Définition combinatoire des polynômes de Kazhdan-Lusztig

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INTRODUCTION

Most of the mathematical objects studied here are algebraic generalizations of tools originally aimed at understanding geometric problems. Thus, when trying to classify regular polytopes (which are a generalization of the Platonic solids), one is naturally led to study the groups of isometries fixing those polytopes; it is in this context that Coxeter found a particularly simple presentation of those groups, generating a group by some of its involutions and taking the order of the product of two involutions as the only additional data to describe the group. This presentation can be isolated and give birth to the concept of a Coxeter group, which by itself yields a flood of intuitive and natural constructions (such as the Bruhat order on the elements of a Coxeter group), making it an algebraic object in its own right.

But as always, the abstract and purely algebraic aspect of the theory should not hide the fact that even the most combinatorial notions were first discovered within their geometric applications. The fact that the Weyl group of a semisimple Lie algebra is a Coxeter group creates a first link in a nicely unified classification theory of semisimple Lie algebras, semisimple linear algebraic groups, crystallographic Coxeter groups, and root systems. The Hecke algebra of a Coxeter group W, which is a deformation of the group algebra of W (and whose "twisted" multiplication rule yields more insight into W than the ordinary group algebra) is a higher-level structure of this algebraic world, originally discovered in the context of finite Chevalley groups (which are a special case of linear algebraic groups; see [5]).

Kazhdan-Lusztig polynomials were first introduced in [18] in 1979, as a family of polynomials appearing naturally as the coordinates of a remarkable basis of the Hecke algebra of a Coxeter group. More precisely, if (W, S) is a Coxeter system and $\mathscr{H}$ is the associated Hecke algebra with basis $(T_w)_{w\in W}$, the vectors of the Kahzdan-Lusztig basis $({C'}_y)_{y\in W}$ of $\mathscr{H}$ are given by

$$C'_y = q^{-\frac{l(y)}{2}} \sum_{x\leq y} P_{x,y}(q) T_x$$

where the $l(y)$ is the length of the element y and $P_{x,y}(q)$ are polynomials in q, called the Kazhdan-Lusztig polynomials of q. An alternative (and less elementary) definition of $P_{x,y}(q)$ can be made, defining the coefficients of $P_{x,y}(q)$ as the dimensions of certain cohomological spaces (see [19]); this is a first result in a deep and not yet fully understood connection between Kazhdan-Lusztig polynomials and algebraic geometry . Those polynomials hold the key to other areas of representation theory, such as Verma module theory and the singularities of Schubert varieties. We refer the interested reader to [1],[2], and [3]. Here we just explain two simple examples.

The degree of $P_{x,y}(q)$ is always $\leq \frac{l(y)-l(x)-1}{2}$, and the coefficient corresponding to degree $\frac{l(y)-l(x)-1}{2}$ in $P_{x,y}(q)$ is an important parameter denoted by $\mu(x, y)$. The action of the endomorphisms T_s on the Hecke algebra $\mathscr{H}$ on the Kazhdan-Lusztig yields a new representation of $\mathscr{H}$. It involves the μ function, as follows: for $s \in S, w \in W$,

$$T_s C_w = \begin{cases} -C_w & \text{if} \quad sw < w \\ qC_w + q^{\frac{1}{2}}(C_{sw} + \sum\limits_{\substack{z<w \\ sz<z \\ \mu(z,w)\neq 0}} \mu(z,w) C_z) & \text{if} \quad sw > w \end{cases}$$

(here as is customary we have replaced the elements (C'_w) by their variant (C_w); the families (C_w) and (C'_w) are both bases of $\mathscr{H}$. See section I.11 for the details). The advantage of this new basis $(C_w)_{w\in W}$ over the ordinary one $(T_w)_{w\in W}$ is that the representation defined above can be canonically decomposed as a direct sum of smaller representations. In fact, there is a canonical partition of W into the so-called left cells (because here we made s act on the left; there is also a partition into right and two-sided cells), such that for each part P the subspace spanned by the C_w for $w \in P$ is invariant by this action of $\mathscr{H}$, yielding an (often finite dimensional) subrepresentation.

Consider the case when W is the Weyl group of a semisimple complex Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{b}$ be a Borel subalgebra containing $\mathfrak{h}$; call S the associated set of simple reflections of W. Let ρ be the linear function on $\mathfrak{h}$ which takes the value 1 on each simple coroot vector. For $w \in W$, denote by M_w the Verma module with highest weight $-w(\rho) - \rho$ and let L_w be its unique irreducible quotient. It was conjectured in [18] (and proved in [2]) that

$$\mathsf{ch}(L_w) = \sum_{z\leq w} (-1)^{l(w)-l(z)} P_{z,w}(1)\mathsf{ch}(M_y)$$
$$\mathsf{ch}(M_w) = \sum_{z\leq w} (-1)^{l(w)-l(z)} P_{w_0z,w_0w}(1)\mathsf{ch}(L_y)$$

Those two examples give a first hint at the various fields in which the Kazhdan-Lusztig polynomials play an important role. Despite those applications, their true nature is not yet fully understood. In particular, the connection between the combinatorics of a Bruhat interval and its associated Kazhdan-Lusztig polynomial is still unclear. It is this latter problem that we address here.

Closely related to the family of Kazhdan-Lusztig polynomials is another family of polynomials, the R-polynomials. They logically precede the Kazhdan-Lusztig polynomials, and the recursion formulas that define them are simpler; they appear as coordinates of the inverses of the ordinary basis elements $T_w (w \in W)$. They are also simpler in many other aspects: the degree of the R-polynomial $R_{x,y}$ is simply $l(y) - l(x)$, while there is no known easy rule to determine the degree of $P_{x,y}$ in general. The central question in this book is whether $P_{x,y}$ only depends on the isomorphism class of the Bruhat interval $[x, y]$. There is an analogous question with $R_{x,y}$ instead of $P_{x,y}$, and from what we have just said it would seem that this latter question is much simpler than the first one. However they turn out to be equivalent (proposition I.11.5), so that a first big reduction in the problem is the replacement of $P_{x,y}$ by $R_{x,y}$. As a matter of fact, all our proofs in parts III and IV only use the R-polynomials.

The "combinatorially invariant" definitions of the R-polynomials that we deal with here proceed in two steps: first, construct an abstract setting in which Bruhat intervals (or at least a large family of Bruhat intervals) can appear as a special case (this is the hard and interesting part), and then define R-polynomials intrisically in this new world (this is the mechanical part, and the proof that the abstract R-polynomials are correctly defined is usually done by a simple induction). Thus, in part III, we show that any lower Bruhat interval can be decomposed as a product of three terms: a "left regular" part, a central "dihedral" part, and a "right regular" part (see definition III.2.4 and theorem III.6.2). In part IV, we extend this decomposition to a larger class of Bruhat intervals, those that enjoy an isomorphism onto a lower Bruhat interval which "preserves" the abovementioned decomposition. In fact, to make things work, some stronger requirements are needed on the isomorphism (see definition IV.1.8); this is essentially a problem of finding the good induction hypothesis.

This report is divided into four parts, the first two dealing with the "old" results and the remaining two with the "new" results.

In part I we reconstruct all the miscellaneous parts of the theory of Coxeter groups, Hecke algebras and Kazhdan-Lusztig polynomials that we need in the sequel. As a result the only prerequisites are some familiarity with linear algebra (including diagonalization of endomorphisms) and with groups defined by generators and relations (our viewpoint is predominantly combinatorial). On many occasions we make a "minimalist" exposition of small parts of a beautiful theory that we cannot dwell on (e.g. the geometry of Coxeter groups, the results in Dyer's thesis, Kazhdan-Lusztig theory, characterizations of the Bruhat order, etc).

In section I.1 we introduce Coxeter groups in a purely algebraic way, as groups defined by generators and relations. In section I.2 we explain the standard geometric realization of a Coxeter group, leading to the fundamental geometric criterion which says that multiplying an element w by a generator s increases or decreases the length of w according to whether the root α_s is "kept on the same side" or not by w (proposition I.2.2). This implies the dichotomy of roots into positive and negative roots, and also the fundamental result (corollary I.2.6), that all the combinatorics of expressions in a Coxeter group can be reduced to the braid relations in dihedral subgroups, i.e. in subgroups of rank two (note that our presentation differs from the usual ones in that we stress corollary I.2.6 instead of the "exchange condition").

This is made more explicit in section I.3, where we explain that the braid relations and the relations $s^2 = e$ for a generator s suffice to reduce any expression in W; furthermore it may be done through a sequence of words of decreasing length (proposition I.3.1). Thus Coxeter groups form a particularly simple subclass of the class of groups defined by generators and relations. Matsumoto's theorem (proposition I.3.3) is an elegant formalization of this fact. At the end of section I.2 we introduce reflections of a Coxeter group (again, from a purely algebraic standpoint, as conjugates of generators), and explain the canonical bijection between reflections and positive roots (in corollary I.2.12).

Section I.4 deals with the properties of parabolic subgroups of Coxeter groups. The results are no surprise and could be guessed even by someone unfamiliar with Coxeter groups: parabolic subgroups are also Coxeter groups, and there is a distinguished representative in each coset (the element of minimal length), yielding a decomposition of W as a direct product. The two-sided decomposition, which is slightly more involved than the left or right ones, is also explained. When those tools to decompose W as a product are pushed to their very end, we obtain the "X-decomposition" of W, which is closely related to the notion of the "Shortlex normal form" of an element $w \in W$; this is the subject of section I.7.

Section I.5 is about the basic properties of the Bruhat order on Coxeter groups. The most fundamental property, from which everything else derives, is the "special matching" property (called the Z-property by Deodhar [14]) (proposition 5.1). Note that later (in part III) we study special matchings in an abstract context, and we could have presented the Bruhat ordering as a special case, but that would have made the presentation more artificial. The Bruhat order is Eulerian (corollary I.5.3) and graded (corollary I.5.5).

In section I.8 we show a link between the notion of algebraic cocycle and Coxeter group. The result, surprising in its simplicity, states that if (W, S) is a pair where W is a group and S is a set of elements of order two generating W, then (W, S) is a Coxeter system if and only if it admits a cocycle (proposition I.8.1). The key idea is to look at the generators $s \in S$ as special cases of reflections $t \in T$; then the cocycle can be interpreted as a descent set. Then follows lemma I.8.2, which shows that reflections can always be written in a simple "palindromic" form. Again, this is very simple compared to the general case in groups defined by generators and relations, where the set of conjugates of an element can be more complicated. In corollary I.8.3 we connect reflections to the Bruhat order more precisely, showing that the Bruhat order can be characterized only in terms of the length function and the set T of all reflections.

Section I.9 is devoted to reflection subgroups, i.e. subgroups of W generated by reflections. They constitute a generalization of the parabolic subgroups of W, and share some properties with them: if W' is a reflection subgroup of W then it is also a Coxeter group, and it even admits a canonical system of generators $S(W')$. This is shown using the cocycle criterion, as the cocycle of $(W', S(W'))$ is simply a restriction of the original cocycle of (W, S). There is a compatibility with respect to Bruhat order, and an element of minimal length in each coset (proposition I.9.3). However, the proofs are more delicate than in the parabolic case, involving a careful use of the interaction between the Bruhat order, the length function and T (the set of reflections). Not all possible identifications turn out to be true (thus the order on W' induced by the Bruhat order of W is strictly stronger than W''s own Bruhat order in general (remark I.9.4)), so that this field can be somewhat confusing to the novice. A major achievement of the theory in section I.9 is the "$K_{3,2}$-avoidance" theorem (I.9.7), which states that in the Bruhat ordering two distinct elements cannot have three coatoms (or three atoms) in common; thus a purely combinatorial statement is deduced from a geometric theory. This theorem in turn has two major corollaries: I.9.8 which states that the dihedral elements are exactly the elements with at most two coatoms, and I.9.9 which states that the coatom set function is injective on the nondihedral elements. Those two corollaries are fundamental in parts III and IV, the last one typically used to prove properties by induction on the length of the element.

In section I.10 we construct the Hecke algebra of a Coxeter group W, which is a deformation of the group algebra of W, and define some low-level tools associated to it, such as the R-polynomials (proposition I.10.5) and the involutions ι and $\jmath$. We then have all the ingredients to define the Kazhdan-Lusztig polynomials in section I.11 (proposition I.11.1), and state the central conjecture of this work (I.11.4), that the Kazhdan-Lusztig polynomial $P_{x,y}$ only depends on the isomorphism class of the Bruhat interval $[x, y]$, along with equivalent reformulations of this conjecture (I.11.5).

Part II is a complement to part I and contains results of a more technical and "local" nature.

Section II.1 develops properties of the B function defined by

$$B(s, i) = s(s-1)(s-2)\dots(s-(i-1))$$
$$B(s, i, j) = B(s, i)B(s+1, i)B(s+2, i)\dots B(s+(j-1), i)$$

The term $B(s,i,j)$ is essentially the elementary block of the normal forms and the X-decomposition (cf. section I.7) in type A. Proposition II.1.7 says that the left descent set of an element can be read off from its X-decomposition. Analogous statements could be made in types other than A, but this is perhaps the only case where such a statement is really useful. Proposition II.1.9 shows that an element has at most one left descent generator and at most one right descent generator if and only if it is of the form $B(s,i,j)$ (the other elements will be a product of two or more $B(s,i,j)$'s). This is used later in section IV.2, where the strategy to tackle a property that we cannot show for all elements is to show it for elements with few descent generators.

Section II.2 details what happens when we decrease some of the coefficients of the Coxeter matrix, passing thus from a Coxeter group W to a "smaller" Coxeter group W'. In this context again, the general case is not more complicated than the dihedral case (when W and therefore W' are both dihedral). It turns out that an expression that is reduced in W' is also reduced in W (proposition II.2.1), and for a certain family of expressions which "mean the same thing in W or W'" (the "absolute" expressions of definition II.2.3) we have canonical isomorphisms between $[e, a_W]$ and $[e, a_{W'}]$ (proposition II.2.4) if a is an absolute expression. Reversing the process and taking an arbitrary $w \in W$, it can be seen that there is an "optimal" W' such that the domain of the canonical isomorphism defined above includes $[e,w]$ (proposition II.2.6). Thus, any $w \in W$ has an "enveloping Coxeter group" W' that is "smaller" than W in general.

Part III is devoted to the proof of the main result of this report, the invariance of Kazhdan-Lusztig polynomials on an interval originating at the identity, which was also discovered independently by Brenti, Caselli& Marietti in [8]. The contents of this part have been put into the article [13].

In order to show an invariance-by-isomorphism result, we naturally seek a purely combinatorial definition of the R-polynomials; this led Du Cloux and Brenti to introduce the notion of a "special matching" (or simply "matching" in this book, as we do not use other types of matchings), which we explain in section III.1. If s is a generator of the Coxeter group, right and left multiplication by s are fundamental examples of special matchings; we call them multiplication matchings and denote them by ρ_s and λ_s. A reasoning used in both [9, definition 6.5] and [7, corollary 5.3] shows that in order to prove conjecture I.11.4 it suffices to check the following rules for any special matching ϕ on a Bruhat interval starting from the origin $[e,v]$ (which are well-known when ϕ is a multiplication matching, cf. (I.10.10)):

$$\forall x, y \in [e,v], \text{ such that } x < \phi(x), y < \phi(y), \qquad \begin{cases} R_{\phi(x),\phi(y)} = R_{x,y} \\ R_{x,\phi(y)} = (q-1)R_{x,y} + qR_{\phi(x),y} \end{cases} \tag{*}$$

We eventually prove this in corollary III.6.3.

The basic idea, already contained in [7], is as follows: let (x,y) be as above, and let $s \in S$ be a (left, say) descent generator for y such that ϕ commutes with (left) multiplication by s. Then we may deduce formula (*) for (x,y) from all the occurrences of formula (*) corresponding to the (x',y') with $l(y') < l(y)$, which provides us with an induction argument on $l(y)$ (proposition 2.5). In general it is not true that any $y \in W$ has such a compatible descent generator. However it will be true for all "sufficiently large" y. We make this precise in III.2.1 when we make the definition that $y \in W$ is "full" if $[e, y]$ contains all the dihedral elements of W. Then we show *in fine* that if W is not a dihedral Coxeter group, any full element in W admits a reduction as above.

For non-full elements $w \in W$, it was shown in section II.2 that the interval $[e, w]$ is isomorphic to an interval $[e, w']$ in a "smaller" (in an appropriate sense) Coxeter group W', the isomorphism preserves R-polynomials, and w' is full, so that we may argue by induction on the "size" of the Coxeter group.

In all our proofs the dihedral elements play a crucial role. In section III.1 we show that a matching is completely determined by its behaviour on dihedral elements; it is even determined by its restriction to the set P of principal dihedral elements (theorem III.1.9). Conversely any matching ϕ defined on P can be extended in an unique way to a matching whose domain is maximal. Similarly, we see in III.3 that commutation between a matching and multiplication by a generator is something that can be read off from their restriction to P (proposition III.3.1).

If $\mathsf{dom}(\phi)$ contains a full element, on each principal dihedral subgroup (which is always stable by ϕ) ϕ cannot "differ too much" from a multiplication matching: we shall see in section III.6 that there is at most one principal dihedral subgroup D such that the restriction of ϕ to D is not a multiplication matching, and even on this D, ϕ must still share some regularity conditions with multiplication matchings.

Technically, a central idea consists in identifying "obstructions" (minimal elements in the complement set of $\mathsf{dom}(\phi)$) whenever ϕ is not a multiplication matching. For example, if $a = \phi(e)$ and $x_0 \in P$ is a minimal element such that $\phi(x_0) \neq x_0 a$, we get obstructions by inserting a well-chosen character in a reduced expression for x_0(this is illustrated in lemmas III.5.3.1 and III.5.4.2). As $\mathsf{dom}(\phi)$ is a decreasing subset of W, any new obstruction erases out a substantial part of W, so that eventually when ϕ is too different from a special matching its domain cannot contain a full element any more.

It is quite remarkable that all the obstructions we need come from rank three subgroups. In section III.4 we describe the simplest types of obstructions and the corresponding restriction on the domain of the matching, appearing in the so-called "mixed" case, which already suffices to treat the case of simply laced Coxeter groups (cf corollary III.4.3). In section III.5 we gather slightly more complicated obstructions that show up in rank three; they are the tools to tackle the general case. The identification of those rank three obstructions was largely guided by computations carried out with a specialized version of the program `Coxeter` [10].

As mentioned before, our result was also found independently by Marietti in his Ph.D. thesis [20], and soon after put into the joint paper [8] by Brenti, Caselli and Marietti, along with other results. The method of proof in [8] is quite close to ours; the main differences are that 1) Brenti, Caselli and Marietti focus on a given interval $[e, w]$, while we try to understand each maximal matching globally; in particular in many cases we are able to determine the domain of a matching, to a large extent (compare lemma III.4.2 and proposition III.6.1) and deduce that it is often rather small; and 2) the "$K_{3,2}$-avoidance" result (theorem I.9.7) allows Brenti, Marietti and Caselli to circumvent some of the lengthy obstruction computations in our section III.5.

Finally, in part IV we discuss some directions in which the theory of part III is likely to be extended; we explain how the original result on interval originating at the identity could perhaps be stretched to cover all completely compressible intervals, by showing that any completely compressible Bruhat interval is isomorphic to a lower Bruhat interval (as in IV.1.3). To achieve this, we develop some tools to construct isomorphisms from an arbitrary interval I onto lower Bruhat intervals. The notion of standard isomorphism (definition IV.1.2), which uniquely defines the image Coxeter group from data intrisic to I, is a first step. But it does not suffice, as there can still exist many distinct standard isomorphisms on the same I. So we also introduce the regularity conditions (definition IV.1.5), which (like the notion of regularity of part III) state that the isomorphism is compatible with multiplication by generators. Even with this additional condition, the isomorphism on I is still nonunique. We then introduce "translations", which are simply isomorphisms θ between Bruhat intervals of the form $\theta(w) = w_1 w w_2$ where w_1, w_2 are fixed elements of W. If we impose compatibility with translations, we obtain the notion of a "totally left-regular" isomorphism (definition IV.1.8), which at last turns out to be unique.

Then we are left with the task of constructing a totally left-regular isomorphism of I. The notion of complete compressibility can be deformed into the notion of "left explicit complete compressibility", by adding into the definition the condition that the various compressions be "explicit", meaning that they must be left multiplications by a generator (up to translation). We prove that a totally left-regular isomorphism does exist on left explicitly completely compressible intervals (proposition IV.1.17). We conjecture that completely compressible intervals coincide with explicitly completely compressible intervals in types A, D, E (conjecture IV.1.21). We provide two different pieces of evidence to support that conjecture: one from massive computations made by the program `Coxeter`[10], treating cases up to types A_8, D_7, E_6, and the other from theoretical results, that we explain in section IV.2.

The main result of that section IV.2 is theorem IV.2.3, which implies that our conjecture is true when the upper extremity y of the interval $I = [x, y]$ is critical (i.e. is an element whose number of coatoms is equal to its length). In type A, this includes the case when y has at most one left descent generator (corollary IV.2.6).

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INDEX OF SPECIAL NOTATIONS

Here we make a list of the notations that are not defined directly in the section or subsection where they appear.

Latin lowercase letters denote integers, words, indeterminates or elements of a Coxeter group. The letter e always stands for the identity element of a Coxeter group, and s and t represent generators or reflections in a Coxeter group. $l(w)$ denotes the length of the element w (or the word w), as in I.1.5, and $m_{s,t}$ is the coefficient of the Coxeter matrix, as in I.1.1.

Uppercase letters denote sets and/or functions.

Sets: (W, S) denotes a Coxeter system, V a real vector space, Φ(resp. Φ^+, Φ^-) denotes the set of (resp. positive, negative) roots, as in I.2.7 and I.2.8, and I stands for a Bruhat interval.

Functions:

$B(\alpha, \beta)$	invariant bilinear form (as in I.(2.6))
$B(s, i, j)$	as in II.(1.1)
$D_l(w)$ ($D_r(w)$)	left (right) descent set of an element w, as in I.1.8
$K(A, B)$	the complete bipartite graph with vertex set $A \cup B$ and edge set $A \times B$
$K_{i,j}$	any $K(A, B)$ with $\lvert A\rvert = i, \lvert B\rvert = j$
$M_{s,t}$	the maximal dihedral element in s and t, as in I.(1.5)
N	the geometric cocycle (as in I.(2.7)) or the algebraic cocycle (as in I.(8.4))
P_s	principal dihedral subgroup associated to the generator s, as in III.1.5
P	the union of all the P_s
$P_{x,y}$ ($R_{x,y}$)	Kazhdan-Lusztig polynomial (R-polynomial), as in proposition I.11.1 (I.10.5)
$Z(\phi, w)$	the set $\{w\} \cup \{\phi(z) \mid z \lhd w, z \lhd \phi(z)\}$

Greek lowercase letters denote mappings (isomorphisms or special matchings). Exceptions: α and β are used to denote roots or indeterminates, and in section I.2, θ represents an angle and λ a real constant.

Other special notations:

$[s, t, j\rangle, \langle j, s, t]$	dihedral words or elements, as in I.(1.1)
$< A >$	the subgroup generated by the subset A
$z = x \cdot y$	means that $z = xy$, $l(z) = l(x) + l(y)$
$A \subseteq B$	A is included in, or equal to, B
$A \subset B$	A is strictly included in B
$A \setminus B$	the elements of A that are not in B
$\vert A \vert$	the number of elements in A
$A \amalg B$	disjoint union of A and B
$\mathcal{P}(A)$	the set of all subsets of A
S^*	the set of all words with letters in the alphabet S
$\{a \vert (...)\}$ or $\{a; (...)\}$	the set of all a's satisfying (...)
$\mathsf{supp}(w)$	the support of the element w, as in I.(3.3)
$[u, v]$	the closed interval between the poset elements u and v
$\mathsf{End}(V)$	the algebra of linear endomorphisms of the real vector space V
$\mathsf{Ker}(f)$	the kernel of the linear mapping f
$x \lhd y$, or $y \rhd x$	$x < y$ and there is no z such that $x < z < y$
$\mathsf{coat}(y)$	the set of coatoms of y, i.e. of x's satisfying $x \lhd y$.
$\mathsf{dom}(\phi)$	domain of the partial special matching ϕ
$\mathcal{I}$	the set of elements of a Coxeter group with a unique reduced representation, or alternatively the associated set of reduced words
λ_a	the special matching defined by $\lambda_a(x) = ax$
ρ_a	the special matching defined by $\rho_a(x) = xa$

Contents

Part I

Preliminaries on Coxeter Groups

1 Coxeter groups

Definition 1.1 *If S is any set, a* **Coxeter matrix** $(m_{st})_{s,t\in S}$ *associated to S is a function $m : S \times S \to \{2, 3, \ldots\} \cup \{\infty\}$ satisfying $m_{ss} = 1, m_{s,t} = m_{t,s}$ for any $s \neq t$ in S.*

The convention $m_{ss} = 1$ is useful to make statements about s, t hold true even when $s = t$. To make our exposition more explicit, however, we will usually discuss whether $s = t$ or not.

Definition 1.2 *Let W be a group and $S \subseteq W$. Denote by e the identity element of W. The pair (W, S) is called a* **Coxeter system** *if there is a Coxeter matrix M associated to S such that W coincides with the group defined by the generators $s \in S$ and the relations*

$$\begin{cases} s^2 = e & (s \in S) \\ (st)^{m_{st}} = e & (s \neq t,\ s, t \in S) \end{cases}$$

(when $m_{st} = \infty$ it is understood that there is no relation imposed on the pair s, t).

In particular, if $m_{st} = 2$ then s commutes with t.

Definition 1.3 *Let W be any group. Then W is said to be a* **Coxeter group** *if there is a subset $S \subseteq W$ such that (W, S) is a Coxeter system.*

In all the examples considered here the set will be finite; the integer $|S|$ is then called the **rank** of the Coxeter system.

Definition 1.4 *Let W be a Coxeter group. We say that W is a* **dihedral Coxeter group** *if there is a subset $S \subseteq W$ such that (W, S) is a Coxeter system and $|S| = 2$.*

Dihedral Coxeter groups are the simplest non-trivial Coxeter groups. They can be simply described: If $S = \{s, t\}$, and if we put

$$[s, t, n\rangle = \underbrace{stst\ldots}_{n\ terms} \tag{1.1}$$

$$\langle n, t, s] = \underbrace{\ldots tsts}_{n\ terms} \tag{1.2}$$

then

$$W = \begin{cases} \{e, s, t, st, ts, sts, tst, \ldots, [s, t, j\rangle, [t, s, j\rangle (j < m) \ldots, [s, t, m\rangle = [t, s, m\rangle\} & \text{if } m = m_{s,t} < \infty \\ \{e, s, t, st, ts, sts, tst, \ldots, [s, t, j\rangle, [t, s, j\rangle (j \in \mathbb{N}), \ldots\} & \text{if } m_{s,t} = \infty \end{cases} \tag{1.3}$$

Note also that the relation $(st)^{m_{st}} = e$ can be written in a more symmetrical form

$$[s, t, m\rangle = [t, s, m\rangle \text{ (with } m = m_{s,t} < \infty) \tag{1.4}$$

This is called a **braid relation**. We will use the notation

$$M_{s,t} = [s, t, m\rangle = [t, s, m\rangle = \langle m, s, t] = \langle m, t, s] (\text{when } m = m_{s,t} < \infty) \tag{1.5}$$

We now consider the ways to write an arbitrary $w \in W$ as a product of generators (i.e. elements of S). For any positive integer n, put

$$E_n(w) = \{(s_1, s_2, \ldots, s_n) \in S^n | w = s_1 s_2 \ldots s_n\}$$

Definition 1.5 *The smallest $n > 0$ such that $E_n(w) \neq \emptyset$ is called the* **length** *of w and is denoted by $l(w)$.*

Definition 1.6 *The elements of $E_{l(w)}(w)$ are called the* **reduced expressions** *of w.*

It is easy to see that for any $s \in S$ and $w \in W$, we have $l(sw) \leq l(w) + 1$, and hence, interchanging w and sw, either $l(sw) = l(w) + 1$ or $l(sw) = l(w) - 1$. In the latter case there is a reduced expression of w starting with s.

Definition 1.7 *Let $s \in S$ and $w \in W$. We say that s is a (left)* **descent generator** *for w if $l(sw) = l(w) - 1$.*

Definition 1.8 *The set of the (left) descent generators of w is called the (left)* **descent set** *of w, and is denoted by $D_l(w)$.*

The right descent set $D_r(w)$ is defined similarly.

Definition 1.9 *Let $x_1, x_2, \ldots, x_n$ be words in W. We say that $x = x_1 x_2 \ldots x_n$ is a* **reduced product** *if $l(x) = l(x_1) + l(x_2) + \ldots + l(x_n)$, and we denote this by $x = x_1 \cdot x_2 \cdot \ldots \cdot x_n$.*

Note that when all the x_i are in S, a reduced product is simply a reduced expression.

Definition 1.10 *Let $x, y \in W$. We say that x is a* **prefix** *(resp.* **suffix***) of y if there is an element $z \in W$ such that $y = x \cdot z$ (resp. $y = z \cdot x$).*

Examples. We give two opposite extremal examples of Coxeter groups; a generic Coxeter group will be somewhere between the two.

When all the coefficients of the Coxeter matrix are equal to ∞ (except those on the diagonal), we obtain the **free Coxeter group** $\mathcal{F}(S)$ on S. In this group, an expression $(s_1, s_2, \ldots, s_n)$ is reduced if and only if $s_{i+1} \neq s_i$ for all i, and if $(s_1, s_2, \ldots, s_n)$ is reduced then $w = s_1 s_2 \ldots s_n$ satisfies $D_l(w) = \{s_1\}, D_r(w) = \{s_n\}$, and the prefixes of w are the elements $s_1, s_1 s_2, \ldots, s_1 s_2 \ldots s_n$.

If, on the contrary, all the (nondiagonal) coefficients of the Coxeter matrix are equal to 2 we obtain a commutative Coxeter group, isomorphic to ${\mathbb{F}_2}^S$. In this group, an expression $(s_1, s_2, \ldots, s_n)$ is reduced if and only if all the s_i are distinct, and if $(s_1, s_2, \ldots, s_n)$ is reduced then $w = s_1 s_2 \ldots s_n$ satisfies $D_l(w) = D_r(w) = \{s_1, s_2, \ldots, s_n\}$, and the prefixes of w are the elements of the form $\prod_{i \in I} s_i$ with $I \subseteq [1, n]$.

2 Geometric representation

One of the nice properties of Coxeter groups is that they enjoy a particularly simple geometric (linear) representation. Indeed, over a vector space V with basis $(\alpha_s)_{s \in S}$ indexed by S, define for each $s \in S$ an endomorphism ϕ_s of V by

$$\begin{cases} \phi_s(\alpha_s) = -\alpha_s \\ \phi_s(\alpha_t) = \alpha_t - \lambda_{s,t}\alpha_s \text{ (for } t \neq s) \end{cases} \tag{2.1}$$

where the $\lambda_{s,t}$ are positive real constants chosen such that $\lambda_{s,t}\lambda_{t,s} = 4\cos^2(\frac{\pi}{m_{s,t}})$. It is easily seen that $\phi_s^2 = id_V$ (in geometric terms ϕ_s is a reflection) and furthermore, if we put $\psi = \phi_s\phi_t (s \neq t)$ we have

$$\begin{cases} \psi(\alpha_s) = (\lambda_{s,t}\lambda_{t,s} - 1)\alpha_s - \lambda_{t,s}\alpha_t \\ \psi(\alpha_t) = \lambda_{s,t}\alpha_s - \alpha_t \end{cases} \tag{2.2}$$

So the subspace $V_0 = \mathsf{Vect}(\alpha_s, \alpha_t)$ is invariant by ψ, and the characteristic polynomial of $\psi_{|V_0}$ is $X^2 - 4\cos^2(\frac{\pi}{m})X + 1 = (X - e^{-\frac{2\pi i}{m}})(X - e^{\frac{2\pi i}{m}})$ (where $m = m_{s,t}$). When m is finite, we have $m \geq 2$ so $\psi_{|V_0}$ has two distinct eigenvalues and is therefore diagonalizable over V_0; there are two vectors β_1 and β_2 such that $\psi(\beta_1) = e^{-\frac{2\pi i}{m}}\beta_1$ and $\psi(\beta_2) = e^{\frac{2\pi i}{m}}\beta_2$. With respect to the basis $\beta_1, \beta_2, (\alpha_u)_{u\notin\{s,t\}}$ of V, the matrix of ψ is upper triangular with $e^{-\frac{2\pi i}{m}}, e^{\frac{2\pi i}{m}}, 1, 1, \ldots 1$ on the diagonal. Thus ψ is diagonalizable and its order is exactly m. When $m = \infty$, the characteristic polynomial of $\psi_{|V_0}$ is $(X-1)^2$ so ψ is non-diagonalizable and of infinite order.

In any case, the order of $\phi_s\phi_t (s \neq t)$ in $GL(V)$ is exactly $m_{s,t}$. We deduce:

Proposition 2.1 *The mapping $s \mapsto \phi_s$ defined in (2.1) can be extended into a linear representation of W in V, i.e. a group homomorphism $W \to GL(V)$.*

The next proposition is a strong property linking Coxeter groups and their geometric representations. Let V_+ be the set of the vectors in V that have nonnegative coordinates (with respect to the basis $(\alpha_s)_{s\in S}$) and $V_- = -V_+$.

Proposition 2.2 *Let $w \in W$ and $s \in S$. Then $\phi_w(\alpha_s) \in V_+$ if $l(ws) = l(w)+1$ and $\phi_w(\alpha_s) \in V_-$ if $l(ws) = l(w) - 1$.*

Proof:

By interchanging w and ws, it suffices to prove the first claim. We achieve this in two steps: first we show it when w is in the dihedral subgroup generated by s and another generator t, and we reduce the general case to this situation.

Lemma 2.3 *Let $s \neq t$ be two generators in S, and $w \in < s, t >$, as in (1.3). Then $\phi_w(\alpha_s) \in V_+$ if $l(ws) = l(w) + 1$.*

Proof: As $l(ws) = l(w) + 1$, w must be of the form $(st)^k$ or $t(st)^k$ for some integer k. But a simple computation yields

$$\begin{cases} (\phi_s\phi_t)^k(\alpha_s) = \frac{\sin(\frac{(2k+1)\theta}{2})}{\sin(\frac{\theta}{2})}\alpha_s + \frac{2\sin(k\theta)\cos(\frac{\theta}{2})}{\lambda\sin(\frac{\theta}{2})}\alpha_t \\ \\ \phi_t(\phi_s\phi_t)^k(\alpha_s) = \frac{\sin(\frac{(2k+1)\theta}{2})}{\sin(\frac{\theta}{2})}\alpha_s + \frac{2\sin((k+1)\theta)\cos(\frac{\theta}{2})}{\lambda\sin(\frac{\theta}{2})}\alpha_t \end{cases} \tag{2.3}$$

where $\theta = \frac{2\pi}{m_{st}}, \lambda = \lambda_{s,t}$. When $m_{st} = \infty$, we have $\theta = 0$ and those formulas become

$$\begin{cases} (\phi_s\phi_t)^k(\alpha_s) = (2k+1)\alpha_s + \frac{4k}{\lambda}\alpha_t \\ \\ \phi_t(\phi_s\phi_t)^k(\alpha_s) = (2k+1)\alpha_s + \frac{4(k+1)}{\lambda}\alpha_t \end{cases} \tag{2.4}$$

so the result follows. Q. E. D.

Lemma 2.4 *Let $s \neq t$ be two generators in S, and $w \in W$. Then there are two words $w' \in W$ and $d \in < s,t >$ such that $w = w' \cdot d$ and neither s nor t are in the right descent set of w'.*

Proof: We argue by induction on $l(w)$. Suppose that the result holds for all elements of length $< l(w)$. If neither s nor t are in the right descent set of w, then we may take $w' = w, d = e$ and we are done (this covers the case $l(w) = 0$). Otherwise, suppose for example that s is in the right descent set of w. Then we can write $w = w_0 s$ with $l(w_0) = l(w) - 1$. By the induction hypothesis, there is a pair $(w_1, d_1) \in W\times < s,t >$ such that $w_0 = w_1 \cdot d_1$ and neither s nor t are in the right descent set of w_1. Then we may take $w' = w_1, d' = d_1 s$, which concludes the proof. Q. E. D.

Let us now finish the proof of proposition 2.2. We reason by induction on $l(w)$. Clearly we may assume $w \neq e$; let t be a generator in the right descent set of w. Then $t \neq s$, and we can write $w = w_1 t$ with $l(w_1) = l(w) - 1$. Using the lemma above (with w_1 instead of w), there is a pair $(w', d_1) \in W\times < s,t >$ such that $w_1 = w' \cdot d_1$ and $D_r(w')$ does not contain s or t. Then $l(d_1 t) \geq l(w' d_1 t) - l(w') = l(w) - l(w') = l(w_1) - l(w') + 1 = l(d_1) + 1$, so if we put $d = d_1 t$, we have $l(d_1 t) = l(d_1) + 1$ and hence $w = w' \cdot d$. Since $D_r(d) \subseteq D_r(w)$, we see that $s \notin D_r(d)$. By lemma 2.3 above, we see that $\phi_d(\alpha_s) = a\alpha_s + b\alpha_t$ with $a, b \geq 0$. Since $l(w') < l(w)$, the induction hypothesis gives $\phi_{w'}(\alpha_s) \in V_+$ and $\phi_{w'}(\alpha_t) \in V_+$; eventually $\phi_w(\alpha_s) = a\phi_{w'}(\alpha_s) + b\phi_{w'}(\alpha_t) \in V_+$. Q. E. D.

Corollary 2.5 *The linear representation defined by (2.1) is faithful, i.e. the morphism $W \to GL(V), w \mapsto \phi_w$ is injective.*

Indeed, if $w \in W \setminus \{e\}$ then there is a generator s in the right descent of w; then $\phi_w(\alpha_s) \in V_-$ whence $\phi_w \neq id_V$.

The next corollary is fundamental.

Corollary 2.6 *Let $s \neq t \in S$ and $w \in W$. Then s and t are both suffixes (resp. prefixes) of w if and only if $m_{st} < \infty$, and M_{st} is a suffix (resp. prefix) of w.*

Proof: Note that the left and right versions of this result are equivalent through the involution $w \mapsto w^{-1}$ of W. Thus, it suffices to show the "suffix" version. The "if" is clear; let us prove the "only if" part. So suppose that s and t are both suffixes of w. Let us write $w = w'd$ as in lemma 2.4.

Suppose that $s \notin D_r(d)$. By lemma 2.4, we have $\phi_d(\alpha_s) = a\alpha_s + b\alpha_t$ with $a, b \geq 0$. Note that a and b cannot both be 0, because the linear map ϕ_d is invertible. Then $\phi_w(\alpha_s) = a\phi_{w'}(\alpha_s) + b\phi_{w'}(\alpha_t) \in V_+$ and $\phi_w(\alpha_s) \neq 0$, whence $\phi_w(\alpha_s) \notin V_-$ which contradicts proposition 2.2.

So $s \in D_r(d)$ and similarly $t \in D_r(d)$. Since $d \in < s,t >$, it is then clear that $m_{st} < \infty$ and $d = M_{st}$, and the result follows. Q. E. D.

From now on we will restrict ourselves to a particular representation in the family of representations defined by (2.1); namely we fix the constants $\lambda_{s,t}$ as follows:

$$\lambda_{st} = \lambda_{ts} = 2|\cos(\frac{\pi}{m_{st}})| \tag{2.5}$$

The immediate consequence of this additional symmetry is that now we have a bilinear symmetric form B on V, defined by (say)

$$\begin{array}{l} B(\alpha_s, \alpha_s) = 1 \\ B(\alpha_s, \alpha_t) = -\frac{\lambda_{st}}{2}(s \neq t) \end{array} \tag{2.6}$$

which is invariant by the action of W on V.

Definition 2.7 *Let (W, S) be a Coxeter system. Let W act on $V = \mathsf{Vect}(\alpha_s)_{s\in S}$ as in 2.1. The elements in the orbit of $\{\alpha_s\}_{s\in S}$ for this action of W (in other words, the elements of the form $\phi_w(\alpha_s)$ with $s \in S, w \in W$) are called* **roots**. *The set of all the roots is denoted by Φ. The α_s are called* **simple roots**.

Definition 2.8 *Let $\Phi_+ = \Phi \cap V_+$ and $\Phi_- = \Phi \cap V_-$. The elements of Φ_+ (Φ_-) are called* **positive** *(resp. negative) roots.*

Thus proposition 2.2 tells us that Φ can be partitioned as $\Phi = \Phi_+ \amalg \Phi_-$. Note that since B is W-invariant, all the roots have norm one, i.e. any $\alpha \in \Phi$ satisfies $B(\alpha, \alpha) = 1$. A natural question that arises is: which roots are sent by a given ϕ_w "to the other side", i.e. what are the elements of

$$N(w) = \{\alpha \in \Phi_+ |\ \phi_w(\alpha) \in \Phi_-\} \tag{2.7}$$

This question can be answered completely:

Proposition 2.9 *Let $w \in W$, and let $w = s_1 s_2 \dots s_r$ be a reduced expression of w. Then*

$$N(w) = \{\alpha_{s_r}, \phi_{s_r}(\alpha_{s_{r-1}}), \phi_{s_r s_{r-1}}(\alpha_{s_{r-2}}), \dots, \phi_{s_r s_{r-1} \dots s_2}(\alpha_{s_1})\}$$

In particular $|N(w)| = l(w)$.

Proof: For $j \in [1, r]$ put $t_j = \phi_{s_r \dots s_{j+2} s_{j+1}}(\alpha_{s_j})$. Notice that the α_{s_j} coordinate of t_k is zero if $k < j$, but nonzero if $k = j$, therefore all the t_j's are distinct. First we show the result for words of length 1.

Lemma 2.10 *If $s \in S$, then $N(\phi_s) = \{\alpha_s\}$.*

Proof: Since $\phi_s(\alpha_s) = -\alpha_s$, it is clear that $\alpha_s \in N(\phi_s)$. Conversely, let $x = \sum_{t\in S} x_t \alpha_t$ be in $N(\phi_s)$. Then $\phi_s(x)$ is of the form $\phi_s(x) = \sum_{t\neq s} x_t \alpha_t + (...)\alpha_s$. Since $x \in V_+$, we have $x_t \geq 0$ for all $t \in S$. Since $\phi_s(x) \in V_-$, we have $x_t \leq 0$ for all $t \neq s$. Consequently x reduces to $x_s \alpha_s$, with $x_s > 0$. Since x has norm 1, we eventually deduce $x = \alpha_s$. Q. E. D.

We now return to the proof of proposition 2.9. Put $w' = s_1 s_2 \dots s_{r-1}$. Using the lemma above and the definition of N, we easily see that

$$N(w) = N(s_r) \cup \phi_{s_r}(N(w')) \tag{2.8}$$

so the result follows by induction on r. Q. E. D.

Definition 2.11 *A* **reflection** *in W is a conjugate of a generator, i.e. an element of the form wsw^{-1} with $w \in W$ and $s \in S$. The generators $s \in S$ are called* **simple reflections**. *The set of all reflections of W is denoted by T.*

Corollary 2.12 *(i) For each $t \in T$ there is a unique $\alpha \in \Phi^+$ such that $t(\alpha) = -\alpha$. Conversely, for each $\alpha \in \Phi^+$ there is a unique $t \in T$ such that $t(\alpha) = -\alpha$.*
Thus, there is a natural bijection between T and Φ^+. We denote by α_t the root associated to the reflection t, and by t_α the reflection associated to the root α.
(ii) Let $\alpha \in \Phi^+$. We have $t_\alpha(x) = x - 2B(x,\alpha)\alpha$ for all $x \in V$.
(iii) Let $s \in S$ and $\beta \in \Phi^+$. Then $B(\alpha_s, \beta) \leq 0$ if $s \notin D_r(t_\beta)$ and $B(\alpha_s, \beta) \geq 0$ if $s \in D_r(t_\beta)$.

Proof: (i) For $t \in T$, define $E(t) = \{\alpha \in \Phi | t(\alpha) = -\alpha\}$. Then it is clear that $E(wtw^{-1}) = wE(t)$ for any $w \in W$. Now any $t \in T$ can be written wsw^{-1} with $w \in W$ and $s \in S$, and lemma 2.10 yields $E(s) = \{\pm\alpha_s\}$. Therefore $E(wsw^{-1}) = \{\pm\phi_w(\alpha_s)\}$, and (i) follows.

(ii) There is a pair $(s, w) \in S \times W$ such that $\alpha = w(\alpha_s)$. If $w = e$ then the result is clear by (2.1). Thus it suffices to show that the result is true for $w_1 w'$ if it is already true for w' (where $w_1 \in S$). So assume that $t_\beta(x) = x - 2B(x,\beta)\beta$ for all $x \in V$, where $\beta = w'(\alpha_s)$. Put $\beta' = w_1(\beta)$. Then for $x \in V$ we have

$$\begin{aligned} t_{\beta'}(x) &= w_1 t_\beta w_1(x) \\ &= w_1(w_1(x) - 2B(w_1(x),\beta)\beta) \\ &= x - 2B(w_1(x),\beta)w_1(\beta) \\ &= x - 2B(x, w_1(\beta))w_1(\beta) \text{ since } B \text{ is invariant} \\ &= x - 2B(x,\beta')\beta' \end{aligned}$$

as required.

(iii) Interchanging β and βs, it suffices to treat the case when $s \notin D_r(t_\beta)$ (say). We can write $\beta = \sum_{u \in S} b_u \alpha_u$ where all the b_u are nonnegative. If β is a multiple of α_s, then $B(\beta,\beta) = B(\alpha_s,\alpha_s) = 1$ yields $\beta = \pm\alpha_s$, hence $\beta = \alpha_s$ since $\beta \in \Phi^+$, and this contradicts $s \notin D_r(t_\beta)$. So β is not a multiple of α_s; then there is a $u_0 \in S \setminus \{s\}$ such that $b_{u_0} > 0$. By proposition 2.2, we have $t_\beta(\alpha_s) \in \Phi^+$. But $t_\beta(\alpha_s) = \alpha_s - 2B(\alpha_s,\beta)\beta$, so the u_0-component of $t_\beta(\alpha_s)$ of $t_\beta(\alpha_s)$ is $-2b_{u_0}B(\alpha_s,\beta)$. We deduce $-2b_{u_0}B(\alpha_s,\beta) \geq 0$, and $B(\alpha_s,\beta) \leq 0$. Q. E. D.

3 Some basic algorithms and combinatorics

Corollary 2.6 alone suffices to answer, and provide algorithms for, most of the elementary and natural questions that can be asked about Coxeter groups viewed as groups defined with generators and relations. Those algorithms, albeit impractical for heavy computations, are quite sufficient for the theoretical or limited-to-simple-examples use we will make of it. We use the standard notation S^* to denote the set of words with letters in the alphabet S. We distinguish the words (in S^*) from the elements (in W) by using the tuple notation for the former ones.

Proposition 3.1 *(i) Two reduced expressions in S^* represent the same element in W if and only if we may obtain one from the other using only the braid relations (1.4) as rewriting rules, through a sequence of words of constant length.*
(ii) Let a be any expression in S^*. Then we may obtain from a a reduced expression a' representing the same element in W, using only the braid relations (1.4) and the relations $s^2 = e$ as rewriting rules, through a sequence of words of decreasing length.

Clearly (i) provides an algorithm that enumerates the reduced expressions equivalent to a given reduced expression, and (ii) tells us how to reduce any expression at all.

Proof: (i) The "if" is clear; let us show the "only if". So let $a = (a_1, a_2, \dots, a_r)$ and $b = (b_1, b_2, \dots b_r)$ be two reduced expressions such that $a_1 a_2 \dots a_r = b_1 b_2 \dots b_r = w$ holds in W; we must show that there is a path from a to b as in the statement of the proposition (we will call such paths admissible). We argue by induction on r, the result being clear if $r = 1$. If $a_1 = b_1$, then $a_2 \dots a_r = b_2 \dots b_r$, the induction hypothesis provides an admissible path between $(a_2, \dots, a_r)$ and $(b_2, \dots, b_r)$, and we are done. Otherwise, by corollary 2.6, $m_{a_1 b_1} < \infty$ and $M_{a_1 b_1}$ is a prefix of w. Thus there are generators $s_1, s_2, \dots s_l$ (with $l = r - m_{a_1 b_1}$) such that $w = M_{a_1 b_1} s_1 s_2 \dots s_l$. Then $a_2 a_3 \dots a_r = [b_1, a_1, m_{a_1 b_1} - 1\rangle s_1 s_2 \dots s_l$, so by the induction hypothesis there is an admissible path from $(a_2, a_3, \dots a_r)$ to $([b_1, a_1, m_{a_1 b_1} - 1\rangle, s_1, s_2, \dots s_l)$, whence an admissible path from a to $([a_1, b_1, m_{a_1 b_1}\rangle, s_1, s_2, \dots, s_l)$. Adding a braid relation, we have an admissible path from a to $c = ([b_1, a_1, m_{a_1 b_1}\rangle, s_1, s_2, \dots s_l)$. So now we only need to find an admissible path between c and b. But since c and b share the same leftmost character, this case has already been treated.

(ii) Write $a = (a_1, a_2, \dots, a_r)$. We argue by induction on r, the result being clear if $r = 1$. By the induction hypothesis, we may assume that $(a_1, a_2, \dots, a_{r-1})$ is reduced.

If a is already reduced, there is nothing to be done. If $a_r = a_{r-1}$, using the rule $a_r^2 = e$ we delete the two rightmost characters and we are done. So we may assume that a is not reduced and that $a_r \neq a_{r-1}$. But since $l(w' a_r) = l(w') - 1$ or $l(w') + 1$ (where $w' = a_1 a_2 \dots a_{r-1}, w = w' a_r$), this implies $l(w) = r - 2$. So there are generators $s_1, s_2, \dots, s_{r-2}$ such that $w = s_1 s_2 \dots s_{r-2}$. Then $s_1 s_2 \dots s_{r-2} a_r$ is reduced expression of w', and we see that $a_r \in D_r(w')$. Then, using (i) we may transform $(a_1, a_2, \dots, a_{r-1})$ into an expression ending with a_r, which brings us back to the case $a_r = a_{r-1}$ that has been dealt with already. Q. E. D.

Corollary 3.2 *Let $a = (a_1, a_2, \dots, a_r)$ be an expression such that the subwords $(a_1, a_2, \dots, a_{r-1})$ and $(a_2, a_3, \dots, a_r)$ are both reduced, but a is not. Then $a_1(a_2 \dots a_{r-1})a_r = a_2 \dots a_{r-1}$. In other words, if $w \in W$ and $s, t \in S$ satisfy $l(sw) = l(w) + 1$ and $l(wt) = l(w) + 1$ but $l(swt) < l(w) + 2$, then $sw = wt$.*

Proof: By (ii) above we can apply braid rewritings a certain number of times and eventually reach an expression which contains identical successive characters. If one of this braid manipulations involves the whole expression, it means that our initial a was of the form M_{st} from the start, and the result is clear in this case. So we may assume that all those braid manipulations involve only strict subwords of a. But then the expression a' obtained after all those braid rewritings still has the property that a' is nonreduced but becomes reduced if the leftmost or rightmost character is deleted. Thus we may assume that $a' = a$, i.e. that $a_j = a_{j+1}$ for some j. Since $(a_1, a_2, \dots, a_{r-1})$ is reduced we have $j = r - 1$, and since $(a_2, a_3, \dots, a_r)$ is reduced we have $j = 1$. Eventually $r = 2$ and we are done. Q. E. D.

Recall that a monoid is a pair $\mathcal{M} = (M, \cdot)$ such that M is a set and $\cdot$ is an associative operation on M. Part (i) of proposition 3.1 can be nicely formalized as follows:

Proposition 3.3 *(***Matsumoto***) Let $\mathcal{M} = (M, \cdot)$ be a monoid. Let $f : S \to M$ be a function such that*

$$[f(s), f(t), m_{st}\rangle = [f(t), f(s), m_{st}\rangle, \text{ whenever } s \neq t \in S, m_{st} < \infty \tag{3.1}$$

Then f can be extended into a unique map $F : W \to M$ such that

$$F(s_1 s_2 s_3 \dots s_r) = f(s_1) f(s_2) \dots f(s_r) \text{ for any reduced expression } (s_1, s_2, \dots s_r). \tag{3.2}$$

Here are some typical applications of this result: if we take $\mathcal{M}$ to be $(\mathcal{P}(S), \cup)$ (the set of subsets of S, with union) and $f(s) = \{s\}$ we obtain a function $\mathsf{supp} : W \to \mathcal{P}(S)$ ($\mathsf{supp}(w)$ is called the **support** of w) such that for any reduced expression $(s_1, s_2, \dots s_r)$,

$$\mathsf{supp}(s_1 s_2 s_3 \dots s_r) = \{s_1, s_2, \dots s_r\} \tag{3.3}$$

Also, if we take $\mathcal{M}$ to be $(\mathcal{P}(W), \star)$ where the star operation is defined as $A \star B = \{ab |\ a \in A, b \in B\}$, and $f(s) = \{e, s\}$, we obtain a function $I : W \to \mathcal{P}(W)$ such that for any reduced expression $(s_1, s_2, \dots s_r)$,

$$I(s_1 s_2 s_3 \dots s_r) = \{s_{i_1} s_{i_2} \dots s_{i_k} |\ 1 \leq i_1 < i_2 < \dots < i_k \leq r\} \tag{3.4}$$

We then define an ordering on W, called the **Bruhat-Chevalley ordering**, or simply Bruhat ordering, by $u \leq v$ if and only if $u \in I(v)$. We develop the properties of this ordering in section 5.

4 Subgroups and cosets of Coxeter groups

Proposition 4.1 *Let (W, S) be a Coxeter system, and let $J \subseteq S$. Then $(< J >, J)$ is again a Coxeter system.*

Proof: Let (W', S') be the Coxeter system defined by $S' = J$ and the Coxeter matrix $m'_{st} = m_{st}$ for $s, t \in J$. Clearly, we have a canonical surjection $\pi : W' \to < J > \subseteq W$, and we must show that π is in fact an isomorphism. Proposition 3.1.(ii) shows that there is a way to decide if an expression is reduced that uses the Coxeter matrix only; therefore $s_1 s_2 \dots s_r$ is reduced in W' if and only if $\pi(s_1)\pi(s_2) \dots \pi(s_r)$ is reduced in W, for any $s_1, s_2, \dots, s_r \in J$. Thus π preserves length and is a isomorphism. Q. E. D.

The subgroups $< J >$ are called **parabolic subgroups** of W. We define the sets

$${}^J W = \{w \in W |\ l(jw) = l(w) + 1 \text{ for any } j \in J\} \tag{4.1}$$

Lemma 4.2 *If $j \in < J >$ and $w \in {}^J W$, then $l(jw) = l(j) + l(w)$. Thus*

$${}^J W = \{w \in W |\ w \text{ has minimal length in its coset } < J > w\}$$

Proof: Let $j = j_1 j_2 \dots j_r$ be a reduced expression for j. We argue by induction on r, the result being clear if $r = 1$. Put $j' = j_2 \dots j_r$. Define a two-periodic sequence (s_k) by $s_1 = j_1$ and $s_2 = j_2$. Using lemma 2.4 and changing $(j_3, \dots, j_r)$ if necessary, we may assume that $j_3 = s_3, j_4 = s_4 \dots$ up to a last index x such that $j_x = s_x$ and $j'' = j_{x+1} j_{x+2} \dots j_r$ is such that neither j_1 nor j_2 are in the left descent set of j''. By the induction hypothesis we have $l(j'w) = l(j') + l(w)$, and hence $l(jw) = l(j) + l(w) - 2$ or $l(j) + l(w)$. Suppose by contradiction that $l(jw) = l(j) + l(w) - 2$.

Then j_1 is in the left descent set of $j'w$, so by corollary 2.6, $m_{j_1j_2}$ is finite and $M_{j_1j_2}$ is a prefix of $j'w$. In other words, (putting $m = m_{j_1j_2}$) $s_2s_3\ldots s_{m+1}$ is a prefix of $s_2s_3s_4\ldots s_xj''w$. Thus $s_{x+1}s_{x+2}\ldots s_{m+1}$ is a prefix of $j''w$, and in particular s_{x+1} is in the left descent set of $j''w$. Then, if we put $j''' = s_{x+1}j''$, we have $l(j''') = l(j'') + 1 \leq l(j) - 1$ (since neither j_1 nor j_2 are in the left descent set of j'') and $l(j'''w) = l(j''w) - 1 < l(j''') + l(w)$, which contradicts the induction hypothesis. Q. E. D.

Definition 4.3 *Let (W, S) be a Coxeter system. A left (resp. right)* **parabolic coset** *in W is a subset of W of the form $< J > w$ (resp. $w < J >$) with $w \in W, J \subseteq S$.*

Proposition 4.4 *Let (W, S) be a Coxeter system.*
(i) A left parabolic coset in W has a unique element of minimal length.
(ii) Any $w \in W$ can be written uniquely $j \cdot w'$ with $j \in< J >$ and $w' \in {}^JW$.

Proof: Clearly (ii) follows immediately from (i) and lemma 4.2, so that we only need to show (i). Suppose by contradiction that there are two minimal elements w_1 and w_2 in the same coset. Then there is a $j \in< J >$ such that $w_2 = jw_1$, whence $l(w_2) = l(j) + l(w_1)$ by lemma 4.2. But $l(w_1) = l(w_2)$ by assumption, so $l(j) = 0$ and $w_1 = w_2$. Q. E. D.

Of course those results can be symmetrized into a "right" version, and we may thus define for $K \subseteq S$

$$W^K = \{w \in W|\ l(wk) = l(w) + 1 \text{ for any } k \in K\} \tag{4.2}$$

We are now going to expound the analogous results for two-sided actions, which are slightly more complicated. Let J, K be two subsets of S. Call a subset K' of K **distinguished** (with respect to J) if it is of the form $(w^{-1}Jw) \cap K$ for some $w \in W$. Let

$${}^JW^K = ({}^JW) \cap (W^K) \tag{4.3}$$

Then lemma 4.2 becomes

Lemma 4.5 *(i) Let $j \in< J >$ and $w \in {}^JW^K$. Let K' be the associated distinguished subset: $K' = (w^{-1}Jw) \cap K$, and let $k_0 \in {}^{K'}K$. Then $l(jwk_0) = l(j) + l(w) + l(k_0)$.*

(ii) Let j, w, K' be as in (i) above and let $k \in< K >$. Write $k = k' \cdot k_0$ with $k' \in< K' >$ and $k_0 \in {}^{K'} < K >$, as in proposition 4.4.(ii). Then the element $j' = wk'w^{-1}$ is in $< J >$, $jwk = jj'wk'$ and $l(jwk) = l(jj') + l(w) + l(k')$. Thus

$${}^JW^K = \{w \in W|\ w \text{ has minimal length in its coset } < J > w < K >\}$$

Proof: (i) We must show that for any $k \in {}^{K'} < K >$, we have $w \cdot k \in {}^JW$. We do this by induction on $r = l(k)$, the result being clear if $r = 0$. Let $k = k_1k_2\ldots k_r$ be a reduced expression for k; put $k^- = k_1k_2\ldots k_{r-1}$. Then $w \cdot k^- \in {}^JW$, by the induction hypothesis. Suppose, by contradiction, that $w \cdot k \notin {}^JW$. Then there is a $j_1 \in J$ such that $j_1 \in D_r(wk)$. But $j_1 \notin D_r(wk^-)$, so that corollary 3.2 yields $wk = j_1wk^-$. Thus, if we put $k'' = k(k^-)^{-1} \in< K >$, we have $j_1w = wk''$. We deduce $l(k'') = 1$ and hence $k'' \in K'$, which contradicts $k \in {}^{K'} < K >$.

(ii) This is now clear, thanks to (i). Q. E. D.

Definition 4.6 *Let (W, S) be a Coxeter system. A* **two-sided parabolic coset** *in W is a subset of W of the form $< J > w < K >$ with $w \in W, J, K \subseteq S$.*

And proposition 4.4 becomes

Proposition 4.7 *Let (W, S) be a Coxeter system.*
(i) A two-sided parabolic coset in W has a unique element of minimal length.
(ii) Any $w \in W$ can be written uniquely $j \cdot w' \cdot k$ with $j \in < J >, w' \in {}^J W^K$, and $k \in {}^{K'} < K >$ (where K' is the distinguished subset of K associated to w: $K' = (w^{-1} J w) \cap K$).

The proof is very similar to the one for proposition 4.4 and we omit it.

5 The Bruhat ordering

The Bruhat-Chevalley ordering on a Coxeter group W was defined at the end of section 3. Note that $x \leq y \Rightarrow l(x) \leq l(y)$ and that for any $y \in W$, $[e, y] = I(y)$ (in the sense of (3.4)) so that $[e, y]$ (and hence any interval $[x, y]$ as well) is finite. Here is one of the basic properties of the Bruhat order (which incidentally provides an algorithm for deciding if two elements are in Bruhat order):

Proposition 5.1 *Let $x, y \in W$ and $s \in D_l(y)$, so that $sy < y$. Then*
(i) If $s \in D_l(x)$, then $x \leq y$ if and only if $sx \leq sy$.
(ii) If $s \notin D_l(x)$, then $x \leq y$ if and only if $x \leq sy$.
In other words, if $x^- = \mathsf{min}(x, sx)$, then $x \leq y$ if and only if $x^- \leq sy$.

Later we will see that this means that left multiplication by s is a "special matching" with respect to the Bruhat ordering.

Proof: Let $sy = y_2 \ldots y_r$ be a reduced expression of sy.
(i) Let $x' = sx$. Then $(x \leq y) \Leftrightarrow$ (Some subexpression of $(s, y_2, \ldots, y_r)$ represents $s \cdot x'$) $\Leftrightarrow$ (Some subexpression of $(y_2, \ldots, y_r)$ represents x') $\Leftrightarrow (x' \leq y)$.
(ii) Since $sy < y$, the "if" is clear. Conversely, suppose $x \leq y$. Putting $y_1 = s$, there is a sequence $1 \leq i_1 < i_2 < \ldots < i_k \leq r$ such that $x = y_{i_1} \ldots y_{i_k}$. Then $i_1 > 1$ since $s \notin D_l(x)$. We then deduce $x \leq y_2 y_3 \ldots y_r = sy$. Q. E. D.

Corollary 5.2 *Let x, y, s be as in the proposition above.*
(i) If $s \notin D_l(x)$, then the interval $[x, y]$ is invariant by left multiplication by s.
(ii) If $s \in D_l(x)$, then $[sx, y] \setminus [x, y] = [sx, sy] \setminus [x, sy]$.

Proof: (i) Let $w \in [x, y]$. If $sw < w$, then $x < sw$ by proposition 5.1.(ii). and $sw < w < y$ by transitivity. If $w < sw$, then $sw < y$ by proposition 5.1.(ii). and $x < w < sw$ by transitivity. In both cases we have $sw \in [x, y]$.

(ii) Let $A = [sx, y] \setminus [x, y]$ and $B = [sx, sy] \setminus [x, sy]$. Then $A = \{w \in [sx, y] | \ w \ngeq x\}$ and $B = \{w \in [sx, sy] | \ w \ngeq x\}$. It is clear then that $B \subseteq A$, and all that remains to be shown is that $w \leq sy$ for any $w \in A$. But if $w \in A$ we have $s \notin D_l(w)$ (otherwise $w \geq x$ by 5.1.(ii)) and hence, by 5.1.(ii) again, $w \leq sy$. Q. E. D.

Next we show that the Bruhat ordering is "Eulerian":

Corollary 5.3 *A Bruhat interval has as many elements of odd length as elements of even length.*

Proof: We use the notation and results of the corollary above; we must show that $[x, y]$ has as many elements of odd length as elements of even length. We argue by induction on $l(y) - l(x)$, the result being clear if $l(y) = l(x)$. If $s \notin D_l(x)$, then left multiplication by s provides a bijection between the elements of odd length and those of even length in $[x, y]$, and we are done. So suppose $s \in D_l(x)$. For any finite $X \subseteq W$, denote by $\delta(X)$ the number of odd elements minus the number of even elements in X (so that we must show $\delta([x, y]) = 0$). Then

$$\delta([sx, y]) - \delta([x, y]) = \delta([sx, sy]) - \delta([x, sy])$$

But $\delta([w, sy]) = 0$ for any $w \in \{x, sx\}$ by the induction hypothesis, and $\delta([sx, y]) = 0$ by the involution argument above. This concludes the proof. Q. E. D.

From this last result we immediately deduce the following:

Corollary 5.4 *Let $I = [x, y]$ be a Bruhat interval of length 2, which means that $x \leq y, l(y) = l(x) + 2$. Then there are two elements w_1, w_2 with $l(w_1) = l(w_2) = l(x) + 1$ such that $I = \{x, w_1, w_2, y\}$.*

The Bruhat order is also "graded":

Proposition 5.5 *If $x \leq y \in W$ there is a finite chain $x = x_1 < x_2 \ldots < x_{n-1} < x_n = y$ such that $l(x_{i+1}) = l(x_i) + 1$ for all $i \in [1, n]$.*

Proof: We argue by induction on $l(x) + l(y) = d$. If $d = 0$ then $x = y = e$ and we are done. So suppose $d > 0$; then $y > e$. Let $s \in D_l(y)$.

If $s \notin D_l(x)$, then $x \leq sy$ and by the induction hypothesis there is a finite chain $x = x_1 < x_2 \ldots < x_{n-1} < x_n = sy$ such that $l(x_{i+1}) = l(x_i) + 1$ for all $i \in [1, n]$. Then all we need to do is extend that chain with $x_{n+1} = y$.

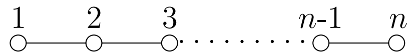
If $s \in D_l(x)$, then $sx < x \leq y$ and by the induction hypothesis there is a finite chain $sx = x_1 < x_2 \ldots < x_{n-1} < x_n = y$ such that $l(x_{i+1}) = l(x_i) + 1$ for all $i \in [1, n]$. Since $s \notin D_l(x_1)$ and $s \in D_l(x_n)$, there is a smallest i such that $s \in D_l(x_i)$. In particular, $i > 1$ and $s \notin D_l(x_{i-1})$. We deduce $x_{i-1} \leq sx_i$, hence $x_{i-1} = sx_i$ since those two words have the same length. If we define a chain $x' = (x'_1, \ldots, x'_{n+1})$ by $x'_j = sx_j$ for $j < i$, $x'_j = x_{j+1}$ for $i \leq j \leq n - 1$. Then x' is a chain joining x to y as required. Q. E. D.

6 Examples of Coxeter groups

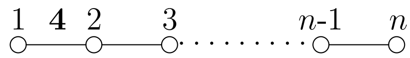
In this very short section we only gather, for completeness' sake, the definition of some classical Coxeter systems that we will use (very briefly), and do not mention the well-known classification results and properties. The reader is referred to [17]. Below n is a positive integer and $S = \{1, 2, \ldots, n\}$. A Coxeter system is uniquely defined by its **Coxeter graph**, which is a labeled graph with vertex set S and edge set $E = \{\{s, s'\} | s, s' \in S, s \neq s', m_{s,s'} \geq 3\}$, and each edge $\{s, s'\} \in E$ is labeled $m_{s,s'}$. When $m_{ss'} = 3$, the most frequent case, the label 3 is omitted on the drawings. We put

$E' = \{\{s, s'\} \in E | m_{s,s'} > 3\}$.

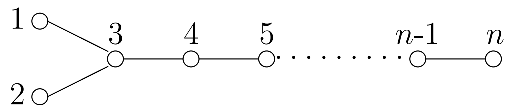
If $E = \{\{j, j+1\} | 1 \leq j \leq n-1\}, E' = \emptyset$ then (W, S) is called a Coxeter system of type A_n. The corresponding Coxeter graph is



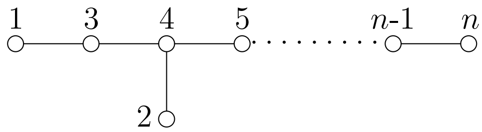
If $E = \{\{j, j+1\} | 1 \leq j \leq n-1\}, E' = \{\{1, 4\}\}$ and $m_{1,2} = 4$, then (W, S) is called a Coxeter system of type B_n. The corresponding Coxeter graph is



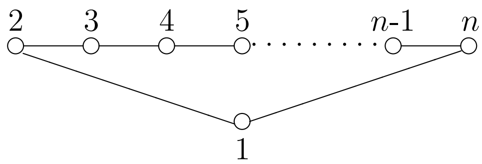
If $n \geq 3, E = \{\{1, 3\}\} \cup \{\{j, j+1\} | 2 \leq j \leq n-1\}, E' = \emptyset$, then (W, S) is called a Coxeter system of type D_n. The corresponding Coxeter graph is



If $6 \leq n \leq 8, E = \{\{1, 3\}, \{2, 4\}\} \cup \{\{j, j+1\} | 3 \leq j \leq n-1\}, E' = \emptyset$, then (W, S) is called a Coxeter system of type E_n. The corresponding Coxeter graph is



If $n \geq 3, E = \{\{1, n\}\} \cup \{\{j, j+1\} | 2 \leq j \leq n-1\}, E' = \emptyset$, then (W, S) is called a Coxeter system of type $\tilde{A}_n$. The corresponding Coxeter graph is



7 The Shortlex normal form

Consider a Coxeter system (W, S) with $S = \{1, 2, \ldots, n\}$. Let $Y_j (1 \leq j \leq n)$ be the parabolic subgroup of W defined by $Y_j =< \{1, 2, \ldots, j\} >$, and $X_j = \{w \in Y_j | D_l(w) \subseteq \{j\}\}$. Then by proposition 4.4, each $w \in Y_j$ can be uniquely written $w = y \cdot x$ with $y \in Y_{j-1}, x \in X_j$. This we denote by

$$Y_j = Y_{j-1} \cdot X_j \tag{7.1}$$

Since $Y_1 = X_1 = \{e, 1\}$, it follows by induction on j that $Y_j = X_1 \cdot X_2 \ldots \cdot X_j$ for each j, so that any $w \in W$ can be written uniquely $w = x_1 x_2 \ldots x_n$ with each x_i in X_i. We call this decomposition the X-decomposition of w (it is of course relative to the numbering of S that we have chosen).

Remember that the Shortlex ordering $\leq_{ShortLex}$ on words with characters in $S = \{1, 2, \ldots, n\}$ is defined recursively as follows: $a = (a_1, a_2, \ldots, a_r) \leq_{ShortLex} b = (b_1, b_2, \ldots, b_s)$ if and only if, $r = 0$, or $r < s$, or $a = b$, or $r = s$ and $a_k < b_k$, where k is the smallest index such that $a_k \neq b_k$. The Shortlex order is a total order on S^*. For each $w \in W$, we define the (Shortlex) normal form of w (denoted $NF(w)$) as the smallest expression representing w in the sense of $\leq_{ShortLex}$ (in particular it is a reduced expression).

There is an interesting compatibility with the X-decomposition. Consider decomposition (7.1) again: write $w \in Y_j$ as $w = yx, y \in Y_{j-1}, x \in X_j$. Suppose that $x \neq e$. Then $D_l(x) = \{j\}$ and any reduced expression of x starts with a j. Let $(w_1, w_2, \dots, w_n)$ be the normal form of w. Since $w \leq x \leq n$, the set $A = \{i \in [1,n] | w_i = j\}$ is nonempty; call i_0 its smallest element. Then $w_i < j$ for $i < i_0$, hence $y' = w_1 w_2 \dots w_{i_0-1} \in Y_{j-1}$. But we also have $x' = w_{i_0+1} w_{i_0+2} \dots w_n \in X_j$ (otherwise there would be a reduced expression a of x' starting with a generator smaller than n, and the concatenation of $(w_1, w_2, \dots, w_{i_0-1})$ and a would be a smaller representant of w than the normal form, which is impossible). So by the uniqueness in (7.1), we have $x' = x, y' = y$. In other words, $NF(w) = NF(y)NF(x)$. This clearly remains true when $x = e$. We deduce the following:

Proposition 7.1 *If $w = x_1 x_2 \dots x_n$ is the X-decomposition of $w \in W$, then*

$$NF(w) = NF(x_1)NF(x_2)\dots NF(x_n).$$

8 Cocycles and Coxeter groups

In this section we will briefly use a special representation of W, rather different from the one we constructed in section 2. Let $\mathfrak{S}$ be the group of permutations of $E = T \times \{-1, 1\}$ (recall the definition of reflections and the set T at the end of section 2). For $s \in S$ define an element $f_s \in \mathfrak{S}$ as follows:

$$f_s(t, \varepsilon) = (sts, (-1)^{\delta_{t,s}} \varepsilon) \tag{8.1}$$

It is easily seen that $f_s^2 = id_E$. Furthermore, we have for any reduced expression $s_1 s_2 \dots s_n$ of an element $w \in W$,

$$\begin{array}{l} f_{s_1} f_{s_2} \dots f_{s_n}(t, \varepsilon) = (wtw^{-1}, (-1)^d \varepsilon), \text{ with} \\ d = \delta_{t,s_n} + \delta_{t,s_n s_{n-1} s_n} + \dots + \delta_{t,s_n s_{n-1} \dots s_2 s_1 s_2 \dots s_{n-1} s_n}. \end{array} \tag{8.2}$$

In the particular case when $m = m_{s_1,s_2}$ is finite, $n = 2m_{s_1,s_2}$ and (s_i) is two-periodic, we have $d = \sum_{k=1}^{n} \delta_{t,u_k}$ where $u_i = s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n$. Note that $l(u_i) = 2(n-i)+1 = 2(2m-i)+1$. Then, for any $k \in [1, m]$, we have $u_{m+k} = u_k (s_{n-2} s_{n-3})^m = u_k$ so that $d = 2(\sum_{k=1}^{m} \delta_{t,u_k})$ and $(-1)^d = 0$. This shows that $(f_{s_1} f_{s_2})^m = id_E$. Eventually the mappings f_s define an action of W on E.

Remember that for any set X, its power set may be viewed as a vector space over $\mathbb{F}_2$ with basis indexed by X, indentifying a subset of X with its characteristic function. Then addition of vectors corresponds to symmetric difference of sets. Thus, we shall use the symbol $+$ to denote symmetric difference. We define a function $N : W \to \mathcal{P}(T)$ by putting for $w \in W$,

$$N(w) = \{t \in T | f_w(t, \varepsilon) = (wtw^{-1}, -\varepsilon) \text{ for any } \varepsilon \in \{-1, 1\}\} \tag{8.3}$$

We claim that N is a cocycle on the Coxeter system (W, S), i.e. it satisfies

$$\begin{cases} (i)\ N(s) = \{s\} \text{ for any } s \in S \\ (ii)\ N(xy) = y^{-1}N(x)y + N(y) \end{cases} \tag{8.4}$$

Indeed, (i) follows from the definition of f_s; this definition also yields (ii) when $x \in S$, the general case following by induction on $l(x)$. Remarkably, the converse is also true:

Theorem 8.1 *Let W be a group and let S be a set of elements of order two generating W. Let $T = \{wsw^{-1} | s \in S, w \in W\}$. Then (W, S) is a Coxeter system if and only if (W, S) admits a cocycle, i.e. a function $N : W \to \mathcal{P}(T)$ satisfying (8.4).*

Proof: The "only if" has been just shown; let us prove the "if". Thus let (W, S) be a pair as above, with a cocyle function N. For $s, t \in S$, let m_{st} be the order of st in W; this Coxeter matrix defines a Coxeter system (W', S), and we have a surjective homomorphism $\pi : W' \to W$. It remains to be shown that π is in fact an isomorphism.

On W, we may redefine the notion of length, reduced expression, descent sets, prefixes and suffixes as we did in section 1. It will suffice to show that corollary 2.6 still holds in W, because it will entail proposition 3.1 for W, showing that π preserves length and reduced expressions.

We have for $s_1 \in S$ and $y \in W$,

$$N(s_1y) = s_1N(y)s_1 + \{s_1\} \tag{8.5}$$

and using this last formula twice,

$$N(s_1s_2y) = s_1s_2N(y)s_2s_1 + \{s_2s_1s_2\} + \{s_2\} \tag{8.6}$$

continuing this way, taking $y = e$ adding more elements $s_3, s_4, \dots s_n \in S$, and putting $w = s_1s_2 \dots s_n$, $t_i = s_ns_{n-1} \dots s_{i+1}s_is_{i+1} \dots s_n (1 \le i \le n)$,

$$N(w) = \{t_1\} + \ldots + \{t_2\} + \{t_n\} \tag{8.7}$$

Suppose now that $(s_1, s_2, \dots, s_n)$ is a reduced expression. We have

$$wt_i = s_1 \dots \hat{s_i} \dots s_n, \text{ so } l(wt_i) < l(w) \tag{8.8}$$

Moreover, for indices $i < j$, if $t_i = t_j$ we deduce $s_js_{j-1} \dots s_{i+1}s_is_{i+1} \dots s_{j-1}s_j = s_j$, $s_is_{i+1} \dots s_{j-1}s_j = s_{i+1} \dots s_{j-2}s_{j-1}$, and hence $w = s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_n$, contradicting the assumption that $(s_1, s_2, \dots, s_n)$ is reduced. Therefore the t_i are distinct, and (8.7) becomes

$$N(w) = \{t_1, t_2, t_3, \dots, t_n\}, \text{ if } (s_1, s_2, \dots, s_n) \text{ is reduced} \tag{8.9}$$

Let $t \in T$. We can write $t = r_1r_2 \dots r_{p-1}r_pr_{p-1} \dots r_1$ with each $r_i \in S$. Let $n = 2p - 1$. Define n elements $s_1, \dots s_n$ of S by $(s_1, s_2, \dots, s_n) = (r_1, r_2, \dots r_{p-1}, r_p, r_{p-1}, \dots r_1)$, so that $t = s_1s_2 \dots s_n$. Then (8.7) holds. But $tt_it = t_{2p-i}(1 \le i \le p)$, so that $(t_i = t) \Leftrightarrow (t_{2p-i} = t)$. Since $t_p = t$, the number of occurrences of t in $t_1, t_2, \dots, t_n$ is odd, whence $t \in \{t_1\} + \ldots + \{t_2\} + \{t_n\} = N(t)$. Thus

$$t \in N(t), \text{ for all } t \in T \tag{8.10}$$

Let $t \in T$. We claim that $(t \in N(w)) \Leftrightarrow (l(wt) < l(w))$. Indeed, if $t \in N(w)$ then $l(wt) < l(w)$ by (8.8) and (8.9), and if $t \notin N(w)$ then $t \in N(t)$ but $t \notin tN(w)t$, whence $t \in tN(w)t + N(t) = N(wt)$, and by what we have just shown $l((wt)t) < l(wt)$, i.e. $l(w) < l(wt)$. Therefore

$$\begin{cases} \text{If } t \in N(w), \text{then } l(wt) < l(w) \\ \text{If } t \notin N(w), \text{then } t \in N(wt), l(w) < l(wt) \end{cases} \tag{8.11}$$

In particular, we have $D_r(w) = S \cap N(w)$. We are now ready to re-show corollary 2.6 (we use the "suffix" version here). We argue by induction on $l(w)$, the result being clear if $l(w) \leq 1$. So suppose $s, t \in D_r(w)$. Then there is a reduced expression for w ending with an s: $w = s_1 s_2 \ldots s_{n-1} s_n$, with $s_n = s$. Also $t \in N(w)$ and by (8.9) there is an index i such that $t = s_n s_{n-1} \ldots s_{i+1} s_i s_{i+1} \ldots s_n$, whence $s_i s_{i+1} \ldots s_n = s_{i+1} \ldots s_n t$. If $i > 1$, then we may apply the induction hypothesis to $w' = s_i s_{i+1} \ldots s_n$, and we are done. We may therefore assume $i = 1$. Then we may replace the original reduced expression $(s_1 \ldots s_{n-1})s$ with $(s_2 \ldots s_{n-1})st$. Restarting the reasoning with that new expression, we obtain $w = (s_3 \ldots s_{n-2})sts$, and continuing this way we eventually see that $m_{s,t}$ is finite and that $M_{s,t}$ is a suffix of w, as wished. Q. E. D.

Lemma 8.2 *(i) Reflections have odd length.*
(ii) Let t be a reflection. Define $n \in \mathbb{N}$ by $l(t) = 2n+1$, and let $s_1 s_2 \ldots s_{2n+1}$ be a reduced expression for t. Then $t = s_1 s_2 \ldots s_{n-1} s_n s_{n-1} \ldots s_1$.

Proof: (i) Let $\varepsilon : S \to \{\pm 1\}$ be the map defined by $\varepsilon(s) = -1$ for all $s \in S$. It is easily seen that ε extends to a homomorphism $W \to \{\pm 1\}$. Then ε has two fibers, the set of elements of even length and the set of element of odd length, hence the result.

(ii) Put $x = s_n \ldots s_1$ and $y = s_{n+2} \ldots s_{2n+1}$, so that $t = x^{-1} s_{n+1} y$. Then $l(x) = l(y) = n < n+1 = l(s_{n+1}x) = l(s_{n+1}y)$, and $s_{n+1}yt = x$. By (8.9) and (8.11), there is an index $i \in [n+1, 2n+1]$ such that $t = s_{2n+1} s_{2n} \ldots s_{i+1} s_i s_{i+1} \ldots s_{2n+1}$. Then $x = s_{n+1} \ldots \hat{s_i} \ldots s_{2n+1}$, and $l(t) \leq 1 + 2(2n+1-i)$, whence $2n+1 \leq 3 + 4n - 2i$ and $2i \leq 2n+2$. We deduce $i = n+1$ and $x = y, t = x^{-1} s_{n+1} x$ as desired. Q. E. D.

We now relate reflections to the Bruhat order:

Corollary 8.3 *(i) Let $x, y \in W$ such that $l(y) = l(x) + 1$. Then $x \leq y$ if and only if $x^{-1}y \in T$.*
(ii) If $w \in W$ and $t \in T$, then $w < wt$ if $t \in N(w)$ and $wt < w$ if $t \notin N(w)$.
(iii) Let $x, y \in W$. Then $x \leq y$ if and only if there is a finite chain $x = x_1 < x_2 \ldots < x_{n-1} < x_n = y$ such that $x_i^{-1} x_{i+1} \in T$, $l(x_{i+1}) = l(x_i) + 1$, for all $i \in [1, n-1]$.

Proof: Let $y = s_1 \ldots s_n$ be a reduced expression for y, and put $t = x^{-1}y$. As in (8.7) and (8.8), put $t_i = s_{n+1} \ldots s_{i+1} s_i s_{i+1} \ldots s_n$ for each $i \in [1, n]$. If $x \leq y$, we must have $x = s_1 s_2 \ldots \hat{s_i} \ldots s_n$ for some i, hence $t = t_i \in T$. Conversely, if $t \in T$ then $t \in N(y)$ by (8.11), hence $t = t_i$ for some i by (8.9), and $x = s_1 s_2 \ldots \hat{s_i} \ldots s_n \leq y$. This proves (i), and (ii) then follows from (i) and (8.11), while (iii) follows from (i) and proposition 5.5. Q. E. D.

The attentive reader will have noticed that we used the same letter N to denote a geometric function in (2.7) and the cocycle of this section. We explain now how the two can be identified:

Remark 8.4 *Let N_{geom} be the function defined in (2.7) and $N_{cocycle}$ the function defined in this section. Then $N_{geom}(w) = \{\alpha_t | t \in N_{cocycle}(w)\}$.*

This follows immediately from the explicit formulas for each "N" function ((2.9) for N_{geom}, and (8.9) for $N_{cocycle}$).

9 Reflection subgroups

A subgroup W' of W is a **reflection subgroup** of W if it is generated by reflections, i.e. if $W' = < W' \cap T >$. One of the main results of this section is that reflection subgroups are Coxeter groups; we even have a canonical system of generators, defined by

$$S(W') = \{t \in T | N(t) \cap W' = \{t\}\} \tag{9.1}$$

where N is the cocycle function defined in the previous section. We will use theorem 8.1, by showing that the function N' defined for all $w \in W$ by

$$N'(w) = N(w) \cap W' \tag{9.2}$$

is a cocycle function on $(W', S(W'))$.

Lemma 9.1 *Let W' be a subgroup of W (not necessarily a reflection subgroup). Then:*
(i) If $s \in S \setminus W'$ then $S(sW's) = sS(W')s$.
(ii) If $t \in W' \cap T$ then there is $m \in \mathbb{N}$ and $t_0, \dots, t_m \in S(W')$ such that $t = t_m \dots t_1 t_0 t_1 \dots t_m$.
(iii) If $x \in W$ and $y \in W'$, then $N'(xy) = y^{-1}N'(x)y + N'(y)$.

Proof: (i) Let $u \in T \cap W'$. Then $N(sus) = \{s\} + \{susus\} + sN(u)s$, so $sN(sus)s = \{s\} + \{usu\} + N(u)$. Now $s \notin W'$ and $usu \notin W'$, whence $(sN(sus)s) \cap W' = N(u) \cap W'$, and $N(sus) \cap (sW's) = s(N(u) \cap W')s$. In particular, if $u \in S(W')$ then $sus \in S(sW's)$. This shows $sS(W')s \subseteq S(sW's)$, and replacing W' with $sW's$ yields the reverse inequality.

(ii) We argue by induction on $l(t)$. If $l(t) = 1$ then $t \in S$ and we may take $m = 0, t_0 = t$. Suppose now that the result is true for all pairs (W'', t'') where W'' is a subgroup of W and $t'' \in W'' \cap T, l(t') < l(t)$. By lemma 8.2, there is a generator $s \in S$ such that $t = st''s$ with $l(t'') = l(t) - 2$. Let $W'' = sW's$. By the induction hypothesis, there is a $m \in \mathbb{N}$ and $t_0, \dots, t_m \in S(W'')$ such that $t'' = t_m \dots t_1 t_0 t_1 \dots t_m$.

If $s \in W'$, then $W'' = W'$. Putting $t_{m+1} = s$, we have $t = t_{m+1} t_m \dots t_1 t_0 t_1 \dots t_m t_{m+1}$ and $t_i \in S(W')$ for all $i \in [0, m+1]$.

If $s \notin W'$, then (i) yields $S(W'') = sS(W')s$. Putting $t'_i = st_is$ for $i \in [0, n]$, we have $t = st_m \dots t_1 t_0 t_1 \dots t_m s = t'_m \dots t'_1 t'_0 t'_1 \dots t'_m$ and $t'_i \in S(W')$ for all $i \in [0, m]$.

(iii) We have

$$\begin{aligned} N'(xy) &= N(xy) \cap W' \\ &= (y^{-1}N(x)y + N(y)) \cap W' \\ &= (y^{-1}N(x)y \cap W') + (N(y) \cap W') \\ &= y^{-1}(N(x) \cap W')y + N'(y) \quad \text{(note that } W' = yW'y^{-1} \text{ since } y \in W') \\ &= y^{-1}N'(x)y + N'(y) \end{aligned}$$

Q. E. D.

Theorem 9.2 *Let W' be a reflection subgroup of W'. Let $S' = S(W')$. Then*
(i) (W', S') is a Coxeter system.
(ii) $W' \cap T = \cup_{w' \in W'} w' S' w'^{-1}$.
(iii) For $w' \in W'$, $N(w') \cap W' = \{t \in W' \cap T | l'(w't) < l'(w')\}$ where l' is the length function of the Coxeter system (W', S').

Proof: Let $W'' =< S' >$ and $T' = \cup_{w' \in W'} w' S' w'^{-1}$. Since $S' \subseteq W'$, we have $W'' \subseteq W'$ and $T' \subseteq T \cap W'$. By lemma 9.1(ii), we have $T \cap W' \subseteq < S' >$, hence $T' = T \cap W'$ and $W' =< W' \cap T >=< T' > \subseteq < S' > \subseteq W''$. This shows that $W' =< S' >$ and that (ii) holds.

For $s' \in S'$, we have $N'(s') = \{s'\}$ by definition of S', and furthermore $N'(x'y') = y'^{-1} N'(x') y' + N'(y')$ for $x', y' \in W'$ by 9.1(iii). Thus N' is a cocycle on (W', S'), and by theorem 8.1 we deduce (i). Eventually, formula (8.11) of the proof of theorem 8.1 yields (iii). Q. E. D.

Now we are compare the Bruhat orders of W and W'. The **Bruhat graph** $\Gamma_{(W,S)}$ of a Coxeter system (W, S) is the directed graph whose set of vertices is W and whose set of edges is

$$E_{(W,S)} = \{(x, y) \in W^2 | x^{-1} y \in T, l(x) < l(y)\} \tag{9.3}$$

For any $X \subseteq W$, there is a corresponding induced subgraph Γ_X whose set of vertices in X and whose set of edges is $E_{W,S} \cap (X \times X)$. Note that the transitive closure of the relation $(x, y) \in E_{W,R}$ is the Bruhat order, according to corollary 8.3(iii).

Proposition 9.3 *Let W' be a reflection subgroup of W, and let $S' = S(W')$. Let $L = xW$ be a left coset of W' in W. Then*
(i) $\Gamma_{(W',S')} = \Gamma_{W'}$
(ii) L contains an unique element x_0 of minimal length.
(iii) The mapping $\theta : W' \to L, w \mapsto x_0 w$ satisfies $N(\theta(w)) \cap W' = N(w) \cap W'$ for $w \in W'$.
(iv) θ is an isomorphism of directed graphs $\Gamma_{W'} \to \Gamma_L$.
(v) If we equip W' with its own Bruhat order and L with the Bruhat order from W, then θ is an injective morphism of partially ordered sets $W' \to L$.

Proof: We use the notations l', N' and T' as in the theorem above.
(i) The graphs $\Gamma_{(W',S')}$ and $\Gamma_{W'}$ both have vertex set W'. To show (i), we must also show that the edges set are equal, i.e. that for $(x, y) \in W' \times W'$ we have $(x, y) \in E_{(W,S)} \Leftrightarrow (x, y) \in E_{(W',S')}$. So take $(x, y) \in W' \times W'$. Then we have

$$\begin{aligned} (x, y) \in E_{(W,S)} &\iff x^{-1} y \in T, l(x) < l(y) \\ &\iff x^{-1} y \in T, x^{-1} y \notin N(x) \text{by corollary 8.3(ii)} \\ &\iff x^{-1} y \in T', x^{-1} y \notin N(x) \text{ by theorem 9.2(ii)} \\ &\iff x^{-1} y \in T', x^{-1} y \notin N'(x) \\ &\iff , x^{-1} y \in T', l'(x) < l'(y) \text{ by theorem 9.2(iii)} \\ &\iff (x, y) \in E_{(W',S')} \end{aligned}$$

This finishes the proof of (i).

Let x_0 be any element of minimal length in L. We defer the proof of the uniqueness of x_0 (which is the content of (ii)) to the end, and prove the other properties first. For any $t \in T'$ we have $l(x_0t) \leq l(x_0)$, and even $l(x_0t) > l(x_0)$ by corollary 8.3(ii). Thus $N'(x_0) = \emptyset$, and hence $N'(x_0y) = y^{-1}N'(x_0)y + N'(y) = N'(y)$ for $y \in W'$. This shows (iii). The mapping θ is clearly a bijection; let us show that it is an isomorphism between $\Gamma_{W'}$ and Γ_L. For $(x, y) \in W' \times W'$, we have $\theta(x)^{-1}\theta(y) = x^{-1}x_0^{-1}x_0y = x^{-1}y$

$$\begin{aligned}(\theta(x), \theta(y)) \in E(\Gamma_L) &\iff \theta(x)^{-1}\theta(y) \in T, \theta(x)^{-1}\theta(y) \notin N(\theta(x)) \\ &\iff x^{-1}y \in T, x^{-1}y \notin N(\theta(x)) \\ &\iff x^{-1}y \in T, x^{-1}y \notin N(x) \text{ by (iii)} \\ &\iff (x, y) \in E(\Gamma_{W'})\end{aligned}$$

which proves (iv). Then (v) follows from (iv) and corollary 8.3(iii).

Finally, let us show (ii). Let $w \in W' \setminus \{e\}$; we must prove that $l(x_0w) > l(x_0)$. Let $w = t_1t_2 \dots t_r$ be a reduced expression for w with respect to the Coxeter system (W', S'). Put $w_0 = x, w_1 = t_1, \dots, w_i = t_1t_2 \dots t_i(1 \leq i \leq n)$. Then for each $i \geq 1$, $(w_i, w_{i+1}) \in E(\Gamma_{W'})$, so $(\theta(w_i), \theta(w_{i+1})) \in E(\Gamma_L)$ by (iv). Thus $l(x_0w_i) < l(x_0w_{i+1})$, and we deduce $l(x_0) < l(x_0w_1) < \dots l(x_0w_n) = l(x_0w)$ as required. Q. E. D.

It is important to realize that in (v) the injective morphism θ is not an isomorphism in general. For example:

Remark 9.4 *Suppose that $S = \{a, b\}$ and $m_{a,b} = \infty$. Then the reflection subgroup $W' =< a, bab >$ satisfies $S(W') = \{a, bab\}$, so a is smaller than bab for the Bruhat ordering of W but not for the Bruhat ordering of W'.*

Lemma 9.5 *Let W' be a reflection subgroup of W. Let t_α, t_β be two distinct reflections of $S(W')$, and let α, β be the corresponding roots in Φ^+. Then $B(\alpha, \beta) \leq 0$ (where B is as in (2.6)).*

Proof: We use induction on $k = l(t_\alpha)$. When $k = 1$, we have $\alpha = \alpha_s$ for some $s \in S$. If $s \in D_r(t_\beta)$, then $\{s, t_\beta\} \subseteq N(t_\beta) \cap W'$, hence $t_\beta = s$, contradicting $t_\beta \neq t_\alpha$. Therefore $s \notin D_r(t_\beta)$, and by corollary 2.12(iii) we are done.

Now assume $k > 1$, and that the result holds for all $k' < k$. Let $u \in D_r(t_\alpha)$, and define $\alpha' = u(\alpha), \beta' = u(\beta), W'' = uW'u$. By lemma 8.2(ii), we have $l(t_{\alpha'}) = l(t_\alpha) - 2$. Note that $u \notin W'$ (otherwise $\{u, t_\alpha\} \subseteq N(t_\alpha) \cap W'$, whence $t_\alpha = u$ contradicting $k > 1$), so that lemma 9.1(i) yields $\{\alpha', \beta'\} \subseteq S(W'')$. Then we may apply the induction hypothesis: $B(\alpha', \beta') \leq 0$. Eventually, since B is invariant, $B(\alpha, \beta) = B(u\alpha', u\beta') = B(\alpha', \beta') \leq 0$. Q. E. D.

A reflection subgroup W' is called **dihedral** if $|S(W')| = 2$.

Lemma 9.6 *Let $t_1, t_2, \dots, t_{2n-1}, t_{2n}$ be $2n$ reflections in W such that*

$$t_1t_2 = t_3t_4 = \dots = t_{2n-1}t_{2n} \neq e$$

Then the reflection subgroup $W' =< t_1, t_2, \dots, t_{2n} >$ is dihedral.

Proof: For each i, denote by α_i the positive root corresponding to t_i. We do not change the problem if we enlarge the Coxeter system (W, S) by adding new elements to S and new coefficients in the Coxeter matrix accordingly. If we add a new generator s_1 along with new coefficients $(m_{s_1 s})_{s\in S}$, then for a root $\alpha \in \Phi^+$, $\alpha = \sum_{s\in S} a_s\alpha_s$ with $a_s \geq 0$, we have

$$B(\alpha_{s_1}, \alpha) = -\sum_{s\in S} a_s |cos(\frac{\pi}{m_{s_1 s}})|$$

Thus, adding two new generators s_1 and s_2, we may choose the coefficients $m_{s_1 s}$ and $m_{s_2 s}$ so that the following determinant is nonzero:

$$\begin{vmatrix} B(\alpha_{s_1}, \alpha_1) & B(\alpha_{s_1}, \alpha_2) \\ B(\alpha_{s_2}, \alpha_1) & B(\alpha_{s_2}, \alpha_2) \end{vmatrix} \neq 0 \tag{9.4}$$

For $i \in [1, n]$ call f_i the endomorphism of V defined by $f_i(x) = x - t_{2i-1}t_{2i}(x)$ for $x \in V$. Then

$$f_i(x) = 2(B(x, t_{\alpha_{2i}}(\alpha_{2i-1}))\alpha_{2i-1} + B(x, \alpha_{2i})\alpha_{2i})$$

so that $\mathsf{Im}(f_i) \subseteq \mathbb{R}\alpha_{2i-1} + \mathbb{R}\alpha_{2i}$. When $i = 1$, this inclusion is in fact an equality by (9.4). But by assumption we have $f_1 = f_2 = \ldots f_n$, so $\mathbb{R}\alpha_1 + \mathbb{R}\alpha_2 = \mathbb{R}\alpha_3 + \mathbb{R}\alpha_4 = \ldots = \mathbb{R}\alpha_{2n-1} + \mathbb{R}\alpha_{2n}$. Let Γ be the sets of positive roots corresponding to the reflections in $S(W')$. Then $\Gamma \subseteq W'\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}\}$ by theorem 9.2(ii), whence $\Gamma \subseteq< t_1, t_2, \ldots, t_{2n} > \{\alpha_1, \alpha_2, \ldots, \alpha_{2n}\} \subseteq \mathbb{R}\alpha_1 + \mathbb{R}\alpha_2 + \ldots \mathbb{R}\alpha_{2n} = \mathbb{R}\alpha_1 + \mathbb{R}\alpha_2$. By lemma 9.5, we have $B(\gamma, \gamma') \leq 0$ for any $\gamma \neq \gamma'$ in Γ. And finally, no nontrivial linear combination with nonnegative coefficients of the elements of Γ equals zero (since $\Gamma \subseteq \Phi^+$). Those three conditions on Γ impose that $|\Gamma| \leq 2$, so that W' is dihedral as required. Q. E. D.

Recall some poset terminology. If $x < y$ and no z satisifies $x < z < y$, we says that y **covers** x, or that x is a **coatom** of y, or that y is an atom of x, and denote this by $x \lhd y$. When there is a length function (which is the case for the Bruhat order), this amounts to $x < y$ and $l(y) = l(x) + 1$. Recall that the **complete bipartite graph** $K(A, B)$ for disjoint sets A, B is the graph with vertex set $A \amalg B$ and edge set $\{\{a, b\} | a \in A, b \in B\}$, and that we denote by $K_{i,j}$ the isomorphism class of any $K(A, B)$ with $|A| = i, |B| = j$. Here is a beautiful consequence of the theory of reflection subgroups:

Theorem 9.7 *The directed graph with edge set W and whose edges are the pairs of the form (x, y) with $x \lhd y$ does not have $K_{3,2}$ or $K_{2,3}$ as an induced subgraph. In other words, if two elements in W have three coatoms (or three atoms) in common, they are equal.*

Proof: We will show that if a, b have three coatoms x, y, z in common then $a = b$; the proof for the "atoms" version is quite similar. Suppose by contradiction that $a \neq b$. Define six reflections as follows (remember corollary 8.3(i)):

$$t_1 = a^{-1}x, t_2 = x^{-1}b, t_3 = a^{-1}y, t_4 = y^{-1}b, t_5 = a^{-1}z, t_6 = z^{-1}b$$

Then $t_1t_2 = t_3t_4 = t_5t_6 = a^{-1}b \neq e$. By lemma 9.6 above, $W' =< t_1, t_2, t_3, t_4, t_5, t_6 >$ is a dihedral reflection subgroup of W. By 9.3(v), there is an injective morphism $\theta : W' \to aW'$ between the Bruhat order in W' and the induced Bruhat order in aW'. In particular, $\theta^{-1}(x), \theta^{-1}(y)$ and $\theta^{-1}(z)$ are three incomparable elements in the dihedral group W', and this is a contradiction. Q. E. D.

A poset is said to be **dihedral** if it is isomorphic to an interval in a dihedral Coxeter group. Thus, for each length n, there is exactly one dihedral poset in length n, which has a minimum and maximum element, and two elements in each intermediate length (see Figure 1).

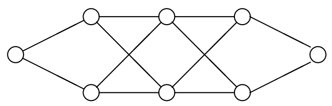


Figure 1: The dihedral poset in length 4

In a poset P with minimum element x, an element $y \in P$ is said to be dihedral when the interval $[x, y]$ (as a subposet of P) is dihedral. The **bud** $B(P)$ of P is defined to be the set of the dihedral elements of P (cf. e.g. [11]).

There are two main corollaries of the important result in theorem 9.7. First, combining it with corollary 5.4, we obtain:

Corollary 9.8 *Let I be a Bruhat interval. The dihedral elements of I are exactly the elements of I that have at most two coatoms.*

Corollary 9.9 *Let I be a Bruhat interval. If two elements in W have the same set of coatoms and one of them is non-dihedral, then they are equal.*

10 The Hecke algebra

The main result of this section is the following:

Theorem 10.1 *Let (W, S) be a Coxeter system and q be an indeterminate. Then there is a unique associative algebra $\mathscr{H}$ over the ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, called the* **Hecke algebra** *of W, that has the following properties:*
(i) The $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module $\mathscr{H}$ has a basis $(T_w)_{w \in W}$ indexed by W.
(ii) The equality $T_w T_{w'} = T_{ww'}$ holds whenever $w, w' \in W, l(ww') = l(w) + l(w')$.
(iii) We have $T_s^2 = (q-1)T_s + qT_e$ for all $s \in S$.

We shall actually prove a slightly stronger result, where $T_s^2 = (q-1)T_s + qT_e$ in (iii) is replaced with $T_s^2 = aT_s + bT_e$ for two independent indeterminates a, b. We start with a real vector space V with basis $(T_w)_{w \in W}$, and we will progressively define the multiplication $V \times V$ that will turn V into an algebra. For each $s \in S$, define an endomorphism $f_1(s)$ of V by

$$f_1(s)(T_w) = \begin{cases} T_{sw} & \text{if } s \notin D_l(w) \\ aT_w + bT_{sw} & \text{if } s \in D_l(w) \end{cases} \tag{10.1}$$

and symmetrically, define another endomorphism $g_1(t)(t \in S)$ by

$$g_1(t)(T_w) = \begin{cases} T_{wt} & \text{if } t \notin D_r(w) \\ aT_w + bT_{wt} & \text{if } t \in D_r(w) \end{cases} \tag{10.2}$$

Lemma 10.2 *The endomorphisms $f_1(s)$ and $g_1(t)$ commute, for any $s, t \in S$.*

Proof: Let $w \in W$ and $s, t \in S$. Put $f = f_1(s), g = g_1(t)$. We must show that $fg(T_w) = gf(T_w)$. By proposition 4.7, the orbit $\Omega =< s > w < t >$ has a unique element w_0 of minimal length. Then $\Omega = \{w_0, sw_0, w_0t, sw_0t\}$. Denote V_0 the subspace of V spanned by the T_u for $u \in \Omega$; then V_0 is invariant by f and g. It will suffice to show that those two endomorphisms commute on V_0.

If $sw_0 = w_0t$, then $\Omega = \{w_0, sw_0\}$, and f and g coincide on V_0, so that they certainly commute in this case.

If $sw_0 \neq w_0t$, then the four elements w_0, sw_0, w_0t, sw_0t are distinct (so that $|\Omega| = 4$) and furthermore $t \notin D_r(sw_0), s \notin D_l(w_0t)$ by corollary 3.2. We then enumerate the various cases:
If $w = w_0$, then $f(T_w) = T_{sw_0}, g(T_w) = T_{w_0t}$ and $fg(T_w) = gf(T_w) = T_{sw_0t}$.
If $w = sw_0$, then $f(T_w) = aT_{sw_0} + bT_{w_0}, g(T_w) = T_{sw_0t}$ and $fg(T_w) = gf(T_w) = aT_{sw_0t} + bT_{w_0t}$.
Symmetrically, if $w = w_0t$ then $fg(T_w) = gf(T_w) = aT_{sw_0t} + bT_{sw_0}$.
Eventually, if $w = sw_0t$, then $f(T_w) = aT_{sw_0t} + bT_{w_0t}, g(T_w) = aT_{sw_0t} + bT_{sw_0}$ and $fg(T_w) = gf(T_w) = a^2T_{sw_0t} + ab(T_{sw_0} + T_{w_0t}) + b^2T_{w_0}$. Q. E. D.

Lemma 10.3 *Let s_1 and s_2 be two distinct generators in S such that $m = m_{s_1s_2}$ is finite. Then the endomorphisms $[f_1(s_1), f_1(s_2), m\rangle$ and $[f_1(s_2), f_1(s_1), m\rangle$ coincide on V.*

Proof: Let $d = [f_1(s_1), f_1(s_2), m\rangle, d' = [f_1(s_2), f_1(s_1), m\rangle$, and $K = \mathsf{Ker}(d - d')$. Since d and d' commute with any $g_1(t)$ by the lemma above, K is invariant by all the $g_1(t)$ for $t \in S$, and furthermore K contains T_e (since $d(T_e) = d'(T_e) = T_{M_{s_1,s_2}}$). Therefore $K = V$ as required. Q. E. D.

By Matsumoto's theorem (3.3), we see that $f_1 : S \to \mathsf{End}(V)$ can be extended to a map $f_2 : W \to \mathsf{End}(V)$, in such a way that

$$f_2(s_1s_2 \dots s_r) = f_1(s_1)f_1(s_2) \dots f_1(s_r), \text{ whenever } (s_1, s_2, \dots, s_r) \text{ is reduced.} \tag{10.3}$$

We then define the multiplication on V by

$$T_w \times T_{w'} = f_2(w)(T_{w'}) \ (w, w' \in W). \tag{10.4}$$

Proposition 10.4 *The operation defined above is associative on V.*

Then theorem 10.1 follows immediately from the construction. The Hecke algebra $\mathscr{H}$ has earned its name.

Proof: Let $x, y, z \in W$; we must show $T_x \times (T_y \times T_z) = (T_x \times T_y) \times T_z$. Using induction on $l(x)$, we are reduced to the case $l(x) = 1$, i.e $x = s \in S$.

If $s \notin D_l(y)$, then $T_x \times (T_y \times T_z) = f_1(s)(T_y \times T_z) = f_1(s)(f_2(y)(T_z))$ by definition. Since $s \notin D_l(y)$, we have $f_1(s)f_2(y) = f_2(sy)$, and hence $T_x \times (T_y \times T_z) = f_2(sy)(T_z) = T_{sy} \times T_z = (T_x \times T_y) \times T_z$ as desired.

If $s \in D_l(y)$, then we can write $y = sy'$ with $l(y') = l(y) - 1$. Then $T_x \times (T_y \times T_z) = f_1(s)(f_2(sy')(T_z)) = f_1(s)^2(f_2(y')(T_z))$ by definition. Now, using (10.1), we have $f_1(s)^2 = af_1(s) +$

bid_V. So $T_x \times (T_y \times T_z) = af_1(s)f_2(y')(T_z) + bf_2(y')(T_z) = a(T_{sy'} \times T_z) + b(T_{y'} \times T_z) = (aT_{sy'} + bT_{y'}) \times T_z = (T_s \times T_{sy'}) \times T_z = (T_x \times T_y) \times T_z$. Q. E. D.

We now derive some formulas for computations inside $\mathscr{H}$. Let $h \in \mathscr{H}$, so that h can be written

$$h = \sum_{w \in W} a(w) T_w$$

where only finitely many of the $a(w)$'s are nonzero. As in (4.1), we put ${}^sW = \{w \in W | s \notin D_l(w)\}$; then ${}^sW \amalg s({}^sW)$ is a partition of W. Then

$$T_s \times h = \sum_{w \in W} a(w) T_s T_w = \sum_{w \in {}^sW} a(w) T_{sw} + \sum_{w \in s({}^sW)} a(w)((q-1)T_w + qT_{sw})$$

and hence

$$\begin{aligned} T_s \times h &= \sum_{w \in {}^sW} a(w) T_{sw} + \sum_{w \in s({}^sW)} (q-1)a(w)T_w + \sum_{w \in s({}^sW)} qa(w)T_{sw} \\ &= \sum_{w \in s({}^sW)} a(sw)T_w + \sum_{w \in s({}^sW)} (q-1)a(w)T_w + \sum_{w \in {}^sW} qa(sw)T_w \\ &= \sum_{w \in {}^sW} qa(sw)T_w + \sum_{w \in s({}^sW)} (a(sw) + (q-1)a(w))T_w \end{aligned}$$

So we have shown the following (where $a(h,w)$ denotes the T_w-coordinate of $h \in \mathscr{H}$)

$$a(T_s \times h, w) = \begin{cases} qa(h,sw) & \text{if } s \notin D_l(w) \\ a(h,sw) + (q-1)a(h,w) & \text{if } s \in D_l(w) \end{cases} \tag{10.5}$$

Part (iii) of theorem 10.1 shows that for $s \in S$ the element T_s is invertible in $\mathscr{H}$, with

$$T_s^{-1} = \frac{T_s - (q-1)T_e}{q} \tag{10.6}$$

We deduce that

$$a(T_s^{-1} \times h, w) = \begin{cases} \frac{qa(h,sw)-(q-1)a(h,w)}{q} & \text{if } s \notin D_l(w) \\ \frac{a(h,sw)}{q} & \text{if } s \in D_l(w) \end{cases} \tag{10.7}$$

If $w \in W$ and $w = s_1 \dots s_n$ is a reduced expression for w, we deduce that T_w is invertible also with $T_{w^{-1}} = T_{s_n}^{-1} \dots T_{s_1}^{-1}$. Explicitly, iterating (10.7) and putting $b(y,x) = (-1)^{l(y)-l(x)} q^{l(y)} a(T_{y^{-1}}^{-1}, x)$ for $x, y \in W$, we have whenever $s \in D_l(y)$,

$$b(y,x) = \begin{cases} qb(sy,sx) + (q-1)b(sy,x) & \text{if } s \notin D_l(x) \\ b(sy,sx) & \text{if } s \in D_l(x) \end{cases} \tag{10.8}$$

Note that $b(e,e) = 1$ and $b(e,x) = 0$ for $x \neq e$. Then, using proposition 5.1 and induction on $l(y)$, we see that $b(y,x) = 0$ if $x \not\leq y$. Similarly, we see by induction on $l(y)$ that $b(y,x)$ is a polynomial in q, of degree $l(y) - l(x)$, and $b(x,x) = 1$ for all x, and $b(y,x) = q$ if $l(y) = l(x) + 1$. Let us gather those results in a single proposition:

Proposition 10.5 *Let $y \in W$. Then there is a family of polynomials $(R_{x,y}(q))_{x\leq y}$ in $\mathbb{Z}[q]$ such that*

$$T_{y^{-1}}^{-1} = \frac{1}{q^{l(y)}} \sum_{x \leq y} (-1)^{l(y)-l(x)} R_{x,y}(q) T_x \tag{10.9}$$

Furthermore, the polynomials $R_{u,v}(q)$ satisfy

(i) If $s \in D_l(y)$,

$$R_{x,y}(q) = \begin{cases} qR_{sx,sy} + (q-1)R_{x,sy} & \text{if } s \notin D_l(x) \\ R_{sx,sy} & \text{if } s \in D_l(x) \end{cases} \tag{10.10}$$

(ii) The degree of $R_{x,y}(q)$ is $l = l(y) - l(x)$.
(iii) If $l \leq 2$ then $R_{x,y}(q) = (q-1)^l$.

Those polynomials are called the R-polynomials associated to the Coxeter system (W, S). There is a unique involutive automorphism of the ground ring $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ (which we will denote by $^-$) such that $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$. There is a unique mapping $\iota : \mathscr{H} \to \mathscr{H}$ that is **antilinear** (i.e. such that $\iota(\sum_{i\in I} P_i(q)h_i) = (\sum_{i\in I} \overline{P_i(q)}\iota(h_i)$ if the P_i are polynomials and the h_i are in $\mathscr{H}$) and such that $\iota(T_w) = T_{w^{-1}}^{-1}$ for all $w \in W$. Then ι is an involution and is called the **canonical involution** of $\mathscr{H}$.

Proposition 10.6 *The mapping ι is antilinear, bijective and preserves the multiplication of $\mathscr{H}$.*

Proof: We must show that $\iota(T_xT_y) = \iota(T_x)\iota(T_y)$ for any $x, y \in W$. Using induction on $l(x)$, we are reduced to the case $l(x) = 1$, i.e. $x = s \in S$.

Suppose $s \notin D_l(y)$. Then $\iota(T_sT_y) = \iota(T_{sy}) = T_{(sy)^{-1}}^{-1} = T_{y^{-1}s}^{-1} = (T_{y^{-1}}T_s)^{-1} = T_s^{-1}T_{y^{-1}}^{-1} = \iota(T_s)\iota(T_y)$.

Suppose $y = s$. Then

$$\begin{aligned}
\iota(T_s)^2 &= (\tfrac{T_s-(q-1)T_e}{q})^2 \\
&= \tfrac{T_s^2-2(q-1)T_s+(q-1)^2T_e}{q^2} \\
&= \tfrac{(1-q)T_s+(q+(q-1)^2)T_e}{q^2} \\
&= \tfrac{(1-q)(qT_s^{-1}+(q-1)T_e)+(q+(q-1)^2)T_e}{q^2} \\
&= \tfrac{q(1-q)T_s^{-1}+qT_e}{q^2} \\
&= \tfrac{(1-q)T_s^{-1}+T_e}{q} \\
&= (\tfrac{1}{q}-1)\iota(T_s) + \tfrac{1}{q}\iota(T_e) \\
&= \iota((q-1)T_s + qT_e) = \iota(T_s^2).
\end{aligned}$$

Suppose $s \in D_l(y)$. Then we can write $y = sy'$ with $l(y') = l(y) - 1$, and $\iota(T_sT_y) = \iota(T_s^2T_{y'}) = \iota(((q-1)T_s + qT_e) \times T_{y'}) = \iota((q-1)T_{sy'} + qT_{y'}) = (\frac{1}{q} - 1)\iota(T_{sy'}) + (\frac{1}{q})\iota(T_{y'}) = (\frac{1}{q})((1-q)\iota(T_{sy'}) + \iota(T_{y'}))$. But by the first case we have treated, $\iota(T_{sy'}) = \iota(T_s)\iota(T_{y'})$. Therefore $\iota(T_sT_y) = (\frac{1}{q})((1-q)\iota(T_s)\iota(T_{y'}) + \iota(T_{y'})) = ((\frac{1}{q}-1)\iota(T_s) + \frac{1}{q}\iota(T_e))\iota(T_{y'}) = \iota((q-1)T_s + qT_e)\iota(T_{y'}) = \iota(T_s^2)\iota(T_{y'})$. But by the second case we have treated, $\iota(T_s^2) = \iota(T_s)^2$, so $\iota(T_sT_y) = \iota(T_s)^2\iota(T_{y'}) = \iota(T_s)\iota(T_{sy'}) = \iota(T_s)\iota(T_y)$. Q. E. D.

Sometimes it is useful to "adjust" the basis (T_w), considering the basis (T'_w) defined by $T'_w = \frac{1}{q^{\frac{l(w)}{2}}} T_w$. It is straightforward to check the following:

Proposition 10.7 *Let $\alpha = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Then the basis T'_w satisfies*
(i)

$$T'_s T'_w = \begin{cases} T'_{sw} & \text{if } s \notin D_l(w) \\ (q^{\frac{1}{2}} - q^{-\frac{1}{2}})T'_w + T'_{sw} & \text{if } s \in D_l(w) \end{cases} \tag{10.11}$$

(ii) $\iota(T'_s) = T'^{-1}_s = T'_s - \alpha T'_e$.
(iii) For $y \in W$,

$$\iota(T'_y) = \sum_{x \leq y} (-1)^{l(y)-l(x)} q^{\frac{l(x)-l(y)}{2}} R_{x,y}(q) T'_x$$

(iv) If $a'(h,w)$ denotes the T'_w-coordinate of $h \in \mathscr{H}$, the following analogue of (10.5) holds:

$$a'(T'_s \times h, w) = \begin{cases} a'(h, sw) & \text{if } s \notin D_l(w) \\ a'(h, sw) + \alpha a'(h, w) & \text{if } s \in D_l(w) \end{cases} \tag{10.12}$$

We now consider a second involutive automorphism of the ground ring $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ (which we will denote by $\widehat{\ }$) such that $\widehat{q^{\frac{1}{2}}} = -q^{-\frac{1}{2}}$. In the following section we will need the unique involution $\jmath : \mathscr{H} \to \mathscr{H}$ that is antilinear with respect to $\widehat{\ }$, and satisfies $\jmath(T'_w) = T'_w$ for all $w \in W$.

Proposition 10.8 *(i) The involution $\jmath$ preserves the multiplication of $\mathscr{H}$.*
(ii) The mappings ι and $\jmath$ commute on $\mathscr{H}$.

Proof: (i) We must show that $\jmath(T'_x T'_y) = \jmath(T'_x)\jmath(T'_y)$ for any $x, y \in W$. Using induction on $l(x)$, we are reduced to the case $l(x) = 1$, i.e. $x = s \in S$.

Suppose $s \notin D_l(y)$. Then $\jmath(T'_s T'_y) = \jmath(T'_{sy}) = T'_{sy} = T'_s T'_y = \jmath(T'_s)\jmath(T'_y)$.

Suppose $y = s$. Then $\jmath(T'_s)^2 = (T'_s)^2 = \alpha T'_s + T'_e = \jmath(\alpha T'_s + T'_e) = \jmath(T'^2_s)$.

Suppose $s \in D_l(y)$. Then we can write $y = sy'$ with $l(y') = l(y) - 1$, and $\jmath(T'_s T'_y) = \jmath(T'^2_s T'_{y'}) = \jmath((\alpha T'_s + T'_e) \times T'_{y'}) = \jmath(\alpha T'_{sy'} + T'_{y'}) = \alpha \jmath(T'_{sy'}) + \jmath(T'_{y'})$. But by the first case we have treated, $\jmath(T'_{sy'}) = \jmath(T'_s)\jmath(T'_{y'})$. Therefore $\jmath(T'_s T'_y) = \alpha \jmath(T'_s)\jmath(T_{y'}) + \jmath(T'_{y'}) = (\alpha T'_s + T'_e) \times \jmath(T'_{y'})$. But by the second case we have treated, $\alpha T'_s + T'_e = \jmath(T'_s)^2$, so $\jmath(T'_s T'_y) = \jmath(T'_s)^2 \jmath(T'_{y'}) = \jmath(T'_s)\jmath(T'_{sy'}) = \jmath(T'_s)\jmath(T'_y)$ as required.

(ii) It suffices to show that $\iota(\jmath(T'_w)) = \jmath(\iota(T'_w))$ for all $w \in W$. Since ι and $\jmath$ both preserve multiplication, it even suffices to show that $\iota(\jmath(T'_s)) = \jmath(\iota(T'_s))$ for $s \in S$. But $\iota(\jmath(T'_s)) = \iota(T'_s) = T'_s - \alpha T'_e$ and $\jmath(\iota(T'_s)) = \jmath(T'_s - \alpha T'_e) = T'_s - \alpha T'_e$, and the result follows. Q. E. D.

11 Kazhdan-Lusztig polynomials

We keep all the notations of the preceding section. The main result of this section is the following:

Proposition 11.1 *Let (W,S) be a Coxeter system, and let $\mathscr{H}$ be the associated Hecke algebra. Then for $y \in W$ there is a unique element $C'_y \in \mathscr{H}$ such that $\iota(C'_y) = C'_y$ and*

$$C'_y = \frac{1}{q^{\frac{l(y)}{2}}} \sum_{x \leq y} P_{x,y}(q) T_x$$

where $P_{y,y}(q) = 1$ and $P_{x,y}(q) \in \mathbb{Z}[q]$ has degree $\leq \frac{l(y)-l(x)-1}{2}$ if $x < y$.

Then $(C'_y)_{y \in W}$ is clearly a basis of $\mathscr{H}$; it is called the **Kazhdan-Lusztig basis** of $\mathscr{H}$, and the $P_{x,y}(q)$ are called the **Kazhdan-Lusztig polynomials** of W.

Proposition 11.2 *Let (W,S) be a Coxeter system, and let $\mathscr{H}$ be the associated Hecke algebra. Let $y \in W$.*
(i) There is a unique element $C_y \in T'_y + \sum_{x<y} q^{\frac{1}{2}} \mathbb{Z}[q^{\frac{1}{2}}] T'_x$ such that $\iota(C_y) = C_y$.
(ii) There is a unique element $C'_y \in T'_y + \sum_{x<y} q^{-\frac{1}{2}} \mathbb{Z}[q^{-\frac{1}{2}}] T'_x$ such that $\iota(C'_y) = C'_y$.

Proof: Note that thanks to proposition 10.8, (i) is equivalent to (ii), since $C'_y = \jmath(C_y)$. Thus it suffices to prove (i). Put $\beta = q^{\frac{1}{2}}, \mathcal{A} = \beta\mathbb{Z}[\beta]$. Let $(Q_{x,y}(\beta))_{x \leq y}$ be a family of elements of $\mathcal{A}$ indexed by the lower interval $[e,y]$, with $Q_{y,y} = 1$, and put

$$C_y = \sum_{x \leq y} Q_{x,y}(\beta) T'_x. \tag{11.1}$$

Then we must solve the equation $(*)$ $\iota(C_y) = C_y$, in the unknowns $(Q_{x,y}(\beta))_{x \leq y}$. But

$$\begin{aligned}
\iota(C_y) &= \sum_{u \leq y} \overline{Q_{u,y}}(\beta) \iota(T'_u) \\
&= \sum_{u \leq y} \overline{Q_{u,y}}(\beta) \sum_{x \leq u} (-1)^{l(x)-l(u)} q^{\frac{l(x)-l(u)}{2}} R_{x,u}(q) T'_x \\
&= \sum_{x \leq y} \Big(\sum_{x \leq u \leq y} (-1)^{l(x)-l(u)} q^{\frac{l(x)-l(u)}{2}} \overline{Q_{u,y}}(\beta) R_{x,u}(q) \Big) T'_x \\
&= \sum_{x \leq y} \Big(\overline{Q_{x,y}}(\beta) + \sum_{x<u \leq y} (-\beta)^{l(x)-l(u)} \overline{Q_{u,y}}(\beta) R_{x,u}(\beta^2) \Big) T'_x
\end{aligned}$$

so that

$$(*) \iff Q_{x,y}(\beta) - \overline{Q_{x,y}}(\beta) = \sum_{x<u \leq y} \Big((-\beta)^{l(x)-l(u)} \overline{Q_{u,y}}(\beta) R_{x,u}(\beta^2) \Big), \text{ for all } x < y. \tag{11.2}$$

Now for any F the equation $Q - \overline{Q} = F, Q \in \mathcal{A}$, has at most one solution (if F is of the form $\sum_{i=1}^{n} a_i(\beta^i - \frac{1}{\beta^i})$ then the unique solution is $\sum_{i=1}^{n} a_i \beta^i$, and if F is not of that form the equation has no solution), so that $Q_{x,y}(\beta)$ is uniquely determined from the values of $Q_{u,y}(\beta)$ for $x < u \leq y$. Thus each $Q_{x,y}(\beta)(x \leq y)$ is unique, and so is C_y.

We now show the existence of C_y, by induction on $l(y)$. Then clearly $C_e = T_e$ and a simple computation yields $C_s = T'_s - \beta T'_e$. Suppose $l(y) \geq 2$, and take $s \in D_l(y)$. Then we can write

$y = sy'$, with $l(y') = l(y) - 1$. By the induction hypothesis, $C_{y'}$ exists, along with the family $(Q_{x,y'}(\beta))$. Since C_s and $C_{y'}$ are fixed by ι, so is $C_sC_{y'}$; this is our first candidate for C_y. The rules in (10.12) allow us to compute $C_sC_{y'}$ explicitly: if we call $I_-(I_+)$ the sets of elements $x \leq y$ that satisfy $s \in D_l(x)(s \notin D_l(x))$, then

$$T'_sC_{y'} = \sum_{x \in I_+} Q_{sx,y'}(\beta)T'_x + \sum_{x \in I_-} \big(Q_{sx,y'}(\beta) + \alpha Q_{x,y'}(\beta)\big)T'_x$$

and hence

$$C_sC_{y'} = \sum_{x \in I_+} \big(Q_{sx,y'}(\beta) - \beta Q_{x,y'}(\beta)\big)T'_x + \sum_{x \in I_-} \big(Q_{sx,y'}(\beta) - \frac{Q_{x,y'}(\beta)}{\beta}\big)T'_x$$

Note that the coefficient before T_y is $Q_{y',y'}(\beta) - \frac{Q_{y,y'}(\beta)}{\beta} = 1$ as expected. So this is almost what we want: the problem is that in the rightmost sum the polynomial $\frac{Q_{x,y'}(\beta)}{\beta}$ need not be in $\mathcal{A}$, i.e. its constant term $q_1(x, y')$ need not be zero. Then all we need to do is readjusting as follows:

$$C_y = C_sC'_y - \sum_{x \in I_-, x \neq y} q_1(x, y')C_x \tag{11.3}$$

Q. E. D.

We now give further properties of the polynomials $Q_{x,y}(\beta)$:

Proposition 11.3 *Let the polynomials $Q_{x,y}(\beta)$ be as in (11.1). Then*
(i) The degree of $Q_{x,y}(\beta)$ is $\leq l(y) - l(x) - 1$.
(ii) The polynomial $Q_{x,y}(\beta)$ is always even or odd, with the same parity as $l(y) - l(x)$.

Proof: (i) According to (11.2), when $x < y$ we have $Q_{x,y}(\beta) - \overline{Q_{x,y}}(\beta) = \sum_{x<u\leq y} t_u$ where $t_u = (-\beta)^{l(x)-l(u)}\overline{Q_{u,y}}(\beta)R_{x,u}(\beta^2)$. Now the degree of t_u with respect to the variable β is at most $l(x) - l(u) - 1 + 2\mathsf{deg}(R_{x,u}) = l(u) - l(x) - 1 \leq l(y) - l(x) - 1$, and (i) follows.

(ii) We argue by induction on $d = l(y) - l(x)$. When $d = 0$, we have $Q_{y,y} = 1$ which is even as required. So suppose $d > 0$. We invoke (11.2) again; by the induction hypothesis, for each u such that $x < u \leq y$ we have $t_u(-\beta) = t_u(\beta)$ or $t_u(-\beta) = -t_u(\beta)$ according to whether $l(y) - l(x)$ is even or odd. So this is also true for $Q_{x,y}(\beta) - \overline{Q_{x,y}}(\beta)$ and for $Q_{x,y}(\beta)$. Q. E. D.

Proof of proposition 11.1 Since $C'_w = \jmath(C_w)$ in proposition 11.2, we have

$$C'_y = \sum_{x \leq y} Q_{x,y}(-\frac{1}{\beta})T'_x.$$

If $l(y) - l(x)$ is odd, by the preceding proposition we can write $Q_{x,y}(\beta) = \sum_{i=0}^{n} a_i\beta^{2i+1}$ with $2n + 1 \leq l(y) - l(x) - 1)$, and we deduce

$$C'_y = \sum_{x \leq y} \frac{1}{\beta^{2n+1}} \sum_{i=0}^{n} a_i\beta^{2(n-i)}T'_x = \frac{1}{q^{\frac{l(y)}{2}}} \sum_{x \leq y} \sum_{i=0}^{n} a_i\beta^{l(y)-l(x)-1-2i}T_x$$

so that $P_{x,y} = \sum_{i=0}^{n} a_i q^{\frac{l(y)-l(x)-1}{2} - i}$ in this case.

When $l(y) - l(x)$ is even the computation is similar. Q. E. D.

The central conjecture here, originally a question asked independently by Dyer and Lusztig, can then be stated as follows:

Conjecture 11.4 *The Kazhdan-Lusztig polynomial $P_{x,y}$ only depends on the isomorphism class of the interval $[x, y]$. In other words, for any poset isomorphism ψ between two Bruhat intervals $[u, v]$ and $[u', v']$ in possibly distinct Coxeter groups W, W' we have*

$$\forall x, y \in [u, v],\ P_{x,y} = P_{\psi(x),\psi(y)} \tag{11.4}$$

Brenti [6] has shown this to be true when $[u, v]$ is adihedral (i.e. when $[u, v]$ does not have a subinterval isomorphic to the full Coxeter group $\mathfrak{S}_3$. The second part of this work shall be devoted to the proof that (11.4) is true when $u = u' = e$:

$$\forall x, y \in [e, v],\ P_{x,y} = P_{\psi(x),\psi(y)} \tag{11.5}$$

Subcases of this particular case have already been dealt with; reference [11] treats the case in which all the connected components of the Coxeter graph of W are trees or of type $\widetilde{A}_n$ while [7] treats the case in which W and W' are both of type A_n.

There are several equivalent forms of this conjecture:

Proposition 11.5 *Let I, I' be two Bruhat intervals, and let $\psi : I \to I'$ be a poset isomorphism. Then the following are equivalent:*
(i) $P_{x,y} = P_{\psi(x),\psi(y)}$ for all $x \le y$ in I.
(ii) $Q_{x,y} = Q_{\psi(x),\psi(y)}$ for all $x \le y$ in I.
(iii) $R_{x,y} = R_{\psi(x),\psi(y)}$ for all $x \le y$ in I.

Proof: Because of the way the polynomial $P_{x,y}$ is constructed from $Q_{x,y}$, it is clear that $(i) \iff (ii)$. Then it suffices to show that $(ii) \iff (iii)$. Recall formula (11.2):

$$Q_{x,y}(\beta) - \overline{Q_{x,y}}(\beta) = \sum_{x<u\le y} \Big((-\beta)^{l(x)-l(u)} \overline{Q_{u,y}}(\beta) R_{x,u}(\beta^2) \Big), \text{ for all } x < y.$$

This shows that $(iii) \implies (ii)$, by induction on $l(y) - l(x)$. Symmetrically, we may rewrite the formula above as (using $Q_{y,y} = 1$)

$$R_{x,y}(\beta^2) = (-\beta)^{l(y)-l(x)} (Q_{x,y}(\beta) - \overline{Q_{x,y}}(\beta)) - \sum_{x<u<y} \Big((-\beta)^{l(y)-l(u)} \overline{Q_{u,y}}(\beta) R_{x,u}(\beta^2) \Big)$$

yielding $(ii) \implies (iii)$, by induction on $l(y) - l(x)$ again. Q. E. D.

The form (iii) is usually the easier to use because the R-polynomials satsify the recurrence relations (10.10) which are much simpler than the recurrence relations satisfied by the Kazhdan-Lusztig polynomials (as in (11.3)).

12 Bibliographical notes

The material in sections 1 to 4 is very classical. On the whole we have followed the modern presentation of Geck&Pfeiffer [16], but from a more combinatorial (as opposed to geometric) standpoint, hence our larger use of results like 2.6, 3.1. The source for section 5 is Deodhar [14]. Section 7 is as in du Cloux[12]. All of sections 8 and 9 comes from Dyer's thesis [15], except for the $K_{3,2}$-avoidance result (theorem 9.7) which first appeared in Brenti,Caselli&Marietti [8]. Section 10 is classical and similar to what is done in Humphreys [17]. The main source for section 11 is of course Kazhdan&Lusztig's fundamental paper [18], although our presentation has been influenced by Dyer's (compare e.g. proposition 11.2). Proposition 11.5 comes from Marietti [20].

Part II

Intermediary Tools

1 Technical results in type A

This section contains some computations in type A that will be needed only in section IV.2.

Here (W,S) is a standard Coxeter system of type A_n: $S=\{1,2,\dots,n\}$, $m_{i,j}=3$ if $|i-j|=1$ and $m_{i,j}=2$ if $|i-j|>1$. For integers i,j and $s\in S$ such that $i\le s$ and $s+j-1\le n$, we define

$$\begin{aligned} B(s,i) &= s(s-1)(s-2)\dots(s-(i-1)) \\ B(s,i,j) &= B(s,i)B(s+1,i)B(s+2,i)\dots B(s+(j-1),i) \end{aligned} \tag{1.1}$$

Note that the word $B(s,i)$ does not contain any braid subword or equal successive characters, so that by proposition I.3.1 the corresponding element has a unique reduced expression. In particular, $B(s,i)$ is a normal form (as in section I.7). The fundamental computational result that we shall use everywhere in this section is the following:

Lemma 1.1 *Let $s,u\in S$ and $i\in\mathbb{N}, i\ge 1$. Then:*
(i) If $u\le s-i-1$ or $u\ge s+2$, then $B(s,i)u=uB(s,i)$.
(ii) If $u=s-i$, then $B(s,i)u=B(s,i+1)$.
(iii) If $u=s-i+1$, then $B(s,i)u=B(s,i-1)$.
(iv) If $s-i+2\le u\le s$, then $B(s,i)u=(u-1)B(s,i)$.

Proof: (i) Under those hypotheses, for any generator g in the word $B(s,i)$ we have $|g-u|\ge 2$ so u commutes with g. Hence u commutes with $B(s,i)$.

(ii) and (iii) are trivial.

(iv) Let $i'=s+2-u$. Then we may write $B(s,i)=B(s,i')B(s-i',i-i')=B(s,i')B(u-2,i-i')$ and by (i) we have $B(u-2,i-i')u=uB(u-2,i,i')$ so $B(s,i)u=B(s,i')uB(u-2,i,i')$. So all we need to show is $B(s,i')u=(u-1)B(s,i')$. In other words, we may replace i with i' and we must show that $B(s,i)(s+2-i)=(s+1-i)B(s,i)$ for any $i\ge 2$. Now

$$\begin{aligned} B(s,i)(s+2-i) &= B(s,i-2)(s+2-i)(s+1-i)(s+2-i) \\ &= B(s,i-2)(s+1-i)(s+2-i)(s+1-i) \\ &= (s+1-i)B(s,i-2)(s+2-i)(s+1-i) \text{ by (i)} \\ &= (s+1-i)B(s,i) \end{aligned}$$

as desired. Q. E. D.

We define the sets X_j, Y_j as in section I.7, and compute X_j explicitly:

Proposition 1.2 *One has $X_j=\{e,j,j(j-1),\dots,j(j-1)\dots 1\}=\{B(j,k)|0\le k\le j\}$.*

Proof: Clearly each $B(j,k)$ is in X_j. Conversely, let $w\in X_j$, and let $w=w_1w_2\dots w_n$ be a reduced expression for w. If $n\le 1$ we are done, so we may assume $n\ge 2$. Then $w_1=j$. Let $I=\{u\in[1,n]|w_u=j+1-u\}$, and suppose by contradiction that $I\neq[1,n]$. Then there is a smallest index i_0 not in I. Let $s=j, i=i_0-1, u=w_{i_0}$. Then $w=B(s,i)uw_{i_0+1}\dots w_n$, and $D_l(B(s,i)u)\subseteq D_l(w)\subseteq\{s\}$. According to lemma 1.1, this is possible only when $u=s+1$ or $u=s-i$. But since $w\in Y_j$, we have $u\le j$. Eventually we obtain $u=s-i=j-i_0+1$, contradicting $i_0\notin I$. Q. E. D.

By proposition I.7.1, we deduce:

Corollary 1.3 *Any expression of the form* $B(s_1,i_1)B(s_2,i_2)\dots B(s_n,i_n)$ *with* $s_1 < s_2 < \dots < s_n$ *is a normal form.*

We also deduce $|X_j| = j+1$ and $|W| = |X_1||X_2|\dots|X_n| = (n+1)!$ Consider, in the symmetric group $\mathfrak{S}_{n+1}$ on $n+1$ letters, the transposition t_i that exchanges i and $i+1$. Then t_i commutes with t_j whenever $|i-j| > 1$, and $t_i t_{i+1}$ is the 3-cycle $i \to i+1 \to i+2 \to i$, so that $(t_i t_{i+1})^3$ is the identity permutation. Thus, the mapping $S \to \mathfrak{S}_{n+1}, i \mapsto t_i$ can be extended into a surjective group homomorphism $\pi : W \to \mathfrak{S}_{n+1}$. Since those two groups have the same number of elements, π is in fact an isomorphism.

Lemma 1.4 *Let* s, s', l, l' *be nonnegative integers.*
(i) If $s-l > s'$*, then* $B(s,l)B(s',l') = B(s',l')B(s,l)$.
(ii) If $l > l'$*, then* $B(s,l)B(s,l') = B(s-1,l')B(s,l)$.
(iii) If $l-l' > s-s' \geq 0$*, then* $B(s,l)B(s',l') = B(s'-1,l')B(s,l)$.

Proof: (i) The smallest generator in $B(s,l)$ is $s+1-l$, and the largest generator in $B(s',l')$ is s'. Thus if u is a generator in $B(s,l)$ and v a generator in $B(s',l')$, we have $u \geq s+1-l \geq s'+2 \geq v+2$, so u commutes with v. Then $B(s,l)$ commutes with $B(s',l')$.

(ii) We use induction on l'. The case $l' = 0$ is trivial. If $l' > 0$, then $B(s,l)B(s,l') = B(s,l)B(s,l'-1)(s+1-l') = B(s-1,l'-1)B(s,l)(s+1-l')$ by the induction assumption. Now the generator $u = s+2-l$ satisfies $s+2-l \leq u \leq s$, so that lemma 1.1(iv) yields $B(s,l)u = (u-1)B(s,l)$, whence $B(s,l)B(s,l') = B(s-1,l'-1)(s-l')B(s,l) = B(s-1,l')B(s,l)$.

(iii) We have $s \geq s' \geq s+1-l+l' \geq s+1-l$, so we can write $B(s,l) = B(s,s-s')B(s',l-(s-s'))$ and hence $B(s,l)B(s',l') = B(s,s-s')B(s',l-(s-s'))B(s',l')$. But $B(s',l-(s-s'))B(s',l') = B(s'-1,l')B(s',l-(s-s'))$ by (ii), and $B(s,s-s')B(s'-1,l') = B(s'-1,l')B(s,s-s')$ by (i). Eventually $B(s,l)B(s',l') = B(s'-1,l')B(s,s-s')B(s',l-(s-s')) = B(s'-1,l')B(s,l)$. Q. E. D.

Lemma 1.5 *Let* s, i, j, l, l' *be nonnegative integers. Then:*

(i) $B(s,i,j)^{-1} = B(s-i+j,j,i)$.

(ii) Let r *be an index in* $\{1,\dots,i\}$*. Denote by* c_r *he word obtained by deleting the* r*-th generator in the normal form* $B(s,i)$*. Then the normal form of* c_r *is* $B(s-r,i-r)B(s,r-1)$.

(iii) Let $q \geq 2$ *and* $r \in [1,i]$*. Then* $B(s,i,q-1)B(s+q-(r+1),i-r) = B(s-r,i-r)B(s,i,q-1)$.

(iv) Let a *be an index in* $\{1,\dots,ij\}$*. Denote by* c_a *the word obtained by deleting the* a*-th generator in the normal form* $B(s,i,j)$*. We can write* $a = qi+r, 1 \leq r \leq i$*. Then the normal form of* c_a *is*

$$B(s-r,i-r)B(s,i,q-1)B(s+q-1,r-1)B(s+q,i,j-q)$$

Proof: (i) Let $w = B(s,i,j)$. We argue by induction on j. If $j=1$ then $w = B(s,i) = s(s-1)(s-2)\dots(s-i+1)$, so $w^{-1} = (s-i+1)\dots(s-2)(s-1)s = B(s-i+1,1,i)$ as required. So suppose $j > 1$. Then $w = B(s,i)B(s+1,i,j-1), w^{-1} = B(s+1,i,j-1)^{-1}B(s,i)^{-1} = B(s-i+j,j-1,i)B(s-i+1,1,i)$ by the induction hypothesis. Thus, if we put $b_k = B(s-i+j+k-1,j-1)$ and $s_k = s-i+k$ for $k \in [1,i]$, we have $w^{-1} = b_1 b_2 \dots b_i s_1 s_2 \dots s_i$. Now, by lemma 1.1(i) we have $b_k s_1 = s_1 b_k$ for each $k \in [2,i]$, so $w^{-1} = b_1 s_1 b_2 \dots b_i s_2 \dots s_i = B(s-i+j,j)b_2 \dots b_i s_2 \dots s_i$. Similarly we have

$b_2 \dots b_i s_2 \dots s_i = B(s-i+j+1,j) b_3 \dots b_i s_3 \dots s_i$, whence $w^{-1} = B(s-i+j,j,2) b_3 \dots b_i s_3 \dots s_i$. It is now clear by induction that $w^{-1} = B(s-i+j,j,k) b_{k+1} \dots b_i s_{k+1} \dots s_i$ for any $k \in [0,i]$. When $k = i$ we eventually obtain $w^{-1} = B(s-i+j,j,i)$ as required.

(ii) By corollary 1.3, $c'_r = B(s-r,i-r)B(s,r-1)$ is a normal form. Thus all we need to show is that $c_r = c'_r$. But lemma 1.4(i) shows that $B(s-r,i-r)$ commutes with $B(s,r-1)$ and (ii) follows.

(iii) Let us show by induction that $H(q) : B(s,i,q-1)B(s+q-(r+1),i-r) = B(s-r,i-r)B(s,i,q-1)$ is true for every $q \geq 2$. Statement $H(2)$ says that $B(s,i)B(s+1-r,i-r) = B(s-r,i-r)B(s,i)$, which follows from lemma 1.4(iii). So assume $q > 2$. Then $B(s,i,q-1) = B(s,i,q-2)B(s+q-2,i)$. Now, $H(2)$ (used with $s+q-2$ in place of s) yields $B(s+q-2,i)B(s+q-(r+1),i-r) = B(s+q-(r+2),i-r)B(s+q-2,i)$, and $H(q-1)$ yields $B(s,i,q-2)B(s+q-(r+2),i-r) = B(s-r,i-r)B(s,i,q-2)$. Combining those two results, $B(s,i,q-1)B(s+q-(r+1),i-r) = B(s-r,i-r)B(s,i,q-2)B(s+q-2,i) = B(s-r,i-r)B(s,i,q-1)$ and this is $H(q)$.

(iv) By corollary 1.3, $c'_a = B(s-r,i-r)B(s,i,q-1)B(s+q-1,r-1)B(s+q,i,j-q)$ is a normal form. Thus all we need to show is that $c_a = c'_a$. We have $B(s,i,j) = B(s,i,q-1)B(s+q-1,i)B(s+q,i,j-q)$. By (ii), $c_a = B(s,i,q-1)B(s+q-(r+1),i-r)B(s+q-1,r-1)B(s+q,i,j-q)$. And (iii) then yields $c_a = B(s-r,i-r)B(s,i,q-1)B(s+q-1,r-1)B(s+q,i,j-q) = c'_a$. Q. E. D.

Lemma 1.6 *Let r,j be integers such that $1 \leq j < r$. Let $y \in W$ be a word such that $r \in D_l((r-1)\dots(r-j)\cdot y)$. Then we can write $y = r(r-1)\dots(r-j)\cdot z$ for some $z \in W$.*

Proof: We use induction on j. If $j = 1$ the result follows from $m_{r,r-1} = 3$ and corollary 2.6. Suppose the result is true for all $j' < j$, and let y, r, j be as above. If we put $y' = (r-j)\cdot y$, the induction hypothesis yields a z' such that $y' = r(r-1)\dots(r-(j-1))\cdot z'$. Then $r-j \in D_l(r(r-1)\dots(r-(j-1))\cdot z')$. Now all the generators in $(r-1)\dots(r-(j-1))$ commute with r. except for the last one $r-(j-1)$. We deduce $r-j \in D_l((r-(j-1))\cdot z')$. As $m_{r-j,r-(j-1)} = 3$, we have $z' = (r-j)(r-(j-1))\cdot z$ for some z, and

$$\begin{aligned}
(r-j)y &= r(r-1)\dots(r-(j-2))(r-(j-1))\cdot((r-j)(r-(j-1))\cdot z) \\
&= r(r-1)\dots(r-(j-2))\cdot(r-(j-1))(r-j)(r-(j-1))\cdot z \\
&= r(r-1)\dots(r-(j-2))\cdot(r-j)(r-(j-1))(r-j)\cdot z \\
&= (r-j)\cdot r(r-1)\dots(r-(j-2))\cdot(r-(j-1))(r-j)\cdot z
\end{aligned}$$

so that

$$y = r(r-1)\dots(r-j)\cdot z$$

which conclude the proof. Q. E. D.

Proposition 1.7 *Consider an X-decomposition $w = x_1 \dots x_n$. Then for r in $[1,n]$: $(r \in D_l(w)) \Leftrightarrow (l(x_r) > l(x_{r-1}))$.*

Proof: Suppose $r \in D_l(w)$. Write $y = x_1 \dots x_{r-2}$, $u = x_{r-1} \dots x_n$. Then $w = y \cdot u$, hence $r \in D_l(y \cdot u)$. But every generator in y commutes with r, so $r \in D_l(u)$, and we see that the normal form of u starts with $r-1$ or with r. If $x_{r-1} = e$, the normal form of u starts with r, so that $x_r \neq e$, whence $l(x_r) = 1 > 0 = l(x_{r-1})$ as desired. If $x_{r-1} \neq e$, there exists an integer j with $1 \leq j < r$ and $x_{r-1} = (r-1)\ldots(r-j)$. If we put $y' = x_r x_{r+1} \ldots x_n$, the preceding lemma yields a z such that $y' = r(r-1)\ldots(r-j)\cdot z$. Let $k = l(x_r)$. Then there is no z such that $y' = r(r-1)\ldots(r-k)\cdot z$ (otherwise $y'' = (r-k)\cdot z$ where $y'' = x_{r+1}\ldots x_n$, so that the first character in the normal form of y'' is both $\leq r-k$ and $\geq r+1$, which is impossible); this forces $k > j$, hence $l(x_r) > l(x_{r-1})$.

Conversely, suppose $l(x_r) > l(x_{r-1})$. If we put $j = l(x_{r-1})$, there is a x such that $x_r = r(r-1)\ldots(r-j)\cdot x$, so that

$$\begin{aligned} w &= x_1x_2\ldots x_{r-2}(r-1)\ldots(r-j)r(r-1)\ldots(r-j)xx_{r+1}x_{r+2}\ldots x_n \\ &= x_1x_2\ldots x_{r-2}(r-1)r(r-1)[(r-2)\ldots(r-j)]^2xx_{r+1}x_{r+2}\ldots x_n \\ &= x_1x_2\ldots x_{r-2}r(r-1)r[(r-2)\ldots(r-j)]^2xx_{r+1}x_{r+2}\ldots x_n \\ &= rx_1x_2\ldots x_{r-2}(r-1)r[(r-2)\ldots(r-j)]^2xx_{r+1}x_{r+2}\ldots x_n \end{aligned}$$

and we see that $r \in D_l(w)$. Q. E. D.

Proposition 1.8 *Let $w \in W$ such that $|D_l(w)| \leq 1$. Then $|\{u < w | l(u) = l(w) - 1\}| = l(w)$.*

Proof: If $D_l(w) = \emptyset$ there is nothing to prove. So suppose that $D_l(w) = \{s\}$ for some $s \in S$. By the preceding proposition, we can write the X-decomposition of w as

$$w = x_s x_{s+1} \ldots x_n \qquad (*)$$

with $l(x_s) \geq l(x_{s+1}) \geq \ldots \geq l(x_n)$. What we must show is that the deletion of any character in the reduced expression above still produces a reduced expression. So let c be the k-th generator in x_j, $j \geq s$. Let w' be the element obtained when we delete c from (*). Put $x = B(j, k-1)$, $y = B(j-k, l_j - k)$, $z = x_{j+1}\ldots x_n$; thus $w' = x_s \ldots x_{j-1}xyz$.

Iterating lemma 1.4(iii), we have

$$\begin{aligned} w' &= x_s x_{s+1}\ldots x_{j-2}x_{j-1}xyz \\ &= x_s x_{s+1}\ldots x_{j-2}x_{j-1}yxz \\ &= x_s x_{s+1}\ldots x_{j-2}y'x_{j-1}xz \text{ (with } y' = B(j-k-1, l_j - k)) \\ &= x_s x_{s+1}\ldots y''x_{j-2}x_{j-1}xz \text{ (with } y'' = B(j-k-2, l_j - k)) \\ &= \ldots \\ &= y^{(j-s)}x_s x_{s+1}\ldots x_{j-2}x_{j-1}xz \text{ (with } y^{(j-s)} = B(s-k, l_j - k)) \\ &= x_{s-k}x_s x_{s+1}\ldots x_{j-2}x_{j-1}x'_j x_{j+1}\ldots x_n \end{aligned}$$

where we put $x_{s-k} = y^{(j-s)}$ and $x'_j = x$. This is clearly the X-decomposition of w', so $l(w') = l(w) - 1$ which concludes the proof. Q. E. D.

Proposition 1.9 *We have:*

1) *If $i > 0, j > 0$, then $D_l(B(s,i,j)) = \{s\}$, $D_r(B(s,i,j)) = \{s - i + j\}$.*
2) *Let $w \in W$. Then $(|D_l(w)| \leq 1, |D_r(w)| \leq 1)$ if and only if w is of the from $B(s,i,j)$ for some s, i, j.*

Proof: By proposition 1.7 we have $D_l(B(s,i,j)) = \{s\}$, and also

$$D_r(B(s,i,j)) = D_l(B(s,i,j)^{-1}) = D_l(B(s-i+j,j,i)) = \{s-i+j\}$$

which yields 1). The "if" part of 2) follows from 1). Conversely, suppose that $w \in W$ is such that $(|D_l(w)| \leq 1, |D_r(w)| \leq 1)$. We may assume $w \neq e$ (since $e = B(s,0,j)$). Let s be the smallest left descent generator for w and r the largest generator in the support of w. Then $D_l(w) = \{s\}$, so that the X-decomposition $w = x_s \ldots x_r$ of w satisfies

$$l(x_s) \geq l(x_{s+1}) \geq l(x_{s+2}) \geq \ldots \geq l(x_r) > 0.$$

What we must show is that all those inequalities are in fact equalities. Suppose that this is not the case, and that $l_s = l_{s+1} = \ldots = l_t > l_{t+1}$ for some index t (we put $l_j = l(x_j)$). Let $x = x_s \ldots x_t$ and $y = x_{t+1} \ldots x_r$, so that $w = xy$. The rightmost generator in x is $a = t - l_t + 1$. Let $j \in [t+1, r]$. We have $l_t > l_{t+1} \geq l_j$, so that the rightmost (and also the smallest) generator in x_j is $j - l_j + 1 > j - l_t + 1 \geq t + 3 - l_t = a + 2$. As this is true for any j, all the generators in y are $\geq a+2$ and hence commute with a. Therefore $ay = ya$ and $a \in D_r(x)$, whence a is a right descent generator for $xy = w$. Now $l_{t+1} \geq l_r > 0$, so $y \neq e$, and we have at least two right descent generators for w (a and any element of $D_r(y)$), contradicting the initial hypothesis. Q. E. D.

2 Decreasing the coefficients of the Coxeter matrix

In all this section we shall be concerned with what happens if, in a Coxeter system (W,S), we replace the Coxeter matrix (m_{st}) by another matrix (m'_{st}) such that $m'_{s,t} \leq m_{s,t}$ for all $s,t \in S, s \neq t$. We then obtain a new Coxeter system (W', S). For a word $w = (w_1, \ldots, w_r)$ in S^*, we denote by $w_W (w_{W'})$ the corresponding element in W (W'). We say that a word w is **absolutely reduced** (with respect to (W, W')) if is reduced in both W and W', or, in other words, $l(w_W) = l(w_{W'}) = l(w)$.

Proposition 2.1 *Let $u = (u_1, u_2, \ldots, u_p), v = (v_1, v_2, \ldots, v_q), w = (w_1, w_2, \ldots, w_r)$ be words in S^*.*
(i) Suppose that u and v are absolutely reduced, and that u_W is a prefix of v_W in W. Then there is an absolutely reduced word z, of length $q-p$, such that the identity $v = uz$ holds in both W and W' (in particular $u_{W'}$ is also a prefix of $v_{W'}$ in W').
(ii) If w is absolutely reduced, then $D_l(w_W) \subseteq D_l(w_{W'})$.
(iii) If w is reduced in W' then w is absolutely reduced.
(iv) If u, v are absolutely reduced and $u_W = v_W$, then $u_{W'} = v_{W'}$.

Proof: (i) We argue by induction on $l = p + q$, the result being clear if $l \leq 2$. We may certainly assume $p > 0$. Then u_1 is a prefix of $v_1 v_2 \ldots v_q$ in W.

Suppose first that u_1 is a prefix of $v_1 v_2 \ldots v_{q-1}$ in W. Then, by the induction hypothesis, there is an absolutely reduced word z' of length $q-2$ such that the equality $v_1 v_2 \ldots v_{q-1} = u_1 z'$ holds in both W and W'. Then $u_2 u_3 \ldots u_p$ is a prefix of $z' v_q$ in both W and W'. By the induction hypothesis again, there is an absolutely reduced word z of length $q-p$ such that the identity $z' v_q = u_2 u_3 \ldots u_p z$ holds in both W and W' and we are done.

Suppose now that u_1 is not a prefix of $v_1v_2\dots v_{q-1}$ in W. By corollary I.3.2, we deduce $u_1(v_1v_2\dots v_{q-1}) = (v_1v_2\dots v_{q-1})v_q$ in W. Then certainly $u_1 \neq v_1$, and by corollary I.2.6 we see that the word $t = [u_1, v_1, m-2\rangle$ (where $m = m_{u_1,v_1}$) is a prefix of $v_2\dots v_{q-1}$ in W. Since v is absolutely reduced, we deduce $m'_{u_1,v_1} \geq m$ and hence $m'_{u_1,v_1} = m$. By the induction hypothesis, there is an absolutely reduced word z' of length $q-m+1$ such that the identity $v_2\dots v_{q-1} = tz'$ holds in both W and W'. Let $v' = [v_1, u_1, m-1\rangle z'$. Then $v = u_1v'$ in both W and W', so $l(v'_{W'}) \geq l(v_{W'}) - 1 = l(v) - 1 = l(v')$, so v' is absolutely reduced. By the induction hypothesis again, there is an absolutely reduced word z of length $q-p$ such that the identity $v' = u_2u_3\dots u_pz$ holds in both W and W', and we are done. This finishes the proof of (i).

(ii) follows from (i), by taking $w = v$ and $u \in S$. And (iii) follows from (ii), using induction on $l(w)$ and the fact that $(w_1, w_2, \dots, w_r)$ is reduced if and only if $(w_2, \dots, w_r)$ is and $w_1 \notin D_l(w_2\dots w_r)$. Eventually (iv) follows from (i) and the fact that for elements u, v, $u = v$ if and only if u is a prefix of v and v is a prefix of u. Q. E. D.

We should now like to strengthen (iv) and construct isomorphisms between lower intervals of W and W'. Unfortunenately, if w is absolutely reduced it is not true in general that the intervals $[e, w]_W$ and $[e, w']_{W'}$ are isomorphic, because $[e, w]_W$ might contain some large dihedral elements that do not exist in the "smaller" group W'. We need to consider another family of words that avoids this situation :

Definition 2.2 *Let s, t be two distinct elements of S. Let $u = (u_1, \dots, u_m)$ be a word in S^*. We say that u is $\{s,t\}$-***dihedral** *if u is two-periodic and either $u_1 = s, u_2 = t$ or $u_1 = t, u_2 = s$.*

Definition 2.3 *Let w be a word in S^* (not necessarily reduced in W or W'). We say that w is* **absolute** *(with respect to the pair (W, W')) if for any $s \neq t$ in S such that $m_{st} > m'_{st}$, any $\{s,t\}$-dihedral subexpression of w has length at most m'_{st}.*

Note that an absolutely reduced word is not necessarily absolute. Note also that the set of absolute words is closed under taking subexpressions.

Proposition 2.4 *(i) Let a be an absolute expression. There exists an absolutely reduced subexpression r of a such that $a = r$ holds in both W and W'.*
(ii) If a and b are absolute and $a_W = b_W$, then $a_{W'} = b_{W'}$.
(iii) If a is reduced in W', b is reduced in W, $a_W = b_W$, and a is absolute, then b is absolute also.
(iv) Let $\mathcal{A}(W)$ ($\mathcal{A}(W')$) denote the set of elements of W (W') that can be represented by an absolute expression. Then there is a canonical map $p : \mathcal{A}(W) \to \mathcal{A}(W')$, defined by $p(a_W) = p(a_{W'})$ for any absolute expression a. The mapping p is surjective and order-preserving. If $w \in \mathcal{A}(W)$, p restricts to an order isomorphism $[e, w]_W \to [e, p(w)]_{W'}$.
(v) Let $s \in S, w \in W$ such that w and sw are both in $\mathcal{A}(W)$. Then $p(sw) = sp(w)$ (similarly $p(ws) = p(w)s$ if w and ws are both in $\mathcal{A}(W)$).

Proof : (i) Write $a = (a_1, \dots, a_n)$; we argue by induction on n. If $n = 0$ there is nothing to prove, so assume $n > 0$. If a is reduced in W', by 2.1.(iii) above a is absolutely reduced so we may take $r = a$. So assume that a is not reduced in W'. Now all we need to do is find a strict subexpression a' of a, such that $a = a'$ holds in both W and W'. Consider the word $a^- = (a_2, \dots, a_n)$. By the induction hypothesis there is a subexpression b of a^- which is absolutely reduced and such that $a^- = b$ holds in both W and W'. If b is a strict subexpression of a^-, taking

$a' = a_1 b$ we are done. So we may assume that $a^- = b$ as words, which means that a^- is absolutely reduced. If $a_2 = a_1$, taking $a' = (a_3, \dots, a_m)$ we are done. So we may assume $a_2 \neq a_1$. Since a is not reduced in W', a_1 is a prefix of $a_2 a_3 a_4 \dots a_n$ in W'. By corollary I.2.6 the element $[a_1, a_2, m'-1\rangle_{W'}$ (with $m' = m'_{a_1 a_2}$) is a prefix of $a_3 \dots a_n$ in W'. In particular $[a_1, a_2, m'-1\rangle$ is a subexpression of $(a_3, \dots, a_n)$ and $[a_1, a_2, m'+1\rangle$ is a subexpression of a. Since a is absolute, we deduce

$$m_{a_1 a_2} = m'_{a_1 a_2} = m' \tag{1}$$

Also a_1 is a prefix of $a_3 \dots a_n$ in W'. Using the induction hypothesis on the word $(a_1, a_3, \dots, a_n)$, we see that there is an increasing sequence $i_1 < i_2 < \dots i_p$ with values in $\{1, 3, 4, \dots, n\}$ such that the expression $(a_{i_1}, a_{i_2}, \dots, a_{i_p})$ is absolutely reduced and $a_1 a_3 \dots a_n = a_{i_1} a_{i_2} \dots a_{i_p}$ holds in both W and W'. If $i_1 = 1$ we would deduce that $a_3 \dots a_n$ is nonreduced in W', which is impossible. So $(a_{i_1}, a_{i_2}, \dots, a_{i_p})$ is in fact a subexpression of $(a_3, \dots, a_n)$. Consider the word

$$a^{(3)} = (a_1, a_2, a_1, a_{i_1}, \dots, a_{i_p})$$

We claim that $a^{(3)}$ is absolute. Indeed, for any pair $s, t \in S, s \neq t$ such that $m_{s,t} > m'_{s,t}$, we have $\{s,t\} \neq \{a_1, a_2\}$ by (1), so any $\{s,t\}$-dihedral subsequence of $a^{(3)}$ is a subsequence of $(a_1, a_{i_1}, \dots, a_{i_p})$ or $(a_2, a_{i_1}, \dots, a_{i_p})$, both of which are subexpressions of a. We know also that $a = a^{(3)}$ holds in both W and W'. Thus we may replace a with $a^{(3)}$; in other words, we may assume $a_3 = a_1$. Continuing in this manner, we may even assume that $[a_1, a_2, m'+1\rangle$ is a prefix of the word a : $a = [a_1, a_2, m'+1\rangle c$ for some word c. Then, by (1), the equality $a = [a_2, a_1, m'-1\rangle c$ holds in both W and W', so we may take $a' = [a_2, a_1, m'-1\rangle c$.

(ii) By (i), there are absolutely reduced expressions r_a and r_b such that the equalities $a = r_a$ and $b = r_b$ hold in both W and W'. By 2.1.(iv), we deduce that $(r_a)_{W'} = (r_b)_{W'}$ and hence $a_{W'} = b_{W'}$.

(iii) By I.2.6, (ii) above and 2.1.(iv), it suffices to show the result when b is obtained from a by applying a braid rewriting in W. In this situation, there are words c and d and generators s_1, s_2 such that $a = c[s_1, s_2, m\rangle d$ and $b = c[s_2, s_1, m\rangle d$ (where $m = m_{s_1 s_2}$). Suppose by contradiction that a is absolute but b is not. Then there are generators $s \neq t$ in S such that $m_{st} > m'_{st}$, and a $\{s,t\}$-dihedral subexpression u of b which has length $> m'_{st}$. We can write $u = u_1 u_2 u_3$, where u_1 is a subexpression of c, u_2 is a subexpression of $[s_2, s_1, m\rangle$, and u_3 is a subexpression of d. Since a is absolute, u is not a subexpression of a, so u_2 is a subexpression of $[s_2, s_1, m\rangle$ but not of $[s_1, s_2, m\rangle$. Now u_2 is $\{s,t\}$-dihedral like u, so this implies $u_2 = [s_2, s_1, m\rangle$. Therefore $\{s_1, s_2\} = \{s, t\}$, hence $m_{s_1, s_2} > m'_{s_1, s_2}$. But then a cannot be absolute.

(iv) The mapping p is correctly defined by (ii). The other properties of p follow from the fact that any subexpression of an absolute expression is also absolute.

(v) Interchanging w and sw, we may assume $w < sw$. By assumption, there is an absolute expression a such that $sw = a$ in W. We may take a absolutely reduced by (i). Similarly, there is an absolutely reduced, absolute expression a' such that $w = a'$ in W. Consider the expression $b = sa'$. This is reduced in W, so that by (iii) b is absolute like a. Therefore $p(b_W) = b_{W'}$, and hence $p(sw) = sa'_{W'} = sp(a'_W) = sp(w)$ as desired. Q. E. D.

Lemma 2.5 *(i) Let $J \subseteq S$ and $w \in W$. Then $[e, w] \cap < J >$ has a largest element $m_J(w)$.*
(ii) Furthermore, $m_J(w)$ may be computed by the following recursion: $m_J(e) = e$ and if $w_1 \in D_l(w)$ then (putting $w' = w_1 w$)

$$m_J(w) = \begin{cases} m_J(w') & if\ w_1 \notin J\ or\ w_1 \in D_l(m_J(w')) \\ w_1 m_J(w') & if\ w_1 \in J, w_1 \notin D_l(m_J(w')) \end{cases} \tag{2.1}$$

Proof: We show (i) and (ii) simultaneously, by induction on $l = l(w)$. If $l = 0$, then clearly $m_J(w) = e$. So assume $l \geq 1$. We can write $w = w_1 w'$, with $w_1 \in S$ and $l(w') = l(w) - 1$. Then $[e, w] = [e, w'] \cup w_1[e, w']$ by proposition I.5.1. We even have $[e, w] = [e, w'] \cup A$, where $A = \{w_1 v | v \in [e, w'], v < w_1 v\}$. By the induction hypothesis $[e, w'] \cap < J >$ has a largest element $m_J(w')$.

Suppose $w_1 \notin J$. Then $A \cap < J >= \{a \in A | \mathsf{supp}(a) \subseteq J\} = \emptyset$ (see I.(3.3)), so $[e, w] \cap J = [e, w']$ and $m_J(w) = m_J(w')$.

Suppose $w_1 \in J$ and $w_1 \notin D_l(m_J(w'))$. Then for $z \in [e, w]$, if we put $z' = \mathsf{min}(z, w_1 z)$ we have $z' \in [e, w']$ and $z \leq w_1 z' \leq w_1 m_J(w')$. So $m_J(w) = w_1 m_J(w')$. Furthermore, if the *word* $m_J(w')$ is absolutely reduced, then the word $w_1 m_J(w')$ will be reduced also.

Eventually, suppose $w_1 \in D_l(m_J(w'))$. Then $[e, w'] \cap < J >= [e, m_J(w')]$. By proposition I.5.1, we deduce that $[e, w'] \cap < J >$ is invariant by the mapping $z \mapsto w_1 z$, so $A \cap < J > \subseteq [e, m_J(w')]$ and $m_J(w) = m_J(w')$. Q. E. D.

In particular, if J is a dihedral subgroup $< s, t >$ then $[e, w] \cap < s, t >$ has a largest element $m_{s,t}(w)$ (do not mistake this for the coefficients of the Coxeter matrix). Put $\mu_{s,t}(w) = l(m_{s,t}(w))$. The final result of this section then goes as follows :

Corollary 2.6 *Let (W, S) be a Coxeter system, and $w \in W$. For $s, t \in S$, define $\mu_{s,t}(w)$ as above. Define a new Coxeter matrix (m'_{st}) on S by putting $m'_{st} = \mu_{s,t}(w)$ for $s \neq t$, and consider the associated Coxeter system (W', S). Then, in the notations and terminology of proposition 2.4, any reduced expression for w_W is absolute, so that we have a canonical isomorphism $p : [e, w_W] \rightarrow [e, w_{W'}]$ satisfying, for $s \in S$ and $u \in [e, w_W]$, $p(su) = sp(u)$ whenever $su \in [e, w_W]$ and $p(us) = p(u)s$ whenever $us \in [e, w_W]$.*

3 Bibliographical notes

Most of the results in section 1 are probably well-known but we do not know of any previous written exposition of those results. Section 2 is a special case of the results in du Cloux [11], but the presentation here is simplified and somewhat different.

Part III

Special Matchings and the main Result

1 General results

Let $(P, <)$ be a poset. We write $x \lhd y$ when we mean that $x < y$ and there is no z such that $x < z < y$. In this case x is a **coatom** of y; we denote by $\mathsf{coat}(\mathbf{y})$ the set of all coatoms of an element $y \in P$. All the posets considered here are **graded**, i.e. they have a function $l \ P \to \mathbb{N}$ such that $l(y) = l(x) + 1$ whenever $x \lhd y$. Actually, the first half of this section contains results that hold for completely general graded posets, while in the rest of the paper we only consider, given a fixed Coxeter system (W, S), the graded poset arising when we equip W with the Bruhat ordering and the usual length function.

If $u \in P$ and ϕ is a partial map defined at least on the coatoms of u, we use the abbreviation

$$Z(\phi, u) = \{u\} \cup \{\phi(v) | \ v \lhd u, \ v \lhd \phi(v)\}$$

Now let $\phi \ \ P \to P$ be a map. We say that ϕ is a **special matching** when the following conditions are fullfilled for any $u \in P$:

(i) ϕ is involutive ($\phi(\phi(u)) = u$)

(ii) $u \lhd \phi(u)$ or $\phi(u) \lhd u$

(iii) $(u \lhd \phi(u)) \Rightarrow (\ \mathsf{coat}(\phi(u)) = Z(\phi, u) \ )$.

Condition (iii) is the most significant, the other two only define the setting. If the terminology is due to Brenti [7], the choice of the definition (among a certain number of equivalent ones) rather comes from du Cloux [9]. Parts (i) and (ii) are common to [7] and [9], while (iii) is expressed explicitly in neither of those two papers, but is easily seen to be equivalent to the versions given in each.

The following easy consequence of (iii) will be used often in the sequel:

Remark 1.1 *Let $w \in P$. If w has at least a coatom u such that $\phi(u) \neq w, u \lhd \phi(u)$, then $w \lhd \phi(w)$.*

We will also use the following:

Remark 1.2 *Let ϕ be a special matching on a graded poset P, and let $x, y \in P$ such that $x \leq y, x \lhd \phi(x), \phi(y) \lhd y$. Then ϕ restricts to a special matching of the interval $[x, y]$.*

Indeed, it suffices to show that under those assumptions we have $\phi(y) \in [x, y]$. To see this, take a path from x to y: $x = x_0 \lhd x_1 \lhd x_2 \lhd \ldots \lhd x_n = y$. Then there is an index i such that $x_i \lhd \phi(x_i)$ but $x_j \rhd \phi(x_j)$ for $j > i$. Then (iii) implies $\phi(x_i) = x_{i+1}$, and hence $x \leq x_i = \phi(x_{i+1}) \lhd \phi(x_{i+2}) \lhd \ldots \phi(x_n) = \phi(y)$ as required.

If Q is a **decreasing** subset of P (i.e. $(x \leq q) \Rightarrow (x \in Q)$ for any $q \in Q, x \in P$), the notion can be relativized as follows: we say that a pair (Q, ϕ) is a **partial** special matching if Q is decreasing and ϕ is a map $Q \to Q$ which gives a special matching on Q. We use the notation $Q = \mathsf{dom}(\phi)$ (so that we often write just ϕ instead of (Q, ϕ) to name the partial matching).

Let $\mathcal{I}(P)$ be the set of all partial matchings of P; there is a natural partial ordering $\leq_{\mathcal{I}}$ on $\mathcal{I}(P)$, namely $(Q_1, \phi_1) \leq_{\mathcal{I}} (Q_2, \phi_2)$ if and only if $Q_1 \subseteq Q_2$ and ϕ_2 extends ϕ_1. The maximal elements of $\leq_{\mathcal{I}}$ are called **maximal matchings**. One can introduce the even more specialized notion of a **Q-maximal matching** of P: this is a partial matching ϕ of P such that $\forall \phi'$ extending ϕ, $\mathsf{dom}(\phi) \cap Q = \mathsf{dom}(\phi') \cap Q$.

If Q is finite, any finite chain $\phi_1, \phi_2, \dots \phi_r$ satisfying for each i between 1 and $r-1$ the condition

$$\phi_{i+1} \text{ extends } \phi_i, \mathsf{dom}(\phi_i) \cap Q \subset \mathsf{dom}(\phi_{i+1}) \cap Q$$

(where $\subset$ denotes strict inclusion) necessarily has length $\leq |Q|$, and if this chain has maximum length its last element must be a Q-maximal matching; therefore:

Remark 1.3 *Let P be a graded poset, Q a finite decreasing subset of P. Then any partial matching of P can be extended into a Q-maximal matching.*

A **maximal matching** of P is a P-maximal matching of P, i.e. a special matching of P whose domain is maximal for inclusion. We then have a "local-to-global"-type result:

Theorem 1.4 *Let P be a locally finite graded poset (i.e. $\{x \in P | \ l(x) = k\}$ is finite for each k). If A is a decreasing subset of P, then any partial matching ϕ defined on A can be extended into a maximal matching on P. If in addition the* coat *function is injective on $P \setminus A$, then this extension is unique.*

Proof: Existence.

For each k we put $B_k = \{x \in P | \ l(x) \leq k\}$. By remark 1.3, ϕ has an extension ϕ_0 which is B_0-maximal. Repeteadly using this remark 1.3, we construct a sequence $(\phi_n)_{n \geq 0}$ of partial matchings on P such that

$$\forall n \geq 1, \phi_n \text{ extends } \phi_{n-1}, \ \phi_n \text{ is } B_n - \text{maximal.}$$

Now we set $Q = \bigcup \mathsf{dom}(\phi_n)_{n \geq 0}$, and define $\psi : Q \to Q$ by $\forall x \in Q, \psi(x) = \phi_n(x)$ if $x \in \mathsf{dom}(\phi_n)$. Then ϕ is well defined and is a maximal matching extending ϕ, as required.

Uniqueness (when coat is injective on $P \setminus A$).

By contradiction, suppose we have two distinct maximal matchings μ_1 and μ_2 extending ϕ. Take w of minimal length such that μ_1 differs from μ_2 at point w, i.e. (interchanging μ_1 and μ_2 if necessary)

$$\begin{array}{ll} \text{Case 1}: & w \in \mathsf{dom}(\mu_1), \ w \in \mathsf{dom}(\mu_2), \ \mu_1(w) \neq \mu_2(w), \text{or else} \\ \text{Case 2}: & w \in \mathsf{dom}(\mu_1), \ w \notin \mathsf{dom}(\mu_2). \end{array}$$

Consider case 1. We certainly have $w \notin A$; by minimality of w we necessarily have $w \lhd \mu_1(w), \ w \lhd \mu_2(w)$, and hence $\mu_1(w) \notin A, \ \mu_2(w) \notin A$. Then condition (iii) gives $\mathsf{coat}(\mu_1(w)) = \mathsf{coat}(\mu_2(w))$ so that $\mu_1(w) = \mu_2(w)$ which is a contradiction.

Now we treat case 2. As in the preceding case, we see that $w \notin A, w \lhd \mu_1(w), \mu_1(w) \notin A$. Also $\mu_1(w) \notin \mathsf{dom}(\mu_2)$ (because $\mathsf{dom}(\mu_2)$ is decreasing and $w \notin \mathsf{dom}(\mu_2)$). Let $Q = \mathsf{dom}(\mu_2) \cup \{w; \mu_1(w)\}$ and $\phi \ Q \to Q$ be defined by $\phi(x) = \mu_2(x)$ if $x \in \mathsf{dom}(\mu_2)$ and $\phi(x) = \mu_1(x)$ if $x \in \{w; \mu_1(w)\}$. The mapping ϕ thus constructed is a partial matching and a nontrivial extension of μ_2, contradicting maximality. Q. E. D.

We now leave the realm of abstract graded posets to stay till the end of this paper within the smaller world of posets P arising from Coxeter systems (W, S) as follows: P will be W equipped with the Bruhat-Chevalley ordering, and the usual (non-weighted) length function, or a decreasing subset of that graded poset (thus we shall not need the symbol P to denote an abstract poset any more; note that just below we will use the notation P for an entirely different thing, and keep the new meaning for P in the sequel).

In this special case, theorem 1.4 can be enunciated in a stronger form. If ϕ is a maximal matching, then $\mathsf{dom}(\phi) \neq \emptyset$, so $e \in \mathsf{dom}(\phi)$ and by rule (ii), $a = \phi(e)$ is a generator: $a \in S$. From now on, we will implicitly consider either a fixed maximal matching or a family of maximal matchings sharing the value $a = \phi(e)$ for a fixed $a \in S$. Corollaries I.9.8 and I.9.9 give a first hint at the importance of dihedral elements. We must also introduce the notion of a principal dihedral element:

Definition 1.5 *A dihedral subgroup of W is a* **parabolic dihedral subgroup** *when it is of the form $< s, t >$ for $s \neq t \in S$. It is a* **principal dihedral subgroup** *when it is of the form $P_s =< s, a >$ for $s \in S \setminus \{a\}$. A* **principal dihedral element** *is defined as an element of*

$$P = \bigcup_{s \in S \setminus \{a\}} P_s.$$

Let us start by describing how a special matching acts on a parabolic dihedral subgroup:

Proposition 1.6 *Let (W, S) be a Coxeter system, ϕ a maximal matching on W and $a = \phi(e)$ as above. Let D be a parabolic dihedral subgroup of W.*
(i) If D is principal, then ϕ is defined on the whole of D and D is invariant by ϕ.
(ii) If D is nonprincipal, then for any $w \in \mathsf{dom}(\phi) \cap D$ such that $l(w) > 1$, we have

$$w \lhd \phi(w), \textit{and } \phi(w) \textit{ is not dihedral}.$$

Proof: Let $s \in S \setminus \{a\}$ and $m = m_{as}$ (integer or infinite coefficient in the Coxeter matrix). Recall that P_s has a unique element in length 0, one element or no element at all in length m (depending on whether m is finite or not) and two elements in length j when $0 < j < m$. For $w \in P_s$ with $0 < l(w) < m$, we denote by $\bar{w}$ the unique element $\neq w$ in P_s that has the same length as w.

Let us show (i). We already have $\phi(e) = a$ and $Z(\phi, s) = \{s; a\}$. If $s \notin \mathsf{dom}(\phi)$, we could extend ϕ by putting $\phi(s) = sa$, contradicting maximality. So $s \in \mathsf{dom}(\phi)$ and $\phi(s) \in \{as; as\}$. If $m = 2$ this reduces to $\phi(s) = as$ and we are done. Otherwise set $u_2 = \phi(s)$, $v_2 = \bar{u_2}$. We have $Z(\phi, v_2) = \{as; sa\}$ so $v_2 \in \mathsf{dom}(\phi), \phi(v_2) \in \{asa; sas\}$. If $m = 3$ this reduces to $\phi(v_2) = asa$ and we are done. Otherwise set $u_3 = \phi(v_2)$, $v_3 = \bar{u_3}$. Continuing this way, it is clear that we eventually get the required result.

Let $w \in D \cap \mathsf{dom}(\phi)$ with $l(w) > 1$. We show (ii) by induction on $l(w)$. Call v and $\bar{v}$ the coatoms of w. We have $v \lhd \phi(v)$, $\bar{v} \lhd \phi(\bar{v})$ (indeed, if $l(w) = 2$ this comes from $\phi(s) \in \{as; sa\}$ for $s \in S$, and otherwise this is the induction hypothesis). Then $Z(\phi, w) = \{\phi(v); \phi(\bar{v}); w\}$. This set has cardinality three (indeed, if $l(w) = 2$ this set is $\{\phi(s); \phi(t); st\}$ where we write $w = st$, and otherwise w is dihedral while $\phi(v)$ and $\phi(\bar{v})$ are not). By remark 1.1, we have $w \lhd \phi(w)$, which completes the proof. Q. E. D.

When we consider larger parabolic subgroups, we have the following :

Lemma 1.7 *Let (W, S) be a Coxeter system, ϕ a maximal matching on W and $a = \phi(e)$ as above. Let J be a subset of S and containing a. Then $< J > \cap \mathsf{dom}(\phi)$ is invariant by ϕ.*

Proof : Put $Q =< J > \cap \mathsf{dom}(\phi)$. Let us show by induction of $l(w)$ that $\phi(w) \in Q$ for all $w \in Q$. If $w = e$ we have $\phi(w) = a \in J \cap \mathsf{dom}(\phi) \subseteq Q$, so assume $l(w) > 0$. We may also assume $w \lhd \phi(w)$. Then $\mathsf{coat}(\phi(w)) = Z(\phi, w) \subseteq Q$ by the induction hypothesis. In particular, $\mathsf{coat}(\phi(w)) \subseteq < J >$, hence $\mathsf{supp}(\phi(w)) \subseteq J$ and $\phi(w) \in < J >$. Q. E. D.

Proposition 1.8 *Let (W, S) be a Coxeter system, ϕ and ψ be two maximal matchings on W, and $w \in W$. Suppose that $\phi(e) = \psi(e)$ and that ϕ coincides with ψ on $[e, w] \cap P$.*
Then

$$\phi(w) = \psi(w)$$
$$(or\ \phi(w)\ and\ \psi(w)\ are\ both\ undefined).$$

Proof: By contradiction, take a minimal counterexample w. Reasoning as in the "uniqueness" part of theorem 1.4, we may assume that $\phi(w)$ and $\psi(w)$ are both defined, that $w \lhd \phi(w)$, $w \lhd \psi(w)$ and that $\phi(w) \neq \psi(w)$, $\mathsf{coat}(\phi(w)) = \mathsf{coat}(\psi(w))$. By corollary I.9.9, this implies that $w, \phi(w)$ and $\psi(w)$ are all elements of some parabolic dihedral subgroup D. This D cannot be principal because ϕ and ψ coincide on $P \cap [e, w]$. Also $l(w) > 1$ since $w \notin P$. But then 1.6.(ii) says that $\phi(w)$ is nondihedral which is a contradiction. Q. E. D.

Putting together the two preceding propositions we get:

Theorem 1.9 *Let (W, S) be a Coxeter system and $a \in S$.*
(i) For any maximal matching ϕ on W such that $\phi(e) = a$, each principal dihedral subgroup P_s is stable by ϕ, hence an induced matching ϕ_s on P_s.
(ii) Conversely, for any family $(\phi_s)_{s \neq a}$ such that ϕ_s is a matching on P_s with $\phi_s(e) = a$, there is a unique maximal matching ϕ that extends the union of the ϕ_s: $\phi_{|P_s} = \phi_s$ for all $s \in S$.

Proof: Of course (i) is just a repetition of 1.6.(i).

Let us now demonstrate the unique extension result. First, since $P_s \cap P_t = \{e; a\}$ when $s \neq t$, the various mappings ϕ_s may be glued together into a mapping $\psi : P \to P$ and ψ will be a partial matching because $\bigcup P_s$ is decreasing; this already gives the existence of ϕ by the "existence" part of theorem 1.4. The uniqueness of ϕ follows from proposition 1.8. Q. E. D.

2 Descent formulas for R-polynomials

For s in S we set $L_s = \{w \in W | l(sw) > l(w)\}$. Recall the following formulas from I.(10.10), which give a method to compute the polynomials $R_{u,v}$: if x and y belong to L_s,

$$
\begin{aligned}
R_{sx,sy} &= R_{x,y} \\
R_{x,sy} &= (q-1)R_{x,y} + qR_{sx,y}
\end{aligned}
\tag{2.1}
$$

The basic idea here is to show that these formulas, that hold for left (or right) multiplication by s, also hold for any special matching. We make the following fundamental definition (recall the definition of M_{st} from formula I.(1.5))

Definition 2.1 *Let (W,S) be a Coxeter system and $w \in W$, $J \subseteq S$. We say that w is J-**full** if for any $s, t \in J$, we have $m_{st} < \infty$, $M_{st} \leq w$. We say that w is **full** if it is S-full.*

Lemma 2.2 *Let (W,S) be a Coxeter system, $J \subseteq S$, and $w \in W$ a J-full element. Then there is a J-full element $v \in < J >$ such that $v \leq w$.*

Proof: We know that $[e,w] \cap < J >$ has a largest element v (this is the element $m_J(w)$ of lemma II.2.5); this v will do. Plainly, $v \leq w$. Furthermore, if s and t are two distinct elements in J and $\mu = M_{st}$ is the corresponding maximal dihedral element, we have $\mu \in [e,w] \cap < J >$ so $\mu \leq v$. As this holds for any s and t, v is J-full. Q. E. D.

Definition 2.3 *If $s \in S$ and ϕ is a matching, we say that s is ϕ(left)-**regular** or that ϕ is s-(left)-regular if*

$$
\begin{aligned}
&(i)\ \mathsf{dom}(\phi) \text{ is stable by } (x \mapsto sx), \text{and} \\
&(ii) \phi(sx) = s\phi(x) \ \text{for all } x \in \mathsf{dom}(\phi).
\end{aligned}
$$

Right-regularity is defined similarly. We shall denote by ρ_s and λ_s respectively the multiplication mappings $x \mapsto xs$ and $x \mapsto sx$. By an abuse of notation, for a partial mapping f we will write $f = \rho_s$ (or λ_s) when f is a restriction of ρ_s and $f \neq \rho_s$ when f is not.

Definition 2.4 *Let (W,S) be a Coxeter system and ϕ a maximal matching on W; let o be an orbit in $\mathsf{dom}(\phi)$ for the action of the involution ϕ. Then o can be written $o = \{m, M\}$ with $m \lhd M$, $\phi(m) = M$. The orbit o is said to be **full** if M is full. We say that o is a **left-reducible orbit** if there is a ϕ-left-regular generator in the left descent set of m. Similarly, we say that o is a **right-reducible orbit** if there is a ϕ-right-regular generator in the right descent set of m. The orbit o is called a **reducible orbit** if it is either left- or right- reducible.*

*Finally, ϕ is called a **reducible matching** if $|S| \leq 2$ or if any full orbit is reducible.*

The usual addition operation and the usual ordering on $\mathbb{N} = \{0; 1; 2; \ldots\}$ can be extended to $\mathbb{N} \cup \{\infty\}$ by putting $x \leq \infty$ and $x + \infty = \infty$ for $x \in \mathbb{N} \cup \{\infty\}$. The main result of this section is the following:

Proposition 2.5 *Let (W,S) be a Coxeter system with the following property: for any Coxeter system (W', S) associated to a Coxeter matrix M' such that $m'_{st} \leq m_{st}$ for any two generators $s \neq t$, we have that any maximal matching on W' is reducible. Let ϕ be a partial matching on W. Put $L_\phi = \{w \in \mathsf{dom}(\phi)\ ;\ w \lhd \phi(w)\}$. Then, for $(x,y) \in {L_\phi}^2, x < y$, we have*

$$R_{\phi(x),\phi(y)} = R_{x,y} \tag{2.5.1}$$
$$R_{x,\phi(y)} = (q-1)R_{x,y} + qR_{\phi(x),y} \tag{2.5.2}$$

Proof of the proposition. First, notice that when $|S| \leq 2$, we have the equivalence $(u < v) \Leftrightarrow (l(u) < l(v))$, whence we deduce easily that $R_{u,v}$ only depends on $l(v) - l(u)$ (for example define the sequence of polynomials $(L_i(q))$ by $L_0 = 1,\ L_1 = q,\ \forall n \geq 2\ L_n = (q-1)L_{n-1} + qL_{n-2}$; by induction on $l(v)$ using (2.1) we get $R_{u,v} = L_{l(v)-l(u)}(q)$ for $u < v$). Thus we get the desired result very quickly when $|S| \leq 2$.

In the remaining cases, we argue by induction both on $l(y)$ and on the size of the Coxeter group: formally, we keep the set S fixed and we show a property of the pair (M, y) by ordinary induction on the quantity $q(M, y) = l(y) + ||M||$, where we set $||M|| = \sum_{s,t \in S} m_{st}$ (*a priori* this works only when $q(M, y)$ is finite; however, it is easily seen that once we are done with the case $q(M, y) < \infty$ the case $q(M, y) = \infty$ easily follows, arguing as in the "reduction to the full case" below).

We claim that we may assume $w = \phi(y)$ is full without loss of generality. Indeed, consider the Coxeter matrix M' defined by $m'_{st} =$ the length of the largest element in $[e, w] \cap < s, t >$ for $s, t \in S$, and consider the Coxeter system (S, W') associated to matrix M'. By proposition II.2.6, the following mapping (where the $s_1 \ldots s_r$ are reduced words)

$$\begin{array}{lcl} \alpha\ \ [e, w] & \rightarrow & W' \\ \{s_1 \ldots s_r\}_W & \mapsto & \{s_1 \ldots s_r\}_{W'} \end{array}$$

is well defined, strictly increasing with respect to the Bruhat orderings and satisfies

$$\forall x \in [e, w],\ \forall s \in S, (xs \in [e, w]) \Rightarrow (\alpha(\{xs\}_W) = \{\alpha(x)s\}_{W'})$$

From which we easily deduce that α gives an isomorphism of graded posets from $[e, w]$ onto $[e, \alpha(w)]$ and that

$$\forall u, v \in [e, w],\ R^W_{u,v} = R^{W'}_{\alpha(u),\alpha(v)}$$

(reason by induction on $l(v)$,using 2.5.1. and 2.5.2 for the right multiplication matchings). So that all the data of the problem on $[e, w] \subseteq W$ are carried isomorphically onto $[e, \alpha(w)] \subseteq W'$ and $\alpha(w)$ is indeed full in that new Coxeter group.

Case $l(y) = 0$:
In this case $y = e$ and all the $R_{u,v}$ under consideration are zero except if $x = e$ or $\phi(e)$; in each of those two cases, formulas 2.5.1 and 2.5.2 are checked directly.

Case $l(y) > 0$:
Thanks to the well-known equalities $R_{u,u} = 1$ and $R_{u,v} = q - 1$ if $u \lhd v$, we may take $x < y$.
Since ϕ is reducible and y is full, we know that there is a (left, say) regular generator g in the (left) descent set of y.

Let $v = gy$. If $m = l(v)$, we have $l(y) = m + 1$, $l(\phi(y)) = m + 2$. If $p = l(\phi(v))$, then on one hand $p - l(v) \in \{-1, 1\}$ and on the other $p - l(g\phi(v)) \in \{-1, 1\}$ so $p = m + 1$, and eventually $v \lhd gv \lhd \phi(y)$, $v \lhd \phi(v) \lhd \phi(y)$.
Suppose first that $gx \lhd x$. Then, putting $x' = gx$, the above reasoning (with x in place of y) gives $x' \lhd gx' \lhd \phi(x)$, $x' \lhd \phi(x') \lhd \phi(x)$. In this case,

$$\begin{aligned} R_{\phi(x),\phi(y)} &= R_{g\phi(x'),g\phi(v)} = R_{\phi(x'),\phi(v)} \\ &= R_{x',v} && \text{(induction hypothesis)} \\ &= R_{gx',gv} = R_{x,y}. \end{aligned}$$

and

$$\begin{aligned} R_{x,\phi(y)} &= R_{gx',g\phi(v)} = R_{x',\phi(v)} \\ &= (q-1)R_{x',v} + qR_{\phi(x'),v} && \text{(induction hypothesis)} \\ &= (q-1)R_{gx',gv} + qR_{g\phi(x'),gv} = (q-1)R_{x,y} + qR_{\phi(x),y}. \end{aligned}$$

Now suppose that $x \lhd gx$. Then, because of $\mathsf{coat}(\phi(x)) = \{x\} \cup \{\phi(z) |\ z \lhd x, z \lhd \phi(z)\}$, we see that $\phi(gx)$ is a coatom of $\phi(x)$ only if $\phi(gx) = x$, i.e. if $\phi(x) = gx$; otherwise $\phi(x) \lhd \phi(gx)$. Thus there are two subcases:

$x \lhd gx$, $\phi(x) = gx$,

$x \lhd gx$, $\phi(x) \lhd \phi(gx)$.

In the first subcase, we have

$$\begin{aligned} R_{\phi(x),\phi(y)} &= R_{gx,g\phi(v)} = R_{x,\phi(v)} \\ &= (q-1)R_{x,v} + qR_{\phi(x),v} && \text{(induction hypothesis)} \\ &= (q-1)R_{x,v} + qR_{gx,v} = R_{x,gv} = R_{x,y}. \end{aligned}$$

and

$$\begin{aligned} R_{x,\phi(y)} = R_{x,g\phi(v)} &= (q-1)R_{x,\phi(v)} + qR_{gx,\phi(v)} = (q-1)R_{x,\phi(v)} + qR_{\phi(x),\phi(v)} \\ &= (q-1)((q-1)R_{x,v} + qR_{\phi(x),v}) + qR_{x,v} && \text{(induction hypothesis)} \\ &= (q-1)((q-1)R_{x,v} + qR_{gx,v}) + qR_{gx,gv} \\ &= (q-1)R_{x,gv} + qR_{gx,gv} = (q-1)R_{x,y} + qR_{\phi(x),y}. \end{aligned}$$

Finally, in the second subcase one has

$$\begin{aligned} R_{\phi(x),\phi(y)} &= R_{\phi(x),\phi(gv)} = R_{\phi(x),g\phi(v)} \\ &= (q-1)R_{\phi(x),\phi(v)} + qR_{g\phi(x),\phi(v)} \\ &= (q-1)R_{x,v} + qR_{gx,v} && \text{(induction hypothesis)} \\ &= R_{x,gv} = R_{x,y}. \end{aligned}$$

and

$$\begin{aligned} R_{x,\phi(y)} &= R_{x,\phi(gv)} = R_{x,g\phi(v)} \\ &= (q-1)R_{x,\phi(v)} + qR_{gx,\phi(v)} \\ &= (q-1)\{(q-1)R_{x,v} + qR_{\phi(x),v}\} + q\{(q-1)R_{gx,v} + qR_{\phi(gx),v}\} \end{aligned}$$

(induction hypothesis)

$$= (q-1)\{(q-1)R_{x,v} + qR_{gx,v}\} + (q-1)\{(q-1)R_{\phi(x),v} + qR_{g\phi(x),v}\}$$

$= (q-1)R_{x,gv} + qR_{\phi(x),gv}$

$= (q-1)R_{x,y} + qR_{\phi(x),y}$. Q. E. D.

Now our aim will be to show that all matchings are reducible.

Definition 2.6 *Let (W,S) be a Coxeter system and ϕ a partial matching on W. We say that ϕ is* **full** *if* $\mathsf{dom}(\phi)$ *contains a full element.*

Note that a non-full matching is trivially reducible. This will be quite a useful fact in the following sections.

3 Regularity criteria

Proposition 3.1 *Let (W,S) be a Coxeter system, ϕ a maximal matching on W, $a = \phi(e)$. Let $w \in \mathsf{dom}(\phi)$ and $s \in S$. (recall that for $s \neq a$ we put $P_s =< s, a >$ and $P = \bigcup_{s\neq a} P_s$)*

If $s \neq a$, and ϕ commutes with λ_s on $[e, w] \cap P_s$, then $sw \in \mathsf{dom}(\phi)$, and $\phi(sw) = s\phi(w)$.
If $s = a$, and ϕ commutes with λ_a on $[e, w] \cap P$, then $sw \in \mathsf{dom}(\phi)$, and $\phi(sw) = s\phi(w)$.

Of course, left may be replaced with right in this proposition.

Note: this result is immediately implied by the much stronger statement in [8, lemma 4.3] about two special matchings on a $K_{3,2}$-avoiding poset.

Proof: Let us show the first assertion.
We argue by induction on the length of w. The case $l(w) = 0$ (or even $w \in P_s$) is trivial. Thus we take $w \notin P_s$. If one of sw or $\phi(w)$ (call it v) is $\lhd w$, then the result is clear by applying the induction hypothesis to v, so we may assume $w \lhd sw, w \lhd \phi(w), \phi(w) \lhd s\phi(w)$. We compute $Z(\phi, sw)$ (using $w < sw$ on the second line and the induction hypothesis on the fourth line)

$$\begin{aligned} Z(\phi, sw) &= \{sw\} \cup \{\phi(z) |\ z \lhd sw, z \lhd \phi(z)\} \\ &= \{sw\} \cup \{\phi(z) |\ (z = w \text{ or } z = su, u \lhd w, u \lhd su), z \lhd \phi(z)\} \\ &= \{sw; \phi(w)\} \cup \{\phi(su) |\ u \lhd w, u \lhd su, su \lhd \phi(su)\} \\ &= \{sw; \phi(w)\} \cup \{s\phi(u) |\ u \lhd w, u \lhd su, su \lhd s\phi(u)\} \end{aligned}$$

Now for any u the assertions $(u \lhd w, u \lhd su, su \lhd s\phi(u))$ and $(u \lhd w, u \lhd \phi(u), \phi(u) \lhd s\phi(u))$ are equivalent (for example if u satisfies the first then $l(\phi(u)) = l(u) + 1$ so u satisfies the second) and so

$$\begin{aligned} Z(\phi, sw) &= \{sw; \phi(w)\} \cup \{s\phi(u) |\ u \lhd w, u \lhd \phi(u), \phi(u) \lhd s\phi(u)\} \\ &= \{\phi(w)\} \cup \{sz |\ (z = w \text{ or } z = \phi(u), u \lhd w, u \lhd \phi(u)), z \lhd sz\} \\ &= \{\phi(w)\} \cup \{sz |\ z \lhd \phi(w), z \lhd sz\} \\ &= \mathsf{coat}(s\phi(w)) \end{aligned}$$

Now, if ϕ were not defined at sw, the formula above shows that we could extend ϕ by putting $\phi(sw) = s\phi(w)$, contradicting the maximality of ϕ. So $sw \in \mathsf{dom}(\phi)$, and $x = \phi(sw)$ satisfies $\mathsf{coat}(x) = \mathsf{coat}(s\phi(w))$. Moreover, $s\phi(w)$ is not dihedral (else there is a dihedral subgroup D such that $s\phi(w) \in D$, so $w \in D$ and $\phi(w) \in D$. Proposition 1.6 shows that D is principal: for some

$t \in S \setminus \{a\}$ we have $D = P_t$. Then $s \in P_t$, so $s = t$ and we get $w \in P_s$ which is impossible) so that corollary I.9.9 gives $x = s\phi(w)$ as required.

The proof of the second assertion is similar : the case $w \in P$ is trivial, we show the equality $Z(\phi, sw) = \mathsf{coat}(s\phi(w))$, and conclude by arguing that $s\phi(w)$ is non-dihedral since $w \notin P$. Q. E. D.

Recalling proposition 1.9, we deduce that

Corollary 3.2 *Let (W,S) be a Coxeter system, ϕ a maximal matching on W, $a = \phi(e)$. Let $s \in S$.*

If $s \neq a$, (ϕ is s-left-regular) $\Leftrightarrow$ ($\phi_{|P_s}$ is s-left-regular)
If $s = a$, (ϕ is a-left-regular) $\Leftrightarrow$ ($\phi_{|P}$ is a-left-regular)

Of course, we may replace left with right in this corollary.

Corollary 3.3 *Let (W,S) be a Coxeter system. Let ϕ be a maximal matching on W, $a = \phi(e)$, X and Y two subsets of S such that:*
(i) $\phi_{|P_x} = \rho_a$, for any $x \in X \setminus \{a\}$
(ii) $\phi_{|P_y}$ is a-left-regular, for any $y \in Y \setminus \{a\}$

Then $< X > (< Y > \cap \mathsf{dom}(\phi)) \subseteq \mathsf{dom}(\phi)$ and for $x \in < X >, y \in < Y > \cap \mathsf{dom}(\phi)$ we have

$$\phi(xy) = x\phi(y).$$

Of course, we may replace left with right in this corollary.

Proof: Let $Q = < X > (< Y > \cap \mathsf{dom}(\phi))$. What we must show is that the restriction of ϕ to Q is x-left-regular for all $x \in X$. This is clear from (i) (and the preceding corollary) if $x \neq a$. And since

$$P \cap Q = \bigcup_{t \in X \cup Y \setminus \{a\}} P_t$$

it also follows immediately from (i) and (ii) when $x = a$. Q. E. D.

Finally we give a practical regularity criterion:

Remark 3.4 *Let ϕ be a special matching defined on a dihedral Coxeter group $< s, t >$. For $i \leq m_{st}$ put*

$$d_i = [t, s, i\rangle$$

Then the following are equivalent:

(1) *ϕ is not s-left-regular*
(2) $\exists i \leq m_{st} - 3,\ \phi(d_i) = d_{i+1},\ \phi(sd_i) \neq sd_{i+1}$*(so $\phi(sd_i) = d_{i+2}$).*

Proof: Put $Z = \{z \in < s, t > \mid \phi(sz) \neq s\phi(z)\}$. As ϕ and $w \mapsto sw$ are involutive, Z is stabilized by those two mappings. So any minimal element z_0 of Z (if there are any) satisfies $z_0 \lhd sz_0$ and $z_0 \lhd \phi(z_0)$ (which implies (2) with $z_0 = d_i$), and the result follows. Q. E. D.

4 Restrictions on the domain in the mixed case

Before proceeding further the reader should recall proposition I.3.1. In particular, part (ii) will implicitly justify all assertions of the form "this element has a unique reduced expression": if we denote by $\mathcal{I}$ the set of elements in W with a unique reduced expression, (or, alternatively, the associated set of reduced words in S^*), a reduced word g is in $\mathcal{I}$ if and only if no braid relation can be applied to g i.e. if and only if g does not contain dihedral subwords that represent maximal dihedral elements. Similarly, we use part (i) of proposition I.3.1 without mention any time we need to know that a certain word is reduced. Note that the words encountered will never be very complex (they will differ from a dihedral word by one character only), which justifies our brievity on that issue.

Given a maximal matching ϕ on a Coxeter group W, for each $s \in S$ we have seen in the proof of 1.6.(i) that $s \in \mathsf{dom}(\phi)$, $\phi(s) \in \{as; sa\}$ (where $a = \phi(e)$). When the restriction of ϕ to the generators does not coincide with a left or right multiplication, which amounts to saying that there are some $s, t \in S$ with $m_{as} > 2$, $m_{at} > 2, \phi(s) = sa, \phi(t) = at$, we say that ϕ is **mixed**.

Define subsets L and R of S by

$$\begin{aligned} L &= \{l \in S |\ \phi(l) = al\} \\ R &= \{r \in S |\ \phi(r) = ra\} \end{aligned}$$

and let $< L >$ and $< R >$ be the associated parabolic subgroups. We show that the following inclusion holds:

Theorem 4.1 $\mathsf{dom}(\phi) \subseteq < R >< L >$.

Proof: Suppose by contradiction that there is a w in $\mathsf{dom}(\phi) \setminus < R >< L >$; take w minimal, so that $[e, w[\subseteq < R >< L >$. First we note that $D_l(w)$ cannot contain an element of R (otherwise we could write $w = rv$ with $r \in R$, $v < w$ and then $v \in < R >< L >$ yields $rv \in < R >< L >$, a contradiction), and because of $S = L \cup R$, we deduce $D_l(w) \subseteq L \setminus R$. Similarly, $D_r(w) \subseteq R \setminus L$.

Let $w_1 \dots w_m$ be a reduced expression for w. Thus we have $w_1 \in L \setminus R$, $w_m \in R \setminus L$. Let $x = w_1 \dots w_{m-1}$. Then $D_l(x) \subseteq D_l(w)$, so $D_l(x) \cap R = \emptyset$. As $x \in < R >< L >$, this imposes $x \in < L >$. Thus we have $\forall i \leq m-1, w_i \in L$. Symmetrically, $\forall i \geq 2, w_i \in R$. So by renaming the w_i,

$$w = lb_1 \dots b_n r, \text{ with}$$

$$\begin{pmatrix} l \in L \setminus R, \\ \forall i \ b_i \in L \cap R, \\ r \in R \setminus L \end{pmatrix} (\dagger)$$

On the one hand $D_l(w) \subseteq \{l, b_1, \dots, b_n, r\}$ and on the other $D_l(w) \subseteq L \setminus R$, so we deduce that $D_l(w) = \{l\}$, and similarly $D_r(w) = \{r\}$. Thus, in any reduced expression for w the characters l and r appear exactly once, at the beginning and at the end respectively.

Now we will show that, for any pair $(l, r) \in (L \setminus R) \times (R \setminus L)$,

(1) If $lr \neq rl$, $lr \notin \mathsf{dom}(\phi)$
(2) In any case, $lar \notin \mathsf{dom}(\phi)$.

For both items we argue by contradiction: if $rl \neq lr$, $lr \in \mathsf{dom}(\phi)$, by remark 1.1 $lr \lhd \phi(lr)$, so that $\mathsf{coat}(\phi(lr)) = \{lr, al, ra\}$; now no element of W has this for a coatom set (if $\mathsf{coat}(z) = \{lr, al, ra\}$ for some z, as $lr \in \mathcal{I}$ and $lr \lhd z$ we have $z = alr, lar$ or lra, but then ar and la cannot both be coatoms of z), hence (1). Now we proceed with the proof of (2), and suppose $lar \in \mathsf{dom}(\phi)$. By (1) and because $\mathsf{dom}(\phi)$ is decreasing, we have $lr = rl$. Then $\phi(la) \in \{ala, lal\}, \phi(ar) \in \{ara, rar\}, \phi(lr) = ral$. By remark 1.1 $lar \lhd \phi(lar)$; put $z = \phi(lar)$. Then both lar and ral (which are in $\mathcal{I}$) are coatoms of z, and this is a contradiction. So (2) holds.

Going back to our initial reasoning, (1) and (2) give $lr = rl$, and all the b_i are distinct from a (otherwise $lar \leq w$, which is impossible because $\mathsf{dom}(\phi)$ is decreasing). Thus $a \not\leq w$, so by remark 1.2 $w \lhd \phi(w)$ (indeed if $\phi(w) \lhd w$ we deduce that ϕ restricts to a special matching of $[e, w]$, and in particular $\phi(e) \in [e, w]$), and if g is a reduced expression for $\phi(w)$ and g' is the word obtained by supressing the unique ocurrence of a in g , then $w = g'$ holds in W. So the three generators a, l and r occur exactly once in g. Now ϕ restricts to a special matching of $[ar, w]$ by remark 1.2, so $ar \leq \phi(ar) \leq \phi(w)$ and similarly $la \leq \phi(w)$, so $lar \leq \phi(w)$ which is impossible by (2). Q. E. D.

The inclusion we have just shown becomes an equality for an important class of matchings which contains almost all matchings on finite or affine Coxeter groups:

Corollary 4.2 *(***Middle multiplication matchings***)Suppose that ϕ is a maximal matching such that $\phi = \rho_a$ on each $P_r (r \in R)$ and $\phi = \lambda_a$ on each $P_l (l \in L)$. Then* $\mathsf{dom}(\phi) = < R >< L >$*, and for $x \in < R >$, $y \in < L >$ we have the middle multiplication formula*

$$\phi(xy) = xay$$

and ϕ is reducible.

Proof: The key remark is that under those hypotheses, the elements of R are left-regular and that the elements of L are right-regular, by corollary 3.2. Then corollary 3.3 makes the inclusion become an equality and yields the middle multiplication formula. Moreover, because of $\mathsf{dom}(\phi) = < R >< L >$, all the orbits (except for the orbit $\{e, a\}$) are reducible, not just the full ones, so that ϕ is *a fortiori* reducible. Q. E. D.

Middle-multiplication matchings first appeared in Brenti's study [7] of special matchings in type A: he found, in fact, that all matchings in type A are right, left, or middle multiplications. This may be generalized as follows:

Corollary 4.3 *Any matching defined on a simply laced Coxeter group is reducible (indeed, it is a middle multiplication matching).*

Proof: Because of the small sizes of the dihedral subgroups we necessarily have $\phi = \rho_a$ for all $r \in R$ and $\phi = \lambda_a$ for all $l \in L$. Then the above corollary applies. Q. E. D.

5 Some results on rank three groups.

In all of this section, we consider a Coxeter system (W, S) of rank 3: $S = \{a, b, b'\}$ and ϕ is a maximal matching on W with $\phi(e) = a$. We denote by β the restriction of ϕ to $< a, b >$.

5.1 Preliminaries.

Lemma 5.1.1 *Let $G =< a, b' >< a, b >$. Then:*
(i) If ($m_{bb'} > 2$ or $m_{ab} = \infty$ or $m_{ab'} = \infty$), then G does not contain any full element.
(ii) If ($m_{bb'} = 2$, $m_{ab} < \infty$, $m_{ab'} < \infty$), then G contains exactly two full elements, namely

$$M_{b',a,b} = \langle m_{ab'} - 1, a, b'][b, a, m_{ab} - 1\rangle, \textit{ and}$$
$$M'_{b',a,b} = \langle m_{ab'} - 1, a, b']a[b, a, m_{ab} - 1\rangle.$$

Proof: Recall that the existence of a full element implies that all the entries of the Coxeter matrix are finite. Moreover, G is decreasing and we have $G \cap < b, b' >= \{e, b, b', b'b\}$. This already proves (i).

Let G satisfy the hypotheses of (ii). Any element g in G can be written xy with $x \in< a, b' >$, $y \in< a, b >$. Putting $x' = min(x, xa)$ and $y' = min(y, ay)$ we see that g can be uniquely rewritten $x'\varepsilon y'$ with $\varepsilon \in \{e; a\}$. As $x' \lhd x'a$, there is a $j \leq m_{ab'} - 1$ such that $x' = \langle j, a, b']$. Similarly, there is a $k \leq m_{ab} - 1$ such that $y' = [b, a, k\rangle$. If g is full, $g \geq M_{ab'}$ so $j = m_{ab'} - 1$. By symmetry $k = m_{ab} - 1$, which completes the proof of (ii). Q. E. D.

Lemma 5.1.2 *Suppose that $m_{ab'} \geq 3$, that $\phi = \rho_a$ on $[e, ab'a]$, and that β is not a-left-regular. By remark 3.4 this forces $m_{ab} \geq 4$, and there is a minimal i such that $\phi([b, a, i\rangle) = [b, a, i + 1\rangle$, $\phi([a, b, i + 1\rangle) = [b, a, i + 2\rangle$, $i \leq m_{ab} - 3$. Then $ab'[b, a, i\rangle$ is a minimal element in $W \setminus \mathsf{dom}(\phi)$.*

Proof: Put $w = ab'[b, a, i\rangle$. Proposition 3.1 yields:

$$\forall x < [b, a, i\rangle,\ \phi(b'x) = b'\phi(x), \phi(ab'x) = ab'\phi(x).$$

In particular $\phi(b'[b, a, i\rangle) = b'[b, a, i + 1\rangle$.

Suppose by contradiction that $w \in \mathsf{dom}(\phi)$. Then by remark 1.1 we have $w \lhd \phi(w)$ so $\mathsf{coat}(\phi(w)) = Z(\phi, w)$. Let g be a reduced expression for $\phi(w)$. As $\phi([a, b, i + 1\rangle) = [b, a, i + 2\rangle \in \mathcal{I}$ and $b' \leq \phi(w)$, we obtain g by inserting the generator b' somewhere in the word $[b, a, i + 2\rangle$. Now, $b'[b, a, i + 1\rangle \lhd \phi(w)$ forces b' to occur before the leftmost a appearing in $[b, a, i + 2\rangle$, so that $\phi(w) = b'[b, a, i + 2\rangle$. But w has at most two reduced expressions, $ab'[b, a, i\rangle$ and $abb'[a, b, i - 1\rangle$ (which reduce to one when $m_{bb'} = 2$), neither of which is a subexpression of $b'[b, a, i + 2\rangle$, and this is a contradiction. Q. E. D.

Lemma 5.1.3 *Suppose that $m_{bb'} \geq 3$, that $\phi = \rho_a$ on $[e, b'a]$, and that β is not b-left-regular. By remark 3.4, this forces $m_{ab} > 3$ and there is a minimal i such that $\phi([a, b, i\rangle) = [a, b, i+1\rangle$, $\phi([b, a, i+1\rangle) = [a, b, i + 2\rangle$, $i \leq m_{ab} - 3$. Then $bb'[a, b, i\rangle$ is a minimal element in $W \setminus \mathsf{dom}(\phi)$.*

Proof: Put $w = bb'[a,b,i\rangle$. Proposition 3.1. yields that any element $< w$ is in $\mathsf{dom}(\phi)$ and

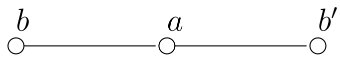
$$\forall x < [a,b,i\rangle,\ \phi(b'x) = b'\phi(x), \phi(bb'x) = bb'\phi(x).$$

In particular $\phi(b'[b,a,i\rangle) = b'[b,a,i+1\rangle$.

Suppose by contradiction that $w \in \mathsf{dom}(\phi)$. Then by remark 1.1 we have $w \lhd \phi(w)$ so $\mathsf{coat}(\phi(w)) = Z(\phi, w)$. Let g be a reduced expression for $\phi(w)$. As $[a,b,i+2\rangle \in \mathcal{I}$ and $b' \leq \phi(w)$, we obtain g by inserting character b' somewhere in the word $[a,b,i+2\rangle$. Now, $w \lhd \phi(w)$ imposes $\phi(w) = abb'[a,b,i\rangle = abb'a[b,a,i-1\rangle$. This is incompatible with $b'[b,a,i+1\rangle \lhd \phi(w)$. Q. E. D.

5.2 Mixed matchings in rank three.

In this subsection, we take $S = \{a; b; b'\}$, $m_{ab} \geq 3$, $m_{ab'} \geq 3$, and $\phi(e) = a$, $\phi(b) = ab$, $\phi(b') = b'a$ (the "mixed" case). We denote by β (β') the restriction of ϕ to $< a, b >$ (respectively $< a, b' >$).



$\phi(e) = a$

$\phi(b) = ab$

$\phi(b') = b'a$

Figure 1: Mixed case

To give the reader an idea of where we are going to, we formulate at once the main and last-to-be-proved result of this subsection:

Proposition 5.2.1 *The matching ϕ is full if and only if $m_{bb'} = 2$, m_{ab} and $m_{ab'}$ are finite, and*

$$(*) \begin{cases} \beta \text{ is } a\text{-left-regular and } \beta' = \rho_a, \text{ or} \\ \beta' \text{ is } a\text{-right-regular and } \beta' = \lambda_a. \end{cases}$$

By theorem 4.1. we have $\mathsf{dom}(\phi) \subseteq < a, b' >< a, b >$. Then, by lemma 5.1.1, the matching can be full only if $m_{bb'} = 2, m_{ab} < \infty, m_{ab'} < \infty$, which we assume for the remainder of the section.

Whenever we find an obstruction $h \notin \mathsf{dom}(\phi)$ with $h \leq M_{b',a,b}$ we may conclude that ϕ is not full. This is the gist of the next three lemmas.

Lemma 5.2.2 *Suppose that $\phi(ab') = ab'a$ (this always holds if $m_{ab'} = 3$) and that β is not a-left-regular. By remark 3.4 there is a minimal i such that $\phi([b,a,i\rangle) = [b,a,i+1\rangle$, $\phi([a,b,i+1\rangle) = [b,a,i+2\rangle$, $i \leq m_{ab} - 3$. Then $ab'[b,a,i\rangle$ is a minimal element in $W \setminus \mathsf{dom}(\phi)$, so ϕ is not full.*

Proof: This is lemma 5.1.2. Q. E. D.

Lemma 5.2.3 *Suppose that $m_{ab'} \geq 4, \phi(ab') = b'ab', \phi(ab'a) = b'ab'a$ (this always holds if $m_{ab'} = 4$) and that $\beta \neq \lambda_a$, so that there is a minimal i such that $\phi([b,a,i\rangle) = [b,a,i+1\rangle$, $i \leq m_{ab} - 2$. Then $ab'[b,a,i\rangle$ is a minimal element in $W \setminus \mathsf{dom}(\phi)$, so ϕ is not full.*

Proof: Put $w = ab'[b,a,i\rangle$. If $w \in \mathsf{dom}(\phi)$, then $w \lhd \phi(w)$ by remark 1.1. Note that $\phi(ab'[a,b,i-1\rangle) = b'ab'[a,b,i-1\rangle$ by proposition 3.1 and the hypotheses on ϕ. If g is a reduced expression for $\phi(w)$ then g can be obtained from a reduced expression of $\phi([a,b,i+1\rangle)$ by inserting a b' somewhere. But then we contradict $b'ab'[b,a,i-1\rangle \leq \phi(w)$ (remember $m_{a,b'} \geq 4$). Q. E. D.

Lemma 5.2.4 *Suppose $m_{a,b'} \geq 5, \phi(ab') = b'ab', \phi(ab'a) = ab'ab'$. Then $ab'ba$ is a minimal element in $W \setminus \mathsf{dom}(\phi)$, so ϕ is not full.*

Proof: Put $w = ab'ba$. We have (by proposition 3.1 again):

$$\begin{aligned}
&\mathsf{coat}(w) = \{b'ba, aba, ab'a, ab'b\}\\
&\phi(b'ba) = b'\phi(ba)\\
&\phi(ab'a) = ab'ab' \text{ (by hypothesis)}\\
&\phi(ab'b) = \phi(ab')b = b'ab'b
\end{aligned}$$

By remark 1.1, if $w \in \mathsf{dom}(\phi)$ then $w \lhd \phi(w)$. If g is a reduced expression for $\phi(w)$, as $ab'ab' \in \mathcal{I}$ (remember $m_{a,b'} \geq 5$) we see that g can be obtained by inserting the generator b somewhere in $ab'ab'$. So $\phi(w) \in \{bab'ab'; abb'ab'; ab'abb'\}$. Since $w \lhd \phi(w)$ and w has exactly two reduced expressions, $ab'ba$ and $abb'a$, we deduce $\phi(w) = abb'ab'$. Since $w' = b'ab'b$ is $\leq \phi(w)$ and w' has exactly two reduced expressions, $b'ab'b$ and $b'abb'$, we deduce $\phi(w) = ab'abb'$. But then $abb'ab = ab'abb'$, i.e. $ab'(bab) = ab'(abb')$ hence $bab = abb'$ which is a contradiction. Q. E. D.

Lemma 5.2.5 *If ϕ is full, then β is a-left-regular and β' is a-right-regular .*

Proof: If we put together lemmas 5.2.2, 5.2.3, and 5.2.4, we see that we have proved that if ϕ is full, then β is a-left-regular. By symmetry, β' in turn is a-right-regular. Q. E. D.

Next we show that in fact one of β, β' must be a multiplication matching:

Lemma 5.2.6 *Suppose that β is a-left-regular, β' is a-right-regular, that $\beta \neq \lambda_a$ and that $\beta' \neq \rho_a$, so that there are minimal i and i' such that $\phi(\langle i', a, b']) = \langle i'+1, a, b']$ and $\phi([b,a,i\rangle) = [b,a,i+1\rangle$. Then $\langle i', a, b'][b,a,i\rangle$ is a minimal element in $W \setminus \mathsf{dom}(\phi)$, so ϕ is not full.*

Proof: Put $w = \langle i', a, b'][b,a,i\rangle$ (note that $i, i' \geq 2$). We have (repeatedly using proposition 3.1 in the last four lines)

$$\begin{aligned}
&\mathsf{coat}(w) = \{\langle i'-1,a,b'][b,a,i\rangle; \langle i'-1,b',a][b,a,i\rangle; \langle i',a,b'][b,a,i-1\rangle; \langle i',a,b'][a,b,i-1\rangle\}\\
&\phi(\langle i'-1,a,b'][b,a,i\rangle) \;=\; \langle i'-1,a,b']\phi([b,a,i\rangle) \;=\; \langle i'-1,a,b'][b,a,i+1\rangle,\\
&\phi(\langle i'-1,b',a][b,a,i\rangle) \;=\; \langle i'-1,b',a]\phi([b,a,i\rangle) \;=\; \langle i'-1,b',a][b,a,i+1\rangle,\\
&\phi(\langle i',a,b'][b,a,i-1\rangle) \;=\; \phi(\langle i',a,b'])[b,a,i-1\rangle \;=\; \langle i'+1,a,b'][b,a,i-1\rangle,\\
&\phi(\langle i',a,b'][a,b,i-1\rangle) \;=\; \phi(\langle i'a,b'])[a,b,i-1\rangle \;=\; \langle i'+1,a,b'][a,b,i-1\rangle.
\end{aligned}$$

Suppose by contradiction that $w \in \mathsf{dom}(\phi)$. Then, by remark 1.1, $w \lhd \phi(w)$ so $\mathsf{coat}(\phi(w)) = Z(\phi, w)$. Notice that $i' \leq m_{ab'} - 2$ because $\phi(\langle i', a, b']) \neq \langle i', a, b']a$ and similarly $i \leq m_{ab} - 2$. Notice also that w has exactly two reduced expressions, namely $\langle i', a, b'][b,a,i\rangle$ and $\langle i'-1, b', a]bb'[a,b,i-1\rangle$. Let g be a reduced expression for $\phi(w)$; we obtain g by inserting a certain generator s into a reduced expression for w. Thus, g is of one of the three forms $x[b,a,i\rangle$ (where x is obtained by inserting

s somewhere in $\langle i', a, b']$), $\langle i', a, b']y$ (where y is obtained by inserting s somewhere in $[b, a, i\rangle$), or $\langle i'-1, b', a]bsb'[a, b, i-1\rangle$. In the first case we do not have $\phi(w) \geq [b, a, i+1\rangle$, in the second we do not have $\phi(w) \geq \langle i'+1, a, b']$, and in the third we have neither. So this is a contradiction. Q. E. D.

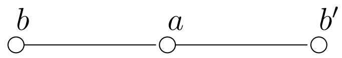
Proof of Proposition 5.2.1: Putting together lemmas 5.2.5 and 5.2.6 we see that if ϕ is full then (*) holds. Conversely, in the (first, say) alternative of (*), corollary 3.3 (with $X = \{a; b'\}$, $Y = \{a; b\}$) yields for any $x \in < a, b' >$, $y \in < a, b >$,

$$xy \in \mathsf{dom}(\phi),\ \phi(xy) = x\phi(y)$$

In particular, we see that $\mathsf{dom}(\phi)$ contains the element $M_{b',a,b}$ (see lemma 5.1.1) and that element is full. Q. E. D.

5.3 Nondegenerate case in rank three.

In this subsection, we suppose $S = \{a, b, b'\}, m_{ab} \geq 3,\ m_{ab'} \geq 3$ (the "nondegenerate" case). As the mixed case has been taken care of in the preceding subsection, here we take $\phi(b) = ba, \phi(b') = b'a$. As before, the case $m_{bb'} > 2$ is simpler.



$$\phi(e) = a$$

$$\phi(b) = ba$$

$$\phi(b') = b'a$$

Figure 2: Nondegenerate, nonmixed case

Lemma 5.3.1 *Suppose that $\beta \neq \rho_a$. Then there is a minimal i such that $\phi(\langle i, a, b]) = \langle i+1, a, b]$ (with $m_{ab} \geq i+2$). Let $H = \{w \in W |\ l(w) = i+1,\ b' \leq w,\ \langle i, a, b] \leq w,\ w \neq b'\langle i, a, b]\}$, and M, M' be the elements defined in lemma 5.1.1. Then we have:*

(1) If $\phi(ab') = ab'a$, then for any $w \in H$, $w \notin \mathsf{dom}(\phi)$.

(2) If $\phi(ab') \neq ab'a$, then $abb' \notin \mathsf{dom}(\phi)$, $ab'b \notin \mathsf{dom}(\phi)$.

(3) The set $\mathsf{dom}(\phi)$ does not contain any full element, except (possibly) when $\phi(ab') = ab'a$, $m_{bb'} = 2$, and i is odd. In that case, any full element in $\mathsf{dom}(\phi)$ is necessarily equal to M or M'.

Note: Statement (1) also follows immediately from lemma 6.2 of [8].

Proof: (1) Define a set T of integers by the following:

$$T = \begin{cases} [0, i-1] & \text{if} \quad m_{bb'} > 2 \\ \{j \in [0, i-2] |\ j \text{ even}\} & \text{if} \quad m_{bb'} = 2 \end{cases}$$

Then H may be described as follows: any $w \in H$ can be written $w = xb'\langle j, a, b]$, (where x is the unique word such that the equality $\langle i, a, b] = x\langle j, a, b]$ holds) with $j \in T$. By remark 1.1, if $w \in \mathsf{dom}(\phi)$ we must have $w \lhd \phi(w)$. Let y be the word obtained by erasing the rightmost character in x. The elements $w_1 = yb'\langle j, a, b]$ and $w_2 = \langle i, a, b]$ are coatoms of w, and as $w_1 \lhd \phi(w_1)(= w_1 a)$, $w_2 \lhd \phi(w_2)$, if $w \in \mathsf{dom}(\phi)$, $z = \phi(w)$ satisfies:

$$yb'\langle j+1, b, a] \lhd z,\ \langle i+1, a, b] \lhd z.$$

Note that $g_2 = \langle i+1, a, b] \in \mathcal{I}$, so if g is a reduced expression for z, then we obtain g by inserting the generator b' somewhere in g_2: we can write $g = ub'v$ and $g_2 = uv$ for some words u, v. Element $\phi(w_1)$ may have several reduced expressions (when $m_{ab'} = 3$) but among those $g_1 = yb'\langle j+1, b, a]$ is the only one that contains at most one ocurrence of b'. In all cases, g_1 must be a subexpression of g. Therefore $u \geq u' = y$ and $v \geq v' = \langle j+1, b, a]$. In addition, $v \neq v'$ (the rightmost characters differ) and because of the length constraints $u = u'$ and $v = v'b$. Then the rightmost character in u coincides with the leftmost character in v, so $g_2 = uv$ is not reduced, which is a contradiction. This finishes the proof of (1).

Let us show (2). The hypotheses imply $\phi(ab') = b'ab'$, $m_{ab'} \geq 4$. Let $w_1 = abb'$; we have (using proposition 1.8. with the special matching ρ_a for the last equality)

$$\begin{array}{l} \mathsf{coat}(abb') = \{ab, ab', bb'\} \\ \phi(ab) \in \{aba, bab\},\ \phi(ab') = b'ab',\ \phi(bb') = bb'a. \end{array}$$

So if $w_1 \in \mathsf{dom}(\phi)$, we must have $w_1 \lhd \phi(w_1)$ and if g_1 is a reduced expression for $\phi(w_1)$, g_1 can be obtained by inserting the generator b somewhere in $b'ab'(\in \mathcal{I})$. Then g_1 has exactly two characters in $\{a; b\}$. This is not consistent with $\phi(ab) \lhd \phi(w_1)$. Therefore $abb' \notin \mathsf{dom}(\phi)$. The proof of $ab'b \notin \mathsf{dom}(\phi)$ is similar.

Now let us proceed with the proof of (3). Suppose that there is a full element $w \in \mathsf{dom}(\phi)$.

Case $\phi(ab') = ab'a$:

Intuitively, the setting is clear: the elements of H tell us that in a reduced expression g of w we cannot have a b' "inside" a long dihedral subword in a and b (such subwords will exist because w is full) so that indeed the $< a, b >$-part and the $< a, b' >$-part are (up to a few generators) separated in g. By lemma 5.1.1, we will be done.

Let w be a full element in $\mathsf{dom}(\phi)$; set $v = min(w, b'w)$. Then $v \geq M_{ab}$ and $v \geq b'$. Consider a reduced expression $v_1 \dots v_n$ for v; for any subset $K = \{k_1 < k_2 < \dots < k_r\}$ of $\{1, \dots, n\}$ we put $v_K = v_{k_1} v_{k_2} \dots v_{k_r}$. Thus there is a $D \subseteq \{1, \dots, n\}$ with cardinality m_{ab} such that $v_D = M_{ab}$ and an index j such that $v_j = b'$. By construction $v < b'v$, so $v_1 \in \{a; b\}$ and hence we may assume $1 \in D$.

If i is even or $m_{bb'} > 2$, then some subword of $v' = v_{D \cup \{j\}}$ belongs to H and $v' \leq v$ which is a contradiction because $\mathsf{dom}(\phi)$ is decreasing. So i is odd, and $m_{bb'} = 2$.

Define a two-periodic sequence (t_i) by $t_1 = a, t_2 = b$. Considering the occurrences of a or b in a reduced expression of w, we can find a decomposition of the form

$$\begin{array}{l} w = (u_0)a(u_1)b(u_2)a(u_3)b(u_4) \dots t_n(u_n), \text{ with} \\ u_0 \ngeq a, u_0 \ngeq b, (\text{so } u_0 \in \{e, b'\}) \\ u_j \ngeq t_{j+1} (\text{so } u_j \in < t_j, b' >) \text{ for each } j \geq 2 \\ l(w) = n + \sum_j l(u_j) \end{array}$$

(we could also start with a b: $w = (u_0)b(u_1)a(u_2)b(u_3)a(u_4) \dots t_n(u_n)$ but this case is similar and simpler). Because w is full, $w \geq M_{ab}$ so $n \geq m_{ab} \geq i+2$. If there is a $j \geq 3$ such that $u_j \geq b'$

then $w' = aba(t_3t_4 \dots t_{j-1}t_j)b'(t_{j+1}t_{j+2} \dots t_{i-1}t_i)$ (or $w' = (t_1t_2 \dots t_{i-1}t_i)b'$ if $j \geq i$) belongs to H and $w' \leq w$ which is a contradiction because $\mathsf{dom}(\phi)$ is decreasing. Therefore for those $j \geq 3$ we have $u_j \in \{e, t_j\}$ whence $u_j = e$. So

$$\begin{aligned} &w = u_0au_1bu_2a[b, a, n-3\rangle \\ &u_0 \in \{e, b'\}, u_1 \in < a, b' >, u_2 \in < b, b' >, \\ &l(w) = n + l(u_0) + l(u_1) + l(u_2) \end{aligned}$$

Because of $m_{bb'} = 2$ we deduce $u_2 \in \{e, b'\}$. Replacing (u_1, u_2) with $(u_1b', b'u_2)$ if necessary, we may assume $u_2 = e$. Then, putting $x = u_0au_1$, $y = [b, a, n-1\rangle$ we have $w = xy, x \in < a, b' >, y \in < a, b >$. By lemma 5.1.1 we are done with the case when $\phi(ab') = ab'a$.

Case $\phi(ab') \neq ab'a$:

As w is full we have $w \geq a$. Hence a decomposition $w = uav$, with $u \in < b, b' >$, $l(w) = l(u) + 1 + l(v)$. Necessarily $v \neq e$ because w is full. So the first character q of v is in $\{b, b'\}$; let $\bar{q}$ be the element defined by $\{b, b'\} = \{q; \bar{q}\}$. We can write $v = qv'$ with $l(v) = 1 + l(v')$. Then, as $w \not\geq aq\bar{q}$ (by hypothesis (2)) we deduce $v' \not\geq \bar{q}$ and so $v' \in < a, q >$. Hence $w = u(aqv') \in < b, b' >< a, q >=< q, \bar{q} >< a, q >$; as $m_{a,\bar{q}} \geq 3$, w cannot be full with respect to $\{a, \bar{q}\}$. Q. E. D.

Using the above lemma twice (interchanging the roles of b' and b the second time) we see that when $m_{bb'} > 2$, ϕ can be full only if β and $\beta^{'}$ are both restrictions of ρ_a; by theorem 1.9 we then obtain:

Lemma 5.3.2 *If $m_{bb'} > 2$, ϕ is full if and only if $\phi = \rho_a$.*

Lemma 5.3.3 *If $\beta \neq \rho_a$ and $\beta^{'} \neq \rho_a$, then ϕ is not full.*

Proof: If $m_{bb'} = 2$ we are done by the lemma above. Suppose to the contrary that ϕ is full. By lemma 5.3.1, $\mathsf{dom}(\phi)$ contains a unique full element in length $m_{ab} + m_{ab'} - 2$ namely $F = M_{b',a,b}$. Interchanging b and b', $\mathsf{dom}(\phi)$ contains a unique full element in length $m_{ab} + m_{ab'} - 2$ namely $F' = M_{b,a,b'}$. As F and F' are different, (notice for example that $bF' \lhd F'$ but $F \lhd bF$) this is a contradiction. Q. E. D.

Lemma 5.3.4 *Suppose that $m_{bb'} = 2$, $\beta^{'} = \rho_a$ and that β is not a-left-regular. Then ϕ is not full.*

Proof: The condition "β is not a-left-regular" technically means that $m_{ab} \geq 4$, and that

$$\exists i \leq m_{ab} - 3,\ \beta([b, a, i\rangle) = [b, a, i+1\rangle, \beta([a, b, i+1\rangle) = [b, a, i+2\rangle$$

Take a minimal such i. Lemma 5.3.1 says that if ϕ is full, then $\mathsf{dom}(\phi)$ contains $M_{b',a,b}$.

In addition, lemma 5.1.2. shows that $ab'[b, a, i\rangle \notin \mathsf{dom}(\phi)$; as $ab'[b, a, i\rangle \leq M_{b',a,b}$, this is impossible because $\mathsf{dom}(\phi)$ is decreasing. Q. E. D.

Proposition 5.3.5 *Suppose $m_{bb'} = 2$, and $\phi \neq \rho_a$. Then ϕ is full if and only if up to interchange of b and b', $\phi_{|<a,b'>} = \rho_a$ and $\phi_{|<a,b>}$ is a-left-regular. Then $\mathsf{dom}(\phi)$ contains exactly two full elements, namely $M_{b',a,b}$ and ${M'}_{b',a,b}$.*

Proof: Corollary 3.3 and lemma 5.1.1 give one half of the equivalence. Conversely, suppose that ϕ is full. By lemma 5.3.3, $\beta^{'}$ (say) coincides with ρ_a. Lemma 5.3.4 ensures then that β is a-left-regular, as required. Eventually, if ϕ is full, as $\phi \neq \rho_a$, we must have $\beta \neq \rho_a$, and thus we can use lemma 5.3.1. to see that the only full elements in $\mathsf{dom}(\phi)$ are (if they exist) M and M'. To see that indeed they are in $\mathsf{dom}(\phi)$, we invoke $< a, b' >< a, b >\subseteq \mathsf{dom}(\phi)$, which comes from corollary 3.3. (with $X = \{a, b'\}$, $Y = \{a, b\}$) Q. E. D.

Now we are left with the degenerate case, when a commutes with one of b, b'. Interchanging b and b' if needed, we may take $m_{ab'} = 2$.

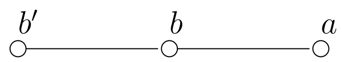
5.4 Degenerate case in rank three.

The degenerate case involves a more complicated family of obstructions than in the former cases. In this subsection we simply gather some of those obstructions that are needed in the general case (section 6) and do not attempt to make an exhaustive study of the degenerate case in itself, although a simple characterization of full matchings in the vein of propositions 5.2.1 and 5.3.5 is perfectly feasible.

The case when $m_{bb'} = 2$ is quickly taken care of by the following obvious remark:

Remark 5.4.1 *Suppose $m_{ab'} = m_{bb'} = 2$. Then $W =< a, b > \amalg b' < a, b >$, and any special matching is b'-left-regular, and so is defined everywhere and full.*

Thus we suppose $m_{bb'} \geq 3$ in the remainder of this section.



$\phi(e) = a$

$\phi(b) = ba$

Figure 3: Degenerate case

Lemma 5.4.2 *Suppose that $\beta \neq \rho_a$. Then there is a minimal $i \geq 2$ such that $\phi(\langle i, a, b]) = \langle i + 1, a, b]$ (so $m_{ab} \geq i + 2$). Let $H = \{w \in W|\ l(w) = i + 1,\ b' \leq w,\ \langle i, a, b] \leq w,\ w \notin \{b'\langle i, a, b]; \langle i, a, b]b'\}\}$. Then for any $h \in H$, we have $h \notin \mathsf{dom}(\phi)$.*

As in lemma 5.3.1 (1), this result also follows immediately from lemma 6.2 of [8].

Proof: Let $w \in H$. Then, since $ab' = b'a$, w can be written $w = xb'\langle 2j, a, b]$, (where x is the unique word such that $\langle i, a, b] = x\langle 2j, a, b]$) with $0 \leq 2j \leq i - 1$ (we could have put $2j + 1$ in place of $2j$ just as well, but the $2j$ is more convenient in the sequel). By remark 1.1, we have $w \lhd \phi(w)$.

The elements $w_1 = xb'\langle 2j-1, a, b]$ and $w_2 = \langle i, a, b]$ are coatoms of w, and as $w_1 \lhd \phi(w_1) (= w_1 a)$, $w_2 \lhd \phi(w_2)$, if $w \in \mathsf{dom}(\phi)$, $y = \phi(w)$ satisfies:

$$xb'\langle 2j, b, a] \lhd y,\ \langle i + 1, a, b] \lhd y.$$

Consider the words $g_1 = xb'\langle 2j, b, a]$ and $g_2 = \langle i+1, a, b]$. If g is a reduced word representing y, then we obtain g by inserting character b' somewhere in g_2: we can write $g = ub'v$ and $g_2 = uv$ for some words u, v. Now, the element $\phi(w_1)$ may have several reduced expressions, but there is only one that has at most one ocurrence of b', namely g_1. Thus $u \geq u' = x$ and $v \geq v' = \langle 2j, b, a]$. In addition, $v \neq v'$ (the rightmost generators differ) and because of the length constraints, $u = u'$ and $v = v'b$. Then the rightmost generator in u coincides with the leftmost generator in v, so $g_2 = uv$ is not reduced, which is a contradiction. Q. E. D.

Lemma 5.4.3 *Suppose $m_{bb'} \geq 4$ and $\phi(ab) = bab$, $m_{ab} \geq 4$. Then $abb'b$ is a (minimal) element in $W \setminus \mathsf{dom}(\phi)$.*

Proof: Put $w = abb'b$. We have:

$$\begin{aligned}
&\mathsf{coat}(w) = \{bb'b, abb', ab'b\} \\
&\phi(bb'b) = bb'b\phi(e) = bb'ba \\
&\phi(abb') = \phi(ab)b' = babb' \\
&\phi(ab'b) = \phi(b'ab) = b'\phi(ab) = b'bab
\end{aligned}$$

so if $w \in \mathsf{dom}(\phi)$ we must have

$$\mathsf{coat}(\phi(w)) = \{b'bab, abb'b, bb'ba, babb'\}$$

which is impossible (for example, using the fact that $bb'ab \in \mathcal{I}, abb'b \in \mathcal{I}$ it is easy to see that there is no y such that $b'bab \lhd y$ and $abb'b \lhd y$ both hold). Q. E. D.

Lemma 5.4.4 *Suppose $m_{bb'} = 3$ and $\phi(ab) = bab, m_{ab} \geq 4$. Then $abb'ab$ is a (minimal) element in $W \setminus \mathsf{dom}(\phi)$.*

Proof: Put $w = abb'ab$. We have:

$$\begin{aligned}
&\mathsf{coat}(w) = \{bb'ab, abab, abb'b, abb'a\} \\
&\phi(bb'ab) = bb'\phi(ab) = bb'bab \\
&\phi(abb'b) = \phi(b'abb') = b'\phi(ab)b' = b'babb' \\
&\phi(abb'a) = \phi(abab') = \phi(aba)b'
\end{aligned}$$

so if $w \in \mathsf{dom}(\phi)$ we must have $w \lhd \phi(w) = y$ and $b'babb' \lhd y$, $bb'bab \lhd y$. As $b'babb' \in \mathcal{I}$, and $bb'bab$ has exactly three reduced expressions (namely $bb'bab$, $b'bb'ab$ and $b'bab'b$), we deduce $y = b'bab'bb'$ which is not consistent with $\phi(aba)b' \lhd y$. Q. E. D.

6 General case.

Now we consider a maximal matching ϕ on a general Coxeter system (W, S). Little by little, we will show that ϕ is reducible in all cases. Naturally we suppose that ϕ is full (by definition any non-full matching is reducible). By lemma 2.2, if $< J >$ is a parabolic subgroup stable by ϕ, then $\phi_{|<J>}$ is full again, which allows us to use the results we obtained in rank three. Put $a = \phi(e)$,

$$E = \{s \in S \ \setminus \{a\} | \ sa = as\}$$
$$L = \{s \in S | \ \phi(s) = as\}, L' = L \setminus (E \cup \{a\})$$
$$R = \{s \in S | \ \phi(s) = sa\}, R' = R \setminus (E \cup \{a\})$$

We start by treating the so-called "mixed" case:

Proposition 6.1 *Suppose that ϕ is a mixed matching (i.e. such that $L' \neq \emptyset, R' \neq \emptyset$). Then ϕ is reducible. More precisely, up to interchange of left and right, we have:*

$$\mathsf{dom}(\phi) =< R > (< L > \cap \mathsf{dom}(\phi))$$
$$\forall (x,y) \in < R > \times (< L > \cap \mathsf{dom}(\phi)), \ \phi(xy) = x\phi(y).$$

Note that the last line above follows from theorem 7.6 of [8].

Proof: We may assume that there is a $r \in R$ such that $\phi \neq \rho_a$ on P_r or that there is a $l \in L$ such that $\phi \neq \lambda_a$ on P_l (otherwise we have a "middle multiplication" matching, cf. corollary 4.2). By symmetry we may assume that $\phi \neq \lambda_a$ on P_{l_0} for some $l_0 \in L \setminus \{a\}$ (then necessarily $l_0 \in L'$). By lemma 5.2.1, (used on the restriction of ϕ to $< \{a, l_0, r\} >$) we see that ϕ is a-left regular on P_{l_0} and $\phi = \rho_a$ on P_r for each $r \in R$. By lemma 5.2.1, (used on the restriction of ϕ to $< \{a, l, r\} >$) we see that ϕ is a-left-regular on P_l for each $l \in L$.

Then corollary 3.3 and theorem 4.1 give an equality for $\mathsf{dom}(\phi)$ by double inclusion: theorem 4.1 gives $\mathsf{dom}(\phi) \subseteq < R >< L >$, hence $\mathsf{dom}(\phi) \subseteq < R > (< L > \cap \mathsf{dom}(\phi))$ because $\mathsf{dom}(\phi)$ is decreasing, and corollary 3.3 gives $< R > (< L > \cap \mathsf{dom}(\phi)) \subseteq \mathsf{dom}(\phi)$, along with the formula $\phi(xy) = x\phi(y)$.

Let us explain why this implies that ϕ is reducible: let o be a full orbit, $o = \{m, M\}$ with $M = \phi(m)$ and M full. Then there is a $(x, y) \in < R > \times < L \cap \mathsf{dom}(\phi) >$ such that $m = xy, \ M = x\phi(y)$. We may assume $l(m) = l(x) + l(y)$ by the cancellation rule. By lemma 1.7, we have $\phi(y) \in < L \cap \mathsf{dom}(\phi) >$. As $R \neq \emptyset$ and M is full, we deduce $x \neq e$. Let $x_1 \in D_l(x)$; then x_1 is left-regular (because $x_1 \in R$) and x_1 is in the left descent set of m, so that the orbit o is left-reducible. Q. E. D.

So we may assume that for example $L' = \emptyset$, i.e. $\phi(s) = sa$ for any $s \in S$.

Using lemma 5.3.3, we can even assume that for any $s \in S \setminus \{a\}$ except at most one element, $\phi_{|<s,a>} = \rho_a$.

Of course, the non-trivial case arises when there is indeed an element (which we will denote b) such that $\phi_{<b,a>}$ does not coincide with right multiplication by a. Now we slightly change the notations in order to work with disjoint subsets of S: we put

$$A = S \setminus (E \cup \{a, b\})$$
$$B = \{b' \in E | \ m_{bb'} \geq 3\}$$
$$C = E \setminus B = \{s \in S | \ sa = as, \ sb = bs\}$$

Using lemma 5.3.2, we see that $a'b = ba'$ for any $a' \in A$. The commutations are summarized by the following picture:

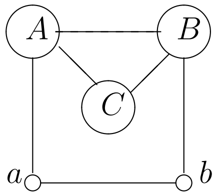


Figure 4

and we have the following regularity data, by corollary 3.2:

x	Is x left-regular?	Is x right-regular?
a	Yes if $A \neq \emptyset$ (proposition 5.3.5)	Unknown
b	Unknown	Unknown
$a' \in A$	Yes	No
$b' \in B$	Yes	Yes
$c \in C$	Yes	Yes

Define $d_j = \langle j, a, b]$ for each integer j. As $\phi \neq \rho_a$ there is a minimal i such that $\phi(d_i) \neq d_i a$, so $i \leq m_{ab} - 2$ and $\phi(d_i) = d_{i+1}$. Define, for $x \in A \cup B$,

$$H_x = \begin{cases} \{w|\ d_i \lhd w,\ a' \leq w,\ w \neq a'd_i\} & if\ x = a' \in A \\ \{w|\ d_i \lhd w,\ b' \leq w,\ w \notin \{b'd_i, d_i b'\}\} & if\ x = b' \in B, i > 2, \\ \{abb'b\} & if\ x = b' \in B, i = 2, m_{bb'} > 3 \\ \{abb'ab\} & if\ x = b' \in B, i = 2, m_{bb'} = 3. \end{cases}$$

Then, by lemmas 5.3.1, 5.4.2, 5.4.3 and 5.4.4:

$$H_x \cap \mathsf{dom}(\phi) = \emptyset \text{ for any } x \in A \cup B \tag{1}$$

Denote by H the union of the H_x. Suppose by contradiction that ϕ is not reducible. Then $|S| > 2$ and there is a nonreducible full orbit, i.e. there is a $w \in \mathsf{dom}(\phi)$ with $w \lhd \phi(w)$, $\phi(w)$ full such that w is **irreducible**, i.e. such that $D_l(w)$ does not contain any left-regular element, and $D_r(w)$ does not contain any right-regular element. Since $\phi(w)$ is full, $\phi(w) \geq M_{ab}$ and hence $w \geq \phi(M_{ab})$.

Note that the subgroup $G =< \{a, b\} \cup C >$ of W is isomorphic to the direct product of $< a, b >$ and $< C >$, and that in addition we have that any element of C is right-regular, so $\phi(xy) = \phi(x)y$ for all $(x, y) \in< a, b > \times < C >$ and the restriction of ϕ to G is reducible. In particular $w \notin G$, so that there is a generator $s \notin \{a, b\} \cup C$ such that $w \geq s$. Thus

$$w \geq \phi(M_{ab}),\ w \text{ irreducible},\ w \geq s, s \in S \setminus (\{a, b\} \cup C) \tag{2}$$

Let $g = w_1 w_2 \dots w_m$ be a reduced expression for w. For any subset $K = \{k_1 < k_2 < \dots < k_r\}$ of $\{1, \dots, n\}$ we put $w_K = w_{k_1} w_{k_2} \dots w_{k_r}$. Thus there is a $J = \{j_1 < j_2 < \dots < j_{m_{ab}-1}\} \subseteq \{1, \dots, n\}$ with cardinality $m_{ab} - 1$ such that w_J is dihedral in a and b and an index j' such that $w_{j'} = s$. Because of $D_l(w) \subseteq \{a, b\}$ we have $w_1 \in \{a, b\}$, and also $w_2 \in \{a, b\}$ (else either $w_1 = a, w_2 \in A$ or

$w_1 = b, w_2 \in B$; in the first case a is a left-regular element in $D_l(w)$ which is excluded, and in the second b is not left-regular so that there is a minimal k with $\phi([a,b,k\rangle) = [a,b,k+1\rangle, \phi([b,a,k+1\rangle) = [a,b,k+2\rangle$, and then $w \geq bw_2[a,b,k\rangle$ contradicts lemma 5.1.3). Thus we may assume $1 \in J, 2 \in J$.

Suppose $s \in A$. Then a is left-regular, and hence $D_l(w) = \{b\}$, so $w_1 = b, w_2 = a$. Putting $h = baw_{\{j_3,j_4,\dots;j_i\}\cup\{j'\}}$ if i is odd and $h = aw_{\{j_3,j_4,\dots;j_i,j_{i+1}\}\cup\{j'\}}$ if i is even, we get $w \geq h \in H$ which contradicts (1). So $s \notin A$, i.e. $s \in B$; the above reasoning clearly also implies that $\mathsf{supp}(w) \cap A = \emptyset$. By reasoning on the right as we did on the left, we see that $w_{m-1} \in \{a,b\}, w_m \in \{a,b\}$.

Suppose that $i > 2$ and that we are not in the case (i even, $w_1 = b$). Put

$$u = \begin{cases} ab & \text{if } \ i \text{ is even and } w_1 = a \\ b & \text{if } \ i \text{ is odd and } w_1 = a \\ ba & \text{if } \ i \text{ is odd and } w_1 = b \end{cases}$$

(so that u is a subword of w_1w_2 that contains b, that has the same leftmost character as $\langle i,a,b]$ and is maximal for this property). Similarly, define $v = b$ if $w_m = a$ and $v = ab$ if $w_m = b$; then v is a subword of $w_{m-1}w_m$ that contains b, that has the same rightmost character as $\langle i,a,b]$ and is maximal for this property. Let $J_{int} = J \cap \{3,4,\dots m-3, m-2\}$. We claim that

$$|J_{int}| \geq i - l(u) - l(v) \tag{*}$$

Indeed, we always have $|J_{int}| \geq |J| - 4 \geq m_{ab} - 5$ and $i - l(u) - l(v) \leq i - 2 \leq m_{ab} - 4$. If J does not contain all of $\{1,2,m-1,m\}$, then the first inequality may be improved to $|J_{int}| \geq m_{ab} - 4$, and if $l(u) = 2$ or $l(v) = 2$ or $i \neq m_{ab} - 2$, then the second inequality may be improved to $i - l(u) - l(v) \leq m_{ab} - 5$. Thus (*) holds in any of those cases, and the only case left is $\{1,2,m-1,m\} \subseteq J, l(u) = l(v) = 1, i = m_{ab} - 2$. From $l(v) = 1$ we deduce that $w_m = a$, and from $l(u) = 1$ we deduce that $w_1 = a$, and that i (and hence m_{ab}) is odd. Then w_J is a dihedral word of even length with identical rightmost and leftmost generators, which is a contradiction.

Let J' be the set of the first $i - l(u) - l(v)$ elements in J_{int}.Then, if $h = uw_{J'\cup\{j'\}}v$ we have $w \geq h \in H$ which contradicts (1).

Suppose that $i > 2$ and that (i is even, $w_1 = b$). Then b is not left-regular (indeed $b \in D_l(w)$) so that there is a minimal j such that $\phi([a,b,j\rangle) = [a,b,j+1\rangle, \phi([b,a,j+1\rangle) = [a,b,j+2\rangle$ If $w_3 = b' \in B$, then then $w \geq bb'w_{\{j_3,j_4,\dots,j_{i-1}\}}$ contradicts lemma 5.1.3. Therefore $w_3 = b$, and the reasoning above may be readjusted (taking a subword u of $w_1w_2w_3$ instead of w_1w_2) so that we get a contradiction in this case also.

Suppose $i = 2$ and $w_1 = a$. If $w_m = b$, then $w \geq w_{\{1,2,j',m-1,m\}} = absab \in H$ contradicts (1). So we have $w_m = a$. Thus a is not right-regular, so there is a $k \leq m_{ab} - 3$ such that $\phi(\langle k,a,b]) = \langle k+1,a,b]$ and $\phi(\langle k+1,b,a]) = \langle k+2,a,b]$. Necessarily $k \geq 2$, so $m_{ab} \geq 5$, hence $|J| \geq 4$. In particular, j_3 exists and satisfies $2 < j_3 < m-1$. Then $w \geq w_{\{1,2,j_3,m-1\}\cup\{j'\}} = abasb \in H$ contradicts (1).

Finally, suppose $i = 2$ and $w_1 = b$. Then b is not left-regular, so that there is a minimal k such that $\phi([a,b,k\rangle) = [a,b,k+1\rangle$ and $\phi([b,a,k+1\rangle) = [a,b,k+2\rangle$. If $w_3 \in B$, then $w \geq w_{\{2,3,j_3,\dots j_{k+2}\}\cup\{j'\}} = bw_3[a,b,k\rangle$ contradicts lemma 5.1.3. Thus $w_3 = b$, and the reasoning above may be readjusted (using $(w_2, w_3) = (a, b)$ instead of $(w_1, w_2) = (a, b)$) to get a similar contradiction. This concludes the proof.

So we have finally shown the following:

Theorem 6.2 *For any Coxeter system (W,S), any special matching on W is reducible.*

Combining this with proposition 2.5, we immediately obtain the following:

Corollary 6.3 *Let (W, S) be a Coxeter system, ϕ a special matching of W, $x, y \in W$ such that $x \lhd \phi(x)$, $y \lhd \phi(y)$. Then*

$$R_{\phi(x),\phi(y)} = R_{x,y} \quad (6.3.1)$$

$$R_{x,\phi(y)} = (q-1)R_{x,y} + qR_{\phi(x),y} \quad (6.3.2)$$

Note that this is exactly theorem 7.8 of [8].

Although we did not need this here, it is interesting to make the following remark (we denote by $\mathcal{M}(W)$ the set of all maximal matchings of a Coxeter group W and for $a \in S$, $\mathcal{M}_a(W) = \{\phi \in \mathcal{M}(W)\ ;\ \phi(e) = a\}$):

Proposition 6.4 *Let (W, S) be a Coxeter system and $a \in S$. Then the only elements of $\mathcal{M}_a(W)$ that are defined on the whole of W are the left- and right- multiplication-by-a matchings, except in the degenerate case*

$$S = \{a, b\} \amalg C,\ \forall c \in C, m_{ac} = m_{bc} = 2.$$

In this case, W is isomorphic to the direct product of the Coxeter groups $< C >$ and $< a, b >$, all the elements of $\mathcal{M}_a(W)$ are $< C >$-regular (i.e. satisfy $cx \in \mathsf{dom}(\phi)$, $\phi(cx) = c\phi(x)$ for any $c \in < C >$, $x \in \mathsf{dom}(\phi)$) and hence defined on the whole of W. In addition, the restriction-to-$< a, b >$ operation provides a bijection between $\mathcal{M}_a(W)$ and $\mathcal{M}_a(< a, b >)$.

Proof: Let $\phi \in \mathcal{M}_a(W)$, $\phi \notin \{\lambda_a; \rho_a\}$ be everywhere defined. Put $E = \{s \in S|\ sa = as, s \neq a\}$, $R = \{s \in S|\ \phi(s) = sa\}$, $L = \{s \in S|\ \phi(s) = as\}$, $R' = R \setminus (E \cup \{a\})$, $L' = L \setminus (E \cup \{a\})$. By theorem 4.1, we have $W =< R >< L >$ and hence $L' = \emptyset$ or $R' = \emptyset$ (otherwise for $l \in L', r \in R'$ we have $lar \notin < R >< L >$). Suppose for example that $L' = \emptyset$, i.e. $\phi(s) = sa$ for all $s \in S$. By theorem 1.9, there is a generator b such that $\phi_{|P_b} \neq \rho_a$, thus $\exists i \leq m_{ab} - 2$, $\phi(\langle i, a, b]) = \langle i+1, a, b])$. Then if $R' \neq \emptyset$, we have for any $r \in R'$, $\langle i, a, b]r \notin \mathsf{dom}(\phi)$ by lemma 5.3.1. Thus we may assume $R' = \emptyset$. Then $S \subseteq \{a, b\} \cup E$.

Assume that there is a $c \in E$ that does not commute with b. If $i > 2$, we have $\langle i-1, b, a]cb \notin \mathsf{dom}(\phi)$ by lemma 5.4.2. If $i = 2$ and $m_{bc} > 3$ we have $abcb \notin \mathsf{dom}(\phi)$ by lemma 5.4.3. If $i = 2$ and $m_{bc} = 3$ we have $abcab \notin \mathsf{dom}(\phi)$ by lemma 5.4.4. So in all cases, $\mathsf{dom}(\phi) \neq W$.

Thus we may assume that any $c \in E$ commutes with b, which means that we are in the degenerate case defined above (with $C = E$). The rest of the proposition is clear. Q. E. D.

Part IV

Some Extensions of the main Result

1 Introduction and statement of the conjecture

Here we will show some (minor) extensions of the main result of the preceding part, namely (III.6.3). Denote by $\mathcal{R}$ the set of all the isomorphisms that preserve R-polynomials; for brievity we call them R-isomorphisms. Then (III.6.3) states that

$$\text{Any isomorphism } \phi : [e, y] \to [e, y'] \text{ is in } \mathcal{R}. \tag{1.1}$$

There are certainly several ways in which this result is likely to be extended; we discuss one of them here. A natural extension of the class of posets isomorphic to a Bruhat interval originating at the identity is the class of **completely compressible** posets, i.e. those that can be reduced to the trivial poset by a sequence of compressions (a compression is simply a special matching ϕ of an interval $[x, y]$, that compresses $[x, y]$ onto $[x, \phi(y)]$. See [9] for more about compressions). More precisely, consider C_1, the class of all completely compressible Bruhat intervals and C_0 the subclass of C_1 consisting of the Bruhat intervals originating at the identity. Then maybe the following extension can be deduced from (1.1):

$$\text{Any isomorphism } \phi : P \to P' \text{ is in } \mathcal{R} \text{ if } P, P' \in C_1. \tag{1.2}$$

To prove (1.2), it would suffice to show that

$$\text{For any } P \in C_1, \text{ there is a } R\text{-isomorphism } \phi : P \to P_0, \text{with } P_0 \in C_0. \tag{1.3}$$

Unfortunately, this is false in general; indeed, the interval $[3, 3(1212)3]$ in type B_3 (which means that $m_{12} = 4,\ m_{13} = 2,\ m_{23} = 3$) is completely compressible, but is not isomorphic to any $[e, y]$ (we explain the reasons for this in the appendix). However, if we restrict our attention to simply laced finite Coxeter groups, (i.e. to the A, D, E types) then (1.3) seems to hold; in fact, for this realm we make a much stronger conjecture, which implies that there is a uniquely defined isomorphism if we impose some additional conditions.

Let us develop some general tools about isomorphisms onto intervals originating at the identity. In a poset P with minimum element x, an element $y \in P$ is said to be dihedral when the interval $[x, y]$ (as a subposet of P) is dihedral. The **bud** $B(P)$ of P is defined to be the set of the dihedral elements of P (cf. e.g. [11]). From now on, $I = [x, y]$ will always denote a Bruhat interval.

It is easy to deduce from corollaries I.9.8 and I.9.9 that for any two atoms a, b of I there is a unique maximal dihedral subposet of I containing $\{x; a; b\}$; we call it $D(a, b)$. Note that there are two possibilities for $D(a, b)$, as shown in figure 2: either it has a maximum element and is a closed Bruhat interval, or it has two maximal elements.

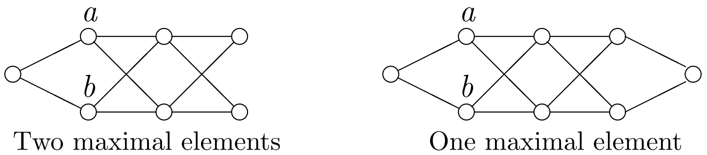


Figure 2: The two cases for $D(a, b)$

If I is completely compressible, however, it is easily seen that all the $D(a, b)$ are closed. Define $m(a, b)$ to be the length of the poset $D(a, b)$. If $\phi : I \to [e, w']$ is an isomorphism onto an interval originating at the identity (in a possibly different Coxeter sytem (W', S')), then restricting S' if necessary we may identify S' with the set $\mathsf{atoms}(I)$ of the atoms of I. Then, because $\phi(D(a,b))$ is dihedral in W', we must have $m'_{a,b} \geq m(a,b)$ for any two atoms a, b of I. In addition, we have the following elementary result from proposition II.2.6:

Proposition 1.1 *Let $[e, w']$ be an interval originating at the identity in some Coxeter system (W', S'). Let (W_{can}, S_{can}) be a Coxeter system defined as follows: $S_{can} = \{s \in S'; s \leq w'\}$ (the support of w'), and for $s, t \in S_{can}$, $m_{can}(s,t)$ is the size of the closed dihedral interval $D(s,t)$, as above (in this special case $D(s,t)$ is the intersection of $[e, w']$ with the dihedral subgroup $< s, t >$). Then there is a uniquely defined isomorphism $\zeta : [e, w'] \to [e, w_{can}]$ (with $w_{can} \in W_{can}$) satisfying*

$$\zeta(su) = s\zeta(u) \ (\zeta(us) = \zeta(u)s) \ \text{whenever } s \in S_{can}, u \in [e, w'], su \in [e, w'] \ (us \in [e, w']).$$

and ζ preserves R-polynomials.

This motivates the following definition:

Definition 1.2 *Let I be a completely compressible Bruhat interval. Define a Coxeter system $\mathcal{S} = (W_{can}, S_{can})$ as follows: $S_{can} = \mathsf{atoms}(I)$ and $m_{can}(a,b) = m(a,b)$ for $a, b \in S'$. We call $\mathcal{S}$ the* **canonical image Coxeter system** *associated to I. A* **standard isomorphism** *ϕ of I is an isomorphism from I onto an interval originating at the identity in W_{can}, such that $\phi(a) = a$ for any a in $\mathsf{atoms}(I)$.*

and we may paraphrase now the aforementioned result:

Proposition 1.3 *For any isomorphism $\phi : I \to [e, w']$ (where the image Coxeter system (W', S') may be arbitrary), there is another isomorphism $\zeta : [e, w'] \to [e, w'']$ such that $\zeta \circ \phi$ is standard.*

In other words, any isomorphism onto a $[e, w]$ may be standardized, so that we need only look for standard isomorphisms. The next proposition (which follows easily from I.9.8 and I.9.9) tells us just how much choice we have in the definition of a standard isomorphism:

Proposition 1.4 *Any standard isomorphism of I is uniquely defined by its restriction to $B(I)$ (which yields an isomorphism between the buds $B(I)$ and $B(W_{can})$).*

For example, if a, b are two generators in W with $m_{a,b} \geq 3$, τ is the transposition which exchanges ab and ba, then τ and id (the identity map) yield two distinct standard isomorphisms of $[e, ab]$. We now introduce some additional conditions on isomorphisms that will enable us to reject τ and accept id.

If $[x, y]$ is a Bruhat interval put $L_{[x,y]} = \{s \in S| \ sx \in [x, y]\}$ and $R_{[x,y]} = \{s \in S| \ xs \in [x, y]\}$.

Definition 1.5 *An isomorphism ϕ between two Bruhat intervals $I = [x, y]$ and I' (an interval originating at the identity in a possibly different Coxeter group W') is* **left-regular** *if there is a (necessarily unique) injection $\lambda : L_{[x,y]} \to S'$ such that $\phi(sw) = \lambda(s)\phi(w)$ holds for any $s \in L_{[x,y]}$ and $w \in [x, y]$ such that $sw \in [x, y]$. It is* **right-regular** *if there is an injection $\rho : R_{[x,y]} \to S'$ such that $\phi(ws) = \phi(w)\rho(s)$ holds for any $s \in R_{[x,y]}$ and $w \in [x, y]$ such that $ws \in [x, y]$. It is* **biregular** *if it is both left- and right- regular.*

Note that λ (or ρ) is "necessarily unique" because we must have $\phi(sx) = \lambda(s)\phi(x) = \lambda(s)$ for any $s \in L_{[x,y]}$. At this point, we naturally ask the question: does the biregularity requirement ensure uniqueness for a standard isomorphism? The answer is no. We provide a counterexample in type A_6 (which is minimal in the sense that no counterexample exists in type A_5, and no example with an x of smaller length exists in type A_6).

Consider the dihedral interval

$$I = [w_L w_R, w_L 343 w_R] = \{w_L w_R, w_L 3 w_R, w_L 4 w_R, w_L 34 w_R, w_L 43 w_R, w_L 343 w_R\}$$

where $w_L = 215, w_R = 265$. The atoms of I (which are the same thing as the atoms of $x = w_L w_R$ in I) are $a = w_L 3 w_R$ and $b = w_L 4 w_R$, so that there are reflections l_a, r_a, l_b, r_b such that $a = l_a x = x r_a$, $b = l_b x = x r_b$ (and for example $l_a x$ is reduced exactly when l_a is a generator). A little computation shows that

$$l_a = r_a = 232$$
$$l_b = r_b = 454.$$

None of those reflections is a generator; we deduce $L_I = R_I = \emptyset$, so that any standard isomorphism on I is trivially biregular. It is easily seen that there are exactly two standard isomorphisms of I (as in the diagram below)

w	$w_L w_R$	$w_L 3 w_R$	$w_L 4 w_R$	$w_L 34 w_R$	$w_L 43 w_R$	$w_L 343 w_R$
$\frac{\phi_1(w)}{\phi_2(w)}$	e	a	b	$\frac{ab}{ba}$	$\frac{ba}{ab}$	$aba = bab$

In this special case, we feel inclined to accept ϕ_1 and reject ϕ_2 because "3 corresponds to a and 4 corresponds to b", but it seems wildly improbable that such a naive approach will be successful with more complicated counterexamples. Surprisingly, this idea can be nicely formalized and provide a conjecture that has withstood the test of reasonably large examples by computer.

Until now, we have combined left and right action of the generators (as in the definition of biregularity). From now on, we shall always act from the left only; the primary reason for this restriction is to circumvent the complex interaction between left and right (this will become clear with proposition 1.10, which holds only for a one-sided action). Some propositions like 1.18 are also valid in a two-sided context, but here we present only the 'left' version for the sake of simplicity.

We now proceed to the formal definitions. The following proposition follows immediately from the "special matching" properties of the Bruhat ordering:

Proposition-Definition. 1.6 *Let $I = [x, y]$ be a Bruhat interval, and let $s \in S$ such that $x > sx, y > sy$. Then the following are equivalent:*

(i) $x \not\leq sy$

(ii) Any $w \in [x, y]$ satisfies $w > sw$.

(iii) Left multiplication by s is a poset isomorphism $[x, y] \to [sx, sy]$.

In this case, we say that $[sx, sy]$ is a **(left) elementary lower translate** *of $[x, y]$. We say that an interval J is a* **(left) lower translate** *of I if there is a chain of intervals $I_0 = I, I_1, \ldots I_n = J$ such that I_{i+1} is a (left) elementary lower translate of I_i; or equivalently if there is an elementt t_L*

in W and an isomorphism $\theta : J \to I$, which satisfies $\theta(w) = t_L w, l(\theta(w)) = l(t_L) + l(w)$ for all $w \in I$. We call such an isomorphism a **(left) translation***. If $\phi : I \to I'$ is any isomorphism onto an interval I' into a possibly different Coxeter group, then we have a new isomorphism*
$\psi = \phi \circ \theta : J \to I'$. We call ψ a **(left) lower translate** *of ϕ. If $c : I \to I$ is any special matching of I, then we have a special matching of J: $c' = \theta^{-1} \circ c \circ \theta : J \to J$. We call c' a* **(left) lower translate** *of c.*

Clearly, left translations preserve the R-polynomials. Note that the left lower translate of a left-regular isomorphism is not necessarily left-regular. Indeed, take W to be the free Coxeter group on the generators l, s_1, s_2, and consider the intervals

$$I = [l, ls_1s_2] = \{l, ls_1, ls_2, ls_1s_2\}$$
$$J = [e, s_1s_2] = \{e, s_1, s_2, s_1s_2\}$$
$$I' = [e, s_2s_1] = \{e, s_1, s_2, s_2s_1\}$$

The interval J is a lower translate of I, with left multiplication by l for a translation. Then the isomorphism $\phi : I \to I'$ defined by $\phi(l) = e, \phi(ls_i) = s_i, \phi(ls_1s_2) = s_2s_1$ is left-regular, but the associated ψ is not left-regular.

Definition 1.7 *A left-regular isomorphism ϕ is* **strongly left-regular** *if all its left lower translates are left-regular.*

Note that if ϕ is a strongly left-regular isomorphism, then a lower restriction of ϕ (i.e. the restriction of ϕ to a lower interval of $[x, y]$) is not necessarily strongly left-regular. To see this, let ϕ be the non-strongly-left-regular example isomorphism on $I = [l, ls_1s_2]$ defined above. Add a new generator l', and consider the free Coxeter group on l, l', s_1, s_2. Extend ϕ to a mapping ϕ' on the larger interval $I_1 = [l, l'ls_1s_2]$ by putting $\phi'(l'w) = l'\phi(w)$ for $w \in I$. Then ϕ' is strongly left-regular (in fact I_1 has no left lower translate other than itself) but its lower restriction ϕ is not.

Definition 1.8 *A strongly left-regular isomorphism ϕ is* **totally left-regular** *if all its left lower restrictions are strongly left-regular.*

Trivially, we have:

Remark 1.9 *If ϕ is totally left-regular, then so are its left lower translates and lower restrictions.*

Next we show a lattice property:

Proposition 1.10 *Let I be a Bruhat interval and let J, J' be two left lower translates of I. Then there is an interval K which is a left lower translate of both J and J'.*

Note that the 'two-sided' version of this does not hold in general: take $I = [12, 121], J = [e, 1], J' = [e, 2]$ in type A_2. Then J is a left lower translate of I and J' is a right lower translate of I, but neither J nor J' can be translated further.

Proof We may assume that J and J' are elementary lower translates of I, the general case following by induction. Clearly, we may also assume that $J \neq J'$. Thus $J = sI, J' = s'I$ with $s, s' \in S, s \neq s'$. Then $s \neq s'$ because $J \neq J'$; for any $w \in I$ both s and s' are in the left descent set of w, so $m = m_{s,s'}$ is finite, and putting $b = ss'ss' \ldots (m \text{ terms}) = s'ss's \ldots (m \text{ terms})$ we can write $w = bw', l(w) = l(b) + l(w')$ for some $w' \in W$. Then we can take $K = (b^{-1})I$. Q. E. D.

Corollary 1.11 *If I is a Bruhat interval there is a smallest left lower translate of I (we call it the left core of I and denote it by* $\mathsf{core}(I)$*). As in 1.6 one can likewise define the left core of an isomorphism defined on I or the left core of a special matching of I.*

As explained above, there is no notion of a 'two-sided core': in general there are several minimal lower translates for a given interval I (e.g. $[e, 1]$ and $[e, 2]$ for $[12, 121]$). In the cases where our final conjecture holds, however, unicity ensures that the 'left' construction yields the same object as the 'right' construction, so that the propositions below hold in a two-sided context, with 'any minimal lower translate' in place of 'left core'.

The following lemma is fundamental:

Lemma 1.12 *Let $I = [x, y]$ be a Bruhat interval, $J = \mathsf{core}(I)$ and θ the associated translation isomorphism $I \to J$. Let s be a generator such that $x < sx < sy < y$. Then there is a generator s' such that $\theta(sx) = s'\theta(x)$ and $\theta(sy) = s'\theta(y)$.*

Proof: We argue by induction on $l(y)$. If $l(y) = 0$ there is nothing to prove, so suppose $l(y) > 0$. Obviously we may assume that $J \neq I$. Since we reason by induction, it is not necessary to show the property on J at once, it suffices to show it on an intermediary left lower translate: in other words, it suffices to find a left lower translate K of I with $K \neq I$ such that for the associated translation isomorphism $\kappa : I \to K$, there is a generator s' such that $\kappa(sx) = s'\kappa(x)$ and $\kappa(sy) = s'\kappa(y)$.

There is an element t_L in W such that $\theta(w) = t_L w, l(\theta(w)) = l(t_L) + l(w)$ for all $w \in J$. Then $t_L \neq e$ since $J \neq I$; let t be a generator in the left descent set of t_L; clearly $t \neq s$. Then both s and t are in the left descent set of y, so $m = m_{s,t}$ is finite, and putting $b = stst\ldots(m \text{ terms}) = tsts\ldots(m \text{ terms})$ we can write $y = by', l(y) = l(b) + l(y')$ for some $y' \in W$. Similarly, there is an element $x' \in W$ such that $sx = bx', l(sx) = l(b) + l(x')$.

Consider the interval $L = t[x, y] = [tx, ty] = [\underbrace{(st\ldots)}_{m-2 \text{ terms}} x', \underbrace{(st\ldots)}_{m-1 \text{ terms}} y']$. Then L is a lower translate of I; in particular, no element of L has t in its left descent set. We deduce $\underbrace{(ts\ldots)}_{m-2 \text{ terms}} y' \notin L$. Thus, $L' = sL$ is a lower translate of L (this is condition (i) of 1.6). Similarly, $L'' = tL'$ is a lower translate of L'. Continuing this way, we eventually obtain a translation decomposition $L = \underbrace{(st\ldots)}_{m-2 \text{ terms}} [x', cy']$ where c is the last generator in $\underbrace{(st\ldots)}_{m-1 \text{ terms}}$ (thus $c = t$ if m is odd and $c = s$ otherwise). Then $K = [x', cy']$ is a lower translate of I, and taking $s' = c$ we are done. Q. E. D.

Here is a typical application of this lemma:

Proposition 1.13 *Let ϕ be an isomorphism from I onto an interval originating at the identity. Then ϕ is totally left-regular if and only if* $\mathsf{core}(\phi)$ *is.*

Proof: Only the "if" part deserves attention, of course. So suppose that $\mathsf{core}(\phi)$ is totally left-regular. We must show that ϕ is totally left-regular, i.e. that for any lower interval I' of I, the restriction of ϕ to I' is strongly left-regular. But if θ is the translation isomorphism $I \to \mathsf{core}(I)$, then $\theta(I')$ is a left lower translate of I', so $\mathsf{core}(I')$ is a left lower translate of $\theta(I')$. Thus we may replace I' with I, and all we need to show is that ϕ is strongly left-regular. For any left lower translate K of I we have $\mathsf{core}(K) = \mathsf{core}(I)$ so we may replace K with I, and all we need to show is that ϕ is left-regular.

$$\begin{array}{ccccc} [x,y] & \longrightarrow & \theta[x,y] & \longrightarrow & \mathsf{core}([x,y]) \\ \downarrow & & \downarrow & \searrow^{\psi} & \downarrow \psi' \\ I & \longrightarrow & \mathsf{core}(I) & \underset{\phi}{\longrightarrow} & \text{Image of } \phi \end{array}$$

Labelled arrows: poset homorphims
For the other arrows:
Horizontal arrows: translations
Vertical arrows: inclusions

Figure 3

Let us show that ϕ is left-regular. So suppose that we have $x < sx < sy < y$ where x is the smallest element of I and $y \in I$. Let θ be the translation isomorphism $I \to \mathsf{core}(I)$; then $\mathsf{core}([x,y])$ is a lower translate of $\theta([x,y])$, whence a translation isomorphism $\alpha : \theta([x,y]) \to \mathsf{core}([x,y])$. By the lemma above there is a generator s' such that $\alpha\theta(sx) = s'\alpha\theta(x)$ and $\alpha\theta(sy) = s'\alpha\theta(y)$. Let ψ be the restriction of $\mathsf{core}(\phi)$ to $\theta([x,y])$. Then ψ is totally left-regular, and so is its left lower translate $\psi' = \psi \circ \alpha^{-1}$. In particular, if we write $\mathsf{core}([x,y]) = [x', y']$, we have $\psi'(sy') = \psi'(sx')\psi'(y')$. In terms of ϕ, it means that $\phi(sy) = \phi(sx)\phi(y)$ as requested. Q. E. D.

Definition 1.14 *A special matching c of a Bruhat interval I is* **explicit** *if some lower translate of it is a (left or right) multiplication matching.*

Thus if c is an explicit matching of I there is a translation $\theta : J \to I$ and an $s \in S$ such that $c(w) = \theta(s\theta^{-1}(w))$ (say). But θ can be defined by $t(w) = t_L w t_R$ for some fixed $t_L, t_R \in W$ and all $w \in J$. Then $c(w) = (t_L s {t_L}^{-1})w$ for any $w \in W$, an explicit formula which justifies the term.

Definition 1.15 *Let I and J be two Bruhat intervals. We say that J is a* **(left) multiplication compression of** *I if we can write $I = [x,y], J = [x, sy]$ for some $x, y \in W$ and some $s \in S$ such that $x < sx, sy < y$.*

The following notion is fundamental:

Definition 1.16 *Let $I = [x,y]$ be a Bruhat interval. We say that I is* **left explicitly completely compressible** *if*

(*) there is a chain of intervals $I_0 = I, I_1, \dots I_n = [z,z]$ such that I_{i+1} is either a left elementary lower translate of I_i or a left multiplication compression of I_i.

Note that if s_1 and s_2 are two generators with $m(s_1, s_2) \geq 4$ and $I = [s_1, s_1 s_2 s_1]$ then I has exactly two special matchings and both are non-explicit. Thus, I is not explicitly completely compressible.

The following two propositions show the usefulness of the notion of (left) explicit complete compressibility:

Proposition 1.17 *If I is left explicitly completely compressible, then there is a unique totally left-regular standard isomorphism ϕ on I.*

Proposition 1.18 *Let $\phi : I \to [e,y]$ be an isomorphism, with ϕ strongly left-regular and I left explicitly completely compressible. Then ϕ preserves R-polynomials.*

The proof of the second proposition is simpler and we begin by this one.

Proof of Proposition 1.18 We argue by induction on the length $l(I)$ of I. If $l(I) = 0$ there is nothing to prove, so suppose $l(I) > 0$. Since I is left explicitly completely compressible, there is a sequence $I_0, \ldots I_n$ as in 1.16.(*). Then $J = I_1$ satisfies $l(J) = l(I) - 1$ and J is either an elementary left lower translate of I or a left elementary compression of I. In the first case, the lower translate isomorphism $\psi = \phi \circ \theta : J \to [e, y]$ is in $\mathcal{R}$, and $\theta \in \mathcal{R}$ also, so $\phi = \psi \circ \theta^{-1} \in \mathcal{R}$. In the second case, we can write $J = [x, sy], I = [x, y]$ as in 1.15. By the induction hypothesis, ϕ_J is in $\mathcal{R}$. Let $u \le v$ in I. If $v \in J$, then $R_{u,v} = R_{\phi(u),\phi(v)}$ by the preceding remark. Otherwise $v' = sv < v$ and $v' \in J$. We have $\phi(v) = \lambda(s)\phi(v')$ by left-regularity, and $\phi(v') < \phi(v)$ since ϕ is an isomorphism. Then

$$\begin{aligned}
R_{u,v} &= R_{u,sv'} \\
&= qR_{u,v'} + (q-1)R_{su,v'} \\
&= qR_{\phi(u),\phi(v')} + (q-1)R_{\phi(su),\phi(v')} (\text{ because } v' \in J) \\
&= qR_{\phi(u),\phi(v')} + (q-1)R_{\lambda(s)\phi(u),\phi(v')} \\
&= R_{\phi(u),\lambda(s)\phi(v')} \\
&= R_{\phi(u),\phi(v)}
\end{aligned}$$

so that in all cases, ϕ preserves R-polynomials as claimed. Q. E. D.

Proof of Proposition 1.17 Unicity is clear: the 'standard' condition defines the image Coxeter group completely, and any sequence sending I to a trivial interval as in 1.16.(*) yields an explicit formula for $\phi(w)$, for each $w \in I$. Also, as standardizing an isomorphism (as explained in 1.1 and 1.3) does not affect the (simple, strong or total) left regularity of it, all we need to show is the existence of a totally left-regular isomorphism of I. This will follow immediately from the next two lemmas:

Lemma 1.19 *Let I be a Bruhat interval and let J be a left lower translate of I, so there is a translation isomorphism $\theta : J \to I$. If ϕ is a totally left-regular isomorphism of J, then $\phi \circ \theta$ is again a totally left-regular isomorphism of I.*

Lemma 1.20 *Let I be a Bruhat interval and let J be a left multiplication compression of I. If ϕ is a totally left-regular isomorphism of J, then ϕ may be extended to a totally left-regular isomorphism of I.*

Proof of Lemma 1.19 Since $\mathsf{core}(I) = \mathsf{core}(J)$, this is clear by proposition 1.13. Q. E. D.

Proof of Lemma 1.20 There is a generator s such that $J = [x, y], I = [x, sy]$ with $x < sx, y < sy$. There are two main cases, according to whether sx is in $[x, y]$ or not. In the first case, the generator s is "already known" inside J and in the second, we have to add a new generator to our image Coxeter system. We only explain the first case here, because the second is similar and easier.

So suppose $sx \in [x, y]$; then the left mapping λ associated to the left-regular mapping ϕ (as in 1.5) is defined at s. We (naturally) extend ϕ to a mapping defined on $[x, sy]$ as follows:

$$\text{For } w \in I, \phi'(w) = \begin{cases} \phi(w) & \text{if } w \le y \\ \lambda(s)\phi(sw) & \text{if } w \not\le y \end{cases}$$

Since ϕ is a left-regular isomorphism, we have for all $u \in J$ that s is in the left descent set of u if and only if $\lambda(s)$ is in the left descent set of $\phi(u)$. We deduce that ϕ' is an isomorphism between two Bruhat intervals.

By proposition 1.13, it suffices to show that $\mathsf{core}(\phi')$ is totally left-regular. By lemma 1.12, there is a generator s' such that if θ is the translation isomorphism $I \to \mathsf{core}(I)$, then $\theta(sx) = s'\theta(x')$ and $\theta(y) = s'\theta(sy)$. Thus we may replace I with $\mathsf{core}(I)$; in other words, we may assume $\mathsf{core}(I) = I$.

We must show that ϕ' is totally left-regular, i.e. that for any lower interval $[x, z]$ of I, the restriction of ϕ' to $[x, z]$ is strongly left-regular. If $z \leq y$ this follows from the total left-regularity of ϕ, otherwise z is of the form $sz' > z'$ with $z' \leq y$, and replacing (z', z) with (y, sy), all we need to show is that ϕ' is strongly left-regular. But since we have assumed $\mathsf{core}(I) = I$, left-regularity is equivalent to strong left-regularity for ϕ'. So all we need to show is that ϕ' is left-regular.

Left-regularity with respect to s is obvious from the construction. Let us show left-regularity with respect to a generator $t \neq s$. So suppose $x < tx < tz < z$ for some $z \in I$, we must show $\phi'(tz) = \phi'(tx)\phi'(z)$. We claim that $tx \leq y$. Otherwise we could write $tx = sx'$ for some $x' \leq y$ with $x' < sx'$; in particular s would be in the left descent set of tx, and of x also since $t \neq s$, so $x > sx$ which is absurd. So $tx \in [x, y]$, we deduce that ϕ is left-regular with respect to both s and t, and the result follows. Q. E. D.

Conjecture 1.21 *If I is a completely compressible Bruhat interval in type A, D or E, then I is left explicitly completely compressible.*

We have checked conjecture 1.21 up to types A_8, D_7, E_6 with a specialized version of the program `Coxeter` [10].

Using proposition 1.18 and 1.17, conjecture 1.21 implies that (1.3) (and hence (1.2)) holds in types A, D, E.

2 Special cases of the conjecture

First we define a reformulation of 1.21 that is easier to handle.

Definition 2.1 *Let $I = [x, y]$ be a Bruhat interval and let s be a generator in the (left) descent set of y (we then say that s is a* **(left) descent generator** *for I). If $x < sx$ (so that $[x, y]$ can be compressed onto $[x, sy]$) we say that s is a* **(left) compression generator** *for I. If $sx < x$ and $x \not< sy$, (so that there is a translation isomorphism $[x, y] \to [sx, sy]$), we say that s is a* **(left) translation generator** *for I. Otherwise $sx < x$ and $x < sy$; in this last case we say that s is a* **(left) nontrivial descent generator** *for I.*

Clearly, in a fixed group W conjecture 1.21 will hold if and only if every completely compressible interval has a left trivial descent generator.

Definition 2.2 *Let W be a Coxeter group and $w \in W$. We say that $w \in W$ is* **critical** *if $|\mathsf{coat}(w)| = l(w)$.*

Our main result is:

Theorem 2.3 *Let $I = [x, y]$ be a Bruhat interval with y critical. If I has at least a nontrivial generator, then $|coat(I)| > l(I)$ so that I is noncompressible.*

Corollary 2.4 *If $I = [x, y]$ is a completely compressible Bruhat interval with y critical, then all the descent generators of I are trivial.*

Proof of the theorem. Let s be a (left, say) nontrivial descent generator for I. Then we can write $x = sx', y = sy'$ with $x' < x, y' < y$. Since s is nontrivial we have $x < y'$. Let $a = y_1 \dots y_n$ be a reduced expression of y'. We know that are r indices $i_1 < i_2 < \dots < i_r$ (with $r = l(y) - l(x)$) such that if we delete the generators $y_{i_1}, \dots, y_{i_r}$ from a, we obtain a reduced expression for x'. Now put $c_k = sy_1 \dots \widehat{y_{i_k}} \dots y_n$ for k between 1 and r. Since y is critical, c_k is a coatom of y. Then $c_1, c_2, \dots c_r, y'$ are all coatoms of I, which provides at least $r + 1$ coatoms as desired. Q. E. D.

Corollary 2.5 *If $[x, y]$ is a completely compressible interval with y critical, then it is left explicitly completely compressible.*

Indeed, if $s_1 s_2 \dots s_m$ is a reduced expression for y, then there is a sequence of intervals as in 1.16(ii), $I_0 = [x, y], I_1 = [x_1, y_1] \dots I_n = [x_n, y_n]$ with $x_n = y_n$, defined as follows:

$$x_i = \mathsf{min}(x_{i-1}, s_i x_{i-1})$$
$$y_i = s_i y_{i-1} (\text{ so that } y_i = s_{i+1} \dots s_m)$$

This sequence stops at the first n for which $x_n = y_n$.

Note that if $D_l(y) \not\subseteq D_l(x)$, then $[x, y]$ has a left compression generator. In addition, in type A we have that if an element's left descent set is a singleton then it is automatically critical (by proposition II.1.8). We deduce:

Corollary 2.6 *If $I = [x, y]$ is a completely compressible Bruhat interval in type A with $|D_l(y)| \leq 1$ (or even $|D_l(x)| \leq 1$), then I has a left trivial descent generator.*

To conclude, let us review to what extent we anwered our original question:

Proposition 2.7 *Consider the following classes of Bruhat intervals:*

- $\mathcal{F}_1 = \{[x, y] | [x, y]$ *is completely compressible, y is critical* $\}$
- $\mathcal{F}_2 = \{$ *Completely compressible intervals in type* A_8, D_7 *or* $E_6\}$
- $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$

Then for each element I in $\mathcal{F}$ there is a (standard totally left-regular) isomorphism from I onto an interval originating at the identity. Any isomorphism between two elements of $\mathcal{F}$ preserves R-polynomials.

Perhaps the ideas explained here could be improved to provide a result that covers all completely compressible intervals. A better understanding of the counterexample in type B_3 might be a starting point.

Appendix: Explanations on the example in type B_3

The interval $I = [3, 32123]$ consists of the following elements (non-dihedral elements in boldface):

		(12)3		
			(212)3	
	23	(21)3		
			3(12)3	
3	13	3(2)3		**3(212)3**
			3(21)3	
	32	3(12)		
			3(212)	
		3(21)		

We will show that I is completely compressible by showing it is isomorphic to an interval originating at the identity (namely, to the interval $I' = [e, abca]$ in type A_3: $m_{a,b} = m_{b,c} = 3, m_{a,c} = 2$). Indeed, define two "monoid homomorphisms" α and β by

$$\alpha(e) = e, \alpha(1) = b, \alpha(2) = a, \alpha(12) = ba, \alpha(21) = ab, \alpha(212) = aba$$
$$\beta(e) = e, \beta(1) = b, \beta(2) = c, \beta(12) = bc, \beta(21) = cb, \beta(212) = cbc$$

and define a mapping $f : I \to I'$ by $f(w3) = \alpha(w), f(3w) = \beta(w)$ if w is dihedral in 1 and 2, and $f(3(12)3) = bac, f(3(21)3) = acb, f(3(212)3) = abca$. It is readily seen that f is a isomorphism. So I is completely compressible.

Now, if we put $J = [3, 3(1212)3]$, then left multiplication by 1 compresses J into I: Thus J is completely compressible.

Suppose that there is an isomorphism $\phi : J \to [e, y]$ where $[e, y]$ is an interval originating at the identity in some Coxeter system (W', S'). Then $a = \phi(23), b = \phi(13), c = \phi(32)$ are elements of S'. We may suppose that the isomorphism is standard, i.e. that $m_{ab} = m_{bc} = 4, m_{ac} = 2$. Then $\phi((1212)3) = abab, \phi(3(1212)) = bcbc, \phi(232) = ac$.

Let $u \in \{121, 212\}$. We have $\phi(3u) \in \{aba, bab\}$ and symetrically $\phi(u3) \in \{bcb, cbc\}$ whence we deduce $\phi(3u3) \in \{bacb, bcab\}$, so $\phi(3u) = bab, \phi(u3) = bcb$, but this must be true for two distinct values of u, which contradicts the fact that ϕ is one-to-one. Q. E. D.

Word Index

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