Minoration de densité pour les diffusions à sauts.Calcul de Malliavin pour processus de sauts purs, applications à la finance.

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THÈSE

pour l'obtention du titre de

Docteur en Mathématiques Appliquées

(Arrêté du 25 avril 2002)

Présentée et soutenue publiquement
par

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le 5 Décembre 2006

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JURY

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A ma mère,

Remerciements

Je commencerai cette page en remerciant mes deux directeurs de thèse Agnès Sulem et Vlad Bally.
En acceptant de m'accueillir à l'INRIA au sein du projet MATHFI, je remercie vivement Agnès de m'avoir fourni un excellent cadre de travail. J'ai pu rencontrer et travailler avec des chercheurs de mon domaine comme Arturo Kohatsu-Higa, Nicolas Privault ou encore Peter Tankov.
Je remercie infiniment Vlad d'avoir accepté d'encadrer ma thèse. Combien d'heures passées dans ce bureau 4B125 de l'université de Marne-la-Vallée où, bloquée (et un peu déprimée à vrai dire...) par un problème, Vlad mettait toute son énergie à m'aider à le résoudre, persuadé que "*ça va marcher*" selon ses termes... Et ça marchait ! !
Merci pour tout ce temps que vous m'avez accordé quelque soit le jour (voire même l'heure...) de la semaine, et surtout merci pour votre gentillesse, votre générosité, et votre humour ! Vos encouragements et votre disponibilité m'ont permis d'aboutir à ce travail dont je vous serai toujours reconnaissante.
Consciente de l'investissement que cela implique, je remercie Mark Davis et Nicolas Privault d'avoir accepté d'être rapporteurs de ma thèse. Je remercie également Laurent Denis et Nizar Touzi de faire partie de mon jury de thèse. Merci à chacun d'avoir donné de votre temps.
Au cours de ces années, j'ai eu l'occasion de travailler avec mon "collègue" et maintenant ami Marouen Messaoud. Nous avons eu des échanges mathématiques très intéressants et je pense motivants pour chacun de nous. J'ai eu aussi l'occasion de rencontrer Arturo Kohatsu-Higa. Je te remercie Arturo pour ta disponibilité, ta "joie de vivre" communicative et pour tous les précieux conseils que tu m'as prodigués tant pour mes travaux de recherche que pour mes voyages au Pérou et au Japon !
Je remercie le CERMICS de m'avoir accueillie pendant ma première année de thèse, ainsi que l'équipe du laboratoire d'Analyse et de Mathématiques Appliquées de l'université de Marne-la-Vallée pour son accueil chaleureux. Je pense tout particulièrement à Mireille pour sa gentillesse et à mes amis doctorants et docteurs : le bureau des filles avec Linda et Margot (jeune et future mariées !), le bureau des garçons avec Vincent, Ahmed, Benoît, Mohammed ; François à l'autre bout du couloir et Etienne, maintenant Maître de Conférence à l'université d'Evry. Merci pour tous les bons moments passés ensemble.

Je pense aussi à mes amis qui d'une façon ou d'une autre ont su rendre cette période de ma vie très agréable : Geneviève, mes Juliette, Aurélia, Christelle, Stéphane, Guillaume, Arnaud, Benoît... Pardon à celles et ceux que j'aurais oubliés !
Je remercie particulièrement ma soeur, mon frère et leurs conjoints de m'avoir soutenue pendant ce travail. J'ai essayé de répondre tant bien que mal à leurs questions "stochastiques"... Comment ne pas remercier mon mari d'avoir supporté mes longues absences lorsque j'étais en conférence, parfois à l'autre bout du monde ? Délaissé, il trouvait refuge chez ses parents et sa belle-mère. Je vous remercie Maman, Régine et Guy d'avoir pris soin de lui avant qu'une solution ne soit trouvée : qu'il vienne avec moi ! Merci mon Flo pour ta gentillesse et ta compréhension.

Je ne peux terminer cette page sans remercier du fond du coeur les deux personnes sans qui cette thèse n'aurait jamais vu le jour : Frédéric, ou plutôt Bobby, pour les maths, et ma mère pour tout le reste...

Avant-Propos

Cette thèse se compose de trois parties, dont la première est indépendante des deux suivantes.

La première partie traite de la minoration de la densité des diffusions à sauts en utilisant un calcul de Malliavin conditionnel par rapport aux sauts, ce qui permet de se ramener au calcul de Malliavin standard basé sur le mouvement Brownien uniquement.

La deuxième partie a pour but d'établir des formules d'intégration par parties du type Malliavin pour les processus de sauts purs.
Pour cela, dans le premier chapitre, nous développons un calcul abstrait basé sur des variables aléatoires de densité localement régulière.
Puis, dans le deuxième chapitre, nous appliquons ce calcul aux amplitudes et temps de sauts de processus à sauts purs.

La troisième partie donne des applications en Mathématiques Financières des intégrations par parties établies dans la deuxième partie : elles sont utilisées dans des algorithmes de Monte-Carlo pour calculer les prix et les Delta d'options européennes, asiatiques et américaines.

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Résumé de la thèse I

1. Introduction

Le calcul sur les variations stochastiques, ou encore calcul de Malliavin, a été introduit dans les années soixante-dix par Paul Malliavin. Depuis, beaucoup de travaux ont été menés dans ce domaine, dont on distingue deux applications majeures.
La première concerne l'étude de l'existence et de la régularité de la densité d'une variable aléatoire par rapport à la mesure de Lebesgue. Quand elle existe, il s'agit de minorer et majorer cette densité et ses dérivées.
Dans son papier fondateur [Mal78], P. Malliavin a utilisé un critère d'absolue continuité pour prouver que, sous la condition de Hörmander, la loi d'un processus de diffusion a une densité régulière. Il a également obtenu des bornes exponentielles pour cette densité et ses dérivées. Ce procédé le mena à une preuve probabiliste du Théorème de Hörmander (voir [Nua95] et [Wat84]). Puis, ce calcul a été utilisé pour d'autres types de processus. En effet, sous certaines hypothèses appropriées, une large classe de fonctionnelles sur l'espace de Wiener (comme les solutions d'équations aux dérivées partielles stochastiques par exemple) ont une loi absolument continue, de densité régulière (voir [Nua95]).
Ces dernières années, depuis les articles fondateurs [FLL+99] et [FLLL01], de nouvelles applications du calcul de Malliavin sont apparues concernant les méthodes probabilistes numériques, plus particulièrement dans le domaine des mathématiques financières. Citons par exemple le calcul des sensibilités d'options (les Grecques) et le calcul d'espérances conditionnelles, qui interviennent dans la programmation dynamique pour calculer le prix d'options américaines.
L'outil principal du calcul de Malliavin est une formule d'intégration par parties du type :

$$\mathrm{E}\left[\phi'(F)\, G\right] = \mathrm{E}\left[\phi(F)\, H(F,G)\right] , \qquad \text{(I.1.1)}$$

où

- F est une variable aléatoire supposée régulière et non dégénérée 'au sens de Malliavin',
- $H(F,G)$ est une variable aléatoire, parfois appelée poids de Malliavin, qui dépend des 'opérateurs de Malliavin' de F et G, mais qui ne dépend pas de la fonction ϕ.

Voyons comment cette intégration par parties (I.1.1) est utilisée dans l'étude de l'existence et de la régularité de densités, et dans les méthodes numériques en Mathématiques Financières.

2. Existence et régularité de densité

La formule d'intégration par parties (I.1.1) et ses itérations permettent d'obtenir, sous des hypothèses appropriées sur la variable aléatoire F, une expression explicite de sa densité et de ses dérivées :

$$p_F(z) = \mathrm{E}\left[\mathbf{1}_{F\geq z}\, H(F,1)\right] , \tag{I.2.1}$$

$$p_F^{(k)}(z) = (-1)^k\, \mathrm{E}\left[\mathbf{1}_{F\geq z}\, H_{k+1}(F,1)\right] ,$$

où $H_{k+1}(F,1)$ est défini par la relation de récurrence :

$$H_0(F,1) = 1 \text{ et } H_{k+1}(F,1) = H(F, H_k(F,1))\,.$$

Remarquons que la représentation intégrale (I.2.1) permet d'obtenir des majorants pour la densité p_F. En effet, si F a des moments d'ordre n et $H(F,G)$ est de carré intégrable, l'inégalité de Bienaimé-Chebychev entraîne

$$p_F(z) \leq \sqrt{\mathbf{P}(F\geq z)}\ \parallel H(F,G)\parallel_2 \leq \frac{C}{z^{n/2}}\,.$$

Ainsi, $\lim_{x\to\infty} p_F(x) = 0$, et la vitesse de convergence est contrôlée par les queues de F. Alors que trouver des majorants pour la densité p_F paraît plutôt simple, minorer p_F s'avère être beaucoup plus complexe.
En effet, dans certains cas, il est possible de montrer que la densité est strictement positive (voir par example les travaux [BAL91], [MS97] ou [Nua95]), mais les techniques utilisées ne donnent que des résultats qualitatifs et non des minorants explicites. Sous une hypothèse d'uniforme ellipticité, Arturo Kohatsu-Higa dans [KH03] a développé une méthode permettant de calculer des minorants pour la densité de fonctionnelles définies sur l'espace de Wiener. Il applique alors ses résultats à l'équation stochastique de la chaleur. Puis, R. Dalang et E. Nualart, dans [DN04], appliquent cette méthode à la théorie du potentiel pour les équations aux dérivées partielles stochastiques hyperboliques.
Vlad Bally, dans [Bal06], a affaibli cette hypothèse d'uniforme ellipticité en la remplaçant par une hypothèse d'ellipticité locale autour d'une courbe déterministe, ce qui permet de traiter d'autres processus que les diffusions uniformément elliptiques, comme les intégrales stochastiques ou les solutions d'équations stochastiques non Markoviennes.

Dans la première partie de cette thèse, nous reprenons la méthode développée par V. Bally afin d'étendre ses résultats aux processus de sauts unidimensionnels contenant une partie continue dirigée par un mouvement Brownien.

3. Mathématiques Financières

3.1. Rappels

Depuis les travaux de F. Black, M. Scholes et R.C. Merton en 1973, les marchés financiers ont connu une expansion considérable et les produits échangés sont de plus en plus nombreux et sophistiqués. Les plus répandus sont les options. Les options de base sont les options d'achat ou de vente, appelées respectivement call et put. Ce sont des contrats passés entre le vendeur et l'acheteur de l'option qui donnent le droit à l'acheteur d'acquérir (pour un call) ou de vendre (pour un put) un bien financier à un prix (prix d'exercice ou strike) et à une date (maturité) convenus au préalable. Si l'option peut être exercée avant sa maturité, on parle d'option américaine, sinon d'option européenne. Puisque l'acheteur n'est pas obligé d'exercer son droit (si cela ne correspond pas à ses intérêts), il gagnera une fonction positive du prix des biens sous-jacents à l'option. Cette fonction est appelée fonction pay-off. Par exemple, pour un call de prix d'exercice K, la fonction pay-off est $\phi(x) = (x - K)_+$.
Bien-sûr, pour obtenir le droit de faire un gain sûr, l'acheteur doit payer une prime au vendeur. Dans la théorie moderne du calcul du prix d'options, on suppose qu'il n'y a pas d'opportunité d'arbitrage, c'est-à-dire de possibilité de faire des bénéfices sans prendre de risques. On considère également que le marché est complet ; autrement dit, on suppose que tout produit échangé sur le marché est réplicable par un portefeuille (dit de couverture) composé uniquement des actifs de base. En reliant ces deux hypothèses à la théorie des martingales, on a observé que l'absence d'opportunité d'arbitrage est équivalente à l'existence d'une probabilité équivalente à la probabilité historique, sous laquelle les processus des prix actualisés des actifs de base sont des martingales (voir par example [HP81], [DMW90] et [DS94]). Dans un marché complet, une telle probabilité est unique, on l'appelle la probabilité risque neutre.
Dans ce cadre, le prix d'une option européenne de sous-jacent $(S_t)_{t\geq 0}$, de maturité T et de fonction pay-off ϕ est donné par

$$P(0, S_0) = \mathrm{E}\left[\phi(S_T)\right] , \tag{I.3.1}$$

où E est l'espérance relative à la probabilité risque neutre.
Concernant le prix à la date t d'une option américaine de sous-jacent $(S_t)_{t\geq 0}$, de maturité T et de fonction pay-off ϕ, A. Bensoussan dans [Ben84] et I. Karatzas dans [Kar88] ont montré qu'il était relié à un problème de temps d'arrêt optimal de la

façon suivante :

$$P(t, S_t) = \sup_{\tau \in \Gamma_{t,T}} \mathrm{E}\left[\phi(S_\tau) \mid S_t\right] , \tag{I.3.2}$$

où $\Gamma_{t,T}$ est l'ensemble des temps d'arrêt à valeurs dans $[t, T]$.
Bien-sûr, pour calculer les prix donnés par les équations (I.3.1) et (I.3.2), nous devons avoir une modélisation des prix des actifs sous-jacents (typiquement, une action ou un indice boursier) à ces options.
F. Black et M. Scholes ont proposé dans [BS73] de modéliser la dynamique du cours du sous-jacent S_t par l'équation différentielle stochastique :

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t\,, \qquad S_0 = x\,, \tag{I.3.3}$$

où W est un mouvement Brownien standard et μ est une constante appelée 'drift'. Dans ce modèle, σ est une constante strictement positive indépendante du temps et du hasard qu'on appelle 'volatilité'. Elle mesure l'intensité du bruit auquel est soumis le sous-jacent.
Ce modèle présente deux avantages.
Il est simple car le processus S est alors un mouvement Brownien géométrique d'expression explicite :

$$S_t = x \exp\left(\sigma\, W_t + \left(\mu - \frac{\sigma^2}{2}\right)\, t\right) .$$

Le logarithme de S_t suit donc une loi gaussienne de moyenne $\mu - \dfrac{\sigma^2}{2}$ et de variance $\sigma^2\, t$.
Ce modèle a aussi l'avantage d'être maniable au sens où il donne lieu à des formules fermées pour le prix des calls et des puts européens. En effet, par example, le prix d'un call Européen de maturité T et de strike K (et donc de fonction pay-off $\phi(x) = (x-K)_+$) est donné au temps t par $P(t, S_t)$, où

$$P(t, y) = y\, N(d_+) - K\, e^{-r\,(T-t)}\, N(d_-)\,,$$

r étant le taux d'intérêt de l'actif sans risque du marché,
$N(d) = \displaystyle\int_{-\infty}^{d} \frac{1}{\sqrt{2\,\pi}}\, e^{-u^2/2}\, du$ désignant la fonction de répartition de la loi gaussienne centrée réduite, et

$$d_+ = \frac{\ln\left(e^{r\,(T-t)}\, \frac{y}{K}\right)}{\sigma\, \sqrt{T-t}} + \frac{1}{2}\, \sigma\, \sqrt{T-t}\,,$$
$$d_- = \frac{\ln\left(e^{r\,(T-t)}\, \frac{y}{K}\right)}{\sigma\, \sqrt{T-t}} - \frac{1}{2}\, \sigma\, \sqrt{T-t}\,.$$

De plus, ce modèle a l' avantage de donner aussi des formules fermées pour les Delta de calls et de puts européens, c'est-à-dire pour les quantités d'actifs risqués que doit contenir le portefeuille de couverture. En effet, pour gérer sa position globale en temps réel, le teneur du marché utilise généralement cinq indicateurs :

– Sensibilité par rapport à la condition initiale (i. e. Delta et Gamma),
– Sensibilité par rapport à la maturité (i. e. Theta),
– Sensibilité par rapport à la volatilité (i.e. Vega),
– Sensibilité par rapport au drift (i. e. Rhô).

En particulier, le Delta d'une position indique la variation de la valeur de la position par rapport à de faibles fluctuations du cours du sous-jacent. En d'autres termes, le Delta d'une option européenne de prix $P(0, S_0)$ donné par l'équation (I.3.1) est défini par :

$$\Delta(0, S_0) := \partial_{S_0} P(0, S_0) = \partial_{S_0} \mathrm{E}\left[\phi(S_T)\right] = \mathrm{E}\left[\phi'(S_T)\, \partial_{S_0} S_T\right] . \qquad \text{(I.3.4)}$$

Si $(S_t)_{t\in[0,T]}$ est modélisé par le modèle de Black-Scholes (I.3.3), alors le Delta vaut $\Delta(t, S_t)$ au temps t, où

$$\Delta(t, y) = N(d_+) .$$

Cependant, les formules fermées citées précédemment dépendent de la volatilité σ qui n'est pas directement observable. Dans la pratique, il est très difficile de déterminer la valeur à donner à cette volatilité constante. En effet, l'idée consiste à utiliser les prix d'options observées sur le marché pour évaluer la constante σ, appelée 'volatilité implicite'. Il s'agit de choisir la constante σ pour laquelle les prix théoriques correspondent aux prix observés sur le marché. Malheureusement, on se heurte vite aux imperfections du modèle de Black-Scholes : les constats empiriques faits à partir des données du marché montrent que contrairement à ce qui est prévu par ce modèle, la volatilité implicite n'est pas constante. Elle semble dépendre du prix d'exercice et de la maturité des options, et sa courbe présente même dans plusieurs cas une convexité par rapport aux prix d'exercice, un phénomène connu sous le nom de *smile*. Pour tenir compte de ces phénomènes empiriques, le modèle de Black-Scholes a dû être étendu.

Une approche largement répandue considère des modèles dits à volatilité locale, modèles où la volatilité σ est une fonction déterministe de la valeur de l'actif sous-jacent et du temps ($\sigma = \sigma(t, x)$), la seule source de bruit restant le mouvement Brownien W. C'est ce que proposa Bruno Dupire dans [Dup95].

Une autre manière d'étendre le modèle de Black-Scholes est d'autoriser la volatilité à être un processus stochastique gouverné par un deuxième bruit, généralement modélisé par un deuxième mouvement Brownien. On parle alors de modèles à volatilité stochastique.

Mais les processus de sauts sont de plus en plus utilisés sur les marchés (voir à ce sujet [CT03]). Par example, M. C. Merton proposa un modèle en 1976 dans [Mer76] et plus tard S. G. Kou en 2002 dans [Kou02] dont l'idée est simple : rajouter une composante de sauts, plus précisément un processus de Poisson composé, au mouvement Brownien qui est fondamentalement un processus continu. Dans le modèle de Merton, la loi des sauts est normale, et dans celui de Kou, les sauts suivent une loi exponentielle double asymétrique. C'est-à-dire, notant $(\Delta_i)_{i\in\mathbb{N}}$ les sauts :

$$\Delta_i = \begin{cases} \eta_1 \,, & \text{avec probabilité } p \,, \\ -\eta_2 \,, & \text{avec probabilité } q \,, \end{cases}$$

où $p, q \geq 0$, $p+q = 1$ et η_1, η_2 sont des variables aléatoires exponentielles de moyenne $1/\lambda_1$ et $1/\lambda_2$ respectivement, avec $\lambda_1 > 0$ et $\lambda_2 > 0$.
Enfin, plus généralement, les processus de Lévy ont été utilisés dans le cadre de modèles dits de Lévy exponentiels (voir par exemple [MCC98]).
Cependant, même dans le cas d'options européennes, il est en général impossible d'avoir une formule fermée d'évaluation du prix et du Delta dès que le sous-jacent ne suit plus un mouvement Brownien géométrique. Il faut donc se tourner vers des solutions numériques.
Une méthode consiste à utiliser le calcul de Malliavin pour calculer numériquement le prix et le Delta d'options. Lorsque les diffusions employées pour modéliser le cours du sous-jacent $(S_t)_{t\in[0,T]}$ sont log-normales, on peut utiliser le calcul de Malliavin standard, c'est-à-dire basé sur le mouvement Brownien contenu dans la diffusion. Ou encore, lorsque les modèles considérés (comme le modèle de Merton par exemple) ont une composante continue gouvernée par un mouvement Brownien et une partie à sauts dirigée par un processus de Poisson composé, on peut utiliser le calcul de Malliavin standard (c'est-à-dire basé sur le mouvement Brownien seulement), après avoir conditionné d'une façon appropriée par rapport à la composante à sauts. Ce procédé a été traité dans [DJ06], [FLT05] et [PD04].
Mais lorsque le cours du sous-jacent $(S_t)_{t\in[0,T]}$ est un processus de sauts purs, il faut utiliser un calcul basé sur les processus ponctuels de Poisson, puisqu'il n'y a plus de mouvement Brownien dans le modèle. [BGJ87] et a développé un tel calcul par rapport aux amplitudes de sauts, [CtP90], [Pri94] et [Den00] par rapport aux temps de sauts, et [Pic96b], [Pic96a] et [NV90] par rapport aux amplitudes et temps de sauts. Récemment, N. Bouleau dans [Bou03] a établi un calcul d'erreur basé sur le langage des formes de Dirichlet, ce qui lui a permis d'unifier les approches de [BGJ87] et [CtP90]. Un autre point de vue basé sur la décomposition en cahos a été traité dans [NkP04] et [VLUS02]. Puis, plusieurs papiers ont utilisé ces calculs dans des applications en finance et assurance : citons par exemple [KP04], [PW05] et [PW04].

3.2. Calcul de Malliavin et méthodes numériques

Delta d'options européennes

Rappelons que le Delta d'une option européenne de prix $P(0, S_0)$ (donné par l'équation (I.3.1)) est défini par l'équation (I.3.4), soit

$$\Delta(0, S_0) := \partial_{S_0} P(0, S_0) = \partial_{S_0} \mathrm{E}\left[\phi(S_T)\right] = \mathrm{E}\left[\phi'(S_T)\, \partial_{S_0} S_T\right] .$$

Si la fonction pay-off ϕ est discontinue (ϕ' est alors une distribution de Dirac par exemple), des problèmes se posent dans les simulations numériques d'un algorithme de Monte-Carlo pour calculer le Delta. Une intégration par parties du type Malliavin (I.1.1) appliquée à $F = S_T$ et $G = \partial_{S_0} S_T$ fait alors disparaître la dérivée de la fonction pay-off ϕ, et la remplace par un poids $H(S_T, \partial_{S_0} S_T)$ indépendant de ϕ :

$$\Delta(0, S_0) = \mathrm{E}\left[\phi(S_T)\, H(S_T, \partial_{S_0} S_T)\right] . \tag{I.3.5}$$

Mais le poids $H(S_T, \partial_{S_0} S_T)$ contient des opérateurs de Malliavin de S_T et $\partial_{S_0} S_T$, ce qui peut lui donner une grande variance. Une méthode de localisation développée dans [FLL+99] et [FLLL01] permet de la réduire.

Options américaines

Numériquement, le prix d'options américaines se calcule par une programmation dynamique (voir [Nev72]) : soit $0 = t_0 < t_1 < \ldots < t_N = T$ une subdivision de l'intervalle $[0, T]$ (où T est la maturité de l'option), et $(\overline{S}_{t_k})_{k=0,\ldots,N}$ une approximation du prix du sous-jacent $(S_t)_{t\in[0,T]}$, c'est-à-dire $\overline{S}_{t_k} \simeq S_{t_k}$. Alors $P(0, S_0) \simeq \overline{P}_0$ où $\overline{P}_0$ est calculé par l'algorithme rétrograde

$$\begin{aligned} &\overline{P}_{t_N} = \phi(\overline{S}_{t_N})\,, \\ &\overline{P}_{t_k} = \max\left\{\phi(\overline{S}_{t_k}), \mathrm{E}\left[\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) \mid \overline{S}_{t_k}\right]\right\}\,,\ k = N-1, \ldots, 0\,. \end{aligned} \tag{I.3.6}$$

Le Delta d'une option américaine de prix $P(0, S_0)$ sera alors approximé par $\overline{\Delta}_0$ calculé par l'algorithme :

$$\overline{\Delta}(\overline{S}_{t_1}) = \begin{cases} \phi'(\overline{S}_{t_1}) & , \text{si } \overline{P}_{t_1} < \phi(\overline{S}_{t_1})\,, \\ \partial_\alpha\, \mathrm{E}\left[\overline{P}_{t_2}(\overline{S}_{t_2}) \mid \overline{S}_{t_1} = \alpha\right]\Big|_{\alpha=\overline{S}_{t_1}} & , \text{si } \overline{P}_{t_1} > \phi(\overline{S}_{t_1})\,. \end{cases} \tag{I.3.7}$$

Et

$$\overline{\Delta}_0 = \mathrm{E}[\overline{\Delta}(\overline{S}_{t_1})]\,.$$

Ainsi, le calcul du prix et du Delta d'options américaines passe par l'évaluation d'espérances conditionnelles du type

$$\mathrm{E}\left[f(S_t) \mid S_s = \alpha\right] \text{ et } \partial_\alpha \mathrm{E}\left[f(S_t) \mid S_s = \alpha\right] . \tag{I.3.8}$$

En utilisant l'intégration par parties (I.1.1) et en l'itérant, dans le cas où le prix du sous-jacent $(S_t)_{t\in[0,T]}$ est modélisé par une diffusion log-normale, [LR00] et [BCZ03] établissent, sous des hypothèses appropriées, des formules de représentations pour les espérances conditionnelles (I.3.8) du type :

$$\mathrm{E}\left[f(S_t) \mid S_s = \alpha\right] = \frac{\mathrm{E}\left[f(S_t)\, H_\alpha(S_s, S_t)\right]}{\mathrm{E}\left[H_\alpha(S_s, S_t)\right]} , \tag{I.3.9}$$

et

$$\begin{aligned}\partial_\alpha \mathrm{E}\left[f(S_t) \mid S_s = \alpha\right] = {} & \frac{\mathrm{E}\left[f(S_t)\, \mathcal{H}_\alpha(S_s, S_t)\right]\, \mathrm{E}\left[H_\alpha(S_s, S_t)\right]}{\mathrm{E}\left[H_\alpha(S_s, S_t)\right]^2} \\ & - \frac{\mathrm{E}\left[f(S_t)\, H_\alpha(S_s, S_t)\right]\, \mathrm{E}\left[\mathcal{H}_\alpha(S_s, S_t)\right]}{\mathrm{E}\left[H_\alpha(S_s, S_t)\right]^2} ,\end{aligned} \tag{I.3.10}$$

où $\mathcal{H}_\alpha$ et H_α sont des poids provenant de la formule (I.1.1) et qui dépendent du paramètre α. Ils mettent ensuite en oeuvre un algorithme de Monte-Carlo pour calculer les représentations (I.3.9) et (I.3.10).

Dans les parties 2 et 3 de cette thèse, nous considèrerons des modèles unidimensionnels à sauts purs. Imitant les méthodes numériques décrites précédemment dans le cas des diffusions continues, nous allons établir un calcul du type Malliavin basé sur le bruit disponible, c'est-à-dire les amplitudes et les temps de sauts (puisqu'il n'y a plus de partie Brownienne dans le modèle), ce qui nous permettra d'obtenir une formule d'intégration par parties du type (I.1.1) et de l'itérer.
Nous pourrons alors calculer les sensibilités d'options européennes et asiatiques (où le sous-jacent $(S_t)_{t\in[0,T]}$ est remplacé par sa moyenne $\frac{1}{T}\int_0^T S_t\, dt$) en utilisant une formule du type (I.3.5), et nous pourrons calculer le prix et les sensibilités d'options américaines en utilisant des représentations d'espérances conditionnelles du type (I.3.9) et (I.3.10) via la programmation dynamique.
Les résultats exposés réfèrent en grande partie à [BM06a], [BBM07] et [BM06b].

4. Plan de la thèse et résultats nouveaux

4.1. Partie 1 : Minoration de densité des diffusions à sauts

Dans cette partie, nous allons minorer la densité d'une diffusion à sauts unidimensionnelle d'équation :

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t \int_{\mathbb{R}} c(s, a, X_{s^-})\, \widetilde{N}(ds, da)\,, \tag{I.4.1}$$

où B est un mouvement Brownien unidimensionnel, $N(dt, da)$ est la mesure associée à un processus ponctuel de Poisson, $ds\,\nu(da)$ son compensateur, et
$\widetilde{N}(ds, da) = N(ds, da) - ds\,\nu(da)$ est la martingale de Poisson compensée correspondante (voir Chapitre II pour plus de précisions).
Les coefficients σ et c vérifient les hypothèses :

Hypothèse I.1. On suppose que les coefficients σ et $(x \to c(s, a, x)) \in \mathcal{C}^5(\mathbb{R})$, et que
i) Il existe une constante $C_0 > 0$ telle que

$$|\sigma(x)| \leq C_0 \text{ et } \max_{n=1,\ldots,5} |\sigma^{(n)}(x)| \leq C_0\,,$$

ii) Il existe une fonction $\overline{c}(a)$ telle que $|c(u, a, x)| \leq \overline{c}(a)$ et
$\max_{n=1,\ldots,5} |\partial_x^n c(u, a, x)| \leq \overline{c}(a)$,
$\int_{\mathbb{R}} \overline{c}(a)^p\, \nu(da) < \infty$ pour tout $p \geq 2$.

Dans le premier chapitre, nous développons un calcul de Malliavin conditionnel par rapport aux sauts, permettant de nous ramener au calcul de Malliavin standard, c'est-à-dire basé sur le mouvement Brownien uniquement.

Dans le deuxième chapitre, nous minorons la densité de X_t en temps petit, c'est-à-dire : nous considérons la filtration

$$\mathcal{F}_t = \sigma(B_s, s \leq t,\, \widetilde{N}(s, A), s \leq t, A \in \mathcal{B}(\mathbb{R}))$$

et pour $0 < t_k < t_{k+1}$ fixés, nous minorons la densité conditionnelle de $X_{t_{k+1}}$ par rapport à $\mathcal{F}_{t_k}$.
C'est dans ce chapitre que la spécificité des sauts apparaît. En effet, l'inégalité de Burkhölder donne des résultats insatisfaisants pour les processus de sauts (voir par exemple [BGJ87], [DM80] ou encore [Pro90]). En effet

$$\mathrm{E}\left| \int_t^{t+\delta} \int_{\mathbb{R}} c(s, a, \omega)\, \widetilde{N}(ds, da) \right|^p \leq C\delta\,,$$

alors que dans le cas d'une intégrale stochastique relative à un mouvement Brownien, on obtient :

$$\mathrm{E}\left|\int_t^{t+\delta} u(s,\omega)\, dB_s\right|^p \leq C\delta^{p/2}\,.$$

On conclut que dans le cas des sauts, on ne peut monter en puissance quand p est grand. Ceci entraîne des difficultés notables et nous oblige alors à des localisations bien plus complexes que celles employées dans [Bal06].
En utilisant les arguments précédents, on obtient une minoration en temps petit.

Une fois cette minoration obtenue entre t_k et t_{k+1}, le troisième chapitre consiste à 'transmettre' ce résultat par 'chaîne' de $t_0 = 0$ à $t_N = T$ le long d'une courbe déterministe $(x_t)_{t\in[0,T]}$. C'est ce qu'on appelle les suites d'évolutions.

Enfin, dans le quatrième chapitre, nous appliquons les résultats précédemment obtenus dans un cadre abstrait à la diffusion (I.4.1), ce qui nous donne une minoration de la densité de X_T en un point fixé $y \in \mathbb{R}$.
Plus précisément, nous établissons le résultat suivant :
• On suppose qu'il existe une courbe continûment différentiable $(x_t)_{t\in[0,T]}$ telle que $x(0) = X_0$, $x(T) = y$, et dont la dérivée vérifie :
il existe $\overline{M} \geq 1$ et $h \geq 0$ tels que

$$\overline{M}\,|\partial_t x_t|^2 \geq |\partial_s x_s|^2 \text{ si } |t-s| \leq h\,.$$

On suppose de plus qu'il existe deux constantes $\overline{\lambda}$ et $\underline{\lambda}$ telles que pour tout $t \in [0,T]$, $0 < 2\,\underline{\lambda} \leq \sigma^2(x_t) \leq \frac{2}{3}\,\overline{\lambda}$.
• On introduit une constante $0 < r \leq \frac{\underline{\lambda}}{2\,C_0^2}$, où C_0 est la constante de lipschitz de σ introduite dans les hypothèses I.1.
• Pour $\zeta \in (0,1/2)$, on note

$$\delta_* = \left(\frac{1}{4\int_{|a|>\varepsilon_*} \overline{c}(a)\,\nu(da)}\right)^{1/(1/2-\zeta)} \wedge \delta(\underline{\lambda},\overline{\lambda})\,,$$

où ε_* vérifie $\displaystyle\int_{|a|\leq\varepsilon_*} \overline{c}^2(a)\,\nu(da) \leq \frac{\underline{\lambda}}{2}$ et $\delta(\underline{\lambda},\overline{\lambda})$ est une constante qui dépend de $\underline{\lambda}$ et $\overline{\lambda}$. On note alors

$$M(r,h) = \delta_* \wedge r \wedge h\,.$$

Alors, si X_T a une densité continue en $y \in \mathbb{R}$, notée $p_T(x_0,y)$, elle est minorée par

$$p_T(x_0,y) \geq \frac{e^{-4/\underline{\lambda}}}{8\sqrt{2\,\pi\,\overline{\lambda}}} \times \exp\left[-\theta\left(\frac{T}{M(r,h)} + \int_0^T 16\,\overline{M}^2\,|\partial_t x_t|^2\,dt\right)\right]\,,$$

où $\theta = \dfrac{4}{\underline{\lambda}} + \ln 32 + \dfrac{\ln(2\,\pi\,\overline{\lambda})}{2} + \ln \overline{M}$.

Remarque 4.1. *Pour avoir l'existence et la continuité de la densité de X_T, il suffit d'ajouter l'hypothèse suivante à notre cadre de travail :*
il existe $\eta > 0$ tel que

$$\forall (t,a,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}\,,\; |1 + \partial_x c(t,a,x)| \geq \eta > 0\,. \tag{I.4.2}$$

En effet, d'après les propriétés de la courbe elliptique $(x_t)_{t\in[0,T]}$, nous avons pour tout $y \in \mathbb{R}$, $|\sigma(x_0)|\,|y|^2 \geq \varepsilon\,|y|^2$, avec $\varepsilon > 0$. Alors, sous l'hypothèse supplémentaire (I.4.2), [BGJ87] (Théorème p. 14) affirme que la densité $p_T(x_0, y)$ existe et est continue.

4.2. Partie 2 : Intégration par parties pour processus de sauts purs

Dans le premier chapitre, nous développons un calcul abstrait du type Malliavin, ce qui nous permet, dans le chapitre suivant, de le baser indifféremment sur les amplitudes de sauts ou les temps de sauts d'un processus de Poisson.
Nous n'établissons pas un calcul infini-dimensionnel, au sens où nous ne considérons que des fonctionnelles simples $F = f(V_1, \ldots, V_n)$, c'est-à-dire d'un nombre fini de variables aléatoires $V_1, \ldots, V_n$. Les algorithmes considérés en finance n'employant que ce genre de fonctions, ceci ne représente pas une restriction gênante dans les applications numériques.
Le point important de ce chapitre est que nous établissons une formule d'intégration par parties du type (I.1.1), à la différence près qu'elle est 'localisée' sur un certain événement A :

$$\mathrm{E}\left[\phi'(F)\, G\, \mathbf{1}_A\right] = \mathrm{E}\left[\phi(F)\, H(F,G)\, \mathbf{1}_A\right]\;. \tag{I.4.3}$$

En effet, pour obtenir une formule d'intégration par parties, on a besoin de bruit qui, dans notre contexte, provient des amplitudes et des temps de sauts. Il faut donc avoir <u>au moins</u> un saut, c'est la signification de A.
De plus, à la différence des accroissements du mouvement Brownien, qui eux sont indépendants et identiquement distribués de loi absolument continue (avec une densité régulière), les temps de sauts n'ont pas de densité régulière par rapport à la mesure de Lebesgue, mais uniforme. Nous traitons donc dans cette thèse un cas plus général : nous ne supposons pas que les variables aléatoires $(V_i)_{i\in\mathbb{N}}$ sont indépendantes, mais nous travaillons avec la loi conditionnelle de V_i par rapport aux autres variables aléatoires V_j, $j \neq i$. De plus, nous supposons que la loi conditionnelle est absolument continue par rapport à la mesure de Lebesgue sur $\mathbb{R}$, et qu'elle a une densité $p_i = p_i(\omega, y)$ différentiable par morceaux en y.
Des termes de bord, correspondant aux points de discontinuités des densités p_i, vont alors apparaître dans l'intégration par parties, et seront gênants pour les simulations numériques. En effet, si par exemple, la loi conditionnelle de V_i a une densité sur

l'intervalle $(0,1)$, une intégration par parties entraîne des termes de bord en 0 et 1 :

$$\int_0^1 (f'\,g)(\omega,y)\,p_i(\omega,y)\,dy = (f\,g)(\omega,1) - (f\,g)(\omega,0) - \int_0^1 f(\omega,y)\ [g' + g\,\partial_y \ln p_i]\,(\omega,y)\,p_i(\omega,y)\,dy\,.$$

Afin de les éliminer, nous allons introduire dans les opérateurs de Malliavin des fonctions poids, notées $(\pi_i)_{i\in\mathbb{N}}$, qui sont nulles aux points de discontinuités des densités conditionnelles p_i. Et, en utilisant ces poids, l'intégration par parties précédente devient

$$\int_0^1 (f'\,g)(\omega,y)\,\pi_i(\omega,y)\,p_i(\omega,y)\,dy = -\int_0^1 f(\omega,y)\ [\pi_i\,(g' + g\,\partial_y \ln p_i) + \pi_i'\,g]\,(\omega,y)\,p_i(\omega,y)\,dy\,. \quad \text{(I.4.4)}$$

Par exemple, si la loi conditionnelle de V_i a une densité uniforme sur $(0,1)$, c'est-à-dire $p_i(\omega,y) = 1_{[0,1]}(y)$, on peut prendre

$$\pi_i(y) = y^\alpha(1-y)^\alpha, \text{ avec } \alpha \in (0,1)\,. \quad \text{(I.4.5)}$$

On obtient alors une relation de dualité entre les dérivées de Malliavin et l'intégrale de Skorohod, qui, au vu de la formule (I.4.4), dépend des poids $(\pi_i)_{i\in\mathbb{N}}$ et de leurs dérivées premières. Ce qui nous permet d'établir, sous des hypothèses appropriées et à la manière du calcul de Malliavin standard, une intégration par parties du type :

$$\mathrm{E}[\phi'(F)\,G\,\mathbf{1}_A] = \mathrm{E}[\phi(F)\,H_\pi(F,G)\,\mathbf{1}_A]\,, \quad \text{(I.4.6)}$$

où $H_\pi(F,G)$ est une variable aléatoire qui dépend des opérateurs de Malliavin et des poids $(\pi_i)_{i\in\mathbb{N}}$, et qui est définie par $H_\pi(F,G) = \delta_\pi(G\,\gamma_{\pi,F}\,DF)$, avec

– D, la dérivée de Malliavin de F,
– $\gamma_{\pi,F}$, l'inverse de la matrice de covariance de F,
– δ_π, l'intégrale de Skorohod de F.

Mais cette intégration par parties (I.4.6) est valide si $H_\pi(F,G)$ est intégrable sur A, ce qui fait apparaître une difficulté liée aux poids $(\pi_i)_{i\in\mathbb{N}}$. En effet, l'expression de $H_\pi(F,G)$ contient l'inverse des poids $\pi_i(V_i)^{-1}$ (dans $\gamma_{\pi,F}$) ainsi que leurs dérivées premières $\pi_i'(V_i)$ (dans δ_π). Reprenant l'exemple d'une densité uniforme sur $(0,1)$ où les poids sont définis par l'équation (I.4.5), nous avons

• $\pi_i'(\omega,y) = \alpha(y^{\alpha-1}(1-y)^\alpha - y^\alpha(1-y)^{\alpha-1})$. Ainsi, pour que $H_\pi(F,G)$ soit intégrable sur A, il ne faut pas que α soit trop petit.
• Par ailleurs, nous avons $\pi_i^{-1}(V_i) = \dfrac{1}{y^\alpha\,(1-y)^\alpha}$, et il ne faut donc pas que α soit

trop grand.
Il nous faut ainsi réaliser un équilibre entre les poids $(\pi_i)_{i\in\mathbb{N}}$ et leurs dérivées premières, ce qui donnera lieu à une condition dite de 'non-dégénérescence' du type : pour tout $i \geq 1$,

$$\mathrm{E}\left[\mathbf{1}_A\,(\det\gamma_{\pi,F})^2\,(1+|\pi_i'(V_i)|)\right] < \infty\,. \tag{I.4.7}$$

Nous nous intéressons ensuite à l'itération de l'intégration par parties (I.4.6) ainsi obtenue, ce qui signifie que nous établissons une formule d'intégration par parties du type :

$$\mathrm{E}\left[\phi'(F)\,H_\pi(F,G)\,\mathbf{1}_A\right] = \mathrm{E}\left[\phi(F)\,\mathcal{H}_\pi(F,G)\,\mathbf{1}_A\right]\,, \tag{I.4.8}$$

où $\mathcal{H}_\pi(F,G) = H_\pi(F, H_\pi(F,G))$.
De la même façon, l'intégration par parties itérée (I.4.8) est valable si la variable aléatoire $\mathcal{H}_\pi(F,G)$ est intégrable sur A. Or l'expression de $\mathcal{H}_\pi(F,G)$ contient les termes $\pi_i(V_i)\,\pi_i''(V_i)$. Reprenant l'exemple de la loi conditionnelle uniforme sur $(0,1)$ où les poids $(\pi_i)_{i\in\mathbb{N}}$ sont définis par l'équation (I.4.5), les dérivées secondes $\pi_i''(\omega,y)$ mettent en jeu les termes $y^{\alpha-2}\,(1-y)^\alpha$, $\alpha\in(0,1)$, qui ne sont jamais intégrables.
Pour résoudre cette difficulté, nous partitionnons en deux intervalles disjoints le support de la densité conditionnelle $p_i(\omega,y)$ des variables V_i. En effet, reprenant l'exemple où $p_i = \mathbf{1}_{[0,1]}$, on pose $[0,1]=[0,1/2]\cup[1/2,1]$ et on considère deux types de poids $(\pi_i^1)_{i\in\mathbb{N}}$ et $(\pi_i^2)_{i\in\mathbb{N}}$ tels que *Supp* $\pi_i^1 \subseteq [0,1/2)$ et *Supp* $\pi_i^2 \subseteq (1/2,1]$ pour tout $i\in\mathbb{N}$. Ce qui revient à prendre :

$$\pi_i^1(y) = \left(\frac{1}{2}-y\right)^\alpha y^\alpha \text{ et } \pi_i^2(y) = (1-y)^\alpha\left(y-\frac{1}{2}\right)^\alpha,\ \alpha\in(0,1)\,.$$

En faisant la première intégration par parties (I.4.6) avec les poids $(\pi_i^1)_{i\in\mathbb{N}}$ et en l'itérant (voir (I.4.8)) avec les poids $(\pi_i^2)_{i\in\mathbb{N}}$, la variable aléatoire $\mathcal{H}_\pi(F,G)$ devient

$$\mathcal{H}_\pi(F,G) = H_{\pi^2}(F, H_{\pi^1}(F,G))\,,$$

et contient les termes $\pi_i^2(V_i)\,(\pi_i^1)''(V_i)$. Puisque les poids $(\pi_i^1)_{i\in\mathbb{N}}$ et $(\pi_i^2)_{i\in\mathbb{N}}$ sont à supports disjoints, ces quantités sont nulles, ce qui éliminent les dérivées secondes des poids $(\pi_i^1)_{i\in\mathbb{N}}$. Mais le prix à payer est que l'on a besoin de plus de bruit, au sens où l'on ne peut traiter que les fonctionnelles simples qui ont aux moins quatre variables aléatoires : $F = f(V_1,\ldots,V_n)$, pour $n\geq 4$.
La fin de ce chapitre est consacrée aux applications de la formule d'intégration par parties (I.4.6) et de son itération (I.4.8). Concernant le calcul de densité, la différence avec le cas Wiener vient de la localisation sur A dans la formule d'intégration par parties. On ne regardera donc pas la loi de F (soit $\mathbf{P}\circ F^{-1}$), mais celle de $(\mathbf{1}_A\,\mathbf{P})\,F^{-1}$, l'image par F de la restriction de la probabilité $\mathbf{P}$ à A. Sous certaines conditions de non dégénérescence du type (I.4.7), on établiera des résultats d'existence et de régularité de la densité de $(\mathbf{1}_A\,\mathbf{P})\,F^{-1}$, et particulièrement des représentations intégrales

de cette densité et de ses dérivées quand elles existent.
Par ailleurs, on montrera comment la formule d'intégration par parties (I.4.6) permet de représenter, sous des hypothèses appropriées, les espérances conditionnelles du type $\mathrm{E}(G\,\mathbf{1}_A \mid F)$:

$$\mathrm{E}(G\,\mathbf{1}_A \mid F = z) = \frac{\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-z)\,H_\pi(F,G)\,\mathbf{1}_A\right)}{\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-z)\,H_\pi(F,1)\,\mathbf{1}_A\right)}\,\mathbf{1}_A\,, \tag{I.4.9}$$

avec la convention que cette quantité est nulle quand
$\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-z)\,H_\pi(F,1)\,\mathbf{1}_A\right) = 0$.

Une fois les intégrations par parties (I.4.6) et (I.4.8) obtenues dans un cadre abstrait, l'objet du deuxième chapitre est de les appliquer aux processus de sauts purs.
L'aléa disponible étant les amplitudes de sauts (notées $(\Delta_i)_{i\in\mathbb{N}}$) et les temps de sauts (notés $(T_i)_{i\in\mathbb{N}}$), trois cas sont alors possibles pour appliquer la formule (I.4.6) : on peut utiliser les amplitudes de sauts seulement (soit $V_i = \Delta_i$), les temps de sauts seulement (soit $V_i = T_i$), ou bien les deux à la fois. Une différence majeure, liée à la vérification de la condition de non dégénérescence (I.4.7), apparaît : l'hypothèse (I.4.7) sera satisfaite pour les temps de sauts s'il y a au moins quatre sauts. Mais pour les amplitudes de sauts, cette hypothèse sera vraie à partir d'un saut. Ce qui signifie que l'on peut appliquer l'intégration par parties (I.4.6) avec les temps de sauts en localisant sur $A =$ "*au moins quatre sauts*", et on peut l'appliquer avec les amplitudes de sauts sur $A =$ "*au moins un saut*".
Nous itérons ensuite l'intégration par parties (I.4.6) en utilisant l'aléa provenant des amplitudes de sauts seulement. Les résultats du chapitre précédent nous disent alors que quatre sauts sont nécessaires, c'est-à-dire la formule itérée (I.4.8) est vraie en localisant sur l'événement $A =$ "*au moins quatre sauts*".
Pour finir, nous appliquons ces résultats au calcul de densité de processus de sauts purs, quand la probabilité $\mathbf{P}$ est restreinte à l'événement $A =$ "*au moins un saut*" ou $A =$ "*au moins quatre sauts*" (puisque les intégrations par parties (I.4.6) et (I.4.8) sont vraies sur ces événements).
Il s'avère que quand la loi des amplitudes de sauts est régulière, nous obtenons des résultats d'existence et de régularité similaires au cas Wiener. Par contre, quand la loi présente des discontinuités, nous montrons que la densité existe et est de classe $\mathcal{C}^1(\mathbb{R})$, sans aller au-delà (les itérations d'intégration par parties étant de plus en plus complexes). Nous obtenons également des représentations intégrales de la densité et sa dérivée.
Enfin, lorsqu'on utilise une intégration par parties basée sur les temps de sauts, nous établissons une représentation intégrale de la densité, et nous montrons qu'elle est continue.

4.3. Partie 3 : Applications au calcul d'options financières

Dans cette partie, nous appliquons les résultats précédemment établis à la Finance. Les modèles considérés pour le cours du sous-jacent $(S_t)_{t\in[0,T]}$ seront du type Vasicek et géométrique, le mouvement Brownien étant remplacé par un processus de Poisson composé. Plus précisément, notant $(T_i)_{i\in\mathbb{N}}$ et $(\Delta_i)_{i\in\mathbb{N}}$ les temps et amplitudes de sauts du processus de Poisson composé, et $J_t := Card\{T_i \leq t\}$, le processus de comptage associé, nous considérons les modèles suivants :

$$S_t = x - \int_0^t r\,(S_u - \alpha)\,du + \sum_{i=1}^{J_t} \sigma\,\Delta_i\,, \tag{I.4.10}$$

et

$$S_t = x + \int_0^t r\,S_u\,du + \sigma \sum_{i=1}^{J_t} S_{T_i^-}\,\Delta_i\,. \tag{I.4.11}$$

Dans les deux chapitres de cette partie, nous traitons deux types d'options : options d'achat (call), dont la fonction pay-off est $\phi_c(x) = (x-K)_+$, et option digitale, dont la fonction pay-off est $\phi_d(x) = \mathbf{1}_{x\geq K}$.

Dans le premier chapitre, nous calculons le Delta d'options européennes et asiatiques. Nous appliquons l'intégration par parties (I.4.6) à $F = S_T$ et $G = \partial_x S_T$, en utilisant les temps de sauts seulement et amplitudes de sauts seulement, pour finalement obtenir une formule du type :

$$\partial_x \mathrm{E}\left[\phi(S_T)\,\mathbf{1}_{A_T}\right] = \mathrm{E}\left[\phi(S_T)\,H_\pi(S_T, \partial_x S_T)\,\mathbf{1}_{A_T}\right], \text{ pour } \phi = \phi_c \text{ ou } \phi = \phi_d\,,$$

et $A_T = \{J_T \geq 1\}$ dans le cas des amplitudes de sauts et $A_T = \{J_T \geq 4\}$ dans le cas des temps de sauts.
Après avoir calculé les estimateurs de Malliavin $H_\pi(S_T, \partial_x S_T)$ pour les modèles (I.4.10) et (I.4.11) considérés, nous mettons en oeuvre un algorithme de Monte-Carlo.
Il s'avère que l'approche par le calcul de Malliavin sera plus 'justifiée' que la méthode des différences finies dans le cas des options digitales. En effet, les différences finies et les estimateurs de Malliavin donnent des résultats numériques très proches pour le calcul du Delta d'options d'achat (call). Mais concernant les options digitales, les estimateurs de Malliavin ont beaucoup moins de variance que ceux obtenus par différences finies, ce qui s'explique par le fait que ϕ_d est plus discontinue que ϕ_c. Plus les pay-offs sont discontinus, plus l'approche par le calcul de Malliavin est performante.
Par ailleurs, on constate que les résultats numériques obtenus en utilisant les amplitudes de sauts seulement sont légèrement plus performants qu'en utilisant les temps de sauts seulement.

Parallèlement, nous regardons le modèle de Merton, au sens où nous ajoutons une composante continue au modèle géométrique (I.4.11), soit

$$S_t = x + \int_0^t r\, S_u\, du \; + \int_0^t \sigma\, S_u\, dW_u + \mu \sum_{i=1}^{J_t} S_{T_i^-}\, \Delta_i\,, \tag{I.4.12}$$

où W est un mouvement Brownien indépendant du processus de Poisson composé. Nous comparons les estimateurs de Malliavin obtenus en utilisant le mouvement Brownien seulement d'une part, et les amplitudes de sauts et le mouvement Brownien d'autre part. En comparant nos résultats à ceux de [PD04] (qui n'utilisait que le mouvement Brownien), il s'avère que plus on utilise de bruit disponible dans le modèle (c'est-à-dire le mouvement Brownien et les sauts via leurs amplitudes), plus les résultats numériques sont performants.

Dans le deuxième chapitre, nous traitons le calcul du prix et du Delta d'options américaines.
Pour cela, nous commençons par établir des formules de représentation d'espérances conditionnelles et de leur gradient du type (I.3.9) et (I.3.10), en appliquant le résultat (I.4.9) à $F = S_s$ et $G = S_t$, pour $0 \le s < t \le T$. La spécificité des sauts apparaît via la localisation de la formule (I.4.9) sur l'événement A. En effet, cette localisation entraîne des représentations localisées du type :

$$\begin{aligned}&\mathrm{E}\left(\phi(S_t)\, \mathbf{1}_{\{0<J_s<J_t\}} \mid S_s = \alpha\right)\\ &\qquad = \frac{\mathrm{E}\left(\phi(S_t)\, \mathbf{1}_{(0,\infty)}(S_s-\alpha)\, H_\pi(S_s,S_t)\, \mathbf{1}_{\{0<J_s<J_t\}}\right)}{\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(S_s-\alpha)\, H_\pi(S_s,S_t)\, \mathbf{1}_{\{0<J_s<J_t\}}\right)}\, \mathbf{1}_{\{0<J_s<J_t\}}\,,\end{aligned} \tag{I.4.13}$$

et notant $A_3 := \{3 < J_s; 3 < J_t - J_s\}$,

$$\begin{aligned}&\partial_\alpha \mathrm{E}\left(\phi(S_t)\, \mathbf{1}_{A_3} \mid S_s = \alpha\right)\\ &\quad = \frac{\mathrm{E}\left(\phi(S_t)\, \mathbf{1}_{(\alpha,\infty)}(S_s)\, \mathcal{H}_\pi(S_s,S_t)\, \mathbf{1}_{A_3}\right)\, \mathrm{E}\left(\mathbf{1}_{(\alpha,\infty)}(S_s)\, H_\pi(S_s,S_t)\, \mathbf{1}_{A_3}\right)}{\mathrm{E}\left(\mathbf{1}_{(\alpha,\infty)}(S_s)\, H_\pi(S_s,S_t)\, \mathbf{1}_{A_3}\right)^2}\, \mathbf{1}_{A_3}\\ &\quad - \frac{\mathrm{E}\left(\phi(S_t)\, \mathbf{1}_{(\alpha,\infty)}(S_s)\, H_\pi(S_s,S_t)\, \mathbf{1}_{A_3}\right)\, \mathrm{E}\left(\mathbf{1}_{(\alpha,\infty)}(S_s)\, \mathcal{H}_\pi(S_s,S_t)\, \mathbf{1}_{A_3}\right)}{\mathrm{E}\left(\mathbf{1}_{(\alpha,\infty)}(S_s)\, H_\pi(S_s,S_t)\, \mathbf{1}_{A_3}\right)^2}\, \mathbf{1}_{A_3}\,.\end{aligned} \tag{I.4.14}$$

Ainsi, pour le calcul du prix d'options américaines par exemple, nous allons approximer l'équation de la programmation dynamique (I.3.6) par une version localisée, c'est-à-dire :

$$\begin{aligned} \overline{u}_{t_N} &= \phi(\overline{S}_{t_N})\,, \\ \overline{u}_{t_k} &= \max\left\{\phi(\overline{S}_{t_k}), \mathrm{E}\left[\overline{u}_{t_{k+1}}(\overline{S}_{t_{k+1}})\,\mathbf{1}_{\{0<J_{t_k}<J_{t_{k+1}}\}} \mid \overline{S}_{t_k}\right]\right\}\,,\; k = N-1, \ldots, 0\,. \end{aligned} \tag{I.4.15}$$

Nous pourrons alors utiliser les représentations (I.4.13) et (I.4.14) dans un algorithme de Monte-Carlo.
Finalement, nous appliquons l'algorithme localisé (I.4.15) précédemment obtenu dans un cadre abstrait au modèle géométrique (I.4.11). Nous calculons pour cela les variables aléatoires $H_\pi(S_s, S_t)$ et $\mathcal{H}_\pi(S_s, S_t)$ qui apparaissent dans les formules de représentation (I.4.13) et (I.4.14), puis les résultats numériques nous permettent de calculer le prix et le Delta d'un call américain dont le prix du sous-jacent suit un modèle géométrique.

Première partie

Minoration de densité des diffusions à sauts

Cadre de travail – Notations II

Introduisons le modèle de diffusion à sauts unidimensionnel $(X_t)_{t\geq 0}$ avec lequel nous allons travailler dans cette partie. Les notations utilisées réfèrent à [IW89].
Soit un espace de probabilités $(\Omega, \widetilde{\mathcal{F}}, \mathbf{P})$ muni d'une filtration $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$. Sur cet espace, on considère un mouvement Brownien $(B_t)_{t\geq 0}$ unidimensionnel et un processus ponctuel de Poisson $(N(t,A))_{t\geq 0, A\in\mathcal{B}(\mathbb{R})}$. Pour tous $t \geq 0, A \in \mathcal{B}(\mathbb{R})$, on note $\widehat{N}(t,A) = \mathrm{E}\left[N(t,A)\right] = t\,\nu(A)$ le compensateur tel que $\widetilde{N}(t,A) := N(t,A) - \widehat{N}(t,A)$ est une $(\widetilde{\mathcal{F}}_t)$-martingale. On considère l'équation différentielle stochastique suivante :

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t \int_{\mathbb{R}} c(s,a,X_{s^-})\, \widetilde{N}(ds,da)\,, \qquad \text{(II.0.1)}$$

où les coefficients σ et c sont mesurables et vérifient : il existe $K > 0$ tel que

$$|\sigma(x)|^2 + \sup_{s\in[0,T]} \int_{\mathbb{R}} |c(s,a,x)|^2\, \nu(da) \leq K\,(1+|x|^2)\,,\ x \in \mathbb{R}\,. \qquad \text{(II.0.2)}$$

Si, de plus, les fonctions σ et c vérifient la condition de Lipschitz :

$$|\sigma(x)-\sigma(y)|^2 + \sup_{s\in[0,T]} \int_{\mathbb{R}} |c(s,a,x)-c(s,a,y)|^2\, \nu(da) \leq K\,|x-y|^2\,,\ x,y \in \mathbb{R}\,, \qquad \text{(II.0.3)}$$

alors, [IW89] (voir Théorème p. 231) affirme qu'il existe un unique processus $(X_t)_{t\geq 0}$ solution de l'équation (II.0.1), $(\widetilde{\mathcal{F}}_t)$-adapté, continu à droite et ayant des limites à gauche, tel que $X_0 = x_0$.
Dans notre cadre, nous supposerons que :

Hypothèse II.1. Les coefficients σ et c vérifient la condition (II.0.2) et les fonctions $(x \to c(s,a,x))$, $\sigma \in \mathcal{C}^5(\mathbb{R})$. De plus :
i) Il existe une constante $C_0 > 0$ telle que

$$|\sigma(x)| \leq C_0 \text{ et } \max_{n=1,\ldots,5} |\sigma^{(n)}(x)| \leq C_0\,,$$

ii) Il existe une fonction mesurable $\overline{c}(a)$ telle que pour tous $(u,x) \in [0,T]\times\mathbb{R}$, $|c(u,a,x)| \leq \overline{c}(a)$ et $\max_{n=1,\ldots,5} |\partial_x^n c(u,a,x)| \leq \overline{c}(a)$, $\int_{\mathbb{R}} \overline{c}(a)^p\, \nu(da) < \infty$, $\forall p \geq 2$.

Le but est de donner un minorant de la densité de X_T en un point fixé $y \in \mathbb{R}$. Présentons le cadre dans lequel nous travaillons.
On se donne la tribu

$$\mathcal{F}_t = \sigma(B_s, s \leq t,\ \widetilde{N}(s, A), s \leq t, A \in \mathcal{B}(\mathbb{R}))\,, \tag{II.0.4}$$

et une subdivision $0 = t_0 < t_1 < \ldots < t_N = T$ de $[0, T]$.
Le point principal de notre démarche est d'obtenir une minoration en temps petit de X_t, c'est-à-dire un minorant de la densité conditionnelle de $X_{t_{k+1}}$ par rapport à $\mathcal{F}_{t_k}$. On note $p_k(\omega, z)$ cette densité conditionnelle. Cette minoration sera valable sur une boule, c'est-à-dire sur un événement $\mathcal{F}_{t_k}$-mesurable A_k vérifiant

$$A_k \subseteq \{\omega / |X_{t_k}(\omega) - z| \leq \sqrt{\delta_k}\}\,,$$

où $\delta_k := t_{k+1} - t_k$ est le pas de temps de la subdivision.
Fixons t_k, un point $z \in \mathbb{R}$ et l'événement A_k correspondant.
Notons $\sigma_k := \sigma(X_{t_k})$. On suppose qu'il existe deux réels strictement positifs $\underline{\lambda}$ et $\overline{\lambda}$ tels que

Hypothèse II.2.

$$(H_1, A_k, z) \qquad \underline{\lambda} \leq \sigma_k^2 \leq \overline{\lambda}\ , \text{ pour tout } \omega \in A_k\,.$$

Cette hypothèse est une condition d'ellipticité. Donnons maintenant une deuxième hypothèse qui exprime essentiellement la 'régularité'. Introduisons pour cela quelques notations.
On choisit $\varepsilon_* > 0$ vérifiant

$$\int_{|a| \leq \varepsilon_*} \overline{c}^2(a)\, \nu(da) \leq \frac{\underline{\lambda}}{2}\,. \tag{II.0.5}$$

On écrit $X_{t_{k+1}} = G_k + R_k$, où G_k et R_k sont respectivement appelés la partie principale et le reste, et sont définis par

$$G_k := X_{t_k} + \int_{t_k}^{t_{k+1}} \sigma(X_{t_k})\, dB_s + \int_{t_k}^{t_{k+1}} \int_{|a| \leq \varepsilon_*} c(s, a, X_{t_k})\, \widetilde{N}(ds, da)\,, \tag{II.0.6}$$

et

$$R_k := \int_{t_k}^{t_{k+1}} [\sigma(X_s) - \sigma(X_{t_k})]\ dB_s + \int_{t_k}^{t_{k+1}} \int_{|a| > \varepsilon_*} c(s, a, X_{s^-})\, \widetilde{N}(ds, da) + \int_{t_k}^{t_{k+1}} \int_{|a| \leq \varepsilon_*} [c(s, a, X_{s^-}) - c(s, a, X_{t_k})]\ \widetilde{N}(ds, da)\,. \tag{II.0.7}$$

On introduit l'hypothèse qui suit afin que le reste R_k soit petit en un sens approprié. Plus précisément, pour $t, \delta > 0$ fixés, notant D^i la dérivée de Malliavin d'ordre i, on introduit les normes de Sobolev :

$$|G|_{t,\delta,i} := \left(\int_{[t,t+\delta)^i} |D^i_{s_1,\dots,s_i} G|^2 \, ds_1 \dots ds_i \right)^{1/2} ,$$

et on suppose que R_k est cinq fois différentiable au sens de Malliavin.

Hypothèse II.3. Soit $\zeta \in (0, 1/2)$. Notons $\overline{\varepsilon} := \dfrac{\zeta}{4\,(1+\zeta)}$.
On note $\mathrm{E}_{\mathcal{F}_{t_k}}$ l'espérance conditionnelle par rapport à $\mathcal{F}_{t_k}$, et on suppose que pour tout $\omega \in A_k$,

$$(H_2, A_k, z) \quad \left[\mathrm{E}_{\mathcal{F}_{t_k}} \left(\left(\sum_{i=0}^{5} |R_k|^2_{t_k,\delta_k,i} \right)^{1+\zeta} (\omega)\, \mathbf{1}_{B_{k,\zeta}}(\omega) \right) \right]^{1/(1+\zeta)} \leq C \delta_k^{1+4\,\overline{\varepsilon}} ,$$

où l'événement $B_{k,\zeta}$ est défini par :

$$B_{k,\zeta} := \{\overline{R}_k \leq \delta_k^{\zeta+1/2}\} , \tag{II.0.8}$$

avec

$$\overline{R}_k := \int_{t_k}^{t_{k+1}} \int_{|a|>\varepsilon_*} \overline{c}(a)\, N(ds, da) . \tag{II.0.9}$$

Revenons à l'évaluation de la densité conditionnelle $p_k(\omega, z)$. On écrit formellement,

$$p_k(\omega, z) = \mathrm{E}_{\mathcal{F}_{t_k}} \left(\delta_0(G_k + R_k - z) \right) , \tag{II.0.10}$$

où δ_0 est la fonction Dirac. La condition (H_2, A_k, z) de l'Hypothèse II.3 nous dit que le reste R_k (localisé sur $B_{k,\zeta}$) est négligeable par rapport à la partie principale G_k (R_k est essentiellement de l'ordre de δ_k alors que G_k est de l'ordre de $\sqrt{\delta_k}$). Un développement de Taylor autour de G_k dans l'équation (II.0.10) entraîne alors que la minoration de $p_k(\omega, z)$ est 'similaire' à celle de $p_{G_k}(\omega, z) := \mathrm{E}_{\mathcal{F}_{t_k}} (\delta_z(G_k))$. Cette dernière est bien contrôlée puisque G_k est une variable aléatoire Gaussienne conditionnellement aux sauts. De plus, ε_* est choisi suffisamment petit dans (II.0.5) pour que la partie à sauts (correspondant à l'intégrale relative à la mesure martingale $\widetilde{N}$) soit petite par rapport à la partie gaussienne.
Une fois cette minoration en temps petit obtenue (c'est-à-dire entre t_k et t_{k+1}), elle est 'transmise' par 'chaîne' de t_0 à $t_N = T$ le long d'une courbe déterministe $(x_t)_{t\in[0,T]}$.
Puis, appliquant les résultats précédents (obtenus dans un cadre abstrait) à la diffusion (II.0.1), on obtient la minoration suivante :

• On suppose qu'il existe une courbe continûment différentiable $(x_t)_{t\in[0,T]}$ telle que $x(0) = X_0$, $x(T) = y$, et dont la dérivée vérifie :
il existe $\overline{M} \geq 1$ et $h \geq 0$ tels que

$$\overline{M}\,|\partial_t x_t|^2 \geq |\partial_s x_s|^2, \text{ si } |t-s| \leq h\,.$$

On suppose de plus qu'il existe deux constantes $\overline{\lambda}$ et $\underline{\lambda}$ telles que pour tout $t \in [0,T]$, $0 < 2\,\underline{\lambda} \leq \sigma^2(x_t) \leq \frac{2}{3}\,\overline{\lambda}$.
• On introduit une constante $0 < r \leq \dfrac{\underline{\lambda}}{2\,C_0^2}$, où C_0 est la constante de lipschitz de σ introduite dans les hypothèses II.1.
• On note

$$\delta_* = \left(\frac{1}{4\int_{|a|>\varepsilon_*}\overline{c}(a)\,\nu(da)}\right)^{1/(1/2-\zeta)} \wedge\,\delta(\underline{\lambda},\overline{\lambda})\,,$$

où ε_* vérifie l'équation (II.0.5) et $\delta(\underline{\lambda},\overline{\lambda})$ est une constante qui dépend de $\underline{\lambda}$ et $\overline{\lambda}$. On note alors

$$M(r,h) = \delta_* \wedge r \wedge h\,.$$

On obtient

Théorème II.1:
Soit $p_T(x_0,y)$ une densité continue de X_T en $y \in \mathbb{R}$. Alors,

$$p_T(x_0,y) \geq \frac{e^{-4/\underline{\lambda}}}{8\sqrt{2\,\pi\,\overline{\lambda}}} \times \exp\left[-\theta\left(\frac{T}{M(r,h)} + \int_0^T 16\,\overline{M}^2\,|\partial_t x_t|^2\,dt\right)\right]\,,$$

où $\theta = \dfrac{4}{\underline{\lambda}} + \ln 32 + \dfrac{\ln(2\,\pi\,\overline{\lambda})}{2} + \ln \overline{M}$.

Calcul de Malliavin conditionnel III

Dans ce chapitre, nous utilisons un calcul de Malliavin basé sur le mouvement Brownien B, ce qui est essentiellement le calcul standard développé dans [Nua95], à la différence près que nous utilisons une version conditionnelle par rapport au processus de sauts N.

1. Opérateurs différentiels

Avant d'introduire les opérateurs différentiels, commençons par des notations :

Notations: *$\mathcal{F}_t^B$ (respectivement $\mathcal{F}_t^N$) est la filtration engendrée par le mouvement Brownien (respectivement le processus de sauts N), soit*

$$\begin{aligned} \mathcal{F}_t^B &= \sigma(B_s, s \le t) \vee \mathcal{N}\,, \\ \mathcal{F}_t^N &= \sigma\left(N(s,A), s \le t, A \in \mathcal{B}(\mathbb{R})\right) \vee \mathcal{N}\,, \end{aligned}$$

où $\mathcal{N}$ est l'ensemble des événements de mesure nulle dans $\widetilde{\mathcal{F}}_\infty$.
On note

$$\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{F}_t^N \text{ et } \mathcal{G}_t = \mathcal{F}_t^B \vee \mathcal{F}_\infty^N\,.$$

• Pour tout $n \in \mathbb{N}$, on note $t_k^n = k\,2^{-n}$ et $\Delta_n^k(B) := B(t_k^n) - B(t_{k-1}^n)$.
Une fonctionnelle simple est une variable aléatoire F qui s'écrit :

$$F = f(\omega, \Delta_n^1(B), \ldots, \Delta_n^{2^n}(B))\,, \tag{III.1.1}$$

avec $f : \Omega \times \mathbb{R}^{d\times 2^n} \to \mathbb{R}$, $\mathcal{F}_\infty^N \times \mathcal{B}(\mathbb{R})$ mesurable et telle que pour tout $\omega \in \Omega$, $(x \to f(\omega, x))$ est de classe $\mathcal{C}_c^\infty(\mathbb{R}^{d\times 2^n}, \mathbb{R})$.
On note $\mathcal{S} \in \mathrm{L}^2(\Omega, \widetilde{\mathcal{F}}, \mathbf{P})$ l'ensemble des fonctionnelles simples.
• Un processus $U : [0,\infty) \times \Omega \to \mathbb{R}$ est dit simple s'il existe $n \in \mathbb{N}^*$ tel que

$$U(t,\omega) = \sum_{k=0}^{\infty} \mathbf{1}_{[t_n^k, t_n^{k+1})}(t)\, F_k(\omega)\,,$$

où F_k sont des fonctionnelles simples.
On note $\mathcal{P} \in \mathrm{L}^2\left(\Omega :\ \mathrm{L}^2([0,\infty))\right)$ l'ensemble des processus simples.

Définition III.1. On définit les dérivées de Malliavin d'ordre $m \geq 1$, $D^m : \mathcal{S} \to \mathrm{L}^2\left(\Omega : \mathrm{L}^2([0,\infty))^m\right)$ par :

$$D^m_{s_1,\ldots,s_m} F = \sum_{k_1,\ldots,k_m=0}^{\infty} \prod_{i=1}^{m} \mathbf{1}_{[t_n^{k_i}, t_n^{k_i+1})}(s_i) \frac{\partial^m f}{\partial x_{k_1} \ldots \partial x_{k_m}}(\omega, \Delta_n^1(B), \ldots, \Delta_n^{2^n}(B)) .$$

• On définit les normes

$$\|F\|_{m,p} = \left(\mathrm{E}\,|F|^p\right)^{1/p} + \sum_{n=1}^{m} \left[\mathrm{E} \left(\int_{[0,\infty)^n} \left| D^n_{s_1,\ldots,s_n} F \right|^2 \, ds_1 \ldots ds_n \right)^{p/2} \right]^{1/p} .$$

$\mathrm{D}^{m,p}$ est l'adhérence de $\mathcal{S}$ pour cette norme.
Jusqu'ici, il n'y a pas de différences avec les définitions standard du calcul de Malliavin. Le processus de sauts N étant indépendant du mouvement Brownien B, il est représenté par l'argument 'ω' dans la définition (III.1.1) des fonctionnelles simples. Mais dans la suite, nous allons utiliser un calcul de Malliavin conditionnel, au sens où pour $t > 0$ et $\delta > 0$ fixés, on veut utiliser une intégration par parties par rapport au mouvement Brownien B_s, $s \in [t, t+\delta)$.
La classe des fonctionnelles dérivables reste inchangée, c'est-à-dire $\mathrm{D}^{m,\infty} := \bigcap_{p \geq 1} \mathrm{D}^{m,p}$.
La localisation apparaît dans le produit scalaire défini sur l'espace des processus simples $\mathcal{P}$, soit

$$\langle U, V \rangle_{t,\delta} = \int_t^{t+\delta} U_s \times V_s \, ds \text{ et } |U|^2_{t,\delta} = \langle U, U \rangle_{t,\delta} . \qquad \text{(III.1.2)}$$

On définit alors les normes de Sobolev pour $F \in \mathrm{D}^{i,2}$ par

$$|F|_{t,\delta,i} := \left(\int_{[t,t+\delta)^i} |D^i_{s_1,\ldots,s_i} F|^2 \, ds_1 \ldots ds_i \right)^{1/2} . \qquad \text{(III.1.3)}$$

On note $\mathrm{E}_{\mathcal{F}_t}$ (respectivement $\mathrm{E}_{\mathcal{G}_t}$) l'espérance conditionnelle sachant $\mathcal{F}_t$ (respectivement $\mathcal{G}_t$), c'est-à-dire $\mathrm{E}_{\mathcal{F}_t}(\Phi) = \mathrm{E}(\Phi \mid \mathcal{F}_t)$. $\mathrm{E}_{\mathcal{G}_t}$ signifie que l'on n'intègre que par rapport au mouvement Brownien B_s, pour $s \geq t$.

2. Intégration par parties conditionnelle

Rappelons que dans le cas du calcul de Malliavin standard, une formule d'intégration par parties s'obtient à partir d'une formule de dualité et d'un opérateur de dérivation. Dans la version conditionnelle que nous voulons mettre en place, ces mêmes conditions demandent d'être remplies.
La définition (III.1.1) des dérivées de Malliavin entraînent que ces opérateurs vérifient bien les formules de dérivation en chaîne.

Soient $t > 0$ et $\delta > 0$ fixés. Afin d'obtenir une formule de dualité en utilisant le mouvement Brownien sur $[t, t+\delta)$ uniquement, nous allons conditionner par rapport à la tribu $\mathcal{G}_t$, ce qui 'enlève' le processus de sauts et le mouvement Brownien B_s pour $s < t$, et nous allons utiliser le produit scalaire localisé défini par l'équation (III.1.2), ce qui 'enlève' le mouvement Brownien B_s pour $s \geq t+\delta$. Il ne 'reste' donc plus que le mouvement Brownien B_s pour $s \in [t, t+\delta)$.
On définit l'intégrale de Skorohod, $\delta_{t,\delta}$, sur $[t, t+\delta)$ comme étant l'adjoint de la dérivée de Malliavin D pour le produit scalaire $\langle ., . \rangle_{t,\delta}$, soit

$$\mathrm{E}_{\mathcal{G}_t}(\langle DF, U \rangle_{t,\delta}) = \mathrm{E}_{\mathcal{G}_t}(F\, \delta_{t,\delta}(U))\,.$$

Et on définit ensuite l'opérateur d'Ornstein-Uhlenbeck $\mathrm{L}_{t,\delta}$ par rapport au mouvement Brownien sur $[t, t+\delta)$ par $\mathrm{L}_{t,\delta}(F) := \delta_{t,\delta}(DF)$, soit encore

$$\mathrm{L}_{t,\delta}(F) = \int_t^{t+\delta} D_s F\, dB_s\,.$$

On a ainsi la formule de dualité conditionnelle suivante :

$$\mathrm{E}_{\mathcal{G}_t}(\langle DF, DG \rangle_{t,\delta}) = \mathrm{E}_{\mathcal{G}_t}(F\, \mathrm{L}_{t,\delta}(G)) = \mathrm{E}_{\mathcal{G}_t}(G\, \mathrm{L}_{t,\delta}(F))\,.$$

Notons que si $F \in \mathrm{D}^{5,p}$, alors $F \in Dom(\mathrm{L})$, où L est l'opérateur d'Ornstein-Uhlenbeck standard, et on a $\mathrm{L}(F) \in \mathrm{D}^{3,p}$. On peut donc appliquer les inégalités de Meyer (voir [Nua95]) : il existe une constante $C_p > 0$ telle que

$$\left[\mathrm{E}_{\mathcal{G}_t}\left(\sum_{i=0}^{3} |\mathrm{L}_{t,\delta}(F)|^2_{t,\delta,i}\right)^{p/2}\right]^{1/p} \leq C_p \left[\mathrm{E}_{\mathcal{G}_t}\left(\sum_{i=0}^{5} |F|^2_{t,\delta,i}\right)^{p/2}\right]^{1/p}.$$

En particulier, puisque $\mathcal{F}_t \subseteq \mathcal{G}_t$,

$$\left[\mathrm{E}_{\mathcal{F}_t}\left(\sum_{i=0}^{3} |\mathrm{L}_{t,\delta}(F)|^2_{t,\delta,i}\right)^{p/2}\right]^{1/p} \leq C_p \left[\mathrm{E}_{\mathcal{F}_t}\left(\sum_{i=0}^{5} |F|^2_{t,\delta,i}\right)^{p/2}\right]^{1/p}. \qquad \text{(III.2.1)}$$

Avant de donner la formule d'intégration par parties conditionnelle, introduisons une dernière définition.

Définition III.2. Soit un événement $\mathcal{F}_t$-mesurable A fixé.
On note D^k_A la classe des variables aléatoires $G \in \bigcap_{p\in\mathbb{N}} \mathrm{D}^{k,p}$ telle que $G(\omega) = 0$ et $D^i G(\omega) = 0$, $i = 1, \ldots, k$, si $\omega \notin A$.

Puisque nous travaillerons par la suite dans un cadre uni-dimensionnel, nous introduisons la matrice de covariance uni-dimensionnelle :

Soit $F \in D^{1,2}$. On définit la matrice de covariance conditionnelle (c'est-à-dire correspondant au mouvement Brownien B_s, $s \in [t, t+\delta)$) par

$$\phi_{t,\delta,F} := \langle DF, DF \rangle_{t,\delta} = \int_t^{t+\delta} |D_s F|^2 \, ds \,.$$

Si $\phi_{t,\delta,F}$ est inversible, on note $\gamma_{t,\delta,F}$ son inverse.
Nous obtenons la formule d'intégration par parties conditionnelle suivante :

Théorème III.1:
Soit $F \in \mathrm{D}^{2,\infty}$. Soient $A \in \mathcal{F}_t$ fixé et $G \in \mathrm{D}^1_A$.
On suppose que $\phi_{t,\delta,F}$ est inversible sur A, et que

$$\left[\mathrm{E}_{\mathcal{G}_t}\left(\gamma^p_{t,\delta,F}\, \mathbf{1}_A\right)\right]^{1/p} < \infty \text{ pour tout } p \in \mathbb{N} \,. \tag{III.2.2}$$

Alors pour toute fonction $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$,

$$\mathrm{E}_{\mathcal{G}_t}(\psi'(F)\, G) = \mathrm{E}_{\mathcal{G}_t}(\psi(F)\, H(F,G)) \,, \tag{III.2.3}$$

où

$$H(F,G) = \delta_{t,\delta}(G\, \gamma_{t,\delta,F}\, DF) = G\, \gamma_{t,\delta,F}\, \mathrm{L}_{t,\delta}(F) + \langle D(G\, \gamma_{t,\delta,F}), DF \rangle_{t,\delta} \,. \tag{III.2.4}$$

En supposant de plus que $F \in \mathrm{D}^{3,\infty}$ et $G \in \mathrm{D}^2_A$, on obtient

$$\mathrm{E}_{\mathcal{G}_t}(\psi''(F)\, G) = \mathrm{E}_{\mathcal{G}_t}(\psi(F)\, H_2(F,G)) \,, \tag{III.2.5}$$

où $H_2(F,G) = H(F, H(F,G))$ et $H(F,G)$ est défini par l'équation (III.2.4).

Afin d'évaluer les poids de Malliavin $H(F,G)$ et $H_2(F,G)$, commençons par un lemme technique.

Lemme III.1:
Pour $i = 1, 2$, il existe une constante universelle $C > 0$ telle que

$$(i) \quad |F\, G|_{t,\delta,i} \le C \left(\sum_{j=0}^{i} |G|_{t,\delta,j} \right) \left(\sum_{j=0}^{i} |F|_{t,\delta,j} \right) ,$$

$$(ii) \quad |\langle DF, DG \rangle_{t,\delta}|_{t,\delta,i} \le C \left(\sum_{j=1}^{i+1} |G|_{t,\delta,j} \right) \left(\sum_{j=1}^{i+1} |F|_{t,\delta,j} \right) .$$

Preuve. (i) Puisque $D_s(F\, G) = F\, D_s G + G\, D_s F$, en intégrant ensuite en $s \in [t, t+\delta)$, il vient

$$|F\, G|_{t,\delta,1} \le |F|\, |G|_{t,\delta,1} + |G|\, |F|_{t,\delta,1} \,. \tag{III.2.6}$$

Et de même, $D^2_{us}(F\,G) = F\,D^2_{us}G + G\,D^2_{us}F + D_uF\,D_sG + D_sF\,D_uG$ entraîne

$$|F\,G|_{t,\delta,2} \leq 2\,|F|\,|G|_{t,\delta,2} + 2\,|G|\,|F|_{t,t+\delta,2} + 2\,|G|_{t,\delta,1}\,|F|_{t,\delta,1}\,.$$

Ce qui prouve (i).
(ii) Par l'inégalité de Cauchy-Schwarz, on a

$$\begin{aligned} D_u\langle DF, DG\rangle_{t,\delta} &= \int_t^{t+\delta} D^2_{ur}F\,D_rG\,dr + \int_t^{t+\delta} D^2_{ur}G\,D_rF\,dr \\ &\leq |G|_{t,\delta,1}\left(\int_t^{t+\delta} D^2_{ur}F\,dr\right)^{1/2} + |F|_{t,\delta,1}\left(\int_t^{t+\delta} D^2_{ur}G\,dr\right)^{1/2}. \end{aligned}$$

En intégrant par rapport à $u \in [t, t+\delta)$, on obtient

$$|\langle DF, DG\rangle_{t,\delta}|_{t,\delta,1} \leq |G|_{t,\delta,1}\,|F|_{t,\delta,2} + |F|_{t,\delta,1}\,|G|_{t,\delta,2}\,. \qquad \text{(III.2.7)}$$

Et de même, puisque

$$\begin{aligned} D^2_{us}\langle DF, DG\rangle_{t,\delta} = \int_t^{t+\delta} D^3_{sur}F\,D_rG\,dr &+ \int_t^{t+\delta} D^3_{sur}G\,D_rF\,dr \\ &+ 2\int_t^{t+\delta} D^2_{ur}F\,D^2_{sr}G\,dr\,, \end{aligned}$$

l'inégalité de Cauchy-Schwarz entraîne

$$|\langle DF, DG\rangle_{t,\delta}|_{t,\delta,2} \leq 2\,|G|_{t,\delta,1}\,|F|_{t,\delta,3} + 2\,|F|_{t,\delta,1}\,|G|_{t,\delta,3} + 2\,|F|_{t,\delta,2}\,|G|_{t,\delta,2}\,.$$

Ce qui achève la preuve. ■

Proposition III.1:
Il existe une constante universelle $C > 0$ telle que

(i) $|H(F,G)|$
$$\leq C\,|G|\,|\gamma_{t,\delta,F}|\,|\mathrm{L}_{t,\delta}F| + C\,|\gamma_{t,\delta,F}|\,|G|_{t,\delta,1}\,|F|_{t,\delta,1} + C\,|G|\,|\gamma_{t,\delta,F}|^2\,|F|^2_{t,\delta,1}\,|F|_{t,\delta,2}\,.$$

(ii)

$$\begin{aligned} |H(F, H(F,G))| \leq C\,(1 \vee |\gamma_{t,\delta,F}|^5)\,(|G| + |G|_{t,\delta,1} + |G|_{t,\delta,2}) \\ \times (1 + |\mathrm{L}_{t,\delta}F| + |\mathrm{L}_{t,\delta}F|_{t,\delta,1})^2\,(1 + |F|_{t,\delta,1} + |F|_{t,\delta,2} + |F|_{t,\delta,3})^6\,. \end{aligned}$$

Preuve. Etape 1. Déduisons du Lemme III.1 des estimations sur l'inverse de la matrice de covariance $\gamma_{t,\delta,F}$.
Nous avons $D_u\gamma_{t,\delta,F} = -\gamma^2_{t,\delta,F}\,D_u\phi_{t,\delta,F}$, avec $D_u\phi_{t,\delta,F} = D_u(\langle DF, DF\rangle_{t,\delta})$. On a donc

d'après l'équation (III.2.7)

$$|\gamma_{t,\delta,F}|_{t,\delta,1} \leq |\gamma_{t,\delta,F}|^2 \, |\langle DF, DF\rangle_{t,\delta}|_{t,\delta,1} \leq 2 \, |\gamma_{t,\delta,F}|^2 \, |F|_{t,\delta,1} \, |F|_{t,\delta,2} \, . \tag{III.2.8}$$

De même, $D^2_{su}\gamma_{t,\delta,F} = -2 \, \gamma_{t,\delta,F} \, D_s \gamma_{t,\delta,F} \, D_u \phi_F - \gamma^2_{t,\delta,F} \, D^2_{su}\phi_F$, et l'équation précédente entraîne donc

$$\begin{aligned} |\gamma_{t,\delta,F}|_{t,\delta,2} &\leq C \, |\gamma_{t,\delta,F}| \, |\gamma_{t,\delta,F}|_{t,\delta,1} \, |\langle DF, DF\rangle_{t,\delta}|_{t,\delta,1} + |\gamma_{t,\delta,F}|^2 \, |\langle DF, DF\rangle_{t,\delta}|_{t,\delta,2} \\ &\leq C \, |\gamma_{t,\delta,F}|^3 \, |F|_{t,\delta,1} \, |F|_{t,\delta,2} \, |\langle DF, DF\rangle_{t,\delta}|_{t,\delta,1} + |\gamma_{t,\delta,F}|^2 \, |\langle DF, DF\rangle_{t,\delta}|_{t,\delta,2} \, . \end{aligned}$$

Et le Lemme III.1 donne

$$|\gamma_{t,\delta,F}|_{t,\delta,2} \leq C \, (1 \vee |\gamma_{t,\delta,F}|^3) \, (1 + |F|_{t,\delta,1} + |F|_{t,\delta,2} + |F|_{t,\delta,3})^4 \, . \tag{III.2.9}$$

Etape 2. Evaluons le poids de Malliavin $H(F,G)$.
Rappelons que $H(F,G) = G \, \gamma_{t,\delta,F} \, \mathrm{L}_{t,\delta} F + \langle D(G \, \gamma_{t,\delta,F}), DF\rangle_{t,\delta}$. L'équation (III.2.6) entraîne donc

$$\begin{aligned} |H(F,G)| &\leq |G| \, |\gamma_{t,\delta,F}| \, |\mathrm{L}_{t,\delta} F| + |\gamma_{t,\delta,F} \, G|_{t,\delta,1} \, |F|_{t,\delta,1} \\ &\leq |G| \, |\gamma_{t,\delta,F}| \, |\mathrm{L}_{t,\delta} F| + |\gamma_{t,\delta,F}| \, |G|_{t,\delta,1} \, |F|_{t,\delta,1} + |\gamma_{k,F}|_{t,\delta,1} \, |G| \, |F|_{t,\delta,1} \, . \end{aligned}$$

Et d'après l'équation (III.2.8), on obtient

$$\begin{aligned} &|H(F,G)| \\ \leq& C \, |G| \, |\gamma_{t,\delta,F}| \, |\mathrm{L}_{t,\delta} F| + C \, |\gamma_{t,\delta,F}| \, |G|_{t,\delta,1} \, |F|_{t,\delta,1} + C \, |G| \, |\gamma_{t,\delta,F}|^2 \, |F|^2_{t,\delta,1} \, |F|_{t,\delta,2} \\ \leq& C \, (1 \vee |\gamma_{t,\delta,F}|^2) \, (|G| + |G|_{t,\delta,1}) \, (1 + |\mathrm{L}_{t,\delta} F|) \, (1 + |F|_{t,\delta,1} + |F|_{t,\delta,2})^3 \, . \end{aligned}$$

Etape 3. Evaluons le poids itéré $H(F, H(F,G))$.
Notons que $H(F, H(F,G)) = H(F,G) \, H(F,1) + \langle DH(F,G), \gamma_{t,\delta,F} \, DF\rangle_{t,\delta}$.
D'après l'étape 2, on a

$$\begin{aligned} |H(F,G) \, H(F,1)| \leq C \, (1 \vee |\gamma_{t,\delta,F}|^4) \, (|G| + |G|_{t,\delta,1}) \\ \times (1 + |\mathrm{L}_{t,\delta} F|)^2 \, (1 + |F|_{t,\delta,1} + |F|_{t,\delta,2})^6 \, . \end{aligned}$$

Regardons maintenant le terme $\langle DH(F,G), \gamma_{t,\delta,F} \, DF\rangle_{t,\delta}$.
On a

$$\begin{aligned} |\langle DH(F,G), \gamma_{t,\delta,F} \, DF\rangle_{t,\delta}| &\leq |\gamma_{t,\delta,F}| \, |\langle DH(F,G), DF\rangle_{t,\delta}| \\ &\leq |\gamma_{t,\delta,F}| \, |H(F,G)|_{t,\delta,1} \, |F|_{t,\delta,1} \, . \end{aligned}$$

Il reste donc à évaluer $|H(F,G)|_{t,\delta,1}$. On a

$$|H(F,G)|_{t,\delta,1} \leq |G\,(\gamma_{t,\delta,F}\,\mathrm{L}_{t,\delta}F)|_{t,\delta,1} + |\langle D(G\,\gamma_{t,\delta,F}), DF\rangle_{t,\delta}|_{t,\delta,1}\,.$$

D'après le Lemme III.1 (i) et l'équation (III.2.8), on a

$$\begin{aligned}
&|G\,(\gamma_{t,\delta,F}\,\mathrm{L}_{t,\delta}F)|_{t,\delta,1}\\
\leq &C\,(|G| + |G|_{t,\delta,1})\,(|\gamma_{t,\delta,F}|\,|\mathrm{L}_{t,\delta}F| + |\gamma_{t,\delta,F}|_{t,\delta,1}\,|\mathrm{L}_{t,\delta}F|_{t,\delta,1})\\
\leq &C\,(|G| + |G|_{t,\delta,1})\,(|\gamma_{t,\delta,F}|\,|\mathrm{L}_{t,\delta}F| + |\mathrm{L}_{t,\delta}F|\,|\gamma_{t,\delta,F}|^2\,|F|_{t,\delta,1}\,|F|_{t,\delta,2})\\
\leq &C\,(|G| + |G|_{t,\delta,1})\,(1\vee|\gamma_{t,\delta,F}|^2)\,(|\mathrm{L}_{t,\delta}F| + |\mathrm{L}_{t,\delta}F|_{t,\delta,1})\,(|F|_{t,\delta,1} + |F|_{t,\delta,2})^2\,.
\end{aligned}$$

D'après le Lemme III.1 (ii), on obtient

$$\begin{aligned}
&|\langle D(G\,\gamma_{t,\delta,F}), DF\rangle_{t,\delta}|_{t,\delta,1}\\
\leq &C\,(|F|_{t,\delta,1} + |F|_{t,\delta,2})\,(|G\,\gamma_{t,\delta,F}|_{t,\delta,1} + |G\,\gamma_{t,\delta,F}|_{t,\delta,2})\\
\leq &C\,(|F|_{t,\delta,1} + |F|_{t,\delta,2})\,(|G|\,|\gamma_{t,\delta,F}| + |G|_{t,\delta,1}\,|\gamma_{t,\delta,F}|_{t,\delta,1} + |G|_{t,\delta,2}\,|\gamma_{t,\delta,F}|_{t,\delta,2})\\
\leq &C\,(|F|_{t,\delta,1} + |F|_{t,\delta,2})\,(|G| + |G|_{t,\delta,1} + |G|_{t,\delta,2})\,(|\gamma_{t,\delta,F}| + |\gamma_{t,\delta,F}|_{t,\delta,1} + |\gamma_{t,\delta,F}|_{t,\delta,2})\,.
\end{aligned}$$

Les équations (III.2.8) et (III.2.9) entraînent

$$\begin{aligned}
&|\gamma_{t,\delta,F}| + |\gamma_{t,\delta,F}|_{t,\delta,1} + |\gamma_{t,\delta,F}|_{t,\delta,2}\\
\leq &|\gamma_{t,\delta,F}| + |\gamma_{t,\delta,F}|^2\,|F|_{t,\delta,1}\,|F|_{t,\delta,2} + (1\vee|\gamma_{t,\delta,F}|^3)\,(1 + |F| + |F|_{t,\delta,1} + |F|_{t,\delta,2} + |F|_{t,\delta,3})^4\\
\leq &C\,(1\vee|\gamma_{t,\delta,F}|^4)\,(1 + |F| + |F|_{t,\delta,1} + |F|_{t,\delta,2} + |F|_{t,\delta,3})^4\,.
\end{aligned}$$

Conclusion : on obtient une majoration du type

$$\begin{aligned}
|\langle DH(F,G), \gamma_{t,\delta,F}\,DF\rangle_{t,\delta}| \leq C\,&(1\vee|\gamma_{t,\delta,F}|^5)\,(|G| + |G|_{t,\delta,1} + |G|_{t,\delta,2})\\
&\times(1 + |\mathrm{L}_{t,\delta}F| + |\mathrm{L}_{t,\delta}F|_{t,\delta,1})^p\,(1 + |F|_{t,\delta,1} + |F|_{t,\delta,2} + |F|_{t,\delta,3})^6\,.
\end{aligned}$$

Ce qui achève la preuve. ■

Terminons ce chapitre par une dernière évaluation qui nous sera utile dans le Chapitre IV, paragraphe 3.1 :

Lemme III.2:
Soit $F \in \mathrm{D}^{i,\infty}$ *pour* $i \geq 1$. *On a*

$$(i)\quad ||F|^2_{t_k,\delta_k,i}|_{t_k,\delta_k,1} \leq 2\,|F|_{t_k,\delta_k,i}\,|F|_{t_k,\delta_k,i+1}\,,$$

et

$$(ii)\quad ||F|^2_{t_k,\delta_k,i}|_{t_k,\delta_k,2} \leq 2\,\left(|F|^2_{t_k,\delta_k,i+1} + |F|_{t_k,\delta_k,i}\,|F|_{t_k,\delta_k,i+2}\right)\,.$$

Preuve. (i) On a $D_{s_1}|F|^2_{t_k,\delta_k,i} = 2\int_{[s_1,t_k+\delta_k)^i}(D^i_{r_1\ldots r_i}F)\,(D^{i+1}_{s_1\,r_1\ldots r_i}F)\,dr_1\ldots dr_i$.
L'inégalité de Cauchy-Schwarz entraîne alors

$$\begin{aligned}\left|D_{s_1}|F|^2_{t_k,\delta_k,i}\right| &\leq 2\int_{[t_k,t_k+\delta_k)^i}|D^i_{r_1\ldots r_i}F|\,|D^{i+1}_{s_1\,r_1\ldots r_i}F|\,dr_1\ldots dr_i \\ &\leq 2\,|F|_{t_k,\delta_k,i}\left(\int_{[t_k,t_k+\delta_k)^i}|D^{i+1}_{s_1\,r_1\ldots r_i}F|^2\,dr_1\ldots dr_i\right)^{1/2}.\end{aligned}$$

En intégrant par rapport à $s_1 \in [t_k, t_k+\delta_k)$, on obtient
$||F|^2_{t_k,\delta_k,i}|_{t_k,\delta_k,1} \leq 2\,|F|_{t_k,\delta_k,i}\,|F|_{t_k,\delta_k,i+1}$.
(ii) De même, nous obtenons

$$\begin{aligned}&\left|D^2_{s_2\,s_1}|F|^2_{t_k,\delta_k,i}\right| \\ \leq& 2\int_{[t_k,t_k+\delta_k)^i}|D^{i+1}_{s_1\,r_1\ldots r_i}F|\,|D^{i+1}_{s_2\,r_1\ldots r_i}F|\,dr_1\ldots dr_i \\ &+2\int_{[t_k,t_k+\delta_k)^i}|D^i_{r_1\ldots r_i}F|\,|D^{i+2}_{s_2\,s_1\,r_1\ldots r_i}F|\,dr_1\ldots dr_i \\ \leq& 2\left(\int_{[t_k,t_k+\delta_k)^i}|D^{i+1}_{s_1\,r_1\ldots r_i}F|^2\,dr_1\ldots dr_i\right)^{1/2}\left(\int_{[t_k,t_k+\delta_k)^i}|D^{i+1}_{s_2\,r_1\ldots r_i}F|^2\,dr_1\ldots dr_i\right)^{1/2} \\ &+2\,|F|_{t_k,\delta_k,i}\left(\int_{[t_k,t_k+\delta_k)^i}|D^{i+2}_{s_2\,s_1\,r_1\ldots r_i}F|\,dr_1\ldots dr_i\right)^{1/2}.\end{aligned}$$

Donc $||F|^2_{t_k,\delta_k,i}|_{t_k,\delta_k,2} \leq 2\,\left(|F|^2_{t_k,\delta_k,i+1} + |F|_{t_k,\delta_k,i}\,|F|_{t_k,\delta_k,i+2}\right)$.

Minoration de la densité en temps petit IV

1. Le résultat principal

Considérons la diffusion à sauts $(X_t)_{t\geq 0}$ définie par l'équation (II.0.1).
Soit $\mathcal{F}_t$ la σ-algèbre définie par l'équation (II.0.4) et une subdivision de l'intervalle $[0,T]$, $0 = t_0 < \ldots < t_N = T$. On note $\delta_k := t_{k+1} - t_k$ le pas de cette subdivision.
Soit un point fixé $z \in \mathbb{R}$. Pour $k = 1, \ldots, N$ fixé, on note

$$A_k \subseteq \{\omega / |X_{t_k}(\omega) - z| \leq \sqrt{\delta_k}\} .$$

Le but de ce chapitre est de minorer la densité conditionnelle de $X_{t_{k+1}}$ sachant $\mathcal{F}_{t_k}$ sur A_k. Pour cela, on considère la régularisation suivante :

$$p_{\eta,k}(z) = \mathrm{E}_{\mathcal{F}_{t_k}}\left(\phi_\eta(X_{t_{k+1}} - z)\right) , \tag{IV.1.1}$$

où la fonction ϕ_η est construite comme suit. Soit une fonction $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ telle que $0 \leq \phi \leq 1$, $\int_{\mathbb{R}} \phi = 1$ et $\phi(y) = 0$ si $|y| > 1$. On définit alors ϕ_η par $\phi_\eta(y) = \dfrac{1}{\eta}\,\phi(\dfrac{y}{\eta})$.
Ainsi, $\phi_\eta \underset{\eta\to 0}{\longrightarrow} \delta_0$.
Rappelons le cadre décrit dans l'introduction et dans lequel nous allons travailler.
• On suppose que la condition (H_1, A_k, z) de l'Hypothèse II.2 est satisfaite.
• On prend ε_* vérifiant l'équation (II.0.5), et on écrit

$$X_{t_{k+1}} = G_k + R_k ,$$

où G_k est la partie principale définie par l'équation (II.0.6) et R_k est le reste défini par l'equation (II.0.7).
• Soit $\zeta \in (0, 1/2)$. On considère l'événement $B_{k,\zeta} := \{|\overline{R}_k| \leq \delta_k^{1/2+\zeta}\}$, où $\overline{R}_k$ est défini par l'équation (II.0.9).
• Soit $\overline{\varepsilon} = \dfrac{\zeta}{4\,(1+\zeta)}$. On suppose que la condition (H_2, A_k, z) de l'Hypothèse II.3 est satisfaite.
Le résultat principal de cette partie est le suivant.

On note

$$\delta_* = \left(\frac{1}{4 \int_{|a|>\varepsilon_*} \overline{c}(a)\, \nu(da)} \right)^{1/(1/2-\zeta)} \wedge \delta(\underline{\lambda}, \overline{\lambda}) \,, \qquad \text{(IV.1.2)}$$

où $\delta(\underline{\lambda}, \overline{\lambda})$ est une constante qui dépend de $\underline{\lambda}$ et $\overline{\lambda}$, et qui sera précisée au court de la preuve du Théorème qui suit par les restrictions (IV.3.10), (IV.3.13) et (IV.3.16). Essentiellement, $\delta(\underline{\lambda}, \overline{\lambda}) = e^{-C/\underline{\lambda}} \left(\frac{\overline{\lambda}}{\underline{\lambda}} \right)^p$, où C et p sont des constantes.

Théorème IV.1:
Supposons que $\delta_k \leq \delta_*$. Alors, pour tout $0 < \eta \leq \sqrt{\delta_k}$, pour tout $\omega \in A_k$, nous avons

$$p_{\eta,k}(z) = \mathrm{E}_{\mathcal{F}_{t_k}} \left(\phi_\eta(X_{t_{k+1}} - z) \right)(\omega) \geq \frac{1}{8 \sqrt{2\,\pi\, \delta_k\, \overline{\lambda}}}\, e^{-4/\underline{\lambda}} \,.$$

Introduisons les normes suivantes : pour $n \geq 2$, et $F \in \mathrm{D}^{n,2}$,

$$N_{k,n}(F) := \left(\sum_{i=0}^{n} |F|^2_{t_k,\delta_k,i} + \sum_{i=0}^{n-2} |\mathrm{L}_{t_k,\delta_k} F|^2_{t_k,\delta_k,i} \right)^{1/2} . \qquad \text{(IV.1.3)}$$

Remarque 1.1. *La condition (H_2, A_k, z) de l'Hypothèse II.3 entraîne*

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(|N_{k,5}(R_k)|^{2\,(1+\zeta)}\, \mathbf{1}_{B_{k,\zeta}} \right) \right)^{\frac{1}{2\,(1+\zeta)}} \leq C\, \delta_k^{1/2+2\,\overline{\varepsilon}} \,.$$

En effet, l'inégalité de Meyer (III.2.1) nous donne

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(\sum_{i=0}^{3} |\mathrm{L}_{t_k,\delta_k}(R_k)|^2_{t_k,\delta_k,i}\, \mathbf{1}_{B_{k,\zeta}} \right)^{1+\zeta} \right)^{1/(1+\zeta)} \leq C \left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(\sum_{i=0}^{5} |R_k|^2_{t_k,\delta_k,i}\, \mathbf{1}_{B_{k,\zeta}} \right)^{(1+\zeta)} \right)^{1/(1+\zeta)} ,$$

et donc

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} (N_{k,5}(R_k)^{2\,(1+\zeta)}\, \mathbf{1}_{B_{k,\zeta}}) \right)^{1/(1+\zeta)} \leq C \left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(\sum_{i=0}^{5} |R_k|^2_{t_k,\delta_k,i}\, \mathbf{1}_{B_{k,\zeta}} \right)^{(1+\zeta)} \right)^{1/(1+\zeta)} \leq C\, \delta_k^{4\,\overline{\varepsilon}+1} \,.$$

Introduisons maintenant une fonction de localisation.
Soit une fonction $\theta \in \mathcal{C}_b^\infty(\mathbb{R}_+)$ telle que $\mathbf{1}_{[0,1/2]} \leq \theta \leq \mathbf{1}_{[0,1]}$. On considère la fonction

de localisation suivante :

$$Q_k = \theta(N_{k,3}^2(R_k)\,\delta_k^{-(2\,\overline{\varepsilon}+1)})\,, \tag{IV.1.4}$$

où $\overline{\varepsilon} = \dfrac{\zeta}{4\,(1+\zeta)}$ a été introduit dans l'hypothèse (H_2, A_k, z).
On note $\eta_k := \dfrac{\eta}{\sqrt{\delta_k}} \in (0,1)$. Puisque $\phi_\eta(\sqrt{\delta_k}\,x) = \dfrac{1}{\sqrt{\delta_k}}\,\phi_{\eta_k}(x)$, il vient

$$p_{\eta,k}(z) = \frac{1}{\sqrt{\delta_k}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left(\phi_{\eta_k}\left(\frac{X_{t_{k+1}}-z}{\sqrt{\delta_k}}\right)\right) \geq \frac{1}{\sqrt{\delta_k}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left(\phi_{\eta_k}\left(\frac{X_{t_{k+1}}-z}{\sqrt{\delta_k}}\right)\,Q_k\,\mathbf{1}_{B_{k,\zeta}}\right)\,.$$

En utilisant un développement de Taylor autour de G_k, on obtient alors pour tout $\omega \in A_k$,

$$\begin{aligned}
p_{\eta,k}(z) &\geq \frac{1}{\sqrt{\delta_k}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left(\phi_{\eta_k}\left(\frac{G_k - z + R_k}{\sqrt{\delta_k}}\right)\,Q_k\,\mathbf{1}_{B_{k,\zeta}}\right) \\
&= \frac{1}{\sqrt{\delta_k}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left[\phi_{\eta_k}\left(\frac{G_k - z}{\sqrt{\delta_k}}\right)\,Q_k\,\mathbf{1}_{B_{k,\zeta}}\right] \\
&+ \frac{1}{\sqrt{\delta_k}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left[\int_0^1 \phi'_{\eta_k}\left(\frac{G_k - z}{\sqrt{\delta_k}} + \rho\,\frac{R_k}{\sqrt{\delta_k}}\right)\,\frac{R_k}{\sqrt{\delta_k}}\,Q_k\,d\rho\,\mathbf{1}_{B_{k,\zeta}}\right] \\
&= \mathrm{E}_{\mathcal{F}_{t_k}}\left[\phi_\eta(G_k - z)\,\mathbf{1}_{B_{k,\zeta}}\right] + \frac{1}{\sqrt{\delta_k}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left[\phi_{\eta_k}\left(\frac{G_k - z}{\sqrt{\delta_k}}\right)\,(Q_k - 1)\,\mathbf{1}_{B_{k,\zeta}}\right] \\
&+ \frac{1}{\sqrt{\delta_k}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left[\int_0^1 \phi'_{\eta_k}\left(\frac{G_k - z}{\sqrt{\delta_k}} + \rho\,\frac{R_k}{\sqrt{\delta_k}}\right)\,\frac{R_k}{\sqrt{\delta_k}}\,Q_k\,d\rho\,\mathbf{1}_{B_{k,\zeta}}\right] \\
&:= \mathrm{E}_{\mathcal{F}_{t_k}}\left[\phi_\eta(G_k - z)\,\mathbf{1}_{B_{k,\zeta}}\right] + \frac{1}{\sqrt{\delta_k}}\,J(\omega) + \frac{1}{\sqrt{\delta_k}}\,J'(\omega)\,,
\end{aligned}$$

où l'on note

$$J(\omega) \;=\; \mathrm{E}_{\mathcal{F}_{t_k}}\left[\phi_{\eta_k}\left(\frac{G_k - z}{\sqrt{\delta_k}}\right)\,(Q_k - 1)\,\mathbf{1}_{B_{k,\zeta}}\right]\,, \tag{IV.1.5}$$

$$J'(\omega) \;=\; \mathrm{E}_{\mathcal{F}_{t_k}}\left[\int_0^1 \phi'_{\eta_k}\left(\frac{G_k - z}{\sqrt{\delta_k}} + \rho\,\frac{R_k}{\sqrt{\delta_k}}\right)\,\frac{R_k}{\sqrt{\delta_k}}\,Q_k\,d\rho\,\mathbf{1}_{B_{k,\zeta}}\right]\,. \tag{IV.1.6}$$

Nous allons minorer la première espérance (qui correspond à la partie principale G_k), puis nous allons montrer que les deux autres sont négligeables par rapport à la première au sens où $J(\omega)$ et $J'(\omega)$ sont de l'ordre de $\delta_k^{\overline{\varepsilon}}$.

2. Minoration de la partie principale

Commençons par un lemme technique.

Lemme IV.1:
Pour tous $x, \delta, \beta, t > 0$, nous avons

$$\mathrm{E}\left[\exp\left(-\beta\left|\int_t^{t+\delta}\int_{|a|\leq\varepsilon_*} c(s,a,x)\,\widetilde{N}(ds,da)\right|^2\right)\right] \geq 1-\beta\,\delta\int_{|a|\leq\varepsilon_*}\overline{c}^2(a)\,\nu(da)\,.$$

<u>**Preuve**</u>. On note $F(x) = e^{-\beta\,x^2}$ et $I_r(x) := \int_t^r\int_{|a|\leq\varepsilon_*} c(s,a,x)\,\widetilde{N}(ds,da)$ pour tout $r \geq t$.
En appliquant le Lemme d'Ito à F, on obtient

$$\begin{aligned}F(I_{t+\delta}) = F(I_t) &+ \int_t^{t+\delta}\int_{|a|\leq\epsilon_*}\left[F(I_{s^-}+c(s,a,x)) - F(I_{s^-})\right]\,\widetilde{N}(ds,da)\\ &+ \int_t^{t+\delta}\int_{|a|\leq\epsilon_*}\left[F(I_s+c(s,a,x)) - F(I_s) - c(s,a,x)\,F'(I_s)\right]\,ds\,\nu(da)\,.\end{aligned}$$

Et passant à l'espérance,

$$\begin{aligned}\mathrm{E}(F(I_{t+\delta})) = 1 + \int_t^{t+\delta}\int_{|a|\leq\epsilon_*} & ds\,\nu(da)\\ & \mathrm{E}\left[F(I_s+c(s,a,x)) - F(I_s) - c(s,a,x)\,F'(I_s)\right]\,.\end{aligned}$$

Utilisons un développement de Taylor d'ordre un :

$$\begin{aligned}F(I_s+c(s,a,x)) &- F(I_s) - c(s,a,x)\,F'(I_s)\\ &= \int_0^1 F''(I_s+\rho\,c(s,a,x))\,c^2(s,a,x)\,(1-\rho)\,d\rho\\ &= 2\,\beta\,c^2(s,a,x)\int_0^1 g\left(\beta\,(I_s+\rho\,c(s,a,x))^2\right)\,(1-\rho)\,d\rho\,,\end{aligned}$$

où $F''(y) = 2\,\beta\,g(\beta\,y^2)$, avec $g(y) = e^{-y}\,(2\,y-1)$. Remarquons pour tout $y \geq 0$ on a $g(y) \geq -1$, il vient donc

$$F(I_s+c(s,a,x)) - F(I_s) - c(s,a,x)\,F'(I_s) \geq -\beta\,c^2(s,a,x)\,.$$

Et puisque $|c(s,a,x)| \leq \overline{c}(a)$ d'après l'hypothèse II.1, on obtient

$$\begin{aligned}\mathrm{E}\left[F(I_{t+\delta}(x))\right] &\geq 1 - \mathrm{E}\left[\beta\int_t^{t+\delta}\int_{|a|\leq\epsilon_*} c^2(s,a,x)\,ds\,\nu(da)\right]\\ &\geq 1-\beta\,\delta\int_{|a|\leq\epsilon_*}\overline{c}^2(a)\,\nu(da)\,.\end{aligned}$$

Ce qui achève la preuve. ■

On a alors la minoration suivante :

Lemme IV.2:
Supposons que $\delta_k \le \delta_*$. Alors, pour tout $0 < \eta \le \sqrt{\delta_k}$, pour tout $\omega \in A_k$, nous avons

$$\mathrm{E}_{\mathcal{F}_{t_k}}(\phi_\eta(G_k - z)\,\mathbf{1}_{B_{k,\zeta}})(\omega) \ge \frac{1}{4\sqrt{2\,\pi\,\delta_k\,\overline{\lambda}}}\,e^{-4/\underline{\lambda}}\,.$$

Avant de commencer la preuve de ce Lemme, rappelons que nous avons introduit dans le chapitre précédent la σ-algèbre

$$\mathcal{G}_{t_k} := \mathcal{F}^B_{t_k} \vee \mathcal{F}^N_\infty = \sigma(B_s\,, s \le t_k) \vee \sigma(\widetilde{N}(s,A)\,, s \ge 0\,, A \in \mathcal{B}(\mathbb{R}))\,. \tag{IV.2.1}$$

Et nous avons noté $\mathrm{E}_{\mathcal{G}_{t_k}}$ l'espérance conditionnelle sachant $\mathcal{G}_{t_k}$, ce qui signifie que l'on n'intègre que par rapport au mouvement Brownien B_s, $s \in [t_k, t_{k+1})$.

<u>**Preuve**</u>. **Première étape.** Regardons le terme $\mathrm{E}_{\mathcal{G}_{t_k}}(\phi_\eta(G_k - z)\,\mathbf{1}_{B_{k,\zeta}})$.
Soit

$$N_k := \int_{t_k}^{t_{k+1}} \int_{|a| \le \varepsilon_*} c(s,a,X_{t_k})\,\widetilde{N}(ds,da)\,.$$

On remarque que conditionnellement à $\mathcal{G}_{t_k}$, G_k est une variable aléatoire Gaussienne d'espérance $X_{t_k} + N_k$ et de variance $\delta_k\,\sigma_k^2$. On obtient donc

$$\mathrm{E}_{\mathcal{G}_{t_k}}(\phi_\eta(G_k - z)) = \int_{\mathbb{R}} \phi_\eta(y)\,\frac{1}{\sqrt{2\,\pi\,\delta_k}\,\sigma_k}\,\exp\left(-\frac{|y - (X_{t_k} + N_k - z)|^2}{2\,\delta_k\,\sigma_k^2}\right)dy\,.$$

On a $|y - X_{t_k} - N_k + z| \le |y| + |z - X_{t_k}| + |N_k| \le |y| + \sqrt{\delta_k} + |N_k|$.
Si $\phi_\eta(y) \ne 0$ alors $|y| \le \eta \le \sqrt{\delta_k}$, et donc
$|y - X_{t_k} - N_k + z|^2 \le 2\,(4\,\delta_k + |N_k|^2) \le 8\,\delta_k + 2\,|N_k|^2$.
Et puisque $\int_{\mathbb{R}} \phi_\eta = 1$, il vient

$$\mathrm{E}_{\mathcal{G}_{t_k}}(\phi_\eta(G_k - z)) \ge \left(\frac{1}{\sqrt{2\,\pi\,\delta_k}\,\sigma_k}\,e^{-4/\sigma_k^2}\right)\exp\left(-\frac{|N_k|^2}{\delta_k\,\sigma_k^2}\right)\,.$$

Il suffit de remarquer que l'événement $B_{k,\zeta}$ est $\mathcal{G}_{t_k}$-mesurable pour finalement obtenir

$$\mathrm{E}_{\mathcal{F}_{t_k}}(\phi_\eta(G_k - z)\,\mathbf{1}_{B_{k,\zeta}})(\omega) \ge \frac{1}{\sqrt{2\,\pi\,\delta_k\,\overline{\lambda}}}\,e^{-4/\underline{\lambda}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left(\exp\left(-\frac{|N_k|^2}{\delta_k\,\underline{\lambda}}\right)\mathbf{1}_{B_{k,\zeta}}\right)\,.$$

Il nous faut donc maintenant minorer le terme $\mathrm{E}_{\mathcal{F}_{t_k}}\left[\exp\left(-\frac{|N_k|^2}{\delta_k\,\underline{\lambda}}\right)\mathbf{1}_{B_{k,\zeta}}\right]$.
Deuxième étape. En utilisant le Lemme IV.1, montrons que

$$\mathrm{E}_{\mathcal{F}_{t_k}}\left[\exp\left(-\frac{|N_k|^2}{\delta_k\,\underline{\lambda}}\right)\mathbf{1}_{B_{k,\zeta}}\right] \ge \frac{1}{4}\,.$$

Et la preuve sera alors complète.
Remarquons que

$$\mathrm{E}_{\mathcal{F}_{t_k}}\left[\exp\left(-\frac{|N_k|^2}{\delta_k\,\underline{\lambda}}\right)\,\mathbf{1}_{B_{k,\zeta}}\right] = \mathrm{E}_{\mathcal{F}_{t_k}}\left[\exp\left(-\frac{|N_k|^2}{\delta_k\,\underline{\lambda}}\right)\right] - \mathrm{E}_{\mathcal{F}_{t_k}}\left[\exp\left(-\frac{|N_k|^2}{\delta_k\,\underline{\lambda}}\right)\,(1-\mathbf{1}_{B_{k,\zeta}})\right].$$

Soient $\beta_k = \dfrac{1}{\delta_k\,\underline{\lambda}}$ et $F_k(x) = \exp(-\beta_k\,x^2)$. Reprenant les notations du Lemme IV.1, on a $N_k = F_k(I_{t_k+\delta_k}(X_{t_k}))$. Le choix de ε_* dans l'équation (II.0.5) entraîne

$$\begin{aligned}\mathrm{E}_{\mathcal{F}_{t_k}}(F_k(N_k)) &= \mathrm{E}(F_k(I_{t_k+\delta_k}(x)))|_{x=X_{t_k}}\\ &\geq 1-\beta_k\,\delta_k\int_{|a|\leq\varepsilon_*}\overline{c}^2(a)\,\nu(da)\\ &= 1-\frac{1}{\underline{\lambda}}\int_{|a|\leq\varepsilon_*}\overline{c}^2(a)\,\nu(da)\\ &\geq \frac{1}{2}\,.\end{aligned}$$

D'autre part, nous avons

$$\begin{aligned}\mathrm{E}_{\mathcal{F}_{t_k}}\left[\exp\left(-\frac{|N_k|^2}{\delta_k\,\underline{\lambda}}\right)\,(1-\mathbf{1}_{B_{k,\zeta}})\right] &\leq \mathrm{P}_{t_k}(\overline{R}_k > \delta_k^{\zeta+1/2})\\ &\leq \frac{1}{\delta_k^{\zeta+1/2}}\,\mathrm{E}_{\mathcal{F}_{t_k}}\left(\int_{t_k}^{t_{k+1}}\int_{|a|>\varepsilon_*}\overline{c}(a)\,N(ds,da)\right)\\ &= \frac{1}{\delta_k^{\zeta+1/2}}\int_{t_k}^{t_{k+1}}\int_{|a|>\varepsilon_*}\overline{c}(a)\,ds\,\nu(da)\\ &= \delta_k^{-\zeta+1/2}\int_{|a|>\varepsilon_*}\overline{c}(a)\,\nu(da)\,.\end{aligned}$$

Prenant $\delta_k \leq \delta_* \leq \left(\int_{|a|>\varepsilon_*}\overline{c}(a)\,\nu(da)\right)^{1/(1/2-\zeta)}$, on obtient

$$\mathrm{E}_{\mathcal{F}_{t_k}}\left[\exp\left(-\frac{|N_k|^2}{\delta_k\,\underline{\lambda}}\right)\,(1-\mathbf{1}_{B_{k,\zeta}})\right] \leq \frac{1}{4}\,.$$

La preuve est ainsi achevée. ■

3. Evaluation du reste

Avant d'évaluer les termes restants $J(\omega)$ et $J'(\omega)$ respectivement définis par les équations (IV.1.5) et (IV.1.6), commençons par quelques évaluations sur la fonction localisante Q_k.

3.1. Evaluations préliminaires sur la fonction localisante

Rappelons tout d'abord la définition de Q_k donnée par l'équation (IV.1.4) :
$Q_k = \theta(N_{k,3}^2(R_k)\,\delta_k^{-(2\bar{\varepsilon}+1)})$. Soit encore

$$Q_k = \theta(N_{k,3}^2(R'_k)\,\delta_k^{-2\bar{\varepsilon}})\,,\ \text{avec}\ R'_k := \delta_k^{-1/2}\,R_k\,.$$

Lemme IV.3:
Il existe une constante universelle $C > 0$ telle que

$$(i)\quad |Q_k|_{t_k,\delta_k,1} \leq C\,\delta_k^{-\bar{\varepsilon}}\,N_{k,4}(R'_k)\ \text{et}\ |Q_k|_{t_k,\delta_k,2} \leq C\,\delta_k^{-2\bar{\varepsilon}}\,N_{k,5}^2(R'_k)\,.$$

(ii) En particulier, pour $\zeta \in (0, 1/2)$,

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(|Q_k|_{t_k,\delta_k,1}^{1+\zeta}\,\mathbf{1}_{B_{k,\zeta}}\right)\right)^{1/(1+\zeta)} \leq C\,\delta_k^{\bar{\varepsilon}}$$
$$\text{et}\ \left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(|Q_k|_{t_k,\delta_k,2}^{1+\zeta}\,\mathbf{1}_{B_{k,\zeta}}\right)\right)^{1/(1+\zeta)} \leq C\,\delta_k^{2\bar{\varepsilon}}\,.$$

Preuve. Montrons tout d'abord le résultat (i).
Etape 1. Pour tous $s_1, s_2 \in [t_k, t_{k+1})$, nous avons
$D_{s_1}Q_k = \delta_k^{-2\bar{\varepsilon}}\,\theta'(N_{k,3}^2(R'_k)\,\delta_k^{-2\bar{\varepsilon}})\,D_{s_1}(N_{k,3}^2(R'_k))$, et donc

$$D_{s_2\,s_1}^2 Q_k = \delta_k^{-4\bar{\varepsilon}}\,\theta^{(2)}(N_{k,3}^2(R'_k)\,\delta_k^{-2\bar{\varepsilon}})\,D_{s_1}(N_{k,3}(R'_k)^2)\,D_{s_2}(N_{k,3}^2(R'_k)) + \delta_k^{-2\bar{\varepsilon}}\,\theta'(N_{k,3}^2(R'_k)\,\delta_k^{-2\bar{\varepsilon}})\,D_{s_2\,s_1}^2(N_{k,3}^2(R'_k))\,.$$

Afin de simplifier les notations, on écrit pour $j = 1, 2$,

$$\theta^{(j)} := \theta^{(j)}(N_{k,3}^2(R'_k)\,\delta_k^{-2\bar{\varepsilon}})\,.$$

On obtient alors $|D_{s_1}Q_k| \leq \delta_k^{-2\bar{\varepsilon}}\,|\theta'|\,|D_{s_1}(N_{k,3}^2(R'_k))|$ et

$$|D_{s_2\,s_1}^2 Q_k| \leq \delta_k^{-4\bar{\varepsilon}}\,|\theta^{(2)}|\,|D_{s_1}(N_{k,3}(R'_k)^2)|\,|D_{s_2}(N_{k,3}^2(R'_k))| + C\,\delta_k^{-2\bar{\varepsilon}}\,|D_{s_2\,s_1}^2(N_{k,3}^2(R'_k))|\,.$$

Conclusion : pour $j = 1, 2$, on a

$$|Q_k|_{t_k,\delta_k,1} \leq \delta_k^{-2\bar{\varepsilon}}\, |\theta'|\, |N_{k,3}^2(R_k')|_{t_k,\delta_k,1} \tag{IV.3.1}$$
$$|Q_k|_{t_k,\delta_k,2} \leq \delta_k^{-4\bar{\varepsilon}}\, |\theta^{(2)}|\, |N_{k,3}^2(R_k')|_{t_k,\delta_k,1}^2 + C\, \delta_k^{-2\bar{\varepsilon}}\, |N_{k,3}^2(R_k')|_{t_k,\delta_k,2}\,. \tag{IV.3.2}$$

Il nous faut donc majorer $|N_{k,3}^2(R_k')|_{t_k,\delta_k,2}$ et $|\theta^{(j)}| \times |N_{k,3}^2(R_k')|_{t_k,\delta_k,1}^j$ pour $j = 1, 2$.
Etape 2. Evaluons $|N_{k,3}^2(R_k')|_{t_k,\delta_k,2}$.
Notons que

$$|N_{k,3}^2(R_k')|_{t_k,\delta_k,2} \leq \sum_{i=0}^{3} ||R_k'|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,2} + \sum_{i=0}^{1} ||\mathrm{L}_{t_k,\delta_k}(R_k')|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,2}\,. \tag{IV.3.3}$$

Appliquant le Lemme III.2 (ii) à $F = R_k'$, on obtient
$||R_k'|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,2} \leq 2\,\left(|R_k'|_{t_k,\delta_k,i+1}^2 + |R_k'|_{t_k,\delta_k,i}\, |R_k'|_{t_k,\delta_k,i+2}\right)$, et donc

$$\sum_{i=0}^{3} ||R_k'|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,2} \leq C\, \sum_{i=0}^{5} |R_k'|_{t_k,\delta_k,i}^2\,.$$

De la même façon, en prenant $F = \mathrm{L}_{t_k,\delta_k}(R_k')$ dans le Lemme III.2 (ii), on obtient

$$\sum_{i=0}^{1} ||\mathrm{L}_{t_k,\delta_k}(R_k')|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,2} \leq C\, \sum_{i=0}^{3} |\mathrm{L}_{t_k,\delta_k}(R_k')|_{t_k,\delta_k,i}^2\,.$$

En additionnant ces deux inégalités, l'équation (IV.3.3) nous donne

$$|N_{k,3}^2(R_k')|_{t_k,\delta_k,2} \leq C\, N_{k,5}^2(R_k')\,.$$

Etape 3. Evaluons $|\theta^{(j)}| \times |N_{k,3}^2(R_k')|_{t_k,\delta_k,1}^j$ pour $j = 1, 2$.
Nous avons

$$|\theta^{(j)}| \times |N_{k,3}^2(R_k')|_{t_k,\delta_k,1} \leq |\theta^{(j)}|\, \sum_{i=0}^{3} ||R_k'|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,1} + |\theta^{(j)}|\, \sum_{i=0}^{1} ||\mathrm{L}_{t_k,\delta_k}(R_k')|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,1}\,. \tag{IV.3.4}$$

Appliquant le Lemme III.2 (i) à $F = R_k'$ et $F = \mathrm{L}_{t_k,\delta_k}(R_k')$, on obtient

$$|\theta^{(j)}|\, ||R_k'|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,1} \leq 2\, |\theta^{(j)}|\, |R_k'|_{t_k,\delta_k,i}\, |R_k'|_{t_k,\delta_k,i+1}\,,$$
$$|\theta^{(j)}|\, ||\mathrm{L}_{t_k,\delta_k}(R_k')|_{t_k,\delta_k,i}^2|_{t_k,\delta_k,1} \leq 2\, |\theta^{(j)}|\, |\mathrm{L}_{t_k,\delta_k}(R_k')|_{t_k,\delta_k,i}\, |\mathrm{L}_{t_k,\delta_k}(R_k')|_{t_k,\delta_k,i+1}\,.$$

Or pour $j = 1, 2$, $\theta^{(j)}(N_{k,3}^2(R'_k)\,\delta_k^{-2\,\overline{\varepsilon}}) \neq 0 \Rightarrow N_{k,3}(R'_k) \leq \delta_k^{\overline{\varepsilon}}$, soit encore

$$|R'_k|_{t_k,\delta_k,i} \leq \delta_k^{\overline{\varepsilon}}\,,\ i = 0, 1, 2, 3 \text{ et } |\mathrm{L}_{t_k,\delta_k}(R'_k)|_{t_k,\delta_k,i} \leq \delta_k^{\overline{\varepsilon}}\,,\ i = 0, 1\,.$$

Ainsi, si $\theta^{(j)} \neq 0$, on a

$$\begin{aligned}|N_{k,3}^2(R'_k)|_{t_k,\delta_k,1} &\leq C\,\delta_k^{\overline{\varepsilon}} \sum_{i=0}^{4} |R'_k|_{t_k,\delta_k,i} + C\,\delta_k^{\overline{\varepsilon}} \sum_{i=0}^{2} |\mathrm{L}_{t_k,\delta_k}(R'_k)|_{t_k,\delta_k,i} \\ &\leq C\,\delta_k^{\overline{\varepsilon}}\,N_{k,4}(R'_k)\,.\end{aligned}$$

La fonction $\theta^{(j)}$ étant bornée, il vient alors

$$|\theta^{(j)}|\,|N_{k,3}^2(R'_k)|_{t_k,\delta_k,1}^j \leq C\,\delta_k^{j\,\overline{\varepsilon}}\,N_{k,4}^j(R'_k)\,,\ j = 1, 2\,.$$

Finalement, l'équation (IV.3.1) devient

$$|Q_k|_{t_k,\delta_k,1} \leq C\,\delta_k^{-\overline{\varepsilon}}\,N_{k,4}(R'_k)\,,$$

et l'équation (IV.3.2) devient

$$|Q_k|_{t_k,\delta_k,2} \leq C\,\delta_k^{-2\,\overline{\varepsilon}}\,N_{k,4}^2(R'_k) + C\,\delta_k^{-2\,\overline{\varepsilon}}\,|N_{k,3}^2(R'_k)|_{t_k,\delta_k,2}\,.$$

Puisque d'après l'étape 2 on a $|N_{k,3}^2(R'_k)|_{t_k,\delta_k,2} \leq C\,N_{k,5}^2(R'_k)$, il vient

$$|Q_k|_{t_k,\delta_k,2} \leq C\,\delta_k^{-2\,\overline{\varepsilon}}\,N_{k,5}^2(R'_k)\,.$$

Ce qui achève le point (i).

Remarque 3.1. *Pour des raisons techniques liées à l'inégalité de Burkhölder pour des processus de sauts, il faut éviter de travailler avec des puissances $p \geq 3$. En effet, une telle inégalité (voir [BGJ87]) donne une évaluation du type :*

$$\left(\mathrm{E}\left|\int_t^{t+\delta}\int_{\mathbb{R}} c(s,a,\omega)\,\widetilde{N}(ds,da)\right|^p\right)^{1/p} \leq C\,\delta^{1/p}\,.$$

Ainsi, si p est grand, $\delta^{1/p}$ donne une mauvaise estimation. C'est pourquoi, dans cette preuve (plus particulièrement dans l'étape 3), nous avons évité une majoration du type $|N_{k,3}^2(R'_k)|_{t_k,\delta_k,1} \leq N_{k,4}^2(R'_k)$, qui aurait donné $|N_{k,3}^2(R'_k)|_{t_k,\delta_k,1}^2 \leq N_{k,4}^4(R'_k)$, et donc des puissances $p = 4$. Cette astuce que nous permet la localisation s'avère être cruciale.

Montrons maintenant le résultat (ii).

D'après la condition (H_2, A_k, z) de l'Hypothèse II.3 et la Remarque 1.1, on a

$$\begin{aligned}\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(N_{k,5}^{2\,(1+\zeta)}(R'_k)\,\mathbf{1}_{B_{k,\zeta}}\right)\right)^{1/(1+\zeta)} &= \frac{1}{\delta_k}\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(N_{k,5}^{2\,(1+\zeta)}(R_k)\,\mathbf{1}_{B_{k,\zeta}}\right)\right)^{1/(1+\zeta)} \\ &\leq \delta_k^{4\,\overline{\varepsilon}}\,.\end{aligned}$$

Donc $\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(N_{k,4}^{1+\zeta}(R'_k)\,\mathbf{1}_{B_{k,\zeta}}\right)\right)^{1/(1+\zeta)} \leq \delta_k^{2\,\overline{\varepsilon}}$. Ce qui achève la preuve. ■

3.2. Evaluation de J

Rappelons que $J(\omega)$ est définie pour tout $\omega \in A_k$ par l'équation (IV.1.5), soit

$$J(\omega) = \mathrm{E}_{\mathcal{F}_{t_k}}\left[\phi_{\eta_k}\left(\frac{G_k - z}{\sqrt{\delta_k}}\right)\,(Q_k - 1)\,\mathbf{1}_{B_{k,\zeta}}\right],\ \text{avec}\ \eta_k = \frac{\eta}{\sqrt{\delta_k}}\,.$$

Voici le résultat de ce paragraphe :

Lemme IV.4:
Supposons que $\delta_k \leq \delta_*$. Alors, pour tout $\omega \in A_k$, nous avons

$$|J(\omega)| \leq \frac{1}{16\,\sqrt{2\,\pi\,\overline{\lambda}}} \times e^{-4/\underline{\lambda}}\,.$$

Preuve. Posons $G_k = V_k + J_k$, avec

$$J_k := \int_{t_k}^{t_{k+1}} \sigma_k\,dB_s,\ \text{où}\ \sigma_k = \sigma(X_{t_k})\,, \tag{IV.3.5}$$

et

$$V_k := X_{t_k} + \int_{t_k}^{t_{k+1}}\int_{|a|\leq\varepsilon_*} c(s,a,X_{t_k})\,\widetilde{N}(ds,da)\,. \tag{IV.3.6}$$

On note $J'_k = \dfrac{J_k}{\sqrt{\delta_k}}$.
Rappelons que nous avons introduit la σ-algèbre $\mathcal{G}_{t_k}$ par l'équation (IV.2.1). En remarquant que V_k est $\mathcal{G}_{t_k}$-mesurable, on obtient

$$\begin{aligned}J(\omega)\,\mathbf{1}_{A_k} &= \mathrm{E}_{\mathcal{F}_{t_k}}\left[\phi_{\eta_k}\left(\frac{V_k - z + J_k}{\sqrt{\delta_k}}\right)\,(Q_k - 1)\,\mathbf{1}_{B_{k,\zeta}}\,\mathbf{1}_{A_k}\right] \\ &= \mathrm{E}_{\mathcal{F}_{t_k}}\left[\mathrm{E}_{\mathcal{G}_{t_k}}\left(\phi_{\eta_k}\left(\frac{V_k - z}{\sqrt{\delta_k}} + J'_k\right)\,(Q_k - 1)\,\mathbf{1}_{A_k}\right)\,\mathbf{1}_{B_{k,\zeta}}\right].\end{aligned}$$

On définit la fonction suivante :

$$\Phi_{\eta_k}(x) = \int_{-\infty}^{x} \phi_{\eta_k}\left(\frac{V_k - z}{\sqrt{\delta_k}} + y\right)\,dy\,. \tag{IV.3.7}$$

On a donc $\Phi'_{\eta_k}(x) = \phi_{\eta_k}\left(\frac{V_k - z}{\sqrt{\delta_k}} + x\right)$, et

$$\mathrm{E}_{\mathcal{G}_{t_k}}\left(\phi_{\eta_k}\left(\frac{V_k - z}{\sqrt{\delta_k}} + J'_k\right)(Q_k - 1)\,\mathbf{1}_{A_k}\right) = \mathrm{E}_{\mathcal{G}_{t_k}}\left(\Phi'_{\eta_k}(J'_k)\,(Q_k - 1)\,\mathbf{1}_{A_k}\right) .$$

Nous allons faire une intégration par parties.
Puisque la condition (H_1, A_k, z) de l'Hypothèse II.2 est satisfaite, la matrice de covariance de J'_k vérifie

$$\phi_{t_k,\delta_k,J'_k} := \int_{t_k}^{t_{k+1}} |D_s J'_k|^2 \, ds = \frac{1}{\delta_k} \times \delta_k\, \sigma_k^2 = \sigma_k^2 \geq \underline{\lambda} > 0 \,.$$

La variabe aléatoire J'_k est donc bien non dégénérée sur A_k au sens de Malliavin, c'est-à-dire elle vérifie la condition (III.2.2), et on peut appliquer l'intégration par parties (III.2.3) du Théorème III.1. On obtient

$$\mathrm{E}_{\mathcal{G}_{t_k}}\left(\phi_{\eta_k}\left(\frac{V_k - z}{\sqrt{\delta_k}} + J'_k\right)(Q_k - 1)\,\mathbf{1}_{A_k}\right) = \mathrm{E}_{\mathcal{G}_{t_k}}\left(\Phi_{\eta_k}(J'_k)\,H(J'_k, Q_k - 1)\,\mathbf{1}_{A_k}\right) .$$

Puisque $0 \leq \Phi_{\eta_k} \leq 1$, on a
$\left|\mathrm{E}_{\mathcal{G}_{t_k}}\left(\Phi_{\eta_k}(I_k)\,H(J'_k, Q_k - 1)\,\mathbf{1}_{A_k}\right)\right| \leq \mathrm{E}_{\mathcal{G}_{t_k}}\left|H(J'_k, Q_k - 1)\,\mathbf{1}_{A_k}\right|$, et donc, pour tout $\omega \in A_k$

$$|J(\omega)| \leq \mathrm{E}_{\mathcal{F}_{t_k}}(|H(J'_k, Q_k - 1)|\ \mathbf{1}_{B_{k,\varsigma}})(\omega) \,. \qquad \text{(IV.3.8)}$$

D'après la Proposition III.1 (i), on a

$$\begin{aligned} |H(J'_k, Q_k - 1)| \leq{} & C\,|Q_k - 1|\,|\phi_{t_k,\delta_k,J'_k}|^{-1}\,|\mathrm{L}_{t_k,\delta_k}(J'_k)| \\ & + C\,|Q_k - 1|_{t_k,\delta_k,1}\,|\phi_{t_k,\delta_k,J'_k}|^{-1}\,|J'_k|_{t_k,\delta_k,1} \\ & + C\,|Q_k - 1|\,|\phi_{t_k,\delta_k,J'_k}|^{-2}\,|J'_k|^2_{t_k,\delta_k,1}\,|J'_k|_{t_k,\delta_k,2} \,. \end{aligned} \qquad \text{(IV.3.9)}$$

Rappelons que sur A_k nous avons $\phi_{t_k,\delta_k,J'_k} \geq \underline{\lambda}$.
De plus, $D_s J'_k = \frac{1}{\sqrt{\delta_k}} D_s J_k = \frac{\sigma_k}{\sqrt{\delta_k}}$, et donc $D^2_{us} J'_k = 0$.
Conclusion :

$$|J'_k|_{t_k,\delta_k,1} = \left(\int_{t_k}^{t_{k+1}} |D_s J'_k|^2 \, ds\right)^{1/2} = |\sigma_k| \leq \sqrt{\overline{\lambda}} \text{ et } |J'_k|_{t_k,\delta_k,2} = 0 \,.$$

D'autre part, nous avons
$|\mathrm{L}_{t_k,\delta_k}(J'_k)| = |\int_{t_k}^{t_{k+1}} D_s J'_k \, dB_s| = \frac{\sigma_k}{\sqrt{\delta_k}} |B_{t_{k+1}} - B_{t_k}| \leq \frac{\sqrt{\overline{\lambda}}}{\sqrt{\delta_k}} |B_{t_{k+1}} - B_{t_k}|$.

En insérant ces évaluations dans l'équation (IV.3.9), on obtient

$$|H(J'_k, Q_k - 1)| \leq C \,\frac{\sqrt{\overline{\lambda}}}{\underline{\lambda}}\, |Q_k - 1| \,\frac{1}{\sqrt{\delta_k}}\, |B_{t_{k+1}} - B_{t_k}| + C \,\frac{\sqrt{\overline{\lambda}}}{\underline{\lambda}}\, |Q_k - 1|_{t_k,\delta_k,1}\,.$$

L'équation (IV.3.8) devient donc

$$\begin{aligned}|J(\omega)| &\leq C \,\frac{\sqrt{\overline{\lambda}}}{\underline{\lambda}}\, \frac{1}{\sqrt{\delta_k}}\, \mathrm{E}_{\mathcal{F}_{t_k}}(|Q_k - 1|\, |B_{t_{k+1}} - B_{t_k}|\, \mathbf{1}_{B_{k,\zeta}}) + C \,\frac{\sqrt{\overline{\lambda}}}{\underline{\lambda}}\, \mathrm{E}_{\mathcal{F}_{t_k}}(|Q_k|_{t_k,\delta_k,1}\, \mathbf{1}_{B_{k,\zeta}}) \\ &\leq C \,\frac{\sqrt{\overline{\lambda}}}{\underline{\lambda}}\, \left(\mathrm{E}_{\mathcal{F}_{t_k}}(|Q_k - 1|^2\, \mathbf{1}_{B_{k,\zeta}})\right)^{1/2} + C \,\frac{\sqrt{\overline{\lambda}}}{\underline{\lambda}}\, \mathrm{E}_{\mathcal{F}_{t_k}}(|Q_k|_{t_k,\delta_k,1}\, \mathbf{1}_{B_{k,\zeta}})\,.\end{aligned}$$

D'après le Lemme IV.3 (ii), on a

$$\mathrm{E}_{\mathcal{F}_{t_k}}(|Q_k|_{t_k,\delta_k,1}\, \mathbf{1}_{B_{k,\zeta}}) \leq C\, \delta_k^{\overline{\varepsilon}}\,.$$

De plus, $Q_k \neq 1 \Rightarrow N^2_{k,3}(R_k) \geq \dfrac{\delta_k^{2\,\overline{\varepsilon}+1}}{2}$. Il vient donc en utilisant la condition (H_2, A_k, z) de l'Hypothèse II.3 et la Remarque 1.1,

$$\begin{aligned}\mathrm{E}_{\mathcal{F}_{t_k}}(|Q_k - 1|^2\, \mathbf{1}_{B_{k,\zeta}}) \leq& \mathbf{P}_{\mathcal{F}_{t_k}}\left(B_{k,\zeta}, |N^2_{k,3}(R_k)| \geq \frac{\delta_k^{2\,\overline{\varepsilon}+1}}{2}\right) \\ =& \mathbf{P}_{\mathcal{F}_{t_k}}\left(|N^2_{k,3}(R_k)|\, \mathbf{1}_{B_{k,\zeta}} \geq \frac{\delta_k^{2\,\overline{\varepsilon}+1}}{2}\right) \\ \leq& 2\, \delta_k^{-2\,\overline{\varepsilon}-1}\, \mathrm{E}_{\mathcal{F}_{t_k}} |N^2_{k,3}(R_k)\, \mathbf{1}_{B_{k,\zeta}}| \\ \leq& C\, \delta_k^{2\,\overline{\varepsilon}}\,.\end{aligned}$$

Conclusion : pour tout $\omega \in A_k$,

$$|J(\omega)| \leq C \,\frac{\sqrt{\overline{\lambda}}}{\underline{\lambda}}\, \delta_k^{\overline{\varepsilon}}\,.$$

En prenant

$$\delta_k \leq \delta_* \leq \delta(\underline{\lambda}, \overline{\lambda}) \leq \frac{\underline{\lambda}}{C\,\sqrt{\overline{\lambda}}} \left(\frac{e^{-4/\underline{\lambda}}}{16\,\sqrt{2\,\pi\,\overline{\lambda}}}\right)^{1/\overline{\varepsilon}}, \qquad \text{(IV.3.10)}$$

la preuve est achevée. ■

3.3. Evaluation de J'

Rappelons que nous avons défini $J'(\omega)$ pour tout $\omega \in A_k$ par l'équation (IV.1.6), soit encore

$$J'(\omega) = \int_0^1 \mathrm{E}_{\mathcal{F}_{t_k}} \left[\phi'_{\eta_k}\left(\frac{G_k - z}{\sqrt{\delta_k}} + \rho \, \frac{R_k}{\sqrt{\delta_k}}\right) \frac{R_k}{\sqrt{\delta_k}} \, Q_k \, \mathbf{1}_{B_{k,\zeta}} \right] d\rho \, .$$

Voici le résultat de ce paragraphe :

Lemme IV.5:
Supposons que $\delta_k \leq \delta_*$. Alors, pour tout $\omega \in A_k$, nous avons

$$|J'(\omega)| \leq \frac{1}{16 \sqrt{2 \, \pi \, \overline{\lambda}}} \times e^{-4/\underline{\lambda}} \, .$$

Preuve. Soient $\eta_k = \dfrac{\eta}{\sqrt{\delta_k}}$, et $R'_k := \dfrac{R_k}{\sqrt{\delta_k}}$. On définit les variables aléatoires V_k et J_k respectivement par les équations (IV.3.6) et (IV.3.5), de telle sorte que $G_k = V_k + J_k$. Posons $J'_k = \dfrac{J_k}{\sqrt{\delta_k}}$. Avec ces notations, on a

$$J'(\omega) \, \mathbf{1}_{A_k} = \int_0^1 \mathrm{E}_{\mathcal{F}_{t_k}} \left[\mathrm{E}_{\mathcal{G}_{t_k}} \left[\phi'_{\eta_k} \left(\frac{V_k - z}{\sqrt{\delta_k}} + (J'_k + \rho \, R'_k) \right) R'_k \, Q_k \, \mathbf{1}_{A_k} \right] \mathbf{1}_{B_{k,\zeta}} \right] d\rho \, . \quad \text{(IV.3.11)}$$

Reprenant la fonction Φ_{η_k} définie par l'équation (IV.3.7), on a

$$\mathrm{E}_{\mathcal{G}_{t_k}} \left[\phi'_{\eta_k} \left(\frac{V_k - z}{\sqrt{\delta_k}} + (J'_k + \rho \, R'_k) \right) R'_k \, Q_k \, \mathbf{1}_{A_k} \right] = \mathrm{E}_{\mathcal{G}_{t_k}} \left[\Phi^{(2)}_{\eta_k}(J'_k + \rho \, R'_k) \, R'_k \, Q_k \, \mathbf{1}_{A_k} \right] \, .$$

Nous allons faire deux intégrations par parties successives. Il nous faut pour cela regarder la condition de non dégénérescence (III.2.2) nécessaire à ces deux intégrations par parties.
Dans le poids $H_2(J'_k + \rho \, R'_k, R'_k \, Q_k)$ qui provient de ces deux intégrations par parties (voir Théorème III.1), apparaissent des termes qui dépendent de la fonction de localisation Q_k et de ses deux premières dérivées de Malliavin. Plus précisément, $H_2(J'_k + \rho \, R'_k, R'_k \, Q_k)$ est une somme dont chaque terme est multiplié par Q_k, DQ_k et $D^2 Q_k$. Ces termes étant nuls si $\theta^{(j)}(N^2_{k,3}(R_k) \, \delta_k^{2\,\bar{\varepsilon}+1}) = 0$, $j = 0, 1, 2$, nous travaillons donc sur l'ensemble

$$\Theta_k := \bigcup_{j=0}^{2} \left\{ \theta^{(j)}(N^2_{k,3}(R_k) \, \delta_k^{2\,\bar{\varepsilon}+1}) \neq 0 \right\} \subseteq \left\{ N^2_{k,3}(R_k) \leq \delta_k^{2\,\bar{\varepsilon}+1} \right\} \, ,$$

et on a $H_2(J'_k + \rho \, R'_k, R'_k \, Q_k) = H_2(J'_k + \rho \, R'_k, R'_k \, Q_k) \, \mathbf{1}_{\Theta_k}$.

Ainsi, dans les calculs qui suivent, on utilise la propriété

$$N_{k,3}(R_k) \le \delta_k^{\bar{\varepsilon}+1/2} \,. \qquad \text{(IV.3.12)}$$

La condition (H_1, A_k, z) de l'Hypothèse II.2 étant satisfaite, et puisque $0 \le \rho \le 1$, nous avons sur A_k,
$\phi_{t_k,\delta_k,J'_k+\rho\, R'_k} \ge \frac{1}{2}\,\phi_{t_k,\delta_k,J'_k} - \rho\,\phi_{t_k,\delta_k,R'_k} \ge \frac{\underline{\lambda}}{2} - \phi_{t_k,\delta_k,R'_k}$.
Par ailleurs, $\phi_{t_k,\delta_k,R'_k} = \frac{1}{\delta_k}\,|R_k|^2_{t_k,\delta_k,1} \le \frac{1}{\delta_k}\,N^2_{k,3}(R_k)$. Donc, d'après la propriété (IV.3.12), il vient $\phi_{t_k,\delta_k,J'_k+\rho\, R'_k} \ge \frac{\underline{\lambda}}{2} - \delta_k^{2\,\bar{\varepsilon}}$.
En prenant

$$\delta_k \le \delta_* \le \delta(\underline{\lambda},\overline{\lambda}) \le \left(\frac{\underline{\lambda}}{4}\right)^{1/(2\,\bar{\varepsilon})} , \qquad \text{(IV.3.13)}$$

il vient

$$\phi_{t_k,\delta_k,J'_k+\rho\, R'_k}(\omega) \ge \frac{\underline{\lambda}}{4}\,, \text{ pour tout } \omega \in A_k \cap \Theta_k \,.$$

La variable aléatoire $J'_k + \rho\, R'_k$ est donc non dégénérée au sens de Malliavin sur $A_k \cap \Theta_k$, c'est-à-dire elle vérifie la condition (III.2.2). Il est donc possible de faire deux intégrations par parties successives sur $A_k \cap \Theta_k$. Le résultat (III.2.5) du Théorème III.1 nous donne alors :

$$\begin{aligned}
&\mathrm{E}_{\mathcal{G}_{t_k}}\left[\phi'_{\eta_k}\left(\frac{V_k - z}{\sqrt{\delta_k}} + (J'_k + \rho\, R'_k)\right)\, R'_k\, Q_k\, \mathbf{1}_{A_k}\right] \\
&\quad = \mathrm{E}_{\mathcal{G}_{t_k}}\left[\Phi^{(2)}_{\eta_k}(J'_k + \rho\, R'_k)\, R'_k\, Q_k\, \mathbf{1}_{A_k}\right] \\
&\quad = \mathrm{E}_{\mathcal{G}_{t_k}}\left[\Phi_{\eta_k}(J'_k + \rho\, R'_k)\, H_2(J'_k + \rho\, R'_k, R'_k\, Q_k)\, \mathbf{1}_{A_k \cap \Theta_k}\right] \,.
\end{aligned}$$

Puisque $0 \le \Phi_{\eta_k} \le 1$, l'équation (IV.3.11) devient pour tout $\omega \in A_k$,

$$|J'(\omega)| \le \int_0^1 \mathrm{E}_{\mathcal{F}_{t_k}}\left(|H_2(J'_k + \rho\, R'_k, R'_k\, Q_k)|\, \mathbf{1}_{B_{k,\varsigma} \cap \Theta_k}\right)(\omega)\, d\rho \,. \qquad \text{(IV.3.14)}$$

D'après la Proposition III.1 (ii), on a

$$\begin{aligned}
|H_2(J'_k + \rho\, R'_k, R'_k\, Q_k)| &\le C F(J'_k + \rho\, R'_k) \\
&\quad \times \left(|R'_k\, Q_k| + |R'_k\, Q_k|_{t_k,\delta_k,1} + |R'_k\, Q_k|_{t_k,\delta_k,2}\right) , \qquad \text{(IV.3.15)}
\end{aligned}$$

avec

$$
\begin{aligned}
F(J'_k + \rho\, R'_k) := (1 \vee |\phi_{t_k,\delta_k,J'_k+\rho R'_k}|^{-5})\,(1 + \sum_{i=0}^{3} |J'_k + \rho\, R'_k|_{t_k,\delta_k,i})^6 \\
\times (1 + |\mathrm{L}_{t_k,\delta_k}(J'_k + \rho\, R'_k)| + |\mathrm{L}_{t_k,\delta_k}(J'_k + \rho\, R'_k)|_{t_k,\delta_k,1})^2 \\
\leq (1 \vee |\phi_{t_k,\delta_k,J'_k+\rho R'_k}|^{-5})\,(1 + N_{k,3}(J'_k + \rho\, R'_k))^8 \,.
\end{aligned}
$$

Regardons le terme $F(J'_k + \rho\, R'_k)$.
On vient de voir que sur $A_k \cap \Theta_k$, $\phi_{t_k,\delta_k,J'_k+\rho R'_k} \geq \dfrac{\lambda}{4}$. Donc

$$
(1 \vee |\phi_{J'_k+\rho R'_k}|^{-5}) \leq C\, \frac{1}{\underline{\lambda}^5}\,.
$$

Puisque $0 \leq \rho \leq 1$, on a

$$
N_{k,3}(J'_k + \rho\, R'_k) \leq N_{k,3}(J'_k) + N_{k,3}(R'_k)\,.
$$

Nous avons vu dans la preuve du Lemme IV.4 que $D^i J'_k = 0$ pour $i = 2, 3$, et $|J'_k|_{t_k,\delta_k,1} = \sigma_k \leq \sqrt{\overline{\lambda}}$. De plus, $\mathrm{L}_{t_k,\delta_k}(J'_k) = J'_k$.
Conclusion :

$$
N_{k,3}(J'_k) \leq C\, \left(|J'_k| + \sqrt{\overline{\lambda}}\right)\,.
$$

D'autre part, en utilisant la propriété (IV.3.12), nous avons sur Θ_k,

$$
N_{k,3}(R'_k) = \frac{1}{\sqrt{\delta_k}}\, N_{k,3}(R_k) \leq \delta_k^{\overline{\varepsilon}} \leq 1\,.
$$

Donc, finalement,

$$
F(J'_k + \rho\, R'_k) \leq \frac{C\,\overline{\lambda}^4}{\underline{\lambda}^5}\,(1 + |J'_k|)^8\,.
$$

Par ailleurs, la propriété (IV.3.12) entraîne $|R'_k| \leq \delta_k^{\overline{\varepsilon}}$ et $|R'_k|_{t_k,\delta_k,i} \leq \delta_k^{\overline{\varepsilon}}$, $i = 1, 2$.
Puisque $|Q_k| \leq 1$, le Lemme III.1 (i) nous donne alors

$$
|R'_k\, Q_k|_{t_k,\delta_k,2} \leq C\, \delta_k^{\overline{\varepsilon}}\,(1 + |Q_k|_{t_k,\delta_k,1} + |Q_k|_{t_k,\delta_k,2})\,.
$$

Conclusion : en insérant ces résultat dans l'équation (IV.3.15), il vient

$$
|H_2(J'_k + \rho\, R'_k, R'_k\, Q_k)| \leq C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}}\,(1 + |J'_k|)^8\,(1 + |Q_k|_{t_k,\delta_k,1} + |Q_k|_{t_k,\delta_k,2})\,.
$$

En utilisant l'inégalité de Hölder, on obtient (pour $q = 8\,(1+\zeta)/\zeta$),

$$
\begin{aligned}
&\mathrm{E}_{\mathcal{F}_{t_k}}\left(|H_2(J'_k + \rho\, R'_k, R'_k\, Q_k)|\ \mathbf{1}_{B_{k,\zeta}\cap\Theta_k}\right) \\
&\quad \le C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}}\, \mathrm{E}_{\mathcal{F}_{t_k}}(1+|J'_k|)^8 + C\, \delta_k^{\overline{\varepsilon}}\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \mathrm{E}_{\mathcal{F}_{t_k}}\left((1+|J'_k|)^8\, |Q_k|_{t_k,\delta_k,1}\, \mathbf{1}_{B_{k,\zeta}}\right) \\
&\quad + C\, \delta_k^{\overline{\varepsilon}}\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \mathrm{E}_{\mathcal{F}_{t_k}}\left((1+|J'_k|)^8\, |Q_k|_{t_k,\delta_k,2}\, \mathbf{1}_{B_{k,\zeta}}\right) \\
&\quad \le C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}} + C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}}\, \left(\mathrm{E}_{\mathcal{F}_{t_k}}(1+|J'_k|)^q\right)^{\zeta/(1+\zeta)}\, \left(\mathrm{E}_{\mathcal{F}_{t_k}}|Q_k|^{1+\zeta}_{t_k,\delta_k,1}\, \mathbf{1}_{B_{k,\zeta}})\right)^{1/(1+\zeta)} \\
&\quad + C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}}\, \left(\mathrm{E}_{\mathcal{F}_{t_k}}(1+|J'_k|)^q\right)^{\zeta/(1+\zeta)}\, \left(\mathrm{E}_{\mathcal{F}_{t_k}}|Q_k|^{1+\zeta}_{t_k,\delta_k,2}\, \mathbf{1}_{B_{k,\zeta}})\right)^{1/(1+\zeta)} \\
&\quad \le C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}} + C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}}\, \left(\mathrm{E}_{\mathcal{F}_{t_k}}|Q_k|^{1+\zeta}_{t_k,\delta_k,1}\, \mathbf{1}_{B_{k,\zeta}})\right)^{1/(1+\zeta)} \\
&\quad + C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}}\, \left(\mathrm{E}_{\mathcal{F}_{t_k}}|Q_k|^{1+\zeta}_{t_k,\delta_k,2}\, \mathbf{1}_{B_{k,\zeta}})\right)^{1/(1+\zeta)}\ .
\end{aligned}
$$

Le Lemme IV.3 (ii) nous donne alors

$$
\mathrm{E}_{\mathcal{F}_{t_k}}\left(|H_2(J'_k + \rho\, R'_k, R'_k\, Q_k)|\ \mathbf{1}_{B_{k,\zeta}\cap\Theta_k}\right) \le C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}}\,.
$$

Finalement, en insérant ce résultat dans l'équation (IV.3.11), il vient pour tout $\omega \in A_k$,

$$
J'(\omega) \le C\, \frac{\overline{\lambda}^4}{\underline{\lambda}^5}\, \delta_k^{\overline{\varepsilon}}\,.
$$

En prenant

$$
\delta_k \le \delta_* \le \delta(\underline{\lambda}, \overline{\lambda}) \le \frac{\underline{\lambda}^5}{C\, \overline{\lambda}^4}\, \left(\frac{e^{-4/\underline{\lambda}}}{16\sqrt{2\,\pi\,\overline{\lambda}}}\right)^{1/\overline{\varepsilon}}\ , \qquad \text{(IV.3.16)}
$$

la preuve est achevée. ■

Suites d'évolution V

Dans ce chapitre, on se donne une grille de temps $0 = t_0 < t_1 < \ldots < t_N = T$, et on note $\delta_k = t_{k+1} - t_k$ le pas de temps. Soit une suite de réels $(x_k)_{k=1,\ldots,N}$ telle que : $x_0 = X_0$ et x_{k+1} satisfait les deux propriétés suivantes,

– $|x_{k+1} - x_k| \leq \dfrac{\sqrt{\delta_k}}{4}$,

– On définit l'événement $\mathcal{F}_{t_k}$-mesurable A_k par

$$A_k = \left\{ \omega / |X_{t_{i-1}} - x_i| < \frac{\sqrt{\delta_{i-1}}}{2}, i = 1, \ldots, k+1 \right\} \subseteq \left\{ |X_{t_k}(\omega) - x_{k+1}| \leq \sqrt{\delta_k} \right\} .$$

On suppose que les conditions (H_1, A_k, x_{k+1}) et (H_2, A_k, x_{k+1}) introduites dans les Hypothèses II.2 et II.3 sont vérifiées.

Le chapitre précédent nous donne le résultat suivant :

Proposition V.1:
Supposons que $\delta_k \leq \delta_*$, où δ_* est défini par l'équation (IV.1.2).
Supposons que $|x_{k+1} - z| \leq \dfrac{\sqrt{\delta_k}}{2}$.
Alors, pour tout $0 < \eta \leq \sqrt{\delta_k}$, pour tout $\omega \in A_k$, on a

$$p_{\eta,k}(\omega, z) \geq \frac{1}{8\sqrt{2\pi\,\delta_k\,\overline{\lambda}}} \times e^{-4/\underline{\lambda}} .$$

Preuve. Pour tout $\omega \in A_k$, on a
$|X_{t_k} - z| \leq |X_{t_k} - x_{k+1}| + |x_{k+1} - z| \leq \dfrac{\sqrt{\delta_k}}{2} + \dfrac{\sqrt{\delta_k}}{2} = \sqrt{\delta_k}$.
Et donc $A_k \subseteq \left\{ \omega / |X_{t_k}(\omega) - z| \leq \sqrt{\delta_k} \right\}$. On peut ainsi appliquer le Théorème IV.1, ce qui nous donne le résultat. ■

En appliquant la Proposition V.1 au point $z = x_k$, la suite $(x_k)_{k=1,\ldots,N}$ nous donne donc une minoration de $p_{\eta,k}(\omega, x_k)$, c'est-à-dire de la régularisation de la densité condionnelle de $X_{t_{k+1}}$ sachant $\mathcal{F}_{t_k}$ au point x_k. Par un argument de récurrence, cette suite va nous permettre de transmette cette minoration pas à pas (c'est-à-dire de t_k à t_{k+1}, $k = 0, \ldots, N-1$), et donc de minorer la densité de X_{t_N} au point x_N. Le résultat principal de ce chapitre est le suivant :

Théorème V.1:
Supposons que la loi de X_{t_N} a une densité continue p_N par rapport à la mesure de Lebesgue sur $\mathbb{R}$.
Supposons que pour $k = 0, \ldots, N$, $\delta_k \leq \delta_*$ et qu'il existe $H_k \geq 1$ tel que

$$\delta_{k-1} \leq H_k^2 \, \delta_k \, .$$

On obtient alors

$$p_N(x_N) \geq \frac{e^{-4/\underline{\lambda}}}{8\sqrt{2\,\pi\,\overline{\lambda}}} \, e^{-(N-1)\,\theta} \, ,$$

où $\theta = \dfrac{4}{\underline{\lambda}} + \ln 32 + \dfrac{\ln(2\,\pi\,\overline{\lambda})}{2} + \dfrac{1}{N-1} \displaystyle\sum_{k=1}^{N-1} \ln H_k$.

__Preuve__. Soit $0 < \eta \leq \sqrt{\delta_{N-1}}$ et $|x - x_N| \leq \sqrt{\delta_{N-1}}/2$. La Proposition V.1 entraîne

$$\begin{aligned}\int_{\mathbb{R}} p_N(x)\,\phi_\eta(x - x_N)\,dx &= \mathrm{E}\left[\mathrm{E}_{\mathcal{F}_{t_{N-1}}}\left(\phi_\eta(X_{t_N} - x_N)\right)\right] \\ &\geq \mathrm{E}\left[p_{\eta,N-1}(x_N)\,\mathbf{1}_{A_{N-1}}\right] \\ &\geq \frac{e^{-4/\underline{\lambda}}}{8\sqrt{2\,\pi\,\delta_{N-1}\,\overline{\lambda}}}\,P(A_{N-1})\,.\end{aligned}$$

Il suffit donc de montrer que $P(A_{N-1}) \geq e^{-(N-1)\,\theta}$ pour obtenir le résultat. En effet, un passage à la limite $\eta \to 0$ et la continuité de p_N permettent ensuite de conclure.
Etape 1. Montrons que pour tout $0 < \eta \leq \dfrac{\sqrt{\delta_{k-1}}}{4\,H_k}$, on a

$$P(A_k) \geq \mathrm{E}\left[\mathbf{1}_{A_{k-1}} \int_{|y - x_k| \leq \frac{\sqrt{\delta_{k-1}}}{4\,H_k} - \eta} p_{\eta,k-1}(y)\,dy\right] .$$

Puisque $\displaystyle\int_{\mathbb{R}} \phi_\eta(X_{t_k} - y)\,dy = \int_{\mathbb{R}} \phi_\eta(y)\,dy = 1$, on obtient

$$\begin{aligned}P(A_k) &= \mathrm{E}(\mathbf{1}_{A_k}) \\ &= \mathrm{E}\left(\mathbf{1}_{A_{k-1}}\,\mathbf{1}_{\{|X_{t_k} - x_{k+1}| < \frac{\sqrt{\delta_k}}{2}\}}\right) \\ &= \mathrm{E}\left[\mathbf{1}_{A_{k-1}}\,\mathrm{E}_{\mathcal{F}_{t_{k-1}}}\left(\mathbf{1}_{\{|X_{t_k} - x_{k+1}| < \frac{\sqrt{\delta_k}}{2}\}}\right)\right] \\ &= \mathrm{E}\left[\mathbf{1}_{A_{k-1}}\,\mathrm{E}_{\mathcal{F}_{t_{k-1}}}\left(\int_{\mathbb{R}} \phi_\eta(X_{t_k} - y)\,\mathbf{1}_{\{|X_{t_k} - x_{k+1}| < \frac{\sqrt{\delta_k}}{2}\}}\,dy\right)\right] .\end{aligned}$$

Or $|X_{t_k} - y| \leq \eta$ si $\phi_\eta(X_{t_k} - y) \neq 0$. De plus, $|x_k - x_{k+1}| \leq \dfrac{\sqrt{\delta_k}}{4}$.

Si $\phi_\eta(X_{t_k} - y) \neq 0$, on obtient donc

$$|X_{t_k} - x_{k+1}| \leq |X_{t_k} - y| + |y - x_k| + |x_k - x_{k+1}| \leq \eta + \frac{\sqrt{\delta_k}}{4} + |y - x_k| \,.$$

Puisque par la définition de H_k on a $\sqrt{\delta_{k-1}} \leq H_k \sqrt{\delta_k}$, il vient $\left\{ |y - x_k| < \frac{\sqrt{\delta_{k-1}}}{4\,H_k} - \eta \right\} \subseteq \left\{ |X_{t_k} - x_{k+1}| < \frac{\sqrt{\delta_k}}{2} \right\}$. Ainsi,

$$\mathrm{E}_{\mathcal{F}_{t_{k-1}}} \left(\int_{\mathbb{R}} \phi_\eta(X_{t_k} - y)\, \mathbf{1}_{\{|X_{t_k} - x_{k+1}| < \frac{\sqrt{\delta_k}}{2}\}}\, dy \right) \geq \int_{\mathbb{R}} \mathbf{1}_{\{|y - x_k| < \frac{\sqrt{\delta_{k-1}}}{4\,H_k} - \eta\}}\, \mathrm{E}_{\mathcal{F}_{t_{k-1}}} \left(\phi_\eta(X_{t_k} - y) \right) dy \,.$$

Ce qui termine cette première étape.

Etape 2. Déduisons maintenant une relation de récurrence.

Prenant $\eta = \frac{\sqrt{\delta_{k-1}}}{8\,H_k}$ dans l'étape précédente, vérifions que les hypothèses de la Proposition V.1 sont satisfaites.

Puisque $H_k \geq 1$, on a $\eta \leq \sqrt{\delta_{k-1}}$, et $\frac{\sqrt{\delta_{k-1}}}{4\,H_k} - \eta = \frac{\sqrt{\delta_{k-1}}}{8\,H_k} \leq \sqrt{\delta_{k-1}}$. Donc, sur l'ensemble $\left\{ |y - x_k| < \frac{\sqrt{\delta_{k-1}}}{4\,H_k} - \eta \right\}$, on a $|x_k - y| \leq \sqrt{\delta_{k-1}}$.

Conclusion : on peut appliquer la Proposition V.1 pour obtenir :

$$\begin{aligned} P(A_k) &\geq \mathrm{E} \left[\mathbf{1}_{A_{k-1}} \int_{\{|y - x_k| < \frac{\sqrt{\delta_{k-1}}}{8\,H_k}\}} p_{\eta,k-1}(y)\, dy \right] \\ &\geq \frac{e^{-4/\underline{\lambda}}}{8\sqrt{2\,\pi\,\delta_{k-1}\,\overline{\lambda}}}\, m \left(\{|y - x_k| < \frac{\sqrt{\delta_{k-1}}}{8\,H_k}\} \right) P(A_{k-1}) \,. \end{aligned}$$

Puisque $m \left(\{|y - x_k| < \frac{\sqrt{\delta_{k-1}}}{8\,H_k}\} \right) = \frac{\sqrt{\delta_{k-1}}}{4\,H_k}$, il vient

$$P(A_k) \geq \frac{e^{-4/\underline{\lambda}}}{32\sqrt{2\,\pi\,\overline{\lambda}}}\, \frac{1}{H_k}\, P(A_{k-1}) \,.$$

Etape 3. Une récurrence nous donne donc

$$P(A_{N-1}) \geq \left(\frac{e^{-4/\underline{\lambda}}}{32\sqrt{2\,\pi\,\overline{\lambda}}} \right)^{N-1} \left(\prod_{k=1}^{N-1} \frac{1}{H_k} \right) P(A_0) \,.$$

Or $P(A_0) = P\left(|X_{t_0} - x_1| < \frac{\sqrt{\delta_0}}{2}\right) = 1$, car $|X_{t_0} - x_1| = |x_0 - x_1| \leq \frac{\sqrt{\delta_0}}{4}$.
Finalement

$$P(A_{N-1}) \geq \left(\frac{e^{-4/\underline{\lambda}}}{32\sqrt{2\,\pi\,\overline{\overline{\lambda}}}}\right)^{N-1} \prod_{k=1}^{N-1} \frac{1}{H_k} \geq e^{-(N-1)\theta}\,.$$

Ce qui achève la preuve. ■

Minoration de la densité VI

Il nous faut tout d'abord vérifier que nous sommes dans le contexte du chapitre précédent, en particulier que le reste R_k défini par l'équation (II.0.7) satisfait la condition (H_2, A_k, z) de l'Hypothèse II.3.
Nous appliquons pour cela l'inégalité de Burkhölder pour les processus de sauts (voir par exemple [BGJ87]). Rappelons la définition de [IW89] :
Une fonction $u : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ est $\mathcal{F}_t$-prévisible si $(t,a,\omega) \to u(t,a,\omega)$ est $\mathcal{S}$-mesurable, où $\mathcal{S}$ désigne la plus petite σ-algèbre sur $[0,\infty) \times \mathbb{R} \times \mathbb{R}$ contenant les fonctions mesurables g telles que :

– pour tout $t > 0$, $(a,\omega) \to g(t,a,\omega)$ est $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$-mesurable
– pour tous (a,ω), $t \to g(t,a,\omega)$ est continue à gauche.

Théorème VI.1:
Soit $u : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ une fonction $\mathcal{F}_t$-prévisible telle que pour tout $t > 0$,
$\mathrm{E}\left[\int_0^t \int_{\mathbb{R}} |u(t,a,\omega)|^2 \, dt \, \nu(da)\right] < \infty.$
On suppose qu'il existe une processus prévisible $(L_t)_{t\in[0,T]}$ et une fonction $\overline{u} \in \bigcap_{p\geq 2} \mathrm{L}^p(\mathbb{R},\nu)$ tels que

$$|u(t,a,\omega)| \leq L_t(\omega)\, \overline{u}(a)\,.$$

Alors, pour tout $p \geq 2$, il existe une constante $C_p > 0$ telle que

$$\mathrm{E}\left|\int_0^t \int_{\mathbb{R}} u(s,a,\omega)\, \widetilde{N}(ds,da)\right|^p \leq C_p \int_0^t \mathrm{E}|L_s(\omega)|^p \, ds\,. \qquad \text{(VI.0.1)}$$

1. Estimation du reste de la diffusion

L'objet de ce paragraphe est de montrer que le reste R_k satisfait la condition (H_2, A_k, z) de l'Hypothèse II.3, soit le Théorème suivant :

Théorème VI.2:
Soit $\zeta \in (0,1/2)$. Notons $\overline{\varepsilon} = \dfrac{\zeta}{4\,(1+\zeta)}$.

Alors, il existe une constante universelle $C > 0$ telle que

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(\sum_{i=0}^{5} |R_k|^{2\,(1+\zeta)}_{t_k,\delta_k,i} \, \mathbf{1}_{B_{k,\zeta}} \right) \right)^{1/(1+\zeta)} \leq C\, \delta_k^{1+4\,\overline{\varepsilon}} \,.$$

Notons

$$R_k^B := \int_{t_k}^{t_{k+1}} \sigma(X_s) - \sigma(X_{t_k})\, dB_s \,, \tag{VI.1.1}$$

$$R_k^N := \int_{t_k}^{t_{k+1}} \int_{|a|\leq \varepsilon_*} [c(s,a,X_{s^-}) - c(s,a,X_{t_k})] \; \widetilde{N}(ds,da) \,, \tag{VI.1.2}$$

$$\overline{R}_k^N := \int_{t_k}^{t_{k+1}} \int_{|a|> \varepsilon_*} c(s,a,X_{s^-}) \, \widetilde{N}(ds,da) \,. \tag{VI.1.3}$$

Puisque $R_k = R_k^B + R_k^N + \overline{R}_k^N$, nous allons regarder chacun de ces termes. Les calculs de la preuve du Théorème VI.2 emploieront toujours les mêmes arguments : inégalités de Burkhölder pour le mouvement Brownien et pour les processus de sauts (voir Théorème VI.1), inégalités de Hölder et hypothèses II.1 portant sur les coefficients de la diffusion. Ces calculs étant très répétitifs, nous allons les présenter jusqu'au dérivées secondes afin de montrer comment ils se mettent en place. Mais commençons par quelques évaluations sur la diffusion $(X_t)_{t\geq 0}$ et ses dérivées $D^i X_t$, $i \geq 1$ (dont l'existence est prouvée dans [BGJ87]).

1.1. Evaluations préliminaires de la diffusion

Lemme VI.1:
Pour tout $n \geq 2$, il existe une constante universelle $C > 0$ telle que,

$$(i)\ \mathrm{E}_{\mathcal{F}_{t_k}} \left(|X_s - X_{t_k}|^{2\,(1+\zeta)} \right) \leq C\, \delta_k \text{ pour tout } t_k \leq s < t_{k+1} \,.$$

$$(ii)\ \mathrm{E}_{\mathcal{F}_{t_k}} \left(\left| D^i_{r_1 \ldots r_i} X_s \right|^{n\,(1+\zeta)} \right) \leq C \text{ pour tout } i \geq 1 \,.$$

Preuve. (i) Pour tous $t_k \leq s < t_{k+1}$, nous avons

$$X_s = X_{t_k} + \int_{t_k}^{s} \sigma(X_r)\, dB_r + \int_{t_k}^{s} \int_{\mathbb{R}} c(r,a,X_{r^-}) \, \widetilde{N}(dr,da) \,.$$

Donc

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left(|X_s - X_{t_k}|^{n\,(1+\zeta)} \right) \leq C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \sigma(X_r)\, dB_r \right|^{n\,(1+\zeta)} + C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \int_{\mathbb{R}} c(r,a,X_{r^-}) \, \widetilde{N}(dr,da) \right|^{n\,(1+\zeta)} .$$

• Les inégalités de Burkhölder et les Hypothèses II.1 nous donnent

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \sigma(X_r)\, dB_r \right|^{n\,(1+\zeta)} \leq C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{s} |\sigma(X_r)|^2\, dr \right)^{n\,(1+\zeta)/2} \leq C\, \delta_k^{n\,(1+\zeta)/2} .$$

• D'autre part, les inégalités de Burkhölder pour les processus de sauts, c'est-à-dire le Théorème VI.1 appliqué à $L_t(\omega) = \mathbf{1}_{[t_k,s]}(t)$ et $\overline{u}(a) = \overline{c}(a)$, donnent

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{s} \int_{\mathbb{R}} c(r,a,X_{r^-})\, \widetilde{N}(dr,da) \right)^{n\,(1+\zeta)} \leq C\, \delta_k .$$

Le résultat (i) est donc démontré.
(ii) Regardons pour commencer la dérivée première. Nous avons

$$D_u X_s = \sigma(X_u) + \int_{t_k}^{s} \sigma'(X_r)\, D_u X_r\, dB_r + \int_{t_k}^{s} \int_{\mathbb{R}} \partial_x c(r,a,X_{r^-})\, D_u X_{r^-} \widetilde{N}(dr,da) .$$

Donc, puisque d'après (i), $\mathrm{E}_{\mathcal{F}_{t_k}} \left(|\sigma(X_u)|^{n\,(1+\zeta)} \right) \leq C$,

$$\begin{aligned} \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D_u X_s|^{n\,(1+\zeta)} \right) \leq C + C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \sigma'(X_r)\, D_u X_r\, dB_r \right|^{n\,(1+\zeta)} \\ + C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \int_{\mathbb{R}} \partial_x c(r,a,X_{r^-})\, D_u X_{r^-} \widetilde{N}(dr,da) \right|^{n\,(1+\zeta)} . \end{aligned}$$

• D'après les inégalités de Burkhölder,

$$\begin{aligned} & \mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \sigma'(X_r)\, D_u X_r\, dB_r \right|^{n\,(1+\zeta)} \\ & \leq C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} |\sigma'(X_r)|^2\, |D_u X_r|^2\, dr \right|^{n\,(1+\zeta)/2} \\ & \leq C\, \delta_k^{n\,(1+\zeta)/2-1} \int_{t_k}^{s} \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D_u X_r|^{n\,(1+\zeta)} \right) dr . \end{aligned}$$

• D'autre part, l'inégalité de Burkhölder pour les processus de sauts, c'est-à-dire le Théorème VI.1 appliqué à $L_t(\omega) = D_u X_{t^-}\, \mathbf{1}_{[t_k,s]}(t)$ et $\overline{u}(a) = \overline{c}(a)$ (on rappelle que $|\partial_x c(r,a,X_r)| \leq \overline{c}(a)$), nous donne

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \int_{\mathbb{R}} \partial_x c(r,a,X_{r^-})\, D_u X_{r^-} \widetilde{N}(dr,da) \right|^{n\,(1+\zeta)} \leq C \int_{t_k}^{s} \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D_u X_r|^{n\,(1+\zeta)} \right) dr .$$

Conclusion : notant $f_u(s) := \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D_u X_s|^{n\,(1+\zeta)} \right)$, nous obtenons

$$f_u(s) \leq C + C \int_{t_k}^{s} f_u(r)\, dr .$$

Le lemme de Gronwall entraîne alors $f_u(s) \leq C$. Ce qui donne (ii) pour les dérivées premières.
Regardons maintenant les dérivées secondes. Nous avons

$$D^2_{r_1\,r_2} X_s = H^2_{r_1,r_2}(s) + \int_{t_k}^s \sigma'(X_r)\, D^2_{r_1\,r_2} X_r\, dB_r \\ + \int_{t_k}^s \int_{\mathbb{R}} \partial_x c(r,a,X_{r^-})\, D^2_{r_1\,r_2} X_{r^-}\, \widetilde{N}(dr,da)\,,$$

avec

$$\begin{aligned} H^2_{r_1,r_2}(s) &= \sigma'(X_{r_1})\, D_{r_2} X_{r_1} + \sigma'(X_{r_2})\, D_{r_1} X_{r_2} \\ &+ \int_{t_k}^s \sigma^{(2)}(X_r)\, D_{r_1} X_r\, D_{r_2} X_r\, dB_r \\ &+ \int_{t_k}^s \int_{\mathbb{R}} \partial_x^2 c(r,a,X_{r^-})\, D_{r_1} X_{r^-}\, D_{r_2} X_{r^-}\, \widetilde{N}(dr,da)\,. \end{aligned}$$

En utilisant les techniques précédentes, c'est-à-dire les inégalités de Burkhölder et de Hölder, et les Hypothèses II.1, nous obtenons :

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^s \sigma'(X_r)\, D^2_{r_1\,r_2} X_r\, dB_r \right|^{n\,(1+\zeta)} \leq C\, \delta_k^{n\,(1+\zeta)/2-1} \int_{t_k}^s \mathrm{E}_{\mathcal{F}_{t_k}} \left| D^2_{r_1\,r_2} X_r \right|^{n\,(1+\zeta)} dr\,,$$

et

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^s \int_{\mathbb{R}} \partial_x c(r,a,X_{r^-})\, D^2_{r_1\,r_2} X_{r^-}\, \widetilde{N}(dr,da) \right|^{n\,(1+\zeta)} \\ \leq C \int_{t_k}^s \mathrm{E}_{\mathcal{F}_{t_k}} \left| D^2_{r_1\,r_2} X_r \right|^{n\,(1+\zeta)} dr\,.$$

Il vient donc

$$\mathrm{E}_{\mathcal{F}_{t_k}} |D^2_{r_1\,r_2} X_s|^{n\,(1+\zeta)} \leq \mathrm{E}_{\mathcal{F}_{t_k}} |H^2_{r_1,r_2}(s)|^{n\,(1+\zeta)} + C \int_{t_k}^s \mathrm{E}_{\mathcal{F}_{t_k}} \left| D^2_{r_1\,r_2} X_r \right|^{n\,(1+\zeta)} dr\,.$$

Evaluons $H^2_{r_1,r_2}(s)$. Remarquons que (i) entraîne

$$\begin{aligned} &\int_{t_k}^s \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D_{r_1} X_r|^{n\,(1+\zeta)}\, |D_{r_2} X_r|^{n\,(1+\zeta)} \right) dr \\ &\quad \leq \left(\int_{t_k}^{t_{k+1}} \mathrm{E}_{\mathcal{F}_{t_k}} |D_{r_1} X_r|^{2\,n\,(1+\zeta)}\, dr \right)^{1/2} \times \left(\int_{t_k}^{t_{k+1}} \mathrm{E}_{\mathcal{F}_{t_k}} |D_{r_2} X_r|^{2\,n\,(1+\zeta)}\, dr \right)^{1/2} \\ &\quad \leq C\, \delta_k\,. \end{aligned}$$

Il vient alors par les inégalités de Burkhölder,

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \sigma^{(2)}(X_r)\, D_{r_1} X_r\, D_{r_2} X_r\, dB_r \right|^{n\,(1+\zeta)} \\ \leq C\, \delta_k^{n\,(1+\zeta)/2-1} \int_{t_k}^{s} \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D_{r_1} X_r|^{n\,(1+\zeta)}\, |D_{r_2} X_r|^{n\,(1+\zeta)} \right) dr \leq C\, \delta_k^{n\,(1+\zeta)/2}\,,$$

et (prendre $L_t(\omega) = D_{r_1} X_{t^-}\, D_{r_2} X_{t^-}\, \mathbf{1}_{[t_k,s]}(t)$ et $\overline{u}(a) = \overline{c}(a)$ dans le Théorème VI.1),

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{s} \int_{\mathbb{R}} \partial_x^2 c(r, a, X_{r^-})\, D_{r_1} X_{r^-}\, D_{r_2} X_{r^-}\, \widetilde{N}(dr, da) \right)^{n\,(1+\zeta)} \\ \leq C \int_{t_k}^{s} \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D_{r_1} X_r|^{n\,(1+\eta)}\, |D_{r_2} X_r|^{n\,(1+\zeta)} \right) dr \leq C\, \delta_k\,.$$

Finalement, puisque d'après (i), on a $\mathrm{E}_{\mathcal{F}_{t_k}} |D_{r_1} X_{r_2}|^{n\,(1+\zeta)} \leq C$, on obtient

$$\mathrm{E}_{\mathcal{F}_{t_k}} |H^2_{r_1,r_2}(s)|^{n\,(1+\zeta)} \leq C\,.$$

Conclusion : notant $f_{r_1\,r_2}(s) := \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D^2_{r_1\,r_2} X_s|^{n\,(1+\zeta)} \right)$, on obtient $f_{r_1\,r_2}(s) \leq C + C \displaystyle\int_{t_k}^{s} f_{r_1\,r_2}(r)\, dr$, et le lemme de Gronwall entraîne $f_{r_1\,r_2}(s) \leq C$. Ce qui prouve (ii) pour les dérivées d'ordre deux.

Pour les dérivées d'ordre supérieur, on note qu'une récurrence se met en place de la façon suivante :

Supposons avoir montré que pour $l \leq i-1$, on a $\mathrm{E}_{\mathcal{F}_{t_k}} |D^l_{r_1 \dots r_l} X_s|^{n\,(1+\zeta)} \leq C$.

On a

$$D^i_{r_1 \dots r_i} X_s = H^i_{r_1,\dots,r_i}(s) + \int_{t_k}^{s} \sigma'(X_r)\, D^i_{r_1 \dots r_i} X_r\, dB_r \\ + \int_{t_k}^{s} \int_{\mathbb{R}} \partial_x c(r, a, X_{r^-})\, D^i_{r_1 \dots r_i} X_r\, \widetilde{N}(dr, da)\,,$$

où $H^i_{r_1,\dots,r_i}(s)$ est un terme dépendant des dérivées d'ordres $j \leq i-1$ (terme analogue à $H^2_{r_1,r_2}(s)$). Les inégalités de Burkhölder donnent

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \sigma'(X_r)\, D^i_{r_1 \dots r_i} X_r\, dB_r \right|^{n\,(1+\zeta)} \leq C\, \delta_k^{n\,(1+\zeta)/2-1} \int_{t_k}^{s} \mathrm{E}_{\mathcal{F}_{t_k}} \left| D^i_{r_1 \dots r_i} X_r \right|^{n\,(1+\zeta)} dr\,,$$

et

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left| \int_{t_k}^{s} \int_{\mathbb{R}} \partial_x c(r, a, X_{r^-})\, D^i_{r_1 \ldots r_i} X_{r^-} \widetilde{N}(dr, da) \right|^{n\,(1+\zeta)} \leq C \int_{t_k}^{s} \mathrm{E}_{\mathcal{F}_{t_k}} \left| D^i_{r_1 \ldots r_i} X_r \right|^{n\,(1+\zeta)} dr\,.$$

Concernant l'évaluation du terme $H^i_{r_1,\ldots,r_i}(s)$, les inégalités de Cauchy-Schwarz nous permettent de nous ramener à l'hypothèse de récurrence et donc de montrer que

$$\mathrm{E}_{\mathcal{F}_{t_k}} |H^i_{r_1,\ldots,r_i}(s)| \leq C\,.$$

Conclusion : notant $f_{r_1 \ldots r_i}(s) := \mathrm{E}_{\mathcal{F}_{t_k}} \left(|D^i_{r_1 \ldots r_i} X_s|^{n\,(1+\zeta)} \right)$, on se ramènera toujours à une majoration du type

$$f_{r_1 \ldots r_i}(s) \leq C + C \int_{t_k}^{s} f_{r_1,\ldots,r_i}(r)\, dr\,.$$

Et le Lemme de Gronwall de conclure que $f_{r_1 \ldots r_i}(s) \leq C$. ■

1.2. Estimation du reste correspondant au mouvement brownien

Rappelons que nous avons noté $R^B_k := \displaystyle\int_{t_k}^{t_{k+1}} \sigma(X_s) - \sigma(X_{t_k})\, dB_s$.

Nous avons :

$$\begin{aligned}
\mathrm{E}_{\mathcal{F}_{t_k}} |R^B_k|^{2\,(1+\zeta)} &= \mathrm{E}_{t_k} \left| \int_{t_k}^{t_{k+1}} \sigma(X_s) - \sigma(X_{t_k})\, dB_s \right|^{2\,(1+\zeta)} \\
&\leq C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{t_{k+1}} |\sigma(X_s) - \sigma(X_{t_k})|^2\, ds \right)^{(1+\zeta)} \\
&\leq C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{t_{k+1}} |X_s - X_{t_k}|^2\, ds \right)^{1+\zeta} \\
&\leq C\, \delta_k^{\zeta} \int_{t_k}^{t_{k+1}} \mathrm{E}_{\mathcal{F}_{t_k}} |X_s - X_{t_k}|^{2\,(1+\zeta)}\, ds \\
&\leq C\, \delta_k^{2+\zeta}\,,
\end{aligned}$$

la dernière inégalité venant du Lemme VI.1 (i).

Conclusion : puisque $\zeta \in (0, 1/2)$, $\overline{\varepsilon} = \dfrac{\zeta}{4\,(1+\zeta)} \leq \dfrac{1}{4\,(1+\zeta)}$, et on a

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(|R^B_k|^{2\,(1+\zeta)} \right) \right)^{1/(1+\zeta)} \leq C\, \delta_k^{1+1/(1+\zeta)} \leq C\, \delta_k^{1+4\,\overline{\varepsilon}}\,.$$

Nous avons

$$D_u^1 R_k^B = D_u^1 \left[\int_{t_k}^{t_{k+1}} \sigma(X_s) - \sigma(X_{t_k})\, dB_s \right]$$
$$= \int_u^{t_{k+1}} \sigma'(X_s)\, D_u^1 X_s\, dB_s + \sigma(X_u) - \sigma(X_{t_k})\,.$$

Donc

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left(|R_k^B|_{t_k,\delta_k,1}^{2\,(1+\zeta)} \right) \leq C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{t_{k+1}} |\sigma(X_u) - \sigma(X_{t_k})|^2\, du \right)^{(1+\zeta)}$$
$$+ C\, \mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{t_{k+1}} \left| \int_u^{t_{k+1}} \sigma'(X_s)\, D_u^1 X_s\, dB_s \right|^2 du \right)^{(1+\zeta)}\,.$$

D'après le Lemme VI.1 (ii), il vient

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{t_{k+1}} |\sigma(X_u) - \sigma(X_{t_k})|^2\, du \right)^{(1+\zeta)}$$
$$\leq C\, \delta_k^{\zeta} \int_{t_k}^{t_{k+1}} \mathrm{E}_{\mathcal{F}_{t_k}} |X_u - X_{t_k}|^{2\,(1+\zeta)}\, du \leq C\, \delta_k^{2+\zeta}\,.$$

De même,

$$\mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{t_{k+1}} \left| \int_u^{t_{k+1}} \sigma'(X_s)\, D_u^1 X_s\, dB_s \right|^2 du \right)^{(1+\zeta)}$$
$$\leq \delta_k^{\zeta} \int_{t_k}^{t_{k+1}} \mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_u^{t_{k+1}} \sigma'(X_s)\, D_u^1 X_s\, dB_s \right)^{2\,(1+\zeta)} du$$
$$\leq C\, \delta_k^{2\,\zeta} \int_{t_k}^{t_{k+1}} \mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{t_{k+1}} |D_u^1 X_s|^{2\,(1+\zeta)}\, ds \right) du$$
$$\leq C\, \delta_k^{2\,\zeta+2}\,.$$

Conclusion :

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(|R_k^B|_{t_k,\delta_k,1}^{2\,(1+\zeta)} \right) \right)^{1/(1+\zeta)} \leq C\, \delta_k^{1+1/(1+\zeta)} \leq C\, \delta_k^{1+4\,\overline{\varepsilon}}\,.$$

Nous avons

$$D_{r_2\,r_1}^2 R_k^B = \sigma'(X_{r_1})\, D_{r_2} X_{r_1} + \sigma'(X_{r_2})\, D_{r_1} X_{r_2}$$
$$+ \int_{t_k}^{t_{k+1}} \sigma^{(2)}(X_r)\, D_{r_2} X_r\, D_{r_1} X_r\, dB_s + \int_{t_k}^{t_{k+1}} \sigma'(X_r)\, D_{r_2\,r_1}^2 X_r\, dB_r\,.$$

Le Lemme VI.1 (ii) entraîne

$$\mathrm{E}_{\mathcal{F}_{t_k}}\left(\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}|\sigma'(X_{r_1})\,D_{r_2}X_{r_1}|^2\,dr_1\,dr_2\right)^{(1+\zeta)} \leq C\,\delta_k^{2\,\eta}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\mathrm{E}_{\mathcal{F}_{t_k}}|D_{r_2}X_{r_1}|^{2\,(1+\eta)}\,dr_1\,dr_2 \leq C\,\delta_k^{2+2\,\zeta}\,.$$

De même,

$$\mathrm{E}_{\mathcal{F}_{t_k}}\left(\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}|\sigma'(X_{r_2})\,D_{r_1}X_{r_2}|^2\,dr_1\,dr_2\right)^{(1+\zeta)} \leq C\,\delta_k^{2+2\,\zeta}\,.$$

$$\begin{aligned}
&\mathrm{E}_{\mathcal{F}_{t_k}}\left(\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\left|\int_{t_k}^{t_{k+1}}\sigma^{(2)}(X_r)\,D_{r_2}X_r\,D_{r_1}X_r\,dB_r\right|^2\,dr_1\,dr_2\right)^{1+\zeta}\\
\leq&\delta_k^{2\,\eta}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\mathrm{E}_{\mathcal{F}_{t_k}}\left|\int_{t_k}^{t_{k+1}}\sigma^{(2)}(X_r)\,D_{r_2}X_r\,D_{r_1}X_r\,dB_r\right|^{2\,(1+\zeta)}\,dr_1\,dr_2\\
\leq&C\,\delta_k^{3\,\eta}\int_{[t_k,t_{k+1})^3}\mathrm{E}_{\mathcal{F}_{t_k}}\left(|D_{r_2}X_r|^{2\,(1+\zeta)}\,|D_{r_1}X_r|^{2\,(1+\zeta)}\right)\,dr_1\,dr_2\,dr\\
\leq&C\,\delta_k^{3+3\,\zeta}\,.
\end{aligned}$$

Enfin,

$$\begin{aligned}
&\mathrm{E}_{\mathcal{F}_{t_k}}\left(\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\left|\int_{t_k}^{t_{k+1}}\sigma'(X_r)\,D^2_{r_2\,r_1}X_r\,dB_r\right|^2\,dr_1\,dr_2\right)^{1+\zeta}\\
\leq&\delta_k^{2\,\eta}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\mathrm{E}_{\mathcal{F}_{t_k}}\left|\int_{t_k}^{t_{k+1}}\sigma'(X_r)\,D^2_{r_2\,r_1}X_r\,dB_r\right|^{2\,(1+\zeta)}\,dr_1\,dr_2\\
\leq&C\,\delta_k^{3\,\zeta}\int_{[t_k,t_{k+1})^3}\mathrm{E}_{\mathcal{F}_{t_k}}|D^2_{r_2\,r_1}X_r|^{2\,(1+\zeta)}\,dr_1\,dr_2\,dr\\
\leq&C\,\delta_k^{3+3\,\zeta}\,.
\end{aligned}$$

Conclusion :

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(|R_k^B|^{2\,(1+\zeta)}_{t_k,\delta_k,2}\right)\right)^{1/(1+\zeta)} \leq C\,\delta_k^{1+1/(1+\zeta)} \leq C\,\delta_k^{1+4\,\bar{\varepsilon}}\,.$$

Finalement, en menant les mêmes calculs pour les dérivées d'ordre supérieur, on obtient le résultat suivant :

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(\sum_{i=0}^{5}|R_k^B|^{2\,(1+\zeta)}_{t_k,\delta_k,i}\right)\right)^{1/(1+\zeta)} \leq C\,\delta_k^{1+4\,\bar{\varepsilon}}\,.$$

1.3. Estimation du reste correspondant aux petits sauts

Rappelons que nous avons noté
$R_k^N := \int_{t_k}^{t_{k+1}} \int_{|a|\le\varepsilon_*} [c(s,a,X_{s^-}) - c(s,a,X_{t_k})] \; \widetilde{N}(ds,da)$.
Dans les calculs qui suivent nous utiliserons constamment le Lemme VI.1.
D'après les Hypothèses II.1, nous avons $|c(s,a,X_s) - c(s,a,X_{t_k})| \le \overline{c}(a)\, |X_s - X_{t_k}|$.
Appliquant donc le Théorème VI.1 à $L_t(\omega) = |X_{t^-} - X_{t_k}|\, \mathbf{1}_{[t_k,t_{k+1}]}(t)$ et $\overline{u}(a) = \overline{c}(a)$, on obtient

$$\mathrm{E}_{\mathcal{F}_{t_k}} |R_k^N|^{2\,(1+\zeta)} \le C \int_{t_k}^{t_{k+1}} \mathrm{E}_{\mathcal{F}_{t_k}} |X_s - X_{t_k}|^{2\,(1+\zeta)}\, ds \le C\, \delta_k^2\, .$$

Conclusion : pour $\zeta \in (0,1/2)$, on a $\overline{\varepsilon} = \dfrac{\zeta}{4\,(1+\zeta)} \le \dfrac{1-\zeta}{4\,(1+\zeta)}$, et donc

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(|R_k^N|^{2\,(1+\zeta)}\right)\right)^{1/(1+\zeta)} \le C\, \delta_k^{2/(1+\zeta)} \le C\, \delta_k^{1+4\,\overline{\varepsilon}}\, .$$

Nous avons $D_s R_k^N = \int_{t_k}^{t_{k+1}} \int_{|a|\le\varepsilon_*} \partial_x c(r,a,X_{r^-})\, D_s X_{r^-} \widetilde{N}(dr,da)$.
Le Théorème VI.1 appliqué à $L_t(\omega) = |D_s X_{t^-}|\, \mathbf{1}_{[t_k,t_{k+1}]}(t)$ et $\overline{u}(a) = \overline{c}(a)$ entraîne donc

$$\begin{aligned}
&\mathrm{E}_{\mathcal{F}_{t_k}} \left(|R_k^N|^{2\,(1+\zeta)}_{t_k,\delta_k,1}\right)\\
=&\mathrm{E}_{\mathcal{F}_{t_k}} \left(\int_{t_k}^{t_{k+1}} \left|\int_{t_k}^{t_{k+1}} \int_{|a|\le\varepsilon_*} \partial_x c(r,a,X_{r^-})\, D_s X_{r^-} \widetilde{N}(dr,da)\right|^2 ds\right)^{1+\zeta}\\
\le& C\, \delta_k^{\zeta} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathrm{E}_{\mathcal{F}_{t_k}} |D_s X_r|^{2\,(1+\zeta)}\, dr\, ds\\
\le& C\, \delta_k^{2+\zeta}\, .
\end{aligned}$$

Conclusion :

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(|R_k^N|^{2\,(1+\eta)}_{t_k,\delta_k,1}\right)\right)^{1/(1+\zeta)} \le C\, \delta_k^{1+1/(1+\zeta)} \le C\, \delta_k^{1+4\,\overline{\varepsilon}}\, .$$

Nous avons

$$\begin{aligned}
D^2_{us} R_k^N = &\int_{t_k}^{t_{k+1}} \int_{|a|\le\varepsilon_*} \partial_x^2 c(r,a,X_{r^-})\, D_u X_{r^-}\, D_s X_{r^-}\, \widetilde{N}(dr,da)\\
&+ \int_{t_k}^{t_{k+1}} \int_{|a|\le\varepsilon_*} \partial_x c(r,a,X_{r^-})\, D^2_{us} X_{r^-} \widetilde{N}(dr,da)\, .
\end{aligned}$$

En appliquant à nouveau les inégalités de Burkhölder du Théorème VI.1, on obtient

$$\begin{aligned}
&\mathrm{E}_{\mathcal{F}_{t_k}}\left[\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\left|\int_{t_k}^{t_{k+1}}\int_{|a|\le\varepsilon_*}\partial_x^2 c(r,a,X_{r^-})\ D_uX_{r^-}\ D_sX_{r^-}\ \widetilde{N}(dr,da)\right|^2 du\,ds\right]^{1+\zeta}\\
&\quad\le C\,\delta_k^{2\,\zeta}\int_{[t_k,t_{k+1})^3}\mathrm{E}_{\mathcal{F}_{t_k}}\left[|D_uX_r|^{2\,(1+\zeta)}\,|D_sX_r|^{2\,(1+\zeta)}\right]\,du\,ds\,dr\\
&\quad\le C\,\delta_k^{2\,\zeta+3}\,.
\end{aligned}$$

De même,

$$\begin{aligned}
&\mathrm{E}_{\mathcal{F}_{t_k}}\left[\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}\left|\int_{t_k}^{t_{k+1}}\int_{|a|\le\varepsilon_*}\partial_x c(r,a,X_{r^-})\ D^2_{us}X_{r^-}\widetilde{N}(dr,da)\right|^2 du\,ds\right]^{1+\zeta}\\
&\quad\le C\,\delta_k^{2\,\zeta}\int_{[t_k,t_{k+1})^3}\mathrm{E}_{\mathcal{F}_{t_k}}|D^2_{us}X_r|^{2\,(1+\zeta)}\,du\,ds\,dr\\
&\quad\le C\,\delta_k^{2\,\zeta+3}\,.
\end{aligned}$$

Conclusion :

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(|R_k^N|^{2\,(1+\zeta)}_{t_k,\delta_k,2}\right)\right)^{1/(1+\zeta)}\le C\,\delta_k^{1+1/(1+\zeta)}\le C\,\delta_k^{1+4\,\overline{\varepsilon}}\,.$$

Finalement, en continuant les mêmes calculs aux dérivées d'ordre supérieur, on obtient le résultat suivant

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(\sum_{i=0}^{5}|R_k^N|^{2\,(1+\zeta)}_{t_k,\delta_k,i}\right)\right)^{1/(1+\zeta)}\le C\,\delta_k^{1+4\,\overline{\varepsilon}}\,.$$

1.4. Estimation du reste correspondant aux grands sauts

Rappelons que nous avons noté $\overline{R}_k^N:=\displaystyle\int_{t_k}^{t_{k+1}}\int_{|a|>\varepsilon_*}c(s,a,X_{s^-})\,\widetilde{N}(ds,da)$.
Remarquons que la localisation sur l'événement $B_{k,\zeta}$ ne nous a pas été utile pour évaluer les restes R_k^B et R_k^N. C'est pour ce terme $\overline{R}_k^N$ que cette localisation joue un rôle important.
En effet, si on applique l'inégalité de Burkhölder pour les processus de sauts sans tenir compte de la localisation, on obtient (prendre $L_t(\omega)=\mathbf{1}_{[t_k,t_{k+1}]}(t)$ et $\overline{u}(a)=\overline{c}(a)$ dans le Théorème VI.1) : $\mathrm{E}_{\mathcal{F}_{t_k}}\left(|\overline{R}_k^N|^{2\,(1+\zeta)}\right)\le C\,\delta_k$. Et dans ce cas,

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}}\left(|\overline{R}_k^N|^{2\,(1+\zeta)}\right)\right)^{1/(1+\zeta)}\le C\,\delta_k^{1/(1+\zeta)}\,.$$

Ce qui ne donne pas une puissance assez grande, et donc la condition (H_2, A_k, z) de l'Hypothèse II.3 ne peut être vérifiée. Il nous faut donc nous y prendre autrement. Puisque $\overline{R}_k^N$ ne prend en compte que les grands sauts, on peut écrire

$$\begin{aligned}
|\overline{R}_k^N| &= \left| \int_{t_k}^{t_{k+1}} \int_{|a|>\varepsilon_*} c(r,a,X_{r^-}) \, (N(dr,da) - dr\,\nu(da)) \right| \\
&\leq \left| \int_{t_k}^{t_{k+1}} \int_{|a|>\varepsilon_*} c(r,a,X_{r^-}) \, N(dr,da) \right| + \left| \int_{t_k}^{t_{k+1}} \int_{|a|>\varepsilon_*} c(r,a,X_r) \, dr\,\nu(da) \right| \\
&\leq \int_{t_k}^{t_{k+1}} \int_{|a|>\varepsilon_*} \overline{c}(a) \, dr\,\nu(da) + \int_{t_k}^{t_{k+1}} \int_{|a|>\varepsilon_*} \overline{c}(a) \, N(dr,da) \\
&= \delta_k \int_{|a|>\varepsilon_*} \overline{c}(a)\,\nu(da) + |\overline{R}_k| \,,
\end{aligned}$$

où on rappelle que $\overline{R}_k$ est défini par l'équation (II.0.9).
Donc sur l'événement $B_{k,\zeta} = \{|\overline{R}_k| \leq \delta_k^{\zeta+1/2}\}$, on obtient

$$|\overline{R}_k^N| \, \mathbf{1}_{B_{k,\zeta}} \leq C\,\delta_k + \delta_k^{\zeta+1/2} \,.$$

Puisque pour $\zeta \in (0,1/2)$, $\overline{\varepsilon} = \dfrac{\zeta}{4\,(1+\zeta)} \leq \dfrac{\zeta}{4} \leq \dfrac{1}{4}$, on obtient donc

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(|\overline{R}_k^N|^{2\,(1+\zeta)} \, \mathbf{1}_{B_{k,\zeta}} \right) \right)^{1/(1+\zeta)} \leq C\,(\delta_k^2 + \delta_k^{1+2\,\zeta}) \leq C\,\delta_k^{1+4\,\overline{\varepsilon}} \,.$$

La condition (H_2, A_k, z) sera alors vérifiée pour $\overline{R}_k^N$.
En ce qui concerne les dérivées de $\overline{R}_k^N$, on reprend les mêmes calculs que ceux menés dans le paragraphe précédent pour R_k^N. En effet, la seule différence est que nous travaillons désormais avec la mesure $\mathbf{1}_{|a|>\varepsilon_*} \, \widetilde{N}(ds,da)$ au lieu de $\mathbf{1}_{|a|\leq\varepsilon_*} \, \widetilde{N}(ds,da)$.
On obtient finalement le résultat suivant :

$$\left(\mathrm{E}_{\mathcal{F}_{t_k}} \left(\sum_{i=0}^{5} |\overline{R}_k^N|^{2\,(1+\zeta)}_{t_k,\delta_k,i} \right) \right)^{1/(1+\zeta)} \leq C\,\delta_k^{1+4\,\overline{\varepsilon}} \,.$$

2. Courbes déterministes elliptiques

Dans ce paragraphe, nous allons réunir les résultats précédents (obtenus dans les Théorèmes V.1 et VI.2) pour minorer la densité de X_T en un point $y \in \mathbb{R}$ fixé. Nous travaillons dans le cadre suivant :
• On suppose que la loi de X_T a une densité continue en $y \in \mathbb{R}$ fixé, notée $p_T(x_0, y)$.
• On suppose qu'il existe une courbe continûment différentiable $(x_t)_{t\in[0,T]}$ telle que $x(0) = X_0$, $x(T) = y$. Et on fait l'hypothèse suivante sur la dérivée :

Hypothèse VI.1. Il existe $\overline{M} \geq 1$ et $h \geq 0$ tels que

$$\overline{M}\, |\partial_t x_t|^2 \geq |\partial_s x_s|^2, \text{ si } |s-t| \leq h\,.$$

On suppose de plus qu'il existe deux constantes $\overline{\lambda}$ et $\underline{\lambda}$ telles que pour tout $t \in [0,T]$, $0 < 2\,\underline{\lambda} \leq \sigma^2(x_t) \leq \frac{2}{3}\,\overline{\lambda}$.

• On introduit une constante $0 < r \leq \frac{\underline{\lambda}}{2\, C_0^2}$, où C_0 est la constante de lipschitz de σ introduite dans les Hypothèses II.1.

• Rappelons que nous avons noté δ_* par (IV.1.2), soit

$$\delta_* = \left(\frac{1}{4\int_{|a|>\varepsilon_*} \overline{c}(a)\, \nu(da)} \right)^{1/(1/2-\zeta)} \wedge \delta(\underline{\lambda}, \overline{\lambda})\,,$$

où ε_* vérifie l'équation (II.0.5), soit

$$\int_{|a|\leq\varepsilon_*} \overline{c}^2(a)\, \nu(da) \leq \frac{\underline{\lambda}}{2}\,.$$

On note alors

$$M(r,h) = \delta_* \wedge r \wedge h\,.$$

La minoration que nous obtenons est la suivante :

Théorème VI.3:

$$p_T(x_0,y) \geq \frac{e^{-4/\underline{\lambda}}}{8\sqrt{2\,\pi\,\overline{\lambda}}} \times \exp\left[-\theta \int_0^T \left(\overline{M}^2\, 16\, |\partial_t x_t|^2 + \frac{1}{M(r,h)} \right) dt \right],$$

où $\theta = \frac{4}{\underline{\lambda}} + \ln 32 + \frac{\ln(2\,\pi\,\overline{\lambda})}{2} + \ln \overline{M}$.

Remarque 2.1. *Remarquons que puisque* $\frac{1}{M(r,h)} = \frac{1}{\delta_* \wedge r \wedge h}$, *la minoration obtenue dans le Théorème* VI.3 *est essentiellement du type :*

$$p_T(x_0,y) \geq \frac{e^{-4/\underline{\lambda}}}{8\sqrt{2\,\pi\,\overline{\lambda}}} \times e^{-C\left(\int_{|a|>\varepsilon_*} \overline{c}(a)\,\nu(da)\right)^\alpha}\,,\ \alpha \in (0, 1/2)\,.$$

Etudions alors l'impact de ε_* *sur cette minoration.*

On a $\int_{|a|>\varepsilon_*} \overline{c}(a)\, \nu(da) \underset{\varepsilon_*\to 0}{\longrightarrow} \infty$. *Donc, plus* ε_* *est petit, plus la minoration devient mauvaise.*

Or rappelons que ε_* *est choisi assez petit pour que l'équation* (II.0.5) *soit vérifiée, c'est-à-dire*

$$\int_{|a|\leq\varepsilon_*} \overline{c}^2(a)\, \nu(da) \leq \frac{\underline{\lambda}}{2}\,.$$

Ainsi, plus $\underline{\lambda}$ est petit, plus ε_* l'est aussi, et plus la minoration est mauvaise. Nous obtenons donc une description de l'effet de la mesure de Lévy $\nu(da)$ des sauts via le rapport entre ε_* et $\underline{\lambda}$ sur la qualité de la minoration du Théorème VI.3.

Preuve. Etape 1. Soit Γ_N une subdivision de $[0,T]$ où les instants de la grille t_k sont construits de la façon suivante :
Définissons

$$\tau_k = \inf\left\{u > 0 / \int_{t_k}^{t_k+u} |\partial_t x_t|^2 \, dt \geq \frac{1}{16\,\overline{M}^2}\right\}.$$

On prend $t_0 = 0$, et t_k étant donné, on pose

$$\text{pour } k = 0, \dots, N-1\,,\ t_{k+1} = t_k + \tau_k \wedge M(r,h)\,,$$

avec

$$N = \min\{k / t_k \geq T\}\,.$$

On note le pas de temps

$$\delta_k = t_{k+1} - t_k = \tau_k \wedge M(r,h)\,.$$

Evaluons N. Pour cela, notons $I_1 = \{k \leq N/\delta_k = \tau_k\}$ et $I_2 = \{k \leq N/\delta_k = M(r,h)\}$.
Remarquons alors que

$$\int_0^T \left(\frac{1}{M(r,h)} + 16\,\overline{M}^2\,|\partial_t x_t|^2\right) dt \geq \sum_{k\in I_1} \int_{t_k}^{t_k+\tau_k} 16\,\overline{M}^2\,|\partial_t x_t|^2 \, dt + \sum_{k\in I_2} \int_{t_k}^{t_k+M(r,h)} \frac{1}{M(r,h)} \, dt\,.$$

Nous avons par la définition de τ_k, pour tout $k \in I_1$, $\displaystyle\int_{t_k}^{t_k+\tau_k} 16\,\overline{M}^2\,|\partial_t x_t|^2\,dt \geq 1$, et clairement, pour tout $k \in I_2$, $\displaystyle\int_{t_k}^{t_k+M(r,h)} \frac{1}{M(r,h)}\,dt = 1$.
Conclusion : nous obtenons

$$N \leq \int_0^T \left(\frac{1}{M(r,h)} + 16\,\overline{M}^2\,|\partial_t x_t|^2\right) dt\,.$$

Pour finir cette étape, montrons que le pas de la grille ainsi définie vérifie

$$\delta_{k-1} \leq \overline{M}^2\,\delta_k\,.$$

Supposons que $\tau_k > M(r,h)$. Alors $\delta_k = M(r,h) \geq M(r,h) \wedge \tau_{k-1} = \delta_{k-1}$. Puisque $\overline{M} \geq 1$, il vient $\delta_{k-1} \leq \overline{M}^2\,\delta_k$.
Supposons maintenant que $\tau_k \leq M(r,h)$. Alors $\delta_k = \tau_k$, et il suffit donc de montrer

que $\dfrac{\delta_{k-1}}{\overline{M}^2} \leq \tau_k$, soit

$$\int_{t_k}^{t_k+\frac{\delta_{k-1}}{\overline{M}^2}} |\partial_t x_t|^2 \, dt \leq \frac{1}{16\,\overline{M}^2} .$$

Puisque $\delta_{k-1} \leq h$ et $\overline{M} \geq 1$, pour tout $t \in [t_k, t_k + \dfrac{\delta_{k-1}}{\overline{M}^2})$, on a $|t - t_k| \leq h$, et pour tout $t \in [t_{k-1}, t_k)$, on a $|t - t_{k-1}| \leq h$. En appliquant alors deux fois l'hypothèse VI.1, il vient

$$\int_{t_k}^{t_k+\frac{\delta_{k-1}}{\overline{M}^2}} |\partial_t x_t|^2 \, dt \leq \frac{\delta_{k-1}}{\overline{M}^2}\, \overline{M}\, |\partial_{t_k} x_{t_k}|^2 \leq \int_{t_{k-1}}^{t_k} |\partial_t x_t|^2 \, dt \leq \frac{1}{16\,\overline{M}^2} .$$

Ce qui achève cette première étape.

Etape 2. Suites d'évolution. On définit la suite des réels $x_k = x(t_k)$, $k = 1, \ldots, N$ et on va montrer que $(x_k)_{k=1,\ldots,N}$ est une suite d'évolution.

• Vérifions tout d'abord que $|x(t_{k+1}) - x(t_k)| \leq \dfrac{\sqrt{\delta_k}}{4}$. D'après la définition du pas de temps δ_k dans l'étape précédente, nous obtenons

$$\begin{aligned} |x(t_{k+1}) - x(t_k)| &\leq \int_{t_k}^{t_{k+1}} |\partial_t x_t| \; dt \\ &\leq \sqrt{\delta_k} \left(\int_{t_k}^{t_{k+1}} |\partial_t x_t|^2 \, dt \right)^{1/2} \\ &\leq \frac{\sqrt{\delta_k}}{4\,\overline{M}} \leq \frac{\sqrt{\delta_k}}{4} \ (\text{car } \overline{M} \geq 1) . \end{aligned}$$

Définissons les événements $\mathcal{F}_{t_k}$-mesurables suivants

$$\begin{aligned} A_k = &\left\{ \omega / \left| X_{t_{i-1}}(\omega) - x_i \right| < \frac{\sqrt{\delta_{i-1}}}{2}, i = 1, \ldots, k+1 \right\} \\ &\subseteq \left\{ \omega / |X_{t_k}(\omega) - x_{k+1}| \leq \frac{\sqrt{\delta_k}}{2} \right\} . \end{aligned}$$

• D'après le Théorème VI.2, la condition (H_2, A_k, x_{k+1}) de l'Hypothèse II.3 est satisfaite.

• Montrons que la condition (H_1, A_k, x_{k+1}) de l'Hypothèse II.2 est vérifiée.
Considérons l'événement $A_t := \{\omega / |X_t(\omega) - x_t| \leq r\}$.

Remarquons que $A_k \subseteq A_{t_k}$. En effet, pour tout $\omega \in A_k$, on a $|X_{t_k}(\omega) - x_{k+1}| \leq \dfrac{\sqrt{\delta_k}}{2}$.

Donc la définition de δ_k dans la première étape entraîne

$$|X_{t_k}(\omega) - x_k| \leq \frac{\sqrt{\delta_k}}{2} + |x_{k+1} - x_k| \leq \sqrt{\delta_k} \leq M(r,h) \leq r\,,$$

et donc $\omega \in A_{t_k}$.
Il suffit donc de montrer que la condition (H_1, A_{t_k}, x_{k+1}) est vraie. Montrons tout d'abord que pour tout $\omega \in A_{t_k}$, on a

$$\left|\sigma^2(X_{t_k}) - \sigma^2(x_k)\right| \leq \underline{\lambda}\,.$$

D'après les Hypothèses II.1, on a

$$\begin{aligned}\left|\sigma^2(X_{t_k}) - \sigma^2(x_k)\right| &\leq C_0\,|\sigma(X_{t_k})|\,|X_{t_k} - x_k| + C_0\,|\sigma(x_k)|\,|X_{t_k} - x_k| \\ &\leq 2\,C_0^2\,|X_{t_k} - x_k| \\ &\leq 2\,C_0^2\,r \\ &\leq \underline{\lambda}\,.\end{aligned}$$

On obtient donc
$\sigma^2(X_{t_k}) \geq \sigma^2(x_k) - (\sigma^2(X_{t_k}) - \sigma^2(x_k)) \geq 2\,\underline{\lambda} - \underline{\lambda} = \underline{\lambda}$, et

$$\begin{aligned}\sigma^2(X_{t_k}) \leq \sigma^2(x_k) + (\sigma^2(X_{t_k}) - \sigma^2(x_k)) \leq \sigma^2(x_k) + \underline{\lambda} &\leq \sigma^2(x_k) + \frac{\sigma^2(x_k)}{2} \\ &\leq \frac{3}{2}\,\sigma^2(x_k) \leq \overline{\lambda}\,.\end{aligned}$$

Conclusion :

$$\text{Pour tout } \omega \in A_{t_k}\,,\ \underline{\lambda} \leq \sigma^2(X_{t_k}) \leq \overline{\lambda}\,.$$

Ce qui signifie que la propriété (H_1, A_{t_k}, x_{k+1}) est satisfaite et donc l'hypothèse (H_1, A_k, x_{k+1}) aussi puisque $A_k \subseteq A_{t_k}$.
$(x_k)_{k=1,\ldots,N}$ est donc bien une suite d'évolution.
Etape 3. Appliquons le Théorème V.1. D'après sa définition dans la première étape, on vérifie bien que $\delta_k \leq \delta_*$. Puisque nous avons montré que $\delta_{k-1} \leq \overline{M}^2\,\delta_k$, on prend $H_k := \overline{M}$. On obtient donc

$$p_T(x_0, y) \geq \frac{e^{-4/\underline{\lambda}}}{8\sqrt{2\,\pi\,\overline{\lambda}}} \times e^{-(N-1)\,\theta}\,,$$

où $\theta = \dfrac{4}{\underline{\lambda}} + \ln 32 + \dfrac{\ln(2\,\pi\,\overline{\lambda})}{2} + \ln\overline{M}$. L'évaluation de N établie dans la première étape nous donne le résultat. ■

Deuxième partie

Integration by parts for pure jump processes

Malliavin calculus for simple functionals VII

Introduction

The standard Malliavin calculus on the Wiener space leads to an integration by parts formula. The aim of this chapter is to settle such a formula, but for locally smooth laws. Let us be more precise.
We will consider functionals of finite number of random variables $V_i, i = 1, ..., n$. In the Wiener space, the random variables V_i would be the increments of the Brownian motion $B(t_i) - B(t_{i-1})$. In this case, $(V_i)_{i\geq 1}$ are independant and identically Gaussian distributed, so their laws are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and have smooth densities. In this chapter, we consider a more general framework. First, we no more assume independancy, but we look at the conditional law of V_i with respect to $V_j, j \neq i$. Then, we assume that this conditional law is absolutely continuous with respect to the Lebesgue measure and has a density $p_i = p_i(\omega, y)$ which is piecewise differentiable with respect to y.
Using integration by parts, one may settle the duality relation which represents the starting point of the Malliavin calculus. But some border terms will appear corresponding to the points in which p_i is not continuous : for example, if V_i has a uniform conditional law on $[0, 1]$, the density is $p_i(\omega, y) = 1_{[0,1]}(y)$ and integration by parts produces border terms in 0 and in 1.
A simple idea allows us to cancel the border terms : we introduce weights π_i which are null at the points of singularity of p_i - in the previous example, we may take $\pi_i(y) = y^\alpha(1 - y)^\alpha$, for some $\alpha \in (0, 1)$. We then obtain a standard duality relation between the Malliavin derivative and the Skorohod integral, and the machinery settled in the Malliavin calculus produces an integration by parts formula.
But there is a difficulty hidden in this procedure : the differential operators involve the weights π_i and their derivatives. In the previous example, we have
$\pi_i'(\omega, y) = \alpha(y^{\alpha-1}(1 - y)^\alpha - y^\alpha(1 - y)^{\alpha-1})$. These derivatives blow up in the neighborhood of the singularity points and this produces some non trivial integrability problems. So one has to realize an equilibrium between the speed of convergence to zero and the speed with which the derivatives of the weights blow up in the singularity points. This leads to a non degeneracy condition which involves the weights and their derivatives.

Once an integration by parts formula is settled, we deal with its iteration. When iterating the integration by parts formula, some terms such as $\pi_i(V_i)\,\pi_i''(V_i)$ appear. But the second order derivatives $\pi_i''(V_i)$ are never integrable -in the previous example, $\pi_i''(\omega, y)$ involves terms as $y^{\alpha-2}\,(1-y)^{\alpha}$, $\alpha \in (0,1)$.
To overcome this difficulty, the idea is to split the support of the conditional density of V_i into two disjoint sets. For example, if V_i has a uniform conditional law on $[0,1]$, we put $[0,1] = [0,1/2] \cup [1/2,1]$ and we consider two kind of weights $(\pi_i^1)_{i\in\mathbb{N}}$ and $(\pi_i^2)_{i\in\mathbb{N}}$, such that π_i^1 (respectively π_i^2) is null on $[1/2,1]$ (respectively $[0,1/2]$). We thus obtain $\pi_i^2(V_i)\,(\pi_i^1)''(V_i) = 0$, and the second order derivatives of π_i^1 disappear. This means that we perform the first integration by parts formula using the weights π_i^1, and the second one using π_i^2.

1. The framework

We consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a sub $\sigma-$algebra $\mathcal{G} \subseteq \mathcal{F}$ and a sequence of random variables $V_i, i \in \mathbb{N}$. We denote

$$\mathcal{G}_i = \mathcal{G} \vee \sigma(V_j, j \neq i)\,.$$

Our aim is to settle an integration by parts formula for functionals of V_i, $i \in \mathbb{N}$, which is analogous to the one in the standard Malliavin calculus. The σ-algebra $\mathcal{G}$ appears to describe all the randomness which is not involved in the differential calculus.
We work on some fixed set $A \in \mathcal{G}$.
We denote by $\mathrm{L}_{(\infty)}(A)$ the space of random variables F such that $\mathrm{E}\left(|F|^p\ \mathbf{1}_A\right) < \infty$ for all $p \in \mathbb{N}$, and by $\mathrm{L}_{(p+)}(A)$ the space of random variables F for which there exists some $\delta > 0$ such that $\mathrm{E}\left(|F|^{p+\delta}\ \mathbf{1}_A\right) < \infty$. We assume that

Hypothesis VII.1. $V_i \in \mathrm{L}_{(\infty)}(A)$, $i \in \mathbb{N}$.

For each $i \in \mathbb{N}$ we consider some $k_i \in \mathbb{N}$ and some $\mathcal{G}_i$-measurable random variables

$$a_i(\omega) = t_i^0(\omega) < t_i^1(\omega) < \ldots < t_i^{k_i}(\omega) < t_i^{k_i+1}(\omega) = b_i(\omega)\,.$$

We denote $B_i(\omega) = \bigcup_{j=0}^{k_i} \left(t_i^j(\omega), t_i^{j+1}(\omega)\right)$. Note that we may take $a_i = -\infty$ and $b_i = \infty$.
We will work with functions defined on $(a_i(\omega), b_i(\omega))$ which are smooth except for the points $t_i^j, j = 1, \ldots, k_i$.

Definition VII.1. We define $\mathcal{C}_k(B_i)$ as the set of the measurable functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that, for every ω, $(y \to f(\omega, y))$ is k times differentiable on $B_i(\omega)$ and for each $j = 1, \ldots, k_i$, the left hand side and the right hand side limits

$f(\omega, t_i^j(\omega)-), f(\omega, t_i^j(\omega)+)$ exist and are finite (for $j=0$ and $j=k_i+1$ we assume that the right hand side, respectively the left hand side limit exists and is finite).

Let us denote

$$\Gamma_i(f) = \sum_{j=1}^{k_i} \left(f(\omega, t_i^j(\omega)-) - f(\omega, t_i^j(\omega)+) \right) + f(\omega, b_i(\omega)-) - f(\omega, a_i(\omega)+) \,. \quad \text{(VII.1.1)}$$

For $f, g \in \mathcal{C}_1(B_i)$, the integration by parts formula gives

$$\int_{(a_i,b_i)} f\, g'(\omega, y)\, dy = \Gamma_i(f\, g) - \int_{(a_i,b_i)} f'\, g(\omega, y)\, dy \,. \quad \text{(VII.1.2)}$$

So Γ_i represents the contribution of the border terms - or, put it otherwise, of the singularities of f or g.

Notation: *Let $n, k \in \mathbb{N}$. We denote by $\mathcal{C}_{n,k}$ the class of the $\mathcal{G} \times \mathcal{B}(\mathbb{R}^n)$ measurable functions $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that $I_i(f) \in \mathcal{C}_k(B_i)$, $i = 1, \dots, n$, where*

$$I_i(f)(\omega, y) := f(\omega, V_1, \dots, V_{i-1}, y, V_{i+1}, \dots, V_n) \,.$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, n\}^k$, we put $\partial_\alpha^k f = \dfrac{\partial^k f}{\partial x_{\alpha_1} \dots \partial x_{\alpha_k}}$.
We then denote by $\mathcal{C}_{n,k}(A)$ the space of functions $f \in \mathcal{C}_{n,k}$ such that for every $0 \le p \le k$ and every $\alpha = (\alpha_1, \dots, \alpha_p) \in \{1, \dots, n\}^p$, $\partial_\alpha^p f(V_1, \dots, V_n) \in \mathrm{L}_{(\infty)}(A)$.

The points $t_i^j, j = 1, \dots, k_i$ represent singularity points for the functions at hand (note that f may be discontinuous in t_i^j) and our main propose is to settle a calculus adapted to such functions.
Our basic hypothesis is the following.

Hypothesis VII.2. For every $i \in \mathbb{N}$ the conditional law of V_i with respect to $\mathcal{G}_i$ is absolutely continuous on (a_i, b_i) with respect to the Lebesgue measure. This means that there exists a $\mathcal{G}_i \times \mathcal{B}(\mathbb{R})$-measurable function $p_i(\omega, x)$ which satisfies

$$\mathrm{E}\left(\Theta\, \psi(V_i)\, \mathbf{1}_{(a_i,b_i)}(V_i)\right) = \mathrm{E}\left(\Theta \int_{\mathbb{R}} \psi(x)\, p_i(\omega, x)\, \mathbf{1}_{(a_i,b_i)}(x)\, dx\right) ,$$

for every positive, $\mathcal{G}_i$-measurable random variable Θ and every positive, measurable function $\psi : \mathbb{R} \to \mathbb{R}$.
We assume that $p_i \in \mathcal{C}_1(B_i)$ and $\partial_y \ln p_i(\omega, y) \in \mathrm{L}_{(\infty)}(A)$.

In the applications, we consider random variables V_i with conditional densities p_i and then we take t_i^j, $i = 0, \dots, k_{i+1}$ as the points of singularities of p_i. This means that we choose B_i in such a way that p_i satisfies hypothesis VII.2 on B_i. This is the significance of B_i (in the case where p_i is smooth on the whole $\mathbb{R}$, we may choose

$B_i = \mathbb{R}$).
For each $i \in \mathbb{N}$ we consider a $\mathcal{G}_i \times \mathcal{B}(\mathbb{R})$-measurable and positive function $\pi_i : \Omega \times \mathbb{R} \to \mathbb{R}_+$ such that $\pi_i(\omega, y) = 0$ for $y \notin (a_i, b_i)$ and $\pi_i \in \mathcal{C}_1(B_i)$.
We assume

Hypothesis VII.3.

$$\pi_i(\omega, V_i)\,\mathbf{1}_{B_i(\omega)}(V_i) \in \mathrm{L}_{(\infty)}(A) \text{ and } \pi_i'(\omega, V_i)\,\mathbf{1}_{B_i(\omega)}(V_i) \in \mathrm{L}_{(1+)}(A)\,.$$

These will be the weights used in our calculus. In the standard Malliavin calculus, they appear as re-normalization constants. On the other hand, p_i may have discontinuities in $t_i^j, j = 1, \ldots, k_i$ and this will produce some border terms in the integration by parts formula - see (VII.1.2). We may choose the weights $(\pi_i)_{i\in\mathbb{N}}$ in order to cancel these border terms (as well as the border terms in a_i and b_i).

2. The differential operators

We introduce in this section the differential operators which represent the analogous of the Malliavin derivative and the Skorohod integral.
We suppose that there exists a partition of $\Omega : \Omega = \bigcup_{n\geq 1} A_n$, where $A_n \in \mathcal{G}$ for all $n \in \mathbb{N}$ and $A_n \cap A_m = \varnothing$ if $n \neq m$.
• **Simple functionals.**
A random variable F is called a simple functional if there exists $N_F \in \mathbb{N}^*$ and a finite sequence of $\mathcal{G} \times \mathcal{B}(\mathbb{R}^n)$-measurable functions $(f_n)_{1\leq n\leq N_F}$ which satisfies :
$f_n : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $F\,\mathbf{1}_{A_n} := f_n(\omega, V_1, \ldots, V_n)\,\mathbf{1}_{A_n}$ for all $n = 1, \ldots, N_F$, that is

$$F = f(\omega, \widetilde{V}) := \sum_{n=1}^{N_F} f_n(\omega, V_1, \ldots, V_n)\,\mathbf{1}_{A_n}, \text{ where } \widetilde{V} := (V_i)_{i\geq 1}\,.$$

Notation: *We denote $\mathcal{S}_k$ the space of simple functionals F such that the corresponding sequence $f_n \in \mathcal{C}_{n,k}$, $n \leq N_F$.*
$\mathcal{S}_k(A)$ is defined as the space of simple functionals such that $f_n \in \mathcal{C}_{n,k}(A \cap A_n)$ for all $n = 1, \ldots, N_F$, which means that $f_n \in \mathcal{C}_{n,k}$ and f_n and its derivative up to order k have finite moments of any order on $A \cap A_n$.

Remark 2.1. *For $F \in \mathcal{S}_k$, we may write*

$$F = \sum_{n=1}^{\infty} f_n(\omega, V_1, \ldots, V_n)\,\mathbf{1}_{A_n}, \text{ with } f_n = 0 \text{ for } n > N_F\,.$$

We will use the notation $\partial_{V_i} F := \dfrac{\partial f}{\partial x_i}(\omega, \widetilde{V})$.

• **Simple processes.**

A simple process is a finite sequence of simple functionals $U = (U_i)_{i=1,\dots,N_U}$, that is there exists $\mathcal{G} \times \mathcal{B}(\mathbb{R}^n)$-measurable functions $u_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be such that for all $i = 1, \dots, N_U$

$$U_i = u_i\left(\omega, \widetilde{V}\right) = \sum_{n=1}^{\infty} u_{i,n}(\omega, V_1, \dots, V_n)\, \mathbf{1}_{A_n}\,, \text{ with } u_{i,n} = 0 \text{ if } i > n\,.$$

We denote by $\mathcal{P}_k$ (respectively $\mathcal{P}_k(A)$) the space of simple processes such that $u_{i,n} \in \mathcal{C}_{n,k}$, $i, n \in \mathbb{N}$ (respectively $u_{i,n} \in \mathcal{C}_{n,k}(A \cap A_n)$, $i, n \in \mathbb{N}$).

Example. *Let us consider the following simple functional* $f(\omega, \widetilde{V}) = \sum_{n=1}^{\infty} f_n(\omega, V_1, \dots, V_n)\, \mathbf{1}_{A_n}$, *with* $f_n \in \mathcal{C}_{n,1}$ *and* $f_n = 0$ *if* $n > N_F$.
We then define the simple process $\partial f = (\partial_{V_i} f)_{i \geq 1}$ *by*

$$\partial_{V_i} f(\omega, \widetilde{V}) := \sum_{n=i}^{\infty} \partial_i f_n(\omega, V_1, \dots, V_n)\, \mathbf{1}_{A_n} = \sum_{n=i}^{N_F} \partial_i f_n(\omega, V_1, \dots, V_n)\, \mathbf{1}_{A_n}\,.$$

• On the space of simple processes we consider the following inner product associated to the weights $(\pi_i)_{i \in \mathbb{N}}$:

$$\langle U, V \rangle_\pi := \sum_{i=1}^{\infty} u_i(\omega, \widetilde{V})\, v_i(\omega, \widetilde{V})\, \pi_i(\omega, V_i)\,. \tag{VII.2.1}$$

Note that since the simple processes U and V are finite sequences of the simple functionals U_i, $i \leq N_U$ and V_i, $i \leq N_V$, the sum defined in equation (VII.2.1) is finite.
Moreover, we have $u_{i,n} = v_{i,n} = 0$ if $i > n$, and in view of Remark 2.1, there exits $N \in \mathbb{N}$ such that $u_{i,n} = v_{i,n} = 0$ if $n > N$. Then, we can write

$$\begin{aligned}\langle U, V \rangle_\pi &= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \pi_i(\omega, V_i)\, (u_{i,n}\, v_{i,n})(\omega, V_1, \dots, V_n)\, \mathbf{1}_{A_n} \\ &= \sum_{n=1}^{N} \sum_{i=1}^{n} \pi_i(\omega, V_i)\, (u_{i,n}\, v_{i,n})(\omega, V_1, \dots, V_n)\, \mathbf{1}_{A_n}\,.\end{aligned}$$

We define now the differential operators which appear in Malliavin calculus.

■ **The Malliavin derivative** $D : \mathcal{S}_1 \to \mathcal{P}_0$:

if $F = f(\omega, \widetilde{V}) = \sum_{n=1}^{\infty} f_n(\omega, V_1, \ldots, V_n)\, \mathbf{1}_{A_n}$, we then define $DF = (D_i F)_{i\in\mathbb{N}} \in \mathcal{P}_0$ by

$$D_i F := \partial_{V_i} f(\omega, \widetilde{V})\, \mathbf{1}_{B_i(\omega)}(V_i) = \mathbf{1}_{B_i(\omega)}(V_i) \sum_{n=1}^{\infty} \frac{\partial f_n}{\partial x_i}(\omega, V_1, \ldots, V_n)\, \mathbf{1}_{A_n}.$$

■ **The Malliavin covariance matrix associated to the inner product $\langle .,.\rangle_\pi$.** Given $F = (F^1, \ldots, F^d)$, with $F^i = f^i(\omega, \widetilde{V}) \in \mathcal{S}_1$, the Malliavin covariance matrix of F is defined by

$$\begin{aligned}\sigma_{\pi,F}^{ij} &:= \langle DF^i, DF^j\rangle_\pi = \sum_{k=1}^{\infty} \pi_k(\omega, V_k)\, \partial_k f^i(\omega, \widetilde{V})\, \partial_k f^j(\omega, \widetilde{V}) \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} \pi_k(\omega, V_k)\, \partial_{V_k} f_n^i\, \partial_{V_k} f_n^j(\omega, V_1, \ldots, V_n)\, \mathbf{1}_{A_n}.\end{aligned}$$

This is a symmetric positive definite matrix.

■ **The Skorohod integral associated to the inner product $\langle .,.\rangle_\pi$**

$\delta_\pi : \mathcal{P}_1 \to \mathcal{S}_0$: if $U = (U_i)_{i\in\mathbb{N}}$, we then define

$$\delta_\pi(U) := -\sum_{i=1}^{\infty} \left(\partial_i(\pi_i\, U_i) + \pi_i\, U_i\, \partial \ln p_i\right)(\omega, \widetilde{V}) = -\sum_{n=1}^{\infty}\sum_{i=1}^{n} \delta_{i,\pi}(U)\, \mathbf{1}_{A_n}, \qquad \text{(VII.2.2)}$$

where on A_n, $\delta_{i,\pi}(U) := \left(\partial_i(\pi_i\, u_{i,n}) + \pi_i\, u_{i,n}\, \partial \ln p_i\right)(\omega, V_1, \ldots, V_n)$.

■ **The border term operator**. For $F = f(\omega, \widetilde{V}) \in \mathcal{S}_0$ and $U = (U_i)_{i\geq 1} \in \mathcal{P}_0$, let us define

$$[F, U]_\pi = \sum_{n=1}^{\infty}\sum_{i=1}^{n} \Gamma_i\left(I_i(F \times U_i) \times \pi_i \times p_i\right) \mathbf{1}_{A_n}. \qquad \text{(VII.2.3)}$$

Put it otherwise, for all $n \in \mathbb{N}^*$, on A_n, we have

$$\begin{aligned}[F, U]_\pi &= \sum_{i=1}^{n} \Gamma_i\left(I_i(f_n \times u_{i,n}) \times \pi_i \times p_i\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k_i} \big((f_n \times u_{i,n})(\omega, V_1, \ldots, V_{j-1}, t_i^j-, V_{j+1}, \ldots, V_n)(\pi_i p_i)(\omega, t_i^j-) \\ &\quad -(f_n \times u_{i,n})(\omega, V_1, \ldots, V_{j-1}, t_i^j+, V_{j+1}, \ldots, V_n)(\pi_i p_i)(\omega, t_i^j+)\big) \\ &\quad +\sum_{i=1}^{n}(f_n \times u_{i,n})(\omega, V_1, \ldots, V_{j-1}, b_i-, V_{j+1}, \ldots, V_n)(\pi_i p_i)(\omega, b_i-) \\ &\quad -\sum_{i=1}^{n}(f_n \times u_{i,n})(\omega, V_1, \ldots, V_{j-1}, a_i+, V_{j+1}, \ldots, V_n)(\pi_i p_i)(\omega, a_i+).\end{aligned}$$

Remark 2.2. *If we choose the weights $(\pi_i)_{i\in\mathbb{N}}$ such that*

$$\left\{\begin{array}{ll} \pi_i(\omega, t_i^j+) & = \pi_i(\omega, t_i^j-) = 0, i \geq 1\,,\, j = 1, \ldots, k_i \\ \pi_i(\omega, a_i+) & = \pi_i(\omega, b_i-) = 0, i \geq 1\,, \end{array}\right. \tag{VII.2.4}$$

then $[F, U]_\pi = 0$ for every $F \in \mathcal{S}_1$ and $U \in \mathcal{P}_1$. Hence, there will be no border terms in the duality formula and in the integration by parts formula. This is - one possible - reason of being of the weights. The other one concerns re-normalization.

■ **The Ornstein Uhlenbeck operator associated to the inner product $\langle ., . \rangle_\pi$.** We introduce $\mathrm{L}_\pi : \mathcal{S}_2 \to \mathcal{S}_0$ defined by : for all $F \in \mathcal{S}_1$, $\mathrm{L}_\pi(F) := \delta_\pi(DF)$. We thus have by (VII.2.2)

$$\mathrm{L}_\pi(F) = -\sum_{i=1}^{\infty} \left(\partial_i(\pi_i\,\partial_i f) + \pi_i\,\partial_i f\,\partial \ln p_i\right)(\omega, \widetilde{V}\,) := -\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathrm{L}_{i,\pi}(F)\,, \tag{VII.2.5}$$

where on A_n, $\mathrm{L}_{i,\pi}(F) = \big(((\pi_i)' + \pi_i\,\partial \ln p_i)\,\partial_i f_n + \pi_i\,\partial_i^2 f_n\big)\,(\omega, V_1, \ldots, V_n)$.
Note that $\pi_i(\omega, y) = 0$ for $y \notin (a_i, b_i)$ and $y \to \ln p_i(\omega, y)$ is differentiable on (a_i, b_i) so that $\pi_i\,\partial_i \ln p_i$ makes sense.

Remark 2.3. *Note that in view of Remark 2.1, the sums with respect to n in equations (VII.2.2), (VII.2.3) and (VII.2.5) are finite.*

In our framework the duality between the Skorohod integral δ_π and the Malliavin derivative D is given by the following Proposition.

Proposition VII.1:
Let $F \in \mathcal{S}_1$ and $U \in \mathcal{P}_1$. Suppose that for every $i \geq 1$, we have

$$\mathrm{E}(|F\,\delta_\pi(U)|\,\mathbf{1}_A) + \mathrm{E}(\pi_i(\omega, V_i)\,|D_i F \times U_i|\,\mathbf{1}_A) < \infty. \tag{VII.2.6}$$

Then $\mathrm{E}(|[F, U]_\pi|\,\mathbf{1}_A) < \infty$ and

$$\mathrm{E}(\langle DF, U\rangle_\pi\;\mathbf{1}_A) = \mathrm{E}(F\,\delta_\pi(U)\,\mathbf{1}_A) + \mathrm{E}([F, U]_\pi\,\mathbf{1}_A)\,. \tag{VII.2.7}$$

If hypothesis (VII.2.4) holds true, then

$$\mathrm{E}(\langle DF, U\rangle_\pi\;\mathbf{1}_A) = \mathrm{E}(F\,\delta_\pi(U)\,\mathbf{1}_A)\,.$$

Proof. Since $\pi_i = 0$ on $(a_i, b_i)^c$ and hypothesis VII.2 holds true, we have

$$\begin{aligned}
&\mathrm{E}(\langle DF, U\rangle_\pi \, \mathbf{1}_A) \\
&= \sum_{n=1}^{\infty} \mathrm{E}\left(\sum_{i=1}^{n} \mathrm{E}\left(\pi_i(\omega, V_i)\, \partial_{V_i} f_n \times u_{i,n} \mid \mathcal{G}_i\right) \mathbf{1}_{A\cap A_n} \right) \\
&= \sum_{n=1}^{\infty} \mathrm{E}\left(\mathbf{1}_{A\cap A_n} \sum_{i=1}^{n} \int_{a_i}^{b_i} (\pi_i\, u_{i,n}\, \partial_i f_n)(\omega, V_1, \ldots, V_{i-1}, y, V_{i+1}, \ldots, V_n)\, p_i(\omega, y)\, dy \right).
\end{aligned}$$

Let us fix $n \in \mathbb{N}^*$. Using integration by parts (see equation (VII.1.2)), we obtain on $A \cap A_n$, for all $i \leq n$,

$$\begin{aligned}
&\int_{a_i}^{b_i} \partial_i f_n \times (\pi_i\, u_{i,n}) \times p_i \\
&= \sum_{j=0}^{k_i} \int_{(t_i^j, t_{i+1}^j)} \partial_i f_n \times (\pi_i\, u_{i,n}) \times p_i \\
&= \Gamma_i\left(I_i(f_n \times u_{i,n})\, \pi_i\, p_i\right) - \sum_{j=0}^{k_i} \int_{(t_i^j, t_{i+1}^j)} f_n \times \left(\partial_i(\pi_i\, u_{i,n}) \times p_i + (\pi_i\, u_{i,n}) \times \partial p_i\right)\big) \\
&= \Gamma_i(I_i(f_n \times u_{i,n})\pi_i\, p_i) - \int_{a_i}^{b_i} f_n \times (\partial_i(\pi_i\, u_{i,n}) + \pi_i\, u_{i,n} \partial \ln p_i) \times p_i\,.
\end{aligned}$$

By hypothesis (VII.2.6) we have for almost every $\omega \in A \cap A_n$,

$$\int_{\mathbb{R}} (|u_{i,n}\, \partial_i f_n|\ \pi_i\, p_i)(\omega, V_1, \ldots, V_{i-1}, y, V_{i+1}, \ldots, V_n)\, dy < \infty\,,$$

$$\int_{\mathbb{R}} (|f_n(\partial_i(\pi_i\, u_{i,n}) + \pi_i\, u_{i,n} \partial \ln p_i)| \times p_i)(\omega, V_1, \ldots, V_{i-1}, y, V_{i+1}, \ldots, V_n) dy < \infty\,,$$

So the above integrals make sense.
Since $\Gamma_i(I_i(F \times U_i)\, \pi_i\, p_i)$ is the sum of these two integrals on $A \cap A_n$, we also obtain $\mathrm{E}\left(|\Gamma_i(I_i(F \times U_i)\, \pi_i\, p_i)|\, \mathbf{1}_{A\cap A_n}\right) < \infty$, so that $\mathrm{E}\left(|[F, U]_\pi|\ \mathbf{1}_{A\cap A_n}\right) < \infty$. In view of Remark 2.3, we get

$$\mathrm{E}\left(|[F, U]_\pi|\ \mathbf{1}_A\right) = \sum_{n=1}^{\infty} \mathrm{E}\left(|[F, U]_\pi|\ \mathbf{1}_{A\cap A_n}\right) = \sum_{n=1}^{N} \mathrm{E}\left(|[F, U]_\pi|\ \mathbf{1}_{A\cap A_n}\right) < \infty\,.$$

Using the definition of the conditional density p_i in hypothesis VII.2, we come back to expectations and we obtain

$$\int_{a_i}^{b_i} (\pi_i\, u_{i,n}\, \partial_i f_n)(\omega, V_1, \ldots, V_{i-1}, y, V_{i+1}, \ldots, V_n)\, p_i(\omega, y)\, dy\, \mathbf{1}_{A\cap A_n}$$
$$= -\mathrm{E}(F\, \delta_{i,\pi}(U) \mid \mathcal{G}_i)\, \mathbf{1}_{A\cap A_n} + \Gamma_i(I_i(F \times U_i)\, \pi_i\, p_i)\, \mathbf{1}_{A\cap A_n}\,.$$

One sums over i and we get

$$\begin{aligned}
&\mathrm{E}(\langle DF, U\rangle_\pi\, \mathbf{1}_A) \\
&= -\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathrm{E}\left[\mathrm{E}(F\, \delta_{i,\pi}(U) \mid \mathcal{G}_i)\, \mathbf{1}_{A\cap A_n}\right] + \mathrm{E}\left[\sum_{n=1}^{\infty}\sum_{i=1}^{n} \Gamma_i(I_i(F \times U_i)\, \pi_i\, p_i)\, \mathbf{1}_{A\cap A_n}\right] \\
&= -\mathrm{E}\left(F \sum_{n=1}^{N}\sum_{i=1}^{n} \delta_{i,\pi}(U)\, \mathbf{1}_{A\cap A_n}\right) + \mathrm{E}\left([F, U]_\pi\, \mathbf{1}_A\right) \qquad \text{(by Remark 2.3)} \\
&= \mathrm{E}\left(F\, \delta_\pi(U)\, \mathbf{1}_A\right) + \mathrm{E}\left([F, U]_\pi\, \mathbf{1}_A\right)\,.
\end{aligned}$$

The result is thus proved. ■

Corollary VII.1:
Let $Q \in \mathcal{S}_1(A)$, that is Q and its firts order derivatives have finite moments of any order on A. Suppose moreover that there exists some $\eta > 0$ such that

$$\mathrm{E}\left(\mathbf{1}_A\ (|\pi_i\, Q| + |\partial_{V_i}(\pi_i\, Q)|)^{1+\eta}\right) < \infty\,,\ i \geq 1\,. \tag{VII.2.8}$$

For every $F \in \mathcal{S}_1(A)$, $U \in \mathcal{P}_1(A)$, one then has

$$\mathrm{E}\left(Q\, \langle DF, U\rangle_\pi\, \mathbf{1}_A\right) = \mathrm{E}\left(F\, \delta_\pi(Q\, U)\, \mathbf{1}_A\right) + \mathrm{E}\left([F, Q\, U]_\pi\ \mathbf{1}_A\right)\,. \tag{VII.2.9}$$

Proof. In order to use the previous Proposition, we just have to check that F and $\widetilde{U} = Q\, U$ satisfy hypothesis (VII.2.6).
We have

$$|\delta_{i,\pi}(Q\, U)| \leq |\partial_{V_i}(\pi_i\, Q)|\ |U_i| + |\pi_i\, Q|\ (|\partial_{V_i} U_i| + |U_i|\ |\partial \ln p_i|)\,.$$

Since $U \in \mathcal{P}_1(A)$, one has U_i, $\partial_{V_i} U_i \in \mathrm{L}_{(\infty)}(A)$, and by hypothesis VII.2, $\partial \ln p_i \in \mathrm{L}_{(\infty)}(A)$. Hence, using hypothesis (VII.2.8), we get $\delta_{i,\pi}(Q\, U) \in \mathrm{L}_{(1+)}(A)$. Moreover, $F \in \mathrm{L}_{(\infty)}(A)$, and we thus obtain $\mathrm{E}\left(F\, \delta_{i,\pi}(Q\, U)|\right) < \infty$.
We have $D_i F, U_i \in \mathrm{L}_{(\infty)}(A)$ and $\pi_i\, Q \in \mathrm{L}_{(1+)}(A)$. Hence, $\mathrm{E}\left(\pi_i\, |D_i F \times (Q\, U_i)|\right) < \infty$.
The proof is thus complete. ■

As an immediate consequence of the duality relation (VII.2.7) we obtain :

Lemma VII.1:
Let $F, G \in \mathcal{S}_2$. Suppose that for every $i \geq 1$, we have

$$\mathrm{E}\left[\left(|F\,\mathrm{L}_{i,\pi}G| + |G\,\mathrm{L}_{i,\pi}F| + \pi_i\,|D_iF \times D_iG|\right)\mathbf{1}_A\right] < \infty\,.$$

Then $\mathrm{E}\left(|[F, DG]_\pi|\,\mathbf{1}_A\right) < \infty$, $\mathrm{E}\left(|[G, DF]_\pi|\,\mathbf{1}_A\right) < \infty$ and

$$\begin{aligned}\mathrm{E}(F\,\mathrm{L}_\pi G\,\mathbf{1}_A) + \mathrm{E}([F, DG]_\pi\,\mathbf{1}_A) &= \mathrm{E}(< DF, DG >_\pi\ \mathbf{1}_A)\\ &= \mathrm{E}(G\,\mathrm{L}_\pi F\,\mathbf{1}_A) + \mathrm{E}([G, DF]_\pi\,\mathbf{1}_A)\,.\end{aligned}$$

We denote by $\mathcal{C}_p^k(\mathbb{R}^d)$ the space of the functions $\phi : \mathbb{R}^d \to \mathbb{R}$ which are k times differentiable and such that ϕ and its derivatives of order less or equal to k have polynomial growth. The standard differential calculus gives the following chain rules.

Lemma VII.2:
i) Let $\phi \in \mathcal{C}_p^1(\mathbb{R}^d)$ and $F = (F^1, \ldots, F^d)$, $F^i \in \mathcal{S}_1(A)$. Then $\phi(F) \in \mathcal{S}_1(A)$ and

$$D\phi(F) = \sum_{k=1}^{d} \partial_k\phi(F)\,DF^k\,. \tag{VII.2.10}$$

ii) If $\phi \in \mathcal{C}_p^2(\mathbb{R}^d)$ and $F^i \in \mathcal{S}_2(A)$ then $\phi(F) \in \mathcal{S}_2(A)$ and

$$\mathrm{L}_\pi\phi(F) = \sum_{k=1}^{d} \partial_k\phi(F)\,\mathrm{L}_\pi F^k - \sum_{k,p=1}^{d} \partial^2_{k,p}\phi(F)\,\left\langle DF^k, DF^p\right\rangle_\pi\,.$$

iii) Let $F \in \mathcal{S}_1(A)$ and $U \in \mathcal{P}_1(A)$. Then $FU \in \mathcal{P}_1(A)$ and

$$\delta_\pi(F\,U) = F\,\delta_\pi(U) - \langle DF, U\rangle_\pi\,.$$

Particulary, if $F \in \mathcal{S}_1(A)$ and $G \in \mathcal{S}_2(A)$ then $F\,DG \in \mathcal{P}_1(A)$ and

$$\delta_\pi(F\,DG) = F\,\mathrm{L}_\pi G - \langle DF, DG\rangle_\pi\,. \tag{VII.2.11}$$

Remark 2.4. *Let us define $\mathcal{L}^2_\pi(A)$ as the closure of $\mathcal{P}_0$ with respect to the norm associated to the scalar product $\langle U, V\rangle_\pi$. If $[F, U]_\pi$ is not null, then the operator $D : \mathcal{S}_1 \subset \mathrm{L}^2(\Omega) \to \mathcal{P}_0 \subset \mathcal{L}^2_\pi(A)$ is not closable.*
Indeed, suppose for example that V_1 is exponentially distributed and V_i, $i \geq 2$ are arbitrary chosen independent of V_1. We take $\pi_1 = 1$ and $\pi_i = 0$, $i \geq 2$. We thus perform our calculus with respect to V_1 only. In this case, $a_1 = 0, b_1 = \infty$ and there are no points t_i^j.
Take now $F_m = f_m(V_1)$, that is $F_m\,\mathbf{1}_{A_n} = f_m(V_1)$ for all $n \geq 1$. We put $f_m(x) = 1 - m\,x$ for $0 < x < 1/m$ and $f_m(x) = 0$ for $x \geq 1/m$.
Take also $U_1 = u_1(V_1)$, that is $U_1\,\mathbf{1}_{A_n} = u_{1,n} = u_1$ for all $n \geq 1$. We put

$u_1(x) = 1 - x$ for $0 < x < 1$ and $u_1(x) = 0$ for $x \geq 1$.
Let us write the duality formula for all $m \in \mathbb{N}$,

$$\mathrm{E}(\langle DF_m, U \rangle_\pi) = \mathrm{E}(F_m\, \delta_\pi(U)) + \mathrm{E}([F_m, U]_\pi)\,.$$

Since $[F_m, U]_\pi = 1$ and $F_m \to 0$ in $\mathrm{L}^2(\Omega)$, we obtain $\lim_{m\to\infty} \mathrm{E}(\langle DF_m, U \rangle_\pi) = 1$. And so $DF_m \nrightarrow 0$ in $\mathcal{L}^2_\pi(A)$. This proves that D is not closable.
But if $[F, U]_\pi = 0$ for every F, U (this happens for example if we choose π_i so that they satisfy hypothesis (VII.2.4)), then the duality formula (VII.2.7) guarantees that the operators D and δ_π are closable. But we stay here in the level of the simple functionals and we do not discuss the extension to the infinite dimensional framework. Hence, the fact that the operators D and δ_π are not closable is not relevant in our framework.

Remark 2.5. *The above differential operators and the duality formula (VII.2.7) represent an abstract version of the operators introduced in Malliavin calculus and of the duality formula used there.*
In order to see it, we consider the simple example of the Euler scheme for a diffusion process, corresponding to the time grid $0 = s_0 < s_1 < \ldots < s_n = s$. This is a simple functional depending on the increments of the Brownian motion B, that is $V_i = B(s_i) - B(s_{i-1})$, $i = 1, \ldots, n$. The variables on which the calculus is based are independent Gaussian variables. It follows that

$$p_i(\omega, y) = (2\,\pi\,(s_i - s_{i-1}))^{-1/2} \exp\left(-y^2/2\,(s_i - s_{i-1})\right)\,.$$

Since p_i is smooth on the whole $\mathbb{R}$ and has null limit at infinity, there will be no border terms coming on, so we take $a_i = -\infty, b_i = \infty$ and $k_i = 0$.
If $F = f(\omega, \widetilde{V})$, then $D_i F = \partial_i f(\omega, \widetilde{V}) = \overline{D}_s F\, \mathbf{1}_{[s_{i-1}, s_i)}(s)$ where $\overline{D}_s F$ is the standard Malliavin derivative. We take $\pi_i = s_i - s_{i-1}$ so that

$$\langle DF, DG \rangle_\pi = \sum_{i=1}^{n} \pi_i\, D_i F\, D_i G = \int_0^s \overline{D}_u F\, \overline{D}_u G\, du\,.$$

Note that here the weights are used in order to obtain the Lebesgue measure. Moreover, we have $\partial_y \ln p_i(y) = -y/(s_i - s_{i-1})$ and so

$$\delta_\pi(U) = -\sum_{i=1}^{\infty} \left(\partial_{V_i} U_i(\omega, \widetilde{V})\,(s_i - s_{i-1}) - u_i(\omega, \widetilde{V})\, V_i\right)\,.$$

We thus find out the standard Malliavin calculus.

Remark 2.6. *If $[F, G]_\pi = 0$, the calculus presented here fits the framework introduced by Bouleau in [Bou03] : in the notation there, the bilinear form $(F, G) \to \langle DF, DG \rangle_\pi$ leads to an error structure.*

3. Integration by parts formulas

3.1. For locally smooth laws

Let $F = (F^1, \ldots, F^d) \in \mathcal{S}_1^d(A)$, that is F^i and their derivatives have finite moments of any order on A. We then define

$$\Theta_F(A) := \left\{ G = \sigma_{\pi,F} \times Q : Q \in \mathcal{S}_1^d(A), Q_i \text{ satisfies hypothesis (VII.2.8)} \right\} .$$

We think to $G \in \Theta_F(A)$ as a random direction in which F is non degenerated (in Malliavin's sense).
The basic integration by parts formula is the following.

Theorem VII.1:
Let $F = (F^1, \ldots, F^d) \in \mathcal{S}_2^d(A)$ and $G \in \Theta_F(A)$, that we write $G = \sigma_{\pi,F} \times Q$. Then $\delta_\pi \left(\sum_{i=1}^d Q^i\, DF^i \right)$, $[\phi(F), \sum_{i=1}^d Q^i\, DF^i]_\pi \in \mathrm{L}_{(1+)}(A)$ and for every $\phi \in \mathcal{C}_p^1(\mathbb{R}^d)$ one has

$$\mathrm{E}\left(\langle \nabla\phi(F), G \rangle\, \mathbf{1}_A \right) = \mathrm{E}\left(\phi(F)\, \delta_\pi \left(\sum_{i=1}^d Q^i\, DF^i \right) \mathbf{1}_A \right) + \mathrm{E}\left([\phi(F), \sum_{i=1}^d Q^i\, DF^i]_\pi\, \mathbf{1}_A \right) . \quad \text{(VII.3.1)}$$

Proof. Using the chain rule (VII.2.10), we get

$$\left\langle D\phi(F), DF^i \right\rangle_\pi = \sum_{j=1}^d \partial_j \phi(F) \left\langle DF^j, DF^i \right\rangle_\pi = \sum_{j=1}^d \partial_j \phi(F)\, \sigma_{\pi,F}^{ij} .$$

Since $G = \sigma_{\pi,F} \times Q$, we obtain

$$\begin{aligned}\langle \nabla\phi(F), G \rangle &= \sum_{j=1}^d \partial_j \phi(F)\, G^j = \sum_{j=1}^d \partial_j \phi(F) \sum_{i=1}^d Q^i\, \sigma_{\pi,F}^{ij} = \sum_{i=1}^d Q^i \sum_{j=1}^d \partial_j \phi(F)\, \sigma_{\pi,F}^{ij} \\ &= \sum_{i=1}^d Q^i \left\langle D\phi(F), DF^i \right\rangle_\pi .\end{aligned}$$

We have $\phi(F) \in \mathcal{S}_1(A)$ and $DF^i \in \mathcal{P}_1(A)$. Moreover $G \in \Theta_F(A)$, and then Q_i satisfies hypothesis (VII.2.8). We thus may use the duality formula (VII.2.9) to obtain the result (VII.3.1). ■

We give now a non degeneracy condition on $\sigma_{\pi,F}$ which guarantees that all the directions are non degenerated for F.

We assume that $\det\sigma_{\pi,F} \neq 0$ on A and we denote $\gamma_{\pi,F} = \sigma^{-1}_{\pi,F}$. We also assume that $\pi_l\,(\det\gamma_{\pi,F})^2$, $\pi'_l\,\det\gamma_{\pi,F}$, $\pi_l\,\pi'_l\,(\det\gamma_{\pi,F})^2 \in \mathrm{L}_{(1+)}(A)$, for every $l \geq 1$. This may be summarized by :

Hypothesis VII.4. There exists $\eta > 0$ such that

$$\mathrm{E}\left[\mathbf{1}_A\,(\det\gamma_{\pi,F})^{2\,(1+\eta)}\,(1+|\pi'_l|)^{1+\eta}\right] < \infty\,. \tag{VII.3.2}$$

In the following, this hypothesis will be called 'The non degeneracy condition'.

Lemma VII.3:
Let $F \in \mathcal{S}^d_2(A)$. Assume that the non degeneracy condition (VII.3.2) holds true. We then have $\mathcal{S}^d_1(A) \subseteq \Theta_F(A)$.

Proof. Let $G \in S^d_1(A)$. We can then write $G = \sigma_{\pi,F} \times Q$, with $Q = \gamma_{\pi,F} \times G$. We have $\gamma^{ij}_{\pi,F} = \widehat{\sigma}^{ij}_{\pi,F} \times \det\gamma_{\pi,F}$, where $\widehat{\sigma}^{ij}_{\pi,F}$ is the algebraic complement. It follows that $Q^i = \det\gamma_{\pi,F} \times S^i$, with $S^i = \sum_{j=1}^{d} G^j\,\widehat{\sigma}^{ij}_{\pi,F}$.
Let us check that hypothesis (VII.2.8) holds true for Q^i, $i = 1,\ldots,d$.
Since $\pi_l \in \mathrm{L}_{(\infty)}(A)$ and $D_l F^i \in \mathrm{L}_{(\infty)}(A)$ one has $\widehat{\sigma}^{ij}_{\pi,F}$ and $\det\sigma_{\pi,F} \in \mathrm{L}_{(\infty)}(A)$. Since $G^j \in \mathrm{L}_{(\infty)}(A)$, we then have $S^i \in \mathrm{L}_{(\infty)}(A)$.
Moreover, by the non degeneracy condition (VII.3.2), we have $\det\gamma_{\pi,F} \in \mathrm{L}_{(1+)}(A)$. Since $\pi_l \in \mathrm{L}_{(\infty)}(A)$, we have $\pi_l\,\det\gamma_{\pi,F} \in \mathrm{L}_{(1+)}(A)$.
Finally,

$$\pi_l\,Q^i = (\pi_l\,\det\gamma_{\pi,F})\,S^i \in \mathrm{L}_{(1+)}(A)\,.$$

We now check that $D_l(\pi_l\,Q^i) \in \mathrm{L}_{(1+)}(A)$.
We write on $A \cap A_n$,

$$D_l\sigma^{ij}_F = \pi'_l\,D_l f^i_n\,D_l f^j_n + \sum_{k=1}^{n} \pi_k\,D_l(D_k f^i_n\,D_k f^j_n)\,.$$

Since $F \in S^d_2(A)$, we have $D_l f^i_n\,D_l f^j_n$, $D_l(D_k f^i_n\,D_k f^j_n) \in \mathrm{L}_{(\infty)}(A\cap A_n)$, and consequently $D_l\sigma^{ij}_{\pi,F} = \theta_1 + \theta_2\,\pi'_l$, with θ_1, $\theta_2 \in \mathrm{L}_{(\infty)}(A)$. Then $D_l(\det\sigma_{\pi,F}) = \mu + \nu\,\pi'_l$ and $D_l S^i = \mu_i + \nu_i\,\pi'_l$, with μ, ν, μ_i, $\nu_i \in \mathrm{L}_{(\infty)}(A)$.
Thus, we obtain

$$\begin{aligned} D_l(\pi_l\,Q^i) &= \pi'_l\,\det\gamma_{\pi,F}\,S^i - \pi_l\,(\det\gamma_{\pi,F})^2\,D_l(\det\sigma_{\pi,F})\,S^i + \pi_l\,\det\gamma_{\pi,F}\,D_l S^i \\ &= \pi'_l\,\det\gamma_{\pi,F}\,S^i - \pi_l\,(\det\gamma_{\pi,F})^2\,(\mu+\nu\,\pi'_l)\,S^i + \pi_l\,\det\gamma_{\pi,F}\,(\mu_i+\nu_i\,\pi'_l)\,. \end{aligned}$$

Since $\pi_l \in \mathrm{L}_{(\infty)}(A)$, the non degeneracy condition (VII.3.2) gives $\pi_l\,(\det\gamma_{\pi,F} + \det\gamma^2_{\pi,F}) \in \mathrm{L}_{(1+)}(A)$.
Moreover, by hypothesis VII.3, we have $\pi'_l \in \mathrm{L}_{(1+)}(A)$, and by the non degeneracy condition (VII.3.2), we have $\pi'_l\,(\det\gamma_{\pi,F})^2 \in \mathrm{L}_{(1+)}(A)$. So, since

$\pi'_l \det \gamma_{\pi,F} = \sqrt{\pi'_l} \times (\sqrt{\pi'_l} \det \gamma_{\pi,F})$, using the Cauchy-Schwarz inequality, we get $\pi'_l \det \gamma_{\pi,F} \in \mathrm{L}_{(1+)}(A)$. And then,

$$D_l(\pi_l Q^i) \in \mathrm{L}_{(1+)}(A) .$$

And the proof is complete. ■

As a consequence we obtain

Theorem VII.2:
Let $F = (F^1, \ldots, F^d) \in \mathcal{S}_2^d(A)$ and $G \in \mathcal{S}_1(A)$, that is F^i and G and their derivatives have moments of any order on A.
Suppose that the non degeneracy condition (VII.3.2) holds true.
Then $\delta_\pi \left(G \sum_{j=1}^d \gamma_{\pi,F}^{ji} \, DF^j \right)$, $\left[\phi(F), G \sum_{j=1}^d \gamma_{\pi,F}^{ji} \, DF^j \right]_\pi \in \mathrm{L}_{(1+)}(A)$,
and for every $\phi \in \mathcal{C}_p^1(\mathbb{R}^d)$, one has for every $i = 1, \ldots, d$,

$$\mathrm{E}(\partial_i \phi(F) \, G \, \mathbf{1}_A) = \mathrm{E}\left[\phi(F) \, \delta_\pi \left(G \sum_{j=1}^d \gamma_{\pi,F}^{ji} \, DF^j \right) \mathbf{1}_A \right] + \mathrm{E}\left(\left[\phi(F), G \sum_{j=1}^d \gamma_{\pi,F}^{ji} \, DF^j \right]_\pi \mathbf{1}_A \right) .$$

Suppose that $\pi_l, l \geq 1$ satisfy the hypothesis (VII.2.4) which cancels the border terms. We then obtain

$$\mathrm{E}(\partial_i \phi(F) \, G \, \mathbf{1}_A) = \mathrm{E}(\phi(F) \, H_{i,\pi}(F,G) \, \mathbf{1}_A) , \tag{VII.3.3}$$

with

$$H_{i,\pi}(F,G) = \delta_\pi \left(G \sum_{j=1}^d \gamma_{\pi,F}^{ji} \, DF^j \right) = \sum_{j=1}^d \left(G \, \gamma_{\pi,F}^{ji} \, \mathrm{L}_\pi F^j - \left\langle D(G \, \gamma_{\pi,F}^{ji}), DF^j \right\rangle_\pi \right) \in \mathrm{L}^{1+\eta}(A) .$$

Proof. We take $\widetilde{G} = (0, \ldots, 0, G, 0, \ldots, 0)$ with G on the place i, so that $\partial_i \phi(F) \, G = \left\langle \nabla \phi(F), \widetilde{G} \right\rangle$. In view of Lemma VII.3, $\widetilde{G} \in \Theta_F(A)$ and $\widetilde{G} = \sigma_{\pi,F} \times Q$, with $Q^j = G \, \gamma_{\pi,F}^{ji}$. One then employes Theorem VII.1 to conclude.
In order to obtain the second equality in the expression of $H_{i,\pi}(F,G)$, one employes the chain rule (VII.2.11). ■

There is one particular situation in which the non degeneracy condition (VII.3.2) does not involve the weights : if F is one dimensional and if the integration by parts formula is based on a single random variable V_i. Then we have the following Proposition.

Proposition VII.2:
Let $F = f(\omega, \widetilde{V}) \in \mathcal{S}_2(A)$ and $G \in \mathcal{S}_1(A)$.
Suppose that there exists some $l \geq 1$ be such that

$$\mathrm{E}\left[\mathbf{1}_A \, (D_l F)^{-6\,(1+\eta)}\right] < \infty, \text{ for some } \eta > 0\,. \tag{VII.3.4}$$

Let us consider the weights $\pi_i = 0$ for $i \neq l$ and π_l an arbitrary function which verifies $\pi_l \in \mathrm{L}_{(\infty)}(A)$ and $\pi_l' \in \mathrm{L}_{(1+)}(A)$.
Then, $\delta_\pi(G\,\gamma_{\pi,F}\,DF)$, $[\phi(F), G\,\gamma_{\pi,F}\,DF]_\pi \in \mathrm{L}_{(1+)}(A)$.
And for every $\phi \in \mathcal{C}_p^1(\mathbb{R})$, one has

$$\mathrm{E}(\phi'(F)\,G\,\mathbf{1}_A) = \mathrm{E}\left(\phi(F)\,\delta_\pi(G\,\gamma_{\pi,F}\,DF)\,\mathbf{1}_A\right) + \mathrm{E}\left([\phi(F), G\,\gamma_{\pi,F}\,DF]_\pi\,\mathbf{1}_A\right)\,. \tag{VII.3.5}$$

Proof. Note that $\sigma_{\pi,F} = \pi_l(V_l)\;|D_l F|^2$.
We then come back to the proof of Theorem VII.1 and we write $G = Q\,\sigma_{\pi,F}$, with

$$\begin{aligned} Q &= \tfrac{G}{\pi_l(V_l)\,|D_l F|^2} && \text{if } \pi_l(V_l)\;|D_l F|^2 \neq 0\,, \\ &= 0 && \text{if } \pi_l(V_l)\;|D_l F|^2 = 0\,. \end{aligned}$$

Hence, $\pi_l(V_l)\,Q = G/|D_l F|^2$ and, as a consequence of hypothesis (VII.3.4), one gets $\pi_l(V_l)\,Q$, $\partial_{V_i}(\pi(V_l)\,Q) \in \mathrm{L}_{(1+)}(A)$, $i \geq 1$. We may thus use the duality relation (VII.2.7) to conclude. ■

On the contrary, there is another particular case where the non degeneracy condition (VII.3.2) does involve nothing but the weights π_i : if $F = f(\omega, \widetilde{V})$ is one dimensional with ∂f elliptic, we have the following Lemma :

Lemma VII.4:
Let $F = f(\omega, \widetilde{V}) \in \mathcal{S}_1(A)$, that is F and its derivatives have finite moments of any order on A. We assume that there exists a positive constant c such that for all $i \in \mathbb{N}$,

$$|\partial_i f(\omega, \widetilde{V})| \geq c > 0\,. \tag{VII.3.6}$$

We suppose that the weights $(\pi_i(\omega, V_i))_{i\in\mathbb{N}}$ and their derivatives $(\pi_i'(\omega, V_i))_{i\in\mathbb{N}}$ are independant.
We also suppose that there exists $\eta > 0$ such that for all $i \in \mathbb{N}$

$$\mathrm{E}\left[\left(\frac{1}{\pi_i(\omega, V_i)}\right)^{2\,(1+\eta)}\right] < \infty\,. \tag{VII.3.7}$$

Then the non-degeneracy condition (VII.3.2) is satisfied.

Proof. Note that, since we deal with the one dimensional case, the non-degeneracy condition (VII.3.2) reads

$$\mathrm{E}\left[\mathbf{1}_A\,\left((\gamma_{\pi,F})^2\,(1+|\pi_l'|)\right)^{1+\eta}\right] < \infty, \text{ for some } \eta > 0\,.$$

We thus have to verify for $l \geq 1$,

$$\mathrm{E}\left(\mathbf{1}_A\,\gamma_{\pi,F}^{2\,(1+\eta)}\right) < \infty \text{ and } \mathrm{E}\left(\mathbf{1}_A\,\left(|(\pi_l)'(V_l)|\,\gamma_{\pi,F}^2\right)^{1+\eta}\right) < \infty\,. \qquad \text{(VII.3.8)}$$

Let us fix $n \in \mathbb{N}^*$. On $A \cap A_n$, we have

$$|\sigma_{\pi,F}| = \mathbf{1}_{A_n}\left|\sum_{j=1}^{n}\pi_j(V_j)\,(\partial_j f_n)^2\right| \geq c^2\,|\pi_1(V_1)|\,.$$

So hypothesis (VII.3.7) gives

$$\mathrm{E}\left(\mathbf{1}_A\,\gamma_{\pi,F}^{2\,(1+\eta)}\right) \leq \frac{1}{c^{2\,(1+\eta)}} \times \mathrm{E}\left(\mathbf{1}_A\,|\pi_1(V_1)|^{-2\,(1+\eta)}\right) < \infty\,.$$

Let us prove that $\mathrm{E}\left(\mathbf{1}_A\,\left(|(\pi_l)'(V_l)|\,\gamma_{\pi,F}^2\right)^{1+\eta}\right) < \infty$.
We fix $l \in \mathbb{N}$. If $n \geq 2$, we can take $j_0 \in \{1,\ldots,n\}$ such that $l \neq j_0$, and we get

$$\mathrm{E}\left(\mathbf{1}_A\,\left(|(\pi_l)'(V_l)|\,\gamma_{\pi,F}^2\right)^{1+\eta}\right) \leq \frac{1}{c^{2\,(1+\eta)}} \times \mathrm{E}\left(\mathbf{1}_A\,\left(\frac{|(\pi_l)'(V_l)|(\omega,V_l)}{(\pi_{j_0})^2(\omega,V_{j_0})}\right)^{1+\eta}\right)\,.$$

Since $\pi_l'(\omega,V_l)$ and $\pi_{j_0}(\omega,V_{j_0})$ are independant, we obtain

$$\begin{aligned}\mathrm{E}\left(\mathbf{1}_A\,\left(\frac{|(\pi_l)'|(\omega,V_l)}{(\pi_{j_0})^2(\omega,V_{j_0})}\right)^{1+\eta}\right) &\leq \mathrm{E}\left(\mathbf{1}_A\,|(\pi_l)'(V_l)|^{1+\eta}\right) \times \mathrm{E}\left(\frac{1}{(\pi_{j_0})^{2\,(1+\eta)}}\right)\\ &< \infty\,.\end{aligned}$$

If $n = 1$, we then have to verify that the condition (VII.3.4) of Proposition VII.2 holds true, that is $\mathrm{E}\left(\mathbf{1}_A\,|f'(V)|^{-6\,(1+\eta)}\right) < \infty$, and this is the case under the ellipticity assumption (VII.3.6). Hence we have

$$\mathrm{E}\left(\mathbf{1}_A\,\left(|(\pi_l)'|\,\gamma_{\pi,F}^2\right)^{1+\eta}\right) < \infty\,.$$

And then, the non degeneracy condition (VII.3.2) holds true. ■

3.2. The case of smooth laws

The aim of this paragraph is to show what the non-degeneracy condition (VII.3.2) and the integration by parts formula (VII.3.3) become when the conditional density of the random variable V_i given $\mathcal{G}_i$ has no discontinuities.
Fitting the notation of the framework given in Section 1, this means that $B_i = \mathbb{R}$, that is $a_i = -\infty$, $b_i = +\infty$ and $k_i = 0$.
As it may remain some border terms in a_i and b_i, we will suppose that the conditional density $p_i(\omega, y)$ vanishes at infinity. Moreover, we have seen in the proof of the duality formula in Proposition VII.1 that we use integration by parts based on $\partial_y \ln p_i(\omega, y)$. That's why the derivatives $\partial_y \ln p_i(\omega, V_i)$ appear in the Malliavin operators (the Skorohod integral and then the Ornstein Uhlenbeck operator). We thus need to keep suitable hypothesis on $\partial_y \ln p_i(\omega, y)$ in order to have appropriate integrability properties for these operators. This leads to the following assumption.

Hypothesis VII.5. For every $i \in \mathbb{N}^*$, the conditional law of V_i given $\mathcal{G}_i$ is absolutely continuous with respect to the Lebesgue measure. We denote $p_i(\omega, y)$ its density.
We suppose that p_i is continuously differentiable on $\mathbb{R}$ and that for all $k \in \mathbb{N}$, $\lim_{y \to \pm\infty} |y|^k \, p_i(y) = 0$.
We also assume that $\partial_y \ln(p_i(y)) = \dfrac{\partial_y p_i(y)}{p_i(y)} \in \mathcal{C}_p^0(\mathbb{R})$.

In this framework, since p_i produces no border terms, we do not need any weights $(\pi_i)_{i \in \mathbb{N}}$, so that we take $\pi_i(\omega, V_i) = 1$ for all $i \geq 1$. Hence, we come back to the classical inner product on the space of the simple processes, say

$$\langle U, V \rangle := \sum_{i=1}^{\infty} U_i(\omega, \widetilde{V}) \, V_i(\omega, \widetilde{V}) \, .$$

The Malliavin operators become
• The Skorohod integral : for all $U \in \mathcal{P}_1$,

$$\delta(U) := -\sum_{i=1}^{\infty} \partial_{V_i} U_i(\omega, \widetilde{V}) + \partial \ln p_i(\omega, V_i) \, U_i(\omega, \widetilde{V}) \, . \tag{VII.3.9}$$

• The Ornstein Uhlenbeck operator : for all $F \in \mathcal{S}_1$,

$$\mathrm{L}F = \delta(DF) = -\sum_{i=1}^{\infty} \partial_{V_i}^2 f(\omega, \widetilde{V}) + \partial \ln p_i(\omega, V_i) \, \partial_{V_i} f(\omega, \widetilde{V}) \, .$$

• The border terms operator $[F, U]_\pi$ disappears.
Concerning the integration by parts formula, let us go back to the integrability

problem of the Malliavin weight obtained in equation (VII.3.3) :

$$H_{i,\pi}(F,G) = \delta_\pi \left(G \sum_{j=1}^{d} \gamma_{\pi,F}^{ji} \, DF^j \right) .$$

This expression involves the derivatives of the weights π (by means of δ_π) as well as their inverse (by means of $\gamma_{\pi,F}$). Hence, we need the non-degeneracy condition (VII.3.2) to realize an equilibrium between these two quantities, which allows us to derive suitable integrability property for the weight $H_{i,\pi}(F,G)$. But in this paragraph, since we have no weights $(\pi_i)_{i\in\mathbb{N}}$, things are much simple. The expression of $H_{i,\pi}(F,G)$ actually becomes

$$\begin{aligned} H_i(F,G) &= \delta \left(G \sum_{j=1}^{d} \gamma_F^{ji} \, DF^j \right) \\ &= \sum_{j=1}^{d} G \, \gamma_F^{ji} \, \mathrm{L} F^j - \gamma_F^{ji} < DF^j, DG > - G < DF^j, D\gamma_F^{ji} > . \end{aligned} \quad \text{(VII.3.10)}$$

Moreover, the Skorohod integral does not contain the term $(\pi_i)' \in \mathrm{L}_{(1+)}(A)$, so we can set the following Lemma :

Lemma VII.5:
For all $U \in \mathcal{P}_1(A)$, that is U and its first order derivatives have moments of any order on A, we have $\delta(U) \in \mathrm{L}_{(\infty)}(A)$.
Hence, for all $F \in \mathcal{S}_1(A)$, we have $\mathrm{L}(F) \in \mathrm{L}_{(\infty)}(A)$.

Proof. By hypotesis VII.5, $\partial \ln p_i$ has polynomial growth. Since $V_i \in \mathrm{L}_{(\infty)}(A)$, we then have $\partial \ln p_i(\omega, V_i) \in \mathrm{L}_{(\infty)}(A)$. Equation (VII.3.9) gives the result. ■

This Lemma allows us to use Cauchy-Schwarz inequalities in equation (VII.3.10), which was not possible with the weights $(\pi_i)_{i\in\mathbb{N}}$: since $(\pi_i)' \in \mathrm{L}_{(1+)}(A)$, we could not have $\mathrm{L}_\pi(F) \in \mathrm{L}_{(\infty)}(A)$ even if F and $\partial F \in \mathrm{L}_{(\infty)}(A)$.
For example, since $D\gamma_F^{ji} = -2\,(\gamma_F^{ji})^2 \, D(\sigma_F^{ji})$, we obtain

$$\begin{aligned} \mathrm{E}\left[|G < DF^j, D\gamma_F^{ji} > |^p \, \mathbf{1}_A \right] &\le 2 \, \mathrm{E}\left[|\gamma_F^{ji}|^{4p} \, \mathbf{1}_A \right]^{1/2} \\ &\quad \times \mathrm{E}\left[|G < DF^j, D\sigma_F^{ji} > |^{2p} \, \mathbf{1}_A \right]^{1/2} . \end{aligned}$$

Hence, if $F \in \mathcal{S}_2(A)$ and $G \in \mathcal{S}_1(A)$ (so that F, G and their derivatives have finite moments of any order on A), we have $D\sigma_F^{ji} \in \mathrm{L}_{(\infty)}(A)$, and then
$\mathrm{E}\left[|H_i(F,G)|^p \, \mathbf{1}_A \right] < \infty$ if $\mathrm{E}\left[(\gamma_F^{ji})^{4p} \, \mathbf{1}_A \right] < \infty$.
Thus, the non-degeneracy condition in the case of smooth conditional laws is the following :

Hypothesis VII.6.

$$(H_q) \qquad \mathrm{E}\left[(\det \gamma_F)^{4\,q}\, \mathbf{1}_A\right] < \infty, \text{ for some } q \in \mathbb{N}^*\,.$$

Let us summarize all these results in the following Theorem :

Theorem VII.3:
Let $F = (F^1, \ldots, F^d) \in S_2(A)^d$, $G \in S_1(A)$, that is G, F and their derivatives have finite moments of any order on A.
We assume that the matrix σ_F is invertible on A, and that its inverse $\gamma_F := \sigma_F^{-1}$ satisfies hypothesis VII.6.
Then for every function $\phi : \mathbb{R}^d \to \mathbb{R} \in \mathcal{C}_p(\mathbb{R})$, for every $i = 1, \ldots, d$, we have

$$E(\partial_i \phi(F)\, G\, \mathbf{1}_A) = E(\phi(F)\, H_i(F,G)\, \mathbf{1}_A)\,, \qquad \text{(VII.3.11)}$$

where $H_i(F,G) \in \mathrm{L}^q(A)$ is given by equation (VII.3.10).

4. Iteration of the integration by parts formula

In this section, we suppose that the weights $(\pi_i)_{i\in\mathbb{N}}$ are chosen so that they cancel the border terms, that is they satisfy hypothesis (VII.2.4) :

$$\pi_i(\omega, t_i^j+) = \pi_i(\omega, t_i^j-) \text{ and } \pi_i(\omega, a_i+) = \pi_i(\omega, b_i-) = 0\,.$$

We want to iterate the previous integration by parts formula (VII.3.3), so that we will have to solve two problems.
The first one comes from the hypothesis of Theorem VII.2. Once equation (VII.3.3) is settled, we actually want to apply again Theorem VII.2 where this time, G is replaced by $H_{i,\pi}(F,G)$. The hypothesis then require $H_{i,\pi}(F,G)$ to be $\mathrm{L}_{(\infty)}(A)$, which is impossible since we just know that $H_{i,\pi}(F,G) \in \mathrm{L}^{1+\eta}(A)$ for small $\eta > 0$ only. So we have to relax the assumption '$G \in \mathrm{L}_{(\infty)}(A)$' by replacing it by '$G \in \mathrm{L}_{(1+)}(A)$'. This gives the following corollary in the one dimensional case :

Corollary VII.2:
Let $F = f(\omega, \widetilde{V}) \in \mathcal{S}_2(A)$. We denote

$$\hat{\sigma}_{\pi,F} := \frac{1}{\|\, 1\,\|_\pi^2}\, \sigma_{\pi,F} = \sigma_{\pi,F} \left(\sum_{n=1}^{\infty} \sum_{i=1}^{n} \pi_i(V_i)\, \mathbf{1}_{A_n} \right)^{-1} \text{ and set } \hat{\gamma}_{\pi,F} := (\hat{\sigma}_{\pi,F})^{-1}.$$

Let $G \in \mathcal{S}_1$. We suppose that $G\,(1 + \hat{\gamma}_{\pi,F}^{1/2}) \in \mathrm{L}_{(1^+)}(A)$ and that

$$G \times H_\pi(F,1) \in \mathrm{L}_{(1^+)}(A) \text{ and } \langle DG, \gamma_{\pi,F}\, DF\rangle_\pi \in \mathrm{L}_{(1^+)}(A)\,. \qquad \text{(VII.4.1)}$$

Theorem VII.2 then still holds true for every $\phi \in \mathcal{C}_p^1(\mathbb{R})$.

Proof. We have $G = \sum_{n\geq 1} g_n(\omega, V_1, \ldots, V_n)\,\mathbf{1}_{A_n}$ with $g_n \in \mathcal{C}_{n,1}$ and $g_n = 0$ for $n > N$.
For all $R > 0$, let us define

$$G^R := \sum_{n\geq 1} \mathbf{1}_{A_n}\, g_n(\omega, V_1, \ldots, V_n) \prod_{i=1}^{n} \phi_R(V_i)\,,$$

where $\phi_R \in \mathcal{C}_b^\infty(\mathbb{R})$, and $\mathbf{1}_{(-R,R)} \leq \phi_R \leq \mathbf{1}_{(-(R+1),R+1)}$.
Then $G^R \in \mathrm{L}_{(\infty)}(A)$ for all $R > 0$.
We denote $g_n^R := g_n(\omega, V_1, \ldots, V_n) \prod_{i=1}^{n} \phi_R(V_i)$. We thus have
$\partial_i G^R = \sum_{n=1}^{\infty} \partial_i g_n^R(\omega, V_1, \ldots, V_n)\,\mathbf{1}_{A_n} \in \mathrm{L}_{(\infty)}(A)$.
Hence, $G^R \in \mathcal{S}_1$ and $G^R, \partial_i G^R \in \mathrm{L}_{(\infty)}(A)$. We can then apply Theorem VII.2 to G^R : for all $R > 0$, for all $\psi \in \mathcal{C}_p^1(\mathbb{R})$, we have

$$\begin{aligned} \mathrm{E}\left(\psi'(F)\, G^R\, \mathbf{1}_A\right) &= \mathrm{E}\left(\psi(F)\, H_\pi(F, G^R)\, \mathbf{1}_A\right) \\ &= \mathrm{E}\left(\psi(F)\, G^R\, H_\pi(F,1)\, \mathbf{1}_A\right) - \mathrm{E}\left(\psi(F)\, \langle DG^R, \gamma_{\pi,F}\, DF\rangle_\pi\, \mathbf{1}_A\right)\,. \end{aligned} \tag{VII.4.2}$$

We take the limit in equation (VII.4.2) as $R \to \infty$ by using Lebesgues' theorem in each term. We have $\lim_{R\to\infty} G^R = G\ a.s$ and $|G^R| \leq |G|$ for all $R > 0$.
• ψ' has polynomial growth and $F \in \mathrm{L}_{(\infty)}(A)$, so $\psi'(F) \in \mathrm{L}_{(\infty)}(A)$. And since $G \in \mathrm{L}_{(1+)}(A)$, we have $\psi'(F)\, G \in \mathrm{L}_{(1+)}(A)$. We then obtain

$$\mathrm{E}\left(\psi'(F)\, G^R\, \mathbf{1}_A\right) \underset{R\to\infty}{\longrightarrow} \mathrm{E}\left(\psi'(F)\, G\, \mathbf{1}_A\right)\,.$$

• We have $G\, H_\pi(F,1) \in \mathrm{L}_{(1+)}(A)$ and $\psi(F) \in \mathrm{L}_{(\infty)}(A)$, so $\psi(F)\, G\, H_\pi(F,1) \in \mathrm{L}_{(1+)}(A)$, and we obtain

$$\mathrm{E}\left(\psi(F)\, G^R\, H_\pi(F,1)\, \mathbf{1}_A\right) \underset{R\to\infty}{\longrightarrow} \mathrm{E}\left(\psi(F)\, G\, H_\pi(F,1)\, \mathbf{1}_A\right)\,.$$

• For all $R > 0$, we have on $A \cap A_n$, $n \geq 1$

$$
\begin{aligned}
&|\langle DG^R, \gamma_{\pi,F}\, DF\rangle_\pi| \\
&\quad = \left| \sum_{i=1}^{n} \pi_i(V_i)\, \gamma_{\pi,F}\, \partial_i f_n\, \partial_i g_n \prod_{j=1}^{n} \phi_R(V_j) \right| \\
&\qquad + \left| \sum_{i=1}^{n} \pi_i(V_i)\, \gamma_{\pi,F}\, \partial_i f_n\, g_n\, \phi_R'(V_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \phi_R(V_j) \right| \\
&\quad \leq \left| \sum_{i=1}^{n} \pi_i(V_i)\, \gamma_{\pi,F}\, \partial_i f_n\, \partial_i g_n \right| + |g_n|\, |\gamma_{\pi,F}| \left| \sum_{i=1}^{n} \pi_i(V_i)\, \partial_i f_n \right| \\
&\quad \leq |\langle DG, \gamma_{\pi,F}\, DF\rangle_\pi| + |g_n|\, |\gamma_{\pi,F}| \left| \sum_{i=1}^{n} |\pi_i(V_i)|\, |\partial_i f_n|^2 \right|^{1/2} \times \left| \sum_{i=1}^{n} |\pi_i(V_i)| \right|^{1/2} \\
&\quad = |\langle DG, \gamma_{\pi,F}\, DF\rangle_\pi| + |g_n|\, |\gamma_{\pi,F}|^{1/2} \left| \sum_{i=1}^{n} |\pi_i(V_i)| \right|^{1/2} \\
&\quad = |\langle DG, \gamma_{\pi,F}\, DF\rangle_\pi| + |G|\, |\hat{\gamma}_{\pi,F}^{1/2}|\,.
\end{aligned}
$$

Hence, by hypothesis (VII.4.1), we obtain $|\langle DG^R, \gamma_{\pi,F}\, DF\rangle_\pi| \in \mathrm{L}_{(1+)}(A)$, and then

$$
\mathrm{E}\left(\psi(F)\, \langle DG^R, \gamma_{\pi,F}\, DF\rangle_\pi\, \mathbf{1}_A\right) \underset{R\to\infty}{\longrightarrow} \mathrm{E}\left(\psi(F)\, \langle DG, \gamma_{\pi,F}\, DF\rangle_\pi\, \mathbf{1}_A\right)\,.
$$

The proof is complete. ■

The second problem concerns the second order derivatives of the weights $(\pi_i)_{i\in\mathbb{N}}$. Let us be more precise. We consider the one dimensional case for more simple notation. Theorem VII.2 allows us to perform an integration by parts formula in the following way :

$$
\mathrm{E}\left(\phi'(F)\, G\, \mathbf{1}_A\right) = \mathrm{E}\left(\phi(F)\, H_\pi(F,G)\, \mathbf{1}_A\right)\,, \tag{VII.4.3}
$$

with $H_\pi(F,G) = G\, \gamma_{\pi,F}\, \mathrm{L}_\pi(F) - \langle D(G\, \gamma_{\pi,F}), DF\rangle_\pi$. Formula (VII.4.3) holds true under the non-degeneracy condition (VII.3.2), which sets that $G\, \gamma_{\pi,F}\, \mathrm{L}_\pi(F) \in \mathrm{L}^{1+\eta}(A)$ and $\langle D(G\, \gamma_{\pi,F}), DF\rangle_\pi \in \mathrm{L}^{1+\eta}(A)$ for some $\eta > 0$.
Suppose that we iterate the integration by parts formula (VII.4.3) using the same weights $(\pi_i)_{i\in\mathbb{N}}$. We then obtain the following formula :

$$
\mathrm{E}\left(\phi'(F)\, H_\pi(F,G)\, \mathbf{1}_A\right) = \mathrm{E}\left(\phi(F)\, \mathcal{H}_\pi(F,G)\, \mathbf{1}_A\right)\,, \tag{VII.4.4}
$$

with

$$\begin{aligned}\mathcal{H}_\pi(F,G) &= H_\pi(F, H_\pi(F,G))\\ &= H_\pi(F,G)\,\gamma_{\pi,F}\,\mathrm{L}_\pi(F) - \langle D(H_\pi(F,G)\,\gamma_{\pi,F}), DF\rangle_\pi\,.\end{aligned}$$

But formula (VII.4.4) holds true if $\mathcal{H}_\pi(F,G) \in \mathrm{L}^1(A)$, which may be a real problem. The expression of $\langle D(H_\pi(F,G)\,\gamma_{\pi,F}), DF\rangle_\pi$ contains some terms such as $\pi_i(\omega, V_i)\, D_{V_i}(\mathrm{L}_\pi(F))$ which involves the second order derivatives $\pi_i''(\omega, V_i) \times \pi_i(\omega, V_i)$ of the weights.
Typically, when $B_i = (\alpha_i, \beta_i)$, the weights π_i are chosen as $\pi_i(y) := (\beta_i - y)^a\,(y-\alpha_i)^a$ if $y \in (\alpha_i, \beta_i)$, and $\pi_i(y) := 0$ if $y \notin (\alpha_i, \beta_i)$, with $a \in (0, 1/2)$. So their second order derivatives are not integrable.
To overcome this difficulty, we split the interval (α_i, β_i) into two disjoint sets (α_i, γ_i) and (γ_i, β_i) (take γ_i as the middle of (α_i, β_i) for example). We define two kinds of weights $(\pi_i^1)_{i\in\mathbb{N}}$ and $(\pi_i^2)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, π_i^1 (resp. π_i^2) satisfies hypothesis VII.3 on (α_i, γ_i) (resp. (γ_i, β_i)), and $\pi_i^1 = 0$ (resp. $\pi_i^2 = 0$) for $y \notin (\alpha_i, \gamma_i)$ (resp. $y \notin (\gamma_i, \beta_i)$). Consequently, since π_i^2 is null on the support of π_i^1, we have $\pi_i^2(V_i)\,\partial_{ii}^2 \pi_i^1(V_i) = 0$ for all $i \in \mathbb{N}$. This removes the above difficulty.
Hence, the method for iterating the Malliavin integration by parts formula is the following : we perform the first integration by parts formula using the weights $(\pi_i^1)_{i\in\mathbb{N}}$ and we perform the second one with the weights $(\pi_i^2)_{i\in\mathbb{N}}$. Then $\langle D\mathrm{L}_{\pi^1}(F), DF\rangle_{\pi^2}$ does not contain any terms with the second order derivatives of π^1.

Theorem VII.4:
Let $F = f(\omega, \widetilde{V}) \in \mathcal{S}_3(A)$ and $G \in \mathcal{S}_2(A)$, that is F, G and their derivatives have finite moments of any order on A. We assume that F statisfies the ellipticity assumption (VII.3.6), that is there exists a positive constant c such that for all $i \in \mathbb{N}$,

$$|\partial_i f(\omega, \widetilde{V})| \geq c > 0\,.$$

We suppose that for $k, l = 1, 2$, $\pi_i^k(\omega, V_i)$, $\pi_j^l(\omega, V_j)$ and their first order derivatives are independant for $i \neq j$.
We also suppose that the weights π_i^k satisfies condition (VII.3.7) for $k = 1, 2$, that is there exists $\eta > 0$ such that for all $i \in \mathbb{N}$

$$\mathrm{E}\left[\left(\frac{1}{\pi_i^k(\omega, V_i)}\right)^{3\,(1+\eta)}\right] < \infty\,,\; k = 1, 2\,.$$

Then the non-degeneracy condition (VII.3.2) is satisfied for the weights π^1 and π^2 and the integration by parts formula (VII.4.3) holds true for all $\phi \in \mathcal{C}_p^1(\mathbb{R})$, that is

$$\mathrm{E}\left[\phi'(F)\,G\,\mathbf{1}_A\right] = \mathrm{E}\left[\phi(F)\,H_{\pi^1}(F,G)\,\mathbf{1}_A\right]\,,\; \textit{with } H_{\pi^1}(F,G) \in \mathrm{L}^{1+\eta}(A)\,.$$

Moreover, we suppose that $A = \bigcup_{n \geq 4} A_n \cap A$, *that is the functionals* F *and* G *depend on four random variables at least :*

$$F = f(\omega, \widetilde{V}) = \sum_{n \geq 4} f_n(\omega, V_1, \ldots, V_n)\, \mathbf{1}_{A_n} .$$

Then, for all $\phi \in \mathcal{C}_p^1(\mathbb{R})$, *we can iterate the formula (VII.4.3), that is*

$$\mathrm{E}\left(\phi'(F)\, H_{\pi^1}(F,G)\, \mathbf{1}_A\right) = \mathrm{E}\left(\phi(F)\, \mathcal{H}_\pi(F,G)\, \mathbf{1}_A\right) ,$$

with $\mathcal{H}_\pi(F,G) = H_{\pi^2}(F, H_{\pi^1}(F,G)) \in \mathrm{L}^{1+\eta}(A)$.

Proof. By Lemma VII.4, we know that if the weights $\pi_i^k(\omega, V_i)$ and their derivatives $(\pi_i^k)'(\omega, V_i)$ are independant, then conditions (VII.3.6) and (VII.3.7) imply that the weights π_i^k satisfy the non degeneracy condition (VII.3.2).
Let us prove that we can iterate the integration by parts formula (VII.4.3). In order to use Corollary VII.2, we have to verify that
$H_{\pi^1}(F,G)\, \hat{\gamma}_{\pi^2,F}^{1/2} \in \mathrm{L}_{(1+)}(A)$,

$$\langle DH_{\pi^1}(F,G), \gamma_{\pi^2,F}\, DF \rangle_{\pi^2} \in \mathrm{L}_{(1+)}(A) \text{ and } H_{\pi^1}(F,G) \times H_{\pi^2}(F,1) \in \mathrm{L}_{(1+)}(A) .$$

By the ellipticity assumption (VII.3.6), we have on $A \cap A_n$,

$$\hat{\gamma}_{\pi^2,F} \leq \gamma_{\pi^2,F} \sum_{m=1}^{n} \pi_m^2(V_m) \leq \frac{1}{c^2} \frac{\sum_{m=1}^{n} \pi_m^2(V_m)}{\sum_{m=1}^{n} \pi_m^2(V_m)} \leq \frac{1}{c^2} .$$

Since $H_{\pi^1}(F,G) \in \mathrm{L}_{(1+)}(A)$, we obtain $H_{\pi^1}(F,G)\, \hat{\gamma}_{\pi^2,F}^{1/2} \in \mathrm{L}_{(1+)}(A)$.
Let us continue with a Lemma.

Lemma VII.6:
Let us define for $i \neq j$, $n \in \mathbb{N}$,

$$\eta_{ij}^n := \sum_{p,q=1,2} \frac{1 + |(\pi_i^1)'(V_i)| + |(\pi_j^2)'(V_j)| + |(\pi_i^1)'(V_i)|\,|(\pi_j^2)'(V_j)|}{\left(\sum_{m=1}^{n} \pi_m^1(V_m)\right)^p \left(\sum_{m=1}^{n} \pi_m^2(V_m)\right)^q} ,$$

and

$$\varepsilon_j^n := \sum_{p=1,2,3} \frac{1 + |(\pi_j^1)'(V_j)|}{\left(\sum_{m=1}^{n} \pi_m^1(V_m)\right)^p} .$$

We then have $\eta_{ij}^n, \varepsilon_j^n \in \mathrm{L}_{(1+)}(A \cap A_n)$ *for all* $n \geq 4$.

Proof. Since $n \geq 4$, we can choose $i \neq j \neq m_0 \neq l_0$ so that $(\pi_i^1)'(\omega, V_i)$, $(\pi_j^2)'(\omega, V_j)$, $\pi_{m_0}^1(\omega, V_{m_0})$ and $\pi_{l_0}^2(\omega, V_{l_0})$ are independant. For $p, q = 1, 2$, we thus get

$$\begin{aligned}
&\mathrm{E}\left[\mathbf{1}_{A\cap A_n}\left(\frac{|(\pi_i^1)'(V_i)|\,|(\pi_j^2)'(V_j)|}{\left(\sum\limits_{m=1}^n \pi_m^1(V_m)\right)^p\left(\sum\limits_{m=1}^n \pi_m^2(V_m)\right)^q}\right)^{1+\eta}\right] \\
&\leq \mathrm{E}\left[\mathbf{1}_A\,|(\pi_i^1)'(V_i)|^{1+\eta}\right]\,\mathrm{E}\left[\mathbf{1}_A\,|(\pi_j^2)'(V_j)|^{1+\eta}\right] \\
&\times \mathrm{E}\left[\mathbf{1}_{A\cap A_n}\left(\frac{1}{\pi_{m_0}^1(V_{m_0})}\right)^{p\,(1+\eta)}\right]\,\mathrm{E}\left[\mathbf{1}_{A\cap A_n}\left(\frac{1}{\pi_{l_0}^2(V_{l_0})}\right)^{q\,(1+\eta)}\right].
\end{aligned}$$

Using hypothesis (VII.3.7) and the fact that $(\pi_i^k)' \in \mathrm{L}_{(1+)}(A)$ for $k = 1, 2$, we then obtain

$$\sum_{p,q=1,2}\mathrm{E}\left[\mathbf{1}_{A\cap A_n}\left(\frac{|(\pi_i^1)'(V_i)|\,|(\pi_j^2)'(V_j)|}{\left(\sum\limits_{m=1}^n \pi_m^1(V_m)\right)^p\left(\sum\limits_{m=1}^n \pi_m^2(V_m)\right)^q}\right)^{1+\eta}\right] < \infty\,.$$

Using exaclty the same computations for each term of η_{ij}^n and ε_j^n, we get the result.■

We now return to the proof of Theorem VII.4.

• Let us prove that $H_{\pi^1}(F, G)\, H_{\pi^2}(F, 1) \in \mathrm{L}_{(1+)}(A)$.

For all $n \in \mathbb{N}$, $k = 1, 2$, we have on $A \cap A_n$,

$$\begin{aligned}
&H_{\pi^k}(F, G) \\
&= \delta_{\pi^k}(G\,\gamma_{\pi^k,F}\,DF) \\
&= \sum_{i=1}^n (\pi_i^k\,\partial_i \ln p_i)(\omega, V_i)\,g_n\,\gamma_{\pi^k,F}\,\partial_i f_n + \sum_{i=1}^n \partial_i\left(\pi_i^k(V_i)\,g_n\,\gamma_{\pi^k,F}\,\partial_i f_n\right)\,.
\end{aligned}$$

Let us denote $\beta_i^n := \partial_i \ln p_i(\omega, V_i)\,g_n\,\partial_i f_n\,\pi_i^k(V_i) \in \mathrm{L}_{(\infty)}(A \cap A_n)$. By the ellipticity assumption (VII.3.6), we get on $A \cap A_n$,

$$\left|\sum_{i=1}^n (\pi_i^k\,\partial_i \ln p_i)(\omega, V_i)\;g_n\,\gamma_{\pi^k,F}\,\partial_i f_n\right| \leq \frac{1}{c^2}\sum_{i=1}^n \frac{\beta_i^n}{\sum\limits_{m=1}^n \pi_m^k(V_m)}\,.$$

For all $i \in \mathbb{N}$, we have

$$\partial_i \sigma_{\pi^k,F} = \partial_i\left(\sum_{m\geq 1}\pi_m^k(V_m)\,\left(\partial_m f_m(\omega, \widetilde{V})\right)^2\right) = \theta_{i,1}^k + \theta_{i,2}^k\,(\pi_i^k)'\,, \qquad \text{(VII.4.5)}$$

where $\theta^k_{i,1} = \sum_{m\geq 1} \pi^k_m(V_m)\,\partial_i\left((\partial_m f)^2\right) = 2\sum_{m\geq 1} \pi^k_m(V_m)\,\partial_m f\,\partial^2_{mi} f \in \mathrm{L}_{(\infty)}(A)$ and $\theta^k_{i,2} = (\partial_i f)^2 \in \mathrm{L}_{(\infty)}(A)$. So

$$\partial_i \gamma_{\pi^k,F} = -\gamma^2_{\pi^k,F}\,(\theta^k_{i,1} + \theta^k_{i,2}\,(\pi^k_i)')(V_i)\,,\ \text{with } \theta^k_{i,1}, \theta^k_{i,2} \in \mathrm{L}_{(\infty)}(A)\,. \qquad \text{(VII.4.6)}$$

Using again the ellipticity assumption (VII.3.6), we thus obtain on $A \cap A_n$

$$\begin{aligned}
&\left|\sum_{i=1}^n \partial_i\left(\pi^k_i(V_i)\,g_n\,\gamma_{\pi^k,F}\,\partial_i f_n\right)\right| \\
&\leq \sum_{i=1}^n |\gamma_{\pi^k,F}|\,\left(|\pi^k_i(V_i)|\,|\partial_i(g_n\,\partial_i f_n)| + |(\pi^k_i)'(V_i)|\,|g_n\,\partial_i f_n|\right) + |\partial_i\gamma_{\pi^k,F}|\,|\pi^k_i(V_i)\,g_n\,\partial_i f_n| \qquad \text{(VII.4.7)} \\
&\leq \frac{1}{c^2}\sum_{i=1}^n \frac{\xi^n_i}{\sum\limits_{m=1}^n \pi^k_m(V_m)} \times \left(1 + |(\pi^k_i)'(V_i)| + \frac{1+|(\pi^k_i)'(V_i)|}{\sum\limits_{m=1}^n \pi^k_m(V_m)}\right)\,,
\end{aligned}$$

where ξ^n_i is a polynom of $g_n, \partial g_n, \partial f_n, \partial^2 f_n$ and $\pi^k_m(V_m)$, so that $\xi^n_i \in \mathrm{L}_{(\infty)}(A \cap A_n)$. Hence, we have on $A \cap A_n$,

$$\begin{aligned}
|H_{\pi^k}(F,G)| \leq \frac{1}{c^2}\sum_{i=1}^n \frac{\beta^n_i}{\sum\limits_{m=1}^n \pi^k_m(V_m)} \\
+ \frac{1}{c^2}\sum_{i=1}^n \frac{\xi^n_i}{\sum\limits_{m=1}^n \pi^k_m(V_m)} \times \left(1 + |(\pi^k_i)'(V_i)| + \frac{1+|(\pi^k_i)'(V_i)|}{\sum\limits_{m=1}^n \pi^k_m(V_m)}\right)\,.
\end{aligned}$$

Finally, since $\pi^1_i(V_i) \times \pi^2_i(V_i) = 0$, we obtain

$$\begin{aligned}
H_{\pi^1}(F,G)\,H_{\pi^2}(F,1) &\leq \frac{1}{c^4}\sum_{i\neq j} \frac{\beta^n_i\,\beta^n_j}{\sum\limits_{m\geq 1}\pi^1_m(V_m)\ \sum\limits_{m\geq 1}\pi^2_m(V_m)} + C\sum_{i\neq j}\Lambda^n_{ij}\,\eta^n_{ij} \\
&\leq C\sum_{i\neq j}(\beta^n_i\,\beta^n_j + \Lambda^n_{ij})\,\eta^n_{ij}\,,
\end{aligned}$$

where Λ^n_{ij} is a polynom of ξ^n_i, ξ^n_j and β^n_i, so that $\Lambda^n_{ij} \in \mathrm{L}_{(\infty)}(A \cap A_n)$. By Lemma VII.6, we have $\eta^n_{ij} \in \mathrm{L}_{(1+)}(A \cap A_n)$, so $H_{\pi^1}(F,G)\,H_{\pi^2}(F,1) \in \mathrm{L}_{(1+)}(A)$.

• Let us prove that $\langle DH_{\pi^1}(F,G), \gamma_{\pi^2,F}\,DF\rangle_{\pi^2} \in \mathrm{L}_{(1+)}(A)$. We have

$$\langle DH_{\pi^1}(F,G), \gamma_{\pi^2,F}\,DF\rangle_{\pi^2} = \sum_{n\geq 1}\sum_{i=1}^n \partial_i H_{\pi^1}(F,G)\,\gamma_{\pi^2,F}\,\partial_i f_n\,\pi^2_i(V_i)\,\mathbf{1}_{A_n}\,,$$

where for all $n \in \mathbb{N}$, on $A \cap A_n$,

$$\partial_i H_{\pi^1}(F,G) = \partial_i \left(\sum_{j=1}^{n} \pi_j^1(V_j)\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n\, \partial \ln p_j \right) + \partial_i \left[\sum_{j=1}^{n} \partial_j \left(\pi_j^1(V_j)\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n \right) \right] . \quad \text{(VII.4.8)}$$

Since $\partial_i f_n \in \mathrm{L}_{(\infty)}(A)$, it is enough to prove that

$$\partial_i H_{\pi^1}(F,G)\, \gamma_{\pi^2,F}\, \pi_i^2(V_i) \in \mathrm{L}_{(1+)}(A \cap A_n) .$$

Let us look at $\gamma_{\pi^2,F}\, \pi_i^2(V_i)\, \partial_i \left(\sum_{j=1}^{n} \pi_j^1(V_j)\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n\, \partial \ln p_j \right)$. Using equation (VII.4.6), we have

$$\begin{aligned}
&\partial_i \left(\sum_{j=1}^{n} \pi_j^1(V_j)\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n\, \partial \ln p_j \right) \pi_i^2(V_i) \\
=&\sum_{j=1}^{n} \pi_i^2(V_i)\, \zeta_{ij}^n \left(\gamma_{\pi^1,F}\, (\pi_1^1)'(V_i) + \pi_j^1(V_j)\, \gamma_{\pi^1,F} + \partial_i \gamma_{\pi^1,F}\, \pi_j^1(V_j) \right) \\
=&\sum_{j=1}^{n} \pi_i^2(V_i)\, \zeta_{ij}^n \left(\gamma_{\pi^1,F}\, (\pi_i^1)'(V_i) + \pi_j^1(V_j)\, \gamma_{\pi^1,F} + \pi_j^1(V_j)\, \gamma_{\pi^1,F}^2 + \pi_j^1(V_j)\, (\pi_i^1)'(V_i) \gamma_{\pi^1,F}^2 \right) ,
\end{aligned}$$

where ζ_{ij}^n is a polynom of $g_n, \partial g_n, \partial f_n, \partial^2 f_n$ and $\pi^1, \partial \ln p_j$, so that $\zeta_{ij}^n \in \mathrm{L}_{(\infty)}(A \cap A_n)$. Since π_i^1 and π_i^2 have disjoint supports, we have $\pi_i^2(V_i) \times \pi_i^1(V_i) = 0$, and then

$$\partial_i \left(\sum_{j=1}^{n} \pi_j^1(V_j)\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n\, \partial \ln p_j \right) \pi_i^2(V_i) = \sum_{j \neq i} \pi_i^2(V_i)\, \zeta_{ij}^n\, \pi_j^1(V_j)\, \gamma_{\pi^1,F} \left(1 + \gamma_{\pi^1,F}\right) .$$

By the ellipticity assumption (VII.3.6), we thus have on $A \cap A_n$,

$$\begin{aligned}
&\left| \partial_i \left(\sum_{j=1}^{n} \pi_j^1(V_j)\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n\, \partial \ln p_j \right) \pi_i^2(V_i)\, \gamma_{\pi^2,F} \right| \\
\leq& C \sum_{j \neq i} \frac{\pi_i^2(V_i)\, \zeta_{ij}^n\, \pi_j^1(V_j)}{\sum_{m=1}^{n} \pi_m^1(V_m) \sum_{m=1}^{n} \pi_m^2(V_m)} \left(1 + \frac{1}{\sum_{m=1}^{n} \pi_m^1(V_m)} \right) \\
\leq& C \sum_{j \neq i} \pi_i^2(V_i)\, \zeta_{ij}^n\, \pi_j^1(V_j)\, \eta_{ij}^n .
\end{aligned}$$

By Lemma VII.6, we have $\eta_{ij}^n \in \mathrm{L}_{(1+)}(A \cap A_n)$, which gives

$$\partial_i \left(\sum_{j=1}^{n} \pi_j^1(V_j)\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n\, \partial \ln p_j \right) \pi_i^2(V_i)\, \gamma_{\pi^2,F} \in \mathrm{L}_{(1+)}(A \cap A_n)\,.$$

Let us look at the second term of equation (VII.4.8), that is
$\gamma_{\pi^2,F}\, \pi_i^2(V_i)\, \partial_i \left[\sum_{j=1}^{n} \partial_j \left(\pi_j^1(V_j)\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n \right) \right]$. By equation (VII.4.7), we have

$$\partial_j \left(\pi_j^1\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n \right) = \xi_j^n \, \left[\gamma_{\pi^1,F}\, (\pi_j^1(V_j) + (\pi_j^1)'(V_j)) + \pi_j^1(V_j)\, \partial_j \gamma_{\pi^1,F} \right] \,,$$

where ξ_j^n is a polynomial of $g_n, \partial g_n, f_n, \partial f_n$ and $\partial^2 f_n$. Hence, $\xi_j^n \in \mathrm{L}_{(\infty)}(A \cap A_n)$ and $\lambda_{ij}^n := \partial_i \xi_j^n \in \mathrm{L}_{(\infty)}(A \cap A_n)$.
Since π^1 and π^2 have disjoint supports, we then obtain on $A \cap A_n$

$$\begin{aligned}
&\pi_i^2(V_i)\, \partial_i \left[\sum_{j=1}^{n} \partial_j \left(\pi_j^1\, g_n\, \gamma_{\pi^1,F}\, \partial_j f_n \right) \right] \\
=&\pi_i^2(V_i) \sum_{\substack{j=1 \\ j>i}}^{n} \lambda_{ij}^n \, \left[\gamma_{\pi^1,F}\, (\pi_j^1(V_j) + (\pi_j^1)'(V_j)) + \pi_j^1(V_j)\, \partial_j \gamma_{\pi^1,F} \right] \qquad \text{(VII.4.9)} \\
+&\pi_i^2(V_i) \sum_{\substack{j=1 \\ j>i}}^{n} \xi_j^n \, \left[\partial_i \gamma_{\pi^1,F}\, (\pi_j^1(V_j) + (\pi_j^1)'(V_j)) + \pi_j^1(V_j)\, \partial_{ij}^2 \gamma_{\pi^1,F} \right] \,. \qquad \text{(VII.4.10)}
\end{aligned}$$

Let us look at the term (VII.4.9). Note that $\pi_i^2(V_i)\, \gamma_{\pi^2,F} \leq 1$. Using the ellipticity assumption (VII.3.6) and equation (VII.4.6), we find a polynom of $\lambda_{ij}^n, \theta_{j,1}^1, \theta_{j,2}^1$ and $\pi_j^1(V_j)$, denoted by $\tilde{\lambda}_{ij}^n$, which satisfies

$$\begin{aligned}
&|\text{(VII.4.9)} \times \gamma_{\pi^2,F}| \\
\leq &\sum_{\substack{j=1 \\ j>i}}^{n} \tilde{\lambda}_{ij}^n \, \left[|\gamma_{\pi^1,F}| \, \left(1 + |(\pi_j^1)'(V_j)| \right) + |\gamma_{\pi^1,F}|^2 \, \left(1 + |(\pi_j^1)'(V_j)| \right) \right] \\
\leq &C \sum_{\substack{j=1 \\ j>i}}^{n} \frac{\tilde{\lambda}_{ij}^n}{\sum_{m=1}^{n} \pi_m^1(V_m)} \left(1 + |(\pi_j^1)'(V_j)| + \frac{1 + |(\pi_j^1)'(V_j)|}{\sum_{m=1}^{n} \pi_m^1(V_m)} \right) \\
\leq &C \sum_{\substack{j=1 \\ j>i}}^{n} \tilde{\lambda}_{ij}^n \, \varepsilon_j^n \,.
\end{aligned}$$

Since $\tilde{\lambda}^n_{ij} \in \mathrm{L}_{(\infty)}(A)$ and $\varepsilon^n_j \in \mathrm{L}_{(1+)}(A)$ by Lemma VII.6, we get

$$\pi^2_i(V_i)\,\gamma_{\pi^2,F} \sum_{\substack{j=1\\ j>i}}^{n} \lambda^n_{ij}\,\left[\gamma_{\pi^1,F}\,(\pi^1_j(V_j) + (\pi^1_j)'(V_j)) + \pi^1_j(V_j)\,\partial_j\gamma_{\pi^1,F}\right] \in \mathrm{L}_{(1+)}(A \cap A_n)\,.$$

Let us look at the term (VII.4.10). Since $\pi^1_i(V_i) \times \pi^2_i(V_i) = 0$, we obtain from equation (VII.4.6),

$$\begin{aligned}\pi^2_i(V_i)\,\partial_i\gamma_{\pi^1,F} &= -\pi^2_i(V_i)\,\gamma^2_{\pi^1,F}\,(\theta^1_{i,1} + \theta^1_{i,2}\,(\pi^1_i)'(V_1)) = -\pi^2_i(V_i)\,\gamma^2_{\pi^1,F}\,\theta^1_{i,1}\\ &= -2\,\pi^2_i(V_i)\,\gamma^2_{\pi^1,F}\,\left(\sum_{m\geq 1}\pi^1_m(V_m)\,\partial_m f_m\,\partial^2_{im} f_m\right)\,.\end{aligned}$$

Hence, for $i \neq j$, we find a polynom $\tau^n_{ij} \in \mathrm{L}_{(\infty)}(A)$ such that

$$\pi^2_i(V_i)\,\partial^2_{ji}\gamma_{\pi^1,F} = \pi^2_i(V_i)\,\left((1 + (\pi^1_j)'(V_1))\,\gamma^3_{\pi^1,F} + (\pi^1_j)'(V_1)\,\gamma^2_{\pi^1,F}\right)\,.$$

Using the ellipticity assumption (VII.3.6) and the fact that $\pi^2_i(V_i)\,\gamma_{\pi^2,F} \leq 1$, we finally get

$$\begin{aligned}&|(\text{VII.4.10}) \times \gamma_{\pi^2,F}|\\ \leq& C\sum_{\substack{j=1\\ j>i}}^{n} \tilde{\xi}^n_{ij}\,\left((1 + (\pi^1_j)'(V_1))\,(\gamma^2_{\pi^1,F} + \gamma^3_{\pi^1,F})\right)\\ \leq& C\sum_{\substack{j=1\\ j>i}}^{n} \frac{\tilde{\xi}^n_{ij}}{\left(\sum\limits_{m=1}^{n}\pi^1_m(V_m)\right)^2}\left((1 + (\pi^1_j)'(V_1) + \frac{(1 + (\pi^1_j)'(V_1)}{\sum\limits_{m=1}^{n}\pi^1_m(V_m)}\right)\\ \leq& C\sum_{\substack{j=1\\ j>i}}^{n} \tilde{\xi}^n_{ij}\,\varepsilon^n_j\,,\end{aligned}$$

where $\tilde{\xi}^n_{ij} \in \mathrm{L}_{(\infty)}(A)$. And since $\varepsilon^n_j \in \mathrm{L}_{(1+)}(A)$ by Lemma VII.6, we obtain

$$\pi^2_i(V_i)\,\gamma_{\pi^2,F} \sum_{\substack{j=1\\ j>i}}^{n} \xi^n_j\,\left[\partial_i\gamma_{\pi^1,F}\,(\pi^1_j(V_j) + (\pi^1_j)'(V_j)) + \pi^1_j(V_j)\,\partial^2_{ij}\gamma_{\pi^1,F}\right] \in \mathrm{L}_{(1+)}(A)\,.$$

The proof is thus complete. ■

5. Applications

In this section, we present two kinds of application of the integration by parts formulas (VII.3.3) and (VII.3.11) : the study of the density of a random variable and the computation of conditional expectations.
We use the following notation in order to unify formulas (VII.3.3) and (VII.3.11) :

Notation: *Let us fix $A \in \mathcal{G}$. Let $F, G \in \mathrm{L}_{(\infty)}(A)$.*
We say that the $IP_A(F,G)$ (integration by parts) property holds true if there exists a random variable $H(F,G) \in \mathrm{L}_{(1+)}(A)$ such that for all $\phi \in \mathcal{C}^1_p(\mathbb{R})$,

$$\mathrm{E}\left(\phi'(F)\, G\, \mathbf{1}_A\right) = \mathrm{E}\left(\phi(F)\, H(F,G)\, \mathbf{1}_A\right) .$$

5.1. Density computation

Let $A \in \mathcal{G}$ be fixed. Since we have settled integration by parts formulas localized on A, we look at the law $(\mathbf{1}_A\, \mathbf{P})\, F^{-1}(dx)$, the image by a random variable F of the restriction of the Probability $\mathbf{P}$ on A, that is : for all measurable and bounded functions ϕ,

$$\mathrm{E}\left(\phi(F)\, \mathbf{1}_A\right) = \int_{\mathbb{R}} \phi(x)\, (\mathbf{1}_A\, \mathbf{P})\, F^{-1}(dx)\, .$$

Notation: *If $(\mathbf{1}_A\, \mathbf{P})\, F^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we denote $p_{F,A}$ its density. This means that, for all measurable and bounded functions ϕ,*

$$\mathrm{E}\left(\phi(F)\, \mathbf{1}_A\right) = \int_{\mathbb{R}} \phi(x)\, p_{F,A}(x)\, dx\, .$$

Let us study the existence of such a density :

Lemma VII.7:
Suppose that the $IP_A(F,1)$ property holds true.
Then, $(\mathbf{1}_A\, \mathbf{P})\, F^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with the following continuous *density $p_{F,A}$:*

$$p_{F,A}(x) = \mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-x)\, H(F,1)\, \mathbf{1}_A\right) .$$

Proof. (i). Let us introduce a regularization function.
Let ϕ be a smooth, symmetric and non-negative function with support in $[-1,1]$, and such that $\int_{\mathbb{R}} \phi(t)\, dt = 1$. Then, we consider for all $\delta > 0$, $\phi_\delta(x) = \frac{1}{\delta}\, \phi\left(\frac{x}{\delta}\right)$ and $\Phi_\delta(x) = \int_{-\infty}^{x} \phi_\delta(t)\, dt$. For all continuous and bounded functions ψ, we define

$\psi_\delta := \psi * \phi_\delta$. We then have

$$\lim_{\delta \to 0} \mathrm{E}\left(\psi_\delta(F) \mathbf{1}_A\right) = \mathrm{E}\left(\psi(F)\, \mathbf{1}_A\right) .$$

For all $\delta > 0$, we write

$$\mathrm{E}\left(\psi_\delta(F)\, \mathbf{1}_A\right) = \mathrm{E}\left(\int_{\mathbb{R}} \phi_\delta(F-y)\, \psi(y)\, dy\, \mathbf{1}_A\right) = \int_{\mathbb{R}} \psi(z)\, \mathrm{E}\left(\phi_\delta(F-z)\, \mathbf{1}_A\right)\, dz\,.$$

Noticing that $\Phi'_\delta = \phi_\delta$, the $IP_A(F,1)$ property gives

$$\mathrm{E}\left(\phi_\delta(F-z)\, \mathbf{1}_A\right) = \mathrm{E}\left(\Phi'_\delta(F-z)\, \mathbf{1}_A\right) = \mathrm{E}\left(\Phi_\delta(F-z)\, H(F,1)\, \mathbf{1}_A\right) .$$

Moreover, $\lim_{\delta \to 0} \Phi_\delta(x) = \tilde{\mathbf{1}}_{[0,\infty)}(x) = \mathbf{1}_{[0,\infty)}(x) + \delta_0(x)/2$. So, using the Lebesgue theorem we obtain

$$\lim_{\delta \to 0} \mathrm{E}\left(\psi_\delta(F)\, \mathbf{1}_A\right) = \int_{\mathbb{R}} \psi(z)\, \mathrm{E}\left(\tilde{\mathbf{1}}_{[0,\infty)}(F-z)\, H(F,1)\, \mathbf{1}_A\right)\, dz\,.$$

Hence, the law $(\mathbf{1}_A\, \mathbf{P})\, F^{-1}(dx)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, and its density has the following representation :

$$p_{F,A}(x) = \mathrm{E}\left(\tilde{\mathbf{1}}_{(0,\infty)}(F-x)\, H(F,1)\, \mathbf{1}_A\right) = \mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-x)\, H(F,1)\, \mathbf{1}_A\right) .$$

Moreover, since $H(F,1) \in \mathrm{L}_{(1+)}(A)$, the Lebesgue theorem proves that the density $p_{F,A}$ is continuous. ∎

Let us apply this abstract result to the framework of section 3. For that, we will consider two different cases :

- Case 1 : The conditional law of the random variables V_i given $\mathcal{G}_i = \mathcal{G} \vee \sigma(V_j, j \neq i)$ has no discontinuities, which means that it satisfies hypothesis VII.5. In this case, the non-degeneracy condition for a simple functional $F = f(\omega, \widetilde{V})$ is given by hypothesis VII.6, say

$$(H_q) \qquad \mathrm{E}\left((\det \gamma_F)^{4\,q}\, \mathbf{1}_A\right) < \infty, \text{ for some } q \geq 1\,.$$

- Case 2 : The conditional law of the random variables V_i given $\mathcal{G}_i$ has some singularities, this means that it satisfies hypothesis VII.2. Since we have introduced some weights $(\pi_i)_{i \in \mathbb{N}}$ to cancel the border terms coming from these singularities, the non-degeneracy condition corresponding this case is given by equation (VII.3.2) , say

$$\mathrm{E}\left[\mathbf{1}_A\, \left((\det \gamma_{\pi,F})^2\, (1 + |\pi'_l|\right)^{1+\eta}\right] < \infty, \text{ for some } \eta > 0\,.$$

Corollary VII.3:
Let $F = f(\omega, \widetilde{V}) \in \mathcal{S}_2(A)$, that is F and its first and second order derivatives have finite moments of any order on A.
Case 1 : suppose that the non-degeneracy condition VII.6 holds true.
Case 2 : suppose that the non-degeneracy condition (VII.3.2) holds true.
Then, $(\mathbf{1}_A \, \mathbf{P}) \, F^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with a continuous density $p_{F,A}$ given by

$$p_{F,A}(x) = \mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-x)\, H(F,1)\, \mathbf{1}_A\right) ,$$

where $H(F,1) \in \mathrm{L}^q(A)$ in Case 1 and $H_\pi(F,1) \in \mathrm{L}_{(1+)}(A)$ in Case 2.

Proof. Under hypothesis VII.6 and VII.4, Theorems VII.3 and VII.2 affirm that the $IP_A(F,1)$ property holds true. We can then apply Lemma VII.7 to conclude. ■

Let us study the regularity of this density. We fist give the following abstract result :

Lemma VII.8:
Suppose that we can iterate the $IP_A(F,1)$ property, which means that the $IP_A(F, H(F,1))$ property holds true.
Then, the density $p_{F,A} \in \mathcal{C}^1(\mathbb{R})$, and we have an explicit expression of its derivative :

$$p'_{F,A}(x) = -\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-x)\, H_2(F,1)\, \mathbf{1}_A\right) , \tag{VII.5.1}$$

where $H_2(F,1) := H(F, H(F,1))$.

Proof. Let us come back to the notation and the proof of Lemma VII.7.
We define $\Psi_\delta(x) := \int_{-\infty}^{x} \Phi_\delta(y)\, dy$, so that $\Psi''_\delta = \phi_\delta$.
Using the $IP_A(F, H(F,1))$ property, we get

$$\mathrm{E}\left(\phi_\delta(F-z)\, \mathbf{1}_A\right) = \mathrm{E}\left(\Phi_\delta(F-z)\, H(F,1)\, \mathbf{1}_A\right) = \mathrm{E}\left(\Psi_\delta(F-z)\, H_2(F,1)\, \mathbf{1}_A\right) .$$

Since $\lim_{\delta\to 0} \Psi_\delta(F-z) = (F-z)_+ := \max(F-z, 0)$, we obtain

$$\mathrm{E}(\psi(F)\, \mathbf{1}_A) = \int_{\mathbb{R}} \psi(z)\, \mathrm{E}\left((F-z)_+\, H_2(F,1)\, \mathbf{1}_A\right)\, dz \, .$$

And then $p_{F,A}(z) = \mathrm{E}\left((F-z)_+\, H_2(F,1)\, \mathbf{1}_A\right)$.
We thus derive a new representation of the density $p_{F,A}$, but here, the function $(z \to (F-z)_+)$ is differentiable. And since $H_2(F,1) \in \mathrm{L}_{(1+)}(A)$, we can differentiate inside the expectation, so that we get

$$p'_{F,A}(x) = -\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-x)\, H_2(F,1)\, \mathbf{1}_A\right) .$$

Let us apply Lemma VII.8 to our framework.
Case 1. If we suppose that the non-degeneracy condition (H_q) holds true for all $q \in \mathbb{N}$, then the random variable $H(F,1)$ coming from the $IP_A(F,1)$ property has finite moments of any order on A. Hence, we can iterate the $IP_A(F,1)$ property : Lemma VII.8 says that $p_{F,A} \in \mathcal{C}^1(\mathbb{R})$ and its first order derivative follows the expression (VII.5.1).
The main point is that $H_2(F,1) := H(F, H(F,1)) \in \mathrm{L}_{(\infty)}(A)$. Hence, we can in fact iterate the $IP_A(F,1)$ property as many times as we want, and straightforward computations (the same as in the standard Malliavin framework, see [Bal03]) give that $p_{F,A} \in \mathcal{C}^\infty(\mathbb{R})$, and

$$p_{F,A}^{(k)}(x) = (-1)^k \,\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-x)\, H_{k+1}(F,1)\, \mathbf{1}_A\right) ,$$

where $H_{k+1}(F,1)$ is defined by the recurrence relation :

$$H_0(F,1) = 1 \text{ and } H_{k+1}(F,1) = H(F, H_k(F,1)) \in \mathrm{L}_{(\infty)}(A)\,.$$

Case 2. The fundamental difference with the previous case comes from the weights $(\pi_i)_{i\in\mathbb{N}}$ that we have introduced to cancel the border terms. Indeed, the random variable $H_\pi(F,1)$ involves the derivatives of these weights, but we have $\pi_i' \in \mathrm{L}_{(1+)}(A)$. Hence, we can not reach finite moments of any order on A for $H_\pi(F,1)$. Moreover, as explained in section 4, we have to avoid the second order derivatives of the weights $\pi_i(\omega, V_i)$. Thus, iterating the $IP_A(F,1)$ property is more complex than in Case 1 : we have to consider two kinds of weights π^1 and π^2 with disjoint supports, and we have to verify that condition (VII.4.1) is satisfied.
Theorem VII.4 allows us to settle an iteration formula but under additional hypothesis on the number of random variables $(V_i)_{i\in\mathbb{N}}$ ($A = \bigcup_{n\geq 4} A \cap A_n$), on the simple functional $F = f(\omega, \widetilde{V})$ (ellipticity of ∂f) and on the weights π_i (independancy and hypothesis (VII.3.7)). Under these assumptions, Lemma VII.8 says that $p_{F,A} \in \mathcal{C}^1(\mathbb{R})$ and its first order derivative follows expression (VII.5.1).
But in this case, $H_2(F,1) := H_{\pi^2}(F, H_{\pi^1}(F,G)) \in \mathrm{L}_{(1+)}(A)$. Hence, for higher order derivatives, the iteration problem is more and more complex : if we want to iterate k times the $IP_A(F,1)$ property, we have to consider $k+1$ kinds of weights with disjoint supports, and we have to verify that condition (VII.4.1) is satisfied for each $H_i(F,1)$, $i = 1, \ldots, k+1$.
Let us summarize these results in the following corollary :

Corollary VII.4:
Case 1 : Let $F \in \mathcal{S}_n(A)$ for all $n \in \mathbb{N}$, that is F is infinitly differentiable, and F and its derivatives have finite moments of any order on A.
Suppose that γ_F has finite moments of any order on A.

Then, $p_{F,A} \in \mathcal{C}^\infty(\mathbb{R})$*, and*

$$p_{F,A}^{(k)}(x) = (-1)^k \,\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-x)\, H_{k+1}(F,1)\,\mathbf{1}_A\right) \,,$$

where $H_{k+1}(F,1)$ *is defined by the recurrence relation :*

$$H_0(F,1) = 1 \text{ and } H_{k+1}(F,1) = H(F, H_k(F,1)) \in \mathrm{L}_{(\infty)}(A)\,.$$

Case 2 : Suppose that $A = \bigcup_{n\geq 4} A \cap A_n$.
Let $F = f(\omega, \widetilde{V}) \in \mathcal{S}_3(A)$ *such that* f *satisfies the ellipticity assumption (VII.3.6). Suppose that the weights* π^1 *and* π^2 *satisfy hypothesis (VII.3.7), and that* $\pi^1(V_i)$ *and* $\pi^2(V_j)$ *and their first order derivatives are independent for* $i \neq j$.
Then, $p_{F,A} \in \mathcal{C}^1(\mathbb{R})$*, and*

$$p'_{F,A}(x) = -\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-x)\,\mathcal{H}_\pi(F,1)\,\mathbf{1}_A\right) \,,$$

where $\mathcal{H}_\pi(F,1) = H_{\pi^2}(F, H_{\pi^1}(F,1)) \in \mathrm{L}_{(1+)}(A)$.

5.2. Conditional expectations computation

We show in this section how the Malliavin integration by parts formulas (VII.3.3) and (VII.3.11) can be used to derive a representation formula for conditional expectations (see [BCZ03], [LR00]) :

Lemma VII.9:
Let us fix $A \in \mathcal{G}$*. Let us denote by* $\Theta_{G,A}(F) := \mathrm{E}(G\,\mathbf{1}_A \mid F)$ *the random variable which satisfies : for all measurable and bounded functions* ϕ

$$\mathrm{E}\left(\phi(F)\,G\,\mathbf{1}_A\right) = \mathrm{E}\left(\phi(F)\,\Theta_{G,A}(F)\right)\,.$$

Suppose that the $IP_A(F,1)$ *and* $IP_A(F,G)$ *properties hold true. Then we have*

$$\Theta_{G,A}(z) = \frac{\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-z)\,H(F,G)\,\mathbf{1}_A\right)}{\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-z)\,H(F,1)\,\mathbf{1}_A\right)}\,\mathbf{1}_A\,,$$

with the convention that the above quantity equals 0 *whenever*
$\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-z)\,H(F,1)\,\mathbf{1}_A\right) = 0$.

Proof. We have to check that for all bounded and measurable functions ψ, we have $\mathrm{E}\left(\psi(F)\,G\,\mathbf{1}_A\right) = \mathrm{E}\left(\psi(F)\,\Theta_{G,A}(F)\right)$.

Using the regularization function defined in the proof of Lemma VII.7, we obtain

$$\begin{aligned} \mathrm{E}\left(\psi(F)\, G\, \mathbf{1}_A\right) &= \lim_{\delta\to 0} \mathrm{E}\left(\psi_\delta(F)\, G\, \mathbf{1}_A\right) \\ &= \lim_{\delta\to 0} \int_{\mathbb{R}} \psi(z)\, \mathrm{E}\left(G\, \phi_\delta(F-z)\, \mathbf{1}_A\right)\, dz \\ &= \lim_{\delta\to 0} \int_{\mathbb{R}} \psi(z)\, \mathrm{E}\left(G\, \Phi'_\delta(F-z)\, \mathbf{1}_A\right)\, dz \\ &= \lim_{\delta\to 0} \int_{\mathbb{R}} \psi(z)\, \mathrm{E}\left(G\, \Phi_\delta(F-z)\, H(F,G)\, \mathbf{1}_A\right)\, dz\,, \end{aligned}$$

the last equality coming from the $IP_A(F,G)$ property. Since the $IP_A(F,1)$ property holds true, we know form Lemma VII.7 that the density $p_{F,A}$ exists. We thus obtain

$$\begin{aligned} \mathrm{E}\left(\psi(F)\, G\, \mathbf{1}_A\right) &= \int_{\mathbb{R}} \psi(z)\, \mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-z)\, H(F,G)\, \mathbf{1}_A\right)\, dz \\ &= \int_{\mathbb{R}} \psi(z)\, \Theta_{G,A}(z)\, \mathrm{E}\left(\mathbf{1}_{(0,\infty)}(F-z)\, H(F,1)\, \mathbf{1}_A\right)\, dz \\ &= \int_{\mathbb{R}} \psi(z)\, \Theta_{G,A}(z)\, p_{F,A}(z)\, dz \\ &= \mathrm{E}\left(\psi(F)\, \Theta_{G,A}(F)\, \mathbf{1}_A\right)\,. \end{aligned}$$

■

Application to pure jump processes VIII

Introduction

In this chapter, we apply the integration by parts formula (VII.3.3) derived in Theorem VII.2 to a pure jump diffusion process $(S_t)_{t\in[0,T]}$.
We use the notation from [IW89]. We consider a Poisson point measure $N(dt, da)$ on $\mathbb{R}$, with positive and finite intensity measure $\mu(da)\times dt$, that is $\mathrm{E}(N([0,t]\times A)) = \mu(A)\,t$. We denote J_t the counting process, that is $J_t := N([0,t]\times\mathbb{R})$, and we denote T_i, $i\in\mathbb{N}$, the jump times of J_t. We represent the above Poisson point measure by means of a sequence Δ_i, $i\in\mathbb{N}$, of independent random variables of law $\nu(da) = \mu(\mathbb{R})^{-1}\times\mu(da)$. This means that

$$N([0,t]\times A) = card\{T_i \le t : \Delta_i \in A\}\,.$$

We look at S_t solution of the following equation

$$\begin{aligned} S_t &= x + \sum_{i=1}^{J_t} c(T_i, \Delta_i, S_{T_i^-}) + \int_0^t g(r, S_r)\,dr\,, \qquad \text{(VIII.0.1)}\\ &= x + \int_0^t\int_{\mathbb{R}} c(s, a, S_{s^-})\,dN(s,a) + \int_0^t g(r, S_r)\,dr\,, \qquad 0\le t\le T\,. \end{aligned}$$

We work under the following hypothesis :

Hypothesis VIII.1. The functions $(a,x)\to c(t,a,x)$ and $x\to g(t,x)$ are twice differentiable and have bounded derivatives of first and second order. The function $t\to c(t,a,x)$ is differentiable with bounded derivative.
Moreover, we assume that there exists a positive constant K be such that

$$\begin{aligned} i)&\quad |c(t,a,x) - c(u,a,y)| \le K\,(|t-u| + |x-y|)\\ ii)&\quad |g(t,x) - g(u,y)| \le K\,(|t-u| + |x-y|)\\ iii)&\quad |c(t,a,x)| + |g(t,x)| \le K\,(1+|x|)\,. \end{aligned}$$

In the first section, we present the deterministic calculus which allows us to express S_t as a simple functional and to compute its Malliavin derivatives. In the following sections, we settle integration by parts formula with respect to jump amplitudes and to jump times separately, and to both of them. In the case of jump amplitudes, we iterate the integration by parts formula. Finally, in the last section, we apply these formulas to the study of the existence and the regularity of a density for S_t.

1. Deterministic equation

Let us fix an increasing sequence $u = (u_n)_{n\in\mathbb{N}}$ such that $u_0 = 0$. We also fix $a = (a_n)_{n\in\mathbb{N}}$, where $a_n \in \mathbb{R}$. To these fixed numbers we associate the deterministic equation

$$s_t = x + \sum_{i=1}^{J_t(u)} c(u_i, a_i, s_{u_i^-}) + \int_0^t g(r, s_r)\, dr\,, \quad 0 \le t \le T \tag{VIII.1.1}$$

where $J_t(u) = k$ if $u_k \le t < u_{k+1}$. We denote by $s_t(u, a)$ or simply by s_t the solution of this equation. This is the deterministic counterpart of the stochastic equation (VIII.0.1).
For all $t \in [0, T]$, on the set $\{J_t \ge 1\}$, the solution S_t of equation (VIII.0.1) is represented as

$$S_t = s_t(\widetilde{T}, \widetilde{\Delta}) = \sum_{n\ge1} s_t(T_1, \dots, T_n, \Delta_1, \dots, \Delta_n)\, \mathbf{1}_{\{J_t=n\}}\,, \tag{VIII.1.2}$$

where $\widetilde{T} := (T_i)_{i\in\mathbb{N}}$ and $\widetilde{\Delta} := (\Delta_i)_{i\in\mathbb{N}^*}$.
In order to solve equation (VIII.1.1), we introduce the flow $\Phi = \Phi_u(t, x)$, $0 \le u \le t$, $x \in \mathbb{R}$, solution of the following ordinary integral equation

$$\Phi_u(t, x) = x + \int_u^t g(r, \Phi_u(r, x))\, dr, \quad t \ge u\,.$$

The solution s of equation (VIII.1.1) is then given by

$$\begin{aligned} s_0 &= x\,, \\ s_t &= \Phi_{u_i}(t, s_{u_i}) \text{ for } u_i \le t < u_{i+1}\,, \\ s_{u_{i+1}} &= s_{u_{i+1}^-} + c(u_{i+1}, a_{i+1}, s_{u_{i+1}^-}) \\ &= \Phi_{u_i}(u_{i+1}, s_{u_i}) + c(u_{i+1}, a_{i+1}, \Phi_{u_i}(u_{i+1}, s_{u_i}))\,. \end{aligned} \tag{VIII.1.3}$$

Let us compute the derivatives of s with respect to u_j and a_j. We first introduce some notation.
We denote

$$e_{u,t}(x) := \exp\left(\int_u^t \partial_x g(r, \Phi_u(r, x))\, dr\right)\,.$$

Since $\Phi_{u_i}(r, s_{u_i}) = s_r$ for $u_i \le r < u_{i+1}$, we have

$$e_{u_i,t}(s_{u_i}) = \exp\left(\int_{u_i}^t \partial_x g(r, s_r)\, dr\right)\ , \text{ for } u_i \le t < u_{i+1}\,.$$

Since

$$\partial_x \Phi_u(t,x) = 1 + \int_u^t \partial_x g(r, \Phi_u(r,x))\, \partial_x \Phi_u(r,x)\, dr\,,$$

it follows that

$$\partial_x \Phi_u(t,x) = e_{u,t}(x)\,.$$

And since

$$\partial_u \Phi_u(t,x) = -g(u,x) + \int_u^t \partial_x g(r, \Phi_u(r,x))\, \partial_u \Phi_u(r,x)\, dr\,,$$

we have

$$\partial_u \Phi_u(t,x) = -g(u,x)\, e_{u,t}(x)\,.$$

We finally denote

$$q(t,\alpha,x) := (\partial_t c + g\, \partial_x c)(t,\alpha,x) + g(t,x) - g(t, x + c(t,\alpha,x))\,.$$

Lemma VIII.1:
Suppose that hypothesis VIII.1 holds true. Then $s_t(u,a)$ is twice differentiable with respect to u_j and a_j, and we have the following explicit expressions of the derivatives.

A. Derivatives with respect to $\mathbf{u_j}$.
For $t < u_j$, $\partial_{u_j} s_t(u,a) = 0$. Moreover,

$$\begin{aligned} \partial_{u_j} s_{u_j -} &= g(u_j, s_{u_j -})\,,\\ \partial_{u_j} s_{u_j} &= (\partial_t c + g\,(1 + \partial_x c))(u_j, a_j, s_{u_j -})\,. \end{aligned}$$

For $u_j < t < u_{j+1}$,

$$\partial_{u_j} s_t = q(u_j, a_j, s_{u_j -})\, e_{u_j,t}(s_{u_j})\,, \tag{VIII.1.4}$$

$$\begin{aligned} \partial_{u_j} s_{u_{j+1}-} &= q(u_j, a_j, s_{u_j -})\, e_{u_j,u_{j+1}}(s_{u_j})\\ \partial_{u_j} s_{u_{j+1}} &= q(u_j, a_j, s_{u_j -})\,(1 + \partial_x c(u_{j+1}, a_{j+1}, s_{u_{j+1}-}))\, e_{u_j,u_{j+1}}(s_{u_j})\,. \end{aligned}$$

Finally, for $p \ge j+1$ and $u_p \le t < u_{p+1}$, we have the recurrence relations

$$\partial_{u_j} s_t = e_{u_p,t}(s_{u_p})\, \partial_{u_j} s_{u_p}\,, \tag{VIII.1.5}$$

$$\partial_{u_j} s_{u_{p+1}} = (1 + \partial_x c(u_{p+1}, a_{p+1}, s_{u_{p+1}-}))\, e_{u_p,u_{p+1}}(s_{u_p})\, \partial_{u_j} s_{u_p}\,.$$

Let us denote $T(f) := \partial_t f + g\partial_x f$. *The second order derivatives are given by*

$$\begin{aligned}\partial^2_{u_j} s_{u_j-} &= T(g)(u_j, a_j, s_{u_j-})\,,\\ \partial^2_{u_j} s_{u_j} &= T(\partial_t c + g\,(1+\partial_x c))(u_j, a_j, s_{u_j-})\,.\end{aligned}$$

We denote

$$\begin{aligned}\rho_j(t) &= \partial_{u_j} e_{u_j,t}(s_{u_j})\\ &= e_{u_j,t}(s_{u_j})\left(-\partial_x g(u_j, s_{u_j}) + q(u_j, a_j, s_{u_j-})\int_{u_j}^t \partial_x^2 g(r, s_r)\, e_{u_j,r}(s_{u_j})\, dr\right)\,.\end{aligned}$$

Then, for $u_j < t < u_{j+1}$,

$$\partial^2_{u_j} s_t(u,a) = T(q)(u_j, a_j, s_{u_j-}(u,a))\, e_{u_j,t}(s_{u_j}) + q(u_j, a_j, s_{u_j-}(u,a))\, \rho_j(t)\,,$$

and

$$\begin{aligned}\partial^2_{u_j} s_{u_{j+1}} &= T(q)(u_j, a_j, s_{u_j-})\,(1+\partial_x c)(u_{j+1}, a_{j+1}, s_{u_{j+1}-})\, e_{u_j,u_{j+1}}(s_{u_j})\\ &\quad + q^2(u_j, a_j, s_{u_j-})\, \partial_x^2 c(u_{j+1}, a_{j+1}, s_{u_{j+1}-})\, e^2_{u_j,u_{j+1}}(s_{u_j})\\ &\quad + q(u_j, a_j, s_{u_j-})\,(1+\partial_x c)(u_{j+1}, a_{j+1}, s_{u_{j+1}-})\, \rho_j(u_j)\,.\end{aligned}$$

For $p \geq j+1$, *we denote*

$$\rho_{j,p}(t) = \partial_{u_j} e_{u_p,t}(s_{u_p}) = e_{u_p,t}(s_{u_p})\, \partial_{u_j} s_{u_p} \int_{u_p}^t \partial_x^2 g(r, s_r)\, e_{u_p,r}(s_{u_p})\, dr\,.$$

Then, for $p \geq j$ *and* $u_p \leq t < u_{p+1}$, *we have the recurrence relations*

$$\begin{aligned}\partial^2_{u_j} s_t &= e_{u_p,t}(s_{u_p})\, \partial^2_{u_j} s_{u_p} + \rho_{j,p}(t,u,a)\, \partial_{u_j} s_{u_p}\,,\\ \partial^2_{u_j} s_{u_{p+1}} &= \partial_x^2 c(u_{p+1}, a_{p+1}, s_{u_{p+1}-})\,(e_{u_p,u_{p+1}}(s_{u_p})\, \partial_{u_j} s_{u_p})^2\\ &+(1+\partial_x c)(u_{p+1}, a_{p+1}, s_{u_{p+1}-})\,(\rho_{j,p}(u_{p+1})\, \partial_{u_j} s_{u_p} + e_{u_p,u_{p+1}}(s_{u_p})\, \partial^2_{u_j} s_{u_p})\,.\end{aligned}$$

B. Derivatives with respect to $\mathbf{a_j}$.
For $t < u_j$, $\partial_{a_j} s_{u_j}(u,a) = 0$, *and for* $t \geq u_j$, $\partial_{a_j} s_t(u,a)$ *satisfies the following equation*

$$\partial_{a_j} s_t = \partial_a c(u_j, a_j, s_{u_j-}) + \sum_{i=j+1}^{J_t(u)} \partial_x c(u_i, a_i, s_{u_i-})\, \partial_{a_j} s_{u_i-} + \int_{u_j}^t \partial_x g(r, s_r)\, \partial_{a_j} s_r\, dr\,. \quad \text{(VIII.1.6)}$$

The second order derivatives are given by

$$\partial^2_{a_j} s_t = \partial^2_a c(u_j, a_j, s_{u_j-}) + \sum_{i=j+1}^{J_t(u)} \partial^2_x c(u_i, a_i, s_{u_i-}) \left(\partial_{a_j} s_{u_i-}\right)^2 \qquad \text{(VIII.1.7)}$$
$$+ \int_{u_j}^t \partial^2_x g(r, s_r) \left(\partial_{a_j} s_r\right)^2 dr$$
$$+ \sum_{i=j+1}^{J_t(u)} \partial_x c(u_i, a_i, s_{u_i-}) \, \partial^2_{a_j} s_{u_i-} + \int_{u_j}^t \partial_x g(r, s_r) \, \partial^2_{a_j} s_r \, dr \,,$$

and for $i < j$

$$\partial^2_{a_j, a_i} s_t = \partial^2_{a,x} c(u_j, a_j, s_{u_j^-}) + \sum_{k=j+1}^{J_t(u)} \partial^2_x c(u_k, a_k, s_{u_k^-}) \, \partial_{a_i} s_{u_k^-} \, \partial_{a_j} s_{u_k^-}$$
$$+ \sum_{k=j+1}^{J_t(u)} \partial_x c(u_k, a_k, s_{u_k^-}) \, \partial^2_{a_j, a_i} s_{u_k^-} + \int_{u_j}^t \partial_x g(r, s_r) \, \partial^2_{a_j, a_i} s_r \, dr$$
$$+ \int_{u_k}^t \partial^2_x g(r, s_r) \, \partial_{a_i} s_r \, \partial_{a_j} s_r \, dr \,.$$

For $i > j$, we derive $\partial^2_{a_j, a_i} s_t$ by symmetry.

Proof. It is clear that for $t < u_j$, s_t does not depend on u_j and so $\partial_{u_j} s_t = 0$. We now compute

$$\partial_{u_j} s_{u_j-} = \partial_{u_j} \Phi_{u_{j-1}}(u_j, s_{u_{j-1}}) = g\left(u_j, \Phi_{u_{j-1}}(u_j, s_{u_{j-1}})\right) = g(u_j, s_{u_j-}) \,.$$

Then,

$$\begin{aligned} \partial_{u_j} s_{u_j} &= \partial_{u_j} (s_{u_j-} + c(u_j, a_j, s_{u_j-})) \\ &= \partial_t c(u_j, a_j, s_{u_j-}) + (1 + \partial_x c(u_j, a_j, s_{u_j-})) \, \partial_{u_j} s_{u_j-} \\ &= \partial_t c(u_j, a_j, s_{u_j-}) + (1 + \partial_x c(u_j, a_j, s_{u_j-})) \, g(u_j, s_{u_j-}) \,. \end{aligned}$$

For $u_j < t < u_{j+1}$, we have

$$\begin{aligned} &\partial_{u_j} s_t \\ =&\partial_{u_j} \Phi_{u_j}(t, s_{u_j}) = e_{u_j,t}(s_{u_j}) \left(-g(u_j, s_{u_j}) + \partial_{u_j} s_{u_j}\right) \\ =&e_{u_j,t}(s_{u_j}) \left(-g(u_j, s_{u_j}) + \partial_t c(u_j, a_j, s_{u_j-}) + (1 + \partial_x c(u_j, a_j, s_{u_j-})) \, g(u_j, s_{u_j-})\right) \\ =&e_{u_j,t}(s_{u_j}) \, q(u_j, a_j, s_{u_j-}) \,. \end{aligned}$$

Similar computations give $\partial_{u_j} s_{u_{j+1}-} = e_{u_j, u_{j+1}}(s_{u_j}) \, q(u_j, a_j, s_{u_j-})$.

Finally,

$$\begin{aligned}\partial_{u_j} s_{u_{j+1}} &= (1+\partial_x c(u_{j+1}, a_{j+1}, s_{u_{j+1}-}))\, \partial_{u_j} s_{u_{j+1}-} \\ &= (1+\partial_x c(u_{j+1}, a_{j+1}, s_{u_{j+1}-}))\, e_{u_j,u_{j+1}}(s_{u_j})\, q(u_j, a_j, s_{u_j-})\,.\end{aligned}$$

We now assume that $u_p \le t < u_{p+1}$, $p \ge j+1$, and we write

$$\partial_{u_j} s_t = \partial_{u_j} \Phi_{u_p}(t, s_{u_p}) = e_{u_p,t}(s_{u_p})\, \partial_{u_j} s_{u_p}\,.$$

Same computations give $\partial_{u_j} s_{u_{p+1}-} = e_{u_p,u_{p+1}}(s_{u_p})\, \partial_{u_j} s_{u_p}$.
We finally have

$$\begin{aligned}\partial_{u_j} s_{u_p} &= \partial_{u_j}(s_{u_p-} + c(u_p, a_p, s_{u_p-})) \\ &= (1+\partial_x c(u_p, a_p, s_{u_p-}))\, \partial_{u_j} s_{u_p-} \\ &= (1+\partial_x c(u_p, a_p, s_{u_p-}))\, e_{u_{p-1},u_p}(s_{u_{p-1}})\, \partial_{u_j} s_{u_{p-1}}\,.\end{aligned}$$

The proof is then complete for the first order derivatives.
The relations concerning the second order derivatives are obtained by direct computations.
B. Using the recurrence relations (VIII.1.3), one verifies that for every $t \in [0,T]$, $(a_j \to s_t(u,a))$ is continuously differentiable and then one may differentiate in equation (VIII.1.1), which was not possible in the case of the derivatives with respect to u_j because these derivatives are not continuous. ■

As an immediate consequence of the above lemma we obtain :

Corollary VIII.1:
Suppose that hypothesis VIII.1 holds true and suppose that the starting point x satisfies $|x| \le K$, for some $K > 0$.
Then for each $n \in \mathbb{N}$ and $T > 0$, there exists a constant $C_n(K,T)$ such that for every $0 < u_1 < \ldots < u_n < T$, $a \in \mathbb{R}^n$ and $0 \le t \le T$,

$$\max_{j=1,\ldots,n} \left(|s_t| + \left|\partial_{u_j} s_t\right| + \left|\partial^2_{u_j} s_t\right| + \left|\partial_{a_j} s_t\right| + \left|\partial^2_{a_j} s_t\right| \right)(u,a) \le C_n(K,T)\,. \tag{VIII.1.8}$$

Finally, we give an useful corollary to control the non degeneracy.

Corollary VIII.2:
Assume that hypothesis VIII.1 holds true and there exists a constant $\eta > 0$ such that for every $(t,a,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$,

$$|1+\partial_x c(t,a,x)| \ge \eta \text{ and } |q(t,a,x)| \ge \eta\,. \tag{VIII.1.9}$$

Let $n \in \mathbb{N}$ be fixed. Then, there exists a constant $\varepsilon_n > 0$ such that for every

$j = 1, \dots, n$ and every $(u,a) \in [0,T]^n \times \mathbb{R}^n$,

$$\inf_{t>u_j} \left| \partial_{u_j} s_t(u,a) \right| \geq \varepsilon_n \,. \tag{VIII.1.10}$$

Proof. Since $\partial_x g$ is bounded, there exists a constant $C > 0$ such that $e_{s,t}(x) \geq e^{-CT}$ for $0 \leq s < t \leq T$. Using then equations (VIII.1.4) and (VIII.1.5), we conclude. ■

2. Formula based on jump amplitudes only

2.1. Locally smooth laws

In this section, we apply the integration by parts formula (VII.3.3) to the pure jump process $(S_t)_{t\in[0,T]}$, which will be regarded as a simple functional of the jump amplitudes $\Delta_i, i \in \mathbb{N}$. Using the notation of Chapter VII, we have $V_i = \Delta_i$. The randomness that we do not use is $\mathcal{G} = \sigma\{T_i : i \in \mathbb{N}\}$, and we put

$$A := \{J_t \geq 1\} \text{ and } A_n := \{J_t = n\}, n \geq 1 \,.$$

We assume that hypothesis VIII.1 and VII.1 (that is $\mathrm{E}\left(|\Delta_i|^p\right) < \infty$ for all $p \in \mathbb{N}$) hold true.

We consider some $q_0 < q_1 < \ldots < q_{k+1}$ and we denote $I = \bigcup_{i=0}^{k} (q_i, q_{i+1})$.

Since the random variables Δ_i are independent and identically distributed, hypothesis VII.2 becomes :

Hypothesis VIII.2. The law of Δ_i is absolutely continuous on I with respect to the Lebesgue measure and has the density $p(y) = e^{\rho(y)}$, that is

$$\mathrm{E}\left(f(\Delta_i)\, \mathbf{1}_I(\Delta_i)\right) = \int_I f(y)\, e^{\rho(y)}\, dy \,,$$

for every measurable and positive function f.

The function ρ is assumed to be continuously differentiable and bounded on I.

Since ρ is not differentiable on the whole $\mathbb{R}$, we work with the following weight. We take $\alpha \in (0,1)$ and $\beta > \alpha$ and we define

$$\pi(y) = \begin{cases} (q_{i+1} - y)^\alpha \, (y - q_i)^\alpha & , \text{for} \quad y \in (q_i, q_{i+1}) \,, \quad i = 0, \dots, k \,, \\ 0 & , \text{for} \quad y \in (q_0, q_{k+1})^c \,. \end{cases} \tag{VIII.2.1}$$

We make the following convention : if $b = q_{k+1} = +\infty$ or $a = q_0 = -\infty$, we define

$$\pi(y) = (y - q_k)^\alpha \, |y|^{-\beta} \,, \text{ for } y > q_k \text{ and } \pi(y) = (q_1 - y)^\alpha \, |y|^{-\beta} \,, \text{ for } y < q_1 \,.$$

Since ρ is bounded on I, elementary computations give that $\pi(\Delta_i) \in \mathrm{L}_{(\infty)}(A)$. Moreover, since $\alpha \in (0,1)$, we can choose $\eta > 0$ such that $(1-\alpha)(1+\eta) < 1$. We thus have

$$\mathrm{E}\left[|(\pi'(\Delta_i))|^{1+\eta}\,\mathbf{1}_A\right] \leq \sum_{i=0}^{k}\int_{q_i}^{q_{i+1}} \alpha\,\frac{(y-q_i)^{\alpha\,(1+\eta)}}{(q_{i+1}-y)^{(1-\alpha)\,(1+\eta)}}\,dy + \sum_{i=0}^{k}\int_{q_i}^{q_{i+1}} \alpha\,\frac{(q_{i+1}-y)^{\alpha\,(1+\eta)}}{(y-q_i)^{(1-\alpha)\,(1+\eta)}}\,dy < \infty\,.$$

That is $\pi'(\Delta_i) \in \mathrm{L}_{(1+)}(A)$. Hence, the weights π satisfy hypothesis VII.3.
In view of Corollary VIII.1, particulary equation (VIII.1.8), the function $(a_1,\dots,a_n) \to s_t(T_1(\omega),\dots,T_n(\omega),a_1,\dots,a_n)$ is twice continuously differentiable and has bounded derivatives, that is, using the notation of Chapter VII, $s_t \in \mathcal{C}_{n,2}(A\cap A_n)$.
Let us fix $M \in \mathbb{N}^*$ be such that there are M jumps on $[0,T]$, that is $J_T = M$. We denote

$$B_M := \{J_T = M\}\,.$$

It then follows from equation (VIII.1.2) that on $\{J_T = M\}$, for all $t \in [0,T]$,

$$S_t = \sum_{n=1}^{M} s_t(T_1,\dots,T_n,\Delta_1,\dots,\Delta_n)\,\mathbf{1}_{\{J_t=n\}}\,. \tag{VIII.2.2}$$

So $S_t \in S_2(A\cap B_M)$, that is S_t is a twice differentiable simple functional, such that S_t and its first and second order derivatives have finite moments of any order on $\{J_t \geq 1; J_T = M\}$.
The differential operators which appear in the integration by parts formula are

$$\begin{aligned}
D_iS_t &= \partial_{a_i}s_t(\widetilde{T},\widetilde{\Delta}) = \sum_{n=i}^{M}\partial_{a_i}s_t(T_1,\dots,T_n,\Delta_1,\dots,\Delta_n)\,\mathbf{1}_{\{J_t=n\}}\,,\\
\mathrm{L}_\pi S_t &= -\sum_{i=1}^{\infty}\pi(\Delta_i)\,\partial^2_{a_i}s_t(\widetilde{T},\widetilde{\Delta}) + (\pi' + \pi\,\frac{\rho'}{\rho})(\Delta_i)\,\partial_{a_i}s_t(\widetilde{T},\widetilde{\Delta})\,,\\
\sigma_{\pi,S_t} &= \sum_{i=1}^{\infty}\pi(\Delta_i)\,|D_iS_t|^2\\
&= \sum_{n=1}^{M}\sum_{i=1}^{n}\pi(\Delta_i)\,|\partial_{a_i}s_t(T_1,\dots,T_n,\Delta_1,\dots,\Delta_n)|^2\,\mathbf{1}_{\{J_t=n\}}\,, \qquad \text{(VIII.2.3)}\\
\gamma_{\pi,S_t} &= \frac{1}{\sigma_{\pi,S_t}} = \frac{1}{\sum_{i=1}^{\infty}\pi(\Delta_i)\,\left|\partial_{a_i}s_t(\widetilde{T},\widetilde{\Delta})\right|^2}\,.
\end{aligned}$$

All these quantities may be computed using equations (VIII.1.6) and (VIII.1.7).
As we want to apply the integration by parts formula (VII.3.3) to the process $(S_t)_{t\in[0,T]}$ following equation (VIII.0.1), we have to verify that the non degeneracy condition (VII.3.2) holds true. Let us give suitable conditions on the coefficient c of equation (VIII.0.1), allowing us to affirm that $(S_t)_{t\in[0,T]}$ satisfies the non-degeneracy condition (VII.3.2) :

Proposition VIII.1:
Suppose that hypothesis VIII.1 and VIII.2 hold true.
We assume that there exists a positive constant ϵ such that for every $(t,a,x) \in [0,T]\times\mathbb{R}\times\mathbb{R}$,

$$|\partial_a c(t,a,x)| \geq \epsilon \text{ and } |1+\partial_x c(t,a,x)| \geq \epsilon\,. \tag{VIII.2.4}$$

Take $\alpha \in (0,1/2)$ and $\beta > \alpha$ in the definition of the weights π.
Then, for all $t\in[0,T]$, S_t satisfies the non-degeneracy condition (VII.3.2) if there is at least one jump on $]0,t]$ and a finite number of jumps on $]0,T]$ (represented here by $M \geq 1$).

Proof. Since the jump amplitudes are independent, we will use Lemma VII.4. For that, we will prove that the deterministic process s_t satisfies the ellipticity assumption (VII.3.6) and that the weights π satisfy condition (VII.3.7).
Recall that in view of Remark 2.1, we write $\sum_{n=1}^{\infty} s_t\,\mathbf{1}_{A_n}$ for $\sum_{n=1}^{M} s_t\,\mathbf{1}_{A_n}$. Then, we have

$$|\partial_i s_t| = \left|\sum_{n=i}^{\infty} \partial_{a_i} s_t(\omega,a_1,\dots,a_n)\,\mathbf{1}_{\{J_t=n\}}\right| = \sum_{n=i}^{\infty} |\partial_{a_i} s_t(\omega,a_1,\dots,a_n)|\;\mathbf{1}_{\{J_t=n\}}\,.$$

Let us fix $1\leq n\leq M$. We compute $\partial_{a_i} s_t(\omega,a_1,\dots,a_n)$ on $\{J_t=n\}$, for $i\leq n$.
Equation (VIII.1.6) of Lemma VIII.1 gives

$$\partial_{a_n} s_t = \partial_a c(u_n,a_n,s_{u_n-}) + \int_{u_n}^{t} \partial_x g(r,s_r)\,\partial_{a_n} s_r\,dr\,.$$

So, using hypothesis (VIII.2.4) and the fact that $\partial_x g$ is bounded, we get

$$|\partial_{a_n} s_t| = |\partial_a c(u_n,a_n,s_{u_n-})|\,\exp\left(\int_{u_n}^{t} \partial_x g(r,s_r)\,dr\right) \geq C > 0\,.$$

Similarly, we have using equation (VIII.1.6),

$$\begin{aligned}
&\partial_{a_{n-1}} s_t \\
=&\partial_a c(u_{n-1}, a_{n-1}, s_{u_{n-1}^-}) + \partial_x c(u_n, a_n, s_{u_n^-})\, \partial_{a_{n-1}} s_{u_n^-} + \int_{u_{n-1}}^t \partial_x g(r, s_r)\, \partial_{a_{n-1}} s_r\, dr \\
=&\partial_a c(u_{n-1}, a_{n-1}, s_{u_{n-1}^-})\, (1 + \partial_x c(u_n, a_n, s_{u_n^-}))\, \exp\left(\int_{u_{n-1}}^t \partial_x g(r, s_r)\, dr \right) .
\end{aligned}$$

So $|\partial_{a_{n-1}} s_t| \geq C > 0$.
An inductive procedure gives that s_t satisfies the ellipticity assumption (VII.3.6) under hypothesis (VIII.2.4).
Since $\alpha \in (0, 1/2)$, we can choose $\delta > 0$ such that $2\,\alpha\,(1+\delta) < 1$, and ρ being bounded on I, we obtain

$$\mathrm{E}\left[|\pi(\Delta_i)|^{-2\,(1+\delta)} \right] \leq \sum_{i=0}^k \int_{q_i}^{q_{i+1}} \frac{dy}{(y - q_i)^{2\,(1+\delta)\,\alpha}\, (q_{i+1} - y)^{2\,(1+\delta)\,\alpha}} < \infty \,.$$

Hence, the weights π satisfy hypothesis (VII.3.7).
Finally, by Lemma VII.4, we obtain that the non degeneracy condition (VII.3.2) holds true on $\{J_t \geq 1; J_T = M\}$. The proof is thus complete. ■

Remark 2.1. *Note that this proof allows us to settle the following properties :*
- *If hypothesis (VIII.2.4) holds true, then S_t satisfies the ellipticity assumption (VII.3.6).*
- *If we take $\alpha \in (0, 1/q)$, $q \geq 1$, in the definition of the weights π, then, there exists $\eta > 0$ such that*
$$\mathrm{E}\left[|\pi(\Delta_i)|^{-q\,(1+\eta)} \right] < \infty \,.$$

By Proposition VIII.1, one may apply integration by parts formula of type (VII.3.3) to $(S_t)_{t \in [0,T]}$ on $\{J_t \geq 1; J_T = M\}$ if hypothesis (VIII.2.4) is satisfied. Let us give a particular example (which will be used in the Sensitivity analysis, see Chapter IX) :

Corollary VIII.3:
Suppose that hypothesis VIII.1 and VIII.2 hold true.
We assume that hypothesis (VIII.2.4) is satisfied.
Take $\alpha \in (0, 1/2)$ and $\beta > \alpha$ in the definition of the weights π.
Then, for every function $\phi \in \mathcal{C}_p^1(\mathbb{R})$, for all $t \in [0, T]$, we have

$$\mathrm{E}(\phi'(S_t)\, \partial_x S_t\, \mathbf{1}_{\{J_t \geq 1; J_T = M\}}) = \mathrm{E}(\phi(S_t)\, H_\pi(S_t, \partial_x S_t)\, \mathbf{1}_{\{J_t \geq 1; J_T = M\}}) \,, \qquad \text{(VIII.2.5)}$$

where $H_\pi(S_t, \partial_x S_t) \in \mathrm{L}_{(1+)}(A \cap B_M)$, $A = \{J_t \geq 1\}$ and $B_M = \{J_T = M\}$, and is *given by*

$$H_\pi(S_t, \partial_x S_t) = \partial_x S_t\, \gamma_{\pi,S_t}\, \mathrm{L}_\pi S_t - \gamma_{\pi,S_t} < DS_t, D(\partial_x S_t) >_\pi \\ - \partial_x S_t < DS_t, D\gamma_{\pi,S_t} >_\pi \,. \quad \text{(VIII.2.6)}$$

Proof. We already know that $S_t \in \mathcal{S}_2(A \cap B_M)$. Then, in order to apply Theorem VII.2, we have to verify that $\partial_x S_t \in \mathcal{S}_1(A \cap B_M)$.
We have $\partial_x S_t = \partial_x s_t(\widetilde{T}, \widetilde{\Delta})$ and, using the deterministic equation (VIII.1.1), $\partial_x s_t$ is computed by the recurrence relations :

$$\partial_x s_0 = 1\,, \quad \text{(VIII.2.7)}$$
$$\partial_x s_t = (1 + \partial_x c(u_i, a_i, s_{u_i-}))\, \partial_x s_{u_i-} + \int_{u_i}^t \partial_x g(r, s_r)\, \partial_x s_r\, dr\,, \quad u_i \le t < u_{i+1}\,.$$

Then, it is easy to check that $\partial_x s_t$ and its derivatives with respect to a_i are bounded on A, and consequently, $\partial_x S_t \in \mathcal{S}_1(A \cap B_M)$. ■

2.2. Smooth laws

In this section, the law of the jump amplitudes are supposed to have no discontinuities. Using the notation of the previous section, we have $V_i = \Delta_i$, $\mathcal{G} = \sigma\{T_i : i \in \mathbb{N}\}$, but $I = \mathbb{R}$. Hypothesis VII.2 becomes

Hypothesis VIII.3. The law of Δ_i is absolutely continuous on $\mathbb{R}$ with respect to the Lebesgue measure and has density p, that is

$$\mathrm{E}\left(f(\Delta_i)\right) = \int_{\mathbb{R}} f(y)\, p(y)\, dy\,,$$

for every measurable and positive function f.
p is assumed to be continuously differentiable, and be such that $\dfrac{p'}{p} \in \mathcal{C}^0_p(\mathbb{R})$, and for all $k \in \mathbb{N}$, $\lim_{y \to \pm\infty} |y|^k\, p(y) = 0$.

As in the previous section, we denote $A = \{J_t \ge 1\}$ and $B_M = \{J_T = M\}$. Recall that for all $t \in [0, T]$, $S_t \in \mathcal{S}_2(A \cap B_M)$, that is S_t and its first and second order derivatives have finite of any moments on $\{J_t \ge 1; J_T = M\}$. Similarly, we have $\partial_x S_t \in \mathcal{S}_1(A \cap B_M)$ (see equation (VIII.2.7)).
We are now in the framework of Chapter VII-section 3.2 where we do not need any

weights π, so that the Malliavin operators are :

$$\begin{aligned}
D_i S_t &= \partial_{a_i} s_t(\widetilde{T}, \widetilde{\Delta}) = \sum_{n=i}^{\infty} \partial_{a_i} s_t(T_1, \dots, T_n, \Delta_1, \dots, \Delta_n)\, \mathbf{1}_{\{J_t = n\}}\,, \\
\mathrm{L} S_t &= -\sum_{i=1}^{\infty} \partial^2_{a_i} s_t(\widetilde{T}, \widetilde{\Delta}) + \frac{p'}{p}(\Delta_i)\, \partial_{a_i} s_t(\widetilde{T}, \widetilde{\Delta})\,, \\
\sigma_{S_t} &= \sum_{i=1}^{\infty} |D_i S_t|^2 = \sum_{n=1}^{\infty}\sum_{i=1}^{n} |\partial_{a_i} s_t(T_1, \dots, T_n, \Delta_1, \dots, \Delta_n)|^2\ \mathbf{1}_{\{J_t = n\}}\,, \\
\gamma_{S_t} &= \frac{1}{\sigma_{S_t}} = \frac{1}{\sum\limits_{i=1}^{\infty} \left|\partial_{a_i} s_t(\widetilde{T}, \widetilde{\Delta})\right|^2}\,.
\end{aligned}$$

All these quantities may be computed using Lemma VIII.1. Since there are no weights, Theorem VII.3 implies that the integration by parts formula (VII.3.11) holds true under the non-degeneracy condition VII.6.

Proposition VIII.2:
Suppose that hypothesis VIII.1 holds true.
We assume that there exists a positive constant ϵ such that for all $(t, a, x) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$|\partial_a c(t, a, x)| \geq \epsilon > 0\,. \tag{VIII.2.8}$$

Then, S_t satisfies the non-degeneracy condition VII.6, more precisely condition (H_q) for all $q \in \mathbb{N}$, if there is at least one jump on $]0, t]$ and a finite number of jumps on $]0, T]$ (represented here by $M \geq 1$).

Proof. Let us verify that the non degeneracy condition (H_q) holds true for all $q \in \mathbb{N}$, that is

$$\mathrm{E}\left((\det \gamma_{S_t})^{4\,q}\, \mathbf{1}_{\{J_t \geq 1; J_T = M\}}\right) < \infty\,.$$

For all $1 \leq n \leq M$, on $\{J_t = n\}$, we have

$$\sigma_{S_t} = \sum_{i=1}^{n} |\partial_{a_i} s_t(t_1, \dots, t_n, \Delta_1, \dots, \Delta_n)|^2 \geq |\partial_{a_n} s_t(T_1, \dots, T_n, \Delta_1, \dots, \Delta_n)|^2\,.$$

Using equation (VIII.1.6) of Lemma VIII.1, we have
$\partial_{a_n} s_t = \partial_a c(t_n, a_n, s_{t_n^-}) + \int_{t_n}^{t} \partial_x g(r, s_r)\, \partial_{a_n} s_r\, dr$, and then

$$|\partial_{a_n} s_t| = \left|\partial_a c(t_n, a_n, s_{t_n^-})\right|\, \exp\left(\int_{t_n}^{t} \partial_x g(r, s_r)\, dr\right) \geq C > 0\,.$$

Hence, the non degeneracy condition (H_q) holds true for all $q \in \mathbb{N}$ on $\{J_t \geq 1; J_T = M\}$. ■

Corollary VIII.4:
Suppose that hypothesis VIII.1 and hypothesis (VIII.2.8) are satisfied.
Then, for every function $\phi \in \mathcal{C}^1_p(\mathbb{R})$, for all $t \in [0,T]$, we have

$$\mathrm{E}(\phi'(S_t)\,\partial_x S_t\,\mathbf{1}_{\{J_t\geq 1;J_T=M\}}) = \mathrm{E}(\phi(S_t)\,H(S_t,\partial_x S_t)\,\mathbf{1}_{\{J_t\geq 1;J_T=M\}})\,,$$

where $H(S_t,\partial_x S_t) \in \mathrm{L}_{(\infty)}(A\cap B_M)$, $A = \{J_t \geq 1\}$ and $B_M = \{J_T = M\}$, is given by

$$H(S_t,\partial_x S_t) = \partial_x S_t\,\gamma_{S_t}\,\mathrm{L}S_t - \gamma_{S_t} < DS_t, D(\partial_x S_t) > -\partial_x S_t < DS_t, D\gamma_{S_t} > \,.$$

Proof. Since S_t satisfies hypothesis VII.6, we can apply Theorem VII.2 to $F = S_t$ and $G = \partial_x S_t$ on $A = \{J_t \geq 1; J_T = M\}$: the integration by parts formula (VII.3.11) gives the result. ■

3. Iteration formula based on jump amplitudes only

In view of conditional expectations computation (which appear in the pricing and hedging problems for American options, see Chapter X), the aim of this section is to settle (and to iterate) the following formula : for ϕ, $\psi \in \mathcal{C}^1_p(\mathbb{R})$,

$$\mathrm{E}\left[\phi'(S_s)\,\psi(S_t)\,\mathbf{1}_A\right] = \mathrm{E}\left[\phi(S_s)\,\psi(S_t)\,H(S_s,S_t)\,\mathbf{1}_A\right]\,, \tag{VIII.3.1}$$

where A and $H(S_s,S_t)$ have to be precised, and $H(S_s,S_t)$ does not depend on the functions ψ and ψ'.
If we use the integration by parts formula (VIII.2.5) by replacing $\partial_x S_t$ by $\psi(S_t)$, the Malliavin weight obtained in equation (VIII.2.6) involves the Malliavin derivative $D(\psi(S_t))$, and then $\psi'(S_t)$. To avoid this term, we will apply again (VIII.2.5) in a suitable way. Let us be more precise.
We assume the framework detailed in section 2.1, that is hypothesis VIII.1, VII.1 and VII.2 are satisfied.
To simplify notation, we work here on $I = (\alpha,\beta)$. It then sufficies to put $(\alpha,\beta) = (q_i,q_{i+1})$, $i = 0,\ldots,k$, to have the results of this section on $I = \bigcup_{i=0}^{k}(q_i,q_{i+1})$.
Let us denote $A_t = \{J_t \geq 1\}$ and recall that $B_M = \{J_T = M\}$. We know from section 2.1 that for all $t \in [0,T]$, $S_t \in \mathcal{S}_2(A_t \cap B_M)$, that is S_t and its first and second order derivatives have finite moments of any order on $\{J_t \geq 1; J_T = M\}$. And similarly, $\partial_x S_t \in \mathcal{S}_1(A_t \cap B_M)$ (see equation (VIII.2.7)).
Let us choose the weights $(\pi_i(\omega,\Delta_i))_{i\in\mathbb{N}}$. Let $0 \leq s < t \leq T$. We suppose that there is at least one jump on $]s,t]$, that is $J_s < J_t$.
In order to iterate the integration by parts formula (VIII.3.1), we split the interval I in two disjoint sets (see Chapter VII, section 4). Let us define $\gamma := \dfrac{\alpha+\beta}{2}$, then we

have a partition of (α,β) : $(\alpha,\beta) = B_1 \cup B_2$, where $B_1 = (\alpha,\gamma]$ and $B_2 = (\gamma,\beta)$ are disjoint sets.
Taking $\delta \in (0,1/3)$, we define for all $i \in \mathbb{N}$, $k = 1,2$

$$\pi^i_{B_k,s,t}(\omega,\Delta_i) := \mathbf{1}_{]s,t]}(T_i(\omega)) \times \pi_k(\Delta_i)\,, \qquad \text{(VIII.3.2)}$$

where π_1 and π_2 are such that $Supp\ \pi_1 \subseteq B_1$ and $Supp\ \pi_2 \subseteq B_2$, are defined by :

$$\pi_1(y) := \begin{cases} (\gamma-y)^\delta\,(y-\alpha)^\delta & \text{for } y \in B_1 \\ 0 & \text{for } y \notin B_1\,, \end{cases}$$

and

$$\pi_2(y) := \begin{cases} (\beta-y)^\delta\,(y-\gamma)^\delta & \text{for } y \in B_2 \\ 0 & \text{for } y \notin B_2\,. \end{cases}$$

Note that the indicative function $\mathbf{1}_{]s,t]}(T_i)$ allows us to settle a calculus involving the jumps occuring between s and t only.
Finally, we assume that hypothesis (VIII.2.4) holds true, that is : there exists a positive constant ε such that for all u, a, x

$$|\partial_a c(u,a,x)| \geq \varepsilon \text{ and } |1 + \partial_x c(u,a,x)| \geq \varepsilon\,.$$

Hence, Proposition VIII.1 implies that the non degeneracy condition (VII.3.2) holds true on $\{J_t \geq 1; J_T = M\}$, so that we can perform an integration by parts formula on $\{J_t \geq 1; J_T = M\}$, using indifferently the weights $\pi_{B_1,s,t}$ or $\pi_{B_2,s,t}$. In the following, we will use the weights $\pi_{B_1,s,t}$ in the first integration by parts formula.
Moreover, Remark 2.1 says that $|\partial_{a_i} S_t| \geq \zeta > 0$, and since $\delta \in (0,1/3)$,

$$\mathrm{E}\left[|\pi_k(\Delta_i)|^{-3\,(1+\eta)}\right] < \infty\,,\ \text{for some } \eta > 0\,.$$

Hence, Theorem VII.4 allows us to iterate the integration by parts formula on $\{J_t \geq 4; ; J_T = M\}$, using the weights $\pi_{B_2,s,t}$ (since we have used $\pi_{B_1,s,t}$ in the first formula).
In the following, we use the triplet (k,s,t), $k = 1,2$, $0 \leq s < t$, in order to indicate that the Malliavin operators are associated to the inner product $\langle .,. \rangle_{\pi_{B_k,s,t}}$. Then we have the following notation :
• The inner product $\langle .,. \rangle_{(k,s,t)}$: for all U, $V \in \mathcal{P}_0$,

$$\langle U,V \rangle_{(k,s,t)} = \sum_{i=1}^{\infty} \mathbf{1}_{]s,t]}(T_i(\omega))\,\pi_k(\Delta_i)\,(u_i\,v_i)(\widetilde{T},\widetilde{\Delta})\,.$$

• The Ornstein Uhlenbeck operator $\mathrm{L}_{(k,s,t)}$: for all $F \in \mathcal{S}_2$, $F = f(\omega, \widetilde{T}, \widetilde{\Delta})$,

$$\mathrm{L}_{(k,s,t)}(F) = -\sum_{i=1}^{\infty} \mathbf{1}_{]s,t]}(T_i) \times \Big[\pi_k(\Delta_i)\, \partial_i^2 f(\omega, \widetilde{T}, \widetilde{\Delta}) \\ +(\pi_k'(\Delta_i) + (\pi_k\, \rho')(\Delta_i))\, \partial_i f(\omega, \widetilde{T}, \widetilde{\Delta})\Big] \,.$$

• The covariance matrix $\left[\sigma_t^{(k,s,t)}\right]_{ij} := \left[\sigma_{S_t}^{(k,s,t)}\right]_{ij} = \langle DS_t^i, DS_t^j\rangle_{(k,s,t)}$.
Let us introduce the operators which will appear in the weight $H(S_s, S_t)$ of equation (VIII.3.1).

Notation: *For $s < t$ and $k = 1, 2$, we denote*

$$U_t^{(k,s,t)} := \gamma_t^{(k,s,t)}\, \mathrm{L}_{(k,s,t)} S_t - \langle DS_t, D\gamma_t^{(k,s,t)}\rangle_{(k,s,t)} \,, \tag{VIII.3.3}$$

$$V_{(k,s,t)} := U_s^{(k,0,s)} - \gamma_s^{(k,0,s)}\, \langle DS_s, DS_t\rangle_{(k,0,s)}\, U_t^{(k,s,t)} \\ + \frac{1}{2}\, \gamma_s^{(k,0,s)}\, \gamma_t^{(k,s,t)}\, \langle DS_s, D\sigma_t^{k,s,t)}\rangle_{(k,0,s)} \,, \tag{VIII.3.4}$$

and

$$\mathcal{H}_{s,t} = V_{(1,s,t)}\, V_{(2,s,t)} + \gamma_s^{(2,0,s)} \\ \times \Big[\gamma_t^{(2,s,t)}\, \langle DS_s, DS_t\rangle_{(2,0,s)}\, \langle DS_t, D(V_{(1,s,t)})\rangle_{(2,s,t)} - \langle DS_s, D(V_{(1,s,t)})\rangle_{(2,0,s)}\Big] \,. \tag{VIII.3.5}$$

Let us finally denote

$$A_{s,t} := \{0 < J_s < J_t; J_T = M\} \text{ and } B_{s,t} := \{3 < J_s; 3 < J_t - J_s; J_T = M\} \,.$$

Proposition VIII.3:
Let $0 < s < t \leq T$. Let $\psi \in \mathcal{C}_p^1(\mathbb{R})$.
(i) For all $\phi \in \mathcal{C}_p^1(\mathbb{R})$, we have

$$\mathrm{E}(\phi'(S_s)\, \psi(S_t)\, \mathbf{1}_{\{0<J_s<J_t;J_T=M\}}) = \mathrm{E}(\phi(S_s)\, \psi(S_t)\, V_{(1,s,t)}\, \mathbf{1}_{\{0<J_s<J_t;J_T=M\}}) \,,$$

where $V_{(1,s,t)} \in \mathrm{L}_{(1+)}(A_{s,t})$ is defined by equation (VIII.3.4).
(ii) For all $\phi \in \mathcal{C}_p^1(\mathbb{R})$, we have

$$\mathrm{E}(\phi'(S_s)\, \psi(S_t)\, V_{(1,s,t)}\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}) = \mathrm{E}(\phi(S_s)\, \psi(S_t)\, \mathcal{H}_{s,t}\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}) \,,$$

where $\mathcal{H}_{s,t} \in \mathrm{L}_{(1+)}(B_{s,t})$ is defined by equation (VIII.3.5).

Proof. (i) The first step consists in removing the derivative of ϕ in the expectation $\mathrm{E}(\phi'(S_s)\,\psi(S_t)\,\mathbf{1}_{\{0<J_s<J_t;J_T=M\}})$. Since S_s involves the jump amplitudes falling in $]0,s]$, we use this randomness in an integration by parts formula. This means that we do not take into account the jumps occuring in $]s,t]$. We thus apply Theorem VII.2 (particulary equation (VII.3.3)) to $F = S_s$, $G = \psi(S_t)$ and $A = A_{s,t}$, using the weights $\pi^i_{B_1,0,s}(\omega,\Delta_i) = \mathbf{1}_{]0,s]}(T_i(\omega)) \times \pi_1(\Delta_i)$. And we obtain

$$\mathrm{E}(\phi'(S_s)\,\psi(S_t)\,\mathbf{1}_{A_{s,t}}) = \mathrm{E}(\phi(S_s)\,H_1(S_s,S_t)\,\mathbf{1}_{A_{s,t}})\,, \tag{VIII.3.6}$$

with

$$H_1(S_s,S_t) = \psi(S_t)\,\left(\gamma_s^{(1,0,s)}\,L_{(1,0,s)}(S_s) - \langle DS_s, D\gamma_s^{(1,0,s)}\rangle_{(1,0,s)}\right) \\ - \gamma_s^{(1,0,s)}\,\psi'(S_t)\,\langle DS_s, DS_t\rangle_{(1,0,s)}\,. \tag{VIII.3.7}$$

Note that by taking the weights $\pi^i_{B_1,0,s}$ in equation (VIII.3.6), we also select the jump amplitudes which belong to B_1.
We now get rid of the derivative of ψ. So we consider the following expectation

$$\mathrm{E}\left(\phi(S_s)\,\gamma_s^{(1,0,s)}\,\psi'(S_t)\,\langle DS_s, DS_t\rangle_{(1,0,s)}\,\mathbf{1}_{A_{s,t}}\right)\,.$$

The point is that the the derivative of ϕ should not appear in the integration by parts formula. This means that we must not use the jumps on $]0,s]$. As S_t involves the jump amplitudes falling in $]0,t]$, we thus take these falling in $]s,t]$ (which is possible since there is at least one jump on $]s,t]$). Hence, we apply again Theorem VII.2 using the weights $\pi^i_{B_1,s,t}(\omega,\Delta_i) = \mathbf{1}_{]s,t]}(T_i(\omega)) \times \pi_1(\Delta_i)$.
Since $\phi(S_s)\,\gamma_s^{(1,0,s)}$ does not depend on the jumps of $]s,t]$, we obtain

$$\mathrm{E}\left(\phi(S_s)\,\psi'(S_t)\,\gamma_s^{(1,0,s)}\,\langle DS_s, DS_t\rangle_{(1,0,s)}\,\mathbf{1}_{A_{s,t}}\right) = \mathrm{E}\left(g(S_t)\,\mathcal{H}_1(S_s,S_t)\,\mathbf{1}_{A_{s,t}}\right)\,,$$

where

$$\begin{aligned}\mathcal{H}_1(S_s,S_t) &= \phi(S_s)\,\gamma_s^{(1,0,s)}\,\Big[\langle DS_s, DS_t\rangle_{(1,0,s)}\,\gamma_t^{(1,s,t)}\,\mathrm{L}_{(1,s,t)}(S_t) \\ &\qquad - \langle D\left(\langle DS_s, DS_t\rangle_{(1,0,s)}\,\gamma_t^{(1,s,t)}\right), DS_t\rangle_{(1,s,t)}\Big] \\ &= \phi(S_s)\,\gamma_s^{(1,0,s)}\,\langle DS_s, DS_t\rangle_{(1,0,s)}\,\left(\gamma_t^{(1,s,t)}\,\mathrm{L}_{(1,s,t)}(S_t) - \langle D\gamma_t^{(1,s,t)}, DS_t\rangle_{(1,s,t)}\right) \\ &\qquad - \gamma_t^{(1,s,t)}\,\phi(S_s)\,\gamma_s^{(1,0,s)}\,\langle D\left(\langle DS_s, DS_t\rangle_{(1,0,s)}\right), DS_t\rangle_{(1,s,t)}\,.\end{aligned}$$

Since DS_s do not depend on the jumps of $]s,t]$, we have

$$\begin{aligned}
&\langle D\left(\langle DS_s, DS_t\rangle_{(1,0,s)}\right), DS_t\rangle_{(1,s,t)} \\
&\qquad = \sum_{i,j=1}^{\infty} \pi^i_{B_1,0,s}(\Delta_i)\, \pi^j_{B_1,s,t}(\Delta_j)\, D_iS_s\, D_jS_t\, D^2_{ij}S_t \\
&\qquad = \sum_{i,j=1}^{\infty} \pi^i_{B_1,0,s}(\Delta_i)\, \pi^j_{B_1,s,t}(\Delta_j)\, D_iS_s \times \frac{1}{2}\, D_i\left(D_jS_t^2\right) \\
&\qquad = \sum_{i=1}^{\infty} \pi^i_{B_1,0,s}(\Delta_i)\, D_iS_s \times \frac{1}{2}\, D_i\sigma_t^{(1,s,t)} \\
&\qquad = \frac{1}{2}\, \langle DS_s, D\sigma_t^{(1,s,t)}\rangle_{(1,0,s)}\,.
\end{aligned}$$

So

$$\begin{aligned}
&\mathcal{H}_1(S_s,S_t) \\
&\quad = \phi(S_s)\, \gamma_s^{(1,0,s)}\, \langle DS_s, DS_t\rangle_{(1,0,s)}\, \left(\gamma_t^{(1,s,t)}\, \mathrm{L}_{(1,s,t)}(S_t) - \langle D\gamma_t^{(1,s,t)}, DS_t\rangle_{(1,s,t)}\right) \\
&\qquad\qquad - \frac{1}{2}\, \gamma_t^{(1,s,t)}\, \gamma_s^{(1,0,s)}\, \phi(S_s)\, \langle DS_s, D\sigma_t^{(1,s,t)}\rangle_{(1,0,s)}\,. \qquad \text{(VIII.3.8)}
\end{aligned}$$

We plug the results (VIII.3.7) and (VIII.3.8) in equation (VIII.3.6) to finally obtain

$$\mathrm{E}(\phi'(S_s)\, \psi(S_t)\, \mathbf{1}_{A_{s,t}}) = \mathrm{E}(\phi(S_s)\, \psi(S_t)\, V_{(1,s,t)}\, \mathbf{1}_{A_{s,t}})\,.$$

(ii) We now iterate the previous integration by parts formula. In view of Theorem VII.4, recall that there will be two changes :
$*$ We need at least four jumps on $]0,s]$ and at least four jumps on $]s,t]$. So we will localize on $B_{s,t} = \{3 < J_s; 3 < J_t - J_s; J_T = M\}$.
$*$ In order to cancel the second order derivatives of $\pi_{B_1,0,s}$ and $\pi_{B_1,s,t}$, we will perform the second integration by parts formula using the weights $\pi_{B_2,0,s}$ and $\pi_{B_2,s,t}$.
This gives, using Theorem VII.4 with the weights $\pi_{B_2,0,s}$:

$$\begin{aligned}
\mathrm{E}\left(\phi'(S_s)\, \psi(S_t)\, V_{(1,s,t)}\, \mathbf{1}_{B_{s,t}}\right) &= \mathrm{E}\left(\phi(S_s)\, \psi(S_t)\, V_{(1,s,t)}\, U_s^{(2,0,s)}\, \mathbf{1}_{B_{s,t}}\right) \\
&\quad - \mathrm{E}\left(\phi(S_s)\, \gamma_s^{(2,0,s)}\, \psi(S_t)\, \langle DS_s, D(V_{(1,s,t)})\rangle_{(2,0,s)}\, \mathbf{1}_{B_{s,t}}\right) \\
&\qquad - \mathrm{E}\left(\phi(S_s)\, \gamma_s^{(2,0,s)}\, \psi'(S_t)\, \langle DS_s, DS_t\rangle_{(2,0,s)}\, V_{(1,s,t)}\, \mathbf{1}_{B_{s,t}}\right)\,.
\end{aligned}$$

Using again Theorem VII.4 with the weights $\pi_{B_{2,s,t}}$ in the last expectation, we obtain

$$\begin{aligned}
&\mathrm{E}\left(\phi(S_s)\,\gamma_{S_s}^{(2,0,s)}\,\psi'(S_t)\,\langle DS_s, DS_t\rangle_{(2,0,s)}\,V_{(1,s,t)}\,\mathbf{1}_{B_{s,t}}\right)\\
&\qquad= \mathrm{E}\left(\phi(S_s)\,\gamma_s^{(2,0,s)}\,\psi(S_t)\,\langle DS_s, DS_t\rangle_{2,0,s}\,V_{(1,s,t)}\,U_t^{(2,s,t)}\,\mathbf{1}_{B_{s,t}}\right)\\
&\quad- \mathrm{E}\left[\phi(S_s)\,\gamma_s^{(2,0,s)}\,\gamma_t^{(2,s,t)}\,\psi(S_t)\,\langle D\left(V_{(1,s,t)}\,\langle DS_s, DS_t\rangle_{(2,0,s)}\right), DS_t\rangle_{(2,s,t)}\,\mathbf{1}_{B_{s,t}}\right].
\end{aligned}$$

Since

$$\begin{aligned}
&\langle D\left(V_{(1,s,t)}\,\langle DS_s, DS_t\rangle_{(2,0,s)}\right), DS_t\rangle_{(2,s,t)}\\
=&\frac{1}{2}\,V_{(1,s,t)}\,\langle DS_s, D\sigma_t^{(2,s,t)}\rangle_{(2,0,s)} + \langle DS_s, DS_t\rangle_{(2,0,s)}\,\langle DS_t, D(V_{(1,s,t)})\rangle_{(2,s,t)}\,,
\end{aligned}$$

the proof is complete. ■

4. Formula based on jump times only

In this section, we will apply the integration by parts formula to the pure jump process $(S_t)_{t\in[0,T]}$ solution of equation (VIII.0.1), which will be regarded as a simple functional of the jump times T_i, $i \in \mathbb{N}$.
It is well known (see [Ber96]) that conditionally to $\{J_t = n\}$, the law of the vector $(T_1, \ldots, T_n)$ is absolutely continuous with respect to the Lebesgue measure and has the following density

$$p(\omega, t_1, \ldots, t_n) = \frac{n!}{t^n}\,\mathbf{1}_{\{0<t_1<\ldots<t_n<t\}}(t_1, \ldots, t_n)\,\mathbf{1}_{\{J_t(\omega)=n\}}\,.$$

In particular, for a given $i = 1, \ldots, n$, conditionally to $\{J_t = n\}$ and to $\{T_j,\, j \neq i\}$, T_i is uniformly distributed on $[T_{i-1}(\omega), T_{i+1}(\omega)]$. So it has the density (with the convention $T_0 = 0, T_{n+1} = t$)

$$p_i(\omega, u) = \frac{1}{T_{i+1}(\omega) - T_{i-1}(\omega)}\,\mathbf{1}_{[T_{i-1}(\omega),T_{i+1}(\omega)]}(u)\,du, \quad i = 1, \ldots, n\,.$$

Using the notation of Chapter VII, we have $V_i = T_i$, $k_i = 2$, with $t_i^1 = T_{i-1}$ and $t_i^2 = T_{i+1}$. We take $\mathcal{G} = \sigma\left(\Delta_i, i \in \mathbb{N}\right) \vee \sigma(J_t)$, and we put $A = \{J_t \geq 1\}$.
Then $T_i \in \mathrm{L}_{(\infty)}(A)$. Hence, hypothesis VII.1 and hypothesis VII.2 are satisfied.
Since p_i is not differentiable with respect to u on the whole $\mathbb{R}$, we use the following weights :

$$\pi_i(\omega, u) = \left(T_{i+1}(\omega) - u\right)^{\alpha}\left(u - T_{i-1}(\omega)\right)^{\alpha}\,\mathbf{1}_{[T_{i-1}(\omega),T_{i+1}(\omega)]}(u), \text{ with } \alpha \in (0, 1)\,. \tag{VIII.4.1}$$

Let us denote $\delta_i = T_i - T_{i-1}$, with the convention that on $\{J_t = n\}$, $\delta_{n+1} = T - T_n$. We then have $\pi_i(\omega, T_i) = \delta_{i+1}^{\alpha}\, \delta_i^{\alpha}$. Since δ_i are independent and exponentially distributed of parameter $\mu(\mathbb{R})$, we have

$$\mathrm{E}\left[|\pi_i(\omega, T_i)|^p\, \mathbf{1}_{\{J_t \geq 1\}}\right] \leq \mathrm{E}\left[|\delta_{i+1}^{\alpha}\, \delta_i^{\alpha}|^p\right] = \mathrm{E}\left(\delta_{i+1}^{\alpha\, p}\right)\, \mathrm{E}\left(\delta_i^{\alpha\, p}\right) < \infty\,,$$

which means that $\pi_i(\omega, T_i) \in \mathrm{L}_{(\infty)}(A)$.
Moreover, since $\alpha \in (0,1)$, we can choose $\eta > 0$ such that $(1-\alpha)\,(1+\eta) < 1$. We thus have $\mathrm{E}\left(\delta_i^{(\alpha-1)\,(1+\eta)}\right) \leq \int_0^\infty \frac{dy}{y^{(1-\alpha)\,(1+\eta)}} < \infty$, and then

$$\begin{aligned}
&\mathrm{E}\left[|\pi_i'(\omega, T_i)|^{1+\eta}\, \mathbf{1}_{\{J_t \geq 1\}}\right] \\
&\quad \leq \alpha\, \mathrm{E}\left(\delta_i^{\alpha\,(1+\eta)}\, \delta_{i+1}^{(\alpha-1)\,(1+\eta)}\right) + \alpha\, \mathrm{E}\left(\delta_{i+1}^{\alpha\,(1+\eta)}\, \delta_i^{(\alpha-1)\,(1+\eta)}\right) \\
&\quad = \alpha\, \mathrm{E}\left(\delta_i^{\alpha\,(1+\eta)}\right)\, \mathrm{E}\left(\delta_{i+1}^{(\alpha-1)\,(1+\eta)}\right) + \alpha\, \mathrm{E}\left(\delta_{i+1}^{\alpha\,(1+\eta)}\right)\, \mathrm{E}\left(\delta_i^{(\alpha-1)\,(1+\eta)}\right) \\
&\quad < \infty\,.
\end{aligned}$$

So $\pi_i'(\omega, T_i) \in \mathrm{L}_{(1+)}(A)$ and the weights $(\pi_i)_{i\in\mathbb{N}}$ satisfy hypothesis VII.3.
Let us fix $M \geq 4$ such that there are M jumps on $]0,T]$, that is $J_T = M$. Let us denote $B_M = \{J_T = M\}$. Corollary VIII.1 and equation (VIII.2.2) give that $S_t \in \mathcal{S}_2(A \cap B_M)$, that is S_t is a twice differentiable simple functional, such that S_t and its derivatives have moments of any order on $\{J_t \geq 1; J_T = M\}$. And similarly, $\partial_x S_t \in \mathcal{S}_1(A \cap B_M)$ (see equation (VIII.2.7)).
The differential operators are

$$\begin{aligned}
D_i S_t &= \partial_{u_i} s_t(\widetilde{T}, \widetilde{\Delta}(\omega)) = \sum_{n=i}^{\infty} \partial_{u_i} s_t(T_1, \ldots, T_n, \Delta_1(\omega), \ldots, \Delta_n(\omega))\, \mathbf{1}_{\{J_t = n\}}\,, \\
\mathrm{L}_\pi S_t &= -\sum_{i=1}^{\infty} \left(\pi_i'\, \partial_{u_i} s_t + \pi_i\, \partial_{u_i}^2 s_t\right) \left(\widetilde{T}, \widetilde{\Delta}(\omega)\right) \\
\sigma_{\pi, S_t} &= \sum_{i=1}^{\infty} \pi_i(\omega, T_i)\, \left|\partial_{u_i} s_t(\widetilde{T}, \widetilde{\Delta}(\omega))\right|^2 \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \pi_i(\omega, T_i)\, \left|\partial_{u_i} s_t(T_1, \ldots, T_n, \Delta_1(\omega), \ldots, \Delta_n(\omega))\right|^2\, \mathbf{1}_{\{J_t = n\}}\,.
\end{aligned}$$

All these quantities may be computed using Lemma VIII.1.
As we want to apply the integration by parts formula (VII.3.3) settled in Theorem VII.2 to the process $(S_t)_{t\in[0,T]}$, we give suitable conditions on the coefficients of equation (VIII.0.1) so that S_t satisfies the non-degeneracy condition (VII.3.2).

Proposition VIII.4:
Suppose that hypothesis VIII.1 holds true. Suppose moreover that condition (VIII.1.9)

is satisfied, that is for some $\epsilon > 0$, for all $(t,a,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$,

$$|q(t,a,x)| \geq \epsilon > 0 \text{ and } |(1+\partial_x c)(t,a,x)| \geq \epsilon > 0 \,.$$

Take $\alpha \in (0,1/2)$ in the definition of the weights $(\pi_i)_{i\in\mathbb{N}}$.
Then, for all $t \in [0,T]$, S_t satisfies the non-degeneracy condition (VII.3.2) if there are at least four jumps on $]0,t]$ and a finite number of jumps on $]0,T]$ (represented here by $M \geq 4$).

Proof. Since the weights π_i are bounded, the non degeneracy condition (VII.3.2) leads to

$$\mathrm{E}\left[\mathbf{1}_{\{J_t\geq 4; J_T=M\}}\,\gamma_{\pi,S_t}^{2(1+\eta)}\right] < \infty \text{ and } \mathrm{E}\left[\mathbf{1}_{\{J_t\geq 4; J_T=M\}}\,\gamma_{\pi,S_t}^{2(1+\eta)}\,|\pi_i'(T_i)|^{1+\eta}\right] < \infty\,,$$

for some $\eta > 0$.
Let us prove that for $4 \leq n \leq M$, we have $\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\gamma_{\pi,S_t}^{2(1+\eta)}\,|\pi_i'(T_i)|^{1+\eta}\right] < \infty$.
Under hypothesis (VIII.1.9), Corollary VIII.2 gives that $|\partial_{u_i} s_t| \geq \varepsilon > 0$. Thus, on $\{J_t = n\}$,

$$\sigma_{\pi,S_t} = \sum_{i=1}^{n} \delta_{i+1}^{\alpha}\,\delta_i^{\alpha}\,|\partial_{u_i} s_t(T_1,\ldots,T_n,\Delta_1,\ldots,\Delta_n)|^2 \geqslant \varepsilon^2 \sum_{i=1}^{n} \delta_{i+1}^{\alpha}\,\delta_i^{\alpha}\,.$$

Since $\pi_i'(T_i) = \alpha(-\delta_{i+1}^{\alpha-1}\,\delta_i^{\alpha} + \delta_{i+1}^{\alpha}\,\delta_i^{\alpha-1})$, we have to check that, for $4 \leq n \leq M$, for every $i = 1,\ldots,n$

$$\mathrm{E}\left[\left(\delta_i^{\alpha-1}\delta_{i+1}^{\alpha} + \delta_i^{\alpha}\delta_{i+1}^{\alpha-1}\right)^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\mathbf{1}_{\{J_t=n\}}\right] < \infty\,.$$

Take $i = 1$ and write

$$\begin{aligned}\mathrm{E}\left[\left(\delta_1^{\alpha-1}\,\delta_2^{\alpha}\right)^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\right] &\leq \mathrm{E}\left[(\delta_1^{\alpha-1}\,\delta_2^{\alpha})^{1+\eta}\,(\delta_2^{\alpha}\,\delta_3^{\alpha})^{-2(1+\eta)}\right] \\ &= \mathrm{E}\left(\delta_1^{(\alpha-1)(1+\eta)}\right)\,\mathrm{E}\left(\delta_2^{-\alpha(1+\eta)}\right)\,\mathrm{E}\left(\delta_3^{-2\alpha(1+\eta)}\right)\,.\end{aligned}$$

Recall that δ_i is exponentially distributed of parameter $\mu(\mathbb{R})$, so that $\mathrm{E}(\delta_i^{-p}) < \infty \iff p < 1$. And since $0 < \alpha < 1/2$, we can choose η small enough such that

$$(1-\alpha)\,(1+\eta) < 1 \text{ and } \alpha\,(1+\eta) < 2\,\alpha\,(1+\eta) < 1\,,$$

which gives $\mathrm{E}\left(\delta_1^{(\alpha-1)(1+\eta)}\right) < \infty$, $\mathrm{E}\left(\delta_2^{-\alpha(1+\eta)}\right) < \infty$ and $\mathrm{E}\left(\delta_3^{-2\alpha(1+\eta)}\right) < \infty$. So

$$\mathrm{E}\left[\left(\delta_1^{\alpha-1}\,\delta_2^{\alpha}\right)^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\right] < \infty\,.$$

We write now

$$\mathrm{E}\left[(\delta_1^{\alpha}\,\delta_2^{\alpha-1})^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\right] \leq \mathrm{E}\left[(\delta_1^{\alpha}\,\delta_2^{\alpha-1})^{1+\eta}\,(\delta_3^{\alpha}\,\delta_4^{\alpha})^{-2(1+\eta)}\right]$$
$$= \mathrm{E}\left(\delta_2^{(\alpha-1)(1+\eta)}\right)\,\mathrm{E}\left(\delta_1^{\alpha(1+\eta)}\right)\,\mathrm{E}\left(\delta_3^{-2\alpha(1+\eta)}\right)\,\mathrm{E}\left(\delta_4^{-2\alpha(1+\eta)}\right)\,.$$

Recalling that δ_i has finite moments of any order, the choice of η then gives

$$\mathrm{E}\left[(\delta_1^{\alpha}\,\delta_2^{\alpha-1})^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\right] < \infty\,.$$

Since $n \geq 4$, the same argument works for $i = 2, \ldots, n$, and leads to
$\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\gamma_{\pi,S_t}^{2\,(1+\eta)}\right] < \infty$. ■

Remark 4.1. *Suppose that $n = 2$. Then*

$$(\delta_1^{\alpha-1}\,\delta_2^{\alpha})^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)} = (\delta_1^{\alpha-1}\,\delta_2^{\alpha})^{1+\eta}\,\delta_2^{-2\alpha(1+\eta)}\,(\delta_1^{\alpha}+\delta_3^{\alpha})^{-2(1+\eta)}$$
$$=\delta_2^{-\alpha\,(1+\eta)} \times \left(\delta_1^{-(\alpha+1)\,(1+\eta)} + \delta_3^{-2\,\alpha\,(1+\eta)}\,\delta_1^{-(1-\alpha)\,(1+\eta)}\right)\,,$$

and this quantity is not integrable for $\alpha > 0$, $\eta > 0$.

Hence, Proposition VIII.4 allows us to settle the following particular integration by parts formula on $\{J_t \geq 4; J_T = M\}$, which will be used for the Greeks computation (see Chapter IX) :

Corollary VIII.5:
Suppose that hypothesis VIII.1 holds true. Suppose moreover that condition (VIII.1.9) is satisfied.
Take $\alpha \in (0, 1/2)$ in the definition of the weights $(\pi_i)_{i\in\mathbb{N}}$.
Then, for every function $\phi \in \mathcal{C}_p^1(\mathbb{R})$, for all $t \in [0, T]$, we have

$$\mathrm{E}(\phi'(S_t)\,\partial_x S_t\,\mathbf{1}_{\{J_t\geq 4;J_T=M\}}) = \mathrm{E}(\phi(S_t)\,H_\pi(S_t,\partial_x S_t)\,\mathbf{1}_{\{J_t\geq 4;J_T=M\}})\,,$$

where $H_\pi(S_t, \partial_x S_t) \in \mathrm{L}_{(1+)}(A \cap B_M)$, $A = \{J_t \geq 4\}$ and $B_M = \{J_T = M\}$, is given *by*

$$H_\pi(S_t, \partial_x S_t) = \partial_x S_t \, \gamma_{\pi,S_t} \, \mathrm{L}_\pi S_t - \gamma_{\pi,S_t} < DS_t, D(\partial_x S_t) >_\pi - \partial_x S_t < DS_t, D\gamma_{\pi,S_t} >_\pi .$$

Example. • *Let us consider the geometrical model :*

$$dS_t = S_t \, (r \, dt + \alpha(t,a) \, dN(t,a)) .$$

In this case $g(t,x) = x \, r$ *and* $c(t,a,x) = x \, \alpha(t,a)$*. It follows that*

$$q(t,a,x) = x \, \partial_t \alpha(t,a) + x \, r \, \alpha(t,a) + x \, r - r(x + x \, \alpha(t,a)) = x \, \partial_t \alpha(t,a) .$$

In particular, if α *does not depend on the time, the model is degenerated from the point of view of the jump times. The non degeneracy condition reads*

$$|\partial_t \alpha(t,a)| \geq \varepsilon .$$

On the other hand, the condition $|1 + \partial_x c(t,a,x)| \geq \eta$ *reads*

$$|1 + \alpha(t,a)| \geq \eta .$$

• *We consider now a Vasicek type model :*

$$dS_t = S_t \, r \, dt + \alpha(t,a) \, dN(t,a) .$$

In this case $g(t,x) = x \, r$ *and* $c(t,a,x) = \alpha(t,a)$*. It follows that*

$$q(t,a,x) = \partial_t \alpha(t,a) + x \, r - r(x + \alpha(t,a)) = \partial_t \alpha(t,a) - r \, \alpha(t,a) .$$

Suppose that α *does not depend on the time so that* $\partial_t \alpha = 0$*. Then the non degeneracy condition reads*

$$|\alpha(a)| \geq \varepsilon .$$

And the condition $|1 + \partial_x c(t,a,x)| \geq \eta$ *reads*

$$|1 + \alpha(a)| \geq \eta .$$

5. Formula based on both jump times and amplitudes

In this section, we present the differential calculus with respect to both noises coming from the jump amplitudes and from the jump times. So, for $n \geq 1$ be fixed, on $\{J_t = n\}$, the random variables will be $(V_1, \ldots, V_{2n}) = (T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)$, that is $V_i = T_i$, $i = 1, \ldots, n$ and $V_{n+i} = \Delta_i, i = 1, \ldots, n$. We have $\mathcal{G} = \sigma(J_t)$.

We put together the results from sections 2 and 4 and we keep the same notation. We assume hypothesis VIII.1 and VIII.2. The differential operators are on $\{J_t = n\}$,

$$D_i S_t = \begin{cases} \partial_{u_i} s_t(u_1, \ldots, u_n, \Delta_1(\omega), \ldots, \Delta_n(\omega))\,, & i = 1, \ldots, n \\ \partial_{a_{i-n}} s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)\,, & i = n+1, \ldots, 2n\,. \end{cases}$$

We use the weights defined in the previous sections, namely for $\alpha \in (0, 1/2)$,

$$\pi_i(\omega, u) = (T_{i+1}(\omega) - u)^\alpha \, (u - T_{i-1}(\omega))^\alpha \, \mathbf{1}_{[T_{i-1}(\omega), T_{i+1}(\omega)]}(u)\,, \quad i = 1, \ldots, n$$
$$\pi_i(y) = \pi(y) = \sum_{p=1}^{k-1} (q_{p+1} - y)^\alpha \, (y - q_p)^\alpha \, \mathbf{1}_{(q_p, q_{p+1})}(y)\,, \quad i = n+1, \ldots, 2n\,.$$

We have on $\{J_t = n\}$, $\mathrm{L}_\pi S_t = \sum_{i=1}^{2n} \mathrm{L}_{i,\pi} S_t$, with

$$\mathrm{L}_{i,\pi} S_t = \begin{cases} -\left(\pi_i'(T_i)\, \partial_{u_i} s_t + \pi_i(T_i)\, \partial^2_{u_i} s_t\right) & \text{, for } i = 1, \ldots, n\,, \\ -\left(\pi(\Delta_i)\, \partial^2_{a_i} s_t + (\pi' + \pi\,\rho')(\Delta_i)\, \partial_{a_i} s_t\right) & \text{, for } i = n+1, \ldots, 2n\,. \end{cases}$$

All these quantities may be computed using the formulas of Lemma VIII.1.

Proposition VIII.5:
Suppose that hypothesis VIII.1 and VIII.2 hold true and that hypothesis (VIII.2.4) is satisfied, that is

$$|\partial_a c(t, a, x)| \geq \varepsilon > 0 \text{ and } |(1 + \partial_x c)(t, a, x)| \geq \varepsilon > 0\,.$$

Then, for every function $\phi \in \mathcal{C}^1_p(\mathbb{R})$, for all $t \in [0, T]$, we have

$$\mathrm{E}(\phi'(S_t) \partial_x S_t \, \mathbf{1}_{\{J_t \geq 1; J_T = M\}}) = \mathrm{E}(\phi(S_t)\, H_\pi(S_t, \partial_x S_t)\, \mathbf{1}_{\{J_t \geq 1; J_T = M\}})\,,$$

where $H_\pi(S_t, \partial_x S_t) \in \mathrm{L}_{(1+)}(A \cap B_M)$, $A = \{J_t \geq 1\}$ and $B_M = \{J_T = M\}$, is given by

$$H_\pi(S_t, \partial_x S_t) = \partial_x S_t \, \gamma_{\pi, S_t} \, \mathrm{L}_\pi S_t - \gamma_{\pi, S_t} < DS_t, D(\partial_x S_t) >_\pi - \partial_x S_t < DS_t, D\gamma_{\pi, S_t} >_\pi \,.$$

Proof. For $1 \leq n \leq M$, on $\{J_t = n\}$, we write

$$\begin{aligned}\sigma_{\pi,S_t} &= \sum_{i=1}^{n} \pi_i(\omega, T_i)\, |\partial_{u_i} s_t|^2 + \sum_{i=n+1}^{2\,n} \pi(\Delta_{i-n})\, |\partial_{a_{i-n}} s_t|^2 \\ &\geq \sum_{i=1}^{n} \pi(\Delta_i)\, |\partial_{a_i} s_t|^2 \\ &:= \sigma^{\Delta}_{\pi,S_t}\,,\end{aligned}$$

where $\sigma^{\Delta}_{\pi,S_t}$ is the covariance matrix corresponding to the jump amplitudes only, that is the one defined in equation (VIII.2.3).
Hence, for $1 \leq n \leq M$, and $i = 1, \ldots, n$, since the jump times and amplitudes are independent, we get

$$\begin{aligned}\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\, \gamma^{2\,(1+\eta)}_{\pi,S_t}\, (1 + |\pi_i'(\omega, T_i)|)^{1+\eta}\right] &\leq \mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\, (\gamma^{\Delta}_{\pi,S_t})^{2\,(1+\eta)}\right] \\ &\quad \times \mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\, (1 + |\pi_i'(\omega, T_i)|)^{1+\eta}\right]\,.\end{aligned}$$

We know that $\pi_i'(\omega, T_i) \in \mathrm{L}_{(1+)}(A)$, with $A = \{J_t \geq 1\}$. Moreover, Proposition VIII.1 says that under hypothesis (VIII.2.4), the non degeneracy condition (VII.3.2) holds true on $\{J_t \geq 1; J_T = M\}$ for the jump amplitudes, that is

$$\mathrm{E}\left[\mathbf{1}_{\{J_t\geq 1; J_T=M\}}\, (\gamma^{\Delta}_{\pi,S_t})^{2\,(1+\eta)}\, (1 + |\pi'(\Delta_i)|)^{1+\eta}\right] < \infty\,.$$

Hence, for all $1 \leq n \leq M$, we have

$$\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\, \gamma^{2\,(1+\eta)}_{\pi,S_t}\, (1 + |\pi_i'(\omega, T_i)|)^{1+\eta}\right] < \infty\,. \qquad \text{(VIII.5.1)}$$

For $i = n + 1, \ldots, 2\,n$, we similarly have

$$\begin{aligned}&\mathrm{E}\left[\mathbf{1}_{\{J_t\geq 1; J_T=M\}}\, \gamma^{2\,(1+\eta)}_{\pi,S_t}\, (1 + |\pi'(\Delta_i)|)^{1+\eta}\right] \\ &\qquad \leq \mathrm{E}\left[\mathbf{1}_{\{J_t\geq 1\}}\, (\gamma^{\Delta}_{\pi,S_t})^{2\,(1+\eta)}\, (1 + |\pi'(\Delta_i)|)^{1+\eta}\right] < \infty\,. \qquad \text{(VIII.5.2)}\end{aligned}$$

Finally, equations (VIII.5.1) and (VIII.5.2) say that the non degeneracy condition(VII.3.2) holds true on $\{J_t \geq 1; J_T = M\}$, and we can perform an integration by parts formula. ■

6. Application to density computation

Let us study in this section the existence of a density for the process $(S_t)_{t\in[0,T]}$ following equation (VIII.0.1).
In this section, we suppose that there is a finite number of jumps on $]0, T]$, that is

there exists $M \in \mathbb{N}^*$ such that $J_T = M$.
Since S_t has a point mass if there is no jump on $]0,t]$, we look at $(\mathbf{1}_{\{J_t>0;J_T=M\}}\,\mathbf{P})\,S_t^{-1}$, the image by S_t of the restriction of the probability $\mathbf{P}$ on $\{J_t > 0; J_T = M\}$.
We will derive two kinds of representation of the density of $(\mathbf{1}_{\{J_t>0;J_T=M\}}\,\mathbf{P})\,S_t^{-1}$: one corresponding to the integration by parts formula based on jump amplitudes (with discontinuous law), and an other one corresponding to the integration by parts formula based on jump times.
Let us start with the jump amplitudes case. We take the weights $\pi_{(k,s,t)}$ as introduced in equation (VIII.3.2), so that they satisfy hypothesis (VII.3.7) of Lemma VII.4.

Proposition VIII.6:
Suppose that the coefficients of equation (VIII.0.1) satisfy hypothesis VIII.1 and that for all $(u, a, x) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$|\partial_a c(u,a,x)| \geq \varepsilon > 0 \text{ and } |1 + \partial_x c(u,a,x)| \geq \varepsilon > 0\,.$$

Then, $(\mathbf{1}_{\{J_t\geq 1;J_T=M\}}\,\mathbf{P})\,S_t^{-1}$ *is absolutely continuous on* $\mathbb{R}$ *with respect to the Lebesgue measure, with a* *<u>continuous</u>* *density* p_t *following the integral representation*

$$p_t(x) = \mathrm{E}\left[\mathbf{1}_{(0,\infty)}(S_t - x)\,U_t^{(1,0,t)}\,\mathbf{1}_{\{J_t\geq 1;J_T=M\}}\right]\,,$$

where $U_t^{(1,0,t)}$ *is defined by equation (VIII.3.3).*

<u>**Proof**</u>. By Proposition VIII.1, we know that the weights $\pi_{(1,0,t)}$ satisfy the non degeneracy condition (VII.3.2) on $\{J_t \geq 1; J_T = M\}$. Hence, Corollary VII.3 (Case 2) gives the result. ■

We have seen in Proposition VIII.3 (ii), that we can iterate the integration by parts formula if there are at least four jumps on $]0,t]$. So, in view of Corollary VII.4 (Case 2), we cannot prove that the previous density is differentiable, unless we replace $\{J_t \geq 1\}$ by $\{J_t \geq 4\}$:

Proposition VIII.7:
Suppose that the coefficients of equation (VIII.0.1) satisfy hypothesis VIII.1 and that for all $(u, a, x) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$|\partial_a c(u,a,x)| \geq \varepsilon > 0 \text{ and } |1 + \partial_x c(u,a,x)| \geq \varepsilon > 0\,.$$

Then, $(\mathbf{1}_{\{J_t\geq 4;J_T=M\}}\,\mathbf{P})\,S_t^{-1}$ *is absolutely continuous on* $\mathbb{R}$ *with respect to the Lebesgue measure, with a density* $q_t \in \mathcal{C}^1(\mathbb{R})$ *such that*

$$q_t(x) = \mathrm{E}\left[\mathbf{1}_{(0,\infty)}(S_t - x)\,U_t^{(1,0,t)}\,\mathbf{1}_{\{J_t\geq 4;J_T=M\}}\right]\,,$$

where $U_t^{(1,0,t)}$ is defined by equation (VIII.3.3). And

$$q_t'(x) = -\mathrm{E}\left[\mathbf{1}_{(0,\infty)}(S_t - x)\,\mathcal{H}_t\,\mathbf{1}_{\{J_t>4;J_T=M\}}\right]\,,$$

where $\mathcal{H}_t = U_t^{(1,0,t)}\,U_t^{(2,0,t)} - \gamma_t^{(2,0,t)}\,\langle DS_t, DU_t^{(1,0,t)}\rangle_{(2,0,t)}$.

Proof. In the proof of Proposition VIII.1, we have seen that hypothesis (VIII.2.4) implies that $\partial_{a_i} s_t$ satisfies the ellipticity assumption (VII.3.6) of Lemma VII.4. Moreover, since the jump amplitudes are independent, $\pi_l(\Delta_i)$ and $\pi_k(\Delta_j)$ are independent for $i \neq j$ and $k, l = 1, 2$. Hence, we can apply Corollary VII.4 (Case 2) to get the result.■

Remark 6.1. *If the law of the jump amplitudes has no discontinuities, let us suppose that hypothesis (VIII.2.8) holds true, say for all $(t,a,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$,*

$$|\partial_a c(t,a,x)| \geq \eta > 0\,.$$

Then, Proposition VIII.2 says that for all $t \in [0,T]$, S_t satisfies the non-degeneracy condition (H_q) for all $q \in \mathbb{N}$ (see hypothesis VII.6), that is γ_{S_t} has finite moments of any order on $\{J_t \geq 1; J_T = M\}$. Hence, Corollary VII.4 (Case 1) gives : $p_t \in \mathcal{C}^\infty(\mathbb{R})$, and

$$p_t^{(k)}(x) = (-1)^k\,\mathrm{E}\left(\mathbf{1}_{(0,\infty)}(S_t - x)\,H_{k+1}(S_t,1)\,\mathbf{1}_{\{J_t\geq 1;J_T=M\}}\right)\,,$$

where $H_{k+1}(S_t,1)$ is defined by the inductive relation :

$$H_0(S_t,1) = 1 \text{ and } H_{k+1}(S_t,1) = H(F, H_k(S_t,1))\,.$$

This case is similar to diffusion processes on the Wiener space.

Let us now give an expression of the density using integration by parts formulas based on jump times.
We take the weights introduced in equation (VIII.4.1). Let us recall that we have denoted

$$q(t,\alpha,x) := (\partial_t c + g\,\partial_x c)(t,\alpha,x) + g(t,x) - g(t, x + c(t,\alpha,x))\,.$$

Proposition VIII.8:
Suppose that the coefficients of equation (VIII.0.1) satisfy hypothesis VIII.1 and hypothesis (VIII.1.9), that is for all $(t,a,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$,

$$|q(t,\alpha,x)| \geq \varepsilon > 0 \text{ and } |1 + \partial_x c(t,a,x)| \geq \varepsilon > 0\,.$$

Then, $(\mathbf{1}_{\{J_t\geq 4;J_T=M\}}\,\mathbf{P})\,S_t^{-1}$ is absolutely continuous on $\mathbb{R}$ with respect to the Lebesgue measure, with a <u>continuous</u> density q_t following the integral representation

$$q_t(x) = \mathrm{E}\left[\mathbf{1}_{(0,\infty)}(S_t - x)\,H(S_t,1)\,\mathbf{1}_{\{J_t\geq 4;J_T=M\}}\right]\,,$$

where $H(S_t, 1)$ involves the Malliavin operators of S_t derived by differentiating with respect to the jump times (see Corollary VIII.5).

Proof. Proposition VIII.4 says that under hypothesis (VIII.1.9), the non-degeneracy condition (VII.3.2) is satisfied on $\{J_t \geq 4; J_T = M\}$. Hence, Corollary VII.3 gives the result. ■

Remark 6.2. *In this framework, under suitable assumptions on the coefficient of the diffusion $(S_t)_{t\in[0,T]}$, we have derived an explicit representation of the density of $(\mathbf{1}_{\{J_t\geq 4; J_T=M\}}\,\mathbf{P})\,S_t^{-1}$. We can moreover state that this density is continuous.*
Let us compare the result of Proposition VIII.8 to the framework developped by Carlen and Pardoux in [CtP90].
Under suitable assumptions on the coefficients of the diffusion equation of $(S_t)_{t\in[0,T]}$, they prove that $(\mathbf{1}_{\{J_t\geq 1\}}\,\mathbf{P})\,S_t^{-1}$ is absolutely continuous on $\mathbb{R}$ with respect to te Lebesgue measure. But they can not derive neither explicit expression nor regularity results for the density. This can be explained by the fact that their approach is not based on an integration by parts formula : the functional S_t is one time, but not twice, differentiable with respect to the jump times (in Malliavin sense), whereas the integration by parts formula involves the Ornstein-Uhlenbeck operator and then the second order derivatives of S_t (see Corollary VIII.5).
By restricting ourselves on a smaller event (that is $\{J_t \geq 4; J_T = M\}$), we get a stronger result : we derive an integral representation for the density of $(\mathbf{1}_{\{J_t\geq 4; J_T=M\}}\,\mathbf{P})\,S_t^{-1}$ as well as an information about its regularity (continuous).

Troisième partie

Applications to Mathematical Finance

Sensitivity analysis for European and Asian options IX

Introduction

In this chapter, we will apply the integration by parts formulas settled in Corollaries VIII.3 and VIII.4 (based on the jump amplitudes), in Corollary VIII.5 (based on the jump times) and in Proposition VIII.5 (based on both jump times and amplitudes), to compute the Delta of two European and Asian options : call option with payoff $\phi(x) = (x - K)_+$ and digitial option with payoff $\phi(x) = \mathbf{1}_{x \geq K}$. This means that, if we denote by $(S_t)_{t \in [0,T]}$ the underlying and T the maturity of the option, we want to compute $\partial_{S_0} \mathrm{E}(\phi(S_T))$ in the case of European options, and $\partial_{S_0} \mathrm{E}(\phi(I_T))$, with $I_T := \dfrac{1}{T} \displaystyle\int_0^T S_t \, dt$, in the case of Asian options.
We denote by Δ_i, $i \in \mathbb{N}$ and T_i, $i \in \mathbb{N}$ the jump amplitudes and times of a compound Poisson process, and we define $(J_t)_{t \in [0,T]}$ the counting process, that is $J_t := Card(T_i \leq t)$.
The asset $(S_t)_{t \in [0,T]}$ is a one dimensional jump diffusion process.
We first deal with two different one dimensional pure jump diffusion equations for modelling the asset $(S_t)_{t \in [0,T]}$.
The first one is motivated by the Vasicek model used for interest rates (but we consider a jump process instead of a Brownian motion) :

$$S_t = x - \int_0^t r \, (S_u - \alpha) \, du + \sum_{i=1}^{J_t} \sigma \, \Delta_i \, . \tag{IX.0.1}$$

And the second one is of Black-Scholes type :

$$S_t = x + \int_0^t r \, S_u \, du + \sigma \sum_{i=1}^{J_t} S_{T_i^-} \, \Delta_i \, . \tag{IX.0.2}$$

Next, we add a continuous part to the geometrical model (IX.0.2), that is we consider the following Merton model :

$$S_t = x + \int_0^t r\, S_u\, du \; + \int_0^t \sigma\, S_u\, dW_u + \mu \sum_{i=1}^{J_t} S_{T_i^-}\, \Delta_i\,, \qquad \text{(IX.0.3)}$$

where W is a one dimensional Brownian motion independent on the compound Poisson process N.
In these models, we take $\Delta_i \sim \mathcal{N}(0,1)$, $i \geq 1$. That is, Δ_i has the density $p(x) = \dfrac{1}{\sqrt{2\,\pi}}\, e^{\rho(x)}$, with $\rho(x) = -\dfrac{x^2}{2}$. And we put $T_i - T_{i-1} \sim \exp(\lambda)$, where λ is called the jump intensity.
The first two pure jump models allow us to compare the Malliavin approach (based on an integration by parts formula used in a Monte Carlo algorithm) to the finite difference method. Moreover, since we use integration by parts formulas using the jump times only or the jump amplitudes only, we can compare the Malliavin estimators corresponding to these two different cases. Adding a continuous part in model (IX.0.3) allows us to compare the Malliavin estimator based on Brownian motion only (obtained in [PD04]) to the one based on Brownian motion and jump amplitudes (obtained in our framework). In other words, using all the noise available in the model does improve the numerical results.
Let us come back to the Delta computation. We write (the following computations hold with I_T)

$$\begin{aligned}\partial_x \mathrm{E}(\phi(S_T)) =& \mathrm{E}\left(\phi'(S_T)\, \partial_x S_T\right) \\ =& \mathrm{E}\left(\phi'(S_T)\, \partial_x S_T\, \mathbf{1}_{\{J_T=0\}}\right) + \mathrm{E}\left(\phi'(S_T)\, \partial_x S_T\, \mathbf{1}_{\{J_T\geq 1\}}\right)\,.\end{aligned}$$

On $\{J_T \geq 1\}$, we use an integration by parts formula such as the one of Corollary VIII.4 for the jump amplitudes (with smooth laws), or of Corollary VIII.5 for the jump times, or of Proposition VIII.5 for both of them. We thus obtain

$$\mathrm{E}\left(\phi'(S_T)\, \partial_x S_T\, \mathbf{1}_{\{J_T\geq 1\}}\right) = \mathrm{E}\left(\phi(S_T)\, H(S_T, \partial_x S_T)\, \mathbf{1}_{\{J_T\geq 1\}}\right)\,,$$

where $H(S_T, \partial_x S_T)$ is a weight involving Malliavin derivatives of S_T and $\partial_x S_T$. Hence, we have

$$\partial_x \mathrm{E}(\phi(S_T)) = \mathrm{E}\left(\phi'(S_T)\, \partial_x S_T\, \mathbf{1}_{\{J_T=0\}}\right) + \mathrm{E}\left(\phi(S_T)\, H(S_T, \partial_x S_T)\, \mathbf{1}_{\{J_T\geq 1\}}\right)\,.$$

In order to compute the two terms in the right hand side of the above equality, we proceed as follows.
On $\{J_T = 0\}$, there is no jump on $]0,T]$, thus S_T and $\partial_x S_T$ solve some deterministic integral equation. In the examples that we considered in this chapter, the solution

of these equations are explicit, so that these terms are explicitly known. Hence, we may use the finite difference method to compute $\mathrm{E}\left(\phi'(S_T)\,\partial_x S_T\,\mathbf{1}_{\{J_T=0\}}\right)$.
For the computation of the term $\mathrm{E}\left(\phi(S_T)\,H(S_T,\partial_x S_T)\,\mathbf{1}_{\{J_T\geq 1\}}\right)$, we use a Monte-Carlo algorithm. We simulate a sample $\left((T_n^k)_{n\in\mathbb{N}},(\Delta_n^k)_{n\in\mathbb{N}}\right)$, $k=1,\ldots,M$ of the times and the amplitudes of the jumps, and we compute the corresponding J_t^k, S_T^k, and $H^k(S_T^k,\partial_x S_T^k)$. Then we write

$$\mathrm{E}\left(\phi(S_T)\,H(S_T,\partial_x S_T)\,\mathbf{1}_{\{J_T\geq 1\}}\right)\simeq\frac{1}{M}\sum_{k=1}^{M}\phi(S_T^k)\,H^k(S_T^k,\partial_x S_T^k)\,\mathbf{1}_{\{J_T^k\geq 1\}}\,.$$

Let us compute now the Malliavin weights $H^k(S_T^k,\partial_x S_T^k)$ for the models (IX.0.1) and (IX.0.2).

1. Malliavin estimators

We may use integration by parts formula with respect to jump amplitudes, times or to both of them.
In the case of jump amplitudes, since their density p has no discontinuities on $\mathbb{R}$, we are in the framework described in Chapter VIII, section 2.2 : the density p satisfies hypothesis VIII.3. As there are no border terms to cancel, we put for all $i\geq 1$, $\pi(\omega,\Delta_i)=1$. We thus use the integration by parts formula derived in Corollary VIII.4, and we get the following Malliavin weight (corresponding to the jump amplitudes only)

$$\begin{aligned}H^{\Delta}(S_T,\partial_x S_T)=\partial_x S_T\,\gamma_{S_T}\,\mathrm{L}S_T-\gamma_{S_T}\;&<DS_T,D(\partial_x S_T)>\\&-\partial_x S_T\;<DS_T,D\gamma_{S_T}>\;.\end{aligned}\tag{IX.1.1}$$

In the case of jump times, we are in the framework described in Chapter VIII, section 4. Recall that we have taken the weights

$$\pi_i(\omega,T_i)=(T_{i+1}-T_i)^{\alpha}\,(T_i-T_{i-1})^{\alpha},\text{ with }\alpha\in(0,1/2)\,.$$

Denoting by $\delta_i=T_i-T_{i-1}$ (with the convention $\delta_{n+1}=T-T_n$ on $\{J_T=n\}$), we then have

$$\pi_i'=\alpha\,\delta_{i+1}^{\alpha-1}\,\delta_i^{\alpha-1}\,(\delta_{i+1}-\delta_i)\,.$$

We thus use the integration by parts formula derived in Corollary VIII.5, and we get the following Malliavin weight (corresponding to the jump times only)

$$\begin{aligned}H^{Tm}(S_T,\partial_x S_T)=\partial_x S_T\,\gamma_{\pi,S_T}\,\mathrm{L}_{\pi}S_T-\gamma_{\pi,S_T}\;&<DS_T,D(\partial_x S_T)>_{\pi}\\&-\partial_x S_T\;<DS_T,D\gamma_{\pi,S_T}>_{\pi}\;.\end{aligned}\tag{IX.1.2}$$

Note that this formula holds true if there is at least four jumps on $]0,T]$. In view of Remark 4.1, we are not able to handle the non degeneracy problem corresponding to the jump times if $J_T \leq 3$. Hence, we will use the noise coming from the first jump amplitude Δ_1 if there is at most three jumps on $]0,T]$.
In the case of both jump times and amplitudes, we choose the weights on $\{J_T = n\}$: for $i =, \ldots, n$, we put $\pi_i(\omega, T_i) = (T_{i+1} - T_i)^\alpha \, (T_i - T_{i-1})^\alpha$, with $\alpha \in (0, 1/2)$, and for $i = n+1, \ldots, 2\,n$, $\pi(\Delta_i) = 1$.
We then use the integration by parts formula derived in Proposition VIII.5, and we get the following Malliavin weight corresponding to the jump times and amplitudes

$$H(S_T, \partial_x S_T) = H^\Delta(S_T, \partial_x S_T) + H^{Tm}(S_T, \partial_x S_T)\,.$$

Let us compute the Malliavin operators involved in the weights $H^\Delta(S_T, \partial_x S_T)$ and $H^{Tm}(S_T, \partial_x S_T)$. One may use Lemma VIII.1, but in the particular cases that we discuss here, we have explicit solutions, so that direct computations are much easier.

1.1. European options

• We first study the Vasicek model (IX.0.1). Let us fix $n \geq 1$. We have an explicit expression of S_T on $\{J_T = n\}$:

$$S_T = x\,e^{-r\,T} + \alpha\,(1 - e^{-r\,T}) + \sigma \sum_{j=1}^{n} \Delta_j\, e^{-r\,(T - T_j)}\,. \tag{IX.1.3}$$

∗ Jump amplitudes : Differentiating with respect to the jump amplitudes in equation (IX.1.3), we get for all $1 \leq i \leq n$,

$$\begin{aligned} D_i S_T &= \sigma e^{-r\,(T-T_i)} \\ D^2_{ii} S_T &= 0 \\ Y_T &:= \frac{\partial S_T}{\partial x} = e^{-r\,T} \\ D_i Y_T &= 0\,, \end{aligned}$$

and the covariance matrix is given by :

$$\sigma_T = \sum_{j=1}^{n} |D_j S_T|^2 = \sigma^2 \sum_{j=1}^{n} e^{-2\,r\,(T-T_j)}\,.$$

Then $\gamma_T = \dfrac{1}{\sigma_T} \Rightarrow D_i \gamma_T = 0$, for all $1 \leq i \leq n$. Since $\dfrac{\partial \ln p(\Delta)}{\partial \Delta} = -\Delta$, one has

$$\mathrm{L} S_T = -\sum_{j=1}^{n} D_j S_T \,\frac{\partial \ln p(\Delta_j)}{\partial \Delta_j} = \sum_{j=1}^{n} \sigma\, e^{-r\,(T-T_j)}\,\Delta_j\,.$$

Finally, putting these results in equation (IX.1.1), we obtain on $\{J_T = n\}$ for $n \geq 1$,

$$H_n^{\Delta}(S_T, \partial_x S_T) = \frac{\sum_{j=1}^{n} e^{r\,T_j}\,\Delta_j}{\sigma \sum_{j=1}^{n} e^{2\,r\,T_j}} \,. \tag{IX.1.4}$$

$*$ Jump times : suppose that $n \geq 4$. Differentiating with respect to the jump times in equation (IX.1.3), we have

$$D_i S_T = \sigma\,\Delta_i\,r\,e^{-r\,(T-T_i)}\,,$$

and then on $\{J_T = n\}$,

$$\sigma_{\pi,S_T} = \sum_{i=1}^{n} \pi_i\,(\sigma\,r)^2\,\Delta_i^2\,e^{-2\,r\,(T-T_i)}\,.$$

On $\{J_T = n\}$, we have $\mathrm{L}_\pi(S_T) = -\sum_{i=1}^{n} \mathrm{L}_{i,\pi}(S_T)$, with

$$\mathrm{L}_{i,\pi} S_T = -\sigma\,r\,\Delta_i\,e^{-r\,(T-T_i)}\,\left(r\,\pi_i + \alpha\,(\delta_{i+1}\,\delta_i)^{\alpha-1}\,(\delta_{i+1} - \delta_i)\right)\,.$$

Let us denote

$$\begin{aligned} A_j &= \alpha\,(\delta_{j+1}\,\delta_j)^{\alpha-1}\,\Delta_j^2\,e^{2\,r\,T_j}\,, \\ B_j &= \Delta_j^2\,e^{2\,r\,T_j}\,\left[2\,r\,\pi_j + \alpha\,(\delta_{j+1}\,\delta_j)^{\alpha-1}\,(\delta_{j+1} - \delta_j)\right]\,. \end{aligned}$$

We then obtain

$$D_j \sigma_{\pi,S_T} = (\sigma\,r)^2\,e^{-2\,r\,T}\,(A_{j-1}\,\delta_{j-1} - A_{j+1}\,\delta_{j+2} + B_j)\,.$$

Moreover $\partial_x S_T = e^{-r\,T}$, so that $D_i \partial_x S_T = 0$ for all $i = 1, \ldots, n$.

We have now the expression of all the terms involved in $H_n^{Tm}(S_T, \partial_x S_T)$ in equation (IX.1.2). For $n \geq 4$, on $\{J_T = n\}$, we obtain

$$H_n^{Tm}(S_T, \partial_x S_T) = \frac{\sum_{i=1}^{n} \Delta_i\,e^{r\,T_i}\,\left(r\,\pi_i + \alpha\,(\delta_{i+1}\,\delta_i)^{\alpha-1}\,(\delta_{i+1} - \delta_i)\right)}{\sigma\,r\,\hat{\sigma}} - \frac{\sum_{i=1}^{n} \pi_i\,\Delta_i\,e^{r\,T_i}\,\left(A_{i-1}\,\delta_{i-1} - A_{i+1}\,\delta_{i+2} + B_i\right)}{\sigma\,r\,\hat{\sigma}^2}\,, \tag{IX.1.5}$$

where $\hat{\sigma} = \sum_{i=1}^{n} \pi_i \, \Delta_i^2 \, e^{2\,r\,T_i}$.

For $n = 1, 2, 3$, we use integration by parts with respect to the first jump amplitude Δ_1 only. Then, similar computations give on $\{J_T = n\}$, for $1 \le n \le 3$:

$$H_n^{T_m}(S_T, \partial_x S_T) = \frac{e^{-r\,T_1}}{\sigma\,\Delta_1}\,.$$

• We now study the geometrical model (IX.0.2). Let us fix $n \ge 1$. On $\{J_T = n\}$, we have

$$S_T = x\, e^{r\,T} \prod_{j=1}^{n} (1 + \sigma\,\Delta_j)\,.$$

We may not use integration by parts with respect to the jump times because S_T depends on $T_1, \ldots, T_n$ by means of J_t only. So we perform integration by parts formula using the jump amplitudes only. Differentiating with respect to the jump amplitudes, we have for all $1 \le i \le n$,

$$D_i S_T = \frac{\sigma\, S_T}{1 + \sigma\,\Delta_i} = \sigma \prod_{j=1,\, j \ne i}^{n} (1 + \sigma\,\Delta_j)\,.$$

Note that if $(1 + \sigma\,\Delta_i) = 0$, then $S_T = 0$. So we use the convention $\frac{0}{0} = 0$. Let us define

$$\widetilde{A}_\sigma = \sum_{j=1}^{n} \frac{1}{(1 + \sigma\,\Delta_j)^2} \tag{IX.1.6}$$

$$\widetilde{B}_\sigma = \sum_{j=1}^{n} \frac{\Delta_j}{(1 + \sigma\,\Delta_j)} \tag{IX.1.7}$$

$$\widetilde{C}_\sigma = \sum_{j=1}^{n} \frac{1}{(1 + \sigma\,\Delta_j)^4}\,. \tag{IX.1.8}$$

We then get, for all $1 \leq i \leq n$

$$
\begin{aligned}
D_{ii}^2 S_T &= 0 \\
Y_T &= \frac{S_T}{S_0} \\
D_i Y_T &= \frac{\sigma\, S_T}{S_0\,(1+\sigma\,\Delta_i)} \\
\sigma_T &= \sigma^2\, S_T^2 \sum_{j=1}^{n} \frac{1}{(1+\sigma\,\Delta_j)^2} = \sigma^2\, S_T^2\, \widetilde{A}_\sigma \qquad \text{(IX.1.9)} \\
D_i \sigma_T &= (\frac{2\,\sigma^3\, S_T^2}{1+\sigma\,\Delta_i}) \left(\widetilde{A}_\sigma - \frac{1}{(1+\sigma\,\Delta_i)^2} \right) \\
D_i \gamma_T &= -\frac{D_i \sigma_T}{\sigma_T^2}\,.
\end{aligned}
$$

Hence, on $\{J_T = n\}$, $n \geq 1$, the Malliavin weight (IX.1.1) for European options is given by

$$
H_n^\Delta(S_T, \partial_x S_T) = \frac{\widetilde{B}_\sigma}{\sigma\, x\, \widetilde{A}_\sigma} + \frac{1}{x} - \frac{2\,\widetilde{C}_\sigma}{x\, \widetilde{A}_\sigma^2}\,. \qquad \text{(IX.1.10)}
$$

1.2. Asian options

In this section, we deal with the geometrical model (IX.0.2). Let us fix $n \geq 1$. On $\{J_T = n\}$, we have

$$
I_T := \frac{1}{T} \int_0^T S_u\, du = \sum_{j=0}^{n} \frac{1}{T} \int_{T_j}^{T_{j+1}} S_u\, du\,,
$$

with the convention $T_0 = 0$ and $T_{n+1} = T$.
On $\{J_T = n\}$, $n \geq 1$, we compute the differential operators involved in the expression of $H_n^\Delta(I_T, \partial_x I_T)$ (take I_T instead of S_T in equation (IX.1.1)).
In order to differentiate I_T, let us first express it as a simple functional.
On $\{J_T = n\}$, $n \geq 1$, we have for all $t \in [T_j, T_{j+1}[$, $S_t = S_{T_j} + \int_{T_j}^t r\, S_u\, du$, so that $S_t = S_{T_j}\, e^{r\,(t-T_j)}$. We thus obtain

$$
I_T = \frac{1}{r\,T} \sum_{j=0}^{n} S_{T_j} \left(e^{r\,(T_{j+1}-T_j)} - 1 \right)\,.
$$

Since we know from Chapter VIII (see equation (VIII.1.2)) that on $\{J_T = n\}$, $S_T = s_T(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)$ (with s_T a twice differentiable function), we can write I_T as a twice differentiable simple functional :

$I_T = \sum_{n=1}^{\infty} i_T\left(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n\right) \mathbf{1}_{\{J_T=n\}}$, where

$$i_t(u_1, \ldots, u_n, a_1, \ldots, a_n) = \frac{1}{r\,t} \sum_{j=0}^{n} s_t(u_1, \ldots, u_j, a_1, \ldots, a_j) \left(e^{r\,(u_{j+1}-u_j)} - 1\right) .$$

So, differentiating with respect to the jump amplitudes, we obtain

$$D_i I_T = \frac{\sigma}{T} K_{i,T}\,, \text{ where } K_{i,T} := \frac{1}{1+\sigma\,\Delta_i} \int_{T_i}^{T} S_u\,du\,.$$

And we get

$$\begin{aligned}
D^2_{ii} I_T &= \frac{1}{r\,T} \sum_{j=0}^{n} D^2_{ii} S_{T_j} \left(e^{r\,(T_{j+1}-T_j)} - 1\right) = 0 \\
Z_T &:= \frac{\partial I_T}{\partial x} = \frac{1}{T} \int_0^T Y_u\,du = \frac{I_T}{x} \\
D_i Z_T &= \frac{\sigma}{T\,x} K_{i,T} \\
\sigma_{I_T} &= \sum_{j=1}^{n} |D_j I_T|^2 = \frac{\sigma^2}{T^2} \sum_{j=1}^{n} K^2_{j,T} \qquad \text{(IX.1.11)} \\
D_i \gamma_{I_T} &= -2\,\gamma^2_{I_T} \sum_{j=1}^{n} D_j I_T\, D^2_{ij} I_T\,,
\end{aligned}$$

with

$$D^2_{ij} I_T = \begin{cases} 0 & \text{if} \quad i = j \\ \frac{\sigma^2}{T\,(1+\sigma\,\Delta_j)} K_{i,T} & \text{if} \quad i > j \\ D^2_{ji} I_T & \text{if} \quad i < j \text{ (by symmetry)}\,. \end{cases}$$

Hence, on $\{J_T = n\}$, $n \geq 1$, the Malliavin weight for Asian options is given by

$$H_n^{\Delta}(I_T, \partial_x I_T) = -\frac{1}{x} + \frac{K_{0,T}}{\sigma\,x\,\overline{K}} \left[\sum_{j=1}^{n} \Delta_j\, K_{j,T} + \frac{4\,\sigma}{\overline{K}} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} K^2_{j,T} \frac{K_{i,T}}{1+\sigma\,\Delta_i} \right], \qquad \text{(IX.1.12)}$$

where $\overline{K} = \sum_{j=1}^{n} K^2_{j,T}$ and $K_{j,T} = \frac{1}{1+\sigma\,\Delta_j} \int_{T_j}^{T} S_u\,du$.

2. Numerical experiments for pure jump processes

In this section, we present several numerical experiments in order to compare the Malliavin approach to the finite difference method.

In arbitrage theory, an expression for the price $u(,)$ of an option, with underlying S, maturity T and payoff ϕ, is given by

$$u(0, S_0) = \mathrm{E}\left[\phi(S_T)|S_0\right] .$$

To compute the Delta (that is $\partial_{S_0} u(0, S_0)$), the finite difference method makes a differentiation using the Taylor expansion of the price with respect to S_0. Indeed, we shift S_0 with ϵ and compute the new price $u(0, S_0 + \epsilon)$, then the first term of the Taylor expansion of the price around S_0 is given by :

$$\frac{\partial u(0, S_0)}{\partial S_0} \simeq \frac{u(0, S_0 + \epsilon) - u(0, S_0 - \epsilon)}{2\epsilon} .$$

We choose the symmetric estimator and we use the same simulated paths in the two "shifted expectation" in order to reduce the variance.

On the other hand, we look at two kinds of Malliavin Monte-Carlo estimators : these obtained using a localization method or not. Let us be more precise about the localization method.
For European and Asian call options, we use the same variance reduction method as the one introduced in [FLL+99]. We have seen that sensitivity analysis using Malliavin calculus leads to terms such as $\phi(S_T)$, $H(S_T, \dfrac{\partial S_T}{\partial S_0})$ (take I_T for S_T in the case of Asian options), which may have a large variance. It is possible to avoid this problem by using a localization function which vanishes out of an interval $[K - \delta\, , \, K + \delta]$, for some $\delta > 0$. In order to develop this idea, let us introduce some notation.

For $\delta > 0$, we consider the following function,

$$\begin{aligned} B_\delta(s) \;&:=\; 0 && \text{if } s \leq K - \delta \\ &:=\; \tfrac{s-(K-\delta)}{2\,\delta} && \text{if } s \in [K - \delta\, , \, K + \delta] \\ &:=\; 1 && \text{if } s \geq K + \delta\, . \end{aligned}$$

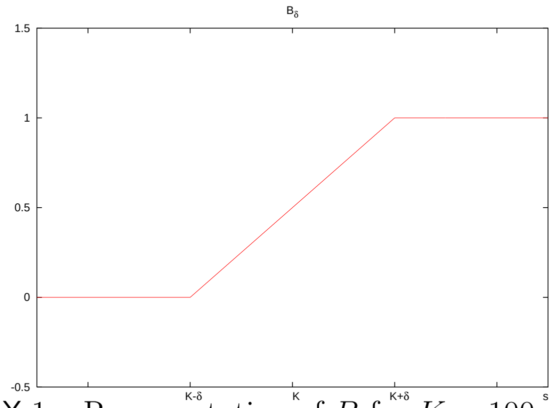

FIG. IX.1 – Representation of B for $K = 100, \delta = 20$

Let the function G_δ be a primitive of B_δ :

$$
\begin{array}{rcll}
G_\delta(t) & := & \int_{-\infty}^{t} B_\delta(s)ds & \\
& := & 0 & \text{if } t \leq K-\delta \\
& := & \frac{(t-(K-\delta))^2}{4\,\delta} & \text{if } t \in [K-\delta\,,\, K+\delta] \\
& := & t-K & \text{if } t \geq K+\delta\,.
\end{array}
$$

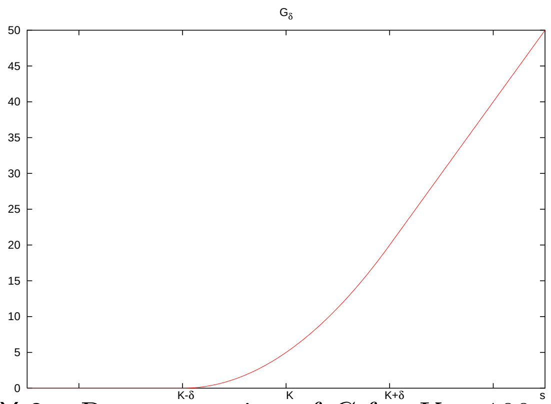

FIG. IX.2 – Representation of G for $K = 100, \delta = 20$

We then define the localization function

$$
\begin{array}{rcll}
F_\delta(t) & := & (t-K)_+ - G_\delta(t) & \\
& := & 0 & \text{if } t \leq K-\delta \\
& := & -\frac{(t-(K-\delta))^2}{4\,\delta} & \text{if } t \in [K-\delta\,,\, K] \\
& := & t-K-\frac{(t-(K-\delta))^2}{4\,\delta} & \text{if } t \in [K,\, K+\delta] \\
& := & 0 & \text{if } t \geq K+\delta\,.
\end{array}
$$

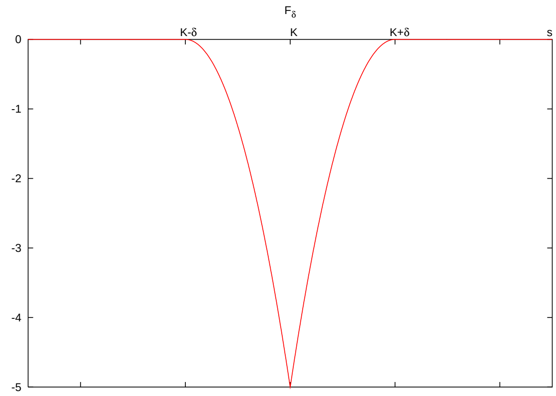

FIG. IX.3 – Representation of F for $K = 100, \delta = 20$

Since $F_\delta(S_T) + G_\delta(S_T) = (S_T - K)_+$, we have on $\{J_T \geq 1\}$,

$$\begin{aligned}\partial_{S_0} \mathrm{E}\left[(S_T - K)_+\right] &= \partial_{S_0} \mathrm{E}\left[G_\delta(S_T)\right] + \partial_{S_0} \mathrm{E}\left[F_\delta(S_T)\right] \\ &= \mathrm{E}\left[B_\delta(S_T)\, \partial_{S_0} S_T\right] + \mathrm{E}\left[F_\delta(S_T)\, H(S_T, \partial_{S_0} S_T)\right] .\end{aligned}$$

Since F_δ vanishes out of $[K-\delta, K+\delta]$, the value of the second expectation does not blow up as $H(S_T, \partial_{S_0} S_T)$ increases.

Remark 2.1. *Since the law p of the jump amplitudes has no discontinuities, Proposition VIII.2 says that we may perform an integration by parts formula using the jump amplitudes under the condition (VIII.2.8), that is*

$$|\partial_a c(t, a, x)| \geq \eta > 0, \text{ for some } \eta .$$

Concerning the geometrical model, we have $\partial_a c(t, a, x) = \sigma\, x$. Hence, condition (VIII.2.8) is not satisfied. Let us show how the localization method allows us to overcome this difficulty.
Let us come back to the relevance of hypothesis (VIII.2.8). In the proof of Proposition VIII.2, it allows us to verify that hypothesis VII.6 is satisfied, that is

$$\mathrm{E}\left(\gamma_{S_T}^4\, \mathbf{1}_{\{J_T \geq 1\}}\right) < \infty .$$

The localization method allows us to settle the non degeneracy condition VII.6 even if condition (VIII.2.8) is not satisfied. Equations (IX.1.9) and (IX.1.11) actually give

$$\begin{aligned}\sigma_T^4 &\geq \sigma^8\, S_T^8\, \frac{1}{(1+\sigma\,\Delta_1)^8} \geq (\sigma\,(K-\delta))^8\, \frac{1}{(1+\sigma\,\Delta_1)^8}, \\ \sigma_{I_T}^4 &\geq \sigma^8\, K_{1,T}^8\, \frac{1}{T^8} \geq \sigma^8\, I_T^8\, \frac{1}{(1+\sigma\,\Delta_1)^8} \geq (\sigma\,(K-\delta))^8\, \frac{1}{(1+\sigma\,\Delta_1)^8} .\end{aligned}$$

Since Δ_1 has moments of any order, we get $\mathrm{E}(\gamma_T^4) < \infty$ and $\mathrm{E}(\gamma_{I_T}^4) < \infty$.

2.1. Comparison of the Malliavin calculus and the finite difference methods

In this section, we compare the results given by Malliavin calculus and finite difference method. We also compare the localized and non localized Malliavin estimators.

Remark 2.2. *We choose the parameter σ in the diffusion models (IX.0.1) and (IX.0.2) in the following way :*

– *For the Geometrical model, the variance of S_t is*

$$Variance(S_t) = x^2\, e^{2\,r\,t}\,\left(e^{\sigma^2\,\lambda\,t} - 1\right)\,.$$

Taking $\lambda = 1$, $r = 0.1$, $T = 5$ and $x = 100$, if $\sigma \in [0.1, 0.6]$, we have $1393.69 \leq Variance(S_T) \leq 137264$. We choose here small values for σ in order to fit the usual values of the volatility taken in the Black-Scholes model.

– *For the Vasicek type model, we have*

$$Variance(S_t) = 2\,\alpha\, e^{-2\,r\,t}\,(x-\alpha) + \frac{\lambda\,\sigma^2}{2\,r}\,\left(1 - e^{-2\,r\,t}\right)\,.$$

Taking $\lambda = 1$, $r = 0.1$, $T = 5$, $\alpha = 10$ and $x = 100$, if $\sigma \in [16, 50]$, we have $1471.3 \leq Variance(S_T) \leq 8563.69$. Note that choosing large values for σ seems to be "sensible" in order to fit the usual values taken by the practiciens in the Vasicek model.

Let us first present the figures obtained for European options using the Vasicek model.

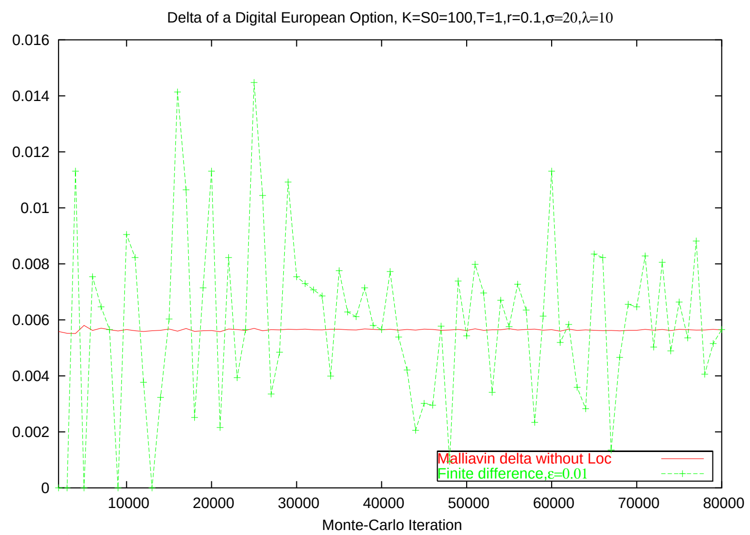

FIG. IX.4 – Delta of an European digital option using Malliavin calculus and finite Difference Method. Vasicek model.

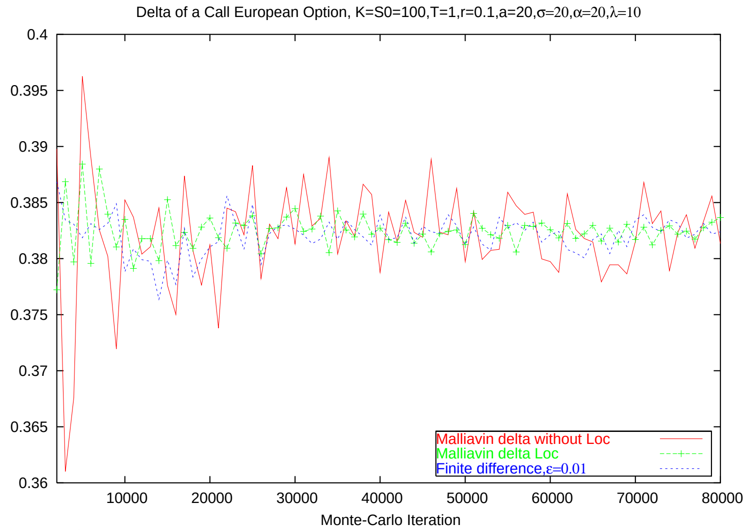

FIG. IX.5 – Delta of an European call option using Malliavin calculus and finite Difference Method. Vasicek model.

We now present the results obtained for European and Asian options using the geometrical model.

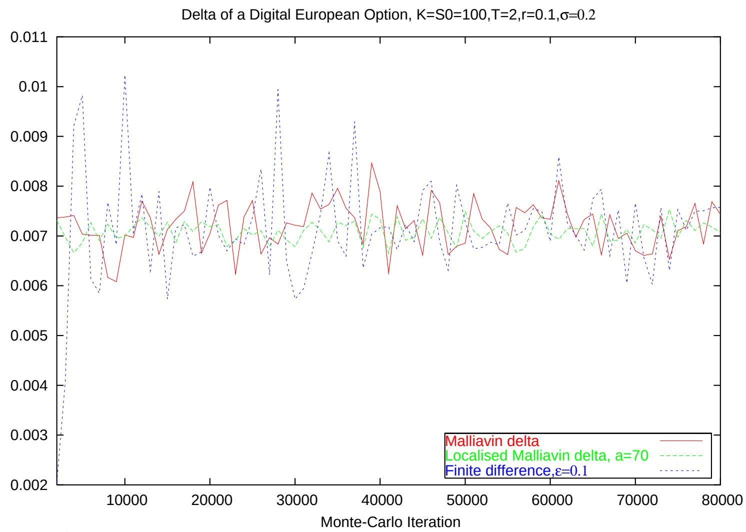

FIG. IX.6 – Delta of an European digital option using Malliavin calculus and finite Difference Method. Geometrical model.

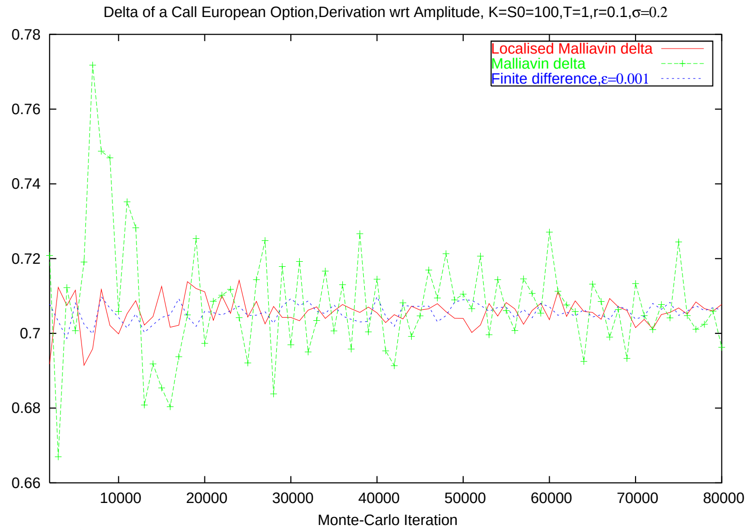

FIG. IX.7 – Delta of an European call option using Malliavin calculus and finite Difference Method. Geometrical model.

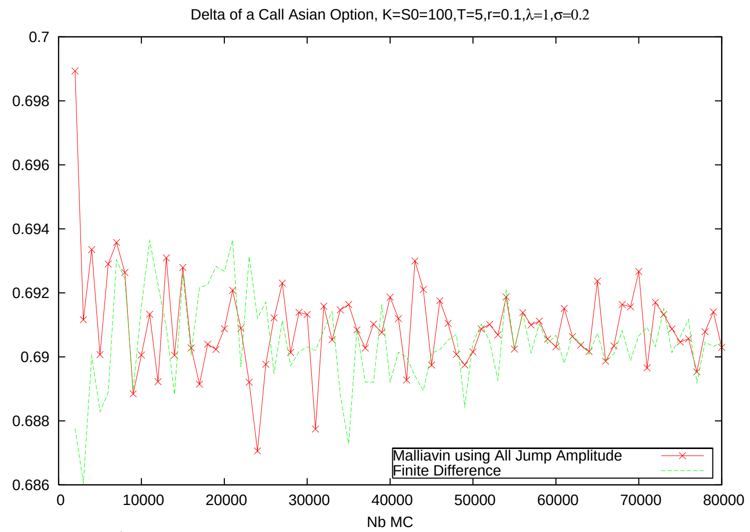

FIG. IX.8 – Delta of an Asian call option using localized Malliavin calculus and finite Difference Method. Geometrical model.

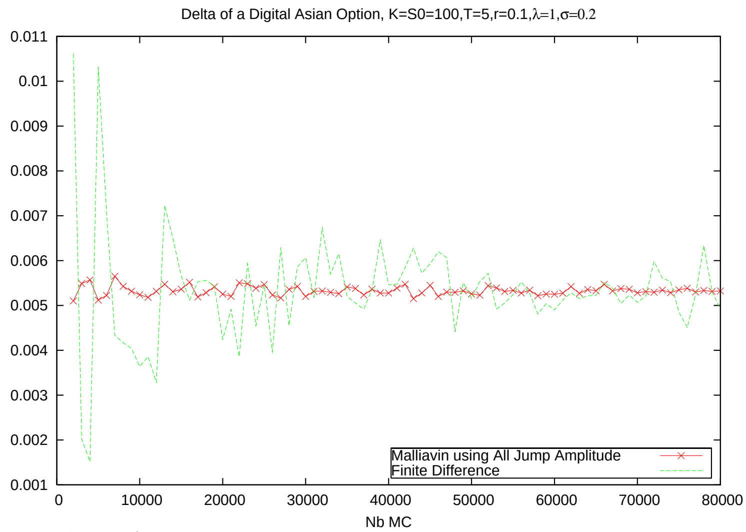

FIG. IX.9 – Delta of an Asian digital option using localized Malliavin calculus and finite Difference Method. Geometrical model.

We can numerically compute the Greeks for European and Asian options with a pure jump underlying process. We obtain numerical results similar to those in the Wiener case ([FLL+99] and [FLLL01]).
For European and Asian call options, the Malliavin estimator has larger variance than the finite difference one (see figures IX.5, IX.7 and IX.8) : the finite difference method approximates the first derivative of the payoff, whereas the Malliavin estimator contains a weight (independent on the payoff), which may increase the variance. The localization method detailed above allows us to reduce the variance of the Malliavin estimator.
On the opposite, the Malliavin estimator of digital options has lower variance than the finite difference one (see figures IX.4 and IX.6) and so does not need to be localized : in this case, the first derivative of the payoff is a Dirac function and, contrary to the finite difference method, the Malliavin calculus allows us to avoid this strong discontinuity.
Finally, note that for both call and digital options, the finite difference method requires to simulate twice more samples of the asset than the Malliavin method does : the finite difference method uses the samples starting from S_0 and those starting from $S_0 + \epsilon$. The Malliavin method is thus less time consuming.

2.2. Comparison jump Amplitudes-jump Times

Since we just noticed that, for call options, the Malliavin estimator is more efficient with localization than without, in all the simulations, we use a variance reduction method based on localization (as the one detailed in the beginning of this section). We compute Malliavin estimators using jump amplitudes or jump times.

In tables IX.1 and IX.2, we give the empirical variance of these estimators : we denote by ‘Var Mall JT’ (respectively ‘Var Mall AJ’) the variance of the Malliavin estimator based on jump times (respectively jump amplitudes). Moreover, we compare them to the finite difference estimator, that we denote by ‘Var Diff’.
We also mention in tables IX.1 and IX.2 the value of the volatility σ that we use and the corresponding variance of the underlying, denoted by $Variance(S_t)$.
We use the following abreviations :

- AJ : Jump Amplitudes
- AJ1 : one jump amplitude only
- JT : Jump times
- FD : Finite difference
- G : Geometrical model
- V : Vasicek model
- Call : Call option
- Dig : digital option.

Then (V/Dig/AJ) means that we deal with the Vasicek model (V), with a digital option (Dig) and we use an algorithm based on the amplitudes of the jumps (AJ). (V/Dig/AJ) versus (V/Dig/JT) means that we compare these two estimators.

Let us compare the variance of the Malliavin estimators based on jump times or amplitudes for the Vasicek model.

• **(V/Call/AJ) versus (V/Call/JT) versus (V/Call/FD)**

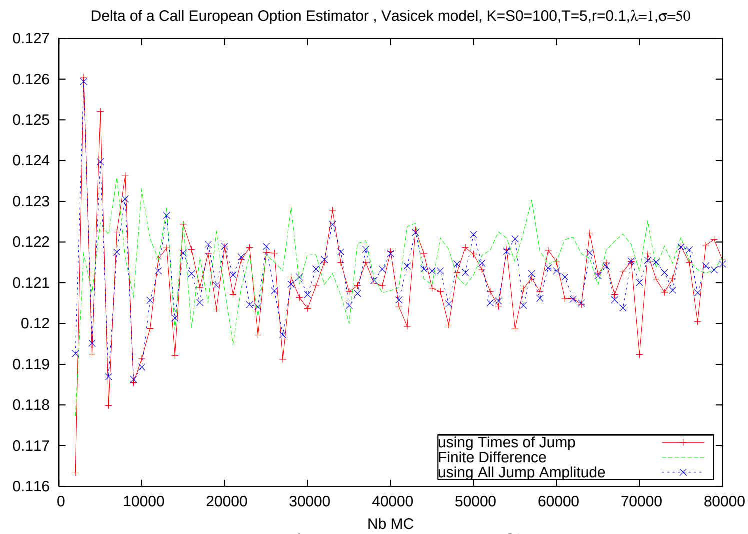

FIG. IX.10 – Vasicek model. Delta of an European Call option using Malliavin calculus based on jump times, on jump amplitudes, and finite difference method.

σ	$Variance(S_T)$	$VarMallJT$	$VarMallAJ$	$VarDiff$
15.8114	796.241	0.0285123	0.0106426	0.0300379
16.6667	897.577	0.0417219	0.0115955	0.0298567
17.6777	991.453	0.0400695	0.013123	0.0298904
18.8982	1134.11	0.0410136	0.0144516	0.0299574
20.4124	1313.42	0.0433065	0.0162378	0.029862
22.3607	1584.9	0.0400481	0.0178726	0.0298987
25	1967.53	0.0407136	0.0202055	0.0299007
28.8675	2604.22	0.0362728	0.0224265	0.0299651
35.3553	3961.31	0.0343158	0.0253757	0.0297775
50	7890.4	0.0333298	0.0287716	0.0299749

TAB. IX.1 – variance of the Malliavin JT estimator, AJ estimator and of the FD for Call option in the Vasicek model.

• **(V/Dig/AJ) versus (V/Dig/JT) versus (V/Dig/FD)**

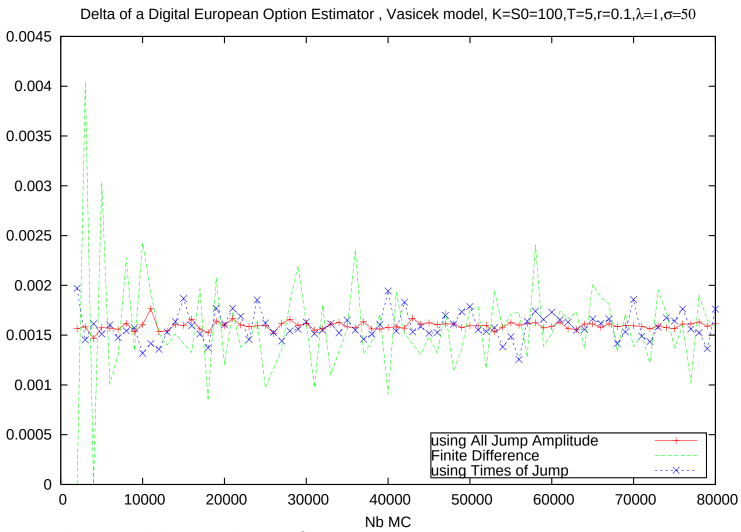

FIG. IX.11 – Vasicek model. Delta of an European Digital option using Malliavin calculus based on the jump amplitudes, on the jump times, and finite difference method.

σ	$Variance(S_T)$	$VarMallJT$	$VarMallAJ$	$VarDiff$
15.8114	796.241	0.00144622	$7.18878e-5$	0.00514743
16.6667	897.577	0.00254652	$7.3629e-5$	0.00459619
17.6777	991.453	0.0018011	$7.85552e-5$	0.00496369
18.8982	1134.11	0.0109864	$8.14005e-5$	0.00477995
20.4124	1313.42	0.00177648	$8.1627e-5$	0.00386111
22.3607	1584.9	0.00152777	$8.06193e-5$	0.00496369
25	1967.53	0.0013786	$7.94341e-5$	0.0062497
28.8675	2604.22	0.00100181	$7.5835e-5$	0.00551488
35.3553	3961.31	0.000617271	$6.95225e-5$	0.00459619
50	7890.4	0.000373802	$5.64325e-5$	0.00533116

TAB. IX.2 – Vasicek model. Variance of the Malliavin JT estimator, AJ estimator and of the FD for Digital option.

As we can see on figure IX.10 and IX.11, the comparison between the finite difference method and the Malliavin estimator using jump times leads to similar conclusions as the comparison of the Malliavin estimator using jump amplitudes with the finite difference method : for call options, these estimators are close, but for digital options, the Malliavin one is the most efficient.
On the other hand, if we look at tables IX.1 and IX.2, we note that $VarMallJT \geq VarMallAJ$. This means that the use of Malliavin calculus with respect to jump amplitudes leads to estimators with lower variance than those based on jump times.

Besides, another question arises : do we improve the numerical results by using as much noise as possible ? In other words, are there significantly differences between the variance of Malliavin estimators using all the jump amplitudes available and those based on one jump amplitude only ?

• **(V/Dig/AJ) versus (V/Dig/AJ1).**

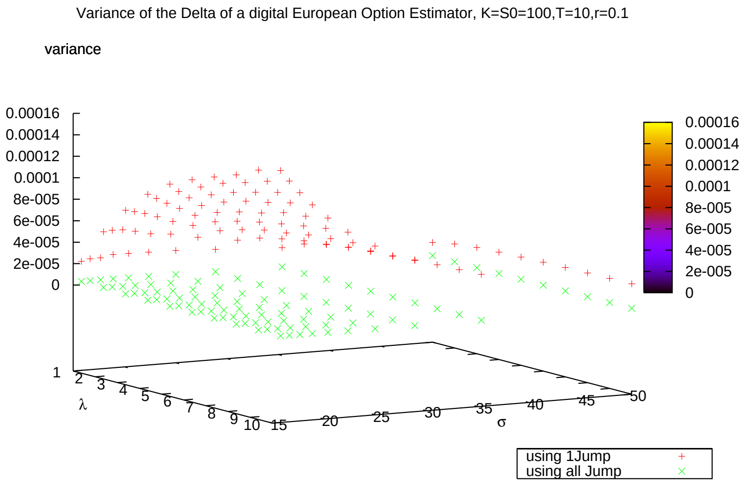

FIG. IX.12 – Variance of the Delta based on all jumps or one jump. Vasicek model.

Figure IX.12 shows that for the Vasicek model, when the jump intensity λ (which represents the quantity of noise available in the system) and the parameter σ (which represents the variance of the jump amplitudes for this model) increase, the Malliavin estimator using all jump amplitudes has a lower variance than the one using one jump only.

• **(G/Call/AJ) versus (G/Call/AJ1).**

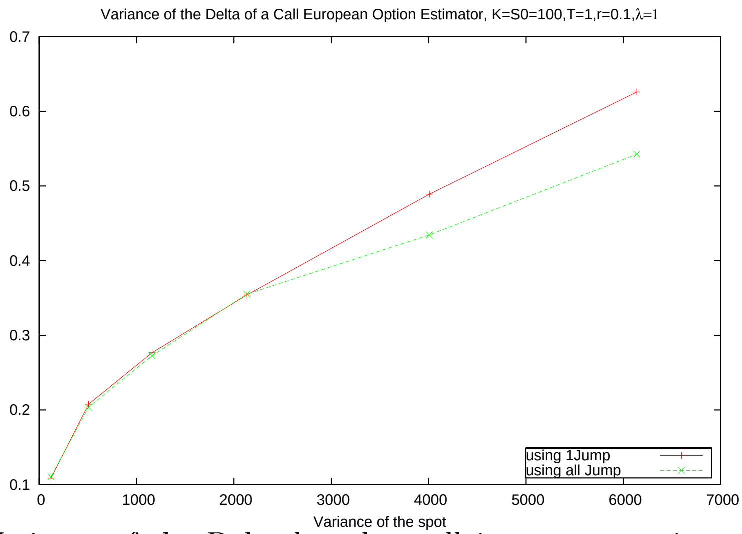

FIG. IX.13 – Variance of the Delta based on all jumps or one jump. Geometrical model.

For the geometrical model, we can observe on figure IX.13 that the variance of the Malliavin estimators increases when $Variance(S_t)$ increases as well. But the estimator using all jump amplitudes has always a lower variance than the other one based on one jump amplitude only.

3. The Merton process

In this section, we add a continuous part to the model (VIII.0.1) that we considered in Chapter VIII, that is we deal with the Merton model :

$$S_t = x + \sum_{i=1}^{J_t} c(T_i, \Delta_i, S_{T_i^-}) + \int_0^t g(u, S_u)\, du + \int_0^t \sigma(u, S_u)\, dW_u\,, \qquad \text{(IX.3.1)}$$

where the coefficients g and c satisfy hypothesis VIII.1. We assume moreover (for the existence and uniqueness of equation (IX.3.1)) that

Hypothesis IX.1. The function $x \to \sigma(u,x)$ is twice continuously differentiable and there exists a constant $C > 0$ such that :

$$|\sigma(u,x)| \leq C\,(1+|x|) \text{ and } |\partial_x \sigma(u,x)| + \left|\partial_x^2 \sigma(u,x)\right| \leq C\,.$$

Concerning the law of the jump amplitudes, we assume that it has a discontinuous density on $\mathbb{R}$, denoted by p, which satisfies hypothesis VIII.2.
We present two alternative calculus for this model. The first one is based on the Brownian motion only, which actually corresponds to the standard Malliavin calculus, and the second one is based on both the Brownian motion and the jump amplitudes. Since the law of the Brownian increments is continuous on the whole $\mathbb{R}$, we may perform an integration by parts formula using the Brownian motion only under the following hypothesis :

Hypothesis IX.2. There exists $\epsilon > 0$ such that

$$|\sigma(u,x)| \geq \epsilon\,.$$

This assumption actually represents a 'non-degeneracy condition' for the Brownian motion, and can be seen as the counterpart of condition (VIII.2.8) settled in Proposition VIII.2 for jump amplitudes with smooth density.
In order to compute the Malliavin operators of S_t, we first express it as a simple functional, which requires to introduce an Euler scheme.

3.1. Merton process and Euler scheme

Suppose that the jump times $T_1 < \ldots < T_n$ are given (this means that we have already simulated $T_1, \ldots, T_n$ in a Monte-Carlo algorithm). We include them in the discretization grid : we consider a time grid
$0 = t_0 < t_1 < \ldots < t_m < \ldots < t_M = T$ and we assume that $T_i = t_{m_i}$, $i = 1, \ldots, n$ for some $m_1 < \ldots < m_n$. For $t > 0$, we denote $m(t) = m$ if $t_m \leq t < t_{m+1}$. Then the corresponding Euler scheme is given by

$$\hat{S}_t = x + \sum_{i=1}^{J_t} c(T_i, \Delta_i, \hat{S}_{T_{i-}}) + \sum_{k=0}^{m(t)-1} \sigma(t_k, \hat{S}_{t_k})\,(W_{t_{k+1}} - W_{t_k}) + \sum_{k=0}^{m(t)-1} g(t_k, \hat{S}_{t_k})(t_{k+1} - t_k)\,.$$

Following the method of Chapter VIII, section 1, we introduce the following deterministic equation :

$$\hat{s}_t = x + \sum_{i=1}^{J_t} c(u_i, a_i, \hat{s}_{u_i^-}) + \sum_{k=0}^{m(t)-1} \sigma(t_k, \hat{s}_{t_k})\,\Delta_k w + \sum_{k=0}^{m(t)-1} g(t_k, \hat{s}_{t_k})\,(t_{k+1} - t_k)\,, \quad \text{(IX.3.2)}$$

where we have denoted by $\Delta_k w = w_{t_{k+1}} - w_{t_k}$. Then equation (IX.3.2) allows us to express $\hat{S}_t$ as a twice differentiable simple functional, say

$$\hat{S}_t = \sum_{k=1}^{\infty} \hat{s}_t(T_1, \ldots, T_k, \Delta_1, \ldots, \Delta_k, \Delta_0 W, \ldots, \Delta_{m(t)-1} W)\,\mathbf{1}_{\{J_t = k\}}\,,$$

where $\Delta_k W = W_{t_{k+1}} - W_{t_k}$. We thus have on $\{J_t = k\}$:

$$\begin{aligned}
\partial_{\Delta_i} \hat{S}_t &= \partial_{a_i} \hat{s}_t(T_1, \ldots, T_k, \Delta_1, \ldots, \Delta_k, \Delta_0 W, \ldots, \Delta_{m(t)-1} W)\,,\\
\partial^2_{\Delta_j, \Delta_i} \hat{S}_t &= \partial^2_{a_j, a_i} \hat{s}_t(T_1, \ldots, T_k, \Delta_1, \ldots, \Delta_k, \Delta_0 W, \ldots, \Delta_{m(t)-1} W)\,,\\
\partial_x \hat{S}_t &= \partial_x \hat{s}_t(T_1, \ldots, T_k, \Delta_1, \ldots, \Delta_k, \Delta_0 W, \ldots, \Delta_{m(t)-1} W)\,,\\
\partial_{\Delta_k W} \hat{S}_t &= \partial_{\Delta_k w} \hat{s}_t(T_1, \ldots, T_k, \Delta_1, \ldots, \Delta_k, \Delta_0 W, \ldots, \Delta_{m(t)-1} W)\,.
\end{aligned}$$

We denote $\delta_k = t_{k+1} - t_k$. The first derivatives of $\hat{s}_t$ satisfy the following equations :

$$\partial_{a_i} \hat{s}_t = \partial_a c(u_i, a_i, \hat{s}_{u_i^-}) + \sum_{k=i+1}^{J_t} \partial_x c(u_k, a_k, \hat{s}_{u_k^-})\,\partial_{a_i} \hat{s}_{u_k^-} + \sum_{k=i}^{m(t)-1} \partial_x g(t_k, \hat{s}_{t_k})\,\partial_{a_i} \hat{s}_{t_k}\,\delta_k + \sum_{k=i}^{m(t)-1} \partial_x \sigma(t_k, \hat{s}_{t_k})\,\partial_{a_i} \hat{s}_{t_k}\,\Delta_k w\,, \quad \text{(IX.3.3)}$$

$$\partial_{\Delta_i w}\hat{s}_t = \sigma(t_i, \hat{s}_{t_i}) + \sum_{k=1}^{J_t} \partial_x c(u_k, a_k, \hat{s}_{u_k^-})\, \partial_{\Delta_i w}\hat{s}_{u_k^-} + \sum_{k=i}^{m(t)-1} \partial_x \sigma(t_k, \hat{s}_{t_k})\, \partial_{\Delta_i w}\hat{s}_{t_k}\, \Delta_k W + \sum_{k=i}^{m(t)-1} \partial_x g(t_k, \hat{s}_{t_k})\, \partial_{\Delta_i w}\hat{s}_{t_k}\, \delta_k\,, \quad \text{(IX.3.4)}$$

For higher order derivatives, one may derive similar equations.
We now have the choice of using integration by parts formula using the Brownian increments $\Delta_i W$ only, or both $\Delta_i W$ and the jump amplitudes Δ_i. In each case we have different forms for the differential operators.
Let us denote $\sigma^{\Delta}_{\pi,t}$ the covariance matrix corresponding to the jump amplitudes, σ^W_t the one corresponding to the Brownian increments, and $\sigma^{\Delta,W}_{\pi,t}$ the one corresponding to both of them. As the density of the jump amplitudes Δ_i may have discontinuities on $\mathbb{R}$, $\sigma^{\Delta}_{\pi,t}$ involves some weights π (see Chapter VIII, section 2.1) introduced in equation (VIII.2.1). We then have on $\{J_t = k\}$, for $k \geq 1$,

$$\begin{aligned}
\sigma^{\Delta,W}_{\pi,t} &:= \sum_{i=1}^{k} \pi(\Delta_i)\ |\partial_{a_i}\hat{s}_t|^2 + \sum_{i=0}^{m(t)-1} |\partial_{\Delta_i w}\hat{s}_t|^2\ , \\
\sigma^{\Delta}_{\pi,t} &:= \sum_{i=1}^{k} \pi(\Delta_i)\ |\partial_{a_i}\hat{s}_t|^2\ , \\
\sigma^W_t &:= \sum_{i=0}^{m(t)-1} |\partial_{\Delta_i w}\hat{s}_t|^2\ .
\end{aligned}$$

The other differential operators will change in a similar way.
Note that $\Delta_i W \sim \mathcal{N}(0, t_{i+1} - t_i)$ so that the corresponding Ornstein Uhlenbeck operator $\mathrm{L}^W := \sum_{i=0}^{m(t)-1} \mathrm{L}^W_i$ is given by

$$\mathrm{L}^W_i \hat{s}_t = \partial^2_{\Delta_i w}\hat{s}_t + \theta^W_i\, \partial_{\Delta_i w}\hat{s}_t,\ \text{with } \theta^W_i = -\frac{\Delta_i W}{t_{i+1} - t_i}\,.$$

The other Ornstein-Uhlenbeck operators will have the following expressions

$$\begin{aligned}
\mathrm{L}^{\Delta}_{\pi}\hat{S}_t &= -\sum_{i=1}^{J_t} \pi(\Delta_i)\, \partial^2_{\Delta_i}\hat{s}_t + \left(\pi'(\Delta_i) + \pi(\Delta_i)\, \partial \ln p(\Delta_i)\right)\, \partial_{\Delta_i}\hat{s}_t\,, \\
\mathrm{L}^{\Delta,W}_{\pi}\hat{S}_t &= \mathrm{L}^{\Delta}_{\pi}\hat{S}_t + \mathrm{L}^W \hat{S}_t\,.
\end{aligned}$$

Note that if $m = m(t)$, then

$$\sigma^{\Delta,W}_{\pi,t} \geq \sigma^W_t \geq \left|\partial_{\Delta_{m-1} W}\hat{s}_t\right|^2 = \left|\sigma(t_{m-1}, \hat{s}_{t_{m-1}})\right|^2 \geq \epsilon^2 > 0\,.$$

Thus, the non degeneracy condition VII.6 (that is $\gamma_t^W \in \mathrm{L}^4(A)$, with $A = \{J_t \geq 1\}$) is satisfied. Hence, Proposition VIII.2 affirms that we can perform an integration by parts formula using the Brownian motion only, as well as both Brownian motion and jump amplitudes (since $\gamma_{\pi,t}^{\Delta,W} \leq \gamma_t^W$). Note that the first case leads to the same calculus as in [DJ06] and [PD04].
Even if the density of the jump amplitudes is smooth (so that $\pi(\Delta_i) = 1$), it is more delicate to prove that the non degeneracy condition VII.6 holds true by using the inequality $\sigma_t^{\Delta,W} \geq \sigma_t^{\Delta} \geq |\partial_{a_n} \hat{s}_t|$. Indeed, in view of equation (IX.3.3), it is not easy to prove that $|\partial_{a_n} \hat{s}_t| \geq c > 0$.

3.2. Malliavin estimators

Concerning the numerical experiments, we deal with the Merton model (IX.0.3), that is

$$S_t = x + \int_0^t r\, S_u\, du \;+ \int_0^t \sigma\, S_u\, dW_u + \mu \sum_{i=1}^{J_t} S_{T_i^-}\, \Delta_i\,,$$

where W is a Brownian motion independent on the compound Poisson process N, whose jump times and amplitudes are denoted by $(T_i)_{i\in\mathbb{N}}$ and $(\Delta_i)_{i\in\mathbb{N}}$. We suppose that the jump amplitudes Δ_i are independent, identically and Gaussian distributed, so that we take $\pi(\Delta_i) = 1$.
Let us compute the Malliavin weight $H(S_T, \partial_x S_T)$ coming from an integration by parts formula using both Brownian motion and jump amplitudes. Let us denote $D_i^{\Delta}(S_t) := \partial_{\Delta_i} S_t$ and $D_i^W(S_t) := \partial_{\Delta_i W} S_t$. Recall that we define

$$H(S_T, \partial_x S_T) = H^{\Delta}(S_T, \partial_x S_T) + H^W(S_T, \partial_x S_T)\,, \tag{IX.3.5}$$

where $H^{\Delta}(S_T, \partial_x S_T)$ (respectively $H^W(S_T, \partial_x S_T)$) is the Malliavin weight using the jump amplitudes (respectively Brownian motion) only. We have

$$\begin{aligned} H^{\Delta}(S_T, \partial_x S_T) = \partial_x S_T\, \gamma_{S_T}^{\Delta}\, \mathrm{L}^{\Delta} S_T - \gamma_{S_T}^{\Delta} \; &< D^{\Delta} S_T, D^{\Delta}(\partial_x S_T) > \\ &- \partial_x S_T \; < D^{\Delta} S_T, D^{\Delta} \gamma_{S_T}^{\Delta} > \;. \end{aligned}$$

Similarly, $H^W(S_T, \partial_x S_T)$ is derived from the previous equation by taking the operators $\gamma_{S_T}^W$, $\mathrm{L}^W S_T$ and D^W. Let us compute all these operators.
We have the following explicit solution :

$$S_T = S_0 \exp\left((r - \frac{\sigma^2}{2})\, T + \sigma\, W_T \right) \prod_{j=1}^{J_T} (1 + \mu\, \Delta_j)\,. \tag{IX.3.6}$$

On $\{J_T = n\}$, $n \in \mathbb{N}^*$, the source of randomness is $(\Delta_1, \ldots, \Delta_n, W_T)$, and then for all $i \in \{1, \ldots, n\}$,

$$\begin{aligned} D_i^{\Delta}(S_T) &:= \frac{\partial S_T}{\partial \Delta_i} = \frac{\mu\, S_T}{1 + \mu\, \Delta_i} \\ D^W(S_T) &:= \frac{\partial S_T}{\partial W_T} = \sigma\, S_T\,. \end{aligned}$$

Then we can compute all the terms involved in the Malliavin weight $H(S_T, \partial_x S_T)$.

$$\begin{aligned} D_i^{\Delta}(D_i^{\Delta} S_T) &= 0 \\ D^W(D^W S_T) &= \sigma^2\, S_T \\ Y_T &= \frac{S_T}{x} \\ D_i^{\Delta}(Y_T) &= \frac{\mu\, S_T}{x\,(1 + \mu\, \Delta_i)} \\ D^W(Y_T) &= \sigma\, Y_T\,. \end{aligned}$$

The covariance matrix corresponding to both jump amplitudes and Brownian motion is

$$\sigma_T = |D^W(S_T)|^2 + \sum_{i=1}^{n} |D_i^{\Delta}(S_T)|^2 = \mu^2\, S_T^2 \sum_{j=1}^{n} \frac{1}{(1 + \mu\, \Delta_j)^2} + \sigma^2\, S_T^2\,.$$

Straightforward computations give

$$\begin{aligned} D_i^{\Delta}(\sigma_T) &= \frac{2\,\mu^3\, S_T^2}{(1 + \mu\, \Delta_i)}(\widetilde{A}_\mu - \frac{1}{(1 + \mu\, \Delta_i)^2}) + \frac{2\,\sigma^2\, \mu\, S_T 2}{1 + \mu\, \Delta_i} \\ D^W(\sigma_T) &= 2\,\sigma\, \mu^2\, S_T^2\, \widetilde{A}_\mu + 2\,\sigma^3\, S_T^2 \\ D_i^{\Delta}(\gamma_T) &= -\frac{D_i^N(\sigma_T)}{\sigma_T 2} \\ D^W(\gamma_T) &= -\frac{D_i^W(\sigma_T)}{\sigma_T 2}\,, \end{aligned}$$

where $\widetilde{A}_\mu$ is given by equation (IX.1.6), that is $\widetilde{A}_\mu := \sum_{j=1}^{n} \frac{1}{(1 + \mu\, \Delta_j)^2}$.

Finally, putting these terms together in equation (IX.3.5), we get the following Malliavin weight :

$$H(S_T, \partial_x S_T) = \frac{\mu\, \widetilde{B}_\mu + \sigma\, \frac{W_T}{T} - \sigma^2}{x\,(\mu^2\, \widetilde{A}_\mu + \sigma^2)} + \frac{1}{x} - \frac{2\,\mu^4\, \widetilde{C}_\mu}{x\,(\mu^2\, \widetilde{A}_\mu + \sigma^2)^2}\,, \qquad \text{(IX.3.7)}$$

where $\widetilde{B}_\mu$ and $\widetilde{C}_\mu$ are defined by equations (IX.1.7) and (IX.1.8), that is

$$\widetilde{B}_\mu := \sum_{j=1}^{n} \frac{\Delta_j}{(1+\mu\,\Delta_j)} \text{ and } \widetilde{C}_\mu := \sum_{j=1}^{n} \frac{1}{(1+\mu\,\Delta_j)^4}\,.$$

3.3. Numerical results

Recently in [DJ06] and [PD04], the Delta of an European option is computed by using Malliavin calculus with respect to the Brownian motion only. Note that if we use our integration by parts formula, just taking into account the derivatives with respect to the Brownian motion, we find $H(S_T, \partial_x S_T) = \dfrac{W_T}{x\,\sigma\,T}$, which is exactly the weight obtained in [PD04] (as well as in Black-Scholes model). So the difference between our algorithm and the one in [PD04] comes from the additional term (corresponding to the derivatives with respect to the jump amplitudes) which appears in our Malliavin weight $H(S_T, \partial_x S_T)$ in equation (IX.3.7).
In figure IX.15, we compare the two algorithms, and in table [IX.3], we give the quotient between the empirical variances of the two algorithms. It turns out that the variance of the Brownian-jump algorithm (presented here) is smaller than the variance of the pure Brownian algorithm (presented in [PD04]). Moreover, the difference increases with the number of jumps up to T : this happens when the maturity T is larger or when the intensity λ of the Poisson measure is larger. We conclude that the more noise one uses in the integration by parts formula, better the algorithm works (there is no theoretical result in this sense, but only numerical evidence).

T \ λ	1	4	8	12
1	2,15	7,27	19,88	16,43
2	1,72	12,17	22,12	36,44
3	2,94	7,15	24,30	35,58

TAB. IX.3 – $\frac{Brownian\ variance}{Brownian-Jump\ variance}$ for Digital delta for various maturities and jump intensities

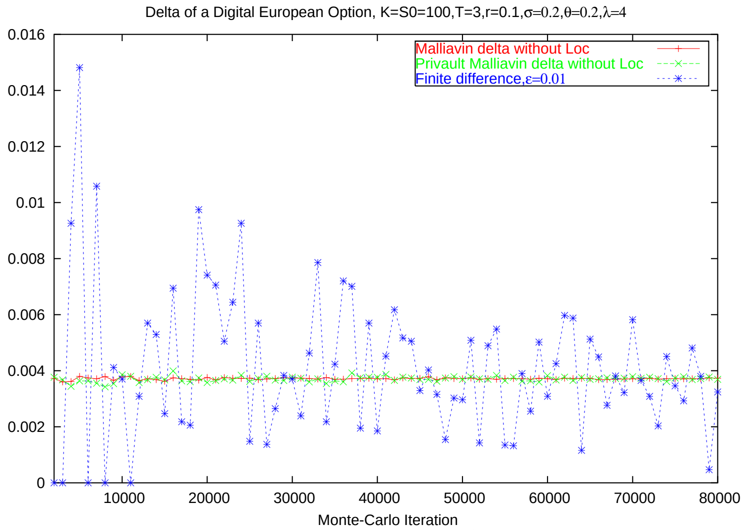

FIG. IX.14 – Delta of Digital option for a Merton Process

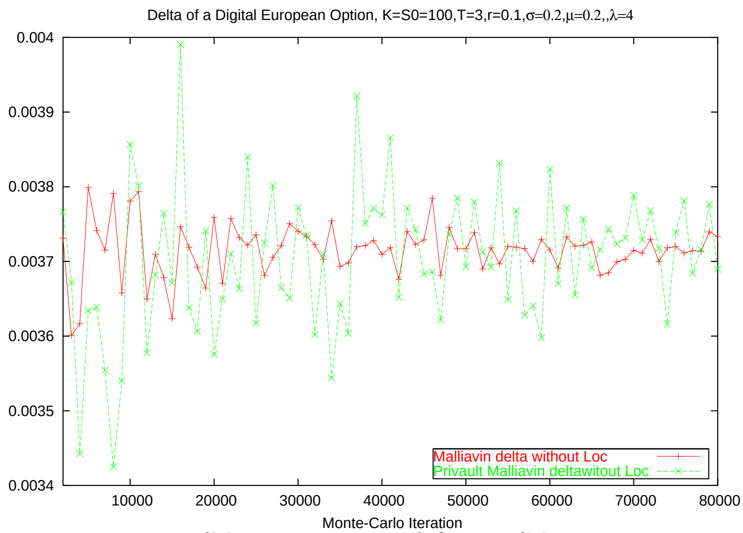

FIG. IX.15 – Zoom of figure IX.14

Pricing and Hedging American Options X

Introduction

The aim of this chapter is to compute the price $P(0,x)$ and the Delta $\Delta(0,x) = \partial_x P(0,x)$ of an American option with payoff function ϕ and maturity T, on an underlying asset whose price $(S_t)_{t\in[0,T]}$ is a pure jump diffusion process.
Let us come back to the beginning of Chapter VIII. We work with the Poisson point measure $N(dt,da)$ defined there, and we suppose that, under the historical probability $\mathbf{P}$, the price $(S_t)_{t\in[0,T]}$ follows the jump diffusion equation (VIII.0.1), that is

$$\begin{aligned} S_t &= x + \sum_{i=1}^{J_t} c(T_i, \Delta_i, S_{T_i^-}) + \int_0^t b(r, S_r)\, dr\,, \\ &= x + \int_0^t \int_{\mathbb{R}} c(s, a, S_{s^-})\, dN(s,a) + \int_0^t b(r, S_r)\, dr\,, \qquad 0 \le t \le T\,. \end{aligned}$$

We assume that the coefficients b and c satisfy hypothesis VIII.1.
We denote by λ the jump intensity, which means that $T_i - T_{i-1}$ are exponentially distributed with parameter λ.
Let $\alpha < \beta$ (we may take $\alpha = -\infty$ and $\beta = +\infty$). We suppose that the law of the jump amplitudes Δ_i is absolutely continuous on (α, β) with respect to the Lebesgue measure. Denoting by $p(y) := e^{\rho(y)}$ its density, we assume that p satisfies hypothesis VIII.2.
Under the hypothesis of absence of arbitrage opportunity, there exists a measure $\mathbf{Q}$ equivalent to the historical probability $\mathbf{P}$ under which the discounted price of the financial asset is a $\mathbf{Q}$-martingale. In Particular, assuming that the spot rate r is constant, the discounted underlying $\widetilde{S}_t = e^{-r\,t}\, S_t$ is a martingale under $\mathbf{Q}$. In the following, we work under the martingale measure $\mathbf{Q}$ which cancels the drift of

$(\widetilde{S}_t)_{t\in[0,T]}$. The (risk-neutral) dynamic of $(S_t)_{t\in[0,T]}$ under $\mathbf{Q}$ is then given by

$$\begin{aligned} S_t &= x + \int_0^t g(u,S_u)\,du + \sum_{i=1}^{J_t} c(T_i,\Delta_i,S_{T_i^-}) \\ &= x + \int_0^t r\,S_u\,du + \int_0^t\int_{\mathbb{R}} c(u,a,S_{u^-})\,\widetilde{N}(du,da)\,, \end{aligned} \tag{X.0.1}$$

where $g(u,S_u) = r\,S_u - \displaystyle\int_{\mathbb{R}} c(u,a,S_u)\,\nu(da)$.
Let us consider the filtration $(\mathcal{F}_t)_{t\geq 0}$ defined by $\mathcal{F}_t = \sigma\left(N(s,A), s\leq t, A\in\mathcal{B}(\mathbb{R})\right)$.
Then the price $P(t,S_t)$ at time t of the American option of payoff ϕ and maturity T is given by

$$P(t,S_t) = \max_{\tau\in\Gamma_{t,T}} \mathrm{E}_{\mathbf{Q}}\left(e^{-r(\tau-t)}\,\phi(S_\tau)\mid\mathcal{F}_t\right)\,, \tag{X.0.2}$$

where $\Gamma_{t,T}$ denotes the set of all the stopping times taking values in $[t,T]$.
In order to compute the price $P(0,x)$ at time 0 and the Delta $\Delta(0,x) := \partial_x P(0,x)$, we will first use the integration by parts formulas (based on jump amplitudes) settled in Proposition VIII.3 to derive representation formulas for conditional expectations and their gradients. We will then use these representations in dynamic programming equations to perform a Monte-Carlo algorithm.
Finally, we apply the previous results obtained in an abstract framework to the computation of the price and the Delta of American call options with payoff $\phi(x) = (x-K)_+$ and American digital options with payoff $\phi(x) = \mathbf{1}_{x\geq K}$, when the asset $(S_t)_{t\in[0,T]}$ follows the geometrical model :

$$S_t = x + \int_0^t r\,S_u\,du + \int_0^t\int_{\mathbb{R}} \sigma\,a\,S_{u^-}\,N(du,da)\,,\ t\in[0,T]\,.$$

1. Representation formulas for conditional expectations and their gradients

As we will apply Proposition VIII.3, we consider the framework described in Chapter VIII, section 3 :
• We suppose that there exists a finite number of jumps on $[0,T]$, that is there exists $M\in\mathbb{N}^*$ such that $J_T = M$.
• We suppose that there exists $\varepsilon > 0$ such that

$$|\partial_a c(u,a,x)| \geq \varepsilon \text{ and } |1+\partial_x c(u,a,x)| \geq \varepsilon\,.$$

• Since the density is not smooth on (α,β), we work with the weights introduced in equation (VIII.3.2) : denoting γ as the middle of (α,β), and taking $\delta\in(0,1/3)$, we put

$$\pi^i_{(k,s,t)}(\omega,\Delta_i) := \mathbf{1}_{]s,t]}(T_i(\omega)) \times \pi_k(\Delta_i)\,,$$

with

$$\pi_1(y) := \begin{cases} (\gamma-y)^\delta\,(y-\alpha)^\delta & \text{for } y\in(\alpha,\gamma) \\ 0 & \text{for } y\notin(\alpha,\gamma)\,, \end{cases}$$

and

$$\pi_2(y) := \begin{cases} (\beta-y)^\delta\,(y-\gamma)^\delta & \text{for } y\in(\gamma,\beta) \\ 0 & \text{for } y\notin(\gamma,\beta)\,. \end{cases}$$

Hence, we can state the following representation formulas :

Theorem X.1:
(i) For all $0\le s<t\le T$, for all $\phi\in\mathcal{C}^1_p(\mathbb{R})$, one has

$$\mathrm{E}\left(\phi(S_t)\,\mathbf{1}_{\{0<J_s<J_t;J_T=M\}}\mid S_s=\alpha\right) = \frac{\mathbb{T}_{s,t}[\phi](\alpha)}{\mathbb{T}_{s,t}[1](\alpha)}\,\mathbf{1}_{\{0<J_s<J_t;J_T=M\}}\,,$$

where for all f,

$$\mathbb{T}_{s,t}[f](\alpha) = \mathrm{E}\left(f(S_t)\,H(S_s-\alpha)\,V_{(1,s,t)}\,\mathbf{1}_{\{0<J_s<J_t;J_T=M\}}\right)\,, \tag{X.1.1}$$

with $H(z)=\mathbf{1}_{z\ge 0}$, $z\in\mathbb{R}$, and $V_{(1,s,t)}$ being introduced in equation (VIII.3.4) .
(ii) For all $0\le s<t\le T$, for all $\phi\in\mathcal{C}^1_p(\mathbb{R})$ and $\alpha>0$, one has

$$\begin{aligned}\partial_\alpha \mathrm{E}&\left(\phi(S_t)\,\mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\mid S_s=\alpha\right)\\ &= \frac{\mathbb{R}_{s,t}[\phi](\alpha)\,\tilde{\mathbb{T}}_{s,t}[1](\alpha)-\tilde{\mathbb{T}}_{s,t}[\phi](\alpha)\,\mathbb{R}_{s,t}[1](\alpha)}{\tilde{\mathbb{T}}^2_{s,t}[1](\alpha)}\,\mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\,, \end{aligned}\tag{X.1.2}$$

where $\tilde{\mathbb{T}}_{s,t}[f](\alpha) := \mathrm{E}\left(f(S_t)\,H(S_s-\alpha)\,V_{(1,s,t)}\,\mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right)$, and

$$\mathbb{R}_{s,t}[f](\alpha) = -\mathrm{E}\left(f(S_t)\,H(S_s-\alpha)\,\mathcal{H}_{s,t}\,\mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right)\,, \tag{X.1.3}$$

where $\mathcal{H}_{s,t}$ is introduced in equation (VIII.3.5).

Proof. The result (i) comes from Lemma VII.9 and Proposition VIII.3 (i).
Let us prove (ii). It sufficies to prove that for all $f\in\mathcal{C}^1_b(\mathbb{R})$,

$$\partial_\alpha\tilde{\mathbb{T}}_{s,t}[f](\alpha) = \mathbb{R}_{s,t}[f](\alpha)\,.$$

Let us define h_δ a $\mathcal{C}^\infty$ probability density function as follows :
we consider $\psi\in\mathcal{C}^\infty(\mathbb{R})$ such that Supp$\psi\in[-1,1]$ and $\displaystyle\int_{\mathbb{R}}\psi(t)\,dt=1$, and we put $h_\delta(t) := \dfrac{1}{\delta}\,\psi(\dfrac{t}{\delta})$ for $\delta>0$. Then h_δ is converging weakly to the Dirac mass δ_0 as $\delta\to 0$.

We denote $H_\delta(x) := \int_{-\infty}^{x} h_\delta(t)\,dt$. Then $H'_\delta = h_\delta$ and H_δ converges to H as $\delta \to 0$.
Let us denote

$$\tilde{\mathbb{T}}^{\delta}_{s,t}[f](\alpha) = \mathrm{E}\left(f(S_t)\, H_\delta(S_s - \alpha)\, V_{(1,s,t)}\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right)\,.$$

We then have

$$\tilde{\mathbb{T}}^{\delta}_{s,t}[f](\alpha) \underset{\delta\to 0}{\longrightarrow} \tilde{\mathbb{T}}_{s,t}[f](\alpha)\,. \tag{X.1.4}$$

Using Proposition VIII.3 (ii), we have

$$\begin{aligned}\partial_\alpha \tilde{\mathbb{T}}^{\delta}_{s,t}[f](\alpha) &= -\mathrm{E}\left(f(S_t)\, h_\delta(S_s - \alpha)\, V_{(1,s,t)}\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right)\\ &= -\mathrm{E}\left(f(S_t)\, H_\delta(S_s - \alpha)\, \mathcal{H}_{s,t}\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right)\,.\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\partial_\alpha \tilde{\mathbb{T}}^{\delta}_{s,t}[f](\alpha) - \mathbb{R}_{s,t}[f](\alpha)\right|\\ &\le \| f \|_\infty\ \mathrm{E}\left[|\mathcal{H}_{s,t}|\, |H_\delta - H|(S_s-\alpha)\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right]\\ &= \| f \|_\infty\ \mathrm{E}\left[|\mathcal{H}_{s,t}|\, |H_\delta - H|(S_s-\alpha)\, \mathbf{1}_{|S_s-\alpha|\le\delta}\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right]\\ &\le 2\ \| f \|_\infty\ \mathrm{E}\left(|\mathcal{H}_{s,t}|^{1+\eta}\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right)^{1/(1+\eta)}\ \mathrm{E}\left(\mathbf{1}_{|S_s-\alpha|\le\delta}\, \mathbf{1}_{\{0<J_s;J_T=M\}}\right)^{1/r}\ ,\end{aligned}$$

where $\eta > 0$ satisfies $\mathrm{E}\left(|\mathcal{H}_{s,t}|^{1+\eta}\, \mathbf{1}_{\{3<J_s;3<J_t-J_s;J_T=M\}}\right)^{1/(1+\eta)} < \infty$ and $1 = \frac{1}{r} + \frac{1}{1+\eta}$.
By Proposition VIII.6, we know that for all $s \in [0,T]$, $(\mathbf{1}_{\{J_s>0;J_T=M\}}\,\mathbf{P})\, S_s^{-1}$ is absolutely continuous on $\mathbb{R}$ with respect to the Lebesgue measure, so that we can write

$$\mathrm{E}\left(\mathbf{1}_{|S_s-\alpha|\le\delta}\, \mathbf{1}_{\{0<J_s;J_T=M\}}\right) = \int_{\alpha-\delta}^{\alpha+\delta} p_s(x)\,dx \le 2\,K_s\,\delta\,.$$

Hence,

$$\partial_\alpha \tilde{\mathbb{T}}^{\delta}_{s,t}[f](\alpha) \underset{\delta\to 0}{\longrightarrow} \mathbb{R}_{s,t}[f](\alpha), \text{ uniformly with respect to } \alpha\,. \tag{X.1.5}$$

Equations (X.1.4) and (X.1.5) finally give $\partial_\alpha \tilde{\mathbb{T}}_{s,t}[f](\alpha) = \mathbb{R}_{s,t}[f](\alpha)$. The proof is thus complete. ∎

2. Algorithms for the price and Delta computation

Let us first construct an approximation scheme $\overline{S}_t$ of S_t.
Recall from Chapter VIII-section 1 (in particular equation (VIII.1.2)), that S_t can be expressed as a simple functional of the jump times and amplitudes, that is $S_t = s_t(\widetilde{T}, \widetilde{\Delta})$, where $\widetilde{T} := (T_i)_{i\in\mathbb{N}}$ and $\widetilde{\Delta} := (\Delta_i)_{i\in\mathbb{N}^*}$, and s_t is the deterministic

equation introduced in (VIII.1.1) :

$$s_t = x + \sum_{i=1}^{J_t(u)} c(u_i, a_i, s_{u_i^-}) + \int_0^t g(r, s_r)\, dr\,, \quad 0 \le t \le T\,,$$

where $J_t(u) = k$ if $u_k \le t < u_{k+1}$. Hence, we will construct an approximation scheme for s_t.

We fix $L \in \mathbb{N}^*$ and we consider $0 = t_0 < t_1 < \ldots < t_L = T$ a discretization grid of the interval $[0, T]$ with step size $\varepsilon_k = t_k - t_{k-1}$. For $k = 0, \ldots, L-1$, we put

$$\overline{s}_{t_k} = x + \sum_{l=1}^{k} g(t_{l-1}, \overline{s}_{t_{l-1}})\, \varepsilon_l + \sum_{l=1}^{k} \sum_{t_{l-1} < u_i \le t_l} c(t_{l-1}, a_i, \overline{s}_{t_{l-1}})\,.$$

We then define

$$\overline{S}_{t_0} = x, \text{ and for all } k = 1, \ldots, L\,,\ \overline{S}_{t_k} = \overline{s}_{t_k}(\widetilde{T}, \widetilde{\Delta})\,.$$

Let us denote $\tau(t) := t_k$ if $t_k < t \le t_{k+1}$. Then, for all $t \ge 0$, we have

$$\overline{S}_t = x + \int_0^t r\, \overline{S}_{\tau(s)}\, ds + \int_0^t \int_{\mathbb{R}} c(\tau(s), a, \overline{S}_{\tau(s)^-})\, \widetilde{N}(ds, da)\,. \qquad \text{(X.2.1)}$$

The approximation error of this scheme is of order $\varepsilon := \max_{k=1,\ldots,L} \varepsilon_k$.

Proposition X.1:
There exists a positive constant C_T such that for all $t \le T$

$$\mathrm{E}_{\mathbf{Q}}\left[\sup_{s \le t} \left|S_s - \overline{S}_s\right|^2\right] \le C_T\, \varepsilon\,.$$

Proof. For all $s \le t$ we have

$$\left|S_s - \overline{S}_s\right|^2 \le 2\, r^2 \int_0^s \left|S_u - \overline{S}_{\tau(u)}\right|^2 du + 2 \sup_{s \le t} \left| \int_0^s \int_{\mathbb{R}} (c(u, a, S_{u^-}) - c(\tau(u), a, \overline{S}_{\tau(u)^-}))\, \widetilde{N}(du, da) \right|^2 .$$

Using Doob's inequality in the last term, we then obtain

$$\mathrm{E}_{\mathbf{Q}}\left(\sup_{s \le t} \left|S_s - \overline{S}_s\right|^2\right) \le 2\, r^2\, \mathrm{E}_{\mathbf{Q}}\left[\int_0^t \left|S_u - \overline{S}_{\tau(u)}\right|^2 du\right] + C\, \mathrm{E}_{\mathbf{Q}}\left[\int_0^t \int_{\mathbb{R}} \left|c(u, a, S_u) - c(\tau(u), a, \overline{S}_{\tau(u)})\right|^2 du\, \nu(da)\right]. \qquad \text{(X.2.2)}$$

By hypothesis VIII.1, we have

$$\begin{aligned}&\left|c(u,a,S_u)-c(\tau(u),a,\overline{S}_{\tau(u)})\right|\\ \leq&\left|c(u,a,S_u)-c(\tau(u),a,S_{\tau(u)})\right|+\left|c(\tau(u),a,S_{\tau(u)})-c(\tau(u),a,\overline{S}_{\tau(u)})\right|\\ \leq&K\,\left(\varepsilon+\left|S_u-S_{\tau(u)}\right|+\left|S_{\tau(u)}-\overline{S}_{\tau(u)}\right|\right)\,.\end{aligned}$$

Let us recall that the solution $(S_t)_{t\in[0,T]}$ of equation (X.0.1) satisfies $\mathrm{E}_{\mathbf{Q}}\left(|S_t-S_u|^2\right)\leq K_T\,|t-u|$. We thus obtain (since $\nu(\mathbb{R})=1$)

- $$\begin{aligned}\mathrm{E}_{\mathbf{Q}}\left[\int_0^t\int_{\mathbb{R}}\left|S_u-S_{\tau(u)}\right|^2\,\nu(da)\,du\right]&=\int_0^t\mathrm{E}_{\mathbf{Q}}\left(\left|S_u-S_{\tau(u)}\right|^2\right)\,du\\ &\leq K_T\,\int_0^t|u-\tau(u)|^2\,du\\ &\leq(K_T\,T)\,\varepsilon\,.\end{aligned}$$

- $$\begin{aligned}\mathrm{E}_{\mathbf{Q}}\left[\int_0^t\int_{\mathbb{R}}\left|\overline{S}_{\tau(u)}-S_{\tau(u)}\right|^2\,\nu(da)\,du\right]&=\int_0^t\mathrm{E}_{\mathbf{Q}}\left(\left|\overline{S}_{\tau(u)}-S_{\tau(u)}\right|^2\right)\,du\\ &\leq\int_0^t\mathrm{E}_{\mathbf{Q}}\left(\sup_{s\leq u}\left|\overline{S}_s-S_s\right|^2\right)\,du\,.\end{aligned}$$

Putting these results in equation (X.2.2), we finally obtain

$$\begin{aligned}\mathrm{E}_{\mathbf{Q}}\left(\sup_{s\leq t}\left|S_s-\overline{S}_s\right|^2\right)&\leq 2\,K\,\varepsilon^2+2\,(r^2+K_T\,T)\times\varepsilon\\ &\quad+2\,(1+r^2)\times\int_0^t\mathrm{E}_{\mathbf{Q}}\left(\sup_{s\leq u}\left|\overline{S}_s-S_s\right|^2\right)\,du\,.\end{aligned}$$

Using Gronwall's lemma, we conclude the proof. ■

Remark 2.1. *Note that if τ and $\tilde{\tau}$ are two stopping times with values in $[0,T]$ such that $\tau\leq\tilde{\tau}\leq\tau+\varepsilon$, we have*

$$\mathrm{E}_{\mathbf{Q}}\left(|S_\tau-S_{\tilde{\tau}}|^2\mid\mathcal{F}_\tau\right)\leq C\,M_{\tau,T}\,\varepsilon,\ \textit{with}\ M_{\tau,T}:=\mathrm{E}_{\mathbf{Q}}\left(\sup_{u\in[0,T]}|S_u|^2\mid\mathcal{F}_\tau\right)\,.$$

Indeed, denoting $\mathrm{E}^\tau_{\mathbf{Q}}:=\mathrm{E}_{\mathbf{Q}}(.\mid\mathcal{F}_\tau)$, we have

$$\begin{aligned}\mathrm{E}^\tau_{\mathbf{Q}}\left(|S_\tau-S_{\tilde{\tau}}|^2\right)&\leq 2\,r^2\,\mathrm{E}^\tau_{\mathbf{Q}}\left[\left|\int_\tau^{\tilde{\tau}}S_u\,du\right|^2\right]\\ &\quad+2\,\mathrm{E}^\tau_{\mathbf{Q}}\left[\left|\int_\tau^{\tilde{\tau}}\int_{\mathbb{R}}c(u,a,S_{u^-})\,\widetilde{N}(du,da)\right|^2\right]\,.\end{aligned}$$

Note that $\mathrm{E}^{\tau}_{\mathbf{Q}}\left[\left|\int_{\tau}^{\tilde{\tau}} S_u\,du\right|^2\right] \leq M_{\tau,T}\,\varepsilon^2$.
Similar computations as above lead to

$$\begin{aligned}\mathrm{E}^{\tau}_{\mathbf{Q}}\left[\left|\int_{\tau}^{\tilde{\tau}}\int_{\mathbb{R}} c(u,a,S_{u^-})\,\widetilde{N}(du,da)\right|^2\right] &= \mathrm{E}^{\tau}_{\mathbf{Q}}\left[\int_{\tau}^{\tilde{\tau}}\int_{\mathbb{R}} |c(u,a,S_u)|^2\;du\,\nu(da)\right]\\ &\leq K\,\mathrm{E}^{\tau}_{\mathbf{Q}}\left(\int_{\tau}^{\tilde{\tau}}(1+|S_u|^2)\,du\right)\\ &\leq K\,(1+M_{\tau,T})\times\varepsilon\,.\end{aligned}$$

2.1. Dynamic programming for the price computation

For all $k=0,\ldots,L$, define the approximated price as

$$\overline{P}_{t_k}(\overline{S}_{t_k}) := \sup_{\tau\in\Theta_k} \mathrm{E}_{t_k}\left(e^{-r\,(\tau-t_k)}\,\phi(\overline{S}_{\tau})\right)\,, \tag{X.2.3}$$

where $\mathrm{E}_{t_k}(.) := \mathrm{E}_{\mathbf{Q}}(.\mid\mathcal{F}_{t_k})$ and Θ_k denotes the set of the stopping times with values in $\{t_k, t_{k+1},\ldots,T\}$.
The approximation error between the price $P(t_k,S_{t_k})$ (introduced in equation (X.0.2)) and $\overline{P}_{t_k}(\overline{S}_{t_k})$ is of order $\varepsilon := \max_{k=1,\ldots,L}\varepsilon_k$.

Proposition X.2:
Suppose that ϕ *is Lipschitz continuous and has at most linear growth. Then, there exists a positive constant* K_T *such that*

$$\mathrm{E}_{\mathbf{Q}}\left[\left|P(t_k,S_{t_k})-\overline{P}_{t_k}(\overline{S}_{t_k})\right|^2\right] \leq K_T\,\varepsilon\,.$$

Proof. Denote

$$P_{t_k} := P(t_k,S_{t_k}) \text{ and } \overline{P}_{t_k} := \overline{P}_{t_k}(\overline{S}_{t_k})\,,$$

and set

$$\widehat{P}_{t_k} := \sup_{\tau\in\Theta_k}\mathrm{E}_{t_k}\left(e^{-r\,(\tau-t_k)}\,\phi(S_\tau)\right)\,.$$

For all $\tau\in\Theta_k$ we have

$$\begin{aligned}\left|\mathrm{E}_{t_k}\left(e^{-r\,(\tau-t_k)}\,\phi(S_\tau)\right)-\mathrm{E}_{t_k}\left(e^{-r\,(\tau-t_k)}\,\phi(\overline{S}_\tau)\right)\right| &\leq \mathrm{E}_{t_k}\left(\left|\phi(S_\tau)-\phi(\overline{S}_\tau)\right|\right)\\ &\leq C\,\mathrm{E}_{t_k}\left(\max_{k\leq i\leq n}\left|S_{t_i}-\overline{S}_{t_i}\right|\right)\,.\end{aligned}$$

Using Theorem X.1, we obtain

$$\mathrm{E}_{\mathbf{Q}}\left[\left|\overline{P}_{t_k}-\widehat{P}_{t_k}\right|^2\right] \leq C\,\mathrm{E}_{\mathbf{Q}}\left[\max_{k\leq i\leq L}\left|S_{t_i}-\overline{S}_{t_i}\right|^2\right] \leq K_T\,\varepsilon\,. \tag{X.2.4}$$

Let $\tau \in \Gamma_{t_k,T}$ and set

$$\tilde{\tau} := \sum_{k=0}^{L-1} t_{k+1}\, \mathbf{1}_{t_k < \tau \leq t_{k+1}} \,.$$

The random variable $\tilde{\tau}$ is a stopping time taking values in Θ_{k+1} such that $\tau \leq \tilde{\tau}$. Moreover, $P_{t_k} \geq \widehat{P}_{t_k}$ since $\Theta_k \subseteq \Gamma_{t_k,T}$. We then obtain

$$0 \leq P_{t_k} - \widehat{P}_{t_k} \leq \sup_{\tau \in \Gamma_{t_k,T}} \mathrm{E}_{t_k} \left| h(\tau, S_\tau) - h(\tilde{\tau}, S_{\tilde{\tau}}) \right| \,,$$

where $h(t, S_t) := e^{-r\,(t-t_k)}\, \phi(S_t)$.
For all $\tau \in \Gamma_{t_k,T}$ we have

$$\begin{aligned}(\mathrm{E}_{t_k} \left| h(\tau, S_\tau) - h(\tilde{\tau}, S_{\tilde{\tau}})\right|)^2 &\leq C\,\varepsilon^2\, \mathrm{E}_{t_k} \left|\phi(S_\tau)\right|^2 + K\, \mathrm{E}_{t_k} \left|S_\tau - S_{\tilde{\tau}}\right|^2 \\ &\leq C\,\varepsilon^2 + K\,\varepsilon^2\, \mathrm{E}_{t_k} \left[\sup_{u\in[0,T]} |S_u|^2\right] + C\, \mathrm{E}_{t_k} \left|S_\tau - S_{\tilde{\tau}}\right|^2 \,.\end{aligned}$$

Remark 2.1 gives

$$\begin{aligned}\mathrm{E}_{\mathbf{Q}} \left(\left|P_{t_k} - \widehat{P}_{t_k}\right|^2\right) &\leq \mathrm{E}_{\mathbf{Q}} \left[\sup_{\tau \in \Gamma_{t_k,T}} (\mathrm{E}_{t_k} \left| h(\tau, S_\tau) - h(\tilde{\tau}, S_{\tilde{\tau}})\right|)^2\right] \\ &\leq C\,\varepsilon^2 + C\, M_T\, \varepsilon^2 + C\, M_T\, \varepsilon \\ &\leq K_T\, \varepsilon \,, \end{aligned} \qquad \text{(X.2.5)}$$

where $M_T := \mathrm{E}_{\mathbf{Q}} \left(\sup_{u\in[0,T]} |S_u|^2\right)$.
Equations (X.2.4) and (X.2.5) conclude the proof. ■

Let us now compute the approximated price $\overline{P}_{t_k}(\overline{S}_{t_k})$.
Let $0 = t_0 < t_1 < \ldots < t_L = T$ be a discretization of the time interval $[0, T]$, with step size $\varepsilon_k = t_k - t_{k-1}$.
Since $\left(\overline{S}_{t_k}\right)_{k=0,\ldots,L}$ is a Markov chain with respect to $(\mathcal{F}_{t_k})_{k=0,\ldots,L}$, the price $P(0, x)$ is approximated by $\overline{P}_0(x)$, where $(\overline{P}_{t_k}(\overline{S}_{t_k})_{k=0,\ldots,L})$ is defined as (see [Nev72]) :
- $\overline{P}_{t_L}(\overline{S}_{t_L}) = \phi(\overline{S}_T)$
- For $k = L-1, \ldots, 1$,

$$\overline{P}_{t_k}(\overline{S}_{t_k}) = \max\left\{\phi(\overline{S}_{t_k}), e^{-r\,\varepsilon_{k+1}}\, \mathrm{E}_{\mathbf{Q}}\left[\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) \mid \overline{S}_{t_k}\right]\right\} \,, \qquad \text{(X.2.6)}$$

–

$$\overline{P}_0(x) = \max\left\{\phi(x), e^{-r\,\varepsilon_1}\, \mathrm{E}_{\mathbf{Q}}\left[\overline{P}_{t_1}(\overline{S}_{t_1})\right]\right\} \,.$$

In view of Theorem X.1 (i), one may compute $\mathrm{E}_{\mathbf{Q}}\left[\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) \mid \overline{S}_{t_k} = \alpha\right]$ if there is at least one jump on $]0, t_k]$ and at least one jump on $]t_k, t_{k+1}]$, and if there is a

finite number of jumps on $]0,T]$. So we will approximate the previous algorithm by a localized one :
we denote

$$A_{k,M} := \left\{ J_{t_{k+1}} - J_{t_k} \geq 1; J_{t_k} \geq 1; J_T = M \right\},$$

and we set

– $u_{t_L}(\overline{S}_{t_L}) = \phi(\overline{S}_T)$
– For $k = L-1, \ldots, 1$,

$$u_{t_k}(\overline{S}_{t_k}) = \max\left\{ \phi(\overline{S}_{t_k}), e^{-r\,\varepsilon_{k+1}}\, \mathrm{E}_{\mathbf{Q}}\left[u_{t_{k+1}}(\overline{S}_{t_{k+1}})\, \mathbf{1}_{A_{k,M}} \mid \overline{S}_{t_k} \right] \right\}, \qquad \text{(X.2.7)}$$

–

$$u_0(x) = \max\left\{ \phi(x), e^{-r\,\varepsilon_1}\, \mathrm{E}_{\mathbf{Q}}\left[u_{t_1}(\overline{S}_{t_1}) \right] \right\}.$$

Let us give the approximation error between the algorithm (X.2.6) and the localized one (X.2.7) :

Lemma X.1:
Let us denote $\underline{\varepsilon} := \min \varepsilon_k$, *we then have*

$$|\overline{P}_0(x) - u_0(x)| \leq C\,\frac{T}{\underline{\varepsilon}}\, \left(e^{-\lambda\,\underline{\varepsilon}} + \mathbf{Q}(J_T \neq M) \right),$$

where $C := 2\,\max \parallel \overline{P}_{t_k} \parallel_\infty$.

Proof. For $k = L-1, \ldots, 1$, $\alpha \geq 0$, we have

$$\begin{aligned}
&|\overline{P}_{t_k}(\alpha) - u_{t_k}(\alpha)| \\
&\quad \leq \left| \mathrm{E}_{\mathbf{Q}}\left[\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) \mid \overline{S}_{t_k} = \alpha \right] - \mathrm{E}_{\mathbf{Q}}\left[u_{t_{k+1}}(\overline{S}_{t_{k+1}})\, \mathbf{1}_{A_{k,M}} \mid \overline{S}_{t_k} = \alpha \right] \right| \\
&\quad \leq \left| \mathrm{E}_{\mathbf{Q}}\left[\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}})\, \mathbf{1}_{A^c_{k,M}} \mid \overline{S}_{t_k} = \alpha \right] \right| \\
&\quad + \mathrm{E}_{\mathbf{Q}}\left[|\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) - u_{t_{k+1}}(\overline{S}_{t_{k+1}})|\, \mathbf{1}_{A_{k,M}} \mid \overline{S}_{t_k} = \alpha \right] \\
&\quad \leq \parallel \overline{P}_{t_{k+1}} \parallel_\infty\, \mathrm{E}_{\mathbf{Q}}\left[\mathbf{1}_{A^c_{k,M}} \mid \overline{S}_{t_k} = \alpha \right] \\
&\quad + \mathrm{E}_{\mathbf{Q}}\left[|\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) - u_{t_{k+1}}(\overline{S}_{t_{k+1}})| \mid \overline{S}_{t_k} = \alpha \right].
\end{aligned}$$

Thus, taking $\alpha = \overline{S}_{t_k}$ and the expectation, we obtain

$$\begin{aligned}
&\mathrm{E}_{\mathbf{Q}}\left| \overline{P}_{t_k}(\overline{S}_{t_k}) - u_{t_k}(\overline{S}_{t_k}) \right| \\
&\quad \leq \parallel \overline{P}_{t_{k+1}} \parallel_\infty\, \mathbf{Q}\left(A^c_{k,M} \right) + \mathrm{E}_{\mathbf{Q}}\left[|\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) - u_{t_{k+1}}(\overline{S}_{t_{k+1}})| \right] \\
&\quad \leq \parallel \overline{P}_{t_{k+1}} \parallel_\infty\, \left(\mathbf{Q}(J_{t_{k+1}} - J_{t_k} = 0) + \mathbf{Q}(J_{t_k} = 0) + \mathbf{Q}(J_{t_k} \neq M) \right) \\
&\quad + \mathrm{E}_{\mathbf{Q}}\left[|\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) - u_{t_{k+1}}(\overline{S}_{t_{k+1}})| \right] \\
&\quad \leq \parallel \overline{P}_{t_{k+1}} \parallel_\infty\, \left(e^{-\lambda\,\underline{\varepsilon}} + e^{-\lambda\, t_k} + \mathbf{Q}(J_{t_k} \neq M) \right) + \mathrm{E}_{\mathbf{Q}}\left[|\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) - u_{t_{k+1}}(\overline{S}_{t_{k+1}})| \right] \\
&\quad = C\, \left(e^{-\lambda\,\underline{\varepsilon}} + \mathbf{Q}(J_{t_k} \neq M) \right) + \mathrm{E}_{\mathbf{Q}}\left[|\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) - u_{t_{k+1}}(\overline{S}_{t_{k+1}})| \right].
\end{aligned}$$

So,

$$\mathrm{E}_{\mathbf{Q}} \left|\overline{P}_{t_k}(\overline{S}_{t_k}) - u_{t_k}(\overline{S}_{t_k})\right| \leq C \, \left(e^{-\lambda \underline{\varepsilon}} + \mathbf{Q}(J_{t_k} \neq M)\right) \\ + \mathrm{E}_{\mathbf{Q}} \left[|\overline{P}_{t_{k+1}}(\overline{S}_{t_{k+1}}) - u_{t_{k+1}}(\overline{S}_{t_{k+1}})|\right] .$$

Since $\overline{P}_{t_L} = u_{t_L}$, we finally get

$$|\overline{P}_0(x) - u_0(x)| \leq \mathrm{E}_{\mathbf{Q}} \left|\overline{P}_{t_1}(\overline{S}_{t_1}) - u_{t_1}(\overline{S}_{t_1})\right| \\ \leq C \sum_{k=0}^{L-1} \left(e^{-\lambda \underline{\varepsilon}} + \mathbf{Q}(J_{t_k} \neq M)\right) = C\,L\, \left(e^{-\lambda \underline{\varepsilon}} + \mathbf{Q}(J_{t_k} \neq M)\right) .$$

Since $T = \sum_{k=1}^{L} \varepsilon_k \geq L \times \underline{\varepsilon}$, the result is proved. ■

Remark 2.2. *As we know that for all $y \geq 0$, $\dfrac{e^{-x} + y}{x} \underset{x \to +\infty}{\longrightarrow} 0$, we can conclude that the error between the algorithms (X.2.6) and (X.2.7) decreases when $\lambda \underline{\varepsilon}$ increases. The parameter λ is the jump intensity, that is $\mathrm{E}(J_T) = \lambda T$, which means that λ represents the noise available in the system : if λ is large (resp. small), there is lots of (resp. few) jumps on $]0, T]$. So, for small noise, we have to take $\underline{\varepsilon}$ very large to have $\lambda \underline{\varepsilon} >> 1$, so that the error coming from the localization is very small.*
In numerical experiments, this means that once λ is fixed (which will fix M), we set a time grid of $]0, T]$ where for $k = 1, \ldots, L$, the step size ε_k is large enough ($\underline{\varepsilon} >> 1/\lambda$) and such that $J_{t_k} - J_{t_{k-1}} \geq 1$.

The conditional expectation $\mathrm{E}_{\mathbf{Q}} \left[u_{t_{k+1}}(\overline{S}_{t_{k+1}}) \, \mathbf{1}_{A_k} \mid \overline{S}_{t_k}\right]$ will be computed using the representation Theorem X.1 (i), by means of suitable empirical means evaluated over N simulated paths. Let us be more precise.
• We fix the intensity of the jumps λ and for $k = 1, \ldots, L$, we choose a step size $\varepsilon_k = t_k - t_{k-1}$ with respect to λ so that there is at least one jump on $]t_k, t_{k+1}]$ (see Remark 2.2).
• We simulate the jump times $(T_i^p)_{i \geq 1}$ (such that $T_i^p - T_{i-1}^p \sim \exp(\lambda)$) and the jump amplitudes $(\Delta_i^p)_{i \geq 1}$, $p = 1, \ldots, N$.
• We then compute the samples $(\overline{S}^p_{t_k}, \overline{J}^p_{t_k})_{k=1,\ldots,L}$, $p = 1, \ldots, N$.
Let us compute $u_{t_k}(\overline{S}^p_{t_k})$ given by the algorithm (X.2.7).

Using the representation Theorem X.1 (i), we obtain

$$\begin{aligned}
&\mathrm{E}_{\mathbf{Q}}\left[u_{t_{k+1}}(\overline{S}_{t_{k+1}})\,\mathbf{1}_{A_{k,M}} \mid \overline{S}_{t_k}=\alpha\right] \\
&= \frac{\mathrm{E}_{\mathbf{Q}}\left[u_{t_{k+1}}(\overline{S}_{t_{k+1}})\,\mathbf{1}_{\overline{S}_{t_k}\geq\alpha}\,\overline{V}_{(1,t_k,t_{k+1})}\,\mathbf{1}_{A_{k,M}}\right]}{\mathrm{E}_{\mathbf{Q}}\left[\mathbf{1}_{\overline{S}_{t_k}\geq\alpha}\,\overline{V}_{(1,t_k,t_{k+1})}\,\mathbf{1}_{A_{k,M}}\right]}\,\mathbf{1}_{A_{k,M}} \\
&\simeq \frac{\sum_{q=1}^{N} u_{t_{k+1}}(\overline{S}^q_{t_{k+1}})\,\mathbf{1}_{\overline{S}^q_{t_k}\geq\alpha}\,\overline{V}^q_{(1,t_k,t_{k+1})}\,\mathbf{1}_{\{\overline{J}^q_{t_{k+1}}-\overline{J}^q_{t_k}\geq 1;\overline{J}^q_{t_k}>0;\overline{J}^q_T=M\}}}{\sum_{q=1}^{N}\mathbf{1}_{\overline{S}^q_{t_k}\geq\alpha}\,\overline{V}^q_{(1,t_k,t_{k+1})}\,\mathbf{1}_{\{\overline{J}^q_{t_{k+1}}-\overline{J}^q_{t_k}\geq 1;\overline{J}^q_{t_k}>0;\overline{J}^q_T=M\}}}\,\mathbf{1}_{A_{k,M}}\,.
\end{aligned}$$

We denote by $\Psi_k(\alpha)$ the fraction :

$$\Psi_k(\alpha) := \frac{\sum_{q=1}^{N} u_{t_{k+1}}(\overline{S}^q_{t_{k+1}})\,\mathbf{1}_{\overline{S}^q_{t_k}\geq\alpha}\,\overline{V}^q_{(1,t_k,t_{k+1})}\,\mathbf{1}_{\{\overline{J}^q_{t_{k+1}}-\overline{J}^q_{t_k}\geq 1;\overline{J}^q_{t_k}>0;\overline{J}^q_T=M\}}}{\sum_{q=1}^{N}\mathbf{1}_{\overline{S}^q_{t_k}\geq\alpha}\,\overline{V}^q_{(1,t_k,t_{k+1})}\,\mathbf{1}_{\{\overline{J}^q_{t_{k+1}}-\overline{J}^q_{t_k}\geq 1;\overline{J}^q_{t_k}>0;\overline{J}^q_T=M\}}}\,. \tag{X.2.8}$$

Thus, we have

$$\mathrm{E}_{\mathbf{Q}}\left[u_{t_{k+1}}(\overline{S}_{t_{k+1}})\,\mathbf{1}_{A_{k,M}} \mid \overline{S}_{t_k}=\alpha\right] \simeq \Psi_k(\alpha)\,\mathbf{1}_{\{\overline{J}_{t_{k+1}}-\overline{J}_{t_k}\geq 1;\overline{J}_{t_k}>0;\overline{J}_T=M\}}\,.$$

Applying this result to $(\overline{J}^p_{t_k})_{k=0,\ldots,L}$ and $\alpha=\overline{S}^p_{t_k}$, we thus obtain for $k=L-1,\ldots,1$,

$$\mathrm{E}_{\mathbf{Q}}\left[u_{t_{k+1}}(\overline{S}_{t_{k+1}})\,\mathbf{1}_{A_{k,M}} \mid \overline{S}_{t_k}=\overline{S}^p_{t_k}\right] \simeq \Psi_k(\overline{S}^p_{t_k})\,\mathbf{1}_{\{\overline{J}^p_{t_{k+1}}-\overline{J}^p_{t_k}\geq 1;\overline{J}^p_{t_k}>0;\overline{J}^p_T=M\}}\,.$$

Hence, we can set up the dynamic programming equation :
– $\hat{u}_{t_L}(\overline{S}^p_{t_L})=\phi(\overline{S}_T)$
– For $k=L-1,\ldots,1$,

$$\hat{u}_{t_k}(\overline{S}^p_{t_k}) = \max\left\{\phi(\overline{S}^p_{t_k}),\, e^{-r\,\varepsilon_{k+1}}\,\Psi_k(\overline{S}^p_{t_k})\,\mathbf{1}_{\{\overline{J}^p_{t_{k+1}}-\overline{J}^p_{t_k}\geq 1;J^p_{t_k}>0;\overline{J}^p_T=M\}}\right\} \tag{X.2.9}$$

– Finally,

$$\hat{u}_0(x) = \max\left\{\phi(x),\, e^{-r\,\varepsilon_1}\,\frac{1}{N}\sum_{p=1}^{N}\hat{u}_{t_1}(\overline{S}^p_{t_1})\right\}\,.$$

2.2. Algorithm for the Delta computation

The Delta $\Delta(0,x)$ is approximated by the following algorithm :
– If $\hat{u}_{t_1}\left(\overline{S}_{t_1}\right) < \phi(\overline{S}_{t_1})$, then

$$\Delta(\overline{S}_{t_1}) = \phi'(\overline{S}_{t_1})\,,$$

– If $\hat{u}_{t_1}\left(\overline{S}_{t_1}\right) > \phi(\overline{S}_{t_1})$, then

$$\Delta(\overline{S}_{t_1}) = e^{-r\,\varepsilon_2}\,\partial_\alpha\,\mathrm{E}\left[\hat{u}_{t_2}\left(\overline{S}_{t_2}\right) \mid \overline{S}_{t_1} = \alpha\right]\Big|_{\alpha=\overline{S}_{t_1}} . \tag{X.2.10}$$

– $\Delta_0(x) = \mathrm{E}(\Delta(\overline{S}_{t_1}))$.

In view of Theorem X.1 (ii), one may compute $\partial_\alpha \mathrm{E}\left[\hat{u}_{t_2}\left(\overline{S}_{t_2}\right) \mid \overline{S}_{t_1} = \alpha\right]$ if there is at least four jumps on $]0, t_1]$ and at least four jumps on $]t_1, t_2]$. Hence, we will not take the same localization as in the pricing algorithm. We will work on :

$$B_M := \{J_{t_2} - J_{t_1} \geq 4; J_{t_1} \geq 4; J_T = M\} .$$

We thus approximate the algorithm (X.2.10) by the localized one :

– If $\hat{u}_{t_1}\left(\overline{S}_{t_1}\right) < \phi(\overline{S}_{t_1})$, then

$$v_1(\overline{S}_{t_1}) = \phi'(\overline{S}_{t_1}) ,$$

– If $\hat{u}_{t_1}\left(\overline{S}_{t_1}\right) > \phi(\overline{S}_{t_1})$, then

$$v_1(\overline{S}_{t_1}) = e^{-r\,\varepsilon_2}\,\partial_\alpha\,\mathrm{E}\left[\hat{u}_{t_2}\left(\overline{S}_{t_2}\right)\,\mathbf{1}_{B_M} \mid \overline{S}_{t_1} = \alpha\right]\Big|_{\alpha=\overline{S}_{t_1}} . \tag{X.2.11}$$

– $v_0(x) = \mathrm{E}(v_1(\overline{S}_{t_1}))$.

Remark 2.3. *In view of Remark 2.2, once λ is fixed, we choose the step size ε_1 and ε_2 with respect to λ and large enough to have at least four jumps on $]0, t_1]$ and at least four jumps on $]t_1, t_2]$.*

Let us compute $v_1(\overline{S}^p_{t_1})$, for $p = 1, \ldots, N$. Using the representation Theorem X.1 (ii), we obtain

$$\begin{aligned}\partial_\alpha \mathrm{E}&\left[\hat{u}_{t_2}\left(\overline{S}_{t_2}\right)\mathbf{1}_{B_M} \mid \overline{S}_{t_1} = \alpha\right] \\ &= \left(\frac{\mathbb{R}_{1,2}[\hat{u}_{t_2}](\alpha)\,\mathbb{T}_{1,2}[1](\alpha) - \mathbb{T}_{1,2}[\hat{u}_{t_2}](\alpha)\,\mathbb{R}_{1,2}[1](\alpha)}{\mathbb{T}^2_{1,2}[1](\alpha)}\right)\mathbf{1}_{B_M} ,\end{aligned}$$

where $\mathbb{R}$ and $\mathbb{T}$ are respectively given by (X.1.3) and (X.1.1), that is for $f = \hat{u}_{t_2}$ or $f = 1$,

$$\mathbb{T}_{1,2}[f](\alpha) = \mathrm{E}\left(f(\overline{S}_{t_2})\,\mathbf{1}_{\overline{S}_{t_1}\geq\alpha}\,\overline{V}_{(1,1,2)}\,\mathbf{1}_{B_M}\right) ,$$

and

$$\mathbb{R}_{1,2}[f](\alpha) = -\mathrm{E}\left(f(\overline{S}_{t_2})\,\mathbf{1}_{\overline{S}_{t_1}\geq\alpha}\,\overline{\mathcal{H}}_{1,2}\,\mathbf{1}_{B_M}\right) .$$

We then take the following approximations $\mathbb{T} \simeq \overline{\mathbb{T}}$ and $\mathbb{R} \simeq \overline{\mathbb{R}}$, where

$$\overline{\mathbb{T}}_{1,2}[f](\alpha) = \frac{1}{N}\sum_{q=1}^{N} f(\overline{S}^q_{t_2})\,\mathbf{1}_{\overline{S}^q_{t_1}\geq\alpha}\,\overline{V}^q_{(1,1,2)}\,\mathbf{1}_{\{\overline{J}^q_{t_2}-\overline{J}^q_{t_1}\geq 4;\overline{J}^q_{t_1}\geq 4;\overline{J}^q_T=M\}} ,$$

$$\overline{\mathbb{R}}_{1,2}[f](\alpha) = -\frac{1}{N}\sum_{q=1}^{N} f(\overline{S}^q_{t_2})\ \mathbf{1}_{\overline{S}^q_{t_1}\geq\alpha}\,\overline{\mathcal{H}}^q_{1,2}\,\mathbf{1}_{\{\overline{J}^q_{t_2}-\overline{J}^q_{t_1}\geq 4;\overline{J}^q_{t_1}\geq 4;\overline{J}^q_T=M\}}\,.$$

We then define $\overline{\Psi}_k(\alpha)$ as

$$\overline{\Psi}_k(\alpha) := \frac{\overline{\mathbb{R}}_{1,2}[\hat{u}_{t_2}](\alpha)\,\overline{\mathbb{T}}_{1,2}[1](\alpha) - \overline{\mathbb{T}}_{1,2}[\hat{u}_{t_2}](\alpha)\,\overline{\mathbb{R}}_{1,2}[1](\alpha)}{\overline{\mathbb{T}}^2_{1,2}[1](\alpha)}\,. \tag{X.2.12}$$

We obtain

$$\partial_\alpha \mathrm{E}\left[\hat{u}_{t_2}\left(\overline{S}_{t_2}\right)\mathbf{1}_B \mid \overline{S}_{t_1}=\alpha\right] \simeq \mathbf{1}_{\{\overline{J}_{t_2}-\overline{J}_{t_1}\geq 4;\overline{J}_{t_1}\geq 4;\overline{J}_T=M\}}\overline{\Psi}_k(\alpha)\,.$$

Finally, applying this result to $(\overline{J}^p_{t_k})_{k=0,\dots,L}$ and $\alpha=\overline{S}^p_{t_1}$, we can set up the following algorithm :

– If $\hat{u}_{t_1}\left(\overline{S}_{t_1^p}\right) < \phi(\overline{S}^p_{t_1})$, then
$$\hat{v}_1(\overline{S}^p_{t_1}) = \phi'(\overline{S}^p_{t_1})\,,$$
– If $\hat{u}_{t_1}\left(\overline{S}^p_{t_1}\right) > \phi(\overline{S}^p_{t_1})$, then
$$\hat{v}_1(\overline{S}^p_{t_1}) = e^{-r\,\varepsilon_2}\,\overline{\Psi}_k(\overline{S}^p_{t_1})\,\mathbf{1}_{\{\overline{J}^p_{t_2}-\overline{J}^p_{t_1}\geq 4;\overline{J}^p_{t_1}\geq 4;\overline{J}^p_T=M\}}\,. \tag{X.2.13}$$

–

$$\hat{v}_0(x) = \frac{1}{N}\sum_{p=1}^{N}\hat{v}_1(\overline{S}^p_{t_1}) = \frac{1}{N}\sum_{p=1}^{N}\phi'(\overline{S}^p_{t_1})\,\mathbf{1}_{\{\hat{u}_{t_1}\left(\overline{S}_{t_1^p}\right)<\phi(\overline{S}^p_{t_1})\}}$$
$$+\frac{1}{N}\sum_{p=1}^{N} e^{-r\,\varepsilon_2}\,\overline{\Psi}_k(\overline{S}^p_{t_1})\,\mathbf{1}_{\{\overline{J}^p_{t_2}-\overline{J}^p_{t_1}\geq 4;\overline{J}^p_{t_1}\geq 4;\overline{J}^p_T=M\}}\,\mathbf{1}_{\{\hat{u}_{t_1}\left(\overline{S}_{t_1^p}\right)>\phi(\overline{S}^p_{t_1})\}}\,.$$

3. Numerical results

We apply the Monte-Carlo algorithms (X.2.9) and (X.2.13) (obtained in an abstract framework) to the geometrical model :

$$S_t = x + \int_0^t r\,S_u\,du + \int_0^t\int_{\mathbb{R}} \sigma\,a\,S_{u^-}\,N(du,da)\,,\ t\in[0,T]\,, \tag{X.3.1}$$

where we represent the Poisson point measure $N(dt,da)$ by means of the jump times $(T_i)_{i\in\mathbb{N}}$ and amplitudes $(\Delta_i)_{i\in\mathbb{N}}$ of a compound Poisson process, which means that $N(t,A) = Card\{T_i\leq t : \Delta_i\in A\}$. We suppose that $T_i - T_{i-1}\sim\exp(\lambda)$ for all $i\geq 1$ and that the law of the jump amplitudes Δ_i is uniform on $(0,1)$. Hence, in view of definition (VIII.3.2), we work with the following weights for $0\leq s\leq t$, $k=1,2$:

$$\pi_{(k,s,t)}(\omega,\Delta_i) := \mathbf{1}_{]s,t]}(T_i)\,\pi_k(\Delta_i)\,,$$

with

$$\pi_1(\Delta_i) = \left(\frac{1}{2} - \Delta_i\right)^{1/4} \Delta_i^{1/4} \text{ and } \pi_2(\Delta_i) = (1-\Delta_i)^{1/4} \left(\Delta_i - \frac{1}{2}\right)^{1/4} .$$

This means that Supp $\pi_1 \subseteq (0, 1/2)$ and Supp $\pi_2 \subseteq (1/2, 1)$, so that

$$\pi_{(1,s,t)}(\omega, \Delta_i) \times \pi_{(2,s,t)}(\omega, \Delta_i) = 0, \text{ for all } i \in \mathbb{N} .$$

Our aim is to perform the Monte-Carlo algorithms (X.2.9) and (X.2.13) to approximate the price $P(0,x)$ and the Delta $\Delta(0,x)$. In equations (X.2.12) and (X.2.8), the functions $\overline{\Psi}_k$ and Ψ_k depend on the Malliavin estimators $V_{(k,s,t)}$, $k = 1, 2$, and $\mathcal{H}_{s,t}$, respectively given by equations (VIII.3.4) and (VIII.3.5). Hence, we have to compute the Malliavin operators (with respect to the jump amplitudes) of S_t involved in their expressions.

3.1. Malliavin estimators

Let $(S_t)_{t\in[0,T]}$ be the solution of the geometrical model (X.3.1). For all $t \in [0, T]$, we have an explicit expression of S_t :

$$S_t = x\, e^{r\,t} \prod_{i=1}^{J_t} (1 + \sigma\, \Delta_i) \; .$$

So the process S can be exactly simulated at each time t_k, and we do not need an approximation $\overline{S}_{t_k}$ of S_{t_k}.

• **Computation of the Malliavin derivatives.**

For all $i = 1, \ldots, J_t$, differentiating with respect to the jump amplitudes Δ_i (see Chapter IV, section 1.1), we have

$$D_i S_t = \frac{\sigma\, S_t}{1 + \sigma\, \Delta_i} \text{ and then } D^2_{ii} S_t = 0 \, .$$

Since the law of the jump amplitude Δ_i is $p(y) = \mathbf{1}_{(0,1)}(y)$, we have $\pi_k(\Delta_i)\, \partial \ln p(\Delta_i) = 0$. So

$$\begin{aligned} &\mathrm{L}_{(k,s,t)}(S_t) \\ &\quad = -\sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i)\; \left[\pi_k(\Delta_i)\, D^2_{ii} S_t + (\pi'_k(\Delta_i) + \pi_k(\Delta_i)\, \partial \ln p(\Delta_i))\, D_i(S_t)\right] \\ &\quad = -\sigma\, S_t \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i)\, \frac{\pi'_k(\Delta_i)}{1 + \sigma\, \Delta_i} \, . \end{aligned}$$

Let us define for $0 \le s \le t$, $k = 1, 2$

$$F_{(k,s,t)} := \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i) \, \frac{\pi_k'(\Delta_i)}{1 + \sigma \, \Delta_i} \, . \tag{X.3.2}$$

We then have

$$\mathrm{L}_{(k,s,t)}(S_t) = -\sigma \, S_t \, F_{(k,s,t)} \, .$$

On the other hand, we have

$$\sigma_t^{(k,s,t)} := \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i) \, \pi_k(\Delta_i) \, |D_i S_t|^2 = \sigma^2 \, S_t^2 \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i) \, \frac{\pi_k(\Delta_i)}{(1 + \sigma \, \Delta_i)^2} \, .$$

Then, denoting by

$$A_{(k,s,t)} := \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i) \, \frac{\pi_k(\Delta_i)}{(1 + \sigma \, \Delta_i)^2} \, , \tag{X.3.3}$$

we have

$$\sigma_t^{(k,s,t)} = \sigma^2 \, S_t^2 \, A_{(k,s,t)} \text{ and then } \gamma_t^{(k,s,t)} = \frac{1}{\sigma^2 \, S_t^2 \, A_{(k,s,t)}} \, .$$

Let us compute now some inner products which are involved in the expression of the Malliavin estimators $V_{(k,s,t)}$.

Lemma X.2:
For all $0 < s < t$, we have

$$(i) \ \langle DS_s, D\sigma_t^{(k,s,t)} \rangle_{(k,0,s)} = 2 \, \sigma^4 \, S_s \, S_t^2 \, A_{(k,s,t)} \, A_{(k,0,s)} \, .$$

Let us denote

$$B_{(k,s,t)} := \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i) \, \frac{\pi_k(\Delta_i) \, \pi_k'(\Delta_i)}{(1 + \sigma \, \Delta_i)^3} \, , \tag{X.3.4}$$

and

$$C_{(k,s,t)} := \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i) \, \frac{\pi_k(\Delta_i)^2}{(1 + \sigma \, \Delta_i)^4} \, . \tag{X.3.5}$$

We then have

$$(ii) \ \langle DS_t, D\sigma_t^{(k,s,t)} \rangle_{(k,s,t)} = 2 \, \sigma^4 \, S_t^3 \, A_{(k,s,t)}^2 + \sigma^3 \, S_t^3 \, (B_{(k,s,t)} - 2 \, \sigma \, C_{(k,s,t)}) \, .$$

Proof. Let us first compute $D_i \sigma_t^{(k,s,t)}$. We have

$$\begin{aligned} D_i \sigma_t^{(k,s,t)} &= 2 \, \sigma^2 \, S_t \, D_i S_t \, A_{(k,s,t)} + \sigma^2 \, S_t^2 \, D_i(A_{(k,s,t)}) \\ &= \frac{2 \, \sigma^3 \, S_t^2}{1 + \sigma \, \Delta_i} \, A_{(k,s,t)} \, \mathbf{1}_{]0,t]}(T_i) + \sigma^2 \, S_t^2 \, D_i(A_{(k,s,t)}) \, \mathbf{1}_{]s,t]}(T_i) \, . \end{aligned}$$

Since $\mathbf{1}_{]s,t]}(T_i) \times \mathbf{1}_{]0,s]}(T_i) = 0$, we get

$$\begin{aligned}\langle DS_s, D\sigma_t^{(k,s,t)}\rangle_{(k,0,s)} &= \sum_{i=0}^{\infty} \mathbf{1}_{]0,s]}(T_i)\,\pi_k(\Delta_i)\,D_iS_s\,D_i\sigma_t^{(k,s,t)} \\ &= \sum_{i=0}^{\infty} \mathbf{1}_{]0,s]}(T_i)\,\pi_k(\Delta_i)\,\frac{\sigma\,S_s}{1+\sigma\,\Delta_i}\,\frac{2\,\sigma^3\,S_t^2}{1+\sigma\,\Delta_i}\,A_{(k,s,t)} \\ &= 2\,\sigma^4\,S_s\,S_t^2\,A_{(k,s,t)}\,A_{(k,0,s)}\,,\end{aligned}$$

which proves (i).
For (ii), the term $\mathbf{1}_{]s,t]}(T_i)$ does not disappear, so that we have to compute $D_iA_{(k,s,t)}$. We have

$$D_iA_{(k,s,t)} = \mathbf{1}_{]s,t]}(T_i)\,\frac{\pi_k'(\Delta_i)}{(1+\sigma\,\Delta_i)^2} - 2\,\sigma\,\mathbf{1}_{]s,t]}(T_i)\,\frac{\pi_k(\Delta_i)}{(1+\sigma\,\Delta_i)^3}\,, \tag{X.3.6}$$

and

$$\sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i)\,\pi_k(\Delta_i)\,\frac{D_iA_{(k,s,t)}}{1+\sigma\,\Delta_i} = B_{(k,s,t)} - 2\,\sigma\,C_{(k,s,t)}\,.$$

We thus obtain

$$\begin{aligned}\langle DS_t, D\sigma_t^{(k,s,t)}\rangle_{(k,s,t)} &= \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i)\,\pi_k(\Delta_i)\,D_iS_t\,D_i\sigma_t^{(k,s,t)} \\ =&\sigma\,S_t \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i)\,\frac{\pi_k(\Delta_i)}{1+\sigma\,\Delta_i}\left(\frac{2\,\sigma^3\,S_t^2}{1+\sigma\,\Delta_i}\,A_{(k,s,t)}\,\mathbf{1}_{]0,t]}(T_i) + \sigma^2\,S_t^2\,D_i(A_{(k,s,t)})\,\mathbf{1}_{]s,t]}(T_i)\right) \\ =&2\,\sigma^4\,S_t^3\,A_{(k,s,t)} \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i)\,\frac{\pi_k(\Delta_i)}{(1+\sigma\,\Delta_i)^2} + \sigma^3\,S_t^3 \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i)\,\pi_k(\Delta_i)\,\frac{D_iA_{(k,s,t)}}{1+\sigma\,\Delta_i} \\ =&2\,\sigma^4\,S_t^3\,A_{(k,s,t)}^2 + \sigma^3\,S_t^3\,(B_{(k,s,t)} - 2\,\sigma\,C_{(k,s,t)})\,,\end{aligned}$$

which completes the proof. ■

• **Computation of the Malliavin estimator $\mathbf{V_{(k,s,t)}}$.**
Let us recall that

$$V_{(k,s,t)} := U_s^{(k,0,s)} - \gamma_s^{(k,0,s)}\,\langle DS_s, DS_t\rangle_{(k,0,s)}\,U_t^{(k,s,t)} + \frac{1}{2}\,\gamma_s^{(k,0,s)}\,\gamma_t^{(k,s,t)}\,\langle DS_s, D\sigma_t^{k,s,t)}\rangle_{(k,0,s)}\,, \tag{X.3.7}$$

with

$$U_t^{(k,s,t)} := \gamma_t^{(k,s,t)}\,L_{(k,s,t)}S_t - \langle DS_t, D\gamma_t^{(k,s,t)}\rangle_{(k,s,t)}\,.$$

We have

$$\gamma_t^{(k,s,t)}\,L_{(k,s,t)}S_t = -\frac{1}{\sigma\,S_t}\,\frac{F_{(k,s,t)}}{A_{(k,s,t)}}\,.$$

Moreover, Lemma X.2 (ii) gives

$$\langle DS_t, D\gamma_t^{(k,s,t)}\rangle_{(k,s,t)} = -(\gamma_t^{(k,s,t)})^2 \langle DS_t, D\sigma_t^{(k,s,t)}\rangle_{(k,s,t)} = -\frac{1}{S_t} - \frac{1}{\sigma S_t} \frac{B_{(k,s,t)} - 2\sigma C_{(k,s,t)}}{A^2_{(k,s,t)}} .$$

Hence,

$$U_t^{(k,s,t)} = \frac{1}{S_t} + \frac{1}{\sigma S_t} \frac{B_{(k,s,t)} - 2\sigma C_{(k,s,t)}}{A^2_{(k,s,t)}} - \frac{1}{\sigma S_t} \frac{F_{(k,s,t)}}{A_{(k,s,t)}} . \quad \text{(X.3.8)}$$

We have

$$\gamma_s^{(k,0,s)} \langle DS_s, DS_t\rangle_{(k,0,s)} U_t^{(k,s,t)} = \frac{1}{\sigma^2 S_s^2 A_{(k,0,s)}} (\sigma^2 S_s S_t A_{(k,0,s)}) U_t^{(k,s,t)} = \frac{1}{S_s} + \frac{1}{\sigma S_s} \frac{B_{(k,s,t)} - 2\sigma C_{(k,s,t)}}{A^2_{(k,s,t)}} - \frac{1}{\sigma S_s} \frac{F_{(k,s,t)}}{A_{(k,s,t)}} . \quad \text{(X.3.9)}$$

Moreover, Lemma X.2 (i) gives

$$\frac{1}{2} \gamma_s^{(k,0,s)} \gamma_t^{(k,s,t)} \langle DS_s, D\sigma_t^{k,s,t)}\rangle_{(k,0,s)} = \frac{1}{2} \frac{1}{\sigma^2 S_s^2 A_{(k,0,s)}} \frac{1}{\sigma^2 S_t^2 A_{(k,s,t)}} (2\sigma^4 S_s S_t^2 A_{(k,s,t)} A_{(k,0,s)}) = \frac{1}{S_s} \quad \text{(X.3.10)}$$

Putting the results (X.3.9) and (X.3.10) together in equation (X.3.7), we obtain finally

$$V_{(k,s,t)} = \frac{1}{S_s} + \frac{1}{\sigma S_s} \left(\frac{B_{(k,0,s)} - 2\sigma C_{(k,0,s)}}{A^2_{(k,0,s)}} - \frac{B_{(k,s,t)} - 2\sigma C_{(k,s,t)}}{A^2_{(k,s,t)}} \right) + \frac{1}{\sigma S_s} \left(\frac{F_{(k,s,t)}}{A_{(k,s,t)}} - \frac{F_{(k,0,s)}}{A_{(k,0,s)}} \right) , \quad \text{(X.3.11)}$$

which may be computed using equations (X.3.3), (X.3.4), (X.3.5) and (X.3.2).

• **Computation of the Malliavin estimator $\mathcal{H}_{s,t}$.**

Let us recall that

$$\mathcal{H}_{s,t} = V_{(1,s,t)} V_{(2,s,t)} - \gamma_s^{(2,0,s)} \langle DS_s, D(V_{(1,s,t)})\rangle_{(2,0,s)} + \gamma_s^{(2,0,s)} \gamma_t^{(2,s,t)} \langle DS_s, DS_t\rangle_{(2,0,s)} \langle DS_t, D(V_{(1,s,t)})\rangle_{(2,s,t)} . \quad \text{(X.3.12)}$$

We thus have to compute $D_i V_{(1,s,t)} \times \pi_2(\Delta_i)$ which appears in the inner products $\langle DS_t, D(V_{(1,s,t)})\rangle_{(2,s,t)}$ and $\langle DS_s, D(V_{(1,s,t)})\rangle_{(2,0,s)}$.

We know from equation (X.3.6) that

$$D_i A_{(1,s,t)} = \mathbf{1}_{]s,t]}(T_i)\, \frac{\pi_1'(\Delta_i)}{(1+\sigma\,\Delta_i)^2} - 2\,\sigma\,\mathbf{1}_{]s,t]}(T_i)\, \frac{\pi_1(\Delta_i)}{(1+\sigma\,\Delta_i)^3}\,.$$

Since π_1 and π_2 have disjoint supports, we have $\pi_1(\Delta_i) \times \pi_2(\Delta_i) = 0$ and $\pi_1'(\Delta_i) \times \pi_2(\Delta_i) = 0$, and then

$$D_i A_{(1,s,t)} \times \pi_2(\Delta_i) = 0\,.$$

Since each term involved in the expressions of $D_i B_{(1,s,t)}$, $D_i C_{(1,s,t)}$ and $D_i F_{(1,s,t)}$ is multiplied by the weights $\pi_1(\Delta_i)$ and their derivatives, we similary derive that

$$(D_i B_{(1,s,t)} + D_i C_{(1,s,t)} + D_i F_{(1,s,t)}) \times \pi_2(\Delta_i) = 0\,.$$

We denote by

$$E_{(1,s,t)} := \frac{B_{(1,s,t)} - 2\,\sigma\,C_{(1,s,t)}}{A^2_{(1,s,t)}}\,. \tag{X.3.13}$$

Hence, differentiating with respect to the jump amplitudes Δ_i in equation (X.3.11), we get

$$\begin{aligned} D_i V_{(1,s,t)} \times \pi_2(\Delta_i) &= -\frac{1}{S_s^2}\, D_i S_s \times \pi_2(\Delta_i) \\ &\quad - \frac{1}{\sigma\, S_s^2}\, D_i S_s\, (E_{(1,0,s)} - E_{(1,s,t)}) \times \pi_2(\Delta_i) \\ &\quad - \frac{1}{\sigma\, S_s^2}\, D_i S_s \left(\frac{F_{(1,s,t)}}{A_{(1,s,t)}} - \frac{F_{(1,0,s)}}{A_{(1,0,s)}} \right) \times \pi_2(\Delta_i)\,, \end{aligned}$$

that is

$$\begin{aligned} D_i V_{(1,s,t)} \times \pi_2(\Delta_i) &= -\frac{\sigma}{S_s}\, \frac{\pi_2(\Delta_i)}{1+\sigma\,\Delta_i} \\ &\quad - \frac{1}{S_s}\, \frac{\pi_2(\Delta_i)}{1+\sigma\,\Delta_i} \left(E_{(1,0,s)} - E_{(1,s,t)} + \frac{F_{(k,s,t)}}{A_{(1,s,t)}} - \frac{F_{(1,0,s)}}{A_{(1,0,s)}} \right)\,. \end{aligned} \tag{X.3.14}$$

We then have

$$\begin{aligned} &\sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i)\, \frac{\pi_2(\Delta_i)}{1+\sigma\,\Delta_i}\, D_i V_{(1,s,t)} \\ &\qquad = -\frac{A_{(2,s,t)}}{S_s} \left(\sigma + E_{(1,0,s)} - E_{(1,s,t)} + \frac{F_{(1,s,t)}}{A_{(1,s,t)}} - \frac{F_{(1,0,s)}}{A_{(1,0,s)}} \right)\,. \end{aligned}$$

We thus obtain

$$\begin{aligned}
&\gamma_s^{(2,0,s)} \langle DS_s, D(V_{(1,s,t)})\rangle_{(2,0,s)} \\
=&\frac{1}{\sigma^2\, S_s^2\, A_{(2,0,s)}} (\sigma\, S_s) \sum_{i=0}^{\infty} \mathbf{1}_{]0,s]}(T_i) \frac{\pi_2(\Delta_i)}{1+\sigma\,\Delta_i} D_i V_{(1,s,t)} \\
=&-\frac{1}{\sigma\, S_s\, A_{(2,0,s)}} \frac{A_{(2,0,s)}}{S_s} \left(\sigma + E_{(1,0,s)} - E_{(1,s,t)} + \frac{F_{(1,s,t)}}{A_{(1,s,t)}} - \frac{F_{(1,0,s)}}{A_{(1,0,s)}}\right) \\
=&-\frac{1}{S_s^2} - \frac{1}{\sigma\, S_s^2} \left(E_{(1,0,s)} - E_{(1,s,t)} + \frac{F_{(1,s,t)}}{A_{(1,s,t)}} - \frac{F_{(1,0,s)}}{A_{(1,0,s)}}\right) .
\end{aligned}$$

Similary,

$$\begin{aligned}
&\gamma_s^{(2,0,s)}\, \gamma_t^{(2,s,t)} \langle DS_s, DS_t\rangle_{(2,0,s)} \langle DS_t, D(V_{(1,s,t)})\rangle_{(2,s,t)} \\
=&\frac{(\sigma^2\, S_t\, S_s\, A_{(2,0,s)})\, (\sigma\, S_t)}{(\sigma^2\, S_s^2\, A_{(2,0,s)})\, (\sigma^2\, S_t^2\, A_{(2,s,t)})} \sum_{i=0}^{\infty} \mathbf{1}_{]s,t]}(T_i) \frac{\pi_2(\Delta_i)}{1+\sigma\,\Delta_i} D_i V_{(1,s,t)} \\
=&-\frac{1}{\sigma\, S_s\, A_{(2,s,t)}} \frac{A_{(2,s,t)}}{S_s} \left(\sigma + E_{(1,0,s)} - E_{(1,s,t)} + \frac{F_{(1,s,t)}}{A_{(1,s,t)}} - \frac{F_{(1,0,s)}}{A_{(1,0,s)}}\right) \\
=&-\frac{1}{S_s^2} - \frac{1}{\sigma\, S_s^2} \left(E_{(1,0,s)} - E_{(1,s,t)} + \frac{F_{(1,s,t)}}{A_{(1,s,t)}} - \frac{F_{(1,0,s)}}{A_{(1,0,s)}}\right) .
\end{aligned}$$

Hence,

$$\gamma_s^{(2,0,s)}\, \gamma_t^{(2,s,t)} \langle DS_s, DS_t\rangle_{(2,0,s)} \langle DS_t, D(V_{(1,s,t)})\rangle_{(2,s,t)} - \gamma_s^{(2,0,s)} \langle DS_s, D(V_{(1,s,t)})\rangle_{(2,0,s)} = 0 \,.$$

Combining with equation (X.3.12), we obtain finally

$$\mathcal{H}_{s,t} = V_{(1,s,t)}\, V_{(2,s,t)} \,,$$

which may be computed using equation (X.3.11).

3.2. Figure and comments

In this section, we compute the price of the American option of maturity $T = 1$ and strike $K = 100$, when the asset $(S_t)_{t\in[0,T]}$ follows the Geometrical model (X.3.1). Figure X.1 shows several values of prices corresponding to different jump intensities $\lambda = 1, 2, 4, 5$. We can observe that the price increases when the jump intensity increases as well, which seems to be intuitive since the jump intensity λ represents the noise available in the system (see Remark 2.2).

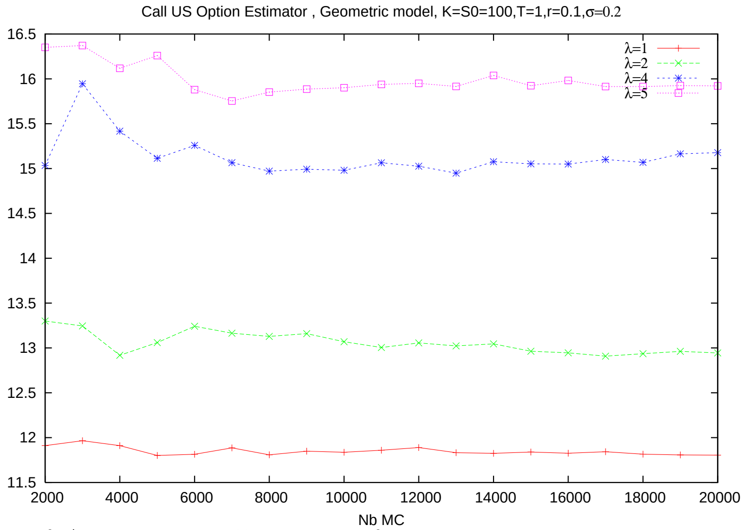

FIG. X.1 – Price of American call options for various jump intensities. Geometrical model.

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