# Etude de quelques EDP non linéaires sans compacité

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# THÈSE

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# **Titre:** *Etude de quelques EDP non linéaires sans compacité*

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# Chapter 1

# Introduction

Cette thèse a pour objet l'étude de quelques équations aux dérivées partielles non linéaires de type Dirichlet ou Neumann, à structure variationnelle, présentant un défaut de compacité. Ce sujet trouve son origine dans des problèmes de géométrie différentielle (problème de Yamabe avec ou sans bord) ou des problèmes de physique (les équations de Yang-Mills).
Etant donnée la structure variationnelle de ces problèmes, les solutions cherchées correspondent aux points critiques d'une fonctionnelle $F$, elles sont solutions de l'équation $F'(u) = 0$. La difficulté principale commune à tous ces problèmes est le manque de compacité. Pour remédier à cette difficulté, on utilise une forme déguisée de compacité : la condition de Palais-Smale (PS).

**Définition 1.0.1.**
*Soit $F$ une fonction de classe $C^1$ sur un Banach $E$. On dit que $F$ vérifie la condition (PS) si pour toute suite $(u_j)$ telle que $F(u_j)$ reste bornée et $\|F'(u_j)\| \to 0$, alors $(u_j)$ est relativement compacte.*

Cette condition empêche les points critiques de $F$ de "fuir à l'infini".
Cependant, la condition (PS) est trop forte, c'est à dire il existe des suites non relativement compactes pour lesquelles $F$ reste bornée et sa dérivée tend vers zero. Aussi, une fonction $F$ peut atteindre son minimum sans vérifier la condition (PS). On introduit, alors, une condition plus faible $(\mathrm{PS})_c$ (voir Brézis-Coron-Nirenberg [BCN]).

**Définition 1.0.2.**
*Soit $c \in I\!R$ fixé. On dit que $F$ vérifie $(PS)_c$ si pour toute suite $(u_j)$ telle que $F(u_j) \to c$ et $\|F'(u_j)\| \to 0$, alors $(u_j)$ est relativement compacte.*

Cette amélioration qui peut paraitre négligeable est en fait extrêmement utile dans les applications. En pratique, on est conduit au programme suivant:

**1)**Identifier les valeurs de $c$ pour lesquelles $(\text{PS})_c$ tombe en défaut.
**2)** Montrer que $\inf F$ n'est pas une telle valeur.
Cette approche conduit à l'existence de solutions pour des équations à structures variationnelles. Le point d'orgue dans cette direction, est l'accomplissement de la preuve de la conjecture de Yamabe par Aubin [A] et Schoen. La résolution du problème de Yamabe marque une étape dans le développement de l'analyse non linéaire.
Depuis, plusieurs travaux fondateurs ont été mis au point, citons par exemples [BN1], [BaC], [C], [CHL], [Le1], [Pa], pour des problèmes avec condition de Dirichlet, et [AM], [AY], [Ch], [E], pour des problèmes avec condition de Neumann.
Cette thèse se répartit en quatre chapitres. Le chapitre 1 est une introduction générale. Dans le chapitre 2, on étudie un problème avec la condition de Dirichlet homogène et avec l'exposant critique de Sobolev.
Dans les chapitres 3 et 4, on s'intéresse a des problèmes de Neumann non linéaires avec une non-linéarité critique au bord.

## 1.1 Problèmes de Dirichlet non linéaires faisant intervenir l'exposant critique du Sobolev

On étudie une équation aux dérivées partielles elliptique non linéaire avec poids faisant intervenir l'exposant critique de Sobolev de type

$$(P1)_\Omega \quad \begin{cases} -div(p(x)\nabla u) = u^{q-1} + \lambda u & \text{dans } \Omega, \\ u > 0 & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où $\Omega$ est un ouvert borné régulier de $I\!R^N$, $N \geq 3$, $p : \bar{\Omega} \longrightarrow I\!R$ est une fonction strictement positive avec $p \in H^1(\Omega) \cap C(\bar{\Omega})$, $\lambda \in I\!R$ et $q = \frac{2N}{N-2}$.
On rappelle que pour $t \in [1, q[$ l'injection de Sobolev $H_0^1(\Omega) \hookrightarrow L^t(\Omega)$ est compacte, et ne l'est plus lorsque $t = q$. L'exposant $q = \frac{2N}{N-2}$ est appelé exposant critique de Sobolev. On introduit alors la meilleure constante de Sobolev

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^q dx\right)^{\frac{2}{q}}}.$$

dont les propriétés essentielles sont (voir [BN1]): $S$ ne dépend pas du domaine $\Omega$, $S$ n'est jamais atteinte si $\Omega \neq I\!R^N$, $S$ est atteinte sur $I\!R^N$ et les fonctions extrémales sont de la forme

$$U_{x_0,\varepsilon}(x) = \alpha_N \frac{\varepsilon^{\frac{N-2}{4}}}{(\varepsilon + |x - x_0|^2)^{\frac{N-2}{2}}}, \quad x \in I\!R^N$$

où $\varepsilon > 0$ est un paramètre de concentration, $x_0 \in I\!R^N$ est un paramètre de translation et $\alpha_N$ est une constante choisie de telle sorte que chacune de ces fonctions vérifie:

$$-\Delta U_{x_0,\varepsilon} = U_{x_0,\varepsilon}^{q-1} \quad \text{sur } I\!R^N.$$

Cette partie repose sur deux résultats fondamentaux d'existence concernant le problème $(\mathrm{P1})_\Omega$ dans le cas $p \equiv 1$. Le premier, dû à Brezis-Nirenberg [BN1], tient compte de la perturbation linéaire $\lambda u$, le second, dû à Coron [C], tient compte de la géometrie de l'ouvert $\Omega$ lorsque $\lambda = 0$.
Les méthodes utilisées dans ce travail, font appel à la recherche de minima de la fonctionnelle associée à $(\mathrm{P1})_\Omega$ et au principe de Concentration-Compacité de Lions [L]. On utilise également une variante du Théorème d'Ambrosetti-Rabinowitz [AR].

### 1.1.1 Influence de la perturbation linéaire

On généralise les résultats de [BN1] concernant $(\mathrm{P1})_\Omega$ au cas où le poids $p \not\equiv 1$. On remarque que l'existence de solutions dans ce cas dépend du paramètre $\lambda$ et du comportement de la fonction $p$ au voisinage de ses minima.
On considère alors $p_0 = \min\{p(x),\ x \in \bar{\Omega}\}$ et on suppose que $p^{-1}(\{p_0\}) \cap \Omega \neq \emptyset$.
Soit $a \in p^{-1}(\{p_0\}) \cap \Omega$, on suppose que, dans un voisinage de $a$

$$(1.1.1) \qquad p(x) = p_0 + \beta_k |x-a|^k + |x-a|^k \theta(x),$$

avec $k > 0$, $\beta_k > 0$ et $\theta(x)$ tend vers 0 quand $x$ tend vers $a$.
Dans le cas où $0 < k \leq 2$ la situation est plus délicate, alors on se restreint au cas où $p$ vérifie la condition supplémentaire

$$(1.1.2) \qquad k\beta_k \leq \frac{\nabla p(x).(x-a)}{|x-a|^k} \quad \text{p.p } x \in \Omega.$$

On a le théorème suivant

**Théorème 1.**
*On suppose que $p \in H^1(\Omega) \cap C(\bar{\Omega})$ vérifie (1.1.1). Soit $\lambda_1^{div}$ la première valeur propre de l'opérateur $\mathrm{div}(p(x)\nabla.)$ sur $\Omega$ avec condition de Dirichlet homogène. On a*
*1)Si $n \geq 4$ et $k > 2$, alors pour tout $\lambda \in ]0, \lambda_1^{div}[$ il existe une solution de $(P1)_\Omega$.*
*2)Si $n \geq 4$ et $k = 2$, alors il existe une constante $\tilde{\gamma}(n) = \frac{(n-2)n(n+2)}{4(n-1)}\beta_2$ telle que pour tout $\lambda \in ]\tilde{\gamma}(n), \lambda_1^{div}[$ il existe une solution de $(P1)_\Omega$.*
*3)Si $n = 3$ et $k \geq 2$, alors il existe une constante $\gamma(k) > 0$ telle que pour tout $\lambda \in ]\gamma(k), \lambda_1^{div}[$ le problème $(P1)_\Omega$ admet une solution.*
*4)Si $n \geq 3$, $0 < k < 2$ et $p$ vérifie la condition (1.1.2) alors il existe $\lambda^* \in [\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$, où $\tilde{\beta}_k = \beta_k \min[(\mathrm{diam}\ \Omega)^{k-2}, 1]$, tel que pour tout $\lambda \in ]\lambda^*, \lambda_1^{div}[$, le problème $(P1)_\Omega$*

*admet une solution.*
*5)Si $n \geq 3$ et $k > 0$, alors pour tout $\lambda \leq 0$ le problème (P1)$_\Omega$ n'a pas de solution minimisante.*
*6)Si $n \geq 3$ et $k > 0$, alors le problème (P1)$_\Omega$ n'admet pas de solution pour tout $\lambda \geq \lambda_1^{div}$.*

Dans la démonstration de ce théorème, l'outil de base est une technique de minimisation. Dans le cas $k > 2$, une estimation fine de l'énergie permet de conclure. Cependant, dans le cas $0 < k \leq 2$, on utilise en plus un lemme de type Hardy (voir [HLP]).
On obtient également grâce à l'identité de Pohozaev, des résultats de non existence pour le problème (P1)$_\Omega$ dans le cas d'un domaine étoilé.

### 1.1.2 Influence de la géométrie du domaine

Dans le cas où $p \equiv 1$ et $\lambda = 0$, l'identité de Pohozaev montre que le problème (P1)$_\Omega$ n'admet pas de solution pour un domaine étoilé. Cependant, Coron [C] a montré l'existence de solutions concernant le problème (P1)$_\Omega$ pour un domaine avec un petit trou.
Dans le cas où $p \not\equiv 1$, on remarque, en utilisant l'identité de Pohozaev, que pour $\lambda = 0$ et pour $p \in H^1(\Omega) \cap C(\bar{\Omega})$ vérifiant $\nabla p(x).(x-a) \geq 0$ p.p sur $\Omega$, le problème (P1)$_\Omega$ n'admet pas de solution pour un domaine étoilé par rapport à $a$. On modifie alors la géométrie du domaine afin d'obtenir une solution dans ce dernier cas.
Soit $\Omega \subset I\!R^N$, $N \geq 3$ un domaine borné régulier et étoilé par rapport à $a$, et soit $\varepsilon > 0$. On étudie le problème (P1)$_{\Omega_\varepsilon}$ pour $\lambda = 0$ avec $\Omega_\varepsilon = \Omega \setminus B(a, \varepsilon)$.
On montre le résultat suivant

**Théorème 2.**
*On suppose que $p \in H^1(\Omega) \cap C(\bar{\Omega})$ vérifie $\nabla p(x).(x-a) \geq 0$ p.p $\Omega$, alors il existe $\varepsilon_0 = \varepsilon_0(\Omega, p) > 0$ tel que pour tout $\varepsilon < \varepsilon_0$, le problème (P1)$_{\Omega_\varepsilon}$ admet une solution.*

On s'inspire des méthodes utilisées dans [C] et dans [H]. L'idée de la démonstration consiste à étudier la fonctionnelle

$$F(u) = \frac{1}{2}\int_{\Omega_\varepsilon} p(x)|\nabla u|^2 dx - \frac{1}{q}\int_{\Omega_\varepsilon} |u|^q dx, \quad u \in H_0^1(\Omega).$$

Les solutions de (P1)$_{\Omega_\varepsilon}$ correspondent aux points critiques non nuls de $F$. En général $F$ ne vérifie pas la condition de Palais Smale (PS), mais on montre que $F$ vérifie une condition plus faible (PS)$_c$ pour $c \in ]\frac{1}{N}(p_0 S)^{\frac{N}{2}}, \frac{2}{N}(p_0 S)^{\frac{N}{2}}[$. On applique alors une variante du Théorème d'Ambrosetti-Rabinowitz [AR] qui permet d'obtenir une solution.

## 1.2 Problèmes de Neumann non linéaires avec une non-linéarité critique au bord

On s'intèresse à des équations de Neumann non-linéaires avec poids et avec une non-linéarité critique au bord du type

$$(P2) \quad \begin{cases} -div(p(x)\nabla u) = \lambda u + f(x,u) & \text{dans } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{sur } \partial\Omega, \end{cases}$$

où $\Omega \subset I\!R^N$, $N \geq 3$, est un domaine borné avec un bord régulier $\partial\Omega$, $\nu$ est la normale extérieure à $\partial\Omega$, $Q$ est une fonction continue et positive dans $\partial\Omega$, $p \in H^1(\Omega)$ est continue et positive dans $\bar{\Omega}$, $q = \frac{2(N-1)}{N-2}$ et $f(.,u)$ est un terme d'ordre inférieur à $q$. On sait que pour $t \in [1, q[$ l'injection de $H^1(\Omega) \hookrightarrow L^t(\partial\Omega)$ est compacte, et ne l'est plus lorsque $t = q$. L'exposant $q = \frac{2(N-1)}{N-2}$ est appelé exposant critique de Sobolev pour l'inclusion de trace de $H^1(\Omega)$ dans $L^q(\partial\Omega)$.
On introduit la meilleure constante de Sobolev de l'injection de trace $H^1(I\!R^N_+)) \hookrightarrow L^q(\partial I\!R^N_+)$

$$S_1 = \inf_{u \in H^1(I\!R^N_+)\setminus\{0\}} \frac{\displaystyle\int_{I\!R^N_+} |\nabla u|^2 dx}{\left(\displaystyle\int_{\partial I\!R^N_+} |u|^q dx\right)^{\frac{2}{q}}}$$

où $R^N_+ = \{x = (x', x_N) : \, x' \in I\!R^{N-1}, \, x_N > 0\}$.

On sait d'aprés Lions [L] que $S_1$ est atteinte, et d'aprés Escobar [E], que les fonctions minimisantes sont de la forme

$$W(x', x_N) = \frac{c_N}{[(1+x_N)^2 + |x'|^2]^{\frac{N-2}{2}}}$$

où $c_N > 0$ est une constante qui ne dépend que de la dimension $N$.
De plus

$$S_1 = \frac{N-2}{2} \omega_N^{\frac{1}{N-1}}$$

où $\omega_N$ = volume de sphère unité $S^{N-1}$.

On considère la condition géometrique, suivante :
On dit que (g.c) est satisfaite au point $y \in \partial\Omega$ s'il existe un voisinage $U(y)$ tel que $\Omega \cap U(y)$ est contenu dans le demi-espace délimité par le plan tangent à $\partial\Omega$ au point $y$. Il s'agit donc d'un point où toutes les courbures principales sont positives.
On note qu'on a toujours l'existence d'un point $y \in \partial\Omega$ pour lequel (g.c) est satisfaite, voir [AY].

On sait d'aprés [AM] que la courbure moyenne de $\partial\Omega$ et la condition géométrique (g.c) en un point $y \in \partial\Omega$ jouent des rôles importants dans la résolution des problèmes de type (P2).

Dans le cas où $p \equiv 1 \equiv Q$ et $f(x,u) \equiv 0$, (P2) a été traité par plusieurs auteurs, on cite par exemple [AY], [Ch], [CY].
On se place dans le cas où $p$ et $Q$ sont des fonctions positives continues, et $f(x,u)$ est, soit de la forme $\alpha u^r$, soit une fonction qui ne dépend pas de $u$.
Soit $x_0 \in \partial\Omega$ vérifiant

$$\frac{p(x_0)}{(Q(x_0))^{N-2}} = \min_{x\in\partial\Omega} \frac{p(x)}{(Q(x))^{N-2}}.$$

On suppose que

$$|p(x) - p(x_0)| = o(|x - x_0|) \tag{1.2.1}$$

et

$$|Q(x) - Q(x_0)| = o(|x - x_0|) \tag{1.2.2}$$

pour $x$ proche de $x_0$.
On définit

$$\tilde{S} = \frac{1}{2(N-1)} \frac{p(x_0)}{Q(x_0)^{N-2}} S_1^{N-1}.$$

### 1.2.1 Le cas où $f(x,u) = \alpha u^r$

On s'intéresse au problème suivant

$$\begin{cases} -div(p(x)\nabla u) = \lambda u + \alpha |u|^{r-2}u & \text{dans } \Omega, \\ u \not\equiv 0 & \text{dans } \Omega \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{sur } \partial\Omega, \end{cases} \tag{1.2.3}$$

avec $\alpha \in I\!R$ et $2 < r < q$.
On considère la fonctionnelle associée au problème (1.2.3)

$$I_\alpha = \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{\alpha}{r}\int_\Omega |u|^r dx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x.$$

On commence par étudier l'existence de solutions au problème (1.2.3) dans le cas où le paramètre $\lambda$ est strictement négatif et $\alpha$ un réel quelconque. Plus précisement on a le théorème suivant

**Théorème 3.**
*Soit $\Omega \in I\!R^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en $x_0$ et $H(x_0) > 0$. On suppose que $p$ vérifie (1.2.1) et $Q$ vérifie (1.2.2). Alors pour tout $\lambda < 0$ et pour tout $\alpha \in I\!R$ le problème (1.2.3) possède une solution.*

Idée de la preuve: On vérifie que l'energie $I_\alpha$ satisfait les hypothèses du Lemme du Col, d'où l'existence d'une suite $(u_n)$ vérifiant $I_\alpha(u_n) \to c$ et $I'_\alpha(u_n) \to 0$ dans $H^{-1}(\Omega)$ avec $c$ donné par le Lemme du Col.
Dans un premier temps, on vérifie que $c < \tilde{S}_1$. Ensuite, on montre que la suite $(u_n)$ converge fortement vers $u$ dans $H^1(\Omega)$. Pour se faire, on est conduit à distinguer les cas $\alpha \leq 0$ et $\alpha \geq 0$. Dans le cas où $\alpha \leq 0$, on conclut en utilisant les méthodes variationelles classiques, par contre dans le cas où $\alpha \geq 0$ on fait appel au principe de Concentration-Compacité de Lions [L]. La limite $u$ de la suite $(u_n)$ sera une solution du problème (1.2.3).

Dans le cas où $\lambda$ est strictement positif. On considère l'opérateur $-div(p(x)\nabla\,.)$ avec la condition de Neumann homogène au bord. On note la suite des valeurs propres correspondantes par $\{\lambda_k,\ k \in \mathbf{N} \setminus \{0\}\}$. Elles vérifient $0 = \lambda_1 < \lambda_2 \leq ....$ De plus les fonctions propres qui correspondent à $\lambda_1$ sont les fonctions constantes.
Soit $k \in \mathbf{N} \setminus \{0\}$ tel que $\lambda_{k-1} \neq \lambda_k$, on montre le résultat suivant

**Théorème 4.**
*Soit $\Omega \in I\!R^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en $x_0$ et $H(x_0) > 0$. On suppose que $p$ vérifie (1.2.1) et $Q$ vérifie (1.2.2). Alors pour tout $\lambda \in ]\lambda_{k-1}, \lambda_k[$ et pour tout $\alpha \geq 0$ le problème (1.2.3) possède une solution.*

Concernant la preuve de Théorème 4, on raisonne comme dans le Théorème 3 et on conclut à l'aide de principe de min-max (voir [W]).

### 1.2.2 Le cas où $f(x,u) = f(x)$

On considère l'équation suivante

$$(1.2.4) \qquad \begin{cases} -div(p(x)\nabla u) = \lambda u + f(x) & \text{dans } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{sur } \partial\Omega, \end{cases}$$

On s'intéresse à l'existence de solutions du problème (1.2.4) qui ne sont pas identiquement nulles sur le bord $\partial\Omega$. On suppose alors que $f$ est telle que
(1.2.5)

$$\text{le problème } \begin{cases} -div(p(x)\nabla u) = \lambda u + f(x) & \text{dans } \Omega, \\ \frac{\partial u}{\partial \nu} = u = 0 & \text{sur } \partial\Omega, \end{cases} \quad \text{n'admet pas de solution.}$$

Notons que l'hypothèse (1.2.5) est satisfaite, par exemple si $f$ ne change pas de signe. En effet, si (1.2.5) n'est pas satisfaite pour une fonction $f$ de signe constant, alors on obtient une contradiction grâce au Lemme de Hopf.
On distingue deux parties principales.

Dans la première partie, on s'intéresse au cas où $\lambda$ est srictement négatif. Pour étudier l'existence de solutions du problème (1.2.4), certaines conditions sur $f$ sont nécessaires pour cette approche. On les résume ici

$$(H1)\begin{cases} f \in H^{-1}(\Omega)\cap C(\bar{\Omega})\setminus\{0\} \text{ satisfait la condition (1.2.5)} \\ \text{et} \\ (*) \displaystyle\int_\Omega f(x)\,u\,dx < \sigma_N\Big(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\Big)^{\frac{N}{2}}, \\ \forall u \in H^1(\Omega),\ \displaystyle\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = 1. \end{cases}$$

avec $\sigma_N = (q-2)\left[\frac{1}{q-1}\right]^{\frac{q-1}{q-2}}$.

On note que $(*)$ est satisfaite par exemple si $\|f\|_{H^{-1}} < M$, où $M$ est une constante connue qui ne dépend que de $\lambda$, de $p$, de $Q$ et de l'injection $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$.

Le premier résultat de cette partie est le suivant

**Théorème 5.**
*Soit $\Omega \in I\!R^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en $x_0$ et tel que $H(x_0) > 0$. On suppose que $f$ vérifie (H1), $p$ vérifie (1.2.1) et $Q$ vérifie (1.2.2). Alors pour tout $\lambda < 0$ on a*

$$I_0 := \inf_{\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}}=1}\left[\sigma_N\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right)^{\frac{N}{2}} - \int_\Omega f(x)\,u\,dx\right]$$

*est atteint et $I_0 > 0$.*

Ce Théorème joue un rôle important dans la preuve de l'existence de solutions du problème (1.2.4).
La démonstration de ce Théorème repose sur des idées de Brezis-Nirenberg [BN2] concernant un problème de minimisation de type Dirichlet.

Le deuxième résultat de cette partie est le Théorème suivant

**Théorème 6.**
*Soit $\Omega \in I\!R^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en $x_0$ et $H(x_0) > 0$. On suppose que $f$ vérifie (H1), $p$ vérifie (1.2.1) et $Q$ vérifie (1.2.2). Lorsque $N = 3$, on suppose de plus que $f$, $p$ et $Q$ sont assez régulieres. On a alors pour tout $\lambda < 0$ le problème (1.2.4) admet au moins deux solutions.*

L'idée de la preuve suit la stratégie dévélloppée par Tarantello dans [T] pour traiter un problème de type Dirichlet.
La fonctionnelle correspondante au problème (1.2.4) est donnée par

$$I(u)=\frac{1}{2}\int_\Omega p(x)|\nabla u|^2dx-\frac{\lambda}{2}\int_\Omega |u|^2dx-\int_\Omega f(x)udx-\frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x,\ \ u\in H^1(\Omega).$$

On note

$$I_1(u)=\int_\Omega p(x)|\nabla u|^2dx-\lambda\int_\Omega |u|^2dx-(q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x.$$

On utilise les ensembles suivants:
$\Lambda=\{u\in H^1(\Omega):\langle I'(u),u\rangle=0\}$, $\Lambda^+=\{u\in\Lambda:\ I_1(u)>0\}$, $\Lambda_0=\{u\in\Lambda:\ I_1(u)=0\}$ et $\Lambda^-=\{u\in\Lambda:\ I_1(u)<0\}$.
Pour montrer l'existence de deux solutions, on considère les problèmes de minimisation suivants

$$c_0=\inf_{\Lambda} I(u) \tag{1.2.6}$$

et

$$c_1=\inf_{\Lambda^-} I(u). \tag{1.2.7}$$

Pour montrer que l'infimum (1.2.6) est atteint pour une certaine fonction $u_0$ on fait appel au principe variationnel d'Ekeland [AE]. Par la suite, on vérifie que $u_0\in\Lambda^+$ et qu'elle correspond à un minimum local. Ainsi on a une première solution.

Dans le but de montrer que (1.2.7) est atteint, on applique le principe variationnel d'Ekeland, on obtient une suite $(u_n)\subset\Lambda^-$ qui vérifie $I(u_n)\to c_1$ et $I'(u_n)\to 0$ dans $H^{-1}(\Omega)$. On montre en utilisant le principe de Concentration-Compacité [L] que si $c_1<c_0+\tilde{S}_1$, alors la suite $(u_n)$ converge fortement vers une fonction $u_1$ dans $H^1(\Omega)$.
La comparaison entre $c_1$ et $c_0+\tilde{S}_1$ présente une grande difficulté. Pour la surmonter, on compare séparement $c_1$, $c_0+\tilde{S}_1$ avec la valeur $c$ donnée par le Lemme du Col. On obtient $c_1\le c<c_0+\tilde{S}_1$.
Finalement, on vérifie que $u_1\in\Lambda^-$. Ainsi on a une deuxième solution du problème (1.2.4).
Dans la deuxième partie, on s'intéresse au cas où le paramètre $\lambda$ est strictement positif. On considère le spectre $\{\lambda_1,\lambda_2,...\}$ de l'opérateur $-div(p(x)\nabla\,.)$ avec la condition de Neumann homogène au bord.
Soit $k\in\mathbf{N}\setminus\{0\}$ tel que $\lambda_{k-1}\neq\lambda_k$. On pose $E_k^-=span\{e_1,...,e_l\}$, où $e_1,...,e_l$ sont les fonctions propres qui correspondent aux valeurs propres $\lambda_1,...,\lambda_{k-1}$. On a

la décomposition orthogonale suivante $H^1(\Omega) = E_k^- \oplus E_k^+$.
Les hypothèses sur $f$ sont données par

$$(H2) \qquad f \in E_k^+ \cap C(\bar{\Omega}) \setminus \{0\} \text{ satisfait (1.2.5).}$$

Le théorème principal de cette partie est le suivant

**Théorème 7.**
*Soit $\Omega \in I\!R^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en $x_0$ et telle que $H(x_0) > 0$. On suppose que $p$ vérifie (1.2.1), $Q$ vérifie (1.2.2) et que $f$ vérifie (H2). Dans les cas où $N = 3$ et $N = 4$, on suppose de plus que $f(x_0) \neq 0$. Alors pour tout $\lambda \in ]\lambda_{k-1}, \lambda_k[$, il existe une constante $\alpha = \alpha(p, Q, \lambda, N) > 0$ telle que si $\|f\|_{L^2} \leq \alpha$ alors le problème (1.2.4) admet au moins une solution.*

Dans ce résultat, le fait que $f$ soit dans $E_k^+$ permet de vérifier les hypothèses du théorème de min-max (voir [W]) et ainsi d'avoir une solution du problème (1.2.4).

# Part I

# Problèmes de Dirichlet non linéaires faisant intervenir l'exposant critique du Sobolev

# Chapter 2

# Problem with critical Sobolev exponent and with weight

(This work in collaboration with R. Hadiji has been submitted to the Chinese Annals of Mathematics serie B)

## 2.1 Introduction

In this paper we study the following problem:

$$(2.1.1) \qquad \begin{cases} -div(p(x)\nabla u) = u^{q-1} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $I\!R^n$, $n \geq 3$, $p : \bar{\Omega} \longrightarrow I\!R$ is a given positive weight such that $p \in H^1(\Omega) \cap C(\bar{\Omega})$, $\lambda$ is a real constant and $q = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$.

In [BN1], Brezis and Nirenberg treated the case where $p$ is constant. They proved, in particular, the existence of a solution of (2.1.1) for $0 < \lambda < \lambda_1$ if $n \geq 4$ and for $\lambda^* < \lambda < \lambda_1$ if $n = 3$, where $\lambda_1$ is the first eigenvalue of $-\Delta$ on $\Omega$ with zero Dirichlet boundary condition and $\lambda^*$ is a positive constant.

In this paper, we extend this result to the general case of where $p$ is not constant. The study of problem (2.1.1), shows that the existence of solutions depends, apart from parameter $\lambda$, on the behavior of p near its minima and on the geometry of the domain $\Omega$.

Set $p_0 = \min\{p(x),\, x \in \bar{\Omega}\}$, we suppose that $p^{-1}(\{p_0\}) \cap \Omega \neq \emptyset$ and let $a \in p^{-1}(\{p_0\}) \cap \Omega$.

In the first part of this work, we study the effect of the behavior of $p$ near its minima on the existence of solution for our problem. The method that is mostly relied upon,

apart from the identities of Pohozeav, is the adaptations to the new context of the arguments developed in [BN].
We assume that, in a neighborhood of $a$, $p$ behaves like

$$p(x) = p_0 + \beta_k |x-a|^k + |x-a|^k \theta(x), \tag{2.1.2}$$

with $k > 0$, $\beta_k > 0$ and $\theta(x)$ tends to 0 when $x$ tends to $a$.
Note that the parameter $k$ will play an essential role in the study of our problem. Indeed, 2 appears as a critical value for $k$. More precisely the case $k > 2$ is treated by a classical procedure, however the case $0 < k \leq 2$ is less easily accessible. Therefore, in this case, we restrict ourself to the case where $p$ satisfies the additional condition

$$k\beta_k \leq \frac{\nabla p(x).(x-a)}{|x-a|^k} \quad \text{a.e } x \in \Omega. \tag{2.1.3}$$

Let us notice that if $p$ is sufficiently smooth, then condition (2.1.2) follows directly from Taylor's expansion of $p$ near $a$.
The fact that 2 is a critical value for $k$ appears clearly in dimension $n = 4$, therefore, in this dimension and with the aim of obtaining more explicit results, we assume moreover that $\theta$ satisfies $\int_{B(a,1)} \frac{\theta(x)}{|x-a|^4} dx < \infty$. Let us emphasize that this last condition is not necessary to prove the existence of solutions.
Moreover, in dimension $n = 3$, the problem is more delicate, then we treat it in a particular case; more precisely for $p(x) = p_0 + \beta_k |x-a|^k$, $k > 0$.
The first result of this work is the following

**Theorem 2.1.1.**
*Assume that $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfies (2.1.2). Let $\lambda_1^{div}$ be the first eigenvalue of $div(p(x)\nabla.)$ on $\Omega$ with zero Dirichlet boundary condition, we have*
*1)If $n \geq 4$ and $k > 2$, then for every $\lambda \in ]0, \lambda_1^{div}[$ there exists a solution of (2.1.1).*
*2)If $n \geq 4$ and $k = 2$, then there exists a constant $\tilde{\gamma}(n) = \frac{(n-2)n(n+2)}{4(n-1)}\beta_2$ such that for every $\lambda \in ]\tilde{\gamma}(n), \lambda_1^{div}[$ there exists a solution of (2.1.1).*
*3)If $n = 3$ and $k \geq 2$, then there exists a constant $\gamma(k) > 0$ such that for every $\lambda \in ]\gamma(k), \lambda_1^{div}[$ there exists a solution of (2.1.1).*
*4)If $n \geq 3$, $0 < k < 2$ and $p$ satisfies the condition (2.1.3) then there exists $\lambda^* \in [\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$, where $\tilde{\beta}_k = \beta_k \min[(\text{diam } \Omega)^{k-2}, 1]$, such that for any $\lambda \in ]\lambda^*, \lambda_1^{div}[$ problem (2.1.1) admits a solution.*
*5)If $n \geq 3$ and $k > 0$, then for every $\lambda \leq 0$ there is no minimizing solution of equation (2.1.1).*
*6)If $n \geq 3$ and $k > 0$, then there is no solution of problem (2.1.1) for every $\lambda \geq \lambda_1^{div}$.*

**Remark 2.1.1.**
*In general, the intervals $]\tilde{\gamma}(n), \lambda_1^{div}[$ in 2) and $[\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$ in 4), may be empty. But*

*there are some sufficient conditions for which the above intervals are nonempty:*
*1) If* $p_0 > \dfrac{n(n-4)}{(n-1)(n-2)^2}\beta_2\,(\mathrm{diam}\ \Omega)^2$, *then* $\tilde{\gamma}(n) < \lambda_1^{div}$.
*Notice that this condition is always true if* $n$ *is rather large.*
*2) If* $p_0 > \dfrac{\tilde{\beta}_k n^2}{(n-2)^2\,(\mathrm{diam}\ \Omega)^2}$, *then* $\tilde{\beta}_k \frac{n^2}{4} < \lambda_1^{div}$.

We postpone the proof of 1) and 2) in this remark until the end of section 2.3.

The second part of this work is dedicated to the study of the effect of the geometry of the domain on the existence of solutions of our problem. More precisely, since for $\lambda = 0$ and $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying $\nabla p(x).(x-a) > 0$ a.e in $\Omega$, the problem (2.1.1) does not have a solution for a starshaped domain about $a$, we will modify the geometry of $\Omega$ in order to find a solution. Therefore, let $\Omega \subset I\!\!R^n$, $n \geq 3$ be a starshaped domain about $a$ and let $\varepsilon > 0$, we will study the existence of solution of the problem

$$(\mathbf{I}_\varepsilon) \quad \begin{cases} -div(p(x)\nabla u) = u^{q-1} & \text{in } \Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \Omega \setminus B(a,\varepsilon)$.
For $p \equiv 1$ and $\lambda = 0$, the problem (2.1.1) has been first investigated in [C] and an interesting result of existence has been proved for domains with holes. In [BaC], this last result is extended to all domains having "nontrivial" topology (in a suitable sense). This nontrivially condition (which covers a large class of domains) is only sufficient for the solvability but not necessary as shown by some examples of contractible domains $\Omega$ for which (2.1.1) has solutions (see [D], [Di], [Pa]).
In other direction, [Le1] shows that the solution of [C], on a domain with a hole of diameter $\varepsilon$ and center $x_0$, concentrates at the point $x_0$. In [H], the author generalized the result of [C] for the case where $u^q$ is replaced by $u^q + \mu u^\alpha$, where $\mu \in I\!\!R$ and $1 < \alpha < q$.
In this work, we consider the case where $p \in H^1(\Omega) \cap C(\bar{\Omega})$ and satisfying $\nabla p(x).(x-a) > 0$ a.e on $\Omega \setminus \{a\}$. The method we use in this part is an adaptation of those used in [C] and [H]. More particularly, we use the min-max techniques and a variant of the Ambrosetti-Rabinowitz theorem [AR].
The second result of this work is the following

**Theorem 2.1.2.**
*There exists* $\varepsilon_0 = \varepsilon_0(\Omega, p) > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *the problem* $(I_\varepsilon)$ *has at least one solution in* $H_0^1(\Omega_\varepsilon)$.

The rest of this paper is divided into three sections. In Section 1 some preliminary results will be established. Section 2 and Section 3 are devoted respectively to the proof of Theorem 2.1.1 and the proof of Theorem 2.1.2.

## 2.2 Some preliminary results

We start by recalling some notations which will be frequently used throughout the rest of this paper. First, we define

$$S = \inf_{u \in H_0^1, \|u\|_q = 1} \|\nabla u\|_2^2$$

that corresponds to the best constant for the Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$. Let us denote by $U_{a,\varepsilon}$ denote an extremal function for the Sobolev inequality

$$U_{a,\varepsilon}(x) = \frac{1}{(\varepsilon + |x-a|^2)^{\frac{n-2}{2}}}, \ x \in I\!R^n.$$

We set

$$u_{a,\varepsilon}(x) = \zeta(x) U_{a,\varepsilon}(x)\,,\ x \in I\!R^n, \tag{2.2.1}$$

where $\zeta \in C_0^\infty(\bar{\Omega})$ is a fixed function such that $0 \leq \zeta \leq 1$, and $\zeta \equiv 1$ in some neighborhood of $a$ included in $\Omega$.
We know from [BN1] that

$$\|\nabla u_{a,\varepsilon}\|_2^2 = \frac{K_1}{\varepsilon^{\frac{n-2}{2}}} + O(1), \tag{2.2.2}$$

$$\| \ u_{a,\varepsilon} \ \|_q^2 = \frac{K_2}{\varepsilon^{\frac{n-2}{2}}} + O(\varepsilon) \tag{2.2.3}$$

and

$$\| \ u_{a,\varepsilon} \ \|_2^2 = \begin{cases} \frac{K_3}{\varepsilon^{\frac{n-4}{2}}} + O(1) \ if \ n \geq 5 \\ \frac{\omega_4}{2} |\log \varepsilon| + O(1) \ if \ n = 4 \end{cases} \tag{2.2.4}$$

where $K_1$ and $K_2$ are positive constants with $\frac{K_1}{K_2} = S$, $\omega_4$ is the area of $S^3$ and $K_3 = \displaystyle\int_{I\!R^n} \frac{1}{(1+|x|^2)^{n-2}} dx$.

We shall state some auxiliary results.
For $p \in C^1(\bar{\Omega})$ or $p \in H^1(\Omega) \cap C(\bar{\Omega})$ and $\nabla p(x).(x-a) \geq 0$ a.e $x \in \Omega$, we consider

$$\alpha(p) = \frac{1}{2} \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_\Omega \nabla p(x).(x-a) |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

We easily see that $\alpha(p) \in [-\infty, +\infty[$, and we have the following result

**Proposition 2.2.1.**
**1)** *If* $p \in C^1(\Omega)$ *and there exists* $b \in \Omega$ *such that* $\nabla p(b)(b-a) < 0$, *then* $\alpha(p) = -\infty$.
**2)** *If* $p \in H^1(\Omega) \cap C(\bar{\Omega})$ *satisfying (2.1.2) and* $\nabla p(x).(x-a) \geq 0$ *a.e* $x \in \Omega$, *we have*
**2.a)** *If* $k > 2$ *and* $p \in C^1(\Omega)$, *then* $\alpha(p) = 0$ *for all* $n \geq 3$.
**2.b)** *If* $0 < k \leq 2$ *and* $p$ *satisfies condition (2.1.3) then for all* $n \geq 3$ *we have*

$$\frac{k}{2}\beta_k \left(\frac{n+k-2}{2}\right)^2 (\text{diam } \Omega)^{k-2} \leq \alpha(p).$$

*Proof.*
We start by proving 1). Set $q(x) = \nabla p(x).(x-a)$ $\forall x \in \Omega$ and let $\varphi \in C_0^\infty(I\!R^n)$ such that $0 \leq \varphi \leq 1$ on $I\!R^n$, $\varphi \equiv 1$ on the ball $\{x, |x| < r\}$, and $\varphi \equiv 0$ outside the ball $\{x, |x| < 2r\}$, where $r < 1$ is a positive constant .
Set $\varphi_j(x) = \varphi(j(x-b))$ for $j \in \mathbf{N}^*$. We have

$$\begin{aligned} \alpha(p) &\leq \frac{1}{2}\frac{\int_\Omega q(x)|\nabla\varphi_j(x)|^2 dx}{\int_\Omega |\varphi_j|^2 dx} \\ &\leq \frac{1}{2}\frac{\int_{B(b,\frac{2r}{j})} q(x)|\nabla\varphi_j(x)|^2 dx}{\int_{B(b,\frac{2r}{j})} |\varphi_j|^2 dx}. \end{aligned}$$

Using the change of variable $y = j(x-b)$, we obtain

$$\alpha(p) \leq \frac{j^2}{2}\frac{\int_{B(0,2r)} q(\frac{y}{j}+b)|\nabla\varphi(x)|^2 dx}{\int_{B(0,2r)} |\varphi|^2 dx}.$$

Applying the Dominated Convergence Theorem, we obtain

$$\alpha(p) \leq \frac{j^2}{2}\left[q(b)\frac{\int_{B(0,2r)} |\nabla\varphi(x)|^2 dx}{\int_{B(0,2r)} |\varphi|^2 dx} + o(1)\right].$$

Letting $j \to \infty$, we deduce the desired result.
Now we will prove 2.a).
Using (2.1.2) and since $p \in C^1(\Omega)$ in a neighborhood $V$ of $a$, we write

(2.2.5) $$p(x) = p_0 + \beta_k|x-a|^k + \theta_1(x),$$

where $\theta_1 \in C^1(V)$ is such that

(2.2.6) $$\lim_{x \to a}\frac{\theta_1(x)}{|x-a|^k} = 0.$$

Looking at (2.2.6), we deduce that there exists $0 < r < 1$, such that

(2.2.7) $$\theta_1(x) \leq |x-a|^k \quad \forall x \in B(a, 2r).$$

Let $\varphi \in C_0^\infty(I\!R^n)$ be a function such that $0 \le \varphi \le 1$ on $I\!R^n$, $\varphi \equiv 1$ on the ball $\{x, |x| < r\}$, and $\varphi \equiv 0$ outside the ball $\{x, |x| < 2r\}$. Set $\varphi_j(x) = \varphi(j(x-a))$ for $j \in \mathbf{N}^*$, we have

$$0 \le \alpha(p) \le \frac{1}{2} \frac{\int_\Omega \nabla p(x).(x-a)|\nabla \varphi_j(x)|^2 dx}{\int_\Omega |\varphi_j|^2 dx}.$$

Using (2.2.5), we obtain

$$0 \le \alpha(p) \le \frac{k\beta_k}{2} \frac{\int_{B(a,\frac{2r}{j})} |x-a|^k |\nabla \varphi_j(x)|^2 dx}{\int_{B(a,\frac{2r}{j})} |\varphi_j|^2 dx} + \frac{1}{2} \frac{\int_{B(a,\frac{2r}{j})} \nabla \theta_1(x).(x-a)|\nabla \varphi_j(x)|^2 dx}{\int_{B(a,\frac{2r}{j})} |\varphi_j|^2 dx}.$$

Performing the change of variable $y = j(x-a)$, and integrating by parts the second term of the right hand side, we obtain

$$0 \le \alpha(p) \le \frac{k\beta_k}{2j^{k-2}} \frac{\int_{B(0,2r)} |y|^k |\nabla \varphi(y)|^2 dx}{\int_{B(0,2r)} |\varphi|^2 dx} + \frac{j}{2} \frac{\int_{B(0,2r)} \theta_1(\frac{y}{j}+a) \nabla (y|\nabla \varphi(y)|^2) dx}{\int_{B(0,2r)} |\varphi|^2 dx}.$$

Using (2.2.7), we write

$$0 \le \alpha(p) \le \frac{k\beta_k}{2j^{k-2}} \frac{\int_{B(0,2r)} |y|^k |\nabla \varphi(y)|^2 dx}{\int_{B(0,2r)} |\varphi|^2 dx} + \frac{1}{2j^{k-1}} \frac{\int_{B(0,2r)} |y|^k \nabla (|\nabla \varphi(y)|^2 y) dx}{\int_{B(0,2r)} |\varphi|^2 dx}.$$

Therefore, for $k > 2$ we deduce that $\alpha(p) = 0$, and this finishes the proof of this case. Now, in order to prove 2.b), we need to recall the following Hardy's inequality, see for example [CKN] or Theorem 330 in [HLP].

**Lemma 2.2.1.**

*Let $t \in I\!R$ such that $t + n > 0$, we have $\forall u \in H_0^1(\Omega)$*

$$\int_\Omega |x|^t |u|^2 dx \le (\frac{2}{n+t})^2 \int_\Omega |x.\nabla u|^2 |x|^t dx.$$

*Moreover the constant $(\frac{2}{n+t})^2$ is optimal and is not achieved.*

Now we prove 2.b). Since $p$ satisfies (2.1.3), we have for all $u \in H_0^1(\Omega) \setminus \{0\}$,

$$\frac{\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx} \ge k\beta_k \frac{\int_\Omega |x-a|^k |\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx}.$$

By applying the last Lemma for $0 < k = 2 + t \le 2$, we find

$$\frac{\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2 dx}{\int_\Omega |u|^2 dx} \ge k\beta_k \left(\frac{n+k-2}{2}\right)^2 (diam\ \Omega)^{k-2}.$$

This implies that $\alpha(p) \ge \frac{k}{2}\beta_k (\frac{n+k-2}{2})^2 (diam\ \Omega)^{k-2}$. □

Let us give the following non-existence result

**Proposition 2.2.2.**
*We assume that $\alpha(p) > -\infty$. There is no solution for (2.1.1) when $\lambda \leq \alpha(p)$ and $\Omega$ is a starshaped domain about $a$.*

*Proof.* This follows from Pohozev's identity. Suppose that $u$ is a solution of (2.1.1). We first multiply (2.1.1) by $\nabla u(x).(x-a)$, next we integrate over $\Omega$ and we obtain

$$\int_\Omega u^{q-1}\nabla u(x).(x-a)dx = -\frac{n-2}{2}\int_\Omega |u(x)|^q dx, \tag{2.2.8}$$

$$\lambda\int_\Omega u\nabla u(x).(x-a)dx = -\frac{n}{2}\lambda\int_\Omega |u(x)|^2 dx \tag{2.2.9}$$

and

$$\begin{aligned} \int_\Omega -div(p(x)\nabla u)\nabla u(x).(x-a)dx &= -\frac{n-2}{2}\int_\Omega p(x)|\nabla u(x)|^2 dx \\ &- \frac{1}{2}\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2 dx \\ &- \frac{1}{2}\int_{\partial\Omega} p(x)(x-a).\nu|\frac{\partial u}{\partial \nu}|^2 dx, \end{aligned} \tag{2.2.10}$$

where $\nu$ denotes the outward normal to $\partial\Omega$.
Before continuing the proof, we give some details about (2.2.10). Indeed

$$\int_\Omega -div(p(x)\nabla u)\nabla u(x).(x-a)dx = -\sum_{i=1}^n\sum_{j=1}^n\int_\Omega \partial_i(p(x)\partial_i u)\partial_j u.(x_j-a_j)dx$$

where $\partial_i. = \frac{\partial.}{\partial x_i}$ and $x_i$ is such that $x = (x_1, ..., x_i, ..., x_n)$
Integrating by parts the right hand side of the above equality, we obtain

$$\begin{aligned} -\int_\Omega \partial_i(p(x)\partial_i u)\partial_j u.(x_j-a_j)dx &= \int_\Omega p(x)\partial_i u\,\partial_i\partial_j u\,(x_j-a_j)dx \\ &+ \textstyle\int_\Omega p(x)\partial_i u\,\partial_j u\,\partial_i(x_j-a_j)dx \\ &- \int_{\partial\Omega} p(x)|\nabla u|^2(x-a).\nu dx. \end{aligned} \tag{2.2.11}$$

On other hand, integrating by parts we see that

$$\begin{aligned} \int_\Omega p(x)\partial_i u\,\partial_i\partial_j u\,(x_j-a_j)dx &= -\frac{1}{2}\int_\Omega \partial_j p(x)\,\partial_i u\,\partial_i u\,.(x_j-a_j)dx \\ &- \tfrac{1}{2}\textstyle\int_\Omega p(x)\partial_i u\,\partial_i u\,\partial_i(x_j-a_j)dx \\ &+ \frac{1}{2}\int_{\partial\Omega} p(x)|\nabla u|^2(x-a).\nu dx. \end{aligned}$$

Inserting the above equality in (2.2.11), we get

$$\begin{aligned} -\int_\Omega \partial_i(p(x)\partial_i u)\partial_j u.(x_j-a_j)dx &= -\frac{1}{2}\int_\Omega \partial_j p(x)\,\partial_i u\,\partial_i u\,.(x_j-a_j)dx \\ &+ \frac{1}{2}\int_\Omega p(x)\partial_i u\,\partial_j u\,\partial_i(x_j-a_j)dx \\ &- \tfrac{1}{2}\int_{\partial\Omega} p(x)|\nabla u|^2(x-a).\nu dx. \end{aligned}$$

By making the sum over $i$ and $j$, knowing that when $u=0$ on the boundary then $\nabla u = \frac{\partial u}{\partial \nu}.\nu$, and using the fact that $\partial_i(x_j-a_j)=1$ if $i=j$ and 0 if not, we obtain (2.2.10).

Now we return to the proof of Proposition 2.2.2.
Combining (2.2.8), (2.2.9) and (2.2.10), we write
(2.2.12)

$$-\tfrac{n-2}{2}\int_\Omega p(x)|\nabla u(x)|^2dx - \tfrac{1}{2}\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2dx =$$

$$-\frac{n-2}{2}\int_\Omega |u(x)|^q dx - \frac{n}{2}\lambda\int_\Omega |u(x)|^2dx.$$

On the other hand, we multiply (2.1.1) by $\frac{n-2}{2}u$ and we integrate by parts, we get

$$\frac{n-2}{2}\int_\Omega p(x)|\nabla u(x)|^2dx = \frac{n-2}{2}\int_\Omega |u(x)|^q dx + \frac{n-2}{2}\lambda\int_\Omega |u(x)|^2dx. \qquad (2.2.13)$$

Combining (2.2.12) and (2.2.13), we obtain

$$\lambda\int_\Omega |u(x)|^2dx - \frac{1}{2}\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2dx - \frac{1}{2}\int_{\partial\Omega} p(x)|\frac{\partial u}{\partial\nu}|^2(x-a).\nu dx = 0.$$

If $\Omega$ is starshaped about $a$, then $(x-a).\nu > 0$ on $\partial\Omega$, and

$$\lambda\int_\Omega |u(x)|^2dx - \frac{1}{2}\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2dx > 0.$$

It follows that

$$\lambda > \frac{1}{2}\frac{\displaystyle\int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2dx}{\displaystyle\int_\Omega |u|^2dx}.$$

Then

$$\lambda \geq \frac{1}{2}\inf_{u\in H_0^1(\Omega),u\neq 0}\frac{\displaystyle\int_\Omega \nabla p(x).(x-a)|\nabla u|^2dx}{\displaystyle\int_\Omega |u|^2dx}$$

and

$$\lambda \geq \alpha(p).$$

Conclusion: If $u$ is a solution of (2.1.1), then $\lambda \geq \alpha(p)$. □

## 2.3 Existence of solutions

Let $\Omega \in I\!R^n$, $n \geq 3$ be a bounded domain. In this section, we show that (2.1.1) possesses a solution of lower energy less than $p_0 S$. We will use a minimization technique.
Set

$$(2.3.1) \qquad Q_\lambda(u) = \frac{\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx}{\| u \|_q^2}$$

the functional associated to (2.1.1).
We define

$$(2.3.2) \qquad S_\lambda(p) = \inf_{u \in H_0^1(\Omega), u \neq 0} Q_\lambda(u).$$

Let us remark that

$$S_\lambda(p) = \inf_{u \in H_0^1(\Omega), \|u\|_q = 1} \int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx.$$

The method used for the proof of Theorem 2.1.1 is the following : First we show that $S_\lambda(p) < p_0 S$, we then prove that the infimum $S_\lambda(p)$ is achieved.
We have the following result

**Lemma 2.3.1.**
*If $S_\lambda(p) < p_0 S$ for $\lambda > 0$, then the infimum in (2.3.2) is achieved.*

*Proof.*
Let $\{u_j\} \subset H_0^1(\Omega)$ be a minimizing sequence for (2.3.2) that is,

$$(2.3.3) \qquad \|u_j\|_q = 1,$$

$$(2.3.4) \quad \int_\Omega p(x)|\nabla u_j(x)|^2 dx - \lambda \int_\Omega |u_j(x)|^2 dx = S_\lambda(p) + o(1) \quad \text{as } j \to \infty.$$

The sequence $u_j$ is bounded in $H_0^1(\Omega)$. Indeed, from (2.3.4), we have

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx = S_\lambda(p) + \lambda \int_\Omega |u_j(x)|^2 dx + o(1).$$

Using the embedding of $L^q(\Omega)$ into $L^2(\Omega)$, there exists a positive constant $C_1$ such that

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx \leq S_\lambda(p) + \lambda\, C_1 \|u_j\|_q^2 + o(1).$$

Using the fact that (see (2.3.3))

$$\|u_j\|_q = 1,$$

we obtain

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx \le S_\lambda(p) + \lambda\, C_1 + o(1).$$

Since $0 < p_0 \le p(x)$ for every $x \in \Omega$, we deduce

$$\int_\Omega |\nabla u_j(x)|^2 dx \le \frac{S_\lambda(p) + \lambda\, C_1}{p_0} + o(1).$$

This gives the desired result.
Since $\{u_j\}$ is bounded in $H_0^1(\Omega)$ we may extract a subsequence still denoted by $u_j$, such that

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),\\ u_j &\to u \quad \text{strongly in } L^2(\Omega),\\ u_j &\to u \qquad \text{a.e. on } \Omega, \end{aligned}$$

with $\|u\|_q \le 1$. Set $v_j = u_j - u$, so that

$$\begin{aligned} v_j &\rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega)\\ v_j &\to 0 \quad \text{strongly in } L^2(\Omega),\\ v_j &\to 0 \qquad \text{a.e. on } \Omega. \end{aligned}$$

Using (2.3.3), the definition of $S$ and the fact that $\min_{\bar\Omega} p(x) = p_0 > 0$, we have

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx \ge p_0 S.$$

From (2.3.4) it follows that $\lambda\|u\|_2^2 \ge p_0 S - S_\lambda(p) > 0$ and therefore $u \ne 0$. Using again (2.3.4) we obtain

$$\int_\Omega p(x)|\nabla u(x)|^2 dx + \int_\Omega p(x)|\nabla v_j(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx = S_\lambda(p) + o(1), \tag{2.3.5}$$

since $v_j \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. On the other hand, it follows from a result of Brezis and Lieb [BL] that

$$\|u + v_j\|_q^q = \|u\|_q^q + \|v_j\|_q^q + o(1),$$

(which holds since $v_j$ is bounded in $L^q$ and $v_j \to 0$ a.e.). Thus, by (2.3.3), we have

$$1 = \|u\|_q^q + \|v_j\|_q^q + o(1)$$

and therefore

$$1 \le \|u\|_q^2 + \|v_j\|_q^2 + o(1),$$

which leads to

$$1 \le \|u\|_q^q + \frac{1}{p_0 S}\int_\Omega p(x)|\nabla v_j(x)|^2 dx + o(1). \tag{2.3.6}$$

We distinguish two cases:
(a) $S_\lambda(p) > 0$, which corresponds to $0 < \lambda < \lambda_1^{div}$,

(b) $S_\lambda(p) \le 0$, which corresponds to $\lambda \ge \lambda_1^{div}$.
In case (a) we deduce from (2.3.6) that

$$S_\lambda(p) \le S_\lambda(p)\|u\|_q^2 + (\frac{S_\lambda(p)}{p_0 S})\int_\Omega p(x)|\nabla v_j(x)|^2 dx + o(1). \tag{2.3.7}$$

Combining (2.3.5) and (2.3.7) we obtain

$$\begin{aligned}\textstyle\int_\Omega p(x)|\nabla u(x)|^2 - \lambda|u(x)|^2 dx + \int_\Omega p(x)|\nabla v_j(x)|^2 dx &\le S_\lambda(p)\|u\|_q^2 \\ &+(\frac{S_\lambda(p)}{p_0 S})\int_\Omega p(x)|\nabla v_j(x)|^2 dx + o(1).\end{aligned}$$

Thus

$$\begin{aligned}\textstyle\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda\int_\Omega |u(x)|^2 dx &\le S_\lambda(p)\|u\|_q^2 \\ &+\left[\frac{S_\lambda(p)}{p_0 S} - 1\right]\int_\Omega p(x)|\nabla v_j(x)|^2 dx + o(1).\end{aligned}$$

Since $S_\lambda(p) < p_0 S$, we deduce

$$\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda\int_\Omega |u(x)|^2 dx \le S_\lambda(p)\|u\|_q^2, \tag{2.3.8}$$

this means that $u$ is a minimum of $S_\lambda(p)$.
In case (b), since $\|u\|_q^2 \le 1$, we have $S_\lambda(p) \le S_\lambda(p)\|u\|_q^2$. Again, we deduce (2.3.8) from (2.3.5). This concludes the proof of Lemma 2.3.1. □

To prove assertion 1) and 2) of Theorem 2.1.1 (case $k \ge 2$), we need the following

**Lemma 2.3.2.**
*a) For $n \ge 4$, we have*

$$S_\lambda(p) < p_0 S \textit{ for all } \lambda > 0 \textit{ and for } k > 2.$$

*b) For $n = 4$ and $k = 2$, we have*

$$S_\lambda(p) < p_0 S \textit{ for all } \lambda > 4\beta_2.$$

*c) For $n \ge 5$ and $k = 2$, we have*

$$S_\lambda(p) < p_0 S \textit{ for all } \lambda > \frac{(n-2)n(n+2)}{4(n-1)}\beta_2.$$

*d) For* $n = 3$ *and* $k \geq 2$*, we have*

$$S_\lambda(p) < p_0 S \quad \textit{for all } \lambda > \gamma(k) \textit{ where } \gamma(k) \textit{ is a positive constant.}$$

*Proof.*
We shall estimate the ratio $Q_\lambda(u)$ defined in (2.3.1), with $u(.) = u_{a,\varepsilon}(.)$.
We claim that, as $\varepsilon \to 0$, we have
(2.3.9)

$$\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx \leq$$

$$\begin{cases} p_0 K_1 + O(\varepsilon^{\frac{n-2}{2}}) & \text{if } \begin{cases} n \geq 4 \quad \text{and} \\ n-2 < k, \end{cases} \\ p_0 K_1 + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } \begin{cases} n \geq 4 \quad \text{and} \\ n-2 > k, \end{cases} \\ p_0 K_1 + \dfrac{(n-2)^2(\beta_{n-2}+M)\omega_n \varepsilon^{\frac{n-2}{2}}|\log\varepsilon|}{2} + o(\varepsilon^{\frac{n-2}{2}}|\log\varepsilon|) & \text{if } \begin{cases} n > 4 \quad \text{and} \\ k = n-2, \end{cases} \\ p_0 K_1 + 2\beta_2 \omega_4 \varepsilon |\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } \begin{cases} n = 4 \quad \text{and} \\ k = 2, \end{cases} \end{cases}$$

with $K_1 = (n-2)^2 \int_{I\!R^n} \frac{|y|^2}{(1+|y|^2)^n} dy$, $s = \min(\frac{k}{2}, \frac{n-2}{2})$, $A_k = (n-2)^2 \beta_k \int_{I\!R^n} \frac{|x|^{k+2}}{(1+|x|^2)^n} dx$ and $M$ is a positive constant.

**Verification of (2.3.9)**

**1. Case $n \geq 4$ and $k > 0$, with $k \neq 2$ if $n = 4$.**

We have

$$\begin{aligned} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= \int_\Omega \frac{p(x)|\nabla\zeta(x)|^2}{(\varepsilon+|x-a|^2)^{n-2}} dx + (n-2)^2 \int_\Omega \frac{p(x)|\zeta(x)|^2|x-a|^2}{(\varepsilon+|x-a|^2)^n} dx \\ &- 2(n-2) \int_\Omega \frac{p(x)\zeta(x)\nabla\zeta(x)(x-a)}{(\varepsilon+|x-a|^2)^{n-1}} dx. \end{aligned}$$

Since $\zeta \equiv 1$ on a neighborhood of $a$, we assume that $\varphi \equiv 1$ on $B(a,l)$ with $l$ is a small positive constant. Therefore we get $|\nabla\varphi|^2 \equiv 0$ on $B(a,l)$ and $\nabla\varphi(x).(x-a) = 0$ on $B(a,l)$.
Thus, we obtain

(2.3.10)

$$\begin{aligned} \int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx =& \int_{\Omega\setminus B(a,l)} \frac{p(x)|\nabla\zeta(x)|^2}{(\varepsilon+|x-a|^2)^{n-2}} dx + (n-2)^2 \int_\Omega \frac{p(x)|\zeta(x)|^2|x-a|^2}{(\varepsilon+|x-a|^2)^n} dx \\ &- 2(n-2) \int_{\Omega\setminus B(a,l)} \frac{p(x)\zeta(x)\nabla\zeta(x)(x-a)}{(\varepsilon+|x-a|^2)^{n-1}} dx. \end{aligned}$$

On the other hand, we have

$$\frac{p(x)|\nabla\varphi|^2}{(\varepsilon+|x-a|^2)^{n-2}} \le \frac{p(x)|\nabla\varphi|^2}{|x-a|^{2(n-2)}} \in L^1(\Omega\setminus B(a,l))$$

and

$$\left|\frac{p(x)\zeta(x)\nabla\zeta(x)(x-a)}{(\varepsilon+|x-a|^2)^{n-1}}\right| \le \left|\frac{p(x)\zeta(x)\nabla\zeta(x)(x-a)}{|x-a|^{2(n-1)}}\right| \in L^1(\Omega\setminus B(a,l)).$$

Therefore, applying the Dominated Convergence Theorem, (2.3.10) becomes

$$\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = (n-2)^2\int_\Omega \frac{p(x)|\zeta(x)|^2|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx + O(1).$$

Using (2.1.2), a direct computation gives

$$\begin{aligned}
\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx &= (n-2)^2p_0\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx\\
&+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\beta_k\int_\Omega\frac{|x-a|^{k+2}}{(\varepsilon+|x-a|^2)^n}dx\\
&+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}\theta(x)}{(\varepsilon+|x-a|^2)^n}dx\\
&+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}(\beta_k+\theta(x))(|\zeta(x)|^2-1)}{(\varepsilon+|x-a|^2)^n}dx\\
&+ O(\varepsilon^{\frac{n-2}{2}}).
\end{aligned}$$

Using again the definition of $\zeta$, and applying the Dominated Convergence Theorem, we obtain

$$\begin{aligned}
\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx &= (n-2)^2p_0\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx\\
&+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\beta_k\int_\Omega\frac{|x-a|^{k+2}}{(\varepsilon+|x-a|^2)^n}\\
&+ (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}\theta(x)}{(\varepsilon+|x-a|^2)^n}dx + O(\varepsilon^{\frac{n-2}{2}}).
\end{aligned}$$

Here we will consider the following three subcases:

**1.1. If** $n-2>k$,

$$\begin{aligned}
&\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx\\
&= p_0(n-2)^2\varepsilon^{\frac{n-2}{2}}\left[\int_{I\!R^n}\frac{|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx - \int_{I\!R^n\setminus\Omega}\frac{|x-a|^2}{(\varepsilon+|x-a|^2)^n}dx\right]\\
&= (n-2)^2\varepsilon^{\frac{n-2}{2}}\left[\int_{I\!R^n}\frac{|x-a|^{k+2}(\beta_k+\theta(x))}{(\varepsilon+|x-a|^2)^n} - \int_{I\!R^n\setminus\Omega}\frac{|x-a|^{k+2}(\beta_k+\theta(x))}{(\varepsilon+|x-a|^2)^n}\right]\\
&= O(\varepsilon^{\frac{n-2}{2}}).
\end{aligned}$$

Using a simple change of variable and applying the Dominated Convergence Theorem, we find

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0\int_{\mathbb{R}^n}\frac{|y|^2}{(1+|y|^2)^n}+(n-2)^2\varepsilon^{\frac{k}{2}}\int_{\mathbb{R}^n}\frac{|y|^{k+2}(\beta_k+\theta(a+\varepsilon^{\frac{1}{2}}y))}{(1+|y|^2)^n}dy + o(\varepsilon^{\frac{k}{2}}).$$

Using the fact that $\theta(x)$ tends to 0 when $x$ tends to $a$, we obtain

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1 + A_k\varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}),$$

with $K_1 = (n-2)^2\int_{\mathbb{R}^n}\frac{|y|^2}{(1+|y|^2)^n}dy$ and $A_k = \beta_k\int_{\mathbb{R}^n}\frac{|y|^{k+2}}{(1+|y|^2)^n}dy$.

**1.2. If** $n-2<k$,

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1+(n-2)^2\beta_k\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}}{(\varepsilon+|x-a|^2)^n}dx + (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega\frac{|x-a|^{k+2}\theta(x)}{(\varepsilon+|x-a|^2)^n}dx + O(\varepsilon^{\frac{n-2}{2}}).$$

Since $\Omega$ is a bounded domain, there exists some positive constant $R$ such that $\Omega\subset B(a,R)$ and thus

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1+O(\varepsilon^{\frac{n-2}{2}}) +(n-2)^2\varepsilon^{\frac{n-2}{2}}\left[\int_{B(a,R)}\frac{|x-a|^{k+2}(\beta_k+\theta(x))}{(\varepsilon+|x-a|^2)^n}dx - \int_{B(a,R)\setminus\Omega}\frac{|x-a|^{k+2}(\beta_k+\theta(x)}{(\varepsilon+|x-a|^2)^n}dx\right].$$

By a simple change of variable, we get

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1+(n-2)^2\varepsilon^{\frac{n-2}{2}}\int_{B(0,R)}\frac{|y|^{k+2}(\beta_k+\theta(a+y))}{(\varepsilon+|y|^2)^n}dy + O(\varepsilon^{\frac{n-2}{2}}).$$

Using the definition of $\theta$ given by (2.1.2), there exists a positive constant $M$ such that

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx \le p_0K_1+(n-2)^2\varepsilon^{\frac{n-2}{2}}(\beta_k+M)\int_{B(0,R)}\frac{|y|^{k+2}}{(\varepsilon+|y|^2)^n}dy + O(\varepsilon^{\frac{n-2}{2}}).$$

Using the Dominated Convergence Theorem we deduce that

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx \le p_0K_1+O(\varepsilon^{\frac{n-2}{2}})$$

and this completes the proof of (2.3.9) in this case.

**1.2. If** $k = n-2$,

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1 + (n-2)^2\beta_{n-2}\varepsilon^{\frac{n-2}{2}}\int_\Omega \frac{|x-a|^n}{(\varepsilon+|x-a|^2)^n}dx + (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_\Omega \frac{|x-a|^n\theta(x)}{(\varepsilon+|x-a|^2)^n}dx + O(\varepsilon^{\frac{n-2}{2}}).$$

Since $\Omega$ is a bounded domain, there exists some positive constant $R$ such that $\Omega\subset B(a,R)$ and thus

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1 + O(\varepsilon^{\frac{n-2}{2}})$$
$$+(n-2)^2\varepsilon^{\frac{n-2}{2}}\left[\int_{B(a,R)}\frac{|x-a|^n(\beta_{n-2}+\theta(x))}{(\varepsilon+|x-a|^2)^n}dx - \int_{B(a,R)\backslash\Omega}\frac{|x-a|^n(\beta_{n-2}+\theta(x))}{(\varepsilon+|x-a|^2)^n}dx\right].$$

Hence

$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx = p_0K_1 + (n-2)^2\varepsilon^{\frac{n-2}{2}}\int_{B(a,R)}\frac{|x-a|^n(\beta_{n-2}+\theta(x))}{(\varepsilon+|x-a|^2)^n}dx + O(\varepsilon^{\frac{n-2}{2}}).$$

Using the definition of $\theta$ given by (2.1.2), there exists a positive constant $M$ such that

(2.3.11)
$$\varepsilon^{\frac{n-2}{2}}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx \le p_0K_1 + (n-2)^2(\beta_{n-2}+M)\varepsilon^{\frac{n-2}{2}}\int_{B(a,R)}\frac{|x-a|^n}{(\varepsilon+|x-a|^2)^n}dx + O(\varepsilon^{\frac{n-2}{2}}).$$

On the other hand, a easy computation gives

$$\varepsilon^{\frac{n-2}{2}}\int_{B(a,R)}\frac{|x-a|^n}{(\varepsilon+|x-a|^2)^n}dx = \omega_n\varepsilon^{\frac{n-2}{2}}\int_0^R\frac{r^{2n-1}}{(\varepsilon+r^2)^n}dr. \tag{2.3.12}$$

We see that, for $r>0$ and for $\varepsilon>0$ small enough

$$\begin{aligned}((\varepsilon+r^2)^n)' = 2\,n\,r(\varepsilon+r^2)^{n-1} &= 2\,n\,r[r^{2n-2}(1+\tfrac{\varepsilon}{r^2})^{n-1}]\\ &= 2\,n\,r^{2n-1}(1+(n-1)\tfrac{\varepsilon}{r^2}+o(\tfrac{\varepsilon}{r^2}))\\ &= 2\,n\,r^{2n-1} + 2n(n-1)\varepsilon r^{2n-3} + r^{2n-3}o(\varepsilon).\end{aligned}$$

Thus

$$r^{2n-1} = \frac{1}{2n}((\varepsilon+r^2)^n)' - (n-1)\varepsilon r^{2n-3} + r^{2n-3}o(\varepsilon).$$

Therefore, inserting the last equality into (2.3.12), we obtain

$$\begin{aligned}\varepsilon^{\frac{n-2}{2}}\int_{B(a,R)}\frac{|x-a|^n}{(\varepsilon+|x-a|^2)^n}dx &= \frac{\omega_n}{2n}\varepsilon^{\frac{n-2}{2}}\int_0^R\frac{((\varepsilon+r^2)^n)'}{(\varepsilon+r^2)^n}dr\\ &-\omega_n(n-1)\varepsilon^{\frac{n}{2}}\int_0^R\frac{r^{2n-3}}{(\varepsilon+r^2)^n}dr + o\left(\varepsilon^{\frac{n}{2}}\int_0^R\frac{r^{2n-3}}{(\varepsilon+r^2)^n}dr\right).\end{aligned} \tag{2.3.13}$$

Using the change of variable $t = \frac{r}{\varepsilon^{\frac{1}{2}}}$, we obtain

$$\varepsilon^{\frac{n}{2}} \int_0^R \frac{r^{2n-3}}{(\varepsilon + r^2)^n} dr = \varepsilon^{\frac{n-2}{2}} \int_0^{\frac{R}{\varepsilon^{\frac{1}{2}}}} \frac{t^{2n-3}}{(1+t^2)^n} dt$$

Applying the Dominated Convergence Theorem, we get

$$\begin{aligned} \varepsilon^{\frac{n}{2}} \int_0^R \frac{r^{2n-3}}{(\varepsilon+r^2)^n} dr &= \varepsilon^{\frac{n-2}{2}} \int_0^{\frac{R}{\varepsilon^{\frac{1}{2}}}} \frac{t^{2n-3}}{(1+t^2)^n} dt = \varepsilon^{\frac{n-2}{2}} \int_0^{+\infty} \frac{t^{2n-3}}{(1+t^2)^n} dt + o(\varepsilon^{\frac{n-2}{2}}) \\ &= O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Thus, combining the above estimate with (2.3.13), we deduce

$$\varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx = \frac{\omega_n}{2n} \varepsilon^{\frac{n-2}{2}} \int_0^R \frac{((\varepsilon + r^2)^n)'}{(\varepsilon + r^2)^n} dr + O(\varepsilon^{\frac{n-2}{2}})$$

and

$$\varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx = \frac{\omega_n}{2} \varepsilon^{\frac{n-2}{2}} |\log \varepsilon| + o(\varepsilon^{\frac{n-2}{2}} |\log \varepsilon|). \tag{2.3.14}$$

Inserting (2.3.14) into (2.3.11) we obtain

$$\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx \le p_0 K_1 + \frac{(n-2)^2(\beta_{n-2} + M)\omega_n}{2} \varepsilon^{\frac{n-2}{2}} |\log \varepsilon| + o(\varepsilon^{\frac{n-2}{2}} |\log \varepsilon|).$$

**2)Case $n = 4$ and $k = 2$.**

As we have announced in the introduction, we assume in this case the following additional condition on $\theta$: $\int_{\mathcal{B}(a,1)} \frac{\theta(x)}{|x-a|^4} dx < \infty$. We have

$$\begin{aligned} \int_\Omega p(x) |\nabla u_{a,\varepsilon}|^2 dx &= \int_\Omega \frac{p(x)|\nabla \zeta(x)|^2}{(\varepsilon + |x-a|^2)^2} dx + 4 \int_\Omega \frac{p(x)|\zeta(x)|^2 |x-a|^2}{(\varepsilon + |x-a|^2)^4} dx \\ &- 4 \int_\Omega \frac{p(x)\zeta(x)\nabla\zeta(x)(x-a)}{(\varepsilon + |x-a|^2)^3} dx. \end{aligned}$$

Using (2.1.2) and the fact that $\zeta \equiv 1$ near $a$, it follows that

$$\begin{aligned} \int_\Omega p(x) |\nabla u_{a,\varepsilon}|^2 dx &= 4p_0 \int_\Omega \frac{|\zeta(x)|^2 |x-a|^2}{(\varepsilon + |x-a|^2)^4} dx + 4\beta_2 \int_\Omega \frac{|\zeta(x)|^2 |x-a|^4}{(\varepsilon + |x-a|^2)^4} dx \\ &+ 4 \int_\Omega \frac{|x-a|^4 \theta(x)}{(\varepsilon + |x-a|^2)^4} dx + O(1), \\ &= \frac{4p_0}{\varepsilon} \int_{I\!R^n} \frac{|y|^2}{(1+|y|^2)^4} dy + 4 \int_\Omega \frac{|x-a|^4 (\beta_k + \theta(x))}{(\varepsilon + |x-a|^2)^4} dx + O(1). \end{aligned}$$

Using the fact that $\int_{B(a,1)} \frac{\theta(x)}{|x-a|^4} dx < \infty$ and applying the Dominated Convergence Theorem , we obtain

$$\begin{aligned} \int_\Omega \frac{|x-a|^4 \theta(x)}{(\varepsilon + |x-a|^2)^4} dx &= \int_\Omega \frac{\theta(x)}{|x-a|^4} dx + o(1) \\ &= O(1). \end{aligned}$$

Consequently

$$\int_\Omega p(x) |\nabla u_{a,\varepsilon}|^2 dx = \frac{4p_0}{\varepsilon} \int_{I\!R^n} \frac{|y|^2}{(1+|y|^2)^4} dy + 4\beta_k \int_\Omega \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx + O(1).$$

Let $R_i > 0$, $i = 1,2$ such that

$$\int_{|x-a| \le R_1} \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx \le \int_\Omega \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx \le \int_{|x-a| \le R_2} \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx.$$

We see that

$$\begin{aligned} \int_{|x-a| \le R} \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx &= \omega_4 \int_0^R \frac{r^7}{(\varepsilon + r^2)^4} dr, \\ &= \frac{1}{8} \omega_4 \int_0^R \frac{((\varepsilon + r^2)^4)'}{(\varepsilon + r^2)^4} dr - \omega_4 \int_0^R \frac{r\varepsilon^3 + 3r^3\varepsilon^2 + 3\varepsilon r^4}{(\varepsilon + r^2)^4} dr, \\ &= \frac{1}{2} \omega_4 |\log \varepsilon| - \omega_4 \int_0^{\frac{R}{\varepsilon^{\frac{1}{2}}}} \frac{t + 3t^3 + 3t^5}{(1+t^2)^4} dt + O(1), \\ &= \frac{1}{2} \omega_4 |\log \varepsilon| + O(1). \end{aligned}$$

Hence, we have

$$\int_\Omega p(x) |\nabla u_{a,\varepsilon}|^2 dx = \frac{p_0 K_1}{\varepsilon} + 2\beta_2 \omega_4 |\log \varepsilon| + O(1),$$

where $K_1 = \int_{I\!R^n} \frac{|y|^2}{(1+|y|^2)^4} dy$. This completes the proof of (2.3.9).
Let us come back to the proof of Lemma 2.3.2.
It is convenient to rewrite (2.3.9) as
(2.3.15)

$$\varepsilon^{\frac{n-2}{2}} \int_\Omega p(x) |\nabla u_{a,\varepsilon}|^2 dx \le \begin{cases} p_0 K_1 + o(\varepsilon) & \text{if } n \ge 5, \text{ and } k > 2 , \\ p_0 K_1 + A_2 \varepsilon + o(\varepsilon) & \text{if } n \ge 5, \text{ and } k = 2 , \\ p_0 K_1 + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } n \ge 4, \text{ and } k < 2 , \\ p_0 K_1 + o(\varepsilon) & \text{if } n = 4, \text{ and } k > 2 , \\ p_0 K_1 + 2\omega_4 \beta_2 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } n = 4, \text{ and } k = 2 . \end{cases}$$

Combining (2.3.15), (2.2.3) and (2.2.4), we obtain

(2.3.16)

$$S_\lambda(p) \le Q_\lambda(u_{a,\varepsilon}) \le \begin{cases} p_0 S - \lambda\frac{K_3}{K_2}\varepsilon + o(\varepsilon) & \text{if } n \ge 5, \text{ and } k > 2\ , \\ p_0 S - (\lambda - C)\frac{K_3}{K_2}\varepsilon + o(\varepsilon) & \text{if } n \ge 5, \text{ and } k = 2\ , \\ p_0 S + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } n \ge 4, \text{ and } k < 2\ , \\ p_0 S - \lambda\frac{\omega_4}{2K_2}\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } n = 4, \text{ and } k > 2\ , \\ p_0 S - \frac{\omega_4}{2K_2}[\lambda - 4\beta_2]\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } n = 4, \text{ and } k = 2\ , \end{cases}$$

with $C = \frac{A_2}{K_3} = \frac{\beta_2(n-2)n(n+2)}{4(n-1)}$.

Assertions a), b) and c) of Lemma 2.3.2 follow directly for $\varepsilon$ small enough.
Now we prove d) in Lemma 2.3.2 (case $n = 3$ and $k \ge 2$). We will estimate the ratio

$$Q_\lambda(u) = \frac{\int_\Omega p(x)|\nabla u|^2 dx - \lambda\|u\|_2^2}{\|u\|_q^2}$$

with

$$u(x) = u_{\varepsilon,a}(r) = \frac{\zeta(r)}{(\varepsilon + r^2)^{\frac{1}{2}}}, r = |x|,\ \varepsilon > 0,$$

where $\zeta$ is a fixed smooth function satisfying $0 \le \zeta \le 1$, $\zeta = 1$ in $\{x,\ |x-a| < \frac{R}{2}\}$ and $\zeta = 0$ in $\{x,\ |x-a| \ge R\}$, where $R$ is a positive constant such that $B(a,R) \subset \Omega$.
We claim that, as $\varepsilon \to 0$,

(2.3.17)

$$\int p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx = \frac{p_0 K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 \int_0^R (p_0 + \beta_k r^k)|\zeta'(r)|^2 dr + \omega_3 k \int_0^R |\zeta|^2 r^{k-2} dr + o(1).$$

And from [BN1], we already have

$$\|\nabla u_{a,\varepsilon}\|_2^2 = \frac{K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 \int_0^R |\zeta'(r)|^2 dr + O(\varepsilon^{\frac{1}{2}}), \tag{2.3.18}$$

$$\|u_{a,\varepsilon}\|_6^2 = \frac{K_2}{\varepsilon^{\frac{1}{2}}} + O(\varepsilon^{\frac{1}{2}}), \tag{2.3.19}$$

$$\|u_{a,\varepsilon}\|_2^2 = \omega_3 \int_0^R \zeta^2(r) dr + O(\varepsilon^{\frac{1}{2}}), \tag{2.3.20}$$

where $K_1$ and $K_2$ are positive constants such that $\frac{K_1}{K_2} = S$ and $\omega_3$ is the area of $S^2$ .

**Verification of (2.3.17).**

Using (2.1.2), (2.3.18) and the fact that $\zeta = 0$ in $\{x,\, |x-a| \geq R\}$, we write

$$\begin{aligned}\int p(x)|\nabla u_{a,\varepsilon}(x)|^2dx &= \frac{p_0 K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 p_0 \int_0^R |\zeta'(r)|^2 dr \\ &+ \omega_3\beta_k \int_0^R \left[\frac{|\zeta'(r)|^2}{\varepsilon + r^2} - \frac{2r\zeta(r)\zeta'(r)}{(\varepsilon+r^2)^2} + \frac{r^2\zeta^2(r)}{(\varepsilon+r^2)^3}\right] r^{k+2}dr \\ &+ O(\varepsilon^{\frac{1}{2}}).\end{aligned}$$

Using the fact that $\zeta = 1$ in $\{x,\, |x-a| < \frac{R}{2}\}$, $\zeta'(0) = 0$ and $\zeta(R) = 0$, we obtain

$$-2\int_0^R \frac{\zeta(r)\zeta'(r)r^{k+3}}{(\varepsilon+r^2)^2}dr = (k+3)\int_0^R \frac{|\zeta(r)|^2 r^{k+2}}{(\varepsilon+r^2)^2}dr - 4\int_0^R \frac{|\zeta(r)|^2 r^{k+4}}{(\varepsilon+r^2)^3}dr.$$

Consequently

$$\begin{aligned}\int_\Omega p(x)|\nabla u_{a,\varepsilon}(x)|^2dx &= \frac{p_0 K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 p_0 \int_0^R |\zeta'(r)|^2 dr + \omega_3\beta_k \int_0^R \frac{|\zeta'(r)|^2 r^{k+2}}{\varepsilon+r^2}dr \\ &- 3\omega_3\beta_k \int_0^R \frac{|\zeta(r)|^2 r^{k+4}}{(\varepsilon+r^2)^3}dr + (k+3)\omega_3\beta_k \int_0^R \frac{|\zeta(r)|^2 r^{k+2}}{(\varepsilon+r^2)^2}dr \\ &+ O(\varepsilon^{\frac{1}{2}}).\end{aligned}$$

By using the Dominated Convergence Theorem, we get the desired result.

Combining (2.3.17), (2.3.19) and (2.3.20), we obtain

$$\begin{aligned}Q_\lambda(u_{a,\varepsilon}) &= p_0 S + \omega_3\left[\int_0^R (p_0 + \beta_k r^k)|\zeta'(r)|^2 dr + k\beta_k \int_0^R |\zeta(r)|^2 r^{k-2}dr - \lambda\int_0^R \zeta^2(r)dr\right]\frac{\varepsilon^{\frac{1}{2}}}{K_2} \\ &+ O(\varepsilon),\end{aligned}$$

thus,

$$\begin{aligned}Q_\lambda(u_{a,\varepsilon}) &= p_0 S + \frac{\omega_3 \int_0^R \zeta^2(r)dr}{K_2}\left[\frac{\int_0^R (p_0+\beta_k r^k)|\zeta'(r)|^2 dr + k\int_0^R |\zeta(r)|^2 r^{k-2}dr}{\int_0^R |\zeta(r)|^2 dr} - \lambda\right]\varepsilon^{\frac{1}{2}} \\ &+ O(\varepsilon).\end{aligned} \tag{2.3.21}$$

Set $\quad D(k,\zeta) = \dfrac{\int_0^R (p_0 + \beta_k r^k)|\zeta'(r)|^2 dr + k\int_0^R |\zeta(r)|^2 r^{k-2}dr}{\int_0^R |\zeta(r)|^2 dr}$ and $\gamma(k) = \inf_H D(k,\zeta)$

where $H$ is defined by

$H = \{\zeta \in C_0^\infty(\bar{\Omega}),\, 0 \leq \zeta \leq 1,\, \zeta = 1$ in $\{x,\, |x-a| < \frac{R}{2}\}$ and $\zeta = 0$ in $\{x,\, |x-a| \geq R\}\}$.

This finishes the proof of Lemma 2.3.2. □

Now, we go back to proof of assertion 3) in Theorem 2.1.1 (case $0 < k < 2$). First of all, let us emphasize that if the domain $\Omega$ is starshaped about $a$, the assertion 3) is more interesting. Indeed, it gives a better estimate of the least value of the parameter $\lambda$ over which there is a solution to problem (2.1.1).
In the case of a non-starshaped domain, combining the fact that $S_0(p) = p_0 S$ with the properties of $S_\lambda(p)$ (see the proof of lemma 2.3.4), we have that there exists $\lambda^* \in [0, \lambda_1^{div}[$ such that for all $\lambda \in ]\lambda^*, \lambda_1^{div}[$, the problem (2.1.1) has a solution. Note that we have no other information on $\lambda^*$.
Therefore, throughout the rest of this proof, we assume that the domain $\Omega$ is starshaped about $a$.
We need two Lemmas. Let us start by the following

**Lemma 2.3.3.**
*Assume* $0 < k \leq 2$. *Then there exists a constant* $\tilde{\beta}_k = \beta_k \min[(\text{diam } \Omega)^{k-2}, 1]$ *such that*

$$S_\lambda(p) = p_0 S \text{ for every } \lambda \in ]-\infty, \tilde{\beta}_k \frac{n^2}{4}] \tag{2.3.22}$$

*and the infimum of* $S_\lambda(p)$ *is not achieved.*

*Proof.* We know from (2.3.16) that

$$S_\lambda(p) \leq Q_\lambda(u_{a,\varepsilon}) \leq p_0 S + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) \quad \text{with } A_k \text{ is a positive constant,}$$

thus

$$S_\lambda(p) \leq p_0 S.$$

On the other hand, we know from Lemma 2.2.2 and Proposition 2.2.1, that for $0 < k \leq 2$, for every $\lambda \leq \frac{k}{2}\beta_k(\frac{n+k-2}{2})^2 \, (diam \ \Omega)^{k-2}$, problem (2.1.1) has no solution. So we exclude the case $S_\lambda(p) < p_0 S$, otherwise, Lemma 2.3.1 will yield in a contradiction.
We conclude that for $0 < k \leq 2$, we have

$$S_\lambda(p) = p_0 S \quad \text{for every} \quad \lambda \leq \frac{k}{2}\beta_k(\frac{n+k-2}{2})^2 \, (diam \ \Omega)^{k-2}. \tag{2.3.23}$$

Now, we consider $\tilde{p}$ defined by

$$\begin{cases} \tilde{p}(x) &= p(x) & \forall x \in \Omega \setminus B(a,r), \\ \tilde{p}(x) &= p_0 + \beta_k |x-a|^2 & \forall x \in B(a, \frac{r}{2}), \\ p(x) &\geq \tilde{p}(x) \geq p_0 + \beta_k |x-a|^2 & \forall x \in B(a,r) \setminus B(a, \frac{r}{2}), \end{cases} \tag{2.3.24}$$

where $r < 1$ is a positive constant.
Since $0 < k \leq 2$, we have $|x-a|^k \geq |x-a|^2$ for every $x \in B(a,r)$ and $p(x) \geq \tilde{p}(x)$

in $\Omega$.
Let $u \in H_0^1(\Omega)$ with $\|u\|_q = 1$, then

$$\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx \geq \int_\Omega \tilde{p}(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx,$$

thus,

(2.3.25)
$$\begin{aligned}\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx \geq & \int_\Omega (p_0 + \frac{1}{2}(\tilde{p}(x) - p_0))|\nabla u(x)|^2 dx \\ & -\lambda \int_\Omega |u(x)|^2 dx + \frac{1}{2}\int_\Omega (\tilde{p}(x) - p_0)|\nabla u(x)|^2 dx.\end{aligned}$$

Set $\tilde{\tilde{p}}(x) = p_0 + \frac{1}{2}(\tilde{p}(x) - p_0)$.
From (2.1.3) we deduce that

$$p(x) - p_0 \geq \beta_k |x-a|^k \text{ a.e in } \Omega. \tag{2.3.26}$$

Indeed,
Set $f(t) = p(a + t(x-a))$, we have

$$p(x) - p(a) = f(t) - f(0) = \int_0^1 f'(t)dt = \int_0^1 \frac{1}{t}\nabla p(a + t(x-a)).t(x-a)dt.$$

From (2.1.3), we write

$$p(x) - p(a) = \int_0^1 \frac{1}{t}\nabla p(a+t(x-a)).t(x-a)dt \geq k\beta_k \int_0^1 t^{k-1}|x-a|^k dt = \beta_k |x-a|^k.$$

Using (2.3.24) and (2.3.26), a simple computation gives $\tilde{p}(x) - p_0 \geq \tilde{\beta_k}|x-a|^2$ a.e in $\Omega$, with $\tilde{\beta_k} = \beta_k \min[(diam\ \Omega)^{k-2}, 1]$.
Applying Lemma 2.2.1, we find

$$\int_\Omega (\tilde{p}(x) - p_0)|\nabla u(x)|^2 dx \geq \tilde{\beta}_k \frac{n^2}{4}\int_\Omega |u(x)|^2 dx.$$

Inequality (2.3.25) becomes for every $u \in H_0^1(\Omega)$,

$$\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx \geq \int_\Omega \tilde{\tilde{p}}(x)|\nabla u|^2 dx - \left(\lambda - \tilde{\beta}_k \frac{n^2}{8}\right)\int_\Omega |u|^2 dx.$$

Thus, we find

$$S_\lambda(p) \geq \inf_{\|u\|_q^2 = 1}\left[\int_\Omega \tilde{\tilde{p}}(x)|\nabla u|^2 dx - (\lambda - \tilde{\beta}_k \frac{n^2}{8})\int_\Omega |u|^2 dx\right].$$

On the other hand $\lambda - \tilde{\beta}_k \frac{n^2}{8} \leq \frac{1}{2}\tilde{\beta}_k \frac{n^2}{4}$ since $\lambda \leq \tilde{\beta}_k \frac{n^2}{4}$, so by (2.3.23), we conclude that

$$\inf_{\|u\|_q=1} \left[ \int_\Omega \tilde{\tilde{p}}(x)|\nabla u|^2 dx - (\lambda - \tilde{\beta}_k \frac{n^2}{8}) \int_\Omega |u|^2 dx \right] = p_0 S,$$

hence, (2.3.22) follows.

Now, we are able to prove that the infimum in (2.3.22) is not achieved. Suppose by contradiction that it is achieved by some $u_0$. Let $\delta$ such that $\tilde{\beta}_k \frac{n^2}{4} \geq \delta > \lambda$. Using $u_0$ as a test function for $S_\delta$, we obtain

$$S_\delta(p) \leq \frac{\int_\Omega p(x)|\nabla u_0|^2 dx - \delta \int_\Omega |u_0|^2 dx}{\|u_0\|_q^2} < \frac{\int_\Omega p(x)|\nabla u_0|^2 dx - \lambda \int_\Omega |u_0|^2 dx}{\|u_0\|_q^2}$$

and thus $S_\delta(p) < S_\lambda(p) = p_0 S$. This is a contradiction since $S_\delta(p) = p_0 S$ for $\delta \leq \tilde{\beta}_k \frac{n^2}{4}$. □

The second Lemma on which the proof of assertion 3) in Theorem 2.1.1 is based is the following

**Lemma 2.3.4.**
*There exists $\lambda^* \in [\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$, such that for all $\lambda \in ]\lambda^*, \lambda_1^{div}[$ we have*

$$S_\lambda(p) < p_0 S.$$

*Proof.* The proof is based on a study of some properties of the function $\lambda \mapsto S_\lambda(p)$. We have $S_{\lambda_1^{div}}(p) = 0$. Indeed let $\varphi_1$ be the eigenfunction of $div(p\nabla.)$ corresponding to $\lambda_1^{div}$, we have

$$S_{\lambda_1^{div}} \leq \frac{\int p(x)|\nabla \varphi_1|^2 dx - \lambda_1^{div} \int |\varphi_1|^2 dx}{(\int |\varphi_1|^q dx)^{\frac{2}{q}}} = 0.$$

Moreover, $\lambda \mapsto S_\lambda(p)$ is continuous and $S_{\tilde{\beta}_k \frac{n^2}{4}}(p) = p_0 S$. Then according to the Mean Value Theorem, there exists $\beta \in ]\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$ such that $0 < S_\beta(p) < p_0 S$. But the function $\lambda \mapsto S_\lambda(p)$ is decreasing hence $\forall \lambda \in [\beta, \lambda_1^{div}[$ we have $S_\lambda(p) < p_0 S$, and the Lemma follows at once. □

Now we have all the necessary ingredients for the proof of Theorem 2.1.1.

**Proof of Theorem 2.1.1 concluded**: Concerning the proof of 1), 2), 3) and 4), let $u \in H_0^1(\Omega)$ be given by Lemma 2.3.1, that is,

$$\|u\|_q = 1 \ \text{ and } \ \int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx = S_\lambda(p).$$

We may as well assume that $u \geq 0$. Since $u$ is a minimizer for (2.3.2) there exists a Lagrange multiplier $\mu \in I\!R$ such that

$$-div(p\nabla u) - \lambda u = \mu u^{q-1} \text{ on } \Omega.$$

In fact, $\mu = S_\lambda(p)$, and $S_\lambda(p) > 0$ since $\lambda < \lambda_1^{div}$. It follows that $\gamma u$ satisfies (2.1.1) for some appropriate constant $\gamma > 0$ ($\gamma = (S_\lambda(p))^{\frac{1}{q-2}}$), note that $u > 0$ on $\Omega$ by the strong maximum principle.

Now we prove the assertion 5) of Theorem 2.1.1. From (2.3.16) and since $\lambda \leq 0$ we have

$$p_0 S \leq S_\lambda(p) \leq Q_\lambda(u_{a,\varepsilon}) \leq p_0 S + o(1).$$

Hence $S_\lambda(p) = p_0 S$ and the infimum is not achieved, indeed we suppose that $S_\lambda(p)$ is achieved by some function $u \in H_0^1(\Omega)$, in that case

$$S_\lambda(p) = \int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx, \text{ with } \|u\|_q = 1.$$

Using the fact that $S$ is not attained and since $\lambda \leq 0$, we deduce

$$p_0 S < p_0 \int_\Omega |\nabla u(x)|^2 dx \leq S_\lambda(p) = p_0 S,$$

then we obtain a contradiction.

Finally we prove assertion 6) in Theorem 2.1.1. Let $\varphi_1$ be the eigenfunction corresponding to $\lambda_1^{div}$ with $\varphi_1 > 0$ on $\Omega$. Suppose that $u$ is a solution of (2.1.1). We have

$$\begin{aligned} -\int_\Omega div(p(x)\nabla u(x))\varphi_1(x)dx &= \lambda_1^{div} \int_\Omega u(x)\varphi_1(x)dx \\ &= \int_\Omega u^{q-1}(x)\varphi_1(x)dx + \lambda \int_\Omega u(x)\varphi_1(x)dx, \end{aligned}$$

thus

$$\lambda_1^{div} \int_\Omega u(x)\varphi_1(x)dx > \lambda \int_\Omega u(x)\varphi_1(x)dx$$

and

$$\lambda_1^{div} > \lambda.$$

This completes the proof of Theorem 2.1.1.

**Proof of Remark 2.1.1**

**Proof of 1)**

In this case, we have $k = 2$ and $n \geq 4$.

We have (see proof of (2.3.22) page 33)

$$p(x) - p_0 \geq \beta_2 |x - a|^2 \quad \text{a.e} \in \Omega.$$

Therefore

$$\frac{\int_\Omega p(x)|\nabla u|^2dx}{\int_\Omega |u|^2dx} \geq p_0\frac{\int_\Omega |\nabla u|^2dx}{\int_\Omega |u|^2dx} + \beta_2\frac{\int_\Omega |x-a|^2|\nabla u|^2dx}{\int_\Omega |u|^2dx}.$$

On one hand, we apply, again, Lemma 2.2.1 (with $t=0$), we obtain

$$\frac{\int_\Omega |x-a|^2|\nabla u|^2dx}{\int_\Omega |u|^2dx} \geq \frac{n^2}{4}.$$

On other hand, we have

$$\frac{\int_\Omega |\nabla u|^2dx}{\int_\Omega |u|^2dx} \geq \frac{1}{(diam\ \Omega)^2}\frac{\int_\Omega |\nabla u|^2dx}{\int_\Omega |x|^{-2}|u|^2dx}$$

We apply Lemma 2.2.1 (with $t=-2$), we deduce

$$\begin{aligned}\frac{\int_\Omega |\nabla u|^2dx}{\int_\Omega |u|^2dx} &\geq \frac{1}{(diam\ \Omega)^2}\frac{\int_\Omega |\nabla u|^2dx}{\int_\Omega |x|^{-2}|u|^2dx}\\ &\geq \frac{1}{(diam\ \Omega)^2}\frac{(n-2)^2}{4}.\end{aligned}$$

Combining the above inequality, we get

$$\frac{\int_\Omega p(x)|\nabla u|^2dx}{\int_\Omega |u|^2dx} \geq p_0\frac{(n-2)^2}{4\,(diam\ \Omega)^2} + \beta_2\frac{n^2}{4}$$

and

$$\lambda_1^{div} \geq p_0\frac{(n-2)^2}{4\,(diam\ \Omega)^2} + \beta_2\frac{n^2}{4}.$$

Therefore, if we have $p_0 > \dfrac{n(n-4)}{(n-1)(n-2)^2}\beta_2\,(diam\ \Omega)^2$, we obtain the desired result.

**Proof of 2)**

Here we have $n \geq 3$ and $0 < k < 2$.

Using the definition of $p_0$, we have

$$\frac{\int_\Omega p(x)|\nabla u|^2dx}{\int_\Omega |u|^2dx} \geq p_0\frac{\int_\Omega |\nabla u|^2dx}{\int_\Omega |u|^2dx}.$$

We apply the Lemma 2.2.1 (with $t = -2$), we obtain

$$\frac{\displaystyle\int_\Omega p(x)|\nabla u|^2 dx}{\displaystyle\int_\Omega |u|^2 dx} \geq p_0 \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\displaystyle\int_\Omega |u|^2 dx} \geq p_0 \frac{(n-2)^2}{4\,(diam\ \Omega)^2}$$

and

$$\lambda_1^{div} \geq p_0 \frac{(n-2)^2}{4\,(diam\ \Omega)^2}.$$

Therefore, if $p_0 > \dfrac{\tilde{\beta}_k n^2}{(n-2)^2\,(diam\ \Omega)^2}$, we deduce that $\tilde{\beta}_k \frac{n^2}{4} < \lambda_1^{div}$.

## 2.4 The effect of the geometry of the domain

Let $\Omega \subset I\!R^n$,$n \geq 3$, be a bounded domain. We study the equation

$$(2.4.1) \qquad \begin{cases} -div(p(x)\nabla u) = u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $q = \frac{2n}{n-2}$ and $p : \bar{\Omega} \longrightarrow I\!R$ is a positive weight belonging to $C(\bar{\Omega}) \cap H_0^1(\Omega)$.
We assume in this section that $p$ is such that $\nabla p(x).(x-a) \geq 0$ a.e $x \in \Omega$ and we set $p_0 = p(a)$.
Let us start by the following non-existence result

**Lemma 2.4.1.**
*There is no solution of (2.4.1) if $\Omega$ is a starshaped domain about $a$.*

*Proof.* This follows from Pohozaev's identity.
Suppose that $u$ is a solution of (2.4.1), we have (see Lemma 2.2.2 Section 1 for $\lambda = 0$),

$$(2.4.2) \qquad \int_\Omega \nabla p(x).(x-a)|\nabla u(x)|^2 dx + \int_{\partial\Omega} p(x)[(x-a).\nu]|\frac{\partial u}{\partial \nu}|^2 dx = 0.$$

Note that $(x-a).\nu > 0$ a.e on $\partial\Omega$ since $\Omega$ is starshaped about $a$.
Since $\nabla p(x).(x-a) \geq 0$ a.e $x \in \Omega$, we deduce from (2.4.2) that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, and then by (2.4.1) we have

$$\int_\Omega u^{q-1}(x) dx = -\int_\Omega div(p(x)\nabla u(x)) dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dx = 0,$$

thus

$$u \equiv 0.$$

$\square$

Suppose that $\Omega$ is starshaped about $a$. In view of Lemma 2.4.1, we will modify the geometry of $\Omega$ in order to find a solution of problem 2.4.1 . For a $\varepsilon > 0$ small enough, we set $\Omega_\varepsilon = \Omega \setminus \bar{B}(a,\varepsilon)$.
We investigate the problem (2.4.1) in the new domain $\Omega_\varepsilon$, and, throughout the rest of this paper, we shall denote this new problem by $(I_\varepsilon)$ .
Since $p$ is a continuous function, then, $\forall\, \theta > 0$, $\exists\, r_0 > 0$ such that $\forall\, \sigma \in \Sigma$, where $\Sigma$ designates the unit sphere of $I\!R^n$, we have $|p(a + r_0\sigma) - p_0| < \frac{\theta}{2S^{\frac{n}{2}}}$.
Throughout the rest of this Section, $\theta > 0$ is fixed, small enough, and $r_0 > 0$ is given as the previous definition.
We recall the main result of this section which we have already stated by theorem 2.1.2 in the introduction

**Theorem 2.4.1.**
*There exists $\varepsilon_0 = \varepsilon_0(\Omega, p) \le r_0$ such that for every $0 < \varepsilon < \varepsilon_0$, the problem $(I_\varepsilon)$ has at least one solution in $H_0^1(\Omega_\varepsilon)$.*

In order to prove the Theorem 2.4.1, we need to apply the following result, see [AR],

**Theorem A 1.**
*Let $E$ be a $C^1$ function defined on a Banach space $X$, and let $K$ a compact metric space. We denote by $K^*$ a nonempty subset of $K$, closed, different from $K$ and we fix $f^* \in C(K^*, X)$.*
*We define $\mathcal{P} = \{f \in C(K, X)/f = f^* \text{on } K^*\}$ and $c = \inf_{f\in\mathcal{P}} \sup_{t\in K} E(f(t))$*
*Suppose that for every $f$ of $\mathcal{P}$, we have*

$$\max_{t\in K} E(f(t)) > \max_{t\in K^*} E(f(t)),$$

*then there exists a sequence $(u_j) \subset X$ such that $E(u_j) \longrightarrow c$ and $E'(u_j) \longrightarrow 0$ in $X^*$.*

We consider the functional

$$E(u) = \frac{1}{2}\int_{\Omega_\varepsilon} p(x)|\nabla u(x)|^2 dx - \frac{1}{q}\int_{\Omega_\varepsilon} |u(x)|^q dx.$$

In addition to Theorem A 1, the proof of Theorem 2.4.1 requires the following result (see [B] and Proposition 2.1 in [S])

**Theorem A 2.**
*Suppose that for some sequence* $(u_j) \subset H_0^1(\Omega_\varepsilon)$ *we have* $E(u_j) \to c \in ]\frac{1}{n}(p_0 S)^{\frac{n}{2}}, \frac{2}{n}(p_0 S)^{\frac{n}{2}}[$ *and* $dE(u_j) \to 0$ *in* $H^{-1}(\Omega_\varepsilon)$*. Then* $(u_j)$ *contains a strongly convergent subsequence.*

See Appendix for more details.
Now, we return to the proof of Theorem 2.4.1.
We shall need the following functions:

$$\begin{aligned} \Gamma \ &: H_0^1(\Omega_\varepsilon) \longrightarrow I\!R, \quad \Gamma(u) = \int_{\Omega_\varepsilon} p(x)|\nabla u(x)|^2 dx - \int_{\Omega_\varepsilon} |u(x)|^q dx. \\ F \ &: H_0^1(\Omega_\varepsilon) \longrightarrow I\!R^n, \quad F(u) = (p_0 S)^{-\frac{n}{2}} \int_{\Omega_\varepsilon} x p(x)|\nabla u(x)|^2 dx. \end{aligned}$$

We have the following result

**Lemma 2.4.2.**
*For every neighborhood* $V$ *of* $\bar{\Omega}_\varepsilon$ *there exists* $\eta > 0$ *such that if* $u \neq 0$, $\Gamma(u) = 0$ *and* $E(u) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta$*, then* $F(u) \in V$*.*

*Proof.*
We proceed by contradiction. We assume that there exists $V$ a compact neighborhood of $\bar{\Omega}_\varepsilon$ not containing $a$, such that $\forall j \in \mathbf{N}^*$, we have

$$\begin{aligned} u_j &\neq 0, \\ \Gamma(u_j) &= 0, \\ E(u_j) &\leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \frac{1}{j}, \\ F(u_j) &\notin V. \end{aligned}$$

Using the fact that $\Gamma(u_j) = 0$, we obtain

$$\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx = \int_{\Omega_\varepsilon} |u_j|^q dx$$

and

$$\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx = \left( \frac{\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx}{\left(\int_{\Omega_\varepsilon} |u_j|^q dx\right)^{\frac{2}{q}}} \right)^{\frac{n}{2}}.$$

Consequently

$$E(u_j) = \frac{1}{n} \int_{\Omega_\varepsilon} p(x)|\nabla u_j(x)|^2 dx.$$

Using the definition of $u_j$, the fact that $p_0 = \min_{\bar\Omega} p(x)$ and the definition of $S$, we write

$$\frac{1}{n}(p_0 S)^{\frac{n}{2}} \leq \frac{1}{n}\left(\frac{p_0 \int_{\Omega_\varepsilon} |\nabla u_j|^2 dx}{\left(\int_{\Omega_\varepsilon} |u_j|^q dx\right)^{\frac{2}{q}}}\right)^{\frac{n}{2}} \leq E(u_j) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \frac{1}{j}$$

and we deduce

$$\int_{\Omega_\varepsilon} p(x)|\nabla u_j(x)|^2 dx = (p_0 S)^{\frac{n}{2}} + o(1).$$

Using Theorem 2 in [C], (see also [L, Lemma I.4 and Lemma I.1]), for a subsequence of $(u_j)_j$ still denoted by $(u_j)_j$, there exists $x_0 \in \bar\Omega_\varepsilon$ such that

$$p(x)|\nabla u_j|^2 \longrightarrow (p_0 S)^{\frac{n}{2}} \delta_{x_0}\ (j \to \infty),$$

where the above convergence is understood for the weak topology of bounded measures on $\bar\Omega_\varepsilon$ and where $\delta_{x_0}$ is the Dirac measure at $x_0$.
As a consequence, $F(u_j) \in \bar\Omega_\varepsilon \subset V$, and this contradicts the hypothesis. □

Let $R_0 > 0$ such that $B(a, 2R_0) \subset \Omega$.
For $k \in \mathbf{N}^*$, let $\varphi_k \in C^\infty(I\!R^n, [0,1])$ such that

$$\begin{cases} \varphi_k(x) = 0 & \text{if } |x-a| \leq \frac{1}{4k^2} \text{ and if } |x-a| \geq 2R_0, \\ \varphi_k(x) = 1 & \text{if } \frac{1}{2k^2} \leq |x-a| \leq R_0. \end{cases}$$

We consider the family of functions

$$u_t^\sigma(x) = \left[\frac{1-t}{(1-t)^2 + |x-a-t\sigma|^2}\right]^{\frac{n-2}{2}},$$

where $t \in [0,1[$, $\sigma \in \Sigma$ and where $\Sigma$ denotes the unit sphere of $I\!R^n$.
We see easily that $\int_{I\!R^n} |\nabla u_t^\sigma|^2 dx$ and $\int_{I\!R^n} |u_t^\sigma|^q dx$ are independent of $t \in [0,1[$ and of $\sigma \in \Sigma$. We also have

$$\int_{I\!R^n} |\nabla u_t^\sigma(x)|^2 dx = S\left(\int_{I\!R^n} |u_t^\sigma(x)|^q dx\right)^{\frac{2}{q}}.$$

Write

$$v_{t,k}^\sigma(x) = \frac{(1-t)^{\frac{n-2}{2}} k^{\frac{n-2}{2}} \varphi_k(x)}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^{\frac{n-2}{2}}},$$

we remark that $v_{t,k}^\sigma \in H_0^1(\Omega_\varepsilon)$. For $r > 0$, let $g(r) = E(rv_{t,k}^\sigma)$, then $rg'(r) = \Gamma(rv_{t,k}^\sigma)$, $g(r) \to -\infty$, when $r \to +\infty$, $g(0) = 0$ and $g(r) > 0$ for $r > 0$ small enough.
We conclude, from the above, that $g$ reaches its maximum at

$$r = \left[\frac{\int_{\Omega_\varepsilon} p(x)|\nabla v_{t,k}^\sigma|^2 dx}{\int_{\Omega_\varepsilon} |v_{t,k}^\sigma|^q dx}\right]^{\frac{1}{q-2}} > 0.$$

We set $w_{t,k}^\sigma = rv_{t,k}^\sigma$. We have

**Lemma 2.4.3.**
*The following two statements are true:*

**a)**$\forall \delta > 0$, $\exists k_0 \geq 1$ *such that* $(\forall\, k \geq k_0)$ *then*
$$(\forall \sigma \in \Sigma \text{ and } \forall t \in [0,1[,\ E(w^\sigma_{t,k}) \leq \tfrac{1}{n}(p_{_0}S)^{\frac{n}{2}} + \delta)$$
**b)**$\forall \alpha > 0$, $\exists \mu > 0$ *such that* $(\mu < t < 1)$ *then*
$$(\forall \sigma \in \Sigma \text{ and } \forall k \geq 1,\ E(w^\sigma_{t,k}) \leq \tfrac{1}{n}(p_{_0}S)^{\frac{n}{2}} + \alpha)$$
$$\text{and} \quad |F(w^\sigma_{t,k}) - (a + r_0\sigma)| \leq \alpha.$$

*Proof.*
Before proving this Lemma, let us remark that the function $v^\sigma_{t,k}$ corresponds to the function $u_{a,\varepsilon}$ defined in the beginning of this paper, so for more details of calculus we refer to section 2.
We start by proving the assertion a). Let $t \in [0,1[$, we have

$$\begin{aligned} E(w^\sigma_{t,k}) &= \frac{1}{2}\int_{\Omega_\varepsilon} p(x)|\nabla w^\sigma_{t,k}|^2 dx - \frac{1}{q}\int_{\Omega_\varepsilon} |w^\sigma_{t,k}|^q dx, \\ &= \frac{r^2}{2}\int_{\Omega_\varepsilon} p(x)|\nabla v^\sigma_{t,k}|^2 dx - \frac{r^q}{q}\int_{\Omega_\varepsilon} |v^\sigma_{t,k}|^q dx. \end{aligned}$$

Using the definition of $r$, the definition of $\varphi_k$ and applying the Dominated Convergence Theorem, we obtain, as $k \to \infty$,

$$E(w^\sigma_{t,k}) = \frac{1}{n}\left[\frac{k^n(n-2)^2(1-t)^{n-2}\int_{\{\frac{1}{2k^2}\leq|x-a|\leq R_0\}} p(x)\frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx}{\left[k^n(n-2)^2(1-t)^n\int_{\{\frac{1}{2k^2}\leq|x-a|\leq R_0\}}\frac{1}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + o(1).$$

By the following change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$, we see that

$$E(w^\sigma_{t,k}) = \frac{1}{n}\left[\frac{(n-2)^2\int_{\{\frac{1}{2k(1-t)}-\frac{tr_0}{1-t}\leq|y|\leq\frac{kR_0}{1-t}+\frac{tr_0}{1-t}\}} p(\frac{y(1-t)}{k}+a+tr_0\sigma)\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{\{\frac{1}{2k(1-t)}-\frac{tr_0}{1-t}\leq|y|\leq\frac{kR_0}{1-t}+\frac{tr_0}{1-t}\}}\frac{1}{(1+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + o(1).$$

Applying again the Dominated Convergence Theorem, we deduce, as $k \to \infty$, that

$$\begin{aligned} E(w^\sigma_{t,k}) &= \frac{1}{n}\left[\frac{(n-2)^2 p(a+tr_0\sigma)\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{I\!R^n}\frac{1}{((1-t)^2+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + o(1), \\ &= \frac{1}{n}(p(a+tr_0\sigma))^{\frac{n}{2}}S^{\frac{n}{2}} + o(1). \end{aligned}$$

Now, using the definition of $r_0$, a simple computation shows that $\forall\delta > 0$, $\exists k_0 \geq 1$ such that $\forall k \geq k_0$, we have

$$E(w^\sigma_{t,k}) \leq \frac{1}{n}(p_{_0}S)^{\frac{n}{2}} + \delta,$$

which finishes the proof of a).
Now we return to the proof of b), let $k \in \mathbf{N}^*$, we have

$$\begin{aligned} E(w^\sigma_{t,k}) &= \frac{1}{2}\int_{\Omega_\varepsilon} p(x)|\nabla w^\sigma_{t,k}|^2 dx - \frac{1}{q}\int_{\Omega_\varepsilon} |w^\sigma_{t,k}|^q dx \\ &= \frac{r^2}{2}\int_{\Omega_\varepsilon} p(x)|\nabla v^\sigma_{t,k}|^2 dx - \frac{r^q}{q}\int_{\Omega_\varepsilon} |v^\sigma_{t,k}|^q dx. \end{aligned}$$

Using the definition of $\varphi_k$ and $r$, we easily see, as $t \to 1$, that

$$E(w^\sigma_{t,k}) = \frac{1}{n}\left[\frac{k^n(n-2)^2(1-t)^{n-2}\int_{I\!R^n} p(x)\frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx}{\left[k^n(1-t)^n\int_{I\!R^n}\frac{1}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n}dx\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + O((1-t)^{n-2}).$$

By the change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$, we get

$$E(w^\sigma_{t,k}) = \frac{1}{n}\left[\frac{(n-2)^2\int_{I\!R^n} p(\frac{(1-t)y}{k} + a + tr_0\sigma)\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{I\!R^n}\frac{1}{(1+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + O((1-t)^{n-2}).$$

Applying the Dominated Convergence Theorem, we obtain

$$\begin{aligned} E(w^\sigma_{t,k}) &= \frac{1}{n}\left[\frac{(n-2)^2 p(a+r_0\sigma)\int_{I\!R^n}\frac{|y|^2}{(1+|y|^2)^n}dy}{\left[\int_{I\!R^n}\frac{1}{(1+|y|^2)^n}dy\right]^{\frac{2}{q}}}\right]^{\frac{n}{2}} + O((1-t)^{n-2}), \\ &= \frac{1}{n}(p(a+r_0\sigma))^{\frac{n}{2}} S^{\frac{n}{2}} + O((1-t)^{n-2}). \end{aligned}$$

Using the definition of $r_0$, a simple computation shows that $\forall\alpha > 0$, $\exists\mu > 0$ such that $\forall\,\mu < t < 1$, we have

$$E(w^\sigma_{t,k}) \leq \frac{1}{n}(p_{_0}S)^{\frac{n}{2}} + \alpha.$$

On the other hand

$$\begin{aligned} F(w^\sigma_{t,k}) &= (p_{_0}S)^{-\frac{n}{2}}\int_{I\!R^n} xp(x)|\nabla w^\sigma_{t,k}(x)|^2 dx, \\ &= (p_{_0}S)^{-\frac{n}{2}} r^2 \int_{I\!R^n} xp(x)|\nabla v^\sigma_{t,k}(x)|^2 dx. \end{aligned}$$

Using the definition of $r$, we get

$$F(w^\sigma_{t,k}) = (p_0 S)^{-\frac{n}{2}} \left[ \frac{\int_{\Omega_\varepsilon} p(x)|\nabla v^\sigma_{t,k}|^2 dx}{\int_{\Omega_\varepsilon} |v^\sigma_{t,k}|^q dx} \right]^{\frac{2}{q-2}} \int_{I\!R^n} x p(x)|\nabla v^\sigma_{t,k}(x)|^2 dx.$$

Using the definition of $v^\sigma_{t,k}$, we write

$$F(w^\sigma_{t,k}) = (p_0 S)^{-\frac{n}{2}} \left[ \frac{(1-t)^{n-2}(n-2)^2 \int_{I\!R^n} p(x) \frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n} dx}{(1-t)^n \int_{I\!R^n} \frac{1}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n} dx} \right]^{\frac{2}{q-2}} \times$$

$$(1-t)^{n-2} k^n (n-2)^2 \int_{I\!R^n} x\, p(x) \frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2+|k(x-a-tr_0\sigma)|^2)^n} dx + o(1-t).$$

Using the change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$, we obtain

$$\begin{aligned} F(w^\sigma_{t,k}) &= (p_0 S)^{-\frac{n}{2}} \left[ \frac{(n-2)^2 \int_{I\!R^n} p\left(\frac{(1-t)y}{k} + a + tr_0\sigma\right) \frac{|y|^2}{(1+|y|^2)^n} dx}{\int_{I\!R^n} \frac{1}{(1+|y|^2)^n} dx} \right]^{\frac{2}{q-2}} \times \\ & \quad (n-2)^2 \int_{I\!R^n} \frac{(\frac{(1-t)y}{k} + a + tr_0\sigma)\, p(\frac{(1-t)y}{k} + a + tr_0\sigma)\, |y|^2}{(1+|y|^2)^n} dx + o(1-t). \end{aligned}$$

Applying the Dominated Convergence Theorem, we deduce that

$$\begin{aligned} F(w^\sigma_{t,k}) &= (p_0 S)^{-\frac{n}{2}} (p(a+r_0\sigma))^{\frac{n}{2}} \left[ \frac{(n-2)^2 \int_{I\!R^n} \frac{|y|^2}{(1+|y|^2)^n} dy}{\left[\int_{I\!R^n} \frac{1}{(1+|y|^2)^n} dy\right]^{\frac{2}{q}}} \right]^{\frac{q}{q-2}} (a+r_0\sigma) + o(1-t), \\ &= (p_0 S)^{-\frac{n}{2}} (p(a+r_0\sigma))^{\frac{n}{2}} S^{\frac{n}{2}} (a+r_0\sigma) + o(1-t). \end{aligned}$$

Using the definition of $r_0$ we get the desired result. □

**Consequences**

Let $V$ be a compact neighborhood of $\bar{\Omega}_\varepsilon$ not containing $a$. Let $0 < \eta < r_0$ small enough, which corresponds to $V$ as in Lemma 2.4.2, verifying $r_0\sigma + \xi \neq a$ for $|\sigma| = 1$ and $|a - \xi| \leq \eta$.

By Lemma 2.4.3, there exists $k_0 \geq 1$ such that :

$$E(w^\sigma_{t,k_0}) \leq \frac{2}{n}(p_0 S)^{\frac{n}{2}} - \eta,\ \forall \sigma \in \Sigma,\ \forall t \in [0,1[. \tag{2.4.3}$$

**Remark 2.4.1.**

*We choose* $\varepsilon_0 = \varepsilon_0(\Omega, p) \leq \frac{1}{4k_0^2}$ *small enough and such that* $\forall \varepsilon < \varepsilon_0$ *we have* $\{x\ |x - a| \leq \varepsilon\} \not\subset V$.

We fix $\lambda > 1$, large enough such that $E(\lambda w^\sigma_{t,k_0}) < 0$, $\forall \sigma \in \Sigma$, $\forall t \in [0,1[$. In order to apply Theorem A 1, we define the sets $K$, $K^*$ and the function $f^*$ as
$K = [0,1] \times \bar{B}(a, r_0)$,
$K^* = \partial K = [0,1] \times \partial\bar{B}(a, r_0) \cup \{0,1\} \times \bar{B}(a, r_0)$ and
$f^* : K \to H^1_0(\Omega_\varepsilon)$,
$f^*(s, tr_0\sigma) = \lambda s w^\sigma_{t,k_0}$.
The conclusion of Theorem 2.4.1 follows from the next

**Lemma 2.4.4.**
*We have*

$$\sup_K E(f) \geq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta, \ \forall f \in \mathcal{P}.$$

We postpone the proof of Lemma 2.4.4 and complete the proof of Theorem 2.4.1. From (2.4.3) we have

$$\max_{r\geq 0} E(r v^\sigma_{t,k_0}) = E(w^\sigma_{t,k_0}) \leq \frac{2}{n}(p_0 S)^{\frac{n}{2}} - \eta \quad \forall \sigma \in \Sigma, \quad \forall t \in [0,1[.$$

From assertion b) of Lemma 2.4.3 there exists $\mu$, we fix $t_0 \in ]\mu, 1[$ such that

$$\max_{r\geq 0} E(r v^\sigma_{t_0,k_0}) = E(w^\sigma_{t_0,k_0}) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \eta, \quad \forall \sigma \in \Sigma.$$

then

$$\max_{\partial K} E(f^*) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \eta \quad \text{ and } \quad \sup_K E(f^*) < \frac{2}{n}(p_0 S)^{\frac{n}{2}}.$$

So, by Lemma 2.4.4,

$$\sup_K E(f) \geq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta > \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \eta \geq \sup_{\partial K} E(f^*)$$

and

$$c = \inf_{f\in\mathcal{P}} \sup_{t\in K} E(f) \in ]\frac{1}{n}(p_0 S)^{\frac{n}{2}}, \frac{2}{n}(p_0 S)^{\frac{n}{2}}[.$$

Applying Theorem A 1 and Theorem A 2, we obtain the conclusion of Theorem 2.4.1.

**Proof of Lemma 2.4.4**. We argue by contradiction. Suppose that there exists $f \in C(K, H^1_0(\Omega_\varepsilon))$ with $f = f^*$ on $\partial K$, and $E(f(s,\xi)) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta$, $\forall (s,\xi) \in K$. We consider the function $G : K \longrightarrow I\!R^{n+1}$, defined by

$$G(s,\xi) = (s, F(f(s,\xi))).$$

We will prove that

$$\deg(G, K, (\lambda^{-1}, a)) = 1. \tag{2.4.4}$$

The map $H : [0,1] \times K \longrightarrow I\!R^{n+1}$, defined by
$H(t,s,\xi) = tG(s,\xi) + (1-t)(s,\xi) = (s, tF(f(s,\xi)) + (1-t)\xi)$
is a homotopy between $G$ and $Id_K$, where $Id_K$ is the Identity application of $K$.
To get (2.4.4), we start by checking that $(\lambda^{-1}, a) \notin H(t, \partial K)$.
If not, there exists $(s,\xi) \in \partial K$ such that $H(t,s,\xi) = (\lambda^{-1}, a)$, as a consequence $s = \lambda^{-1}$ and $a = tF(f(\lambda^{-1},\xi)) + (1-t)\xi = t(F(w^{\sigma}_{t_0 k_0}) - \xi) + \xi$.
Since $s = \lambda^{-1} \in ]0,1[$, we have $\xi \in \partial \bar{B}(a, r_0)$. But, since $|F(w^{\sigma}_{t_0,k_0}) - (a + r_0\sigma)| < \eta$ $\forall \sigma \in \Sigma$ (Lemma 2.4.3, b)), the fact that $t(F(w^{\sigma}_{t_0,k_0}) - \xi) + \xi = a$, $\xi \in \partial\bar{B}(a, r_0)$ leads to a contradiction. Then, we deduce that $(\lambda^{-1}, a) \notin H(t, \partial K)$ and consequently $\forall t \in [0,1]$, $\deg(H(t,.), K, (\lambda^{-1}, a))$ is well defined.
We consider the following sets:
$K^+ = \{(s,\xi) \in K | \Gamma(f(s,\xi)) > 0\} \cup (0,\xi)$, $K^- = \{(s,\xi) \in K | \Gamma(f(s,\xi)) < 0\}$ and $K^0 = \{(s,\xi) \in K | \Gamma(f(s,\xi)) = 0\}$.
If $(s,\xi) \in \partial K$ then we have $f(s,\xi) = f^*(s,\xi) = \lambda s w^{\sigma}_{t_0,k_0}$ and

$$\begin{aligned}
\Gamma(f(s,\xi)) &= (s\lambda)^2 \int_{\Omega_\varepsilon} p(x) |\nabla w^{\sigma}_{t_0,k_0}(x)|^2 dx - (s\lambda)^q \int_{\Omega_\varepsilon} |w^{\sigma}_{t_0,k_0}(x)|^q dx \\
\Gamma(f(s,\xi)) &= [(s\lambda)^2 - (s\lambda)^q] \int_{\Omega_\varepsilon} p(x) \nabla w^{\sigma}_{t_0,k_0}(x)|^2 dx.
\end{aligned}$$

Since $\int_{\Omega_\varepsilon} p(x) |\nabla w^{\sigma}_{t_0,k_0}(x)|^2 dx > 0$, we see that

(2.4.5) $\quad$ If $\quad (s,\xi) \in \partial K \quad$ and if $\quad 0 \leq s < \lambda^{-1}, \quad$ then $\quad (s,\xi) \in K^+$

(2.4.6) $\quad$ If $\quad (s,\xi) \in \partial K \quad$ and if $\quad \lambda^{-1} < s \leq 1, \quad$ then $\quad (s,\xi) \in K^-$

(2.4.7) $$(\lambda^{-1}, \xi) \in K^0, \quad \forall \xi \in \partial \bar{B}(a, r_0).$$

Let $(s,\xi) \in K^0$, we have $\Gamma(f(s,\xi)) = 0$. Moreover, since $E(f(s,\xi)) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta$, looking at Lemma 2.4.2, we deduce that

$$F(f(s,\xi)) \in V.$$

Consequently $\forall (s,\xi) \in K^0$, $F(f(s,\xi)) \neq a$ since $a \notin V$.
Hence $(\lambda^{-1}, a) \notin G(K^0) = G(K \setminus (K^+ \cup K^-))$, then

(2.4.8) $\deg(G, K^+, (\lambda^{-1}, a)) + \deg(G, K^-, (\lambda^{-1}, a)) = \deg(G, K, (\lambda^{-1}, a))$.

On the other hand, since $(\lambda^{-1}, a) \notin H(t, \partial K)$ $\forall t \in [0,1]$ we have

$$\deg(H(1,.), K, (\lambda^{-1}, a)) = \deg(H(0,.), K, (\lambda^{-1}, a)).$$

Using the fact that $H(0,.) = G$, $H(1,.) = Id_K$ and $\deg(Id_K, K, (\lambda^{-1}, a)) = 1$, we deduce (2.4.4).
Now, to finish we will prove that

$$\deg(G, K^+, (\lambda^{-1}, a)) = 0 \tag{2.4.9}$$

$$\deg(G, K^-, (\lambda^{-1}, a)) = 0. \tag{2.4.10}$$

Fix $R > \lambda^{-1}$ let $y \in \mathbb{R}^{n+1}$ such that $|y| \geq R$ then $y \notin G(K)$.
We define the path $r(t) = (tR + (1-t)\lambda^{-1}, a)$, for $t \in [0,1]$.
We claim that $r(t) \notin G(\partial K^+)$ $\forall\, t \in [0,1]$.
If not, there exists $(s,\xi) \in \partial K^+$ with $(Rt+(1-t)\lambda^{-1}, a) = (s, F(f(s,\xi)))$. Hence $s = tR+(1-t)\lambda^{-1} \geq \lambda^{-1}$ and $a = F(f(s,\xi))$. But $\forall (s,\xi) \in K^0$, we have $F(f(s,\xi)) \neq a$, then $(s,\xi) \notin K^0$. Hence $(s,\xi) \in \partial K \cap K^+$, (2.4.5) implies that $s < \lambda^{-1}$ and this contradicts the fact that $s \geq \lambda^{-1}$. Thus $r(t) \notin G(\partial K^+)$ $\forall\, t \in [0,1]$. Hence $\deg(G, K^+, r(t))$ is well defined and is independent of $t$.
Since $(R, a) \notin G(K)$ we obtain

$$\deg(G, K^+, (R, a)) = 0.$$

Using the fact that

$$\deg(G, K^+, r(t)) = \deg(G, K^+, (R, a)) \quad \forall\, t \in [0,1],$$

we deduce (2.4.9).
Similarly, we prove (2.4.10) by using the path $q(t) = (-tR + (1-t)\lambda^{-1}, a)$, $t \in [0,1]$. We have that $\deg(G, K^-, q(t))$ is independent of $t$. Using the fact that $(-R, a) \notin G(K)$, we conclude that

$$\deg(G, K^-, (\lambda^{-1}, a)) = \deg(G, K^-, (-R, a)) = 0.$$

From (2.4.4), (2.4.8), (2.4.9) and (2.4.10) we obtain a contradiction, and Lemma 2.4.4 is proved.

## 2.5 Appendix

In this appendix, $\Omega_\varepsilon$ will be noted $\Omega$.

### 2.5.1 The limiting problem of (2.4.1)

Let $u \in H_0^1(\Omega)$, a solution of

$$\begin{cases} -div(p(x)\nabla u) = |u|^{q-2}u(x), & x \in \Omega, \\ u = 0 & \text{sur } \partial\Omega \end{cases}$$

We have

$$(2.5.1)\qquad -p(x)\Delta u(x) - \nabla p(x).\nabla u(x) = |u|^{q-2}u(x), \quad x \in \Omega.$$

Let $r > 0$, we set

$$u(x) = r^{-\frac{n-2}{r}} v_r(\frac{x-a}{r}),$$

($u$ is extended by 0 outside $\Omega$).
One can see that $v_r$ satisfies

$$-r^{-\frac{n+2}{2}} p(x)\Delta v_r(\frac{x-a}{r}) - r^{-\frac{n-2}{2}-1}\nabla p(x).(\nabla v_r)(\frac{x-a}{r}) = r^{-\frac{n+2}{2}}|v|^{q-2}v_r(\frac{x-a}{r}), \quad x \in \Omega.$$

We set $y = \frac{x-a}{r}$ and we write the above equality according to $y$, we obtain

$$-r^{-\frac{n+2}{2}} p(a+r\,y)\Delta v_r(y) - r^{-\frac{n-2}{2}}(\nabla p)(a+r\,y).(\nabla v_r)(y) = r^{-\frac{n+2}{2}}|v_r|^{q-2}v_r(y), \quad y \in \Omega_r,$$

with $\Omega_r = \frac{\Omega - a}{r}$.
We multiply the above equality by $r^{\frac{n+2}{2}}$, we get

$$(2.5.2)\qquad -p(a+r\,y)\Delta v_r(y) - r^2(\nabla p)(a+r\,y).(\nabla v_r)(y) = |v_r|^{q-2}v_r(y), \quad y \in \Omega_r,$$

We see that $\{v_r\}_r$ is bounded in $H^1(I\!R^n)$, indeed

$$\int_{\Omega_r} |\nabla v_r(y)|^2 dy = \int_\Omega |\nabla u(x)|^2 dx.$$

Therefore $v_r \rightharpoonup w$ weakly in $H^1_{loc}(I\!R^n)$ and $v_r \rightharpoonup w$ weakly in $L^q_{loc}(I\!R^n)$, as $r \to 0$.
We claim that the limiting problem is

$$(2.5.3)\qquad \begin{aligned} -p(a)\Delta w(y) &= |w|^{q-2}w(y), \quad y \in I\!R^n \\ w(x) &\to 0 \qquad |y| \to +\infty. \end{aligned}$$

Indeed, let $\varphi \in C_c^\infty(\Omega_r)$. There exists $r_0 > 0$ such that $K = \textit{support}\,\varphi \subset \Omega_{r_0} \subset \Omega_r$ for every $r \le r_0$.
We multiply (2.5.2) by $\varphi$ and we integrate by part, we obtain

$$\int_K p(a+r\,y)\nabla v_r(y)\nabla\varphi(y)dy + r\int_K \nabla p(a+r\,y).\nabla v_r(y)\varphi(y)dy$$
$$-r^2\int_K \nabla p(a+r\,y).\nabla v_r(y)\varphi(y)dy = \int_K |v_r|^{q-2}v_r(y)\varphi(y)dy.$$

Since $v_r \rightharpoonup w$ weakly in $H^1_{loc}(I\!R^n)$, we have $\int_K \nabla v_r.\nabla\varphi(y)dy \to \int_K \nabla w.\nabla\varphi(y)dy$.
Therefore we apply [Bre, Théorème IV.9, page 58], and we obtain for a subsequence,
(2.5.4)

$$\begin{cases} \nabla v_r(y).\nabla\varphi(y) \to \nabla w(y).\nabla\varphi(y) \text{ a.e. on } K \\ \\ \text{and} \\ \\ \text{there exists } h \in L^1(K), \text{ with } h \ge 0, \text{ such that } |\nabla v_r(y).\nabla\varphi(y)| \le h(y) \text{ a.e. on } K. \end{cases}$$

On the one hand, since $p \in C(\bar{\Omega})$, there exists $\beta > 0$ such that $p(x) \leq \beta \ \forall x \in \bar{\Omega}$, moreover using (2.5.4), we obtain

$$p(a + r\,y)\nabla v_r(y)\nabla\varphi(y) \to p(a)\nabla w(y)\nabla\varphi(y) \quad \text{a.e. on } K,$$

and

$$|p(a + r\,y)\nabla v_r(y)\nabla\varphi(y)| \leq \beta h(y) \in L^1(K).$$

We apply the Dominated Convergence Theorem to get

$$\int_K p(a + r\,y)\nabla v_r(y)\nabla\varphi(y)dy \to \int_K p(a)\nabla w(y)\nabla\varphi(y)dy. \tag{2.5.5}$$

On the other hand, Using Hölder's inequality, the fact that $p \in H^1(\Omega)$ and the fact that $v_r$ is bounded in $H^1_{loc}(I\!R^n)$, we obtain

$$r\left|\int_K \nabla p(a + r\,y).\nabla v_r(y)\varphi(y)dy\right| \leq r\|\nabla p\|_{L^2(\Omega)}\|\varphi\nabla v_r\|_{L^2(K)} \to 0 \tag{2.5.6}$$

and

$$r^2\left|\int_K \nabla p(a + r\,y).\nabla v_r(y)\varphi(y)dy\right| \to 0. \tag{2.5.7}$$

Moreover, since $v_r \rightharpoonup w$ in $L^q_{loc}(I\!R^n)$, we see that

$$\int_K |v_r|^{q-2}v_r(y)\varphi(y)dy \to \int_K |w|^{q-2}w(y)\varphi(y)dy. \tag{2.5.8}$$

Finally, combining (2.5.5), (2.5.6), (2.5.7) and (2.5.8) we deduce, as $r \to 0$,

$$\int_K p(a)\nabla w(y)\nabla\varphi(y)dy = \int_K |w|^{q-2}w(y)\varphi(y)dy,$$

and we deduce (2.5.3).

### 2.5.2 Proof of Theorem A 2

We recall that $p_0 = p(a) = \min_{\bar{\Omega}} p(x)$.
We have also, (see page 35 line 5), that

$$\inf_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega p(x)|\nabla u|^2dx}{\left(\displaystyle\int_\Omega |u|^q dx\right)^{\frac{2}{q}}} = p_0 S.$$

**Theorem A 2**
Suppose that $\{u_j\} \subset H^1_0(\Omega)$ satisfies:
$E(u_j) \to c$, with $\frac{1}{n}(p_0 S)^{\frac{n}{2}} < c < \frac{2}{n}(p_0 S)^{\frac{n}{2}}$

and
$dE(u_j) \to 0$ strongly in $H^{-1}(\Omega)$ $(j \to \infty)$.
Then $\{u_j\}$ contains a strongly convergent subsequence.
We start by proving the following

**Lemma 2.5.1.**
*Suppose* $\{u_j\} \subset H_0^1(\Omega)$ *satisfies:*
$E(u_j) \to l$, *with* $l < \frac{1}{n}(p_0 S)^{\frac{n}{2}}$
*and*
$dE(u_j) \to 0$ *strongly in* $H^{-1}(\Omega)$ $(j \to \infty)$.
*Then* $\{u_j\} \to 0$ *strongly in* $H_0^1(\Omega)$, *as* $j \to \infty$, *and* $l = 0$.

*Proof.*
Let $\{u_j\}$ be a sequence such that $E(u_j) \to l < \frac{1}{n}(p_0 S)^{\frac{n}{2}}$ and $dE(u_j) \to 0$.
Then we have

$$\frac{1}{2}\int_\Omega p(x)|\nabla u_j|^2 dx - \frac{1}{q}\int_\Omega |u_j|^q dx = l + o(1) \tag{2.5.9}$$

and

$$-div(p(x)\nabla u_j) = |u_j|^{q-2}u_j + \xi_j, \quad \text{with } \xi_j \to 0 \text{ in } H^{-1}. \tag{2.5.10}$$

We start by showing that $\{u_j\}$ is bounded in $H_0^1(\Omega)$.
Testing (2.5.10) with $u_j$, we obtain

$$\int_\Omega p(x)|\nabla u_j|^2 dx - \int_\Omega |u_j|^q dx = \langle \xi_j, u_j \rangle. \tag{2.5.11}$$

Computing (2.5.9)$-\frac{1}{q}$(2.5.11), we get

$$\frac{1}{n}\int_\Omega p(x)|\nabla u_j|^2 dx = l - \frac{1}{q}\langle \xi_j, u_j \rangle + o(1)$$

and

$$\frac{1}{q}p_0\|u_j\|^2 \le l + \|\xi_j\|_{H^{-1}}\|u_j\| + o(1).$$

From the above inequality we deduce that $\{u_j\}$ is bounded in $H_0^1(\Omega)$.
We may extract a subsequence still denoted by $u_j$, such that

$$u_j \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),$$

$$u_j \to u \quad \text{weakly in } L^q(\Omega).$$

We take, respectively, (2.5.9)$-\frac{1}{q}$(2.5.11), and (2.5.9)$-\frac{1}{2}$(2.5.11), we obtain

$$\int_\Omega p(x)|\nabla u_j|^2 dx = n\, l + o(1) \quad \text{and} \quad \int_\Omega |u_j|^q dx = n\, l + o(1).$$

Then

$$\lim_{j\to\infty}\int_\Omega p(x)|\nabla u_j|^2dx = n\,l \quad \text{and} \quad \lim_{j\to\infty}\int_\Omega |u_j|^q dx = n\,l.$$

On the other hand, using Sobolev's inequality and the fact that $p(x) \geq p_0$ for all $x \in \bar{\Omega}$, we write

$$\begin{aligned}\int_\Omega p(x)|\nabla u_j|^2dx &\geq p_0 \int_\Omega |\nabla u_j|^2 dx \\ &\geq p_0 S\left(\int_\Omega |u_j|^q dx\right)^{\frac{2}{q}}.\end{aligned}$$

Letting $j$ tend to $+\infty$, we obtain

$$n\,l \geq p_0 S(n\,l)^{\frac{2}{q}}.$$

From the above inequality, we have either $l = 0$ or $l \geq \frac{1}{n}(p_0 S)^{\frac{n}{2}}$.
By assumption, $l < \frac{1}{n}(p_0 S)^{\frac{n}{2}}$, we deduce that $l = 0$ and $u_j \to 0$ strongly in $H_0^1(\Omega)$.
□

**Proof of Theorem A 2**

We use the procedure of the proof of Lemma 2 in ([B], page S33), (see also [S]).
Let $\{u_j\}$ be given as in the hypothesis of Theorem A 2. Therefore we have

$$\frac{1}{2}\int_\Omega p(x)|\nabla u_j|^2dx - \frac{1}{q}\int_\Omega |u_j|^q dx = c + o(1) \tag{2.5.12}$$

and

$$-div(p(x)\nabla u_j) = |u_j|^{q-2}u_j + \xi_j, \quad \text{with } \xi_j \to 0 \text{ in } H^{-1}. \tag{2.5.13}$$

As in proof of Lemma 2.5.1, we see that $\{u_j\}$ is bounded in $H_0^1(\Omega)$ and therefore $u_j \rightharpoonup u$ weakly in $H_0^1(\Omega)$.
We will obtain Theorem A 2, by contradiction. We suppose that

$$u_j \text{ does not strongly converge to } u \text{ in } H_0^1(\Omega).$$

Set $v_j = u_j - u$, we have $\int_\Omega |\nabla v_j|^2 dx \not\to 0$. Therefore, as in [L], (see also [B], [S] and [Le1]), there exists a sequence $\{x_j\}$ in $\Omega$ and a sequence $r_j \to 0$ such that

$$\int_{B(x_j,r_j)} p(x)|\nabla v_j|^2 dx = \mathbf{v} \quad \text{with } \mathbf{v} > 0 \text{ small enough.}$$

Since $\{x_j\} \subset \Omega$ and $\Omega$ is bounded, then for a subsequence, still denoted by $x_j$, we can assume that $x_j \to x_0$ in $\bar{\Omega}$. Following [Le1, Corollary 6], we have that $x_0 = a$ and then

$$x_j \to a \quad \text{in } \bar{\Omega}. \tag{2.5.14}$$

Set

$$\tilde{v}_j(x) = r^{\frac{n-2}{2}} v_j(r_j x + x_j), \quad x \in I\!R^n,$$

($v_j$ is extended by 0 outside $\Omega$), since $\{v_j\}$ is bounded in $H_0^1(\Omega)$, then $\{\tilde{v}_j\}$ is bounded in $H^1(I\!R^n)$.

We have, (see the limiting problem), that $\tilde{v}_j$ satisfies

(2.5.15)

$$-p(x_j + r_j y)\Delta\tilde{v}_j(y) - r_j^2(\nabla p)(x_j + r_j y).(\nabla\tilde{v}_j)(y) = |\tilde{v}_j|^{q-2}\tilde{v}_j(y) + \tilde{\xi}_j \quad \text{on } \frac{\Omega - x_j}{r_j},$$

with $\tilde{\xi}_j \to 0$.

Passing to the limit, on a convenient subsequence of $\tilde{v}_j$, in (2.5.15), (see the limiting problem for more details), we obtain $\tilde{v}_j \rightharpoonup w$ weakly in $L^q_{loc}(I\!R^n)$, $\tilde{v}_j \rightharpoonup w$ weakly in $H^1_{loc}(I\!R^n)$, and $w$ satisfies

$$-p_0 \Delta w(y) = |w|^{q-2} w(y) \quad \text{on } I\!R^n, \tag{2.5.16}$$

and as in [Le1, Proposition 3], (see also [Le2]), we have $w \neq 0$.

It is easy to see that $\tilde{w} = (p_0)^{-\frac{1}{q-2}} w$ satisfies

$$-\Delta\tilde{w}(y) = |\tilde{w}|^{q-2}\tilde{w}(y) \quad \text{on } I\!R^n, \tag{2.5.17}$$

Therefore by the uniqueness of solution of problem (2.5.17), (see Aubin [A]), we know that $\tilde{w}$ is of the form $\dfrac{\gamma_n}{(1+|x|^2)^{\frac{n-2}{2}}}$, where $\gamma_n$ is a positive constant depending only on $n$.

Now, we set

$$w_j(x) = v_j(x) - \frac{1}{r_j^{\frac{n-2}{2}}} w\left(\frac{x - x_j}{r_j}\right), \quad x \in \Omega.$$

We have

$$E(w_j) = E(u_j) - E(u) - E\left(\frac{1}{r_j^{\frac{n-2}{2}}} w\left(\frac{x - x_j}{r_j}\right)\right) + o(1) \tag{2.5.18}$$

and

$$\begin{aligned} dE(w_j) &= dE(u_j) - dE(u) - dE\left(\frac{1}{r_j^{\frac{n-2}{2}}} w\left(\frac{x-x_j}{r_j}\right)\right) + o(1) \\ dE(w_j) &= o(1). \end{aligned} \tag{2.5.19}$$

On other hand, we see that

$$E\left(\frac{1}{r_j^{\frac{n-2}{2}}} w\left(\frac{x - x_j}{r_j}\right)\right) = \frac{1}{2}\int_{I\!R^n} \frac{p(x)}{r_j^n}|\nabla w|^2(\frac{x - x_j}{r_j})dx - \frac{1}{q}\int_{I\!R^n} \frac{|w(\frac{x-x_j}{r_j})|^q}{r_j^n}dx.$$

Using the change of variable $y = \frac{x-x_j}{r_j}$, we write

$$E\left(\frac{1}{r_j^{\frac{n-2}{2}}}\, w\left(\frac{x-x_j}{r_j}\right)\right) = \frac{1}{2}\int_{\mathbb{R}^n} p(x_j + r_j y)|\nabla w(y)|^2 dy - \frac{1}{q}\int_{\mathbb{R}^n} |w(y)|^q dy.$$

Since $p$ is a continuous function and $x_j \to a$ in $\bar{\Omega}$, applying the Dominated Convergence Theorem, we obtain

$$E\left(\frac{1}{r_j^{\frac{n-2}{2}}}\, w\left(\frac{x-x_j}{r_j}\right)\right) = \frac{1}{2}p_0 \int_{\mathbb{R}^n} |\nabla w(y)|^2 dy - \frac{1}{q}\int_{\mathbb{R}^n} |w(y)|^q dy + o(1).$$

Now, we use the fact that $w = (p_0)^{\frac{1}{q-2}}\tilde{w}$, we obtain

$$E\left(\frac{1}{r_j^{\frac{n-2}{2}}}\, w\left(\frac{x-x_j}{r_j}\right)\right) = (p_0)^{\frac{q}{q-2}}\left[\frac{1}{2}\int_{\mathbb{R}^n} |\nabla \tilde{w}(y)|^2 dy - \frac{1}{q}\int_{\mathbb{R}^n} |\tilde{w}(y)|^q dy\right] + o(1).$$

By definition of $\tilde{w}$, we know that $\left[\frac{1}{2}\int_{\mathbb{R}^n} |\nabla \tilde{w}(y)|^2 dy - \frac{1}{q}\int_{\mathbb{R}^n} |\tilde{w}(y)|^q dy\right] = \frac{1}{n}S^{\frac{n}{2}}$. Therefore, we deduce that

$$E\left(\frac{1}{r_j^{\frac{n-2}{2}}}\, w\left(\frac{x-x_j}{r_j}\right)\right) = \frac{1}{n}(p_0 S)^{\frac{n}{2}}. \tag{2.5.20}$$

Inserting (2.5.20) into (2.5.18), we get

$$\begin{aligned} E(w_j) &= E(u_j) - E(u) - E\left(\frac{1}{r_j^{\frac{n-2}{2}}}\, w\left(\frac{x-x_j}{r_j}\right)\right) + o(1) \\ &= c - E(u) - \tfrac{1}{n}(p_0 S)^{\frac{n}{2}} + o(1). \end{aligned}$$

From the assumption in Theorem A 2, namely, $c \in ]\frac{1}{n}(p_0 S)^{\frac{n}{2}}, \frac{2}{n}(p_0 S)^{\frac{n}{2}}[$, and the fact that $E(u) \geq 0$ we have

$$c - E(u) - \frac{1}{n}(p_0 S)^{\frac{n}{2}} < \frac{1}{n}(p_0 S)^{\frac{n}{2}}. \tag{2.5.21}$$

Therefore, we have

$$E(w_j) \to l = c - E(u) - \tfrac{1}{n}(p_0 S)^{\frac{n}{2}} < \tfrac{1}{n}(p_0 S)^{\frac{n}{2}}$$

and

$$dE(w_j) \to 0 \quad \text{in } H^{-1}.$$

Applying Lemma 2.5.1, we deduce that $l = 0$ and

$$c = E(u) + \frac{1}{n}(p_0 S)^{\frac{n}{2}}. \tag{2.5.22}$$

We distinguish two cases:

**Case 1**: If $u = 0$, then $E(u) = 0$ and (2.5.22) becomes $c = \frac{1}{n}(p_0 S)^{\frac{n}{2}}$ which is a contradiction since $c \in ]\frac{1}{n}(p_0 S)^{\frac{n}{2}}, \frac{2}{n}(p_0 S)^{\frac{n}{2}}[$.

**Case 2**: If $u \neq 0$, then, since $c < \frac{2}{n}(p_0 S)^{\frac{n}{2}}$, from (2.5.22), we see that $E(u) < \frac{1}{n}(p_0 S)^{\frac{n}{2}}$ which is a contradiction.

Indeed, since $u_j \rightharpoonup u$, by the weak continuity of $dE$, we have $dE(u) = 0$.

Multiplying this last equality by $u$, we obtain

$$\int_\Omega p(x)|\nabla u|^2 dx = \int_\Omega |u|^q dx.$$

Therefore, since $u \neq 0$, we have

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 dx - \frac{1}{q}\int_\Omega |u|^q dx \\ &= \frac{1}{n}\int_\Omega p(x)|\nabla u|^2 dx \\ &= \frac{1}{n}\left(\frac{\int_\Omega p(x)|\nabla u|^2 dx}{\left(\int_\Omega |u|^q dx\right)^{\frac{2}{q}}}\right)^{\frac{n}{2}} \\ &> \frac{1}{n}(p_0 S)^{\frac{n}{2}}. \end{aligned}$$

Conclusion: $\{u_j\}$, for a subsequence, converge strongly to $u$.

This finishes the proof of Theorem A 2. □

# Part II

# Problèmes de Neumann non linéaires avec une non-linéarité critique au bord

# Chapter 3

# On some nonlinear Neumann problem with weight and with critical Sobolev trace maps

(This work is a paper in preparation)

## 3.1 Introduction

In this paper, we study the existence of solutions to the problem

$$(3.1.1) \qquad \begin{cases} -div(p(x)\nabla u) = \lambda u + \alpha |u|^{r-2}u & \text{in } \Omega, \\ u \not\equiv 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset I\!R^N$, $N \geq 3$, is a bounded domain with the smooth boundary $\partial\Omega$, $\nu$ is the outer normal on $\partial\Omega$, the coefficient $Q$ is continuous and positive on $\partial\Omega$ and the coefficient $p \in H^1(\Omega)$ is continuous and positive in $\bar{\Omega}$. $q = \frac{2(N-1)}{N-2}$, $2 < r < q$, $\lambda \in I\!R \setminus \{0\}$ and $\alpha \in I\!R$. The exponent $q$ is the critical Sobolev exponent for the trace embedding of the space $H^1(\Omega)$ into $L^q(\partial\Omega)$. The embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$ is continuous, but not compact.
The case $p \equiv 1 \equiv Q$ and $\alpha = 0$ has an extensive literature and we refer to [AM], [AY], [CK] and [CY]. It was shown that the mean curvature of $\partial\Omega$ and the geometrical condition (g.c) at some point on $\partial\Omega$ play an important role in solving similar problems of (3.1.1).
The condition (g.c) is defined as follows (see [AM] and [AY]),
(g.c): $\partial\Omega$ is said to satisfy the geometrical condition at $y \in \partial\Omega$ if there exists a neighborhood $U(y)$ of $y$ such that $\Omega \cap U(y)$ lies on one side of the tangent plane at

$y$.
Let us remark that there always exists at least one point $y \in \partial\Omega$ for which (g.c) is satisfied. Let $x_0 \in \partial\Omega$ satisfy

$$\frac{p(x_0)}{(Q(x_0))^{N-2}} = \min_{x\in\partial\Omega} \frac{p(x)}{(Q(x))^{N-2}}.$$

We assume that

$$|p(x) - p(x_0)| = o(|x - x_0|) \tag{3.1.2}$$

and

$$|Q(x) - Q(x_0)| = o(|x - x_0|) \tag{3.1.3}$$

for $x$ near $x_0$.
Throughout this paper, we suppose that $\partial\Omega$ satisfies the geometrical condition (g.c) at the point $x_0$. This condition guarantees that all the principal curvatures at $x_0$ with respect to unit outward normal are non-negative.
By $\{\lambda_k(p)\}$ we denote the sequence of eigenvalues for $-div(p(x)\nabla.)$ with the Neumann boundary conditions

$$\begin{cases} -div(p(x)\nabla u) = \mu u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that $0 = \lambda_1(p) < \lambda_2(p) \leq ....$
In this paper, we treat the case where $\lambda < 0$ and the case where $\lambda_{k-1}(p) < \lambda < \lambda_k(p)$ for some $k \in \mathbf{N} \setminus \{0\}$.
Our main result is the following

**Theorem 3.1.1.**
*Let $\Omega \subset I\!R^N$, $N \geq 3$ such that $\partial\Omega$ satisfies the geometrical condition (g.c) at $x_0$ and $H(x_0) > 0$. We assume that $p$ and $Q$ respectively satisfy (3.1.2) and(3.1.3). Then*
*1) for every $\lambda < 0$ and for every $\alpha \in I\!R$ there exists a solution of (3.1.1),*
*2) for every $\lambda \in ]\lambda_{k-1}(p), \lambda_k(p)[$ and for every $\alpha \geq 0$ the problem (3.1.1) possesses a solution.*

The rest of this paper is divided into three sections. In Section 1 we give some notations and some estimates which will be used throughout this paper. Section 2, Section 3 and section 4 are devoted respectively to the study of the case $\lambda \in ]\lambda_{k-1}(p), \lambda_k(p)[$ and $\alpha \geq 0$, the case $\lambda < 0$ and $\alpha \geq 0$ and the case $\lambda < 0$ and $\alpha \leq 0$.

## 3.2 Preliminary and notations

Let

$$S_1 = \inf\left\{\int_{\mathbb{R}^N_+} |\nabla u|^2 dx;\ u \in H^1(\mathbb{R}^N_+),\ \int_{\partial\mathbb{R}^N_+} |u|^q dx = 1\right\}$$

be the best constant in the trace embedding for the space $H^1(\mathbb{R}^N_+)$ into $L^q(\partial\mathbb{R}^N_+)$, where $R^N_+ = \{x = (x', x_N) : x' \in \mathbb{R}^{N-1},\ x_N > 0\}$. Due to a result of [L], $S_1$ is achieved and from a result of [E], $S_1$ is given by

$$S_1 = \frac{N-2}{2}(\omega_N)^{\frac{1}{N-1}}, \quad \text{where } \omega_N \text{ is the area of } S^{N-1}.$$

The minimizing functions are of the form

$$W(x) = \frac{\gamma_N}{\left[|x'|^2 + (1+x_N)^2\right]^{\frac{N-2}{2}}}, \tag{3.2.1}$$

where $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $\gamma_N$ is a positive constant depending on $N$. We set

$$W_{\varepsilon,x_0}(x) = \varepsilon^{-\frac{N-2}{2}}\phi(x)W\left(\frac{x-x_0}{\varepsilon}\right),$$

where $x_0 \in \partial\Omega$ and $\phi$ is a radial $C^\infty$-function such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x-x_0| \le \frac{R}{4} \\ 0 & \text{if } |x-x_0| > \frac{R}{2} \end{cases}$$

with $R > 0$ is a small constant.

Using (3.1.2) and (3.1.3), from [AY], we have the following estimates

(3.2.2)

$$\int_\Omega p(x)|\nabla W_{\varepsilon,x_0}|^2 dx = p(x_0)A_1 - p(x_0)H(x_0)\begin{cases} A_2'\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } N = 3 \\ A_2\varepsilon + o(\varepsilon) & \text{if } N \ge 4, \end{cases}$$

$$\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x = p(x_0)Q(x_0)(B_1 - H(x_0)B_2\varepsilon) + o(\varepsilon) \tag{3.2.3}$$

and

$$\int_\Omega |W_{\varepsilon,x_0}|^2 dx = \begin{cases} O(\varepsilon) & \text{if } N = 3 \\ O(\varepsilon^2|\log\varepsilon|) & \text{if } N = 4 \\ D\varepsilon^2 + o(\varepsilon^2) & \text{if } N \ge 5. \end{cases} \tag{3.2.4}$$

where $A_1 = \dfrac{(\Gamma(\frac{N-1}{2}))^2}{2\Gamma(N-2)}\omega_{N-1}$, $A_2' = \dfrac{\omega_2}{2}$, $A_2 = \dfrac{(N-1)(N-2)\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{8\Gamma(N-2)}\omega_{N-1}$,

$B_1 = \dfrac{(\Gamma(\frac{N-1}{2}))^2}{2\Gamma(N-1)}\omega_{N-2}$, $B_2 = \dfrac{(N-1)(\Gamma(\frac{N-1}{2}))^2}{4\Gamma(N-1)}\omega_{N-2}$ and $D = \dfrac{\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{2(N-4)\Gamma(N-2)}\omega_{N-1}$.
The function $\Gamma$ is defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt \quad \text{for every } z \in ]0,+\infty[.$$

Let us notice that

$$S_1 = \frac{A_1}{B_1^{\frac{2}{q}}}.$$

We claim that, as $\varepsilon \to 0$, we have

$$\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^{q-1}ds_x = p(x_0)Q(x_0)K_1\varepsilon^{\frac{N-2}{2}} + o(\varepsilon^{\frac{N-2}{2}}). \tag{3.2.5}$$

and

$$\int_\Omega |W_{\varepsilon,x_0}|^r dx = \begin{cases} o(\varepsilon|\log\varepsilon|) & \text{if } N=3 \\ o(\varepsilon) & \text{if } N \geq 4. \end{cases} \tag{3.2.6}$$

The proof of these two claims are given in the appendix.
The norm in the Lebesgue space $L^t(\Omega)$ is denoted by $\|.\|_t$. We endow the Sobolev space $H^1(\Omega)$ with the norm

$$\|u\| = \left[\int_\Omega (|\nabla u|^2 + |u|^2)dx\right]^{\frac{1}{2}}.$$

## 3.3 Case $\lambda_{k-1}(p) < \lambda < \lambda_k(p)$ and $\alpha \geq 0$

The main result of this section is the following

**Theorem 3.3.1.**
*Let $\Omega \subset I\!\!R^N$, $N \geq 3$ such that $\partial\Omega$ satisfies the geometrical condition (g.c) at $x_0$ and such that $H(x_0) > 0$. We assume that $p$ and $Q$ respectively satisfy (3.1.2) and(3.1.3). Then for every $\lambda \in ]\lambda_{k-1}(p), \lambda_k(p)[$ and for every $\alpha \geq 0$, the problem (3.1.1) has a solution.*

The proof of Theorem 3.3.1 relies on the following min-max principle based on a topological linking, see [W].

**Theorem 3.3.2.**
*Let $X = E^- \oplus E^+$ be a Banach space with $\dim E^- < \infty$. Let $\rho > \theta > 0$ and let $w \in E^+$ be such that $\|w\| = \theta$. Define*

$$M := \{u = v + sw \,:\, \|u\| \leq \rho,\, s \geq 0,\, v \in E^-\},$$
$$M_0 := \{u = v + sw \,:\, v \in E^-, \|u\| = \rho \text{ and } s \geq 0 \text{ or } \|u\| \leq \rho \text{ and } s = 0\},$$
$$N := \{u \in E^+ \,:\, \|u\| = \theta\}.$$

*Let $\varphi \in C^1(X, I\!R)$ be such that*

$$b := \inf_N \varphi > a := \max_{M_0} \varphi.$$

*If $\varphi$ satisfies the Palais-Smale condition with*

$$c := \inf\nolimits_{\gamma \in \Gamma} \max\nolimits_{u \in M} \varphi(\gamma(u)), \quad \text{where } \Gamma := \{\gamma \in C(M, X) \,:\, \gamma_{|M_0} = id\}.$$

*Then $c$ is a critical value of $\varphi$.*

Let $I_\alpha$ be a variational functional for (3.1.1) given by

$$I_\alpha(u) = \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{\alpha}{r}\int_\Omega |u|^r dx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x.$$

We have the following lemma

**Lemma 3.3.1.**
*Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying*

$$I_\alpha(u_n) \to c < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1} \tag{3.3.1}$$

*and*

$$I'_\alpha(u_n) \to 0 \ \text{ in } \ H^{-1}(\Omega). \tag{3.3.2}$$

*Then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.*

*Proof.* We start by showing that $\{u_n\}$ is bounded in $H^1(\Omega)$. Arguing by contradiction, we assume that $\|u_n\| \to \infty$. We set $v_n = \frac{u_n}{\|u_n\|}$. Since $\{v_n\}$ is bounded in $H^1(\Omega)$, we may assume that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$.
Using (3.3.2), for every $\phi \in H^1(\Omega)$, we obtain
(3.3.3)

$$\int_\Omega (p(x)\nabla v_n \nabla\phi - \lambda v_n\phi)\, dx = \|u_n\|^{-1}\left[\int_{\partial\Omega} Q(x)|u_n|^{q-2}u_n\phi ds_x + \alpha\int_\Omega |u_n|^{r-2}u_n\phi dx\right] + o(1).$$

On the other hand, we have

$$\int_{\partial\Omega} p(x)Q(x)|u_n|^{q-2}u_n\phi ds_x \le C_1 \left(\int_{\partial\Omega} |u_n|^q ds_x\right)^{\frac{q-1}{q}} \left(\int_{\partial\Omega} |\phi|^q ds_x\right)^{\frac{1}{q}} \tag{3.3.4}$$

and

$$\int_{\Omega} |u_n|^{r-2}u_n\phi dx \le \left(\int_{\Omega} |u_n|^r dx\right)^{\frac{r-1}{r}} \left(\int_{\Omega} |\phi|^r dx\right)^{\frac{1}{r}}. \tag{3.3.5}$$

Combining the fact that $H^1(\Omega) \subset L^r(\Omega)$ with the relations (3.3.1) and (3.3.2), an easy computation yields

$$\frac{1}{2(N-1)}\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \le c + \alpha(\frac{1}{r} - \frac{1}{2})\int_{\Omega} |u_n|^r dx + o(\|u_n\|).$$

Since $\alpha \ge 0$ and $2 < r < q$, there exists a positive constant $C_2$ such that

$$\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \le C_2 + o(\|u_n\|). \tag{3.3.6}$$

Inserting (3.3.4), (3.3.5) and (3.3.6) into (3.3.3), we deduce that, for a positive constant $C_3$ we have

$$\int_{\Omega} (p(x)\nabla v_n \nabla\phi - \lambda v_n\phi)\, dx \le C_3 \left[\|u_n\|^{\frac{(q-1)}{q}-1} + \|u_n\|^{-\frac{1}{r}}\right] + o(1).$$

We obtain after letting $n \to \infty$

$$\int_{\Omega} (p(x)\nabla v \nabla\phi - \lambda v\phi)\, dx = 0$$

for every $\phi \in H^1(\Omega)$. Since $\lambda \in ]\lambda_{k-1}(p), \lambda_k(p)[$, we see that $v \equiv 0$ on $\Omega$. Consequently, we may assume that $v_n \to 0$ strongly in $L^2(\Omega)$ and in $L^r(\Omega)$ since $r < q < \frac{2N}{N-2}$. Hence, from (3.3.1) and (3.3.2), we can write

$$\frac{1}{2}\int_{\Omega} p(x)|\nabla v_n|^2 dx = \frac{\|u_n\|^{q-2}}{q}\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1)$$

and

$$\int_{\Omega} p(x)|\nabla v_n|^2 dx = \|v_n\|^{q-2}\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1).$$

These two relations imply that $\int_{\Omega} p(x)|\nabla v_n|^2 dx = o(1)$ and thus $v_n \to 0$ strongly in $H^1(\Omega)$, which is impossible since $\|v_n\| = 1$. Consequently $\{u_n\}$ is bounded in $H^1(\Omega)$ and we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.

By the concentration-compactness principle [L], there exists some, at most countable, set $J$ and a family $(x_j)_{j\in J}$ of distinct points in $\partial\Omega$, $(\mu_j)_{j\in J}$ and $(\nu_j)_{j\in J}$ in $]0, +\infty[$ such that

$$|\nabla u_n|^2 \rightharpoonup \mu \ge |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}$$

and

$$|u_n|^q \rightharpoonup |u|^q + \sum_{j\in J} \nu_j \delta_{x_j}$$

weakly in the sense of measures. Moreover, for every $j \in J$ we have $S_1(\nu_j)^{\frac{2}{q}} \leq \mu_j$.
Let $x_{j_0}$ be fixed. Testing (3.3.2) by a family of $C^1$-functions concentrating at $x_{j_0}$ we get

$$p(x_{j_0})\mu_{j_0} \leq p(x_{j_0})Q(x_{j_0})\nu_{j_0}.$$

If $\nu_{j_0} > 0$, then

$$\frac{S_1^{N-1}}{Q(x_{j_0})^{N-1}} \leq \nu_{j_0}. \tag{3.3.7}$$

On the other hand we have

$$I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle = \frac{1}{2(N-1)}\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x + \alpha(\frac{1}{2} - \frac{1}{r})\int_\Omega |u_n|^r dx + o(1).$$

Letting $n \to \infty$ we obtain

$$c = \frac{1}{2(N-1)}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + \alpha(\frac{1}{2} - \frac{1}{r})\int_\Omega |u|^r dx + \frac{1}{2(N-1)}\sum_{j\in J} p(x_j)Q(x_j)\nu_j,$$

Using (3.3.7), the fact that $\alpha \geq 0$, $2 < r < q$ and also $\min_{x\in\partial\Omega} \frac{p(x)}{(Q(x))^{N-2}} = \frac{p(x_0)}{Q(x_0)^{N-2}}$, we get

$$c \geq \frac{p(x_{j_0})S_1^{N-1}}{2(N-1)(Q(x_{j_0}))^{N-2}} \geq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$$

and we have a contradiction with the hypothesis $c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$. Hence $\nu_j = 0$ for every $j \in J$. This gives that $u_n \to u$ strongly in $L^q(\partial\Omega)$. Since $u_n \to u$ strongly in $L^2(\Omega)$ and in $L^r(\Omega)$, by testing (3.3.2) successively by $u_n$ and $u$, an easy computation yields

$$\lim_{n\to\infty}\int_\Omega p(x)|\nabla u_n|^2 dx = \int_\Omega p(x)|\nabla u|^2 dx.$$

Since $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$, we deduce that $u_n \to u$ strongly in $H^1(\Omega)$ and $\{u_n\}$ is relatively compact in $H^1(\Omega)$. □

We now apply the Theorem 3.3.2 to our case, with $X = H^1(\Omega)$, $\varphi = I_\alpha$ and $E^- = span\{e_1(p), ..., e_l(p)\}$, where $e_1(p), ..., e_l(p)$ are eigenfunctions corresponding to the eigenvalues $\lambda_1(p), ..., \lambda_{k-1}(p)$. We have the orthogonal decomposition $X = H^1(\Omega) = E^- \oplus E^+$.
We have the following

**Lemma 3.3.2.**
*There exist constants $\beta > 0$, $\rho > 0$ and $\rho > \theta$ such that*

$$I_\alpha(u) \geq \beta \ \textit{for all} \quad u \in N$$

*and*

$$I_\alpha(u) \leq 0 \quad \textit{for all } u \in M_0$$

*where $M_0 = \{u = v + sw \ : \ v \in E^-, \|u\| = \rho$ and $s \geq 0$ or $\|u\| \leq \rho$ and $s = 0\}$ and $N = \{u \in E^+ \ : \ \|u\| = \theta\}$.*

*Proof.*
We start by a useful remark: For every $u \in E^+$, we have

$$\int_\Omega p(x)|\nabla u|^2 dx \geq \lambda_k(p) \int_\Omega |u|^2 dx.$$

Therefore, there exists a positive constants $C_1$ and $C_2$, such that for every $u \in E^+$ we have

$$\int_\Omega p(x)|\nabla u|^2 dx \geq C_1\|u\|^2 \geq C_2 \int_\Omega p(x)|\nabla u|^2 dx. \tag{3.3.8}$$

Using (3.3.8) and Hölder's inequality, we write

$$\begin{aligned} I_\alpha(u) &= \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \frac{1}{\alpha}\int_\Omega |u|^r dx \\ &\geq \frac{(1 - \frac{\lambda}{\lambda_k(p)})}{2} C_1\|u\|^2 - C_3\|u\|^q - C_4\|u\|^r, \end{aligned}$$

where $C_3$ and $C_4$ are some positive constants.
Therefore

$$I_\alpha(u) \geq \|u\|^2 \left( \frac{(1 - \frac{\lambda}{\lambda_k(p)})}{2} C_1 - C_3\|u\|^{q-2} - C_4\|u\|^{r-2} \right)$$

and it is easy to see that there exists $\beta > 0$ and $\theta > 0$ such that

$$I_\alpha(u) \geq \beta \quad \forall u \in N.$$

On the other hand, we have that

$$I_\alpha(u) \to -\infty, \text{ as } \|u\| \to +\infty,$$

thus, there exists $\rho > \theta$ such that $\max\limits_{M_0} I_\alpha(u) \leq 0$. □

We now define

$$Z_\varepsilon = E^- \oplus I\!RW_{\varepsilon,x_0} = E^- \oplus I\!RW^+_{\varepsilon,x_0},$$

where $W^+_{\varepsilon,x_0}$ denotes the projection of $W_{\varepsilon,x_0}$ onto $E^+$. From now on, we use $W^+_{\varepsilon,x_0}$ in place of $w$.

**Proof of Theorem** 3.3.1

We start by checking that $c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$.

Since $\alpha \geq 0$, we have

$$I(u) \leq \frac{1}{2}\left[\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx\right] - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x,$$

then

$$\max_{0\leq t<\infty} I(tu) \leq \frac{\left[\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx\right]^{N-1}}{2(N-1)\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{N-2}}$$

for $u \in H^1(\Omega)$ with $u \neq 0$ on $\partial\Omega$.

Set

$$m_\varepsilon = \sup_{\substack{u \in Z_\varepsilon \\ \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = 1}} \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx.$$

Therefore if

$$m_\varepsilon < \frac{(p(x_0))^{\frac{1}{N-1}} S_1}{2(N-1)(Q(x_0))^{\frac{N-2}{N-1}}}, \tag{3.3.9}$$

then

$$\sup_{u\in M} I(u) < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}.$$

Hence it is sufficient to show that (3.3.9) holds.

Suppose that $\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = 1$. We write $u = u^- + sW_{\varepsilon,x_0} = (u^- + sW^-_{\varepsilon,x_0}) + sW^+_{\varepsilon,x_0}$.

Since $\lambda \in ]\lambda_{k-1}(p), \lambda_k(p)[$, we have

$$\int_\Omega p(x)|\nabla W^-_{\varepsilon,x_0}|^2 - \lambda |W^-_{\varepsilon,x_0}|^2 dx \leq 0,$$

and

$$\int_\Omega p(x)|\nabla W^-_{\varepsilon,x_0}|^2 dx \leq \lambda \int_\Omega |W^-_{\varepsilon,x_0}|^2 dx \leq \int_\Omega |W_{\varepsilon,x_0}|^2 dx \to 0$$

as $\varepsilon \to 0$. Moreover

$$\int_{\partial\Omega} p(x)Q(x)|W^-_{\varepsilon,x_0}|^q ds_x \leq C\left(\int_\Omega (|\nabla W^-_{\varepsilon,x_0}|^2 + |W^-_{\varepsilon,x_0}|^2 dx\right)^{\frac{q}{2}} \to 0$$

as $\varepsilon \to 0$.

On the other hand, it is easy to see that $\int_{\partial\Omega} |u^-|^q ds_x \leq C_1$ and $0 < s < C_2$ with $C_1$

and $C_2$ are a positive constants.
Using the convexity of $u \to \int_{\partial\Omega} p(x)Q(x)|u|^q dx$, we can write

$$\begin{aligned} 1 &= \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \\ &\geq \int_{\partial\Omega} p(x)Q(x)|sW_{\varepsilon,x_0}|^q ds_x + q\int_{\partial\Omega} p(x)Q(x)u^-(sW_{\varepsilon,x_0})^{q-1} ds_x \\ &\geq \int_{\partial\Omega} p(x)Q(x)|sW_{\varepsilon,x_0}|^q ds_x - C_3\Big(\int_{\partial\Omega} |W_{\varepsilon,x_0}|^{q-1} ds_x\Big)^{\frac{1}{q}}\Big(\int_{\partial\Omega} |u^-|^q ds_x\Big)^{\frac{1}{q}} \end{aligned}$$

where $C_3$ is a positive constant.
Since $\int_{\partial\Omega} |W_{\varepsilon,x_0}|^{q-1} ds_x = O(\varepsilon^{\frac{N-2}{2}})$ (see (3.2.5)), we deduce from the above inequality that

$$\int_{\partial\Omega} p(x)Q(x)|sW_{\varepsilon,x_0}|^q dx \leq 1 + C_4\varepsilon^{\frac{N-2}{2}} \tag{3.3.10}$$

for some constant $C_4 > 0$.
Since all the norms on $E_k^-$ are equivalent, there exists a positive constant $C_5$ such that
(3.3.11)

$$\begin{aligned} \int_\Omega (p(x)Q(x)\nabla W_{\varepsilon,x_0}\nabla u^- - \lambda W_{\varepsilon,x_0}u^-)dx &\leq C_5(\|\nabla W_{\varepsilon,x_0}\|_1 + \lambda\|W_{\varepsilon,x_0}\|_1)\|u^-\| \\ &= O(\varepsilon^{\frac{N-2}{2}})\|u^-\|. \end{aligned}$$

Finally, using the above estimate, we can write

$$\begin{aligned} \int_\Omega p(x)|\nabla u|^2 - \lambda\int_\Omega |u|^2 dx &\leq \int_\Omega (p(x)|\nabla u^-|^2 - \lambda|u^-|^2)dx + O(\varepsilon^{\frac{N-2}{2}})\|u^-\| \\ &\quad + s^2\int_\Omega (p(x)|\nabla W_{\varepsilon,x_0}|^2 - \lambda|W_{\varepsilon,x_0}|^2)dx \\ &\leq (\lambda_{k-1}(p) - \lambda)\int_\Omega |u^-|^2 dx + O(\varepsilon^{\frac{N-2}{2}})\|u^-\| \\ &\quad + \frac{\int_\Omega (p(x)|\nabla W_{\varepsilon,x_0}|^2 - \lambda|W_{\varepsilon,x_0}|^2)dx}{\Big(\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x\Big)^{\frac{2}{q}}}\Big(s^q\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x\Big)^{\frac{2}{q}}. \end{aligned}$$

Combining (3.2.2), (3.2.3), (3.2.4) and (3.3.10), an easy computation yields $m_\varepsilon < \dfrac{(p(x_0))^{\frac{1}{N-1}} S_1}{2(N-1)(Q(x_0))^{\frac{N-2}{N-1}}}$ for $\varepsilon > 0$ sufficiently small.
Combining Lemma 3.3.2, Theorem 3.3.2 and Lemma 3.3.1, we deduce the conclusion of Theorem 3.3.1.

## 3.4 Case $\lambda < 0$ and $\alpha \geq 0$

The main result of this section is the following.

**Theorem 3.4.1.**
*Let $\Omega \subset I\!R^N$, $N \geq 3$ such that $\partial\Omega$ satisfies the geometrical condition (g.c) at $x_0$ and $H(x_0) > 0$. We assume that $p$ and $Q$ respectively satisfy (3.1.2) and(3.1.3). Then for every $\lambda < 0$ and for every $\alpha \geq 0$ there exists a solution of (3.1.1).*

The proof of Theorem 3.4.1 relies on the following variant of the mountain pass Theorem of Ambrosetti and Rabinowitz without the (PS) condition:

**Theorem 3.4.2.**
*Let $\Phi$ be a $C^1$ function on a Banach space $E$. Suppose that :*
(3.4.1)
*There exists a neighborhood $U$ of $0$ in $E$ and a constant $\rho$ such that $\Phi(u) \geq \rho$ for every $u$ in the boundary of $U$,*

(3.4.2) $$\Phi(0) < \rho \quad and \quad \Phi(v) < \rho \quad for\, some\, v \notin U.$$

*Set*

(3.4.3) $$c = \inf_{\mathcal{P} \in \mathcal{A}} \max_{w \in \mathcal{P}} \Phi(w) \geq \rho,$$

*where $\mathcal{A}$ denotes the class of continuous paths joining $0$ to $v$.*

*Then there is a sequence $\{u_j\}$ in $E$ such that*
$$\Phi(u_j) \to c \quad and \quad \Phi'(u_j) \to 0 \quad in \quad E^*.$$

In addition to Theorem 3.4.2, the proof of Theorem 3.4.1 requires the following two Lemmas

**Lemma 3.4.1.**
*Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying*

(3.4.4) $$I_\alpha(u_n) \to c < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1}$$

*and*

(3.4.5) $$I'_\alpha(u_n) \to 0 \quad in \quad H^{-1}(\Omega).$$

*Then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.*

The proof of Lemma 3.4.1 is identical to that of Lemma 3.3.1.

**Lemma 3.4.2.**
*There exists* $\varepsilon_0 > 0$ *such that*

$$\sup_{t\geq 0} I_\alpha(tW_{\varepsilon,x_0}) < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1}$$

*for every* $0 < \varepsilon < \varepsilon_0$.

*Proof.*
We have

$$\begin{aligned} I_\alpha(tW_{\varepsilon,x_0}) \ =& \frac{t^2}{2}\int_\Omega p(x)|\nabla W_{\varepsilon,x_0}|^2 dx - \frac{\lambda t^2}{2}\int_\Omega |W_{\varepsilon,x_0}|^2 dx - \frac{\alpha\, t^r}{r}\int_\Omega |W_{\varepsilon,x_0}|^r dx \\ &- \frac{t^q}{q}\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x. \end{aligned}$$

Set

$$I_0(tW_{\varepsilon,x_0}) = \frac{t^2}{2}\int_\Omega p(x)|\nabla W_{\varepsilon,x_0}|^2 dx - \frac{\lambda t^2}{2}\int_\Omega |W_{\varepsilon,x_0}|^2 dx - \frac{t^q}{q}\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x.$$

Since $\alpha \geq 0$, it is easy to see that

$$\sup_{t\geq 0} I_\alpha(tW_{\varepsilon,x_0}) \leq \sup_{t\geq 0} I_0(tW_{\varepsilon,x_0}). \tag{3.4.6}$$

Using (3.2.2), (3.2.3) and (3.2.4), an easy computation gives

$$I_0(tW_{\varepsilon,x_0}) = \begin{cases} \frac{t^2}{2}p(x_0)\left(A_1 - A_2'\varepsilon|\log\varepsilon|\right) - \frac{t^q}{q}p(x_0)Q(x_0)B_1 + o(\varepsilon|\log\varepsilon|) & \text{if } N=3 \\ \frac{t^2}{2}p(x_0)\left(A_1 - A_2\varepsilon\right) - \frac{t^q}{q}p(x_0)Q(x_0)\left(B_1 - B_2\varepsilon\right) + o(\varepsilon). & \text{if } N\geq 4 \end{cases}$$

Set

$$g(t) = \begin{cases} \frac{t^2}{2}p(x_0)\left(A_1 - A_2'\varepsilon|\log\varepsilon|\right) - \frac{t^q}{q}p(x_0)Q(x_0)B_1 & \text{if } N=3 \\ \frac{t^2}{2}p(x_0)\left(A_1 - A_2\varepsilon\right) - \frac{t^q}{q}p(x_0)Q(x_0)\left(B_1 - B_2\varepsilon\right). & \text{if } N\geq 4 \end{cases}$$

It is easy to see that $g$ achieves its maximum at

$$t_0 = \begin{cases} \left(\frac{A_1 - A_2'\varepsilon|\log\varepsilon|}{Q(x_0)B_1}\right)^{\frac{1}{q-2}} & \text{if } N=3 \\ \left(\frac{A_1 - A_2\varepsilon}{Q(x_0)(B_1 - B_2\varepsilon)}\right)^{\frac{1}{q-2}} & \text{if } N\geq 4. \end{cases}$$

By a standard computation we obtain

$$t_0 = \begin{cases} \left(\dfrac{A_1}{Q(x_0)B_1}\right)^{\frac{1}{q-2}}\left[1 - \dfrac{1}{q-2}\dfrac{A_2'}{A_1}\varepsilon|\log\varepsilon|\right] + o(\varepsilon|\log\varepsilon|) & \text{if } N=3 \\ \left(\dfrac{A_1}{Q(x_0)B_1}\right)^{\frac{1}{q-2}}\left[1 - \dfrac{1}{(q-2)A_1}\left(A_2 - \dfrac{A_1B_2}{B_1}\right)\varepsilon\right] + o(\varepsilon) & \text{if } N\geq 4. \end{cases} \tag{3.4.7}$$

Hence, for $\varepsilon$ small enough we have

$$I_0(t_0W_{\varepsilon,x_0}) =$$

$$\begin{cases} \frac{1}{2(N-1)}\frac{p(x_0)}{(Q(x_0))^{N-2}}(\frac{A_1}{B_1^{\frac{2}{q}}})^{\frac{q}{q-2}} - \left(\frac{A_1}{Q(x_0)B_1}\right)^{\frac{2}{q-2}} p(x_0)A_2'\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } N=3, \\ \frac{1}{2(N-1)}\frac{p(x_0)}{(Q(x_0))^{N-2}}(\frac{A_1}{B_1^{\frac{2}{q}}})^{\frac{q}{q-2}} - \left(\frac{A_1}{Q(x_0)B_1}\right)^{\frac{2}{q-2}} \left[\frac{p(x_0)}{(q-2)}\left(A_2 - \frac{A_1B_2}{B_1}\right) + \frac{p(x_0)A_2}{2}\right]\varepsilon & \\ \quad + \left(\frac{A_1}{Q(x_0)B_1}\right)^{\frac{q}{q-2}} \left[\frac{p(x_0)Q(x_0)B_1}{(q-2)A_1}(A_2 - \frac{A_1B_2}{B_1}) + \frac{p(x_0)Q(x_0)B_2}{q}\right]\varepsilon + o(\varepsilon) & \text{if } N\geq 4. \end{cases}$$

Therefore
(3.4.8)

$$I_0(t_0W_{\varepsilon,x_0}) = \begin{cases} \frac{1}{4}\frac{p(x_0)}{(Q(x_0))}S_1^2 - \left(\frac{A_1}{Q(x_0)B_1}\right)^{\frac{2}{q-2}} A_2'\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } N=3, \\ \frac{1}{2(N-1)}\frac{p(x_0)}{(Q(x_0))^{N-2}}S_1^{N-1} - \frac{p(x_0)}{2}\left(A_2 - \frac{2}{q}\frac{A_1B_2}{B_1}\right)\varepsilon & \\ \qquad +o(\varepsilon) & \text{if } N\geq 4. \end{cases}$$

Using the fact that $\Gamma(z+1) = z\Gamma(z)$ $\forall z \in ]0,+\infty[$ together with the definitions of the constants $A_1$, $A_2$, $B_1$ and $B_2$, we get

$$A_2 - \frac{A_1B_2}{B_1} > 0. \tag{3.4.9}$$

Combining (3.4.9), the fact that $\frac{2}{q} < 1$ and (3.4.8) we deduce that there exists $\varepsilon_0 > 0$ such that

$$I_0(t_0W_{\varepsilon,x_0}) < \frac{1}{2(N-1)}\frac{p(x_0)}{(Q(x_0))^{N-2}}S_1^{N-1} \tag{3.4.10}$$

for $0 < \varepsilon < \varepsilon_0$.
Finally, this together with (3.4.6) gives the desired result. □

**Proof of Theorem** 3.4.1
We start by verifying the assumptions of the Theorem 3.4.2 with $E = H^1(\Omega)$ and $\Phi = I_\alpha$.
**Verification of (3.4.1)**:
Let $u \in H^1(\Omega)$, we have

$$I_\alpha(u) = \frac{1}{2}\int_\Omega p(x)|\nabla u|^2dx - \frac{\lambda}{2}\int_\Omega |u|^2dx - \frac{\alpha}{r}\int_\Omega |u|^rdx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^qds_x.$$

Using the embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$ and $L^r(\Omega)$ and the fact that the norm $\left(\int_\Omega p(x)|\nabla\,.\,|^2dx - \lambda\int_\Omega |\,.\,|^2dx\right)$ is equivalent to $\|.\|$ we obtain

$$I_\alpha(u) \geq M_1\|u\|^2 - \alpha M_2\|u\|^r - M_3\|u\|^q$$

where $M_1$, $M_2$ and $M_3$ are some positive constants.
This implies (3.4.1) with some $\rho > 0$ and $U$ a small ball in $H^1(\Omega)$.
**Verification of (3.4.2)**:
For any $u \in H^1(\Omega)$, $u \not\equiv 0$ on $\bar{\Omega}$, we have $\lim_{t\to+\infty} I_\alpha(tu) = -\infty$. Thus there are many $v$'s satisfying (3.4.2). However, it will be important for later purpose to use Theorem 3.4.2 with a special $v$, namely $v = t_1 W_{\varepsilon,x_0}$, where $t_1$ is chosen large enough so that $v \notin U$ and $I_\alpha(v) \leq 0$.
It follows from Lemma 3.4.2 that for $0 < \varepsilon < \varepsilon_0$ we have

$$\sup_{t\geq 0} I_\alpha(tv) < \frac{1}{2(N-1)}\frac{p(x_0)}{(Q(x_0))^{N-2}}S_1^{N-1}$$

and therefore we deduce

$$c < \frac{1}{2(N-1)}\frac{p(x_0)}{(Q(x_0))^{N-2}}S_1^{N-1}. \tag{3.4.11}$$

Applying Theorem 3.4.2 we obtain a sequence $\{u_n\}$ in $H^1(\Omega)$ such that $I_\alpha(u_n) \to c$ and $I'_\alpha(u_n) \to 0$ in $H^{-1}(\Omega)$.
From Lemma 3.4.1, for a subsequence of $\{u_n\}$ (which we still call $\{u_n\}$), we have $u_n \to u$ strongly in $H^1(\Omega)$ and $I_\alpha(u) = c$. Thus, $u$ is a solution of (3.1.1). This finishes the proof of Theorem 3.4.1.

## 3.5 Case $\lambda < 0$ and $\alpha \leq 0$

The main result of this section is the following

**Theorem 3.5.1.**
*Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ such that $\partial\Omega$ satisfies the geometric condition (g.c) at $x_0$ and $H(x_0) > 0$. We assume that $p$ and $Q$ respectively satisfy (3.1.2) and(3.1.3). Then for every $\lambda < 0$ and for every $\alpha \leq 0$ there exists a solution of (3.1.1).*

As in section 3, the proof of Theorem 3.5.1 is a consequence of the following two Lemmas and Theorem 3.4.2.

**Lemma 3.5.1.**
*Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying*

$$I_\alpha(u_n) \to c < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}}S_1^{N-1} \tag{3.5.1}$$

*and*

(3.5.2) $$I'_\alpha(u_n) \to 0 \quad in \;\; H^{-1}(\Omega).$$

*Then* $\{u_n\}$ *is relatively compact in* $H^1(\Omega)$.

*Proof.*
We rewrite (3.5.1) as
(3.5.3)
$$\frac{1}{2}\int_\Omega p(x)|\nabla u_n|^2dx-\frac{\lambda}{2}\int_\Omega |u_n|^2dx-\frac{\alpha}{r}\int_\Omega |u_n|^r dx-\frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x=c+o(1)$$

and multiplying (3.5.2) by $u_n$, we find
(3.5.4)
$$\int_\Omega p(x)|\nabla u_n|^2dx=\lambda\int_\Omega |u_n|^2dx+\alpha\int_\Omega |u_n|^r dx+\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x+\langle\varepsilon_n,u_n\rangle$$

with $\varepsilon_n \to 0$ in $H^{-1}(\Omega)$.
We start by showing that $(u_n)$ is bounded in $H^1(\Omega)$.
Computing (3.5.3)$-\frac{1}{q}$(3.5.4), we obtain

$$\frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u_n|^2dx-\lambda\int_\Omega |u_n|^2dx\right)=c+\alpha(\frac{1}{r}-\frac{1}{q})\int_\Omega |u_n|^r dx+\langle\varepsilon_n,u_n\rangle+o(1).$$

Since $\alpha \leq 0$ and $2 < r < q$, we have, for large $n$,

$$\begin{aligned}\frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u_n|^2dx-\lambda\int_\Omega |u_n|^2dx\right) &\leq c+\langle\varepsilon_n,u_n\rangle+o(1)\\ &\leq c+\|u_n\|+o(1).\end{aligned}$$

Since $\lambda < 0$, the norm $\left(\int_\Omega p(x)|\nabla .|^2dx-\lambda\int_\Omega |.|^2dx\right)^{\frac{1}{2}}$ is equivalent to $\|.\|$, therefore there exists a positive constant $C$ such that

$$\|u_n\| \leq C$$

and $\{u_n\}$ is bounded in $H^1(\Omega)$. Hence we may extract a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{aligned}u_n &\rightharpoonup u \quad \text{weakly in } H^1(\Omega),\\ u_n &\to u \quad \text{strongly in } L^2(\Omega),\\ u_n &\to u \quad \text{pointwise a.e in } \;\Omega,\\ u_n &\rightharpoonup u \quad \text{weakly in } L^q(\partial\Omega),\\ u_n &\to u \quad \text{pointwise a.e in } \;\partial\Omega,\end{aligned}$$

From (3.5.2) we have

$$\langle I'_\alpha(u), v\rangle = 0 \quad \text{for every } w \in H^1(\Omega)$$

and then $u$ satisfies

$$\begin{cases} -div(p(x)\nabla u) = \lambda u + \alpha |u|^{r-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

Hence we also have

$$\int_\Omega p(x)|\nabla u|^2 dx = \lambda \int_\Omega |u|^2 dx + \alpha \int_\Omega |u|^r dx + \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x. \tag{3.5.5}$$

We write

$$u_n = u + v_j.$$

From a result of [BL] we have

$$\int_{\partial\Omega} p(x)Q(x)|u_j|^q ds_x = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x + o(1). \tag{3.5.6}$$

Combining (3.5.3) with (3.5.6) and (3.5.4) with (3.5.6) we obtain

$$I_\alpha(u) + \frac{1}{2}\int_\Omega p(x)|\nabla v_j|^2 dx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x = c + o(1) \tag{3.5.7}$$

and

(3.5.8)

$$\begin{aligned} \int_\Omega p(x)|\nabla u|^2 dx + \int_\Omega p(x)|\nabla v_j|^2 dx \; &= \lambda \int_\Omega |u|^2 dx + \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \\ &+ \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x + \alpha \int_\Omega |u|^r dx + o(1). \end{aligned}$$

Then, using (3.5.5), we deduce that

$$\int_\Omega p(x)|\nabla v_j|^2 dx = \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x + o(1). \tag{3.5.9}$$

We may therefore assume that, as $n \to +\infty$

$$\int_\Omega p(x)|\nabla v_j|^2 dx \to l \quad \text{and} \quad \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x \to l.$$

From the result of [Z, Theorem 02] we know that there exists a constant $C(\Omega) > 0$, such that for every $w \in H^1(\Omega)$

$$\int_\Omega |\nabla w|^2 dx + C(\Omega)\int_\Omega |w|^k dx \geq S_1 \left(\int_{\partial\Omega} |w|^q dx\right)^{\frac{2}{q}}$$

with $k = \dfrac{2N}{N-1}$ if $N \geq 4$ and $k > 3 = \frac{2N}{N-1}$ if $N = 3$.
We apply this result for $w_n = (p(x))^{\frac{1}{2}} v_n$, and in particular for $N = 3$ we take $k$ such that $6 = \frac{2N}{N-2} > k > 3$, we obtain, for $n$ large enough

$$\begin{aligned}\int_\Omega p(x)|\nabla v_n|^2 dx + C(\Omega)\int_\Omega |(p(x))^{\frac{1}{2}} v_n|^k dx &= \int_\Omega |\nabla[(p(x))^{\frac{1}{2}} v_n]|^2 dx \\ &\quad + C(\Omega)\int_\Omega |(p(x))^{\frac{1}{2}} v_n|^k dx + o(1) \\ &\geq S_1 \left(\int_{\partial\Omega} |(p(x))^{\frac{1}{2}} v_n|^q ds_x\right)^{\frac{2}{q}} + o(1).\end{aligned}$$

Since $k < \frac{2N}{N-2}$ for every $N \geq 3$, thanks to the compact embedding $H^1(\Omega) \hookrightarrow L^k(\Omega)$ we have $v_n \to 0$ strongly in $L^k(\Omega)$, and we deduce

$$\int_\Omega p(x)|\nabla v_n|^2 dx + o(1) \geq S_1 \left(\int_{\partial\Omega} |(p(x))^{\frac{1}{2}} v_n|^q ds_x\right)^{\frac{2}{q}} + o(1).$$

Using the fact that

$$\frac{p(x)}{(Q(x))^{N-2}} \geq \frac{p(x_0)}{(Q(x_0))^{N-2}} \quad \forall x \in \partial\Omega,$$

an easy computation yields

$$\int_\Omega p(x)|\nabla v_n|^2 dx \geq \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x\right)^{\frac{2}{q}} + o(1) \tag{3.5.10}$$

and at the limit we have $l \geq S_1 l^{\frac{2}{q}}$. It follows that either $l = 0$ or $l \geq S_1^{N-1}$.
We shall now see that $l \geq S_1^{N-1}$ is excluded, which implies that $l = 0$, i.e., $u_n \to u$ strongly in $H^1(\Omega)$ and the proof of Lemma 3.5.1 will be complete.
Suppose that $l \geq S_1^{N-1}$. Passing to the limit in (3.5.7) we obtain

$$I_\alpha(u) + \frac{1}{2(N-1)} l = c$$

and with the assumption on $c$ in (3.5.1) we find

$$I_\alpha(u) < 0. \tag{3.5.11}$$

On the other hand we have, by definition of $I_\alpha$ and in view of (3.5.5),

$$\begin{aligned} I_\alpha(u) &= \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{\alpha}{r}\int_\Omega |u|^r dx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \\ &= \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) + \alpha\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega |u|^r dx. \end{aligned}$$

Since $\alpha \leq 0$ and $r < q$ we obtain

$$I_\alpha(u) \geq 0$$

which contradicts (3.5.11). This achieves the proof of Lemma 3.5.1. □

**Lemma 3.5.2.**
*There exists $\varepsilon_0 > 0$ such that*

$$\sup_{t \geq 0} I_\alpha(tW_{\varepsilon,x_0}) < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1}$$

*for every $0 < \varepsilon < \varepsilon_0$.*

*Proof.*
We have

$$\begin{aligned} I_\alpha(tW_{\varepsilon,x_0}) &= \frac{t^2}{2}\int_\Omega p(x)|\nabla W_{\varepsilon,x_0}|^2 dx - \frac{\lambda t^2}{2}\int_\Omega |W_{\varepsilon,x_0}|^2 dx - \frac{\alpha\, t^r}{r}\int_\Omega |W_{\varepsilon,x_0}|^r dx \\ &- \tfrac{t^q}{q}\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x. \end{aligned}$$

Using (3.2.2), (3.2.3), (3.2.4) and (3.2.6), an easy computation gives

$$I_\alpha(tW_{\varepsilon,x_0}) = \begin{cases} \tfrac{t^2}{2} p(x_0)\left(A_1 - A_2'\varepsilon|\log\varepsilon|\right) - \tfrac{t^q}{q} p(x_0)Q(x_0)B_1 + o(\varepsilon|\log\varepsilon|) & \text{if } N = 3 \\ \tfrac{t^2}{2} p(x_0)\left(A_1 - A_2\varepsilon\right) - \tfrac{t^q}{q} p(x_0)Q(x_0)\left(B_1 - B_2\varepsilon\right) + o(\varepsilon). & \text{if } N \geq 4. \end{cases}$$

For the rest of this proof, we proceed as in the proof of Lemma 3.4.2 and we obtain the desired result. □

Now, as in proof of Theorem 3.3.2, we apply Theorem 3.4.2 together with Lemma 3.5.1 and Lemma 3.5.2 we obtain the conclusion of Theorem 3.5.1.

Finally, combining Theorem 3.3.1, Theorem 3.3.2 and Theorem 3.4.1 we obtain the conclusion of Theorem 3.1.1.

## 3.6 Appendix

In this appendix we give the proof of (3.2.5) and (3.2.6).
Let $R > 0$ (see the definition of $R$ in Section 1), using the ideas developed in [AY],

we introduce the following notations

$$x = (x', x_N) \in I\!R^{N-1} \times I\!R \quad and \quad x_0 = (x'_0, x_{0N}),$$

$$B(x_0, R) = \{x \in I\!R^N,\ |x - x_0| < R\},$$

$$B(x_0, R) \cap \Omega = \{(x', x_N) \in B(x_0, R),\ x_N - x_{0N} > \rho(x' - x'_0)\},$$

$$B(x_0, R) \cap \partial\Omega = \{x', x_N) \in B(x_0, R),\ x_N - x_{0N} = \rho(x' - x'_0)\},$$

$$B(x_0, R)^+ = B(x_0, R) \cap \{x_N - x_{0N} > 0\}$$

$$\sum = \{(x', x_N) \in B(x_0, R),\ 0 < x_N - x_{0N} < \rho(x' - x'_0)\},$$

where

$$\rho(x' - x'_0) = \tfrac{1}{2} \textstyle\sum_{i=1}^{N-1} \alpha_i (x_i - x_{0i})^2 + o(|x' - x_0|^2),$$

where $\{\alpha_i\}_{1 \le i \le N-1}$ are the principal curvatures at $x_0$ and $H(x_0) = \frac{1}{n-1} \displaystyle\sum_{i=1}^{N-1} \alpha_i$.
Since $\partial\Omega$ satisfies (e.g), then $\rho(.) \geq 0$.
We recall the following relation

$$\int_0^{+\infty} \frac{r^\theta}{(1 + r^2)^\beta} dr = \frac{\Gamma(\frac{\theta+1}{2})\Gamma(\frac{2\beta-\theta-1}{2})}{2\Gamma(\beta)} \quad \text{for } 2\beta - \theta > 1. \tag{3.6.1}$$

**Proof of (3.2.5):**
Set $v(x) = p(x)Q(x)$, $x \in \bar{\Omega}$ and $v = 0$ outside $\partial\Omega$. Then we have

$$\begin{aligned} \int_{\partial\Omega} |W_{\varepsilon,x_0}|^{q-1} v\, ds_z &= \varepsilon^{\frac{N}{2}} \int_{|z'-x_0| \le a} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}} v(z', x_{0N} + \rho(z' - x'_0))}{[(\varepsilon + \rho(z' - x'_0))^2 + |z' - x'_0|^2]^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}) \\ &= \varepsilon^{\frac{N}{2}} \int_{|x'| \le a} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}} v(x'_0 + x', x_{0N} + \rho(x'))}{[(\varepsilon + \rho(x'))^2 + |x'|^2]^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}). \end{aligned}$$

Using Taylor's expansion of $\rho$ at $x'$, we obtain

$$\frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}}}{[(\varepsilon + \rho(x'))^2 + |x'|^2]^{\frac{N}{2}}} =$$

$$\frac{1 + \frac{1}{2}|\nabla\rho|^2 + o(|\nabla\rho|^2)}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} \left[ 1 - \frac{\frac{N}{2}\rho(x')(2\varepsilon + \rho(x'))}{(\varepsilon^2 + |x'|^2)} + o\left( \frac{\frac{N}{2}\rho(x')(2\varepsilon + \rho(x'))}{(\varepsilon^2 + |x'|^2)} \right) \right].$$

Consequently

$$\begin{aligned}
\varepsilon^{\frac{N}{2}}\int_{\partial\Omega}|W_{\varepsilon,y}|^{q-1}v\,ds_z =& \varepsilon^{\frac{N}{2}}\int_{|x'|\le a}\frac{v(x'_0+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' \\
&+\frac{\varepsilon^{\frac{N}{2}}}{2}\int_{|x'|\le a}\frac{|\nabla\rho|^2 v(x'_0+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' \\
&+\varepsilon^{\frac{N+2}{2}}\int_{|x'|\le a}\frac{N\rho(x')v(x'_0+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx' \\
&+o\left(\varepsilon^{\frac{N+2}{2}}\int_{|x'|\le a}\frac{N\rho(x')v(x'_0+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx'\right) \\
&= T_1+T_2+T_3+o(T_3)+O(\varepsilon^{\frac{N}{2}}).
\end{aligned}$$

We have

$$\begin{aligned}
T_1 &= \varepsilon^{\frac{N}{2}}\int_{|x'|\le a}\frac{v(x'_0+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' \\
&= \varepsilon^{\frac{N}{2}}\int_{I\!R^{N-1}}\frac{v(x'_0+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx'+O(\varepsilon^{\frac{N}{2}}) \\
&= \varepsilon^{\frac{N-2}{2}}v(x_0)\int_{I\!R^{N-1}}\frac{1}{(1+|z'|^2)^{\frac{N}{2}}}dz'+o(\varepsilon^{\frac{N-2}{2}})
\end{aligned}$$

By changing to polar coordinates and using (3.6.1), we get

$$T_1=\omega_{N-2}\varepsilon^{\frac{N-2}{2}}v(x_0)\int_0^{+\infty}\frac{r^{N-2}}{(1+r^2)^{\frac{N}{2}}}dr+o(\varepsilon^{\frac{N-2}{2}}),$$

$$T_1=\frac{\omega_{N-2}\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})v(x_0)\varepsilon^{\frac{N-2}{2}}}{2\Gamma(\frac{N}{2})}+o(\varepsilon^{\frac{N-2}{2}}). \tag{3.6.2}$$

Since $|\nabla\rho|^2=\sum_{i=1}^{N-1}\alpha_i^2x_i^2+o(|x'|^2)$, then

$$\begin{aligned}
T_2 &= \varepsilon^{\frac{N}{2}}\sum_{i=1}^{N-1}\alpha_i^2\int_{|x'|^2\le a}\frac{x_i^2v(x'_0+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' \\
&+ o\left(\varepsilon^{\frac{N}{2}}\int_{|x'|^2\le a}\frac{x_i^2v(x'_0+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx'\right).
\end{aligned}$$

We have

$$\varepsilon^{\frac{N}{2}}\sum_{i=1}^{N-1}\alpha_i^2\int_{|x'|\le a}\frac{x_i^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx'$$

$$\le \frac{\sum_{i=1}^{N-1}\alpha_i\varepsilon^{\frac{N}{2}}}{(N-1)}\int_{|x'|\le a}\frac{|x'|^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx'.$$

Applying the Dominated Convergence Theorem we find that

$$\int_{|x'|\le a}\frac{|x'|^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' = O(1)$$

and we deduce that

$$T_2 = O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2}{2}}). \tag{3.6.3}$$

By definition of $\rho$, we have

$$T_3 = \varepsilon^{\frac{N+2}{2}}\frac{N}{2}\sum_{i=1}^{N-1}\alpha_i\int_{|x'|\le a}\frac{x_i^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx'$$

$$+o\left(\varepsilon^{\frac{N+2}{2}}\int_{|x'|\le a}\frac{x_i^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx'\right).$$

We have

$$\varepsilon^{\frac{N+2}{2}}\frac{N}{2}\sum_{i=1}^{N-1}\alpha_i\int_{|x'|\le a}\frac{x_i^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx'$$

$$\le \frac{N\sum_{i=1}^{N-1}\alpha_i}{2(N-1)}\varepsilon^{\frac{N}{2}}v(x_0)\int_0^{\frac{a}{\varepsilon}}\frac{r^N}{(1+r^2)^{\frac{N+2}{2}}}dr$$

$$= \frac{N\,v(x_0)\sum_{i=1}^{N-1}\alpha_i}{2(N-1)}\varepsilon^{\frac{N}{2}}\int_0^{+\infty}\frac{r^N}{(1+r^2)^{\frac{N+2}{2}}}dr + o(\varepsilon^{\frac{N}{2}})$$

$$= \frac{N\,v(x_0)\Gamma(\frac{N+1}{2})\Gamma(\frac{1}{2})\sum_{i=1}^{N-1}\alpha_i}{4(N-1)\Gamma(\frac{N+2}{2})}\varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}).$$

Thus,

$$T_3 \le \frac{Nv(x_0)\Gamma(\frac{N+1}{2})\Gamma(\frac{1}{2})\sum_{i=1}^{N-1}\alpha_i}{4(N-1)\Gamma(\frac{N+2}{2})}\varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2}{2}}). \tag{3.6.4}$$

Finally, combining (3.6.2), (3.6.3) and (3.6.4) we obtain

$$\int_{\partial\Omega}|W_{\varepsilon,x_0}|^{q-1}v\,ds_z = \frac{\omega_{N-2}\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})u(x_0)\varepsilon^{\frac{N-2}{2}}}{2\Gamma(\frac{N}{2})} + o(\varepsilon^{\frac{N-2}{2}}).$$

**Proof of (3.2.6):**
We have

$$\int_{\Omega}|W_{\varepsilon,x_0}|^r dx = \varepsilon^{\frac{r(N-2)}{2}}\int_{B(x_0,R)^+}\frac{1}{[(\varepsilon+x_N-x_{0N})^2+|x'-x'_0|^2]^{\frac{r(N-2)}{2}}}dx$$

$$-\varepsilon^{\frac{r(N-2)}{2}}\int_{\Sigma}\frac{1}{[(\varepsilon+x_N-x_{0N})^2+|x'-x'_0|^2]^{\frac{r(N-2)}{2}}}dx+O(\varepsilon^{\frac{r(N-2)}{2}})$$

$$= L_1-L_2+O(\varepsilon^{\frac{r(N-2)}{2}}).$$

Now, in order to estimate $L_1$ we distinguish two cases:

**Case $N\geq 4$:**
By a change of variables and using polar coordinates, we obtain

$$L_1 = \varepsilon^{\frac{2N-r(N-2)}{2}}\int_{\mathbb{R}^N}\frac{1}{[(1+x_N)^2+|x'|^2]^{\frac{r(N-2)}{2}}}dx$$

$$+o(\varepsilon^{\frac{2N-r(N-2)}{2}})$$

$$= \varepsilon^{\frac{2N-r(N-2)}{2}}\left(\int_0^{+\infty}\frac{1}{(1+x_N)^{r(N-2)-(N-1)}}dx_N\right)\left(\int_0^{+\infty}\frac{\omega_{N-2}t^{N-2}}{(1+t^2)^{\frac{r(N-2)}{2}}}dt\right)$$

$$+o(\varepsilon^{\frac{2N-r(N-2)}{2}}).$$

Since $2<r<q$, we get

$$L_1 = \varepsilon^{\frac{2N-r(N-2)}{2}}\frac{\omega_{N-2}}{N-2}\int_0^{+\infty}\frac{\omega_{N-2}t^{N-2}}{(1+t^2)^{\frac{r(N-2)}{2}}}dt+o(\varepsilon^{\frac{2N-r(N-2)}{2}}) \tag{3.6.5}$$

$$= o(\varepsilon).$$

**Case $N=3$:**
Here $q=4$ and $2<r<4$. The cases $r>3$, $r=3$ and $r<3$ are different, therefore we distinguish three cases.

**If $r>3$:**
As in the above estimates we have

$$L_1 = \varepsilon^{\frac{6-r}{2}}\int_{\mathbb{R}^3}\frac{1}{[(1+x_3)^2+|x'|^2]^{\frac{r}{2}}}dx+o(\varepsilon^{\frac{6-r}{2}})$$

$$= \varepsilon^{\frac{6-r}{2}}\left(\int_0^{+\infty}\frac{1}{(1+x_3)^{r-2}}dx_3\right)\left(\int_0^{+\infty}\frac{\omega_1 t}{(1+t^2)^{\frac{r}{2}}}dt\right)+o(\varepsilon^{\frac{6-r}{2}})=o(\varepsilon)$$

and

$$L_1=o(\varepsilon)=o(\varepsilon|\log\varepsilon|). \tag{3.6.6}$$

**If $r = 3$:**

We have

$$\begin{aligned} L_1 &= \varepsilon^{\frac{3}{2}} \int_{B^+(x_0,R)} \frac{1}{[(\varepsilon + x_N - x_{0N})^2 + |x' - x_0'|^2]^{\frac{3}{2}}} dx \\ &= \varepsilon^{\frac{3}{2}} \int_0^a \int_{\{|y'| \le a\}} \frac{1}{[(\varepsilon + y_N)^2 + |y'|^2]^{\frac{3}{2}}} dy' dy_N \\ &= \varepsilon^{\frac{3}{2}} \int_0^a \frac{1}{(\varepsilon + y_N)} \int_0^{\frac{a}{\varepsilon + y_N}} \frac{t}{(1+t^2)^{\frac{3}{2}}} dt dy_N \\ &= \varepsilon^{\frac{3}{2}} \int_0^a \frac{1}{\varepsilon + y_N} dy_N - \varepsilon^{\frac{3}{2}} \int_0^a \frac{1}{(a^2 + (\varepsilon + y_N)^2)^{\frac{1}{2}}} dy_N \end{aligned}$$

and

(3.6.7) $$L_1 = o(\varepsilon |\log \varepsilon|).$$

**If $r < 3$:**

Using the Dominated Convergence Theorem, we obtain

$$\begin{aligned} L_1 &= \varepsilon^{\frac{r}{2}} \int_{B^+(x_0,R)} \frac{1}{[(\varepsilon + x_N - x_{0N})^2 + |x' - x_0'|^2]^{\frac{r}{2}}} dx \\ &= \varepsilon^{\frac{r}{2}} \int_{B^+(x_0,R)} \frac{1}{|x - x_0|^r} dx + o(\varepsilon^{\frac{r}{2}}) \end{aligned}$$

and

(3.6.8) $$L_1 = o(\varepsilon |\log \varepsilon|).$$

Now, we go back to estimate $L_2$.

We have

$$\begin{aligned} L_2 &= \varepsilon^{\frac{r(N-2)}{2}} \int_\Sigma \frac{1}{[(\varepsilon + x_N - x_{0N})^2 + |x' - x_0'|^2]^{\frac{r(N-2)}{2}}} dx \\ &= \varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'| \le a\}} \int_0^{\rho(y')} \frac{1}{[(\varepsilon + y_N)^2 + |y'|^2]^{\frac{r(N-2)}{2}}} dy_N dy'. \end{aligned}$$

Applying Taylor's formula to $\dfrac{1}{[(\varepsilon+(y_N)^2+|y'|^2]^{\frac{r(N-2)}{2}}}$ with respect to $y_N$, we obtain

$$
\begin{aligned}
L_2 &= \varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'|\leq a\}} \frac{\rho(y')}{(\varepsilon^2+|y'|^2)^{\frac{r(N-2)}{2}}} dy' \\
&-r(N-2)\varepsilon^{\frac{r(N-2)}{2}+1} \int_{\{|y'|\leq a\}} \frac{\rho(y')}{(\varepsilon^2+|y'|^2)^{\frac{r(N-2)}{2}+1}} dy' \\
&+o\left(\varepsilon^{\frac{r(N-2)}{2}+1} \int_{\{|y'|\leq a\}} \frac{|y'|^2}{(\varepsilon^2+|y'|^2)^{\frac{r(N-2)}{2}+1}} dy'\right) \\
&= L_{21} - L_{22} + o(\varepsilon^{\frac{r(N-2)}{2}+1} \int_{\{|y'|\leq a\}} \frac{|y'|^2}{(\varepsilon^2+|y'|^2)^{\frac{r(N-2)}{2}+1}} dy').
\end{aligned}
$$

On one hand, we have

$$
\begin{aligned}
L_{21} &= \frac{\sum_{i=1}^{N-1}\alpha_i}{2(N-1)} \varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'|\leq a\}} \frac{|y'|^2}{(\varepsilon^2+|y'|^2)^{\frac{r(N-2)}{2}}} dy' \\
&+o\left(\varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'|\leq a\}} \frac{|y'|^2}{(\varepsilon^2+|y'|^2)^{\frac{r(N-2)}{2}}} dy'\right).
\end{aligned}
$$

**If $N \geq 5$:**
By changing to polar coordinates, we obtain

$$
\begin{aligned}
\varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'|\leq a\}} \frac{|y'|^2}{(\varepsilon^2+|y'|^2)^{\frac{r(N-2)}{2}}} dy' &= \varepsilon^{\frac{2(N+1)-r(N-2)}{2}} \int_0^{+\infty} \frac{t^N}{(1+t^2)^{\frac{r(N-2)}{2}}} dt \\
&= o(\varepsilon)
\end{aligned}
$$

and

$$L_{21} = o(\varepsilon). \tag{3.6.9}$$

**If $N = 4$:**
We distinguish three subcases:
**If $r > \frac{5}{2}$ :**
As in the above case, we have

$$
\begin{aligned}
\varepsilon^r \int_{\{|y'|\leq a\}} \frac{|y'|^2}{(\varepsilon^2+|y'|^2)^r} dy' &= \varepsilon^{5-r} \int_0^{+\infty} \frac{t^4}{(1+t^2)^r} dt \\
&= o(\varepsilon)
\end{aligned}
$$

and

$$L_{21} = o(\varepsilon). \tag{3.6.10}$$

**If $r = \frac{5}{2}$ :**
Using polar coordinates, we obtain

$$\begin{aligned} L_{21} &= \varepsilon^{\frac{5}{2}} \int_{\{|y'| \le a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{5}{2}}} dy' \\ &= \omega_2 \, \varepsilon^{\frac{5}{2}} \int_0^a \frac{t^4}{(\varepsilon^2 + t^2)^{\frac{5}{2}}} dt \end{aligned}$$

and

$$L_{21} = O(\varepsilon^{\frac{5}{2}} |\log \varepsilon|) = o(\varepsilon). \tag{3.6.11}$$

**If $r < \frac{5}{2}$ :**
Applying the Dominated Convergence Theorem, we get

$$\begin{aligned} L_{21} &= \varepsilon^r \int_{\{|y'| \le a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^r} dy' \\ &= \varepsilon^r \int_{\{|y'| \le a\}} \frac{|y'|^2}{|y'|^{2r-2}} dy' + o(\varepsilon^r) \end{aligned}$$

and

$$L_{21} = o(\varepsilon). \tag{3.6.12}$$

**If $N = 3$**
Using the Dominated Convergence Theorem, we obtain

$$\begin{aligned} L_{21} &= \frac{\sum_{i=1}^{2} \alpha_i}{4} \varepsilon^{\frac{r}{2}} \int_{\{|y'| \le a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r}{2}}} dy' \\ &= \frac{\sum_{i=1}^{N-1} \alpha_i}{2(N-1)} \varepsilon^{\frac{r}{2}} \int_{\{|y'| \le a\}} \frac{1}{|y'|^{r-2}} dy' \end{aligned}$$

and

$$L_{21} = o(\varepsilon |\log \varepsilon|). \tag{3.6.13}$$

On other hand, using the definition of $\rho$, we write

$$\begin{aligned} L_{22} &= \frac{r(N-2)\varepsilon^{\frac{r(N-2)}{2}+1} \sum_{i=1}^{N-1} \alpha_i^2}{2(N-1)} \int_{\{|y'| \le a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}+1}} dy' \\ &+ o\left( \int_{\{|y'| \le a\}} \frac{\varepsilon^{\frac{r(N-2)}{2}+1} |y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}+1}} dy' \right). \end{aligned}$$

By a change of variable and using polar coordinates, we obtain

$$L_{22} = \frac{r(N-2)\varepsilon^{\frac{2N-r(N-2)}{2}}\sum_{i=1}^{N-1}\alpha_i^2}{2(N-1)}\int_{\mathbb{R}^{N-1}}\frac{|x'|^2}{(1+|x'|^2)^{\frac{r(N-2)}{2}+1}}dx' + o(\varepsilon^{\frac{2N-r(N-2)}{2}})$$

and

$$L_{22} = o(\varepsilon). \tag{3.6.14}$$

Finally, combining all the above estimates, we deduce

$$\int_\Omega |W_{\varepsilon,x_0}|^r dx = \begin{cases} o(\varepsilon|\log\varepsilon|) & \text{if } N=3 \\ o(\varepsilon) & \text{if } N\geq 4 \end{cases}$$

which is the desired result.

# Chapter 4

# On nonhomogeneous Neumann problem with weight and with critical nonlinearity in the boundary

(This work is submitted to Nonlinear Analysis: Theory, Methods and Applications, series A. )

## 4.1 Introduction

In this paper, we are concerned with the problem of the form

$$(4.1.1) \qquad \begin{cases} -div(p(x)\nabla u) = \lambda u + f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset I\!R^N$, $N \geq 3$, is a bounded domain with the smooth boundary $\partial\Omega$, $\nu$ is the outer normal on $\partial\Omega$, the coefficient $Q$ is continuous and positive on $\partial\Omega$ and the coefficient $p \in H^1(\Omega)$ is continuous and positive in $\bar{\Omega}$. $q = \frac{2(N-1)}{N-2}$, $\lambda$ is a constant and $f \not\equiv 0$. The assumptions on $f$ will be formulated later. The exponent $q$ is the critical Sobolev exponent for the trace embedding of the space $H^1(\Omega)$ into $L^q(\partial\Omega)$. The embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$ is continuous, but not compact.
The case $p \equiv 1 \equiv Q$ and $f \equiv 0$ is considered in [AM], [AY], [CK] and [CY]. As in Chapiter 3, the geometrical condition (g.c) at some point on $\partial\Omega$ play an important role in solving problems similar to (4.1.1). It is convenient to recall the definition.
(g.c): $\partial\Omega$ is said to satisfy the geometrical condition at $y \in \partial\Omega$ if there exists a neighborhood $U(y)$ of $y$ such that $\Omega \cap U(y)$ lies on one side of the tangent plane at

$y$.
Let $x_0 \in \partial\Omega$ satisfy

$$\frac{p(x_0)}{(Q(x_0))^{N-2}} = \min_{x\in\partial\Omega} \frac{p(x)}{(Q(x))^{N-2}}.$$

We assume that

$$|p(x) - p(x_0)| = o(|x - x_0|) \tag{4.1.2}$$

and

$$|Q(x) - Q(x_0)| = o(|x - x_0|) \tag{4.1.3}$$

for $x$ near $x_0$.
Throughout this paper, we suppose that $\partial\Omega$ satisfies the geometrical condition (g.c) at the point $x_0$. This condition guarantees that the mean curvature at $x_0$ is non-negative.
In this work, we are interested in the existence of a solution of (4.1.1) which satisfies $u \not\equiv 0$ on $\partial\Omega$. Therefore we assume that $f$ is chosen so that

$$\text{(4.1.4)} \quad \text{the equation} \quad \begin{cases} -div(p(x)\nabla u) = \lambda u + f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{has no solution.}$$

Note that assumption (4.1.4) certainly holds if $f$ has a constant sign on $\Omega$, otherwise we obtain a contradiction thanks to Hopf's Lemma.

In the first part of this work we study the case where $\lambda$ is negative. The assumptions on $f$ are summarized in the following

$$(H1) \begin{cases} f \in H^{-1}(\Omega) \cap C(\bar{\Omega}) \setminus \{0\} \text{ satisfies (4.1.4)} \\ \text{and} \\ (*) \displaystyle\int_\Omega f(x)\, u\, dx < \sigma_N \left( \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx \right)^{\frac{N}{2}}, \\ \forall u \in H^1(\Omega), \displaystyle\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = 1. \end{cases}$$

with $\sigma_N = (q-2)\left[\frac{1}{q-1}\right]^{\frac{q-1}{q-2}}$.

Let us remark that $(*)$ certainly holds if

$$\|f\|_{H^{-1}} < \sigma_N \frac{\left[\min(-\lambda, \min_{\bar{\Omega}} p(x))\right]^{\frac{N}{2}}}{\left[\max_{\partial\Omega} p(x)Q(x)\right]^{\frac{N-1}{q}}} C^{N-1}$$

where

$$C = C(\Omega) = \inf_{H^1(\Omega)\backslash H_0^1(\Omega)} \frac{\left(\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx\right)^{\frac{1}{2}}}{\left(\displaystyle\int_{\partial\Omega} |u|^q ds_x\right)^{\frac{1}{q}}}$$

is the best constant in the trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$.
To obtain the existence of solutions of problem (4.1.1) we use the strategy developed by Tarantello, in [T], to study a nonhomogeneous Dirichlet problem with critical Sobolev exponent. The main result is the following

**Theorem 4.1.1.**
*Let $\Omega \in I\!R^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at $x_0$ and $H(x_0) > 0$. We assume that $f$ satisfies (H1) and the functions $p$ and $Q$ respectively satisfy (4.1.2) and (4.1.3). When $N = 3$, we further assume that $f$, $p$ and $Q$ are sufficiently smooth. Then for every $\lambda < 0$ the problem (4.1.1) admits at least two weak solutions.*

The second part of this work is dedicated to the case where $\lambda$ interferes with the spectrum $\{\lambda_1, \lambda_2, ...\}$ of the operator $div(p(x)\nabla\, .)$ with the Neumann boundary conditions.
It is known that $0 = \lambda_1 < \lambda_2 \leq ...$ and that the eigenfunction corresponding to $\lambda_1$ is constant function.
Let $k \in \mathbf{N} \setminus \{0\}$ such that $\lambda_{k-1} \neq \lambda_k$. Set $E_k^- = span\{e_1, ..., e_l\}$, where $e_1, ..., e_l$ are eigenfunctions corresponding to eigenvalues $\lambda_1, ..., \lambda_{k-1}$. We have the orthogonal decomposition $H^1(\Omega) = E_k^- \oplus E_k^+$.
The assumption on $f$ is the following

$(H2)$ $\quad f \in E_k^+ \setminus \{0\} \cap C(\bar{\Omega})$ satisfies (4.1.4).

Applying the min-max principle argument based on the topological linking [W], we obtain the following result

**Theorem 4.1.2.**
*Let $\Omega \in I\!R^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at $x_0$ and $H(x_0) > 0$. We assume that $f$ satisfies (H2) and the functions $p$ and $Q$ respectively satisfy (4.1.2) and (4.1.3). When $N = 3$ and $N = 4$ we further assume that $f(x_0) \neq 0$. For every $\lambda \in ]\lambda_{k-1}, \lambda_k[$, there is a constant $\alpha = \alpha(p, Q, \lambda, N) > 0$ such that if $\|f\|_{L^2} \leq \alpha$ then the problem (4.1.1) admits at least one solution.*

The rest of this paper is divided into three sections. In section 1 we give some notations which will be used throughout this paper. Section 2 and Section 3 are devoted respectively to the proof of Theorem 4.1.1 and the proof of Theorem 4.1.2.

## 4.2 Notation

We denote by $H(x_0)$ the mean curvature of $\partial\Omega$ at $x_0$ with respect to the unit outward normal. Let us recall that since $\partial\Omega$ satisfies (g.c) then $H(x_0) \geq 0$.
Let

$$S_1 = \inf \left\{ \int_{I\!R^N_+} |\nabla u|^2 dx;\ u \in H^1(I\!R^N_+),\ \int_{I\!R^{N-1}} |u|^q dx = 1 \right\}$$

be the best constant in the trace embedding for the space $H^1(I\!R^N_+)$ into $L^q(\partial I\!R^N_+)$, where $R^N_+ = \{x = (x', x_N) : x' \in I\!R^{N-1},\ x_N > 0\}$. By a result of [L], $S_1$ is achieved and by a result of [E], $S_1$ is given by

$$S_1 = \frac{N-2}{2}(\omega_N)^{\frac{1}{N-1}}, \quad \text{where } \omega_N \text{ is the area of } S^{N-1}.$$

The minimizing functions are of the form

$$W(x) = \frac{\gamma_N}{\left[|x'|^2 + (1+x_N)^2\right]^{\frac{N-2}{2}}}, \tag{4.2.1}$$

where $x = (x', x_N)$ with $x' \in I\!R^{N-1}$ and $\gamma_N$ is a positive constant depending only on $N$. We set

$$W_{\varepsilon,x_0}(x) = \varepsilon^{-\frac{N-2}{2}} \phi(x) W(\frac{x-x_0}{\varepsilon}),$$

where $\phi$ is a radial $C^\infty$-function such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x-x_0| \leq \frac{R}{4} \\ 0 & \text{if } |x-x_0| > \frac{R}{2} \end{cases}$$

with $R > 0$ is a small constant.
Using (4.1.2) and (4.1.3), from [AY] we obtain the following estimates

$$\int_\Omega p(x)|\nabla W_{\varepsilon,x_0}|^2 dx = p(x_0)A_1 - p(x_0)H(x_0) \begin{cases} A_2'\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{if } N=3 \\ A_2\varepsilon + o(\varepsilon) & \text{if } N \geq 4, \end{cases} \tag{4.2.2}$$

$$\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x = p(x_0)Q(x_0)(B_1 - H(x_0)B_2\varepsilon) + o(\varepsilon) \tag{4.2.3}$$

and

$$\int_\Omega |W_{\varepsilon,x_0}|^2 dx = \begin{cases} O(\varepsilon) & \text{if } N=3 \\ O(\varepsilon^2|\log\varepsilon|) & \text{if } N=4 \\ D\varepsilon^2 + o(\varepsilon^2) & \text{if } N \geq 5. \end{cases} \tag{4.2.4}$$

where $A_1 = \dfrac{(\Gamma(\frac{N-1}{2}))^2}{2\Gamma(N-2)}\omega_{N-1}$, $A_2' = \dfrac{\omega_2}{2}$, $A_2 = \dfrac{(N-1)(N-2)\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{8\Gamma(N-2)}\omega_{N-1}$,

$B_1 = \dfrac{(\Gamma(\frac{N-1}{2}))^2}{2\Gamma(N-1)}\omega_{N-2}$, $B_2 = \dfrac{(N-1)(\Gamma(\frac{N-1}{2}))^2}{4\Gamma(N-1)}\omega_{N-2}$ and $D = \dfrac{\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{2(N-4)\Gamma(N-2)}\omega_{N-1}$.
The function $\Gamma$ is defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt \quad \text{for every } z \in ]0, +\infty[.$$

Let us notice that

$$S_1 = \frac{A_1}{B_1^{\frac{2}{q}}}.$$

We endow the Sobolev space $H^1(\Omega)$ with the norm

$$\|u\| = \left[\int_\Omega (|\nabla u|^2 + |u|^2)dx\right]^{\frac{1}{2}}.$$

The norms in the Lebesgue space $L^t(\Omega)$ are denoted by $\|.\|_t$.

## 4.3 Case $\lambda < 0$

In this section, we study the following problem

(4.3.1)
$$\begin{cases} -div(p(x)\nabla u) = \lambda u + f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

Throughout this section we assume that $f$ satisfies $(H1)$, $p$ satisfies (4.1.2), $Q$ satisfies (4.1.3), $\lambda < 0$ and $\partial\Omega$ satisfies (g.c) at $x_0$. We start by the following Theorem which plays an essential role in the study of existence of a solution of the problem (4.3.1)

**Theorem 4.3.1.**
*Let $\Omega \in \mathbb{R}^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at $x_0$ and $H(x_0) > 0$. We assume that $f$ satisfies (H1) and the functions $p$ and $Q$ respectively satisfy (4.1.2) and (4.1.3). Then for every $\lambda < 0$ we have*
(4.3.2)
$$I_0 := \inf_{\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}}=1} \left[\sigma_N \left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right)^{\frac{N}{2}} - \int_\Omega f(x)\, u\, dx\right]$$

*is achieved and $I_0 > 0$.*

*Proof.*
We use a straightforward adaptation of the proof given in [BN2] for an analogous

minimization problem.
Let $\{u_n\}$ be a minimizing sequence for (4.3.2). It is easy to see that $\{u_n\}$ is bounded in $H^1(\Omega)$, thus we may extract a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{aligned}
u_n &\rightharpoonup u \quad \text{weakly in } H^1(\Omega),\\
u_n &\to u \quad \text{strongly in } L^2(\Omega),\\
u_n &\to u \quad \text{pointwise a.e in } \ \Omega,\\
u_n &\rightharpoonup u \quad \text{weakly in } L^q(\partial\Omega),\\
u_n &\to u \quad \text{pointwise a.e in } \ \partial\Omega,
\end{aligned}$$

with $\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}} \leq 1.$
It remains to prove that $\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}} = 1.$
Suppose, by contradiction, that

$$\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}} < 1.$$

Set $v_n = u_n - u$. Using the Brezis-Lieb Lemma [BL], we obtain

$$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + \int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1)$$

(which holds since $v_n$ is bounded in $L^q(\partial\Omega)$ and $v_n \to 0$ a.e on $\partial\Omega$).
We deduce that

$$\left(\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x\right)^{\frac{2}{q}} = \left[1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)\right]^{\frac{2}{q}} + o(1). \tag{4.3.3}$$

On the other hand, we have
(4.3.4)

$$\begin{aligned}
I_0 + o(1) &= \sigma_N \left(\int_\Omega p(x)|\nabla u_n|^2 dx - \lambda \int_\Omega |u_n|^2 dx\right)^{\frac{N}{2}} - \int_\Omega f(x)\, u_n\, dx\\
&= \sigma_N \left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx + \int_\Omega p(x)|\nabla v_n|^2 dx\right)^{\frac{N}{2}} - \int_\Omega f(x)\, u\, dx.
\end{aligned}$$

From the result of [Z, Theorem 02] we have the existence of a constant $C(\Omega) > 0$, such that for every $w \in H^1(\Omega)$ we have

$$\int_\Omega |\nabla w|^2 dx + C(\Omega) \int_\Omega |w|^k dx \geq S_1 \left(\int_{\partial\Omega} |w|^q dx\right)^{\frac{2}{q}}$$

with $k = \dfrac{2N}{N-1}$ if $N \geq 4$ and $k > 3 = \frac{2N}{N-1}$ if $N = 3$.
We apply this result for $w_n = (p(x))^{\frac{1}{2}} v_n$ and particulary for $N = 3$ we take $k$ such that $6 = \frac{2N}{N-2} > k > 3$, we obtain, for $n$ large enough

$$\int_\Omega p(x)|\nabla v_n|^2 dx + C(\Omega)\int_\Omega |(p(x))^{\frac{1}{2}} v_n|^k dx$$

$$= \int_\Omega |\nabla[(p(x))^{\frac{1}{2}} v_n]|^2 dx + C(\Omega)\int_\Omega |(p(x))^{\frac{1}{2}} v_n|^k dx + o(1)$$

$$\geq S_1 \left(\int_{\partial\Omega} |(p(x))^{\frac{1}{2}} v_n|^q ds_x\right)^{\frac{2}{q}} + o(1).$$

Since $k < \frac{2N}{N-2}$ for every $N \geq 3$, thanks to the compact injection of $H^1(\Omega)$ into $L^k(\Omega)$ we have $v_n \to 0$ strongly in $L^k(\Omega)$, and we deduce

$$\int_\Omega p(x)|\nabla v_n|^2 dx + o(1) \geq S_1 \left(\int_{\partial\Omega} |(p(x))^{\frac{1}{2}} v_n|^q ds_x\right)^{\frac{2}{q}} + o(1).$$

Using the fact that

$$\frac{p(x)}{(Q(x))^{N-2}} \geq \frac{p(x_0)}{(Q(x_0))^{N-2}} \quad \forall x \in \partial\Omega,$$

an easy computation yields

$$\int_\Omega p(x)|\nabla v_n|^2 dx \geq \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x\right)^{\frac{2}{q}} + o(1). \tag{4.3.5}$$

Inserting (4.3.3) and (4.3.5) into (4.3.4), we obtain after letting $n$ tend to infinity
(4.3.6)

$$I_0 \geq -\int_\Omega f(x)\, u\, dx + \sigma_N \times$$

$$\left[\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx + \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)\right)^{\frac{2}{q}}\right]^{\frac{N}{2}}.$$

We shall now prove the opposite inequality.
Since $\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}} < 1$ there exists $t_\varepsilon > 0$ such that

$$\left(\int_{\partial\Omega} p(x)Q(x)|u + t_\varepsilon W_{\varepsilon,x_0}|^q ds_x\right)^{\frac{1}{q}} = 1.$$

For $\varepsilon$ small enough we have

$$\int_\Omega p(x)|\nabla(u + t_\varepsilon W_{\varepsilon,x_0})|^2 dx - \lambda\int_\Omega |u + t_\varepsilon W_{\varepsilon,x_0}|^2 dx$$

$$= \int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx + t_\varepsilon^2\left(\int_\Omega p(x)|\nabla W_{\varepsilon,x_0}|^2 dx - \lambda\int_\Omega |W_{\varepsilon,x_0}|^2 dx\right) + o(1)$$

and

$$1 = \int_{\partial\Omega} p(x)Q(x)|u + t_\varepsilon W_{\varepsilon,x_0}|^q ds_x$$

$$= \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x + o(1).$$

From (4.2.2), (4.2.3) and (4.2.4) we deduce that

$$\int_\Omega p(x)|\nabla(u + t_\varepsilon W_{\varepsilon,x_0})|^2 dx - \lambda \int_\Omega |u + t_\varepsilon W_{\varepsilon,x_0}|^2 dx$$

(4.3.7)

$$= \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx + t_\varepsilon^2 p(x_0)A_1 + o(1)$$

and

(4.3.8) $$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q p(x_0)Q(x_0)B_1 + o(1).$$

We use $u + t_\varepsilon W_{\varepsilon,x_0}$ as a test function in (4.3.2), we deduce

$$I_0 \le c_N \Big[\int_\Omega p(x)|\nabla(u + t_\varepsilon W_{\varepsilon,x_0})|^2 dx - \lambda \int_\Omega |u + t_\varepsilon W_{\varepsilon,x_0}|^2 dx\Big]^{\frac{N}{2}}$$

$$- \int_\Omega f(x)\,(u + c_\varepsilon W_{\varepsilon,x_0}) dx$$

$$\le - \int_\Omega f(x)\,u\,dx + \sigma_N \Big[\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx$$

$$- \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)\right)^{\frac{2}{q}}\Big]^{\frac{N}{2}} + o(1).$$

Passing to the limit as $\varepsilon \to 0$, we get

$$I_0 \le - \int_\Omega f(x)\,u\,dx + \sigma_N \Big[\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx$$

$$- \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)\right)^{\frac{2}{q}}\Big]^{\frac{N}{2}}.$$

Combining the last inequality with (4.3.6) we obtain

$$I_0 = - \int_\Omega f(x)\,u\,dx + \sigma_N \Big[\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx$$

(4.3.9)

$$- \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(1 - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{2}{q}}\Big]^{\frac{N}{2}}.$$

Set

$$K_1 = N\sigma_N\Big[\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx$$

$$- \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(1 - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{2}{q}}\Big]^{\frac{N-2}{2}}$$

and

$$K_2 = \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(1 - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{2-q}{q}}.$$

For every $w \in H^1(\Omega)$, $t \in I\!R$, we consider

$$J(u+tw) = -\int_\Omega f(x)(u+tw)dx + \sigma_N \Big[\int_\Omega p(x)|\nabla(u+tw)|^2 dx - \lambda\int_\Omega |u+tw|^2 dx$$

$$- \left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}} S_1 \left(1 - \left(\int_{\partial\Omega} p(x)Q(x)|u+tw|^q ds_x\right)\right)^{\frac{2}{q}}\Big]^{\frac{N}{2}}.$$

Differentiating with respect to $t$ and passing to the limit as $t \to 0$, from (4.3.9) we get

$$K_1\left[\int_\Omega p(x)\nabla u.\nabla w\, dx - \lambda\int_\Omega u\, w\, dx - K_2\int_{\partial\Omega} p(x)Q(x)|u|^{q-2}u\, w\, ds_x\right] - \int_\Omega f(x)\, w\, dx = 0$$

for every $w \in H^1(\Omega)$.
We obtain that $u$ satisfies weakly

$$\text{(4.3.10)} \qquad \begin{cases} -div(p(x)\nabla u) = \lambda u + \frac{1}{K_1} f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = K_2 Q(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

Since $f \not\equiv 0$ we easily see that $u \not\equiv 0$ on $\Omega$. Moreover, since $f$ satisfies (4.1.4), then $u \not\equiv 0$ on $\partial\Omega$.

**Remark 4.3.1.**
*If $p \in C^\infty(\bar\Omega)$, $Q \in C^\infty(\partial\Omega)$ and $f \in C^\infty(\bar\Omega)$, then by [Ch], any solution $u$ of (4.3.10) belongs to $C^\infty(\bar\Omega)$.*

Let us note that we do not know if $u(x_0) \neq 0$ or not, but if $u(x_0) \neq 0$ we can assume that $u(x_0) > 0$ (replace $u$ with $-u$ and $f$ with $-f$ if necessary). Therefore, throughout the rest of proof, we assume that $u(x_0) \geq 0$. Set

$$J(v) = \sigma_N \left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right)^{\frac{N}{2}} - \int_\Omega f(x)\, u\, dx,$$

we will obtain a contradiction by showing that

$$\text{(4.3.11)} \qquad J(u + t_\varepsilon W_{\varepsilon,x_0}) < I_0$$

for $\varepsilon > 0$ small enough.
At this stage, we need to give a more explicit expression of $t_\varepsilon$.
From (4.3.8) and (4.2.3) it follows that $t_\varepsilon \to t_0$ as $\varepsilon \to 0$ where

$$t_0 = \left[\frac{1 - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}{p(x_0)Q(x_0)B_1}\right]^{\frac{1}{q}}. \tag{4.3.12}$$

Set

$$t_\varepsilon = t_0(1+\delta_\varepsilon), \quad \text{with } \delta_\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{4.3.13}$$

We shall give the precise rate at which $\delta_\varepsilon \to 0$.
We remark that if $q \leq 3$ then $N \geq 4$ and if $q > 3$ then $N < 4$, consequently we shall distinguish the cases $N \geq 4$ and $N = 3$.
**Case** $N \geq 4$
We have the following inequality :
For every $1 \leq q \leq 3$, there exists a constant $C = C(q)$ such that, for $x$, $y \in I\!R$, we have
(4.3.14)

$$||x+y|^q - |x|^q - |y|^q| - qxy(|x|^{q-2} + |y|^{q-2})| \leq \begin{cases} C|x|^{q-1}|y| & \text{if } |x| \leq |y| \\ C|y|^{q-1}|x| & \text{if } |x| \geq |y|. \end{cases}$$

We apply (4.3.14) with $x = u$ and $y = W_{\varepsilon,x_0}$.
We need the following estimates which are proved in the appendix.
(4.3.15)

$$\begin{aligned}\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^{q-2}W_{\varepsilon,x_0}u\, ds_z &= \frac{\omega_{N-2}\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})p(x_0)Q(x_0)u(x_0)\varepsilon^{\frac{N-2}{2}}}{2\Gamma(\frac{N}{2})} + o(\varepsilon^{\frac{N-2}{2}}) \\ &= u(x_0)D\varepsilon^{\frac{N-2}{2}} + o(\varepsilon^{\frac{N-2}{2}}),\end{aligned}$$

(4.3.16)

$$\int_{\partial\Omega} p(x)Q(x)|u|^{q-2}u\, W_{\varepsilon,x_0} ds_x = \varepsilon^{\frac{N-2}{2}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^{N-2}} ds_x + o(\varepsilon^{\frac{N-2}{2}}),$$

$$t_\varepsilon \int_{\partial\Omega\cap\{|u|\leq t_\varepsilon W_{\varepsilon,x_0}\}} p(x)Q(x)|u|^{q-1}W_{\varepsilon,x_0} ds_x = o(\varepsilon^{\frac{N-2}{2}}) \tag{4.3.17}$$

and

$$t_\varepsilon^{q-1} \int_{\partial\Omega\cap\{|u|\geq t_\varepsilon W_{\varepsilon,x_0}\}} p(x)Q(x)|u||W_{\varepsilon,x_0}|^{q-1} ds_x = o(\varepsilon^{\frac{N-2}{2}}). \tag{4.3.18}$$

Looking at (4.2.3) and the above estimates, direct computation yields that the cases $N = 4$ and $N \geq 5$ are different and will be treated separately. Therefore we distinguish two cases:

**If** $N \geq 5$

Combining, (4.3.13), (4.3.14), (4.3.15), (4.3.16), (4.3.17) and (4.3.18) we deduce that

$$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x + O(\varepsilon^{\frac{N-2}{2}}).$$

Using (4.2.3), we obtain

$$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q \left[p(x_0)Q(x_0)(B_1 - H(x_0)B_2\varepsilon)\right] + o(\varepsilon).$$

Consequently

$$t_\varepsilon^2 = t_0^2(1 + \frac{2}{q}H(x_0)\frac{B_2}{B_1}\varepsilon) + o(\varepsilon) \tag{4.3.19}$$

and

$$\delta_\varepsilon = \frac{1}{q}H(x_0)\frac{B_2}{B_1}\varepsilon + o(\varepsilon). \tag{4.3.20}$$

We have

$$\begin{aligned} J(u + t_\varepsilon W_{\varepsilon,x_0}) &= \sigma_N \Bigg[\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) \\ &+2t_\varepsilon \left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0} dx - \lambda\int_\Omega u\, W_{\varepsilon,x_0} dx\right) \\ &+t_\varepsilon^2 \left(\int_\Omega p(x)|\nabla W_{\varepsilon,x_0}|^2 dx - \lambda\int_\Omega |W_{\varepsilon,x_0}|^2 dx\right)\Bigg]^{\frac{N}{2}} \\ &- \int_\Omega f(x)\, u\, dx - t_\varepsilon \int_\Omega f(x)\, W_{\varepsilon,x_0} dx. \end{aligned}$$

Using (4.2.2) and (4.2.4), we write

$$\begin{aligned} J(u + t_\varepsilon W_{\varepsilon,x_0}) &= \sigma_N \Bigg[\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) \\ &+2t_\varepsilon \left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0} dx - \lambda\int_\Omega u\, W_{\varepsilon,x_0} dx\right) \\ &+t_0^2(1 + 2\delta_\varepsilon)A_1 - H(x_0)t_0^2 A_2\varepsilon + o(\varepsilon) + o(\delta_\varepsilon)\Bigg]^{\frac{N}{2}} \\ &- \int_\Omega f(x)\, u\, dx - \int_\Omega f(x)\, W_{\varepsilon,x_0} dx. \end{aligned} \tag{4.3.21}$$

From (4.3.20) we have

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = -\int_\Omega f(x)\,u\,dx + o(\varepsilon)$$

$$+\sigma_N\left[\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right) + t_0^2A_1 + \frac{2}{q}t_0^2H(x_0)\frac{B_2}{B_1}A_1\varepsilon - t_0^2A_2H(x_0)\varepsilon + o(\varepsilon)\right]^{\frac{N}{2}}.$$

Therefore

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = \sigma_N\left[\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right) + t_0^2A_1\right]^{\frac{N}{2}} - \int_\Omega f(x)\,u\,dx$$

$$+\frac{N}{2}\sigma_N\left[\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right) + t_0^2A_1\right]^{\frac{N-2}{2}} t_0^2\left(\frac{2}{q}\frac{B_2}{B_1}A_1 - A_2\right)H(x_0)\varepsilon + o(\varepsilon).$$

Combining the fact that $\Gamma(z+1) = z\Gamma(z)$ $\forall z \in ]0,+\infty[$ with the definition of the constants $A_1$, $A_2$, $B_1$ and $B_2$, we deduce that

$$A_2 - \frac{A_1B_2}{B_1} > 0. \tag{4.3.22}$$

Using (4.3.9), (4.3.22), the fact that $\frac{2}{q} < 1$, the expression of $t_0$ and the expression of $K_1$ we deduce that, for $\varepsilon > 0$ sufficiently small

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = I_0 + \tfrac{K_1}{2}t_0^2\left(\tfrac{2}{q}\tfrac{B_2}{B_1}A_1 - A_2\right)H(x_0)\varepsilon + o(\varepsilon) < I_0$$

which gives a contradiction in this case.

**If** $N = 4$

From (4.3.13), (4.3.14), (4.3.15), (4.3.16), (4.3.17) and (4.3.18) we have

$$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x$$

$$-\varepsilon\left(qu(x_0)Dt_0^{q-1} + qt_0\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2}ds_x\right) + o(\varepsilon).$$

Using (4.2.3) we obtain

$$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q\left[p(x_0)Q(x_0)(B_1 - H(x_0)B_2\varepsilon)\right]$$

$$-\varepsilon\left(qu(x_0)Dt_0^{q-1} + q\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2}ds_x\right) + o(\varepsilon).$$

Consequently

(4.3.23)

$$t_\varepsilon^2 = t_0^2\left[1 + \frac{2}{q}\left(H(x_0)\frac{B_2}{B_1} - \frac{qu(x_0)Dt_0^{q-1} + q\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2}ds_x}{B_1t_0^q}\right)\varepsilon\right]$$

$$+o(\varepsilon)$$

and

(4.3.24)

$$\delta_\varepsilon=\frac{1}{q}\left(H(x_0)\frac{B_2}{B_1}-\frac{qu(x_0)Dt_0^{q-1}+q\displaystyle\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2}ds_x}{B_1t_0^q}\right)\varepsilon+o(\varepsilon).$$

Therefore

$$\begin{aligned}J(u+t_\varepsilon W_{\varepsilon,x_0}) &= \sigma_N\Bigg[\left(\int_\Omega p(x)|\nabla u|^2dx-\lambda\int_\Omega|u|^2dx\right)\\ &\quad +2t_\varepsilon\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx-\lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right)\\ &\quad +t_0^2(1+2\delta_\varepsilon)A_1-H(x_0)t_0^2A_2\varepsilon+o(\varepsilon)+o(\delta_\varepsilon)\Bigg]^{\frac{N}{2}}\\ &\quad -\int_\Omega f(x)\,u\,dx-t_\varepsilon\int_\Omega f(x)\,W_{\varepsilon,x_0}dx\end{aligned}$$

and

$$\begin{aligned}&J(u+t_\varepsilon W_{\varepsilon,x_0})=\\ &\sigma_N\left[\left(\int_\Omega p(x)|\nabla u|^2dx-\lambda\int_\Omega|u|^2dx\right)+t_0^2A_1\right]^{\frac{N}{2}}\int_\Omega f(x)\,u\,dx-t_0\int_\Omega f(x)\,W_{\varepsilon,x_0}dx\\ &+\left[2t_0^2\delta_\varepsilon A_1-H(x_0)t_0^2A_2\varepsilon+o(\varepsilon)+2t_0\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx-\lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right)\right]^{\frac{N}{2}}\\ &\times\sigma_N\left[\left(\int_\Omega p(x)|\nabla u|^2dx-\lambda\int_\Omega|u|^2dx\right)+t_0^2A_1\right]^{\frac{N-2}{2}}.\end{aligned}$$

Using (4.3.9) we get

$$\begin{aligned}J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0+t_0\left[K_1\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx-\lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right)\right.\\ &\quad\left.-\int_\Omega f(x)\,W_{\varepsilon,x_0}dx\right]+t_0^2K_1\delta_\varepsilon A_1-H(x_0)\frac{K_1}{2}t_0^2A_2\varepsilon+o(\varepsilon).\end{aligned}$$

Thus from equation (4.3.10) we derive

$$\begin{aligned}J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0+t_0K_1K_2\int_{\partial\Omega}p(x)Q(x)|u|^{q-2}u\,W_{\varepsilon,x_0}ds_x\\ &\quad +t_0^2K_1\delta_\varepsilon A_1-H(x_0)\tfrac{K_1}{2}t_0^2A_2\varepsilon+o(\varepsilon).\end{aligned}$$

Inserting (4.3.16) in the above equality, we obtain

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = I_0 + t_0K_1K_2\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2}ds_x\,\varepsilon$$

$$+t_0^2K_1\delta_\varepsilon A_1 - H(x_0)\tfrac{K_1}{2}t_0^2A_2\varepsilon + o(\varepsilon).$$

Therefore

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = I_0$$

$$+K_1\left[t_0K_2\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2}ds_x\,\varepsilon + t_0^2\delta_\varepsilon A_1 - \frac{H(x_0)t_0^2A_2}{2}\varepsilon\right]+o(\varepsilon)$$

A combination of the definition of $t_0$ together with the definition of $S_1$, namely $S_1 = \frac{A_1}{B_1^{\frac{2}{q}}}$, and the definition of $K_2$, in the last inequality, allows to write

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = I_0 + \frac{K_1A_1}{B_1t_0^{q-2}}\left[\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2}ds_x\,\varepsilon + t_0^q\delta_\varepsilon B_1 - H(x_0)t_0^qB_1\frac{A_2}{2A_1}\varepsilon\right]$$

$$+o(\varepsilon).$$

It follows from (4.3.24) that

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0 + \frac{K_1A_1}{B_1t_0^{q-2}}\left[\frac{1}{q}H(x_0)t_0^qB_2\varepsilon - u(x_0)t_0^{q-1}D\varepsilon - H(x_0)t_0^qB_1\frac{A_2}{2A_1}\varepsilon\right] \\ &\quad +o(\varepsilon) \\ &= I_0 - \frac{H(x_0)t_0^2K_1}{2}\left(A_2 - \frac{2}{q}\frac{A_1B_2}{B_1}\right)\varepsilon - u(x_0)D\frac{t_0K_1A_1}{B_1}\varepsilon + o(\varepsilon). \end{aligned}$$

Since $H(x_0) > 0$ and $u(x_0) \geq 0$, from (4.3.22) and the fact that $\frac{2}{q} < 1$ we deduce the desired result in this case.

**Case** $N = 3$

In this case, we have $q = 4$ therefore we use the simple identity

$$|x+y|^4 = |x|^4 + |y|^4 + 4|x|^2xy + 4xy|y|^2 + 6|x|^2|y|^2. \tag{4.3.25}$$

By opposition with the other cases, in dimension $N = 3$, the case $u(x_0) = 0$ and $u(x_0) > 0$ will be treated separately. We distinguish two subcases.

**Case 1: If** $u(x_0) > 0$.

In order to apply (4.3.25) with $x = u$ and $y = W_{\varepsilon,x_0}$, we need the following new estimate which will be proved in the appendix.

$$\int_{\partial\Omega} p(x)Q(x)|u(x)|^2|W_{\varepsilon,x_0}|^2ds_x \leq M_1\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) = o(\varepsilon^{\frac{1}{2}}), \tag{4.3.26}$$

$$\text{(4.3.27)} \qquad \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x = M_2 u(x_0)\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}})$$

and

(4.3.28)

$$\int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x)\, W_{\varepsilon,x_0} ds_x = \varepsilon^{\frac{1}{2}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^2 u(x)\phi(x)}{|x-x_0|} ds_x + o(\varepsilon^{\frac{1}{2}}).$$

where $M_1$ and $M_2$ are positive constants.
From (4.3.25), (4.3.26), (4.3.27) and (4.3.28) we have

$$\begin{aligned} 1 &= \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x \\ &\quad -q\varepsilon^{\frac{1}{2}} \left( u(x_0)Dt_0^{q-1} + t_0 \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|} ds_x \right) + o(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

Using (4.2.3) we obtain

$$\begin{aligned} 1 &= \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q p(x_0)Q(x_0)B_1 \\ &\quad -q\varepsilon^{\frac{1}{2}} \left( u(x_0)Dt_0^{q-1} + \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|} ds_x \right) + o(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

Consequently

(4.3.29)

$$t_\varepsilon^2 = t_0^2 \left( 1 - \frac{2}{q} \frac{u(x_0)Dt_0^{q-1} + \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|} ds_x}{B_1 t_0^q} \varepsilon^{\frac{1}{2}} \right) + o(\varepsilon^{\frac{1}{2}})$$

and

$$\text{(4.3.30)} \qquad \delta_\varepsilon = - \frac{u(x_0)Dt_0^{q-1} + \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|} ds_x}{B_1 t_0^q} \varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}}).$$

Therefore

$$J(u+t_\varepsilon W_{\varepsilon,x_0})=$$

$$\sigma_3\left[\left(\int_\Omega p(x)|\nabla u|^2dx-\lambda\int_\Omega |u|^2dx\right)+t_0^2(1+2\delta_\varepsilon)A_1-H(x_0)t_0^2A_2'\varepsilon|\log\varepsilon|+o(\varepsilon|\log\varepsilon|)\right.$$

$$\left.+o(\delta_\varepsilon)+2t_\varepsilon\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx-\lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right)\right]^{\frac{3}{2}}-\int_\Omega f(x)\,u\,dx-\int_\Omega f(x)\,W_{\varepsilon,x_0}dx$$

$$=\sigma_3\left[\left(\int_\Omega p(x)|\nabla u|^2dx-\lambda\int_\Omega |u|^2dx\right)+t_0^2A_1\right]^{\frac{N}{2}}-\int_\Omega f(x)\,u\,dx-t_0\int_\Omega f(x)\,W_{\varepsilon,x_0}dx$$

$$+\left[2t_0^2\delta_\varepsilon A_1+o(\delta_\varepsilon)+2t_0\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx-\lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right)\right]$$

$$\times\tfrac{N}{2}\sigma_3\left[\left(\int_\Omega p(x)|\nabla u|^2dx-\lambda\int_\Omega |u|^2dx\right)+t_0^2A_1\right]^{\frac{N-2}{2}}.$$

Using (4.3.9) we get

$$J(u+t_\varepsilon W_{\varepsilon,x_0})=I_0+t_0^2K_1\delta_\varepsilon A_1+o(\delta_\varepsilon)$$

$$+t_0\left[K_1\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx-\lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right)-\int_\Omega f(x)\,W_{\varepsilon,x_0}dx\right].$$

Thus from equation (4.3.10) we derive

$$J(u+t_\varepsilon W_{\varepsilon,x_0})=I_0+t_0K_1K_2\int_{\partial\Omega}p(x)Q(x)|u|^{q-2}u\,W_{\varepsilon,x_0}ds_x+t_0^2K_1\delta_\varepsilon A_1+o(\delta_\varepsilon).$$

Inserting (4.3.16) in the above equality , we write

$$\begin{aligned}J(u+t_\varepsilon W_{\varepsilon,x_0})\ &=I_0+t_0K_1K_2\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|}ds_x\,\varepsilon^{\frac{1}{2}}\\ &+t_0^2K_1\delta_\varepsilon A_1+o(\varepsilon^{\frac{1}{2}}).\end{aligned}$$

Therefore

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = I_0 +$$

$$K_1\left[t_0K_2\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|}ds_x\,\varepsilon^{\frac{1}{2}}+t_0^2\delta_\varepsilon A_1\right]+o(\varepsilon^{\frac{1}{2}})$$

$$= I_0 + o(\varepsilon^{\frac{1}{2}}) +$$

$$K_1\left[\frac{\left(\frac{p(x_0)}{(Q(x_0))^{N-2}}\right)^{\frac{1}{N-1}}S_1t_0\varepsilon^{\frac{1}{2}}}{\left(1-\left(\int_{\partial\Omega}p(x)Q(x)|u|^q ds_x\right)\right)^{\frac{q-2}{q}}}\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|}ds_x+t_0^2\delta_\varepsilon A_1\right].$$

Combining the definition of $t_0$ together with the definition of $S_1$, namely $S_1=\frac{A_1}{B_1^{\frac{2}{q}}}$, we write

$$J(u+t_\varepsilon W_{\varepsilon,x_0})=I_0+\frac{K_1A_1}{B_1t_0^{q-2}}\left[\int_{\partial\Omega}\frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|}ds_x\,\varepsilon^{\frac{1}{2}}+t_0^q\delta_\varepsilon B_1\right]+o(\varepsilon^{\frac{1}{2}}).$$

It follows from (4.3.30) that

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = I_0 - u(x_0)D\frac{t_0K_1A_1}{B_1}\varepsilon^{\frac{1}{2}}+o(\varepsilon^{\frac{1}{2}}).$$

Since $u(x_0)>0$ we deduce that for $\varepsilon>0$ sufficiently small we have

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) < I_0$$

and we obtain a contradiction in this case.

**Case 2: If** $u(x_0)=0$

From (4.3.13) we have

$$t_\varepsilon = t_0(1+\delta_\varepsilon) \quad \text{with } \delta_\varepsilon\to 0 \text{ as } \varepsilon\to 0.$$

We write

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = -\int_\Omega f(x)\,u\,dx - \int_\Omega f(x)\,W_{\varepsilon,x_0}dx + \sigma_3\left[\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right)\right.$$

$$+2t_\varepsilon\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx - \lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right) + t_0^2(1+2\delta_\varepsilon+\delta_\varepsilon^2)p(x_0)A_1$$

$$\left.-H(x_0)t_0^2p(x_0)A_2'\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|)\right]^{\frac{3}{2}}$$

$$= \sigma_3\left[\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right) + t_0^2A_1\right]^{\frac{3}{2}} - \int_\Omega f(x)\,u\,dx - t_0\int_\Omega f(x)\,W_{\varepsilon,x_0}dx$$

$$+\tfrac{3}{2}\sigma_3\left[\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right) + t_0^2A_1\right]^{\frac{1}{2}}\left[2t_0^2\delta_\varepsilon p(x_0)A_1 + t_0^2\delta_\varepsilon^2p(x_0)A_1\right.$$

$$\left.-H(x_0)t_0^2p(x_0)A_2'\varepsilon|\log\varepsilon| + 2t_0\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx - \lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right)\right] + o(\varepsilon|\log\varepsilon|).$$

Using (4.3.9) we get

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = I_0 + t_0^2K_1\delta_\varepsilon p(x_0)A_1 + t_0^2\frac{K_1}{2}\delta_\varepsilon^2p(x_0)A_1 - H(x_0)t_0^2p(x_0)A_2'\varepsilon|\log\varepsilon|$$

$$+t_0\left[K_1\left(\int_\Omega p(x)\nabla u.\nabla W_{\varepsilon,x_0}dx - \lambda\int_\Omega u\,W_{\varepsilon,x_0}dx\right) - \int_\Omega f(x)\,W_{\varepsilon,x_0}dx\right]$$

$$+o(\varepsilon|\log\varepsilon|).$$

Thus from equation (4.3.10) we derive
(4.3.31)

$$J(u+t_\varepsilon W_{\varepsilon,x_0}) = I_0 + t_0K_1K_2\int_{\partial\Omega} p(x)Q(x)|u|^{q-2}u\,W_{\varepsilon,x_0}ds_x + t_0^2K_1\delta_\varepsilon p(x_0)A_1$$

$$+t_0^2\tfrac{K_1}{2}p(x_0)A_1\delta_\varepsilon^2 - H(x_0)t_0^2p(x_0)A_2'\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|).$$

On the other hand, we apply (4.3.25) with $x = u$ and $y = W_{\varepsilon,x_0}$, we obtain

$$
\begin{aligned}
& t_\varepsilon \int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x)\, W_{\varepsilon,x_0} ds_x \\
& \leq \frac{1}{4}\left[1 - \int_{\partial\Omega} p(x)Q(x)|u(x)|^4 ds_x - t_\varepsilon^4 \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^4 ds_x\right] \\
& -t_\varepsilon^3 \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x \\
& \leq \frac{1}{4}\left[1 - \int_{\partial\Omega} p(x)Q(x)|u(x)|^4 ds_x - t_0^4(1+4\delta_\varepsilon+6\delta_\varepsilon^2)p(x_0)Q(x_0)(B_1 - B_2\varepsilon)\right] \\
& -t_\varepsilon^3 \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x + o(\varepsilon) + o(\delta_\varepsilon^2).
\end{aligned}
$$

Using the expression of $t_0$ given by (4.3.12) we get

$$
\begin{aligned}
t_\varepsilon \int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x)\, W_{\varepsilon,x_0} ds_x \leq & -t_0^4 p(x_0)Q(x_0)B_1\delta_\varepsilon - \tfrac{3}{2}t_0^4 p(x_0)Q(x_0)B_1\delta_\varepsilon^2 + t_0^4 B_2\varepsilon \\
& -t_\varepsilon^3 \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x + o(\delta_\varepsilon^2) + o(\varepsilon).
\end{aligned}
$$

Moreover we have (see appendix)

$$
\int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x = o(\varepsilon|log\varepsilon|). \tag{4.3.32}
$$

Therefore

(4.3.33)

$$
\begin{aligned}
t_\varepsilon \int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x)\, W_{\varepsilon,x_0} ds_x \leq & -t_0^4 p(x_0)Q(x_0)B_1\delta_\varepsilon - \tfrac{3}{2}t_0^4 p(x_0)Q(x_0)B_1\delta_\varepsilon^2 \\
& + o(\delta_\varepsilon^2) + o(\varepsilon|\log\varepsilon|).
\end{aligned}
$$

Inserting (4.3.33) into (4.3.31), we obtain

$$
\begin{aligned}
J(u + t_\varepsilon W_{\varepsilon,x_0}) \ & \leq I_0 - \left(\frac{3K_1}{2}K_2 t_0^4 p(x_0)Q(x_0)B_1 - t_0^2\frac{K_1}{2}p(x_0)A_1\right)\delta_\varepsilon^2 + o(\delta_\varepsilon^2) \\
& -H(x_0)t_0^2 p(x_0)A_2'\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|).
\end{aligned}
$$

Using the expression of $K_1$ and $t_0$ we deduce that

$$
\begin{aligned}
J(u + t_\varepsilon W_{\varepsilon,x_0}) \ & \leq I_0 - K_1 t_0^2 p(x_0)A_1\delta_\varepsilon^2 + o(\delta_\varepsilon^2) - H(x_0)t_0^2 p(x_0)A_2'\varepsilon|\log\varepsilon| \\
& + o(\varepsilon|\log\varepsilon|).
\end{aligned}
$$

Since $H(x_0) > 0$, we conclude that for $\varepsilon > 0$ small enough we have $J(u + t_\varepsilon W_{\varepsilon,x_0}) < I_0$, which gives a contradiction in this case.

This ends the proof of Theorem 4.3.1. □

Now we return to the proof of Theorem 4.1.1. As announced in the introduction, we follow the steps of [T] in studying a Dirichlet problem with critical Sobolev exponent.
We consider the functional

$$I(u) = \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \int_\Omega f(x)\, u dx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x,\ u \in H^1(\Omega).$$

We define the following manifolds

$$\begin{aligned}
\Lambda \quad &= \{u \in H^1(\Omega) : \langle I'(u), u\rangle = 0\},\\
\Lambda^+ \quad &= \left\{u \in \Lambda : \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx - (q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x > 0\right\},\\
\Lambda_0 \quad &= \left\{u \in \Lambda : \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx - (q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = 0\right\},\\
\Lambda^- \quad &= \left\{u \in \Lambda : \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx - (q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x < 0\right\}.
\end{aligned}$$

We start by studying the existence of a first solution of (4.3.1) which will be a local minimum of $I$ on $\Lambda$. We have the following

**Theorem 4.3.2.**
*Let $\Omega \in I\!R^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at $x_0$ and $H(x_0) > 0$. We assume that $f$ satisfies (H1) and the functions $p$ and $Q$ respectively satisfy (4.1.2) and (4.1.3). Then for every $\lambda < 0$,*

$$c_0 = \inf_\Lambda I \tag{4.3.34}$$

*is achieved*

In order to prove Theorem 4.3.2, we need some Lemmas

**Lemma 4.3.1.**
*Let $f$ satisfy $(H1)$. For every $u \in H^1(\Omega) \setminus \{0\}$ such that $u \not\equiv 0$ in $\partial\Omega$, there exists a unique $t^+ = t^+(u) > 0$ such that $t^+u \in \Lambda^-$. In particular:*

$$t^+ > \left[\frac{\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}\right]^{\frac{1}{q-2}} := t_{max}$$

*and $I(t^+u) = \max_{t \geq t_{max}} I(tu)$*
*Moreover, if $\int_\Omega f(x)udx > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^-u \in \Lambda^+$.*
*In particular,*

$$t^- < \left[\frac{\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}\right]^{\frac{1}{q-2}}$$

*and $I(t^-u) \leq I(tu)$, $\forall t \in [0, t^+]$.*

*Proof.*
Set $\varphi(t) = t\left[\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx\right] - t^{q-1}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x$.
It is easy to see that $\varphi$ is concave and achieves its maximum at

$$t_{max} = \left[\frac{\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}\right]^{\frac{1}{q-2}}.$$

Moreover

$$\varphi(t_{max}) = (q-2)\left[\frac{1}{q-1}\right]^{\frac{q-1}{q-2}}\left[\frac{\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx\right)^{q-1}}{\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}\right]^{\frac{1}{q-2}}$$

and

$$\varphi(t_{max}) = \sigma_N \frac{\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx\right)^{\frac{N}{2}}}{\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{N-1}}$$

with

$$\sigma_N = (q-2)\left[\frac{1}{q-1}\right]^{\frac{q-1}{q-2}}. \tag{4.3.35}$$

Therefore if $\int_\Omega f(x)u\,dx \le 0$ then there exists a unique $t^+ > t_{max}$ such that

$$\varphi(t^+) = \int_\Omega f(x)u\,dx \qquad \text{and} \qquad \varphi'(t^+) < 0.$$

This gives that $t^+u \in \Lambda^-$ and $I(t^+u) \ge I(tu)\ \forall\, t \ge t_{max}$.
In case $\int_\Omega f(x)u\,dx > 0$, by assumption $(*)$ of (H1) we have that necessarily $\int_\Omega f(x)u\,dx < \varphi(t_{max})$ and there exist $t^-$ and $t^+$ unique such that

$$0 < t^- < t_{max} < t^+, \quad \varphi(t^+) = \int_\Omega f(x)\,u\,dx = \varphi(t^-) \quad \text{and} \quad \varphi'(t^-) > 0 > \varphi'(t^+).$$

This implies that $t^+u \in \Lambda^-$ and $t^-u \in \Lambda^+$. Moreover $I(t^+u) \ge I(tu),\ \forall\, t \ge t^-$ and $I(t^-u) \le I(tu),\ \forall\, t \in [0, t^+]$. □

For $u \not\equiv 0$ on $\partial\Omega$, let

$$\psi(u) = \sigma_N \frac{\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx\right)^{\frac{N}{2}}}{\left[\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}}\right]^{N-1}} - \int_\Omega f(x)\,u\,dx. \tag{4.3.36}$$

Let $t > 0$ and $\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = 1$ we have:

$$\psi(tu) = t\left[\sigma_N\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx\right)^{\frac{N}{2}} - \int_\Omega f(x)\,u\,dx\right],$$

given $\gamma > 0$, from Theorem 4.3.1 we deduce that

$$\inf_{\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}} \geq \gamma} \psi(u) \geq \gamma I_0 > 0. \tag{4.3.37}$$

**Lemma 4.3.2.**
*For every $u \in \Lambda \setminus \{0\}$, $u \not\equiv 0$ on $\partial\Omega$ we have*

$$\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx - (q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \neq 0$$

*Proof.*
Arguing by contradiction assume that for $u \in \Lambda \setminus \{0\}$, $u \not\equiv 0$ on $\partial\Omega$ we have

$$\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx - (q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = 0. \tag{4.3.38}$$

Combining the last equality with the fact that $u \in \Lambda$, we write

$$0 = (q-2)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \int_\Omega f(x)\, u\, dx \tag{4.3.39}$$

Condition (4.3.38) gives that

$$\begin{aligned}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x &= \tfrac{1}{q-1}\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx\right)\\ &\geq \frac{M_1}{q-1}\left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx\right) = \frac{M_1}{q-1}\|u\|^2\end{aligned}$$

where $M_1 = \min\left(\min_{x\in\bar{\Omega}} p(x), \dfrac{-\lambda}{\min_{x\in\bar{\Omega}} p(x)}\right) > 0.$
Using the embedding of $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ there exists a positive constant $M_2$ such that

$$\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \geq M_2\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{2}{q}}.$$

Thus

$$\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}} \geq M_2^{\frac{1}{q-2}} := \gamma.$$

Therefore, from (4.3.37) and (4.3.39) we obtain

$$
\begin{aligned}
0 < I_0\gamma \leq & \\
\psi(u) = & \left[\frac{1}{q-1}\right]^{\frac{q-1}{q-2}}(q-2)\left[\frac{\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right)^{\frac{q-1}{q-2}}}{\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}\right]^{\frac{1}{q-2}} - \int_\Omega f(x)\,u\,dx \\
\leq (q-2) & \left(\left[\frac{1}{q-1}\right]^{\frac{q-1}{q-2}}\left[\frac{\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right)^{\frac{q-1}{q-2}}}{\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}\right]^{\frac{1}{q-2}} - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right) \\
\leq (q-2) & \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\left(\left[\frac{\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}\right]^{\frac{q-1}{q-2}} - 1\right) = 0
\end{aligned}
$$

which yields a contradiction. □

As a consequence of Lemma 4.3.2 we have the

**Lemma 4.3.3.**
*Given $u \in \Lambda \setminus \{0\}$, $u \not\equiv 0$ on $\partial\Omega$ there exist $\varepsilon > 0$ and a differentiable function $t = t(w) > 0$, $w \in H^1(\Omega)$ and $\|w\| < \varepsilon$ satisfying*

$$
t(0) = 1, \quad t(w)(u-w) \in \Lambda, \quad \textit{for } \|w\| < \varepsilon,
$$

*and*

(4.3.40)

$$
\langle t'(0), w\rangle = \frac{2\left(\int_\Omega p(x)\nabla u.\nabla w dx - \lambda\int_\Omega u\,w\,dx\right) - \int_\Omega f(x)\,w\,dx - q\int_{\partial\Omega} p(x)Q(x)|u|^{q-2}u\,w\,ds_x}{\left[\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx - (q-1)\int_{\partial\Omega}|u|^q ds_x\right]}.
$$

*Proof.*
Define $G : I\!R \times H^1(\Omega) \to I\!R$ as follows

$$
\begin{aligned}
G(t,w) = & \, t\left(\int_\Omega p(x)|\nabla(u-w)|^2dx - \lambda\int_\Omega |u-w|^2dx\right) - t^{q-1}\int_{\partial\Omega} p(x)Q(x)|u-w|^q ds_x \\
& - \int_\Omega f(x)\,(u-w)\,dx.
\end{aligned}
$$

Since $G(1,0) = 0$ and

$$
\frac{\partial G}{\partial t}(1,0) = \left(\int_\Omega p(x)|\nabla(u)|^2dx - \lambda\int_\Omega |u|^2dx\right) - (q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \neq 0
$$

by Lemma 4.3.2, we can apply the Implicit Function Theorem at the point $(1,0)$ and get the desired result. □

We are now in a position to give the

**Proof of Theorem 4.3.2**

We start by showing that $I$ is bounded from below in $\Lambda$.

Let $u \in \Lambda$, we have

$$\int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \int_\Omega f(x)\, u\, dx = 0.$$

Thus

$$\begin{aligned} I(u) &= \frac{1}{2(N-1)} \left( \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx \right) - \frac{N}{2(N-1)} \int_\Omega f(x)\, u\, dx \\ &\geq \frac{1}{2(N-1)} \left( \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx \right) - \frac{N}{2(N-1)} \|f\|_{H^{-1}} \|u\| \\ &\geq \frac{M}{2(N-1)} \|u\|^2 - \frac{N}{2(N-1)} \|f\|_{H^{-1}} \|u\| \end{aligned}$$

with $M$ is a positive constant.

Consequently

$$c_0 \geq -\frac{N^2}{8(N-1)} M \|f\|^2_{H^{-1}}. \tag{4.3.41}$$

In order to obtain an upper bound for $c_0$, let $v \in H^1(\Omega)$ be a solution for the following Neumann problem

$$\begin{cases} -div(p(x)\nabla v) = \lambda v + f(x) & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us remark that, since $f$ satisfies (H1), namely (4.1.4), then $v \not\equiv 0$ on $\partial\Omega$.

We have

$$\int_\Omega f(x)\, v\, dx = \int_\Omega p(x)|\nabla v|^2 dx - \lambda \int_\Omega |v|^2 dx.$$

Let $t_0 = t^-(v) > 0$ as defined by Lemma 4.3.1, we write

$$\begin{aligned} I(t_0 v) &= \frac{t_0^2}{2} \left( \int_\Omega p(x)|\nabla v|^2 dx - \lambda \int_\Omega |v|^2 dx \right) - \frac{t_0^q}{q} \int_{\partial\Omega} p(x)Q(x)|v|^q ds_x \\ &\quad - t_0 \left( \int_\Omega p(x)|\nabla v|^2 dx - \lambda \int_\Omega |v|^2 dx \right). \end{aligned}$$

Using the fact that $t_0 v \in \Lambda$ we obtain

$$I(t_0 v) = -\frac{t_0^2}{2} \left( \int_\Omega p(x)|\nabla v|^2 dx - \lambda \int_\Omega |v|^2 dx \right) + \frac{q-1}{q} t_0^q \int_{\partial\Omega} p(x)Q(x)|v|^q ds_x.$$

Since $t_0 v \in \Lambda^+$, from the above inequality we deduce

$$I(t_0 v) < -\frac{t_0^2}{2(N-1)} \left( \int_\Omega p(x)|\nabla v|^2 dx - \lambda \int_\Omega |v|^2 dx \right) = -\frac{t_0^2}{2(N-1)} \|f\|^2_{H^{-1}}$$

This implies that

$$c_0 < -\frac{t_0^2}{2(N-1)}\|f\|_{H^{-1}}^2. \tag{4.3.42}$$

Applying Ekeland's Variational principle (see [AE]) to the minimization problem (4.3.34), we obtain a minimizing sequence$\{u_n\} \subset \Lambda$ with the following properties

$$I(u_n) < c_0 + \frac{1}{n} \tag{4.3.43}$$

(4.3.44)

$$I(w) \geq I(u_n) - \frac{1}{n}\left(\int_\Omega p(x)|\nabla(w-u_n)|^2 dx - \lambda\int_\Omega |w-u_n|^2 dx\right)^{\frac{1}{2}}, \ \forall\, w \in \Lambda.$$

Combining (4.3.42) and (4.3.43), we obtain for $n$ large

(4.3.45)

$$\begin{aligned} I(u_n) &= \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u_n|^2 dx - \lambda\int_\Omega |u_n|^2 dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\, u_n\, dx \\ &< -\tfrac{t_0^2}{2(N-1)}\|f\|_{H^{-1}}^2 \end{aligned}$$

This yields

$$\int_\Omega f(x)\, u_n\, dx \geq \frac{t_0^2}{N}\|f\|_{H^{-1}}^2 > 0. \tag{4.3.46}$$

Consequently $u_n \not\equiv 0$ in $\Omega$. Since $\lambda < 0$, putting together (4.3.45) and (4.3.46) we derive

$$\frac{t_0^2}{N}\|f\|_{H^{-1}} \leq \|u_n\| \leq \frac{M}{2}\|f\|_{H^{-1}} \tag{4.3.47}$$

with $M$ is a positive constant.

We now prove that $\|I'(u_n)\| \to 0$ as $n \to +\infty$.

We argue by contradiction and assume that $\|I'(u_n)\| \geq \alpha > 0$, with $\alpha$ a constant. Applying Lemma 4.3.3 with $u = u_n$ and $w = \delta\frac{I'(u_n)}{\|I'(u_n)\|}$, with $\delta > 0$ a small constant, there exists $t_n(\delta) := t\left(\delta\frac{I'(u_n)}{\|I'(u_n)\|}\right)$ such that

$$w_\delta = t_n(\delta)\left[u_n - \delta\frac{I'(u_n)}{\|I'(u_n)\|}\right] \in \Lambda.$$

From (4.3.44) we have

$$\frac{1}{n}\left(\int_\Omega p(x)|\nabla(w_\delta - u_n)|^2 dx - \lambda\int_\Omega |w_\delta - u_n|^2 dx\right)^{\frac{1}{2}} \geq I(u_n) - I(w_\delta).$$

Using Taylor's expansion we obtain

$$\begin{aligned} \frac{1}{n}\left(\int_\Omega p(x)|\nabla(w_\delta - u_n)|^2 dx - \lambda\int_\Omega |w_\delta - u_n|^2 dx\right)^{\frac{1}{2}} &\geq \langle I'(w_\delta), u_n - w_\delta\rangle + o(\delta) \\ &\geq (1 - t_n(\delta))\langle I'(w_\delta), u_n\rangle + \delta t_n(\delta)\langle I'(w_\delta), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle + o(\delta). \end{aligned}$$

Dividing by $\delta > 0$ and passing to the limit as $\delta \to 0$ we derive

$$\frac{1}{n}\left[1+|t_n'(0)|\left(\int_\Omega p(x)|\nabla u_n|^2dx-\lambda\int_\Omega |u_n|^2dx\right)^{\frac{1}{2}}\right] \geq -t_n'(0)\langle I'(u_n),u_n\rangle + \|I'(u_n)\|$$

where we have set $t_n' = \langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle$.
Since $u_n \in \Lambda$ we obtain

$$\frac{1}{n}\left[1+|t_n'(0)|\left(\int_\Omega p(x)|\nabla (u_n)|^2dx-\lambda\int_\Omega |u_n|^2dx\right)^{\frac{1}{2}}\right] \geq \|I'(u_n)\|.$$

Using (4.3.47) we conclude

$$\|I'(u_n)\| \leq \frac{C}{n}\left(1+|t_n'(0)|\right) \tag{4.3.48}$$

where $C$ is a positive constant.
Now, to conclude it remains to show that $|t_n'(0)|$ is uniformly bounded in $n$.
From (4.3.40) and the estimate (4.3.47) we obtain

$$|t_n'(0)| \leq \frac{C_1}{|\int_\Omega p(x)|\nabla u_n|^2dx - \lambda\int_\Omega |u_n|^2dx - (q-1)\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x|}$$

where $C_1$ is a positive constant.
Hence we need to show that

$$\left|\int_\Omega p(x)|\nabla u_n|^2dx - \lambda\int_\Omega |u_n|^2dx - (q-1)\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x\right|$$

is bounded away from zero.
Arguing by contradiction, assume that for a subsequence (still denoted by $u_n$) we have

$$\int_\Omega p(x)|\nabla u_n|^2dx - \lambda\int_\Omega |u_n|^2dx - (q-1)\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x = o(1). \tag{4.3.49}$$

Using (4.3.47) we obtain

$$\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \geq \gamma \qquad (\gamma > 0 \text{ is a suitable constant})$$

and

$$\left[\frac{\int_\Omega p(x)|\nabla u_n|^2dx - \lambda\int_\Omega |u_n|^2dx}{q-1}\right]^{\frac{q-1}{q-2}} - \left[\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x\right]^{\frac{q-1}{q-2}} = o(1).$$

On the other hand, combining (4.3.49) with the fact that $u_n \in \Lambda$ we see that

$$\int_\Omega f(x)\,u_n\,dx = (q-2)\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x + o(1).$$

Finally, using the above together with (4.3.35), (4.3.36) and (4.3.37) we are led to

$$
\begin{aligned}
0 \ < I_0\gamma^N &\leq \left[\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x\right]^{\frac{1}{q-2}} \psi(u_n)\\
&\leq (q-2)\left[\left(\frac{\int_\Omega p(x)|\nabla u_n|^2dx-\lambda\int_\Omega |u_n|^2dx}{q-1}\right)^{\frac{q-1}{q-2}}-\left(\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x\right)^{\frac{q-1}{q-2}}\right]\\
&= o(1)
\end{aligned}
$$

which is impossible and we conclude that

$$
\|I'(u_n)\| \to 0 \quad \text{as}\, n \to +\infty. \tag{4.3.50}
$$

Let $u_0 \in H^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(\Omega)$.
From (4.3.50) we deduce that

$$
\langle I'(u_0), w\rangle = 0, \qquad \forall w \in H^1(\Omega)
$$

and thus $u_0 \in \Lambda$ is a weak solution of (4.3.1).
Therefore

$$
\begin{aligned}
c_0 \leq I(u_0) \ &= \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u_0|^2dx - \lambda\int_\Omega |u_0|^2dx\right) - \int_\Omega f(x)\, u_0\, dx\\
&\leq \lim_{n\to+\infty} I(u_n) = c_0.
\end{aligned}
$$

Consequently $u_n \to u_0$ strongly in $H^1(\Omega)$ and also in $L^q(\partial\Omega)$.
Thus $\inf_\Lambda I = c_0 = I(u_0)$ and from Lemma 4.3.1 and (4.3.46) we deduce that $u_0 \in \Lambda^+$.
Now we claim that $u_0$ is a local minimum for $I$. Indeed we know from Lemma 4.3.1 that for every $u \in H^1(\Omega)$ with $u \not\equiv 0$ on $\partial\Omega$ and $\int_\Omega f(x)\, u\, dx > 0$ there exists $t^- > 0$ such that

$$
I(tu) \geq I(t^-u) \quad \text{for every } 0 < t < \left(\frac{\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}\right)^{\frac{1}{q-2}}. \tag{4.3.51}
$$

Since $u_0 \in \Lambda^+$,

$$
t^- = 1 < \left(\frac{\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u_0|^2dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u_0|^q ds_x}\right)^{\frac{1}{q-2}}. \tag{4.3.52}
$$

Let $\varepsilon > 0$ sufficiently small so that

$$
1 < \frac{\int_\Omega p(x)|\nabla(u_0-w)|^2dx - \lambda\int_\Omega |u_0-w|^2dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u_0-w|^q ds_x} \qquad \text{for } \|w\| < \varepsilon.
$$

From Lemma 4.3.3, let $t(w) > 0$ such that $t(w)(u_0 - w) \in \Lambda$ for every $\|w\| < \varepsilon$.
Since $t(w) \to 1$ as $\|w\| \to 0$, we can always assume that

$$
t(w) < \left(\frac{\int_\Omega p(x)|\nabla(u_0-w)|^2dx - \lambda\int_\Omega |u_0-w|^2dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u_0-w|^q ds_x}\right)^{\frac{1}{q-2}} \quad \text{for every}\, w:\ \|w\| < \varepsilon.
$$

Moreover $t(w)(u_0 - w) \in \Lambda^+$ and we have

$$I(s(u_0 - w)) \geq I(t(w)(u_0 - w)) \geq I(u_0)$$

$$\text{for evry} \quad 0 < s < \left( \frac{\int_\Omega p(x)|\nabla(u_0 - w)|^2 dx - \lambda \int_\Omega |u_0 - w|^2 dx}{(q-1)\int_{\partial\Omega} p(x)Q(x)|u_0 - w|^q ds_x} \right)^{\frac{1}{q-2}}.$$

From (4.3.52) we can take $s = 1$ and we obtain the desired conclusion.
This completes the proof. We now turn to the study of the existence of a second solution of problem (4.3.1). Then we are led to investigate a second minimization problem. Namely

$$c_1 = \inf_{\Lambda^-} I. \tag{4.3.53}$$

We have the following result

**Theorem 4.3.3.**
*Let $\Omega \in I\!R^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at $x_0$ and $H(x_0) > 0$. We assume that $f$ satisfies (H1) and that the functions $p$ and $Q$ respectively satisfy (4.1.2) and (4.1.3). Then for every $\lambda < 0$ the infimum in (4.3.53) is achieved and we have $c_1 > c_0$.*

In order to prove this last result we start by the following lemma

**Lemma 4.3.4.**
*Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying*

$$I(u_n) \to c < c_0 + \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} \tag{4.3.54}$$

*and*

$$I'(u_n) \to 0 \quad in\, H^{-1}(\Omega) \tag{4.3.55}$$

*then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.*

*Proof.*
We start by showing that $\{u_n\}$ is bounded in $H^1(\Omega)$. Arguing by contradiction, we assume that $\|u_n\| \to \infty$. We set $v_n = \frac{u_n}{\|u_n\|}$. Since $\{v_n\}$ is bounded in $H^1(\Omega)$, we may assume that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$.
Using (4.3.55), for every $\phi \in H^1(\Omega)$, we have
(4.3.56)

$$\int_\Omega (p(x)\nabla v_n \nabla\phi - \lambda v_n\phi)\, dx = \|u_n\|^{-1} \left[ \int_{\partial\Omega} p(x)Q(x)|u_n|^{q-2}u_n\phi\, ds_x + \int_\Omega f(x)\,\phi\, dx \right] + o(1).$$

On the one hand, we have

$$(4.3.57)\qquad \int_{\partial\Omega} p(x)Q(x)|u_n|^{q-2}u_n\phi\, ds_x \le C_1 \left(\int_{\partial\Omega}|u_n|^q ds_x\right)^{\frac{q-1}{q}}\left(\int_{\partial\Omega}|\phi|^q ds_x\right)^{\frac{1}{q}}.$$

On the other hand, we compute (4.3.54) $-\frac{1}{2}$(4.3.55), which gives

$$\frac{1}{2(N-1)}\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x - \frac{1}{2}\int_\Omega f(x)\, u_n\, dx \le c + o(\|u_n\|).$$

Therefore

$$\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \le M_1\left[1+\|f\|_{H^{-1}}\|u_n\|\right] + o(\|u_n\|)$$

$$\left(\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x\right)^{\frac{q-1}{q}} \le M_2^{\frac{q-1}{q}}\|f\|_{H^{-1}}^{\frac{q-1}{q}}\|u_n\|^{\frac{q-1}{q}}\left[1+\frac{1}{\|u_n\|}+o(1)\right]^{\frac{q-1}{q}}$$

where $M_1$, $M_2$ are some positive constants.
Since $\|u_n\|\to\infty$, we have

$$\left(\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x\right)^{\frac{q-1}{q}} \le M_2^{\frac{q-1}{q}}\|f\|_{H^{-1}}^{\frac{q-1}{q}}\|u_n\|^{\frac{q-1}{q}} + \frac{q-1}{q}M_2^{\frac{q-1}{q}}\|f\|_{H^{-1}}^{\frac{q-1}{q}}\|u_n\|^{\frac{-1}{q}} + o(\|u_n\|^{\frac{q-1}{q}})$$

and

(4.3.58)

$$\|u_n\|^{-1}\left(\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x\right)^{\frac{q-1}{q}} \le M^{\frac{q-1}{q}}\|f\|_{H^{-1}}^{\frac{q-1}{q}}\|u_n\|^{\frac{-1}{q}} + M^{\frac{q-1}{q}}\|f\|_{H^{-1}}^{\frac{q-1}{q}}\|u_n\|^{\frac{-(q+1)}{q}} + o(\|u_n\|^{\frac{-1}{q}}).$$

Combining (4.3.57) with (4.3.58) we obtain

$$(4.3.59)\ \|u_n\|^{-1}\left[\int_{\partial\Omega} Q(x)|u_n|^{q-2}u_n\phi ds_x + \int_\Omega f(x)\,\phi dx\right]\to 0 \quad \text{as } \varepsilon\to 0.$$

Inserting (4.3.59) into (4.3.56), letting $n\to\infty$, we deduce that

$$\int_\Omega (p(x)\nabla v\nabla\phi - \lambda v\phi)\, dx = 0$$

for every $\phi\in H^1(\Omega)$. Since $\lambda<0$ we see that $v\equiv 0$ on $\Omega$. Consequently, we may assume that $v_n\to 0$ strongly in $L^2(\Omega)$. Hence, from (4.3.54) and (4.3.55), we can write

$$\frac{1}{2}\int_\Omega p(x)|\nabla v_n|^2 dx = \frac{\|u_n\|^{q-2}}{q}\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1)$$

and

$$\int_\Omega p(x)|\nabla v_n|^2 dx = \|v_n\|^{q-2}\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1).$$

These two relations imply that $\int_\Omega p(x)|\nabla v_n|^2 dx = o(1)$ and thus $v_n \to 0$ strongly in $H^1(\Omega)$, which is impossible since $\|v_n\| = 1$. Consequently $\{u_n\}$ is bounded in $H^1(\Omega)$ and we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.
Hence, by going to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.
By the concentration-compactness principle [L] we have

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j}$$

and

$$|u_n|^q \rightharpoonup |u|^q + \sum_{j \in J} \nu_j \delta_{x_j}$$

weakly in the sense of measures, where $x_j \in \partial\Omega$, $\mu_j$ and $\nu_j$ are some positive constants satisfying

$$S_1 \nu_j^{\frac{2}{q}} \leq \mu_j \tag{4.3.60}$$

and $J$ is a countable set.
Testing (4.3.55) by a family of $C^1$-functions concentrating at $x_j$ with some $j \in J$, we get

$$p(x_j)\mu_j \leq p(x_j)Q(x_j)\nu_j. \tag{4.3.61}$$

We claim that $\nu_j = 0$ for every $j \in J$, otherwise there exists $j_0 \in J$ such that $\nu_{j_0} > 0$ and from (4.3.60) and (4.3.61) we have

$$\frac{S_1^{N-1}}{Q(x_{j_0})^{N-1}} \leq \nu_{j_0}. \tag{4.3.62}$$

On the other hand we have

$$I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle = \frac{1}{2(N-1)} \int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x - \frac{1}{2}\int_\Omega f(x)\, u_n\, dx + o(1).$$

Letting $n \to +\infty$ we obtain

$$c = \frac{1}{2(N-1)} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \frac{1}{2}\int_\Omega f(x)\, u\, dx + \frac{1}{2(N-1)} \sum_{j \in J} p(x_j)Q(x_j)\nu_j.$$

The fact that $u$ is a weak solution of problem (4.3.1) gives

$$c \geq I(u) + \frac{1}{2(N-1)} p(x_{j_0})Q(x_{j_0})\nu_{j_0}.$$

Using (4.3.62) and the definition of $c_0$ we obtain

$$c \geq c_0 + \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$$

which contradicts (4.3.54). Hence $\nu_j = 0$ for every $j \in J$. This yields that $u_n \to u$ strongly in $L^q(\partial\Omega)$. Easily we deduce that $u_n \to u$ strongly in $H^1(\Omega)$, which is the desired conclusion. □

Analogously to the proof of Theorem 4.3.2, the Ekeland's Variational Principle (see [AE], Corollary 7.5.3) applied to minimizing problem (4.3.53) gives a sequence $\{u_n\} \subset \Lambda^-$ satisfying $I(u_n) \to c_1$ and $\|I'(u_n)\| \to 0$. Then the conclusion of Theorem follows from Lemma 4.3.4 if we have

$$c_1 < c_0 + \frac{p(x_0) S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}. \tag{4.3.63}$$

However it appears difficult to derive (4.3.63) directly. Then, as in [T], we shall obtain it by comparison with a mountain-pass value.
Let $(\mathcal{F})$ be the class of continuous paths joining $u_0$ to $u_0 + R_0 U_{\varepsilon,x_0}$, with $R_0 > 0$ fixed. We have the following result

**Lemma 4.3.5.**
*For a suitable choice of $R_0 > 0$ and $\varepsilon > 0$, the value*

$$c = \inf_{h \in \mathcal{F}} \max_{t \in [0,1]} I(h(t))$$

*defines a critical value for $I$, and $c \geq c_1$.*

The proof of this last Lemma relies on the following variant of the Mountain Pass Theorem of Ambroseti and Rabinowitz [AR]

**Theorem 4.3.4.**
*Let $\Phi$ be a $C^1$ function on a Banach space $E$. Suppose that :*
(4.3.64)
*There exists a neighborhood $U$ of $0$ in $E$ and a constant $\rho$ such that $\Phi(u) \geq \rho$ for every $u$ in the boundary of $U$,*

$$\Phi(0) < \rho \quad and \quad \Phi(v) < \rho \quad for\, some\; v \notin U. \tag{4.3.65}$$

*Set*

$$c = \inf_{\mathcal{P} \in \mathcal{A}} \max_{w \in \mathcal{P}} \Phi(w) \geq \rho, \tag{4.3.66}$$

*where $\mathcal{A}$ denotes the class of continuous paths joining $0$ to $v$.*
*Conclusion :*

*There exists a sequence $\{u_j\}$ in $E$ such that*

$$\Phi(u_j) \to c \quad and \quad \Phi'(u_j) \to 0 \quad in \quad E^*.$$

**Lemma 4.3.6.**
*For every $R > 0$, there exists $\varepsilon_0 = \varepsilon_0(R) > 0$ such that*

$$I(u_0 + RW_{\varepsilon,x_0}) < c_0 + \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$$

*for every $0 < \varepsilon < \varepsilon_0$*

*Proof.*
The proof is similar to the proof of (4.3.11) in Theorem 4.3.1. □

Now, we are in position to give the
**Proof of Lemma** 4.3.5
We start by verifying the assumptions of the Theorem 4.3.4 with $u_0$ in the place of $0$, $\Psi = I - I(u_0)$, $E = H^1(\Omega)$ and $\mathcal{A} = \mathcal{F}$.
**Verification of (4.3.64)**:
Let $u \in H^1(\Omega)$. For $t$ small enough we have

$$\begin{aligned} I(u_0 + tu) - I(u_0) &= t\langle I'(u_0), u\rangle + t^2\langle I''(u_0)u, u\rangle + o(t^2) \\ &= t^2\langle I''(u_0)u, u\rangle + o(t^2). \end{aligned}$$

Since $u_0$ is a local minimum for $I$ we deduce (4.3.64) with some $\rho > 0$ (and $U$ a small ball centered at $u_0$ in $H^1$).
**Verification of (4.3.65)**:
From the proof of Theorem 4.3.3, we have $I(u_0) = c_0 < 0$. On the other hand, it is easy to see that, for any $u \in H^1(\Omega) \setminus \{0\}$, $u \not\equiv 0$ on $\partial\Omega$, we have $\lim\limits_{t\to+\infty} I(u_0 + tu) = -\infty$. Thus, there are many $v$'s satisfying (4.3.65). However, it will be important for later purpose to use Theorem 4.3.4 with a special $v$, namely $v = u_0 + R_0 W_{\varepsilon,x_0}$, where $R_0 > 0$ is chosen large enough so that $v \notin U$ and $I(v) \leq 0$.
Applying Theorem 4.3.4 we obtain a sequence $\{u_j\}$ in $H^1(\Omega)$ such that $I(u_j) \to c$ and $I'(u_j) \to 0$ in $H^{-1}(\Omega)$.
Combining the results of Lemma 4.3.6 and Lemma 4.3.4 we deduce that $c$ is a critical value for $I$.
Now, we will prove that $c \geq c_1 = \inf\limits_{\Lambda^-} I(u)$. The proof of this last fact is based on two claims
**Claim 1**: $\Lambda^-$ disconnects $H^1(\Omega)$ in exactly two connected components $U_1$ and $U_2$.
Indeed, from Lemma 4.3.1, for every $u \in H^1(\Omega)$ there exists a unique $t^+(u) > 0$ such that $t^+(u)u \in \Lambda^-$ and $I(t^+(u)u) = \max\limits_{t \geq t_{max}} I(tu)$.
The uniqueness of $t^+(u)$ and its extremal property give that $t^+(u)$ is a continuous function of $u$.

Define for $u \in H^1(\Omega)$

$$\|u\|_\Omega = \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx.$$

Let us remark that since $\lambda < 0$, $\|.\|_\Omega$ is equivalent to the standard norm $\|.\|$.
Set

$$U_1 = \left\{ u \equiv 0 \text{ or } u \,:\, \|u\|_\Omega < t^+ \left( \frac{u}{\|u\|_\Omega} \right) \right\}$$

and

$$U_2 = \left\{ u \,:\, \|u\|_\Omega > t^+ \left( \frac{u}{\|u\|_\Omega} \right) \right\}.$$

We have $H^1(\Omega) \setminus \Lambda^- = U_1 \cup U_2$ and $\Lambda^+ \subset U_1$.
Since $u_0 \in \Lambda^+$ we deduce that $u_0 \in U_1$.
**Claim 2**: For a suitable choice of $R_0$, $u_0 + R_0 W_{\varepsilon,x_0} \in U_2$.
Indeed, let $u \not\equiv 0$ on $\partial\Omega$ with $\|u\|_\Omega = 1$, looking at the proof of Lemma 4.3.1, we have $t_{max} < t^+$ and

$$\varphi(t^+ u) = t^+ - (t^+)^{q-1} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = \int_\Omega f(x)\, u\, dx.$$

Therefore, using the embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ and the fact that $t_{max} < t^+$, a standard computation yields that there exists a constant $M_1 > 0$ such that $0 < t^+ < M_1$.
We choose $R_0$ such that $R_0 > \left( \frac{1}{p(x_0)A_1} \left| M_1 - \|u_0\|_\Omega^2 \right| \right)^{\frac{1}{2}}$. It follows from (4.2.2) and (4.2.4) that, for $\varepsilon$ small enough we have

$$\|u_0 + R_0 W_{\varepsilon,x_0}\|_\Omega^2 = \|u_0\|_\Omega^2 + R_0^2 p(x_0) A_1 + o(1) > M_1 > \left[ t^+ \left( \frac{u_0 + R_0 W_{\varepsilon,x_0}}{\|u_0 + R_0 W_{\varepsilon,x_0}\|_\Omega} \right) \right]^2,$$

which gives the result of Claim 2.
Consequently, since $u_0 \in \Lambda^+ \subset U_1$ and $u_0 + R_0 W_{\varepsilon,x_0} \in U_2$, we see that the range of any $h \in \mathcal{F}$ intersects $\Lambda^-$. Therefore $c \geq c_1 = \inf_{\Lambda^-} I(u)$ and this completes the proof of Lemma 4.3.5. From Lemma 4.3.5 and Lemma 4.3.6 we obtain (4.3.63) and this gives the conclusion of Theorem 4.3.3.
Finally, combining the conclusions of Theorem 4.3.2 and Theorem 4.3.3 we obtain Theorem 4.1.1. This completes this section.

## 4.4 Case $\lambda > 0$

It is convenient to rewrite (4.1.1) as

$$\text{(4.4.1)} \qquad \begin{cases} -div(p(x)\nabla u) = \lambda u + f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

Throughout this section we assume that $f$ satisfies $(H2)$, $p$ satisfies (4.1.2), $Q$ satisfies (4.1.3) and $\partial\Omega$ satisfies (g.c) at $x_0$ and $H(x_0) > 0$.
We recall some notations given in the introduction. By $\{\lambda_i\}$ we denote the sequence of eigenvalues for $div(p(x)\nabla\,.)$ with Neumann boundary conditions

$$\begin{cases} -div(p(x)\nabla u) = \beta u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $k \in \mathbf{N} \setminus \{0\}$ such that $\lambda_{k-1} \neq \lambda_k$. Set $E_k^- = span\{e_1, ..., e_l\}$, where $e_1, ..., e_l$ are eigenfunctions corresponding to eigenvalues $\lambda_1, ..., \lambda_{k-1}$. We have the orthogonal decomposition $H^1(\Omega) = E_k^- \oplus E_k^+$.

**Remark 4.4.1.**
*Since $f$ satisfies (H2), namely $f \in E_k^+ \setminus \{0\}$, it is easy to see that, if there exists a solution $u$ of (4.4.1), then $u \in E_k^+$.*

The proof of Theorem 4.1.2 relies on the following min-max principle based on a topological linking, see [W],

**Theorem 4.4.1.**
*Let $X = X^- \oplus X^+$ be a Banach space with $\dim X^- < \infty$. Let $\rho > r > 0$ and let $w \in X^+$ be such that $\|w\| = \delta$. Define*

$$M := \{u = v + sw \,:\, \|u\| \le \rho,\, s \ge 0,\, v \in X^-\},$$
$$M_0 := \{u = v + sw \,:\, v \in X^-, \|u\| = \rho \text{ and } s \ge 0 \text{ or } \|u\| \le \rho \text{ and } s = 0\},$$
$$N := \{u \in X^+ \,:\, \|u\| = r\}.$$

*Let $\varphi \in C^1(X, I\!R)$ be such that*

$$b := \inf_N \varphi > a := \max_{M_0} \varphi.$$

*If $\varphi$ satisfies the Palais-Smale condition with*

$$c := \inf_{\gamma\in\Gamma} \max_{u\in M} \varphi(\gamma(u)), \quad \text{where } \Gamma := \{\gamma \in C(M, X) \,:\, \gamma_{|M_0} = id\},$$

*then $c$ is a critical value of $\varphi$.*

We consider the functional

$$I(u) = \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \int_\Omega f(x)\, u dx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x,\ u \in H^1(\Omega).$$

Set

$$K = \frac{N^2\|f\|_2^2}{4(N-1)(\lambda_k - \lambda)}.$$

We need the following two Lemmas

**Lemma 4.4.1.**
*Let* $\{u_n\} \subset H^1(\Omega)$ *be a sequence satisfying*

$$I(u_n) \to c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - K \tag{4.4.2}$$

*and*

$$I'(u_n) \to 0 \quad in\, H^{-1}(\Omega) \tag{4.4.3}$$

*then* $\{u_n\}$ *is relatively compact in* $H^1(\Omega)$*.*

*Proof.*
We argue as in the proof of Lemma 4.3.4 and we obtain that $\{u_n\}$ is bounded in $H^1(\Omega)$. Hence, by choosing a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.
By the Concentration-Compactness principle [L] we have

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}$$

and

$$|u_n|^q \rightharpoonup |u|^q + \sum_{j\in J} \nu_j \delta_{x_j}$$

weakly in the sense of measures. Here $x_j \in \partial\Omega$, $\mu_j$ and $\nu_j$ are some positive constants satisfying

$$S_1 \nu_j^{\frac{2}{q}} \leq \mu_j, \tag{4.4.4}$$

and $J$ is a countable set.
Testing (4.4.3) by a family of $C^1$-functions concentrating at $x_j$ with the same $j \in J$ we get

$$p(x_j)\mu_j \leq p(x_j)Q(x_j)\nu_j. \tag{4.4.5}$$

We claim that $\nu_j = 0$ for every $j \in J$, otherwise there exists $j_0 \in J$ such that $\nu_{j_0} > 0$ and from (4.4.4) and (4.4.5) we have

$$\frac{S_1^{N-1}}{Q(x_{j_0})^{N-1}} \leq \nu_{j_0}. \tag{4.4.6}$$

We write

$$\begin{aligned} I(u_n) - \tfrac{1}{q}\langle I'(u_n), u_n\rangle \ &= \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u_n|^2 dx - \lambda \int_\Omega |u_n|^2 dx\right) \\ &\quad - \tfrac{N}{2(N-1)} \int_\Omega f(x)\, u_n dx + o(1). \end{aligned}$$

Letting $n \to +\infty$ we obtain

$$\begin{aligned} c \quad &\geq \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\, u dx \\ &+\frac{1}{2(N-1)}\sum_{j\in J} p(x_j)\mu_j \\ &\geq \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\, u dx \\ &+\frac{1}{2(N-1)}\sum_{j\in J} p(x_j)S_1\nu_j^{\frac{2}{q}} \\ &\geq \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\, u dx \\ &+\frac{1}{2(N-1)} p(x_{j_0})S_1\nu_{j_0}^{\frac{2}{q}}. \end{aligned}$$

Inserting (4.4.6) into the above inequality and using the definition of $x_0$ we find

$$\begin{aligned} c \quad &\geq \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\, u dx \\ &+\frac{1}{2(N-1)}\frac{p(x_{j_0})}{Q(x_{j_{j_0}})^{N-2}}S_1^{N-1} \end{aligned}$$

and

$$\begin{aligned} c \quad &\geq \frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\, u dx \\ &+\frac{1}{2(N-1)}\frac{p(x_0)}{Q(x_0)^{N-2}}S_1^{N-1}. \end{aligned} \tag{4.4.7}$$

On the other hand, combining the fact that $\langle I'(u), v\rangle = 0$ for every $v \in H^1(\Omega)$ and the Remark 4.4.1 we deduce that $u \in E_k^+$ and we have

$$\begin{aligned} &\frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2 dx - \lambda\int_\Omega |u|^2 dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\, u dx \\ &\qquad\qquad \geq \frac{1}{2(N-1)}(\lambda_k - \lambda)\int_\Omega |u|^2 dx - \frac{N}{2(N-1)}\int_\Omega f(x)\, u dx. \end{aligned}$$

Using the Hölder inequality we write

$$\frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\,udx$$

$$\geq \frac{(\lambda_k-\lambda)}{2(N-2)}\|u\|_2^2 - \frac{N}{2(N-2)}\|f\|_2\|u\|_2.$$

Thus

(4.4.8)

$$\frac{1}{2(N-1)}\left(\int_\Omega p(x)|\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right) - \frac{N}{2(N-1)}\int_\Omega f(x)\,udx \geq h(\|u\|),$$

where

$$h(t) = \frac{(\lambda_k-\lambda)}{2(N-1)}t^2 - \frac{N}{2(N-1)}\|f\|_2 t \quad \text{for } t \geq 0.$$

Since $\lambda_k - \lambda > 0$, it follows that $h$ achieves its minimum at $t_0 = \dfrac{N}{2(\lambda_k-\lambda)}\|f\|_2$ and

$$h(t_0) = \min_{t\geq 0} h(t) = -K. \qquad (4.4.9)$$

Inserting (4.4.9) into (4.4.7) we obtain

$$c \geq \frac{1}{2(N-1)}\frac{p(x_0)}{Q(x_0)^{N-2}}S_1^{N-1} - K.$$

This contradicts the hypothesis. Consequently, $\nu_j = 0$ for each $j$ and we deduce the desired result as in the proof of Lemma 4.3.4. □

We now apply the Theorem 4.4.1 with $X = H^1(\Omega)$, $X^- = E_k^-$, $X^+ = E_k^+$ and $\varphi = I$.

We need the following Lemma

**Lemma 4.4.2.**
*There exist constants $\beta > 0$, $\rho > 0$ and $\rho > r$ such that*

$$I(u) \geq \beta \ \textit{for all } u \in N$$

*and*

$$I(u) \leq 0 \ \textit{for all } u \in M_0.$$

*Proof.*
We start by this useful remark: For every $u \in E_k^+$, we have

$$\int_\Omega p(x)|\nabla u|^2 dx \geq \lambda_k \int_\Omega |u|^2 dx.$$

Therefore, there exists constants $C_1 > 0$ and $C_2 > 0$ such that for every $u \in E_k^+$ we have

$$\int_\Omega p(x)|\nabla u|^2 dx \geq C_1 \|u\|^2 \geq C_2 \int_\Omega p(x)|\nabla u|^2 dx. \tag{4.4.10}$$

Using (4.4.10) and the Hölder inequality, we write

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{1}{q}\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \int_\Omega f(x)u dx \\ &\geq \frac{(1-\frac{\lambda}{\lambda_k})}{2} C_1 \|u\|^2 - \frac{C_3}{q}\|u\|^q - \|f\|_2 \|u\|, \end{aligned}$$

where $C_3$ is positive constant.
Thus, we have

$$I(u) \geq h(\|u\|), \tag{4.4.11}$$

where

$$h(t) = \frac{(1-\frac{\lambda}{\lambda_k})}{2} C_1 t^2 - \frac{C_3}{q} t^q - \|f\|_2 t.$$

We see easily that

$$h'(t) = (1 - \frac{\lambda}{\lambda_k(p)}) C_1 t - C_3 t^{q-1} - \|f\|_2$$

and

$$h''(t) = (1 - \frac{\lambda}{\lambda_k(p)}) C_1 - (q-1) C_3 t^{q-2}.$$

We have

$$h''(t_0) = 0 \quad \text{with} \quad t_0 = \left[\frac{(1-\frac{\lambda}{\lambda_k(p)})C_1}{(q-2)C_3}\right]^{\frac{1}{q-2}}.$$

If we let

$$\|f\|_2 < \frac{(1-\frac{\lambda}{\lambda_k})C_1(q-3)}{2(q-2)} t_0, \tag{4.4.12}$$

a easy computation gives that

$$h(t_0) > (q-2) C_1 t_0^2.$$

Consequently, since $q > 2$, using this last inequality and (4.4.11) we obtain the conclusion of Lemma 4.4.2 with $\beta = (q-2)C_1 t_0^2$ and $\delta = t_0$. □

**Proof of Theorem** 4.1.2
We start by checking that

$$c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - K.$$

We have

$$\begin{aligned} I(tW_{\varepsilon,x_0}) &= \frac{t^2}{2}\int_\Omega p(x)|\nabla W_{\varepsilon,x_0}|^2dx - \frac{t^2\lambda}{2}\int_\Omega |W_{\varepsilon,x_0}|^2dx - t\int_\Omega f(x)W_{\varepsilon,x_0}dx \\ &- \tfrac{t^q}{q}\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x. \end{aligned}$$

Using the Dominated Convergence Theorem, we easily see that

$$\int_\Omega f(x)W_{\varepsilon,x_0}dx = \varepsilon^{\frac{N-2}{2}}\int_\Omega \frac{f(x)\phi(x)}{|x-x_0|^{N-2}}dx + o(\varepsilon^{\frac{N-2}{2}}) \tag{4.4.13}$$

Looking at (4.2.2), (4.2.3), (4.2.4) and (4.4.13), a direct computation shows that the cases $N = 3$, $N = 4$ and $N \geq 5$ are different. Therefore we distinguish three cases.
**When** $N \geq 5$**:**
We have
(4.4.14)

$$I(t\,W_{\varepsilon,x_0}) = \frac{t^2}{2}p(x_0)\left(A_1 - A_2H(x_0)\varepsilon\right) - \frac{t^q}{q}p(x_0)Q(x_0)\left(B_1 - B_2H(x_0)\varepsilon\right) + o(\varepsilon)$$

Define

$$g(t) = \frac{t^2}{2}p(x_0)\left(A_1 - A_2H(x_0)\varepsilon\right) - \frac{t^q}{q}p(x_0)Q(x_0)\left(B_1 - B_2H(x_0)\varepsilon\right).$$

It is easy to see that $g$ achieves its maximum at

$$t_0 = \left[\frac{(A_1 - A_2H(x_0)\varepsilon))}{Q(x_0)(B_1 - B_2H(x_0)\varepsilon)}\right]^{\frac{1}{q-2}}.$$

An easy computation yields

$$t_0 = (\frac{A_1}{Q(x_0)B_1})^{\frac{1}{q-2}} - \frac{H(x_0)}{(q-2)A_1}(\frac{A_1}{Q(x_0)B_1})^{\frac{1}{q-2}}\left(A_2 - \frac{A_1B_2}{B_1}\right)\varepsilon + o(\varepsilon). \tag{4.4.15}$$

From (4.3.22) we have $A_2 - \dfrac{A_1B_2}{B_1} > 0$.
Consequently, for $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned} \sup_{t\geq 0} I(t\,W_{\varepsilon,x_0}) &\leq I(t_0\,W_{\varepsilon,x_0}) = \tfrac{t_0^2}{2}\left(A_1 - A_2H(x_0)\varepsilon\right) - \tfrac{t_0^q}{q}\left(B_1 - H(x_0)B_2\varepsilon)\right) + o(\varepsilon) \\ &\leq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - MH(x_0)\varepsilon + o(\varepsilon) \end{aligned}$$

where $M$ is a positive constant.
Therefore, for $\varepsilon > 0$ sufficiently small, if we let

$$\|f\|_2^2 < \varepsilon \frac{2H(x_0)M(N-1)(\lambda_k - \lambda)}{N^2} \tag{4.4.16}$$

we obtain

$$\sup_{t\geq 0} I(t\,W_{\varepsilon,x_0}) < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - K.$$

Thus we deduce that

$$c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - K$$

which is the desired result in this case.
**When** $N = 4$
Combining (4.2.2), (4.2.3), (4.2.4) and (4.4.13), we obtain

$$\begin{aligned} I(t\,W_{\varepsilon,x_0}) &= \tfrac{t^2}{2}p(x_0)\,(A_1 - A_2H(x_0)\varepsilon) - \tfrac{t^q}{q}p(x_0)Q(x_0)\,(B_1 - B_2H(x_0)\varepsilon) \\ &\quad -\varepsilon t\int_\Omega \frac{f(x)\phi(x)}{|x-x_0|^2}dx + o(\varepsilon). \end{aligned} \tag{4.4.17}$$

Since in the case where $N = 4$ we have supposed that $f(x_0) \neq 0$, and using the fact that $f \in C(\bar{\Omega})$, without loss of generality, we can suppose that $f > 0$ in some neighborhood of $x_0$. Then (4.4.17) becomes

$$I(t\,W_{\varepsilon,x_0}) \leq \frac{t^2}{2}p(x_0)\,(A_1 - A_2H(x_0)\varepsilon) - \frac{t^q}{q}p(x_0)Q(x_0)\,(B_1 - B_2H(x_0)\varepsilon) + o(\varepsilon)$$

and we conclude as in the above case.
**When** $N = 3$**:**
From (4.2.2), (4.2.3), (4.2.4) and (4.4.13), we have

$$I(t\,W_{\varepsilon,x_0}) = \frac{t^2}{2}p(x_0)A_1 - \frac{t^q}{q}p(x_0)Q(x_0)B_1 - \varepsilon^{\frac{1}{2}}t\int_\Omega \frac{f(x)\phi(x)}{|x-x_0|}dx + o(\varepsilon^{\frac{1}{2}}). \tag{4.4.18}$$

For $\varepsilon > 0$ sufficiently small, we set

$$h(t) = \frac{t^2}{2}p(x_0)A_1 - \frac{t^q}{q}p(x_0)Q(x_0)B_1 - \varepsilon^{\frac{1}{2}}t\int_\Omega \frac{f(x)\phi(x)}{|x-x_0|}dx \quad \text{for } t \geq 0.$$

Since $h(t)$ goes to $-\infty$ as $t$ goes to $+\infty$, $\sup_{t\geq 0} h(t)$ is achieved at some $t_\varepsilon \geq 0$.
We have

$$h'(t_\varepsilon) = t_\varepsilon p(x_0)(A_1 - t_\varepsilon^{q-2}Q(x_0)B_1) - \varepsilon^{\frac{1}{2}}\int_\Omega \frac{f(x)\phi(x)}{|x-x_0|}dx = 0 \quad \textit{and} \quad h''(t_\varepsilon) \leq 0,$$

thus

$$\left[\frac{A_1}{(q-1)Q(x_0)B_1}\right]^{\frac{1}{q-2}} \leq t_\varepsilon \leq \left[\frac{A_1}{Q(x_0)B_1}\right]^{\frac{1}{q-2}}. \tag{4.4.19}$$

Let

$$t_0 = \frac{1}{2}\left[\frac{A_1}{(q-1)Q(x_0)B_1}\right]^{\frac{1}{q-2}} < t_\varepsilon. \tag{4.4.20}$$

On the other hand, since the function $t \to \frac{t^2}{2}p(x_0)A_1 - \frac{t^q}{q}p(x_0)Q(x_0)B_1$ is increasing on the interval $[0, \left(\frac{A_1}{Q(x_0)B_1}\right)^{\frac{1}{q-2}}]$, we get

$$h(t_\varepsilon) \leq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - \varepsilon^{\frac{1}{2}}t_\varepsilon \int_\Omega \frac{f(x)\phi(x)}{|x-x_0|}dx.$$

Using (4.4.20) we deduce

$$h(t_\varepsilon) \leq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - \varepsilon^{\frac{1}{2}}t_0 \int_\Omega \frac{f(x)\phi(x)}{|x-x_0|}dx. \tag{4.4.21}$$

Inserting (4.4.21) into (4.4.18) we obtain

$$I(t\,W_{\varepsilon,x_0}) \leq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - \varepsilon^{\frac{1}{2}}t_0 \int_\Omega \frac{f(x)\phi(x)}{|x-x_0|}dx + o(\varepsilon^{\frac{1}{2}}).$$

Since in the case where $N = 3$ we have supposed that $f(x_0) \neq 0$ and using the fact that $f \in C(\bar{\Omega})$, without loss of generality, we can suppose that $f > 0$ in some neighborhood of $x_0$. Then, for $\varepsilon > 0$ small enough, if we let

$$\frac{8(\lambda_k - \lambda)t_0}{9}\varepsilon^{\frac{1}{2}} \int_\Omega \frac{f(x)\phi(x)}{|x-x_0|}dx > \|f\|_2^2,$$

we deduce that

$$\sup_{t\geq 0} I(t\,W_{\varepsilon,x_0}) < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}.$$

This completes the proof in this case.

Finally, Theorem 4.1.2 follows from the combination of Lemma 4.4.1, Lemma 4.4.2 and Theorem 4.4.1. This completes the proof.

## 4.5 Appendix

Let $R > 0$ (see the definition of $R$ in Section 1), using the ideas developed in Adimurthi-Yadava [AY], we introduce the following notations

$$x = (x', x_N) \in I\!R^{N-1} \times I\!R \quad and \quad x_0 = (x'_0, x_{0N}),$$

$$B(x_0, R) = \{x \in I\!R^N,\ |x - x_0| < R\},$$

$$B(x_0, R) \cap \Omega = \{(x', x_N) \in B(x_0, R),\ x_N - x_{0N} > \rho(x' - x'_0)\},$$

$$B(x_0, R) \cap \partial\Omega = \{x', x_N) \in B(x_0, R),\ x_N - x_{0N} = \rho(x' - x'_0)\},$$

$$B(x_0, R)^+ = B(x_0, R) \cap \{x_N - x_{0N} > 0\}$$

$$\sum = \{(x', x_N) \in B(x_0, R),\ 0 < x_N - x_{0N} < \rho(x' - x'_0)\},$$

where

$$\rho(x' - x'_0) = \tfrac{1}{2} \textstyle\sum_{i=1}^{N-1} \alpha_i (x_i - x_{0i})^2 + o(|x' - x_0|^2),$$

and $\{\alpha_i\}_{1 \le i \le N-1}$ are the principal curvatures at $x_0$ and $H(x_0) = \frac{1}{n-1} \sum\limits_{i=1}^{N-1} \alpha_i$.

The following relation will be used frequently

$$\int_0^{+\infty} \frac{r^\theta}{(1 + r^2)^\beta} dr = \frac{\Gamma(\frac{\theta+1}{2})\Gamma(\frac{2\beta-\theta-1}{2})}{2\Gamma(\beta)} \quad \text{for } 2\beta - \theta > 1. \tag{4.5.1}$$

Let $a = \frac{R}{k}$ for some $k \in [1, N^2]$ and set $v(x) = p(x)Q(x)u(x)$, $x \in \bar{\Omega}$ and $v = 0$ outside $\partial\Omega$.

### 4.5.1 Appendix 1

Here we give the proof of (4.3.15), (4.3.16), (4.3.17), (4.3.18).

**Proof of (4.3.15):**

We have

$$\begin{aligned}
\int_{\partial\Omega} |W_{\varepsilon,x_0}|^{q-1} v \, ds_z &= \varepsilon^{\frac{N}{2}} \int_{|z'-x_0| \le a} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}} v(z', x_{0N} + \rho(z' - x'_0))}{[(\varepsilon + \rho(z' - x'_0))^2 + |z' - x'_0|^2]^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}) \\
&= \varepsilon^{\frac{N}{2}} \int_{|x'| \le a} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}} v(x'_0 + x', x_{0N} + \rho(x'))}{[(\varepsilon + \rho(x'))^2 + |x'|^2]^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}).
\end{aligned}$$

Using Taylor's expansion of $\rho$, we obtain

$$\frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}}}{[(\varepsilon + \rho(x'))^2 + |x'|^2]^{\frac{N}{2}}} =$$

$$\frac{1 + \frac{1}{2}|\nabla\rho|^2 + o(|\nabla\rho|^2)}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} \left[ 1 - \frac{\frac{N}{2}\rho(x')(2\varepsilon + \rho(x'))}{(\varepsilon^2 + |x'|^2)} + o\left( \frac{\frac{N}{2}\rho(x')(2\varepsilon + \rho(x'))}{(\varepsilon^2 + |x'|^2)} \right) \right].$$

Consequently

$$\begin{aligned}
\varepsilon^{\frac{N}{2}}\int_{\partial\Omega}|W_{\varepsilon,y}|^{q-1}v\,ds_z \; =& \varepsilon^{\frac{N}{2}}\int_{|x'|\leq a}\frac{u(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' \\
&+\frac{\varepsilon^{\frac{N}{2}}}{2}\int_{|x'|\leq a}\frac{|\nabla\rho|^2u(x_0'+x',x_{0N}+\rho(x')}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' \\
&+\varepsilon^{\frac{N+2}{2}}\int_{|x'|\leq a}\frac{N\rho(x')v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx' \\
&+o\left(\varepsilon^{\frac{N+2}{2}}\int_{|x'|\leq a}\frac{N\rho(x')v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx'\right) \\
&= T_1+T_2+T_3+o(T_3).
\end{aligned}$$

We have

$$\begin{aligned}
T_1 &= \varepsilon^{\frac{N}{2}}\int_{|x'|\leq a}\frac{v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' \\
&= \varepsilon^{\frac{N}{2}}\int_{I\!R^{N-1}}\frac{v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx'+O(\varepsilon^{\frac{N}{2}}) \\
&= \varepsilon^{\frac{N-2}{2}}v(x_0)\int_{I\!R^{N-1}}\frac{1}{(1+|z'|^2)^{\frac{N}{2}}}dz'+o(\varepsilon^{\frac{N-2}{2}})
\end{aligned}$$

By changing to polar coordinates and using (4.5.1), we get

$$T_1 = \omega_{N-2}\varepsilon^{\frac{N-2}{2}}v(x_0)\int_0^{+\infty}\frac{r^{N-2}}{(1+r^2)^{\frac{N}{2}}}dr+o(\varepsilon^{\frac{N-2}{2}})$$

$$= \frac{\omega_{N-2}\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})v(x_0)\varepsilon^{\frac{N-2}{2}}}{2\Gamma(\frac{N}{2})}+o(\varepsilon^{\frac{N-2}{2}}). \tag{4.5.2}$$

Since $|\nabla\rho|^2=\sum_{i=1}^{N-1}\alpha_i^2x_i^2+o(|x'|^2)$, then

$$\begin{aligned}
T_2 \;=\; & \varepsilon^{\frac{N}{2}}\sum_{i=1}^{N-1}\alpha_i^2\int_{|x'|^2\leq a}\frac{x_i^2v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' \\
+\; & o\left(\varepsilon^{\frac{N}{2}}\int_{|x'|^2\leq a}\frac{x_i^2v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx'\right).
\end{aligned}$$

We have

$$\varepsilon^{\frac{N}{2}}\sum_{i=1}^{N-1}\alpha_i^2\int_{|x'|\le a}\frac{x_i^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx'$$

$$\le \frac{\sum_{i=1}^{N-1}\alpha_i^2\varepsilon^{\frac{N}{2}}}{(N-1)}\int_{|x'|\le a}\frac{|x'|^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx'.$$

Applying the Dominated Convergence Theorem we find that

$$\int_{|x'|\le a}\frac{|x'|^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N}{2}}}dx' = O(1)$$

and we deduce that

$$T_2 = O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2}{2}}). \tag{4.5.3}$$

By the definition of $\rho$, we have

$$\begin{aligned} T_3 &= \varepsilon^{\frac{N+2}{2}}\frac{N}{2}\sum_{i=1}^{N-1}\alpha_i\int_{|x'|\le a}\frac{x_i^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx' \\ &+o\left(\varepsilon^{\frac{N+2}{2}}\int_{|x'|\le a}\frac{|x'|^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx'\right). \end{aligned}$$

We have

$$\varepsilon^{\frac{N+2}{2}}\frac{N}{2}\sum_{i=1}^{N-1}\alpha_i\int_{|x'|\le a}\frac{x_i^2 v(x_0'+x',x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{N+2}{2}}}dx'$$

$$\begin{aligned} &= \frac{N\sum_{i=1}^{N-1}\alpha_i}{2(N-1)}\varepsilon^{\frac{N}{2}}v(x_0)\int_0^{\frac{a}{\varepsilon}}\frac{r^N}{(1+r^2)^{\frac{N+2}{2}}}dr \\ &\le \frac{N\,v(x_0)\sum_{i=1}^{N-1}\alpha_i}{2(N-1)}\varepsilon^{\frac{N}{2}}\int_0^{+\infty}\frac{r^N}{(1+r^2)^{\frac{N+2}{2}}}dr + o(\varepsilon^{\frac{N}{2}}) \\ &= \frac{N\,v(x_0)\Gamma(\frac{N+1}{2})\Gamma(\frac{1}{2})\sum_{i=1}^{N-1}\alpha_i}{4(N-1)\Gamma(\frac{N+2}{2})}\varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}). \end{aligned}$$

Thus,

$$T_3 \le \frac{Nv(x_0)\Gamma(\frac{N+1}{2})\Gamma(\frac{1}{2})\sum_{i=1}^{N-1}\alpha_i}{4(N-1)\Gamma(\frac{N+2}{2})}\varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2}{2}}). \tag{4.5.4}$$

Finally, combining (4.5.2), (4.5.3) and (4.5.4) we obtain

$$\int_{\partial\Omega}|W_{\varepsilon,x_0}|^{q-1}v\,ds_z = \frac{\omega_{N-2}\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})u(x_0)\varepsilon^{\frac{N-2}{2}}}{2\Gamma(\frac{N}{2})} + o(\varepsilon^{\frac{N-2}{2}}).$$

**Proof of (4.3.16):**

Using the Dominated Convergence Theorem, it follows easily that

$$\int_{\partial\Omega} p(x)Q(x)|u(x)|^{q-2}u(x)\, W_{\varepsilon,x_0} ds_x = \varepsilon^{\frac{N-2}{2}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^{N-2}} ds_x + o(\varepsilon^{\frac{N-2}{2}}).$$

**Proof of (4.3.17) and (4.3.18):**

Since

$$\{|u| \le t_\varepsilon W_{\varepsilon,x_0}\} \subset \{|x| \le k_1 \varepsilon^{\frac{1}{2}}\}$$

and

$$\{|u| \ge t_\varepsilon W_{\varepsilon,x_0}\} \subset \{k_2 \ge |x| \ge k_1 \varepsilon^{\frac{1}{2}}\}$$

following the arguments of the previous calculus, we write

$$\begin{aligned}
\int_{\partial\Omega \cap \{|u| \ge t_\varepsilon W_{\varepsilon,x_0}\}} |u(x)|^{q-2}|v(x)| W_{\varepsilon,x_0} ds_x &\le M_1 \varepsilon^{\frac{N}{2}} \int_0^{k_1\varepsilon^{\frac{1}{2}}} \frac{r^{N-2}}{(\varepsilon^2+r^2)^{\frac{N-2}{2}}} dr \\
&= M_2 \varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}),
\end{aligned}$$

$$\begin{aligned}
\int_{\partial\Omega \cap \{|u| \le t_\varepsilon W_{\varepsilon,x_0}\}} |v||W_{\varepsilon,x_0}|^{q-1} ds_x &\le M_3 \varepsilon^{\frac{N}{2}} \int_{k_1\varepsilon^{\frac{1}{2}}}^{k_2} \frac{r^{N-2}}{(\varepsilon^2+r^2)^{\frac{N}{2}}} dr \\
&= M_3 \varepsilon^{\frac{N-2}{2}} \int_{k_1\varepsilon^{-\frac{1}{2}}}^{k_2\varepsilon^{-1}} \frac{r^{N-2}}{(1+r^2)^{\frac{N}{2}}} dr \\
&= o(\varepsilon^{\frac{N-2}{2}}),
\end{aligned}$$

for some constants $M_i$, $i = 1, 2, 3, 4$. This gives the desired result.

### 4.5.2 Appendix 2

Here we suppose that $N = 3$ and $q = 4$.

We distinguish two cases:

**Case** $u(x_0) > 0$

**Proof of (4.3.26), (4.3.27) and (4.3.28):**

Using the arguments of the derivation of the estimates in Appendix 1, we can write

$$\begin{aligned}
\int_{\partial\Omega} p(x)Q(x)|u(x)|^2|W_{\varepsilon,x_0}|^2 ds_x &\le M_0 \varepsilon \int_{\{|x'| \le a\}} \frac{1}{\varepsilon^2 + |x'|} dx' \\
&\le M_0 \omega_2 \varepsilon \int_0^a \frac{r}{\varepsilon^2 + r^2} dr \\
&\le M_1 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) = o(\varepsilon^{\frac{1}{2}}),
\end{aligned}$$

where $M_0$ and $M_1$ are some positive constants. This gives (4.3.26).

$$\int_{\partial\Omega} v(x)|W_{\varepsilon,x_0}|^3 ds_x = \varepsilon^{\frac{3}{2}} \int_{|x'|\le a} \frac{v(x'_0+x', x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{3}{2}}} dx'$$

$$+o\left(\varepsilon^{\frac{3}{2}} \int_{|x'|\le a} \frac{v(x'_0+x', x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{3}{2}}} dx'\right)$$

$$= \varepsilon^{\frac{3}{2}} \int_{I\!R^{N-1}} \frac{v(x'_0+x', x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{3}{2}}} dx'$$

$$+o\left(\varepsilon^{\frac{3}{2}} \int_{|x'|\le a} \frac{v(x'_0+x', x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{3}{2}}} dx'\right)$$

$$+\varepsilon^{\frac{3}{2}} \left[\int_{I\!R^{N-1}\setminus\{|x'|\le a\}} \frac{p(x)Q(x)u}{|x'|^3} dx' + \int_{\partial\Omega} \frac{p(x)Q(x)(\phi(x)-1)u}{|x'|^3} dx'\right]$$

$$+o(\varepsilon^{\frac{3}{2}})$$

Using a simple change of variable and applying the Dominated Convergence Theorem we deduce that

$$\int_{\partial\Omega} v(x)|W_{\varepsilon,x_0}|^3 ds_x = M_2 v(x_0)\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}})$$

where $M_2$ is a positive constant. This gives (4.3.27).

By the Dominated Convergence Theorem we have

$$\int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x) W_{\varepsilon,x_0} ds_x = \varepsilon^{\frac{1}{2}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^2 u(x)\phi(x)}{|x-x_0|} ds_x + o(\varepsilon^{\frac{1}{2}}).$$

and this gives (4.3.28).

**Case** $u(x_0)=0$

**Proof of (4.3.32)**

Using the arguments of Appendix 1, we remark that in order to estimate

$$\int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x$$

it is enough to estimate

$$\varepsilon^{\frac{3}{2}} \int_{I\!R^{N-1}} \frac{p(x'_0+x', x_{0N}+\rho(x'))Q(x'_0+x', x_{0N}+\rho(x'))u(x'_0+x', x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{3}{2}}} dx'.$$

We have

(4.5.5)

$$\varepsilon^{\frac{3}{2}} \int_{I\!R^2} \frac{p(x'_0+x', x_{0N}+\rho(x'))Q(x'_0+x', x_{0N}+\rho(x'))u(x'_0+x', x_{0N}+\rho(x'))}{(\varepsilon^2+|x'|^2)^{\frac{3}{2}}} dx' =$$

$$\varepsilon^{\frac{1}{2}} \int_{I\!R^2} \frac{p((x'_0+\varepsilon y', x_{0N}+\rho(\varepsilon y'))Q(x'_0+\varepsilon y', x_{0N}+\rho(\varepsilon y'))u(x'_0+\varepsilon y', x_{0N}+\rho(\varepsilon y'))}{(1+|y'|^2)^{\frac{3}{2}}} dy'.$$

Since $p$, $Q$ and $f$ are smooth enough then by the result of [Ch] $u$ is smooth enough. Therefore, we have

$$|u(x) - u(x_0)| \leq C\|u\|_{W^{1,6}(I\!R^3)}|x - x_0|^{\frac{1}{2}} \quad \forall\, x \in I\!R^3 \tag{4.5.6}$$

where $C$ is positive constant and $W^{1,6}(I\!R^3)$ is the Sobolev space.
Inserting (4.5.6) into (4.5.5) and using the fact that $u(x_0)$, we obtain

$$\varepsilon^{\frac{3}{2}} \int_{I\!R^2} \frac{p(x_0' + x', x_{0N} + \rho(x'))Q(x_0' + x', x_{0N} + \rho(x'))u(x_0' + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx'$$

$$\leq \varepsilon p(x_0)Q(x_0)C\|u\|_{W^{1,6}} \int_{I\!R^2} \frac{|y'|^{\frac{1}{2}}}{(1 + |y'|^2)^{\frac{3}{2}}} dy' + o(\varepsilon)$$

Therefore

$$\varepsilon^{\frac{3}{2}} \int_{I\!R^{N-1}} \frac{p(x_0' + x', x_{0N} + \rho(x'))Q(x_0' + x', x_{0N} + \rho(x'))u(x_0' + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx'$$

$$= o(\varepsilon|\log \varepsilon|)$$

and

$$\int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x = o(\varepsilon|\log \varepsilon|)$$

which is the desired result.

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**Résumé**

Cette thèse est consacrée à l'étude de quelques équations aux dérivées partielles non linéaires de type Dirichlet ou Neumann sur un domaine borné régulier, qui sont à structure variationnelle, et qui présentent un défaut de compacité. Dans la première partie, nous étudions une EDP homogène avec un opérateur non linéaire faisant intervenir un poids strictement positif, une non-linéarité critique au sens de Sobolev et un paramètre $\lambda$. Nous établissons des résultats d'existence et de non-existence de solutions qui dépendent du comportement du poids au voisinage de ses minima, du paramètre $\lambda$ et de la géométrie du domaine. Dans la seconde partie, nous nous intéressons à des EDP non homogènes avec poids et avec une non-linéarité critique au bord au sens de l'inclusion de trace. Nous montrons des résultats d'existence qui dépendent des différents coefficients des EDP étudiées et de la courbure moyenne en un point minimum de poids.

**Mots-clés**

Exposant critique de Sobolev, non-linéarité critique au bord, principe de Concentration-Compacité, principe variationnel d'Ekeland.

**Abstract**

This thesis is devoted to the study of some nonlinear partial differential equations of Dirichelet or Neumann type, with a non compact variational structure. In the first part, we study homogeneous PDE with a positive weight, with the critical Sobolev exponent and a parameter $\lambda$. We establish some existence and non-existence results which depend on the behavior of the weight near its minima, the parameter $\lambda$ and the geometry of the domain. In the second part, we are interested in some non-homogeneous PDE with weight and with a critical nonlinearity on the boundary. We show some existence results which depend on the various coefficients of the studied PDE, and of the mean curvature of the boundary of the domain.