Diffusions in random environments and multi-excited random walks

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Chapitre I

Introduction

Le présent travail comporte deux parties. Dans la première partie, nous considérons deux modèles stochastiques en milieux hétérogènes : la marche aléatoire unidimensionnelle en milieu aléatoire et son analogue continu, la diffusion dans un potentiel aléatoire. L'étude de certaines propriétés de ces processus fait l'objet de trois articles publiés ou soumis pour publication, présentés dans les chapitres II - IV.

- **Chapitre II.** On étudie le comportement asymptotique d'une diffusion dans un potentiel asymptotiquement stable en recherchant en particulier des lois du logarithme itéré pour différentes fonctionnelles du processus. Les résultats obtenus se transcrivent également pour le modèle discret de la marche aléatoire en milieu aléatoire.
- **Chapitre III.** On caractérise les différents régimes de transience d'une diffusion dans un potentiel de type Lévy sans sauts positifs. On met en évidence, pour la diffusion, un comportement analogue à celui observé, dans le cas transient, pour la marche aléatoire en milieu aléatoire.
- **Chapitre IV.** On présente, dans ce court chapitre, un exemple de diffusion transiente dans un potentiel stable avec dérive. Le processus a, dans ce cas, une vitesse de déplacement d'ordre logarithmique qui contraste avec les régimes polynomiaux habituellement observés.

La seconde partie de la thèse est consacrée à un autre modèle stochastique introduit récemment : la marche multi-excitée. L'étude de ce processus utilise des méthodes similaires à celles employées pour les marches aléatoires en milieux aléatoires. Cette partie inclut deux articles réalisés en collaboration avec Anne-Laure Basdevant.

- **Chapitre V.** On donne un critère sur l'environnement initial caractérisant la stricte positivité de la vitesse asymptotique d'une marche multi-excitée. En particulier, une marche 2-excitée a nécessairement une vitesse nulle tandis qu'une marche 3-excitée peut, par contre, avoir une vitesse strictement positive.

- **Chapitre VI.** On poursuit l'étude commencée dans le chapitre précédent par une recherche des différents régimes de transience d'une marche multi-excitée lorsque la vitesse asymptotique est nulle. Ces résultats mettent en évidence, pour la marche multi-excitée, un comportement similaire à celui observé pour la marche aléatoire en milieu aléatoire.

Dans le suite de cette introduction, nous présentons rapidement les différents modèles étudiés et nous décrivons les principaux résultats des travaux exposés dans les chapitres suivants.

1 Deux modèles en milieu aléatoire

Les modèles dits en milieu aléatoire sont communément utilisés pour représenter certains systèmes physiques ou biologiques pour lesquels il est indispensable de tenir compte de l'hétérogénéité spatiale de l'environnement. En effet, la présence d'impuretés dans le milieu peut entraîner un comportement très différent de celui observé dans le cadre classique d'un environnement homogène. L'étude mathématique de tels modèles connaît un essor considérable depuis une vingtaine d'années. Cet essor est dû, d'une part, au désir de répondre aux questions posées par les physiciens et les biologistes et, d'autre part, à la richesse des comportements observés qui requièrent souvent pour leur analyse l'introduction d'outils nouveaux.

1.1 La marche aléatoire unidimensionnelle en milieu aléatoire

La marche aléatoire en milieu aléatoire (abrégée en M.A.M.A.) compte parmi les modèles en environnement aléatoire les plus élémentaires. Il semble que ce modèle ait été introduit pour la première fois par le biologiste Chernov [Che67] en 1967 afin d'étudier certains phénomènes de duplication de l'A.D.N. Plus récemment, Lubensky et Nelson [LN02] font intervenir des M.A.M.A. pour étudier la micro-manipulation de brins d'A.D.N. D'un autre côté, Temkin [Tem72] utilise ce modèle dans le cadre de la métallurgie pour étudier la transition de phase dans divers alliages. On pourra aussi se référer à Fisher, Le Doussal et Monthus [FLDM99, FLDM01] pour d'autres applications des M.A.M.A. en physique.

1.1.1 Le modèle

Nous considérons ici le modèle de la M.A.M.A. en dimension 1, aux plus proches voisins, dans un environnement indépendant et identiquement distribué (i.i.d.). Un tel processus est défini de la manière suivante. On se fixe un espace probabilisé $(\Omega, \mathcal{A}, \mathbf{P})$ sur lequel on considère une famille $\omega = (\omega_i)_{i\in\mathbb{Z}}$ de variables aléatoires i.i.d. à valeurs dans $]0,1[$. On dit que ω représente l'*environnement*. Plus précisément, ω_i correspond à la probabilité de transition du site i au site $i+1$ tandis que $1-\omega_i$ correspond à la probabilité de sauter du site i au site $i-1$.

FIG. I.1 : Probabilités de transition.

La marche aléatoire $S = (S_n)_{n\in\mathbb{N}}$ dans l'environnement ω est un processus à valeurs dans les entiers relatifs, issu de 0, qui pour chaque réalisation de l'environnement ω est une simple chaîne de Markov dont les probabilités de transition sont données par

$$\mathbf{P}\left\{S_{n+1} = S_n + e \mid S_0, \dots, S_n, \omega\right\} = \begin{cases} \omega_{S_n} & \text{si } e = 1, \\ 1 - \omega_{S_n} & \text{si } e = -1, \\ 0 & \text{autrement.} \end{cases}$$

Étant fixé un environnement ω, la loi conditionnelle $\mathbf{P}\{\cdot \mid \omega\}$ est appelée loi *quenched* (ce terme signifiant « trempé » est emprunté au lexique de la métallurgie). La probabilité $\mathbf{P}$ représente la loi du processus lorsque l'environnement est lui même inconnu ; on l'appelle loi *annealed* (soit loi « recuite »).

Notons que lorsque l'environnement ω est déterministe (*i.e.* quand la loi de ω_0 est un dirac), S est une marche usuelle et nous exclurons implicitement ce cas dégénéré. Dans le cas général, par contre, la M.A.M.A., sous la loi annealed $\mathbf{P}$, n'est plus une chaîne de Markov car la connaissance de la trajectoire passée fournit des informations sur la configuration de l'environnement aux sites déjà visités et permet ainsi de mieux prédire le comportement futur de la marche. Les résultats classiques concernant la théorie des processus de Markov n'étant pas directement accessibles, il est nécessaire de trouver des approches alternatives.

Le modèle de M.A.M.A. présenté ici est le plus simple possible. Précisons toutefois que d'autres modèles plus sophistiqués sont aussi largement étudiés. Ainsi, on peut considérer une marche dans un environnement non i.i.d. ou encore permettre à la marche de faire des sauts de taille variable (pour des travaux sur de tels modèles, on pourra par exemple consulter [Key84, Der99, Ali99, Bré02, Bré04, MWRZ04, Roi05]).

La M.A.M.A. unidimensionnelle est actuellement relativement bien comprise. Ce n'est pas le cas des dimensions supérieures où l'étude de la marche se révèle extraordinairement plus délicate. Cette difficulté est due, en particulier, à l'irréversibilité du modèle. Un bon compromis entre les dimensions 1 et 2 semble être l'étude des M.A.M.A. sur des arbres. Quelques résultats sur de tels modèles peuvent être trouvés dans [LP92, PP95, MP02, HS06].

En ce qui concerne les M.A.M.A. en dimensions supérieures, on ne connaît même pas de critère permettant de caractériser la transience ou la récurrence de la marche (citons néanmoins la condition de Kalikow [Kal81] ainsi que les conditions T et T' de Sznitman [Szn04]). Toutefois, des progrès ont été réalisés récemment, concernant en particulier des lois du $0-1$ [ZM01, RA05], des théorèmes de lois des grands nombres [SZ99, Szn01, CZ04, Sab04] ou des théorèmes de type limite centrale [Szn00, CZ05].

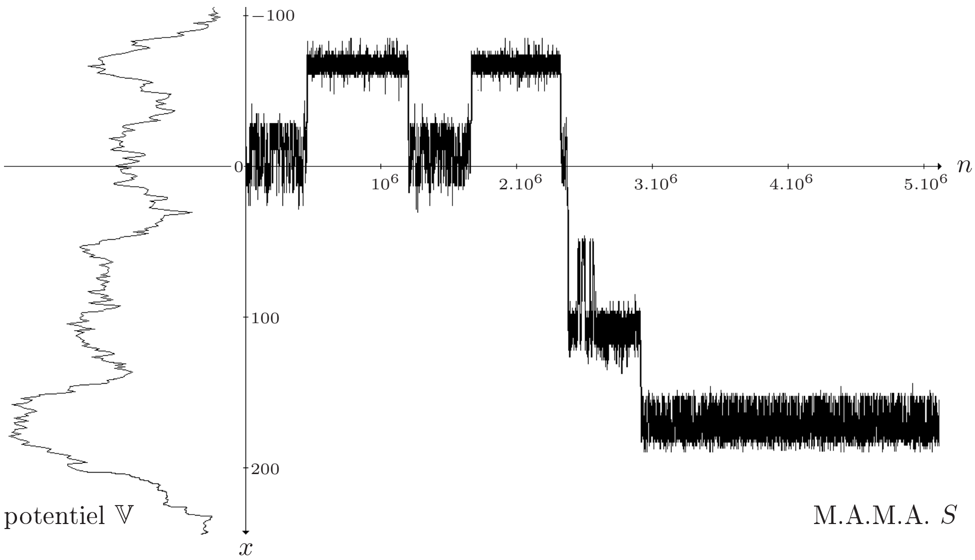

Fig. I.2 : Simulation d'une M.A.M.A. récurrente.

Les références citées ici ne représentent bien évidemment qu'une toute petite partie de la littérature sur le sujet. Pour une description plus complète des différents modèles de M.A.M.A. et des résultats connus, du moins jusqu'en 2004, on pourra consulter les cours de Sznitman [Szn04] et Zeitouni [Zei04] ainsi que le livre de Révész [Rév05].

1.1.2 Quelques résultats importants

L'outil fondamental associé à la M.A.M.A. unidimensionnelle est son *potentiel* $\mathbb{V} = (\mathbb{V}_n)_{n\in\mathbb{Z}}$ que l'on définit de la manière suivante :

$$\mathbb{V}_n \stackrel{\text{def}}{=} \begin{cases} \sum_{i=1}^{n} \log\left(\frac{1-\omega_i}{\omega_i}\right) & \text{si } n \geqslant 1. \\ 0 & \text{si } n = 0. \\ -\sum_{i=n+1}^{0} \log\left(\frac{1-\omega_i}{\omega_i}\right) & \text{si } n \leqslant -1. \end{cases} \tag{1.1}$$

Il s'agit donc d'une marche simple (indexée sur $\mathbb{Z}$) qui caractérise l'environnement. D'une certaine manière, ce processus joue le rôle d'une énergie en physique. La M.A.M.A. tend à se déplacer vers les zones d'énergie les plus faibles : là où le potentiel est petit. En particulier, lorsque le potentiel présente des « puits », la marche a tendance à y rester piégée un temps très long. Ce phénomène appelé *localisation* est bien visible sur la figure I.2.

Le potentiel étant une marche aléatoire simple, soit il oscille, soit il dérive (vers $+\infty$ ou vers $-\infty$). En 1975, Solomon [Sol75] montre que la nature du potentiel détermine le caractère récurrent ou transient de la M.A.M.A. :

- si $\mathbb{V}$ oscille, alors S est récurrente ;
- si $\mathbb{V}$ dérive vers $+\infty$ (resp. $-\infty$), alors S est transiente vers $-\infty$ (resp. $+\infty$).

En particulier, lorsque $\mathbf{E}[\log(\frac{1-\omega_0}{\omega_0})]$ est bien définie, la position de cette quantité par rapport à zéro détermine la récurrence ou la transience de la marche.

Solomon montre, de plus, que la vitesse asymptotique de la M.A.M.A. est bien définie mais peut être nulle, même lorsque S est transiente. Ce résultat indique un déplacement plus lent que pour la marche usuelle. Ce comportement original, conséquence de la localisation de la marche au fond de certaines vallées du potentiel, est précisé par Sinai [Sin82] en 1982. Il montre que dans le cas récurrent et lorsque l'environnement vérifie une propriété d'ellipticité, en posant $\sigma^2 = \mathbf{Var}[\mathbb{V}_1]$,

$$\sigma^2 \frac{S_n}{\log^2 n} \xrightarrow[n\to\infty]{\text{loi}} b_\infty \quad \text{(loi non dégénérée)}. \tag{1.2}$$

L'expression explicite de la loi b_∞ a ensuite été déterminée indépendamment par Kesten [Kes86] et Golosov [Gol86], ce dernier précisant également le phénomène de localisation dans [Gol84].

L'hypothèse d'ellipticité faite sur l'environnement par Sinai revient à supposer que le potentiel se comporte approximativement comme un mouvement brownien. Pour des potentiels plus irréguliers, la marche se révèle encore plus lente : dans le cas d'un potentiel qui se comporte asymptotiquement comme un processus stable d'indice α, Kawazu, Tamura et Tanaka [KTT89, KTT92] montrent que le déplacement de la marche est d'ordre $\log^\alpha n$.

En ce qui concerne le comportement de la marche dans le cas transient, Kesten, Kozlov et Spitzer [KKS75] ont déterminé en 1975 tous les régimes possibles de croissance en fonction de la valeur κ, unique solution de l'équation $\mathbf{E}[e^{\kappa \mathbb{V}_1}] = 1$. En particulier, quand $\kappa \in]0,1[$, ils montrent (sous certaines hypothèses techniques supplémentaires),

$$\frac{S_n}{n^\kappa} \xrightarrow[n\to\infty]{\text{loi}} \mathcal{L}_\kappa \quad \text{(loi de Mittag-Leffler d'indice } \kappa).$$

Notons qu'une preuve différente de ce résultat, donnant une expression plus précise des paramètres de la loi limite, a récemment été établie par Enriquez, Sabot et Zindy [ESZ07].

Il faut ajouter, enfin, que beaucoup d'autres propriétés de la M.A.M.A. on été étudiées, telles que des questions concernant les grandes déviations (voir par exemple [GDH94, CGZ00, Dev06b]) ou encore des problèmes reliés au temps local et aux sites favoris *c.f.* [Rév88, HS98b, GS02, HS00, SZ06].

1.2 Un analogue continu : la diffusion en potentiel aléatoire

Nous introduisons ici un modèle en temps continu étroitement relié à la M.A.M.A. : la diffusion en potentiel aléatoire. Pour cela, on considère un potentiel aléatoire $\mathbb{V} = (\mathbb{V}_x)_{x\in\mathbb{R}}$ désormais indexé par $\mathbb{R}$, avec $\mathbb{V}_0 = 0$. On appelle diffusion dans le potentiel $\mathbb{V}$ une solution formelle $X = (X_t)_{t\geqslant 0}$ de l'équation différentielle stochastique :

$$\begin{cases} dX_t = d\beta_t - \frac{1}{2}\mathbb{V}'_{X_t}dt, \\ X_0 = 0, \end{cases}$$

où β est un mouvement brownien indépendant de $\mathbb{V}$. Lorsque le potentiel n'est pas dérivable, cette équation n'a pas de sens et, plus rigoureusement, on considérera X comme une diffusion dont le générateur infinitésimal, conditionnellement à $\mathbb{V}$, est donné par

$$\frac{1}{2}e^{\mathbb{V}_x}\frac{d}{dx}\left(e^{-\mathbb{V}_x}\frac{d}{dx}\right).$$

Une telle diffusion (sous des hypothèses minimales de régularité du potentiel) peut être explicitement construite, par changement de temps et changement d'échelle, à partir d'un mouvement brownien.

L'un des intérêts du modèle est de permettre, pour un bon choix du potentiel, un couplage avec une M.A.M.A. En effet, supposons que le potentiel $\mathbb{V}$ de la diffusion soit une marche aléatoire de la forme décrite par (1.1), prolongé par $\mathbb{V}_x = \mathbb{V}_n$ pour $x \in [n, n+1[$. On définit la suite de temps d'arrêt :

$$\begin{cases} \tau_0 & \stackrel{\text{def}}{=} 0, \\ \tau_{n+1} & \stackrel{\text{def}}{=} \inf\{t > \tau_n,\ |X_t - X_{\tau_n}| = 1\}. \end{cases}$$

D'après Schumacher [Sch85], le processus $(X_{\tau_n})_{n \geqslant 0}$ est une M.A.M.A. de potentiel $\mathbb{V}$. Grâce à ce couplage, Hu et Shi [HS98a, HS98b, HS00] ont obtenu, en utilisant des outils issus du calcul stochastique et via l'étude de la diffusion associée, des résultats précis concernant la M.A.M.A. lorsque son potentiel se comporte asymptotiquement comme un mouvement brownien. Dans le second chapitre de cette thèse, nous utilisons une méthode similaire afin d'étendre les résultats décrits dans [HS98a] à des potentiels plus généraux.

Lorsque le potentiel associé à la diffusion n'est pas une marche aléatoire, il n'existe plus de couplage évident avec une M.A.M.A. Cependant, le comportement de la diffusion reste, dans un certain nombre de cas, similaire à celui de la M.A.M.A. Ainsi, quand le potentiel $\mathbb{V}$ est un mouvement brownien, la diffusion est récurrente et Brox [Bro86] a établi en 1986 un résultat analogue au théorème de Sinai :

$$\frac{X_t}{\log^2 t} \xrightarrow[t\to\infty]{\text{loi}} b_\infty.$$

De manière remarquable, la loi limite est la même que pour la M.A.M.A. (*c.f.* (1.2)). Comme pour le modèle discret, ce déplacement très lent est la conséquence d'une localisation de la diffusion dans certains puits du potentiel, phénomène mis en évidence par Kawazu, Tamura et Tanaka [KTT92] et Hu [Hu00].

On peut aussi construire des diffusions transientes. L'un des procédés les plus naturels consiste à choisir un potentiel brownien auquel on ajoute une dérive :

$$\mathbb{V}_t = \mathbb{B}_t - \frac{\kappa}{2}t \quad (\mathbb{B} \text{ mouvement brownien et } \kappa > 0). \tag{1.3}$$

Dans ce cas, la diffusion est transiente vers $+\infty$ et peut, de même que la M.A.M.A., avoir une vitesse asymptotique nulle selon la valeur de κ. En 1997, Kawazu et Tanaka [KT97] ont

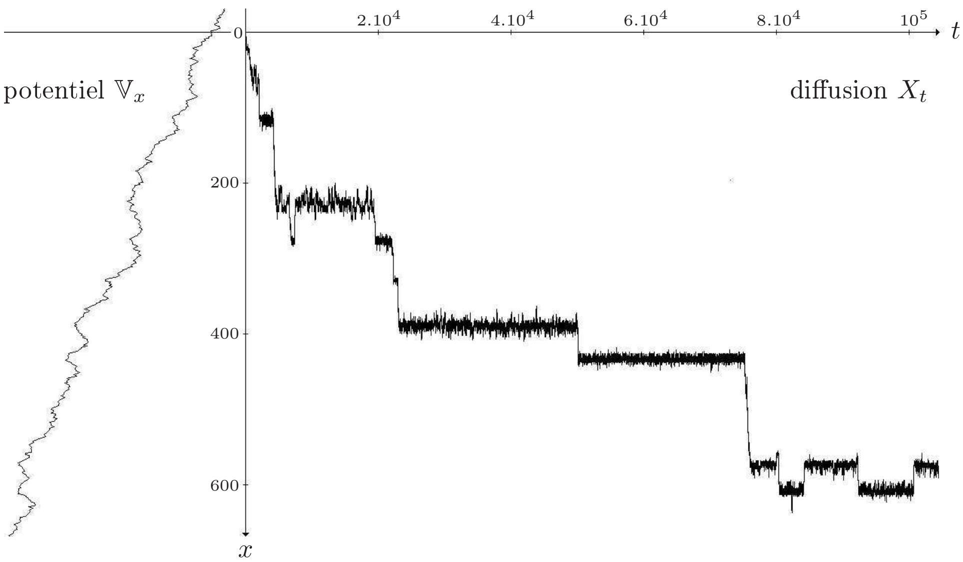

Fig. I.3 : Simulation d'une diffusion dans un potentiel de la forme (1.3) avec $\kappa = \frac{1}{2}$.

décrit les différents régimes de transience d'une telle diffusion. Leur résultat est analogue à celui obtenu par Kesten, Kozlov et Spitzer [KKS75] pour le modèle discret (mais en mettant en œuvre des techniques très différentes). Ils montrent que, selon la valeur de κ :

- si $\kappa < 1$ alors $\frac{X_t}{t^\kappa}$ converge en loi vers une distribution de Mittag-Leffler d'indice κ ;
- si $\kappa = 1$, alors $\frac{\log t}{t} X_t$ converge en probabilité vers $\frac{1}{4}$;
- si $\kappa > 1$, alors $\frac{X_t}{t}$ converge presque sûrement vers $\frac{\kappa-1}{4}$.

Ce résultat a été par la suite affiné par Tanaka [Tan97] et Hu, Shi et Yor [HSY99] qui distinguent plusieurs régimes dans le cas linéaire ($\kappa > 1$) et obtiennent, en particulier, des théorèmes de limite centrale.

Comme pour le modèle discret de la M.A.M.A., les diffusions dans des potentiels de type brownien sont relativement bien comprises et des résultats fins ont été obtenus. On pourra par exemple se référer à : [Tal01, Dev06b] pour des théorèmes liés aux grandes déviations, [HS98b, Dev06a] pour des résultats concernant le temps local, [HS00, Che06a] pour des questions en rapport avec les sites favoris.

Bien que la plupart des travaux portent sur des potentiels de type brownien, il peut être intéressant d'étudier des diffusions dans des potentiels plus généraux. Dans cet esprit, Carmona [Car97] considère une diffusion dans un potentiel Lévy et obtient un théorème de loi des grands nombres. Il conjecture également que lorsque l'exposant de Laplace du potentiel satisfait une condition de Cramer, on doit retrouver, pour la diffusion, des régimes de transience similaires à ceux observés quand le potentiel est un brownien avec dérive. Dans le chapitre 3, nous apportons une réponse positive lorsque le potentiel est un processus de Lévy spectralement négatif.

Finalement, mentionnons qu'à l'instar du modèle discret de la M.A.M.A., il est aussi possible de considérer des diffusions dans des potentiels aléatoires en dimensions supérieures à 1. Nous pensons, par exemple, aux travaux de Mathieu [Mat94, Mat95]).

2 Présentation des résultats en milieu aléatoire

2.1 Chapitre II. Comportement asymptotique d'une diffusion dans un potentiel asymptotiquement stable[1]

Hu et Shi [HS98a] ont déterminé en 1998 toutes les classes de Lévy d'une diffusion (ainsi que celles de la M.A.M.A.) lorsque le potentiel se comporte approximativement comme un mouvement brownien. Dans ce chapitre, nous cherchons à généraliser ces résultats à des environnements possiblement plus irréguliers. Nous faisons sur $\mathbb{V}$ les hypothèses

(a) Le processus $(\mathbb{V}_n)_{n\in\mathbb{Z}}$ est une marche aléatoire et $\mathbb{V}$ est constant sur les intervalles $]n, n+1[$, $n \in \mathbb{Z}$.

(b) Il existe une fonction $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ (que l'on choisit continue et croissante) telle que

$$\frac{\mathbb{V}_x}{a(x)} \xrightarrow[x\to\infty]{\text{loi}} \mathbb{S}$$

où $\mathbb{S}$ est une variable stable d'indice $\alpha \in]0, 2]$ dont le support est $\mathbb{R}$ en entier.

Ces hypothèses sont similaires à celles de Kawazu, Tamura et Tanaka [KTT92] qui ont montré, dans ce cas, que la diffusion X est récurrente et que $X_t/a^{-1}(\log t)$ converge vers une limite non dégénérée (on note a^{-1} l'inverse de a).

Le premier résultat de ce chapitre établit une loi du logarithme itéré décrivant le comportement en limite supérieure de la diffusion.

Théorème 2.1

Il existe une constante $C > 0$, qui ne dépend que de la loi de $\mathbb{S}$, telle que

$$\limsup_{t\to\infty} \frac{X_t}{a^{-1}(\log t)\log\log\log t} = C \qquad \textit{presque sûrement.}$$

En outre, nous déterminons la valeur exacte de la constante C lorsque $\mathbb{S}$ est complètement asymétrique (*c.f.* Chap. II, Théorème 1.2). En particulier, nous retrouvons dans le cas brownien la valeur $C = 8/\pi^2$ obtenue dans [HS98a].

Le théorème précédent reste inchangé si l'on remplace X_t par $\sup_{s\leqslant t} X_s$. Par symétrie, on obtient un résultat similaire pour $\inf_{s\leqslant t} X_s$ et on déduit ainsi du théorème le comportement asymptotique en limite supérieure de $\sup_{s\leqslant t}|X_s|$.

Introduisons maintenant le temps d'échelle $T \stackrel{\text{def}}{=} \inf(n > 0, \mathbb{V}_n < 0)$. Un théorème de Rogozin [Rog71] affirme que, sous nos hypothèses, T est dans le domaine d'attraction

[1] A. Singh, *Limiting behavior of a diffusion in an asymptotically stable environment*, Ann. Inst. H. Poinc. Probab. Statist., **43**(1), 101–138, 2007.

d'une loi stable positive d'indice $q \stackrel{\text{def}}{=} \mathbf{P}\{\mathbb{S} < 0\}$. Ainsi la fonction $b(x) \stackrel{\text{def}}{=} \mathbf{P}\{T > x\}$ varie régulièrement d'indice q.

Le comportement en limite inférieure du supremun unilatéral est donné par :

Théorème 2.2

Pour toute fonction positive et croissante f, on a, presque sûrement,

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)} \sup_{s\leqslant t} X_s = \begin{cases} 0 \\ \infty \end{cases} \iff \int^{\infty} \frac{b(a^{-1}(\log t)/f(t))dt}{b(a^{-1}(\log))t\log t} \begin{cases} = \infty \\ < \infty \end{cases}$$

(on ne précise pas la borne inférieure de l'intégrale car seule la convergence en $+\infty$ importe). En particulier,

$$\liminf_{t\to\infty} \frac{(\log\log t)^{\beta}}{a^{-1}(\log t)} \sup_{s\leqslant t} X_s = \begin{cases} 0 & \text{si } \beta < 1/q, \\ \infty & \text{si } \beta > 1/q. \end{cases}$$

En ce qui concerne le supremum bilatéral, le problème est plus complexe : il semble que plusieurs comportements soient possibles, selon les différents modes de décroissance des queues de distribution $\mathbf{P}\{\mathbb{V}_1 > x\}$ et $\mathbf{P}\{\mathbb{V}_1 < -x\}$. Cependant, lorsque la loi de $\mathbb{S}$ n'est pas complètement asymétrique, ces queues de distribution varient, toutes deux, régulièrement d'indice α et nous obtenons un critère très simple.

Théorème 2.3

Lorsque la variable limite $\mathbb{S}$ n'est pas complètement asymétrique, pour toute fonction positive et croissante f, on a, presque sûrement,

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)} \sup_{s\leqslant t} |X_s| = \begin{cases} 0 \\ \infty \end{cases} \iff \int^{\infty} \frac{dt}{tf(t)^2\log t} \begin{cases} = \infty \\ < \infty. \end{cases}$$

En particulier,

$$\liminf_{t\to\infty} \frac{(\log\log t)^{\beta}}{a^{-1}(\log t)} \sup_{s\leqslant t} |X_s| = \begin{cases} 0 & \text{si } \beta \leqslant 1/2, \\ \infty & \text{si } \beta > 1/2. \end{cases}$$

Remarquons que ce résultat est très différent de celui obtenu par Hu et Shi dans le cadre brownien (*c.f.* Théorème 1.7, [HS98a]).

Les preuves de ces théorèmes reposent, comme dans [HS98a], sur l'utilisation conjointe de la méthode de Laplace et des théorèmes de Ray-Knight qui permettent de ramener l'étude de la diffusion à l'étude de certaines fonctionnelles du potentiel. La difficulté majeure dans notre cas consiste à étudier finement les fluctuations d'une marche aléatoire dans le domaine d'attraction d'une loi stable. En particulier, plusieurs résultats précis concernant les fluctuations d'une telle marche sont obtenus dans la seconde section de ce chapitre et peuvent présenter un intérêt propre.

Notons également que tous les résultats de l'article restent valides lorsque le potentiel de la diffusion est un processus strictement stable d'indice $\alpha \in]0,2]$, en choisissant maintenant $a(x) = x^{1/\alpha}$ et $b(x) = x^q$.

Bien que nous nous intéressions principalement au comportement asymptotique presque sûr de la diffusion, la méthode employée permet également d'établir la convergence en loi de $\sup_{s \leqslant t} X_s / a^{-1}(\log t)$ vers une variable non dégénérée (*c.f.* Chap. II, Théorème 1.5). Nous calculons de plus la loi de cette variable lorsque $\mathbb{S}$ est asymétrique, ce qui complète un résultat de Cheliotis [Che06b] qui, lorsque le potentiel est un processus stable asymétrique, a déterminé la distribution limite de $X_t / \log^\alpha t$.

L'hypothèse (a) a pour objet principal de permettre le couplage de la diffusion avec une M.A.M.A. et, donc, de transcrire facilement pour le modèle discret les résultats obtenus pour la diffusion. Par exemple, si S est une M.A.M.A. dont le potentiel vérifie la condition (b), pour toute suite (c_n) croissante et positive, on obtient l'équivalence

$$\liminf_{n\to\infty} \frac{c_n}{a^{-1}(\log n)} \sup_{i \leqslant n} |S_i| = \begin{cases} 0 \\ \infty \end{cases} \iff \sum_{n \geqslant 2} \frac{1}{c_n^2 n \log n} \begin{cases} = \infty \\ < \infty. \end{cases}$$

Les autres théorèmes s'adaptent de manière similaire.

2.2 Chapitre III. Régimes de transience d'une diffusion dans un potentiel Lévy sans sauts positifs [2]

Nous nous intéressons à la vitesse de transience d'une diffusion dans un potentiel de type Lévy. Cette étude, quand le potentiel est un mouvement brownien avec dérive de la forme

$$\mathbb{V}_x = \mathbb{B}_x - \frac{\kappa}{2} x \qquad (\kappa > 0), \tag{2.1}$$

a été réalisée par Kawazu et Tanaka [KT97] puis précisée par Tanaka [Tan97] et Hu, Shi et Yor [HSY99]. On distingue alors cinq régimes selon la valeur de κ :

$$\kappa < 1, \qquad \kappa = 1, \qquad 1 < \kappa < 2, \qquad \kappa = 2, \qquad \kappa > 2.$$

En particulier, la diffusion admet une vitesse asymptotique non nulle pour $\kappa > 1$ et vérifie un théorème de type limite centrale pour $\kappa > 2$.

Plus généralement, lorsque le potentiel $\mathbb{V}$ est un processus de Lévy, nous introduisons son exposant de Laplace Φ défini par

$$\mathbf{E}[e^{\lambda \mathbb{V}_x}] = e^{x\Phi(\lambda)} \qquad \text{pour } x, \lambda \geqslant 0.$$

[2] A. Singh, *Rates of convergence of a transient diffusion in a spectrally negative Lévy potential*, à paraître dans Ann. Probab.

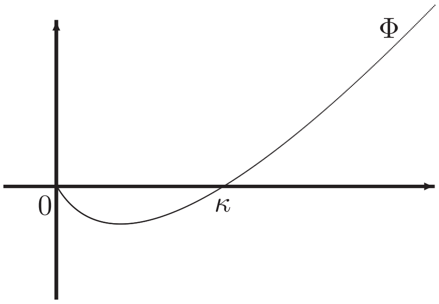

FIG. I.4 : L'exposant de Laplace Φ.

Dans le cas du potentiel décrit par (2.1), on obtient $\Phi(\lambda) = \lambda(\lambda - \kappa)/2$. En particulier, κ est l'unique racine positive de Φ ou, de manière équivalente, l'unique solution de l'équation

$$\mathbf{E}[e^{\kappa \mathbb{V}_1}] = 1. \tag{2.2}$$

On retrouve ainsi la condition (dite de Cramer) introduite par Kesten, Kozlov et Spitzer [KKS75] qui caractérise les régimes de transience d'une M.A.M.A. Or, les résultats concernant le modèle discret de la M.A.M.A. sont valides pour des potentiels très généraux, il paraît donc intéressant de voir si la condition (2.2) caractérise aussi les différents régimes de transience de la diffusion pour des potentiels plus généraux.

Dans cet esprit, Carmona [Car97] prouve une loi des grands nombres quand le potentiel est un processus Lévy. Il conjecture, de plus, que si le potentiel vérifie la condition (2.2), on doit effectivement observer des régimes de transience similaires à ceux de la M.A.M.A.

Dans ce chapitre, nous apportons lorsque le potentiel est spectralement négatif une réponse positive à cette conjecture. Plus précisement, nous faisons les hypothèses suivantes :

(a) Le potentiel $(\mathbb{V}_x)_{x \geqslant 0}$ est un processus de Lévy, sans saut positif, qui n'est pas l'opposé d'un subordinateur, et qui dérive vers $-\infty$.

(b) La diffusion X est transiente vers $+\infty$.

Sous l'hypothèse (a), l'exposant de Laplace Φ du potentiel $\mathbb{V}$ est bien défini. C'est une fonction continue, convexe (*c.f.* figure I.4) et il existe, de plus, un unique réel $\kappa > 0$ solution de l'equation (2.2).

Le résultat principal de cet article caractérise les taux de convergence des temps d'atteinte de la diffusion.

Théorème 2.4

On définit, pour $r \geqslant 0$,

$$H(r) \stackrel{def}{=} \inf\{t \geqslant 0,\ X_t = r\}.$$

Lorsque $\kappa > 1$, *on pose également* $\mathbf{m} \stackrel{def}{=} \frac{-2}{\Phi(1)}$. *On a selon la valeur de* κ *:*

- *Si* $0 < \kappa < 1$,
$$\frac{1}{r^{1/\kappa}} H(r) \xrightarrow[r\to\infty]{\text{loi}} \mathcal{S}_\kappa.$$
où $\mathcal{S}_\kappa$ *est une loi stable positive d'indice* κ.
- *Si* $\kappa = 1$, *il existe une fonction* f *vérifiant* $f(r) \sim \frac{2}{\Phi'(1)} r \log r$ *telle que*
$$\frac{1}{r}\left(H(r) - f(r)\right) \xrightarrow[r\to\infty]{\text{loi}} \mathcal{C}.$$
où $\mathcal{C}$ *est une loi de Cauchy complètement asymétrique.*
- *Si* $1 < \kappa < 2$,
$$\frac{1}{r^{1/\kappa}}\left(H(r) - \mathbf{m}r\right) \xrightarrow[r\to\infty]{\text{loi}} \mathcal{S}_\kappa.$$
où $\mathcal{S}_\kappa$ *est une loi stable complètement asymétrique d'indice* κ.
- *Si* $\kappa = 2$,
$$\frac{1}{\sqrt{r\log r}}\left(H(r) - \mathbf{m}r\right) \xrightarrow[r\to\infty]{\text{loi}} \left(\frac{-4}{\Phi(1)\sqrt{\Phi'(2)}}\right)\mathcal{N}.$$
où $\mathcal{N}$ *est une loi gaussienne, centrée et réduite.*
- *Si* $\kappa > 2$,
$$\frac{1}{\sqrt{r}}\left(H(r) - \mathbf{m}r\right) \xrightarrow[r\to\infty]{\text{loi}} \sqrt{\frac{8\left(\Phi(2) - 4\Phi(1)\right)}{\Phi(1)^3\Phi(2)}}\mathcal{N}.$$

Nous retrouvons en particulier, lorsque le potentiel est de la forme (2.1), les résultats de Hu, Shi et Yor [HSY99], à l'exception du cas $\kappa = 1$ qui est un peu moins précis.

Notons que ces estimations des temps d'atteinte se transcrivent directement sur la diffusion. Par exemple, pour $0 < \kappa < 1$, on obtient
$$\frac{X_t}{t^\kappa} \xrightarrow[t\to\infty]{\text{loi}} \mathcal{L}_\kappa \quad \text{(loi de Mittag-Leffler d'indice } \kappa\text{)}.$$

Dans le cas d'un potentiel brownien avec dérive, Kawazu et Tanaka étudient les temps d'atteinte de la diffusion à l'aide du lemme de Kotani et de la théorie spectrale de Krein. Hu, Shi et Yor, quant à eux, utilisent la transformation de Lamperti et exploitent certaines propriétés fines des processus de Jacobi. Ces deux approches se fondent sur des résultats spécifiques au mouvement brownien et ne semblent pas adaptées aux cas de potentiels plus généraux.

Notre méthode est, en un sens, plus proche de celle employée par Kesten, Kozlov et Spitzer dans le cadre des M.A.M.A. Nous ramenons l'étude des temps d'atteinte de la diffusion à l'étude des excursions d'un processus Ornstein-Uhlenbeck généralisé qui, sous l'hypothèse (a), est un processus de Markov « sympathique » pour lequel on peut définir un temps local et une mesure d'excursion. Le résultat clef dans la preuve du théorème est une estimation précise de l'aire d'une excursion générique de ce processus.

2.3 Chapitre IV. Une diffusion transiente très lente [3]

L'objectif de ce court chapitre est de fournir un exemple de diffusion X dans un potentiel aléatoire $\mathbb{V}$ dont le régime de transience est très différent de celui observé lorsque le potentiel est un mouvement brownien avec dérive (ou plus généralement un processus de Lévy sans saut positif). Nous considérons ici un potentiel de la forme :

$$\mathbb{V}_x = \mathbb{S}_x - \delta x$$

où $\mathbb{S}$ est un processus stable d'indice $\alpha \in]1,2[$ et $\delta > 0$. La mesure de Lévy Π du processus stable est donnée par

$$\Pi(dx) = \left(c^+ \mathbf{1}_{\{x>0\}} + c^- \mathbf{1}_{\{x<0\}}\right) \frac{dx}{|x|^{\alpha+1}} \qquad \text{avec } c^+, c^- \geqslant 0 \text{ et } c^+ + c^- > 0.$$

En particulier, $\mathbb{S}$ possède des sauts positifs (resp. négatifs) si et seulement si $c^+ > 0$ (resp. $c^- > 0$). Lorsque $c^+ = 0$, la vitesse de transience de la diffusion dans le potentiel $\mathbb{V}$ a été caractérisée dans la section précédente. Nous supposons maintenant $c^+ > 0$. Dans ce cas, $\mathbb{V}$ n'admet pas de moments exponentiels et la condition de Cramer (2.2) n'est plus vérifiée. Le comportement asymptotique de la diffusion est alors original :

Théorème 2.5

La diffusion X est transiente vers $+\infty$ et

$$\frac{X_t}{\log^\alpha t} \xrightarrow[t\to\infty]{\text{loi}} \mathcal{E}\left(\frac{c^+}{\alpha}\right),$$

où $\mathcal{E}(c^+/\alpha)$ est une loi exponentielle de paramètre c^+/α. Ce résultat demeure valable si l'on remplace X_t par $\sup_{s\leqslant t} X_s$ ou $\inf_{s\geqslant t} X_s$.

Notons que la valeur de $\delta > 0$ n'influe ni sur le régime de transience, ni sur la loi limite. De plus, lorsque $\delta = 0$, la diffusion est récurrente et $X_t/\log^\alpha t$ converge vers une loi non dégénérée (*c.f.* [Sch85]). Nous avons donc ici un exemple de diffusion transiente qui se déplace aussi lentement que dans le cas récurrent. La preuve de ce théorème utilise, d'ailleurs, des méthodes employées traditionellement pour l'étude de la diffusion dans le cas récurrent.

Ce résultat admet une heuristique simple : si l'on voit le potentiel comme une « énergie », le temps nécessaire à la diffusion pour toucher un niveau $r > 0$ est, à peu près, exponentiellement proportionnel à la plus grande « barrière » de $\mathbb{V}$ sur l'intervalle $[0, r]$. Pour un mouvement brownien, l'adjonction d'une dérive diminue fortement la hauteur de ces barrières. Par contre, pour un processus stable avec sauts, la plus grande barrière sur l'intervalle $[0, r]$ est du même ordre de grandeur que son plus grand saut positif sur cet intervalle. Comme l'ajout d'une dérive ne modifie en rien les sauts du processus, les barrières

[3] A. Singh, *A slow transient diffusion in a drifted stable potential*, J. Theoret. Probab., **20**(2), 153–166, 2007.

du potentiel $\mathbb{V}$ sont comparables à celles observées pour un potentiel stable sans dérive, ce qui explique le déplacement très lent de la diffusion.

3 La marche multi-excitée

Dans la seconde partie de cette thèse, nous nous intéressons au modèle stochastique de la marche excitée et de sa généralisation : la marche multi-excitée.

3.1 Présentation du modèle et de quelques résultats

La marche excitée, introduite en 2003 par Benjamini et Wilson [BW03] peut être considérée comme un analogue en temps discret des modèles de mouvements browniens perturbés aux extrema (*c.f.* [PW97, Dav99]). Il s'agit d'une marche $X = (X_n)_{n \geqslant 0}$ aux plus proches voisins sur $\mathbb{Z}^d$ qui possède, lors de son premier passage en un site, un biais dans une direction spécifique, mais qui se comporte, lors de son retour en un site déjà visité, comme une simple marche symétrique.

Benjamini et Wilson [BW03] ont montré que la marche excitée est récurrente en dimension $d = 1$, mais devient transiente, dans la direction imposée par le biais, dès que $d \geqslant 2$. Il ont de plus établi que, pour $d \geqslant 4$, le régime de transience de la marche est linéaire :

$$\liminf_{n\to\infty} \frac{X_n \cdot \mathbf{e}}{n} > 0 \quad \text{presque sûrement,} \tag{3.1}$$

où $\mathbf{e}$ représente un vecteur indiquant la direction du biais. Plus récemment, Kozma a montré que (3.1) reste valide en dimensions 2 et 3 [Koz05, Koz03].

Dans le cas unidimensionnel, le biais fourni lors de la première visite d'un site ne suffit pas à rendre la marche transiente. Il paraît donc intéressant d'autoriser plusieurs degrés d'excitation. Dans cet esprit, Zerner [Zer05] introduit, d'abord dans le cas de la dimension 1 [Zer05], puis pour les dimensions supérieures [Zer06], un modèle appelé marche multi-excitée ou, de manière plus imagée, « cookie random walk ».

Le modèle décrit dans [Zer05] est très général, il permet en particulier à « l'environnement de cookies » d'être lui même aléatoire. Toutefois, nous ne considérons, dans ce travail, que le cadre d'un environnement initial déterministe. Le modèle est le suivant. On choisit $M \in \mathbb{N}^*$ ainsi qu'un vecteur

$$\bar{p} = (p_1, p_2, \ldots, p_M) \in \left[\frac{1}{2}, 1\right[^M.$$

L'entier M représente le nombre de cookies par site et $\bar{p}$ décrit l'environnement de cookies. Plus précisément, p_i correspond à la force du i-ème cookie placé initialement en chaque site de $\mathbb{Z}$. On définit alors la marche $\bar{p}$-excitée $X = (X_n)_{n \geqslant 0}$ comme une marche aux plus proches voisins sur $\mathbb{Z}$, issue de 0, qui « mange » les cookies qu'elle rencontre sur son chemin en se déplaçant de la manière suivante :

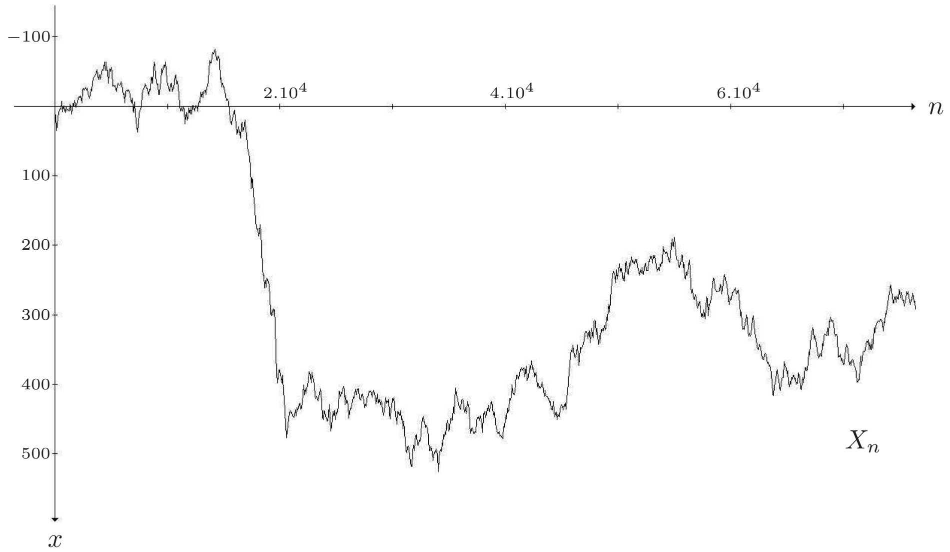

FIG. I.5 : Simulation d'une marche multi-excitée avec $M = 2$ et $p_1 = p_2 = 0.7$.

- si $X_n = x$ et s'il ne reste plus aucun cookie au site x, alors X se déplace, au temps $n+1$ vers les sites $x+1$ ou $x-1$ avec une égale probabilité $\frac{1}{2}$;
- si $X_n = x$ et s'il reste, au site x, les cookies de forces $p_j, p_{j+1}, \dots, p_M$, alors la marche « mange » le premier cookie disponible (*i.e.* celui de force p_j) puis se déplace vers $x+1$ avec probabilité p_j ou vers $x-1$ avec probabilité $1-p_j$.

Rigoureusement, la marche $\bar{p}$-excitée est un processus, issu de zéro, dont les probabilités de transition vérifient :

$$\begin{aligned} \mathbf{P}\{X_{n+1} = X_n \pm 1 \,|\, X_0, \dots, X_n\} &= 1, \\ \mathbf{P}\{X_{n+1} = X_n + 1 \,|\, X_0, \dots, X_n\} &= \begin{cases} p_j & \text{si } j = \sharp\{0 \leqslant i \leqslant n,\, X_i = X_n\} \leqslant M, \\ \frac{1}{2} & \text{sinon.} \end{cases} \end{aligned}$$

Il ne s'agit donc pas, sauf cas dégénérés, d'un processus de Markov puisque le déplacement de la marche dépend, non seulement de sa position actuelle, mais également du nombre de visites précédentes à ce site.

Dans le cas d'un unique cookie par site ($M = 1$), on retrouve le modèle de la marche excitée unidimensionnelle décrit par Benjamini et Wilson [BW03]. Cependant, en autorisant plusieurs degrés d'excitation (*i.e.* plusieurs cookies par site), on enrichit considérablement le modèle. Ainsi, la marche multi-excitée peut être récurrente ou transiente selon le choix $\bar{p}$. Plus précisément, Zerner [Zer05] a montré, pour le modèle présenté ici, que si l'on définit

$$\alpha(\bar{p}) \stackrel{\text{def}}{=} \sum_{i=1}^{M} (2p_i - 1) - 1, \tag{3.2}$$

alors, selon la valeur de cette quantité :

- si $\alpha(\bar{p}) \leqslant 0$, la marche $\bar{p}$-excitée est récurrente ;
- si $\alpha(\bar{p}) > 0$, la marche $\bar{p}$-excitée est transiente vers $+\infty$.

Lorsque $\bar{p}$ varie, $\alpha(\bar{p})$ décrit l'intervalle $[-1, M-1[$. On retrouve donc que la marche excitée est récurrente, mais, par contre, la marche multi-excitée peut être transiente à partir de deux cookies par site. Zerner a de plus établi que la vitesse asymptotique de la marche est bien définie : il existe une constante $v = v(\bar{p}) \geqslant 0$ telle que

$$\lim_{n\to\infty} \frac{X_n}{n} = v \quad \text{presque sûrement.}$$

Toutefois, la vitesse limite v peut, comme dans le cas des M.A.M.A., être nulle même lorsque la marche est transiente. Zerner [Zer05] montre d'ailleurs que la vitesse est forcément nulle lorsque $M = 2$. D'un autre côté, Mountford, Pimentel et Valle [MPV06] ont établi que la vitesse de la marche peut, dans certains cas, être strictement positive. Notre travail porte sur les deux questions suivantes :

(1) Pour quels environnements $\bar{p}$ la vitesse asymptotique de la marche est-elle strictement positive ?

(2) Dans le cas d'une marche transiente de vitesse asymptotique nulle, quel est le régime de transience ?

3.2 Chapitre V. Vitesse d'une marche multi-excitée [4]

Dans [MPV06], Mountford, Pimentel et Valle considèrent une marche $\bar{p}$-excitée dans le cas où tous les cookies ont la même force *i.e.* $p_1 = p_2 = \ldots = p_M = p \in [\frac{1}{2}, 1[$. Ils montrent que :

- si $M(2p-1) < 2$, alors la vitesse limite de la marche est nulle ;
- pour tout $p \in]\frac{1}{2}, 1[$, la vitesse limite de la marche devient strictement positive dès que le nombre initial M de cookies est suffisamment grand.

De plus, ils conjecturent que, lorsque $M(2p-1) > 2$, la vitesse de la marche doit être strictement positive. Notons que dans le cas considéré par Mountford, Pimentel et Valle, on a $M(2p-1) = \alpha(\bar{p}) + 1$. Dans ce chapitre, nous montrons que la position de $\alpha(\bar{p})$ par rapport à 1 détermine la stricte positivité de la vitesse.

Théorème 3.1

Soit X une marche $\bar{p}$-excitée. On note $v(\bar{p})$ sa vitesse asymptotique. On a l'équivalence

$$v(\bar{p}) > 0 \quad \Longleftrightarrow \quad \alpha(\bar{p}) > 1$$

où $\alpha(\bar{p})$ est défini par (3.2).

[4] A.-L. Basdevant et A. Singh, *On the speed of a cookie random walk*, article soumis

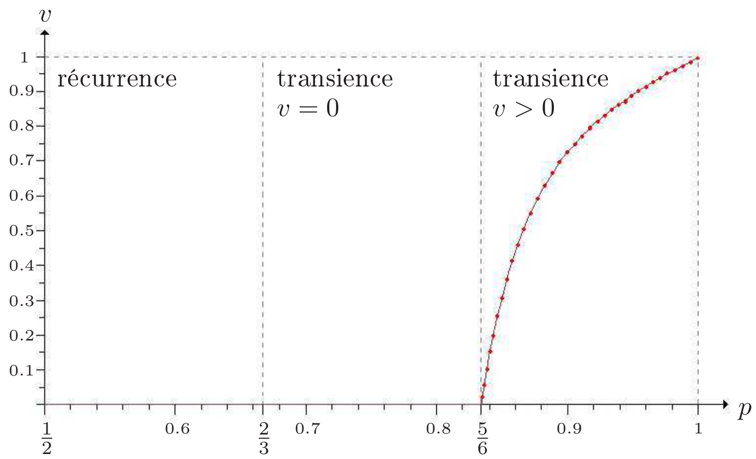

FIG. I.6 : Simulation de la vitesse d'une marche (p, p, p)-excitée.

Ce théorème montre qu'une vitesse limite non nulle peut être obtenue avec seulement trois cookies par site. De plus, le critère de Zerner [Zer05] affirme que la marche est transiente si et seulement si $\alpha(\bar{p}) > 0$. On constate donc une seconde « transition de phase » aux points critiques $\alpha(\bar{p}) = 1$.

Il semble délicat d'obtenir une expression explicite de la vitesse. Nous montrons toutefois que la vitesse est une fonction continue de $\bar{p}$ et admet une « dérivée à droite » en tout point critique.

Théorème 3.2

Soit $\Omega_M \stackrel{def}{=} [\frac{1}{2}, 1[^M$, l'ensemble des environnements avec au plus M cookies par site.

(a) Pour chaque $M \geqslant 1$, la vitesse $v(\bar{p})$ de la marche $\bar{p}$-excitée est une fonction continue de $\bar{p}$ dans Ω_M.

(b) Pour tout environnement (critique) $\bar{p}_c \in \Omega_M$ tel que $\alpha(\bar{p}_c) = 1$, il existe une constante $C(\bar{p}_c) > 0$ telle que

$$\lim_{\substack{\bar{p} \to \bar{p}_c \\ \bar{p} \in \Omega_M \\ \alpha(\bar{p}) > 1}} \frac{v(\bar{p})}{\alpha(\bar{p}) - 1} = C(\bar{p}_c).$$

Lorsque $M = 3$ et $p_1 = p_2 = p_3 = p$, la marche $\bar{p}$-excitée est récurrente si $p \in [\frac{1}{2}, \frac{2}{3}]$, transiente mais de vitesse nulle si $p \in]\frac{2}{3}, \frac{5}{6}]$, et admet une vitesse strictement positive si $p \in]\frac{5}{6}, 1[$. Une simulation de la courbe de vitesse pour cette marche, illustrant le théorème précédent, est proposée sur la figure I.6.

Afin d'établir ces théorèmes, nous introduisons un processus auxiliaire Z étroitement relié aux temps d'atteinte de la marche multi-excitée. La construction de ce processus s'inspire de la construction classique du processus de Galton-Watson critique associé à une marche symétrique simple. Dans notre cas, le processus Z obtenu est un processus de branchement avec migration aléatoire. Il s'agit, lorsque la marche est transiente, d'une chaîne de Markov récurrente positive. L'étude de sa distribution invariante, en particulier l'existence ou non d'un premier moment pour cette loi, permet d'établir le Théorème 3.1.

La preuve du théorème 3.2 repose, quant à elle, d'une part sur une expression implicite de la vitesse obtenue lors de l'étude de la loi invariante du processus Z, et d'autre part, sur un résultat de Zerner [Zer05] concernant la monotonie des temps d'atteinte de la marche vis à vis de l'environnement initial.

3.3 Chapitre VI. Régime de transience d'une marche multi-excitée[5]

Nous étudions, dans le dernier chapitre de cette thèse, le comportement asymptotique d'une marche multi-excitée X, transiente et de vitesse limite nulle. Des simulations numériques effectuées par Antal et Redner [AR05] indiquent que X_n est alors d'ordre n^ν pour un certain exposant $\nu \in]0,1[$.

Nous montrons que $\nu = \frac{\alpha(\bar{p})+1}{2}$, où $\alpha(\bar{p})$ est le paramètre défini par (3.2). Notre résultat principal décrit les différents régimes de transience de la marche.

Théorème 3.3

Soit X une marche $\bar{p}$-excitée. On suppose que X est transiente et de vitesse limite nulle (i.e. $0 < \alpha(\bar{p}) \leqslant 1$). On pose

$$\nu \stackrel{\text{def}}{=} \frac{\alpha(\bar{p})+1}{2},$$

alors

- *si $\alpha(\bar{p}) < 1$,*
$$\frac{X_n}{n^\nu} \xrightarrow[n\to\infty]{\text{loi}} \mathcal{L}_\nu$$
où $\mathcal{L}_\nu$ est une loi de Mittag-Leffler d'indice ν ;

- *si $\alpha(\bar{p}) = 1$, il existe une constante $C(\bar{p}) > 0$ telle que*
$$\frac{\log n}{n} X_n \xrightarrow[n\to\infty]{\text{prob.}} C(\bar{p}).$$

Ces résultats demeurent inchangés si l'on remplace X_n par $\sup_{i\leqslant n} X_i$ ou $\inf_{i\geqslant n} X_i$.

Remarquons que les régimes de transience de la marche multi-excitée sont analogues à ceux de la M.A.M.A. La preuve du théorème utilise d'ailleurs une approche similaire à celle employée par Kesten, Kozlov et Spitzer [KKS75] lors de l'étude des régimes de transience de la M.A.M.A. Comme dans le chapitre précédent, on ramène l'étude de la marche multi-excitée à l'étude d'un processus de branchement avec migration Z. Le résultat clef est alors

[5] A.-L. Basdevant et A. Singh, *Rate of growth of a transient cookie random walk*, article soumis.

une estimation précise de la queue de distribution de la population totale du processus lors d'une excursion :

$$\mathbf{P}\left\{\sum_{k=0}^{\sigma} Z_k > x\right\} \underset{x\to\infty}{\sim} \begin{cases} C/x^{\nu} & \text{si } \nu < 1, \\ C(\ln x)/x & \text{si } \nu = 1. \end{cases} \tag{3.3}$$

où σ est le premier temps de retour en 0 pour Z. Dans le cadre de la M.A.M.A., Kesten, Kozlov et Spitzer montrent un résultat similaire à (3.3) mais Z est, dans leur cas, un processus de branchement en environnement aléatoire. Toutefois, la technique employée dans ce chapitre pour obtenir (3.3) est très différente de celle utilisée dans [KKS75].

Notre méthode se base principalement sur un argument de martingale et ne repose pas (comme c'est habituellement le cas pour l'étude des processus de branchement avec migration) sur l'étude de fonctions génératrices. Notons enfin que, bien que nous n'étudions que le processus Z associé à la marche multi-excitée, l'approche utilisée peut permettre d'étudier, plus généralement, la population totale d'une classe assez large de processus de branchement avec migration aléatoire.

Part

Diffusions en milieux aléatoires

Chapter II

Limiting behavior of a diffusion in an asymptotically stable environment[1]

Abstract. We study the almost sure asymptotics of a diffusion in an asymptotically stable potential. The results also translate for the corresponding one-dimensional random walk in random environment.

1 Introduction

Let $(\mathbb{V}_x,\, x \in \mathbb{R})$ be a càdlàg, real-valued locally bounded stochastic process on some probability space $(\Omega, \mathbb{P})$ with $\mathbb{V}_0 = 0$ almost surely. Let also $(X_t,\, t \geqslant 0)$ denote the coordinate process on the space of continuous functions $C([0,\infty))$ equipped with the topology of uniform convergence on compact sets and the associated σ-field. For each realization of $\mathbb{V}$, let $P_{\mathbb{V}}$ be a probability on $C([0,\infty))$ such that X is a diffusion process with $X_0 = 0$ and generator

$$\frac{1}{2} e^{\mathbb{V}_x} \frac{d}{dx} \left(e^{-\mathbb{V}_x} \frac{d}{dx} \right).$$

It is well known, see for instance [IM65], that such a diffusion may be constructed from a standard Brownian motion by a change of scale and a change of time. We consider the annealed probability $\mathbf{P}$ on $\mathbf{\Omega} = \Omega \times C([0,\infty))$ defined as the semi-direct product $\mathbf{P} = \mathbb{P} \times P_{\mathbb{V}}$. The process X under $\mathbf{P}$ is called a diffusion in the random potential $\mathbb{V}$. This process was first studied by Schumacher [Sch85] and Brox [Bro86] who proved that, when $\mathbb{V}$ is a Brownian motion, the diffusion is recurrent and $X_t/\log^2 t$ converges in law as t goes to infinity to some non-degenerate distribution. Extension of this result when $\mathbb{V}$ is a stable process may be found in [Che06b, KTT92, Sch85]. In this paper, we consider the case where $\mathbb{V}$ is a

[1]This chapter is a slightly modified version of the article: A. Singh, *Limiting behavior of a diffusion in an asymptotically stable environment*, Ann. Inst. H. Poinc. Probab. Statist., **43**(1), 101–138, 2007.

two-sided random walk. More precisely, $(\mathbb{V}_x,\ x \in \mathbb{R})$ satisfies:

$$\begin{cases} \mathbb{V} \text{ is identically } 0 \text{ on } (-1,1), \\ \mathbb{V} \text{ is flat on } (n, n+1) \text{ for all } n \in \mathbb{Z}, \\ \mathbb{V} \text{ is right continuous on } [0,\infty) \text{ and left continuous on } (-\infty, 0], \\ (\mathbb{V}_{n+1} - \mathbb{V}_n,\ n \in \mathbb{Z}) \text{ is a sequence of i.i.d. random variables.} \end{cases}$$

The reason for chosing a potential flat on integer intervals is that the diffusion is, in this case, "close" to a random walk in random environment and the results obtained for the diffusion X will also translate for the discrete time model.

Our aim is to describe the almost sure asymptotics of X_t, $\sup_{s \leqslant t} X_s$ and $\sup_{s \leqslant t} |X_s|$. This has been done by Hu and Shi [HS98a] when the process $\mathbb{V}$ behaves roughly like a Brownian motion. Here, we consider the more general setting where a typical step of the random walk is in the domain of attraction of a strictly stable law. Precisely, we make the following assumption which is similar to that of Kawazu, Tamura and Tanaka [KTT92].

Assumption 1.1

There exists a positive sequence $(a_n,\ n \geqslant 0)$ such that

$$\frac{\mathbb{V}_n}{a_n} \xrightarrow[n\to\infty]{\text{law}} \mathbb{S}$$

where $\mathbb{S}$ is a random variable whose law is strictly stable with index $\alpha \in (0,2]$ and whose density is everywhere positive on $\mathbb{R}$.

This assumption implies, of course, that $\mathbb{V}_{-n}/a_n$ converges in law toward $-\mathbb{S}$. It is also well known that the norming sequence (a_n) is regularly varying with index $1/\alpha$ and we can without loss of generality assume that (a_n) is strictly increasing with $a_1 = 1$. We now denote by $a(\cdot)$ a continuous, strictly increasing interpolation of (a_n) and $a^{-1}(\cdot)$ stands for its inverse. It is to be noted that $a(\cdot)$ and $a^{-1}(\cdot)$ are respectively regularly varying with index $1/\alpha$ and α. Let p denote the positivity parameter of $\mathbb{S}$ and q its negativity parameter,

$$p \stackrel{\text{def}}{=} \mathbf{P}\{\mathbb{S} > 0\} = 1 - \mathbf{P}\{\mathbb{S} < 0\} \stackrel{\text{def}}{=} 1 - q.$$

The assumption that $\mathbb{S}$ has a positive density in the whole of $\mathbb{R}$ implies that $p, q \in (0,1)$. More precisely, for $\alpha > 1$, we have $1 - 1/\alpha \leqslant p, q \leqslant 1/\alpha$ (see section 2.6 of [Zol86] or p218 of [Ber96a]) and in any case:

$$0 < \alpha p, \alpha q \leqslant 1.$$

Note also that the Fourier transform of $\mathbb{S}$ has the form

$$\mathbf{E}\left[e^{i\lambda\mathbb{S}}\right] = e^{-\gamma|\lambda|^\alpha \left(1 - i\frac{\lambda}{|\lambda|}\tan\left(\pi\alpha\left(p-\frac{1}{2}\right)\right)\right)} \tag{1.1}$$

where γ is some strictly positive constant. Let us now extend $\mathbb{S}$ into a two-sided strictly stable process $(\mathbb{S}_x,\ x \in \mathbb{R})$ such that $\mathbb{S}_1$ has the same law as $\mathbb{S}$. By two-sided, we mean that the processes $(\mathbb{S}_t,\, t \geqslant 0)$ and $(-\mathbb{S}_{-t},\, t \geqslant 0)$ are independent, are both càdlàg, and have the same law. Notice in particular that, when $\alpha = 1$, $\mathbb{S}$ is a symmetric Cauchy process with drift, whereas for $\alpha = 2$ we have $p = 1/2$ and $\mathbb{S}$ is a Brownian motion. Furthermore, the extremal cases $\alpha p = 1$ (resp. $\alpha q = 1$) can only happen when $\alpha > 1$ and are equivalent to the assumption that $\mathbb{S}$ has no positive jumps (resp. no negative jumps). When $\mathbb{S}$ has no positive jumps, it admits finite exponential moments of any order and the Fourier transform can be extended so that

$$\mathbf{E}\Big[e^{\lambda \mathbb{S}_1}\Big] = e^{\gamma' \lambda^\alpha} \text{for all } \lambda \geqslant 0 \tag{1.2}$$

where γ' is a positive constant that we will always assume to be 1 (we can reduce to this case by changing the norming sequence (a_n)). Similarly, when $\mathbb{S}$ has no negative jumps, we will assume $\mathbf{E}[\exp(-\lambda \mathbb{S}_1)] = \exp(\lambda^\alpha)$ for all $\lambda \geqslant 0$. Let also $\mathcal{M}_\alpha$ denote the Mittag-Leffler function with parameter α:

$$\mathcal{M}_\alpha(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)} \quad \text{for } x \in \mathbb{R}.$$

Define $-\rho_1(\alpha)$ to be the first negative root of $\mathcal{M}_\alpha$ and $-\rho_2(\alpha)$ to be the first negative root of $\alpha x \mathcal{M}''_\alpha(x) + (\alpha - 1)\mathcal{M}'_\alpha(x)$. The first result of this paper is a law of the iterated logarithm for the limsup of the diffusion X in the random environment $\mathbb{V}$.

Theorem 1.2

We have, almost surely,

$$\limsup_{t\to\infty} \frac{X_t}{a^{-1}(\log t) \log\log\log t} = \frac{1}{K^\#}$$

where $K^\# \in (0,\infty)$ is a constant that only depends on the limit law $\mathbb{S}$ and is given by

$$K^\# \stackrel{\text{def}}{=} -\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}\Big\{ \sup_{0 \leqslant u \leqslant v \leqslant t} (\mathbb{S}_v - \mathbb{S}_u) \leqslant 1 \Big\}.$$

Furthermore, when $\mathbb{S}$ is completely asymmetric, the value of $K^\#$ is given by

$$K^\# = \begin{cases} \rho_1(\alpha) & \text{when } \mathbb{S} \text{ has no positive jumps,} \\ \rho_2(\alpha) & \text{when } \mathbb{S} \text{ has no negative jumps.} \end{cases}$$

Note that X_t and $\sup_{s \leqslant t} X_s$ have the same running maximum, hence Theorem 1.2 also holds with $\sup_{s \leqslant t} X_s$ in place of X_t. A symmetry argument yields

$$\limsup_{t\to\infty} \frac{-\inf_{s \leqslant t} X_t}{a^{-1}(\log t) \log\log\log t} = \frac{1}{\widetilde{K}^\#} \text{ a.s.}$$

where $\widetilde{K}^\# = -\lim_{t\to\infty} \log \mathbf{P}\{\sup_{0 \leqslant u \leqslant v \leqslant t} (\mathbb{S}_{-v} - \mathbb{S}_{-u}) \leqslant 1\}/t$. Hence, we also get a iterated logarithm law for the bilateral supremum:

$$\limsup_{t\to\infty} \frac{\sup_{s\leqslant t}|X_t|}{a^{-1}(\log t)\log\log\log t} = \frac{1}{\widetilde{K}^{\#}\wedge K^{\#}} \quad \text{a.s.}$$

When $\alpha = 2$, we have $\mathcal{M}_\alpha(-x) = \cos(\sqrt{x})$ for all $x \geqslant 0$. Therefore, $\widetilde{K}^{\#} = K^{\#} = \pi^2/4$ and we recover the law of the iterated logarithm of Theorem 1.6 of Hu and Shi [HS98a].

Let $\mathbf{T}_n$ denote the n^{th} strictly descending ladder time of the random walk $\mathbb{V}$,

$$\begin{cases} \mathbf{T}_0 \stackrel{\text{def}}{=} 0, \\ \mathbf{T}_{n+1} \stackrel{\text{def}}{=} \min\left\{k > \mathbf{T}_n,\, \mathbb{V}_k < \mathbb{V}_{\mathbf{T}_n}\right\}. \end{cases}$$

Since $\mathbb{V}$ is oscillatory, $\mathbf{T}_n$ is finite for all n. Theorem 4 of Rogozin [Rog71] states that $\mathbf{T}_1$ is in the domain of attraction of a positive stable law with index q. Moreover, $\mathbf{T}_1$ is in the domain of normal attraction of this distribution if and only if

$$\sum_{n=1}^{\infty} \frac{\mathbf{P}\{\mathbb{V}_n < 0\} - q}{n} < \infty. \tag{1.3}$$

Let (b_n) denote a strictly increasing sequence of norming constants for $\mathbf{T}_1$ and let $b(\cdot)$ stand for a continuous, strictly increasing interpolation of this sequence. The function $b^{-1}(\cdot)$ is therefore regularly varying with index q. The next result characterizes the liminf behavior of $\sup_{s\leqslant t} X_s$.

Theorem 1.3

For any positive, non-decreasing function f define

$$J(f) \stackrel{\text{def}}{=} \int^{\infty} \frac{b^{-1}\left(a^{-1}(\log t)/f(t)\right)dt}{b^{-1}\left(a^{-1}(\log t)\right) t \log t}$$

(we do not specify the lower bound since we are only concerned with the convergence of the integral at infinity). We have, almost surely,

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)} \sup_{s\leqslant t} X_s = \begin{cases} 0 \\ \infty \end{cases} \iff J(f) \begin{cases} = \infty \\ < \infty. \end{cases}$$

In particular, with probability 1,

$$\liminf_{t\to\infty} \frac{(\log\log t)^\beta}{a^{-1}(\log t)} \sup_{s\leqslant t} X_s = \begin{cases} 0, & \text{if } \beta < 1/q, \\ \infty, & \text{if } \beta > 1/q. \end{cases} \tag{1.4}$$

Notice that (1.3) holds whenever $\mathbb{V}_1$ is strictly stable or when $\mathbf{E}[\mathbb{V}_1^2] < \infty$ (according to Theorem 1 of Feller [Fel71], p 575). In those two cases, $\mathbb{V}_1$ is also in the domain of normal attraction of $\mathbb{S}$ so that we can choose both $a(x) = x^{1/\alpha}$ and $b(x) = x^{1/q}$ and the last theorem is simplified:

$$\liminf_{t\to\infty} \frac{f(t)}{(\log t)^\alpha} \sup_{s \leqslant t} X_s = \begin{cases} 0 \\ \infty \end{cases} \iff \int^\infty \frac{dt}{f^q(t) t \log t} \begin{cases} = \infty \\ < \infty. \end{cases}$$

In particular, the liminf for the critical case $\beta = 1/q$ in (1.4) is infinite.

We are also interested in the asymptotic behavior of the bilateral supremum $\sup_{s \leqslant t} |X_s|$. We have already said that the limsup behavior of this process may be deduced from Theorem 1.2. Concerning the liminf behavior, although we were not able to deal with the general case as it seems that many different behaviors may occur in the completely asymmetric case, depending on the distribution tail of $\mathbb{V}_1$, we still obtain, when the limiting process has jumps of both signs, the following integral test.

Theorem 1.4

When the limiting stable process $\mathbb{S}$ has jumps of both signs, we have, for any non-decreasing positive function f, almost surely,

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)} \sup_{s \leqslant t} |X_s| = \begin{cases} 0 \\ \infty \end{cases} \iff \int^\infty \frac{dt}{t f(t)^2 \log t} \begin{cases} = \infty \\ < \infty. \end{cases}$$

In particular, with probability 1,

$$\liminf_{t\to\infty} \frac{(\log\log t)^\beta}{a^{-1}(\log t)} \sup_{s \leqslant t} |X_s| = \begin{cases} 0, & \text{if } \beta \leqslant 1/2, \\ \infty, & \text{if } \beta > 1/2. \end{cases}$$

It is worth noticing that in this case the limiting behavior does not depend on the symmetry parameter. We also point out that this liminf is very unlike the Brownian case (*c.f.* Theorem 1.7 of [HS98a]). This may be informally explained from the fact that, when the limiting process has jumps of both signs, typical valleys of the potential $\mathbb{V}$ are much deeper than in the Brownian case.

Although we are mainly concerned with the almost sure behavior of the diffusion, our approach also allows us to prove a convergence in law for the supremum process.

Theorem 1.5

There exists a non-degenerate random variable Ξ depending only on the limiting process $\mathbb{S}$ such that under the annealed probability $\mathbf{P}$,

$$\frac{1}{a^{-1}(\log t)} \sup_{s \leqslant t} X_s \xrightarrow[t\to\infty]{\text{law}} \Xi.$$

Moreover, when $\mathbb{S}$ has no positive jumps the law of Ξ is characterized by its Laplace *transform,*

$$\mathbf{E}\left[e^{-q\Xi}\right] = \Gamma\left(\alpha+1\right)\frac{\mathcal{M}_\alpha'(q)}{\mathcal{M}_\alpha(q)} \quad \textit{for } q \geqslant 0,$$

and when $\mathbb{S}$ *has no negative jumps*

$$\mathbf{E}\left[e^{-q\Xi}\right] = \frac{(\alpha-1)\mathcal{M}_\alpha'(q)}{\alpha q \mathcal{M}_\alpha''(q) + (\alpha-1)\mathcal{M}_\alpha'(q)} \quad \textit{for } q \geqslant 0.$$

The remainder of this paper is organized as follows: in Section 2, we prove sharp results concerning the fluctuations of the potential $\mathbb{V}$ as well as on the limiting stable process $\mathbb{S}$. These estimates, which may be found of independent interest, ultimately play an important role in the proof of the main theorems. In Section 3, we reduce the study of the hitting times of the diffusion to the study of some functionals of the potential process. This step is similar to [HS98a]; we make use of Laplace's method and the reader may refer to [Shi01] for an overview of the key ideas. The proofs of the main theorems are given in Section 4. We shall eventually discuss these results in the last section, in particular, we explain why all the results mentioned above still hold when the potential $\mathbb{V}$ is itself a strictly stable process. We also explain how one can use these results to deduce similar theorems for a one-dimensional random walk in a random environment with an asymptotically stable potential.

2 Fluctuations of $\mathbb{V}$ and $\mathbb{S}$

We now give several estimates concerning the fluctuations of the random walk $\mathbb{V}$ and its limiting stable process $\mathbb{S}$. In the first subsection, we recall elementary properties of the stable process $\mathbb{S}$ as well as a result of functional convergence of the random walk toward the limiting stable process.

In the following, for any process Z, we shall use indifferently the notation Z_x or $Z(x)$.

2.1 Preliminaries and functional convergence in $\mathbb{D}$

We introduce the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg functions $Z : \mathbb{R}_+ \to \mathbb{R}$ equipped with the Skorohod topology. Let θ stand for the shift operator, *i.e.* for any $Z \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ and any $x_0 \geqslant 0$, we have

$$((\theta_{x_0} Z)_x,\, x \geqslant 0) \overset{\text{def}}{=} (Z_{x+x_0} - Z_{x_0},\, x \geqslant 0)\,. \tag{2.1}$$

Since our processes are double-sided, we will also need the space $\mathbb{D}(\mathbb{R}, \mathbb{R})$ of functions Z : $\mathbb{R} \to \mathbb{R}$ which are right continuous with left limits on $[0, \infty)$ and left continuous with right limits on $(-\infty, 0]$ considered jointly with the associated Skorohod topology. Recall that $\mathbb{S}$ and $\mathbb{V}$ have paths on $\mathbb{D}(\mathbb{R}, \mathbb{R})$. We are interested in the following functionals: for $a \in \mathbb{R}$ and for $Z \in \mathbb{D}(\mathbb{R},\mathbb{R})$, define (we give two notations for each definition)

$$\begin{aligned}
\overline{Z}_a &= F_a^{(1)}(Z) \stackrel{\text{def}}{=} \begin{cases} \sup_{y\in[0,a]} Z_y, & \text{for } a \geqslant 0,\\ \sup_{y\in[a,0]} Z_y, & \text{for } a < 0,\end{cases}\\
\underline{Z}_a &= F_a^{(2)}(Z) \stackrel{\text{def}}{=} \begin{cases} \inf_{y\in[0,a]} Z_y, & \text{for } a \geqslant 0,\\ \inf_{y\in[a,0]} Z_y, & \text{for } a < 0,\end{cases}\\
Z^*_a &= F_a^{(3)}(Z) \stackrel{\text{def}}{=} \begin{cases} \sup_{y\in[0,a]} |Z_y|, & \text{for } a \geqslant 0,\\ \sup_{y\in[a,0]} |Z_y|, & \text{for } a < 0,\end{cases}\\
Z^R_a &= F_a^{(4)}(Z) \stackrel{\text{def}}{=} Z_a - \underline{Z}_a,\\
Z^\#_a &= F_a^{(5)}(Z) \stackrel{\text{def}}{=} \begin{cases} \sup_{0\,\leqslant\, y\,\leqslant\, a} Z^R_y, & \text{for } a \geqslant 0,\\ \sup_{a\,\leqslant\, y\,\leqslant\, 0} Z^R_y, & \text{for } a < 0,\end{cases}\\
\sigma_Z(a) &= F_a^{(6)}(Z) \stackrel{\text{def}}{=} \begin{cases} \inf\{x \geqslant 0,\ Z_x \geqslant a\}, & \text{for } a \geqslant 0,\\ \inf\{x \geqslant 0,\ Z_x \leqslant a\}, & \text{for } a < 0,\end{cases}\\
\widetilde{\sigma}_Z(a) &= F_a^{(7)}(Z) \stackrel{\text{def}}{=} \begin{cases} \inf\{x \geqslant 0,\ Z_{-x} \geqslant a\}, & \text{for } a \geqslant 0,\\ \inf\{x \geqslant 0,\ Z_{-x} \leqslant a\}, & \text{for } a < 0,\end{cases}\\
U_Z(a) &= F_a^{(8)}(Z) \stackrel{\text{def}}{=} a - \underline{Z}(\sigma_Z(a)),\ \text{for } a \geqslant 0,\\
\widetilde{U}_Z(a) &= F_a^{(9)}(Z) \stackrel{\text{def}}{=} a - \underline{Z}\left(\widetilde{\sigma}_Z(a)\right),\ \text{for } a \geqslant 0,\\
\widetilde{G}_Z(a) &= F_a^{(10)}(Z) \stackrel{\text{def}}{=} \widetilde{U}_Z(\overline{Z}_a) \vee Z^\#_a,\ \text{for } a \geqslant 0.
\end{aligned}$$

Let $\mathcal{D}_i(a)$, $i \in \{1,\cdots,10\}$ denote the set of discontinuity points in $\mathbb{D}(\mathbb{R},\mathbb{R})$ of $F_a^{(i)}$. For $v \geqslant 1$, define $\mathbb{V}^{(v)} \stackrel{\text{def}}{=} (\mathbb{V}_{vx}/a(v),\ x\in\mathbb{R})$. Following a theorem of Skorohod [Sko57], Assumption 1.1 implies that the family of processes $(\mathbb{V}^{(v)},\ v \geqslant 1)$ converges in law, in the Skorohod space, towards $\mathbb{S}$ as v goes to infinity. It remains to check that the previously defined functionals have nice continuous properties (with respect to $\mathbb{S}$) in order to obtain results such as the convergence in law of $F_a^{(i)}(\mathbb{V}^{(v)})$ towards $F_a^{(i)}(\mathbb{S})$ as v tends to infinity.

For $Z \in \mathbb{D}(\mathbb{R},\mathbb{R})$ and $a \in \mathbb{R}$, we say that

$$\begin{aligned}
&Z \text{ is oscillating at } a^- \text{ if for all } \varepsilon > 0,\quad \inf_{(a-\varepsilon,a)} Z < Z_{a-} < \sup_{(a-\varepsilon,a)} Z,\\
&Z \text{ is oscillating at } a^+ \text{ if for all } \varepsilon > 0,\quad \inf_{(a,a+\varepsilon)} Z < Z_{a+} < \sup_{(a,a+\varepsilon)} Z.
\end{aligned}$$

The following lemma collects some easy results about the sample path of $\mathbb{S}$.

Lemma 2.1

(1) $\sup_{[0,\infty)} \mathbb{S} = \sup_{(-\infty,0]} \mathbb{S} = \infty$ *almost surely.*

(2) *With probability* 1*, any path of* $\mathbb{S}$ *is such that, if* $\mathbb{S}$ *is discontinuous at a point* x*, then* $\mathbb{S}$ *is oscillating at* x^- *and* x^+.

(3) *For any fixed* $a \in \mathbb{R}$*, the process* $\mathbb{S}$ *is almost surely continuous at* a *and oscillating at* a^- *and* a^+*.*

Proof. Assertions (1) and (2) come from Lemma 3.1 of [KTT92], p531. As for (3), it is well known that $\mathbb{S}$ is almost surely continuous at any given point and the fact that it is oscillating follows from the assumption that $|\mathbb{S}|$ is not a subordinator. ■

Note that (2) of the lemma implies that, almost surely, $\mathbb{S}$ is continuous at all its local extrema. It also implies that, with probability 1, $\mathbb{S}$ attains its bound on any compact interval. These facts enable us to prove the following:

Proposition 2.2

For any $a \in \mathbb{R}$ *and* $i \in \{1, \cdots, 10\}$*, we have*

$$\mathbf{P}\{\mathbb{S} \in \mathcal{D}_i(a)\} = 0.$$

Proof. Let a be fixed. The functionals $F_a^{(i)}, i \in \{1, 2, 3, 4, 5\}$ are continuous at all $Z \in \mathbb{D}(\mathbb{R}, \mathbb{R})$ such that Z is continuous at point a (refer to Proposition 2.11 on p305 of [JS87] for further details) and the result follows from (3) of the previous lemma. It is also easily checked from the definition of the Skorohod topology that the functionals $F_a^{(i)}, i \in \{6, 8\}$ are continuous at all Z having the following properties:

(a) $\sigma_Z(a) < \infty$,

(b) Z is oscillating at $\sigma_Z(a)+$,

(c) Z attains its bounds on any compact interval.

Using again the previous lemma, we see that (a) and (c) hold for almost any path of $\mathbb{S}$. From the Markov property of the stable process, assertion (3) of the lemma is unchanged when a is replaced by an arbitrary stopping time. Hence, (b) is also true for almost any path of $\mathbb{S}$. The proofs for the functionnals $F_a^{(i)}, i \in \{7, 9\}$ are, of course, similar. Finally, the result for $F_a^{(10)}$ is easily deduced from previous ones by using the independence of $(\mathbb{S}_x,\ x \geqslant 0)$ and $(\mathbb{S}_{-x},\ x \geqslant 0)$. ■

We will also use the fact that the random variables $F_a^{(i)}$ have continuous cumulative functions (except for the degenerated case $a = 0$).

Proposition 2.3

For all $a \neq 0$*,* $b \in \mathbb{R}$ *and* $i \in \{1, \cdots, 10\}$*, we have*

$$\mathbf{P}\{F_a^{(i)}(\mathbb{S}) = b\} = 0.$$

We shall skip the proof of this proposition since it is an easy consequence of the fact that $\mathbb{S}$ has a continuous density and the assumption that it is not a subordinator.

Finally, throughout the rest of this paper, the notation c_i will always denote a finite strictly positive constant depending only on our choice of $\mathbf{P}$. In the case of a constant depending on some other parameters, these parameters will appear in the subscript. We will also repeatedly use the following lemma easily deduced from the uniform convergence theorem for regularly varying functions (*c.f.* [BGT89], p22) combined with the monotonicity property.

Lemma 2.4

Let $f : [1,\infty) \mapsto \mathbb{R}_+$ be a strictly positive non-decreasing function which is regularly varying at infinity with index $\beta \geqslant 0$. Then, for any $\varepsilon > 0$ there exist $c_{1,\varepsilon,f}, c_{2,\varepsilon,f} > 0$ such that for any $1 \leqslant x \leqslant y$,

$$c_{1,\varepsilon,f}\left(\frac{x}{y}\right)^{\beta+\varepsilon} \leqslant \frac{f(x)}{f(y)} \leqslant c_{2,\varepsilon,f}\left(\frac{x}{y}\right)^{\beta-\varepsilon}.$$

2.2 Supremum of the reflected process

We now give some bounds and asymptotics concerning $\mathbb{V}^{\#}$. These estimates which may seem quite technical play a central role in the proof of Theorem 1.2. This subsection is devoted to proving the following three propositions.

Proposition 2.5

We have

$$\lim_{\substack{x\to\infty \\ v/a^{-1}(x)\to\infty}} \frac{a^{-1}(x)}{v}\log \mathbf{P}\{\mathbb{V}^{\#}_v \leqslant x\} = -K^{\#}$$

where $K^{\#} = -\lim_{v\to\infty}\frac{1}{v}\log\mathbf{P}\{\mathbb{S}^{\#}_v \leqslant 1\}$ is strictly positive and finite.

Proposition 2.6

For all $0 < b < 1$, there exists a constant $c_{3,b} > 0$ such that for all x large enough (depending on b) and all $v > 0$,

$$c_{3,b}\mathbf{P}\{\mathbb{V}^{\#}_v \leqslant x\} \leqslant \mathbf{P}\{\mathbb{V}^{\#}_v \leqslant x, \overline{\mathbb{V}}_v \leqslant bx\} \leqslant \mathbf{P}\{\mathbb{V}^{\#}_v \leqslant x\}.$$

Proposition 2.7

There exists $c_4 > 0$ such that for all x large enough and all $v_1, v_2 > 0$,

$$c_4\mathbf{P}\{\mathbb{V}^{\#}_{v_1} \leqslant x\}\mathbf{P}\{\mathbb{V}^{\#}_{v_2} \leqslant x\} \leqslant \mathbf{P}\{\mathbb{V}^{\#}_{v_1+v_2} \leqslant x\}.$$

Notice that, in view of Proposition 2.6, we deduce that Proposition 2.5 remains unchanged if we replace $\mathbf{P}\{\mathbb{V}_v^\# \leqslant x\}$ by $\mathbf{P}\{\mathbb{V}_v^\# \leqslant x, \overline{\mathbb{V}}_v \leqslant bx\}$ for any $b > 0$. The proof of the first proposition relies on the following lemma.

Lemma 2.8

There exists a constant $K^\# \in (0, \infty)$ such that, for any $a, c > 0$ and any $b \geqslant 0$,

$$\lim_{t\to\infty} \frac{a^\alpha}{t} \log \mathbf{P}\left\{\mathbb{S}_t^\# \leqslant a,\, \underline{\mathbb{S}}_t \leqslant -b,\, \mathbb{S}_t - \underline{\mathbb{S}}_t \leqslant c\right\} = -K^\#.$$

In particular $K^\# = -\lim_{v\to\infty} \frac{1}{v} \log(\mathbf{P}\{\mathbb{S}_v^\# \leqslant 1\})$.

Proof. Using the scaling property of the stable process, we can assume without loss of generality that $a = 1$. For the sake of clarity, let

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \left\{\mathbb{S}_t^\# \leqslant 1,\, \underline{\mathbb{S}}_t \leqslant -b,\, \mathbb{S}_t - \underline{\mathbb{S}}_t \leqslant c\right\},$$

and define $f(t) \stackrel{\text{def}}{=} \log \mathbf{P}\{\mathbb{S}_t^\# \leqslant 1\}$. Using the Markov property of the stable process $\mathbb{S}$, we see that $f(t+s) \leqslant f(t) + f(s)$ for any $s, t \geqslant 0$. Since f is subadditive, elementary analysis shows that the limit $K^\# = -\lim_{t\to\infty} f(t)/t$ exists and furthermore $K^\# \in (0, \infty]$. In order to prove that $K^\# < \infty$, notice that $\{\mathbb{S}_t^\# \leqslant 1\} \supset \{\mathbb{S}_t^* \leqslant 1/2\}$ which implies $f(t)/t \geqslant \log \mathbf{P}\{\mathbb{S}_t^* \leqslant 1/2\}/t$. Using Proposition 3 of [Ber96a], p220, the r.h.s. of this last inequality converges to some finite constant when t converges to infinity. Therefore $K^\#$ must be finite and we have shown that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{P}\{\mathcal{E}_1\} \leqslant \lim_{t\to\infty} \frac{1}{t} f(t) \leqslant -K^\#.$$

It remains to prove the lower bound. Let $0 < \varepsilon < \min(c, 1)$ and let $t > 1$. Define

$$\begin{aligned} \mathcal{E}_2 &\stackrel{\text{def}}{=} \left\{\mathbb{S}_{t-1}^\# \leqslant 1 - \varepsilon\right\}, \\ \mathcal{E}_3 &\stackrel{\text{def}}{=} \left\{(\theta_{t-1}\mathbb{S})_1^\# \leqslant \varepsilon,\, \underline{(\theta_{t-1}\mathbb{S})}_1 \leqslant -b-1\right\}. \end{aligned}$$

We have $\mathcal{E}_1 \supset \mathcal{E}_2 \cap \mathcal{E}_3$. Since $\mathbb{S}$ has independent increments, $\mathcal{E}_2$ and $\mathcal{E}_3$ are independent. Therefore $\mathbf{P}\{\mathcal{E}_1\} \geqslant \mathbf{P}\{\mathcal{E}_2\}\mathbf{P}\{\mathcal{E}_3\}$. Furthermore, $\mathbf{P}\{\mathcal{E}_2\} = f\left((t-1)/(1-\varepsilon)^\alpha\right)$. Hence

$$\frac{1}{t} \log \mathbf{P}\{\mathcal{E}_1\} \geqslant \frac{\log \mathbf{P}\{\mathcal{E}_3\}}{t} + \frac{1}{t} f\left(\frac{t}{(1-\varepsilon)^\alpha}\right), \tag{2.2}$$

and $\mathbf{P}\{\mathcal{E}_3\} = \mathbf{P}\{\mathbb{S}_1^\# \leqslant \varepsilon,\, \underline{\mathbb{S}}_1 \leqslant -b-1\}$ does not depend on t and is not zero (this is easy to check from the assumption that $\mathbb{S}$ is not a subordinator). Letting t go to infinity in (2.2), we conclude that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbf{P}\{\mathcal{E}_1\} \geqslant \lim_{t\to\infty} \frac{1}{t} f\left(\frac{t}{(1-\varepsilon)^\alpha}\right) = \frac{-K^\#}{(1-\varepsilon)^\alpha}.$$

■

Proof of Proposition 2.5. Pick $\varepsilon > 0$, the previous lemma and the scaling property of $\mathbb{S}^\#$ give

$$K^\# = -\lim_{y\to\infty} \frac{1}{y^\alpha} \log \mathbf{P}\Big\{\mathbb{S}_1^\# < \frac{1}{y}\Big\}.$$

Hence, we can choose $y_0 > 0$ such that $\log \mathbf{P}\{\mathbb{S}_1^\# \leqslant 1/y_0\} \leqslant -(K^\# - \varepsilon)y_0^\alpha$. Combining results of Proposition 2.2 and 2.3 for the functional $F^{(3)}$, we get

$$\lim_{k\to\infty} \log \mathbf{P}\Big\{\frac{1}{a(k)}\mathbb{V}_k^\# \leqslant \frac{1}{y_0}\Big\} = \log \mathbf{P}\Big\{\mathbb{S}_1^\# \leqslant \frac{1}{y_0}\Big\} \leqslant -(K^\# - \varepsilon)y_0^\alpha.$$

Therefore, for all k large enough,

$$\log \mathbf{P}\Big\{\frac{1}{a(k)}\mathbb{V}_k^\# \leqslant \frac{1}{y_0}\Big\} \leqslant -(K^\# - 2\varepsilon)y_0^\alpha. \tag{2.3}$$

We use the notation $\lfloor\cdot\rfloor$ to denote the integer part of a number. Let us choose $k = \lfloor a^{-1}(xy_0)\rfloor + 1$, thus (2.3) holds whenever x is large enough. Notice that

$$\Big\{\mathbb{V}_v^\# \leqslant x\Big\} \subset \bigcap_{n=0}^{\lfloor \frac{v}{k}\rfloor - 1} \Big\{(\theta_{nk}\mathbb{V})_k^\# \leqslant x\Big\},$$

hence, from the independence and stationarity of the increments of the random walk at integer times, we get

$$\mathbf{P}\{\mathbb{V}_v^\# \leqslant x\} \leqslant \Big(\mathbf{P}\{\mathbb{V}_k^\# \leqslant x\}\Big)^{\lfloor \frac{v}{k}\rfloor}. \tag{2.4}$$

Since $a(\cdot)$ is non-decreasing, our choice of k implies $x/a(k) \leqslant 1/y_0$, therefore

$$\mathbf{P}\{\mathbb{V}_k^\# \leqslant x\} \leqslant \mathbf{P}\Big\{\frac{\mathbb{V}_k^\#}{a(k)} \leqslant \frac{1}{y_0}\Big\}.$$

The combination of this inequality with (2.3) and (2.4) yields

$$\log \mathbf{P}\{\mathbb{V}_v^\# \leqslant x\} \leqslant -\Big\lfloor \frac{v}{k}\Big\rfloor y_0^\alpha (K^\# - 2\varepsilon).$$

It is easy to check from the regular variation of $a^{-1}(\cdot)$ with index α that $\lfloor v/k\rfloor y_0^\alpha \sim v/a^{-1}(x)$ when x and $v/a^{-1}(x)$ both go to infinity, therefore

$$\limsup \frac{a^{-1}(x)}{v} \log \mathbf{P}\{\mathbb{V}_v^\# \leqslant x\} \leqslant -K^\#.$$

The proof of the lower bound is quite similar yet slightly more technical. Using Lemma 2.8 and the scaling property, we can find $y_0 > 0$ such that

$$\log \mathbf{P}\Big\{\mathbb{S}_1^\# \leqslant \frac{1-\varepsilon}{y_0}, \underline{\mathbb{S}}_1 \leqslant -\frac{2\varepsilon}{y_0}, \mathbb{S}_1 - \underline{\mathbb{S}}_1 \leqslant \frac{\varepsilon}{y_0}\Big\} \geqslant -\frac{K^\# y_0^\alpha}{(1-2\varepsilon)^\alpha}. \tag{2.5}$$

Let us set

$$\mathcal{E}_4(k) \stackrel{\text{def}}{=} \left\{\frac{\mathbb{V}_k^\#}{a(k)} \leqslant \frac{1-\varepsilon}{y_0}, \frac{\underline{\mathbb{V}}_k}{a(k)} \leqslant -\frac{2\varepsilon}{y_0}, \frac{\mathbb{V}_k - \underline{\mathbb{V}}_k}{a(k)} \leqslant \frac{\varepsilon}{y_0}\right\}.$$

Using Proposition 2.2 and 2.3, we check that

$$\lim_{k\to\infty} \mathbf{P}\{\mathcal{E}_4(k)\} = \mathbf{P}\left\{\mathbb{S}_1^{\#} \leqslant \frac{1-\varepsilon}{y_0},\, \underline{\mathbb{S}}_1 \leqslant -\frac{2\varepsilon}{y_0},\, \mathbb{S}_1 - \underline{\mathbb{S}}_1 \leqslant \frac{\varepsilon}{y_0}\right\}.$$

Hence, it follows from (2.5) that for all k large enough,

$$\log \mathbf{P}\{\mathcal{E}_4(k)\} \geqslant \frac{-K^{\#} y_0^{\alpha}}{(1-3\varepsilon)^{\alpha}}. \tag{2.6}$$

We now choose $k = \lfloor a^{-1}(xy_0)\rfloor$. Notice that $1/y_0 \leqslant x/a(k) \leqslant 2/y_0$ for all x large enough, thus

$$\mathcal{E}_4(k) \subset \left\{\mathbb{V}_k^{\#} \leqslant (1-\varepsilon)x,\, \underline{\mathbb{V}}_k \leqslant -\varepsilon x,\, \mathbb{V}_k - \underline{\mathbb{V}}_k \leqslant \varepsilon x\right\}.$$

One may check by induction that

$$\left\{\mathbb{V}_v^{\#} \leqslant x\right\} \supset \bigcap_{n=0}^{\lfloor \frac{v}{k}\rfloor} \left\{(\theta_{nk}\mathbb{V})_k^{\#} \leqslant (1-\varepsilon)x,\, \underline{(\theta_{nk}\mathbb{V})}_k \leqslant -\varepsilon x,\, (\theta_{nk}\mathbb{V})_k - \underline{(\theta_{nk}\mathbb{V})}_k \leqslant \varepsilon x\right\},$$

thus, using the independence and stationarity of the increments of $\mathbb{V}$ at integer times,

$$\begin{aligned}\mathbf{P}\{\mathbb{V}_v^{\#} \leqslant x\} &\geqslant \mathbf{P}\left\{\mathbb{V}_k^{\#} \leqslant (1-\varepsilon)x,\, \underline{\mathbb{V}}_k \leqslant -\varepsilon x,\, \mathbb{V}_k - \underline{\mathbb{V}}_k \leqslant \varepsilon x\right\}^{\lfloor \frac{v}{k}\rfloor+1}\\ &\geqslant \mathbf{P}\{\mathcal{E}_4(k)\}^{\lfloor \frac{v}{k}\rfloor+1}.\end{aligned}$$

This inequality combined with (2.6) shows that for all x large enough,

$$\log \mathbf{P}\{\mathbb{V}_v^{\#} \leqslant x\} \geqslant \frac{-K^{\#}}{(1-3\varepsilon)^{\alpha}} \left(\left\lfloor \frac{v}{k}\right\rfloor + 1\right) y_0^{\alpha}.$$

We finally note that $(\lfloor v/k\rfloor + 1)y_0^{\alpha} \sim v/a^{-1}(x)$ as x and $v/a^{-1}(x)$ go to infinity simultaneously, so the proof in complete. ■

Proof of Proposition 2.6. The upper bound is trivial. Let $0 < b < 1$ and define $r = \lfloor a^{-1}(x)\rfloor$ and set $c = (b-1)x$. We have

$$\begin{aligned}\left\{\mathbb{V}_v^{\#} \leqslant x,\, \overline{\mathbb{V}}_v \leqslant bx\right\} &\supset \left\{\mathbb{V}_v^{\#} \leqslant x,\, \overline{\mathbb{V}}_v \leqslant bx,\, \sigma_{\mathbb{V}}(c) \leqslant r\right\}\\ &\supset \left\{\mathbb{V}_{\sigma_{\mathbb{V}}(c)}^{\#} \leqslant bx,\, \sigma_{\mathbb{V}}(c) \leqslant r\right\} \cap \left\{\left(\theta_{\sigma_{\mathbb{V}}(c)}\mathbb{V}\right)_v^{\#} \leqslant x\right\},\end{aligned}$$

thus

$$\begin{aligned}\mathbf{P}\left\{\mathbb{V}_v^{\#} \leqslant x,\, \overline{\mathbb{V}}_v \leqslant bx\right\} &\geqslant \mathbf{P}\left\{\mathbb{V}_{\sigma_{\mathbb{V}}(c)}^{\#} \leqslant bx,\, \sigma_{\mathbb{V}}(c) \leqslant r\right\} \mathbf{P}\left\{\mathbb{V}_v^{\#} \leqslant x\right\}\\ &\geqslant \mathbf{P}\left\{\mathbb{V}_r^{\#} \leqslant bx,\, \underline{\mathbb{V}}_r \leqslant c\right\} \mathbf{P}\left\{\mathbb{V}_v^{\#} \leqslant x\right\}.\end{aligned}$$

Just like in the previous proof, $\mathbf{P}\{\mathbb{V}_r^{\#} \leqslant bx,\, \underline{\mathbb{V}}_r \leqslant c\}$ converges, as x tends to infinity, towards $\mathbf{P}\{\mathbb{S}_1^{\#} \leqslant b,\, \underline{\mathbb{S}}_1 \leqslant b-1\}$, which quantity is strictly positive since $|\mathbb{S}|$ is not a subordinator.

■

Proof of Proposition 2.7. Notice that

$$\begin{aligned}\left\{\mathbb{V}^{\#}_{v_1+v_2} \leqslant x\right\} &\supset \left\{\mathbb{V}^{\#}_{\lfloor v_1\rfloor+\lfloor v_2\rfloor+2} \leqslant x\right\} \\ &\supset \left\{\mathbb{V}_1 \leqslant 0,\, \mathbb{V}_2 - \mathbb{V}_1 \leqslant 0\right\} \cap \left\{(\theta_2\mathbb{V})^{\#}_{\lfloor v_1\rfloor} \leqslant x,\, (\theta_2\mathbb{V})_{\lfloor v_1\rfloor} - \underline{(\theta_2\mathbb{V})}_{\lfloor v_1\rfloor} \leqslant \frac{x}{2}\right\} \\ &\qquad \cap \left\{(\theta_{2+\lfloor v_1\rfloor}\mathbb{V})^{\#}_{\lfloor v_2\rfloor} \leqslant x,\, \overline{(\theta_{2+\lfloor v_1\rfloor}\mathbb{V})}_{\lfloor v_2\rfloor} \leqslant \frac{x}{2}\right\}.\end{aligned}$$

Using the independence and stationarity of the increments of $\mathbb{V}$ at integer times, and setting $c_5 = \mathbf{P}\{\mathbb{V}_1 \leqslant 0\} > 0$, we deduce that $\mathbf{P}\{\mathbb{V}^{\#}_{v_1+v_2} \leqslant x\}$ is larger than

$$c_5^2 \mathbf{P}\left\{\mathbb{V}^{\#}_{\lfloor v_1\rfloor} \leqslant x,\, \mathbb{V}_{\lfloor v_1\rfloor} - \underline{\mathbb{V}}_{\lfloor v_1\rfloor} \leqslant \frac{x}{2}\right\} \mathbf{P}\left\{\mathbb{V}^{\#}_{\lfloor v_2\rfloor} \leqslant x,\, \overline{\mathbb{V}}_{\lfloor v_2\rfloor} \leqslant \frac{x}{2}\right\}.$$

Moreover, time reversal of the random walk $\mathbb{V}$ induces

$$\mathbf{P}\left\{\mathbb{V}^{\#}_{\lfloor v_1\rfloor} \leqslant x,\, \mathbb{V}_{\lfloor v_1\rfloor} - \underline{\mathbb{V}}_{\lfloor v_1\rfloor} \leqslant x/2\right\} = \mathbf{P}\left\{\mathbb{V}^{\#}_{\lfloor v_1\rfloor} \leqslant x,\, \overline{\mathbb{V}}_{\lfloor v_1\rfloor} \leqslant x/2\right\},$$

therefore, using Proposition 2.6, we conclude that

$$\begin{aligned}\mathbf{P}\left\{\mathbb{V}^{\#}_{v_1+v_2} \leqslant x\right\} &\geqslant (c_{3,\frac{1}{2}}c_5)^2 \mathbf{P}\left\{\mathbb{V}^{\#}_{\lfloor v_1\rfloor} \leqslant x\right\} \mathbf{P}\left\{\mathbb{V}^{\#}_{\lfloor v_2\rfloor} \leqslant x\right\} \\ &\geqslant (c_{3,\frac{1}{2}}c_5)^2 \mathbf{P}\left\{\mathbb{V}^{\#}_{v_1} \leqslant x\right\} \mathbf{P}\left\{\mathbb{V}^{\#}_{v_2} \leqslant x\right\}.\end{aligned}$$

■

2.3 The case where $\mathbb{S}$ is completely asymmetric

One certainly wishes to calculate the value of the constant $K^{\#}$ that appears in the last section. Unfortunately, we do not know its value in general. However, the completely asymmetric case is a particularly nice setting where calculations may be carried out to their full extend. We now assume throughout this section that the stable process $(\mathbb{S}_x,\, x \geqslant 0)$ either has no positive jumps hence the exponential moments of $\mathbb{S}$ are finite and (1.2) holds (recall that we assume $\gamma' = 1$) or $\mathbb{S}$ has no negative jumps and $\mathbf{E}[\exp(-\lambda\mathbb{S}_t)] = \exp(t\lambda^\alpha)$ for all $t, \lambda \geqslant 0$. For $a, b > 0$, define the stopping times:

$$\begin{aligned}\tau_b &\stackrel{\text{def}}{=} \inf\{t \geqslant 0,\, \mathbb{S}_t \geqslant b\} = \sigma_{\mathbb{S}}(b), \\ \tau^{\#}_b &\stackrel{\text{def}}{=} \inf\{t \geqslant 0,\, \mathbb{S}^{\#}_t \geqslant b\} = \sigma_{\mathbb{S}^{\#}}(b), \\ \tau^*_{a,b} &\stackrel{\text{def}}{=} \inf\{t \geqslant 0,\, \mathbb{S}_t \text{ not in } (-a, b)\}.\end{aligned}$$

Recall that $\mathcal{M}_\alpha$ stands for the Mittag-Leffler function with parameter α.

Proposition 2.9

When $\mathbb{S}$ has no positive jumps, we have

$$\mathbf{E}\left[e^{-q\tau^{\#}_1}\right] = \frac{1}{\mathcal{M}_\alpha(q)},$$

and when $\mathbb{S}$ has no negative jumps, we have

$$\mathbf{E}\Big[e^{-q\tau_1^{\#}}\Big] = \mathcal{M}_\alpha(q) - \frac{\alpha q(\mathcal{M}'_\alpha(q))^2}{\alpha q\mathcal{M}''_\alpha(q) + (\alpha-1)\mathcal{M}'_\alpha(q)}.$$

This proposition is a particular case of Proposition 2 of [Pis04], p191. Still, we give here a simpler proof when $\mathbb{S}$ is stable using the solution of the two-sided exit problem given by Bertoin [Ber96b].

Proof. We suppose that $\mathbb{S}$ has no negative jumps. Let $\eta(q)$ be an exponential random time of parameter q independent of $\mathbb{S}$. Let also a, b be strictly positive real numbers such that $a + b = 1$. We may without loss of generality assume any path of $\mathbb{S}$ attains its bounds on any compact interval and is continuous at all local extrema (because this happens with probability 1 according to Lemma 2.1). Thus, on the one hand, the event $\{\tau_1^{\#} > \eta(q)\}$ contains

$$\Big\{\tau_{a,b}^* > \eta(q)\Big\} \cup \Big(\Big\{\tau_{a,b}^* \leqslant \eta(q), \mathbb{S}_{\tau_{a,b}^*} \leqslant -a\Big\} \cap \Big\{(\theta_{\tau_{a,b}^*}\mathbb{S})^{\#}_{\eta(q)-\tau_{a,b}^*} < 1\Big\}\Big).$$

Using the strong Markov property of $\mathbb{S}$, the lack of memory, and the independence of the exponential time, it follows that $\mathbf{P}\{\tau_1^{\#} > \eta(q)\}$ is larger than

$$\mathbf{P}\{\tau_{a,b}^* > \eta(q)\} + \mathbf{P}\{\tau_{a,b}^* \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau_{a,b}^*} \leqslant -a\}\mathbf{P}\{\tau_1^{\#} > \eta(q)\},$$

therefore

$$\mathbf{P}\{\tau_1^{\#} > \eta(q)\} \geqslant \frac{\mathbf{P}\{\tau_{a,b}^* > \eta(q)\}}{1 - \mathbf{P}\{\tau_{a,b}^* \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau_{a,b}^*} \leqslant -a\}}. \tag{2.7}$$

On the other hand, one may check that the event $\{\tau_1^{\#} > \eta(q)\}$ is a subset of

$$\Big\{\tau_{a,b}^* > \eta(q)\Big\} \cup \Big(\Big\{\tau_{a,b}^* \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau_{a,b}^*} \leqslant -a\Big\} \cap \Big\{(\theta_{\tau_{a,b}^*}\mathbb{S})^{\#}_{\eta(q)-\tau_{a,b}^*} < b\Big\}\Big),$$

and similarly we deduce

$$\mathbf{P}\{\tau_b^{\#} > \eta(q)\} \leqslant \frac{\mathbf{P}\{\tau_{a,b}^* > \eta(q)\}}{1 - \mathbf{P}\{\tau_{a,b}^* \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau_{a,b}^*} \leqslant -a\}}. \tag{2.8}$$

Obviously $\tau_b^{\#}$ converges to $\tau_1^{\#}$ almost surely as b converges to 1. Combining this observation with (2.7) and (2.8), we find

$$\mathbf{P}\{\tau_1^{\#} > \eta(q)\} = \lim_{b\nearrow 1} \frac{\mathbf{P}\{\tau_{1-b,b}^* > \eta(q)\}}{1 - \mathbf{P}\{\tau_{1-b,b}^* \leqslant \eta(q), \mathbb{S}_{\tau_{1-b,b}^*} \leqslant b-1\}}. \tag{2.9}$$

The probabilities of the r.h.s. of this equation have been calculated by Bertoin [Ber96b]:

$$\mathbf{P}\{\tau_{1-b,b}^* > \eta(q)\} = 1 - \mathcal{M}_\alpha(b^\alpha) + \tfrac{b^{\alpha-1}\mathcal{M}'_\alpha(qb^\alpha)}{\mathcal{M}'_\alpha(q)}\left(\mathcal{M}_\alpha(q) - 1\right), \tag{2.10}$$

$$\mathbf{P}\{\tau_{1-b,b}^* \leqslant \eta(q)\ ,\ \mathbb{S}_{\tau_{1-b,b}^*} \leqslant b-1\} = \tfrac{b^{\alpha-1}\mathcal{M}'_\alpha(qb^\alpha)}{\mathcal{M}'_\alpha(q)}. \tag{2.11}$$

Taylor expansions of $\mathcal{M}_\alpha$ and $\mathcal{M}'_\alpha$ near point q enable us to calculate the limit in (2.9) in term of $\mathcal{M}_\alpha$ and its first and second derivatives. After a few lines of elementary calculus, we get

$$\mathbf{P}\{\tau_1^\# > \eta(q)\} = 1 - \mathcal{M}_\alpha(q) + \frac{\alpha q(\mathcal{M}'_\alpha(q))^2}{\alpha q\mathcal{M}''_\alpha(q) + (\alpha-1)\mathcal{M}'_\alpha(q)}.$$

We complete the proof using the classical relation $\mathbf{E}[\exp(-q\tau_1^\#)] = 1 - \mathbf{P}\{\tau_1^\# > \eta(q)\}$. The proof when $\mathbb{S}$ has no positive jumps is similar (and the calculation of the limit is even easier). We skip the details. ■

Corolary 2.10

Recall that $-\rho_1(\alpha)$ denotes the first negative root of $\mathcal{M}_\alpha$ and $-\rho_2(\alpha)$ denotes the first negative root of $\alpha x\mathcal{M}''_\alpha(x)+(\alpha-1)\mathcal{M}'_\alpha(x)$. The constant $K^\#$ of Proposition 2.5 is given by

$$K^\# = \begin{cases} \rho_1(\alpha) & \text{when } \mathbb{S} \text{ has no positive jumps,} \\ \rho_2(\alpha) & \text{when } \mathbb{S} \text{ has no negative jumps.} \end{cases}$$

Proof. Recall that $K^\# = -\lim_{t\to\infty} \mathbf{P}\{\mathbb{S}_t^\# \leqslant 1\}/t$. Using the previous proposition and the same argument as in Corollary 1 of [Ber96b], we see that, when $\mathbb{S}$ has no positive jumps, $-K^\#$ is equal to the first negative pole of the analytic function $x \to 1/\mathcal{M}_\alpha(x)$. Therefore, $K^\# = \rho_1(\alpha)$. Similarly, when $\mathbb{S}$ has no negative jumps $-K^\#$ is equal to the first negative pole of

$$g(x) \stackrel{\text{def}}{=} \frac{\alpha x(\mathcal{M}'_\alpha(x))^2}{\alpha x\mathcal{M}''_\alpha(x) + (\alpha-1)\mathcal{M}'_\alpha(x)} = \mathcal{M}_\alpha(x) - \mathbf{E}\left[e^{-x\tau_1^\#}\right].$$

Let $-x_0$ be the first negative root of $\mathcal{M}'_\alpha$. Since $\mathcal{M}'_\alpha(0) > 0$, the function $\mathcal{M}_\alpha$ is strictly increasing on $[-x_0, 0]$. Notice also that $x \mapsto -\mathbf{E}[\exp(-x\tau_1^\#)]$ is increasing on $(-K^\#, 0]$, thus $g(x)$ is strictly increasing on $(-(K^\# \wedge x_0), 0]$. Since $g(-x_0) = g(0) = 0$ (this holds even when $-x_0$ is a zero of multiple order) we deduce from the monotonicity of g that $K^\# < x_0$ and this shows that the first negative pole of g is indeed $-\rho_2(\alpha)$. ■

We conclude this subsection by calculating the Laplace transform of $\tau_1^\# \wedge \tau_b$. This will be useful for the determination of the limiting law in the proof of Theorem 1.5.

Corolary 2.11

For any $0 < b \leqslant 1$, when $\mathbb{S}$ has no positive jumps, we have

$$\mathbf{E}\left[e^{-q\tau_1^\# \wedge \tau_b}\right] = \frac{\mathcal{M}_\alpha(q(1-b)^\alpha)}{\mathcal{M}_\alpha(q)},$$

and when $\mathbb{S}$ has no negative jumps,

$$\mathbf{E}\left[e^{-q\tau_1^\# \wedge \tau_b}\right] = \mathcal{M}_\alpha(qb^\alpha) - b^{\alpha-1}\frac{\alpha q\mathcal{M}'_\alpha(qb^\alpha)\mathcal{M}'_\alpha(q)}{\alpha q\mathcal{M}''_\alpha(q) + (\alpha-1)\mathcal{M}'_\alpha(q)}.$$

Proof. Let $\eta(q)$ still denote an exponential time with parameter q independent of $\mathbb{S}$. Suppose that $\mathbb{S}$ has no negative jumps, using the Markov property and the lack of memory of the exponential law, we get

$$\mathbf{P}\{\tau_1^\# \wedge \tau_b > \eta(q)\} = \mathbf{P}\{\tau^*_{1-b,b} \leqslant \eta(q), \mathbb{S}_{\tau^*_{1-b,b}} \leqslant b-1\}\mathbf{P}\{\tau_1^\# > \eta(q)\} + \mathbf{P}\{\tau^*_{1-b,b} > \eta(q)\}.$$

The r.h.s. of the last equality may be calculated explicitly using again (2.10), (2.11), and Proposition 2.9. After simplification, we obtain

$$\mathbf{P}\{\tau_1^\# \wedge \tau_b > \eta(q)\} = 1 - \mathcal{M}_\alpha(qb^\alpha) + b^{\alpha-1}\frac{\alpha q \mathcal{M}'_\alpha(qb^\alpha)\mathcal{M}'_\alpha(q)}{\alpha q \mathcal{M}''_\alpha(q) + (\alpha-1)\mathcal{M}'_\alpha(q)}.$$

The no positive jumps case may be treated the same way. ■

2.4 The exit problem for the random walk $\mathbb{V}$

Let us define for $x, y > 0$ the following events:

$$\begin{aligned}
\Lambda(x,y) &\stackrel{\text{def}}{=} \left\{(\mathbb{V}_s)_{s \geqslant 0} \text{ hits } (y,\infty) \text{ before it hits } (-\infty,-x)\right\},\\
\Lambda'(x,y) &\stackrel{\text{def}}{=} \left\{(\mathbb{V}_s)_{s \geqslant 0} \text{ hits } [y,\infty) \text{ before it hits } (-\infty,-x]\right\},\\
\widetilde{\Lambda}'(x,y) &\stackrel{\text{def}}{=} \left\{(\mathbb{V}_{-s})_{s \geqslant 0} \text{ hits } (-\infty,-y] \text{ before it hits } [x,\infty)\right\}.
\end{aligned}$$

We are interested in the behavior of the probabilities of these events for large x, y. In the case of a fixed x, when y goes to infinity, this study was done by Bertoin and Doney [BD94]. Here, we need to study this quantities when both x and y go to infinity with the ratio y/x also going to infinity. Recall the definition of the sequence $(\mathbf{T}_n)_{n \geqslant 0}$ of strictly descending ladder times defined in the introduction. We now consider the associated ladder heights:

$$\mathbf{H}_n \stackrel{\text{def}}{=} -\mathbb{V}_{\mathbf{T}_n} \quad \text{for } n \geqslant 0.$$

We also introduce the sequence $(\mathbf{M}_n)_{n \geqslant 1}$ defined by

$$\mathbf{M}_n \stackrel{\text{def}}{=} \max\left\{\mathbb{V}_k + \mathbf{H}_{n-1},\ \mathbf{T}_{n-1} \leqslant k < \mathbf{T}_n\right\}.$$

Note that the random variables $(\mathbf{T}_{n+1} - \mathbf{T}_n, \mathbf{H}_{n+1} - \mathbf{H}_n, \mathbf{M}_n)_{n \geqslant 1}$ are i.i.d. We have already said that $\mathbf{T}_1$ is in the domain of attraction of a positive stable law of index q with norming constants (b_n). Moreover, Corollary 3 of [Don85] states that $\mathbf{P}\{\mathbf{M}_1 > x\}$ is regularly varying with index $-\alpha q$. More precisely, we have

$$\mathbf{P}\{\mathbf{M}_1 > x\} \underset{x\to\infty}{\sim} \frac{c_6}{b^{-1}\left(a^{-1}(x)\right)}. \tag{2.12}$$

In particular, $\mathbf{M}_1$ is in the domain of attraction of a positive stable law when $\alpha q < 1$ and $\mathbf{M}_1$ is relatively stable when $\alpha q = 1$ (relatively stable meaning that $\frac{1}{a(b(n))}\sum_{k \leqslant n}\mathbf{M}_k$ converges in probability to some strictly positive constant).

Concerning $\mathbf{H}_1$, Theorem 9 of [Rog71] states that $\mathbf{H}_1$ is in the domain of attraction of a positive stable law with index αq when $\alpha q < 1$ and that $\mathbf{H}_1$ is relatively stable when $\alpha q = 1$. Furthermore, the lemma on p358 of [Don85] states that we can even choose $a(b(n))$ as norming constant for $\mathbf{H}_1$, *i.e.*

$$\frac{\mathbf{H}_n}{a(b(n))} \text{ converges } \begin{cases} \text{in probability to some positive constant } c_7 \text{ when } \alpha q = 1, \\ \text{in law to a positive stable law of index } \alpha q \text{ otherwise.} \end{cases}$$

When $\alpha q < 1$, this implies that (2.12) holds with $\mathbf{H}_1$ in place of $\mathbf{M}_1$ (for a different value of c_6) but when $\alpha q = 1$, the relative stability of $\mathbf{H}_1$ does not imply the regular variation of $\mathbf{P}\{\mathbf{H}_1 > x\}$ (look at the counter example in [Rog71], p 576). Yet, we can still prove a smooth behavior for the associated renewal function

$$\mathbf{R}(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \mathbf{P}\{\mathbf{H}_n \leqslant x\}.$$

Lemma 2.12

There exists a constant $c_8 > 0$ such that

$$\mathbf{R}(x) \underset{x\to\infty}{\sim} c_8 b^{-1}\left(a^{-1}(x)\right).$$

Proof. When $\alpha q < 1$ we have already mentioned that $\mathbf{P}\{\mathbf{H}_1 > x\} \sim c_9/b^{-1}\left(a^{-1}(x)\right)$ thus the asymptotic behavior of $\mathbf{R}$ follows from a Tauberian Theorem just as in the lemma on p446 of [Fel71]. We now consider the case $\alpha q = 1$. Let $L(\lambda) \stackrel{\text{def}}{=} \mathbf{E}\left[e^{-\lambda \mathbf{H}_1}\right]$ stand for the Laplace transform of $\mathbf{H}_1$. We know that

$$\frac{\mathbf{H}_n}{a(b(n))} \underset{n\to\infty}{\overset{\text{prob.}}{\longrightarrow}} c_7.$$

Therefore, for any $\lambda \geqslant 0$ and when n ranges through the set of integers, we get

$$\left(L\left(\frac{\lambda}{a(b(n))}\right)\right)^n \underset{n\to\infty}{\longrightarrow} e^{-c_7\lambda}. \tag{2.13}$$

Since L is continuous at 0 with $L(0) = 1$, setting $\lambda = 1$ and taking the logarithm in (2.13) yield

$$n\left(1 - L\left(\frac{1}{a(b(n))}\right)\right) \underset{n\to\infty}{\longrightarrow} c_7. \tag{2.14}$$

Using the monotonicity of L and $a(b(\cdot))$, we check that (2.14) still holds when n now ranges through the set of real numbers, thus

$$1 - L\left(\frac{1}{x}\right) \underset{x\to\infty}{\sim} \frac{c_7}{b^{-1}(a^{-1}(x))}. \tag{2.15}$$

Let us set $\widehat{\mathbf{R}}(y) \stackrel{\text{def}}{=} \int_0^\infty e^{-yx}\mathbf{R}(dx)$. The well-known relation $\widehat{\mathbf{R}}(y) = 1/(1 - L(y))$ combined with (2.15) shows that $\widehat{\mathbf{R}}$ is regularly varying near 0. We conclude the proof of the lemma using Karamata's Tauberian/Abelian theorem. ■

Proposition 2.13

There exists c_{10} such that, when $x \to \infty$ and $y/x \to \infty$ simultaneously,

$$\mathbf{P}\{\Lambda(x,y)\} \sim c_{10} \frac{b^{-1}\left(a^{-1}(x)\right)}{b^{-1}\left(a^{-1}(x+y)\right)}.$$

This result also holds for $\mathbf{P}\{\Lambda'(x,y)\}$ and $\mathbf{P}\{\widetilde{\Lambda}'(x,y)\}$.

Proof. Since the two processes $(\mathbb{V}_s)_{s \geqslant 0}$ and $(-\mathbb{V}_{-s})_{s \geqslant 0}$ have the same law, we have $\mathbf{P}\{\Lambda'(x,y)\} = \mathbf{P}\{\widetilde{\Lambda}'(x,y)\}$. Notice also that $\Lambda(x-1,y) \subset \Lambda'(x,y) \subset \Lambda(x,y-1)$. Thus, we just need to prove the proposition for $\Lambda(x,y)$. The first part of the proof is borrowed from Bertoin and Doney [BD94], p2157: the probability $\mathbf{P}\{\Lambda(x,y)\}$ is equal to

$$\mathbf{P}\{\mathbf{M}_1 > y\} + \sum_{k=1}^{\infty} \mathbf{P}\Big\{\mathbf{M}_1 \leqslant y + \mathbf{H}_0, \cdots, \mathbf{M}_k \leqslant y + \mathbf{H}_{k-1}, \mathbf{H}_k \leqslant x, \mathbf{M}_{k+1} > y + \mathbf{H}_k\Big\}, \tag{2.16}$$

thus

$$\begin{aligned} \mathbf{P}\{\Lambda(x,y)\} &\leqslant \mathbf{P}\{\mathbf{M}_1 > y\} + \sum_{k=1}^{\infty} \mathbf{P}\{\mathbf{H}_k \leqslant x, \mathbf{M}_{k+1} > y + \mathbf{H}_k\} \\ &\leqslant \mathbf{P}\{\mathbf{M}_1 > y\} + \sum_{k=1}^{\infty} \mathbf{P}\{\mathbf{H}_k \leqslant x, \mathbf{M}_{k+1} > y\} \\ &\leqslant \mathbf{P}\{\mathbf{M}_1 > y\}\mathbf{R}(x). \end{aligned}$$

Using (2.12), Lemma 2.12, and the equivalence $\mathbf{P}\{\mathbf{M}_1 > y\} \sim \mathbf{P}\{\mathbf{M}_1 > x+y\}$ when x and y/x go to infinity, we obtain the desired upper bound with $c_{10} \stackrel{\text{def}}{=} c_6 c_8$. We now prove the result pertaining to the lower bound. Let $k_0 \in \mathbb{N}^*$. From (2.16), we see that $\mathbf{P}\{\Lambda(x,y)\}$ is bigger than

$$\begin{aligned} &\mathbf{P}\{\mathbf{M}_1 > y\} + \sum_{k=1}^{\infty} \mathbf{P}\{\mathbf{M}_1 \leqslant y, \cdots, \mathbf{M}_k \leqslant y, \mathbf{H}_k \leqslant x, \mathbf{M}_{k+1} > x+y\} \\ &\qquad \geqslant \mathbf{P}\{\mathbf{M}_1 > x+y\}\Big(1 + \sum_{k=1}^{k_0} \mathbf{P}\{\mathbf{M}_1 \leqslant y, \cdots, \mathbf{M}_k \leqslant y, \mathbf{H}_k \leqslant x\}\Big), \end{aligned}$$

hence

$$\mathbf{P}\{\Lambda(x,y)\} \geqslant \mathbf{P}\{\mathbf{M}_1 > x+y\}\Big(\mathbf{R}(x) - \mathbf{R}_{k_0}(x) - \mathbf{W}_{k_0}(y)\Big), \tag{2.17}$$

with

$$\begin{aligned} \mathbf{R}_{k_0}(x) &\stackrel{\text{def}}{=} \sum_{k=k_0+1}^{\infty} \mathbf{P}\{\mathbf{H}_k \leqslant x\}, \\ \mathbf{W}_{k_0}(y) &\stackrel{\text{def}}{=} \sum_{k=1}^{k_0} \mathbf{P}\{\mathbf{M}_1 > y \text{ or } \cdots \text{ or } \mathbf{M}_k > y\}. \end{aligned}$$

On the one hand, in view of (2.12) and Lemma 2.12, we get, for y large enough,

$$\mathbf{W}_{k_0}(y) \leqslant \sum_{k=1}^{K} k_0 \mathbf{P}\{\mathbf{M}_1 > y\} \leqslant k_0^2 \mathbf{P}\{\mathbf{M}_1 > y\} \leqslant \frac{c_{11} k_0^2}{\mathbf{R}(y)}.$$

On the other hand, we have

$$\begin{aligned}\mathbf{R}_{k_0}(x) &= \sum_{k=0}^{\infty} \mathbf{P}\{\mathbf{H}_{k_0+1} + (\mathbf{H}_{k+k_0+1} - \mathbf{H}_{k_0+1} \leqslant x)\} \\ &\leqslant \sum_{k=0}^{\infty} \mathbf{P}\{\mathbf{H}_{k+k_0+1} - \mathbf{H}_{k_0+1} \leqslant x\} \mathbf{P}\{\mathbf{H}_{k_0+1} \leqslant x\} \\ &\leqslant \mathbf{R}(x)\, \mathbf{P}\{\mathbf{H}_{k_0} \leqslant x\}.\end{aligned}$$

Combining these two bounds with (2.17) yields, for all x, y large enough,

$$\mathbf{P}\{\Lambda(x,y)\} \geqslant \mathbf{P}\{\mathbf{M}_1 > x + y\} \mathbf{R}(x) \left(1 - \mathbf{P}\{\mathbf{H}_{k_0} \leqslant x\} - \frac{c_{11} k_0^2}{(\mathbf{R}(y))^2}\right).$$

It only remains to show that for a good choice of $k_0 = k_0(x,y)$, we have

$$\mathbf{P}\{\mathbf{H}_{k_0} \leqslant x\} + \frac{c_{11} k_0^2}{(\mathbf{R}(y))^2} \underset{x, \frac{y}{x} \to \infty}{\longrightarrow} 0. \tag{2.18}$$

Let $k_0 = \lfloor b^{-1}(a^{-1}(x \log(y/x))) \rfloor$. Note that k_0 is such that $k_0 \to \infty$, when x and y/x go to infinity simultaneously, and we know that

$$\frac{\mathbf{H}_{k_0}}{a(b(k_0))} \underset{k_0 \to \infty}{\overset{\text{law}}{\longrightarrow}} J_\infty$$

where J_∞ is either a positive stable law (when $\alpha q < 1$) or a strictly positive constant (when $\alpha q = 1$). In either cases $\mathbf{P}\{J_\infty = 0\} = 0$. Since $x/a(b(k_0)) \to 0$ when x and y/x go to infinity simultaneously, we deduce that

$$\mathbf{P}\{\mathbf{H}_{k_0} \leqslant x\} = \mathbf{P}\left\{\frac{\mathbf{H}_{k_0}}{a(b(k_0))} \leqslant \frac{x}{a(b(k_0))}\right\} \underset{x, \frac{y}{x} \to \infty}{\longrightarrow} 0. \tag{2.19}$$

Finally, using Lemmas 2.4 and 2.12, we verify that

$$\frac{c_{11} k_0^2}{(\mathbf{R}(y))^2} \underset{x, \frac{y}{x} \to \infty}{\sim} \frac{c_{11}}{c_8^2} \left(\frac{\mathbf{R}(x \log \frac{y}{x})}{\mathbf{R}(y)}\right)^2 \underset{x, \frac{y}{x} \to \infty}{\longrightarrow} 0. \tag{2.20}$$

The combination of (2.19) and (2.20) yields (2.18). ∎

2.5 Other estimates

We conclude the section about the fluctuations of $\mathbb{V}$ by collecting several results concerning the functionals $\overline{\mathbb{V}}$ and $\underline{\mathbb{V}}$. We start with a reflection principle for $\mathbb{V}$.

Lemma 2.14

There exists c_{12} such that for all $v, x > 0$,

$$\mathbf{P}\{\overline{\mathbb{V}}_v \geqslant x\} \leqslant c_{12}\mathbf{P}\{\mathbb{V}_v \geqslant x\},$$

and similarly

$$\mathbf{P}\{\underline{\mathbb{V}}_v \leqslant -x\} \leqslant c_{12}\mathbf{P}\{\mathbb{V}_v \leqslant -x\}.$$

Proof. We only need to prove the first inequality (the second inequality can be obtained in the same way, with a possibly enlarged value for c_{12}).

$$\begin{aligned}\mathbf{P}\{\overline{\mathbb{V}}_v \geqslant x\} &= \mathbf{P}\{\sigma_{\mathbb{V}}(x) \leqslant \lfloor v \rfloor\} \\ &\leqslant \mathbf{P}\{\sigma_{\mathbb{V}}(x) \leqslant \lfloor v \rfloor,\, \mathbb{V}_{\lfloor v \rfloor} < x\} + \mathbf{P}\{\mathbb{V}_v \geqslant x\} \\ &\leqslant \sum_{k=1}^{\lfloor v \rfloor} \mathbf{P}\{\sigma_{\mathbb{V}}(x) = k,\, \mathbb{V}_{\lfloor v \rfloor} < x\} + \mathbf{P}\{\mathbb{V}_v \geqslant x\}.\end{aligned}$$

Using the Markov property, we check that $\mathbf{P}\{\sigma_{\mathbb{V}}(x) = k, \mathbb{V}_{\lfloor v \rfloor} < x\}$ is equal to

$$\begin{aligned}\mathbf{P}\{\sigma_{\mathbb{V}}(x) = k\} \int_{y \,\geqslant\, x} \mathbf{P}\{\mathbb{V}_{\lfloor v \rfloor - k} < x - y\}\mathbf{P}\{\mathbb{V}_{\sigma_{\mathbb{V}}(x)} = dy | \sigma_{\mathbb{V}}(x) = k\} \\ \leqslant \mathbf{P}\{\sigma_{\mathbb{V}}(x) = k\}\mathbf{P}\{\mathbb{V}_{\lfloor v \rfloor - k} < 0\}.\end{aligned}$$

Our assumption on $\mathbb{V}$ implies that $\lim_{n\to\infty} \mathbf{P}\{\mathbb{V}_n < 0\} = \mathbf{P}\{\mathbb{S} < 0\} = q < 1$. Thus, there exists $c_{13} > 0$ such that $\sup_n \mathbf{P}\{\mathbb{V}_n < 0\} = c_{13} < 1$. Therefore

$$\begin{aligned}\mathbf{P}\{\overline{\mathbb{V}}_v \geqslant x\} &\leqslant c_{13} \sum_{k=1}^{\lfloor v \rfloor} \mathbf{P}\{\sigma_{\mathbb{V}}(x) = k\} + \mathbf{P}\{\mathbb{V}_v \geqslant x\} \\ &\leqslant c_{13}\mathbf{P}\{\sigma_{\mathbb{V}}(x) \leqslant v\} + \mathbf{P}\{\mathbb{V}_v \geqslant x\} \\ &\leqslant \frac{1}{1 - c_{13}}\mathbf{P}\{\mathbb{V}_v \geqslant x\}.\end{aligned}$$

■

We now estimate the large deviations of $\mathbf{P}\{\mathbb{V}_v > x\}$. The characterization of the domains of attraction (see Chapter IX,8 of [Fel71]) and Assumption 1.1 imply

$$a^{-1}(x)\mathbf{P}\{\mathbb{V}_1 > x\} \underset{x\to\infty}{\longrightarrow} \begin{cases} c_{14} > 0 & \text{if } \mathbb{S} \text{ has positive jumps,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

$$a^{-1}(x)\mathbf{P}\{\mathbb{V}_1 < -x\} \underset{x\to\infty}{\longrightarrow} \begin{cases} c_{15} > 0 & \text{if } \mathbb{S} \text{ has negative jumps,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.22}$$

The following proposition strengthens this result:

Proposition 2.15

There exists $c_{16} > 0$ such that for all $v \geqslant 1$ and all $x \geqslant 1$,

$$\mathbf{P}\{\mathbb{V}_v > x\} \leqslant c_{16} \frac{v}{a^{-1}(x)}. \tag{2.23}$$

Moreover, if $\mathbb{S}$ has positive jumps,

$$\mathbf{P}\{\mathbb{V}_v > x\} \underset{\substack{v \to \infty \\ \frac{a^{-1}(x)}{v} \to \infty}}{\sim} v\mathbf{P}\{\mathbb{V}_1 > x\} \underset{\substack{v \to \infty \\ \frac{a^{-1}(x)}{v} \to \infty}}{\sim} c_{14} \frac{v}{a^{-1}(x)}. \tag{2.24}$$

There is of course a similar result for $\mathbf{P}\{\mathbb{V}_v < -x\}$.

Proof. The equivalence (2.24) is already known and is stated in [Bor03], but we have not been able to find a proof of this result in English. A weaker result is proved by Heyde [Hey68]. Yet, a slight modification of his argument will enable us to prove the proposition. Let us choose $1/2 < \delta < 1$ and set $z \stackrel{\text{def}}{=} (x/a(v))^{\delta} a(v)$. Define, for $k \geqslant 1$,

$$\zeta_{k,z} \stackrel{\text{def}}{=} \begin{cases} \mathbb{V}_k - \mathbb{V}_{k-1} & \text{if } |\mathbb{V}_k - \mathbb{V}_{k-1}| \leqslant z, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$ and define

$$\begin{aligned} \mathcal{E}_5 &\stackrel{\text{def}}{=} \Big\{ \mathbb{V}_k - \mathbb{V}_{k-1} > (1-\varepsilon)x \text{ for at least one } k \text{ in } \{1, \ldots, \lfloor v \rfloor\} \Big\}, \\ \mathcal{E}_6 &\stackrel{\text{def}}{=} \Big\{ \mathbb{V}_k - \mathbb{V}_{k-1} > z \text{ for at least two } k\text{'s in } \{1, \ldots, \lfloor v \rfloor\} \Big\}, \\ \mathcal{E}_7 &\stackrel{\text{def}}{=} \Big\{ \zeta_{1,z} + \ldots + \zeta_{\lfloor v \rfloor, z} > \varepsilon x \Big\}. \end{aligned}$$

We first observe that $\{\mathbb{V}_v > x\} \subset \mathcal{E}_5 \cap \mathcal{E}_6 \cap \mathcal{E}_7$, hence

$$\mathbf{P}\{\mathbb{V}_v > x\} \leqslant \mathbf{P}\{\mathcal{E}_5\} + \mathbf{P}\{\mathcal{E}_6\} + \mathbf{P}\{\mathcal{E}_7\}. \tag{2.25}$$

We deal with each term on the r.h.s. of (2.25) separately. Let us choose $C > c_{14}$ if $\mathbb{S}$ has positive jumps and set $C = 1$ otherwise. We now assume that v and $a^{-1}(x)/v$ are very large. According to (2.21) and using the regular variation of $a^{-1}(\cdot)$, we get

$$\mathbf{P}\{\mathcal{E}_5\} \leqslant v\mathbf{P}\{\mathbb{V}_1 > (1-\varepsilon)x\} \leqslant \frac{C}{(1-\varepsilon)^{\alpha}} \frac{v}{a^{-1}(x)}. \tag{2.26}$$

We now deal with $\mathbf{P}\{\mathcal{E}_6\}$. Let $\eta > 0$. Lemma 2.4 gives for all v and $a^{-1}(x)/v$ large enough,

$$\frac{v a^{-1}(x)}{(a^{-1}(z))^2} = \frac{a^{-1}\left(a(v)\frac{x}{a(v)}\right)}{a^{-1}(a(v))} \left(\frac{a^{-1}(a(v))}{a^{-1}\left(a(v)\left(\frac{x}{a(v)}\right)^{\delta}\right)} \right)^2 \leqslant \left(\frac{x}{a(v)}\right)^{\alpha+\eta} \left(\frac{a(v)}{x}\right)^{2\delta(\alpha-\eta)}.$$

Since $\delta > 1/2$, we can assume η small enough such that $2\delta(\alpha - \eta) - (\alpha + \eta) > \eta$, then

$$\frac{v a^{-1}(x)}{(a^{-1}(z))^2} \leqslant \left(\frac{a(v)}{x}\right)^{\eta}. \tag{2.27}$$

Using (2.21) and (2.27), we deduce that

$$\mathbf{P}\{\mathcal{E}_6\} \leqslant v^2\mathbf{P}\{\mathbb{V}_1 > z\}^2 \leqslant C\frac{v^2}{(a^{-1}(z))^2} \leqslant C\frac{v}{a^{-1}(x)}\left(\frac{a(v)}{x}\right)^{\eta}. \tag{2.28}$$

We now turn our attention to $\mathbf{P}\{\mathcal{E}_7\}$. From Tchebychev's inequality, we have

$$\mathbf{P}\{\mathcal{E}_7\} \leqslant \frac{1}{\varepsilon^2x^2}\mathbf{E}\big[(\zeta_{1,z}+\ldots+\zeta_{\lfloor v\rfloor,z})^2\big] \leqslant \frac{v}{\varepsilon^2x^2}\mathbf{E}\big[\zeta_{1,z}^2\big] + \frac{v^2}{\varepsilon^2x^2}\mathbf{E}[\zeta_{1,z}]^2. \tag{2.29}$$

Let $f(z) = \mathbf{E}\big[(\zeta_{1,z})^2\big] = \int_{-z}^{z} y^2\mathbf{P}\{\mathbb{V}_1 \in dy\}$. This function is non-decreasing and non-zero for z large enough. It is also known from the characterization of the domains of attraction (*c.f.* (8.14) of [Fel71], p304) that the norming constants (a_n) are such that $nf(a_n)/a_n^2 \to c_{17} > 0$. Hence, $f(z) \sim c_{17}z^2/a^{-1}(z)$ as z goes to infinity (f is regularly varying with index $2-\alpha$). Therefore, for v and $a^{-1}(x)/v$ large enough, we get

$$\frac{v}{\varepsilon^2x^2}\mathbf{E}\big[(\zeta_{1,z})^2\big] = \frac{vf(z)}{\varepsilon x^2} \leqslant c_{18,\varepsilon}\frac{v}{a^{-1}(x)}\frac{f(z)}{f(x)} \leqslant c_{18,\varepsilon}\frac{v}{a^{-1}(x)}. \tag{2.30}$$

We can sharpen this estimate when $\alpha < 2$. Indeed, in this case, f is regularly varying with index $2-\alpha > 0$. Thus, using Lemma 2.4 and setting $\eta' = (1-\delta)(2-\alpha)/2$,

$$\frac{f(z)}{f(x)} \leqslant \left(\frac{z}{x}\right)^{(2-\alpha)/2} = \left(\frac{a(v)(x/a(v))^{\delta}}{x}\right)^{(2-\alpha)/2} = \left(\frac{a(v)}{x}\right)^{\eta'}.$$

When $\alpha < 2$, we therefore obtain

$$\frac{v}{\varepsilon^2x^2}\mathbf{E}[(\zeta_{1,z}] \leqslant c_{18,\varepsilon}\frac{v}{a^{-1}(x)}\left(\frac{a(v)}{x}\right)^{\eta'}. \tag{2.31}$$

Let $g(z) \stackrel{\text{def}}{=} \mathbf{E}[\zeta_{1,z}] = \int_{-z}^{z} y\mathbf{P}\{\mathbb{V}_1 \in dy\}$. Since $\mathbb{V}_1$ is in the domain of attraction of a stable law, it is known that the centering constants $c(n)$ such that $\mathbb{V}_n/a(n) - c(n)$ converge to a stable law may be chosen to be $c(n) = ng(a(n))/a(n)$ (see [Fel71], p305). The main assumption of this paper states that the sequence $c(n)$ may also be chosen to be identically 0. Thus the sequence $ng(a(n))/a(n)$ is bounded and so there exists $c_{19} > 0$ such that

$$|g(z)| \leqslant c_{19}\frac{z}{a^{-1}(z)} \quad \text{for all } z \geqslant 1.$$

Using this inequality, we get, for v and $a^{-1}(x)/v$ large enough,

$$\begin{aligned}
\frac{v^2}{\varepsilon^2x^2}\mathbf{E}[\zeta_{1,z}]^2 &\leqslant c_{20,\varepsilon}\frac{v^2z^2}{x^2(a^{-1}(z))^2}\\
&= c_{20,\varepsilon}\frac{v}{a^{-1}(x)}\frac{va^{-1}(x)}{(a^{-1}(z))^2}\left(\frac{z}{x}\right)^2\\
&\leqslant c_{20,\varepsilon}\frac{v}{a^{-1}(x)}\frac{va^{-1}(x)}{(a^{-1}(z))^2}\\
&\leqslant c_{20,\varepsilon}\frac{v}{a^{-1}(x)}\left(\frac{a(v)}{x}\right)^{\eta}
\end{aligned} \tag{2.32}$$

where we used (2.27) for the last inequality. Putting the pieces together, (2.25)-(2.26)-(2.28)-(2.29)-(2.30) and (2.32) yield (2.23). Moreover, when $\mathbb{S}$ has positive jumps, we have $\alpha < 2$. Hence, we can use (2.31) instead of (2.30) and we deduce that

$$\limsup_{\substack{v\to\infty \\ \frac{a^{-1}(x)}{v}\to\infty}} \frac{a^{-1}(x)\mathbf{P}\{\mathbb{V}_v > x\}}{v} \leqslant c_{14}.$$

It remains to prove that the lower bound holds. Assume that $\mathbb{S}$ has positive jumps and notice that the event $\{\mathbb{V}_v > x\}$ contains

$$\bigcap_{k=0}^{\lfloor v\rfloor -1} \Big\{\mathbb{V}^*_k \leqslant \varepsilon x,\, \mathbb{V}_{k+1}-\mathbb{V}_k > (1+2\varepsilon)x,\, (\theta_{k+1}\mathbb{V})^*_{\lfloor v\rfloor -k-1} \leqslant \varepsilon x\Big\}.$$

Moreover, the events of the last formula are disjoints. The independence and the stationarity of the increments of the random walk $\mathbb{V}$ yield

$$\begin{aligned}\mathbf{P}\{\mathbb{V}_v > x\} &\geqslant \sum_{k=0}^{\lfloor v\rfloor -1} \mathbf{P}\Big\{\mathbb{V}^*_k \leqslant \varepsilon x\Big\}\mathbf{P}\Big\{\mathbb{V}_1 > (1+2\varepsilon)x\Big\}\mathbf{P}\Big\{\mathbb{V}^*_{\lfloor v\rfloor -k-1} \leqslant \varepsilon x\Big\} \\ &\geqslant \lfloor v\rfloor \mathbf{P}\Big\{\mathbb{V}^*_v \leqslant \varepsilon x\Big\}^2 \mathbf{P}\Big\{\mathbb{V}_1 > (1+2\varepsilon)x\Big\}.\end{aligned}$$

From (2.21) and the regular variation of $a^{-1}(\cdot)$ we see that

$$\lfloor v\rfloor \mathbf{P}\{\mathbb{V}_1 > (1+2\varepsilon)x\} \underset{\substack{v\to\infty \\ \frac{a^{-1}(x)}{v}\to\infty}}{\sim} \frac{c_{14} v}{(1+2\varepsilon)^\alpha a^{-1}(x)}.$$

We also know from the results of Section 2.1 that $\mathbb{V}^*_v/a(v)$ converges in law towards $\mathbb{S}^*_1$. Therefore,

$$\lim_{\substack{v\to\infty \\ \frac{a^{-1}(x)}{v}\to\infty}} \mathbf{P}\Big\{\mathbb{V}^*_v \leqslant \varepsilon x\Big\} = \lim_{\substack{v\to\infty \\ \frac{a^{-1}(x)}{v}\to\infty}} \mathbf{P}\Big\{\frac{\mathbb{V}^*_v}{a(v)} \leqslant \varepsilon \frac{x}{a(v)}\Big\} = 1.$$

We conclude that

$$\liminf_{\substack{v\to\infty \\ \frac{a^{-1}(x)}{v}\to\infty}} \frac{a^{-1}(x)\mathbf{P}\{\mathbb{V}_v > x\}}{v} \geqslant \frac{c_{14}}{(1+2\varepsilon)^\alpha}.$$

■

Corolary 2.16

By possibly extending the value of c_{16}, equation (2.23) also holds with $\overline{\mathbb{V}}_v$, $-\underline{\mathbb{V}}_v$, $\mathbb{V}^\#_v$ or even with $\mathbb{V}^*_v$ in place of $\mathbb{V}_v$.

Proof. The results for $\overline{\mathbb{V}}_v$ and $-\underline{\mathbb{V}}_v$ are straightforward using Lemma 2.14. To prove the result for $\mathbb{V}^*$ and $\mathbb{V}^\#$, we simply notice that $\{\mathbb{V}^\#_v \geqslant 2x\} \subset \{\mathbb{V}^*_v \geqslant x\} \subset \{\overline{\mathbb{V}}_v \geqslant x\} \cup \{-\underline{\mathbb{V}}_v \geqslant x\}$. ■

Corolary 2.17

For any $0 < \delta < \alpha$, we have

$$\lim_{v\to\infty} \mathbf{E}\left[\left(\frac{\overline{\mathbb{V}}_v}{a(v)}\right)^\delta\right] = \mathbf{E}\left[(\overline{\mathbb{S}}_1)^\delta\right] \text{ and } \lim_{v\to\infty} \mathbf{E}\left[\left|\frac{\underline{\mathbb{V}}_v}{a(v)}\right|^\delta\right] = \mathbf{E}\left[(-\underline{\mathbb{S}}_1)^\delta\right].$$

Proof. It follows from the previous corollary and the regular variation of $a^{-1}(\cdot)$ with index α that for any $0 < \delta < \alpha$,

$$\sup_{v \geqslant 1} \mathbf{E}\left[\left(\frac{\overline{\mathbb{V}}_v}{a(v)}\right)^\delta\right] < \infty.$$

The family $\big((\overline{\mathbb{V}}_v/a(v))^\delta, v \geqslant 1\big)$ is therefore uniformly integrable for all $0 < \delta < \alpha$. We also know that $\overline{\mathbb{V}}_v/a(v)$ converges in law toward $\overline{\mathbb{S}}_1$ as v goes to infinity. These two facts combined yield the first assertion. The proof of the second part of the corollary is similar. ■

Proposition 2.18

For all $0 < \delta < q$ (recall that q is the negativity parameter of $\mathbb{S}$) there exists $c_{21,\delta}$ such that, for all $v, x \geqslant 1$,

$$\mathbf{P}\{-\underline{\mathbb{V}}_v \leqslant x\} \leqslant c_{21,\delta}\left(\frac{a^{-1}(x)}{v}\right)^\delta.$$

A similar result holds for $\mathbf{P}\{\overline{\mathbb{V}}_v \leqslant x\}$ on replacing the condition $\delta < q$ by $\delta < p$.

Proof. We just prove the result for $\underline{\mathbb{V}}_v$. By possibly extending the value of $c_{21,\delta}$, it suffices to prove the inequality for x and $v/a^{-1}(x)$ large enough. Let us choose δ' such that $\delta < \delta' < q < 1$ and notice that for any $y > 0$,

$$\{-\underline{\mathbb{V}}_v \leqslant x\} \;\subset\; \Lambda(x,y) \cup (\{-\underline{\mathbb{V}}_v \leqslant x\} \cap \Lambda(x,y)^c) \;\subset\; \Lambda(x,y) \cup \{\mathbb{V}_v^\# \leqslant x + y\},$$

thus

$$\mathbf{P}\{-\underline{\mathbb{V}}_v \leqslant x\} \leqslant \mathbf{P}\{\Lambda\,(x,y)\} + \mathbf{P}\{\mathbb{V}_v^\# \leqslant x + y\}. \tag{2.33}$$

On the one hand, for x and y/x large enough, using Proposition 2.13 and Lemma 2.4, we get

$$\mathbf{P}\{\Lambda\,(x,y)\} \;\leqslant\; c_{22}\frac{b^{-1}\left(a^{-1}(x)\right)}{b^{-1}\left(a^{-1}(x+y)\right)} \;\leqslant\; c_{23,\delta'}\left(\frac{a^{-1}(x)}{a^{-1}(x+y)}\right)^{\delta'}. \tag{2.34}$$

On the other hand, for $x + y$ and $v/a^{-1}(x + y)$ large enough, using Proposition 2.5, we obtain

$$\mathbf{P}\{\mathbb{V}_v^\# \leqslant x + y\} \leqslant \exp\left(-\frac{K^\#}{2}\frac{v}{a^{-1}(x+y)}\right). \tag{2.35}$$

Let us choose $y = a\left(\frac{K^{\#}v}{2\log(v/a^{-1}(x))}\right) - x$. It is easy to check that (2.34) and (2.35) hold whenever x and $v/a^{-1}(x)$ are large enough, thus, (2.33) yields

$$\begin{aligned} \mathbf{P}\{-\underline{\mathbb{V}}_v \leqslant x\} &\leqslant c_{23,\delta'}\left(\frac{2}{K^{\#}}\right)^{\delta'}\left(\frac{a^{-1}(x)}{v}\left(\log\frac{v}{a^{-1}(x)}\right)\right)^{\delta'} + \frac{a^{-1}(x)}{v} \\ &\leqslant c_{24,\delta'}\left(\frac{a^{-1}(x)}{v}\right)^{\delta}. \end{aligned}$$

■

3 Behavior of X

We now study the diffusion X in the random potential $\mathbb{V}$. Let us first recall the classical representation of X from a Brownian motion through a (random) change of scale and a (random) change of time (*c.f.* [Bro86, HS98a, IM65]). Indeed, we can assume that X is of the form

$$X_t = \mathbb{A}^{-1}\left(B_{\mathbb{T}^{-1}(t)}\right) \tag{3.1}$$

where B is a standard Brownian motion independent of $\mathbb{V}$ and where $\mathbb{A}^{-1}$ and $\mathbb{T}^{-1}$ are the respective inverses of

$$\mathbb{A}(x) \stackrel{\text{def}}{=} \int_0^x e^{\mathbb{V}_y} dy \qquad \text{and} \qquad \mathbb{T}(t) \stackrel{\text{def}}{=} \int_0^t e^{-2\mathbb{V}_{\mathbb{A}^{-1}(B_s)}} ds$$

Note that our assumption on $\mathbb{V}$ implies, with probability 1, that $\mathbb{A}$ is an increasing homeomorphism on $\mathbb{R}$ and that $\mathbb{T}$ is an increasing homeomorphism on $\mathbb{R}_+$. Thus, $\mathbb{A}^{-1}$ and $\mathbb{T}^{-1}$ are well defined. Let $v > 0$ and recall the definition of σ_X given in Section 2.1. Using (3.1), we obtain

$$\sigma_X(v) = \mathbb{T}\left(\sigma_B\left(\mathbb{A}(v)\right)\right).$$

Let now $(L(t,x),\ t \geqslant 0,\ x \in \mathbb{R})$ stand for the bi-continuous version of the local time process of the Brownian motion B. The last equality may be rewritten:

$$\begin{aligned} \sigma_X(v) &= \int_0^{\sigma_B(\mathbb{A}(v))} e^{-2\mathbb{V}_{\mathbb{A}^{-1}(B_s)}} ds \\ &= \int_{-\infty}^{\mathbb{A}(v)} e^{-2\mathbb{V}_{\mathbb{A}^{-1}(x)}} L(\sigma_B(\mathbb{A}(v)), x) dx \\ &= \int_{-\infty}^{v} e^{-\mathbb{V}_y} L(\sigma_B(\mathbb{A}(v)), \mathbb{A}(y)) dy \end{aligned}$$

where we have used the change of variable $x = \mathbb{A}(y)$. We split this integral into two parts:

$$I_1(v) \stackrel{\text{def}}{=} \int_0^v e^{-\mathbb{V}_y} L(\sigma_B(\mathbb{A}(v)), \mathbb{A}(y)) dy, \tag{3.2}$$

$$I_2(v) \stackrel{\text{def}}{=} \int_0^\infty e^{-\mathbb{V}_{-y}} L(\sigma_B(\mathbb{A}(v)), \mathbb{A}(-y)) dy. \tag{3.3}$$

Using the definition of σ_X, we get

$$\left\{\overline{X}_t \geqslant v\right\} = \{I_1(v) + I_2(v) \leqslant t\}. \tag{3.4}$$

The next two propositions show the connection between $\mathbb{V}$ and X. These estimates will enable us to reduce the study of the limiting behavior of X to the study of some functionals of the potential $\mathbb{V}$. The streamline of the proofs are essentially the same as those of Lemmas 4.1 and 4.2 of Hu and Shi [HS98a] and one may refer to the proof of these two lemmas for further details.

Proposition 3.1

There exists c_{25} such that for all v large enough

$$\mathbb{V}^{\#}_{v-\frac{1}{2}} - (\log v)^4 \leqslant \log I_1(v) \leqslant \mathbb{V}^{\#}_v + (\log v)^4 \quad \text{on } \mathcal{E}_8(v),$$

where $\mathcal{E}_8(v)$ is a measurable set such that

$$\mathbf{P}\{\mathcal{E}_8(v)^c\} \leqslant c_{25} e^{-(\log v)^2}.$$

Proposition 3.2

There exists c_{26} such that for all v large enough

$$\begin{aligned} \log I_2(v) &\leqslant \widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + (\log v)^4\right) \quad \text{on } \mathcal{E}_9(v), \\ \log I_2(v) &\geqslant \widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_{v-\frac{1}{2}} - (\log v)^4\right) \quad \text{on } \mathcal{E}_9(v) \cap \left\{\overline{\mathbb{V}}_{v-\frac{1}{2}} > (\log v)^4\right\}, \end{aligned}$$

where $\widetilde{U}$ was defined in Section 2.1 and where $\mathcal{E}_9(v)$ is a measurable set such that

$$\mathbf{P}\{\mathcal{E}_9(v)^c\} \leqslant c_{26} e^{-(\log v)^2}.$$

Proof of Proposition 3.1. For $v > 0$, let $\mathcal{R}^2$ be defined by

$$\mathcal{R}^2(t) \stackrel{\text{def}}{=} \frac{L\left(\sigma_B(\mathbb{A}(v)), \mathbb{A}(v) - t\mathbb{A}(v)\right)}{\mathbb{A}(v)} \qquad \text{for } 0 \leqslant t \leqslant 1.$$

Let $\mathcal{R}$ stand for the positive square root of $\mathcal{R}^2$. Just as in [HS98a], p1498, we see, using the first Ray-Knight Theorem and the scaling property of the Brownian motion, that for any fixed v the process $(\mathcal{R}(t), 0 \leqslant t \leqslant 1)$ has the law of a two-dimensional Bessel process starting from 0. Moreover, $\mathcal{R}$ is independent of $\mathbb{V}$. We can now rewrite (3.2) as

$$I_1(v) = \mathbb{A}(v) \int_0^v e^{-\mathbb{V}_s} \mathcal{R}^2\left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) ds.$$

Define

$$\mathcal{E}_{10} \stackrel{\text{def}}{=} \left\{\sup_{0 < t \leqslant 1} \frac{\mathcal{R}(t)}{\sqrt{t \log(8/t)}} \leqslant \sqrt{v}\right\}.$$

Using Lemma 6.1 p1497 of [HS98a], we get $\mathbf{P}\{\mathcal{E}_{10}^c\} \leqslant c_{27}e^{-v/2}$. On $\mathcal{E}_{10}$, we have

$$I_1(v) \leqslant v \int_0^v e^{-\mathbb{V}_s} \left(\mathbb{A}(v) - \mathbb{A}(s)\right) \log\left(\frac{8\mathbb{A}(v)}{\mathbb{A}(v) - \mathbb{A}(s)}\right) ds,$$

and for all $s \leqslant v$,

$$e^{-\mathbb{V}_s} \left(\mathbb{A}(v) - \mathbb{A}(s)\right) = \int_s^v e^{\mathbb{V}_y - \mathbb{V}_s} dy \leqslant v e^{\mathbb{V}_v^{\#}}.$$

This implies

$$I_1(v) \leqslant v^2 e^{\mathbb{V}_v^{\#}} \int_0^v \log\left(\frac{8\mathbb{A}(v)}{\mathbb{A}(v) - \mathbb{A}(s)}\right) ds. \tag{3.5}$$

We also have

$$\mathbb{A}(v) = \int_0^v e^{\mathbb{V}_s} ds \leqslant v e^{\overline{\mathbb{V}}_v} \quad \text{and} \quad \mathbb{A}(v) - \mathbb{A}(s) = \int_s^v e^{\mathbb{V}_y} dy \geqslant (v-s) e^{\underline{\mathbb{V}}_v},$$

thus

$$\begin{aligned} \int_0^v \log\left(\frac{8\mathbb{A}(v)}{\mathbb{A}(v) - \mathbb{A}(s)}\right) ds &\leqslant v\left(\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v\right) + \int_0^v \log\left(\frac{8v}{v-s}\right) ds \\ &\leqslant v\left(\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v + 1 + \log(8)\right). \end{aligned}$$

Combining this with (3.5) yields $\log(I_1(v)) \leqslant \mathbb{V}_v^{\#} + \log\left(\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v\right) + 4\log(v)$ for all v large enough. We now define $\mathcal{E}_{11} \stackrel{\text{def}}{=} \left\{\log\left(\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v\right) \leqslant \log^3(v)\right\}$. On $\mathcal{E}_{10} \cap \mathcal{E}_{11}$, for all v large enough, we get the upper bound,

$$\log(I_1(v)) \leqslant \mathbb{V}_v^{\#} + \log^4(v).$$

Moreover, we have $\{\overline{\mathbb{V}}_v - \underline{\mathbb{V}}_v > a\} \subset \{\mathbb{V}_v^* > a/2\}$. Using Corollary 2.16 and the regular variation of $a^{-1}(\cdot)$, we get $\mathbf{P}\{\mathcal{E}_{11}^c\} \leqslant \exp(-\log^2(v))$ for any v large enough.

We now prove the lower bound. For the sake of clarity, we will use the notation $l \stackrel{\text{def}}{=} \log(v)$ and $\delta \stackrel{\text{def}}{=} \exp(-l^2)$. For $v > 1/2$, there exist two integers $0 \leqslant k^- \leqslant k^+ \leqslant v - \frac{1}{2}$ such that $\mathbb{V}_{v-\frac{1}{2}}^{\#} = \mathbb{V}_{k^+} - \mathbb{V}_{k^-}$. Let us define the sets:

$$\begin{aligned} \mathcal{E}_{12} &\stackrel{\text{def}}{=} \left\{ \inf_{k^- \leqslant s \leqslant k^- + \frac{1}{2}} \mathcal{R}\left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) > \delta \sqrt{\frac{\mathbb{A}(v) - \mathbb{A}(k^-)}{\mathbb{A}(v)}} \right\}, \\ \mathcal{E}_{13} &\stackrel{\text{def}}{=} \left\{ \mathbb{V}_{v-\frac{1}{2}}^{\#} \geqslant 3l^2 \right\}. \end{aligned}$$

Using again Lemma 6.1 p1497 of [HS98a] combined with the independence of $\mathcal{R}$ and $\mathbb{V}$, we get

$$\mathbf{P}\{(\mathcal{E}_{12} \cap \mathcal{E}_{13})^c\} \leqslant \mathbf{P}\{\mathcal{E}_{13}^c\} + 2\delta + 2\mathbf{E}\Big[e^{-\frac{\delta^2}{2}J(v)} \mathbb{1}_{\mathcal{E}_{13}}\Big], \tag{3.6}$$

where J is given by

$$J(v) \stackrel{\text{def}}{=} \frac{\mathbb{A}(v) - \mathbb{A}(k^-)}{\mathbb{A}\left(k^- + \frac{1}{2}\right) - \mathbb{A}(k^-)}.$$

On the one hand, we have

$$\mathbb{A}(v) - \mathbb{A}(k^-) = \int_{k^-}^{v} e^{\mathbb{V}_s} ds \geqslant \int_{k^+}^{k^+ + \frac{1}{2}} e^{\mathbb{V}_s} ds = \frac{1}{2} e^{\mathbb{V}_{k^+}}.$$

On the other hand, since k^- is an integer and $\mathbb{V}$ is flat on $[k^-, k^- + 1)$, we also have

$$\mathbb{A}\left(k^- + \frac{1}{2}\right) - \mathbb{A}(k^-) = \int_{k^-}^{k^- + \frac{1}{2}} e^{\mathbb{V}_s} ds = \frac{1}{2} e^{\mathbb{V}_{k^-}}.$$

This implies $J(v) \geqslant \exp(\mathbb{V}^{\#}_{v-1/2})$. Using this inequality combined with (3.6), we get

$$\mathbf{P}\{(\mathcal{E}_{12} \cap \mathcal{E}_{13})^c\} \leqslant \mathbf{P}\{\mathcal{E}_{13}^c\} + 2\delta + 2\exp(-\delta^2 \exp(3l^2)/2).$$

Therefore, we have $\mathbf{P}\{(\mathcal{E}_{12} \cap \mathcal{E}_{13})^c\} \leqslant \mathbf{P}\{\mathcal{E}_{13}^c\} + 3\exp(-l^2)$ for all v large enough. Using Proposition 2.5, it is easily seen that $\mathbf{P}\{\mathcal{E}_{13}^c\} \leqslant e^{-l^2}$ for all large enough v's. Let us finally set $\mathcal{E}_8 \stackrel{\text{def}}{=} \mathcal{E}_{10} \cap \mathcal{E}_{11} \cap \mathcal{E}_{12} \cap \mathcal{E}_{13}$. We have proved that there exists $c_{25} > 0$ such that $\mathbf{P}\{\mathcal{E}_8^c\} \leqslant c_{25} \exp(-l^2)$. Notice that

$$\begin{aligned} I_1(v) &= \mathbb{A}(v) \int_0^v e^{-\mathbb{V}_s} \mathcal{R}^2 \left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) ds \\ &\geqslant \mathbb{A}(v) e^{-\mathbb{V}_{k^-}} \int_{k^-}^{k^- + \frac{1}{4}} \mathcal{R}^2 \left(\frac{\mathbb{A}(v) - \mathbb{A}(s)}{\mathbb{A}(v)}\right) ds, \end{aligned}$$

therefore, on $\mathcal{E}_8$,

$$I_1(v) \geqslant \delta^2 e^{-\mathbb{V}_{k^-}} \int_{k^-}^{k^- + \frac{1}{4}} (\mathbb{A}(v) - \mathbb{A}(s))\, ds,$$

but for all s such that $k^- \leqslant s \leqslant k^- + \frac{1}{4}$ we also have

$$\mathbb{A}(v) - \mathbb{A}(s) \geqslant \mathbb{A}(v) - \mathbb{A}\left(k^- + \frac{1}{4}\right) = \int_{k^- + \frac{1}{4}}^{v} e^{\mathbb{V}_y} dy \geqslant \int_{k^+ + \frac{1}{4}}^{k^+ + \frac{1}{2}} e^{\mathbb{V}_y} dy = \frac{1}{4} e^{\mathbb{V}_{k^+}},$$

hence

$$\int_{k^-}^{k^- + \frac{1}{4}} (\mathbb{A}(v) - \mathbb{A}(s))\, ds \geqslant \frac{1}{16} e^{\mathbb{V}_{k^+}}.$$

We finally get

$$I_1(v) \geqslant \frac{\delta^2}{16} e^{\mathbb{V}^{\#}_{v-1/2}} \quad \text{on } \mathcal{E}_8.$$

We conclude the proof of the proposition by taking the logarithm. ■

Proof of Proposition 3.2. For $v > 0$, we define the process $\mathcal{Z}$ by

$$\mathcal{Z}(t) \stackrel{\text{def}}{=} \frac{L\left(\sigma_B\left(\mathbb{A}(v)\right), -t\mathbb{A}(v)\right)}{\mathbb{A}(v)} \qquad \text{for } t \geqslant 0.$$

Using the second Ray-Knight Theorem and the scaling property of the Brownian motion, we see that for any fixed v the process $\mathcal{Z}$ has the law of a squared Bessel process of dimension

0 such that $\mathcal{Z}(0)$ has an exponential distribution with mean 2. Moreover, $\mathcal{Z}$ is independent of $\mathbb{V}$. We can now rewrite (3.3):

$$I_2(v) = \mathbb{A}(v)\int_0^{\infty} e^{-\mathbb{V}_{-s}}\mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right)ds.$$

Recall that 0 is an absorbing state for $\mathcal{Z}$. Let $\zeta \overset{\text{def}}{=} \inf\left(s \geqslant 0,\ \mathcal{Z}_s = 0\right)$ be the absorption time of $\mathcal{Z}$ and let us also define

$$\zeta(v) \overset{\text{def}}{=} \inf\left\{s \geqslant 0\ ,\ \mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right) = 0\right\}.$$

We can now write

$$I_2(v) = \mathbb{A}(v)\int_0^{\zeta(v)} e^{-\mathbb{V}_{-s}}\mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right)ds.$$

We keep the notation $l \overset{\text{def}}{=} \log(v)$. Note that $\mathbb{A}(v) = \int_0^v e^{\mathbb{V}_s}ds \leqslant \exp(\overline{\mathbb{V}}_v + l)$. Therefore,

$$\begin{aligned} I_2(v) &\leqslant e^{\overline{\mathbb{V}}_v + l}\zeta(v) \sup_{0\,\leqslant\, s\,\leqslant\,\zeta(v)}\left(e^{-\mathbb{V}_{-s}}\right)\sup_{s\,\geqslant\,0}\mathcal{Z}(s) \\ &\leqslant \zeta(v)\sup_{s\,\geqslant\,0}\mathcal{Z}(s)e^{l + \overline{\mathbb{V}}(v) - \underline{\mathbb{V}}(-\zeta(v))}. \end{aligned}$$

Let us define $\mathcal{E}_{14} \overset{\text{def}}{=} \{\sup_{s\,\geqslant\,0}\mathcal{Z}(s) \leqslant \exp(l^2)\}$. Using Lemma 7.1, p1501 of [HS98a], we get $\mathbf{P}\{\mathcal{E}_{14}^c\} \leqslant 4\exp(-l^2)$. Thus, on $\mathcal{E}_{14}$, we have

$$I_2(v) \leqslant \zeta(v)e^{2l^2 + \overline{\mathbb{V}}(v) - \underline{\mathbb{V}}(-\zeta(v))}. \tag{3.7}$$

Let $\mathcal{E}_{15} \overset{\text{def}}{=} \left\{\zeta(v) \leqslant \widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + l^4\right) + \frac{1}{2}\right\}$ and notice that for all $a \geqslant 0$,

$$\{\zeta(v) > a\} = \left\{\frac{-\mathbb{A}(-a)}{\mathbb{A}(v)} < \zeta\right\}.$$

Therefore

$$\mathbf{P}\{\mathcal{E}_{15}^c\} = \mathbf{P}\left\{\frac{-\mathbb{A}\left(-\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + l^4\right) - \frac{1}{2}\right)}{\mathbb{A}(v)} < \zeta\right\},$$

but

$$-\mathbb{A}\left(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) - \frac{1}{2}\right) \geqslant \int_{-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) - \frac{1}{2}}^{-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4)} e^{\mathbb{V}_s}ds \geqslant \frac{1}{2}e^{\overline{\mathbb{V}}_v + l^4},$$

and we have already checked that $\mathbb{A}_v \leqslant \exp(\overline{\mathbb{V}}_v + l)$. Combining these two inequalities, we get, for all v large enough,

$$\frac{-\mathbb{A}\left(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + l^4) - \frac{1}{2}\right)}{\mathbb{A}(v)} \geqslant e^{l^3},$$

hence

$$\mathbf{P}\{\mathcal{E}_{15}^c\} \leqslant \mathbf{P}\{\zeta > e^{l^3}\} \leqslant e^{-l^3},$$

where we have used Lemma 7.1 on p1501 of [HS98a] for the last inequality. On $\mathcal{E}_{14}\cap\mathcal{E}_{15}$ and for v large enough, we deduce from (3.7) that

$$I_2(v)\leqslant \zeta(v)e^{2l^2+\overline{\mathbb{V}}(v)-\underline{\mathbb{V}}(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v+l^4)+\frac12)}.$$

But we also have $\overline{\mathbb{V}}(v)-\underline{\mathbb{V}}(-\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+l^4\right)+\frac12)=\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v+l^4)-l^4$ (recall that $\mathbb{V}$ is flat on the intervals $(-n-1,-n]$, $n\in N$). Therefore, on $\mathcal{E}_{14}\cap\mathcal{E}_{15}$,

$$I_2(v)\leqslant \zeta(v)e^{-l^3+\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v+l^4)}.$$

Let $\mathcal{E}_{16}\overset{\text{def}}{=}\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v+l^4)+\frac12\leqslant\exp(l^3)\}$. On $\mathcal{E}_{17}\overset{\text{def}}{=}\mathcal{E}_{14}\cap\mathcal{E}_{15}\cap\mathcal{E}_{16}$, we have $\zeta(v)\leqslant\exp(l^3)$. Hence, on $\mathcal{E}_{17}$ and for all v large enough,

$$\log\left(I_2(v)\right)\leqslant\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right).$$

This gives the upper bound on $\mathcal{E}_{17}$. Let us check that

$$\mathbf{P}\{\mathcal{E}_{16}^c\}\leqslant c_{28}\exp(-l^2). \tag{3.8}$$

We have $\mathbf{P}\{\mathcal{E}_{16}^c\}\leqslant\mathbf{P}\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v+l^4)>\exp(l^3)/2\}$, thus

$$\mathbf{P}\{\mathcal{E}_{16}^c\}\leqslant\mathbf{P}\Big\{\overline{\mathbb{V}}(-\frac12e^{l^3})\leqslant2\overline{\mathbb{V}}(v)\Big\}+\mathbf{P}\Big\{\overline{\mathbb{V}}(-\frac12e^{l^3})\leqslant2l^4\Big\}.$$

We also have

$$\mathbf{P}\Big\{\overline{\mathbb{V}}(-\frac12e^{l^3})\leqslant2\overline{\mathbb{V}}(v)\Big\}\leqslant\mathbf{P}\Big\{\overline{\mathbb{V}}(-\frac12e^{l^3})\leqslant e^{l^{5/2}}\Big\}+\mathbf{P}\Big\{\overline{\mathbb{V}}(v)>\frac12e^{l^{5/2}}\Big\}.$$

Using Corollary 2.16 and the regular variation of $a^{-1}(\cdot)$, for all v large enough,

$$\mathbf{P}\Big\{\overline{\mathbb{V}}(v)>\frac12e^{l^{5/2}}\Big\}\leqslant e^{-l^2}.$$

Recall that $(\mathbb{V}(x),x\geqslant0)$ and $(-\mathbb{V}(-x),x\geqslant0)$ have the same law, thus Proposition 2.18 yields

$$\mathbf{P}\Big\{\overline{\mathbb{V}}(-\frac12e^{l^3})\leqslant2l^4\Big\}\leqslant\mathbf{P}\Big\{\overline{\mathbb{V}}(-\frac12e^{l^3})\leqslant e^{l^{5/2}}\Big\}\leqslant e^{-l^2}.$$

These inequalities give $\mathbf{P}\{\mathcal{E}_{16}^c\}\leqslant3e^{-l^2}$, hence $\mathbf{P}\{\mathcal{E}_{17}^c\}\leqslant8e^{-l^2}$. We now prove the lower bound. Notice that

$$\mathbb{A}(v)\geqslant\int_{\sigma_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac12))}^{\sigma_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac12))+\frac12}e^{\mathbb{V}_s}ds=\frac12e^{\overline{\mathbb{V}}(v-\frac12)}, \tag{3.9}$$

and for all $x\leqslant\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_{v-\frac12}-l^4)\leqslant\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)$,

$$-\mathbb{A}(-x)=\int_{-x}^0e^{\mathbb{V}(s)}ds\leqslant e^{\overline{\mathbb{V}}(v-\frac12)-l^4}\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v). \tag{3.10}$$

Thus, for all $x\leqslant\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac12)-l^4)$ we have $-\mathbb{A}_{-x}/\mathbb{A}_v\leqslant\exp(-l^4)\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)$. Let $\mathcal{E}_{18}\overset{\text{def}}{=}\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant\exp(l^3)\}$. As for the estimate of $\mathbf{P}\{\mathcal{E}_{16}^c\}$, it is easily checked that for all v large enough, $\mathbf{P}\{\mathcal{E}_{18}^c\} \leqslant 3\exp(-l^2)$. Moreover, on the set $\mathcal{E}_{18}$, combining (3.9) and (3.10), we have $-\mathbb{A}(-x)/\mathbb{A}(v) \leqslant e^{-\frac{1}{2}l^4}$ for all $0 \leqslant x \leqslant \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2}) - l^4)$. Let us now define

$$\mathcal{E}_{19} \stackrel{\text{def}}{=} \Big\{ \inf_{0 \leqslant s \leqslant e^{-l^4/2}} \mathcal{Z}(s) \geqslant e^{-l^2} \Big\}.$$

Using Lemma 7.1 on p1501 of [HS98a], we see that $\mathbf{P}\{\mathcal{E}_{19}^c\} \leqslant 2e^{-l^2}$. Recall that

$$I_2(v) = \mathbb{A}(v) \int_0^\infty e^{-\mathbb{V}_{-s}} \mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right) ds \;\geqslant\; \mathbb{A}(v) \int_0^{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2})-l^4)} e^{-\mathbb{V}_{-s}} \mathcal{Z}\left(\frac{-\mathbb{A}(-s)}{\mathbb{A}(v)}\right) ds,$$

therefore, on $\mathcal{E}_{20} \stackrel{\text{def}}{=} \mathcal{E}_{18} \cap \mathcal{E}_{19}$,

$$I_2(v) \geqslant \widetilde{\sigma}_{\mathbb{V}}\Big(\overline{\mathbb{V}}\big(v - \frac{1}{2}\big) - l^4\Big)\mathbb{A}(v) e^{-\underline{\mathbb{V}}\left(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2})-l^4)\right)-l^2}.$$

Using (3.9) again, on $\mathcal{E}_{20}$,

$$\begin{aligned} I_2(v) \;&\geqslant\; \frac{1}{2}\widetilde{\sigma}_{\mathbb{V}}\Big(\overline{\mathbb{V}}\big(v - \frac{1}{2}\big) - l^4\Big) e^{\overline{\mathbb{V}}(v-\frac{1}{2}) - \underline{\mathbb{V}}\left(-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-\frac{1}{2})-l^4)\right)-l^2} \\ &\geqslant\; \frac{1}{2}\widetilde{\sigma}_{\mathbb{V}}\Big(\overline{\mathbb{V}}\big(v - \frac{1}{2}\big) - l^4\Big) e^{\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}(v-\frac{1}{2})-l^4\right)+l^4-l^2}. \end{aligned}$$

Notice that on $\{\overline{\mathbb{V}}\big(v-1/2\big) > l^4\}$, we have $\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}(v-1/2) - l^4) \geqslant 1$ because $\mathbb{V}$ is identically 0 on $(-1, 0]$. This implies that on $\mathcal{E}_{20} \cap \{\overline{\mathbb{V}}(v-1/2) > l^4\}$,

$$I_2(v) \geqslant e^{\widetilde{U}_{\mathbb{V}}\;\;\overline{\mathbb{V}}\left(v-\frac{1}{2}\right)-l^4} \quad,$$

which yields the lower bound by taking the logarithm. Finally, set $\mathcal{E}_9 \stackrel{\text{def}}{=} \mathcal{E}_{20} \cap \mathcal{E}_{17}$, we have, for all large enough v's

$$\mathbf{P}\{\mathcal{E}_9^c\} \leqslant \mathbf{P}\{\mathcal{E}_{17}^c\} + \mathbf{P}\{\mathcal{E}_{20}^c\} \leqslant 13 e^{-(\log v)^2}$$

The upper bound holds on $\mathcal{E}_9$ and the lower bound on $\mathcal{E}_9 \cap \{\overline{\mathbb{V}}\big(v-1/2\big) > l^4\}$. ■

4 Proof of the main theorems

4.1 Proof of Theorem 1.2

We first state two lemmas before we give the proof of the theorem.

Lemma 4.1

For any $c > 0$, we have

$$\limsup_{t\to\infty} \frac{\log \mathbf{P}\{\overline{X}_t \geqslant ca^{-1}(\log t)\log\log\log t\}}{\log\log\log t} \leqslant -cK^{\#},$$

where $K^{\#}$ is the constant of Proposition 2.5.

Proof. Let $v \stackrel{\text{def}}{=} ca^{-1}(\log t)\log\log\log t$. Using (3.4) and Proposition 3.1 we get, for all t large enough,

$$\begin{aligned}\mathbf{P}\{\overline{X}_t \geqslant v\} &\leqslant \mathbf{P}\{I_1(v) \leqslant t\}\\ &\leqslant \mathbf{P}\{\mathbb{V}^{\#}_{v-\frac{1}{2}} \leqslant \log t + (\log v)^4\} + c_{25}\exp\left(-(\log v)^2\right).\end{aligned}$$

Using Proposition 2.5, for any $\varepsilon > 0$ and for all t large enough (depending on ε), we obtain

$$\begin{aligned}\mathbf{P}\{\mathbb{V}^{\#}_{v-\frac{1}{2}} \leqslant \log t + (\log v)^4\} &\leqslant \exp\left(-(K^{\#}-\varepsilon)\frac{v-1/2}{a^{-1}(\log t + (\log v)^4)}\right)\\ &\leqslant \exp\left(-c(K^{\#}-2\varepsilon)\log\log\log t\right)\end{aligned}$$

where we used the regular variation of $a^{-1}(\cdot)$ to check that $a^{-1}(\log t + (\log v)^4) \sim a^{-1}(\log t)$. Therefore, for all t large enough,

$$\begin{aligned}\mathbf{P}\{\overline{X}_t \geqslant v\} &\leqslant \exp\left(-c(K^{\#}-2\varepsilon)\log\log\log t\right) + \exp\left(-(\log v)^2\right)\\ &\leqslant 2\exp\left(-c(K^{\#}-2\varepsilon)\log\log\log t\right).\end{aligned}$$

■

Lemma 4.2

For any $c > 0$ and for all t large enough (depending on c) we have

$$\begin{aligned}\{\overline{X}_t \geqslant v\} \supset& \left\{\mathbb{V}^{\#}_v \leqslant \log t - \sqrt{\log t},\ \overline{\mathbb{V}}_v \leqslant \frac{\log t}{5}\right\}\\ &\cap\left\{\widetilde{U}_{\mathbb{V}}\left(\frac{\log t}{4}\right) \leqslant \frac{\log t}{2}\right\} \cap \mathcal{E}_{21}(v)\end{aligned}$$

where $v \stackrel{\text{def}}{=} ca^{-1}(\log t)\log\log\log t$ and where $\mathcal{E}_{21}(v)$ is a measurable set such that

$$\mathbf{P}\{\mathcal{E}^c_{21}(v)\} \leqslant c_{29}e^{-(\log v)^2}.$$

Proof. Using (3.4) combined with Proposition 3.1 and 3.2, for t sufficiently large,

$$\begin{aligned}\{\overline{X}_t \geqslant v\} &= \{I_1(v) + I_2(v) \leqslant t\}\\ &\supset \left\{e^{\mathbb{V}^{\#}_v + (\log v)^4} + e^{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + (\log v)^4)} \leqslant t\right\} \cap \mathcal{E}_{21}(v)\end{aligned}$$

with $\mathcal{E}_{21}(v) \stackrel{\text{def}}{=} \mathcal{E}_8(v) \cap \mathcal{E}_9(v)$, thus $\mathbf{P}\{\mathcal{E}^c_{21}(v)\} \leqslant c_{29}e^{-(\log v)^2}$. Notice also that

$$\left\{\mathbb{V}^{\#}_v \leqslant \log t - \sqrt{\log t}\right\} \subset \left\{\mathbb{V}^{\#}_v + \log^4 v \leqslant \log\frac{t}{2}\right\}.$$

Hence, $\{\overline{X}_t \geqslant v\}$ contains

$$\left\{\mathbb{V}^{\#}_v \leqslant \log t - \sqrt{\log t}\right\} \cap \left\{\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + (\log v)^4\right) \leqslant \log\left(\frac{t}{2}\right)\right\} \cap \mathcal{E}_{21}(v). \tag{4.1}$$

We also have $\left\{\overline{\mathbb{V}}_v \leqslant \frac{\log t}{5}\right\} \subset \left\{\overline{\mathbb{V}}_v + (\log v)^4 \leqslant \frac{\log t}{4}\right\}$, therefore

$$\left\{\overline{\mathbb{V}}_v \leqslant \frac{\log t}{5}, \widetilde{U}_{\mathbb{V}}\left(\frac{\log t}{4}\right) \leqslant \frac{\log t}{2}\right\} \subset \left\{\widetilde{U}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v + (\log v)^4\right) \leqslant \frac{\log t}{2}\right\}.$$

This inclusion combined with (4.1) completes the proof of the lemma. ■

Proof of Theorem 1.2. As already mentioned in the introduction, X and $\overline{X}$ have the same upper function so we only need to prove the theorem for $\overline{X}$. Let us choose K such that $K < K^{\#}$ and $\varepsilon > 0$. Define the sequence $t_i \stackrel{\text{def}}{=} \exp(\exp(\varepsilon i))$. We also use the notation $f(x) \stackrel{\text{def}}{=} a^{-1}(\log x) \log\log\log x$. Using regular variation of $a(\cdot)$ we easily check that $f(t_i)/f(t_{i+1})$ converges to $\exp(-\alpha\varepsilon)$. Thus, for all i large enough

$$\mathbf{P}\left\{\overline{X}_{t_{i+1}} \geqslant \frac{f(t_i)}{K}\right\} \leqslant \mathbf{P}\left\{\overline{X}_{t_{i+1}} \geqslant \frac{f(t_{i+1})}{e^{2\varepsilon}K}\right\}.$$

Using Lemma 4.1, we get

$$\limsup_{i\to\infty} \frac{1}{\log(\varepsilon(i+1))} \log \mathbf{P}\left\{\overline{X}_{t_{i+1}} \geqslant \frac{f(t_i)}{K}\right\} \leqslant -\frac{K^{\#}}{e^{2\varepsilon}K}.$$

Since $K < K^{\#}$, we can choose ε small enough such that $K^{\#}/(K\exp(2\varepsilon)) < 1$ and we deduce from the last inequality that the sum $\sum \mathbf{P}\{\overline{X}_{t_{i+1}} \geqslant f(t_i)/K\}$ converges. Using Borel-Cantelli's Lemma, with probability 1, for all i large enough, $\overline{X}_{t_{i+1}} \leqslant f(t_i)/K$. For $t \in [t_i, t_{i+1}]$, using monotonicity of f and $\overline{X}$,

$$\overline{X}_t \leqslant \overline{X}_{t_{i+1}} \leqslant \frac{f(t_i)}{K} \leqslant \frac{f(t)}{K}.$$

This holds for all $K < K^{\#}$. Hence, we have proved that

$$\limsup_{t\to\infty} \frac{\overline{X}_t}{f(t)} \leqslant \frac{1}{K^{\#}} \quad \text{a.s.}$$

We now establish the lower bound. Choose $K > K^{\#}$ and change the sequence (t_i) to $t_i \stackrel{\text{def}}{=} \exp(\exp i)$. From Lemma 4.2, for i large enough,

$$\left\{\overline{X}_{t_i} \geqslant \frac{f(t_i)}{K}\right\} \supset \mathcal{E}_{21}(f(t_i)/K) \cap \mathcal{E}_{22}(i)$$

where $\mathcal{E}_{21}$ was defined in Lemma 4.2 and where $\mathcal{E}_{22}(i) \stackrel{\text{def}}{=} \mathcal{E}_{23}(i) \cap \mathcal{E}_{24}(i) \cap \mathcal{E}_{25}(i)$ with

$$\begin{aligned} \mathcal{E}_{23}(i) &\stackrel{\text{def}}{=} \left\{\widetilde{U}_{\mathbb{V}}\left(e^i/4\right) \leqslant e^i/2\right\}, \\ \mathcal{E}_{24}(i) &\stackrel{\text{def}}{=} \left\{\mathbb{V}^{\#}_{f(t_i)/K} \leqslant e^i - e^{i/2}\right\}, \\ \mathcal{E}_{25}(i) &\stackrel{\text{def}}{=} \left\{\overline{\mathbb{V}}_{f(t_i)/K} \leqslant e^i/5\right\}. \end{aligned}$$

Moreover, $\sum \mathbf{P}\{\mathcal{E}^c_{21}(f(t_i)/K)\} < \infty$. So it only remains to prove that the events $\mathcal{E}_{22}(i)$ happen infinitely often. It follows from the results of Section 2.1 that $\lim_{i\to\infty} \mathbf{P}\{\mathcal{E}_{23}(i)\} =$

$\mathbf{P}\{\widetilde{U}_{\mathbb{S}}(1/4) \leqslant 1/2\}$ and it is clear that this quantity is not 0. Since the events $\mathcal{E}_{24}(i) \cap \mathcal{E}_{25}(i)$ and $\mathcal{E}_{23}(i)$ are independent, we have $\mathbf{P}\{\mathcal{E}_{22}(i)\} \geqslant c_{30}\mathbf{P}\{\mathcal{E}_{24}(i) \cap \mathcal{E}_{25}(i)\}$ for all i large enough. Thus, we deduce from Proposition 2.6 that for all large enough i's,

$$c_{31}\mathbf{P}\{\mathcal{E}_{24}(i)\} \leqslant \mathbf{P}\{\mathcal{E}_{22}(i)\} \leqslant \mathbf{P}\{\mathcal{E}_{24}(i)\}. \tag{4.2}$$

We now use Proposition 2.5 to obtain

$$\log \mathbf{P}\{\mathcal{E}_{24}(i)\} \underset{i\to\infty}{\sim} -\frac{K^{\#}}{K} \frac{f(t_i)}{a^{-1}\left(e^i - e^{i/2}\right)} \underset{i\to\infty}{\sim} -\frac{K^{\#}}{K}\log i, \tag{4.3}$$

where we used the regular variation of $a(\cdot)$ for the last equivalence. In particular, combining (4.2), (4.3) and using the fact that $K^{\#}/K < 1$, we get

$$\sum_i \mathbf{P}\{\mathcal{E}_{22}(i)\} = \infty.$$

We now estimate $\mathbf{P}\{\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j)\}$ for i large enough and for $j > i$.

$$\begin{array}{rcl} \mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j) & \subset & \mathcal{E}_{24}(i) \cap \mathcal{E}_{24}(j) \\ & \subset & \mathcal{E}_{24}(i) \cap \left\{ \left(\theta_{f(t_i)/K}\mathbb{V}\right)^{\#}_{f(t_j)/K - f(t_i)/K} \leqslant e^j - e^{j/2} \right\}. \end{array}$$

Hence, from the independence and the stationarity of the increments of $\mathbb{V}$ (at integer times), combined with Proposition 2.7, for all i large enough (*i.e.* all j large enough), we get

$$\begin{array}{rcl} \mathbf{P}\{\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j)\} & \leqslant & \mathbf{P}\{\mathcal{E}_{24}(i)\}\mathbf{P}\{\mathbb{V}^{\#}_{f(t_j)/K - f(t_i)/K} \leqslant e^j - e^{j/2}\} \\ & \leqslant & c_{32} \dfrac{\mathbf{P}\{\mathcal{E}_{24}(i)\}\mathbf{P}\{\mathcal{E}_{24}(j)\}}{\mathbf{P}\{\mathbb{V}^{\#}_{f(t_i)/K} \leqslant e^j - e^{j/2}\}}. \end{array}$$

Using Lemma 2.4, we find, after a few lines of calculus, that for all i sufficiently large $\exp(j) - \exp(j/2) \geqslant a^{-1}(f(t_i)/K)$ whenever $j - i \geqslant \log i$. Thus,

$$\mathbf{P}\{\mathbb{V}^{\#}_{f(t_i)/K} \leqslant e^j - e^{j/2}\} \geqslant \mathbf{P}\Big\{\frac{\mathbb{V}^{\#}_{f(t_i)/K}}{a\left(f(t_i)/K\right)} \leqslant 1\Big\}.$$

Since the r.h.s. of the last inequality converges to $\mathbf{P}\{\mathbb{S}^{\#}_1 \leqslant 1\} \neq 0$ as i goes to infinity, we deduce that for all i large enough and all $j - i \geqslant \log i$,

$$\mathbf{P}\{\mathbb{V}^{\#}_{f(t_i)/K} \leqslant e^j - e^{j/2}\} \geqslant c_{33} > 0.$$

Finally, for all i large enough and for all $j \geqslant i$,

$$\mathbf{P}\{\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j)\} \leqslant \begin{cases} \mathbf{P}\{\mathcal{E}_{22}(i)\}, & \text{if } 0 \leqslant j - i < \log i, \\ c_{34}\mathbf{P}\{\mathcal{E}_{22}(i)\}\mathbf{P}\{\mathcal{E}_{24}(j)\}, & \text{if } j - i \geqslant \log i. \end{cases} \tag{4.4}$$

Combining (4.2),(4.3) and (4.4), we see that

$$\liminf_{n\to\infty} \sum_{i,j \leqslant n} \mathbf{P}\{\mathcal{E}_{22}(i) \cap \mathcal{E}_{22}(j)\} \Big/ \Big(\sum_{i \leqslant n} \mathbf{P}\{\mathcal{E}_{22}(i)\}\Big)^2 \leqslant c_{35}.$$

Thus, the Borel-Cantelli Lemma of [KS64] yields $\mathbf{P}\{\mathcal{E}_{22}(i) \text{ i.o.}\} > 1/c_{35}$. From a classical 0-1 argument (compare with [HS98a], p1511 for details), we conclude that $\mathbf{P}\{\mathcal{E}_{22}(i) \text{ i.o.}\} = 1$. Hence, with probability 1,

$$\limsup_{t\to\infty} \frac{\overline{X}_t}{f(t)} \geqslant \frac{1}{K^{\#}}.$$

Moreover, the value of $K^{\#}$ when the limiting process $\mathbb{S}$ is completely asymmetric was calculated in Corollary 2.10. ■

4.2 Proof of Theorem 1.3

Lemma 4.3

Let $\rho > 0$, for all t large enough (depending on ρ) and all $1 \leqslant \lambda \leqslant (\log\log t)^{\rho}$, we have

$$\mathbf{P}\Big\{\overline{X_t} < \frac{a^{-1}(\log t)}{\lambda}\Big\} \leqslant c_{36}\frac{b^{-1}(a^{-1}(\log t)/\lambda)}{b^{-1}(a^{-1}(\log t))}.$$

Proof. With the notation $v \stackrel{\text{def}}{=} a^{-1}(\log t)/\lambda$, the bounds on λ give

$$\frac{a^{-1}(\log t)}{(\log\log t)^{\rho}} \leqslant v \leqslant a^{-1}(\log t).$$

We assume that t is very large, hence v is also large. From (3.4) combined with Proposition 3.1 and Proposition 3.2, we deduce that

$$\begin{aligned}\mathbf{P}\{\overline{X}_t < v\} &\leqslant \mathbf{P}\Big\{I_1(v) \geqslant \frac{t}{2}\Big\} + \mathbf{P}\Big\{I_2(v) \geqslant \frac{t}{2}\Big\} \\ &\leqslant \mathbf{P}\Big\{\mathbb{V}_v^{\#} \geqslant \log\frac{t}{2} - (\log v)^4\Big\} + \mathbf{P}\Big\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + (\log v)^4) \geqslant \log\frac{t}{2}\Big\} + c_{37}e^{-(\log v)^2}.\end{aligned}$$

Remind that $b^{-1}(\cdot)$ is regularly varying with index $q < 1$. Therefore, using Corollary 2.16 and Lemma 2.4,

$$\begin{aligned}\mathbf{P}\Big\{\mathbb{V}_v^{\#} \geqslant \log\frac{t}{2} - (\log v)^4\Big\} &\leqslant \mathbf{P}\Big\{\mathbb{V}_v^{\#} \geqslant \frac{1}{2}\log t\Big\} \\ &\leqslant c_{38}\frac{v}{a^{-1}(\log t)} \\ &\leqslant c_{39}\frac{b^{-1}(v)}{b^{-1}(a^{-1}(\log t))}.\end{aligned}$$

It is also easy to check from the bounds on v and the regular variations of $a^{-1}(\cdot)$ and $b^{-1}(\cdot)$ that

$$e^{-(\log v)^2} \leqslant \frac{b^{-1}(v)}{b^{-1}\left(a^{-1}(\log t)\right)}.$$

We still have to prove a similar bound for $\mathbf{P}\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + (\log v)^4) \geqslant \log(t/2)\}$. Notice that for any $y > x > 0$, $\{\widetilde{U}_{\mathbb{V}}(x) \geqslant y\} = \widetilde{\Lambda}'(x, y-x)$. Hence, using Proposition 2.13 and the independence of $(\mathbb{V}_x)_{x\geqslant 0}$ and $(\mathbb{V}_{-x})_{x\geqslant 0}$, we get

$$\mathbf{P}\Big\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v+(\log v)^4)\geqslant \log\frac{t}{2}\Big\} \leqslant c_{40}\mathbf{E}\left[\frac{b^{-1}(a^{-1}(\overline{\mathbb{V}}_v+(\log v)^4))}{b^{-1}(a^{-1}(\log\frac{t}{2}))}\right]$$
$$\leqslant c_{40}\frac{b^{-1}(v)}{b^{-1}(a^{-1}(\log\frac{t}{2}))}\mathbf{E}\left[\frac{b^{-1}(a^{-1}(\overline{\mathbb{V}}_v+(\log v)^4))}{b^{-1}(a^{-1}(a(v)))}\right]. \quad (4.5)$$

Pick $\varepsilon>0$, we now use Lemma 2.4 for the regularly varying function $b^{-1}(a^{-1}(\cdot))$ to verify that (4.5) is smaller than

$$c_{41,\varepsilon}\frac{b^{-1}(v)}{b^{-1}(a^{-1}(\log\frac{t}{2}))}\mathbf{E}\left[\left(\frac{\overline{\mathbb{V}}_v+(\log v)^4}{a(v)}\right)^{\alpha q+\varepsilon}+1\right].$$

Finally, since $q<1$, we can choose ε small enough such that $\alpha q+\varepsilon<\alpha$, therefore Corollary 2.17 implies

$$\mathbf{E}\left[\left(\frac{\overline{\mathbb{V}}_v+(\log v)^4}{a(v)}\right)^{\alpha q+\varepsilon}\right]\leqslant \mathbf{E}\left[\left(\frac{\overline{\mathbb{V}}_v}{a(v)}+1\right)^{\alpha q+\varepsilon}\right]\leqslant c_{42,\varepsilon}.$$

We conclude the proof by noticing that $b^{-1}(a^{-1}(\log\frac{t}{2}))\sim b^{-1}(a^{-1}(\log t))$. ■

Lemma 4.4

Let $\rho>0$, for all t large enough (depending on ρ) and for all $1\leqslant\lambda\leqslant(\log\log t)^{\rho}$, we have

$$\left\{\overline{X}_t<\frac{a^{-1}(\log t)}{\lambda}\right\}\supset\left\{\widetilde{U}_{\mathbb{V}}(a(v))\geqslant \log t\right\}\cap\{\overline{\mathbb{V}}_{v/2}\geqslant 2a(v)\}\cap\mathcal{E}_9(v),$$

with $v\overset{def}{=}a^{-1}(\log t)/\lambda$, and where $\mathcal{E}_9(v)$ is the event defined in Proposition 3.2 and satisfies

$$\mathbf{P}\{\mathcal{E}_9(v)^c\}\leqslant c_{26}e^{-(\log v)^2}.$$

Proof. Recall that Relation (3.4) gives $\{\overline{X}_t<v\}=\{I_1(v)+I_2(v)>t\}$ and notice that $I_1(v)>0$ for all $v>0$, thus, $\{\overline{X}_t<v\}\supset\{I_2(v)\geqslant t\}$. We now use Proposition 3.2 to see that for all t large enough (*i.e.* v large enough), the event $\{\overline{X}_t<v\}$ contains

$$\left\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_{v-\frac12}-(\log v)^4)\geqslant \log t\right\}\cap\left\{\overline{\mathbb{V}}_{v-\frac12}>(\log v)^4\right\}\cap\mathcal{E}_9(v)$$
$$\supset\left\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_{v/2}-a(v))\geqslant \log t\right\}\cap\left\{\overline{\mathbb{V}}_{v/2}\geqslant 2a(v)\right\}\cap\mathcal{E}_9(v)$$
$$\supset\left\{\widetilde{U}_{\mathbb{V}}(a(v))\geqslant \log t\right\}\cap\left\{\overline{\mathbb{V}}_{v/2}\geqslant 2a(v)\right\}\cap\mathcal{E}_9(v),$$

where we used the fact that $x\mapsto\widetilde{U}_{\mathbb{V}}(x)$ is a non-decreasing function and the trivial inequalities $\overline{\mathbb{V}}_{v/2}\leqslant\mathbb{V}_{v-1/2}$ and $(\log v)^4\leqslant a(v)$ which hold for all large enough v's. ■

Proof of Theorem 1.3. For any positive non-decreasing function f, recall that

$$J(f) = \int^{\infty} \frac{b^{-1}(a^{-1}(\log t)/f(t))dt}{b^{-1}(a^{-1}(\log t))t\log t}$$

(we do not specify the lower bound for the integral since we are only concerned with the convergence of $J(f)$ at infinity).

Let us first prove the theorem when $J(f) < \infty$. Since f is non-decreasing, it is clear that $f(t) \to \infty$ as $t \to \infty$. Thus, $f(t) \geqslant e^{2\alpha}$ for all t large enough. Let $f_0(t) \stackrel{\text{def}}{=} (\log\log t)^{2/q}$. Since $J(f_0) < \infty$, we may assume without loss of generality that

$$f(t) \leqslant f_0(t) = (\log\log t)^{2/q} \quad \text{for all large } t \tag{4.6}$$

(compare with the argument given in the beginning of the proof of Theorem 1 in [Erd42]). Let us set $t_i \stackrel{\text{def}}{=} \exp(\exp i)$. Since $a^{-1}(\cdot)$ is regularly varying with index α, for all i large enough, we have

$$a^{-1}(\log t_{i+1}) \leqslant e^{2\alpha} a^{-1}(\log t_i). \tag{4.7}$$

Hence, Lemma 4.3 yields, i still being very large,

$$\begin{aligned}
\mathbf{P}\Big\{\overline{X}_{t_i} < \frac{a^{-1}(\log t_{i+1})}{f(t_i)}\Big\} &\leqslant \mathbf{P}\Big\{\overline{X}_{t_i} < \frac{a^{-1}(\log t_i)}{e^{-2\alpha}f(t_i)}\Big\} \\
&\leqslant c_{36}\frac{b^{-1}(e^{2\alpha}a^{-1}(\log t_i)/f(t_i))}{b^{-1}(a^{-1}(\log t_i))} \\
&\leqslant c_{43}\frac{b^{-1}(a^{-1}(\log t_{i-1})/f(t_i))}{b^{-1}(a^{-1}(\log t_i))} \\
&\leqslant c_{43}\int_{t_{i-1}}^{t_i} \frac{b^{-1}(a^{-1}(\log t)/f(t))dt}{b^{-1}(a^{-1}(\log t))t\log t},
\end{aligned}$$

where we again used (4.7) and the regular variation of b^{-1} for the third inequality and the monotonicity of a^{-1},b^{-1} and f for the last inequality. Since $J(f) < \infty$, we conclude that $\sum_i \mathbf{P}\{\overline{X}_{t_i} < a^{-1}(\log t_{i+1})/f(t_i)\} < \infty$ and Borel-Cantelli's Lemma implies that, almost surely,

$$\overline{X}_{t_i} \geqslant \frac{a^{-1}(\log t_{i+1})}{f(t_i)} \quad \text{for all } i \text{ large enough.}$$

But for $t_i \leqslant t \leqslant t_{i+1}$, we have $a^{-1}(\log t_{i+1})/f(t_i) > a^{-1}(\log t)/f(t)$ and $\overline{X}_t \geqslant \overline{X}_{t_i}$, therefore, with probability 1,

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)}\overline{X}_t \geqslant 1 \text{ a.s.} \tag{4.8}$$

Changing f to Cf for any $C > 0$ does not alter the convergence of $J(f)$. Thus, the liminf in (4.8) is in fact infinite.

We now prove the second part of the theorem. Let f be a positive, non-decreasing function such that $J(f) = \infty$. Again, we may without loss of generality assume that (4.6) holds (compare with the argument given in the proof of Theorem 3 in [Erd42]). Moreover, the theorem is straightforward for any bounded function f provided that we prove the theorem for at least one function h going to infinity with $J(h)=\infty$ (for example $h(t)=(\log\log t)^{1/(2q)}$). Thus, we now also assume that $f(t)\to\infty$ as $t\to\infty$. We use the notation $v_i \stackrel{\text{def}}{=} a^{-1}(\log t_i)/f(t_i)$. Our assumptions on f yield the following estimates:

$$\frac{a^{-1}(\log t_i)}{(\log\log t_i)^{2/q}} \leqslant v_i \leqslant a^{-1}(\log t_i) \text{ for } i \text{ large enough,} \tag{4.9}$$

and

$$\lim_{i\to\infty} v_i = \infty, \qquad \lim_{i\to\infty} \frac{v_i}{a^{-1}(\log t_i)} = 0. \tag{4.10}$$

From now on, we assume that i is very large. Using Lemma 4.4, we get

$$\left\{ \overline{X}_{t_i} \leqslant \frac{a^{-1}(\log t_i)}{f(t_i)} \right\} \supset \mathcal{E}_9(v_i)\cap\mathcal{E}_{26}(i),$$

where $\mathcal{E}_{26}(i) \stackrel{\text{def}}{=} \mathcal{E}_{27}(i)\cap\mathcal{E}_{28}(i)$ with

$$\begin{aligned} \mathcal{E}_{27}(i) &\stackrel{\text{def}}{=} \big\{\widetilde{U}_{\mathbb{V}}(a(v_i)) \geqslant \log(t_i)\big\}, \\ \mathcal{E}_{28}(i) &\stackrel{\text{def}}{=} \big\{\overline{\mathbb{V}}_{v_i/2} \geqslant 2a(v_i)\big\}. \end{aligned}$$

Since $\mathbf{P}\{\mathcal{E}_9(v_i)^c\} \leqslant c_{26}\exp(-\log^2 v_i)$, we deduce from (4.9) that

$$\sum_i \mathbf{P}\{\mathcal{E}_9(v_i)^c\} < \infty.$$

Thus, it only remains to prove that $\mathbf{P}\{\mathcal{E}_{26}(i) \text{ i.o.}\}=1$. Since $v_i\to\infty$ as $i\to\infty$, results of Section 2.1 imply that

$$\lim_{i\to\infty} \mathbf{P}\{\mathcal{E}_{28}(i)\} = \mathbf{P}\{\overline{\mathbb{S}}_{1/2} \geqslant 2\} > 0.$$

Therefore, the independence of $\mathcal{E}_{27}(i)$ and $\mathcal{E}_{28}(i)$ yields

$$c_{43}\mathbf{P}\{\mathcal{E}_{27}(i)\} \leqslant \mathbf{P}\{\mathcal{E}_{26}(i)\} \leqslant \mathbf{P}\{\mathcal{E}_{27}(i)\}. \tag{4.11}$$

Recall that $\big\{\widetilde{U}_{\mathbb{V}}(a(v_i)) \geqslant \log(t_i)\big\} = \widetilde{\Lambda}'\big(a(v_i),\log(t_i)-a(v_i)\big)$. Keeping in mind (4.10), we can estimate $\mathbf{P}\{\mathcal{E}_{27}(i)\}$ using Proposition 2.13:

$$c_{44}\frac{b^{-1}(v_i)}{b^{-1}(a^{-1}(\log t_i))} \leqslant \mathbf{P}\{\mathcal{E}_{27}(i)\} \leqslant c_{45}\frac{b^{-1}(v_i)}{b^{-1}(a^{-1}(\log t_i))}. \tag{4.12}$$

Combining (4.11), (4.12) and the assumption that $J(f)=\infty$, we obtain

$$\sum_i \mathbf{P}\{\mathcal{E}_{26}(i)\} = \infty.$$

We now estimate $\mathbf{P}\{\mathcal{E}_{26}(i)\cap\mathcal{E}_{26}(j)\}$. Let $g(i) \stackrel{\text{def}}{=} \log(t_i)-a(v_i)$. It is easy to check from (4.10) that g is ultimately increasing. Let us pick $j>i$. We can rewrite

$$\mathcal{E}_{27}(i)\cap\mathcal{E}_{27}(j) = \widetilde{\Lambda}'\big(a(v_i),g(i)\big)\cap\widetilde{\Lambda}'\big(a(v_i),g(j)\big).$$

There are two cases (which are not disjoint):

1. $(\mathbb{V}_{-n})_{n\geqslant 0}$ hits $(-\infty, -g(j)]$ before hitting $[a(v_i), \infty)$. Using Proposition 2.13, we see that the probability of this case is less than $c_{46} b^{-1}(v_i)/b^{-1}(a^{-1}(\log t_j))$.

2. $(\mathbb{V}_{-n})_{n\geqslant 0}$ hits $(-\infty, -g(i)]$ before hitting $[a(v_i), \infty)$ (*i.e.* $\mathcal{E}_{27}(i)$ happens) and also the shifted random walk $(\mathbb{V}_{-\widetilde{\sigma}_{\mathbb{V}}(a(v_i))-n})_{n\geqslant 0}$ hits $(-\infty, -g(j)]$ before hitting $[a(v_j), +\infty)$ (the probability of this event is clearly smaller than $\mathbf{P}\{\mathcal{E}_{27}(j)\}$). Using the Markov property for the random walk $(\mathbb{V}_{-n})_{n\geqslant 0}$ we conclude that the probability of this case is smaller than $\mathbf{P}\{\mathcal{E}_{27}(i)\}\mathbf{P}\{\mathcal{E}_{27}(j)\}$.

Combining 1. and 2. we deduce that $\mathbf{P}\{\mathcal{E}_{27}(i)\cap\mathcal{E}_{27}(j)\}$ is smaller than

$$\begin{aligned}&\mathbf{P}\{\mathcal{E}_{27}(i)\}\mathbf{P}\{\mathcal{E}_{27}(j)\} + c_{46}\frac{b^{-1}(v_i)}{b^{-1}(a^{-1}(\log t_j))}\\ &\qquad\leqslant \mathbf{P}\{\mathcal{E}_{27}(i)\}\mathbf{P}\{\mathcal{E}_{27}(j)\} + \frac{c_{46}}{c_{44}}\mathbf{P}\{\mathcal{E}_{27}(i)\}\frac{b^{-1}(a^{-1}(\log t_i))}{b^{-1}(a^{-1}(\log t_j))},\end{aligned}$$

where we used (4.12) for the second inequality. Finally, using Lemma 2.4 and (4.11), we conclude that for all i large enough and all $j > i$,

$$\begin{aligned}\mathbf{P}\{\mathcal{E}_{26}(i)\cap\mathcal{E}_{26}(j)\} &\leqslant \mathbf{P}\{\mathcal{E}_{27}(i)\cap\mathcal{E}_{27}(j)\}\\ &\leqslant c_{47}\left(\mathbf{P}\{\mathcal{E}_{26}(i)\}\mathbf{P}\{\mathcal{E}_{26}(j)\} + \mathbf{P}\{\mathcal{E}_{26}(i)\}e^{-c_{48}(j-i)}\right),\end{aligned}$$

hence

$$\liminf_{n\to\infty} \sum_{i,j\leqslant n} \mathbf{P}\{\mathcal{E}_{26}(i)\cap\mathcal{E}_{26}(j)\}\Big/\Big(\sum_{i\leqslant n}\mathbf{P}\{\mathcal{E}_{26}(i)\}\Big)^2 \leqslant c_{47}.$$

As in the proof of Theorem 1.2, we apply the Borel-Cantelli Lemma of [KS64] and a standard 0-1 argument to conclude that $\mathbf{P}\{\mathcal{E}_{26}(i) \text{ i.o.}\} = 1$. Since this result still holds on changing f to Cf for any $C > 0$, we have proved that, almost surely,

$$\liminf_{t\to\infty}\frac{\overline{X}_t}{f(t)} = 0.$$

■

4.3 Proof of Theorem 1.4

The proof is again based on two lemmas.

Lemma 4.5

Let $\rho > 0$. For all t large enough and all $1 \leqslant \lambda \leqslant (\log\log t)^\rho$ we have

$$\mathbf{P}\{X_t^* < \frac{a^{-1}(\log t)}{\lambda}\} \leqslant \frac{c_{49}}{\lambda^2}.$$

Proof. We use the notation $v \stackrel{\text{def}}{=} a^{-1}(\log t)/\lambda$. Let also $Y \stackrel{\text{def}}{=} -X$, it is clear from a symmetry argument that Y is a diffusion in the "reversed" environment $\mathbb{W} \stackrel{\text{def}}{=} (\mathbb{V}_{-x}, x \in \mathbb{R})$. Let us notice that

$$\begin{aligned}\mathbf{P}\{X_t^* < v\} &= \mathbf{P}\{\overline{X}_t < v, \overline{Y}_t < v\} \\ &\leqslant \mathbf{P}\{\overline{X}_t < v, \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}\} + \mathbf{P}\{\overline{Y}_t < v\ , \overline{\mathbb{W}}_v \leqslant \overline{\mathbb{W}}_{-v}\}.\end{aligned}$$

Let us also note that all the assumptions we made on $\mathbb{V}$ also hold for $\mathbb{W}$. Hence, we only need to prove the upper bound for the first member on the r.h.s. of the last inequality. According to (3.4), we have

$$\big\{\overline{X}_t < v\big\} = \big\{I_1(v) + I_2(v) > t\big\},$$

thus,

$$\begin{aligned}\mathbf{P}\big\{\overline{X}_t < v, \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}\big\} &\leqslant \mathbf{P}\Big\{\frac{1}{4}\log t \leqslant \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}\Big\} &(4.13)\\ &\quad + \mathbf{P}\Big\{I_1(v) \geqslant \frac{t}{2}, \overline{\mathbb{V}}_v \leqslant \frac{\log t}{4}\Big\} &(4.14)\\ &\quad + \mathbf{P}\Big\{I_2(v) \geqslant \frac{t}{2}, \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}, \overline{\mathbb{V}}_v \leqslant \frac{\log t}{4}\Big\}. &(4.15)\end{aligned}$$

We deal with each term separately. First, using independence of $(\mathbb{V}_x)_{x\geqslant 0}$ and $(\mathbb{V}_{-x})_{x\geqslant 0}$ we see that the right term of (4.13) is smaller than

$$\mathbf{P}\Big\{\overline{\mathbb{V}}_v \geqslant \frac{1}{4}\log t\Big\}\mathbf{P}\Big\{\overline{\mathbb{V}}_{-v} \geqslant \frac{1}{4}\log t\Big\} \leqslant \frac{c_{49}}{\lambda^2},$$

where we used Corollary 2.16 for the last inequality. We now turn our attention to (4.14). Using Proposition 3.1, we check that for t large enough, this probability is smaller than

$$\mathbf{P}\Big\{\mathbb{V}_v^{\#} \geqslant \log\frac{t}{2} - \log^4 v\ , \overline{\mathbb{V}}_v \leqslant \frac{1}{4}\log t\Big\} + c_{25}e^{-\log^2 v}.$$

For t large enough, using the Markov property, we also have

$$\begin{aligned}\mathbf{P}\Big\{\mathbb{V}_v^{\#} \geqslant \log\frac{t}{2} - \log^4 v, \overline{\mathbb{V}}_v \leqslant \frac{\log t}{4}\Big\} &\leqslant \mathbf{P}\Big\{\mathbb{V}_v^{\#} \geqslant \frac{\log t}{2}, \overline{\mathbb{V}}_v \leqslant \frac{\log t}{4}\Big\}\\ &\leqslant \mathbf{P}\Big\{\sigma_{\mathbb{V}}\Big(-\frac{\log t}{4}\Big) \leqslant v, \Big(\theta_{\sigma_{\mathbb{V}}(-\frac{\log t}{4})}\mathbb{V}\Big)_v^{\#} \geqslant \frac{\log t}{2}\Big\}\\ &\leqslant \mathbf{P}\Big\{\underline{\mathbb{V}}_v \leqslant -\frac{\log t}{4}\Big\}\mathbf{P}\Big\{\mathbb{V}_v^{\#} \geqslant \frac{\log t}{2}\Big\}\\ &\leqslant \frac{c_{50}}{\lambda^2},\end{aligned}$$

where we again used Corollary 2.16 for the last line. It is also clear from our bounds on λ that $e^{-\log^2 v} \leqslant 1/\lambda^2$ for all t large enough. This gives the desired bound for (4.14). It remains to prove a similar inequality for (4.15). We first use Proposition 3.2 to see that, for all t large enough, (4.15) is smaller than

$$\mathbf{P}\Big\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v + \log^4 v) \geqslant \log\frac{t}{2}, \overline{\mathbb{V}}_v \leqslant \overline{\mathbb{V}}_{-v}, \overline{\mathbb{V}}_v \leqslant \frac{1}{4}\log t\Big\} + c_{26}e^{-\log^2 v}.$$

We can rewrite:

$$\begin{aligned}
&\Big\{\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v+\log^4 v)\geqslant \log\frac{t}{2},\ \overline{\mathbb{V}}_v\leqslant \overline{\mathbb{V}}_{-v},\ \overline{\mathbb{V}}_v\leqslant \frac{1}{4}\log t\Big\}\\
&\quad=\Big\{\widetilde{\sigma}_{\mathbb{V}}\Big(\overline{\mathbb{V}}_v+\log^4 v-\log\frac{t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right),\ \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v,\ \overline{\mathbb{V}}_v\leqslant\frac{1}{4}\log t\Big\}\\
&\quad\subset\Big\{\widetilde{\sigma}_{\mathbb{V}}\Big(-\frac{\log t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right),\ \widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\Big\}\\
&\quad\subset\Big\{\widetilde{\sigma}_{\mathbb{V}}\Big(-\frac{\log t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\Big\}\cup\Big\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)<\widetilde{\sigma}_{\mathbb{V}}\Big(-\frac{\log t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right)\Big\}.
\end{aligned}$$

Notice that on the event $\{\widetilde{\sigma}_{\mathbb{V}}(-(\log t)/2)<\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\}$, the process $(\mathbb{V}_{-x})_{x\,\geqslant\,0}$ hits $(-\infty,-(\log t/2)]$ before time v, and then hits $[0,\infty)$, again before time v. The Markov property with the stopping time $\widetilde{\sigma}_{\mathbb{V}}(-(\log t)/2)$ and Corollary 2.16 yield

$$\mathbf{P}\Big\{\widetilde{\sigma}_{\mathbb{V}}\Big(-\frac{\log t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\leqslant v\Big\}\leqslant\mathbf{P}\Big\{\underline{\mathbb{V}}_{-v}\leqslant-\frac{\log t}{2}\Big\}\mathbf{P}\Big\{\overline{\mathbb{V}}_{-v}\geqslant\frac{\log t}{2}\Big\}\leqslant\frac{c_{51}}{\lambda^2}.$$

We also check from the Markov property of $(\mathbb{V}_{-x})_{x\,\geqslant\,0}$ applied to the stopping time $\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)$ that the probability of $\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)<\widetilde{\sigma}_{\mathbb{V}}(-(\log t)/2)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right)\}$ is smaller than the probability that the random walk $(\mathbb{V}_{-x})_{x\,\geqslant\,0}$ hits $(-\infty,-(\log t)/2]$ before it hits $[\log^4 v,\infty)$. Using the estimate for the exit problem (Proposition 2.13) and the regular variation of $b^{-1}(a^{-1}(\cdot))$, for t large enough,

$$\mathbf{P}\Big\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v)<\widetilde{\sigma}_{\mathbb{V}}\Big(-\frac{\log t}{2}\Big)<\widetilde{\sigma}_{\mathbb{V}}\left(\overline{\mathbb{V}}_v+\log^4 v\right)\Big\}\leqslant c_{52}\frac{b^{-1}\left(a^{-1}\left((\log v)^4\right)\right)}{b^{-1}\left(a^{-1}\left(\frac{\log t}{2}\right)\right)}\leqslant\frac{1}{\lambda^2},$$

so we conclude that (4.15) is indeed smaller than c_{53}/λ^2. ■

Lemma 4.6

Let $\rho>0$. For all t large enough and all $1\leqslant\lambda\leqslant(\log\log t)^{\rho}$, we have

$$\Big\{X_t^*<\frac{a^{-1}(\log t)}{\lambda}\Big\}\supset\Big\{\mathbb{V}_{v-\frac{1}{2}}\geqslant 2\log t,\ \mathbb{V}_{-v+\frac{1}{2}}\geqslant 2\log t\Big\}\cap\mathcal{E}_{29}(v),$$

with the notation $v=a^{-1}(\log t)/\lambda$ and where $\mathcal{E}_{29}(v)^c$ is a measurable set such that

$$\mathbf{P}\{\mathcal{E}_{29}(v)^c\}\leqslant c_{54}e^{-\log^2 v}.$$

Proof. Recall that X is given by the formula

$$X_t=\mathbb{A}^{-1}\left(B_{\mathbb{T}^{-1}(t)}\right),$$

where B is a Brownian motion independent of $\mathbb{V}$. Let $\widetilde{B}\overset{\text{def}}{=}-B$ and let $\widetilde{L}$ denote the bi-continuous version of the local time process of $\widetilde{B}$. Recall also that $\mathbb{W}$ stands for the reversed process $(\mathbb{V}_{-x},x\in\mathbb{R})$. At the beginning of Section 3, we stated that

$$\sigma_X(v)=I_1(v)+I_2(v)\quad\text{for all } v>0. \tag{4.16}$$

It is easily checked, using similar arguments, that

$$\sigma_X(-v) = \widetilde{I}_1(v) + \widetilde{I}_2(v) \quad \text{for all } v > 0, \tag{4.17}$$

with

$$\begin{aligned}
\widetilde{I}_1(v) &\stackrel{\text{def}}{=} \int_0^v e^{-\mathbb{W}_y} \widetilde{L}\big(\sigma_{\widetilde{B}}(\widetilde{\mathbb{A}}(v)), \widetilde{\mathbb{A}}(y)\big) dy, \\
\widetilde{I}_2(v) &\stackrel{\text{def}}{=} \int_0^\infty e^{-\mathbb{W}_{-y}} \widetilde{L}\big(\sigma_{\widetilde{B}}(\widetilde{\mathbb{A}}(v)), \widetilde{\mathbb{A}}(-y)\big) dy,
\end{aligned}$$

and where

$$\widetilde{\mathbb{A}}(x) \stackrel{\text{def}}{=} \int_0^x e^{\mathbb{W}_y} dy.$$

Thus, $\widetilde{I}_1$ and $\widetilde{I}_2$ are given by the same formulas as I_1 and I_2 by simply changing the process B to $\widetilde{B}$ and $\mathbb{V}$ to $\mathbb{W}$. Notice that $\widetilde{B}$ is again a Brownian motion independent of $\mathbb{W}$ and that $\mathbb{W}$ fulfills all the assumptions we made on $\mathbb{V}$. Therefore, Propositions 3.1 and 3.2 also hold for $\widetilde{I}_1$ and $\widetilde{I}_2$ with $\mathbb{W}$ instead of $\mathbb{V}$ (with different events and different values of the constants). In particular, for all v large enough,

$$\log \widetilde{I}_1(v) \geqslant \mathbb{V}^{\#}_{-v+1/2} - (\log v)^4 \quad \text{on } \mathcal{E}_{30}(v), \tag{4.18}$$

where $\mathcal{E}_{30}(v)$ is a measurable set such that $\mathbf{P}\{\mathcal{E}_{30}^c(v)\} \leqslant c_{55}\exp(-\log^2 v)$. In view of Proposition 3.1, we also have

$$\log I_1(v) \geqslant \mathbb{V}^{\#}_{v-1/2} - (\log v)^4 \quad \text{on } \mathcal{E}_8(v). \tag{4.19}$$

Let $\mathcal{E}_{29}(v) \stackrel{\text{def}}{=} \mathcal{E}_8(v) \cap \mathcal{E}_{30}(v)$, then $\mathbf{P}\{\mathcal{E}_{29}^c(v)\} \leqslant c_{54}\exp\big(-\log^2 v\big)$. Combining (4.16), (4.17), (4.18) and (4.19), we get

$$\begin{aligned}
\big\{X_t^* < v\big\} &= \big\{\sigma_X(v) > t\big\} \cap \big\{\sigma_X(-v) > t\big\} \\
&\supset \big\{I_1(v) > t\big\} \cap \big\{\widetilde{I}_1(v) > t\big\} \\
&\supset \big\{\mathbb{V}^{\#}_{v-1/2} > \log t + \log^4 v\big\} \cap \big\{\mathbb{V}^{\#}_{-v+1/2} > \log t + \log^4 v\big\} \cap \mathcal{E}_{29}(v) \\
&\supset \big\{\mathbb{V}^{\#}_{v-1/2} \geqslant 2\log t\big\} \cap \big\{\mathbb{V}^{\#}_{-v+1/2} \geqslant 2\log t\big\} \cap \mathcal{E}_{29}(v) \\
&\supset \big\{\mathbb{V}_{v-1/2} \geqslant 2\log t\big\} \cap \big\{\mathbb{V}_{-v+1/2} \geqslant 2\log t\big\} \cap \mathcal{E}_{29}(v).
\end{aligned}$$

■

Proof of Theorem 1.4. This theorem is now an easy consequence, using similar techniques as in the proof of Theorem 1.3 of the last two lemmas and of Proposition 2.15 when the limiting process has jumps of both signs. We feel free to omit it. ■

4.4 Proof of Theorem 1.5

Proposition 4.7

We have

$$\frac{1}{a(v)}\left(\log\sigma_X(v) - \mathbb{V}_v^{\#}\vee\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_v)\right) \xrightarrow[v\to\infty]{prob.} 0.$$

The proof of this proposition is very similar to that of Proposition 11.1 of [HS98a] using the estimates for I_1 and I_2 obtained in Propositions 3.1 and 3.2. We therefore skip the details.

Proof of Theorem 1.5. Let $\lambda > 0$ and let v be a large number,

$$\begin{aligned}\mathbf{P}\left(\frac{\overline{X}_v}{a^{-1}(\log v)} \geqslant \lambda\right) &= \mathbf{P}\left(\log\sigma_X(\lambda a^{-1}(\log v)) \leqslant \log v\right)\\ &= \mathbf{P}\left(\frac{\log\sigma_X(x)}{c(x)} \leqslant \frac{1}{\lambda^{1/\alpha}}\right),\end{aligned}$$

with the change of variable $x = \lambda a^{-1}(\log v)$ and where

$$c(x) \stackrel{\text{def}}{=} \lambda^{1/\alpha} a(x/\lambda) \underset{x\to\infty}{\sim} a(x). \tag{4.20}$$

Results of Section 2.1 ensure that $(\mathbb{V}_x^{\#}\vee\widetilde{U}_{\mathbb{V}}(\overline{\mathbb{V}}_x))/a(x)$ converges in law as $x\to\infty$ towards the random variable $\mathbb{S}_1^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_1)$ whose cumulative function is continuous. Hence, it follows from Proposition 4.7 and from (4.20) that

$$\lim_{v\to\infty}\mathbf{P}\left(\frac{\overline{X}_v}{a^{-1}(\log v)} \geqslant \lambda\right) = \mathbf{P}\left\{\mathbb{S}_1^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_1) \leqslant \frac{1}{\lambda^{1/\alpha}}\right\}.$$

Thus, we have proved the convergence in law of $\overline{X}_v/a^{-1}(\log v)$ towards the non-degenerate random variable $\Xi \stackrel{\text{def}}{=} (\mathbb{S}_1^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_1))^{-\alpha}$. We now calculate the Laplace transform of this law when $\mathbb{S}$ is completely asymmetric. Recall the notation $\tau_x^{\#}$ and τ_x defined in Section 2.3. Let r_1 be the stopping time:

$$r_1 \stackrel{\text{def}}{=} \inf\left(x \geqslant 0,\ (\mathbb{S}_{-t})_{t\,\geqslant\,0} \text{ hits } (-\infty,-(1-x)) \text{ before it hits } (x,\infty)\right).$$

From the scaling property of $\mathbb{S}$,

$$\begin{aligned}\mathbf{P}\{(\mathbb{S}_1^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_1))^{-\alpha} \leqslant \lambda\} &= \mathbf{P}\{\mathbb{S}_\lambda^{\#}\vee\widetilde{U}_{\mathbb{S}}(\overline{\mathbb{S}}_\lambda) \geqslant 1\}\\ &= \mathbf{P}\{\tau_1^{\#}\wedge\tau_{r_1} \leqslant \lambda\},\end{aligned}$$

therefore Ξ and $\tau_1^{\#}\wedge\tau_{r_1}$ have the same law. Let us first assume that $\mathbb{S}$ has no positive jumps and recall that $(-\mathbb{S}_{-t},\, t \geqslant 0)$ and $(\mathbb{S}_t,\, t \geqslant 0)$ have the same law. It follows from the well known solution of the exit problem for a completely asymmetric Levy process via its scale function W (*c.f.* [Ber96a] , p194) that

$$\begin{aligned}\mathbf{P}\{r_1 > x\} &= \mathbf{P}\{(\mathbb{S}_{-t})_{t\,\geqslant\,0} \text{ hits } (x,\infty) \text{ before it hits } (-\infty,-(1-x))\}\\ &= 1-\mathbf{P}\{(\mathbb{S}_t)_{t\,\geqslant\,0} \text{ hits } (1-x,\infty) \text{ before it hits } (-\infty,-x)\}\\ &= 1-\frac{W(x)}{W(1)},\end{aligned}$$

and it is known that in our case $W(x) = x^{\alpha-1}/\Gamma(\alpha)$. Hence, the density of r_1 is

$$\mathbf{P}\{r_1 \in dx\} = \frac{\alpha-1}{x^{2-\alpha}}dx \quad \text{for } x \in (0,1).$$

Using Corollary 2.11 and the independence of $(\mathbb{S}_t)_{t\geqslant 0}$ and $(\mathbb{S}_{-t})_{t\geqslant 0}$, for $q \geqslant 0$,

$$\begin{aligned}
\mathbf{E}\Big[e^{-q\tau_1^{\#}\wedge\tau_{r_1}}\Big] &= \int_0^1 \mathbf{E}\Big[e^{-q\tau_1^{\#}\wedge\tau_x}\Big]\frac{\alpha-1}{x^{2-\alpha}}dx \\
&= \frac{\alpha-1}{\mathcal{M}_\alpha(q)}\int_0^1 \frac{\mathcal{M}_\alpha(q(1-x)^\alpha)}{x^{2-\alpha}}dx \\
&= \frac{\alpha-1}{\mathcal{M}_\alpha(q)}\sum_{n=0}^{\infty}\frac{q^n}{\Gamma(1+\alpha n)}\int_0^1 \frac{(1-x)^{\alpha n}}{x^{2-\alpha}}dx,
\end{aligned}$$

but

$$\frac{1}{\Gamma(1+\alpha n)}\int_0^1 \frac{(1-x)^{\alpha n}}{x^{2-\alpha}}dx = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha(n+1))},$$

hence

$$\mathbf{E}\Big[e^{-q\tau_1^{\#}\wedge\tau_{r_1}}\Big] = \frac{\Gamma(\alpha)}{\mathcal{M}_\alpha(q)}\sum_{n=0}^{\infty}\frac{q^n}{\Gamma(\alpha(n+1))} = \Gamma(\alpha+1)\frac{\mathcal{M}'_\alpha(q)}{\mathcal{M}_\alpha(q)}.$$

We now assume that $\mathbb{S}$ has no negative jumps. As in the previous case, we can again calculate the density of r_1 from the scale function and we find

$$\mathbf{P}\{r_1 \in dx\} = \frac{\alpha-1}{(1-x)^{2-\alpha}}$$

Thus, using Corollary 2.11 we get

$$\begin{aligned}
\mathbf{E}\Big[e^{-q\tau_1^{\#}\wedge\tau_{r_1}}\Big] &= \int_0^1 \mathbf{E}\Big[e^{-q\tau_1^{\#}\wedge\tau_x}\Big]\frac{\alpha-1}{x^{2-\alpha}}dx \\
&= (\alpha-1)\int_0^1 \frac{\mathcal{M}_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx - \frac{\mathcal{M}'_\alpha(q)\alpha(\alpha-1)q}{\alpha q\mathcal{M}''_\alpha(q)+(\alpha-1)\mathcal{M}'_\alpha(q)}\int_0^1 \frac{x^{\alpha-1}\mathcal{M}'_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx.
\end{aligned}$$

We have already calculated the first integral:

$$\int_0^1 \frac{\mathcal{M}_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx = \int_0^1 \frac{\mathcal{M}_\alpha(q(1-y)^\alpha)}{y^{2-\alpha}}dy = \frac{\Gamma(\alpha+1)}{\alpha-1}\mathcal{M}'_\alpha(q).$$

As for the second integral,

$$\int_0^1 \frac{x^{\alpha-1}\mathcal{M}'_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx = \sum_{n=0}^{\infty}\frac{(n+1)q^n}{\Gamma(\alpha(n+1)+1)}\int_0^1 \frac{x^{\alpha(n+1)-1}}{(1-x)^{2-\alpha}}dx,$$

and it is known that

$$\int_0^1 \frac{x^{\alpha(n+1)-1}}{(1-x)^{2-\alpha}}dx = \frac{\Gamma(\alpha(n+1))\Gamma(\alpha-1)}{\Gamma(\alpha(n+2)-1)},$$

hence

$$\begin{aligned}\int_0^1 \frac{x^{\alpha-1}\mathcal{M}'_\alpha(qx^\alpha)}{(1-x)^{2-\alpha}}dx &= \frac{\Gamma(\alpha-1)}{\alpha}\sum_{n=0}^{\infty}\frac{q^n}{\Gamma(\alpha(n+2)-1)}\\ &= \Gamma(\alpha-1)\sum_{n=0}^{\infty}\frac{(n+2)(\alpha(n+2)-1)q^n}{\Gamma(\alpha(n+2)+1)}\\ &= \Gamma(\alpha-1)\left(\alpha\sum_{n=0}^{\infty}\frac{(n+1)(n+2)q^n}{\Gamma(\alpha(n+2)+1)}+(\alpha-1)\sum_{n=0}^{\infty}\frac{(n+2)q^n}{\Gamma(\alpha(n+2)+1)}\right)\\ &= \frac{\Gamma(\alpha-1)}{q}\left(q\alpha\mathcal{M}''_\alpha(q)+(\alpha-1)\mathcal{M}'_\alpha(q)-\frac{\alpha-1}{\Gamma(\alpha+1)}\right).\end{aligned}$$

Putting the pieces together, we conclude that

$$\mathbf{E}\left[e^{-q\tau_1^{\#}\wedge\tau_{r_1}}\right] = \frac{(\alpha-1)\mathcal{M}'_\alpha(q)}{\alpha q\mathcal{M}''_\alpha(q)+(\alpha-1)\mathcal{M}'_\alpha(q)}.$$

■

5 Comments

5.1 The case where $\mathbb{V}$ is a stable process

Throughout the paper, we assumed $\mathbb{V}$ to be a random walk in the domain of attraction of a stable process $\mathbb{S}$. Let us now assume that $\mathbb{V}$ itself is a strictly stable process (such that $|\mathbb{V}|$ is not a subordinator) and let us explain why Theorems 1.2 – 1.5 still hold in this case. It is clear that all the results dealing with the fluctuations of $\mathbb{V}$ remain unchanged. In fact, they even take a nicer form since we can now choose $a(x) = x^{1/\alpha}$ and $b(x) = x^{1/q}$. Notice also that we did not use the fact that $\mathbb{V}$ was a random walk in the proofs of the theorems in Section 4. Indeed, the only time we really used the assumption that $\mathbb{V}$ was flat on the intervals $(n, n+1)$, $n \in \mathbb{Z}$ was in the proofs of Propositions 3.1 and 3.2 because we needed to make sure that $\mathbb{V}$ spends "enough" time around its local extrema. Looking closely at those two proofs, we see that they will still hold if we can show that there exists a measurable event $\mathcal{E}_{31}(v)$ such that:

(a) there exists c_{56} such that $\mathbf{P}\{\mathcal{E}_{31}(v)^c\} \leqslant c_{56}\exp(-\log^2 v)$.

(b) On $\mathcal{E}_{31}(v)$, any path of $\mathbb{V}$ is such that for all $x \in [-\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v + \log^4 v), v]$, we have $|\mathbb{V}_y - \mathbb{V}_x| \leqslant 1$ either for all y in $[x, x+\exp(-\log^3 v)]$ or for all y in the interval $[x-\exp(-\log^3 v), x]$.

Let us quickly explain how we can construct such an event. Define the sequence of random variables $(\gamma_n)_{n\in\mathbb{Z}}$ by

$$\begin{cases}\gamma_0 \overset{\text{def}}{=} 0,\\ \gamma_{n+1} \overset{\text{def}}{=} \inf\{t > \gamma_n,\ |\mathbb{V}_t - \mathbb{V}_{\gamma_n}| \geqslant \frac{1}{2}\} \text{ for } n \geqslant 0,\\ \gamma_{-n-1} \overset{\text{def}}{=} \inf\{t < \gamma_{-n},\ |\mathbb{V}_t - \mathbb{V}_{\gamma_{-n}}| \geqslant \frac{1}{2}\} \text{ for } n \geqslant 0.\end{cases}$$

Define

$$\begin{aligned}
\mathcal{E}_{32}(v) &\stackrel{\text{def}}{=} \left\{\gamma_{i+1}-\gamma_i > 2e^{-\log^3 v} \text{ for all } -e^{\frac12\log^3 v} \leqslant i \leqslant e^{\frac12\log^3 v}\right\},\\
\mathcal{E}_{33}(v) &\stackrel{\text{def}}{=} \left\{\gamma_{-[e^{\frac12\log^3 v}]} > e^{\log^{5/2} v},\ \gamma_{[e^{\frac12\log^3 v}]} > e^{\log^{5/2} v}\right\},\\
\mathcal{E}_{34}(v) &\stackrel{\text{def}}{=} \left\{\widetilde{\sigma}_{\mathbb{V}}(\overline{\mathbb{V}}_v+\log^4 v) \leqslant e^{\log^{5/2} v}\right\},\\
\mathcal{E}_{31}(v) &\stackrel{\text{def}}{=} \mathcal{E}_{32}(v)\cap\mathcal{E}_{33}(v)\cap\mathcal{E}_{34}(v).
\end{aligned}$$

It is clear that (b) holds for $\mathcal{E}_{31}$. We now assume that v is very large. We have

$$\mathbf{P}\{\mathcal{E}_{32}(v)^c\} \leqslant 2e^{\frac12\log^3 v}\mathbf{P}\{\gamma_1 \leqslant 2e^{-\log^3 v}\} \leqslant c_{57}e^{-\frac12\log^3 v},$$

where we used the relation $\mathbf{P}\{\gamma_1 \leqslant x\} = \mathbf{P}\{\mathbb{V}^*_x \geqslant \frac12\}$ and Corollary 2.16 for the last inequality. Using Cramer's large deviation theorem, it is easy to check that $\mathbf{P}\{\mathcal{E}_{33}(v)^c\} \leqslant e^{-v}$ (in fact, we can obtain a much better bound). We also have $\mathbf{P}\{\mathcal{E}_{34}(v)^c\} \leqslant 3e^{-\log^2 v}$, compare with the proof of the inequality (3.8) for details. Thus, (a) holds.

5.2 Non-symmetric environments

Throughout the paper, to avoid even more complicated notations, we assumed that the processes $(\mathbb{V}_x,\, x \geqslant 0)$ and $(-\mathbb{V}_{-x},\, x \leqslant 0)$ have the same law. However it is easy to see that this assumption can be relaxed. Indeed, we may swap Assumption 1.1 for the following assumption.

Assumption 5.1

$(\mathbb{V}_n)_{n\geqslant 0}$ and $(\mathbb{V}_{-n})_{n\geqslant 0}$ are independent random walks and there exists a positive sequence $(a_n)_{n\geqslant 0}$ such that

$$\frac{\mathbb{V}_n}{a_n} \xrightarrow[n\to\infty]{law} \mathbb{S}^1 \text{ and } \frac{-\mathbb{V}_{-n}}{a_n} \xrightarrow[n\to\infty]{law} \mathbb{S}^2,$$

where $\mathbb{S}^1$ and $\mathbb{S}^2$ are random variables whose laws are strictly stable with respective parameters (α, p_1) and (α, p_2) and whose densities are everywhere positive on $\mathbb{R}$.

It is crucial to assume that the norming sequence (a_n) may be chosen to be the same for both random walks (in order to keep the results of functional convergence of Section 2.1) but the positivity parameters p_1 and p_2 may differ. Theorem 1.2-1.5 must be adapted in consequences. For example, Theorem 1.2 now takes the form:

Theorem 5.2

Under the annealed probability $\mathbf{P}$, almost surely,

$$\limsup_{t\to\infty} \frac{X_t}{a^{-1}(\log t)\log\log\log t} = \frac{1}{K_1^{\#}},$$

where $K_1^\#$ depends only on $\mathbb{S}^1$ and is given by

$$K_1^\# \stackrel{\text{def}}{=} -\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}\Big\{ \sup_{0\,\leqslant\, u\,\leqslant\, v\,\leqslant\, t} \big(\mathbb{S}_v^1 - \mathbb{S}_u^1\big) \leqslant 1 \Big\}.$$

Furthermore, when $\mathbb{S}^1$ is completely asymmetric, $K_1^\#$ is given by

$$K_1^\# = \begin{cases} \rho_1(\alpha) & \textit{when } \mathbb{S}^1 \textit{ has no positive jumps,} \\ \rho_2(\alpha) & \textit{when } \mathbb{S}^1 \textit{ has no negative jumps.} \end{cases}$$

Now let $(\mathbf{T}_n)$ stand for the sequence of strictly ascending ladder indices of the random walk $(\mathbb{V}_{-x})_{x\,\geqslant\,0}$,

$$\begin{cases} \mathbf{T}_0 \stackrel{\text{def}}{=} 0, \\ \mathbf{T}_{n+1} \stackrel{\text{def}}{=} \min\left(k > \mathbf{T}_n,\, \mathbb{V}_{-k} > \mathbb{V}_{-\mathbf{T}_n}\right). \end{cases}$$

Hence, $\mathbf{T}_1$ is in the domain of attraction of a positive stable law with index p_2 and we choose $b(\cdot)$ to be a continuous positive increasing function such that $(b(n))_{n\,\geqslant\,1}$ is a norming sequence for $\mathbf{T}_1$. Theorem 1.3 now takes the form:

Theorem 5.3

For any positive, non-decreasing function f define

$$L(f) \stackrel{\text{def}}{=} \int^\infty \frac{b^{-1}\left(a^{-1}(\log t)/f(t)\right) dt}{b^{-1}\left(a^{-1}(\log t)\right) t \log t}.$$

We have, almost surely,

$$\liminf_{t\to\infty} \frac{f(t)}{a^{-1}(\log t)} \sup_{s\,\leqslant\, t} X_s = \begin{cases} 0 \\ \infty \end{cases} \iff L(f) \begin{cases} = \infty \\ < \infty. \end{cases}$$

In particular, with probability 1,

$$\liminf_{t\to\infty} \frac{(\log\log t)^\beta}{a^{-1}(\log t)} \sup_{s\,\leqslant\, t} X_s = \begin{cases} 0, & \textit{if } \beta < 1/p_2, \\ \infty, & \textit{if } \beta > 1/p_2. \end{cases}$$

Theorems 1.4 and 1.5 must be adapted similarly. Notice that we can again calculate the Laplace transform of the limiting distribution as in Theorem 1.5 when the processes $\mathbb{S}^1$ and $\mathbb{S}^2$ have both completely asymmetric laws.

5.3 Random walk in random environment

We finally recall the connection between the model of the diffusion in random potential and the discrete model of the random walk in random environment. Let $\omega = (\omega_i)_{i\in\mathbb{Z}}$ be an i.i.d. family of random variables in $(0,1)$ and define for each realization of this family a Markov chain $(Z_n)_{n\,\geqslant\,0}$ taking values in $\mathbb{Z}$ such that $Z_0 = 0$ and

$$\mathbf{P}\big\{Z_{n+1} = Z_n + e \mid Z_n = x,\ (\omega_i)_{i\in\mathbb{Z}}\big\} = \begin{cases} \omega_x & \text{if } e = 1, \\ 1-\omega_x & \text{if } e = -1. \end{cases}$$

(Z_n) is a random walk in the random environment ω. Its associated potential is the random walk $(\mathbb{V}_n)_{n\in\mathbb{Z}}$ defined by $\mathbb{V}_0 \stackrel{\text{def}}{=} 0$ and $\mathbb{V}_{n+1} - \mathbb{V}_n \stackrel{\text{def}}{=} \log\left((1-\omega_n)/\omega_n\right)$ for all $n \in \mathbb{Z}$. Let X still denote the random diffusion in the random potential $\mathbb{V}$. The following result from Schumacher relates the two processes X and Z.

Proposition 5.4 (*Schumacher, [Sch85]*)

Define the sequence $(\mu_n)_{n\,\geqslant\,0}$ by

$$\begin{cases} \mu_0 \stackrel{\text{def}}{=} 0, \\ \mu_{n+1} \stackrel{\text{def}}{=} \inf\left(t > \mu_n,\ |X_{\mu_{n+1}} - X_{\mu_n}| = 1\right). \end{cases}$$

Under the annealed probability $\mathbf{P}$, the sequence $(\mu_{n+1} - \mu_n)_{n\,\geqslant\,0}$ is i.i.d. and μ_1 is distributed as the first hitting time of level 1 of a reflected standard Brownian motion. Moreover, for each realization of the environment ω, the processes $(X_{\mu_n})_{n\,\geqslant\,0}$ and $(Z_n)_{n\,\geqslant\,0}$ have the same law.

Using this proposition, we can easily adapt Theorem 1.2-1.5 for the random walk in random environment Z in the case where $\mathbb{V}_1 \stackrel{\text{def}}{=} \log\left((1-\omega_0)/\omega_0\right)$ satisfies Assumption 1.1. Compare with Section 10 of [HS98a] for details. For example, Theorem 1.4 for Z takes the form:

Theorem 5.5

Assume that $\mathbb{S}$ has jumps of both signs. We have, with probability 1, for any non-decreasing positive sequence $(c_n)_{n\,\geqslant\,0}$,

$$\liminf_{n\to\infty} \frac{c_n}{a^{-1}(\log n)} \sup_{k\,\leqslant\,n} |Z_k| = \begin{cases} 0 \\ \infty \end{cases} \Longleftrightarrow \sum_{n\,\geqslant\,2} \frac{1}{(c_n)^2 n \log n} \begin{cases} = \infty \\ < \infty. \end{cases}$$

In particular, almost surely,

$$\liminf_{n\to\infty} \frac{(\log\log n)^\beta}{a^{-1}(\log n)} \sup_{k\,\leqslant\,n} |Z_k| = \begin{cases} 0, & \text{if } \beta \leqslant 1/2, \\ \infty, & \text{if } \beta > 1/2. \end{cases}$$

Chapter III

Rates of convergence of a transient diffusion in a spectrally negative Lévy potential[1]

Abstract. We study the rate of transience of a diffusion X in a Lévy potential which does not possess positive jumps. We generalize the previous results of Hu-Shi-Yor for drifted Brownian potentials. In particular, we prove a conjecture of Carmona: provided that there exists $0 < \kappa < 1$ such that $\mathbf{E}[e^{\kappa \mathbb{V}_1}] = 1$, then X_t/t^κ converges, as t goes to infinity, towards a Mittag-Leffler distribution with parameter κ. These results are in a way analogous to those obtained by Kesten-Kozlov-Spitzer for the transient random walk in a random environment.

1 Introduction

Let $(\mathbb{V}(x),\, x \in \mathbb{R})$ be a càdlàg, real-valued stochastic process with $\mathbb{V}(0) = 0$, defined on some probability space $(\Omega, \mathbf{P})$. We consider a diffusion process X, solution of the informal stochastic differential equation

$$\begin{cases} dX_t = d\beta_t - \frac{1}{2}\mathbb{V}'(X_t)dt \\ X_0 = 0, \end{cases}$$

where $(\beta_s,\, s \geqslant 0)$ is a standard Brownian motion independent of $\mathbb{V}$. Formally, one can see X as a diffusion process whose conditional generator given $\mathbb{V}$ is

$$\frac{1}{2}e^{\mathbb{V}(x)}\frac{d}{dx}\left(e^{-\mathbb{V}(x)}\frac{d}{dx}\right).$$

We call X a diffusion in the random potential $\mathbb{V}$. Somehow, this process may be thought as the continuous analogue of the random walk in random environment (see Schumacher

[1]This chapter is a slightly modified version of the article: A. Singh, *Rates of convergence of a transient diffusion in a spectrally negative Lévy potential*, to appear in Ann. Probab.

[Sch85] or Shi [Shi01] for a connection between the two models). In particular, both models exhibit similar interesting features such as asymptotic sub-linear growth.

For instance, if $\mathbb{V}$ is a two-sided Brownian motion, then X is recurrent and Brox [Bro86] proved an equivalent of Sinai's theorem [Sin82] for random walk in random environment, that is: $X_t/\log^2 t$ converges to some non-degenerate distribution as t goes to infinity.

When the potential process $\mathbb{V}$ is a drifted Brownian motion ($\mathbb{V}_x = B_x - \frac{\kappa}{2}x$, with $\kappa > 0$ and B a two-sided Brownian motion), the diffusion is transient towards $+\infty$. More precisely, Kawazu and Tanaka [KT97] showed that the rate of convergence to infinity depends on the value of κ.

- If $0 < \kappa < 1$, then $\frac{1}{t^\kappa}X_t$ converges in law, as t goes to infinity, towards a non-degenerate positive random variable.
- If $\kappa = 1$, then $\frac{\log t}{t}X_t$ converges in probability towards $\frac{1}{4}$.
- If $\kappa > 1$, then $\frac{1}{t}X_t$ converges almost surely towards $\frac{\kappa-1}{4}$.

Refined results were later obtained by Tanaka [Tan97] and Hu *et al.* [HSY99], in particular, they proved a central-limit type theorem when $\kappa > 1$. We point out that these results are the analogues, when the potential is a drifted Brownian motion, of those previously obtained by Kesten *et al.* [KKS75] for the discrete model of the random walk in a random environment. However, the results of Kesten *et al.* hold for a wide class of environments, whereas few results are available in the continuous setting for general potentials. One would certainly like to extend the results of [HSY99] and [Tan97] for drifted Brownian motion to a wider class of potentials. In this spirit, Carmona [Car97] considered the case where $\mathbb{V}$ is a two-sided Lévy process and proved, by use of ergodic theorems that, if Φ denotes the Laplace exponent of $\mathbb{V}$,

$$\mathbf{E}\left[e^{\lambda \mathbb{V}_t}\right] = e^{t\Phi(\lambda)} \quad t \geqslant 0\ ,\ \lambda \in \mathbb{R} \tag{1.1}$$

(note that $\Phi(\lambda)$ may be infinite), then

- If $\Phi(1) < 0$ then X_t/t converges almost surely towards $-\Phi(1)/2$.
- If $\Phi(-1) < 0$ then X_t/t converges almost surely towards $\Phi(-1)/2$.
- Otherwise, X_t/t converges almost surely towards 0.

Carmona also conjectured that when the limiting velocity is zero, assuming that there exists $0 < \kappa < 1$ such that $\Phi(\kappa) = 0$, then one should observe the same asymptotic behavior as in the case of a drifted Brownian potential, *i.e.* the rate of growth of X_t should again be of order t^κ. We prove that this is the case when $\mathbb{V}$ is a spectrally negative Lévy process (*i.e.* a Lévy process without positive jumps).

Throughout this paper, we will make the following assumption on $\mathbb{V}$:

Assumption 1.1

The following hold:

(a) $(\mathbb{V}_x, x \in \mathbb{R})$ *is a càdlàg locally bounded process with* $\mathbb{V}_0 = 0$ *and the two processes* $(\mathbb{V}_x, x \geqslant 0)$ *and* $(\mathbb{V}_{-x}, x \geqslant 0)$ *are independent.*

(b) $(\mathbb{V}_x, x \geqslant 0)$ *is a Lévy process with no positive jumps, which is not the opposite of a subordinator, and is such that* $\lim_{x\to\infty} \mathbb{V}_x = -\infty$ *almost surely.*

(c) $(\mathbb{V}_{-x}, x \geqslant 0)$ *is such that* $\int_0^\infty e^{\mathbb{V}_{-x}} dx = \infty$ *almost surely.*

Let us first make some comments concerning our assumptions.

- Notice that (c) is a weak condition. For instance, it is fulfilled whenever $(\mathbb{V}_{-x}, x \geqslant 0)$ is a Lévy process which does not diverge to $-\infty$. In fact, (c) is only to ensure that the diffusion X does not go to $-\infty$ with positive probability but we are not really concerned about the behavior of $\mathbb{V}$ for negative x's. In particular, the process $(\mathbb{V}_{-x}, x \geqslant 0)$ may have jumps of both signs.

- Since $(\mathbb{V}_x, x \geqslant 0)$ has no positive jumps, its Laplace exponent Φ given by (1.1) is finite at least for all $\lambda \in [0, \infty)$. The assumption that $\mathbb{V}$ is not the opposite of a subordinator implies that $\Phi(\lambda) \to \infty$ as $\lambda \to \infty$. Moreover, since $\mathbb{V}$ is transient towards $-\infty$, the right derivative of Φ at $0+$ is such that $\Phi'(0+) = \mathbf{E}[\mathbb{V}_1] \in [-\infty, 0)$. Thus, the strict convexity of Φ implies that $\mathbb{V}$ fulfills the so-called Cramer's condition: there exists a unique $\kappa > 0$ such that
$$\Phi(\kappa) = 0. \tag{1.2}$$
In particular, $\Phi(x) < 0$ for all $x \in (0, \kappa)$ whereas $\Phi(x) > 0$ for all $x > \kappa$.

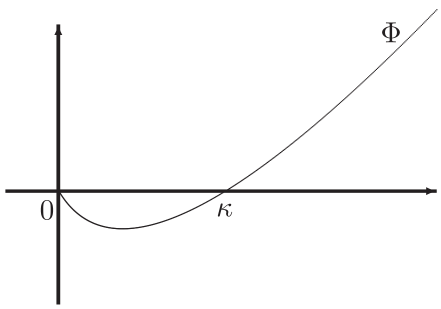

Figure III.1 : The Laplace exponent Φ.

We introduce the scale function of the diffusion X:
$$A(x) \stackrel{\text{def}}{=} \int_0^x e^{\mathbb{V}_y} dy \quad \text{for } x \in [-\infty, \infty]. \tag{1.3}$$

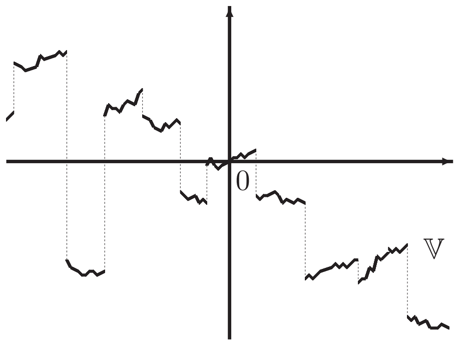

Figure III.2 : Sample path of $\mathbb{V}$.

On the one hand, Assumption (c) implies that

$$\lim_{x \to -\infty} A(x) = A(-\infty) = -\infty \quad \mathbf{P}\text{-a.s.} \tag{1.4}$$

On the other hand, in view of Assumption (b), for $0 < \delta < -\mathbf{E}[\mathbb{V}_1]$, the Lévy process $(\mathbb{V}_x + \delta x, x \geqslant 0)$ also diverges towards $-\infty$. This entails

$$\lim_{x \to +\infty} A(x) = A(+\infty) < \infty \quad \mathbf{P}\text{-a.s.} \tag{1.5}$$

Combining (1.4) and (1.5), classical results concerning diffusion processes show that X is almost surely transient towards $+\infty$ (see [Shi01] for details). We now introduce the hitting time of level $r \geqslant 0$ for the diffusion:

$$H(r) \stackrel{\text{def}}{=} \inf \{t \geqslant 0, X_t = r\} . \tag{1.6}$$

Let $\mathcal{N}$ stand for a Gaussian $\mathcal{N}(0,1)$ variable. For $\alpha \in (0,1) \cup (1,2)$, let $\mathcal{S}_\alpha^{ca}$ be a completely asymmetric stable variable with characteristic function

$$\mathbf{E}\left[e^{it\mathcal{S}_\alpha^{ca}}\right] = \exp\left(-|t|^\alpha \left(1 - i\,\text{sgn}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right)$$

(S_α^{ca} is positive when $\alpha < 1$). Let also $\mathcal{C}^{ca}$ denote a completely asymmetric Cauchy variable with characteristic function

$$\mathbf{E}\left[e^{it\mathcal{C}^{ca}}\right] = \exp\left(-\left(|t| + it\frac{2}{\pi}\log|t|\right)\right).$$

The main result of this paper characterizes all the possible rates of transience of the diffusion X in the Lévy potential $\mathbb{V}$.

Theorem 1.2

Recall that κ defined by (1.2) is the unique positive root of the Laplace exponent Φ of $\mathbb{V}$. We denote by Φ' the derivative of Φ. Set

$$\mathbf{K} \stackrel{\text{def}}{=} \mathbf{E}\left[\left(\int_0^\infty e^{\mathbb{V}_y}dy\right)^{\kappa-1}\right].$$

This constant (which only depends on the potential $\mathbb{V}$) is finite. When $\kappa > 1$ (i.e. when $\Phi(1) < 0$), set $\mathbf{m} \stackrel{\text{def}}{=} -2/\Phi(1) > 0$. We have, depending on the value of κ:

(a) *If $0 < \kappa < 1$,*

$$\frac{1}{r^{1/\kappa}}H(r) \xrightarrow[r\to\infty]{\text{law}} 2\left(\frac{\pi\kappa^2\mathbf{K}^2}{2\sin\left(\frac{\pi\kappa}{2}\right)\Phi'(\kappa)}\right)^{\frac{1}{\kappa}} \mathcal{S}^{ca}_\kappa.$$

(b) *If $\kappa = 1$, there exists a function f with $f(r) \sim \frac{2}{\Phi'(1)}r\log r$ such that*

$$\frac{1}{r}\left(H(r) - f(r)\right) \xrightarrow[r\to\infty]{\text{law}} \left(\frac{\pi}{\Phi'(1)}\right)\mathcal{C}^{ca}.$$

(c) *If $1 < \kappa < 2$,*

$$\frac{1}{r^{1/\kappa}}\left(H(r) - \mathbf{m}r\right) \xrightarrow[r\to\infty]{\text{law}} 2\left(\frac{\pi\kappa^2\mathbf{K}^2}{2\sin\left(\frac{\pi\kappa}{2}\right)\Phi'(\kappa)}\right)^{\frac{1}{\kappa}} \mathcal{S}^{ca}_\kappa.$$

(d) *If $\kappa = 2$,*

$$\frac{1}{\sqrt{r\log r}}\left(H(r) - \mathbf{m}r\right) \xrightarrow[r\to\infty]{\text{law}} \left(\frac{-4}{\Phi(1)\sqrt{\Phi'(2)}}\right)\mathcal{N}.$$

(e) *If $\kappa > 2$,*

$$\frac{1}{\sqrt{r}}\left(H(r) - \mathbf{m}r\right) \xrightarrow[r\to\infty]{\text{law}} \sqrt{\frac{8\left(\Phi(2) - 4\Phi(1)\right)}{\Phi(1)^3\Phi(2)}}\mathcal{N}.$$

This theorem gives precise asymptotics for $H(r)$. It is well known that these estimates may in turn be used to obtain asymptotics for X_t, $\sup_{s\leqslant t} X_s$ and $\inf_{s\geqslant t} X_s$ (see [KT97] for details). For example, when $0 < \kappa < 1$, (a) of the theorem entails

$$\frac{X_t}{t^\kappa} \xrightarrow[t\to\infty]{\text{law}} \frac{2^{1-\kappa}\sin\left(\frac{\pi\kappa}{2}\right)\Phi'(\kappa)}{\pi\kappa^2\mathbf{K}^2}\left(\frac{1}{\mathcal{S}^{ca}_\kappa}\right)^\kappa.$$

The same result also holds for $\sup_{s\leqslant t} X_s$ or $\inf_{s\geqslant t} X_s$ in place of X_t.

One would certainly wish to express the value of the constant $\mathbf{K}$ in term of the characteristics of the Lévy process. Although there is to our knowledge no explicit formula for this constant, there are some cases where the calculations may be carried to their full extent.

Example 1 : We consider a potential of the form $\mathbb{V}_x = B_x - \frac{\kappa}{2}x$ with $\kappa > 0$ and where B is a two-sided standard Brownian motion. According to Dufresne [Duf90] (see also Proposition 2.2 of [CPY01]), the random variable $\int_0^\infty e^{\mathbb{V}_s}ds$ has the same law as $\frac{2}{\gamma_\kappa}$ where γ_κ denotes a gamma variable with parameter κ. Therefore, the constant $\mathbf{K}$ may be explicitly calculated:

$$\mathbf{K} = \frac{2^{\kappa-1}}{\Gamma(\kappa)}$$

(Γ denotes Euler's Gamma function). Thus, we recover the results of Hu *et al.* [HSY99] and Tanaka [Tan97], except for $\kappa = 1$ where we do not have the explicit form of the centering function f.

Example 2 : We consider a potential of the form

$$\mathbb{V}_x = cx - \tau_x \quad \text{for } x \geqslant 0,$$

with $c > 0$ and where τ is a subordinator without drift whose Lévy measure ν has the form $\nu[x, \infty) = ae^{-bx}$ with $a, b > 0$. Then, the Laplace exponent of $\mathbb{V}$ is given by

$$\Phi(\lambda) = c\lambda - \frac{a\lambda}{\lambda + b} \quad \text{for all } \lambda \geqslant 0.$$

Since $\mathbf{E}[\mathbb{V}_1] = c - \frac{a}{b}$, Assumption 1.1 is fulfilled whenever $c < \frac{a}{b}$, in which case Theorem 1.2 holds with $\kappa = \frac{a}{c} - b$. According to Proposition 2.1 of [CPY97], the density k of the integral functional $\int_0^\infty e^{\mathbb{V}_x}dx$ satisfies the differential equation

$$(1 + cx)k(x) = a\int_x^\infty \left(\frac{x}{u}\right)^b k(u)du.$$

This equation may be explicitly solved and we find

$$k(x) = \left(\frac{c^{b+1}\Gamma\left(\frac{a}{c}+1\right)}{\Gamma\left(\frac{a}{c}-b\right)\Gamma\left(b+1\right)}\right)\frac{x^b}{(1+cx)^{1+\frac{a}{c}}}.$$

Thus, we can again calculate the value of the constant of Theorem 1.2,

$$\mathbf{K} = \int_0^\infty x^{\kappa-1}k(x)dx = \frac{\Gamma\left(\frac{a}{c}\right)}{c^{\frac{a}{c}-b-1}\Gamma\left(\frac{a}{c}-b\right)\Gamma\left(b+1\right)}.$$

In the case of a drifted Brownian potential, in order to obtain the rates of transience of the diffusion, Kawazu and Tanaka [KT97, Tan97] made use of Kotani's formula whereas Hu *et al.* [HSY99] made use of Lamperti's representation combined with the study of Jacobi processes. Unfortunately, both methods fail for more general potentials. Our approach consists in reducing the study of $H(r)$ to that of an additive functional of a Markov process.

More precisely, the remainder of this paper is organized as follow: in Section 2, we show that $H(r)$ has the same rates of convergence as $\int_0^r Z_s ds$ where Z is a generalized Ornstein-Uhlenbeck process. In Section 3, we study the basic properties of Z. Section 4 is devoted to the study of the hitting times of Z and we will prove that this process is recurrent. In Section 5, we define the local time and excursion measure associated with the excursions of Z away from level 1. The main result of that section is an estimate of the distribution tail of the area of a generic excursion. Section 6 is devoted to the calculus of the second moment of the area of an excursion when $\kappa > 2$. Once all these results have been obtained, the rest of the proof is very classical and is given in the last section.

2 The process Z

We first construct X from a Brownian motion through a random change of time and a random change of scale. Let B denote a standard Brownian motion independent of $\mathbb{V}$ and, for $x \in \mathbb{R}$, set $\sigma_B(x) \stackrel{\text{def}}{=} \inf\{t \geqslant 0\ ,\ B_t = x\}$. Recall that the scale function A was defined in (1.3). The process A is continuous and strictly increasing. Let $A^{-1} : (-\infty, A(+\infty)) \mapsto \mathbb{R}$ denote the inverse of A. We define

$$T(t) \stackrel{\text{def}}{=} \int_0^t \exp\left(-2\mathbb{V}(A^{-1}(B_s))\right) ds \quad \text{for } 0 \leqslant t < \sigma_B(A(+\infty)). \tag{2.1}$$

The process T is strictly increasing on $[0, \sigma_B(A(+\infty))\,)$. Let T^{-1} denote the inverse of T and set

$$X_t = A^{-1}(B(T^{-1}(t))) \quad \text{for all } t \geqslant 0. \tag{2.2}$$

According to Brox [Bro86], the process $(X_t\,,\, t \geqslant 0)$ is a diffusion in the random potential $\mathbb{V}$. Recall that $H(r)$ defined by (1.6) stands for the hitting time of level r for X. Using the representation (2.2), we obtain

$$H(r) = T(\sigma_B(A(r))). \tag{2.3}$$

Now, let $L_B(x,t)$ denote the (bi-continuous) local time of B at level $x \in \mathbb{R}$ and time $t \geqslant 0$. Substituting (2.1) in (2.3), we get

$$\begin{aligned} H(r) &= \int_0^{\sigma_B(A(r))} \exp\left(-2\mathbb{V}(A^{-1}(B_s))\right) ds \\ &= \int_{-\infty}^{A(r)} \exp\left(-2\mathbb{V}(A^{-1}(y))\right) L_B(y, \sigma_B(A(r)))dy. \end{aligned}$$

Making use of the change of variable $A(x) = y$,

$$H(r) = \int_{-\infty}^{r} \exp\left(-\mathbb{V}_x\right) L_B(A(x), \sigma_B(A(r)))dx = J_1(r) + J_2(r),$$

where

$$\begin{aligned} J_1(r) &\stackrel{\text{def}}{=} \int_{-\infty}^{0} \exp\left(-\mathbb{V}_x\right) L_B(A(x), \sigma_B(A(r)))dx, \\ J_2(r) &\stackrel{\text{def}}{=} \int_0^r \exp\left(-\mathbb{V}_x\right) L_B(A(x), \sigma_B(A(r)))dx. \end{aligned}$$

We first deal with J_1. Since $x \mapsto L_B(x,t)$ has compact support for all t and since $\lim_{x\to-\infty} A(x) = -\infty$, we see that

$$J_1(\infty) \stackrel{\text{def}}{=} \int_{-\infty}^{0} \exp\left(-\mathbb{V}_x\right) L_B(A(x), \sigma_B(A(+\infty)))dx < \infty \qquad \mathbf{P}\text{-a.s.}$$

Moreover, $J_1(r) \leqslant J_1(\infty)$ for all $r \geqslant 0$. Thus, we only need to prove Theorem 1.2 for $J_2(r)$ in place of $H(r)$.

According to the first Ray-Knight theorem, for all $a > 0$, $(L_B(a-t, \sigma(a))\,,\, 0 \leqslant t \leqslant a)$ has the law of a two-dimensional squared Bessel process starting from 0 and is independent of $\mathbb{V}$. Let $(U(x)\,,\, x \geqslant 0)$ under $\mathbf{P}$ be a two-dimensional squared Bessel process starting from 0, independent of $\mathbb{V}$. Then, for each fixed $r > 0$,

$$\begin{aligned} J_2(r) &\stackrel{\text{law}}{=} \int_0^r e^{-\mathbb{V}_x} U(A(r) - A(x))dx \\ &\stackrel{\text{law}}{=} \int_0^r e^{-\mathbb{V}_{r-y}} U(A(r) - A(r-y))dy \\ &\stackrel{\text{law}}{=} \int_0^r e^{-\mathbb{V}_{r-y}} U\left(\int_0^y e^{\mathbb{V}_{r-s}} ds\right) dy \\ &\stackrel{\text{law}}{=} \int_0^r e^{-\mathbb{V}_{(r-y)-}} U\left(\int_0^y e^{\mathbb{V}_{(r-s)-}} ds\right) dy \end{aligned}$$

(where $\mathbb{V}_{x-}$ denotes the left limit of $\mathbb{V}$ at point x). For any fixed $r > 0$, we define $\widehat{\mathbb{V}}_t^r \stackrel{\text{def}}{=} \mathbb{V}_{(r-t)-} - \mathbb{V}_r$ for all $0 \leqslant t \leqslant r$. Therefore, the scaling property of U yields

$$\begin{aligned} J_2(r) &\stackrel{\text{law}}{=} \int_0^r e^{-\widehat{\mathbb{V}}_y^r - \mathbb{V}_r} U\left(e^{\mathbb{V}_r} \int_0^y e^{\widehat{\mathbb{V}}_s^r} ds\right) dy \\ &\stackrel{\text{law}}{=} \int_0^r e^{-\widehat{\mathbb{V}}_y^r} U\left(\int_0^y e^{\widehat{\mathbb{V}}_s^r} ds\right) dy. \end{aligned}$$

Time reversal of the Lévy process $\mathbb{V}$ (see Lemma 2, p45 of [Ber96a]) states that for each $r > 0$, the two processes $(\widehat{\mathbb{V}}_t^r\,,\, 0 \leqslant t \leqslant r)$ and $(-\mathbb{V}_t\,,\, 0 \leqslant t \leqslant r)$ have the same law. Thus, for each fixed r, under $\mathbf{P}$,

$$J_2(r) \stackrel{\text{law}}{=} \int_0^r e^{\mathbb{V}_y} U\left(\int_0^y e^{-\mathbb{V}_s} ds\right) dy = \int_0^r Z_s ds, \tag{2.4}$$

with the notations

$$Z_t \stackrel{\text{def}}{=} e^{\mathbb{V}_t} U\left(a(t)\right), \tag{2.5}$$

where

$$a(x) \stackrel{\text{def}}{=} \int_0^x e^{-\mathbb{V}_s} ds. \tag{2.6}$$

According to (2.4), we only need to prove Theorem 1.2 for the additive functional $\int_0^r Z_s ds$ instead of dealing directly with $H(r)$.

The rest of the proof now relies on the study of the process Z. As we will see in the next sections, Z is a 'nice' recurrent Markov process for which we may define a local time L at any positive level, say 1. We may therefore also consider the associated excursion measure $\mathbf{n}$ of its excursions away from 1. Given a generic excursion (ϵ_t) with lifetime ζ, we define the functional

$$\widetilde{I}(\epsilon) \stackrel{\text{def}}{=} \int_0^\zeta \epsilon_s ds.$$

The key step consists in proving that $\widetilde{I}(\epsilon)$, under the excursion measure $\mathbf{n}$, has a regularly varying tail of the form

$$\mathbf{n}\left\{\widetilde{I}(\epsilon) > x\right\} \underset{x\to\infty}{\sim} \frac{C}{x^\kappa}.$$

Then, as we may write

$$\int_0^t Z_s ds \;\approx \sum_{\substack{\text{excursion } \epsilon \\ \text{starting before } t}} \widetilde{I}(\epsilon),$$

the asymptotics of $\int_0^t Z_s ds$ will follow from classical results on the characterization of the domains of attraction to a stable law.

3 Basic properties of Z

Recall that U under $\mathbf{P}$ is a two-dimensional squared Bessel process starting from 0 and is independent of $\mathbb{V}$. We now consider a family of probabilities $(\mathbf{P}_x\,,\, x \geqslant 0)$ such that U under $\mathbf{P}_x$ is a two-dimensional squared Bessel process starting from x and is independent of $\mathbb{V}$. In particular $\mathbf{P} = \mathbf{P}_0$. We will use the notation $\mathbf{E}_x$ for the expectation under $\mathbf{P}_x$ (and $\mathbf{E} = \mathbf{E}_0$ for the expectation under $\mathbf{P} = \mathbf{P}_0$). Of course, the law of $\mathbb{V}$ is the same under all $\mathbf{P}_x$ and, when dealing with probabilities that do not depend on the starting point x of U, we will use the notation $\mathbf{P}$.

Let us first observe that the process Z defined by (2.5) is non-negative and does not possess positive jumps because $\mathbb{V}$ has no positive jumps. Moreover, under $\mathbf{P}_x$, the process Z starts from x. We define the filtration

$$\mathcal{F}_t \stackrel{\text{def}}{=} \sigma(\mathbb{V}_s, U(a(s)), s \leqslant t).$$

Our first lemma states that Z is an $\mathcal{F}$-Markov process.

Lemma 3.1

$((Z_t)_{t\,\geqslant\,0}, (\mathbf{P}_x)_{x\,\geqslant\,0})$ is an $\mathcal{F}$-Markov process whose semigroup fulfills the Feller property. Moreover, for each $x > 0$, the process $(Z_t\,,\, t \geqslant 0)$ under $\mathbf{P}_x$ (i.e. starting from x) has the same law as the process $(\widetilde{Z}^x_t\,,\, t \geqslant 0)$ under $\mathbf{P}_1$ where

$$\widetilde{Z}^x_t \stackrel{\text{def}}{=} x e^{\mathbb{V}_t} U\left(\frac{a(t)}{x}\right). \tag{3.1}$$

Proof. The process U is a squared Bessel process. Therefore, our process Z is a generalized Ornstein-Uhlenbeck process in the sense of [CPY97] and Proposition 5.5 of [CPY97] states that Z is indeed a Markov process in the filtration $\mathcal{F}$. Let $(P_t)_{t\geqslant 0}$ and $(Q_t)_{t\geqslant 0}$ stand for the respective semi-groups of U and Z. The independence of U and $\mathbb{V}$ yields the relation

$$Q_t f(x) = \mathbf{E}_x\left[f(Z_t)\right] = \mathbf{E}\left[P_{a(t)}\left(f(e^{\mathbb{V}_t}\cdot)\right)(x)\right]. \tag{3.2}$$

Since U is a squared Bessel process, its semi-group fulfills the Feller property. Moreover, $a(\cdot)$ is continuous with $a(0) = 0$ and $\lim_{t\to 0+} e^{\mathbb{V}_t} = 1$ $\mathbf{P}$-a.s. These facts combined with (3.2) readily show that (Q_t) is also a Fellerian semigroup. Finally, (3.1) is an immediate consequence of the scaling property of U. ■

For $x \geqslant 0$, we say that x is instantaneous for Z if the process Z starting from x leaves x instantaneously with probability 1. Moreover, we say x is regular (for itself) for Z if Z starting from x returns to x at arbitrarily small times with probability 1.

Lemma 3.2

Any $x > 0$ is regular and instantaneous for Z.

Proof. We only prove the result for $x = 1$, the general case may be treated in the same way. Since U under $\mathbf{P}_1$ is a squared Bessel process of dimension 2 starting from 1, it has the same law as $(B^2(t)+\widetilde{B}^2(t)+2B(t)+1\ ,\ t\geqslant 0)$ where B and $\widetilde{B}$ are two independent standard Brownian motions. It is therefore easy to check, using classical results on Brownian motion, that

(a) For any strictly decreasing sequence $(t_i)_{i\geqslant 0}$ of (non-random) real numbers such that $\lim_{i\to\infty} t_i = 0$, we have:

$$\mathbf{P}_1\left\{U(t_i) > 1 \text{ i.o.}\right\} = \mathbf{P}_1\left\{U(t_i) < 1 \text{ i.o.}\right\} = 1.$$

(b) $\liminf\limits_{t\to 0+} \frac{U(t)-1}{t} = -\infty$ $\mathbf{P}_1$-a.s.

Let us now prove that Z starting from 1 visits $(1,\infty)$ at arbitrarily small times. Recall that $(\mathbb{V}_x\ ,\ x\geqslant 0)$ is a Lévy process with no positive jumps which is not the opposite of a subordinator. According to Theorem 1, p189 of [Ber96a], the process $\mathbb{V}$ visits $(0,\infty)$ at arbitrarily small times with probability 1. Thus, for almost any fixed path of $\mathbb{V}$, we can find a strictly positive decreasing sequence $(u_i)_{i\geqslant 0}$ with limit 0 such that $\mathbb{V}_{u_i} > 0$ for all i. But, conditionally on $\mathbb{V}$, under $\mathbf{P}_1$, U is still a squared Bessel process of dimension 2 starting from 1 and

$$Z_{u_i} = e^{\mathbb{V}_{u_i}} U(a(u_i)) > U(a(u_i)).$$

Since $a(\cdot)$ is continuous with $\lim_{t\to 0} a(t) = 0$, the sequence $(a(u_i))_{i\geqslant 0}$ is positive, strictly decreasing with limit 0. Using (a), we conclude that Z starting from 1 visits $(1,\infty)$ at arbitrarily small times almost surely.

When 0 is regular for $(-\infty, 0)$ for the Lévy process $\mathbb{V}$, a similar argument shows that Z starting from 1 visits $(0,1)$ at arbitrarily small times almost surely. Let us therefore assume that 0 is irregular for $(-\infty, 0)$ for $\mathbb{V}$. According to Corollary 5, p192 of [Ber96a], this implies that $\mathbb{V}$ has bounded variations, thus there exists $d \geqslant 0$ such that $\lim_{x\to 0+} \mathbb{V}_x/x = d$ a.s. (*c.f.* Proposition 11, p166 of [Ber96a]). Let $a^{-1}(\cdot)$ denote the inverse of $a(\cdot)$. Since $a(t) \sim t$ as $t \to 0+$, we have $e^{\mathbb{V}_{a^{-1}(t)}} \leqslant 1 + 2dt$ for all t small enough, almost surely. In consequence,

$$Z(a^{-1}(t)) = e^{\mathbb{V}_{a^{-1}(t)}} U(t) \leqslant (1 + 2dt)\, U(t) \quad \text{for } t \text{ small enough, } \mathbf{P}_1\text{-a.s.}$$

Using (b), we conclude that the process $(Z(a^{-1}(t))\,, t \geqslant 0)$ visits $(0,1)$ at arbitrarily small times $\mathbf{P}_1$-a.s. Since $a^{-1}(\cdot)$ is continuous, increasing and $a(0) = 0$, this result also holds for Z.

We proved that Z starting from 1 visits $(0,1)$ and $(1,\infty)$ at arbitrarily small times almost surely. Since Z has no positive jumps, starting from 1, it returns to 1 at arbitrarily small times almost surely. ■

Lemma 3.3

For all $x, y \geqslant 0$ and all $t > 0$, we have $\mathbf{P}_x\{Z_t = y\} = 0$. In consequence,

$$\int_0^\infty \mathbf{1}_{\{Z_t = y\}} dt = 0 \quad \mathbf{P}_x\text{-a.s. } \textit{for all } x, y \geqslant 0.$$

Proof. Since a squared Bessel process has a continuous density, $\mathbf{P}_x\{U(a) = b\} = 0$ for all $b, x \geqslant 0$ and all $a > 0$. Recalling that $\mathbb{V}$ and U are independent and $a(t) > 0$ for all $t > 0$, we get

$$\mathbf{P}_x\{Z_t = y\} = \mathbf{E}\left[\mathbf{P}_x\left\{U(a(t)) = ye^{-\mathbb{V}_t} \middle| \mathbb{V}\right\}\right] = 0.$$

■

The following easy lemma will be found very useful in the remainder of this paper.

Lemma 3.4

*For all $0 \leqslant x \leqslant y$, the process Z under $\mathbf{P}_x$ (*i.e.* starting from x) is stochastically dominated by Z under $\mathbf{P}_y$ (*i.e.* starting from y).*

Proof. The process U is a two-dimensional squared Bessel process and a theorem of comparison for diffusion process (*c.f.* Theorem IX.3.7 of [RY99]) states that U under $\mathbf{P}_x$ is stochastically dominated by U under $\mathbf{P}_y$ whenever $x \leqslant y$. Thus, the lemma is a direct consequence of the independence of U and $\mathbb{V}$. ■

We conclude this section by proving the convergence of Z at infinity.

Proposition 3.5

For $x > 0$, under $\mathbf{P}_x$, the process Z_t converges as t goes to infinity towards a non-degenerate random variable Z_∞ whose law does not depend on the starting point. The distribution of Z_∞ is the same as that of the random variable

$$U(1)\int_0^\infty e^{\mathbb{V}_s}ds \quad \text{under } \mathbf{P}_0. \tag{3.3}$$

In particular, the law of Z_∞ has a strictly positive continuous density on $(0,\infty)$ and

$$\mathbf{P}\{Z_\infty > x\} \underset{x\to\infty}{\sim} \frac{2^\kappa \Gamma(\kappa+1)\mathbf{K}}{\Phi'(\kappa)x^\kappa} \tag{3.4}$$

where κ is the constant of (1.2) and where $\mathbf{K} = \mathbf{E}\left[A(+\infty)^{\kappa-1}\right] \in (0,\infty)$ is the constant defined in the statement of Theorem 1.2.

Proof. According to Proposition 5.7 of [CPY97], under the assumption that $\mathbf{E}[\mathbb{V}_1] < 0$, the generalized Ornstein-Uhlenbeck process Z converges in law towards a random variable Z_∞ whose distribution is given by (3.3). In our case, we may also have $\mathbf{E}[\mathbb{V}_1] = -\infty$. However, in the proof of Proposition 5.7 of [CPY97], the assumption that $\mathbf{E}[\mathbb{V}_1] < 0$ is required only to ensure that

$$\lim_{t\to\infty} \mathbb{V}_t = -\infty \quad \text{and} \quad \int_0^\infty e^{\mathbb{V}_t}dt = A(+\infty) < \infty \quad \text{a.s.}$$

Since we have already established these two results, Proposition 5.7 of [CPY97] is also true in our case. The process U under $\mathbf{P}_0$ is a squared Bessel process of dimension 2 starting from 0. Therefore, $U(1)$ under $\mathbf{P}_0$ has an exponential distribution with mean 2. Keeping in mind that $\mathbb{V}$ and U are independent, we find

$$\begin{aligned}\mathbf{P}\{Z_\infty > x\} &= \mathbf{P}_0\{U(1)A(+\infty) > x\} \\ &= \mathbf{E}\left[\mathbf{P}_0\{U(1)A(+\infty) > x \,|\, A(+\infty)\}\right] \\ &= \mathbf{E}\left[\exp\left(-\frac{x}{2A(+\infty)}\right)\right].\end{aligned} \tag{3.5}$$

It is now clear that Z_∞ has a continuous density, everywhere positive on $(0,\infty)$. Moreover, in view of the Abelian/Tauberian theorem (see for instance chapter VIII of [Fel71]), we deduce from (3.5) that the estimate (3.4) on the tail distribution of Z_∞ is equivalent to

$$\mathbf{P}\{A(+\infty) > x\} \underset{x\to\infty}{\sim} \frac{\mathbf{E}\left[A(+\infty)^{\kappa-1}\right]}{\Phi'(\kappa)x^\kappa}. \tag{3.6}$$

This result is proved in Lemma 4 of [Riv05] in the case $0 < \kappa < 1$. Another proof, valid for any $\kappa > 0$ is given in Theorem 3.1 of [MZ06] under the restrictive assumption that $\mathbb{V}_1$ admits a finite first moment. However, one may check that, in the proof of Theorem 3.1 of [MZ06], the assumption $\mathbf{E}[|\mathbb{V}_1|] < \infty$ is only required for $0 < \kappa < 1$. Thus, in our setting, (3.6) holds for any $\kappa > 0$. We point out that Lemma 4 of [Riv05] and Theorem 3.1 of

[MZ06] are both based on a theorem of Goldie [Gol91] which is, in turn, a refined version in the one-dimensional case of a famous result of Kesten [Kes73] on the affine equation for random matrices. ■

4 Hitting times of Z

Given a stochastic process Y and a set A we define the hitting times

$$\tau_A(Y) = \inf\{t \geqslant 0\ ,\ Y_t \in A\} \quad \text{(with the convention } \inf\emptyset = \infty). \tag{4.1}$$

For simplicity, we will use the notation $\tau_x(Y)$ instead of $\tau_{\{x\}}(Y)$. When referring to the process Z, we will also simply write τ_A instead of $\tau_A(Z)$. We now show that the hitting times of Z are finite almost-surely and we give estimates on their distribution tail. In particular, this will show that Z is recurrent. The rest of this section is devoted to proving the following four propositions. These estimates are quite technical and, on a first reading, the details of the proof may be skipped after glancing at the statements of the propositions.

Proposition 4.1

For any $0 \leqslant x < y$, there exist $c_{1,y}, c_{2,y} > 0$ (depending on y) such that

$$\mathbf{P}_x\left\{\tau_{[y,\infty)} > t\right\} \leqslant c_{1,y} e^{-c_{2,y}t} \quad \textit{for all } t \geqslant 0.$$

Proposition 4.2

There exist $y_0, c_3, c_4 > 0$ such that for all $y_0 \leqslant y < x$:

$$\mathbf{P}_x\left\{\tau_{[0,y]} > t\right\} \leqslant c_3\left(\log(x/y)+1\right) e^{-\frac{c_4}{\log(x/y)+1}t} \quad \textit{for all } t \geqslant 0.$$

Proposition 4.3

For all $x \geqslant 0$ and all $y > 0$, there exist $c_{5,x,y}, c_{6,x,y} > 0$ (depending on x and y) such that

$$\mathbf{P}_x\left\{\tau_y > t\right\} \leqslant c_{5,x,y} e^{-c_{6,x,y}t} \quad \textit{for all } t \geqslant 0.$$

In particular, Z starting from $x \geqslant 0$ ultimately hits any positive level.

Proposition 4.4

We have

$$\lim_{\lambda\to\infty} \sup_{y \geqslant 1} \mathbf{P}_y\left\{\tau_{\lambda y} < \tau_1\right\} = 0.$$

Proof of Proposition 4.1. Let $0 \leqslant x \leqslant y$. According to Lemma 3.4, $\tau_{[y,\infty)}$ under $\mathbf{P}_0$ is stochastically dominated by $\tau_{[y,\infty)}$ under $\mathbf{P}_x$, thus we only need to prove the proposition for $x = 0$. Let $\lfloor t \rfloor$ stand for the integer part of t. We have

$$\mathbf{P}_0\left\{\tau_{[y,\infty)} > t\right\} \leqslant \mathbf{P}_0\left\{Z_1 < y\,,\, Z_2 < y\,,\,\ldots\,,\, Z_{\lfloor t \rfloor} < y\right\} \leqslant \mathbf{P}_0\{Z_1 < y\}^{\lfloor t \rfloor}$$

where we repeatedly used the Markov property of Z combined with the stochastic monotonicity of Z (Lemma 3.4) for the last inequality. Since $Z_1 = e^{\mathbb{V}_1}U(a(1))$, it is clear that $\mathbf{P}_0\{Z_1 < y\} < 1$ for all $y > 0$. Thus, setting $c_{2,y} = -\log\left(\mathbf{P}_0\{Z_1 < y\}\right) > 0$ and $c_{1,y} = e^{c_{2,y}}$, we find

$$\mathbf{P}_0\left\{\tau_{[y,\infty)} > y\right\} \leqslant e^{-c_{2,y}\lfloor t \rfloor} \leqslant c_{1,y}e^{-c_{2,y}t}.$$

■

The proof of Proposition 4.2 relies on

Lemma 4.5

There exist $c_7, c_8, x_0 > 0$ such that, for all $x \geqslant x_0$,

$$\mathbf{P}_x\left\{\tau_{[0,x/2]} > t\right\} \leqslant c_7e^{-c_8t} \quad \textit{for all } t > 0.$$

Proof of Lemma 4.5. Pick $\eta > 0$ and let $(\mathbb{V}^{(\eta)}_t\,,\, t \geqslant 0)$ stand for the Lévy process $\mathbb{V}^{(\eta)}_t = \mathbb{V}_t + \eta t$. Recall that Φ denotes the Laplace exponent of $\mathbb{V}$. Thus, the Laplace exponent $\Phi^{(\eta)}$ of $\mathbb{V}^{(\eta)}$ is given by $\Phi^{(\eta)}(x) = \Phi(x) + \eta x$. Since $\Phi(\kappa/2) < 0$, we can choose η small enough such that $\Phi^{(\eta)}(\kappa/2) < 0$. Then $\mathbb{V}^{(\eta)}_t$ diverges to $-\infty$ as t goes to infinity and we can define the sequence

$$\begin{cases} \gamma_0 & \stackrel{\text{def}}{=} \quad 0, \\ \gamma_{n+1} & \stackrel{\text{def}}{=} \quad \inf\left\{t > \gamma_n\,,\, \mathbb{V}^{(\eta)}_t - \mathbb{V}^{(\eta)}_{\gamma_n} < -\log(8)\right\}. \end{cases}$$

The sequence $(\gamma_{n+1} - \gamma_n)_{n \geqslant 0}$ is i.i.d. and distributed as γ_1. We have

$$\begin{aligned} \mathbf{P}\{\gamma_1 > t\} \leqslant \mathbf{P}\left\{\mathbb{V}^{(\eta)}_t \geqslant -\log(8)\right\} &\leqslant \mathbf{P}\left\{\exp\left(\frac{\kappa}{2}\mathbb{V}^{(\eta)}_t\right) \geqslant \frac{1}{8^{\frac{\kappa}{2}}}\right\} \\ &\leqslant 8^{\frac{\kappa}{2}}\mathbf{E}\left[\exp\left(\frac{\kappa}{2}\mathbb{V}^{(\eta)}_t\right)\right] = 8^{\frac{\kappa}{2}}e^{t\Phi^{(\eta)}(\kappa/2)}. \end{aligned}$$

Since $\Phi^{(\eta)}(\kappa/2) < 0$, we deduce from Cramer's large deviation theorem that there exist $c_9, c_{10}, c_{11} > 0$ such that

$$\mathbf{P}\left\{\gamma_n > c_9n\right\} \leqslant c_{10}e^{-c_{11}n} \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

Notice that, from the definition of γ_1,

$$e^{\mathbb{V}_{\gamma_1}}a(\gamma_1) = \int_0^{\gamma_1} e^{\mathbb{V}^{(\eta)}_{\gamma_1} - \mathbb{V}^{(\eta)}_s - \eta(\gamma_1 - s)}ds \leqslant \int_0^{\gamma_1} e^{-\eta(\gamma_1 - s)}ds \leqslant \frac{1}{\eta}, \tag{4.3}$$

and also

$$e^{\mathbb{V}_{\gamma_1}} \leqslant \frac{1}{8}. \tag{4.4}$$

The process U under $\mathbf{P}_x$ is a squared Bessel process of dimension 2 starting from x. Therefore, U under $\mathbf{P}_x$ is stochastically dominated by $2(x+U)$ under $\mathbf{P}_0$. Using the independence of $\mathbb{V}$ and U, we deduce that Z_{γ_1} under $\mathbf{P}_x$ is stochastically dominated by $2e^{\mathbb{V}_{\gamma_1}}(x+U(a(\gamma_1)))$ under $\mathbf{P}_0$. Moreover, the scaling property of U combined with (4.3) and (4.4) yields, under $\mathbf{P}_0$,

$$2e^{\mathbb{V}_{\gamma_1}}(x+U(a(\gamma_1))) \overset{\text{law}}{=} 2xe^{\mathbb{V}_{\gamma_1}} + 2e^{\mathbb{V}_{\gamma_1}}a(\gamma_1)U(1) \leqslant \frac{x}{4} + \frac{2}{\eta}U(1).$$

Thus, Z_{γ_1} under $\mathbf{P}_x$ is stochastically dominated by the random variable $\frac{x}{4} + \frac{2}{\eta}U(1)$ under $\mathbf{P}_0$. Now, let $(\chi_n\,,\, n \geqslant 1)$ denote a sequence of i.i.d. random variables with the same distribution as $\frac{2}{\eta}U(1)$ under $\mathbf{P}_0$. Define also the sequence $(R^x_n\,,\, n \geqslant 0)$ by

$$\begin{cases} R^x_0 & \overset{\text{def}}{=} & x, \\ R^x_{n+1} & \overset{\text{def}}{=} & \frac{1}{4}R^x_n + \chi_{n+1}. \end{cases}$$

The process $(Z_{\gamma_n}\,,\, n \geqslant 0)$ under $\mathbf{P}_x$ is a Markov chain starting from x. We have already proved that Z_{γ_1} is stochastically dominated by R^x_1. By induction and with the help of Lemma 3.4, we conclude with similar arguments that the sequence $(Z_{\gamma_n}\,,\, n \geqslant 0)$ under $\mathbf{P}_x$ is stochastically dominated by $(R^x_n\,,\, n \geqslant 0)$. In particular, choosing $n = \lfloor t/c_9 \rfloor$ and using (4.2), we find

$$\begin{aligned} \mathbf{P}_x\left\{\tau_{[0,x/2]} > t\right\} &\leqslant \mathbf{P}\left\{\gamma_n > c_9 n\right\} + \mathbf{P}_x\left\{Z_{\gamma_1} > \frac{x}{2}\,,\,\ldots\,,\, Z_{\gamma_n} > \frac{x}{2}\right\} \\ &\leqslant c_{10}e^{-c_{11}t} + \mathbf{P}\left\{R^x_1 > \frac{x}{2}\,,\,\ldots\,,\, R^x_n > \frac{x}{2}\right\}. \end{aligned}$$

Thus, it only remains to prove that there exist $c_{12}, x_0 > 0$ such that

$$\mathbf{P}\left\{R^x_1 > \frac{x}{2}\,,\,\ldots\,,\, R^x_n > \frac{x}{2}\right\} \leqslant e^{-c_{12}n} \quad \text{for all } n \in \mathbb{N} \text{ and all } x \geqslant x_0.$$

Expanding the definition of R^x, we get

$$R^x_n = \frac{x}{4^n} + \frac{1}{4^{n-1}}\chi_1 + \ldots + \frac{1}{4}\chi_{n-1} + \chi_n.$$

Let us set $c = 8/\eta$. We have

$$R^x_n - \frac{4}{3}c \leqslant \frac{x}{4^n} + \frac{1}{4^{n-1}}(\chi_1 - c) + \ldots + \frac{1}{4}(\chi_{n-1} - c) + (\chi_n - c)\,. \tag{4.5}$$

Let also S denote the random walk given by $S_0 \overset{\text{def}}{=} 0$ and $S_{n+1} \overset{\text{def}}{=} S_n + (\chi_{n+1} - c)$. We can rewrite (4.5) in the form

$$R^x_n - \frac{4}{3}c \leqslant \frac{x}{4^n} + S_n - \frac{3}{4}\sum_{k=1}^{n-1}\frac{1}{4^{n-1-k}}S_k. \tag{4.6}$$

Let $\mu \stackrel{\text{def}}{=} \inf\{n \geqslant 1\ ,\ S_n < 0\}$ stand for the first strict descending ladder index of the random walk S. We have

$$\mathbf{P}\{\mu > n\} \leqslant \mathbf{P}\{S_n \geqslant 0\} \leqslant \mathbf{E}\left[e^{\frac{\eta}{8}S_n}\right] = \mathbf{E}\left[e^{\frac{\eta}{8}S_1}\right]^n = \left(\frac{2}{e}\right)^n$$

where we used the fact that S_1 has the same distribution as the random variable $\frac{2}{\eta}U(1) - \frac{8}{\eta}$ under $\mathbf{P}_0$ (and $U(1)$ under $\mathbf{P}_0$ has an exponential distribution with mean 2). Therefore, μ is almost surely finite. Setting $c_{12} = 1 - \log 2 > 0$, we get

$$\mathbf{P}\{\mu > n\} \leqslant e^{-c_{12}n} \quad \text{for all } n \in \mathbb{N}.$$

Finally, from the definition of μ, we have $S_\mu < 0$ and $S_n \geqslant 0$ for all $0 \leqslant n < \mu$ hence

$$S_\mu - \frac{3}{4}\sum_{k=1}^{\mu-1} \frac{1}{4^{n-1-k}} S_k \leqslant 0.$$

Combining this inequality with (4.6) and the fact that $\mu \geqslant 1$, we obtain, whenever $x \geqslant x_0 \stackrel{\text{def}}{=} \frac{16}{3}c$:

$$R^x_\mu \leqslant \frac{4}{3}c + \frac{x}{4^\mu} \leqslant \frac{4}{3}c + \frac{x}{4} \leqslant \frac{x}{2}.$$

Thus, for all $x \geqslant x_0$,

$$\mathbf{P}\left\{R^x_1 > \frac{x}{2}\ ,\ \ldots\ ,\ R^x_n > \frac{x}{2}\right\} \leqslant \mathbf{P}\{\mu > n\} \leqslant e^{-c_{12}n}.$$

This completes the proof of the lemma. ∎

Proof of Proposition 4.2. Set $y_0 \stackrel{\text{def}}{=} x_0/2$ where x_0 is the constant of the previous lemma. This lemma ensures that for all x, y such that $y_0 < y < x$ and $\frac{x}{y} \leqslant 2$, we have

$$\mathbf{P}_x\left\{\tau_{[0,y]} > t\right\} \leqslant c_7 e^{-c_8 t} \quad \text{for all } t > 0. \tag{4.7}$$

Let us now fix x, y such that $y_0 \leqslant y < x$. Define the sequence (z_n) by $z_0 \stackrel{\text{def}}{=} x$ and $z_{n+1} \stackrel{\text{def}}{=} z_n/2$. Set $m \stackrel{\text{def}}{=} 1 + \lfloor \log(x/y)/\log(2) \rfloor$, then

$$x = z_0 \geqslant z_1 \geqslant \ldots \geqslant z_{m-1} \geqslant y \geqslant z_m$$

Thus,

$$\mathbf{P}_x\left\{\tau_{[0,y]} > t\right\} \leqslant \mathbf{P}_x\left\{\tau_{[0,z_1]} > \frac{t}{m}\right\} + \sum_{i=1}^{m-2} \mathbf{P}_x\left\{\tau_{[0,z_{i+1}]} - \tau_{[0,z_i]} > \frac{t}{m}\right\} + \mathbf{P}_x\left\{\tau_{[0,y]} - \tau_{[0,z_{m-1}]} > \frac{t}{m}\right\}.$$

Making use of the Markov property of Z for the stopping times $\tau_{[0,x_i]}$ combined with Lemma 3.4, we get

$$\mathbf{P}_x\left\{\tau_{[0,y]} > t\right\} \leqslant \mathbf{P}_{z_0}\left\{\tau_{[0,z_1]} > \frac{t}{m}\right\} + \sum_{i=1}^{m-2}\mathbf{P}_{z_i}\left\{\tau_{[0,z_{i+1}]} > \frac{t}{m}\right\} + \mathbf{P}_{z_{m-1}}\left\{\tau_{[0,y]} > \frac{t}{m}\right\}.$$

According to (4.7) each term on the r.h.s. of this last inequality is smaller than $c_7 e^{-c_8 t/m}$. Hence, choosing c_3, c_4 large enough, we get

$$\mathbf{P}_x\left\{\tau_{[0,y]} > t\right\} \leqslant m c_7 e^{-c_8 \frac{t}{m}} \leqslant c_3\left(\log(x/y)+1\right) e^{-\frac{c_4}{\log(x/y)+1}t}.$$

■

Proof of Proposition 4.3. We have already proved Proposition 4.1 and Proposition 4.2. Recall that Z has no positive jumps. In view of Lemma 3.4, it simply remains to prove that for any $0 < y < y_0$ (y_0 is the constant of Proposition 4.2), we have

$$\mathbf{P}_{y_0}\left\{\tau_{[0,y]} > t\right\} \leqslant c_{13,y_0,y} e^{-c_{14,y_0,y} t} \quad \text{for all } t > 0. \tag{4.8}$$

Fix $y < y_0$, pick $z > y_0$ and define the sequence (ν_n^z) by

$$\begin{cases} \nu_0^z & \overset{\text{def}}{=} \; 0, \\ \nu_{n+1}^z & \overset{\text{def}}{=} \; \inf\left\{t > \nu_n^z\ ,\ Z_t = y_0 \text{ and } \sup_{\nu_n^z \leqslant s \leqslant t} Z_s \geqslant z\right\}. \end{cases}$$

Making use of Proposition 4.1 and Proposition 4.2, we check that ν_n^z is finite for all n, $\mathbf{P}_{y_0}$-a.s. More precisely, these propositions yield

$$\mathbf{P}_{y_0}\left\{\nu_1^z > t\right\} \leqslant c_{15,y_0,z} e^{-c_{16,y_0,z} t} \quad \text{for all } t > 0.$$

Since the sequence $(\nu_{n+1}^z - \nu_n^z)_{n \geqslant 0}$ is i.i.d, Cramer's large deviation theorem ensure that there exist $c_{17,y_0,z}, c_{18,y_0,z}, c_{19,y_0,z} > 0$ such that

$$\mathbf{P}_{y_0}\left\{\nu_n^z > c_{17,y_0,z} n\right\} \leqslant c_{18,y_0,z} e^{-c_{19,y_0,z} n} \quad \text{for all } n \in \mathbb{N}. \tag{4.9}$$

Notice that $\lim_{z\to\infty} \nu_1^z = \infty$ $\mathbf{P}_{y_0}$-a.s, thus

$$\mathbf{P}_{y_0}\left\{\tau_{[0,y]} < \nu_1^z\right\} \underset{z\to\infty}{\longrightarrow} \mathbf{P}_{y_0}\left\{\tau_{[0,y]} < \infty\right\}. \tag{4.10}$$

According to Proposition 3.5, we have $\mathbf{P}_{y_0}\left\{Z_\infty \in (0,y]\right\} > 0$ so the limit in (4.10) is strictly positive. Thus, we may choose z large enough such that $\mathbf{P}_{y_0}\left\{\tau_{[0,y]} > \nu_1^z\right\} = d < 1$. Repeated use of the Markov property of Z for the stopping times ν_i^z yields

$$\mathbf{P}_{y_0}\left\{\tau_{[0,y]} > \nu_n^z\right\} = \mathbf{P}_{y_0}\left\{\tau_{[0,y]} > \nu_1^z\right\}^n = d^n. \tag{4.11}$$

Finally, setting $n = \lfloor t/c_{15,y_0,z}\rfloor$, we get from (4.9) and (4.11):

$$\begin{aligned} \mathbf{P}_{y_0}\left\{\tau_{[0,y]} > t\right\} & \leqslant \mathbf{P}_{y_0}\left\{\nu_n^z > t\right\} + \mathbf{P}_{y_0}\left\{\tau_{[0,y]} > \nu_n^z\right\} \\ & \leqslant c_{18,y_0,z} e^{-c_{19,y_0,z} n} + d^n \\ & \leqslant c_{20,y_0,y} e^{-c_{21,y_0,y} t}. \end{aligned}$$

■

We need the following lemma before giving the proof of Proposition 4.4.

Lemma 4.6

There exist $k_0 > 1$ and $y_1 > 1$ such that

$$\mathbf{P}_y\left\{\tau_{k_0 y} < \tau_{y/k_0}\right\} \leqslant \frac{1}{4} \quad \textit{for all } y \geqslant y_1.$$

Proof of Lemma 4.6. Let us choose $k > 1$ and y such that

$$6^4 k^5 < y. \tag{4.12}$$

We use the notations $m \overset{\text{def}}{=} \frac{1}{4}\log\left(\frac{y}{k}\right)$ and $\gamma_m \overset{\text{def}}{=} \inf\{t \geqslant 0, \mathbb{V}_t < -m\}$. Set $\mathcal{E}_1 \overset{\text{def}}{=} \{\gamma_m \leqslant e^m\}$. Since $\Phi(\kappa/2) < 0$, we deduce that

$$\mathbf{P}\{\mathcal{E}_1^c\} \leqslant \mathbf{P}\{\mathbb{V}_{e^m} > -m\} \leqslant e^{\frac{\kappa}{2}m}\mathbf{E}\left[e^{\frac{\kappa}{2}\mathbb{V}_{e^m}}\right] = e^{\frac{\kappa}{2}m + e^m\Phi(\kappa/2)} \underset{y/k\to\infty}{\longrightarrow} 0.$$

Define $\mathcal{E}_2 \overset{\text{def}}{=} \{\sup_{s\geqslant 0}\mathbb{V}_s < \log(k/7)\}$. Since $\mathbb{V}$ diverges to $-\infty$, its overall supremum is finite (it has an exponential distribution with parameter κ), therefore

$$\mathbf{P}\{\mathcal{E}_2^c\} \underset{k\to\infty}{\longrightarrow} 0.$$

Define also

$$\mathcal{E}_3 \overset{\text{def}}{=} \left\{U(t) \leqslant 2\Big(y + \frac{y}{k} + t^2\Big) \quad \text{for all } t \geqslant 0\right\}.$$

We noticed in the proof of Lemma 4.5 that U under $\mathbf{P}_y$ is stochastically dominated by $2(y + B^2 + \widetilde{B}^2)$ where B and $\widetilde{B}$ are two independent squared Brownian motions. Therefore, the law of the iterated logarithm for Brownian motion (see for instance chap II of [RY99]) entails

$$\mathbf{P}_y\{\mathcal{E}_3^c\} \leqslant \mathbf{P}_0\left\{\text{there exists } t \geqslant 0 \text{ with } U(t) > \frac{y}{k} + t^2\right\} \underset{y/k\to\infty}{\longrightarrow} 0.$$

We finally set $\mathcal{E}_4 \overset{\text{def}}{=} \mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3$. Our previous estimates ensure that $\mathbf{P}_y\{\mathcal{E}_4^c\} < 1/4$ whenever k and y/k are both large enough. Moreover, on the set $\mathcal{E}_4$, for all $0 \leqslant t \leqslant \gamma_m$,

$$a(t)^2 = \left(\int_0^t e^{-\mathbb{V}_s}ds\right)^2 \leqslant \left(\int_0^{\gamma_m} e^{-\mathbb{V}_s}ds\right)^2 \leqslant (\gamma_m e^m)^2 \leqslant e^{4m} = \frac{y}{k}.$$

Thus, on the one hand, on $\mathcal{E}_4$, for $k \geqslant 1$ and for all $0 \leqslant t \leqslant \gamma_m$

$$Z_t = e^{\mathbb{V}_t}U(a(t)) \leqslant e^{(\sup_{s\geqslant 0}\mathbb{V}_s)}2\left(y + \frac{y}{k} + a(t)^2\right) \leqslant \frac{2k}{7}\left(y + \frac{y}{k} + \frac{y}{k}\right) < ky.$$

On the other hand, on $\mathcal{E}_4$, since $\mathbb{V}_{\gamma_m} \leqslant -m$,

$$Z_{\gamma_m} \leqslant e^{-m}2\left(y + \frac{y}{k} + \frac{y}{k}\right) \leqslant 6ye^{-m} \leqslant \frac{y}{k}$$

where we used (4.12) for the last inequality. Therefore,

$$\mathbf{P}_y\left\{\tau_{[ky,\infty)} < \tau_{[0,y/k]}\right\} \leqslant \mathbf{P}_y\{\mathcal{E}_4^c\} < \frac{1}{4} \qquad \text{for all } k,\ \tfrac{y}{k} \text{ large enough.}$$

Finally, since Z has no positive jumps, we also have

$$\mathbf{P}_y\{\tau_{[ky,\infty)} < \tau_{[0,y/k]}\} = \mathbf{P}_y\{\tau_{ky} < \tau_{y/k}\}.$$

■

Proof of Proposition 4.4. Let y_1 and k_0 denote the constants of the previous lemma and let $y \geqslant y_1$. Define the sequence (μ_n) of stopping times for Z:

$$\begin{cases} \mu_0 & \stackrel{\text{def}}{=} \; 0, \\ \mu_{n+1} & \stackrel{\text{def}}{=} \; \inf\left\{t > \mu_n \, , \, Z_t = k_0 Z_{\mu_n} \text{ or } Z_t = \frac{1}{k_0} Z_{\mu_n}\right\}. \end{cases}$$

Proposition 4.1 ensures that $\mu_n < \infty$ for all n, $\mathbf{P}_y$-a.s. The Markov property of Z also implies that the sequence $(Z_{\mu_n}\, , \, n \in \mathbb{N})$ is, under $\mathbf{P}_y$, a Markov chain starting from y and taking values in $\{k_0^n y \, , \, n \in \mathbb{Z}\}$. Moreover, according to the previous lemma

$$\mathbf{P}_y\Big\{Z_{\mu_{n+1}} = k_0 Z_{\mu_n} | Z_{\mu_n} > y_1\Big\} = 1 - \mathbf{P}_y\Big\{Z_{\mu_{n+1}} = \frac{1}{k_0} Z_{\mu_n} | Z_{\mu_n} > y_1\Big\} < \frac{1}{4}.$$

Thus, if $(S_n \, , \, n \geqslant 0)$ now denotes a random walk such that

$$\begin{cases} \mathbf{P}\{S_0 = 0\} = 1, \\ \mathbf{P}\left\{S_{n+1} = S_n + 1\right\} = 1 - \mathbf{P}\left\{S_{n+1} = S_n - 1\right\} = \frac{1}{4}, \end{cases}$$

then we deduce from the previous lemma that $\big(Z_{\mu_n}\big)_{0 \leqslant n \leqslant \inf\{n \geqslant 0, Z_{\mu_n} \leqslant y_1\}}$ under $\mathbf{P}_y$ is stochastically dominated by $\big(y k_0^{S_n}\big)_{0 \leqslant n \leqslant \inf\{n \geqslant 0, y k_0^{S_n} \leqslant y_1\}}$. In particular, for all $y \geqslant y_1$ and all $p \in \mathbb{N}^*$,

$$\mathbf{P}_y\Big\{(Z_{\mu_n}) \text{ hits } [k_0^p y, \infty) \text{ before it hits } [0, y_1]\Big\} \leqslant \mathbf{P}\Big\{\sup_n S_n \geqslant p\Big\}.$$

Since Z has no positive jumps, we obtain, for all $y \geqslant y_1$ and all $p \in \mathbb{N}^*$

$$\mathbf{P}_y\left\{\tau_{y k_0^p} < \tau_{[0,y_1]}\right\} \leqslant \mathbf{P}\Big\{\sup_n S_n \geqslant p\Big\}. \tag{4.13}$$

Note that the last inequality is trivial when $y \leqslant y_1$. Also, since S is transient towards $-\infty$, its overall supremum is finite and, given $\varepsilon > 0$, we may find p_0 such that $\mathbf{P}\{\sup_n S_n \geqslant p_0\} \leqslant \varepsilon$. Setting $\lambda_0 \stackrel{\text{def}}{=} k_0^{p_0}$, we deduce from (4.13) that

$$\sup_{\lambda \geqslant \lambda_0} \sup_{y \geqslant 1} \mathbf{P}_y\left\{\tau_{y\lambda} < \tau_{[0,y_1]}\right\} \leqslant \varepsilon.$$

Note that $\tau_{[0,y_1]} \leqslant \tau_1$ (because $y_1 \geqslant 1$) and recall that Z has no positive jumps. Using the Markov property of Z combined with Lemma 3.4, we get, for all $y \geqslant 1$ and all $\lambda > \lambda_0$,

$$\begin{aligned} \mathbf{P}_y\left\{\tau_{\lambda y} < \tau_1\right\} &\leqslant \mathbf{P}_y\left\{\tau_{\lambda y} < \tau_{[0,y_1]}\right\} + \mathbf{P}_y\left\{\tau_{[0,y_1]} \leqslant \tau_{\lambda y}\right\} \mathbf{P}_{y_1}\left\{\tau_{\lambda y} < \tau_1\right\} \\ &\leqslant \varepsilon + \mathbf{P}_{y_1}\left\{\tau_\lambda < \tau_1\right\}. \end{aligned}$$

Since $\mathbf{P}_{y_1}\{\tau_\lambda < \tau_1\}$ converges to 0 as λ tends to infinity, there exists $\lambda_1 > \lambda_0$ such that $\mathbf{P}_{y_1}\{\tau_\lambda < \tau_1\} \leqslant \varepsilon$ for all $\lambda > \lambda_1$. Thus, we have proved that

$$\sup_{y \geqslant 1} \mathbf{P}_y\left\{\tau_{\lambda y} < \tau_1\right\} \leqslant 2\varepsilon \qquad \text{for all } \lambda > \lambda_1.$$

■

5 Excursion of Z

5.1 The local time and the associated excursion measure

According to Lemma 3.1 and Lemma 3.2, the Markov process Z is a Feller process in the filtration $\mathcal{F}$ for which 1 is regular for itself and instantaneous. It is therefore a 'nice' Markov process in the sense of chap IV of [Ber96a] and we may consider a local time process $(L_t\,,\,t\geqslant 0)$ of Z at level 1. Precisely, the local time process is such that

- $(L_t\,,\,t\geqslant 0)$ is a continuous, $\mathcal{F}$-adapted process which increases on the closure of the set $\{t\geqslant 0\,,\,Z_t=1\}$.
- For any stopping time T such that $Z_T=1$ a.s, the shifted process $(Z_{t+T},L_{T+t}-L_T)_{t\,\geqslant\,0}$ is independent of $\mathcal{F}_t$ and has the same law as $(Z_t,L_t)_{t\,\geqslant\,0}$ under $\mathbf{P}_1$.

We can also consider the associated excursion measure $\mathbf{n}$ of the excursions of Z away from 1 which we define as in IV.4 of [Ber96a]. We denote by $(\epsilon_t\,,\,0\leqslant t\leqslant\zeta)$ a generic excursion with lifetime ζ. Let L^{-1} stand for the right continuous inverse of L:

$$L_t^{-1}\stackrel{\text{def}}{=}\inf\{s\geqslant 0\,,\,L_s>t\}\qquad\text{for all } t\geqslant 0, \tag{5.1}$$

Note that $L_t^{-1}<\infty$ for all t since Z is recurrent.

Lemma 5.1

Under $\mathbf{P}_1$, the process L^{-1} is a subordinator whose Laplace exponent φ defined by $\mathbf{E}_1\left[e^{-\lambda L_t^{-1}}\right]\stackrel{\text{def}}{=}e^{-t\varphi(\lambda)}$ has the form

$$\varphi(\lambda)=\lambda\int_0^\infty e^{-\lambda r}\mathbf{n}\{\zeta>r\}\,dr.$$

Moreover, there exist $c_{22},c_{23}>0$ such that $\mathbf{n}\{\zeta>r\}\leqslant c_{22}e^{-c_{23}r}$ for all $r\geqslant 1$. In particular, $\mathbf{n}[\zeta]<\infty$.

Proof. According to Theorem 8, p114 of [Ber96a], L^{-1} is a subordinator and its Laplace exponent φ has the form

$$\varphi(\lambda)=\lambda\mathbf{d}+\lambda\int_0^\infty e^{-\lambda r}\mathbf{n}\{\zeta>r\}\,dr. \tag{5.2}$$

Moreover, the drift coefficient $\mathbf{d}$ is such that $\mathbf{d}L(t)=\int_0^t\mathbf{1}_{\{Z_t=1\}}dt$ $\mathbf{P}_1$-a.s. (*c.f.* Corollary 6, p112 of [Ber96a]). Thus, Lemma 3.3 implies that $\mathbf{d}=0$. We now estimate the tail distribution of ζ under $\mathbf{n}$. Recall that $\tau_A(\epsilon)$ stands for the hitting time of the set A for the excursion ϵ:

$$\tau_A(\epsilon)\stackrel{\text{def}}{=}\inf\{t\in[0,\zeta]\,,\,\epsilon_t\in A\}\qquad\text{(with the convention }\inf\emptyset=\infty).$$

Since a generic excursion ϵ has no positive jumps, the Markov property yields, for $r > 1$,

$$\begin{aligned}\mathbf{n}\{\zeta > r\} &\leqslant \mathbf{n}\{\tau_2(\epsilon) \leqslant 1\ ,\ \zeta > r\} + \mathbf{n}\{\epsilon_1 \leqslant 2\ ,\ \zeta > r\} \\ &\leqslant \mathbf{n}\{\tau_2(\epsilon) \leqslant 1\ ,\ \zeta > 1\}\,\mathbf{P}_2\{\tau_1 > r-1\} + \mathbf{n}\{\epsilon_1 \leqslant 2\ ,\ \zeta > 1\} \sup_{x\in(0,2)} \mathbf{P}_x\{\tau_1 > r-1\} \\ &\leqslant 2\mathbf{n}\{\zeta > 1\} \sup_{x\in(0,2]} \mathbf{P}_x\{\tau_1 > r-1\}.\end{aligned}$$

Combining Lemma 3.4 and Proposition 4.3, we also have

$$\sup_{x\in(0,2]} \mathbf{P}_x\{\tau_1 > r-1\} \leqslant \max\left(\mathbf{P}_0\{\tau_1 > r-1\}, \mathbf{P}_2\{\tau_1 > r-1\}\right) \leqslant c_{24} e^{-c_{25}(r-1)}.$$

This yields our estimate for $\mathbf{n}\{\zeta > r\}$. Finally, any excursion measure fulfills the condition $\int_0^1 \mathbf{n}\{\zeta > r\}\,dr < \infty$ thus $\mathbf{n}[\zeta] = \int_0^\infty \mathbf{n}\{\zeta > r\}\,dr < \infty$. ■

Lemma 5.2

Let f be a non-negative measurable function. For all $\lambda > 0$, we have

$$\text{(a)}\qquad \mathbf{E}_1\left[\int_0^\infty e^{-\lambda t} f(Z_t)dt\right] = \frac{1}{\varphi(\lambda)}\mathbf{n}\left[\int_0^\zeta e^{-\lambda t} f(\epsilon_t)dt\right],$$

$$\begin{aligned}\text{(b)}\qquad \mathbf{E}_1\left[\left(\int_0^\infty e^{-\lambda t} f(Z_t)dt\right)^2\right] &= \frac{1}{\varphi(2\lambda)}\mathbf{n}\left[\left(\int_0^\zeta e^{-\lambda t} f(\epsilon_t)dt\right)^2\right] \\ &+ \frac{2}{\varphi(\lambda)\varphi(2\lambda)}\mathbf{n}\left[\int_0^\zeta e^{-\lambda t} f(\epsilon_t)dt\right]\mathbf{n}\left[e^{-\lambda\zeta}\int_0^\zeta e^{-\lambda t} f(\epsilon_t)dt\right].\end{aligned}$$

Proof. Assertion (a) is a direct application of the compensation formula in excursion theory combined with the fact that the set $\{t \geqslant 0, Z_t = 1\}$ has 0 Lebesgue measure under $\mathbf{P}_1$ (Lemma 3.3). Compare with the example p120 of [Ber96a] for details. We now prove (b). We use the notation $G_\lambda f(x) = \mathbf{E}_x\left[\int_0^\infty e^{-\lambda t} f(Z_t)dt\right]$. By a change of variable and with the help of the Markov property of Z,

$$\begin{aligned}\mathbf{E}_1\left[\left(\int_0^\infty e^{-\lambda t} f(Z_t)dt\right)^2\right] &= 2\mathbf{E}_1\left[\int_0^\infty e^{-\lambda t} f(Z_t)\int_t^\infty e^{-\lambda s} f(Z_s)ds\,dt\right] \\ &= 2\mathbf{E}_1\left[\int_0^\infty e^{-2\lambda t} f(Z_t) G_\lambda f(Z_t)dt\right].\end{aligned}$$

Thus, using (a) with the function $x \mapsto f(x)G_\lambda f(x)$, we get that

$$\mathbf{E}_1\left[\left(\int_0^\infty e^{-\lambda t} f(Z_t)dt\right)^2\right] = \frac{2}{\varphi(2\lambda)}\mathbf{n}\left[\int_0^\zeta e^{-2\lambda t} f(\epsilon_t) G_\lambda f(\epsilon_t)dt\right]. \tag{5.3}$$

We also have, with the help of the Markov property,

$$\begin{aligned} G_\lambda f(z) &= \mathbf{E}_z\left[\int_0^{\tau_1} e^{-\lambda s} f(Z_s)ds\right] + \mathbf{E}_z\left[\int_{\tau_1}^{\infty} e^{-\lambda s} f(Z_s)ds\right] \\ &= \mathbf{E}_z\left[\int_0^{\tau_1} e^{-\lambda s} f(Z_s)ds\right] + \mathbf{E}_z\left[e^{-\lambda\tau_1}\right] G_\lambda f(1). \end{aligned}$$

Therefore, we may rewrite (5.3) as

$$\frac{2}{\varphi(2\lambda)}\mathbf{n}\left[\int_0^{\zeta} e^{-2\lambda t} f(\epsilon_t)\mathbf{E}_{\epsilon_t}\left[\int_0^{\tau_1} e^{-\lambda s} f(Z_s)ds\right]dt\right] + \frac{2}{\varphi(2\lambda)}\mathbf{n}\left[\int_0^{\zeta} e^{-2\lambda t} f(\epsilon_t)\mathbf{E}_{\epsilon_t}\left[e^{-\lambda\tau_1}\right]dt\right] G_\lambda f(1). \quad (5.4)$$

We deal with each term separately. Making use of the Markov property of the excursion ϵ at time t under $\mathbf{n}(\cdot|\zeta > t)$ and with a change of variable, the first term of the last sum is equal to

$$\frac{2}{\varphi(2\lambda)}\mathbf{n}\left[\int_0^{\zeta} e^{-2\lambda t} f(\epsilon_t)\int_t^{\zeta} e^{-\lambda(s-t)} f(\epsilon_s)dsdt\right] = \frac{1}{\varphi(2\lambda)}\mathbf{n}\left[\left(\int_0^{\zeta} e^{-\lambda t} f(\epsilon_t)dt\right)^2\right]. \quad (5.5)$$

Similarly, the second term of (5.4) may be rewritten

$$\frac{2G_\lambda f(1)}{\varphi(2\lambda)}\mathbf{n}\left[\int_0^{\zeta} e^{-2\lambda t} f(\epsilon_t)e^{-\lambda(\zeta-t)}dt\right] = \frac{2}{\varphi(\lambda)\varphi(2\lambda)}\mathbf{n}\left[\int_0^{\zeta} e^{-\lambda t} f(\epsilon_t)dt\right]\mathbf{n}\left[e^{-\lambda\zeta}\int_0^{\zeta} e^{-\lambda t} f(\epsilon_t)dt\right], \quad (5.6)$$

where we used (a) for the expression of $G_\lambda f(1)$ for the last equality. The combination of (5.3),(5.4),(5.5) and (5.6) yields (b). ■

Corolary 5.3

Let g be a measurable, non-negative function which is continuous almost everywhere with respect to the Lebesgue measure. Then

$$\mathbf{n}\left[\int_0^{\zeta} g(\epsilon_t)dt\right] = \mathbf{n}[\zeta]\,\mathbf{E}[g(Z_\infty)].$$

Proof. In view of the monotone convergence theorem, we assume that g is bounded. First, using (a) of the previous lemma with the function $f = 1$,

$$\frac{\varphi(\lambda)}{\lambda} = \mathbf{n}\left[\int_0^{\zeta} e^{-\lambda t}dt\right] \underset{\lambda\to 0+}{\longrightarrow} \mathbf{n}[\zeta]. \quad (5.7)$$

Thus, using again (a) of Lemma 5.2 but now with the function g, and with the help of the monotone convergence theorem, we find

$$\mathbf{n}\left[\int_0^\zeta g(\epsilon_t)dt\right] = \lim_{\lambda\to 0+} \varphi(\lambda)\mathbf{E}_1\left[\int_0^\infty e^{-\lambda t} g(Z_t)dt\right] = \mathbf{n}[\zeta] \lim_{\lambda\to 0+} \mathbf{E}_1\left[\lambda\int_0^\infty e^{-\lambda t} g(Z_t)dt\right].$$

By a change of variable and using Fubini's theorem, we also have

$$\mathbf{E}_1\left[\lambda\int_0^\infty e^{-\lambda t} g(Z_t)dt\right] = \int_0^\infty \mathbf{E}_1\left[g(Z_{y/\lambda})\right] e^{-y}dy.$$

For any $y > 0$, $Z_{y/\lambda}$ converges in law towards Z_∞ as $\lambda \to 0^+$. Moreover, according to Proposition 3.5, Z_∞ has a continuous density with respect to the Lebesgue measure and g is continuous almost everywhere, hence $\lim_{\lambda\to 0+} \mathbf{E}_1[g(Z_{y/\lambda})] = \mathbf{E}[g(Z_\infty)]$. Making use of the dominated convergence theorem, we conclude that

$$\mathbf{E}_1\left[\lambda\int_0^\infty e^{-\lambda t} g(Z_t)dt\right] \underset{\lambda\to 0+}{\longrightarrow} \int_0^\infty \mathbf{E}\left[g(Z_\infty)\right] e^{-y}dy = \mathbf{E}\left[g(Z_\infty)\right].$$

■

Corolary 5.4

Recall that $\mathbf{m} \overset{\text{def}}{=} \frac{-2}{\Phi(1)}$. *When* $\kappa > 1$ *(*i.e. *when* $\Phi(1) < 0$*), we have*

$$\mathbf{n}\left[\int_0^\zeta \epsilon_t dt\right] = \mathbf{n}\left[\zeta\right]\mathbf{m}.$$

Proof. Corollary 5.3 yields $\mathbf{n}\left[\int_0^\zeta \epsilon_t dt\right] = \mathbf{n}\left[\zeta\right]\mathbf{E}\left[Z_\infty\right]$. According to Proposition 3.5, Z_∞ has the same law as $U(1)\int_0^\infty e^{\mathbb{V}_s}ds$ under $\mathbf{P}_0$. Moreover, $U(1)$ under $\mathbf{P}_0$ has an exponential distribution with mean 2 and is independent of $\mathbb{V}$, hence

$$\mathbf{E}\left[Z_\infty\right] = 2\int_0^\infty \mathbf{E}\left[e^{\mathbb{V}_s}\right]ds = 2\int_0^\infty e^{s\Phi(1)}ds = -\frac{2}{\Phi(1)}.$$

■

5.2 Maximum of an excursion

The goal of this subsection is to study the distribution of the supremum of an excursion. Our main result is contained in the following proposition.

Proposition 5.5

We have

$$\mathbf{n}\left\{\tau_z(\epsilon) < \infty\right\} \underset{z\to\infty}{\sim} \mathbf{n}\left[\zeta\right]\frac{2^\kappa\Gamma(\kappa)\kappa^2\mathbf{K}}{z^\kappa}.$$

Of course, this estimate may be rewritten

$$\mathbf{n}\left\{\sup_{[0,\zeta]} \epsilon > z\right\} \underset{z\to\infty}{\sim} \mathbf{n}\,[\zeta]\, \frac{2^{\kappa}\Gamma(\kappa)\kappa^2\mathbf{K}}{z^{\kappa}}.$$

The proof relies on two lemmas.

Lemma 5.6

We have

$$\mathbf{E}\left[\int_0^{\infty} \mathbf{1}_{\{\mathbb{V}_t>0\}}dt\right] = \mathbf{E}\left[\int_0^{\infty} \mathbf{1}_{\{\mathbb{V}_t \geqslant 0\}}dt\right] = \frac{1}{\kappa\Phi'(\kappa)}.$$

Lemma 5.7

We have

$$\lim_{z\to\infty} \mathbf{E}_z \left[\int_0^{\tau_1} \mathbf{1}_{\{Z_t \geqslant z\}}dt\right] = \frac{1}{\kappa\Phi'(\kappa)}.$$

Let us for the time being admit the lemmas and give the proof of the proposition.

Proof of Proposition 5.5. Since a generic excursion ϵ under $\mathbf{n}$ has no positive jumps, the Markov property yields

$$\begin{aligned}\mathbf{n}\left[\int_0^{\zeta} \mathbf{1}_{\{\epsilon_s>z\}}ds\right] &= \mathbf{n}\left[\mathbf{1}_{\{\tau_z(\epsilon)<\infty\}} \int_{\tau_z(\epsilon)}^{\zeta} \mathbf{1}_{\{\epsilon_s>z\}}ds\right] \\ &= \mathbf{n}\left\{\tau_z(\epsilon) < \infty\right\} \mathbf{E}_z\left[\int_0^{\tau_1} \mathbf{1}_{\{Z_s>z\}}ds\right]. \end{aligned} \tag{5.8}$$

On the one hand, from Corollary 5.3 and Proposition 3.5,

$$\mathbf{n}\left[\int_0^{\zeta} \mathbf{1}_{\{\epsilon_s>z\}}ds\right] = \mathbf{n}\,[\zeta]\,\mathbf{P}\left\{Z_\infty > z\right\} \underset{z\to\infty}{\sim} \mathbf{n}\,[\zeta]\, \frac{2^{\kappa}\Gamma(\kappa+1)\mathbf{K}}{\Phi'(\kappa)z^{\kappa}}. \tag{5.9}$$

On the other hand, according to Lemma 5.7,

$$\mathbf{E}_z\left[\int_0^{\tau_1} \mathbf{1}_{\{Z_s>z\}}ds\right] \underset{z\to\infty}{\longrightarrow} \frac{1}{\kappa\Phi'(\kappa)}. \tag{5.10}$$

The proposition follows from the combination of (5.8), (5.9) and (5.10). ■

Proof of Lemma 5.6. Since $\mathbb{V}$ has no positive jumps, it is not a compound Poisson process, therefore Proposition 15, p30 of [Ber96a] states that the resolvent measures of $\mathbb{V}$ are diffuse *i.e.* $\mathbf{E}\left[\int_0^{\infty} \mathbf{1}_{\{\mathbb{V}_t=0\}}dt\right] = 0$. Thus,

$$\mathbf{E}\left[\int_0^{\infty} \mathbf{1}_{\{\mathbb{V}_t \geqslant 0\}}dt\right] = \mathbf{E}\left[\int_0^{\infty} \mathbf{1}_{\{\mathbb{V}_t>0\}}dt\right].$$

Let $\Psi : [0,\infty) \mapsto [\kappa,\infty)$ denote the right inverse of the Laplace exponent Φ such that $\Phi \circ \Psi(\lambda) = \lambda$ for all $\lambda \geqslant 0$ (in particular, $\Psi(0) = \kappa$). Then, Exercise 1 p212 of [Ber96a] which is an easy consequence of Corollary 3, p190 of [Ber96a] states that

$$\mathbf{E}\left[\int_0^\infty e^{-\lambda t}\mathbf{1}_{\{\mathbb{V}_t \geqslant 0\}}dt\right] = \frac{\Psi'(\lambda)}{\Psi(\lambda)} \qquad \text{for all } \lambda > 0.$$

Taking the limit as $\lambda \to 0$, we conclude that

$$\mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{\mathbb{V}_t \geqslant 0\}}dt\right] = \frac{\Psi'(0)}{\Psi(0)} = \frac{1}{\kappa\Phi'(\kappa)}.$$

■

Proof of Lemma 5.7. Assume that $z > 1$ and let $\varepsilon > 0$. Note that for $1 < b < z$, we have $\tau_{[0,z/b]} \leqslant \tau_1$ $\mathbf{P}_z$-a.s. Thus, on the one hand

$$\mathbf{E}_z\left[\int_0^{\tau_{[0,\frac{z}{b}]}} \mathbf{1}_{\{Z_t \geqslant z\}}dt\right] \leqslant \mathbf{E}_z\left[\int_0^{\tau_1} \mathbf{1}_{\{Z_t \geqslant z\}}dt\right]. \tag{5.11}$$

On the other hand, the Markov property of Z combined with Lemma 3.4 yield

$$\begin{aligned}
\mathbf{E}_z\left[\int_0^{\tau_1}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right] &= \mathbf{E}_z\left[\int_0^{\tau_{[0,\frac{z}{b}]}}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right] + \mathbf{E}_z\left[\int_{\tau_{[0,\frac{z}{b}]}}^{\tau_1}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right]\\
&\leqslant \mathbf{E}_z\left[\int_0^{\tau_{[0,\frac{z}{b}]}}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right] + \mathbf{E}_{\frac{z}{b}}\left[\int_0^{\tau_1}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right]\\
&= \mathbf{E}_z\left[\int_0^{\tau_{[0,\frac{z}{b}]}}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right] + \mathbf{P}_{\frac{z}{b}}\{\tau_z < \tau_1\}\,\mathbf{E}_z\left[\int_0^{\tau_1}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right].
\end{aligned} \tag{5.12}$$

According to Proposition 4.4, there exists $b_1 > 1$ such that for any $b > b_1$, we have $\sup_{z \geqslant b}\mathbf{P}_{z/b}\{\tau_z < \tau_1\} \leqslant \varepsilon$. Therefore, combining (5.11) and (5.12), for all $z > b > b_1$,

$$\mathbf{E}_z\left[\int_0^{\tau_{[0,\frac{z}{b}]}}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right] \leqslant \mathbf{E}_z\left[\int_0^{\tau_1}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right] \leqslant \frac{1}{1-\varepsilon}\mathbf{E}_z\left[\int_0^{\tau_{[0,\frac{z}{b}]}}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right].$$

Thus, we just need to prove that we may find $b_2 > b_1$ and $z_0 > 0$ such that

$$\frac{1}{\kappa\Phi'(\kappa)} - \varepsilon \leqslant \mathbf{E}_z\left[\int_0^{\tau_{[0,\frac{z}{b_2}]}}\mathbf{1}_{\{Z_t \geqslant z\}}dt\right] \leqslant \frac{1}{\kappa\Phi'(\kappa)} + \varepsilon \quad \text{for all } z \geqslant z_0. \tag{5.13}$$

Recall from Lemma 3.1 that the process Z under $\mathbf{P}_z$ has the same law as the process $(ze^{\mathbb{V}_t}U(a(t)/z)\ ,\ t \geqslant 0)$ under $\mathbf{P}_1$. Thus,

$$\mathbf{P}_z\left\{Z_t \geqslant z\ ,\ \tau_{[0,\frac{z}{b}]} \geqslant t\right\} = \mathbf{P}_1\left\{e^{\mathbb{V}_t}U\left(\frac{a(t)}{z}\right) \geqslant 1\ ,\ \forall s \in [0,t)\ e^{\mathbb{V}_s}U\left(\frac{a(s)}{z}\right) > \frac{1}{b}\right\}. \tag{5.14}$$

Since U is continuous at 0 and starting from 1 under $\mathbf{P}_1$, we also have

$$\sup_{0 \leqslant s \leqslant t}\left|U\left(\frac{a(s)}{z}\right) - 1\right| \xrightarrow[z\to\infty]{\mathbf{P}_1\text{-a.s.}} 0 \qquad \text{for all } t \geqslant 0. \tag{5.15}$$

Combining (5.14) and (5.15), we get, for all fixed $t \geqslant 0$, that

$$\begin{aligned}\liminf_{z\to\infty} \mathbf{P}_z \left\{ Z_t \geqslant z \, , \, \tau_{[0,\frac{z}{b}]} \geqslant t \right\} &\geqslant \mathbf{P}\left\{ e^{\mathbb{V}_t} > 1 \, , \forall s \in [0,t) \; e^{\mathbb{V}_s} > \frac{1}{b} \right\} \\ &= \mathbf{P}\{ \mathbb{V}_t > 0 \, , \, \tau_{(-\infty,-\log b]}(\mathbb{V}) \geqslant t \}.\end{aligned}$$

Thus, by inversion of the sum and from Fatou's Lemma

$$\begin{aligned}\liminf_{z\to\infty} \mathbf{E}_z \left[\int_0^{\tau_{[0,\frac{z}{b}]}} \mathbf{1}_{\{Z_t \geqslant z\}} dt \right] &= \liminf_{z\to\infty} \int_0^\infty \mathbf{P}_z \left\{ Z_t \geqslant z \, , \, \tau_{[0,\frac{z}{b}]} \geqslant t \right\} dt \\ &\geqslant \int_0^\infty \liminf_{z\to\infty} \mathbf{P}_z \left\{ Z_t \geqslant z \, , \, \tau_{[0,\frac{z}{b}]} \geqslant t \right\} dt \\ &\geqslant \int_0^\infty \mathbf{P}\left\{ \mathbb{V}_t > 0 \, , \, \tau_{(-\infty,-\log b]}(\mathbb{V}) \geqslant t \right\} dt \\ &= \mathbf{E}\left[\int_0^{\tau_{(-\infty,-\log b]}(\mathbb{V})} \mathbf{1}_{\{\mathbb{V}_t > 0\}} dt \right].\end{aligned}$$

By use of the monotone convergence theorem, we also have

$$\lim_{b\to\infty} \mathbf{E}\left[\int_0^{\tau_{(-\infty,-\log b]}(\mathbb{V})} \mathbf{1}_{\{\mathbb{V}_t > 0\}} dt \right] = \mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{\mathbb{V}_t > 0\}} dt \right] = \frac{1}{\kappa \Phi'(\kappa)},$$

where we used Lemma 5.6 for the last equality. We may therefore find $b_2 > b_1$ such that for all z large enough

$$\mathbf{E}_z \left[\int_0^{\tau_{[0,\frac{z}{b_2}]}} \mathbf{1}_{\{Z_t \geqslant z\}} dt \right] \geqslant \frac{1}{\kappa \Phi'(\kappa)} - \varepsilon.$$

We still have to prove the upper bound in (5.13). Keeping in mind (5.15), we notice that for all fixed $t \geqslant 0$,

$$\limsup_{z\to\infty} \mathbf{P}_z \left\{ Z_t \geqslant z \, , \, \tau_{[0,\frac{z}{b_2}]} \geqslant t \right\} \leqslant \limsup_{z\to\infty} \mathbf{P}_z \left\{ Z_t \geqslant z \right\} \leqslant \mathbf{P}\left\{ \mathbb{V}_t \geqslant 0 \right\}.$$

Moreover, Proposition 4.2 states that there exist $c_{26,b_2}, c_{27,b_2} > 0$ such that for all z large enough and all $t \geqslant 0$,

$$\mathbf{P}_z \left\{ Z_t \geqslant z \, , \, \tau_{[0,\frac{z}{b_2}]} \geqslant t \right\} \leqslant \mathbf{P}_z \left\{ \tau_{[0,\frac{z}{b_2}]} > t \right\} \leqslant c_{26,b_2} e^{-c_{27,b_2} t}.$$

This domination result enables us to use Fatou's Lemma for the limsup. Thus, just as for the liminf, we now find

$$\begin{aligned}\limsup_{z\to\infty} \mathbf{E}_z \left[\int_0^{\tau_{[0,\frac{z}{b_2}]}} \mathbf{1}_{\{Z_t \geqslant z\}} dt \right] &\leqslant \int_0^\infty \limsup_{z\to\infty} \mathbf{P}_z \left\{ Z_t \geqslant z \, , \, \tau_{[0,\frac{z}{b_2}]} \geqslant t \right\} dt \\ &\leqslant \int_0^\infty \mathbf{P}\left\{ \mathbb{V}_t \geqslant 0 \right\} dt \\ &= \mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{\mathbb{V}_t \geqslant 0\}} dt \right] = \frac{1}{\kappa \Phi'(\kappa)}.\end{aligned}$$

This completes the proof of the lemma. ■

5.3 Integral of an excursion

We now estimate the tail distribution of the area of an excursion. The next proposition is the key to the proof of our main theorem.

Proposition 5.8

We have

$$\mathbf{n}\left\{\int_0^\zeta \epsilon_s ds > x\right\} \underset{x\to\infty}{\sim} \mathbf{n}\left[\zeta\right] \frac{2^\kappa\Gamma(\kappa)\kappa^2\mathbf{K}^2}{\Phi'(\kappa)x^\kappa}.$$

In the rest of this subsection, we assume x to be a large number and we will use the notations

$$m \stackrel{\text{def}}{=} \log^3 x, \tag{5.16}$$

$$y \stackrel{\text{def}}{=} \frac{x}{m} = \frac{x}{\log^3 x}. \tag{5.17}$$

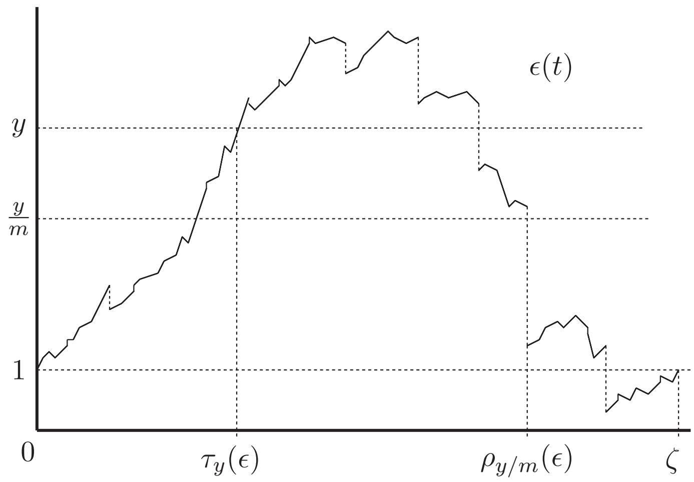

Figure III.3 : An excursion ϵ.

The idea of the proof of the proposition is to decompose the integral of an excursion ϵ such that $\tau_y(\varepsilon) < \infty$ in the form (see figure III.3)

$$\int_0^\zeta \epsilon_s ds = \int_0^{\tau_y(\epsilon)} \epsilon_s ds + \int_{\tau_y(\varepsilon)}^{\rho_{y/m}(\epsilon)} \epsilon_s ds + \int_{\rho_{y/m}(\epsilon)}^{\zeta} \epsilon_s ds \tag{5.18}$$

where $\rho_{y/m} = \inf\{t > \tau_y(\epsilon)\ ,\ \epsilon_t \leqslant y/m\}$. We will show that the contributions of the first and last term on the r.h.s. of (5.18) are negligible. As for the second term, we will show that its distribution is well approximated by the distribution of the random variable $y\int_0^\infty e^{\mathbb{V}_t}dt$. This will give

$$\mathbf{n}\left\{\int_0^\zeta \epsilon_s ds > x\right\} \approx \mathbf{n}\left\{\tau_y(\epsilon) < \infty\right\} \mathbf{P}\left\{y\int_0^\infty e^{\mathbb{V}_t} > x\right\}$$

and the proposition will follow from the estimates obtained in the previous sections. We start with a lemma:

Lemma 5.9

Recall the notations (5.16) and (5.17). We have

$$\mathbf{P}_y\left\{\int_0^{\tau_{[0,\frac{y}{m}]}} Z_s ds > x\right\} \underset{x\to\infty}{\sim} \frac{\mathbf{K}}{\Phi'(\kappa)}\left(\frac{y}{x}\right)^{\kappa}.$$

Proof of Lemma 5.9. Let $(\widetilde{Z}_t\ ,\ t \geqslant 0)$ denote the process

$$\widetilde{Z}_t = ye^{\mathbb{V}_t}U\left(\frac{a(t)}{y}\right).$$

We have already proved in Lemma 3.1 that $\widetilde{Z}$ under $\mathbf{P}_1$ has the same law as Z under $\mathbf{P}_y$. Let $\widetilde{\tau}_A$ denote the hitting time of the set A for the process $\widetilde{Z}$. We must prove that

$$\mathbf{P}_1\left\{\int_0^{\widetilde{\tau}_{[0,\frac{y}{m}]}} e^{\mathbb{V}_t}U\left(\frac{a(t)}{y}\right)dt > \frac{x}{y}\right\} \underset{x\to\infty}{\sim} \frac{\mathbf{K}}{\Phi'(\kappa)}\left(\frac{y}{x}\right)^{\kappa}.$$

We define

$$\begin{aligned}\gamma &\overset{\text{def}}{=} \inf\{t\geqslant 0\ ,\ \mathbb{V}_t < -\log(2m)\}\,,\\ \gamma' &\overset{\text{def}}{=} \inf\{t\geqslant 0\ ,\ \mathbb{V}_t < -\log(m/2)\}\,,\end{aligned}$$

and for $0<\varepsilon<\frac{1}{2}$, set

$$\mathcal{E} \overset{\text{def}}{=} \left\{|U(z)-1|\leqslant\varepsilon \text{ for all } 0\leqslant z\leqslant \frac{2m\gamma}{y}\right\}.$$

Let us first notice that for all $0\leqslant t\leqslant\gamma$, we have $a(t)=\int_0^t e^{-\mathbb{V}_s}ds \leqslant 2m\gamma$ and $e^{\mathbb{V}_\gamma}\leqslant\frac{1}{2m}$. Thus, on $\mathcal{E}$, we have

$$\widetilde{Z}_\gamma = ye^{\mathbb{V}_\gamma}U\left(\frac{a(\gamma)}{y}\right) < \frac{y}{2m}(1+\varepsilon) < \frac{y}{m}. \tag{5.19}$$

We also have $e^{\mathbb{V}_t}\geqslant\frac{2}{m}$ for all $t<\gamma'\leqslant\gamma$. Thus, on $\mathcal{E}$,

$$\widetilde{Z}_t = ye^{\mathbb{V}_t}U\left(\frac{a(t)}{y}\right) \geqslant \frac{2y}{m}(1-\varepsilon) > \frac{y}{m} \quad \text{for all } t<\gamma'. \tag{5.20}$$

Combining (5.19) and (5.20), we deduce that

$$\mathcal{E}\subset\left\{\gamma'\leqslant\widetilde{\tau}_{[0,\frac{y}{m}]}\leqslant\gamma\right\}.$$

Let us for the time being admit that

$$\lim_{x\to\infty}\left(\frac{x}{y}\right)^{\kappa}\mathbf{P}_1\{\mathcal{E}^c\}=0. \tag{5.21}$$

We now write

$$\begin{aligned}\mathbf{P}_1\left\{\int_0^{\widetilde{\tau}_{[0,\frac{y}{m}]}} e^{\mathbb{V}_t} U\left(\frac{a(t)}{y}\right) dt > \frac{x}{y}\right\} &\leqslant \mathbf{P}_1\left\{\int_0^{\widetilde{\tau}_{[0,\frac{y}{m}]}} e^{\mathbb{V}_t} U\left(\frac{a(t)}{y}\right) dt > \frac{x}{y}\,,\,\mathcal{E}\right\} + \mathbf{P}_1\left\{\mathcal{E}^c\right\} \\ &\leqslant \mathbf{P}_1\left\{\int_0^{\widetilde{\tau}_{[0,\frac{y}{m}]}} e^{\mathbb{V}_t}(1+\varepsilon)\, dt > \frac{x}{y}\,,\,\mathcal{E}\right\} + \mathbf{P}_1\left\{\mathcal{E}^c\right\} \\ &\leqslant \mathbf{P}\left\{\int_0^{\infty} e^{\mathbb{V}_t} dt > \frac{x}{(1+\varepsilon)y}\right\} + \mathbf{P}_1\left\{\mathcal{E}^c\right\}.\end{aligned}$$

We have already checked in the proof of Proposition 3.5 that

$$\mathbf{P}\left\{\int_0^{\infty} e^{\mathbb{V}_t} dt > \frac{x}{(1+\varepsilon)y}\right\} \underset{x\to\infty}{\sim} \frac{\mathbf{K}}{\Phi'(\kappa)}\left(\frac{(1+\varepsilon)y}{x}\right)^{\kappa}.$$

Therefore,

$$\limsup_{x\to\infty}\left(\frac{x}{y}\right)^{\kappa}\mathbf{P}_1\left\{\int_0^{\widetilde{\tau}_{[0,\frac{y}{m}]}} e^{\mathbb{V}_t} U\left(\frac{a(t)}{y}\right) dt > \frac{x}{y}\right\} \leqslant \frac{\mathbf{K}(1+\varepsilon)^{\kappa}}{\Phi'(\kappa)}.$$

We now prove the liminf. Since $\gamma' \leqslant \widetilde{\tau}_{[0,\frac{y}{m}]}$ on $\mathcal{E}$,

$$\begin{aligned}\mathbf{P}_1\left\{\int_0^{\widetilde{\tau}_{[0,\frac{y}{m}]}} e^{\mathbb{V}_t} U\left(\frac{a(t)}{y}\right) dt > \frac{x}{y}\right\} &\geqslant \mathbf{P}_1\left\{\int_0^{\widetilde{\tau}_{[0,\frac{y}{m}]}} e^{\mathbb{V}_t} U\left(\frac{a(t)}{y}\right) dt > \frac{x}{y}\,,\,\mathcal{E}\right\} - \mathbf{P}_1\left\{\mathcal{E}^c\right\} \\ &\geqslant \mathbf{P}_1\left\{\int_0^{\gamma'} e^{\mathbb{V}_t}(1-\varepsilon) dt > \frac{x}{y}\,,\,\mathcal{E}\right\} - \mathbf{P}_1\left\{\mathcal{E}^c\right\} \\ &\geqslant \mathbf{P}\left\{\int_0^{\gamma'} e^{\mathbb{V}_t} dt > \frac{x}{(1-\varepsilon)y}\right\} - 2\mathbf{P}_1\left\{\mathcal{E}^c\right\}.\end{aligned}$$

Since $\mathbb{V}_{\gamma'} < -\log(m/2)$, it follows from the Markov property of $\mathbb{V}$ that

$$\begin{aligned}\mathbf{P}\left\{\int_0^{\gamma'} e^{\mathbb{V}_t} dt > \frac{x}{(1-\varepsilon)y}\right\} &\underset{x\to\infty}{\sim} \mathbf{P}\left\{\int_0^{\infty} e^{\mathbb{V}_t} dt > \frac{x}{(1-\varepsilon)y}\right\} \\ &\underset{x\to\infty}{\sim} \frac{\mathbf{K}}{\Phi'(\kappa)}\left(\frac{(1-\varepsilon)y}{x}\right)^{\kappa},\end{aligned}$$

so we obtain the lower bound

$$\liminf_{x\to\infty}\left(\frac{x}{y}\right)^{\kappa}\mathbf{P}_1\left\{\int_0^{\widetilde{\tau}_{[0,\frac{y}{m}]}} e^{\mathbb{V}_t} U\left(\frac{a(t)}{y}\right) dt > \frac{x}{y}\right\} \geqslant \frac{\mathbf{K}\,(1-\varepsilon)^{\kappa}}{\Phi'(\kappa)}.$$

It remains to prove (5.21). To this end, notice that

$$\begin{aligned}\mathbf{P}_1\left\{\mathcal{E}^c\right\} &\leqslant \mathbf{P}\left\{\frac{2m\gamma}{y} \geqslant \frac{m^2}{y}\right\} + \mathbf{P}_1\left\{\sup_{z\in[0,m^2/y]}|U(z)-1| > \varepsilon\right\} \\ &\leqslant \mathbf{P}\left\{\mathbb{V}_{m/2} \geqslant -\log(2m)\right\} + \mathbf{P}_1\left\{\sup_{z\in[0,m^2/y]}|U(z)-1| > \varepsilon\right\}.\end{aligned}$$

Recall that $\Phi(\kappa/2) < 0$. Thus, on the one hand

$$\mathbf{P}\big\{\mathbb{V}_{m/2} \geqslant -\log(2m)\big\} \leqslant (2m)^{\frac{\kappa}{2}}\mathbf{E}\Big[e^{\frac{\kappa}{2}\mathbb{V}_{m/2}}\Big] = (2m)^{\frac{\kappa}{2}}e^{\frac{m}{2}\Phi(\frac{\kappa}{2})} = o\left(\left(\frac{y}{x}\right)^{\kappa}\right).$$

On the other hand, U under $\mathbf{P}_1$ is a squared Bessel process of dimension 2 starting from 1. Thus, it has the same law as $B^2 + \widetilde{B}^2 + 2B + 1$ where B and $\widetilde{B}$ are two independent Brownian motions. Hence,

$$\begin{aligned}\mathbf{P}_1\Big\{\sup_{[0,\frac{m^2}{y}]} |U-1| > \varepsilon\Big\} &\leqslant 2\mathbf{P}\Big\{\sup_{[0,\frac{m^2}{y}]} |B|^2 > \frac{\varepsilon}{4}\Big\} + \mathbf{P}\Big\{\sup_{[0,\frac{m^2}{y}]} |B| > \frac{\varepsilon}{4}\Big\}\\ &\leqslant 3\mathbf{P}\Big\{\sup_{z\in[0,\frac{m^2}{y}]} |B(z)| > \frac{\varepsilon}{4}\Big\}.\end{aligned}$$

Finally, from the exact distribution of $\sup_{[0,1]}|B|$ and the usual estimate on gaussian tails,

$$\mathbf{P}_1\left\{\sup_{[0,t]}|B| > a\right\} \leqslant \frac{2\sqrt{t}}{a}e^{-\frac{a^2}{2t}} \quad \text{for all } a,t>0.$$

Therefore,

$$\mathbf{P}_1\Big\{\sup_{z\in[0,m^2/y]} |U(z)-1| > \varepsilon\Big\} \leqslant \frac{24m}{\varepsilon\sqrt{y}}\exp\left(-\frac{\varepsilon^2 y}{32m^2}\right) = o\left(\left(\frac{y}{x}\right)^{\kappa}\right).$$

This completes the proof of the lemma ■

Proof of Proposition 5.8. We first deal with the liminf, we have

$$\mathbf{n}\left\{\int_0^{\zeta}\epsilon_s ds > x\right\} \geqslant \mathbf{n}\left\{\tau_y(\epsilon) < \infty\ ,\ \int_{\tau_y(\epsilon)}^{\tau_{[0,\frac{y}{m}]}(\epsilon)} \epsilon_s ds > x\right\}.$$

Using the Markov property and the fact that the excursion ϵ does not possess positive jumps, the r.h.s. of this inequality is equal to

$$\mathbf{n}\left\{\tau_y(\epsilon)<\infty\right\}\mathbf{P}_y\left\{\int_0^{\tau_{[0,\frac{y}{m}]}} Z_s ds > x\right\} \underset{x\to\infty}{\sim} \mathbf{n}\left[\zeta\right]\frac{2^{\kappa}\Gamma(\kappa)\kappa^2\mathbf{K}^2}{\Phi'(\kappa)x^{\kappa}},$$

where we used Lemma 5.9 and Proposition 5.5 for the equivalence. Therefore,

$$\liminf_{x\to\infty} x^{\kappa}\mathbf{n}\left\{\int_0^{\zeta}\epsilon_s ds > x\right\} \geqslant \frac{2^{\kappa}\Gamma(\kappa)\kappa^2\mathbf{K}^2}{\Phi'(\kappa)}.$$

We now prove the upper bound. Let $\varepsilon > 0$. We need only to show that

$$\limsup_{x\to\infty} x^{\kappa}\mathbf{n}\left\{\int_0^{\zeta}\epsilon_s ds > (1+2\varepsilon)x\right\} \leqslant \frac{2^{\kappa}\Gamma(\kappa)\kappa^2\mathbf{K}^2}{\Phi'(\kappa)}.$$

According to Lemma 5.1, we have $\mathbf{n}\left\{\zeta \geqslant \log^2 x\right\} = o(x^{-\kappa})$, thus

$$\mathbf{n}\left\{\int_0^{\zeta}\epsilon_s ds > (1+2\varepsilon)x\right\} = \mathbf{n}\left\{\zeta < \log^2 x\, ,\ \int_0^{\zeta}\epsilon_s ds > (1+2\varepsilon)x\right\} + o(x^{-\kappa}).$$

We also note that $\int_0^\zeta \epsilon_s ds \leqslant \zeta \sup_{s\in[0,\zeta]} \epsilon_s$. Since $y = x/\log^3 x$, we deduce that for all x large enough,

$$\begin{aligned}&\left\{\zeta < \log^2 x\ ,\ \int_0^\zeta \epsilon_s ds > (1+2\varepsilon)x\right\}\\ &\qquad = \left\{\tau_y(\varepsilon) < \zeta < \log^2 x\ ,\ \int_0^\zeta \epsilon_s ds > (1+2\varepsilon)x\ ,\ \int_0^{\tau_y(\epsilon)} \epsilon_s ds < \varepsilon x\right\}\\ &\qquad \subset \left\{\tau_y(\varepsilon) < \zeta < \log^2 x\ ,\ \int_{\tau_y(\epsilon)}^\zeta \epsilon_s ds > (1+\varepsilon)x\right\}.\end{aligned}$$

Thus, making use of the Markov property of ϵ for the stopping time $\tau_y(\epsilon)$,

$$\begin{aligned}\mathbf{n}\left\{\int_0^\zeta \epsilon_s ds > (1+2\varepsilon)x\right\} &\leqslant \mathbf{n}\left\{\tau_y(\varepsilon) < \zeta < \log^2 x\ ,\ \int_{\tau_y(\epsilon)}^\zeta \epsilon_s ds > (1+\varepsilon)x\right\} + o(x^{-\kappa})\\ &\leqslant \mathbf{n}\Big\{\tau_y(\varepsilon) < \infty\Big\}\mathbf{P}_y\left\{\int_0^{\tau_1} Z_s ds > (1+\varepsilon)x\ ,\ \tau_1 < \log^2 x\right\} + o(x^\kappa).\end{aligned}$$

In view of Proposition 5.5, it remains to prove that

$$\limsup_{x\to\infty}\left(\frac{x}{y}\right)^\kappa \mathbf{P}_y\left\{\int_0^{\tau_1} Z_s ds > (1+\varepsilon)x\ ,\ \tau_1 < \log^2 x\right\} \leqslant \frac{\mathbf{K}}{\Phi'(\kappa)}.$$

We have

$$\begin{aligned}&\mathbf{P}_y\left\{\int_0^{\tau_1} Z_s ds > (1+\varepsilon)x\ ,\ \tau_1 < \log^2 x\right\}\\ &\qquad \leqslant \mathbf{P}_y\left\{\int_0^{\tau_{[0,\frac{y}{m}]}} Z_s ds > x\right\} + \mathbf{P}_y\left\{\int_{\tau_{[0,\frac{y}{m}]}}^{\tau_1} Z_s ds > \varepsilon x\ ,\ \tau_1 < \log^2 x\right\}.\end{aligned}$$

On the one hand, according to Lemma 5.9,

$$\lim_{x\to\infty}\left(\frac{x}{y}\right)^\kappa \mathbf{P}_y\left\{\int_0^{\tau_{[0,\frac{y}{m}]}} Z_s ds > x\right\} = \frac{\mathbf{K}}{\Phi'(\kappa)}.$$

On the other hand,

$$\begin{aligned}\mathbf{P}_y\left\{\int_{\tau_{[0,\frac{y}{m}]}}^{\tau_1} Z_s ds > \varepsilon x\ ,\ \tau_1 < \log^2 x\right\} &\leqslant \mathbf{P}_y\left\{\sup_{s\in[\tau_{[0,\frac{y}{m}]},\tau_1]} Z_s > \frac{\varepsilon x}{\log^2 x}\right\}\\ &\leqslant \mathbf{P}_{\frac{y}{m}}\left\{\tau_{\frac{\varepsilon x}{\log^2 x}} < \tau_1\right\},\end{aligned} \tag{5.22}$$

where we used the Markov property of Z for the stopping time $\tau_{[0,\frac{y}{m}]}$ combined with Lemma 3.4 and the absence of positive jumps for the last inequality. Since $\frac{y}{m} < \frac{x}{\log^2 x}$, we notice that

$$\begin{aligned}\mathbf{n}\left\{\tau_{\frac{\varepsilon x}{\log^2 x}}(\epsilon) < \infty\right\} &= \mathbf{n}\left\{\tau_{\frac{y}{m}}(\epsilon) < \tau_{\frac{\varepsilon x}{\log^2 x}}(\epsilon) < \infty\right\}\\ &= \mathbf{n}\left\{\tau_{\frac{y}{m}}(\epsilon) < \infty\right\}\mathbf{P}_{\frac{y}{m}}\left\{\tau_{\frac{\varepsilon x}{\log^2 x}} < \tau_1\right\}.\end{aligned}$$

Therefore, (5.22) is also equal to

$$\frac{\mathbf{n}\left\{\tau_{\varepsilon x/\log^2 x}(\epsilon)<\infty\right\}}{\mathbf{n}\left\{\tau_{y/m}(\epsilon)<\infty\right\}} \underset{x\to\infty}{\sim} \left(\frac{y\log^2 x}{\varepsilon x m}\right)^{\kappa} = o\left(\left(\frac{y}{x}\right)^{\kappa}\right),$$

where we used Proposition 5.5 for the equivalence. This concludes the proof. ■

6 The second moment

Recall that $\mathbf{m}=-2/\Phi(1)$. The aim of this section is to calculate $\mathbf{n}\left[(\int_0^\zeta(\epsilon_t-\mathbf{m})dt)^2\right]$ when $\kappa>2$ in term of the Laplace exponent Φ of $\mathbb{V}$. We start with:

Lemma 6.1

When $\kappa>2$, for all $t,z\geqslant 0$,

(a) $\mathbf{E}_z\left[Z_t\right]=\mathbf{m}+(z-\mathbf{m})e^{t\Phi(1)}$.

(b) $$\mathbf{E}_0\left[Z_t^2\right]=\frac{16\left(1-e^{t\Phi(2)}\right)}{\Phi(1)\Phi(2)}+\begin{cases}\frac{16t}{\Phi(1)}e^{t\Phi(1)} & \text{if } \Phi(1)=\Phi(2),\\ \frac{16\left(e^{t\Phi(2)}-e^{t\Phi(1)}\right)}{\Phi(1)(\Phi(2)-\Phi(1))} & \text{otherwise.}\end{cases}$$

Proof. U under $\mathbf{P}_z$ is a squared Bessel process of dimension 2 starting from z, therefore $\mathbf{E}_z[U(x)]=z+2x$. Making use of the independence of U and $\mathbb{V}$, we get

$$\mathbf{E}_z\left[Z_t\right]=\mathbf{E}_z\left[e^{\mathbb{V}_t}U(a(t))\right]=\mathbf{E}\left[e^{\mathbb{V}_t}\mathbf{E}_z[U(a(t))\,|\,\mathbb{V}]\right]=\mathbf{E}\left[e^{\mathbb{V}_t}\left(z+2a(t)\right)\right].$$

We have already noticed that time reversal of the Lévy process $\mathbb{V}$ implies that $e^{\mathbb{V}_t}a(t)$ and $\int_0^t e^{\mathbb{V}_s}ds$ have the same law, therefore

$$\begin{aligned}\mathbf{E}_z\left[Z_t\right] &= z\mathbf{E}\left[e^{\mathbb{V}_t}\right]+2\int_0^t\mathbf{E}\left[e^{\mathbb{V}_s}\right]ds = ze^{t\Phi(1)}+\frac{2}{\Phi(1)}\left(e^{t\Phi(1)}-1\right)\\ &= \mathbf{m}+(z-\mathbf{m})e^{t\Phi(1)}.\end{aligned}$$

We now prove (b). First, the scaling property of U shows that, under $\mathbf{P}_0$, the random variables Z_t and $e^{\mathbb{V}_t}a(t)U(1)$ have the same law. Second, $e^{\mathbb{V}_t}a(t)$ and $\int_0^t e^{\mathbb{V}_s}ds$ also have the same law. Therefore,

$$\mathbf{E}_0\left[Z_t^2\right]=\mathbf{E}_0\left[U(1)^2\right]\mathbf{E}\left[\left(\int_0^t e^{\mathbb{V}_s}ds\right)^2\right]=8\mathbf{E}\left[\left(\int_0^t e^{\mathbb{V}_s}ds\right)^2\right] \tag{6.1}$$

where we used the fact that $\mathbf{E}_0[U(1)^2]=8$ because $U(1)$ under $\mathbf{P}_0$ has an exponential law with mean 2. By a change of variable and making use of the stationarity and the independence of the increments of $\mathbb{V}$, we get

$$\begin{aligned}\mathbf{E}\left[\left(\int_0^t e^{\mathbb{V}_s}ds\right)^2\right] &= 2\int_0^t \mathbf{E}\left[e^{2\mathbb{V}_x}\int_x^t e^{\mathbb{V}_y-\mathbb{V}_x}dy\right]dx \\ &= 2\int_0^t \mathbf{E}\left[e^{2\mathbb{V}_x}\right]\int_0^{t-x}\mathbf{E}\left[e^{\mathbb{V}_y}\right]dy\,dx \\ &= 2\int_0^t e^{x\Phi(2)}\int_0^{t-x}e^{y\Phi(1)}dy\,dx \\ &= \frac{2\left(1-e^{t\Phi(2)}\right)}{\Phi(1)\Phi(2)} + \begin{cases}\frac{2t}{\Phi(1)}e^{t\Phi(1)} & \text{if } \Phi(1)=\Phi(2), \\ \frac{2\left(e^{t\Phi(2)}-e^{t\Phi(1)}\right)}{\Phi(1)(\Phi(2)-\Phi(1))} & \text{otherwise.}\end{cases}\end{aligned}$$

This equality combined with (6.1) completes the proof of (b). ■

Lemma 6.2

When $\kappa > 2$,

$$\lim_{\lambda\to 0+}\lambda\mathbf{E}_0\left[\left(\int_0^\infty (Z_t-\mathbf{m})e^{-\lambda t}dt\right)^2\right] = \frac{4\left(\Phi(2)-4\Phi(1)\right)}{\Phi(1)^3\Phi(2)}.$$

This limit is strictly positive because Φ is a convex function with $\Phi(0)=\Phi(\kappa)=0$.

Proof. We write, for $\lambda > 0$,

$$\mathbf{E}_0\left[\left(\int_0^\infty (Z_t-\mathbf{m})e^{-\lambda t}dt\right)^2\right] = \mathbf{E}_0\left[\left(\int_0^\infty Z_t e^{-\lambda t}dt\right)^2\right] - \frac{2\mathbf{m}}{\lambda}\mathbf{E}_0\left[\int_0^\infty Z_t e^{-\lambda t}dt\right] + \frac{\mathbf{m}^2}{\lambda^2}. \tag{6.2}$$

Making use of (a) of Lemma 6.1, we find, for any $z \geqslant 0$,

$$\begin{aligned}\mathbf{E}_z\left[\int_0^\infty Z_t e^{-\lambda t}dt\right] &= \int_0^\infty \mathbf{E}_z[Z_t]e^{-\lambda t}dt \\ &= \int_0^\infty \left(\mathbf{m}+(z-\mathbf{m})e^{t\Phi(1)}\right)e^{-\lambda t}dt = \frac{z\lambda+2}{\lambda(\lambda-\Phi(1))}.\end{aligned} \tag{6.3}$$

This equality for $z=0$ combined with (6.2) yields

$$\mathbf{E}_0\left[\left(\int_0^\infty (Z_t-\mathbf{m})e^{-\lambda t}dt\right)^2\right] = \mathbf{E}_0\left[\left(\int_0^\infty Z_t e^{-\lambda t}dt\right)^2\right] + \frac{4(\lambda+\Phi(1))}{\lambda^2\Phi(1)^2(\lambda-\Phi(1))}. \tag{6.4}$$

We also have

$$\begin{aligned}\mathbf{E}_0\left[\left(\int_0^\infty Z_t e^{-\lambda t}dt\right)^2\right] &= 2\mathbf{E}_0\left[\int_0^\infty Z_x e^{-\lambda x}\int_x^\infty Z_y e^{-\lambda y}dydx\right] \\ &= 2\int_0^\infty \mathbf{E}_0\left[Z_x e^{-\lambda x}\int_0^\infty Z_{x+y}e^{-\lambda(x+y)}dy\right]dx \\ &= 2\int_0^\infty e^{-2\lambda x}\mathbf{E}_0\left[Z_x\mathbf{E}_{Z_x}\left[\int_0^\infty Z_y e^{-\lambda y}dy\right]\right]dx,\end{aligned}$$

where we used the Markov property of Z for the last equality. Thus, with the help of (6.3), we find

$$\mathbf{E}_0\left[\left(\int_0^\infty Z_t e^{-\lambda t}dt\right)^2\right] = \frac{2}{\lambda(\lambda-\Phi(1))}\int_0^\infty e^{-2\lambda x}\left(\lambda\mathbf{E}_0[Z_x^2]+2\mathbf{E}_0[Z_x]\right)dx.$$

This integral can now be explicitly computed with the help of Lemma 6.1. After a few lines of elementary calculus, we obtain

$$\mathbf{E}_0\left[\left(\int_0^\infty Z_t e^{-\lambda t}dt\right)^2\right] = \frac{4(6\lambda-\Phi(2))}{\lambda^2(\lambda-\Phi(1))(4\lambda^2-2\lambda(\Phi(1)+\Phi(2))+\Phi(1)\Phi(2))}$$

(this result does not depend on whether or not $\Phi(1)=\Phi(2)$). Substituting this equality in (6.4), we get

$$\lambda\mathbf{E}_0\left[\left(\int_0^\infty (Z_t-\mathbf{m})e^{-\lambda t}dt\right)^2\right] = \frac{4\Phi(1)\left(\Phi(2)-4\Phi(1)\right)-4\lambda\left(4\lambda+2(\Phi(1)-\Phi(2))\right)}{\Phi(1)^2\left(\Phi(1)-\lambda\right)\left(4\lambda^2-2\lambda(\Phi(1)+\Phi(2))+\Phi(1)\Phi(2)\right)}.$$

We conclude the proof of the lemma by taking the limit as λ tends to $0+$. ■

Lemma 6.3

When $\kappa > 2$,

$$\lim_{\lambda\to0+}\lambda\mathbf{E}_1\left[\left(\int_0^\infty (Z_t-\mathbf{m})e^{-\lambda t}dt\right)^2\right] = \frac{1}{2\mathbf{n}\left[\zeta\right]}\mathbf{n}\left[\left(\int_0^\zeta(\epsilon_t-\mathbf{m})dt\right)^2\right].$$

Proof. Recall that φ stands for the Laplace exponent of the inverse of the local time L^{-1}. We first use (b) of Lemma 5.2 with the function $f(x)=|x-\mathbf{m}|$:

$$\mathbf{E}_1\left[\left(\int_0^\infty |Z_t-\mathbf{m}|e^{-\lambda t}dt\right)^2\right] = \frac{1}{\varphi(2\lambda)}\mathbf{n}\left[\left(\int_0^\zeta|\epsilon_t-\mathbf{m}|e^{-\lambda t}dt\right)^2\right] + \frac{2}{\varphi(\lambda)\varphi(2\lambda)}\mathbf{n}\left[\int_0^\zeta|\epsilon_t-\mathbf{m}|e^{-\lambda t}dt\right]\mathbf{n}\left[e^{-\lambda\zeta}\int_0^\zeta|\epsilon_t-\mathbf{m}|e^{-\lambda t}dt\right]. \quad (6.5)$$

Note also that Lemma 5.1 and Proposition 5.8 readily show that

$$\mathbf{n}\left[\left(\int_0^\zeta|\epsilon_t-\mathbf{m}|dt\right)^\beta\right] < \infty \quad \text{for all } \beta<\kappa. \quad (6.6)$$

Thus, the three expectations under $\mathbf{n}$ on the r.h.s. of (6.5) are finite because $\kappa > 2$. Therefore, we can also use (b) of Lemma 5.2 with the function $f(x)=x-\mathbf{m}$:

$$\mathbf{E}_1\left[\left(\int_0^\infty (Z_t-\mathbf{m})e^{-\lambda t}dt\right)^2\right] = \frac{1}{\varphi(2\lambda)}\mathbf{n}\left[\left(\int_0^\zeta(\epsilon_t-\mathbf{m})e^{-\lambda t}dt\right)^2\right] + \frac{2}{\varphi(\lambda)\varphi(2\lambda)}\mathbf{n}\left[\int_0^\zeta(\epsilon_t-\mathbf{m})e^{-\lambda t}dt\right]\mathbf{n}\left[e^{-\lambda\zeta}\int_0^\zeta(\epsilon_t-\mathbf{m})e^{-\lambda t}dt\right]. \quad (6.7)$$

Recall that $\varphi(\lambda) \sim \mathbf{n}[\zeta]\,\lambda$ (*c.f.* (5.7) in the proof of Corollary 5.3). Thus, keeping in mind (6.6), the dominated convergence theorem yields

$$\lim_{\lambda\to 0+} \frac{\lambda}{\varphi(2\lambda)}\mathbf{n}\left[\left(\int_0^\zeta (\epsilon_t - \mathbf{m})e^{-\lambda t}dt\right)^2\right] = \frac{1}{2\mathbf{n}[\zeta]}\mathbf{n}\left[\left(\int_0^\zeta (\epsilon_t - \mathbf{m})dt\right)^2\right]$$

and

$$\lim_{\lambda\to 0+} \mathbf{n}\left[e^{-\lambda\zeta}\int_0^\zeta (\epsilon_t - \mathbf{m})e^{-\lambda t}dt\right] = \mathbf{n}\left[\int_0^\zeta (\epsilon_t - \mathbf{m})dt\right] = 0$$

where we used Corollary 5.4 for the last equality. Finally, (a) of Lemma 5.2 combined with (6.3) give

$$\begin{aligned}\frac{2\lambda}{\varphi(\lambda)\varphi(2\lambda)}\mathbf{n}\left[\int_0^\zeta (\epsilon_t - \mathbf{m})e^{-\lambda t}dt\right] &= \frac{2\lambda}{\varphi(2\lambda)}\mathbf{E}_1\left[\int_0^\infty (Z_t - \mathbf{m})e^{-\lambda t}dt\right] \\ &= \frac{2\lambda}{\varphi(2\lambda)}\left(\frac{\Phi(1)+2}{\Phi(1)(\lambda - \Phi(1))}\right) \underset{\lambda\to 0+}{\longrightarrow} -\frac{2+\Phi(1)}{\mathbf{n}[\zeta]\,\Phi(1)^2}.\end{aligned}$$

These last three estimates combined with (6.7) entail the lemma. ■

We can now easily obtain the calculation of the second moment.

Proposition 6.4

When $\kappa > 2$,

$$\mathbf{n}\left[\left(\int_0^\zeta (\epsilon_t - \mathbf{m})dt\right)^2\right] = \mathbf{n}[\zeta]\,\frac{8\,(\Phi(2) - 4\Phi(1))}{\Phi(1)^3\Phi(2)}.$$

Proof. In view of Lemma 6.2 and Lemma 6.3, it suffices to prove that

$$\lim_{\lambda\to 0+} \lambda\mathbf{E}_1\left[\left(\int_0^\infty (Z_t - \mathbf{m})e^{-\lambda t}dt\right)^2\right] = \lim_{\lambda\to 0+} \lambda\mathbf{E}_0\left[\left(\int_0^\infty (Z_t - \mathbf{m})e^{-\lambda t}dt\right)^2\right].$$

Indeed, the Markov property of Z for the stopping time τ_1 yields

$$\begin{aligned}&\mathbf{E}_0\left[\left(\int_0^\infty (Z_t - \mathbf{m})e^{-\lambda t}dt\right)^2\right] \\ &\quad= \mathbf{E}_0\left[\left(\int_0^{\tau_1} (Z_t - \mathbf{m})e^{-\lambda t}dt + \int_{\tau_1}^\infty (Z_t - \mathbf{m})e^{-\lambda t}dt\right)^2\right] \\ &\quad= \mathbf{E}_0\left[\left(\int_0^{\tau_1} (Z_t - \mathbf{m})e^{-\lambda t}dt\right)^2\right] + \mathbf{E}_0\left[e^{-2\lambda\tau_1}\right]\mathbf{E}_1\left[\left(\int_0^\infty (Z_t - \mathbf{m})e^{-\lambda t}dt\right)^2\right] \\ &\qquad+ 2\mathbf{E}_0\left[e^{-\lambda\tau_1}\int_0^{\tau_1} (Z_t - \mathbf{m})e^{-\lambda t}dt\right]\mathbf{E}_1\left[\int_0^\infty (Z_t - \mathbf{m})e^{-\lambda t}dt\right].\end{aligned} \tag{6.8}$$

Proposition 4.1 and the absence of positive jumps for Z give

$$\mathbf{E}_0\left[\left(\int_0^{\tau_1} (Z_t - \mathbf{m})e^{-\lambda t}dt\right)^2\right] \leqslant (\mathbf{m}+1)^2\mathbf{E}_0[\tau_1^2] < \infty. \tag{6.9}$$

Similarly,

$$\left| \mathbf{E}_0 \left[e^{-\lambda \tau_1} \int_0^{\tau_1} (Z_t - \mathbf{m}) e^{-\lambda t} dt \right] \right| \leqslant (\mathbf{m}+1) \mathbf{E}_0[\tau_1] < \infty. \tag{6.10}$$

Note also that, according to (6.3),

$$\lambda \mathbf{E}_1 \left[\int_0^{\infty} (Z_t - \mathbf{m}) e^{-\lambda t} dt \right] = \frac{\lambda(\Phi(1)+2)}{\Phi(1)(\lambda - \Phi(1))} \underset{\lambda \to 0+}{\longrightarrow} 0. \tag{6.11}$$

Thus, (6.8)-(6.9)-(6.10)-(6.11) and the fact that $\lim_{\lambda \to 0+} \mathbf{E}_0 \left[e^{-2\lambda \tau_1} \right] = 1$ conclude the proof of the proposition. ∎

7 End of the proof of the main theorem

We showed in Section 2 that we only need to prove Theorem 1.2 for the additive functional

$$I(r) \overset{\text{def}}{=} \int_0^r Z_s ds$$

under $\mathbf{P} = \mathbf{P}_0$ in place of $H(r)$. Moreover Proposition 4.3 states that the hitting time of level 1 by Z is $\mathbf{P}_0$-almost surely finite, therefore it is sufficient to prove this result for $I(r)$ under $\mathbf{P}_1$. The remaining portion of the proof is now quite standard and very similar to the argument given p166,167 of [KKS75]. Let us first deal with the case $\kappa < 1$. Recall that L^{-1} stands for the inverse of the local time of Z at level 1. Since I is an additive functional of Z, the process $(I(L_t^{-1}), t \geqslant 0)$ under $\mathbf{P}_1$ is a subordinator (without drift thanks to Lemma 3.3), whose Laplace transform is given by

$$\mathbf{E}_1 \left[e^{-\lambda I(L_t^{-1})} \right] = \exp \left(-t\lambda \int_0^{\infty} e^{-\lambda x} \mathbf{n} \left\{ \widetilde{I}(\epsilon) > x \right\} dx \right), \tag{7.1}$$

where we used the notation $\widetilde{I}(\epsilon) \overset{\text{def}}{=} \int_0^{\zeta} \epsilon_t dt$. We now define

$$\xi_n \overset{\text{def}}{=} \int_{L_{n-1}^{-1}}^{L_n^{-1}} Z_s ds = I(L_n^{-1}) - I(L_{n-1}^{-1}).$$

The sequence $(\xi_n\,,\, n \geqslant 1)$ under $\mathbf{P}_1$ is i.i.d. Moreover, in view of Proposition 5.8, we deduce from (7.1) that

$$\mathbf{P}_1 \left\{ \xi_1 > x \right\} \underset{x \to \infty}{\sim} \mathbf{n} \left\{ \widetilde{I}(\epsilon) > x \right\} \underset{x \to \infty}{\sim} \mathbf{n}\left[\zeta\right] \frac{2^{\kappa} \Gamma(\kappa) \kappa^2 \mathbf{K}^2}{\Phi'(\kappa) x^{\kappa}}. \tag{7.2}$$

The characterization of the domains of attraction to a stable law (see for instance chap. IX.8 of [Fel71]) implies that

$$\frac{I(L_n^{-1})}{n^{1/\kappa}} = \frac{\xi_1 + \ldots + \xi_n}{n^{1/\kappa}} \underset{n \to \infty}{\overset{\text{law}}{\longrightarrow}} 2 \left(\frac{\mathbf{n}\left[\zeta\right] \pi \kappa^2 \mathbf{K}^2}{2 \sin\left(\frac{\pi\kappa}{2}\right) \Phi'(\kappa)} \right)^{\frac{1}{\kappa}} \mathcal{S}_{\kappa}^{ca}.$$

Moreover, according to Lemma 5.1, we have $\mathbf{E}\left[L_1^{-1}\right] = \mathbf{n}\left[\zeta\right] < \infty$ so the strong law of large numbers for subordinators (*c.f.* p92 of [Ber96a]) yields

$$\frac{L_t^{-1}}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \mathbf{n}\left[\zeta\right]. \tag{7.3}$$

We can therefore use Theorem 8.1 of [Ser75] with the change of time L^{-1} to check that, under $\mathbf{P}_1$,

$$\frac{I(t)}{t^{1/\kappa}} \xrightarrow[t\to\infty]{\text{law}} 2\left(\frac{\pi\kappa^2\mathbf{K}^2}{2\sin\left(\frac{\pi\kappa}{2}\right)\Phi'(\kappa)}\right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca}.$$

This concludes the proof of the theorem when $\kappa < 1$. Let us now assume that $\kappa = 1$. In this case, $\mathbf{K} = \mathbf{E}\left[\left(\int_0^\infty e^{\mathbb{V}_s}ds\right)^0\right] = 1$ hence (7.2) takes the form

$$\mathbf{P}_1\left\{\xi_1 > x\right\} \underset{x\to\infty}{\sim} \frac{2\mathbf{n}\left[\zeta\right]}{\Phi'(1)x}.$$

The characterization of the domains of attraction now states that there exists a constant c_{28} such that

$$\frac{I(L_n^{-1}) - ng(n)}{n} = \frac{\xi_1 + \ldots + \xi_n}{n} - g(n) \xrightarrow[n\to\infty]{\text{law}} c_{28} + \frac{\pi\mathbf{n}\left[\zeta\right]}{\Phi'(1)}\mathcal{C}^{ca} \tag{7.4}$$

where $g(x) \overset{\text{def}}{=} \int_0^x \mathbf{P}_1\{\xi_1 > y\}dy$. Note also that our estimate on $\mathbf{n}\left\{\zeta > x\right\}$ (Lemma 5.1) entails an iterated logarithm law for the subordinator L^{-1}, in particular

$$\frac{L_n^{-1}}{\mathbf{n}\left[\zeta\right]} \in [n - n^{2/3}, n + n^{2/3}] \text{ for all } n \text{ large enough.}$$

Using this result and the fact that $I(\cdot)$ is non-decreasing, it is not difficult to deduce from (7.4) that

$$\frac{1}{t}\left(I(t) - \frac{t}{\mathbf{n}\left[\zeta\right]}g\left(\frac{t}{\mathbf{n}\left[\zeta\right]}\right)\right) \xrightarrow[t\to\infty]{\text{law}} \frac{c_{28}}{\mathbf{n}\left[\zeta\right]} + \frac{\pi}{\Phi'(1)}\mathcal{C}^{ca}$$

(compare with the argument given on p166 of [KKS75] for details). Thus, setting

$$f(t) \overset{\text{def}}{=} \frac{t}{\mathbf{n}\left[\zeta\right]}\left(g\Big(\frac{t}{\mathbf{n}\left[\zeta\right]}\Big) - c_{28}\right),$$

we get the desired limiting law for $(I(t) - f(t))/t$ and also

$$f(t) \underset{t\to\infty}{\sim} \frac{t}{\mathbf{n}\left[\zeta\right]}\int_0^{\frac{t}{\mathbf{n}[\zeta]}} \mathbf{P}_1\left\{\xi_1 > y\right\}dy \underset{t\to\infty}{\sim} \frac{2t\log t}{\Phi'(1)}.$$

The proof of the theorem when $\kappa > 1$ is very similar to that in the case $\kappa < 1$, considering now the sequence $(\xi_n', n \geqslant 1)$ instead of $(\xi_n, n \geqslant 1)$, defined by

$$\xi_n' \overset{\text{def}}{=} \int_{L_{n-1}^{-1}}^{L_n^{-1}} (Z_s - \mathbf{m})ds = \xi_n - \mathbf{m}(L_n^{-1} - L_{n-1}^{-1}).$$

These random variables are i.i.d. and are centered under $\mathbf{P}_1$ because

$$\mathbf{E}_1\left[\xi'_1\right] = \mathbf{n}\left[\int_0^\zeta (\epsilon_s - \mathbf{m})ds\right] = \mathbf{n}\left[\int_0^\zeta \epsilon_s ds\right] - \mathbf{n}\left[\zeta\right]\mathbf{m} = 0$$

(we used Corollary 5.4 for the last equality). Moreover, when $\kappa > 2$, Proposition 6.4 yields

$$\mathbf{E}_1\left[{\xi'_1}^2\right] = \mathbf{n}\left[\left(\int_0^\zeta (\epsilon_s - \mathbf{m})ds\right)^2\right] = \mathbf{n}\left[\zeta\right]\frac{8\left(\Phi(2) - 4\Phi(1)\right)}{\Phi(1)^3\Phi(2)}.$$

Since the tail distribution of ζ under $\mathbf{n}$ has (at least) an exponential decrease, we see that the estimate (7.2) still holds with ξ'_1 in place of ξ_1. Thus, the characterization of the domains of attraction to a stable law insures that, when $\kappa \in (1,2)$,

$$\frac{I(L_n^{-1}) - \mathbf{m}L_n^{-1}}{n^{1/\kappa}} = \frac{\xi'_1 + \ldots + \xi'_n}{n^{1/\kappa}} \xrightarrow[n\to\infty]{\text{law}} 2\left(\frac{\mathbf{n}\left[\zeta\right]\pi\kappa^2\mathbf{K}^2}{2\sin\left(\frac{\pi\kappa}{2}\right)\Phi'(\kappa)}\right)^{\frac{1}{\kappa}}\mathcal{S}_\kappa^{ca}.$$

Similarly, when $\kappa = 2$ and since $\mathbf{K} = \mathbf{E}\left[\int_0^\infty e^{\mathbb{V}_s}ds\right] = \frac{-1}{\Phi(1)}$,

$$\frac{I(L_n^{-1}) - \mathbf{m}L_n^{-1}}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\text{law}} \frac{-4\sqrt{\mathbf{n}\left[\zeta\right]}}{\Phi(1)\sqrt{\Phi'(2)}}\mathcal{N},$$

and when $\kappa > 2$,

$$\frac{I(L_n^{-1}) - \mathbf{m}L_n^{-1}}{\sqrt{n}} \xrightarrow[n\to\infty]{\text{law}} \sqrt{\mathbf{E}_1\left[{\xi'_1}^2\right]}\mathcal{N} = \sqrt{\frac{\mathbf{n}\left[\zeta\right]8\left(\Phi(2) - 4\Phi(1)\right)}{\Phi(1)^3\Phi(2)}}\mathcal{N}.$$

Just as in the case $\kappa < 1$, we easily check that the hypotheses of Theorem 8.1 of [Ser75] are fulfilled. Thus the change of time $L_t^{-1} \sim \mathbf{n}\left[\zeta\right]t$ is legitimate and concludes the proof of the theorem.

Chapter IV

A slow transient diffusion in a drifted stable potential[1]

Abstract. We consider a diffusion process X in a random potential $\mathbb{V}$ of the form $\mathbb{V}_x = \mathbb{S}_x - \delta x$, where δ is a positive drift and $\mathbb{S}$ is a strictly stable process of index $\alpha \in (1, 2)$ with positive jumps. Then the diffusion is transient and $X_t / \log^\alpha t$ converges in law towards an exponential distribution. This behaviour contrasts with the case where $\mathbb{V}$ is a drifted Brownian motion and provides an example of a transient diffusion in a random potential which is as "slow" as in the recurrent setting.

1 Introduction

Let $(\mathbb{V}(x),\ x \in \mathbb{R})$ be a two-sided stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We call a diffusion in the random potential $\mathbb{V}$ an formal solution X of the S.D.E:

$$\begin{cases} dX_t = d\beta_t - \frac{1}{2}\mathbb{V}'(X_t)dt \\ X_0 = 0, \end{cases}$$

where β is a standard Brownian motion independent of $\mathbb{V}$. Of course, the process $\mathbb{V}$ may not be differentiable (for example when $\mathbb{V}$ is a Brownian motion) and we should formally consider X as a diffusion whose conditional generator given $\mathbb{V}$ is

$$\frac{1}{2} e^{\mathbb{V}(x)} \frac{d}{dx} \left(e^{-\mathbb{V}(x)} \frac{d}{dx} \right).$$

Such a diffusion may be explicitly constructed from a Brownian motion through a random change of time and a random change of scale. This class of processes has been widely studied for the last twenty years and bears a close connection with the model of the random walk in

[1] This chapter is a slightly modified version of the article: A. Singh, *A slow transient diffusion in a drifted stable potential*, J. Theoret. Probab., **20**(2), 153–166, 2007.

random environment (RWRE), see [Zei04] and [Shi01] for a survey on RWRE and [Sch85], [Shi01] for the connection between the two models.

This model exhibits many interesting features. For instance, when the potential process $\mathbb{V}$ is a Brownian motion, the diffusion X is recurrent and Brox [Bro86] proved that $X_t/\log^2 t$ converges to a non-degenerate distribution. Thus, the diffusion is much "slower" than in the trivial case $\mathbb{V} = 0$ (then X is simply a Brownian motion).

We point out that Brox's theorem is the analogue of Sinai's famous theorem for RWRE [Sin82] (see also [Gol86] and [Kes86]). Just as for the RWRE, this result is a consequence of a so-called "localization phenomenon": the diffusion is trapped in some valleys of its potential $\mathbb{V}$. Brox's theorem may also be extended to a wider class of potentials. For instance, when $\mathbb{V}$ is a strictly stable process of index $\alpha \in (0, 2]$, Schumacher [Sch85] proved that

$$\frac{X_t}{\log^\alpha t} \xrightarrow[t\to\infty]{\text{law}} b_\infty,$$

where b_∞ is a non-degenerate random variable, whose distribution depends on the parameters of the stable process $\mathbb{V}$.

There is also much interest concerning the behaviour of X in the transient case. When the potential is a drifted Brownian motion *i.e.* $\mathbb{V}_x = \mathbb{B}_x - \frac{\kappa}{2}x$ where $\mathbb{B}$ is a two-sided Brownian motion and $\kappa > 0$, then the associated diffusion X is transient toward $+\infty$ and its rate of growth is polynomial and depends on κ. Precisely, Kawazu and Tanaka [KT97] proved that

- If $0 < \kappa < 1$, then $\frac{1}{t^\kappa} X_t$ converges in law towards a Mittag-Leffler distribution of index κ.
- If $\kappa = 1$, then $\frac{\log t}{t} X_t$ converges in probability towards $\frac{1}{4}$.
- If $\kappa > 1$, then $\frac{1}{t} X_t$ converges almost surely towards $\frac{\kappa-1}{4}$.

In particular, when $\kappa < 1$, the rate of growth of X is sub-linear. Refined results on the rates of convergence for this process were later obtained by Tanaka [Tan97] and Hu *et al.* [HSY99].

In fact, this behaviour is not specific to diffusions in a drifted Brownian potential. More generally, it is proved in [Sin06] (*c.f.* Chapter III) that, if $\mathbb{V}$ is a two-sided Lévy process with no positive jumps and if there exists $\kappa > 0$ such $\mathbf{E}[e^{\kappa \mathbb{V}_1}] = 1$, then the rate of growth of X_t is linear when $\kappa > 1$ and of order t^κ when $0 < \kappa < 1$. See also [Car97] for a law of large numbers in more general Lévy potential. These results are the analogues of those previously obtained by Kesten *et al.* [KKS75] for the discrete model of the RWRE.

In this paper, we study the asymptotic behaviour of a diffusion in a drifted stable potential. Precisely, let $(\mathbb{S}_x,\ x \in \mathbb{R})$ denote a two-sided càdlàg stable process with index $\alpha \in (1, 2)$. By two-sided, we mean that

(a) The process $(\mathbb{S}_x,\ x \geqslant 0)$ is strictly stable with index $\alpha \in (1, 2)$, in particular $\mathbb{S}_0 = 0$.

(b) For all $x_0 \in \mathbb{R}$, the process $(\mathbb{S}_{x+x_0} - \mathbb{S}_{x_0},\ x \in \mathbb{R})$ has the same law as $\mathbb{S}$.

It is well known that the Lévy measure Π of $\mathbb{S}$ has the form

$$\Pi(dx) = \left(c^+ \mathbf{1}_{\{x>0\}} + c^- \mathbf{1}_{\{x<0\}}\right) \frac{dx}{|x|^{\alpha+1}} \tag{1.1}$$

where c^+ and c^- are two non-negative constants such that $c^+ + c^- > 0$. In particular, the process $(\mathbb{S}_x\,,\ x \geqslant 0)$ has no positive jumps (resp. no negative jumps) if and only if $c^+ = 0$ (resp. $c^- = 0$). Given $\delta > 0$, we consider a diffusion X in the random potential

$$\mathbb{V}_x = \mathbb{S}_x - \delta x.$$

Since the index α of the stable process $\mathbb{S}$ is larger than 1, we have $\mathbf{E}[\mathbb{V}_x] = -\delta x$, and therefore

$$\lim_{x \to +\infty} \mathbb{V}_x = -\infty \quad \text{and} \quad \lim_{x \to -\infty} \mathbb{V}_x = +\infty \quad \text{almost surely.}$$

This implies that the random diffusion X is transient toward $+\infty$. We have already mentioned that, when $\mathbb{S}$ has no positive jumps (*i.e.* $c^+ = 0$), the rate of growth of X is polynomial. Thus, we now assume that $\mathbb{S}$ possesses positive jumps.

Theorem 1.1

Assume that $c^+ > 0$, then

$$\frac{X_t}{\log^\alpha t} \xrightarrow[t \to \infty]{\text{law}} \mathcal{E}\left(\frac{c^+}{\alpha}\right),$$

where $\mathcal{E}(c^+/\alpha)$ denotes an exponential law with parameter c^+/α. This result also holds with $\sup_{s \leqslant t} X_s$ or $\inf_{s \geqslant t} X_s$ in place of X_t.

The asymptotic behaviour of X is in this case very different from the one observed when $\mathbb{V}$ is a drifted Brownian motion. Here, the rate of growth is very slow: it is the same as in the recurrent setting. We also note that neither the rate of growth nor the limiting law depend on the value of the drift parameter δ.

Theorem 1.1 has a simple heuristic explanation: the "localisation phenomenon" for the diffusion X tells us that the time needed to reach a positive level x is approximatively exponentially proportional to the biggest ascending barrier of $\mathbb{V}$ on the interval $[0, x]$. In the case of a Brownian potential, or more generally a spectrally negative Lévy potential, the addition of a negative drift somehow "kills" the ascending barriers, thus accelerating the diffusion and leading to a polynomial rate of transience. However, in our setting, the biggest ascending barrier on $[0, x]$ of the stable process $\mathbb{S}$ is of the same order as its biggest jump on this interval. Since the addition of a drift has no influence on the jumps of the potential process, the time needed to reach level x still remains of the same order as in the recurrent case (*i.e.* when the drift is zero) and yields a logarithmic rate of transience.

2 Proof of the theorem

2.1 Representation of X and of its hitting times

In the remainder of this paper, we indifferently use the notation $\mathbb{V}_x$ or $\mathbb{V}(x)$. Let us first recall the classical representation of the diffusion X in the random potential $\mathbb{V}$ from a Brownian motion through a random change of scale and a random change of time (see [Bro86] or [Shi01] for details). Let $(B_t,\ t \geqslant 0)$ denote a standard Brownian motion independent of $\mathbb{V}$ and let σ stand for its hitting times:

$$\sigma(x) \stackrel{\text{def}}{=} \inf(t \geqslant 0\,,\, B_t = x).$$

Define the scale function of the diffusion X,

$$\mathbb{A}(x) \stackrel{\text{def}}{=} \int_0^x e^{\mathbb{V}_y} dy \qquad \text{for } x \in \mathbb{R}. \tag{2.1}$$

Since $\lim_{x\to+\infty} \mathbb{V}_x/x = -\delta$ and $\lim_{x\to-\infty} \mathbb{V}_x/x = \delta$ almost surely, it is clear that

$$\mathbb{A}(\infty) = \lim_{x\to+\infty} \mathbb{A}(x) < \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \mathbb{A}(x) = -\infty \qquad \text{almost surely.}$$

Let $\mathbb{A}^{-1} : (-\infty, \mathbb{A}(\infty)) \mapsto \mathbb{R}$ denote the inverse of $\mathbb{A}$ and define

$$\mathbb{T}(t) \stackrel{\text{def}}{=} \int_0^t e^{-2\mathbb{V}(\mathbb{A}^{-1}(B_s))} ds \qquad \text{for } 0 \leqslant t < \sigma(\mathbb{A}(\infty)).$$

Similarly, let $\mathbb{T}^{-1}$ denote the inverse of $\mathbb{T}$. According to Brox [Bro86] (see also [Shi01]), the diffusion X in the random potential $\mathbb{V}$ may be represented in the form

$$X_t = \mathbb{A}^{-1}\Big(B_{\mathbb{T}^{-1}(t)}\Big). \tag{2.2}$$

It is now clear that, under our assumptions, the diffusion X is transient toward $+\infty$. We will study X via its hitting times H defined by

$$H(r) \stackrel{\text{def}}{=} \inf(t \geqslant 0\,,\, X_t = r) \qquad \text{for } r \geqslant 0.$$

Let $(L(t,x),\ t \geqslant 0,\ x \in \mathbb{R})$ stand for the bi-continuous version of the local time process of B. In view of (2.2), we can write

$$H(r) = \mathbb{T}\left(\sigma(\mathbb{A}(r))\right) = \int_0^{\sigma(\mathbb{A}(r))} e^{-2\mathbb{V}(\mathbb{A}^{-1}(B_s))} ds = \int_{-\infty}^{\mathbb{A}(r)} e^{-2\mathbb{V}(\mathbb{A}^{-1}(x))} L(\sigma(\mathbb{A}(r)), x) dx.$$

Making use of the change of variable $x = \mathbb{A}(y)$, we get

$$H(r) = \int_{-\infty}^{r} e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy = I_1(r) + I_2(r) \tag{2.3}$$

where

$$I_1(r) \stackrel{\text{def}}{=} \int_0^r e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy,$$
$$I_2(r) \stackrel{\text{def}}{=} \int_{-\infty}^0 e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy.$$

2.2 Proof of Theorem 1.1

Given a càdlàg process $(Z_t\ ,\ t \geqslant 0)$, we denote by $\Delta_t Z = Z_t - Z_{t-}$ the size of the jump at time t. We also use the notation $Z_t^\natural$ to denote the largest positive jump of Z before time t,

$$Z_t^\natural \stackrel{\text{def}}{=} \sup_{0 \leqslant s \leqslant t} \Delta_s Z.$$

Let $Z_t^\#$ stand for the largest ascending barrier before time t, namely:

$$Z_t^\# \stackrel{\text{def}}{=} \sup_{0 \leqslant x \leqslant y \leqslant t} (Z_y - Z_x).$$

We also define the functionals:

$$\overline{Z}_t \stackrel{\text{def}}{=} \sup_{s\in[0,t]} Z_s \qquad \text{(running unilateral maximum)}$$

$$\underline{Z}_t \stackrel{\text{def}}{=} \inf_{s\in[0,t]} Z_s \qquad \text{(running unilateral minimum)}$$

$$Z_t^* \stackrel{\text{def}}{=} \sup_{s\in[0,t]} |Z_s| \qquad \text{(running bilateral supremum)}$$

We start with a simple lemma concerning the fluctuations of the potential process.

Lemma 2.1

There exist two constants $c_1, c_2 > 0$ such that for all $a, x > 0$

$$\mathbf{P}\{\mathbb{V}_x^\# \leqslant a\} \leqslant e^{-c_1 \frac{x}{a^\alpha}}, \tag{2.4}$$

and whenever $\frac{a}{x}$ is sufficiently large,

$$\mathbf{P}\{\mathbb{V}_x^* > a\} \leqslant c_2 \frac{x}{a^\alpha}. \tag{2.5}$$

Proof. Recall that $\mathbb{V}_x = \mathbb{S}_x - \delta x$. In view of the form of the density of the Lévy measure of $\mathbb{S}$ given in (1.1), we get

$$\mathbf{P}\{\mathbb{V}_x^\# \leqslant a\} \leqslant \mathbf{P}\{\mathbb{V}_x^\natural \leqslant a\} = \exp\left(-x\int_a^\infty \frac{c^+}{y^{\alpha+1}}dy\right) = \exp\left(-\frac{c^+}{\alpha}\frac{x}{a^\alpha}\right).$$

This yields (2.4). From the scaling property of the stable process $\mathbb{S}$, we also have

$$\mathbf{P}\{\mathbb{V}_x^* > a\} = \mathbf{P}\left\{x^{\frac{1}{\alpha}} \sup_{t\in[0,1]} |\mathbb{S}_t - \delta x^{1-\frac{1}{\alpha}} t| > a\right\} \leqslant \mathbf{P}\left\{\mathbb{S}_1^* > \frac{a}{x^{\frac{1}{\alpha}}} - \delta x^{1-\frac{1}{\alpha}}\right\}.$$

Notice further that $a/x^{1/\alpha} - \delta x^{1-1/\alpha} > a/(2x^{1/\alpha})$ whenever a/x is large enough. Therefore, making use of a classical estimate concerning the tail distribution of the stable process $\mathbb{S}$ (*c.f.* Proposition 4, p221 of [Ber96a]), we find that

$$\mathbf{P}\{\mathbb{V}_x^* > a\} \leqslant \mathbf{P}\left\{\mathbb{S}_1^* > \frac{a}{2x^{\frac{1}{\alpha}}}\right\} \leqslant \mathbf{P}\left\{\overline{\mathbb{S}}_1 > \frac{a}{2x^{\frac{1}{\alpha}}}\right\} + \mathbf{P}\left\{\underline{\mathbb{S}}_1 < -\frac{a}{2x^{\frac{1}{\alpha}}}\right\} \leqslant c_2 \frac{x}{a^\alpha}.$$

■

Proposition 2.2

There exists a constant $c_3 > 0$ such that, for all r sufficiently large and all $x \geqslant 0$,

$$\mathbf{P}\{\mathbb{V}_r^{\#} \geqslant x + \log^4 r\} - c_3 e^{-\log^2 r} \leqslant \mathbf{P}\{\log I_1(r) \geqslant x\} \leqslant \mathbf{P}\{\mathbb{V}_r^{\#} \geqslant x - \log^4 r\} + c_3 e^{-\log^2 r}.$$

Proof. This estimate was first proved by Hu and Shi (see Lemma 4.1 of [HS98a]) when the potential process is close to a standard Brownian motion. A similar result is given in Proposition 3.1 of [Sin07a] (*c.f.* Chapter II) when $\mathbb{V}$ is a random walk in the domain of attraction of a stable law. As explained by Shi [Shi01], the key idea is the combined use of Ray-Knight's Theorem and Laplace's method. However, in our setting, additional difficulties appear since the potential process is neither flat on integer interval nor continuous. We shall therefore give a complete proof but one can still look in [HS98a] for additional details. Recall that

$$I_1(r) = \int_0^r e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y)) dy,$$

where L is the local time of the Brownian motion B (independent of $\mathbb{V}$). Let $(U(t),\ t \geqslant 0)$ denote a two-dimensional squared Bessel process starting from zero and independent of $\mathbb{V}$. According to the first Ray-Knight Theorem (*c.f.* Theorem 2.2 p455 of [RY99]), for any $x > 0$ the process $(L(\sigma(x), x - y),\ 0 \leqslant y \leqslant x)$ has the same law as $(U(y),\ 0 \leqslant y \leqslant x)$. Therefore, making use of the scaling property of the Brownian motion and the independence of $\mathbb{V}$ and B, for each fixed $r > 0$, the random variable $I_1(r)$ has the same law as

$$\widetilde{I}_1(r) \stackrel{\text{def}}{=} \mathbb{A}(r) \int_0^r e^{-\mathbb{V}_y} U\left(\frac{\mathbb{A}(r) - \mathbb{A}(y)}{\mathbb{A}(r)}\right) dy.$$

We simply need to prove the proposition for $\widetilde{I}_1$ instead of I_1. In the rest of the proof, we assume that r is very large. We start with the upper bound. Define the event

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \left\{ \sup_{t \in (0,1]} \frac{U(t)}{t \log\left(\frac{8}{t}\right)} \leqslant r \right\}.$$

According to Lemma 6.1 of [HS98a], $\mathbf{P}\{\mathcal{E}_1^c\} \leqslant c_4 e^{-r/2}$ for some constant $c_4 > 0$. On $\mathcal{E}_1$, we have

$$\begin{aligned}
\widetilde{I}_1(r) &\leqslant r \int_0^r e^{-\mathbb{V}_y} (\mathbb{A}(r) - \mathbb{A}(y)) \log\left(\frac{8\mathbb{A}(r)}{\mathbb{A}(r) - \mathbb{A}(y)}\right) dy \\
&= r \int_0^r \left(\int_y^r e^{\mathbb{V}_z - \mathbb{V}_y} dz\right) \log\left(\frac{8\mathbb{A}(r)}{\mathbb{A}(r) - \mathbb{A}(y)}\right) dy \\
&\leqslant r^2 e^{\mathbb{V}_r^{\#}} \int_0^r \log\left(\frac{8\mathbb{A}(r)}{\mathbb{A}(r) - \mathbb{A}(y)}\right) dy.
\end{aligned}$$

Notice also that $\mathbb{A}(r)=\int_0^r e^{\mathbb{V}_z}dz \leqslant re^{\overline{\mathbb{V}}_r}$ and similarly $\mathbb{A}(r)-\mathbb{A}(y)\geqslant (r-y)e^{\underline{\mathbb{V}}_r}$. Therefore

$$\begin{aligned}\int_0^r \log\left(\frac{8\mathbb{A}(r)}{\mathbb{A}(r)-\mathbb{A}(y)}\right)dy &\leqslant r(\overline{\mathbb{V}}_r-\underline{\mathbb{V}}_r)+\int_0^r \log\left(\frac{8r}{r-y}\right)dy\\ &= r(\overline{\mathbb{V}}_r-\underline{\mathbb{V}}_r+1+\log 8).\end{aligned}$$

Define the set $\mathcal{E}_2 \stackrel{\text{def}}{=} \{\overline{\mathbb{V}}_r-\underline{\mathbb{V}}_r\leqslant e^{\log^3 r}\}$. In view of Lemma 2.1,

$$\mathbf{P}\{\mathcal{E}_2^c\}\leqslant \mathbf{P}\left\{\mathbb{V}_r^*>\frac{1}{2}e^{\log^3 r}\right\}\leqslant e^{-\log^2 r}.$$

Therefore, $\mathbf{P}\{(\mathcal{E}_1\cap\mathcal{E}_2)^c\}\leqslant 2e^{-\log^2 r}$ and on $\mathcal{E}_1\cap\mathcal{E}_2$,

$$\widetilde{I}_1(r)\leqslant r^3(e^{\log^3 r}+1+\log 8)e^{\mathbb{V}_r^{\#}}\leqslant e^{\log^4 r+\mathbb{V}_r^{\#}}.$$

This completes the proof of the upper bound. We now deal with the lower bound. Define the sequence $(\gamma_k,\ k\geqslant 0)$ by induction

$$\begin{cases}\gamma_0 &\stackrel{\text{def}}{=} 0,\\ \gamma_{k+1} &\stackrel{\text{def}}{=} \inf(t>\gamma_n,\ |\mathbb{V}_t-\mathbb{V}_{\gamma_k}|\geqslant 1).\end{cases}$$

The sequence $(\gamma_{k+1}-\gamma_k,\ k\geqslant 0)$ is *i.i.d.* and distributed as $\gamma_1=\inf(t>0,\ |\mathbb{V}_t|\geqslant 1)$. We denote by $[x]$ the integer part of x. We also use the notation $\epsilon\stackrel{\text{def}}{=} e^{-\log^3 r}$. Consider the following events

$$\begin{aligned}\mathcal{E}_3 &\stackrel{\text{def}}{=} \left\{\gamma_{[r^2]}>r\right\},\\ \mathcal{E}_4 &\stackrel{\text{def}}{=} \left\{\gamma_k-\gamma_{k-1}\geqslant 2\epsilon \text{ for all } k=1,2\ldots,[r^2]\right\}.\end{aligned}$$

In view of Cramer's large deviation Theorem and since r is very large, we get $\mathbf{P}\{\mathcal{E}_3^c\}\leqslant e^{-r}$. We also have

$$\begin{aligned}\mathbf{P}\{\mathcal{E}_4^c\}\leqslant\sum_{k=1}^{[r^2]}\mathbf{P}\{\gamma_k-\gamma_{k-1}<2\epsilon\} &\leqslant [r^2]\mathbf{P}\{\gamma_1<2\epsilon\}\\ &\leqslant [r^2]\mathbf{P}\{\mathbb{V}_{2\epsilon}^*\geqslant 1\}\\ &\leqslant e^{-\log^2 r},\end{aligned}$$

where we used Lemma 2.1 for the last inequality. Define also

$$\mathcal{E}_5\stackrel{\text{def}}{=}\left\{|\mathbb{V}_x-\mathbb{V}_r|<1 \text{ for all } x\in[r-2\epsilon,r]\right\}.$$

From time reversal, the processes $(\mathbb{V}_t,\ 0\leqslant t\leqslant 2\epsilon)$ and $(\mathbb{V}_r-\mathbb{V}_{(r-t)^-},0\leqslant t\leqslant 2\epsilon)$ have the same law. Thus,

$$\mathbf{P}\{\mathcal{E}_5^c\}\leqslant\mathbf{P}\{\mathbb{V}_{2\epsilon}^*\geqslant 1\}\leqslant e^{-\log^2 r}.$$

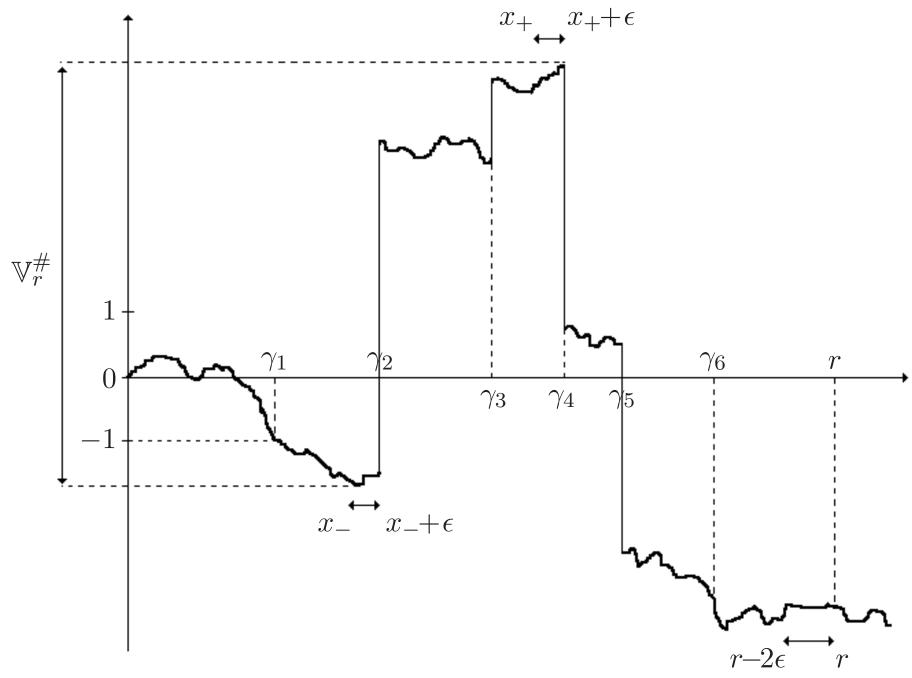

Figure IV.1 : Sample path of $\mathbb{V}$ on $\mathcal{E}_6$.

Setting $\mathcal{E}_6 \stackrel{\text{def}}{=} \mathcal{E}_3 \cap \mathcal{E}_4 \cap \mathcal{E}_5$, we get $\mathbf{P}\{\mathcal{E}_6^c\} \leqslant 3e^{-\log^2 r}$. Moreover, it is easy to check (see figure IV.1) that on $\mathcal{E}_6$, we can always find x_-, x_+ such that:

$$\begin{cases} 0 \leqslant x_- \leqslant x_+ \leqslant r - 2\epsilon, \\ \text{for any } a \in [x_-, x_- + \epsilon],\ |\mathbb{V}_{x_-} - \mathbb{V}_a| \leqslant 2, \\ \text{for any } b \in [x_+, x_+ + \epsilon],\ |\mathbb{V}_{x_+} - \mathbb{V}_b| \leqslant 2, \\ \mathbb{V}_{x_+} - \mathbb{V}_{x_-} \geqslant \mathbb{V}_r^{\#} - 4. \end{cases}$$

Let us also define

$$\begin{aligned} \mathcal{E}_7 &\stackrel{\text{def}}{=} \mathcal{E}_6 \cap \left\{ \inf_{y \in [x_-, x_- + \epsilon]} U\left(\frac{\mathbb{A}(r) - \mathbb{A}(y)}{\mathbb{A}(r)}\right) \geqslant \frac{\mathbb{A}(r) - \mathbb{A}(x_-)}{\mathbb{A}(r)} e^{-2\log^2 r} \right\}, \\ \mathcal{E}_8 &\stackrel{\text{def}}{=} \left\{ \mathbb{V}_r^{\natural} \geqslant 3\log^2 r \right\}. \end{aligned}$$

We finally set $\mathcal{E}_9 \stackrel{\text{def}}{=} \mathcal{E}_7 \cap \mathcal{E}_8$. Then on $\mathcal{E}_9$, we have, for all r large enough,

$$\begin{aligned}
\widetilde{I}_1(r) &\geqslant \mathbb{A}(r) \int_{x_-}^{x_-+\epsilon} e^{-\mathbb{V}_y} U\left(\frac{\mathbb{A}(r)-\mathbb{A}(y)}{\mathbb{A}(r)}\right) dy \\
&\geqslant e^{-\mathbb{V}_{x_-}-2-2\log^2 r} \int_{x_-}^{x_-+\epsilon} (\mathbb{A}(r)-\mathbb{A}(x_-)) dy \\
&= e^{-\mathbb{V}_{x_-}-2-2\log^2 r - \log^3 r} \int_{x_-}^{r} e^{\mathbb{V}_y} dy \\
&\geqslant e^{-\mathbb{V}_{x_-}-2-2\log^2 r - \log^3 r} \int_{x_+}^{x_++\epsilon} e^{\mathbb{V}_y} dy \\
&\geqslant e^{\mathbb{V}_{x_+}-\mathbb{V}_{x_-}-4-2\log^2 r-2\log^3 r} \\
&\geqslant e^{\mathbb{V}_r^{\#}-\log^4 r}.
\end{aligned}$$

This proves the lower bound on $\mathcal{E}_9$. It simply remains to show that $\mathbf{P}\{\mathcal{E}_9^c\} \leqslant c_5 e^{-\log^2 r}$. According to Lemma 6.1 of [HS98a], for any $0 < a < b$ and any $\eta > 0$, we have

$$\mathbf{P}\left\{\inf_{a<t<b} U(t) \leqslant \eta b\right\} \leqslant 2\sqrt{\eta} + 2\exp\left(-\frac{\eta}{2(1-a/b)}\right).$$

Therefore, making use of the independence of $\mathbb{V}$ and U, we find

$$\begin{aligned}
\mathbf{P}\{\mathcal{E}_9^c\} &\leqslant \mathbf{P}\{\mathcal{E}_6^c\} + \mathbf{P}\{\mathcal{E}_8^c\} + \mathbf{P}\{\mathcal{E}_7^c \cap \mathcal{E}_6 \cap \mathcal{E}_8\} \\
&\leqslant \mathbf{P}\{\mathcal{E}_6^c\} + \mathbf{P}\{\mathcal{E}_8^c\} + 2e^{-\log^2 r} + 2\mathbf{E}\left[e^{-\frac{1}{2}\mathbb{J}(r)e^{-2\log^2 r}} \mathbf{1}_{\mathcal{E}_6\cap\mathcal{E}_8}\right],
\end{aligned}$$

where

$$\mathbb{J}(r) \stackrel{\text{def}}{=} \frac{\mathbb{A}(r)-\mathbb{A}(x_-)}{\mathbb{A}(x_-+\epsilon)-\mathbb{A}(x_-)}.$$

We have already proved that $\mathbf{P}\{\mathcal{E}_6^c\} \leqslant 3e^{-\log^2 r}$. Using Lemma 2.1, we also check that $\mathbf{P}\{\mathcal{E}_8^c\} \leqslant e^{-\log^2 r}$. Thus, it remains to show that

$$\mathbf{E}\left[e^{-\frac{1}{2}\mathbb{J}(r)e^{-2\log^2 r}} \mathbf{1}_{\mathcal{E}_6\cap\mathcal{E}_8}\right] \leqslant c_6 e^{-\log^2 r}. \tag{2.6}$$

Notice that, on $\mathcal{E}_6$,

$$\mathbb{A}(r) - \mathbb{A}(x_-) = \int_{x_-}^{r} e^{\mathbb{V}_y} dy \geqslant \int_{x_+}^{x_++\epsilon} e^{\mathbb{V}_y} dy \geqslant e^{\log^3 r + \mathbb{V}_{x_+}-2},$$

and also

$$\mathbb{A}(x_-+\epsilon) - \mathbb{A}(x_-) = \int_{x_-}^{x_-+\epsilon} e^{\mathbb{V}_y} dy \leqslant e^{\log^3 r + \mathbb{V}_{x_-}+2}.$$

Therefore, on $\mathcal{E}_6 \cap \mathcal{E}_8$,

$$\mathbb{J}(r) \geqslant e^{\mathbb{V}_{x_+}-\mathbb{V}_{x_-}-4} \geqslant e^{\mathbb{V}_r^{\#}-8} \geqslant e^{\mathbb{V}_r^{\natural}-8} \geqslant e^{3\log^2 r-8}$$

which clearly yields (2.6) and the proof of the proposition is complete. ■

Lemma 2.3

We have

$$\frac{\mathbb{V}_r^{\#}}{r^{1/\alpha}} \xrightarrow[r\to\infty]{\text{law}} \mathbb{S}_1^{\natural}.$$

Proof. Let $f:[0,1]\mapsto\mathbb{R}$ be a deterministic càdlàg function. For $\lambda\geqslant 0$, define

$$f_\lambda(x)\stackrel{\text{def}}{=}f(x)-\lambda x.$$

We first show that

$$\lim_{\lambda\to\infty} f_\lambda^{\#}(1)=f^{\natural}(1). \tag{2.7}$$

It is clear that $f^{\natural}(1)=f_\lambda^{\natural}(1)\leqslant f_\lambda^{\#}(1)$ for any $\lambda>0$. Thus, we simply need to prove that $\limsup f_\lambda^{\#}(1)\leqslant f^{\natural}(1)$. Let $\eta>0$ and set

$$\begin{aligned} A(\eta,\lambda) &\stackrel{\text{def}}{=} \sup\left(f_\lambda(y)-f_\lambda(x)\ ,\ 0\leqslant x\leqslant y\leqslant 1 \text{ and } y-x\leqslant\eta\right), \\ B(\eta,\lambda) &\stackrel{\text{def}}{=} \sup\left(f_\lambda(y)-f_\lambda(x)\ ,\ 0\leqslant x\leqslant y\leqslant 1 \text{ and } y-x>\eta\right), \end{aligned}$$

so that

$$f_\lambda^{\#}(1)=\max(A(\eta,\lambda),B(\eta,\lambda)). \tag{2.8}$$

Notice that $A(\eta,\lambda)\leqslant A(\eta)$ where

$$A(\eta)\stackrel{\text{def}}{=}A(\eta,0)=\sup\left(f(y)-f(x)\ ,\ 0\leqslant x\leqslant y\leqslant 1 \text{ and } y-x\leqslant\eta\right).$$

Since f is càdlàg, we have $\lim_{\eta\to 0}A(\eta)=f^{\natural}(1)$. Thus, for any $\varepsilon>0$, we can find $\eta_0>0$ small enough such that

$$\limsup_{\lambda\to\infty} A(\eta_0,\lambda)\leqslant f^{\natural}(1)+\varepsilon. \tag{2.9}$$

Notice also that

$$\begin{aligned} B(\eta_0,\lambda) &\leqslant \sup\left(f(y)-f(x)-\eta_0\lambda\ ,\ 0\leqslant x\leqslant y\leqslant 1 \text{ and } y-x>\eta_0\right) \\ &\leqslant f^{\#}(1)-\eta_0\lambda \end{aligned}$$

which implies

$$\lim_{\lambda\to\infty} B(\eta_0,\lambda)=-\infty. \tag{2.10}$$

The combination of (2.8), (2.9) and (2.10) yields (2.7). Making use of the scaling property of the stable process $\mathbb{S}$, for any fixed $r>0$,

$$(\mathbb{V}_y\ ,\ 0\leqslant y\leqslant r)\stackrel{\text{law}}{=}(r^{1/\alpha}\mathbb{S}_y-\delta r y\ ,\ 0\leqslant y\leqslant 1).$$

Therefore, setting $\mathbb{R}(z)=(\mathbb{S}_{\cdot}-z\cdot)_1^{\#}$, we get the equality in law:

$$\frac{\mathbb{V}_r^{\#}}{r^{1/\alpha}}\stackrel{\text{law}}{=}\mathbb{R}(\delta r^{1-1/\alpha}). \tag{2.11}$$

Making use of (2.7), we see that $\mathbb{R}(z)$ converges almost surely towards $\mathbb{S}_1^\natural$ as z goes to infinity. Since $\alpha > 1$ and $\delta > 0$, we also have $\delta r^{1-1/\alpha} \to \infty$ as r goes to infinity and we conclude from (2.11) that

$$\frac{\mathbb{V}_r^\#}{r^{1/\alpha}} \xrightarrow[r\to\infty]{\text{law}} \mathbb{S}_1^\natural.$$

■

Proof of Theorem 1.1. Recall that the random variable $\mathbb{S}_1^\natural$ denotes the largest positive jump of $\mathbb{S}$ over the interval $[0,1]$. Thus, according to the density of the Lévy measure of $\mathbb{S}$,

$$\mathbf{P}\{\mathbb{S}_1^\natural \leqslant x\} = \exp\left(-\int_x^\infty \frac{c^+}{y^{\alpha+1}}dy\right) = \exp\left(-\frac{c^+}{\alpha y^\alpha}\right). \tag{2.12}$$

On the one hand, the combination of Proposition 2.2 and Lemma 2.3 readily shows that

$$\frac{\log(I_1(r))}{r^{1/\alpha}} \xrightarrow[r\to\infty]{\text{law}} \mathbb{S}_1^\natural. \tag{2.13}$$

On the other hand, the random variables $\mathbb{A}(\infty) = \lim_{x\to\infty} \mathbb{A}(x)$ and $\int_{-\infty}^0 e^{-\mathbb{V}_y}dy$ have the same law. Moreover, we have already noticed that these random variables are almost surely finite. Since the function $L(t,\cdot)$ is, for any fixed t, continuous with compact support, we get

$$I_2(r) = \int_{-\infty}^0 e^{-\mathbb{V}_y} L(\sigma(\mathbb{A}(r)), \mathbb{A}(y))dy \leqslant \sup_{z\in(-\infty,0]} L(\sigma(\mathbb{A}(\infty)), z) \int_{-\infty}^0 e^{-\mathbb{V}_y}dy < \infty.$$

Therefore,

$$\sup_{r\,\geqslant\,0} I_2(r) < \infty \quad \text{almost surely.} \tag{2.14}$$

Combining (2.3), (2.13) and (2.14), we deduce that

$$\frac{\log(H(r))}{r^{1/\alpha}} \xrightarrow[r\to\infty]{\text{law}} \mathbb{S}_1^\natural$$

which, from the definition of the hitting times H, yields

$$\frac{\sup_{s\,\leqslant\,t} X_s}{\log^\alpha t} \xrightarrow[t\to\infty]{\text{law}} \left(\frac{1}{\mathbb{S}_1^\natural}\right)^\alpha.$$

According to (2.12), the random variable $(1/\mathbb{S}_1^\natural)^\alpha$ has an exponential distribution with parameter c^+/α, so the proof of the theorem for $\sup_{s\,\leqslant\,t} X_s$ is complete. We finally use the classical argument given by Kawazu and Tanaka, p201 [KT97] to obtain the corresponding results for X_t and $\inf_{s\,\geqslant\,t} X_s$. ■

Part

Marches multi-excitées

Chapter V

On the speed of a cookie random walk[1]

Abstract. We consider the model of the one-dimensional cookie random walk when the initial cookie distribution is spatially uniform and the number of cookies per site is finite. We give a criterion to decide whether the limiting speed of the walk is non-zero. In particular, we show that a positive speed may be obtained for just 3 cookies per site. We also prove a result on the continuity of the speed with respect to the initial cookie distribution.

1 Introduction

We consider the model of the multi-excited random walk, also called cookie random walk, introduced by Zerner in [Zer05] as a generalization of the model of the excited random walk described by Benjamini and Wilson in [BW03] (see also Davis [Dav99] for a continuous time analogue). The aim of this paper is to study under which conditions the speed of a cookie random walk is strictly positive. In dimension $d \geq 2$, this problem was solved by Kozma [Koz03, Koz05], who proved that the speed is always non-zero. In the one-dimensional case, the speed can either be zero or strictly positive. We give here a necessary and sufficient condition to determine if the walk's speed is strictly positive when the initial cookie environment is deterministic, spatially uniform and with a finite number of cookies per site. Let us start with an informal definition of such a process:

Let us put $M \geqslant 1$ "cookies" at each site of $\mathbb{Z}$ and let us pick $p_1, p_2, \dots, p_M \in [\frac{1}{2}, 1)$. We say that p_i represents the "strength" of the i^{th} cookie at any given site. Then, a cookie random walk $X = (X_n)_{n \geqslant 0}$ is simply a nearest neighbour random walk, eating the cookies it finds along its path by behaving in the following way:

[1]This chapter is a slightly modified version of the joint work: A.-L. Basdevant and A. Singh, *On the speed of a cookie random walk*, submitted

- If $X_n = x$ and there is no remaining cookie at site x, then X jumps at time $n+1$ to $x+1$ or $x-1$ with equal probability $\frac{1}{2}$.
- If $X_n = x$ and there remain the cookies with strengths $p_j, p_{j+1}, \dots, p_M$ at this site, then X eats the cookie with attached strength p_j (which therefore disappears from this site) and then jumps at time $n+1$ to $x+1$ with probability p_j and to $x-1$ with probability $1-p_j$.

This model is a particular case of self-interacting random walk: the position of X at time $n+1$ depends not only of its position at time n but also on the number of previous visits to its present site. Therefore, X is not a Markov process.

Let us now give a formal description of the general model. We define the set of cookie environments by $\Omega = [\frac{1}{2}, 1]^{\mathbb{N}^* \times \mathbb{Z}}$. Thus, a cookie environment is of the form $\omega = (\omega(i,x))_{i \geqslant 1, x \in \mathbb{Z}}$ where $\omega(i,x)$ represents the strength of the i^{th} cookie at site x. Given $x \in \mathbb{Z}$ and $\omega \in \Omega$, a cookie random walk starting from x in the cookie environment ω is a process $(X_n)_{n \geqslant 0}$ on some probability space $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P}_{\omega,x})$ such that:

$$\begin{cases} \mathbf{P}_{\omega,x}\{X_0 = z\} = 1, \\ \mathbf{P}_{\omega,x}\{|X_{n+1} - X_n| = 1\} = 1, \\ \mathbf{P}_{\omega,x}\{X_{n+1} = X_n + 1 \mid X_1, \dots, X_n\} = \omega(j, X_n) \text{ where } j = \sharp\{0 \leqslant i \leqslant n \ , \ X_i = X_n\}. \end{cases}$$

In this paper, we restrict our attention to the set of environments $\Omega_M^u \subset \Omega$ which are spatially uniform with at most $M \geqslant 1$ cookies per site:

$$\omega \in \Omega_M^u \quad \Longleftrightarrow \quad \begin{cases} \text{for all } x \in \mathbb{Z} \text{ and all } i \geqslant 1 \ \omega(i,x) = \omega(i,0), \\ \text{for all } i > M \ \omega(i,0) = \frac{1}{2}, \\ \text{for all } i \geqslant 1 \ \omega(i,0) < 1. \end{cases}$$

The last condition $\omega(i,0) < 1$ is introduced only to exclude some possible degenerate cases but can be relaxed (see Remark 2.4). A cookie environment $\omega \in \Omega_M^u$ may be represented by $(M, \bar{p})$ where

$$\bar{p} = (p_1, \dots, p_M) = (\omega(1,0), \dots, \omega(M,0)).$$

In this case, we shall say that the associated cookie random walk is an $(M, \bar{p})$-cookie random walk and we will use the notation $\mathbf{P}_{(M,\bar{p})}$ instead of $\mathbf{P}_\omega$.

The question of the recurrence or transience of a cookie random walk was solved by Zerner in [Zer05] for general cookie environments (even in the case where the initial cookie environment may itself be random). In particular, he proved that, if X is an $(M, \bar{p})$-cookie random walk, there is a phase transition according to the value of

$$\alpha = \alpha(M, \bar{p}) \stackrel{\text{def}}{=} \sum_{i=1}^{M} (2p_i - 1) - 1. \tag{1.1}$$

- If $\alpha \leqslant 0$ then the walk is recurrent *i.e.* $\limsup X_n = -\liminf X_n = +\infty$ a.s.

- If $\alpha > 0$ then X is transient toward $+\infty$ *i.e.* $\lim X_n = +\infty$ a.s.

In particular, for $M = 1$, the cookie random walk is always recurrent for any choice of $\bar{p}$. However, as soon as $M \geqslant 2$, the cookie random walk can either be transient or recurrent, depending on $\bar{p}$. Zerner [Zer05] also proved that the speed of a $(M, \bar{p})$-cookie random walk X is always well defined (but may or may not be zero). Precisely,

- there exists a constant $v(M, \bar{p}) \geqslant 0$ such that
$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} v(M, \bar{p}) \quad \mathbf{P}_{(M,\bar{p})}\text{-almost surely.}$$

- The speed is monotonic in $\bar{p}$: if $\bar{p} = (p_1, \ldots, p_M)$ and $\bar{q} = (q_1, \ldots, q_M)$ are two cookie environments such that $p_i \leqslant q_i$ for all i, then $v(M, \bar{p}) \leqslant v(M, \bar{q})$.

- The speed of a $(2, \bar{p})$-cookie random walk is always 0.

The question of whether one can construct a $(M, \bar{p})$-cookie random walk with strictly positive speed was affirmatively answered by Mountford, Pimentel and Valle [MPV06] who considered the case where all the cookies have the same strength $p \in [\frac{1}{2}, 1)$ *i.e.* the cookie vector $\bar{p}$ has the form $[p]_M \overset{\text{def}}{=} (p, \ldots, p)$. They showed that:

- For any $p \in (\frac{1}{2}, 1)$, there exists an M_0 such that for all $M > M_0$ the speed of the $(M, [p]_M)$-cookie random walk is strictly positive.

- If $M(2p - 1) < 2$, then the speed of the $(M, [p]_M)$-cookie random walk is zero.

They also conjectured that when $M(2p - 1) > 2$, the speed should be non-zero. The aim of this paper is to prove that such is indeed the case.

Theorem 1.1

Let X denote a $(M, \bar{p})$-cookie random walk, then
$$\lim_{n\to\infty} \frac{X_n}{n} = v(M, \bar{p}) > 0 \quad \Longleftrightarrow \quad \alpha(M, \bar{p}) > 1$$
where $\alpha(M, \bar{p})$ is given by (1.1).

In particular, we see that a non-zero speed may be achieved for as few as 3 cookies per site. Comparing this result with the transience/recurrence criteria, we have a second order phase transition at the critical value $\alpha = 1$. In fact, it is proved in [BS07] (chapter VI of this thesis) that, in the zero speed case $0 < \alpha < 1$, the rate of transience of X_n is of order $n^{\frac{\alpha+1}{2}}$.

One would certainly like an explicit calculation of the limiting velocity in term of the cookie environment $(M, \bar{p})$ but this seems a challenging problem (one can still look at Corollary 3.7 where we give an implicit formula for the speed). However, one can prove that the speed is continuous in $\bar{p}$ and has a positive right derivative at all its critical points:

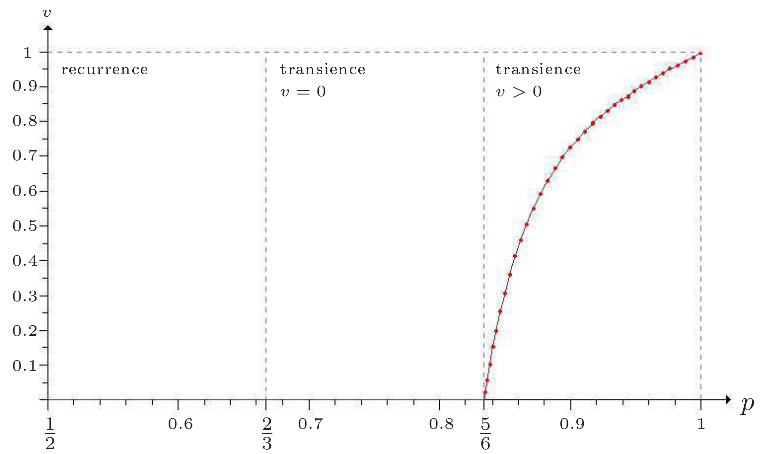

Figure V.1 : Simulation of the speed of a $(3, [p]_3)$-cookie random walk.

Theorem 1.2

- *For each M, the speed $v(M, \bar{p})$ is a continuous function of $\bar{p}$ in Ω^u_M.*
- *For any environment $(M, \bar{p}_c)$ with $\alpha(M, \bar{p}_c) = 1$, there exists a constant $C > 0$ (depending on $(M, \bar{p}_c)$) such that*

$$\lim_{\substack{\bar{p}\to\bar{p}_c \\ \bar{p}\in\Omega^u_M \\ \alpha(\bar{p})>1}} \frac{v(M, \bar{p})}{\alpha(M, \bar{p}) - 1} = C.$$

In particular, for $M \geqslant 3$, the (unique) critical value for an $(M, [p]_M)$-cookie random walk is $p_c = \frac{1}{M} + \frac{1}{2}$ and the function $v(p)$ is continuous, non-decreasing, zero for $p \leqslant p_c$, and admits a finite strictly positive right derivative at p_c (see the figure above).

The remainder of this paper is organized as follow. In the next section, we construct a Markov process associated with the hitting time of the cookie random walk. The method is similar to that used by Kesten, Kozlov and Spitzer [KKS75] for the determination of the rates of transience of a random walk in a one-dimensional random environment. The same method was also used by Tóth [Tót95, Tót96] for studying another class of self interacting random walk on $\mathbb{Z}$. It turns out that, in our setting, the resulting process is a branching process with random migration. The study of this process and of its stationary distribution is undertaken in Section 3. This enables us to complete the proof of Theorem 1.1. Finally, the last section is dedicated to the proof of Theorem 1.2.

2 An associated branching process with migration

In the remainder of this paper, $X = (X_n)_{n \geqslant 0}$ will denote a $(M, \bar{p})$-cookie random walk. Since the speed of a recurrent cookie random walk is zero, we shall also assume that we are in the transient regime *i.e.*

$$\alpha(M, \bar{p}) = \sum_{i=1}^{M} (2p_i - 1) - 1 > 0. \tag{2.1}$$

For the sake of brevity, we simply write $\mathbf{P}_x$ for $\mathbf{P}_{(M,\bar{p}),x}$ and $\mathbf{P}$ instead of $\mathbf{P}_0$ (the process starting from 0). Let T_n stand for the hitting time of level $n \geqslant 0$ by X:

$$T_n = \inf(k \geqslant 0\ ,\ X_k = n). \tag{2.2}$$

For $0 \leqslant k \leqslant n$, let U_i^n denote the number of jumps of the cookie random walk from site i to site $i-1$ before reaching level n

$$U_i^n = \sharp\{0 \leqslant k < T_n,\ X_k = i \text{ and } X_{k+1} = i-1\}.$$

Let K_n stand for the total time spent by X in the negative half-line up to time T_n

$$K_n = \sharp\{0 \leqslant k \leqslant T_n,\ X_k < 0\}.$$

A simple combinatorial argument readily yields

$$T_n = K_n - U_0^n + n + 2\sum_{k=0}^{n} U_k^n.$$

Notice that, as n tends to infinity, the random variable K_n increases toward K_∞, the total time spent by the cookie random walk in the negative half line. Similarly, U_0^n increases toward U_0^∞, the total number of jumps from 0 to -1. Since X is transient, $K_\infty + U_0^\infty$ is almost-surely finite and therefore

$$T_n \underset{n\to\infty}{\sim} n + 2\sum_{k=0}^{n} U_k^n. \tag{2.3}$$

Let us now prove that, for each n, the reverse process $(U_n^n, U_{n-1}^n, \ldots, U_1^n, U_0^n)$ has the same law as the n first steps of some branching process Z with random migration. We first need to introduce some notations. Let $(B_i)_{i \geqslant 1}$ denote a sequence of independent Bernoulli random variables under $\mathbf{P}$ with distribution:

$$\mathbf{P}\{B_i = 1\} = 1 - \mathbf{P}\{B_i = 0\} = \begin{cases} p_i & \text{if } i \leqslant M, \\ \frac{1}{2} & \text{if } i > M. \end{cases} \tag{2.4}$$

For $j \in \mathbb{N}$, define

$$k_j = \min(k \geqslant 1, \sharp\{1 \leqslant i \leqslant k, B_i = 1\} = j+1)$$

and

$$A_j = \sharp\{1 \leqslant i \leqslant k_j, B_i = 0\} = k_j - j - 1.$$

We have the following easy lemma:

Lemma 2.1

- *For any $i, j \geqslant 0$, we have $\mathbf{P}\{A_j = i\} > 0$.*
- *For all $j \geqslant M$, we have*

$$A_j \stackrel{\text{law}}{=} A_{M-1} + \xi_1 + \ldots + \xi_{j-M+1} \tag{2.5}$$

where $(\xi_i)_{i\,\geqslant\,0}$ are i.i.d. random variables independent of A_{M-1} with geometrical distribution starting from 0 and with parameter $\frac{1}{2}$ i.e. $\mathbf{P}\{\xi_1 = i\} = (1/2)^{i+1}$.

Proof. The first part of the lemma is a direct consequence of the assumption that $\bar{p}$ is such that $p_k < 1$ for all k. To prove the second part, we simply notice that $k_{M-1} \geqslant M$ so that, for $j \geqslant M$, the random variable $A_j - A_{M-1}$ has the same law as the random variable

$$\min(k \geqslant 1, \sharp\{1 \leqslant i \leqslant k, \widetilde{B}_i = 1\} = j + 1 - M) - j - 1 + M \tag{2.6}$$

where $(\widetilde{B}_i)_{i\,\geqslant\,0}$ is a sequence of i.i.d. random variables independent of A_{M-1}, with common Bernoulli distribution $\mathbf{P}\{\widetilde{B}_i = 0\} = \mathbf{P}\{\widetilde{B}_i = 1\} = \frac{1}{2}$. It is clear that (2.6) has the same law as $\xi_1 + \ldots + \xi_{j-M+1}$. ■

We now consider a process $Z = (Z_n, n \geq 0)$ and a family of probabilities $(\mathbb{P}_z)_{z\,\geqslant\,0}$ such that, under $\mathbb{P}_z$, the process Z is a Markov chain starting from z, with transition probabilities:

$$\begin{cases} \mathbb{P}_z\{Z_0 = z\} = 1, \\ \mathbb{P}_z\{Z_{n+1} = k \mid Z_n = j\} = \mathbf{P}\{A_j = k\}. \end{cases}$$

Since the family of probabilities $(\mathbb{P}_z)$ depends on the law of the cookie environment $(M, \bar{p})$, we should rigourously write $\mathbb{P}_{(M,\bar{p}),z}$ instead of $\mathbb{P}_z$. However, when there is no possibility of confusion, we shall keep using the abbreviated notation. Furthermore, we simply write $\mathbb{P}$ for $\mathbb{P}_0$ and $\mathbb{E}$ stands for the expectation with respect to $\mathbb{P}$.

Let us now notice that, in view of the previous lemma, Z_n under $\mathbb{P}_z$ may be interpreted as the number of particles alive at time n of a branching process with random migration starting from z, that is a branching process which allows immigration and emigration (see Vatutin and Zubkov [VZ93] for a survey of these processes). Indeed:

- If $Z_n = j \geqslant M - 1$ then, according to Lemma 2.1, Z_{n+1} has the same law as $\sum_{k=1}^{j-M+1} \xi_k + A_{M-1}$, *i.e.* $M - 1$ particles emigrate and the remaining particles reproduce according to a geometrical law with parameter $\frac{1}{2}$ and there is also an immigration of A_{M-1} new particles.
- If $Z_n = j \in \{0, \ldots, M - 2\}$ then Z_{n+1} has the same law as A_j *i.e.* all the j particles emigrate and A_j new particles immigrate.

We can now state the main result of this section:

Proposition 2.2

For each $n \in \mathbb{N}$, $(U_n^n, U_{n-1}^n, \dots, U_0^n)$ under $\mathbf{P}$ has the same law as $(Z_0, Z_1, \dots, Z_n)$ under $\mathbb{P}$.

Proof. The argument is similar to the one given by Kesten *et al.* in [KKS75]. Recall that U_i^n represents the numbers of jumps of the cookie random walk X from i to $i-1$ before reaching n. Then, conditionally on $(U_n^n, U_{n-1}^n, \dots, U_{i+1}^n)$, the number of jumps U_i^n from i to $i-1$ depends only on the number of jumps from $i+1$ to i, that is, depends only on U_{i+1}^n. This shows that $(U_n^n, U_{n-1}^n, \dots, U_0^n)$ is indeed a Markov process.

By definition, $Z_0 = 0$ $\mathbb{P}$-a.s. and $U_n^n = 0$ $\mathbf{P}$-a.s. It remains to compute $\mathbf{P}\{U_i^n = k \mid U_{i+1}^n = j\}$. Note that the number of jumps from i to $i-1$ before reaching level n is equal to the number of jumps from i to $i-1$ before reaching $i+1$ for the first time plus the sum of the number of jumps from i to $i-1$ between two consecutive jumps from $i+1$ to i which occur before reaching level n. Thus, conditionally on $\{U_{i+1}^n = j\}$, the random variable U_i^n has the same law as the number of failures (*i.e.* $B_k = 0$) in the Bernoulli sequence $(B_1, B_2, B_3, \dots)$ defined by (2.4) before obtaining exactly $j+1$ successes. This is precisely the definition of A_j and therefore $\mathbf{P}\{U_i^n = k \mid U_{i+1}^n = j\} = \mathbb{P}_j\{Z_1 = k\}$. ■

Since U_0^n is the number of jumps from 0 to -1 of the cookie random walk X before reaching level n and since we assumed that the cookie random walk X is transient, U_0^n increases almost surely toward the total number U_0^∞ of jumps of X from 0 to -1. In view of the previous proposition, this implies that under $\mathbb{P}$, Z_n converges in law toward a random variable which we denote by Z_∞.

Let us also note that Z is an irreducible Markov chain (this is a consequence of part 1 of Lemma 2.1). Since Z converges in law toward a limiting distribution, this shows that Z is in fact a positive recurrent Markov chain. In particular, Z_n converges in law toward Z_∞ independently of its starting point (*i.e.* the law of Z_∞ is the same under any $\mathbb{P}_x$) and the law of Z_∞ is also the unique invariant probability for Z.

Corolary 2.3

Recall that $v(M, \bar{p})$ denotes the limiting speed of the cookie random walk X. We have

$$v(M, \bar{p}) = \frac{1}{1 + 2\mathbb{E}[Z_\infty]} \quad \text{(with the convention } 0 = \tfrac{1}{+\infty}\text{)}.$$

In particular, the speed of an $(M, \bar{p})$-cookie random walk is non zero i.i.f. the limiting random variable Z_∞ of its associated process Z has a finite expectation.

Proof. Since X is transient, we have the well known equivalence valid for $v \in [0, \infty]$:

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} v \quad \mathbf{P}\text{-a.s.} \qquad \Longleftrightarrow \qquad \frac{T_n}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{v} \quad \mathbf{P}\text{-a.s.} \tag{2.7}$$

On the one hand, this equivalence and (2.3) yield

$$\frac{1}{n}\sum_{k=0}^{n} U_k^n \underset{n\to\infty}{\longrightarrow} \frac{1}{2v(M,\bar{p})} - \frac{1}{2} \quad \mathbf{P}\text{-a.s.} \tag{2.8}$$

On the other hand, making use of an ergodic theorem for the positive recurrent Markov chains Z with stationary limiting distribution Z_∞ (see for instance Theorem 1.10.2 on p53 of [Nor98]), we find that

$$\frac{1}{n}\sum_{i=1}^{n} Z_k \underset{n\to\infty}{\to} \mathbb{E}[Z_\infty] \quad \mathbb{P}\text{-a.s.} \tag{2.9}$$

(note that this result is valid even if $\mathbb{E}[Z_\infty] = \infty$). Proposition 2.2 implies that the limits in (2.8) and (2.9) are the same. This completes the proof of the corollary. ■

Remark 2.4

We assumed in the definition of an $(M,\bar{p})$ cookie environment that

$$p_i \neq 1 \quad \textit{for all } 1 \leqslant i \leqslant M.$$

This hypothesis is intended only to ensure that Z, starting from 0, is not almost surely bounded (for instance, if $p_1 = 1$ then 0 is a absorbing state for Z). More generally, one may check from the definition of the random variables A_j that Z starting from 0 is almost surely unbounded i.i.f.

$$\sharp\{1 \leqslant j \leqslant i \;,\; p_j = 1\} \leqslant \frac{i}{2} \quad \textit{for all } 1 \leqslant i \leqslant M. \tag{2.10}$$

When this condition fails, Z starting from 0 is almost surely bounded by $M-1$, thus $\mathbb{E}[Z_\infty] < \infty$ and the speed of the associated cookie random walk is strictly positive. Otherwise, when (2.10) is fulfilled, Z ultimately hits any level $x \in \mathbb{N}$ with probability 1 and the proof of Theorem 1.1 below remains valid.

3 Study of Z_∞

We proved in the previous section that the strict positivity of the speed of the cookie random walk X is equivalent to the existence of a finite first moment for the limiting distribution of its associated Markov chain Z. We shall now show that, for any cookie environment $(M,\bar{p})$ (with $\alpha(M,\bar{p}) > 0$), we have

$$\mathbb{E}[Z_\infty] \overset{\text{def}}{=} \mathbb{E}_{(M,\bar{p})}[Z_\infty] < \infty \qquad \Longleftrightarrow \qquad \alpha(M,\bar{p}) > 1.$$

This will complete the proof of Theorem 1.1. We start by proving that Z_∞ cannot have moments of any order.

Proposition 3.1

We have

$$\mathbb{E}\left[Z_\infty^{M-1}\right] = +\infty.$$

Proof. Let us introduce the first return time to 0 for Z:

$$\sigma = \inf(n \geqslant 1\ ,\ Z_n = 0).$$

Since Z is a positive recurrent Markov chain, we have $1 \leqslant \mathbb{E}_0[\sigma] < \infty$ and the invariant probability measure is given for any $y \in \mathbb{N}$ by

$$\mathbb{P}\{Z_\infty = y\} = \frac{\mathbb{E}_0\left[\sum_{k=0}^{\sigma-1} \mathbb{1}_{\{Z_k = y\}}\right]}{\mathbb{E}_0[\sigma]}.$$

A monotone convergence argument yields

$$\mathbb{E}_0\left[\sum_{k=0}^{\sigma-1} Z_k^{M-1}\right] = \mathbb{E}_0[\sigma]\mathbb{E}[Z_\infty^{M-1}] \tag{3.1}$$

(where both side of this equality may be infinite). We can find $n_0 \in \mathbb{N}^*$ such that $\mathbb{P}_0\{Z_{n_0} = M,\ n_0 < \sigma\} > 0$ (in fact, since we assume that $p_i < 1$ for all i, we can choose $n_0 = 1$). Therefore, making use of the Markov property of Z, we find that

$$\begin{aligned}\mathbb{E}_0\left[\sum_{k=0}^{\sigma-1} Z_k^{M-1}\right] &\geqslant \mathbb{P}_0\{Z_{n_0} = M,\ n_0 < \sigma\}\mathbb{E}_M\left[\sum_{k=0}^{\sigma-1} Z_k^{M-1}\right] \\ &= \mathbb{P}_0\{Z_{n_0} = M,\ n_0 < \sigma\}\sum_{k=0}^{\infty}\mathbb{E}_M\left[Z_{k\wedge\sigma}^{M-1}\right].\end{aligned} \tag{3.2}$$

In view of (3.1) and (3.2), we just need to prove that

$$\sum_{k=0}^{\infty}\mathbb{E}_M\left[Z_{k\wedge\sigma}^{M-1}\right] = \infty. \tag{3.3}$$

We now use a coupling argument. Let again $(\xi_i)_{i\geq 1}$ denote a sequence of i.i.d. geometrical random variables with parameter 1/2. We define an inhomogeneous Markov chain $\widetilde{Z}$ such that, under $\mathbb{P}_z$:

- $\widetilde{Z}_0 = z$.
- $\widetilde{Z}_1$ has the same law as $\sum_{i=1}^{\widetilde{Z}_0} \xi_i$.
- For $n \geqslant 1$, $\widetilde{Z}_{n+1}$ has the same law as $\sum_{i=1}^{\min(0,\widetilde{Z}_n-(M-1))} \xi_i$ (with the convention $\sum_1^0 = 0$).

Thus, $\widetilde{Z}$ is a branching process with $\min(\widetilde{Z}_n, M-1)$ emigrants at each unit of time, except at time $n = 0$ where no emigration occurs.

Recall that Z is a branching process with migration, where at most $\min(Z_n, M-1)$ particles emigrate at each unit of time, and has the same offspring reproduction law as $\widetilde{Z}$. Therefore, for any $z \geqslant 0$, the process $\widetilde{Z}$ under $\mathbb{P}_z$ is stochastically dominated by Z under $\mathbb{P}_{z+M-1}$ (we need to shift the starting point by $M-1$ because $\widetilde{Z}$ has no emigration at time $n = 0$). Since 0 is an absorbing state for $\widetilde{Z}$, this implies that, for all $n \geqslant 0$,

$$\mathbb{E}_1[\widetilde{Z}_n^{M-1}] \leqslant \mathbb{E}_M[Z_{n\wedge\sigma}^{M-1}]. \tag{3.4}$$

The process $\widetilde{Z}$ belongs to the class of processes studied by Vinokurov in [Vin87]. Moreover, all the assumptions of Theorem 2 and 3 of [Vin87] are fulfilled (in the notation of [Vin87], we have $\theta = M-1$). Therefore, there exist two constants $c_1, c_2 > 0$, such that, as n tends to infinity,

$$\mathbb{P}_1\{\widetilde{Z}_n > 0\} \sim \frac{c_1}{n^M} \quad \text{and} \quad \mathbb{P}_1\{\widetilde{Z}_n > n \mid \widetilde{Z}_n > 0\} \sim c_2.$$

Thus

$$\begin{aligned} \mathbb{E}_1[\widetilde{Z}_n^{M-1}] &= \mathbb{E}_1[\widetilde{Z}_n^{M-1} \,|\widetilde{Z}_n > 0]\mathbb{P}_1\{\widetilde{Z}_n > 0\} \\ &\geq n^{M-1}\mathbb{P}_1\{\widetilde{Z}_n > n \mid \widetilde{Z}_n > 0\}\mathbb{P}_1\{\widetilde{Z}_n > 0\} \sim \frac{c_1 c_2}{n}. \end{aligned} \tag{3.5}$$

The combination of (3.4) and (3.5) yields (3.3). ■

Remark 3.2

In view of the last proposition and Corollary 2.3, we recover the fact that for $M = 2$, the speed of the cookie random walk is always zero.

In order to study more precisely the distribution of Z_∞, we need the following lemma:

Lemma 3.3

We have

$$\mathbf{E}\left[A_{M-1}\right] = 2\sum_{i=1}^{M}(1-p_i).$$

Proof. Recall that $(B_i)_{i\,\geqslant\,1}$ denotes a sequence of independent Bernoulli random variables with distribution given by (2.4). Let $L = \sharp\{1 \leqslant i \leqslant M,\ B_i = 1\} = \sum_{i=1}^{M} B_i$, we have

$$\mathbf{E}[L] = \sum_{i=1}^{M} p_i.$$

Recall also that A_{M-1} denotes the number of failures in the sequence $(B_i)_{i\geq 1}$ before obtaining M successes. Furthermore, $M - L$ represents the number of failures in the subsequence

$(B_i)_{1 \leqslant i \leqslant M}$. So we may rewrite A_{M-1} in the form

$$\begin{aligned} A_{M-1} &= M - L + \left(\inf \left\{ j \geq 0, \sum_{i=M+1}^{M+j} B_i = M - L \right\} - (M - L) \right) \\ &= \inf \left\{ j \geq 0, \sum_{i=M+1}^{M+j} B_i = M - L \right\} \end{aligned}$$

(with the convention $\sum_{M+1}^{M} = 0$). Therefore, given L, the random variable A_{M-1} represents the number of trials needed to get $M-L$ successes along the unbiased coin tossing sequence $(B_i)_{i \geq M+1}$. Thus, given L, the random variable A_{M-1} has a negative binomial distribution with parameters $M-L$ and $p = 1/2$. In particular, we have $\mathbf{E}[A_{M-1} \mid L] = 2(M-L)$ and we conclude that

$$\mathbf{E}[A_{M-1}] = \mathbf{E}[\mathbf{E}[A_{M-1} \mid L]] = \mathbf{E}[2(M-L)] = 2\sum_{i=1}^{M}(1-p_i).$$

■

We now study the law of the limiting distribution Z_∞ of the Markov chain Z. This is done via the study of its probability generating function (p.g.f.)

$$G(s) = \mathbb{E}\left[s^{Z_\infty}\right] \qquad \text{for } s \in [0,1].$$

Lemma 3.4

The p.g.f. G of Z_∞ is the unique p.g.f. solution of the following equation

$$1 - G\left(\frac{1}{2-s}\right) = a(s)(1 - G(s)) + b(s) \qquad \textit{for all } s \in [0,1], \tag{3.6}$$

with

$$a(s) = \frac{1}{(2-s)^{M-1}\mathbf{E}\left[s^{A_{M-1}}\right]},$$

and

$$b(s) = 1 - \frac{1}{(2-s)^{M-1}\mathbf{E}\left[s^{A_{M-1}}\right]} + \sum_{k=0}^{M-2} \frac{G^{(k)}(0)}{k!} \left(\frac{\mathbf{E}\left[s^{A_k}\right]}{(2-s)^{M-1}\mathbf{E}\left[s^{A_{M-1}}\right]} - \frac{1}{(2-s)^k} \right).$$

Proof. The law of Z_∞ is a stationary distribution for the Markov chain Z, therefore

$$\begin{aligned} G(s) = \mathbb{E}\left[\mathbb{E}_{Z_\infty}\left[s^{Z_1}\right]\right] &= \sum_{k=0}^{\infty} \mathbb{P}\{Z_\infty = k\} \mathbb{E}_k\left[s^{Z_1}\right] \\ &= \sum_{k=0}^{M-2} \mathbb{P}\{Z_\infty = k\} \mathbb{E}_k\left[s^{Z_1}\right] + \sum_{k=M-1}^{\infty} \mathbb{P}\{Z_\infty = k\} \mathbb{E}_k\left[s^{Z_1}\right]. \end{aligned}$$

By the definition of Z, the random variable Z_1 under $\mathbb{P}_k$ has the same law as A_k under $\mathbf{P}$. Moreover, according to Lemma 2.1, for $k \geqslant M-1$, A_k has the same law as $A_{M-1}+\xi_1+\ldots+\xi_{k-M+1}$ where $(\xi_i)_{i\geq 1}$ is a sequence of i.i.d. random variables independent of A_{M-1} and with geometric distribution with parameter $\frac{1}{2}$. Thus,

$$\begin{aligned}
G(s) &= \sum_{k=0}^{M-2}\mathbb{P}\{Z_\infty=k\}\mathbf{E}\left[s^{A_k}\right]+\sum_{k=M-1}^{\infty}\mathbb{P}\{Z_\infty=k\}\mathbf{E}\left[s^{A_{M-1}+\xi_1+\ldots+\xi_{k+1-M}}\right]\\
&= \sum_{k=0}^{M-2}\mathbb{P}\{Z_\infty=k\}\mathbf{E}\left[s^{A_k}\right]+\frac{\mathbf{E}\left[s^{A_{M-1}}\right]}{\mathbf{E}\left[s^{\xi}\right]^{M-1}}\sum_{k=M-1}^{\infty}\mathbb{P}\{Z_\infty=k\}\mathbf{E}\left[s^{\xi}\right]^{k}\\
&= \sum_{k=0}^{M-2}\mathbb{P}\{Z_\infty=k\}\left(\mathbf{E}\left[s^{A_k}\right]-\mathbf{E}\left[s^{A_{M-1}}\right]\mathbf{E}\left[s^{\xi}\right]^{k+1-M}\right)+\frac{\mathbf{E}\left[s^{A_{M-1}}\right]}{\mathbf{E}\left[s^{\xi}\right]^{M-1}}G\left(\mathbf{E}\left[s^{\xi}\right]\right).
\end{aligned}$$

Since $\mathbf{E}\left[s^{\xi}\right]=\frac{1}{2-s}$, and $k!\mathbb{P}\{Z_\infty=k\}=G^{(k)}(0)$, we get

$$G(s)=\sum_{k=0}^{M-2}\frac{G^{(k)}(0)}{k!}\left(\mathbf{E}\left[s^{A_k}\right]-\mathbf{E}\left[s^{A_{M-1}}\right](2-s)^{M-1-k}\right)+\mathbf{E}\left[s^{A_{M-1}}\right](2-s)^{M-1}G\left(\frac{1}{2-s}\right),$$

from which we deduce that G solves (3.6).

Furthermore, using the same arguments as above and going backward, we can check that if some p.g.f. satisfies (3.6), then the associated probability distribution is stationary for the irreducible Markov chain Z. In view of the uniqueness of the stationary distribution, we conclude that G is indeed the unique p.g.f. satisfying equation (3.6). ■

Given two functions f and g, we use the classical notation $f(x)=O(g(x))$ in the neighbourhood of zero if $|f(x)|\leqslant C|g(x)|$ for some constant C and all $|x|$ small enough.

Lemma 3.5

The functions a and b of Lemma 3.4 are analytic on $(0,2)$. In particular, they admit a Taylor expansion of any order near point 1 and, as x goes to 0:

$$\begin{aligned}
a(1-x) &= 1-\alpha x+O(x^2),\\
b(1-x) &= O(x).
\end{aligned}$$

Proof. Recall the definitions of the random variables A_k given in Section 2. Since a geometric random variable with parameter $\frac{1}{2}$ admits exponential moments of order strictly smaller than 2, it follows that the p.g.f. $s\mapsto\mathbf{E}[s^{A_k}]$ are strictly positive and analytic on $(0,2)$. From the explicit form of the functions a and b given in the previous lemma, we conclude that these two functions are indeed analytic on $(0,2)$. A Taylor expansion of a near 1 gives

$$a(1-x)=1-(M-1-\mathbf{E}[A_{M-1}])\,x+O(x^2)=1-\alpha x+O(x^2), \tag{3.7}$$

where we used Lemma 3.3 for the last equality. Since G is a p.g.f. we have $G(1) = 1$ which, in view of (3.6), yields $b(1) = 0$ and therefore $b(1-x) = O(x)$. ■

The following proposition relies on a careful study of equation (3.6) and is the key to the proof of Theorem 1.1.

Proposition 3.6

Recall that

$$\alpha = \sum_{i=1}^{M}(2p_i - 1) - 1 > 0.$$

The p.g.f. G of Z_∞ is such that, as $x > 0$ goes to 0:

- *if $0 < \alpha < 1$, then $1 - G(1-x) \sim cx^\alpha$, for some constant $c > 0$.*

  In particular $\mathbf{E}[Z_\infty] = +\infty$.
- *if $\alpha = 1$, then $1 - G(1-x) \sim cx|\ln x|$, for some constant $c > 0$.*

  In particular $\mathbf{E}[Z_\infty] = +\infty$.
- *if $\alpha > 1$, then $1 - G(1-x) = \frac{b''(1)}{2(\alpha-1)}x + O(x^{2\wedge\alpha})$.*

  In particular $\mathbf{E}[Z_\infty] = \frac{b''(1)}{2(\alpha-1)} < +\infty$.

Proof. Since G is a p.g.f, it is completely monotonic and we just need to prove the proposition along the sequence $x = \frac{1}{n}$ with $n \in \mathbb{N}^*$. Making use of Lemma 3.4 with $s = 1 - \frac{1}{n}$, we get, for all $n \geqslant 1$

$$1 - G\left(1 - \frac{1}{n+1}\right) = a\left(1 - \frac{1}{n}\right)\left(1 - G\left(1 - \frac{1}{n}\right)\right) + b\left(1 - \frac{1}{n}\right).$$

Let us define the sequence $(u_n)_{n\geq 1}$ by

$$\begin{cases} u_1 \stackrel{\text{def}}{=} 1 - G(0) = 1 - \mathbf{P}\{Z_\infty = 0\} > 0, \\ u_n \stackrel{\text{def}}{=} \frac{1-G(1-1/n)}{\prod_{i=1}^{n-1} a(1-1/i)} \quad \text{for } n \geq 2. \end{cases} \tag{3.8}$$

We also use the notation

$$r_n \stackrel{\text{def}}{=} \frac{b(1-1/n)}{\prod_{i=1}^{n} a(1-1/i)}.$$

Hence, (u_n) is a sequence of positive numbers which satisfies the relation

$$u_{n+1} = u_n + r_n,$$

thus

$$u_n = u_1 + \sum_{j=1}^{n-1} r_j.$$

This equality may be rewritten

$$1 - G\left(1 - \frac{1}{n}\right) = \prod_{i=1}^{n-1} a\left(1 - \frac{1}{i}\right)\left(1 - G(0) + \sum_{j=1}^{n-1} r_j\right). \tag{3.9}$$

In view of Lemma 3.5, we can write the Taylor expansion of a of order M near 1 in the form

$$a(1-x) = 1 - \alpha x + a_2 x^2 + \ldots + a_M x^M + O(x^{M+1}).$$

Using the classical result

$$\sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma_0 + \ldots + \frac{\gamma_M}{n^M} + O\left(\frac{1}{n^{M+1}}\right),$$

we deduce that

$$\prod_{i=1}^{n} a\left(1 - \frac{1}{i}\right) = \frac{C}{n^\alpha}\left(1 + \frac{a'_1}{n} + \frac{a'_2}{n^2} + \ldots + \frac{a'_{M-1}}{n^{M-1}} + O\left(\frac{1}{n^M}\right)\right) \quad \text{with } C > 0. \tag{3.10}$$

Lemma 3.5 also states that, when b is not identically 0, there exists a unique $k \in \{1, 2, \ldots\}$ such that

$$b(1-x) = D_k x^k + O(x^{k+1}), \quad \text{with } D_k \neq 0. \tag{3.11}$$

If b is identically 0, we use the convention $k = +\infty$. In particular, when k is finite, combining (3.10) and (3.11) , we deduce that

$$r_n = D_k C^{-1} n^{\alpha-k} + O(n^{\alpha-k-1}). \tag{3.12}$$

This implies, whenever $\alpha - k > -1$ that

$$\sum_{j=1}^{n-1} r_j = \frac{D_k C^{-1}}{\alpha - k + 1} n^{\alpha-k+1} + O(1 \vee n^{\alpha-k}). \tag{3.13}$$

Let us now assume that $k = 1$. Combining (3.9), (3.10) and (3.13) we find that $1 - G(1 - \frac{1}{n})$ converges towards $\frac{D_1}{\alpha} \neq 0$ as n tends to infinity but this cannot happen because G is continuous at 1^- with $G(1) = 1$. Thus, we have shown that in fact

$$k \geqslant 2.$$

We now consider the three cases $\alpha > 1$, $\alpha = 1$, $\alpha < 1$ separately.

$\boxed{\alpha > 1}$

We have three sub-cases: either $\alpha > k - 1$, or $\alpha < k - 1$, or $\alpha = k - 1$ with $k \geqslant 3$.

- $\underline{\alpha > k-1}$: Just as before, combining (3.9), (3.10) and (3.13), we now get

$$1 - G\left(1 - \frac{1}{n}\right) = \frac{D_k}{(\alpha - k + 1)n^{k-1}} + O\left(\frac{1}{n^{k\wedge\alpha}}\right).$$

If k were strictly larger than 2, we would have

$$\lim_{n\to\infty} n(1 - G(1 - 1/n)) = 0$$

and therefore $G'(1) = \mathbb{E}[Z_\infty] = 0$ which cannot be true because Z is a positive random variable which is not equal to zero almost surely. Thus k must be equal to 2 and

$$1 - G\left(1 - \frac{1}{n}\right) = \frac{D_2}{(\alpha - 1)n} + O\left(\frac{1}{n^{2\wedge\alpha}}\right). \tag{3.14}$$

Using the equality $D_2 = \frac{b''(1)}{2}$, we conclude that

$$\mathbf{E}[Z_\infty] = \frac{b''(1)}{2(\alpha - 1)} < +\infty.$$

- $\underline{\alpha < k-1}$: We prove that this case never happens. Indeed, in view of (3.12) we find that

$$\sum_{j=1}^{\infty} r_j < \infty$$

(this result also trivially holds when $k = \infty$ since r_j is equally zero in this case). Combining this with (3.9) and (3.10) we see that

$$1 - G\left(1 - \frac{1}{n}\right) = O\left(\frac{1}{n^\alpha}\right).$$

Since $\alpha > 1$, this implies, just as in the previous case, that $\mathbb{E}[Z_\infty] = 0$, which is absurd.

- $\underline{\alpha = k-1 \text{ and } k \geqslant 3}$: Again, we prove that this case is empty. Using (3.12), we get

$$r_n \sim \frac{D_k C^{-1}}{n}.$$

With the help of (3.9) and (3.10), we conclude that

$$1 - G\left(1 - \frac{1}{n}\right) \sim D_k \frac{\ln n}{n^{k-1}}.$$

Since $k \geqslant 3$, we again obtain $\mathbb{E}[Z_\infty] = 0$, which is unacceptable.

Thus, we have completed the proof of the proposition when $\alpha > 1$ and we proved by the way that k must be equal to 2 and that $b''(1) > 0$.

$\boxed{\alpha = 1}$

We first prove, just as in the previous cases, that $k=2$. Let us suppose that $k \geqslant 3$. In view of Lemma 3.5, we can write the Taylor expansion of b of order M near 1 in the form

$$b(1-x) = D_3 x^3 + \ldots + D_M x^M + O(x^{M+1}) \tag{3.15}$$

where $D_i \in \mathbb{R}$ for $i \in \{3, 4, \ldots, M\}$. Combining (3.10) and (3.15) we deduce that

$$\sum_{j=1}^{n-1} r_j = g_0 + \frac{g_1}{n} + \frac{g_2}{n^2} + \ldots + \frac{g_{M-2}}{n^{M-2}} + O\left(\frac{1}{n^{M-1}}\right). \tag{3.16}$$

Therefore, in view of (3.9), (3.10) and (3.16), we get

$$1 - G\left(1 - \frac{1}{n}\right) = \frac{\lambda_1}{n} + \frac{\lambda_2}{n^2} + \ldots + \frac{\lambda_{M-1}}{n^{M-1}} + O\left(\frac{1}{n^M}\right).$$

Comparing with the Taylor expansion of the p.g.f. G, we conclude that $\mathbf{E}(Z_\infty^{M-1}) < \infty$ which contradicts Proposition 3.1. Thus, $k=2$ and (3.12) yields

$$r_n \sim \frac{D_2 C^{-1}}{n} \quad \text{with } D_2 \neq 0. \tag{3.17}$$

In view of (3.9) and (3.10), this estimate implies

$$1 - G\left(1 - \frac{1}{n}\right) \sim D_2 \frac{\ln n}{n},$$

and therefore

$$\mathbb{E}[Z_\infty] = +\infty. \tag{3.18}$$

$\boxed{\alpha < 1}$

Since $k \geqslant 2$, equation (3.12) yields

$$\sum_{j=1}^{\infty} r_j < \infty$$

(of course, this is trivially true when $k = \infty$). Thus, the sequence (u_n) defined by (3.8) converges to a constant $c_1 \geqslant 0$. Suppose first that $c_1 = 0$. In this case, k cannot be infinite (because when $k = \infty$, the sequence (u_n) is constant and then $c_1 = u_1 > 0$). From (3.12) we deduce that

$$u_n = -\sum_{j=n}^{\infty} r_j \sim \frac{D_k C^{-1}}{(k-\alpha-1)n^{k-\alpha-1}},$$

therefore, with the help of (3.10) we get

$$1 - G\left(1 - \frac{1}{n}\right) = u_n \prod_{i=1}^{n-1} a\left(1 - \frac{1}{i}\right) \sim \frac{D_k}{(k-\alpha-1)n^{k-1}}.$$

Since $k \geqslant 2$, this implies that $n(1 - G(1 - 1/n))$ converges to a finite constant and so $\mathbf{E}[Z_\infty] < \infty$. We have already noticed that this implies a strictly positive speed for the cookie random walk in the associated cookie environment $(M, \bar{p})$. But (by possibly extending the value of M) we can always construct a cookie environment $(M, \bar{q})$ such that $\bar{p} \leqslant \bar{q}$ and $\alpha(\bar{q}) = 1$. In view of (3.18), the associated cookie random walk has zero speed and this contradicts a monotonicity result of Zerner (*c.f.* Theorem 17 of [Zer05]). Therefore c_1 cannot be 0 and by (3.8) and (3.10), we get

$$1 - G\left(1 - \frac{1}{n}\right) = u_n \prod_{i=1}^{n-1} a\left(1 - \frac{1}{i}\right) \sim \frac{c_1 C}{n^\alpha}.$$

■

Theorem 1.1 is now a direct consequence of the last proposition and Corollary 2.3. Moreover, in view of the expression of $\mathbf{E}[Z_\infty]$ given in the previous proposition, we get the following expression for the limiting speed:

Corolary 3.7

For any cookie environnement such that $\alpha \geq 1$, we have $b''(1) > 0$ and the speed of the walk is given by the formula

$$v = \frac{\alpha - 1}{\alpha - 1 + b''(1)}.$$

In view of a classical Abelian/Tauberian Theorem (*c.f.* section XIII.5 of [Fel71]), we also deduce from Proposition 3.6 the following estimate concerning the tail distribution of Z_∞ in the zero speed case:

Corolary 3.8

When $\alpha \leqslant 1$, there exists a constant $c > 0$ such that

$$\mathbb{P}\{Z_\infty > n\} \underset{n\to\infty}{\sim} \begin{cases} c/n^\alpha & \text{if } 0 < \alpha < 1, \\ (c \ln n)/n & \text{if } \alpha = 1. \end{cases} \tag{3.19}$$

Remark 3.9

Recall that the random variable Z_∞ has the same distribution as the total number of jumps from 0 to -1 for the cookie random walk. We may also relate this quantity to the total number R of returns to the origin. Indeed, since U_0^n (resp. U_1^n) stands for the respective total number of jumps from 0 to -1 (resp. from 1 to 0) before reaching level n, the total number of returns to the origin before reaching level n is $U_0^n + U_1^n$ which, under $\mathbf{P}$, has the same distribution as $Z_n + Z_{n-1}$ under $\mathbb{P}$. Therefore, we may express the p.g.f. H of the random variable R in term of the p.g.f. G of Z_∞:

$$\begin{aligned} H(s) &= \mathbb{E}\left[s^{Z_\infty}\mathbb{E}_{Z_\infty}\left[s^{Z_\infty}\right]\right] \\ &= \frac{1}{a(s)}G\left(\frac{s}{2-s}\right)+\sum_{k=0}^{M-2}\frac{G^{(k)}(0)}{k!}s^k\left(\mathbb{E}\left[s^{A_k}\right]-\frac{1}{a(s)(2-s)^k}\right). \end{aligned}$$

In particular, Proposition 3.6 holds for H and the tail distribution of the total number of returns to the origin when $\alpha \leqslant 1$ has the same form as in (3.19).

Remark 3.10

In the particular case $M = 2$ (there are at most 2 cookies per site), the only unknown in the definition of the function b is $G(0)$. Since we know that $b'(1) = 0$ (c.f. the beginning of the proof of Proposition 3.6) we can explicitly calculate $G(0)$, that is the probability that the cookie random walk never jumps from 0 to 1, which is also the probability that the cookie random walk never hits -1. According to the previous remark, we can also calculate the probability that the cookie random walk never returns to 0. Hence, we recover Theorem 18 of [Zer05] in the case of a deterministic cookie environment.

4 Continuity of the speed and differentiability at the critical point

The aim of this section is to prove Theorem 1.2. Recall that Corollary 3.7 states that

$$v(M,\bar{p}) = \begin{cases} 0 & \text{if } \alpha(M,p) \leqslant 1, \\ \frac{\alpha-1}{\alpha-1+b''(1)} & \text{if } \alpha(M,p) > 1, \end{cases}$$

where $b''(1)$ stands for the second derivative at point 1 of the function b defined in Lemma 3.4. Furthermore, when $\alpha(M,\bar{p}) = 1$, then $b''(1)$ is strictly positive (*cf.* (3.17)). Hence, in order to prove Theorem 1.2, we just need to show that $b''(1) = b''_{(M,\bar{p})}(1)$ is a continuous function of $\bar{p}$ in Ω^u_M. It is also clear from the definition of the random variables A_k that the functions

$$\bar{p} \to \left(\mathbf{E}_{(M,\bar{p})}\left[s^{A_k}\right]\right)^{(i)}(1) \quad (i.e.\ \text{the } i^{\text{th}} \text{ derivative at point } 1)$$

are continuous in $\bar{p}$ in Ω^u_M for all $k \geqslant 0$ and all $i \geqslant 0$ (they are polynomial functions in $p_1, \dots, p_M$). Therefore, it simply remains to prove that, for all $k \geqslant 0$, the functions

$$\bar{p} \to \mathbb{P}_{(M,\bar{p})}\left\{Z_\infty = k\right\}$$

are continuous in Ω^u_M. The following lemma is based on the monotonicity of the hitting times of a cookie random walk with respect to the environment.

Lemma 4.1

Let $(M,\bar{p})$ be a cookie environment such that $\alpha(M,\bar{p})>0$. Then there exist $\varepsilon>0$ and $f:\mathbb{N}\mapsto\mathbb{R}_+$ with $\lim_{n\to+\infty} f(n)=0$ such that

$$\forall \bar{q}\in B(\bar{p},\varepsilon)\ \ \forall j\in\mathbb{N}\ \ \forall n\in\mathbb{N}\quad |\mathbb{P}_{(M,\bar{q})}\left\{Z_\infty=j\right\}-\mathbb{P}_{(M,\bar{q})}\left\{Z_n=j\right\}|\le f(n),$$

where

$$B(\bar{p},\varepsilon)=\Big\{\bar{q}=(q_1,\ldots,q_M),\ \frac{1}{2}\le q_i<1,\ \alpha(M,\bar{q})>0 \text{ and } \sum_{i=1}^{M}|p_i-q_i|\le\varepsilon\Big\}.$$

Proof. Let us fix $(M,\bar{p})$ with $\alpha(M,\bar{p})>0$. For $\varepsilon>0$, define the vector $\bar{p}^\varepsilon=(p_1^\varepsilon,\ldots,p_M^\varepsilon)$ by $p_i^\varepsilon=\max(\frac{1}{2},p_i-\varepsilon)$. We can choose $\varepsilon>0$ such that $\alpha(M,\bar{p}^\varepsilon)>0$. Then, for all $\bar{q}\in B(\bar{p},\varepsilon)$, we have

$$\bar{p}^\varepsilon\leqslant\bar{q} \tag{4.1}$$

(where $\leqslant$ denotes the canonical partial order on $\mathbb{R}^M$). Let us now pick $\bar{q}\in B(\bar{p},\varepsilon)$, $j\in\mathbb{N}$ and $n\in\mathbb{N}$. Recall that U_0^∞ denotes the total number of jumps of the cookie random walk from 0 to -1 and

$$\mathbb{P}_{(M,\bar{q})}\{Z_\infty=j\}=\mathbf{P}_{(M,\bar{q})}\{U_0^\infty=j\}=\mathbf{P}_{(M,\bar{q})}\{X \text{ jumps } j \text{ times from 0 to -1}\},$$

and

$$\begin{aligned}\mathbb{P}_{(M,\bar{q})}\{Z_n=j\}&=\mathbf{P}_{(M,\bar{q})}\{U_0^n=j\}\\&=\mathbf{P}_{(M,\bar{q})}\{X \text{ jumps } j \text{ times from 0 to -1 before reaching } n\}.\end{aligned}$$

Hence

$$\begin{aligned}|\mathbb{P}_{(M,\bar{q})}\{Z_\infty=j\}-\mathbb{P}_{(M,\bar{q})}\{Z_n=j\}| &= |\mathbf{P}_{(M,\bar{q})}\{U_0^\infty=j\}-\mathbf{P}_{(M,\bar{q})}\{U_0^n=j\}|\\ &\le \mathbf{P}_{(M,\bar{q})}\{U_0^n\neq U_0^\infty\}\\ &= \mathbf{P}_{(M,\bar{q})}\{A\},\end{aligned} \tag{4.2}$$

where A is the event "X visits -1 at least once after reaching level n". Recall the notation $\omega=\omega(i,x)_{i\,\geqslant\,1,x\in\mathbb{Z}}$ for a general cookie environment given in the introduction. Let now $\omega_{X,n}$ denote the (random) cookie-environment obtained when the cookie random walk X hits level n for the first time and shifted by n, *i.e.* for all $x\in\mathbb{Z}$ and $i\ge1$, if the initial cookie environment is ω, then

$$\omega_{X,n}(i,x)=\omega(j,x+n)\qquad\text{where } j=i+\sharp\{0\le k<T_n, X_k=x+n\}.$$

With this notation we have

$$\mathbf{P}_{(M,\bar{q})}\left\{A\right\}=\mathbf{E}_{(M,\bar{q})}\left[\mathbf{P}_{\omega_{X,n}}\{X \text{ visits } -(n+1) \text{ at least once}\}\right].$$

Besides, X has not eaten any cookie at the sites $x \geqslant n$ before time T_n. Thus, the environment $\omega_{X,n}$ satisfies

$$\omega_{X,n}(i,x) = q_i, \quad \text{for all } x \geq 0 \text{ and } i \geqslant 1 \text{ (with the convention } q_i = \tfrac{1}{2} \text{ for } i > M).$$

Hence, in view of (4.1), the random cookie environment $\omega_{X,n}$ is larger (for the canonical partial order) than the deterministic environment $\omega_{\bar{p}^\varepsilon}$ defined by

$$\begin{cases} \omega_{\bar{p}^\varepsilon}(i,x) = \frac{1}{2}, & \text{for all } x < 0 \text{ and } i \geq 1, \\ \omega_{\bar{p}^\varepsilon}(i,x) = p_i^\varepsilon, & \text{for all } x \geq 0 \text{ and } i \geq 1 \text{ (with the convention } p_i^\varepsilon = \frac{1}{2} \text{ for } i \geqslant M). \end{cases}$$

Thus, Lemma 15 of [Zer05] yields

$$\mathbf{P}_{\omega_{X,n}}\{X \text{ visits } -(n+1) \text{ at least once}\} \leqslant \mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X \text{ visits } -(n+1) \text{ at least once}\}$$

In view of (4.2) we deduce that

$$|\mathbb{P}_{(M,\bar{q})}\{Z_\infty = j\} - \mathbb{P}_{(M,\bar{q})}\{Z_n = j\}| \leq f(n),$$

where $f(n) = \mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X \text{ visits } -(n+1) \text{ at least once}\}$ does not depend of $\bar{q}$. It remains to prove that $f(n)$ tends to 0 as n goes to infinity. Let us first notice that

$$\mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{\forall n \geq 0 \quad X_n \geq 0\} = \mathbf{P}_{(M,\bar{p}^\varepsilon)}\{\forall n \geq 0 \quad X_n \geq 0\},$$

since these probabilities depend only on the environments on the half line $[0, +\infty)$. Recall also that the cookie random walk in the environment $(M, \bar{p}^\varepsilon)$ is transient (we have chosen ε such that $\alpha(M, \bar{p}^\varepsilon) > 0$), thus

$$\mathbf{P}_{(M,\bar{p}^\varepsilon)}\{\forall n \geq 0 \quad X_n \geq 0\} = \mathbf{P}_{(M,\bar{p}^\varepsilon)}\{U_0^\infty = 0\} = \mathbb{P}_{(M,\bar{p}^\varepsilon)}\{Z_\infty = 0\} > 0.$$

Hence

$$\mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{\forall n \geq 0 \quad X_n \geq 0\} > 0,$$

which implies

$$\mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X_n = 0 \text{ infinitely often}\} < 1,$$

and a 0-1 law (*c.f.* Proposition 5 of [Zer05]) yields

$$\mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X_n = 0 \text{ infinitely often}\} = \mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X_n \leqslant 0 \text{ infinitely often}\} = 0.$$

Therefore, $\lim_{n\to\infty} f(n) = 0$. ■

Recall that the transition probabilities of the Markov chain Z are given by the law of the random variables A_k:

$$\mathbb{P}_{(M,\bar{p})}\{Z_{n+1} = j \mid Z_n = i\} = \mathbf{P}_{(M,\bar{p})}\{A_i = j\}.$$

It is therefore clear that for each fixed n and each k, the function $\bar{p} \to \mathbb{P}_{(M,\bar{p})}\{Z_n = k\}$ is continuous in $\bar{p}$ in Ω_M^u. Writing

$$\begin{aligned}|\mathbb{P}_{(M,\bar{q})}\{Z_\infty = k\} - \mathbb{P}_{(M,\bar{p})}\{Z_\infty = k\}| &\le |\mathbb{P}_{(M,\bar{q})}\{Z_\infty = k\} - \mathbb{P}_{(M,\bar{q})}\{Z_n = k\}| \\ &+ |\mathbb{P}_{(M,\bar{q})}\{Z_n = k\} - \mathbb{P}_{(M,\bar{p})}\{Z_n = k\}| + |\mathbb{P}_{(M,\bar{p})}\{Z_\infty = k\} - \mathbb{P}_{(M,\bar{p})}\{Z_n = k\}|\end{aligned}$$

and in view of the previous lemma, we conclude that for each k the function

$$\bar{p} \to \mathbb{P}_{(M,\bar{p})}\{Z_\infty = k\}$$

is also continuous in $\bar{p}$ in Ω_M^u, which completes the proof of Theorem 1.2.

Chapter VI

Rate of growth of a transient cookie random walk[1]

Abstract. We consider a one-dimensional transient cookie random walk. It is known from a previous paper [BS06] (Chapter V of this thesis) that the cookies random walk (X_n) has positive or zero speed according to some positive parameter $\alpha > 1$ or ≤ 1. In this article, we give the exact rate of growth of (X_n) in the zero speed regime, namely: for $0 < \alpha < 1$, $X_n/n^{\frac{\alpha+1}{2}}$ converges in law to a Mittag-Leffler distribution whereas for $\alpha = 1$, $X_n(\log n)/n$ converges in probability to some positive constant.

1 Introduction

Let us pick a strictly positive integer M. An M-cookie random walk (also called multi-excited random walk) is a walk on $\mathbb{Z}$ which has a bias to the right upon its M first visits at a given site and evolves like a symmetric random walk afterwards. This model was introduced by Zerner [Zer06] as a generalization, in the one-dimensional setting, of the model of the excited random walk studied by Benjamini and Wilson [BW03]. In this paper, we consider the case where the initial cookie environment is spatially homogeneous. Formally, let $(\Omega, \mathbf{P})$ be some probability space and choose a vector $\bar{p} = (p_1, \ldots, p_M)$ such that $p_i \in [\frac{1}{2}, 1)$ for all $i = 1, \ldots, M$. We say that p_i represents the strength of the i^{th} cookie at a given site. Then, an $(M, \bar{p})$-cookie random walk $(X_n,\, n \in \mathbb{N})$ is a nearest neighbour random walk, starting from 0, and with transition probabilities:

$$\mathbf{P}\{X_{n+1} = X_n + 1 \,|\, X_0, \ldots, X_n\} = \begin{cases} p_j & \text{if } j = \sharp\{0 \leqslant i \leqslant n,\, X_i = X_n\} \leqslant M, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

In particular, the future position X_{n+1} of the walk after time n depends on the whole trajectory $X_0, X_1, \ldots, X_n$. Therefore, X is not, unless in degenerated cases, a Markov process. The cookie random walk is a rich stochastic model. Depending on the cookie

[1] This chapter is a slightly modified version of the joint work: A.-L. Basdevant and A. Singh, *Rate of growth of a transient cookie random walk*, submitted.

environment $(M, \bar{p})$, the process can either be transient or recurrent. Precisely, Zerner [Zer06] (who considered an even more general setting) proved, in our case, that if we define

$$\alpha = \alpha(M, \bar{p}) \stackrel{\text{def}}{=} \sum_{i=1}^{M} (2p_i - 1) - 1, \tag{1.1}$$

- if $\alpha \leqslant 0$, the cookie random walk is recurrent,
- if $\alpha > 0$, the cookie random walk is transient towards $+\infty$.

Thus, a 1-cookie random walk is always recurrent but, for two or more cookies, the walk can either be transient or recurrent. Zerner also proved that the limiting velocity of the walk is well defined. That is, there exists a deterministic constant $v = v(M, \bar{p}) \geqslant 0$ such that

$$\lim_{n \to \infty} \frac{X_n}{n} = v \quad \text{almost surely.}$$

However, we may have $v = 0$. Indeed, when there are at most two cookies per site, Zerner proved that v is always zero. On the other hand, Mountford *et al.* [MPV06] showed that it is possible to have $v > 0$ if the number of cookies is large enough. In a previous paper [BS06] (*c.f.* Chapter V), the authors showed that, in fact, the strict positivity of the speed depends on the position of α with respect to 1:

- if $\alpha \leqslant 1$, then $v = 0$,
- if $\alpha > 1$, then $v > 0$.

In particular, a positive speed may be obtained with just three cookies per site. The aim of this paper is to find the exacte rate of growth of a transient cookie random walk in zero speed regime. In this perspective, numerical simulations of Antal and Redner [AR05] indicated that, for a transient 2-cookies random walk, the expectation of X_n is of order n^ν, for some constant $\nu \in (\frac{1}{2}, 1)$ depending on the strength of the cookies. We shall prove that, more generally, $\nu = \frac{\alpha+1}{2}$.

Theorem 1.1

Let X be a $(M, \bar{p})$-cookie random walk and let α be defined by (1.1)*. Then, when the walk is transient with zero speed,* i.e. *when $0 < \alpha \leqslant 1$,*

- *If $\alpha < 1$,*
$$\frac{X_n}{n^{\frac{\alpha+1}{2}}} \xrightarrow[n \to \infty]{\text{law}} \mathcal{L}_{\frac{\alpha+1}{2}}$$
where $\mathcal{L}_{\frac{\alpha+1}{2}}$ denotes a Mittag-Leffler distribution of index $\frac{\alpha+1}{2}$.
- *If $\alpha = 1$, there exists a constant $c > 0$ such that*
$$\frac{\log n}{n} X_n \xrightarrow[n \to \infty]{\text{prob.}} c.$$

These results also hold with $\sup_{i \leqslant n} X_i$ and $\inf_{i \geqslant n} X_i$ in place of X_n.

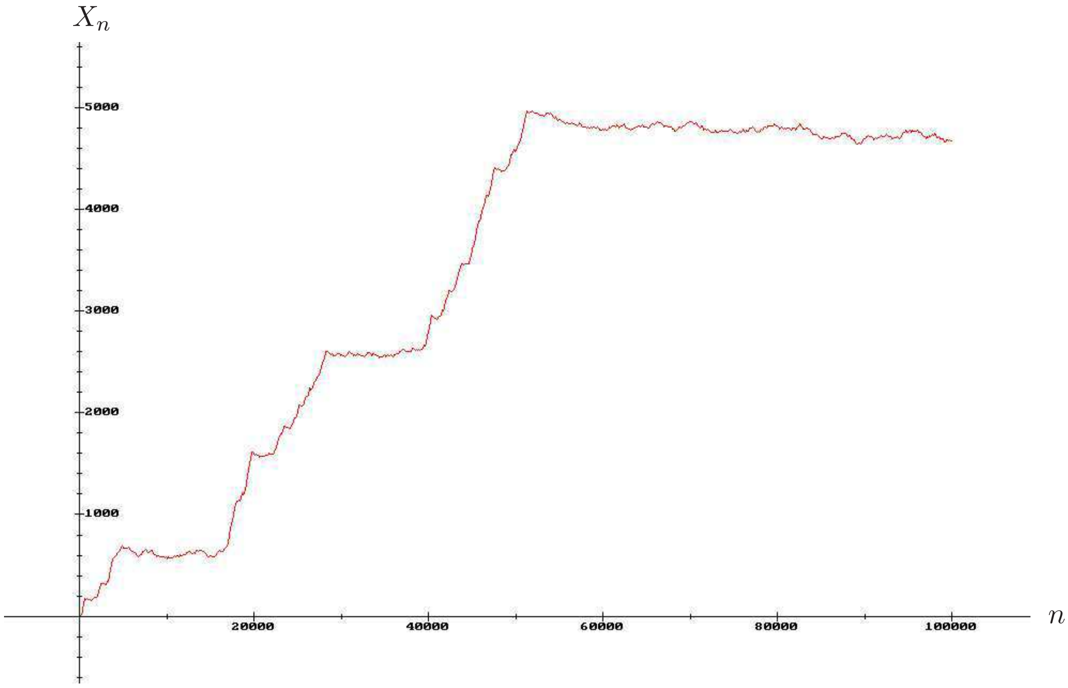

Figure VI.1 : Simulation of the 100000 first steps of a cookie random walk with $M = 3$ and $p_1 = p_2 = p_3 = \frac{3}{4}$ (*i.e.* $\alpha = \frac{1}{2}$ and $\nu = \frac{3}{4}$).

This theorem bears many likenesses to the famous result of Kesten *et al.* [KKS75] concerning the rate of transience of a one-dimensional random walk in random environment. Indeed, following the method initiated in [BS06], we can reduce the study of the walk to that of an auxiliary Markov process Z. In our setting, Z is a branching process with migration. By comparison, Kesten *et al.* obtained the rates of transience of the random walk in random environment via the study of an associated branching process in random environment. However, the process Z considered here and the process introduced in [KKS75] have quite dissimilar behaviours and the methods used for their study are fairly different.

Let us also note that, as α tends to zero, the rate of growth $n^{(1+\alpha)/2}$ tends to $\sqrt{n}$. This suggests that, when the cookie walk is recurrent (*i.e.* $-1 < \alpha \leqslant 0$), its growth should not be much larger than that of a simple symmetric random walk. In fact, we believe that, in the recurrent setting, $\sup_{i \leqslant n} X_i$ should be of order $l(n)\sqrt{n}$ for some slowly varying function l.

The remainder of this paper is organized as follow. In the next section, we recall the construction of the associated process Z described in [BS06] as well as some important results concerning this process. In section 3, we study the tail distribution of the return time to zero of the process Z. Section 4 is devoted to estimating the tail distribution of the total progeny of the branching process over an excursion away from 0. The proof of this result is based on technical estimates whose proofs are given in section 5. Once all these results obtained, the proof of the main theorem is quite straightforward and is finally given in the last section.

2 The process Z

In the rest of this paper, X will denote an $(M,\bar{p})$-cookie random walk. We will also always assume that we are in the transient regime and that the speed of the walk is zero, that is

$$0 < \alpha \leqslant 1.$$

The proof of Theorem 1.1 is based on a careful study of the hitting times of the walk:

$$T_n \stackrel{\text{def}}{=} \inf\{k \geqslant 0,\ X_k = n\}.$$

We now introduce a Markov process Z closely connected with these hitting times. Indeed, we can summarize Proposition 2.2 and equation (2.3) of [BS06] (the previous chapter of this thesis) as follows:

Proposition 2.1

There exist a Markov process $(Z_n,\ n \in \mathbb{N})$ starting from 0 and a sequence of random variables $(K_n,\ n \geqslant 0)$ converging in law towards a finite random variable K such that, for each n

$$T_n \stackrel{\text{law}}{=} n + 2\sum_{k=0}^{n} Z_k + K_n.$$

Therefore, a careful study of Z will enable us to obtain precise estimates on the distribution of the hitting times. In the rest of this section, we shall recall the construction of Z and some important results obtained in [BS06].

For each $i = 1, 2, \ldots$, let B_i be a Bernoulli random variable with distribution

$$\mathbf{P}\{B_i = 1\} = 1 - \mathbf{P}\{B_i = 0\} = \begin{cases} p_i & \text{if } 1 \leqslant i \leqslant M, \\ \frac{1}{2} & \text{if } i > M. \end{cases}$$

We define the random variables $A_0, A_1, \ldots, A_{M-1}$ by

$$A_j \stackrel{\text{def}}{=} \sharp\{1 \leqslant i \leqslant k_j,\ B_i = 0\} \quad \text{where} \quad k_j \stackrel{\text{def}}{=} \inf\Big(i \geqslant 1,\ \sum_{l=1}^{i} B_l = j+1\Big).$$

Therefore, A_j represents the number of "failures" before having $j+1$ "successes" along the sequence of coin tossings (B_i). It is to be noted that the random variables A_j admit some exponential moments:

$$\mathbf{E}[s^{A_j}] < \infty \quad \text{for all } s \in [0, 2). \tag{2.1}$$

According to Lemma 3.3 of [BS06], we also have

$$\mathbf{E}[A_{M-1}] = 2\sum_{i=1}^{M}(1 - p_i) = M - 1 - \alpha. \tag{2.2}$$

Let $(\xi_i,\ i \in \mathbb{N}^*)$ be a sequence of i.i.d. geometric random variables with parameter $\frac{1}{2}$ (*i.e.* with mean 1), independent of the A_j. The process Z mentioned above is a Markov process with transition probabilities given by

$$\mathbf{P}\big\{Z_{n+1} = j \,|\, Z_n = i\big\} = \mathbf{P}\Big\{\mathbb{1}_{\{i \leqslant M-1\}} A_i + \mathbb{1}_{\{i > M-1\}}\Big(A_{M-1} + \sum_{k=1}^{i-M+1} \xi_k\Big) = j\Big\}. \quad (2.3)$$

As usual, we will use the notation $\mathbf{P}_x$ to describe the law of the process starting from $x \in \mathbb{N}$ and $\mathbf{E}_x$ the associated expectation, with the conventions $\mathbf{P} = \mathbf{P}_0$ and $\mathbf{E} = \mathbf{E}_0$. Let us notice that Z may be interpreted as a branching process with random migration, that is, a branching process which allows both immigration and emigration components.

- If $Z_n = i \in \{M, M+1, \ldots\}$, then Z_{n+1} has the law of $\sum_{k=1}^{i-M+1} \xi_k + A_{M-1}$, *i.e.* $M-1$ particles emigrate from the system and the remaining particles reproduce according to a geometrical law with parameter $\frac{1}{2}$ and there is also an immigration of A_{M-1} new particles.
- If $Z_n = i \in \{0, \ldots, M-1\}$, then Z_{n+1} has the same law as A_i, *i.e.* all the i particles emigrate the system and A_i new particles immigrate.

Since we assume that the cookie vector $\bar{p}$ is such that $p_i < 1$ for all i, the process Z is an irreducible Markov process. More precisely,

$$\mathbf{P}_x\{Z_1 = y\} > 0 \quad \text{for all } x, y \in \mathbb{N}.$$

From the construction of the random variables A_i, we have $A_0 \leqslant A_1 \leqslant \ldots \leqslant A_{M-1}$. This fact easily implies that, for any $x \leqslant y$, the process Z under $\mathbf{P}_x$ (starting from x) is stochastically dominated by Z under $\mathbf{P}_y$ (starting from y). Let us also note that, for any $k \geqslant M-1$,

$$\mathbf{E}[Z_{n+1} - Z_n \,|\, Z_n = k] = \mathbf{E}[A_{M-1}] - M + 1 = -\alpha. \quad (2.4)$$

This quantity is negative and we say that emigration dominates immigration. In view of (2.4), a simple martingale argument shows that Z is recurrent. More precisely, according to section 2 of [BS06], the process Z is, in fact, positive recurrent and thus converges in law, independently of its starting point, towards a random variable Z_∞ whose law is the unique invariant probability for Z. Moreover, according to Remark 3.7 of [BS06], the tail distribution of Z_∞ is regularly varying with index α:

Proposition 2.2

There exists a constant $c > 0$ such that

$$\mathbf{P}\{Z_\infty > x\} \underset{x\to\infty}{\sim} \begin{cases} c/x^\alpha & \text{if } \alpha \in (0,1), \\ c \log x / x & \text{if } \alpha = 1. \end{cases}$$

Let now σ denote the first return time to 0 for the process Z,

$$\sigma \stackrel{\text{def}}{=} \inf\{n \geqslant 1,\ Z_n = 0\}.$$

According to the classical expression of the invariant probability, for any non negative function f, we have

$$\mathbf{E}\left[\sum_{i=0}^{\sigma-1} f(Z_i)\right] = \mathbf{E}[\sigma]\mathbf{E}[f(Z_\infty)]. \tag{2.5}$$

In particular, we deduce the following corollary which will be found very useful:

Corolary 2.3

We have, for $\beta \geqslant 0$,

$$\mathbf{E}\left[\sum_{i=0}^{\sigma-1} Z_i^\beta\right] \begin{cases} < \infty & \text{if } \beta < \alpha, \\ = \infty & \text{if } \beta \geqslant \alpha. \end{cases}$$

3 The return time to zero

We have already stated that Z is a positive recurrent Markov chain, thus the return time σ to zero has finite expectation. The aim of this section is to strengthen this result by giving the asymptotic of the tail distribution of σ. Precisely, we will show that

Proposition 3.1

For any initial starting point $x \geqslant 1$, there exists $c = c(x) > 0$ such that

$$\mathbf{P}_x\{\sigma > n\} \underset{n\to\infty}{\sim} \frac{c}{n^{\alpha+1}}.$$

Notice that we do not allow the starting point x to be 0. In fact, this assumption could be dropped but it would unnecessarily complicate the proof of the proposition which is technical enough already. Yet, we have already mentioned that Z starting from 0 is stochastically dominated by Z starting from 1, thus $\mathbf{P}\{\sigma > n\} \leqslant \mathbf{P}_1\{\sigma > n\}$. We also have $\mathbf{P}\{\sigma > n\} \geqslant \mathbf{P}\{Z_1 = 1\}\mathbf{P}_1\{\sigma > n-1\}$. Therefore, we deduce that

$$\frac{c_1}{n^{\alpha+1}} \leqslant \mathbf{P}\{\sigma > n\} \leqslant \frac{c_2}{n^{\alpha+1}}$$

where c_1 and c_2 are two strictly positive constants. In particular, we obtain the following corollary which will be sufficient for our needs.

Corolary 3.2

We have

$$\mathbf{E}[\sigma^\beta] \begin{cases} < \infty & \text{if } \beta < \alpha+1, \\ = \infty & \text{if } \beta \geqslant \alpha+1. \end{cases} \tag{3.1}$$

The method used in the proof of the proposition is classical and based on the study of probability generating functions. Proposition 3.1 was first proved by Vatutin [Vat77] who considered a branching process with exactly one emigrant at each generation. This result was later generalized for branching processes with more than one emigrant by Vinokurov [Vin87] and also by Kaverin [Kav90]. However, in our setting, we deal with a branching process with migration, that is, where both immigration and emigration are allowed. More recently, Yanev and Yanev proved similar results for such a class of processes, under the assumption that, either there is at most one emigrant per generation [YY04] or that immigration dominates emigration [YY95] (in our setting, this would correspond to $\alpha < 0$).

For the process Z, the emigration component dominates the immigration component and this leads to some additional technical difficulties. Although there is a vast literature on the subject (see the authoritative survey of Vatutin and Zubkov [VZ93] for additional references), we did not find a proof of Proposition 3.1 in our setting. We shall therefore provide here a complete argument but we invite the reader to look in the references mentioned above for additional details.

Recall the definition of the random variables A_i and ξ_i defined in section 2. We introduce, for $s \in [0,1]$,

$$\begin{aligned} F(s) &\stackrel{\text{def}}{=} \mathbf{E}[s^{\xi_1}] = \frac{1}{2-s}, \\ \delta(s) &\stackrel{\text{def}}{=} (2-s)^{M-1}\mathbf{E}[s^{A_{M-1}}], \\ H_k(s) &\stackrel{\text{def}}{=} (2-s)^{M-1-k}\mathbf{E}[s^{A_{M-1}}] - \mathbf{E}[s^{A_k}] \quad \text{for } 1 \leqslant k \leqslant M-2. \end{aligned}$$

Let $F_j(s) \stackrel{\text{def}}{=} F \circ \ldots \circ F(s)$ stand for the j-fold of F (with the convention $F_0 = \mathrm{Id}$). We also define by induction

$$\begin{cases} \gamma_0(s) \stackrel{\text{def}}{=} 1, \\ \gamma_{n+1}(s) \stackrel{\text{def}}{=} \delta(F_n(s))\gamma_n(s). \end{cases}$$

We use the abbreviated notations $F_j \stackrel{\text{def}}{=} F_j(0)$, $\gamma_n \stackrel{\text{def}}{=} \gamma_n(0)$. We start with a simple lemma.

Lemma 3.3

(a) $F_n = 1 - \frac{1}{n+1}$.

(b) $H_k(1-s) = -H_k'(1)s + \mathcal{O}(s^2)$ *when* $s \to 0$ *for all* $1 \leqslant k \leqslant M-2$.

(c) $\delta(1-s) = 1 + \alpha s + \mathcal{O}(s^2)$ *when* $s \to 0$.

(d) $\gamma_n \sim_\infty c_3 n^\alpha$ *with* $c_3 > 0$.

Proof. Assertion (a) is straightforward. According to (2.1), the functions H_k are analytic on $(0,2)$ and (b) follows from a Taylor expansion near 1. Similarly, (c) follows from a Taylor expansion near 1 of the function δ combined with (2.2). Finally, γ_n can be expressed in the form

$$\gamma_n = \prod_{j=0}^{n-1} \delta(F_j) \underset{n\to\infty}{\sim} \prod_{j=1}^{n} \left(1 + \frac{\alpha}{j}\right) \underset{n\to\infty}{\sim} c_3 n^{\alpha},$$

which yields (d). ■

Let $\widetilde{Z}$ stand for the process Z absorbed at 0:

$$\widetilde{Z}_n \stackrel{\text{def}}{=} Z_n \mathbb{1}_{\{n \leqslant \inf(k,\, Z_k=0)\}}.$$

We also define, for $x \geqslant 1$ and $s \in [0,1]$,

$$\begin{aligned} J_x(s) &\stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \mathbf{P}_x\{\widetilde{Z}_i \neq 0\} s^i, \\ G_{n,x}(s) &\stackrel{\text{def}}{=} \mathbf{E}_x[s^{\widetilde{Z}_n}], \end{aligned} \tag{3.2}$$

and for $1 \leqslant k \leqslant M-2$,

$$g_{k,x}(s) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \mathbf{P}_x\{\widetilde{Z}_i = k\} s^{i+1}.$$

Lemma 3.4

For any $1 \leqslant k \leqslant M-2$, we have

(a) $\sup_{x \geqslant 1} g_{k,x}(1) < \infty$.

(b) *for all $x \geqslant 1$, $g'_{k,x}(1) < \infty$.*

Proof. The value $g_{x,k}(1)$ represents the expected number of visits to site k before hitting 0 for the process Z starting from x. Thus, an easy application of the Markov property yields

$$g_{k,x}(1) = \frac{\mathbf{P}_x\{Z \text{ visits } k \text{ before } 0\}}{\mathbf{P}_k\{Z \text{ visits } 0 \text{ before returning to } k\}} < \frac{1}{\mathbf{P}_k\{Z_1 = 0\}} < \infty.$$

This proves (a). We now introduce the return times $\sigma_k \stackrel{\text{def}}{=} \inf(n \geqslant 1,\, Z_n = k)$. In view of the Markov property, we have

$$\begin{aligned} g'_{k,x}(1) &= g_{k,x}(1) + \mathbf{E}_x\Big[\sum_{n=1}^{\infty} n \mathbb{1}_{\{\widetilde{Z}_n = k\}}\Big] \\ &= g_{k,x}(1) + \sum_{i=1}^{\infty} \mathbf{P}_x\{\sigma_k = i,\, \sigma_k < \sigma\} \mathbf{E}_k\Big[\sum_{n=0}^{\infty} (i+n) \mathbb{1}_{\{\widetilde{Z}_n = k\}}\Big] \\ &= g_{k,x}(1) + \mathbf{E}_x[\sigma_k \mathbb{1}_{\{\sigma_k < \sigma\}}] g_{k,k}(1) + \mathbf{P}_x\{\sigma_k < \sigma\} \mathbf{E}_k\Big[\sum_{n=0}^{\infty} n \mathbb{1}_{\{\widetilde{Z}_n = k\}}\Big]. \end{aligned}$$

Since Z is a positive recurrent Markov process, we have $\mathbf{E}_x[\sigma_k \mathbb{1}_{\{\sigma_k<\sigma\}}] \leqslant \mathbf{E}_x[\sigma] < \infty$. Thus, it simply remains to show that $\mathbf{E}_k\Big[\sum_{n=0}^{\infty} n\mathbb{1}_{\{\widetilde{Z}_n=k\}}\Big] < \infty$. Using the Markov property, as above, but considering now the partial sums, we get, for any $N \geqslant 1$,

$$\begin{aligned}\mathbf{E}_k\left[\sum_{n=1}^{N} n\mathbb{1}_{\{\widetilde{Z}_n=k\}}\right] &= \sum_{i=1}^{N}\mathbf{P}_k\{\sigma_k=i,\,\sigma_k<\sigma\}\mathbf{E}_k\left[\sum_{n=0}^{N-i}(i+n)\mathbb{1}_{\{\widetilde{Z}_n=k\}}\right]\\ &\leqslant \mathbf{E}_k\left[\sigma_k\mathbb{1}_{\{\sigma_k<\sigma\}}\right]g_{k,k}(1)+\mathbf{P}_k\{\sigma_k<\sigma\}\mathbf{E}_k\left[\sum_{n=1}^{N} n\mathbb{1}_{\{\widetilde{Z}_n=k\}}\right].\end{aligned}$$

Since $\mathbf{P}_k\{\sigma<\sigma_k\} \geqslant \mathbf{P}_k\{Z_1=0\} > 0$, we deduce that

$$\mathbf{E}_k\left[\sum_{n=1}^{N} n\mathbb{1}_{\{\widetilde{Z}_n=k\}}\right] \leqslant \frac{\mathbf{E}_k\left[\sigma_k\mathbb{1}_{\{\sigma_k<\sigma\}}\right]g_{k,k}(1)}{\mathbf{P}_k\{\sigma<\sigma_k\}} < \infty.$$

and we conclude the proof letting N tend to $+\infty$. ■

Lemma 3.5

The function J_x defined by (3.2) may be expressed in the form

$$J_x(s) = \widehat{J}_x(s) + \sum_{k=1}^{M-2}\widetilde{J}_{k,x}(s) \quad \textit{for } s\in[0,1),$$

where

$$\widehat{J}_x(s) \stackrel{\text{def}}{=} \frac{\sum_{n=0}^{\infty}\gamma_n(1-(F_n)^x)s^n}{(1-s)\sum_{n=0}^{\infty}\gamma_n s^n} \quad \textit{and} \quad \widetilde{J}_{k,x}(s) \stackrel{\text{def}}{=} \frac{g_{k,x}(s)\sum_{n=0}^{\infty}\gamma_n H_k(F_n)s^n}{(1-s)\sum_{n=0}^{\infty}\gamma_n s^n}.$$

Proof. From the definition (2.3) of the branching process Z, we get, for $n \geqslant 0$,

$$\begin{aligned}&G_{n+1,x}(s) = \mathbf{E}_x\Big[\mathbf{E}_{\widetilde{Z}_n}[s^{\widetilde{Z}_1}]\Big]\\ &= \mathbf{P}_x\{\widetilde{Z}_n=0\} + \sum_{k=1}^{M-2}\mathbf{P}_x\{\widetilde{Z}_n=k\}\mathbf{E}[s^{A_k}] + \sum_{k=M-1}^{\infty}\mathbf{P}_x\{\widetilde{Z}_n=k\}\mathbf{E}[s^{\xi}]^{k-(M-1)}\mathbf{E}[s^{A_{M-1}}]\\ &= \left(1-\frac{\mathbf{E}[s^{A_{M-1}}]}{\mathbf{E}[s^{\xi}]^{M-1}}\right)\mathbf{P}_x\{\widetilde{Z}_n=0\} - \sum_{k=1}^{M-2}\mathbf{P}_x\{\widetilde{Z}_n=k\}H_k(s) + \frac{\mathbf{E}[s^{A_{M-1}}]}{\mathbf{E}[s^{\xi}]^{M-1}}\sum_{k=0}^{\infty}\mathbf{P}_x\{\widetilde{Z}_n=k\}\mathbf{E}[s^{\xi}]^k.\end{aligned}$$

Since $\mathbf{E}[s^{\xi}] = F(s)$ and $G_{n,x}(0) = \mathbf{P}_x\{\widetilde{Z}_n = 0\}$, using the notation introduced in the beginning of the section, the last equality may be rewritten

$$G_{n+1,x}(s) = \delta(s)G_{n,x}(F(s)) + (1-\delta(s))G_{n,x}(0) - \sum_{k=1}^{M-2}\mathbf{P}_x\{\widetilde{Z}_n=k\}H_k(s).$$

Iterating this equation then setting $s = 0$ and using the relation $G_{0,x}(F_{n+1}) = (F_{n+1})^x$, we deduce that, for any $n \geqslant 0$,

$$G_{n+1,x}(0) = \sum_{i=0}^{n}(1-\delta(F_i))\gamma_i G_{n-i,x}(0) \ + \ \gamma_{n+1}(F_{n+1})^x - \sum_{k=1}^{M-2}\sum_{i=0}^{n}\mathbf{P}_x\{\widetilde{Z}_{n-i} = k\}\gamma_i H_k(F_i). \tag{3.3}$$

Notice also that $\mathbf{P}_x\{\widetilde{Z}_n \neq 0\} = 1 - G_{n,x}(0)$. In view of (3.3) and making use of the relation $(1-\delta(F_i))\gamma_i = \gamma_i - \gamma_{i+1}$, we find, for all $n \geqslant 0$ (with the convention $\sum_0^{-1} = 0$)

$$\begin{aligned}\mathbf{P}_x\{\widetilde{Z}_n \neq 0\} \ &= \ \gamma_n(1-(F_n)^x) + \sum_{i=0}^{n-1}(\gamma_i - \gamma_{i+1})\mathbf{P}_x\{\widetilde{Z}_{n-1-i} \neq 0\} \\ &\quad + \sum_{k=1}^{M-2}\sum_{i=0}^{n-1}\mathbf{P}_x\{\widetilde{Z}_{n-1-i} = k\}\gamma_i H_k(F_i).\end{aligned}$$

Therefore, summing over n, for $s < 1$,

$$\begin{aligned}J_x(s) &= \sum_{n=0}^{\infty}\mathbf{P}_x\{\widetilde{Z}_n \neq 0\}s^n \\ &= \sum_{n=0}^{\infty}\gamma_n(1-(F_n)^x)s^n + \sum_{n=0}^{\infty}\sum_{i=0}^{n}(\gamma_i - \gamma_{i+1})\mathbf{P}_x\{\widetilde{Z}_{n-i} \neq 0\}s^{n+1} \\ &\quad + \sum_{k=1}^{M-2}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\mathbf{P}_x\{\widetilde{Z}_{n-i} = k\}\gamma_i H_k(F_i)s^{n+1} \\ &= \sum_{n=0}^{\infty}\gamma_n(1-(F_n)^x)s^n + J_x(s)\sum_{n=0}^{\infty}(\gamma_n - \gamma_{n+1})s^{n+1} + \sum_{k=1}^{M-2} g_{k,x}(s)\sum_{n=0}^{\infty}\gamma_n H_k(F_n)s^n.\end{aligned}$$

We conclude the proof noticing that $\sum_{n=0}^{\infty}(\gamma_n - \gamma_{n+1})s^{n+1} = (s-1)\sum_{n=0}^{\infty}\gamma_n s^n + 1$. ■

We can now give the proof of the proposition.

Proof of Proposition 3.1. Recall that the parameter α is such that $0 < \alpha \leqslant 1$. We first assume $\alpha < 1$. Fix $x \geqslant 1$ and $1 \leqslant k \leqslant M-2$. In view of Lemma 3.3 and with the help of an Abelian/Tauberian theorem (*c.f.* Chap VIII of [Fel71]), we check that

$$(1-s)\sum_{n=0}^{\infty}\gamma_n s^n \underset{s\to 1^-}{\sim} \frac{c_3\Gamma(\alpha+1)}{(1-s)^\alpha} \quad \text{and} \quad \sum_{n=0}^{\infty}\gamma_n H_k(F_n)s^n \underset{s\to 1^-}{\sim} -\frac{c_3 H_k'(1)\Gamma(\alpha)}{(1-s)^\alpha}.$$

These two equivalences show that $\widetilde{J}_{k,x}(1) \overset{\text{def}}{=} \lim_{s\to 1^-}\widetilde{J}_{k,x}(s)$ is finite. More precisely, we get

$$\widetilde{J}_{k,x}(1) = -\frac{g_{k,x}(1)H_k'(1)}{\alpha},$$

so that we may write

$$\frac{\widetilde{J}_{k,x}(1) - \widetilde{J}_{k,x}(s)}{1-s} = \left(\frac{g_{k,x}(1) - g_{k,x}(s)}{1-s}\right)\frac{\widetilde{J}_{k,x}(s)}{g_{k,x}(s)} + \frac{g_{k,x}(1)\widetilde{B}_k(s)}{(1-s)^2\sum_{n=0}^{\infty}\gamma_n s^n} \tag{3.4}$$

with the notation

$$\widetilde{B}_k(s) \stackrel{\text{def}}{=} \frac{H_k'(1)}{\alpha}(s-1)\sum_{n=0}^{\infty}\gamma_n s^n - \sum_{n=0}^{\infty}\gamma_n H_k(F_n)s^n.$$

The first term on the r.h.s. of (3.4) converges towards $-g_k'(1)H_k'(1)/\alpha$ as s tends to 1 (this quantity is finite thanks to Lemma 3.4). Making use of the relation $\gamma_{n+1} = \delta(F_n)\gamma_n$, we can also rewrite $\widetilde{B}_k$ in the form

$$\widetilde{B}_k(s) = \sum_{n=1}^{\infty}\gamma_{n-1}\left[\frac{H_k'(1)}{\alpha}(1-\delta(F_{n-1})) - \delta(F_{n-1})H_k(F_n)\right]s^n - \frac{H_k'(1)}{\alpha} - H_k(0).$$

With the help of Lemma 3.3, it is easily check that

$$\gamma_{n-1}\left[\frac{H_k'(1)}{\alpha}(1-\delta(F_{n-1})) - \delta(F_{n-1})H_k(F_n)\right] = \mathcal{O}\left(\frac{1}{n^{2-\alpha}}\right).$$

Since $\alpha < 1$, we conclude that

$$\widetilde{B}_k(1) = \lim_{s\to 1^-}\widetilde{B}_k(s) \quad \text{is finite.} \tag{3.5}$$

We also have

$$(1-s)^2\sum_{n=0}^{\infty}\gamma_n s^n \underset{s\to 1^-}{\sim} \frac{c_3\Gamma(\alpha+1)}{(1-s)^{\alpha-1}}. \tag{3.6}$$

Thus, combining (3.4), (3.5) and (3.6), as $s \to 1^-$,

$$\frac{\widetilde{J}_{k,x}(1) - \widetilde{J}_{k,x}(s)}{1-s} = \frac{g_{k,x}(1)\widetilde{B}_k(1)}{c_3\Gamma(\alpha+1)}(1-s)^{\alpha-1} + o\big((1-s)^{\alpha-1}\big). \tag{3.7}$$

We can deal with $\widehat{J}_x$ in exactly the same way. We now find $\widehat{J}_x(1) = \frac{x}{\alpha}$ and setting

$$\widehat{B}_x(1) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty}\gamma_{n-1}\left[\frac{x}{\alpha}(\delta(F_{n-1})-1) - \delta(F_{n-1})(1-(F_n)^x)\right] + \frac{x}{\alpha} - 1, \tag{3.8}$$

we also find that, as $s \to 1^-$,

$$\frac{\widehat{J}_x(1) - \widehat{J}_x(s)}{1-s} = \frac{\widehat{B}_x(1)}{c_3\Gamma(\alpha+1)}(1-s)^{\alpha-1} + o\big((1-s)^{\alpha-1}\big). \tag{3.9}$$

Putting together (3.7) and (3.9) and using Lemma 3.5, we obtain

$$\frac{J_x(1) - J_x(s)}{1-s} = C_x(1-s)^{\alpha-1} + o\big((1-s)^{\alpha-1}\big) \tag{3.10}$$

with

$$C_x \stackrel{\text{def}}{=} \frac{1}{c_3\Gamma(\alpha+1)}\left(\widehat{B}_x(1) + \sum_{k=1}^{M-2} g_{k,x}(1)\widetilde{B}_k(1)\right). \tag{3.11}$$

Since $x \neq 0$, we have $\mathbf{P}_x\{\widetilde{Z}_n \neq 0\} = \mathbf{P}_x\{\sigma > n\}$ and, from the definition of J_x, we deduce

$$\sum_{n=0}^{\infty}\Big(\sum_{k=n+1}^{\infty}\mathbf{P}_x\{\sigma > k\}\Big)s^n = \frac{J_x(1) - J_x(s)}{1-s}. \tag{3.12}$$

Combining (3.10) and (3.12), we see that $C_x \geqslant 0$. Moreover, the use of two successive Tauberian theorems yields

$$\mathbf{P}_x\{\sigma > n\} = \frac{C_x \alpha}{\Gamma(1-\alpha)n^{\alpha+1}} + o\Big(\frac{1}{n^{\alpha+1}}\Big).$$

It remains to prove that $C_x \neq 0$. To this end, we first notice that, for $x, y \geqslant 0$, we have $\mathbf{P}_y\{Z_1 = x\} > 0$ and

$$\mathbf{P}_y\{\sigma > n\} \geqslant \mathbf{P}_y\{Z_1 = x\}\mathbf{P}_x\{\sigma > n-1\}.$$

Thus, $C_y \geqslant \mathbf{P}_y\{Z_1 = x\}C_x$ so it suffices to show that C_x is not zero for some x. In view of (a) of Lemma 3.4, the quantity

$$\sum_{k=1}^{M-2} g_{k,x}(1)\widetilde{B}_k(1)$$

is bounded in x. Looking at the expression of C_x given in (3.11), it just remains to prove that $\widehat{B}_x(1)$ can be arbitrarily large. In view of (3.8), we can write

$$\widehat{B}_x(1) = xS(x) + \frac{x}{\alpha} - 1$$

where

$$S(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty}\gamma_{n-1}\left[\frac{1}{\alpha}(\delta(F_{n-1}) - 1) - \delta(F_{n-1})\frac{(1-(F_n)^x)}{x}\right].$$

But for each fixed n, the function

$$x \to \delta(F_{n-1})\frac{(1-(F_n)^x)}{x}$$

decreases to 0 as x tends to infinity, so the monotone convergence theorem yields

$$S(x) \underset{x\to\infty}{\uparrow} \sum_{n=1}^{\infty}\frac{\gamma_{n-1}}{\alpha}(\delta(F_{n-1}) - 1) \sim c_3\sum_{n=1}^{\infty}\frac{1}{n^{1-\alpha}} = +\infty.$$

Thus, $\widehat{B}_x(1)$ tends to infinity as x tends to infinity and the proof of the proposition for $\alpha < 1$ is complete. The case $\alpha = 1$ may be treated in a similar fashion (and it is even easier to prove that the constant is not zero). We skip the details. ■

Remark 3.6

The study of the tail distribution of the return time is the key to obtaining conditional limit theorems for the branching process, see for instance [Kav90, Vat77, Vin87, YY04].

Indeed, following Vatutin's scheme [Vat77] and using Proposition 3.1, it can now be proved that Z_n/n conditioned on not hitting 0 before time n converges in law towards an exponential distribution. Precisely, for each $x = 1, 2, \dots$ and $r \in \mathbb{R}_+$,

$$\lim_{n\to\infty} \mathbf{P}_x \left\{ \frac{Z_n}{n} \leqslant r \mid \sigma > n \right\} = 1 - e^{-r}.$$

It is to be noted that this result is exactly the same as that obtained for a classical critical Galton-Watson process (i.e. when there is no migration). Although, in our setting, the return time to zero has a finite expectation, which is not the case for the critical Galton-Watson process, the behaviours of both processes conditionally on their non-extinction are still quite similar.

4 Total progeny over an excursion

The aim of this section is to study the distribution of the total progeny of the branching process Z over an excursion away from 0. We will constantly use the notation

$$\nu \stackrel{\text{def}}{=} \frac{\alpha+1}{2}.$$

In particular, ν ranges through $(\frac{1}{2}, 1]$. The main result of this section is the key to the proof of Theorem 1.1 and states as follows.

Proposition 4.1

There exists a constant $c > 0$ such that

$$\mathbf{P}\left\{ \sum_{k=0}^{\sigma-1} Z_k > x \right\} \underset{x\to\infty}{\sim} \begin{cases} c/x^{\nu} & \text{if } \alpha \in (0,1) \\ c \log x/x & \text{if } \alpha = 1. \end{cases}$$

Let us first give an informal explanation for this polynomial decay with exponent ν. In view of Remark 3.6, we can expect the shape of a large excursion away from zero of the process Z to be quite similar to that of a Galton-Watson process. Indeed, if H denotes the height of an excursion of Z (and σ denotes the length of the excursion), numerical simulations show that, just as in the case of a classical branching process without migration, $H \approx \sigma$ and the total progeny $\sum_{k=0}^{\sigma-1} Z_k$ is of the same order as $H\sigma$. Since the decay of the tail distribution of σ is polynomial with exponent $\alpha+1$, the tail distribution of $\sum_{k=0}^{\sigma-1} Z_k$ should then decrease with exponent $\frac{\alpha+1}{2}$. In a way, this proposition tells us that the shape of an excursion is very "squared".

Although there is a vast literature on the subject of branching processes, it seems that there has not been much attention given to the total progeny of the process. Moreover, the classical machinery of generating functions and analytic methods, often used as a rule in the study of branching processes seems, in our setting, inadequate for the study of the total progeny.

The proof of Proposition 4.1 uses a somewhat different approach and is mainly based on a martingale argument. The idea of the proof is fairly simple but, unfortunately, since we are dealing with a discrete time model, a lot of additional technical difficulties appear and the complete argument is quite lengthy. For the sake of clarity, we shall first provide the skeleton of the proof of the proposition, while postponing the proof of the technical estimates to section 5.2.

Let us also note that, although we shall only study the particular branching process associated with the cookie random walk, the method presented here could be used to deal with a more general class of branching processes with migration.

We start with an easy lemma stating that $\mathbf{P}\{\sum_{k=0}^{\sigma-1} Z_k > x\}$ cannot decrease much faster than $\frac{1}{x^\nu}$.

Lemma 4.2

For any $\beta > \nu$, we have

$$\mathbf{E}\left[\Big(\sum_{k=0}^{\sigma-1} Z_k\Big)^\beta\right] = \infty.$$

Proof. When $\alpha = \nu = 1$, the result is a direct consequence of Corollary 2.3 of section 2. We now assume $\alpha < 1$. Hölder's inequality gives

$$\sum_{n=0}^{\sigma-1} Z_n^\alpha \leq \sigma^{1-\alpha}(\sum_{n=0}^{\sigma-1} Z_n)^\alpha.$$

Taking the expectation and applying again Hölder's inequality, we obtain, for $\varepsilon > 0$ small enough

$$\mathbf{E}\left[\sum_{n=0}^{\sigma-1} Z_n^\alpha\right] \leqslant \mathbf{E}[\sigma^{1+\alpha-\varepsilon}]^{\frac{1}{p}}\,\mathbf{E}\left[(\sum_{n=0}^{\sigma-1} Z_n)^{\alpha q}\right]^{\frac{1}{q}},$$

with $p = \frac{1+\alpha-\varepsilon}{1-\alpha}$ and $\alpha q = \frac{1+\alpha-\varepsilon}{2-\varepsilon/\alpha}$. Moreover, Corollary 2.3 states that $\mathbf{E}[\sum_{n=0}^{\sigma-1} Z_n^\alpha] = \infty$ and thanks to Corollary 3.2, $\mathbf{E}[\sigma^{1+\alpha-\varepsilon}] < \infty$. Therefore,

$$\mathbf{E}\left[(\sum_{n=0}^{\sigma-1} Z_n)^{\alpha q}\right] = \mathbf{E}\left[(\sum_{n=0}^{\sigma-1} Z_n)^{\nu+\varepsilon'}\right] = \infty.$$

This result is valid for any ε' small enough and completes the proof of the lemma. ■

Proof of Proposition 4.1. Let us first note that, in view of an Abelian/Tauberian theorem, Proposition 4.1 is equivalent to

$$\mathbf{E}\left[1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}\right] \underset{\lambda\to 0^+}{\sim} \begin{cases} C\lambda^\nu & \text{if } \alpha \in (0,1), \\ C\lambda \log \lambda & \text{if } \alpha = 1, \end{cases}$$

where C is a positive constant. We now construct a martingale in the following way. Let K_ν denote the modified Bessel function of second kind with parameter ν. For $\lambda > 0$, we define

$$\phi_\lambda(x) \stackrel{\text{def}}{=} (\sqrt{\lambda}x)^\nu K_\nu(\sqrt{\lambda}x), \quad \text{for } x > 0. \tag{4.1}$$

We shall give some important properties of ϕ_λ in section 5.1. For the time being, we simply recall that ϕ_λ is an analytic, positive, decreasing function on $(0,\infty)$ such that ϕ_λ and ϕ'_λ are continuous at 0 with

$$\phi_\lambda(0) = 2^{\nu-1}\Gamma(\nu) \quad \text{and} \quad \phi'_\lambda(0) = 0. \tag{4.2}$$

Our main interest in ϕ_λ is that it satisfies the following differential equation, for $x > 0$:

$$-\lambda x\phi_\lambda(x) - \alpha\phi'_\lambda(x) + x\phi''_\lambda(x) = 0. \tag{4.3}$$

Now let $(\mathcal{F}_n, n \geq 0)$ denote the natural filtration of the branching process Z *i.e.* $\mathcal{F}_n \stackrel{\text{def}}{=} \sigma(Z_k, 0 \leq k \leq n)$ and define, for $n \geqslant 0$ and $\lambda > 0$,

$$W_n \stackrel{\text{def}}{=} \phi_\lambda(Z_n)e^{-\lambda\sum_{k=0}^{n-1} Z_k}. \tag{4.4}$$

Setting

$$\mu(n) \stackrel{\text{def}}{=} \mathbf{E}[W_n - W_{n+1} \mid \mathcal{F}_n], \tag{4.5}$$

it is clear that the process

$$Y_n \stackrel{\text{def}}{=} W_n + \sum_{k=0}^{n-1} \mu(k)$$

is an $\mathcal{F}$-martingale. Furthermore, this martingale has bounded increments since

$$|Y_{n+1} - Y_n| \leq |W_{n+1} - W_n| + |\mu(n)| \leq 4||\phi_\lambda||_\infty.$$

Therefore, the use of the optional sampling theorem is legitimate with any stopping time with finite mean. In particular, applying the optional sampling theorem with the first return time to 0, we get

$$\phi_\lambda(0)\mathbf{E}[e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k}] = \phi_\lambda(0) - \mathbf{E}[\sum_{k=0}^{\sigma-1} \mu(k)],$$

which we may be rewritten, using that $\phi_\lambda(0) = 2^{\nu-1}\Gamma(\nu)$,

$$\mathbf{E}[1 - e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k}] = \frac{1}{2^{\nu-1}\Gamma(\nu)}\mathbf{E}[\sum_{k=0}^{\sigma-1} \mu(k)]. \tag{4.6}$$

The proof of Proposition 4.1 now relies on a careful study of the expectation of $\sum_{k=0}^{\sigma-1} \mu(k)$. To this end, we shall decompose μ into several terms using a Taylor expansion of ϕ_λ. We first need the following lemma:

Lemma 4.3

(a) *There exists a function* f_1 *with* $f_1(x) = 0$ *for all* $x \geq M-1$ *such that*

$$\mathbf{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] = -\alpha + f_1(Z_n).$$

(b) *There exists a function* f_2 *with* $f_2(x) = f_2(M-1)$ *for all* $x \geq M-1$ *such that*

$$\mathbf{E}[(Z_{n+1} - Z_n)^2 \mid \mathcal{F}_n] = 2Z_n + 2f_2(Z_n).$$

(c) *For* $p \in \mathbb{N}^*$, *there exists a constant* D_p *such that*

$$\mathbf{E}[|Z_{n+1} - Z_n|^p \mid \mathcal{F}_n] \leq D_p(Z_n^{p/2} + \mathbb{1}_{\{Z_n=0\}}).$$

Proof. Assertion (a) is just a rewriting of equation (2.4). Recall the notations introduced in section 2. Recall in particular that $\mathbf{E}[A_{M-1}] = M-1-\alpha$. Thus, for $j \geq M-1$, we have

$$\begin{aligned}
\mathbf{E}[(Z_{n+1} - Z_n)^2 \mid Z_n = j] &= \mathbf{E}\big[\big(A_{M-1} + \xi_1 + \ldots + \xi_{j-M+1} - j\big)^2\big] \\
&= \mathbf{E}\Big[\Big(\alpha + (A_{M-1} - \mathbf{E}[A_{M-1}]) + \sum_{k=1}^{j-M+1} (\xi_k - \mathbf{E}[\xi_k])\Big)^2\Big] \\
&= \alpha^2 + \mathbf{V}\mathrm{ar}(A_{M-1}) + (j-M+1)\mathbf{V}\mathrm{ar}(\xi_1) \\
&= 2Z_n + \alpha^2 + \mathbf{V}\mathrm{ar}(A_{M-1}) - 2(M-1).
\end{aligned}$$

This proves (b). When p is an even integer, we have $\mathbf{E}[|Z_{n+1} - Z_n|^p \mid \mathcal{F}_n] = \mathbf{E}[(Z_{n+1} - Z_n)^p \mid \mathcal{F}_n]$ and assertion (c) can be proved by developing $(Z_{n+1} - Z_n)^p$ in the same manner as for (b). Finally, when p is an odd integer, Hölder's inequality gives

$$\mathbf{E}[|Z_{n+1} - Z_n|^p \mid Z_n = j > 0] \leq \mathbf{E}[|Z_{n+1} - Z_n|^{p+1} \mid Z_n = j > 0]^{\frac{p}{p+1}} \leq D_{p+1}^{\frac{p}{p+1}} Z_n^{\frac{p}{2}}.$$

■

Continuation of the proof of Proposition 4.1. For $n \in [1, \sigma - 2]$, the random variables Z_n and Z_{n+1} are both non zero and, since ϕ_λ is infinitely differentiable on $(0, \infty)$, a Taylor expansion yields

$$\phi_\lambda(Z_{n+1}) = \phi_\lambda(Z_n) + \phi'_\lambda(Z_n)(Z_{n+1} - Z_n) + \frac{1}{2}\phi''_\lambda(Z_n)(Z_{n+1} - Z_n)^2 + \theta_n, \tag{4.7}$$

where θ_n is given by Taylor's integral remainder formula

$$\theta_n \stackrel{\text{def}}{=} (Z_{n+1} - Z_n)^2 \int_0^1 (1-t)(\phi''_\lambda(Z_n + t(Z_{n+1} - Z_n)) - \phi''_\lambda(Z_n))dt. \tag{4.8}$$

When $n = \sigma - 1$, this result is *a priori* incorrect because then $Z_{n+1} = 0$. However, according to (4.2) and (4.3), the functions $\phi_\lambda(t)$, $\phi'_\lambda(t)$ and $t\phi''_\lambda(t)$ have finite limits as t tends to 0^+, thus equation (4.7) still holds when $n = \sigma - 1$. Therefore, for $n \in [1, \sigma - 1]$,

$$\mathbf{E}[e^{\lambda Z_n}\phi_\lambda(Z_n) - \phi_\lambda(Z_{n+1}) \mid \mathcal{F}_n] =$$
$$(e^{\lambda Z_n} - 1)\phi_\lambda(Z_n) - \phi'_\lambda(Z_n)\mathbf{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] - \frac{1}{2}\phi''_\lambda(Z_n)\mathbf{E}[(Z_{n+1} - Z_n)^2 \mid \mathcal{F}_n] - \mathbf{E}[\theta_n \mid \mathcal{F}_n].$$

In view of (a) and (b) of Lemma 4.3 and recalling the differential equation (4.3) satisfied by ϕ_λ, the r.h.s. of the previous equality may be rewritten

$$(e^{\lambda Z_n} - 1 - \lambda Z_n)\phi_\lambda(Z_n) - \phi'_\lambda(Z_n)f_1(Z_n) - \phi''_\lambda(Z_n)f_2(Z_n) - \mathbf{E}[\theta_n \mid \mathcal{F}_n].$$

On the other hand, in view of (4.4) and (4.5), we have

$$\mu(n) = e^{-\lambda\sum_{k=0}^{n} Z_k}\mathbf{E}[e^{\lambda Z_n}\phi_\lambda(Z_n) - \phi_\lambda(Z_{n+1}) \mid \mathcal{F}_n]. \tag{4.9}$$

Thus, for each $n \in [1, \sigma - 1]$, we may decompose $\mu(n)$ in the form

$$\mu(n) = \mu_1(n) + \mu_2(n) + \mu_3(n) + \mu_4(n), \tag{4.10}$$

where

$$\begin{aligned}
\mu_1(n) &\stackrel{\text{def}}{=} e^{-\lambda\sum_{k=0}^{n} Z_k}(e^{\lambda Z_n} - 1 - \lambda Z_n)\phi_\lambda(Z_n)\\
\mu_2(n) &\stackrel{\text{def}}{=} -e^{-\lambda\sum_{k=0}^{n} Z_k}\phi'_\lambda(Z_n)f_1(Z_n)\\
\mu_3(n) &\stackrel{\text{def}}{=} -e^{-\lambda\sum_{k=0}^{n} Z_k}\phi''_\lambda(Z_n)f_2(Z_n)\\
\mu_4(n) &\stackrel{\text{def}}{=} -e^{-\lambda\sum_{k=0}^{n} Z_k}\mathbf{E}[\theta_n \mid \mathcal{F}_n].
\end{aligned}$$

In particular, we can rewrite (4.6) in the form (we have to treat $\mu(0)$ separately since (4.8) does not hold for $n = 0$)

$$\mathbf{E}[1 - e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k}] = \frac{1}{2^{\nu-1}\Gamma(\nu)}\left(\mathbf{E}\big[\mu(0)\big] + \sum_{i=1}^{4}\mathbf{E}\Big[\sum_{n=1}^{\sigma-1}\mu_i(n)\Big]\right). \tag{4.11}$$

We now state the main estimates:

Lemma 4.4

There exist $\varepsilon > 0$ and eight finite constants $(C_i, C'_i, i = 0, 2, 3, 4)$ such that, as λ tends to 0^+,

(a) $\mathbf{E}\left[\mu(0)\right] = \begin{cases} C_0\lambda^\nu + \mathcal{O}(\lambda) & \text{if } \alpha \in (0,1) \\ C_0\lambda\log\lambda + C'_0\lambda + o(\lambda) & \text{if } \alpha = 1, \end{cases}$

(b) $\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\mu_1(n)\right] = o(\lambda)$ *for* $\alpha \in (0,1]$,

(c) $\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\mu_2(n)\right] = \begin{cases} C_2\lambda^\nu + o(\lambda^{\nu+\varepsilon}) & \textit{if } \alpha\in(0,1) \\ C_2\lambda\log\lambda + C_2'\lambda + o(\lambda) & \textit{if } \alpha = 1, \end{cases}$

(d) $\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\mu_3(n)\right] = \begin{cases} C_3\lambda^\nu + o(\lambda^{\nu+\varepsilon}) & \textit{if } \alpha\in(0,1) \\ C_3\lambda\log\lambda + C_3'\lambda + o(\lambda) & \textit{if } \alpha = 1, \end{cases}$

(e) $\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\mu_4(n)\right] = \begin{cases} C_4\lambda^\nu + o(\lambda^{\nu+\varepsilon}) & \textit{if } \alpha\in(0,1) \\ C_4'\lambda + o(\lambda) & \textit{if } \alpha = 1. \end{cases}$

Let us for the time being postpone the long and technical proof of these estimates until section 5.2 and complete the proof of Proposition 4.1. In view of (4.11), using the previous lemma, we deduce that there exist some constants C, C' such that

$$\mathbf{E}\left[1-e^{-\lambda\sum_{k=0}^{\sigma-1}Z_k}\right] = \begin{cases} C\lambda^\nu + o(\lambda^{\nu+\varepsilon}) & \text{if } \alpha\in(0,1), \\ C\lambda\log\lambda + C'\lambda + o(\lambda) & \text{if } \alpha = 1. \end{cases} \tag{4.12}$$

with

$$C \stackrel{\text{def}}{=} \begin{cases} 2^{1-\nu}\Gamma(\nu)^{-1}(C_0+C_2+C_3+C_4) & \text{when } \alpha<1, \\ 2^{1-\nu}\Gamma(\nu)^{-1}(C_0+C_2+C_3) & \text{when } \alpha=1. \end{cases}$$

It simply remains to check that the constant C is not zero. Indeed, suppose that $C=0$. We first assume $\alpha = 1$. Then, from (4.12),

$$\mathbf{E}\left[1-e^{-\lambda\sum_{k=0}^{\sigma-1}Z_k}\right] = C'\lambda + o(\lambda)$$

which implies $\mathbf{E}[\sum_{k=0}^{\sigma-1}Z_k] < \infty$ and contradicts Corollary 2.3. Similarly, when $\alpha\in(0,1)$ and $C=0$, we get from (4.12),

$$\mathbf{E}\left[1-e^{-\lambda\sum_{k=0}^{\sigma-1}Z_k}\right] = o(\lambda^{\nu+\varepsilon}).$$

This implies, for any $0<\varepsilon'<\varepsilon$, that

$$\mathbf{E}\left[\left(\sum_{n=0}^{\sigma-1}Z_n\right)^{\nu+\varepsilon'}\right] < \infty$$

which contradicts Lemma 4.2. Therefore, C cannot be zero and the proposition is proved. ■

5 Technical estimates

5.1 Some properties of modified Bessel functions

We now recall some properties of modified Bessel functions. All the results cited here may be found in [AS64] (section 9.6) or [Leb72] (section 5.7). For $\eta\in\mathbb{R}$, the modified Bessel function of the first kind I_η is defined by

$$I_\eta(x) \stackrel{\text{def}}{=} \left(\frac{x}{2}\right)^\eta \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{\Gamma(k+1)\Gamma(k+1+\eta)}$$

and the modified Bessel function of the second kind K_η is given by the formula

$$K_\eta(x) \stackrel{\text{def}}{=} \begin{cases} \frac{\pi}{2}\frac{I_{-\eta}(x)-I_\eta(x)}{\sin \pi\eta} & \text{for } \eta \in \mathbb{R}-\mathbb{Z}, \\ \lim_{\eta'\to\eta} K_{\eta'}(x) & \text{for } \eta \in \mathbb{Z}. \end{cases}$$

We are particularly interested in

$$F_\eta(x) \stackrel{\text{def}}{=} x^\eta K_\eta(x) \quad \text{for } x > 0.$$

Thus, the function ϕ_λ defined in (4.1) may be expressed in the form

$$\phi_\lambda(x) = F_\nu(\sqrt{\lambda}x). \tag{5.1}$$

Fact 5.1

For $\eta \geqslant 0$, the function F_η is analytic, positive and strictly decreasing on $(0,\infty)$. Moreover

1. *Behaviour at 0*
   (a) *If $\eta > 0$, the function F_η is defined by continuity at 0 with $F_\eta(0) = 2^{\eta-1}\Gamma(\eta)$.*
   (b) *If $\eta = 0$, then $F_0(x) = -\log x + \log 2 - \gamma + o(1)$ as $x \to 0^+$ where γ denotes Euler's constant.*

2. *Behaviour at infinity*

$$F_\eta(x) \underset{x\to\infty}{\sim} \sqrt{\frac{\pi}{2x}}e^{-x}.$$

   In particular, for every $\eta > 0$, there exists $c_\eta \in \mathbb{R}$ such that

$$\forall x \geq 0, \quad F_\eta(x) \leq c_\eta e^{-x}. \tag{5.2}$$

3. *Formula for the derivative*

$$F'_\eta(x) = -x^{2\eta-1}F_{1-\eta}(x). \tag{5.3}$$

   In particular, F_η solves the differential equation

$$xF''_\eta(x) - (2\eta-1)F'_\eta(x) - xF_\eta(x) = 0.$$

Concerning the function ϕ_λ, in view of (5.1), we deduce

Fact 5.2

For each $\lambda > 0$, the function ϕ_λ is analytic, positive and strictly decreasing on $(0, \infty)$. Moreover

(a) *ϕ_λ is continuous and differentiable at 0 with $\phi_\lambda(0) = 2^{\nu-1}\Gamma(\nu)$ and $\phi'_\lambda(0) = 0$.*

(b) *For $x > 0$, we have*

$$\begin{aligned} \phi'_\lambda(x) &= -\lambda^\nu x^\alpha F_{1-\nu}(\sqrt{\lambda} x), \\ \phi''_\lambda(x) &= \lambda F_\nu(\sqrt{\lambda} x) - \alpha \lambda^\nu x^{\alpha-1} F_{1-\nu}(\sqrt{\lambda} x). \end{aligned}$$

In particular, ϕ_λ solves the differential equation

$$-\lambda x \phi_\lambda(x) - \alpha \phi'_\lambda(x) + x \phi''_\lambda(x) = 0.$$

5.2 Proof of Lemma 4.4

The proof of Lemma 4.4 is long and tedious but requires only elementary methods. We shall treat, in separate subsections the assertions (a) - (e) when $\alpha < 1$ and explain, in a last subsection, how to deal with the case $\alpha = 1$.

We will use the following result extensively throughout the proof of Lemma 4.4.

Lemma 5.3

There exists $\varepsilon > 0$ such that

$$\mathbf{E}\left[\sigma(1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k})\right] = o(\lambda^\varepsilon) \quad \text{as } \lambda \to 0^+.$$

Proof. Let $\beta < \alpha \leqslant 1$, the function $x \to x^\beta$ is concave, thus

$$\mathbf{E}\left[(\sum_{k=0}^{\sigma-1} Z_k)^\beta\right] \leqslant \mathbf{E}\left[\sum_{k=0}^{\sigma-1} Z_k^\beta\right] \stackrel{\text{def}}{=} c_1 < \infty,$$

where we used Corollary 2.3 to conclude on the finiteness of c_1. From Markov's inequality, we deduce that $\mathbf{P}\left\{\sum_{k=0}^{\sigma-1} Z_k > x\right\} \leqslant \frac{c_1}{x^\beta}$ for all $x \geqslant 0$. Therefore,

$$\mathbf{E}\left[1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}\right] \leqslant (1 - e^{-\lambda x}) + \mathbf{P}\left\{\sum_{k=0}^{\sigma-1} Z_k > x\right\} \leqslant \lambda x + \frac{c_1}{x^\beta}.$$

Choosing $x = \lambda^{-\frac{1}{\beta+1}}$ and setting $\beta' \stackrel{\text{def}}{=} \frac{\beta}{\beta+1}$, we deduce

$$\mathbf{E}\left[1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}\right] \leqslant (1 + c_1)\lambda^{\beta'}.$$

According to Corollary 3.2, for $\delta < \alpha$, we have $\mathbf{E}[\sigma^{1+\delta}] < \infty$, so Hölder's inequality gives

$$\begin{aligned}\mathbf{E}\left[\sigma(1-e^{-\lambda\sum_{k=0}^{\sigma-1}Z_k})\right] &\leqslant \mathbf{E}[\sigma^{1+\delta}]^{\frac{1}{1+\delta}}\mathbf{E}\left[(1-e^{-\lambda\sum_{k=0}^{\sigma-1}Z_k})^{\frac{1+\delta}{\delta}}\right]^{\frac{\delta}{1+\delta}} \\ &\leqslant \mathbf{E}[\sigma^{1+\delta}]^{\frac{1}{1+\delta}}\mathbf{E}\left[1-e^{-\lambda\sum_{k=0}^{\sigma-1}Z_k}\right]^{\frac{\delta}{1+\delta}} \leqslant c_2\lambda^{\frac{\beta'\delta}{1+\delta}},\end{aligned}$$

which completes the proof of the lemma. ■

5.2.1 Proof of (a) of Lemma 4.4 when $\alpha < 1$

Using the expression of $\mu(0)$ given by (4.9) and the relation (5.3) between of F'_ν and $F_{1-\nu}$, we have

$$\mathbf{E}[\mu(0)] = \mathbf{E}[F_\nu(0) - F_\nu(\sqrt{\lambda}Z_1)] = -\mathbf{E}\left[\int_0^{\sqrt{\lambda}Z_1} F'_\nu(x)dx\right] = \lambda^\nu \mathbf{E}\left[\int_0^{Z_1} y^\alpha F_{1-\nu}(\sqrt{\lambda}y)dy\right].$$

Thus, using the dominated convergence theorem,

$$\lim_{\lambda\to 0}\frac{1}{\lambda^\nu}E[\mu(0)] = \mathbf{E}\left[\int_0^{Z_1} y^\alpha F_{1-\nu}(0)dy\right] = \frac{F_{1-\nu}(0)}{1+\alpha}\mathbf{E}[Z_1^{1+\alpha}] \stackrel{\text{def}}{=} C_0 < \infty.$$

Furthermore, using again (5.3), we get

$$\begin{aligned}\left|\frac{1}{\lambda^\nu}E[\mu(0)] - C_0\right| &= \mathbf{E}\left[\int_0^{Z_1} y^\alpha \left(F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda}y)\right)dy\right] \\ &= \mathbf{E}\left[\int_0^{Z_1} y^\alpha \int_0^{\sqrt{\lambda}y} x^{-\alpha}F_\nu(x)dxdy\right] \\ &\leq \frac{||F_\nu||_\infty}{1-\alpha}\lambda^{\frac{1-\alpha}{2}}\mathbf{E}\left[\int_0^{Z_1} ydy\right] = \frac{||F_\nu||_\infty \mathbf{E}[Z_1^2]}{2(1-\alpha)}\lambda^{\frac{1-\alpha}{2}}.\end{aligned}$$

Therefore, we obtain

$$E[\mu(0)] = C_0\lambda^\nu + \mathcal{O}(\lambda)$$

which proves (a) of Lemma 4.4.

5.2.2 Proof of (b) of Lemma 4.4 when $\alpha < 1$

Recall that

$$\mu_1(n) = e^{-\lambda\sum_{k=0}^n Z_k}(e^{\lambda Z_n} - 1 - \lambda Z_n)\phi_\lambda(Z_n) = e^{-\lambda\sum_{k=0}^n Z_k}(e^{\lambda Z_n} - 1 - \lambda Z_n)F_\nu(\sqrt{\lambda}Z_n).$$

Thus, $\mu_1(n)$ is almost surely positive and

$$\mu_1(n) \leqslant (1 - e^{-\lambda Z_n} - \lambda Z_n e^{-\lambda Z_n})F_\nu(\sqrt{\lambda}Z_n).$$

Moreover, for any $y > 0$, we have $1 - e^{-y} - ye^{-y} \leqslant \min(1, y^2)$, thus

$$\begin{aligned}
\mu_1(n) &\leqslant (1 - e^{-\lambda Z_n} - \lambda Z_n e^{-\lambda Z_n}) F_\nu(\sqrt{\lambda} Z_n) \left(\mathbb{1}_{\{Z_n > \frac{-2\log\lambda}{\sqrt{\lambda}}\}} + \mathbb{1}_{\{Z_n \leq \frac{-2\log\lambda}{\sqrt{\lambda}}\}} \right) \\
&\leqslant F_\nu(\sqrt{\lambda} Z_n) \mathbb{1}_{\{Z_n > \frac{-2\log\lambda}{\sqrt{\lambda}}\}} + ||F_\nu||_\infty \lambda^2 Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2\log\lambda}{\sqrt{\lambda}}\}} \\
&\leqslant F_\nu(-2\log\lambda) + ||F_\nu||_\infty \lambda^2 Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2\log\lambda}{\sqrt{\lambda}}\}},
\end{aligned}$$

where we used the fact that F_ν is decreasing for the last inequality. In view of (5.2), we also have $F_\nu(-2\log\lambda) \leq c_\nu \lambda^2$ and therefore

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} \mu_1(n)\right] \leqslant \lambda^2 c_\nu \mathbf{E}[\sigma] + \lambda^2 ||F_\nu||_\infty \mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2\log\lambda}{\sqrt{\lambda}}\}}\right]. \tag{5.4}$$

On the one hand, according to (2.5), we have

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2\log\lambda}{\sqrt{\lambda}}\}}\right] = \mathbf{E}\left[Z_\infty^2 \mathbb{1}_{\{Z_\infty \leq \frac{-2\log\lambda}{\sqrt{\lambda}}\}}\right] \mathbf{E}[\sigma]. \tag{5.5}$$

On the other hand, Proposition 2.2 states that $\mathbf{P}(Z_\infty \geqslant x) \sim \frac{C}{x^\alpha}$ as x tends to infinity, thus

$$\mathbf{E}\left[Z_\infty^2 \mathbb{1}_{\{Z_\infty \leq x\}}\right] \underset{x\to\infty}{\sim} 2\sum_{k=1}^{x} k\mathbf{P}(Z_\infty \geq k) \underset{x\to\infty}{\sim} \frac{2C}{2-\alpha} x^{2-\alpha}.$$

This estimate and (5.5) yield

$$\lambda^2 \mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2\log\lambda}{\sqrt{\lambda}}\}}\right] \underset{\lambda\to 0^+}{\sim} c_3 \lambda^{1+\frac{\alpha}{2}} |\log\lambda|^{2-\alpha}. \tag{5.6}$$

Combining (5.4) and (5.6), we finally obtain

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} \mu_1(n)\right] = o(\lambda),$$

which proves (b) of Lemma 4.4.

5.2.3 Proof of (c) of Lemma 4.4 when $\alpha < 1$

Recall that

$$\mu_2(n) = -e^{-\lambda\sum_{k=0}^n Z_k} \phi'_\lambda(Z_n) f_1(Z_n) = \lambda^\nu Z_n^\alpha F_{1-\nu}(\sqrt{\lambda} Z_n) f_1(Z_n) e^{-\lambda\sum_{k=0}^n Z_k}.$$

Since $f_1(x) = 0$ for $x \geqslant M - 1$ (*c.f.* Lemma 4.3), the quantity $|\mu_2(n)|/\lambda^\nu$ is smaller than $M^\alpha ||f_1||_\infty ||F_{1-\nu}||_\infty$. Thus, using the dominated convergence theorem, we get

$$\lim_{\lambda\to 0} \frac{1}{\lambda^\nu} \mathbf{E}\left[\sum_{n=1}^{\sigma-1} \mu_2(n)\right] = \mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^\alpha F_{1-\nu}(0) f_1(Z_n)\right] \stackrel{\text{def}}{=} C_2 \in \mathbb{R}.$$

It remains to prove that, for $\varepsilon > 0$ small enough, as $\lambda \to 0^+$

$$\Big|\frac{1}{\lambda^\nu}\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\mu_2(n)\right] - C_2\Big| = o(\lambda^\varepsilon). \tag{5.7}$$

We can rewrite the l.h.s. of (5.7) in the form

$$\Big|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^\alpha f_1(Z_n)(F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda}Z_n))\right] \\ + \mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^\alpha f_1(Z_n)F_{1-\nu}(\sqrt{\lambda}Z_n)(1-e^{-\lambda\sum_{k=0}^n Z_k})\right]\Big|. \tag{5.8}$$

On the one hand, the first term is bounded by

$$\begin{aligned}
\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^\alpha |f_1(Z_n)|(F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda}Z_n))\right] &\leq M^\alpha ||f_1||_\infty \mathbf{E}[\sigma]\int_0^{\sqrt{\lambda}M}|F'_{1-\nu}(x)|dx \\
&\leq M^\alpha ||f_1||_\infty \mathbf{E}[\sigma]||F_\nu||_\infty \int_0^{\sqrt{\lambda}M} x^{1-2\nu}dx \\
&\leq c_4\lambda^{1-\nu},
\end{aligned}$$

where we used formula (5.3) for the expression of $F'_{1-\nu}$ for the second inequality. On the other hand the second term of (5.8) is bounded by

$$\begin{aligned}
\mathbf{E}\Bigg[\sum_{n=1}^{\sigma-1} Z_n^\alpha |f_1(Z_n)|F_{1-\nu}(\sqrt{\lambda}Z_n)(1-e^{-\lambda\sum_{k=0}^n Z_k})\Bigg] &\leqslant M^\alpha ||f_1||_\infty ||F_{1-\nu}||_\infty \mathbf{E}\big[\sigma(1-e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k})\big] \\
&\leqslant c_5\lambda^\varepsilon
\end{aligned}$$

where we used Lemma 5.3 for the last inequality. Putting the pieces together, we conclude that (5.7) holds for $\varepsilon > 0$ small enough.

5.2.4 Proof of (d) of Lemma 4.4 when $\alpha < 1$

Recall that

$$\begin{aligned}
\mu_3(n) &= -e^{-\lambda\sum_{k=0}^n Z_k}\phi''_\lambda(Z_n)f_2(Z_n) \\
&= -e^{-\lambda\sum_{k=0}^n Z_k}f_2(Z_n)\left(\lambda F_\nu(\sqrt{\lambda}Z_n) + \alpha\lambda^\nu Z_n^{\alpha-1}F_{1-\nu}(\sqrt{\lambda}Z_n)\right).
\end{aligned}$$

Note that, since $\alpha \leqslant 1$, we have $Z_n^{\alpha-1} \leqslant 1$ when $Z_n \neq 0$. The quantities $f_2(Z_n)$, $F_\nu(\sqrt{\lambda}Z_n)$ and $F_{1-\nu}(\sqrt{\lambda}Z_n))$ are also bounded, so we check, using the dominated convergence theorem, that

$$\lim_{\lambda\to 0}\frac{1}{\lambda^\nu}\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\mu_3(n)\right] = -\alpha\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^{\alpha-1}F_{1-\nu}(0)f_2(Z_n)\right] \stackrel{\text{def}}{=} C_3 \in \mathbb{R}.$$

Furthermore we have

$$\frac{1}{\lambda^\nu}\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\mu_3(n)\right] - C_3 = -\lambda^{1-\nu}\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^n Z_k} f_2(Z_n)F_\nu(\sqrt{\lambda}Z_n)\right] \tag{5.9}$$
$$+\alpha\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} f_2(Z_n)\left(F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda}Z_n)\right)\right]$$
$$+\alpha\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} f_2(Z_n)F_{1-\nu}(\sqrt{\lambda}Z_n)\left(1-e^{-\lambda\sum_{k=0}^n Z_k}\right)\right].$$

The first term is clearly bounded by $c_6\lambda^{1-\nu}$. We turn our attention to the second term. In view of (5.3), we have

$$F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda}Z_n) = \int_0^{\sqrt{\lambda}Z_n} x^{1-2\nu}F_\nu(x)dx \leqslant \frac{||F_\nu||_\infty}{2-2\nu}\lambda^{1-\nu}Z_n^{2-2\nu} = \frac{||F_\nu||_\infty}{1-\alpha}\lambda^{1-\nu}Z_n^{1-\alpha},$$

where we used $2-2\nu = 1-\alpha$ for the last equality. Therefore,

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} f_2(Z_n)(F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda}Z_n))\right]\right| \leqslant \frac{||F_\nu||_\infty||f_2||_\infty}{1-\alpha}\lambda^{1-\nu}\mathbf{E}\left[\sum_{n=1}^{\sigma-1}1\right]$$
$$\leqslant \frac{||F_\nu||_\infty||f_2||_\infty\mathbf{E}[\sigma]}{1-\alpha}\lambda^{1-\nu}.$$

As for the third term of (5.9), with the help of Lemma 5.3, we find

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} f_2(Z_n)F_{1-\nu}(\sqrt{\lambda}Z_n)(1-e^{-\lambda\sum_{k=0}^n Z_k})\right]\right| \leqslant ||f_2||_\infty||F_{1-\nu}||_\infty\mathbf{E}\left[\sigma(1-e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k})\right]$$
$$\leqslant c_7\lambda^\varepsilon.$$

Putting the pieces together, we conclude that

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\mu_3(n)\right] = C_3\lambda^\nu + o(\lambda^{\nu+\varepsilon}).$$

5.2.5 Proof of (e) of Lemma 4.4 when $\alpha < 1$

Recall that

$$\mu_4(n) = -e^{-\lambda\sum_{k=0}^n Z_k}\mathbf{E}[\theta_n \mid \mathcal{F}_n]. \tag{5.10}$$

This term is clearly the most difficult to deal with. We first need the next lemma stating that Z_{n+1} cannot be too "far" from Z_n.

Lemma 5.4

There exist two constants $K_1, K_2 > 0$ such that for all $n \geq 0$,

(a) $\mathbf{P}(Z_{n+1} \leq \frac{1}{2}Z_n \mid \mathcal{F}_n) \leq K_1 e^{-K_2 Z_n}$,

(b) $\mathbf{P}(Z_{n+1} \geq 2Z_n \mid \mathcal{F}_n) \leq K_1 e^{-K_2 Z_n}$.

Proof. This lemma follows from large deviation estimates. Indeed, with the notation of section 2, in view of Cramer's theorem, we have, for any $j \geqslant M-1$,

$$\begin{aligned}\mathbf{P}\Big\{Z_{n+1} \leq \frac{1}{2}Z_n \,|Z_n = j\Big\} &= \mathbf{P}\Big\{A_{M-1} + \xi_1 + \ldots + \xi_{j-M+1} \leq \frac{j}{2}\Big\} \\ &\leqslant \mathbf{P}\Big\{\xi_1 + \ldots + \xi_{j-M+1} \leq \frac{j}{2}\Big\} \leqslant K_1 e^{-K_2 j},\end{aligned}$$

where we used the fact that (ξ_i) is a sequence of i.i.d geometric random variables with mean 1. Similarly, recalling that A_{M-1} admits exponential moments of order $\beta < 2$, we also deduce, for $j \geqslant M-1$, with possibly extended values of K_1 and K_2 that

$$\begin{aligned}\mathbf{P}\Big\{Z_{n+1} \geq 2Z_n \,|Z_n = j\Big\} &= \mathbf{P}\Big\{A_{M-1} + \xi_1 + \ldots + \xi_{j-M+1} \geq 2j\Big\} \\ &\leqslant \mathbf{P}\Big\{A_{M-1} \geq \frac{j}{2}\Big\} + \mathbf{P}\Big\{\xi_1 + \ldots + \xi_{j-M+1} \geq \frac{3j}{2}\Big\} \leqslant K_1 e^{-K_2 j}.\end{aligned}$$

■

Throughout this section, we use the notation, for $t \in [0,1]$ and $n \in \mathbb{N}$,

$$V_{n,t} \stackrel{\text{def}}{=} Z_n + t(Z_{n+1} - Z_n).$$

In particular $V_{n,t} \in [Z_n, Z_{n+1}]$ (with the convention that for $a > b$, $[a,b]$ means $[b,a]$). With this notation, we can rewrite the expression of θ_n given in (4.8) in the form

$$\theta_n = (Z_{n+1} - Z_n)^2 \int_0^1 (1-t)\Big(\phi_\lambda''(V_{n,t}) - \phi_\lambda''(Z_n)\Big)dt.$$

Therefore, using the expression of ϕ_λ' and ϕ_λ'' stated in Fact (5.2), we get

$$\mathbf{E}[\theta_n \mid \mathcal{F}_n] = \int_0^1 (1-t)(I_n^1(t) + I_n^2(t))dt, \tag{5.11}$$

with

$$\begin{aligned}I_n^1(t) &\stackrel{\text{def}}{=} \lambda \mathbf{E}\left[(Z_{n+1} - Z_n)^2 \Big(F_\nu(\sqrt{\lambda}V_{n,t}) - F_\nu(\sqrt{\lambda}Z_n)\Big) \,\Big|\, \mathcal{F}_n\right], \\ I_n^2(t) &\stackrel{\text{def}}{=} -\alpha\lambda^\nu \mathbf{E}\left[(Z_{n+1} - Z_n)^2 \left(V_{n,t}^{\alpha-1} F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - Z_n^{\alpha-1} F_{1-\nu}(\sqrt{\lambda}Z_n)\right) \,\Big|\, \mathcal{F}_n\right].\end{aligned}$$

Recall that we want to estimate

$$\begin{aligned}\mathbf{E}\left[\sum_{n=1}^{\sigma-1} \mu_4(n)\right] &= \mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^1(t) dt\right] \\ &\quad + \mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^2(t) dt\right].\end{aligned}$$

We deal with each term separately.

Dealing with I^1: We prove that the contribution of this term is negligible, *i.e.*

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^{n} Z_k}\int_0^1 (1-t)I_n^1(t)dt\right]\right| \leqslant c_8\lambda^{\nu+\varepsilon}. \tag{5.12}$$

To this end, we first notice that

$$\begin{aligned}
|I_n^1(t)| &\leqslant \lambda^{\frac{3}{2}}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \max_{x\in[Z_n,Z_{n+1}]} |F'_\nu(\sqrt{\lambda}x)| \;\middle|\; \mathcal{F}_n\right]\\
&= \lambda^{\frac{3}{2}}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \max_{x\in[Z_n,Z_{n+1}]} (\sqrt{\lambda}x)^\alpha F_{1-\nu}(\sqrt{\lambda}x) \;\middle|\; \mathcal{F}_n\right]\\
&\leq c_{1-\nu}\lambda^{\frac{3}{2}}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \max_{x\in[Z_n,Z_{n+1}]} (\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x} \;\middle|\; \mathcal{F}_n\right],
\end{aligned} \tag{5.13}$$

where we used (5.2) to find $c_{1-\nu}$ such that $F_{1-\nu}(x) \leq c_{1-\nu}e^{-x}$. We now split (5.13) according to whether

$$\text{(a) } \frac{1}{2}Z_n \leq Z_{n+1} \leq 2Z_n \quad \text{ or } \quad \text{(b) } Z_{n+1} < \frac{1}{2}Z_n \text{ or } Z_{n+1} > 2Z_n.$$

One the one hand, Lemma 4.3 states that

$$\mathbf{E}\left[|Z_{n+1}-Z_n|^p \mid \mathcal{F}_n\right] \leq D_p Z_n^{\frac{p}{2}} \qquad \text{for all } p\in\mathbb{N} \text{ and } Z_n\neq 0.$$

Hence, for $1 \leqslant n \leqslant \sigma-1$, we get

$$\begin{aligned}
&\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \max_{x\in[Z_n,Z_{n+1}]} (\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x}\mathbb{1}_{\{\frac{1}{2}Z_n\leq Z_{n+1}\leq 2Z_n\}} \;\middle|\; \mathcal{F}_n\right]\\
&\qquad\leq \mathbf{E}\left[|Z_{n+1}-Z_n|^3 \max_{x\in[\frac{1}{2}Z_n,2Z_n]} (\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x}\mathbb{1}_{\{\frac{1}{2}Z_n\leq Z_{n+1}\leq 2Z_n\}} \;\middle|\; \mathcal{F}_n\right]\\
&\qquad\leq \mathbf{E}\left[|Z_{n+1}-Z_n|^3 (2\sqrt{\lambda}Z_n)^\alpha e^{-\frac{1}{2}\sqrt{\lambda}Z_n} \;\middle|\; \mathcal{F}_n\right]\\
&\qquad\leq c_9 Z_n^{\frac{3}{2}}(\sqrt{\lambda}Z_n)^\alpha e^{-\frac{1}{2}\sqrt{\lambda}Z_n}\\
&\qquad\leq c_9 \lambda^{\frac{3\alpha-6}{8}} Z_n^{\frac{3\alpha}{4}}(\sqrt{\lambda}Z_n)^{\frac{6+\alpha}{4}} e^{-\frac{1}{2}\sqrt{\lambda}Z_n}\\
&\qquad\leq c_{10}\lambda^{\frac{3\alpha-6}{8}} Z_n^{\frac{3\alpha}{4}},
\end{aligned} \tag{5.14}$$

where we used the fact that the function $x^{\frac{6+\alpha}{4}}e^{-\frac{x}{2}}$ is bounded on $\mathbb{R}_+$ for the last inequality. On the other hand,

$$\begin{aligned}
&\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \max_{x\in[Z_n,Z_{n+1}]}(\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x}\mathbb{1}_{\{Z_{n+1}<\frac{1}{2}Z_n \text{ or } Z_{n+1}>2Z_n\}}\;\Big|\;\mathcal{F}_n\right]\\
&\qquad\le \mathbf{E}\left[|Z_{n+1}-Z_n|^3 \max_{x\ge 0}(\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x}\mathbb{1}_{\{Z_{n+1}<\frac{1}{2}Z_n \text{ or } Z_{n+1}>2Z_n\}}\;\Big|\;\mathcal{F}_n\right]\\
&\qquad\le c_{11}\mathbf{E}\left[|Z_{n+1}-Z_n|^6 \mid \mathcal{F}_n\right]^{1/2}\mathbf{P}\left\{Z_{n+1}<\frac{1}{2}Z_n \text{ or } Z_{n+1}>2Z_n\;\Big|\;\mathcal{F}_n\right\}^{\frac{1}{2}} \qquad (5.15)\\
&\qquad\le c_{12}Z_n^{\frac{3}{2}}e^{-\frac{K_2}{2}Z_n}\\
&\qquad\le c_{13}.
\end{aligned}$$

Combining (5.13), (5.14) and (5.15), we get

$$|I_n^1(t)| \;\leqslant\; c_{1-\nu}c_{13}\lambda^{\frac{3}{2}} + c_{1-\nu}c_{10}\lambda^{\frac{3\alpha+6}{8}}Z_n^{\frac{3\alpha}{4}} \;\leqslant\; c_{14}\lambda^{\nu+\frac{2-\alpha}{8}}Z_n^{\frac{3\alpha}{4}}.$$

And therefore

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1}e^{-\lambda\sum_{k=0}^n Z_k}\int_0^1(1-t)I_n^1(t)dt\right]\right| \leqslant c_{14}\lambda^{\nu+\frac{2-\alpha}{8}}\mathbf{E}\left[\sum_{n=1}^{\sigma-1}Z_n^{\frac{3\alpha}{4}}\right].$$

Corollary 2.3 states that $\mathbf{E}[\sum_{n=1}^{\sigma-1}Z_n^{\frac{3\alpha}{4}}]$ is finite so the proof of (5.12) is complete.

Dealing with I^2: It remains to prove that

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1}e^{-\lambda\sum_{k=0}^n Z_k}\int_0^1(1-t)I_n^2(t)dt\right] = C_4\lambda^\nu + o(\lambda^{\nu+\varepsilon}). \qquad (5.16)$$

To this end, we write

$$I_n^2(t) = -\alpha\lambda^\nu(J_n^1(t)+J_n^2(t)+J_n^3(t)), \qquad (5.17)$$

with

$$\begin{aligned}
J_n^1(t) &\overset{\text{def}}{=} \mathbf{E}\left[(Z_{n+1}-Z_n)^2(F_{1-\nu}(\sqrt{\lambda}V_{n,t}))-F_{1-\nu}(\sqrt{\lambda}Z_n))Z_n^{\alpha-1}\mid\mathcal{F}_n\right],\\
J_n^2(t) &\overset{\text{def}}{=} \mathbf{E}\left[(Z_{n+1}-Z_n)^2(V_{n,t}^{\alpha-1}-Z_n^{\alpha-1})(F_{1-\nu}(\sqrt{\lambda}V_{n,t})-F_{1-\nu}(0))\mid\mathcal{F}_n\right],\\
J_n^3(t) &\overset{\text{def}}{=} F_{1-\nu}(0)\mathbf{E}\left[(Z_{n+1}-Z_n)^2(V_{n,t}^{\alpha-1}-Z_n^{\alpha-1})\mid\mathcal{F}_n\right].
\end{aligned}$$

Again, we shall study each term separately. In view of (5.16) and (5.17), the proof of (e) of Lemma 4.4, when $\alpha<1$, will finally be complete once we established the following three estimates:

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^{n} Z_k}\int_0^1 (1-t)J_n^1(t)dt\right] = \mathcal{O}(\lambda^{\frac{1-\alpha}{4}}), \quad (5.18)$$

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^{n} Z_k}\int_0^1 (1-t)J_n^2(t)dt\right] = o(\lambda^{\varepsilon}), \quad (5.19)$$

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^{n} Z_k}\int_0^1 (1-t)J_n^3(t)dt\right] = C + o(\lambda^{\varepsilon}). \quad (5.20)$$

Proof of (5.18): Using a technique similar to that used for I^1, we split J^1 into three different terms according to whether

$$\text{(a) } \frac{1}{2}Z_n \leqslant Z_{n+1} \qquad \text{(b) } 1 \leq Z_{n+1} < \frac{1}{2}Z_n \qquad \text{(c) } Z_{n+1} = 0.$$

For the first case (a), we write, for $1 \leqslant n \leqslant \sigma - 1$, recalling that $V_{n,t} \in [Z_n, Z_{n+1}]$,

$$\begin{aligned}
&\left|\mathbf{E}\left[(Z_{n+1}-Z_n)^2\left(F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(\sqrt{\lambda}Z_n)\right)Z_n^{\alpha-1}\mathbb{1}_{\{\frac{1}{2}Z_n\leq Z_{n+1}\}}\;\Big|\;\mathcal{F}_n\right]\right| \\
&\leq \lambda^{\frac{1}{2}}\mathbf{E}\Big[|Z_{n+1}-Z_n|^3 Z_n^{\alpha-1}\max_{x\geq\frac{1}{2}Z_n}|F'_{1-\nu}(\sqrt{\lambda}x)|\;\Big|\;\mathcal{F}_n\Big] \\
&= \lambda^{\frac{1}{2}}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \mid \mathcal{F}_n\right] Z_n^{\alpha-1}\max_{x\geq\frac{1}{2}Z_n}\left((\sqrt{\lambda}x)^{-\alpha}F_\nu(\sqrt{\lambda}x)\right) \\
&\leq c_{15}\lambda^{\frac{1}{2}}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \mid \mathcal{F}_n\right] Z_n^{\alpha-1}\max_{x\geq\frac{1}{2}Z_n}\left((\sqrt{\lambda}x)^{-\alpha}e^{-\sqrt{\lambda}x}\right) \\
&= c_{15}\lambda^{\frac{1}{2}}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \mid \mathcal{F}_n\right] Z_n^{-1}(\frac{1}{2}\sqrt{\lambda})^{-\alpha}e^{-\frac{1}{2}\sqrt{\lambda}Z_n} \\
&\leq c_{16}Z_n^{\frac{1}{2}}\lambda^{\frac{1-\alpha}{2}}e^{-\frac{1}{2}\sqrt{\lambda}Z_n} \\
&= c_{16}\lambda^{\frac{1-\alpha}{4}}Z_n^{\frac{\alpha}{2}}\left((\sqrt{\lambda}Z_n)^{\frac{1-\alpha}{2}}e^{-\frac{1}{2}\sqrt{\lambda}Z_n}\right) \\
&\leq c_{17}\lambda^{\frac{1-\alpha}{4}}Z_n^{\frac{\alpha}{2}},
\end{aligned} \quad (5.21)$$

where we used Lemma 4.3 to get an upper bound for the conditional expectation. For the second case (b), keeping in mind Lemma 5.4, we get

$$\begin{aligned}
&\mathbf{E}\Big[(Z_{n+1}-Z_n)^2\left(F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(\sqrt{\lambda}Z_n)\right)Z_n^{\alpha-1}\mathbb{1}_{\{1\leq Z_{n+1}<\frac{1}{2}Z_n\}}\;\Big|\;\mathcal{F}_n\Big] \\
&\leq c_{18}\lambda^{\frac{1}{2}}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 Z_n^{\alpha-1}\mathbb{1}_{\{1\leq Z_{n+1}<\frac{1}{2}Z_n\}} \mid \mathcal{F}_n\right]\max_{x\geq 1}\left((\sqrt{\lambda}x)^{-\alpha}e^{-\sqrt{\lambda}x}\right) \\
&\leq c_{19}\lambda^{\frac{1}{2}}\mathbf{E}\left[Z_n^{\alpha+2}\mathbb{1}_{\{1\leq Z_{n+1}<\frac{1}{2}Z_n\}} \mid \mathcal{F}_n\right]\lambda^{-\frac{\alpha}{2}} \\
&\leq c_{19}\lambda^{\frac{1-\alpha}{2}}Z_n^{\alpha+2}\mathbf{P}\{Z_{n+1} < \frac{1}{2}Z_n \mid \mathcal{F}_n\} \\
&\leq c_{19}K_1\lambda^{\frac{1-\alpha}{2}}Z_n^{\alpha+2}e^{-K_2 Z_n} \\
&\leq c_{20}\lambda^{\frac{1-\alpha}{2}}.
\end{aligned} \quad (5.22)$$

For the last case (c), we note that when $Z_{n+1} = 0$, then $V_{n,t} = (1-t)Z_n$, therefore

$$\begin{aligned}
&\mathbf{E}\Big[(Z_{n+1}-Z_n)^2 \left(F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(\sqrt{\lambda}Z_n)\right) Z_n^{\alpha-1}\mathbb{1}_{\{Z_{n+1}=0\}} \;\Big|\; \mathcal{F}_n\Big] \\
&\qquad = Z_n^2(F_{1-\nu}(\sqrt{\lambda}(Z_n(1-t))) - F_{1-\nu}(\sqrt{\lambda}Z_n))Z_n^{\alpha-1}\mathbf{P}\{Z_{n+1}=0 \mid \mathcal{F}_n\} \\
&\qquad \leq c_{21}\lambda^{\frac{1}{2}} Z_n^{2+\alpha} e^{-K_2 Z_n} \max_{x\in[Z_n(1-t),Z_n]} (\sqrt{\lambda}x)^{-\alpha} \\
&\qquad \leq c_{21}\lambda^{\frac{1-\alpha}{2}}(1-t)^{-\alpha} Z_n^2 e^{-K_2 Z_n} \\
&\qquad \leqslant c_{22}\lambda^{\frac{1-\alpha}{2}}(1-t)^{-\alpha}.
\end{aligned} \tag{5.23}$$

Combining (5.21), (5.22) and (5.23), we deduce that, for $1 \leqslant n \leqslant \sigma - 1$,

$$\int_0^1 (1-t)|J_n^1(t)|dt \leq c_{23}\lambda^{\frac{1-\alpha}{4}} Z_n^{\frac{\alpha}{2}}.$$

Moreover, according to Corollary 2.3, we have $\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^{\frac{\alpha}{2}}\right] < \infty$, therefore

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^n Z_k} \int_0^1 (1-t)J_n^1(t)dt\right]\right| \leq \mathbf{E}\left[\sum_{n=1}^{\sigma-1}\int_0^1 (1-t)|J_n^1(t)|dt\right] \leq c_{24}\lambda^{\frac{1-\alpha}{4}} \tag{5.24}$$

which yields (5.18).

Proof of (5.19): We write

$$J_n^2(t) = \mathbf{E}[R_n(t) \mid \mathcal{F}_n]$$

with

$$R_n(t) \stackrel{\text{def}}{=} (Z_{n+1}-Z_n)^2 \left(V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}\right)\left(F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(0)\right).$$

Again, we split the expression of J^2 according to four cases:

$$\begin{aligned}
J_n^2(t) \;=\;& \mathbf{E}[R_n(t)\mathbb{1}_{\{Z_{n+1}=0\}} \mid \mathcal{F}_n] + \mathbf{E}[R_n(t)\mathbb{1}_{\{1\leq Z_{n+1}<\frac{1}{2}Z_n\}} \mid \mathcal{F}_n] \\
&+\mathbf{E}[R_n(t)\mathbb{1}_{\{\frac{1}{2}Z_n\leq Z_{n+1}\leq 2Z_n\}} \mid \mathcal{F}_n] + \mathbf{E}[R_n(t)\mathbb{1}_{\{Z_{n+1}>2Z_n\}} \mid \mathcal{F}_n].
\end{aligned} \tag{5.25}$$

We do not detail the cases $Z_{n+1} = 0$ and $1 \leq Z_{n+1} < \frac{1}{2}Z_n$ which may be treated with the same method used in (5.22) and (5.23) and yields similar bounds which do not depend on Z_n:

$$\begin{aligned}
\mathbf{E}[R_n(t)\mathbb{1}_{\{Z_{n+1}=0\}} \mid \mathcal{F}_n] &\leqslant c_{25}\lambda^{\frac{1-\alpha}{2}}(1-t)^{-\alpha} \\
\mathbf{E}[R_n(t)\mathbb{1}_{\{1\leq Z_{n+1}<\frac{1}{2}Z_n\}} \mid \mathcal{F}_n] &\leqslant c_{26}\lambda^{\frac{1-\alpha}{2}}.
\end{aligned}$$

In particular, the combination of these two estimates gives:

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^n Z_k} \int_0^1 (1-t)\mathbf{E}[R_n(t)\mathbb{1}_{\{Z_{n+1}<\frac{Z_n}{2}\}} \mid \mathcal{F}_n]dt\right]\right| \leq c_{27}\lambda^{\frac{1-\alpha}{2}}. \tag{5.26}$$

In order to deal with the third term on the r.h.s. of (5.25), we write

$$\begin{aligned}
&|\mathbf{E}[R_n(t)\mathbb{1}_{\{\frac{1}{2}Z_n\le Z_{n+1}\le 2Z_n\}} \mid \mathcal{F}_n]| \\
&= \left|\mathbf{E}\Big[(Z_{n+1}-Z_n)^2(V_{n,t}^{\alpha-1}-Z_n^{\alpha-1})(F_{1-\nu}(\sqrt{\lambda}V_{n,t})-F_{1-\nu}(0))\mathbb{1}_{\{\frac{1}{2}Z_n\le Z_{n+1}\le 2Z_n\}} \mid \mathcal{F}_n\Big]\right| \\
&\le c_{28}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \max_{x\ge \frac{Z_n}{2}} x^{\alpha-2}\int_0^{2\sqrt{\lambda}Z_n}|F'_{1-\nu}(y)|dy \;\Big|\; \mathcal{F}_n\right] \\
&\le c_{29}\mathbf{E}\left[|Z_{n+1}-Z_n|^3 \mid \mathcal{F}_n\right]\max_{x\ge \frac{Z_n}{2}} x^{\alpha-2}\int_0^{2\sqrt{\lambda}Z_n} y^{-\alpha}dy \\
&\le c_{30}\lambda^{\frac{1-\alpha}{2}}Z_n^{\frac{1}{2}}.
\end{aligned}$$

According to Corollary 2.3, when $\frac{1}{2}<\alpha<1$, we have $\mathbf{E}\left[\sum_{n=1}^{\sigma-1}Z_n^{1/2}\right]<\infty$. In this case, we get

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1}e^{-\lambda\sum_{k=0}^{n}Z_k}\int_0^1(1-t)\mathbf{E}[R_n(t)\mathbb{1}_{\{\frac{1}{2}Z_n\le Z_{n+1}\le 2Z_n\}} \mid \mathcal{F}_n]dt\right]\right| \le c_{31}\lambda^{\frac{1-\alpha}{2}}. \tag{5.27}$$

When $0<\alpha\leqslant\frac{1}{2}$, the function $x^{\frac{2-3\alpha}{4}}e^{-x}$ is bounded on $\mathbb{R}_+$, so

$$\begin{aligned}
e^{-\lambda Z_n}\int_0^1(1-t)|\mathbf{E}[R_n(t)\mathbb{1}_{\{\frac{1}{2}Z_n\le Z_{n+1}\le 2Z_n\}} \mid \mathcal{F}_n]|dt &\le c_{30}\lambda^{\frac{\alpha}{4}}Z_n^{\frac{3\alpha}{4}}(\lambda Z_n)^{\frac{2-3\alpha}{4}}e^{-\lambda Z_n} \\
&\le c_{31}\lambda^{\frac{\alpha}{4}}Z_n^{\frac{3\alpha}{4}}.
\end{aligned}$$

Therefore, when $\alpha\leqslant\frac{1}{2}$, the estimate (5.27) still holds by changing $\lambda^{\frac{1-\alpha}{2}}$ to $\lambda^{\frac{\alpha}{4}}$. Hence, for every $\alpha\in(0,1)$, we can find $\varepsilon>0$ such that

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1}e^{-\lambda\sum_{k=0}^{n}Z_k}\int_0^1(1-t)\mathbf{E}[R_n(t)\mathbb{1}_{\{\frac{1}{2}Z_n\le Z_{n+1}\le 2Z_n\}} \mid \mathcal{F}_n]dt\right]\right| \le c_{32}\lambda^{\varepsilon}. \tag{5.28}$$

We now give the upper bound for the last term on the r.h.s. of (5.25). We have

$$\begin{aligned}
\mathbf{E}\Big[R_n(t)\mathbb{1}_{\{Z_{n+1}\ge 2Z_n\}} \;\Big|\; \mathcal{F}_n\Big] &= \mathbf{E}\Big[R_n(t)\mathbb{1}_{\{2Z_n\leqslant Z_{n+1}\leqslant \lambda^{-\frac{1}{4}}\}} \;\Big|\; \mathcal{F}_n\Big] \\
&\quad +\mathbf{E}\Big[R_n(t)\mathbb{1}_{\{Z_{n+1}>\max(\lambda^{-\frac{1}{4}},2Z_n)\}} \;\Big|\; \mathcal{F}_n\Big].
\end{aligned}$$

On the one hand, when $Z_n \neq 0$ and $Z_{n+1} \neq 0$, we have $|V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}| \leqslant 2$ thus, for $1 \leqslant n \leqslant \sigma - 1$,

$$\begin{aligned}
&\Big|\mathbf{E}\Big[R_n(t)\mathbb{1}_{\{2Z_n \leqslant Z_{n+1} \leqslant \lambda^{-\frac{1}{4}}\}} \Bigm| \mathcal{F}_n\Big]\Big| \\
&= \Big|\mathbf{E}\left[(Z_{n+1} - Z_n)^2\Big(V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}\Big)\Big(F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(0)\Big)\mathbb{1}_{\{2Z_n<Z_{n+1}\leq\lambda^{-\frac{1}{4}}\}} \Bigm| \mathcal{F}_n\right]\Big| \\
&\leqslant 2\mathbf{E}\Big[(Z_{n+1} - Z_n)^2 \int_0^{\sqrt{\lambda}Z_{n+1}} x^{-\alpha}F_\nu(x)dx\mathbb{1}_{\{2Z_n<Z_{n+1}\leq\lambda^{-\frac{1}{4}}\}} \mid \mathcal{F}_n\Big] \\
&\leqslant c_{33}\mathbf{E}\Big[(Z_{n+1} - Z_n)^2 \int_0^{\lambda^{\frac{1}{4}}} x^{-\alpha}dx\mathbb{1}_{\{Z_{n+1}>2Z_n\}} \mid \mathcal{F}_n\Big] \\
&\leqslant c_{34}\lambda^{\frac{1-\alpha}{4}}\mathbf{E}\Big[(Z_{n+1} - Z_n)^2\mathbb{1}_{\{Z_{n+1}>2Z_n\}} \mid \mathcal{F}_n\Big] \\
&\leqslant c_{34}\lambda^{\frac{1-\alpha}{4}}\mathbf{E}\Big[(Z_{n+1} - Z_n)^4 \mid \mathcal{F}_n\Big]^{\frac{1}{2}}\mathbf{P}\Big\{Z_{n+1} > 2Z_n \mid \mathcal{F}_n\Big\}^{\frac{1}{2}} \\
&\leqslant c_{35}\lambda^{\frac{1-\alpha}{4}},
\end{aligned}$$

where we used Lemma 4.3 and Lemma 5.4 for the last inequality. On the other hand,

$$\begin{aligned}
&\mathbf{E}\Big[R_n(t)\mathbb{1}_{\{Z_{n+1}>\max(\lambda^{-\frac{1}{4}},2Z_n)\}} \Bigm| \mathcal{F}_n\Big] \\
&= \Big|\mathbf{E}\Big[(Z_{n+1} - Z_n)^2(V_{n,t}^{\alpha-1} - Z_n^{\alpha-1})\big(F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(0)\big)\mathbb{1}_{\{Z_{n+1}>\max(\lambda^{-\frac{1}{4}},2Z_n)\}} \mid \mathcal{F}_n\Big]\Big| \\
&\leq 2||F_{1-\nu}||_\infty\mathbf{E}\Big[(Z_{n+1} - Z_n)^2\mathbb{1}_{\{Z_{n+1}>\max(\lambda^{-\frac{1}{4}},2Z_n)\}} \mid \mathcal{F}_n\Big] \\
&\leq c_{36}\mathbf{E}\Big[(Z_{n+1} - Z_n)^4\mathbb{1}_{\{Z_{n+1}>2Z_n\}} \mid \mathcal{F}_n\Big]^{\frac{1}{2}}\mathbf{P}\{Z_{n+1} > \lambda^{-\frac{1}{4}} \mid \mathcal{F}_n\}^{\frac{1}{2}}. \\
&\leqslant c_{37}Z_n e^{-\frac{K_2}{4}Z_n}\mathbf{P}\{Z_{n+1} > \lambda^{-\frac{1}{4}} \mid \mathcal{F}_n\}^{\frac{1}{2}} \\
&\leqslant c_{37}Z_n e^{-\frac{K_2}{4}Z_n}\mathbf{E}[Z_{n+1} \mid \mathcal{F}_n]^{\frac{1}{2}}\lambda^{\frac{1}{8}} \\
&\leqslant c_{38}\lambda^{\frac{1}{8}}.
\end{aligned}$$

These two bounds yield

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^n Z_k}\int_0^1 (1-t)\mathbf{E}[R_n(t)\mathbb{1}_{\{\frac{1}{2}Z_n\leq Z_{n+1}>2Z_n\}} \mid \mathcal{F}_n]dt\right]\right| \leq c_{39}\lambda^\beta \tag{5.29}$$

with $\beta = \min(\frac{1-\alpha}{4}, \frac{1}{8})$. Combining (5.26), (5.28) and (5.29), we finally obtain (5.19).

Proof of (5.20): Recall that

$$J_n^3(t) \stackrel{\text{def}}{=} F_{1-\nu}(0)\mathbf{E}\left[(Z_{n+1} - Z_n)^2(V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}) \mid \mathcal{F}_n\right].$$

In particular, $J_n^3(t)$ does not depend on λ. We want to show that there exist $C \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^n Z_k}\int_0^1 (1-t)J_n^2(t)dt\right] = C + o(\lambda^\varepsilon). \tag{5.30}$$

We must first check that

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1}\int_0^1 (1-t)|J_n^2(t)|dt\right] < \infty.$$

This may be done, using the same method as before by distinguishing three cases:

$$\text{(a) } Z_{n+1} \geq \frac{1}{2}Z_n \qquad \text{(b) } 1 \leq Z_{n+1} < \frac{1}{2}Z_n \qquad \text{(c) } Z_{n+1} = 0.$$

Since the arguments are very similar to those provided above, we feel free to skip the details. We find, for $1 \leqslant n \leqslant \sigma - 1$,

$$\int_0^1 (1-t)|J_n^2(t)|dt \;\leq\; c_{40}Z_n^{\alpha-\frac{1}{2}} + c_{41} \;\leq\; c_{42}Z_n^{\frac{\alpha}{2}}.$$

Since $\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^{\frac{\alpha}{2}}\right] < \infty$, with the help of the dominated convergence theorem, we get

$$\lim_{\lambda\to 0}\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^{n} Z_k}\int_0^1 (1-t)J_n^2(t)dt\right] = \mathbf{E}\left[\sum_{n=1}^{\sigma-1}\int_0^1 (1-t)J_n^2(t)dt\right] \stackrel{\text{def}}{=} C \in \mathbb{R}.$$

Furthermore we have

$$\begin{aligned}\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda\sum_{k=0}^{n} Z_k}\int_0^1 (1-t)J_n^2(t)dt\right] - C\right| &= \left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1}(1-e^{-\lambda\sum_{k=0}^{n} Z_k})\int_0^1 (1-t)J_n^2(t)dt\right]\right| \\ &\leq c_{42}\mathbf{E}\left[(1-e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k})\sum_{n=1}^{\sigma-1} Z_n^{\frac{\alpha}{2}}\right].\end{aligned}$$

And using Hölder's inequality, we get

$$\begin{aligned}\mathbf{E}\left[(1-e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k})\sum_{n=1}^{\sigma-1} Z_n^{\frac{\alpha}{2}}\right] &\leq \mathbf{E}\left[(1-e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k})\sigma^{\frac{1}{3}}(\sum_{n=1}^{\sigma-1} Z_n^{\frac{3\alpha}{4}})^{\frac{2}{3}}\right] \\ &\leq \mathbf{E}\left[(1-e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k})^3\sigma\right]^{\frac{1}{3}}\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^{\frac{3\alpha}{4}}\right]^{\frac{2}{3}} \\ &\leq c_{43}\mathbf{E}\left[(1-e^{-\lambda\sum_{k=0}^{\sigma-1} Z_k})\sigma\right]^{\frac{1}{3}} \\ &\leqslant c_{44}\lambda^{\varepsilon}\end{aligned}$$

where we used Lemma 5.3 for the last inequality. This yields (5.20) and completes, at last, the proof of (e) of Lemma 4.4 when $\alpha \in (0,1)$.

5.2.6 Proof of Lemma 4.4 when $\alpha = 1$

The proof of the lemma when $\alpha = 1$ is quite similar to that of the case $\alpha < 1$. Giving a complete proof would be lengthy and redundant. We shall therefore provide only the arguments which differ from the case $\alpha < 1$.

For $\alpha = 1$, the main difference from the previous case comes from the fact that the function $F_{1-\nu} = F_0$ is not bounded near 0 anymore, a property that was extensively used in the course of the proof when $\alpha < 1$. To overcome this new difficulty, we introduce the function G defined by

$$G(x) \stackrel{\text{def}}{=} F_0(x) + F_1(x)\log x \qquad \text{for } x > 0. \tag{5.31}$$

Using the properties of F_0 and F_1 stated in section 5.1, we easily check that the function G satisfies

(1) $G(0) \stackrel{\text{def}}{=} \lim_{x\to 0^+} G(x) = \log(2) - \gamma$ (where γ denotes Euler's constant).

(2) There exists $c_G > 0$ such that $G(x) \leqslant c_G e^{-x}$ for all $x \geq 0$.

(3) $G'(x) = -xF_0(x)\log x$, so $G'(0) = 0$.

(4) There exists $c_{G'} > 0$ such that $|G'(x)| \leq c_{G'}\sqrt{x}e^{-x/2}$ for all $x \geq 0$.

Thus, each time we encounter $F_0(x)$ in the study of $\mu_k(n)$, we will write $G(x) - F_1(x)\log x$ instead. Let us also notice that F_1 and F_1' are also bounded on $[0, \infty)$.

We now point out, for each assertion (a) - (e) of Lemma 4.4, the modification required to handle the case $\alpha = 1$.

Assertion (a): $\mathbf{E}[\mu(0)] = C_0\lambda\log\lambda + C_0'\lambda + o(\lambda)$

As in section 5.2.1, we have

$$\begin{aligned}
\mathbf{E}[\mu(0)] &= \lambda\mathbf{E}\left[\int_0^{Z_1} xF_0(\sqrt{\lambda}x)dx\right]\\
&= \lambda\mathbf{E}\left[\int_0^{Z_1} xG(\sqrt{\lambda}x)dx\right] - \lambda\mathbf{E}\left[\int_0^{Z_1} xF_1(\sqrt{\lambda}x)\log(\sqrt{\lambda}x)dx\right]\\
&= \lambda\mathbf{E}\left[\int_0^{Z_1} x\left(G(\sqrt{\lambda}x) - F_1(\sqrt{\lambda}x)\log x\right)dx\right] - \frac{1}{2}\lambda\log\lambda\mathbf{E}\left[\int_0^{Z_1} xF_1(\sqrt{\lambda}x)dx\right]
\end{aligned}$$

and by dominated convergence,

$$\lim_{\lambda\to 0}\mathbf{E}\left[\int_0^{Z_1} x\left(G(\sqrt{\lambda}x) - F_1(\sqrt{\lambda}x)\log x\right)dx\right] = \mathbf{E}\left[\int_0^{Z_1} x\Big(G(0) - F_1(0)\log x\Big)dx\right].$$

Furthermore, using the fact that F_1' is bounded, we get

$$\mathbf{E}\left[\int_0^{Z_1} xF_1(\sqrt{\lambda}x)dx\right] = \frac{F_1(0)}{2}\mathbf{E}[Z_1^2] + \mathcal{O}(\sqrt{\lambda})$$

so that

$$\mathbf{E}[\mu(0)] = C_0\lambda\log\lambda + C_0'\lambda + o(\lambda).$$

Assertion (b): $\mathbf{E}[\sum_{n=1}^{\sigma-1}\mu_1(n)] = o(\lambda)$

This result is the same as when $\alpha < 1$, the only difference being that now

$$\mathbf{P}\{Z_\infty > x\} \underset{x\to\infty}{\sim} \frac{C\log x}{x}.$$

Thus, equality (5.6) becomes

$$\lambda^2 \mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n^2 \mathbb{1}_{\{Z_n \le \frac{-2\log\lambda}{\sqrt{\lambda}}\}}\right] \underset{\lambda\to 0^+}{\sim} c_{45}\lambda^{\frac{3}{2}}|\log\lambda|^2$$

and the same upper bound holds.

Assertion (c): $\mathbf{E}[\sum_{n=1}^{\sigma-1}\mu_2(n)] = C_2\lambda\log\lambda + C_2'\lambda + o(\lambda)$

Using the definition of G, we now have

$$\begin{aligned}\mu_2(n) &= \lambda Z_n F_0(\sqrt{\lambda}Z_n) f_1(Z_n) e^{-\lambda\sum_{k=0}^n Z_k}\\ &= \lambda Z_n f_1(Z_n) e^{-\lambda\sum_{k=0}^n Z_k}\left[\left(G(\sqrt{\lambda}Z_n) - F_1(\sqrt{\lambda}Z_n)\log(Z_n)\right) - \frac{1}{2}\log\lambda F_1(\sqrt{\lambda}Z_n)\right].\end{aligned}$$

Since $f_1(x)$ is equal to 0 for $x \ge M-1$, we get the following (finite) limit

$$\lim_{\lambda\to 0}\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n f_1(Z_n) e^{-\lambda\sum_{k=0}^n Z_k}(G(\sqrt{\lambda}Z_n) - F_1(\sqrt{\lambda}Z_n)\log(Z_n))\right] = \mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n f_1(Z_n)(G(0) - F_1(0)\log(Z_n))\right].$$

Using the same idea as in (5.8), using also Lemma 5.3 and the fact that F_1' is bounded, we deduce that

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n f_1(Z_n) e^{-\lambda\sum_{k=0}^n Z_k} F_1(\sqrt{\lambda}Z_n))\right] = \mathbf{E}\left[\sum_{n=1}^{\sigma-1} Z_n f_1(Z_n) F_1(0)\right] + o(\lambda^\varepsilon)$$

which completes the proof of the assertion.

Assertion (d): $\mathbf{E}[\sum_{n=1}^{\sigma-1}\mu_3(n)] = C_3\lambda\log\lambda + C_3'\lambda + o(\lambda)$

We do not detail the proof of this assertion since it is very similar to the proof of (c).

Assertion (e): $\mathbf{E}[\sum_{n=1}^{\sigma-1}\mu_4(n)] = C_4'\lambda + o(\lambda)$

It is worth noticing that, when $\alpha = 1$, the contribution of this term is negligible compared to (a) (c) (d) and does not affect the value of the constant in Proposition 4.1. This differs from the case $\alpha < 1$. Recall that

$$\mu_4(n) = -e^{-\lambda\sum_{k=0}^n Z_k}\mathbf{E}[\theta_n \mid \mathcal{F}_n],$$

where θ_n is given by (4.8). Recall also the notation $V_{n,t} \overset{\text{def}}{=} Z_n + t(Z_{n+1} - Z_n)$. Just as in (5.11), we write

$$\mathbf{E}[\theta_n \mid \mathcal{F}_n] = \int_0^1 (1-t)(I_n^1(t) + I_n^2(t))dt,$$

with

$$\begin{aligned} I_n^1(t) &\overset{\text{def}}{=} \lambda \mathbf{E}\left[(Z_{n+1} - Z_n)^2\Big(F_1(\sqrt{\lambda}V_{n,t}) - F_1(\sqrt{\lambda}Z_n)\Big) \mid \mathcal{F}_n\right] \\ I_n^2(t) &\overset{\text{def}}{=} -\lambda \mathbf{E}\left[(Z_{n+1} - Z_n)^2(F_0(\sqrt{\lambda}V_{n,t}) - F_0(\sqrt{\lambda}Z_n)) \mid \mathcal{F}_n\right], \end{aligned}$$

It is clear that inequality (5.13) still holds *i.e.*

$$|I_n^1(t)| \leq \lambda^{\frac{3}{2}} \mathbf{E}\left[|Z_{n+1} - Z_n|^3 \max_{x \in [Z_n, Z_{n+1}]} \sqrt{\lambda} x F_0(\sqrt{\lambda}x) \mid \mathcal{F}_n\right].$$

In view of the relation

$$F_0(\sqrt{\lambda}x) = G(\sqrt{\lambda}x) - F_1(\sqrt{\lambda}x)\log x - \frac{1}{2}F_1(\sqrt{\lambda}x)\log\lambda,$$

and with similar techniques to those used in the case $\alpha < 1$, we can prove that

$$\left|\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^1(t)dt\right]\right| \leq c_{46}\lambda^{\frac{9}{8}}|\log\lambda| = o(\lambda). \tag{5.32}$$

It remains to estimate $I_n^2(t)$ which we now decompose into four terms:

$$I_n^2(t) = -\lambda(\tilde{J}_n^1(t) + \tilde{J}_n^2(t) + \tilde{J}_n^3(t) + \tilde{J}_n^4(t)),$$

with

$$\begin{aligned} \tilde{J}_n^1(t) &\overset{\text{def}}{=} \mathbf{E}\left[(Z_{n+1} - Z_n)^2(G(\sqrt{\lambda}V_{n,t}) - G(\sqrt{\lambda}Z_n)) \mid \mathcal{F}_n\right] \\ \tilde{J}_n^2(t) &\overset{\text{def}}{=} -\frac{1}{2}\log\lambda \mathbf{E}\left[(Z_{n+1} - Z_n)^2(F_1(\sqrt{\lambda}V_{n,t}) - F_1(\sqrt{\lambda}Z_n)) \mid \mathcal{F}_n\right] \\ \tilde{J}_n^3(t) &\overset{\text{def}}{=} -\mathbf{E}\left[(Z_{n+1} - Z_n)^2 \log Z_n(F_1(\sqrt{\lambda}V_{n,t}) - F_1(\sqrt{\lambda}Z_n)) \mid \mathcal{F}_n\right] \\ \tilde{J}_n^4(t) &\overset{\text{def}}{=} -\mathbf{E}\left[(Z_{n+1} - Z_n)^2(\log V_{n,t} - \log(Z_n))F_1(\sqrt{\lambda}V_{n,t}) \mid \mathcal{F}_n\right]. \end{aligned}$$

We can obtain an upper bound of order λ^ε for $\tilde{J}_n^1(t)$ by considering again three cases:

$$(1)\ \frac{1}{2}Z_n < Z_{n+1} < 2Z_n \qquad (2)\ Z_{n+1} \leq \frac{1}{2}Z_n \qquad (3)\ Z_{n+1} \geq 2Z_n.$$

For (1), we use that $|G'(x)| \leqslant c_{G'}\sqrt{x}e^{-x/2}$ for all $x \geq 0$. We deal with (2) combining Lemma 5.4 and the fact that G' is bounded. Finally, the case (c) may be treated by similar methods as those used for dealing with $J_n^2(t)$ in he proof of (e) when $\alpha < 1$ (*i.e.* we separate into two terms according to whether $Z_{n+1} \leq \lambda^{-1/4}$ or not).

Keeping in mind that F_1 is bounded and that $|F_1'(x)| = xF_0(x) \leq c_{47}\sqrt{x}e^{-x}$, the same method enables us to deal with $\tilde{J}_n^2(t)$ and $\tilde{J}_n^3(t)$. Combining these estimates, we get

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^{n} Z_k} \int_0^1 (1-t)\left(\tilde{J}_n^1(t) + \tilde{J}_n^2(t) + \tilde{J}_n^3(t)\right) dt\right] = o(\lambda^{\varepsilon}).$$

for $\varepsilon > 0$ small enough. Therefore, it simply remains to prove that

$$\lim_{\lambda \to 0^+} \mathbf{E}\left[\sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^{n} Z_k} \int_0^1 (1-t)\tilde{J}_n^4(t)dt\right] \tag{5.33}$$

exists and is finite. In view of the dominated convergence theorem, it suffice to prove that

$$\mathbf{E}\left[\sum_{n=1}^{\sigma-1} \int_0^1 (1-t)\mathbf{E}\Big[(Z_{n+1}-Z_n)^2 |\log V_{n,t} - \log(Z_n)| \;\Big|\; \mathcal{F}_n\Big] dt\right] < \infty. \tag{5.34}$$

We consider separately the cases $Z_{n+1} > Z_n$ and $Z_{n+1} \leq Z_n$. On the one hand, using the inequality $\log(1+x) \leq x$, we get

$$\begin{aligned} &\mathbf{E}\Big[\mathbb{1}_{\{Z_{n+1}>Z_n\}}(Z_{n+1}-Z_n)^2 |\log V_{n,t} - \log(Z_n)| \;\Big|\; \mathcal{F}_n\Big] \\ &\qquad\leq \mathbf{E}\Big[\mathbb{1}_{\{Z_{n+1}>Z_n\}}(Z_{n+1}-Z_n)^2 \log\Big(1 + \frac{t(Z_{n+1}-Z_n)}{Z_n}\Big) \;\Big|\; \mathcal{F}_n\Big] \leq t\sqrt{Z_n}. \end{aligned}$$

On the other hand, we find

$$\begin{aligned} &\mathbf{E}\Big[\mathbb{1}_{\{Z_{n+1}\leq Z_n\}}(Z_{n+1}-Z_n)^2 |\log V_{n,t} - \log(Z_n)| \;\Big|\; \mathcal{F}_n\Big] \\ &\qquad\leq \mathbf{E}\Big[\mathbb{1}_{\{Z_{n+1}\leq Z_n\}}(Z_{n+1}-Z_n)^2 \log\Big(1 + \frac{t(Z_n-Z_{n+1})}{Z_n - t(Z_n-Z_{n+1})}\Big) \;\Big|\; \mathcal{F}_n\Big] \leq \frac{t}{1-t}\sqrt{Z_n}. \end{aligned}$$

Since $\mathbf{E}[\sum_{n=1}^{\sigma-1}\sqrt{Z_n}]$ is finite, we deduce (5.34) and the proof of assertion (e) is complete.

6 Proof of Theorem 1.1

Recall that X stands for the $(M,\bar{p})$-cookie random walk and Z stands for its associated branching process. We define the sequence of return times $(\sigma_n)_{n\geqslant 0}$ by

$$\begin{cases} \sigma_0 \stackrel{\text{def}}{=} 0, \\ \sigma_{n+1} \stackrel{\text{def}}{=} \inf\{k > \sigma_n \,,\, Z_k = 0\}. \end{cases}$$

In particular, $\sigma_1 = \sigma$ with the notation of the previous sections. We write

$$\sum_{k=0}^{\sigma_n} Z_k = \sum_{k=\sigma_0}^{\sigma_1 - 1} Z_k + \ldots + \sum_{k=\sigma_{n-1}}^{\sigma_n - 1} Z_k.$$

The random variables $(\sum_{k=\sigma_i}^{\sigma_{i+1}-1} Z_k\,,\, i \in \mathbb{N})$ are i.i.d. In view of Proposition 4.1, the characterization of the domains of attraction to a stable law implies

$$\begin{cases} \frac{\sum_{k=0}^{\sigma_n} Z_k}{n^{1/\nu}} \xrightarrow[n\to\infty]{\text{law}} \mathcal{S}_\nu & \text{when } \alpha \in (0,1), \\ \frac{\sum_{k=0}^{\sigma_n} Z_k}{n \log n} \xrightarrow[n\to\infty]{\text{prob}} c & \text{when } \alpha = 1. \end{cases} \tag{6.1}$$

where $\mathcal{S}_\nu$ denotes a positive, strictly stable law with index $\nu \overset{\text{def}}{=} \frac{\alpha+1}{2}$ and where c is a strictly positive constant. Moreover, the random variables $(\sigma_{n+1} - \sigma_n\,,\, n \in \mathbb{N})$ are i.i.d. with finite expectation $\mathbf{E}[\sigma]$, thus

$$\frac{\sigma_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} \mathbf{E}[\sigma]. \tag{6.2}$$

The combination of (6.1) and (6.2) easily gives

$$\begin{cases} \frac{\sum_{k=0}^{n} Z_k}{n^{1/\nu}} \xrightarrow[n\to\infty]{\text{law}} \mathbf{E}[\sigma]^{-\frac{1}{\nu}}\mathcal{S}_\nu & \text{when } \alpha \in (0,1), \\ \frac{\sum_{k=0}^{n} Z_k}{n \log n} \xrightarrow[n\to\infty]{\text{prob}} c\mathbf{E}[\sigma]^{-1} & \text{when } \alpha = 1. \end{cases}$$

Concerning the hitting times of the cookie random walk $T_n = \inf\{k \geqslant 0\,,\, X_k = n\}$, making use of Proposition 2.1, we now deduce that

$$\begin{cases} \frac{T_n}{n^{1/\nu}} \xrightarrow[n\to\infty]{\text{law}} 2\mathbf{E}[\sigma]^{-\frac{1}{\nu}}\mathcal{S}_\nu & \text{when } \alpha \in (0,1), \\ \frac{T_n}{n \log n} \xrightarrow[n\to\infty]{\text{prob}} 2c\mathbf{E}[\sigma]^{-1} & \text{when } \alpha = 1. \end{cases}$$

Since T_n is the inverse of $\sup_{k \leqslant n} X_k$, we conclude that

$$\begin{cases} \frac{1}{n^\nu} \sup_{k \leqslant n} X_k \xrightarrow[n\to\infty]{\text{law}} \mathcal{L}_\nu & \text{when } \alpha \in (0,1), \\ \frac{\log n}{n} \sup_{k \leqslant n} X_k \xrightarrow[n\to\infty]{\text{prob}} C & \text{when } \alpha = 1, \end{cases}$$

where $C \overset{\text{def}}{=} (2c)^{-1}\mathbf{E}[\sigma] > 0$ and $\mathcal{L}_\nu \overset{\text{def}}{=} 2^{-\nu}\mathbf{E}[\sigma]\mathcal{S}_\nu^{-\nu}$ is a Mittag-Leffler random variable with index ν. This completes the proof of the theorem for $\sup_{k \leqslant n} X_k$. It remains to prove that this result also holds for X_n and for $\inf_{k \geq n} X_k$. We need the following lemma.

Lemma 6.1

Let X be a transient cookie random walk. There exists a function $f : \mathbb{N} \mapsto \mathbb{R}_+$ with $\lim_{K\to+\infty} f(K) = 0$ such that, for every $n \in \mathbb{N}$,

$$\mathbf{P}\{n - \inf_{i \geq T_n} X_i > K\} \leq f(K).$$

Proof. The proof of this lemma is very similar to that of Lemma 4.1 of [BS06]. For $n \in \mathbb{N}$, let $\omega_{X,n} = (\omega_{X,n}(i,x))_{i \geqslant 1, x \in \mathbb{Z}}$ denote the random cookie environment at time T_n "viewed from the particle", *i.e.* the environment obtained at time T_n and shifted by n. With this notation, $\omega_{X,n}(i,x)$ denotes the strength of the i^{th} cookies at site x:

$$\omega_{X,n}(i,x) = \begin{cases} p_j & \text{if } j = i + \sharp\{0 \leq k < T_n, X_k = x+n\} \leqslant M, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Since the cookie random walk X has not visited the half line $[n,\infty)$ before time T_n, the cookie environment $\omega_{X,n}$ on $[0,\infty)$ is the same as the initial cookie environment, that is, for $x \geqslant 0$,

$$\omega_{X,n}(i,x) = \begin{cases} p_i & \text{if } 1 \leqslant i \leqslant M, \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{6.3}$$

Given a cookie environment ω, we denote by $\mathbf{P}_\omega$ a probability under which X is a cookie random walk starting from 0 in the cookie environment ω. Therefore, with these notations,

$$\mathbf{P}\{n - \inf_{i\geq T_n} X_i > K\} \leq \mathbf{E}\left[\mathbf{P}_{\omega_{X,n}}\{X \text{ visits } -K \text{ at least once}\}\right]. \tag{6.4}$$

Consider now the deterministic (but non-homogeneous) cookie environment $\omega_{\bar{p},+}$ obtained from the classical homogeneous $(M,\bar{p})$ environment by removing all the cookies situated on $(-\infty,-1]$:

$$\begin{cases} \omega_{\bar{p},+}(i,x) = \frac{1}{2}, & \text{for all } x<0 \text{ and } i\geq 1, \\ \omega_{\bar{p},+}(i,x) = p_i, & \text{for all } x\geq 0 \text{ and } i \geq 1 \text{ (with the convention } p_i = \frac{1}{2} \text{ for } i \geqslant M). \end{cases}$$

According to (6.3), the random cookie environment $\omega_{X,n}$ is almost surely larger than the environment $\omega_{\bar{p},+}$ for the canonical partial order, *i.e.*

$$\omega_{X,n}(i,x) \geq \omega_{\bar{p},+}(i,x) \quad \text{for all } i \geqslant 1,\ x \in \mathbb{Z}, \text{ almost surely.}$$

The monotonicity result of Zerner stated in Lemma 15 of [Zer05] yields

$$\mathbf{P}_{\omega_{X,n}}\{X \text{ visits } -K \text{ at least once}\} \leqslant \mathbf{P}_{\omega_{\bar{p},+}}\{X \text{ visits } -K \text{ at least once}\} \quad \text{almost surely.}$$

Combining this with (6.4), we get

$$\mathbf{P}\{n - \inf_{i\geq T_n} X_i > K\} \leq \mathbf{P}_{\omega_{\bar{p},+}}\{X \text{ visits } -K \text{ at least once}\}. \tag{6.5}$$

This upper bound does not depend on n. Moreover, it is shown in the proof of Lemma 4.1 of [BS06] that the walk in the cookie environment $\omega_{\bar{p},+}$ is transient which implies, in particular,

$$\mathbf{P}_{\omega_{\bar{p},+}}\{X \text{ visits } -K \text{ at least once}\} \underset{K\to\infty}{\longrightarrow} 0.$$

■

We now complete the proof of Theorem 1.1. Let $n,r,p \in \mathbb{N}$, using that $\{T_{r+p} \leqslant n\} = \{\sup_{k\leq n} X_k \geqslant r+p\}$, we get

$$\{\sup_{k\leq n} X_k < r\} \subset \{\inf_{k\geq n} X_k < r\} \subset \{\sup_{k\leq n} X_k < r+p\} \cup \{\inf_{k\geq T_{r+p}} X_k < r\}.$$

Taking the probability of these sets, we obtain

$$\mathbf{P}\{\sup_{k\leq n} X_k < r\} \leq \mathbf{P}\{\inf_{k\geq n} X_k < r\} \leq \mathbf{P}\{\sup_{k\leq n} X_k < r+p\} + \mathbf{P}\{\inf_{k\geq T_{r+p}} X_k < r\}.$$

But, using Lemma 6.1, we have

$$\mathbf{P}\{\inf_{k\geq T_{r+p}} X_k < r\} = \mathbf{P}\{r + p - \inf_{k\geq T_{r+p}} X_k > p\} \leq f(p) \underset{p\to\infty}{\longrightarrow} 0.$$

Choosing $x \geqslant 0$ and $r = \lfloor xn^\nu \rfloor$ and $p = \lfloor \log n \rfloor$, we get, for $\alpha < 1$, as n tends to infinity

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\inf_{k\geq n} X_k}{n^\nu} < x\right\} = \lim_{n\to\infty} \mathbf{P}\left\{\frac{\sup_{k\leq n} X_i}{n^\nu} < x\right\} = \mathbf{P}\left\{\mathcal{L}_\nu < x\right\}.$$

Of course, the same method also works when $\alpha = 1$. This proves Theorem 1.1 for $\inf_{k\geq n} X_k$. Finally, the result for X_n follows from

$$\inf_{k\geq n} X_k \leq X_n \leq \sup_{k\leq n} X_k.$$

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