# Deformations of functions and F-manifolds

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# Declaration

I declare that, to the best of my knowledge, this thesis is my own work, except where explicitly stated otherwise.
I confirm that this thesis has not been submitted for a degree at any other university.

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# Abstract


In this thesis we study deformations of functions on singular varieties with a view toward Frobenius manifolds.

Chapter 2 is mainly introductory. We prove standard results in deformation theory for which we do not know a suitable reference. We also give a construction of the miniversal deformation of a function on a singular space that to the best of our knowledge does not appear in this form in literature.

In Chapter 3 we find a sufficient condition for the dimension of the base space of the miniversal deformation to be equal to the number of critical points into which the original singularity splits. We show that it holds for functions on smoothable and unobstructed curves and for function on isolated complete intersections singularities, unifying under the same argument previously known results.

In Chapter 4 we use the previous results to construct a multiplicative structure known as $F$-manifold on the base space of the miniversal deformation. We relate our construction to the theory of Frobenius manifolds by means of an example.

The appendix is joint work with D. Mond.

# Contents

# Chapter 1

# Introduction

This thesis is devoted to the study of germs of functions on singular varieties. The main aim of this work is to produce a natural singularity theory framework in which to find Frobenius manifolds.

**Frobenius manifolds.** Frobenius manifolds were defined by B. Dubrovin around 1990 in his work on Topological Field Theories (amply summarised in [11]). They are complex manifolds with at least[1] three extra data in the tangent bundle:

1. Each tangent space $T_pM$ is a commutative and associative algebra with unity. These pointwise multiplications glue together to yield a holomorphic tensor $\star\colon \Theta_M \times \Theta_M \to \Theta_M$ on the sheaf of sections on the tangent bundle. The local unities are also required to glue together to yield a global vector field.
2. A flat metric compatible with the multiplication, more precisely, a non-degenerate symmetric tensor $<,>\colon \Theta_M \times \Theta_M \to \mathcal{O}_M$ with zero curvature[2] such that
$$< u \star v, w >=< u, v \star w >$$
3. At each point $p$, a holomorphic function germ $\Phi_p$ whose third derivatives in flat coordinates define the multiplication. More precisely, for any three flat vector fields $u, v$ and $w$
$$(uvw)(\Phi_p) =< u \star v, w >$$
The germ $\Phi_p$ is usually referred to as the (local) potential of the Frobenius structure. It is unique up to addition of quadratic polynomials.

The potential is undoubtedly the most mysterious of all three elements making up the definition. It was however one of the early objects of research that led to the concept of Frobenius manifold. Suppose for a moment that, at a point

[1] The original definition required that the global unit should be flat and the existence of a distinguished vector field $E$ that rescales both the multiplication and the metric, in the sense that $\mathrm{Lie}_E(\star) = d \cdot \star$ and $\mathrm{Lie}_E(<,>) = D\cdot <,>$ for constants $d, D \in \mathbb{C}$.

[2] Or equivalently, a subsheaf $\Theta_M^{\text{flat}} \subset \Theta_M$ of linear subspaces defining an integrable distribution such that $\Theta_M^{\text{flat}} \otimes \mathcal{O}_M = \Theta_M$

$p \in M$, we are given a germ $\Phi$ and flat coordinates $(x_1, \ldots, x_m)$. As $<,>$ is non-degenerate, we can define the product $\star$ by the formula

$$\frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_k} = < \frac{\partial}{\partial x_i} \star \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} >, \ 1 \leq i, j \leq m.$$

Whereas the commutativity of the product so defined can be read off as the equality of the mixed partial derivatives, the associativity becomes a non-linear system of partial differential equations:

$$\frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_k} g^{kl} \frac{\partial^3 \Phi}{\partial x_l \partial x_m \partial x_n} = \frac{\partial^3 \Phi}{\partial x_j \partial x_m \partial x_k} g^{kl} \frac{\partial^3 \Phi}{\partial x_l \partial x_i \partial x_n}$$

These equations had been introduced previously by R. Dijkgraaf, H. Verlinde, E. Verlinde ([8]) and, independently, E. Witten ([40]) in the context of Topological Field Theories. The definition of Frobenius manifolds is therefore a coordinate-free description of the above equations.

Although originating from in Mathematical Physics, Frobenius manifolds now occur in different and seemingly unrelated branches of mathematics: most remarkably in Quantum Cohomology and Singularity Theory, but also in Integrable Systems, Symplectic Geometry and others. In Quantum Cohomology the flat structure is rather trivial: the underlying manifold $M$ is simply the total cohomology ring $H^*(X, \mathbb{C})$ as a complex vector space and the bilinear pairing is given by Poincaré duality. On the other hand the construction of the potential, and hence the product, requires a formidable amount of machinery in the form of moduli spaces of curves and Gromov-Witten invariants. In Singularity Theory the situation is exactly the opposite. Whereas the multiplication is defined straightforwardly, the existence of a metric with the required properties requires sophisticated techniques of algebraic analysis and variations of mixed Hodge structure. For function-germs on smooth spaces, it was first conjectured, and proved in some cases, by K. Saito ([32]) and by M. Saito [33] in general.

**F-manifolds.** As indicated above, the multiplication in the case of the miniversal unfolding of an isolated singularity is obtained in a rather straightforward manner. It is natural to study it on its own without any reference to the metric. This study appears for first time in the joint work of C. Hertling and Y. Manin ([18]) and it is further pursued by the former in the first part of his book [19]. They define the notion of $F$-manifold as a manifold $M$ with a multiplication $\star$ in the tangent bundle satisfying the following condition for any two vector fields $v, w \in \Theta_M$:

$$\mathrm{Lie}_{u \star v}(\star) = u \star \mathrm{Lie}_v(\star) + \mathrm{Lie}_u(\star) \star v$$

To explain the meaning of this condition, we need to explain in more detail how the multiplication is defined in the case of unfolding[3] of isolated singularities.

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ with an isolated singularity. If $F\colon (\mathbb{C}^{n+1} \times (\mathbb{C}^\mu, 0) \to (\mathbb{C}, 0)$ is a miniversal unfolding of $f$, the initial velocities of $F$ respect to the parameters of the deformation form a basis over $\mathbb{C}$ of the Jacobian algebra

[3] Unfolding and miniversal unfoldings always refers to respect to the right equivalence of germs.

of $f$, i.e.,

$$\mathbb{C} < \frac{\partial F}{\partial t_1}(x,0), \ldots, \frac{\partial F}{\partial t_\mu}(x,0) >= \mathcal{O}_{\mathbb{C},0} / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}} \right)$$

The algebra structure on the right hand side is pulled back to define the multiplication at least on $T_0\mathbb{C}^\mu$. If we choose appropriate representatives of all the germs involve, say $F\colon \mathfrak{X} \to S$ for $F$ and $g\colon \mathfrak{X} \to B$ for the projection on the parameter space, we can sheafify the above isomorphism

$$\Theta_B \ni \frac{\partial}{\partial t_i} \mapsto \frac{\partial F}{\partial t_i} \in g_* \mathcal{O}_{\Sigma_F^r} \tag{1.1}$$

where $\Sigma_F^r$ denotes the relative critical locus of $F$. As the restriction $g\colon \Sigma_F^r \to B$ is finite, the variety $\Sigma_F^r$ is a complete intersection and hence $g_*\mathcal{O}_{\Sigma_F^r}$ is a free $\mathcal{O}_B$-module of rank $\mu$. The morphism 1.1 is then an isomorphism of free sheaves and defines a multiplication in $\Theta_M$. The key point is to note that if $d$ denotes the differential respect to the parameter $t$, the map $\Sigma_F^r \to T^*M$ defined as

$$\Sigma_F^r \ni (x,t) \mapsto (t, d_{(x,t)}F) \in T_t^*M$$

embeds $\Sigma_F^r$ as a Lagrangian subvariety $L$. Let $\pi\colon T^*M \to M$ the canonical projection. It is easy to see that the composition

$$\Theta_M \xrightarrow{\rho_F} g_*\mathcal{O}_{\Sigma_F^r} \xrightarrow{\simeq} \pi_*\mathcal{O}_L$$

is just induced by the evaluation map $\Theta_M \to \mathcal{O}_{T^*M}$. Moreover, $F$ thought of as a function on $L$ satisfies

$$dF = \alpha|_L$$

where $\alpha$ is the canonical 1-form on the cotangent bundle. At a point $t$ such that $F(-,t)$ has only Morse singularities at points, the Lagrangian property of $L$ is equivalent to the critical values of $F$ defining a coordinate system in a neighbourhood of $t$.

If $M$ is now a manifold with a multiplication on the tangent bundle, the variety $L$ can be defined as the subset of $T^*M$ consisting of 1-forms that are $\mathbb{C}$-algebra homomorphisms from $T_pM$ onto $\mathbb{C}$. If near a point $p \in M$, there exist a frame of vector fields $e_1, \ldots, e_m$ such that

$$e_i \star e_j = \delta_{ij} e_i$$

then the following statements are equivalent:

1. $L$ is Lagrangian
2. there exist a coordinate system $(u_1, \ldots, u_m)$ such that $\frac{\partial}{\partial u_i} = e_i$ for all $i = 1, \ldots, m$
3. for any $u, v \in \Theta_M$, $\mathrm{Lie}_{u\star v}(\star) = u \star \mathrm{Lie}_v(\star) + \mathrm{Lie}_u(\star) \star v$

The last condition is then taken as the definition of $F$-manifold even if the frame of idempotent vector fields $e_1, \ldots, e_m$ does not exist. In this case, it still ensures the integrability of certain multiplicatively closed distributions in the tangent bundle.

**Description of results.** We now describe briefly the main results in this thesis. We wish to generalise the above construction to the case of germs of functions defined on singular varieties. We consider a function germ $f\colon (X,0) \to (\mathbb{C},0)$ with an isolated singularity, i.e., away from 0, $X$ is smooth and $f$ is a submersive. An unfolding of $f$ in this situation is an extension of $f$ to the total space of a (flat) deformation of $(X,0)$, that is, a commutative diagram

$$\begin{array}{ccccc} (X,0) & \overset{i}{\hookrightarrow} & & & (\mathfrak{X},0) \\ \downarrow & \searrow^{f} & & \swarrow^{F} & \downarrow {\scriptstyle g \text{ flat}} \\ & & (\mathbb{C},0) & & \\ \{0\} & \hookrightarrow & & & (B,0) \end{array}$$

The conditions on $f$ and $(X,0)$ ensure the finite dimensionality of the space of first order infinitesimal deformations $T^1_{X/\mathbb{C}}$ and hence the existence of versal deformations. If $\varphi = (F,g)\colon (\mathfrak{X},0) \to (\mathbb{C}\times B,0)$ denotes the miniversal deformation with base space $B$, the corresponding Kodaira-Spencer map

$$\rho(0)_\varphi\colon T_0B \longrightarrow T^1_{X/\mathbb{C}}$$

though an isomorphism, cannot be used to induce a multiplication on $T_0B$ for $T^1_{X/\mathbb{C}}$ is not an algebra (unless of course $(X,0)$ was smooth). Instead we consider the module $\mathcal{L}_{\mathfrak{X}/B,0}$ of liftable vector fields

$$\mathcal{L}_{\mathfrak{X}/B,0} = \{u \in \Theta_{B,0}\colon \exists\ \tilde{u} \in \Theta_{\mathfrak{X},0} \text{ such that } tg(\tilde{u}) = u \circ g\}$$

For a liftable vector field $u$, we can differentiate $F$ with respect to a lift $\tilde{u}$. As two different lifts differ by a "vertical" vector field (denoted by $\Theta_{\mathfrak{X}/B,0}$), we have a map:

$$t'F\colon \mathcal{L}_{\mathfrak{X}/B,0} \longrightarrow \frac{F^*\Theta_{\mathbb{C},0}}{tF(\Theta_{\mathfrak{X}/B,0})} :\ = M_{\varphi,0}$$

In chapter 2 we characterise versal deformations of $f$ as those whose fibration $g$ is versal as a deformation for $(X,0)$ and $t'F$ is surjective. We then prove standard results of deformation theory for which although they are in some sense standard, we do not know a suitable reference.

For a miniversal deformation, the map $t'F$ induces the algebra structure of $M_{\varphi,0}$ on the module $\mathcal{L}_{\mathfrak{X}/B,0}$. If the generic fibre of $g$ was smooth, at points $p \in B$ outside the discriminant $\Delta_g$ and the singular locus of $B$, we would have $\mathcal{L}_{\mathfrak{X}/B,p} = \Theta_{B,p}$, and the sheafified $t'F$ would have the chance to define a multiplication at least on $\Theta_{B-\Delta_g}$. In chapter 3 we isolate a sufficient condition for the isomorphism $t'F$ to extend to an isomorphism of sheaves in the case where $B$ is smooth: if the generic fibre of $g$ is smooth, and any vector field tangent to the fibres of $f$ can be extended to a vector field tangent simultaneously to the fibres of $g$ and $F$, then $t'F$ extends to an isomorphism of free sheaves

$$t'F\colon \mathcal{L}_{\mathfrak{X}/B} \longrightarrow g_*M_\varphi$$

As a consequence we see that the number of critical points of $F$ on the smooth fibre, say $\mu$, and the dimension of the base space of the miniversal deformation,

$\tau$, are equal. We also see that, if $\mathcal{L}_{\mathfrak{X}/B}$ coincides with the sheaf $\Theta(\log \Delta_g)$ of vector fields tangent to the regular part of the discriminant, then $\Delta_g$ is a free divisor.

The above extendibility condition is trivially satisfied if $(X, 0)$ is a curve (the fibres of $f$ are just points). Note that we need both the base space $B$ and the generic fibre to be smooth. This happens, for example, if $(X, 0)$ is a curve in $(\mathbb{C}^3, 0)$ or a Gorenstein curve in $(\mathbb{C}^4, 0)$. The equality of $\mu$ and $\tau$ was first proved by D. Mond and D. van Straten in [27] and the freeness of the discriminant by the latter in [38]. We also show that the extendibility condition holds true for functions on isolated complete intersection singularities. In this case, the equality $\mu = \tau$ is due to V. Goryunov [15] and the freeness of $\Delta_g$ to K. Saito in the hypersurface case and E.J.N. Looijenga [24] in general. In both the case of functions on smoothable curves and functions on isolated complete intersections we show that

$$\mu = \dim_{\mathbb{C}} \frac{\omega_{X,0}}{df \wedge \Omega^{n-1}_{X,0}}$$

where $\omega_{X,0}$ denotes the dualising module. In the case of isolated complete intersection, we use a result of D.T. Lê to express $\mu$ as certain vanishing homology: if $X_b$ is a Milnor fibre of $g$ and $Y_s$ that of $f$, then

$$\mu = \operatorname{rk} H_{n-1}(X_b, Y_s)$$

In chapter 4 we show that the multiplication so defined in $\mathcal{L}_{\mathfrak{X}/B}$ satisfies the integrability condition and hence at least in $B - \Delta_g$ we obtain the structure of $F$-manifold. In fact we show something else: if the generic singularity over the discriminant is quadratic (something that holds both in the case of space curves and complete intersections), then each stratum of the logarithmic stratification of $B$ associated to $\Theta(\log \Delta_g)$ is an $F$-manifold. It is then natural to ask if it is possible to define metrics that induce the structure of Frobenius manifolds in the different strata. Although in general the algebras $g_* M_\varphi$ are not sum of Gorenstein rings (a necessary condition for the existence of a non-degenerate and multiplicatively invariant bilinear pairing), there is a case in which this is easily seen to hold: if $(X, 0)$ is a complete intersection curve, the module $\omega_{X,0}/\mathcal{O}_{X,0} df$ supports such a bilinear form and so does its parametrised version. A choice of generator for $\omega_{\mathfrak{X}/B}$ induces therefore the desired pairing on $\Theta(\log \Delta_g)$ and we conjectured the existence of a relative dualising form for which the induced metric is flat. The evidence for this conjecture is compelling: the miniversal base spaces of deformations of curves can be interpreted as partial closures of Hurwitz spaces, and B. Dubrovin [11] show the existence of Frobenius structures on those spaces. We prove it for the simplest case of all, namely, functions on the double point $xy = 0$. This fact is well-known to specialists although perhaps not stated in this context: for example, A. Douai and C. Sabbah show Frobenius structures in the base space of the miniversal deformation of a Laurent polynomial in several variables (see [10]).

# Chapter 2

# Deformations of functions

In this first chapter we begin by introducing the standard definitions and notations of deformations of functions on analytic spaces. We then review the properties of the main tool to study deformations, the cotangent cohomology, and use it to provide a method to construct the miniversal deformation of a function with an isolated singularity. We end the chapter with some remarks on finite determinacy of functions with isolated singularities on non-smooth varieties.

Although most of the results included in this chapter are well known to specialists, there are some that, to the best of my knowledge, do not appear in this form in literature. These include the criterion for versality given in corollary 2.4.1 and the consequent construction of a versal (or miniversal) deformation of a function on a singular space explained in section 2.5.

We work on the analytic category all throughout this thesis.

## 2.1 Definitions and notations

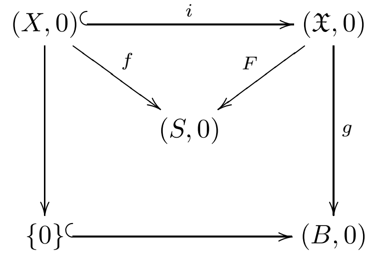
Let $(X,0)$ be a reduced analytic space germ and let $\mathcal{O}_{X,0}$ denote the local ring of analytic functions on $(X,0)$. A *deformation of* $(X,0)$ *over* $(B,0)$ is simply the realisation of $(X,0)$ as the fibre of a flat analytic map germ $g\colon (\mathfrak{X},0) \to (B,0)$. That is, a commutative diagram of analytic maps

$$\begin{array}{ccc} (X,0) & \stackrel{i}{\hookrightarrow} & (\mathfrak{X},0) \\ \downarrow & & \downarrow g \\ \{0\} & \hookrightarrow & (B,0) \end{array}$$

where $i$ is an inclusion into the distinguished fibre of $g$ and $\mathcal{O}_{\mathfrak{X},0}$ is a flat $\mathcal{O}_{B,0}$-module. Note that we always denote by 0 the base point of any analytic space germ. This should not lead to confusion.

If $f \in \mathcal{O}_{X,0}$ is an analytic function, a *deformation of* $f$ is a deformation of $(X,0)$, say $g\colon (\mathfrak{X},0) \to (B,0)$, together with an extension $F \in \mathcal{O}_{\mathfrak{X},0}$ of $f$.

As $(X,0)$ is assumed to be reduced, we can think of $f$ as a map from $(X,0)$ to $(\mathbb{C}, f(0))$. Writing $(S,0)$ for the germ of the complex plane $(\mathbb{C}, f(0))$, a deformation of $f$ is then a commutative diagram

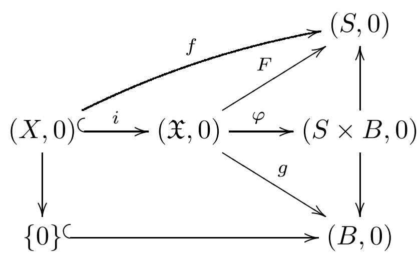



where $i\colon (X,0) \hookrightarrow (\mathfrak{X},0)$ is an inclusion and $g$ is a flat analytic map. The space $(\mathfrak{X},0)$, resp. $(B,0)$, is called the *total space*, resp. the *base space*, of the deformation.

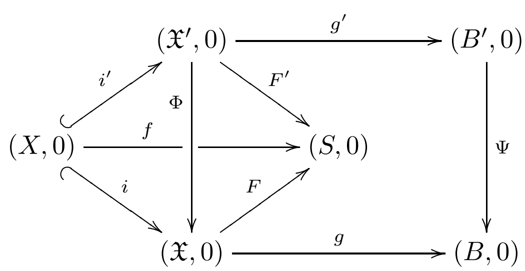
Given a deformation of $f$, we can consider the map

$$\varphi = (F,g)\colon (\mathfrak{X},0) \to (S \times B, 0)$$

and recover $F$ and $g$ by means of the projections from $(S \times B, 0)$ onto $(S,0)$ and $(B,0)$ respectively. For this reason, we will call the pair $(i,\varphi)$ a deformation of $f$. We will denote it simply by $\varphi$ if the inclusion $i$ is unambiguous. The following diagram helps visualise the relations between the different maps.




If $(i',\varphi')$ is another deformation with total space $(\mathfrak{X}',0)$ and base space $(B',0)$, it is said to be *induced from* $\varphi$ *by* $\Psi\colon (B',0) \to (B,0)$ if there exists a map $\Phi\colon (\mathfrak{X}',0) \to (\mathfrak{X},0)$ such that the following diagram is commutative:




where $\varphi' = (g',F')$. In terms of $\varphi$ and $\varphi'$, the above diagram can be expressed as:

$$\Phi \circ i' = i$$
$$\varphi \circ \Phi = (\Psi \times \mathbb{I}_{S,0}) \times \varphi'$$

If $\Psi$ (and hence $\Phi$) is an isomorphism the deformations are said *isomorphic*. A deformation is called *versal* if any other deformation can be induced from it.

**Example 2.1.1.** Let $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a holomorphic function. As $(\mathbb{C}^{n+1},0)$ is smooth, any deformation of $f$ is isomorphic to a deformation of the form:

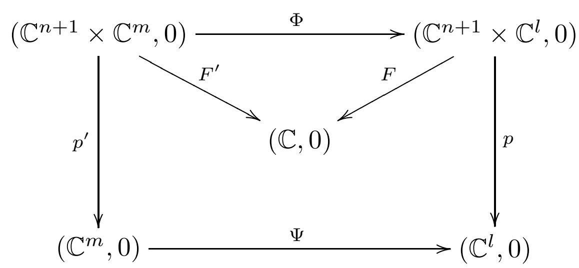
$$\varphi = F \times \mathbb{I}_{\mathbb{C}^l} : (\mathbb{C}^{n+1} \times \mathbb{C}^l, 0) \to (\mathbb{C} \times \mathbb{C}^l, 0)$$

where $F(x,t) = f(x) + g(x,t)$ for some $g \in \mathcal{O}_{\mathbb{C}^{n+1+l},0}$ with $g(x,0) = 0$.

Another deformation:

$$\varphi' = F' \times \mathbb{I}_{\mathbb{C}^m} : (\mathbb{C}^{n+1} \times \mathbb{C}^m, 0) \to (\mathbb{C} \times \mathbb{C}^m, 0)$$

is induced from $\varphi$ if there exist maps $\Phi$ and $\Psi$ making the following diagram commutative:




and the restriction $\Phi|_{(\mathbb{C}^{n+1}\times\{0\},0)}$ is an analytic diffeomorphism of $(\mathbb{C}^{n+1},0)$. Hence we recover the notion of unfolding and equivalence of unfoldings for the right equivalence as described for example in [39].

Forgetting the functions $f, F$ and $F'$ in the above definitions, we obtain the analogous notions of induced, isomorphic and versal deformations for the analytic space $(X,0)$. Thus, from a deformation of a function $f$ on an analytic space $(X,0)$, we can obtain a deformation of $(X,0)$ just by 'forgetting' the function itself. It is clear that this operation preserves isomorphism classes and sends versal deformations of $f$ to versal deformations of $(X,0)$. This is just an example of a much more general theory. We have considered deformations of functions between (reduced) analytic germs but the same definitions also work for example, in the algebraic setting. The deformation theory we have defined is usually denoted by $Def(X/S)$: deformations and isomorphism classes of deformations have been defined by diagrams that do not alter the target space $S$. This condition can also be relaxed. Given a map of analytic (or algebraic) spaces, it is possible to consider up to six different notions of deformations (and/or isomorphism classes). Each of them naturally forms a *fibred category* and they are related by several functors that can be put in the vertices of an octahedron (see [5]).

## 2.2 The cotangent complex

For any ring $A$, for simplicity commutative and noetherian, and for any $A$-algebra $B$, there is a complex $L^{\cdot}_{B/A}$, called the *cotangent complex*, that plays an important role in deformation theory (see [20] for a complete definition). If $M$ is a $B$-module, the cotangent cohomology modules are defined as:

$$T^i(B/A; M) = H^i(\mathrm{Hom}_B(L^{\cdot}_{B/A}, M))$$

There is also an analogous homology theory but we do not define it as we will not make any use of it.

Of special importance in deformation theory are the modules $T^i(B/A;M)$ for $i=0,1,2$. They can be computed from a truncation of $L^{\cdot}_{B/A}$, known as the *Lichtenbaum-Schlessinger complex*, that we now describe.

Let us take a presentation of $B$ by free $A$-algebras, i.e., an exact sequence:

$$F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} B \to 0$$

The free algebra $F_1$ contains the submodule $R$ of relations between the generators of the ideal $I$, where $F_0/I = B$. Let $R_0$ be the submodule of *Koszul or trivial relations*. These are generated by the relations of the form $d_1(r')r - d_1(r)r'$. We then have a complex:

$$L^{\leq 2}_{B/A} \colon R/R_0 \xrightarrow{\partial_1} \Omega_{F_0/A} \xrightarrow{\partial_0} \Omega_{B/A} \to 0$$

where $\Omega_{-/A}$ denotes the module of relative Kähler differentials. The modules $T^i(B/A;M)$ for $i=0,1$ and 2 are the cohomology modules of the dual complex $\mathrm{Hom}_B(L^{\leq 2}_{B/A}, M)$. For example:

$$T^0(B/A;M) = \mathrm{Hom}_B(\Omega_{B/A}, M)$$

and $T^1(B/A;M)$ is the cokernel of the map

$$\mathrm{Der}_A(F_0, M) \to \mathrm{Hom}_B(I/I^2, M)$$

Let us discuss briefly the role of these modules in deformation theory and how they appear. Let $(X,0)$ be an analytic germ in $(\mathbb{C}^N,0)$ defined by the ideal $I_{X,0} = (g_1,\dots,g_l) \subset \mathcal{O}_{\mathbb{C}^N,0}$. A *first order infinitesimal deformation of* $(X,0)$ is simply a deformation over Spec $\mathbb{C}[\epsilon]/(\epsilon^2)$. For such a deformation the total space $(\mathfrak{X},0)$ is defined by an ideal

$$I_{\mathfrak{X},0} = (g_1+\epsilon h_1,\dots,g_l+\epsilon h_l) \subset \mathcal{O}_{\mathbb{C}^N,0}[\epsilon]/(\epsilon^2)$$

for some $(h_1,\dots,h_l) \in \mathcal{O}_{\mathbb{C}^N,0}$.

The flatness of $\mathcal{O}_{\mathfrak{X},0}$ over $\mathbb{C}[\epsilon]/(\epsilon^2)$ means that it is possible to lift the relations between the generators of $I_{X,0}$. That is, if $r_1,\dots,r_l \in \mathcal{O}_{\mathbb{C}^N,0}$ are such that:

$$r_1g_1 + \ldots + r_lg_l = 0$$

then there exist $s_1,\dots,s_l \in \mathcal{O}_{\mathbb{C}^N,0}$ such that:

$$(r_1+\epsilon s_1)(g_1+\epsilon h_1) + \ldots + (r_l+\epsilon s_l)(g_l+\epsilon h_l) = 0$$

It follows that the homomorphism $I_{X,0} \to \mathcal{O}_{X,0}$ sending $g_i$ to $h_i$ is well defined and induces a homomorphism of $\mathcal{O}_{X,0}$-modules $I_{X,0}/I^2_{X,0} \to \mathcal{O}_{X,0}$. Reciprocally, an element of $\mathrm{Hom}_{\mathcal{O}_{X,0}}(I_{X,0}/I^2_{X,0}, \mathcal{O}_{X,0})$ defines a deformation over Spec $\mathbb{C}[\epsilon]/(\epsilon^2)$ and therefore first order infinitesimal deformations of $(X,0)$ are in one-to-one correspondence with the *normal module*

$$\mathrm{Hom}_{\mathcal{O}_{X,0}}(I_{X,0}/I^2_{X,0}, \mathcal{O}_{X,0})$$

We now want to identify the submodule of $\mathrm{Hom}_{\mathcal{O}_{X,0}}(I_{X,0}/I_{X,0}^2, \mathcal{O}_{X,0})$ that corresponds under this identification to those isomorphic to the *trivial deformation* given by the product

$$(X,0) \times \mathrm{Spec}\ \mathbb{C}[\epsilon]/(\epsilon^2) \longrightarrow \mathrm{Spec}\ \mathbb{C}[\epsilon]/(\epsilon^2)$$

Such a deformation will be induced from an automorphism of $(\mathbb{C}^N,0)\times \mathrm{Spec}\ \mathbb{C}[\epsilon]/(\epsilon^2)$ and after a change of coordinates in $\mathbb{C}^N$, we can assume that it is of the form:

$$x_i \mapsto x_i + \epsilon\phi_i(x)$$

where $x = (x_1, \dots, x_N)$ denotes coordinates in a neighbourhood of 0 in $\mathbb{C}^N$.

Using the series development of $g_1, \dots, g_l$ we see that the generators of $I_{\mathfrak{X},0}$ are:

$$g_i(x) + \epsilon h_i(x) = g_i(x) + \epsilon \sum_{j=1}^{N} \frac{\partial g_i}{\partial x_j}(x)\phi_j(x)$$

To describe the deformations that occur in this way, we notice that a derivation $D\colon \mathcal{O}_{\mathbb{C}^N,0} \to \mathcal{O}_{X,0}$ defines an homomorphism $\bar{D}\colon\ I_{X,0}/I_{X,0}^2 \to \mathcal{O}_{X,0}$. The above equality shows that the isomorphism class of the trivial deformation is represented by derivations. Hence the set of isomorphism classes of first order infinitesimal deformations of $(X,0)$ is in one-to-one correspondence with

$$T^1(\mathcal{O}_{X,0}/\mathbb{C};\mathcal{O}_{X,0}) = \frac{\mathrm{Hom}_{\mathcal{O}_{X,0}}(I_{X,0}/I_{X,0}^2, \mathcal{O}_{X,0})}{\mathrm{Der}(\mathcal{O}_{\mathbb{C}^N,0}, \mathcal{O}_{X,0})}$$

The module $T^2(\mathcal{O}_{X,0}/\mathbb{C};\mathcal{O}_{X,0})$ is naturally encountered by investigating when a first order infinitesimal deformation can be extended to a *second order infinitesimal deformation*, that is, a deformation over Spec $\mathbb{C}[\epsilon]/(\epsilon^3)$. If $(\mathfrak{X},0) \to \mathrm{Spec}\ \mathbb{C}[\epsilon]/(\epsilon^2)$ is a first order infinitesimal deformation of $(X,0)$, any relation $(r_1,\dots,r_l)$ among the generators of $I_{X,0}$ can be lifted to a relation $(r_1+\epsilon s_1,\dots,r_l+\epsilon s_l)$ among the generators $(g_1+\epsilon h_1,\dots,g_l+\epsilon h_l)$ of $I_{\mathfrak{X},0}$. If $R$ denotes the module of relations between the generators of $I_{X,0}$ and $R_0$ the submodule generated by the trivial relations, a straightforward calculation shows that the map

$$o\colon\ R/R_0 \longrightarrow \mathcal{O}_{X,0}$$

defined by $o(r_1,\dots,r_l) = \sum_{i=1}^{l} s_i h_i$ is well defined and the projection $(\mathfrak{X},0) \to$ Spec $\mathbb{C}[\epsilon]/(\epsilon^2)$ can be extended to a deformation over Spec $\mathbb{C}[\epsilon]/(\epsilon^3)$ if and only if there exist $h_1',\dots,h_l' \in \mathcal{O}_{\mathbb{C}^N,0}$ such that

$$o(r_1,\dots,r_l) = \sum_{i=1}^{l} r_i h_i'$$

The last statement can be rephrased as follows: the module $R$ defines a presentation of the ideal $I_{X,0}$:

$$0 \to R \to \mathcal{O}_{\mathbb{C}^N,0}^k \xrightarrow{(g_1,\dots,g_l)} I_{X,0} \to 0$$

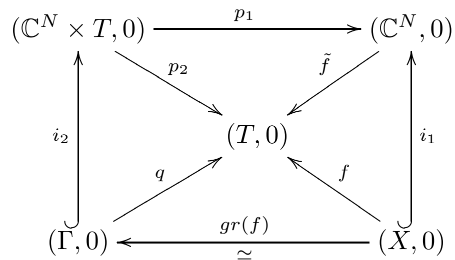
Note that $R/R_0$ is naturally an $\mathcal{O}_{X,0}$-module so that we have a homomorphism of $\mathcal{O}_{X,0}$-modules

$$R/R_0 \to \mathcal{O}_{\mathbb{C}^N,0}^k \otimes \mathcal{O}_{X,0}$$

The condition $o(r_1, \dots, r_l) = \sum_{i=1}^{l} r_i h'_i$ for some $h'_1, \dots, h'_l \in \mathcal{O}_{\mathbb{C}^N,0}$ is equivalent to the map $o$ being zero in the cokernel of the $\mathcal{O}_{X,0}$-dual of the above homomorphism. This cokernel is exactly $T^2(\mathcal{O}_{X,0}/\mathbb{C}; \mathcal{O}_{X,0})$.

It can be shown that the obstructions for a *third order infinitesimal deformation*, that is a deformation over Spec $\mathbb{C}[\epsilon]/(\epsilon)^3$, to be induced from another deformation over Spec $\mathbb{C}[\epsilon]/(\epsilon)^4$ lies again in $T^2(\mathcal{O}_{X,0}/\mathbb{C}; \mathcal{O}_{X,0})$. This justifies referring to $T^2(\mathcal{O}_{X,0}/\mathbb{C}; \mathcal{O}_{X,0})$ as the module of *obstructions*.

Before going into reviewing the properties of the cotangent cohomology modules that will be used, we mention that for the case of a map between analytic spaces $f\colon (X,0) \to (T,0)$, the cohomology modules $T^1(\mathcal{O}_{X,0}/\mathcal{O}_{T,0}; \mathcal{O}_{X,0})$ resp. $T^2(\mathcal{O}_{X,0}/\mathcal{O}_{T,0}; \mathcal{O}_{X,0})$, also have an interpretation as isomorphism classes of infinitesimal deformations, resp. obstructions. The case where $(T,0)$ is smooth is specially simple. Let $(T,0) = (\mathbb{C}^k, 0)$ with coordinates $(t_1, \dots, t_k)$ and let $f = (f_1, \dots, f_k)$. Embedding $(X,0)$ as the graph of $f$ and choosing a representative of $f$ in $\mathcal{O}^k_{\mathbb{C}^N,0}$, say $\tilde{f}$, we obtain a commutative diagram




and a presentation of $\mathcal{O}_{X,0}$ as the quotient of a free $\mathcal{O}_{T,0}$-algebra, namely:

$$\mathcal{O}_{X,0} \simeq \mathcal{O}_{\Gamma,0} = \frac{\mathcal{O}_{\mathbb{C}^N \times T,0}}{(g_1, \dots, g_l, f_1 - t_1, \dots, f_k - t_k)}$$

(if $(T,0)$ was not smooth, we would have to add generators for the ideal of $(T,0)$ to obtain the ideal of $(\Gamma,0)$). Let us introduce the following notation: for an analytic space germ $(A,0)$ we write $\Theta_{A,0}$ for the module of vector fields on $(A,0)$ ($\mathbb{C}$-derivations $v\colon \mathcal{O}_{A,0} \to \mathcal{O}_{A,0}$) and for an analytic map $h\colon (A,0) \to (B,0)$, we denote by $\Theta(h)_0$ the module of vector fields along $h$, i.e., $\mathbb{C}$-derivations $v\colon \mathcal{O}_{B,0} \to \mathcal{O}_{A,0}$. The tangent map of $h$ is thus the map

$$\begin{array}{rcl} th\colon \Theta_{A,0} & \longrightarrow & \Theta(h)_0 \\ v & \mapsto & th(v)\colon \mathcal{O}_{B,0} \ni b \mapsto v(b \circ h) \end{array}$$

**Proposition 2.2.1.** *For $(T,0)$ smooth, there is an identification*

$$T^1(\mathcal{O}_{X,0}/\mathcal{O}_{T,0}; \mathcal{O}_{X,0}) = \frac{Hom_{\mathcal{O}_{X,0}}(I_{X,0}/I^2_{X,0}, \mathcal{O}_{X,0}) \oplus \Theta(f)_0}{\Theta_{X,0}}$$

*where each element $v$ of the denominator is identified with $(\phi_v, tf(v))$ in the numerator. Here $\phi_v(h + I^2_{X,0}) = v(h) + I^2_{X,0}$.*

*Proof.* We show that there is a *split* short exact sequence

$$0 \to \Theta(f)_0 \to \mathrm{Hom}_{\mathcal{O}_{\Gamma,0}}(I_{\Gamma,0}/I^2_{\Gamma,0}, \mathcal{O}_{\Gamma,0}) \to \mathrm{Hom}_{\mathcal{O}_{X,0}}(I_{X,0}/I^2_{X,0}, \mathcal{O}_{X,0}) \to 0$$

The map on the left is given by the composition

$$\Theta(f)_0 \simeq \Theta(p_2)_0 \otimes \mathcal{O}_{\Gamma,0} \hookrightarrow \Theta_{\mathbb{C}^N \times T,0} \otimes \mathcal{O}_{\Gamma,0} \to \mathrm{Hom}_{\mathcal{O}_{\Gamma,0}}(I_{\Gamma,0}/I^2_{\Gamma,0}, \mathcal{O}_{\Gamma,0})$$

To see that this map is injective, we take coordinates $(t_1, \ldots, t_k)$ in $(T, 0)$. The action of a derivation $v = \sum_{i=1}^k a_i \frac{\partial}{\partial t_i}$ on the generators of $I_{\Gamma,0}$ is given by

$$v(g_i) = 0 \text{ for } i = 1, \ldots, l$$
$$v(f_j - t_j) = a_j \text{ for } j = 1, \ldots, k$$

so that $v$ is zero if and only if $a_j \in I_{\Gamma,0}$ for all $j = 1, \ldots, k$.
Given a $\mathcal{O}_{\Gamma,0}$-homomorphism $\phi\colon I_{\Gamma,0} \to \mathcal{O}_{\Gamma,0}$, we define a homomorphism by the composition

$$p_1^* \circ \phi \circ gr(f)^*\colon I_{X,0} \xrightarrow{p_1^*} I_{\Gamma,0} \xrightarrow{\phi} \mathcal{O}_{\Gamma,0} \xrightarrow{gr(f)^*} \mathcal{O}_{X,0}$$

Such a composition is zero if and only if $\phi(g_i) = 0$ for all $i = 1, \ldots, l$. If this is the case, $\phi$ can be written as the homomorphism associated to the vector field

$$v = \sum_{i=1}^k \phi(f_i - t_i) \frac{\partial}{\partial t_i}$$

This shows that the sequence is exact in the middle term. To conclude the exactness of the sequence, we construct a left inverse of this homomorphism, showing at once that the sequence is split. Let $\psi\colon I_{X,0}/I^2_{X,0} \to \mathcal{O}_{X,0}$ be a $\mathcal{O}_{X,0}$-homomorphism. We define $\tilde{\psi}\colon I_{\Gamma,0}/I^2_{\Gamma,0} \to \mathcal{O}_{\Gamma,0}$ on the generators of $I_{\Gamma,0}$ by

$$\begin{cases} \tilde{\psi}(g_i) = gr(f)^{*-1}(\psi(g_i)) \\ \tilde{\psi}(f_i - t_i) = 0 \end{cases}$$

and extend it by linearity. We need to check that $\tilde{\psi}$ is well defined. For this, let us consider a relation between the generators of $I_{\Gamma,0}$ in $\mathcal{O}_{\mathbb{C}^N \times T,0}$,

$$\sum_{i=1}^l r_i g_i + \sum_{j=1}^k s_j (f_j - y_j) = 0$$

Applying $\tilde{\psi}$ to the left hand side of the above equation, we see that $\tilde{\psi}$ is well defined if

$$\sum_{i=1}^l r_i \tilde{\psi}(g_i) = \sum_{i=1}^l r_i gr(f)^{*-1}(\psi(g_i)) = 0 \in \mathcal{O}_{\Gamma,0}$$

As $gr(f)^*$ is an isomorphism, this is equivalent to

$$\sum_{i=1}^l gr(f)^*(r_i) \psi(g_i) = 0 \in \mathcal{O}_{X,0}$$

The left hand side of the above equation is equal to the image by $\phi$ of the image by $gr(f)^*$ of the original relation, that is,

$$0 = \psi(gr(f)^*(\sum_{i=1}^l r_i g_i + \sum_{j=1}^k s_j (f_j - y_j)))$$

Finally, the action of the $\mathcal{O}_{T,0}$-derivations in $\Theta_{\mathbb{C}^N\times T,0}$ on the module $\mathrm{Hom}_{\mathcal{O}_{\Gamma,0}}(I_{\Gamma,0}/I^2_{\Gamma,0},\mathcal{O}_{\Gamma,0})$ clearly corresponds under the decomposition $\mathrm{Hom}_{\mathcal{O}_{X,0}}(I_{X,0}/I^2_{X,0},\mathcal{O}_{X,0})\oplus\Theta(f)_0$ to that described in the proposition. □

The description given in preceding proposition tallies with the usual notation used in the theory of unfolding of isolated singularities. If $(X,0)$ is smooth with coordinates $(x_1,\dots,x_n)$ and $(T,0)$ is one dimensional, we see that there is an identification

$$T^1(\mathcal{O}_{X,0}/\mathcal{O}_{T,0};\mathcal{O}_{X,0})=\mathcal{O}_{X,0}/(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n})$$

Therefore in this case $T^1(\mathcal{O}_{X,0}/\mathcal{O}_{T,0};\mathcal{O}_{X,0})$ is just the usual Jacobian algebra. It is this algebra that provides the multiplicative structure on the tangent bundle of the base space of the miniversal unfolding of $f$. In the case of arbitrary $(X,0)$, the module $T^1(\mathcal{O}_{X,0}/\mathcal{O}_{T,0};\mathcal{O}_{X,0})$ is not an algebra even if $(T,0)$ is one-dimensional. Instead, we have an exact sequence:

$$0\to\frac{\Theta(f)_0}{tf(\Theta_{X,0})}\to\ T^1(\mathcal{O}_{X,0}/\mathcal{O}_{T,0};\mathcal{O}_{X,0})\to\ T^1(\mathcal{O}_{X,0}/\mathbb{C};\mathcal{O}_{X,0})\to 0$$

This will be the crucial point to extend the multiplicative structure for deformations of functions with isolated critical points on isolated singularities.

The above exact sequence can be derived from the standard properties of the cotangent complex (and we will do so) which we now list. For proofs we refer to [20].

*Vanishing*. If $B$ is a smooth $A$-algebra then $T^i(B/A;M)=0$ for $i>0$ and any $B$-module $M$.

*Long exact sequence*. If $0\to M'\to M\to M''\to 0$ is a short exact sequence of $B$-modules, there is a long exact sequence:

$$\begin{aligned}\dots\to T^i(B/A;M')\to T^i(B/A;M)\to T^i(B/A;M'')\to\\ \to T^{i+1}(B/A;M')\to T^{i+1}(B/A;M)\to T^{i+1}(B/A;M'')\to\dots\end{aligned}$$

*Base change*. If $B$ is flat as $A$-module and $A'$ is an $A$-algebra, then for any module $M'$ over $B'=B\otimes_A A'$ there is a natural isomorphism

$$T^i(B'/A';M')\simeq T^i(B/A;M')$$

Moreover, if $A'$ is flat as $A$-module then for any $B$-module $M$,

$$T^i(B'/A';M\otimes_B B')\simeq T^i(B/A;M)\otimes_B B'$$

*Zariski-Jacobi sequence*. If $B\to C$ is a map of $A$-algebras and $M$ is a $C$-module, there is a long exact sequence:

$$\dots\to T^i(C/B;M)\to T^i(C/A;M)\to T^i(B/A;M)\to T^{i+1}(C/B;M)\to\dots$$

**Notation**: Let $f\colon(X,0)\to(T,0)$ be a morphism of analytic spaces. We will use the following notation: for a $\mathcal{O}_{X,0}$-module $M$, we will write:

$$T^i_{X/T,0}(M)\colon=T^i(\mathcal{O}_{X,0}/\mathcal{O}_{T,0};M)$$

If $M = \mathcal{O}_{X,0}$, we will drop the module $M$:

$$T^i_{X/T,0}\colon = T^i(\mathcal{O}_{X,0}/\mathcal{O}_{T,0}; \mathcal{O}_{X,0})$$

In the absolute case where $T = \{0\}$, we write

$$T^i_{X,0}(M)\colon = T^i_{X/\mathbb{C}}(M)$$

And finally a notation we have already used, if $(T, 0)$ is smooth, we will write:

$$\Theta(f)_0\colon = T^0_{T,0}(\mathcal{O}_{X,0})$$

A last remark before going into the next section. There exists a global counterpart to the theory of the cotangent complex. Enough to say that for any map of analytic spaces $f\colon\ X \to T$ and any coherent sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules, it provides sheaves $\mathcal{T}^i_{X/Y}(\mathcal{M})$ whose stalks coincide with our local definitions.

## 2.3 The Kodaira-Spencer map and versal deformations

Associated to a deformation $g\colon (\mathfrak{X}, 0) \to (B, 0)$ of an analytic germ $(X, 0)$, there is a diagram of ring homomorphisms:

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X},0} & \longrightarrow & \mathcal{O}_{X,0} \\ \uparrow & & \uparrow \\ \mathcal{O}_{B,0} & \longrightarrow & \mathbb{C} \end{array}$$

and a long exact sequence:

$$0 \to T^0_{\mathfrak{X}/B,0} \to T^0_{\mathfrak{X},0} \to T^0_{B,0}(\mathcal{O}_{\mathfrak{X}}) \to T^1_{\mathfrak{X}/B,0} \to \dots$$

A derivation of $\mathcal{O}_{B,0}$ can be composed with $g^*\colon \mathcal{O}_{B,0} \to \mathcal{O}_{\mathfrak{X},0}$ to obtain a derivation of $\mathcal{O}_{B,0}$ with values in $\mathcal{O}_{\mathfrak{X},0}$. This yields a map

$$wg\colon \Theta_{B,0} \to T^0_{B,0}(\mathcal{O}_{\mathfrak{X},0})$$

The composition with the connecting homomorphism $T^0_{B,0}(\mathcal{O}_{\mathfrak{X},0}) \to T^1_{\mathfrak{X}/B,0}$ gives the *Kodaira-Spencer map*:

$$\rho_{\mathfrak{X}/B}\colon \Theta_{B,0} \longrightarrow T^1_{\mathfrak{X}/B,0}$$

Reducing modulo $\mathfrak{m}_{B,0}$ we obtain the *reduced Kodaira-Spencer map*:

$$\rho_{\mathfrak{X}/B}(0)\colon T_0B \longrightarrow T^1_{X,0}$$

This map interprets a tangent vector as the first order infinitesimal deformation defined by $g$ in the direction of that vector. If a deformation is versal, it must contain all the first order infinitesimal deformations so that its Kodaira-Spencer map must be surjective. It is an essential result in deformation theory that this is also a sufficient condition for versality. The infinitesimal-global step is essentially the Malgrange's Preparation Theorem (for example, [25], pg. 134).

If $(X, 0)$ is an isolated singularity, the module $T^1_{X,0}$ is Artinian and hence a finite dimensional vector space over $\mathbb{C}$. Its dimension is called the *Tjurina number* of $(X, 0)$ and denoted by $\tau(X, 0)$. A deformation whose reduced Kodaira-Spencer map is an isomorphism is called *miniversal*. If $(X, 0)$ is unobstructed, i.e. $T^2_{X,0} = 0$, then all the first order infinitesimal deformations extend to actual deformations and $(B, 0)$ is the whole of $(\mathbb{C}^k, 0)$ with $k = \tau(X, 0)$.

Note that by Nakayama's lemma, the surjectivity of the $\rho_{\mathfrak{X}/B}(0)$ is equivalent to that of $\rho_{\mathfrak{X}/B}$, so that for a versal deformation, we have a short exact sequence:

$$0 \to T^0_{\mathfrak{X}/B,0} \to T^0_{\mathfrak{X},0} \to T^0_{B,0}(\mathcal{O}_{\mathfrak{X}}) \to T^1_{\mathfrak{X}/B,0} \to 0$$

and thus:

$$T^1_{\mathfrak{X}/B,0} = \frac{T^0_{B,0}(\mathcal{O}_{\mathfrak{X}})}{\text{Im }(T^0_{\mathfrak{X},0} \to T^0_{B,0}(\mathcal{O}_{\mathfrak{X}}))}$$

In the case where $(B, 0)$ is smooth, we have the identification

$$T^1_{\mathfrak{X}/B,0} = \frac{\Theta(g)_0}{tg(\Theta_{\mathfrak{X},0})}$$

If $f \in \mathcal{O}_{X,0}$ is now an analytic function on a reduced analytic space, we can assume $f(0) = 0$ and think of $f$ as a map $f\colon (X, 0) \to (S, 0)$ where $(S, 0)$ denotes the germ of the complex plane $(\mathbb{C}, 0)$. As usual, we can write $\varphi = (F, g)\colon (\mathfrak{X}, 0) \to (S \times B, 0)$ and reason as above to obtain a map:

$$\rho_{\mathfrak{X}/S\times B}\colon \Theta_{S\times B,0} \longrightarrow T^1_{\mathfrak{X}/S\times B,0}$$

A derivation of $\mathcal{O}_{B,0}$ induces a $\mathcal{O}_{S,0}$-derivation of $\mathcal{O}_{S\times B,0}$. Composing with $\rho_{\mathfrak{X}/S\times B}$ we obtain the Kodaira-Spencer map of the deformation $\varphi$ as a deformation of $f$:

$$\rho_{\mathfrak{X}/B,F}\colon \Theta_{B,0} \longrightarrow T^1_{\mathfrak{X}/S\times B,0}$$

Or reducing modulo $\mathfrak{m}_{B,0}$, we obtain the reduced Kodaira-Spencer map:

$$\rho_{\mathfrak{X}/B,F}(0)\colon T_0B \longrightarrow T^1_{X/S,0}$$

Note that the reduction of $\rho_{\mathfrak{X}/S\times B}$ module $\mathfrak{m}_{S\times B,0}$ does not yield the reduced Kodaira-Spencer map for the 0-fibre of $f$ as we have not assumed that $f$ is flat. Analogously to the case of analytic spaces, a deformation of a function is versal if and only if its reduced Kodaira-Spencer map is surjective. We will prove this statement in the next section, due in part to a lack of suitable reference, and because we wish to put our main objects of study into the right framework.

## 2.4 Liftable and relative vector fields

From the very definition of the Kodaira-Spencer map, given a deformation $g\colon (\mathfrak{X}, 0) \to (B, 0)$ we have a long exact sequence and a commutative triangle:

$$\begin{array}{ccccccccc} & & & & \Theta_{B,0} & & & & \\ & & & & \downarrow{\scriptstyle wg} & \searrow{\scriptstyle \rho_{\mathfrak{X}/B}} & & & \\ 0 \longrightarrow & T^0_{\mathfrak{X}/B,0} & \longrightarrow & T^0_{\mathfrak{X},0} & \xrightarrow{tg} & T^0_{B,0}(\mathcal{O}_{\mathfrak{X},0}) & \longrightarrow & T^1_{\mathfrak{X}/B,0} & \longrightarrow \cdots \end{array} \quad (2.1)$$

where $tg$, resp. $wg$, is defined for $v \in \Theta_{\mathfrak{X},0}$, resp. for $u \in \Theta_{B,0}$, by the composition:

$$tg(v)\colon \mathcal{O}_{B,0} \xrightarrow{g^*} \mathcal{O}_{\mathfrak{X},0} \xrightarrow{v} \mathcal{O}_{\mathfrak{X},0} \quad \text{, resp.} \quad wg(u)\colon \mathcal{O}_{B,0} \xrightarrow{u} \mathcal{O}_{B,0} \xrightarrow{g^*} \mathcal{O}_{\mathfrak{X},0}$$

If $u \in \Theta_{B,0}$ is in the kernel of $\rho_{\mathfrak{X}/B}$, it follows from the exactness of the sequence 2.1 that $wg(u)$ is in the image of $tg$, i.e., there exists $\tilde{u} \in \Theta_{\mathfrak{X},0}$ such that $tg(\tilde{u}) = wg(u)$. In this situation we will say that $\tilde{u}$ is a *lift* of $u$ and $u$ is *liftable*. We will call the kernel of $\rho_{\mathfrak{X}/B}$ the module of *liftable vector fields* and denote it by $\mathcal{L}_{\mathfrak{X}/B,0}$. We then have an exact sequence:

$$0 \longrightarrow \mathcal{L}_{\mathfrak{X}/B,0} \longrightarrow \Theta_{B,0} \longrightarrow T^1_{\mathfrak{X}/B,0}$$

The kernel of $tg$ corresponds to lifts of $0 \in \Theta_{B,0}$; it is the $\mathcal{O}_{\mathfrak{X},0}$-dual of the module of relative differentials $\Omega_{\mathfrak{X}/B,0}$. We will denote it by $\Theta_{\mathfrak{X}/B,0}$.

We now consider the case of a function $f \in \mathcal{O}_{X,0}$. As usual, we assume $f(0) = 0$ and think of $f$ as a map $f\colon (X,0) \to (S,0)$ where $(S,0) = (\mathbb{C},0)$. Let $F$ be an extension of $f$ to $(\mathfrak{X},0)$ and write $\varphi = (F,g)$ The ring homomorphisms $\mathcal{O}_{B,0} \to \mathcal{O}_{S\times B,0} \to \mathcal{O}_{\mathfrak{X},0}$ induce a long exact sequence:

$$\begin{aligned} 0 \longrightarrow & T^0_{\mathfrak{X}/S\times B,0} \longrightarrow T^0_{\mathfrak{X}/B,0} \longrightarrow T^0_{S\times B/B,0}(\mathcal{O}_{\mathfrak{X},0}) \longrightarrow \\ & T^1_{\mathfrak{X}/S\times B,0} \longrightarrow T^1_{\mathfrak{X}/B,0} \longrightarrow T^1_{S\times B/B,0}(\mathcal{O}_{\mathfrak{X},0}) \longrightarrow \dots \end{aligned} \tag{2.2}$$

Note that $T^0_{\mathfrak{X}/S\times B,0} = \Theta_{\mathfrak{X}/S\times B,0}$ and similarly for $T^0_{\mathfrak{X}/B,0}$. On the other hand, the diagram:

$$\begin{array}{ccc} \mathcal{O}_{S,0} & \longrightarrow & \mathcal{O}_{S\times B,0} \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & \mathcal{O}_{B,0} \end{array}$$

induces natural isomorphisms $T^i_{S\times B/B,0}(M) = T^i_{S,0}(M)$ for any module over $\mathcal{O}_{S\times B,0}$. As $(S,0)$ is smooth, $T^0_{S,0}(\mathcal{O}_{\mathfrak{X},0}) = \Theta(F)_0$ and $T^i_{S,0}(M) = 0$ for $i \geq 1$.

Hence the sequence 2.2 becomes:

$$0 \to \Theta_{\mathfrak{X}/S\times B,0} \to \Theta_{\mathfrak{X}/B,0} \to \Theta(F)_0 \to T^1_{\mathfrak{X}/S\times B,0} \to T^1_{\mathfrak{X}/B,0} \to 0$$

and we have an exact sequence:

$$0 \to \frac{\Theta(F)_0}{tF(\Theta_{\mathfrak{X}/B,0})} \to T^1_{\mathfrak{X}/S\times B,0} \to T^1_{\mathfrak{X}/B,0} \to 0$$

**Notation.** Given a deformation $\varphi = (F,g)\colon (\mathfrak{X},0) \to (S\times B,0)$ of a function $f\colon (X,0) \to (S,0)$, we will denote by $M_{\varphi,0}$ the module $\Theta(F)_0/tF(\Theta_{\mathfrak{X}/B,0})$. Note that in the case of unfolding of isolated singularities, i.e., if $(X,0)$ is smooth, the module $M_{\varphi,0}$ is just the relative Jacobian algebra of an unfolding of $f$.

**Lemma 2.4.1.** *Given a deformation $\varphi = (F,g)\colon (\mathfrak{X},0) \to (S\times B,0)$ of $f\colon (X,0) \to (S,0)$, there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_{\mathfrak{X}/B,0} & \longrightarrow & \Theta_{B,0} & \xrightarrow{\rho_{\mathfrak{X}/B}} & T^1_{\mathfrak{X}/B,0} & \longrightarrow & T^1_{\mathfrak{X},0} \\ & & \downarrow{\scriptstyle -t'F} & & \downarrow{\scriptstyle \rho_{\mathfrak{X}/B,F}} & & \| {\scriptstyle \mathbb{I}} & & \\ 0 & \longrightarrow & M_{\varphi,0} & \longrightarrow & T^1_{\mathfrak{X}/S\times B,0} & \longrightarrow & T^1_{\mathfrak{X}/B,0} & \longrightarrow & 0 \end{array}$$

*where $t'F$ is given as follows: for $u \in \mathcal{L}_{\mathfrak{X}/B,0}$, let $\tilde{u} \in \Theta_{\mathfrak{X},0}$ be a lift of $u$. Then $t'F(v)$ is the class of $tF(\tilde{u})$ in $M_{\varphi,0}$.*

*Proof.* It is enough to check the commutativity for the first square, for the second it clearly follows from the definitions. We first show that $t'F$ is well defined: if $\tilde{u}'$ is another lift of $u$ to $\Theta_{\mathfrak{X},0}$, then:

$$tg(\tilde{u} - \tilde{u}') = wg(u) - wg(u) = 0$$

so that $\tilde{u} - \tilde{u}' \in \Theta_{\mathfrak{X}/B,0}$. Hence $tF(\tilde{u}) - tF(\tilde{u}') \in tF(\Theta_{\mathfrak{X}/B,0})$.

For the commutativity, we see that $-tF(\tilde{u})$ is mapped to the class:

$$\begin{aligned}-tF(\tilde{u}) + t\varphi(\Theta_{\mathfrak{X},0}) &= -tF(\tilde{u}) + t\varphi(\tilde{u}) + t\varphi(\Theta_{\mathfrak{X},0})\\ &= tg(\tilde{u}) + t\varphi(\Theta_{\mathfrak{X},0}) = wg(u) + t\varphi(\Theta_{\mathfrak{X},0})\\ &= \rho_{\mathfrak{X}/B,F}(u)\end{aligned}$$

□

**Corollary 2.4.1.** *The Kodaira-Spencer map $\rho_{\mathfrak{X}/B,F}$ is surjective if and only if $\rho_{\mathfrak{X}/B}$ and $t'F$ are surjective.*

*Proof.* The surjectivity of $\rho_{\mathfrak{X}/B,F}$ clearly implies that of $\rho_{\mathfrak{X}/B}$. Hence the sequence on top in Lemma 2.4 remains exact if $T^1_{\mathfrak{X},0}$ is replaced by 0. The Snake Lemma shows that $t'F$ is also surjective. The other implication is analogous. □

By Nakayama's lemma, an analogous result holds for the reduced Kodaira-Spencer map. Note that the top exact sequence may no longer be exact after reducing modulo $\mathfrak{m}_{B,0}$. The reduction of $t'F$ modulo $\mathfrak{m}_{B,0}$ will be denoted by $t'F(0)$.

We can now prove the versality criterion for deformation of functions with isolated critical points on isolated singularities. We will say that a function germ $f$ on a analytic germ $(X,0)$ has a isolated singularity if $(X,0)$ is reduced and there exists a representative $\tilde{f}\colon X \longrightarrow S$ such that $X - \{0\}$ is smooth and $F$ is a submersion at any point of $X - \{0\}$.

**Theorem 2.4.1.** *Let $f\colon (X,0) \longrightarrow (S,0)$ be analytic function germ with an isolated singularity. Then a deformation of $f$ is versal if and only if its reduced Kodaira-Spencer map is surjective.*

*Proof.* Let $\varphi = (F,g)\colon (\mathfrak{X},0) \longrightarrow (S \times B,0)$ be a versal deformation of $f$. From the definition of versality it follows that $g$ must be versal as a deformation of $(X,0)$. We can then assume that $g$ and $F$ are of the form:

$$\begin{aligned}g\colon (\mathfrak{X},0) = (\mathfrak{X}_0 \times \mathbb{C}^K,0) &\xrightarrow{g_0 \times \mathbb{I}_{\mathbb{C}^K}} (B_0 \times \mathbb{C}^K,0) = (B,0)\\ F = \tilde{f}(x) + f_1(x,y) &+ \ldots + f_K(x,y)\end{aligned}$$

where $g_0\colon (\mathfrak{X}_0,0) \subset (\mathbb{C}^N,0) \to (B_0,0)$ is a miniversal deformation of $(X,0)$, $\tilde{f}$ is an extension of $f$ to $(\mathbb{C}^N,0)$ and $x = (x_1,\ldots,x_N)$, resp. $y = (y_1,\ldots,y_K)$, denotes coordinates in $(\mathbb{C}^N,0)$, resp. in $(\mathbb{C}^K,0)$.

Let $h \in \mathcal{O}_{X,0}$ be arbitrary and consider the deformation of $f$ given by

$$\chi = (f + uh, p)\colon (X \times \mathbb{C},0) \longrightarrow (S \times \mathbb{C},0)$$

where $p$ is the projection and $u$ is a parameter in $\mathbb{C}$. By versality, there exist $\Phi$ and $\Psi$ making the following diagram commutative:

$$\begin{array}{ccccc}
(X,0)\times\mathbb{C} & & \xrightarrow{\quad\Phi\quad} & & (\mathfrak{X}_0\times\mathbb{C}^K,0) \\
\Big\downarrow p & \searrow^{f+uh} & & \swarrow^{F} & \Big\downarrow g \\
 & & (S,0) & & \\
(\mathbb{C},0) & & \xrightarrow{\quad\Psi\quad} & & (B_0\times\mathbb{C}^K,0)
\end{array}$$

As $p$ is a trivial deformation of $(X,0)$ and $g_0$ is miniversal, it follows that the image of $\Psi$ is contained in $\{0\}\times(\mathbb{C}^K,0)$. Then $\Phi$ is of the form:

$$(\phi_u(x),\Psi(u))$$

being $\phi_u(x)$ the germ of a family of automorphisms of $(X,0)$. In particular, the tangent vector:

$$\left.\frac{d\phi_u}{du}\right|_{u=0}$$

belongs to $\Theta_{\mathfrak{X}/B,0}/\mathfrak{m}_{B,0}\Theta_{\mathfrak{X}/B,0}$. It follows from the chain rule that

$$h=\left.\frac{\partial(f+uh)}{\partial u}\right|_{u=0}=tF(t\Phi(\left.\frac{\partial}{\partial u}\right|_{u=0}))\ (\text{mod. } \mathfrak{m}_{B,0}tF(\Theta_{\mathfrak{X}/B,0}))$$

Noting that the coordinate vector fields associated to $(y_1,\dots,y_K)$ are liftable, it follows that $t'F(0)$ is surjective.

We now show the reciprocal statement, i.e., if the reduced Kodaira-Spencer map of a deformation is surjective then the deformation is versal. Keeping the notation above, if the reduced Kodaira-Spencer map of $\varphi=(F,g)$ is surjective, then $g$ is a versal deformation of $(X,0)$ so that we can assume that $\varphi$ has the same form as before, i.e., it is the product of a miniversal deformation of $(X,0)$ with a smooth space. It is clear that any deformation of $f$ can be induced from one of the form $F'=\tilde{f}+h(x,u_1,\dots,u_{K'})$ and the obvious deformation of $(X,0)$ with total space $(\mathfrak{X}_0\times\mathbb{C}^{K'},0)$ (as before $g_0\colon(\mathfrak{X}_0,0)\longrightarrow(B_0,0)$ denotes a miniversal deformation of $(X,0)$). It then suffices to show that it is possible to induce such a deformation from $\varphi$. From a deformation of that form, we can construct a deformation "sum": the deformation of $(X,0)$ is given by:

$$g_1\colon(\mathfrak{X}_1,0)=(\mathfrak{X}_0\times\mathbb{C}^{K+K'},0)\longrightarrow(B_0\times\mathbb{C}^{K+K'},0)=(B_1,0)$$

and the extension of $f$ defined by the sum $F+F'-f$. Let us denote it by $\chi=(F_1,g_1)$. We are going to show that it is possible to induce $\chi$ from $\varphi$. The key point is following lemma. For notational simplicity, we state it for a coordinate vector field, but it holds for any liftable vector field that does not vanish at the origin:

**Lemma 2.4.2.** *Let $(z_1,\dots,z_{K'})$ be a coordinate system in $(\mathbb{C}^{K'},0)$ and $u$ a coordinate vector field, for example $\partial_{z_{K'}}$. If $t'F_1(u)=0$ then there exist $\Phi$ and*

$\Psi$ *making the following diagram commutative:*

$$\begin{array}{ccccc}
(\mathfrak{X}_0 \times \mathbb{C}^{K+K'},0) & & \xrightarrow{\Phi} & & (\mathfrak{X}_0 \times \mathbb{C}^{K+K'-1},0) \\
 & \searrow^{F_1} & & \swarrow^{F_{1,K'}} & \\
\downarrow{g_1} & & (S,0) & & \downarrow \\
(B_0 \times \mathbb{C}^{K+K'},0) & & \xrightarrow{\Psi} & & (B_0 \times \mathbb{C}^{K+K'-1},0)
\end{array}$$

*Here* $F_{1,K'}$ *stands for the restriction of* $F_1$ *to the hyperplane section of* $(\mathfrak{X}_1,0)$ *given by* $z_{K'}=0$*.*

*Proof.* The condition $t'F_1(u)=0$ means that for a lift $\tilde{u}\in\Theta_{\mathfrak{X}_1,0}$, there exists $\tilde{u}'=\Theta_{\mathfrak{X}_1/B_1,0}$ such that $tF_1(\tilde{u})=tF_1(\tilde{u}')$. Then $\tilde{u}''=\tilde{u}-\tilde{u}'$ is another lift of $u$ such that $tF_1(\tilde{u}'')=0$. For $z_{K'}$ small enough, we map a point $(p,z_{K'})$ of a (small enough representative of) $(B_0\times\mathbb{C}^{K+K'},0)$ to $\gamma(-z_{K'})$, being $\gamma$ the integral curve of $u$ with $\gamma(0)=(p,z_{K'})$. This way we obtain $\Psi\colon (B_0\times\mathbb{C}^{K+K'},0)\longrightarrow (B_0\times\mathbb{C}^{K+K'-1},0)$. Analogously, integration along $\tilde{u}''$ gives the desired map $\Phi$. The fact that $\tilde{u}''$ is a lifting of $u$ gives the commutativity of the outer square and $tF_1(\tilde{u}'')=0$ implies the commutativity of the inner triangle. □

We now go on with the proof of the theorem. By Malgrange's Preparation Theorem (for example, [25]), the surjectivity of $t'F(0)$ implies

$$\mathcal{O}_{\mathfrak{X}_1,0}=tF_1(\Theta_{\mathfrak{X}_1/B_1,0})+\mathcal{O}_{\mathbb{C}^{K+K'},0}<\frac{\partial F_1}{\partial y_1},\ldots,\frac{\partial F_1}{\partial y_K}>$$

so that we can write

$$\frac{\partial F_1}{\partial z_{K'}}=tF_1(u)+\sum_{i=1}^{K'}a_i(y,z)\frac{\partial F_1}{\partial y_i}$$

for $u\in\Theta_{\mathfrak{X}_1/B_1,0}$ and $a_i\in\mathcal{O}_{\mathbb{C}^{K+K'}}$. Hence the vector field:

$$\frac{\partial}{\partial z_{K'}}-\sum_{i=1}^{K}a_i(y,z)\frac{\partial}{\partial y_i}$$

is liftable and does not vanish at the origin. Change coordinates so that the above vector field is the last coordinate. Applying the above lemma we can induce $F_1$ from $F_{1,K'}$. The theorem now follows from the obvious induction argument. □

We will call a deformation of $f$ *miniversal* if its Kodaira-Spencer map is an isomorphism. In this case the reduced Kodaira-Spencer map

$$\rho_{\mathfrak{X}/B,F}\colon\ T_0B\longrightarrow\ T^1_{X/S,0}$$

is an isomorphism of finite dimensional vector spaces. This dimension is called the *Tjurina number* of $f$ and it will be denoted by $\tau(X/S,0)$.

## 2.5 Construction of versal deformations

According to the previous section, a versal deformation of a function with an isolated singularity can be obtained as follows: as $(X,0)$ is an isolated singularity, we can take a versal deformation, say $g_0\colon (\mathfrak{X}_0,0) \longrightarrow (B_0,0)$. Let $\tilde{f}$ be any extension of $f$ to $(\mathfrak{X}_0,0)$ and consider the deformation $\varphi_0 = (\tilde{f}, g_0)$. Let $\mathcal{C}_{\varphi_0,0}$ the cokernel of $t'\tilde{f}$:

$$\mathcal{L}_{\mathfrak{X}_0/B_0,0} \xrightarrow{t'\tilde{f}} M_{\varphi_0,0} \to \mathcal{C}_{\varphi_0,0} \to 0$$

As $f$ is assumed to have an isolated critical point, the $\mathcal{O}_{X,0}$-module

$$\frac{\mathcal{C}_{\varphi_0,0}}{\mathfrak{m}_{B_0,0}\mathcal{C}_{\varphi_0,0}}$$

is Artinian. Choose $f_1,\dots,f_k \in \mathcal{O}_{X,0}$ so that project onto a system of generators over $\mathbb{C}$ of the above quotient. Then the deformation $\varphi = (F,g)$ given by:

$$g = g_0 \times \mathbb{I}_{\mathbb{C}^k,0}\colon (\mathfrak{X}_0 \times \mathbb{C}^k) \longrightarrow (B_0 \times \mathbb{C}^k, 0)$$
$$F(x,y) = \tilde{f}(x) + y_1 f_1(x) + \ldots + y_k f_k(x)$$

is versal, where the extensions of the functions are understood. It is miniversal if and only if the versal deformation $(\mathfrak{X}_0,0) \to (B_0,0)$ of $(X,0)$ is miniversal and $f_1,\dots,f_k$ project onto a $\mathbb{C}$-basis of $\mathcal{C}_{\varphi_0,0}/\mathfrak{m}_{B,0}\mathcal{C}_{\varphi_0,0}$.

## 2.6 Finite determinacy

We can use the ideas in the previous section to prove the results referring to finite determinacy in the case of functions on isolated singularities.

**Definition 2.6.1.** Let $(X,0)$ be an analytic germ with an isolated singularity and $f_1, f_2 \in \mathfrak{m}_{X,0} \subset \mathcal{O}_{X,0}$. We say that $f_1$ and $f_2$ are *right equivalent* if there exists an automorphism $\phi\colon (X,0) \longrightarrow (X,0)$ such that $f_2 \circ \phi = f_1$.

A function $f \in \mathcal{O}_{X,0}$ is called *right $k$-determined* if for any $h \in \mathfrak{m}_{X,0}^{k+1}$, $f$ and $f+h$ are right equivalent. It is *finitely right determined* if it is right $k$-determined for some $k < \infty$.

The following theorem is the analogous to the determinacy theorem for functions on smooth spaces ([26], [39]). It is also shown to hold for functions on hypersurfaces in [9] with a different technique. It can also be deduced from the results found in [4], although there the point of view is slightly different. We focus on the non-smooth case.

**Theorem 2.6.1.** *Let $(X,0)$ be a* non smooth *germ with an isolated singularity. If $tf(\Theta_{X,0}) \supset \mathfrak{m}_{X,0}^k\Theta(f)_0$ then $f$ is right $k+1$-determined. Therefore if $f\colon (X,0) \to (S,0)$ has an isolated singularity, it is finitely right determined.*

*Proof.* Let $h \in \mathfrak{m}_{X,0}^{k+1}$ and consider the deformation $F\colon (X \times \mathbb{C}, 0) \longrightarrow (S,0)$ of $f$ given by

$$F(x,u) = f + uh$$

and the projection $g\colon (X \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$. As $(X, 0)$ is non-smooth, any vector field $v \in \Theta_{X,0}$ vanishes at 0. Then for $h \in \mathfrak{m}_{X,0}^{k+1}$ we have $th(v) \in \mathfrak{m}_{X,0}^{k+1}\Theta(h)_0$ and for any $u_0 \in \mathbb{C}$

$$t(f + u_0h)(\Theta_{X,0}) + \mathfrak{m}_{X,0}^{k+1}\Theta(f + u_0h)_0 = tf(\Theta_{X,0}) + \mathfrak{m}_{X,0}^{k+1}\Theta(f)_0 \supset \mathfrak{m}_{X,0}^{k}\Theta(f)_0$$

By Nakayama's lemma it follows

$$t(f + u_0h)(\Theta_{X,0}) \supset \mathfrak{m}_{X,0}^{k}\Theta(f + u_0h)_0 \tag{2.3}$$

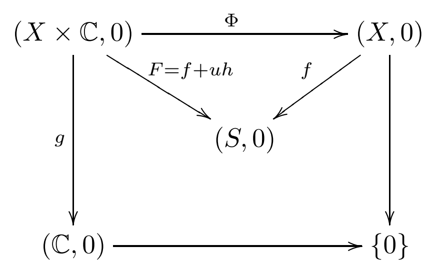
Since the deformation $g$ of $(X, 0)$ is trivial, we have

$$\Theta_{X\times\mathbb{C}/\mathbb{C},0} = \Theta_{X,0} \otimes_{\mathcal{O}_{X,0}} \mathcal{O}_{X\times\mathbb{C},0}$$

so that 2.3 implies

$$tF(\Theta_{X\times\mathbb{C}/\mathbb{C},0}) \supset \mathfrak{m}_{X,0}^{k}\Theta(F)_0$$

and in particular it contains $h = tF(\frac{\partial}{\partial u})$. Then, according to lemma 2.4.2, it follows we can find $\Phi$ making the following diagram commutative:




and hence $f$ and $f + uh$ are right equivalent for any $u$. □

Note that it is not clear which power of the maximal ideal is contained in $tf(\Theta_{X,0})$ for a $k$-determined germ $f$. It will depend on the vanishing order at 0 of vector fields on $(X, 0)$. In the weighted homogeneous case we can say something about this order. Recall that a germ $(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$ is weighted homogeneous if it can be defined by equations $g_1, \ldots, g_k$ such that there exist integers (called *weights*) $w_1, \ldots, w_N$ and $d_1, \ldots, d_k$ such that for all $t \in \mathbb{C}$

$$g_j(t^{w_1}x_1, \ldots, t^{w_N}x_N) = t^{d_j}g_j(x_1, \ldots, x_N) \text{ for } j = 1, \ldots, k$$

**Proposition 2.6.1.** *Let $(X, 0)$ a be weighted homogeneous isolated singularity defined by $g_1, \ldots, g_l$ with weights $w_i > 0$. If $f\colon (X, 0) \longrightarrow (S, 0)$ is right $k$-determined, then*

$$tf(\Theta_{X,0}) \supset \mathfrak{m}_{X,0}^{k+1}\Theta(f)_0$$

*Proof.* Let $h = x_1^{l_1} \ldots x_N^{l_N} \in \mathfrak{m}_{X,0}^{k+1}$. Then $f$ and $f + h$ are right equivalent so that there exists an automorphism $\phi\colon (X, 0) \to (X, 0)$ such that $f \circ \phi = f + h$. Applying to this equation the Euler vector field given by

$$E = w_1x_1\frac{\partial}{\partial x_1} + \ldots + w_Nx_N\frac{\partial}{\partial x_N} \in \Theta_{X,0}$$

we obtain

$$tf(t\phi(E) - E) = th(E) = (\sum_{i=1}^{N} w_il_i)h\frac{\partial}{\partial s}$$

As the weights $w_i$ are strictly positive, the result follows. □

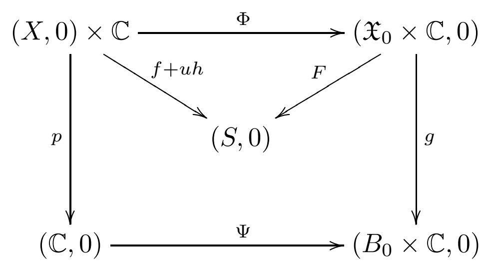
The following proposition will be used in Chapter 2 to characterise functions with minimal Milnor number.

**Proposition 2.6.2.** *Let* $f\colon (X,0) \to (S,0)$ *be a function with an isolated singularity,* $(X,0)$ *non-smooth and let* $\varphi_0 = (\tilde{f}, g)\colon (\mathfrak{X}_0, 0) \to (S \times B_0, 0)$ *be a deformation of* $f$ *such that* $g\colon (\mathfrak{X}_0,0) \to (B_0,0)$ *is a miniversal deformation of* $(X,0)$*. Let* $\mathcal{C}_{\varphi_0,0}$ *denote the cokernel of* $t'\tilde{f}\colon \mathcal{L}_{\mathfrak{X}_0/B_0,0} \to M_{\varphi_0,0}$*. If* $\dim_{\mathbb{C}} \mathcal{C}_{\varphi_0,0}/\mathfrak{m}_{B_0,0}\mathcal{C}_{\varphi_0,0} = 1$ *then* $f$ *is right* 0*-determined.*

*Proof.* As $(X,0)$ is non-smooth, any relative vector field $v \in \Theta_{\mathfrak{X}_0/B_0}$ vanishes at the singular point 0 so that 1 is a $\mathbb{C}$-basis for $\mathcal{C}_{\varphi_0,0}/\mathfrak{m}_{B,0}\mathcal{C}_{\varphi_0,0}$. Accordingly to the previous section, a miniversal deformation of $f$ is given by

$$g = g_0 \times \mathbb{I}_{\mathbb{C},0}\colon (\mathfrak{X}_0 \times \mathbb{C}) \longrightarrow (B_0 \times \mathbb{C}, 0)$$
$$F(x,y) = \tilde{f} + y$$

Let $h \in \mathfrak{m}_{X,0}$ and consider the deformation of $f$ given by $f + uh$ on $(X \times \mathbb{C}, 0)$ and the projection $p\colon (X \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$. As $\varphi = (F, g)$ is miniversal, we can find a commutative diagram




and reasoning as in the proof of theorem 2.4.1, we see that $\Phi$ is of the form $(\phi_u(x), \Psi(u))$ with $\phi_u$ an automorphism of $(X,0)$. Hence

$$\tilde{f} \circ \phi_u(x) + \Psi(u) = f(x) + uh(x)$$

and as both $f$ and $h$ belong to $\mathfrak{m}_{X,0}$, it follows $\Psi \equiv 0$ so that $f$ is right equivalent to $f + uh$ for all $u$. □

# Chapter 3

# The Milnor number

In this chapter we investigate the relation between the minimal number of parameters to versally unfold a function (i.e. the Tjurina number of $f$) and the number of critical points into which the original singularity splits (we will call this number the Milnor number). We give a sufficient condition for those two numbers to coincide and show that the condition is satisfied for the case of functions on space curves (or more generally, smoothable and unobstructed curves) and function on isolated complete intersection singularities. This provides a unified treatment of previously known results. To finish we interpret the Milnor number as the dimension as complex vector space of certain quotient of the dualising module of the variety.

## 3.1 Milnor and Tjurina numbers

Given a deformation $\varphi = (F, g)\colon (\mathfrak{X}, 0) \to (S \times B, 0)$ of a function germ $f\colon (X, 0) \to (S, 0)$ with an isolated singularity, the module $M_{\varphi,0}$ was introduced in the previous chapter as

$$M_{\varphi,0} = \frac{\Theta(F)_0}{tF(\Theta_{\mathfrak{X}/B,0})}$$

As remarked before, if $(X, 0) = (\mathbb{C}^n, 0)$ with coordinates $(x_1, \ldots, x_n)$, then $M_{\varphi,0}$ is simply the relative Jacobian algebra $\mathcal{O}_{\mathbb{C}^n \times B,0}/(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$. Its rank as $\mathcal{O}_{B,0}$-module is the so-called *Milnor number* of $f$, the module $M_{\varphi,0}$ is *free* over $\mathcal{O}_{B,0}$ and its rank equals the number of Morse critical points in a generic deformation of $f$.

We begin with an example to show, that if $(X, 0)$ is a singular germ, the rank of $M_{\varphi,0}$ is *not* in general independent of the deformation $\varphi$ and it does *not* necessarily coincide with the number of Morse critical points in a generic deformation of $f$.

**Example 3.1.1.** Let us consider the function $f = x_0 + x_1 + x_2 + x_3 + x_4$ on the germ $(X, 0)$ of the cone over the rational normal curve of degree 4. The germ $(X, 0)$ is defined by the $2 \times 2$-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}$$

The base space of the miniversal deformation of $(X,0)$ has embedding dimension 4 and is given by the union of a 3-dimensional linear subspace $B_1$ and a line $B_2$ intersecting transversally (see [29]). Over $B_1$, the total space $(\mathfrak{X}_1,0)$ is defined by the maximal minors of

$$\begin{pmatrix} x_0 & x_1+t_1 & x_2+t_2 & x_3+t_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}$$

where $(t_1,t_2,t_3)$ are the parameters of the deformation. Let us consider $\varphi_1 = (f_1,g_1)\colon (\mathfrak{X}_1,0) \to (S\times B,0)$ the deformation of $f$ defined by the extension $f_1$ of $f$ to $(\mathfrak{X}_1,0)$ given by the same linear function $x_0+x_1+x_2+x_3+x_4$. A calculation carried out with SINGULAR ([17]) shows the following:

1. $f$ has four Morse critical points on the smooth fibre of the map $g_1\colon (\mathfrak{X}_1,0)\to(B_1,0)$ whereas the dimension over $\mathbb{C}$ of $M_{\varphi_1,0}/\mathfrak{m}_{B_1,0}M_{\varphi_1,0}$ is 10.

2. If $\varphi_{1,1}$ denotes the induced deformation over the line $L_1=\{(t_1,0,0)\in B_1\}$ then $\dim_{\mathbb{C}} M_{\varphi_{1,1},0}/\mathfrak{m}_{L_1,0}M_{\varphi_{1,1},0} = 4$. Therefore $M_{\varphi_{1,1},0}$ is free of rank 4 over $\mathcal{O}_{L_1,0}$. The same holds for the deformation induced over $L_2 = \{(0,t_2,0)\in B_1\}$. However, if $L_3=\{(0,0,t_3)\in B_1\}$, then $\dim_{\mathbb{C}} M_{\varphi_{1,3},0}/\mathfrak{m}_{L_3,0}M_{\varphi_{1,3},0} = 5$

3. The deformation over $B_2$ is defined by the $2\times 2$-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2+s & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

   In this case, the trivial extension $f_2$ of $f$ to $(\mathfrak{X}_2,0)$ has 3 Morse critical points on a smooth fibre and $\dim_{\mathbb{C}} M_{\varphi_2,0}/\mathfrak{m}_{B_2,0}M_{\varphi_2,0}=3$ so that $M_{\varphi_2,0}$ is again free as $\mathcal{O}_{B_2,0}$-module.

In order to avoid the dependence of the dimension of $M_{\varphi,0}$ on the deformation of $f$ we use the miniversal deformation of $(X,0)$.

**Definition 3.1.1.** For a function $f\colon (X,0)\to(S,0)$ with an isolated singularity and let $\varphi=(F,g_0)\colon(\mathfrak{X}_0,0)\to(S\times B_0,0)$ a deformation of $f$ such that $g_0\colon(\mathfrak{X}_0,0)\to(B_0,0)$ is the miniversal deformation of $(X,0)$. We denote by $M_f$ the $\mathcal{O}_{X,0}$-module

$$M_f = \frac{M_{\varphi,0}}{\mathfrak{m}_{B_0,0}M_{\varphi,0}}$$

Its dimension over $\mathbb{C}$ will be denoted by $\mu(X/S,0)$ and called the *Milnor number of $f$*.

As two extensions of $f$ to $(\mathfrak{X}_0,B_0)$ differ by an element of $\mathfrak{m}_{B_0,0}\mathcal{O}_{\mathfrak{X}_0,0}$, this definition does not depend on the choice of $F$.

The following theorem gives sufficient conditions for $M_{\varphi,0}$ to be free as a module over the base space of the deformation and for its rank to coincide with the dimension of the base space of the miniversal deformation of $f$.

**Theorem 3.1.1.** *Let $f\colon(X,0)\longrightarrow(S,0)$ be a function with an isolated singularity and let $\varphi=(F,g)\colon(\mathfrak{X},0)\to(S\times B,0)$ be a 1-parameter deformation of $f$ (i.e., $(B,0)=(\mathbb{C},0)$). Assume that the following* extendability condition *is satisfied:*

any relative vector field $u \in \Theta_{X/S,0}$ can be extended to $u' \in \Theta_{\mathfrak{X}/S\times B,0}$.

*Then both $T^1_{\mathfrak{X}/S\times B,0}$ and $M_{\varphi,0}$ are free $\mathcal{O}_{B,0}$-modules. Moreover, if $T^2_{X,0} = 0$ and the generic fibre of $g$ is smooth, then their rank is equal and hence*

$$\mu(X/S,0) = \tau(X/S,0)$$

*Proof.* Let $y$ be a parameter in $(B,0)$. The exact sequence

$$0 \to \mathcal{O}_{\mathfrak{X},0} \xrightarrow{\cdot\, y} \mathcal{O}_{\mathfrak{X},0} \to \mathcal{O}_{X,0} \to 0$$

induces a long exact sequence:

$$\begin{aligned} 0 &\longrightarrow T^0_{\mathfrak{X}/S\times B,0} \xrightarrow{\cdot y} T^0_{\mathfrak{X}/S\times B,0} \longrightarrow T^0_{\mathfrak{X}/S\times B}(\mathcal{O}_{X,0}) \\ &\longrightarrow T^1_{\mathfrak{X}/S\times B,0} \xrightarrow{\cdot y} T^1_{\mathfrak{X}/S\times B,0} \longrightarrow T^1_{\mathfrak{X}/S\times B}(\mathcal{O}_{X,0}) \longrightarrow \dots \end{aligned} \tag{3.1}$$

It follows from the diagram

$$\begin{array}{ccc} (X,0) & \longrightarrow & (\mathfrak{X},0) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (S,0) & \longrightarrow & (S\times B,0) \end{array}$$

that $T^i_{\mathfrak{X}/S\times B}(\mathcal{O}_{X,0}) = T^i_{X/S,0}$.

The map $T^0_{\mathfrak{X}/S\times B,0} \to T^0_{X/S,0}$ is surjective as any relative vector field in $\Theta_{X/S,0} = T^0_{X/S,0}$ can be extended to $\Theta_{\mathfrak{X}/S\times B,0} = T^0_{\mathfrak{X}/S\times B,0}$. Hence the long exact sequence 3.1 contains the exact sequence:

$$0 \to T^1_{\mathfrak{X}/S\times B,0} \xrightarrow{\cdot y} T^1_{\mathfrak{X}/S\times B,0} \to T^1_{X/S,0} \to T^2_{\mathfrak{X}/S\times B,0} \xrightarrow{\cdot y} T^2_{\mathfrak{X}/S\times B,0} \to \dots \tag{3.2}$$

This shows that $T^1_{\mathfrak{X}/S\times B,0} \xrightarrow{\cdot y} T^1_{\mathfrak{X}/S\times B,0}$ is injective and hence $T^1_{\mathfrak{X}/S\times B,0}$ is flat over $\mathcal{O}_{B,0}$. As $(B,0) = (\mathbb{C},0)$, freeness is equivalent to flatness. The algebra $M_{\varphi,0}$ is a submodule of $T^1_{\mathfrak{X}/S\times B,0}$ so that is also free.

For the second statement, we first show that the condition $T^2_{X,0} = 0$ also implies $T^2_{X/S,0} = 0$. Associated to $\mathbb{C} \to \mathcal{O}_{S,0} \to \mathcal{O}_{X,0}$ we have a long exact sequence:

$$\dots \to T^i_{X/S,0} \to T^i_{X,0} \to T^i_S(\mathcal{O}_{X,0}) \to T^{i+1}_{X/S,0} \to \dots$$

As $(S,0)$ is smooth, $T^i_S(\mathcal{O}_{X,0}) = 0$ for $i \geq 1$, so that $T^i_{X/S,0} = T^i_{X,0}$ for $i \geq 2$. If the generic fibre of $g$ is a smooth, then $T^2_{\mathfrak{X}/S\times B,0}$ is annihilated by a power of the maximal ideal $\mathfrak{m}_{B,0}$, and hence it is Artinian. The exact sequence 3.2 then contains the short exact sequence:

$$0 \to T^1_{\mathfrak{X}/S\times B,0} \xrightarrow{\cdot y} T^1_{\mathfrak{X}/S\times B,0} \to T^1_{X/S,0} \to 0$$

It follows that $\operatorname{rk} T^1_{\mathfrak{X}/S\times B,0} = \dim_{\mathbb{C}}\, T^1_{X/S,0}$. To see that this is also the rank of $M_{\varphi,0}$ we write one more exact sequence:

$$0 \to M_{\varphi,0} \to T^1_{\mathfrak{X}/S\times B,0} \to T^1_{\mathfrak{X}/B,0} \to 0$$

and notice that $T^1_{\mathfrak{X}/B,0}$ is supported at 0. □

**Remark 3.1.1.** Note from the proof that the extendibility of relative vector fields is actually equivalent to the freeness of $T^1_{\mathfrak{X}/S\times B,0}$.

We briefly recall the notions of *critical space* and *discriminant space* of a analytic map $g\colon\ X \longrightarrow\ Y$ as in [24], Ch.4. Assume that all the fibres of $g$ have the same pure dimension $n$. Set-theoretically, the critical space $C_g$ is the set where either $X$ is singular or $g$ has a critical point (i.e. neither immersive nor submersive). It is equipped with the following (possible non reduced) analytic structure: assume that $X \subset\ U \subset\ \mathbb{C}^N$ and let $h_1, \dots, h_l$ generators of the ideal of $X$ in an open subset $U' \subset\ U$. The critical space $C_g$ is endowed with the analytic structure given by the ideal generated by the $(N-n)\times(N-n)$ minors of the Jacobian map of $(g, h_1, \dots, h_l)$. Assume now that $g$ is finite when restricted to $C_g$. The *discriminant space* $\Delta_g$ is, set-theoretically, the image $g(C_g)$ but endowed with the analytic structure $\mathcal{O}_B/\mathcal{F}_0(g_*\mathcal{O}_{C_g})$, where $\mathcal{F}_0(g_*\mathcal{O}_{C_g})$ denotes the *0-th Fitting ideal* of $g_*\mathcal{O}_{C_g}$. The reason to consider these analytic structures instead of those inherited from $X$ and $Y$ is that they are compatible with base change, whereas the latter may not be.

In the local case, if $g\colon (X,0) \to (\mathbb{C}^k, 0)$ has an isolated singularity, it is always possible to choose a representative $g\colon X \to Y$, such that $g|_{C_g}$ is finite. It then follows that the singular locus of $C_g$ is of dimension $\leq\ k$ and in fact, $C_g - X_{\text{sing}}$ is of pure dimension $k-1$. Moreover, if $C_g - X_{\text{sing}}$ is dense in $C_g$, then the discriminant is a hypersurface (or empty).

We recall that an analytic space $(X,0)$ is called *unobstructed* if $T^2_{X,0} = 0$ and *smoothable* if there exists a deformation with smooth generic fibre. Such a deformation is called a *smoothing*. If an isolated singularity is unobstructed and smoothable, it follows that both the base space and the generic fibre of its miniversal deformation are smooth.

After all these general remarks, we go back to our study of deformations of functions. We extract some corollaries from the previous theorem.

**Proposition 3.1.1.** *Let $f\colon (X,0) \to (S,0)$ be a function with an isolated singularity and $\varphi = (F,g)\colon (\mathfrak{X},0) \to (S\times B, 0)$ a deformation of $f$ such that $(B,0)$ and the generic fibre of $g$ are both smooth. Suppose that any relative vector field $u \in \Theta_{X/S,0}$ can be extended to a relative vector field $u' \in \Theta_{\mathfrak{X}/S\times B,0}$. Then*

$$0 \to M_{\varphi,0} \to T^1_{\mathfrak{X}/S\times B,0} \to T^1_{\mathfrak{X}/B,0} \to 0$$

*is a free resolution of $T^1_{\mathfrak{X}/B,0}$ as $\mathcal{O}_{B,0}$-module. As a consequence $T^1_{\mathfrak{X}/B,0}$ is Cohen-Macaulay $\mathcal{O}_{B,0}$ of dimension* $\dim B - 1$ *and the module of liftable vector field $\mathcal{L}_{\mathfrak{X}/B,0}$ is free.*

*Proof.* We can choose a representative of $g\colon \mathfrak{X} \to B$ such that $g|_{C_g}$ is finite. As $g$ is a smoothing, the discriminant $\Delta_g$ is a hypersurface (or empty if $(X,0)$ is smooth). We can then choose coordinates on $B$ such that no coordinate axis is contained in $\Delta_g$. For any of these axis $l$, the deformation induced from the inclusion $l \hookrightarrow B$ satisfies the hypothesis of the previous proposition. It follows $M_{\varphi,0}$ and $T^1_{\mathfrak{X}/S\times B,0}$ are free of rank $\mu(X/S,0)$ and the sequence is a free resolution. As $\mathcal{O}_{B,0}$ is a regular local ring, it follows from the Auschlander-Buchsbaum formula ([13], Th. 19.9) that $\operatorname{depth}_{\mathcal{O}_{B,0}} T^1_{\mathfrak{X}/B,0} = k-1$ and hence $T^1_{\mathfrak{X}/B,0}$ is Cohen-Macaulay as $\mathcal{O}_{B_0,0}$-module.

As $T^1_{\mathfrak{X}/B,0}$ is Cohen-Macaulay of dimension dim $B-1$, the fact that $\mathcal{L}_{\mathfrak{X}/B}$ is free follows from the exactness of the sequence

$$0 \to \mathcal{L}_{\mathfrak{X}/B} \to \Theta_{\mathfrak{X}/B,0} \to T^1_{\mathfrak{X}/B} \to 0$$

□

We recall that an analytic space $X$ is called *Stein* if the cohomology groups $H^i(X;\mathcal{F}) = 0$ for all $i > 0$ and all coherent sheaf $\mathcal{F}$. An analytic map $g\colon X \to Y$ is called Stein if the inverse image of any Stein subspace is again Stein. The main feature of such maps is that the direct image functor $g_*$ is exact. For functions with isolated singularities, it is always possible to choose a Stein representative ([24], 8.3).

**Corollary 3.1.1.** *In the conditions of the previous proposition, the Milnor number $\mu(X/S,0)$ is equal to the number of critical points of $F$ restricted to a smooth fibre of $g$, counted with multiplicity.*

*Proof.* Choosing a Stein representative of $g$, we have a short exact sequence of sheaves of $\mathcal{O}_{B,0}$-modules:

$$0 \to g_*M_\varphi \to g_*T^1_{\mathfrak{X}/S\times B} \to g_*T^1_{\mathfrak{X}/B} \to 0$$

If the fibre $X_y = g^{-1}(y)$ is smooth, then the dimension of the $\mathbb{C}$-vector space $g_*M_{\varphi,y}/\mathfrak{m}_B g_*M_{\varphi,y}$ is clearly the number of critical points of $F|_{X_y}$, counted with multiplicity. As $g_*M_{\varphi,0}$ is free, the result follows. □

Given a hypersurface $(D,0) \in (B,0)$ defined by an equation $\delta$, we consider the module of *logarithmic vector fields along* $D$ defined by the condition

$$\Theta(\log D) = \{u \in \Theta_B :\ u(h) \in (h)\}$$

It consists of those vector fields tangent to the regular part of $(D,0)$ and it is a *reflexive module*, i.e., isomorphic to its bidual. We refer to the original article [31] for a study of these modules.

**Corollary 3.1.2.** *In the conditions of the proposition 3.1.1, if the fibre $X_b = g^{-1}(b)$ for generic $b \in \Delta_g$ has only a quadratic singularity, then*

$$\mathcal{L}_{\mathfrak{X}/B} = \Theta(\log \Delta_g)$$

*As a consequence, $\Delta_g$ is a free divisor.*

*Proof.* We have an inclusion $\mathcal{L}_{\mathfrak{X}/B} \subset \Theta(\log \Delta_g)$, as for a vector field to be liftable it must be tangent to $\Delta_g$. An explicit (and easy) calculation shows that $\mathcal{L}_{\mathfrak{X}/B,b} = \Theta(\log\ \Delta_g)_b$ for those points $b \in \Delta_g$ where $X_b$ has only one quadratic singularity. Hence, if we write a short exact sequence:

$$0 \to \mathcal{L}_{\mathfrak{X}/B} \to \Theta(\log \Delta_g) \to \mathcal{C} \to 0$$

the module $\mathcal{C}$ is supported on a subset of codimension at least 2. Then $\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{C},\mathcal{O}_B) = 0$ (this is easy, we refer to [24], Lemma 9.2, for an argument based on depth). As $\mathcal{L}_{\mathfrak{X}/B}$ is free and $\Theta(\log\ \Delta_g)$ is reflexive, the result follows dualising twice the above exact sequence. □

## 3.2 Functions on curves

The case where $(X,0)$ is a (reduced) one-dimensional space is specially simple: if $f$ is a function with an isolated critical point, the fibres of $f$ are just points, so that $\Theta_{X/S,0} = 0$ and the extendibility condition holds trivially. A function $f\colon (X,0) \to (S,0)$ with an isolated singularity is just a finite map. If $(X,0)$ is unobstructed and smoothable, we have the equality

$$\mu(X/S,0) = \tau(X/S,0)$$

This is the main result of [27], first conjectured by V. Goryunov in the preprint versions of [16]. The definition of the Milnor number in [27] is a priori different from ours. The authors defined the number $\mu(X/S,0)$ in terms of the *dualising module* $\omega_{X,0}$.

For a Cohen-Macaulay variety $(X,0)$ of dimension $n$ embedded in $(\mathbb{C}^N,0)$, the module $\omega_{X,0}$ is defined as:

$$\omega_{X,0} = \operatorname{Ext}^{N-n}_{\mathcal{O}_{\mathbb{C}^N,0}}(\mathcal{O}_{X,0}, \Omega^N_{\mathbb{C}^N,0})$$

where $\Omega^N_{\mathbb{C}^N,0}$ denotes the module of Kähler $N$-forms in $(\mathbb{C}^N,0)$.

If $c = \operatorname{codim}\ (X,0) = N-n$, the ring $\mathcal{O}_{X,0}$ has a resolution by free $\mathcal{O}_{\mathbb{C}^N,0}$-modules of length $c$:

$$0 \to F_c \xrightarrow{M_c} \ldots \xrightarrow{M_2} F_1 \xrightarrow{M_1} F_0 = \mathcal{O}_{\mathbb{C}^N,0} \to \mathcal{O}_{X,0} \to 0$$

Applying $\operatorname{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(-,\Omega^N_{\mathbb{C}^N,0})$, we get a free resolution of the dualising module $\omega_{X,0}$ ([13], Cor. 21.16):

$$0 \to F_0^* \xrightarrow{M_0^t} \ldots \xrightarrow{M_{c-2}^t} F_{c-1}^* \xrightarrow{M_c^t} F_c^* \to \omega_{X,0} \to 0$$

A $n$-form $\alpha \in \Omega^n_{\mathbb{C}^N,0}$ can be multiplied with

$$\frac{1}{c!} dM_1 \wedge \ldots \wedge dM_c$$

to obtain an element of:

$$F_c^* = \operatorname{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(F_c, \Omega_{\mathbb{C}^N,0})$$

This element represents a class in $\omega_{X,0}$. It can be shown that this yields a map, known as the *class map* ([2]):

$$\operatorname{cl}\colon \Omega^n_{X,0} \longrightarrow \omega_{X,0}$$

**Remark 3.2.1.** For curves there is another interpretation of both the dualising module and the class map, in terms of meromorphic forms (see [6]). Although it will not be used until the next chapter, we briefly recall it here. Let $n\colon \overline{(X,0)} \to (X,0)$ denote the normalization of $(X,0)$ (here $\overline{(X,0)}$ denotes a multigerm), and $\Omega_{\overline{X,0}}(*)$ denotes the module of meromorphic forms on $\overline{(X,0)}$ with poles at most at $n^{-1}(0)$. Then

$$\omega_{X,0} = \{\alpha \in n_*\Omega_{\overline{X,0}}(*)\colon \sum_{x\in n^{-1}(0)} \operatorname{Res}_x(h\circ n)\alpha = 0\ \forall\ h \in \mathcal{O}_{X,0}\}$$

Here $\mathrm{Res}_x$ is the usual residue of a meromophic form at the smooth point $x \in n^{-1}(0)$. The class map in this context simply interprets a holomorphic 1-form as a meromorphic 1-form.

The Milnor number of $f\colon (X,0) \longrightarrow (S,0)$ according to D. Mond and D. van Straten ([27]), is

$$\mu_{MvS}(X/S,0) = \dim_{\mathbb{C}} \frac{\omega_{X,0}}{\mathcal{O}_{X,0}df}$$

It clearly coincides with the Milnor number as defined previously in the case of smooth $(X,0)$. The authors of [27] show that for $(X,0)$ unobstructed and smoothable:

$$\mu_{MvS}(X/S,0) = \tau(X/S,0)$$

It follows that in this case it indeed coincides with $\mu(X/S,0)$. This in particular holds for the case of space curves, i.e. curves $(X,0) \hookrightarrow (\mathbb{C}^3,0)$. Such a curve is a Cohen-Macaulay space of codimension 2. As such, there is a resolution by $\mathcal{O}_{\mathbb{C}^3,0}$-modules of length 2:

$$0 \to F_2 \xrightarrow{M} F_1 \xrightarrow{\Delta} F_0 = \mathcal{O}_{\mathbb{C}^3,0} \to \mathcal{O}_{X,0} \to 0$$

where rk $F_2 = l-1$, rk $F_1 = l$ and the entries of $\Delta$ are given by the maximal minors $\Delta_i$ of $M$ obtained by deleting the $i$-th row (see [7]).

Deformations of Cohen-Macaulay varieties of codimension 2 are studied in [34]: if $\dim(X,0) = n \leq 3$, they are smoothable, their miniversal deformations have smooth base space and their total space is defined by the maximal minors of a perturbation of the matrix $M$.

It is proved in [27] that the image of the dual of the class map:

$$\mathrm{cl}^*\colon \omega_{X,0} \to \Theta_{X,0}$$

is generated by the vector fields defined by:

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial \Delta_i}{\partial x_1} & \frac{\partial \Delta_1}{\partial x_2} & \frac{\partial \Delta_i}{\partial x_3} \\ \frac{\partial \Delta_j}{\partial x_1} & \frac{\partial \Delta_j}{\partial x_2} & \frac{\partial \Delta_j}{\partial x_3} \end{vmatrix}, \quad 1 \leq i < j \leq l$$

and the Milnor number can be computed by differentiating $f$ with respect to those vector fields:

$$\mu(X/S,0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,0}}{\omega^*_{X,0}(f)}$$

The next propostion shows that in fact the right hand side of the above equality is $M_f$:

**Proposition 3.2.1.** *For a function $f\colon (X,0) \to (S,0)$ with an isolated singularity on a (reduced) space curve,*

$$M_f \simeq \frac{\mathcal{O}_{X,0}}{\omega^*_{X,0}(f)}$$

*Proof.* Let $g_0\colon (\mathfrak{X}_0, 0) \to (B_0, 0) = (\mathbb{C}^k, 0)$ be the miniversal deformation of $(X, 0)$ and consider the *relative dualising module* $\omega_{\mathfrak{X}_0/B_0,0}$ defined analogously to $\omega_{X,0}$ as

$$\omega_{\mathfrak{X}_0/B_0,0} = \operatorname{Ext}^2_{\mathcal{O}_{\mathbb{C}^{3+k},0}}(\mathcal{O}_{\mathfrak{X}_0,0}, \Omega^3_{\mathbb{C}^{3+k}/\mathbb{C}^k,0})$$

We also have a resolution of $\mathcal{O}_{\mathfrak{X}_0,0}$ by free $\mathcal{O}_{\mathbb{C}^{3+k},0}$-modules:

$$0 \to \tilde{F}_2 \xrightarrow{\tilde{M}} \tilde{F}_1 \xrightarrow{\tilde{\Delta}} \tilde{F}_0 = \mathcal{O}_{\mathbb{C}^{3+k},0} \to \mathcal{O}_{\mathfrak{X}_0,0} \to 0$$

and a relative class map:

$$\mathrm{cl}_{\mathfrak{X}_0/B_0}\colon \Omega_{\mathfrak{X}_0/B_0} \longrightarrow \omega_{\mathfrak{X}_0/B_0}$$

given as before, but using the relative differential $d_{\mathfrak{X}_0/B_0}$ instead of $d$. Note that both the resolution of $\mathcal{O}_{\mathfrak{X}_0,0}$ and $\mathrm{cl}_{\mathfrak{X}_0/B_0}$ become the corresponding objects for $(X, 0)$ taking modulo $\mathfrak{m}_{B_0,0}$.

As the class map cl is an isomorphism if the curve is smooth, both the kernel and cokernel of $\mathrm{cl}_{\mathfrak{X}_0/B_0}$ are supported on a subset of codimension at least 2. The dual $\mathrm{cl}^*_{\mathfrak{X}_0/B_0}$ is therefore an isomorphism ([38]):

$$\omega^*_{\mathfrak{X}_0/B_0,0} \xrightarrow{\simeq} \Theta_{\mathfrak{X}_0/B_0,0}$$

We now use an explicit calculation like in [27] to compute the image of $\mathrm{cl}^*_{\mathfrak{X}_0/B_0}$. Given $\tilde{M}$, we consider the map:

$$\Gamma\colon \wedge^2 \tilde{F}_1 \longrightarrow \tilde{F}_2$$

whose entries are the $(l-2) \times (l-2)$-minors of $\tilde{M}$. The composition:

$$\wedge^2 \tilde{F}_1 \xrightarrow{\Gamma} \tilde{F}_2 \xrightarrow{\tilde{M}} \tilde{F}_1$$

maps the generator $e_i \wedge e_j \in \wedge^2 \tilde{F}_1$ to the Koszul relation $\tilde{\Delta}_i e_j - \tilde{\Delta}_j e_i$. The cokernel of $\Gamma$ is then the module $H_1 = R/R_0$ of relations among $\Delta_i$ modulo Koszul relations.

On the other hand, we have the map $\wedge^2 \tilde{M}\colon \wedge^2 \tilde{F}_2 \to \wedge^2 \tilde{F}_1$ and the complex:

$$0 \to \wedge^2 \tilde{F}_2 \xrightarrow{\wedge^2 \tilde{M}} \wedge^2 \tilde{F}_1 \xrightarrow{\Gamma} \tilde{F}_2 \to H_1 \to 0$$

This complex is exact (see [27] and the references therein). In particular $H_1$ is Cohen-Macaulay and torsion free.

Note that the composition:

$$\wedge^2 \tilde{F}_1 \xrightarrow{\Gamma} \tilde{F}_2 \xrightarrow{\tilde{M}} \tilde{F}_1$$

is zero modulo $\tilde{\Delta}_1, \dots, \tilde{\Delta}_l$. On the other hand, applying $\mathrm{Hom}(-, \mathcal{O}_{\mathfrak{X}_0,0})$ to the free resolution of $\omega_{\mathfrak{X}_0/B_0,0}$, we obtain the dual $\omega^*_{\mathfrak{X}_0/B_0,0}$ as a kernel:

$$0 \to \omega^*_{\mathfrak{X}_0/B_0,0} \to \tilde{F}_2 \otimes \mathcal{O}_{\mathfrak{X}_0,0} \xrightarrow{\tilde{M}} \tilde{F}_1 \otimes \mathcal{O}_{\mathfrak{X}_0,0}$$

It follows that the map $\Gamma$ factorises through $\omega^*_{\mathfrak{X}_0/B_0,0}$. The cokernel of $\Gamma$ injects into $H_1$ and it is supported on the critical locus of $g_0$. As $H_1$ is torsion free, it follows we have a surjection

$$\Gamma\colon \wedge^2 \tilde{F}_1 \otimes \mathcal{O}_{\mathfrak{X}_0,0} \longrightarrow \omega^*_{\mathfrak{X}_0/B_0,0} \to 0$$

Back to the class map, it is now possible to carry out an explicit description of $\Theta_{\mathfrak{X}_0/B_0,0}$ using the image of the generators of $\wedge^2 \tilde{F}_1 \otimes \mathcal{O}_{\mathfrak{X}_0,0}$ in $\omega^*_{\mathfrak{X}_0/B_0,0}$. For a relative differential $\alpha \in \Omega_{\mathbb{C}^{3+k}/\mathbb{C}^k,0}$, the element

$$\frac{1}{2}\alpha \wedge\ d_{\mathfrak{X}_0/B_0}\tilde{\Delta} \wedge\ d_{\mathfrak{X}_0/B_0}\tilde{M}$$

is interpreted as an element of $\tilde{F}_2^* \otimes \Omega^3_{\mathbb{C}^{3+k}/\mathbb{C}^k,0}$ representing a class in $\omega_{\mathfrak{X}_0/B_0,0}$. Fixing a relative volume form $dx_1 \wedge\ dx_2 \wedge\ dx_3$, we can think of $\phi \in \omega_{\mathfrak{X}_0/B_0,0}$ as a vector field in $\Theta_{\mathfrak{X}_0/B_0,0}$ that acts as:

$$\phi(\mathrm{cl}_{\mathfrak{X}_0/B_0}(dx_1))\frac{\partial}{\partial x_1} + \phi(\mathrm{cl}_{\mathfrak{X}_0/B_0}(dx_2))\frac{\partial}{\partial x_2} + \phi(\mathrm{cl}_{\mathfrak{X}_0/B_0}(dx_3))\frac{\partial}{\partial x_3}$$

For the generators $e_i \wedge\ e_j$ of $\omega^*_{\mathfrak{X}_0/B_0,0}$ and $\alpha \in \Omega_{\mathfrak{X}_0/B_0,0}$, we have (summation understood over the repeated indexes):

$$\Gamma(e_i \wedge\ e_j)(\mathrm{cl}_{\mathfrak{X}_0/B_0}(\alpha)) = \frac{1}{2}\alpha \wedge\ d_{\mathfrak{X}_0/B_0}\tilde{\Delta}_p \wedge\ d_{\mathfrak{X}_0/B_0}(M^p_k)\Gamma^k_{ij}$$

where $\Gamma^k_{ij}$ denotes the $l-1$ minor of $\tilde{M}$ obtained by deleting the $i$-th and $j$-th rows and the $k$-th column.

The relation $\tilde{\Delta}_p \tilde{M}^p_{ij} = 0$ implies

$$d_{\mathfrak{X}_0/B_0}\tilde{\Delta}_p \wedge\ d_{\mathfrak{X}_0/B_0}(\tilde{M}^p\Gamma^k_{ij}) = d_{\mathfrak{X}_0/B_0}\tilde{\Delta}_p \wedge\ d_{\mathfrak{X}_0/B_0}(\tilde{M}^p)\Gamma^k_{ij} \ (\mathrm{mod}\ (\tilde{\Delta}_1, \ldots, \tilde{\Delta}_l))$$

and from $\tilde{M}^p_k \Gamma^k_{ij} = \tilde{\Delta}_i \delta^p_j - \tilde{\Delta}_j \delta^p_i$, it follows that

$$d_{\mathfrak{X}_0/B_0}\tilde{\Delta}_p \wedge\ d_{\mathfrak{X}_0/B_0}(\tilde{M}^p\Gamma^k_{ij}) = 2d_{\mathfrak{X}_0/B_0}\tilde{\Delta}_j \wedge\ d_{\mathfrak{X}_0/B_0}\tilde{\Delta}_i$$

Finally we obtain:

$$\Gamma(e_i \wedge\ e_j)(\mathrm{cl}_{\mathfrak{X}_0/B_0}(\alpha)) = \alpha \wedge\ d_{\mathfrak{X}_0/B_0}\tilde{\Delta}_i \wedge\ d_{\mathfrak{X}_0/B_0}\tilde{\Delta}_j$$

so that $\Theta_{\mathfrak{X}_0/B_0,0}$ is generated by the vector fields:

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial \tilde{\Delta}_i}{\partial x_1} & \frac{\partial \tilde{\Delta}_i}{\partial x_2} & \frac{\partial \tilde{\Delta}_i}{\partial x_3} \\ \frac{\partial \tilde{\Delta}_j}{\partial x_1} & \frac{\partial \tilde{\Delta}_j}{\partial x_2} & \frac{\partial \tilde{\Delta}_j}{\partial x_3} \end{vmatrix},\ 1 \le i < j \le l$$

and $M_f$ is:

$$\mathcal{O}_{X,0}/(\{\frac{\partial(f, \Delta_i, \Delta_j)}{\partial(x_1, x_2, x_3)}:\ 1 \le\ i < j \le\ l\})$$

□

Finally, we remark that the Cohen-Macaulay property of $T^1_{\mathfrak{X}_0/B_0,0}$ for space curves is also proved in [38]. As a consequence, the author deduces that the discriminant $\Delta_g$ is a free divisor, for the generic singularity over $\Delta_g$ is a local complete intersection.
Unfoldings of functions on space curves have been studied in [16]. The author classified all the simple functions according to (a priori) different equivalence relation. We illustrate the construction of the miniversal deformation of a function on a space curve with the following example.

**Example 3.2.1.** Let $(X,0) \in (\mathbb{C}^3,0)$ be the union of the three coordinate axis defined by the equations

$$\Delta_1 = x_2x_3 \ \Delta_2 = x_1x_3 \ \Delta_3 = x_1x_2$$

which are the maximal minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_2 & x_3 \end{pmatrix}$$

The vector fields extendable to the versal deformation are given by

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ 0 & x_3 & x_2 \\ x_2 & x_1 & 0 \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{vmatrix}$$

Modulo the equations of $(X,0)$ they become (up to sign)

$$x_3^2\frac{\partial}{\partial x_3}, x_2^2\frac{\partial}{\partial x_2}, x_1^2\frac{\partial}{\partial x_1}$$

Any function that is non-constant on any branch of $(X,0)$ can be written as $f(x_1,x_2,x_3) = x_1^p + x_2^q + x_3^r$ for some $p,q$ and $r$. Hence

$$M_f = \mathcal{O}_{\mathbb{C}^3,0}/(x_2x_3, x_1x_3, x_1x_2, rx_3^{r+1}, qx_2^{q+1}, px_1^{p+1})$$

so that $\mu(X/S) = \tau(X/S) = p+q+r+1$ and a $\mathbb{C}$-basis of $M_f$ is given by $1, x_1, \dots, x_1^p, x_2, \dots, x_2^q, x_3, \dots, x_3^r$. To construct the miniversal deformation of $f$, we first take the miniversal deformation of $(X,0)$. Such a deformation is well-known: the total space $(\mathfrak{X}_0,0)$ is the 5-dimensional variety in $(\mathbb{C}^6,0)$ defined by the maximal minors of the matrix

$$\begin{pmatrix} x & y & \alpha \\ \beta & y+\gamma & z \end{pmatrix}$$

and $g\colon (\mathfrak{X}_0,0) \to (\mathbb{C}^3,0)$ is the projection of the parameters $(\alpha,\beta,\gamma)$. The discriminant is defined by the equation $\Delta : \alpha\beta\gamma = 0$ and a free basis for $\Theta(\log \Delta)$ is given by

$$\alpha\frac{\partial}{\partial\alpha}, \beta\frac{\partial}{\partial\beta}, \gamma\frac{\partial}{\partial\gamma}$$

which lift to

$$x_3\frac{\partial}{\partial x_3} + \alpha\frac{\partial}{\partial\alpha}, x_1\frac{\partial}{\partial x_1} + \beta\frac{\partial}{\partial\beta}, x_2\frac{\partial}{\partial x_2} + \gamma\frac{\partial}{\partial\gamma},$$

Differentiating $f$ with respect to these vector fields we get the terms $x_1^p, x_2^q, x_3^r$ so that the miniversal deformation is given by the function

$$\begin{aligned} F = \quad & x_1^p + a_{p-1}x_1^{p-1} + \ldots + a_1x_1 + \\ & x_2^q + b_{q-1}x_2^{q-1} + \ldots + b_1x_2 + \\ & x_3^r + c_{r-1}x_3^{r-1} + \ldots + c_1x_3 + d \end{aligned}$$

## 3.3 Functions on isolated complete intersection singularities

The next proposition computes the relative vector fields for a function on a isolated complete intersection singularity. It will follow that in this case the extendibility condition is satisfied.

**Proposition 3.3.1.** *Let $f\colon (X,0) \to (S,0)$ be a function with an isolated singularity on a complete intersection $(X,0)$ defined by a regular sequence $(h_1,\dots,h_l)$ in $(\mathbb{C}^N,0)$. If $\tilde f$ denotes a representative of $f$ in $\mathcal{O}_{\mathbb{C}^N,0}$, the module $\Theta_{X/S,0}$ is generated by the maximal minors of the matrix:*

$$\begin{pmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_N} \\ \frac{\partial \tilde f}{\partial x_1} & \cdots & \frac{\partial \tilde f_1}{\partial x_N} \\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_N} \\ \cdots & \cdots & \cdots \\ \frac{\partial h_l}{\partial x_1} & \cdots & \frac{\partial h_l}{\partial x_N} \end{pmatrix}$$

*Proof.* Consider the map:

$$\Phi = (\tilde f, h_1, \dots, h_l)\colon (\mathfrak{X},0) \longrightarrow (\mathbb{C}^{1+l},0)$$

The module of relative vector fields $\Theta_{X/S,0}$ is the kernel of the map of free $\mathcal{O}_{\mathbb{C}^N,0}$-modules:

$$\Theta_{\mathbb{C}^N,0} \otimes \mathcal{O}_{\mathfrak{X},0} \xrightarrow{t\Phi} \Theta_{\mathbb{C}^{1+l},0} \otimes \mathcal{O}_{\mathfrak{X},0}$$

The ideal generated by the maximal minors of the Jacobian matrix of $\Phi$ defines the critical locus $C_f = \{0\}$. As $(X,0)$ is a complete intersection, it is in particular Cohen-Macaulay. Then:

$$\begin{aligned} n &= C_g\text{- depth}\,\mathcal{O}_{X,0} \\ &= \operatorname{codim} C_f = \dim\ (X,0) \\ &= \operatorname{rk}\Theta_{\mathbb{C}^N,0} - \operatorname{rk}\Theta_{\mathbb{C}^{1+l},0} + 1 \end{aligned}$$

It follows that the Eagon-Northcott complex is exact ([12]) and thus the kernel is generated by the above vector fields. □

Note that those vector fields are always extendable. One might expect a similar result to hold for a function $f\colon (X,0) \to (S,0)$ with an isolated singularity defined on a Cohen-Macaulay space. If $(X,0)$ is defined by $h_1,\dots,h_l$ in $(\mathbb{C}^N,0)$, and $\dim\ (X,0) = n$, the critical locus of $f$ will be defined by the minors of order $N-(n-1)$ of the Jacobian matrix of $\Phi = (\tilde f, h_1,\dots,h_k)$:

$$\Theta_{\mathbb{C}^N,0} \otimes \mathcal{O}_{\mathfrak{X},0} \xrightarrow{t\Phi} \Theta_{\mathbb{C}^{1+l},0} \otimes \mathcal{O}_{\mathfrak{X},0}$$

The quotient ring $\mathcal{O}_{C_f}$ would have a free resolution given by the generalisations of the Eagon-Northcott complex (see [23] and [30]) if:

$$\begin{aligned} n = C_f - \operatorname{depth}\mathcal{O}_{X,0} &= \operatorname{codim}\ C_f = \dim\ (X,0) \\ &= (\operatorname{rk}\Theta_{\mathbb{C}^N,0} - \text{ord. of minors} + 1)(\operatorname{rk}\Theta_{\mathbb{C}^{1+l},0} - \text{ord. of minors} + 1) \end{aligned}$$

i.e. if $n = n(n+l-N+1)$. It would follow that $N = n+l$ and $(X,0)$ is a complete intersection.

We can use the same argument as that of the above proposition to compute the relative vector fields of the miniversal deformation of $(X,0)$.

**Proposition 3.3.2.** *If* $g\colon (\mathfrak{X},0) = (\mathbb{C}^{n+l},0) \to (\mathbb{C}^l,0) = (B,0)$ *is a deformation of an isolated complete intersection singularity, then* $\Theta_{\mathfrak{X}/B,0}$ *is generated by the maximal minors of*

$$\begin{pmatrix} \dfrac{\partial}{\partial x_1} & \cdots & \dfrac{\partial}{\partial x_{n+l}} \\ \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_{n+l}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial g_l}{\partial x_1} & \cdots & \dfrac{\partial g_l}{\partial x_{n+l}} \end{pmatrix}$$

*Proof.* The critical locus of $g$ has dimension $l-1$ and is defined by the maximal minors of $tg\colon \Theta_{\mathfrak{X},0} \to \Theta_{B,0} \otimes \mathcal{O}_{\mathfrak{X},0}$. Again:

$$C_g - \operatorname{depth} \mathcal{O}_{\mathfrak{X},0} = \operatorname{codim}\ C_g = n+1 = (n+l) - l + 1$$

and the result follows from the exactness of the Eagon-Northcott complex. □

**Corollary 3.3.1.** *For a function* $f\colon (X,0) \to (S,0)$ *with an isolated singularity on a complete intersection of dimension* $n$*:*

$$M_f \simeq \frac{\omega_{X,0}}{df \wedge \Omega^{n-1}_{X,0}}$$

*Proof.* If $(X,0) \hookrightarrow (\mathbb{C}^{n+l},0)$ is defined by a regular sequence $g_1,\dots,g_l$:

$$\omega_{X,0} = \operatorname{Ext}^l_{\mathcal{O}_{\mathbb{C}^{n+l},0}}(\mathcal{O}_{X,0}, \Omega^{n+l}_{\mathbb{C}^{n+l},0})$$

As $g_1,\dots,g_l$ is regular, the Koszul complex of $g_1,\dots,g_l$:

$$0 \to \wedge^l \mathcal{O}^l_{\mathbb{C}^{n+l},0} \to \ldots \to \wedge^2 \mathcal{O}^l_{\mathbb{C}^{n+l},0} \to \mathcal{O}^l_{\mathbb{C}^{n+l},0} \to \mathcal{O}_{\mathbb{C}^{n+l},x} \to 0$$

is a free resolution of $\mathcal{O}_{X,0}$ as $\mathcal{O}_{\mathbb{C}^{n+l},0}$-module, so that $\omega_{X,0} \simeq \mathcal{O}_{X,0}$. On the other hand, exterior multiplication with $dg_1 \wedge \ldots \wedge\ dg_l$ gives the class map:

$$\mathrm{cl}\colon \Omega^n_{X,0} \longrightarrow \omega_{X,0}$$

and fixing a volume form, we obtain an isomorphism:

$$\Omega^{n+l}_{\mathbb{C}^{n+l},0} \otimes \mathcal{O}_{X,0} \xrightarrow{\simeq} \omega_{X,0}$$

This ismorphism identifies the submodule $df \wedge \Omega^{n-1}_{X,0}$ with

$$\mathcal{O}_{X,0}/(\{\frac{\partial(f,g_1,\dots,g_l)}{\partial(x_{i_1},\dots,x_{i_{l+1}})}\colon\ 1 \le i_1 < \ldots < i_{l+1} \le n+l\})$$

and the result follows form the previous proposition. □

There is another remarkable interpretation of $\mu(X/S,0)$. Recall that for an isolated complete intersection the Milnor fibre, that is, the generic fibre of a small enough representative of a smoothing of $(X,0)$, has the homotopy type of a bouquet of $n$-spheres $\mathbb{S}^n \vee \ldots \vee \mathbb{S}^n$. The number of spheres occurring in the bouquet is called the *Milnor number* $(X,0)$ and denoted by $\mu(X,0)$.

Note that that the 0-fibre of $f$ is again a complete intersection singularity, say $(Y,0)$, and $f$ is a smoothing of $(Y,0)$.

**Corollary 3.3.2.** *For a function with an isolated singularity on an isolated complete intersection singularity,*

$$\mu(X/S,0) = \mu(X,0) + \mu(Y,0)$$

*Proof.* ([37] or [24] 5.11) Multiplying by some $(\mathbb{C}^k,0)$ both the total and base space of the miniversal deformation of $(X,0)$, we can obtain a miniversal deformation of $f$. Denote it by $\varphi = (F,g)\colon (\mathfrak{X},0) \longrightarrow (S \times B,0)$. The key point is that we can choose a representative of $\varphi\colon \mathfrak{X} \to S \times B$ such that the composition with the projection $S \times B \to B$ is a representative of $g$ whose generic fibre is a Milnor fibre of $(X,0)$. As $f$ has an isolated singularity, the line $S\times\{0\}$ intersects the discriminant $\Delta_g$ only at the origin. As $\varphi$ is a versal deformation of $f$, it is also a versal deformation of its fibre $Y$ over the origin $0 \in S \times B$ and hence its discriminant $\Delta_\varphi$ is reduced. The multiplicity of intersection of $S \times \{0\}$ and $\Delta_\varphi$ is therefore given by:

$$\mathcal{O}_{\mathfrak{X},0}/(\left\{\frac{\partial(F,g_1,\ldots,\ g_l)}{\partial(x_{i_1},\ldots,x_{i_{l+1}})}\colon\ 1 \le\ i_1 < \ldots\ < i_{l+1} \le\ n+l\right\}, g_1,\ldots,g_l)$$

and hence is equal to $\mu(X/S,0)$.

For a generic $y \in B - \Delta_g$, the line $S\times\{y\}$ will meet $\Delta_\varphi$ at $\mu(X/S,0)$ regular points, one for each quadratic singularity of $F$ on $X_y$. If $(s,y) \in S \times B - \Delta_\varphi$, the fibre $X_y$ is obtained from $Y_{s,y}$ by attaching $\mu(X/S,0)$ $n$-spheres. Therefore the only (possibly) non-zero terms of the exact (reduced) homology sequence of the pair $(X_y, Y_{s,y})$ are:

$$\ldots \to 0 \to H_n(X_y) \to H_{n-1}(X_y, Y_{s,y}) \to \tilde{H}_{n-1}(Y_{s,y}) \to 0 \to \ldots$$

have respectively ranks $\mu(X,0)$, $\mu(X/S,0)$ and $\mu(Y,0)$ . The result then follows. $\square$

**Remark 3.3.1.** Note that the proof shows that $\mu(X/S) = H_{n-1}(X_s, Y_{s,y})$. This can be understood as a generalization of the Milnor number of a hypersurface as the rank of the vanishing homology.

**Corollary 3.3.3.** *Let $f\colon (X,0) \to (S,0)$ be a function with an isolated singularity on a complete intersection, $g_0\colon (\mathfrak{X}_0,0) \to (S,0)$ the miniversal deformation of $(X,0)$ and $\tilde{f}$ an extension of $f$ to $(\mathfrak{X}_0,0)$. The following statements are equivalent:*

1. *$\varphi = (F = \tilde{f} + c, g_0 \times \mathbb{I}_{\mathbb{C}})\colon (\mathfrak{X}_0 \times \mathbb{C},0) \longrightarrow (S \times B_0 \times \mathbb{C},0)$ is the miniversal deformation of $f$,*

2. *$f$ defines a quadratic singularity in $(X,0)$,*

3. *there exists an embedding* $(X,0) \hookrightarrow (\mathbb{C}^{n+1},0)$ *such that* $(X,0)$ *is defined by* $x_1^{\mu+1} + x_2^2 + \ldots + x_{n+1}^2$, $\mu = \mu(X,0)$, *and* $f$ *is the restriction to* $(X,0)$ *of a function in* $(\mathbb{C}^{n+1},0)$ *with* $\frac{\partial f}{\partial x_1}(0) \neq 0$

*Proof.* (1) $\Leftrightarrow$ (2). According to the previous proposition and 2.6.2, $\varphi$ is miniversal if and only if

$$\mu(X,0) + \mu(Y,0) = \tau(X,0) + 1$$

As $\mu(Y,0) \geq 1$ and $\mu(X,0) \geq \tau(X,0)$ (for example [24], pg. 174), the above equation is equivalent to $\mu(Y,0) = 1$, which characterizes the quadratic singularity.

(1) $\Rightarrow$ (3). Let us embed $(X,0)$ in some $(\mathbb{C}^{n+k},0)$ by equations $(h_1,\ldots,h_k)$. Again according to 2.6.2, if $\varphi$ is the miniversal defined deformation of $f$, then $f$ is 0-determined, so that it can be seen as the restriction to $(X,0)$ of a function $\bar{f}$ in $(\mathbb{C}^{n+k},0)$ whose differential $d\bar{f}$ has maximal rank at 0. We can take coordinates in $(\mathbb{C}^N,0)$ such that $\bar{f} = x_1$. On the other hand, $f$ defines a quadratic singularity in $(X,0)$, so that we can find coordinates $(x_2,\ldots,x_{n+k})$ in $x_1 = 0$ such that

$$\begin{aligned} h_1(0,x_2,\ldots,x_{n+k}) &= x_2^2 + \ldots + x_{n+1}^2 \\ h_2(0,x_2,\ldots,x_{n+k}) &= x_{n+2} \\ &\vdots \\ h_k(0,x_2,\ldots,x_{n+k}) &= x_{n+k} \end{aligned}$$

or equivalently, there exist functions $h_i'$ with $h_i'(0,x_2,\ldots,x_{n+k}) \neq 0$ such that

$$\begin{aligned} h_1(x_1,x_2,\ldots,x_{n+k}) &= x_1^{\mu_1} h_1' + x_2^2 + \ldots + x_{n+1}^2 \\ h_2(x_1,x_2,\ldots,x_{n+k}) &= x_1^{\mu_2} h_2' + x_{n+2} \\ &\vdots \\ h_k(x_2,x_2,\ldots,x_{n+k}) &= x_1^{\mu_k} h_k' + x_{n+k} \end{aligned}$$

Then $(x_1,\ldots,x_{n+1},h_2,\ldots,h_k)$ form a coordinate system in $(\mathbb{C}^{n+k},0)$ and (3) follows.

(3) $\Rightarrow$ (1) is an easy and direct computation. □

### 3.3.1 Milnor number as multiplicity of intersection

We have seen in the proof of 3.3.2 that $\mu(X/S,0)$ is the multiplicity of intersection of the line $S \times \{0\}$ and the discriminant $\Delta_\varphi$. In [4] the authors propose different definitions of the Milnor number of a function on a singular variety as multiplicity of intersections of some subvarieties in the cotangent bundle of $\mathbb{C}^{n+l}$. We want to end this section relating our definition to one of those numbers.

Keeping the notations used in the proof of the above theorem, let $(L,0)$ the germ of the cotangent bundle $(T^*\mathbb{C}^{n+l},0)$ defined by:

$$(L,0) = \{\alpha \in\ T^*(\mathbb{C}^{n+l},0)\colon\ \alpha(v) = 0 \text{ for all } v \in \Theta_{\mathfrak{X}/B,0}\}$$

The differential $dF$ can be regarded as map between $(\mathbb{C}^{n+l},0)$ and the cotangent bundle:

$$DF\colon (\mathbb{C}^{n+l},0) \longrightarrow (T^*\mathbb{C}^{n+l},0)$$

Let $(DF(\mathbb{C}^{n+l}),0)$ denote its image. Then

**Proposition 3.3.3.** *$\mu(X/S,0)$ is equal to the multiplicity of intersection of $(L,0)$ and $(DF(\mathbb{C}^{n+l}),0)$.*

*Proof.* The induced homomorphism $DF^*\colon \mathcal{O}_{T^*\mathbb{C}^{n+l},0} \longrightarrow \mathcal{O}_{\mathbb{C}^{n+l},0}$ is surjective and sends the defining ideal $\mathcal{I}_{L,0}$ to $tF(\Theta_{\mathfrak{X}/B,0})$. Therefore it induces an isomorphism:

$$\frac{\mathcal{O}_{T^*\mathbb{C}^{n+l},0}}{\mathcal{I}_{L,0}} \xrightarrow{\simeq} \frac{\mathcal{O}_{\mathbb{C}^{n+l},0}}{tF(\Theta_{\mathfrak{X}/B,0})}$$

From [14], Prop. 7.1, the left hand side is the required intersection multiplicity if $(L,0)$ is Cohen-Macaulay and this is also proved in [4], Prop. 7.10. □

# Chapter 4

# F-Manifolds

In this chapter we use the construction carried out in the two previous chapters to endow the sheaf of liftable vector fields with a multiplicative structure satisfying certain integrability condition. We then show that under reasonable assumptions, each stratum of the logarithmic stratification defined by the discriminant has the structure of $F$-manifold.

We finally defined a multiplicatively invariant metric in the case of functions on complete intersection curves and illustrate the construction of Frobenius manifolds in the case of functions on the double point.

## 4.1 The structure of F-manifold

We describe in this section the structure of F-manifold as defined in [19]. Let $M$ be a complex manifold of dimension $n$ and $\Theta_M$ the sheaf of sections of the tangent bundle $TM$.

**Definition 4.1.1.** Let $\star\colon TM \times TM \longrightarrow TM$ be an analytic tensor of type $(2,1)$ and let us write $v \star w$ for $\star(v,w)$. $M$ is called an $F$-manifold if for any open subset $U \subset M$ and any $u, v, w \in \Theta_M(U)$, the following conditions are satisfied:

1. *Symmetry*: $u \star v = v \star u$.
2. *Associativity*: $(u \star v) \star w = u \star (v \star w)$.
3. *Unity*: There exists a global vector field $e \in \Theta_M(M)$ such that $e \star u = u$.
4. *Integrability*: $\mathrm{Lie}_{u\star v}(\star) = \mathrm{Lie}_u(\star) \star v + u \star \mathrm{Lie}_v(\star)$

The three first conditions make each tangent space $T_pM$ into a commutative and associative $\mathbb{C}$-algebra with unity $e(p)$. Then $T_pM$ can be uniquely decomposed as a sum:

$$Q_p^1 \oplus \ldots \oplus Q_p^{l(p)}$$

with the following properties (we refer to [19], Lemma 2.1):

1. $Q_p^i \star Q_p^j = 0$ for $i \neq j$

2. For each $v(p) \in T_pM$, the endomorphism $v(p)\star\colon Q_p^i \to Q_p^i$ has a unique eigenvalue. The function $\lambda_i(p)\colon T_pM \to \mathbb{C}$ that associates to each $v(p)$ that unique eigenvalue of $v(p)\star$ in $Q_p^i$ is an algebra homomorphism. Moreover, they are all the $\mathbb{C}$-algebra homomorphisms onto $\mathbb{C}$.

3. Each $Q_p^i$ is an irreducible algebra with unity $e_i(p)$, where

$$e(p) = e_1(p) + \ldots + e_{l(p)}(p)$$

is the decomposition of $e(p)$ according to that of $T_pM$.

This decomposition is called the *eigenspace decomposition* of the algebra $T_pM$.

In order to explain the meaning of the integrability condition we take coordinates $(y_1, \ldots, y_n)$ in a neighbourhood $U$ of $p$ with $\frac{\partial}{\partial y_1} = e$ and write the tensor $\star$ in those coordinates:

$$\frac{\partial}{\partial y_i} \star \frac{\partial}{\partial y_j} = \sum_{k=1}^{n} a_{ij}^k \frac{\partial}{\partial y_k}$$

Let $(y_1, \ldots, y_n, \partial_1, \ldots, \partial_n)$ be the coordinates induced by $(y_1, \ldots, y_n)$ on the the restriction of the cotangent bundle $T^*M|_U$. The analytic subset $L$ of $T^*M|_U$ defined by the ideal:

$$(\partial_1 - 1, \partial_i\partial_j - \sum_{k=1}^{n} a_{ij}^k \partial_k)$$

consists exactly of those 1-forms $\omega$ that are $\mathbb{C}$-algebra homomorphisms on each $T_qM$, $q \in M$. As there are only a finite number of such homomorphisms, it follows the projection $\pi\colon L \to U$ is finite. The restriction of the canonical map $\Theta_M \hookrightarrow \pi_*\mathcal{O}_{T^*M}$ to $L$ induces an isomorphism of $\mathcal{O}_M$-algebras $\Theta_M \xrightarrow{\simeq} \pi_*\mathcal{O}_L$ and the eigenspace decomposition at $p$ correspond to:

$$(\pi_*\mathcal{O}_L)_p = \bigoplus_{\omega \in \pi^{-1}(p)} \mathcal{O}_{L,\omega}$$

Taking small neigbourhoods of each $\omega \in \pi^{-1}(p)$ we see that, shrinking $U$ if necessary, the eigenspace decomposition at $p$ induces a decomposition of $TM|_U$ into subalgebras

$$\bigoplus_{i=1}^{l(p)} Q_p^i(U)$$

that coincides with the eigenspace decomposition at $p$. The integrability condition ensures that this decomposition is integrable ([19], Th. 2.11).

Although the decomposition induced by a point $p$ on a neighbourhood $U$ is not necessarily the eigenspace decomposition at each point of $U$, this is the case when the eigenspace decomposition at $p$ consists of one dimensional linear subspaces. In this situation, $M$ is said to be *semisimple* at $p$. The integrability condition is then equivalent to the fact that there exist coordinates $(t_1, \ldots, t_n)$ in a neigbourhood of $p$ such that the vector fields $e_i = \frac{\partial}{\partial t_i}$ are *idempotent*, i.e.,

$$e_i \star e_j = \delta_{ij} e_i \text{ and } e = e_1 + \ldots + e_n$$

Such coordinates are called *canonical coordinates*. The set of points where $M$ is semisimple is thus open. If it is not empty, $M$ is called *massive*. The set of points where $M$ is not semisimple is called the *caustic* and is a hypersurface (or empty) ([19], Prop. 2.6 although there the caustic is defined by means of partitions even for the non-massive case). The next proposition gives an equation of the caustic for a massive $F$-manifold.

**Proposition 4.1.1.** *Let $(y_1, \ldots, y_n)$ be coordinates in an open subset $U \subset M$. For each $1 \leq i, j \leq n$, consider the endomorphism $(\frac{\partial}{\partial y_i} \star \frac{\partial}{\partial y_j})\star \colon TU \to TU$ and the matrix:*

$$A_{ij} = Trace((\frac{\partial}{\partial y_i} \star \frac{\partial}{\partial y_j})\star)$$

*Then the determinant $det(A)$ is a (perhaps non-reduced) equation of the caustic $K$ in $U$.*

*Proof.* The matrix $A$ represents the trace form written with respect to the basis $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$, that is, for any pair of vector fields $v, w$ in $U$,

$$vAw = \text{Trace}(v \star w\star)$$

Let $U \cap K \neq \emptyset$ and take $p \in U \cap K$. As $T_pM$ does not decompose into one-dimensional (unitary) algebras, there exists a vector field $v$ on $U$ such that $v(p) \in T_pM$ is a non-zero nilpotent element, i.e., $(v \star \ldots \star v)(p) = v^k(p) = 0$ for some $k \in \mathbb{N}$. Hence for any vector field $w$ on $U$, $(v \star w)(p)$ is also nilpotent and all the eigenvalues of the endomorphism $(v \star w)(p) \colon T_pM \longrightarrow T_pM$ are zero. Therefore $v(p)A(p) = 0$ and $A(p)$ is degenerate from which $\det(A(p)) = 0$ follows. The reverse implication is obvious: for a semisimple algebra the matrix $A$ cannot be degenerate (just write it with respect to the basis of idempotents). $\square$

For a massive $F$-manifold the integrability condition is also equivalent to the analytic spectrum $L$ being a *Lagrangian subvariety* of the cotangent bundle $T^*M$ ([19], Th. 3.2, also [3]). Let $\tilde{\pi} \colon T(T^*M) \to T^*M$ denote the canonical projection and $\alpha$ the canonical 1-form on $T^*M$ given by

$$\begin{array}{rcl} \alpha_{(p,\omega)} \colon T_{(p,\omega)}T^*M & \longrightarrow & \mathbb{C} \\ \tilde{u} & \longrightarrow & \omega(u) \text{ where } u = d_{(p,\omega)}\tilde{\pi}(\tilde{u}) \end{array}$$

For each $p$ where $M$ is semisimple, there exists a local function $F$ on $L$, called a *generating function*, such that locally, $dF = \alpha|_{L_{\text{reg}}}$. Note that the isomorphism of $\mathcal{O}_M$-algebras $\Theta_M \simeq \pi_*\mathcal{O}_L$ can be expressed by means of $\alpha$ as $u \hookrightarrow \alpha(\tilde{u})|_L$ where $\tilde{u}$ denotes a lift of $u$ to the tangent bundle of $T^*M$. If $F$ is a generating function, then $F$ corresponds to a vector field $E$ such that, on the semisimple part,

$$\text{Lie}_E(\star) = d\,\star \text{ for some } d \in \mathbb{C}$$

A vector field on $M$ with satisfying this equation everywhere is called an *Euler vector field of weight $d$*. Such a vector field on $M$ yields a holomorphic function on $L$ by $\alpha(\tilde{E})$, which is generating function on the regular part of $L$. Thus Euler vector fields are in 1-to-1 correspondence with holomorphic functions on $L$ such that $dF = \alpha|_L$ on the regular part of $L$.

The main examples of $F$-manifolds are given by the miniversal deformations of functions $f \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with isolated singularities. If $F = f + y_1 f_1 + \ldots + y_l f_l$ is such deformation, the map:

$$\Theta_{\mathbb{C}^l,0} \ni \frac{\partial}{\partial y_i} \xrightarrow{\rho_F} \frac{\partial F}{\partial y_i} \in \mathcal{O}_{\mathbb{C}^{n+1}}/(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n+1}})$$

is an isomorphism. The algebra structure of the right hand side can be pulled back making $\Theta_{\mathbb{C}^k,0}$ into a germ of an $F$-manifold. To see that the integrability condition holds, we identify the analytic spectrum with the relative critical locus of $F$ by means of the relative differential $dF = f_1 dy_1 + \ldots + f_l dy_l$:

$$V(\frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_l}) \ni x \longrightarrow dF(x) \in \Omega^1_{\mathbb{C}^l,0}$$

The versality of $F$ implies that for a generic point $y \in B$, the eigenspace decomposition of $\mathcal{O}_{\mathbb{C}^{n+1}}/(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n+1}})$ is the sum of 1-dimensional subspaces, one for each critical point of $F(-, y)$. The multiplicative structure is then semisimple and the above identification can be understood as $dF = \alpha|_L$ on the regular part of $L$. Hence $(\mathbb{C}^l, 0)$ is a germ of an $F$-manifold and the class of $F$ in $\mathcal{O}_{\mathbb{C}^{n+1}}/(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n+1}})$ corresponds to an Euler vector field.

### 4.1.1 Multiplication of liftable vector fields

Let $(X, 0)$ be an smoothable and unobstructed singularity and $f \in \mathcal{O}_{X,0}$ a function satisfying the extendability condition (3.1.1). For its miniversal deformation $\varphi = (g, F)\colon (\mathfrak{X}, 0) \to (S \times B, 0)$, the associated map (see 2.4.1)

$$t'F\colon \mathcal{L}_{\mathfrak{X}/B,0} \to M_{\varphi,0}$$

induces a algebra structure in the module $\mathcal{L}_{\mathfrak{X}/B,0}$. For a Stein good representative of $g$, we can sheafify this map obtaining an isomorphism

$$t'F\colon \mathcal{L}_{\mathfrak{X}/B} \to g_* M_\varphi$$

of free sheaves of $\mathcal{O}_B$-modules.

**Proposition 4.1.2.** *The map $t'F$ endows the sheaf of liftable vector fields $\mathcal{L}_{\mathfrak{X}/B}$ with the structure of commutative and associative $\mathcal{O}_B$-algebra $\star$ such that, for any $u, v \in \mathcal{L}_{\mathfrak{X}/B}$:*

$$\mathrm{Lie}_{u\star v}(\star) = \mathrm{Lie}_u(\star) \star v + u \star \mathrm{Lie}_v(\star)$$

*The class of $F$ in $g_* M_\varphi$ corresponds to a vector field $E$ such that*

$$\mathrm{Lie}_E(\star) = \star$$

*Proof.* For a generic point $y \in B$, the line $S \times \{y\}$ intersects the discriminant $\Delta_\varphi$ at $\mu(X/S, 0)$ regular points, for the fibre $X_y = g^{-1}(y)$ is smooth and $F$ restricted to $X_y$ has only Morse type singularities. Hence $g_* M_\varphi$ decomposes in $\mu(X/S, 0)$ 1-dimensional algebras. To prove the proposition is then enough to show that there exists a generating function locally around such a $y$. Let $U$ be neighbourhood of $y$ such that:

$$g\colon g^{-1}(U) \cap C_\varphi \longrightarrow U$$

is $\mu(X/S, 0)$-to-one covering. A calculation like that for unfoldings of functions on smooth spaces (see [19], Th 4.20), shows that the map

$$g^{-1}(U) \cap C_\varphi \longrightarrow T^*B|_U$$

that sends a point $x$ to the 1-form $u \mapsto t'F(x)(u)$ identifies $g^{-1}(U) \cap C_\varphi$ with the analytic spectrum $L|_U$ and the 1-form

$$dF|_{g^{-1}(U)\cap C_\varphi}$$

with the restriction to $L|_U$ of the canonical 1-form $\alpha$ on $T^*B|_U$. The result then follows for a generic point of $B$ and hence everywhere and for any liftable vector field. □

The previous proposition endows the complement $B - \Delta_g$ with the structure of $F$-manifold with Euler vector field $E$ of weight 1. If we now suppose that for generic $y \in \Delta_g$, the fibre $X_y$ has only one quadratic singulirity, then by 3.1.2, we have

$$\mathcal{L}_{\mathfrak{X}/B} = \Theta(\log \Delta_g)$$

The image of $\Theta(\log \Delta_g)$ in $\Theta_B$ defines a distribution whose integral submanifolds correspond to the logarithmic stratification of the discriminant $\Delta_g$ (see [31]). We now show that in fact, each stratum has such a structure. We first prove the following lemma:

**Lemma 4.1.1.** *For any ideal sheaf $I \subset \mathcal{O}_B$, the kernel of the map*

$$\mathcal{L}_{\mathfrak{X}/B}/I\mathcal{L}_{\mathfrak{X}/B} \longrightarrow \Theta_B/I\Theta_B$$

*is identified by $t'F$ with an ideal of $g_*M_\varphi/Ig_*M_\varphi$.*

*Proof.* We have maps between two free resolutions of $g_*T^1_{\mathfrak{X}/B}$ as $\mathcal{O}_B$-modules (see 2.4.1):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_{\mathfrak{X}/B} & \longrightarrow & \Theta_B & \xrightarrow{\rho_{\mathfrak{X}/B}} & g_*T^1_{\mathfrak{X}/B} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t'F} & & \downarrow{\scriptstyle \rho_{\mathfrak{X}/B,F}} & & \| \, {\scriptstyle \mathbb{I}} & & \\
0 & \longrightarrow & g_*M_\varphi & \longrightarrow & g_*T^1_{\mathfrak{X}/S\times B} & \longrightarrow & g_*T^1_{\mathfrak{X}/B} & \longrightarrow & 0
\end{array}$$

where the second row is actually a complex of $g_*\mathcal{O}_{\mathfrak{X},0}$-modules. Hence

$$\ker\,(\mathcal{L}_{\mathfrak{X}/B}/I\mathcal{L}_{\mathfrak{X}/B} \longrightarrow \Theta_B/I\Theta_B) = \mathrm{Tor}_1^{\mathcal{O}_B}(g_*T^1_{\mathfrak{X}/B}, \mathcal{O}_B/I)$$

is mapped by $t'F$ to the kernel of

$$g_*M_\varphi/Ig_*M_\varphi \to g_*T^1_{\mathfrak{X}/S\times B}/Ig_*T^1_{\mathfrak{X}/S\times B}$$

and hence its an ideal. □

**Theorem 4.1.1.** *If the generic singularity over $\Delta_g$ has only quadratic singularities, then each stratum of the logarithmic stratification has the structure of $F$-manifold with an Euler vector field of weight* 1.

*Proof.* Let $b \in B$ and $S_b$ the stratum in which $b$ lies. Let $V$ be a open neighbourhood of $b$ in which $S_b \cap V$ is an analytic subset of $V$ defined by the ideal $I_{S_b}$. The sheaf $\Theta_{S_b\cap V}$ of sections of the tangent bundle of $S_b \cap V$ is naturally identify with the quotient

$$\frac{\mathrm{im}\,(\mathcal{L}_{\mathfrak{X}/B}|_V \to \Theta_B|_V)}{I_{S_b}\mathrm{im}\,(\mathcal{L}_{\mathfrak{X}/B}|_V \to \Theta_B|_V)}$$

Let $\mathcal{K}$ denote the sheaf $\mathrm{Tor}_1^{\mathcal{O}_B}(g_*T^1_{\mathfrak{X}/B}, \mathcal{O}_B/I_{S_b})$. The map $t'F$ descends to the above quotient and it yields an isomorphism of $\mathcal{O}_{S_b\cap V}$-modules

$$\Theta_{S_b\cap V} \xrightarrow{\simeq} \frac{g_*M_\varphi|_V}{I_{S_b}g_*M_\varphi|_V + t'F(\mathcal{K}|_V)}$$

According to the previous lemma, the right hand side is a $\mathcal{O}_B$-algebra. The above isomorphism defines the multiplication on the tangent bundle of the stratum $S_b$. From propostion 4.1.2 it follows that it is an $F$-manifold with Euler vector field of weight 1 given by the class of $F$ in the corresponding algebra. □

An interesting consequence of the stratified structure of $F$-manifold in the base space of the miniversal deformation is the following fact. Assume for a moment that $M$ is a semisimple $F$-manifold with canonical coordinates $(u_1, \dots, u_m)$ and an Euler vector field $E$ of weight 1. In the induced coordinates $(u, \partial) = (u_1, \dots, u_m, \partial_1, \dots, \partial_m)$ on $T^*M$, the analytic spectrum is the smooth subvariety

$$\begin{aligned} L &= \{(u, \partial) \colon \partial_1 + \cdots + \partial_m = 1, \partial_i\partial_j = \delta_{ij}\partial_i\} \\ &= \{(u, 1, 0, \dots, 0)\} \cup \{(u, 0, 1, \dots, 0)\} \cup \cdots \cup \{(u, 0, \dots, 1)\} \end{aligned}$$

As explained before, $E$ corresponds to a function $F$ on $L$ whose differential $dF$ is the restriction to $L$ of the canonical 1-form $\alpha$. The 1-form $\alpha$ is written as

$$\alpha = \partial_1 du_1 + \cdots + \partial_m du_m$$

Therefore, the condition $dF = \alpha|_L$ means that, up to addition of constants, the canonical coordinates are the values of $F$ on the points of $L$ lying over $u$.

**Corollary 4.1.1.** *If the stratum $S_b$ is a massive $F$-manifold, then the critical values of $F$ are generically local coordinates on $S_b$. In particular, this always holds on the stratum $B - \Delta_g$.*

**Remark 4.1.1.** 1. Off $\Delta_g$, the set of points where the critical values of $F$ are not local coordinates is the *bifurcation set*, that is, the set of points where $F$ has a non-Morse singularity.

2. In the case of space curves, that the critical values of $F$ off the bifurcation diagram are local coordinates is shown for simple functions in [16]. The author also conjectured an equivalent result for non-simple functions.

### 4.1.2 Meromorphic multiplication

It is possible to extend the multiplication meromorphically as follows. If $\delta$ is an equation for the discriminant $\Delta$ of $g$, then there exist $m \in \mathbb{N}$ such that

$$\delta^m\Theta_B \subset \mathcal{L}_{\mathfrak{X}/B}$$

For $v, w \in \Theta_{\mathfrak{X}/B}$ we define

$$v \star w = \delta^{-2m}\{(\delta^m v) \star (\delta^m w)\}$$

To compute the order of the poles of the multiplication in the case where the generic singularity over the discriminant is a quadratic singularity, we first look at a linear section of the $A_1$-singularity.

**Example 4.1.1.** Let $X\colon = \{x_1^2 + \cdots + x_{n+1}^2 = 0$ and $f$ the restriction to $X$ of the linear function $x_1 + \cdots + x_{n+1}$. The miniversal deformation of $f$ is given by $g = (x_1^2 + \cdots + x_{n+1}^2, c)\colon \mathbb{C}^{n+2} \to \mathbb{C}^2$ and the function $F = x_1 + \cdots + x_{n+1} + c$. We use coordinates $(\epsilon, c)$ in the base space $\mathbb{C}^2$. The module $M_\varphi$ is

$$\begin{aligned} M_\varphi &= \mathcal{O}_{\mathbb{C}^{n+2}}/(\frac{\partial(g,f)}{\partial(x_1,\ldots,x_{n+1},c)}) \\ &= \mathcal{O}_{\mathbb{C}^{n+2}}/(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_{n+1}) \end{aligned}$$

The liftable vector fields are generated by $\epsilon\frac{\partial}{\partial\epsilon}$ and $\frac{\partial}{\partial c}$. A lift of the former is $2^{-1}(x_1\frac{\partial}{\partial x_1} + \ldots + x_{n+1}\frac{\partial}{\partial x_{n+1}})$ and the multiplication table is given by:

$$\left(\epsilon\frac{\partial}{\partial\epsilon}\right) \star \left(\epsilon\frac{\partial}{\partial\epsilon}\right) = \frac{n+1}{4}\epsilon\frac{\partial}{\partial c}$$
$$\frac{\partial}{\partial c} = \text{unit}$$

In order to reduce the general case to that of the previous example, we also need a slight modification of Th. 2.11 in [19].

**Proposition 4.1.3.** *Assume that $\mathcal{L}_{\mathfrak{X}/B}$ can be decomposed as a sum $N_1 \oplus N_2$ of multiplicatively closed submodules with unities $e_1$ and $e_2$ respectively. Then $e_1$ and $e_2$ are nowhere vanishing and, if $N_1$ defines a distribution, it is integrable.*

*Proof.* The decomposition $N_1 \oplus N_2$ induces a decomposition of $g_*M_\varphi$ as a sum of algebras whose unities are given by $t'F(e_i)$. If at a point $y \in B$, $e_i(y) = 0$ then we must also have $\tilde{e}_i(x) = 0$ for a lift $\tilde{e}_i$ of $e_i$ at any point $x \in C_\varphi \cap X_y$, where $X_y = g^{-1}(y)$. Hence $tF(\tilde{e}_i)(x) = 0$ for all $x \in C_\varphi \cap X_y$. On the other hand $t'F(e_i)$ is the unity of $N_i$, so that there exists $x' \in C_\varphi \cap X_y$ such that $tF(\tilde{e}_i) = 1 + tF(u)$ for some $u \in \Theta_{\mathfrak{X}/B,x'}$. But this is impossible because the point $x'$ is either a singular point of $X_y$, and then $u(x) = 0$ or a smooth point of $X_y$, in which case $F|_{X_y}$ would not have a critical point. This proves the first assertion. Assume now that $N_1$ defines a distribution. According to the Frobenius Integrability Theorem (for example, [36], Th 6.5), to show that $N_1$ is integrable is enough that $N_1$ is closed under the Lie bracket, i.e., for any $v, w \in N_1$, we have $[v,w] \in N_1$. Let $e_1$ and $e_2$ be the unities of $N_1$ and $N_2$ respectively and $v = v_1 + v_2 \in \mathcal{L}_{\mathfrak{X}/B}$ with $v_i \in N_i$. Write $e = e_1 + e_2$. Then

$$e \star v = (e_1 + e_2) \star (v_1 + v_2) = v_1 + v_2$$

from which follows that $e$ is the unity of $\Theta(\log\Delta)$ and the endomorphisms

$$e_i\star\colon \mathcal{L}_{\mathfrak{X}/B} \to \mathcal{L}_{\mathfrak{X}/B}$$

are the projections associated to the decomposition $N_1 \oplus N_2$. For $v, w \in N_1$ we have:

$$\begin{aligned} 0 &= \mathrm{Lie}_{e_2\star v}(\star)(e_2,w) = e_2 \star \mathrm{Lie}_v(\star)(e_2,w) + v \star \mathrm{Lie}_{e_2}(\star)(e_2,w) \\ &= e_2 \star ([v, e_2 \star w] - [v,e_2] \star w - [v,w] \star e_2) \\ &\quad + v \star ([e_2, e_2 \star w] - [e_2,e_2] \star w - [e_2,w] \star e_2) \\ &= -e_2 \star [v,w] \end{aligned}$$

Therefore $[v,w] \in N_1$. □

**Proposition 4.1.4.** *If the generic fibre over $\Delta_g$ has only quadratic singularities and $\delta$ is a reduced equation of $\Delta_g$, then*

$$\begin{array}{l} \mathcal{L}_{\mathfrak{X}/B} \star \Theta_B \subset \Theta_B \\ \textit{For any } v, w \in \Theta_B, \delta(v \star w) \in \Theta_B \end{array}$$

*Proof.* Let $v \in \mathcal{L}_{\mathfrak{X}/B}$ and $w \in \Theta_B$. If $v \star w$ is holomorphic outside a set of codimension $\geq 2$, then it is holomorphic everywhere. As $v \star w$ is holomorphic outside $\Delta_g$, it is enough to check that $v \star w \in \Theta_B$ in a dense open subset of $\Delta_g$. At a regular point $y \in \Delta_g$, the fibre $X_y = g^{-1}(y)$ has only one quadratic singularity at, say $x_0 \in X_y$. Let $x_1, \ldots, x_r \in X_y - \{x_0\}$ be the critical points of $F|_{X_y - \{x_0\}}$. The stalk $(g_* M_\varphi)_y$ decomposes as a sum of algebras with unity

$$\oplus_{i=1}^r M_{\varphi, x_i} \oplus M_{\varphi, x_0}$$

that induces a decomposition $N_1 \oplus N_2$ of $\Theta(\log \Delta_g)_y$ into multiplicatively closed submodules with unity. If $y$ is sufficiently generic, we can assume $F|_{X_y}$ has non-degenerate singularities at $x_1, \ldots, x_r$ and it is the restriction of a submersive function to $(X_y, x_0)$. Then the unities of $N_1$ do not vanish, and therefore $N_1$ generates a subbundle of $TB$ that, according to the previous proposition, is integrable. Let then $(y_1, y_2, \ldots, y_k)$ be coordinates adapted to the subbundle $N_1$ in the sense that $\frac{\partial}{\partial y_i}, i = 3, \ldots, k$ generate $N_1$. As $\frac{\partial}{\partial y_i}$ are tangent to $\Delta_g$, it follows the germ of the $\delta$ at $y$ is a function depending only on $y_1, y_2$. After a new change of coordinates involving only $y_1$ and $y_2$, and renaming again these new coordinates $(y_1, \ldots, y_k)$, we can assume that $\Delta_g$ is defined by $y_1 = 0$ near $y$. Then we need to check that

$$(y_1 \frac{\partial}{\partial y_1}) \star (y_1 \frac{\partial}{\partial y_1}) = y_1 u \in \Theta_B \text{ and}$$
$$(y_1 \frac{\partial}{\partial y_1}) \star \frac{\partial}{\partial y_2} \in \Theta_B$$

This follows from the calculation carried out in the previous example. □

### 4.1.3 A non-isolated singularity: Mirror of $\mathbb{P}^n$

So far, we have defined an $F$-manifold structure on the base space of the miniversal deformation of a function with an isolated singularity satisfying the extendibility condition. There may be situations in which even if the miniversal deformation might not satisfy the extendibility condition, it still holds for some particular deformations. We illustrate here how to use the construction in the previous section to obtain $F$-manifolds with an example relevant in Mirror Symmetry.

Let $X$ be the variety in $\mathbb{C}^{n+1}$ consisting of the union of hyperplanes defined by the equation $x_0 \ldots x_n = 0$ and consider the function $f = x_0 + \ldots + x_n$. The map $g(x) = x_0 \ldots\ x_n$ can be regarded as a deformation of $X$ with smooth generic fibre. The sheaf of liftable vector fields is freely generated by $u = y\frac{\partial}{\partial y}$, where $y$ denotes a parameter in $B = \mathbb{C}$. A lift is given by the Euler vector field

$$\tilde{u} = \frac{1}{n+1} \left( x_0 \frac{\partial}{\partial x_0} + \ldots + x_n \frac{\partial}{\partial x_n} \right)$$

As $X$ is a homogeneous free divisor, the sheaf of logarithmic vector fields is freely generated by $\tilde{u}$ and:

$$v_i = \frac{x_0 x_i}{g}\left(\frac{\partial g}{\partial x_i}\frac{\partial}{\partial x_0} - \frac{\partial g}{\partial x_0}\frac{\partial}{\partial x_i}\right) = x_0\frac{\partial}{\partial x_0} - x_i\frac{\partial}{\partial x_i} \text{ for } 1 \leq i \leq n$$

The vector fields $v_i$ are clearly tangent to all the fibres of $g$ whereas $\tilde{u}$ is tangent only to the fibre $g^{-1}(0)$. It then follows that $v_i$ generate $\Theta_{\mathbb{C}^{n+1}/B}$. Differentiating $f$ with respect to $v_i$ we find

$$M_{(f,g)} \simeq \frac{\mathcal{O}_{\mathbb{C}^{n+1}}}{(\{x_0 - x_i\colon 1 \leq i \leq n\})}$$

is a free sheaf of $\mathcal{O}_B$-modules of rank $n+1$ and a free basis is given by $1, f, f^2, \ldots, f^n$. We also see that:

$$t'f(u) = tf(\tilde{u}) = (n+1)^{-1}f \pmod{tf(\Theta_{\mathbb{C}^{n+1}/B})}$$

Then, if we consider $n$ new parameters $y_2, \ldots, y_n$ and rename the old $y$ as $y_1$, the function:

$$F(x, y_2, \ldots, y_n) = f + y_2 f^2 + \cdots + y_n f^n$$

together with the family:

$$G(x, y_2, y_3) = (g(x), y_2, \ldots, y_n)$$

has the property that $t'F$ is an isomorphism between the modules $\Theta(\log \Delta_G)$ and $G_* M_{(F,G)}$.

**Remark 4.1.2.** It is somehow striking that the powers of $f$ generate $M_f$ as an $\mathcal{O}_B$-module. Let us consider the fibre $X_1 : x_0 \ldots x_n = 1$. Then $f$ is a Morse function on $X_1$ in the strict sense, that is, all its critical points $P_0, \ldots, P_n$ are non-degenerate and the critical values are distinct. Then if $J_{f,P_i}$ denotes the jacobian ideal of at $P_i$, we have an isomorphism

$$g_* M_{(f,g)}/\mathfrak{m}_{B,0}\, g_* M_{(f,g)} \longrightarrow \oplus_{i=0}^{m} \frac{\mathcal{O}_{X_1,P_i}}{J_{f,P_i}}$$

given by evaluation at the critical points. If we consider the deformation

$$F(x, y_0, \ldots, y_m) = f(x) + y_0 f(x) + \ldots + y_m f(x)^m$$

then the partial derivatives $\frac{\partial F}{\partial y_i}$ evaluated at the origin $(y_0, \ldots, y_m) = 0$ become by the above isomorphism $n+1$ $(n+1)$-tuples of complex numbers. The determinant that tests their linear independence is the van der Monde determinant of the critical values of $f$. Hence $F$ is versal (assuming that infinitesimal versality implies versality in this semi-local case) precisely because $f$ is Morse in the strict sense.

## 4.2 Frobenius manifolds

**Definition 4.2.1.** A *Frobenius manifold* $M$ is an $F$-manifold together with a bilinear form $g\colon TM \times\ TM \longrightarrow \mathcal{O}_B$ satisfying:

1. *Symmetry*: $g(u, v) = g(v, u)$ for any $u, v \in \Theta_B$.
2. *Multiplication invariant*: $g(u \star v, w) = g(u, v \star w)$ for any $u, v, w \in \Theta_B$.
3. *Flatness*: The *Levi-Civita connection* $\nabla$ associated to $g$ is flat, i.e., $\nabla^2 = 0$
4. *Potentiality*: If $A(u, v, w) = g(u \star v, w)$, the tensor $\nabla A$ is symmetric in all four arguments.

**Remark 4.2.1.** The last condition is equivalent to the existence of a *local potential* at any $p \in B$, that is, a function germ $\Phi_p \in \mathcal{O}_{B,p}$ such that for any $u, v$ and $w$ with $\nabla u = \nabla v = \nabla w = 0$, we have

$$uvw(\Phi_p) = g(u \star v, w)$$

The bilinear pairing $g$ is usually called a *metric*. In the case of unfoldings of singularities on smooth spaces, the existence of a pairing with those properties has been established in [33] after a conjecture of K.Saito ([32]), who also proved it for some cases. It is a deep result based on the study of the *Gauss-Manin system* associated to the singularity.

The existence of a bilinear pairing with the properties described above implies that the algebras $T_pM$ are *Frobenius*, and in particular, if irreducible, *Gorenstein*. ([22]). In the case of functions on isolated singularities, we do not know if $M_\varphi$ is Gorenstein. However there are some very special cases in which they trivially are, namely, the case of functions on complete intersection curves and hypersurface sections of a general complete intersection.

### 4.2.1 Functions on curves

Let $(X, 0)$ be a curve singularity. The following construction is taken from [28]. Let $\Omega_{X,0}(*)$ denote the module of meromorphic differentials with poles at most at the singular point 0. The *residue* of $\omega \in \Omega_{X,0}$ at 0 is defined as

$$\mathrm{Res}_0(\alpha) \colon = \frac{1}{2\pi i} \int_{\partial X} \omega$$

where $\partial X$ denotes the boundary of a representative of $(X, 0)$ such that $\omega$ converges on $X - \{0\}$. If $\mathcal{O}_{X,0}(*)$ denotes the total ring of fractions of $\mathcal{O}_{X,0}$, we obtain a pairing, also denoted by Res, by defining

$$\mathcal{O}_{X,0}(*) \times \Omega_{X,0}(*) \ni (h, \omega) \mapsto \mathrm{Res}_0(h\omega)$$

The dualising module $\omega_{X,0}$ is naturally identified with the orthogonal space to $\mathcal{O}_{X,0}$ respect to this pairing (cf. [6] and [35], IV.9)

$$\omega_{X,0} \simeq \mathcal{O}_{X,0}^{\perp} \colon = \{\omega \in \Omega_{X,0}(*) \colon \mathrm{Res}_0(h\omega) = 0 \text{ for all } h \in \mathcal{O}_{X,0}\}$$

and with this identification we also have (see [28])

$$\omega_{X,0}^{\perp} = \mathcal{O}_{X,0}$$

Let now $f \colon (X, 0) \to (S, 0)$ be a function with an isolated singularity. As $f$ is not constant on any branch of $(X, 0)$, we have an isomorphism

$$\mathcal{O}_{X,0}(*) \xrightarrow{\cdot df} \Omega_{X,0}(*)$$

whose inverse we denote by multiplication by $1/df$. Composing with the residue pairing we obtain a symmetric bilinear form $\Psi\colon \Omega_{X,0}(*) \times \Omega_{X,0}(*) \to \mathbb{C}$ given by

$$\Psi(\omega_1, \omega_2) \colon = \mathrm{Res}_0(\frac{\omega_1 \omega_2}{df})$$

that we can restrict to $\omega_{X,0}$ with the above identification. As $\omega_{X,0}^{\perp} = \mathcal{O}_{X,0}$, it follows that $\Psi$ descends to a well-defined and non-degenerate pairing

$$<,>\colon \frac{\omega_{X,0}}{\mathcal{O}_{X,0}df} \times \frac{\omega_{X,0}}{\mathcal{O}_{X,0}df} \longrightarrow \mathbb{C}$$

In the case where $(X,0)$ is a complete intersection defined by $(g_1,\ldots,g_n)$ in $(\mathbb{C}^{n+1},0)$, exterior multiplication with $dg_1 \wedge \ldots \wedge dg_n$ defines an isomorphism (see 3.3.1)

$$\frac{\omega_{X,0}}{\mathcal{O}_{X,0}df} \simeq M_f \otimes_{\mathcal{O}_{\mathbb{C}^{n+1}}} \Omega^{n+1}_{\mathbb{C}^{n+1},0}$$

and fixing a volume form on $(\mathbb{C}^{n+1},0)$ we obtain a non-degenerate bilinear pairing on $M_f$. For a fixed volume form $V$, the pairing can be written as

$$M_f \times M_f \ni (h_1,h_2) \mapsto \int_{\partial X} \frac{h_1 h_2}{\frac{\partial(g_1,\ldots,g_n,f)}{\partial(x_1,\ldots,x_{n+1})}} \alpha$$

where $dg_1 \wedge \ldots \wedge dg_n \wedge \alpha = V$.

The above construction can also be carried out in a family.
If $(F,g)\colon (\mathfrak{X},0) \to (S \times B, 0)$ is a deformation of $f\colon (X,0) \to (S,0)$ we obtain a pairing

$$<,>\colon \frac{\omega_{\mathfrak{X}/B,0}}{\mathcal{O}_{\mathfrak{X},0}dF} \times \frac{\omega_{\mathfrak{X}/B,0}}{\mathcal{O}_{\mathfrak{X},0}dF} \longrightarrow \mathcal{O}_{B,0}$$

We need to justify that this pairing indeed maps to $\mathcal{O}_{B,0}$. The following lemma is a modification of the standard argument of the Milnor fibration of a hypersurface singularity (for example, [21], Th. 4.1.7).

**Lemma 4.2.1.** *Let $g\colon \mathfrak{X} \to B$ be a good representative of a deformation of complete intersection singularity with $(\mathfrak{X},0) = (\mathbb{C}^{n+k},0)$ and $(B,0) = (\mathbb{C}^k,0)$. Let $\omega \in \omega_{\mathfrak{X}/B,0}$ and let $\gamma_y \subset X_y = g^{-1}(y)$ be a family of $n$-cycles depending continuously on $y \in B$ and such that $\gamma_y \cap C_g = \emptyset$ for all $y \in B$. Then the function*

$$I(y) = \int_{\gamma_y} \omega$$

*is holomorphic.*

*Proof.* First we note the condition $\gamma_y \cap C_g = \emptyset$ implies that $I(y)$ is well defined; it is continuous as the family of cycles depends continuously on $y$.
Write $g = (g_1,\ldots,g_k)$ for the components of $g$ and let $\pi_i\colon \mathbb{C}^k \to \mathbb{C}^{k-i}$ be the projection onto the first $k-i$ components of $\mathbb{C}^k$. By Sard's theorem, the set of points $y \in B$ such that $\pi_i(y)$ is a regular value of $\pi_i \circ g$ for all $i$ is open and dense. Let $y = (y_1,\ldots,y_k)$ be such a point and denote by $X_{(y_1,\ldots,y_{k-i})}$ the fibre $(\pi_i \circ g)^{-1}(y_1,\ldots,y_{k-i})$. By constructing a tubular neighbourhood of $X_{(y_1,\ldots,y_{k-i+1})}$ inside $X_{(y_1,\ldots,y_{k-i})}$, we obtain the Leray coboundary operator

$$\delta_{i+1}\colon H_{n+i}(X_{(y_1,\ldots,y_{k-i})}) \longrightarrow H_{n+i+1}(X_{(y_1,\ldots,y_{k-i-1})} - X_{(y_1,\ldots,y_{k-i})})$$

The operator $\delta_i$ and $\delta_{i+1}$ can be composed via the homomorphisms in homology induced by the inclusions $X_{(y_1,\ldots,y_{k-i-1})} - X_{(y_1,\ldots,y_{k-i})} \hookrightarrow X_{(y_1,\ldots,y_{k-i-1})}$. Iterating Leray's residue theorem we obtain:

$$\int_{\gamma_y} \omega = \frac{1}{2\pi i}\int_{\delta_1\gamma_y} \frac{g_k}{g_k - y_k} \wedge \omega =$$
$$= \frac{1}{(2\pi i)^2}\int_{\delta_2\delta_1\gamma_y} \frac{g_k}{g_k - y_k} \wedge \frac{g_{k-1}}{g_{k-1} - y_{k-1}} \wedge \omega = \ldots$$
$$= \frac{1}{(2\pi i)^k}\int_{\delta_k\cdots\delta_1\delta_1\gamma_y} \frac{g_k}{g_k - y_k} \wedge \ldots \wedge \frac{g_1}{g_1 - y_1} \wedge \omega$$

For $y'$ in a small enough neigbourhood of $y$, we can assume that $\delta_k\cdots\delta_1\delta_1\gamma_y = \delta_k\cdots\delta_1\delta_1\gamma_{y'}$ so that we can see the above integral as a family of forms varying holomorphically respect to $y$ along a fixed cycle. Therefore it is holomorphic near $y$ and continuouos everywhere, so that it is holomorphic. □

Let $(X, 0)$ be a complete intersection curve and $(F, g)\colon \mathfrak{X} \to S\times B$ a representative of the miniversal deformation of $f\colon (X, 0) \to (S, 0)$, such that $g\colon \mathfrak{X} \to B$ is a good representative and the boundary $\partial\mathfrak{X}$ does not intersect the critical locus of $\varphi = (F, g)$. A choice of a generator of $\omega_{\mathfrak{X}/B}$ yields an identification $g_*M_\varphi \simeq g_*(\omega_{\mathfrak{X}/B}/\mathcal{O}_{\mathfrak{X}}dF)$. The pairing $<,>$ can be pulled back via the isomorphism $t'F\colon \Theta(\log\Delta) \to g_*M_\varphi$ to define a non-degenerate, multiplicatively invariant bilinear form on $\Theta(\log\Delta)$.

### 4.2.2 Functions on the double point

The aim of this section is to show the existance of a Frobenius structure on the base space of the miniversal deformation of a function on the double point curve singularity.

The curve $(X, 0)$ is defined by $xy = 0$ in $\mathbb{C}^2$ and $f\colon (X, 0) \to (S, 0)$ is the restriction to $(X, 0)$ of the function on $\mathbb{C}^2$ given by $x^p + y^q$. To construct the miniversal deformation of $f$ we first take a miniversal deformation of $(X, 0)$, which is simply given by the equation of $(X, 0)$, i.e., $g_0\colon (\mathbb{C}^2, 0) = (\mathfrak{X}_0, 0) \to (B_0, 0) = (\mathbb{C}, 0)$ defined by $g_0(x, y) = xy$. Let $\epsilon$ be a coordinate around the origin in $B_0$. The module of relative vector fields $\Theta_{\mathfrak{X}_0/B_0,0}$ is generated by the single vector field

$$r_0 = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

and the discriminant is just the origin. The sheaf $\Theta(\log\Delta_{g_0})$ is generated by $u = \epsilon\frac{\partial}{\partial\epsilon}$ and lift of $u$ is given by

$$\tilde{u} = \frac{1}{2}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$$

The cokernel of $t'f$ is

$$\frac{\mathcal{O}_{\mathfrak{X}_0,0}}{\Theta_{\mathfrak{X}_0/B_0}(f) + (tf(\tilde{u})) + g^*\mathfrak{m}_{B_0,0}} = \frac{\mathbb{C}\{x, y\}}{(px^p - qy^q, px^p + y^q, xy)}$$

and a $\mathbb{C}$-basis is given by $x^{p-1}, \ldots, x^2, x, y^{q-1}, \ldots, y^2, y, 1$. Writing $B_1 = \mathbb{C}^{p-1} \times \mathbb{C}^{q-1} \times \mathbb{C}$ with coordinates $(a, b, c) = (a_{p-1}, \ldots, a_1, b_{q-1}, \ldots, b_1, c)$, the miniversal deformation of $f$ is written as

$$g = g_0 \times \mathbb{I}_{B_1} \colon (\mathfrak{X}, 0) = (\mathfrak{X}_0 \times B_1, 0) \longrightarrow (B_0 \times B_1, 0) = (B, 0)$$
$$F(x, y, a, b, c) = x^p + \sum_{i=1}^{p-1} a_{p-i} x^{p-i} + q y^q + \sum_{i=1}^{q-1} b_{q-i} y^{q-i} + c$$

We now fix a Gorenstein generator of $\omega_{\mathfrak{X}/B}$, for example

$$\alpha = \frac{dx \wedge dy}{dg_0} = \frac{dx}{x} = -\frac{dy}{y}$$

The following notation is useful:

$$F_x \colon = \frac{\partial F}{\partial x}, \ F_y \colon = \frac{\partial F}{\partial y}, \ H \colon = \begin{vmatrix} F_x & F_y \\ y & x \end{vmatrix} = xF_x - yF_y$$

To simplify certain computations, it will also prove useful to fix representatives of $t'F(u)$ for $u \in \Theta(\log \Delta_g)$. This is achieved simply by fixing the lifts of logarithmic vector fields. For $u = \epsilon \frac{\partial}{\partial \epsilon}$ we take the above defined $\tilde{u}$ and for the rest, for example $\frac{\partial}{\partial a_i}$, we take themselves seen as vector fields on $\mathfrak{X}$. If now $v \in \Theta(\log \Delta_g)$, we denote by $h_v$ the representative of $t'F(v)$ induced by this fixed lifts.

For later reference, we collect the results of several calculations in the next lemma:

**Lemma 4.2.2.** *For arbitrary logarithmic vector fields $u, v$, we have:*

*1.*
$$< u, v >= -\operatorname{Res}_{x=0} \frac{h_u h_v}{H} \frac{dx}{x} + \operatorname{Res}_{y=0} \frac{h_u h_v}{H} \frac{dy}{y}$$

*2.*
$$< \frac{\partial}{\partial a_i}, u > = \operatorname{Res}_{y=0} \frac{x^i h_u}{H} dy = \operatorname{Res}_{y=0} \frac{x^{i-1} h_u}{F_x} \frac{dy}{y}$$
$$< \frac{\partial}{\partial b_j}, u > = \operatorname{Res}_{y=0} \frac{y^j h_u}{H} dx = -\operatorname{Res}_{x=0} \frac{y^{j-1} h_u}{F_y} \frac{dx}{x}$$

*and in particular*

$$< \frac{\partial}{\partial a_i}, 2\epsilon \frac{\partial}{\partial \epsilon} >=< \frac{\partial}{\partial b_j}, 2\epsilon \frac{\partial}{\partial \epsilon} >=< \frac{\partial}{\partial a_i}, \frac{\partial}{\partial c} >=< \frac{\partial}{\partial b_j}, \frac{\partial}{\partial c} >= 0$$

*3.*
$$< 2\epsilon \frac{\partial}{\partial \epsilon}, 2\epsilon \frac{\partial}{\partial \epsilon} >= 0, \quad < 2\epsilon \frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial c} >= 2$$

*Proof.* 1. A priori, we need to integrate along the boundary of an annulus that encircles only those $p + q$ critical points of $F$ that come from the original singularity of $f$ at $0 \in X$. If we regard $F$ as a family of meromorphic functions globally defined on $\mathbb{P}^1 \colon = \{xy = \epsilon\} \cup \{\infty\}$, its critical points $(x, y)$ in the affine part $\{xy = \epsilon\}$ are characterized by the fact $x$ is a root of the polynomial equation

$$\begin{aligned} 0 &= x^q (xF_x - yF_y) \\ &= px^{p+q} + (p-1)a_{p-1}x^{p-1} + \cdots + a_1 x^{1+q} \\ &\quad - q\epsilon^q - (q-1)b_{q-1}\epsilon^{q-1}x - \cdots - b_1 \epsilon x^{q-1} \end{aligned}$$

There are at most $p+q$ such points, so that we can take the radii defining the annulus arbitrarily large and small respectively. As the sum of the residues of any member of $F$ on $\mathbb{P}^1$ vanishes, we obtain (1).

2. We show the first part, the other part is analogous. We begin by noticing that

$$\begin{aligned} x^q H = x^q(xF_x - yF_y) = px^{p+q} + (p-1)a_{p-1}x^{p-1} + \cdots + a_1 x^{1+q} \\ - q\epsilon^q - (q-1)b_{q-1}\epsilon^{q-1}x - \cdots - b_1\epsilon x^{q-1} \end{aligned}$$

is holomorphic and not vanishing at $x=0$. Similarly, $y^p H$ is holomorphic and not vanishing at $y=0$. Then

$$\mathrm{Res}_{x=0} \frac{x^i h_u}{H}\frac{dx}{x} = \mathrm{Res}_{x=0} \frac{h_u}{x^q H} x^{q+i-1} dx = 0$$

because $i \geq 1$ and the order of the pole of $h_u$ at $x=0$ is at worst $q$ (this is the reason why we fixed the lifts of $u$). Accordingly we have

$$\begin{aligned} < \frac{\partial}{\partial a_i}, u > &= \mathrm{Res}_{y=0} \frac{x^i h_u}{H}\frac{dy}{y} = \mathrm{Res}_{y=0} \frac{x^i h_u}{xF_x}\left(1 - 1 + \frac{xF_x}{H}\right)\frac{dy}{y} \\ &= \mathrm{Res}_{y=0} \frac{x^{i-1}h_u}{F_x}\left(1 + \frac{yF_y}{H}\right)\frac{dy}{y} \\ &= \mathrm{Res}_{y=0} \frac{x^{i-1}h_u}{F_x}\left(1 + y^{p+1}\frac{F_y}{y^p H}\right)\frac{dy}{y} \\ &= \mathrm{Res}_{y=0} \frac{x^{i-1}h_u}{F_x}\frac{dy}{y} \end{aligned}$$

The rest of the equalities are all straightforward. We show for instance those involving $\frac{\partial}{\partial \epsilon}$. To compute residues at $x=0$, the "trick" is the following expression:

$$\begin{aligned} \frac{tF(\tilde{\partial}_\epsilon)}{H} &= \frac{xF_x + yF_y}{H} = \left(1 + \frac{xF_x}{yF_y}\right)\left(-1 + 1 + \frac{yF_y}{H}\right) \\ &= \left(1 + x^{q+1}\frac{F_x}{x^q yF_y}\right)\left(-1 + \frac{xF_x}{H}\right) \\ &= \left(1 + x^{q+1}\frac{F_x}{x^q yF_y}\right)\left(-1 + x^{q+1}\frac{F_x}{x^{q+1}H}\right) \end{aligned}$$

As $x^q yF_y$ is holomorphic and it does not vanishes at $x=0$, this implies

$$\mathrm{Res}_{x=0} \frac{xF_x + yF_y}{H} x^i \frac{dx}{x} = \begin{cases} -1 & \text{if } i = 0 \\ 0 & \text{if } i = -q, -(q-1), \ldots, -1, 1, \ldots, p \end{cases}$$

Analogously, we compute the residues at $y=0$:

$$\mathrm{Res}_{y=0} \frac{xF_x + yF_y}{H} y^j \frac{dy}{y} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j = -p, -(p-1), \ldots, -1, 1, \ldots, q \end{cases}$$

and we obtain

$$
\begin{aligned}
< 2\epsilon\frac{\partial}{\partial\epsilon}, 2\epsilon\frac{\partial}{\partial\epsilon} > = 0 \\
< 2\epsilon\frac{\partial}{\partial\epsilon}, \frac{\partial}{\partial a_i} > =< 2\epsilon\frac{\partial}{\partial\epsilon}, \frac{\partial}{\partial b_j} >= 0 \\
< 2\epsilon\frac{\partial}{\partial\epsilon}, \frac{\partial}{\partial c} > = 2
\end{aligned}
$$

□

**Proposition 4.2.1.** *The pairing* $<,>$ *is flat.*

*Proof.* We will construct coordinates respect to which the matrix of $<,>$ is constant. The restriction of the function $F(x,y)$ to $\mathbb{P}^1 = \{xy = \epsilon\} \cup \{\infty\}$ has a pole of order $p$ at $y = 0$. Hence we can find a coordinate $u$ near $y = 0$ such that

$$F(x,y) = u^{-p}$$

As $xy = \epsilon$, the function $xu$ is holomorphic and not vanishing at $y = 0$. Hence, if log denotes a fixed branch of the logarithm $\log xu$ is holomorphic at $y = 0$ and we can expand it as a power series respect to $u$:

$$\log xu = t_0 + t_1 u + \cdots + t_{p-1}u^{p-1} + O(u^p)$$

Similarly, we can find a coordinate $v$ near $x = 0$ such that

$$F(x,y) = v^{-q}$$

and expand $\log yv$ as a series in $v$:

$$\log yv = s_0 + s_1 v + \cdots + s_{q-1}v^{q-1} + O(v^q)$$

Write $t = (t_1, \dots, t_{p-1})$ and analogously $s = (s_1, \dots, s_{q-1})$.
*Claim 1*: The functions ($\epsilon' = \log\epsilon, t, s, c' = c$) form a coordinate system. Moreover $t$ (resp. $s$) are functions depending only on $a$ (resp. only on $b$).
*Proof*: To see that they are indeed coordinates, we write $x$ as a power series with respect to $u$:

$$x = \frac{1}{u}\exp\left(\sum_{i\geq 0} t_i u^i\right)$$

Although elementary, we do it carefully. First expand $\left(\sum_{i\geq 0} t_i u^i\right)^k$:

$$\left(\sum_{i\geq 0} t_i u^i\right)^k = \sum_{i\geq 0} A_i^k u^i$$

where

$$
\begin{aligned}
A_0^k &= t_0^k \\
A_i^k &= A_i^k(t_i, \dots, t_0)
\end{aligned}
$$

In order to avoid special cases, set also

$$A_i^0 = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now expand $\exp\left(\sum_{i\geq 0} t_i u^i\right)$:

$$\begin{aligned}\exp\left(\sum_{i\geq 0} t_i u^i\right) &= 1 + \frac{1}{1!}\left(\sum_{i\geq 0} t_i u^i\right) + \frac{1}{2!}\left(\sum_{i\geq 0} t_i u^i\right)^2 + \cdots \\ &= \sum_{i\geq 0} A_i^0 u^i + \frac{1}{1!}\sum_{i\geq 0} A_i^1 u^i + \frac{1}{2!}\sum_{i\geq 0} A_i^2 u^i + \cdots \\ &= \sum_{i\geq 0}\left(\sum_{j\geq 0} A_i^j\right) u^i =: \sum_{i\geq 0} B_i u^i\end{aligned}$$

where

$$\begin{aligned}B_0 &= e^{t_0}\\ B_i &= B_i(t_i, \ldots, t_0)\end{aligned}$$

Next, we expand $\left(\exp \sum_{i\geq 0} t_i u^i\right)^k$:

$$\begin{aligned}\left(\exp \sum_{i\geq 0} t_i u^i\right)^k &= \left(\sum_{i\geq 0} B_i u^i\right)^k \\ &=: \sum_{i\geq 0} C_i^k u^i\end{aligned}$$

where

$$\begin{aligned}C_0^k &= e^{kt_0}\\ C_i^k &= C_i^k(t_i, \ldots, t_0)\end{aligned}$$

Set also

$$C_i^0 = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise}\end{cases}$$

Substituting $x = u^{-1}\exp(\sum_{i\geq 0} t_i u^i)$ in the equation $F(x, y) = u^{-p}$ we find:

$$\begin{aligned}\frac{1}{u^p} = \frac{1}{u^p}\sum_{i\geq 0} C_i^p u^i + a_{p-1}\frac{1}{u^{p-1}}\sum_{i\geq 0} C_i^{p-1} u^i + \cdots + a_1 \frac{1}{u}\sum_{i\geq 0} C_i^1 u^i + \\ \text{“holomorphic terms in } u\text{”}\end{aligned}$$

Equaling coefficients we obtain the following equations:

$$\begin{aligned}1 &= C_0^p\\ 0 &= C_1^p + a_{p-1}C_0^{p-1}\\ 0 &= C_2^p + a_{p-1}C_1^{p-1} + a_{p-2}C_0^{p-2}\\ &\vdots\\ 0 &= C_{p-1}^p + a_{p-1}C_{p-1}^{p-2} + \cdots + a_1 C_0^1\end{aligned}$$

As $C_0^p = e^{pt_0}$, it follows from the first equation that $t_0 = 0$ and hence $C_0^k = e^{kt_0} = 1$ for all $k$. The above set of equations becomes then

$$\begin{aligned}
0 &= C_1^p(t_1) + a_{p-1} \\
0 &= C_2^p(t_2, t_1) + a_{p-1}C_1^{p-1}(t_1) + a_{p-2} \\
&\vdots \\
0 &= C_{p-1}^p(t_p, \dots, t_1) + a_{p-1}C_{p-1}^{p-2}(t_{p-1}, \dots, t_1) + \cdots + a_1
\end{aligned}$$

These equations and those analogous for $s = (s_1, \dots, s_{q-1})$ show that the functions $(\epsilon', t, s, c')$ are indeed coordinates with the properties described in the claim.

*Claim 2*: The matrix of $<,>$ in the coordinates $(\epsilon', t, s, c')$ is constant.

*Proof*: We first compute lifts of the coordinate vector fields associated to the coordinate system $(\epsilon', t, s, c')$. Near $\{\infty\} \times B$, we have a commutative diagram:

$$\begin{array}{ccccc}
(x, y, a, b, c) \simeq & \xrightarrow{\Phi=(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5)} & & & (u, \epsilon', t, s, c') \\
\downarrow & \searrow^{F} & & {}^{F}\swarrow & \downarrow \\
& & F(x, y, a, b, c) = u^{-p} & & \\
(\epsilon, a, b, c) & \xrightarrow[\simeq]{\Psi=(\Psi_1,\Psi_2,\Psi_3,\Psi_4)} & & & (\epsilon', t, s, c')
\end{array}$$

Let us consider, for instance, the vector field $\frac{\partial}{\partial t} \in \Theta_B$ associated to the coordinate system $(\epsilon', t, s, c')$. From the change of coordinates formula we obtain

$$\frac{\partial}{\partial t} = \frac{\partial a}{\partial t}\frac{\partial}{\partial a}$$

because $t = \Psi_2(a)$ is only a function of $a$. A lift of $\frac{\partial}{\partial t}$ is given by the coordinate vector field corresponding to the function $t$ in the coordinate system defined by $(u, \epsilon', t, s, c')$. Let us denote this coordinate vector field by $\frac{\tilde{\partial}}{\partial t}$. On the other hand, from the equation of change of coordinates, we can lift the right-hand side of the above equation. Using the same self-explanatory notation we have a diagram of lifts:

$$\begin{array}{ccc}
\frac{\partial a}{\partial t}\frac{\tilde{\partial}}{\partial a} & \longrightarrow & \frac{\tilde{\partial}}{\partial t} \\
\downarrow & & \downarrow \\
\frac{\partial a}{\partial t}\frac{\partial}{\partial a} & \longrightarrow & \frac{\partial}{\partial t}
\end{array}$$

Therefore $\frac{\partial a}{\partial t}\frac{\tilde{\partial}}{\partial a}$ is a lifting of $\frac{\partial}{\partial t}$. On the other hand we have the expression

$$F(x, y, a, b, c) = u^{-p} = F \circ \Phi = F(u, \epsilon', t, s, c')$$

We apply the vector field $\frac{\tilde{\partial}}{\partial t}$ to this expression. On the left hand side, we obtain through change of coordinates:

$$\frac{\partial F}{\partial x}\frac{\tilde{\partial} x}{\partial t} + \frac{\partial F}{\partial y}\frac{\tilde{\partial} y}{\partial t} + \frac{\partial F}{\partial a}\frac{\tilde{\partial} a}{\partial t} + \frac{\partial F}{\partial b}\frac{\tilde{\partial} b}{\partial t} + \frac{\partial F}{\partial c}\frac{\tilde{\partial} c}{\partial t} = 0$$

As $b, c$ do not depend on $t$, it follows

$$tF(\frac{\partial a}{\partial t}\frac{\partial}{\partial a}) = tF(\frac{\tilde{\partial}}{\partial t}) = -\frac{\partial F}{\partial x}\frac{\tilde{\partial x}}{\partial t} - \frac{\partial F}{\partial y}\frac{\tilde{\partial y}}{\partial t}$$

and hence

$$t'F(\frac{\partial}{\partial t}) = -\frac{\partial F}{\partial x}\frac{\tilde{\partial x}}{\partial t} - \frac{\partial F}{\partial y}\frac{\tilde{\partial y}}{\partial t}$$

We now show that the matrix of the metric is constant in the coordinates $(\epsilon', t, s, c')$. Differentiating the equation $\log xu = \sum_{i\geq 0} t_i u^i$ respect to $t_i$ we obtain

$$\frac{1}{x}\frac{\partial x}{\partial t_i} = u^i,\ \frac{1}{y}\frac{\partial y}{\partial t_i} = -u^i$$

$$F_x\frac{\tilde{\partial x}}{\partial t} + F_y\frac{\tilde{\partial y}}{\partial t} = (xF_x - yF_y)u^i = Hu^i$$

As $t$ only depends on $a$ we see according to the previous lemma

$$< \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} > = \mathrm{Res}_{y=0}\frac{(Hu^i)(Hu^j)}{H}\frac{dy}{y} = \mathrm{Res}_{y=0}\, u^{i+j}H\frac{dy}{y}$$
$$= -\mathrm{Res}_{x=\infty}\, u^{i+j}dF = \mathrm{Res}_{u=0}\, pu^{i+j-(p+1)}du = p\delta_p^{i+j}$$

As $s$ only depends on $b$ we obtain similarly

$$< \frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j} > = q\delta_q^{i+j}$$
$$< \frac{\partial}{\partial t_i}, \frac{\partial}{\partial s_j} > = 0$$

To finish the proof, we simply note that

$$\frac{\partial}{\partial \epsilon'} = \epsilon\frac{\partial}{\partial \epsilon}$$
$$\frac{\partial}{\partial c'} = \frac{\partial}{\partial c}$$

and again it follows from the previous lemma. □

**Proposition 4.2.2.** *The Euler vector field $E$ corresponding to the class of $F$ in $M_\varphi$ satisfies*

$$\mathrm{Lie}_E(g) = g$$

*Proof.* The class of $F$ corresponds to the vector field

$$E = \left(\frac{1}{p} + \frac{1}{q}\right)\epsilon\frac{\partial}{\partial \epsilon} + \sum_{i=1}^{p-1}\frac{p-i}{p}a_i\frac{\partial}{\partial a_i} + \sum_{i=1}^{q-1}\frac{q-i}{q}b_i\frac{\partial}{\partial b_i} + c\frac{\partial}{\partial c}$$

Giving weights $1/p + 1/q$ to the variable $\epsilon$, $p - i/p$ to $a_i$, $q - i/q$ to $b_i$ and 1 to $c$, we have that a polynomial $h(\epsilon, a, b, c)$ is quasi-homogeneous of degree $d$ if and only if $\mathrm{Lie}_E(h) = d \cdot h$. We will compute the matrix of $<,>$ to show that each entry is quasi-homogeneous. Writing $u = \epsilon^{-1}y$, we see that

$$\begin{aligned} < \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} > &= \mathrm{Res}_{y=0} \frac{x^{i+j-2}}{F_x} \frac{dy}{y} \\ &= \mathrm{Res}_{u=0} \frac{u^{p-(i+j+1)}}{p + (p-1)a_{p-1}u + \cdots + a_1 u^{p-1}} du \end{aligned}$$

Expanding as a power series:

$$\frac{1}{p + (p-1)a_{p-1}u + \cdots + a_1 u^{p-1}} = \frac{1}{p} + l_1 u + \cdots l_{p-2}u^{p-2} + O(u^{p-1}) \tag{4.1}$$

we see that the submatrix of $<,>$ corresponding to the coordinates $a_i$ is

$$\begin{pmatrix} 0 & \dots & 0 & 0 & p^{-1} \\ 0 & \dots & 0 & p^{-1} & l_1 \\ 0 & \dots & p^{-1} & l_1 & l_2 \\ \dots & \dots & \dots & \dots & \dots \\ p^{-1} & l_1 & \dots & \dots & l_{p-2} \end{pmatrix}$$

On the other hand

$$\begin{aligned} \left(p + (p-1)a_{p-1}u + \cdots + a_1 u^{p-1}\right) \left(\tfrac{1}{p} + l_1 u + \cdots l_{p-2}u^{p-2} + O(u^{p-1})\right) \\ = 1 + (pl_1 + (p-1)a_{p-1})u + \cdots = 1 \end{aligned}$$

so that the above matrix is the inverse of

$$M_a = \begin{pmatrix} 2a_2 & 3a_3 & 4a_4 & \dots & (p-1)a_{p-1} & p \\ 3a_3 & 4a_4 & \dots & & p & 0 \\ 4a_4 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

An analogous reasoning applies to the submatrix corresponding to the coordinates $b_j$. Together with the orthogonality relations of the previous lemma we see that the matrix of $<,>$ is

$$\begin{pmatrix} 0 & 0 & 0 & 2^{-1} \\ 0 & M_a & 0 & 0 \\ 0 & 0 & M_b & 0 \\ 2^{-1} & 0 & 0 & 0 \end{pmatrix}^{-1}$$

Using 4.1, we see that the entry in the position $ij$ of $M_a^{-1}$ (resp. $M_b^{-1}$) is (if not constant) quasi-homogeneous of degree $(i+j-p)/p$ (resp. $(i+j-q)/q$). Hence

$$\begin{aligned} &\mathrm{Lie}_E(< \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} >) - < \mathrm{Lie}_E \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} > - < \frac{\partial}{\partial a_i}, \mathrm{Lie}_E \frac{\partial}{\partial a_j} >= \\ &\frac{i+j-p}{p} < \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} > + \frac{p-i}{p} < \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} > + \frac{p-j}{p} < \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} >= \\ &< \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} > \end{aligned}$$

and similarly for $< \frac{\partial}{\partial b_i}, \frac{\partial}{\partial b_j} >$. To finish the proof we simply notice that $\mathrm{Lie}_E(\epsilon\frac{\partial}{\partial\epsilon}) = 0$. □

**Remark 4.2.2.** There is at least on more situation in which the algebra $M_f$ is Gorenstein and hence supports a non-degenerate bilinear pairing. If $f$ the restriction of a linear function, say $f = x_1$, to an isolated hypersurface singularity $(X, 0)$ defined by $h(x_1, \dots, x_{n+1}) = 0$, we have

$$M_f = \mathcal{O}_{\mathbb{C}^{n+1}} / \left( h, \frac{\partial h}{\partial x_2}, \dots, \frac{\partial h}{\partial x_{n+1}} \right)$$

and hence its a 0-dimensional complete intersection. As such, it supports the Grothendieck's residue pairing. We have checked with Maple ([1]) the flatness of this pairing for the function $f = x_1$ on the $A_2$-singularity $x_1^3 + x_2^2 + \cdots + x_{n+1}$. As the addition of sum of squares in new variable affects neither the $F$-manifold structure nor the metric, this is indeed equivalent to the case of the linear function $x_1$ on the cusp singularity curve.

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# Appendix A

# F-manifolds from composed functions

*Ignacio de Gregorio and David Mond*

## A.1 Introduction

In this note we show how to endow the base of a versal deformation of a composite singularity with an F-manifold structure, as defined by Hertling and Manin in [8], and in particular with a pointwise, integrable multiplication on the tangent bundle. This is closely related to, but not a special case of, K.Saito's construction of a Frobenius manifold structure on the base-space of a versal deformation of a function with isolated critical point.

Let us clarify the notion of versality we are concerned with. Consider a function $f:(Y,y_0)\to\mathbf{C}$ and a map-germ $F:(X,x_0)\to(Y,y_0)$. Deformations of $F$ give rise to some, but not all, of the possible deformations of the composite $f\circ F$. In this context, a deformation $\mathscr{F}$ of $F$ is versal if, up to the usual notion of equivalence, it contains all the deformations of $f\circ F$ which can be achieved by deforming $F$. A precise definition is given below. For now, we point out that even when $f\circ F$ has non-isolated singularity, $F$ may have a finite-dimensional versal deformation in the sense being considered. Exactly this is the case in Damon's theory of almost free divisors, [3].

The ideas of the previous paragraph are made precise by means of the subgroup $\mathcal{K}_f$ of the contact group $\mathcal{K}$, acting on the space of map-germs $(X,x_0)\to(Y,y_0)$. The $\mathcal{K}_f$-equivalence of $F_1$ and $F_2$ equivalence implies (but is not implied by) right-equivalence of the composites $f\circ F_1$ and $f\circ F_2$. A $\mathcal{K}_f$-miniversal deformation $\mathscr{F}:(X\times S,(x_0,0))\to(Y,y_0)$ of $F:(X,x_0)\to(Y,y_0)$ can be constructed by the usual procedures of singularity theory; loosely speaking, the tangent space $T_0S$ is isomorphic to the quotient $T^1_{\mathcal{K}_f}F$ of the space $\theta(F)$ of infinitesimal deformations of $F$ by the tangent space to the $\mathcal{K}_f$-orbit of $F$.

Write $F_s(x)=\mathscr{F}(x,s)$. The multiplicative structure on $TS$ is defined at those points $s \in S$ over which the natural epimorphism

$$T^1_{\mathcal{K}_f} F_s \longrightarrow \frac{F_s^*(J_f)}{J_{f \circ F_s}},$$

defined by contracting with $F^*(df)$, is injective. Although the ring $F_s^*(J_f)/J_{f\circ F_s}$ does not appear to have a unit, and although contracting with $F_s^*(df)$ does not always give an injection, under favourable circumstances there is a Zariski-open set in the $S$ where both dificulties vanish. In particular,

(∗) there is a proper analytic subset $B$ of $S$, such that for $s \in S \setminus B$, $\operatorname{supp} T^1_{\mathcal{K}_f} F_s \cap V(F_s^*(J_f)) = \emptyset$,

so that at the very least $T^1_{\mathcal{K}_f} F_s$ maps onto the ring $\mathcal{O}_X/J_{f\circ F_s}$. In Section A.3 we prove a transversality lemma which allows us to show that there is an open set in $S$, the critical points of $f \circ F_s$ off $F_s^{-1}(E)$ are generically non-degenerate and that the critical values are generically pairwise distinct.

To transfer the multiplicative structure to the tangent sheaf of the base, we need the relative Kodaira-Spencer map

$$\theta_S \to \pi_*(T^1_{\mathcal{K}_f/S}\mathscr{F})$$

to be an isomorphism, and in particular we require $T^1_{\mathcal{K}_f/S}\mathscr{F}$ to be free over the base. Section A.4 recalls the arguments given in [4] and [5] to prove freeness in three cases: where $E := f^{-1}(0)$ is a free divisor, and where $\dim X$ is equal to $m_0 := \dim Y - \dim E_{Sing}$ or to $m_0 - 1$.

If $\dim X \geq m_0$, the generic fibres $D_s = F_s^{-1}(E)$ will contain singularities; they are only *partial* smoothings of $D := F^{-1}(E)$. The most extreme case is where $E$ is a free divisor (and not smooth). In this case, $D_s$, like $E$, will be singular in codimension 1. Nevertheless, in all cases every fibre $D_s$ has the homotopy type of a wedge of spheres of middle dimension. This is because we have "triviality at the boundary" (recall that under the assumption that $\operatorname{supp} T^1_{\mathcal{K}_f} = \{0\}$, $F$ is transverse to $E$ away from 0), and thus the vanishing homology of $D_s$ is accounted for by the isolated critical points which move off $D$ as $s$ moves away from 0 (cf [10]). The number of spheres in this wedge for a generic parameter-value $s$ is called by Damon the "singular Milnor number" of $D$. We will denote it by $\mu_E(F)$. In Section 5 of [4] it is shown that if

1. $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over $\mathcal{O}_S$, and
2. condition (∗) holds,

then $\mu_E(F) = \dim_{\mathbf{C}} T^1_{\mathcal{K}_f} F$. Thus, in these favourable circumstances, our $F$-manifold $S$ supports a locally trivial holomorphic fibration whose fibre $D_s$ has homology concentrated in middle dimension, where its rank is equal to the dimension of $S$.

## A.2 Background and notation

Throughout this paper, $X$ and $Y$ will denote the germs $(\mathbf{C}^m, 0)$ and $(\mathbf{C}^n, 0)$ respectively. We consider a fixed map $f : Y \to \mathbf{C}$, and classify map-germs

$X \to Y$ as follows: $F_1 : X \to Y$ and $F_2 : X \to Y$ are $\mathcal{K}_f$-equivalent if there exists a germ of diffeomorphism $\Phi : X \times Y \to X \times Y$ such that

1. $\Phi$ covers a diffeomorphism $\phi : X \to X$ (i.e. there is a germ of diffeomorphism $\phi : X \to X$ such that $\pi_X \circ \Phi = \phi \circ \pi_X$)

2. $\Phi$ preserves the level sets of $f$; more precisely, $f \circ \pi_Y \circ \Phi = f \circ \pi_Y$, and

3. $\Phi(\mathrm{graph}(F_1)) = \mathrm{graph}(F_2)$.

Observe that $\mathcal{K}_f$ contains the group $\mathscr{R}$ of right-equivalences. One calculates that the extended tangent space to the group action on a germ $F : X \to Y$ is

$$T\mathcal{K}_f F = tF(\theta_X) + F^*(\mathrm{Der}(-\log f))$$

where $\mathrm{Der}(-\log f)$ is the $\mathcal{O}_Y$-module of germs of vector-fields tangent to all the level sets of $f$. We denote the quotient $\theta(F)/T\mathcal{K}_f F$ by $T^1_{\mathcal{K}_f} F$. It is easy to show that

$$F_1 \sim_{\mathcal{K}_f} F_2 \quad \Rightarrow f \circ F_1 \sim_{\mathscr{R}} f \circ F_2,$$

but the converse does not always hold.

The group $\mathcal{K}_f$ is geometric, in the sense of Damon [1], and so the usual properties hold; in particular, if $T^1_{\mathcal{K}_f} F$ has finite length then a deformation $\mathscr{F} : X \times U \to Y$ of $F$ is versal if the initial velocities $\partial \mathscr{F} / \partial s_i|_{s=0}$ generate $T^1_{\mathcal{K}_f} F$ over $\mathbf{C}$, and miniversal if they form a basis.

Closely related to $\mathcal{K}_f$ is the group $\mathcal{K}_E$, in which part 2 of the definition above is weakened to the requirement that $\Phi$ preserve only the level set $X \times E$ of $f \circ \pi_Y$. It is an immediate consequence of Nakayama's lemma that

$$\mathrm{supp}\, T^1_{\mathcal{K}_E} F = \{x \in X : F \not\pitchfork E \text{ at } x\},$$

where transversality is understood to mean transversality to the distribution $\mathrm{Der}(-\log E)$.

Damon showed in [2] that if

$$\begin{array}{ccc} Y_0 & \xrightarrow{H} & Y_1 \\ \uparrow & & \uparrow F \\ X_0 & \xrightarrow{h} & X_1 \end{array}$$

is a fibre square in which $H$ is a right-left stable map-germ with discriminant $E$ (or image, if $\dim Y_0 < \dim Y_1$), and $F \pitchfork H$, then

$$T^1_{\mathcal{K}_E} F \simeq T^1_{h : X_0 \to X} := \frac{\theta(h)}{th(\theta_{X_0}) + \omega h(\theta_Y)}.$$

Our $F$-manifold structure is therefore closely related to the theory of right-left equivalence of map-germs, and of right-left versal unfoldings.

## A.3 Transversality

To prove a number of properties of $\mathcal{K}_f$-versal deformations, we will use a local transversality lemma. Recall that $X^{(r)}$ is the subset of the $r$-fold cartesian product $X^r$ consisting of $r$-tuples of pairwise distinct points, that ${}_rJ^k(X,Y)$ is the restriction of $(J^k(X,Y))^r$ to $X^{(r)}$, and that

$$ {}_rj^k_x\mathscr{F} : X^{(r)} \times S \to {}_rJ^k(X,Y), $$

the *relative r-fold multi-jet extension map*, is defined by

$$ {}_rj^k_x\mathscr{F}(x_1,\dots,x_r,s) = \big(j^kF_s(x_1),\dots,j^kF_s(x_r)\big). $$

**Lemma A.3.1.** *Let $W \subset {}_rJ^k(X,Y)$ be a $\mathcal{K}_f$ invariant submanifold. If $\mathscr{F} : X \times S \to Y$ is a $\mathcal{K}_f$-versal deformation of a germ $F : X \to Y$, then ${}_rj^k_x\mathscr{F} \pitchfork W$.*

*Proof.* It is possible to find a deformation $\widetilde{\mathscr{F}} : X \times S \times U \to Y$ of $\mathscr{F}$ such that ${}_rj^k_x\widetilde{\mathscr{F}} : X \times S \times U \to J^k(X,Y)$ is transverse to $W$. For example, identifying $X$ and $Y$ with $\mathbf{C}^m$ and $\mathbf{C}^n$ respectively, we take as $U$ the space of polynomial maps $p : \mathbf{C}^m \to \mathbf{C}^n$ with each component of degree $\leq N$, and define

$$ \widetilde{\mathscr{F}}(x,s,p) = \mathscr{F}(x,s) + p(x). $$

If $N$ is sufficiently large then ${}_rj^k_x\widetilde{\mathscr{F}}$ is a submersion, and in particular transverse to $W$. [1]

As $\mathscr{F}$ is $\mathcal{K}_f$-versal, so is $\widetilde{\mathscr{F}}$. Versality of $\mathscr{F}$ implies that $\widetilde{\mathscr{F}}$ is $\mathcal{K}_{f\mathrm{un}}$-equivalent to the deformation $i^*\mathscr{F}$ induced from $\mathscr{F}$ by some map of base-spaces $i : U \times V \to U$. Versality of $\widetilde{\mathscr{F}}$ implies that $i$ is a submersion, and in particular locally surjective. As $W$ is $\mathcal{K}_f$-invariant, the transversality of $j^k_x\widetilde{\mathscr{F}}$ to $W$ implies that $j^k_x i^*(\mathscr{F}) : X \times U \times V \to J^k(X,Y)$ is also transverse to $W$. But $j^k_x i^*(\mathscr{F}) = (j^k_x\mathscr{F})\circ(\mathrm{id}_X \times i)$. As $\mathrm{id}_X \times i$ is surjective, it follows that $j^k_x\mathscr{F}$ is transverse to $W$. □

In what follows, when we consider a map $F : X \to Y$, we write $D = V(f\circ F)$. If $\mathscr{F} : X \times S \to Y$ is a deformation of $F$, we write $F_s(x) := \mathscr{F}(x,s)$, $\mathscr{D} := \mathscr{F}^{-1}(E)$ and $D_s = F_s^{-1}(E)$.

From our transversality lemma we derive first a statement about the behaviour of perturbations $F_s$ of $F$ off the level set $D_s$ of $f\circ F_s$. The reason for this is that typically, $D_s$ will have non-isolated singularities.

Here is a simple example. Define $f : \mathbf{C}^4 \to \mathbf{C}$ by $f(y_1,y_2,y_3,y_4) = y_1y_2y_3y_4$, let $E = f^{-1}(0)$ and let $F : \mathbf{C}^3 \to \mathbf{C}^4$ be given by $F(x_1,x_2,x_3) = (x_1,x_2,x_3,x_1+x_2+x_3)$. $\mathrm{Der}(-\log E)$ is well known to be generated by the vector fields $y_i\partial/\partial y_i$ for $i = 1,\dots,4$, and $\mathrm{Der}(-\log f)$ consists of all linear combinations $\sum a_iy_i\partial/\partial y_i$ where $\sum_i a_i = 0$. Thus $T\mathcal{K}_fF$ is generated over $\mathcal{O}_X$ by

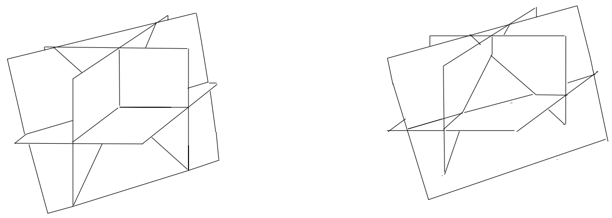
$$ \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_4},\ \cdots,\ \frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_4} $$

and by

$$ x_1\frac{\partial}{\partial y_1} - x_2\frac{\partial}{\partial y_2},\ x_1\frac{\partial}{\partial y_1} - x_3\frac{\partial}{\partial y_3},\ x_1\frac{\partial}{\partial y_1} - (x_1+x_2+x_3)\frac{\partial}{\partial y_4} $$

[1] It is interesting to note that the extension of ${}_rj^k_x\widetilde{\mathscr{F}}$ to $X^r \times S \times U$ can *not* be a submersion, since ${}_rj^k_x\widetilde{\mathscr{F}}$ is equivariant with respect to the obvious symmetric group actions on $X^r \times U \times V$ and on $\big(J^k(X,Y)\big)^r$.

The quotient $T^1_{\mathcal{K}_f}F$ has length 1, and is generated by the class of $\partial/\partial y_4$. The two drawings below show the real part of $D = F^{-1}(E)$ and $D_s = F_s^{-1}(E)$, where $F_s$ is the deformation $F_s = F + s\partial/\partial y_4$, for $s < 0$. Both surfaces have non-isolated singularities; the defining equation $x_1x_2x_3(x_1+x_2+x_3+s) = 0$ of the second also has an isolated singularity at $(-s/4, -s/4, -s/4)$ with Milnor number 1, inside the chamber which has opened up as $s$ moves away from zero.



**Proposition A.3.1.** *If $\mathscr{F} : X \times S \to Y$ is a $\mathcal{K}_f$-miniversal deformation of $F$, then*
*(i)* $\sum^{rel}_{f\circ\mathscr{F}}$ *is non-singular off $\mathscr{D}$.*
*Moreover, there is are analytic hypersurfaces $B_1$ and $B_2$ of $S$ such that*
*(ii) for $s \in S \setminus B_1$, each critical point of $f{\circ}F_s$ off $D_s$ is non-degenerate, and*
*(iii) for $s \in S \setminus B_2$, the values of $f{\circ}F_s$ at these critical points are all distinct.*

*Proof.* Apply A.3.1 taking as $W$ the submanifold of $J^1(X,Y)$ consisting of 1-jets $(x,y,A)$ with the property that $y \notin E$ and the image of the linear map $A$ lies in $\ker d_y f$. For each $y \notin E$, $\ker d_y f$ has dimension $n-1$, and so the space of admissible matrices $A$ has dimension $m(n-1)$. Thus the codimension of $W$ in $J^1(X,Y)$ is $m$. Clearly $\sum^{\text{rel}}_{\mathscr{F}} \setminus \mathscr{D} = (j^1_x f \circ \mathscr{F})^{-1}(W)$ and is therefore smooth by A.3.1. This proves (i).

Let $W_1 \subset J^2(X,Y)$ be the set consisting of jets $j^2H(x)$ such that $H(x) \notin E$, $d_x(f{\circ}H) = 0$ and the Hessian determinant of $f{\circ}H$ vanishes at $x$. Although not a manifold, $W_1$ is an analytic set and can be stratified. Its open stratum has codimension $m+1$. The set $B_1$ is the closure of $\pi_S(j^2\mathscr{F})^{-1}(W_1)$.

The projection $\pi : \sum^{\text{rel}}_{f\circ\mathscr{F}} \to S$ is finite. If it were not, then for some $s \in S$, $f{\circ}F_s$ would have a non-isolated singularity off $D_s$. But the length of $T^1_{\mathcal{K}_f}F_s$ is upper semi-continuous, and for $s = 0$ it is finite. Thus, $B_1$ is a hypersurface.

To ensure that the critical values of the critical points off the zero level are all distinct, let $W_2 \subset {}_2J^1(X,Y)$ be the submanifold consisting of jets $(x_1, y_1, A_1, x_2, y_2, A_2)$ such that $f(y_1) = f(y_2) \neq 0$, and $d_{y_i} f{\circ}A_i = 0$ for $i = 1, 2$. As $\mathcal{K}_f$ leaves the level sets of $f$ unchanged, $W_2$ is indeed $\mathcal{K}_f$-invariant. The codimension of $W_2$ in ${}_2J^1(X,Y)$ is $2m+1$, so transversality of ${}_2j^1_x\mathscr{F}$ to $W_2$ means that the set $(x_1, x_2, s) \in X^{(2)} \times S$ such that $x_1, x_2$ are critical points of $f{\circ}F_s$ not in $D_s$ and with equal critical values, is empty or has dimension $\dim S - 1$. In particular, the closure $B_2$ of its projection to $S$ is a hypersurface (or empty), and if $s$ is not in $B_2$ then the values of $f{\circ}F_s$ at its critical points off $D_s$ are pairwise distinct. □

A divisor $E$ is *holonomic* at $x$ if the logarithmic partition of $E$ is locally finite (and thus a stratification) in some neighbourhood $U$ of $E$. Holonomicity is an analytic condition, and thus the set of points where it fails is an analytic subset of $E$. We say that $E$ is *holonomic in codimension* $k$ if this subset has codimension at least $k+1$.

In similar vein, $E$ is *quasi-homogeneous* at $x$ if there is a local analytic coordinate system centred at $x$ with respect to which $E$ has a defining equation which is weighted homogeneous with respect to some set of strictly positive weights. We say that $E$ is *locally weighted homogeneous in codimension* $k$ if there is a Whitney stratification of $E$ such that $E$ is quasi-homogeneous at every point of each stratum of codimension $\leq k$.

**Proposition A.3.2.** *(J.N.Damon, [3]) Suppose that $E = V(f)$ is holonomic and locally quasihomogeneous in codimension $m$. Then there is a proper analytic subset $B$ of the semi-universal bases-space $S$ of $\mathscr{F}$ such that if $s \notin B$,*

$$\operatorname{supp}\left(T^1_{\mathcal{K}_f} F_s\right) \cap \ D_s = \emptyset.$$

*Proof.* Let $\mathscr{S} = \{E_\alpha\}$ be a Whitney stratification of $E$, in which each stratum of codimension $\leq k$ is logarithmic. Any $\mathcal{K}_f$-versal deformation of $F$ is logarithmically transverse to $E$, and thus is transverse to $\mathscr{S}$. By Sard's Theorem, the set $\Delta_\alpha$ of critical values in $S$ of the projection $\mathscr{F}^{-1}(E_\alpha) \to S$ has measure zero, and so also does $\Delta := \bigcup_\alpha \Delta_\alpha$. If $s \notin \Delta$ then $F_s$ is transverse to each stratum $E_\alpha$ of $\mathscr{S}$. Since $E$ is holonomic in codimension $m$, this means that $F_s$ meets only holonomic strata of $E$, and so in fact $F_s$ is logarithmically transverse to $E$ itself.

For such a point $s \in S$, we have

$$d_x F_s(T_x X) + T^{-\log}_{F_s(x)} E = T_{F_s(x)} Y \tag{A.1}$$

for all $x \in F_s^{-1}(E)$. Let $\chi \in \operatorname{Der}(-\log E)_{F_s(x)}$ be a germ of Euler vector field, vanishing at $F_s(x)$, such that $\chi \cdot f = f$. Then $\operatorname{Der}(-\log E)_{F_s(x)} = \langle \chi \rangle_{F_s(x)} + \operatorname{Der}(-\log f)_{F_s(x)}$, where $\langle \chi \rangle$ is $\mathcal{O}_Y$ module generated by $\chi$. Since $\chi$ vanishes at $F_s(x)$, from (A.1) we obtain

$$d_x F_s(T_x X) + \operatorname{Der}(-\log f)(F_s(x)) = T_{F_s(x)} Y \tag{A.2}$$

and now Nakayama's Lemma gives $(T\mathcal{K}_f F_s)_x = \theta(F_s)_x$. ☐

**Remark A.3.1.** In this proof the only role played by local quasihomogeneity is to guarantee that at every point $y \in E$, $T^{-\log}_x E = \operatorname{Der}(-\log f)(y)$.

## A.4 When is the relative $T^1$ free over the base?

Suppose that $\mathcal{O}_Y / J_f$ is Cohen-Macaulay, and let $m_0$ be the codimension of $V(J_f)$ in $Y$. Let $\mathscr{F} : X \times S \to Y$ be a deformation of a germ $F : X \to Y$ for which $\operatorname{supp} T^1_{\mathcal{K}_f} F = \{0\}$. There are three cases where we can show that under these circumstances the relative module $T^1_{\mathcal{K}_f / S} \mathscr{F}$ is free over $S$. These are

1. where $f$ is an Euler-homogeneous defining equation for $E$ — one for which $f \in J_f$ — for a free divisor $E$,

2. where $\dim X = m_0$, and

3. where $\dim X = m_0 - 1$.

Proofs of all three are straightforward, and may be found, for example, in [4], [5], though for completeness we sketch them here.

*1. Free divisors* $\mathrm{Der}(-\log f)$ is a direct summand of the free module $\mathrm{Der}(-\log E)$, with complementary summand generated by a vector field $\chi$ such that $\chi \cdot f = f$, and so is free on $n-1$ generators. Thus, the presentation

$$\theta_{X\times S/S} \oplus \mathscr{F}^*(\mathrm{Der}(-\log f)) \to \theta(\mathscr{F}/S) \to T^1_{\mathcal{K}_f/S}\mathscr{F} \to 0$$

can be read as

$$\mathcal{O}^m_{X\times S} \oplus \mathcal{O}^{n-1}_{X\times S} \to \mathcal{O}^n_{X\times S} \to T^1_{\mathcal{K}_f/S}\mathscr{F} \to 0. \tag{A.3}$$

Since $\operatorname{supp} T^1_{\mathcal{K}_f/S}\mathscr{F}$ is finite over $S$, $\dim T^1_{\mathcal{K}_f/S}\mathscr{F} \leq \dim S$, which is the minimum possible given (A.3). From (A.3) it now follows, by the theorem of Eagon-Northcott, that $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is a free $\mathcal{O}_S$-module.

*2. The cases $m = m_0$ and $m = m_0 - 1$*

**Lemma A.4.1.** *If $m \leq m_0$ and $T^1_{\mathcal{K}_f}F$ has finite length then $f\circ F$ has an isolated singularity.*

*Proof.* First, because $\operatorname{supp} T^1_{\mathcal{K}_f}F = \{0\}$, the restriction of $F$ to $X \setminus \{0\}$ is transverse to every level set of $f$. We have $\dim X \leq \operatorname{codim} V(J_f) \leq \operatorname{codim} E_\alpha$ for each stratum $E_\alpha$ of any Whitney stratification of $E$ contained in $V(J_f)$, and so $F^{-1}(E_\alpha)$ must consist of isolated points. Thus the germ of $F^{-1}(V(J_f))$ consists at most of $\{0\}$. At every point $x \notin F^{-1}(V(J_f))$, the transversality of $F$ to the level set of $f$ through $F(x)$ means that $x$ is not a critical point of $f\circ F$. □

The multiplicity, $\mu$, of the critical point of $f\circ F$ is preserved in any deformation. When $m = m_0$, the exact sequence of Corollary 1.3 of [5] reduces to

$$0 \to T^1_{\mathcal{K}_f}F \to \mathcal{O}_X/J_{f\circ F} \to \mathcal{O}_X/F^*(J_f) \to 0. \tag{A.4}$$

The lengths of the second and third non-trivial terms in this short exact sequence are conserved (the latter because $\mathcal{O}_Y/J_f$ is Cohen-Macaulay), and hence so is the length of the first. This implies that $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over $\mathcal{O}_S$. When $m = m_0 - 1$, the exact sequence acquires an extra term, and becomes

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f, \mathcal{O}_X) \to T^1_{\mathcal{K}_f}F \to \mathcal{O}_X/J_{f\circ F} \to \mathcal{O}_X/F^*(J_f) \to 0 \tag{A.5}$$

An easy argument ([5] Lemma 4.3(i)) shows that the lengths of the modules $\mathrm{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f, \mathcal{O}_X)$ and $\mathcal{O}_X/F^*(J_f)$ are equal, so that the length of $T^1_{\mathcal{K}_f}F$ is equal to $\mu$. As $\mu$ is conserved, so is the length of $T^1_{\mathcal{K}_f}F$, and so once again $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over $\mathcal{O}_S$.

## A.5 Multiplication on the tangent bundle of the base

Let $\mathscr{F}$ be a $\mathcal{K}_f$-miniversal deformation of some germ $F\colon X \to Y$ for which $T^1_{\mathcal{K}_f}F$ has finite length. Suppose that $E$ is locally quasihomogeneous and holonomic in codimension $m$, and that $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over $S$. By Prop. A.3.2, there is a proper analytic subset $B$ of the base space $S$, such that for $s \in S\backslash B$, $\operatorname{supp} T^1_{\mathcal{K}_f}F_s$ does not meet $D_s$. For such $s$,

$$(T^1_{\mathcal{K}_f}F_s)_x \simeq \mathcal{O}_{X,x}/J_{f\circ F_s}$$

for each $x$, and indeed

$$\pi_*(T^1_{\mathcal{K}_f/S}\mathscr{F}) \simeq \pi_*(\mathcal{O}_{X\times S}/J^{\text{rel}}_{f\circ\mathscr{F}}). \tag{A.6}$$

Because $T^1_{\mathcal{K}_f/S}\mathscr{F}$ is free over $S$, the Kodaira-Spencer map gives an isomorphism of free sheaves

$$\theta_S \simeq \pi_*(T^1_{\mathcal{K}_f/S}\mathscr{F})$$

on all of $S$. Composing this with the isomorphism (A.6) we get an isomorphism

$$\theta_{S\backslash B} \simeq \pi_*(\mathcal{O}_{X\times S}/J^{\text{rel}}_{f\circ\mathscr{F}})\big|_{S\backslash B} \tag{A.7}$$

and it is this that we use to define a multiplication on the tangent sheaf, just as in the case of deformations of isolated hypersurface singularities.

A complex manifold with an associative and commutative multiplication $\star$ on the tangent bundle is called an $F$-manifold if:

1. *(unity)* there exists a *global* vector field $e$ such that $e \star u = u$ for any $u \in \theta_M$ and,
2. *(integrability)* $\text{Lie}_{u\star v}(\star) = u \star \text{Lie}_v(\star) + \text{Lie}_u(\star) \star v$ for any $u, v \in \theta_M$.

The main consequence of this definition is the integrability of multiplicative subbundles of $TM$, namely, if in a neigbourhood $U$ of a point $p \in M$ we can decompose $TU$ as a sum $A \oplus B$ of multiplicatively closed subbundles with unity, then $A$ and $B$ are integrable.

An *Euler vector field* $E$ for $M$ is defined by the condition

$$\text{Lie}_E(\star) = \star$$

**Theorem A.5.1.** *The complement $S \setminus B$ with the multiplication induced from (A.7) is an $F$-manifold with Euler vector field $E_S$ given by the class of $f \circ \mathscr{F}$ in $\mathcal{O}_{X\times S}/J^{\text{rel}}_{f\circ\mathscr{F}}$.*

*Proof.* It is enough to show that the integrability conditions holds in an open and dense subset of $S \setminus B$. According to Prop. A.3.1, there exists a proper analytic subvariety $B_1$ such that for $s \in S \setminus B_1$, the composite $f \circ F_s$ has only non-degenerate critical points. In a neigbourhood $U \subset S \setminus B$ of such a point, the integrability condition is equivalent to the image $L$ of the map

$$\text{supp } T^1_{K_f}\mathscr{F} \ni (x, s) \mapsto d_{(x,s)}(f \circ \mathscr{F}) \in T^*_s S \tag{A.8}$$

being a Lagrangian subvariety of $T^*S$ (see [7], Th. 3.2). If $\alpha$ denotes the canonical 1-form on $T^*S$ and $p\colon T^*S \to S$ the projection, it is easy to check that the diagram

$$\begin{array}{ccc} \pi_*(\text{supp } T^1_{K_f}\mathscr{F}) & \longrightarrow & p_*\mathcal{O}_L \\ & \nwarrow \quad \nearrow & \\ & \theta_S & \end{array}$$

is commutative. The homomorphism on the right hand side is given by evaluation, so that it can also be expressed as $\alpha(\tilde{u})$ where $\tilde{u}$ is a lift of $u \in \theta_S$ to $\theta_{T*S}$. Hence $\alpha_L$ is the relative differential of $(f \circ \mathscr{F})$ when thought of as a map on $L$ via the identification (A.8). It follows that $\alpha_L$ is the exact and hence closed, so that $L$ is Lagrangian.
The statement about the Euler vector is an easy calculation that we leave to the reader (see [7], Th. 3.3.). □

**Remark A.5.1.** It follows that the critical values of $f \circ \mathscr{F}$ are local coordinates around a generic point in the base. For a point $s$ where $\text{supp}\, T^1_{\mathcal{K}_f} F_s$ consists of $m$ different points, the algebra $T_sS$ decomposes in 1-dimensional subalgebras with unity. Hence there exist coordinates $(u_1, \ldots, u_m)$ such that $(\partial/\partial_i) \star (\partial/\partial_j) = \delta_{ij}\partial/\partial_i$. These special coordinates are known as *canonical coordinates.* Writing the Euler vector field $E$ in these coordinates and using the fact that $d(f \circ \mathcal{F}) = \alpha|_L$, we see that the canonical coordinates coincide, up to a constant, with the critical values of $f \circ \mathscr{F}$.

In the cases where $f \circ F$ has an isolated singularity, we can compare its $\mathcal{R}$-miniversal deformation with $\mathscr{F}$. Let us assume that $f \circ \mathscr{F}$, thought of as a deformation of $f \circ \mathscr{F}$ is (up to $\mathcal{R}_{\text{e-un}}$-equivalence) induced from some other, say $G$. Then we have a fiber square

$$\begin{array}{ccccc} X \times S & & \xrightarrow{\ \phi\ } & & X \times T \\ & \searrow^{f\circ\mathscr{F}} & & {}^{G}\swarrow & \\ \Big\downarrow \pi_S & & \mathbf{C} & & \Big\downarrow \pi_T \\ S & & \xrightarrow[\ i\ ]{} & & T \end{array} \tag{A.9}$$

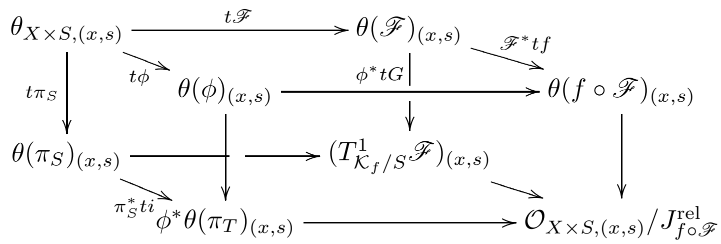
where $\phi$ is the $\mathcal{R}_{\text{e-un}}$-equivalence and $i$ is the inducing map from the base-space of $f \circ \mathscr{F}$ to the base space of $G$.

**Lemma A.5.1.** *There is a commutative diagram*

$$\begin{array}{ccc} (\pi_S)_*(T^1_{\mathcal{K}_f/S}\mathscr{F}) & \longrightarrow & (\pi_S)_*(\mathcal{O}_{X\times S}/J^{rel}_{f\circ\mathscr{F}}) \\ \uparrow & & \uparrow \\ \theta_S & \xrightarrow{\ ti\ } & \theta(i) \end{array} \tag{A.10}$$

*The vertical arrow on the left hand side is the Kodaira-Spencer map of $\mathscr{F}$ as a $\mathcal{K}_f$-deformation whereas the one on the right hand side is the pull-back of that of $G$ as an $\mathcal{R}$-deformation of $f \circ \mathscr{F}$.*

*Proof.* The diagram (A.9) induces a commutative cube at every $(x,s) \in X \times S$,




(A.11)

Let us say few words to explain the diagram. The commutativity for the top layer is simply the chain rule. In the vertical slice on the right, we have the canonical projections onto the respective quotients. Commutavity there simply follows from the fact that

$$(\mathscr{F}^* tf)(\theta\mathscr{F}(\theta_{X\times S}) + F^*(\mathrm{Der}(-\log f))) \subset t(f\circ\mathscr{F})(\theta_{X\times S}).$$

Finally on the bottom layer, we obtain the left-to-right maps by lifting vector fields and then projecting onto the respective quotients. For example, we lift $u \in \theta_S \otimes \mathcal{O}_{X\times S}$ to $\tilde{u}$ and apply $t\mathcal{F}$. As two lifts differ by a relative vector field, this defines a map $\theta(\pi_S) \to T^1_{\mathcal{K}_f}\mathscr{F}$. If we now compose with $\omega\pi_S\colon \theta_S \to \theta(\pi_S)$ we obtain the Kodaira-Spencer map of the deformation $\mathscr{F}$. Analogous reasoning defines the other map and the claim follows. □

**Proposition A.5.1.** *Assume that $G$ in (A.9) is an $\mathcal{R}_e$-miniversal deformation of the isolated singularity $f\circ F$. Then*

1. *if $m = m_0$, $i$ is an immersion into the discriminant $\Delta$ of $G$,*
2. *if $m = m_0 - 1$, the critical locus $\mathcal{C}$ of $i$ is $\pi_S(V(\mathscr{F}^* J_f))$ and $i\colon S\backslash\mathcal{C} \to T\backslash\Delta$ is an unramified covering.*

*Proof.* In any of two cases, since $G$ is a $\mathcal{R}_\mathrm{e}$-miniversal deformation, the vertical arrows of (A.10) are both isomorphisms. The exact sequences (A.5) and (A.4) show that the support of the cokernel of $ti$ is exactly the projection onto $S$ of $\mathcal{O}_{X\times S}/\mathscr{F}^* J_f$. As this last module is Cohen-Macaulay and supported inside $\mathscr{D} = V(f\circ\mathscr{F})$ we see that $D_s = V(f\circ F_s)$ is singular. Hence the set of values where $i$ is not submersive is contained in $\Delta$.
To conclude, the exact sequence (A.5) says that $i$ is injective in the case $m = m_0$, whereas for $m = m_0 - 1$, (A.4) says the critical locus $\mathcal{C}$ is the projection by $\pi_S$ of $\operatorname{supp}\operatorname{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f, \mathcal{O}_{X\times S})$ and hence equal to $\pi_S(V(\mathscr{F}^* J_f))$. □

**Remark A.5.2.** The Kodaira-Spencer map $\rho_G\colon \theta_T \to \mathcal{O}_{X\times T}/J^{\mathrm{rel}}_G$ defines Saito's $F$-manifold structure on $T$, and hence on $\theta(i) = \theta_T \otimes \mathcal{O}_S$. The Euler vector field $E_T$ of this $F$-manifold is given by the class of $G$ in the relative Jacobian algebra. The mapping $i$ respects the multiplication and Euler vector field in the sense that

$$\begin{aligned} ti(u \star v) &= ti(u) \star ti(v) \\ ti(E_S) &= E_T \circ i \end{aligned} \tag{A.12}$$

Note that in the case $m = m_0$ the multiplication is defined through the identification $T^1_{\mathcal{K}_f/S}\mathscr{F} \simeq \mathscr{F}^*(J_f)/J_{\mathscr{F}\circ f}$ (this is not an $F$-manifold as it lacks the unity). The second equation of (A.12) still holds if $f$ is quasi-homogeneous.

**Remark A.5.3.** The statement (2) in A.5.1 implies a conjecture in [6]. Let $F\colon \mathbf{C}^2 \to \mathrm{Sym}_n$ be a family of $n \times n$-symmetric matrices and $f\colon \mathrm{Sym}_n \to \mathbf{C}$ the determinant. Then the subvariety $\sum \subset S$ corresponding to values of the parameter space for which $F_s$ intersects the set of matrices of corank at least 2 is $(\pi_S)_* V(\mathscr{F}^* J_f)$ and hence coincides with $\mathcal{C}$.

**Remark A.5.4.** For simple matrix singularities, it is shown in [6] that $i$ is indeed finite. We conjecture this is the case for any $\mathcal{K}_f$-simple singularity.

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