Théorie de Ramsey Structurale des Espaces Métriques et Dynamique Topologique des Groupes d'Isométries.

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4 Décembre 2006

Théorie de Ramsey :

Etude de certains objets au sein desquels des structures organisées apparaissent lorsque les objets en question deviennent grands.

Ici : Des **espaces métriques**.

Dynamique topologique :

Etude de certains ensembles de transformations via leurs actions sur des espaces géométriques.

Ici : Des **groupes d'isométries** agiront sur des **espaces topologiques compacts**.

Motivation :

L'étude de propriétés de type Ramsey peut aider à comprendre la structure de certains espaces géométriques très homogènes.

Definition:
A metric space X is **ultrahomogeneous** when every isometry between finite subsets extends to an isometry of X onto itself.

Prototype examples:

- The unit sphere $\mathbb{S}^\infty$ of ℓ_2.

- The Baire space $\mathcal{N}$.

- The Urysohn sphere $\mathbb{S}$:

  Up to isometry, the unique metric space with distances in $[0, 1]$ which is:

  i) Complete, separable.

  ii) Ultrahomogeneous.

  iii) Universal for the class of all finite metric spaces with distances in $[0, 1]$.

First general idea:

The structure of countable ultrahomogeneous metric spaces can be studied from a combinatorial point of view.

Theorem (Fraïssé, Bogatyi):

Every countable ultrahomogeneous metric space is the generic universal object associated to a particular class $\mathcal{M}_0$ of finite metric spaces (Fraïssé class).

Second general idea:

The structure of complete separable ultrahomogeneous metric spaces can be studied via the structure of countable ultrahomogeneous spaces.

Theorem 1:

Every separable ultrahomogeneous metric space has a countable ultrahomogeneous dense subspace.

Finite Ramsey calculus, extreme amenability and universal minimal flows

Pestov's theorem on the extreme amenability of iso($\mathbb{S}$)

Theorem (Pestov):

Equip iso($\mathbb{S}$) with the pointwise convergence topology. Then every continuous action of iso($\mathbb{S}$) on a compact Hausdorff space K has a fixed point, ie:

$$\exists x \in K \quad \forall g \in \text{iso}(\mathbb{S}) \quad g \cdot x = x$$

Remark:

iso($\mathbb{S}$) is said to be **extremely amenable**.

Definition:
Let $(\mathbf{X}, <^{\mathbf{X}}), (\mathbf{Y}, <^{\mathbf{Y}}), (\mathbf{Z}, <^{\mathbf{Z}})$ be finite ordered metric spaces and $k \in \omega, k > 0$.

$$(\mathbf{Z}, <^{\mathbf{Z}}) \longrightarrow (\mathbf{Y}, <^{\mathbf{Y}})_k^{(\mathbf{X}, <^{\mathbf{X}})}$$

means: For every coloring of the copies of $(\mathbf{X}, <^{\mathbf{X}})$ in $(\mathbf{Z}, <^{\mathbf{Z}})$ with k colors, there is a copy $(\widetilde{\mathbf{Y}}, <^{\widetilde{\mathbf{Y}}})$ of $(\mathbf{Y}, <^{\mathbf{Y}})$ in which all copies of $(\mathbf{X}, <^{\mathbf{X}})$ in $(\widetilde{\mathbf{Y}}, <^{\widetilde{\mathbf{Y}}})$ have same color.

Definition:
A class $\mathcal{M}$ of finite ordered metric spaces has the **Ramsey property** when:

For every $(\mathbf{X}, <^{\mathbf{X}}), (\mathbf{Y}, <^{\mathbf{Y}})$ in $\mathcal{M}$, there is $(\mathbf{Z}, <^{\mathbf{Z}})$ in $\mathcal{M}$ such that

$$(\mathbf{Z}, <^{\mathbf{Z}}) \longrightarrow (\mathbf{Y}, <^{\mathbf{Y}})_k^{(\mathbf{X}, <^{\mathbf{X}})}$$

Definition:

Let $\mathcal{M}$ be a class of finite ordered metric spaces and $\mathcal{M}_0$ be the class of finite metric spaces $\mathbf{X}$ such that for some linear ordering $<^{\mathbf{X}}$ on $\mathbf{X}$: $(\mathbf{X}, <^{\mathbf{X}}) \in \mathcal{M}$.

$\mathcal{M}$ has the **Ordering property** when:

For any $(\mathbf{X}, <^{\mathbf{X}})$ in $\mathcal{M}$, there is $\mathbf{Y}$ in $\mathcal{M}_0$ such that for every linear ordering $\prec$ on $\mathbf{Y}$:

$$(\mathbf{Y}, \prec) \in \mathcal{M} \Rightarrow (\mathbf{X}, <^{\mathbf{X}}) \text{ embeds in } (\mathbf{Y}, \prec).$$

Theorem (Nešetřil):

The class $\mathcal{M}^{<}_{\mathbb{Q}\cap[0,1]}$ of finite ordered metric spaces with rational distances in $[0,1]$ has the Ramsey and the Ordering properties.

Corollary (Kechris-Pestov-Todorcevic):

iso($\mathbb{S}$) is extremely amenable.

Ultrametric spaces

Definition:

A linear ordering $<$ on a metric space is **convex** when all the metric balls are $<$-convex.

Theorem 2:

Let $S \subset]0, +\infty[$. The class of finite convexly ordered ultrametric spaces with distances in S has the Ramsey and the Ordering properties.

Definition:
Let G be a topological group. A **minimal** **G-flow** is a compact Hausdorff space K together with a continuous action of G on K where the orbit of every $x \in K$ is dense in K.

Fact:
Let G be a topological group. Then there is a minimal G-flow $M(G)$ such that for every minimal G-flow K, there is a continuous onto $\pi : M(G) \longrightarrow K$ such that

$$\forall g \in G, \forall x \in K, \pi(g \cdot x) = g \cdot \pi(x).$$

$M(G)$ is unique up to isomorphism.

Theorem 3:
$M(\text{iso}(\mathcal{N}))$ is the set of all convex linear orderings on $\mathcal{N}$ together with the action $(g, <) \longmapsto <^g$ defined by

$$x <^g y \leftrightarrow g^{-1}(x) < g^{-1}(y).$$

Other examples

Notation:

Let $S \subset]0,+\infty[$, $s \in]0,+\infty[$, $\mathcal{M}_S$ the class of all finite metric space with distances in S, $\mathbf{X} \in \mathcal{M}_S$. Define $E_s^{\mathbf{X}}$ by

$$\forall x, y \in \mathbf{X} \quad x E_s^{\mathbf{X}} y \leftrightarrow d^{\mathbf{X}}(x,y) \leqslant s.$$

Definition:

s is **critical for** $\mathcal{M}_S$ when for every $\mathbf{X}$ in $\mathcal{M}_S$, $E_s^{\mathbf{X}}$ is an equivalence relation on $\mathbf{X}$.

Definition:

Let $\mathbf{X} \in \mathcal{M}_S$, $<$ a linear ordering on $\mathbf{X}$. $<$ is **metric** when for every critical s, the $E_s^{\mathbf{X}}$-equivalence classes are $<$-convex.

Fact(Delhommé-Laflamme-Pouzet-Sauer):

There is a condition characterizing those sets $S \subset]0, +\infty[$ for which the class $\mathcal{M}_S$ of finite metric spaces with distances in S is a Fraïssé class: The **4 values condition**.

Theorem 4:

Let $S \subset]0, +\infty[$ of size $|S| \leqslant 3$ satisfying the 4 values condition. Then the class of all finite metrically ordered metric spaces with distances in S has the Ramsey and the Ordering properties.

Ramsey degrees and Big Ramsey degrees

Definition: Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be metric.

$$\mathbf{Z} \longrightarrow (\mathbf{Y})^{\mathbf{X}}_{k,l}$$

means: Given a coloring of the copies of $\mathbf{X}$ in $\mathbf{Z}$ with k colors, there is a copy $\widetilde{\mathbf{Y}}$ of $\mathbf{Y}$ such that at most l colors are taken on the set of copies of $\mathbf{X}$ in $\widetilde{\mathbf{Y}}$.

Definition: Let $\mathcal{M}_0$ be a class of finite metric spaces, $\mathbf{X} \in \mathcal{M}_0$. The **Ramsey degree of X in** $\mathcal{M}_0$ is the least $l \in \omega \cup \{\omega\}$ such that: For every $\mathbf{Y} \in \mathcal{M}_0$ and $k \in \omega$, there is $\mathbf{Z} \in \mathcal{M}_0$ such that

$$\mathbf{Z} \longrightarrow (\mathbf{Y})^{\mathbf{X}}_{k,l}.$$

Notation:
LO($\mathbf{X}$): Linear orderings on $\mathbf{X}$.
cLO($\mathbf{X}$): Convex linear orderings on $\mathbf{X}$.
mLO($\mathbf{X}$): Metric linear orderings on $\mathbf{X}$.

Corollary 4:
Every $\mathbf{X} \in \mathcal{M}_{\mathbb{Q}\cap[0,1]}$ has a finite Ramsey degree in $\mathcal{M}_{\mathbb{Q}\cap[0,1]}$ equal to

$$|\mathrm{LO}(\mathbf{X})|/|\mathrm{iso}(\mathbf{X})|$$

Corollary 5:
Every $\mathbf{X} \in \mathcal{U}_S$ has a finite Ramsey degree in $\mathcal{U}_S$ equal to

$$|\mathrm{cLO}(\mathbf{X})|/|\mathrm{iso}(\mathbf{X})|$$

Corollary 6:
Let $S \subset]0, +\infty[$ satisfying the 4 values condition, $|S| \leqslant 3$. Then every $\mathbf{X} \in \mathcal{M}_S$ has a finite Ramsey degree in $\mathcal{M}_S$ equal to

$$|\mathrm{mLO}(\mathbf{X})|/|\mathrm{iso}(\mathbf{X})|$$

Big Ramsey degrees

Definition:

Let $\mathcal{M}_0$ be a Fraïssé class of finite metric spaces and $\mathbb{U}_{\mathcal{M}_0}$ its associated Urysohn space. Let $\mathbf{X} \in \mathcal{M}_0$. The **Big Ramsey Degree of X in** $\mathcal{M}_0$ is the least $l \in \omega \cup \{\omega\}$ such that: For every $k \in \omega$,

$$\mathbb{U}_{\mathcal{M}_0} \longrightarrow (\mathbb{U}_{\mathcal{M}_0})^{\mathbf{X}}_{k,l}$$

Theorem 7:

Let $S \subset]0, +\infty[$ be finite. Then any $\mathbf{X} \in \mathcal{U}_S$ has a finite Big Ramsey Degree in $\mathcal{U}_S$ which can be computed explicitly.

Theorem 8:

Let $S \subset]0, +\infty[$ be infinite. Then any $\mathbf{X} \in \mathcal{U}_S$ with $|\mathbf{X}| \geqslant 2$ has a Big Ramsey Degree in $\mathcal{U}_S$ equal to ω.

Question:

What if S is infinite and $|\mathbf{X}| = 1$?

Indivisibility and metric oscillation stability

Definition:

A metric space $\mathbf{X}$ is **indivisible** when for every partition $\mathbf{X} = A_1 \cup \ldots \cup A_k$, there are $\widetilde{\mathbf{X}}$ isometric to $\mathbf{X}$ and $i \leqslant k$ such that:

$$\widetilde{\mathbf{X}} \subset A_i$$

Definition:

A complete separable ultrahomogeneous metric space $\mathbf{X}$ is **metrically oscillation stable** when for every bounded, 1-Lipschitz $f : \mathbf{X} \longrightarrow \mathbb{R}$ and every $\varepsilon > 0$, there is $\widetilde{\mathbf{X}}$ isometric to $\mathbf{X}$ such that:

$$\forall x, y \in \widetilde{\mathbf{X}}, \quad |f(y) - f(x)| < \varepsilon$$

Equivalently:

Whenever $\mathbf{X} = A_1 \cup \ldots \cup A_k$ and $\varepsilon > 0$, there are $\widetilde{\mathbf{X}}$ isometric to $\mathbf{X}$ and $i \leqslant k$ such that:

$$\widetilde{\mathbf{X}} \subset (A_i)_\varepsilon$$

The distortion problem for ℓ_2

Theorem (Odell-Schlumprecht):

The unit sphere $\mathbb{S}^\infty$ of the Hilbert space ℓ_2 is not metrically oscillation stable.

Problem:

The intrinsic geometry of $\mathbb{S}^\infty$ does not appear in the proof !

Ultrahomogeneous ultrametric spaces

Theorem (Folklore):

The Baire space $\mathcal{N}$ is metrically oscillation stable.

Theorem 9:

Let $\mathbf{X}$ be a countable ultrahomogeneous ultrametric space with distance set $D_{\mathbf{X}}$. Then $\mathbf{X}$ is indivisible iff $(D_{\mathbf{X}}, >)$ is a well-ordering.

Corollary 10:

Let $\mathbf{Y}$ be a complete separable ultrahomogeneous ultrametric space with distance set $D_{\mathbf{Y}}$. Then $\mathbf{Y}$ is metrically oscillation stable iff $(D_{\mathbf{Y}}, >)$ is a well-ordering.

The oscillation stability problem for $\mathbb{S}$

1st attempt:

Study the indivisibility of the rational Urysohn sphere $\mathbb{S}_{\mathbb{Q}}$.

Thm(Delhommé-Laflamme-Pouzet-Sauer): $\mathbb{S}_{\mathbb{Q}}$ is not indivisible.

Key: The distance set of $\mathbb{S}_{\mathbb{Q}}$ is too rich.

General question:

What if we replace it by a simpler set S? For example, what if S is finite ?

Case $|S| = 1$: Trivial.

Case $|S| = 2$:

- $S = \{1, 2\}$: Random graph. Indivisible.
- $S = \{1, 3\}$: Ultrametric. Indivisible.

Case $|S| = 3$:
7 cases to check with the 4 values condition.
6 possible distance sets S.
All of them provide an indivisible space.

Case $|S| = 4$:
About **50** essential cases to check with the 4 values condition.
22 possible distance sets S.
21 cases provide an indivisible space.
1 case remains open: $\{1, 2, 3, 4\}$.

This is quite unfortunate. . .

Theorem 11:

The following are equivalent:

i) $\mathbb{S}$ is metrically oscillation stable.

ii) $\forall m \in \omega$, $\mathbb{U}_{\{1,\ldots,m\}}$ is indivisible.

Thm(Delhommé-Laflamme-Pouzet-Sauer):
$\mathbb{U}_{\{1,2,3\}}$ is indivisible.

Remark:

i) and ii) are also equivalent to:
iii) $\forall m \in \omega$, $\mathbb{U}_{\{1,\ldots,m\}}$ is 1-indivisible.

Theorem 12:

$\mathbb{U}_{\{1,\ldots,m\}}$ is 1-indivisible whenever $m \leqslant 9$.

Theorem 13:

Let $m \in \omega$, $\varepsilon > 0$. Assume $\mathbb{U}_{\{1,\ldots,m\}}$ indivisible. Then $\mathbb{S}$ is $(1/2m + \varepsilon)$-metrically oscillation stable.

Corollary 14:

$\mathbb{S}$ is $(1/6 + \varepsilon)$-metrically oscillation stable for every $\varepsilon > 0$.

Concluding questions:

- Can the extreme amenability of $\mathbb{U}(\ell_2)$ (Gromov-Milman, Pestov) be proved via Ramsey property for a class of finite Euclidean metric spaces ?

- Can combinatorial methods help understanding the Distortion property for ℓ_2. Why is $\mathbb{S}^\infty$ NOT metrically oscillation stable (Odell-Schlumprecht)?

- For $m \in \omega$, is $\mathbb{U}_{\{1,\ldots,m\}}$ indivisible ?