Random walks in random environment on Z:localization studies in the recurrent and transient cases

Olivier Zindy

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par **Olivier ZINDY**

Marches aléatoires en milieu aléatoire sur $\mathbb{Z}$: études de localisation dans les cas récurrent et transient

Rapporteurs :	Mme	Nina	**GANTERT**	Universität Münster
	M.	Ofer	**ZEITOUNI**	Technion, Haïfa

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M. Jean	**BERTOIN**	Université Paris–VI	Examinateur
M. Francis	**COMETS**	Université Paris–VII	Examinateur
Mme Nina	**GANTERT**	Universität Münster	Rapporteur
M. Yueyun	**HU**	Université Paris–XIII	Examinateur
M. Christophe	**SABOT**	Université Lyon–I	Examinateur
M. Zhan	**SHI**	Université Paris–VI	Directeur

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CHAPITRE 1

Introduction et présentation des résultats

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1. Marches aléatoires en milieu aléatoire

1.1. Motivations et applications. Les milieux aléatoires constituent un modèle permettant de décrire des phénomènes de diffusion et de transport en milieux inhomogènes, possédant néanmoins des propriétés de régularité à grande échelle (statistique). Depuis leur apparition, l'étude des phénomènes aléatoires en milieu aléatoire intéresse physiciens théoriciens, biologistes et mathématiciens.

Les milieux aléatoires sont liés à des modèles utilisés notamment par les physiciens de la matière molle pour les polymères et les gels ; on se reportera à De Gennes [28] pour l'un des premiers modèles de ce type. Les modèles multi-dimensionnels étant souvent hors de portée, on se ramène à l'étude de modèles simplifiés : les marches aléatoires en milieu aléatoire, par la présence même de "pièges" (aussi appelées "trappes") présentent les mêmes types de phénomènes que les modèles multi-dimensionnels. Il y a deux sources d'aléa : le mouvement de la particule (correspondant à l'agitation thermique) et le milieu, qui dicte les règles de déplacement. La combinaison de ces deux aléas de natures différentes fait que les marches aléatoires en milieu aléatoire exhibent des propriétés asymptotiques surprenantes et très différentes de la marche aléatoire simple.

Les marches aléatoires en milieu aléatoire uni-dimensionnelles ne constituent pas qu'un modèle jouet. Le modèle est introduit, en 1967, par le biophysicien Chernov [22], soucieux de comprendre les mécanismes de duplication de l'ADN et de dommages causés à l'ADN. En 1972, Temkin [111] reprend le modèle motivé par des problèmes de génétique et de métallurgie, notamment pour traîter la cinétique des transitions de phase dans des alliages.

On peut présenter le modèle de la façon informelle suivante :

– Premièrement, on associe un taux de saut aléatoire $\omega_x \in]0,1[$ à chaque site $x \in \mathbb{Z}$. La famille de coefficients obtenue est appelée *environnement* ou *milieu.*
– Deuxièmement, la réalisation de l'environnement étant fixée, on considère une particule partant de 0 au temps 0. Alors, au temps 1, la particule saute en 1 avec probabilité ω_0 et en -1 avec probabilité $1-\omega_0$. Et ainsi de suite si, au temps n, la particule visite le site x, elle saute, au temps $n+1$, en $x+1$ avec probabilité ω_x et en $x-1$ avec probabilité $1-\omega_x$.

Fig. 1. Probabilités de transition.

Ces dernières années, le modèle a bénéficié d'un regain d'intérêts en biologie moléculaire, dû à l'explosion de techniques monomoléculaires pour explorer la matière biologique. En contraste avec des expériences plus traditionnelles, ces nouvelles approches permettent d'accéder aux variations à l'échelle de la molécule, sans avoir besoin de faire la moyenne sur un échantillon macroscopique. Citons Lubensky et Nelson [75] qui s'intéressent, d'un point de vue théorique, à des expériences de micromanipulation réalisées par Essevaz-Roulet, Bockelmann et Heslot [43]. Ils analysent la séparation d'un double brin d'ADN grâce à des micro-pinces. Le nombre de bases rompues est modélisé par une marche aléatoire en environnement aléatoire, dont le rôle est joué par l'énergie libre du double-brin d'ADN (celle-ci est aléatoire car la séquence des bases A-T et G-C est supposée aléatoire). Récemment, Monasson et Cocco [23] étudient le nombre de trajectoires d'une marche aléatoire dans un environnement aléatoire transiente à considérer pour reconstruire l'environnement. On renvoit à Le Doussal, Monthus et Fisher [74] et la bibliographie associée en ce qui concerne les résultats obtenus par les physiciens théoriciens.

1.2. Le modèle uni-dimensionnel. On définit les marches aléatoires en milieu aléatoire uni-dimensionnelles, qui font l'objet principal de ce manuscrit, de la façon suivante.

DÉFINITION 1.1. *Soit* $\omega = (\omega_x)_{x\in\mathbb{Z}}$ *une famille de variables aléatoires indépendantes et identiquement distribuées à valeurs dans* $]0,1[$. *On appelle* ω *l' environnement. Etant donné une réalisation de l'environnement* $\omega = (\omega_x)_{x\in\mathbb{Z}}$, *on appelle marche aléatoire en milieu aléatoire la chaîne de Markov* $(X_n)_{n\geq 0}$ *définie par* $X_0 = 0$ *et pour* $n \geq 0$ *par*

$$P_\omega\left(X_{n+1} = x+1 \,|\, X_n = x\right) = \omega_x = 1 - P_\omega\left(X_{n+1} = x-1 \,|\, X_n = x\right).$$

On note P *la loi de l'environnement et* P_ω *la loi de la marche aléatoire dans l'environnement aléatoire* ω, *appelée loi quenched. Enfin on note* $\mathbb{P}$ *et appelle loi annealed la moyenne de la loi quenched sur tous les environnements, i.e.* $\mathbb{P}(\cdot) := \int P_\omega(\cdot)P(d\omega)$.

Remarquons que si la loi de ω_0 est une masse de dirac, alors $(X_n)_{n\geq 0}$ correspond à la marche aléatoire simple. C'est pourquoi nous éviterons ce cas dégénéré dans la suite.

Le terme quenched, qui est associé à la loi de la marche aléatoire conditionnellement à l'environnement, signifie "trempé" dans la terminologie issue de la métallurgie. La loi quenched a la propriété d'être markovienne, mais n'est pas invariante par translation. La loi annealed ("recuite"), quant à elle, est invariante par translation mais non-markovienne. Nous verrons que les comportements de la marche aléatoire sous la loi quenched et sous la loi annealed dépendent fortement de la loi P de l'environnement.

Dans la partie suivante, nous verrons que les marches aléatoires en milieu aléatoire ont des propriétés très différentes de la marche aléatoire simple. Nous y rappelons les principaux résultats historiques concernant les marches aléatoires en milieu aléatoire uni-dimensionnelles et soulignons ainsi la richesse des comportements en milieu aléatoire, richesse résultant de la compétition entre les deux sources d'aléa : l'environnement et l'agitation thermique.

2. Rappel de certains résultats connus

2.1. Récurrence-transience et loi des grands nombres. En 1975, Solomon [102] obtient un critère de récurrence-transience pour les marches aléatoires en milieu aléatoire. Définissons $\rho_0 := (1-\omega_0)/\omega_0$. Solomon montre que, dans le cas où $E[\log \rho_0]$ est défini, la marche aléatoire est récurrente si et seulement si $E[\log \rho_0] = 0$. De plus, il établit une loi des grands nombres : il existe une vitesse $v \in [-1, 1]$, ne dépendant que de la loi de l'environnement, telle que, $\mathbb{P}$-presque sûrement,

$$\frac{X_n}{n} \longrightarrow v, \qquad n \to \infty,$$

où v vérifie

$$v := \begin{cases} \frac{1-E[\rho_0]}{1+E[\rho_0]} > 0 & \text{si } E[\rho_0] < 1, \\ 0 & \text{si } (E[\rho_0^{-1}])^{-1} \leq 1 \leq E[\rho_0], \\ \frac{E[\rho_0^{-1}]-1}{E[\rho_0^{-1}]+1} < 0 & \text{si } (E[\rho_0^{-1}])^{-1} > 1. \end{cases}$$

On remarque notamment qu'il est possible que la marche aléatoire en milieu aléatoire soit transiente et de vitesse nulle, contrairement à la marche aléatoire simple. Le caractère sous-diffusif de la marche dans ce cas sera étudié plus en détails dans les sous-sections 2.3 et 4.4.

2.2. Cas récurrent : marche de Sinai et localisation. Dans le cas récurrent, Sinai [98] montre, en 1982, que la marche aléatoire en milieu aléatoire est nettement plus lente que la marche aléatoire simple. Avant de préciser le résultat obtenu par Sinai, nous donnons les hypothèses (ne concernant que l'environnement) sous lesquelles la marche aléatoire en milieu aléatoire est appelée *marche de Sinai :* il existe $\delta > 0$ tel que

$$P(\delta \leq \omega_0 \leq 1-\delta) = 1, \tag{1.1}$$

$$E[\log(\frac{1-\omega_0}{\omega_0})] = 0, \tag{1.2}$$

$$\sigma^2 := \mathrm{Var}[\log(\frac{1-\omega_0}{\omega_0})] > 0. \tag{1.3}$$

La première est une hypothèse technique, la seconde assure que la marche aléatoire en milieu aléatoire est récurrente d'après le critère de Solomon [102] et la troisième permet d'éviter le cas de la marche aléatoire simple. Alors, sous les hypothèse (1.1)-(1.3), Sinai [98] montre que

$$\sigma^2 \frac{X_n}{\log^2 n} \xrightarrow{\text{loi}} b_\infty, \qquad n \to \infty,$$

où b_∞ est une variable aléatoire non-dégénérée et non-gaussienne, qui ne dépend pas de la loi de l'environnement. En 1986, Golosov [51] et Kesten [69] explicitent la loi de cette variable aléatoire. Il est intéressant d'observer que la renormalisation en

$\log^2 n$ contraste avec le comportement asymptotique de la marche aléatoire simple en $\sqrt{n}$, dans le cas récurrent.

Par ailleurs, la démonstration de Sinai fait apparaître un processus qui ne dépend que de l'environnement ω et appelé *potentiel.* Ce potentiel, noté $V = (V(x), x \in \mathbb{Z})$, est défini par :

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log(\frac{1-\omega_i}{\omega_i}) & \text{si } x \geq 1, \\ 0 & \text{si } x = 0, \\ -\sum_{i=x+1}^{0} \log(\frac{1-\omega_i}{\omega_i}) & \text{si } x \leq -1. \end{cases}$$

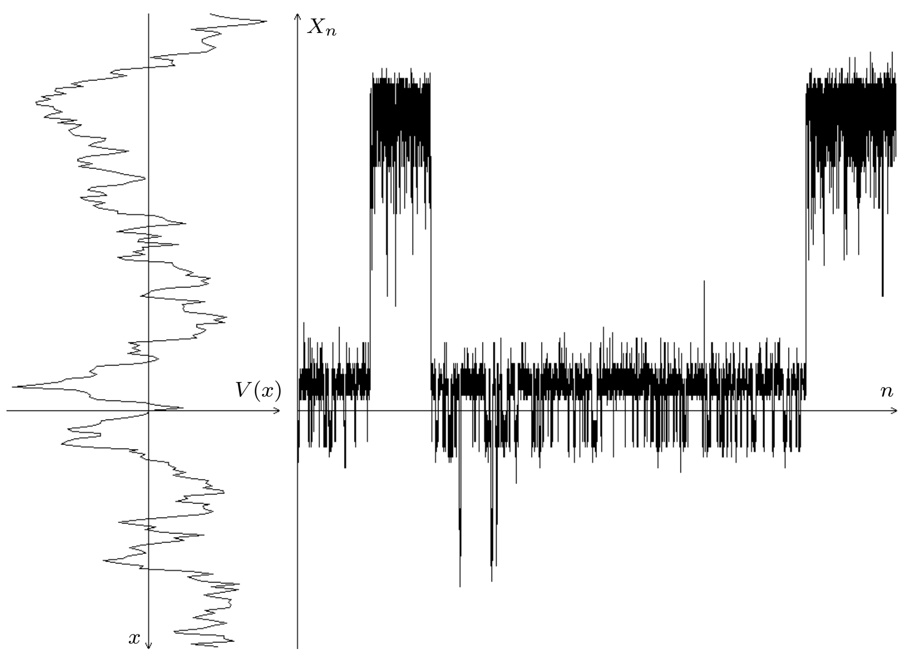

FIG. 2. Potentiel donné à gauche et marche aléatoire en milieu aléatoire associée à droite (cas récurrent).

Il s'avère que cette fonctionnelle de l'environnement joue le rôle d'une énergie en physique. Plus précisément, le caractère sous-diffusif démontré par Sinai dans le cas récurrent est dû à l'existence de puits de potentiel, correspondant à des minimas locaux pour le processus $(V(x), x \in \mathbb{Z})$. On parle de *vallées* piégeant la marche aléatoire $(X_n)_{n\geq 0}$, voir la Figure 2. En outre, Golosov [50] précise ce phénomène en donnant des estimées précises concernant la *localisation* de la marche de Sinai dans une vallée donnée. Nous renvoyons à Bovier et Faggionato [14] pour une analyse spectrale de la marche de Sinai, permettant de raffiner le théorème de Sinai.

2.3. Cas transient de vitesse nulle. Le comportement de la marche aléatoire en milieu aléatoire dans le cas transient de vitesse nulle a été précisé par Kesten, Kozlov et Spitzer [70], qui considèrent le cas transient vers $+\infty$. Ils introduisent un processus de branchement en milieu aléatoire avec immigration, qui tient compte des deux sources d'aléa (l'environnement et le mouvement de la marche aléatoire) et utilisent un résultat de renouvellement sophistiqué, dû à Kesten [68], faisant apparaître l'indice κ tel que $E\left[\rho_0^\kappa\right] = 1$. Il s'avère que cet indice détermine le comportement asymptotique de la marche aléatoire en milieu aléatoire. Avant de donner le théorème correspondant, nous énonçons les hypothèses faites par Kesten, Kozlov et Spitzer, que nous reprendrons dans les sous-sections 4.3 et 4.4 :

(a) il existe $0 < \kappa < 1$ pour lequel $E\left[\rho_0^\kappa\right] = 1$ et $E\left[\rho_0^\kappa \log^+ \rho_0\right] < \infty$,

(b) la loi de $\log \rho_0$ est non-arithmétique.

Le comportement de la marche aléatoire en milieu aléatoire dans le cas transient de vitesse nulle est alors décrit par le résultat suivant.

Théorème 1.1 (Kesten, Kozlov et Spitzer [70]). *Sous les hypothèses (a)–(b), on a les convergences en loi suivantes :*

$$\begin{aligned} \lim_{n\to\infty} P\left\{n^{-1/\kappa}\tau(n) \le x\right\} &= L_\kappa(x), \\ \lim_{n\to\infty} P\left\{n^{-\kappa}X_n \le x\right\} &= 1 - L_\kappa(x^{-1/\kappa}), \end{aligned}$$

où $L_\kappa(\cdot)$ est la fonction de répartition d'une loi stable complètement asymétrique d'indice κ (L_κ est concentrée sur $[0,\infty)$).

Remarquons que, comme la marche aléatoire en milieu aléatoire est transiente vers $+\infty$, la convergence en loi de $n^{-\kappa}X_n$ est une conséquence de la convergence en loi de $n^{-1/\kappa}\tau(n)$. De plus, les trois auteurs obtiennent un comportement en $n/\log n$ dans le cas $\kappa = 1$ et traitent les fluctuations dans le cas balistique correspondant à $\kappa > 1$. Ce travail a été récemment étendu à des environnements markoviens par Mayer-Wolf, Roitershtein et Zeitouni [80].

2.4. Temps local, sites favoris et concentration. Ayant observé des phénomènes de ralentissement, dus à la localisation de la marche aléatoire en milieu aléatoire au fond de vallées, il paraît naturel de s'intéresser au *temps local*. Le temps local en x au temps n, noté $L(n,x)$, correspond au nombre de visites de la marche aléatoire au site x avant le temps n et est défini par

$$L(n,x) \quad := \quad \#\left\{0 \le i \le n :\ X_i = x\right\}, \qquad n \ge 0,\ x \in \mathbb{Z}.$$

Grâce au phénomène de localisation, on peut raisonnablement penser que le *maximum du temps local* de la marche aléatoire en milieu aléatoire, défini par

$$L^*(n) \quad := \quad \max_{x\in\mathbb{Z}} L(n,x), \qquad n \ge 0,$$

sera considérablement plus grand que celui de la marche aléatoire simple. Dans ce sens, Révész [88], pour un environnement particulier, et Shi [93], sous les hypothèses

(1.1)-(1.3), montrent que

$$\limsup_{n\to\infty} \frac{L^*(n)}{n} = c_1 > 0, \qquad \mathbb{P}\text{-p.s.}$$

En outre, le comportement asymptotique en " lim sup " du maximum du temps local est étudié par Gantert et Shi [45] dans le cas transient. La " lim inf " dans le cas récurrent est obtenue par Dembo, Gantert, Peres et Shi [30], qui montrent que

$$\liminf_{n\to\infty} \frac{L^*(n)}{n/\log_{(3)} n} = c_2 > 0, \qquad \mathbb{P}\text{-p.s.},$$

où $\log_{(j)}$ désigne la j-ème itérée de la fonction logarithme. La même question dans le cas transient reste un problème ouvert.

Par ailleurs, Hu et Shi [59] caractérisent les classes de Lévy du temps local dans le cas récurrent grâce à des tests intégraux. Dans [60], ils traitent également le problème des *sites favoris* (ou sites les plus visités), que l'on définit par

$$\mathbb{V}(n) \;:=\; \{x \in \mathbb{Z}_+ :\; L(n,x) = L^*(n)\}\,, \qquad n \geq 0. \tag{1.4}$$

Ils prouvent notamment que le processus défini par le maximum (ou l'infimum) de $\mathbb{V}(n)$ est transient vers $+\infty$.

Andreoletti [3] s'intéresse à la propriété de *concentration* de la marche de Sinai, en regardant la longueur ℓ suffisante pour que la marche de Sinai passe infiniment souvent plus de la moitié de son temps dans un intervalle de taille 2ℓ. Il obtient aussi, pour tout $0 < \beta < 1$, l'existence d'une longueur $\ell(\beta)$ telle que

$$\limsup_{n\to\infty} \frac{\sup_{x\in\mathbb{Z}} L\left(n, [x-\ell(\beta), x+\ell(\beta)]\right)}{n} \geq \beta, \qquad \mathbb{P}\text{-p.s.}, \tag{1.5}$$

où $L(n,A) := \sum_{x\in A} L(n,x)$, pour tout sous-ensemble A de $\mathbb{Z}$. Ce résultat confirme, encore une fois, l'importance du phénomène de localisation mis en exergue par Sinai [98]. Grosso modo, l'intervalle de longueur 2ℓ concerné correspond à un voisinage du fond de la vallée la plus profonde visitée par la marche aléatoire.

2.5. Grandes déviations. L'étude des grandes déviations traîte le comportement asymptotique de la probabilité de certains évènements rares, i.e. dont la probabilité tend vers 0, du type $\mathbb{P}\{X_n/n \in A\}$ ou $P_\omega\{X_n/n \in A\}$.

En 1994, Greven et den Hollander [52] démontrent, pour les marches aléatoires en milieu aléatoire, un principe de grandes déviations sous la loi quenched, i.e. conditionnellement à l'environnement. Il est intéressant de noter que la fonction de taux obtenue est déterministe, i.e. ne dépend pas de la réalisation de l'environnement.

Grâce à une approche différente, Comets, Gantert et Zeitouni [24] obtiennent, en plus, un principe de grandes déviations sous la loi annealed et établissent l'existence d'un principe de grandes déviations fonctionnel. Ils prouvent notamment que les fonctions de taux sous les lois quenched et annealed diffèrent dès lors que la fonction de taux sous la loi annealed est non-nulle. Observons que Dembo, Gantert, Peres et Zeitouni [31] montrent que ce résultat n'est plus vrai dans le cas des marches aléatoires sur un arbre aléatoire.

Citons, entre autres, [32], [37], [46], [47], [82] et [83] pour une littérature plus détaillée concernant les grandes déviations. Nous renvoyons à [52] et [34] pour des questions encore ouvertes à propos de grandes déviations pour les marches aléatoires en milieu aléatoire uni-dimensionnelles.

3. Trois autres modèles

3.1. Diffusions dans un potentiel aléatoire. En 1985, Schumacher [92] introduit un équivalent continu à la marche aléatoire en milieu aléatoire. Etant donnée la réalisation d'un processus $(V(x),\ x \in \mathbb{R})$, on appelle *diffusion aléatoire dans le potentiel* V la diffusion vérifiant $\mathcal{X}(0) = 0$ et définie comme la solution formelle de l'équation différentielle stochastique

$$d\mathcal{X}_t = d\beta_t - \frac{1}{2}V'(\mathcal{X}_t)dt,$$

où $(\beta_t,\ t \geq 0)$ désigne un mouvement brownien indépendant de V. Observons que si V n'est pas dérivable l'équation différentielle précédente n'admet qu'un sens formel. En réalité, on définit, plus rigoureusement, la diffusion aléatoire $\mathcal{X}$ dans le potentiel V, càdlàg et localement bornée, comme le processus de Markov, dont le générateur infinitésimal est donné par

$$\frac{1}{2}\mathrm{e}^{V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

De la même façon que l'on peut construire la marche aléatoire simple (récurrente) à partir du mouvement brownien par l'argument de Skorokhod, Schumacher [92] montre qu'il est possible de construire la marche de Sinai à partir d'une diffusion dont le potentiel aléatoire est défini par $V(x) := 0$ si $x \in [0, 1)$, $V(x) := \sum_{i=1}^{n} \log \frac{1-\omega_i}{\omega_i}$ si $x \in [n, n+1[$ avec $n \geq 1$ et $V(x) := \sum_{i=n+1}^{0} \log \frac{1-\omega_i}{\omega_i}$ si $x \in [n, n+1[$ avec $n \leq -1$, où $(\omega_i)_{i\in\mathbb{Z}}$ désigne l'environnement associé à la marche de Sinai. Etant donnée la diffusion aléatoire $\mathcal{X}$ associée à ce potentiel, on considère la suite de temps d'arrêt définie par $\mu_0 := 0$ et

$$\mu_n \quad := \quad \inf\left\{t > \mu_{n-1} : |\mathcal{X}(t) - \mathcal{X}(\mu_{n-1})| = 1\right\}, \qquad n \geq 1.$$

Introduisons également la fonction d'échelle associée à $\mathcal{X}$, à V fixé (i.e. à ω fixé), définie par $A_x := \int_0^x \mathrm{e}^{V(y)}\, \mathrm{d}y$ pour tout x réel. On peut alors écrire

$$P_V\left\{\mathcal{X}(\mu_{n+1}) = i+1 | \mathcal{X}(\mu_n) = i\right\} = \frac{A_i - A_{i-1}}{A_{i+1} - A_{i-1}} = \omega_i.$$

Autrement dit, le processus $(\mathcal{X}(\mu_n)\,;\, n \geq 0)$ est distribué comme la marche de Sinai $(X_n\,;\, n \geq 0)$. En outre, Schumacher montre que $(\mu_n - \mu_{n-1})_{n\geq 1}$ est une famille de variables aléatoires indépendantes et identiquement distribuées ayant même loi que

$\inf\{t > 0 : |B_t| = 1\}$, où B désigne le mouvement brownien standard. Comme nous savons, de plus, que μ_1 admet un moment exponentiel fini et est de moyenne égale à 1, la loi des grands nombres nous dit que $\mu_n \approx n$, presque sûrement. On peut ainsi s'intéresser au comportement de la marche de Sinai en étudiant le processus $\mathcal{X}(n)$ par des méthodes de calcul stochastique. Un grand nombre de résultats récents concernant les marches aléatoires en milieu aléatoire utilisent ce genre de techniques. Citons, entre autres, Hu [56], [57], Hu et Shi [58], [59], [60], Shi [93] et Singh [100].

En utilisant un argument de type Donsker, on constate que le potentiel d'une marche aléatoire en milieu aléatoire peut, sous de bonnes hypothèses, être vu asymptotiquement comme un mouvement brownien avec ou sans drift. C'est pourquoi il paraît judicieux d'étudier la diffusion $\mathcal{X}$ dans le potentiel $W_\kappa(x) := W(x) - \frac{\kappa}{2}x$, où W désigne un mouvement brownien indépendant de β et $\kappa \in \mathbb{R}$. Brox [16] est le premier, en 1986, à s'intéresser à un tel processus dans le cas particulier où $\kappa = 0$. Il montre que le comportement asymptotique de ce processus est le même que celui de la marche de Sinai. Par ailleurs, Kawazu, Tamura et Tanaka [65] et Hu [57] obtiennent des résultats de localisation analogues à ceux de Golosov. Comme Kesten, Kozlov et Spitzer, Kawazu et Tanaka [66] traitent le cas transient ($\kappa \neq 0$). Ces résultats sont précisés par les travaux de Kawazu et Tanaka [67], Tanaka [110] puis Hu, Shi et Yor [61]. Nous renvoyons à Taleb [109] pour ce qui concerne l'étude des grandes déviations, à Devulder [36] pour l'étude du maximum du temps local et à Cheliotis pour des précisions concernant la localisation [19] ainsi que l'étude des sites favoris [20].

Des diffusions dans d'autres types de potentiels sont également étudiées. Nous citons, entre autres, Carmona [17], Cheliotis [21], Kawazu, Tamura et Tanaka [65], Mathieu [77], [78], [79] et Singh [99], [101].

3.2. Le modèle multi-dimensionnel. Contrairement au cas uni-dimensionnel, dont la littérature est abondante, il y a peu d'articles avant 1998 concernant les marches aléatoires en milieu aléatoire multi-dimensionnelles, nous renvoyons à Bricmont et Kupiainen [15], Kalikow [64] et Lawler [73]. Une difficulté importante dans l'étude de ce modèle résulte de son caractère non-reversible et du fait qu'on ne dispose que de très peu de formules explicites. En particulier, il n'y a pas de formule permettant de décrire le comportement balistique ou non de la marche aléatoire, comme celle démontrée par Solomon [102] dans le cas $d = 1$. Au cours des dernières années, le domaine a connu un considérable regain d'intérêts et des avancées considérables, cf. [8], [10], [11], [26], [27], [84], [85], [86], [87], [91], [97], [103], [104], [105], [106], [107], [108], [112], [113], [114], [115], [116], [117]. Des progrès ont notamment été réalisés concernant le comportement balistique et les propriétés de grandes déviations associées et concernant des lois du 0-1.

3.3. Marches aléatoires en paysage aléatoire. Les marches aléatoires en paysage aléatoire constituent une classe de processus stationnaires admettant une riche diversité de comportements asymptotiques. Le modèle peut être décrit de la façon suivante : étant donnée une chaîne de Markov portée par un espace d'états $\mathcal{S}$, il est possible d'associer un champs aléatoire à cet espace d'états, appelé *paysage aléatoire*. Au fur et à mesure que la marche aléatoire évolue, elle observe le paysage du site qu'elle visite.

On définit le modèle uni-dimensionnel en considérant une marche aléatoire sur $\mathbb{Z}$, notée $S = (S_n,\ n \geq 0)$, et une famille de variables aléatoires indépendantes et identiquement distribuées, notée $\xi = (\xi_x,\ x \in \mathbb{Z})$ et indépendante de S. La *marche aléatoire en paysage aléatoire* est alors définie par

$$Y_n := \sum_{i=0}^{n} \xi(S_i), \qquad n \geq 0.$$

Ce processus est également appelé marche aléatoire en paysage aléatoire de Kesten-Spitzer. On peut en faire l'interprétation suivante : si un marcheur doit payer ξ_x à chaque fois qu'il visite le site x, alors Y_n correspond à la quantité totale qu'il a payée pendant l'intervalle de temps $[0, n]$. Une remarque importante dans l'étude des marches aléatoires en paysage aléatoire consiste à observer que l'on peut, dans un certain sens, dissocier les deux sources d'aléa en réécrivant Y_n en fonction du temps local de S. En effet, il est facile de voir que

$$Y_n = \sum_{x \in \mathbb{Z}} \xi_x \, L(n, x), \qquad n \geq 0,$$

où, exceptionnellement ici, $L(n, x)$ désigne le temps local de la marche aléatoire S au site x et au temps n, et non pas le temps local de la marche aléatoire en milieu aléatoire.

En 1979, ce modèle est introduit et étudié par Kesten et Spitzer [71] pour les dimensions $d = 1$ et $d \geq 3$ (la définition est analogue dans le cadre multi-dimensionnel). Ils prouvent, en dimension $d = 1$, que si S et ξ appartiennent aux domaines d'attraction de lois stables d'indice α et β respectivement, alors il existe δ, fonction de α et β, tel que $n^{-\delta} Y_{\lfloor nt \rfloor}$ converge faiblement. Dans le cas simple où $\alpha = \beta = 2$, ils montrent que

$$\left(n^{-3/4}\, Y_{\lfloor nt \rfloor};\, 0 \leq t \leq 1\right) \xrightarrow{\text{law}} (\Lambda(t);\, 0 \leq t \leq 1)\,,$$

où « $\xrightarrow{\text{law}}$ » désigne la convergence en loi (dans un certain espace fonctionnel ; par exemple l'ensemble des fonctions bornées sur $[0, 1]$ muni de la topologie uniforme). Le processus $(\Lambda(t),\, t \geq 0)$, appelé mouvement brownien en paysage aléatoire, est défini par $\Lambda(0) = 0$ et $\Lambda(t) := \int_{\mathbb{R}} \ell(t, x)\, \mathrm{d}W(x)$ pour $t > 0$, où $(W(x);\, x \in \mathbb{R})$ désigne un mouvement brownien réel et $(\ell(t, x),\, t \geq 0, x \in \mathbb{R})$ la version bicontinue du temps local d'un mouvement brownien $(B(t),\, t \geq 0)$, indépendant de $(W(x);\, x \in \mathbb{R})$.

Notons également qu'en 1979, Borodin analyse les cas de la marche aléatoire simple récurrente [13] et transiente [12].

Bolthausen [7] étudie le cas de la dimension $d = 2$. Il montre que, si S est une marche aléatoire récurrente et si ξ_0 est de moyenne nulle et de variance finie, alors $(n \log n)^{-1/2} Y_{\lfloor nt \rfloor}$ satisfait un théorème de la limite centrale fonctionnel. Nous renvoyons à Den Hollander et Steif [35] pour un "survey" récent concernant les marches aléatoires en paysage aléatoire et à Asselah et Castell [4] pour des résultats de grandes déviations dans le cas de dimensions $d \geq 5$.

4. Présentation des résultats

4.1. Un paradoxe lié à la localisation pour la marche de Sinai. Le deuxième chapitre de ce manuscrit est consacré à la résolution d'une conjecture d'Erdös et Révész [42] (rappelée dans [89]) initialement posée pour la marche aléatoire simple. Ce travail a été réalisé en collaboration avec Zhan Shi, et a été accepté pour publication à *Annals of Probability* [95]. Il s'agit de savoir si le fait que la marche passe quasiment tout son temps sur $\mathbb{Z}_+$ implique que les sites favoris appartiennent également à $\mathbb{Z}_+$. Pour préciser ce problème, introduisons d'abord la notion de *suite positive :* une suite (aléatoire) d'entiers $0 < n_1 < n_2 < \dots$ est appellée suite positive pour la marche aléatoire si

$$\lim_{k\to\infty} \frac{\# \{0 \leq i \leq n_k : \ X_i > 0\}}{n_k} = 1.$$

Du fait que la marche de Sinai est connue pour un phénomène de localisation forte, les sites favoris doivent être situés au fond de la vallée du potentiel dans laquelle la marche a passé presque tout son temps. Il est alors naturel de penser que, le long d'une suite positive, les vallées qui piègent la marche aléatoire, et donc les sites favoris, sont portés par $\mathbb{Z}_+$. Rappelons que l'ensemble des sites favoris $\mathbb{V}(n)$ est défini en (1.4). Nous pouvons maintenant formuler rigoureusement la conjecture de la manière suivante.

Problème 1.1. *Est-il vrai que, $\mathbb{P}$-presque sûrement, pour toute suite positive (n_k), on ait $\mathbb{V}(n_k) \subset \mathbb{Z}_+$ pour tout k assez grand ?*

Cependant le comportement de la marche aléatoire peut être tout autre : nous obtenons le résultat suivant.

Théorème 1.2. *Sous les hypothèses* (1.1)–(1.3),

$$\mathbb{P}\{\forall \text{ suite positive } (n_k), \text{ on ait } \mathbb{V}(n_k) \subset \mathbb{Z}_+, \text{ pour } k \text{ assez grand}\} = 0.$$

En fait, la raison pour laquelle l'heuristique précédente est incorrecte est que même si la marche est fortement localisée au fond d'une vallée, il se peut qu'il y ait un grand nombre de sites au voisinage du fond de cette vallée. Dans ce cas, aucun site n'est nécessairement favori, car le temps passé au fond de la vallée est partagé entre tous ces sites.

4.2. Limites supérieures de la marche de Sinai en paysage aléatoire. Le troisième chapitre de ce manuscrit concerne une conjecture de Révész ([90], p. 353) sur la marche de Sinai en paysage aléatoire. Des problèmes combinant environnement aléatoire et paysage aléatoire ont déjà été étudiés pour des modèles plus généraux. Par exemple, remplaçant $\mathbb{Z}$ par un groupe dénombrable, Lyons et Schramm [76] obtiennent, sous certaines conditions, l'existence d'une mesure stationnaire du point de vue de la particule, pour la marche aléatoire en environnement aléatoire et paysage aléatoire. Quant à Häggström [53], puis Häggström et Peres [54], ils utilisent ce modèle pour traiter des problèmes de percolation. Voyant la percolation comme un paysage aléatoire, ils considèrent l'environnement aléatoire déterminé par ce paysage et construisent la marche aléatoire en milieu aléatoire associée, qu'ils étudient afin d'obtenir des informations sur l'environnement, i.e. l'amas de percolation.

Pour introduire le modèle de marche de Sinai en paysage aléatoire, nous considérons d'abord la marche de Sinai $(X_n,\, n \geq 0)$ sous les hypothèses (1.1)–(1.3) et rappelons les notations P_ω et $\mathbb{P}$ associées respectivement à la loi quenched et à la loi annealed. Comme dans la sous-section 3.3, on considère une famille de variables aléatoires indépendantes et identiquement distribuées pour définir le paysage aléatoire, noté $\xi = (\xi_x,\, x \in \mathbb{Z})$. On suppose, de plus, que ξ est indépendant du couple $(\omega, (X_n)_{n\geq 0})$. On note alors $(Z_n,\, n \geq 0)$ la marche de Sinai en paysage aléatoire définie par

$$Z_n := \sum_{i=0}^{n} \xi(X_i), \qquad n \geq 0.$$

Comme précédemment, remarquons que Z_n peut être réécrit en terme du temps local de la marche de Sinai

$$Z_n = \sum_{x\in\mathbb{Z}} \xi(x)\, L(n,x), \qquad n \geq 0. \tag{1.6}$$

Dans un certain sens, on dissocie ainsi les deux sources d'aléa liées à la marche de Sinai et au paysage aléatoire.

On s'intéresse ici au comportement de la limite supérieure de $(Z_n/n,\, n \geq 0)$ dans le cas où le support de la loi de ξ_0 est borné supérieurement, i.e. $a := \operatorname{ess\,sup} \xi_0 < \infty$. Nous rappelons ici la propriété de concentration d'ordre β démontrée par Andreoletti [3], déjà précisée en (1.5),

$$\limsup_{n\to\infty} \frac{\sup_{x\in\mathbb{Z}} L\left(n, [x-\ell(\beta), x+\ell(\beta)]\right)}{n} \geq \beta, \qquad \mathbb{P}\text{-p.s.}$$

Le cas où β est très proche de 1 associé au caractère "i.i.d." du paysage aléatoire nous permet de formuler la conjecture de Révész ([90], p. 353) : l'hypothèse $a :=$

$\operatorname{ess\,sup} \xi_0 < \infty$ implique-t-elle que, $\mathbb{P} \otimes Q$-presque sûrement,

$$\limsup_{n\to\infty} \frac{Z_n}{n} = a\,?$$

Il s'avère que la conjecture n'est vérifiée que si l'on fait une hypothèse supplémentaire. En effet, une étude fine de la compétition entre propriété de concentration pour la marche de Sinai et records négatifs pour le paysage aléatoire nous permet d'obtenir le théorème suivant [118], qui donne une solution au problème en fonction du comportement asymptotique de la queue de distribution de $\xi_0^- := \max\{-\xi_0, 0\}$.

Théorème 1.3. *Supposons* (1.1)–(1.3) *et* $a := \operatorname{ess\,sup} \xi_0 < \infty$.
(i) Si $Q\{\xi_0^- > \lambda\} \leq \frac{1}{(\log \lambda)^{2+\varepsilon}}$, *pour un certain* $\varepsilon > 0$ *et tout* λ *assez grand, alors*

$$\mathbb{P} \otimes Q \left\{ \limsup_{n\to\infty} \frac{Z_n}{n} = a \right\} = 1.$$

(ii) Si $Q\{\xi_0^- > \lambda\} \geq \frac{1}{(\log \lambda)^{2-\varepsilon}}$, *pour un certain* $\varepsilon > 0$ *et tout* λ *assez grand, alors*

$$\mathbb{P} \otimes Q \left\{ \lim_{n\to\infty} \frac{Z_n}{n} = -\infty \right\} = 1.$$

En fait, il est possible de donner plus de précisions dans le cas *(ii)*. En effet, on peut montrer que, si $Q\{\xi_0^- > \lambda\} \geq \frac{1}{(\log \lambda)^{\alpha}}$, pour un certain $\alpha < 2$, alors on a que $\lim_{n\to\infty} n^{-\frac{2}{\alpha}+\varepsilon'} Z_n = -\infty$, $\mathbb{P} \otimes Q$-presque sûrement, pour tout $\varepsilon' > 0$. Par ailleurs, le cas $\varepsilon = 0$ reste une question ouverte.

4.3. Une représentation probabiliste des constantes du théorème de renouvellement de Kesten. Le quatrième chapitre de ce manuscrit est un travail réalisé en collaboration avec Nathanaël Enriquez et Christophe Sabot [41].

En 1973, Kesten publie un papier célèbre [68] concernant le comportement de la queue de distribution de séries de renouvellement de la forme $\sum_{i\geq 1} A_1 \dots A_{i-1} B_i$, où $(A_i)_{i\geq 0}$ est une suite de matrices aléatoires $d \times d$ positives et i.i.d., et $(B_i)_{i\geq 1}$ une suite de vecteurs aléatoires i.i.d. de $\mathbb{R}^d$. Son résultat montre que la queue de distribution d'une projection quelconque de ce vecteur aléatoire est équivalente à $Ct^{-\kappa}$, quand t tend vers l'infini et où C et κ sont des constantes positives. La constante κ est définie comme la solution de l'équation $k(s) = 1$, avec $k(s) := \lim_{n\to\infty} \mathbb{E}[\| A_1 \dots A_n \|^s]^{1/n}$.

Même si nous nous intéressons ici au cas uni-dimensionnel, mentionnons la généralisation du résultat de Kesten dans le cas multi-dimensionnel, récemment obtenue par de Saporta, Guivarc'h et Le Page [29].

En dimension 1, Goldie [48] se passe de l'hypothèse de positivité et simplifie la preuve de Kesten. En outre, il obtient une formule pour la constante implicite C de Kesten dans le cas particulier où A_i est positif et κ entier. En 1991, Chamayou et Letac [18] montrent qu'en dimension $d = 1$, si A_i a même loi que $(1 - X_i)/X_i$, où X_i suit une loi Beta sur $(0, 1)$, alors la loi de la série elle-même peut être calculée de telle sorte que la constante C est explicite. On peut alors se poser la question suivante. Comment peut-on évaluer la constante C ?

Dans ce travail, nous considérons le cas $d = 1$ et supposons que : $\rho_i = A_i$ est une famille de variables aléatoires i.i.d., $B_i = 1$ et qu'il existe κ tel que $\mathbb{E}[\rho_1^\kappa] = 1$. On suppose, de plus, une condition d'intégrabilité et que la loi de $\log \rho_i$, qui a une moyenne négative sous les hypothèses précédentes, est non-arithmétique. On s'intéresse alors à la série

$$R := 1 + \sum_{k\geq 1} \rho_1 \cdots \rho_k.$$

Ces hypothèses impliquent que la queue de distribution de la série de renouvellement est équivalente à $C_K t^{-\kappa}$, quand t tend vers l'infini. Notre but ici est de donner une repésentation probabiliste de la constante C_K.

En outre, ce travail est relié à l'étude de marches aléatoires en milieu aléatoire uni-dimensionnelles. Dans [70], Kesten, Kozlov et Spitzer montrent, en utilisant le résultat de Kesten [68], que si la marche aléatoire en milieu aléatoire transiente est de vitesse nulle, alors son comportement dépend d'un indice $\kappa \leq 1$: la marche aléatoire en milieu aléatoire X_n renormalisée par $n^{1/\kappa}$ converge en loi vers $C_\kappa \left(\frac{1}{\mathcal{S}_\kappa}\right)^\kappa$, où $\mathcal{S}_\kappa$ est une variable aléatoire stable complètement asymétrique d'indice κ. Cependant, ils n'obtiennent pas d'expression explicite de C_κ. Dans [40], nous obtenons une expression explicite, soit en fonction de la constante de Kesten C_K lorsqu'elle est connue, soit en fonction de l'espérance d'une série aléatoire lorsque C_K n'est pas connue. Pour cela, nous avons besoin d'estimer le comportement de la queue de distribution d'une variable aléatoire intimement reliée à la série aléatoire R. Ceci est le second objectif de ce quatrième chapitre.

Soyons plus précis et considérons une famille de variables aléatoires positives i.i.d. $(\rho_i)_{i\in\mathbb{Z}}$ de loi $Q = \mu^{\otimes\mathbb{Z}}$. On associe à la suite $(\rho_i)_{i\in\mathbb{Z}}$ le potentiel $(V_k)_{k\in\mathbb{Z}}$ défini par

$$V_n := \begin{cases} \sum_{k=1}^n \log \rho_k & \text{si } n \geq 1, \\ 0 & \text{si } n = 0, \\ -\sum_{k=n+1}^0 \log \rho_k & \text{si } n \leq -1. \end{cases}$$

Notons ρ une variable aléatoire de loi μ. On suppose alors que la loi μ vérifie

(a) il existe $\kappa > 0$ pour lequel $\mathbb{E}^\mu[\rho^\kappa] = 1$ et $\mathbb{E}^\mu[\rho^\kappa \log^+ \rho] < \infty$,

(b) la loi de $\log \rho$ est non-arithmétique.

On appelle S le maximum absolu de $(V_k)_{k\geq 0}$, défini par $S := \max\{V_k,\ k \geq 0\}$.

Des résultats classiques de théorie des fluctuations [44] donnent le comportement de la queue de distribution de S :

$$\mathbb{P}^Q\{\mathrm{e}^S \geq t\} \sim C_F t^{-\kappa},$$

quand $t \to \infty$, où
$$C_F := \frac{1 - \mathbb{E}^Q[\mathrm{e}^{\kappa V(T_{\mathbb{R}_-})}]}{\kappa \mathbb{E}^\mu[\rho^\kappa \log \rho]\mathbb{E}^Q[T_{\mathbb{R}_-}]}.$$

On introduit maintenant la variable aléatoire R, qui nous intéresse, définie par
$$R := \sum_{n\geq 0} \mathrm{e}^{V_n}.$$
On remarque alors que R satisfait l'équation de renouvellement
$$R \overset{\text{loi}}{=} 1 + \rho R,$$
où ρ est une variable aléatoire de loi μ, indépendante de R. Dans [68], Kesten montre (en fait, son résultat est plus général et concerne le cas multi-dimensionnel) qu'il existe une constante positive C_K telle que
$$\mathbb{P}^Q\{R \geq t\} \sim C_K t^{-\kappa},$$
quand $t \to \infty$. Le but de chapitre est d'obtenir une expression de cette constante en terme de l'espérance d'une variable aléatoire simple. A cet effet, on a besoin d'introduire une certaine transformée de Girsanov de Q. Grâce à l'hypothèse (a), on peut définir
$$\tilde{\mu} := \rho^\kappa \mu,$$
ainsi que $\tilde{Q} =: \tilde{\mu}^{\otimes \mathbb{Z}}$. On définit alors M par
$$M := \sum_{k<0} \mathrm{e}^{-V_k} + \sum_{k\geq 0} \mathrm{e}^{-V_k}, \tag{1.7}$$
où $(V_k)_{k<0}$ est distribué sous $Q\{\cdot|V_k \geq 0, \forall k < 0\}$ et indépendant de $(V_k)_{k\geq 0}$, qui est distribué sous $\tilde{Q}\{\cdot|V_k > 0, \forall k > 0\}$. On a le résultat suivant qui donne une expression probabiliste de C_K en fonction de $\mathbb{E}[M^\kappa]$, qui est fini.

THÉORÈME 1.4. *Sous les hypothèses (a)-(b), on a le résultat suivant*
$$\mathbb{P}^Q\{R \geq t\} \sim C_K t^{-\kappa},$$
quand $t \to \infty$, *où*
$$C_K = C_F \mathbb{E}[M^\kappa].$$

4.4. Lois limites pour les marches aléatoires en milieu aléatoire transientes de vitesse nulle. Le cinquième chapitre de ce manuscrit est un travail réalisé en collaboration avec Nathanaël Enriquez et Christophe Sabot [40]. Il correspond à une nouvelle démonstration du théorème obtenu par Kesten, Kozlov et Spitzer [70] dans le cas non-balistique. En utilisant le potentiel associé à l'environnment ω, nous obtenons une expression du paramètre de la loi stable limite. En outre, cette expression est explicite dans le cas d'environnements de Dirichlet.

La preuve présentée ici choisit une approche radicalement différente de celles utilisées dans [70] et [80]. Alors que les preuves de [70] et [80] sont basées sur une représentation de la trajectoire de la marche aléatoire en termes de processus de branchement en milieu aléatoire (avec immigration), notre approche repose sur l'interprétation faite par Sinai d'une particule dans un potentiel aléatoire. Cependant, dans le cas récurrent, le potentiel concerné correspond à une marche aléatoire récurrente et Sinai introduit une notion de vallée qui n'a plus de sens dans notre contexte, où le potentiel est une marche aléatoire (disons négativement) biaisée. On introduit alors une notion de vallée différente, qui est étroitement liée aux excursions de cette marche aléatoire au dessus de son minimum passé. Par ailleurs, Iglehart [63] donne un équivalent de la queue de distribution de la hauteur de ces excursions. Dans un premier temps, on prouve que l'étude du temps d'atteinte du niveau n peut être réduite à celle du temps mis par la marche aléatoire en milieu aléatoire pour gravir les hautes excursions du potentiel au dessus de son minimum passé entre 0 et n. Comme ces hautes excursions sont bien séparées en espace, on peut montrer qu'elles admettent une propriété "i.i.d." Du coup, le problème peut être réduit à l'étude de la queue de distribution du temps mis par la marche aléatoire pour gravir une telle excursion.

Il s'avère que cette queue de distribution fait apparaître l'espérance d'une fonctionnelle d'un certain méandre associé à la marche aléatoire qui définit le potentiel. En outre, l'espérance de cette fonctionnelle est, elle-même, reliée à la constante du théorème de renouvellement de Kesten [68]. Ces deux derniers résultats sont contenus dans le chapitre 4. Enfin, dans le cas où les probabilités de transition suivent une loi Beta, un résultat de Chamayou et Letac [18] donne une expression explicite de cette constante, qui nous permet d'obtenir une formule explicite pour le paramètre de la loi stable limite.

Avant d'énoncer les principaux résultats de ce chapitre, nous introduisons certaines notations. Le temps d'atteinte $\tau(x)$ du site x pour la marche aléatoire $(X_n,\, n \geq 0)$ est défini par

$$\tau(x) := \inf\{n \geq 1 :\, X_n = x\}, \quad x \in \mathbb{Z}.$$

Pour $\alpha \in (0,1)$, on note $\mathcal{S}_\alpha^{ca}$ une variable aléatoire stable complètement asymétrique d'indice α, ayant pour transformée de Laplace

$$E[\mathrm{e}^{-\lambda \mathcal{S}_\alpha^{ca}}] = \mathrm{e}^{-\lambda^\alpha},$$

pour $\lambda > 0$. On introduit, de plus, la constante C_K décrivant le comportement de la queue de distribution de la série de renouvellement de Kesten, voir [68], définie par $R := \sum_{k\geq 0} \mathrm{e}^{V(k)}$:

$$P\{R > x\} \sim \frac{C_K}{x^\kappa}, \qquad x \to \infty.$$

On a alors le résultat suivant, où le symbole "$\xrightarrow{\text{loi}}$" désigne la convergence en loi.

THÉORÈME 1.5. *Soit* $\omega := (\omega_i,\, i \in \mathbb{Z})$ *une famille de variables aléatoires indépendantes et identiquement distribuées telle que*

(a) il existe $0 < \kappa < 1$ *pour lequel* $E[\rho_0^\kappa] = 1$ *et* $E\left[\rho_0^\kappa \log^+ \rho_0\right] < \infty$,

(b) la loi de $\log \rho_0$ *est non-arithmétique.*

Alors on a, quand n *tend vers l'infini,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{loi}} 2\left(\frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]\right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{loi}} \frac{\sin(\pi\kappa)}{2^\kappa \pi\kappa^2 C_K^2 E[\rho_0^\kappa \log \rho_0]} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa.$$

Le Théorème 1.5 est vraiment intéressant lorsque C_K est explicitement connue. Par exemple, dans le cas d'environnements de Dirichlet, i.e. les environnements dont la loi satisfait $\omega_1(\,\mathrm{d}x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{[0,1]}(x)\,\mathrm{d}x$, où $\alpha, \beta > 0$ et $B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{[0,1]}(x)\,\mathrm{d}x$, le Théorème 1.5 peut être nettement plus explicite. Les hypothèses du Théorème 1.5 correspondent aux cas où $0 < \alpha - \beta < 1$ et un rapide calcul montre que $\kappa = \alpha - \beta$.

COROLLAIRE 1.1. *Dans le cas d'environnements* ω *tels que* ω_1 *suit une loi* $\mathrm{Beta}(\alpha, \beta)$ *avec* $0 < \alpha - \beta < 1$, *le Théorème 1.5 s'applique avec* $\kappa = \alpha - \beta$. *Alors on a, quand* n *tend vers l'infini,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{loi}} 2\left(\frac{\pi}{\sin(\pi(\alpha-\beta))} \frac{\psi(\alpha) - \psi(\beta)}{B(\alpha,\beta)^2}\right)^{\frac{1}{\alpha-\beta}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{loi}} \frac{\sin(\pi(\alpha-\beta))}{2^{\alpha-\beta}\pi} \frac{B(\alpha,\beta)^2}{\psi(\alpha)-\psi(\beta)} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa,$$

où ψ *désigne la fonction Digamma définie par* $\psi(z) := (\log \Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Si C_K est inconnue, il est possible de donner une repésentation probabiliste du paramètre caractérisant la loi stable limite. En réalité, on montre d'abord le Théorème 1.6 qui suit, puis on en déduit le Théorème 1.5.

Avant dénoncer le Théorème 1.6, on introduit le premier record négatif du potentiel défini par

$$e_1 := \inf\{k > 0 :\ V(k) \le 0\}.$$

Remarquons que des résultats classiques de théorie des fluctuations (voir [44], p. 396), garantissent que sous les hypothèses (a)-(b) du Théorème 1.5, on a

$$E[e_1] < \infty.$$

On introduit, en plus, la loi classique $\tilde{P}$ associée au potentiel $(V(x), x \in \mathbb{Z})$ sous P vu comme une marche aléatoire (cette loi est notée ${}^aP$ dans [44], p. 406). A cet effet, si μ désigne la loi de $\log \rho_0$, alors grâce à l'hypothèse (a) du Théorème 1.5 on peut définir la loi $\tilde{\mu} = \rho_0^\kappa \mu$ et la loi $\tilde{P} = \tilde{\mu}^{\otimes \mathbb{Z}}$ qui correspond à la loi d'une suite de variables aléatoires i.i.d. de loi $\tilde{\mu}$. La définition de κ implique que $\int \log \rho \, \tilde{\mu}(d\rho) > 0$.

THÉORÈME 1.6. *Soit $\omega := (\omega_i,\ i \in \mathbb{Z})$ une famille de variables aléatoires indépendantes et identiquement distribuées satisfaisant les hypothèses (a)–(b) du Théorème 1.5. Alors on a, quand n tend vers l'infini,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{loi}} 2 \left(\frac{\pi}{\sin(\pi\kappa)} \frac{E[M^\kappa]^2}{E[e_1]^2} \frac{(1 - E[\mathrm{e}^{\kappa V(e_1)}])^2}{E[\rho_0^\kappa \log \rho_0]} \right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{loi}} \frac{\sin(\pi\kappa)}{2^\kappa \pi} \frac{E[e_1]^2}{E[M^\kappa]^2} \frac{E[\rho_0^\kappa \log \rho_0]}{(1 - E[\mathrm{e}^{\kappa V(e_1)}])^2} \left(\frac{1}{\mathcal{S}_\kappa^{ca}} \right)^\kappa .$$

où M a la loi de l'exponentielle d'un méandre définie par

$$M \stackrel{\text{loi}}{=} \sum_{k<0} \mathrm{e}^{-V'_k} + \sum_{k \geq 0} \mathrm{e}^{-V''_k},$$

avec $(V'_k)_{k<0}$ sous $P\{\cdot | V'_k \geq 0, \forall k < 0\}$ et indépendant de $(V''_k)_{k \geq 0}$ sous $\tilde{P}\{\cdot | V''_k > 0, \forall k > 0\}$.

REMARQUE 1.1. *Lorsque C_K n'est pas explicite, il est plus intéressant d'utiliser l'expression du paramètre de la loi stable limite en termes de $E[M^\kappa]$, qui peut facilement être estimé numériquement.*

Part 1

The recurrent case

CHAPTER 2

A weakness in strong localization for Sinai's walk

Sinai's walk is a recurrent one-dimensional nearest-neighbour random walk in random environment. It is known for a phenomenon of strong localization, namely, the walk spends almost all time at or near the bottom of deep valleys of the potential. Our main result shows a weakness of this localization phenomenon: with probability one, the zones where the walk stays for the most time can be far away from the sites where the walk spends the most time. In particular, this gives a negative answer to a problem of Erdős and Révész [42], originally formulated for the usual homogeneous random walk.

Contents

The material of this chapter is a joint work with Z. Shi and has been accepted for publication in *Annals of Probability* [95].

1. Introduction

Let $\omega = (\omega_x,\ x \in \mathbb{Z})$ be a collection of independent and identically distributed random variables taking values in $(0, 1)$. The distribution of ω is denoted by P. Given the value of ω, we define $(X_n,\ n \geq 0)$ as a random walk in random environment (RWRE), which is a Markov chain whose distribution is denoted by P_ω. The transition probabilities of $(X_n,\ n \geq 0)$ are as follows: for $x \in \mathbb{Z}$,

$$P_\omega\left(X_{n+1} = x+1 \,|\, X_n = x\right) = \omega_x = 1 - P_\omega\left(X_{n+1} = x-1 \,|\, X_n = x\right).$$

We denote by $\mathbb{P}$ the joint distribution of $(\omega,\ (X_n))$.

Throughout the paper, we assume that there exists $0 < \delta < \frac{1}{2}$ such that

$$P\left(\delta \leq \omega_0 \leq 1-\delta\right) = 1, \tag{2.1}$$

and that

$$\mathbb{E}[\,\log(\frac{1-\omega_0}{\omega_0})\,] \quad = \quad 0, \tag{2.2}$$

$$\sigma^2 := \mathrm{Var}[\,\log(\frac{1-\omega_0}{\omega_0})\,] \quad > \quad 0. \tag{2.3}$$

Assumption (2.1) is a commonly adopted technical condition, and can for example be replaced by the existence of exponential moments of $\log(\frac{1-\omega_0}{\omega_0})$. It implies that, P-a.s., $|\log(\frac{1-\omega_0}{\omega_0})| \leq M := \log(\frac{1-\delta}{\delta})$. Condition (2.2) ensures, according to Solomon [102], that for P-almost all ω, (X_n) is recurrent, i.e., it hits any site infinitely often. Finally, (2.3) simply excludes the case of a usual homogeneous random walk.

Recurrent RWRE is known for its slow movement: indeed, under (2.1)–(2.3), it is proved by Sinai [98] that $X_n/(\log n)^2$ converges in distribution to a non-degenerate limit. Recurrent RWRE will thus be referred to as Sinai's walk. We will from now on assume (2.1)–(2.3).

For an overview of RWRE, see Zeitouni [113]. Although the understanding of one-dimensional RWRE reached a high level in the last decade, there are still some important questions that remain unanswered. See den Hollander [34] for those concerning large deviations.

Let

$$\xi(n,x) \quad := \quad \#\left\{0 \leq i \leq n:\ X_i = x\right\}, \qquad n \geq 0,\ x \in \mathbb{Z}, \tag{2.4}$$

$$\mathbb{V}(n) \quad := \quad \left\{x \in \mathbb{Z}:\ \xi(n,x) = \max_{y\in\mathbb{Z}} \xi(n,y)\right\}, \qquad n \geq 0. \tag{2.5}$$

In words, $\xi(n,x)$ records the number of visits at site x by the walk in the first n steps, and $\mathbb{V}(n)$ is the set of sites that are the most visited. Note that $\mathbb{V}(n)$ is not empty. Following Erdős and Révész [42], any element in $\mathbb{V}(n)$ is called a "favourite site".

The basic question we are addressing is: if we know that the walk spends almost all time in $\mathbb{Z}_+$, does it imply that favourite sites would also lie in $\mathbb{Z}_+$?

To formulate the problem more precisely, let us introduce the notion of "positive sequence": a (random) sequence $0 < n_1 < n_2 < \ldots$ of positive numbers is called a "positive sequence" (for the walk (X_n)) if

$$\lim_{k\to\infty} \frac{\#\left\{0 \leq i \leq n_k:\ X_i > 0\right\}}{n_k} = 1. \tag{2.6}$$

In words, the walk spends an overwhelming time in $\mathbb{Z}_+$ along any positive sequence.

PROBLEM 2.1. *Is it true that $\mathbb{P}$-almost surely for any positive sequence (n_k), we have $\mathbb{V}(n_k) \subset \mathbb{Z}_+$ for all large k?*

Problem 2.1 was raised by Erdős and Révész [42] (also stated as Problem 10 on page 131 of Révész [89]), originally formulated for the usual homogeneous random walk.

It turns out for the homogeneous walk that the answer is no. Roughly speaking, it is because there is too much "freedom" for the homogeneous walk, so that with probability one, it is possible to find a (random) positive sequence along which the walk does not spend much time in any of the sites of $\mathbb{Z}_+$ – typically, the homogeneous walk makes excursions in $\mathbb{Z}_+$ without spending much time in any sites of $\mathbb{Z}_+$, thus the favourite sites are still in $\mathbb{Z}_-$.

When the environment is random, there is a phenomenon of strong localization (Golosov [50]); indeed, Sinai's walk spends almost all time at the bottom of some special zones, called (deep) "valleys". If we know that Sinai's walk spends almost all time in $\mathbb{Z}_+$, then these deep valleys are likely to be located in $\mathbb{Z}_+$, and the favourite sites – which should be located at or near to the bottom of these deep valleys – would also lie in $\mathbb{Z}_+$. In other words, due to strong localization, it looks natural to conjecture that the answer to Problem 2.1 would be yes.

However, things do not go like this. Here is the main result of the paper.

THEOREM 2.1. *Under assumptions* (2.1)–(2.3),

$$\mathbb{P}\{\forall \text{ positive sequence } (n_k), \text{ we have } \mathbb{V}(n_k) \subset \mathbb{Z}_+ \text{ for all large } k\} = 0.$$

The reason for which the aforementioned heuristics are wrong is that even though Sinai's walk is strongly localized around the bottom of deep valleys, it can happen that a (relatively) big number of sites are around the bottom. In such situations, none of these sites is necessarily favourite, since the visits are shared more or less equally by all these sites.

The main steps in the proof of Theorem 2.1 can be briefly described as follows.

Step A. For P-almost all environment ω, we define a special sequence, denoted by $(m_k)_{k\geq 1}$. This is the starting point in our construction of a positive sequence (n_k) such that for any k, $\mathbb{V}(n_k) \subset \mathbb{Z}_-$.

We mention that the special sequence (m_k) depends only on the environment.

Step B. Based on the special sequence (m_k) and on the movement of the walk, we construct in Section 4 another sequence (n_k). We prove that (n_k) is a positive sequence for (X_n), i.e., condition (2.6) is satisfied.

Step C. Let (n_k) be the positive sequence constructed in Step B. We prove in Section 5 that $\mathbb{P}$-almost surely for all large k, $\mathbb{V}(n_k) \subset \mathbb{Z}_-$.

Clearly, Steps B and C together yield Theorem 2.1.

The rest of the paper is organized as follows. In Section 2, we present some elementary facts about Sinai's walk. These facts will be frequently used throughout the paper. A detailed description of Step A is given in Section 3, but the proof of the main result of the section, Proposition 2.1, is postponed to Section 6. Sections 4 and 5 are devoted to Steps B and C, respectively. Finally, in Section 7, we make some comments on the concentration of Sinai's walk.

We use C_i $(1 \le i \le 22)$ to denote finite and positive constants.

2. Preliminaries on Sinai's walk

We list some basic estimates about hitting times and excursions of Sinai's walk.

In the study of Sinai's walk, an important role is played by a process called the potential, denoted by $V = (V(x),\ x \in \mathbb{Z})$. The potential is a function of the environment ω, and is defined as follows:

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log(\frac{1-\omega_i}{\omega_i}) & \text{si } x \ge 1, \\ 0 & \text{si } x = 0, \\ -\sum_{i=x+1}^{0} \log(\frac{1-\omega_i}{\omega_i}) & \text{si } x \le -1. \end{cases}$$

By (2.1), we have $|V(x) - V(x-1)| \le M$ for any $x \in \mathbb{Z}$.

2.1. Hitting times. For any $x \in \mathbb{Z}$, we define

$$\tau(x) := \min\{n \ge 1 :\ X_n = x\}, \qquad \min \emptyset := \infty. \tag{2.7}$$

(Attention, if $X_0 = x$, then $\tau(x)$ is the first *return* time to x.) Throughout the paper, we write $P_\omega^x(\cdot) := P_\omega(\,\cdot \mid X_0 = x)$ (thus $P_\omega^0 = P_\omega$) and write E_ω^x for expectation with respect to P_ω^x.

It is known (Zeitouni [113], formula (2.1.4)) that for $r < x < s$,

$$P_\omega^x\{\tau(r) < \tau(s)\} = \sum_{j=x}^{s-1} \mathrm{e}^{V(j)} \left(\sum_{j=r}^{s-1} \mathrm{e}^{V(j)} \right)^{-1}. \tag{2.8}$$

The next lemma, which gives a simple bound for the expectation of $\tau(r) \wedge \tau(s)$ when the walk starts from a site $x \in (r, s)$, is essentially contained in Golosov [50].

Lemma 2.1. *For any integers $r < s$, we have*

$$\max_{x \in (r,\, s) \cap \mathbb{Z}} E_\omega^x\left[\tau(r)\, \mathbf{1}_{\{\tau(r) < \tau(s)\}}\right] \le (s-r)^2 \exp\left[\max_{r \le i \le j \le s} (V(i) - V(j))\right]. \tag{2.9}$$

PROOF. Given $\{\tau(r) < \tau(s)\}$, the walk does not hit site s during time interval $[0, \tau(r)]$. Therefore, $\tau(r)$ under $P_\omega^x\{\,\cdot \mid \tau(r) < \tau(s)\}$ is stochastically smaller than the first hitting time of site r by a walk starting from s with a reflecting barrier (to the left) at site s. The expected value of this latter random variable is, according to (A1) of Golosov [50], bounded by $(s-r)^2 \exp\{\max_{r\le i\le j\le s}(V(i)-V(j))\}$. This yields the lemma. □

We will also use the following estimate borrowed from Lemma 7 of Golosov [50]: for $\ell \ge 1$ and $x < y$,

$$P_\omega^x\{\tau(y) < \ell\} \le \ell \exp\left(-\max_{x\le i<y}[V(y-1)-V(i)]\right). \tag{2.10}$$

Looking at the environment backwards, we get: for $\ell \ge 1$ and $w < x$,

$$P_\omega^x\{\tau(w) < \ell\} \le \ell \exp\left(-\max_{w<i\le x}[V(w+1)-V(i)]\right). \tag{2.11}$$

2.2. Excursions. We quote some elementary facts about excursions of Sinai's walk (for detailed discussions, see Section 3 of [30]). Let $b \in \mathbb{Z}$ and $x \in \mathbb{Z}$, and consider $\xi(\tau(b), x)$ under P_ω^b. In words, we look at the number of visits to x of the walk (starting from b) at the first return to b. Then there exist constants C_1, C_2 and C_3 such that

$$C_1\, \mathrm{e}^{-[V(x)-V(b)]} \le E_\omega^b[\xi(\tau(b),x)] \le C_2\, \mathrm{e}^{-[V(x)-V(b)]}, \tag{2.12}$$

and that

$$\mathrm{Var}_\omega^b[\xi(\tau(b),x)] \le C_3\, |x-b| \exp\left(\max_{b\le y\le x}[V(y)-V(x)]\right) \mathrm{e}^{-[V(x)-V(b)]}, \tag{2.13}$$

where $\max_{b\le y\le x}$ should be replaced by $\max_{x\le y\le b}$ if $x < b$.

3. Step A: a special sequence

Recall the constant δ from condition (2.1). We write

$$C_4 := \frac{\delta^3}{2}.$$

For any $j > 0$, we define

$$d^+(j) := \min\{n \ge 0 :\ V(n) \ge j\}, \tag{2.14}$$

$$b^+(j) := \min\left\{n \ge 0 :\ V(n) = \min_{0\le x\le d^+(j)} V(x)\right\}. \tag{2.15}$$

Similarly, we define

$$d^-(j) := \max\{n \le 0 :\ V(n) \ge j\}, \tag{2.16}$$

$$b^-(j) := \max\left\{n \le 0 :\ V(n) = \min_{d^-(j)\le x\le 0} V(x)\right\}. \tag{2.17}$$

In the next sections, we will be frequently using the following elementary estimates: for any $\varepsilon > 0$, P-almost surely for all large j,

$$j^{2-\varepsilon} \le |b^{\pm}(j)| < |d^{\pm}(j)| \le j^{2+\varepsilon}. \tag{2.18}$$

To introduce the announced special sequence in Step A, we define the events (the constant C_5 will be defined in (2.56)):

$$E_1^+(j) := \left\{-2j \le V(b^+(j)) \le -j\right\}, \tag{2.19}$$

$$E_2^+(j) := \left\{\max_{0\le x\le y\le b^+(j)} [V(y) - V(x)] \le \frac{j}{4}\right\}, \tag{2.20}$$

$$E_3^+(j) := \left\{\max_{b^+(j)\le x\le y\le d^+(j)} [V(x) - V(y)] \le j\right\}, \tag{2.21}$$

$$E_4^+(j) := \left\{\sum_{0\le x\le d^+(j)} \mathrm{e}^{-[V(x)-V(b^+(j))]} \ge C_4 \log\log j\right\}, \tag{2.22}$$

and

$$E_1^-(j) := \left\{V(b^-(j)) \le -3j\right\}, \tag{2.23}$$

$$E_2^-(j) := \left\{\max_{b^-(j)\le x\le 0} V(x) \ge \frac{j}{3}\right\}, \tag{2.24}$$

$$E_3^-(j) := \left\{\max_{b^-(j)\le x\le y\le 0} [V(x) - V(y)] \le \frac{j}{2}\right\}, \tag{2.25}$$

$$E_4^-(j) := \left\{\frac{j}{3} \le \max_{d^-(j)\le x\le y\le b^-(j)} [V(y) - V(x)] \le j\right\}, \tag{2.26}$$

$$E_5^-(j) := \left\{\sum_{d^-(j)\le x\le 0} \mathrm{e}^{-[V(x)-V(b^-(j))]} \le 1 + C_5\right\}. \tag{2.27}$$

We set

$$E^+(j) := \bigcap_{i=1}^{4} E_i^+(j), \qquad E^-(j) := \bigcap_{i=1}^{5} E_i^-(j). \tag{2.28}$$

In words, $E_1^+(j)$, $E_2^+(j)$ and $E_3^+(j)$ require $(V(x),\ 0 \le x \le d^+(j))$ to behave "normally" (i.e., without excessive minimum, nor excessive fluctuations), whereas $E_4^+(j)$ requires the potential to have a "relatively large" number of sites near the minimum.

Similarly, $E_1^-(j)$ and $E_2^-(j)$ require $(V(y),\ d^-(j) \le y \le 0)$ to have no excessive extreme values, $E_3^-(j)$ and $E_4^-(j)$ no excessive fluctuations, $E_5^-(j)$ no excessive concentration around the minimum.

Later, we will see that $P\{E_1^+(j) \cap E_2^+(j) \cap E_3^+(j) \cap E^-(j)\}$ is greater than a positive constant, while $P\{E_4^+(j)\}$ tends to 0 (as $j \to \infty$) "sufficiently slowly".

See Figure 1 for an example of $\omega \in E^+(j) \cap E^-(j)$.

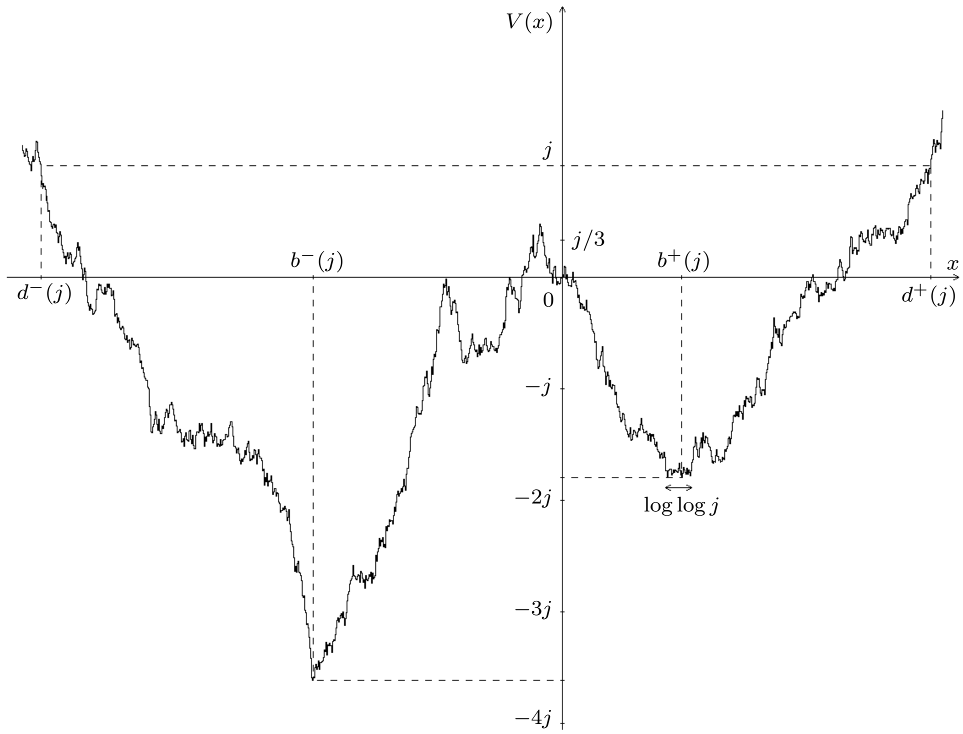

FIGURE 1. Example of $\omega \in E^+(j) \cap E^-(j)$

For future use, let us note that for $\omega \in E_3^-(j) \cap E_1^+(j) \cap E_2^+(j) \cap E_3^+(j)$, we have

$$\max_{b^-(j) \le x \le y \le d^+(j)} [V(x) - V(y)] \le \frac{5j}{2}. \tag{2.29}$$

The proof of the following proposition is postponed until Section 6.

PROPOSITION 2.1. *Under assumptions* (2.1)–(2.3)*, for P-almost all environment ω, there exists a random sequence (m_k) such that $\omega \in E^+(m_k) \cap E^-(m_k)$ for all $k \ge 1$.*

By admitting Proposition 2.1, we will complete Steps B and C in the next two sections.

4. Step B: a positive sequence

Let (m_k) be the special sequence introduced in Proposition 2.1. Without loss of generality, we can assume $m_k \ge k^{3k}$ for all $k \ge 1$. For brevity, we write throughout the paper,

$$b_k^+ := b^+(m_k), \qquad d_k^+ := d^+(m_k), \qquad \tau_k^+ := \tau(b_k^+), \tag{2.30}$$

$$b_k^- := b^-(m_k), \qquad d_k^- := d^-(m_k), \qquad \tau_k^- := \tau(b_k^-). \tag{2.31}$$

We define

$$n_k := (1+(\log k)^{-1/4})\tau_k^- \,. \tag{2.32}$$

We prove in this section that $\mathbb{P}$-almost surely, (n_k) is a positive sequence for (X_n), which means that, $\mathbb{P}$-almost surely,

$$\frac{1}{n_k}\#\{0 \le i \le n_k :\; X_i > 0\} \to 1, \qquad k \to \infty.$$

We start with a few lemmas.

LEMMA 2.2. *We have, P-almost surely, for all large k,*

$$P_\omega\left\{\tau_k^- < \tau_k^+\right\} \;\le\; m_k^3\,\mathrm{e}^{-m_k/12}, \tag{2.33}$$

$$P_\omega\left\{\tau(d_k^+) < \tau_k^-\right\} \;\le\; m_k^3\,\mathrm{e}^{-m_k/2}. \tag{2.34}$$

As a consequence, $\mathbb{P}$-almost surely for all large k,

$$\tau_k^+ < \tau_k^- < \tau(d_k^+). \tag{2.35}$$

PROOF. By (2.8), $P_\omega\{\tau_k^- < \tau_k^+\} = \sum_{j=0}^{b_k^+-1}\mathrm{e}^{V(j)}/\sum_{j=b_k^-}^{b_k^+-1}\mathrm{e}^{V(j)}$. Moreover, since $\max_{b_k^-\le j\le 0} V(j) \ge \frac{m_k}{3}$ (see (2.24)), we have $\sum_{j=b_k^-}^{b_k^+-1}\mathrm{e}^{V(j)} \ge \mathrm{e}^{m_k/3}$. On the other hand, $\max_{0\le j\le b_k^+} V(j) \le \frac{m_k}{4}$ according to (2.20). Therefore, $P_\omega\{\tau_k^- < \tau_k^+\} \le b_k^+\mathrm{e}^{-m_k/12}$. Since $b_k^+ \le m_k^3$ for large k (see (2.18)), this yields (2.33). The proof of (2.34) is along the same lines, using the fact that $\max_{b_k^-\le j\le 0} V(j) \le \frac{m_k}{2}$ (a consequence of (2.25)).

Since $m_k \ge k$, (2.33) and (2.34) yield, respectively, $\sum_k P_\omega\{\tau_k^- < \tau_k^+\} < \infty$, $\sum_k P_\omega\{\tau(d_k^+) < \tau_k^-\} < \infty$, P-almost surely. Now (2.35) follows from the Borel–Cantelli lemma. □

LEMMA 2.3. *Let $A_-(n) := \#\{i :\; 0 \le i \le n,\; X_i \le 0\}$. There exists a constant C_6 such that P-almost surely, for all large k,*

$$E_\omega\left[A_-(\tau_k^-)\right] \le C_6\,(b_k^-)^2\,\mathrm{e}^{m_k/2}. \tag{2.36}$$

PROOF. Let $x \in (b_k^-, 0] \cap \mathbb{Z}$. Recall $\xi(n,x)$ from (2.4). Recall that $P_\omega^x(\cdot) := P_\omega(\,\cdot\,|\,X_0 = x)$. Clearly, $P_\omega\{\xi(\tau_k^-, x) = \ell\} = (1-\pi_x)^{\ell-1}\pi_x$, $\ell \ge 1$, where

$$\begin{aligned}\pi_x &:= P_\omega^x\left\{\tau(x) > \tau_k^-\right\}\\ &= (1-\omega_x)\,P_\omega^{x-1}\left\{\tau(x) > \tau_k^-\right\}\\ &= \frac{1-\omega_x}{\sum_{j=b_k^-}^{x-1}\mathrm{e}^{V(j)-V(x-1)}},\end{aligned} \tag{2.37}$$

the last identity being a consequence of (2.8). In view of assumption (2.1), this yields

$$\frac{1}{\pi_x} \le C_6\, |b_k^-| \, \exp\left(\max_{b_k^- \le j \le i \le 0} (V(j) - V(i)) \right).$$

Since $\max_{b_k^- \le j \le i \le 0}(V(j) - V(i)) \le \frac{m_k}{2}$, and $E_\omega[\xi(\tau_k^-, x)] = \frac{1}{\pi_x}$, this yields

$$E_\omega[\xi(\tau_k^-, x)] \le C_6\, |b_k^-| \, \mathrm{e}^{m_k/2}.$$

Summing over $x \in (b_k^-, 0] \cap \mathbb{Z}$ completes the proof of the lemma. □

REMARK 2.1. *A similar argument shows that for all* $x \in [0, b_k^+]$,

$$E_\omega[\xi(\tau_k^+, x)] \le C_6\, b_k^+ \, \mathrm{e}^{m_k/4}. \tag{2.38}$$

LEMMA 2.4. *For any* $k \ge 1$ *and* $N \ge 1$, *we have*

$$P_\omega\{\tau_k^+ < \tau_k^- < N\} \le C_6\, \mathrm{e}^{-4m_k/3} N. \tag{2.39}$$

Furthermore, P-almost surely, for all large k,

$$E_\omega\left(\frac{1}{\tau_k^-} \mathbf{1}_{\{\tau_k^+ < \tau_k^-\}} \right) \le C_7\, m_k\, \mathrm{e}^{-4m_k/3}. \tag{2.40}$$

PROOF. We observe that

$$P_\omega\left\{ \xi(\tau_k^-, b_k^+) = \ell \right\} = q_k (1 - \pi_{b_k^+})^{\ell - 1} \pi_{b_k^+}, \qquad \ell \ge 1, \tag{2.41}$$

where $\pi_{b_k^+}$ is as in (2.37), $q_k := P_\omega\{\tau_k^+ < \tau_k^-\}$, and $P_\omega\{\xi(\tau_k^-, b_k^+) = 0\} = 1 - q_k$. Therefore, for any $N \ge 1$, $P_\omega\{1 \le \xi(\tau_k^-, b_k^+) \le N\} \le \pi_{b_k^+} N$. Note that,

$$\pi_{b_k^+} \le \exp\left(V(b_k^+ - 1) - \max_{b_k^- \le j \le 0} V(j) \right) \le C_6\, \mathrm{e}^{-4m_k/3}, \tag{2.42}$$

the second inequality following from (2.19) and (2.24). In view of the trivial relations $\tau_k^- \ge \xi(\tau_k^-, b_k^+) + 1$ and $\{\tau_k^+ < \tau_k^-\} = \{\xi(\tau_k^-, b_k^+) \ge 1\}$, this implies (2.39).

To prove the second inequality in the lemma, we note that by the strong Markov property, $E_\omega[\frac{\mathbf{1}_{\{\tau_k^+ < \tau_k^-\}}}{1 + \xi(\tau_k^-, b_k^+)}] = q_k\, E_\omega^{b_k^+}[\frac{1}{1 + \xi(\tau_k^-, b_k^+)}]$. Since $P_\omega^{b_k^+}\{\xi(\tau_k^-, b_k^+) = \ell\} = (1 - \pi_{b_k^+})^{\ell - 1} \pi_{b_k^+}$, $\ell \ge 1$, it follows that

$$\begin{aligned} E_\omega\left(\frac{\mathbf{1}_{\{\tau_k^+ < \tau_k^-\}}}{1 + \xi(\tau_k^-, b_k^+)} \right) &= \frac{q_k\, \pi_{b_k^+}}{(1 - \pi_{b_k^+})^2} \left(\log(\frac{1}{\pi_{b_k^+}}) - (1 - \pi_{b_k^+}) \right) \\ &\le \frac{\pi_{b_k^+}}{(1 - \pi_{b_k^+})^2} \log(\frac{1}{\pi_{b_k^+}}). \end{aligned}$$

The function $u \mapsto \frac{u}{(1-u)^2} \log(\frac{1}{u})$ is increasing in the (positive) neighbourhood of 0. Therefore, by (2.42), $\frac{\pi_{b_k^+}}{(1-\pi_{b_k^+})^2} \log(\frac{1}{\pi_{b_k^+}}) \leq C_7 \, m_k \, \mathrm{e}^{-4m_k/3}$ (for large k). Now (2.40) follows again by means of the trivial inequality $\tau_k^- \geq \xi(\tau_k^-, b_k^+) + 1$. □

LEMMA 2.5. *We have, $\mathbb{P}$-almost surely, for all large k,*

$$\max_{\tau_k^- \leq i \leq n_k} X_i < 0. \tag{2.43}$$

PROOF. By the strong Markov property, $\{X(i+\tau_k^-), \ i \geq 0\}$ is independent (under P_ω) of τ_k^-. Recall that

$$n_k = (1 + (\log k)^{-1/4})\tau_k^-.$$

Therefore, we have, for any $\ell \geq 1$,

$$P_\omega \left\{ \max_{\tau_k^- \leq i \leq n_k} X_i \geq 0 \,\middle|\, \tau_k^- = \ell \right\} = P_\omega^{b_k^-} \left\{ \tau(0) \leq \frac{\ell}{(\log k)^{1/4}} \right\} \leq C_6 \frac{\ell}{(\log k)^{1/4}} \, \mathrm{e}^{-3m_k},$$

the last inequality being a consequence of (2.10) (together with (2.23) and (2.24)). As a consequence,

$$P_\omega \left\{ \max_{\tau_k^- \leq i \leq n_k} X_i \geq 0, \, \tau_k^- < \tau(d_k^+) \right\} \leq \frac{C_6}{(\log k)^{1/4}} \, \mathrm{e}^{-3m_k} E_\omega \left(\tau_k^- \, \mathbf{1}_{\{\tau_k^- < \tau(d_k^+)\}} \right).$$

By (2.9) and (2.29),

$$E_\omega \left(\tau_k^- \, \mathbf{1}_{\{\tau_k^- < \tau(d_k^+)\}} \right) \leq (d_k^+ - b_k^-)^2 \mathrm{e}^{5m_k/2} \tag{2.44}$$

Since $d_k^+ - b_k^- \leq m_k^3$, P-almost surely, for all large k (see (2.18)) and since $m_k \geq k$, it follows that

$$\sum_k P_\omega \left\{ \max_{\tau_k^- \leq i \leq n_k} X_i \geq 0, \, \tau_k^- < \tau(d_k^+) \right\} < \infty, \qquad P\text{-a.s.}$$

Recall from (2.35) that $\tau_k^- < \tau(d_k^+)$ $\mathbb{P}$-almost surely, for all large k. Lemma 2.5 now follows from the Borel–Cantelli lemma. □

It is now time to complete the argument for Step B by showing that (n_k) is a positive sequence for (X_n).

Combining (2.36) with (2.39) yields that

$$\begin{aligned} P_\omega \left\{ \frac{A_-(\tau_k^-)}{\tau_k^-} \geq \mathrm{e}^{-m_k/3}, \, \tau_k^+ < \tau_k^- \right\} \quad &\leq \quad P_\omega \left\{ A_-(\tau_k^-) \geq \mathrm{e}^{-m_k/3} N \right\} \\ &\qquad + P_\omega \left\{ \tau_k^+ < \tau_k^- < N \right\} \\ &\leq \quad \frac{C_6 \, (b_k^-)^2 \, \mathrm{e}^{m_k/2}}{\mathrm{e}^{-m_k/3} N} + C_6 \mathrm{e}^{-4m_k/3} N. \end{aligned}$$

Recall that $|b_k^-| \le m_k^3$ P-almost surely, for all large k (see (2.18)). Choosing $N := \mathrm{e}^{m_k}$, and we have, for large k,

$$P_\omega \left\{ \frac{A_-(\tau_k^-)}{\tau_k^-} \ge \mathrm{e}^{-m_k/3}, \ \tau_k^+ < \tau_k^- \right\} \le C_8\, m_k^6\, \mathrm{e}^{-m_k/6}.$$

Since $m_k \ge k$, this yields $\sum_k P_\omega\{A_-(\tau_k^-) \ge \mathrm{e}^{-m_k/3}\tau_k^-, \ \tau_k^+ < \tau_k^-\} < \infty$, P-almost surely. On the other hand, by (2.35), we have $\tau_k^+ < \tau_k^-$ $\mathbb{P}$-almost surely, for all large k. Therefore, the Borel–Cantelli lemma shows that $\mathbb{P}$-almost surely when $k \to \infty$,

$$\frac{A_-(\tau_k^-)}{\tau_k^-} \le \mathrm{e}^{-m_k/3} \to 0. \tag{2.45}$$

Since for large k, $A_-(n_k) = A_-(\tau_k^-) + (\log k)^{-1/4}\, \tau_k^-$ (Lemma 2.5) and $\tau_k^- < n_k$ by definition, we have proved that $\frac{A_-(n_k)}{n_k} \to 0$, $\mathbb{P}$-almost surely. In words, (n_k) is a positive sequence for the walk. □

5. Step C: negative favourite sites along a positive sequence

Let (n_k) be the positive sequence defined in (2.32). In this section, we prove that $\mathbb{P}$-almost surely for all large k, $\mathbb{V}(n_k) \subset \mathbb{Z}_-$. As before, we use the notation $b_k^\pm$, $d_k^\pm$ and $\tau_k^\pm$ as in (2.30)–(2.31).

We will prove that $\mathbb{P}$-almost surely, for all large k,

$$\xi(n_k, b_k^-) \ \ge \ \frac{\tau_k^-}{(\log k)^{1/3}}, \tag{2.46}$$

$$\max_{x\in[1,\, d_k^+]} \xi(\tau_k^-, x) \ \le \ \frac{\tau_k^-}{(\log k)^{1/2}}. \tag{2.47}$$

Observe that $\mathbb{P}$-almost surely, we have $\max_{x\in[1,d_k^+]} \xi(\tau_k^-, x) = \max_{x\ge 1} \xi(\tau_k^-, x)$, for all large k (by (2.35)), and $\max_{x\ge1} \xi(\tau_k^-, x) = \max_{x\ge1} \xi(n_k, x)$ (Lemma 2.5). It is now clear that (2.46) and (2.47) together will complete Step C, and thus the proof of Theorem 2.1.

The rest of the section is devoted to the proof of (2.46) and (2.47). For the sake of clarity, they are proved in distinct subsections.

5.1. Proof of (2.46). Let $T_0^- := \tau_k^-$ and

$$T_j^- = T_j^-(k) := \min\left\{n > T_{j-1}^- : \ X_n = b_k^-\right\}, \qquad j = 1, 2, \dots$$

We define, for any $j \ge 1$,

$$\begin{aligned} Y_j^-(x) \ &:= \ \xi(T_j^-, x) - \xi(T_{j-1}^-, x), \qquad x \in \mathbb{Z}, \\ Z_j^- \ &:= \ \sum_{x\in(d_k^-,\, 0]} Y_j^-(x). \end{aligned}$$

By the strong Markov property, $(Z_j^-, \ j \ge 1)$ is a sequence of iid random variables (under P_ω). Recall that $n_k = (1 + (\log k)^{-1/4})\tau_k^-$. Let $\ell \ge 1$. By the strong Markov property,

$$P_\omega\{\xi(n_k, b_k^-) < \frac{\ell}{(\log k)^{1/3}} \,|\, \tau_k^- = \ell\} = P_\omega^{b_k^-}\{\xi(\frac{\ell}{(\log k)^{1/4}}, b_k^-) < \frac{\ell}{(\log k)^{1/3}}\}.$$

Under probability $P_\omega^{b_k^-}$, if $\tau(d_k^-) \wedge \tau(0) > \frac{\ell}{(\log k)^{1/4}}$, then the walk stays in $(d_k^-, 0)$ during time interval $[0, \frac{\ell}{(\log k)^{1/4}}]$; if moreover $\xi(\frac{\ell}{(\log k)^{1/4}}, b_k^-) < \frac{\ell}{(\log k)^{1/3}}$, then

$$\sum_{j=1}^{\ell/(\log k)^{1/3}} Z_j^- \geq \frac{\ell}{(\log k)^{1/4}}.$$

Accordingly,

$$\begin{aligned} & P_\omega\left\{\xi(n_k, b_k^-) < \frac{\ell}{(\log k)^{1/3}} \,\Big|\, \tau_k^- = \ell\right\} \\ \leq\ & P_\omega\left\{\sum_{j=1}^{\ell/(\log k)^{1/3}} Z_j^- \geq \frac{\ell}{(\log k)^{1/4}}\right\} + P_\omega^{b_k^-}\left\{\tau(d_k^-) \wedge \tau(0) \leq \frac{\ell}{(\log k)^{1/4}}\right\}. \end{aligned}$$

By (2.12), $E_\omega(Z_j^-) \leq C_2 \sum_{x \in (d_k^-, 0]} \mathrm{e}^{-[V(x) - V(b_k^-)]}$, which is bounded by $C_2(1 + C_5) =: C_9$, according to (2.27). Therefore

$$\begin{aligned} & P_\omega\left\{\sum_{j=1}^{\ell/(\log k)^{1/3}} Z_j^- \geq \frac{\ell}{(\log k)^{1/4}}\right\} \\ \leq\ & P_\omega\left\{\sum_{j=1}^{\ell/(\log k)^{1/3}} (Z_j^- - E_\omega Z_j^-) \geq ((\log k)^{-1/4} - C_9\,(\log k)^{-1/3})\ell\right\} \\ \leq\ & \frac{\mathrm{Var}_\omega(Z_1^-)}{((\log k)^{-1/4} - C_9\,(\log k)^{-1/3})^2 (\log k)^{1/3}\,\ell}. \end{aligned}$$

We have $\mathrm{Var}_\omega(Z_1^-) \leq |d_k^-| \sum_{x \in (d_k^-, 0]} \mathrm{Var}_\omega(Y_1^-(x))$. By (2.13) and (2.25)–(2.26), $\mathrm{Var}_\omega(Y_1^-(x))$ is bounded by $C_3\,|d_k^-|\,\mathrm{e}^{m_k}$ for all $x \in (d_k^-, 0]$. Thus $\mathrm{Var}_\omega(Z_1^-) \leq C_3\,(d_k^-)^2\,\mathrm{e}^{m_k}$. Accordingly, for large k,

$$P_\omega\left\{\sum_{j=1}^{\ell/(\log k)^{1/3}} Z_j^- \geq \frac{\ell}{(\log k)^{1/4}}\right\} \leq C_{10}\,\frac{(\log k)^{1/6}\,(d_k^-)^2\,\mathrm{e}^{m_k}}{\ell}.$$

We now estimate $P_\omega^{b_k^-}\{\tau(d_k^-) \wedge \tau(0) \leq \frac{\ell}{(\log k)^{1/4}}\}$. There is nothing to estimate if $\ell < (\log k)^{1/4}$, so let us assume $\ell \geq (\log k)^{1/4}$. By (2.11) and (2.23),

$$P_\omega^{b_k^-}\left\{\tau(d_k^-) \leq \frac{\ell}{(\log k)^{1/4}}\right\} \leq \left(\frac{\ell}{(\log k)^{1/4}} + 1\right) \mathrm{e}^{-[V(d_k^- + 1) - V(b_k^-)]} \leq \frac{C_6\,\ell}{(\log k)^{1/4}}\,\mathrm{e}^{-4m_k},$$

whereas by (2.10) and (2.23),

$$P_\omega^{b_k^-}\left\{\tau(0) \leq \frac{\ell}{(\log k)^{1/4}}\right\} \leq \left(\frac{\ell}{(\log k)^{1/4}} + 1\right) \mathrm{e}^{-[V(-1) - V(b_k^-)]} \leq \frac{C_6\,\ell}{(\log k)^{1/4}}\,\mathrm{e}^{-3m_k}.$$

Thus, for all $\ell \geq 1$,

$$P_\omega^{b_k^-}\left\{\tau(d_k^-)\wedge\tau(0) \leq \frac{\ell}{(\log k)^{1/4}}\right\} \leq \frac{2C_6\,\ell}{(\log k)^{1/4}}\,\mathrm{e}^{-3m_k} =: \frac{C_{11}\,\ell}{(\log k)^{1/4}}\,\mathrm{e}^{-3m_k}.$$

As a consequence, we have proved that

$$P_\omega\left\{\xi(n_k, b_k^-) \leq \frac{\ell}{(\log k)^{1/3}}\,\Big|\,\tau_k^- = \ell\right\} \leq \frac{C_{10}\,(\log k)^{1/6}\,(d_k^-)^2\,\mathrm{e}^{m_k}}{\ell} + \frac{C_{11}\,\ell}{(\log k)^{1/4}}\,\mathrm{e}^{-3m_k}.$$

Therefore,

$$\begin{aligned} & P_\omega\{\xi(n_k, b_k^-) \leq (\log k)^{-1/3}\tau_k^-,\ \tau_k^+ < \tau_k^- < \tau(d_k^+)\} \\ \leq\ & C_{10}\,(\log k)^{1/6}\,(d_k^-)^2\,\mathrm{e}^{m_k} E_\omega\left[\frac{\mathbf{1}_{\{\tau_k^+<\tau_k^-\}}}{\tau_k^-}\right] + \frac{C_{11}}{(\log k)^{1/4}}\,\mathrm{e}^{-3m_k} E_\omega\left[\tau_k^-\,\mathbf{1}_{\{\tau_k^-<\tau(d_k^+)\}}\right]. \end{aligned}$$

The two expectations, $E_\omega(\frac{1}{\tau_k^-}\mathbf{1}_{\{\tau_k^+<\tau_k^-\}})$ and $E_\omega(\tau_k^-\,\mathbf{1}_{\{\tau_k^-<\tau(d_k^+)\}})$, are estimated by means of (2.40) and (2.44), respectively. We have therefore proved that, for large k, $P_\omega\{\xi(n_k, b_k^-) \leq (\log k)^{-1/3}\tau_k^-,\ \tau_k^+ < \tau_k^- < \tau(d_k^+)\}$ is bounded by

$$C_{10}C_7\,(\log k)^{1/6}\,(d_k^-)^2\,m_k\,\mathrm{e}^{-m_k/3} + \frac{C_{11}}{(\log k)^{1/4}}\,(d_k^+ - b_k^-)^2\mathrm{e}^{-m_k/2}.$$

Since $|d_k^-| \leq m_k^3$ and $d_k^+ - b_k^- \leq m_k^3$, P-almost surely, for all large k (see (2.18)), and since $m_k \geq k$, this implies

$$\sum_k P_\omega\left\{\xi(n_k, b_k^-) \leq (\log k)^{-1/3}\tau_k^-,\ \tau_k^+ < \tau_k^- < \tau(d_k^+)\right\} < \infty, \qquad P\text{-a.s.}$$

The proof of (2.46) is now completed by means of the Borel–Cantelli lemma and (2.35). □

5.2. Proof of (2.47). The proof of (2.47) bears many similarities to the proof of (2.46), the basic idea being again via excursions.

Let $T_0^+ := \tau_k^+$ and

$$T_j^+ = T_j^+(k) := \inf\left\{n > T_{j-1}^+ :\ X_n = b_k^+\right\}, \qquad j = 1, 2, \ldots$$

We write, for any $j \geq 1$,

$$\begin{aligned} Y_j^+(y) &:= \xi(T_j^+, y) - \xi(T_{j-1}^+, y), \qquad y \in \mathbb{Z}, \\ Z_j^+ &:= \sum_{y\in[1,\,d_k^+]} Y_j^+(y). \end{aligned}$$

Let $M = M(k) := \max\{j :\ T_j^+ < \tau_k^-\}$. In words, M denotes the number of excursions (away from b_k^+) completed by the walk before hitting b_k^-.

Let $x \in [1, d_k^+]$. We have

$$\xi(\tau_k^-, x) \leq \xi(\tau_k^+, x) + \sum_{j=1}^{M+1} Y_j^+(x)$$

and

$$\#\{i \le \tau_k^- : X_i \ge 0\} \ge \sum_{j=1}^{M} Z_j^+.$$

Note that $\{M \ge 1\} = \{\tau_k^+ < \tau_k^-\}$. Therefore, for any $\ell \ge 1$ and $k_r := \ell\, 2^r$,

$$\begin{aligned} p(x) &:= P_\omega\{(\log k)^{1/2}\xi(\tau_k^-, x) > \#\{i \le \tau_k^- : X_i \ge 0\},\ \tau_k^+ < \tau_k^-\} \\ &\le P_\omega\{1 \le M < \ell\} + \sum_{r=0}^{\infty} P_\omega\{(\log k)^{1/2}\xi(\tau_k^-, x) > \sum_{j=1}^{M} Z_j^+,\ k_r \le M < k_{r+1}\} \\ &\le P_\omega\{1 \le M < \ell\} + P_\omega\{\xi(\tau_k^+, x) > \ell\} + \sum_{r=0}^{\infty} I_r, \end{aligned}$$

where

$$I_r := P_\omega\left\{(\log k)^{1/2}\ell + (\log k)^{1/2}\sum_{j=1}^{k_{r+1}} Y_j^+(x) > \sum_{j=1}^{k_r} Z_j^+\right\}.$$

By (2.41), we have $P_\omega\{1 < \xi(\tau_k^-, b_k^+) \le \ell\} \le \pi_{b_k^+}\ell$, whereas $P_\omega\{\xi(\tau_k^+, x) > \ell\} \le \frac{1}{\ell} E_\omega[\xi(\tau_k^+, x)] \le \frac{C_6 b_k^+}{\ell}\mathrm{e}^{m_k/4}$, by (2.38). Thus,

$$p(x) \le \pi_{b_k^+}\ell + \frac{C_6 b_k^+}{\ell}\mathrm{e}^{m_k/4} + \sum_{r=0}^{\infty} I_r. \tag{2.48}$$

We now estimate I_r. Recall that $Y_j^+(x)$ is the number of visits at site x by an excursion (away from b_k^+). According to (2.12), $E_\omega[Y_1^+(x)] \le C_2\mathrm{e}^{-[V(x)-V(b_k^+)]} \le C_2$. On the other hand, it follows from (2.12) and then (2.22) that $E_\omega(Z_1^+) \ge C_1\sum_{y\in[1,\, d_k^+]}\mathrm{e}^{-[V(y)-V(b_k^+)]} \ge C_1C_4\log\log m_k$. Since $(\log k)^{1/2}\ell - C_1C_4k_r\log\log m_k + C_2\,(\log k)^{1/2}k_{r+1} \le -\frac{C_1C_4}{2}k_r\log\log m_k$ (for large k; recalling that $m_k \ge k^{3k}$), we see that, P-almost surely, for all large k, the probability I_r is bounded (uniformly in all $r \ge 0$) by

$$P_\omega\{\sum_{j=1}^{k_r}(Z_j^+ - E_\omega Z_j^+) - (\log k)^{1/2}\sum_{j=1}^{k_{r+1}}(Y_j^+(x) - E_\omega Y_j^+(x)) < -\frac{C_1C_4}{2}(\log_{(2)} m_k)k_r\}$$

where $\log_{(2)} x := \log\log x$, for all $x \in \mathbb{R}$. Moreover, this term is bounded by

$$\frac{8}{(C_1C_4\log\log m_k)^2\, k_r}\left[\mathrm{Var}_\omega(Z_1^+) + 2(\log k)\mathrm{Var}_\omega(Y_1^+(x))\right].$$

By means of (2.13) and (2.20)–(2.21), $\mathrm{Var}_\omega(Y_1^+(x)) \le C_3\, d_k^+\mathrm{e}^{m_k}$; it follows that

$$\mathrm{Var}_\omega(Z_1^+) \le d_k^+\sum_{y\in[1,\, d_k^+]}\mathrm{Var}_\omega(Y_1^+(y)) \le C_3\,(d_k^+)^3\mathrm{e}^{m_k} * .$$

Accordingly,

$$I_r \le \frac{8C_3\, d_k^+[(d_k^+)^2 + 2\log k]\mathrm{e}^{m_k}}{(C_1C_4\log\log m_k)^2\, k_r}.$$

Plugging this into (2.48), and using the fact that $\sum_r k_r^{-1} = 2\ell^{-1}$, we get that for any $\ell \geq 1$,

$$\max_{x\in[1,d_k^+]} p(x) \leq \pi_{b_k^+}\ell + \frac{C_6 b_k^+}{\ell}\mathrm{e}^{m_k/4} + C_{12}\frac{d_k^+[(d_k^+)^2 + 2\log k]\mathrm{e}^{m_k}}{(\log\log m_k)^2\,\ell}.$$

Recall from (2.42) that $\pi_{b_k^+} \leq C_6\,\mathrm{e}^{-4m_k/3}$. Now we choose $\ell := \mathrm{e}^{5m_k/4}$, to see that by (2.18),

$$\sum_k d_k^+ \max_{x\in[1,d_k^+]} p(x) < \infty, \qquad P\text{-a.s.}$$

This implies that, P-almost surely,

$$\sum_k P_\omega\left\{(\log k)^{1/2}\max_{x\in[1,d_k^+]}\xi(\tau_k^-, x) > \#\{i \leq \tau_k^- :\ X_i \geq 0\},\ \tau_k^+ < \tau_k^-\right\} < \infty,$$

which yields (2.47) by an application of the Borel–Cantelli lemma and (2.35). □

6. Proof of Proposition 2.1

We now prove that, for P-almost all environment ω, there exists a sequence (m_k) such that $\omega \in E^+(m_k) \cap E^-(m_k)$, $\forall k \geq 1$, where $E^+(m_k)$ and $E^-(m_k)$ are defined in (2.28).

Let $j_k := k^{3k}$ $(k \geq 1)$ and $\mathcal{F}_{j_{k-1}} := \sigma\{V(x),\, 0 \leq x \leq d^+(j_{k-1})\}$.

Recall that (E_j^+) and (E_j^-) are independent events. If we are able to show that

$$\sum_k P\left\{E^+(j_k)\,|\,\mathcal{F}_{j_{k-1}}\right\} = \infty, \qquad P\text{-a.s.,} \tag{2.49}$$

and that for some $C_- > 0$ and all large j,

$$P\left\{E^-(j)\right\} \geq C_-, \tag{2.50}$$

then Lévy's Borel–Cantelli lemma ([96], p. 518) will tell us that with positive probability, there are infinitely many k such that $\omega \in E^+(j_k) \cap E^-(j_k)$. An application of the Hewitt–Savage zero–one law (Feller [44], Theorem IV.6.3) will then yield Proposition 2.1.

The rest of the section is devoted to the proof of (2.49) and (2.50), presented in distinct subsections.

6.1. Proof of (2.49). Recall that $|V(x) - V(x-1)| \leq M = \log\frac{1-\delta}{\delta}$ for any $x \in \mathbb{Z}$.

To bound $P\{E^+(j_k)\,|\,\mathcal{F}_{j_{k-1}}\}$ from below, we start with the trivial inequality $E^+(j_k) \supset E^+(j_k) \cap B^+(j_{k-1})$, for any set $B^+(j_{k-1})$. We choose

$$B^+(j_{k-1}) := \left\{\inf_{0\leq y\leq d^+(j_{k-1})} V(y) \geq -j_{k-1}\log^2 j_{k-1}\right\}.$$

Clearly, $B^+(j_{k-1})$ is $\mathcal{F}_{j_{k-1}}$-measurable. Moreover, on $B^+(j_{k-1}) \cap E^+(j_k)$, we have $d^+(j_{k-1}) \leq b^+(j_k)$.

Recall that $E^+(j_k) = \cap_{i=1}^4 E_i^+(j_k)$. Let

$$F_2^+(j_k) := \left\{ \max_{0 \le x \le y \le b^+(j_k)} [V(y) - V(x)] \le \frac{j_k}{4} - j_{k-1} \log^2 j_{k-1} - j_{k-1} - M \right\}.$$

We consider

$$F^+(j_k) := E_1^+(j_k) \cap E_3^+(j_k) \cap E_4^+(j_k) \cap F_2^+(j_k).$$

Since $V(d^+(j_{k-1})) \in I_{j_{k-1}} := [j_{k-1}, j_{k-1} + M]$, we have, by applying the strong Markov property at $d^+(j_{k-1})$,

$$P\left\{ E^+(j_k) \,|\, \mathcal{F}_{j_{k-1}} \right\} \ge \left(\inf_{z \in I_{j_{k-1}}} P_z \left\{ F^+(j_k) \right\} \right) \mathbf{1}_{B^+(j_{k-1})},$$

where $P_z(\cdot) := P(\,\cdot\,|\, V(0) = z)$, for any $z \in \mathbb{R}$; thus $P = P_0$. (Strictly speaking, we should be working in a canonical space for V, with P_z defined as the image measure of P under translation.)

Clearly, $\mathbf{1}_{B^+(j_{k-1})} = 1$, P-almost surely for all large k. Therefore, the proof of (2.49) boils down to showing the existence of a positive constant C^+ such that P-almost surely for all large k,

$$\inf_{z \in I_{j_{k-1}}} P_z \left\{ F^+(j_k) \right\} \ge \frac{C^+}{k}. \tag{2.51}$$

Let, for any Borel set $A \subset \mathbb{R}$,

$$d^+(A) := \inf \left\{ i \ge 0 :\, V(i) \in A \right\}.$$

A simple martingale argument yields that, whenever $x < y < z$,

$$P_y \left\{ d^+([z, \infty)) < d^+((-\infty, x]) \right\} \ \ge \ \frac{y - x}{z - x + M}, \tag{2.52}$$

$$P_y \left\{ d^+((-\infty, x)) < d^+([z, \infty)) \right\} \ \ge \ \frac{z - y}{z - x + M}. \tag{2.53}$$

We now proceed to prove (2.51). Let

$$a_\ell := -2j_k + 3M\ell, \qquad G_1^+(j_k, \ell) := \left\{ a_\ell \le V(b^+(j_k)) < a_{\ell+1} \right\}.$$

Then

$$\begin{aligned} P_z \left\{ F^+(j_k) \right\} &= P_z \left\{ E_1^+(j_k)\,,\, F_2^+(j_k)\,,\, E_3^+(j_k)\,,\, E_4^+(j_k) \right\} \\ &\ge \sum_{\ell=0}^{\lfloor j_k/(3M) \rfloor - 1} P_z \left\{ G_1^+(j_k, \ell)\,,\, F_2^+(j_k)\,,\, E_3^+(j_k)\,,\, E_4^+(j_k) \right\} \\ &=: \sum_{\ell=0}^{\lfloor j_k/(3M) \rfloor - 1} P_{k,\ell}^+. \end{aligned} \tag{2.54}$$

Let $L(k,\ell) := \#\{0 \le i \le d^+(j_k) : V(i) \in [a_\ell, a_{\ell+1})\}$. On $G_1^+(j_k,\ell)$, we clearly have

$$\mathrm{e}^{-3M} L(k,\ell) \le \sum_{0 \le x \le d^+(j_k)} \mathrm{e}^{-[V(x)-V(b^+(j_k))]}.$$

Therefore,

$$\begin{aligned} P_{k,\ell}^+ &\ge P_z\left\{G_1^+(j_k,\ell)\,,\, F_2^+(j_k)\,,\, E_3^+(j_k)\,,\, \mathrm{e}^{-3M}L(k,\ell) \ge C_4 \log\log j_k\right\} \\ &\ge P_z\left\{G_1^+(j_k,\ell)\,,\, F_2^+(j_k)\,,\, E_3^+(j_k)\,,\, L(k,\ell) \ge \frac{1}{2}\log\log j_k\right\}, \end{aligned}$$

the last inequality following from the values of $M := \log \frac{1-\delta}{\delta}$ and $C_4 := \frac{\delta^3}{2}$.

We define $T_0 := 0$, and by induction,

$$\begin{aligned} \tau_p &:= \min\{i \ge T_{p-1} : V(i) < a_{\ell+1}\}, \\ T_p &:= \min\{i \ge \tau_p : V(i) \ge a_{\ell+1}\}, \qquad p = 1, 2, \dots \end{aligned}$$

Let

$$\alpha = \alpha(k) := \lfloor \frac{1}{2}\log\log j_k \rfloor, \qquad \widetilde{T} := \min\{i \ge \tau_1 : V(i) \ge a_{\ell+2}\}.$$

Since $G_1^+(j_k,\ell) \cap \{L(k,\ell) \ge \alpha\} \supset \{\tau_\alpha < \widetilde{T} < d^+(j_k) < d^+((-\infty, a_\ell])\}$, we have

$$P_{k,\ell}^+ \ge P_z\left\{\tau_\alpha < \widetilde{T} < d^+(j_k) < d^+((-\infty,a_\ell]),\, F_2^+(j_k),\, E_3^+(j_k)\right\}.$$

Consider now the events

$$\begin{aligned} H_2^+(j_k) &:= \left\{\max_{0 \le x \le y \le \tau_1} [V(y) - V(x)] \le \frac{j_k}{5}\right\}, \\ H_3^+(j_k) &:= \left\{\max_{\widetilde{T} \le x \le y \le d^+(j_k)} [V(x) - V(y)] \le j_k\right\}. \end{aligned}$$

We have, for large k, $\{\tau_\alpha < \widetilde{T} < d^+(j_k) < d^+((-\infty,a_\ell])\} \cap H_2^+(j_k) \subset F_2^+(j_k)$, and $\{\tau_\alpha < \widetilde{T} < d^+(j_k) < d^+((-\infty,a_\ell])\} \cap H_3^+(j_k) \subset E_3^+(j_k)$. Therefore, for large k,

$$P_{k,\ell}^+ \ge P_z\left\{\tau_\alpha < \widetilde{T} < d^+(j_k) < d^+((-\infty,a_\ell])\,,\, H_2^+(j_k)\,,\, H_3^+(j_k)\right\}.$$

We apply the strong Markov property at time $\widetilde{T}$. Since $V(\widetilde{T}) \in I_{a_{\ell+2}} := [a_{\ell+2},\, a_{\ell+2} + M]$, we have, for large k,

$$\begin{aligned} P_{k,\ell}^+ &\ge P_z\left\{\tau_\alpha < \widetilde{T} < d^+(j_k),\, \widetilde{T} < d^+((-\infty,a_\ell])\,,\, H_2^+(j_k)\right\} \times \\ &\quad \times \inf_{v \in I_{a_{\ell+2}}} P_v\left\{d^+(j_k) < d^+((-\infty,a_\ell])\,,\, \max_{0 \le x \le y \le d^+(j_k)} [V(x)-V(y)] \le j_k\right\}. \end{aligned} \tag{2.55}$$

Of course, $\{\tau_\alpha < \widetilde{T}\} = \{\tau_1 < T_1 < \tau_2 < \cdots < T_{\alpha-1} < \tau_\alpha < \widetilde{T}\}$. To estimate $P_z\{\cdots\}$ on the right hand side, we apply the strong Markov property successively at τ_α, $T_{\alpha-1}$, $\tau_{\alpha-1}$, $\dots$, T_1 and τ_1. At time τ_α, we use the following inequality (see (2.52)):

for $v \in [a_{\ell+1} - M,\ a_{\ell+1})$,

$$P_v\left\{d^+([a_{\ell+2}, \infty)) < d^+((-\infty, a_\ell])\right\} \geq \frac{(a_{\ell+1} - M) - a_\ell}{a_{\ell+2} - a_\ell + M} = \frac{2}{7}.$$

At times τ_p and T_p ($1 \leq p < \alpha$), we use (see (2.52) and (2.53)), respectively, for $v \in [a_{\ell+1} - M,\ a_{\ell+1})$ and $u \in [a_{\ell+1},\ a_{\ell+1} + M]$,

$$\begin{aligned} P_v\left\{d^+([a_{\ell+1}, \infty)) < d^+((-\infty, a_\ell])\right\} &\geq \frac{(a_{\ell+1} - M) - a_\ell}{a_{\ell+1} - a_\ell + M} = \frac{1}{2}, \\ P_u\left\{d^+((-\infty, a_{\ell+1})) < d^+([a_{\ell+2}, \infty))\right\} &\geq \frac{a_{\ell+2} - (a_{\ell+1} + M)}{a_{\ell+2} - a_{\ell+1} + M} = \frac{1}{2}. \end{aligned}$$

Accordingly,

$$\begin{aligned} & P_z\left\{\tau_\alpha < \widetilde{T} < d^+(j_k),\ \widetilde{T} < d^+((-\infty, a_\ell])\,,\ H_2^+(j_k)\right\} \\ \geq\ & \frac{2/7}{2^{2\alpha-2}} P_z\left\{\tau_1 < d^+(j_k),\ H_2^+(j_k)\right\}. \end{aligned}$$

By Donsker's theorem, $\inf_{z \in I_{j_{k-1}}} P_z\{\tau_1 < d^+(j_k),\ H_2^+(j_k)\}$ is greater than a constant (for large k, and uniformly in ℓ). Thus

$$P_z\left\{\tau_\alpha < \widetilde{T} < d^+(j_k),\ \widetilde{T} < d^+((-\infty, a_\ell])\,,\ H_2^+(j_k)\right\} \geq \frac{C_{13}}{2^{2\alpha}} \geq \frac{C_{14}}{k},$$

the last inequality following from the definition of $\alpha := \lfloor \frac{1}{2} \log\log j_k \rfloor$. Plugging this into (2.55) gives that for large k,

$$\begin{aligned} P_{k,\ell}^+ &\geq \frac{C_{14}}{k} \inf_{v \in I_{a_{\ell+2}}} P_v\left\{d^+(j_k) < d^+((-\infty, a_\ell]),\ \max_{0 \leq x \leq y \leq d^+(j_k)} [V(x) - V(y)] \leq j_k\right\} \\ &\geq \frac{C_{14}}{k} \inf_{v \in I_{a_{\ell+2}}} P_v\{A_\ell^{(+1)}\} \prod_{2 \leq p \leq 5} \inf_{v \in [\frac{p-4}{2} j_k,\ \frac{p-4}{2} j_k + M]} P_v\{A_\ell^{(+p)}\}, \end{aligned}$$

where

$$\begin{aligned} A_\ell^{(+1)} &:= \left\{d^+([-j_k, \infty)) < d^+((-\infty, a_\ell])\right\}, \\ A_\ell^{(+p)} &:= \left\{d^+([\frac{p-3}{2} j_k, \infty)) < d^+((-\infty, \frac{p-5}{2} j_k])\right\}, \quad 2 \leq p \leq 5. \end{aligned}$$

(The last inequality was obtained by applying the strong Markov property successively at the stopping times $d^+([j_k/2, \infty))$, $d^+([0, \infty))$, $d^+([-j_k/2, \infty))$ and at $d^+([-j_k, \infty))$.) It is clear that there exist constants $C_{15} > 0$ and $C_{16} > 0$ such that

$$\inf_{v \in I_{a_{\ell+2}}} P_v\{A_\ell^{(+1)}\} \geq \frac{C_{15}}{j_k}, \qquad \min_{2 \leq p \leq 5} \inf_{v \in [\frac{p-4}{2} j_k,\ \frac{p-4}{2} j_k + M]} P_v\{A_\ell^{(+p)}\} \geq C_{16}.$$

Therefore,

$$P_{k,\ell}^+ \geq \frac{C_{14}}{k} \frac{C_{15}}{j_k} (C_{16})^4 =: \frac{C_{17}}{k\, j_k}.$$

Plugging this into (2.54) gives

$$P_z\left\{F^+(j_k)\right\} \geq \left\lfloor \frac{j_k}{3M} \right\rfloor \frac{C_{17}}{k\, j_k} \geq \frac{C_{18}}{k},$$

which implies (2.51), and completes the proof of (2.49). □

6.2. Proof of (2.50). We write $V_-(n) := V(-n)$, $\forall n \geq 0$. Let as before $P_z(\cdot) := P(\,\cdot\,|\, V(0) = z)$. Under P_z, for $r > z$, we define $d^-(r)$ exactly as in (2.16), i.e., $|d^-(r)| := \min\{i \geq 0 :\ V_-(i) \geq r\}$, whereas for $s < z$, we define

$$|d^-(s)| := \min\left\{i \geq 0 :\ V_-(i) \leq s\right\}.$$

We start with the following estimate: there exist positive constants, denoted by C_5 and C_{19}, such that

$$\inf_{r\geq 1} P\left\{\sum_{0\leq x\leq |d^-(r)|} \mathrm{e}^{-[V_-(x)-V_-(|b^-(r)|)]} \leq C_5\,,\ |d^-(r)| < |d^-(-\frac{r}{2})|\right\} \geq C_{19} > 0. \tag{2.56}$$

This is essentially a consequence of Theorem 2.1 of Bertoin [5], which is a path decomposition for $(V_-(s), s \leq n)$, when n is deterministic. For more details, we refer to Lemma 3.2 of [93], which, by means of an elementary argument, extends Bertoin's theorem for hitting times. Inequality (2.56) then follows from this lemma via the observation that it is possible to choose $1+c_{11} > 2c_{13}$ in [93] (notation of [93]) such that when $E_1(t)\cap E_2(r)$ is true (notation of [93]), we have $\min_{0\leq x\leq |d^-(r)|} V_-(x) = \min_{0\leq x\leq t} V_-(x) \geq -\frac{r}{2}$ (our notation).

To prove (2.50), we write $\beta := 3 - \frac{1}{1000}$ and $\gamma := 3 + \frac{1}{1000}$, and define

$$\begin{aligned}
\overline{T} &:= \min\left\{i \geq |d^-(-3j)| :\ V_-(i) \geq -\beta j\right\},\\
\widetilde{T} &:= \min\left\{i \geq \overline{T} :\ V_-(i) \leq -3j\right\},\\
\Theta^-(j) &:= \left\{|d^-(\frac{j}{3})| < |d^-(-\frac{j}{12})| < \overline{T} < |d^-(j)| < \widetilde{T} < |d^-(-\gamma j)|\right\}.
\end{aligned}$$

See Figure 2 for an example of $\omega \in \Theta^-(j)$.

Recall that $E^-(j) = \cap_{i=1}^5 E_i^-(j)$. Clearly, $E_1^-(j) \cap E_2^-(j) \supset \Theta^-(j)$. Thus

$$E^-(j) \supset \Theta^-(j) \cap E_3^-(j) \cap E_4^-(j) \cap E_5^-(j).$$

Let

$$\begin{aligned}
F_3^-(j) &:= \left\{\max_{|d^-(\frac{j}{3})|\leq x\leq y\leq |d^-(-3j)|} [V_-(y) - V_-(x)] \leq \frac{j}{12}\right\},\\
F_4^-(j) &:= \left\{\frac{j}{3} \leq \max_{\overline{T}\leq x\leq y\leq |d^-(j)|} [V_-(x) - V_-(y)] \leq j\right\}.
\end{aligned}$$

Then $E_3^-(j) \supset \Theta^-(j) \cap F_3^-(j)$, and $E_4^-(j) \supset \Theta^-(j) \cap F_4^-(j)$. Thus

$$E^-(j) \ \supset\ \Theta^-(j) \cap F_3^-(j) \cap F_4^-(j) \cap E_5^-(j).$$

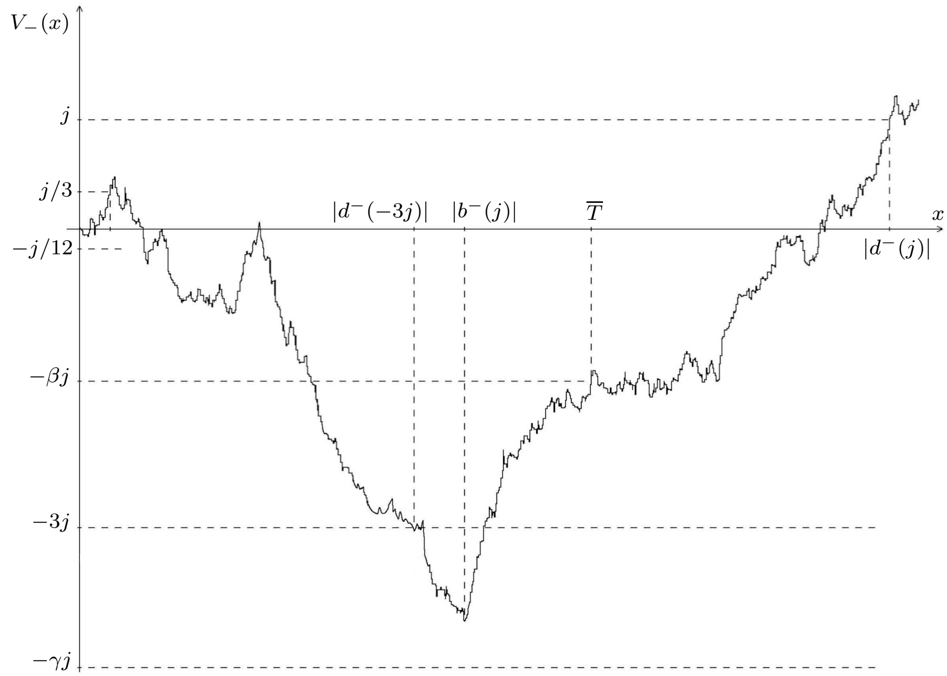

FIGURE 2. Example of $\omega \in \Theta^-(j)$

On $\Theta^-(j) \cap \{|d^-(j)| \le j^3\} \cap \{V_-(|b^-(j)|) \le -3j - j^{1/2}\}$, we have

$$\sum_{x \in [0, |d^-(j)|] \setminus [|d^-(-3j)|, \overline{T}]} \mathrm{e}^{-[V_-(x) - V_-(|b^-(j)|)]} \le j^3 \mathrm{e}^{-j^{1/2}} \le 1,$$

(for large j). Thus $E_5^-(j) \supset F_5^-(j) \cap \Theta^-(j) \cap \{|d^-(j)| \le j^3\} \cap \{V_-(|b^-(j)|) \le -3j - j^{1/2}\}$ (for large j), where

$$F_5^-(j) := \left\{ \sum_{|d^-(-3j)| \le x \le \overline{T}} \mathrm{e}^{-[V_-(x) - V_-(|\widehat{b}^-(-\beta j)|)]} \le C_5 \right\},$$

and $|\widehat{b}^-(-\beta j)| := \min\{n \ge |d^-(-3j)| : V_-(n) = \min_{x \in [|d^-(-3j)|, \overline{T}]} V_-(x)\}$.

For $j \to \infty$, we have $P\{|d^-(j)| > j^3\} \to 0$ and $P\{V_-(|b^-(j)|) \in (-3j - j^{1/2}, -3j]\} \to 0$. Therefore,

$$P\left\{E^-(j)\right\} \ge P\left\{\Theta^-(j),\, F_3^-(j),\, F_4^-(j),\, F_5^-(j)\right\} - o(1), \tag{2.57}$$

where $o(1)$ denotes a term which tends to 0 (when $j \to \infty$). The value of $o(1)$ may vary from line to line.

We apply the strong Markov property at time $\overline{T}$. Since $V_-(\overline{T}) \in I_{-\beta j} := [-\beta j, -\beta j + M]$, this leads to: for large j,

$$P\left\{\Theta^-(j),\, F_3^-(j),\, F_4^-(j),\, F_5^-(j)\right\} \ge P^{(1)} \inf_{v \in I_{-\beta j}} P_v^{(2)},$$

where

$$P^{(1)} := P\{|d^-(\frac{j}{3})| < |d^-(-\frac{j}{12})| < \overline{T} < |d^-(j)|,\, \overline{T} < |d^-(-\gamma j)|,\, F_3^-(j),\, F_5^-(j)\},$$

$$P_v^{(2)} := P_v\{|d^-(j)| < |d^-(-3j)|,\, \frac{j}{3} \le \max_{0\le x\le y\le |d^-(j)|} [V_-(x) - V_-(y)] \le j\}.$$

By Donsker's theorem, $\inf_{v\in I_{-\beta j}} P_v^{(2)} \ge C_{20} > 0$ (for large j). Therefore, for large j,

$$P\left\{\Theta^-(j),\, F_3^-(j),\, F_4^-(j),\, F_5^-(j)\right\} \ge C_{20}\, P^{(1)}.$$

To obtain a lower bound for $P^{(1)}$, we apply the strong Markov property at time $d^-(-3j)$. Since $V_-(|d^-(-3j)|) \in I_{-3j-M} := [-3j-M,\, -3j]$, we have

$$P\left\{\Theta^-(j),\, F_3^-(j),\, F_4^-(j),\, F_5^-(j)\right\} \ge C_{20}\, P^{(3)} \inf_{v\in I_{-3j-M}} P_v^{(4)},$$

where

$$P^{(3)} := P\left\{|d^-(\frac{j}{3})| < |d^-(-\frac{j}{12})| < |d^-(-3j)| < |d^-(j)|,\, F_3^-(j)\right\},$$

$$P_v^{(4)} := P_v\left\{|d^-(-\beta j)| < |d^-(-\gamma j)|,\, \sum_{x=0}^{|d^-(-\beta j)|} \mathrm{e}^{-[V_-(x) - V_-(|b^-(-\beta j)|)]} \le C_5\right\}.$$

We recall that $|b^-(-\beta j)| := \min\{n\ge 0:\ V_-(n) = \min_{x\in[0,\,|d^-(-\beta j)|]} V_-(x)\}$.

By Donsker's theorem, $P^{(3)}$ is greater than a positive constant (for all large j), whereas according to (2.56), $P_v^{(4)} \ge C_{19}$ (for large j, uniformly in $v\in I_{-3j-M}$). As a consequence, for large j,

$$P\left\{\Theta^-(j),\, F_3^-(j),\, F_4^-(j),\, F_5^-(j)\right\} \ge C_{21} > 0.$$

Plugging this into (2.57) completes the proof of (2.50). □

7. A remark

For any set A, let $\xi(n, A) := \sum_{x\in A} \xi(n,x) = \#\{i:\ 0\le i\le n,\ X_i \in A\}$.

The recent work of Andreoletti [3] focuses on:

$$Y_n := \inf_{x\in\mathbb{Z}} \min\left\{k\ge 0:\ \xi(n, [x-k,\, x+k]) \ge an\right\},$$

where $a\in[0,1)$ is an arbitrary but fixed constant. In words, Y_n is (half) the minimal size of an interval where the walk hits at least na times in the first n steps.

It is proved in [3] that under (2.1)–(2.3), there exists a constant $c\in(0,\infty)$ such that

$$\liminf_{n\to\infty} Y_n \le c, \qquad \mathbb{P}\text{-a.s.}$$

A close look at our argument in Section 5 reveals that for some constant $c_* > 0$,

$$\limsup_{n\to\infty} \frac{Y_n}{\log\log\log n} \ge c_*, \qquad \mathbb{P}\text{-a.s.} \tag{2.58}$$

In fact, the proof of (2.47) shows that $\max_{x\in[1,\, d_k^+]} \xi(\tau_k^-, x) \le C_{22} \frac{\tau_k^-}{\log\log m_k}$, for some constant $C_{22} > 0$, ($\mathbb{P}$-almost surely, for all large k; ditto for all the other inequalities stated in this paragraph). In view of (2.35) and (2.45), this implies $\max_{x\in\mathbb{Z}} \xi(\tau_k^-, x) \le C_{22} \frac{\tau_k^-}{\log\log m_k}$. On the other hand, $\sum_k P_\omega\{\tau_k^- \ge \mathrm{e}^{3m_k}, \tau_k^- < \tau(d_k^+)\} < \infty$, by (2.44). Since $\tau_k^- < \tau(d_k^+)$ (Lemma 2.2), we have $\tau_k^- \le \mathrm{e}^{3m_k}$. Thus, we get $\max_{x\in\mathbb{Z}} \xi(\tau_k^-, x) \le 2C_{22} \frac{\tau_k^-}{\log\log\log \tau_k^-}$. As a result, (2.58) follows, with $c_* := \frac{1}{2C_{22}}$.

It is, however, not clear whether inequality "$\le$" would hold in (2.58) with an enlarged value of the constant c_*.

CHAPTER 3

Upper limits for Sinai's walk in random scenery

We consider Sinai's walk in i.i.d. random scenery and focus our attention on a conjecture of Révész [90] concerning the upper limits of Sinai's walk in random scenery when the scenery is bounded from above. A close study of the competition between the concentration property for Sinai's walk and negative values for the scenery enables us to prove that the conjecture is true if the scenery has "thin" negative tails and is false otherwise.

Contents

The material of this chapter has been submitted for publication [118].

1. Introduction

1.1. Random walk in random environment. Problems involving random environments arise in different domains of physics and biology. Originally, one-dimensional random walk in random environment appeared as a simple model for DNA transcription. In the following, we consider the elementary model of one-dimensional Random Walk in Random Environment (RWRE), defined as follows. Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of independent and identically distributed (i.i.d.)

random variables defined on Ω, which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation.

Conditioning on ω (i.e., choosing an environment), we define the RWRE $(X_n,\ n \geq 0)$ as a nearest-neighbor random walk on $\mathbb{Z}$ with transition probabilities given by ω: $(X_n,\ n \geq 0)$ the Markov chain satisfying $X_0 = 0$ and for $n \geq 0$,

$$P_\omega\{X_{n+1} = x+1 \,|\, X_n = x\} = \omega_x = 1 - P_\omega\{X_{n+1} = x-1 \,|\, X_n = x\}.$$

We denote by P_ω the law of $(X_n,\ n \geq 0)$, by E_ω the corresponding expectation, and by $\mathbb{P}$ the joint law of $(\omega, (X_n)_{n\geq 0})$. We refer to Zeitouni [113] for an overview of random walks in random environment.

Throughout the paper, we make the following assumptions on ω:

$$\exists\, \delta \in (0, 1/2): \qquad P\{\delta \leq \omega_0 \leq 1-\delta\} = 1, \tag{3.1}$$

$$E[\log(\frac{1-\omega_0}{\omega_0})] = 0, \tag{3.2}$$

$$\sigma^2 := \mathrm{Var}[\log(\frac{1-\omega_0}{\omega_0})] > 0. \tag{3.3}$$

Assumption (3.1) implies that $|\log(\frac{1-\omega_0}{\omega_0})|$ is, P-a.s., bounded by the constant $L := \log(\frac{1-\delta}{\delta})$. It is a technical assumption, which can be replaced by an exponential moment of $\log(\frac{1-\omega_0}{\omega_0})$. According to a recurrence-transience result due to Solomon [102], assumption (3.2) ensures that $(X_n)_{n\geq 0}$ is $\mathbb{P}$-almost surely recurrent, i.e., the random walk hits any site infinitely often. Assumption (3.3) excludes the case of deterministic environment, which corresponds to the homogeneous symmetric random walk.

Under assumptions (3.1)–(3.3), the RWRE is referred to as Sinai's walk. Sinai [98] proves that $X_n/(\log n)^2$ converges in law, under $\mathbb{P}$, toward a non-degenerate random variable, whose distribution is explicitly computed by Kesten [69] and Golosov [51]. This result contrasts with the usual central limit theorem which gives the convergence in law of $X_n/\sqrt{n}$.

Let

$$\begin{aligned} L(n,x) &:= \#\{0 \leq i \leq n :\ X_i = x\}, \qquad n \geq 0,\ x \in \mathbb{Z}, \\ L(n,A) &:= \sum_{x\in A} L(n,x), \qquad n \geq 0,\ A \subset \mathbb{Z}. \end{aligned} \tag{3.4}$$

In words, the quantity $L(n, A)$ measures the number of visits to the set A by the walk in the first n steps.

The maximum of local time is studied by Révész ([90], p. 337) and Shi [93]: under assumptions (3.1)–(3.3), there exists $c_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{\max_{x\in\mathbb{Z}} L(n,x)}{n} \geq c_0, \qquad \mathbb{P}\text{-a.s.}$$

It means that the walk spends, infinitely often, a positive part of its life on a single site. The liminf behavior is analyzed by Dembo, Gantert, Peres and Shi [30], who prove that

$$\liminf_{n\to\infty} \frac{\max_{x\in\mathbb{Z}} L(n,x)}{n/\log\log\log n} = c_0', \qquad \mathbb{P}\text{-a.s.},$$

for some $c_0' \in (0,\infty)$. A concentration property is obtained by Theorem 1.3 of Andreoletti [3], which says that, under assumptions (3.1)–(3.3) and for any $0 < \beta < 1$, there exists $\ell(\beta) > 0$ such that

$$\limsup_{n\to\infty} \frac{\sup_{x\in\mathbb{Z}} L\left(n, [x-\ell(\beta), x+\ell(\beta)]\right)}{n} \geq \beta, \qquad \mathbb{P}\text{-a.s.} \tag{3.5}$$

In words, for any β close to 1, it is possible to find a length $\ell(\beta)$ such that, $\mathbb{P}$-almost surely, the particle spends, infinitely often, more than a β-fraction of its life in an interval of length $2\ell(\beta)$.

1.2. Random walk in random scenery. Random Walk in Random Scenery (RWRS) is a simple model of diffusion in disordered media, with long-range correlations. It is a class of stationary random processes exhibiting rich behavior. It can be described as follows: given a Markov chain on a state space, there may be a random field indexed by the state space, called a random scenery. As the random walk moves on this state space, he observes the scenery at his location. For a survey of recent results about RWRS, we refer to den Hollander and Steif [35], and to Asselah and Castell [4] for large deviations results in dimension $d \geq 5$.

Let us now define the model of one-dimensional RWRS: consider $S = (S_n,\, n \geq 0)$ a random walk on $\mathbb{Z}$ and $\xi := (\xi(x),\, x\in\mathbb{Z}) = (\xi_x,\, x\in\mathbb{Z})$, a family of i.i.d. random variables defined on a probability space Ξ. We refer to ξ as the random scenery and denote by Q its law. Then, define the process $(Y_n,\, n\geq 0)$ by

$$Y_n := \sum_{i=0}^{n} \xi(S_i),$$

called RWRS or the Kesten-Spitzer Random Walk in Random Scenery. An interpretation is the following: if a random walker has to pay ξ_x each time he visits x, then Y_n stands for the total amount he has paid in the time interval $[0,n]$.

The model is introduced and studied by Kesten and Spitzer [71] in dimensions $d = 1$ and $d \geq 3$. They prove in dimension $d = 1$ that, when S and ξ belong to the domains of attraction of stable laws of indice α and β respectively, then there exists δ, depending on α and β, such that $n^{-\delta} Y_{\lfloor nt\rfloor}$ converges weakly. In the simple case where $\alpha = \beta = 2$, they show that

$$\left(n^{-3/4}\, Y_{\lfloor nt\rfloor};\, 0\leq t\leq 1\right) \xrightarrow{\text{law}} \left(\Lambda(t);\, 0\leq t\leq 1\right),$$

where " $\xrightarrow{\text{law}}$ " stands for weak convergence in law (in some functional space; for example in the space of bounded functions on $[0,1]$ endowed with the uniform topology). The process $(\Lambda(t),\, t \geq 0)$, called Brownian motion in Brownian scenery, is defined by $\Lambda(0) = 0$ and $\Lambda(t) := \int_{\mathbb{R}} \ell(t,x)\, \mathrm{d}W(x)$ for $t > 0$, where $(W(x);\, x \in \mathbb{R})$ denotes a two-sided Brownian motion and $(\ell(t,x),\, t \geq 0, x \in \mathbb{R})$ denotes the jointly continuous version of the local time process of a Brownian motion $(B(t),\, t \geq 0)$, independent of $(W(x);\, x \in \mathbb{R})$.

Independently, Borodin analyzes the case of one-dimensional nearest-neighbor random walk in random scenery, see [13] and [12]. Bolthausen [7] studies the case $d = 2$. He proves that, if S is a recurrent random walk and ξ_0 has zero expectation and finite variance, then $(n \log n)^{-1/2}\, Y_{\lfloor nt \rfloor}$ satisfies a functional central limit theorem.

1.3. Random environment and random scenery. In this paper, we consider Sinai's Walk in Random Scenery. Problems combining random environment and random scenery have been examined for more general models. Replacing $\mathbb{Z}$ by a more general countable state space, Lyons and Schramm [76] exhibit, under certain conditions, a stationary measure for Random Walks in a Random Environment with Random Scenery (RWRERS) from the viewpoint of the random walker. Häggström [53], Häggström and Peres [54] treat the case where the scenery arises from percolation on a graph. In this particular case, the scenery determines the random environment of the associated RWRE, which is used by the authors to obtain information about the scenery.

Let us first describe the model of Sinai's walk in random scenery. We consider Sinai's walk $(X_n,\, n \geq 0)$ under assumptions (3.1)–(3.3), and recall that the environment ω is defined on (Ω, P). For the scenery, we consider a family of i.i.d. random variables $\xi := (\xi(x),\, x \in \mathbb{Z}) = (\xi_x,\, x \in \mathbb{Z})$, defined on (Ξ, Q), independent of ω and $(X_n,\, n \geq 0)$. To translate independence between ω and ξ, we consider the probability space $(\Omega \times \Xi, P \otimes Q)$, on which we define (ω, ξ). Moreover, we denote by $\mathbb{P} \otimes Q$ the law of $(\omega, (X_n)_{n \geq 0}, \xi)$. Then we define as Sinai's walk in random scenery the process $(Z_n,\, n \geq 0)$:

$$Z_n := \sum_{i=0}^{n} \xi(X_i).$$

Observe that Z_n can be written using local time notation:

$$Z_n = \sum_{x \in \mathbb{Z}} \xi(x)\, L(n,x), \qquad n \geq 0, \tag{3.6}$$

where $L(n,x)$ stands for the local time of the random walk at site x until time n, see (3.4).

We are interested in the upper limit of Z_n in the case where $a := \operatorname{ess\,sup} \xi_0$ is finite. We consider the concentration property of order β for Sinai's walk with β

close to 1 (see (3.5)), which enables us to formulate the conjecture of Révész ([90], p. 353): does the assumption that $a := \operatorname{ess\,sup} \xi_0$ is finite imply that, $\mathbb{P} \otimes Q$-almost surely,

$$\limsup_{n\to\infty} \frac{Z_n}{n} = a\,?$$

It turns out that the conjecture holds only under some additional assumptions on the distribution of the random scenery. A close study of the competition between the concentration property for Sinai's walk and negative values for the scenery enables us to obtain the following theorem, which gives a solution to this problem, depending on the tail decay of $\xi_0^- := \max\{-\xi_0, 0\}$.

THEOREM 3.1. *Assume (3.1)–(3.3) and* $a := \operatorname{ess\,sup} \xi_0 < \infty$.

(i) *If* $Q\{\xi_0^- > \lambda\} \le \frac{1}{(\log \lambda)^{2+\varepsilon}}$, *for some* $\varepsilon > 0$ *and all large* λ, *then*

$$\mathbb{P} \otimes Q \left\{ \limsup_{n\to\infty} \frac{Z_n}{n} = a \right\} = 1.$$

(ii) *If* $Q\{\xi_0^- > \lambda\} \ge \frac{1}{(\log \lambda)^{2-\varepsilon}}$, *for some* $\varepsilon > 0$ *and all large* λ, *then*

$$\mathbb{P} \otimes Q \left\{ \lim_{n\to\infty} \frac{Z_n}{n} = -\infty \right\} = 1.$$

It is possible to give more precision in the case (*ii*), see Remark 3.1. On the other hand, the case $\varepsilon = 0$ is still open.

The paper is organized as follows: in Section 2, we present some key results for the environment and for Sinai's walk when the environment is fixed (i.e., quenched results). In Section 3, we define precisely the notion of "good" environment-scenery and prove Theorem 3.1 by accepting two intermediate propositions. The first one, proved in Section 4, is devoted to the study of the RWRE within the "good" environment-scenery. The second one, proved in Section 5, does not concern the RWRE, but only the environment-scenery. We show that, $P \otimes Q$-almost surely, (ω, ξ) is a "good" environment-scenery.

In the following, we use c_i $(1 \le i \le 33)$ to denote finite and positive constants.

2. Preliminaries

In this section, we collect some basic properties of random walk in random environment that will be useful in the forthcoming sections.

2.1. About the environment. In the study of one-dimensional RWRE, an important role is played by a function of the environment ω, called the potential.

This process, noted $V = (V(x),\ x \in \mathbb{Z})$, is defined on (Ω, P) by:

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log(\frac{1-\omega_i}{\omega_i}) & \text{si } x \geq 1, \\ 0 & \text{si } x = 0, \\ -\sum_{i=x+1}^{0} \log(\frac{1-\omega_i}{\omega_i}) & \text{si } x \leq -1. \end{cases} \tag{3.7}$$

By (3.1), we observe that $|V(x) - V(x-1)| \leq L$ for any $x \in \mathbb{Z}$. Moreover, we define $P_z\{\cdot\} := P\{\,\cdot\,|\,V(0) = z\}$, for any $z \in \mathbb{R}$; thus $P = P_0$. (Strictly speaking, we should be working in a canonical space for V, with P_z defined as the image measure of P under translation.)

Let us define, for any Borel set $A \subset \mathbb{R}$,

$$\begin{aligned} d^+(A) &:= \min\{n \geq 0 :\ V(n) \in A\}, \\ d^-(A) &:= \max\{n \leq 0 :\ V(n) \in A\}. \end{aligned}$$

We recall the following result, whose proof is given by a simple martingale argument.

Lemma 3.1. *For any $x < y < z$, we have*

$$\frac{y-x}{z-x+L} \leq P_y\{d^+([z,\infty)) < d^+((-\infty,x])\} \quad \leq \quad \frac{y-x+L}{z-x}.$$

Proof. Since (3.1) and (3.2) imply that $(V(n);\, n \geq 0)$ is a martingale with bounded jumps, we can apply the optional stopping theorem ([38], p. 270) at $d^+([z,\infty)) \wedge d^+((-\infty,x])$ to get

$$\begin{aligned} y = E_y[X_0] \quad &= \quad E_y[X_{d^+([z,\infty))}\,;\, d^+([z,\infty)) < d^+((-\infty,x])] \\ &\quad +E_y[X_{d^+((-\infty,x])}\,;\, d^+([z,\infty)) > d^+((-\infty,x])]. \end{aligned}$$

Since $X_{d^+([z,\infty))} \in [z, z+L]$ and $X_{d^+((-\infty,x])} \in [x-L, x]$ by ellipticity, we obtain

$$y \geq zP_y\{d^+([z,\infty)) < d^+((-\infty,x])\} + (x-L)(1 - P_y\{d^+([z,\infty)) < d^+((-\infty,x])\}),$$

which yields the right inequality. The left inequality is obtained by similar arguments. □

Moreover, we recall a result of Hirsch [55], which, under assumptions (3.1)–(3.3), takes the following simplified form: for any $0 < \varepsilon' < \frac{1}{34}$, there exists $c_1 > 0$ such that

$$P\{\max_{0 \leq x \leq N} V(x) < N^{\frac{1}{2}-\varepsilon'}\} \sim c_1 N^{-\varepsilon'}, \qquad N \to \infty. \tag{3.8}$$

2.2. Quenched results. We define, for any $x \in \mathbb{Z}$,

$$\tau(x) := \min\{n \geq 1 : \, X_n = x\}, \qquad \min \emptyset := \infty.$$

(Note in particular that when $X_0 = x$, then $\tau(x)$ is the first *return* time to x.) Throughout the paper, we write $P_\omega^x\{\cdot\} := P_\omega\{\,\cdot \mid X_0 = x\}$ (thus $P_\omega^0 = P_\omega$) and denote by E_ω^x the expectation with respect to P_ω^x.

Recalling that $\omega_i/(1-\omega_i) = \mathrm{e}^{-(V(i)-V(i-1))}$, we get, for any $r < x < s$,

$$P_\omega^x\{\tau(r) < \tau(s)\} = \sum_{j=x}^{s-1} \mathrm{e}^{V(j)} \left(\sum_{j=r}^{s-1} \mathrm{e}^{V(j)} \right)^{-1}. \tag{3.9}$$

This result is proved in [113], see formula (2.1.4).

The next result, which gives a simple bound for the expectation of $\tau(r) \wedge \tau(s)$ when the walk starts from a site $x \in (r, s)$, is essentially contained in Golosov [50]; its proof can be found in [95]. For any integers $r < s$, we have

$$\max_{x \in (r,\,s) \cap \mathbb{Z}} E_\omega^x[\tau(s)\, \mathbf{1}_{\{\tau(s) < \tau(r)\}}] \leq (s-r)^2 \exp\left[\max_{r \leq i \leq j \leq s} (V(j) - V(i)) \right]. \tag{3.10}$$

We will also use the following estimate borrowed from Lemma 7 of Golosov [50]: for $\ell \geq 1$ and $x < y$,

$$P_\omega^x\{\tau(y) < \ell\} \leq \ell \exp\left(- \max_{x \leq i < y} [V(y-1) - V(i)] \right). \tag{3.11}$$

Looking at the environment backwards, we get: for $\ell \geq 1$ and $w < x$,

$$P_\omega^x\{\tau(w) < \ell\} \leq \ell \exp\left(- \max_{w < i \leq x} [V(w+1) - V(i)] \right). \tag{3.12}$$

Finally we quote an important result about excursions of Sinai's walk (for detailed discussions, see Section 3 of [30]). Let $b \in \mathbb{Z}$ and $x \in \mathbb{Z}$, and consider $L(\tau(b), x)$ under P_ω^b. In words, we look at the number of visits to the site x by the random walk (starting from b) until the first return to b. Then there exist constants c_2 and c_3 such that

$$c_2\, \mathrm{e}^{-[V(x)-V(b)]} \leq E_\omega^b[L(\tau(b), x)] \leq c_3\, \mathrm{e}^{-[V(x)-V(b)]}. \tag{3.13}$$

3. Good environment-scenery and proof of Theorem 3.1

For any $j \in \mathbb{N}^*$, we define

$$\begin{aligned} d^+(j) &:= \min\{n \geq 0 : \, V(n) \geq j\}, \\ b^+(j) &:= \min\left\{ n \geq 0 : \, V(n) = \min_{0 \leq x \leq d^+(j)} V(x) \right\}. \end{aligned}$$

These variables enable us to consider the valley $(0, b^+(j), d^+(j))$. Similarly, we define

$$d^-(j) := \max\{n \leq 0 : \, V(n) \geq j\},$$

$$b^-(j) \quad := \quad \max\left\{n \le 0:\ V(n) = \min_{d^-(j)\le x\le 0} V(x)\right\}.$$

In the next sections, we will be frequently using the following elementary estimates.

LEMMA 3.2. *for any $\varepsilon' > 0$, we have, P-almost surely for all large j,*

$$j^{2-\varepsilon'} \le |b^\pm(j)| < |d^\pm(j)| \le j^{2+\varepsilon'}.$$

PROOF. Fix $\varepsilon' > 0$. Let us consider the sequence $(j_p)_{p\ge1}$ defined by $j_p := p^{12/\varepsilon'}$ for all $p \ge 1$. Using (3.8), we obtain $\sum_{p\ge1} P\{d^+(j_p) > 1/3 j_p^{2+\varepsilon'}\} < \infty$. Therefore, Borel-Cantelli lemma implies that, P-almost surely, $d^+(j_p) \le 1/3 j_p^{2+\varepsilon'}$ for all large p, say $p \ge p_0$. We fix a realization of ω and consider $j_p \le j \le j_{p+1}$ with $p \ge p_0$. Since $d^+(j) \le d^+(j_{p+1})$, we get

$$d^+(j) \le \frac{1}{3} j_{p+1}^{2+\varepsilon'} \le j^{2+\varepsilon'}\frac{1}{3}\left(\frac{j_{p+1}}{j}\right)^{2+\varepsilon'} \le j^{2+\varepsilon'}\frac{1}{3}\left(\frac{j_{p+1}}{j_p}\right)^{2+\varepsilon'} = j^{2+\varepsilon'}\frac{1}{3}(1+p^{-1})^{\frac{12(2+\varepsilon')}{\varepsilon'}},$$

which yields $d^+(j) \le j^{2+\varepsilon'}$ for all large j. In a similar way, we can prove that $j^{2-\varepsilon'} \le d^+((-\infty, -j^{1-\nu}]) \le d^+(j)$ for some $\nu > 0$ and all large j, which implies $j^{2-\varepsilon'} \le b^+(j)$ for all large j. Moreover, the arguments are the same to prove that, P-almost surely, $j^{2-\varepsilon'} \le |b^-(j)| < |d^-(j)| \le j^{2+\varepsilon'}$ for all large j. □

To introduce the announced "good" environment-scenery, we fix $\varepsilon > 0$ such that assumption of Part *(i)* of Theorem 3.1 holds. For $\alpha \in (0,1)$ (which will depend on ε), $0 < c_4 < 1/6$, and $j \ge 100$, we define

$$\gamma_0(j) \quad := \quad j,$$

$$\gamma_i(j) \quad := \quad j^{(1-\alpha)^i} = (\gamma_{i-1}(j))^{1-\alpha}, \qquad i \ge 1, \tag{3.14}$$

$$\varepsilon_i(j) \quad := \quad \exp\left\{ -c_4\gamma_{i+2}(j)\right\}, \qquad i \ge 0. \tag{3.15}$$

For convenience of notation we define $\varepsilon_{-1}(j) := \varepsilon_0(j)$. In words, $(\gamma_i(j))_{i\ge0}$ represents a decreasing sequence of distances, which enables us to classify the sites according to the value of $V(x) - V(b^+(j))$.

Write $\log_p$ for the p-th iterative logarithmic function. Fix $\varepsilon' := \min\{1/35, \varepsilon/2\} > 0$, and introduce, for $j \ge 100$,

$$M(j) := \inf\left\{n \ge 0 : \gamma_n(j) \le (\log_2 j)^{\frac{1-\alpha}{2+\varepsilon'}}\right\}. \tag{3.16}$$

By definition of $M(j)$, we have

$$\gamma_{M(j)-1}(j) \in \left[(\log_2 j)^{\frac{1-\alpha}{2+\varepsilon'}}, (\log_2 j)^{\frac{1}{2+\varepsilon'}}\right].$$

Moreover, in view of (3.14) and since $\gamma_M(j)$ belongs to $[(\log_2 j)^{\frac{(1-\alpha)^2}{2+\varepsilon'}}, (\log_2 j)^{\frac{1-\alpha}{2+\varepsilon'}}]$, we get that

$$M(j) \sim \frac{1}{|\log(1-\alpha)|} \log_2 j, \qquad j \to \infty. \tag{3.17}$$

Note that we choose α small enough such that

$$\beta := (1-\alpha)^2 (2+\varepsilon) - (2+\varepsilon') > 0, \tag{3.18}$$

$$\beta' := \frac{\varepsilon'}{2} - \alpha > 0. \tag{3.19}$$

Then we introduce the set (the constant c_5 will be chosen small enough in (3.52))

$$\overline{\Theta}_{M(j)-1}(j) := \left[b^+(j) - c_5 \left(\gamma_{M(j)-1}(j)\right)^{2+\varepsilon'},\, b^+(j) + c_5 \left(\gamma_{M(j)-1}(j)\right)^{2+\varepsilon'}\right],$$

and, for $i = M(j)-2, ..., 1, 0$, the sets (the constant $c_6 \geq 1$ will be chosen large enough in (3.60))

$$\overline{\Theta}_i(j) := \left[b^+(j) - c_6 \left(\gamma_i(j)\right)^{2+\varepsilon'},\, b^+(j) + c_6 \left(\gamma_i(j)\right)^{2+\varepsilon'}\right] \setminus \bigcup_{p=i+1}^{M(j)-1} \overline{\Theta}_p(j).$$

Observe that the sets $(\overline{\Theta}_i(j))_{0\leq i\leq M(j)-1}$ form a partition of the interval $[b^+(j) - c_6\, j^{2+\varepsilon'},\, b^+(j) + c_6\, j^{2+\varepsilon'}]$. The final sets we consider are given, for $0 \leq i \leq M(j)-1$, by

$$\Theta_i(j) := \overline{\Theta}_i(j) \cap I(j),$$

where $I(j) := [d^+((-\infty,-j]), d^+(j)]$. Note that $d^+((-\infty,-j]) < d^+(j)$ on $A(j)$ which will be defined in (3.27). In this case, the sets $(\Theta_i(j))_{0\leq i\leq M(j)-1}$ form a partition of $I(j)$ into annuli (since $c_6 \geq 1$). Loosely speaking, the set $\Theta_i(j)$ contains the sites x satisfying $V(x) - V(b^+(j)) \approx \gamma_i(j)$. To cover $[d^-(j), d^+(j)]$, we define

$$\Theta_{-1}(j) := \left[-j^{2+\varepsilon'}, j^{2+\varepsilon'}\right] \cap \left[d^-(j), d^+((-\infty,-j])\right]. \tag{3.20}$$

Moreover, for the environment on $\mathbb{Z}^+$, we introduce the events

$$A_1^{env}(j) := \left\{-4j \leq V(b^+(j)) \leq -3j\right\}, \tag{3.21}$$

$$A_2^{env}(j) := \left\{\max_{0\leq x\leq y\leq b^+(j)} [V(y) - V(x)] \leq \frac{j}{4}\right\}. \tag{3.22}$$

The first event ensures that the valley considered is "deep enough" and the second one that the particle reaches the bottom of the valley "fast enough". To control the time spent by the particle in different $\Theta_i(j)$ during an excursion from $b^+(j)$ to $b^+(j)$, we define

$$A_{ann}^{env}(j) := \bigcap_{i=0}^{M(j)-2} \Big\{ \sum_{x\in\Theta_i(j)} \mathrm{e}^{-[V(x)-V(b^+(j))]} \leq (\varepsilon_i(j))^2 \Big\} =: \bigcap_{i=0}^{M(j)-2} A_{ann,i}^{env}(j). \tag{3.23}$$

For the environment on $\mathbb{Z}^-$, let

$$B^{env}(j) := \left\{ V(b^-(j)) \le -\frac{j}{6}, \max_{d^-(j)\le x\le y\le 0} [V(y)-V(x)] \le \frac{j}{3} \right\}, \tag{3.24}$$

which ensures that the particle will not spent too much time on $\mathbb{Z}^-$.

Recalling that $\xi_x^- = \max\{-\xi_x, 0\}$, we define for the scenery

$$A_i^{sce}(j) := \left\{ \max_{x\in\Theta_i(j)} \xi_x^- < (\varepsilon_i(j))^{-1/2} \right\}, \qquad -1 \le i \le M(j)-2, \tag{3.25}$$

which ensures that the scenery does not reach excessive negative value in each $\Theta_i(j)$. In order to force the scenery in a neighborhood of the bottom (where the particle is concentrated), to be close to $a = \operatorname{ess\,sup} \xi_0$, we fix $\rho \in (0,1)$ and introduce

$$A_{M(j)-1}^{sce}(j) := \left\{ \min_{x\in\Theta_{M(j)-1}(j)} \xi_x \ge a-\rho \right\}. \tag{3.26}$$

We set

$$A^{env}(j) := A_1^{env}(j) \cap A_2^{env}(j) \cap A_{ann}^{env}(j), \qquad A^{sce}(j) := \bigcap_{i=-1}^{M(j)-1} A_i^{sce}(j).$$

Moreover, we define

$$A(j) := A^{env}(j) \cap B^{env}(j) \cap A^{sce}(j). \tag{3.27}$$

A pair (ω, ξ) is a "good" environment-scenery if $(\omega,\xi) \in A(j)$ for infinitely many $j \in \mathbb{N}$.

For future use, let us note that for $\omega \in B^{env}(j) \cap A_2^{env}(j)$, we have

$$\max_{d^-(j)\le x\le y\le b^+(j)} [V(y)-V(x)] \le \frac{2j}{3}. \tag{3.28}$$

To prove Theorem 3.1, we need two propositions, whose proofs are respectively postponed until Sections 5 and 4. The first one ensures that almost all pair (ω,ξ) is a "good" environment-scenery, while the second one describes the behavior of the particle in a "good" environment.

Proposition 3.1. *Under assumptions* (3.1)–(3.3), *we have that $P\otimes Q$-almost all (ω,ξ) is a "good" environment-scenery. More precisely, $P\otimes Q$-almost surely, there exists a random sequence $(m_k)_{k\ge1}$ such that $m_k \ge k^{3k}$ and (ω,ξ) is a good environment-scenery along $(m_k)_{k\ge1}$, i.e., $(\omega,\xi)\in A(m_k)$, for all $k\ge 1$.*

In fact $(m_k)_{k\ge1}$ is constructed in the following way. Let us first introduce the sequence $j_p := p^{3p}$ for $p \ge 0$. We define then $(m_k)_{k\ge1}$ by $m_1 := \inf\{j_p \ge 0 : (\omega,\xi) \in A(j_p)\}$ and $m_k := \inf\{j_p > m_{k-1} : (\omega,\xi) \in A(j_p)\}$ for $k \ge 2$. Then, Proposition 3.1 means that $m_k \to \infty$, $k \to \infty$, $P\otimes Q$-almost surely. Before establishing the proposition about the behavior of the particle, we extract a random sequence $(n_k)_{k\ge1}$

from $(m_k)_{k\geq 1}$ such that

$$\sum_{k\geq 1} \varepsilon_{M(n_k)}(n_k) < \infty. \tag{3.29}$$

Actually, we consider the sequence defined by $n_1 := \inf\{m_p \geq 1 : \varepsilon_{M(m_p)}(m_p) \leq 1\}$ and $n_k := \inf\{m_p > n_{k-1} : \varepsilon_{M(m_p)}(m_p) \leq \frac{1}{k^2}\}$ for $k \geq 2$.

To ease notations, we write throughout the paper, $d_k^+ := d^+(n_k)$, $\tau_k^+ := \tau(d_k^+)$, $b_k^+ := b^+(n_k)$ and $d_k^- := d^-(n_k)$, $\tau_k^- := \tau(d_k^-)$. Moreover, we define, for all $k \geq 1$,

$$t_k := \lfloor \mathrm{e}^{n_k} \rfloor. \tag{3.30}$$

Proposition 3.2. *For $P \otimes Q$ almost all (ω, ξ), we have that, P_ω-a.s., for all large k,*

$$\begin{aligned} L\left(t_k, \Theta_{-1}(n_k)\right) &\leq \varepsilon_{-1}(n_k)\, t_k, & \quad & (3.31)\\ L\left(t_k, \Theta_{i}(n_k)\right) &\leq \varepsilon_{i}(n_k)\, t_k, \qquad 0 \leq i \leq M(n_k) - 2, & & (3.32)\\ \tau_k^+ \wedge \tau_k^- &> t_k. & & (3.33) \end{aligned}$$

Proof. *(of Theorem 3.1).*

Proof. *(of Part (i)).* For any $\delta > 0$, we define $\varepsilon^{(\delta)}(j) := \sum_{i=-1}^{M(j)-2} \varepsilon_i^\delta(j)$. Recalling (3.6), we use Proposition 3.2 and Lemma 3.2 to obtain, for $\mathbb{P} \otimes Q$-almost all realization of ω, ξ and $(X_j)_{j\geq 0}$,

$$\sum_{j=0}^{t_k} \xi(X_j) \quad \geq \quad (1 - \varepsilon^{(1)}(n_k))\, t_k \left(\min_{x\in\Theta_{M(n_k)-1}(n_k)} \xi_x \right) - \sum_{i=-1}^{M(n_k)-2} \varepsilon_i(n_k)\, t_k\, (\max_{x\in\Theta_i(n_k)} \xi_x^-),$$

for all large k. Then, Proposition 3.1 implies

$$\begin{aligned} \sum_{j=0}^{t_k} \xi(X_j) \quad &\geq \quad (1 - \varepsilon^{(1)}(n_k))\, t_k\, (a - \rho) - \sum_{i=-1}^{M(n_k)-2} \sqrt{\varepsilon_i(n_k)}\, t_k \\ &\geq \quad (1 - \varepsilon^{(1)}(n_k))\, t_k\, (a - \rho) - \varepsilon^{(1/2)}(n_k)\, t_k, \end{aligned} \tag{3.34}$$

for all large k. We claim that, for any $\delta > 0$ and all large j,

$$\varepsilon^{(\delta)}(j) \leq \sum_{i=-1}^{M(j)} \varepsilon_i^\delta(j) \leq 2 \left(1 + \frac{1}{\delta}\right) \varepsilon_{M(j)}^\delta(j). \tag{3.35}$$

To prove (3.35), we observe that

$$\sum_{i=-1}^{M(j)} \varepsilon_i^\delta(j) \leq 2\, \varepsilon_{M(j)}^\delta(j) + \sum_{i=0}^{M(j)-1} \int_{\varepsilon_i(j)}^{\varepsilon_{i+1}(j)} \frac{\varepsilon_i^\delta(j)}{\varepsilon_{i+1}(j) - \varepsilon_i(j)}\, \mathrm{d}x.$$

Recalling (3.15), we have that $\varepsilon_{i+1}(j) - \varepsilon_i(j) = \varepsilon_{i+1}(j) \left(1 - \mathrm{e}^{-c_4(\gamma_{i+2}(j) - \gamma_{i+3}(j))}\right)$. Recalling (3.14) we get that $\gamma_{i+2}(j) - \gamma_{i+3}(j) = \gamma_{i+2}(j)(1 - \gamma_{i+2}^{-\alpha}(j))$. Since (3.14) and (3.16) imply $\gamma_{i+2}(j) \geq \gamma_{M(j)+2}(j) \geq (\log_2 j)^{\frac{(1-\alpha)^4}{2+\varepsilon'}}$ for $0 \leq i \leq M(j)$, we obtain that $\gamma_{i+2}(j) - \gamma_{i+3}(j) \geq \gamma_{i+2}(j)/2$, for all large j and for $0 \leq i \leq M(j)$. Therefore, we get $\varepsilon_{i+1}(j) - \varepsilon_i(j) \geq \varepsilon_{i+1}(j)/2$, implying that

$$\varepsilon^{(\delta)}(j) \leq 2\,\varepsilon^{\delta}_{M(j)}(j) + 2 \sum_{i=0}^{M(j)-1} \int_{\varepsilon_i(j)}^{\varepsilon_{i+1}(j)} \frac{\varepsilon_i^{\delta}(j)}{\varepsilon_{i+1}(j)}\,\mathrm{d}x.$$

Moreover, $\sum_{i=0}^{M(j)-1} \int_{\varepsilon_i(j)}^{\varepsilon_{i+1}(j)} \frac{\varepsilon_i^{\delta}(j)}{\varepsilon_{i+1}(j)}\,\mathrm{d}x \leq \sum_{i=0}^{M(j)-1} \int_{\varepsilon_i(j)}^{\varepsilon_{i+1}(j)} x^{\delta-1}\,\mathrm{d}x = \int_{\varepsilon_0(j)}^{\varepsilon_{M(j)}(j)} x^{\delta-1}\,\mathrm{d}x$, which is less than $\varepsilon^{\delta}_{M(j)}(j)/\delta$. This implies (3.35).

Combining (3.34) and (3.35) and recalling that $\varepsilon^{\delta}_{M(j)}(j) \to 0$ when $j \to \infty$, we get

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \xi(X_i) \geq a - \rho, \qquad \mathbb{P}\otimes Q\text{-a.s.} \tag{3.36}$$

To conclude the proof, it remains only to observe that (3.36) is true for all $\rho > 0$ and that the definition of a implies that $\mathbb{P}\otimes Q$-a.s., $\frac{1}{n}\sum_{i=0}^{n} \xi(X_i) \leq a$, for all $n \geq 0$. □

Proof. *(of Part (ii)).* Using Theorem 1.5 of [58], we have that, for any $\varepsilon'' > 0$, $\mathbb{P}$-almost surely, $\max_{0\leq i\leq n} X_i \geq (\log n)^{2-\varepsilon''} + 1$, for all large n. This implies

$$\sum_{i=0}^{n} \xi(X_i) \leq a\,n - \max_{0\leq x\leq \lceil (\log n)^{2-\varepsilon''}\rceil} \xi_x^{-}. \tag{3.37}$$

By assumption, there exists $\varepsilon > 0$ such that $Q\{\xi_0 < -\lambda\} \geq (\log\lambda)^{-2+\varepsilon}$. Therefore, fixing $\varepsilon'' < \varepsilon$, we get for $k \geq 1$ and all $N \geq 1$,

$$Q\left\{\max_{0\leq x\leq N} \xi_x^{-} < k\,a\,\mathrm{e}^{N^{\frac{1}{2-\varepsilon''}}}\right\} \leq \exp\left\{-c_7 N^{\delta}\right\}, \tag{3.38}$$

where $\delta := 1 - \frac{2-\varepsilon}{2-\varepsilon''} > 0$.

We choose $N_p := \lfloor (\log p)^T \rfloor$ for $p \geq 1$ with T large enough such that $T\delta > 1$. Therefore, (3.38) and the Borel–Cantelli lemma imply that, Q-almost surely, there exists $p_0(\xi)$ such that

$$\max_{0\leq x\leq N_p} \xi_x^{-} \geq k\,a\,\mathrm{e}^{N_p^{\frac{1}{2-\varepsilon''}}}, \tag{3.39}$$

for $p \geq p_0(\xi)$. Fixing a realization of ξ, we define $p(n)$ by

$$N_{p(n)} \leq \lceil (\log n)^{2-\varepsilon''}\rceil \leq N_{p(n)+1}, \tag{3.40}$$

for all n such that $p(n) \geq p_0(\xi)$. This yields

$$\max_{0\leq x\leq \lceil (\log n)^{2-\varepsilon''}\rceil} \xi_x^{-} \geq \max_{0\leq x\leq N_{p(n)}} \xi_x^{-} \geq k\,a\,\mathrm{e}^{N_{p(n)}^{\frac{1}{2-\varepsilon''}}},$$

the last inequality being a consequence of (3.39). Therefore, we obtain

$$\begin{aligned} &\max_{0\le x\le \lceil (\log n)^{2-\varepsilon''}\rceil} \xi_x^- \\ \ge\quad & ka\exp\{\lceil (\log n)^{2-\varepsilon''}\rceil^{\frac{1}{2-\varepsilon''}}\}\exp\{-(\lceil (\log n)^{2-\varepsilon''}\rceil^{\frac{1}{2-\varepsilon''}} - N_{p(n)}^{\frac{1}{2-\varepsilon''}})\} \\ \ge\quad & kan\exp\{-(N_{p(n)+1}^{\frac{1}{2-\varepsilon''}} - N_{p(n)}^{\frac{1}{2-\varepsilon''}})\}, \end{aligned}$$

the second inequality being a consequence of (3.40). Moreover, we easily get that $N_{p(n)+1}^{\frac{1}{2-\varepsilon''}} - N_{p(n)}^{\frac{1}{2-\varepsilon''}} \to 0$, when $n\to\infty$, implying that for all large n,

$$\max_{0\le x\le \lceil (\log n)^{2-\varepsilon''}\rceil} \xi_x^- \ge \frac{k}{2}\,an. \tag{3.41}$$

Assembling (3.37) and (3.41), we get that $\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^n \xi(X_i)\le a\,(1-\frac{k}{2})$, $\mathbb{P}\otimes Q$-almost surely. We conclude the proof by sending k to infinity. □

This concludes the proof of Theorem 3.1. □

REMARK 3.1. *It is possible to give more precision in the case* (*ii*). *Indeed, using the same arguments, we can prove that if* $Q\{\xi_0^- > \lambda\}\ge \frac{1}{(\log\lambda)^\alpha}$, *for some* $\alpha<2$, *then we have, for any* $\varepsilon'>0$, *that* $\lim_{n\to\infty} n^{-\frac{2}{\alpha}+\varepsilon'} Z_n = -\infty$, $\mathbb{P}\otimes Q$*-almost surely.*

4. Proof of Proposition 3.2

To get (3.33), we observe that

$$P_\omega\left\{\tau_k^+\wedge\tau_k^-\le t_k\right\}\le P_\omega\left\{\tau_k^+\le t_k\right\}+P_\omega\left\{\tau_k^-\le t_k\right\}.$$

Then using (3.11), (3.12) and (3.21), (3.24) we obtain

$$P_\omega\left\{\tau_k^+\wedge\tau_k^-\le t_k\right\}\le t_k\left(\mathrm{e}^{-4n_k}+\mathrm{e}^{-7n_k/6}\right)\le 2\,\mathrm{e}^{-n_k/6},$$

Since $n_k\ge k$, this yields

$$\sum_{k\ge 0} P_\omega\left\{\tau_k^+\wedge\tau_k^-\le t_k\right\}\le 2\sum_{k\ge 0}\mathrm{e}^{-n_k/6}<\infty.$$

We conclude by using the Borel–Cantelli lemma.

To prove (3.32), we apply the strong Markov property at $\tau(b_k^+)$ and obtain that, for any $\lambda_k\ge 0$, $P_\omega\{L(t_k,\Theta_i(n_k))\ge\varepsilon_i(n_k)\,t_k\}$ is less or equal than

$$P_\omega^{b_k^+}\{L(t_k,\Theta_i(n_k))\ge\varepsilon_i(n_k)\,t_k-\lambda_k\}+P_\omega\{\lambda_k\le\tau(b_k^+)\le\tau_k^-\}+P_\omega\{\tau_k^-\le\tau(b_k^+)\},$$

for $0\le i\le M(n_k)-2$. By (3.9), (3.22) and Lemma 3.2, we get, for all large k,

$$P_\omega\{\tau_k^-\le\tau(b_k^+)\}\le\frac{b_k^+\mathrm{e}^{n_k/3}}{\mathrm{e}^{n_k}}\le n_k^{2+\varepsilon'}\mathrm{e}^{-2n_k/3}.$$

Since $P_\omega\{\lambda_k \le \tau(b_k^+) \le \tau_k^-\} \le \lambda_k^{-1} E_\omega[\tau(b_k^+)\, \mathbf{1}_{\{\tau(b_k^+)\le\tau_k^-\}}]$, (3.10) and (3.28) yield

$$P_\omega\{\lambda_k \le \tau(b_k^+) \le \tau_k^-\} \le \frac{(b_k^+ - d_k^-)^2}{\lambda_k}\, \mathrm{e}^{2n_k/3} \le \frac{2n_k^{2\,(2+\varepsilon')}}{\lambda_k}\, \mathrm{e}^{2n_k/3},$$

for all large k, the second inequality being a consequence of Lemma 3.2. Choosing $\lambda_k := \mathrm{e}^{5n_k/6}$, we obtain, for all large k,

$$P_\omega\left\{\lambda_k \le \tau(b_k^+) \le \tau_k^-\right\} + P_\omega\left\{\tau_k^- \le \tau(b_k^+)\right\} \le \mathrm{e}^{-n_k/7}. \tag{3.42}$$

To treat $P_{k,i} := P_\omega^{b_k^+}\left\{L\left(t_k, \Theta_i(n_k)\right) \ge \varepsilon_i(n_k)\, t_k - \lambda_k\right\}$, we observe that (3.30) implies $\lambda_k \le 2\,\mathrm{e}^{-n_k/6}\, t_k$. Therefore, we obtain

$$P_{k,i} \quad\le\quad P_\omega^{b_k^+}\left\{L\left(t_k, \Theta_i(n_k)\right) \ge \left(\varepsilon_i(n_k) - 2\,\mathrm{e}^{-n_k/6}\right) t_k\right\}.$$

Then, by Chebyshev's inequality, we get

$$P_{k,i} \quad\le\quad \frac{1}{(\varepsilon_i(n_k) - 2\,\mathrm{e}^{-n_k/6})\, t_k}\, E_\omega^{b_k^+}\left[L\left(t_k, \Theta_i(n_k)\right)\right].$$

Furthermore, observe that Sinai's walk can not make more than t_k excursions from b_k^+ to b_k^+ before t_k. Since these excursions are i.i.d., we obtain

$$P_{k,i} \quad\le\quad \frac{t_k}{(\varepsilon_i(n_k) - 2\,\mathrm{e}^{-n_k/6})\, t_k}\, E_\omega^{b_k^+}\left[L\left(\tau(b_k^+), \Theta_i(n_k)\right)\right].$$

Now we recall (3.13), which yields $E_\omega^{b_k^+}[L(\tau(b_k^+), \Theta_i(n_k))] \le c_3 \sum_{x\in\Theta_i(n_k)} \mathrm{e}^{-(V(x)-V(b_k^+))}$, for all $0 \le i \le M(n_k) - 2$. Moreover, by (3.23), we get for all large k and for $0 \le i \le M(n_k) - 2$,

$$P_{k,i} \le \frac{c_3\, \left(\varepsilon_i(n_k)\right)^2}{\left(\varepsilon_i(n_k) - 2\,\mathrm{e}^{-n_k/6}\right)} \le c_8\, \varepsilon_i(n_k),$$

for some $c_8 > 0$. The second inequality is a consequence of $\varepsilon_i(n_k) \ge \varepsilon_0(n_k)$ and the fact that $c_4 < 1/6$ implies $\mathrm{e}^{-n_k/6} = o(\varepsilon_0(n_k))$.

Summing from 0 to $M(n_k) - 2$ and using (3.35), we get, with $c_9 := 2(1 + \frac{1}{\delta})\, c_8$,

$$\sum_{i=0}^{M(n_k)-2} P_{k,i} \le c_8 \sum_{i=0}^{M(n_k)-2} \varepsilon_i(n_k) \le c_9\, \varepsilon_{M(n_k)}(n_k). \tag{3.43}$$

Assembling (3.42), (3.43) and recalling (3.17), (3.29) we obtain

$$\sum_{k\ge1} \sum_{i=0}^{M(n_k)-2} P_\omega\{L(t_k, \Theta_i(n_k)) \ge \varepsilon_i(n_k)\, t_k\} \le \sum_{k\ge1} (c_9\, \varepsilon_{M(n_k)}(n_k) + M(n_k)\, \mathrm{e}^{-n_k/7}) < \infty.$$

This implies (3.32) by an application of the Borel–Cantelli lemma.

We get (3.31) by an argument very similar to the one used in the proof of (3.32), the main ingredients being the facts that $V(x) - V(b_k^+) \ge 2\, n_k$, for $x \in \Theta_{-1}(n_k)$

(which is a consequence of (3.24), (3.21) and the definition of $d^+(j)$), and that $\Theta_{-1}(n_k)$ contains less than $2\, n_k^{2+\varepsilon'}$ sites (by (3.20)). We feel free to omit the details.

5. Proof of Proposition 3.1

We now prove that, for $P \otimes Q$-almost all (ω, ξ), there exists a sequence (m_k) such that $(\omega, \xi) \in A(m_k)$, $\forall k \geq 1$, where $A(m_k)$ is defined in (3.27).

Let $j_k := k^{3k}$ $(k \geq 1)$ and $\mathcal{F}_{j_{k-1}} := \sigma\{V(x), \xi_z,\, d^-(j_{k-1}) \leq x, z \leq d^+(j_{k-1})\}$. In the following, we ease notations by using γ_i, ε_i and M instead of $\gamma_i(j_k)$, $\varepsilon_i(j_k)$ and $M(j_k)$.

If we are able to show that

$$\sum_k P \otimes Q \left\{ A(j_k) \,|\, \mathcal{F}_{j_{k-1}} \right\} = \infty, \qquad P \otimes Q\text{-a.s.,} \tag{3.44}$$

then Lévy's Borel–Cantelli lemma ([38], p. 237) will tell us that $P \otimes Q$-almost surely there are infinitely many k such that $(\omega, \xi) \in A(j_k)$.

To bound $P \otimes Q\{A(j_k) \,|\, \mathcal{F}_{j_{k-1}}\}$ from below, we start with the trivial inequality $A(j_k) \supset A(j_k) \cap C(j_{k-1})$, for any set $C(j_{k-1})$. We choose $C(j_{k-1}) := C^{env}(j_{k-1}) \cap D^{env}(j_{k-1}) \cap C^{sce}(j_{k-1})$, where

$$\begin{aligned}
C^{env}(j_{k-1}) &:= \left\{ \inf_{0 \leq y \leq d^+(j_{k-1})} V(y) \geq -j_{k-1} \log^2 j_{k-1} \right\}, \\
D^{env}(j_{k-1}) &:= \left\{ \inf_{d^-(j_{k-1}) \leq y \leq 0} V(y) \geq -j_{k-1} \log^2 j_{k-1} \right\}, \\
C^{sce}(j_{k-1}) &:= \left\{ \max_{d^-(j_{k-1}) \leq x \leq d^+(j_{k-1})} \xi_x^- < (\varepsilon_{-1}(j_k))^{-1/2} \right\}.
\end{aligned}$$

Clearly, $C(j_{k-1})$ is $\mathcal{F}_{j_{k-1}}$-measurable. Moreover on $C^{env}(j_{k-1}) \cap A^{env}(j_k)$, we have $d^+(j_{k-1}) \leq d^+((-\infty, -j_k]) \leq b^+(j_k)$.

Let

$$E_{-1}^{sce}(j_k) := \left\{ \max_{x \in \Theta_{-1} \setminus [d^-(j_{k-1}), d^+(j_{k-1})]} \xi_x^- < (\varepsilon_{-1}(j_k))^{-1/2} \right\},$$

and consider

$$E^{sce}(j_k) := \bigcap_{i=0}^{M-1} A_i^{sce}(j_k) \cap E_{-1}^{sce}(j_k).$$

Since $C^{sce}(j_{k-1}) \cap E_{-1}^{sce}(j_k) \subset A_{-1}^{sce}(j_k)$, it follows that

$$\begin{aligned}
& P \otimes Q \left\{ A(j_k) \,|\, \mathcal{F}_{j_{k-1}} \right\} \\
\geq\ & P \otimes Q \left\{ P \otimes Q \left\{ A^{env}(j_k)\,,\, B^{env}(j_k)\,,\, E^{sce}(j_k)\,,\, C(j_{k-1}) \,|\, \mathcal{F}_{j_{k-1}}, \omega \right\} \,|\, \mathcal{F}_{j_{k-1}} \right\} \\
\geq\ & P \otimes Q \left\{ \mathbf{1}_{\{A^{env}(j_k),\ B^{env}(j_k)\,, C(j_{k-1})\}} P \otimes Q \left\{ E^{sce}(j_k) \,|\, \omega \right\} \,|\, \mathcal{F}_{j_{k-1}} \right\}.
\end{aligned}$$

Now, we suppose for the moment that we are able to prove that there exists $c_{10} > 0$ such that, for P-almost all ω,

$$P \otimes Q \{E^{sce}(j_k) \,|\, \omega\} \geq \frac{c_{10}}{k^{1/4}}. \tag{3.45}$$

We get

$$\begin{aligned} P \otimes Q \left\{A(j_k) \,|\, \mathcal{F}_{j_{k-1}}\right\} &\geq \frac{c_{10}}{k^{1/4}}\, P \otimes Q \left\{A^{env}(j_k)\,,\ B^{env}(j_k)\,,\, C(j_{k-1}) \,|\, \mathcal{F}_{j_{k-1}}\right\} \\ &\geq \frac{c_{10}}{k^{1/4}}\, P_k^+\, P_k^-\, \mathbf{1}_{C^{sce}(j_{k-1})}, \end{aligned} \tag{3.46}$$

where we use the fact that $(V(x),\, x \geq 0)$ and $(V(x),\, x < 0)$ are independent processes and introduce

$$\begin{aligned} P_k^+ &:= P\left\{A^{env}(j_k)\,,\ C^{env}(j_{k-1}) \,|\, \mathcal{F}_{j_{k-1}}\right\}, \\ P_k^- &:= P\left\{B^{env}(j_k)\,,\ D^{env}(j_{k-1}) \,|\, \mathcal{F}_{j_{k-1}}\right\}. \end{aligned}$$

To bound P_k^+ from below, we introduce

$$E_2^{env}(j_k) := \left\{ \max_{0 \leq x \leq y \leq b^+(j_k)} [V(y) - V(x)] \leq \frac{j_k}{4} - j_{k-1} \log^2 j_{k-1} - j_{k-1} - L \right\},$$

and consider

$$E^{env}(j_k) := A_1^{env}(j_k) \cap A_{ann}^{env}(j_k) \cap E_2^{env}(j_k).$$

Observe that $C^{env}(j_{k-1}) \cap \{\max_{d^+(j_{k-1}) \leq x \leq y \leq b^+(j_k)} [V(y) - V(x)] \leq \frac{j_k}{4} - j_{k-1} \log^2 j_{k-1} - j_{k-1} - L\} \subset A_2^{env}(j_k)$. Thus, since $V(d^+(j_{k-1})) \in I_{j_{k-1}} := [j_{k-1},\, j_{k-1} + L]$, we have, by applying the strong Markov property at $d^+(j_{k-1})$,

$$P_k^+ \geq \left(\inf_{z \in I_{j_{k-1}}} P_z \{E^{env}(j_k)\} \right) \mathbf{1}_{C^{env}(j_{k-1})}. \tag{3.47}$$

To bound P_k^- from below, we observe the following inclusion

$$B^{env}(j_k) \supset \left\{ \max_{d^-(j_k) \leq x \leq y \leq d^-(j_{k-1})} [V(y) - V(x)] \leq \frac{j_k}{3} \right\} \cap D^{env}(j_{k-1}).$$

Then since $V(d^-(j_{k-1}))$ belongs to $I_{j_{k-1}}$, the strong Markov property applied at $d^-(j_{k-1})$ yields

$$P_k^- \geq \left(\inf_{z \in I_{j_{k-1}}} P_z \{B^{env}(j_k)\} \right) \mathbf{1}_{D^{env}(j_{k-1})}. \tag{3.48}$$

Observe that an easy calculation yields $\mathbf{1}_{C(j_{k-1})} = 1$, $P \otimes Q$-almost surely for all large k. Therefore, recalling (3.46), (3.47) and (3.48), the proof of (3.44) boils down to showing that

$$\liminf_{k \to \infty} \inf_{z \in I_{j_{k-1}}} P_z \{E^{env}(j_k)\} > 0, \tag{3.49}$$

$$\liminf_{k \to \infty} \inf_{z \in I_{j_{k-1}}} P_z \{B^{env}(j_k)\} > 0. \tag{3.50}$$

The rest of the section is devoted to the proof of (3.45) and (3.49), whereas (3.50) is an immediate consequence of Donsker's theorem.

5.1. Proof of (3.45). Since the sets $\{\Theta_i\}_{-1\le i\le M-1}$ are disjoint, the events $E^{sce}_{-1}(j_k)$ and $\{A^{sce}_i(j_k)\}_{0\le i\le M-1}$ are mutually independent. We write

$$P\otimes Q\left\{E^{sce}(j_k)\,|\,\omega\right\}=\prod_{i=0}^{M-1}P\otimes Q\left\{A^{sce}_i(j_k)\,|\,\omega\right\}\times P\otimes Q\left\{E^{sce}_{-1}(j_k)\,|\,\omega\right\}.$$

Thus, (3.45) will be a consequence of the two following lemmas.

Lemma 3.3. *For P-almost all ω, we have*

$$P\otimes Q\left\{A^{sce}_{M-1}(j_k)\,|\,\omega\right\}\ge\frac{1}{k^{1/4}}.$$

Lemma 3.4. *There exists $c_{11}>0$ such that, for P-almost all ω,*

$$\liminf_{k\to\infty}\prod_{i=0}^{M-2}P\otimes Q\left\{A^{sce}_i(j_k)\,|\,\omega\right\}\times P\otimes Q\left\{E^{sce}_{-1}(j_k)\,|\,\omega\right\}\ge c_{11}. \tag{3.51}$$

Proof. *(of Lemma 3.3).* Recalling (3.26), (3.20) and (3.17), we get, P-almost surely,

$$P\otimes Q\left\{A^{sce}_{M-1}(j_k)\,|\,\omega\right\}\ge\exp\left\{c_5\,\log q\,\log_2 j_k\right\},$$

where $q:=Q\{\xi_0\ge a-\rho\}$. Note that the definition of a implies $-\infty<\log q<0$. Therefore, it remains only to observe that $\log_2 j_k=\log k+\log_2 k+\log 3$ and to choose c_5 small enough such that

$$c_5\,\log q>-1/5, \tag{3.52}$$

to conclude the proof. □

Proof. *(of Lemma 3.4).* Recalling (3.25) and that $(\xi^-_x)_{x\in\mathbb{Z}}$ is a family of i.i.d. random variables, we get, P-almost surely, for $0\le i\le M-2$,

$$\begin{aligned}P\otimes Q\left\{A^{sce}_i(j_k)\,|\,\omega\right\} &\ge \left(Q\{\xi^-_0\le\varepsilon_i^{-1/2}\}\right)^{2c_6\,\gamma_i^{2+\varepsilon'}}\\ &\ge \exp\left\{2c_6\,\gamma_i^{2+\varepsilon'}\,\log\left(1-Q\{\xi^-_0\ge\varepsilon_i^{-1/2}\}\right)\right\}.\end{aligned}$$

Then, since $Q\{\xi^-_0\ge\varepsilon_i^{-1/2}\}$ tends to 0 when k tends to ∞ and using the fact that $\log(1-x)\ge -c_{12}\,x$ for $x\in[0,1/2)$ with $c_{12}:=2\log 2>0$, it follows that

$$P\otimes Q\left\{A^{sce}_i(j_k)\,|\,\omega\right\}\ge\exp\left\{-\,c_{13}\,\gamma_i^{2+\varepsilon'}\,Q\{\xi^-_0\ge\varepsilon_i^{-1/2}\}\right\},$$

for all large k, with $c_{13} := 2\, c_6\, c_{12}$. Recalling that $Q\left\{\xi_0^- \geq \lambda\right\} \leq (\log \lambda)^{-(2+\varepsilon)}$ for $\lambda \geq \lambda_0 > 0$ and (3.15) we get for k large enough and uniformly in $0 \leq i \leq M-2$,

$$P \otimes Q\left\{A_i^{sce}(j_k) \,|\, \omega\right\} \geq \exp\left\{ -\, c_{14}\, \gamma_i^{-\beta}\right\}, \tag{3.53}$$

where $\beta := (1-\alpha)^2\,(2+\varepsilon)-(2+\,\varepsilon') > 0$ by (3.18), and $c_{14} := c_{13}\left(\frac{2}{c_4}\right)^{2+\varepsilon}$. Similarly, since $E_{-1}^{sce}(j_k) \subset A_{-1}^{sce}(j_k)$ and recalling (3.25), we obtain

$$P \otimes Q\left\{E_{-1}^{sce}(j_k) \,|\, \omega\right\} \geq \exp\left\{ -\, c_{15}\, \gamma_0^{-\beta}\right\}, \tag{3.54}$$

for some $c_{15} > 0$. Combining (3.53) and (3.54), we get

$$\prod_{i=0}^{M-2} P \otimes Q\left\{A_i^{sce}(j_k) \,|\, \omega\right\} \times P \otimes Q\left\{E_{-1}^{sce}(j_k) \,|\, \omega\right\} \geq \exp\{-c_{16}\, \sigma_\beta\},$$

with $c_{16} := \max\{c_{14}, c_{15}\}$ and $\sigma_\beta := \gamma_0^{-\beta} + \sum_{i=0}^{M-2} \gamma_i^{-\beta}$. By the same way we proved (3.35), we obtain that, for any $\beta > 0$, there exists $c_{17} \leq 1 + 2/\beta$ such that $\sigma_\beta \leq c_{17}\gamma_{M-1}^{-\beta}$. Recalling (3.17), it follows that $\sigma_\beta \to 0$ when $k \to \infty$, which implies (3.51). □

5.2. Proof of (3.49). To prove (3.49), we need the following preliminary result.

LEMMA 3.5. *For any $\delta > 0$, $k \geq 1$ and any $0 \leq p \leq M$, we have*

$$\sum_{i=p}^{M} \gamma_i^\delta \leq \left(1 + \frac{2}{\delta}\right) \gamma_p^\delta.$$

PROOF. Observe that we easily get

$$\sum_{i=p}^{M} \gamma_i^\delta \leq \gamma_p^\delta + \sum_{i=p+1}^{M} \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1} - \gamma_i} dx.$$

Recalling that $\gamma_{i-1} - \gamma_i \geq \gamma_{i-1}/2$, for all large j and for $1 \leq i \leq M$, we get

$$\sum_{i=p}^{M} \gamma_i^\delta \leq \gamma_p^\delta + 2 \sum_{i=p+1}^{M} \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1}} dx.$$

Then, $\sum_{i=p+1}^{M} \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1}} dx \leq \sum_{i=p+1}^{M} \int_{\gamma_i}^{\gamma_{i-1}} x^{\delta-1} dx = \int_{\gamma_M}^{\gamma_p} x^{\delta-1} dx \leq \gamma_p^\delta/\delta$ concludes the proof of Lemma 3.5. □

We now proceed to prove (3.49). Let

$$a_\ell := -3j_k - \ell\, \gamma_M, \qquad F_1^{env}(j_k, \ell) := \left\{a_{\ell+1} \leq V(b^+(j_k)) < a_\ell\right\}.$$

Denoting $\theta_k := \lfloor j_k/\gamma_M \rfloor - 1$, the inclusion $\bigsqcup_{\ell=0}^{\theta_k} F_1^{env}(j_k,\ell) \subset A_1^{env}(j_k)$ yields

$$P_z\{E^{env}(j_k)\} \geq \sum_{\ell=0}^{\theta_k} P_z\{F_1^{env}(j_k,\ell)\,,\,A_{ann}^{env}(j_k)\,,\,E_2^{env}(j_k)\} =: \sum_{\ell=0}^{\theta_k} P_{k,\ell}^{+}. \tag{3.55}$$

To bound $P_{k,\ell}^{+}$ by below for $0 \leq \ell \leq \theta_k$, we define the following levels,

$$\eta_i = \eta_i(j_k,\ell) := a_\ell + \gamma_i, \qquad 0 \leq i \leq M, \tag{3.56}$$

$$\eta_{M+1} = \eta_{M+1}(j_k,\ell) := a_\ell, \tag{3.57}$$

$$\eta_{M+2} = \eta_{M+2}(j_k,\ell) := a_{\ell+1}. \tag{3.58}$$

In the following, we introduce stopping times for the potential, which will enable us to consider a valley having "good" properties. Let us write

$$\begin{aligned} T = T(j_k,\ell) &:= d^+((-\infty,\eta_{M+1}]),\\ \tilde{T} = \tilde{T}(j_k,\ell) &:= d^+((-\infty,\eta_{M+2}]). \end{aligned}$$

Then, let us define the following stopping times, for $0 \leq i \leq M$,

$$\begin{aligned} T_i = T_i(j_k,\ell) &:= d^+((-\infty,\eta_i]),\\ T_i' = T_i'(j_k,\ell) &:= \min\{n \geq T : V(n) \geq \eta_{M-i}\},\\ R_i = R_i(j_k,\ell) &:= \min\{n \geq T_i' : V(n) \leq \eta_{M-i+1}\}. \end{aligned}$$

We introduce the events , for $0 \leq i \leq M-1$,

$$\begin{aligned} G_i(j_k) &:= \{T_{i+1} - T_i \leq \gamma_i^{2+\varepsilon'}\,,\, \max_{T_i \leq x \leq y \leq T_{i+1}} [V(y) - V(x)] \leq \frac{j_k}{5}\},\\ G_M(j_k) &:= \{T - T_M \leq \gamma_M^{2+\varepsilon'}\,,\, \max_{T_M \leq x \leq y \leq T} [V(y) - V(x)] \leq \frac{j_k}{5}\}, \end{aligned}$$

and

$$\begin{aligned} G_0'(j_k) &:= \{T_0' - T \leq \gamma_M^{2+\varepsilon'}\,,\, T_0' < \tilde{T}\},\\ G_i'(j_k) &:= \{T_i' - T_{i-1}' \leq \gamma_{M-i}^{2+\varepsilon'}\,,\, T_i' < R_{i-1}\}, \qquad 1 \leq i \leq M. \end{aligned}$$

Moreover, we set

$$G(j_k,\ell) := \bigcap_{i=0}^{M} G_i(j_k), \qquad G'(j_k,\ell) := \bigcap_{i=0}^{M} G_i'(j_k),$$

and

$$\begin{aligned} H(j_k,\ell) &:= \left\{ \max_{0 \leq x \leq y \leq T_0} [V(y) - V(x)] \leq \frac{j_k}{5} \right\},\\ H'(j_k,\ell) &:= \left\{ d^+(j_k) < R_M \right\}. \end{aligned}$$

See Figure 1 for an example of $\omega \in G(j_k,\ell) \cap G'(j_k,\ell) \cap H(j_k,\ell) \cap H'(j_k,\ell)$.

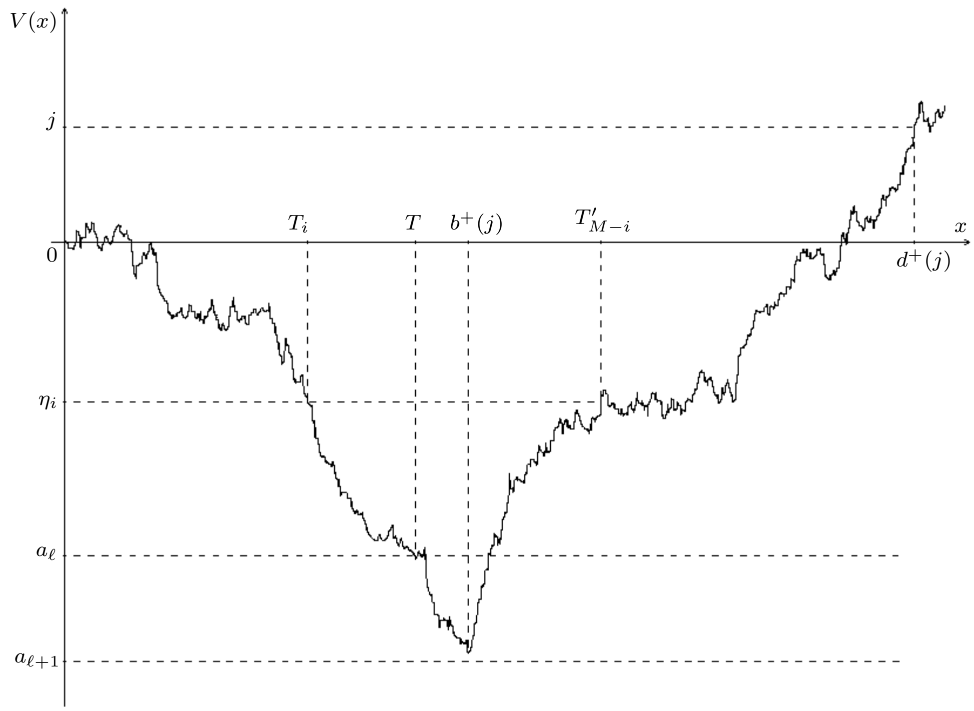

FIGURE 1. Example of $\omega \in G(j_k,\ell) \cap G'(j_k,\ell) \cap H(j_k,\ell) \cap H'(j_k,\ell)$

Observe that on $G(j_k,\ell) \cap G'(j_k,\ell) \cap H'(j_k,\ell)$, we have, for $0 \le i \le M-1$,

$$[T_i, T'_{M-i}] \supset \left\{x \in [0, d^+(j_k)] : V(x) - V(b^+(j_k)) \le \gamma_{i+1}\right\}. \tag{3.59}$$

Moreover, on $G(j_k,\ell) \cap G'(j_k,\ell)$,

$$T'_{M-i} - T_i \le 2 \sum_{p=i}^{M} \gamma_p^{2+\varepsilon'}, \qquad 0 \le i \le M.$$

If we choose c_6 such that

$$c_6 \ge 2(1 + \frac{2}{2+\varepsilon'}), \tag{3.60}$$

then Lemma 3.5 yields

$$[T_i, T'_{M-i}] \subset [b^+(j_k) - c_6\, \gamma_i^{2+\varepsilon'},\, b^+(j_k) + c_6\, \gamma_i^{2+\varepsilon'}], \qquad 0 \le i \le M-2. \tag{3.61}$$

Recall now definition of $\Theta_i(j_k)$, so that, by assembling (3.59) and (3.61), we have on $G(j_k,\ell) \cap G'(j_k,\ell) \cap H'(j_k,\ell)$,

$$\Theta_i(j_k) \subset \left\{x \in \mathbb{Z} : V(x) - V(b^+(j_k)) \ge \gamma_{i+2}\right\}, \qquad 0 \le i \le M-2.$$

An easy calculation yields $\sum_{x\in\Theta_i(j_k)} \exp\{-[V(x) - V(b^+(j_k))]\} \le 2c_6\, \gamma_i^{2+\varepsilon'} \mathrm{e}^{-\gamma_{i+2}}$, for all $0 \le i \le M-2$, on $G(j_k,\ell) \cap G'(j_k,\ell) \cap H'(j_k,\ell)$. On the other hand, since $6\,c_4 < 1$, we get $2c_6\, \gamma_i^{2+\varepsilon'} \mathrm{e}^{-\gamma_{i+2}} \le \varepsilon_i^2$, for all large k and uniformly in $0 \le i \le M-2$. This implies that $G(j_k,\ell)\cap G'(j_k,\ell)\cap H'(j_k,\ell)\subset F_1^{env}(j_k,\ell)\cap A_{ann}^{env}(j_k)$. We easily observe that $G(j_k,\ell)\cap H(j_k,\ell)\subset E_2^{env}(j_k)$, for all large k. Thus we obtain

$$G(j_k,\ell)\cap G'(j_k,\ell)\cap H(j_k,\ell)\cap H'(j_k,\ell)\subset F_1^{env}(j_k,\ell)\cap A_{ann}^{env}(j_k)\cap E_2^{env}(j_k).$$

Recalling (3.55), we get

$$P_{k,\ell}^{+}\geq P_z\left\{G(j_k,\ell)\,,\,G'(j_k,\ell)\,,\,H(j_k,\ell)\,,\,H'(j_k,\ell)\right\}.$$

To bound $P_z\{G(j_k,\ell)\,,\,G'(j_k,\ell)\,,\,H(j_k,\ell)\,,\,H'(j_k,\ell)\}$ by below, we will apply the strong Markov property several times.

Since $V(T'_M)\in I_{\eta_0}:=[\eta_0,\,\eta_0+L]$, the strong Markov property applied at T'_M implies, for $z\in I_{j_{k-1}}$,

$$P_{k,\ell}^{+}\geq P_z\left\{G(j_k,\ell)\,,\,G'(j_k,\ell)\,,\,H(j_k,\ell)\right\}\inf_{y\in I_{\eta_0}}P_y\left\{d^+(j_k)\leq d^+((-\infty,\eta_1])\right\}.$$

To bound by below $P_y\{\cdots\}$ on the right hand side, observe that $P_y\{\cdots\}$ is greater than $P_{\eta_0}\{\cdots\}$. Moreover since $\eta_1\geq -4j_k$ implies $j_k-\eta_1\leq 5j_k$, Lemma 3.1 yields

$$P_{\eta_0}\{d^+(j_k)\leq d^+((-\infty,\eta_1])\}\geq\frac{\eta_0-\eta_1}{5j_k+L}=\frac{\eta_0(1-\eta_0^{-\alpha})}{5j_k+L}\geq c_{18},$$

for all large k and some $c_{18}>0$, which implies

$$P_{k,\ell}^{+}\geq c_{18}\,P_z\left\{G(j_k,\ell)\,,\,G'(j_k,\ell)\,,\,H(j_k,\ell)\right\}.$$

We now apply the strong Markov property successively at $(T'_{M-i})_{1\leq i\leq M}$ and T, such that

$$P_{k,\ell}^{+}\geq c_{18}\,P_z\left\{G(j_k,\ell)\,,\,H(j_k,\ell)\right\}\inf_{y\in I_{\eta_{M+1}-L}}Q'_{0,y}\prod_{p=1}^{M}\inf_{y\in I_{\eta_{M-p+1}}}Q'_{p,y},\tag{3.62}$$

where

$$Q'_{p,y}\;:=\;P_y\{d^+(\eta_{M-p})\leq\gamma_{M-p}^{2+\varepsilon'},\,d^+(\eta_{M-p})<d^+((-\infty,\eta_{M-p+2}])\},\qquad 0\leq p\leq M.$$

First, observe that $\inf_{y\in I_{\eta_{M-p+1}}}Q'_{p,y}\geq Q'_{p,\eta_{M-p+1}}=:Q'_p$, for all $1\leq p\leq M$ and similarly $\inf_{y\in I_{\eta_{M+1}-L}}Q'_{0,y}\geq Q'_{0,\eta_{M+1}-L}:=Q'_0$. Therefore we only have to bound from below Q'_p for $1\leq p\leq M$ and Q'_0. Recalling that $P\{A\,,\,B\}\geq P\{A\}-P\{B^c\}$, we get, for $1\leq p\leq M$,

$$Q'_p\geq P_{\eta_{M-p+1}}\{d^+(\eta_{M-p})<d^+((-\infty,\eta_{M-p+2}])\}-P_{\eta_{M-p+1}}\{d^+(\eta_{M-p})\geq\gamma_{M-p}^{2+\varepsilon'}\},$$

and

$$Q'_0\geq P_{\eta_{M+1}-L}\left\{d^+(\eta_M)<d^+((-\infty,\eta_{M+2}])\right\}-P_{\eta_{M+1}-L}\left\{d^+(\eta_M)\geq\gamma_M^{2+\varepsilon'}\right\}.$$

By Lemma 3.1, we obtain, for $1\leq p\leq M$,

$$P_{\eta_{M-p+1}}\left\{d^+(\eta_{M-p})<d^+((-\infty,\eta_{M-p+2}])\right\}\geq\frac{\eta_{M-p+1}-\eta_{M-p+2}}{\eta_{M-p}-\eta_{M-p+2}+L},$$

and

$$P_{\eta_{M+1}-L}\left\{d^+(\eta_M) < d^+((-\infty, \eta_{M+2}])\right\} \geq \frac{\eta_{M+1} - L - \eta_{M+2}}{\eta_M - \eta_{M+2} + L}.$$

Recalling (3.56) and (3.57), we bound $P_{\eta_{M-p+1}}\left\{d^+(\eta_{M-p}) < d^+((-\infty, \eta_{M-p+2}])\right\}$ by below (for all $1 \leq p \leq M$) by

$$\begin{aligned}\frac{\gamma_{M-p+1}}{\gamma_{M-p}} \frac{1-\gamma_{M-p}^{-\alpha(1-\alpha)}}{1+L\gamma_{M-p}^{-1}} &\geq \frac{\gamma_{M-p+1}}{\gamma_{M-p}}(1-\gamma_{M-p}^{-\alpha(1-\alpha)})(1-L\gamma_{M-p}^{-1}) \\ &\geq \frac{\gamma_{M-p+1}}{\gamma_{M-p}}(1-2\,\gamma_{M-p}^{-\alpha(1-\alpha)}),\end{aligned}$$

for all large k. The first inequality is a consequence of $(1+x)^{-1} \geq 1-x$ for any $x \in (0,1)$ and the second one is a consequence of $0 < \alpha < 1$. Similarly, recalling (3.56), (3.57) and (3.58), we get, for all large k,

$$P_{\eta_{M+1}-L}\left\{d^+(\eta_M) < d^+((-\infty, \eta_{M+2}])\right\} \geq \frac{\gamma_M - L}{2\,\gamma_M + L} \geq c_{18},$$

with $c_{18} > 0$. Moreover, combining (3.8) and the fact that $\gamma_{M-p} \leq \gamma_{M-p}^{(2+\varepsilon')(\frac{1}{2}-\frac{\varepsilon'}{6})}$ for $0 \leq p \leq M$ yields

$$\begin{aligned}P_{\eta_{M-p+1}}\left\{d^+(\eta_{M-p}) \geq \gamma_{M-p}^{2+\varepsilon'}\right\} &\leq c_{19}\gamma_{M-p}^{-\varepsilon'/6}, \qquad 1 \leq p \leq M, \\ P_{\eta_{M+1}-L}\left\{d^+(\eta_M) \geq \gamma_M^{2+\varepsilon'}\right\} &\leq c_{19}\gamma_M^{-\varepsilon'/6},\end{aligned}$$

for all large k and for some $c_{19} > 0$. Therefore, we obtain $Q'_0 \geq c_{20}$ for some $c_{20} > 0$ and recalling (3.19) we get, for $1 \leq p \leq M$,

$$Q'_p \geq \frac{\gamma_{M-p+1}}{\gamma_{M-p}}(1 - c_{21}\gamma_{M-p}^{-\beta''}),$$

where $\beta'' := \min\{\alpha(1-\alpha), \beta'\} > 0$ (β' is defined in (3.19)) and $c_{21} > 0$. Observe that $\gamma_{M-p}^{-\beta''} \leq \gamma_M^{-\beta''}$ for $1 \leq p \leq M$, and that $\gamma_M^{-\beta''} \to 0$, $k \to \infty$. Recalling the fact that $\log(1-x) \geq -c_{12}\,x$, for $x \in [0, 1/2)$, we obtain

$$\inf_{y \in I_{\eta_{M+1}-L}} Q'_{0,y} \prod_{p=1}^{M} \inf_{y \in I_{\eta_{M-p+1}}} Q'_{p,y} \geq c_{20}\frac{\gamma_M}{\gamma_0} \exp\left\{-c_{22}\sum_{p=1}^{M}\gamma_{M-p}^{-\beta''}\right\},$$

where $c_{22} := c_{12}c_{21}$.

Recall that for any $\beta'' > 0$, there exists $c_{23} > 0$ such that $\sum_{p=1}^{M}\gamma_{M-p}^{-\beta''} \leq c_{23}\,\gamma_M^{-\beta''}$. Then, recalling (3.62), this yields, for all large k,

$$P_{k,\ell}^{+} \geq c_{24}\,\frac{\gamma_M}{\gamma_0}\,P_z\left\{G(j_k,\ell)\,,\,H(j_k,\ell)\right\}, \tag{3.63}$$

with $c_{24} > 0$. To bound $P_z\{G(j_k,\ell)\,,\,H(j_k,\ell)\}$ from below, we apply successively the strong Markov property at $(T_{M-i})_{0 \leq i \leq M}$ such that

$$P_z\left\{G(j_k,\ell)\,,\,H(j_k,\ell)\right\} \geq P_z\left\{H(j_k,\ell)\right\}\prod_{p=0}^{M} Q_p,$$

where

$$Q_p \quad := \quad P_{\eta_p}\left\{d^+((-\infty,\eta_{p+1}]) \le \min\left\{d^+(\eta_{p+1}+j_k/5),\gamma_p^{2+\varepsilon'}\right\}\right\}, \qquad 0\le p\le M.$$

Recall that $P\{A\,,\,B\}\ge P\{A\}-P\{B^c\}$. Then (3.8) yields, for $1\le p\le M$,

$$P_{\eta_p}\left\{d^+((-\infty,\eta_{p+1}])\le \gamma_p^{2+\varepsilon'}\right\}\ge 1-c_{25}\gamma_p^{-\varepsilon'/6},$$

with $c_{25}>0$. Moreover, using Lemma 3.1, we get, for $1\le p\le M$,

$$P_{\eta_p}\left\{d^+(\eta_{p+1}+j_k/5)\le d^+((-\infty,\eta_{p+1}])\right\}\le c_{26}\,\frac{\gamma_p}{j_k},$$

with $c_{26}>0$. Therefore, observing that, for $1\le p\le M$, we have $\frac{\gamma_p}{j_k}\le\frac{\gamma_1}{j_k}=j_k^{-\alpha}\to 0$, $k\to\infty$, and using the fact that $\log(1-x)\ge -c_{12}\,x$, for $x\in[0,1/2)$, we get that

$$\prod_{p=1}^{M} Q_p\ge \exp\left\{-c_{27}\sum_{p=1}^{M}\left(\gamma_p^{-\varepsilon'/6}+\frac{\gamma_p}{j_k}\right)\right\}, \tag{3.64}$$

where $c_{27}:=c_{12}\,\max\{c_{25},c_{26}\}$.

Recalling that $\sum_{p=1}^{M}\gamma_p^{-\varepsilon'/6}\le c_{28}\,\gamma_M^{-\varepsilon'/6}$ and $\sum_{p=1}^{M}\gamma_p\le c_{29}\,\gamma_1=o(j_k)$ for some $c_{28},c_{29}>0$, (3.64) yields

$$P_z\left\{G(j_k,\ell)\,,\,H(j_k,\ell)\right\}\ge c_{30}\,Q_0\,P_z\left\{H(j_k,\ell)\right\}, \tag{3.65}$$

for some $c_{30}>0$. Observe that Donsker's theorem implies that there exists $c_{31}>0$ such that $\min\{P_z\{H(j_k,\ell)\},Q_0\}\ge c_{31}$. Therefore, assembling (3.63) and (3.65), we get

$$P^+_{k,\ell}\ge c_{32}\,\frac{\gamma_M}{\gamma_0},$$

where $c_{32}:=c_{24}\,c_{30}\,c_{31}^2$.

Recalling (3.55) and $\theta_k=\lfloor\, j_k/\gamma_M\rfloor-1$, we get, uniformly in $z\in I_{j_{k-1}}$,

$$P_z\left\{E^{env}(j_k)\right\}\ge c_{32}\,\theta_k\,\frac{\gamma_M}{\gamma_0}\ge c_{33},$$

for all large k and for some $c_{33}>0$, which concludes the proof of (3.49). □

Part 2

The transient case

CHAPTER 4

A probabilistic representation of constants in Kesten's renewal theorem

The aims of this chapter are twofold. Firstly, we derive some probabilistic representation for the constant which appears in the one-dimensional case of Kesten's renewal theorem. Secondly, we estimate the tail of some related random variable which plays an essential role in the description of the stable limit law of one-dimensional transient sub-ballistic random walks in random environment.

Contents

The material of this chapter is a joint work with N. Enriquez and C. Sabot and has been submitted for publication, see [41].

1. Introduction

In 1973, Kesten published a famous paper [68] about the tail estimates of renewal series of the form $\sum_{i\geq 1} A_1 \dots A_{i-1} B_i$, where $(A_i)_{i\geq 0}$ is a sequence of non-negative i.i.d. $d \times d$ random matrices and $(B_i)_{i\geq 1}$ is a sequence of i.i.d. random vectors of $\mathbb{R}^d$. His result states that the tail of the projection of this random vector on every direction is equivalent to $Ct^{-\kappa}$, when t tends to infinity, where C and κ are positive constants. The constant κ is defined as the solution of the equation $k(s) = 1$, with $k(s) := \lim_{n\to\infty} \mathbb{E}(\| A_1 \dots A_n \|^s)^{1/n}$. The proof of his result in the one-dimensional case, even if it is much easier than in dimension $d \geq 2$, is already rather complicated.

Even if we are concerned by the one-dimensional case in this paper, let us mention that a significant generalization of Kesten's result, in the multi-dimensional case, was recently achieved by de Saporta, Guivarc'h and Le Page [29], who relaxed the assumption of positivity on A_i.

In 1991, Goldie [48] relaxed, in dimension $d = 1$, the assumption of positivity on the A_i and simplified Kesten's proof. Furthermore, he obtained a formula for the implicit constant C in the special case where A_i is non-negative and κ is an integer.

In 1991, Chamayou and Letac [18] observed that, in dimension $d = 1$, if A_i has the same law as $(1 - X_i)/X_i$, with X_i following a Beta distribution on $(0, 1)$, then the law of the series itself is computable so that the constant C is explicit in this special case also. The following question was then asked. How does one effectively compute the constant C?

In our framework, we consider the case $d = 1$ and we make the following assumptions: $\rho_i = A_i$ is a sequence of i.i.d. positive random variables, $B_i = 1$ and there exists κ such that $\mathbb{E}(\rho_1^\kappa) = 1$. Moreover, we assume a weak integrability condition and that the law of $\log \rho_i$, which has a negative expectation by the previous assumptions, is non-arithmetic. In this context we are interested in the random series

$$R = 1 + \sum_{k\geq 1} \rho_1 \cdots \rho_k.$$

The previous assumptions ensure that the tail of the renewal series is equivalent to $C_K t^{-\kappa}$, when t tends to infinity. We are now aiming at finding a probabilistic representation of the constant C_K.

Besides, this work is motivated by the study of one-dimensional random walks in random environment. In [70], Kesten, Kozlov and Spitzer proved, using the tail estimate derived in [68], that when the RWRE is transient with null asymptotic speed, then the behavior depends on an index $\kappa \leq 1$: the RWRE X_n normalized by $n^{1/\kappa}$ converges in law to $C_\kappa \left(\frac{1}{\mathcal{S}_\kappa}\right)^\kappa$ where $\mathcal{S}_\kappa$ is a positive stable random variable with index κ. The computation of the explicit value of C_κ was left open. In [40], the authors derive an explicit expression, either in terms of the Kesten's constant C_K when it is explicit, or in terms of the expectation of a random series when C_K is not explicit. To this end, we need to obtain a tail estimate for a random variable closely related to the random series R, and to relate it to Kesten's constant. This is the other aim of this paper.

The strategy of our proof is based on a coupling argument in the spirit of the coupling argument used to derive the renewal theorem (cf [38], 4.3). We first interpret $\rho_1 \dots \rho_n$ as the exponential of a random walk $(V_n\,,\, n \geq 0)$, which is negatively drifted, since $\mathbb{E}(\log \rho_1) < 0$. We have now to deal with the series $R := \sum_{n\geq 0} \mathrm{e}^{V_n}$.

One can write

$$R = \mathrm{e}^S \sum_{n\geq 0} \mathrm{e}^{V_n - S},$$

where S is the maximum of $(V_n\,,\, n \geq 0)$. The heuristic is that S and $\sum_{n\geq 0} \mathrm{e}^{V_n-S}$ are asymptotically independent. The coupling argument is used to derive this asymptotic independence. But, in order to implement this strategy, several difficulties have to be overcome: we first need to condition S to be large. Moreover, we have to couple conditioned processes: this requires to describe precisely the part of the process $(V_0, \ldots, V_{T_S})$, where T_S is the first hitting time of the level S.

2. Notations and statement of the results

Let $(\rho_i)_{i\in\mathbb{Z}}$ be a sequence of i.i.d. positive random variables with law $Q = \mu^{\otimes\mathbb{Z}}$. With the sequence $(\rho_i)_{i\in\mathbb{Z}}$ we associate the potential $(V_k)_{k\in\mathbb{Z}}$ defined by

$$V_n := \begin{cases} \sum_{k=1}^n \log \rho_k & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -\sum_{k=n+1}^0 \log \rho_k & \text{if } n \leq -1. \end{cases}$$

Let ρ having law μ. Suppose now that the law μ is such that there is $\kappa > 0$ satisfying

$$\mathbb{E}^\mu(\rho^\kappa) = 1 \qquad \text{and} \qquad \mathbb{E}^\mu(\rho^\kappa \log^+ \rho) < \infty. \tag{4.1}$$

Moreover, we assume that the distribution of $\log \rho$ is non-lattice. Then the law μ is such that $\log \rho$ is integrable and that

$$\mathbb{E}^\mu(\log \rho) < 0, \tag{4.2}$$

which implies

$$\lim_{n\to\infty} \frac{V_n}{n} = \int \log \rho \,\mathrm{d}\mu < 0,$$

under Q.

We set

$$S := \max\{V_k,\ k \geq 0\},$$

and

$$H := \max\{V_k, 0 \leq k \leq T_{\mathbb{R}_-}\},$$

where $T_{\mathbb{R}_-}$ is the first positive hitting time of $\mathbb{R}_-$:

$$T_{\mathbb{R}_-} := \inf\{k > 0,\ V_k \leq 0\}.$$

The random variable S is the absolute maximum of the path $(V_k)_{k\geq 0}$ while H is the maximum of the first positive excursion. We also set

$$T_S := \inf\{k \geq 0,\ V_k = S\}, \quad T_H := \inf\{k \geq 0,\ V_k = H\}.$$

We clearly have, Q-almost surely,

$$H \le S < \infty, \quad T_H \le T_S < \infty.$$

The following tail estimate for S is a classical consequence of renewal theory, see [44],

$$\mathbb{P}^Q(\mathrm{e}^S \ge t) \sim C_F t^{-\kappa}, \tag{4.3}$$

when $t \to \infty$, where

$$C_F = \frac{1 - \mathbb{E}^Q(\mathrm{e}^{\kappa V(T_{\mathbb{R}_-})})}{\kappa \mathbb{E}^\mu(\rho^\kappa \log \rho)\mathbb{E}^Q(T_{\mathbb{R}_-})}.$$

The tail estimate of H is derived by Iglehart, in [63],

$$\mathbb{P}^Q(\mathrm{e}^H \ge t) \sim C_I t^{-\kappa}, \tag{4.4}$$

when $t \to \infty$, where

$$C_I = \frac{(1 - \mathbb{E}^Q(\mathrm{e}^{\kappa V(T_{\mathbb{R}_-})}))^2}{\kappa \mathbb{E}^\mu(\rho^\kappa \log \rho)\mathbb{E}^Q(T_{\mathbb{R}_-})} = (1 - \mathbb{E}^Q(\mathrm{e}^{\kappa V(T_{\mathbb{R}_-})}))C_F.$$

Consider now the random variable

$$R := \sum_{n=0}^{\infty} \mathrm{e}^{V_n}.$$

This random variable clearly satisfies the following renewal equation

$$R \overset{\text{law}}{=} 1 + \rho R,$$

where ρ is a random variable with law μ independent of R. In [68], Kesten proved (actually his result was more general and concerned by the multidimensional version of this one) that there exists a positive constant C_K such that

$$\mathbb{P}^Q(R \ge t) \sim C_K t^{-\kappa}, \tag{4.5}$$

when $t \to \infty$. The constant C_K has been made explicit in some particular cases: for κ integer by Goldie, see [48], and when $\rho \overset{\text{law}}{=} \frac{W}{1-W}$ where W is a beta variable, by Chamayou and Letac [18]. One aim of this paper is to derive an expression of this constant in terms of the expectation of some simple random variable.

We need now to introduce some Girsanov transform of Q. Thanks to (4.1) we can define the law

$$\tilde{\mu} = \rho^\kappa \mu,$$

and the law $\tilde{Q} = \tilde{\mu}^{\otimes \mathbb{Z}}$ which is the law of a sequence of i.i.d. random variables with law $\tilde{\mu}$. The definition of κ implies that

$$\int \log \rho \, \tilde{\mu}(\,\mathrm{d}\rho) > 0,$$

and thus that

$$\lim_{n\to\infty}\frac{V_n}{n}=\int\log\rho\,\mathrm{d}\tilde\mu>0,$$

under $\tilde Q$. Moreover, $\tilde Q$ is a Girsanov transform of Q, i.e. we have for all n

$$\mathbb{E}^Q\left(\phi(V_0,\dots,V_n)\right)=\mathbb{E}^{\tilde Q}\left(e^{-\kappa V_n}\phi(V_0,\dots,V_n)\right),$$

for any bounded test function ϕ. Let us now introduce the random variable M defined by

$$M=\sum_{k<0}\mathrm{e}^{-V_k}+\sum_{k\geq 0}\mathrm{e}^{-V_k}, \tag{4.6}$$

where $(V_k)_{k<0}$ is distributed under $Q(\cdot|V_k\geq 0,\forall k<0)$ and independent of $(V_k)_{k\geq 0}$ which is distributed under $\tilde Q(\cdot|V_k>0,\forall k>0)$.

THEOREM 4.1. *i) We have the following tail estimate*

$$\mathbb{P}^Q(R\geq t)\sim C_K t^{-\kappa},$$

when $t\to\infty$, *where*

$$C_K=C_F\mathbb{E}(M^\kappa).$$

ii) We have

$$\mathbb{P}^Q(R\geq t\,;\,H=S)\sim C_{KI}t^{-\kappa},$$

when $t\to\infty$, *where*

$$C_{KI}:=C_I\mathbb{E}(M^\kappa).$$

Remark 4.1 : The conditioning $H=S$ means that the path $(V_k)_{k\geq 0}$ never goes above the height of its first excursion. This conditioning appears naturally in [40], and is useful to return the path, cf Section 4.

In [40], we need some tail estimate on a random variable of the type of R but with an extra term. Let us introduce the event

$$\mathcal{I}:=\{H=S\}\cap\{V_k\geq 0\,,\,\forall k\leq 0\}, \tag{4.7}$$

and the random variable

$$Z:=\mathrm{e}^S M_1 M_2,$$

where

$$M_1:=\sum_{k=-\infty}^{T_S}\mathrm{e}^{-V_k},$$

$$M_2:=\sum_{k=0}^{\infty}\mathrm{e}^{V_k-S}.$$

Theorem 4.2. *We have the following tail estimate*

$$\mathbb{P}^Q(Z \geq t|\mathcal{I}) \sim \frac{1}{\mathbb{P}^Q(H = S)} C_U t^{-\kappa},$$

when $t \to \infty$*, where*

$$C_U = C_I \mathbb{E}(M^\kappa)^2 = \frac{C_I}{C_F}(C_K)^2.$$

Remark 4.2 : The factor $1/\mathbb{P}^Q(H = S)$ in the last formula just comes from the fact that the event $\{H = S\}$ is now in the conditioning. But this term will not play any role in [40].

Remark 4.3 : In [40] we need to evaluate C_U: when C_K is not explicit it is better to use the expression of C_U in terms of $\mathbb{E}(M^\kappa)$ which is easy to evaluate numerically. When C_K is explicit, we use the expression in terms of C_K.

Let us now discuss the case where the B_i's are not necessarily equal to 1. Let $(B_i)_{i\geq 0}$ be a sequence of positive i.i.d. random variables, which is independent of the sequence $(\rho_i)_{i\geq 0}$, and denote by R^B the random series $R^B := B_0 + \sum_{k\geq 1} B_k \rho_1 \cdots \rho_k$. The result of Theorem 4.1, i), is then generalized into the following result.

Theorem 4.3. *If there exists* $\varepsilon > 0$ *such that* $\mathbb{E}(|B_1|^{\kappa+\varepsilon}) < \infty$*, then*

$$\mathbb{P}^Q(R^B \geq t) \sim C_{KB} t^{-\kappa},$$

when $t \to \infty$*, where*

$$C_{KB} = C_F \mathbb{E}((M^B)^\kappa)$$

and where M^B *is defined by*

$$M^B = \sum_{k<0} \mathrm{e}^{-V_k} \tilde{B}_k + \sum_{k\geq 0} \mathrm{e}^{-V_k} \tilde{B}_k,$$

with $(V_k)_{k<0}$ *distributed under* $Q(\cdot|V_k \geq 0, \forall k < 0)$ *and independent of* $(V_k)_{k\geq 0}$ *which is distributed under* $\tilde{Q}(\cdot|V_k > 0, \forall k > 0)$ *while* $(\tilde{B}_k)_{k\in\mathbb{Z}}$ *is a sequence of i.i.d. random variables having the same distribution as* B_1 *and independent of* $(V_k)_{k\in\mathbb{Z}}$.

Sketch of the proof and organization of the paper

The intuition behind Theorem 4.1 and Theorem 4.2 is the following. Let us first consider $\mathbb{P}^Q(R \geq t|H = S)$. The law $Q(\cdot|\mathcal{I})$ has a symmetry property which implies that the variable $R = M_2 \mathrm{e}^H$ has the same distribution as $M_1 \mathrm{e}^H$ (cf Section 4). Then, the proof of Theorem 4.1 is based on the following arguments.

Firstly, we prove that the variables M_1 and e^H are asymptotically independent. To this end, we use a delicate coupling argument which works only when H is conditioned to be large. Therefore, we need to restrict ourselves to large values of

H. To this end, we need to control the value of R conditioned by H; this is done in Section 6. Then, a second difficulty is that we have to couple conditioned processes (namely, the process (V_k) conditioned to have a first high excursion). We overcome this difficulty by using an explicit description of the law of the path $(V_0, \dots, V_{T_H})$. Namely, the path $(V_0, \dots, V_{T_H})$ behaves like V under $\tilde{Q}(\cdot | V_k > 0, \forall k > 0)$ stopped at some random time.

Secondly, we observe that the distribution of M_1 is close to the distribution of M as a consequence of the above description of the law of $(V_0, \dots, V_{T_H})$.

From these two facts, we deduce that $\mathbb{P}^Q(R \geq t \mid \mathcal{I}) \simeq \mathbb{P}^Q(M e^H \geq t \mid \mathcal{I})$, where M and H are roughly independent. Using the tail estimate for H we get the part ii) of Theorem 4.1. For Theorem 4.2, we proceed similarly: the variable Z writes $M_1 R$ and, for large H, the variables M_1 and R are asymptotically independent and the law of M_1 is close to the law of M. Then the estimate on the tail of R allows to conclude.

Let us now describe the organization of the proofs. In Section 3, we prove that M has finite moments of all orders and we estimate the rest of the series M. Section 4 contains some preliminary properties of the law $Q(\cdot|\mathcal{I})$, and Section 5 presents a representation of the law of the process $(V_0, \dots, V_S)$ in terms of the law $\tilde{Q}$. Section 6 contains crucial estimates which will allow to restrict ourselves to large values of H. In Section 7, we detail the coupling arguments which roughly give the asymptotic independence of M_1 and $e^H M_2$. Finally, in Section 8 we compile the arguments of the previous sections to prove Theorem 4.2 and Theorem 4.1. In Section 9, we present a Tauberian version of these tail estimates, which is the version we ultimately use in [40].

3. Moments of M

Here is a series of three lemmas about the moment of the exponential functional of the meanders. In this section, we denote by $\{V \geq -L\}$ the event $\{V_k \geq -L,\, k \geq 0\}$.

Lemma 4.1. *There exists $C > 0$ such that, for all $L \geq 0$,*

$$\mathbb{E}^{\tilde{Q}}\Big(\sum_{k \geq 0} e^{-V_k} \,|\, V \geq -L\Big) \leq C\, e^L.$$

Proof. Using Markov inequality, we get

$$\mathbb{E}^{\tilde{Q}}\Big(\sum_{k \geq 0} e^{-V_k} \,|\, V \geq -L\Big) \leq \sum_{k \geq 1} \frac{1}{k^2} + \sum_{k \geq 1} \tilde{Q}\Big(e^{-V_k} \geq \frac{1}{k^2} \,|\, V \geq -L\Big) e^L.$$

Since $\mathbb{P}^{\tilde{Q}}(V \geq -L) \geq \mathbb{P}^{\tilde{Q}}(V \geq 0) > 0$, for all $L \geq 0$,

$$\tilde{Q}\Big(e^{-V_k} \geq \frac{1}{k^2} \,|\, V \geq -L\Big) = \tilde{Q}(V_k \leq 2 \log k \,|\, V \geq -L) \leq C\, \tilde{Q}(V_k \leq 2 \log k).$$

Now, since large deviations do occur, we get, from Cramer's theory, see [33], that $\mathbb{E}^{\tilde{Q}}(\log \rho_0) > 0$ implies that the sequence $\tilde{Q}(V_k \leq 2 \log k)$ is exponentially decreasing.

The sum $\sum_{k \geq 1} \tilde{Q}\big(\mathrm{e}^{-V_k} \geq \frac{1}{k^2} \,|\, V \geq -L\big)$ is therefore bounded uniformly in L, and the result follows. □

Lemma 4.2. *Under* $\tilde{Q}^{\geq 0} := \tilde{Q}(\cdot \,|\, V_k \geq 0\,,\, \forall k \geq 0)$, *all the moments of* $\sum_{k \geq 0} \mathrm{e}^{-V_k}$ *are finite.*

Proof. Let us treat, for more readability, the case of the second moment. Observe first that

$$\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\Big(\sum_{i \geq 0} \mathrm{e}^{-V_i}\Big)^2\right) \leq 2\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i \geq 0} \mathrm{e}^{-2V_i}\Big(\sum_{j \geq i} \mathrm{e}^{-(V_j - V_i)}\Big)\right).$$

Applying the Markov property to the process V under $\tilde{Q}$ at time i, we get

$$\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\Big(\sum_{i \geq 0} \mathrm{e}^{-V_i}\Big)^2\right) \leq 2\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i \geq 0} \mathrm{e}^{-2V_i}\mathbb{E}^{\tilde{Q}}\Big[\sum_{l \geq 0} \mathrm{e}^{-V'_l} \,|\, V' \geq -V_i\Big]\right),$$

where V' is a copy of V independent of $(V_k)_{0 \leq k \leq i}$. Now, we use Lemma 4.1 to get the upper bound

$$C\mathbb{E}^{\tilde{Q}^{\geq 0}}\Big(\sum_{i \geq 0} \mathrm{e}^{-2V_i} \times \mathrm{e}^{V_i}\Big) \leq C\mathbb{E}^{\tilde{Q}^{\geq 0}}\Big(\sum_{i \geq 0} \mathrm{e}^{-V_i}\Big),$$

which is finite, again by applying Lemma 4.1. This scheme is then easily extended to higher moments. □

We will need further a finer result than Lemma 4.1, we state here.

Lemma 4.3. *There exists* $C > 0$ *such that, for all* $L > 0$ *and for all* $\varepsilon' > 0$, *we have*

- *if* $\kappa < 1$,
$$\mathbb{E}^{\tilde{Q}}\Big(\sum_{i \geq 0} \mathrm{e}^{-V(i)} \,|\, V \geq -L\Big) \leq C\mathrm{e}^{(1-\kappa+\varepsilon')L},$$
- *if* $\kappa \geq 1$,
$$\mathbb{E}^{\tilde{Q}}\Big(\sum_{i \geq 0} \mathrm{e}^{-V(i)} \,|\, V \geq -L\Big) \leq C\mathrm{e}^{\varepsilon' L}.$$

Proof. Let $\alpha \in [0,1]$ and define $T_{(-\infty,-\alpha L]} := \min\{i \geq 0 : V_i \leq -\alpha L\}$. Let us write

$$\sum_{i \geq 0} \mathrm{e}^{-V_i} = \Big(\sum_{i \geq 0} \mathrm{e}^{-V_i}\Big)\mathbf{1}_{\{V > -\alpha L\}} + \Big(\sum_{i=0}^{T_{(-\infty,-\alpha L]}-1} \mathrm{e}^{-V_i} + \sum_{i=T_{(-\infty,-\alpha L]}}^{\infty} \mathrm{e}^{-V_i}\Big)\mathbf{1}_{\{T_{(-\infty,-\alpha L]} < \infty\}}.$$

Now, since $\tilde{Q}(V \geq -A)$ is uniformly bounded by below, for $A > 0$, by $\tilde{Q}(V > 0) > 0$, we obtain that $\mathbb{E}^{\tilde{Q}}\big(\sum_{i\geq 0} \mathrm{e}^{-V_i} \,|\, V \geq -L\big)$ is less or equal than

$$C\mathbb{E}^{\tilde{Q}}\big(\sum_{i\geq 0} \mathrm{e}^{-V_i} \,|\, V \geq -\alpha L\big) \tag{4.8}$$
$$+ \quad C\mathbb{E}^{\tilde{Q}}\bigg(\sum_{i<T_{(-\infty,-\alpha L]}} \mathrm{e}^{-V_i} \,;\, T_{(-\infty,-\alpha L]} < \infty \,;\, V \geq -L\bigg)$$
$$+ \quad C\mathbb{E}^{\tilde{Q}}\bigg(\sum_{i\geq T_{(-\infty,-\alpha L]}} \mathrm{e}^{-V_i} \,;\, T_{(-\infty,-\alpha L]} < \infty \,;\, V \geq -L\bigg).$$

Lemma 4.1 bounds the first term in (4.8) from above by $C\mathrm{e}^{\alpha L}$. Furthermore, $i < T_{(-\infty,-\alpha L]}$ implies $\mathrm{e}^{-V_i} \leq \mathrm{e}^{\alpha L}$. Therefore, $C\mathrm{e}^{\alpha L}\mathbb{E}^{\tilde{Q}}(T_{(-\infty,-\alpha L]}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}})$ is an upper bound for the second term in (4.8), which is treated as follows,

$$\begin{aligned}
\mathbb{E}^{\tilde{Q}}\big(T_{(-\infty,-\alpha L]}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}\big) &\leq \sum_{k\geq 0} k\tilde{Q}\big(T_{(-\infty,-\alpha L]} = k\big)\\
&\leq \sum_{k\geq 0} k\tilde{Q}\big(V_k \leq -\alpha L\big)\\
&\leq \sum_{k\geq 0} k\mathrm{e}^{-k\theta\tilde{I}(-\frac{\alpha L}{k})}\mathrm{e}^{-k(1-\theta)\tilde{I}(-\frac{\alpha L}{k})},
\end{aligned}$$

where $0 < \theta < 1$ and $\tilde{I}$ denotes the rate function associated with $\tilde{P}$ which is positive convex and admits a unique minimum on $\mathbb{R}_+$. We can therefore bound by below all the terms $\tilde{I}(-\frac{\alpha h}{k})$ by $\tilde{I}(0) > 0$. Moreover, a more sophisticated result yields $\sup_{x\leq 0} \tilde{I}(x)/x \leq -\kappa$ (see definition of κ and formula (2.2.10) in ([33], p. 28)). Therefore, we obtain

$$\mathbb{E}^{\tilde{Q}}\big(T_{(-\infty,-\alpha L]}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}\big) \leq \mathrm{e}^{-\theta\kappa\alpha L}\sum_{k\geq 0} k\mathrm{e}^{-k(1-\theta)\tilde{I}(0)} \leq C\mathrm{e}^{-\theta\kappa\alpha L}.$$

As a result, the second term in (4.8) is bounded by $C\mathrm{e}^{(1-\theta\kappa\alpha)L}$.

Finally, concerning the third term in (4.8), we have that

$$\begin{aligned}
& C\,\mathbb{E}^{\tilde{Q}}\bigg(\sum_{i\geq T_{(-\infty,-\alpha L]}} \mathrm{e}^{-V_i} \,;\, T_{(-\infty,-\alpha L]} < \infty \,;\, V \geq -L\bigg)\\
\leq\; & C\,\mathbb{E}^{\tilde{Q}}\bigg(\mathrm{e}^{-V_{T_{(-\infty,-\alpha L]}}}\sum_{i\geq T_{(-\infty,-\alpha L]}} \mathrm{e}^{-(V_i - V_{T_{(-\infty,-\alpha L]}})} \,;\, T_{(-\infty,-\alpha L]} < \infty \,;\, V \geq -L\bigg)\\
\leq\; & C\,\mathbb{E}^{\tilde{Q}}\bigg(\mathrm{e}^{-V_{T_{(-\infty,-\alpha L]}}}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}\mathbb{E}^{\tilde{Q}}\Big(\sum_{i\geq 0} \mathrm{e}^{-V'_i} \,|\, V' \geq -(L + V_{T_{(-\infty,-\alpha L]}})\Big)\bigg),
\end{aligned}$$

where $V'_i := V_{T_{(-\infty,-\alpha L]}+i} - V_{T_{(-\infty,-\alpha L]}}$, for $i \geq 0$. The last inequality is a consequence of the strong Markov property applied at $T_{(-\infty,-\alpha L]}$, which implies that $(V'_i,\, i \geq 0)$ is a copy of $(V_i,\, i \geq 0)$ independent of $(V_i,\, 0 \leq i \leq T_{(-\infty,-\alpha L]})$. Then, Lemma 4.1

yields that the third term in (4.8) is less than

$$\begin{aligned} & C\mathbb{E}^{\tilde{Q}}\left(\mathrm{e}^{-V_{T_{(-\infty,-\alpha L]}}}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}\mathrm{e}^{L+V_{T_{(-\infty,-\alpha L]}}}\right) \\ \leq \quad & C\mathrm{e}^{L}\tilde{Q}(T_{(-\infty,-\alpha L]}<\infty)\leq C\mathrm{e}^{(1-\kappa\alpha)L}. \end{aligned}$$

Since $\theta < 1$ implies $1-\theta\kappa\alpha \geq 1-\kappa\alpha$, we optimize the value of α by taking $\alpha = -\alpha\kappa\theta + 1$, i.e. $\alpha = 1/(1+\kappa\theta)$. As a result, we get already a finer result than Lemma 4.1 with a bound $\mathrm{e}^{\frac{L}{1+\kappa\theta}}$ instead of e^{L}.

Now, the strategy is to use this ameliorated estimation instead of Lemma 4.1 and repeat the same procedure. Like that, we obtain recursively a sequence of bounds, we denote by $C\mathrm{e}^{u_n L}$. The first term in (4.8) is bounded by $C\mathrm{e}^{\alpha u_n L}$ whereas the second and the third term are still bounded respectively by $C\mathrm{e}^{(1-\kappa\alpha\theta)L}$ and $C\mathrm{e}^{(1-\kappa\alpha)L}$.

Optimizing in α again, one chooses $\alpha u_n = -\alpha\kappa\theta+1$, i.e. $\alpha = \frac{1}{u_n+\kappa\theta}$. The new exponent is therefore $u_{n+1} = \alpha u_n = \frac{u_n}{u_n+\kappa\theta}$. Thus, the sequence u_n is homographic and converges to a limit satisfying $l = \frac{l}{l+\kappa\theta}$. For $\kappa\theta \leq 1$, the limit is $l = 1-\kappa\theta$ and for $\kappa\theta \geq 1$, the limit is 0. Since this result holds for any $0<\theta<1$, it concludes the proof of Lemma 4.3. □

Remark 4.4 : Analogous results as in Lemma 4.1, Lemma 4.2 and Lemma 4.3 apply for $\sum_{k\geq 0}\mathrm{e}^{V_k}$ under Q and conditionally on the event $\{V_k \leq L\,,\,\forall k\geq 0\}$.

4. A time reversal

Let us denote by $Q^{\mathcal{I}}$ the conditional law $Q^{\mathcal{I}}(\cdot) := Q(\cdot|\mathcal{I})$, where $\mathcal{I}$ is defined in (4.7). The law $Q^{\mathcal{I}}$ has the following symmetry property.

Lemma 4.4. *Under $Q^{\mathcal{I}}$ we have the following equality in law*

$$(V_k)_{k\in\mathbb{Z}} \overset{law}{=} (V_{T_H} - V_{T_H-k})_{k\in\mathbb{Z}}.$$

Proof. Let ϕ be a positive test function. We have

$$\begin{aligned} & \mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_{T_H}-V_{T_H-k})_{k\geq 0})) \\ = \quad & \sum_{p=0}^{\infty}\mathbb{E}^{Q^{\mathcal{I}}}(\mathbf{1}_{T_H=p}\phi((V_p-V_{p-k})_{k\geq 0})) \\ = \quad & \frac{1}{\mathbb{P}^Q(\mathcal{I})}\sum_{p=0}^{\infty}\mathbb{E}^{Q}(\mathbf{1}_{\{V_k\geq 0\,,\,\forall k\leq 0\}}\mathbf{1}_{\{V_k\leq V_p\,,\,\forall k\geq p\}}\mathbf{1}_{\{0<V_k<V_p\,,\,\forall 0<k<p\}}\phi((V_p-V_{p-k})_{k\geq 0})). \end{aligned}$$

By construction we have, for all $p\geq 0$,

$$(V_p - V_{p-k})_{k\in\mathbb{Z}} \overset{\text{law}}{=} (V_k)_{k\in\mathbb{Z}}.$$

This implies that

$$\begin{aligned}
&\mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_{T_H}-V_{T_H-k})_{k\geq 0}))\\
=\ &\frac{1}{\mathbb{P}^Q(\mathcal{I})}\sum_{p=0}^{\infty}\mathbb{E}^Q(\mathbf{1}_{\{V_k\geq 0\,,\forall k\leq 0\}}\mathbf{1}_{\{V_k\leq V_p\,,\forall k\geq p\}}\mathbf{1}_{\{0<V_k<V_p\,,\forall 0<k<p\}}\phi((V_k)_{k\geq 0}))\\
=\ &\mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_k)_{k\in\mathbb{Z}})).
\end{aligned}$$

□

This implies that under $Q^{\mathcal{I}}$, R has the law of $\mathrm{e}^H M_1$. This last formula will be useful since the asymptotic independence of e^H and M_1, in the limit of large H, is more visible than the asymptotic independence of H and M_2 and will be easier to prove.

5. The two faces of the mountain

It will be convenient to introduce the following notations: we denote by $Q^{\leq 0}$ the conditional law

$$Q^{\leq 0}(\cdot)=Q(\cdot|V_k\leq 0,\ \forall k\geq 0),$$

and by $\tilde{Q}^{>0}$ the conditional law

$$\tilde{Q}^{>0}(\cdot)=\tilde{Q}(\cdot|V_k>0,\ \forall k>0).$$

It will be useful to describe the law of the part of the path $(V_0,\ldots,V_{T_S})$. Let us introduce some notations. If $(Y_k)_{k\geq 0}$ is a random process under the law $\tilde{Q}$, then $Y_k\to+\infty$ a.s. and we can define its strict increasing ladder times $(e_k)_{k\geq 0}$ by: $e_0:=0$, and

$$e_{k+1}:=\inf\{n>e_k,\ Y_n>Y_{e_k}\}.$$

We define a random variable $((Y_k)_{k\geq 0},\Theta)$ with values in $\mathbb{R}^{\mathbb{N}}\times\mathbb{N}$ as follows: the random process $(Y_k)_{k\geq 0}$ has a law with density with respect to $\tilde{Q}$ given by

$$\frac{1}{\mathcal{Z}}\Big(\sum_{k=0}^{\infty}\mathrm{e}^{-\kappa Y_{e_k}}\Big)\tilde{Q},$$

where $\mathcal{Z}$ is the normalizing constant given by

$$\mathcal{Z}=\frac{1}{1-\mathbb{E}^{\tilde{Q}}(\mathrm{e}^{-\kappa Y_{e_1}})}.$$

Then, conditionally on $(Y_k)_{k\geq 0}$, Θ takes one of the value of the strict ladder times with probability

$$\mathbb{P}(\Theta=e_p\mid\sigma((Y_k)_{k\geq 0}))=\frac{\mathrm{e}^{-\kappa Y_{e_p}}}{\sum_{k=0}^{\infty}\mathrm{e}^{-\kappa Y_{e_k}}}.$$

We denote by $\hat{Q}$ the law of $((Y_k)_{k\geq 0}, \Theta)$. Otherwise stated, it means that, for all test function ϕ,

$$\mathbb{E}^{\hat{Q}}(\phi(\Theta, (Y_n)_{n\geq 0})) = \frac{1}{\mathcal{Z}}\mathbb{E}^{\tilde{Q}}\Big(\sum_{k=0}^{\infty} \mathrm{e}^{-\kappa Y_{e_k}}\phi(e_k, (Y_n)_{n\geq 0})\Big).$$

Lemma 4.5. *The processes* $(V_0, \ldots, V_{T_S})$ *and* $(V_{T_{S+k}} - V_{T_S})_{k\geq 0}$ *are independent and have the following laws:* $(V_{T_{S+k}} - V_{T_S})_{k\geq 0}$ *has the law* $Q^{\leq 0}$ *and*

$$(V_0, \ldots, V_{T_S}) \overset{law}{=} (Y_0, \ldots, Y_\Theta),$$

where $((Y_k)_{k\geq 0}, \Theta)$ *has the law* $\hat{Q}$.

Proof. Let ψ, θ be positive test functions. Let us compute

$$\begin{aligned}
& \mathbb{E}^{Q}\left(\psi((V_{T_S+k} - V_{T_S})_{k\geq 0})\theta((V_0, \ldots, V_{T_S}))\right) \\
= & \sum_{p=0}^{\infty} \mathbb{E}^{Q}\left(\mathbf{1}_{T_S=p}\psi((V_{p+k} - V_p)_{k\geq 0})\theta((V_0, \ldots, V_p))\right) \\
= & \sum_{p=0}^{\infty} \mathbb{E}^{Q}\left(\mathbf{1}_{\{V_k<V_p\,,\,\forall 0\leq k<p\}}\mathbf{1}_{\{V_k\leq V_p\,,\,\forall k\geq p\}}\psi((V_{p+k} - V_p)_{k\geq 0})\theta((V_0, \ldots, V_p))\right) \\
= & \sum_{p=0}^{\infty} \mathbb{E}^{Q}\left(\mathbf{1}_{\{V_k<V_p\,,\,\forall k<p\}}\theta((V_0, \ldots, V_p))\right) \mathbb{E}^{Q}\left(\mathbf{1}_{\{V_k\leq 0\,,\,\forall k\geq 0\}}\psi((V_k)_{k\geq 0})\right),
\end{aligned}$$

using Markov property at time p. The second term is equal to

$$\mathbb{P}^{Q}(V_k \leq 0\,,\, \forall k \geq 0)\, \mathbb{E}^{Q^{\leq 0}}\left(\psi((V_k)_{k\geq 0})\right).$$

Let us now consider only the first term. Using the Girsanov property of Q and $\tilde{Q}$ we get

$$\begin{aligned}
\sum_{p=0}^{\infty} \mathbb{E}^{Q}\left(\mathbf{1}_{\{V_k<V_p\,,\,\forall k<p\}}\theta((V_0, \ldots, V_p))\right) &= \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}\left(\mathbf{1}_{\{V_k<V_p\,,\,\forall k<p\}}\mathrm{e}^{-\kappa V_p}\theta((V_0, \ldots, V_p))\right) \\
&= \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}\left(\mathrm{e}^{-\kappa V_{e_p}}\theta((V_0, \ldots, V_{e_p}))\right),
\end{aligned}$$

where $(e_p)_{p\geq 0}$ are the strict increasing ladder times of $(V_k\,,\, k \geq 0)$ as defined above. The last formula is exactly the one we need, and also imply that

$$\frac{1}{\mathcal{Z}} = \mathbb{P}^{Q}(V_k \leq 0\,,\, \forall k \geq 0) = 1 - \mathbb{E}^{\tilde{Q}}(\mathrm{e}^{-\kappa V_{e_1}}),$$

(which can also be obtained directly). □

Denote now by $\hat{Q}^{>0}$ the law

$$\hat{Q}^{>0} = \hat{Q}(\cdot \mid Y_k > 0, \forall k > 0).$$

We will need the following result.

LEMMA 4.6. *There exists a positive constant $c > 0$ such that, for all positive test function ψ,*

$$\mathbb{E}^{Q^{\mathcal{I}}}(\psi(V_0, \ldots, V_{T_H})) \leq c\mathbb{E}^{\hat{Q}^{>0}}(\psi(Y_0, \ldots, Y_\Theta)).$$

PROOF. Let Ψ be a positive test function. Thank to the previous lemma, we have

$$\begin{aligned}
& \mathbb{E}^{Q^{\mathcal{I}}}(\Psi(V_0, \ldots, V_{T_H})) \\
= \quad & \frac{1}{\mathbb{P}^Q(H=S)} \mathbb{E}^Q(\mathbf{1}_{H=S} \Psi(V_0, \ldots, V_{T_H})) \\
= \quad & \frac{1}{\mathcal{Z}\mathbb{P}^Q(H=S)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}(\mathbf{1}_{Y_k > 0\,,\, \forall 0 < k \leq e_p} \mathrm{e}^{-\kappa Y_{e_p}} \Psi(Y_0, \ldots, Y_{e_p})) \\
= \quad & \frac{1}{\mathcal{Z}\mathbb{P}^Q(H=S)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}(\mathbf{1}_{Y_k > 0\,,\, \forall k > 0} \frac{1}{\mathbb{P}^{\tilde{Q}}(V_k > -Y_{e_p}\,,\, \forall k > 0)} \mathrm{e}^{-\kappa Y_{e_p}} \Psi(Y_0, \ldots, Y_{e_p})) \\
\leq \quad & \frac{1}{\mathcal{Z}\mathbb{P}^Q(H=S)\mathbb{P}^{\tilde{Q}}(V_k > 0\,,\, \forall k > 0)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}(\mathbf{1}_{Y_k > 0\,,\, \forall k > 0} \mathrm{e}^{-\kappa Y_{e_p}} \Psi(Y_0, \ldots, Y_{e_p})) \\
\leq \quad & \frac{1}{\mathbb{P}^Q(H=S)\mathbb{P}^{\tilde{Q}}(V_k > 0\,,\, \forall k > 0)} \mathbb{E}^{\hat{Q}^{>0}}(\Psi(Y_0, \ldots, Y_\Theta)),
\end{aligned}$$

using the Markov property at time e_p in the fourth line. This is exactly what we want. □

6. A preliminary estimate

To derive the tail estimate of R or Z we need to restrict to large values of H: this will be possible, thank to the following estimate.

LEMMA 4.7. *For all $\eta > 0$ there exists a positive constant c_η such that*

$$\mathbb{E}^{Q^{\mathcal{I}}}((M_1)^\eta \mid \lfloor H \rfloor) \leq c_\eta, \quad Q^{\mathcal{I}}\text{- a.s.},$$

where $\lfloor H \rfloor$ is the integer part of H.

PROOF. Since $(V_k)_{k \leq 0}$ is independent of H under $Q^{\mathcal{I}}$, we have, for all $p \in \mathbb{N}$,

$$\mathbb{E}^{Q^{\mathcal{I}}}((M_1)^\eta \mid \lfloor H \rfloor = p) \leq 2^\eta \left(\mathbb{E}^{Q^{\leq 0}}\left((\sum_{k=0}^{\infty} \mathrm{e}^{V_k})^\eta \right) + \mathbb{E}^{Q^{\mathcal{I}}}\left((\sum_{k=0}^{T_H} \mathrm{e}^{-V_k})^\eta \mid \lfloor H \rfloor = p \right) \right).$$

The first term on the left-hand side is finite for all $\eta > 0$ as proved in Section 3. Consider now the last term. Using Lemma 4.6, we get

$$\begin{aligned}
&\mathbb{E}^{Q^{\mathcal{I}}}\Big((\sum_{k=0}^{T_H} \mathrm{e}^{-V_k})^\eta \mid \lfloor H \rfloor = p\Big) \\
\leq\ & \frac{c}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\hat{Q}^{>0}}\Big((\sum_{k=0}^{T_H} \mathrm{e}^{-Y_k})^\eta \mathbf{1}_{\lfloor H \rfloor = p}\Big) \\
\leq\ & \frac{c'}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\Big((\sum_{k=0}^{\infty} \mathrm{e}^{-\kappa Y_{e_k}} \mathbf{1}_{Y_{e_k} \in [p,p+1[}) (\sum_{j=0}^{\infty} \mathrm{e}^{-V_j})^\eta\Big).
\end{aligned}$$

Now, using Cauchy-Schwarz inequality in the last expression, we get

$$\begin{aligned}
&\mathbb{E}^{Q^{\mathcal{I}}}\Big((\sum_{k=0}^{T_H} \mathrm{e}^{-V_k})^\eta \mid \lfloor H \rfloor = p\Big) \\
\leq\ & \frac{c'}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\Big((\sum_{k=0}^{\infty} \mathrm{e}^{-\kappa Y_{e_k}} \mathbf{1}_{Y_{e_k} \in [p,p+1[})^2\Big)^{\frac{1}{2}} \mathbb{E}^{\tilde{Q}^{>0}}\Big((\sum_{k=0}^{\infty} \mathrm{e}^{-V_k})^{2\eta}\Big)^{\frac{1}{2}} \\
\leq\ & \frac{c'\mathrm{e}^{-\kappa p}}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\Big((\sum_{k=0}^{\infty} \mathbf{1}_{Y_{e_k} \in [p,p+1[})^2\Big)^{\frac{1}{2}} \mathbb{E}^{\tilde{Q}^{>0}}\Big((\sum_{k=0}^{\infty} \mathrm{e}^{-V_k})^{2\eta}\Big)^{\frac{1}{2}}.
\end{aligned}$$

But the last term is independent of p and finite by Lemma 4.2. On the other hand, since $\tilde{Q}(V_k > 0\,,\, \forall k > 0) > 0$ and from Markov property, we obtain

$$\mathbb{E}^{\tilde{Q}^{>0}}\Big((\sum_{k=0}^{\infty} \mathbf{1}_{Y_{e_k} \in [p,p+1[})^2\Big) \leq c\mathbb{E}^{\tilde{Q}}\Big((\sum_{k=0}^{\infty} \mathbf{1}_{Y_{e_k} \in [p,p+1[})^2\Big) \leq c\mathbb{E}^{\tilde{Q}}\Big((\sum_{k=0}^{\infty} \mathbf{1}_{Y_{e_k} \in [0,1[})^2\Big),$$

which is finite since $(Y_k)_{k\geq 0}$ has a positive drift under $\tilde{Q}$. Finally, using the tail estimate on H, we know that

$$\begin{aligned}
\lim_{p\to\infty} \mathrm{e}^{\kappa p}\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p) &= \lim_{p\to\infty} \mathrm{e}^{\kappa p}\Big(\mathbb{P}^{Q^{\mathcal{I}}}(H \geq p) - \mathbb{P}^{Q^{\mathcal{I}}}(H \geq p+1)\Big) \\
&= C_I(1 - \mathrm{e}^{-\kappa}).
\end{aligned} \tag{4.9}$$

Hence, $(\mathrm{e}^{\kappa p}\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p))^{-1}$ is a bounded sequence (we do not have to consider the cases where eventually $\mathbb{P}(\lfloor H \rfloor = p) = 0$ since it is a conditioning by an event of null probability which can be omitted). □

COROLLARY 4.1. *We have, $Q^{\mathcal{I}}$-almost surely,*

$$\mathbb{E}^{Q^{\mathcal{I}}}(Z \mid \lfloor H \rfloor) \leq \mathrm{e}c_2\mathrm{e}^{\lfloor H \rfloor}.$$

PROOF. We have $Z = M_1 M_2 \mathrm{e}^H$. Using Cauchy-Schwarz inequality and Lemma 4.7 we get

$$\mathbb{E}^{Q^{\mathcal{I}}}(Z \mid \lfloor H \rfloor) \leq \mathrm{e}^{\lfloor H \rfloor + 1}\Big(\mathbb{E}^{Q^{\mathcal{I}}}((M_1)^2 \mid \lfloor H \rfloor)\mathbb{E}^{Q^{\mathcal{I}}}((M_2)^2 \mid \lfloor H \rfloor)\Big)^{\frac{1}{2}}$$

$$\leq \quad \mathrm{e}c_2\mathrm{e}^{\lfloor H \rfloor},$$

since M_1 and M_2 have the same law under $Q^{\mathcal{I}}$. □

COROLLARY 4.2. *Let $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a function such that*

$$\lim_{t\to\infty} t^{-1}\mathrm{e}^{h(t)} = 0.$$

Then, we have

$$\mathbb{P}^{Q^{\mathcal{I}}}(R \geq t,\ H \leq h(t)) = o(t^{-\kappa}),$$
$$\mathbb{P}^{Q^{\mathcal{I}}}(Z \geq t,\ H \leq h(t)) = o(t^{-\kappa}),$$

when t tends to infinity.

PROOF. Let us do the proof for Z. Let η be a positive real such that

$$\eta > \kappa.$$

We have (all expectations are relative to the measure $Q^{\mathcal{I}}$; so, to simplify the reading, we remove the reference to $Q^{\mathcal{I}}$ in the following)

$$\begin{aligned}
\mathbb{P}^{Q^{\mathcal{I}}}(Z \geq t,\ H \leq h(t)) &= \mathbb{E}\left(\mathbb{P}\left(Z \geq t,\ H \leq h(t) \mid \lfloor H \rfloor\right)\right) \\
&\leq \mathbb{E}\left(\mathbf{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor}\mathbb{P}\left(Z \geq t \mid \lfloor H \rfloor\right)\right) \\
&\leq \mathbb{E}\left(\mathbf{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor}\mathbb{P}\left(M_1M_2 \geq t\mathrm{e}^{-(\lfloor H \rfloor+1)} \mid \lfloor H \rfloor\right)\right) \\
&\leq \mathrm{e}^{\eta}\mathbb{E}\left(\mathbf{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor}t^{-\eta}\mathrm{e}^{\eta\lfloor H \rfloor}\mathbb{E}\left((M_1M_2)^{\eta} \mid \lfloor H \rfloor\right)\right) \\
&\leq \mathrm{e}^{\eta}\mathbb{E}\left(\mathbf{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor}t^{-\eta}\mathrm{e}^{\eta\lfloor H \rfloor}\mathbb{E}\left((M_1)^{2\eta} \mid \lfloor H \rfloor\right)\right).
\end{aligned}$$

In the last formula, we used Cauchy-Schwarz inequality and the symmetry property of $Q^{\mathcal{I}}$, see Lemma 4.4, to obtain

$$\mathbb{E}((M_2)^{2\eta} \mid \lfloor H \rfloor) = \mathbb{E}((M_1)^{2\eta} \mid \lfloor H \rfloor).$$

We can now use the estimate of Lemma 4.7, which gives

$$\begin{aligned}
\mathbb{P}^{Q^{\mathcal{I}}}(Z \geq t,\ H \leq h(t)) &\leq \mathrm{e}^{\eta}c_{2\eta}t^{-\eta}\sum_{p=0}^{\lfloor h(t) \rfloor}\mathrm{e}^{\eta p}\mathbb{P}(\lfloor H \rfloor = p) \\
&\leq c't^{-\eta}\sum_{p=0}^{\lfloor h(t) \rfloor}\mathrm{e}^{(\eta-\kappa)p}.
\end{aligned}$$

In the last formula, we used the fact that $\mathbb{P}(\lfloor H \rfloor = p) = O(\mathrm{e}^{-\kappa p})$, see (4.9). Since we chose $\eta > \kappa$ we can bound uniformly

$$\mathbb{P}^{Q^{\mathcal{I}}}(Z \geq t,\ H \leq h(t)) \leq c''t^{-\eta}\mathrm{e}^{(\eta-\kappa)h(t)} = c''t^{-\kappa}\left(\frac{\mathrm{e}^{h(t)}}{t}\right)^{\eta-\kappa}.$$

This gives the result for Z. Since $R \leq Z$, we get the result for R. □

7. The coupling argument

We set

$$I(t) := \mathbb{P}^{Q^{\mathcal{I}}}\left(\mathrm{e}^{H} M_1 M_2 \geq t\right),$$
$$J(t) := \mathbb{P}^{Q^{\mathcal{I}}}\left(\mathrm{e}^{H} M_2 \geq t\right),$$
$$K(t) := \mathbb{P}^{Q^{\mathcal{I}}}\left(\mathrm{e}^{H} \geq t\right).$$

From the estimate of Iglehart, see [63], we know that

$$K(t) \sim \frac{1}{\mathbb{P}^{Q}(H=S)} C_I t^{-\kappa},$$

when $t \to \infty$. Indeed, we have

$$\mathbb{P}^{Q^{\mathcal{I}}}\left(\mathrm{e}^{H} \geq t\right) = \frac{1}{\mathbb{P}^{Q}(H=S)}(\mathbb{P}^{Q}(\mathrm{e}^{H} \geq t) - P^{Q}(\mathrm{e}^{H} \geq t,\ S > H)).$$

The second term is clearly of order $O(t^{-2\kappa})$, the first term is estimated in [63], cf (4.4).

We will prove the following key estimates.

Proposition 4.1. *For all $\xi > 0$ there exists a function $\epsilon_\xi(t) > 0$ such that $\lim_{t\to\infty} \epsilon_\xi(t) = 0$ and*

$$\mathrm{e}^{-3\xi}\mathbb{E}\left(J(\mathrm{e}^{3\xi} t M^{-1})\right)(1-\epsilon_\xi(t)) \leq I(t) \leq \mathrm{e}^{3\xi}\mathbb{E}\left(J(\mathrm{e}^{-3\xi} t M^{-1})\right)(1+\epsilon_\xi(t)),$$
$$\mathrm{e}^{-2\xi}\mathbb{E}\left(K(\mathrm{e}^{2\xi} t M^{-1})\right)(1-\epsilon_\xi(t)) \leq J(t) \leq \mathrm{e}^{2\xi}\mathbb{E}\left(K(\mathrm{e}^{-2\xi} t M^{-1})\right)(1+\epsilon_\xi(t)),$$

where M is the random variable defined in (4.6).

We see that Theorem 4.1 ii) is a direct consequence of the second estimate and of the tail estimate for $K(t)$. Theorem 4.2 is a consequence of the estimate i) and of the estimate for J.

Proof. *Step 1:* We first restrict the expectations to large values of H. Let $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be any increasing function such that

$$\lim_{t\to\infty} t^{-1}\mathrm{e}^{h(t)} = 0, \tag{4.10}$$

$$h(t) \geq \frac{9}{10}\log t. \tag{4.11}$$

From Corollary 4.2, we know that

$$\mathbb{P}^{Q^{\mathcal{I}}}\left(\mathrm{e}^{H} M_1 M_2 \geq t,\ H \leq h(t)\right) = o(t^{-\kappa}) = o(K(t)). \tag{4.12}$$

Hence, we can restrict ourselves to consider

$$I_h(t) := \mathbb{P}^{Q^{\mathcal{I}}}\left(\mathrm{e}^{H} M_1 M_2 \geq t \mid H \geq h(t)\right),$$
$$J_h(t) := \mathbb{P}^{Q^{\mathcal{I}}}\left(\mathrm{e}^{H} M_2 \geq t \mid H \geq h(t)\right),$$

Step 2: (Truncation of M_1, M_2). We need to truncate the sums M_1 and M_2 so that they do not overlap. Under $Q^{\mathcal{I}}(\cdot|H \geq h(t))$ we consider the random variables

$$\tilde{M}_1 := \sum_{-\infty}^{t_1} \mathrm{e}^{-V_k}, \tag{4.13}$$

$$\tilde{M}_2 := \sum_{t_2}^{\infty} \mathrm{e}^{V_k - S}, \tag{4.14}$$

where

$$t_1 := \inf\{k \geq 0,\ V_k \geq \frac{1}{3}\log t\} - 1,$$

$$t_2 := \sup\{k \leq T_H,\ V_k \leq H - \frac{1}{3}\log t\} + 1.$$

Since $h(t) \geq \frac{9}{10}\log t$, we have

$$0 \leq t_1 < t_2 \leq T_H.$$

Clearly, by the symmetry property of $Q^{\mathcal{I}}$, $\tilde{M}_1$ and $\tilde{M}_2$ have the same law under $Q^{\mathcal{I}}(\cdot|H \geq h(t))$. (Observe that the random variables $\tilde{M}_1$ and $\tilde{M}_2$ are implicitly defined in terms of the variable t.)

LEMMA 4.8. *Let ξ be a positive real. There exists a constant $c_\xi > 0$ such that*

$$\mathbb{P}^{Q^{\mathcal{I}}}\left(\tilde{M}_1 \leq \mathrm{e}^{-\xi} M_1 \mid H \geq h(t)\right) \leq \begin{cases} c_\xi t^{-\kappa/6} & \textit{for } \kappa \leq 1, \\ c_\xi t^{-1/6} & \textit{for } \kappa \geq 1. \end{cases}$$

PROOF. We have, since $M_1 \geq 1$

$$\begin{aligned} & \mathbb{P}^{Q^{\mathcal{I}}}\left(\tilde{M}_1 \leq \mathrm{e}^{-\xi} M_1 \mid H \geq h(t)\right) \\ \leq\ & \mathbb{P}^{Q^{\mathcal{I}}}\left(M_1 - \tilde{M}_1 \geq 1 - \mathrm{e}^{-\xi} \mid H \geq h(t)\right) \\ \leq\ & \frac{1}{1-\mathrm{e}^{-\xi}} \mathbb{E}^{Q^{\mathcal{I}}}\left(M_1 - \tilde{M}_1 \mid H \geq h(t)\right) \\ \leq\ & c\frac{\mathrm{e}^{-\kappa h(t)}}{\mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t))} \mathbb{E}^{\tilde{Q}^{>0}}\left(\sum_{k=t_1+1}^{\infty} \mathrm{e}^{-Y_k}\left(\sum_{\substack{e_p \geq k, \\ Y_{e_p} \geq h(t)}} \mathrm{e}^{-\kappa(Y_{e_p} - h(t))}\right)\right), \end{aligned}$$

where in the last expression we used the result of Lemma 4.6, and the notations of the related section, and where c is a constant depending on ξ and on the parameters of the model. Using the fact that $\mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t)) \sim C\mathrm{e}^{-\kappa h(t)}$, when $t \to \infty$, the Markov property and the fact that

$$\mathbb{E}^{\tilde{Q}^{>0}}\left(\sum_{\substack{e_p \geq k, \\ Y_{e_p} \geq h(t)}} \mathrm{e}^{-\kappa(Y_{e_p} - h(t))}\right) \leq \frac{1}{\mathbb{P}^{\tilde{Q}}(Y_n > 0, \forall n > 0)(1 - \mathbb{E}^{\tilde{Q}}(\mathrm{e}^{-\kappa Y_{e_1}}))},$$

independently of k, we see that

$$\begin{aligned}\mathbb{P}^{Q^{\mathcal{I}}}\Big(\tilde{M}_1 \le \mathrm{e}^{-\xi} M_1 \mid H \ge h(t)\Big) &\le c\mathbb{E}^{\tilde{Q}^{>0}}\big(\sum_{k=t_1+1}^{\infty} \mathrm{e}^{-Y_k}\big) \\ &\le c_\xi t^{-\frac{\kappa\wedge 1}{6}},\end{aligned}$$

using the estimate of Lemma 4.3. □

Step 3: (A small modification of the conditioning.) We set

$$\mathcal{I}_h^{(t)} := \mathcal{I} \cap \{S \ge h(t)\} = \{V_k \ge 0\,,\, \forall k \le 0\} \cap \{S = H\} \cap \{S \ge h(t)\},$$

the event by which we condition in $S_h(t)$, $R_h(t)$. We set

$$\tilde{\mathcal{I}}_h^{(t)} := \{S \ge h(t)\} \cap \{V_k \ge 0\,,\, \forall k \le 0\} \cap \{V_k > 0,\ \forall 0 < k < T_{\frac{1}{3}\log t}\},$$

where

$$T_{\frac{1}{3}\log t} := \inf\{k \ge 0,\ V_k \ge \frac{1}{3}\log t\}.$$

Clearly, we have $\mathcal{I}_h^{(t)} \subset \tilde{\mathcal{I}}_h^{(t)}$ and

$$\mathbb{P}(\tilde{\mathcal{I}}_h^{(t)} \setminus \mathcal{I}_h^{(t)} \mid \tilde{\mathcal{I}}_h^{(t)}) \le ct^{-\kappa/3},$$

for a constant $c > 0$ depending only on the parameters of the model. We set

$$\begin{aligned}\tilde{I}_h(t) &:= \mathbb{P}^{Q}\left(\mathrm{e}^H \tilde{M}_1 \tilde{M}_2 \ge t \mid \tilde{\mathcal{I}}_h^{(t)}\right),\\ \tilde{J}_h(t) &:= \mathbb{P}^{Q}\left(\mathrm{e}^H \tilde{M}_2 \ge t \mid \tilde{\mathcal{I}}_h^{(t)}\right),\\ \tilde{K}_h(t) &:= \mathbb{P}^{Q}\left(\mathrm{e}^H \ge t \mid \tilde{\mathcal{I}}_h^{(t)}\right).\end{aligned}$$

From Step 2 (Lemma 4.8) and Step 3, we see that we have, for all $\xi > 0$, the following estimate

$$I_h(\mathrm{e}^{2\xi}t) - c_\xi t^{-\frac{\kappa\wedge 1}{6}} \le \tilde{I}_h(t) \le I_h(t) + ct^{-\frac{\kappa}{3}}, \tag{4.15}$$

$$J_h(\mathrm{e}^{\xi}t) - c_\xi t^{-\frac{\kappa\wedge 1}{6}} \le \tilde{J}_h(t) \le J_h(t) + ct^{-\frac{\kappa}{3}}. \tag{4.16}$$

Step 4: (The coupling strategy.)

Let $(Y'_k)_{k\ge 0}$ and $(Y''_k)_{k\ge 0}$ be two independent processes with law

$$\tilde{Q}(\cdot \mid Y_k > 0\,,\, 0 < k \le T_{\frac{1}{3}\log t}).$$

Let us define, for all $u > 0$, the hitting times

$$T'_u := \inf\{k \ge 0,\ Y'_k \ge u\},\quad T''_u := \inf\{k \ge 0,\ Y''_k \ge u\}.$$

Set

$$N'_0 := T'_{\frac{1}{3}\log t},\quad N''_0 := T''_{\frac{1}{3}\log t}.$$

We couple the processes $(Y'_{N'_0+k})_{k\geq 0}$ and $(Y''_{N''_0+k})_{k\geq 0}$ as in Durrett (cf [38], (4.3), p. 204): we construct some random times $K' \geq N'_0$ and $K'' \geq N''_0$ such that

$$|Y'_{K'} - Y''_{K''}| \leq \xi,$$

and such that $(Y'_{K'+k} - Y'_{K'})_{k\geq 0}$ and $(Y''_{K''+k} - Y''_{K''})_{k\geq 0}$ are independent of the σ-field generated by $Y'_0, \dots, Y'_{K'}$ and $Y''_0, \dots, Y''_{K''}$. The method for this ξ-coupling is the following: we consider some independent Bernoulli random variables $(\eta'_i)_{i\in\mathbb{N}}$ and $(\eta''_i)_{i\in\mathbb{N}}$ (with $\mathbb{P}(\eta'_i = 1) = \mathbb{P}(\eta''_i = 1) = \frac{1}{2}$) and we define

$$(Z'_k) = (Y'_{N'_0+\sum_{i=1}^k \eta'_i}), \quad (Z''_k) = (Y''_{N''_0+\sum_{i=1}^k \eta''_i}).$$

This extra randomization ensures that the process $(Z'_k - Z''_k)$ is non arithmetic. Since its expectation is null, there exists a positive random time for which they are at a distance at most ξ (cf the proof of Chung-Fuchs theorem (2.7), p. 188 and theorem (2.1), p. 183 in [38]). Then we define

$$Y_k = \begin{cases} Y'_k, & \text{when } k \leq K', \\ (Y''_{K''+(k-K')} - Y''_{K''}) + Y'_{K'}, & \text{when } k > K'. \end{cases}$$

Clearly, by construction, since the processes Y' and Y'' are no longer conditioned when they reach the level $\frac{1}{3}\log t$, $(Y_k)_{k\geq 0}$ has the law

$$\tilde{Q}(\cdot \mid Y_k > 0, \ \forall\, 0 < k < T_{\frac{1}{3}\log t}).$$

We want that Y' and Y'' couple before they reach the level $\frac{1}{2}\log t$, so we set

$$\mathcal{A} = \{K' < T'_{\frac{1}{2}\log t}\} \cap \{K'' < T''_{\frac{1}{2}\log t}\}.$$

Clearly, since the distribution of $Y'_{N'_0} - \frac{1}{3}\log t$ converges (and the same for Y'', cf limit theorem (4.10), p. 370 in [44]) and since for all starting points $Y'_{N'_0}$ and $Y''_{N''_0}$, Z' and Z'' couple in a finite time almost surely, we have the following result (whose proof is postponed to the end of the section).

LEMMA 4.9.

$$\lim_{t\to\infty} \mathbb{P}(\mathcal{A}^c) = 0.$$

We set

$$\eta(t) := \mathbb{P}(\mathcal{A}^c),$$

and we choose $h(t)$ in terms of η by

$$h(t) = (\log t + \frac{1}{2\kappa}\log \eta(t)) \vee (\frac{9}{10}\log t) \vee ((1 - \frac{1}{7\kappa})\log t), \tag{4.17}$$

where $\vee$ stands for the maximum of the three values. Clearly, $h(t)$ satisfies the hypotheses (4.10), (4.11).

Consider now two independent processes $(W_k)_{k\geq 0}$ and $(W'_k)_{k\geq 0}$ (and independent of Y', Y'') with the same law $Q^{\leq 0}$ (cf Section 5). Let e be a strict increasing ladder time of Y and define the process $V(W,W',Y,e)=(V_k)_{k\in\mathbb{Z}}$ by

$$\begin{cases} (V_k)_{k\le 0}=(-W_{-k})_{k\le 0},\\ \\ (V_k)_{k\ge 0}=(Y_0,\dots,Y_e,Y_e+W'_1,\dots,Y_e+W'_k,\dots). \end{cases}$$

If $Y_e\ge h(t)$ then clearly $(V_k)_{k\in\mathbb{Z}}$ belongs to the event $\tilde{\mathcal{I}}_h^{(t)}$, and the functional $\tilde M_1$ defined in (4.13) depends only on W and Y'; we denote it by $\tilde M_1(W,Y')$. The functional $\tilde M_2$ depends only on Y,W',e; we denote it by $\tilde M_2(Y,W',e)$. Using Lemma 4.5, we see that

$$\tilde I_h(t)=\frac{1}{\mathcal{Z}_h(t)}\mathbb{E}\Big(\sum_{p=0}^{\infty}\mathrm{e}^{-\kappa Y_{e_p}}\mathbf{1}_{Y_{e_p}\ge h(t)}\mathbf{1}_{\tilde M_1(W,Y')\tilde M_2(Y,W',e_p)\mathrm{e}^{Y_{e_p}}\ge t}\Big),$$

where $(e_p)_{p\ge 0}$ is the set of strict increasing ladder times of Y (cf Section 5) and where $\mathcal{Z}_h(t)$ is the normalizing constant

$$\mathcal{Z}_h(t)=\mathbb{E}\Big(\sum_{p=0}^{\infty}\mathrm{e}^{-\kappa Y_{e_p}}\mathbf{1}_{Y_{e_p}\ge h(t)}\Big).$$

Clearly, $\mathcal{Z}_h(t)\sim_{t\to\infty}c\mathrm{e}^{-\kappa h(t)}$. The variable $Y_{T_{h(t)}}-h(t)$ is indeed the residual waiting time of the renewal process defined by the values of the process Y at the successive increasing ladder epochs. Hence, it converges in distribution by the limit theorem (4.10) in ([44], p. 370).

On the coupling event $\mathcal{A}$, we have

$$\begin{aligned} Y''_{e_p-K'+K''}-\xi\le Y_{e_p}&\le Y''_{e_p-K'+K''}+\xi,\\ \tilde M_2(Y,W',e_p)&=\tilde M_2(Y'',W',e_p-K'+K''), \end{aligned}$$

for all ladder time e_p such that $Y_{e_p}\ge h(t)$ (indeed $h(t)\ge\frac{9}{10}\log t$) and where $\tilde M_2(Y'',W',e_p-K'+K'')$ is the functional obtained from the concatenation of the processes Y'' and W' at time $e_p-K'+K''$, as it is done for $\tilde M_2(Y,W',e_p)$. The first set of inequalities implies that, on the coupling event $\mathcal{A}$, the set $\{e_p-K'+K'',\ Y_{e_p}\ge h(t)\}$ is included in the set of strict increasing ladder times of Y'' larger than $h(t)-\xi$. So we have

$$\begin{aligned}\tilde I_h(t)\quad\le\quad&\frac{\mathrm{e}^{\kappa\xi}}{\mathcal{Z}_h(t)}\mathbb{E}\Big(\mathbf{1}_{\mathcal{A}}\big(\sum_{p=0}^{\infty}\mathrm{e}^{-\kappa Y''_{e''_p}}\mathbf{1}_{Y''_{e''_p}\ge h(t)-\xi}\mathbf{1}_{\tilde M_1(W,Y')\tilde M_2(Y'',W',e''_p)\exp(Y''_{e''_p})\ge t\mathrm{e}^{-\xi}}\big)\Big)\\ &+\frac{\mathrm{e}^{-\kappa h(t)}}{\mathcal{Z}_h(t)}\mathbb{E}\Big(\mathbf{1}_{\mathcal{A}^c}\big(\sum_{p=0}^{\infty}\mathrm{e}^{-\kappa(Y_{e_p}-h(t))}\mathbf{1}_{Y_{e_p}\ge h(t)}\big)\Big),\end{aligned}$$

where $(e''_p)_{p\ge 0}$ denote the strict increasing ladder times for the process Y''. Since the process $\{Y_{e_p},\ Y_{e_p}\ge h(t)\}$ depends on the event $\mathcal{A}$ only through the value of $Y_{T_{h(t)}}$, we see that the second term is less or equal than

$$\frac{1}{1-\mathbb{E}^{\tilde{Q}}(\mathrm{e}^{-\kappa Y_{e_1}})}\frac{\mathrm{e}^{-\kappa h(t)}}{\mathcal{Z}_h(t)}\mathbb{P}(\mathcal{A}^c) \le c\mathbb{P}(\mathcal{A}^c). \tag{4.18}$$

Now, the first term is lower than

$$\mathrm{e}^{\kappa\xi}\frac{\mathcal{Z}_{h-\xi}(t)}{\mathcal{Z}_h(t)}\mathbb{P}\big(\mathrm{e}^{S''}\tilde{M}_1(W,Y')\tilde{M}_2'' \ge t\mathrm{e}^{-\xi}\big) \le \mathrm{e}^{3\kappa\xi}\mathbb{P}\big(\mathrm{e}^{S''}\tilde{M}_1(W,Y')\tilde{M}_2'' \ge t\mathrm{e}^{-\xi}\big), \tag{4.19}$$

for t large enough (using the equivalent of $\mathcal{Z}_h(t)$), where S'' and $\tilde{M}_2''$ are relative to a process V'' independent of W, Y' and with law $Q(\cdot \mid \tilde{\mathcal{I}}^{(t)}_{h-\xi})$. Moreover, let us introduce $M_2'' := \sum_{k=0}^{\infty}\mathrm{e}^{V_k''-S''}$. We need now to replace the truncated sum $\tilde{M}_1$ by the meander M. Using the fact that $\mathbb{P}(\exists k>0 : Y_k' \le 0) \le ct^{-\kappa/3}$, we see that

$$\begin{aligned}\mathbb{P}\big(\mathrm{e}^{S''}\tilde{M}_1(W,Y')\tilde{M}_2'' \ge t\mathrm{e}^{-\xi}\big) &\le \mathbb{P}\big(\mathrm{e}^{S''}M_2''M \ge t\mathrm{e}^{-\xi}\big) + ct^{-\kappa/6}\\ &\le \mathbb{E}\big(J_{h-\xi}(\mathrm{e}^{-\xi}t/M)\big) + c't^{-\kappa/6},\end{aligned} \tag{4.20}$$

the second inequality being a consequence of $\mathbb{P}(\tilde{\mathcal{I}}^{(t)}_h \setminus \mathcal{I}^{(t)}_h \mid \tilde{\mathcal{I}}^{(t)}_h) \le ct^{-\kappa/3}$ and M the random variable defined in (4.6) and independent of V''. Finally, considering the choice made for $h(t)$ (cf (4.17)), we have

$$t^{-\frac{\kappa\wedge 1}{6}}\mathbb{P}^{Q^{\mathcal{I}}}(H \ge h(t)) = o(t^{-\kappa}),$$

$$\mathbb{P}(\mathcal{A}^c)\mathbb{P}^{Q^{\mathcal{I}}}(H \ge h(t)) \le ct^{-\kappa}\sqrt{\mathbb{P}(\mathcal{A}^c)} = o(t^{-\kappa}).$$

Putting everything together (i.e., the estimates (4.12), (4.15), (4.18), (4.19), (4.20))

$$\begin{aligned} I(t) &\le \mathbb{P}(H \ge h(t))I_h(t) + o(t^{-\kappa})\\ &\le \mathbb{P}(H \ge h(t))(\tilde{I}_h(\mathrm{e}^{-2\xi}t) + ct^{-\frac{\kappa\wedge 1}{6}}) + o(t^{-\kappa})\\ &\le \mathbb{P}(H \ge h(t))(\mathrm{e}^{3\kappa\xi}\mathbb{E}(J_{h-\xi}(\mathrm{e}^{-3\xi}t/M)) + c\mathbb{P}(\mathcal{A}^c)) + o(t^{-\kappa})\\ &\le \mathrm{e}^{3\kappa\xi}\mathbb{P}\left(RM \ge t\mathrm{e}^{-3\xi},\ H \ge h(t)-\xi\right) + o(t^{-\kappa}),\end{aligned}$$

where R and M are independent processes with laws defined in Section 2 (indeed, in the last inequality, $\mathbb{P}(H \ge h(t))\mathbb{P}(\mathcal{A}^c) \le \sqrt{\mathbb{P}(\mathcal{A}^c)}t^{-\kappa} = o(t^{-\kappa})$). Now, proceeding exactly as in Corollary 4.2, we see that

$$\mathbb{P}(RM \ge t,\ H < h(t)-\xi) = o(t^{-\kappa}),$$

(indeed, the only difference is that M_1 is replaced by M and that M and R are independent). Finally, we proved that

$$I(t) \le \mathrm{e}^{3\kappa\xi}\mathbb{E}(J(\mathrm{e}^{-3\xi}t/M)) + o(t^{-\kappa}).$$

The lower estimate is similar. We first have, since the set $\{e_p - K' + K'',\ Y_{e_p} \ge h(t)\}$ includes the set of strict increasing ladder times of Y'' larger than $h(t)+\xi$:

$$\tilde{I}_h(t) \ \ge\ \frac{\mathrm{e}^{-\kappa\xi}}{\mathcal{Z}_h(t)}\mathbb{E}\bigg(\mathbf{1}_{\mathcal{A}}\Big(\sum_{p=0}^{\infty}\mathrm{e}^{-\kappa Y''_{e''_p}}\mathbf{1}_{Y''_{e''_p}\ge h(t)+\xi}\mathbf{1}_{\tilde{M}_1(W,Y')\tilde{M}_2(Y'',W',e''_p)\exp(Y''_{e''_p})\ge t\mathrm{e}^{\xi}}\Big)\bigg).$$

Hence, by the same argument as above

$$\tilde{I}_h^{(t)} \geq \mathrm{e}^{-3\kappa\xi}\mathbb{P}\big(\mathrm{e}^{S''}\tilde{M}_1(W,Y')\tilde{M}_2'' \geq \mathrm{e}^{\xi}t\big) + c\mathbb{P}(\mathcal{A}^c),$$

where S'' and $\tilde{M}_2''$ are relative to a process V'' independent of W and Y' and with law $Q(\cdot \mid \tilde{I}_{h+\xi}^{(t)})$. Using, now the fact that $Y_k' > 0$ for all $k > 0$ with probability at least $1 - ct^{-\kappa/3}$ and the fact that $\tilde{M}_2 \geq \mathrm{e}^{-\xi}M_2$ with probability at least $1 - ct^{-\kappa/6}$, and the estimate on the tail of the sum $\sum \mathrm{e}^{-Y_k'}$ (of Section 3) we see that

$$\tilde{I}_h^{(t)} \geq \mathrm{e}^{-3\kappa\xi}\mathbb{P}\left(M\mathrm{e}^{S''}M_2'' \geq \mathrm{e}^{3\xi}t\right) + o(t^{-\kappa/6}) + c\mathbb{P}(\mathcal{A}^c),$$

where M is the random variable defined in (4.6) and independent of V''. Then, we conclude as previously.

To prove the estimate on $J(t)$ and $K(t)$ we proceed exactly in the same way: we first observe that by the property of time reversal (see Lemma 4.4), we have

$$J(t) = \mathbb{P}^{Q^{\mathcal{I}}}(\mathrm{e}^H M_1 \geq t).$$

The situation is then even simpler, we just have to decouple M_1 and e^H. □

Proof. (of Lemma 4.9). Denote by $F_{y',y''}(u)$ the probability that Z' and Z'' couple before the level $\frac{1}{3}\log t+u$ knowing that $Y'_{N_0'} = \frac{1}{3}\log t+y'$ and $Y''_{N_0''} = \frac{1}{3}\log t+y''$. By the arguments above, $F_{y',y''}(u)$ tends to 1 when u tends to infinity. Let $A > 0$, we first prove that this convergence is uniform in y', y'' on the compact $y' \leq A$, $y'' \leq A$. For this we consider the set $\mathcal{S} = (\mathbb{N}\cdot\frac{\xi}{4}) \cap [0, A]$, and for y', y'' in $\mathcal{S}\times\mathcal{S}$ the function $\widehat{F}_{y',y''}(u)$ the probability that Z' and Z'' starting form the points $Y'_{N_0'} = \frac{1}{3}\log t + y'$ and $Y''_{N_0''} = \frac{1}{3}\log t + y''$ couple at a distance $\xi/2$, before the level $\frac{1}{3}\log t + u - \xi$. Let

$$\phi(u) = \inf_{y'\in\mathcal{S},\, y''\in\mathcal{S}} \widehat{F}_{y',y''}(u).$$

Clearly $\phi(u) \to 1$ when $u \to \infty$ and $F_{y',y''}(u) \geq \phi(u)$, whenever y' and y'' are in $[0, A]$. This implies that

$$\liminf_{t\to\infty}\mathbb{P}(\mathcal{A}) \geq \liminf_{A\to\infty}\liminf_{t\to\infty}\left(\mathbb{P}(Y'_{N_0'} - \frac{1}{3}\log t \leq A)\right)^2.$$

Moreover, $\mathbb{P}^{\tilde{Q}}(Y_k' > 0\,,\, 0 < k \leq T_{\frac{1}{3}\log t}) \geq \mathbb{P}^{\tilde{Q}}(Y_k' > 0\,,\, k \geq 0) > 0$ implies

$$\mathbb{P}(Y'_{N_0'} - \frac{1}{3}\log t \geq A) = \mathbb{P}^{\tilde{Q}}(V_{T_{\frac{1}{3}\log t}} - \frac{1}{3}\log t \geq A \,|\, V > 0) \leq c\mathbb{P}^{\tilde{Q}}(V_{T_{\frac{1}{3}\log t}} - \frac{1}{3}\log t \geq A),$$

where here V is the canonical process under $\tilde{Q}$. Therefore, since $V_{T_{\frac{1}{3}\log t}} - \frac{1}{3}\log t$ converges in law (under $\tilde{Q}$) to a finite random variable when t tends to infinity (see limit theorem (4.10), p. 370 in [44] or Example 4.4 part II, page 214 in [38]), this yields $\liminf_{t\to\infty}\mathbb{P}(\mathcal{A}) = 1$. □

8. Proof of Theorem 4.1, Theorem 4.2 and Theorem 4.3

PROOF. (of Theorem 4.1, *ii*) and Theorem 4.2). Let $\xi > 0$. By Proposition 4.1, we have, for all $A > 0$ and for t large enough,

$$J(t) \leq \mathrm{e}^{3\xi}\left(\mathbb{E}(K(\mathrm{e}^{-2\xi}tM^{-1})\mathbf{1}_{M\leq A}) + \mathbb{E}(K(\mathrm{e}^{-2\xi}tM^{-1})\mathbf{1}_{M>A})\right).$$

On the first term, for t large enough, we can bound from above $K(\mathrm{e}^{-2\xi}tM^{-1})$ by $(\frac{C_I}{\mathbb{P}^Q(H=S)}+\xi)(t\mathrm{e}^{-2\xi}M^{-1})^{-\kappa}$. For the second term we can use a uniform bound $K(t) \leq ct^{-\kappa}$. Thus we get

$$J(t) \leq \mathrm{e}^{3(1+\kappa)\xi}(\frac{C_I}{\mathbb{P}^Q(H=S)} + \xi)t^{-\kappa}(\mathbb{E}(M^\kappa)\mathbf{1}_{M\leq A}) + ct^{-\kappa}\mathbb{E}(M^\kappa\mathbf{1}_{M>A})).$$

Since M^κ is integrable, letting A tend to ∞, then ξ tends to 0, we get the upper bound

$$\limsup_{t\to\infty} t^\kappa J(t) \leq \frac{C_{KI}}{\mathbb{P}^Q(H=S)}.$$

For the lower bound it is the same. The proof of Theorem 4.2 is the same: we use the estimate i) of Proposition 4.1 and the tail estimate for J. □

PROOF. (of Theorem 4.1, *i*)).

Let us first recall (4.5) and Theorem 4.1, *ii*), which tells that

$$Q(R > t\,;\, H = S) = \frac{C_{KI}}{t^\kappa} + o(t^{-\kappa}), \qquad t \to \infty, \tag{4.21}$$

where $C_{KI} = C_I\mathbb{E}(M^\kappa)$. Then, introducing

$$KI := \sum_{0\leq k\leq T_{\mathbb{R}_-}} \mathrm{e}^{V_k}, \qquad O_1 := -V_{T_{\mathbb{R}_-}},$$

Theorem 4.1, *i*) is a consequence of Theorem 4.1, *ii*) together with the two following lemmas.

LEMMA 4.10. *We have*

$$Q(KI > t) = \frac{C_{KI}}{t^\kappa} + o(t^{-\kappa}), \qquad t \to \infty. \tag{4.22}$$

PROOF. Firstly, observe that $KI \leq R$ implies $Q(KI > t\,;\, H = S) \leq Q(R > t\,;\, H = S)$. Moreover, Corollary 4.2 implies $Q(KI > t\,;\, \mathrm{e}^H = \mathrm{e}^S \leq t^{2/3}) = o(t^{-\kappa})$, $t \to \infty$, since $KI \leq R$. Furthermore, we have $0 \leq Q(KI > t\,;\, \mathrm{e}^H > t^{2/3}) - Q(KI > t\,;\, \mathrm{e}^H = \mathrm{e}^S > t^{2/3}) \leq Q(H \neq S\,;\, \mathrm{e}^H > t^{2/3}) = o(t^{-\kappa})$, $t \to \infty$. Therefore, we obtain, when $t \to \infty$,

$$Q(R > t\,;\, H = S) \geq Q(KI > t\,;\, \mathrm{e}^H > t^{2/3}) + o(t^{-\kappa}). \tag{4.23}$$

Since, by Corollary 4.2, $Q(KI > t\,;\, \mathrm{e}^H \leq t^{2/3}) = o(t^{-\kappa})$, $t \to \infty$, we get

$$Q(KI > t\,;\, \mathrm{e}^H > t^{2/3}) = Q(KI > t) + o(t^{-\kappa}), \tag{4.24}$$

when $t \to \infty$. Then, assembling (4.23) and (4.24) yields

$$Q(R > t\,;\, H = S) \geq Q(KI > t) + o(t^{-\kappa}), \qquad t \to \infty. \tag{4.25}$$

On the other hand, observe that Corollary 4.2 implies that $Q(R > t\,;\, H = S) = Q(R > t\,;\, \mathrm{e}^H = \mathrm{e}^S > t^{2/3}) + o(t^{-\kappa})$, $t \to \infty$. Moreover, since we have $R = KI + \mathrm{e}^{O_1} R'$, with R' a random variable independent of KI and O_1, having the same law as R, we obtain that $Q(R > t\,;\, \mathrm{e}^H = \mathrm{e}^S > t^{2/3}) \leq Q_1 + Q_2$, where

$$\begin{aligned} Q_1 &:= Q(KI \leq t - t^{2/3}\,;\, R' > t^{2/3}\,;\, \mathrm{e}^H > t^{2/3}), \\ Q_2 &:= Q(KI > t - t^{2/3}\,;\, R > t\,;\, \mathrm{e}^H = \mathrm{e}^S > t^{2/3}). \end{aligned}$$

Now, since R' and H are independent, we get $Q_1 \leq Q(\mathrm{e}^H > t^{2/3})Q(R' > t^{2/3}) = o(t^{-\kappa})$, $t \to \infty$. Moreover, we easily have $Q_2 \leq Q(KI > t - t^{2/3})$. Therefore

$$Q(R > t\,;\, H = S) \leq Q(KI > t - t^{2/3}) + o(t^{-\kappa}), \qquad t \to \infty. \tag{4.26}$$

Recalling (4.21) and assembling (4.25) and (4.26) concludes the proof of Lemma 4.10. □

LEMMA 4.11. *C_{KI} satisfies*

$$C_{KI} = (1 - \mathbb{E}^Q(\mathrm{e}^{-\kappa O_1}))C_K.$$

PROOF. First, observe that $Q(R > t) = Q(KI > t) + P_1 + P_2$, where

$$\begin{aligned} P_1 &:= Q(KI + \mathrm{e}^{-O_1} R' > t\,;\, t^{1/2} < KI \leq t), \\ P_2 &:= Q(KI + \mathrm{e}^{-O_1} R' > t\,;\, KI \leq t^{1/2}), \end{aligned}$$

with R' a random variable independent of KI and O_1, with the same law as R.

Now, let us prove that P_1 is negligible. Observe first that, since $O_1 \geq 0$ by definition, we have $P_1 \leq Q(R' > t - KI\,;\, t^{1/2} < KI \leq t)$. Therefore $0 \leq P_1 \leq P_1' + P_1''$, where

$$\begin{aligned} P_1' &:= Q(R' > t - KI\,;\, t - t^{2/3} < KI \leq t), \\ P_1'' &:= Q(R' > t - KI\,;\, t^{1/2} < KI \leq t - t^{2/3}). \end{aligned}$$

Since R' and KI are independent, (4.5) and (4.22) yield $P_1'' \leq Q(R' > t^{2/3})Q(KI > t^{1/2}) = o(t^{\kappa})$, $t \to \infty$. Furthermore, we have

$$\begin{aligned} P_1' &\leq Q(t - t^{2/3} < KI \leq t) \\ &\leq Q(KI > t - t^{2/3}) - Q(KI > t) \\ &= Q(KI > t)\left(\frac{Q(KI > t - t^{2/3})}{Q(KI > t)} - 1\right). \end{aligned}$$

Therefore (4.22) implies $P_1' = o(t^{-\kappa})$, $t \to \infty$. Then, we obtain $P_1 = o(t^{-\kappa})$, $t \to \infty$.

Now, let us estimate P_2. Observe that $\underline{P}_2 \le P_2 \le \overline{P}_2$, where

$$\begin{aligned} \underline{P}_2 &:= Q(\mathrm{e}^{-O_1}R' > t\,;\; KI \le t^{1/2}), \\ \overline{P}_2 &:= Q(\mathrm{e}^{-O_1}R' > t - t^{1/2}). \end{aligned}$$

Since R' and O_1 are independent, (4.5) yields

$$\overline{P}_2 = \frac{\mathbb{E}^Q(\mathrm{e}^{-\kappa O_1})C_K}{t^\kappa} + o(t^{-\kappa}), \qquad t \to \infty. \tag{4.27}$$

Therefore, it only remains to estimate $\underline{P}_2$. Since R' is independent of KI and O_1, we obtain for any $\varepsilon > 0$ and t large enough,

$$(1-\varepsilon)\, C_K \mathbb{E}^Q\left(\mathbf{1}_{\{KI \le t^{1/2}\}}\frac{\mathrm{e}^{-\kappa O_1}}{t^\kappa}\right) \le \underline{P}_2 \le (1+\varepsilon)\, C_K \mathbb{E}^Q\left(\mathbf{1}_{\{KI \le t^{1/2}\}}\frac{\mathrm{e}^{-\kappa O_1}}{t^\kappa}\right).$$

Moreover,

$$\mathbb{E}^Q\left(\mathbf{1}_{\{KI \le t^{1/2}\}}\frac{\mathrm{e}^{-\kappa O_1}}{t^\kappa}\right) \quad = \quad \frac{\mathbb{E}^Q(\mathrm{e}^{-\kappa O_1})}{t^\kappa} - \mathbb{E}^Q\left(\mathbf{1}_{\{KI > t^{1/2}\}}\frac{\mathrm{e}^{-\kappa O_1}}{t^\kappa}\right),$$

and the second term on the right-hand side is less or equal than $t^{-\kappa}Q(KI > t^{1/2}) = o(t^{-\kappa})$, $t \to \infty$. Thus

$$\underline{P}_2 = \frac{\mathbb{E}^Q(\mathrm{e}^{-\kappa O_1})C_K}{t^\kappa} + o(t^{-\kappa}), \qquad t \to \infty. \tag{4.28}$$

Assembling (4.27) and (4.28) yields $P_2 = \frac{\mathbb{E}^Q(\mathrm{e}^{-\kappa O_1})C_K}{t^\kappa} + o(t^{-\kappa})$, $t \to \infty$. Therefore, recalling (4.5), (4.22) and $Q(R > t) = Q(KI > t) + P_1 + P_2$, we obtain $C_{KI} = (1 - \mathbb{E}^Q(\mathrm{e}^{-\kappa O_1}))C_K$, which concludes the proof of Lemma 4.11. □

Since Theorem 4.1, *ii*) together with Lemma 4.10 and Lemma 4.11 yield $C_{KI} = C_I \mathbb{E}^Q(M^\kappa) = (1 - \mathbb{E}^Q(\mathrm{e}^{-\kappa O_1}))C_K$, we get $C_K = C_I \mathbb{E}^Q(M^\kappa)(1 - \mathbb{E}^Q(\mathrm{e}^{-\kappa O_1}))^{-1}$. Now, recalling that $C_I = (1 - \mathbb{E}^Q(\mathrm{e}^{-\kappa O_1}))C_F$, this concludes the proof of Theorem 4.1, *i*). □

PROOF. (of Theorem 4.3).

The proof of Theorem 4.3 is based on the same arguments as in the proof of Theorem 4.1, *i*). We mainly have to check analogous statements as Lemma 4.7 and Corollary 4.2. Namely, we check that there exists $c > 0$ such that

$$\mathbb{E}^{Q^{\mathcal{I}}}((M_1^B)^{\kappa + \frac{\varepsilon}{2}} \mid \lfloor H \rfloor) \le c, \quad Q^{\mathcal{I}}\text{- a.s.},$$

where $M_1^B := \sum_{k=-\infty}^{T_S} \mathrm{e}^{-V_k}\tilde{B}_k$. Using Hölder inequality instead of Cauchy-Schwarz inequality in the proof of Lemma 4.7 we are led to check the integrability of $(M^B)^{\kappa+\varepsilon}$. This is used in the proof of

$$\mathbb{P}^{Q^{\mathcal{I}}}\left(R^B \ge t,\ H \le h(t)\right) = o(t^{-\kappa}),$$

when t tends to infinity, which is analogous to the proof of Corollary 4.2 (in its R version), choosing $\eta = \kappa + \frac{\varepsilon}{2}$.

Now, it only remains to check the integrability of $(M^B)^{\kappa+\varepsilon}$. To this aim, we prove that $\mathbb{E}^{\tilde{Q}^{>0}}((\sum_{k\geq 0} \mathrm{e}^{-V_k}\tilde{B}_k)^{\kappa+\varepsilon}) < \infty$, the case of $\mathbb{E}^{Q^{>0}}((\sum_{k<0} \mathrm{e}^{-V_k}\tilde{B}_k)^{\kappa+\varepsilon})$ being similar.

If $\kappa \geq 1$, Minkowski inequality yields

$$\begin{aligned}
\mathbb{E}^{\tilde{Q}^{>0}}\Big((\sum_{k\geq 0} \mathrm{e}^{-V_k}\tilde{B}_k)^{\kappa+\varepsilon}\Big) &\leq \left(\sum_{k\geq 0} \mathbb{E}^{\tilde{Q}^{>0}}\Big((\mathrm{e}^{-V_k}\tilde{B}_k)^{\kappa+\varepsilon}\Big)^{\frac{1}{\kappa+\varepsilon}}\right)^{\kappa+\varepsilon} \\
&\leq C\left(\sum_{k\geq 0} \mathbb{E}^{\tilde{Q}^{>0}}(\mathrm{e}^{-(\kappa+\varepsilon)V_k})^{\frac{1}{\kappa+\varepsilon}}\right)^{\kappa+\varepsilon} \\
&\leq C\left(\sum_{k\geq 0} \mathbb{E}^{\tilde{Q}^{>0}}(\mathrm{e}^{-V_k})^{\frac{1}{\kappa+\varepsilon}}\right)^{\kappa+\varepsilon}, \qquad (4.29)
\end{aligned}$$

the second inequality being a consequence of the independence between $(\tilde{B}_i)_{i\geq 0}$ and $(V_i)_{i\geq 0}$, while the third inequality is due to the fact that $V_i \geq 0$ for $i \geq 0$ under $\tilde{Q}^{>0}$ together with $\kappa + \varepsilon \geq 1$. Choosing p such that $p/(\kappa+\varepsilon) > 1$, let us write

$$\mathbb{E}^{\tilde{Q}^{>0}}(\mathrm{e}^{-V_k}) \leq \frac{1}{k^p} + \mathbb{P}^{\tilde{Q}^{>0}}(\mathrm{e}^{-V_k} \geq k^{-p})$$

Now, as in the proof of Lemma 4.1, since large deviations do occur, we get from Cramer's theory, see [33], that the sequence $(\mathbb{P}^{\tilde{Q}^{>0}}(\mathrm{e}^{-V_k} \geq k^{-p}))_{k\geq 1}$ is exponentially decreasing. This yields that the sum in (4.29) is finite.

If $\kappa < 1$, observe that we can restrict our attention to the case where $\kappa + \varepsilon < 1$. Then, let us write

$$\begin{aligned}
\mathbb{E}^{\tilde{Q}^{>0}}\Big((\sum_{k\geq 0} \mathrm{e}^{-V_k}\tilde{B}_k)^{\kappa+\varepsilon}\Big) &\leq \mathbb{E}^{\tilde{Q}^{>0}}\Big(\sum_{k\geq 0}(\mathrm{e}^{-V_k}\tilde{B}_k)^{\kappa+\varepsilon}\Big) \\
&\leq C\sum_{k\geq 0} \mathbb{E}^{\tilde{Q}^{>0}}(\mathrm{e}^{-(\kappa+\varepsilon)V_k}),
\end{aligned}$$

the second inequality being a consequence of the independence between $(\tilde{B}_i)_{i\geq 0}$ and $(V_i)_{i\geq 0}$. Now, the conclusion is the same as in the case $\kappa \geq 1$. □

9. A Tauberian result

COROLLARY 4.3. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be such that*

$$\lim_{\lambda\to 0} \lambda \mathrm{e}^{h(\lambda)} = 0, \quad \lim_{\lambda\to 0} h(\lambda) = \infty.$$

Then, for $\kappa < 1$,

$$\mathbb{E}^{Q}\Big(1 - \frac{1}{1+\lambda Z} \mid \mathcal{I}_h^{(\lambda)}\Big) \sim \frac{1}{\mathbb{P}^{Q}(H \geq h(\lambda))}\, \frac{\pi\kappa}{\sin(\pi\kappa)} C_U \lambda^{\kappa},$$

when $\lambda \to 0$, *where* $\mathcal{I}_h^{(\lambda)}$ *is the event*

$$\mathcal{I}_h^{(\lambda)} = \mathcal{I} \cap \{H \geq h(\lambda)\} = \{V_k \geq 0\,,\, \forall k \leq 0\} \cap \{H = S \geq h(\lambda)\}.$$

PROOF. Clearly, we have

$$\mathbb{E}^Q\Big(1 - \frac{1}{1+\lambda Z} \mid \mathcal{I}_h^{(\lambda)}\Big) = \frac{\mathbb{P}^Q(H = S)}{\mathbb{P}^Q(H = S \geq h(\lambda))} \mathbb{E}^{Q^{\mathcal{I}}}\Big(\mathbf{1}_{H \geq h(\lambda)}(1 - \frac{1}{1+\lambda Z})\Big)$$

Since $\mathbb{P}^Q(H = S \geq h(\lambda)) \sim \mathbb{P}^Q(H \geq h(\lambda))$ we consider now

$$\mathbb{E}^{Q^{\mathcal{I}}}\Big(\mathbf{1}_{H \geq h(\lambda)}(1 - \frac{1}{1+\lambda Z})\Big)$$

We will forget in the following the reference to the law $Q^{\mathcal{I}}$, and simply write $\mathbb{E}$ for the expectation with respect to $Q^{\mathcal{I}}$. We have

$$\begin{aligned} &\mathbb{E}\Big(\mathbf{1}_{H \geq h(\lambda)}(1 - \frac{1}{1+\lambda Z})\Big) \\ = \; &\mathbb{E}\Big(\mathbf{1}_{Z \geq \mathrm{e}^{h(\lambda)}}(1 - \frac{1}{1+\lambda Z})\Big) - \mathbb{E}\Big(\mathbf{1}_{\mathrm{e}^H < \mathrm{e}^{h(\lambda)} \leq Z}(1 - \frac{1}{1+\lambda Z})\Big). \end{aligned} \tag{4.30}$$

For $\kappa < 1$, the second term can be bounded by

$$\begin{aligned} \mathbb{E}\Big(\mathbf{1}_{\mathrm{e}^H < \mathrm{e}^{h(\lambda)} \leq Z}(1 - \frac{1}{1+\lambda Z})\Big) &\leq \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E}\Big(\mathbf{1}_{\lfloor H \rfloor = p} \frac{\lambda Z}{1+\lambda Z}\Big) \\ &= \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E}\Big(\mathbf{1}_{\lfloor H \rfloor = p} \mathbb{E}\Big(\frac{\lambda Z}{1+\lambda Z} \mid \lfloor H \rfloor = p\Big)\Big) \\ &\leq \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E}\Big(\mathbf{1}_{\lfloor H \rfloor = p} \frac{c\lambda \mathrm{e}^p}{1 + c\lambda \mathrm{e}^p}\Big), \end{aligned}$$

where, in the last inequality, we used Jensen inequality and Corollary 4.7, and where c denotes a constant independent of λ (which may change from line to line). Now, since $\mathbb{P}(\lfloor H \rfloor = p) \leq c\mathrm{e}^{-\kappa p}$ for a positive constant c, we get that

$$\begin{aligned} \mathbb{E}\Big(\mathbf{1}_{\mathrm{e}^H < \mathrm{e}^{h(\lambda)} \leq Z}(1 - \frac{1}{1+\lambda Z})\Big) &\leq c\lambda \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathrm{e}^{(1-\kappa)p} \leq c'\lambda \mathrm{e}^{(1-\kappa))h(\lambda)} \\ &\leq c'\lambda^\kappa(\lambda \mathrm{e}^{h(\lambda)})^{1-\kappa} = o(\lambda^\kappa), \end{aligned}$$

for $\kappa < 1$, since $\lambda \mathrm{e}^{h(\lambda)} \to 0$, $\lambda \to 0$.

By integration by part, we see that the first term of (4.30) is equal to

$$\begin{aligned} &\mathbb{E}\Big(\mathbf{1}_{Z \geq h(\lambda)}(1 - \frac{1}{1+\lambda Z})\Big) \\ = \; &\Big[\frac{\lambda z}{1+\lambda z}\mathbb{P}(Z \geq z)\Big]_{\mathrm{e}^{h(\lambda)}}^{\infty} + \int_{\mathrm{e}^{h(\lambda)}}^{\infty} \frac{\lambda}{(1+\lambda z)^2}\mathbb{P}(Z \geq z)\,\mathrm{d}z. \end{aligned}$$

The first term is lower than

$$c\lambda \mathrm{e}^{(1-\kappa)h(\lambda)} = c\lambda^{\kappa}(\lambda \mathrm{e}^{h(\lambda)})^{1-\kappa} = o(\lambda^{\kappa}),$$

for $\kappa < 1$. For the second term, let us suppose first that

$$h(\lambda) \to \infty.$$

We can estimate $\mathbb{P}(Z \geq z)$ by

$$(\frac{C_U}{\mathbb{P}^Q(H=S)} - \eta)z^{-\kappa} \leq \mathbb{P}(Z \geq z) \leq (\frac{C_U}{\mathbb{P}^Q(H=S)} + \eta)z^{-\kappa},$$

for any η, when λ is sufficiently small. Hence we are lead to compute the integral

$$\int_{\mathrm{e}^{h(\lambda)}}^{\infty} \frac{\lambda}{1+\lambda z} z^{-\kappa}\, \mathrm{d}z = \lambda^{\kappa} \int_{\frac{\lambda \mathrm{e}^{h(\lambda)}}{1+\lambda \mathrm{e}^{h(\lambda)}}}^{1} x^{-\kappa}(1-x)^{\kappa}\, \mathrm{d}x,$$

(making the change of variables $x = \lambda z/(1+\lambda z)$). For $\kappa < 1$ this integral converges, when $\lambda \to 0$, to

$$\Gamma(\kappa+1)\Gamma(-\kappa+1) = \frac{\pi\kappa}{\sin(\pi\kappa)}.$$

If $h(\lambda)$ does not converge to ∞ when λ tends to 0, then we can take $\overline{h}(\lambda)$ such that $h(\lambda) \leq \overline{h}(\lambda)$, and such that $\overline{h}(\lambda) \to \infty$, $\lambda \mathrm{e}^{\overline{h}(\lambda)} \to 0$. The part of the integral between $h(\lambda)$ and $\overline{h}(\lambda)$ is of order $o(\lambda^{\kappa})$. For the part between $\overline{h}(\lambda)$ and ∞ it is the previous estimate (this is essentially the same as proving Tauber's theorem). □

Remark 4.5 : Let us make a final remark useful for [40]. If we truncate the series M_1 on the right and on the left when V_k reaches the level $A > 0$, and if we truncate M_2 when $H - V_k$ reaches the level A then the results of Theorem 4.2 and Corollary 4.3 remain valid just by replacing in the tail estimate M by the meander truncated at level A. More precisely, let $A > 0$ and consider

$$\overline{M}_1 = \sum_{k=t_1^-}^{t_1^+} \mathrm{e}^{-V_k}, \quad \overline{M}_2 = \sum_{k=t_2^-}^{t_2^+} \mathrm{e}^{V_k - H}$$

where

$$t_1^- = \sup\{k \leq 0,\ V_k \geq A\}, \quad t_1^+ = \inf\{k \geq 0,\ V_k \geq A\} \wedge T_H$$
$$t_2^- = \sup\{k \leq T_H,\ H - V_k \geq A\} \vee 0, \quad t_2^+ = \inf\{k \geq T_H,\ V_k \geq A\}$$

then the results of Theorem 4.2 and Corollary 4.3 remain valid when we consider $\overline{Z} = \mathrm{e}^H \overline{M}_1 \overline{M}_2$ instead of Z, if we replace in the tail estimate M by $\overline{M} = \sum_{t_-}^{t_+} \mathrm{e}^{-V_k}$ where t_- and t_+ are the hitting times of the level A on the left and on the right. Indeed, in the proof of Theorem 4.2 we see that considering the truncated $\overline{M}_1$ and $\overline{M}_2$ only simplifies the proof: we don't need to truncate M_1 and M_2 as we did. In particular, it implies that in Corollary 4.3 we can truncate M_1 and M_2 at a level $\underline{h}(\lambda) \leq h(\lambda)$: if $\underline{h}(\lambda)$ tends to ∞, we have exactly the same result.

CHAPTER 5

Limit laws for transient random walks in random environment on $\mathbb{Z}$

We consider transient random walks in random environment on $\mathbb{Z}$ with zero asymptotic speed. A classical result of Kesten, Kozlov and Spitzer says that the hitting time of the level n converges in law, after a proper normalization, towards a positive stable law, but they do not obtain a description of its parameter. A different proof of this result is presented, that leads to a complete characterization of this stable law. The case of Dirichlet environment turns out to be remarkably explicit.

Contents

The material of this chapter is a joint work with N. Enriquez and C. Sabot and has been submitted for publication, see [40].

1. Introduction

One-dimensional random walks in random environment to the nearest neighbors have been introduced in the sixties in order to give a model of DNA replication. Recently, this model has known a strong revival in view of applications to the detection of genetics anomalies (see for instance [23] or [75]). In 1975, Solomon gives, in a seminal work [102], a criterion of transience-recurrence for these walks, and shows that three different regimes can be distinguished: the random walk may be recurrent, or transient with a positive asymptotic speed, but it may also be transient with zero asymptotic speed. This last regime, which does not exist among usual random walks, is probably the one which is the less well understood and its study is the purpose of the present paper.

Let us first remind the main existing results concerning the other regimes. In his paper, Solomon computes the asymptotic speed of transient regimes. In 1982, Sinai states, in [98], a limit theorem in the recurrent case. It turns out that the motion in this case is unusually slow since the position of the walk at time n has to be normalized by $(\log n)^2$ in order to present a non trivial limit. In 1986, the limiting law is characterized independently by Kesten [69] and Golosov [51]. Let us notice here that, beyond the interest of his result, Sinai introduces a very powerful and intuitive tool in the study of one-dimensional random walks in random environment. This tool is the potential, which is a function on $\mathbb{Z}$ canonically associated to the random environment. It turns out to be an usual random walk when the transition probabilities at each site are independent and identically distributed (i.i.d.).

Let us now focus on the works about the transient walk with zero asymptotic speed. The main result was obtained by Kesten, Kozlov and Spitzer in [70] who proved that, when normalized by a suitable power of n, the hitting time of the level n converges towards a positive stable law whose index corresponds to the power of n lying in the normalization. Recently, Mayer-Wolf, Roitershtein and Zeitouni [80] generalized this result to the case when the environment is defined by an irreducible Markov chain.

Our purpose is to characterize the positive stable law in the case of i.i.d. transition probabilities. Let us mention here that the stable limiting law has been characterized in the case of diffusions in random potential when the potential is either a Brownian motion with drift [66], [61] or a Lévy process [99], but we remind here that despite the similarities of both models one cannot transport results from the continuous model to the discrete one.

The proof chooses a radically different approach than previous ones dealing with the transient case. While the proofs in [70] and [80] are mainly based on the representation of the trajectory of the walk in terms of branching processes in random environment (with immigration), our approach relies heavily on Sinai's interpretation of a particle living in a random potential. However, in the recurrent case, the potential one has to deal with is a recurrent random walk and Sinai introduces a notion of valleys which does not make sense anymore in our setting where the potential is a (let's say negatively) drifted random walk. Therefore, we introduce a different notion of valley which is closely related to the excursions of this random walk above its past minimum. It turns out that a result of Iglehart [63] gives an equivalent of the tail of the height of these excursions. Now, as soon as one can prove that the hitting time of the level n can be reduced to the time spent by the random walk to cross the high excursions of the potential above its past minimum, between 0 and n, which are well separated in space, an i.i.d. property comes out, and the problem is reduced to the study of the tail of the time spent by the walker to cross a single excursion.

It turns out that this tail involves the expectation of the functional of some meander associated with the random walk defining the potential. Now, this functional is itself related to the constant that appears in Kesten's renewal theorem [68]. These last two facts are contained in [41]. Now, in the case when the transition probabilities follow some Beta distribution a result of Chamayou and Letac [18] gives an explicit formula for this constant which yields finally an explicit formula for the parameter of the positive stable law which is obtained at the limit.

Soon after finishing this article, we learnt of an independent work, by Peterson and Zeitouni [81], which, by the study of the fluctuations of the potential, showed that a quenched stable limit law is not possible in the zero asymptotic speed regime.

The paper is organized as follows: the results are stated in Section 2, a detailed sketch of the proof is presented in Section 3, and the rest of the paper is devoted to proofs.

2. Notations and main results

Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of i.i.d. random variables taking values in $(0,1)$ defined on Ω, which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation. Conditioning on ω (i.e. choosing an environment), we define the random walk in random environment $(X_n,\, n \geq 0)$ as a nearest-neighbor random walk on $\mathbb{Z}$ with transition probabilities given by ω: $(X_n,\, n \geq 0)$ is the Markov chain satisfying $X_0 = 0$ and for $n \geq 0$,

$$P_\omega\left(X_{n+1} = x+1 \,|\, X_n = x\right) = \omega_x = 1 - P_\omega\left(X_{n+1} = x-1 \,|\, X_n = x\right).$$

We denote by P_ω the law of $(X_n,\, n \geq 0)$ and E_ω the corresponding expectation. We denote by $\mathbb{P}$ the joint law of $(\omega, (X_n)_{n\geq 0})$. We refer to Zeitouni [113] for an overview of results on random walks in random environment.

In the study of one-dimensional random walks in random environment, an important role is played by a process called the potential, denoted by $V = (V(x),\ x \in \mathbb{Z})$. Let us introduce

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \qquad i \in \mathbb{Z}.$$

Then, the potential is a function of the environment ω, and is defined as follows:

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^{0} \log \rho_i & \text{if } x \leq -1. \end{cases}$$

Furthermore, we consider the weak descending ladder epochs for the potential defined by $e_0 := 0$ and

$$e_i := \inf\{k > e_{i-1} :\ V(k) \leq V(e_{i-1})\}, \qquad i \geq 1,$$

which play a crucial role in our proof. Observe that $(e_i - e_{i-1})_{i\geq 1}$ is a family of i.i.d. random variables. Moreover, classical results of fluctuation theory (see [44], p. 396), tell us that, under assumptions (a)-(b) of Theorem 5.1,

$$E[e_1] < \infty. \tag{5.1}$$

Now, observe that the $((e_i, e_{i+1}])_{i\geq 0}$ stand for the set of excursions of the potential above its past minimum. Let us introduce H_i, the height of the excursion $(e_i, e_{i+1}]$ defined by $H_i := \max_{e_i \leq k \leq e_{i+1}} (V(k) - V(e_i))$, for $i \geq 0$. Note that the $(H_i)_{i\geq 0}$'s are i.i.d. random variables.

We now introduce the hitting time $\tau(x)$ of level x for the random walk $(X_n,\ n \geq 0)$,

$$\tau(x) := \inf\{n \geq 1 :\ X_n = x\}, \quad x \in \mathbb{Z}. \tag{5.2}$$

For $\alpha \in (0,1)$, let $\mathcal{S}_\alpha^{ca}$ be a completely asymmetric stable random variable of index α with Laplace transform, for $\lambda > 0$,

$$E[\mathrm{e}^{-\lambda \mathcal{S}_\alpha^{ca}}] = \mathrm{e}^{-\lambda^\alpha}.$$

Moreover, let us introduce the constant C_K describing the tail of Kesten's renewal series, see [68], defined by $R := \sum_{k\geq 0} \mathrm{e}^{V(k)}$:

$$P\{R > x\} \sim \frac{C_K}{x^\kappa}, \qquad x \to \infty. \tag{5.3}$$

Then the main result of the paper can be stated as follows. The symbol "$\xrightarrow{\text{law}}$" denotes the convergence in distribution.

THEOREM 5.1. *Let $\omega := (\omega_i,\ i \in \mathbb{Z})$ be a family of independent and identically distributed random variables such that*

(a) there exists $0 < \kappa < 1$ for which $E\left[\rho_0^\kappa\right] = 1$ and $E\left[\rho_0^\kappa \log^+ \rho_0\right] < \infty$,

(b) the distribution of $\log\rho_0$ *is non-lattice.*

Then, we have, when n *goes to infinity,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} 2\left(\frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log\rho_0]\right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} \frac{\sin(\pi\kappa)}{2^\kappa \pi\kappa^2 C_K^2 E[\rho_0^\kappa \log\rho_0]} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa.$$

REMARK 5.1. *We think that the method used in this paper could also treat the case* $\kappa = 1$ *(see Section 9 for conjecture and comments).*

The result of Theorem 5.1 is interesting when C_K is explicitly known. In the case of Dirichlet environment, i.e. when the law of the environment satisfies $\omega_1(\,\mathrm{d}x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{[0,1]}(x)\,\mathrm{d}x$, with $\alpha, \beta > 0$ and $B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$, things can be made much more explicit. The assumption of Theorem 5.1 correspond to the case where $0 < \alpha - \beta < 1$ and an easy computation leads to $\kappa = \alpha - \beta$.

COROLLARY 5.1. *In the case when* ω_1 *has a distribution* $\mathrm{Beta}(\alpha,\beta)$, *with* $0 < \alpha - \beta < 1$, *Theorem 5.1 applies with* $\kappa = \alpha - \beta$. *Then, we have, when* n *goes to infinity,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} 2\left(\frac{\pi}{\sin(\pi(\alpha-\beta))} \frac{\psi(\alpha)-\psi(\beta)}{B(\alpha,\beta)^2}\right)^{\frac{1}{\alpha-\beta}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} \frac{\sin(\pi(\alpha-\beta))}{2^{\alpha-\beta}\pi} \frac{B(\alpha,\beta)^2}{\psi(\alpha)-\psi(\beta)} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa,$$

where ψ *denotes the classical Digamma function,* $\psi(z) := (\log\Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

In the case where C_K is unknown, it is possible to give a probabilistic representation of the parameter. Actually, we obtain first Theorem 5.2, from which we deduce Theorem 5.1. In this aim, let us introduce the classical distribution $\tilde{P}$ associated with the random walk $(V(x), x \in \mathbb{Z})$ under P (denoted by ${}^aP$ in [44], p. 406). If μ denotes the law of $\log\rho_0$, thanks to assumption (a) of Theorem 5.1 we can define the law $\tilde{\mu} = \rho_0^\kappa \mu$, and the law $\tilde{P} = \tilde{\mu}^{\otimes\mathbb{Z}}$ which is the law of a sequence of i.i.d. random variables with law $\tilde{\mu}$. The definition of κ implies that $\int \log\rho\,\tilde{\mu}(d\rho) > 0$.

THEOREM 5.2. *Let* $\omega := (\omega_i,\, i \in \mathbb{Z})$ *be a family of independent and identically distributed random variables satisfying assumptions* (a)–(b) *of Theorem 5.1. Then, we have, when* n *goes to infinity,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} 2\left(\frac{\pi}{\sin(\pi\kappa)} \frac{E[M^\kappa]^2}{E[e_1]^2} \frac{(1-E[\mathrm{e}^{\kappa V(e_1)}])^2}{E[\rho_0^\kappa \log\rho_0]}\right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} \frac{\sin(\pi\kappa)}{2^\kappa \pi} \frac{E[e_1]^2}{E[M^\kappa]^2} \frac{E[\rho_0^\kappa \log \rho_0]}{(1 - E[\mathrm{e}^{\kappa V(e_1)}])^2} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa .$$

where M has the law of the exponential of a meander, i.e.

$$M \stackrel{\text{law}}{=} \sum_{k<0} \mathrm{e}^{-V'_k} + \sum_{k\geq 0} \mathrm{e}^{-V''_k},$$

with $(V'_k)_{k<0}$ under $P\{\cdot | V'_k \geq 0, \forall k < 0\}$ and independent of $(V''_k)_{k\geq 0}$ under $\tilde{P}\{\cdot | V''_k > 0, \forall k > 0\}$.

Remark 5.2. *When C_K is not explicit it is better to use the expression of the parameter in terms of $E[M^\kappa]$ which is easy to evaluate numerically.*

In the following, the constant C stands for a positive constant large enough, whose value can change from line to line.

3. Sketch of the proof

Let us start now with the outlines of our proof.

Since assumption (a) of Theorem 5.1 implies $E[\log \rho_0] < 0$, the random walk describing the potential is negatively drifted, so that the random walker will converge almost surely to the region of lowest potential, i.e. to infinity. Along its way, it will have to overcome some obstacles which are represented by the *excursions* of the random potential above its past minimum.

Now, a result of Iglehart [63] says that, under assumptions (a)-(b) of Theorem 5.1, the tail of the height H of an excursion above its past minimum is given by

$$P\{H > h\} \sim C_I \,\mathrm{e}^{-\kappa h}, \qquad h \to \infty, \tag{5.4}$$

where

$$C_I = \frac{(1 - E[\mathrm{e}^{\kappa V(e_1)}])^2}{\kappa E[\rho_0^\kappa \log \rho_0] E[e_1]}, \tag{5.5}$$

with e_1 denoting the endpoint of the first excursion, so that $V(e_1) \leq 0$. Iglehart's result is actually deduced from a former well-known result of Cramer, whose proof was later simplified by Feller [44], concerning the tail of the maximum S of a $\mathbb{N}$-time indexed random walk which claims that

$$P\{S > h\} \sim C_F \mathrm{e}^{-\kappa h}, \qquad h \to \infty. \tag{5.6}$$

Since S is stochastically bigger than H, C_I must be smaller than C_F, and a rather straight argument of Iglehart shows that the ratio between both constants is equal to $1 - E[\mathrm{e}^{\kappa V(e_1)}]$.

Recalling (5.1), the law of large numbers implies that the number of excursions between 0 and n is almost surely equivalent to $n/E[e_1]$. We will be therefore interested in the asymptotic of the hitting time of the n-th excursion, we will denote by $\tau(e_n)$.

3.1. The general case. In a first step, we show (see Lemma 5.10) that $\tau(e_n)$ reduces to the time spent by the walker to climb high excursions, namely, higher than $h_n := \frac{(1-\varepsilon)}{\kappa}\log n$. Let us notice here, that, statistically, by Iglehart's result, no excursion of height larger than $\frac{(1+\varepsilon)}{\kappa}\log n$ can be found among the first n excursions.

It turns out that these excursions are spatially well separated (see Lemma 5.3), and that there are asymptotically $nP\{H \geq h_n\}$ of these, i.e. $C_I n^\varepsilon$ (see Lemma 5.2). One can therefore define boxes around, we shall denote by $([a_k, d_k])_{0\leq k\leq C_I n^\varepsilon}$, such that the random walker will have a small probability to go back to a box which was already visited. More precisely, let b_k and c_k denote respectively the starting point of the k-th high excursion and the first time this excursion reaches its maximum, so that the following ranking $a_k \leq b_k \leq c_k \leq d_k$ holds. With an overwhelming probability, for all $k \in [0, C_I n^\varepsilon]$, the walker, once arrived at b_k, will never visit a_k again (see Lemma 5.9).

In addition, one can prove that the portions of potential between a_k and d_k, we call "deep valleys" are almost i.i.d. The proof of this fact requires the introduction of what we call "$*$-valleys" which are i.i.d., and coincide with the sequence of "deep valleys" with a high probability (see Lemma 5.5).

Now, gathering these two previous facts, we get that $\tau(e_n)$ can be roughly written:

$$\tau(e_n) = \tau(b_1, d_1) + \ldots + \tau(b_{C_I n^\varepsilon}, d_{C_I n^\varepsilon}),$$

where the $\tau(b_k, d_k)$'s are i.i.d. random variables representing the time spent by the walker to cross the k-th excursion, i.e. to go from b_k to d_k.

Consequently, considering the Laplace transform of $n^{-1/\kappa}\tau(e_n)$, we are led to the study of the asymptotic when λ goes to 0 of $\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(b_1,d_1)}]^{C_I n^\varepsilon}$ (see Proposition 5.1).

Now, the passage from b_1 to d_1 can be decomposed into the sum of a random geometrically distributed number of unsuccessful attempts to cross the excursion, followed by a successful attempt. The accurate estimation of the time spent by each (successful and unsuccessful) attempt leads us to consider two $h-$processes where the random walker evolves in two modified potentials, one corresponding to the conditioning on a failure (potential $\widehat{V}$, see Lemma 5.11), and the other to the conditioning on a success (potential $\overline{V}$, see Lemma 5.12).

It turns out that the contribution of the last successful attempt to the quantity $\tau(b_1,d_1)$ is negligible so that $\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(b_1,d_1)}]^{C_I n^\varepsilon}$ is approximately equal to

$$E\Bigg[\sum_{k\geq 0}(1-p(\omega))E_\omega[\mathrm{e}^{-\lambda F}]^k p(\omega)^k\Bigg]^{C_I n^\varepsilon} = E\Bigg[\frac{1-p(\omega)}{1-p(\omega)\,E_\omega[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}F}]}\Bigg]^{C_I n^\varepsilon},$$

where F denotes the time of an unsuccessful attempt (failure), and $1-p(\omega)$ denotes the (small) probability of success which is known, by classical arguments, to be equal to $\omega_b \frac{\mathrm{e}^{V(b)}}{\sum_{x=b}^{d-1}\mathrm{e}^{V(x)}}$ (a first step of probability ω_b to go to $b+1$ and then, starting at $b+1$, a probability $\frac{\mathrm{e}^{V(b)}}{\sum_{x=b}^{d-1}\mathrm{e}^{V(x)}}$ to hit d before b).

Now, a key step consists in the fact that the linearization $E_\omega[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}F}] \sim 1 - \frac{\lambda}{n^{1/\kappa}}E_\omega[F]$ can be justified. The error is expressed in terms of $E_\omega[F^2]$ which is explicitly computed (see Lemma 5.11) and dominated by a function of the maximal fall of the potential during its rise from $V(b)$ to $V(c)$, and the maximal rise of the potential during its fall from $V(c)$ to $V(d)$ which can be uniformly controlled on all the $C_I n^\varepsilon$ boxes (see Lemma 5.13). We are therefore led to the study of

$$\left(E\Bigg[\frac{1}{1+\frac{\lambda}{n^{1/\kappa}}\frac{p}{1-p}E_\omega[F]}\Bigg]\right)^{C_I n^\varepsilon}.$$

Now, $E_\omega[F]$ is known to be equal to $2\omega_b\sum_{a+1}^{d-1}\mathrm{e}^{-(\widehat{V}(x)-\widehat{V}(b))}$. Therefore we are back to the study of

$$\left(E\Bigg[\frac{1}{1+\frac{2\lambda}{n^{1/\kappa}}\mathrm{e}^H\widehat{M}_1 M_2}\Bigg]\right)^{C_I n^\varepsilon},$$

where $H = \mathrm{e}^{V(c)-V(b)}$ denotes the height of an high excursion and where $\widehat{M}_1 := \sum_{a+1}^{d-1}\mathrm{e}^{-(\widehat{V}(x)-\widehat{V}(b))}$ and $M_2 := \sum_{b}^{d-1}\mathrm{e}^{-(V(x)-V(c))}$ are two functionals of the potential that depends very locally on the potential respectively around the local minimum b and the local maximum c.

Since $V(b)$ and $V(c)$ are locally extremal, these functionals can be assimilated to two functionals of *meanders* associated to the random walk defining the potential. Furthermore, a reversal time argument and the proximity of V and $\widehat{V}$ around b show that these two quantities are asymptotically the same functionals of the same meander. It is defined as follows $M := \sum_{n\in\mathbb{Z}}\mathrm{e}^{-Y_n}$, where Y_n is the random walk of step $\log\rho$, conditioned to be positive on all $\mathbb{Z}$. This conditioning has to be understood as follows: on $\mathbb{Z}_-$ it is the natural one (we condition on an event having a strictly positive probability), whereas on $\mathbb{Z}_+$ it represents the limit in law of random walks of step $\log\rho$ that are conditioned to overshoot a high level before visiting $\mathbb{R}_-$ (see for instance the paper of Bertoin and Doney [6] and the references therein for detailed discussions on the subject) .

Furthermore, it turns out that the three quantities e^H, $\widehat{M}_1$ and M_2 are asymptotically independent. This delicate step based on coupling arguments, which are adapted from the proof of the renewal theorem for the sum of i.i.d. variables, is treated in Chapter 4, see Proposition 4.1. As a consequence, the tail of $\mathrm{e}^H \widehat{M}_1 M_2$ can be derived, see Theorem 4.2 in Chapter 4, as well as a Tauberian result about $\frac{1}{1+\lambda \mathrm{e}^H \widehat{M}_1 M_2}$, see Corollary 4.3 in Chapter 4. This Tauberian result yields to

$$\left(E\left[\frac{1}{1+\frac{2\lambda}{n^{1/\kappa}}\mathrm{e}^H \widehat{M}_1 M_2}\right]\right)^{C_I n^{\varepsilon}} = \exp\left\{-\left(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} E[M^\kappa]^2 C_I\right)\lambda^\kappa\right\} + o(1).$$

where C_I is given in (5.5). Now, one can be tempted to express the functional $E[M^\kappa]$ in terms of the more usual constant C_K, see (5.3). This is the content of Theorem 4.1 in Chapter 4, which yields

$$C_K = E[M^\kappa] C_F = E[M^\kappa] \frac{(1-E[\mathrm{e}^{\kappa V(e_1)}])}{\kappa E[\rho_0^\kappa \log \rho_0] E[e_1]}.$$

Therefore, the Laplace transform of $n^{-1/\kappa}\tau(e_n)$ writes

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(e_n)}] &= \exp\left\{-\left(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} \frac{C_K^2 C_I}{C_F^2}\right)\lambda^\kappa\right\} + o(1) \\
&= \exp\left\{-\left(2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0] E[e_1]\right)\lambda^\kappa\right\} + o(1).
\end{aligned}$$

Finally, since, by the law of large numbers, e_n/n converges a.s. to $E[e_1]$, we conclude that

$$\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(n)}] = \exp\left\{-\left(2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]\right)\lambda^\kappa\right\} + o(1).$$

Hence, we obtain that the limit is the positive stable law with index κ and parameter $2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]$.

3.2. The case of a Dirichlet environment. In the case of a Dirichlet environment, namely when $\omega_1(dx) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} 1_{[0,1]}(x)dx$, $(\alpha, \beta > 0)$ things can be made much more explicit. The assumptions of Theorem 5.1 correspond to the case when $0 < \alpha - \beta < 1$ and an easy computation shows that $\kappa = \alpha - \beta$. Now, a classical argument of derivation under the sign integral shows that

$$E[\rho_0^\kappa \log \rho_0] = \psi(\alpha) - \psi(\beta),$$

where ψ denotes the classical Digamma function $\psi(z) := (\log \Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Furthermore, a work of Chamayou and Letac [18] shows that C_K can be made explicit. Indeed, with the notations of [18], ρ_0 follows the law $\beta^{(2)}_{p,q}(dx) := \frac{1}{B(p,q)} x^{p-1}(1+x)^{-p-q} 1_{\mathbb{R}_+}(x)dx$ with $p = \beta$ and $q = \alpha$. Then, Example 9 of [18] says that $\sum_{k\geq 1} \mathrm{e}^{V(k)}$ follows the law of $\beta^{(2)}_{\beta,\alpha-\beta}$ having density $\frac{1}{B(\alpha,\beta)} x^{\beta-1}(1+x)^{-\alpha} 1_{\mathbb{R}_+}(x)$. But we have $\beta^{(2)}_{\beta,\alpha-\beta}([t,+\infty[) \sim \frac{1}{(\alpha-\beta)B(\alpha,\beta)} \frac{1}{t^{\alpha-\beta}}$, $t \to \infty$. Hence, $C_K = \frac{1}{(\alpha-\beta)B(\alpha,\beta)}$.

The expression of the parameter can be simplified into

$$2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0] = \frac{\pi 2^{\alpha-\beta}}{\sin(\pi(\alpha-\beta))} \frac{\psi(\alpha)-\psi(\beta)}{B(\alpha,\beta)^2}.$$

4. Two notions of valleys

Sinai introduced in [98] the notion of valley in a context where the random walk defining the potential was recurrent. We have to do a similar job in our framework where the random walk defining the potential is negatively drifted. The deep valleys we introduce here are closely related to the excursions of the random walk above its past minimum which are higher than a critical height. They consist actually in some portion of potential including these excursions. When the critical height is taken sufficiently large, the excursions are quite seldom and the valleys are likely to be disjoint. In order to deal with almost sure disjoint valleys, we also introduce $*$-valleys which coincide with deep valleys with high probability.

4.1. The deep valleys. Let us define the maximal variations of the potential before site x by:

$$V^{\uparrow}(x) := \max_{0\le i\le j\le x}(V(j)-V(i)), \qquad x\in\mathbb{N},$$
$$V^{\downarrow}(x) := \min_{0\le i\le j\le x}(V(j)-V(i)), \qquad x\in\mathbb{N}.$$

By extension, we introduce

$$V^{\uparrow}(x,y) := \max_{x\le i\le j\le y}(V(j)-V(i)), \qquad x<y,$$
$$V^{\downarrow}(x,y) := \min_{x\le i\le j\le y}(V(j)-V(i)), \qquad x<y.$$

In order to define deep valleys, we extract from the first n excursions of the potential above its minimum, these whose heights are greater than a critical height h_n, defined by

$$h_n := \frac{(1-\varepsilon)}{\kappa}\log n, \tag{5.7}$$

for some $0<\varepsilon<1/3$. Let $(\sigma(i))_{i\ge1}$ be the successive indexes of excursions, whose heights are greater than h_n. More precisely,

$$\begin{aligned}
\sigma(1) &:= \inf\{i\ge 0 : H_i\ge h_n,\},\\
\sigma(j) &:= \inf\{i>\sigma(j-1) : H_i\ge h_n\}, \qquad j\ge 2,\\
K_n &:= \max\{j\ge 0 : \sigma(j)\le n\}.
\end{aligned}$$

We consider now some random variables depending only on the environment, which define the deep valleys.

Definition 5.1. *For $1\le j\le K_n+1$, let us introduce*

$$\begin{aligned}
b_j &:= e_{\sigma(j)},\\
a_j &:= \sup\{k\le b_j : V(k)-V(b_j)\ge D_n\},\\
T_j^{\uparrow} &:= \inf\{k\ge b_j : V(k)-V(b_j)\ge h_n\},
\end{aligned}$$

$$\begin{aligned}
\overline{d}_j &:= e_{\sigma(j)+1}, \\
c_j &:= \inf\{k \geq b_j : V(k) = \max_{b_j \leq x \leq \overline{d}_j} V(x)\}, \\
d_j &:= \inf\{k \geq \overline{d}_j : V(k) - V(\overline{d}_j) \leq -D_n\}.
\end{aligned}$$

where $D_n := (1 + \frac{1}{\kappa}) \log n$. *We call* (a_j, b_j, c_j, d_j) *a deep valley and denote by* $H^{(j)}$ *the height of the* j*-th deep valley.*

REMARK 5.3. *It may happen that two different deep valleys are not disjoint, even if this event is highly improbable as it will be shown in Lemma 5.3 and Lemma 5.4 in Subsection 5.1.*

4.2. The $*$-valleys. Let us introduce now a subsequence of the deep valleys defined above. It will turn out that both sequences coincide with probability tending to 1 as n goes to infinity. This will be specified in Lemma 5.5. Let us first introduce

$$\begin{aligned}
\gamma_1^* &:= \inf\{k \geq 0 : V(k) \leq -D_n\}, \\
T_1^* &:= \inf\{k \geq \gamma_1^* : V^{\uparrow}(\gamma_1^*, k) \geq h_n\}, \\
b_1^* &:= \sup\{k \leq T_1^* : V(k) = \min_{0 \leq x \leq T_1^*} V(x)\}, \\
a_1^* &:= \sup\{k \leq b_1^* : V(k) - V(b_1^*) \geq D_n\}, \\
\overline{d}_1^* &:= \inf\{k \geq T_1^* : V(k) \leq V(b_1^*)\}, \\
c_1^* &:= \inf\{k \geq b_1^* : V(k) = \max_{b_1^* \leq x \leq \overline{d}_1^*} V(x)\}, \\
d_1^* &:= \inf\{k \geq \overline{d}_1^* : V(k) - V(\overline{d}_1^*) \leq -D_n\}.
\end{aligned}$$

Let us define the following sextuplets of points by iteration

$$(\gamma_j^*, a_j^*, b_j^*, T_j^*, c_j^*, \overline{d}_j^*, d_j^*) := (\gamma_1^*, a_1^*, b_1^*, T_1^*, c_1^*, \overline{d}_1^*, d_1^*) \circ \theta_{d_{j-1}^*}, \qquad j \geq 2,$$

where θ_i denotes the i-shift operator.

DEFINITION 5.2. *We call a $*$-valley any quadruplet* $(a_j^*, b_j^*, c_j^*, d_j^*)$ *for* $j \geq 1$. *Moreover, we shall denote by* K_n^* *the number of such $*$-valleys before* e_n, *i.e.* $K_n^* := \sup\{j \geq 0 : T_j^* \leq e_n\}$.

It will be made of independent and identically distributed portions of potential (up to some translation).

5. Reduction to a single valley

This section is devoted to the proof of Proposition 5.1 which tells that the study of $\tau(e_n)$ can be reduced to the analysis of the time spent by the random walk to cross the first deep valley. To ease notations, we introduce $\lambda_n := \frac{\lambda}{n^{1/\kappa}}$.

Proposition 5.1. *For all n large enough, we have*

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_n\,\tau(e_n)}\right] \quad\in\quad \left[E\left[E^{b_1}_{\omega,|a_1}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]^{\overline{K}_n}+o(1)\,,\,E\left[E^{b_1}_{\omega,|a_1}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]^{\underline{K}_n}+o(1)\right].$$

where $\underline{K}_n := \lfloor nq_n(1-n^{-\varepsilon/4})\rfloor$, $\overline{K}_n := \lceil nq_n(1+n^{-\varepsilon/4})\rceil$, $q_n := P\{H_0\geq h_n\}$ *and where* $E^x_{\omega,|y}$ *denotes the quenched law of the random walk in the environment* ω, *starting at* x *and reflected at site* y.

5.1. Introducing "good" environments. Let us define the four following events, that concern exclusively the potential V. The purpose of this subsection is to show that they are realized with an asymptotically overwhelming probability when n goes to infinity. These results will then make it possible to restrict the study of $\tau(e_n)$ to these events.

$$\begin{aligned}
A_1(n) &:= \{e_n < C'n\}\,,\\
A_2(n) &:= \left\{\lfloor nq_n(1-n^{-\varepsilon/4})\rfloor \leq K_n \leq \lceil nq_n(1+n^{-\varepsilon/4})\rceil\right\},\\
A_3(n) &:= \cap_{j=0}^{K_n}\left\{\sigma(j+1)-\sigma(j)\geq n^{1-3\varepsilon}\right\},\\
A_4(n) &:= \cap_{j=1}^{K_n+1}\left\{d_j-a_j\leq C''\log n\right\},
\end{aligned}$$

where $\sigma(0):=0$ (for convenience of notation) and C', C'' stand for positive constants which will be specified below.

In words, $A_1(n)$ allows us to bound the total length of the first n excursions. The event $A_2(n)$ gives a control on the number of deep valleys. The event $A_3(n)$ ensures that the deep valleys are well separated, while $A_4(n)$ bounds finely the length of each of them.

Let us introduce the following hitting times (for the potential)

$$\begin{aligned}
T_h &:= \min\{x\geq 0:\ V(x)\geq h\}, \qquad h>0,\\
T_A &:= \min\{x\geq 0:\ V(x)\in A\}, \qquad A\subset\mathbb{R}.
\end{aligned}$$

Then, we obtain the following results.

Lemma 5.1. *The probability* $P\{A_1(n)\}$ *converges to* 1 *when* n *goes to infinity.*

Proof. It is a direct consequence of the law of large numbers as soon as C' is taken bigger than $E[e_1]$. □

Lemma 5.2. *The probability* $P\{A_2(n)\}$ *converges to* 1 *when* n *goes to infinity.*

In words, Lemma 5.2 means that K_n "behaves" like $C_I n^{\varepsilon}$, when n tends to infinity. In particular, (5.4), which yields $q_n\sim\frac{C_I}{n^{1-\varepsilon}}$, and Lemma 5.2 imply

$$P\{K_n+1\geq 2C_I n^{\varepsilon}\}\to 0, \qquad n\to\infty. \tag{5.8}$$

PROOF. At first, observe that

$$P\{\frac{K_n}{nq_n} \geq 1 + n^{-\varepsilon/4}\} = P\{K_n - nq_n \geq n^{1-\varepsilon/4} q_n\} \leq \frac{\mathrm{Var}(K_n)}{n^{2(1-\varepsilon/4)} q_n^2},$$

the inequality being a consequence of Markov inequality and the fact that K_n follows a binomial distribution of parameter (n, q_n). Moreover, $\mathrm{Var}(K_n) = nq_n(1-q_n) \leq nq_n$ implies

$$P\{\frac{K_n}{nq_n} \geq 1 + n^{-\varepsilon/4}\} \leq \frac{1}{n^{1-\varepsilon/2} q_n}.$$

Now, Iglehart's result (see (5.4)) implies $q_n \sim \frac{C_I}{n^{1-\varepsilon}}$, $n \to \infty$. Therefore we get that $P\{\frac{K_n}{nq_n} \leq 1 + n^{-\varepsilon/4}\}$ converges to 1 when n goes to infinity. Using similar arguments, we get the convergence to 1 of $P\{\frac{K_n}{nq_n} \geq 1 - n^{-\varepsilon/4}\}$. □

LEMMA 5.3. *The probability* $P\{A_3(n)\}$ *converges to* 1 *when* n *goes to infinity.*

PROOF. We make first the trivial observation that

$$\begin{aligned} P\{A_3(n)\} &\geq P\{\sigma(j+1) - \sigma(j) \geq n^{1-3\varepsilon},\, 0 \leq j \leq \lfloor 2C_I n^{\varepsilon} \rfloor\, ;\, K_n \leq 2C_I n^{\varepsilon}\} \\ &\geq P\{\sigma(j+1) - \sigma(j) \geq n^{1-3\varepsilon},\, 0 \leq j \leq \lfloor 2C_I n^{\varepsilon} \rfloor\} - P\{K_n \geq 2C_I n^{\varepsilon}\}, \end{aligned}$$

the second inequality being a consequence of $P\{A\,;B\} \geq P\{A\} - P\{B^c\}$, for any couple of events A and B. Therefore, recalling (5.8) and using the fact that $(\sigma(j+1) - \sigma(j))_{0 \leq j \leq \lfloor 2C_I n^{\varepsilon} \rfloor}$ are i.i.d. random variables, it remains to prove that

$$P\{\sigma(1) \geq n^{1-3\varepsilon}\}^{\lfloor 2C_I n^{\varepsilon} \rfloor} \to 1, \qquad n \to \infty.$$

Since $\sigma(1)$ is a geometrical random variable with parameter q_n, $P\{\sigma(1) \geq n^{1-3\varepsilon}\}$ is equal to $(1-q_n)^{\lceil n^{1-3\varepsilon} \rceil}$, which implies

$$P\{\sigma(1) \geq n^{1-3\varepsilon}\}^{\lfloor 2C_I n^{\varepsilon} \rfloor} = (1-q_n)^{\lfloor 2C_I n^{\varepsilon} \rfloor \lceil n^{1-3\varepsilon} \rceil} \geq \exp\left\{-Cn^{1-2\varepsilon} q_n\right\}.$$

Then, the conclusion follows from (5.4), which implies that $q_n \sim C_I / n^{1-\varepsilon}$, $n \to \infty$. □

LEMMA 5.4. *For* C'' *large enough, The probability* $P\{A_4(n)\}$ *converges to* 1 *when* n *goes to infinity.*

PROOF. Looking at the proof of Lemma 5.3, we have to prove that $P\{d_j - a_j \geq C'' \log n\}$ is equal to a $o(n^{-\varepsilon})$, $n \to \infty$. Moreover, observing that $d_j - a_j = (d_j - \overline{d}_j) + (\overline{d}_j - T_j^{\uparrow}) + (T_j^{\uparrow} - b_j) + (b_j - a_j)$, the proof of Lemma 5.4 boils down to showing that, for C'' large enough,

$$P\{d_j - \overline{d}_j \geq \frac{C''}{4} \log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.9}$$

$$P\{\overline{d}_j - T_j^{\uparrow} \geq \frac{C''}{4}\log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.10}$$

$$P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4}\log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.11}$$

$$P\{b_j - a_j \geq \frac{C''}{4}\log n\} = o(n^{-\varepsilon}), \qquad n \to \infty. \tag{5.12}$$

To prove (5.9), we apply the strong Markov property at time $\overline{d}_j$ such that we get $P\{d_j - \overline{d}_j \geq \frac{C''}{4}\log n\} \leq P\{T_{(-\infty,-D_n]} \geq \frac{C''}{4}\log n\}$. Therefore, we have

$$P\{d_j - \overline{d}_j \geq \frac{C''}{4}\log n\} \leq P\{\inf_{0\leq x\leq \frac{C''}{4}\log n} V(x) > -D_n\} \leq P\{V(\frac{C''}{4}\log n) > -D_n\}.$$

Recalling that $D_n := (1+\frac{1}{\kappa})\log n$ and observing that large deviations do occur, we obtain, from Cramer's theory, that $P\{V(\frac{C''}{4}\log n) > -D_n\} \leq \mathrm{e}^{-\frac{C''}{4}\log n\, I(-\frac{4}{C''}(1+\frac{1}{\kappa}))}$, with $I(\cdot)$ the convex rate function associated to V. This inequality implies (5.9) by choosing C'' large enough such that $\frac{C''}{4} I(-\frac{4}{C''}(1+\frac{1}{\kappa})) > \varepsilon$, which is possible since $I(0) > 0$.

To prove (5.10), observe first that (5.4) implies $P\{H^{(j)} > \frac{(1+\varepsilon')}{\kappa}\log n\} \sim n^{-(\varepsilon'+\varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$. Therefore, we obtain that $P\{\overline{d}_j - T_j^{\uparrow} \geq \frac{C''}{4}\log n\}$ is less or equal than $P\{T_{(-\infty,-\frac{1+\varepsilon'}{\kappa}\log n]} \geq \frac{C''}{4}\log n\} + o(n^{-\varepsilon})$ and conclude the proof with the same arguments we used to treat (5.9).

To get (5.11), observe first that

$$\begin{aligned} P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4}\log n\} &= P\{T_{h_n} \geq \frac{C''}{4}\log n \,|\, H_0 \geq h_n\} \\ &\leq P\{\frac{C''}{4}\log n \leq T_{h_n} < \infty\}/P\{H_0 \geq h_n\}. \end{aligned}$$

Therefore, Cramer's theory, see [33], yields

$$\begin{aligned} P\{\frac{C''}{4}\log n \leq T_{h_n} < \infty\} &\leq \sum_{k\geq \frac{C''}{4}\log n} P\{V(k) \geq h_n\} \leq \sum_{k\geq \frac{C''}{4}\log n} \mathrm{e}^{-k\, I(\frac{h_n}{k})} \\ &\leq \sum_{k\geq \frac{C''}{4}\log n} \mathrm{e}^{-k\, I(0)} \leq \frac{C}{n^{\frac{C''}{4}I(0)}}, \end{aligned}$$

the second inequality being a consequence of the fact that the convex rate function $I(\cdot)$ is an increasing function on $(m, +\infty)$. Using (5.4), we get, for all large n,

$$P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4}\log n\} \leq \frac{C}{n^{\frac{C''}{4}I(0)-(1-\varepsilon)}},$$

which yields (5.11), by choosing C'' large enough such that $C'' > \frac{4}{I(0)}$.

For (5.12), observe first that $((V(k-b_j)-V(b_j))_{a_j\leq k\leq b_j}, a_j, b_j)$ has the same distribution as $((V(k))_{a^-\leq k\leq 0}, a^-, 0)$ under $P\{\cdot | V(k) \geq 0, a^- \leq k \leq 0\}$, where $a^- := \sup\{k \leq 0 : V(k) \geq D_n\}$. Then, since $P\{V(k) \geq 0, k \leq 0\} > 0$ and since $(V(-k),\, k \geq 0)$ has the same distribution as $(-V(k),\, k \geq 0)$, we obtain

$$P\{b_j - a_j \geq \frac{C''}{4} \log n\} \leq CP\{T_{(-\infty,-D_n]} > \frac{C''}{4} \log n\} \leq CP\{V(\frac{C''}{4} \log n) > -D_n\}.$$

Now, the arguments are the same as in the proof of (5.9). □

Defining $A(n) := A_1(n) \cap A_2(n) \cap A_3(n) \cap A_4(n)$, a consequence of Lemma 5.1, Lemma 5.2, Lemma 5.3 and Lemma 5.4, is that

$$P\{A(n)\} \to 1. \tag{5.13}$$

The following lemma tells us that the $*$-valleys coincide with the sequence of deep valleys with an overwhelming probability when n goes to infinity.

LEMMA 5.5. *If* $A^*(n) := \{K_n = K_n^*;\, (a_j, b_j, c_j, d_j) = (a_j^*, b_j^*, c_j^*, d_j^*),\, 1 \leq j \leq K_n\}$, *then we have that the probability* $P\{A^*(n)\}$ *converges to* 1, *when* n *goes to infinity.*

PROOF. Since, by definition, the $*$-valleys constitute a subsequence of the deep valleys, Lemma 5.5 is a consequence of Lemma 5.3 together with Lemma 5.4. □

REMARK 5.4. *Another meaning of this result is that, with probability tending to* 1, *two deep valleys are necessarily disjoint.*

5.2. Preparatory lemmas. In this subsection, we develop some technical tools allowing us to improve our understanding of the random walk's behavior. In Lemma 5.8, we prove that, after exiting a deep valley, the random walk will not come back to another deep valley it has already visited, with probability tending to one. Moreover, Lemma 5.9 specifies that the random walk typically exits from a $*$-valley on the right, while Lemma 5.10 shows that the time spent between two deep valleys is negligible.

5.2.1. *Preliminary estimates for inter-arrival times.* Let us first give a preliminary result concerning large deviations, more precisely about the convex rate function associated to the potential $V(\cdot)$, denoted by $I(\cdot)$.

LEMMA 5.6. *Under assumptions* (a)–(b), *we have*

$$\inf_{x \geq 0} \frac{I(x)}{x} = \kappa.$$

Moreover, the minimum is reached at $x_0 := \Lambda'(\kappa)$, *with* $\Lambda(t) := \log E[\rho_0^t]$.

Proof. Recalling that $I(\cdot)$ is defined by $I(x) := \sup_{t\geq 0}\{tx-\Lambda(t)\}$, for $x \geq 0$, we have $I(x) \geq \kappa x - \Lambda(\kappa) = \kappa x$, since $\Lambda(\kappa) = 0$. Moreover, under assumption (a)–(b), formula (2.2.10) in ([33], p. 28) implies $I(\Lambda'(\kappa)) = \kappa\Lambda'(\kappa)$, which concludes the proof of Lemma 5.6. □

Let us introduce

$$\begin{aligned} T^{\uparrow}(h) &:= \min\{x \geq 0 : V^{\uparrow}(x) \geq h\}, \qquad h > 0, \\ T^{\downarrow}(h) &:= \min\{x \geq 0 : V^{\downarrow}(x) \leq -h\}, \qquad h > 0. \end{aligned}$$

Lemma 5.7. *Under assumptions (a)–(b), we have, for h large enough,*

$$\mathbb{E}_{|0}[\tau_h] \leq C\,\mathrm{e}^h,$$

where $\mathbb{E}_{|0}$ denotes the expectation under the law $\mathbb{P}_{|0}$ of the random walk in the random environment ω (under P) reflected at 0 and $\tau_h := \tau(T^{\uparrow}(h)-1)$.

Proof. Using (Zeitouni [113], formula (2.1.14)), we obtain that $\mathbb{E}_{|0}[\tau_h]$ is bounded from above by $E\big[\sum_{0\leq i\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)-V(i)}\big]$. Therefore, since $T^{\uparrow}(h) \leq T^{\uparrow}(h)\circ\theta_i$, for any $i \geq 0$, we obtain

$$\mathbb{E}_{|0}[\tau_h] \leq \sum_{i\geq 0} E\Big[\mathbf{1}_{\{i<T^{\uparrow}(h)\}} \sum_{i\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)-V(i)}\Big] \leq \beta_1(h)\,\beta_2(h), \tag{5.14}$$

where

$$\begin{aligned} \beta_1(h) &:= E\big[T^{\uparrow}(h)\big], \\ \beta_2(h) &:= E\Big[\sum_{0\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)}\Big]. \end{aligned}$$

To bound $\beta_1(h)$, let us introduce the number N of complete excursions before $T^{\uparrow}(h)$, defined by $N = N(h) := \sup\{i \geq 0 : e_i < T^{\uparrow}(h)\}$. Then, we can write $\beta_1(h) = E[\sum_{i=0}^{N-1}(e_i - e_{i-1}) + (T^{\uparrow}(h) - e_N)]$. Observe that the definition of $T^{\uparrow}(h)$ implies that N is a geometrical random variable with parameter $q = q(h) := P\{H \geq h\}$ and recall that, by (5.4), we have $q \sim C_I\,\mathrm{e}^{-\kappa h}$, $h \to \infty$. Therefore, we get, for h large enough,

$$\begin{aligned} \beta_1(h) &\leq \sum_{k\geq 0}(1-q)^k q\big(kE[e_1|H<h] + E[T_h|H\geq h]\big) \\ &\leq C\sum_{k\geq 0}(1-q)^k q\big(kE[e_1] + E[T_h|H\geq h]\big), \end{aligned}$$

the second inequality being a consequence of the fact that $E[e_1] < \infty$ (see (5.1)) together with $P\{H < h\} \to 1$, $h \to \infty$, by (5.4). By obvious calculations, this yields $\beta_1(h) \leq C(1-q)q^{-1}E[e_1] + E[T_h|H \geq h]$, which implies with (5.4) that

$$\beta_1(h) \leq C\mathrm{e}^{\kappa h} + E[T_h|H \geq h]. \tag{5.15}$$

Now, let us bound $E[T_h|H\geq h]$. To this aim, we observe first that $E[T_h|H\geq h]\leq Ce^{\kappa h}\sum_{k\geq 0}(k+1)P\{T_h=k+1\,;\,H\geq h\}$. Then, applying the Markov property at time k, we get

$$\begin{aligned} E[T_h|H\geq h] &\leq Ce^{\kappa h}\sum_{k\geq 0}(k+1)E[\mathbf{1}_{\{0<V(k)<h\}}e^{-\kappa(h-V(k))}] \\ &\leq C\sum_{k\geq 0}(k+1)\sum_{j=0}^{\lfloor h\rfloor}e^{\kappa(j+1)}P\{V(k)\geq j\}. \end{aligned}$$

Since large deviations do occur, Cramer's theory, see [33], implies $P\{V(k)\geq j\}\leq e^{-kI(\frac{j}{k})}$. Now, recalling that $I(\cdot)$ is an increasing function on $\mathbb{R}^+$ together with Lemma 5.6, we obtain

$$P\{V(k)\geq j\}\leq e^{-k\frac{I(0)}{2}}e^{-\kappa\frac{j}{2}}.$$

Since $I(0)>0$, this yields that there exists $C>0$ such that, for all large h,

$$E[T_h|H\geq h]\leq Ce^{\frac{\kappa}{2}h}. \tag{5.16}$$

Assembling (5.15) and (5.16) implies, for h large enough,

$$\beta_1(h)\leq Ce^{\kappa h}. \tag{5.17}$$

In a second step, we bound $\beta_2(h)$. Let us first introduce $\mathcal{E}_k:=\{\max_{0\leq j\leq k-1}H_j<h\,;\,H_k\geq h\}$ and write

$$\begin{aligned} \beta_2(h) &= \sum_{k\geq 0}E\Big[\mathbf{1}_{\mathcal{E}_k}\sum_{0\leq j<T^{\uparrow}(h)}e^{V(j)}\Big] \\ &= \sum_{k\geq 0}\Big(\sum_{i=0}^{k-1}E\Big[\mathbf{1}_{\mathcal{E}_k}e^{V(e_j)}J_i\Big]+E\Big[\mathbf{1}_{\mathcal{E}_k}e^{V(e_k)}\overline{J}_k\Big]\Big), \end{aligned}$$

where $J_i:=\sum_{j=e_i}^{e_{i+1}}e^{V(j)-V(e_i)}$ for $i\geq 0$ and $\overline{J}_k:=\sum_{j=e_k}^{T^{\uparrow}(h)-1}e^{V(j)-V(e_k)}$ which is well defined on $\mathcal{E}_k$. Observe that $\mathcal{E}_k=\{N(h)=k\}$ and recall that $N(h)$ is a geometrical random variable with parameter $q=q(h)=P\{H\geq h\}$. Then, the Markov property applied at times $(e_j)_{1\leq j\leq k}$ yields that $\beta_2(h)$ is less or equal than

$$\sum_{k\geq 0}(1-q)^kq\Big(E[J_0|H_0<h]\sum_{j=0}^{k-1}E[e^{V(e_1)}|H_0<h]^j+E[\overline{J}_0|H_0\geq h]E[e^{V(e_1)}|H_0<h]^k\Big),$$

which implies that $\beta_2(h)$ is bounded by

$$\frac{1}{1-E[e^{V(e_1)}|H_0<h]}E[J_0|H_0<h]+\frac{q}{1-(1-q)E[e^{V(e_1)}|H_0<h]}E[\overline{J}_0|H_0\geq h].$$

Now, since V is transient to $-\infty$, then H_0 is almost surely finite and $E[e^{V(e_1)}|H_0<h]\to E[e^{V(e_1)}]<1$, when $h\to\infty$. Recalling that $q=q(h)\to 0$, $h\to\infty$, it follows that

$$\beta_2(h)\quad\leq\quad C\big(E[J_0|H_0<h]+qE[\overline{J}_0|H_0\geq h]\big), \tag{5.18}$$

for h large enough.

Let us first bound $E[\overline{J}_0|H_0 \geq h]$. Recall that if μ denotes the law of $\log \rho_0$, thanks to assumption (a) of Theorem 5.1 we can define the law $\tilde{\mu} = \rho_0^\kappa \mu$, and the law $\tilde{P} = \tilde{\mu}^{\otimes \mathbb{Z}}$ which is the law of a sequence of i.i.d. random variables with law $\tilde{\mu}$. The definition of κ implies that $\int \log \rho \, \tilde{\mu}(d\rho) > 0$. Then, using the Girsanov property between P and $\tilde{P}$, we can write

$$\begin{aligned} E[\overline{J}_0|H_0 \geq h] &\leq Ce^{\kappa h}\tilde{E}[e^{-\kappa V(T_h)}\overline{J}_0 \mathbf{1}_{\{H_0\geq h\}}] \\ &\leq C\tilde{E}\Big[e^{-\kappa(V(T_h)-h)}\sum_{k=0}^{T_h-1} e^{V(k)}\mathbf{1}_{\{H_0\geq h\}}\Big] \\ &\leq C\tilde{E}\Big[\sum_{k=0}^{T_h-1} e^{V(k)}\mathbf{1}_{\{\min_{0<i<T_h} V(k)>0\}}\Big] \\ &\leq C\tilde{E}\Big[\sum_{k\geq 0}\sum_{p=0}^{\lfloor h\rfloor} e^{V(k)}\mathbf{1}_{\{p\leq V(k)<p+1\}}\Big] \\ &\leq C\sum_{p=0}^{\lfloor h\rfloor} e^{p+1}\tilde{E}\Big[\sum_{k\geq 0}\mathbf{1}_{\{p\leq V(k)<p+1\}}\Big]. \end{aligned}$$

Moreover, Markov property yields $\tilde{E}[\sum_{k\geq 0}\mathbf{1}_{\{p\leq V(k)<p+1\}}] \leq \tilde{E}[\sum_{k\geq 0}\mathbf{1}_{\{0\leq V(k)<1\}}]$, which is finite since $(V(k))_{k\geq 0}$ has a positive drift under $\tilde{P}$.

Therefore, recalling (5.18) and (5.4), we get

$$\beta_2(h) \leq C(E[J_0|H_0 < h] + e^{(1-\kappa)h}) \tag{5.19}$$

and only have to bound $E[J_0|H_0 < h]$. Recall that $R = \sum_{k\geq 0} e^{V(k)}$ and observe that $J_0 \leq R$. Moreover, let us denote by $E^{\mathcal{I}}[\cdot]$ the expectation under $P^{\mathcal{I}}\{\cdot\} := P\{\cdot|\mathcal{I}\}$, with $\mathcal{I} := \{H = S\}$. Then, we first observe that $E^{\mathcal{I}}[R|H < h] \geq E[R\,\mathbf{1}_{\{H=S<h\}}] \geq E[J_0\mathbf{1}_{\{H=S<h\}}]$. Furthermore, since J_0 depends only on $(V(k)\,;\, 0 \leq k \leq e_1)$ and since $P\{V(k) \leq 0\,;\, k \geq 0\} > 0$, we get, by applying the strong Markov property at time e_1, that $E[J_0\mathbf{1}_{\{H<h\}}] \leq CE^{\mathcal{I}}[R|H < h]$, which implies

$$E[J_0|H < h] \leq CE^{\mathcal{I}}[R|H < h].$$

Therefore, we only have to prove that $E^{\mathcal{I}}[R|H < h] \leq Ce^{(1-\kappa)h}$. To this aim, we recall first that Corollary 4.1 in Chapter 4 implies that, $P^{\mathcal{I}}$-almost surely,

$$E^{\mathcal{I}}[R|\lfloor H\rfloor] \leq Ce^{\lfloor H\rfloor}. \tag{5.20}$$

Now, observe that $E^{\mathcal{I}}[R|H < h] \leq CE^{\mathcal{I}}[R\,\mathbf{1}_{\{H<h\}}]$ and let us write

$$E^{\mathcal{I}}[R\,\mathbf{1}_{\{H<h\}}] \leq \sum_{k=0}^{\lfloor h\rfloor} E^{\mathcal{I}}\Big[\mathbf{1}_{\{\lfloor H\rfloor=k\}}E^{\mathcal{I}}[R|\lfloor H\rfloor = k]\Big]$$

$$\leq \quad C \sum_{k=0}^{\lfloor h \rfloor} E^{\mathcal{I}} \left[\mathbf{1}_{\{\lfloor H \rfloor = k\}} \mathrm{e}^{\lfloor H \rfloor} \right]$$

$$\leq \quad C \sum_{k=0}^{\lfloor h \rfloor} \mathrm{e}^{k} P^{\mathcal{I}} \{ \lfloor H \rfloor = k \}$$

$$\leq \quad C \sum_{k=0}^{\lfloor h \rfloor} \mathrm{e}^{(1-\kappa)k} \leq C \mathrm{e}^{(1-\kappa)h}, \tag{5.21}$$

the second inequality is a consequence of (5.20) and the fourth inequality due to the fact that $P^{\mathcal{I}}\{\lfloor H \rfloor = k\} \leq c\mathrm{e}^{-\kappa p}$ for some positive constant c. Now assembling (5.14), (5.17), (5.19) and (5.21) concludes the proof of Lemma 5.7. □

5.2.2. *Important preliminary results.* Before establishing the announced lemmas, we introduce, for any $x, y \in \mathbb{Z}$,

$$\tau(x, y) := \inf\{k \geq 0 : \ X_{\tau(x)+k} = y\}.$$

Then, we have the following results.

LEMMA 5.8. *Defining* $DT(n) := A(n) \cap \bigcap_{j=1}^{K_n} \left\{ \tau(d_j, b_{j+1}) < \tau(d_j, \overline{d}_j) \right\}$, *we have*

$$P\left\{DT(n)\right\} \to 1, \qquad n \to \infty.$$

PROOF. Recalling (5.13), we only have to prove that

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_{\omega}^{d_j} \{ \tau(b_{j+1}) > \tau(\overline{d}_j) \} \right] \to 0. \tag{5.22}$$

By (Zeitouni [113], formula (2.1.4)), we get, for $1 \leq j \leq K_n$ and for all ω in $A(n)$:

$$P_{\omega}^{d_j} \left\{ \tau(b_{j+1}) > \tau(\overline{d}_j) \right\} = \frac{\sum_{k=d_j}^{b_{j+1}-1} \mathrm{e}^{V(k)}}{\sum_{k=\overline{d}_j}^{b_{j+1}-1} \mathrm{e}^{V(k)}} \leq (b_{j+1} - d_j) \mathrm{e}^{V(d_j) - V(\overline{d}_j) + h_n}.$$

Combining (5.17) and Markov inequality, we easily get that $b_{K_n+1} - d_{K_n} = o(n)$ with probability tending to 1. Moreover, by definition, $V(d_j) - V(\overline{d}_j) \leq -D_n$ for $1 \leq j \leq K_n$, and $b_{j+1} - d_j \leq e_n \leq C' n$, for $1 \leq j \leq K_n - 1$ on $A_1(n)$. Therefore, we have

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_{\omega}^{d_j} \{ \tau(b_{j+1}) > \tau(\overline{d}_j) \} \right] \leq C\, n E[K_n] \mathrm{e}^{-D_n + h_n}.$$

Recalling that $D_n = (1 + \frac{1}{\kappa}) \log n$, $h_n = \frac{1-\varepsilon}{\kappa} \log n$ and since $E[K_n] \leq C\, n^{\varepsilon}$, we obtain

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_{\omega}^{d_j} \{ \tau(b_{j+1}) > \tau(\overline{d}_j) \} \right] \quad \leq \quad C\, \mathrm{e}^{\varepsilon(1-1/\kappa) \log n},$$

which implies (5.22). □

Lemma 5.9. *Defining* $DT^*(n) := \bigcap_{j=1}^{K_n^*} \left\{\tau(b_j^*, d_j^*) < \tau(b_j^*, \gamma_j^*)\right\}$, *we have*

$$P\{DT^*(n)\} \to 1, \qquad n \to \infty.$$

Proof. Since, by definition, the $*$-valleys correspond to the K_n deep valleys on $A^*(n)$, we consider $A^\dagger(n) := A^*(n) \cap A_3(n) \cap A_4^*(n)$ to control the $*$-valleys, where $A_4^*(n)$ is defined by $A_4^*(n) := \cap_{j=1}^{K_n^*} \left\{\gamma_{j+1}^* - a_j^* \le C'' \log n\right\} \cap \left\{\gamma_1^* \le C'' \log n\right\}$. Using the same arguments as in the proof of Lemma 5.4, we can prove that $P\{A_4^*(n)\} \to 1$, $n \to \infty$, for C'' large enough. Then, recalling that Lemma 5.3 and Lemma 5.5 imply $P\{A^*(n) \cap A_3(n)\} \to 1$, $n \to \infty$, it remains only to prove that

$$E\left[\mathbf{1}_{A^\dagger(n)} \sum_{j=1}^{K_n} P_\omega^{b_j}\{\tau(d_j) > \tau(\gamma_j^*)\}\right] \to 0. \tag{5.23}$$

Observe that by (Zeitouni [113], formula (2.1.4)) we get, for $1 \le j \le K_n$,

$$\begin{aligned} P_\omega^{b_j}\{\tau(d_j) > \tau(\gamma_j^*)\} &\le (d_j - b_j)\mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))} \\ &\le C \log n \, \mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))}, \end{aligned}$$

the second inequality being a consequence of $\omega \in A^*(n) \cap A_4^*(n)$. Then, to bound $\mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))}$ from above, observe that (5.4) implies $P\{H^{(j)} > \frac{(1+\varepsilon')}{\kappa} \log n\} \sim n^{-(\varepsilon'+\varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$, for any $\varepsilon' > 0$, which yields that $P\{\bigcap_{j=1}^{K_n}\{H^{(j)} < \frac{(1+\varepsilon')}{\kappa} \log n\}\}$ tends to 1, when n tends to ∞. Therefore, recalling (5.23), we only have to prove that

$$C \log n \; n^{\frac{(1+\varepsilon')}{\kappa}} E\left[\mathbf{1}_{A^\dagger(n)} \sum_{j=1}^{K_n} \mathrm{e}^{-(V(\gamma_j^*) - V(b_j))}\right] \to 0. \tag{5.24}$$

Since $\gamma_j^* - b_{j-1} \le C'' \log n$ on $A_4^*(n)$ and $b_j - b_{j-1} \ge n^{1-3\varepsilon}$ on $A_3(n)$, we get $b_j - \gamma_j^* \ge \frac{1}{2} n^{1-3\varepsilon}$ for $2 \le j \le K_n$ on $A^\dagger(n)$, for all large n. Similarly, $\gamma_0^* \le C'' \log n$ on $A_4^*(n)$ and $b_1 \ge n^{1-3\varepsilon}$ on $A_3(n)$ yield $b_1 - \gamma_1^* \ge \frac{1}{2} n^{1-3\varepsilon}$ on $A^\dagger(n)$. Therefore, by definition of b_j and since large deviations do occur, we obtain from Cramer's theory, see [33],

$$\begin{aligned} P\{A^\dagger(n)\,;\, V(b_j) - V(\gamma_{j-1}^*) \ge -n^{\frac{1-3\varepsilon}{2}}\} &\le P\{V(\frac{1}{2} n^{1-3\varepsilon}) \ge -n^{\frac{1-3\varepsilon}{2}}\} \\ &\le \mathrm{e}^{-\frac{n^{1-3\varepsilon}}{2} I\left(n^{\frac{-1-3\varepsilon}{2}}\right)} = o(n^{-\varepsilon}), \end{aligned}$$

for any $1 \le j \le K_n$. This result implies that the term on the left-hand side in (5.24) is bounded from above by $C \log n \; n^{\frac{(1+\varepsilon')}{\kappa}} E[K_n] \mathrm{e}^{-\frac{n^{1-3\varepsilon}}{2}}$. Then, since $E[K_n] \le C n^\varepsilon$, this concludes the proof of Lemma 5.9. □

Lemma 5.10. *For any* $0 < \eta < \varepsilon(\frac{1}{\kappa} - 1)$, *let us introduce the following event* $IA(n) := A(n) \cap \left\{\sum_{j=1}^{K_n} \tau(d_j, b_{j+1}) < n^{1/\kappa - \eta}\right\}$. *Then, we have*

$$P\{IA(n)\} \to 1, \qquad n \to \infty.$$

PROOF. Recalling that $P\{K_n \geq 2C_I n^\varepsilon\} \to 0$, $n \to \infty$, and that Lemma 5.8 implies that $P\{DT(n)\} \to 1$, $n \to \infty$, it only remains to prove

$$\mathbb{P}\left\{DT(n) \cap \left\{\sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} \tau(d_j, b_{j+1}) \geq n^{1/\kappa-\eta}\right\}\right\} \to 0, \qquad n \to \infty.$$

Using Markov inequality, we have to prove that

$$\mathbb{E}\left[\mathbf{1}_{DT(n)} \sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} \tau(d_j, b_{j+1})\right] = o\left(\frac{1}{n^{1/\kappa-\eta}}\right), \qquad n \to \infty. \tag{5.25}$$

Furthermore, by definition of the event DT (see Lemma 5.8), we get

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{DT(n)} \sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} \tau(d_j, b_{j+1})\right] &\leq E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} E_{\omega,|\overline{d}_j}^{d_j}[\tau(b_{j+1})]\right] \\ &\leq E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} E_{\omega,|\overline{d}_j}^{\overline{d}_j}[\tau(b_{j+1})]\right].\end{aligned}$$

Applying successively the strong Markov property at $\overline{d}_{\lfloor 2C_I n^\varepsilon \rfloor}, \ldots, \overline{d}_2, \overline{d}_1$, this implies

$$\mathbb{E}\left[\mathbf{1}_{DT(n)} \sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} \tau(d_j, b_{j+1})\right] \leq 2C_I n^\varepsilon \mathbb{E}_{|0}[\tau(T^\uparrow(h_n) - 1)].$$

Therefore, Lemma 5.7 implies

$$\mathbb{E}\left[\mathbf{1}_{DT(n)} \sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} \tau(d_j, b_{j+1})\right] \leq C n^\varepsilon \mathrm{e}^{h_n} \leq C n^{\frac{1}{\kappa} - \varepsilon\left(\frac{1}{\kappa}-1\right)},$$

which yields (5.25) and concludes the proof, since $0 < \eta < \varepsilon(\frac{1}{\kappa} - 1)$. □

5.3. Proof of Proposition 5.1. Since the time spent on $\mathbb{Z}_-$ is almost surely finite, we reduce our study to the random walk in random environment reflected at 0 and observe that

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] = \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] + o(1), \qquad n \to \infty,$$

where $\mathbb{E}_{|0}$ denotes the expectation under the law $\mathbb{P}_{|0}$ of the random walk in the random environment ω (under P) reflected at 0.

Furthermore, by definition, $\tau(e_n)$ satisfies

$$\tau(b_1) + \sum_{j=1}^{K_n-1}\{\tau(b_j, d_j) + \tau(d_j, b_{j+1})\} \leq \tau(e_n) \leq \tau(b_1) + \sum_{j=1}^{K_n}\{\tau(b_j, d_j) + \tau(d_j, b_{j+1})\},$$

such that we easily get that $\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right]$ belongs to

$$\left[\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right], \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n-1}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right]\right].$$

Let us first recall that Lemma 5.8 and Lemma 5.10 imply that $P\{DT(n)\cap IA(n)\} \to 1$, $n\to\infty$. Then, we get that the lower bound in the previous interval is equal to

$$\begin{aligned}
& \mathbb{E}_{|0}\left[\mathbf{1}_{DT(n)\cap IA(n)} \mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right] + o(1) \\
= \;& \mathbb{E}_{|0}\left[\mathbf{1}_{DT(n)\cap IA(n)}\, \mathrm{e}^{-\lambda_n \sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] + o(1) \\
= \;& \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] + o(1).
\end{aligned}$$

Then, applying the strong Markov property for the random walk successively at $\tau(b_{K_n})$, $\tau(b_{K_n-1})$, $\dots$, $\tau(b_2)$ and $\tau(b_1)$ we get

$$\begin{aligned}
\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] &= E\left[\prod_{j=1}^{K_n} E_{\omega,|0}^{b_j}\left[\mathrm{e}^{-\lambda_n\tau(d_j)}\right]\right] \\
&= E\left[\mathbf{1}_{A^*(n)} \prod_{j=1}^{K_n^*} E_{\omega,|0}^{b_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1) \\
&= E\left[\prod_{j=1}^{K_n^*} E_{\omega,|0}^{b_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1),
\end{aligned}$$

the second equality being a consequence of Lemma 5.5. Then, since Lemma 5.9 implies $\mathbb{P}\{DT^*(n)\}\to 1$, we have

$$\begin{aligned}
\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] &= E\left[\prod_{j=1}^{K_n^*} E_{\omega,|0}^{b_j^*}\left[\mathbf{1}_{DT^*(n)}\, \mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1) \\
&= E\left[\prod_{j=1}^{K_n^*} E_{\omega,|\gamma_j^*}^{b_j^*}\left[\mathbf{1}_{DT^*(n)}\, \mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1) \\
&= E\left[\prod_{j=1}^{K_n^*} E_{\omega,|\gamma_j^*}^{b_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1),
\end{aligned}$$

Since $\mathbb{P}\{K_n = K_n^*\}\to 1$, and $\mathbb{P}\{K_n \le \overline{K}_n\}\to 1$, with $\overline{K}_n = \lceil nq_n(1+n^{-\varepsilon/4})\rceil$, we get

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\,\tau(e_n)}\right] \ge E\left[\prod_{j=1}^{\overline{K}_n} E_{\omega,|\gamma_j^*}^{b_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1).$$

Then, applying the strong Markov property (for the potential V) successively at times $\gamma_{\overline{K}_n}^*$, $\dots$, γ_2^* and observing that the $\left(E_{\omega,|\gamma_j^*}^{b_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right)_{1\le j\le \overline{K}_n}$ are i.i.d. random variables, we obtain that

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\,\tau(e_n)}\right] \ge E\left[E_{\omega,|\gamma_1^*}^{b_1^*}\mathrm{e}^{-\lambda_n\tau(d_1^*)}\right]^{\overline{K}_n} + o(1).$$

Since we can easily prove that $P\{(a_1, b_1, c_1, d_1) \neq (a_1^*, b_1^*, c_1^*, d_1^*)\} = o(n^{-\varepsilon})$, and since $\overline{K}_n = O(n^\varepsilon)$, $n \to \infty$, the strong Markov property applied at γ_1^* yields

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] \geq E\left[E^{b_1}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right]^{\overline{K}_n} + o(1).$$

Using similar arguments for the upper bound in the aforementioned interval, we get

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] \in \left[E\left[E^{b_1}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right]^{\overline{K}_n} + o(1)\,,\, E\left[E^{b_1}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right]^{\underline{K}_n} + o(1)\right].$$

Furthermore, observe that $E\left[E^{b_1}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right] = E\left[E^{b_1}_{\omega,|a_1}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right] + o(n^{-\varepsilon})$. This is a consequence of Lemma 5.4, definition of a and the fact that (5.4) implies $P\{H^{(1)} > \frac{(1+\varepsilon')}{\kappa} \log n\} \sim n^{-(\varepsilon'+\varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$, for any $\varepsilon' > 0$, which gives

$$E\left[P^{b_1}_{\omega}\left\{\tau(a_1) < \tau(d_1)\right\}\right] \leq C \log n\, \mathrm{e}^{\frac{(1+\varepsilon')}{\kappa}\log n - D_n} = o(n^{-\varepsilon}).$$

This concludes the proof of Proposition 5.1. □

6. Annealed Laplace transform for the exit time from a deep valley

This section is devoted to the proof of the linearization. It involves h-processes theory and "sculpture" of a typical deep valley. To ease notations, we shall use a, b, c, and d instead of a_1, b_1, c_1 and d_1. Moreover, let us introduce, for any random variable $Z \geq 0$,

$$R_n(\lambda, Z) := E\left[\frac{1}{1 + \frac{\lambda}{n^{1/\kappa}} Z}\right]. \tag{5.26}$$

Then, the result can be expressed in the following way.

Proposition 5.2. *For any $\xi > 0$, we have, for all large n,*

$$R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) + o(n^{-\varepsilon}) \leq E\left[E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n \tau(d)}]\right] \leq R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) + o(n^{-\varepsilon}).$$

where $\widehat{M}_1 := \sum_{x=a+1}^{d-1} \mathrm{e}^{-(\widehat{V}(x) - \widehat{V}(b))}$ and $M_2 := \sum_{x=b}^{d-1} \mathrm{e}^{V(x)-V(c)}$. Note that $\widehat{V}$ is defined in the following subsection.

6.1. Two h-processes. In order to estimate $E^b_{\omega,|a}\left[\mathrm{e}^{-\lambda_n \tau(d)}\right]$, we decompose the passage from b to d into the sum of a random geometrically distributed number, denoted by N, of unsuccessful attempts to cross the excursion, followed by a successful attempt. More precisely, since N is a geometrically distributed random variable with parameter $1 - p$ satisfying

$$1 - p \quad = \quad \omega_b \frac{\mathrm{e}^{V(b)}}{\sum_{x=b}^{d-1} \mathrm{e}^{V(x)}}, \tag{5.27}$$

we can write $\tau(d) = \sum_{i=1}^{N} F_i + G$, where the F_i's are the successive i.i.d. failures and G the first success. The accurate estimation of the time spent by each (successful and unsuccessful) attempt leads us to consider two h-processes where the random walker evolves in two modified potentials, one corresponding to the conditioning on a failure (see the potential $\widehat{V}$ and Lemma 5.11) and the other to the conditioning on a success (see the potential $\bar{V}$ and Lemma 5.12).

6.1.1. *The failure case: the h-potential $\widehat{V}$.* Let us fix a realization of ω. To introduce the h-potential $\widehat{V}$, we consider the valley $a < b < c < d$ and define $h(x) := P_\omega^x\{\tau(b) < \tau(d)\}$. Therefore, for any $b < x < d$, we define $\widehat{\omega}_x := \omega_x \frac{h(x+1)}{h(x)}$ and similarly $(1-\widehat{\omega}_x) := (1-\omega_x)\frac{h(x-1)}{h(x)}$. We obtain for any $b \le x < y < d$,

$$\widehat{V}(y) - \widehat{V}(x) = (V(y) - V(x)) + \log\left(\frac{h(x)\,h(x+1)}{h(y)\,h(y+1)}\right). \tag{5.28}$$

Using (Zeitouni [113], formula (2.1.4)), we get

$$\frac{h(x)\,h(x+1)}{h(y)\,h(y+1)} = \frac{\sum_{j=x}^{d-1} \mathrm{e}^{V(j)} \sum_{j=x+1}^{d-1} \mathrm{e}^{V(j)}}{\sum_{j=y}^{d-1} \mathrm{e}^{V(j)} \sum_{j=y+1}^{d-1} \mathrm{e}^{V(j)}} \ge 1. \tag{5.29}$$

Thus we obtain for any $b \le x < y \le c$,

$$\widehat{V}(y) - \widehat{V}(x) \ge V(y) - V(x). \tag{5.30}$$

Lemma 5.11. *For any environment ω, we have*

$$E_\omega\left[F_1\right] = 2\,\omega_b\Bigg(\sum_{i=a+1}^{b-1} \mathrm{e}^{-(V(i)-V(b))} + \sum_{i=b}^{d-1} \mathrm{e}^{-(\widehat{V}(i)-\widehat{V}(b))}\Bigg), \tag{5.31}$$

and

$$E_\omega\left[F_1^2\right] = 4\omega_b\, R^+ + 4(1-\omega_b)\, R^-, \tag{5.32}$$

where

$$\begin{aligned} R^+ &:= \sum_{i=b+1}^{d-1}\Bigg(1 + 2\sum_{j=b}^{i-2} \mathrm{e}^{\widehat{V}(j)-\widehat{V}(i-1)}\Bigg)\Bigg(\mathrm{e}^{-(\widehat{V}(i-1)-\widehat{V}(b))} + 2\sum_{j=i+1}^{d-1} \mathrm{e}^{-(\widehat{V}(j-1)-\widehat{V}(b))}\Bigg), \\ R^- &:= \sum_{i=a+1}^{b-1}\Bigg(1 + 2\sum_{j=i+2}^{b} \mathrm{e}^{V(j)-V(i+1)}\Bigg)\Bigg(\mathrm{e}^{-(V(i+1)-V(b))} + 2\sum_{j=a+1}^{i-1} \mathrm{e}^{-(V(j+1)-V(b))}\Bigg). \end{aligned}$$

Remark 5.5. *Alili [2] and Goldsheid [49] prove a similar result for a non-conditioned hitting time. Here we give the proof in order to be self-contained.*

Proof. Let us first introduce

$$\begin{aligned} N_i^+ &:= \sharp\{k < \tau(b) : X_k = i-1, X_{k+1} = i\}, \qquad i > b, \\ N_i^- &:= \sharp\{k < \tau(b) : X_k = i+1, X_{k+1} = i\}, \qquad i < b. \end{aligned}$$

Observe that, under $P_{\widehat{\omega}}$, for $i > b$ and conditionally on $N_i^+ = x$, N_{i+1}^+ is the sum of x independent geometrical random variables with parameter $\widehat{\omega}_i$. It means that $E_{\widehat{\omega}}[N_{i+1}^+ | N_i^+ = x] = \frac{x}{\widehat{\rho}_i}$ and $\mathrm{Var}_{\widehat{\omega}}[N_{i+1}^+ | N_i^+ = x] = \frac{x}{\widehat{\omega}_i \widehat{\rho}_i^2}$. Similarly, under P_ω, for $i < b$ and conditionally on $N_i^- = x$, N_{i-1}^- is the sum of x independent geometrical random variables with parameter $1 - \omega_i$. It means that $E_\omega[N_{i-1}^- | N_i^- = x] = x\rho_i$ and $\mathrm{Var}_\omega[N_{i-1}^- | N_i^- = x] = \frac{x\rho_i^2}{(1-\omega_i)}$.
Since

$$E_\omega[F_1] = 2\omega_b\, E_{\widehat{\omega}}[\sum_{b+1}^{d-1} N_i^+] + 2(1-\omega_b)\, E_\omega[\sum_{a+1}^{b-1} N_i^-],$$

an easy calculation yields (5.31).

To calculate $E_\omega[F_1^2]$, observe first that

$$E_\omega[F_1^2] = 4\omega_b\, E_{\widehat{\omega}}\left[(\sum_{i=b+1}^{d-1} N_i^+)^2\right] + 4(1-\omega_b)\, E_\omega\left[(\sum_{i=a+1}^{b-1} N_i^-)^2\right].$$

Then, it remains to prove that $E_{\widehat{\omega}}[(\sum_{b+1}^{d-1} N_i^+)^2] = R^+$ and $E_\omega[(\sum_{a+1}^{b-1} N_i^-)^2] = R^-$. We will only treat $E_{\widehat{\omega}}[(\sum_{b+1}^{d-1} N_i^+)^2]$, the case of $E_\omega[(\sum_{a+1}^{b-1} N_i^-)^2]$ being similar. We get first

$$E_{\widehat{\omega}}\left[(\sum_{b+1}^{d-1} N_i^+)^2\right] = \sum_{i=b+1}^{d-1} E_{\widehat{\omega}}[(N_i^+)^2] + 2\sum_{i=b+1}^{d-1}\sum_{j=i+1}^{d-1} E_{\widehat{\omega}}[N_i^+ N_j^+]. \tag{5.33}$$

Observe that $E_{\widehat{\omega}}\left[N_i^+ N_j^+\right] = E_{\widehat{\omega}}\left[N_i^+ E_{\widehat{\omega}}\left[N_j^+ \mid N_i^+, \dots, N_{j-1}^+\right]\right] = E_{\widehat{\omega}}\left[N_i^+ \frac{N_{j-1}^+}{\widehat{\rho}_{j-1}}\right]$, for $i < j$, so that we get, by iterating,

$$E_{\widehat{\omega}}\left[N_i^+ N_j^+\right] = E_{\widehat{\omega}}\left[(N_i^+)^2\right] \frac{1}{\widehat{\rho}_{j-1} \dots \widehat{\rho}_i}.$$

Recalling (5.33), this yields

$$\begin{aligned} E_{\widehat{\omega}}\left[(\sum_{b+1}^{d-1} N_i^+)^2\right] &= \sum_{i=b+1}^{d-1} E_{\widehat{\omega}}\left[(N_i^+)^2\right]\left(1 + 2\sum_{j=i+1}^{d-1} \frac{1}{\widehat{\rho}_i \dots \widehat{\rho}_{j-1}}\right) \\ &= \sum_{i=b+1}^{d-1} E_{\widehat{\omega}}\left[(N_i^+)^2\right]\left(1 + 2\sum_{j=i+1}^{d-1} \mathrm{e}^{-(\widehat{V}(j-1) - \widehat{V}(i-1))}\right). \end{aligned} \tag{5.34}$$

Now, observe that $E_{\widehat{\omega}}\left[(N_i^+)^2\right] = E_{\widehat{\omega}}\left[E_{\widehat{\omega}}\left[(N_i^+)^2 | N_{i-1}^+\right]\right]$, which implies

$$E_{\widehat{\omega}}\left[(N_i^+)^2\right] = E_{\widehat{\omega}}\left[\sum_{k\geq 1} E_{\widehat{\omega}}[G_1^{(i)} + \dots + G_k^{(i)}]\mathbf{1}_{\{N_{i-1}^+ = k\}}\right].$$

Since the $G_\cdot^{(i)}$'s are i.i.d., we get $E_{\widehat{\omega}}[G_1^{(i)} + \dots + G_k^{(i)}] = k\mathrm{Var}_{\widehat{\omega}}[G_1^{(i)}] + k^2 E_{\widehat{\omega}}[G_1^{(i)}]^2$. Recalling that $E_{\widehat{\omega}}[G_1^{(i)}] = \frac{1}{\widehat{\rho}_{i-1}}$ and $\mathrm{Var}_{\widehat{\omega}}[G_1^{(i)}] = \frac{1}{\widehat{\omega}_{i-1}\widehat{\rho}_{i-1}^2}$, this yields

$$E_{\widehat{\omega}}\left[(N_i^+)^2\right] = \frac{E_{\widehat{\omega}}\left[N_{i-1}^+\right]}{\widehat{\omega}_{i-1}\widehat{\rho}_{i-1}^2} + \frac{E_{\widehat{\omega}}\left[(N_{i-1}^+)^2\right]}{\widehat{\rho}_{i-1}^2}$$

$$= \frac{1}{\widehat{\omega}_{i-1}\widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-2}\widehat{\rho}^2_{i-1}} + \frac{E_{\widehat{\omega}}\left[(N^+_{i-1})^2\right]}{\widehat{\rho}^2_{i-1}}. \tag{5.35}$$

Denoting $W_{b+1} := 1$ and $W_i := (\widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-1})^2 E_{\widehat{\omega}}\left[(N^+_i)^2\right]$ for $b+1 < i < d$, (5.35) becomes

$$W_i - W_{i-1} = \frac{\widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-1}}{\widehat{\omega}_{i-1}} = \widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-1} + \widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-2},$$

the second equality being a consequence of $1/\widehat{\omega}_{i-1} = \widehat{\rho}_{i-1} + 1$. Therefore, we have $W_i = \sum_{b+2}^{i}(W_j - W_{j-1}) + W_{b+1} = \widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-1} + 2(1 + \sum_{b+1}^{i-2}\widehat{\rho}_{b+1}\dots\widehat{\rho}_j)$, which implies

$$E_{\widehat{\omega}}\left[(N^+_i)^2\right] = \frac{1}{\widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-1}} + 2\sum_{j=b}^{i-2}\frac{\widehat{\rho}_{b+1}\dots\widehat{\rho}_j}{(\widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-1})^2}$$

$$= \mathrm{e}^{-(\widehat{V}(i-1)-\widehat{V}(b))} + 2\sum_{j=b}^{i-2}\mathrm{e}^{\widehat{V}(j)-2\widehat{V}(i-1)+\widehat{V}(b)}. \tag{5.36}$$

Assembling (5.34) and (5.36) yields (5.32). □

6.1.2. *The success case: the h-potential* $\bar{V}$. In a similar way, we introduce the h-potential $\bar{V}$ by considering the valley $a < b < c < d$ and defining $g(x) := P^x_\omega\{\tau(d) < \tau(b)\}$. Therefore, for any $b < x < d$, we define $\bar{\omega}_x := \omega_x \frac{g(x+1)}{g(x)}$ and similarly $(1-\bar{\omega}_x) := (1-\omega_x)\frac{g(x-1)}{g(x)}$. We obtain for any $b < x < y \le d$,

$$\bar{V}(y) - \bar{V}(x) = (V(y) - V(x)) + \log\left(\frac{g(x)\,g(x+1)}{g(y)g(y+1)}\right). \tag{5.37}$$

Recalling (Zeitouni [113], formula (2.1.4)), we have

$$\frac{g(x)\,g(x+1)}{g(y)\,g(y+1)} = \frac{\sum_{j=b}^{x-1}\mathrm{e}^{V(j)}\,\sum_{j=b}^{x}\mathrm{e}^{V(j)}}{\sum_{j=b}^{y-1}\mathrm{e}^{V(j)}\,\sum_{j=b}^{y}\mathrm{e}^{V(j)}} \le 1. \tag{5.38}$$

Therefore, we obtain for any $c \le x < y \le d$,

$$\bar{V}(y) - \bar{V}(x) \le V(y) - V(x). \tag{5.39}$$

Using the same arguments as in the failure case, we get the following result.

Lemma 5.12. *For any environment* ω, *we have*

$$E_\omega[G] \le 1 + \sum_{i=b+1}^{d}\sum_{j=i}^{d}\mathrm{e}^{\bar{V}(j)-\bar{V}(i)}. \tag{5.40}$$

6.2. Preparatory lemmas. The study of a typical deep valley involves the following event

$$A_5(n) := \left\{\max\{V^{\uparrow}(a,b)\,;\,-V^{\downarrow}(b,c)\,;\,V^{\uparrow}(c,d)\} \le \delta\log n\right\},$$

where $\delta > \varepsilon/\kappa$. In words, $A_5(n)$ ensures that the potential does not have excessive fluctuations in a typical box. Moreover, we have the following result.

LEMMA 5.13. *For any* $\delta > \varepsilon/\kappa$,

$$P\{A_5(n)\} = 1 - o(n^{-\varepsilon}), \qquad n \to \infty.$$

PROOF. We easily observe that the proof of Lemma 5.13 boils down to showing that

$$P\{V^{\uparrow}(a,b) \geq \delta \log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.41}$$

$$P\{-V^{\downarrow}(b,c) \geq \delta \log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.42}$$

$$P\{V^{\uparrow}(c,d) \geq \delta \log n\} = o(n^{-\varepsilon}), \qquad n \to \infty. \tag{5.43}$$

In order to prove (5.43), let us first observe the following trivial inequality

$$P\{V^{\uparrow}(c,d) \geq \delta \log n\} \leq P\{V^{\uparrow}(T_1^{\uparrow}, d) \geq \delta \log n\}.$$

Looking at the proof of (5.10), we observe that $P\{d - T_1^{\uparrow} \geq C \log n\} = o(n^{-\varepsilon'})$, for any $\varepsilon' > 0$, by choosing C large enough, depending on ε'. Therefore, we only have to prove that $P\{V^{\uparrow}(T_1^{\uparrow}, T_1^{\uparrow} + C\log n) \geq \delta \log n\} = o(n^{-\varepsilon})$. Then, applying the strong Markov property at time $T_1^{\uparrow}$, we have to prove that $P\{V^{\uparrow}(0, C\log n) \geq \delta \log n\} = o(n^{-\varepsilon})$. Now, by Cramer's theory, see [33], and Lemma 5.6, we get

$$\begin{aligned} P\{V^{\uparrow}(0, C\log n) \geq \delta \log n\} &\leq (C\log n)^2 \max_{0 \leq k \leq C \log n} P\{V(k) \geq \delta \log n\} \\ &\leq (C\log n)^2 \max_{0 \leq k \leq C \log n} \mathrm{e}^{-kI\left(\frac{\delta \log n}{k}\right)} \\ &\leq (C\log n)^2 \exp\{-\kappa \delta \log n\}. \end{aligned}$$

Since $\delta > \varepsilon/\kappa$, this yields (5.43).

To get (5.42), observe first that

$$P\{-V^{\downarrow}(b,c) \geq \delta \log n\} \leq P\{-V^{\downarrow}(b, T_1^{\uparrow}) \geq \delta \log n\} + P\{-V^{\downarrow}(T_1^{\uparrow}, c) \geq \delta \log n\}.$$

The first term on the right-hand side is equal to $P\{V^{\downarrow}(0, T^{\uparrow}(h_n)) \geq \delta \log n | H_0 > h_n\}$. Recalling that (5.4) implies $P\{H_0 > h_n\} \leq Cn^{-(1-\varepsilon)}$ for all large n and observing the trivial inclusion $\{V^{\downarrow}(0, T^{\uparrow}(h_n)) \geq \delta \log n\,;\, H_0 > h_n\} \subset \{T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\}$, it follows that $P\{-V^{\downarrow}(b, T_1^{\uparrow}) \geq \delta \log n\}$ is less or equal than

$$\begin{aligned} & Cn^{1-\varepsilon} P\{T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\} \\ \leq\ & Cn^{1-\varepsilon} \sum_{p=\lfloor \delta \log n \rfloor}^{\lfloor h_n \rfloor} P\{M_\delta \in [p, p+1)\,;\, T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\}, \end{aligned}$$

where $M_\delta := \max\{V(k);\, 0 \leq k \leq T^{\downarrow}(\delta \log n)\}$. Applying the strong Markov property at time $T^{\downarrow}(\delta \log n)$ and recalling (5.6) we bound the term of the previous sum, for

$\lfloor \delta \log n \rfloor \le p \le \lfloor h_n \rfloor$ and all large n, by

$$P\{S \ge p\}\, P\{S \ge h_n - (p - \delta \log n)\} \le C\mathrm{e}^{-\kappa p}\mathrm{e}^{-\kappa(h_n - p + \delta \log n))},$$

where $S := \sup\{V(k);\, k \ge 0\}$. Thus, we get $P\{-V^{\downarrow}(b, T_1^{\uparrow}) \ge \delta \log n\} \le C\lfloor h_n \rfloor n^{-\kappa\delta}$, for all large n, which yields $P\{-V^{\downarrow}(b, T_1^{\uparrow}) \ge \delta \log n\} = o(n^{-\varepsilon})$, $n \to \infty$, since $\delta > \varepsilon/\kappa$. Furthermore, applying the strong Markov property at $T_1^{\uparrow}$, we obtain that $P\{-V^{\downarrow}(T_1^{\uparrow}, c) \ge \delta \log n\} \le P\{-V^{\downarrow}(0, V_{max}) \ge \delta \log n\}$. In a similar way we used before (but easier), we get, by applying the strong Markov property at $T^{\downarrow}(\delta \log n)$, that $P\{-V^{\downarrow}(T_1^{\uparrow}, c) \ge \delta \log n\} \le n^{-\kappa\delta}$ for all large n. Since $\delta > \varepsilon/\kappa$ this yields (5.42).

For (5.41), observe first that $((V(k-b) - V(b))_{a \le k \le b}, a, b)$ has the same distribution as $((V(k))_{a^- \le k \le 0}, a^-, 0)$ under $P\{\cdot | V(k) \ge 0\,,\, a^- \le k \le 0\}$, where $a^- := \sup\{k \le 0 : V(k) \ge D_n\}$. Then, since $P\{V(k) \ge 0\,,\, k \le 0\} > 0$ and since $(V(-k)\,,\, k \ge 0)$ has the same distribution as $(-V(k)\,,\, k \ge 0)$, we obtain

$$P\{V^{\uparrow}(a, b) \ge \delta \log n\} \le C P\{V^{\uparrow}(0, T_{(-\infty, -D_n]}) \ge \delta \log n\}.$$

Now, the arguments are the same as in the proof of (5.43). □

6.3. Proof of Proposition 5.2. Recall that we can write $\tau(d) = \sum_{i=1}^{N} F_i + G$, where the F_i's are the successive i.i.d. failures and G the first success. Then, denoting F_1 by F, we have

$$\begin{aligned} E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n \tau(d)}] &= E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}] \sum_{k \ge 0} E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]^k (1-p)p^k \\ &= E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}] \frac{1-p}{1 - p\, E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}. \qquad (5.44) \end{aligned}$$

In order to replace $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]$ by $1 - \lambda_n E^b_{\omega,|a}[F]$, we observe that $1 - \lambda_n E^b_{\omega,|a}[F] \le E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}] \le 1 - \lambda_n E^b_{\omega,|a}[F] + \frac{\lambda_n^2}{2} E^b_{\omega,|a}[F^2]$, which implies that $E[\frac{1-p}{1-p\, E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}]$ belongs to

$$\left[E\left[\frac{1-p}{1 - p(1 - \lambda_n E^b_{\omega,|a}[F])}\right] ; E\left[\frac{1-p}{1 - p(1 - \lambda_n E^b_{\omega,|a}[F] + \frac{\lambda_n^2}{2} E^b_{\omega,|a}[F^2])}\right]\right].$$

Now, we have to bound $\lambda_n E^b_{\omega,|a}[F^2]$ from above. Then, recalling (5.32), which implies $E^b_{\omega,|a}[F^2] \le 4(R^+ + R^-)$, we only have to bound R^+ and R^-. By definition of R^+, we obtain

$$R^+ \le (d-b)\left(1 + 2(d-b)\mathrm{e}^{-\widehat{V}^{\downarrow}(b,d)}\right)\left(3(d-b) \max_{b \le j \le d} \mathrm{e}^{-(\widehat{V}(j) - \widehat{V}(b))}\right). \qquad (5.45)$$

Recalling that the proof of Lemma 5.4 contains the fact that $P\{d - a \ge C'' \log n\} = o(n^{-\varepsilon})$ and that Lemma 5.13 tells that $P\{A_5(n)\} = 1 - o(n^{-\varepsilon})$, we can consider the event $A^{\ddagger}(n) := \{d - a \le C'' \log n\} \cap A_5(n)$, whose probability is greater than $1 - o(n^{-\varepsilon})$ for n large enough. It allows us to sculpt the deep valley (a, b, c, d), such that we can bound R^+. We are going to show that the fluctuations of $\widehat{V}$ are, in a sense, related to the fluctuations of V controlled by $A_5(n)$. Indeed, (5.30) yields $\widehat{V}^{\downarrow}(b,c) \geq V^{\downarrow}(b,c) \geq -\delta \log n$ on $A^{\ddagger}(n)$. Moreover, (5.28) together with (5.29) imply that $\widehat{V}(y) - \widehat{V}(x)$ is greater than

$$[V(y) - \max_{y \leq j \leq d-1} V(j)] - [V(x) - \max_{x \leq j \leq d-1} V(j)] - O(\log_2 n),$$

for any $c \leq x \leq y \leq d$, on $A^{\ddagger}(n)$. Since $V(x) - \max_{x \leq j \leq d-1} V(j) \leq 0$ and $V(y) - \max_{y \leq j \leq d-1} V(j) \geq -\delta \log n$ on $A^{\ddagger}(n)$, this yields $\widehat{V}^{\downarrow}(c,d) \geq -\delta \log n - O(\log_2 n)$. Furthermore, since (5.28) and (5.29) imply that $\widehat{V}(c)$ is larger than $\max_{b \leq j \leq c} \widehat{V}(j) - O(\log_2 n)$, assembling $\widehat{V}^{\downarrow}(b,c) \geq -\delta \log n$ with $\widehat{V}^{\downarrow}(c,d) \geq -\delta \log n - O(\log_2 n)$ yield

$$\widehat{V}^{\downarrow}(b,d) \geq -\delta \log n - O(\log_2 n), \tag{5.46}$$

on $A^{\ddagger}(n)$. Therefore, we have, on $A^{\ddagger}(n)$ and for all large n,

$$R^+ \leq C(\log n)^3 n^{\delta} \max_{b \leq j \leq d} \mathrm{e}^{-(\widehat{V}(j) - \widehat{V}(b))}. \tag{5.47}$$

Since $\widehat{V}(b) = V(b)$ and (5.29) implies $\widehat{V}(x) \geq V(x)$, for all $b \leq x \leq c$ (in particular $\widehat{V}(c) \geq V(c)$), it follows from (5.46) that $\widehat{V}(j) - \widehat{V}(b) = (\widehat{V}(j) - \widehat{V}(c)) + (\widehat{V}(c) - \widehat{V}(b)) \geq h_n - \delta \log n - O(\log_2 n)$, which is greater than 0 for n large enough whenever $\delta < (1-\varepsilon)/\kappa$ (it is possible since $\delta > \varepsilon/\kappa$ and $0 < \varepsilon < 1/3$). Therefore, recalling (5.47), we obtain, on $A^{\ddagger}(n)$,

$$R^+ \leq C(\log n)^3 n^{\delta}. \tag{5.48}$$

In a similar way, we prove that $R^- \leq C(\log n)^3 n^{\delta}$, on $A^{\ddagger}(n)$, which implies that $\lambda_n E^b_{\omega,|a}[F^2] \leq C(\log n)^3 n^{\delta - \frac{1}{\kappa}}$. Now, observe that, for any $\xi > 0$, $\{\lambda_n E^b_{\omega,|a}[F^2] \leq 2(1 - \mathrm{e}^{-\xi})\}$ is included in $A^{\ddagger}(n)$, such that $\lambda_n E^b_{\omega,|a}[F^2] \leq 2(1-\mathrm{e}^{-\xi}) E^b_{\omega,|a}[F]$ with probability larger than $1 - o(n^{-\varepsilon})$. Then, introducing

$$R'_n(\lambda) := E\left[\frac{1}{1 + \frac{\lambda}{n^{1/\kappa}} \frac{p}{1-p} E^b_{\omega,|a}[F]}\right],$$

we get, for n large enough,

$$R'_n(\lambda) + o(n^{-\varepsilon}) \leq E\left[\frac{1-p}{1 - p\, E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}\right] \leq R'_n(\mathrm{e}^{-\xi}\lambda) + o(n^{-\varepsilon}). \tag{5.49}$$

In order to bound $E^b_{\omega,|a}\left[\mathrm{e}^{-\lambda_n G}\right]$ by below, we observe that $\mathrm{e}^{-x} \geq 1 - x$, for any $x \geq 0$, such that $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}] \geq 1 - \lambda_n E^b_{\omega,|a}[G]$. Therefore, we only have to bound $E^b_{\omega,|a}[G]$ from above. Recalling (5.40), we get $E^b_{\omega,|a}[G] \leq (d-b)^2 \mathrm{e}^{\bar{V}^{\uparrow}(b,d)}$. Now, let us bound $\bar{V}^{\uparrow}(b,d)$. We observe first that (5.39) implies $\bar{V}^{\uparrow}(c,d) \leq V^{\uparrow}(c,d)$, which yields $\bar{V}^{\uparrow}(c,d) \leq \delta \log n$ on $A^{\ddagger}(n)$. Moreover, (5.37) together with (5.38) imply that $\bar{V}(y) - \bar{V}(x)$ is less or equal than

$$[V(y) - \max_{b \leq j \leq y} V(j)] - [V(x) - \max_{b \leq j \leq x} V(j)] + O(\log_2 n),$$

for any $b \le x \le y \le c$, on $A^{\ddagger}(n)$. Since $V(y) - \max_{b\le j\le y} V(j) \le 0$ and $V(x) - \max_{b\le j\le x} V(j) \ge -\delta \log n$ on $A^{\ddagger}(n)$, this yields $\bar{V}^{\uparrow}(b,c) \le \delta \log n + O(\log_2 n)$. Furthermore, (5.39) and the fact that $V(y) \le V(c)$, for $c \le y \le d$, imply that $\bar{V}(y) \le \bar{V}(c)$ for $c \le y \le d$. Therefore, we have

$$\bar{V}^{\uparrow}(b,d) \le \delta \log n + O(\log_2 n),$$

on $A^{\ddagger}(n)$. It means that $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}]$ is greater than $1 - o(n^{-\varepsilon})$ on $A^{\ddagger}(n)$ whenever $\delta < \frac{1}{\kappa} - \varepsilon$, which is possible since $\delta > \varepsilon/\kappa$ and $0 < \varepsilon < 1/3$. Therefore, recalling (5.49), we obtain

$$R'_n(\lambda) + o(n^{-\varepsilon}) \le E\big[E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n \tau(d)}]\big] \le R'_n(\mathrm{e}^{-\xi}\lambda) + o(n^{-\varepsilon}). \tag{5.50}$$

Recalling (5.31) and (5.27), we get

$$R_n(\lambda, 2\widehat{M}_1(\mathrm{e}^{H^{(1)}}M_2 + \omega_b)) \le R'_n(\lambda) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2),$$

where $\widehat{M}_1 := \sum_{x=a+1}^{d-1} \mathrm{e}^{-(\widehat{V}(x)-\widehat{V}(b))}$, $M_2 := \sum_{x=b}^{d-1} \mathrm{e}^{V(x)-V(c)}$ and $R_n(\lambda, Z)$ is defined in (5.26). Furthermore, since $\mathrm{e}^{H^{(1)}} \ge n^{\frac{1-\varepsilon}{\kappa}}$, $M_2 \ge 1$ and $\omega_b \le 1$ we obtain that, for any $\xi > 0$ and n large enough, $\omega_b \le (\mathrm{e}^{\xi} - 1)\mathrm{e}^{H^{(1)}} M_2$. Therefore, we have for all large n,

$$R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) \le R'_n(\lambda) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2). \tag{5.51}$$

Now, assembling (5.50) and (5.51) concludes the proof of Proposition 5.2. □

7. Back to canonical meanders

Let us set $S := \max\{V(k)\,;\, k \ge 0\}$, $H := \max\{V(k)\,;\, 0 \le k \le T_{\mathbb{R}_-}\} = H_0$, and $T_S := \inf\{k \ge 0 : V(k) = S\}$. Moreover, we define $\mathcal{I}_n := \{H = S \ge h_n\} \cap \{V(k) \ge 0, \forall\, k \le 0\}$, and introduce the random variable $Z := e^S M_1^+ M_2^+$, where $M_1^+ := \sum_{k=a^-}^{T_{h_n/2}} e^{-V(k)}$ and $M_2^+ := \sum_{k=0}^{d^+} e^{V(k)-S}$, with $a^- = \sup\{k \le 0 : V(k) \ge D_n\}$ and $d^+ := \inf\{k \ge e_1 : V(k) - V(e_1) \le -D_n\}$. Then, denoting

$$\mathcal{R}_n(\lambda) := E\Big[\frac{1}{1 + n^{-\frac{1}{\kappa}} 2\lambda Z} | \mathcal{I}_n\Big],$$

we get the following result.

Proposition 5.3. *For any $\xi > 0$, we have, for n large enough,*

$$\mathcal{R}_n(\mathrm{e}^{\xi}\lambda) + o(n^{-\varepsilon}) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) \le \mathcal{R}_n(\mathrm{e}^{-\xi}\lambda) + o(n^{-\varepsilon}).$$

Proof. *Step 1: we replace $\widehat{M}_1$ by $\widehat{M}_1^T$.*

Recall that $A^{\ddagger}(n) = \{d - a \le C'' \log n\} \cap A_5(n)$ and that $P\{A^{\ddagger}(n)\} \ge 1 - o(n^{-\varepsilon})$, for all large n. Now, let us introduce $T(\frac{h_n}{2}) := \inf\{k \ge b : V(k) - V(b) \ge h_n/2\}$ and $\widehat{M}_1^T := \sum_{k=a+1}^{T(\frac{h_n}{2})} e^{-(\widehat{V}(k)-\widehat{V}(b))}$. Recalling (5.46), we observe that $\widehat{M}_1 \le \widehat{M}_1^T + C'' \log n e^{-\frac{h_n}{2} + \delta \log n}$ on $A^{\ddagger}(n)$. This implies that, for any $\xi > 0$, we have $\widehat{M}_1 - \widehat{M}_1^T \le$

$(\mathrm{e}^{\xi}-1)\widehat{M}_1^T$ for all large n, whenever $\delta < \frac{1-\varepsilon}{2\kappa}$, which is possible since $\delta > \varepsilon/\kappa$ and $0<\varepsilon<1/3$. Therefore, we obtain, for n large enough,

$$R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1^T M_2) + o(n^{-\varepsilon}) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1^T M_2).$$

Step 2: we replace $\widehat{M}_1^T$ by M_1^T.

Let us denote $M_1^T := \sum_{k=a+1}^{T(\frac{h_n}{2})} e^{-(V(k)-V(b))}$. Since $T(\frac{h_n}{2}) \le c$, (5.30) implies that $\widehat{M}_1^T \le M_1^T$. Observe that (5.28) with (5.29) imply that $\widehat{V}(y)-\widehat{V}(b)-(V(y)-V(b))$ is less or equal than

$$\log\left(\frac{\sum_{j=b}^{d-1}\mathrm{e}^{V(j)}}{\sum_{j=y}^{d-1}\mathrm{e}^{V(j)}}\frac{\sum_{j=b+1}^{d-1}\mathrm{e}^{V(j)}}{\sum_{j=y+1}^{d-1}\mathrm{e}^{V(j)}}\right) \le \frac{\sum_{j=b}^{y-1}\mathrm{e}^{V(j)}}{\sum_{j=y}^{d-1}\mathrm{e}^{V(j)}} + \frac{\sum_{j=b+1}^{y}\mathrm{e}^{V(j)}}{\sum_{j=y+1}^{d-1}\mathrm{e}^{V(j)}},$$

for any $b \le y \le d$. Therefore, on $A^{\ddagger}(n)$, we obtain $\widehat{V}(y)-\widehat{V}(b) \le (V(y)-V(b)) + C\log n\mathrm{e}^{-\frac{h_n}{2}}$ for any $b \le y \le T(\frac{h_n}{2})$, which yields $\widehat{M}_1^T \ge \exp\{C\log n\,\mathrm{e}^{-\frac{h_n}{2}}\}M_1^T$. Then, for any $\xi>0$, we obtain that $\widehat{M}_1^T \ge \mathrm{e}^{-\xi}M_1^T$, on $A^{\ddagger}(n)$ and for all large n. This implies

$$R_n(\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1^T M_2) \le R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) + o(n^{-\varepsilon}).$$

Now, assembling Step 1 and Step 2, we get that, for any $\xi>0$ and n large enough, $R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2)$ belongs to

$$\left[R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) + o(n^{-\varepsilon})\,;\, R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) + o(n^{-\varepsilon})\right]. \tag{5.52}$$

Step 3: the "good" conditioning.

Let us first observe that $((V(k-b)-V(b))_{a\le k\le d}, a, b, c, d)$ has the same law as $((V(k))_{a^-\le k\le d^+}, a^-, 0, T_H, d^+)$ under $P\{\cdot|\mathcal{I}_n'\}$, where $\mathcal{I}_n' := \{H \ge h_n\,;\, V^{\uparrow}(a^-,0) \le h_n\,;\, V(k)\ge 0\,,\, a^- \le k \le 0\}$. Moreover, we easily obtain that $P\{\{V(k)\ge 0\,,\, a^- \le k \le 0\} \setminus \{V(k)\ge 0\,,\, k\le 0\}\} = O(n^{-(1+\kappa)}) = o(n^{-\varepsilon})$, that $P\{\{H\ge h_n\}\setminus\{H = S\}\} = O(n^{-2(1-\varepsilon)}) = o(n^{-\varepsilon})$ and that $P\{V^{\downarrow}(a^-,0) > h_n\} \le P\{V^{\downarrow}(a^-,0) > \delta\log n\} = o(n^{-\varepsilon})$, with the same arguments as in the proof of Lemma 5.13. Therefore, we have $P\{\mathcal{I}_n' \,\triangle\, \mathcal{I}_n\} = o(n^{-\varepsilon})$. Since $0 \le R_n(\lambda, Y) \le 1$, for any $\lambda>0$ and any positive random variable Y, this yields

$$R_n(\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) = \mathcal{R}_n(\lambda) + o(n^{-\varepsilon}). \tag{5.53}$$

Now, assembling (5.52) and (5.53) concludes the proof of Proposition 5.3. □

8. Proof of Theorem 5.2

Observe first that $\mathcal{R}_n(\lambda)$ can be written

$$\mathcal{R}_n(\lambda) = 1 - E\left[1 - \frac{1}{1+2\lambda_n Z}|\mathcal{I}_n\right].$$

Then, we can use Corollary 4.3 and Remark 4.5 in Chapter 4, which imply

$$E\Big[1-\frac{1}{1+2\lambda_n Z}\,\Big|\,\mathcal{I}_n\Big] \sim 2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)}\,\frac{E[M^\kappa]^2 C_I}{nP\{H\geq h_n\}}\,\lambda^\kappa, \qquad n\to\infty.$$

Therefore, assembling Proposition 5.1, Proposition 5.2, Proposition 5.3 and recalling that $q_n := P\{H \geq h_n\}$, we get that, for any $\xi > 0$,

$$\liminf_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n\,\tau(e_n)}] \;\geq\; \exp\Big\{-\Big(2^\kappa\frac{\pi\kappa}{\sin(\pi\kappa)}E[M^\kappa]^2C_I\Big)(\mathrm{e}^{\xi}\lambda)^\kappa\Big\},$$

$$\limsup_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n\,\tau(e_n)}] \;\leq\; \exp\Big\{-\Big(2^\kappa\frac{\pi\kappa}{\sin(\pi\kappa)}E[M^\kappa]^2C_I\Big)(\mathrm{e}^{-\xi}\lambda)^\kappa\Big\}.$$

Since this result holds for any $\xi > 0$, we get,

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n\,\tau(e_n)}] = \exp\Big\{-\Big(2^\kappa\frac{\pi\kappa}{\sin(\pi\kappa)}E[M^\kappa]^2C_I\Big)\lambda^\kappa\Big\}.$$

Now, for the conclusion of the proof of Theorem 5.2 and for the proofs of Theorem 5.1 and Corollary 5.1, we refer to the detailed sketch of the proof, see Section 3. □

9. Toward the case $\kappa = 1$

We intend to treat soon the critical case $\kappa = 1$ between the transient ballistic and sub-ballistic cases. This case turns out to be more delicate. Indeed, Lemma 5.7 is replaced by a weaker statement, which says that $\tau(e_n)$ reduces to the time spent by the walker to climb excursions which are higher than $\alpha \log n$ for α arbitrarily small. Due to this reduced height, the new "high" excursions are much more numerous and are not anymore well separated. The definition of the valleys should then be adapted as well as the "linearization" argument, which is more difficult to carry out. Moreover, a result of Goldie [48] gives an explicit formula for the Kesten's renewal constant, namely $C_K = \frac{1}{E[\rho_0 \log \rho_0]}$. As a result, we should obtain, in this case, the following result, which takes a remarkably simple form: $X_n/(\frac{n}{\log n})$ converges in probability to $E[\rho_0 \log \rho_0]/2$.

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Deuxième trimestre 2007

Marches aléatoires en milieu aléatoire sur $\mathbb{Z}$: études de localisation dans les cas récurrent et transient

Résumé : Les marches aléatoires en milieu aléatoire constituent un modèle permettant de décrire des phénomènes de diffusion et de transport en milieux inhomogènes, possédant néanmoins des propriétés de régularité à grande échelle. Le premier chapitre, introductif, illustre la richesse de comportements des marches aléatoires en milieu aléatoire. Le second chapitre concerne la marche de Sinai (cas récurrent) et répond négativement à une conjecture d'Erdős et Révész initialement posée pour la marche aléatoire simple. Il révèle un paradoxe lié au phénomène de localisation obtenu par Sinai. Dans le troisième chapitre, nous nous intéressons à la limite supérieure de la marche de Sinai en paysage aléatoire et traitons une conjecture de Révész. Les quatrième et cinquième chapitres concernent les marches aléatoires en milieu aléatoire transientes de vitesse nulle. Un résultat classique de Kesten, Kozlov et Spitzer dit que le temps d'atteinte du niveau n converge en loi, après renormalisation, vers une variable aléatoire positive stable, mais ils n'obtiennent pas la description de son paramètre. Nous présentons ici une nouvelle preuve de ce résultat: une analyse fine du potentiel associé à l'environnement nous permet d'obtenir une caractérisation complète de la loi stable limite. Le cas d'environnements de Dirichlet s'avère être particulièrement explicite.

Mots-clés : marches aléatoires en milieu aléatoire sur $\mathbb{Z}$, marche de Sinai, localisation, temps local, lois stables, théorie des fluctuations pour une marche aléatoire, h-processus, lois Beta, série de renouvellement, couplage.

Random walks in random environment on $\mathbb{Z}$: localization studies in the recurrent and transient cases

Abstract: Random walks in random environment is a suitable model for diffusion and transport in inhomogeneous media that have regularity properties on a macroscopic scale. The first introductive chapter illustrates the wide variety of behaviors that are captured by the random walks in random environment model. The second chapter concerns Sinai's walk (the recurrent case), which is known for a phenomenon of strong localization. Our main result shows a weakness of this localization phenomenon. In particular, we give a negative answer to a problem of Erdős and Révész, originally formulated for the usual homogeneous random walk. In the third chapter, we focus our attention on the upper limits of Sinai's walk in random scenery and treat a conjecture of Révész. The fourth and fifth chapters deal with transient random walks in random environment with zero asymptotic speed. A classical result of Kesten, Kozlov and Spitzer says that the hitting time of the level n converges in law, after a proper normalization, towards a positive stable law, but they do not obtain a description of its parameter. A different proof of this result is presented: a close study of the potential associated to the environment leads to a complete characterization of this stable law. The case of Dirichlet environment turns out to be remarkably explicit.

Keywords: random walks in random environment on $\mathbb{Z}$, Sinai's walk, localization, local time, stable laws, fluctuations theory for random walks, h-processes, Beta distribution, renewal series, coupling.