation régulières d'indice α. Sous l'hypothèse (**HA**), Grimvall [**Grim74**] a montré que si Z est un processus de Galton-Watson associé à la loi de reproduction μ, alors

$$\left(\frac{1}{a_n} Z_{[nt/a_n]},\, t \geq 0\right) \Rightarrow (\overline{Z}_t,\, t \geq 0)\,, \quad \text{as } n \to +\infty,$$

où $(\overline{Z}_t,\, t \geq 0)$ est un processus de branchement à espace d'états continu. Dans la suite, $\Rightarrow$ désigne la convergence faible dans l'espace de Skohorod des trajectoires càdlàg. Sous la memê hypothèse, on a d'après Corollaire 2.5.1 dans Duquesne et Le Gall [**DuLG02**] que

$$\left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt}\right),\, t \geq 0\right] \Rightarrow \left[(X_t, \bar{H}_t, \bar{H}_t),\, t \geq 0)\right]\,, \quad \text{quand } n \to +\infty.$$

où X est le processus de Lévy stable ayant pour loi θ et $\bar{H}$ est le processus des hauteurs associé.
Fixons un réel $s > 0$ et considèrons une suite d'entiers positifs (k_n) telle que

$$\frac{k_n}{a_n} \to s\,, \quad \text{quand } n \to +\infty.$$

Pour chaque $n \geq 1$, soient $(X^{br,n}, \bar{H}^{br,n}, C^{br,n})$ les processus dont les lois sont celles des

$$\left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt}\right),\, 0 \leq t \leq 1\right],$$

sous $\mathbb{Q}_\mu(\,\cdot \mid \zeta(\tau_1) + \cdots + \zeta(\tau_{k_n}) = n)$.

THÉORÈME 8. *Quand n tend vers $+\infty$, on a*

$$(X^{br,n}, \bar{H}^{br,n}, C^{br,n}) \Longrightarrow (X^{br}, \bar{H}^{br}, \bar{H}^{br}).$$

Nous avons supposé que la taille de la forêt s est strictement positive. Ceci signifie que notre résultat ne comprend pas le cas particulier d'un arbre conditionné par sa masse étudié par Duquesne et que se traduirait par $s = 0$ dans le cas continu. Toutefois des arguments proches de ceux que nous utilisons permettraient de démontrer le résultat dans ce cas.

Part 1

Asymptotic behaviour of positive self-similar Markov processes.

Introduction.

A real self-similar Markov process $X^{(x)}$, starting from x is a càdlàg Markov process which fulfills a scaling property, i.e., there exists a constant $\alpha > 0$ such that for any constant $k > 0$,

$$\left(kX^{(x)}_{k^{-\alpha}t},\, t \geq 0\right) \stackrel{(d)}{=} \left(X^{(kx)}_t,\, t \geq 0\right). \tag{0.14}$$

For each $x \in \mathbb{R}$ we denote by $\mathbb{P}_x$ the law of the self-similar Markov process starting from the state x.
Self-similar Markov processes often arise in various parts of probability theory as limit of rescaled processes. Their properties were studied by the early sixties under the impulse of Lamperti's work [**Lamp62**]. The Markov property added to self-similarity provides some interesting features, as noted by Lamperti himself in [**Lamp72**], where the particular case of positive self-similar Markov processes is studied. These processes appear in certain domains of probability theory, for instance we mention branching processes theory, fragmentation theory and exponential functionals of Lévy processes. Here, we will consider *positive* self-similar Markov process and refer to them as *pssMp*. Some particularly well known examples are Bessel processes, stable subordinators or more generally, stable Lévy processes conditioned to stay positive.

Our aim is to describe the lower and the upper envelope at 0 and at $+\infty$ through integral tests and laws of the iterated logarithm of a large class of pssMp and some related processes, as their future infimum and the pssMp reflected at its future infimum. A crucial point in our arguments is the famous Lamperti representation of self-similar $\mathbb{R}_+$-valued Markov processes. This transformation enables us to construct the paths of any such process starting from $x > 0$, say $X^{(x)}$, from those of a Lévy process. More precisely, Lamperti [**Lamp72**] found the representation

$$X^{(x)}_t = x \exp\left\{\xi_{\tau(tx^{-\alpha})}\right\}, \quad 0 \leq t \leq x^{\alpha} I(\xi)\,, \tag{0.15}$$

under $\mathbb{P}_x$, for $x > 0$, where

$$\tau_t = \inf\left\{s : I_s(\xi) \geq t\right\}, \quad I_s(\xi) = \int_0^s \exp\left\{\alpha\xi_u\right\} \mathrm{d}u\,, \quad I(\xi) = \lim_{t\to+\infty} I_t(\xi)\,,$$

and where ξ is a real Lévy process which is possibly killed at independent exponential time. Note that for $t < I(\xi)$, we have the equality

$$\tau_t = \int_0^{x^\alpha t} \left(X^{(x)}_s\right)^{-\alpha} ds,$$

so that (0.15) is invertible and yields a one to one relation between the class of positive self-similar Markov processes up to their first hitting time of 0 and the one of Lévy processes.

In this work, we consider pssMp's which drift towards $+\infty$, i.e.

$$\lim_{t\to+\infty} X^{(x)}_t = +\infty, \qquad \text{almost surely,}$$

and which fulfills the Feller property on $[0,\infty)$, so that we may define the law of a pssMp, which we will call $X^{(0)}$, starting from 0 and with the same transition function as $X^{(x)}$, for $x > 0$. Bertoin and Caballero [**BeCa02**] and Bertoin and Yor [**BeYo02**] proved that a sufficient condition for the convergence of the family of processes $X^{(x)}$, as $x \downarrow 0$, in the sense of finite dimensional distributions towards a non degenerate process, denoted by $X^{(0)}$, is that the underlying Lévy process ξ in the Lamperti representation satisfies

$$(\mathbf{H}) \qquad \xi \quad \text{is non lattice} \quad \text{and} \quad 0 < m \overset{(\text{def})}{=} \mathbb{E}(\xi_1) \le \mathbb{E}(|\xi_1|) < +\infty\,.$$

As proved by Caballero and Chaumont in [**CaCh06**], the latter condition is also a NASC for the weak convergence of the family $(X^{(x)})$, $x \ge 0$ on the Skohorod's space of càdlàg trajectories. In the same article, the authors also provided a path construction of the process $X^{(0)}$ that we will discuss in the following chapter. The entrance law of $X^{(0)}$ has been described in [**BeCa02**] and [**BeYo02**] as follows: for every $t > 0$ and for every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathbb{E}\left(f\left(X_t^{(0)}\right)\right) = \frac{1}{m}\mathbb{E}\Big(I(-\alpha\xi)^{-1} f\big(tI(-\alpha\xi)^{-1}\big)\Big)\,. \tag{0.16}$$

where

$$I(-\alpha\xi) = \int_0^s \exp\Big\{-\alpha\xi_u\Big\}\,\mathrm{d}u.$$

From our hypothesis, it is clear that $I(-\alpha\xi) < \infty$ a.s.

Several partial results on the lower and upper envelope of $X^{(0)}$ have been established before, particularly for the case of Bessel processes, stable subordinators and stable Lévy processes conditioned to stay positive. The lower and upper envelope of Bessel processes have been studied by Dvoretsky and Erdős [**DvEr51**]. They characterized the class of lower and upper functions throughout integral tests. In a later work, Motoo [**Moto58**] gave a simple and elegant proof of these results using the diffusion equation. More precisely, when $X^{(0)}$ is a Bessel process with dimension $\delta > 2$, we have the following integral test for its lower envelope at 0: if f is an increasing positive and unbounded function as t goes to $+\infty$ then

$$\mathbb{P}\Big(X_t^{(0)} < f(t),\ \text{i.o., as } t \to 0\Big) = 0 \text{ or } 1,$$

according as,

$$\int_{0+} \left(\frac{f(t)}{t}\right)^{\frac{\delta-2}{4}} \frac{\mathrm{d}t}{t} \quad \text{is finite or infinite.}$$

The time inversion property of Bessel processes induces the same integral test for the behaviour at $+\infty$, it is enough to replace

$$\int_{0+} \left(\frac{f(t)}{t}\right)^{\frac{\delta-2}{4}} \frac{\mathrm{d}t}{t} \qquad \text{by} \qquad \int^{+\infty} \left(\frac{f(t)}{t}\right)^{\frac{\delta-2}{4}} \frac{\mathrm{d}t}{t}.$$

According to Dvoretsky and Erdős [**DvEr51**], the upper envelope of Bessel processes is as follows: let $X^{(0)}$ be a Bessel process of dimension $\delta > 2$. If f is a nondecreasing, positive and unbounded function as t goes to $+\infty$ then

$$\mathbb{P}\Big(X_t^{(0)} > f(t),\ \text{i.o., as } t \to 0\Big) = 0 \text{ or } 1,$$

according as,

$$\int_{0+} \left(\frac{f(t)}{t}\right)^{\delta} \exp\Big\{-f^2(t)/2\Big\}\frac{\mathrm{d}t}{t} \quad \text{is finite or infinite.}$$

Similarly as for the lower envelope, the inversion property of Bessel processes induces the same integral test for the behaviour at $+\infty$. This integral test is known as the Kolmogorov-Dvoretzky-Erdős integral test.

It is important to note that the upper envelope of a Bessel process of dimension $\delta > 2$ is much smoother than its lower envelope. For example, let $f(t) = t^\beta$ for $\beta > 1$, hence the lower envelope satisfies

$$\lim_{t\to 0} \frac{X_t^{(0)}}{t^\beta} = \infty, \quad \text{and} \quad \liminf_{t\to+\infty} \frac{X_t^{(0)}}{t^\beta} = 0 \qquad \text{almost surely.}$$

and when $\beta < 1$ we have

$$\liminf_{t\to 0} \frac{X_t^{(0)}}{t^\beta} = 0, \quad \text{and} \quad \lim_{t\to+\infty} \frac{X_t^{(0)}}{t^\beta} = \infty \qquad \text{almost surely.}$$

On the other hand, for the upper envelope we may obtain the following law of the iterated logarithm,

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\sqrt{2t\log\log t}} = 1, \quad \text{and} \quad \limsup_{t\to+\infty} \frac{X_t^{(0)}}{\sqrt{2t\log\log t}} = 1 \qquad \text{almost surely.}$$

Now, we turn our attention to the future infimum of the Bessel process $X^{(y)}$ at time $t \geq 0$, defined by

$$J_t^{(y)} = \inf_{s\geq t} X_s^{(y)}.$$

Note that $J^{(y)} = (J_t^{(y)}, t \geq 0)$, the future infimum process associated to the Bessel process starting from y, inherits the scaling property and when the Bessel process is transient, i.e. that it drifts towards $+\infty$, the process $J^{(y)}$ drifts towards $+\infty$ as well. From the above discussion, we deduce that the future infimum process associated to a transient Bessel process is an increasing self-similar process which drift towards $+\infty$.
The process $J^{(0)}$ has been investigated for the first time by Erdős and Taylor [**ErTa62**] who were interested in the rate of escape of a random walk (Brownian motion) in space. Okoroafor and Ugbebor [**OkUg91**] and Khoshnevisan, Lewis and Li [**Khal94**] studied independently the asymptotic behaviour of the future infimum process associated with a Bessel process. Khoshnevisan et al. [**Khal94**] also studied the upper envelope of Bessel processes reflected at their future infimum. In section 4 of [**Khal94**], the authors described the upper envelope at $+\infty$ of the future infimum process and that of Bessel processes reflected at their future infimum throughout the following laws of the iterated logarithm:

$$\limsup_{t\to+\infty} \frac{J_t^{(0)}}{\sqrt{2t\log\log t}} = 1, \qquad \text{and} \qquad \limsup_{t\to+\infty} \frac{X_t^{(0)} - J_t^{(0)}}{\sqrt{2t\log\log t}} = 1 \quad \text{almost surely.}$$

In the same work, Khoshnevisan et al. gave an integral test that describes the class of functions that are bigger than the future infimum for sufficently large times. More precisely, let $\phi(t) = \sqrt{t}\psi(t)$ be a nondecreasing function of $t > 0$ and assume that $\phi(t) \to +\infty$ as $t \to +\infty$. The condition

$$\int_1^{+\infty} \big(\phi(t)\big)^{\delta-2} \exp\Big\{ -\phi^2(t)/2\Big\} \frac{\mathrm{d}t}{t} < \infty,$$

implies that

$$\mathbb{P}\Big(J_t^{(0)} > \phi(t), \text{ i.o., as } t \to +\infty\Big) = 0.$$

Stable subordinators are increasing self-similar Markov processes with scaling index $\alpha \in (0,1)$. It is well-known that if $X^{(0)}$ is a stable subordinator, its Laplace transform is given by

$$\mathbb{E}\Big(\exp\Big\{-\lambda X^{(0)}_t\Big\}\Big) = \exp\Big\{-t\int_0^{+\infty}\big(1-e^{-\lambda x}\big)x^{-(1+\alpha)}\mathrm{d}x\Big\},$$

see for instance Bertoin [**Bert96**]. Khinchin [**Khin38**] studied for the first time, the upper envelope of stable processes, in particular if $X^{(0)}$ is a stable subordinator with index $\alpha \in (0,1)$, the upper envelope of $X^{(0)}$ is as follows: suppose that h is an increasing positive function such that the function $t \to h(t)/t$ increases as well. Then

$$\mathbb{P}\Big(X^{(0)}_t > h(t), \text{ i.o., as } t \to 0\Big) = 0 \text{ or } 1,$$

according as,

$$\int_{0+}\big(h(t)\big)^{-\alpha}\mathrm{d}t \qquad \text{is finite or infinite.}$$

The same integral test holds at $+\infty$, it is enough to replace

$$\int_{0+}\big(h(t)\big)^{-\alpha}\mathrm{d}t \qquad \text{by} \qquad \int^{+\infty}\big(h(t)\big)^{-\alpha}\mathrm{d}t.$$

Friested [**Fris64**] studied the lower envelope of stable subordinators and he found the following law of the iterated logarithm:

$$\liminf_{t\to 0}\frac{X^{(0)}_t}{t^{1/\alpha}(\log|\log t|)^{1-1/\alpha}} = \alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}, \qquad \text{almost surely.}$$

Note that the same law of the iterated logarithm is satisfied for large times.

Bertoin [**Bert95**] studied the upper envelope of Lévy processes with no positive jumps at a local minimum through a law of the iterated logarithm. Under the assumption that Lévy processes do not have positive jumps, Bertoin [**Bert95**] proved that the sample path behaviour of a Lévy process after a local minimum is the same as that of a Lévy process conditioned to stay positive at the origin. In particular, we have the following law of the iterated logarithm for stable Lévy processes conditioned to stay positive which are themselves positive self-similar Markov processes: let $X^{(0)}$ be such a process with index $\alpha \in (1,2]$. Then there exists a positive constant c such that

$$\limsup_{t\to 0}\frac{X^{(0)}_t}{t^{1/\alpha}(\log|\log t|)^{1-1/\alpha}} = c, \qquad \textit{almost surely.}$$

In [**Lamp72**] (see Theorem 7.1), Lamperti used his representation to describe the asymptotic behaviour of a pssMp starting from $x > 0$ in terms of the underlying Lévy process. More precisely, let ξ be a Lévy process. Suppose ξ admits a law of the iterated logarithm, this is for some function $g : [0,+\infty) \to [0,+\infty)$ and some constant $c \in \mathbb{R}$

$$\liminf_{t\to 0}\frac{\xi_t}{g(t)} = c \quad \text{or} \quad \limsup_{t\to 0}\frac{\xi_t}{g(t)} = c, \quad \text{almost surely.}$$

Then for $x > 0$, $X^{(x)}$ its associated positive self-similar Markov process by (0.15) satisfies

$$\liminf_{t\to 0}\frac{X^{(x)}_t - x}{g(t)} = C(x,c) \quad \text{or} \quad \limsup_{t\to 0}\frac{X^{(x)}_t - x}{g(t)} = C(x,c), \quad \text{almost surely,}$$

where $C(x,c)$ is a constant that only depends on x and c. We also cite Xiao [**Xiao98**] who studied the asymptotic behaviour of self-similar Markov processes taking values in $\mathbb{R}^d$ or $\mathbb{R}^d_+$.

The most recent result concerns increasing pssMp and is due to Rivero [**Rive03**] who gave the following law of the iterated logarithm: suppose that ξ is a subordinator who satisfies condition (**H**) (see page 22) and whose Laplace exponent ϕ is regularly varying at $+\infty$ with index $\beta \in (0,1)$. Suppose that the density ρ, of the Lévy exponential functional $I(-\xi)$ of ξ satisfies that is decreasing in a neighborhood of $+\infty$, and bounded. For $\alpha > 0$ and $x \geq 0$, let $X^{(x)}$ be the increasing positive self-similar Markov process associated to ξ with scaling index α. Define

$$f(t) = \frac{\phi(\log|\log t|)}{\log|\log t|}, \qquad t \neq e, \quad t > 0,$$

then

$$\liminf_{t\to 0} \frac{X_t^{(0)}}{(tf(t))^{1/\alpha}} = \alpha^{\beta/\alpha}(1-\beta)^{(1-\beta)/\alpha} \qquad \text{almost surely,}$$

and for any $x \geq 0$

$$\liminf_{t\to +\infty} \frac{X_t^{(x)}}{(tf(t))^{1/\alpha}} = \alpha^{\beta/\alpha}(1-\beta)^{(1-\beta)/\alpha} \qquad \text{almost surely.}$$

The examples presented above belong to the class of pssMp that drift towards $+\infty$. It is important to note that the underlying Lévy process in the Lamperti representation of a pssMp that belongs to such class satisfies condition (**H**) (see page 22). Our aim is to obtain general results on the asymptotic behaviour of such processes. With this purpose, we will begin in Chapter 1 with some path properties of pssMp which are important tools for the development of this work. In particular, we present the construction of Caballero and Chaumont [**CaCh06**], which allows us to decompose the path of pssMp $X^{(0)}$ at their first passage times, and also a path decomposition of $X^{(0)}$ at their last passage times. Finally, we give special attention to the case of absence of positive jumps on the paths of pssMp where we will present an analogous construction of $X^{(0)}$ and a time reversal property of $X^{(0)}$ at its first passage times.

Chapter 2 is devoted to our general integral tests. We first present integral test for the lower envelope of pssMp and we go further with the study of the upper envelope of its future infimum. We will note that the upper envelope of pssMp is not easy to determine in a complete form, except for the increasing case and under the assumption of absence of positive jumps. Using the upper envelope of the future infimum, we will give an integral test for the upper envelope of pssMp which will be very useful for our applications.

Chapters 3 and 4, are devoted to the applications of our main integral test to the regular and log-regular cases, respectively. In Chapter 3, we will suppose that the tail probabilities that appear in our integral test are regularly varying functions and in Chapter 4 , we will assume that the logarithm of the mentioned tail probabilities are regularly varying functions.

Finally in Chapter 5, we will present some new results on the upper envelope of the future infimum of transient Bessel processes and we also give a variant of the Kolmogorov-Dvoretsky-Erdős integral test for the upper envelope of transient Bessel processes.

CHAPTER 1

Path properties of positive self-similar Markov processes.

In this chapter, we present path properties of positive self-similar Markov processes which will be important tools for the study of their asymptotic behaviour. In particular, a path decomposition of the pssMp $X^{(0)}$ at its last passage times is established via Nagasawa's time reversal theory. Under the assumption of absence of positive jumps, we also establish a new construction of $X^{(0)}$ at its last passage times and a time reversal property of pssMp at its first passage time via Caballero and Chaumont's construction.

1. Preliminaries and Caballero and Chaumont's construction.

Let $\mathcal{D}$ be the space of Skorokhod of càdlàg paths with a probability measure $\mathbb{P}$ under which ξ will always denote a real Lévy process such that $\xi_0 = 0$. Let Π be the Lévy measure of ξ, that is the measure satisfying

$$\int_{(-\infty,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty,$$

and such that the characteristic exponent Ψ, defined by

$$\mathbb{E}\Big(\exp\big\{iu\xi_t\big\}\Big) = \exp\big\{-t\Psi(u)\big\}, \qquad t \geq 0,$$

is given, for some $b \geq 0$ and $a \in \mathbb{R}$, by

$$\Psi(u) = iau + \frac{1}{2}b^2u^2 + \int_{(-\infty,\infty)} \Big(1 - e^{iux} + iux\mathbb{I}_{\{|x|\leq 1\}}\Big)\Pi(\mathrm{d}x), \quad u \in \mathbb{R}.$$

Then according to Caballero and Chaumont [**CaCh06**],

$$(\mathbf{H}) \quad \xi \text{ is not arithmetic} \qquad \text{and} \qquad 0 < \mathbb{E}(\xi_1) \leq \mathbb{E}\big(|\xi_1|\big) < \infty$$

is a necessary and sufficient condition for the weak convergence of the family of pssMp which drifts towards $+\infty$, $(X^{(x)}, x > 0)$, as $x \downarrow 0$, towards $X^{(0)}$ on the Skorokhod space. In the sequel, we will assume that condition $(\mathbf{H})$ is always satisfied.

A crucial point on the Caballero and Chaumont construction is the overshoot of Lévy processes. The overshoot of Lévy processes is defined by $(\xi_{T_z} - z, z \geq 0)$, where T_z is the first passage time of ξ above z, i.e. $T_z = \inf\{t : \xi_t \geq z\}$. According to Doney and Maller [**DoMa02**], $(\mathbf{H})$ is a sufficient condition for the weak convergence of the overshoot $\xi_{T_z} - z$ towards the law of a finite random variable as z goes to $+\infty$.

Doney and Maller [**DoMa02**] noted that condition $(\mathbf{H})$ may be expressed in terms of the upward ladder height process σ associated with ξ (see Chap. VI in [**Bert96**] for a proper definition). In fact condition $(\mathbf{H})$ can be stated as

$$\sigma \quad \text{is not arithmetic} \quad \text{and} \quad \mathbb{E}(\sigma_1) < \infty.$$

In the sequel θ will denote the weak limit of the overshoot of ξ. This weak limit has the same law as $\mathcal{U}Z$, where $\mathcal{U}$ and Z are independent random variables, $\mathcal{U}$ is uniformly distributed over $[0,1]$ and the law of Z is given by

$$P(Z>t)=E(\sigma_1)^{-1}\int_{(t,\infty)} s\nu(\mathrm{d}s), \quad t\geq 0, \tag{1.1}$$

where ν is the Lévy measure of σ.

Let $\alpha>0$ be the scaling coefficient of the pssMp $(X,\mathbb{P}_x)$. Note that from the scaling property, the process $(X^\alpha,\mathbb{P}_x)$, $x>0$ is a pssMp whose scaling coefficient is equal to one. Moreover, the function $x\mapsto x^\alpha$ is a continuous functional of the càdlàg paths, hence we do not lose any generality in the sequel by assuming that α is equal to one.

Let (x_n) be an infinite decreasing sequence of positive real numbers which converges towards 0. According to Caballero and Chaumont, under condition (**H**), there exists a random sequence $(\theta_n,\xi^{(n)})$ of $(\mathbb{R}_+\times\mathcal{D})^{\mathbb{N}}$ such that for each n, θ_n and $\xi^{(n)}$ are independent and have the same distribution as θ and ξ, respectively. Moreover, for any i,j such that $1\leq i\leq j$:

$$\xi^{(i)} \stackrel{\text{(a.s.)}}{=} \left(\xi^{(j)}_{T^{(j)}_{\log(x_ie^{-\theta_j}/x_j)}+t}-\xi^{(j)}_{T^{(j)}_{\log(x_ie^{-\theta_j}/x_j)}}, t\geq 0\right), \tag{1.2}$$

$$\theta_i \stackrel{\text{(a.s.)}}{=} \xi^{(j)}_{T^{(j)}_{\log(x_ie^{-\theta_j}/x_j)}}-\log(x_ie^{-\theta_j}/x_j), \tag{1.3}$$

where $T^{(j)}_z=\inf\{t\geq 0:\xi^{(j)}_t\geq z\}$, for $z\in\mathbb{R}_+$. Furthermore, for any n, the Lévy process $\xi^{(n)}$ is independent of $(\theta_k,k\geq n)$ and (θ_n) is a Markov chain.
From the sequence $(\theta_n,\xi^{(n)})$ defined above, we introduce a sequence of pssMp defined by

$$X^{(\bar{x}_n)}_t=\bar{x}_n\exp\left\{\xi^{(n)}_{\tau^{(n)}(t/\bar{x}_n)}\right\}, \quad t\geq 0, \quad n\geq 1,$$

where $\bar{x}_n=x_ne^{\theta_n}$ and with the natural definition

$$\tau^{(n)}_t \stackrel{\text{(def)}}{=} \inf\left\{s\geq 0:\int_0^s\exp\left\{\xi^{(n)}_u\right\}\mathrm{d}u>t\right\}.$$

Let also

$$S^{(n-1)}=\inf\left\{t\geq 0:X^{\bar{x}_n}_t\geq x_{n-1}\right\}, \quad n\geq 2,$$

The hypothesis (**H**) ensures that $\Sigma_n=\sum_{k\geq n}S^{(k)}<\infty$, a.s., then we can construct a process, that we will denote by $X^{(0)}$, as the concatenation of the processes $X^{(\bar{x}_n)}$ on each interval $\left[0,S^{(n)}\right]$, i.e.,

$$X^{(0)}_t=\begin{cases} X^{(\bar{x}_1)}_{t-\Sigma_2} & \text{if } t\in[\Sigma_2,\infty[,\\ X^{(\bar{x}_2)}_{t-\Sigma_3} & \text{if } t\in[\Sigma_3,\Sigma_2[,\\ \vdots & \\ X^{(\bar{x}_n)}_{t-\Sigma_{n+1}} & \text{if } t\in[\Sigma_{n+1},\Sigma_n[,\\ \vdots & \end{cases}$$

Note that from the definition of the process $X^{(0)}$, we have

$$\Sigma_n=\inf\left\{t\geq 0:X^{(0)}_t\geq x_{n-1}\right\}.$$

Caballero and Chaumont proved that this construction makes sense, it does not depend on the sequence (x_n), and $X_0^{(0)} = 0$. They also showed, that $X^{(0)}$ is a càdlàg self-similar Markov process defined in $[0,\infty[$ with the same semi-group as $(X,\mathbb{P}_x)$ for $x>0$ and that the family of probability measures $(\mathbb{P}_x, x\geq 0)$ converges weakly in $\mathcal{D}$ to the law of $X^{(0)}$, as x tends to 0.

2. Time reversal and last passage time of $X^{(0)}$

Let us define the family of positive self-similar Markov processes $\hat{X}^{(x)}$ whose Lamperti's representation is given by

$$\hat{X}^{(x)} = \left(x\exp\left\{\hat{\xi}_{\hat{\tau}(t/x)}\right\},\, 0\leq t\leq xI(\hat{\xi})\right)\,,\quad x>0\,, \tag{1.4}$$

where

$$\hat{\xi} = -\xi,\qquad \hat{\tau}_t = \inf\left\{s : \int_0^s \exp\left\{\hat{\xi}_u\right\}\mathrm{d}u \geq t\right\},\quad \text{and}\quad I(\hat{\xi}) = \int_0^\infty \exp\left\{\hat{\xi}_s\right\}\mathrm{d}s.$$

We recall that $\hat{\xi}$ is well-known as the dual process of ξ which is of course a Lévy process. We emphasize that the random variable $xI(\hat{\xi})$, corresponds to the first time at which the process $\hat{X}^{(x)}$ hits 0, i.e.

$$xI(\hat{\xi}) = \inf\left\{t : \hat{X}^{(x)}_t = 0\right\}, \tag{1.5}$$

moreover, for each $x>0$, the process $\hat{X}^{(x)}$ hits 0 continuously.
We now fix a decreasing sequence (x_n) of positive real numbers which tends to 0 and we set

$$U_y = \sup\left\{t : X_t^{(0)} \leq y\right\}.$$

The aim of this section is to establish a path decomposition of the process $X^{(0)}$ reversed at time U_{x_1} in order to get a representation of this time in terms of the exponential functional $I(\hat{\xi})$, see Corollaries 2 and 3 below.

To simplify the notations, we set $\Gamma = X^{(0)}_{U_{x_1}-}$ and we will denote by K the support of the law of Γ. We will see in Lemma 1 that actually $K=[0,x_1]$. For any process X that we consider here (that is satisfying condition (**H**)), we make the convention that $X_{0-}=X_0$.

PROPOSITION 1. *Fix $x\in K$; then the law of the process $\hat{X}^{(x)}$ is a regular version of the law of the process*

$$\hat{X} \overset{(def)}{=} \left(X^{(0)}_{(U_{x_1}-t)-},\, 0\leq t\leq U_{x_1}\right),$$

conditionally on $\Gamma = x$.

Proof: The result is a consequence of Nagasawa's theory of time reversal for Markov processes. First, it follows from Lemma 2 in [**BeYo02**] that the resolvent operators of $X^{(x)}$ and $\hat{X}^{(x)}$, $x>0$ are in duality with respect to the Lebesgue measure. That is, for every measurable functions $f,g:(0,\infty)\to \mathbb{R}_+$ and $q\geq 0$, with

$$V^q f(x) \overset{(\mathrm{def})}{=} \mathbb{E}\left(\int_0^\infty e^{-qt} f(X_t^{(x)})\,\mathrm{d}t\right)\quad\text{and}\quad \hat{V}^q f(x) \overset{(\mathrm{def})}{=} \mathbb{E}\left(\int_0^\zeta e^{-qt} f(\hat{X}_t^{(x)})\,\mathrm{d}t\right),$$

we have

$$\int_0^\infty f(x)\hat{V}^q g(x)\,\mathrm{d}x = \int_0^\infty g(x) V^q f(x)\,\mathrm{d}x\,. \tag{1.6}$$

Let $p_t(\mathrm{d}x)$ be the entrance law of $X^{(0)}$ at time t, then it follows from the scaling property that for any $t > 0$, $p_t(\mathrm{d}x) = p_1(\mathrm{d}x/t)$, hence

$$\int_0^\infty p_t(\mathrm{d}x)\,\mathrm{d}t = \int_0^\infty p_1(\mathrm{d}y)/y\,\mathrm{d}x \qquad \text{for all} \quad x > 0,$$

where from (0.16),

$$\int_0^\infty p_1(\mathrm{d}y)/y\,\mathrm{d}y = m^{-1}.$$

In other words, the resolvent measure of $\delta_{\{0\}}$ is proportional to the Lebesgue measure, i.e.,

$$m^{-1}\int_0^\infty f(x)\,\mathrm{d}x = \mathbb{E}\left(\int_0^\infty f\left(X_t^{(0)}\right)\mathrm{d}t\right). \tag{1.7}$$

Conditions of Nagasawa's theorem are satisfied as shown in (1.6) and (1.7), then it remains to apply this result to U_{x_1} which is a return time such that

$$0 < U_{x_1} < \infty \qquad \mathbb{P}-\text{a.s.},$$

and the proposition is proved. ■

Another way to state Proposition 1 is to say that for any $x \in K$, the returned process $(\hat{X}_{(xI(\hat{\xi})-t)-}, 0 \le t \le xI(\hat{\xi}))$, has the same law as $(X_t^{(0)}, 0 \le t < U_{x_1})$ given $\Gamma = x$. In [**BeYo02**], the authors show that when the semigroup operator of $X^{(0)}$ is absolutely continuous with respect to the Lebesgue measure with density $p_t(x,y)$, this process is an h-process of $X^{(0)}$, the corresponding harmonic function being

$$h(x) = \int_0^\infty p_t(x,1)\,\mathrm{d}t.$$

For $y > 0$, we set

$$\hat{S}_y = \inf\left\{t : \hat{X}_t \le y\right\}.$$

COROLLARY 1. *Between the passage times $\hat{S}_{x_n}$ and $\hat{S}_{x_{n+1}}$, the process $\hat{X}$ may be described as follows:*

$$\left(\hat{X}_{\hat{S}_{x_n}+t},\ 0 \le t \le \hat{S}_{x_{n+1}} - \hat{S}_{x_n}\right) = \left(\Gamma_n \exp\left\{\hat{\xi}^{(n)}_{\hat{\tau}^{(n)}(t/\Gamma_n)}\right\},\ 0 \le t \le H_n\right), \quad n \ge 1,$$

where the processes $\hat{\xi}^{(n)}$, $n \ge 1$ are independent between themselves and have the same law as $\hat{\xi}$. Moreover the sequence $(\hat{\xi}^{(n)})$ is independent of Γ defined above and

$$\begin{aligned}
\hat{\tau}_t^{(n)} &= \inf\left\{s : \int_0^s \exp\left\{\hat{\xi}_u^{(n)}\right\}\mathrm{d}u \ge t\right\}\\
H_n &= \Gamma_n \int_0^{\hat{T}^{(n)}(\log(x_{n+1}/\Gamma_n))} \exp\left\{\hat{\xi}_s^{(n)}\right\}\mathrm{d}s\\
\Gamma_{n+1} &= \Gamma_n \exp\left\{\hat{\xi}^{(n)}_{\hat{T}^{(n)}(\log x_{n+1}/\Gamma_n)}\right\}, \quad n \ge 1, \qquad \Gamma_1 = \Gamma,\\
\hat{T}_z^{(n)} &= \inf\left\{t : \hat{\xi}_t^{(n)} \le z\right\}.
\end{aligned}$$

For each n, Γ_n is independent of $\xi^{(n)}$ and

$$x_n^{-1}\Gamma_n \overset{(d)}{=} x_1^{-1}\Gamma\,. \tag{1.8}$$

Proof: From (1.4) and Proposition 1, the process $\hat{X}$ may be described as

$$\hat{X} = \left(\Gamma \exp\left\{\hat{\xi}^{(1)}_{\hat{\tau}^{(1)}(t/\Gamma)}\right\},\, 0 \le t \le U_{x_1}\right)\,,$$

where $\hat{\xi}^{(1)} \overset{(d)}{=} \hat{\xi}$ is independent of $\Gamma = X^{(0)}_{U(x_1)-}$ and

$$\hat{\tau}^{(1)}_t = \inf\left\{s : \int_0^s \exp\left\{\hat{\xi}^{(1)}_u\right\} \mathrm{d}u \ge t\right\}.$$

Note that $\Gamma \le x_1$, a.s., so between the passages times $\hat{S}_{x_1} = 0$ and $\hat{S}_{x_2}$, the process $\hat{X}$ is clearly described as in the statement with $\hat{\xi}^{(1)} = \hat{\xi}$ and

$$\hat{S}_{x_2} - \hat{S}_{x_1} = H_1 = \Gamma \int_0^{\hat{T}^{(1)}_{\log(x_2/\Gamma)}} \exp\left\{\hat{\xi}^{(1)}_s\right\} \mathrm{d}s.$$

Now if we set

$$\hat{\xi}^{(2)} \overset{(\mathrm{def})}{=} \left(\hat{\xi}^{(1)}_{\hat{T}^{(1)}_{\log(x_2/\Gamma_1)}+t} - \hat{\xi}^{(1)}_{\hat{T}^{(1)}_{\log(x_2/\Gamma_1)}},\, t \ge 0\right),$$

then with the definitions of the statement,

$$\left(\hat{X}_{\hat{S}_{x_2}+t},\, t \ge 0\right) = \left(\Gamma_2 \exp \hat{\xi}^{(2)}_{\hat{\tau}^{(2)}(t/\Gamma_2)},\, t \ge 0\right) \quad \text{and} \tag{1.9}$$

$$\hat{S}_{x_3} - \hat{S}_{x_2} = \inf\left\{t : \hat{X}_{\hat{S}_{x_2}+t} \le x_3\right\} = H_2\,.$$

The process $\hat{\xi}^{(2)}$ is independent of $\left\{(\hat{\xi}^{(1)}_t,\, 0 \le t \le \hat{T}^{(1)}_{\log(x_2/\Gamma_1)}),\, \Gamma_1\right\}$, hence it is clear that we do not change the law of $\hat{X}$ if, by reconstructing it according to this decomposition, we replace $\hat{\xi}^{(2)}$ by a process with the same law which is independent of $[\hat{\xi}^{(1)}, \Gamma_1]$. Moreover, $\hat{\xi}^{(2)}$ is independent of Γ_2. Relation (1.8) is a consequence of the scaling property. Indeed, we have

$$\left(\frac{x_2}{x_1} X^{(0)}_{tx_1/x_2},\, 0 \le t \le \frac{x_2}{x_1} U_{x_1}\right) \overset{(d)}{=} \left(X^{(0)}_t,\, 0 \le t \le U_{x_2}\right)\,,$$

which implies the identities in law

$$x_1^{-1} X^{(0)}_{U_{x_1}-} \overset{(d)}{=} x_2^{-1} X^{(0)}_{U_{x_2}-}\,, \quad \text{and} \quad x_1^{-1} U_{x_1} \overset{(d)}{=} x_2^{-1} U_{x_2}\,. \tag{1.10}$$

On the other hand, we see from the definition of $\hat{X}$ in Proposition 1 that

$$\left(\hat{X}_{\hat{S}_{x_2}+t},\, 0 \le t \le U_{x_1} - \hat{S}_{x_2}\right) = \left(X^{(0)}_{(U_{x_2}-t)-},\, 0 \le t \le U_{x_2}\right)\,.$$

Then, we obtain (1.8) for $n = 2$ from this identity, (1.9) and (1.10). The proof follows by induction. ■

COROLLARY 2. *With the same notations as in Corollary* 1*, the time* U_{x_n} *may be decomposed into the sum*

$$U_{x_n} = \sum_{k \ge n} \Gamma_k \int_0^{\hat{T}^{(k)}_{\log(x_{k+1}/\Gamma_k)}} \exp\left\{\hat{\xi}^{(k)}_s\right\} \mathrm{d}s\,, \quad a.s. \tag{1.11}$$

In particular, for all $z_n > 0$*, we have*

$$z_n \mathbb{I}_{\{\Gamma_n \ge z_n\}} \int_0^{\hat{T}^{(n)}_{\log(x_{n+1}/z_n)}} \exp\left\{\hat{\xi}^{(n)}_s\right\} \mathrm{d}s \le U_{x_n} \le x_n I(\overline{\xi}^{(n)})\,, \quad a.s., \tag{1.12}$$

where $\overline{\xi}^{(n)}$, $n \geq 1$ *are Lévy processes with the same law as* $\hat{\xi}$.

Proof: Identity (1.11) is a consequence of Corollary 1 and the fact that

$$U_{x_n} = \sum_{k \geq n} (\hat{S}_{k+1} - \hat{S}_k).$$

The first inequality in (1.12) is a consequence of (1.11), which implies that

$$\Gamma_n \int_0^{\hat{T}^{(n)}_{\log(x_{n+1}/\Gamma_n)}} \exp\left\{\hat{\xi}^{(n)}_s\right\} \mathrm{d}s \leq U_{x_n}.$$

To prove the second inequality in (1.12), it suffices to note that by Proposition 1 and the strong Markov property at time $\hat{S}_{x_n}$, for any $n \geq 1$, we have the representation

$$\left(\hat{X}_{\hat{S}_{x_n}+t}, \, 0 \leq t \leq U_{x_1} - \hat{S}_{x_n}\right) = \left(\Gamma_n \exp\left\{\overline{\xi}^{(n)}_{\overline{\tau}^{(n)}(t/\Gamma_n)}\right\}, \, 0 \leq t \leq U_{x_1} - \hat{S}_{x_n}\right),$$

where

$$\overline{\tau}^{(n)}_t = \inf\left\{s : \int_0^s \exp\left\{\overline{\xi}^{(n)}_u\right\} \mathrm{d}u > t\right\},$$

and the process $\overline{\xi}^{(n)}$ is described as follows,

$$\overline{\xi}^{(n)}_t = \begin{cases} \hat{\xi}^{(n)}_t & \text{if } t \in [0, \Xi^{(n)}_1[, \\ \hat{\xi}^{(n+1)}_{t-\Xi^{(n)}_1} & \text{if } t \in [\Xi^{(n)}_1, \Xi^{(n)}_2[, \\ \vdots & \\ \hat{\xi}^{(n+k)}_{t-\Xi^{(n)}_k} & \text{if } t \in [\Xi^{(n)}_k, \Xi^{(n)}_{k+1}[, \\ \vdots & \end{cases} \tag{1.13}$$

where $\Xi^{(n)}_k = \sum_{j=n}^{n+k-1} \hat{T}^{(j)}$ and $\hat{T}^{(j)} = \hat{T}^{(j)}_{\log(x_{j+1}/\Gamma_j)}$.
From Corollary, we get that $1\Gamma_n = \hat{X}_{\hat{S}_{x_n}}$ is independent of $\overline{\xi}^{(n)}$ which clearly has the same law as $\hat{\xi}$. It remains to note from (1.5) that $U_{x_1} - \hat{S}_{x_n} = U_{x_n} = \Gamma_n I(\overline{\xi}^{(n)})$ and that $\Gamma_n \leq x_n$.
■

The same reasoning as we used for a sequence that tends to 0 can be applied to sequences that tends to $+\infty$ as we show in the following result.

COROLLARY 3. *Let* (y_n) *be an increasing sequence of positive real numbers which tend to* $+\infty$. *There exist some sequences* $(\check{\xi}^{(n)})$, $(\tilde{\xi}^{(n)})$ *and* $(\check{\Gamma}_n)$, *such that for each* n, $\check{\xi}^{(n)} \stackrel{(d)}{=} \tilde{\xi}^{(n)} \stackrel{(d)}{=} \hat{\xi}$, $\check{\Gamma}_n \stackrel{(d)}{=} \Gamma$, $\check{\Gamma}_n$ *and* $\check{\xi}^{(n)}$ *are independent; moreover the Lévy processes* $(\check{\xi}^{(n)})$ *are independent between themselves and we have for all* $z_n > 0$,

$$z_n \mathbb{I}_{\{\check{\Gamma}_n \geq z_n\}} \int_0^{\check{T}^{(n)}_{\log(y_{n-1}/z_n)}} \exp\left\{\check{\xi}^{(n)}_s\right\} \mathrm{d}s \leq U_{y_n} \leq y_n I(\tilde{\xi}^{(n)}), \quad a.s. \tag{1.14}$$

where $\check{T}^{(n)}_z = \inf\{t : \check{\xi}^{(n)}_t \leq z\}$.

Proof: Fix an integer $n \geq 1$ and define the decreasing sequence $x_1, \ldots, x_n$ as follows $x_n = y_1, x_{n-1} = y_2, \ldots, x_1 = y_n$, then construct the sequences $\hat{\xi}^{(1)}, \ldots, \hat{\xi}^{(n)}$ and

$\Gamma_1,\dots,\Gamma_n$ from $x_1,\dots,x_n$ as in Corollary 1 and construct the sequence $\overline{\xi}^{(1)},\dots,\overline{\xi}^{(n)}$ as in Corollary 2. Now define

$$\check{\xi}^{(1)}=\hat{\xi}^{(n)},\check{\xi}^{(2)}=\hat{\xi}^{(n-1)},\dots,\check{\xi}^{(n)}=\hat{\xi}^{(1)},$$

and

$$\tilde{\xi}^{(1)}=\overline{\xi}^{(n)},\tilde{\xi}^{(2)}=\overline{\xi}^{(n-1)},\dots,\tilde{\xi}^{(n)}=\overline{\xi}^{(1)},$$

and $\check{\Gamma}_1=\Gamma_n,\check{\Gamma}_2=\Gamma_{n-1},\dots,\check{\Gamma}_n=\Gamma_1$.
Then from (1.12), we deduce that for any $k=2,\dots,n$,

$$z_k\,\mathbb{I}_{\{\check{\Gamma}_k\ge z_k\}}\int_0^{\check{T}^{(k)}_{\log(y_{k-1}/z_k)}}\exp\left\{\check{\xi}^{(k)}_s\right\}\mathrm{d}s\le U_{y_k}\le y_k I(\tilde{\xi}^{(k)})\,,\quad a.s.$$

Hence the whole sequences $(\tilde{\xi}^{(n)})$, $(\check{\xi}^{(n)})$ and $(\check{\Gamma}_n)$ are well constructed and fulfill the desired properties. ■

Remark: We emphasize that

$$\hat{T}^{(n)}_{\log(x_{n+1}/\Gamma_n)}=0,\ \text{a.s.}\quad \text{on the event}\quad \{\Gamma_n\le x_{n+1}\}\,,$$

moreover, we have $\Gamma_n\le x_n$, a.s., so the first inequality in (1.12) is relevant only when $x_{n+1}<z_n<x_n$. Similarly, in Corollary 3, the first inequality in (1.14) is relevant only when $y_{n-1}<z_n<y_n$.

We end this section with the computation of the law of Γ. Recall that the upward ladder height process $(\sigma_t,t\ge 0)$ associated to ξ is the subordinator which corresponds to the right continuous inverse of the local time at 0 of the reflected process $(\xi_t-\sup_{s\le t}\xi_s,t\ge 0)$, see [**Bert96**] Chap. VI for a proper definition. We denote by ν the Lévy measure of $(\sigma_t,t\ge 0)$.

LEMMA 1. *The law of Γ is characterized as follows:*

$$\log(x_1^{-1}\Gamma)\overset{(d)}{=}-\mathcal{U}Z\,,$$

where $\mathcal{U}$ and Z are independent r.v.'s, $\mathcal{U}$ is uniformly distributed over $[0,1]$ and the law of Z is given by:

$$\mathbb{P}(Z>u)=\mathbb{E}(\sigma_1)^{-1}\int_{(u,\infty)}s\,\nu(\mathrm{d}s),\quad u\ge 0\,. \tag{1.15}$$

In particular, for all $\eta<x_1$, $\mathbb{P}(\Gamma>\eta)>0$.

Proof. It is proved in [**DoMa02**] that under the hypothesis (**H**), the overshoot process of ξ converges in law, that is

$$\hat{\xi}_{\hat{T}_x}-x\longrightarrow-\mathcal{U}Z,\quad\text{in law as } x \text{ tends to } -\infty,$$

and the limit law is computed in [**Chow86**] in terms of the upward ladder height process $(\sigma_t,t\ge 0)$.

On the other hand, we proved in Corollary 1, that

$$\begin{aligned}x_{n+1}^{-1}\Gamma_{n+1} &= \exp\left\{\hat{\xi}^{(n)}_{\hat{T}^{(n)}_{\log(x_{n+1}/\Gamma_n)}}-\log(x_{n+1}/\Gamma_n)\right\}\overset{(d)}{=}x_1^{-1}\Gamma\\ &\overset{(d)}{=}\exp\left\{\hat{\xi}_{\hat{T}_{\log(x_{n+1}/x_n)+\log(x_1^{-1}\Gamma)}}-\log(x_{n+1}/x_n)-\log(x_1^{-1}\Gamma)\right\}.\end{aligned}$$

Then by taking $x_n = e^{-n^2}$, we deduce from these equalities that $\log x_1^{-1}\Gamma$ has the same law as the limit overshoot of the process $\hat{\xi}$, i.e.

$$\hat{\xi}_{\hat{T}_x} - x \longrightarrow \log(x_1^{-1}\Gamma), \quad \text{in law as } x \text{ tends to } -\infty.$$

■

As a consequence of the above results we have the following identity in law:

$$U_x \overset{(d)}{=} \frac{x}{x_1} \Gamma I(\hat{\xi}) \,, \tag{1.16}$$

(Γ and $I(\hat{\xi})$ being independent) which has been proved in [**BeCa02**], Proposition 3 in the special case where the process $X^{(0)}$ is increasing.

3. Positive self-similar Markov processes with no positive jumps

Positive self-similar Markov processes with no positive jumps form a remarkable class of pssMp. Path properties of such processes can be developed in a simple and complete form. In particular, we obtain a new construction of $X^{(0)}$ using the last passage times and an interesting time reversal property at its first passage time. This last property will allow us to determine the law of the first passage time. Moreover, we will see that the first and last passage time processes are positive increasing self-similar processes with independent increments. This last remarkable property is lost in the general case.

In the rest of this section we may assume that ξ is a Lévy process with no positive jumps satisfying condition $(\mathbf{H})$ (see page 27). From the general theory of Lévy processes (see [**Bert96**] for background), we know that the exponential moments of ξ are finite and that we can obtain an explicit form for them. In particular

$$\mathbb{E}\Big(\exp\{u\xi_t\}\Big) = \exp\big\{t\psi(u)\big\}, \qquad u \geq 0,$$

where the Laplace exponent ψ satisfies

$$\psi(u) = au + \frac{1}{2}\sigma^2 u^2 + \int_{]-\infty,0)} \big(e^{ux} - 1 - ux\mathbb{I}_{\{x>-1\}}\big)\Pi(\mathrm{d}x), \quad u \geq 0.$$

It is important to note that assumption $(\mathbf{H})$ is equivalent to

$$m = \mathbb{E}(\xi_1) = \psi'(0+) \in]0, \infty[.$$

We recall the definitions of the first and last passage times of $X^{(0)}$,

$$S_y = \inf\Big\{t \geq 0 : X_t^{(0)} \geq y\Big\} \quad \text{and} \quad U_y = \sup\Big\{t \geq 0 : X_t^{(0)} \leq y\Big\},$$

for $y > 0$. Note that due to the absence of positive jumps and since the process $X^{(0)}$ drifts to $+\infty$, for all $x \geq 0$

$$S_x \quad \text{and} \quad U_x \quad \text{are finite} \quad \text{and} \quad X_{S_x}^{(0)} = X_{U_x}^{(0)} = x, \qquad \text{a.s.}$$

From the definition of S_x and U_x, we deduce that the first passage time process $S = (S_x, x \geq 0)$ and the last passage time process $U = (U_x, x \geq 0)$ are increasing self-similar processes and their scaling index is the inverse of that of $X^{(0)}$. We remark that S and U are increasing self-similar processes in general, i.e. when $X^{(0)}$ has positive jumps. From the path properties of $X^{(0)}$ we easily see that both processes start from 0 and go to $+\infty$ as x increases.

3.1. Another construction of $X^{(0)}$. The main idea of the following construction is to divide the limit process at its last passage times. In this way, if we set (x_n) a decreasing sequence of strictly positive real numbers which converges to 0, then we can define a sequence of processes $\big(X^{(0,n)}\big)$ such that for each $n \geq 1$ the process $X^{(0,n)}$ starts at x_n, never returns to its starting point and it is killed at the last passage time above x_{n-1}. Given that $X^{(0)}$ has the same semi-group as $(X, \mathbb{P}_x)$ for $x > 0$, it is then clear that for every $y \in (0, x_n]$ the concatenation of the processes $(X^{(0,k)}, k \leq n)$ has the same law as the process $X^{(y)}$ shifted at its last passage time below x_n. This last property plays an important role in this new construction.

For every $x \in \mathbb{R}_+$, let us define

$$\gamma_x = \sup\{t \geq 0, \xi_t \leq x\} \qquad \text{and} \qquad \hat{\gamma}_x = \sup\{t \geq 0, \hat{\xi}_t \geq -x\}.$$

It is clear, by the absence of positive jumps and since the process ξ derives towards $+\infty$, that

$$\gamma_x < \infty \quad \text{and} \quad \xi_{\gamma_x} = x, \qquad \mathbb{P} - \text{a.s.}$$

The following lemma is an obvious consequence of Nagasawa's theory of time reversal and the duality between ξ and $\hat{\xi}$, see for instance Prop. II.1 in [**Bert96**]. It is for that reason that we state it without a proof.

LEMMA 2. *For every $x > 0$, the law of the process $\big(x + \hat{\xi}_t, 0 \leq t \leq \hat{\gamma}_x\big)$ is the same as that of the law of the time reversed process $\big(\xi_{(\gamma_x - t)-}, 0 \leq t \leq \gamma_x\big)$. Moreover, the process $\big(\xi_{(\gamma_x - t)-}, \gamma_0 \leq t \leq \gamma_x\big)$ has the same law as that of $\big(x + \hat{\xi}_t, 0 \leq t \leq \hat{T}_x\big)$.*

The following path decomposition of the process $\xi^{\uparrow} = \big(\xi_{\gamma_0 + t}, t \geq 0\big)$ can be easily deduced from Lemma 2.

COROLLARY 4. *For every $x > 0$, the process $\big(\xi_t, \gamma_0 \leq t \leq \gamma_x\big)$ and the shifted process $\big(\xi_{\gamma_x + t} - x, t \geq 0\big)$ are independent, and the latter has the same law as that of the process $\bar{\xi}$. Moreover, both processes are independent of $\big(\xi_t, 0 \leq t \leq \gamma_0\big)$*

Proof: Fix $x > 0$ and take $y > 0$. Let us consider the dual process of ξ started at $x + y$ and killed as it enters $(-\infty, 0)$, this is $\big((x+y) + \hat{\xi}_t, 0 \leq t \leq \hat{T}_{x+y}\big)$. We can decompose this process at the first time at which it reaches the state x, this is in $\big((x + y) + \hat{\xi}_t, 0 \leq t \leq \hat{T}_y\big)$ and $\big((x + y) + \hat{\xi}_{\hat{T}_y + t}, 0 \leq t \leq \hat{T}_{x+y} - \hat{T}_y\big)$. On the other hand, it is clear that

$$\hat{T}_{x+y} - \hat{T}_y = \inf\big\{t \geq 0, \hat{\xi}_{\hat{T}_y + t} + y = -x\big\},$$

and from the Markov property and the reversed identity of Lemma 2, we have that the processes $\big(\xi_t, \gamma_0 \leq t \leq \gamma_x\big)$ and $\big(\xi_{\gamma_x + t} - x, 0 \leq t \leq \gamma_{x+y} - \gamma_x\big)$ are independent and that the latter have the same law as $\big(\xi_t, \gamma(0) \leq t \leq \gamma_y\big)$. Then we get the desired result, letting y go towards ∞.
The last statement of this corollary is consequence of a simple application of the Markov property to the process $\big((x + y) + \hat{\xi}_t, t \geq 0\big)$ at $\hat{T}_{x+y}$ and Lemma 2. ■

Let us define for $x > 0$, the pssMp $X^{(x)} = \big(X^{(x)}_t, t \geq 0\big)$ by the Lamperti's representation (0.15) and denote its last passage time below y by

$$\sigma_y = \sup\Big\{t \geq 0 : X^{(x)}_t \leq y\Big\}, \qquad \text{for any} \quad y \geq x.$$

The next proposition gives us a path decomposition of $X^{(x)}$ at the random time σ_y.

PROPOSITION 2. *For every $y \geq x > 0$, the process killed at its last passage time below the level y, $\big(X_t^{(x)}, t \leq \sigma_y\big)$, and the shifted process $\big(X_{t+\sigma_y}^{(x)}, t \geq 0\big)$ are independent; the latter has the same law as $\big(X_{t+\sigma_y}^{(y)}, t \geq 0\big)$ which will be denoted by $\mathbb{Q}_y$. Moreover, if we set $z = \log(y/x)$, we have the following path representation*

$$\Big(X_{\sigma_y+t}^{(x)}, t \geq 0\Big) = \Big(y \exp\big\{\xi'_{\tau'(t/y)}\big\}, t \geq 0\Big), \tag{1.17}$$

where $\xi' = \big(\xi_{\gamma_z+t} - z, t \geq 0\big)$ and τ' is the right continuous inverse of the exponential functional

$$I'_s = \int_0^s \exp\Big\{\xi'_u\Big\} \mathrm{d}u,$$

this is $\tau'(t) = \inf\{t \geq 0, I'_s > t\}$.

Note that when $y = x$ the processes ξ' and $\xi^{\uparrow}$ are the same. In this case we denote the exponential functional I' by $I^{\uparrow}$ and the right continuous inverse of $I^{\uparrow}$ by $\tau^{\uparrow}$.

Proof: Fix $y \geq x > 0$, and let $z = \log(y/x)$. From relation (0.15), we observe that the random time $\sigma_y = xI_{\gamma_z}$. It is then clear, that the killed process at its last passage time below y, $\big(X_t^{(x)}, 0 \leq t \leq \sigma_y\big)$, only depends on $\big(\xi_t, 0 \leq t \leq \gamma_z\big)$.
Next, again from (0.15), we have

$$\Big(X_{\sigma_y+t}^{(x)}, t \geq 0\Big) = \Big(x \exp\big\{\xi_{\tau(t/x+I_{\gamma_z})}\big\}, t \geq 0\Big).$$

From elementary calculations, we see

$$\begin{aligned}\tau\big(t/x + I_{\gamma_z}\big) &= \inf\big\{s \geq 0, I_s > t/x + I_{\gamma_z}\big\}\\ &= \gamma_z + \inf\Bigg\{s \geq 0 : \int_0^s \exp\Big\{\xi_{\gamma_z+u} - z\Big\}\mathrm{d}u > t/y\Bigg\}.\end{aligned}$$

Consequently, if we denote by $\xi' = (\xi_{\gamma_z+t} - z, t \geq 0)$, therefore

$$X_{t+\sigma_y}^{(x)} = y \exp\Big\{\xi'_{\tau'(t/y)}\Big\}, \qquad \text{for } t \geq 0,$$

where

$$\tau'(t/x) = \inf\Big\{s \geq 0 : I'_s > t/x\Big\} \quad \text{and} \quad I'_s = \int_0^s \exp\Big\{\xi'_u\Big\}\mathrm{d}u.$$

Clearly, the shifted process at its last passage time below y, $\big(X_{t+\sigma_y}^{(x)}, t \geq 0\big)$, depends only on ξ'. It is from here that the independence of the stated processes emerges. From Corollary 4 and equality (1.17), we can easily deduce that the processes $\big(X_{t+\sigma_y}^{(x)}, t \geq 0\big)$ and $\big(X_{t+\sigma_y}^{(y)}, t \geq 0\big)$ possess the same law. ■

An immediate consequence of this Proposition is the independence between the processes $\big(X_t^{(x)}, 0 \leq t \leq \sigma_x\big)$, $\big(X_{\sigma_x+t}^{(x)}, 0 \leq t \leq \sigma_y - \sigma_x\big)$ and $\big(X_{\sigma_y+t}^{(x)}, t \geq 0\big)$.

Let (x_n) be a decreasing sequence of strictly positive real numbers which converges toward 0 and $(\xi^{\uparrow(n)})$ a sequence of independent processes with the same distribution as $\xi^{\uparrow}$. We define a sequence of processes as follows,

$$Y_t^{(x_n)} = x_n \exp\Big\{\xi^{\uparrow(n)}_{\tau^{\uparrow(n)}(t/x_n)}\Big\} \quad \text{for } t \geq 0, \quad n \geq 1,$$

where for each $n \geq 1$

$$\tau^{\uparrow(n)}(t/x_n) = \inf\Big\{s \geq 0 : I_s^{\uparrow(n)} > t/x_n\Big\} \quad \text{and} \quad I_s^{\uparrow(n)} = \int_0^s \exp\Big\{\xi_u^{\uparrow(n)}\Big\}\mathrm{d}u.$$

Let also

$$\sigma^{(n)} = \sup\Big\{t \geq 0 : Y_t^{(x_n)} \leq x_{n-1}\Big\}, \qquad n \geq 2.$$

From our assumptions, the random times $\sigma^{(n)}$ are a.s. finite. We are interested in verifying when $\Sigma'_n = \sum_{k\geq n} \sigma^{(k)}$ is a.s. finite, thus to be able to define the concatenation of the processes $\Big(Y^{(x_n)}_{t-\Sigma'_{n+1}}, \Sigma'_{n+1} \leq t < \Sigma'_n\Big)$, $n \geq 2$.

LEMMA 3. *For any $n \geq 2$, we have that $0 < \Sigma'_n < \infty$ a.s.*

Proof: It is clear that if $\Sigma'_n = 0$ then for all $k \geq n$, $\sigma^{(k)} = 0$. Given that (x_n) is a decreasing sequence then we deduce that for all $k \geq n$, $x_k = x_n$, which contradicts the fact that the sequence (x_n) converges to 0.
Now, let us observe that $\sigma^{(n)} = x_n I^{\uparrow(n)}_{\gamma^{(n)}}$, where

$$\gamma^{(n)} = \sup\Big\{t \geq 0 : \xi^{\uparrow(n)} \geq \log(x_{n-1}/x_n)\Big\}.$$

Then, we can express the sum Σ'_n in the following way

$$\Sigma'_n = \sum_{k\geq n} x_k \int_0^{\gamma^{(k)}} \exp\Big\{\xi_u^{\uparrow(k)}\Big\}\mathrm{d}u.$$

From Lemma 2, we deduce that for each $n \geq 1$

$$x_n \int_0^{\gamma^{(n)}} \exp\Big\{\xi_u^{\uparrow(n)}\Big\}\mathrm{d}u \overset{(d)}{=} x_{n-1} \int_0^{\widetilde{T}^{(n)}} \exp\Big\{\zeta_u^{(n)}\Big\}\mathrm{d}u,$$

where $\big(\zeta^{(n)}\big)$ is a sequence of independent Lévy processes with same distribution as $\hat{\xi}$ and for each $n \geq 1$, $\widetilde{T}^{(n)} = \inf\big\{t \geq 0, \zeta^{(n)} \leq \log(x_n/x_{n-1})\big\}$. Hence,

$$\Sigma'_n \overset{(d)}{=} \sum_{k\geq n} x_{k-1} \int_0^{\widetilde{T}^{(k)}} \exp\Big\{\zeta_u^{(k)}\Big\}\mathrm{d}u. \tag{1.18}$$

On the other hand, we define for every $n \geq 1$,

$$\hat{T}^{(n)} = \inf\Big\{t \geq 0 : \hat{\xi} \leq \log\left(x_1/x_n\right)\Big\}$$

then by a simple application of the Markov property, we have that

$$\begin{aligned}
\int_0^\infty \exp\big\{-\xi_u\big\}\mathrm{d}u &= \sum_{k\geq 1} \int_{\hat{T}^{(k)}}^{\hat{T}^{(k+1)}} \exp\big\{-\xi_u\big\}\mathrm{d}u \\
&= \frac{1}{x_1}\sum_{k\geq 1} x_k \int_0^{\hat{T}^{(k+1)}-\hat{T}^{(k)}} \exp\Big\{-\big(\xi_{\hat{T}^{(k)}+u} - \log(x_1/x_k)\big)\Big\}\mathrm{d}u \\
&\overset{(d)}{=} \frac{1}{x_1}\sum_{k\geq 1} x_k \int_0^{\widetilde{T}^{(k+1)}} \exp\Big\{\zeta_u^{(k+1)}\Big\}\mathrm{d}u.
\end{aligned}$$

But, since ξ derives toward $+\infty$, we get that

$$\int_0^\infty \exp\big\{-\xi_u\big\}\mathrm{d}u < \infty \qquad \text{a.s.}$$

This and the equality (1.18) implies that Σ'_n is a.s. finite. ■

Now, with all these results, we are able to give the following construction. As we will see in Theorem 1, this construction is the weak limit process of $(X, \mathbb{P}_x)$, as x approaches 0.

PROPOSITION 3. *Let $\Sigma'_n = \sum_{k\geq n}\sigma^{(k)}$, then for any n, $0 < \Sigma'_n < \infty$ a.s. In addition, the following concatenation of processes*

$$Y^{(0)}_t = \begin{cases} Y^{(x_1)}_{t-\Sigma'_2} & \text{if } t \in [\Sigma'_2, \infty[, \\ Y^{(x_2)}_{t-\Sigma'_3} & \text{if } t \in [\Sigma'_3, \Sigma'_2[, \\ \vdots & \\ Y^{(x_n)}_{t-\Sigma'_{n+1}} & \text{if } t \in [\Sigma'_{n+1}, \Sigma'_n[, \\ \vdots & \end{cases}, \quad Y^{(0)}_0 = 0, \tag{1.19}$$

makes sense and it defines a càdlàg stochastic process on the real half-line $[0, \infty)$ with the following properties:

i) *The paths of the process $Y^{(0)}$ are such that $\lim_{t\to\infty} Y^{(0)}_t = +\infty$, a.s. and $Y^{(0)} > 0$, a.s. for any $t \geq 0$.*
ii) *The law of $Y^{(0)}$ does not depend on the sequence (x_n).*
iii) *The family of probability measures $(\mathbb{Q}_x, x > 0)$ converges weakly in $\mathcal{D}$ to the law of the process $Y^{(0)}$, as x approaches 0.*

Proof: From the previous lemma, we see that the definition of the process $Y^{(0)}$ makes sense. It is also clear that the process $Y^{(0)}$ is well defined on $(0, \infty)$ and that the limit of $Y^{(0)}_t$ as t goes to 0 is equal to 0. Hence $Y^{(0)}$ is a càdlàg process defined on $[0, \infty)$ which is strictly positive on the open interval $(0, \infty)$. The first part of (i) is consequence of the path properties of the sequence of processes $(Y^{(x_n)})$.
Now, let (y_k) be another decreasing sequence of strictly positive real numbers which converges toward 0 and define

$$\tilde{\Sigma}_k = \sup\Big\{t \geq 0 : Y^{(0)} \leq y_{k-1}\Big\} \quad \text{and} \quad \sigma^{(n)}(z) = \sup\Big\{t \geq 0 : Y^{(x_n)} \leq z\Big\}.$$

We also recall that $\sigma^{(n)} = \sigma^{(n)}(x_{n-1})$.
For indices m, l and k such that $l \geq m+2$ and $x_l \leq y_k \leq x_{l-1} \leq y_{k-1} \leq x_m$, we define

$$\tilde{Y}^{(y_k)}_t = \begin{cases} Y^{(x_l)}_{t+\sigma^{(l)}(y_k)} & \text{if } t \in [0, t_{l,k}[, \\ Y^{(x_{l-1})}_{t-t_{l,k}} & \text{if } t \in [t_{l,k}, t_{l,l-1,k}[, \\ \vdots & \\ Y^{(x_{m+1})}_{t-t_{l,m+2,k}} & \text{if } t \in [t_{l,m+2,k}, t_{l,m+2,k} + \sigma^{(m+1)}(y_{k-1})[, \end{cases} \tag{1.20}$$

where $t_{l,k} = \sigma^{(l)} - \sigma^{(l)}(y_k)$ and $t_{l,j,k} = t_{l,k} + \sum_{i=j}^{l-1}\sigma^{(i)}$. An application of Propositon 2, show us that the law of the process defined above is the same as the law of the shifted process $(X^{(y_k)}_{\sigma_{y_k}+t}, 0 \leq t \leq \sigma_{y_{k-1}})$.
From Propostion 2 and construction (1.19), we get that $Y^{(0)}$ may be represented as the concatenation of the processes $\Big(\tilde{Y}^{(y_k)}_{t-\tilde{\Sigma}_{k+1}}, \tilde{\Sigma}_{k+1} \leq t < \tilde{\Sigma}_k\Big)$. Note that the law of these processes does not depend on the sequence (x_n), obviously the same property is also satisfied by $Y^{(0)}$ and hence part (ii) is proved.
In order to prove part (iii), we define for each $n \geq 1$, the process $Y^{(n)} = \Big(Y^{(0)}_{\Sigma'_{n+1}+t}, t \geq 0\Big)$.

From the independence of the processes $\xi^{\uparrow(1)}, \ldots, \xi^{\uparrow(n)}$, it is then clear that the law of $Y^{(n)}$ is $\mathbb{Q}_{x_n}$. From the construction of $Y^{(n)}$, it is also obvious that $\lim_{n\to\infty} Y^{(n)} = Y^{(0)}$ a.s. on the Skorokhod space $\mathcal{D}$. Hence, we have

$$\mathbb{Q}_{x_n}(H) \xrightarrow[n\to\infty]{} \mathbb{E}\Big(H\big(Y^{(0)}\big)\Big),$$

for any bounded, continuous functional H defined on $\mathcal{D}$.
From the proof of part (ii), we note that we can obtain the above convergence to any decreasing sequence (z_k) which converge to 0. This implies that the family of probability measures $(\mathbb{Q}_x, x > 0)$ converges weakly in $\mathcal{D}$ to the law of the process $Y^{(0)}$, as x approaches 0. With this argument we complete the proof of this proposition. ■

THEOREM 1. *The processes $Y^{(0)}$ and $X^{(0)}$, defined by Caballero and Chaumont's construction, have the same distribution. Moreover, it satisfies the following conditions:*

i) *The process $Y^{(0)}$ satisfies the scaling property, that is for any $k > 0$,*

$$\big(kY^{(0)}_{k^{-1}t}, t \geq 0\big) \quad \textit{has the same law as} \quad Y^{(0)}.$$

ii) *The process $Y^{(0)}$ satisfies the strong Markov property and has the same semigroup as $(X, \mathbb{P}_x)$ for $x > 0$.*

Proof: Let (y_n) be a decreasing sequence of strictly positive real numbers which converges to 0. We choose $(\xi^{\uparrow(n)})$, a sequence of independent Lévy processes with the same distribution as $\xi^{\uparrow}$. Then, we construct a process $Y^{(0)}$, as in (1.19) and the sequence $\big(Y^{(n)}\big)$ as in the proof of the previous Proposition, i.e., for each $n \geq 1$, $Y^{(n)} = \Big(Y^{(0)}_{\Sigma'_n + t}, t \geq 0\Big)$.
We recall that

$$\lim_{n\to\infty} Y^{(n)} = Y^{(0)} \quad \text{a.s. on the space} \quad \mathcal{D}.$$

Now, we choose a sequence $\big(X^{(n)}\big)$ of pssMp which is independent of the random sequence $(\xi^{\uparrow(n)})$ and for each $n \geq 1$, the law of $X^{(n)}$ is $\mathbb{P}_{y_n}$. Let us define the last passage time of the process $X^{(n)}$ below y, by $\rho^{(n)}_y = \sup\big\{t \geq 0, X^{(n)} \leq y\big\}$. From the scaling property, we see that the law of $\rho^{(n)}_{y_n}$ under $\mathbb{P}_{y_n}$ and the law of $y_n\rho_1$ under $\mathbb{P}_1$ are the same, where $\rho_y = \sup\big\{t \geq 0, X^{(1)} \leq y\big\}$. This implies that $\rho^{(n)}_{y_n}$ converge almost surely towards 0, as n goes to ∞.
On the other hand for each $n \geq 1$, we construct a process $Z^{(n)}$ as follows

$$Z^{(n)}_t = \begin{cases} X^{(n)}_t & \text{if } t < \rho^{(n)}_{y_n}, \\ Y^{(n)}_{t-\rho^{(n)}_{y_n}} & \text{if } t \geq \rho^{(n)}_{y_n}. \end{cases}$$

By the independence between the sequences $\big(Y^{(n)}\big)$ and $(X^{(n)})$, and the fact that, for each $n \geq 1$, the process $\big(X^{(n)}_{\rho^{(n)}_{y_n}+t}, t \geq 0\big)$ has the same law as $Y^{(n)}$; it is clear that the law of $Z^{(n)}$ is $\mathbb{P}_{y_n}$.
From the previous discussions we have that $Z^{(n)}$ converges almost surely on the space $\mathcal{D}$ towards $Y^{(0)}$, as n goes to ∞. This implies that

$$\mathbb{E}_{y_n}\Big(H\big(Z^{(n)}\big)\Big) \xrightarrow[n\to\infty]{} \mathbb{E}\Big(H\big(Y^{(0)}\big)\Big),$$

for any bounded, continuous functional H defined on $\mathcal{D}$.
From Theorem 2 of [**CaCh06**], we know that the family $(\mathbb{P}_x, x > 0)$ converges weakly to the law of $X^{(0)}$. Then we conclude that the processes $X^{(0)}$ and $Y^{(0)}$ have the same

distribution. The properties (i) and (ii) above, follows from the properties of the process $X^{(0)}$. ■

3.2. Time reversal and first and last passage times of $X^{(0)}$. The aim of this section is to describe the law of the process $(X^{(0)}_{(S_x-t)-}, 0 \leq t \leq S_x)$. This allow us to obtain the law of the first passage time of $X^{(0)}$ in terms of its associated Lévy process.

Now, for every $y > 0$ let us define

$$\tilde{X}^{(y)}_t = y \exp\Big\{\tilde{\xi}_{\tilde{\tau}(t/y)}\Big\} \qquad t \geq 0,$$

where

$$\tilde{\xi} = -\xi^{\uparrow}, \quad \tilde{\tau}_t = \inf\Big\{s \geq 0 : I_s\big(\tilde{\xi}\big) > t\Big\} \quad \text{and} \quad I_s\big(\tilde{\xi}\big) = \int_0^s \exp\big\{\tilde{\xi}_u\big\}\mathrm{d}u.$$

Since ξ derives towards $+\infty$, we deduce that $\tilde{X}^{(y)}$ reaches 0 at an almost surely finite random time, denoted by $\tilde{\rho}^{(y)} = \inf\{t \geq 0, \tilde{X}^{(y)}_t = 0\}$.

PROPOSITION 4. *The law of the process time-reversed at its first passage time below x, $\big(X^{(0)}_{(S_x-t)-}, 0 \leq t \leq S_x\big)$ is the same as that of the process $(\tilde{X}^{(x)}_t, 0 \leq t \leq \tilde{\rho}^{(x)})$.*

Proof: Let us take any decreasing sequence (x_n) of positive real numbers which converges to 0 and such that $x_1 = x$.

By Corollary 4, we can divide the process $(\tilde{X}^{(x)}_t, 0 \leq t \leq \tilde{\rho})$ into the sequence

$$\Big(x_1 \exp\big\{\tilde{\xi}_{\tilde{\tau}(t/x_1)}\big\}, x_1 I_{\tilde{\gamma}^{(n)}}\big(\tilde{\xi}\big) \leq t \leq x_1 I_{\tilde{\gamma}^{(n+1)}}\big(\tilde{\xi}\big)\Big), \quad n \geq 1,$$

where $\tilde{\gamma}^{(n)} = \sup\{t \geq 0 : \tilde{\xi}_t \leq \log x_n/x\}$.
Then to prove this result, it is enough to show that, for each $n \geq 1$

$$\Big(X^{(0)}_{(S_{x_n}-t)-}, 0 \leq t \leq S_n\Big) \overset{(d)}{=} \Big(x_1 \exp\big\{\tilde{\xi}_{\tilde{\tau}(t/x_1)}\big\}, x_1 I_{\tilde{\gamma}^{(n)}}\big(\tilde{\xi}\big) \leq t \leq x_1 I_{\tilde{\gamma}^{(n+1)}}\big(\tilde{\xi}\big)\Big),$$

where $S_n = S_{x_n} - S_{x_{n+1}}$.

Fix $n \geq 1$, from the Caballero and Chaumont's construction, we know that the left-hand side of the above identity has the same law as

$$\Big(x_{n+1} \exp\Big\{\xi^{(n+1)}_{\tau^{(n+1)}\big(I_{T^{(n+1)}}\big(\xi^{(n+1)}\big)-t/x_{n+1}\big)}\Big\}, 0 \leq t \leq x_{n+1} I_{T^{(n+1)}}\big(\xi^{(n+1)}\big)\Big), \tag{1.21}$$

where $T^{(n+1)}$ is the first passage time of the process $\xi^{(n+1)}$ above $\log(x_n/x_{n+1})$.

On the other hand, by Corollary 4 we know that $(\tilde{\xi}_t, 0 \leq t \leq \tilde{\gamma}^{(n)})$ is independent of $\tilde{\xi}^{(n)} = (\log(x/x_n) + \tilde{\xi}_{\tilde{\gamma}^{(n)}+t}, t \geq 0)$ and that the latter has the same law as $\tilde{\xi}$. Since

$$\tilde{\tau}\Big(I_{\tilde{\gamma}^{(n)}}\big(\tilde{\xi}\big) + t/x\Big) = \tilde{\gamma^{(n)}} + \inf\left\{s \geq 0 : \int_0^s \exp\Big\{\tilde{\xi}^{(n)}_u\Big\}\mathrm{d}u \geq t/x_n\right\},$$

it is clear that the right-hand side of the above identity in distribution has the same law as,

$$\Big(x_n \exp\big\{\tilde{\xi}_{\tilde{\tau}(t/x_n)}\big\}, 0 \leq t \leq x_n I_{\tilde{\gamma}(\log(x_n/x_{n+1}))}\Big). \tag{1.22}$$

Therefore, it is enough to show that (1.21) and (1.22) have the same distribution.
Now, let us define the exponential functional of $\big(\xi^{(n+1)}_{(T^{(n+1)}-t)-}, 0 \leq t \leq T^{(n+1)}\big)$ as follows,

$$B^{(n+1)}_s = \int_0^s \exp\Big\{\xi^{(n+1)}_{T^{(n+1)}-u}\Big\}\mathrm{d}u \qquad \text{for } s \in [0, T^{(n+1)}],$$

and $\mathcal{H}(t) = \inf\big\{0 \le s \le T^{(n+1)}, B_s^{(n+1)} > t\big\}$, the right continuous inverse of the exponential functional $B^{(n+1)}$.
By a change of variable, it is clear that $B_s^{(n+1)} = I_{T^{(n+1)}}\big(\xi^{(n+1)}\big) - I_{T^{(n+1)}-s}\big(\xi^{(n+1)}\big)$, and if we set $t = x_{n+1}B_s^{(n+1)}$, then $s = \mathcal{H}(t/x_{n+1})$ and hence

$$\tau^{(n+1)}\Big(I_{T^{(n+1)}}\big(\xi^{(n+1)}\big) - t/x_{n+1}\Big) = \tau^{(n+1)}\Big(I_{T^{(n+1)}-s}\big(\xi^{(n+1)}\big)\Big)$$
$$= T^{(n+1)} - \mathcal{H}(t/x_{n+1}).$$

Therefore, we can rewrite (1.21) as follows

$$\left(x_{n+1}\exp\left\{\xi^{(n+1)}_{T^{(n+1)}-\mathcal{H}(t/x_{n+1})}\right\}, 0 \le t \le x_{n+1}B^{(n+1)}_{T^{(n+1)}}\right), \tag{1.23}$$

and applying Lemma 2, we get that (1.23) has the same law as that of the process defined in (1.22). ■

It is important to note that under the absence of positive jumps, we can give a similar proof to Proposition 1 using our new construction of $X^{(0)}$ and Lemma 2. An important consequence of this proposition is the following time-reversed identity. For any $y < x$,

$$\Big(X^{(0)}_{(S_x-t)-}, S_y \le t \le S_x\Big) \overset{(d)}{=} \big(\tilde{X}^{(x)}_t, 0 \le t \le \tilde{U}_y\big),$$

where $\tilde{U}_y = \sup\big\{t \ge 0, \tilde{X}^{(x)}_t \le y\big\}$.
For the next results we need to recall the notion of self-decomposable random variable. Such concept is an extension of the notion of stable distributions (see for instance Sato [**Sato99**])

DEFINITION 1. *We say that a random variable X is self-decomposable if for every $0 < c < 1$ there exists a variable Y_c which is independent of X and such that $Y_c + cX$ has the same law as X.*

COROLLARY 5. *For every $x > 0$, the first passage time S_x above x of the process $X^{(0)}$, has the same law as $xI\big(\tilde{\xi}\big)$, where*

$$I\big(\tilde{\xi}\big) = \int_0^\infty \exp\{\tilde{\xi}_u\}\mathrm{d}u = \int_{\gamma(0)}^\infty \exp\{-\xi_u\}\mathrm{d}u.$$

Moreover, S_1 is self-decomposable.

Proof: From Proposition 4, we see that the laws of S_x and $\tilde{\rho}^{(x)}$ are the same. By the Lamperti representation of $\tilde{X}^{(x)}$, we deduce that $\tilde{\rho}^{(x)} = xI\big(\tilde{\xi}\big)$ and then the identity in law follows.
Now, let $0 < c < 1$. From Corollary 4, we know that $(\xi_t, \gamma_0 \le t \le \gamma_{\log(1/c)})$ is independent of $(\xi_{\gamma_{\log(1/c)}+t} + \log c, t \ge 0)$ and that the latter has the same as the process $(\xi_{\gamma_0+t}, t \ge 0)$, then

$$I\big(\tilde{\xi}\big) = \int_{\gamma_0}^{\gamma_{\log(1/c)}} \exp\{-\xi_u\}\mathrm{d}u + c\int_0^{+\infty} \exp\big\{-\xi_{\gamma_{\log(1/c)}+u} - \log c\big\}\mathrm{d}u,$$

the self-decomposability follows. ■

To end this chapter, we establish the following proposition which give us the self-decomposable property of the last passage times.

PROPOSITION 5. *For every $x > 0$, the last passage time U_x below x of the process $X^{(0)}$, has the same law as $xI\big(\hat{\xi}\big)$. Moreover, U_1 is self-decomposable.*

Proof: The first part of this Lemma is consequence of Proposition 1. Let $0 < c < 1$. From the Markov property, we know that $(\xi_t, 0 \le t \le T_{\log(1/c)})$ is independent of the shifted process $(\xi_{T_{\log(1/c)}+t} + \log c, t \ge 0)$ and that the latter has the same distribution as ξ, then

$$I\big(\hat{\xi}\big) = \int_0^{T_{\log(1/c)}} \exp\big\{-\xi_u\big\}\mathrm{d}u + c\int_0^{+\infty} \exp\big\{-\xi_{T_{\log(1/c)}+u} - \log c\big\}\mathrm{d}u,$$

the self-decomposability follows. ■

As we mentioned at the beginning of this section, S and U are increasing self-similar processes and from Caballero and Chaumont construction and the construction presented here; we deduce that they also have independent increments and moreover they are self-decomposable processes since S_1 and U_1 are self-decomposable. These properties were studied by the first time by Getoor in [**Geto79**] for the last passage time of a Bessel process of index $\delta \ge 3$ and later by Jeanblanc, Pitman and Yor [**JePY02**] for $\delta > 2$.

From Theorem 53.1 in [**Sato99**], we deduce that the distribution of S and U are unimodal in $[0, \infty)$. We recall that an unimodal distribution in $[0, \infty)$ is absolutely continuous with respect to the Lebesgue mesure and that its density satisfies that there exists $b > 0$ such that it is increasing in $(0, b)$ and decreasing in (b, ∞).

CHAPTER 2

Integral tests for positive self-similar Markov processes.

The purpose of this chapter is to study the lower and upper envelope at 0 and at $+\infty$ of positive self-similar Markov processes and some related processes. Our main results extend the integral tests for transient Bessel processes obtained by Dvoretzky and Erdös [**DvEr51**] and the integral test for the future infimum of transient Bessel processes due to Khoshnevisan et al. [**Khal94**].

1. The lower envelope

The aim of this section is to study the lower envelope at 0 and $+\infty$ of $X^{(0)}$. When no confusion is possible, we set

$$I \overset{\text{(def)}}{=} I(\hat{\xi}) = \int_0^\infty \exp\left\{\hat{\xi}_s\right\} \mathrm{d}s.$$

The main results of this section means in particular that the asymptotic behaviour of $X^{(0)}$ only depends on the tail behaviour of the law of I, and on this of the law of

$$I_q \overset{\text{(def)}}{=} \int_0^{\hat{T}_{-q}} \exp\left\{\hat{\xi}_s\right\} \mathrm{d}s,$$

with $\hat{T}_x = \inf\{t : \hat{\xi}_t \leq x\}$, for $x \leq 0$. So also we set

$$F(t) \overset{\text{(def)}}{=} \mathbb{P}(I > t) \qquad \text{and} \qquad F_q(t) \overset{\text{(def)}}{=} \mathbb{P}(I_q > t)\,.$$

The following lemma will be used to show that actually, in many particular cases, F suffices to describe the lower envelope of $X^{(0)}$.

LEMMA 4. *Assume that there exists $\gamma > 1$ such that*

$$\limsup_{t\to+\infty} \frac{F(\gamma t)}{F(t)} < 1.$$

For any $q > 0$ and $\delta > \gamma e^{-q}$,

$$\liminf_{t\to+\infty} \frac{F_q((1-\delta)t)}{F(t)} > 0\,.$$

Proof: It follows from the decomposition of ξ into the two independent processes $(\hat{\xi}_s,\, s \leq \hat{T}_{-q})$ and $\hat{\xi}' \overset{\text{(def)}}{=} (\hat{\xi}_{s+\hat{T}_{-q}} - \hat{\xi}_{\hat{T}_{-q}},\, s \geq 0)$ that

$$I = I_q + e^{\hat{\xi}_{\hat{T}_{-q}}}\hat{I}' \leq I_q + e^{-q}\hat{I}'$$

where

$$\hat{I}' = \int_0^\infty \exp\left\{\hat{\xi}'_s\right\} \mathrm{d}s,$$

is a copy of I which is independent of I_q. Then we can write for any $q > 0$ and $\delta \in (0,1)$, the inequalities

$$\begin{aligned} \mathbb{P}(I > t) &\le \mathbb{P}(I_q + e^{-q}\hat{I}' \ge t) \\ &\le \mathbb{P}(I_q > (1-\delta)t) + \mathbb{P}(e^{-q}I > \delta t)\,, \end{aligned}$$

so that if moreover, $\delta > \gamma e^{-q}$ then

$$1 - \frac{\mathbb{P}(I > \gamma t)}{\mathbb{P}(I > t)} \le 1 - \frac{\mathbb{P}(I > e^q \delta t)}{\mathbb{P}(I > t)} \le \frac{\mathbb{P}(I_q > (1-\delta)t)}{\mathbb{P}(I > t)}\,.$$

■

We start by stating the integral test at time 0.

THEOREM 2. *The lower envelope of $X^{(0)}$ at 0 is described as follows:*
Let f be an increasing function.

(i) *If*

$$\int_{0+} F\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} < \infty\,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}\Big(X_t^{(0)} < (1-\varepsilon)f(t), \textit{i.o., as } t \to 0\Big) = 0\,.$$

(ii) *If for all $q > 0$,*

$$\int_{0+} F_q\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} = \infty\,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}\Big(X_t^{(0)} < (1+\varepsilon)f(t), \textit{i.o., as } t \to 0\Big) = 1\,.$$

(iii) *Suppose that $t \mapsto f(t)/t$ is increasing. If there exists $\gamma > 1$ such that,*

$$\limsup_{t\to+\infty} \frac{\mathbb{P}(I > \gamma t)}{\mathbb{P}(I > t)} < 1 \quad \textit{and if} \quad \int_{0+} F\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} = \infty\,,$$

then for all $\varepsilon > 0$,

$$\mathbb{P}\Big(X_t^{(0)} < (1+\varepsilon)f(t), \textit{i.o., as } t \to 0\Big) = 1\,.$$

Proof: Let (x_n) be a decreasing sequence such that $\lim_n x_n = 0$. Recall the notations of Chapter 1. We define the events

$$A_n = \Big\{\text{There exists } t \in [U_{x_{n+1}}, U_{x_n}] \text{ such that } X_t^{(0)} < f(t)\Big\}\,.$$

Since U_{x_n} tends to 0, almost surely when n goes to $+\infty$, we have:

$$\Big\{X_t^{(0)} < f(t), \text{i.o., as } t \to 0\Big\} = \limsup_n A_n\,. \tag{2.1}$$

Since f is increasing, the following inclusions hold:

$$\Big\{x_n \le f(U_{x_n})\Big\} \subset A_n \subset \Big\{x_{n+1} \le f(U_{x_n})\Big\}\,. \tag{2.2}$$

Then we prove the convergent part (i). Let us choose $x_n = r^{-n}$ for $r > 1$, and recall from relation (1.12) in Corollary 2 that $U_{r^{-n}} \le r^{-n} I(\overline{\xi}^{(n)})$. From this inequality and (2.2), we can write:

$$A_n \subset \left\{ r^{-(n+1)} \le f\big(r^{-n} I(\overline{\xi}^{(n)})\big) \right\}. \tag{2.3}$$

Let us denote $I(\hat{\xi})$ simply by I. From Borel-Cantelli Lemma, (2.3) and (2.1),

$$\text{if} \quad \sum_n \mathbb{P}\Big(r^{-(n+1)} \le f(r^{-n} I)\Big) < \infty \quad \text{then} \quad \mathbb{P}(X_t^{(0)} < f(t), \text{i.o., as } t \to 0) = 0. \tag{2.4}$$

Note that

$$\int_1^{+\infty} \mathbb{P}(r^{-t} \le f(r^{-t} I))\, \mathrm{d}t = \int_{0+}^{+\infty} \frac{\mathbb{P}(s < f(s) I,\ s < I/r)}{s \log r}\, \mathrm{d}s,$$

hence since f is increasing, we have the inequalities:

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}\Big(r^{-n} \le f(r^{-(n+1)} I)\Big) &\le \int_{0+}^{+\infty} \mathbb{P}\left(\frac{s}{f(s)} < I,\ s < \frac{I}{r}\right) \frac{\mathrm{d}s}{s \log r} \\ &\le \sum_{n=1}^{\infty} \mathbb{P}(r^{-(n+1)} \le f(r^{-n} I)). \end{aligned} \tag{2.5}$$

With no loss of generality, we can restrict ourself to the case $f(0) = 0$, so it is not difficult to check that for any $r > 1$,

$$\int_{0+} \mathbb{P}\left(\frac{s}{f(s)} < I, s < \frac{I}{r}\right) \frac{\mathrm{d}s}{s} < +\infty \quad \text{iff} \quad \int_{0+} \mathbb{P}\left(\frac{s}{f(s)} < I\right) \frac{\mathrm{d}s}{s} < +\infty. \tag{2.6}$$

Suppose the latter condition holds, then from (2.5), for all $r > 1$,

$$\sum_{n=2}^{\infty} \mathbb{P}\Big(r^{-(n+1)} \le r^{-2} f(r^{-n} I)\Big) < +\infty$$

and from (2.4), for all $r > 1$,

$$\mathbb{P}\Big(X_t^{(0)} < r^{-2} f(t), \text{i.o., as } t \to 0\Big) = 0$$

which proves the desired result.

Now we prove the divergent part (ii). Again, we choose $x_n = r^{-n}$ for $r > 1$, and $z_n = k r^{-n}$, where $k = 1 - \varepsilon + \varepsilon/r$ and $0 < \varepsilon < 1$, (so that $x_{n+1} < z_n < x_n$). We set

$$B_n = \left\{ r^{-n} \le f_{r,\varepsilon}(k r^{-n} \mathbb{I}_{\{\Gamma_n \ge k r^{-n}\}} I^{(n)}) \right\},$$

where, $f_{r,\varepsilon}(t) = r f(t/k)$ and with the same notations as in Corollary 2, for each n,

$$I^{(n)} \overset{(\text{def})}{=} \int_0^{\hat{T}^{(n)}_{\log(x_{n+1}/z_n)}} \exp\left\{\hat{\xi}_s^{(n)}\right\} \mathrm{d}s \overset{(d)}{=} \int_0^{\hat{T}_{\log(1/rk)}} \exp\left\{\hat{\xi}_s\right\} \mathrm{d}s \tag{2.7}$$

is independent of Γ_n, and Γ_n is such that $x_n^{-1} \Gamma_n \overset{(d)}{=} x_1^{-1} \Gamma$. Moreover the random variables $I^{(n)}$, $n \ge 1$ are independent between themselves and identity (2.7) shows that they have the same law as I_q defined in Lemma 4, where $q = -\log(1/rk)$. With no loss of generality, we may assume that $f(0) = 0$, so that we can write

$$B_n = \left\{ r^{-n} \le f_{r,\varepsilon}(k r^{-n} I^{(n)}),\ \Gamma_n \ge k r^{-n} \right\}$$

and from the above arguments we deduce

$$\mathbb{P}(B_n) = \mathbb{P}\Big(r^{-n} \le f_{r,\varepsilon}(kr^{-n}I_q)\Big)\mathbb{P}(\Gamma \ge kr^{-1})\,. \tag{2.8}$$

The arguments which are developed above to show (2.5) and (2.6), are also valid if we replace I by I_q. Hence from the hypothesis, since

$$\int_{0+} \mathbb{P}\Big(s < f(s)I_q\Big)\,\frac{\mathrm{d}s}{s} = +\infty,$$

then from (2.5) and (2.6) applied to I_q, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(r^{-(n+1)} \le f(r^{-n}I_q)\Big) = \sum_{n=1}^{\infty} \mathbb{P}\Big(r^{-n} \le f_{r,\varepsilon}(kr^{-n}I_q)\Big) = \infty,$$

and from (2.8) we have $\sum_n \mathbb{P}(B_n) = +\infty$. Then another application of (2.8), gives for any n and m,

$$\begin{aligned}
\mathbb{P}(B_n \cap B_m) &\le \mathbb{P}\Big(r^{-n} \le f_{r,\varepsilon}(kr^{-n}I_q)\Big)\mathbb{P}\Big(r^{-m} \le f_{r,\varepsilon}(kr^{-m}I_q)\Big)\\
\mathbb{P}(B_n \cap B_m) &\le \mathbb{P}(\Gamma \ge kr^{-1})^{-2}\mathbb{P}(B_n)\mathbb{P}(B_m)\,,
\end{aligned}$$

where $\mathbb{P}(\Gamma \ge kr^{-1}) > 0$, from (1.15). Hence from the extension of Borel-Cantelli lemma given in [**KoSt64**],

$$\mathbb{P}(\limsup B_n) \ge \mathbb{P}(\Gamma \ge kr^{-1})^2 > 0\,. \tag{2.9}$$

Then recall from Corollary 2 in Chapter 1, the inequality

$$kr^{-n}\,\mathrm{1\!I}_{\{\Gamma_n \ge kr^{-n}\}} I^{(n)} \le U_{r^{-n}}$$

which implies from (2.2) that $B_n \subset A_n$, (where in the definition of A_n we replaced f by $f_{r,\varepsilon}$). So, from (2.9), $\mathbb{P}(\limsup_n A_n) > 0$, but since $X^{(0)}$ is a Feller process and since $\limsup_n A_n$ is a tail event, we have $\mathbb{P}(\limsup_n A_n) = 1$. We deduce from the scaling property of $X^{(0)}$ and (2.1) that

$$\begin{aligned}
\mathbb{P}\Big(X^{(0)}_t \le f_{r,\varepsilon}(t), \text{i.o., as } t \to 0.\Big) &= \mathbb{P}\Big(X^{(0)}_{kt} \le rf(t), \text{i.o., as } t \to 0.\Big)\\
&= \mathbb{P}\Big(X^{(0)}_t \le k^{-1}rf(t), \text{i.o., as } t \to 0.\Big) = 1\,.
\end{aligned}$$

Since $k = 1 - \varepsilon + \varepsilon/r$, with $r > 1$ and $0 < \varepsilon < 1$ arbitrary chosen, we obtain (ii).

Now we prove the divergent part (iii). The sequences (x_n) and (z_n) are defined as in the proof of (ii) above. Recall that $q = -\log(1/rk)$ and take $\delta > \gamma e^{-q}$ as in Lemma 4. With no loss of generality, we may assume that $f(t)/t \to 0$, as $t \to 0$. Then from the hypothesis in (iii) and Lemma 4, we have

$$\int_{0+} F_q\left(\frac{(1-\delta)t}{f(t)}\right)\frac{\mathrm{d}t}{t} = \infty\,.$$

As already noticed above, this is equivalent to

$$\int_1^{+\infty} \mathbb{P}((1-\delta)r^{-t} \le f(r^{-t}I_q))\,\mathrm{d}t = \infty.$$

Since $t \mapsto f(t)/t$ increases,

$$\int_1^{+\infty} \mathbb{P}\Big((1-\delta)r^{-t} \le f(r^{-t}I_q)\Big)\,\mathrm{d}t \le \sum_1^{\infty} \mathbb{P}\Big((1-\delta)r^{-n} \le f(r^{-n}I_q)\Big) = \infty.$$

Set $f_r^{(\delta)}(t)=(1-\delta)^{-1}f(t/k)$, then

$$\sum_1^\infty \mathbb{P}\Big(r^{-n}\le f_r^{(\delta)}(kr^{-n}I_q)\Big)=\infty\,.$$

Similarly as in the proof of (ii), define

$$B_n'=\Big\{r^{-n}\le f_r^{(\delta)}(kr^{-n}I^{(n)}),\;\Gamma_n\ge kr^{-n}\Big\}.$$

Then $B_n'\subset A_n$, (where in the definition of A_n we replaced f by $f_r^{(\delta)}$). From the same arguments as above, since $\sum_n \mathbb{P}(B_n')=\infty$, we have $\mathbb{P}(\limsup_n A_n)=1$, hence from the scaling property of $X^{(0)}$ and (2.1)

$$\begin{aligned}\mathbb{P}\Big(X_t^{(0)}\le f_r^{(\delta)}(t),\text{i.o., as }t\to 0.\Big) &= \mathbb{P}\Big(X_{kt}^{(0)}\le (1-\delta)^{-1}f(t),\text{i.o., as }t\to 0.\Big)\\ &= \mathbb{P}\Big(X_t^{(0)}\le k^{-1}(1-\delta)^{-1}f(t),\text{i.o., as }t\to 0.\Big)=1\,.\end{aligned}$$

Since $k=1-\varepsilon+\varepsilon/r$, with $r>1$ and $0<\varepsilon<1$ and $\delta>\gamma e^{-q}=\gamma/(r+\varepsilon(1-r))$, by choosing r sufficiently large and ε sufficiently small, δ can be taken sufficiently small so that $k^{-1}(1-\delta)^{-1}$ is arbitrary close to 1. ■

The divergent part of the integral test at $+\infty$ requires the following Lemma.

LEMMA 5. *For any Lévy process ξ such that $0<\mathbb{E}(\xi_1)\le\mathbb{E}(|\xi_1|)<\infty$, and for any $q\ge 0$,*

$$\mathbb{E}\Big(\,\big|\inf_{t\le T_q}\xi_t\big|\,\Big)<\infty\,,$$

where, $T_q=\inf\{t:\xi_t\ge q\}$.

Proof. The proof bears upon a result on stochastic bounds for Lévy processes due to Doney [**Done04**] which we briefly recall. Let ν_n be the time at which the n-th jump of ξ whose value lies in $[-1,1]^c$, occurs and define

$$I_n=\inf_{\nu_n\le t<\nu_{n+1}}\xi_t\,.$$

Theorem 1.1 in [**Done04**] asserts that the sequence (I_n) admits the representation

$$I_n=S_n^{(-)}+\tilde{\imath}_0,\quad n\ge 0\,,$$

where $S^{(-)}$ is a random walk with the same distribution as $(\xi(\nu_n),\,n\ge 0)$ and $\tilde{\imath}_0$ is independent of $S^{(-)}$. For $a\ge 0$, let $\varsigma(a)=\min\{n:S_n^{(-)}>a\}$, then for any $q\ge 0$, we have the inequality

$$\min_{n\le\varsigma(q+|\tilde{\imath}_0|)}(S_n^{(-)}+\tilde{\imath}_0)\le\inf_{t\le T_q}\xi_t\,. \tag{2.10}$$

On the other hand, it follows from our hypothesis on ξ that

$$0<\mathbb{E}(S_1^{(-)})\le\mathbb{E}(|S_1^{(-)}|)<+\infty,$$

hence from Theorem 2 of [**Jans86**] and its proof, there exists a finite constant C which depends only on the law of $S^{(-)}$ such that for any $a\ge 0$,

$$\mathbb{E}\left(\Big|\min_{n\le\varsigma(a)}S_n^{(-)}\Big|\right)\le C\mathbb{E}(\varsigma(a))\mathbb{E}(|S_1^{(-)}|)\,. \tag{2.11}$$

Moreover from (1.5) in [**Jans86**], there are finite constants A and B depending only on the law of $S^{(-)}$ such that for any $a \geq 0$

$$\mathbb{E}(\varsigma(a)) \leq A + Ba\,. \tag{2.12}$$

Since $\tilde{\imath}_0$ is integrable (see [**Done04**]), the result follows from (2.10), (2.11), (2.12) and the independence between $\tilde{\imath}_0$ and $S^{(-)}$. ■

THEOREM 3. *The lower envelope of $X^{(x)}$ at $+\infty$ is described as follows: Let f be an increasing function.*

(i) *If*

$$\int^{+\infty} F\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} < \infty\,,$$

then for all $\varepsilon > 0$, and for all $x \geq 0$,

$$\mathbb{P}\Big(X_t^{(x)} < (1-\varepsilon)f(t), \textit{i.o., as } t \to +\infty\Big) = 0\,.$$

(ii) *If for all $q > 0$,*

$$\int^{+\infty} F_q\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} = \infty\,,$$

then for all $\varepsilon > 0$, and for all $x \geq 0$,

$$\mathbb{P}\Big(X_t^{(x)} < (1+\varepsilon)f(t), \textit{i.o., as } t \to +\infty\Big) = 1\,.$$

(iii) *Assume that there exists $\gamma > 1$ such that,*

$$\limsup_{t\to+\infty}\frac{\mathbb{P}(I > \gamma t)}{\mathbb{P}(I > t)} < 1.$$

Assume also that $t \mapsto f(t)/t$ is decreasing. If

$$\int^{+\infty} F\left(\frac{t}{f(t)}\right)\frac{\mathrm{d}t}{t} = \infty\,,$$

then for all $\varepsilon > 0$, and for all $x \geq 0$,

$$\mathbb{P}\Big(X_t^{(x)} < (1+\varepsilon)f(t), \textit{i.o., as } t \to +\infty\Big) = 1\,.$$

Proof: We first consider the case where $x = 0$. The proof is very similar to this of Theorem 2. We can follow the proofs of (i), (ii) and (iii) line by line, replacing the sequences $x_n = r^{-n}$ and $z_n = kr^{-n}$ respectively by the sequences $x_n = r^n$ and $z_n = kr^n$, and replacing Corollary 2 by Corollary 3. Then with the definition

$$A_n = \Big\{\text{There exists } t \in [U_{r^n}, U_{r^{n+1}}] \text{ such that } X_t^{(0)} < f(t)\Big\}\,,$$

we see that the event $\limsup A_n$ belongs to the tail sigma-field $\cap_t\sigma\{X_s^{(0)} : s \geq t\}$ which is trivial from the representation (0.15) and the Markov property.

The only thing which has to be checked more carefully is the counterpart at $+\infty$ of the equivalence (2.6). Indeed, since in that case

$$\int_1^{\infty}\mathbb{P}\Big(rt < f(r^tI)\Big)\,\mathrm{d}t = \int_{0+}^{\infty}\mathbb{P}\left(\frac{s}{f(s)} < I_q,\, s > rI_q\right)\frac{\mathrm{d}s}{s\log r},$$

in the proof of (ii) and (iii), we need to make sure that for any $r > 1$,

(2.13)
$$\int^{+\infty} \mathbb{P}\left(\frac{s}{f(s)} < I_q\right) \frac{\mathrm{d}s}{s} = +\infty \quad \text{implies} \quad \int^{+\infty} \mathbb{P}\left(\frac{s}{f(s)} < I_q < sr\right) \frac{\mathrm{d}s}{s} = +\infty\,.$$

To this aim, note that

$$\int_1^{\infty} \mathbb{P}\left(\frac{s}{f(s)} < I_q < sr\right) \frac{\mathrm{d}s}{s} = \int_1^{\infty} \mathbb{P}\left(\frac{s}{f(s)} < I_q\right) - \mathbb{P}\left(\frac{s}{f(s)} < I_q,\, sr < I_q\right) \frac{\mathrm{d}s}{s}\,,$$

and since f is increasing, we have

$$\int_1^{\infty} \mathbb{P}\left(\frac{s}{f(s)} < I_q,\, sr < I_q\right) \frac{\mathrm{d}s}{s} < +\infty \quad \text{if and only if} \quad \int_1^{\infty} \mathbb{P}\left(s < I_q\right) \frac{\mathrm{d}s}{s} < +\infty\,.$$

But

$$\int_1^{\infty} \mathbb{P}\left(s < I_q\right) \frac{\mathrm{d}s}{s} = \mathbb{E}(\log^+ I_q)\,.$$

Note that from our hypothesis on ξ, we have $\mathbb{E}(\hat{T}_{-q}) < +\infty$, then the conclusion follows from the inequality

$$\mathbb{E}(\log^+ I_q) \le \mathbb{E}\left(\sup_{0\le s\le \hat{T}_{-q}} \hat{\xi}_s\right) + \mathbb{E}(\hat{T}_{-q})$$

and Lemma 5. This achieves the proof of the theorem for $x = 0$.

Now we prove (i) for any $x > 0$. Let f be an increasing function such that

$$\int^{+\infty} F\left(\frac{t}{f(t)}\right) \frac{\mathrm{d}t}{t} < +\infty.$$

Let $x > 0$, put $S_x = \inf\{t : X_t^{(0)} \ge x\}$ and denote by μ_x the law of $X_{S_x}^{(0)}$. From the Markov property at time S_x, we have for all $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\Big(X_t^{(0)} < (1-\varepsilon)f(t-S_x), \text{i.o., as } t \to +\infty\Big) \\
&= \int_{[x,\infty)} \mathbb{P}\Big(X_t^{(y)} < (1-\varepsilon)f(t), \text{i.o., as } t \to +\infty\Big)\, \mu_x(\mathrm{d}y) \\
&\le \mathbb{P}\Big(X_t^{(0)} < (1-\varepsilon)f(t), \text{i.o., as } t \to +\infty\Big) = 0\,. \qquad (2.14)
\end{aligned}$$

If x is an atom of μ_x, then the inequality (2.14) shows that

$$\mathbb{P}\Big(X_t^{(x)} < (1-\varepsilon)f(t), \text{i.o., as } t \to +\infty\Big) = 0$$

and the result is proved. Suppose that x is not an atom of μ_x. Recall from Lemma 1 that $\log(x_1^{-1}\Gamma)$ is the limit in law of the overshoot process $\hat{\xi}_{\hat{T}_z} - z$, as $z \to +\infty$. Moreover, it follows from **[CaCh06]**, Theorem 1 that $X_{S_x}^{(0)} \overset{(d)}{=} \frac{xx_1}{\Gamma}$. Hence, again from Lemma 1, we have for any $\eta > 0$, $\mu_x(x, x+\eta) > 0$. Then, the inequality (2.14) implies that for any $\eta > 0$, there exists $y \in (x, x+\eta)$ such that

$$\mathbb{P}\Big(X_t^{(y)} < (1-\varepsilon)f(t), \text{i.o., as } t \to +\infty\Big) = 0,$$

for all $\varepsilon > 0$. It allows us to conclude.

Parts (ii) and (iii) can be proved through the same way. ∎

Recall that we are assuming that the scaling index $\alpha = 1$. In order to obtain these above integral tests for pssMp with any scaling index $\alpha > 0$, it is enough to consider the process

$(X^{(0)})^{1/\alpha}$ *in the above theorems. The same remark holds for the results of the next sections.*

Now, we introduce $J^{(x)} = (J_t^{(x)}, t \geq 0)$, the future infimum process of $X^{(x)}$, defined by

$$J_t^{(x)} \stackrel{\text{(def)}}{=} \inf_{s \geq t} X_s^{(x)}, \qquad \text{for} \quad t \geq 0.$$

Note that the future infimum process $J^{(x)}$, is an increasing self-similar process with the same scaling coefficient as $X^{(x)}$. It is clear that when the pssMp $X^{(x)}$ starts from $x = 0$, the process $J^{(0)}$ starts also from 0. When the pssMp $X^{(x)}$ starts from $x > 0$, the future infimum $J^{(x)}$ starts from the global infimum, that is from $\inf_{t \geq 0} X_t^{(x)}$. In both cases, the future infimum process $J^{(x)}$ tends to $+\infty$ as t increases.

The lower envelope of $X^{(x)}$ are based on the study of its last passage times. Since the future infimum process $J^{(x)}$ can be seen as the right inverse of the last passage times of $X^{(x)}$, it is not difficult to deduce that we can replace $X^{(x)}$ by its future infimum in all the above results. In other words, we will obtain the same integral tests for the lower envelope of $J^{(x)}$ at 0 (when $x = 0$) and at $+\infty$ (for all $x \geq 0$), which means that the process $X^{(0)}$ and its future infimum have the same lower functions.

2. The lower envelope of the last passage time.

In Chapter 1, we mentioned that $U = (U_x, x \geq 0)$ is an increasing self-similar process whose scaling coefficient is inversely proportional to the scaling coefficient of $X^{(0)}$. Moreover, since $X^{(0)}$ starts at 0 and drifts towards $+\infty$, we deduce that U also starts at 0 and tends to infinity as x increases.

Here, we are interested in the lower envelope of the last passage time process U at 0 and at $+\infty$. As we will see later, the lower envelope of U is related to the upper envelope of the future infimum of $X^{(0)}$.

The following result will give us integral test at 0 for the lower envelope of U. With the same notation as in the precedent section, we define

$$\bar{F}_\nu(t) \stackrel{\text{(def)}}{=} \mathbb{P}\Big(\nu I < t\Big) \qquad \text{and} \qquad \bar{F}(t) \stackrel{\text{(def)}}{=} \mathbb{P}\Big(I < t\Big),$$

where ν is independent of I and has the same law as $x_1^{-1}\Gamma$. Note that the support of the distribution of ν is the interval $[0, 1]$.

Let us denote by $\mathcal{H}_0^{-1}$, the totality of positive increasing functions $h(x)$ on $(0, \infty)$ that satisfy

i) $h(0) = 0$, and

ii) there exists $\beta \in (0, 1)$ such that $\displaystyle\sup_{x<\beta} \frac{h(x)}{x} < \infty$.

THEOREM 4. *Let* $h \in \mathcal{H}_0^{-1}$.

i) *If*

$$\int_{0+} \bar{F}_\nu \left(\frac{h(x)}{x} \right) \frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1 - \epsilon)h(x), \text{ i.o., as } x \to 0\Big) = 0.$$

ii) *If*

$$\int_{0+} \bar{F} \left(\frac{h(x)}{x} \right) \frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1+\epsilon)h(x), \textit{ i.o., as } x \to 0\Big) = 1.$$

Proof: We first prove the convergent part. Let (x_n) be a decreasing sequence of positive numbers which converges to 0 and let us define the events

$$A_n = \Big\{U_{x_{n+1}} < h(x_n)\Big\}.$$

Now, we choose $x_n = r^n$, for $r < 1$. From the first Borel-Cantelli Lemma, if we have that $\sum_n \mathbb{P}(A_n) < \infty$, it follows

$$U_{r^{n+1}} \geq h\big(r^n\big) \qquad \mathbb{P}-\text{ a.s.},$$

for all large n. Since the function h and the process U are increasing, we have

$$U_x \geq h(x) \qquad \text{for} \quad r^{n+1} \leq x \leq r^n.$$

From the identity in law (1.16), we get the following inequality

$$\sum_n \mathbb{P}\Big(U_{r^n} < h\big(r^{n+1}\big)\Big) \leq \int_1^\infty \mathbb{P}\Big(r^t \nu I < h\big(r^t\big)\Big)\mathrm{d}t$$
$$= -\frac{1}{\log r}\int_0^r \bar{F}_\nu\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x}.$$

From our hypothesis, this last integral is finite. Then from the above discussion, there exist x_0 such that for every $x \leq x_0$

$$U_x \geq r^2 h(x), \qquad \text{for all} \quad r < 1.$$

Clearly, this implies that

$$\mathbb{P}\Big(U_x < r^2 h(x), \text{ i.o., as } x \to 0\Big) = 0,$$

which proves part (i).

Now we prove the divergent part. First, we assume that h satisfies

$$\int_{0+} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \infty.$$

Let us take, again $x_n = r^n$ for $r < 1$, and define the events

$$C_n = \Big\{U_x < r^{-2}h(x), \text{ for some } x \in (0, r^n)\Big\}.$$

Note that the family (C_n) is decreasing, then

$$C = \bigcap_{n\geq 1} C_n = \Big\{U_x < r^{-2}h(x), \text{ i.o., as } x \to 0\Big\}.$$

If we prove that $\lim \mathbb{P}(C_n) > 0$, then since $X^{(0)}$ is a Feller process and by Blumenthal's 0-1 law we will have that

$$\mathbb{P}\Big(U_x < r^{-2}h(x), \text{ i.o., as } x \to 0\Big) = 1,$$

which will prove part (ii).
In this direction, we define the following events. For $n \leq m-1$,

$$D_{(n,m)} = \Big\{r^{j+1}\bar{I}_{(j+1,m+1)} \geq h(r^j), \text{ for all } n \leq j \leq m-1\Big\},$$

and for $r < k < 1$ and $n \le m-2$

$$E_{(n,m-1)} = \Big\{ r^{j+1}\bar{I}_{(j+1,m)} + r^{j+1}R_{(j+1,m)}\bar{I}_{(m,m+1)} \ge h(r^j), \text{ for all } n \le j \le m-2 \Big\} \text{ and}$$

$$E^{(k)}_{(n,m-1)} = \Big\{ r^{j+1}\bar{I}_{(j+1,m)} + r^{j+1}R_{(j+1,m)}\bar{I}^{(k)}_m \ge h(r^j), \text{ for all } n \le j \le m-2 \Big\},$$

where

$$\bar{I}_{(j+1,m+1)} = \int_0^{\bar{T}^{(j+1)}_{\log(r^{m+1}/\Gamma_{j+1})}} \exp\Big\{\bar{\xi}^{(j+1)}_s\Big\} \mathrm{d}s,$$

$$\bar{I}^{(k)}_m = \int_0^{\bar{T}^{(m)}_{\log(r^{m+1}/kr^m)}} \exp\Big\{\bar{\xi}^{(m)}_s\Big\} \mathrm{d}s \qquad \text{and}$$

$$R_{(j+1,m)} = \exp\Big\{\bar{\xi}^{(j+1)}_{\bar{T}^{(j+1)}_{\log(r^m/\Gamma_{j+1})}}\Big\},$$

and for $n \le j \le m-1$, $\bar{\xi}^{(j+1)}$ is a Lévy process defined as in Corollary 2.
From the definition of $\bar{\xi}^{(j+1)}$, we can deduce that for $j < m$

$$\bar{\xi}^{(m)} = \Big(\bar{\xi}^{(j+1)}_{\bar{T}^{(j+1)}_{\log(r^m/\Gamma_{j+1})}+t} - \bar{\xi}^{(j+1)}_{\bar{T}^{(j+1)}_{\log(r^m/\Gamma_{j+1})}}, t \ge 0\Big) \quad \text{and}$$

$$\Gamma_m = \Gamma_{j+1}\exp\Big\{\bar{\xi}^{(j+1)}_{\bar{T}^{(j+1)}_{\log(r^m/\Gamma_{j+1})}}\Big\},$$

then it is straightforward that

$$\bar{T}^{(j+1)}_{\log(r^{m+1}/\Gamma_{j+1})} = \bar{T}^{(j+1)}_{\log(r^m/\Gamma_{j+1})} + \inf\Big\{t \ge 0; \bar{\xi}^{(m)}_t \le \log(r^{m+1}/\Gamma_m)\Big\}.$$

The above decomposition allows us to determine the following identity

$$\bar{I}_{(j+1,m+1)} = \bar{I}_{(j+1,m)} + R_{(j+1,m)}\bar{I}_{(m,m+1)}. \tag{2.15}$$

In the same way we can also get that,

$$I(\bar{\xi}^{(j+1)}) = \bar{I}_{(j+1,m+1)} + R_{(j+1,m+1)}I(\bar{\xi}^{(m+1)}). \tag{2.16}$$

By Corollaries 1 and 2, it follows that $I\big(\bar{\xi}^{(m+1)}\big)$ is independent of $(\bar{I}_{(j+1,m+1)}, R_{(j+1,m+1)})$ and distributed as I.
From (2.15) and since

$$\Big\{r^m\bar{I}_{(m,m+1)} \ge h(r^{m-1})\Big\} \subset \big\{\Gamma_m > r^{m+1}\big\},$$

we conclude that

$$D_{(n,m)} = E_{(n,m-1)} \bigcap \big\{r^m\bar{I}_{(m,m+1)} \ge h(r^{m-1})\big\} \bigcap \big\{\Gamma_m > r^{m+1}\big\}.$$

Now, for $n \le m-1$, we define

$$H(n,m) = \mathbb{P}\Big(E^{(k)}_{(n,m-1)}, r^m\bar{I}^{(k)}_m \ge h(r^{m-1}), \Gamma_m > r^m k\Big).$$

On the event $\{\Gamma_m > r^m k\}$, we have that $\bar{I}^{(k)}_m \le \bar{I}_{(m,m+1)}$. Hence since $k > r$, we deduce that $\mathbb{P}(D_{(n,m)}) \ge H(n,m)$.
For our purpose, we will prove that there exist (n_l) and (m_l), two increasing sequences such that $0 \le n_l \le m_l - 1$, and n_l, m_l go to ∞ and $H(n_l, m_l)$ tends to 0 as l goes to infinity. In this direction, we define the events

$$B_n = \Big\{r^{n+1}I\big(\bar{\xi}^{(n+1)}\big) < h(r^n)\Big\}.$$

If we suppose the contrary, this is that there exists $\delta > 0$ such that $H(n,m) \geq \delta$ for all sufficiently large integers m and n, we see from identity (2.16) that

$$
\begin{aligned}
1 \geq \mathbb{P}\left(\bigcup_{m=n+1}^{\infty} B_m\right) &\geq \sum_{m=n+1}^{\infty} \mathbb{P}\left(B_m \bigcap \left(\bigcap_{j=n}^{m-1} B_j^c\right)\right) \\
&= \sum_{m=n+1}^{\infty} \mathbb{P}\left(r^{m+1} I\big(\bar{\xi}^{(m+1)}\big) < h(r^m), \bigcap_{j=n}^{m-1} \left\{r^{j+1} I\big(\bar{\xi}^{(j+1)}\big) \geq h(r^j)\right\}\right) \\
&\geq \sum_{m=n+1}^{\infty} \mathbb{P}\left(r^{m+1} I\big(\bar{\xi}^{(m+1)}\big) < h(r^m)\right) \mathbb{P}\big(D_{(n,m)}\big) \\
&\geq \sum_{m=n+1}^{\infty} \mathbb{P}\Big(r^{m+1} I\big(\bar{\xi}^{(m+1)}\big) < h(r^m)\Big) H(n,m) \geq \delta \sum_{m=n+1}^{\infty} \mathbb{P}\Big(r^{m+1} I < h(r^m)\Big),
\end{aligned}
$$

but this last sum diverges, since

$$
\begin{aligned}
\sum_{m=n+1}^{\infty} \mathbb{P}\big(r^{m+1} I < h(r^m)\big) &\geq \int_{n+1}^{\infty} \mathbb{P}\Big(r^t I < h(r^t)\Big) \mathrm{d}t \\
&= -\frac{1}{\log r} \int_0^{r^{n+1}} \bar{F}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x}.
\end{aligned}
$$

Hence our assertion is true.

Next, we denote $\mathbb{P}(I \in \mathrm{d}x) = \mu(\mathrm{d}x)$ and $\mathbb{P}(I_{r/k} \in \mathrm{d}x) = \bar{\mu}(\mathrm{d}x)$ for $k > r$, where

$$
I_{r/k} = \int_0^{\hat{T}_{\log(r/k)}} \exp\{\hat{\xi}_s\} \mathrm{d}s,
$$

and we define

$$
\rho_{n_l,m_l}(x) = \mathbb{P}\left(\bigcap_{j=n_l}^{m_l-2} \left\{r^{j+1} \bar{I}_{(j+1,m_l)} + r^{j+1} R_{j+1,m_l} x \geq r^{-1} h(r^j)\right\}, \Gamma_{m_l} > k r^{m_l}\right),
$$

and

$$
G(n_l,m_l) = \mathbb{P}\left(\bigcap_{j=n_l}^{m_l-1} \left\{r^{j+1} I(\bar{\xi}^{(j+1)}) \geq h(r^j)\right\}, \Gamma_{m_l} > k r^{m_l}\right).
$$

Note that $\rho_{n_l,m_l}(x)$ is increasing in x.

Hence, $H(n_l,m_l)$ and $G(n_l,m_l)$ are expressed as follows

$$
\begin{aligned}
H(n_l,m_l) &= \int_{r^{-m_l} h(r^{m_l-1})}^{\infty} \bar{\mu}(\mathrm{d}x) \rho_{n_l,m_l}(x) \quad \text{and} \\
G(n_l,m_l) &= \int_{r^{-m_l} h(r^{m_l-1})}^{\infty} \mu(\mathrm{d}x) \rho_{n_l,m_l}(x).
\end{aligned}
$$

The equality for $H(n_l,m_l)$ is evident since the random variable $\bar{I}_m^{(k)}$ is independent from $\big\{\Gamma_{m_l}, (\bar{I}_{(j+1,m_l)}, R_{(j+1,m_l)}; n_l \leq j \leq m_l - 2)\big\}$. To show the second one, we use (2.16) in the following form

$$
I\big(\bar{\xi}^{(j+1)}\big) = \bar{I}_{(j+1,m_l)} + R_{(j+1,m_l)} I\big(\bar{\xi}^{(m_l)}\big),
$$

and the independence between $I\big(\bar{\xi}^{(m_l)}\big)$ and $\big\{\Gamma_{m_l}, (\bar{I}_{(j+1,m_l)}, R_{(j+1,m_l)}; n_l \leq j \leq m_l - 2)\big\}$. In particular, it follows that for l sufficiently large

$$H(n_l, m_l) \geq \rho_{n_l,m_l}(N) \int_N^\infty \bar{\mu}(\mathrm{d}x) \qquad \text{for} \qquad N \geq rC,$$

where $C = \sup_{x \leq \beta} x^{-1} h(x)$.
Since $H(n_l, m_l)$ converges to 0, as l goes to $+\infty$ and $\bar{\mu}$ does not depend on l, then $\rho_{n_l,m_l}(N)$ also converges to 0 when l goes to $+\infty$, for every $N \geq rC$.
On the other hand, we have

$$G(n_l, m_l) \leq \rho_{n_l,m_l}(N) \int_0^N \mu(\mathrm{d}x) + \int_N^\infty \mu(\mathrm{d}x),$$

then, letting l and N go to infinity, we get that $G(n_l, m_l)$ goes to 0.
Note that the set C_{n_l} satisfies

$$\mathbb{P}(C_{n_l}) \geq 1 - \mathbb{P}\left(r^{j+1} I(\bar{\xi}^{(j+1)}) \geq h(r^j), \text{ for all } n_l \leq j \leq m_l - 1\right)$$

and it is not difficult to see that

$$\mathbb{P}\left(r^{j+1} I(\bar{\xi}^{(j+1)}) \geq h(r^j), \text{ for all } n_l \leq j \leq m_l - 1\right) \leq \mathbb{P}\big(\Gamma_{m_l} \leq kr^{m_l}\big) + G(n_l, m_l).$$

Then,

$$\mathbb{P}(C_{n_l}) \geq \mathbb{P}\big(\Gamma_{m_l} > kr^{m_l}\big) - G(n_l, m_l),$$

and since $\mathbb{P}\big(\Gamma_{m_l} > kr^{m_l}\big) = \mathbb{P}\big(\Gamma > kr\big) > 0$ (see Corollary 1 and the properties of Γ in Lemma 1), we conclude that $\lim \mathbb{P}(C_n) > 0$ and with this we finish the proof. ■

For the integral tests at $+\infty$, we define $\mathcal{H}_\infty^{-1}$ the totality of positive increasing functions $h(x)$ on $(0, \infty)$ that satisfy

i) $\lim_{t \to \infty} h(x) = +\infty$, and

ii) there exists $\beta \in (1, +\infty)$ such that $\displaystyle\sup_{x > \beta} \frac{h(x)}{x} < \infty$,

THEOREM 5. *Let $h \in \mathcal{H}_\infty^{-1}$.*

i) *If*

$$\int^{+\infty} \bar{F}_\nu \left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1 - \epsilon) h(x), \text{ i.o., as } x \to +\infty\Big) = 0.$$

ii) *If*

$$\int^{+\infty} \bar{F} \left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1 + \epsilon) h(x), \text{ i.o., as } x \to +\infty\Big) = 1.$$

Proof: The proof is very similar to that in Theorem 4. First, note that we have the same results as Corollary 2 for x large (see Corollary 3), then we get the integral test following the same arguments for the proof of (i) and (ii) for the sequence $x_n = r^n$, for $r > 1$, and noticing that if we define

$$\begin{aligned} C_n &= \Big\{U_x < h_r(x),\ \text{for some } x \in (r^n, +\infty)\Big\} \\ &= \Big\{J_t^{(0)} > h_r^{-1}(t),\ \text{for some } t \in (U_{r^n}, +\infty)\Big\}, \end{aligned}$$

where $h_r(t) = r^2 h(t)$, then the event $C = \cap_{n\geq 1} C_n$ is in the upper-tail sigma-field

$$\bigcap_t \sigma\Big\{X_s^{(0)} : s \geq t\Big\},$$

which is trivial. ■

In some cases, it will prove complicated to find sharp estimations of the tail probability of νI, given that we will not have enough information about the distribution of ν. However, if we can determine the law of I then by (0.16), we will also determine the law of $X_1^{(0)}$ and sometimes it will be possible to have sharp estimations of its tail probability. For this reason, we will give another integral test for the convergence cases in Theorems 4 and 5, in terms of the tail probability of $X_1^{(0)}$.
Let us define

$$H(t) = \mathbb{P}_0\big(X_1 > t\big).$$

COROLLARY 6. i) *Let* $h \in \mathcal{H}_0^{-1}$. *If*

$$\int_{0+} H\left(\frac{x}{h(x)}\right)\frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1-\epsilon)h(x),\ \textit{i.o., as } x \to 0\Big) = 0.$$

ii) *Let* $h \in \mathcal{H}_\infty^{-1}$. *If*

$$\int^{+\infty} H\left(\frac{x}{h(x)}\right)\frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1-\epsilon)h(x),\ \textit{i.o., as } x \to \infty\Big) = 0.$$

Proof: The proof of this corollary is consequence of the following inequality. By the scaling property,

$$\bar{F}_\nu\big(h(x)/x\big) = \mathbb{P}\big(U_1 < h(x)/x\big) = \mathbb{P}\big(J_1^{(0)} > x/h(x)\big) \leq \mathbb{P}_0\big(X_1 > x/h(x)\big),$$

and then applying Theorem 4 part (i) for the integral test at 0 and Theorem 5 part (i) for the integral test at $+\infty$, we obtain the desired result. ■

3. The upper envelopes of the future infimum and increasing positive self-similar Markov processes.

The aim of this section is to determine the upper envelope of the future infimum of pssMp at 0 and at $+\infty$. With this purpose, we will use similar arguments to the used ones in the lower envelope of the last passage time process.
We first note that if the pssMp is increasing; then its supremum , its past infimum and its

future infimum are the same. Moreover, its first passage time over the level $y > 0$ is the same as the last passage time below y. Therefore, with the following integral tests for the future infimum we may also describe the upper envelope of increasing pssMp.

Let us denote by $\mathcal{H}_0$ the totality of positive increasing functions $h(t)$ on $(0, \infty)$ that satisfy

i) $h(0) = 0$, and

ii) there exists $\beta \in (0, 1)$ such that $\sup_{t<\beta} \dfrac{t}{h(t)} < \infty$.

THEOREM 6. *Let* $h \in \mathcal{H}_0$.

i) *If*

$$\int_{0+} \bar{F}_\nu \left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_0\Big(J_t > (1+\epsilon)h(t), \textit{ i.o., as } t \to 0\Big) = 0.$$

ii) *If*

$$\int_{0+} \bar{F} \left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_0\Big(J_t > (1-\epsilon)h(t), \textit{ i.o., as } t \to 0\Big) = 1.$$

Proof: Let (x_n) be a decreasing sequence which converges to 0. We define the events

$$A_n = \Big\{\text{There exists } t \in [U_{x_{n+1}}, U_{x_n}] \text{ such that } J^{(0)}_t > h(t)\Big\}.$$

From the fact that U_{x_n} tends to 0, a.s. when n goes to $+\infty$, we see

$$\Big\{J^{(0)}_t > h(t), \text{ i.o., as } t \to 0\Big\} = \limsup_{n\to+\infty} A_n.$$

Since h is an increasing function and $J^{(0)}_{U_{x_n}} \geq x_n$ a.s., the following inclusions hold

$$\Big\{x_n > h\big(U_{x_n}\big)\Big\} \subset A_n \subset \Big\{x_n > h\big(U_{x_{n+1}}\big)\Big\}. \tag{2.17}$$

Now, we prove the convergent part. We choose $x_n = r^n$, for $r < 1$ and $h_r(t) = r^{-2}h(t)$. Since h is increasing, we deduce that

$$\sum_n \mathbb{P}\Big(r^n > h_r\big(U_{r^{n+1}}\big)\Big) \leq -\frac{1}{\log r}\int_0^r \mathbb{P}\Big(t > h\big(U_t\big)\Big)\frac{\mathrm{d}t}{t}.$$

Replacing h by h_r in (2.17), we see that we can obtain our result if

$$\int_0^r \mathbb{P}\Big(t > h\big(U_t\big)\Big)\frac{\mathrm{d}t}{t} < \infty.$$

From elementary calculations, we deduce that

$$\int_0^r \mathbb{P}\Big(t > h\big(U_t\big)\Big)\frac{\mathrm{d}t}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{I}_{\{t/r<\nu I<t/h(t)\}}\frac{\mathrm{d}t}{t}\right),$$

where $h^{-1}(s) = \inf\{t > 0, h(t) > s\}$, the right inverse function of h. Then, this integral converges if

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(\nu I < \frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} < \infty.$$

This proves part (i).
Next, we prove the divergent case. We suppose that h satisfies

$$\int_{0+} \bar{F}\left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} = \infty.$$

Take, again, $x_n = r^n$, for $r < 1$ and note that,

$$B_n = \bigcup_{m=n}^{\infty} A_m = \Big\{\text{There exist } t \in (0, U_{r^n}] \text{ such that } J_t^{(0)} > h_r(t)\Big\}$$
$$= \Big\{\text{There exist } x \in (0, r^n] \text{ such that } U_x < h_r^{-1}(x)\Big\}$$

where $h_r(t) = rh(t)$ and h_r^{-1} its right inverse function. Hence, by analogous arguments to the proof of Theorem 4 part (ii) it is enough to prove that $\lim \mathbb{P}(B_n) > 0$ to obtain our result. With this purpose, we will follow the proof of Theorem 4.
From inclusion (2.17) and the inequality (1.12) in Corollary 2, we see

$$\mathbb{P}(B_n) \geq 1 - \mathbb{P}\Big(r^j \leq rh\big(r^j I(\bar{\xi}^{(j)})\big), \text{ for all } n \leq j \leq m-1\Big),$$

where m is chosen arbitrarily $m \geq n+1$.
Now, we define the events

$$C_n = \left\{r^n > rh\Big(r^n I\big(\bar{\xi}^{(n)}\big)\Big)\right\},$$

and we will prove that $\sum \mathbb{P}(C_n) = \infty$. Since the function h is increasing, it is straightforward that

$$\sum_{n\geq 1} \mathbb{P}(C_n) \geq \int_0^{+\infty} \mathbb{P}\left(r^t > h\big(r^t I\big)\right) \mathrm{d}t = -\frac{1}{\log r} \int_0^1 \mathbb{P}\Big(t > h\big(tI\big)\Big) \frac{\mathrm{d}t}{t}.$$

Hence, it is enough to prove that this last integral is infinite. In this direction, we have that

$$\int_0^r \mathbb{P}\Big(t > h\big(tI\big)\Big) \frac{\mathrm{d}t}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{I}_{\big\{t/r < I < t/h(t)\big\}} \frac{\mathrm{d}t}{t}\right).$$

On the other hand, we see that

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(I < \frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} = \int_0^{h^{-1}(r)} \mathbb{P}\left(\frac{t}{r} < I < \frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t}$$
$$+ \int_0^{h^{-1}(r)} \mathbb{P}\left(I < \frac{t}{r}\right) \frac{\mathrm{d}t}{t},$$

and since $e^{-1}\hat{T}_1 \leq I$ almost surely, then

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(I < \frac{t}{r}\right) \frac{\mathrm{d}t}{t} \leq \mathbb{E}\left(\log^+ \frac{h^{-1}(r)}{r} I^{-1}\right)$$
$$\leq 1 + \log^+ \frac{h^{-1}(r)}{r} + \mathbb{E}\left(|\log \hat{T}_1|\right),$$

which is clearly finite from our assumptions. Then, we deduce that

$$\mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{I}_{\left\{t/r<I<t/h(t)\right\}}\frac{\mathrm{d}t}{t}\right) = \infty,$$

and hence $\sum \mathbb{P}(C_n) = \infty$.
Next following the same notation as in the proof of Theorem 4, we define the following events. For $n \leq m-1$

$$D_{(n,m)} = \Big\{r^j \leq rh\big(r^j \bar{I}_{(j,m)}\big), \text{ for all } n \leq j \leq m-1\Big\},$$

and, for $r < k < 1$ and $n \leq m-2$

$$E_{n,m-1} = \Big\{r^j \leq rh\big(r^j \bar{I}_{(j,m-1)} + r^j R_{(j,m-1)}\bar{I}_{(m-1,m)}\big), \text{ for all } n \leq j \leq m-2\Big\} \quad \text{and}$$

$$E^{(k)}_{n,m-1} = \Big\{r^j \leq rh\big(r^j \bar{I}_{(j,m-1)} + r^j R_{(j,m-1)}\bar{I}^{(k)}_{m-1}\big), \text{ for all } n \leq j \leq m-2\Big\}.$$

Again, we have that

$$D_{(n,m)} = E_{(n,m-1)} \bigcap \Big\{r^{m-1} \leq rh\big(r^{m-1}\bar{I}_{(m-1,m)}\big)\Big\} \bigcap \big\{\Gamma_{m-1} > r^m k\big\}.$$

Now, for $n \leq m-1$, we define

$$H(n,m) = \mathbb{P}\Big(E^{(k)}_{(m,m-1)}, r^{m-1} \leq rh\big(r^{m-1}\bar{I}_{(m-1,m)}\big), \Gamma_{m-1} > r^m k\Big).$$

Since $k > r$, we deduce that $\mathbb{P}(D_{(n,m)}) \geq H(n,m)$. Then similarly as in the proof of Theorem 4, we will prove that there exist (n_l) and (m_l), two increasing sequences such that $0 \leq n_l \leq m_l - 1$, and n_l, m_l go to $+\infty$ and $H(n_l, m_l)$ tends to 0 as l goes to infinity.
We suppose the contrary, i.e., there exist $\delta > 0$ such that $H(n,m) \geq \delta$ for all sufficiently large integers m and n, hence

$$\begin{aligned}
1 \geq \mathbb{P}\left(\bigcup_{m=n+1}^{\infty} C_m\right) &\geq \sum_{m=n+1}^{\infty} \mathbb{P}\left(C_m \bigcap \left(\bigcap_{j=n}^{m-1} C_j^c\right)\right) \\
&\geq \sum_{m=n+1}^{\infty} \mathbb{P}\Big(r^m > rh\Big(r^m I\big(\bar{\xi}^{(m)}\big)\Big)\Big)\mathbb{P}(D_{(n,m)}) \\
&\geq \sum_{m=n+1}^{\infty} \mathbb{P}\Big(r^m > rh\Big(r^m I\big(\bar{\xi}^{(m)}\big)\Big)\Big)H(n,m) \geq \delta \sum_{m=n+1}^{\infty} \mathbb{P}\big(C_m\big),
\end{aligned}$$

but since $\sum \mathbb{P}(C_n)$ diverges, we see that our assertion is true.
Next, we define

$$\rho_{n_l,m_l}(x) = \mathbb{P}\left(\bigcap_{j=n_l}^{m_l-2} \Big\{r^j \leq rh\big(r^j \bar{I}_{(j,m-1)} + r^j R_{(j,m-1)}x\big)\Big\}, \Gamma_{m_l-1} > kr^{m_l-1}\right)$$

and

$$G(n_l,m_l) = \mathbb{P}\left(\bigcap_{j=n_l}^{m_l-1} \Big\{r^j \leq rh\Big(r^j I\big(\bar{\xi}^{(j)}\big)\Big)\Big\}, \Gamma_{m_l-1} > kr^{m_l-1}\right).$$

Since h is increasing, we see that $\rho_{n_l,m_l}(x)$ is increasing in x.
Again, we express $H(n_l,m_l)$ and $G(n_l,m_l)$ as follows

$$H(n_l,m_l) = \int_0^{+\infty} \bar{\mu}(\mathrm{d}x)\, \mathbb{I}_{\left\{h(r^{m_l-1}x) \geq r^{m_l}\right\}} \rho_{n_l,m_l}(x) \quad \text{and,}$$

$$G(n_l, m_l) = \int_0^{+\infty} \mu(\mathrm{d}x) \mathbb{I}_{\left\{h(r^{m_l-1}x)\geq r^{m_l}\right\}} \rho_{n_l,m_l}(x).$$

In particular, we get that for l sufficiently large

$$H(n_l, m_l) \geq \rho_{n_l,m_l}(N) \int_N^{+\infty} \bar{\mu}(\mathrm{d}x)\rho_{n_l,m_l}(x) \qquad for \quad N \geq rC,$$

where $C = \sup_{x\leq\beta} x/h(x)$. Hence following the same arguments of the proof of Theorem 4, it is not difficult to see that $G(n_l, m_l)$ goes to 0 as l goes to infinity and that

$$\lim_{l\to+\infty} 1 - \mathbb{P}\Big(r^{j+1} \leq rh\big(r^{j+1}I(\bar{\xi}^{(j+1)})\big), \text{ for all } n_l \leq j \leq m_l - 1\Big) > 0.$$

Then, we conclude that $\lim \mathbb{P}(B_n) > 0$ and with this we finish the proof. ■

For the integral tests at $+\infty$, we define $\mathcal{H}_\infty$, the totality of positive increasing functions $h(t)$ on $(0, \infty)$ that satisfy

i) $\lim_{t\to\infty} h(t) = \infty$, and
ii) there exists $\beta > 1$ such that $\sup_{t>\beta} \dfrac{t}{h(t)} < \infty$.

Then the upper envelope of $J^{(x)}$ at $+\infty$ is given by the following result.

THEOREM 7. *Let* $h \in \mathcal{H}_\infty$.

i) *If*

$$\int^{+\infty} \bar{F}_\nu\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$ *and for all* $x \geq 0$,

$$\mathbb{P}_x\Big(J_t > (1+\epsilon)h(t), \textit{ i.o., as } t \to +\infty\Big) = 0.$$

ii) *If*

$$\int^{+\infty} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty,$$

then for all $\epsilon > 0$ *and for all* $x \geq 0$

$$\mathbb{P}_x\Big(J_t > (1-\epsilon)h(t), \textit{ i.o., as } t \to +\infty\Big) = 1.$$

Proof: We first consider the case where $x = 0$. In this case the proof of the tests at $+\infty$ is almost the same as that of the tests at 0. It is enough to apply the same arguments to the sequence $x_n = r^n$, for $r > 1$.
Now, we prove (i) for any $x > 0$. Let $h \in \mathcal{H}_\infty$ such that

$$\int^{+\infty} \bar{F}_\nu\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} \qquad \text{is finite.}$$

Let $x > 0$ and $S_x = \inf\{t \geq 0 : X_t^{(0)} \geq x\}$ and note by μ_x the law of $X_{S_x}^{(0)}$. Since clearly

$$\int^{+\infty} \bar{F}_\nu\left(\frac{t}{h(t - S_x)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

from the Markov property at time S_x, we have for all $\epsilon > 0$

$$
\begin{aligned}
&\mathbb{P}_0\Big(J_t > (1+\epsilon)h(t-S_x),\ \text{i. o., as } t\to\infty\Big)\\
(2.18)\qquad &= \int_{[x,+\infty)} \mathbb{P}_y\Big(J_t > (1+\epsilon)h(t),\ \text{i. o., as } t\to\infty\Big)\mu_x(\mathrm{d}y) = 0.
\end{aligned}
$$

If x is an atom of μ_x, then equality (2.18) shows that

$$\mathbb{P}\Big(J_t^{(x)} > (1+\epsilon)h(t),\ \text{i. o., as } t\to\infty\Big) = 0$$

and the result is proved. Suppose that x is not an atom of μ_x. Recall from Lemma 1, that $\log(x_1^{-1}\Gamma)$ is the limit in law of the overshoot process $\hat{\xi}_{\hat{T}_x} - x$, as $x \to +\infty$. So, it follows from [**CaCh06**], Theorem 1 that $X^{(0)}_{S_x} \stackrel{(d)}{=} \frac{xx_1}{\Gamma}$, and since $\mathbb{P}(\Gamma > z)$ for $z < x_1$, we have for any $\alpha > 0$, $\mu_x(x, x+\alpha) > 0$. Hence (2.18) shows that there exists $y > x$ such that

$$\mathbb{P}\Big(J_t^{(y)} > (1+\epsilon)h(t),\ \text{i. o., as } t\to\infty\Big) = 0,$$

for all $\epsilon > 0$. The previous allows us to conclude part (i).
Part (ii) can be proved in the same way. ■

Similarly as for the lower envelope of U, we may obtain integral tests for the convergence cases of Theorems 6 and 7 in terms of the large tail probability of $X_1^{(0)}$ that we denoted by H.

COROLLARY 7. i) *Let $h \in \mathcal{H}_0$. If*

$$\int_{0+} H\left(\frac{h(t)}{t}\right)\frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(J_t^{(0)} > (1+\epsilon)h(t),\ \textit{i.o., as } t\to 0\Big) = 0.$$

ii) *Let $h \in \mathcal{H}_\infty$. If*

$$\int^{+\infty} H\left(\frac{h(t)}{t}\right)\frac{\mathrm{d}t}{t} < \infty,$$

then and for all $\epsilon > 0$

$$\mathbb{P}\Big(J_t^{(x)} > (1+\epsilon)h(t),\ \textit{i.o., as } t\to\infty\Big) = 0.$$

Proof: As in Corollary 6, the proof of this result is consequence of the following inequality. By the scaling property,

$$\bar{F}_\nu\big(t/h(t)\big) = \mathbb{P}\big(U_1 < t/h(t)\big) = \mathbb{P}\big(J_1^{(0)} > h(t)/t\big) \le \mathbb{P}_0\big(X_1 > h(t)/t\big),$$

and then applying Theorem 6 part (i) for the integral test at 0 and Theorem 7 part (i) for the integral test at $+\infty$, we obtain the desired result. ■

4. The upper envelope of positive self-similar Markov processes with no positive jumps

In the precedent section, we noted that the upper envelope of the future infimum is determined by the lower envelope of the last passage times and also that the same arguments describe the upper envelope of increasing pssMp since in this case the first and last passage times are the same. Following the same type of reasonings, we deduce that the upper envelope of $X^{(0)}$ (and that of its past supremum) is determined by its first passage times and a natural question that we may raise is: *could we use similar arguments, as for the future infimum, to determine the upper envelope of positive self-similar Markov processes?* In general, we do not know how to determine the law of the first passage time and even how to establish an integral test, since from the Caballero and Chaumont's construction we have that the first passage time depends on the sequence (θ_n) which is a Markov chain.

In Chapter 1, under the assumption of absence of positive jumps, we determined the law of the first passage time in terms of its associated Lévy process and moreover that S_1 is a self-decomposable random variable. The self-decomposability of the first passage time will allow us to obtain in a complete and satisfactory way integral tests for the upper envelope of pssMp in this case.

4.1. The lower envelope of the first and last passage times. In Chapter 1, we showed that the first and last passage time processes are increasing and positive self-similar processes with independent increments (*ipsspii* to simplify the notation). Watanabe [**Wata96**] established integral tests and laws of the iterated logarithm for this type of processes. Here, we will use the integral tests found by Watanabe to describe the lower envelope of the first and last passage time of pssMp with no positive jumps.

Let Y be an ipsspii starting from 0 and define

$$R(t) \stackrel{\text{(def)}}{=} \mathbb{P}(Y_1 < t).$$

We recall that $\mathcal{H}_0^{-1}$ is the totality of positive increasing functions $h(x)$ on $(0,\infty)$ that satisfy

i) $h(0) = 0$, and

ii) there exists $\beta \in (0,1)$ such that $\displaystyle\sup_{x<\beta} \frac{h(x)}{x} < \infty$.

LEMMA 6 (Watanabe [**Wata96**]). *Let $h \in \mathcal{H}_0^{-1}$ and Y an isspii.*

i) *If*

$$\int_{0+} R\left(\frac{h(t)}{t}\right)\frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(Y_t < (1-\epsilon)h(t), \text{ i.o., as } t \to 0\Big) = 0.$$

ii) *If*

$$\int_{0+} R\left(\frac{h(t)}{t}\right)\frac{\mathrm{d}t}{t} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(Y_t < (1+\epsilon)h(t), \text{ i.o., as } t \to 0\Big) = 1.$$

We have the same integral tests at $+\infty$, for $h \in \mathcal{H}_\infty^{-1}$, we only need to exchange

$$\int_{0+} R\left(\frac{h(t)}{t}\right)\frac{\mathrm{d}t}{t} \qquad \text{by} \qquad \int^{+\infty} R\left(\frac{h(t)}{t}\right)\frac{\mathrm{d}t}{t}.$$

Since U and S are ipsspii starting from 0, we will have integral tests for the lower envelope at 0 and at $+\infty$ of both processes. By Corollary 5 and Proposition 5, we see that the integral test of U and S depends on the distribution functions of $I(\hat{\xi})$ and $I(\tilde{\xi})$, respectively.

4.2. The upper envelope. Now, we will establish integral tests for the upper envelope of $X^{(0)}$ at 0 and at $+\infty$. The following theorem means in particular that the asymptotic behaviour of $X^{(0)}$ only depends on the tail behaviour of the law of

$$I(\tilde{\xi}) = \int_0^{+\infty} \exp\Big\{\tilde{\xi}_u\Big\}\mathrm{d}u = \int_{\gamma(0)}^{+\infty} \exp\Big\{-\xi_u\Big\}\mathrm{d}u,$$

and on the additional hypothesis

$$\mathbb{E}\Big(\log^+ I\big(\tilde{\xi}\big)^{-1}\Big) < \infty. \tag{2.19}$$

Let us define

$$\tilde{F}(t) \overset{\text{(def)}}{=} \mathbb{P}\Big(I\big(\tilde{\xi}\big) < t\Big).$$

THEOREM 8. *Let $h \in \mathcal{H}_0$.*

i) *If*

$$\int_{0+} \tilde{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t),\ \textit{i.o., as } t \to 0\Big) = 0.$$

ii) *Assume that (2.19) is satisfied. If*

$$\int_{0+} \tilde{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1-\epsilon)h(t),\ \textit{i.o., as } t \to 0\Big) = 1.$$

Proof: The proof is very similar to this of Theorem 6. We only need to make some remarks in part (ii) that we will explain below. We recall from Corollary 5, the following identity in law $S_x \overset{(d)}{=} xI(\tilde{\xi})$, for $x > 0$.
The proof of part (i) follows line by line, replacing the future infimum process $J^{(0)}$ by the pssMp $X^{(0)}$ and the last passage time U by the first passage time S.
The proof of part (ii) is much easier to this of Theorem 6, since in this case the first passage process S has independent increments. In order to proof this part, we follow again line by line the arguments in the proof of part (ii) in Theorem 6 replacing the future infimum process $J^{(0)}$ by the pssMp $X^{(0)}$, the last passage time U by the first passage time S, and noting the following assertions:

- First, we note that to prove $\sum \mathbb{P}(C_n) = \infty$ the additional hypothesis is required since
$$\int_0^{h^{-1}(r)} \mathbb{P}\left(I(\tilde{\xi}) < \frac{t}{r}\right) \frac{\mathrm{d}t}{t} \leq \mathbb{E}\left(\log^+ \frac{h^{-1}(r)}{r} I(\tilde{\xi})^{-1}\right).$$
- Since the process S has independent increments, we will not need to define the sets $D_{(n,m)}, E_{n,m-1}$ and $E^{(k)}_{n,m-1}$. Then $H(n,m)$ becomes
$$H(n,m) \overset{\text{(def)}}{=} \mathbb{P}\Big(r^j \leq rh\big(S_{r^j} - S_{r^m}\big),\ \text{for all } n \leq j \leq m-1\Big).$$
- The existence of the two increasing sequences $(n_l, l \geq 1)$ and $(m_l, l \geq 1)$, is also much easier. In fact, if we assume that there exist $\delta > 0$ such that $H(n,m) \geq \delta$ for all sufficiently large integers m, n; from the independence of the increments of S, we have
$$\begin{aligned} 1 \geq \mathbb{P}\left(\bigcup_{m=n+1}^{\infty} C_m\right) &\geq \sum_{m=n+1}^{\infty} \mathbb{P}\left(C_m \bigcap \left(\bigcap_{j=n}^{m-1} C_j^c\right)\right) \\ &\geq \sum_{m=n+1}^{\infty} \mathbb{P}\Big(r^m > rh\Big(S_{r^m}\Big)\Big) H(n,m) \geq \delta \sum_{m=n+1}^{\infty} \mathbb{P}\big(C_m\big). \end{aligned}$$
- The definitions of $\rho_{n_l,m_l}(x)$ and $G(n_l, m_l)$ become,
$$\rho_{n_l,m_l}(x) \overset{\text{(def)}}{=} \mathbb{P}\Big(r^j \leq rh\big(S_{r^j} - S_{r^{m_l-1}} + x\big) \text{ for, } n_l \leq j \leq m_l - 2\Big), \qquad x \geq 0,$$
and
$$G(n_l, m_l) \overset{\text{(def)}}{=} \mathbb{P}\Big(r^j \leq rh\big(S_{r^j}\big) \text{ for, } n_l \leq j \leq m_l - 1\Big).$$
- Finally, we note that the probability measures μ and $\bar{\mu}$ in the decomposition of $H(n_l, m_l)$ and $G(n_l, m_l)$ are the laws of S_1 and $S_1 - S_r$ and hence the proof follows. ■

The upper envelope of $X^{(x)}$ at $+\infty$ is given by the following result.

THEOREM 9. *Let $h \in \mathcal{H}_\infty$.*

i) *If*
$$\int^{+\infty} \tilde{F}\left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} < \infty,$$
then for all $\epsilon > 0$ and for all $x \geq 0$,
$$\mathbb{P}_x\Big(X_t > (1+\epsilon)h(t),\ \textit{i.o., as } t \to +\infty\Big) = 0.$$

ii) *Assume that (2.19) is satisfied. If*
$$\int^{+\infty} \tilde{F}\left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} = \infty,$$
then for all $\epsilon > 0$ and for all $x \geq 0$
$$\mathbb{P}_x\Big(X_t > (1-\epsilon)h(t),\ \textit{i.o., as } t \to +\infty\Big) = 1.$$

Proof: We first consider the case where $x = 0$. In this case the proof of the tests at $+\infty$ is almost the same as that of the tests at 0. It is enough to apply the same arguments to the sequence $x_n = r^n$, for $r > 1$.
Now, we prove (i) for any $x > 0$. Let $h \in \mathcal{H}_\infty$ such that

$$\int^{+\infty} \tilde{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} \qquad \textit{is finite.}$$

Let $x > 0$ and S_x as usual. Since clearly

$$\int^{+\infty} \tilde{F}\left(\frac{t}{h(t-S_x)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

from the Markov property at time S_x, we have for all $\epsilon > 0$

$$\begin{aligned} 0 &= \mathbb{P}_0\Big(X_t > (1+\epsilon)h(t-S_x), \text{ i. o., as } t \to \infty\Big) \\ &= \mathbb{P}_x\Big(X_t > (1+\epsilon)h(t), \text{ i. o., as } t \to \infty\Big), \end{aligned}$$

which proves part (i).
Part (ii) can be proved in the same way. ■

5. Describing the upper envelope of a positive self-similar Markov process using its future infimum.

In this section, we will use the integral tests which describe the upper envelope of the future infimum of pssMp to determine the upper envelope of pssMp, under general hypothesis. The integral tests that we will find here are related with the tail probabilities of the exponential functional I and S_1. Note that the tail probability of I can be smaller than the tail probability of S_1 but for our applications (see Chapters 3 and 4) these integral test will be very useful. In fact, in the following chapters we will compare the behaviour of these tail probabilities under different conditions.

5.1. Lower envelope of the first passage time. Recall that $S = (S_x, x \geq 0)$ is an increasing self-similar process whose scaling coefficient is the inverse of the scaling coefficient of $X^{(0)}$. Since the process $X^{(0)}$ starts at 0 and drifts towards $+\infty$, we deduce that the process S also starts at 0 and tends to $+\infty$ as x increases.

In this section, we are interested in the lower envelope of the process S at 0 and at $+\infty$. As we will see later, the asymptotic behaviour of the process S is related to the asymptotic behaviour of $X^{(0)}$. Let us define

$$G(t) := \mathbb{P}\big(S_1 < t\big).$$

PROPOSITION 6. *Let* $h \in \mathcal{H}_0^{-1}$.

i) *If*

$$\int_{0+} G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1-\epsilon)h(x), \textit{ i.o., as } x \to 0\Big) = 0.$$

ii) *If*

$$\int_{0+} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1+\epsilon)h(x), \text{ i.o., as } x \to 0\Big) = 1.$$

Proof: We first prove the convergent part. Let (x_n) be a decreasing sequence of positive numbers which converges to 0 and let us define the events

$$A_n = \Big\{S_{x_{n+1}} < h(x_n)\Big\}.$$

Next, we choose $x_n = r^n$, for $r < 1$. From the first Borel-Cantelli Lemma, if we have that $\sum_n \mathbb{P}(A_n) < \infty$, it follows

$$S_{r^{n+1}} \geq h\big(r^n\big) \qquad \mathbb{P}- \text{ a.s.},$$

for all large n. Since the function h and the process S are increasing, we have

$$S_x \geq h(x) \qquad \text{for} \quad r^{n+1} \leq x \leq r^n.$$

Hence from the scaling property, we get that

$$\sum_n \mathbb{P}\Big(S_{r^n} < h\big(r^{n+1}\big)\Big) \leq \int_1^\infty \mathbb{P}\Big(r^t S_1 < h\big(r^t\big)\Big)\mathrm{d}t$$
$$= -\frac{1}{\log r}\int_0^r G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x}.$$

From our hypothesis, this last integral is finite. Then from the above discussion, there exist x_0 such that for every $x \geq x_0$

$$S_x \geq r^2 h(x), \qquad \text{for all} \quad r < 1.$$

Clearly, this implies that

$$\mathbb{P}_0\Big(S_x < r^2 h(x), \text{ i.o., as } x \to 0\Big) = 0,$$

which proves part (i).
The divergent part is a natural consequence from the integral test of lower envelope of the last passage time see section 2 (Theorem 4, part (ii)) since $S_x \leq U_x$ for all $x \geq 0$. ■

The integral test at $+\infty$ is as follows;

PROPOSITION 7. *Let* $h \in \mathcal{H}_0^{-1}$.

i) *If*

$$\int^{+\infty} G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1-\epsilon)h(x), \text{ i.o., as } x \to +\infty\Big) = 0.$$

ii) *If*

$$\int^{+\infty} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1+\epsilon)h(x), \text{ i.o., as } x \to +\infty\Big) = 1.$$

Proof: The proof is very similar to that in Proposition 7. We get the integral test following the same arguments for the proof of part (i) and (ii) for the sequence $x_n = r^n$, with $r > 1$. ■

5.2. The upper envelope. The first result that we present here establishes the integral test at 0 for the upper envelope of $X^{(0)}$

PROPOSITION 8. *Let* $h \in \mathcal{H}_0$.

i) *If*
$$\int_{0+} G\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty,$$
then for all $\epsilon > 0$
$$\mathbb{P}\Big(X_t^{(0)} > (1+\epsilon)h(t),\ \textit{i.o., as } t \to 0\Big) = 0.$$

ii) *If*
$$\int_{0+} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty,$$
then for all $\epsilon > 0$
$$\mathbb{P}\Big(X^{(0)} < (1-\epsilon)h(t),\ \textit{i.o., as } t \to 0\Big) = 1.$$

Proof: Let (x_n) be a decreasing sequence which converges to 0. We define the events
$$A_n = \Big\{\text{There exists } t \in [S_{x_{n+1}}, S_{x_n}) \text{ such that } X_t^{(0)} > h(t)\Big\}.$$
From the fact that S_{x_n} tends to 0, a.s. when n goes to $+\infty$, we see
$$\Big\{X_t^{(0)} > h(t),\ \text{i.o., as } t \to 0\Big\} = \limsup_{n\to+\infty} A_n.$$
Since h is an increasing function the following inclusion hold
$$A_n \subset \Big\{x_n > h\big(S_{x_{n+1}}\big)\Big\}. \tag{2.20}$$
Now, we prove the convergent part. We choose $x_n = r^n$, for $r < 1$ and $h_r(t) = r^{-2}h(t)$. Since h is increasing, we deduce that
$$\sum_n \mathbb{P}\Big(r^n > h_r\big(S_{r^{n+1}}\big)\Big) \le -\frac{1}{\log r}\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)\frac{\mathrm{d}t}{t}.$$
Replacing h by h_r in (2.20), we see that we can obtain our result if
$$\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)\frac{\mathrm{d}t}{t} < \infty.$$
From elementary calculations, we deduce that
$$\int_0^r \mathbb{P}\Big(t > h\big(S_t\big)\Big)\frac{\mathrm{d}t}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{I}_{\big\{t/r < S_1 < t/h(t)\big\}}\frac{\mathrm{d}t}{t}\right),$$
where $h^{-1}(s) = \inf\{t > 0, h(t) > s\}$, the right inverse function of h. Then, this integral converges if
$$\int_0^{h^{-1}(r)} \mathbb{P}\left(S_1 < \frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty.$$

This proves part (i).
The divergent part follows from the integral test for the upper envelope of the future infimum of pssMp, see section 3 (Theorem 6, part (ii)) since $X_t \geq J_t$, for all $t \geq 0$. ■

The upper envelope at $+\infty$ for pssMp is as follows;

PROPOSITION 9. *Let* $h \in \mathcal{H}_\infty$.

i) *If*

$$\int_{+\infty} G\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$ *and for all* $x \geq 0$

$$\mathbb{P}\Big(X_t^{(x)} > (1+\epsilon)h(t), \text{ i.o., as } t \to +\infty\Big) = 0.$$

ii) *If*

$$\int_{+\infty} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty,$$

then for all $\epsilon > 0$ *and for all* $x \geq 0$

$$\mathbb{P}\Big(X^{(x)} < (1-\epsilon)h(t), \text{ i.o., as } t \to +\infty\Big) = 1.$$

Proof: We first consider the case where $x = 0$. In this case the proof of the tests at $+\infty$ is almost the same as that of the tests at 0. It is enough to apply the same arguments to the sequence $x_n = r^n$, for $r > 1$.
Now, we prove (i) for any $x > 0$. Let $h \in \mathcal{H}_\infty$ such that

$$\int^{+\infty} G\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty.$$

Let $x > 0$ and S_x and note by μ_x the law of $X^{(0)}_{S_x}$. Since clearly

$$\int^{+\infty} G\left(\frac{t}{h(t-S_x)}\right)\frac{\mathrm{d}t}{t} < \infty,$$

from the Markov property at time S_x, we have for all $\epsilon > 0$

$$\begin{aligned} &\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t-S_x), \text{ i. o., as } t \to \infty\Big) \\ &= \int_{[x,+\infty)} \mathbb{P}_y\Big(X_t > (1+\epsilon)h(t), \text{ i. o., as } t \to \infty\Big)\mu_x(\mathrm{d}y) = 0. \end{aligned} \tag{2.21}$$

If x is an atom of μ_x, then equality (2.21) shows that

$$\mathbb{P}\Big(X_t^{(x)} > (1+\epsilon)h(t), \text{ i. o., as } t \to \infty\Big) = 0$$

and the result is proved. Suppose that x is not an atom of μ_x. From Caballero and Chaumont **[CaCh06]**, Theorem 1 we know that $X^{(0)}_{S_x} \stackrel{(d)}{=} xe^\theta$, (see also Chapter 1).
In Chapter 1, we also determined the law of θ. Hence from (1.1), we can easily deduce that

$$\mathbb{P}(e^\theta > z) > 0 \qquad \text{for} \quad z > 1,$$

and for any $\alpha > 0$, $\mu_x(x, x+\alpha) > 0$. Then (2.21) shows that there exists $y > x$ such that

$$\mathbb{P}\Big(X_t^{(y)} > (1+\epsilon)h(t), \text{ i. o., as } t \to \infty\Big) = 0,$$

for all $\epsilon > 0$. The previous allows us to conclude part (i).
Part (ii) can be proved in the same way. ■

CHAPTER 3

Regular cases.

The aim of this chapter is to provide interesting applications of the general integral tests established in Chapter 2. Here, we will assume that each tail probability that we considered in our main integral tests satisfies a regular condition either in 0 or in ∞ depending on the case. We also provide some explicit examples.

1. The lower envelope

Recall that

$$I = \int_0^{+\infty} \exp\left\{\hat{\xi}_s\right\} \mathrm{d}s \quad \text{and} \quad F(t) = \mathbb{P}(I > t).$$

In this section, we consider that F is regularly varying at infinity, i.e.

$$F(t) \sim \lambda t^{-\gamma} L(t)\,, \quad t \to +\infty\,, \tag{3.1}$$

where $\gamma > 0$ and L is a slowly varying function at $+\infty$. As shown in the following lemma, under this assumption, for any $q > 0$ the functions F_q and F are equivalent, i.e. $F_q \asymp F$.

Examples that satisfy the above condition are given by transient Bessel processes raised to any power and more generally when the process ξ satisfies the so called Cramér's condition, that is,

$$\text{there exists } \gamma > 0 \text{ such that } E(e^{-\gamma\xi_1}) = 1. \tag{3.2}$$

In that case, Rivero [**Rive05**] and Maulik and Zwart [**MaZw06**] proved by using results of Kesten and Goldie on tails of solutions of random equations that the behavior of $\mathbb{P}(I > t)$ is given by

$$F(t) \sim Ct^{-\gamma}\,, \quad \text{as } t \to +\infty, \tag{3.3}$$

where the constant C is explicitly computed in [**Rive05**] and [**MaZw06**].

Stable Lévy processes conditioned to stay positive are themselves positive self-similar Markov processes which belong to the regular case. These processes are defined as h-processes of the initial process when it starts from $x > 0$ and killed at its first exit time of $(0, \infty)$. Denote by (q_t) the semigroup of a stable Lévy process Y with index $\alpha \in (0, 2]$, killed at time $R = \inf\{t : Y_t \leq 0\}$. The function $h(x) = x^{\alpha(1-\rho)}$, where $\rho = \mathbb{P}(Y_1 \geq 0)$, is invariant for the semi-group (q_t), i.e. for all $x \geq 0$ and $t \geq 0$, $E_x(h(Y_t)\mathbb{1}_{\{t<R\}}) = h(x)$, ($E_x$ denotes the law of $Y + x$). The Lévy process Y conditioned to stay positive is the strong Markov process whose semigroup is

$$p_t^{\uparrow}(x, dy) := \frac{h(y)}{h(x)} q_t(x, dy),\; x > 0, y > 0,\; t \geq 0\,. \tag{3.4}$$

We will denote this process by $X^{(x)}$ when it is issued from $x > 0$. We refer to [**Chau96**] for more on the definition of Lévy processes conditioned to stay positive and for a proof of the above facts. It is easy to check that the process $X^{(x)}$ is self-similar and drifts towards $+\infty$. Moreover, it is proved in [**Chau96**], Theorem 6 that $X^{(x)}$ converges weakly as $x \to 0$ towards a non degenerated process $X^{(0)}$ in the Skorohod's space, so from [**CaCh06**], the underlying Lévy process in the Lamperti representation of $X^{(x)}$ satisfies condition $(\mathbf{H})$.

We can check that the law of $X^{(x)}$ belongs to the regular case by using the equality in law (1.17). Indeed, it follows from Proposition 1 and Theorem 4 in [**Chau96**] that the law of the exponential functional I is given by

$$\mathbb{P}(t < x^\alpha I) = x^{1-\alpha\rho} E_{-x}\left(\hat{Y}_t^{\alpha\rho-1} \mathbb{I}_{\{t<\hat{R}\}}\right), \tag{3.5}$$

where $\hat{Y} = -Y$ and $\hat{R} = \inf\{t : \hat{Y}_t \le 0\}$. If Y (and thus $X^{(0)}$) has no positive jumps, then $\alpha\rho = 1$ and it follows from (3.5) and Lemma 1 in [**Chau97**] that

$$\mathbb{P}(t < I) = Ct^{-\rho}. \tag{3.6}$$

We conjecture that (3.6) is also valid when Y has positive jumps. We also emphasize the possibility that the underlying Lévy process in the Lamperti representation of $X^{(x)}$ even satisfies (3.2) with $\gamma = \rho$.

LEMMA 7. *Recall that*

$$I_q = \int_0^{T_{-q}} \exp\left\{\hat{\xi}_s\right\} \mathrm{d}s \qquad \textit{and} \qquad F_q(t) = \mathbb{P}(I_q > t).$$

If (3.1) *holds then for all* $q > 0$*,*

$$(1 - e^{-\gamma q})F(t) \le F_q(t) \le F(t), \tag{3.7}$$

for all t *large enough.*

Proof: Recall from Lemma 4, that if $(\hat{\xi}_s,\, s \le \hat{T}_{-q})$ and $\hat{\xi}' \overset{\text{(def)}}{=} (\hat{\xi}_{s+\hat{T}_{-q}} - \hat{\xi}_{\hat{T}_{-q}},\, s \ge 0)$ then

$$I = I_q + \exp(\hat{\xi}_{\hat{T}_{-q}})\hat{I}' \le I_q + e^{-q}\hat{I}' \quad \text{where} \quad \hat{I}' = \int_0^\infty \exp\left\{\hat{\xi}'_s\right\} \mathrm{d}s. \tag{3.8}$$

The exponential functional $\hat{I}'$ is a copy of I which is independent of I_q. It yields the second equality of the lemma.

To show the first inequality, we write for all $\delta > 0$,

$$\mathbb{P}(I > (1+\delta)t) \le \mathbb{P}(I_q + e^{-q}\hat{I}' \ge (1+\delta)t)$$

which implies that,

$$\begin{aligned} \mathbb{P}(I > (1+\delta)t) &\le \mathbb{P}(I_q > t) + \mathbb{P}(e^{-q}I > t) + \mathbb{P}(I_q > \delta t)\mathbb{P}(e^{-q}I > \delta t) \\ &\le \mathbb{P}(I_q > t) + \mathbb{P}(e^{-q}I > t) + \mathbb{P}(I > \delta t)\mathbb{P}(e^{-q}I > \delta t), \end{aligned}$$

so that

$$\liminf_{t\to+\infty} \frac{\mathbb{P}(I_q > t)}{\mathbb{P}(I > t)} \ge (1+\delta)^{-\gamma} - e^{-q\gamma},$$

and the result follows since δ can be chosen arbitrary small. ■

The regularity of the behaviour of F allows us to drop the ε of Theorems 2 and 3 in the next integral test.

THEOREM 10. *Under condition* (3.1)*, the lower envelope of* $X^{(0)}$ *at* 0 *and at* $+\infty$ *is as follows:*
Let f *be an increasing function, such that either* $\lim_{t\downarrow 0} f(t)/t = 0$*, or* $\liminf_{t\downarrow 0} f(t)/t > 0$*, then:*

$$\mathbb{P}\Big(X_t^{(0)} < f(t), \textit{i.o., as } t \to 0\Big) = 0 \textit{ or } 1,$$

according as

$$\int_{0+} F\left(\frac{t}{f(t)}\right)\frac{dt}{t} \quad \textit{is finite or infinte}.$$

Let g *be an increasing function, such that either* $\lim_{t\uparrow\infty} g(t)/t = 0$*, or* $\liminf_t g(t)/t > 0$*, then for all* $x \geq 0$*,*

$$\mathbb{P}\Big(X_t^{(x)} < g(t), \textit{i.o., as } t \to +\infty\Big) = 0 \textit{ or } 1,$$

according as

$$\int^{+\infty} F\left(\frac{t}{g(t)}\right)\frac{dt}{t} \quad \textit{is finite or infinte}.$$

Proof: First let us check that for any constant $\beta > 0$:

$$(3.9) \qquad \int_{0+}^{\lambda} F\left(\frac{s}{f(s)}\right)\frac{ds}{s} < \infty \quad \text{if and only if} \quad \int_{0+}^{\lambda} F\left(\frac{\beta s}{f(s)}\right)\frac{ds}{s} < \infty\,.$$

From the hypothesis, either $\lim_{t\downarrow 0} f(t)/t = 0$, or $\liminf_{t\downarrow 0} f(t)/t > 0$. In the first case, we deduce (3.9) from (3.1). In the second case, since

$$\mathbb{P}(I > \lambda) > 0, \qquad \text{for any} \quad 0 < \lambda < \infty$$

and $\limsup_{u\downarrow 0} u/f(u) < +\infty$, we have for any s,

$$0 < \mathbb{P}\left(\limsup_{u\downarrow 0}\frac{u}{f(u)} < I\right) < \mathbb{P}\left(\frac{s}{f(s)} < I\right),$$

so both of the integrals above are infinite.

Now it follows from Theorem 2 part (i) that if

$$\int_{0+} F\left(\frac{t}{f(t)}\right)\frac{dt}{t} < \infty$$

then for all $\varepsilon > 0$,

$$\mathbb{P}\Big(X_t^{(0)} < (1-\varepsilon)f(t), \text{i.o., as } t \to 0\Big) = 0.$$

If

$$\int_{0+} F\left(\frac{t}{f(t)}\right)\frac{dt}{t} = \infty$$

then from Lemma 7, for all $q > 0$,

$$\int_{0+} F_q\left(\frac{t}{f(t)}\right)\frac{dt}{t} = \infty,$$

and it follows from Theorem 2 part (ii) that for all $\varepsilon > 0$,

$$\mathbb{P}\Big(X_t^{(0)} < (1+\varepsilon)f(t), \text{i.o., as } t \to 0\Big) = 1.$$

Then the equivalence (3.9) allows us to drop ε in these implications.

The test at $+\infty$ is proven through the same way. ∎

Remarks: **1.** It is possible to obtain the divergent parts of Theorem 10 by applying parts (iii) of Theorems 2 and 3 but then, one has to assume that $f(t)/t$ is an increasing (respectively a decreasing) function for the test at 0 (respectively at $+\infty$), which is slightly stronger than the hypothesis on f of Theorem 10.
2. This result is due to Dvoretzky and Erdős [**DvEr51**] and Motoo [**Moto58**] when $X^{(0)}$ is a transient Bessel process, i.e. the square root of the solution of the SDE:

$$Z_t = 2\int_0^t \sqrt{Z_s}\, dB_s + \delta t\,, \tag{3.10}$$

where $\delta > 2$ and B is a standard Brownian motion. We recall that when δ is an integer, $X^{(0)} = \sqrt{Z}$ has the same law as the norm of the δ-dimensional Brownian motion. Processes $X^{(0)} = \sqrt{Z}$ such that Z satisfies the equation (3.10) with $\delta > 2$ are the only continuous self-similar Markov process with index $\alpha = 2$, which drifts towards $+\infty$. In this particular case, thanks to the time-inversion property, i.e.:

$$(X_t,\, t > 0) \overset{(d)}{=} (tX_{1/t},\, t > 0),$$

we may deduce the test at $+\infty$ from the test at 0.
3. A possible way to improve the test at ∞ in the general case (that is in the setting of Theorem 2) would be to first establish it for the Ornstein-Uhlenbeck process associated to $X^{(0)}$, i.e. $(e^{-t}X^{(0)}(e^t),\, t \geq 0)$, as Motoo did for Bessel processes in [**Moto58**]. This would allow us to consider test functions which are not necessarily increasing.

2. The lower envelope of the first and last passage times

We begin this section with the study of the lower envelope of the last passage time process. Recall that

$$\bar{F}(t) = \mathbb{P}(I < t) \quad \text{and} \quad \bar{F}_\nu(t) = \mathbb{P}(\nu I < t),$$

where ν is independent of I and has the same las as $x_1^{-1}\Gamma$. We also recall that the support of the distribution of ν is the interval $[0, 1]$.

Here, we consider that $\bar{F}$ and $\bar{F}_\nu$ satisfy

$$ct^\alpha L(t) \leq \bar{F}(t) \leq \bar{F}_\nu(t) \leq Ct^\alpha L(t) \qquad \text{as} \quad t \to 0, \tag{3.11}$$

where $\alpha > 0$, c and C are two positive constants such that $c \leq C$ and L is a slowly varying function at 0. An important example included in this case is when $\bar{F}$ and $\bar{F}_\nu$ are regularly varying functions at 0.
The "regularity" of the behaviour of $\bar{F}$ and $\bar{F}_\nu$ gives us the following integral tests for the lower envelope of the last passage time process at 0.

THEOREM 11. *Under condition (3.11), the lower envelope of U at 0 and at $+\infty$ is as follows:*

i) *Let $h \in \mathcal{H}_0^{-1}$, such that either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$, then*

$$\mathbb{P}\Big(U(x) < h(x),\ \textit{i.o., as } x \to 0\Big) = 0 \textit{ or } 1,$$

according as

$$\int_{0^+} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \qquad \textit{is finite or infinite.}$$

ii) *Let $h \in \mathcal{H}_\infty^{-1}$, such that either* $\lim_{x\to+\infty} h(x)/x = 0$ *or* $\liminf_{x\to+\infty} h(x)/x > 0$, *then*

$$\mathbb{P}\Big(U(x) < h(x), \text{ i.o., as } x \to \infty\Big) = 0 \text{ or } 1,$$

according as

$$\int^{+\infty} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \qquad \text{is finite or infinite}.$$

Proof: First let us check that under condition (3.11) we have

$$\int_0^\lambda \bar{F}_\nu\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{if and only if} \quad \int_0^\lambda \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty. \tag{3.12}$$

Since $\nu I \leq I$ a.s., it is clear that we only need to prove that

$$\int_0^\lambda \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{implies that} \quad \int_0^\lambda \bar{F}_\nu\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty.$$

From the hypothesis, either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$. In the first case, from condition (3.11) there exists $\lambda > 0$ such that, for every $x < \lambda$

$$c\left(\frac{h(x)}{x}\right)^\alpha L\left(\frac{h(x)}{x}\right) \leq F\left(\frac{h(x)}{x}\right) \leq C\left(\frac{h(x)}{x}\right)^\alpha L\left(\frac{h(x)}{x}\right).$$

Since, we suppose that

$$\int_0^\lambda F\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty, \quad \text{then} \quad \int_0^\lambda \left(\frac{h(x)}{x}\right)^\alpha L\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty,$$

and again from condition (3.11), we get that

$$\int_0^\lambda F_\nu\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \quad \text{is also finite.}$$

In the second case, since

$$\mathbb{P}\big(I < \delta\big) > 0, \quad \text{for any} \quad 0 < \delta < \infty,$$

and $\liminf_{x\to 0} h(x)/x > 0$, we have for any y

$$0 < \mathbb{P}\left(I < \liminf_{x\to 0}\frac{h(x)}{x}\right) < \mathbb{P}\left(I < \frac{h(y)}{y}\right). \tag{3.13}$$

Hence, since for every $t \geq 0$, $F(t) \leq F_\nu(t)$, we deduce that

$$\int_0^\lambda F\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \int_0^\lambda F_\nu\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \infty.$$

Now, let us check that for any constant $\beta > 0$,

$$\int_0^\lambda \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{if and only if} \quad \int_0^\lambda \bar{F}\left(\frac{\beta h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty, \tag{3.14}$$

Again, from the hypothesis either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$. In the first case, we deduce (3.14) from (3.11). In the second case, from (3.13) both of the integrals in (3.14) are infinite.

Next, it follows from Theorem 4 part (i) and (3.12) that if

$$\int_{0+} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \quad \text{is finite,}$$

then for all $\epsilon > 0$,

$$\mathbb{P}\big(U(x) < (1-\epsilon)h(t), \text{ i.o., as } t \to 0\big) = 0.$$

If

$$\int_{0+} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \quad \text{diverges,}$$

then from Theorem 4 part (ii) that for all $\epsilon > 0$,

$$\mathbb{P}\big(U(x) < (1+\epsilon)h(t), \text{ i.o., as } t \to 0\big) = 1.$$

Then (3.14) allows us to drop ϵ in this implications.

The tests at $+\infty$ are proven through the same way. ■

Now, we turn our attention to the lower envelope of the first passage time. Recall that

$$G(t) = \mathbb{P}(S_1 < t).$$

The following Proposition shows that under condition (3.11), the functions $\bar{F}, \bar{F}_\nu$ and G have a similar behaviour at 0.

PROPOSITION 10. *Under condition (3.11), we have that*

$$ct^\alpha L(t) \le G(t) \le C_\epsilon t^\alpha L(t) \qquad \textit{as} \quad t \to 0,$$

where C_ϵ is a positive constant bigger than C.

Proof: The lower bound is clear since $\bar{F}(t) \le G(t)$, for all $t \ge 0$ and our assumption. Now, let us define $M^{(0)}_t = \sup_{0\le s\le t} X^{(0)}_s$ and fix $\epsilon > 0$. Then, by the Markov property and the fact that $J^{(x)}$ is an increasing process, we have

$$\begin{aligned}
\mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}\right) &\ge \mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}, M_1 \ge \frac{1}{t}\right)\\
&= \mathbb{E}\left(S_{1/t} \le 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_{1-S_{1/t}} > \frac{1-\epsilon}{t}\right)\right)\\
&\ge \mathbb{E}\left(S_{1/t} \le 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_0 > \frac{1-\epsilon}{t}\right)\right).
\end{aligned} \tag{3.15}$$

Since $X^{(0)}_{S_{1/t}} \ge 1/t$ a.s., and the Lamperti representation (0.15), we deduce that

$$\mathbb{E}\left(S_{1/t} \le 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_0 > \frac{1-\epsilon}{t}\right)\right) \ge \mathbb{P}\big(S_{1/t} < 1\big)\mathbb{P}\Big(\inf_{s\ge 0}\xi_s > \log(1-\epsilon)\Big).$$

On the other hand, under the assumption that ξ drifts towards $+\infty$, we know from Section 2 of Chaumont and Doney [**ChDo05**] that for all $\epsilon > 0$

$$K_\epsilon := \mathbb{P}\Big(\inf_{s\ge 0}\xi_s > \log(1-\epsilon)\Big) > 0.$$

Hence

$$K_\epsilon^{-1}\mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}\right) \ge \mathbb{P}\big(S_1 < t\big)$$

which implies that

$$CK_\epsilon^{-1}\left(\frac{t}{1-\epsilon}\right)^\alpha L(t) \ge K_\epsilon^{-1}\mathbb{P}\left(U_1 < \frac{t}{1-\epsilon}\right) \ge \mathbb{P}\big(S_1 < t\big), \qquad \text{as} \quad t \to 0,$$

then the proposition is proved. ■

The next result give us integral tests for the lower envelope of S at 0 and at ∞, under condition (3.11). In particular, we can deduce that the first and the last passage time processes have the same upper functions.

THEOREM 12. *Under condition (3.11), the lower envelope of S at 0 and at $+\infty$ is as follows:*

i) *Let $h \in \mathcal{H}_0^{-1}$, such that either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$, then*
$$\mathbb{P}\Big(S_x < h(t), \textit{ i.o., as } x \to 0\Big) = 0 \textit{ or } 1,$$
according as
$$\int_{0+} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \qquad \textit{is finite or infinite}.$$

ii) *Let $h \in \mathcal{H}_\infty^{-1}$, such that either $\lim_{x\to +\infty} h(x)/x = 0$ or $\liminf_{x\to +\infty} h(x)/x > 0$, then*
$$\mathbb{P}\Big(S_x < h(x), \textit{ i.o., as } x \to \infty\Big) = 0 \textit{ or } 1,$$
according as
$$\int^{+\infty} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \qquad \textit{is finite or infinite}.$$

Proof: First let us check that under condition (3.11) we have

$$\int_0^\lambda \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{if and only if} \quad \int_0^\lambda G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty. \tag{3.16}$$

Since $\bar{F}(t) \le G(t)$ for all $t \ge 0$, it is clear that we only need to prove that
$$\int_0^\lambda \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{implies that} \quad \int_0^\lambda G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty.$$

From the hypothesis, either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$. In the first case, from condition (3.11) there exists $\lambda > 0$ such that, for every $x < \lambda$
$$c\left(\frac{h(x)}{x}\right)^\alpha L\left(\frac{h(x)}{x}\right) \le \bar{F}\left(\frac{h(x)}{x}\right) \le C\left(\frac{h(x)}{x}\right)^\alpha L\left(\frac{h(x)}{x}\right).$$

Since, we suppose that
$$\int_0^\lambda \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \quad \text{is finite,}$$
then
$$\int_0^\lambda \left(\frac{h(x)}{x}\right)^\alpha L\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty,$$
hence from Proposition 10, we get that
$$\int_0^\lambda G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \quad \text{is also finite.}$$

In the second case, since for any $0 < \delta < \infty$, $\mathbb{P}\big(I < \delta\big) > 0$, and $\liminf_{x\to 0} h(x)/x > 0$, we have for any y

$$0 < \mathbb{P}\left(I < \liminf_{x\to 0}\frac{h(x)}{x}\right) < \mathbb{P}\left(I < \frac{h(y)}{y}\right). \tag{3.17}$$

Hence, since for every $t \geq 0$, $\bar{F}(t) \leq P(t)$, we deduce that

$$\int_0^\lambda \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \int_0^\lambda G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \infty.$$

On the other hand from (3.14), we recall that for $\beta > 0$,

$$\int_0^\lambda \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty \quad \text{if and only if} \quad \int_0^\lambda \bar{F}\left(\frac{\beta h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty. \tag{3.18}$$

Next, it follows from Proposition 6 part (i) and (3.16) that if

$$\int_{0+} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \quad \text{is finite,}$$

then for all $\epsilon > 0$,

$$\mathbb{P}\big(S_x < (1-\epsilon)h(x), \text{ i.o., as } x \to 0\big) = 0.$$

If

$$\int_{0+} \bar{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \quad \text{diverges,}$$

then from Proposition 6 part (ii) and (3.16) that for all $\epsilon > 0$,

$$\mathbb{P}\big(S_x < (1+\epsilon)h(x), \text{ i.o., as } x \to 0\big) = 1.$$

Then (3.18) allows us to drop ϵ in this implications.
The tests at $+\infty$ are proven through the same way. ■

2.1. Example. Let ξ a subordinator with zero drift and Lévy measure

$$\Pi(\mathrm{d}x) = \frac{\beta e^x}{\Gamma(1-\beta)(e^x-1)^{1+\beta}}\mathrm{d}x,$$

with $\beta \in (0,1)$. The pssMp $X^{(x)}$ associated to ξ is the stable subordinator of index β (see for instance Rivero [**Rive03**]).
From Zolotarev [**Zolo86**], we know that there exists k a positive constant such that

$$\mathbb{P}_0(X_1 > x) \sim kx^{-\beta} \qquad x \to +\infty.$$

It is well-known that the law of $X_1^{(0)}$ has a density ρ_1 with respect to the Lebesgue measure and that this density is unimodal, i.e., there exist $b > 0$ such that $\rho_1(x)$ is increasing in $(0,b)$ and decreasing in $(b,+\infty)$ (see for instance Sato [**Sato99**]). Hence ρ_1 is monotone in a neighborhood of $+\infty$, then by the monotone density Theorem (see Theorem 1.7.2 in Bingham et al.[**Bial89**] page 38) we get

$$\rho_1(x) \sim k\beta x^{-\beta-1} \qquad x \to +\infty.$$

On the other hand, from Proposition 2.1 in Carmona et al. [**CaPY97**] provided that $m < \infty$, we know that the law of I admits a density ρ which is infinitely differentiable on $(0,\infty)$. Moreover from (0.16), we have the following relation

$$\rho_1(x) = \frac{1}{mx}\rho\left(\frac{1}{x}\right) \quad \text{for} \quad x \in (0,\infty). \tag{3.19}$$

Hence, we can easily deduce that

$$\rho(x) \sim mk\beta x^{\beta} \qquad x \to 0,$$

and it is also easy to see that

$$\mathbb{P}\Big(I < x\Big) \sim mk\beta x^{\beta+1} \qquad x \to 0.$$

Note that in this example, we can not apply Theorem 11. The jumps of the stable subordinator contribute a lot on the estimate of $\bar{F}_\nu$ and have a different index of regularity as $\bar{F}$. A simple application of Theorems 4 and 5 gives us the following integral test.

COROLLARY 8. *Let ξ be a subordinator without drift and such that its Lévy mesure Π satisfies*

$$\Pi(\mathrm{d}x) = \frac{\beta e^x}{\Gamma(1-\beta)(e^x-1)^{1+\beta}}\mathrm{d}x.$$

The lower envelope of S, the first passage time of the pssMp $X^{(0)}$, at 0 and at $+\infty$ is as follows:

i) *Let $h \in \mathcal{H}_0^{-1}$, such that either $\lim_{x\to 0} h(x)/x = 0$ or $\liminf_{x\to 0} h(x)/x > 0$, then*

$$\mathbb{P}\Big(S_x < h(x), \text{ i.o., as } x \to 0\Big) = 0 \text{ or } 1,$$

according as

$$\int_{0+} \left(\frac{h(x)}{x}\right)^\beta \frac{\mathrm{d}x}{x} \qquad \text{is finite or infinite.}$$

ii) *Let $h \in \mathcal{H}_\infty^{-1}$, such that either $\lim_{x\to+\infty} h(x)/x = 0$ or $\liminf_{x\to+\infty} h(x)/x > 0$, then*

$$\mathbb{P}\Big(S_x < h(x), \text{ i.o., as } x \to \infty\Big) = 0 \text{ or } 1,$$

according as

$$\int^{+\infty} \left(\frac{h(x)}{x}\right)^{\beta+1} \frac{\mathrm{d}x}{x} \qquad \text{is finite or infinite.}$$

It is important to note that if we suppose that the$t \mapsto h(t)/t$ is also increasing, hence we may recover an integral test where the divergent part only depends on the index β.

3. The upper envelopes of positive self-similar Markov processes and its future infimum.

We begin this section describing the upper envelope of the future infimum.

THEOREM 13. *Under condition (3.11), the upper envelope of the future infimum at 0 and at $+\infty$ is as follows:*

i) *Let $h \in \mathcal{H}_0$, such that either $\lim_{t\to 0} t/h(t) = 0$ or $\liminf_{t\to 0} t/h(t) > 0$, then*

$$\mathbb{P}\Big(J_t^{(0)} > h(t), \text{ i.o., as } t \to 0\Big) = 0 \text{ or } 1,$$

according as

$$\int_{0+} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty \qquad \text{is finite or infinite.}$$

ii) *Let* $h \in \mathcal{H}_\infty$*, such that either* $\lim_{t\to+\infty} t/h(t) = 0$ *or* $\liminf_{t\to+\infty} t/h(t) > 0$*, then for all* $x \geq 0$

$$\mathbb{P}\Big(J_t^{(x)} > h(t),\ \textit{i.o., as } t \to \infty\Big) = 0 \textit{ or } 1,$$

according as

$$\int^{+\infty} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty \qquad \textit{is finite or infinite}.$$

Proof: We prove this result by following the same arguments as the proof of Theorem 11. ■

Now, we turn our attention to the upper envelope of positive self-similar Markov processes. In particular, we deduce that under condition (3.11) a pssMp and its future infimum have the same upper functions.

THEOREM 14. *Under condition (3.11), the upper envelope of the pssMp at* 0 *and at* $+\infty$ *is as follows:*

i) *Let* $h \in \mathcal{H}_0$*, such that either* $\lim_{t\to0} t/h(t) = 0$ *or* $\liminf_{t\to0} t/h(t) > 0$*, then*

$$\mathbb{P}\Big(X_t^{(0)} > h(t),\ \textit{i.o., as } t \to 0\Big) = 0 \textit{ or } 1,$$

according as

$$\int_{0+} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty \qquad \textit{is finite or infinite}.$$

ii) *Let* $h \in \mathcal{H}_\infty$*, such that either* $\lim_{t\to+\infty} t/h(t) = 0$ *or* $\liminf_{t\to+\infty} t/h(t) > 0$*, then for all* $x \geq 0$

$$\mathbb{P}\Big(X_t^{(x)} > h(t),\ \textit{i.o., as } t \to \infty\Big) = 0 \textit{ or } 1,$$

according as

$$\int^{+\infty} \bar{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} < \infty \qquad \textit{is finite or infinite}.$$

Proof: Similarly as the above theorem, we prove this result following the same arguments as the proof of Theorem 12. ■

3.1. Example. Let ξ a subordinator with zero drift and Lévy measure

$$\Pi(\mathrm{d}x) = \frac{\beta e^x}{\Gamma(1-\beta)(e^x-1)^{1+\beta}}\mathrm{d}x,$$

with $\beta \in (0,1)$. In Example 2.1, we noted that the pssMp $X^{(x)}$ associated to ξ is the stable subordinator of index β and that

$$\mathbb{P}(\nu I < x) \sim kx^\beta \quad \text{and} \quad \mathbb{P}\Big(I < x\Big) \sim mk\beta x^{\beta+1} \quad \text{as } x \to 0.$$

Then we have the following result.

COROLLARY 9. *Let* ξ *be a subordinator as in Corollary 8, the upper envelope of* $X^{(x)}$ *at* 0 *(*$x = 0$*) and at* $+\infty$ *(*$x \geq 0$*) is as follows:*

i) *Let* $h \in \mathcal{H}_0$, *such that either* $\lim_{t\to 0} t/h(t) = 0$ *or* $\liminf_{t\to 0} t/h(t) > 0$, *then*

$$\mathbb{P}\Big(X_t^{(0)} > h(t), \text{ i.o., as } t \to 0\Big) = 0 \text{ or } 1,$$

according as

$$\int_{0^+} \left(\frac{x}{h(x)}\right)^{\beta} \frac{\mathrm{d}x}{x} \quad \text{is finite or infinite}.$$

ii) *Let* $h \in \mathcal{H}_\infty$, *such that either* $\lim_{t\to +\infty} t/h(t) = 0$ *or* $\liminf_{t\to +\infty} t/h(t) > 0$, *then for all* $x \geq 0$

$$\mathbb{P}\Big(X_t^{(x)} > h(t), \text{ i.o., as } t \to \infty\Big) = 0 \text{ or } 1,$$

according as

$$\int^{+\infty} \left(\frac{x}{h(x)}\right)^{\beta+1} \frac{\mathrm{d}x}{x} \quad \text{is finite or infinite}.$$

We recall that if we suppose that thefunction $t \mapsto h(t)/t$ is also increasing, hence we may recover an integral test where the divergent part only depends on the index β.

CHAPTER 4

Log-regular cases.

Similarly as the previous chapter, our aim is to provide interesting applications of our main theorems, but here we will assume that the logarithm of each tail probability satisfies a regular condition either in 0 or in $+\infty$. Under this condition, the behaviour of the lower or upper envelope (depending on the case) is much smoother.

1. The lower envelope.

We first recall that

$$F(t) = \mathbb{P}(I > t) \quad \text{and} \quad I = \int_0^{+\infty} \exp\big\{-\xi_s\big\}\mathrm{d}s.$$

The type of behaviour that we shall consider here is when $\log F$ is regularly varying at $+\infty$, more precisely

$$-\log F(t) \sim \lambda t^{\beta} L(t)\,, \quad \text{as } t \to \infty, \tag{4.1}$$

where $\lambda > 0$, $\beta > 0$ and L is a function which varies slowly at $+\infty$. Define the function Φ by

$$\Phi(t) \stackrel{(\mathrm{def})}{=} \frac{t}{\inf\{s : 1/F(s) > |\log t|\}}\,, \quad t > 0\,,\ t \neq 1\,. \tag{4.2}$$

Then the lower envelope of $X^{(0)}$ may be described as follows:

THEOREM 15. *Under condition* (4.1)*, the process $X^{(0)}$ satisfies the following law of the iterated logarithm:*

(i)

$$\liminf_{t\to 0} \frac{X_t^{(0)}}{\Phi(t)} = 1\,, \quad \textit{almost surely.} \tag{4.3}$$

(ii) *For all $x \geq 0$,*

$$\liminf_{t\to +\infty} \frac{X_t^{(x)}}{\Phi(t)} = 1\,, \quad \textit{almost surely.} \tag{4.4}$$

Proof: We shall apply Theorem 2. We first have to check that under hypothesis (4.1), the conditions of part (iii) in Theorem 2 are satisfied. On the one hand, from (4.1) we deduce that for any $\gamma > 1$, $\limsup F(\gamma t)/F(t) = 0$. On the other hand, it is easy to see that both $\Phi(t)$ and $\Phi(t)/t$ are increasing in a neighbourhood of 0.

Let $\overline{L}$ be a slowly varying function such that

$$-\log F(\lambda^{-1/\beta} t^{1/\beta}\overline{L}(t)) \sim t\,, \quad \text{as } t \to +\infty. \tag{4.5}$$

Th. 1.5.12, p.28 in [**Bial89**] ensures that such a function exists and that

$$\inf\{s : -\log F(s) > t\} \sim \lambda^{-1/\beta} t^{1/\beta} \overline{L}(t)\,, \quad \text{as } t \to +\infty. \tag{4.6}$$

Then we have for all $k_1 < 1$ and $k_2 > 1$ and for all t sufficiently large,

$$k_1 \lambda^{-1/\beta} t^{1/\beta} \overline{L}(t) \le \inf\{s : -\log F(s) > t\} \le k_2 \lambda^{-1/\beta} t^{1/\beta} \overline{L}(t)$$

so that for Φ defined above and for all $k_2' > 0$,

$$-\log F\left(\frac{t\,k_2'}{k_2 \Phi(t)}\right) \le -\log F(k_2' \lambda^{-1/\beta} (\log|\log t|)^{1/\beta} \overline{L}(\log|\log t|)) \tag{4.7}$$

for all t sufficiently small. But from (4.5), for all $k_2'' > 1$ and for all t sufficiently small,

$$-\log F(k_2' \lambda^{-1/\beta} (\log|\log t|)^{1/\beta} \overline{L}(\log|\log t|)) \le k_2'' {k_2'}^{\beta} \log|\log t|\,,$$

hence

$$F\left(\frac{t\,k_2'}{k_2 \Phi(t)}\right) \ge (|\log t|)^{-k_2'' {k_2'}^{\beta}}\,.$$

By choosing $k_2' < 1$ and $k_2'' < (k_2')^{-\beta}$, we obtain the convergence of the integral

$$\int_{0+} F\left(\frac{t\,k_2'}{k_2 \Phi(t)}\right) \frac{\mathrm{d}t}{t}\,,$$

for all $k_2 > 1$ and $k_2' < 1$, which proves that for all $\varepsilon > 0$,

$$\mathbb{P}\Big(X_t^{(0)} < (1+\varepsilon)\Phi(t), \text{i.o., as } t \to 0\Big) = 1$$

from Theorem 2 (iii). The convergent part is proven through the same way so that from Theorem 2 (i), one has for all $\varepsilon > 0$,

$$\mathbb{P}\Big(X_t^{(0)} < (1-\varepsilon)\Phi(t), \text{i.o., as } t \to 0\Big) = 0$$

and the conclusion follows.

Condition (4.8) implies that $\Phi(t)$ is increasing in a neighbourhood of $+\infty$ whereas $\Phi(t)/t$ is decreasing in a neighbourhood of $+\infty$. Hence, the proof of the result at $+\infty$ is done through the same way as at 0, by using Theorem 3, (i) and (iii). ∎

1.1. Example. An example of such a behaviour is provided by the case where the process $X^{(0)}$ is increasing, that is when the underlying Lévy process ξ is a subordinator. Then Rivero [**Rive03**], see also Maulik and Zwart [**MaZw06**], proved that when the Laplace exponent ϕ of ξ which is defined by

$$\exp(-t\phi(\lambda)) = \mathbb{E}\Big(\exp\left\{\lambda \hat{\xi}_t\right\}\Big)\,, \quad \lambda > 0, \ \ t \ge 0$$

is regularly varying at $+\infty$ with index $\beta \in (0,1)$, the upper tail of the law of I and the asymptotic behavior of ϕ at $+\infty$ are related as follows:

PROPOSITION 11. *Suppose that ξ is a subordinator whose Laplace exponent ϕ varies regularly at infinity with index $\beta \in (0,1)$, then*

$$-\log F(t) \sim (1-\beta)\phi^{\leftarrow}(t)\,, \quad \textit{as } t \to \infty,$$

where $\phi^{\leftarrow}(t) = \inf\{s > 0 : s/\phi(s) > t\}$.

Then by using an argument based on the study of the associated Ornstein-Uhlenbeck process $(e^{-t}X^{(0)}(e^t),\, t \geq 0)$ Rivero [**Rive03**] derived from Proposition 11 the following result. Define

$$\varphi(t) = \frac{\phi(\log|\log t|)}{\log|\log t|}, \quad t > e\,.$$

COROLLARY 10. *If ϕ is regularly varying at infinity with index $\beta \in (0,1)$ then*

$$\liminf_{t\downarrow 0} \frac{X^{(0)}}{t\varphi(t)} = (1-\beta)^{1-\beta} \quad \textit{and} \quad \liminf_{t\uparrow +\infty} \frac{X^{(0)}}{t\varphi(t)} = (1-\beta)^{1-\beta}, \quad a.s.$$

This corollary is also a consequence of Proposition 11 and Theorem 15. To establish Corollary 10, Rivero assumed moreover that the density of the law of the exponential functional I is decreasing and bounded in a neighbourhood of $+\infty$. This additional assumption is actually needed to establish an integral test which involves the density of I and which implies Corollary 10.

2. The lower envelope of the first and last passage times.

Recall that

$$\bar{F}(t) = \mathbb{P}(I < t), \quad \bar{F}_\nu(t) = \mathbb{P}(\nu I < t) \quad \text{and} \quad G(t) = \mathbb{P}(S_1 < t),$$

where ν is independent of I and has the same law as $x_1^{-1}\Gamma$ (see section 2 in Chapter 1 for the definition of Γ). We also recall that the support of the distribution of ν is the interval $[0,1]$.

In this section, we will study two types of behaviour for $\bar{F}$ and $\bar{F}_\nu$. The first case that we shall consider is when $\log\bar{F}$ and $\log\bar{F}_\nu$ are regularly varying at 0, i.e

$$-\log\bar{F}_\nu(1/t) \sim -\log\bar{F}(1/t) \sim \lambda t^\beta L(t), \quad \text{as } t \to +\infty, \tag{4.8}$$

where $\lambda > 0$, $\beta > 0$ and L is a slowly varying function at $+\infty$. The second type of behaviour is when $\log\bar{F}$ and $\log\bar{F}_\nu$ satisfy that

$$-\log\bar{F}_\nu(1/t) \sim -\log\bar{F}(1/t) \sim K(\log t)^\gamma, \quad \text{as } t \to +\infty, \tag{4.9}$$

where K and γ are strictly positive constants.

Our next result shows that under conditions (4.8) and (4.9), the functions $\log G$, $\log\bar{F}_\nu$ and $\log\bar{F}$ are asymptotically equivalents.

PROPOSITION 12. *Under condition (4.8), we have that*

$$-\log G(1/t) \sim \lambda t^\beta L(t) \qquad \textit{as} \quad t \to +\infty. \tag{4.10}$$

Similarly, under condition (4.9), we have that

$$-\log G(1/t) \sim K(\log t)^\gamma \qquad \textit{as} \quad t \to +\infty. \tag{4.11}$$

Proof: First, we prove the upper bound of (4.10). We recall that $J_1^{(0)} = \inf_{t\geq 1} X_t^{(0)}$ and $M_1^{(0)} = \sup_{t\leq 1} X_t^{(0)}$. Hence, it is clear that,

$$-\log\mathbb{P}\Big(\nu I < 1/t\Big) = -\log\mathbb{P}_0\Big(J_1 > t\Big) \geq -\log\mathbb{P}_0\Big(M_1 > t\Big),$$

which implies that

$$1 \geq \limsup_{t\to\infty} \frac{-\log\mathbb{P}_0\Big(M_1 > t\Big)}{\lambda t^\beta L(t)},$$

and since $\mathbb{P}_0\big(M_1 > t\big) = \mathbb{P}\big(S_1 < 1/t\big)$, we get the upper bound.
Now, fix $\epsilon > 0$. From the inequality (3.15) found in the proof of Proposition 10, we have that

$$\mathbb{P}_0\big(J_1 > (1-\epsilon)t\big) \geq \mathbb{P}\big(S_t < 1\big)\mathbb{P}\Big(\inf_{s\geq 0} \xi_s > \log(1-\epsilon)\Big).$$

On the other hand, we know that

$$K_\epsilon := \mathbb{P}\Big(\inf_{s\geq 0} \xi_s > \log(1-\epsilon)\Big) > 0,$$

Hence,

$$-\log \mathbb{P}_0\big(J_1 > (1-\epsilon)t\big) \leq -\log \mathbb{P}\Big(S_1 < 1/t\Big) - \log K_\epsilon,$$

which implies that

$$(1-\epsilon)^\beta \leq \liminf_{t\to\infty} \frac{-\log \mathbb{P}\Big(S_1 < 1/t\Big)}{\lambda t^\beta L(t)},$$

and since ϵ can be chosen arbitrarily small, (4.10) is proved.
The upper bound of tail behaviour (4.11) is proven through the same way as in the proof of (4.10). For the lower bound, we follow the same arguments as above and we get that

$$-\log \mathbb{P}_0\big(J_1 > (1-\epsilon)t\big) \leq -\log \mathbb{P}\Big(S_1 < 1/t\Big) - \log K_\epsilon,$$

which implies that

$$1 = \lim_{t\to\infty} \left(\frac{\log(1-\epsilon)t}{\log t}\right)^\gamma \leq \liminf_{t\to\infty} \frac{-\log \mathbb{P}\Big(S_1 < 1/t\Big)}{K(\log t)^\gamma},$$

then the proposition is proved. ■

Define the functions

$$\bar{\Phi}(x) \stackrel{\text{(def)}}{=} \frac{x}{\inf\big\{s : 1/\bar{F}(1/s) > |\log x|\big\}}, \quad x > 0, \quad x \neq 1,$$

and

$$\hat{\Phi}(x) \stackrel{\text{(def)}}{=} x \exp\Big\{ -\big(K^{-1}\log|\log x|\big)^{1/\gamma}\Big\}, \quad x > 0, \quad x \neq 1.$$

Then the lower envelope of the first and last passage time processes may be described as follows.

THEOREM 16. *Under condition (4.8), we have the following laws of the iterated logarithm:*

i) *For the first passage time, we have*

$$\limsup_{x\to 0} \frac{S_x}{\bar{\Phi}(x)} = 1, \qquad \limsup_{x\to\infty} \frac{S_x}{\bar{\Phi}(x)} = 1 \quad \textit{almost surely.}$$

ii) *For the last passage time, we have*

$$\limsup_{x\to 0} \frac{U_x}{\bar{\Phi}(x)} = 1 \quad \textit{and} \quad \limsup_{x\to+\infty} \frac{U_x}{\bar{\Phi}(x)} = 1 \quad \textit{almost surely.}$$

Similarly, under condition (4.9), we have the following laws of the iterated logarithm:

iii) *For the first passage time, we have*

$$\limsup_{x\to 0} \frac{S_x}{\hat{\Phi}(x)} = 1, \qquad \limsup_{x\to\infty} \frac{S_x}{\hat{\Phi}(x)} = 1 \quad \textit{almost surely.}$$

iv) *For the last passage time, we have*

$$\limsup_{x\to 0} \frac{U_x}{\hat{\Phi}(x)} = 1 \quad \textit{and} \quad \limsup_{x\to +\infty} \frac{U_x}{\hat{\Phi}(x)} = 1 \quad \textit{almost surely.}$$

Proof: We first prove part (ii). This law of the iterated logarithm is a consequence of Theorems 4 and 5, and it is proven in the same way as Theorem 15, we only need to emphasize that we can replace $\log \bar{F}_\nu$ by $\log \bar{F}$, since they are asymptotically equivalent.

The proof of part (i) is very similar. Here, we will use Propositions 6, 7 and 12, and following the same arguments of Theorem 15. We only need to emphasize that we can replace $\log G$ by $\log \bar{F}$, since they are asymptotically equivalent.

Now, we prove part (iii). Here, we shall apply again Propositions 5,6 and 11. It is easy to check that both $\hat{\Phi}(x)$ and $\hat{\Phi}(x)/x$ are increasing in a neighbourhood of 0, moreover the function $\hat{\Phi}(x)/x$ is bounded by 1, for $x \in [0, 1)$.
From condition (4.9), we have for all $k_1 < 1$ and $k_2 > 1$ and for all t sufficiently large,

$$k_1 K(\log t)^\gamma \le -\log G(1/t) \le k_2 K(\log t)^\gamma,$$

so that for $\hat{\Phi}$ defined above,

$$k_1 \log|\log t| \le -\log G\left(\hat{\Phi}(x)/x\right) \le k_2 \log|\log t|,$$

hence

$$G\left(\frac{\hat{\Phi}(x)}{x}\right) \ge (|\log t|)^{-k_2}.$$

Since $k_2 > 1$, we obtain the convergence of the integral

$$\int_{0+} G\left(\frac{\hat{\Phi}(x)}{x}\right)\frac{\mathrm{d}t}{t},$$

which proves that for all $\varepsilon > 0$,

$$\mathbb{P}\Big(S_x < (1-\varepsilon)\hat{\Phi}(x), \text{i.o., as } x \to 0\Big) = 0$$

from Proposition 6 part (i). The divergent part is proven through the same way so that from Proposition 6 part (ii), one has for all $\varepsilon > 0$,

$$\mathbb{P}\Big(S_x < (1+\varepsilon)\hat{\Phi}(x), \text{i.o., as } x \to 0\Big) = 1$$

and the conclusion follows.
Condition (4.9) implies that $\hat{\Phi}(x)$ is increasing in a neighbourhood of $+\infty$ whereas $\hat{\Phi}(x)/x$ is decreasing in a neighbourhood of $+\infty$. Hence, the proof of the result at $+\infty$ is done through the same way as at 0, by using Proposition 7.
The laws of the iterated logarithm for the last passage time (part (iv))are proven in the same way using the integral tests for the lower envelope of process U (see Theorems 4 and 5). ■

2.1. Examples. 1. Let $X_t^{(0)}$ be a stable Lévy process conditioned to stay positive with no positive jumps and with index $1 < \alpha \le 2$, (see Example 2 in Chapter 3.1). From Theorem VII.18 in [**Bert96**], we know that the process time-reversed at its last passage time below x, $(x - X^{(0)}_{(U_x-t)^-}, 0 \le t \le U_x)$, has the same law as the killed process at its first passage time above x, $(\xi_t, 0 \le t \le T_x)$, where ξ is a stable Lévy process with no positive jumps and with the same index as $X^{(0)}$.
From Theorem VII.1 in [**Bert96**], we know that $(T_x, x \ge 0)$ is a subordinator with Laplace

exponent $\phi(\lambda) = \lambda^{1/\alpha}$. Hence by the previous argument, we will have that $X^{(0)}$ drifts towards $+\infty$ and that the process $(U_x, x \geq 0)$ is a stable subordinator with index $1/\alpha$. Hence an application of the Tauberian theorem of de Brujin (see for instance Theorem 5.12.9 in Bingham et al. [**Bial89**]) gives us the following estimate

$$-\log \bar{F}(x) \sim \frac{\alpha-1}{\alpha}\left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} x^{-1/(\alpha-1)} \quad \text{as} \quad x \to 0.$$

Note that due to the absence of positive jumps $\nu = 1$ a.s.
Then applying Theorem 16, we get the following laws of the iterated logarithm.

COROLLARY 11. *Let $X^{(0)}$ be a stable Lévy process conditioned to stay positive with no positive jumps and $\alpha > 1$. Then, its related first passage time process satisfies*

$$\liminf_{x\to 0} \frac{S_x\big(\log|\log x|\big)^{\alpha-1}}{x^\alpha} = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}, \qquad \textit{almost surely}.$$

Similarly, the last passage time process associated to $X^{(0)}$ satisfies

$$\liminf_{x\to 0} \frac{U(x)\big(\log|\log x|\big)^{\alpha-1}}{x^\alpha} = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}, \qquad \textit{almost surely}.$$

We have the same law of the iterated logarithm for large times.

2. Let $\xi = N$ be a standard Poisson process. From Proposition 3 in Bertoin and Yor [**BerY02**], we know that

$$-\log \mathbb{P}\big(I < t\big) \sim \frac{1}{2}(\log 1/t)^2, \qquad \text{as } t \to 0,$$

and also that

$$-\log \rho(x) \sim \frac{1}{2}(\log 1/x)^2, \qquad \text{as } x \to 0.$$

From (3.19) we get that

$$-\log \rho_1(x) \sim \frac{1}{2}(\log x)^2, \qquad \text{as } x \to +\infty.$$

Now, applying Theorem 4.12.10 in Bingham et al. [**Bial89**] and doing some elementary calculations, we obtain that

$$-\log \int_x^{+\infty} \rho_1(y)\mathrm{d}y \sim \frac{1}{2}(\log x)^2 \qquad \text{as } x \to +\infty.$$

These estimations allow us the following laws of the iterated logarithm. Let us define

$$f(x) = x \exp\Big\{ -\sqrt{2\log|\log x|}\Big\}.$$

Note that we cannot construct a weak limit process $X^{(0)}$ using the arguments of Chaumont and Caballero, since N is arithmetic. According to Bertoin and Caballero [**BeCa02**] a limit process (in the sense of finite dimensional distribution) can be defined. By an abuse of notation, we will denote such process by $X^{(0)}$.

COROLLARY 12. *Let N be a Poisson process and $X^{(0)}$ its associated pssMp starting from 0. Then, the first passage time process S associated to $X^{(0)}$ satisfies the following law of the iterated logarithm,*

$$\liminf_{x\to 0} \frac{S_x}{f(x)} = 1, \quad \textit{and} \quad \liminf_{x\to+\infty} \frac{S_x}{f(x)} = 1 \quad \textit{a. s.}$$

3. Let ξ be a subordinator with zero drift and Lévy measure $\Pi(\mathrm{d}x) = abe^{-bx}\mathrm{d}x$, with $a, b > 0$, i.e. a compound Poisson process with jumps having an exponential distribution. Carmona, Petit and Yor showed that the density ρ of I is given by

$$\rho(x) = \frac{a^{1+b}}{\Gamma(1+b)} x^b e^{-ax}, \qquad \text{for } x > 0.$$

The pssMp associated to ξ by the Lamperti representation is the well-know generalized Watanabe process. From (3.19), we get that

$$\mathbb{P}_0\big(X_1 > y\big) = \frac{ba^{1+b}}{\Gamma(1+b)} \int_0^{1/y} z^{b-1} e^{-az} \mathrm{d}z.$$

On the other hand, It is clear that

$$\mathbb{P}\Big(I < y\Big) = \frac{a^{1+b}}{\Gamma(1+b)} \int_0^y x^b e^{-ax} \mathrm{d}x$$

Elementary calculations give us the following inequality,

$$e^{-ax} \frac{x^{b+1}}{b+1} \leq \int_0^x z^b e^{-az} \mathrm{d}z \leq x^b \frac{(1 - e^{-ax})}{a}.$$

Hence for x sufficiently small, there exists c_b a positive constant such that

$$\mathbb{P}\Big(I < x\Big) \sim c_b \frac{a^{1+b}}{\Gamma(1+b)} x^{b+1} e^{-ax},$$

and for y sufficiently large there exist C_b such that

$$\mathbb{P}_0\Big(X_1 > y\Big) \sim C_b \frac{a^{1+b}}{\Gamma(1+b)} (1/y)^b e^{-a/y}.$$

Then applying Corollary 6 and Theorem 16, we get the following law of the iterated logarithm for the first passage time process of the generalized Watanabe process. Let us define

$$g(x) = a^{-1} t \log|\log x|.$$

COROLLARY 13. *Let ξ be a compound Poisson process with jumps having and exponential distribution as above and $X^{(0)}$ its associated pssMp starting from 0. Then the first passage time process S associated to $X^{(0)}$ satisfies the following law of the iterated logarithm,*

$$\liminf_{x \to 0} \frac{S_x}{g(x)} = 1, \quad \textit{and} \quad \liminf_{x \to 0} \frac{S_x}{g(x)} = 1 \quad \textit{a. s.}$$

3. The upper envelope

Now, we turn our attention to the upper envelopes of pssMp and their future infimum. With this purpose we define the functions,

$$\bar{\Psi}(t) \overset{\text{(def)}}{=} t \inf\big\{s : 1/\bar{F}(1/s) > |\log t|\big\}, \quad t > 0, \quad t \neq 1,$$

and

$$\hat{\Psi}(t) \overset{\text{(def)}}{=} t \exp\Big\{\big(K^{-1} \log|\log t|\big)^{1/\gamma}\Big\}, \quad t > 0, \quad t \neq 1.$$

THEOREM 17. *Under condition (4.8), we have the following laws of the iterated logarithm:*

i)

$$\limsup_{t\to 0}\frac{X_t^{(0)}}{\bar{\Psi}(t)}=1 \quad \textit{and} \quad \limsup_{t\to 0}\frac{J_t^{(0)}}{\bar{\Psi}(t)}=1 \quad \textit{almost surely.}$$

ii) *For all* $x\geq 0$,

$$\limsup_{t\to +\infty}\frac{X_t^{(x)}}{\bar{\Psi}(t)}=1 \quad \textit{and} \quad \limsup_{t\to 0}\frac{J_t^{(x)}}{\bar{\Psi}(t)}=1 \quad \textit{almost surely.}$$

Similarly, under condition (4.9), we have the following laws of the iterated logarithm:

iii)

$$\limsup_{t\to 0}\frac{X_t^{(0)}}{\hat{\Psi}(t)}=1 \quad \textit{and} \quad \limsup_{t\to 0}\frac{J_t^{(0)}}{\hat{\Psi}(t)}=1 \quad \textit{almost surely.}$$

iv) *For all* $x\geq 0$,

$$\limsup_{t\to +\infty}\frac{X_t^{(x)}}{\hat{\Psi}(t)}=1 \quad \textit{and} \quad \limsup_{t\to 0}\frac{J_t^{(x)}}{\hat{\Psi}(t)}=1 \quad \textit{almost surely.}$$

Proof: We prove this theorem by following the same arguments as in the proof of Theorem 16. ■

It is important to note that even if that under condition(4.8) a pssMp and its future infimum satisfy the same law of the iterated logarithm, they do not necessarily have the same upper functions.

3.1. Examples. **1.** Let $X_t^{(0)}$ be a stable Lévy process conditioned to stay positive with no positive jumps. The absence of positive jumps implies that $\alpha\geq 1$, here we exclude the case $\alpha=1$ which corresponds to the symmetric Cauchy process. Recall that the function $\bar{F}$ satisfies

$$-\log\bar{F}(x)\sim\frac{\alpha-1}{\alpha}\left(\frac{1}{\alpha}\right)^{1/(\alpha-1)}x^{-1/(\alpha-1)} \quad \text{as} \quad x\to 0.$$

Note that due to the absence of positive jumps $\nu=1$ a.s.
Then applying Theorem 17, we get the following laws of the iterated logarithm.

COROLLARY 14. *Let $X^{(0)}$ be a stable Lévy process conditioned to stay positive with no positive jumps and $\alpha>1$. Then the future infimum process of $X^{(0)}$ satisfies*

$$\limsup_{t\to 0}\frac{J_t^{(0)}}{t^{1/\alpha}\big(\log|\log x|\big)^{1-1/\alpha}}=\alpha(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \quad \textit{almost surely},$$

and for all $x\geq 0$,

$$\limsup_{t\to +\infty}\frac{J_t^{(x)}}{t^{1/\alpha}\big(\log|\log x|\big)^{1-1/\alpha}}=\alpha(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \quad \textit{almost surely}.$$

Similarly the pssMp satisfies the following law of the iterated logarithm:

$$\limsup_{t\to 0}\frac{X_t^{(0)}}{t^{1/\alpha}\big(\log|\log x|\big)^{1-1/\alpha}}=\alpha(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \quad \textit{almost surely},$$

and for all $x\geq 0$,

$$\limsup_{t\to +\infty}\frac{X_t^{(x)}}{t^{1/\alpha}\big(\log|\log x|\big)^{1-1/\alpha}}=\alpha(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \quad \textit{almost surely}.$$

2. Let $\xi = N$ be a standard Poisson process. Recall that

$$-\log \mathbb{P}\Big(I < t\Big) \sim \frac{1}{2}(\log 1/t)^2, \qquad \text{as } t \to 0,$$

and also that

$$-\log \int_x^{+\infty} \rho_1(y)\mathrm{d}y \sim \frac{1}{2}(\log x)^2 \qquad \text{as } x \to +\infty.$$

These estimations allow us the following laws of the iterated logarithm. Let us define

$$f(t) = t\exp\Big\{ - \sqrt{2\log|\log t|}\Big\}.$$

COROLLARY 15. *Let N be a Poisson process, then the pssMp $X^{(x)}$ associated to N by the Lamperti representation satisfies the following law of the iterated logarithm,*

$$\limsup_{t\to 0} \frac{X_t^{(0)} f(t)}{t^2} = 1, \quad \textit{almost surely.}$$

For all $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{X_t^{(x)} f(t)}{t^2} = 1, \quad \textit{almost surely.}$$

3. Let ξ be a subordinator with zero drift and Lévy measure $\Pi(\mathrm{d}x) = abe^{-bx}\mathrm{d}x$, with $a, b > 0$, i.e. a compound Poisson process with jumps having an exponential distribution. Recall from example 3 in Section 2.1 that the function $\bar{F}$ satisfies for t sufficiently small

$$F(t) = \mathbb{P}\Big(I < t\Big) \sim c_b \frac{a^{1+b}}{\Gamma(1+b)} t^{b+1} e^{-at},$$

where c_b is a positive constant, and for s sufficiently large there exist C_b such that

$$H(s) = \mathbb{P}_0\Big(X_1 > s\Big) \sim C_b \frac{a^{1+b}}{\Gamma(1+b)} (1/s)^b e^{-a/s}.$$

Then applying Corollary 7 and Theorem 17, we get the following law of the iterated logarithm for the generalized Watanabe process. Let us define

$$g(t) = a^{-1} t \log|\log t|.$$

COROLLARY 16. *Let ξ be a compound Poisson process with jumps having and exponential distribution as above, then the pssMp $X^{(x)}$ associated to ξ by the Lamperti representation satisfies the following law of the iterated logarithm,*

$$\limsup_{t\to 0} \frac{X_t^{(0)} g(t)}{t^2} = 1, \quad \textit{almost surely.}$$

For all $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{X_t^{(x)} g(t)}{t^2} = 1, \quad \textit{almost surely.}$$

4. The case when ξ has finite exponential moments

In this section, we suppose that the Lévy process ξ associated to a pssMp $X^{(x)}$ by the Lamperti representation, has finite exponential moments of arbitrary positive order. This condition is satisfied, for example, when the jumps of ξ are bounded from above by some fixed number, in particular when ξ is a Lévy process with no positive jumps. Then, we have

$$\mathbb{E}\big(e^{\lambda\xi_t}\big) = \exp\big\{t\psi(\lambda)\big\} < +\infty \qquad t, \lambda \geq 0.$$

From Theorem 25.3 in Sato [**Sato99**], we know that this hypothesis is equivalent to assume that the Lévy measure Π of ξ satisfies

$$\int_{[1,\infty)} e^{\lambda x}\Pi(\mathrm{d}x) < +\infty \quad \text{for every} \quad \lambda > 0.$$

Under this hypothesis, Bertoin and Yor [**BerY02**] gave a formula for the negative moments of the exponential functional I

$$\mathbb{E}\Big(I^{-k}\Big) = m\frac{\psi(1)\cdots\psi(k-1)}{(k-1)!} \quad \text{for} \quad k \geq 1, \tag{4.12}$$

where $m = \mathbb{E}(\xi_1)$ and with the convention that the right-hand side equals m for $k = 1$. Moreover they proved that if ξ has no positive jumps, then I^{-1} admits some exponential moments, this means that the distribution of I is determined by its negative entire moments. From the entrance law of $X^{(x)}$ at 0 (see (0.16)), and the above equality (4.12), we get the following formula for the positive moments of $X_1^{(0)}$,

$$\mathbb{E}_0\big(X_1^k\big) = \frac{\psi(1)\cdots\psi(k)}{k!} \quad \text{for} \quad k \geq 1. \tag{4.13}$$

Now, if we suppose that the Laplace exponent ψ is regularly varying at $+\infty$ with index $\beta \in (1, 2)$, i.e. $\psi(x) = x^\beta L(x)$, where L is a slowly varying function at $+\infty$; then from equation (4.12), we see

$$\mathbb{E}\Big(I^{-k}\Big) = m\big((k-1)!\big)^{\beta-1}L(1)\cdots L(k-1),$$

and from (4.13),

$$\mathbb{E}_0\big(X_1^k\big) = \big(k!\big)^{\beta-1}L(1)\cdots L(k).$$

In consequence, we can easily deduce that

$$\mathbb{E}\Big(\exp\big\{\lambda I^{-1}\big\}\Big) < +\infty \quad \text{and} \quad \mathbb{E}_0\Big(\exp\{\lambda X_1\}\Big) < +\infty \quad \text{for all } \lambda > 0.$$

This allows us to apply the Kasahara's Tauberian Theorem (see Theorem 4.12.7 in Bingham et al. [**Bial89**]) and get the following estimate.

PROPOSITION 13. *Let I be the exponential functional associated to the Lévy process ξ. Suppose that ψ, the Laplace exponent of ξ, varies regularly at $+\infty$ with index $\beta \in (1, 2)$. Then*

$$-\log \mathbb{P}_0(X_1 > x) \sim -\log \mathbb{P}\,(I < 1/x) \sim (\beta-1)\overleftarrow{H}(x) \quad \textit{as} \quad x \to +\infty, \tag{4.14}$$

where

$$\overleftarrow{H}(x) = \inf\Big\{s > 0\,,\, \psi(s)/s > x\Big\}.$$

Recall that if the process ξ has no positive jumps then the fact that the Laplace exponent ψ is regularly varying at ∞ with index $\beta \in (1,2)$ is equivalent to that ξ satisfies the Spitzer's condition (see Proposition VII.6 in Bertoin [**Bert96**]), this is

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \mathbb{P}(\xi_s \geq 0)\mathrm{d}s = \frac{1}{\beta}.$$

Proof: As we see above, the moment generating functions of I^{-1} and $X_1^{(0)}$ are well defined for all $\lambda > 0$. We will only prove the case of I^{-1}, the proof of the estimate of the tail probability of $X_1^{(0)}$ is similar.
From the main result of Geluk [**Gelu84**], we know that if ϕ is a regularly varying function at $+\infty$ with index $\sigma \in (0,1)$, then the following are equivalent

$$\text{(i)} \qquad \Big(\mathbb{E}\Big(I^{-n}\Big)/n!\Big)^{1/n} \sim e^{\sigma}/\phi(n) \qquad \text{as } n \to +\infty,$$

$$\text{(ii)} \qquad \log\mathbb{E}\bigg(\exp\Big\{\lambda I^{-1}\Big\}\bigg) \sim \sigma\overleftarrow{\phi}(\lambda) \qquad \text{as } \lambda \to +\infty,$$

where $\overleftarrow{\phi}(\lambda) = \inf\Big\{s > 0\,,\, \phi(s) > \lambda\Big\}$.
If we have (ii), then a straightforward application of Kasahara's Tauberian Theorem gives us

$$-\log\mathbb{P}\Big(I^{-1} > x\Big) \sim (1-\sigma)\overleftarrow{\varphi}(x) \qquad \text{as } x \to +\infty,$$

where $\overleftarrow{\varphi}$ is the asymptotic inverse of $s/\phi(s)$. Therefore, it is enough to show (i) with $\phi(s) = s^2/\psi(s)$ to obtain the desired result.
Let us recall that if ψ is regularly varying at ∞ with index β, it can be expressed as $\psi(x) = x^{\beta}L(x)$, where $L(x)$ is a slowly varying function. By the formula (4.12) of negative moments of I and the fact that ψ is regularly varying, we have

$$\Big(\mathbb{E}\big(I^{-n}\big)/n!\Big)^{1/n} = m^{1/n}(n!)^{\frac{\beta-2}{n}}n^{\frac{1-\beta}{n}}\big(L(1)\cdots L(n-1)\big)^{\frac{1}{n}},$$

due to the fact that $(n!)^{1/n} \sim ne^{-1}$ for n sufficiently large, then

$$\Big(\mathbb{E}\big(I^{-n}\big)/n!\Big)^{1/n} \sim \big(ne^{-1}\big)^{\beta-2}\exp\left\{\frac{1}{n}\sum_{k=1}^{n}\log L(k) - \frac{1}{n}\log L(n)\right\}.$$

On the other hand, from the proof of Proposition 2 of Rivero [**Rive03**] we have that

$$\frac{1}{n}\sum_{k=1}^{n}\log L(k) \sim \log L(n) \qquad \text{as} \quad n \to +\infty.$$

This implies that

$$\Big(\mathbb{E}\big(I^{-n}\big)/n!\Big)^{1/n} \sim e^{2-\beta}\frac{\psi(n)}{n^2}.$$

This last relation proves the Proposition. ■

Since the tail probability of I^{-1} and $X_1^{(0)}$ have the same asymptotic behaviour, it is logical to think that the tail probability of $(\nu I)^{-1}$ could have the same behaviour. The next Corollary confirms this last argument.

COROLLARY 17. *Let I be the exponential functional associated to the Lévy process ξ. Suppose that ψ, the Laplace exponent of ξ, varies regularly at $+\infty$ with index $\beta \in (1,2)$. Then*

$$-\log \mathbb{P}\Big(\nu I < 1/x\Big) \sim (\beta-1)\overleftarrow{H}(x) \quad \text{as} \quad x \to +\infty,$$

where

$$\overleftarrow{H}(x) = \inf\Big\{s > 0\,,\, \psi(s)/s > x\Big\}.$$

Proof: Since $\nu I \le I$ a.s., then

$$-\log \mathbb{P}\Big(\nu I < 1/x\Big) \le -\log \mathbb{P}\Big(I < 1/x\Big).$$

On the other hand, from the scaling property and since $X_1^{(0)} \ge J_1^{(0)}$ a.s., we see

$$-\log \mathbb{P}\Big(\nu I < 1/x\Big) = -\log \mathbb{P}\big(U(1) < 1/x\big) \ge -\log \mathbb{P}_0(X_1 > x).$$

Hence, from the estimate (4.14) we have that

$$-\log \mathbb{P}\Big(\nu I < 1/x\Big) \sim (\beta-1)\overleftarrow{H}(x) \quad \text{as} \quad x \to +\infty,$$

and this finishes the proof. ■

These estimates will allow us to obtain laws of iterated logarithm for the first and last passage time processes, for the pssMp $X^{(x)}$ and their future infimum process in terms of the following function.
Let us define the function

$$h(t) = \frac{\log|\log t|}{\psi(\log|\log t|)} \qquad \text{for} \qquad t > 1, \quad t \ne e.$$

By integration by parts, we can see that the function $\psi(\lambda)/\lambda$ is increasing, hence it is straightforward that the function $th(t)$ is also increasing in a neighbourhood of ∞.

COROLLARY 18. *If ψ is regularly varying at $+\infty$ with index $\beta \in (1,2)$, then*

$$\liminf_{x\to 0} \frac{S_x}{xh(x)} = (\beta-1)^{\beta-1} \qquad \textit{almost surely}$$

and,

$$\liminf_{x\to +\infty} \frac{S_x}{xh(x)} = (\beta-1)^{\beta-1} \qquad \textit{almost surely.}$$

Similarly, for the last passage time process, we have

$$\liminf_{x\to 0} \frac{U_x}{xh(x)} = (\beta-1)^{\beta-1} \qquad \textit{almost surely}$$

and,

$$\liminf_{x\to +\infty} \frac{U_x}{xh(x)} = (\beta-1)^{\beta-1} \qquad \textit{almost surely.}$$

Proof: It is enough to see that for t sufficiently small and t sufficiently large the functions $th(t)$ and $\bar{\Phi}(t)$ are asymptotically equivalent, but this is clear from (4.8). Now, applying Theorem 16 parts (i) and (ii), we obtain the desired result. ■

Let us define

$$f(t) = \frac{\psi(\log|\log t|)}{\log|\log t|} \qquad \text{for} \quad t > 1, \quad t \ne e.$$

COROLLARY 19. *If ψ is regularly varying at $+\infty$ with index $\beta \in (1,2)$, then*

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{tf(t)} = (\beta-1)^{-(\beta-1)} \qquad \textit{almost surely}$$

and for $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{J_t^{(x)}}{tf(t)} = (\beta-1)^{-(\beta-1)} \qquad \textit{almost surely}.$$

Similarly, for the process $X^{(x)}$, we have

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{tf(t)} = (\beta-1)^{-(\beta-1)} \qquad \textit{almost surely}$$

and,

$$\limsup_{t\to +\infty} \frac{X^{(x)}(t)}{tf(t)} = (\beta-1)^{-(\beta-1)} \qquad \textit{almost surely}.$$

Proof: This proof follows from similar arguments of the last corollary and using Theorem 17 parts (i) and (ii). ■

4.1. Example. Let us suppose that $\xi = (Y_t + ct, t \geq 0)$, where Y is a stable Lévy process of index $\beta \in (1,2)$ with no positive jumps and c a positive constant. Its Laplace exponent has the form

$$\mathbb{E}\big(e^{\lambda \xi_t}\big) = \exp\{t(\lambda^\beta + c\lambda)\}, \qquad \text{for } t \geq 0, \text{ and } \lambda > 0,$$

where $c > 0$. Note that under the hypothesis that Y has no positive jumps, $\nu = 1$ a.s.

Let us define by $X^{(x)}$, the pssMp associated to ξ starting from x and with scaling index $\alpha > 0$, then when $x = 0$ its first and last passage time processes satisfies

$$\liminf_{x\to 0} \frac{S_x}{x^\alpha\big(\log|\log x|\big)^{(1-\beta)\alpha}} = \alpha^{-\beta\alpha}(\beta-1)^{\alpha(\beta-1)}, \quad \text{almost surely,}$$

and,

$$\liminf_{x\to 0} \frac{U_x}{x^\alpha\big(\log|\log x|\big)^{(1-\beta)\alpha}} = \alpha^{-\beta\alpha}(\beta-1)^{\alpha(\beta-1)}, \quad \text{almost surely,}$$

Note that we have the same law of the iterated logarithm at $+\infty$.
The pssMp $X^{(x)}$ satisfies that

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{t^{\frac{1}{\alpha}}\big(\log|\log t|\big)^{\frac{(\beta-1)}{\alpha}}} = \alpha^{\frac{\beta}{\alpha}}(\beta-1)^{-\frac{\beta-1}{\alpha}}, \quad \text{almost surely,}$$

and for all $x \geq 0$

$$\limsup_{t\to +\infty} \frac{X_t^{(x)}}{t^{\frac{1}{\alpha}}\big(\log|\log t|\big)^{\frac{\beta-1}{\alpha}}} = \alpha^{\frac{\beta}{\alpha}}(\beta-1)^{-\frac{\beta-1}{\alpha}}, \quad \text{almost surely.}$$

Finally, its future infimum process also satisfies that

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{t^{\frac{1}{\alpha}}\big(\log|\log t|\big)^{\frac{(\beta-1)}{\alpha}}} = \alpha^{\frac{\beta}{\alpha}}(\beta-1)^{-\frac{\beta-1}{\alpha}}, \quad \text{almost surely,}$$

and for all $x \geq 0$

$$\limsup_{t\to+\infty} \frac{J_t^{(x)}}{t^{\frac{1}{\alpha}}\big(\log|\log t|\big)^{\frac{\beta-1}{\alpha}}} = \alpha^{\frac{\beta}{\alpha}}(\beta-1)^{-\frac{\beta-1}{\alpha}}, \quad \text{almost surely.}$$

Note that when $\alpha = \beta$, the processes $X^{(x)}$ and $J^{(x)}$ have the same asymptotic behaviour as ξ, this is

$$\limsup_{t\to 0 (\text{or } +\infty)} \frac{\xi_t}{t^{\frac{1}{\beta}}\big(\log|\log t|\big)^{\frac{\beta-1}{\beta}}} = \beta(\beta-1)^{-\frac{\beta-1}{\beta}}, \quad \text{almost surely,}$$

see Zolotarev [**Zolo64**] for details, and also the same asymptotic behaviour of the stable Lévy process conditioned to stay positive with no positive jumps (see Corollary 14).

5. The case with no positive jumps

We finish this chapter with some remarkable asymptotic properties of pssMp with no positive jumps. The following result means in particular that if there exist a positive function that describes the upper envelope of $X^{(x)}$ by a law of the iterated logarithm then the same function describes the upper envelope of the future infimum of $X^{(x)}$ and the pssMp $X^{(x)}$ reflected at its future infimum.

THEOREM 18. *Let us suppose that*

$$\limsup_{t\to 0} \frac{X^{(0)}}{F(t)} = 1 \qquad \textit{almost surely,}$$

then

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{F(t)} = 1 \quad \textit{and} \quad \limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{F(t)} = 1 \quad \textit{almost surely.}$$

Moreover, if we suppose that for all $x \geq 0$

$$\limsup_{t\to+\infty} \frac{X^{(x)}}{F(t)} = 1 \qquad \textit{almost surely,}$$

then

$$\limsup_{t\to+\infty} \frac{J_t^{(x)}}{F(t)} = 1 \quad \textit{and} \quad \limsup_{t\to+\infty} \frac{X_t^{(x)} - J_t^{(x)}}{F(t)} = 1 \quad \textit{almost surely.}$$

Proof: First, we prove part (i) and (ii) for large times. Let $x \geq 0$. Since $J_t^{(x)} \geq X_t^{(x)}$ for every $t \geq 0$ and our hypothesis, it is clear that

$$\limsup_{t\to+\infty} \frac{J_t^{(x)}}{F(t)} \leq 1 \qquad \text{almost surely.}$$

Now, fix $\epsilon \in (0, 1/2)$ and define

$$R_n = \inf\left\{ s \geq n : \frac{X_s^{(x)}}{F(s)} \geq (1-\epsilon) \right\}.$$

From the above definition, it is clear that $R_n \geq n$ and that R_n diverge a.s. as n goes to $+\infty$. From our hypothesis, we deduce that R_n is finite, a.s.

Now, since $X^{(x)}$ has no positive jumps and applying the strong Markov property, we have that

$$\begin{aligned}\mathbb{P}\left(\frac{J^{(x)}_{R_n}}{F(R_n)}\geq(1-2\epsilon)\right)&=\mathbb{P}\left(J^{(x)}_{R_n}\geq\frac{(1-2\epsilon)X^{(x)}_{R_n}}{(1-\epsilon)}\right)\\&=\mathbb{E}\left(\mathbb{P}\left(J^{(x)}_{R_n}\geq\frac{(1-2\epsilon)X^{(x)}_{R_n}}{(1-\epsilon)}\Bigg|X^{(x)}_{R_n}\right)\right)\\&=\mathbb{P}\left(\inf_{t\geq0}\xi\geq\log\frac{(1-2\epsilon)}{(1-\epsilon)}\right)=cW\left(\log\frac{1-\epsilon}{1-2\epsilon}\right)>0,\end{aligned}$$

where $W:[0,+\infty)\to[0,+\infty)$ is the unique absolutely continuous increasing function with Laplace exponent

$$\int_0^{+\infty}e^{-\lambda x}W(x)\mathrm{d}x=\frac{1}{\psi(\lambda)}\qquad\text{for }\lambda>0,$$

and $c=1/W(+\infty)$, (see Bertoin [**Bert96**] Theorem VII.8).
Since $R_n\geq n$,

$$\mathbb{P}\left(\frac{J^{(x)}_t}{F(t)}\geq(1-2\epsilon),\ \text{for some }t\geq n\right)\geq\mathbb{P}\left(\frac{J^{(x)}_{R_n}}{F(R_n)}\geq(1-2\epsilon)\right).$$

Therefore, for all $\epsilon\in(0,1/2)$,

$$\mathbb{P}\left(\frac{J^{(x)}_t}{F(t)}\geq(1-2\epsilon),\text{i.o., as }t\to+\infty\right)\geq\lim_{n\to+\infty}\mathbb{P}\left(\frac{J^{(x)}_{R_n}}{F(R_n)}\geq(1-2\epsilon)\right)>0.$$

The event of the left hand side is in the upper-tail sigma-field $\cap_t\sigma\{X^{(x)}_s:s\geq t\}$ which is trivial, then

$$\limsup_{t\to+\infty}\frac{J^{(x)}_t}{F(t)}\geq1-2\epsilon\qquad\text{almost surely.}$$

The proof of part (ii) is very similar, in fact

$$\begin{aligned}\mathbb{P}\left(\frac{X^{(x)}_{R_n}-J^{(x)}_{R_n}}{F(R_n)}\geq(1-2\epsilon)\right)&=\mathbb{P}\left(J^{(x)}_{R_n}\leq\frac{\epsilon X^{(x)}_{R_n}}{1-\epsilon}\right)\\&=\mathbb{E}\left(\mathbb{P}\left(J^{(x)}_{R_n}\leq\frac{\epsilon X^{(x)}_{R_n}}{1-\epsilon}\Bigg|X^{(x)}_{R_n}\right)\right)\\&=\mathbb{P}\left(\inf_{t\geq0}\xi\leq\log\frac{\epsilon}{1-\epsilon}\right)=1-cW\left(\log\frac{1-\epsilon}{\epsilon}\right)>0,\end{aligned}$$

Since $R_n\geq n$,

$$\mathbb{P}\left(\frac{X^{(x)}_t-J^{(x)}_t}{F(t)}\geq(1-2\epsilon),\ \text{for some }t\geq n\right)\geq\mathbb{P}\left(\frac{X^{(x)}_{R_n}-J^{(x)}_{R_n}}{F(R_n)}\geq(1-2\epsilon)\right).$$

Therefore, for all $\epsilon\in(0,1/2)$,

$$\mathbb{P}\left(\frac{X^{(x)}_t-J^{(x)}_t}{F(t)}\geq(1-2\epsilon),\text{i.o., as }t\to+\infty\right)\geq\lim_{n\to+\infty}\mathbb{P}\left(\frac{X^{(x)}_{R_n}-J^{(x)}_{R_n}}{F(R_n)}\geq(1-2\epsilon)\right)>0.$$

The event of the left hand side of the above inequality is in the upper-tail sigma-field $\cap_t \sigma\{X_s^{(x)} : s \geq t\}$ which is trivial and this establishes part (ii) for large times.
In order to prove the LIL for small times, we now define the following stopping time

$$R_n = \inf\left\{ \frac{1}{n} \leq s : \frac{X_s^{(0)}}{\Lambda(s)} \geq (1-\epsilon) \right\}.$$

Following same arguments as above, we get that for a fixed $\epsilon \in (0, 1/2)$ and n sufficiently large

$$\mathbb{P}\left(\frac{J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1-2\epsilon) \right) > 0 \quad \text{and} \quad \mathbb{P}\left(\frac{X_{R_n}^{(0)} - J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1-2\epsilon) \right) > 0.$$

Next, we note

$$\mathbb{P}\left(\frac{J_{R_p}^{(0)}}{\Lambda(R_p)} \geq (1-2\epsilon), \text{ for some } p \geq n \right) \geq \mathbb{P}\left(\frac{J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1-2\epsilon) \right),$$

and

$$\mathbb{P}\left(\frac{X_{R_p}^{(0)} - J_{R_p}^{(0)}}{\Lambda(R_p)} \geq (1-2\epsilon), \text{ for some } p \geq n \right) \geq \mathbb{P}\left(\frac{X_{R_n}^{(0)} - J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1-2\epsilon) \right).$$

Since R_n converge a.s. to 0 as n goes to ∞, the conclusion follows taking the limit when n goes towards to $+\infty$ ■

Hence from Theorem 17, we deduce the following result.

COROLLARY 20. *Under condition* (4.8)*, we have the following law of the iterated logarithm for all* $x \geq 0$

$$\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{\bar{\Psi}(t)} = 1 \quad \textit{and} \quad \limsup_{t\to +\infty} \frac{X_t^{(x)} - J_t^{(x)}}{\bar{\Psi}(t)} = 1 \quad \textit{almost surely.}$$

Similarly, under condition (4.9)*, we have that*

$$\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{\hat{\Psi}(t)} = 1 \quad \textit{and} \quad \limsup_{t\to +\infty} \frac{X_t^{(x)} - J_t^{(x)}}{\hat{\Psi}(t)} = 1 \quad \textit{almost surely.}$$

5.1. Examples. **1.** Let $X^{(0)}$ be a stable Lévy process conditioned to stay positive with no positive jumps and index $1 < \alpha \leq 2$. In Section 2.1, we noted that

$$-\log \bar{F}(1/t) \sim \frac{\alpha-1}{\alpha} \left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} t^{1/\alpha - 1} \quad \text{as} \quad t \to +\infty.$$

Then applying Corollary 20, we get the following law of the iterated logarithm.

COROLLARY 21. *Let $X^{(0)}$ be a stable Lévy process conditioned to stay positive with no positive jumps and $\alpha > 1$. Then, the processes $X^{(x)} - J^{(x)}$ satisfy the following law of the iterated logarithm*

$$\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{t^{1/\alpha}\big(\log|\log t|\big)^{1-1/\alpha}} = \alpha\,(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \qquad \textit{almost surely,}$$

and for all $x \geq 0$,

$$\limsup_{t\to+\infty} \frac{X_t^{(x)} - J_t^{(x)}}{t^{1/\alpha}\big(\log\log t\big)^{1-1/\alpha}} = \alpha\,(\alpha-1)^{-\frac{\alpha-1}{\alpha}}, \qquad \textit{almost surely,}$$

2. Let ξ be a Lévy process with no positive jumps and suppose that its Laplace exponent ψ is regularly varying at $+\infty$ with index $\gamma \in (1,2)$. Recall from the previous section that,

$$-\log \mathbb{P}\Big(I\big(\hat{\xi}\big) < 1/t\Big) \sim (\beta-1)\overleftarrow{H}(t) \quad \text{as} \quad t \to +\infty,$$

where

$$\overleftarrow{H}(t) = \inf\Big\{s > 0, \psi(s)/s > t\Big\}.$$

Let us define the function

$$f(t) = \frac{\psi(\log|\log t|)}{\log|\log t|} \quad \text{for} \quad t > 1, \quad t \neq e.$$

COROLLARY 22. *Let ξ be a Lévy process with no positive jumps such that its Laplace exponent ψ is regularly varying at $+\infty$ with index $\gamma \in (1,2)$. The process $X^{(x)}$ denotes the pssMp starting from $x > 0$ associated to ξ by the Lamperti relation (0.15). Then, the processes $X^{(x)} - J^{(x)}$ satisfies the following laws of the iterated logarithm*

$$\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{tf(t)} = (\beta-1)^{-(\beta-1)}, \qquad \textit{almost surely,}$$

and for all $x \geq 0$,

$$\limsup_{t\to+\infty} \frac{X_t^{(x)} - J_t^{(x)}}{tf(t)} = (\beta-1)^{-(\beta-1)}, \qquad \textit{almost surely.}$$

3. Sato [**Sato91**] (see also Sato [**Sato99**]) studied some interesting properties of ipsspii (increasing and positive self-similar processes with independent increments). In particular, he showed that if $Y = (Y_t, t \geq 0)$ is an ipsspii with 0 as starting point, we can represent its Laplace transform by

$$\mathbb{E}\Big[\exp\big\{-\lambda Y_1\big\}\Big] = \exp\big\{-\bar{\phi}(\lambda)\big\} \qquad \text{for } \lambda > 0,$$

where

$$\bar{\phi}(\lambda) = c\lambda + \int_0^{+\infty} \big(1 - e^{-\lambda x}\big)\frac{k(x)}{x}\mathrm{d}x,$$

$c \geq 0$ and $k(x)$ is a nonnegative decreasing function on $(0,+\infty)$ with

$$\int_0^{+\infty} \frac{k(x)}{1+x}\mathrm{d}x < +\infty.$$

From its definition, it is clear that the Laplace exponent $\bar{\phi}$ is an increasing continuous function and more precisely a concave function.
Under the assumption that $\bar{\phi}$ varies regularly at $+\infty$ with index $\alpha \in (0,1)$, we will have the following sharp estimate for the distribution of Y_1.
Let us define the function

$$h(t) = \frac{t\log|\log t|}{\bar{\varphi}(\log|\log t|)}, \qquad \text{for} \quad t \neq e, \quad t > 1,$$

where $\bar{\varphi}$ is the inverse function of $\bar{\phi}$.

PROPOSITION 14. *Let* $(Y_t, t \geq 0)$ *be an isspii and suppose that* $\bar{\phi}$*, its Laplace exponent, varies regularly at* $+\infty$ *with index* $\alpha \in (0,1)$*. Then for every* $c > 0$*, we have*

$$-\log \mathbb{P}\left(Y_1 \leq \frac{ch(t)}{t}\right) \sim \big(\alpha/c\big)^{\frac{\alpha}{1-\alpha}}(1-\alpha)\log|\log t| \quad \textit{as} \quad t \to 0 \quad (t \to \infty).$$

Proof: From de Brujin's Tauberian Theorem (see for instance Theorem 4.12.9 in [**Bial89**]), we have that if $\bar{\phi}$ varies regularly at $+\infty$ with index $\alpha \in (0,1)$ then

$$-\log \mathbb{P}\big(Y_1 \leq x\big) \sim \frac{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{\overleftarrow{\Theta}(1/x)}, \quad \text{for} \quad x \to 0,$$

where $\overleftarrow{\Theta}$ is the asymptotic inverse of Θ, a regularly varying function at $+\infty$ with index $(\alpha-1)/\alpha$ and that satisfies

$$\frac{\lambda}{\bar{\phi}(\lambda)} \sim \Theta\left(\frac{1}{\bar{\phi}(\lambda)}\right) \quad \text{for} \quad \lambda \to +\infty. \tag{4.15}$$

Hence, taking $x = ch(t)/t$ and $\lambda = \bar{\varphi}(\log|\log t|)$, and doing some calculations we get the desired result. ■

This estimate and Lemma 6 allow us to get the following law of the iterated logarithm.

COROLLARY 23. *Let* $(Y_t, t \geq 0)$ *be an ipsspii and suppose that* $\bar{\phi}$*, its Laplace exponent, satisfies the conditions of the previous proposition. Then, we have*

$$\liminf_{t \to 0} \frac{Y_t}{h(t)} = \alpha(1-\alpha)^{(1-\alpha)/\alpha}, \qquad \textit{almost surely.}$$

The same las of iterated logarithm is satisfied for large times.

Now, we denote by ϕ_1 and ϕ_2 the Laplace exponents of the last and first passage time processes, respectively. Since $S_1 \leq U_1$ a.s., it is clear that $\phi_2(\lambda) \leq \phi_1(\lambda)$ for all $\lambda \geq 0$. Let us suppose that ϕ_1 and ϕ_2 are regularly varying at $+\infty$ with index α_1 and α_2 respectively, such that $0 < \alpha_2 \leq \alpha_1 < 1$. By Theorem 16 and Proposition 12, we can deduce that in fact ϕ_1 and ϕ_2 are asymptotically equivalents and that $\alpha_1 = \alpha_2$. Then by the above corollary, we have that

$$h_1(t) = \frac{t\log|\log t|}{\varphi_1(\log|\log t|)} \quad \text{and} \quad h_2(t) = \frac{t\log|\log t|}{\varphi_2(\log|\log t|)}, \quad \text{for} \quad t \neq e, \quad t > 1,$$

where φ_1 and φ_2 are the inverse of ϕ_1 and ϕ_2, respectively, the processes U and S satisfy

$$\liminf_{t \to 0} \frac{U_t}{h_1(t)} = \alpha_1(1-\alpha_1)^{(1-\alpha_1)/\alpha_1} \qquad \text{almost surely,}$$

and

$$\liminf_{t \to 0} \frac{S_t}{h_1(t)} = \alpha_1(1-\alpha_1)^{(1-\alpha_1)/\alpha_1} \qquad \text{almost surely.}$$

Note that we can replace h_1 by h_2 and that we also have the same laws of the iterated logarithm for large times.
By the sharp estimation in Proposition 14 of the tail probability of S_1, we deduce the following law of the iterated logarithm.
Let us define

$$f_2(t) = \frac{t\varphi_2(\log|\log t|)}{\log|\log t|}, \quad \text{for} \quad t \neq e, \quad t > 1.$$

COROLLARY 24. *Let ϕ_2 be the Laplace exponent of S_1 and φ_2 its inverse. If ϕ_2 is regularly varying at $+\infty$ with index $\alpha_2 \in (0,1)$, then*

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{f_2(t)} = \alpha_2^{-1}(1-\alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely,}$$

and for any $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{X_t^{(x)}}{f_2(t)} = \alpha_2^{-1}(1-\alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely.}$$

On the other hand, from Theorem 18 we get the following Corollary.

COROLLARY 25. *Let ϕ_2 be the Laplace exponent of S_1 and φ_2 its inverse. If ϕ_2 is regularly varying at $+\infty$ with index $\alpha_2 \in (0,1)$, then*

i)

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{f_2(t)} = \alpha_2^{-1}(1-\alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely,}$$

and for any $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{J_t^{(x)}}{f_2(t)} = \alpha_2^{-1}(1-\alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely.}$$

ii)

$$\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{f_2(t)} = \alpha_2^{-1}(1-\alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely,}$$

and for any $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{X_t^{(x)} - J_t^{(x)}}{f_2(t)} = \alpha_2^{-1}(1-\alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely.}$$

CHAPTER 5

Transient Bessel processes.

Bessel processes form the sub-class of continuous positive self-similar Markov processes. In this chapter, the upper envelope of the future infimum of transient Bessel processes is completely described through integral test. This result improve the results found by Khoshnevisan et al. [**Khal94**]. We also establish an integral test for the upper envelope of transient Bessel processes which is a variant of the Kolmogorov-Dvoretsky-Erdös integral test.

1. The future infimum.

In this section we will suppose that $\xi = (2(B_t + at), t \geq 0)$, where B is a standard Brownian motion and $a > 0$.
We define the process $Z = (Z_t, t \geq 0)$, the square of the δ-dimensional Bessel processes starting at $x \geq 0$, as the unique strong solution of the stochastic differential equation

$$Z_t = x + 2\int_0^t \sqrt{|Z_s|}\mathrm{d}\beta_s + \delta t, \qquad \text{for} \quad \delta \geq 0, \tag{5.1}$$

where β is a standard Brownian motion.
By the Lamperti representation, we know that we can define $X^{(x)}$ a pssMp starting at $x > 0$, such that

$$X^{(x)}_{xI_t(\xi)} = x \exp\big\{\xi_t\big\} \quad \text{for} \quad t \geq 0.$$

Then, applying the Itô's formula and Dubins-Schwartz's Theorem (see for instance, Revuz and Yor [**ReYo99**]), we get

$$X^{(x)}_{xI_t(\xi)} = x + 2\int_0^{xI_t(\xi)} \sqrt{X^{(x)}_s}\mathrm{d}B_s + 2(a+1)xI_t(\xi).$$

Hence it follows that $X^{(x)}$ satisfies (5.1) with $\delta = 2(a+1)$ and therefore $X^{(x)}$ is the square of the δ-dimensional Bessel processes starting at $x > 0$. From the main result of Caballero and Chaumont [**CaCh06**], we may define $X^{(x)}$ at $x = 0$, and from (0.16) we can computed its entrance law. Since, we suppose that $a > 0$, we deduce that $X^{(x)}$ is a transient process and that $\delta > 2$.
From the formula of negative moments (4.12) of the exponential functional $I(\hat{\xi})$, we can deduce (see Example 3 in Bertoin and Yor [**BerY02**]) the following identity in distribution

$$\int_0^\infty \exp\big\{-2(B_s + as)\big\}\mathrm{d}s \overset{(d)}{=} \frac{1}{2\gamma_a}, \tag{5.2}$$

where γ_a is a gamma random variable with index $a > 0$. In fact, we can also deduce that $X^{(0)}_1$ is distributed as $2\gamma_{a+1}$.

We recall that the distribution of γ_a for $a > 0$, is given by

$$(5.3) \qquad \mathbb{P}(\gamma_a \leq x) = \frac{1}{\Gamma(a)} \int_0^x e^{-y} y^{a-1} \mathrm{d}y, \quad \text{where} \quad \Gamma(a) = \int_0^\infty e^{-y} y^{a-1} \mathrm{d}y.$$

It is important to note that due the continuity of the paths of $X^{(0)}$, we have that $\nu = 1$ almost surely.
The following Lemma will be helpful for the application of our general results to the case of transient Bessel processes.

LEMMA 8. *Let $a > 0$, then there exist c and C, two positive constants such that*

$$ce^{-x} x^{a-1} \leq \int_x^\infty e^{-y} y^{a-1} \mathrm{d}y \leq C e^{-x} x^{a-1}, \quad \textit{for} \quad x \geq \frac{C(a-1)}{C-1}.$$

Proof: First, we prove the lower bound for $a > 1$. For $x > 0$, we see

$$\int_x^\infty e^{-y} y^{a-1} \mathrm{d}y = x^a \int_1^\infty e^{-xy} y^{a-1} \mathrm{d}y \geq x^{\alpha-1} e^x.$$

For $a \in (0,1)$, we have

$$\begin{aligned} \int_x^\infty e^{-y} y^{a-1} \mathrm{d}y &= (1-a) \int_x^\infty e^{-y} \left(\int_y^\infty z^{a-2} \mathrm{d}z \right) \mathrm{d}y \\ &= x^{a-1} e^{-x} - (1-a) \int_x^\infty z^{a-2} e^{-z} \mathrm{d}z \\ &\geq x^{a-1} e^{-x} - (1-a) \int_x^\infty z^{a-1} e^{-z} \mathrm{d}z, \end{aligned}$$

then,

$$\int_x^\infty e^{-y} y^{a-1} \mathrm{d}y \geq \frac{1}{2-a} x^{a-1} e^{-x}.$$

Next, we prove the upper bound for $a \in (0,1)$. For $x > 0$,

$$\int_x^\infty e^{-y} y^{a-1} \mathrm{d}y = x^a \int_1^\infty e^{-xy} y^{a-1} \mathrm{d}y \leq x^{\alpha-1} e^x.$$

For $a > 1$, we see

$$\begin{aligned} \int_x^\infty e^{-y} y^{a-1} \mathrm{d}y &= (a-1) \int_x^\infty e^{-y} \left(\int_0^y z^{a-2} \mathrm{d}z \right) \mathrm{d}y \\ &= x^{a-1} e^{-x} + (a-1) \int_x^\infty e^{-y} y^{a-2} \mathrm{d}y \\ &\leq x^{a-1} e^{-x} + \frac{(a-1)}{x} \int_x^\infty e^{-y} y^{a-1} \mathrm{d}y. \end{aligned}$$

Now, let $b \in (0,1)$. Then, for $x \geq \frac{a-1}{1-b}$ it follows

$$\beta \int_x^\infty e^{-y} y^{a-1} \mathrm{d}y \leq \left(1 - \frac{a-1}{x}\right) \int_x^\infty e^{-y} y^{a-1} \mathrm{d}y \leq x^{a-1} e^{-x},$$

and therefore, we have the upper bound with $C = b^{-1}$.
The case $a = 1$ is evident. ■

From this Lemma, we deduce the following integral tests for the last passage time process of transient square Bessel process.

THEOREM 19. *Let $h \in \mathcal{H}_0^{-1}$, then:*

i) *If*

$$\int_{0+} \Big(x/2h(x)\Big)^{\frac{\delta-4}{2}} \exp\Big\{-x/2h(x)\Big\}\frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1-\epsilon)h(x),\ \textit{i.o., as } x \to 0\Big) = 0.$$

ii) *If*

$$\int_{0+} \Big(x/2h(x)\Big)^{\frac{\delta-4}{2}} \exp\Big\{-x/2h(x)\Big\}\frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1+\epsilon)h(x),\ \textit{i.o., as } x \to 0\Big) = 1.$$

Proof: The proof of this Theorem follows from the fact that

$$\mathbb{P}(I < x) = \mathbb{P}\big(\gamma_{(\delta-2)/2} > 1/2x\big) \quad \text{for} \quad x > 0, \tag{5.4}$$

and an application of Theorem 4 and Lemma 8. ■

THEOREM 20. *Let $h \in \mathcal{H}_\infty^{-1}$, then:*

i) *If*

$$\int^{+\infty} \Big(x/2h(x)\Big)^{\frac{\delta-4}{2}} \exp\Big\{-x/2h(x)\Big\}\frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1-\epsilon)h(x),\ \textit{i.o., as } x \to +\infty\Big) = 0.$$

ii) *If*

$$\int^{+\infty} \Big(x/2h(x)\Big)^{\frac{\delta-4}{2}} \exp\Big\{-x/2h(x)\Big\}\frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(U_x < (1+\epsilon)h(x),\ \textit{i.o., as } x \to +\infty\Big) = 1.$$

Proof: The proof of these integral tests is very similar to the proof of the previous result, it is enough to apply Lemma 8 and Theorem 5 to the tail probability (5.4). ■

From these integral tests, we get the following law of iterated logarithm.

$$\liminf_{x\to 0} U_x \frac{2\log|\log x|}{x} = 1 \quad \text{and} \quad \liminf_{x\to+\infty} U_x \frac{2\log\log x}{x} = 1 \quad \text{almost surely.}$$

Note that we are also in the "log-regular" case and we can apply Theorem 16 to get the same law of the iterated logarithm.
For the upper envelope of the future infimum process, we have the following integral tests.

THEOREM 21. *Let $h \in \mathcal{H}_0$, then:*

i) *If*

$$\int_{0+}\Big(h(t)/2t\Big)^{\frac{\delta-4}{2}}\exp\Big\{-h(t)/2t\Big\}\frac{\mathrm{d}t}{t}<\infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(J^{(0)}_t > (1+\epsilon)h(t), \textit{ i.o., as } t\to 0\Big)=0.$$

ii) *If*

$$\int_{0+}\Big(h(t)/2t\Big)^{\frac{\delta-4}{2}}\exp\Big\{-h(t)/2t\Big\}\frac{\mathrm{d}t}{t}=\infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(J^{(0)}_t > (1-\epsilon)h(t), \textit{ i.o., as } t\to 0\Big)=1.$$

Proof: We get this result applying Theorem 6 and the estimate of Lemma 8 to the tail probability (5.4). ■

THEOREM 22. *Let* $h\in\mathcal{H}_\infty$*, then for all* $x\geq 0$*:*

i) *If*

$$\int^{+\infty}\Big(h(t)/2t\Big)^{\frac{\delta-4}{2}}\exp\Big\{-h(t)/2t\Big\}\frac{\mathrm{d}t}{t}<\infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(J^{(x)}_t > (1+\epsilon)h(t), \textit{ i.o., as } t\to +\infty\Big)=0.$$

ii) *If*

$$\int^{+\infty}\Big(h(t)/2t\Big)^{\frac{\delta-4}{2}}\exp\Big\{-h(t)/2t\Big\}\frac{\mathrm{d}t}{t}=\infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(J^{(x)}_t > (1-\epsilon)h(t), \textit{ i.o., as } t\to +\infty\Big)=1.$$

Proof: The proof of these integral test is similar to the previous Theorem. We only replace Theorem 6 by Theorem 7. ■

From these integral tests, we get the following laws of iterated logarithm,

$$\limsup_{t\to 0}\frac{J^{(0)}_t}{2t\log|\log t|}=1 \quad\text{and}\quad \limsup_{t\to+\infty}\frac{J^{(x)}_t}{2t\log\log t}=1 \quad\text{almost surely,}$$

for $x\geq 0$. Here we can also obtain the same laws of the iterated logarithm applying Theorem 17.

2. The upper envelope of transient Bessel processes.

Gruet and Shi [**GrSh96**] proved that there exist a finite constant $K>1$, such that for any $0<s\leq 2$,

$$K^{-1}s^{1-\delta/2}\exp\left\{-\frac{1}{2s}\right\}\leq\mathbb{P}(S_1<s)\leq Ks^{1-\delta/2}\exp\left\{-\frac{1}{2s}\right\}. \tag{5.5}$$

Hence we establish the following integral test for the lower envelope of the first passage time process of the square Bessel process $X^{(0)}$.

THEOREM 23. *Let* $h \in \mathcal{H}_0^{-1}$,

i) *If*

$$\int_{0+} \left(\frac{t}{h(t)}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{t}{2h(t)}\right\} \frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_t < (1-\epsilon)h(t), \text{ i.o., as } t \to 0\Big) = 0.$$

ii) *If*

$$\int_{0+} \frac{\mathrm{d}t}{t} \left(\frac{t}{h(t)}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{t}{2h(t)}\right\} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_t < (1+\epsilon)h(t), \text{ i.o., as } t \to 0\Big) = 1.$$

Proof: The proof of this Theorem is a simple application of (5.5) to Lemma 6. ∎

Similarly, we have the same integral test for large times.

THEOREM 24. *Let* $h \in \mathcal{H}_\infty^{-1}$,

i) *If*

$$\int^{+\infty} \left(\frac{t}{h(t)}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{t}{2h(t)}\right\} \frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_t < (1-\epsilon)h(t), \text{ i.o., as } t \to +\infty\Big) = 0.$$

ii) *If*

$$\int_{+\infty} \frac{\mathrm{d}t}{t} \left(\frac{t}{h(t)}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{t}{2h(t)}\right\} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_t < (1+\epsilon)h(t), \text{ i.o., as } t \to +\infty\Big) = 1.$$

From these integral tests, we get the following law of the iterated logarithm

$$\liminf_{t\to 0} S_t \frac{2\log|\log t|}{t} = 1 \quad \text{and} \quad \liminf_{t\to +\infty} S_t \frac{2\log\log t}{t} = 1 \quad \text{almost surely.}$$

For the upper envelope of $X^{(0)}$, we have the following integral tests.

THEOREM 25. *Let* $h \in \mathcal{H}_0$,

i) *If*

$$\int_{0+} \left(\frac{h(t)}{t}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{h(t)}{2t}\right\} \frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(X_t^{(0)} > (1+\epsilon)h(t), \text{ i.o., as } t \to 0\Big) = 0.$$

ii) *If*

$$\int_{0+}\left(\frac{h(t)}{t}\right)^{\frac{\delta-2}{2}}\exp\left\{-\frac{h(t)}{2t}\right\}\frac{\mathrm{d}t}{t}=\infty,$$

then for all $\epsilon>0$

$$\mathbb{P}\Big(X_t^{(0)}>(1-\epsilon)h(t),\ \textit{i.o., as } t\to 0\Big)=1.$$

Proof: The proof of this Theorem follows from a simple application of (5.5) to Theorem 8. The proof of the additional hypothesis (2.19), is clear from (5.5). ■

Similarly, we have the same integral tests for large times.

THEOREM 26. *Let* $h\in\mathcal{H}_\infty$

i) *If*

$$\int^{+\infty}\left(\frac{h(t)}{t}\right)^{\frac{\delta-2}{2}}\exp\left\{-\frac{h(t)}{2t}\right\}\frac{\mathrm{d}t}{t}<\infty,$$

then for all $\epsilon>0$ *and for all* $x\geq 0$

$$\mathbb{P}\Big(X_t^{(x)}>(1+\epsilon)h(t),\ \textit{i.o., as } t\to +\infty\Big)=0.$$

ii) *If*

$$\int^{+\infty}\left(\frac{h(t)}{t}\right)^{\frac{\delta-2}{2}}\exp\left\{-\frac{h(t)}{2t}\right\}\frac{\mathrm{d}t}{t}=\infty,$$

then for all $\epsilon>0$ *and for all* $x\geq 0$

$$\mathbb{P}\Big(X_t^{(x)}>(1-\epsilon)h(t),\ \textit{i.o., as } t\to +\infty\Big)=1.$$

Recall from the Kolmogorov and Dvoretzky-Erdös (KDE for short) integral test that for h a nondecreasing, positive and unbounded function as t goes to $+\infty$, the upper envelope of $X^{(0)}$ at 0 may be described as follows:

$$\mathbb{P}\big(X_t^{(0)}>h(t),\ \text{i.o., as } t\to 0\big)=0 \text{ or } 1,$$

according as,

$$\int_0\left(\frac{h(t)}{t}\right)^{\frac{\delta}{2}}\exp\left\{-\frac{h(t)}{2t}\right\}\frac{\mathrm{d}t}{t}\qquad\text{is finite or infinite.}$$

Note that the class of functions that satisfy the divergent part of Theorems 25 and 26 implies the divergent part of the KDE integral test, hence ϵ can also take the value 0. The convergent part of the KDE integral test obviously implies the convergent part of Theorems 25 and 26. From these integral tests, we get the following laws of the iterated logarithm: for $x\geq 0$,

$$\limsup_{t\to 0}\frac{X_t^{(0)}}{2t\log|\log t|}=1\quad\text{and}\quad\limsup_{t\to+\infty}\frac{X^{(x)}}{2t\log\log t}=1\quad\text{almost surely.}$$

On the other hand, it is not difficult to deduce from Lemma 8 that

$$-\log\bar{F}(x)\sim x,\qquad x\to 0,$$

then the square of a transient Bessel process satisfies condition (4.8) and Theorem 18, which implies that for $x\geq 0$,

$$\limsup_{t\to 0}\frac{X_t^{(0)}-J_t^{(0)}}{2t\log|\log t|}=1\quad\text{and}\quad\limsup_{t\to+\infty}\frac{X^{(x)}J_t^{(x)}}{2t\log\log t}=1\quad\text{almost surely.}$$

Now, we will apply some of the results of the third example of Section 5.1. Here we employ the usual Bessel functions I_a and K_a, as in Kent [**Kent78**] and Jeanblanc, Pitman and Yor [**JePY02**]. It is well-known that

$$\mathbb{E}\Big(\exp\Big\{-\lambda S_1\Big\}\Big) = \lambda^{a/2}\frac{1}{2^{a/2}\Gamma(a+1)I_a(\sqrt{2\lambda})}, \quad \lambda > 0,$$

and

$$\mathbb{E}\Big(\exp\Big\{-\lambda U_1\Big\}\Big) = \frac{\lambda^{a/2}}{2^{a/2-1}\Gamma(a)}K_a(\sqrt{2\lambda}), \quad \lambda > 0,$$

where Γ is the well-known gamma function (see for instance Jeanblanc, Pitman and Yor [**JePY02**]).

Now, we define for $\lambda > 0$

$$\phi_1(\lambda) = \log(2^{a/2-1}\Gamma(a)) - \log K_a(\sqrt{2\lambda}) - \log \lambda^{a/2},$$
$$\phi_2(\lambda) = \log I_a(\sqrt{2\lambda}) + \log(2^{a/2}\Gamma(a+1)) - \log \lambda^{a/2}.$$

Since, we have the following asymptotic behaviour

$$I_a(x) \sim (2\pi x)^{-1/2}e^x \quad \text{and} \quad K_a(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \quad \text{when } x \to +\infty,$$

(see Kent [**Kent78**] for instance), we deduce that ϕ_1 and ϕ_2 are regularly varying at $+\infty$ with index $1/2$. From Propositions 11 and 13 and Theorem 18, we deduce that they are asymptotically equivalent.

From Corollaries 23, 24 and 25, we have that

$$\liminf_{t\to 0}\frac{U_t}{h_1(t)} = 1/4, \qquad \liminf_{t\to 0}\frac{S_t}{h_2(t)} = 1/4 \quad \text{almost surely,}$$

$$\limsup_{t\to 0}\frac{X_t^{(0)}}{f_2(t)} = 4, \qquad \limsup_{t\to 0}\frac{J_t^{(0)}}{f_1(t)} = 4 \quad \text{almost surely,}$$

and

$$\limsup_{t\to 0}\frac{J_t^{(0)}}{f_2(t)} = 4, \qquad \limsup_{t\to 0}\frac{X_t^{(0)} - J_t^{(0)}}{f_2(t)} = 4 \quad \text{almost surely,}$$

where

$$h_1(t) = \frac{t\log|\log t|}{\varphi_1(\log|\log t|)}, \quad h_2(t) = \frac{t\log|\log t|}{\varphi_2(\log|\log t|)}, \quad f_1(t) = \frac{t^2}{h_1(t)}, \quad f_2(t) = \frac{t^2}{h_2(t)}$$

and, φ_1 and φ_2 are the inverse functions of ϕ_1 and ϕ_2, respectively.

Similarly, we have all these laws of the iterated logarithm for large times and for $x \geq 0$,

$$\liminf_{t\to \infty}\frac{U_t}{h_1(t)} = 1/4, \qquad \liminf_{t\to \infty}\frac{S_t}{h_2(t)} = 1/4 \quad \text{almost surely,}$$

$$\limsup_{t\to \infty}\frac{X_t^{(x)}}{f_2(t)} = 4, \qquad \limsup_{t\to \infty}\frac{J_t^{(x)}}{f_1(t)} = 4 \quad \text{almost surely,}$$

and

$$\limsup_{t\to \infty}\frac{J_t^{(x)}}{f_2(t)} = 4, \qquad \limsup_{t\to \infty}\frac{X_t^{(x)} - J_t^{(x)}}{f_2(t)} = 4 \quad \text{almost surely.}$$

Part 2

Conditioned stable Lévy forest.

Introduction.

Continuous state branching processes or CB-processes are Markov processes taking values in the half-line $[0, \infty]$, with càdlàg paths and satisfying the branching property. Such processes have been introduced by Jirina [**Jiri58**] and studied by many authors including Bingham [**Bing76**], Grey [**Grey74**], Grimval [**Grim74**], Lamperti [**Lamp67, Lam67a, Lam67b**], etc... An important property of this class of Markov processes is that they appear as limit of rescaled Galton-Watson processes (see for instance [**Lamp67, Lam67b**] and [**Grim74**]). At the end of the sixties, Lamperti [**Lam67a**] stated that CB-processes are connected with Lévy processes with no negative jumps by a simple time-change. The Laplace exponent ψ of a Lévy process is well known as the branching mechanism of its related CB-process by the Lamperti transform. The branching mechanism ψ solves a differential equation that characterizes the law of the CB-process.
Motivated in extending the notion of the Brownian snake, Le Gall and Le Jan [**LGLJ98**] studied the genealogical structure of CB-processes. In [**LGLJ98**], the authors proposed a coding of the genealogy of CB-processes via a real-valued random process called the height process. In the case of the Feller branching diffusion (i.e. when $\psi(u) = u^2$), the height process is the reflected Brownian motion. Le Gall and Le Jan also observed that for a general critical or subcritical CB-process, there is an explicit formula expressing the height process as a functional of its related Lévy process with no negative jumps. Recently in the monograph [**DuLG02**], Duquesne and Le Gall studied the genealogical structure of CB-processes in connection with limit theorems of discrete branching trees well known as Galton-Watson trees.
The basic object here is the Galton-Watson tree with offspring distribution μ. It can be seen as the underlying family tree of the corresponding Galton-Watson process started with one ancestor and offspring distribution μ. This random tree is chosen to be rooted and ordered (see chapter 6). It is well-known that if μ is critical or subcritical, the Galton-Watson process is almost surely finite and therefore so is its corresponding Galton-Watson tree. The Galton-Watson tree can be coded by two different discrete real valued processes: the height process and the contour process (see chapter 6 for a proper definition). These two processes are not Markovian but they can be written as functionals of a certain left-continuous random walk whose jump distribution depends on μ.
When the sequence of rescaled Galton-Watson processes converges towards the CB-process with branching mechanism ψ, Duquesne and Le Gall [**DuLG02**] have shown that the genealogical structure of the Galton-Watson processes converges too, i.e. that the corresponding rescaled sequences of contour processes and height processes, converge respectively towards $(\bar{H}_{t/2}, t \geq 0)$ and $(\bar{H}_t, t \geq 0)$, where the limit process $(\bar{H}_t, t \geq 0)$ is the height process in continuous time that has been introduced by Le Gall and Le Jan in [**LGLJ98**]. Similarly as the discrete case, the height process is not Markovian in general but it can be described as a functional of a Lévy process with no negative jumps.
Real trees or $\mathbb{R}$-trees have been studied for a long time for algebraic or geometric purpose (see [**DMTe96**] for instance), in probability theory their use seems to be quite recent. The

precise definition of an $\mathbb{R}$-tree is recalled in chapter 6. Informally an $\mathbb{R}$-tree is a metric space $(\mathcal{T}, d)$ such that for any two points σ and σ' in $\mathcal{T}$ there is a unique arc with endpoints σ and σ' and furthermore this arc is isometric to a compact interval of the real line. A rooted $\mathbb{R}$-tree is an $\mathbb{R}$-tree with a distinguished vertex called the root. In a recent paper [**EPWi06**] of Evans, Pitman and Winter, $\mathbb{R}$-trees are studied from the point of view of measure theory and establishes in particular that the space $\mathbb{T}_c$ of equivalent classes of (rooted) compact real trees, endowed with the Gromov-Hausdorff metric, is a Polish space. This makes it very natural to consider random variables or even random processes taking values in the space $\mathbb{T}_c$. Our presentation owes a lot to the recent paper of Duquesne and Le Gall [**DuLG05**], which uses the formalism of $\mathbb{R}$-trees to define the so-called Lévy trees that were implicit in [**LGLJ98**] or [**DuLG02**]. Lévy trees are the continuous analogues of discrete Galton-Watson trees. We may consider Lévy trees as random variables taking values in the space of compact rooted $\mathbb{R}$-trees.

Aldous [**Aldo91, Ald91a, Aldo93**] developed the theory of the Continuum Random Tree or CRT which can be naturally viewed as a $\mathbb{R}$-tree, but this interpretation was not made explicit in Aldous' work. In particular, Aldous showed that this object is the limit as n increases, in a suitable sense, of rescaled critical Galton-Watson trees conditioned to have n vertices whose offspring distribution has a finite variance. Although the CRT was first defined as a particular random subset of the space l^1, it was identified in [**Aldo93**] as the tree coded by the normalized Brownian excursion. Recently, Duquesne [**Duqu03**] extended such result to Galton-Watson trees with offspring distribution in the domain of attraction of a stable law with index α in $(1, 2]$. Then, Duquesne showed that the discrete height process of the Galton Watson tree conditioned to have a large fixed progeny, converges on the space of Skorokhod of càdlàg paths to the normalized excursion of the height process associated with the α-stable CB-process. Note that in the case when $\alpha = 2$ such result coincides with Aldous' CRT.

In a natural way, Galton-Watson forest and Lévy forest are a finite or infinite collection of independent Galton-Watson trees and independent Lévy trees, respectively. Our aim is to study the genealogy of the stable Lévy forest of a given size and conditioned by its mass and also prove an invariance principle for this conditioned forest by considering k independent Galton-Watson trees whose offspring distribution is in the domain of attraction of any stable law conditioned on their total progeny to be equal to n. More precisely, when n and k go towards ∞, under suitable rescaling, the associated coding random walk, contour and height processes converge in law on the space of Skorokhod towards the first passage bridge of a stable Lévy process with no negative jumps and its height process, respectively. With this purpose in Chapter 6, we recall some basic results on Galton-Watson trees and Lévy trees. In particular, we introduce the notion of a Galton-Watson tree and define their related coding random walk, height process and contour process. We also introduce the conditioned Galton-Watson forest and their related coding first passage bridge, conditioned height process and contour process which is the starting point of our work. We will finish this second part with Chapter 7 were we will construct the Lévy forest of a given size s and conditioned by its mass and prove the invariance principle stated above.

CHAPTER 6

Galton-Watson and Lévy forests.

In this Chapter, we introduce the concept of Galton-Watson and Lévy forests. In particular, we define the contour and the height process of a Galton-Watson forest and also its continuous analogue, the height process of a Lévy forest. We will remark that the height process can be written as a simple functional of a left-continuous random walk in the discret case and in the continuous case as a functional of a Lévy process with no negative jumps.

1. Discrete trees.

In this section, we are interested in finite and rooted ordered trees. Let us denote by $\mathbb{N}^*$ the set of strictly positive integers, i.e. $\mathbb{N}^* = \{1, 2, \ldots\}$. In all the sequel, an element u of $(\mathbb{N}^*)^n$ is written as $u = (u_1, \ldots, u_n)$ and we set $|u| = n$. Now, we introduce the set of labels

$$\mathbb{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. The concatenation of two elements of $\mathbb{U}$, let us say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$ is denoted by

$$uv = (u_1, \ldots u_n, v_1, \ldots, v_m).$$

A discrete rooted tree is an element τ of the set $\mathbb{U}$ which satisfies:

(i) $\emptyset \in \tau$,
(ii) If $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$,
(iii) For every $u \in \tau$, there exists a number $k_u(\tau) \geq 0$, such that $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

In this definition, $k_u(\tau)$ represents the number of children of the vertex u. We denote by $\mathbb{T}$ the set of all rooted trees. The total cardinality of an element $\tau \in \mathbb{T}$ will be denoted by $\zeta(\tau)$, we emphasize that the root is counted in $\zeta(\tau)$. If $\tau \in \mathbb{T}$ and $u \in \tau$, then we define the shifted tree at the vertex u by

$$\theta_u(\tau) = \big\{ v \in \mathbb{U} : uv \in \tau \big\} .$$

We say that $u \in \tau$ is a leaf of τ if $k_u(\tau) = 0$. The last common ancestor between two elements of τ, say u and v, is denoted by $u \wedge v$.

We will now explain how discrete trees can be coded by three different functions. We first introduce the so-called *height function* associated with the rooted tree τ. With this purpose, let us denote by $u_\tau(0) = 0, u_\tau(1) = 1, \ldots, u_\tau(\zeta(\tau) - 1)$ the elements of the tree τ ordered in lexicographical order. The height function $(H_n(\tau), 0 \leq n < \zeta(\tau))$ is defined by

$$H_n(\tau) = |u_\tau(n)|, \qquad \text{for} \quad 0 \leq n < \zeta(\tau).$$

Hence the height function is the sequence of the generations of the elements of the discrete tree τ listed with the lexicographical order. There is another way to present the height function which is more natural. For two vertices u and v of a tree τ, the distance $d_\tau(u,v)$ is the number of edges of the unique elementary path from u to v. Then, we may define the height function in terms of the distance from the root $\emptyset$, i.e. $H_n(\tau) = d_\tau(\emptyset, u_\tau(n))$. In particular, we have the following relation

$$d_\tau(u_\tau(n), u_\tau(m)) = H_n(\tau) + H_m(\tau) - 2H_{k(n,m)}(\tau)\,, \tag{6.1}$$

were $k(n,m)$ is the integer satisfying that $u_\tau(k(n,m)) = u_\tau(n) \wedge u_\tau(m)$. It is not difficult to see that the height function of a tree allows us to recover the entire structure of this tree. We say that it *codes* the genealogy of the tree. The *contour function* or *Dick path* gives another characterization of the tree which is easier to visualize. We suppose that the tree is embedded in a half-plane in such a way that edges have length one. Informally, we imagine the motion of a particle that starts at time 0 from the root of the tree and then explores the tree from the left to the right continuously along each edge of τ at unit speed until all edges have been explored and the particle has come back to the root. Note that if $u_\tau(n)$ is a leaf, the particle goes to $u_\tau(n+1)$, taking the shortest way that consist first to move backward on the line of descent from $u_\tau(n)$ to their last common ancestor $u_\tau(n) \wedge u_\tau(n+1)$ and then to move forward along the single edge between $u_\tau(n) \wedge u_\tau(n+1)$ to $u_\tau(n+1)$. Since it is clear that each edge will be crossed twice, the total time needed to explore the tree is $2(\zeta(\tau)-1)$. The value $C_s(\tau)$ of the contour function at time $s \in [0, 2(\zeta(\tau)-1)]$ is the distance (on the continuous tree not the distance d_τ) between the position of the particle at time s and the root. More precisely, let us denote by $l_1 < l_2 < \cdots < l_p$ the p leaves of τ listed in lexicographical order. Hence, the contour function $(C_t(\tau), 0 \le t \le 2(\zeta(\tau)-1))$ is the piecewise linear continuous path with slope equal to +1 or -1, that takes successive local extremes with values: $0, |l_1|, |l_1 \wedge l_2|, |l_2|, \ldots |l_{p-1} \wedge l_p|, |l_p|$ and 0. It is important to note that the contour function can be recovered from the height function through the following transform: set $K_n = 2n - H_n(\tau)$, then

$$C_t(\tau) = \begin{cases} (H_n(\tau) - (t - K_n))^+ & \text{if } t \in [K_n, K_{n+1}-1), \\ (t - K_{n+1} + H_{n+1}(\tau))^+, & \text{if } t \in [K_{n+1}-1, K_{n+1}), \end{cases} \tag{6.2}$$

There is still another way of coding the tree. We denote by $\mathbb{S}$ the set of all sequences of nonnegative integers $m_1, \ldots, m_p$ (with $p \ge 1$) such that

- $m_1 + m_2 + \cdots + m_i \ge i$, for all $i \in \{1, \ldots, p-1\}$;
- $m_1 + m_2 + \cdots + m_p = p - 1$.

The mapping

$$\Phi : \tau \to \left(k_{u_\tau(0)}, k_{u_\tau(1)}, \ldots, k_{u_\tau(\zeta(\tau)-1)}\right),$$

defines a bijection from $\mathbb{T}$ onto $\mathbb{S}$. Rather than the sequence $\Phi(\tau)$, we will consider the Lukasiewicz path (see figure 1) defined by

$$x_n = \sum_{i=1}^{n} (k_{u_\tau(i)} - 1), \qquad 0 \le n \le \zeta(\tau),$$

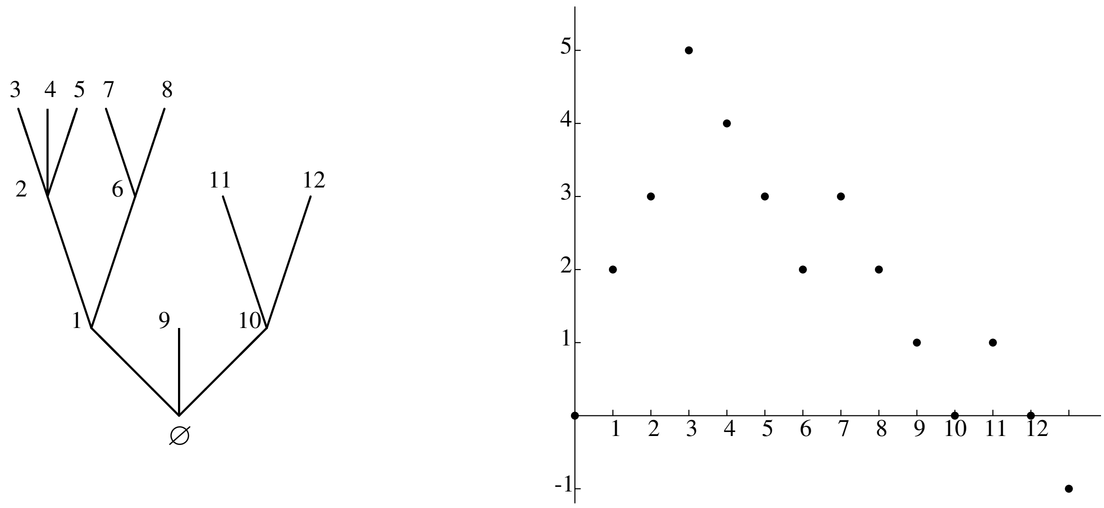
where $k_{u_\tau(\zeta(\tau))} = 0$. The Lukasiewicz path satisfies the following properties

- $x_0 = 0$ and $x_{\zeta(\tau)} = -1$.
- $x_n \ge 0$ for every $0 \le n \le p-1$.
- $x_i - x_{i-1} \ge -1$ for every $1 \le i \le p$.

Obviously the mapping Φ induces a bijection between trees and the Lukasiewicz path. Finally we note that we can recover the height function from the Lucasiewicz path by the following formula

$$H_n(\tau) = \mathrm{card}\Big\{ j \in \{0, 1, \ldots, n-1\} : x_j = \inf_{j \le l \le n} x_l \Big\},$$

for every $n \in \{0, 1, \ldots, \zeta(\tau) - 1\}$.



Rooted tree τ Lukasiewicz path of τ

Figure 1

2. Galton-Watson trees and forest.

Now let us consider a probability measure μ on $\mathbb{Z}_+$, such that

$$\sum_{k=0}^{\infty} k\mu(k) \le 1 \quad \text{and} \quad \mu(1) < 1\,.$$

A probability measure satisfying such conditions is called critical or subcritical offspring distribution.

We will use the approach of discrete trees to construct our basic objects, the Galton-Watson trees. Let $(R_u, u \in \mathbb{U})$ be a family of independent random variables with law μ, indexed by $\mathbb{U}$. Denote by Δ the random subset of $\mathbb{U}$ defined by

$$\Delta = \Big\{ u = (u_1, \ldots, u_n) \in \mathbb{U} : u_j \le R_{(u_1, \ldots, u_{j-1})} \text{ for every } 1 \le j \le n \Big\}.$$

Note that Δ is almost surely a discrete tree and if we define

$$Z_n = \mathrm{card}\big\{ u \in \Delta : |u| = n \big\},$$

it is not difficult to show that $(Z_n, n \ge 0)$ is a Galton-Watson process with offspring distribution μ and initial value $Z_0 = 1$. By the definition of Δ, it is clear that $k_u(\Delta) = R_u$ for every $u \in \Delta$.

The tree Δ, or any other random tree with the same distribution, will be called a Galton-Watson tree with offspring distribution μ and its law is the unique probability measure $\mathbb{Q}_\mu$ on $\mathbb{T}$ satisfying:

(i) $\mathbb{Q}_\mu(k_\emptyset(\Delta) = j) = \mu(j)$, $j \in \mathbb{Z}_+$.

(ii) For every $j \geq 1$, with $\mu(j) > 0$, the shifted trees $\theta_1(\Delta), \ldots, \theta_j(\Delta)$ are independent under the conditional distribution $\mathbb{Q}_\mu(\,\cdot \mid k_\emptyset = j)$ and their conditional law is $\mathbb{Q}_\mu$.

A Galton-Watson forest with offspring distribution μ is a finite or infinite sequence of independent Galton-Watson trees with offspring distribution μ. In the sequel, we will denote by τ for a Galton-Watson tree and by $\mathcal{F} = (\tau_k)$ for a Galton-Watson forest, respectively. With a misuse of notation, we will denote by $\mathbb{Q}_\mu$ the law on $(\mathbb{T})^{\mathbb{N}^*}$ of a Galton-Watson forest with offspring distribution μ.

Since τ is a random discrete tree, we may code its genealogy by its associated height function, contour function and Lukasiewicz path which obviously became random processes. The definition of such objects are the same as in the previous section, but here we will introduce them for the case of Galton-Watson forests.

The height process of a Galton-Watson forest $\mathcal{F} = (\tau_k)_{k\geq 1}$ is defined by

$$n \mapsto H_n(\mathcal{F}) = H_{n-(\zeta(\tau_0)+\cdots+\zeta(\tau_{k-1}))}(\tau_k), \quad \text{if } \zeta(\tau_0) + \cdots + \zeta(\tau_{k-1}) \leq n \leq \zeta(\tau_0) + \cdots + \zeta(\tau_k) - 1,$$

for $k \geq 1$, and with the convention that $\zeta(\tau_0) = 0$.

Although this process is natural and simple to define from discrete trees, its law is rather complicated to characterize. In particular, H is neither a Markov process nor a martingale.

In a similar way, we may introduce the contour process related to a Galton-Watson forest. The contour process for a Galton-Watson forest $\mathcal{F} = (\tau_k)_{k\geq 1}$ is the concatenation of the processes $C(\tau_1), \ldots, C(\tau_k), \ldots$, i.e. for $k \geq 1$

$$C_t(\mathcal{F}) = C_{t-2(\zeta(\tau_0)+\cdots+\zeta(\tau_{k-1}))}(\tau_k), \text{ if } 2(\zeta(\tau_0)+\cdots+\zeta(\tau_{k-1})) \leq t \leq 2(\zeta(\tau_0)+\cdots+\zeta(\tau_k)).$$

If there is a finite number of trees, say j, in the forest, we set $C_t(\mathcal{F}) = 0$, for $t \geq 2(\zeta(\tau_0) + \cdots + \zeta(\tau_j))$. Note that for each tree τ_k, $[2(\zeta(\tau_k) - 1), 2\zeta(\tau_k)]$ is the only nontrivial subinterval of $[0, 2\zeta(\tau_k)]$ on which $C(\tau_k)$ vanishes. This convention ensures that the contour process $C(\mathcal{F})$ also codes the genealogy of the forest. However, it has no "good properties" in law either.

The Lukasiewicz path of a Galton-Watson tree is a process with nice properties. In fact, it is a random walk killed at its first passage time below the negative half-line. Such process is also well known as the coding random walk.

Here, we will denote by $S(\tau)$ for the coding random walk associated to a Galton-Watson tree τ and from its definition it satisfies that:

$$S_0 = 0\,, \quad S_{n+1}(\tau) - S_n(\tau) = k_{u(n)}(\tau) - 1, \quad 0 \leq n \leq \zeta(\tau) - 1\,.$$

Note that for each n, $S_n(\tau)$ is the sum of all the younger brother of each of the ancestor of $u(n)$ including $u(n)$ itself.

For a forest $\mathcal{F} = (\tau_k)$, the process $S(\mathcal{F})$ is the concatenation of $S(\tau_1), \ldots, S(\tau_n), \ldots$:

$$S_n(\mathcal{F}) = S_{n-(\zeta(\tau_0)+\cdots+\zeta(\tau_{k-1}))}(\tau_k) - k + 1, \quad \text{if } \zeta(\tau_0) + \cdots + \zeta(\tau_{k-1}) \leq n \leq \zeta(\tau_0) + \cdots + \zeta(\tau_k).$$

If there is a finite number of trees j, then we set $S_n(\mathcal{F}) = S_{\zeta(\tau_0)+\cdots+\zeta(\tau_j)}(\mathcal{F})$, for $n \geq \zeta(\tau_0) + \cdots + \zeta(\tau_j)$. From the construction of $S(\tau_1)$ it appears that $S(\tau_1)$ is a random walk with initial value $S_0 = 0$ and step distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, \ldots$ which is killed when it first enters into the negative half-line. Hence, when the number of trees is infinite, $S(\mathcal{F})$ is a downward skip free random walk on $\mathbb{Z}$ with the law described above.

Let us denote $H(\mathcal{F})$, $C(\mathcal{F})$ and $S(\mathcal{F})$ respectively by H, C and S when no confusion is possible. We recall the identity

$$H_n = \text{card}\left\{0 \le k \le n : S_k = \inf_{k \le j \le n} S_j\right\}$$

which is established in Section 1 for any discrete tree.
For any integer $k \ge 1$, we denote by $\mathcal{F}^{k,n}$ a G-W forest with k trees conditioned to have n vertices, that is a forest with the same law as $\mathcal{F} = (\tau_l, \dots, \tau_k)$ under the conditional law $\mathbb{Q}_\mu(\,\cdot \,|\, \zeta(\tau_1) + \dots + \zeta(\tau_k) = n)$. The starting point of our work is the observation $\mathcal{F}^{k,n}$ can be coded by a downward skip free random walk conditioned to first reach $-k$ at time n. An interpretation of this result may be found in **[Pitm02]**, Lemma 6.3 for instance.

PROPOSITION 15. *Let $\mathcal{F} = (\tau_j)$ be a forest with offspring distribution μ and S and H be respectively its coding walk and its height process. Let W be a random walk defined on a probability space $(\Omega, \mathcal{F}, P)$ with the same law as S. We define $T_i^W = \inf\{j : W_j = -i\}$, for $i \ge 1$. Take k and n such that $P(T_k^W = n) > 0$. Then under the conditional law $\mathbb{Q}_\mu(\,\cdot \,|\, \zeta(\tau_1) + \dots + \zeta(\tau_k) = n)$,*

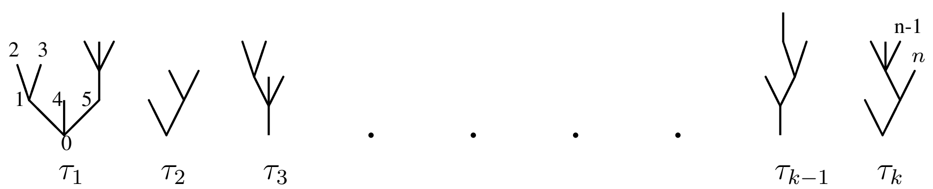
(1) *The process $(S_j,\, 0 \le j \le \zeta(\tau_1) + \dots + \zeta(\tau_k))$ has the same law as the killed random walk $(W_j,\, 0 \le j \le T_k^W)$.*

Moreover, define the processes $H_n^W = \text{card}\left\{k \in \{0, \dots, n-1\} : W_k = \inf_{k \le j \le n} W_j\right\}$ and C^W using the height process H^W as in (6.2), then

(2) *the process $(H_j,\, 0 \le j \le \zeta(\tau_1) + \dots + \zeta(\tau_k))$ has the same law as the process $(H_j^W,\, 0 \le j \le T_k^W)$.*

(3) *the process $(C_t,\, 0 \le 0 \le t \le 2(\zeta(\tau_1) + \dots + \zeta(\tau_k) - k))$ has the same law as the process $(C_t^W,\, 0 \le t \le 2(T_k^W - k))$.*

It is also straightforward that the identities in law involving separately the processes H, S and C in the above proposition also hold for the triple (H, S, C). In the figure below, we have represented an occurrence of the forest $\mathcal{F}^{k,n}$ and its associated *coding first passage bridge*.



Conditioned random forest: k trees, n vertices.

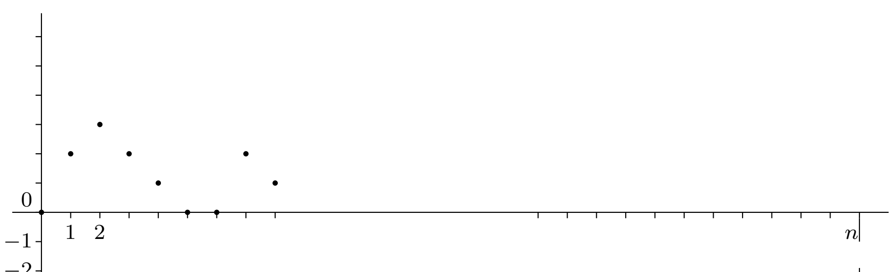


Figure 2

In chapter 7, we will present a continuous time version of this result, but before we need to introduce the continuous time setting of Lévy trees and forests.

3. Real trees.

Discrete trees may be considered in an obvious way as compact metric spaces with no loops. Such metric spaces are special cases of IR-trees which are defined hereafter. Similarly to the discrete case, an IR-forest is any collection of IR-trees. In this section we keep the same notations as in Duquesne and Le Gall's articles [**DuLG02**] and [**DuLG05**]. The following formal definition of IR-trees is now classical and originates from T-theory. It may be found in [**DMTe96**].

DEFINITION 2. *A metric space $(\mathcal{T},d)$ is an* IR*-tree if for every $\sigma_1,\sigma_2\in\mathcal{T}$,*

1. *There is a unique map f_{σ_1,σ_2} from $[0,d(\sigma_1,\sigma_2)]$ into $\mathcal{T}$ such that $f_{\sigma_1,\sigma_2}(0)=\sigma_1$ and $f_{\sigma_1,\sigma_2}(d(\sigma_1,\sigma_2))=\sigma_2$.*
2. *If g is a continuous injective map from $[0,1]$ into $\mathcal{T}$ such that $g(0)=\sigma_1$ and $g(1)=\sigma_2$, we have*

$$g([0,1])=f_{\sigma_1,\sigma_2}([0,d(\sigma_1,\sigma_2)])\,.$$

A rooted IR*-tree is an* IR*-tree $(\mathcal{T},d)$ with a distinguished vertex $\rho=\rho(\mathcal{T})$ called the root. An* IR*-forest is any collection of rooted* IR*-trees: $\mathcal{F}=\{(\mathcal{T}_i,d_i),\,i\in\mathcal{I}\}$.*

Let us explain in a more detailed way this definition. The range of the mapping f_{σ_1,σ_2} in (1), denoted by $l(\sigma_1,\sigma_2)$, is the line segment between σ_1 and σ_2 in the tree. In particular, for every $\sigma\in\mathcal{T}$, $l(\rho,\sigma)$ is the path going from the root to σ, such line can be interpreted as the ancestral line of vertex σ. In fact, we may define a partial order on $\mathcal{T}$ in the following way: let σ and ς be two elements of the tree, σ is an ancestor of ς if and only if $\sigma\in l(\rho,\varsigma)$. If $\sigma,\varsigma\in\mathcal{T}$, there is a unique $\eta\in\mathcal{T}$ such that $l(\rho,\varsigma)\cap l(\rho,\sigma)=l(\rho,\eta)$, such element of the tree is called the last common ancestor of σ and ς. The multiplicity of a vertex $\sigma\in\mathcal{T}$ is defined as the number of connected components of $\mathcal{T}\setminus\{\sigma\}$. Vertices of $\mathcal{T}\setminus\{\rho\}$ which have multiplicity 1 are called leaves.

Now, we discuss some important points on IR-trees. Two rooted real trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are called equivalent if there is a root-preserving isometry that maps $\mathcal{T}_1$ into $\mathcal{T}_2$. The space of all equivalent classes of rooted compact IR-trees will be denoted by $\mathbb{T}_c$. It is endowed with the Gromov-Hausdorff distance, d_{HG} which we briefly recall now. For a metric space (E,δ) and K, K' two subspaces of E, $\delta_{\text{Haus}}(K,K')$ will denote the Hausdorff distance between K and K'. Then we define the distance between $\mathcal{T}$ and $\mathcal{T}'$ by:

$$d_{GH}(\mathcal{T},\mathcal{T}')=\inf\left(\delta_{\text{Haus}}(\varphi(\mathcal{T}),\varphi'(\mathcal{T}'))\vee\delta(\varphi(\rho),\varphi'(\rho'))\right)\,,$$

where the infimum is taken over all isometric embeddings $\varphi:\mathcal{T}\to E$ and $\varphi':\mathcal{T}'\to E$ of $\mathcal{T}$ and $\mathcal{T}'$ into a common metric space (E,δ). We refer to Chapter 3 of Evans [**Evan05**] and the references therein for a complete description of the Gromov-Hausdorff topology. We only emphasize that from Theorem 3.23 of [**Evan05**], the space $(\mathbb{T}_c,D_{GH})$ is complete and separable.

A construction of some particular cases of such metric spaces has been introduced by Aldous [**Aldo91**] and may be found in [**DuLG05**] in a more general setting. Let f be a positive-continuous function with compact support defined on $[0,\infty)$, such that $f(0)=0$.

For $0 \le s \le t$, we define

$$d_f(s,t) = f(s) + f(t) - 2 \inf_{u \in [s,t]} f(u) \tag{6.3}$$

and the equivalence relation by

$$s \sim t \quad \text{if and only if} \quad d_f(s,t) = 0\,.$$

(Note that $d_f(s,t) = 0$ if and only if $f(s) = f(t) = \inf_{u\in[s,t]} f(u)$.) We easily check that the projection of d_f on the quotient space

$$\mathcal{T}_f = [0,\infty)/\sim$$

defines a distance. This distance will be denoted by d_f.

THEOREM 27. *The metric space $(\mathcal{T}_f, d_f)$ is a compact* $\mathbb{R}$*-tree.*

Denote by $p_f : [0,\infty) \to \mathcal{T}_f$ the canonical projection. The vertex $\rho = p_f(0)$ will be chosen as the root of $\mathcal{T}_f$. It has recently been proved by Duquesne [**Duqu06**] that any $\mathbb{R}$-tree (satisfying some rather weak assumptions) may by represented as $(\mathcal{T}_f, d_f)$ where f is a left continuous function with right limits and without positive jumps.

4. Lévy trees.

Now, we will introduce the random process which codes, in the sense of section 2, the genealogical structure of a continuous state branching process (or CB-process). As we mentioned in the introduction, the CB-process is a Markov process $Z = (Z_t, t \ge 0)$ taking values in $[0,\infty]$ with Feller semigroup $(Q_t, t \ge 0)$ which satisfies the following property: for every $t \ge 0$ and $x, y \ge 0$,

$$Q_t(x,\cdot) * Q_t(y,\cdot) = Q_t(x+y,\cdot),$$

where $*$ denotes the convolution. This is the well known branching property.
The Laplace functional of the semigroup $(Q_t, t \ge 0)$ can be written in the following form:

$$\int_{[0,\infty]} e^{-\lambda y} Q_t(x, \mathrm{d}y) = \exp\Big\{ - x u_t(\lambda)\Big\}, \qquad \text{for } \lambda \ge 0,$$

where the function $u_t(\lambda)$ is determined by the following partial differential equation

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)), \qquad u_0(\lambda) = \lambda,$$

and ψ is a function of the type

$$\psi(\lambda) = a\lambda + \beta\lambda^2 + \int_{(0,\infty)} \big(e^{-x} - 1 + \lambda x\big)\Pi(\mathrm{d}x),$$

where a, β are positive real numbers and Π is a σ-finite measure such that

$$\int_{(0,\infty)} \big(x \wedge x^2\big)\Pi(\mathrm{d}x) < \infty.$$

The process Z is called the CB-process with branching mechanism ψ. It is well known that Z may be obtained as a time change of a Lévy process with no negatives jumps. We remark that if the CB-process with branching mechanism satisfies that

$$\int_1^\infty \frac{du}{\psi(u)} < \infty\,, \tag{6.4}$$

hence Z have a finite time extinction almost surely.
In the remainder of this section, we will recall from [**DuLG05**] the definition of Lévy trees

and given this of the Lévy forests. Let $(\mathbb{P}_x)$, $x \in \mathbb{R}$ be a sequence of probability measures on the Skorokhod space $\mathcal{D}$ of càdlàg paths from $[0, \infty)$ to $\mathbb{R}$ such that for each $x \in \mathbb{R}$, the canonical process X is a Lévy process with no negative jumps. Set $\mathbb{P} = \mathbb{P}_0$, so $\mathbb{P}_x$ is the law of $X + x$ under $\mathbb{P}$. Here, we suppose that the characteristic exponent ψ of X, defined by

$$\mathbb{E}(e^{-\lambda X_t}) = e^{t\psi(\lambda)}, \qquad \lambda \in \mathbb{R}$$

satisfies the condition (6.4).
By analogy with the discrete case, the continuous time height process $\mathcal{H}$ is the measure (in a sense which is to be defined) of the set

$$\{s \le t : X_s = \inf_{s \le r \le t} X_r\}.$$

A rigorous meaning to this measure is given by the following result due to Le Jan and Le Gall [**LGLJ98**], see also [**DuLG02**]. Define $I_t^s = \inf_{s \le u \le t} X_u$. There is a sequence of positive real numbers (ε_k) which decreases to 0 such that for any t, the limit

$$\bar{H}_t \overset{\text{(def)}}{=} \lim_{k \to +\infty} \frac{1}{\varepsilon_k} \int_0^t 1\!\!\mathrm{I}_{\{X_s - I_t^s < \varepsilon_k\}} \, ds \tag{6.5}$$

exists a.s. It is also proved in [**LGLJ98**] that under assumption (6.4), $\bar{H}$ is a continuous process, so that each of its positive excursion codes a real tree in the sense of Aldous. We easily deduce from this definition that the height process $\bar{H}$ is a functional of the Lévy process reflected at its minimum, i.e. $X - I$, where $I := I^0$. In particular, when $\alpha = 2$, $\bar{H}$ is equal to the reflected process multiplied by a constant. It is well known that $X - I$ is a strong Markov process. Moreover, under our assumptions, 0 is regular for itself for this process and we can check that the process $-I$ is a local time at level 0. We denote by N the corresponding Itô measure of the excursions away from 0.
Now in order to define the Lévy forest, we need to introduce the local times of the height process $\bar{H}$. It is proved in [**DuLG02**] that for any level $a \ge 0$, there exists a continuous increasing process $(L_t^a,\, t \ge 0)$ which is defined by the approximation:

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left(\sup_{0 \le s \le t} \left| \frac{1}{\varepsilon} \int_0^s du\, 1\!\!\mathrm{I}_{\{a < \bar{H}_u \le a + \varepsilon\}} - L_s^a \right| \right) = 0\,.$$

The support of the measure dL_t^a is contained in the set $\{t \ge 0 : \bar{H}_t = a\}$ and its is not difficult to check that $L^0 = -I$. Then we may define the Poisson point process of the excursions away from 0 of the process $\bar{H}$ as follows. Let $T_s = \inf\{t : -I_t \ge s\}$ be the right continuous inverse of the local time at 0 of the reflected process $X - I$. The time T_u corresponds to the first passage time of X bellow $-u$. Set $T_{0-} = 0$ and for all $u \ge 0$,

$$e_u(v) = \begin{cases} \bar{H}_{T_{u-}+v}\,, & \text{if } 0 \le v \le T_u - T_{u-} \\ 0\,, & \text{if } v > T_u - T_{u-} \end{cases}.$$

For each $u \ge 0$, we may define the tree $(\mathcal{T}_{e_u}, d_{e_u})$ under $\mathbb{P}$ as in Theorem 27. We easily deduce from the Markov property of $X - I$ that under the probability measure $\mathbb{P}$, the process $\{(\mathcal{T}_{e_u}, d_{e_u}), u \ge 0\}$ is a Poisson point process whose characteristic measure is the law of the random real tree $(\mathcal{T}_{\bar{H}}, d_{\bar{H}})$ under N. By analogy to the discrete case, this Poisson point process, as a $\mathbb{T}_c$-valued process, provides a natural definition for the Lévy forest.

DEFINITION 3. *The Lévy tree is the real tree $(\mathcal{T}_{\bar{H}}, d_{\bar{H}})$ coded by the function $\bar{H}$ under the measure N. We denote by $\Theta(d\mathcal{T})$ the σ-finite measure on $\mathbb{T}_c$ which is the law of the Lévy tree $\mathcal{T}_{\bar{H}}$ under N. The Lévy forest $\mathcal{F}_{\bar{H}}$ is the Poisson point process*

$$(\mathcal{F}_{\bar{H}}(u),\, u \ge 0) \overset{(def)}{=} \{(\mathcal{T}_{e_u}, d_{e_u}), u \ge 0\}$$

which has for characteristic measure $\Theta(d\mathcal{T})$ under $\mathbb{P}$. For $s>0$, the process

$$\mathcal{F}^s_{\bar H} \overset{(def)}{=} \{(\mathcal{T}_{e_u}, d_{e_u}), 0 \le u \le s\},$$

under $\mathbb{P}$ will be called the Lévy forest of size s.

Such a definition of a Lévy forest has already been introduced in [**Pitm02**], Proposition 7.8 in the Brownian setting. In this work, it is observed that this forests may also be simply defined as the real tree coded by the function $\bar H$ under the law $\mathbb{P}$. One may also see [**PiWi05**] for the case of Lévy forests. Similarly, the Lévy forest with size s may be defined as the compact real tree coded by the continuous function with compact support $(\bar H_u,\, 0 \le u \le T_s)$. These definitions are more natural when considering convergence of sequences of real forest and we will make appeal to them in section 5, see Corollary 26.

We will simply denote the Lévy tree and the Lévy forests respectively by $\mathcal{T}_{\bar H}$, $\mathcal{F}_{\bar H}$ or $\mathcal{F}^s_{\bar H}$, the corresponding distances being implicit. When X is stable, condition (6.4) is satisfied if and only if its index α satisfies $\alpha \in (1,2)$. Then it follows from (6.5) that $\bar H$ is a self-similar process with index $\alpha/(\alpha-1)$, i.e.:

$$(\bar H_t,\, t \ge 0) \overset{(d)}{=} (k^{1/\alpha-1}\bar H_{kt},\, t \ge 0)\,, \quad \text{for all } k>0.$$

In this case, the Lévy tree $\mathcal{T}_{\bar H}$ associated to the stable mechanism is called the α-stable Lévy tree and its law will be denoted by $\Theta_\alpha(d\mathcal{T})$. This random metric space also inherits from X a scaling property which may be stated as follows: for any $a>0$, we denote by $a\mathcal{T}_{\bar H}$ the Lévy tree $\mathcal{T}_{\bar H}$ endowed with the distance $ad_{\bar H}$, i.e.

$$a\mathcal{T}_{\bar H} \overset{(\mathrm{def})}{=} (\mathcal{T}_{\bar H}, ad_{\bar H})\,. \tag{6.6}$$

Then the law of $a\mathcal{T}_{\bar H}$ under $\Theta_\alpha(d\mathcal{T})$ is $a^{\frac{1}{\alpha-1}}\Theta_\alpha(d\mathcal{T})$. This property is stated in [**DuLG05**] where other fractal properties of stable trees are considered.

CHAPTER 7

Conditioned stable Lévy forests.

In this Chapter, we define the total mass of the Lévy forest of a given size s. Then we define the Lévy forest of size s conditioned by its total mass. In the stable case, we give a construction of this conditioned forest from the unconditioned forest and we prove an invariance principle for this conditioned forest by considering k_n independent Galton-Watson trees whose offspring distribution is in the domain of attraction of any stable law conditioned on their total progeny to be equal to n.

1. Construction of the conditioned Lévy forest

In this section we present the continuous analogue of the forest $\mathcal{F}^{k,n}$ introduced in the previous chapter. In particular, we define the total mass of the Lévy forest of a given size s. Then we define the Lévy forest of size s conditioned by its total mass. In the stable case, we give a construction of this conditioned forest from the unconditioned forest.

We begin with the definition of the measure $\ell^{a,u}$ which represents a local time at level $a > 0$ for the Lévy tree $\mathcal{T}_{e_u}$. For every level $a > 0$, and every bounded and continuous function φ on $\mathcal{T}_{e_u}$, the finite measure $\ell^{a,u}$ is defined by:

$$\langle \ell^{a,u}, \varphi \rangle = \int_0^{T_u - T_{u-}} dL^a_{T_u+v} \varphi(p_{e_u}(v)) \,, \tag{7.1}$$

where we recall from the previous section that p_{e_u} is the canonical projection from $[0,\infty)$ onto $\mathcal{T}_{e_u}$ for the equivalence relation $\sim$ and (L^a_u) is the local time at level a of $\bar{H}$. Then the mass measure of the Lévy tree $\mathcal{T}_{e_u}$ is

$$\mathbf{m}_u = \int_0^\infty da \, l^{a,u} \tag{7.2}$$

and the total mass of the tree is $\mathbf{m}_u(\mathcal{T}_{e_u})$. Now we fix $s > 0$ and $t > 0$. The total mass of the forest of size s, $\mathcal{F}^s_{\bar{H}}$ is naturally given by

$$\mathbf{M}_s = \sum_{0 \le u \le s} \mathbf{m}_u(\mathcal{T}_{e_u}) \,.$$

PROPOSITION 16. *$\mathbb{P}$-almost surely $T_s = \mathbf{M}_s$.*

Proof. It follows from the definitions (7.1) and (7.2) that for each tree $\mathcal{T}_{e_u}$, the mass measure $\mathbf{m}_u$ coincides with the image of the Lebesgue measure on $[0, T_u - T_{u-}]$ under the mapping $v \mapsto p_{e_u}(v)$. Thus, the total mass of each tree $\mathcal{T}_{e_u}$ is $T_u - T_{u-}$. This implies the result. ■

Then we will construct processes which encode the genealogy of the Lévy forest of size s conditioned to have a mass equal to t. From the analogy with the discrete case in Proposition 15, the natural candidates may be informally defined as:

$$X^{br} \stackrel{\text{(def)}}{=} [(X_u,\, 0 \le u \le T_s)\,|\, T_s = t]$$
$$\bar{H}^{br} \stackrel{\text{(def)}}{=} [(\bar{H}_u,\, 0 \le u \le T_s)\,|\, T_s = t]\,.$$

When X is the Brownian motion, the process X^{br} is called the *first passage bridge*, see [**BeCP03**]. In order to give a proper definition in the general case, we need the additional assumption:

The semigroup of $(X, \mathbb{P})$ is absolutely continuous with respect to the Lebesgue measure.

Then denote by $p_t(\cdot)$ the density of the semigroup of X, by $\mathcal{G}_u^X \stackrel{\text{(def)}}{=} \sigma\{X_v, v \le u\}$, $u \ge 0$ the σ-field generated by X and set $\hat{p}_t(x) = p_t(-x)$.

LEMMA 9. *The probability measure defined on each $\mathcal{G}_u^X$ by*

$$\mathbb{P}(X^{br} \in \Lambda_u) = \mathbb{E}\left(1\!\!\mathrm{I}_{\{X\in\Lambda_u\,,u<T_s\}} \frac{t(s+X_u)}{s(t-u)} \frac{\hat{p}_{t-u}(s+X_u)}{\hat{p}_t(s)} \right)\,, \quad u<t \quad \Lambda_u \in \mathcal{G}_u^X\,, \tag{7.3}$$

is a regular version of the conditional law of $(X_u,\, 0 \le u \le T_s)$ given $T_s = t$, in the sense that for all $u > 0$, for λ-a.e. $s > 0$ and λ-a.e. $t > u$,

$$\mathbb{P}(X^{br} \in \Lambda_u) = \lim_{\varepsilon\downarrow 0} \mathbb{P}(X \in \Lambda_u \,|\, |T_s - t| < \varepsilon)\,,$$

where λ is the Lebesgue measure.

Proof. Let $u < t$, $\Lambda_u \in \mathcal{G}_u^X$ and $\varepsilon < t - u$. From the Markov property, we may write

$$\begin{aligned} \mathbb{P}(X \in \Lambda_u \,|\, |T_s - t| < \varepsilon) &= \mathbb{E}\left(1\!\!\mathrm{I}_{\{X\in\Lambda_u\}} \frac{1\!\!\mathrm{I}_{\{\,|T_s-t|<\varepsilon\}}}{\mathbb{P}(|T_s - t| < \varepsilon)} \right) \\ &= \mathbb{E}\left(1\!\!\mathrm{I}_{\{X\in\Lambda_u, u<T_s\}} \frac{\mathbb{P}_{X_u}(|T_s - (t-u)| < \varepsilon)}{\mathbb{P}(|T_s - t| < \varepsilon)} \right)\,. \end{aligned} \tag{7.4}$$

On the other hand, from Corollary VII.3 in [**Bert96**] one has for λ-a.e. $s > 0$ and λ-a.e. $t > 0$,

$$t\mathbb{P}(T_s \in dt)\, ds = s\hat{p}_t(s)\, dt\, ds\,. \tag{7.5}$$

Hence, for all $x \in \mathbb{R}$, for all $u > 0$, for λ-a.e. $s > 0$ and λ-a.e. $t > u$,

$$\lim_{\varepsilon\downarrow 0} \frac{\mathbb{P}_x(|T_s - (t-u)| < \varepsilon)}{\mathbb{P}(|T_s - t| < \varepsilon)} = \frac{t(s+x)}{s(t-u)} \frac{\hat{p}_{t-u}(s+x)}{\hat{p}_t(s)}\,.$$

Moreover we can check from (7.5) that $\mathbb{E}\left(\frac{t(s+X_u)}{s(t-u)} \frac{\hat{p}_{t-u}(s+X_u)}{\hat{p}_t(s)} \right) < +\infty$ for λ-a.e. t, so the result follows from (7.4) and Fatou's lemma. ∎

We may now construct a height process $\bar{H}^{br}$ from the path of the first passage bridge X^{br} exactly as $\bar{H}$ is constructed from X in (6.5) or in Definition 1.2.1 of [**DuLG02**] and check that the law of $\bar{H}^{br}$ is a regular version of the conditional law of $(\bar{H}_u, 0 \le u \le T_s)$ given $T_s = t$. Call $(e_v^{s,t},\, 0 \le v \le s)$ the excursion process of $\bar{H}^{br}$, that is in particular

$$(e_v^{s,t},\, 0 \le v \le s) \text{ has the same law as } (e_v,\, 0 \le v \le s) \text{ given } T_s = t\,.$$

The following proposition is a straightforward consequence of the above definition and Proposition 16.

PROPOSITION 17. *The law of the process* $\{(\mathcal{T}_{e_v^{s,t}}, d_{e_v^{s,t}}),\, 0 \le v \le s\}$ *is a regular version of the law of the forest of size* s, $\mathcal{F}^s_{\bar{H}}$ *given* $\mathbf{M}_s = t$.

We will denote by $(\mathcal{F}^{s,t}_{\bar{H}}(u),\, 0 \le u \le s)$ a process with values in $\mathbb{T}_c$ whose law under $\mathbb{P}$ is this of the Lévy forest of size s conditioned by $\mathbf{M}_s = t$, i.e. conditioned to have a mass equal to t.
In the remainder of this section, we will consider the case when the driving Lévy process is stable. We suppose that its index α belongs to $(1, 2]$ so that condition

$$\int_1^\infty \frac{\mathrm{d}u}{\psi(u)} < \infty,$$

is satisfied. We will give a pathwise construction of the processes $(X^{br}, \bar{H}^{br})$ from the path of the original processes $(X, \bar{H})$. This result leads to the following realization of the Lévy forest of a given size conditioned by its mass. From now on, with no loss of generality, we suppose that $t = 1$.

THEOREM 28. *Define* $g = \sup\{u \le 1 : T_{u^{1/\alpha}} = s \cdot u\}$.

(1) $\mathbb{P}$*-almost surely,*

$$0 < g < 1\,.$$

(2) *Under* $\mathbb{P}$*, the rescaled process*

$$(g^{(1-\alpha)/\alpha} \bar{H}(gu),\, 0 \le u \le 1) \tag{7.6}$$

has the same law as $\bar{H}^{br}$ *and is independent of* g.

(3) *The forest* $\mathcal{F}^{s,1}_{\bar{H}}$ *of size* s *and mass* 1 *may be constructed from the rescaled process defined in* (7.6)*, i.e. if we denote by* $u \mapsto \epsilon_u \overset{(def)}{=} (g^{(1-\alpha)/\alpha} e_u(gv), v \ge 0)$ *its process of excursions away from* 0*, then under* $\mathbb{P}$, $\mathcal{F}^{s,1}_{\bar{H}} \overset{(d)}{=} \{(\mathcal{T}_{\epsilon_u}, d_{\epsilon_u}),\, 0 \le u \le s\}$.

Proof. The process $T_u = \inf\{v : I_v \le -u\}$ is a stable subordinator with index $1/\alpha$. Therefore,

$$T_u < su^\alpha\,, \quad \text{i.o. as } u \downarrow 0 \quad \text{and} \quad T_u > su^\alpha\,, \quad \text{i.o. as } u \downarrow 0.$$

Indeed, if $u_n \downarrow 0$ then $\mathbb{P}(T_{u_n} < su_n^\alpha) = \mathbb{P}(T_1 < s) > 0$, so that

$$\mathbb{P}\left(\limsup_{n\to\infty}\{T_{u_n} < su_n^\alpha\}\right) \ge \mathbb{P}(T_1 < s) > 0.$$

But T satisfies Blumenthal 0-1 law, so this probability is 1. The same arguments prove that $\mathbb{P}(\limsup_n\{T_{u_n} > su_n^\alpha\}) = 1$ for any sequence $u_n \downarrow 0$. Since T has only positive jumps, we deduce that $T_u = su^\alpha$ infinitely often as u tends to 0, so we have proved the first part of the theorem.

The rest of the proof is a consequence of the following lemma.

LEMMA 10. *The first passage bridge* X^{br} *fulfills the following path construction:*

$$X^{br} \overset{(d)}{=} (g^{-1/\alpha} X(gu),\, 0 \le u \le 1)\,.$$

Moreover, the process $(g^{-1/\alpha} X(gu),\, 0 \le u \le 1)$ *is independent of* g.

Proof. First note that for any $t > 0$ the bivariate random variable (X_t, I_t) under $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure and there is a version of its density which is continuous. Indeed from the Markov property and (7.5), one has for all $x \in \mathbb{R}$ and $y \geq 0$,

$$\begin{aligned} \mathbb{P}(I_t \leq y \,|\, X_t = x) &= \mathbb{P}\left(\mathrm{1\!I}_{\{T_y \leq t\}} \frac{p_{t-T_y}(x-y)}{p_t(0)} \right) \\ &= \int_0^t \frac{y}{s} \hat{p}_s(y) \frac{p_{t-s}(x-y)}{p_t(0)} \, ds \,. \end{aligned}$$

Looking at the expression of $\hat{p}_t(x)$ and $p_t(x)$ obtained from the Fourier inverse of the characteristic exponent, we see that theses functions are continuously differentiable and that their derivatives are continuous in t. It allows us to conclude.

Now let us consider the two dimensional self-similar strong Markov process $Y \stackrel{\text{(def)}}{=} (X, I)$ with state space $\{(x, y) \in \mathbb{R}^2 : y \leq x\}$. From our preceding remark, the semigroup $q_t((x, y), (dx', dy')) = \mathbb{P}(X_t + x \in dx', y \wedge (I_t + x) \in dy')$ of Y is absolutely continuous with respect to the Lebesgue measure and there is a version of its density which is continuous. Denote by $q_t((x, y), (x', y'))$ this version. We derive from (7.5) that for all $-s \leq x$,

$$q_t((x, y), (-s, -s)) = \mathrm{1\!I}_{\{y \geq -s\}} \frac{1}{t} \hat{p}_t(s + x) \,. \tag{7.7}$$

Then we may apply a result due to Fitzsimmons, Pitman and Yor [**FPYo92**] which asserts that the inhomogenous Markov process defined on $[0, t]$, whose law is defined by

$$\mathbb{E}\left(H(Y_u, v \leq u) \frac{q_{t-u}(Y_u, (x', y'))}{q_t((x, y), (x', y'))} \,|\, Y_0 = (x, y) \right), \quad 0 \leq u < t \,, \tag{7.8}$$

where H is a measurable functional on $C([0, u], \mathbb{R}^2)$, is a regular version of the conditional law of $(Y_v,\, 0 \leq v \leq t)$ given $Y_t = (x', y')$, under $\mathbb{P}(\,\cdot\,|\, Y_0 = (x, y))$. This law is called the law of the bridge from (x, y) to (x', y') with length t. Then from (7.7), the law which is defined in (7.8), when specifying it on the first coordinate and for $(x, y) = (0, 0)$ and $(x', y') = (-s, -s)$, corresponds to the law of the first passage bridge which is defined in (7.3).

It remains to apply another result which may also be found in [**FPYo92**]: observe that g is a backward time for Y in the sense which is defined in this paper. Indeed g may also be defined as $g = \sup\{u \leq 1 : X_u = -su^{1/\alpha},\, X_u = I_u\}$, so that for all $u > 0$, $\{g > u\} \in \sigma(Y_v : v \geq u)$. Then from Corollary 3 in [**FPYo92**], conditionally on g, the process $(Y_u,\, 0 \leq u \leq g)$ under $\mathbb{P}(\,\cdot\,|\, Y_0 = (0, 0))$ has the law of a bridge from $(0, 0)$ to Y_g with length g. (This result has been obtained and studied in a greater generality in [**ChUr06**].) But from the definition of g, we have $Y_g = (-sg^{1/\alpha}, -sg^{1/\alpha})$, so from the self-similarity of Y, under $\mathbb{P}$ the process

$$(g^{-1/\alpha} Y(g \cdot u)\,, 0 \leq u \leq 1)$$

has the law of the bridge of Y from $(0, 0)$ to $(-s, -s)$ with length 1. The lemma follows by specifying this result on the first coordinate. ■

The second part of the theorem is a consequence of Lemma 10, the construction of $\bar{H}^{br}$ from X^{br} and the scaling property of $\bar{H}$. The third part follows from the definition of the conditioned forest $\mathcal{F}_{\bar{H}}^{s,1}$ in Proposition 16 and the second part of this theorem. ■

2. Invariance principles

We know from Lamperti that the only possible limits of sequences of re-scaled G-W processes are continuous state branching processes. Then a question which arises is: when can we say that the whole genealogy of the tree or the forest converges ? In particular, do the height process, the contour process and the coding walk converge after a suitable re-scaling ? This question has now been completely solved by Duquesne and Le Gall [**DuLG02**]. Then one may ask the same for the trees or forests conditioned by their size and their mass. In [**Duqu03**], Duquesne proved that when the law ν is in the domain of attraction of a stable law, the height process, the contour process and the coding excursion of the corresponding G-W tree converge in law in the Skorohod space of càdlàg paths. This work generalizes Aldous' result [**Aldo91**] which concerned the brownian case. In this section we will prove that in the stable case, an invariance principle also holds when we consider a G-W forest conditioned by its size and its mass.

Recall from section 2 that for an offspring distribution μ we have set $\nu(k) = \mu(k+1)$, for $k = -1, 0, 1, \dots$. We make the following assumption:

$$\text{(H)} \qquad \begin{cases} \mu \text{ is aperiodic and there is an increasing sequence } (a_n)_{n\geq 0} \\ \text{such that } a_n \to +\infty \text{ and } S_n/a_n \text{ converges in law as } n \to +\infty \\ \text{toward the law of a non-degenerated r.v. } \theta. \end{cases}$$

Note that we are necessarily in the critical case, i.e. $\sum_k k\mu(k) = 1$, and that the law of θ is stable. Moreover, since $\nu(-\infty, -1) = 0$, the support of the Lévy measure of θ is $[0, \infty)$ and its index α is such that $1 < \alpha \leq 2$. Also (a_n) is a regularly varying sequence with index α. Under hypothesis (H), it has been proved by Grimvall [**Grim74**] that if Z is the G-W process associated to a tree or a forest with offspring distribution μ, then

$$\left(\frac{1}{a_n} Z_{[nt/a_n]},\, t \geq 0\right) \Rightarrow (\overline{Z}_t,\, t \geq 0)\,, \quad \text{as } n \to +\infty,$$

where $(\overline{Z}_t,\, t \geq 0)$ is a continuous state branching process. Here and in the sequel, $\Rightarrow$ will stand for the weak convergence in the Skohorod space of càdlàg trajectories. Recall from chapter 6 the definition of the discrete process (S, H). Then under the same hypothesis, it follows from Corollary 2.5.1 in Duquesne and Le Gall [**DuLG02**] that

$$\text{(7.9)} \quad \left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt}\right),\, t \geq 0\right] \Rightarrow \left[(X_t, \bar{H}_t, \bar{H}_t),\, t \geq 0)\right]\,, \quad \text{as } n \to +\infty,$$

where X is a stable Lévy process with law θ and $\bar{H}$ is the associated height process, as defined in section 4 of chapter 6.

Now we fix a real $s > 0$ and we consider a sequence of positive integers (k_n) such that

$$\text{(7.10)} \qquad \frac{k_n}{a_n} \to s\,, \quad \text{as } n \to +\infty.$$

Recall the notations of chapter 6. For any $n \geq 1$, let $(X^{br,n}, \bar{H}^{br,n}, C^{br,n})$ be the process whose law is this of

$$\left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt}\right),\, 0 \leq t \leq 1\right],$$

under $\mathbb{Q}_\mu(\,\cdot \mid \zeta(\tau_1) + \cdots + \zeta(\tau_{k_n}) = n)$. Note that we could also define this three dimensional process over the whole halfline $[0, \infty)$, rather than on $[0, 1]$. However, from the definitions in section 2, $\bar{H}^{br,n}$ and $C^{br,n}$ would simply vanish over $[1, \infty)$ and $X^{br,n}$ would be constant. Here is the conditional version of the invariance principle that we have recalled in (7.9).

THEOREM 29. *As n tends to $+\infty$, we have*

$$(X^{br,n}, \bar{H}^{br,n}, C^{br,n}) \Longrightarrow (X^{br}, \bar{H}^{br}, \bar{H}^{br}) .$$

In order to give a sense to the convergence of the Lévy forest, we may consider the trees $\mathcal{T}^{br,n}$ and $\mathcal{T}^{br}$ which are coded respectively by the continuous processes with compact support, $C_u^{br,n}$ and $\bar{H}_u^{br}$, in the sense given at the beginning of section 3 (here we suppose that these processes are defined on $[1,\infty)$ and both equal to 0 on this interval). Roughly speaking the trees $\mathcal{T}^{br,n}$ and $\mathcal{T}^{br}$ are obtained from the original (conditioned) forests by rooting all the trees of these forests to a same root.

COROLLARY 26. *The sequence of trees $\mathcal{T}^{br,n}$ converges weakly in the space $\mathbb{T}_c$ endowed with the Gromov-Hausdorff topology towards $\mathcal{T}^{br}$.*

Proof. This results is a consequence of the weak convergence of the contour function $C^{br,n}$ toward $\bar{H}^{br}$ and the inequality

$$d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \le 2\|g - g'\| ,$$

which is proved in [**DuLG05**], see Lemma 2.3. (We recall that d_{GH} the Gromov-Hausdorff distance which has been defined in chapter 6.) ∎

A first step for the proof of Theorem 29 is to obtain the weak convergence of $(X^{br,n}, \bar{H}^{br,n})$ restricted to the Skorokhod space $\mathbb{D}([0,t])$ for any $t < 1$. Then we will derive the convergence on $\mathcal{D}([0,1])$ from an argument of cyclic exchangeability. The convergence of the third coordinate $C^{br,n}$ is a consequence of its particular expression as a functional of the process $\bar{H}^{br,n}$. In the remainder of the proof, we suppose that S is defined on the same probability space as X and has step distribution ν under $\mathbb{P}$. Define also $T_k = \inf\{i : S_i = -k\}$, for all integer $k \ge 0$.

LEMMA 11. *For any $t < 1$, as n tends to $+\infty$, we have*

$$\left[(X_u^{br,n}, \bar{H}_u^{br,n}),\, 0 \le u \le t\right] \Longrightarrow \left[(X_u^{br}, \bar{H}_u^{br}),\, 0 \le y \le t\right] .$$

Proof. From Feller's combinatorial lemma, see [**Fell71**], we have

$$\mathbb{P}(T_k = n) = \frac{k}{n}\mathbb{P}(S_n = -k), \qquad \text{for all } n \ge 1,\ k \ge 0.$$

Let F be any bounded and continuous functional on $\mathbb{D}([0,t])$. By the Markov property at time $[nt]$,

$$\mathbb{E}[F(X_u^{br,n}, \bar{H}_u^{br,n}; 0 \le u \le t)] = \mathbb{E}\left[F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \le u \le t\right) \mid T_{k_n} = n\right]$$

$$= \mathbb{E}\left(\mathbb{I}_{\{[nt] \le T_{k_n}\}}\frac{\mathbb{P}_{S_{[nt]}}(T_{k_n} = n - [nt])}{\mathbb{P}(T_{k_n} = n)} \times F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \le u \le t\right)\right)$$

$$= \mathbb{E}\left(\mathbb{I}_{\{\frac{1}{a_n}\underline{S}_{[nt]} \ge -\frac{k_n}{a_n}\}}\frac{n(k_n + S_{[nt]})}{k_n(n - [nt])}\frac{\mathbb{P}_{S_{[nt]}}(S_{n-[nt]} = -k_n)}{\mathbb{P}(S_n = -k_n)}\right.$$

$$\left. \times F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \le u \le t\right)\right) . \tag{7.11}$$

where $\underline{S}_k = \inf_{i \le k} S_i$. To simplify the computations in the remainder of this proof, we set $P^{(n)}$ for the law of the process $\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; u \ge 0\right)$ and P will stand for the law of the process $(X_u, \bar{H}_u; u \ge 0)$. Then $Y = (Y^1, Y^2)$ is the canonical process of the coordinates on the Skorohod space $\mathcal{D}^2$ of càdlàg paths from $[0,\infty)$ into $\mathbb{R}^2$. We will also use special notations for the densities introduced in (7.3) and (7.11):

$$\begin{aligned} D_t &= 1\!\!\mathrm{I}_{\{\underline{Y}^1_t \geq -s\}} \frac{s - Y^1_t}{s(1-t)} \frac{p_{1-t}(Y^1_t, -s)}{p_1(0,-s)}\,, \quad \text{and} \\ D^{(n)}_t &= 1\!\!\mathrm{I}_{\{\underline{Y}^1_{[nt]} \geq -\frac{k_n}{a_n}\}} \frac{n(k_n + a_n Y^1_{[nt]})}{k_n(n-[nt])} \frac{\mathbb{P}_{a_n Y^1_{[nt]}}(S_{n-[nt]} = -k_n)}{\mathbb{P}(S_n = -k_n)}\,, \end{aligned}$$

where $\underline{Y}^1_s = \inf_{u \leq s} Y^1_u$. Put also F_t for $F(Y_u, 0 \leq u \leq t)$. To obtain our result, we have to prove that

$$\lim_{n\to+\infty} |E^{(n)}(F_t D^{(n)}_t) - E(F_t D_t)| = 0\,. \tag{7.12}$$

Let $M > 0$ and set $I_M(x) \stackrel{\text{(def)}}{=} 1\!\!\mathrm{I}_{[-s,M]}(x)$. By writing

$$E^{(n)}(F_t D^{(n)}_t) = E^{(n)}(F_t D^{(n)}_t I_M(Y^1_t)) + E^{(n)}(F_t D^{(n)}_t (1 - I_M(Y^1_t))$$

and by doing the same for $E(F_t D_t)$, we have the following upper bound for the term in (7.12)

$$\begin{aligned} |E^{(n)}(F_t D^{(n)}_t) - E(F_t D_t)| \;\leq\; & |E^{(n)}(F_t D^{(n)}_t I_M(Y^1_t)) - E(F_t D_t I_M(Y^1_t))| \\ & + C E^{(n)}(D^{(n)}_t (1 - I_M(Y^1_t))) + C E(D_t (1 - I_M(Y^1_t)))\,, \end{aligned}$$

where C is an upper bound for the functional F. But since D_t and $D^{(n)}_t$ are densities, $E^{(n)}(D^{(n)}_t) = 1$ and $E(D_t) = 1$, hence

$$\begin{aligned} |E^{(n)}(F_t D^{(n)}_t) - E(F_t D_t)| \;\leq\; & |E^{(n)}(F_t D^{(n)}_t I_M(Y^1_t)) - E(F_t D_t I_M(Y^1_t))| \\ & + C[1 - E^{(n)}(D^{(n)}_t I_M(Y^1_t))] + C[1 - E(D_t I_M(Y^1_t))]\,. \end{aligned} \tag{7.13}$$

Now it remains to prove that the first term of the right hand side of the inequality (7.13) tends to 0, i.e.

$$|E^{(n)}(F_t D^{(n)}_t I_M(Y^1_t)) - E(F_t D_t I_M(Y^1_t))| \to 0\,, \tag{7.14}$$

as $n \to +\infty$. Indeed, suppose that (7.14) holds, then by taking $F_t \equiv 1$, we see that the second term of the right hand side of (7.13) converges towards the third one. Moreover, $E(D_t I_M(Y^1_t))$ tends to 1 as M goes to $+\infty$. Therefore the second and the third terms in (7.12) tend to 0 as n and M go to $+\infty$.

Let us prove (7.14). From the triangle inequality and the expression of the densities D_t and $D^{(n)}_t$, we have

$$\begin{aligned} |E^{(n)}(F_t D^{(n)}_t I_M(Y^1_t)) - E(F_t D_t I_M(Y^1_t))| \leq & \sup_{x \in [-s,M]} |g_n(x) - g(x)| + \\ & |E^{(n)}(F_t D_t I_M(Y^1_t)) - E(F_t D_t I_M(Y^1_t))|\,, \end{aligned} \tag{7.15}$$

where $g_n(x) = \frac{n(k_n+x)}{k_n(n-[nt])} \frac{\mathbb{P}_x(S_{n-[nt]} = -k_n)}{\mathbb{P}(S_n = -k_n)}$ and $g(x) = \frac{s-x}{s(1-t)} \frac{p_{1-t}(x,-s)}{p_1(0,-s)}$. But thanks to Gnedenko local limit theorem and the fact that $k_n/a_n \to s$, we have

$$\lim_{n\to+\infty} \sup_{x\in[-s,M]} |g_n(x) - g(x)| = 0\,.$$

Moreover, recall that from Corollary 2.5.1 of Duquesne and Le Gall [**DuLG02**],

$$P^{(n)} \Rightarrow P\,,$$

as $n \to +\infty$, where $\Rightarrow$ stands for the weak convergence of measures on $\mathcal{D}^2$. Finally, note that the discontinuity set of the functional $F_tD_tI_M(Y_t^1)$ is negligible for the probability measure P so that the last term in (7.15) tends to 0 as n goes to $+\infty$. ■

The next lemmas are needed to prove the tightness of the sequence, $(X^{br,n}, \bar{H}^{br,n})$. Define the height process associated to any downward skip free chain $x = (x_0, x_1, \dots, x_n)$, i.e. $x_0 = 0$ and $x_i - x_{i-1} \geq -1$, as follows:

$$H_n^{(x)} = \text{card}\left\{i \in \{0, \dots, n-1\} : x_k = \inf_{i \leq j \leq n} x_j\right\}.$$

Define also the first passage time of x by $t(k) = \inf\{i : x_i = -k\}$ and for $n \geq k$, define the shifted chain:

$$\theta_{t(k)}(x)_i = \begin{cases} x_{i+t(k)} + k, & \text{if } i \leq n - t(k) \\ x_{t(k)+i-n} + x_n + k, & \text{is } n - t(k) \leq i \leq n \end{cases}, \quad i = 0, 1, \dots, n,$$

which consists in inverting the pre-$t(k)$ and the post-$t(k)$ parts of x and sticking them together.

LEMMA 12. *For any $k \geq 0$, we have almost surely*

$$H^{(\theta_{t(k)}(x))} = \theta_{t(k)}(H^{(x)}).$$

Proof. It is just a consequence of the fact that $t(k)$ is a zero of $H^{(x)}$. ■

LEMMA 13. *Let u_{k_n} be a random variable which is uniformly distributed over $\{0, 1, \dots, k_n\}$ and independent of S. Under $\mathbb{P}(\,\cdot\,|\,T(k_n) = n)$, the first passage time $T(u_{k_n})$ is uniformly distributed over $\{0, 1, \dots, n\}$.*

Proof. It follows from elementary properties of random walks that for all $k \in \{0, 1, \dots, k_n\}$, under $\mathbb{P}(\,\cdot\,|\,T(k_n) = n)$, the chain $\theta_{T(k_n)}(S)$ has the same law as $(S_i,\, 0 \leq i \leq n)$. As a consequence, for all $j \in \{0, 1, \dots, n\}$

$$P(T(k) = j\,|\,T(k_n) = n) = P(T(k_n - k) = n - j\,|\,T(k_n) = n).$$

which allows us to conclude. ■

LEMMA 14. *The family of processes*

$$(X^{br,n}, \bar{H}^{br,n}), \quad n \geq 1$$

is tight.

Proof. Let $\mathcal{D}([0,t])$ be the Skorokhod space of càdlàg paths from $[0,t]$ to $\mathbb{R}$. In Lemma 11 we have proved the weak convergence of $(X^{br,n}, \bar{H}^{br,n})$ restricted to the space $\mathcal{D}([0,t])$ for each $t > 0$. Therefore, from Theorem 15.3 of **[Bill99]**, it suffices to prove that for all $\delta \in (0,1)$ and $\eta > 0$,

(7.16)

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} \mathbb{P}\left(\sup_{s,t \in [1-\delta,1]} |X_t^{br,n} - X_s^{br,n}| > \eta,\ \sup_{s,t \in [1-\delta,1]} |\bar{H}_t^{br,n} - \bar{H}_s^{br,n}| > \eta\right) = 0.$$

Recall from Lemma 13 the definition of the r.v. u_{k_n}. Put $V_n = T(u_{k_n})/n$. Since from this lemma, V_n is uniformly distributed over $\{0, 1/n, \dots, 1 - 1/n, 1\}$, we have for any

$\varepsilon < 1-\delta$,

$$\mathbb{P}\left(\sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right)\le\varepsilon+\delta+$$
$$\mathbb{P}\left(V_n\in[\varepsilon,1-\delta],\ \sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right).$$

Now for a càdlàg path ω defined on $[0,1]$ and $t\in[0,1]$, define the shift:

$$\theta_t(\omega)_u=\left\{\begin{array}{ll}\omega_{s+t}+u, & \text{if } s\le 1-t\\ x_{t+u-1}+\omega_u+k, & \text{is } 1-t\le s\le 1\end{array}\right.,\quad u\in[0,1],$$

which consists in inverting the paths $(\omega_u, 0\le u\le t)$ and $(\omega_u, t\le u\le 1)$ and sticking them together. We can check on a picture the inclusion:

$$\{V_n\in[\varepsilon,1-\delta],\ \sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\}\subset$$
$$\{\sup_{s,t\in[0,1-\varepsilon]}|\theta_{V_n}(X^{br,n})_t-\theta_{V_n}(X^{br,n})_s|>\eta,\ \sup_{s,t\in[0,1-\varepsilon]}|\theta_{V_n}(\bar{H}^{br,n})_t-\theta_{V_n}(\bar{H}^{br,n})_s|>\eta\}.$$

From Lemma 12 and the straightforward identity in law $X^{br,n}\overset{(d)}{=}\theta_{V_n}(X^{br,n})$, we deduce the two dimensional identity in law $(X^{br,n},\bar{H}^{br,n})\overset{(d)}{=}(\theta_{V_n}(X^{br,n}),\theta_{V_n}(\bar{H}^{br,n}))$ which implies

$$\mathbb{P}\left(\sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right)\le\varepsilon+\delta+$$
$$\mathbb{P}\left(\sup_{s,t\in[0,1-\varepsilon]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[0,1-\varepsilon]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right).$$

But from Lemma 11 and Theorem 15.3 in [**Bill99**], we have

$$\lim_{\delta\to0}\limsup_{n\to+\infty}\mathbb{P}\left(\sup_{s,t\in[0,1-\varepsilon]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[0,1-\varepsilon]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right)=0.$$

which yields (7.16). ■

Proof of Theorem 29. Lemma 11 shows that the sequence of processes $(X^{br,n},\bar{H}^{br,n})$ converges toward $(X^{br},\bar{H}^{br})$ in the sense of finite dimensional distributions. Moreover tightness of this sequence has been proved in Lemma 14, so we conclude from Theorem 15.1 of [**Bill99**]. The convergence of the two first coordinates in Theorem 29 is proved, i.e. $(X^{br,n},\bar{H}^{br,n})\Longrightarrow(X^{br},\bar{H}^{br})$. Then we may deduce the functional convergence of the third coordinates from this convergence in law following similar arguments as in Theorem 2.4.1 in [**DuLG02**]:

From (6.2), we can recover the contour process of $X^{br,n}$ as follows set $K_i=2i-\bar{H}_i^{br,n}$, for $0\le i<n$. For $i<n-1$ and $t\in[K_i,K_{i+1})$

$$C_{t/2}^{br,n}=\left\{\begin{array}{ll}(\bar{H}_i^{br,n}-(t-K_i))^+ & \text{if } t\in[K_i,K_{i+1}-1),\\ (t-K_{i+1}+\bar{H}_{i+1}^{br,n})^+, & \text{if } t\in[K_{i+1}-1,K_{i+1}),\end{array}\right.$$

Hence for $0\le i<n$,

$$\sup_{K_i\le t<K_{n+1}}\left|C_{t/2}^{br,n}-\bar{H}_i^{br,n}\right|\le\left|\bar{H}_{i+1}^{br,n}-\bar{H}_i^{br,n}\right|+1 \tag{7.17}$$

Now, we define $h_n(t) = i$, if $t \in [K_i, K_{i+1})$ and $i < n$, and $h_n(t) = n$, if $t \in [2n-2, 2n]$. The definitions of K_i and h_n implies

$$\sup_{0\le t\le 2n}\left|h_n(t) - \frac{t}{2}\right| \le \frac{1}{2}\sup_{0\le k\le n}\bar{H}_k^{br,n} + 1.$$

Next, we set $f_n(t) = h_n(nt)/n$. By (7.17), we have

$$\sup_{0\le t\le 2}\frac{a_n}{n}\left|C^{br,n}_{nt/2} - \bar{H}^{br,n}_{nf_n(t)}\right| \le \frac{a_n}{n}\sup_{0\le t\le 1}\left|\bar{H}^{br,n}_{[nt]+1} - \bar{H}^{br,n}_{[nt]}\right| + \frac{a_n}{n},$$

and

$$\sup_{0\le t\le 2}\left|f_n(t) - \frac{t}{2}\right| \le \frac{1}{2a_n}\sup_{0\le k\le n}\frac{a_n}{n}\bar{H}_k^{br,n} + \frac{1}{p}.$$

From our hypothesis, we get

$$\frac{a_n}{n}\sup_{0\le t\le 1}\left|\bar{H}^{br,n}_{[nt]+1} - \bar{H}^{br,n}_{[nt]}\right| + \frac{a_n}{n} \to 0 \quad \text{as } n\to\infty,$$

and

$$\frac{1}{2a_n}\sup_{0\le k\le n}\frac{a_n}{n}\bar{H}_k^{br,n} + \frac{1}{p} \to 0 \quad \text{as } n\to\infty,$$

in probabiity. Hence, from Theorem 4.1 in [**Bill99**] and Skorokhod representation theorem we obtain the convergence,

$$(X^{br,n}, \bar{H}^{br,n}, C^{br,n} \Rightarrow (X^{br}, \bar{H}^{br}, \bar{H}^{br}).$$

■

Remarks: By a classical time reversal argument, the weak convergence of the first coordinate in Theorem 29 implies the main result of Bryn-Jones and R.A. Doney [**BrDo06**]. Indeed, when X is the standard Brownian motion, it is straightforward to prove that the returned first passage bridge $(s + X^{br}_{t-u},\ 0 \le u \le t)$ is the bridge of a three dimensional Bessel process from 0 to s with length t. Similarly, the returned discrete first passage bridge whose law is this of $(k_n + S_{T(k_n)-i},\ 0 \le i \le n)$ under $\mathbb{P}(\,\cdot\,|\,T(k_n) = n)$ has the same law as $(S_i,\ 0 \le i \le n)$ given $S_n = k_n$ and conditioned to stay positive. Then integrating with respect to the terminal values and applying Theorem 29 gives the result contained in [**BrDo06**].

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