Dual mixed finite element method of the elasticity and elastodynamic problems: a priori and a posteriori error analysis.

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Méthode des éléments finis mixte duale pour les problèmes de l'élasticité et de l'élastodynamique: analyse d'erreur à priori et à posteriori

THÈSE

présentée et soutenue publiquement le 10 juillet 2006

pour l'obtention du

Doctorat de l'université de Valenciennes et de Hainaut Cambrésis
(spécialité: Mathématiques Appliquées)

par

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Composition du jury

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Invité :	Philippe Bouillard	Université Libre de Bruxelles

Laboratoire de Mathématiques et ses appliquations de Valenciennes — EA 4015

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Je dédie cette thèse
à mon père.
à ma mère.

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Introduction

It is well-known that the solution of a elliptic problem posed in a polygonal domain Ω of the plane, present singularities in the vicinity of the vertices; let us quote for example the works of Kondratiev, Maz' ya-Plamenevski, Grisvard, Dauge, Stupelis, Kozlov-Maz' ya- Rossmann$\cdots$. These singularities lead to a nonoptimal order of convergence if a finite elements method P^1 or P^2 is used for the Laplace operator or if one uses the finite element method of Hood-Taylor or the mini finite element for the Stokes system.

To cure this disadvantage, various methods have been proposed to restore the optimal order of convergence. Let us quote for example the method of addition of singular functions to the trial space (Strang and Fix, 1973), the method of the dual singular functions (Blum-Doborowolski 1982) and the methods of refinement of grids (Babuska 1970, Raugel 1978, Doborowolski 1982). Lately, a particular interest has been devoted for the method of the a posteriori error estimators. This interest is mainly due to the need to obtain precise numerical results without high calculation effort. Indeed, the a posteriori error analysis avoids the analytical study of singularities and allows to determine explicitly if the computed approximate solution of the exact solution is an approximation with sufficient precision for the needs of the applications. Moreover, the a posteriori error estimates allows to use an adaptive procedure for automatic mesh refinement.

In this work, we study the refinement of grids for the dual mixed finite element method for two types of problems: the first one concerns the linear elasticity problem and the second one the linear elastodynamic problem. A number of reasons have been put forth to prefer mixed methods over classical ones [3]. A commonly stated reason to prefer mixed methods is that the dual variable (stress -strain in elasticity, flux for thermal problems ...) is often the variable of most interest. For the classical methods this variable is not a fundamental unknown and must be obtained a posteriori by differentiation, which entails a loss of accuracy.

Mixed methods are also a valid alternative to locking phenomena, as it is well known that for nearly incompressible materials, i.e. for values of the Lamé coefficient λ near to infinity $(\lambda \to \infty)$, finite element computations based on standard displacement formulation fail.

For these two types of problems and in nonregular domains, the mixed finite element methods analyzed until present relate to the primal mixed methods. Here, we analyze the dual mixed formulation in $\big((\sigma := 2\mu\varepsilon(u), p := -\lambda div\ u), (u, \omega := \frac{1}{2} curl\ u)\big)$, where the equations of the linear elasticity problem are rewritten in the following way:

$$\begin{cases} -div\ (\sigma - p\delta) & = & f & \text{in} & \Omega, \\ \sigma & = & 2\mu\varepsilon(u) & \text{in} & \Omega, \\ p & = & -\lambda div\ u & \text{in} & \Omega, \\ \omega & = & \frac{1}{2}curl\ u & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \Gamma_D, \\ (\sigma - p\delta).n & = & 0 & \text{on} & \Gamma_N, \end{cases}$$

and in $\big((\sigma(t,.) := 2\mu\varepsilon(u(t,.)), p(t,.) := -\lambda div\ u(t,.)), (u(t,.), \omega(t,.) := \frac{1}{2}curl\ u(t,.))\big)$, where the equations of the linear elastodynamic problem are rewritten in the following way:

$$\begin{cases} u_{tt} - div\ (\sigma(t,.) - p(t,.)\delta) & = & f & \text{in} & [0,T]\times\Omega, \\ \sigma(t,.) & = & 2\mu\varepsilon(u(t,.)) & \text{in} & [0,T]\times\Omega, \\ p(t,.) & = & -\lambda div\ u(t,.) & \text{in} & [0,T]\times\Omega, \\ \omega(t,.) & = & \frac{1}{2}curl\ u(t,.) & \text{in} & [0,T]\times\Omega, \\ u(t,.) & = & 0 & \text{on} & [0,T]\times\Gamma_D, \\ (\sigma(t,.) - p(t,.)\delta).n & = & 0 & \text{on} & [0,T]\times\Gamma_N, \\ u(0,.) & = & u_0 & & \text{in} \quad \Omega, \\ u_t(0,.) & = & u_1 & & \text{in} \quad \Omega, \end{cases}$$

where, n means the normal vector along Γ oriented outside Ω. These two methods are justified primarily by the use of the stabilized BDM_1 element for approximating the strain tensor σ in the case of the linear elasticity problem and for approximating the strain tensor $\sigma(t,.)$ in the case of the linear elastodynamic problem.

The main difficulty appearing in these problems is finding a way to take into account the symmetry of the strain tensor. In our approach, the symmetry of of the strain tensor is relaxed by a Lagrange multiplier, which is noting else than the rotation ω in the case of the linear elasticity problem or $\omega(t,.)$ in the case of the linear elastodynamic problem.

For the elasticity problem, we are concerned firstly by an a priori error analysis when using finite element approximation by stabilized BDM_1 element. In order to derive optimal a priori error estimates, we lay down refinement rules on the regular family of triangulations which will be used to locally refine the triangulation and consequently, derive some optimal interpolation error estimates which will be used to derive optimal and global error estimates.

Then, we make an a posteriori error analysis for the dual mixed finite element method for both a simply and a multiply-connected domain. In fact we establish a residue based reliable and efficient error estimator for the dual mixed finite element method. This estimator is then used in an adaptive algorithm for automatic mesh refinement. Consequently, we refine locally and with precision the areas on which the error is significant, so restoring the optimal order of convergence with lower cost of calculations.

For the elastodynamic problem, we make an a priori error analysis when using the same finite element as for the elasticity problem, using a dual mixed formulation for the discretization in the spatial variables and the explicit or implicit Newmark scheme for the discretization in time. By adequate refinement rules on the regular family of triangulations we derive optimal a priori error estimates.

In this context, this thesis is made up of three chapters whose contents can be summarized as follows:

Chapter1
In this chapter, we introduce the considered linear elasticity problem and give the dual-mixed formulation. We then show the equivalence between this formulation and the standard one and establish an inf-sup condition and a coerciveness result uniform with respect to Lamé coefficient λ. We then discretize the mixed variational formulation by conforming finite elements based on regular family of triangulations and show again a uniform inf-sup condition and a coerciveness result uniform with respect to Lamé coefficient λ. By studying some regularity results of the solution of our elasticity problem in terms of weighted Sobolev spaces, we deduce the optimal order of convergence for a regular family of triangulations satisfying adequate refinement rules. Finally, we present some numerical tests which confirm our theoretical analysis.

Chapter2
In this chapter, we make the a posteriori error analysis for the considered linear elasticity problem. We begin by establishing, in the case of a simply connected domain, some results on tensor fields like some particular Helmholtz decomposition and a generalization of the results of [8] concerninig the estimation of the trace of tensor fields. Then we recall some standard tools, namely some inverse inequalities and interpolation error estimates for Clément's interpolant and finish by establishing the efficiency and reliability of our error indicator η. Then we treat the case of a multiply-connected domain by using an adapted Helmholtz decomposition of tensor fields. To our knowledge this decomposition seems to be new. Finally, we establish appropriate adaptive mesh-refinement algorithms and present some numerical tests which confirm our theoretical analysis.

Chapter3
In this chapter we establish optimal a priori error estimates for the linear elastodynamic problem. We begin by presenting the model evolution problem and recall two comparison results concerning continuous and discrete Grounwall's inequalities. We define the new dual mixed formulation of the elastodynamic problem. We give some regularity results of the solution of our elastodynamic system in terms of weighted Sobolev spaces. Then we introduce the semi-discrete mixed formulation and prove the existence and uniqueness of the solution. We then recall some results concerning the inf-sup and coercivety conditions. Under some adequate refinement rules of grids, we establish some error estimates on some interpolation operators and we prove an inverse inequality for the divergence operator. By establishing some error estimates between the exact solution of the mixed problem and the solution of the elliptic projection problem we derive the error estimates between the exact and the semi-discrete solution. We then discretize completely the mixed finite element problem. By the energy estimates we study the stability of both our explicit and implicit scheme in time , and give an appropriate CFL condition for the explicit scheme. We then establish optimal error estimates for the fully discrete problem. Finally , we give numerical experiments to confirm our theoretical predictions.

1

Dual MFE for the elasticity problem with mixed boundary conditions

Contents

1.1 Introduction

A number of reasons have been put forth to prefer mixed methods over classical ones [3], a commonly stated reason to prefer mixed methods is that the dual variable (stress - strain in elasticity, flux for thermal problems ...) is often the variable of most interest. For the classical methods this variable is not a fundamental unknown and must be obtained a posteriori by differentiation, which entails a loss of accuracy.

Mixed methods are also a valid alternative to locking phenomenon, as it is well known that for nearly incompressible materials, i.e. for a value of the Lamé coefficient near to infinity ($\lambda \to \infty$), finite element computations based on standard displacement formulation fail.

Over the last two decades there has been considerable interest in the area of mixed finite element discretizations of the system of linear elasticity; let us quote [24, 8, 22, 23]. The main difficulty appearing in this problem is finding a way to take into account the symmetry of the stress tensor. In our approach, the symmetry of of the stress tensor is relaxed by a Lagrange multiplier, which is noting else than the rotation.

The goal of this chapter is mainly to establish an optimal rate of convergence of the numerical solution of the new dual mixed finite element method of the elasticity problem in a polygonal domain, introduced by M. Farhloul and M. Fortin in [22] elasticity problem with mixed boundary conditions in polygonal domains. The results shown during this chapter are generalization of the results already obtained in the case of homogeneous Direchlet boundary condition (see [24]).

The schedule of this chapter is the following one: In section 1.2 we introduce the considered boundary value problem and give the dual-mixed formulation. We then show the equivalence between this formulation and the standard one. We end this section by establishing an inf-sup condition and a coerciveness result uniform with respect to Lamé coefficient λ. Section 1.3 is devoted to some regularity results of the solution of our elasticity problem in terms of weighted Sobolev spaces. In section 1.4, one discretized the mixed variational formulation by conforming finite element based one regular family of triangulations and show again a uniform inf-sup condition and a coerciveness result uniform with respect to Lamé coefficient λ. In section 1.5, with an aim of restoring optimal order of convergence of our method we locally refine the triangulation and we show some interpolation error estimates. In section 1.6 we present conclusions. In section 1.7, we present some numerical tests which confirm our theoretical analysis.

1.2 The new dual mixed variational formulation

This section introduces the dual mixed formulation of the elasticity boundary value problem and gives the equivalence between this formulation and the standard one. The existence and unicity of the solution is proved by establishing an inf-sup condition and a coerciveness result uniform with respect to the Lamé coefficient λ. Let us fix a bounded plane domain Ω with a polygonal boundary. More precisely, we assume that Ω is a simply connected domain and that its boundary Γ is the union of a finite number of linear segments $\bar{\Gamma}_j$, $1 \leq j \leq n_e$ (Γ_j is assumed to be an open segment). We also fix a partition of $1, 2, \cdots, n_e$ into two subsets $I_{\mathcal{N}}$ and $I_{\mathcal{D}}$. The union Γ_D of the Γ_j, j running over $I_{\mathcal{D}}$, is the part of the boundary Γ, where we assume zero displacement field. We assume that $meas(\Gamma_D) > 0$. The union Γ_N, of the Γ_j, $j \in I_{\mathcal{N}}$ is the part of the boundary Γ where we assume zero traction field.

In this domain we consider anisotropic elastic homogeneous material. Let $u = (u_1, u_2)$ be the displacement field and $f = (f_1, f_2) \in [L^2(\Omega)]^2$ the body force by unit of mass. Thus the displacement field $u = (u_1, u_2)$ satisfies the following equations and boundary conditions:

$$\left\{ \begin{array}{rcll} -div\, \sigma_s(u) & = & f & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \Gamma_D, \\ \sigma_s(u).n & = & 0 & \text{on} \quad \Gamma_N. \end{array} \right. \tag{1.2.1}$$

where the stress tensor $\sigma_s(u)$ is defined by

$$\sigma_s(u) \ := \ 2\mu\epsilon\,(u) \ + \ \lambda tr\ \epsilon(u)\delta. \qquad (1.2.2)$$

The positive constants μ and λ are called Lamé coefficients. We assume that [7]

$$(\lambda,\mu) \ \in \ [\lambda_0,+\infty[\ \times \ [\mu_1,\mu_2] \qquad (1.2.3)$$

where

$$0 \ < \ \mu_1 \ < \ \mu_2 \text{ and } \lambda_0 > 0.$$

As usual, $\epsilon(u)$ denotes the linearized strain tensor ($i.e., \epsilon(u) \ = \ \frac{1}{2}(\nabla u + \nabla u^T)$) and δ the identity tensor. The classical variational formulation of the boundary value problem (1.2.1) is the following [7, 11]:

$$\text{find } \ u \in [H^1_{\Gamma_D}(\Omega)]^2 := \{v \in [H^1(\Omega)]^2, \quad v_{|\Gamma_D} = 0\} \ \text{ such that :}$$

$$\int_\Omega (2\mu\epsilon(u) : \epsilon(v) + \lambda tr\ \epsilon(u) tr\ \epsilon(v))\ dx = \int_\Omega f.v\ dx, \quad \forall v \in [H^1_{\Gamma_D}(\Omega)]^2. \qquad (1.2.4)$$

Due to our hypothesis $meas(\Gamma_D) > 0$ and Korn's inequality (cf. [7], corollary 11.2.22 p.285) , equation (1.2.4) possesses one and only one solution for every $f \in [H^1_{\Gamma_D}(\Omega)]^*$.

Introducing as new unknowns:

$$\sigma := 2\mu\epsilon(u), \ p := -\lambda div\ (u) \ \text{ and } \ \omega := \frac{1}{2} rot\ (u) := \frac{1}{2}(\frac{\partial u_2}{\partial x_1} \ - \ \frac{\partial u_1}{\partial x_2})$$

and the spaces:

$$\Sigma_0 := \Big\{(\tau,q) \ \in \ [L^2(\Omega)]^{2\times 2} \times [L^2(\Omega)]; \ div\ (\tau - q\delta) \in [L^2(\Omega)]^2, \\ (\tau - q\delta).n \ = \ 0 \text{ on } \Gamma_N\Big\} \qquad (1.2.5)$$

$$V \times W \ := \ \Big\{(\mathrm{v},\theta) \in \ [L^2(\Omega)]^2 \times L^2(\Omega)\Big\}, \qquad (1.2.6)$$

we state the dual mixed formulation [22], [23], [24] :
Find $(\sigma,p) \in \Sigma_0$ and $(u,\omega) \in V \times W$ such that :

$$\left\{ \begin{array}{l} \frac{1}{2\mu}(\sigma,\tau) \ + \ \frac{1}{\lambda}(p,q) + \ (div\ (\tau - q\delta), u) \ + \ (as\ (\tau),\omega) \ = \ 0, \quad \forall\ (\tau,q) \ \in \ \Sigma_0 \\ \\ (div\ (\sigma - p\delta), v) \ + \ (as\ (\sigma),\theta) \ + \ (f,v) \ = \ 0, \quad \forall\ (v,\theta) \ \in \ V \times W. \end{array} \right. \qquad (1.2.7)$$

Here, for a tensor τ, $(.,.)$ means L^2-scalar product, and $(\sigma,\tau) \ = \ \int_\Omega \sigma : \tau\ dx$ where $\sigma : \tau$ means the standard notation for the contraction of two tensors

$$\sigma : \tau \ = \ \Sigma^2_{i,j=1}\sigma_{ij}\tau_{ij}.$$

In the present approach, the symmetry of the strain tensor σ is relaxed by a Lagrange multiplier which is nothing else than the rotation ω. Problem (1.2.7) will be approximated by conforming finite element spaces $\Sigma_{0,h} \times V_h \times W_h$ of $\Sigma_0 \times V \times W$ based on a triangulation $\mathcal{T}_h$ of the domain Ω from a regular family (regular in Ciarlet's sense [11]).

Finally, let us precise some notations that will be used subsequently. For any tensor field $\tau \in [H^1(\Omega)]^{2\times 2}$, for any vector field $v = (v_1, v_2) \in [H^1(\Omega)]^2$ and for any scalar function $\phi \in H^1(\Omega)$, we define :

$$
\begin{aligned}
tr\ (\tau) &:= \tau_{11} + \tau_{22}, \\
div\ (\tau) &:= \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2}, \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2}\right)^T, \\
as\ (\tau) &:= \tau_{21} - \tau_{12}, \\
Sym\ (\tau) &:= \frac{1}{2}(\tau + \tau^T), \\
rot\ (\tau) &:= \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2}\right)^T, \\
Curl\ (v) &:= \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix}, \\
Curl\ (\phi) &:= \left(\frac{\partial \phi}{\partial x_2} \quad -\frac{\partial \phi}{\partial x_1}\right)^T, \\
rot\ (v) &:= \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.
\end{aligned}
$$

Before proving the equivalence between the classical variational formulation (1.2.4) with the mixed formulation (1.2.7), let us recall the following formula of Green formula (see [21]):

Lemma 1.2.1. *Let* $\tau \in [H(div\ ,\Omega)]^2 := \left\{\tau \in [L^2(\Omega)]^{2\times 2}; div\ \tau \in [L^2(\Omega)]^2\right\}$ *and* $v \in [H^1(\Omega)]^2$*. Then*

$$(\epsilon(v), \tau) =< \tau n, v > -(div\ \tau, v) - \frac{1}{2}(as\ (\tau), curl\ v),$$

where $\tau n = (\tau_{11}n_1 + \tau_{12}n_2, \tau_{21}n_1 + \tau_{22}n_2)$.

For shortness we often write the pairs (σ, p), $(\tau, q) \in \Sigma_0$ by $\underset{\sim}{\sigma} = (\sigma, p)$, $\underset{\sim}{\tau} = (\tau, q)$ and similarly the pairs (u, ω), $(v, \theta) \in V \times W$ by $\underset{\sim}{u} = (u, \omega)$, $\underset{\sim}{v} = (v, \theta)$. We introduce two bilinear forms $a : \Sigma_0 \times \Sigma_0 \to \mathbb{R}$, $b : \Sigma_0 \times (V \times W) \to \mathbb{R}$ and the linear form $\mathcal{F} : V \times W \to \mathbb{R}$ defined as follows

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) := \frac{1}{2\mu}(\sigma, \tau) + \frac{1}{\lambda}(p, q), \quad \forall \underset{\sim}{\sigma},\ \underset{\sim}{\tau} \in \Sigma_0, \tag{1.2.8}$$

$$b(\underset{\sim}{\tau}, \underset{\sim}{v}) := (div\ (\tau - q\delta), v) + (as\ (\tau), \theta), \quad \forall \underset{\sim}{\tau} \in \Sigma_0, \quad \forall \underset{\sim}{v} \in V \times W, \tag{1.2.9}$$

$$\mathcal{F}(\underset{\sim}{v}) := (f, v).$$

With these notations the mixed variational formulation of problem (1.2.7) may be rewritten: Find $\underset{\sim}{\sigma} \in \Sigma_0$ and $\underset{\sim}{u} \in V \times W$ such that

$$\left\{ \begin{array}{l} a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) + b(\underset{\sim}{\tau}, \underset{\sim}{u}) = 0, \quad \forall \underset{\sim}{\tau} \in \Sigma_0, \\ \\ b(\underset{\sim}{\sigma}, \underset{\sim}{v}) + \mathcal{F}(\underset{\sim}{v}) = 0, \quad \forall \underset{\sim}{v} \in V \times W \end{array} \right. \tag{1.2.10}$$

Let us first show the following equivalent result between the classical variational formulation (1.2.4) with the mixed formulation (1.2.7).

Theorem 1.2.2. *$u \in [H^1_{\Gamma_D}(\Omega)]^2$ is solution of (1.2.4) if and only if $\big((\sigma, p), (u, \omega)\big) \in \Sigma_0 \times (V \times W)$ is solution of (1.2.7), where*

$$\sigma = 2\mu\epsilon(u), \ p = -\lambda div\ u, \ \omega = \frac{1}{2} curl\ u$$

Proof: The proof is essentially the same as the proof of theorem 3.2 in [24]. Only some small adaptation is necessary due to the boundary condition in the definition of the space Σ_0 (see 1.2.5) which was not present in the definition of Σ (see (5) in [24]) ■

The previous result guaranties in particular the well posedness of problem (1.2.7). But for further purposes, we need that the so-called inf-sup condition holds for $b(.,.)$, as well as uniform coerciveness for$a(.,.)$ with respect to the Lamé coefficient λ on the kernel of $b(.,.)$ (see Proposition 1.2.4). Before giving the result about the uniqueness of the solution of (1.2.7), let us first check the inf-sup condition.

Proposition 1.2.3. *There exists a positive constant C such that*

$$\sup_{\underset{\sim}{\tau}=(\tau,q)\in \ \Sigma_0} \frac{b(\underset{\sim}{\tau}, \underset{\sim}{v})}{\|\underset{\sim}{\tau}\|_{0,\Omega}} \geq C\big(\|v\|_{0,\Omega} + \|\theta\|_{0,\Omega}\big), \quad \forall \underset{\sim}{v} = (v, \theta) \in V \times W, \tag{1.2.11}$$

where

$$\|\underset{\sim}{\tau}\|^2_{0,\Omega} = \|\tau\|^2_{0,\Omega} + \|q\|^2_{0,\Omega}.$$

Proof: The main idea of the constitutive proof is the following: given an arbitrary element $\underset{\sim}{v} = (v, \theta) \in V \times W$, one may construct an element $\underset{\sim}{\tau} = (\tau, 0) \in \Sigma_0$ such that

$$b(\underset{\sim}{\tau}, \underset{\sim}{v}) = \|\underset{\sim}{v}\|^2_{0,\Omega} \quad \text{and } \|\underset{\sim}{\tau}\|_{0,\Omega} \lesssim \|\underset{\sim}{v}\|_{0,\Omega}.$$

In the whole proof we fix λ^* and μ^* independently of λ, μ and of $\underset{\sim}{v}$

i)Let $\omega \in [H^1_{\Gamma_D}(\Omega)]^2$ be the unique solution of the problem:

$$\left\{ \begin{array}{rcll} div\ (2\mu^*\epsilon(\omega) + \lambda^* div\ \omega\delta) & = & v & \text{in } \Omega, \\ \omega & = & 0 & \text{on } \Gamma_D, \\ (2\mu^*\epsilon(\omega) + \lambda^* div\ \omega\delta).n & = & 0 & \text{on } \Gamma_N. \end{array} \right.$$

By Korn's inequality, ω satisfies (the constant below depends on λ^*, μ^* and on Ω)

$$|\omega|_{1,\Omega} \lesssim \|v\|_{0,\Omega}. \tag{1.2.12}$$

Setting $\tau^1 = 2\mu^*\epsilon(\omega) + \lambda^* div\, \omega\delta$ and $\underset{\sim}{\tau}^1 = (\tau^1, 0)$, it's clear that $\underset{\sim}{\tau}^1$ belongs to Σ_0 and is symmetric (i.e. $as\ (\tau^1) = 0$). Consequently

$$\begin{aligned} b(\underset{\sim}{\tau}^1, \underset{\sim}{v}) &= (div\ \tau^1, v) + (as\ (\tau^1), \theta) \\ &= (div\ \tau^1, v). \end{aligned}$$

And by the above problem solved by ω, we get

$$b(\underset{\sim}{\tau}^1, \underset{\sim}{v}) = \|v\|_{0,\Omega}^2. \tag{1.2.13}$$

The definition of $\underset{\sim}{\tau}^1$, and the inequality (1.2.12) yields

$$\|\underset{\sim}{\tau}^1\|_{0,\Omega} \lesssim \|v\|_{0,\Omega}. \tag{1.2.14}$$

ii) Since $D(\Omega)$ is dense in $L^2(\Omega)$, there exists a sequence $(\theta_k)_{k\geq 0} \in D(\Omega)$ such that

$$\theta_k \to \theta \text{ in } L^2(\Omega), \text{ as } k \to \infty.$$

Now, for each $k \in \mathbb{N}$, consider $\omega_k \in [H^1_{\Gamma_D}(\Omega)]^2$ the unique solution of the problem:

$$\left\{ \begin{array}{rcll} div\ (2\mu^*\epsilon(\omega_k) + \lambda^* div\ \omega_k\delta) &=& \frac{1}{2} Curl\ \theta_k & \text{in} \quad \Omega, \\ \omega_k &=& 0 & \text{on} \quad \Gamma_D, \\ (2\mu^*\epsilon(\omega_k) + \lambda^* div\ \omega_k\delta).n &=& 0 & \text{on} \quad \Gamma_N. \end{array} \right.$$

The variational formulation of this problem is:

$$\int_\Omega (2\mu^*\epsilon(\omega_k) : \epsilon(v) + \lambda^* tr\ \epsilon(\omega_k) tr\ \epsilon(v))\ dx = -\frac{1}{2}\int_\Omega \theta_k rot\ v\ dx, \quad \forall v \in [H^1_{\Gamma_D}(\Omega)]^2.$$

Consequently by Korn's inequality we have

$$|\omega_k|_{1,\Omega} \lesssim \|\theta_k\|_{0,\Omega}. \tag{1.2.15}$$

Then we set

$$\tau_k^2 = 2\mu^*\epsilon(\omega_k) + \lambda^* div\ \omega_k\delta + \frac{1}{2}\theta_k\chi,$$

where χ is the antisymmetric matrix defined by:

$$\chi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

By the above problem we remark that

$$\begin{aligned} div\ \tau_k^2 = 0 \quad &\text{in } \Omega, \\ as\ (\tau_k^2) = \theta_k \quad &\text{in } \Omega, \\ \tau_k^2.n = 0 \quad &\text{on } \Gamma_N. \end{aligned}$$

Moreover from (1.2.15), we clearly have

$$\|\tau_k^2 - \tau_l^2\|_{0,\Omega} \lesssim \|\theta_k - \theta_l\|_{0,\Omega}, \quad \forall k, l \in \mathbb{N}.$$

This means that the sequence $(\tau_k^2)_{k\geq 0}$ is a Cauchy sequence in $H(div\,,\Omega)^2$. Denote by τ^2 its limit. By the above properties of τ_k^2, τ^2 satisfies

$$\begin{aligned} div\,\tau^2 &= 0 \quad \text{in } \Omega, \\ as\,(\tau^2) &= \theta \quad \text{in } \Omega, \\ \tau^2.n &= 0 \quad \text{on } \Gamma_N, \\ \|\tau^2\|_{0,\Omega} &\lesssim \|\theta\|_{0,\Omega}. \end{aligned}$$

Now setting $\underset{\sim}{\tau}^2 = (\tau^2, 0)$, it's clear that $\underset{\sim}{\tau}^2$ belongs to Σ_0 and we then have

$$\begin{aligned} b(\underset{\sim}{\tau}^2, \underset{\sim}{v}) &= (div\,\tau^2, v) + (as\,(\tau^2), \theta) \\ &= (\theta, \theta) = \|\theta\|_{0,\Omega}^2, \end{aligned} \tag{1.2.16}$$

as well as

$$\|\underset{\sim}{\tau}^2\|_{0,\Omega} = \|\tau^2\|_{0,\Omega} \lesssim \|\theta\|_{0,\Omega}. \tag{1.2.17}$$

iii) The two preceding points suggest to set

$$\tau = \tau^1 + \tau^2.$$

Indeed the identities (1.2.13) and (1.2.16) leads to

$$b(\underset{\sim}{\tau}, \underset{\sim}{v}) = \|\underset{\sim}{v}\|_{0,\Omega}^2,$$

while the estimates (1.2.14) and (1.2.17) show that

$$\|\underset{\sim}{\tau}\|_{0,\Omega} \lesssim \|\underset{\sim}{v}\|_{0,\Omega}$$

Therefore we may conclude that

$$\sup_{\underset{\sim}{\tau}=(\tau,q)\in\,\Sigma_0} \frac{b(\underset{\sim}{\tau}, \underset{\sim}{v})}{\|\underset{\sim}{\tau}\|_{0,\Omega}} \gtrsim \|\underset{\sim}{v}\|_{0,\Omega},$$

which means that the bilinear form b satisfies the inf-sup condition. ■

It remains to prove the uniform coerciveness of the bilinear form a on the kernel K of the bilinear form b in Σ_0 defined by

$$K := \left\{\underset{\sim}{\tau} = (\tau, q) \in \Sigma_0;\ b(\underset{\sim}{\tau}, \underset{\sim}{v}) = 0, \quad \forall\ \underset{\sim}{v} = (v, \theta) \in V \times W\right\},$$

Proposition 1.2.4. *The bilinear form a is coercive uniformly with respect to λ on K in other words*

$$a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) \geq c\|\underset{\sim}{\tau}\|^2_{0,\Omega}, \quad \forall \underset{\sim}{\tau} \in K,$$

where c is a strictly positive constant independent of λ.

Proof: Let us consider $\underset{\sim}{\tau} = (\tau, q) \in K$. In particular

$$\begin{aligned} div\ (\tau - q\delta) &= 0, \\ as\ (\tau) &= 0. \end{aligned} \tag{1.2.18}$$

By Lemma 3.3 of [8], it follows that

$$\|tr\ (\tau - q\delta)\|_{0,\Omega} \lesssim \|(\tau - q\delta)^D\|_{0,\Omega},$$

where we recall that $\tau^D = \tau - \frac{1}{2} tr\ (\tau)\delta$ denotes the deviatoric of τ. Now

$$\begin{aligned} (\tau - q\delta)^D &= \tau - q\delta - \frac{1}{2}(tr\ (\tau)\delta - 2q\delta) \\ &= \tau^D. \end{aligned}$$

Combined with the previous inequality, this gives us :

$$\|tr\ (\tau - q\delta)\|_{0,\Omega} \lesssim \|\tau^D\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}.$$

By the triangle inequality we get

$$\|q\|_{0,\Omega} \lesssim \|q\delta\|_{0,\Omega} \lesssim \|\tau - q\delta\|_{0,\Omega} + \|\tau\|_{0,\Omega},$$

and by the above estimate there exists a positive constant C depending only on Ω such that

$$\|q\|_{0,\Omega} \leq C\|\tau\|_{0,\Omega}. \tag{1.2.19}$$

This inequality implies that for $\underset{\sim}{\tau} = (\tau, q) \in K$ we have:

$$\begin{aligned} a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) &= \frac{1}{2\mu}\|\tau\|^2_{0,\Omega} + \frac{1}{\lambda}\|q\|^2_{0,\Omega} \\ &\geq \frac{1}{2\mu}\|\tau\|^2_{0,\Omega} \geq \frac{1}{2\mu_2}\|\tau\|^2_{0,\Omega} \\ &\geq \frac{1}{4\mu_2}\|\tau\|^2_{0,\Omega} + \frac{1}{4\mu_2}\|\tau\|^2_{0,\Omega}. \end{aligned}$$

By using the estimate (1.2.19), we conclude that

$$\begin{aligned} a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) &\geq \frac{1}{4\mu_2}\|\tau\|^2_{0,\Omega} + \frac{1}{4C^2\mu_2}\|q\|^2_{0,\Omega} \\ &\geq C(\mu_2)(\|\tau\|^2_{0,\Omega} + \|\tau\|^2_{0,\Omega}). \end{aligned}$$

Therefore the coerciveness of the bilinear form a in K holds uniformly in λ. ■

The inf-sup condition and the coerciveness being satisfied, a straightforward application of Theorem II.1 of [8] leads to

Theorem 1.2.5. *There exists a unique solution* $(\underset{\sim}{\sigma}, \underset{\sim}{u}) \in \Sigma_0 \times (V \times W)$ *of the mixed variational formulation (1.2.7) such that*

$$\|\underset{\sim}{\sigma}\|_{0,\Omega} + \|\underset{\sim}{u}\|_{0,\Omega} \lesssim (1 + \frac{1}{\lambda})^2 \|f\|_{0,\Omega}.$$

1.3 Regularity of the solutions

It is well-known (see [25] or [27, 28, 14]) that the solution of problem (1.2.1) presents vertex singularities. To describe them, we need to introduce the following notations:

Definition 1.3.1. *Let* S_j $(1 \leq j \leq n_e)$ *the vertex of our polygonal domain* Ω*, at intersection of sides* Γ_j *and* Γ_{j+1} $(\Gamma_{n_e+1} := \Gamma_1)$*. Les us denote by* ω_j *the interior opening of* Ω *at* S_j *and by* (r_j, θ_j) *the polar coordinates centered at the vertex* S_j*. By the characteristic equation associated to the vertex* S_j*, we mean the transcendental equation in the complex variable* α*:*

$$\sin^2(\alpha\omega_j) = \left[\frac{\lambda + \mu}{\lambda + 3\mu}\right]^2 \alpha^2 \sin^2 \omega_j, \tag{1.3.1}$$

if S_j *is a vertex of Dirichlet type i.e.* $j,\ j+1 \in I_{\mathcal{D}}$

$$\sin^2(\alpha\omega_j) = \alpha^2 \sin^2 \omega_j, \tag{1.3.2}$$

if S_j *is a vertex of Neumann type i.e.* $j,\ j+1 \in I_{\mathcal{N}}$

$$\sin^2(\alpha\omega_j) = \frac{(\lambda + 2\mu)^2 - (\lambda + \mu)^2 \alpha^2 \sin^2 \omega_j}{(\lambda + \mu)(\lambda + 3\mu)}, \tag{1.3.3}$$

if S_j *is a vertex of mixed type i.e.* $j \in I_{\mathcal{D}},\ j+1 \in I_{\mathcal{N}}$ *or* $j \in I_{\mathcal{N}},\ j+1 \in I_{\mathcal{D}}$*.*

Let us recall the following theorem (see [25],P.53):

Theorem 1.3.2. *Let us suppose that* $f \in [H^m(\Omega)]^2$ $(m = 0, 1)$ *and for every* j $(1 \leq j \leq n_e)$ *the characteristic equations (1.3.1 - 1.3.3) has no root on the vertical line* $Re\alpha = m+1$ *in the complex plane. Then the weak solution* u *of the problem (1.2.1) admits the following decomposition:*

$$u = u_R + \sum_{1 \leq j \leq n_e} \sum_{0 < Re\alpha < m+1} r_j^\alpha \sum_{0 < k < \nu(\alpha) - 1} c_{j,\alpha,k} (\ln r_j)^k \varphi_{j,\alpha,k}(\theta_j), \tag{1.3.4}$$

where u_R *belongs to* $[H^2(\Omega)]^2$ *is the regular part of* u*,* $c_{j,\alpha,k} \in \mathbb{C}$ *is so-called coefficient of singularity and* $\varphi_{j,\alpha,k}$ *is a smooth function (explicitly known, cf. [27]).*

The above decomposition allows to show that u belongs to appropriate weighted Sobolev spaces that we next define.

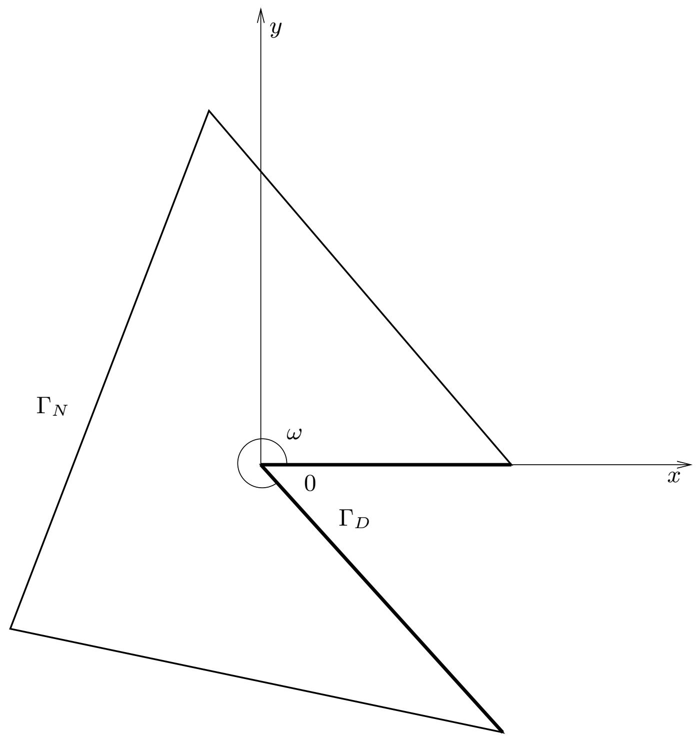


Figure 1.1: Polygonal domains with singularity

Definition 1.3.3. *Let $\phi \in C^0(\bar{\Omega})$ such that $\phi(x) > 0$ for every $x \in \bar{\Omega} \backslash \{S_1, S_2, \cdots, S_{ne}\}$, and any $(m,k) \in I\!N^2$, we define*

$$H_\phi^{m,k}(\Omega) = \big\{ v \in H^m(\Omega); \quad \phi D^\beta v \in L^2(\Omega), \ \forall \beta \in I\!N^2 \ \ \textit{such that} \ \ m < |\beta| \leq m+k \big\}.$$

$H_\phi^{m,k}(\Omega)$ is a Hilbert space equipped with the norm:

$$||v||_{m,k;\phi,\Omega} = \big(||v||^2_{m,\Omega} + \sum_{m<|\beta|\leq m+k} ||\phi D^\beta v|^2_{0,\Omega} \big)^{1/2}.$$

On this space, we also define the semi-norm:

$$|v|_{m,k;\phi,\Omega} = \big(\sum_{|\beta|=m+k} ||\phi D^\beta v||^2_{0,\Omega} \big)^{1/2}.$$

Let us pick some $\alpha_j \in]m+1-\xi_j, m+1/2[$ if $m+1-\xi_j \geq 0$, and let us take $\alpha_j = 0$ if $m+1-\xi_j < 0$, where

$\xi_j = \inf_k \{Re\, \alpha_{j,k};\ \alpha_{j,k}$ is solution of trancendantal equations (1.3.1 - 1.3.3) and $Re\, \alpha_{j,k} > 0\}$.

Then we have the following corollary of Theorem 1.3.2.

Corollary 1.3.4. *Let us suppose that $f \in [H^m(\Omega)]^2$ $(m = 0, 1)$ and for every j $(1 \leq j \leq n_e)$ the characteristic equations (1.3.1 - 1.3.3) has no root on the vertical line $Re\alpha = m+1$*

in the complex plan. Let $\phi \in C^0(\bar{\Omega})$ *such that* $\phi(x) > 0$ *for every* $x \in \bar{\Omega}\backslash\{S_1, S_2, \cdots, S_{ne}\}$ *and such that* $\phi(x) = r_j(x)^{\alpha_j}$ *in a neighborhood of the vertex* S_j *of the polygonal domain* Ω *for every* $j = 1, \cdots, n_e$ *where* $r_j(x) = |x - S_j|$ ($|.|$ *means Euclidian norm). Then* $u \in [H_\phi^{1,m+1}(\Omega)]^2$.
Consequently $\sigma \in [H_\phi^{0,m+1}(\Omega)]^{2\times 2}$, $p \in H_\phi^{0,m+1}(\Omega)$ *and* $\omega \in H_\phi^{0,m+1}(\Omega)$.

For further purposes, we need to give a meaning to the traces of functions in $H_\phi^{0,m+1}(\Omega)$, namely we show the following Lemma:

Lemma 1.3.5. *Let* ϕ *be a function like in Corollary 1.3.4. If* $\omega \in H_\phi^{0,m+1}(\Omega)$, *then for all* $K \in \mathcal{T}_h$, *it holds*

$$\omega|_E \in L^1(E), \quad \forall E \in \partial K.$$

Proof: If $\omega \in H_\phi^{0,1}(\Omega)$, then by Hölder's inequality using the fact that $\alpha_j < \frac{1}{2}$ ($j = 1, 2\cdots, n_e$) it follows that there exists some $\delta > 0$ such that $\omega \in W^{1,\frac{4}{3}+\delta}(\Omega)$. But by Sobolev embedding Theorems $W^{1,\frac{4}{3}+\delta}(\Omega) \hookrightarrow H^{\frac{1}{2}+\epsilon}(\Omega)$ for $\epsilon > 0$ sufficiently small. Thus $H_\phi^{0,1}(\Omega) \hookrightarrow H^{\frac{1}{2}+\epsilon}(\Omega)$ for $\epsilon > 0$ sufficiently small. Now if $\omega \in H_\phi^{0,2}(\Omega)$, then by Hardy's inequality (see [26], P. 28) $\omega \in H_\psi^{0,1}(\Omega)$, where $\psi \in C^0(\bar{\Omega})$, $\psi(x) > 0$ for every $x \in \bar{\Omega}\backslash\{S_1, S_2, \cdots, S_{n_e}\}$ and $\psi(x) = r_j(x)^{\beta_j}$ in neighborhood of S_j with $\beta_j = \alpha_j - 1$ if $1 \leq \alpha_j \leq \frac{3}{2}$, and 0 if not. Thus by the first case $\omega \in H^{\frac{1}{2}+\epsilon}(\Omega)$ for some $\epsilon > 0$. In conclusion, if $\omega \in H_\phi^{0,\, m+1}(\Omega)$ ($m = 0$ or 1), $\omega|_E$ has sense in the sense of the trace theory and is in $L^1(E)$. ∎

1.4 Discretization

1.4.1 Discretization of the domain Ω

We assume that Ω is discretized by a regular family of triangulations $(\mathcal{T}_h)_{h>0}$ in the sense of [11]. As usual as indicated by the context, the letter h will also denote $h = \max h_K, K \in \mathcal{T}_h$. The set of all interior and boundary edges of the skeleton of the triangulation $\mathcal{T}_h$ will be denoted by $\mathcal{E}_h$. We then have $\mathcal{E}_h = \mathcal{E}_\Omega \cup \mathcal{E}_D \cup \mathcal{E}_N$ where $\mathcal{E}_\Omega$ denotes the set of all interior edges and $\mathcal{E}_D$, $\mathcal{E}_N$ denotes the collection of all edges contained in Γ_D and Γ_N respectively.

1.4.2 Discrete mixed formulation with mixed boundary conditions

This section concerns the approximation of the dual mixed problem (1.2.7) by conforming finite element. For each fixed triangulation $\mathcal{T}_h$, we introduce the finite dimensional spaces $\Sigma_{0,h}$ and $V_h \times W_h$ of Σ and $V \times W$ respectively, defined in the following way:

$$\Sigma_{0,h} := \{(\tau_h, q_h) \in \Sigma_0;\ \forall T \in \mathcal{T}_h:\ q_{h|T} \in \mathbb{P}_1(T) \text{ and } (\tau_h - q_h\delta)_{|T} \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\ Curl\ b_T]^2\}, \quad (1.4.1)$$

$$V_h \times W_h := \{(v_h, \theta_h) \in V \times W;\ \forall T \in \mathcal{T}_h :\ v_{h|T} \in [\mathbb{P}_0(T)]^2 \ \text{ and } \theta_{h|T} \in [\mathbb{P}_1(T)\}. \quad (1.4.2)$$

Note that by $(\tau_h - q_h\delta)_{|T} \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\ Curl\, b_T]^2$, we mean that there exist polynomials of degree ≤ 1 : $p_{11},\ p_{12},\ p_{21},\ p_{22}$ and two real numbers $\alpha_1,\ \alpha_2$ such that

$$(\tau_h - q_h\delta)_{|T} = \begin{bmatrix} p_{11|T} + \alpha_1 \frac{\partial b_T}{\partial x_2} & p_{12|T} - \alpha_1 \frac{\partial b_T}{\partial x_1} \\ p_{21|T} + \alpha_2 \frac{\partial b_T}{\partial x_2} & p_{22|T} - \alpha_2 \frac{\partial b_T}{\partial x_1} \end{bmatrix},$$

where b_T denotes the bubble function for the actual element T defined by

$$b_T \ = \ 27\lambda_1\lambda_2\lambda_3.$$

$\lambda_1, \lambda_2, \lambda_3$ denote the barycentric coordinates on T. We introduce the discretized problem: find $(\sigma_h, p_h)\ \in\ \Sigma_{0,h}$ and $(u_h, \omega_h)\ \in\ V_h \times W_h$ such that

$$\begin{cases} \frac{1}{2\mu}(\sigma_h, \tau_h) \ + \ \frac{1}{\lambda}(p_h, q_h) + \ (div\ (\tau_h - q_h\delta), u_h) \ + \ (as\ (\tau_h), \omega_h) = 0, \quad \forall\ (\tau_h, q_h) \in \Sigma_{0,h} \\ (div\ (\sigma_h - p_h\delta), v_h) \ + \ (as\ (\sigma_h), \theta_h) \ + \ (f, v_h) = 0, \quad \forall\ (v_h, \theta_h) \in V_h \times W_h. \end{cases} \quad (1.4.3)$$

With the notations (1.2.8) and (1.2.9) the above discrete problem can be restated:

Find $\underset{\sim h}{\sigma} \in \Sigma_{0,h}$ and $\underset{\sim h}{u} \in V_h \times W_h$ such that

$$\begin{cases} a(\underset{\sim h}{\sigma}, \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{u}) = 0, \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h}, \\ b(\underset{\sim h}{\sigma}, \underset{\sim h}{v}) + \mathcal{F}(\underset{\sim h}{v}) = 0, \quad \forall \underset{\sim h}{v} \in V_h \times W_h \end{cases} \quad (1.4.4)$$

To get appropriate error estimates, we need to show that the discrete inf-sup condition holds, as well as uniform coerciveness on the discrete kernel of b. For these purposes, we need a Lemma analogous to Lemma 4.1 of [24]. But the proof given in [24] is no more valid so that we give another proof. We will use the BDM_1 interpolation operator I_h defined as follows (see [1, 8, 35]): for any $\epsilon \in (0, 1)$,

$$I_h : [H^\epsilon(\Omega)]^{2\times 2} \cap [H(div\ , \Omega)]^2 \rightarrow H_h : \quad \tau \longmapsto I_h(\tau),$$

where $I_h(\tau) \in H_h$ is uniquely determined by the condition

$$\int_{\partial K} I_h(\tau) n.p_1\ ds = \int_{\partial K} \tau n.p_1\ ds, \quad \forall p_1 \in [R_1(\partial K)]^2,\ \forall K \in \mathcal{T}_h,$$

where

$$R_1(\partial K) = \{\phi \in L^2(\partial K) :\ \phi_{|E} \in \mathbb{P}_1(E),\ \forall E \in \partial K\},$$

and

$$H_h = \{\tau_h \in [H(div\ , \Omega)]^2 :\ \tau_{h|K} \in BDM_1(K)^2 = [\mathbb{P}_1(K)]^{2\times 2},\ \forall K \in \mathcal{T}_h\}.$$

If P_h^0 denotes the L^2-orthogonal projection from $L^2(\Omega)$ onto the subspace V_h, then we recall that the following diagram is commuting (see [8]):

$$\begin{array}{ccc} [H^\epsilon(\Omega)]^{2\times 2} \cap [H(div\,,\Omega)]^2 & \xrightarrow{div} & [L^2(\Omega)]^2 \\ I_h \big\downarrow & & \big\downarrow P_h^0 \\ H_h & \xrightarrow{div} & V_h \end{array}$$

Consequently

$$div\; I_h(\tau) = P_h^0(div\;\tau), \quad \forall \tau \in [H^\epsilon(\Omega)]^{2\times 2} \cap [H(div\,,\Omega)]^2. \tag{1.4.5}$$

In addition, the following approximation property holds (see Theorems 3.2 and 3.3 of [1])

$$\|\tau - I_h(\tau)\|_{0,\Omega} \lesssim h^\epsilon ||\tau||_{[H^\epsilon(\Omega)]^{2\times 2}}, \quad \forall \tau \in [H^\epsilon(\Omega)]^{2\times 2} \cap [H(div\,,\Omega)]^2. \tag{1.4.6}$$

Let us now prove the following inf-sup inequality:

Theorem 1.4.1. *There exists a constant β^*, independent of h, such that*

$$\sup_{\underset{\sim h}{\tau}=(\tau_h,q_h)\in\; \Sigma_{0,h}} \frac{b(\underset{\sim h}{\tau}\,,\underset{\sim h}{v}\,)}{\|\underset{\sim h}{\tau}\|_{0,\Omega}} \geq \beta^* \|\underset{\sim h}{v}\|_{0,\Omega}, \quad \forall \underset{\sim h}{v} = (v_h,\theta_h) \in V_h \times W_h. \tag{1.4.7}$$

Proof: Let us fix $\underset{\sim h}{v} = (v_h,\theta_h) \in V_h \times W_h$. As in the continuous case, we further fix $\lambda^* > 0$ and $\mu^* > 0$ independently of $\underset{\sim h}{v}$ and of λ,μ.

i) Let $\omega \in [H^1_{\Gamma_D}(\Omega)]^2$ be the unique solution of the problem:

$$\left\{\begin{array}{rcll} div\;(2\mu^*\epsilon(\omega) + \lambda^*\, div\;\omega\delta) & = & v_h & \text{in} \quad \Omega, \\ \omega & = & 0 & \text{on} \quad \Gamma_D, \\ (2\mu^*\epsilon(\omega) + \lambda^*\, div\;\omega\delta).n & = & 0 & \text{on} \quad \Gamma_N. \end{array}\right.$$

By the elliptic regularity of the Lamé system (see for instance [28]), there exists $\epsilon_0 \in (0,1)$ such that $\omega \in [H^{1+\epsilon}(\Omega)]^2$, for all $\epsilon \in (0,\epsilon_0)$ and satisfies

$$||\omega||_{[H^{1+\epsilon}(\Omega)]^2} \lesssim \|v_h\|_{0,\Omega}. \tag{1.4.8}$$

We now fix $\epsilon \in (0,\epsilon_0), \epsilon \neq \frac{1}{2}$ and set $\tau^1 = 2\mu^*\epsilon(\omega) + \lambda^*\, div\;\omega\delta$. As $div\;\tau_1 = v_h \in L^2(\Omega)$, we deduce that τ^1 belongs to $[H^\epsilon(\Omega)]^{2\times 2} \cap [H(div\,,\Omega)]^2$. Therefore we may set $\tau_h^1 = I_h\tau^1$, which then belongs to H_h. By the above commuting diagram we get

$$div\;\tau_h^1 = P_h^0(div\;\tau^1) = P_h^0(v_h) = v_h = div\;\tau^1.$$

By the triangular inequality we have

$$\begin{aligned} \|\tau_h^1\|_{0,\Omega} &\leq \|\tau_h^1 - \tau^1\|_{0,\Omega} + \|\tau^1\|_{0,\Omega} \\ &\lesssim h^\epsilon ||\tau^1||_{[H^\epsilon(\Omega)]^{2\times 2}} + \|\tau^1\|_{0,\Omega}. \end{aligned} \tag{1.4.9}$$

But, owing to (1.4.8), we have

$$||\tau^1||_{[H^\epsilon(\Omega)]^{2\times 2}} \lesssim ||\omega||_{[H^{1+\epsilon}(\Omega)]^2} \lesssim \|v_h\|_{0,\Omega}. \tag{1.4.10}$$

On the other hand, by Korn's inequality, we have

$$\begin{aligned} \|\tau^1\|_{0,\Omega} &\lesssim ||\omega||_{[H^1(\Omega)]^2} + \|v_h\|_{0,\Omega} \\ &\lesssim \|v_h\|_{0,\Omega}. \end{aligned} \tag{1.4.11}$$

Therefore using (1.4.9), (1.4.10) and (1.4.11), we get

$$\begin{aligned} \|\tau_h^1\|_{0,\Omega} &= \|I_h(\tau^1)\|_{0,\Omega} \\ &\lesssim \|v_h\|_{0,\Omega}. \end{aligned} \tag{1.4.12}$$

Besides, by the definition of I_h

$$\int_e I_h(\tau^1)n.p_1\ ds = \int_e \tau^1 n.p_1\ ds, \quad \forall p_1 \in [\mathbb{P}_1(e)]^2,\ \forall e \in \partial K,$$

Taking $e \subset \Gamma_N$ and recalling that $\tau^1 n = 0$ on Γ_N, we deduce that

$$\tau_h^1 n = I_h(\tau^1)n = 0 \quad \text{ on } \Gamma_N.$$

ii) Now we will construct a strain tensor τ_h^2 which depend on τ_h^1, since $as\ (\tau_h^1)$ does not necessarily vanish; moreover τ_h^2 must satisfy the following conditions

$$\begin{aligned} (as\ (\tau_h^2), \mu_h) &= (\theta_h - as\ (\tau_h^1), \mu_h) \quad \forall \mu_h \in Q_h, \\ \tau_h^2 n &= 0 \quad \text{on } \Gamma_N, \end{aligned}$$

where Q_h denotes the following subspace:

$$Q_h := \{\mu \in L^2(\Omega); \mu|_K \in \mathbb{P}_1(K), \quad \forall K \in \mathcal{T}_h\}.$$

Let us then set $\gamma_h = \theta_h - as\ (\tau_h^1)$, where τ_h^1 is previously determined. Contrary to [24] γ_h is no more vanishing mean. To correct that one will use the Lagrange interpolant of a smooth function. For this purpose we consider the finite dimensional space:

$$X_h = \{v_h \in H^1(\Omega) : v_{h|K} \in \mathbb{P}_2(K) \oplus \mathbb{R} b_K\}.$$

Let us fix a nonempty open subset Γ_0 of Γ_D such that Γ_0 is included into one edge of Γ. Moreover by an eventual change of variables, we may suppose that the outward normal vector along Γ_0 is the vector $(0, -1)$. Fix further another nonempty open subset Γ_{00} such that $\overline{\Gamma}_{00} \subset \Gamma_0$. Now fix a smooth function η defined on $\bar{\Omega}$ such that $0 \leq \eta \leq 1$ and satisfying

$$\begin{cases} \eta = 1 & \text{on } \overline{\Gamma \backslash \Gamma_0}, \\ \eta = 0 & \text{on } \overline{\Gamma_{00}}. \end{cases}$$

Let us consider $\eta_h \in X_h$ the Lagrange interpolant of η, which then fulfils $0 \leq \eta_h \leq 1$ and $\eta_h = 1$ on $\Gamma \backslash \Gamma_0$ (remark that $\Gamma_N \subset \Gamma \backslash \Gamma_0$ and consequently $\eta_h = 1$ on Γ_N). We now fix the constant vector field

$$c = \begin{pmatrix} 0 \\ e \end{pmatrix} \in \mathbb{R}^2,$$

such that

$$\int_\Omega \left(\gamma_h - c.\nabla \eta_h\right)\, dx = 0. \tag{1.4.13}$$

Indeed this condition is equivalent to

$$\int_\Omega \gamma_h \, dx = \int_\Gamma \eta_h c.n \, ds. \tag{1.4.14}$$

That is always possible one choosing the following condition

$$e = \frac{\int_\Omega \gamma_h \, dx}{\int_{\Gamma_0} (1 - \eta_h)\, ds}. \tag{1.4.15}$$

Indeed by property of η_h, we have

$$\begin{aligned} \int_\Gamma \eta_h c.n \, ds &= \int_{\Gamma \backslash \Gamma_0} \eta_h c.n \, ds + \int_{\Gamma_0} \eta_h c.n \, ds \\ &= \int_{\Gamma \backslash \Gamma_0} c.n \, ds + \int_{\Gamma_0} \eta_h c.n \, ds. \end{aligned}$$

Moreover the property $\int_\Omega div\ c\ dx = 0$ and Green's formula yield

$$\int_\Gamma c.n \, ds = 0,$$

and therefore

$$\int_{\Gamma \backslash \Gamma_0} c.n \, ds = -\int_{\Gamma_0} c.n \, ds.$$

Together with (1.4.14) lead to the condition

$$\int_{\Gamma_0} (\eta_h - 1) c.n \, ds = \int_\Omega \gamma_h \, dx.$$

By the choice of the normal vector along Γ_0, we get the condition (1.4.15). Now as (1.4.13) means that $\gamma_h - c.\nabla \eta_h$ is vanishing mean in Ω. By Corollary I.2.4 of [25], there exists $\rho \in [H_0^1(\Omega)]^2$ such that

$$div\ \rho = \gamma_h - c.\nabla \eta_h,$$

$$|\rho|_{1,\Omega} \lesssim \|\gamma_h - c.\nabla \eta_h\|_{0,\Omega}.$$

By the definition of e, this estimate becomes

$$|\rho|_{1,\Omega} \lesssim \|\gamma_h\|_{0,\Omega}.$$

Using the definition of γ_h and the estimates (1.4.12), we finally obtain

$$|\rho|_{1,\Omega} \lesssim \|\theta_h\|_{0,\Omega} + \|v_h\|_{0,\Omega}.$$

We now look for $\omega_h \in [X_h]^2$ such that

$$\begin{cases} (div\ \omega_h - \gamma_h, \mu_h) &= 0 \quad \forall\ \mu_h \in Q_h, \\ \omega_h &= c \quad \text{on } \Gamma_N, \end{cases} \tag{1.4.16}$$

with the above vector c. For that purpose, using the fact that the discretization of the Stokes problem (in primitive variables) by the pair $\left([X_h]^2 \cap [H_0^1(\Omega)]^2, Q_h \cap L_0^2(\Omega)\right)$ is stable (section II.2.2 of [25]) and making use Fortin's lemma (Lemma II.1.1 of [8]), there exists $\rho_h \in [X_h]^2 \cap [H_0^1(\Omega)]^2$ such that
$(div\ (\rho - \rho_h), \mu_h) = 0 \quad \forall\ \mu_h \in Q_h$ and $|\rho_h|_{1,\Omega} \lesssim |\rho|_{1,\Omega}$. Let

$$\omega_h = \rho_h + \eta_h c.$$

Hence ω_h belongs to $[X_h]^2$ and satisfies, owing to the properties of ρ:

$$\begin{aligned} |\omega_h|_{1,\Omega} &\lesssim |\rho_h|_{1,\Omega} + |c| \\ &\lesssim \|\theta_h\|_{0,\Omega} + \|v_h\|_{0,\Omega}. \end{aligned} \tag{1.4.17}$$

Further for any $\mu_h \in Q_h$, one has

$$\begin{aligned} (div\ \omega_h, \mu_h) &= (div\ \rho_h + c.\nabla\eta_h, \mu_h) \\ &= (\gamma_h, \mu_h). \end{aligned} \tag{1.4.18}$$

Besides

$$\omega_h = \rho_h + \eta_h c = 0 + c = c \quad \text{on } \Gamma_N,$$

hence ω_h satisfies (1.4.16) . Setting

$$\tau_h^2 = Curl\ \omega_h,$$

by (1.4.16), we remark that it fulfills

$$\begin{cases} (as\ (\tau_h^2), \mu_h) &= (div\ \omega_h, \mu_h) = (\gamma_h, \mu_h) \quad \forall \mu_h \in Q_h, \\ \tau_h^2 n &= 0 \quad \text{on } \Gamma_N. \end{cases}$$

iii) We are now able to define τ_h:

$$\tau_h = \tau_h^1 + \tau_h^2,$$

which satisfies

$$\begin{cases} div\ (\tau_h) &= v_h, \\ (as\ (\tau_h), \mu_h) &= (as\ (\tau_h^1) + as\ (\tau_h^2), \mu_h) = (\theta_h, \mu_h) \quad \forall \mu_h \in Q_h, \end{cases}$$

by the definition of γ_h. Choosing $\underset{\sim h}{\tau} = (\tau_h, 0) \in \Sigma_{0,h}$, and using the definition of b we obtain

$$b(\underset{\sim h}{\tau}, \underset{\sim h}{v}) = \|v_h\|_{0,\Omega}^2 + \|\theta_h\|_{0,\Omega}^2. \tag{1.4.19}$$

Finally the estimates (1.4.12) and (1.4.17), lead to

$$\begin{aligned}\|\underset{\sim h}{\tau}\|_{0,\Omega} &= \|\tau_h\|_{0,\Omega}\\ &\le \|\tau_h^1\|_{0,\Omega} + \|\tau_h^2\|_{0,\Omega}\\ &\lesssim \|v_h\|_{0,\Omega} + \|\theta_h\|_{0,\Omega}.\end{aligned}$$

In conclusion the inf-sup condition follows from the identity (1.4.19) and the estimate (1.4.20) ■

We end this section by proving the discrete uniform coerciveness of the bilinear form a on the kernel of b

Proposition 1.4.2. *The bilinear form a is coercive uniformly with respect to λ on*

$$K_h := \left\{\underset{\sim h}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h};\ b(\underset{\sim h}{\tau}, \underset{\sim h}{v}) = 0, \quad \forall\ \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h\right\},$$

in other words

$$a(\underset{\sim h}{\tau}, \underset{\sim h}{\tau}) \ge c\|\underset{\sim h}{\tau}\|^2_{0,\Omega}, \quad \forall \underset{\sim h}{\tau} \in K_h, \tag{1.4.20}$$

where c is a strictly positive constant independent of λ.

Proof: Let us consider $\underset{\sim h}{\tau} = (\tau_h, q_h) \in K_h$. In particular $(div\ (\tau_h - q_h\delta), v_h) = 0$ for every $v_h \in \{v_h \in L^2(\Omega);\ \forall K \in \mathcal{T}_h :\ v_{h|K} \in \mathbb{P}_0(K)\}$. But $div\ (\tau_h - q_h\delta)$ is piecewise constant, hence $div\ (\tau_h - q_h\delta) = 0$. The rest of the proof now follows the proof of the continuous case (cf. Proposition 1.2.4). ■

In particular this Proposition and Theorem 1.4.1 guarantee the existence and uniqueness of a solution to problem (1.2.7).

1.5 A priori error estimates

In this section, we take advantage of previous results and will establish some interpolation error estimates. In order to derive optimal a priori error estimates we lay down refinement rules on the regular family of triangulations. Introducing first a kind of Fortin interpolation operator (compare with Proposition 4.5 of [24]):

Proposition 1.5.1. *Let $\phi = \phi_\alpha$ be a function like in corollary (1.3.4). Then there exists an operator*

$$\Pi_h : \Sigma_0 \cap ([H_\phi^{0,m+1}(\Omega)]^2 \times H_\phi^{0,m+1}(\Omega)) \longrightarrow \Sigma_{0,h}$$

$$\underset{\sim}{\tau} = (\tau, q) \longmapsto \Pi_h \underset{\sim}{\tau} = (\tau_h, q_h)$$

($m = 0$ or 1) such that

$$b(\underset{\sim}{\tau} - \Pi_h \underset{\sim}{\tau}, \underset{\sim h}{v}) = 0 \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h. \tag{1.5.1}$$

Proof: Les us fix $(\tau, q) \in \Sigma_0 \cap [H_\phi^{0,m+1}(\Omega)]^2 \times H_\phi^{0,m+1}(\Omega)$. First let us define q_h. Denote by P_h^1 the L^2-orthogonal projection from $L^2(\Omega)$ onto the subspace Q_h. We set $q_h = P_h^1 q$. Before defining τ_h, let us first define the intermediate tensor τ_h^*. For each $K \in \mathcal{T}_h$: $\tau_{h|K}^* \in [\mathbb{P}_1(K)]^{2\times 2}$ and is uniquely determined by the requirement:

$$\int_{\partial K} [(\tau_{h|K}^* - q_h\delta) - (\tau - q\delta)]n.p_1 \; ds = 0, \quad \forall p_1 \in [R_1(\partial K)]^2. \tag{1.5.2}$$

That the term in the left-hand side of (1.5.2) has sense follows from the fact that $(\tau - q\delta) \in [H_\phi^{0,m+1}(\Omega)]^2$, and by Lemma 1.3.5, $(\tau - q\delta)_{|e} \in [L^1(e)]^2$, for all edges e of the triangulation. Let us set $\gamma = as\ (\tau - \tau_h^*)$. Since γ is non necessarily of vanishing mean (contrary to the one in the proof of Proposition 4.4 of [24]), as in Theorem 1.4.1 , we fix the constant vector field

$$c = \begin{pmatrix} 0 \\ e \end{pmatrix} \in \mathbb{R}^2,$$

such that (η_h being defined in Theorem 1.4.1)

$$\int_\Omega \big(\gamma - c.\nabla \eta_h\big)\; dx = 0. \tag{1.5.3}$$

Indeed this condition is equivalent to

$$e = \frac{\int_\Omega \gamma_h \; dx}{\int_{\Gamma_0} (1 - \eta_h)\; ds}.$$

Now as (1.5.3) means that $\gamma - c.\nabla\eta_h$ is of vanishing mean in Ω. By Corollary I.2.4 of [25], there exists $\rho \in [H_0^1(\Omega)]^2$ such that

$$div\ \rho = \gamma - c.\nabla\eta_h,$$

with the estimate

$$|\rho|_{1,\Omega} \lesssim \|\gamma\|_{0,\Omega} \le \|\tau - \tau_h^*\|_{0,\Omega},$$

with the above expression of e. Using once again the fact that the discretization of the Stokes problem (in primitive variables) by the pair $\big([X_h]^2 \cap [H_0^1(\Omega)]^2, Q_h \cap L_0^2(\Omega)\big)$ is stable (section II.2.2 of [25]) and making use of Fortin's lemma (Lemma II.1.1 of [8]), there exists $\rho_h \in [X_h]^2 \cap [H_0^1(\Omega)]^2$ such that

$(div\ (\rho - \rho_h), \mu_h) = 0 \quad \forall\ \mu_h \in Q_h$ and $|\rho_h|_{1,\Omega} \lesssim |\rho|_{1,\Omega}$. Let

$$\omega_h = \rho_h + \eta_h c.$$

Hence ω_h belongs to $[X_h]^2$ and satisfies, owing to the properties of ρ:

$$\begin{aligned} |\omega_h|_{1,\Omega} &\lesssim |\rho_h|_{1,\Omega} + |c| \\ &\lesssim \|\gamma\|_{0,\Omega} \\ &\lesssim \|\tau - \tau_h^*\|_{0,\Omega}. \end{aligned} \tag{1.5.4}$$

Further for any $\mu_h \in Q_h$, one has

$$\begin{aligned}(div\ \omega_h, \mu_h) &= (div\ \rho_h + c.\nabla\eta_h, \mu_h)\\ &= (\gamma, \mu_h).\end{aligned} \tag{1.5.5}$$

Besides

$$\omega_h = \rho_h + \eta_h c = 0 + c = c \quad \text{on } \Gamma_N.$$

We finally set $\Pi_h(\tau, q) = (\tau_h, q_h)$, where

$$\tau_h = \tau_h^* + Curl\ \omega_h,$$

Clearly $\Pi_h(\tau, q)$ belongs to $\Sigma_{0,h}$ and satisfies

$$\begin{cases} div\ (\tau_h - q_h\delta) &= div\ (\tau_h^* - q_h\delta),\\ as\ (\tau_h) &= as\ (\tau_h^*) + div\ \omega_h,\\ \tau_h n &= \tau_h^* n \quad \text{on } \Gamma_N.\end{cases}$$

Now using all these proprieties we get

$$b(\underset{\sim}{\tau} - \Pi_h\underset{\sim}{\tau}, \underset{\sim h}{v}) = (div\ (\tau - \tau_h^* - (q - q_h)\delta), v_h) + (as\ (\tau - \tau_h^*) - div\ \omega_h, \theta_h).$$

But Green's formula and the definition of τ_h^* (1.5.2) yield

$$\begin{aligned}(div\ (\tau - \tau_h^* - (q - q_h)\delta), v_h) &= \sum_{K\in\mathcal{T}_h}\int_K div\ (\tau - \tau_h^* - (q - q_h)\delta).v_h\ dx\\ &= \sum_{K\in\mathcal{T}_h}\int_{\partial K} (\tau - \tau_h^* - (q - q_h)\delta)n.v_h\ ds = 0.\end{aligned}$$

On the other hand, by (1.5.5), we have

$$(as\ (\tau - \tau_h^*) - div\ \omega_h, \theta_h) = (\gamma - div\ \omega_h, \theta_h) = 0.$$

Together these identities complete the proof of the Proposition. ■

Corollary 1.5.2. *We keep the same context as in the preceding proposition. Then*

$$\|\underset{\sim}{\tau} - \Pi_h\underset{\sim}{\tau}\|_{0,\Omega} \lesssim \|(\tau - q\delta) - (\tau_h^* - q_h\delta)\|_{0,\Omega} + \|q - q_h\|_{0,\Omega}. \tag{1.5.6}$$

Proof: By the construction of the operator Π_h (see the proof of the Proposition 1.5.1) and (1.5.4), we have

$$\begin{aligned}\|\underset{\sim}{\tau} - \Pi_h\underset{\sim}{\tau}\|_{0,\Omega} &\leq \|\tau - \tau_h\|_{0,\Omega} + \|q - q_h\|_{0,\Omega}\\ &\lesssim \|\tau - \tau_h^*\|_{0,\Omega} + |\omega_h|_{1,\Omega} + \|q - q_h\|_{0,\Omega}\\ &\lesssim \|\tau - \tau_h^*\|_{0,\Omega} + \|q - q_h\|_{0,\Omega}.\end{aligned}$$

The conclusion follows from the triangular inequality. ■

We now need to define some local weighted Sobolev spaces:

Definition 1.5.3. *Let K be an arbitrary triangle in the plane and A a vertex of K. Let $k = 1$ or 2 and $\beta \in [0, k - \frac{1}{2}[$, we will denote*

$$H_A^{m,k;\beta}(K) = \{\psi \in H^m(K); \quad |x - A|^\beta D^\alpha \psi \in L^2(K), \ \forall \alpha \in I\!N^2 \ \text{ such that } \ 1 \leq |\alpha| \leq k\},$$

equipped with the norm:

$$||\psi||_{m,k;\beta,K} = \big(||\psi||_{m,K}^2 + \sum_{1 \leq |\alpha| \leq k} |||x - A|^\beta D^\alpha \psi||_{0,K}^2\big)^{1/2}.$$

On this space, we also define the semi-norm:

$$|\psi|_{m,1;\beta,K} = \big(\sum_{|\alpha| = k} |||x - A|^\beta D^\alpha \psi||_{0,K}^2\big)^{1/2}.$$

Thanks to the Lemma 1.3.5, the trace of an element of $H_A^{m,k;\beta}(K)$ with $\beta \in [0, k - \frac{1}{2}[$ is well defined and is in $L^1(\partial K)$. Thus given a vector-field $v \in [H_A^{m,k;\beta}(K)]^2$, its Brezzi-Douglas-Marini interpolate $\rho_K v \in BDM_1(K) = [I\!P_1(K)]^2$ (see [8], p.125) is well defined by the relations:

$$\int_{\partial K} \rho_K v.np_1 \ ds = \int_{\partial K} v.np_1 \ ds, \quad \forall p_1 \in R_1(\partial K).$$

Lemma 1.5.4. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$. For $k = 1$ or 2 and for any $\beta \in [0, k - \frac{1}{2}[$, it holds*

$$||v - \rho_K v||_{0,K} \lesssim h_K^{k-\beta} |v|_{0,k;\beta,K}, \quad \forall v \in [H_A^{0,k;\beta}(K)]^2.$$

Direct consequences of this Lemma are the next global interpolation error estimates between a vector field $v \in [H_\phi^{0,m+1}(\Omega)]^2$ and its Brezzi-Douglas-Marini interpolate $\rho_h v$, under appropriate refinement rules of grids (imposing constraints on the diameter of the triangle of the triangulations according to the geometrical situation of the triangle) in order to recapture optimal order of convergence of the interpolates (see Theorem 4.13 and its Corollary in [24]):

Theorem 1.5.5. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations on Ω and $m = 0$ or 1. We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies the two following refinement rules:*

R_1: *if K is a triangle of T_h admitting S_j as a vertex, then*

$$h_K \lesssim h^{(m+1)/(m+1-\alpha_j)}, \tag{1.5.7}$$

where α_j has been defined just before Corollary 1.3.4,

R_2: *if K is a triangle of T_h admitting no $S_j (j = 1, \cdots, n_e)$ as a vertex, then*

$$h_K \lesssim h(\inf_K \phi)^{1/(m+1)}, \tag{1.5.8}$$

(ϕ has been defined in Corollary 1.3.4).

Then for every vector field $v \in [H_\phi^{0,m+1}(\Omega)]^2$, *it holds*

$$\|v - \rho_h v\|_{0,\Omega} \lesssim h^{m+1}|v|_{0,m+1;\phi,\Omega}, \tag{1.5.9}$$

where $\rho_h v$ *denotes the* BDM_1 *interpolate of* v, *i.e., for all* $K \in \mathcal{T}_h$, $(\rho_h v)|_K = \rho_K v$.

Similarly for every $q \in H_\phi^{0,m+1}(\Omega)$, it holds

$$\|q - P_h^1 q\|_{0,\Omega} \lesssim h^{m+1}|q|_{0,m+1;\phi,\Omega}, \tag{1.5.10}$$

where we recall that P_h^1 denotes the L^2-orthogonal projection on Q_h.

Corollary 1.5.6. *Under the same hypotheses as in Theorem 1.5.5 and for* $m = 0$ *or* 1, *and for every* $\underset{\sim}{\tau} \in [H_\phi^{0,m+1}(\Omega)]^{2\times 2} \times H_\phi^{0,m+1}(\Omega)$

$$\|\underset{\sim}{\tau} - \Pi_h \underset{\sim}{\tau}\|_{0,\Omega} \lesssim h^{m+1}(|\tau|_{0,m+1;\phi,\Omega} + |q|_{0,m+1;\phi,\Omega}). \tag{1.5.11}$$

Proof: By Corollary 1.5.2, we have

$$\|\underset{\sim}{\tau} - \Pi_h \underset{\sim}{\tau}\|_{0,\Omega} \lesssim \|(\tau - q\delta) - (\tau_h^* - q_h\delta)\|_{0,\Omega} + \|q - q_h\|_{0,\Omega},$$

where (τ_h^*, q_h) satisfies (1.5.2). Notice that the definition of τ_h^* in (1.5.2) means that each line of $\tau_h^* - q_h\delta$ is the BDM_1-interpolate of the corresponding line of the tensor $\tau - q\delta$. Thus, by the theorem 1.5.5

$$\begin{aligned}\|(\tau - q\delta) - (\tau_h^* - q_h\delta)\|_{0,\Omega} &\lesssim h^{m+1}|\tau - q\delta|_{0,m+1;\phi,\Omega} \\ &\lesssim h^{m+1}\big(|\tau|_{0,m+1;\phi,\Omega} + |q|_{0,m+1;\phi,\Omega}\big).\end{aligned}$$

The estimate (1.5.10) and (1.5.6) lead to the conclusion.

■

Remark 1.5.7. *Regular families of meshes satisfying the refinement conditions* $(R_1 - R_2)$ *are easily built, see for instance [34].*

We are now in a position to establish optimal error estimates. In the following, we estimate the error between $\underset{\sim}{\sigma} = (\sigma, p)$, $\underset{\sim}{u} = (u, \omega)$ the exact solution of the mixed problem (1.2.7) or equivalently (1.2.10) and $\underset{\sim h}{\sigma} = (\sigma_h, p_h)$, $\underset{\sim h}{u} = (u_h, \omega_h)$ the solution of the discrete problem (1.4.3) or equivalently (1.4.4).

Theorem 1.5.8. *Let* $(\mathcal{T}_h)_{h>0}$ *be a regular family of triangulations on* Ω *and* $m = 0$ *or* 1. *We suppose that* $(\mathcal{T}_h)_{h>0}$ *satisfies to conditions R1 and R2 . We suppose that* $f \in [H^m(\Omega)]^2$ *and that the characteristic equations (1.3.1 -1.3.3) have no root on the vertical line Re* $\alpha = m + 1$ *for each* $j = 1, 2, \cdots, n_e$. *Then the following error estimate holds for every* $\lambda \in [\lambda_0, +\infty[$

$$\|\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|_{0,\Omega} \lesssim \big(1 + \frac{1}{\lambda}\big) h^{m+1}\big[|u|_{1,m+1;\phi,\Omega} + |p|_{0,m+1;\phi,\Omega}\big]. \tag{1.5.12}$$

$$\|u - u_h\|_{0,\Omega} + \|\omega - \omega_h\|_{0,\Omega} \lesssim \big(1 + \frac{1}{\lambda}\big)^2 h\big[|u|_{1,m+1;\phi,\Omega} + |p|_{0,m+1;\phi,\Omega}\big]. \tag{1.5.13}$$

Proof: If we subtract (1.2.10) from (1.4.4), we get the system with the errors

$$\begin{cases} a\big(\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \underset{\sim h}{\tau}\big) + b\big(\underset{\sim h}{\tau}, \underset{\sim}{u} - \underset{\sim h}{u}\big) = 0, \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h}, \\ b\big(\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \underset{\sim h}{v}\big) = 0, \quad \forall \underset{\sim h}{v} \in V_h \times W_h. \end{cases} \tag{1.5.14}$$

Let us set $\Pi_h \underset{\sim}{\sigma} = \Pi_h(\sigma, p) = (\sigma_h^*, p_h^*)$. Taking $\underset{\sim h}{\tau} = \Pi_h \underset{\sim}{\sigma} - \underset{\sim h}{\sigma}$ in (1.5.14), we get

$$\begin{aligned} a\big(\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \Pi\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\big) &+ \big(div\ (\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), u - u_h\big) \\ &+ \big(as\ (\sigma_h^* - \sigma_h), \omega - \omega_h\big) = 0. \end{aligned}$$

Let $(P_h^0 u, P_h^1 \omega)$ denote the L^2-orthogonal projection of (u, ω) on the space $V_h \times W_h$, we have

$$\begin{aligned} & a\big(\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \Pi\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\big) + \big(div\ (\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), u - P_h^0 u\big) \\ & + \big(div\ (\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), P_h^0 u - u_h\big) + \big(as\ (\sigma_h^* - \sigma_h), \omega - P_h^1 \omega\big) \\ & + \big(as\ (\sigma_h^* - \sigma_h), P_h^1 \omega - \omega_h\big) = 0. \end{aligned} \tag{1.5.15}$$

Since $div\ (\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta)$ is a constant vector field on each triangle $K \in \mathcal{T}_h$ and $u - P_h^0 u$ is of mean value 0 on each triangle $K \in \mathcal{T}_h$, we deduce that

$$\big(div\ (\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), u - P_h^0 u\big) = 0.$$

From (1.5.1) and the second equation of (1.5.14), we have

$$\big(div\ (\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), P_h^0 u - u_h\big) + \big(as\ (\sigma_h^* - \sigma_h), P_h^1 \omega - \omega_h\big) = 0$$

Together with (1.5.15) we get

$$a\big(\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \Pi\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\big) + \big(as\ (\sigma_h^* - \sigma_h), \omega - P_h^1 \omega\big) = 0,$$

which yields

$$a\big(\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \Pi\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\big) = \big(as\ (\sigma_h - \sigma_h^*), \omega - P_h^1 \omega\big). \tag{1.5.16}$$

Using (1.5.16) and the triangular inequality we get

$$\begin{aligned} a\big(\Pi_h\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \Pi_h\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\big) &= a\big(\Pi_h\underset{\sim}{\sigma} - \underset{\sim}{\sigma}, \Pi_h\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\big) + a\big(\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \Pi_h\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\big) \\ &= \frac{1}{2\mu}\big(\sigma_h^* - \sigma, \sigma_h^* - \sigma_h\big) + \frac{1}{\lambda}\big(p_h^* - p, p_h^* - p_h\big) \\ &\quad + \big(as\ (\sigma_h - \sigma_h^*), \omega - P_h^1 \omega\big). \end{aligned} \tag{1.5.17}$$

From (1.5.1) and the second equation of (1.5.14), we remark that $\Pi_h\underset{\sim}{\sigma} - \underset{\sim}{\sigma}$ belongs to K_h and by Lemma 1.4.2, we get

$$\|\Pi_h\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|^2_{0,\Omega} \lesssim a\big(\Pi_h\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \Pi_h\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\big). \tag{1.5.18}$$

It remains to estimate the tree terms of the right-hand side of (1.5.17). By Cauchy-Schwarz's inequality, we have

$$|(\sigma_h^* - \sigma, \sigma_h^* - \sigma_h)| \leq \|\sigma_h^* - \sigma\|_{0,\Omega} \|\sigma_h^* - \sigma_h\|_{0,\Omega} \leq \|\Pi_h \underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|_{0,\Omega} \|\Pi_h \underset{\sim}{\sigma} - \underset{\sim}{\sigma}\|_{0,\Omega},$$

$$|(p_h^* - p, p_h^* - p_h)| \leq \|p_h^* - p\|_{0,\Omega} \|p_h^* - p_h\|_{0,\Omega} \leq \|\Pi_h \underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|_{0,\Omega} \|\Pi_h \underset{\sim}{\sigma} - \underset{\sim}{\sigma}\|_{0,\Omega},$$

$$\left|\left(as\ (\sigma_h - \sigma_h^*), \omega - P_h^1 \omega\right)\right| \leq \|\sigma_h - \sigma_h^*\|_{0,\Omega} \|\omega - P_h^1 \omega\|_{0,\Omega} \leq \|\Pi_h \underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|_{0,\Omega} \|\omega - P_h^1 \omega\|_{0,\Omega}.$$

Using the interpolation error estimates (1.5.10) and (1.5.11), we get

$$|(\sigma_h^* - \sigma, \sigma_h^* - \sigma_h)| \lesssim h^{m+1} \|\Pi_h \underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|_{0,\Omega} \left(|u|_{1,m+1;\phi,\Omega} + |p|_{0,m+1;\phi,\Omega}\right),$$

$$|(p_h^* - p, p_h^* - p_h)| \lesssim h^{m+1} \|\Pi_h \underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|_{0,\Omega} \left(|u|_{1,m+1;\phi,\Omega} + |p|_{0,m+1;\phi,\Omega}\right),$$

$$\left|\left(as\ (\sigma_h - \sigma_h^*), \omega - P_h^1 \omega\right)\right| \lesssim h^{m+1} \|\Pi_h \underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|_{0,\Omega} |u|_{1,m+1;\phi,\Omega}.$$

Together with (1.5.17) and using the estimate (1.5.18) lead to

$$\|\Pi_h \underset{\sim}{\sigma} - \underset{\sim h}{\sigma}\|_{0,\Omega} \lesssim (1 + \frac{1}{\lambda}) h^{m+1} \left(|u|_{1,m+1;\phi,\Omega} + |p|_{0,m+1;\phi,\Omega}\right).$$

Using once again the interpolation error estimates (1.5.11) and the triangle inequality, we obtain (1.5.12). To prove (1.5.13), we shall use the uniform inf-sup condition (1.4.7). First observe that

$$b\left(\underset{\sim h}{\tau}, (P_h^0 u, P_h^1 \omega) - \underset{\sim h}{u}\right) = \begin{array}{l} -a\left(\underset{\sim}{\sigma} - \underset{\sim h}{\sigma}, \underset{\sim h}{\tau}\right) \\ +\left(as\ (\tau_h), P_h^1 \omega - \omega\right), \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h} \end{array}$$

Thus by the uniform inf-sup condition (1.4.7), we have

$$\|P_h^0 u - u_h\|_{0,\Omega} + \|P_h^1 \omega - \omega_h\|_{0,\Omega} \lesssim \sup_{\underset{\sim h}{\tau} = (\tau_h, q_h) \in\ \Sigma_{0,h}} \frac{b(\underset{\sim h}{\tau}, (P_h^0 u, P_h^1 \omega) - \underset{\sim h}{u})}{\|\underset{\sim h}{\tau}\|_{0,\Omega}}. \tag{1.5.19}$$

It follows from the first equation of the system with errors (1.5.14) that:

$$\left(div\ (\tau_h - q_h \delta), u - u_h\right) + \left(as\ (\tau_h), \omega - \omega_h\right) = \frac{1}{2\mu}\left(\sigma_h - \sigma, \tau_h\right) + \frac{1}{\lambda}\left(p_h - p, q_h\right).$$

Introducing $P_h^0 u$ and $P_h^1 \omega$ as intermediate quantities in the left member of this last equality we get

$$\begin{aligned} &\left(div\ (\tau_h - q_h \delta), u - P_h^0 u\right) + \left(div\ (\tau_h - q_h \delta), P_h^0 u - u_h\right) + \left(as\ (\tau_h), P_h^1 \omega - \omega_h\right) \\ &\qquad = \frac{1}{2\mu}\left(\sigma_h - \sigma, \tau_h\right) + \frac{1}{\lambda}\left(p_h - p, q_h\right) + \left(as\ (\tau_h), P_h^1 \omega - \omega\right). \end{aligned} \tag{1.5.20}$$

Remark that the first term in the left member of equality (1.5.21) is 0. Moreover

$$
\begin{aligned}
b(\underset{\sim h}{\tau}, (P_h^0 u, P_h^1 \omega) - \underset{\sim h}{u}) &= b(\underset{\sim h}{\tau}, (P_h^0 u - u_h, P_h^1 \omega - \omega_h)) \\
&= \big(div\ (\tau_h - q_h \delta), P_h^0 u - u_h\big) + \big(as\ (\tau_h), P_h^1 \omega - \omega_h\big) \\
\text{by (1.5.19)} &= \frac{1}{2\mu}\big(\sigma_h - \sigma, \tau_h\big) + \frac{1}{\lambda}\big(p_h - p, q_h\big) + \big(as\ (\tau_h), P_h^1 \omega - \omega\big).
\end{aligned}
$$

We have thus by Cauchy-Schwarz's inequality

$$
\begin{aligned}
|b(\underset{\sim h}{\tau}, (P_h^0 u, P_h^1 \omega) - \underset{\sim h}{u})| &\le \frac{1}{2\mu}\|\sigma_h - \sigma\|_{0,\Omega}\|\tau_h\|_{0,\Omega} + \frac{1}{\lambda}\|p_h - p\|_{0,\Omega}\|q_h\|_{0,\Omega} \\
&\quad + \sqrt{2}\|P_h^1 \omega - \omega\|_{0,\Omega}\|\tau_h\|_{0,\Omega} \\
&\lesssim \big(1 + \frac{1}{\lambda}\big)\big(\|\sigma_h - \sigma\|_{0,\Omega} + \|p_h - p\|_{0,\Omega} + \|P_h^1 \omega - \omega\|_{0,\Omega}\big)\|\tau_h\|_{0,\Omega}.
\end{aligned}
$$

Hence by the estimates (1.5.12) we get

$$
|b(\underset{\sim h}{\tau}, (P_h^0 u, P_h^1 \omega) - \underset{\sim h}{u})| \lesssim \big(1 + \frac{1}{\lambda}\big)^2 h^{m+1}\big[|u|_{1,m+1;\phi,\Omega} + |p|_{0,m+1;\phi,\Omega}\big]\|\tau_h\|_{0,\Omega}
$$

This estimate combined with (1.5.19) show that

$$
\|P_h^0 u - u_h\|_{0,\Omega} + \|P_h^1 \omega - \omega_h\|_{0,\Omega} \lesssim \big(1 + \frac{1}{\lambda}\big)^2 h^{m+1}\big[|u|_{1,m+1;\phi,\Omega} + |p|_{0,m+1;\phi,\Omega}\big]. \tag{1.5.21}
$$

Moreover by standard scaling arguments (see [36], p.27) it holds

$$
\|u - P_h^0 u\|_{0,\Omega} \lesssim h|u|_{1,\Omega}. \tag{1.5.22}
$$

Finally, (1.5.13) follows from (1.5.10), (1.5.21) and (1.5.22). ■

1.6 Conclusion

Optimal and uniform error estimates in the Lamé coefficient λ are obtained for nearly incompressible materials, i.e. for λ very large. The main ingredient is the knowledge of singular behavior of the solution which allows to use appropriate refinement rules of grids in order to recapture optimal order of convergence of the interpolates. By relaxing the symmetry of the discrete stain tensor σ_h in the dual mixed finite element method we obtain a stable numerical scheme.

1.7 Numerical Experiments

For the implementation of the mixed elasticity problem 1.2.7, a so called "Hybrid formulation" [8, 15, 33] should be used. In this hybrid form, the continuity of the normal trace $(\sigma_h - p_h\delta).n$ across interelement edges of the triangulation is relaxed by using a

Lagrange multiplier λ_h. The Lagrange multiplier λ_h is an approximation of the trace of the displacement field on the edges of the triangulation. This technique enables us to eliminate the approximations of σ, u, and ω at the element level and leads to a linear system that involves only the Lagrange multiplier λ_h and p_h as degrees of freedom. It allows also us to treat non homogeneous boundary conditions on Γ_N [32].

The numerical results are presented on a L-shaped domain. Given $f : \Omega \longmapsto \mathbb{R}^2$ and a surface force density $g : \Gamma_N \longmapsto \mathbb{R}^2$, the displacement field $u = (u_1, u_2)$ satisfies the following equations :

$$\left\{ \begin{array}{rcll} -div\, \sigma_s(u) & = & f & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \Gamma_D, \\ \sigma_s(u).n & = & g & \text{on} \quad \Gamma_N, \end{array} \right. \tag{1.7.1}$$

1.7.1 Hybrid formulation

We first introduce the enlarged space $\widetilde{\Sigma}_h$ (with respect to $\Sigma_{h,0}$) by suppressing the requirement for its elements to have continuous normal component in the interfaces of the triangulation $\mathcal{T}_h$:

$$\begin{aligned} \widetilde{\Sigma}_h = S_h \times Q_h := \{(\tau_h, q_h) \in [L^2(\Omega)]^{2\times 2} \times L^2(\Omega);\ \forall T \in \mathcal{T}_h :\ q_{h|T} \in \mathbb{P}_1(T) \ \text{ and} \\ (\tau_h - q_h\delta)_{|T} \in [\mathbb{P}_1(K)]^{2\times 2} \oplus [\mathbb{R}\, Curl\, b_T]^2\}, \end{aligned}$$

and the space of Lagrangian multiplier:

$$\Lambda_h := \{\mu_h \in [L^2(\mathcal{E}_h)]^2;\ \mu_{h|e} \in [\mathbb{P}_1(e)]^2\ \forall e \in \mathcal{E}_h \ \text{ and } \mu_{h|e} = 0,\ \forall e \in \Gamma_D\}.$$

We introduce the following Hybrid formulation: Find $(\widetilde{\sigma}_h, \widetilde{p}_h, \lambda_h) \in \widetilde{\Sigma}_h \times \Lambda_h$ and $(\widetilde{u}_h, \widetilde{\omega}_h) \in V_h \times W_h$ such that

$$\left\{ \begin{array}{l} \dfrac{1}{2\mu}(\widetilde{\sigma}_h, \tau_h) \ + \ \dfrac{1}{\lambda}(\widetilde{p}_h, q_h) + \displaystyle\sum_{K\in\mathcal{T}_h} \int_K div\, (\tau_h - q_h\delta).\widetilde{u}_h\, dx \ + \ (as\, (\tau_h), \widetilde{\omega}_h) \\[2ex] - \displaystyle\sum_{K\in\mathcal{T}_h} \int_{\partial K} \lambda_h(\tau_h - q_h\delta).n_K\, ds = 0, \quad \forall\ (\tau_h, q_h) \in \widetilde{\Sigma}_h \\[2ex] \displaystyle\sum_{K\in\mathcal{T}_h} \int_K div\, (\widetilde{\sigma}_h - \widetilde{p}_h\delta).v_h\, dx \ + \ (as\, (\widetilde{\sigma}_h), \theta_h) \ + \ (f, v_h) = 0, \quad \forall\ (v_h, \theta_h) \in V_h \times W_h \\[2ex] \displaystyle\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu_h(\widetilde{\sigma}_h - \widetilde{p}_h\delta).n_K\, ds = \int_{\partial K\cap\Gamma_N} \mu_h.g\, ds \quad \forall\ \mu_h \in \Lambda_h. \end{array} \right. \tag{1.7.2}$$

It is easily proved that $\widetilde{\sigma}_h = \sigma_h$, $\widetilde{p}_h = p_h$, $\widetilde{u}_h = u_h$ and $\widetilde{\sigma}_h = \omega_h$ here $((\sigma_h, p_h), (u_h, \omega_h))$ is the solution of the non-hybridized mixed formulation. Taking the advantage of the fact that $\widetilde{\Sigma}_h$ is a product space, we can thus uncouple the first equation of the system 1.7.2 and we get:

$$\begin{cases} \dfrac{1}{2\mu}\displaystyle\int_K \sigma_K : \tau_K \; dx + |K| div\,(\tau_K).u_K - \int_{\partial K} \lambda_{\partial K}\tau_K.n_K \; ds + \int_K as\,(\tau_K)\omega_K \; dx = 0 \\ \quad \forall \tau_K \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\,Curl\; b_T]^2 \\ \dfrac{1}{\lambda}\displaystyle\int_K p_K q_K \; dx - |K|\nabla q_K.u_K - \int_{\partial K} (\lambda_{\partial K}.n_K) q_K \; ds = 0, \quad \forall q_K \in \mathbb{P}_1(K) \\ \int_K as\,(\sigma_K)\theta_K \; dx = 0, \quad \forall \theta_K \in \mathbb{P}_1(K) \\ |K| div\,(\sigma_K).v_K - |K|\nabla p_K.v_K = \int_K f.v_K \; dx, \quad \forall v_K \in \mathbb{P}_0(K)^2 \\ \displaystyle\int_a ((\sigma_{K_1^a} - p_{K_1^a}\delta).n_{K_1^a} + (\sigma_{K_2^a} - p_{K_2^a}\delta).n_{K_2^a}).\mu_a \; ds = \int_a \mu_a.g \; ds. \\ \quad \forall \mu_a \in [\mathbb{P}_1(a)]^2 \text{ if } a \in \mathcal{E}_h / \Gamma_D \;\; (\mu_a = 0 \text{ if } a \in \Gamma_D). \end{cases}$$

The basis functions implemented here are explicitly described in Annex. On each element we consider the corresponding linear and bilinear form:

- $A_K(\sigma_K, \tau_K) \; := \; \dfrac{1}{2\mu}\displaystyle\int_K \sigma_K : \tau_K \; dx,$
- $B_K(p_K, v_K) \; := \; |K|\nabla p_K.v_K,$
- $C_K(\sigma_K, v_K) \; := \; |K| div\,(\sigma_K).v_K,$
- $H_K(\sigma_K, \theta_K) \; := \; \displaystyle\int_K as\,(\sigma_K)\;\theta_K \; dx,$
- $P_K(p_K, q_K) \; := \; \dfrac{1}{\lambda}\displaystyle\int_K p_K q_K \; dx.$
- $F_K(v_K) \; := \; \displaystyle\int_K f_K.v_K \; dx.$

The corresponding linear and bilinear form on the collection of internal edges and edges contained on Γ_N are of the form :

- $E_K^a(\sigma_K, \mu_a) \; := \; \displaystyle\int_a (\sigma_K.n_K)\mu_a \; ds,$
- $G_K^a(p_K, \mu_a) \; := \; \displaystyle\int_a p_K n_K.\mu_a \; ds,$

- $T^e(\mu_e) \ := \ \int_e g.\mu_e \, ds.$

With these notation we can thus write the system 1.7.2 locally on the form:

$$\begin{cases} A^K\sigma^K + (C^K)^T u^K - (E^{K,a})^T\lambda^a - (E^{K,b})^T\lambda^b - (E^{K,c})^T\lambda^c + (H^K)^T\omega^K = 0, \\ P^K p^K - (B^K)^T u^K + (G^{K,a})^T\lambda^a + (G^{K,b})^T\lambda^b + (G^{K,c})^T\lambda^c = 0, \\ H^K\sigma^K = 0, \\ E^{K_1,e}\sigma^{K_1} + E^{K_2,e}\sigma^{K_2} - G^{K_1,e}p^{K_1} - G^{K_2,e}p^{K_2} = T^e, \\ C^K\sigma^K - B^K p^K = -F^K, \end{cases} \tag{1.7.3}$$

where $A^K,\ B^K,\ C^K,\ E^{K,e}, G^{K,e}\ H^K$, and P^K denote there corresponding local stiffness matrices corresponding to the previously defined bilinear forms. $\sigma^K,\ p^K,\ u^K,\ \omega^K$ and λ^e denote vectors of degrees of freedom corresponding to $\sigma_K,\ p_K,\ u_K,\ \omega_K$ and λ_e. The explicit forms of these local stiffness matrices are shown in Annex A. Still noting $\sigma,\ p,\ u,\ \omega$ and λ the vectors of the degrees of freedom of the unknowns $\sigma,\ p,\ u,\ \omega$ and λ , the global algebraic system generated by (1.7.3) becomes in the following form:

$$\begin{cases} A\sigma + C^T u - E^T\lambda + H^T\omega = 0, \\ Pp - B^T u + G^T\lambda = 0, \\ H\sigma = 0, \\ E\sigma - Gp = T, \\ C\sigma - Bp = -F. \end{cases} \tag{1.7.4}$$

In the system (1.7.4), we start by eliminating σ and u and after we eliminate ω. These eliminations are made element by element. After this procedure, we end to the following system:

$$\begin{cases} \mathbf{\Lambda}\lambda + \mathbf{B}^T p = \mathbf{F_1}, \\ \mathbf{B}\lambda - \mathbf{C}p = \mathbf{F_2}, \end{cases} \tag{1.7.5}$$

where the matrix appearing in this last algebraic system are of the forms

- $\mathbf{\Lambda} := \Lambda - MW^{-1}M^T$,

- $\mathbf{B} := P_2 - MW^{-1}P_1$,

- $\mathbf{C} := P_3 - P_1^T W^{-1}P_1$,

- $F_1 := I_4 - MW^{-1}I_3$,

- $F_2 := I_5 - P_1^T W^{-1} I_3$,

whith

$$\begin{aligned}
\Lambda &= EA^{-1}E^T - EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}E^T,\\
M &= EA^{-1}C^T(CA^{-1}C^T)^{-1}CA^{-1}H^T - EA^{-1}H^T,\\
W &= HA^{-1}C^T(CA^{-1}C^T)^{-1}CA^{-1}H^T - HA^{-1}H^T,\\
P_1 &= HA^{-1}C^T(CA^{-1}C^T)^{-1}B,\\
P_2 &= EA^{-1}C^T(CA^{-1}C^T)^{-1}B - G,\\
P_3 &= B^T(CA^{-1}C^T)^{-1}B + P,\\
I_4 &= T - EA^{-1}C^T(CA^{-1}C^T)^{-1}F,\\
I_3 &= HA^{-1}C^T(CA^{-1}C^T)^{-1}F,\\
I_5 &= B^T(CA^{-1}C^T)^{-1}F.
\end{aligned}$$

Note that the matrix W is bloc diagonal and that each bloc is positive definite. Thus W is invertible (see Annex). The system of algebraic equations (1.7.5) is then resolved by the use of the following extension of the Augmented Lagrangian algorithm (see [41]) :

$$\left\{\begin{aligned}
&\mathbf{\Lambda}\lambda_m + \mathbf{B}^T p_m = \mathbf{F_1},\\
&\mathbf{B}\lambda_m - (\mathbf{C} + \epsilon\mathbf{D})p_m = \mathbf{F_2} - \epsilon\mathbf{D}p_{m-1}.
\end{aligned}\right.$$

The convergence of this scheme is $O(\epsilon^m)$, for $m = 1, 2, \cdots$. Thus the parameter ϵ does not have to be chosen too much small so that the conditioning of the system can be improved and a few iterations can reduce the error due to penalization. The implementation issues is as follows:

1: Start with any p_0 and fix a tolerance $Tol > 0$.

2: p_{m-1} being given we calculate λ_m by

$$(\mathbf{\Lambda} + \mathbf{B}^T(\mathbf{C} + \epsilon\mathbf{D})^{-1}\mathbf{B})\lambda_m = \mathbf{F_1} + \mathbf{B}^T(\mathbf{C} + \epsilon\mathbf{D})^{-1}(\mathbf{F_2} - \epsilon\mathbf{D}p_{m-1}),$$

3: λ_m being now known, we calculate p_m by

$$p_m = (\mathbf{C} + \epsilon\mathbf{D})^{-1}(\epsilon\mathbf{D}p_{m-1} + \mathbf{B}\lambda_m - \mathbf{F_2}),$$

4: if $||p_m - p_{m-1}||/||p_m|| < Tol$, stop. Else, $p_{m-1} \leftarrow p_m$, and we come back to step 2.

1.7.2 Numerical test

We now present some numerical results on a test problem in the L-shaped domain $\Omega =]-1,1[^2 \backslash [0,1[\times]-1,0]$ which models one singularity arising at the re-entrant corner as shown in figure 1.2. Using polar coordinates (r,θ), $0 \leq \theta \leq \omega := \frac{3\pi}{2}$, which are centered at the re-entrant corner (see figure 1.2), the analytical solution is

$$u(r,\theta) = r^{\alpha}\overrightarrow{\phi}_{\alpha}(\theta), \tag{1.7.6}$$

where

$$\overrightarrow{\phi}_{\alpha}(\theta)_1 = C_1(\rho+\tau)\{\cos(\alpha-2)\theta - \cos(\alpha\theta)\} + C_2((\rho+\tau)sin(\alpha-2)\theta + (\tau-3\rho)sin(\alpha\theta)),$$

$$\overrightarrow{\phi}_{\alpha}(\theta)_2 = C_1(-(\rho+\tau)\sin(\alpha-2)\theta + (3\rho-\tau)\sin(\alpha\theta)) + C_2(\rho+\tau)\{\cos(\alpha-2) - \cos(\alpha\theta)\}.$$

The parameters are

$$C_1 = (\rho+\tau)\sin(\alpha-2)\omega - (3\tau-\rho)\sin(\alpha\omega),$$

$$C_2 = (\rho+\tau)\{\cos(\alpha\omega) - \cos(\alpha-2)\omega\},$$

$$\rho = \frac{\lambda+\mu}{\mu}(\alpha-1) - 2, \quad \tau = \frac{\lambda+\mu}{\mu}(\alpha+1) + 2,$$

where α is the smallest strictly positive solution of the transcendental equation (1.3.1) for $\omega = \frac{3\pi}{2}$. With an aim of corroborating the robustness of our mixed method, we fix the Lamé coefficient $\mu = 1000$ and takes different values of λ as shown below:

λ	α
1.E+014	0.544483736824905
1.E+012	0.544483737042583
1.E+010	0.544483758810418
1.E+008	0.544485935531526

We use two kinds of meshes. The first one (uniform) is obtained by dividing the intervals $[0,1]$ and $[-1,0]$ into n subintervals of length $\frac{1}{n}$, and then each square is divided into triangles (see figure 1.3 where we have chosen n=10). The second kind of meshes (refined) is obtained from the first one by refinement near $(0,0)$ according to Raugel's procedure [34]. Namely, Ω is divided into six big triangles; on the three ones which do not contain $(0,0)$, a uniform mesh is used; each big triangle containing $(0,0)$ is divided according to the ratios $(\frac{i}{n})^{\beta}$, $1 \leq i \leq n$, where $\beta \geq \frac{1}{(1-\alpha)}$ along the sides which end at $(0,0)$ and finally divide uniformly each of these strips (see figure 1.3 where we have chosen n=10 and $\beta = 1.8$). We then represent the variations of the errors $\|\sigma_h - \sigma\|_{0,\Omega}$, $\|p_h - p\|_{0,\Omega}$, $\|u_h - u\|_{0,\Omega}$ and $\|\omega_h - \omega\|_{0,\Omega}$, with respect to the mesh size h, in figure 1.5 and figure 1.6. A double logarithmic scale was used such that the slope of the curves yields the order of convergence $O(h)$ for refined meshes (see figure 1.6) according to the theoretical results, and $O(h^{\frac{2}{3}})$ for uniform meshes (see figure 1.5) due to the singular behavior of the solution. In Table 1.5 - 1.8 we summarize the results of the errors for

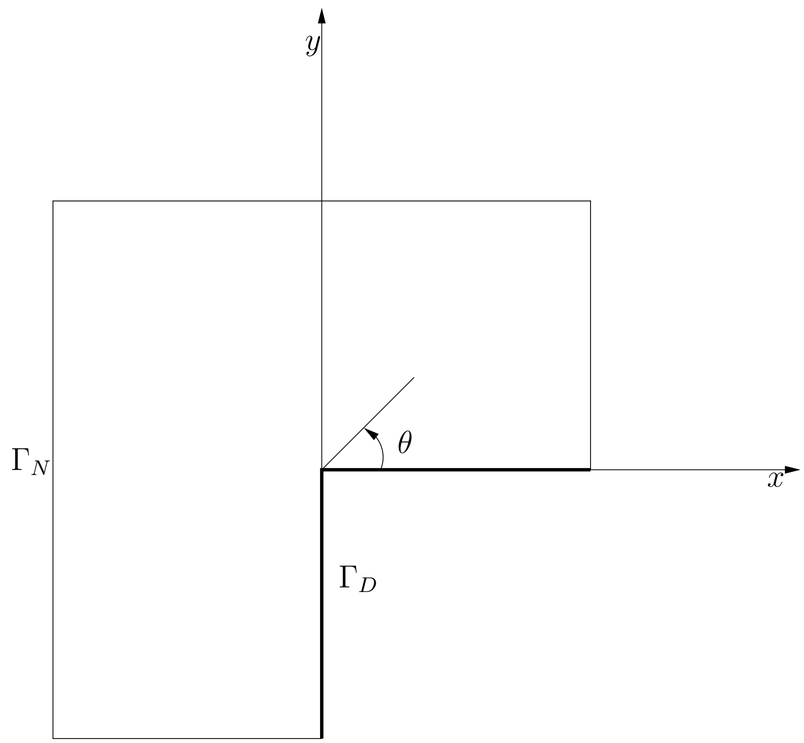


Figure 1.2: L-shaped domain

Table 1.1: Convergence results when using uniform meshes with $\lambda = 1.E + 014$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
1.414214e-001	5.163859e-002	2.601827e-001	3.090242e-002	8.579546e-002
9.428090e-002	4.354295e-002	2.089409e-001	2.182452e-002	6.957428e-002
6.428243e-002	3.375359e-002	1.680380e-001	1.399730e-002	5.509209e-002
5.656854e-002	3.149908e-002	1.566222e-001	1.231011e-002	5.132011e-002
4.714045e-002	2.853792e-002	1.416981e-001	1.026146e-002	4.640283e-002

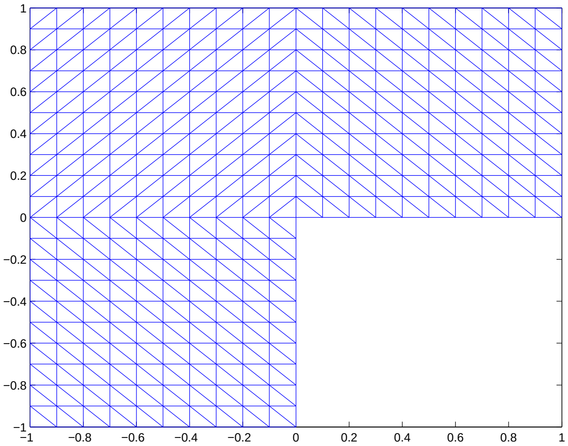


Figure 1.3: Uniform meshes

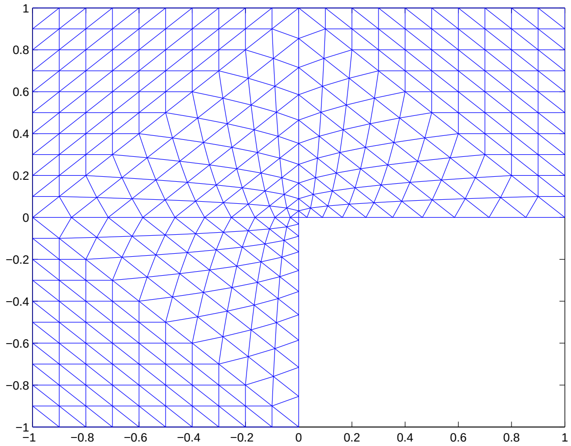


Figure 1.4: Refined meshes

Table 1.2: Convergence results when using uniform meshes with $\lambda = 1.E + 012$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
1.414214e-001	5.163858e-002	2.601830e-001	3.090301e-002	8.579590e-002
9.428090e-002	4.008911e-002	2.003295e-001	2.137504e-002	6.579243e-002
6.428243e-002	3.675379e-002	1.780374e-001	1.599629e-002	5.509133e-002
5.656854e-002	3.149934e-002	1.566211e-001	1.230785e-002	5.131864e-002
4.714045e-002	2.853807e-002	1.416971e-001	1.025871e-002	4.640141e-002

the refined meshes and in Table 1.1 - 1.4 the results of the errors for the uniform meshes. Finally we display the streamlines of the computational solutions (see figure 1.7, figure 1.8, figure 1.9, figure 1.10, figure 1.11, figure 1.12, figure 1.13 and figure 1.14).

Table 1.3: Convergence results when using uniform meshes with $\lambda = 1.E + 010$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
1.414214e-001	5.163953e-002	2.601799e-001	3.090034e-002	8.579277e-002
9.428090e-002	4.151113e-002	2.076128e-001	2.055004e-002	6.820973e-002
7.071068e-002	3.679594e-002	1.769789e-001	1.574209e-002	5.817971e-002
5.656854e-002	3.150377e-002	1.566104e-001	1.229235e-002	5.130741e-002
4.714045e-002	2.870350e-002	1.423981e-001	1.050813e-002	4.797506e-002

Table 1.4: Convergence results when using uniform meshes with $\lambda = 1.E + 008$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
1.414214e-001	5.159269e-002	2.597236e-001	3.087883e-002	8.570153e-002
9.428090e-002	4.146930e-002	2.072321e-001	2.053559e-002	6.813217e-002
7.071068e-002	3.550324e-002	1.767544e-001	1.537506e-002	5.800856e-002
5.656854e-002	3.146816e-002	1.563070e-001	1.228360e-002	5.124399e-002
4.714045e-002	2.870350e-002	1.423981e-001	1.220813e-002	4.797506e-002

Table 1.5: Convergence results when using refined meshes with $\lambda = 1.E + 014$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
1.727505e-001	2.479220e-002	1.039259e-001	2.709075e-002	3.218993e-002
1.167855e-001	1.672979e-002	6.856479e-002	1.803318e-002	2.146092e-002
8.819391e-002	1.261839e-002	5.175323e-002	1.351628e-002	1.619474e-002
7.084489e-002	1.016007e-002	4.169446e-002	1.094806e-002	1.328097e-002
5.919820e-002	8.550739e-003	3.480747e-002	8.992277e-003	1.085695e-002

Table 1.6: Convergence results when using refined meshes with $\lambda = 1.E + 012$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
1.727505e-001	2.467295e-002	1.021431e-001	2.702967e-002	3.193313e-002
1.167855e-001	1.663992e-002	6.863181e-002	1.798801e-002	2.139741e-002
8.819391e-002	1.255935e-002	5.177089e-002	1.348680e-002	1.612946e-002
7.084489e-002	1.016007e-002	4.169446e-002	1.094806e-002	1.328097e-002
5.919820e-002	8.446944e-003	3.479323e-002	8.988435e-003	1.083321e-002

Table 1.7: Convergence results when using refined meshes with $\lambda = 1.E + 010$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
1.727505e-001	2.467223e-002	1.021420e-001	2.702977e-002	3.193291e-002
1.167855e-001	1.663157e-002	6.863619e-002	1.799182e-002	2.140394e-002
8.819391e-002	1.255929e-002	5.176968e-002	1.348637e-002	1.612834e-002
7.084489e-002	1.010000e-002	4.159641e-002	1.078532e-002	1.295108e-002
5.919820e-002	8.448403e-003	3.479281e-002	8.987573e-003	1.083218e-002

Table 1.8: Convergence results when using refined meshes with $\lambda = 1.E + 008$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
1.727505e-001	2.464250e-002	1.019259e-001	2.702076e-002	3.188992e-002
1.167855e-001	1.660903e-002	6.848025e-002	1.798592e-002	2.137182e-002
8.819391e-002	1.254092e-002	5.164626e-002	1.348198e-002	1.610233e-002
7.084489e-002	1.008196e-002	4.149613e-002	1.078278e-002	1.293147e-002
5.919820e-002	8.434122e-003	3.470283e-002	8.984371e-003	1.081153e-002

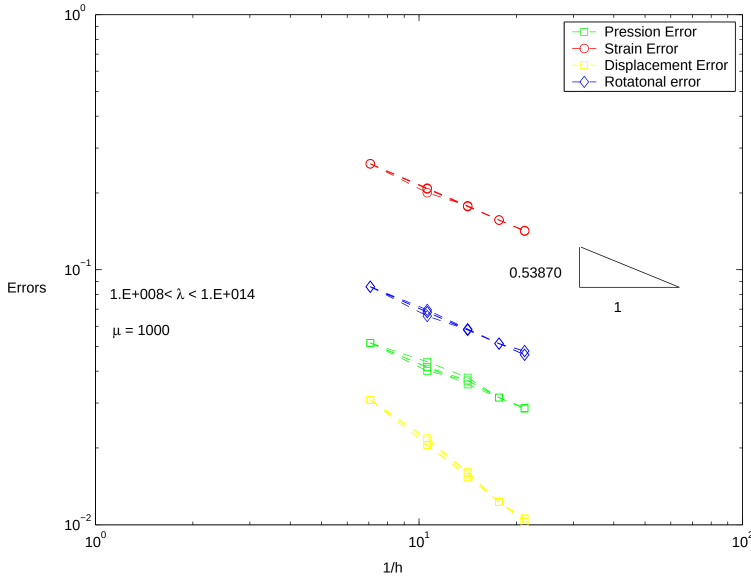


Figure 1.5: Error on uniform meshes

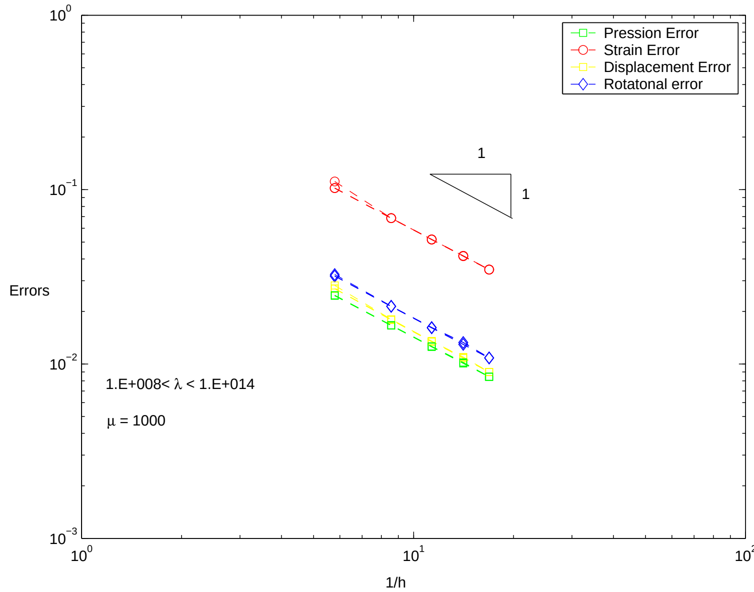


Figure 1.6: Error on refined meshes

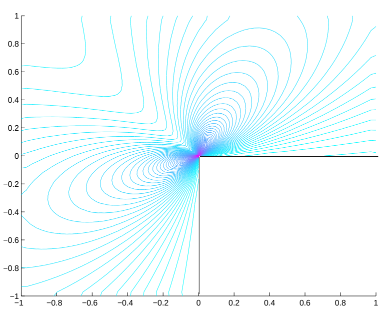


Figure 1.7: Streamlines of the Strain in y direction

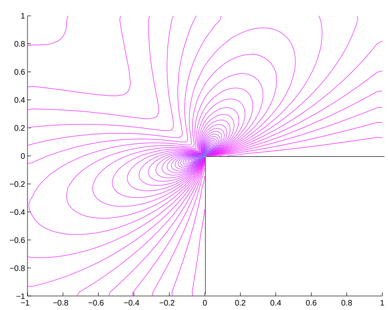


Figure 1.8: Streamlines of the Strain in x direction

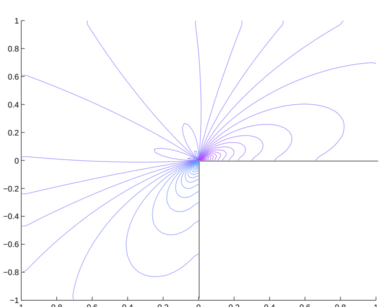


Figure 1.9: Streamlines of $\sigma_{1,2}$

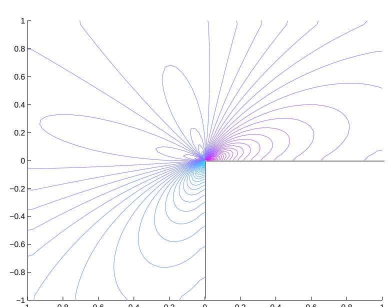


Figure 1.10: Streamlines of $\sigma_{2,1}$

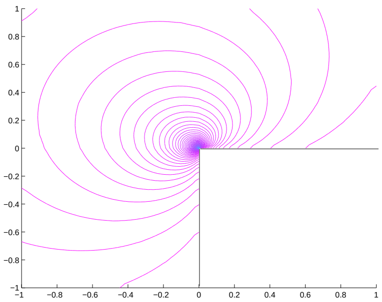


Figure 1.11: Streamlines of the Rotational

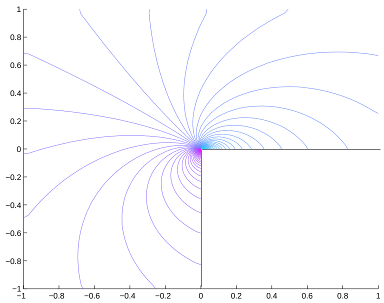


Figure 1.12: Streamlines of the Pressure

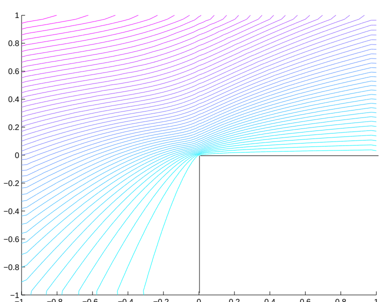


Figure 1.13: Streamlines of the Displacement in x direction

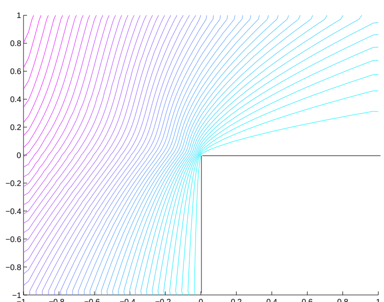


Figure 1.14: Streamlines of the Displacement in y direction

2

A POSTERIORI ERROR ESTIMATION

Contents

The a posteriori error estimates were introduced in 1978 by Babuška and Rheinboldt [4, 5]. At the difference of the a priori error estimates, the a posteriori error estimates allows to control the exact error by a quantity depending only on the triangulation, of the data of problem (member of right-hand side, boundary conditions, parameters of the physical models) and of the computed solution (thus known). Since the work of Babuška and Rheinboldt, the interest for such estimates has increased considerably. This interest is mainly due to the need to obtain precise numerical results without high calculation effort. Indeed, the a posteriori error analysis allows to determine explicitly if the computed approximate solution of the exact solution is an approximation with sufficient precision for the needs of the engineers. Moreover closely connected to optimize calculations, the a posteriori error estimates allow to refine certain parts of the triangulation according to the approximate solution.

In this chapter, we propose a residue based reliable and efficient error estimator for the dual mixed finite element method. With the help of a specific generalized Helmholtz decomposition of the error on the strain tensor and the classical decomposition of the error on the gradient of the displacements, we show that our global error estimator is reliable. Efficiency of our estimator follows by using classical inverse estimates.

The lower and upper error bounds obtained are uniform with respect to the Lamé coefficient λ, in particular avoiding locking phenomena.

2.1 Introduction

Several works have already been made on some various mixed finite elements methods concerning a posteriori error estimators. D. Braess and R. Verfürth [6] is apparently the first paper introducing an error estimator for the dual mixed finite element discretization of the Poisson equation. Similarly an error estimator for a mixed formulation which is based on the mixed variational principle of Hellinger and Reisner for the linear elasticity is established by D. Braess and al. in [37]. These estimators are efficient with respect to mesh-dependent norms but only reliable in the standard norms of the given problem. A. Alonso [2], C. Carstensen [10, 9] and M. Lonsing and R. Verfürth [29] circumvent this difficulty by the use of the Helmholtz decomposition of square-integrable tensor. In contrast to C.Carstensen [10], M. Lonsing and R. Verfürth [29] do not use differently weighted norms for the upper and lower error bound and have found still supposing the H^2-regularity hypothesis of the solution an error estimator reliable and efficient and furthermore robust for nearly incompressible materials. Wholmuth and Hoppe in [40] compare four different kinds of error estimators for the dual mixed finite element discretization of linear second order boundary value problems. In this section, we are concerned by the construction of an efficient and reliable a posteriori error estimator for the new dual mixed formulation introduced by M. Farhloul and M. Fortin [22]. A priori optimal error estimates uniform in λ have been proved in chapter 2 by imposing appropriate refinement rules near the corners. In this chapter, we propose an a posteriori error analysis for the errors:

$$\varepsilon := \sigma - \sigma_h, \quad P := p - p_h, \quad r := \omega - \omega_h \quad \text{and} \quad e := u - u_h.$$

Our analysis release on a residual error indicator η, which is based on residues on each triangle $T \in \mathcal{T}_h$ and jumps across the interelement boundaries $E \in \mathcal{E}_h$. Our goal in this chapter is to prove reliabilty and efficiency of the indicator η uniformly in λ and h, in particular avoiding locking phenomena.

The proof of the reliability of our indicator is based on some generalized Helmohltz decomposition of tensor fields. Efficiency follows by using classical inverse estimates [38], see paragraph 2.3.3 for more details.

Let us note that contrarily to [10] and [9], we do not suppose the H^2-regularity of the displacement field and we estimate the error with respect to the L^2-norm and not a weighted norm.

The schedule of this chapter is the following one: Section 2.2 recalls the discretization of our problem and we give some preliminaries and notations. In section 2.3 we establish some results on tensor fields like some particular Helmholtz decomposition and a generalization of the results of [8] concerninig the estimation of the trace of tensor fields. At the end of the section we recall some standard tools, namely some inverse inequalities and interpolation error estimates for Clément's interpolant. In section 2.4 we establish the efficiency and reliability of our error indicator η. Section 2.5 treats the case of a multiply-connected domain by using an adapted Helmholtz decomposition of tensor fields. To our knowledge this decomposition seems to be new. Section 2.6 describes an appropriate adaptive mesh-refinement algorithm. In section 2.7, we present some numerical tests which confirm our theoretical analysis. We end this chapter by a conclusion in section 2.8.

2.2 Preliminaries and notations

Let us first recall that Ω is discretized by a regular family of triangulations $(\mathcal{T}_h)_{h>0}$ in the sense of [11]. The set of all interior and boundary edges of the skeleton of the triangulation $\mathcal{T}_h$ will be denoted by $\mathcal{E}_h$. We then have $\mathcal{E}_h = \mathcal{E}_\Omega \cup \mathcal{E}_D \cup \mathcal{E}_N$ where $\mathcal{E}_\Omega$ denotes the set of all interior edges and $\mathcal{E}_D$, $\mathcal{E}_N$ denotes the collection of all edges contained in Γ_D and Γ_N respectively. The measure of an element or edge is denoted by $|T|$ and $|E|$, respectively. For each $E \in \mathcal{E}_h$, we fix a normal n to E such that n_E coincides with the exterior normal to $\partial\Omega$ if $E \subset \partial\Omega$. Additionally, we denote by t_E, the tangent vector $t_E = n_E^\perp$ such that (n_E, t_E) is a direct basis of $\mathbb{R}^2$. The *jump* of some (scalar or vector valued) function v across an edge E at a point $y \in E$ is then defined as

$$[\![v(y)]\!]_E := \begin{cases} \lim\limits_{\alpha \to +0} v(y+\alpha n) - v(y-\alpha n) & \text{for an interior edge E,} \\ v(y) & \text{for a boundary edge E .} \end{cases}$$

Given an $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$ and $x \in \mathcal{N}$ we then define ω_T, ω_E, and ω_x the union of all elements having a common edge with T, of all elements admitting E as an edge and of all elements admitting x as an vertex respectively (see figure 2.1).

We then state the continuous dual mixed formulation Find $(\sigma, p) \in \Sigma_0$ and $(u, \omega) \in$

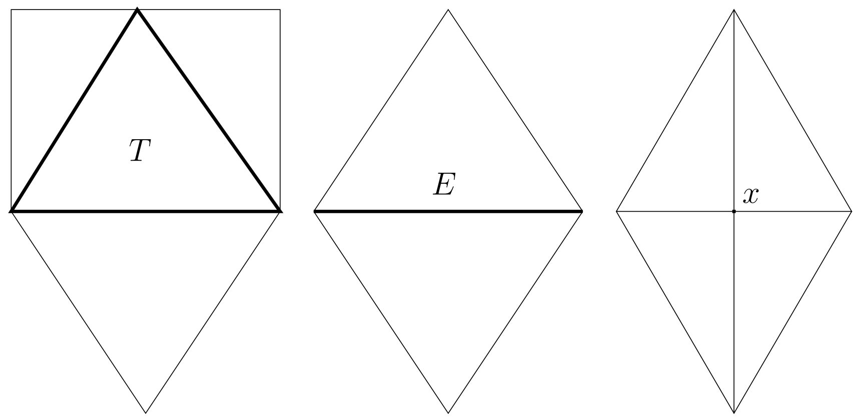


Figure 2.1: domains $\omega_T,\ \omega_E$ and ω_x left to right

M such that :

$$\begin{cases} \frac{1}{2\mu}(\sigma,\tau) \ + \ \frac{1}{\lambda}(p,q) + \ (div\ (\tau - q\delta), u) \ + \ (as\ (\tau),\omega) \ = \ 0, \quad \forall\ (\tau,q)\ \in\ \Sigma_0 \\ \\ (div\ (\sigma - p\delta), v) \ + \ (as\ (\sigma),\theta) \ + \ (f,v) \ = \ 0, \quad \forall\ (v,\theta)\ \in\ M. \end{cases} \tag{2.2.1}$$

For each fixed triangulation $\mathcal{T}_h$, we introduce the finite dimensional spaces $\Sigma_{0,h}$ and $V_h \times W_h$ of Σ and M respectively defined in the following way

$$\begin{aligned} \Sigma_{0,h} := \{(\tau_h, q_h) \in \Sigma_0;\ \forall T \in \mathcal{T}_h :\ & q_{h|T} \in \mathbb{P}_1(T) \ \text{ and} \\ & (\tau_h - q_h\delta)_{|T} \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\ curl\ b_T]^2\}, \end{aligned} \tag{2.2.2}$$

$$V_h \times W_h := \{(v_h,\theta_h) \in V \times W;\ \forall T \in \mathcal{T}_h :\ v_{h|T} \in [\mathbb{P}_0(T)]^2 \ \text{ and } \theta_{h|T} \in [\mathbb{P}_1(T)\}. \tag{2.2.3}$$

We introduce the discretized problem: find $(\sigma_h, p_h)\ \in\ \Sigma_{0,h}$ and $(u_h,\omega_h)\ \in\ V_h \times W_h$ such that

$$\begin{cases} \frac{1}{2\mu}(\sigma_h,\tau_h) \ + \ \frac{1}{\lambda}(p_h,q_h) + \ (div\ (\tau_h - q_h\delta), u_h) \ + \ (as\ (\tau_h),\omega_h) = 0, \quad \forall\ (\tau_h,q_h) \in \Sigma_{0,h} \\ \\ (div\ (\sigma_h - p_h\delta), v_h) \ + \ (as\ (\sigma_h),\theta_h) \ + \ (f,v_h) = 0, \quad \forall\ (v_h,\theta_h) \in V_h \times W_h. \end{cases} \tag{2.2.4}$$

Recall that

$$\sigma := 2\mu\epsilon(u),\ \ p := -\lambda div\ (u) \ \text{ and } \ \omega := \frac{1}{2} rot\ (u) := \frac{1}{2}(\frac{\partial u_2}{\partial x_1} \ - \ \frac{\partial u_1}{\partial x_2})$$

We close this section by introducing, for any bounded domain Ω in $\mathbb{R}^2$ with Lipschitz boundary, the space

$$H(rot\ ,\Omega) \ := \ \{\tau \ \in \ [L^2(\Omega)]^{2\times 2};\ rot\ (\tau) \ \in \ [L^2(\Omega)]^2\}.$$

We recall the following formula of integration by parts: for all $\rho\ \in\ H(rot\ ,\Omega)$ and for all $\varphi\ \in\ [H^1(\Omega)]^2$

$$\int_\Omega rot\ (\rho).\varphi\ dx - \int_\Omega \rho : Curl\ \varphi \ = \ \int_{\partial\Omega} \rho.t\ \varphi\ ds. \tag{2.2.5}$$

2.3 Analytical tools

2.3.1 Decomposition for tensor fields

For our further analysis, we require the following results on the decomposition of tensor fields which are essential for the subsequent proofs:

Proposition 2.3.1. *Let $\tau \in [L^2(\Omega)]^{2\times 2}$. Then there exist $p \in [H^1(\Omega)]^2$ with $p = 0$ on Γ_D and $\varphi \in [H^1(\Omega)]^2$ with $\varphi =$ constant on each connected component of Γ_N such that*

$$\tau = \nabla p + Curl\ \varphi. \tag{2.3.1}$$

with the estimate

$$\|\nabla p\|_{0,\Omega} + \|\nabla\varphi\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}. \tag{2.3.2}$$

Proof: Let $p \in [H^1_{\Gamma_D}(\Omega)]^2$ be the unique solution of the following variational problem:

$$\int_\Omega (\tau - \nabla p) : \nabla\psi\ dx = 0 \quad \text{for all } \psi \in [H^1_{\Gamma_D}(\Omega)]^2. \tag{2.3.3}$$

This last equation implies in particular, that $\tau - \nabla p$ is divergence free in the sense of distributions and moreover by Green's formula, we can now write

$$\begin{aligned} \langle(\tau - \nabla p).n, \psi\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} &= \int_\Omega (\tau - \nabla p) : \nabla\psi\ dx + \int_\Omega div\ (\tau - \nabla p).\psi\ dx \\ &= 0 \quad \text{for all } \psi \in [H^1_{\Gamma_D}(\Omega)]^2, \quad (\text{by } (2.5.5)). \end{aligned}$$

Thus

$$\langle(\tau - \nabla p).n, \psi\rangle_{H^{-1/2}(\Gamma_N),H^{1/2}(\Gamma_N)} = 0 \quad \text{for all } \psi \in [H^1_{\Gamma_D}(\Omega)]^2.$$

Applying Theorem 3.1 p.37 of [25] line by line to the tensor $\tau - \nabla p$ we conclude that there exist a function $\varphi \in [H^1(\Omega)]^2$ such that

$$\tau - \nabla p = Curl\ \varphi.$$

It remains to prove that $\varphi = const$ on Γ_N. We have

$$\begin{aligned} \langle(\tau - \nabla p).n, \psi\rangle_{H^{-1/2}(\Gamma_N),H^{1/2}(\Gamma_N)} &= \langle Curl\ \varphi.n, \psi\rangle_{H^{-1/2}(\Gamma_N),H^{1/2}(\Gamma_N)} \\ &= \langle\frac{\partial\varphi}{\partial\tau}, \psi\rangle_{H^{-1/2}(\Gamma_N),H^{1/2}(\Gamma_N)} \\ &= 0 \quad \text{for all } \psi \in [H^1_{\Gamma_D}(\Omega)]^2. \end{aligned}$$

Thus, in the sense of the distributions

$$\frac{\partial\varphi}{\partial\tau} = 0 \quad on\ \Gamma_N,$$

that is

$$\varphi = const \quad on\ \Gamma_N.$$

Now, let us prove the estimate (2.3.2). Taking as test function $\psi = p$ in (2.3.3), we obtain:

$$\begin{aligned} \|\nabla p\|_{0,\Omega}^2 &= \int_\Omega \tau : \nabla p \\ &\leq \|\nabla p\|_{0,\Omega}\|\tau\|_{0,\Omega} \\ &\leq \tfrac{1}{2}\|\nabla p\|_{0,\Omega}^2 + \tfrac{1}{2}\|\tau\|_{0,\Omega}^2. \end{aligned}$$

Thus

$$\|\nabla p\|_{0,\Omega} \leq \|\tau\|_{0,\Omega}.$$

On the other hand, we have

$$\|\nabla\varphi\|_{0,\Omega} = \|Curl\ \varphi\|_{0,\Omega} = \|\tau - \nabla p\|_{0,\Omega} \leq \|\nabla p\|_{0,\Omega} + \|\tau\|_{0,\Omega} \leq 2\|\tau\|_{0,\Omega}.$$

Consequently we have proved (2.3.2). ■

Proposition 2.3.2. *Let $\tau \in [L^2(\Omega)]^{2\times 2}$. Then there exist $z \in [H^1(\Omega)]^2$ with $z = 0$ on Γ_D, $\psi \in [H^1(\Omega)]^2$ with $\psi =$ constant on each connected component of Γ_N and $q \in L^2(\Omega)$ such that*

$$\begin{cases} \tau = 2\mu\epsilon(z) - q\delta + Curl\ (\psi) \\ \frac{1}{\lambda}q + div\ (z) = 0. \end{cases}$$

Moreover the following estimate holds:

$$\|\epsilon(z)\|_{0,\Omega} + \|\nabla\psi\|_{0,\Omega} + \|q\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}. \tag{2.3.4}$$

Proof: Let $z \in [H^1_{\Gamma_D}(\Omega)]^2$ be the unique solution of the following variational problem:

$$\int_\Omega (-2\mu\epsilon(z) - \lambda div\ (z)\delta + \tau) : \nabla\varphi\ dx = 0 \quad \text{for all } \varphi \in [H^1_{\Gamma_D}(\Omega)]^2. \tag{2.3.5}$$

This implies that $-2\mu\epsilon(z) + q\delta + \tau$, with $q = -\lambda div\ u$, is a divergence free tensor in the sense of distributions. By Green's formula, we can write

$$\begin{aligned} \langle(-2\mu\epsilon(z) + q\delta + \tau).n, \varphi\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} &= \int_\Omega (-2\mu\epsilon(z) + q\delta + \tau) : \nabla\varphi\ dx + \\ &\quad \int_\Omega div\ (-2\mu\epsilon(z) + q\delta + \tau).\varphi\ dx \\ &= 0 \ \text{ for all } \varphi \in [H^1_{\Gamma_D}(\Omega)]^2. \end{aligned}$$

Thus

$$\langle(-2\mu\epsilon(z) + q\delta + \tau).n, \varphi\rangle_{H^{-1/2}(\Gamma_N),H^{1/2}(\Gamma_N)} = 0 \quad \text{for all } \varphi \in [H^1(\Omega)]^2.$$

Applying Theorem 3.1 p.37 of [25] line by line to the tensor $-2\mu\epsilon(z) + q\delta + \tau$ we conclude that there exists a function $\psi \in [H^1_{\Gamma_D}(\Omega)]^2$ so that

$$-2\mu\epsilon(z) + q\delta + \tau = Curl\ \psi.$$

It remains to prove that $\psi = const$ on Γ_N. We have

$$\begin{aligned}\langle(-2\mu\epsilon(z) + q\delta + \tau).n, \varphi\rangle_{H^{-1/2}(\Gamma_N),H^{1/2}(\Gamma_N)} &= \langle(Curl\ \psi).n, \varphi\rangle_{H^{-1/2}(\Gamma_N),H^{1/2}(\Gamma_N)} \\ &= \langle\frac{\partial\psi}{\partial\tau}, \varphi\rangle_{H^{-1/2}(\Gamma_N),H^{1/2}(\Gamma_N)} \\ &= 0 \quad \text{for all } \varphi \in [H^1_{\Gamma_D}(\Omega)]^2.\end{aligned}$$

Thus, in the sense of the distributions

$$\frac{\partial\psi}{\partial\tau} = 0 \quad \text{on } \Gamma_N,$$

that is

$$\psi = const \quad \text{on } \Gamma_N.$$

To prove the estimate (2.3.4), we take as test function $\varphi = z$ in equation (2.3.5) and we obtain

$$-2\mu\|\epsilon(z)\|^2_{0,\Omega} - \lambda\int_\Omega |div\ z|^2\ dx + \int_\Omega \tau : \nabla z\ dx = 0.$$

By Cauchy-Schwarz's and Korn's inequalities, we derive from the previous equation

$$2\mu\|\epsilon(z)\|^2_{0,\Omega} \leq c\|\tau\|_{0,\Omega}\|\epsilon(z)\|_{0,\Omega},$$

which gives us by simplifying

$$\|\epsilon(z)\|_{0,\Omega} \leq \tfrac{c}{2\mu}\|\tau\|_{0,\Omega}.$$

By Lemma 3.4 [30], there exists $v \in [H^1_{\Gamma_D}(\Omega)]^2$ such that

$$div\ v = div\ z,$$

and

$$\|\nabla v\|_{0,\Omega} \lesssim \|div\ z\|_{0,\Omega}.$$

Equation (2.3.5) with $\varphi = v$ yields

$$\begin{aligned}\lambda\int_\Omega |div\ z|^2\ dx &= \int_\Omega \tau : \nabla v\ dx - 2\mu\int_\Omega \epsilon(z) : \nabla v\ dx \\ &\lesssim \|\tau\|_{0,\Omega}\|\nabla v\|_{0,\Omega} + 2\mu\|\epsilon(z)\|_{0,\Omega}\|\nabla v\|_{0,\Omega} \\ &\lesssim \|\tau\|_{0,\Omega}\|div\ z\|_{0,\Omega} \quad \text{by the above bound on } \|\epsilon(z)\|_{0,\Omega}.\end{aligned}$$

Thus

$$\|q\|_{0,\Omega} = \|\lambda div\ z\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}.$$

By the triangle inequality, we get

$$\begin{aligned}\|Curl\ \psi\|_{0,\Omega} &\leq \|\tau\|_{0,\Omega} + 2\mu\|\epsilon(z)\|_{0,\Omega} + \sqrt{2}\|q\|_{0,\Omega} \\ &\lesssim \|\tau\|_{0,\Omega}.\end{aligned}$$

Consequently, we have proved (2.3.4).

■

In the following, one will need to estimate the trace of a tensor field which is not necessarily of null average. To do that, we adapt the proof of proposition 3.1 p.161 of [8].

2.3.2 Estimate of the trace of a tensor field

The following estimate of the trace of a tensor field holds. We recall that $\tau^D := \tau - \frac{1}{2} tr\,(\tau)\delta$.

Lemma 2.3.3. *For every $\tau \in [H(div\,,\Omega)]^2 := \{\tau \in [L^2(\Omega)]^{2\times 2}, div\,(\tau) \in [L^2(\Omega)]^2\}$ such that $\tau.n = 0$ on Γ_N, the following estimate holds*

$$\|tr\,(\tau)\|_{0,\Omega} \lesssim \|\tau^D\|_{0,\Omega} + \|div\,(\tau)\|_{0,\Omega}. \tag{2.3.6}$$

Proof: Let us consider $\tau \in [H(div\,,\Omega)]^2$ such that $\tau.n = 0$ on Γ_N. By Lemma 3.4 of [30], there exists $v \in [H^1(\Omega)]^2$ solution of the problem

$$\begin{cases} div\,(v) = tr\,(\tau) & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_D \end{cases}$$

and

$$|v|_{1,\Omega} \lesssim \|tr\,(\tau)\|_{0,\Omega}. \tag{2.3.7}$$

Therefore

$$\begin{aligned}
\|tr\,(\tau)\|^2_{0,\Omega} &= \int_\Omega tr\,(\tau) div\,(v)\;dx \\
&= \int_\Omega (\tau : \delta) div\,(v)\;dx \\
&= 2\int_\Omega \tau : \frac{1}{2} tr\,(\nabla v)\delta\;dx \\
&= 2\int_\Omega \tau : (\nabla v - \nabla v^D)\;dx \\
&= -2\int_\Omega \tau^D : \nabla v\;dx - 2\int_\Omega div\,(\tau) v\;dx \qquad \text{by Green's formula} \\
&\lesssim 2\|\tau^D\|_{0,\Omega}|v|_{1,\Omega} + 2\|div\,(\tau)\|_{0,\Omega}\|v\|_{0,\Omega}. \qquad (2.3.8)
\end{aligned}$$

The estimates (2.3.8) and (2.3.7) give immediately (2.3.6). ■

2.3.3 Bubble functions, extension operator, and inverse inequalities

We briefly summarize the relevant results (see [17] for more details). We need two types of bubble functions, namely b_T and b_E associated with an element T and an edge E, respectively. Let $\mathcal{N}$ be the set of all (internal and boundary) nodes of the mesh. Denoting by $\lambda^T_{x_i},\ x_i \in \mathcal{N} \cap \partial T,\ i = 1, 2, 3$, the barycentric coordinates of an element T and by $x^E_i \in \mathcal{N} \cap E,\ i = 1, 2$, the vertices of the edge $E \subset \partial T$, we define

$$b_T = 27\lambda^T_{x_1}\lambda^T_{x_2}\lambda^T_{x_3} \text{ and } b_E = 4\lambda^{T_i}_{x^E_1}\lambda^{T_i}_{x^E_2} \text{ if } x \in T_i\ (i = 1, 2)$$

where T_1 and T_2 are the adjacent triangles to the edge E. One recalls that

$$b_T = 0 \ on\ \partial T,\ b_E = 0 \text{ on } \partial\omega_E,\ \|b_T\|_{\infty,T} = \|b_E\|_{\infty,\omega_E} = 1.$$

For an edge $E \subset \partial T$ using temporarily the local orthogonal coordinates system (x, y) such that E is included into the x-axis, then we define the extension operator $\mathrm{F}_{\mathrm{ext}}$ (see figure 2.2)

$$\begin{array}{rcl} \mathrm{F}_{\mathrm{ext}} : C(E) & \longrightarrow & C(\omega_E) \\ v_E & \longmapsto & \mathrm{F}_{\mathrm{ext}}(v_E). \end{array}$$

With $\mathrm{F}_{\mathrm{ext}}(v_E)(x, y) = v_E(x)$ for every $(x, y) \in \omega_E,\ \omega_E$ denoting the union of the triangles

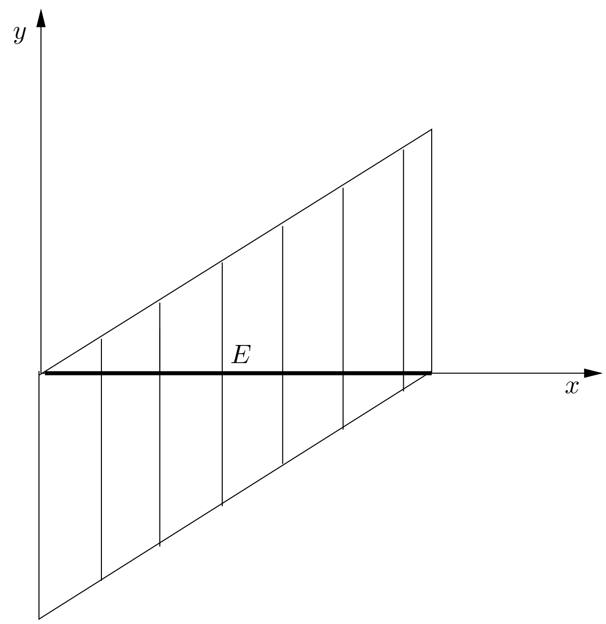


Figure 2.2: Level lines of $F_{ext}(v_E)$

admitting E as an edge. Now we may recall the so-called inverse inequalities that are proved using classical scaling argument techniques [38].

Lemma 2.3.4. *Let $v_T \in I\!P_{k_0}(T)$ and $v_E \in I\!P_{k_1}(E)$, for some nonnegative integers k_0 and k_1. Then the following inequalities hold, the inequality constants depending on the polynomial degree k_0 or k_1 but not on T, E or v_T, v_E:*

$$\|v_T b_T^{1/2}\|_{0,T} \sim \|v_T\|_{0,T} \tag{2.3.9}$$

$$\|\nabla(v_T b_T)\|_{0,T} \lesssim h_T^{-1}\|v_T\|_{0,T} \tag{2.3.10}$$

$$\|v_E b_E^{1/2}\|_{0,E} \sim \|v_E\|_{0,E} \tag{2.3.11}$$

$$\|\ F_{ext}(v_E) b_E\|_{0,T} \lesssim h_E^{1/2}\|v_E\|_{0,E}, \quad \forall\ T \subset \omega_E \tag{2.3.12}$$

$$\|\nabla(\ F_{ext}(v_E) b_E)\|_{0,T} \lesssim h_E^{1/2} h_T^{-1}\|v_E\|_{0,E}, \quad \forall\ T \subset \omega_E \tag{2.3.13}$$

2.3.4 Some interpolation operators

For the analysis we require some interpolation operator that maps a function from $H^1(\Omega)$ to some continuous, piecewise polynomial functions $S(\Omega, \mathcal{T}_h)$. Hence Lagrange interpolation is unsuitable, but Clément like interpolant is more appropriate. Recall that the nodal basis function $\varphi_x \in S(\Omega, \mathcal{T}_h)$ associated with a node x is uniquely determined by the condition

$$\varphi_x(y) = \delta_{x,y} \quad \forall y \in \mathcal{N}.$$

Now, let us recall the definition of the Clément interpolation opetrator

Definition 2.3.5 (Clément interpolation operator). *We define the Clément interpolation operator* $I_{Cl} : H^1(\Omega) \to S(\Omega, \mathcal{T}_h)$ *by*

$$I_{Cl}v := \sum_{x\in\mathcal{N}} \frac{1}{|\omega_x|}\Big(\int_{\omega_x} v\Big)\varphi_x.$$

Finally we may state the interpolation estimates [12].

Lemma 2.3.6 (Clément interpolation estimates). *Let* $v \in H^1(\Omega)$. *If the triangulation* $\mathcal{T}_h$ *is regular then for any* $E \in \mathcal{E}_h$ *and for any* $T \in \mathcal{T}_h$ *it holds:*

$$h_E^{-\frac{1}{2}}\|v - I_{Cl}v\|_{0,E} \lesssim \|\nabla v\|_{\omega_E}, \tag{2.3.14}$$

$$h_T^{-1}\|v - I_{Cl}v\|_{0,T} \lesssim \|\nabla v\|_{\omega_T}. \tag{2.3.15}$$

Now we recall that for any $\tau \in [H^1(\Omega)]^{2\times 2}$ the Brezzi Douglas Marini interpolate $BDM_1\tau \in H_h$ of τ is uniquely determined on each element $T \in \mathcal{T}_h$ by the condition

$$\int_{\partial K} BDM_1(\tau)n.p_1\; ds = \int_{\partial K} \tau n.p_1\; ds, \quad \forall p_1 \in [R_1(\partial K)]^2,\ \forall K \in \mathcal{T}_h,$$

where we recall that

$$R_1(\partial K) = \big\{\phi \in L^2(\partial K) :\ \phi_{|E} \in \mathbb{P}_1(E),\ \forall E \in \partial K\big\},$$

and

$$H_h = \big\{\tau_h \in [H(div\,, \Omega)]^2 :\ \tau_{h|K} \in BDM_1(K)^2 = [\mathbb{P}_1(K)]^{2\times 2},\ \forall K \in \mathcal{T}_h\big\}.$$

Now, let us recall the following interpolation estimate (see [8]).

Lemma 2.3.7. *Let* $\tau \in [H^1(\Omega)]^{2\times 2}$. *If the triangulation* $\mathcal{T}_h$ *is regular then the following estimate holds*

$$\|\tau - BDM_1(\tau)\|_{0,T} \lesssim h_T|\tau|_{1,T}, \quad \forall T \in \mathcal{T}_h.$$

2.4 Residual error estimators

In this section we propose a residual based error indicator for efficient and reliable which we would like to use to perform local grid refinement where it is needed always with an aim of increasing the accuracy of the finite element approximation. Different strategies exist for the refinement of the triangulation , which are in detail presented in [38]. Note that the characteristic to have a lower error bound for the error of the approximation is called efficiency of the a posteriori error indicator. In addition the error indicator must have additionally global upper error bound allowing it to be used in an adaptive procedure of automatic mesh refinement.

We propose an a posteriori error estimator for the errors $\varepsilon = \sigma - \sigma_h$, $P = p - p_h$, $r = \omega - \omega_h$ and $e = u - u_h$. For our further analysis, we will use the generalized Helmholtz decompositon of tensor fields studied in subsection 2.3.1. The local estimator accounts for the residues on the triangle T and the jumps across the edges $E \subset \partial T$. In the following , we denote the jump in the tangential direction of a discrete tensor ρ_h by $\left[\!\left[\rho_h.t_E\right]\!\right]_E$. For any $T \in \mathcal{T}_h$, the local residual error estimator η_T is defined by:

$$\begin{aligned}\eta_T^2 : \; = \; & \|f - P_h^0 f\|_{0,T}^2 + \tfrac{1}{4}\|as\ (\sigma_h)\|_{0,T}^2 + \|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\|_{0,T}^2 + h_T^2\|\sigma_h + 2\mu\omega_h\chi\|_{0,T}^2 + \\ & h_T^2\|rot\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T}^2 + \sum_{E\subset\partial T} h_E\|\left[\!\left[(\sigma_h + 2\mu\omega_h\chi).t_E\right]\!\right]_E\|_{0,E}^2. \end{aligned} \tag{2.4.1}$$

Here P_h^0 denotes the L^2-orthogonal projection onto the space of piecewise constant functions on the triangulation $\mathcal{T}_h$. Let us recall that χ denotes the antisymmetric matrix defined by: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let us observe in (2.4.1) that $f - P_h^0 f$ is the residual of $\sigma_h - p_h\delta$ with respect to the div operator and that the two following terms have their analogues null for the exact solution. Also $rot\ (\sigma_h + 2\mu\omega_h\chi)$ and $\left[\!\left[(\sigma_h + 2\mu\omega_h\chi).t_E\right]\!\right]_E$ have corresponding terms zero for the exact solution. The global residual error estimator is simply defined by:

$$\eta^2 := \sum_{T\in\mathcal{T}_h} \eta_T^2.$$

2.4.1 Proof of the reliability of the estimator

In this section we establish the global upper error bound by the estimator η.

Theorem 2.4.1 (upper error bound). *The next estimate holds*

$$\|\varepsilon\|_{0,\Omega} + \|r\|_{0,\Omega} + \|P\|_{0,\Omega} + \|e\|_{0,\Omega} \lesssim \eta. \tag{2.4.2}$$

The proof of this theorem is direct consequences of the following propositions. We begin by bounding the error $\epsilon := \sigma - \sigma_h$. Let us point out that all the estimates which will be established are independent of the Lamé coefficient λ for $\lambda > \lambda_0$.

Proposition 2.4.2. *The following estimate holds*

$$\|\varepsilon\|_{0,\Omega} \lesssim \eta. \tag{2.4.3}$$

Proof: Proposition 2.3.2 implies the existence of $z \in [H^1(\Omega)]^2$ with $z = 0$ on Γ_D, $q \in L^2(\Omega)$ and $\psi \in [H^1(\Omega)]^2$ with $\psi = const$ on Γ_N such that

$$\begin{aligned} \varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi &= 2\mu\epsilon(z) - q\delta + Curl\ (\psi) && (2.4.4) \\ \tfrac{1}{\lambda} q + div\ (z) &= 0. && (2.4.5) \end{aligned}$$

Moreover the following estimate holds:

$$\|\epsilon(z)\|_{0,\Omega} + \|\nabla\psi\|_{0,\Omega} + \|q\|_{0,\Omega} \lesssim \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}. \quad (2.4.6)$$

It follows from $\epsilon := \sigma - \sigma_h$ and equality (2.4.4) that $Curl\ (\psi)$ is a symmetric tensor field which implies that $div\ \psi = 0$. By triangle inequality we have

$$\begin{aligned} \|\varepsilon\|_{0,\Omega} &\leq \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega} + \|\tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega} \\ &= \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega} + \tfrac{1}{\sqrt{2}}\|as\ (\sigma_h)\|_{0,\Omega}. && (2.4.7) \end{aligned}$$

In view of the definition of the error estimator η, it suffices to bound $\|\varepsilon - \frac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}$. The above decomposition allows to write

$$\begin{aligned} \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}^2 &= \int_\Omega (\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi) : (2\mu\epsilon(z) - q\delta + Curl\ (\psi))\ dx \\ &= \int_\Omega \varepsilon : (2\mu\epsilon(z) - q\delta + Curl\ (\psi))\ dx \\ &\quad -\tfrac{1}{2}\int_\Omega as\ (\sigma_h)\chi : (2\mu\epsilon(z) - q\delta + Curl\ (\psi))\ dx \\ &= \int_\Omega \varepsilon : (2\mu\epsilon(z) - q\delta + Curl\ (\psi))\ dx \\ &\quad \text{as } 2\mu\epsilon(z) - q\delta + Curl\ (\psi) \quad \text{is a symmetric tensor field.} \end{aligned}$$

By (2.4.5) we may write

$$\begin{aligned} \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}^2 &= \int_\Omega \varepsilon : 2\mu\epsilon(z)\ dx - \int_\Omega 2\mu P(\tfrac{1}{\lambda} q + div\ (z))\ dx - \int_\Omega \varepsilon : q\delta\ dx \\ &\quad + \int_\Omega \varepsilon : Curl\ (\psi)\ dx \\ &= 2\mu \int_\Omega \varepsilon : \epsilon(z)\ dx - 2\mu \int_\Omega P div\ (z)\ dx - \tfrac{2\mu}{\lambda}\int_\Omega Pq\ dx - \int_\Omega tr\ (\varepsilon) q\ dx \\ &\quad + \int_\Omega \varepsilon : Curl\ (\psi)\ dx \\ &= 2\mu \int_\Omega \varepsilon : \epsilon(z)\ dx - 2\mu \int_\Omega P\delta : \epsilon(z)\ dx - \tfrac{2\mu}{\lambda}\int_\Omega Pq\ dx - \int_\Omega tr\ (\varepsilon) q\ dx \\ &\quad + \int_\Omega \varepsilon : Curl\ \psi\ dx \\ &= 2\mu \int_\Omega (\varepsilon - P\delta) : \epsilon(z)\ dx - \tfrac{2\mu}{\lambda}\int_\Omega Pq\ dx + \tfrac{2\mu}{\lambda}\int_\Omega pq\ dx + \int_\Omega tr\ (\sigma_h) q\ dx \\ &\quad + \int_\Omega \varepsilon : Curl\ (\psi)\ dx. \end{aligned}$$

Using Green's formula we may write:

$$\begin{aligned}\|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|^2_{0,\Omega} &= -2\mu \int_\Omega div\ (\varepsilon - P\delta).z\ dx + \mu \int_\Omega as\ (\sigma_h) rot\ z\ dx - \tfrac{2\mu}{\lambda} \int_\Omega Pq\ dx \\ &\quad + \tfrac{2\mu}{\lambda} \int_\Omega pq\ dx + \int_\Omega tr\ (\sigma_h) q\ dx + \int_\Omega \varepsilon : Curl\ \psi\ dx \\ &= 2\mu \int_\Omega (f - P_h^0 f).z\ dx + \mu \int_\Omega as\ (\sigma_h) rot\ (z)\ dx - \tfrac{2\mu}{\lambda} \int_\Omega Pq\ dx + \tfrac{2\mu}{\lambda} \int_\Omega pq\ dx \\ &\quad + \int_\Omega tr\ (\sigma_h) q\ dx + \int_\Omega \varepsilon : Curl\ (\psi)\ dx\end{aligned}$$

By $P = p - p_h$, we obtain the following decomposition formula:

$$\begin{aligned}\|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|^2_{0,\Omega} &= 2\mu \int_\Omega (f - P_h^0 f).z\ dx + \mu \int_\Omega as\ (\sigma_h) rot\ z\ dx + \tfrac{2\mu}{\lambda} \int_\Omega p_h q\ dx \\ &\quad + \int_\Omega tr\ (\sigma_h) q\ dx + \int_\Omega \varepsilon : Curl\ (\psi)\ dx \\ &= 2\mu \int_\Omega (f - P_h^0 f).z\ dx + \mu \int_\Omega as\ (\sigma_h) rot\ z\ dx + \int_\Omega \big(\tfrac{2\mu}{\lambda} p_h + tr\ (\sigma_h)\big)\ q\ dx \\ &\quad + \int_\Omega \varepsilon : Curl\ (\psi)\ dx. \qquad (2.4.8)\end{aligned}$$

To transform the last term of the right-hand side, let us consider $\psi_h := \mathrm{I}_{\mathrm{Cl}} \psi$. By the first equality of the continuous problem (2.2.1) with $(\tau, 0) = (Curl\ (\psi), 0) \in \Sigma_0$, we get

$$\tfrac{1}{2\mu} \int_\Omega \sigma : Curl\ (\psi)\ \ dx = 0.$$

Thus

$$\begin{aligned}\int_\Omega \varepsilon : Curl\ \psi\ dx &= -\int_\Omega \sigma_h : Curl\ \psi\ dx \\ &= \int_\Omega \sigma_h : Curl\ (\psi_h - \psi)\ dx - \int_\Omega \sigma_h : Curl\ (\psi_h)\ dx. \qquad (2.4.9)\end{aligned}$$

We now estimate separately the two terms of the right-hand side of (2.4.9) . For the first one, using Green's formula we get

$$\begin{aligned}\int_\Omega \sigma_h : Curl\ (\psi_h - \psi)\ dx &= \sum_{T \in \mathcal{T}_h} \int_T rot\ (\sigma_h).(\psi_h - \psi)\ dx - \sum_{T \in \mathcal{T}_h} \int_{\partial T} \sigma_h.t.(\psi_h - \psi)\ ds \\ &= \sum_{T \in \mathcal{T}_h} \int_T rot\ (\sigma_h).(\psi_h - \psi)\ dx - \sum_{E \in \mathcal{E}_h} \int_E [[\sigma_h.t_E]]_E.(\psi_h - \psi)\ ds.\end{aligned}$$

For the second term of the right-hand side of (2.4.9), taking as a test function $(Curl\ \psi_h, 0) \in \Sigma_{0,h}$ in the first equation of (2.2.4), we get

$$\begin{aligned}\int_\Omega \sigma_h : Curl\ \psi_h\ dx &= -2\mu \int_\Omega as\ (Curl\ \psi_h)\omega_h\ dx \\ &= -2\mu \int_\Omega div\ (\psi_h)\omega_h\ dx\end{aligned}$$

Remembering that $div\ (\psi) = 0$, it follows that:

$$\begin{aligned}
\int_\Omega \sigma_h : Curl\ \psi_h\ dx &= -2\mu \int_\Omega div\ (\psi_h - \psi)\omega_h\ dx \\
&= \sum_{T\in\mathcal{T}_h} \int_T 2\mu\nabla\omega_h : (\psi_h - \psi)\ dx - \sum_{T\in\mathcal{T}_h} \int_{\partial T} 2\mu\omega_h(\psi_h - \psi).n\ ds \\
&= \sum_{T\in\mathcal{T}_h} \int_T 2\mu\nabla\omega_h : (\psi_h - \psi)\ dx - \sum_{T\in\mathcal{T}_h} \int_{\partial T} 2\mu\omega_h\delta.n.(\psi_h - \psi)\ ds \\
&= \sum_{T\in\mathcal{T}_h} \int_T 2\mu\nabla\omega_h : (\psi_h - \psi)\ dx + \sum_{T\in\mathcal{T}_h} \int_{\partial T} 2\mu\omega_h\chi.t.(\psi_h - \psi)\ ds \\
&= -\sum_{T\in\mathcal{T}_h} \int_T rot\ (2\mu\omega_h\chi).(\psi_h - \psi)\ dx + \sum_{E\in\mathcal{E}_h} \int_E [[2\mu(\omega_h\chi).t_E]]_E.(\psi_h - \psi)\ ds.
\end{aligned}$$

Combined with the previous equality and (2.4.9), we obtain:

$$\begin{aligned}
\int_\Omega \varepsilon : Curl\ (\psi)\ dx &= \sum_{T\in\mathcal{T}_h} \int_T rot\ (\sigma_h + 2\mu\omega_h\chi).(\psi_h - \psi)\ dx \\
&\quad - \sum_{E\in\mathcal{E}_h} \int_E [[(\sigma_h + 2\mu\omega_h\chi).t_E]]_E.(\psi_h - \psi)\ ds \\
&\leq \sum_{T\in\mathcal{T}_h} \|rot\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T}\|\psi_h - \psi\|_{0,T} \\
&\quad + \sum_{E\in\mathcal{E}_h} \|[[(\sigma_h + 2\mu\omega_h\chi).t_E]]_E\|_{0,E}\|\psi_h - \psi\|_{0,E}.
\end{aligned}$$

Discrete Cauchy-Schwarz's inequality and Clément inequalities (2.3.14), (2.3.15) yield

$$\begin{aligned}
\int_\Omega \varepsilon : Curl\ (\psi)\ dx \lesssim &\Bigg[\sum_{T\in\mathcal{T}_h} h_T^2\|rot\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T}^2 \\
&+ \sum_{E\in\mathcal{E}_h} h_E\|[[(\sigma_h + 2\mu\omega_h\chi).t_E]]_E\|_{0,E}^2\Bigg]^{\frac{1}{2}} \|\nabla\psi\|_{0,\Omega}. \quad (2.4.10)
\end{aligned}$$

The first three terms of the right hand side of (2.4.8) can be estimated using simply continuous and discrete Cauchy-Schwarz's inequalities:

$$
\begin{aligned}
2\mu\int_\Omega (f - P_h^0 f).z\ dx + \mu\int_\Omega as\ (\sigma_h) rot\ (z)\ dx + \int_\Omega \big(\tfrac{2\mu}{\lambda} p_h + tr\ (\sigma_h)\big)\ q\ dx \\
&= \sum_{T\in\mathcal{T}_h}\Big[2\mu\int_T (f-P_h^0 f).z\ dx + \mu\int_T as\ (\sigma_h) rot\ (z)\ dx + \int_T \big(\tfrac{2\mu}{\lambda} p_h + tr\ (\sigma_h)\big)\ q\ dx\Big] \\
&\le \sum_{T\in\mathcal{T}_h}\Big[\|2\mu(f-P_h^0 f)\|_{0,T}\|z\|_{0,T} + \|\mu as\ (\sigma_h)\|_{0,T}\|rot\ (z)\|_{0,T} \\
&\quad + \|\tfrac{2\mu}{\lambda} p_h + tr\ (\sigma_h)\|_{0,T}\|q\|_{0,T}\Big] \\
&\le 2\mu\Big\{\Big[\sum_{T\in\mathcal{T}_h}\|(f-P_h^0 f)\|_{0,T}^2\Big]^{\frac12}\|z\|_{0,\Omega} + \Big[\sum_{T\in\mathcal{T}_h}\tfrac14\|as\ (\sigma_h)\|_{0,T}^2\Big]^{\frac12}\|rot\ (z)\|_{0,\Omega} \\
&\quad + \Big[\sum_{T\in\mathcal{T}_h}\|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\|_{0,T}^2\Big]^{\frac12}\|q\|_{0,\Omega}\Big\}. \qquad (2.4.11)
\end{aligned}
$$

By Korn's, Poincare's and (2.4.6) inequalities, we obtain

$$\|rot\ (z)\|_{0,\Omega} \lesssim \|\nabla z\|_{0,\Omega} \lesssim \|\epsilon(z)\|_{0,\Omega} \lesssim \|\varepsilon - \tfrac12 as\ (\sigma_h)\chi\|_{0,\Omega}, \qquad (2.4.12)$$

$$\|z\|_{0,\Omega} \lesssim \|\nabla z\|_{0,\Omega} \lesssim \|\epsilon(z)\|_{0,\Omega} \lesssim \|\varepsilon - \tfrac12 as\ (\sigma_h)\chi\|_{0,\Omega}. \qquad (2.4.13)$$

Equations (2.4.8), (2.4.10), (2.4.11), (2.4.12), (2.5.28) and (2.4.6), with the help of the discrete Cauchy-Schwarz's inequality, yield

$$
\begin{aligned}
\|\varepsilon - \tfrac12 as\ (\sigma_h)\chi\|_{0,\Omega} &\lesssim \Big[\sum_{T\in\mathcal{T}_h}\Big\{\|(f-P_h^0 f)\|_{0,T}^2 + \tfrac14\|as\ (\sigma_h)\|_{0,T}^2 + \|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\|_{0,T}^2 \\
&\quad + h_T^2\|rot\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T}^2\Big\} + \sum_{E\in\mathcal{E}_h} h_E\|[[(\sigma_h + 2\mu\omega_h\chi).t_E]]_E\|_{0,E}^2\Big]^{\frac12}.
\end{aligned}
$$

Now using the estimate (2.5.20) and this last bound, we obtain

$$\|\varepsilon\|_{0,\Omega} \lesssim \eta. \qquad (2.4.14)$$

■

We turn now to bound the error term $P := p - p_h$ by the error estimator η.

Proposition 2.4.3. *The following estimate holds*

$$\|P\|_{0,\Omega} \lesssim \eta \qquad (2.4.15)$$

Proof: From the second equations of the continuous problem (2.2.1) and the discrete problem (2.2.4), we get

$$div\ (\varepsilon - P\delta) + f - P_h^0 f = 0. \tag{2.4.16}$$

By Lemma 3.4 of [30], there exists a tensor field $\xi \in [H^1(\Omega)]^{2\times 2}$ such that $\xi_{|\Gamma_N} = 0$ and

$$div\ (\xi) = f - P_h^0 f, \tag{2.4.17}$$

with the estimate

$$\|\xi\|_{H(div\ ,\Omega)} \lesssim \|f - P_h^0 f\|_{0,\Omega}. \tag{2.4.18}$$

From (2.4.16) and (2.4.17) we obtain

$$div\ (\varepsilon + \xi - P\delta) = 0.$$

Applying Lemma 2.3.3, we obtain

$$\begin{aligned}\|tr\ (\varepsilon + \xi - P\delta)\|_{0,\Omega} &\lesssim \|(\varepsilon + \xi - P\delta)^D\|_{0,\Omega}\\ &= \|(\varepsilon + \xi)^D\|_{0,\Omega}\\ &\lesssim \|\varepsilon + \xi\|_{0,\Omega}\\ &\lesssim \|\varepsilon\|_{0,\Omega} + \|f - P_h^0 f\|_{0,\Omega}\end{aligned}$$

But

$$\begin{aligned}\|tr\ (\varepsilon - \xi - P\delta)\|_{0,\Omega} &\geq \|tr\ (P\delta)\|_{0,\Omega} - \|tr\ (\varepsilon - \xi)\|_{0,\Omega}\\ &\gtrsim 2\|P\|_{0,\Omega} - \|\varepsilon - \xi\|_{0,\Omega}\end{aligned}$$

Combining with the previous inequality and (2.4.18), we obtain

$$\begin{aligned}\|P\|_{0,\Omega} &\lesssim \|\varepsilon\|_{0,\Omega} + \|f - P_h^0 f\|_{0,\Omega}\\ &\lesssim \eta.\end{aligned}$$

■

It remains to bound the errors $e := u - u_h$ and $r := \omega - \omega_h$. Throughout the rest of this section, we use the notations $\beta_h := \sigma_h + 2\mu\omega_h\chi$, $\beta := \sigma + 2\mu\omega\chi = 2\mu\nabla u$. Since proposition 2.4.2 bounds $\|\varepsilon\|_{0,\Omega} = \|\sigma - \sigma_h\|_{0,\Omega}$ by a constant times η, it suffices to bound $\|\beta - \beta_h\|_{0,\Omega}$ in order to obtain an estimate for $\|\omega - \omega_h\|_{0,\Omega}$. This will be also the main ingredient to bound $\|u - u_h\|_{0,\Omega}$ in terms of η as we will see.

Lemma 2.4.4. *The following estimate holds*

$$\|\beta - \beta_h\|_{0,\Omega} \lesssim \eta.$$

Proof: In view of proposition 2.3.1, there exists $v \in [H^1(\Omega)]^2$ with $v = 0$ on Γ_D, and $\phi \in [H^1(\Omega)]^2$ with $\phi = constant$ on Γ_N such that

$$\tfrac{1}{2\mu}(\beta - \beta_h) = \nabla v + Curl\ (\phi), \tag{2.4.19}$$

with the estimate

$$\|\nabla v\|_{0,\Omega} + \|\nabla \phi\|_{0,\Omega} \lesssim \|\beta - \beta_h\|_{0,\Omega}. \tag{2.4.20}$$

By Green's formula, we have:

$$\begin{aligned}
\|\mathit{Curl}\ \phi\|^2_{0,\Omega} &= \int_\Omega \mathit{Curl}\ \phi : (\tfrac{1}{2\mu}(\beta - \beta_h) - \nabla v)\ dx \\
&= \int_\Omega \mathit{Curl}\ \phi : \nabla u\ dx - \tfrac{1}{2\mu}\int_\Omega \mathit{Curl}\ \phi : \beta_h\ dx - \int_\Omega \mathit{Curl}\ \phi : \nabla v\ dx \\
&= -\tfrac{1}{2\mu}\int_\Omega \mathit{Curl}\ \phi : \beta_h\ dx.
\end{aligned}$$

Let $\phi_h = \mathrm{I}_{\mathrm{Cl}}\phi$ be the Clément interpolation of ϕ. By the first equation of the discrete problem (2.2.4) with $\tau_h = \mathit{Curl}\ (\phi_h)$ and $q_h = 0$, we have

$$\begin{aligned}
-\int_\Omega \beta_h : \mathit{Curl}\ \phi\ dx &= \int_\Omega \beta_h : \mathit{Curl}\ (\phi_h - \phi)\ dx \\
&= \sum_{T\in\mathcal{T}_h}\int_T (\mathit{rot}\ (\beta_h)).(\phi_h - \phi)\ dx - \sum_{E\in\mathcal{E}_h}\int_E [[(\beta_h).t_E]]_E.(\phi_h-\phi)\ ds \\
&\lesssim \sum_{T\in\mathcal{T}_h} h_T\|\mathit{rot}\ (\beta_h)\|_{0,T}\|\nabla\phi\|_{\omega_T} + \sum_{E\in\mathcal{E}_h} h_E^{\frac{1}{2}}\|[[(\beta_h).t_E]]_E\|_{0,E}\|\nabla\phi\|_{\omega_E} \\
&\qquad \text{by (2.3.14) and (2.3.15)} \\
&\lesssim \left[\sum_{T\in\mathcal{T}_h} h_T^2\|\mathit{rot}\ (\beta_h)\|^2_{0,T} + \sum_{E\in\mathcal{E}_h} h_E\|[[(\beta_h).t_E]]_E\|^2_{0,E}\right]^{\frac{1}{2}}\|\nabla\phi\|_{0,\Omega} \\
&= \left[\sum_{T\in\mathcal{T}_h} h_T^2\|\mathit{rot}\ (\beta_h)\|^2_{0,T} + \sum_{E\in\mathcal{E}_h} h_E\|[[(\beta_h).t_E]]_E\|^2_{0,E}\right]^{\frac{1}{2}}\|\mathit{Curl}\ \phi\|_{0,\Omega}.
\end{aligned}$$

Combined with the previous equality, this gives us

$$\begin{aligned}
\|\mathit{Curl}\ \phi\|_{0,\Omega} &\lesssim \left[\sum_{T\in\mathcal{T}_h} h_T^2\|\mathit{rot}\ (\beta_h)\|^2_{0,T} + \sum_{E\in\mathcal{E}_h} h_E\|[[(\beta_h).t_E]]_E\|^2_{0,E}\right]^{\frac{1}{2}} \\
&\lesssim \eta.
\end{aligned} \tag{2.4.21}$$

Taking the symmetric parts in (2.4.4) and (2.4.19) we get

$$\mathit{Sym}\ (\varepsilon) = 2\mu\epsilon(z) - q\delta + \mathit{Curl}\ \psi,$$

as due to (2.4.4) $\mathit{Curl}\ \psi$ is a symmetric tensor field and

$$\mathit{Sym}\ (\beta - \beta_h) = \mathit{Sym}\ (\varepsilon) = 2\mu\epsilon(v) + 2\mu\ \mathit{Sym}\ (\mathit{Curl}\ (\phi)).$$

Hence

$$2\mu\epsilon(z - v) - q\delta = 2\mu\ \mathit{Sym}\ (\mathit{Curl}\ (\phi)) - \mathit{Curl}\ \psi.$$

Thus we may estimate

$$\begin{aligned}
\|2\mu\epsilon(z-v)-q\delta\|_{0,\Omega}^2 &= \int_\Omega (2\mu\ Sym\ (Curl\ \phi) - Curl\ \psi) : (2\mu\epsilon(z-v)-q\delta)\ dx \\
&= \int_\Omega (2\mu Curl\ \phi) : (2\mu\epsilon(z-v)-q\delta)\ dx + \int_\Omega Curl\ \psi : q\delta\ dx \\
&\le 2\mu \| Curl\ (\phi)\|_{0,\Omega} \|2\mu\epsilon(z-v)-q\delta\|_{0,\Omega} + \sqrt{2}\| Curl\ (\psi)\|_{0,\Omega}\|q\|_{0,\Omega} \\
&\le 2\mu^2 \| Curl\ (\phi)\|_{0,\Omega}^2 + \tfrac{1}{2}\|2\mu\epsilon(z-v)-q\delta\|_{0,\Omega}^2 + \sqrt{2}\| Curl\ (\psi)\|_{0,\Omega}\|q\|_{0,\Omega}
\end{aligned}$$

The estimates above imply

$$\|2\mu\epsilon(z-v)-q\delta\|_{0,\Omega}^2 \lesssim \| Curl\ \phi\|_{0,\Omega}^2 + \| Curl\ \psi\|_{0,\Omega}\|q\|_{0,\Omega}$$

By Korn's and triangular inequalities we get

$$\begin{aligned}
\|\nabla v\|_{0,\Omega} &\lesssim \|\epsilon(v)\|_{0,\Omega} \\
&\lesssim \|2\mu\epsilon(z-v)-q\delta\|_{0,\Omega} + \|\epsilon(z)\|_{0,\Omega} + \|q\|_{0,\Omega} \\
&\lesssim \left[\| Curl\ \phi\|_{0,\Omega}^2 + \| Curl\ \psi\|_{0,\Omega}\|q\|_{0,\Omega}\right]^{\frac{1}{2}} + \|\epsilon(z)\|_{0,\Omega} + \|q\|_{0,\Omega} \\
&\lesssim \| Curl\ \phi\|_{0,\Omega} + \|\varepsilon\|_{0,\Omega} + \eta \text{ by (2.4.6) and (2.4.1)} \\
&\lesssim \eta \text{ by (2.4.3) and (2.4.21).}
\end{aligned} \tag{2.4.22}$$

From (2.4.21), (2.4.22) and (2.4.19) we obtain by triangle inequality:

$$\|\beta-\beta_h\|_{0,\Omega} \lesssim \eta.$$

■

From the preceding Lemma and Proposition 2.4.2, we have immediately:

Proposition 2.4.5. *The following bound holds*

$$\|\omega-\omega_h\|_{0,\Omega} \lesssim \eta.$$

It remains to prove also that $\|u-u_h\|_{0,\Omega} \lesssim \eta$ in order to conclude that our estimator η is reliable. To be able to prove such an estimate, we will make some geometric assumption on Ω and Γ_N. We have the following result:

Proposition 2.4.6. *Let us suppose that at each vertex $s \in \overline{\Gamma}_N$, the angle is convex. Then the following estimate holds*

$$\|u-u_h\|_{0,\Omega} \lesssim \eta.$$

Proof: Let D be a bounded domain such that $\Gamma_N \subset \partial D$, $\overline{D} \supset \overline{\Omega}$, $D\backslash\overline{\Omega} \neq \varnothing$ and every point of $\partial D\backslash\overline{\Gamma}_N$ is a smooth boundary point. We consider $\tilde{e} \in [L^2(D)]^2$ such that $\tilde{e}_{|\Omega} = e$ and $\tilde{e}$ is of mean value 0. Let $z \in [H^2(D)]^2$ be a solution of the elasticity problem

$$\begin{cases} div\ (2\mu\epsilon(z) + \lambda tr\ \epsilon(z)\delta) = \tilde{e} & \text{in } D \\ (2\mu\epsilon(z) + \lambda tr\ \epsilon(z)\delta).n = 0 & \text{on } \partial D. \end{cases}$$

Let $\tau := 2\mu\epsilon(z) + \lambda tr\ \epsilon(z)\delta$. Integration by parts yield (recall that $u = 0$ on Γ_D):

$$\begin{aligned}\|e\|^2_{0,\Omega} &= \int_\Omega (u - u_h) div\ (\tau)\ dx \\ &= -\int_\Omega \nabla u : \tau\ dx - \int_\Omega u_h div\ (\tau)\ dx.\end{aligned}$$

Now considering the global BDM_1 interpolation operator I_h, setting $(\tau_h, 0) := (I_h\tau, 0) \in \Sigma_{0,h}$, and using the fact that u_h is constant on each triangle of the triangulation, we get

$$\begin{aligned}\|e\|^2_{0,\Omega} &= -\int_\Omega \nabla u : \tau\ dx - \int_\Omega u_h div\ (\tau_h)\ dx \\ &= -\int_\Omega \nabla u : \tau\ dx + \tfrac{1}{2\mu}\int_\Omega \sigma_h : \tau_h\ dx + \int_\Omega \omega_h as\ (\tau_h)\ dx \\ &\quad \text{by the first equation of the discrete problem (2.2.4)} \\ &= -\tfrac{1}{2\mu}\int_\Omega \beta : \tau\ dx + \tfrac{1}{2\mu}\int_\Omega (\sigma_h + 2\mu\omega_h\chi) : \tau_h\ dx \\ &= -\tfrac{1}{2\mu}\int_\Omega (\beta - \beta_h) : \tau\ dx + \tfrac{1}{2\mu}\int_\Omega \beta_h : (\tau_h - \tau)\ dx\end{aligned}$$

Now using Cauchy-Schwarz inequality and Lemma 2.3.7 allow us to obtain

$$\begin{aligned}\|e\|^2_{0,\Omega} &\lesssim \|\beta - \beta_h\|_{0,\Omega}\|\tau\|_{0,\Omega} + \sum_{T\in\mathcal{T}_h} \|\beta_h\|_{0,T} h_T |\tau|_{1,T} \\ &\lesssim \left[\|\beta - \beta_h\|_{0,\Omega} + \left(\sum_{T\in\mathcal{T}_h} h_T^2 \|\beta_h\|^2_{0,T}\right)^{\frac{1}{2}}\right] ||\tau||_{1,\Omega}.\end{aligned}$$

But

$$||\tau||_{1,\Omega} \lesssim ||\tau||_{1,D} \lesssim \|z\|_{2,D} \lesssim \|\tilde{e}\|_{0,D} \lesssim \|e\|_{0,\Omega}.$$

In conclusion we have proved that

$$\|e\|_{0,\Omega} \lesssim \eta$$

■

Comparing the above Proposition with Lemma 5.4 of [10] (see also [9]), we remark that we have avoided the H^2-regularity requirement of the solution of (1.2.1), but that nevertheless we need some geometric hypothesis on the boundary part Γ_N of Ω.

2.4.2 Proof of the efficiency of the estimator

Recall further the notations $\beta = \sigma + 2\mu\omega\chi$, $\beta_h = \sigma_h + 2\mu\omega_h\chi$, $\varepsilon = \sigma - \sigma_h$, $P = p - p_h$, $r = \omega - \omega_h$, and $e = u - u_h$. We treat separately the various contributions appearing in the estimator η.

Theorem 2.4.7 (Local lower error bound). *For all $T \in \mathcal{T}_h$ the following local lower error bound holds:*

$$\eta_T \lesssim \|f - P_h^0 f\|_{0,T} + \|e\|_{0,T} + \|\varepsilon\|_{0,\omega_T} + \|r\|_{0,\omega_T} + \|P\|_{0,T}. \tag{2.4.23}$$

Lemma 2.4.8. *The following estimate holds:*

$$\|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\|_{0,T} \lesssim \|p - p_h\|_{0,T} + \|\sigma - \sigma_h\|_{0,T}.$$

Proof: The Cauchy-Schwarz's inequality and the fact that $\frac{1}{\lambda}p + \frac{1}{2\mu}tr\ (\sigma) = 0$ yield

$$\begin{aligned}
\|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\|^2_{0,T} &= -\int_\Omega \left[\tfrac{1}{\lambda}(p - p_h) + \tfrac{1}{2\mu}tr\ (\sigma - \sigma_h)\right]\left[\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\right]\, dx \\
&\leq \|\tfrac{1}{\lambda}(p - p_h) + \tfrac{1}{2\mu}tr\ (\sigma - \sigma_h)\|_{0,T}\|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\|_{0,T} \\
&\lesssim \big(\|p - p_h\|_{0,T} + \|\sigma - \sigma_h\|_{0,T}\big)\|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\|_{0,T}
\end{aligned}$$

■

Lemma 2.4.9. *The following estimate holds:*

$$h_T\|rot\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T} \lesssim \|\sigma - \sigma_h\|_{0,T} + \|\omega - \omega_h\|_{0,T}$$

Proof: Inverse inequalities and Green's formula yield

$$\begin{aligned}
\|rot\ (\sigma_h + 2\mu\omega_h\chi)\|^2_{0,T} &\lesssim \int_T b_T\|rot\ (\sigma_h + 2\mu\omega_h\chi)\|^2\, dx \quad \text{by (2.3.9)} \\
&= -\int_T b_T rot\ ((\sigma - \sigma_h) + 2\mu(\omega - \omega_h)\chi).rot\ (\sigma_h + 2\mu\omega_h\chi)\, dx \\
&= -\int_T rot\ (\beta - \beta_h).b_T rot\ (\beta_h)\, dx \\
&= -\int_T (\beta - \beta_h) : Curl\ (b_T rot\ (\beta_h))\, dx \\
&\leq \|\beta - \beta_h\|_{0,T}\|Curl\ (b_T rot\ (\beta_h))\|_{0,T} \\
&\lesssim \|\beta - \beta_h\|_{0,T} h_T^{-1}\|rot\ (\beta_h)\|_{0,T} \quad \text{by (2.3.10)} \\
&\lesssim (\|\sigma - \sigma_h\|_{0,T} + \|\omega - \omega_h\|_{0,T}) h_T^{-1}\|rot\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T}.
\end{aligned}$$

This proves the Lemma. ■

Lemma 2.4.10. *For all $E \in \mathcal{E}_h$ the following bound of the tangential jump error holds:*

$$h_E^{1/2}\|\left[\!\left[(\sigma_h + 2\mu\omega_h\chi).t_E\right]\!\right]_E\|_{0,E} \lesssim \|\sigma - \sigma_h\|_{0,\omega_E} + \|\omega - \omega_h\|_{0,\omega_E}$$

Proof: Let us set $\psi_E := \mathrm{F}_{\text{ext}}(\left[\!\left[\beta_h.t_E\right]\!\right]_E).b_E$, which belongs to $[H_0^1(\omega_E)]^2$. As $\beta_{|\omega_E} \in [H(rot\ ,\omega_E)]^2$, by integration by parts with ψ_E, we obtain

$$\int_{\omega_E} (rot\ (\beta)).\psi_E\, dx - \int_{\omega_E} \beta : Curl\ (\psi_E)\, dx = \int_{\partial\omega_E} (\beta.t_E).\psi_E ds$$

As $rot\ (\beta) = 0$ and $\psi_{E}/_{\partial\omega_E} = 0$, this gives us:

$$\int_{\omega_E} \beta : Curl\ (\psi_E)\ dx = 0$$

For β_h we integrate elementarily and obtain

$$\begin{aligned}
\|\,[[\beta_h.t_E]]_E b_E^{1/2}\|_{0,E}^2 &= \int_E [[\beta_h.t_E]]_E.\psi_E\ ds \\
&= \sum_{T\subset\omega_E} \int_{\partial T} \beta_h.t.\psi_E\ ds \\
&= \sum_{T\subset\omega_E} \left[\int_T (rot\ (\beta_h)).\psi_E\ dx - \int_T \beta_h : Curl\ (\psi_E)\ dx\right] \\
&= \sum_{T\subset\omega_E} \left[\int_T (rot\ (\beta_h)).\psi_E\ dx + \int_T (\beta - \beta_h) : Curl\ (\psi_E)\ dx\right] \\
&\leq \sum_{T\subset\omega_E} \left[\|rot\ (\beta_h)\|_{0,T}\|\psi_E\|_{0,T} + \|\beta - \beta_h\|_{0,T}\|Curl\ (\psi_E)\|_{0,T}\right].
\end{aligned}$$

Lemma 2.4.9 and inverse inequalities (2.3.12), (2.3.13) lead to

$$\begin{aligned}
\|\,[[\beta_h.t_E]]_E b_E^{1/2}\|_{0,E}^2 \lesssim \sum_{T\subset\omega_E} \Big[&\ h_T^{-1} \quad (\|\sigma - \sigma_h\|_{0,T} + \|\omega - \omega_h\|_{0,T})\ h_E^{1/2}\|\,[[\beta_h.t_E]]_E\|_{0,E} \\
+ &\quad (\|\sigma - \sigma_h\|_{0,T} + \|\omega - \omega_h\|_{0,T})\ h_E^{1/2} h_T^{-1}\|\,[[\beta_h.t_E]]_E\|_{0,E}\Big]
\end{aligned}$$

The regularity of the triangulation enables us to bound $h_E^{1/2}h_T^{-1} \lesssim h_E^{-1/2}$ for all $E \in \partial T$ with $T \in \mathcal{T}_h$. Thus

$$\|\,[[\beta_h.t_E]]_E b_E^{1/2}\|_{0,E}^2 \lesssim h_E^{-1/2}(\|\sigma - \sigma_h\|_{0,\omega_E} + \|\omega - \omega_h\|_{0,\omega_E})\|\,[[\beta_h.t_E]]_E\|_{0,E}.$$

We conclude by using the equivalence (2.3.11). ■

Lemma 2.4.11. *The following estimate holds*

$$h_T\|\sigma_h + 2\mu\omega_h\chi\|_{0,T} \lesssim h_T(\|\sigma - \sigma_h\|_{0,T} + \|\omega - \omega_h\|_{0,T}) + \|u - u_h\|_{0,T}.$$

Proof: Recall that $\beta = \sigma + 2\mu\omega\chi = 2\mu\nabla u$. Now, we have

$$\begin{aligned}
\|\sigma_h + 2\mu\omega_h\chi\|_{0,T}^2 &\lesssim \int_T b_T\beta_h : \beta_h \, dx \qquad \text{by (2.3.9)} \\
&= -\int_T b_T(\beta - \beta_h) : \beta_h \, dx + \int_T b_T\beta : \beta_h \, dx \\
&= -\int_T b_T(\beta - \beta_h) : \beta_h \, dx + 2\mu \int_T b_T(\nabla(u - u_h)) : \beta_h \, dx \\
&= -\int_T b_T(\beta - \beta_h) : \beta_h \, dx - 2\mu \int_T (u - u_h).div\ (b_T\beta_h) \, dx \\
&\leq \|\beta - \beta_h\|_{0,T}\|\beta_h\|_{0,T} + 2\mu\|u - u_h\|_{0,T}\|div\ (b_T\beta_h)\|_{0,T} \\
&\lesssim \|\beta - \beta_h\|_{0,T}\|\beta_h\|_{0,T} + h_T^{-1}\|u - u_h\|_{0,T}\|\beta_h\|_{0,T} \qquad \text{by (2.3.10)} \\
&\lesssim (\|\sigma - \sigma_h\|_{0,T} + \|\omega - \omega_h\|_{0,T} + h_T^{-1}\|u - u_h\|_{0,T})\|\sigma_h + 2\mu\omega_h\chi\|_{0,T}.
\end{aligned}$$

In conclusion we have proved that

$$h_T\|\sigma_h + 2\mu\omega_h\chi\|_{0,T} \lesssim h_T(\|\sigma - \sigma_h\|_{0,T} + \|\omega - \omega_h\|_{0,T}) + \|u - u_h\|_{0,T}.$$

■

We are now able to proof Theorem 2.4.7. Indeed it results from the sequence of Lemmas 2.4.8 - 2.4.11, and the fact that $\|as\ (\sigma_h)\|_{0,T}$ appearing in the estimator η_T is easily estimated as

$$\begin{aligned}
\|as\ (\sigma_h)\|_{0,T} &= \|-as\ (\sigma - \sigma_h)\|_{0,T} \\
&= \|-as\ (\varepsilon)\|_{0,T} \\
&\leq 2\|\varepsilon\|_{0,T}.
\end{aligned}$$

2.5 The case of a multiply-connected domain

The case of a multiply-connected domain is frequently used in mechanics. In practice the majority of mechanic parts contains for examples holes. It is thus interesting to treat this kind of domain by checking that our error indicator still reliable and efficient this more general setting. We suppose in the whole of this section that every vertex of $\overline{\Gamma}_N$ is a convex angle. Let us first introduce some notations (see Figure 2.3):

Let us denote by Γ_0 the exterior boundary of Ω (i.e. Γ_0 is the boundary of the only unbounded connected component of $\mathbb{R}^2\backslash\Omega$) and by Γ_i, $1 \leq i \leq p$, the other connected components of Γ. We further fix a bounded simply connected open domain D such that $\Gamma_N \subset \partial D$, $\overline{D} \supset \overline{\Omega}$, $D\backslash\overline{\Omega} \neq \varnothing$ and every boundary point of D is regular or convex angle. Then the set $D\backslash\overline{\Omega}$ is not connected and for any $i \in \{1, \cdots, p\}$, we denote by Ω_i the connected component of this set bounded by Γ_i. For all $i \in \{1, \cdots, p\}$, we now fix a function $v_i \in \mathcal{D}(D)$ such that

$$\begin{aligned}
v_i &= 0 \quad \text{on } \overline{\Omega}, \\
v_i &= 0 \quad \text{on } \overline{\Omega}_j, \quad \forall j \in \{1, \cdots, p\} : j \neq i.
\end{aligned}$$

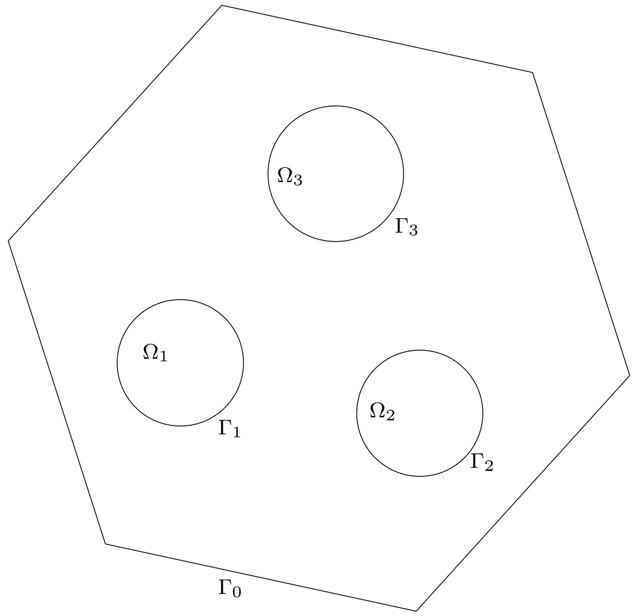


Figure 2.3: Multiply-connected domain

Moreover each function v_i must satisfy

$$\int_{\Omega_i} v_i \; dx = 1 \text{ and } \int_D v_i \; dx = 0.$$

Let $\Gamma_D = \partial\Omega \backslash \Gamma_N$ and $q_i \in H^1_{\Gamma_D}(D) \cap L^2_0(D)$ be the unique variational solution of mean 0 of the Poisson equation with homogeneous Neumann boundary condition:

$$\left\{ \begin{array}{rcll} -\Delta q_i & = & v_i & \text{in } D \\ \frac{\partial q_i}{\partial n} & = & 0 & \text{on } \partial D. \end{array} \right. \tag{2.5.1}$$

Since every boundary point of D is regular or convex angle, it is well known that q_i belongs to $H^2(D)$. From the choice of v_i, we remark that q_i is harmonic in Ω and by Green's formula we have for every $i \in \{1, \cdots, p\}$

$$\int_{\Gamma_j} (\nabla q_i).n \; ds = \int_{\Gamma_j} \frac{\partial q_i}{\partial n} \; ds = \int_{\Omega_j} \Delta q_i \; dx = \int_{\Omega_j} v_i \; dx = \delta_{ij}, \tag{2.5.2}$$

where we recall that n means the normal vector along Γ oriented outside Ω. Let $\beta = (\beta_1, \beta_2)^T \in \mathbb{R}^2$. In the following, we will use the following notation

$$\beta * \nabla q_i = \left[\begin{array}{c} \beta_1 \nabla q_i^T \\ \beta_2 \nabla q_i^T \end{array} \right].$$

With these notations, we are now able to formulate the Helmholtz like decomposition we have in mind.

2.5.1 Decomposition for tensor fields

To establish reliability of the residual based error indicator, we need to modify the Helmholtz decomposition for tensor fields due to the multiply-connected geometry of the domain. To our knowledge this decomposition seems to be new. Let us observe that the proof of the efficiency of the error indicator must not be modified because we use the Helmholtz decomposition only in the proof of the reliability. Although we will use only in the sequel the Helmholtz decomposition mentioned in the Proposition 2.5.2, it is worthwhile to note that we have also the following decomposition of Helmholtz type for tensor fields.

Proposition 2.5.1. *Let $\tau \in [L^2(\Omega)]^{2\times 2}$. Then there exist $p \in [H^1(\Omega)]^2$ with $p = 0$ on Γ_D, $\varphi \in [H^1(\Omega)]^2$ with $\varphi =$ constant on each connected component of Γ_N and $\beta_i = (\beta_i^1, \beta_i^2)^T \in \mathbb{R}^2$ for all $i \in \{1, \cdots, p\}$ such that*

$$\tau = \nabla p + Curl\ \varphi - \sum_{i=1}^{p} \beta_i * \nabla q_i, \tag{2.5.3}$$

with the estimate

$$\|\nabla p\|_{0,\Omega} + \|\nabla \varphi\|_{0,\Omega} + \sum_{i=1}^{p} |\beta_i| \lesssim \|\tau\|_{0,\Omega}. \tag{2.5.4}$$

Proof: As in the proof of Lemma 2.3.1 let $p \in [H^1_{\Gamma_D}(\Omega)]^2$ be the variational formulation of

$$\int_\Omega (\tau - \nabla p) : \nabla \psi \, dx = 0 \quad \text{for all } \psi \in [H^1_{\Gamma_D}(\Omega)]^2. \tag{2.5.5}$$

This last equation implies in particular, that $\tau - \nabla p$ is divergence free in Ω in the sense of distributions but it does not necessarily satisfy

$$\langle (\tau - \nabla p).n, \psi \rangle_{H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i)} = 0, \quad \forall i \in \{1, \cdots, p\}, \quad \forall \psi \in [H^1_{\Gamma_D}(\Omega)]^2.$$

Therefore we search vectors $\beta_i = (\beta_i^1, \beta_i^2)^T \in \mathbb{R}^2$ such that

$$v := (\tau - \nabla p) - \sum_{i=1}^{p} \beta_i * \nabla q_i,$$

satisfies this property, namely

$$\langle v.n, 1 \rangle_{H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i)} = 0, \quad \forall i \in \{1, \cdots, p\}. \tag{2.5.6}$$

By property (2.5.2), this is equivalent to

$$\beta_i = \langle (\tau - \nabla p).n, 1 \rangle_{H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i)}.$$

Condition (2.5.6) allows us to apply Theorem I.3.1 of [25] line by line to the tensor $\tau - \nabla p - \sum_{i=1}^{p} \beta_i * \nabla q_i$ which yields a vectorial stream function $\varphi \in [H^1(\Omega)]^2$ such that

$$\tau - \nabla p - \sum_{i=1}^{p} \beta_i * \nabla q_i = Curl\ \varphi.$$

This proves the expansion (2.5.3). Applying the same ingredient as in the proof of Lemma 2.3.1, we see easily that $\varphi = const$ on Γ_N. The estimate

$$\|\nabla p\|_{0,\Omega} \leq \|\tau\|_{0,\Omega}, \tag{2.5.7}$$

follows immediately from the variational equation (2.5.5) by taking $\psi = p$ as a test function.

Moreover from the expression of β_i and the trace estimate [25], we have

$$\begin{aligned}|\beta_i| &\lesssim ||(\tau - \nabla p).n||_{H^{-1/2}(\Gamma_i)}\\ &\lesssim \|\tau - \nabla p\|_{0,\Omega}\\ &\lesssim \|\tau\|_{0,\Omega} + \|\nabla p\|_{0,\Omega},\end{aligned}$$

reminding that $\tau - \nabla p$ is divergence free. Therefore estimate (2.5.7) allows to conclude that

$$|\beta_i| \lesssim \|\tau\|_{0,\Omega}. \tag{2.5.8}$$

Thus it remains only to prove that

$$\|\nabla\varphi\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}.$$

Since

$$\begin{aligned}\|\nabla\varphi\|_{0,\Omega} = \|Curl\ \varphi\|_{0,\Omega} &= \|\tau - \nabla p - \sum_{i=1}^{p} \beta_i * \nabla q_i\|_{0,\Omega}\\ &\leq \|\nabla p\|_{0,\Omega} + \|\tau\|_{0,\Omega} + \sum_{i=1}^{p} |\beta_i|\|\nabla q_i\|_{0,\Omega},\end{aligned}$$

the two estimates (2.5.7) and (2.5.8) allow to conclude that

$$\|\nabla\varphi\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}.$$

■

Proposition 2.5.2. *Let $\tau \in [L^2(\Omega)]^{2\times 2}$. Then there exist $z \in [H^1(\Omega)]^2$ with $z = 0$ on Γ_D, $\psi \in [H^1(\Omega)]^2$ with $\psi = constant$ on each connected component of Γ_N, $q \in L^2(\Omega)$ and $\beta_i = (\beta_i^1, \beta_i^2)^T \in I\!R^2$ for all $i \in \{1, \cdots, p\}$ such that*

$$\begin{cases} \tau = 2\mu\epsilon(z) - q\delta + Curl\ (\psi) - \sum_{i=1}^{p} \beta_i * \nabla q_i \\ \frac{1}{\lambda} q + div\ (z) = 0. \end{cases} \tag{2.5.9}$$

Moreover the following estimate holds:

$$\|\epsilon(z)\|_{0,\Omega} + \|\nabla\psi\|_{0,\Omega} + \|q\|_{0,\Omega} + \sum_{i=1}^{p} |\beta_i| \lesssim \|\tau\|_{0,\Omega}. \tag{2.5.10}$$

Proof: As in the proof of Lemma 2.3.2 let $z \in [H^1_{\Gamma_D}(\Omega)]^2$ denotes the solution to the variational formulation of the Lamé system

$$\int_\Omega (-2\mu\epsilon(z) - \lambda div\ (z)\delta + \tau) : \nabla\varphi\ dx = 0 \quad \text{for all } \varphi \in [H^1_{\Gamma_D}(\Omega)]^2. \tag{2.5.11}$$

This last equation implies in particular, that $-2\mu\epsilon(z) + q\delta + \tau$, with $q = -\lambda div\ z$, is a divergence free tensor in Ω, but z does not necessarily satisfy

$$\langle(-2\mu\epsilon(z) + q\delta + \tau).n, \psi\rangle_{H^{-1/2}(\Gamma_i),H^{1/2}(\Gamma_i)} = 0, \forall i \in \{1, \cdots, p\} \quad \forall\psi \in [H^1_{\Gamma_D}(\Omega)]^2.$$

As before we fix the vector

$$\beta_i = \langle(-2\mu\epsilon(z) + q\delta + \tau).n, 1\rangle_{H^{-1/2}(\Gamma_i),H^{1/2}(\Gamma_i)},$$

such that

$$\langle(-2\mu\epsilon(z) + q\delta + \tau + \sum_{i=1}^{p} \beta_i * \nabla q_i).n, 1\rangle_{H^{-1/2}(\Gamma_i),H^{1/2}(\Gamma_i)} = 0, \forall i \in \{1, \cdots, p\}. \tag{2.5.12}$$

Applying Theorem I.3.1 p.37 of [25] line by line to the tensor

$$-2\mu\epsilon(z) + q\delta + \tau + \sum_{i=1}^{p} \beta_i * \nabla q_i,$$

we conclude that there exists a stream function $\psi \in [H^1_{\Gamma_D}(\Omega)]^2$ so that

$$-2\mu\epsilon(z) + q\delta + \tau + \sum_{i=1}^{p} \beta_i * \nabla q_i = Curl\ \psi.$$

This yields the expansion (2.5.9). Similarly as before we can prove that

$$\psi = const \quad \text{on } \Gamma_N.$$

To prove the estimate (2.5.10), we take as test function $\varphi = z$ in equation (2.5.11)and by Cauchy-Schwarz's and Korn's inequalities, we get

$$\|\epsilon(z)\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}. \tag{2.5.13}$$

Now by using Lemma 3.4 [30], we can show as in Lemma 2.3.2 that

$$\|q\|_{0,\Omega} = \|\lambda div\ z\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}. \tag{2.5.14}$$

As before, from the expression of β_i and the trace estimate [25], we have

$$\begin{aligned}|\beta_i| &\lesssim ||(-2\mu\epsilon(z) + q\delta + \tau).n||_{H^{-1/2}(\Gamma_i)}\\ &\lesssim \|-2\mu\epsilon(z) + q\delta + \tau\|_{0,\Omega}\\ &\lesssim \|\tau\|_{0,\Omega} + \|\epsilon(z)\|_{0,\Omega} + \|q\|_{0,\Omega}.\end{aligned}$$

Therefore the estimates (2.5.13) and (2.5.14) allow to conclude that

$$|\beta_i| \lesssim \|\tau\|_{0,\Omega}. \tag{2.5.15}$$

Since

$$\begin{aligned}\|\nabla\psi\|_{0,\Omega} = \|\mathit{Curl}\ \psi\|_{0,\Omega} &= \|-2\mu\epsilon(z) + q\delta + \tau + \sum_{i=1}^{p}\beta_i * \nabla q_i\|_{0,\Omega} \\ &\lesssim \|\epsilon(z)\|_{0,\Omega} + \|q\|_{0,\Omega} + \|\tau\|_{0,\Omega} + \sum_{i=1}^{p}|\beta_i|\|\nabla q_i\|_{0,\Omega},\end{aligned}$$

the three estimates (2.5.13), (2.5.14) and (2.5.15) yield

$$\|\nabla\psi\|_{0,\Omega} \lesssim \|\tau\|_{0,\Omega}.$$

Consequently, we have proved inequality (2.5.10).

■

2.5.2 Proof of the reliability of the estimator

The new decompositions of tensor fields involve modifications in the proof of the reliability of the error estimator. As only the upper error bound of $\|\varepsilon\|_{0,\Omega} = \|\varepsilon - \varepsilon_h\|_{0,\Omega}$ and $\|\beta - \beta_h\|_{0,\Omega}$ by η use the Helmholtz type decomposition, it is thus only on this level that one must make the necessary modifications. Let us note that the upper error bound of $\|r\|_{0,\Omega} = \|p - p_h\|_{0,\Omega}$ and $\|u - u_h\|_{0,\Omega}$ remain still valid in spite of the new decompositions of tensor fields. We begin with the following estimate:

Proposition 2.5.3. *The following estimate holds*

$$\|\varepsilon\|_{0,\Omega} \lesssim \eta. \tag{2.5.16}$$

Proof: Proposition 2.5.2 implies the existence of $z \in [H^1(\Omega)]^2$ with $z = 0$ on Γ_D, $q \in L^2(\Omega)$, $\psi \in [H^1(\Omega)]^2$ with $\psi = \mathit{const}$ on each connected component of Γ_N and $\beta_i \in \mathbb{R}^2$ such that

$$\varepsilon - \tfrac{1}{2}as\ (\sigma_h)\chi = 2\mu\epsilon(z) - q\delta - \sum_{i=1}^{p}\beta_i * \nabla q_i + \mathit{Curl}\ \psi \tag{2.5.17}$$

$$\tfrac{1}{\lambda}q + \mathit{div}\ (z) = 0. \tag{2.5.18}$$

Moreover the following estimate holds:

$$\|\epsilon(z)\|_{0,\Omega} + \|\nabla\psi\|_{0,\Omega} + \|q\|_{0,\Omega} + \sum_{i=1}^{p}|\beta_i| \lesssim \|\varepsilon - \tfrac{1}{2}as\ (\sigma_h)\chi\|_{0,\Omega}. \tag{2.5.19}$$

Setting

$$\phi = -\sum_{i=1}^{p} \beta_i * \nabla q_i,$$

it follows from $\epsilon := \sigma + \sigma_h$ and equality (2.5.17) that $Curl\ (\psi) + \phi$ is a symmetric tensor field. By the triangle inequality, we have

$$\begin{aligned} \|\varepsilon\|_{0,\Omega} &\leq \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega} + \|\tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega} \\ &= \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega} + \tfrac{1}{\sqrt{2}}\|as\ (\sigma_h)\|_{0,\Omega}. \end{aligned} \tag{2.5.20}$$

In view of the definition of the error estimator η, it suffices to bound $\|\varepsilon - \frac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}$. The above decomposition allows to write

$$\begin{aligned} \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}^2 &= \int_\Omega (\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi) : (2\mu\epsilon(z) - q\delta + Curl\ \psi + \phi)\ dx \\ &= \int_\Omega \varepsilon : (2\mu\epsilon(z) - q\delta + Curl\ \psi + \phi)\ dx \\ &\quad -\tfrac{1}{2}\int_\Omega as\ (\sigma_h)\chi : (2\mu\epsilon(z) - q\delta + Curl\ \psi + \phi)\ dx \\ &= \int_\Omega \varepsilon : (2\mu\epsilon(z) - q\delta + \phi)\ dx \\ &\quad \text{as } 2\mu\epsilon(z) - q\delta + \phi + Curl\ \psi \quad \text{is a symmetric tensor field.} \end{aligned}$$

By (2.5.18) we may write

$$\begin{aligned} \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}^2 &= \int_\Omega \varepsilon : 2\mu\epsilon(z)\ dx - \int_\Omega 2\mu P(\tfrac{1}{\lambda} q + div\ (z))\ dx - \int_\Omega \varepsilon : q\delta\ dx \\ &\quad + \int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx \\ &= 2\mu \int_\Omega \varepsilon : \epsilon(z)\ dx - 2\mu \int_\Omega P\,div\ (z)\ dx - \tfrac{2\mu}{\lambda}\int_\Omega Pq\ dx - \int_\Omega tr\ (\varepsilon) q\ dx \\ &\quad + \int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx \\ &= 2\mu \int_\Omega \varepsilon : \epsilon(z)\ dx - 2\mu \int_\Omega P\delta : \epsilon(z)\ dx - \tfrac{2\mu}{\lambda}\int_\Omega Pq\ dx - \int_\Omega tr\ (\varepsilon) q\ dx \\ &\quad + \int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx \\ &= 2\mu \int_\Omega (\varepsilon - P\delta) : \epsilon(z)\ dx - \tfrac{2\mu}{\lambda}\int_\Omega Pq\ dx + \tfrac{2\mu}{\lambda}\int_\Omega pq\ dx + \int_\Omega tr\ (\sigma_h) q\ dx \\ &\quad + \int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx. \end{aligned}$$

Using Green's formula, we may write:

$$\begin{aligned}
\|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|^2_{0,\Omega} &= -2\mu \int_\Omega div\ (\varepsilon - P\delta).z\ dx + \mu \int_\Omega as\ (\sigma_h) rot\ z\ dx - \tfrac{2\mu}{\lambda} \int_\Omega Pq\ dx \\
&\quad + \tfrac{2\mu}{\lambda} \int_\Omega pq\ dx + \int_\Omega tr\ (\sigma_h) q\ dx + \int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx \\
&= 2\mu \int_\Omega (f - P_h^0 f).z\ dx + \mu \int_\Omega as\ (\sigma_h) rot\ z\ dx - \tfrac{2\mu}{\lambda} \int_\Omega Pq\ dx + \tfrac{2\mu}{\lambda} \int_\Omega pq\ dx \\
&\quad + \int_\Omega tr\ (\sigma_h) q\ dx + \int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx.
\end{aligned}$$

By $P = p - p_h$, we obtain the following decomposition formula:

$$\begin{aligned}
\|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|^2_{0,\Omega} &= 2\mu \int_\Omega (f - P_h^0 f).z\ dx + \mu \int_\Omega as\ (\sigma_h) rot\ z\ dx + \tfrac{2\mu}{\lambda} \int_\Omega p_h q\ dx \\
&\quad + \int_\Omega tr\ (\sigma_h) q\ dx + \int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx \\
&= 2\mu \int_\Omega (f - P_h^0 f).z\ dx + \mu \int_\Omega as\ (\sigma_h) rot\ z\ dx + \int_\Omega \big(\tfrac{2\mu}{\lambda} p_h + tr\ (\sigma_h) \big)\ q\ dx \\
&\quad + \int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx. \qquad (2.5.21)
\end{aligned}$$

To transform the last term of the right-hand side, let us consider $\psi_h := \mathrm{I}_{\mathrm{Cl}} \psi$ and $\phi_h = I_h \phi$ the BDM_1 interpolate of ϕ. By the first equality of the continuous problem (2.2.1) with $(\tau, 0) = (\phi + Curl\ \psi, 0) \in \Sigma_0$, we get

$$\frac{1}{2\mu} \int_\Omega \sigma : (\phi + Curl\ \psi)\ dx = 0.$$

Thus

$$\begin{aligned}
&\int_\Omega \varepsilon : (\phi + Curl\ \psi)\ dx \\
&\quad = - \int_\Omega \sigma_h : (\phi + Curl\ \psi)\ dx \\
&\quad = - \int_\Omega \sigma_h : \big(\phi - \phi_h + Curl\ (\psi - \psi_h)\big)\ dx - \int_\Omega \sigma_h : (\phi_h + Curl\ \psi_h)\ dx \qquad (2.5.22) \\
&\quad = - \int_\Omega \sigma_h : Curl\ (\psi - \psi_h)\ dx - \int_\Omega \sigma_h : (\phi - \phi_h)\ dx \\
&\qquad - \int_\Omega \sigma_h : (\phi_h + Curl\ \psi_h)\ dx.
\end{aligned}$$

We now treat separately the terms of the right-hand side of (2.5.22) . For the first one, using Green's formula we get

$$\begin{aligned}
&-\int_\Omega \sigma_h : \mathit{Curl}\ (\psi-\psi_h)\ dx \\
&\quad = -\sum_{T\in\mathcal{T}_h}\int_T \mathit{rot}\ (\sigma_h).(\psi-\psi_h)\ dx + \sum_{T\in\mathcal{T}_h}\int_{\partial T} \sigma_h.t.(\psi-\psi_h)\ ds \\
&\quad = -\sum_{T\in\mathcal{T}_h}\int_T \mathit{rot}\ (\sigma_h).(\psi-\psi_h)\ dx + \sum_{E\in\mathcal{E}_h}\int_E [[\sigma_h.t_E]]_E.(\psi-\psi_h)\ ds.
\end{aligned}$$

For the third term of the right-hand side of (2.5.22), taking as a test function $(\phi_h + \mathit{Curl}\ \psi_h, 0) \in \Sigma_{0,h}$ in the first equation of (2.2.4), we get

$$-\int_\Omega \sigma_h : (\phi_h + \mathit{Curl}\ \psi_h)\ dx \quad = \quad 2\mu\int_\Omega \mathit{as}\ (\phi_h + \mathit{Curl}\ \psi_h)\omega_h\ dx$$

Remembering that $\mathit{as}\ (\phi + \mathit{Curl}\ \psi) = 0$, it follows that:

$$\begin{aligned}
-\int_\Omega \sigma_h : (\phi_h + \mathit{Curl}\ \psi_h)\ dx &= 2\mu\int_\Omega \mathit{as}\ \big((\phi_h + \mathit{Curl}\ \psi_h) - (\phi + \mathit{Curl}\ \psi)\big)\omega_h\ dx \\
&= 2\mu\int_\Omega \mathit{as}\ \big(\phi_h - \phi + \mathit{Curl}\ (\psi_h - \psi)\big)\omega_h\ dx \\
&= -2\mu\int_\Omega \mathit{div}\ (\psi-\psi_h)\omega_h\ dx - 2\mu\int_\Omega \mathit{as}\ (\phi-\phi_h)\omega_h\ dx \\
&= -2\mu\int_\Omega \mathit{div}\ (\psi-\psi_h)\omega_h\ dx - 2\mu\int_\Omega (\phi-\phi_h) : \omega_h\chi\ dx.
\end{aligned} \tag{2.5.23}$$

For the first term in the right-hand side of (2.5.23), using Green's formula we get

$$\begin{aligned}
&-2\mu\int_\Omega \mathit{div}\ (\psi-\psi_h)\omega_h\ dx \\
&\quad = \sum_{T\in\mathcal{T}_h}\int_T 2\mu\nabla\omega_h : (\psi-\psi_h)\ dx - \sum_{T\in\mathcal{T}_h}\int_{\partial T} 2\mu\omega_h(\psi-\psi_h).n\ ds \\
&\quad = \sum_{T\in\mathcal{T}_h}\int_T 2\mu\nabla\omega_h : (\psi-\psi_h)\ dx - \sum_{T\in\mathcal{T}_h}\int_{\partial T} 2\mu\omega_h\delta.n.(\psi-\psi_h)\ ds \\
&\quad = \sum_{T\in\mathcal{T}_h}\int_T 2\mu\nabla\omega_h : (\psi-\psi_h)\ dx + \sum_{T\in\mathcal{T}_h}\int_{\partial T} 2\mu\omega_h\chi.t.(\psi-\psi_h)\ ds \\
&\quad = -\sum_{T\in\mathcal{T}_h}\int_T \mathit{rot}\ (2\mu\omega_h\chi).(\psi-\psi_h)\ dx + \sum_{E\in\mathcal{E}_h}\int_E [[2\mu(\omega_h\chi).t_E]]_E.(\psi-\psi_h)\ ds.
\end{aligned}$$

Altogether in the equality (2.5.22) yields

$$\begin{aligned}\int_\Omega \varepsilon : (\phi + \mathit{Curl}\ \psi)\ dx = &- \sum_{T\in\mathcal{T}_h} \int_T \mathit{rot}\ (\sigma_h + 2\mu\omega_h\chi).(\psi - \psi_h)\ dx \\ &+ \sum_{E\in\mathcal{E}_h} \int_E [[(\sigma_h + 2\mu\omega_h\chi).t_E]]_E.(\psi - \psi_h)\ ds \\ &- \sum_{T\in\mathcal{T}_h} \int_T (\sigma_h + 2\mu\omega_h\chi).(\phi - \phi_h)\ dx \\ \leq &\sum_{T\in\mathcal{T}_h} \|\mathit{rot}\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T}\|\psi - \psi_h\|_{0,T} \\ &+ \sum_{E\in\mathcal{E}_h} \|[[(\sigma_h + 2\mu\omega_h\chi).t_E]]_E\|_{0,E}\|\psi - \psi_h\|_{0,E} \\ &+ \sum_{T\in\mathcal{T}_h} \|\sigma_h + 2\mu\omega_h\chi\|_{0,T}\|\phi - \phi_h\|_{0,T}.\end{aligned}$$

Now applying Cauchy-Schwarz's inequality, Lemmas 2.3.6 and 2.3.7, we obtain

$$\begin{aligned}\int_\Omega \varepsilon : (\phi + \mathit{Curl}\ \psi)\ dx \lesssim &\Big[\sum_{T\in\mathcal{T}_h} h_T^2\|\mathit{rot}\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T}^2 \\ &+ \sum_{E\in\mathcal{E}_h} h_E\|[[(\sigma_h + 2\mu\omega_h\chi).t_E]]_E\|_{0,E}^2\Big]^{\frac{1}{2}}\|\nabla\psi\|_{0,\Omega} \\ &+ \sum_{T\in\mathcal{T}_h} h_T\|\sigma_h + 2\mu\omega_h\chi\|_{0,T}\|\nabla\phi\|_{0,\Omega}.\end{aligned} \tag{2.5.24}$$

The first three terms of the right hand side of (2.5.21) can be estimated using simply continuous and discrete Cauchy-Schwarz's inequalities:

$$\begin{aligned}&2\mu\int_\Omega (f - P_h^0 f).z\ dx + \mu\int_\Omega \mathit{as}\ (\sigma_h)\mathit{rot}\ (z)\ dx + \int_\Omega \big(\tfrac{2\mu}{\lambda}p_h + \mathit{tr}\ (\sigma_h)\big)\ q\ dx \\ &= \sum_{T\in\mathcal{T}_h}\Big[2\mu\int_T (f - P_h^0 f).z\ dx + \mu\int_T \mathit{as}\ (\sigma_h)\mathit{rot}\ (z)\ dx + \int_T \big(\tfrac{2\mu}{\lambda}p_h + \mathit{tr}\ (\sigma_h)\big)\ q\ dx\Big] \\ &\leq \sum_{T\in\mathcal{T}_h}\Big[\|2\mu(f - P_h^0 f)\|_{0,T}\|z\|_{0,T} + \|\mu\,\mathit{as}\ (\sigma_h)\|_{0,T}\|\mathit{rot}\ (z)\|_{0,T} \\ &\quad + \|\tfrac{2\mu}{\lambda}p_h + \mathit{tr}\ (\sigma_h)\|_{0,T}\|q\|_{0,T}\Big] \\ &\leq 2\mu\Big\{\Big[\sum_{T\in\mathcal{T}_h}\|(f - P_h^0 f)\|_{0,T}^2\Big]^{\frac{1}{2}}\|z\|_{0,\Omega} + \Big[\sum_{T\in\mathcal{T}_h}\tfrac{1}{4}\|\mathit{as}\ (\sigma_h)\|_{0,T}^2\Big]^{\frac{1}{2}}\|\mathit{rot}\ (z)\|_{0,\Omega} \\ &\quad + \Big[\sum_{T\in\mathcal{T}_h}\|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}\mathit{tr}\ (\sigma_h)\|_{0,T}^2\Big]^{\frac{1}{2}}\|q\|_{0,\Omega}\Big\}.\end{aligned} \tag{2.5.25}$$

Since $q_i \in H^2(\Omega)$, for all $i \in \{1, \cdots, p\}$ we have

$$\|\nabla \phi\|_{0,\Omega} \lesssim \sum_{i=1}^{p} |\beta_i|. \tag{2.5.26}$$

By Korn's, Poincare's and (2.5.19) inequalities, we obtain

$$\|rot\ (z)\|_{0,\Omega} \lesssim \|\nabla z\|_{0,\Omega} \lesssim \|\epsilon(z)\|_{0,\Omega} \lesssim \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}, \tag{2.5.27}$$

$$\|z\|_{0,\Omega} \lesssim \|\nabla z\|_{0,\Omega} \lesssim \|\epsilon(z)\|_{0,\Omega} \lesssim \|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega}. \tag{2.5.28}$$

Equations (2.5.26), (2.5.21), (2.5.24), (2.5.25), (2.5.27), (2.5.28) and (2.5.19), with the help of the discrete Cauchy-Schwarz's inequality, yield

$$\begin{aligned}\|\varepsilon - \tfrac{1}{2} as\ (\sigma_h)\chi\|_{0,\Omega} \lesssim & \Bigg[\sum_{T\in\mathcal{T}_h}\Big\{\|(f - P_h^0 f)\|_{0,T}^2 + \tfrac{1}{4}\|as\ (\sigma_h)\|_{0,T}^2 + \|\tfrac{1}{\lambda}p_h + \tfrac{1}{2\mu}tr\ (\sigma_h)\|_{0,T}^2 \\ & + h_T^2\|\sigma_h + 2\mu\omega_h\chi\|_{0,T}^2 + h_T^2\|rot\ (\sigma_h + 2\mu\omega_h\chi)\|_{0,T}^2\Big\} \\ & + \sum_{E\in\mathcal{E}_h} h_E\|[[(\sigma_h + 2\mu\omega_h\chi).t_E]]_E\|_{0,E}^2\Bigg]^{\frac{1}{2}}.\end{aligned}$$

Now using the estimate (2.5.20) and this last bound, we obtain

$$\|\varepsilon\|_{0,\Omega} \lesssim \eta. \tag{2.5.29}$$

■

The proof that the error $\|p - p_h\|_{0,\Omega}$ is bounded by the error estimator η up to a multiplicative constant remains the same as in the Proposition 2.4.3. Thus we have the following proposition:

Proposition 2.5.4. *The following estimate holds*

$$\|P\|_{0,\Omega} \lesssim \eta \tag{2.5.30}$$

It remains to bound the errors $e := u - u_h$ and $r := \omega - \omega_h$. As previously , we use the notations $\beta_h := \sigma_h + 2\mu\omega_h\chi$, $\beta := \sigma + 2\mu\omega\chi = 2\mu\nabla u$. Since proposition 2.5.3 bounds $\|\varepsilon\|_{0,\Omega} = \|\sigma - \sigma_h\|_{0,\Omega}$ by a constant times η, it suffices to bound $\|\beta - \beta_h\|_{0,\Omega}$ in order to obtain an estimate for $\|\omega - \omega_h\|_{0,\Omega}$. This will be also the main ingredient to bound $\|u - u_h\|_{0,\Omega}$ in terms of η as we will see. The preceding demonstration of the upper bound of the term $\|\beta - \beta_h\|_{0,\Omega}$ by the error indicator η does not work any more because of the additional terms which appear in the decomposition of tensor field in the case of multiply-connected domain. For this reason we propose a new shorter demonstration based on the decomposition stated in Proposition 2.5.2.

Lemma 2.5.5. *The following estimate holds*

$$\|\beta - \beta_h\|_{0,\Omega} \lesssim \eta.$$

Proof: In view of proposition 2.5.2, there exists $v \in [H^1(\Omega)]^2$ with $v = 0$ on Γ_D, $q \in [H^1(\Omega)]^2$, $\gamma_i \in \mathbb{R}^2$, $\gamma_i = (\gamma_i^1, \gamma_i^2)^T \in \mathbb{R}^2$ for all $i \in \{1, \cdots, p\}$ and $\phi \in [H^1(\Omega)]^2$ with $\phi = constant$ on Γ_N such that

$$\beta - \beta_h = 2\mu\epsilon(v) - q\delta + Curl\ \phi - \sum_{i=1}^{p} \gamma_i * \nabla q_i, \tag{2.5.31}$$

with the estimate

$$\|\epsilon(v)\|_{0,\Omega} + \|\nabla\phi\|_{0,\Omega} + \sum_{i=1}^{p} |\gamma_i| \lesssim \|\beta - \beta_h\|_{0,\Omega}. \tag{2.5.32}$$

Let us set

$$\tau = \sum_{i=1}^{p} \gamma_i * \nabla q_i.$$

Hence

$$\begin{aligned} \|\beta - \beta_h\|_{0,\Omega}^2 &= \int_\Omega (\beta - \beta_h) : (2\mu\epsilon(v) - q\delta + Curl\ \phi - \tau)\ dx \\ &= \int_\Omega (\beta - \beta_h) : (2\mu\epsilon(v) - q\delta)\ dx - \int_\Omega \beta_h : (Curl\ \phi - \tau)\ dx. \end{aligned} \tag{2.5.33}$$

For the first term of the right-hand side of (2.5.33) we use the definition of $\beta - \beta_h$ and by Cauchy Schwarz inequality we get

$$\begin{aligned} \int_\Omega (\beta - \beta_h) : (2\mu\epsilon(v) - q\delta)\ dx &= \int_\Omega (\varepsilon + 2\mu r\chi) : (2\mu\epsilon(v) - q\delta)\ dx \\ &= \int_\Omega \varepsilon : (2\mu\epsilon(v) - q\delta)\ dx \\ &\leq \|\varepsilon\|_{0,\Omega} \|2\mu\epsilon(v) - q\delta\|_{0,\Omega}. \end{aligned}$$

For the second term of the right-hand side (2.5.33) we use the Clément and BDM_1 interpolation. Let $\phi_h = \mathrm{I}_{\mathrm{Cl}}\phi$ be the Clément interpolation of ϕ and $\tau_h^* = I_h\tau$ the global BDM_1 interpolate of τ. By the first equation of the discrete problem (2.2.4) with

$\tau_h = Curl\ (\phi_h) - \tau_h^*$ and $q_h = 0$, we have

$$
\begin{aligned}
-\int_\Omega \beta_h : (Curl\ \phi - \tau)\ dx &= \int_\Omega \beta_h : \big(Curl\ (\phi_h - \phi) - (\tau_h^* - \tau)\big)\ dx \\
&= \sum_{T \in \mathcal{T}_h} \int_T (rot\ (\beta_h)).(\phi_h - \phi)\ dx - \sum_{E \in \mathcal{E}_h} \int_E \big[\big[(\beta_h).t_E\big]\big]_E.(\phi_h - \phi)\ ds \\
&\quad - \sum_{T \in \mathcal{T}_h} \int_T \beta_h : (\tau_h^* - \tau)\ dx \\
&\lesssim \sum_{T \in \mathcal{T}_h} h_T \|rot\ (\beta_h)\|_{0,T} \|\nabla \phi\|_{\omega_T} + \sum_{E \in \mathcal{E}_h} h_E^{\frac{1}{2}} \|\big[\big[(\beta_h).t_E\big]\big]_E\|_{0,E} \|\nabla \phi\|_{\omega_E} \\
&\quad + \sum_{T \in \mathcal{T}_h} h_T \|\beta_h\|_{0,T} \|\nabla \tau\|_{0,T} \quad \text{by (2.3.14) and (2.3.15).}
\end{aligned}
$$

By discrete Cauchy-Schwarz inequality we get

$$
\begin{aligned}
-\int_\Omega \beta_h : (Curl\ \phi - \tau)\ dx \lesssim \Big[&\sum_{T \in \mathcal{T}_h} h_T^2 \|rot\ (\beta_h)\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h} h_E \|\big[\big[(\beta_h).t_E\big]\big]_E\|_{0,E}^2 + \\
&\sum_{T \in \mathcal{T}_h} h_T^2 \|\beta_h\|_{0,T}^2\Big]^{\frac{1}{2}} \times \big(\|\nabla \phi\|_{0,\Omega} + \|\nabla \tau\|_{0,\Omega}\big).
\end{aligned} \tag{2.5.34}
$$

Since $q_i \in H^2(\Omega)$, for all $i \in \{1, \cdots, p\}$ we have

$$
\|\nabla \tau\|_{0,\Omega} \lesssim \sum_{i=1}^{p} |\gamma_i|. \tag{2.5.35}
$$

Now the inequalities (2.5.35), (2.5.34), (2.5.34) in (2.5.33) yield

$$
\begin{aligned}
&\|\beta - \beta_h\|_{0,\Omega}^2 \lesssim \|\varepsilon\|_{0,\Omega} \|\beta - \beta_h\|_{0,\Omega} + \\
&\quad \Big[\sum_{T \in \mathcal{T}_h} h_T^2 \|rot\ (\beta_h)\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h} h_E \|\big[\big[(\beta_h).t_E\big]\big]_E\|_{0,E}^2 + \sum_{T \in \mathcal{T}_h} h_T^2 \|\beta_h\|_{0,T}^2\Big]^{\frac{1}{2}} \|\beta - \beta_h\|_{0,\Omega}
\end{aligned}
$$

This last inequality with Proposition 2.5.3 yield

$$
\|\beta - \beta_h\|_{0,\Omega} \lesssim \eta.
$$

■

Remark 2.5.6. *The preceding demonstration is also valid in the case of a simply connected domain Ω. Thus we can also use the above technique to prove Proposition 2.4.4.*

From the preceding Lemma and proposition 2.5.3, we have immediately:

Proposition 2.5.7. *The following bound holds*

$$\|\omega - \omega_h\|_{0,\Omega} \lesssim \eta.$$

With the same techniques and geometric assumption on Ω and Γ_N as in the preceding proof of Proposition 2.4.6, we get the following result:

Proposition 2.5.8. *Let us suppose that each vertex $s \in \overline{\Gamma}_N$ is convex. Then the following estimate holds*

$$\|u - u_h\|_{0,\Omega} \lesssim \eta.$$

2.5.3 Proof of the efficiency of the estimator

As it were noted above, no modification in the proof of the efficiency of the estimator will be necessary. Indeed, no decomposition of tensor fields was used in its proof. Hence efficiency of our estimator η still hold. Consequently Theorem 2.4.7 remains true.

2.6 Adaptive algorithm

From our theoretical considerations, an adaptive mesh-refinement algorithm has to use appropriately our global and local refinement indicators η and η_K respectively. A relatively general mesh-refinement algorithm is as follows:

1: Start with initial mesh $\mathcal{T}_{h,0}$. Set $i = 0$ and fix a tolerance $Tol > 0$.

2: Solve the discrete problem with respect to the present mesh $\mathcal{T}_{h,i}$.

3: On each element we compute a local error indicator.

4: Compute the global error indicator

$$\eta = \Big(\sum_{K \in \mathcal{T}_{h,i}} \eta_K \Big)^{\frac{1}{2}}$$

corresponding to the mesh $\mathcal{T}_{h,i}$. Terminate if $\eta \leq Tol$.

5: Else, with the help of these local error indicators, mark the elements K which need further refinement.

6: Perform Red-Green-Blue refinement to avoid hanging nodes.

7: Generate the new mesh $\mathcal{T}_{h,i+1}$, set $i \leftarrow i + 1$ and come back to step 2.

Several strategies are possible at the step 5 when the new mesh is built starting from the local indicators. A strategy frequently used in the applications consists in to distribute in a balanced way the error on the mesh. The criterion of refinement is to refin those triangles $K \in \mathcal{T}_{h,i}$ for which :

$$\eta_K \geq \frac{1}{2}\Big(\max_{T \in \mathcal{T}_{h,i}} \eta_T \Big).$$

It is this strategy which is actually used in our adaptive algorithm. Details on the Red-Green-Blue refinement can be found in [38]. The numerical performance of this algorithm will be illustrated in Section 2.8.

2.7 Conclusion

A new a posteriori error estimator for a dual mixed finite element method of the elasticity problem is introduced and analyzed. It is shown that this error estimator is reliable and efficient for simply-connected domains and also for multiply-connected domains. The lower and upper error bounds obtained are uniform with respect to the Lamé coefficient λ (thus avoiding the so-called locking effect). The estimator allows an adaptive finite element scheme which refines a given grid only in regions where the error is relatively large. Finally, the technique developed to establish this estimator can be extended to the three-dimensional case.

2.8 Numerical test

In this section we will corroborate our theoretical analysis by numerical tests. We investigate three model problems to provide experimental evidence of the robustness, reliability and efficiency of our a posteriori error estimator. On each example we exhibit in evidence the good performance of the adaptive strategy of Section 2.6 in comparison with a uniform mesh-refinement.

2.8.1 Analytic solution on a L-shaped domain

We consider as a first model example, the L-shaped domain shown in figure 1.2. With the same analytical solution (1.7.6) used in the numerical test 1.7.2. We first investigate the main theoretical results which are the upper and lower error bounds (2.4.2) and (2.4.23).

First, we define the ratio of the left-hand side and the right-hand side of the inequality (2.4.2).

$$q_{up} = \frac{\|\sigma - \sigma_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} + \|\omega - \omega_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega}}{\eta}.$$

The stability index q_{up} measures the reliability of the estimator and is related to the global upper error bound. In figure 2.5, Table 2.1 and Table 2.2, the quotient q_{up} is seen to be nicely bounded from above. This confirms numerically the theoretical results obtained in Theorem 2.4.1.

Now, we define the ratio of the left-hand side and the right-hand side of the inequality (2.4.23):

$$q_{low} = \max_{T\in\mathcal{T}_h} \frac{\eta_T}{\|f - P_h^0 f\|_{0,T} + \|u - u_h\|_{0,T} + \|\sigma - \sigma_h\|_{0,\omega_T} + \|\omega - \omega_h\|_{0,\omega_T} + \|p - p_h\|_{0,T}}.$$

The efficiency index q_{low} measures the efficiency of the estimator. In figure 2.4, Table 2.1 and Table 2.2, the quotient q_{low} is seen to be nicely bounded from above as theoretically predicted in Theorem 2.4.7.

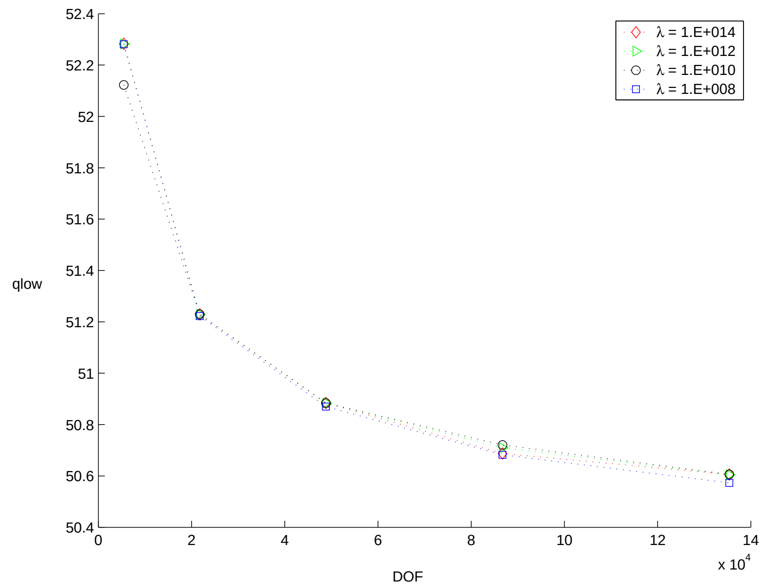


Figure 2.4: q_{low} wrt DOF for uniform meshes

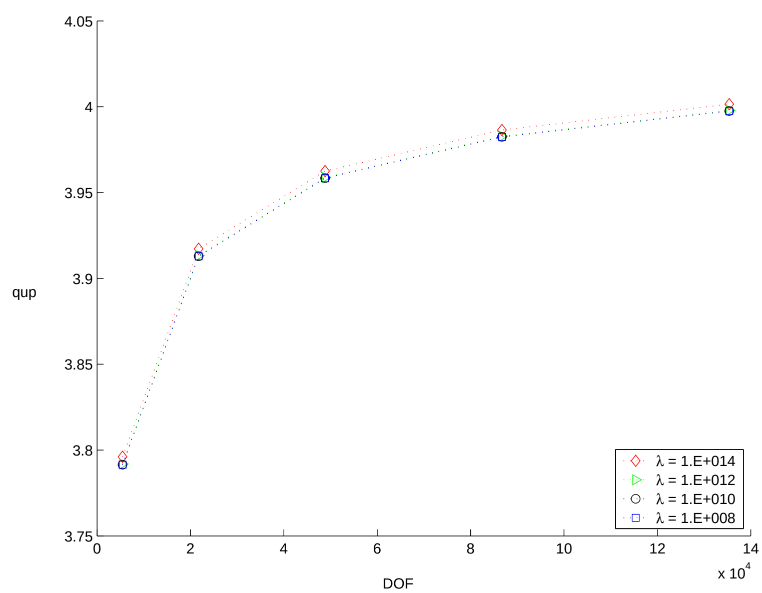


Figure 2.5: q_{up} and wrt DOF for uniform meshes

Table 2.1: q_{low} and q_{up} wrt DOF for uniform meshes with λ = 1.E+06

h	DOF	q_{low}	q_{up}
1.414214e-001	5480	52.2825	3.796093
7.071068e-002	21760	51.2292	3.917217
4.714045e-002	48840	50.8841	3.962506
3.535534e-002	48840	50.6875	3.986473
2.828427e-002	135400	50.6055	4.001465

Table 2.2: q_{low} and q_{up} wrt DOF for uniform meshes with λ = 1.E+12

h	DOF	q_{low}	q_{up}
1.414214e-001	5480	52.2808	3.791462
7.071068e-002	21760	51.2224	3.912900
4.714045e-002	48840	50.8700	3.958350
3.535534e-002	48840	50.6821	3.982366
2.828427e-002	135400	50.5731	3.997484

The identical behavior of the errors $\|\sigma - \sigma_h\|_{0,\Omega}$, $\|p - p_h\|_{0,\Omega}$, $\|u - u_h\|_{0,\Omega}$, $\|\omega - \omega_h\|_{0,\Omega}$ and the error estimator η with respect to h shown on the figure 2.6, highlights the reliability and the efficiency of our error indicator η.

The final mesh after 12 refinement steps is shown in Figure 2.7. A very strong refinement is produced by the algorithm around the re-entrant corner.

2.8.2 Cook's membrane problem

The well known Cook's membrane problem [13] is shown in Figure 2.8. Here we consider a panel maintained on an end and subjected to a shear loading on the opposite end with $\lambda = 1000$, $\mu = 500$, $f = 0$, $g = (0, 1000)$ on the vertical component of Γ_N and $g = (0, 0)$ on the remaining components of Γ_N.

This problem is useful for the comparative study of accuracy because there are available numerical results from other sources. In figure 2.8 we summarize numerical results for the vertical displacement at point B, calculated by using uniform or adaptive meshes. The final and initial mesh after 10 refinement steps are shown in Figure 2.10.

2.8.3 Plate with an elliptic notch

As a final test case, we consider a plate with an elliptic notch with a = 10 mm and b = 2 mm, subject to the traction load $g = (0, 1000)$ on the top and $g = (0, -1000)$ on the bottom. $g = (0, 0)$ on the remaining components of Γ_N and $f = 0$. We set $\lambda = 7.5E{+}003$,

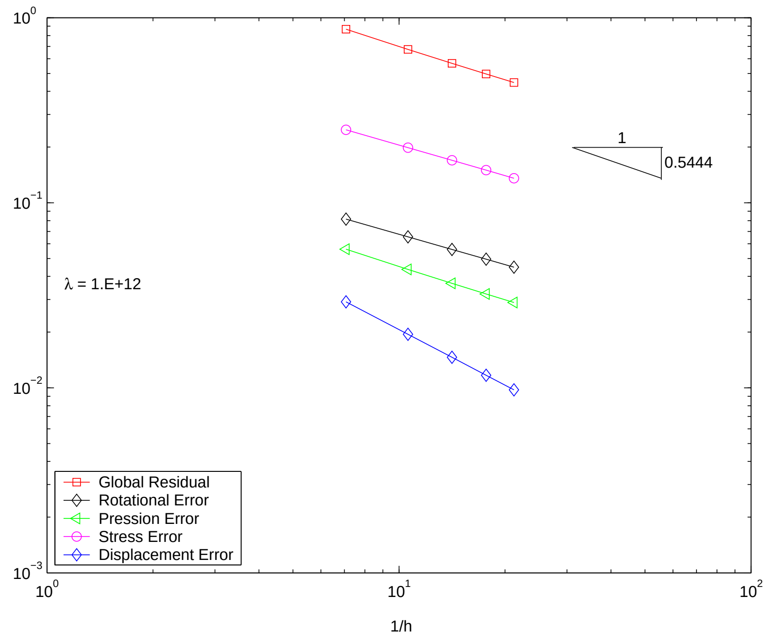


Figure 2.6: Errors and error estimator for uniform meshes with λ = 1.E+12

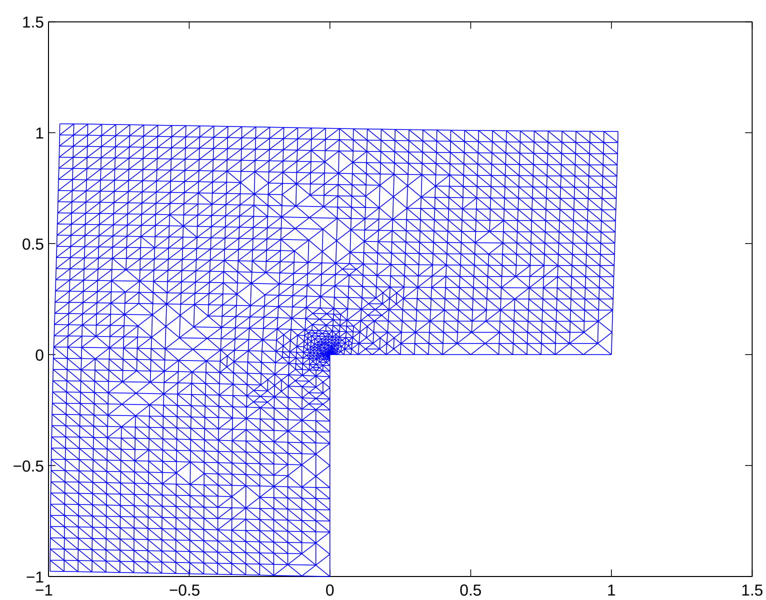


Figure 2.7: Adapted deformed mesh after 12 refinement steps with Algorithm of Section 2.6 (displacement magnified by factor 1.E+05) for λ = 1.E+08

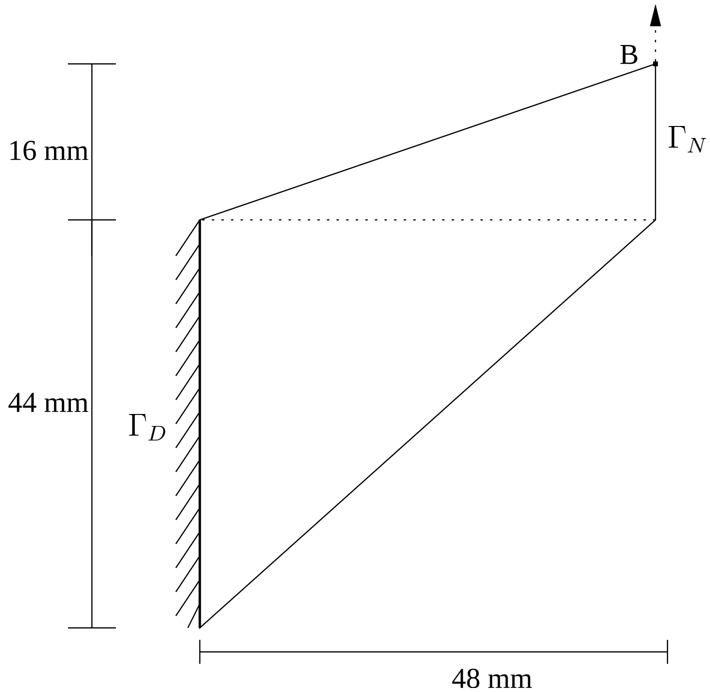


Figure 2.8: Cook's membrane problem

$\mu = 3750$ (see figure 2.11). Inhomogeneous Direchlet boundary conditions are enforced on the lips of the notch: $u_D(x, y) = (0, 20\sqrt{a^2 - x^2})$ if $y > 0$ and $u_D(x, y) = (0, -20\sqrt{a^2 - x^2})$ if $y < 0$.

In figure 2.13 we have plotted the value $\sigma_{1,1}$ at point P computed by an averaged strain approximation using uniform meshes and adapted meshes. We observe that the Algorithm of Section 2.6 generates a strong refinement around the notch. We may conclude that the adaptively refined mesh is more efficient than the uniform one.

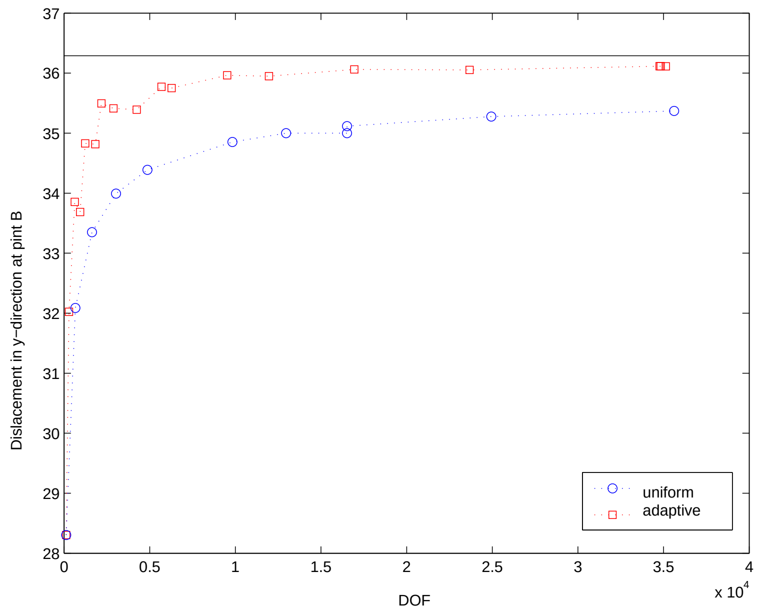


Figure 2.9: Vertical displacement at B wrt DOF

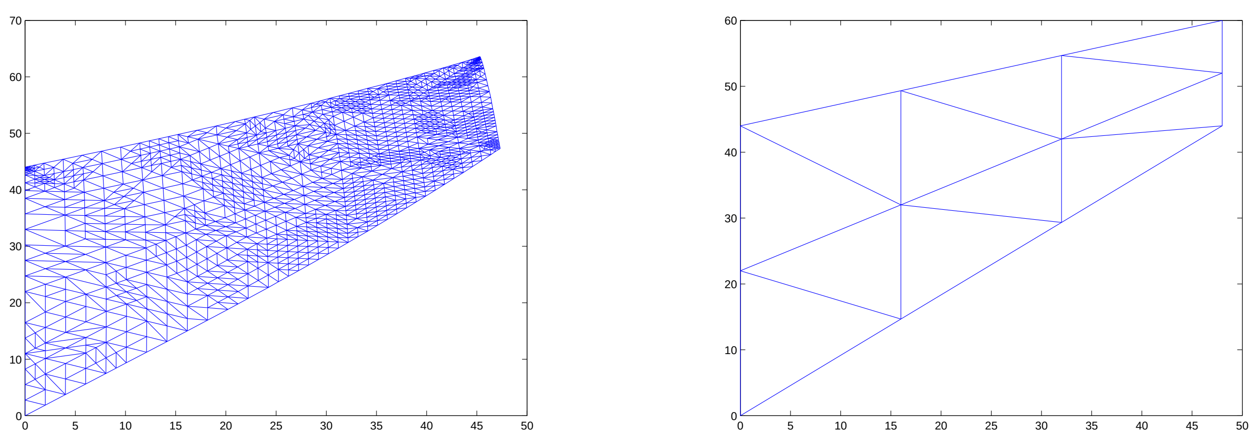


Figure 2.10: Deformed (left) and initial (right) mesh (displacement magnified by factor $\frac{1}{10}$) after 10 refinements with Algorithm of Section 2.6

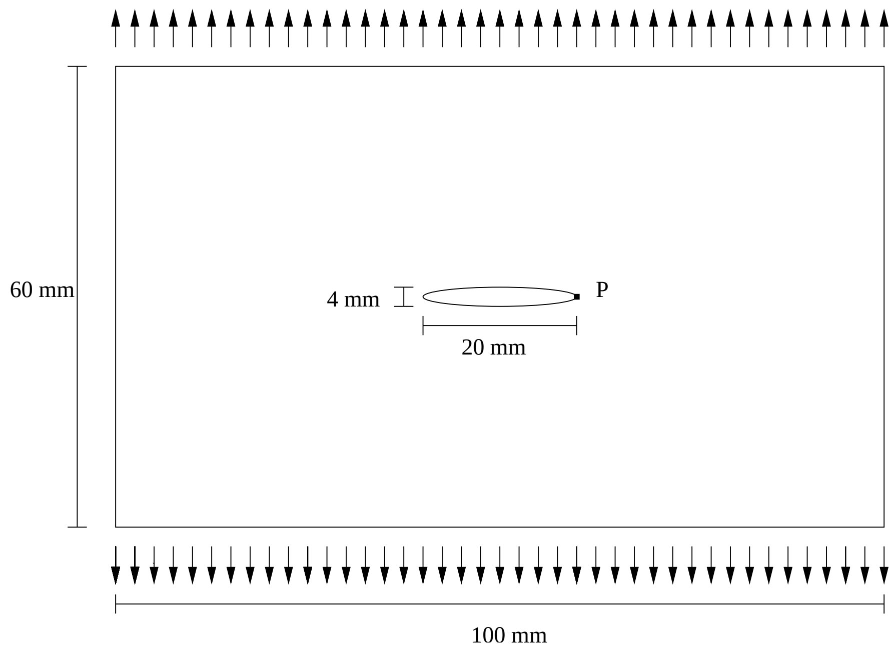


Figure 2.11: Plate with an elliptic notch

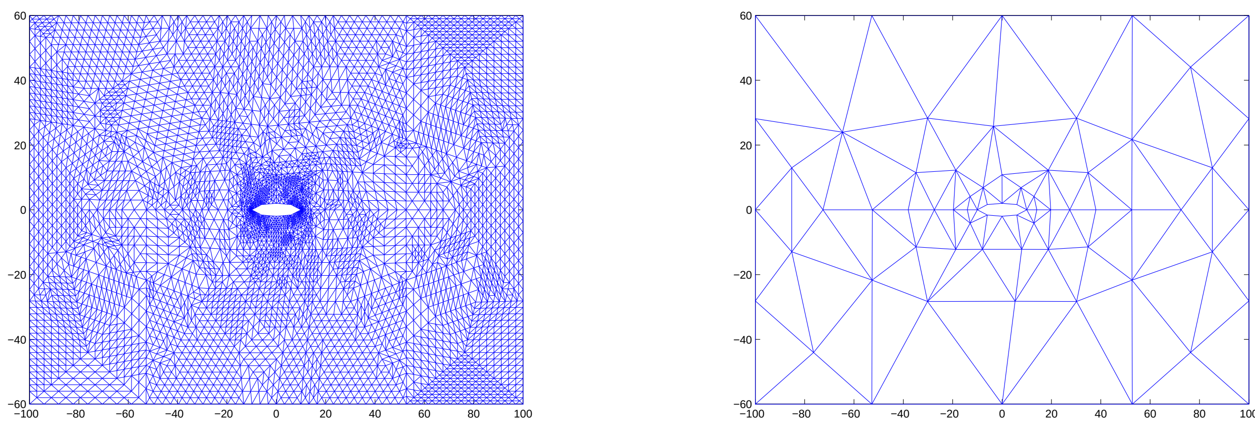


Figure 2.12: Initial(right) and adaptive (left) mesh after 15 refinements with Algorithm of Section 2.6

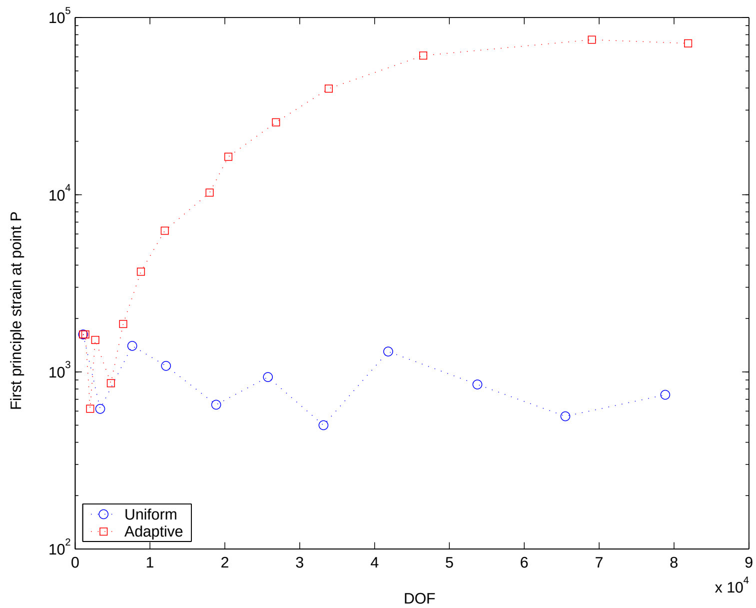


Figure 2.13: $\sigma_{1,1}$ component of the strain tensor at point P as a function of the number of degrees of freedom for uniform and adaptive meshes

3

DUAL MFE METHOD FOR THE ELASTODYNAMIC PROBLEM

Contents

3.1 Introduction

The purpose of this chapter is the analysis of a finite element method for approximating the linear elastodynamic system using a new dual mixed formulation for the discretization in the spatial variables and the explicit or implicit Newmark scheme for the discretization in time. The explicit scheme is shown to be stable under an appropriate CFL condition.

The analysis of a priori error estimates for the mixed finite element method of a second order hyperbolic problem was initiated in [16, 18, 31] but to our knowledge a similar analysis for the dual mixed formulation of the linear elastodynamic system was not yet done. Therefore our goal in this chapter is to make this analysis.

The outline of this chapter is as follows: Section 3.2 defines some notations, present the model evolution problem we shall consider and recall two comparison results concerning continuous and discrete Gronwall's inequalities. In section 3.3 we define the new dual mixed formulation of the model evolution problem. Section 3.4 is devoted to some regularity results of the solution of our elastodynamic system in terms of weighted Sobolev spaces. In section 3.5 we introduce the semi-discrete mixed formulation and prove the existence and uniqueness of the solution, we begin by recalling some results concerning the inf-sup and coercivety conditions. Then, under some adequate refinement rules of grids, we establish some error estimates on some interpolation operators and we prove an inverse inequality for the divergence operator. In subsection 3.5.1 we derive some error estimates between the exact solution of the mixed problem and the solution of the elliptic projection problem which will be used to derive the error estimates between the exact and the semi-discrete solution. Section 3.6 is concerned with the fully discrete mixed finite element scheme. We introduce in subsection 3.6.1 some notations for the discrete derivatives in time and present the fully discrete problem. Subsection 3.6.2 is devoted to analyze the stability of our explicit scheme in time , by establishing an appropriate CFL condition. In subsection 3.6.3, we establish optimal error estimates for the fully discrete problem. Subsection 3.6.4 is devoted to the implicit scheme. In section 3.7 we present conclusions. The numerical experiments of section 3.8 confirm our theoretical predictions.

3.2 Preliminaries and notations

3.2.1 The model problem

Let us fix a bounded plane domain Ω with a polygonal boundary. More precisely, we assume that Ω is a simply connected domain and that its boundary Γ is the union of a finite number of linear segments $\bar{\Gamma}_j$, $1 \le j \le n_e$ (Γ_j is assumed to be an open segment). We also fix a partition of $\{1, 2, \cdots, n_e\}$ into two subsets $I_{\mathcal{N}}$ and $I_{\mathcal{D}}$. The union Γ_D of the Γ_j, j running over $I_{\mathcal{D}}$, is the part of the boundary Γ, where we assume zero displacement field. The union Γ_N, of the Γ_j, $j \in I_{\mathcal{N}}$ is the part of the boundary Γ where we assume zero traction field.

In this domain Ω we consider anisotropic elastic homogeneous material. Let $u = (u_1, u_2)$ be the displacement field and $f = (f_1, f_2) \in L^2([0,T]; [L^2(\Omega)]^2)$ the body force by unit of mass. Thus the displacement field $u = (u_1, u_2)$ satisfies the following equations :

$$\begin{cases} u_{tt} - div\, \sigma_s(u) &= f \quad \text{in} \quad [0,T] \times \Omega, \\ u &= 0 \quad \text{on} \quad [0,T] \times \Gamma_D, \\ \sigma_s(u).n &= 0 \quad \text{on} \quad [0,T] \times \Gamma_N, \\ u(0,.) &= u_0 \quad \text{in} \quad \Omega, \\ u_t(0,.) &= u_1 \quad \text{in} \quad \Omega, \end{cases} \tag{3.2.1}$$

where u_0 and u_1 be the initial conditions on displacements and velocities. The stress tensor $\sigma_s(u)$ is defined by

$$\sigma_s(u) \; := \; 2\mu\epsilon(u) \; + \; \lambda tr\, \epsilon(u)\delta. \tag{3.2.2}$$

The positive constants μ and λ are called the Lamé coefficients. We assume that

$$(\lambda, \mu) \; \in \; [\lambda_0, \lambda_1] \; \times \; [\mu_1, \mu_2] \tag{3.2.3}$$

where

$$0 \; < \; \mu_1 \; < \; \mu_2 \text{ and } 0 \; < \; \lambda_0 \; < \; \lambda_1.$$

As usual, $\epsilon(u)$ denotes the linearized strain tensor ($i.e., \epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$) and δ the identity tensor.

For reasons of simplicity we have chosen her homogeneous boundary conditions in both Dirichlet and Newman boundary, the treatment of the inhomogeneous boundary conditions is done without difficulty. Let us note in passing that the numerical test (see section 7) are made under the inhomogeneous surface traction . In the sequel, we will recall the following notations. For $\tau = (\tau_{ij}) \in (H(div\,;\Omega))^2$, we denote by

$$div\ (\tau) = \left(\frac{\partial \tau_{1,1}}{\partial x_1} + \frac{\partial \tau_{1,2}}{\partial x_2}, \frac{\partial \tau_{2,1}}{\partial x_1} + \frac{\partial \tau_{2,2}}{\partial x_2} \right),$$
$$as\ (\tau) = \tau_{2,1} - \tau_{1,2}.$$

For $v = (v_1, v_2) \in [H^1(\Omega)]^2$, we recall that

$$rot\ v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

As usual, we denote by $L^2(.)$ the Lebesgue space and by $H^s(.),\ s \geq 0$, the standard Sobolev spaces. The usual norm and seminorm of $H^s(D)$ are denoted by $||\ .\ ||_{s,D}$ and $|\ .\ |_{s,D}$. The inner product in $[L^2(\Omega)]^2$ will be written $(.,.)$. If $\sigma = (\sigma_{ij}),\ \tau = (\tau_{ij}) \in [L^2(\Omega)]^{2\times 2}$, then we denote by

$$\sigma : \tau = \sum_{i,j} \sigma_{ij}\tau_{ij},$$
$$(\sigma, \tau) = \int_\Omega \sigma_{ij}\tau_{ij}\ dx.$$

We now introduce the Hilbert space

$$[H^1_{\Gamma_D}(\Omega)]^2 := \{v \in [H^1(\Omega)]^2; \quad v|_{\Gamma_D} = 0\}$$

Finally, in order to avoid excessive use of constants, we will use the following notations: $a \lesssim b$ stand for $a \leq cb$, with positive constants c independent of $a,\ b,\ h$ and Δt.

3.2.2 Gronwall's inequalities

In this section, we recall two comparison results, which will be useful in the stability and convergence analysis of our problem. Let $\phi \geq 0$ such that $\phi_t(t) \leq \rho\phi(t) + \eta(t)$ for $0 \leq t \leq T$, where $\rho \geq 0$ is some constant and $\eta(t) \geq 0$, $\eta \in L^1([0,T])$, then

$$\phi(t) \leq e^{\rho T}\bigg(\phi(0) + \int_0^T \eta(s)\ ds\bigg). \tag{3.2.4}$$

Let $(k_n)_{n\geq 0}$, $(p_n)_{n\geq 0}$ two non-negative sequences be given, $g_0 \geq 0$ given also and let us suppose that the sequence $(\phi_n)_{n\geq 0}$ satisfies:

$$\begin{cases} \phi_0 & \leq g_0, \\ \phi_n & \leq g_0 \quad + \displaystyle\sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s\phi_s, \quad \forall n \geq 1. \end{cases} \tag{3.2.5}$$

Then

$$\phi_n \leq \bigg(g_0 + \sum_{s=0}^{n-1} p_s\bigg) \exp\bigg(\sum_{s=0}^{n-1} k_s\bigg), \quad \forall n \geq 1. \tag{3.2.6}$$

3.3 The dual mixed formulation

We recall from chapter 1 the unknowns:

$$\sigma := 2\mu\epsilon(u),\ p := -\lambda div\ (u) \ \text{ and } \ \omega := \frac{1}{2}rot\ (u),$$

and the spaces:

$$\begin{aligned} \Sigma_0 := \{(\tau, q) \ \in \ [L^2(\Omega)]^{2\times 2} \times [L^2(\Omega)];\ div\ (\tau - q\delta) \in [L^2(\Omega)]^2, \\ (\tau - q\delta).n \ = \ 0 \text{ on } \Gamma_N\} \end{aligned} \tag{3.3.1}$$

$$V \times W \ := \ \{(\mathrm{v}, \theta) \in \ [L^2(\Omega)]^2 \times L^2(\Omega)\}. \tag{3.3.2}$$

We state the dual mixed finite element method for our model hyperbolic equation (3.2.1): Find $(\sigma(.), p(.)) \in L^2([0,T]; \Sigma_0)$, $u(.) \in H^2([0,T]; [L^2(\Omega)]^2)$ and $\omega(.) \in L^2([0,T]; L^2(\Omega))$ such that for all $(\tau, q) \in \Sigma_0$, for all $(v, \theta) \in V \times W$ and for a.e. $t \in [0,T]$ we have:

$$\begin{cases} \dfrac{1}{2\mu}(\sigma(t), \tau) + \dfrac{1}{\lambda}(p(t), q) + (div\ (\tau - q\delta), u(t)) + (as\ (\tau), \omega(t)) = 0, \\ (u_{tt}(t), v) - (div\ (\sigma(t) - p(t)\delta), v) - (as\ (\sigma(t)), \theta) - (f(t), v) = 0, \\ u(0) = u_0, u_t(0) = u_1. \end{cases} \tag{3.3.3}$$

We conclude this section by introducing some notations. We set

$$\underset{\sim}{\sigma}(t) = (\sigma(t), p(t)),\ \underset{\sim}{\tau} = (\tau, q),\ \underset{\sim}{u}(t) = (u(t), \omega(t)),\ \underset{\sim}{v} = (v, \theta),$$

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) := \frac{1}{2\mu}(\sigma, \tau) + \frac{1}{\lambda}(p, q), \quad \forall \underset{\sim}{\sigma},\ \underset{\sim}{\tau} \in \Sigma_0, \tag{3.3.4}$$

$$b(\underset{\sim}{\tau}, \underset{\sim}{v}) := (div\ (\tau - q\delta), v) + (as\ (\tau), \theta), \quad \forall \underset{\sim}{\tau} \in \Sigma_0, \quad \forall \underset{\sim}{v} \in [L^2(\Omega)]^2 \times L^2(\Omega). \tag{3.3.5}$$

With these notations, the mixed formulation (3.3.3) may be rewritten: Find $\underset{\sim}{\sigma}(.) = (\sigma(.), p(.)) \in L^2([0,T]; \Sigma_0)$ and $\underset{\sim}{u}(.) = (u(.), \omega(.)) \in H^2([0,T]; [L^2(\Omega)]^2) \times L^2([0,T]; L^2(\Omega))$ such that

$$\begin{cases} a(\underset{\sim}{\sigma}(t), \underset{\sim}{\tau}) + b(\underset{\sim}{\tau}, \underset{\sim}{u}(t)) = 0, \quad \forall \underset{\sim}{\tau} \in \Sigma_0, \text{ for a.e. } t \in [0,T] \\ b(\underset{\sim}{\sigma}(t), \underset{\sim}{v}) + (\mathcal{F}(t), \underset{\sim}{v}) = (u_{tt}(t), v), \quad \forall \underset{\sim}{v} \in [L^2(\Omega)]^2 \times L^2(\Omega), \text{ for a.e. } t \in [0,T] \end{cases} \tag{3.3.6}$$

where $(\mathcal{F}(t), \underset{\sim}{v}) := (f(t), v)$.

3.4 Regularity of the solutions

Let $u \in L^2(0,T; [H^1_{\Gamma_D}(\Omega)]^2)$ such that $\frac{du}{dt} \in L^2(0,T; [L^2(\Omega)]^2)$ be the solution of (3.2.1). We consider the Lamé operator defined below

$$L := -\mu\Delta - (\lambda + \mu)\nabla div\ ().$$

Thus, equivalently u is solution of the problem

$$\begin{cases} u_{tt} + L(u) &= f \quad \text{in} \quad [0,T] \times \Omega, \\ u &= 0 \quad \text{on} \quad [0,T] \times \Gamma_D, \\ \sigma_s(u).n &= 0 \quad \text{on} \quad [0,T] \times \Gamma_N, \\ u(0,.) &= u_0 \quad \text{in} \quad \Omega, \\ u_t(0,.) &= u_1 \quad \text{in} \quad \Omega. \end{cases} \tag{3.4.1}$$

It is wellknown (see [25] or [27, 28, 14]) that the weak solution of the corresponding Lamé system of (3.4.1) presents vertex singularities. We returns to the definition 1.3.1 for the description of these singularities. Let us recall from the Definition 1.3.3 the following weighted Sobolev Space:

For any scalar function $\phi \in C^0(\overline{\Omega})$ such that $\phi(x) > 0$ for every $x \in \bar{\Omega}\backslash\{S_1, S_2, \cdots, S_{ne}\}$ and any $m,\ k \in \mathbb{N}$, we define

$$H^{m,k}_\phi(\Omega) = \{v \in H^m(\Omega); \quad \phi D^\beta v \in L^2(\Omega),\ \forall \beta \in \mathbb{N}^2 \ \text{ such that } \ m < |\beta| \le m + k\}.$$

$H^{m,k}_\phi(\Omega)$ is equiped with the norm:

$$||v||_{m,k;\phi,\Omega} = \left(||v||^2_{m,\Omega} + \sum_{m<|\beta|\le m+k} ||\phi D^\beta v|^2_{0,\Omega} \right)^{1/2}.$$

and the semi-norm:

$$|v|_{m,k;\phi,\Omega} = \left(\sum_{|\beta|=m+k} ||\phi D^\beta v||^2_{0,\Omega} \right)^{1/2}.$$

The time space-norm is defined as

$$||v||_{L^2(H_\phi^{m,k})} = \left(\int_0^T ||v||^2_{m,k;\phi,\Omega}\, dt\right)^{1/2}.$$

The usual modification is made for the time-space norm $||v||_{L^\infty(H_\phi^{m,k})}$. Let us set $\xi = \min_{j=1,\cdots,n_e} \xi_j$ where

$$\xi_j = \inf_k\{Re\ \alpha_{j,k};\quad Re\ \alpha_{j,k} > 0\},$$

where $\alpha_{j,k}$ is solution of the appropriate trancendantal equation appearing in definition 1.3.1. Let us pick some $\alpha \in]1-\xi, 1/2[$ if $\xi \leq 1$, and let us take $\alpha = 0$ if $\xi > 1$.

Now we can give the following regularity result:

Proposition 3.4.1. *Let us suppose that the appropriate characteristic equation among 1.3.1 - 1.3.3 for each vertex of* Ω *has no root on the vertical line* $Re\alpha = 1$ *in the complex plane. Let* $\phi \in C^0(\overline{\Omega})$ *like above in Corollary 1.3.4.*

Let us suppose that:

$$\left\{\begin{array}{ccc} f & \in & H^3(0,T;[L^2(\Omega)]^2) \\ u_0,\ u_1,\ f(0)-Lu_0 & \in & [H^1_{\Gamma_D}(\Omega)]^2 \\ f_t(0)-Lu_1,\ f_{tt}(0)-Lf(0)+L^2u_0 & \in & [H^1_{\Gamma_D}(\Omega)]^2 \end{array}\right. \tag{3.4.2}$$

Then $u \in C(0,T;[H_\phi^{1,1}(\Omega)]^2)$ *and* $u_{tt} \in L^2(0,T;[H_\phi^{1,1}(\Omega)]^2)$.
Consequently $\sigma \in L^\infty(0,T;[H_\phi^{0,1}(\Omega)]^{2\times 2})$, $p \in L^\infty(0,T;H_\phi^{0,1}(\Omega))$ *and* $\omega \in L^\infty(0,T;H_\phi^{0,1}(\Omega))$.
Moreover $\sigma_{tt} \in L^2(0,T;[H_\phi^{0,1}(\Omega)]^{2\times 2})$, $p_{tt} \in L^2(0,T;H_\phi^{0,1}(\Omega))$ *and* $\omega_{tt} \in L^2(0,T;H_\phi^{0,1}(\Omega))$, $u \in L^\infty(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ *and* $u_{tt} \in L^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$.

Proof: According to Theorem 30.1 p.442-443 of [39] we have $u \in H^3(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$, $u^{(4)} \in L^2(0,T;[L^2(\Omega)]^2)$ and $u^{(5)} \in L^2(0,T;[H^{-1}(\Omega)]^2)$. In particular $u_{tt} \in L^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $(u_{tt})_{tt} \in L^2(0,T;[L^2(\Omega)]^2)$. Knowing that $Lu = f - u_{tt}$ we have $Lu_{tt} = f_{tt} - (u_{tt})_{tt} \in L^2(0,T;[L^2(\Omega)]^2)$. Thus $u_{tt} \in L^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $Lu_{tt} \in L^2(0,T;[L^2(\Omega)]^2)$. That is $u_{tt} \in L^2(0,T;D(L))$ where $D(L)$ denote the domain of the Lamé's operator. But $D(L) \hookrightarrow [H_\phi^{1,1}(\Omega)]^2$ by adapting corollary 2.4 p.326 of [24]. Thus $u_{tt} \in L^2(0,T;[H_\phi^{1,1}(\Omega)]^2)$ and consequently $\sigma_{tt} \in L^2(0,T;[H_\phi^{0,1}(\Omega)]^{2\times 2})$, $p_{tt} \in L^2(0,T;H_\phi^{0,1}(\Omega))$, $\omega_{tt} \in L^2(0,T;H_\phi^{0,1}(\Omega))$. On the other hand $u_t \in H^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $Lu_t = f_t - u_{ttt} \in L^2(0,T;[L^2(\Omega)]^2)$. So that $u_t \in L^2(0,T;D(L))$. and hence

$$u_t \in L^2(0,T;[H_\phi^{1,1}(\Omega)]^2). \tag{3.4.3}$$

Similarly we have $u \in H^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $Lu = f - u_{tt} \in L^2(0,T;[L^2(\Omega)]^2)$, so that $u \in L^2(0,T;D(L))$, and hence also $u \in L^2(0,T;[H_\phi^{1,1}(\Omega)]^2)$. From this and (3.4.3) we get

$$u \in C(0,T;[H_\phi^{1,1}(\Omega)]^2) \subset L^\infty(0,T;[H_\phi^{1,1}(\Omega)]^2).$$

Thus $\sigma \in L^\infty(0,T;[H_\phi^{0,1}(\Omega)]^{2\times 2})$, $p \in L^\infty(0,T;H_\phi^{0,1}(\Omega))$ and $\omega \in L^\infty(0,T;H_\phi^{0,1}(\Omega))$. Moreover $u, u_t \in L^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ implies $u \in C(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$. Thus

$$u \in L^\infty(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$$

.

■

Proposition 3.4.2. *Let us suppose that the appropriate characteristic equation among 1.3.1 - 1.3.3 for each vertex of* Ω *has no root on the vertical line* $Re\alpha = 1$ *in the complex plane. Let* $\phi \in C^0(\overline{\Omega})$ *like in Proposition 3.4.1 Let us suppose that:*

$$\left\{\begin{array}{ccc} f & \in & H^6(0,T;[L^2(\Omega)]^2) \\ f^{(4)} & \in & L^2(0,T;[H^1(\Omega)]^2) \\ u_0,\ u_1,\ f(0)-Lu_0 & \in & [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(1)}(0)-Lu_1 & \in & [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(2)}(0)-Lf(0)+L^2u_0 & \in & [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(3)}(0)-Lf^{(1)}(0)+L^2u_1 & \in & [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(4)}(0)-Lf^{(2)}(0)+L^2f(0)-L^3u_0 & \in & [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(5)}(0)-Lf^{(3)}(0)+L^2f^{(1)}(0)-L^3u_1 & \in & [L^2(\Omega)]^2 \end{array}\right. \tag{3.4.4}$$

Then $\sigma_{tttt} \in L^2(0,T;[H^{0,1}_\phi(\Omega)]^{2\times 2})$, $p_{tttt} \in L^2(0,T;H^{0,1}_\phi(\Omega))$ *and* $\omega_{tttt} \in L^2(0,T;H^{0,1}_\phi(\Omega))$, $u_{tttt} \in L^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ *and* $u_{tttt} \in L^\infty(0,T;[L^2(\Omega)]^2)$.

Proof: By once Theorem 30.1 p.442-443 of [39], it follows that $u \in H^6(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$. By the equation $Lu^{(4)} = f^{(4)} - u^{(6)}$ and the hypothesis $f^{(4)} \in L^2(0,T;[H^1(\Omega)]^2)$, it follows by corollary 2.4 p.326 of [24] that $u^{(4)} \in L^2(0,T;H^{1,1}_\phi(\Omega))$. This implies the above assertions. ■

3.5 The semi-discrete mixed formulation

We assume that Ω is discretized by a regular family of triangulations $(\mathcal{T}_h)_{h>0}$ in the sense of [11]. We recall the finite dimensional subpaces $\Sigma_{0,h}$ and $V_h \times W_h$ of Σ_0 and $V \times W$ respectively defined by

$$\Sigma_{0,h} := \{(\tau_h, q_h) \in \Sigma_0;\ \forall T \in \mathcal{T}_h:\ q_{h|T} \in \mathbb{P}_1(T) \text{ and } (\tau_h - q_h\delta)_{|T} \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\ Curl\ b_T]^2\}, \tag{3.5.1}$$

$$V_h \times W_h := \{(v_h, \theta_h) \in M;\ \forall T \in \mathcal{T}_h:\ v_{h|T} \in [\mathbb{P}_0(T)]^2 \text{ and } \theta_{h|T} \in [\mathbb{P}_1(T)\}. \tag{3.5.2}$$

Recall that by $(\tau_h - q_h\delta)_{|T} \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}Curl\ b_T]^2$, we mean that there exist polynomials of degree $\leq 1:\ p_{11},\ p_{12},\ p_{21},\ p_{22}$ and two real numbers $\alpha_1,\ \alpha_2$ such that

$$(\tau_h - q_h\delta)_{|T} = \begin{bmatrix} p_{11|T} + \alpha_1 \frac{\partial b_T}{\partial x_2} & p_{12|T} - \alpha_1 \frac{\partial b_T}{\partial x_1} \\ p_{21|T} + \alpha_2 \frac{\partial b_T}{\partial x_2} & p_{22|T} - \alpha_2 \frac{\partial b_T}{\partial x_1} \end{bmatrix},$$

where b_T denotes the bubble function for the actual element T defined by

$$b_T = 27\lambda_1\lambda_2\lambda_3.$$

$\lambda_1, \lambda_2, \lambda_3$ denote the barycentric coordinates on T. Now we introduce the following semi-discretized problem: Find $(\sigma_h(.), p_h(.)) \in L^2([0,T]; \Sigma_{0,h})$, $u_h(.) \in H^2([0,T]; V_h)$ and $\omega_h(.) \in L^2([0,T]; W_h)$ such that for all $(\tau_h, q_h) \in \Sigma_{0,h}$, for all $(v_h, \theta_h) \in V_h \times W_h$ and for a.e. $t \in [0,T]$ we have:

$$\begin{cases} \dfrac{1}{2\mu}(\sigma_h(t), \tau_h) + \dfrac{1}{\lambda}(p_h(t), q_h) + (div\ (\tau_h - q_h\delta), u_h(t)) + (as\ (\tau_h), \omega_h(t)) = 0, \\ (u_{h,tt}(t), v_h) - (div\ (\sigma_h(t) - p_h(t)\delta), v_h) - (as\ (\sigma_h(t)), \theta_h) - (f(t), v_h) = 0, \\ u_h(0) = u_{0,h}, u_{h,t}(0) = u_{1,h}. \end{cases} \tag{3.5.3}$$

We may think to $u_{0,h}$ and $u_{1,h}$ as approximations in V_h of u_0 and u_1 respectively . The precise definitions of $u_{0,h}$ and $u_{1,h}$ will be given later. With the notations (3.3.4) and (3.3.5), the mixed formulation (3.5.3) may be rewritten: Find $\underset{\sim h}{\sigma}(.) = (\sigma_h(.), p_h(.)) \in L^2([0,T]; \Sigma_{0,h})$ and $\underset{\sim h}{u}(.) = (u_h(.), \omega_h(.)) \in H^2([0,T]; V_h) \times L^2([0,T]; W_h)$ such that

$$\begin{cases} a(\underset{\sim h}{\sigma}(t), \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{u}(t)) = 0, \quad \forall \underset{\sim}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h}, \text{ a.e. } t \in [0,T] \\ b(\underset{\sim h}{\sigma}(t), \underset{\sim h}{v}) + (\mathcal{F}(t), \underset{\sim h}{v}) = (u_{h,tt}(t), v_h), \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h, \text{ a.e. } t \in [0,T], \\ u_h(0) = u_{0,h}, \quad u_{h,t}(0) = u_{1,h}. \end{cases} \tag{3.5.4}$$

The existence and uniqueness of a solution $(\sigma_h(.), p_h(.))$ and $(u_h(.), \omega_h(.))$ to (3.5.3) or equivalently to (3.5.4) is shown in the following lemma:

Lemma 3.5.1. *a solution $(\sigma_h(.), p_h(.))$ and $(u_h(.), \omega_h(.))$ of (3.5.3) or equivalently to (3.5.4) exist and is unique.*

Proof: The evolution problem (3.5.4) can be rewritten as

$$\begin{cases} a(\underset{\sim h}{\sigma}(t), \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{u}(t)) = 0, \quad \forall \underset{\sim}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h}, \text{ a.e. } t \in [0,T] \\ b(\underset{\sim h}{\sigma}(t), \underset{\sim h}{v}) = (f(t) - u_{h,tt}(t), v_h), \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h, \text{ a.e. } t \in [0,T]. \end{cases} \tag{3.5.5}$$

We may think the solution $(\underset{\sim h}{\sigma}, \underset{\sim h}{u}) \in \Sigma_{0,h} \times V_h \times W_h$ of (3.5.5), for a fixed time, as a solution of the stationary problem:

$$\begin{cases} a(\underset{\sim h}{\sigma}, \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{u}) = 0, \quad \forall \underset{\sim}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h}, \\ b(\underset{\sim h}{\sigma}, \underset{\sim h}{v}) = (g, v_h), \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h, \end{cases} \tag{3.5.6}$$

where $g = f(t) - u_{h,tt}(t) \in [L^2(\Omega)]^2$. We consider the pair of operators (T_h, S_h) defined by

$$\begin{aligned} (T_h, S_h) : [L^2(\Omega)]^2 &\longrightarrow \Sigma_{0,h} \times (V_h \times W_h) \\ g &\longmapsto (\underset{\sim h}{\sigma}, \underset{\sim h}{u}). \end{aligned}$$

The evolution problem (3.5.5) can be rewritten as

$$\begin{cases} \underset{\sim h}{u}(t) = T_h\big(P_h^0 f(t) - \frac{d^2u_h}{dt^2}(t)\big), \\ \underset{\sim h}{\sigma}(t) = S_h\big(P_h^0 f(t) - \frac{d^2u_h}{dt^2}(t)\big). \end{cases}$$

In particular:

$$u_h(t) = T_{h,1}\big(P_h^0 f(t) - \frac{d^2u_h}{dt^2}(t)\big). \tag{3.5.7}$$

Let us show that the operator $T_{h,1}|V_h : V_h \longmapsto V_h$ is invertible. Suppose that $\int_\Omega g.T_{h,1}g\;dx = 0$. Then from (3.6.44) we get $a(\underset{\sim h}{\sigma}, \underset{\sim h}{\sigma}) = 0$ i.e.

$$\frac{1}{2\mu}\int_\Omega |\sigma_h|^2\;dx + \frac{1}{\lambda}\int_\Omega |p_h|^2\;dx = 0.$$

Hence $\sigma_h = 0$ and $p_h = 0$ i.e.

$$\underset{\sim h}{\sigma} = 0. \tag{3.5.8}$$

By the first equation of (3.6.44) it now follows that:

$$b(\underset{\sim h}{\tau}, \underset{\sim h}{u}) = 0 \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h}.$$

The inf-sup inequality (1.4.7) yields

$$\underset{\sim h}{u} = (u_h, \omega_h) = 0.$$

Thus in particular, if $\int_\Omega g.T_{h,1}g\;dx = 0$, then $T_{h,1}g = 0$.

Now, if $g_h \in V_h$ and $\int_\Omega g_h.T_{h,1}g_h\;dx = 0$, then by (3.5.8) we have $\underset{\sim h}{\sigma} = 0$ and by the second equation of (3.6.44) we have $(g_h, v_h) = 0, \quad \forall v_h \in V_h$. Thus

$$g_h = 0.$$

Finally, we have proved that $T_{h,1}|V_h : V_h \longrightarrow V_h$ is injective, thus invertible. From (3.6.43) follows:

$$P_h^0 f(t) - \frac{d^2u_h}{dt^2}(t) = (T_{h,1}|V_h)^{-1}(u_h(t)). \tag{3.5.9}$$

Hence

$$\frac{d^2u_h}{dt^2}(t) = P_h^0 f(t) - (T_{h,1}|V_h)^{-1}(u_h(t)).$$

If we consider a basis of the subspace V_h, we obtain an homogeneous linear system of differential equations, and if furthermore we fix the initial conditions $u_h(0)$ and $\frac{du_h}{dt}(0)$ in V_h, problem (3.5.9) has a unique solution. ■

Before discussing some error estimates between the exact solution and its elliptic projection, we suppose that $(\mathcal{T}_h)_{h>0}$ satisfies the following refinement rules:

R_1: if K is a triangle of T_h admitting S_j as a vertex, then

$$h_K \lesssim h^{1/(1-\alpha)},$$

where α has been defined just before Proposition 3.4.1,

R_2: if K is a triangle of T_h admitting no S_j $(j = 1, \cdots, n_e)$ as a vertex, then

$$h_K \lesssim h \inf_{x \in K} \phi(x),$$

(ϕ has been defined in Proposition 3.4.1);

R_3: for all $K \in \mathcal{T}_h$

$$h_K \gtrsim h^\beta,$$

where $\beta \geq 1/(1-\alpha)$.

Remark 3.5.2. *Regular families of meshes satisfying the refinement conditions $R_1 - R_3$ are easily built, see for instance [34].*

We now recall, from the chapter 1, the two following Corollary:

Corollary 3.5.3. *Under the hypotheses R_1, R_2, the following error estimate hold for every $q \in H_\phi^{0,1}(\Omega)$,*

$$\|q - P_h^1 q\|_{0,\Omega} \lesssim h|q|_{0,1;\phi,\Omega}, \tag{3.5.10}$$

where P_h^1 denotes the L^2-orthogonal projection on $\{\theta_h \in L^2(\Omega); \theta_h|_K \in I\!P_1(K), \forall K \in \mathcal{T}_h\}$.

Corollary 3.5.4. *Under the hypotheses R1 and R2, the following error estimate hold for every $\underset{\sim}{\tau} = (\tau, q) \in [H_\phi^{0,1}(\Omega)]^{2\times 2} \times H_\phi^{0,1}(\Omega)$*

$$\|\underset{\sim}{\tau} - \Pi_h \underset{\sim}{\tau}\|_{0,\Omega} \lesssim h(|\tau|_{0,1;\phi,\Omega} + |q|_{0,1;\phi,\Omega}). \tag{3.5.11}$$

By the help of the refinement rule R_3, we show the following global inverse property:

Lemma 3.5.5. *Let X_h be a finite dimensional subspace of vectofields of $H(div\,, \Omega)$ such that $\forall K \in \mathcal{T}_h \quad \forall v_h \in X_h$: $v_h|_K$ is the image by the Piola transformation of a fixed finite dimensional subspace of $H(div, \hat{K})$. Then under the hypotheses R3, there exists a constant $C_0 > 0$ such that, for every $v_h \in X_h$*

$$\|div\, v_h\|_{0,\Omega} \leq C_0 h^{-\beta} \|v_h\|_{0,\Omega}. \tag{3.5.12}$$

Proof: Let K be an arbitrary triangle in the plane and $\hat{K}$ be the reference triangle. The affine mapping

$$F_K : \hat{K} \longrightarrow K : \hat{x} \longmapsto A + B_K \hat{x},$$

where A is a vertex of the triangle K and B_K is the matrix of the of the affine transformation F_K with respect to the triangle K, is a bijection from $\hat{K}$ onto K. Let us recall the definition of the Piola transformation 4.7 p.333 of [24]. The image by the Piola transformation of any vector field v on K is the vector $\hat{v}$ on $\hat{K}$ defined by

$$\hat{v}(\hat{x}) = detB_K B_K^{-1} v(F_K(\hat{x})), \quad \forall \hat{x} \in \hat{K}.$$

Reciprocally, the image by the Piola transformation of any vector field $\hat{v}$ on $\hat{K}$ is the vector v on K defined by

$$v(x) = \frac{1}{detB_K} B_K \hat{v}(F_K^{-1}(x)), \quad \forall x \in K.$$

It is easy to see that

$$div\ \hat{v}(\hat{x}) = (div\ v)(F_K(\hat{x}))detB_K.$$

Hence

$$\begin{aligned}
\int_K |div\ v|^2(x)\ dx &= \int_{\hat{K}} |div\ v|^2(F_K(\hat{x}))|detB_K|\ d\hat{x} \\
&= \frac{1}{|detB_K|} \int_{\hat{K}} |div\ \hat{v}(\hat{x})|^2\ d\hat{x} \\
&\leq \frac{c}{|detB_K|} \int_{\hat{K}} |\hat{v}(\hat{x})|^2\ d\hat{x} \\
&= \frac{c}{|detB_K|^2} \int_K |\hat{v} o F_K^{-1}(x)|^2\ dx \\
&= c \int_K |B_K^{-1} v(x)|^2\ dx \\
&\leq c|B_K^{-1}|^2 \int_K |v(x)|^2\ dx \\
&\leq c h_K^{-2} \int_K |v(x)|^2\ dx
\end{aligned}$$

Hypothesis R_3 yields

$$\int_K |div\ v|^2(x)\ dx \leq c h^{-2\beta} \int_K |v(x)|^2\ dx, \quad \forall K \in \mathcal{T}_h.$$

Hence

$$\int_\Omega |div\ v|^2(x)\ dx \leq c h^{-2\beta} \int_\Omega |v(x)|^2\ dx,$$

i.e.

$$\|div\ v\|_{0,\Omega} \leq C_0 h^{-\beta} \|v\|_{0,\Omega}, \forall v \in X_h$$

■

3.5.1 Continuous in time a priori error estimates

The elliptic projection error estimates

Our next purpose is to derive error estimates for both $(\sigma_h(t), p_h(t))$ and $(u_h(t), \omega_h(t))$. Firstly, we consider the "elliptic projection" of the exact solution. Let us introduce the following discrete elliptic projection problem:
Find $\underset{\sim h}{\widehat{\sigma}}(.) = (\widehat{\sigma}_h(.), \widehat{p}_h(.)) \in L^2([0,T]; \Sigma_{0,h})$, $\underset{\sim h}{\widehat{u}}(.) = (\widehat{u}_h(.), \widehat{\omega}_h(.)) \in L^2([0,T]; V_h \times W_h)$ such that for all $(\tau_h, q_h) \in \Sigma_{0,h}$, for all $(v_h, \theta_h) \in V_h \times W_h$ and for a.e. $t \in [0,T]$ we have:

$$\begin{cases} \dfrac{1}{2\mu}(\widehat{\sigma}_h(t), \tau_h) + \dfrac{1}{\lambda}(\widehat{p}_h(t), q_h) + (div\ (\tau_h - q_h\delta), \widehat{u}_h(t)) + (as\ (\tau_h), \widehat{\omega}_h(t)) = 0, \\ \\ (u_{tt}(t), v_h) - (div\ (\widehat{\sigma}_h(t) - \widehat{p}_h(t)\delta), v_h) - (as\ (\widehat{\sigma}_h(t)), \theta_h) - (f(t), v_h) = 0 \end{cases} \tag{3.5.13}$$

With the notations (3.3.4) and (3.3.5), the discrete elliptic projection formulation (3.5.13) may be rewritten:
Find $\underset{\sim h}{\widehat{\sigma}}(.) = (\widehat{\sigma}_h(.), \widehat{p}_h(.)) \in L^2([0,T]; \Sigma_{0,h})$, $\underset{\sim h}{\widehat{u}}(.) = (\widehat{u}_h(.), \widehat{\omega}_h(.)) \in L^2([0,T]; V_h \times W_h)$ such that for all $\underset{\sim h}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h}$, for all $\underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h$ and for a.e. $t \in [0,T]$ we have

$$\begin{cases} a(\underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{\widehat{u}}(t)) = 0, \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h}, \\ \\ b(\underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{v}) + (\mathcal{F}(t), \underset{\sim h}{v}) = (u_{tt}(t), v_h), \quad \forall \underset{\sim h}{v} \in [L^2(\Omega)]^2 \times L^2(\Omega). \end{cases} \tag{3.5.14}$$

We are now in a position to establish optimal error estimates. In the following, we estimate the error between $(\sigma(.), p(.))$, $(u(.), \omega(.))$ the exact solution of the mixed problem (3.3.3) or equivalently (3.3.6) and $(\widehat{\sigma}_h(.), \widehat{p}_h(.))$, $(\widehat{u}_h(.), \widehat{\omega}_h(.))$ the solution of the discrete elliptic projection problem (3.5.13) or equivalently (3.5.14).

Proposition 3.5.6. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations on Ω. We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies to conditions R1 and R2 . Under the hypotheses of the Proposition 3.4.1 the following error estimate holds for a.e. $t \in [0,T]$:*

$$\|\sigma(t) - \widehat{\sigma}_h(t)\|_{0,\Omega} + \|p(t) - \widehat{p}_h(t)\|_{0,\Omega} \lesssim h|u(t)|_{1,1;\phi,\Omega}. \tag{3.5.15}$$

Proof: If we subtract (3.5.14) from (3.3.6), we get the system in the errors

$$\begin{cases} a\big(\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{\tau}\big) + b\big(\underset{\sim h}{\tau}, \underset{\sim}{u}(t) - \underset{\sim h}{\widehat{u}}(t)\big) = 0, \quad \forall\ \underset{\sim h}{\tau} \in \Sigma_{0,h}, \text{ for a.e. } t \in [0,T] \\ \\ b\big(\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{v}\big) = 0, \quad \forall\ \underset{\sim h}{v} \in V_h \times W_h, \text{ for a.e. } t \in [0,T] \end{cases} \tag{3.5.16}$$

Let $(P_h^0 u(t), P_h^1 \omega(t))$ denote the L^2-orthogonal projection of $(u(t), \omega(t))$ on the space $V_h \times W_h$ and let us set $\Pi_h \underset{\sim}{\sigma}(t) = \Pi_h(\sigma(t), p(t)) = (\sigma_h^*(t), p_h^*(t))$. The equation (1.5.1) and the relation (3.5.16) yields

$$\begin{cases} a\Big(\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \Pi_h\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t)\Big) = \big(as\;(\widehat{\sigma}_h(t) - \sigma_h^*(t)), \omega(t) - P_h^1\omega(t)\big), \text{ for a.e. } t \in [0,T] \\ \\ b\Big(\Pi_h\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{v}\Big) = 0, \quad \forall \underset{\sim h}{v} \in V_h \times W_h \text{ for a.e. } t \in [0,T] \end{cases} \tag{3.5.17}$$

The first equation of (3.5.17) implies

$$\begin{aligned} a\Big(\Pi_h\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \Pi_h\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t)\Big) &= \frac{1}{2\mu}\big(\sigma_h^*(t) - \sigma(t), \sigma_h^*(t) - \widehat{\sigma}_h(t)\big) \\ &\quad + \frac{1}{\lambda}\big(p_h^*(t) - p(t), p_h^*(t) - \widehat{p}_h(t)\big) \\ &\quad + \big(as\;(\widehat{\sigma}_h(t) - \sigma_h^*(t)), \omega(t) - P_h^1\omega(t)\big). \end{aligned} \tag{3.5.18}$$

Thus due to proposition 1.4.2, we have

$$\|\Pi_h\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t)\|_{0,\Omega} \lesssim \Big[\|\Pi_h\underset{\sim}{\sigma}(t) - \underset{\sim}{\sigma}(t)\|_{0,\Omega} + \|\omega(t) - P_h^1\omega(t)\|_{0,\Omega}\Big]. \tag{3.5.19}$$

Using (3.5.10) and corollary 3.5.4, we get

$$\begin{aligned} \|\Pi_h\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t)\|_{0,\Omega} &\lesssim h\Big[|\sigma(t)|_{0,1;\phi,\Omega} + |p(t)|_{0,1;\phi,\Omega} + |\omega(t)|_{0,1;\phi,\Omega}\Big] \\ &\lesssim h|u(t)|_{1,1;\phi,\Omega}. \end{aligned} \tag{3.5.20}$$

Finally, (3.5.15) follows from (3.5.20), Corollary 3.5.4 and the triangle inequality.

■

Proposition 3.5.7. *Under the hypothesis of the Proposition 3.5.6 , the following error estimate holds:*

$$\|\omega_{tt}(t) - \widehat{\omega}_{h,tt}(t)\|_{0,\Omega} + \|P_h^0 u_{tt}(t) - \widehat{u}_{h,tt}(t)\|_{0,\Omega} \lesssim h|u_{tt}(t)|_{1,1;\phi,\Omega}, \tag{3.5.21}$$

$$\|u_{tt}(t) - \widehat{u}_{h,tt}(t)\|_{0,\Omega} \lesssim h\Big[|u_{tt}(t)|_{1,1;\phi,\Omega} + |u_{tt}(t)|_{1,\Omega}\Big]. \tag{3.5.22}$$

Proof: First we consider the second derivative of the system with the errors (3.5.16)

$$\begin{cases} a\Big(\underset{\sim tt}{\sigma}(t) - \underset{\sim h,tt}{\widehat{\sigma}}(t), \underset{\sim h}{\tau}\Big) + b\Big(\underset{\sim h}{\tau}, \underset{\sim tt}{u}(t) - \underset{\sim h,tt}{\widehat{u}}(t)\Big) = 0, \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h}, \\ \\ b\Big(\underset{\sim tt}{\sigma}(t) - \underset{\sim h,tt}{\widehat{\sigma}}(t), \underset{\sim h}{v}\Big) = 0, \quad \forall \underset{\sim h}{v} \in V_h \times W_h. \end{cases} \tag{3.5.23}$$

Firstly, let us observe that with the same techniques as in the Proposition (3.5.6) we get the following results

$$\|\underset{\sim tt}{\sigma}(t) - \underset{\sim h,tt}{\widehat{\sigma}}(t)\|_{0,\Omega} \lesssim h\Big[|u_{tt}(t)|_{1,1;\phi,\Omega} + |p_{tt}(t)|_{0,1;\phi,\Omega}\Big]. \tag{3.5.24}$$

To prove (3.5.21), we shall use the uniform inf-sup condition (1.4.7). Firstly, it follows from the second equation of (3.5.16) and (3.3.5) that

$$\begin{aligned} b\Big(\underset{\sim h}{\tau}, \big(P_h^0 u_{tt}(t), P_h^1\omega_{tt}(t)\big) - \underset{\sim h,tt}{\widehat{u}}(t)\Big) = \quad & -a\Big(\underset{\sim tt}{\sigma}(t) - \underset{\sim h,tt}{\widehat{\sigma}}(t), \underset{\sim h}{\tau}\Big) \\ & + \big(as\ (\tau_h), P_h^1\omega_{tt}(t) - \omega_{tt}(t)\big), \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h} \end{aligned}$$

Thus by the uniform inf-sup condition (1.4.7), we have

$$\begin{aligned} \|P_h^0 u_{tt}(t) - \widehat{u}_{h,tt}(t)\|_{0,\Omega} + \|P_h^1\omega_{tt}(t) - \widehat{\omega}_{h,tt}(t)\|_{0,\Omega} \lesssim \Big[& \|\underset{\sim tt}{\sigma}(t) - \underset{\sim h,tt}{\widehat{\sigma}}(t)\|_{0,\Omega} \\ & + \|P_h^1\omega_{tt}(t) - \omega_{tt}(t)\|_{0,\Omega}\Big]. \end{aligned} \tag{3.5.25}$$

Finally, (3.5.21) and (3.5.22) follow from (3.5.24), (3.5.25), (3.5.10), (1.47) p.27 of [36] and the triangle inequality. ∎

Remark 3.5.8. *Under the hypothesis of Proposition 3.4.2, if we consider fourth order derivatives with respect to t instead of second ones of the system of errors (3.5.16) and using similar techniques as above, we obtain the following estimate:*

$$\|u_{ttt}(t) - \widehat{u}_{h,ttt}(t)\|_{0,\Omega} \lesssim h\Big[|u_{ttt}(t)|_{1,1;\phi,\Omega} + |u_{ttt}(t)|_{1,\Omega}\Big]. \tag{3.5.26}$$

Error estimates for the evolution problem

Before given optimal error estimates for our mixed method, we shall fix the two discrete initial conditions $u_{0,h}$ and $u_{1,h}$ in problem (3.5.3) or equivalently (3.5.4) as the elliptic projection of the two continuous initial conditions u_0 and u_1 respectively. We can now derive the following error estimates:

Theorem 3.5.9. *Under the hypothesis of Proposition 3.5.6, the following error estimate holds*

$$||\sigma - \sigma_h||_{L^\infty(L^2)} + ||p - p_h||_{L^\infty(L^2)} \lesssim h\Big[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)} + |u|_{L^\infty(H^{1,1}_\phi)}\Big], \tag{3.5.27}$$

$$||\omega - \omega_h||_{L^\infty(L^2)} + ||P_h^0 u - u_h||_{L^\infty(L^2)} \lesssim h\big[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)} + |u|_{L^\infty(H^{1,1}_\phi)}\big], \tag{3.5.28}$$

$$||u - u_h||_{L^\infty(L^2)} \lesssim h\big[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)} + |u|_{L^\infty(H^{1,1}_\phi)} + |u|_{L^\infty(L^2)}\big]. \tag{3.5.29}$$

Proof: Let $(\widehat{\sigma}_h(t), \widehat{p}_h(t))$, $(\widehat{\omega}_h(t), \widehat{u}_h(t))$ be the elliptic projection of $(\sigma(t), p(t))$, $(\omega(t), u(t))$ and set

$$\varepsilon_h(t) = \sigma_h(t) - \widehat{\sigma}_h(t),\ \ \chi_h(t) = u_h(t) - \widehat{u}_h(t),\ \ \psi_h(t) = \omega_h(t) - \widehat{\omega}_h(t) \text{ and } r_h(t) = p_h(t) - \widehat{p}_h(t).$$

We may then write the error system in the form

$$\begin{cases} \dfrac{1}{2\mu}(\varepsilon_h(t), \tau_h) + \dfrac{1}{\lambda}(r_h(t), q_h) + (div\ (\tau_h - q_h\delta), \chi_h(t)) + (as\ (\tau_h), \psi_h(t)) = 0, \\ \\ (div\ (\varepsilon_h(t) - r_h(t)\delta), v_h) + (as\ (\varepsilon_h(t)), \theta_h) = (u_{h,tt}(t) - u_{tt}(t), v_h). \end{cases} \quad (3.5.30)$$

Note that $\chi_h(0) = 0$, $\chi_{h,t}(0) = 0$, $\varepsilon_h(0) = 0$ and $r_h(0) = 0$ from the initial conditions and the first equation of (3.5.30) at time $t = 0$ with $(\tau_h, q_h) = (\varepsilon_h(0), r_h(0))$.

We then differentiate the first equation of (3.5.30) with respect to time to obtain

$$\frac{1}{2\mu}(\varepsilon_{h,t}(t), \tau_h) + \frac{1}{\lambda}(r_{h,t}(t), q_h) + (div\ (\tau_h - q_h\delta), \chi_{h,t}(t)) + (as\ (\tau_h), \psi_{h,t}(t)) = 0. \quad (3.5.31)$$

Now taking $(\tau_h, q_h) = (\varepsilon_h(t), r_h(t))$ in this last equality, we obtain

$$\frac{1}{2\mu}(\varepsilon_{h,t}(t), \varepsilon_h(t)) + \frac{1}{\lambda}(r_{h,t}(t), r_h(t)) + (div\ (\varepsilon_h(t) - r_h(t)\delta), \chi_{h,t}(t)) + (as\ (\varepsilon_h(t)), \psi_{h,t}(t)) = 0. \quad (3.5.32)$$

The second equation of (3.5.30) with $(v_h, \theta_h) = (\chi_{h,t}(t), \psi_{h,t}(t))$ gives

$$(div\ (\varepsilon_h(t) - r_h(t)\delta), \chi_{h,t}(t)) + (as\ (\varepsilon_h(t)), \psi_{h,t}(t)) = (u_{h,tt}(t) - u_{tt}(t), \chi_{h,t}(t)). \quad (3.5.33)$$

Substracting (3.5.33) from (3.5.32) gives

$$\begin{aligned} \frac{1}{2\mu}\frac{d}{dt}\|\varepsilon_h(t)\|^2_{0,\Omega} + \frac{1}{\lambda}\frac{d}{dt}\|r_h(t)\|^2_{0,\Omega} &= 2(u_{tt}(t) - u_{h,tt}(t), \chi_{h,t}(t)) \\ &= 2(u_{tt}(t) - \widehat{u}_{h,tt}(t), \chi_{h,t}(t)) - \frac{d}{dt}\|\chi_{h,t}(t)\|^2_{0,\Omega}. \end{aligned} \quad (3.5.34)$$

Using the Cauchy-Schwarz inequality to bound the right-hand side of (3.5.34) we obtain

$$\begin{aligned} \frac{1}{2\mu}\frac{d}{dt}\|\varepsilon_h(t)\|^2_{0,\Omega} + \frac{1}{\lambda}\frac{d}{dt}\|r_h(t)\|^2_{0,\Omega} + \frac{d}{dt}\|\chi_{h,t}(t)\|^2_{0,\Omega} &= 2(u_{tt}(t) - \widehat{u}_{h,tt}(t), \chi_{h,t}(t)) \\ &\le \|u_{tt}(t) - \widehat{u}_{h,tt}(t)\|^2_{0,\Omega} + \|\chi_{h,t}(t)\|^2_{0,\Omega}. \end{aligned} \quad (3.5.35)$$

Now applying the Gronwall's inequality (3.2.4) to (3.5.35) we get

$$\frac{1}{2\mu}\|\varepsilon_h(t)\|^2_{0,\Omega} + \frac{1}{\lambda}\|r_h(t)\|^2_{0,\Omega} + \|\chi_{h,t}(t)\|^2_{0,\Omega} \le e^T \int_0^T \|u_{tt}(s) - \widehat{u}_{h,tt}(s)\|^2_{0,\Omega}\ ds.$$

Thanks to (3.5.22) one can write

$$\frac{1}{2\mu}\|\varepsilon_h(t)\|^2_{0,\Omega} + \frac{1}{\lambda}\|r_h(t)\|^2_{0,\Omega} + \|\chi_{h,t}(t)\|^2_{0,\Omega} \lesssim h^2\left[\int_0^T |u_{tt}(s)|^2_{1,1;\phi,\Omega}\ ds + \int_0^T |u_{tt}(s)|^2_{1,\Omega}\ ds\right]. \quad (3.5.36)$$

Taking the square root of (3.5.36) and using assumption (3.2.3) on λ and μ gives us

$$\|\varepsilon_h(t)\|_{0,\Omega} + \|r_h(t)\|_{0,\Omega} \lesssim h\Big[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)}\Big]. \tag{3.5.37}$$

Therefore, (3.5.37), (3.5.15) and the triangle inequality, we get

$$\|\sigma(t) - \sigma_h(t)\|_{0,\Omega} + \|p(t) - p_h(t)\|_{0,\Omega} \lesssim h\Big[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)} + |u(t)|_{1,1;\phi,\Omega}\Big]. \tag{3.5.38}$$

Taking the supremum over all $t \in [0, T]$ in this last inequality we get (3.5.27). Proceeding similarly as in the prof of the Proposition (3.5.7) we get

$$\begin{aligned}\|\omega(t) - \omega_h(t)\|_{0,\Omega} + \|P^0_h u(t) - u_h(t)\|_{0,\Omega} \lesssim \Big[&\|\sigma(t) - \sigma_h(t)\|_{0,\Omega} \\ &+ \|p(t) - p_h(t)\|_{0,\Omega} \\ &+ \|P^1_h\omega(t) - \omega(t)\|_{0,\Omega}\Big].\end{aligned} \tag{3.5.39}$$

This last inequality combined with (3.5.38), (3.5.10) and the triangle inequality we get

$$\|\omega(t) - \omega_h(t)\|_{0,\Omega} + \|P^0_h u(t) - u_h(t)\|_{0,\Omega} \lesssim h\Big[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)} + |u(t)|_{1,1;\phi,\Omega}\Big]. \tag{3.5.40}$$

Taking the supremum over all $t \in [0, T]$ in this last inequality we get (3.5.28) . Using furthermore the bound on the error of the P^0_h projection (1.47) p.27 of [36] and the triangle inequality, we obtain (3.5.29).

■

3.6 The fully discrete mixed finite element scheme

3.6.1 Notation

let $\Delta t := \frac{T}{N} > 0$ denote the time step size and define $t_i = i\Delta t$ $(i = 0, 1, \cdots, N)$, $t_N = T$ and $t_0 = 0$. For any function ϕ of time, let ϕ^n denotes $\phi(t_n)$. We denote by

$$\phi^{n+\frac{1}{2}} := \frac{\phi^n + \phi^{n+1}}{2}, \quad \phi^{n,\frac{1}{4}} := \frac{\phi^{n+1} + 2\phi^n + \phi^{n-1}}{4},$$

and we define the following discrete temporal derivatives:

$$\Delta_t\phi^n := \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t}, \quad \Delta_t\phi^{n+\frac{1}{2}} := \frac{\phi^{n+1} - \phi^n}{\Delta t}, \quad \Delta_t^2\phi^n := \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{\Delta t^2}.$$

We can easily see that we have

$$\Delta_t^2\phi^n := \frac{\Delta_t\phi^{n+\frac{1}{2}} - \Delta_t\phi^{n-\frac{1}{2}}}{\Delta t} \quad \text{and} \quad \Delta_t\phi^n := \frac{\Delta_t\phi^{n+\frac{1}{2}} + \Delta_t\phi^{n-\frac{1}{2}}}{2}. \tag{3.6.1}$$

The fully discrete mixed formulation is as follows: Find $(\sigma_h^{n+1}, p_h^{n+1}) \in \Sigma_{0,h}$, and $(u_h^{n+1}, \omega_h^{n+1}) \in V_h \times W_h$ such that

$$u_h^0 = \hat{u}_h(0), \tag{3.6.2}$$

$$u_h^{-1} = \hat{u}_h(-\Delta t) \simeq \hat{u}_h(0) - \Delta t\ \hat{u}_{h,t}(0) + \frac{\Delta t^2}{2}\hat{u}_{h,tt}(0), \tag{3.6.3}$$

and for all $(\tau_h, q_h) \in \Sigma_{0,h}$, for all $(v_h, \theta_h) \in V_h \times W_h$ we have

$$\begin{cases} \frac{1}{2\mu}(\sigma_h^{n+1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1}, q_h) + (div\ (\tau_h - q_h\delta), u_h^{n+1}) + (as\ (\tau_h), \omega_h^{n+1}) = 0, \quad n \geq -1, \\ (as\ (\sigma_h^{n+1}), \theta_h) = 0, \quad n \geq -1, \\ (\Delta_t^2 u_h^n, v_h) - (div\ (\sigma_h^n - p_h^n\delta), v_h) - (f^n, v_h) = 0, \quad n \geq 0. \end{cases} \tag{3.6.4}$$

The existence and uniqueness of a solution to problem (3.6.4) is provided by the following lemma:

Lemma 3.6.1. *a solution $(\sigma_h^{n+1}, p_h^{n+1})$ and $(u_h^{n+1}, \omega_h^{n+1})$ of (3.6.4) exist and is unique.*

Proof: To every $\big((\sigma_h, p_h), \omega_h\big) \in \Sigma_{0,h} \times V_h$, we associate the element of its dual $\Sigma_{0,h}' \times V_h'$:

$$\begin{pmatrix} (\tau_h, q_h) & \longmapsto & \frac{1}{2\mu}(\sigma_h, \tau_h) + \frac{1}{\lambda}(p_h, q_h) + (as\ (\tau_h), \omega_h) \\ \theta_h & \longmapsto & (as\ (\sigma_h), \theta_h) \end{pmatrix}.$$

Let as call this mapping T_h; it is a linear mapping from $\Sigma_{0,h} \times V_h$ into its dual. We have to prove that T_h is bijective. But the arrival and departure spaces has the same dimension. Thus by a well known theorem of linear algebra it suffices to prove that T_h is injective. Thus let $\big((\sigma_h, p_h), \omega_h\big) \in \Sigma_{0,h} \times V_h$ such that:

$$\frac{1}{2\mu}(\sigma_h, \tau_h) + \frac{1}{\lambda}(p_h, q_h) + (as\ (\tau_h), \omega_h) = 0, \quad \forall(\tau_h, q_h) \in \Sigma_{0,h} \tag{3.6.5}$$

$$(as\ (\sigma_h), \theta_h) = 0. \quad \forall\theta_h \in W_h \tag{3.6.6}$$

From (3.6.44) it follows that $(as\ (\sigma_h), \omega_h) = 0$ and then by taking $(\tau_h, q_h) = (\sigma_h, p_h)$ in (3.6.43) we get

$$\frac{1}{2\mu}\|\sigma_h\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h\|_{0,\Omega}^2 = 0.$$

Which implies that

$$\sigma_h = 0, \quad p_h = 0.$$

Thus (3.6.43) reduces to:

$$(as\ (\tau_h), \omega_h) = 0.$$

By inf-sup inequality (1.4.7), with $(v_h, \theta_h) = (0, \omega_h)$, we get $\omega_h = 0$. Furthermore u_h is given explicitly by the last equation of (3.6.4). ■

Note that it follows by uniqueness from the first two equations of system (3.6.4) with $n = -1$ that

$$\sigma_h^0 = \hat{\sigma}_h(0),\ p_h^0 = \hat{p}_h(0) \text{ and } \omega_h^0 = \hat{\omega}_h(0). \tag{3.6.7}$$

3.6.2 Stability of the explicit scheme

Theorem 3.6.2. *Under the hypotheses (3.5), the explicit scheme defined by 3.6.2 - 3.6.4 is stable if the following CFL condition is satisfied* $\Delta t < \frac{1}{C_0\sqrt{2\mu+\lambda}} h^\beta$ *that is*

$$\begin{aligned} &\alpha_0 \|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 \\ &\quad \le \left[\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 + T\big(\max_{t\in[0,T]} \|f(t)\|_{0,\Omega}\big)^2\right] \times \exp\left(T\alpha_0^{-1}\right), \end{aligned} \tag{3.6.8}$$

$$\beta^*(\|u_h^{N+\frac{1}{2}}\|_{0,\Omega} + \|\omega_h^{N+\frac{1}{2}}\|_{0,\Omega}) \le \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega} + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}, \tag{3.6.9}$$

where

$$\alpha_0 = \frac{1}{2} - \frac{C_0^2}{2}(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}} \ (\beta \text{ is defined in } R_3)$$

and β^* *is the constant of the inf-sup condition defined in (1.4.7).*

Proof: Subtracting the first equation of (3.6.4) from itself in step $n-1$, we get for all $(\tau_h, q_h) \in \Sigma_{0,h}$

$$\begin{aligned} \frac{1}{2\mu}(\sigma_h^{n+1} - \sigma_h^{n-1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1} - p_h^{n-1}, q_h) + (div\ (\tau_h - q_h\delta), u_h^{n+1} - u_h^{n-1}) \\ + (as\ (\tau_h), \omega_h^{n+1} - \omega_h^{n-1}) = 0, \end{aligned} \tag{3.6.10}$$

Taking $(\tau_h, q_h) = \frac{1}{2\Delta t}(\sigma_h^n, p_h^n)$ in (3.6.10) and the fact that $(as\ (\sigma_h^n), \theta_h) = 0 \quad \forall \theta_h \in W_h$ we get

$$\frac{1}{2\mu}(\Delta_t \sigma_h^n, \sigma_h^n) + \frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) + (div\ (\sigma_h - p_h\delta), \Delta_t u_h^n) = 0, \tag{3.6.11}$$

The second equation of (3.6.4) with $v_h = \Delta_t u_h^n$, $\theta_h = 0$ become

$$(\Delta_t^2 u_h^n, \Delta_t u_h^n) - (div\ (\sigma_h^n - p_h^n\delta), \Delta_t u_h^n) = (f^n, \Delta_t u_h^n), \tag{3.6.12}$$

Adding (3.6.12) and (3.6.11), we get

$$(\Delta_t^2 u_h^n, \Delta_t u_h^n) + \frac{1}{2\mu}(\Delta_t \sigma_h^n, \sigma_h^n) + \frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) = (f^n, \Delta_t u_h^n), \tag{3.6.13}$$

from (3.6.1) we get

$$\frac{1}{2\Delta t}(\Delta_t u_h^{n+\frac{1}{2}} - \Delta_t u_h^{n-\frac{1}{2}}, \Delta_t u_h^{n+\frac{1}{2}} + \Delta_t u_h^{n-\frac{1}{2}}) + \frac{1}{2\mu}(\Delta_t \sigma_h^n, \sigma_h^n) + \frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) = (f^n, \Delta_t u_h^n). \tag{3.6.14}$$

Thus

$$\frac{1}{2\Delta t}(\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\Delta_t u_h^{n-\frac{1}{2}}\|_{0,\Omega}^2) + \frac{1}{2\mu}(\Delta_t \sigma_h^n, \sigma_h^n) + \frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) = (f^n, \Delta_t u_h^n) \tag{3.6.15}$$

Now we inspect the two last terms in the left hand side of this last equation, we have

$$\Delta_t \sigma_h^n = \frac{\sigma_h^{n+1} - \sigma_h^{n-1}}{2\Delta t} = \frac{\sigma_h^{n+1} + \sigma_h^n - \sigma_h^n + \sigma_h^{n-1}}{2\Delta t} = \frac{\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}}{\Delta t}$$

$$\sigma_h^n = \frac{\sigma_h^{n+1} + \sigma_h^n}{4} + \frac{\sigma_h^{n-1} + \sigma_h^n}{4} - \frac{\Delta t^2}{4}(\frac{\sigma_h^{n+1} - 2\sigma_h^n + \sigma^{n-1}}{\Delta t^2}),$$

with these last equalities we can write

$$\begin{aligned}
\frac{1}{2\mu}(\Delta_t \sigma_h^n, \sigma_h^n) &= \frac{1}{2\mu}\left(\frac{\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}}{\Delta t}, \frac{\sigma_h^{n+1} + \sigma_h^n}{4} + \frac{\sigma_h^{n-1} + \sigma_h^n}{4} - \frac{\Delta t^2}{4}(\frac{\sigma_h^{n+1} - 2\sigma_h^n + \sigma^{n-1}}{\Delta t^2})\right) \\
&= \frac{1}{2\mu}\left(\frac{\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}}{\Delta t}, \frac{\sigma_h^{n+1} + \sigma_h^n}{4} + \frac{\sigma_h^{n-1} + \sigma_h^n}{4} - \frac{\Delta t^2}{4}\Delta_t^2 \sigma_h^n\right) \\
&= \frac{1}{2\mu}\left(\frac{\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}}{\Delta t}, \frac{\sigma_h^{n+1} + \sigma_h^n}{4} + \frac{\sigma_h^{n-1} + \sigma_h^n}{4} - \frac{\Delta t^2}{4}\frac{\Delta_t \sigma_h^{n+\frac{1}{2}} - \Delta_t \sigma_h^{n-\frac{1}{2}}}{\Delta t}\right) \\
&= \frac{1}{2\mu}\frac{1}{2\Delta t}(\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}, \sigma_h^{n+\frac{1}{2}} + \sigma_h^{n-\frac{1}{2}}) - \frac{1}{2\mu}\frac{\Delta t^2}{4}\left(\Delta_t \sigma_h^n, \frac{\Delta_t \sigma_h^{n+\frac{1}{2}} - \Delta_t \sigma_h^{n-\frac{1}{2}}}{\Delta t}\right) \\
&= \frac{1}{4\mu\Delta t}(\|\sigma_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\sigma_h^{n-\frac{1}{2}}\|_{0,\Omega}^2) \\
&\quad - \frac{1}{2\mu}\frac{\Delta t^2}{4}\left(\frac{\Delta_t \sigma_h^{n+\frac{1}{2}} + \Delta_t \sigma_h^{n-\frac{1}{2}}}{2}, \frac{\Delta_t \sigma_h^{n+\frac{1}{2}} - \Delta_t \sigma_h^{n-\frac{1}{2}}}{\Delta t}\right) \quad \text{form } (3.6.1) \\
&= \frac{1}{4\mu\Delta t}(\|\sigma_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\sigma_h^{n-\frac{1}{2}}\|_{0,\Omega}^2) - \frac{\Delta t}{16\mu}\left(\|\Delta_t \sigma_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\Delta_t \sigma_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\right).
\end{aligned}$$

In the same way we get

$$\frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) = \frac{1}{2\lambda\Delta t}(\|p_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|p_h^{n-\frac{1}{2}}\|_{0,\Omega}^2) - \frac{\Delta t}{8\lambda}\left(\|\Delta_t p_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\Delta_t p_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\right).$$

The equation (3.6.15) become

$$\begin{aligned}
&\frac{1}{2\Delta t}\big(\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\Delta_t u_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\big) + \frac{1}{4\mu\Delta t}\big(\|\sigma_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\sigma_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\big)\\
&\quad - \frac{\Delta t}{16\mu}\left(\|\Delta_t\sigma_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\Delta_t\sigma_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\right) + \frac{1}{2\lambda\Delta t}\big(\|p_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|p_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\big)\\
&\quad - \frac{\Delta t}{8\lambda}\left(\|\Delta_t p_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\Delta_t p_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\right) = (f^n, \Delta_t u_h^n).
\end{aligned} \tag{3.6.16}$$

We then sum (3.6.16) from $n = 1, \cdots, N$ and we get

$$\begin{aligned}
&\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 - \frac{\Delta t^2}{8\mu}\|\Delta_t\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 - \frac{\Delta t^2}{4\lambda}\|\Delta_t p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\
&\qquad = 2\Delta t\sum_{n=1}^{N}(f^n, \Delta_t u_h^n) + \|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 - \frac{\Delta t^2}{8\mu}\|\Delta_t\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2\\
&\qquad\quad + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2 - \frac{\Delta t^2}{4\lambda}\|\Delta_t p_h^{\frac{1}{2}}\|_{0,\Omega}^2.
\end{aligned} \tag{3.6.17}$$

We recall from (3.6.11) that

$$\begin{aligned}
&\frac{1}{2\mu}(\sigma_h^{N+1} - \sigma_h^N, \tau_h) + \frac{1}{\lambda}(p_h^{N+1} - p_h^N, q_h)\\
&\qquad + (div\ (\tau_h - q_h\delta), u_h^{N+1} - u_h^N)\\
&\qquad + (as\ (\tau_h), \omega_h^{N+1} - \omega_h^N) = 0,
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{2\mu}\Delta t(\Delta_t\sigma_h^{N+\frac{1}{2}}, \tau_h) + \frac{1}{\lambda}\Delta t(\Delta_t p_h^{N+\frac{1}{2}}, q_h)\\
&\qquad + \Delta t(div\ (\tau_h - q_h\delta), \Delta_t u_h^{N+\frac{1}{2}})\\
&\qquad + \Delta t(as\ (\tau_h), \Delta_t\omega_h^{N+\frac{1}{2}}) = 0.
\end{aligned}$$

We choose $(\tau_h, q_h) = (\Delta_t\sigma_h^{N+\frac{1}{2}}, \Delta_t p_h^{N+\frac{1}{2}})$ and then by Cauchy-Schwarz inequality and (3.5.12)we get

$$\begin{aligned}
&\frac{1}{2\mu}(\Delta_t\sigma_h^{N+\frac{1}{2}}, \Delta_t\sigma_h^{N+\frac{1}{2}}) + \frac{1}{\lambda}(\Delta_t p_h^{N+\frac{1}{2}}, \Delta_t\sigma_h^{N+\frac{1}{2}})\\
&\qquad = -\big(div\ (\Delta_t\sigma_h^{N+\frac{1}{2}} - \Delta_t p_h^{N+\frac{1}{2}}\delta), \Delta_t u_h^{N+\frac{1}{2}}\big)\\
&\qquad \le \|div\ (\Delta_t\sigma_h^{N+\frac{1}{2}} + \Delta_t p_h^{N+\frac{1}{2}}\delta)\|_{0,\Omega}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}\\
&\qquad \le C_0 h^{-\beta}\|-\Delta_t\sigma_h^{N+\frac{1}{2}} + \Delta_t p_h^{N+\frac{1}{2}}\delta\|_{0,\Omega}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}\\
&\qquad \le C_0\sqrt{2}h^{-\beta}\big(\|\Delta_t\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega} + \|\Delta_t p_h^{N+\frac{1}{2}}\|_{0,\Omega}\big)\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}\\
&\qquad \le C_0\sqrt{2}h^{-\beta}\sqrt{\lambda + 2\mu}\left(\frac{1}{2\mu}\|\Delta_t\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|\Delta_t p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\right)^{\frac{1}{2}}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}.
\end{aligned}$$

Thus

$$\frac{1}{2\mu}\|\Delta_t\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}+\frac{1}{\lambda}\|\Delta_t p_h^{N+\frac{1}{2}}\|_{0,\Omega}\lesssim\sqrt{2}h^{-\beta}\sqrt{\lambda+2\mu}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}.$$

From (3.6.53) we get

$$\begin{aligned}\big(1-\frac{1}{2}C_0^2(2\mu{+}\lambda)\frac{\Delta t^2}{h^{2\beta}}\big)\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2&+\frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\ &\le 2\Delta t\sum_{n=1}^{N}(f^n,\Delta_t u_h^n)+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2\\ &\quad-\frac{\Delta t^2}{8\mu}\|\Delta_t\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2-\frac{\Delta t^2}{4\lambda}\|\Delta_t p_h^{\frac{1}{2}}\|_{0,\Omega}^2\\ &\le \Delta t\sum_{n=1}^{N}\|f^n\|_{0,\Omega}^2+\Delta t\sum_{n=1}^{N}\|\Delta_t u_h^n\|_{0,\Omega}^2+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2\\ &\quad-\frac{\Delta t^2}{8\mu}\|\Delta_t\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2-\frac{\Delta t^2}{4\lambda}\|\Delta_t p_h^{\frac{1}{2}}\|_{0,\Omega}^2\quad\text{By Young's inequality.}\end{aligned}\tag{3.6.18}$$

Moreover

$$\begin{aligned}\sum_{n=1}^{N}\|\Delta_t u_h^n\|_{0,\Omega}^2&=\sum_{n=1}^{N}\Big\|\frac{\Delta_t u_h^{n+\frac{1}{2}}+\Delta_t u_h^{n-\frac{1}{2}}}{2}\Big\|_{0,\Omega}^2\\ &\le\frac{1}{2}\Big(\sum_{n=1}^{N}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2\Big)\\ &=\frac{1}{2}\Big(2\sum_{n=1}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2\Big)\\ &\le\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2.\end{aligned}\tag{3.6.19}$$

Thus (3.6.18) becomes

$$\begin{aligned}\Big(\frac{1}{2}-\frac{1}{2}C_0^2(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}}\Big)\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2&+\frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\ \le\Delta t\sum_{n=1}^{N}\|f^n\|_{0,\Omega}^2+\Delta t\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2&+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2\\ +\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2-\frac{\Delta t^2}{8\mu}\|\Delta_t\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2&-\frac{\Delta t^2}{4\lambda}\|\Delta_t p_h^{\frac{1}{2}}\|_{0,\Omega}^2.\end{aligned}\tag{3.6.20}$$

If we suppose that $\alpha_0:=\frac{1}{2}-\frac{1}{2}C_0^2(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}}>0$ that is

$$\frac{\Delta t}{h^\beta}<\frac{1}{C_0\sqrt{2\mu+\lambda}},\tag{3.6.21}$$

we obtain

$$\begin{aligned}
&\Big(\frac{1}{2}-\frac{1}{2}C_0^2(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}}\Big)\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 \\
&\quad\leq \Delta t\sum_{n=1}^{N}\|f^n\|_{0,\Omega}^2+\Delta t\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2.
\end{aligned} \tag{3.6.22}$$

Discrete Gronwwall's inequality (3.2.6) to (3.6.22) yields

$$\begin{aligned}
&\alpha_0\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 \\
&\quad\leq\Big[\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2+\Delta t\sum_{n=1}^{N}\|f^n\|_{0,\Omega}^2\Big]\times\exp\big(\Delta t\sum_{n=0}^{N-1}\alpha_0^{-1}\big) \\
&\quad\leq\Big[\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2+T\big(\max_{t\in[0,T]}\|f(t)\|_{0,\Omega}\big)^2\Big]\times\exp\big(T\alpha_0^{-1}\big).
\end{aligned}$$

Finally (3.6.9) comes from the inf-sup (1.4.7) condition and thus the temporal iterates are bounded by the initial data. ∎

3.6.3 Discrete in time a priori error estimates

Let us first precise that in the following the notation σ_h means the function $n\longmapsto\sigma_h^n$ and not σ_h from the semi-discrete problem. similarly for p_h, ω_h and u_h. Now, we shall prove the following optimal estimate for the error between the fully discrete and continuous problems:

Theorem 3.6.3. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations on Ω . We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies to all conditions R1 - R3. Under the hypothesis of the Proposition 3.4.2 such that if $\Delta t<\frac{1}{C_0\sqrt{2\mu+\lambda}}h^\beta$, the next error estimates holds*

$$\begin{aligned}
||\sigma-\sigma_h||_{L^\infty(L^2)}+||p-p_h||_{L^\infty(L^2)}\lesssim h\Big[&|u|_{L^\infty(H_\phi^{1,1})}+|u_{tt}|_{L^\infty(H_\phi^{1,1})}+|u_{tt}|_{L^\infty(H^1)} \\
&+|u_{ttt}|_{L^2(H_\phi^{1,1})}+|u_{ttt}|_{L^\infty(H^1)}\Big] \\
&+\Delta t^2||u_{tttt}||_{L^\infty(L^2)},
\end{aligned} \tag{3.6.23}$$

$$\begin{aligned}
||\omega-\omega_h||_{L^\infty(L^2)}+||P_h^0u-u_h||_{L^\infty(L^2)}\lesssim h\Big[&|u|_{L^\infty(H_\phi^{1,1})}+|u_{tt}|_{L^\infty(H_\phi^{1,1})}+|u_{tt}|_{L^\infty(H^1)} \\
&+|u_{ttt}|_{L^2(H_\phi^{1,1})}+|u_{ttt}|_{L^\infty(H^1)}\Big] \\
&+\Delta t^2||u_{tttt}||_{L^\infty(L^2)},
\end{aligned} \tag{3.6.24}$$

$$||u-u_h||_{L^\infty(L^2)} \lesssim h\Big[|u|_{L^\infty(H^{1,1}_\phi)} + |u|_{L^\infty(H^1)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^2(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)}\Big] + \Delta t^2||u_{tttt}||_{L^\infty(L^2)}. \tag{3.6.25}$$

Proof: Let $(\widehat{\sigma}_h(t_n), \widehat{p}_h(t_n))$, $(\widehat{\omega}_h(t_n), \widehat{u}_h(t_n))$ be the elliptic projection of $(\sigma(t_n), p(t_n))$, $(\omega(t_n), u(t_n))$ and set

$$\varepsilon_h^n = \sigma_h^n - \widehat{\sigma}_h(t_n),\quad \chi_h^n = u_h^n - \widehat{u}_h(t_n),\quad \psi_h^n = \omega_h^n - \widehat{\omega}_h(t_n) \ \text{ and } \ r_h^n = p_h^n - \widehat{p}_h(t_n).$$

We may then write the error system in the form

$$\begin{cases} \dfrac{1}{2\mu}(\varepsilon_h^{n+1}, \tau_h) + \dfrac{1}{\lambda}(r_h^{n+1}, q_h) + (\mathit{div}\ (\tau_h - q_h\delta), \chi_h^{n+1}) + (\mathit{as}\ (\tau_h), \psi_h^{n+1}) = 0, \\ (\mathit{as}\ (\varepsilon_h^{n+1}), \theta_h) = 0, \\ (\mathit{div}\ (\varepsilon_h^n - r_h^n\delta), v_h) = (-u_{tt}(t_n) + \Delta_t^2\widehat{u}_h(t_n) + \Delta_t^2\chi_h^n, v_h). \end{cases} \tag{3.6.26}$$

Note that $\varepsilon_h^0 = 0$ and $r_h^0 = 0$ from (3.6.7) . Furthermore $\chi_h^0 = 0$ and $\chi_h^{-1} = 0$ from (3.6.2) and (3.6.3) respectively and thus $\Delta_t\chi_h^{-\frac{1}{2}} = 0$.

If we differentiate the first equation of (3.6.26) we obtain

$$\frac{1}{2\mu}(\Delta_t\varepsilon_h^{n+\frac{1}{2}}, \tau_h) + \frac{1}{\lambda}(\Delta_t r_h^{n+\frac{1}{2}}, q_h) + \big(\mathit{div}\ (\tau_h - q_h\delta), \Delta_t\chi_h^{n+\frac{1}{2}}\big) + \big(\mathit{as}\ (\tau_h), \Delta_t\psi_h^{n+\frac{1}{2}}\big) = 0. \tag{3.6.27}$$

Now taking $(\tau_h, q_h) = \big(\varepsilon_h^{n+\frac{1}{2}}, r_h^{n+\frac{1}{2}}\big)$ in this last equality and using the second equation of (3.6.26), we obtain

$$\frac{1}{2\mu}\big(\Delta_t\varepsilon_h^{n+\frac{1}{2}}, \varepsilon_h^{n+\frac{1}{2}}\big) + \frac{1}{\lambda}\big(\Delta_t r_h^{n+\frac{1}{2}}, r_h^{n+\frac{1}{2}}\big) + \big(\mathit{div}\ (\varepsilon_h^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}}\delta), \Delta_t\chi_h^{n+\frac{1}{2}}\big) = 0. \tag{3.6.28}$$

The last equation of (3.6.26) with $v_h = \frac{1}{2}(\Delta_t\chi_h^{n+\frac{1}{2}} + \Delta_t\chi_h^{n-\frac{1}{2}})$ gives

$$\begin{aligned}\big(\mathit{div}\ (\varepsilon_h^n - r_h^n\delta), \tfrac{1}{2}(\Delta_t\chi_h^{n+\frac{1}{2}} + \Delta_t\chi_h^{n-\frac{1}{2}})\big)& \\ = \big(-u_{tt}(t_n) + \Delta_t^2\widehat{u}_h(t_n) + \Delta_t^2\chi_h^n, \tfrac{1}{2}(\Delta_t\chi_h^{n+\frac{1}{2}} + \Delta_t\chi_h^{n-\frac{1}{2}})\big).&\end{aligned} \tag{3.6.29}$$

Subtracting (3.6.29) from (3.6.28) we get

$$\begin{aligned}&\frac{1}{2\mu}\big(\Delta_t\varepsilon_h^{n+\frac{1}{2}}, \varepsilon_h^{n+\frac{1}{2}}\big) + \frac{1}{\lambda}\big(\Delta_t r_h^{n+\frac{1}{2}}, r_h^{n+\frac{1}{2}}\big) + \frac{1}{2}\big(\mathit{div}\ (\varepsilon_h^{n+1} - r_h^{n+1}\delta), \Delta_t\chi_h^{n+\frac{1}{2}}\big) \\ &\quad - \frac{1}{2}\big(\mathit{div}\ (\varepsilon_h^n - r_h^n\delta), \Delta_t\chi_h^{n-\frac{1}{2}}\big) = \big(u_{tt}(t_n) - \Delta_t^2\widehat{u}_h(t_n) - \Delta_t^2\chi_h^n, \tfrac{1}{2}(\Delta_t\chi_h^{n+\frac{1}{2}} + \Delta_t\chi_h^{n-\frac{1}{2}})\big).\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2\mu}\big(\Delta_t\varepsilon_h^{n+\frac12},\varepsilon_h^{n+\frac12}\big)+\frac{1}{\lambda}\big(\Delta_t r_h^{n+\frac12},r_h^{n+\frac12}\big)+\frac12\big(div\ (\varepsilon_h^{n+1}-r_h^{n+1}\delta),\Delta_t\chi_h^{n+\frac12}\big)\\ -\frac12\big(div\ (\varepsilon_h^{n}-r_h^{n}\delta),\Delta_t\chi_h^{n-\frac12}\big)+\big(\Delta_t^2\chi_h^n,\tfrac12(\Delta_t\chi_h^{n+\frac12}+\Delta_t\chi_h^{n-\frac12})\big)\\ =\big(u_{tt}(t_n)-\Delta_t^2\widehat{u}_h(t_n),\tfrac12(\Delta_t\chi_h^{n+\frac12}+\Delta_t\chi_h^{n-\frac12})\big).\end{aligned} \tag{3.6.30}$$

We expand (3.6.30) to get

$$\begin{aligned}\frac{1}{2\mu}\left(\frac{\varepsilon_h^{n+1}-\varepsilon_h^n}{\Delta t},\frac{\varepsilon_h^{n+1}+\varepsilon_h^n}{2}\right)+\frac{1}{\lambda}\left(\frac{r_h^{n+1}-r_h^n}{\Delta t},\frac{r_h^{n+1}+r_h^n}{2}\right)+\frac12\big(div\ (\varepsilon_h^{n+1}-r_h^{n+1}\delta),\Delta_t\chi_h^{n+\frac12}\big)\\ -\frac12\big(div\ (\varepsilon_h^{n}-r_h^{n}\delta),\Delta_t\chi_h^{n-\frac12}\big)+\left(\frac{\Delta_t\chi_h^{n+\frac12}-\Delta_t\chi_h^{n-\frac12}}{\Delta t},\frac{\Delta_t\chi_h^{n+\frac12}+\Delta_t\chi_h^{n-\frac12}}{2}\right)\\ =\big(u_{tt}(t_n)-\Delta_t^2\widehat{u}_h(t_n),\tfrac12(\Delta_t\chi_h^{n+\frac12}+\Delta_t\chi_h^{n-\frac12})\big).\end{aligned}$$

So that

$$\begin{aligned}\frac{1}{2\Delta t}\Big(\frac{1}{2\mu}(\|\varepsilon_h^{n+1}\|_{0,\Omega}^2-\|\varepsilon_h^{n}\|_{0,\Omega}^2)+\frac{1}{\lambda}(\|r_h^{n+1}\|_{0,\Omega}^2-\|r_h^{n}\|_{0,\Omega}^2)\Big)+\frac12\big(div\ (\varepsilon_h^{n+1}-r_h^{n+1}\delta),\Delta_t\chi_h^{n+\frac12}\big)\\ -\frac12\big(div\ (\varepsilon_h^{n}-r_h^{n}\delta),\Delta_t\chi_h^{n-\frac12}\big)+\frac{1}{2\Delta t}\Big(\|\Delta_t\chi_h^{n+\frac12}\|_{0,\Omega}^2-\|\Delta_t\chi_h^{n-\frac12}\|_{0,\Omega}^2\Big)\\ =\big(u_{tt}(t_n)-\Delta_t^2\widehat{u}_h(t_n),\tfrac12(\Delta_t\chi_h^{n+\frac12}+\Delta_t\chi_h^{n-\frac12})\big).\end{aligned} \tag{3.6.31}$$

Replacing n by j and then sum over time levels and multiplying (3.6.31) by $2\Delta t$ to get

$$\begin{aligned}\frac{1}{2\mu}(\|\varepsilon_h^{n}\|_{0,\Omega}^2-\|\varepsilon_h^{0}\|_{0,\Omega}^2)+\frac{1}{\lambda}(\|r_h^{n}\|_{0,\Omega}^2-\|r_h^{0}\|_{0,\Omega}^2)+\Delta t\big(div\ (\varepsilon_h^{n}-r_h^{n}\delta),\Delta_t\chi_h^{n-\frac12}\big)\\ -\Delta t\big(div\ (\varepsilon_h^{0}-r_h^{0}\delta),\Delta_t\chi_h^{-\frac12}\big)+\|\Delta_t\chi_h^{n-\frac12}\|_{0,\Omega}^2-\|\Delta_t\chi_h^{-\frac12}\|_{0,\Omega}^2\\ =\Delta t\sum_{j=0}^{n-1}\Delta t\big(u_{tt}(t_j)-\Delta_t^2\widehat{u}_h(t_j),\Delta_t\chi_h^{j+\frac12}+\Delta_t\chi_h^{j-\frac12}\big).\end{aligned} \tag{3.6.32}$$

Recall that $\varepsilon_h^0=0$, $r_h^0=0$ and $\Delta_t\chi_h^{\frac{-1}{2}}=0$. Thus (3.6.32) become

$$\begin{aligned}\frac{1}{2\mu}\|\varepsilon_h^{n}\|_{0,\Omega}^2+\frac{1}{\lambda}\|r_h^{n}\|_{0,\Omega}^2-\Delta t\big(div\ (-\varepsilon_h^{n}+r_h^{n}\delta),\Delta_t\chi_h^{n-\frac12}\big)+\|\Delta_t\chi_h^{n-\frac12}\|_{0,\Omega}^2\\ =\Delta t\sum_{j=0}^{n-1}\big(u_{tt}(t_j)-\Delta_t^2\widehat{u}_h(t_j),\Delta_t\chi_h^{j+\frac12}+\Delta_t\chi_h^{j-\frac12}\big).\end{aligned} \tag{3.6.33}$$

By Cauchy-Schwarz and inverse inequalities (3.5.12) and choosing $\Delta t < \frac{h^\beta}{C_0\sqrt{2\mu+\lambda}}$ we get

$$\begin{aligned}
|\Delta t(\mathbf{div}\,(-\varepsilon_h^n + r_h^n\delta), \Delta_t\chi_h^{n-\frac{1}{2}})| &\leq \Delta t\|\mathbf{div}\,(-\varepsilon_h^n + r_h^n\delta)\|_{0,\Omega}\|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega} \\
&\leq \Delta t C_0 h^{-\beta}\|-\varepsilon_h^n + r_h^n\delta\|_{0,\Omega}\|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega} \\
&\leq \sqrt{2}\Delta t C_0 h^{-\beta}(\|\varepsilon_h^n\|_{0,\Omega} + \|r_h^n\|_{0,\Omega})\|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega} \\
&\leq \sqrt{2}\Delta t C_0 h^{-\beta}\sqrt{2\mu+\lambda}\left(\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2\right)^{\frac{1}{2}}\|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega} \\
&< \frac{\sqrt{2}}{2}\left(\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2 + \|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\right)
\end{aligned} \tag{3.6.34}$$

Thus we have

$$\begin{aligned}
\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2 + \|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2 &\leq C\Delta t\sum_{j=0}^{n-1}|\big(u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j), \Delta_t\chi_h^{j+\frac{1}{2}} + \Delta_t\chi_h^{j-\frac{1}{2}}\big)| \\
&\leq C\Delta t\sum_{j=0}^{n-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\|\Delta_t\chi_h^{j+\frac{1}{2}} + \Delta_t\chi_h^{j-\frac{1}{2}}\|_{0,\Omega} \\
&\leq 2C\Delta t\sum_{j=0}^{n-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}||\Delta_t\chi_h||_{L^\infty(L^2)},
\end{aligned}$$

since $\|\Delta_t\chi^{j+\frac{1}{2}}\|_{0,\Omega} \leq ||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)} := \sup\limits_{0\leq j\leq N-1}\|\Delta_t\chi_h^{j+\frac{1}{2}}\|_{0,\Omega}$. Then

$$\begin{aligned}
\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2 + \|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2 &\leq 2C\Delta t||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}\sum_{j=0}^{N-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega} \\
&\leq ||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}^2 + C^2\Delta t^2\left(\sum_{j=0}^{N-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\right)^2.
\end{aligned} \tag{3.6.35}$$

Taking the supremum on n on the left-hand side of (3.6.35) we get

$$\begin{aligned}
&\frac{1}{2\mu}||\varepsilon_h||_{L^\infty(L^2)}^2 + \frac{1}{\lambda}||r_h||_{L^\infty(L^2)}^2 + ||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}^2 \\
&\qquad\leq ||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}^2 + C\Delta t^2\left(\sum_{j=0}^{N-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j\|_{0,\Omega}\right)^2.
\end{aligned}$$

This last inequality and assumption on μ and λ yields

$$||\varepsilon_h||_{L^\infty(L^2)} + ||r_h||_{L^\infty(L^2)} \lesssim \Delta t\sum_{j=0}^{N-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}. \tag{3.6.36}$$

Now it remains to bound $\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}$. We write

$$u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j) = u_{tt}(t_j) - \Delta_t^2 u(t_j) + \Delta_t^2 u(t_j) - \Delta_t^2\widehat{u}_h(t_j)$$

If we denote by R_h the elliptic projection operator we can write

$$\Delta_t^2 u(t_j) - \Delta_t^2 \widehat{u}_h(t_j) = (I - R_h)\Delta_t^2 u(t_j) = (I - R_h)\frac{u(t_{j+1}) - 2u(t_j) + u(t_{j-1})}{\Delta t^2}.$$

By Taylor expansion we have

$$u(t_{j-1}) = u(t_j) - \Delta t \; u_t(t_j) + \frac{\Delta t^2}{2} u_{tt}(t_j) + \frac{1}{2}\int_{t_j}^{t_{j-1}} (t_{j-1} - s)^2 u_{ttt}(s) \; ds.$$

and

$$u(t_{j+1}) = u(t_j) + \Delta t \; u_t(t_j) + \frac{\Delta t^2}{2} u_{tt}(t_j) + \frac{1}{2}\int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 u_{ttt}(s) \; ds.$$

So that

$$\begin{aligned} \Delta_t^2 u(t_j) &= u_{tt}(t_j) + \frac{1}{2\Delta t^2}\Big(\int_{t_j}^{t_{j-1}} (t_{j-1} - s)^2 u_{ttt}(s) \; ds + \int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 u_{ttt}(s) \; ds\Big) \\ &= u_{tt}(t_j) + \frac{1}{2\Delta t^2}\Big(\int_0^{\Delta t} (\Delta t - t)^2 u_{ttt}(t + t_j) \; dt - \int_{-\Delta t}^{0} (\Delta t + t)^2 u_{ttt}(t + t_j) \; dt\Big). \end{aligned} \tag{3.6.37}$$

Thus

$$\begin{aligned} (I - R_h)\Delta_t^2 u(t_j) = (I - R_h)u_{tt}(t_j) + \frac{1}{2\Delta t^2}\Big(&- \int_{-\Delta t}^{0} (\Delta t + t)^2 (I - R_h)u_{ttt}(t + t_j) \; dt \\ &+ \int_0^{\Delta t} (\Delta t - t)^2 (I - R_h)u_{ttt}(t + t_j) \; dt\Big). \end{aligned}$$

So that

$$\begin{aligned} \|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} &\leq \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{1}{2}\frac{\sqrt{\Delta t}}{\sqrt{5}}\Big[\Big(\int_{-\Delta t}^{0} \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2 \; dt\Big)^{\frac{1}{2}} \\ &\qquad + \Big(\int_0^{\Delta t} \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2 \; dt\Big)^{\frac{1}{2}}\Big] \\ &\leq \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{2}}{2}\frac{\sqrt{\Delta t}}{\sqrt{5}}\Big(\int_{-\Delta t}^{\Delta t} \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2 \; dt\Big)^{\frac{1}{2}} \\ &= \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{\Delta t}}{\sqrt{10}}\Big(\int_{-\Delta t + t_j}^{\Delta t + t_j} \|(I - R_h)u_{ttt}(s)\|_{0,\Omega}^2 \; ds\Big)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} \leq \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{\Delta t}}{\sqrt{10}}\Big(\int_{-\Delta t + t_j}^{\Delta t + t_j} \|(I - R_h)u_{ttt}(s)\|_{0,\Omega}^2 \; ds\Big)^{\frac{1}{2}}.$$

From the estimate (3.5.22) for the elliptic projection we obtain:

$$\|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} \lesssim h\Big[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)}\Big],$$

and in view of remark (3.5.8)

$$\Delta t \int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I-R_h)u_{ttt}(s)\|_{0,\Omega}^2 \, ds \lesssim (\Delta t)h^2 \left[\int_{-\Delta t+t_j}^{\Delta t+t_j} \left(|u_{ttt}(s)|_{1,1;\phi,\Omega}^2 + |u_{ttt}(s)|_{1,\Omega}^2 \right) ds \right]$$
$$\lesssim \Delta t^2 h^2 \left[|u_{ttt}|_{L^\infty(H_\phi^{1,1})}^2 + |u_{ttt}|_{L^\infty(H^1)}^2 \right].$$

By the stability condition (3.6.21) we can write

$$\sqrt{\Delta t} \left(\int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I-R_h)u_{ttt}(s)\|_{0,\Omega}^2 \, ds \right)^{\frac{1}{2}} \lesssim h^{\beta+1} \left[|u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} \right]$$
$$\lesssim h \left[|u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} \right].$$

Combining these last inequalities, we get

$$\Delta t \sum_{j=0}^{N-1} \|(I-R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} \lesssim h \Big[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})}$$
$$+ |u_{ttt}|_{L^\infty(H^1)} \Big] \times \sum_{j=0}^{N-1} \Delta t$$
$$\lesssim h \left[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} \right].$$

Using once again Taylor expansion we get

$$u(t_{j-1}) = u(t_j) - \Delta t \; u_t(t_j) + \frac{\Delta t^2}{2} u_{tt}(t_j) - \frac{\Delta t^3}{6} u_{ttt}(t_j) + \frac{1}{6} \int_{t_j}^{t_{j-1}} (t_{j-1}-s)^3 u_{tttt}(s) \, ds.$$

and

$$u(t_{j+1}) = u(t_j) + \Delta t \; u_t(t_j) + \frac{\Delta t^2}{2} u_{tt}(t_j) + \frac{\Delta t^3}{6} u_{ttt}(t_j) + \frac{1}{6} \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^3 u_{tttt}(s) \, ds.$$

Thus

$$\Delta_t^2 u(t_j) - u_{tt}(t_j) = \frac{1}{6\Delta t^2} \left(\int_{-\Delta t}^{0} (\Delta t + t)^3 u_{tttt}(t+t_j) \, dt + \int_0^{\Delta t} (\Delta t - t)^3 u_{tttt}(t+t_j) \, dt \right).$$

So that

$$\|\Delta_t^2 u(t_j) - u_{tt}(t_j)\|_{0,\Omega} \le \frac{1}{6} \frac{\Delta t^{\frac{3}{2}}}{\sqrt{7}} \left[\left(\int_{-\Delta t}^{0} \|u_{tttt}(t+t_j)\|_{0,\Omega}^2 \, dt \right)^{\frac{1}{2}} + \left(\int_0^{\Delta t} \|u_{tttt}(t+t_j\|_{0,\Omega}^2 \, dt \right)^{\frac{1}{2}} \right]$$
$$\lesssim \Delta t^{\frac{3}{2}} \left(\int_{-\Delta t}^{\Delta t} \|u_{tttt}(t+t_j)\|_{0,\Omega}^2 \, dt \right)^{\frac{1}{2}}.$$

Hence

$$\|\Delta_t^2 u(t_j) - u_{tt}(t_j)\|_{0,\Omega} \lesssim \Delta t^{\frac{3}{2}} \left(\int_{-\Delta t + t_j}^{\Delta t + t_j} \|u_{tttt}(s)\|_{0,\Omega}^2 \, ds \right)^{\frac{1}{2}}. \tag{3.6.38}$$

So that

$$\Delta t \sum_{j=0}^{N-1} \|\Delta_t^2 u(t_j) - u_{tt}(t_j)\|_{0,\Omega} \lesssim \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)} \sum_{j=0}^{N-1} \Delta t$$
$$\lesssim \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}.$$

Then (3.6.36) become

$$||\varepsilon_h||_{L^\infty(L^2)} + ||r_h||_{L^\infty(L^2)} \lesssim h\Big[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)}\Big]$$
$$+ \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}.$$

By the triangle inequality and taking the supremum on t in (3.5.15) we then find

$$||\sigma - \sigma_h||_{L^\infty(L^2)} + ||p - p_h||_{L^\infty(L^2)} \lesssim h\Big[|u|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)}$$
$$+ |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)}\Big] + \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}.$$

By the inf-sup and the triangle inequality, we get

$$\|\omega(t_n) - \omega_h^n\|_{0,\Omega} + \|P_h^0 u(t_n) - u_h^n\|_{0,\Omega} \lesssim \Big[\|\sigma(t_n) - \sigma_h^n\|_{0,\Omega} + \|p(t_n) - p_h^n\|_{0,\Omega}$$
$$+ \|P_h^1 \omega(t_n) - \omega(t_n)\|_{0,\Omega}\Big].$$

Taking the supremum on n we get

$$||\omega - \omega_h||_{L^\infty(L^2)} + ||P_h^0 u - u_h||_{L^\infty(L^2)} \lesssim \Big[||\sigma - \sigma||_{L^\infty(L^2)} + ||p - p_h||_{L^\infty(L^2)}$$
$$+ ||P_h^1 \omega - \omega||_{L^\infty(L^2)}\Big].$$

By the propriety of the projection operator (3.5.10) we hence find

$$||\omega - \omega_h||_{L^\infty(L^2)} + ||P_h^0 u - u_h||_{L^\infty(L^2)} \lesssim h\Big[|u|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)}$$
$$+ |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)}\Big] + \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}.$$

The triangle inequality and the propriety of the projection operator (1.47) p.27 of [36], give

$$||u-u_h||_{L^\infty(L^2)} \lesssim h\Big[|u|_{L^\infty(H^{1,1}_\phi)} + |u|_{L^\infty(H^1)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)}\Big] + \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}.$$

■

3.6.4 Implicit scheme

The implicit-in-time discrete mixed formulation is as follows: Find $(\sigma_h^{n+1}, p_h^{n+1}) \in \Sigma_{0,h}$, and $(u_h^{n+1}, \omega_h^{n+1}) \in V_h \times W_h$ such that

$$u_h^0 = \hat{u}_h(0), \quad u_h^1 = \hat{u}_h(\Delta t), \tag{3.6.39}$$

and for all $(\tau_h, q_h) \in \Sigma_{0,h}$, for all $(v_h, \theta_h) \in V_h \times W_h$ we have

$$\begin{cases} \dfrac{1}{2\mu}(\sigma_h^{n+1}, \tau_h) + \dfrac{1}{\lambda}(p_h^{n+1}, q_h) + (div\ (\tau_h - q_h\delta), u_h^{n+1}) + (as\ (\tau_h), \omega_h^{n+1}) = 0, \quad n \geq -1, \\ (as\ (\sigma_h^{n+1}), \theta_h) = 0, \quad n \geq -1, \\ (\Delta_t^2 u_h^n, v_h) - (div\ (\sigma_h^{n,\frac14} - p_h^{n,\frac14}\delta), v_h) - (f^{n,\frac14}, v_h) = 0, \quad n \geq 1. \end{cases} \tag{3.6.40}$$

Remark 3.6.4. *Note that by the first two equations of system (3.6.40) with $n=-1$ (see the proof of Lemma (3.6.1)) σ_h^0, p_h^0 and ω_h^0 are defined. Similarly by taking $n=0$, σ_h^1, p_h^1 and ω_h^1 are defined. Moreover we have*

$$\sigma_h^0 = \hat{\sigma}_h(0),\ p_h^0 = \hat{p}_h(0) \text{ and } \omega_h^0 = \hat{\omega}_h(0), \tag{3.6.41}$$

and

$$\sigma_h^1 = \hat{\sigma}_h(\Delta t),\ p_h^1 = \hat{p}_h(\Delta t) \text{ and } \omega_h^1 = \hat{\omega}_h(\Delta t). \tag{3.6.42}$$

The existence and uniqueness of a solution to problem (3.6.40) is provided by the following lemma:

Lemma 3.6.5. *a solution $(\sigma_h^{n+1}, p_h^{n+1})$ and $(u_h^{n+1}, \omega_h^{n+1})$ of (3.6.40) exist and is unique.*

Proof:

To every $\big((\sigma_h, p_h), (u_h, \omega_h)\big) \in \Sigma_{0,h} \times (V_h \times W_h)$, we associate the element of its dual $\Sigma'_{0,h} \times V'_h \times W'_h$:

$$\begin{pmatrix} (\tau_h, q_h) & \longmapsto & \frac{1}{2\mu}(\sigma_h, \tau_h) + \frac{1}{\lambda}(p_h, q_h) + (div\ (\tau_h - q_h\delta), u_h) + (as\ (\tau_h), \omega_h) \\ \theta_h & \longmapsto & (as\ (\sigma_h), \theta_h) \\ v_h & \longmapsto & \frac{1}{\Delta t^2}(u_h, v_h) - \frac{1}{4}(div\ (\sigma_h - p_h\delta), v_h) \end{pmatrix}.$$

Let as call this mapping T_h; it is a linear mapping from $\Sigma_{0,h} \times V_h$ into its dual. We have to prove that T_h is bijective. But the arrival and departure spaces has the same dimension. Thus by a well known theorem of linear algebra it suffices to prove that T_h is injective. Thus let $\big((\sigma_h, p_h), (u_h, \omega_h)\big) \in \Sigma_{0,h} \times (V_h \times W_h)$ such that:

$$\frac{1}{2\mu}(\sigma_h, \tau_h) + \frac{1}{\lambda}(p_h, q_h) + (div\ (\tau_h - q_h\delta), u_h) + (as\ (\tau_h), \omega_h) = 0, \quad \forall (\tau_h, q_h) \in \Sigma_{0,h} \tag{3.6.43}$$

$$(as\ (\sigma_h), \theta_h) = 0, \quad \forall \theta_h \in W_h \tag{3.6.44}$$

$$\frac{1}{\Delta t^2}(u_h, v_h) - \frac{1}{4}(div\ (\sigma_h - p_h\delta), v_h) = 0. \quad \forall \theta_h \in V_h \tag{3.6.45}$$

Then by taking $(\tau_h, q_h) = (\sigma_h, p_h)$ in (3.6.43) and $v_h = u_h$ in (3.6.45) and the fact that $(as\ (\sigma_h), \omega_h) = 0$, we get

$$\frac{1}{2\mu}\|\sigma_h\|^2_{0,\Omega} + \frac{1}{\lambda}\|p_h\|^2_{0,\Omega} + \frac{1}{\Delta t^2}\|u_h\|^2_{0,\Omega} = 0.$$

Which implies that

$$\sigma_h = 0, \quad p_h = 0 \quad \text{and} \quad u_h.$$

Thus (3.6.43) reduces to:

$$(as\ (\tau_h), \omega_h) = 0.$$

By inf-sup inequality (1.4.7), with $(v_h, \theta_h) = (0, \omega_h)$, we get $\omega_h = 0$.

■

3.6.5 Stability of the implicit scheme

As expected for such an implicit scheme this method is unconditionally stable, as provided by the following theorem:

Theorem 3.6.6. *The implicit -in-time scheme defined by 3.6.40 is unconditionally stable, that is*

$$\begin{aligned} &\frac{1}{2}\|\Delta_t u_h^{N+\frac{1}{2}}\|^2_{0,\Omega} + \frac{1}{2\mu}\frac{1}{2}\|\sigma_h^{N+\frac{1}{2}}\|^2_{0,\Omega} + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|^2_{0,\Omega} \\ &\qquad \leq \left[\|\Delta_t u_h^{\frac{1}{2}}\|^2_{0,\Omega} + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|^2_{0,\Omega} + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|^2_{0,\Omega} + T\big(\max_{t\in[0,T]}\|f(t)\|_{0,\Omega}\big)^2\right] \times \exp\big(2T\big), \end{aligned} \tag{3.6.46}$$

$$\beta^*(\|u_h^{N+\frac{1}{2}}\|_{0,\Omega} + \|\omega_h^{N+\frac{1}{2}}\|_{0,\Omega}) \leq \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega} + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}, \tag{3.6.47}$$

where β^* is the constant of the inf-sup condition defined in (1.4.7).

Proof:

Subtracting the first equation of (3.6.40) from itself in step $n-1$, we get for all $(\tau_h, q_h) \in \Sigma_{0,h}$

$$\begin{aligned}\frac{1}{2\mu}(\sigma_h^{n+1}-\sigma_h^{n-1},\tau_h)+\frac{1}{\lambda}(p_h^{n+1}-p_h^{n-1},q_h)+(div\ (\tau_h-q_h\delta),u_h^{n+1}-u_h^{n-1})\\ +(as\ (\tau_h),\omega_h^{n+1}-\omega_h^{n-1})=0,\end{aligned} \tag{3.6.48}$$

Taking $(\tau_h, q_h) = \frac{1}{2\Delta t}(\sigma_h^{n,\frac{1}{4}}, p_h^{n,\frac{1}{4}})$ in (3.6.48) and the fact that

$$(as\ (\sigma_h^n), \theta_h) = 0 \quad \forall \theta_h \in W_h,$$

we get

$$\frac{1}{2\mu}(\Delta_t\sigma_h^n,\sigma_h^{n,\frac{1}{4}})+\frac{1}{\lambda}(\Delta_t p_h^n,p_h^{n,\frac{1}{4}})+(div\ (\sigma_h^{n,\frac{1}{4}}-p_h^{n,\frac{1}{4}}\delta),\Delta_t u_h^n)=0, \tag{3.6.49}$$

The third equation of (3.6.40) with $v_h = \Delta_t u_h^n$, become

$$(\Delta_t^2 u_h^n,\Delta_t u_h^n)-(div\ (\sigma_h^{n,\frac{1}{4}}-p_h^{n,\frac{1}{4}}\delta),\Delta_t u_h^n)=(f^{n,\frac{1}{4}},\Delta_t u_h^n), \tag{3.6.50}$$

Adding (3.6.50) and (3.6.49), we get

$$(\Delta_t^2 u_h^n,\Delta_t u_h^n)+\frac{1}{2\mu}(\Delta_t\sigma_h^n,\sigma_h^{n,\frac{1}{4}})+\frac{1}{\lambda}(\Delta_t p_h^n,p_h^{n,\frac{1}{4}})=(f^{n,\frac{1}{4}},\Delta_t u_h^n), \tag{3.6.51}$$

Now we inspect separately the three terms in the left hand side of this last equation, we have from (3.6.1)

$$\begin{aligned}(\Delta_t^2 u_h^n,\Delta_t u_h^n)&=\frac{1}{2\Delta t}(\Delta_t u_h^{n+\frac{1}{2}}-\Delta_t u_h^{n-\frac{1}{2}},\Delta_t u_h^{n+\frac{1}{2}}+\Delta_t u_h^{n-\frac{1}{2}})\\ &=\frac{1}{2\Delta t}(\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2-\|\Delta_t u_h^{n-\frac{1}{2}}\|_{0,\Omega}^2)\end{aligned}$$

and

$$\Delta_t\sigma_h^n=\frac{\sigma_h^{n+1}-\sigma_h^{n-1}}{2\Delta t}=\frac{\sigma_h^{n+1}+\sigma_h^n-\sigma_h^n+\sigma_h^{n-1}}{2\Delta t}=\frac{\sigma_h^{n+\frac{1}{2}}-\sigma_h^{n-\frac{1}{2}}}{\Delta t},$$

$$\sigma_h^{n,\frac{1}{4}}=\frac{\sigma_h^{n+1}+\sigma_h^n}{4}+\frac{\sigma_h^{n-1}+\sigma_h^n}{4}=\frac{1}{2}(\sigma_h^{n+\frac{1}{2}}+\sigma_h^{n-\frac{1}{2}}),$$

with these last equalities we can write

$$\begin{aligned}\frac{1}{2\mu}(\Delta_t\sigma_h^n,\sigma_h^{n,\frac{1}{4}})&=\frac{1}{2\mu}\left(\frac{\sigma_h^{n+\frac{1}{2}}-\sigma_h^{n-\frac{1}{2}}}{\Delta t},\frac{\sigma_h^{n+\frac{1}{2}}+\sigma_h^{n-\frac{1}{2}}}{2}\right)\\ &=\frac{1}{4\mu\Delta t}\big(\|\sigma_h^{n+\frac{1}{2}}\|_{0,\Omega}^2-\|\sigma_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\big).\end{aligned}$$

In the same way we get

$$\frac{1}{\lambda}(\Delta_t p_h^n, p_h^{n,\frac{1}{4}}) = \frac{1}{2\lambda\Delta t}\big(\|p_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|p_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\big).$$

The equation (3.6.51) become

$$\begin{aligned}\frac{1}{2\Delta t}\big(\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\Delta_t u_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\big) + \frac{1}{4\mu\Delta t}\big(\|\sigma_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|\sigma_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\big)&\\ + \frac{1}{2\lambda\Delta t}\big(\|p_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 - \|p_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\big) = (f^{n,\frac{1}{4}}, \Delta_t u_h^n).&\end{aligned} \tag{3.6.52}$$

We then sum (3.6.52) from $n = 1, \cdots, N$ and we get

$$\begin{aligned}&\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\ &\qquad = 2\Delta t\sum_{n=1}^{N}(f^{n,\frac{1}{4}}, \Delta_t u_h^n) + \|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2.\end{aligned} \tag{3.6.53}$$

By Cauchy-Schwarz inequality we get

$$\begin{aligned}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 \le \Delta t\sum_{n=1}^{N}\|f^{n,\frac{1}{4}}\|_{0,\Omega}^2 + \Delta t\sum_{n=1}^{N}\|\Delta_t u_h^n\|_{0,\Omega}^2&\\ + \|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2.&\end{aligned} \tag{3.6.54}$$

Moreover from (3.6.19) we get

$$\sum_{n=1}^{N}\|\Delta_t u_h^n\|_{0,\Omega}^2 \le \sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2$$

Thus

$$\begin{aligned}\frac{1}{2}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 \le \Delta t\sum_{n=1}^{N}\|f^{n,\frac{1}{4}}\|_{0,\Omega}^2 + \Delta t\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2&\\ + \|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2.&\end{aligned} \tag{3.6.55}$$

Discrete Gronwall's inequality (3.2.6) applied to (3.6.55) yields

$$\begin{aligned}&\frac{1}{2}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\ &\qquad \le \Big[\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 + T\big(\max_{t\in[0,T]}\|f(t)\|_{0,\Omega}\big)^2\Big] \times \exp\big(2T\big).\end{aligned}$$

This is inequality (3.6.46). Finally (3.6.47) comes from the inf-sup (1.4.7) condition and the first equation of the system (3.6.40). Thus the temporal iterates are bounded by the initial data. ■

3.6.6 Discrete in time a priori error estimates

The following theorem provide a priori error estimates for the implicit-in-time scheme:

Theorem 3.6.7. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations on Ω . We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies to conditions R1 - R2. Under the hypotheses of Proposition 3.4.2, the next error estimates holds*

$$\begin{aligned}||\sigma-\sigma_h||_{L^\infty(L^2)}+||p-p_h||_{L^\infty(L^2)} \lesssim h\Big[&|u|_{L^\infty(H^{1,1}_\phi)}+|u_{tt}|_{L^\infty(H^{1,1}_\phi)}+|u_{tt}|_{L^\infty(H^1)}\\ &+|u_{ttt}|_{L^2(H^{1,1}_\phi)}+|u_{ttt}|_{L^\infty(H^1)}\Big]\\ &+\Delta t^2||u_{tttt}||_{L^\infty(L^2)},\end{aligned} \tag{3.6.56}$$

$$\begin{aligned}||\omega-\omega_h||_{L^\infty(L^2)}+||P^0_h u-u_h||_{L^\infty(L^2)} \lesssim h\Big[&|u|_{L^\infty(H^{1,1}_\phi)}+|u_{tt}|_{L^\infty(H^{1,1}_\phi)}+|u_{tt}|_{L^\infty(H^1)}\\ &+|u_{ttt}|_{L^2(H^{1,1}_\phi)}+|u_{ttt}|_{L^\infty(H^1)}\Big]\\ &+\Delta t^2||u_{tttt}||_{L^\infty(L^2)},\end{aligned} \tag{3.6.57}$$

$$\begin{aligned}||u-u_h||_{L^\infty(L^2)} \lesssim h\Big[&|u|_{L^\infty(H^{1,1}_\phi)}+|u|_{L^\infty(H^1)}+|u_{tt}|_{L^\infty(H^{1,1}_\phi)}+|u_{tt}|_{L^\infty(H^1)}\\ &+|u_{ttt}|_{L^2(H^{1,1}_\phi)}+|u_{ttt}|_{L^\infty(H^1)}\Big]+\Delta t^2||u_{tttt}||_{L^\infty(L^2)}.\end{aligned} \tag{3.6.58}$$

Proof: Let $\big(\widehat{\sigma}_h(t_n),\widehat{p}_h(t_n)\big)$, $\big(\widehat{\omega}_h(t_n),\widehat{u}_h(t_n)\big)$ be the elliptic projection of $\big(\sigma(t_n),p(t_n)\big)$, $\big(\omega(t_n),u(t_n)\big)$ and set

$$\varepsilon^n_h=\sigma^n_h-\widehat{\sigma}_h(t_n),\quad \chi^n_h=u^n_h-\widehat{u}_h(t_n),\quad \psi^n_h=\omega^n_h-\widehat{\omega}_h(t_n)\ \text{ and }\ r^n_h=p^n_h-\widehat{p}_h(t_n).$$

We may then write the error system in the form

$$\begin{cases}\dfrac{1}{2\mu}(\varepsilon^{n+1}_h,\tau_h)+\dfrac{1}{\lambda}(r^{n+1}_h,q_h)+(div\ (\tau_h-q_h\delta),\chi^{n+1}_h)+(as\ (\tau_h),\psi^{n+1}_h)=0,\\ (as\ (\varepsilon^{n+1}_h),\theta_h)=0,\\ (div\ (\varepsilon^{n,\frac14}_h-r^{n,\frac14}_h\delta),v_h)=(-(u_{tt})^{n,\frac14}+\Delta^2_t\widehat{u}_h(t_n)+\Delta^2_t\chi^n_h,v_h).\end{cases} \tag{3.6.59}$$

Note that $\varepsilon^{\frac12}_h=0$ and $r^{\frac12}_h=0$ from Remark 3.6.4 . Furthermore $\chi^0_h=0$ and $\chi^1_h=0$ from (3.6.39) and thus $\Delta_t\chi^{\frac12}_h=0$.

If we differentiate the first equation of (3.6.59) we obtain

$$\frac{1}{2\mu}\big(\Delta_t\varepsilon_h^n,\tau_h\big)+\frac{1}{\lambda}\big(\Delta_t r_h^n,q_h\big)+\big(div\,(\tau_h-q_h\delta),\Delta_t\chi_h^n\big)+\big(as\,(\tau_h),\Delta_t\psi_h^n\big)=0. \tag{3.6.60}$$

Now taking $(\tau_h,q_h)=\big(\varepsilon_h^{n,\frac14},r_h^{n,\frac14}\big)$ in this last equality and using the second equation of (3.6.59), we obtain

$$\frac{1}{2\mu}\big(\Delta_t\varepsilon_h^n,\varepsilon_h^{n,\frac14}\big)+\frac{1}{\lambda}\big(\Delta_t r_h^n,r_h^{n,\frac14}\big)+\big(div\,(\varepsilon_h^{n,\frac14}-r_h^{n,\frac14}\delta),\Delta_t\chi_h^n\big)=0. \tag{3.6.61}$$

The last equation of (3.6.59) with $v_h=\Delta_t\chi_h^n$ gives

$$\big(div\,(\varepsilon_h^{n,\frac14}-r_h^{n,\frac14}\delta),\Delta_t\chi_h^n\big)=\big(-(u_{tt})^{n,\frac14}+\Delta_t^2\widehat{u}_h(t_n)+\Delta_t^2\chi_h^n,\Delta_t\chi_h^n\big). \tag{3.6.62}$$

Substracting (3.6.62) from (3.6.61) we get

$$\frac{1}{2\mu}\big(\Delta_t\varepsilon_h^n,\varepsilon_h^{n,\frac14}\big)+\frac{1}{\lambda}\big(\Delta_t r_h^n,r_h^{n,\frac14}\big)=\big((u_{tt})^{n,\frac14}-\Delta_t^2\widehat{u}_h(t_n)-\Delta_t^2\chi_h^n,\Delta_t\chi_h^n\big).$$

Thus

$$\frac{1}{2\mu}\big(\Delta_t\varepsilon_h^n,\varepsilon_h^{n,\frac14}\big)+\frac{1}{\lambda}\big(\Delta_t r_h^n,r_h^{n,\frac14}\big)+\big(\Delta_t^2\chi_h^n,\Delta_t\chi_h^n\big)=\big((u_{tt})^{n,\frac14}-\Delta_t^2\widehat{u}_h(t_n),\Delta_t\chi_h^n\big). \tag{3.6.63}$$

We expand (3.6.63) to get

$$\begin{aligned}\frac{1}{2\mu}\left(\frac{\varepsilon_h^{n+\frac12}-\varepsilon_h^{n-\frac12}}{\Delta t},\frac{\varepsilon_h^{n+\frac12}+\varepsilon_h^{n-\frac12}}{2}\right)&+\frac{1}{\lambda}\left(\frac{r_h^{n+\frac12}-r_h^{n-\frac12}}{\Delta t},\frac{r_h^{n+\frac12}+r_h^{n-\frac12}}{2}\right)\\&+\left(\frac{\Delta_t\chi_h^{n+\frac12}-\Delta_t\chi_h^{n-\frac12}}{\Delta t},\frac{\Delta_t\chi_h^{n+\frac12}+\Delta_t\chi_h^{n-\frac12}}{2}\right)\\&=\Big((u_{tt})^{n,\frac14}-\Delta_t^2\widehat{u}_h(t_n),\frac{\Delta_t\chi_h^{n+\frac12}+\Delta_t\chi_h^{n-\frac12}}{2}\Big).\end{aligned}$$

So that

$$\begin{aligned}\frac{1}{2\Delta t}\Big(\frac{1}{2\mu}(\|\varepsilon_h^{n+\frac12}\|_{0,\Omega}^2-\|\varepsilon_h^{n-\frac12}\|_{0,\Omega}^2)\Big)&+\frac{1}{2\Delta t}\Big(\frac{1}{\lambda}(\|r_h^{n+\frac12}\|_{0,\Omega}^2-\|r_h^{n-\frac12}\|_{0,\Omega}^2)\Big)\\&+\frac{1}{2\Delta t}\Big(\|\Delta_t\chi_h^{n+\frac12}\|_{0,\Omega}^2-\|\Delta_t\chi_h^{n-\frac12}\|_{0,\Omega}^2\Big)\\&=\Big((u_{tt})^{n,\frac14}-\Delta_t^2\widehat{u}_h(t_n),\frac{\Delta_t\chi_h^{n+\frac12}+\Delta_t\chi_h^{n-\frac12}}{2}\Big).\end{aligned} \tag{3.6.64}$$

Replacing n by j and then summing over time levels and multiplying (3.6.64) by $2\Delta t$ we get

$$\begin{aligned}\frac{1}{2\mu}(\|\varepsilon_h^{n+\frac12}\|_{0,\Omega}^2-\|\varepsilon_h^{\frac12}\|_{0,\Omega}^2)+\frac{1}{\lambda}(\|r_h^{n+\frac12}\|_{0,\Omega}^2-\|r_h^{\frac12}\|_{0,\Omega}^2)+\|\Delta_t\chi_h^{n+\frac12}\|_{0,\Omega}^2-\|\Delta_t\chi_h^{\frac12}\|_{0,\Omega}^2\\=\sum_{j=1}^{n}\Delta t\big((u_{tt})^{j,\frac14}-\Delta_t^2\widehat{u}_h(t_j),\Delta_t\chi_h^{j+\frac12}+\Delta_t\chi_h^{j-\frac12}\big).\end{aligned} \tag{3.6.65}$$

Let us recall that $\varepsilon_h^{\frac{1}{2}} = 0$, $r_h^{\frac{1}{2}} = 0$ and $\Delta_t \chi_h^{\frac{1}{2}} = 0$. Thus (3.6.65) become

$$\frac{1}{2\mu}\|\varepsilon_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|r_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t\chi_h^{n+\frac{1}{2}}\|_{0,\Omega}^2=\Delta t\sum_{j=1}^{n}\left((u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j),\Delta_t\chi_h^{j+\frac{1}{2}}+\Delta_t\chi_h^{j-\frac{1}{2}}\right). \tag{3.6.66}$$

Thus we have

$$\begin{aligned}\frac{1}{2\mu}\|\varepsilon_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|r_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t\chi_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 &\le C\Delta t\sum_{j=1}^{n}\left|\left((u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j),\Delta_t\chi_h^{j+\frac{1}{2}}+\Delta_t\chi_h^{j-\frac{1}{2}}\right)\right|\\ &\le C\Delta t\sum_{j=1}^{n}\|(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\|\Delta_t\chi_h^{j+\frac{1}{2}}+\Delta_t\chi_h^{j-\frac{1}{2}}\|_{0,\Omega}\\ &\le 2C\Delta t\sum_{j=1}^{n}\|(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{2\mu}\|\varepsilon_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|r_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t\chi_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 &\le 2C\Delta t||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}\sum_{j=1}^{n}\|(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_{j,\frac{1}{4}})\|_{0,\Omega}\\ &\le ||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}^2+C^2\Delta t^2\left(\sum_{j=1}^{n}\|(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\right)^2.\end{aligned} \tag{3.6.67}$$

Taking the supremum on n on the left-hand side of (3.6.67) we get

$$\begin{aligned}&\frac{1}{2\mu}||\varepsilon_h||_{L^\infty(L^2)}^2+\frac{1}{\lambda}||r_h||_{L^\infty(L^2)}^2+||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}^2\\ &\qquad\le ||\Delta_t^{\frac{1}{2}}\chi_h||_{L^\infty(L^2)}^2+C^2\Delta t^2\left(\sum_{j=1}^{N}\|(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\right)^2.\end{aligned}$$

This last inequality and assumption on μ and λ yields

$$||\varepsilon_h||_{L^\infty(L^2)}+||r_h||_{L^\infty(L^2)}\lesssim\Delta t\sum_{j=1}^{N}\|(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}. \tag{3.6.68}$$

Now it remains to bound $\|(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}$. We write

$$(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2\widehat{u}_h(t_j)=(u_{tt})^{j,\frac{1}{4}}-\Delta_t^2u(t_j)+\Delta_t^2u(t_j)-\Delta_t^2\widehat{u}_h(t_j)$$

Like in the proof of Theorem (3.6.3), we denote by R_h the elliptic projection operator. We can write

$$\Delta_t^2u(t_j)-\Delta_t^2\widehat{u}_h(t_j)=(I-R_h)\Delta_t^2u(t_j)=(I-R_h)\frac{u(t_{j+1})-2u(t_j)+u(t_{j-1})}{\Delta t^2}.$$

By Taylor expansion we have

$$u(t_{j-1})=u(t_j)-\Delta t\ u_t(t_j)+\frac{\Delta t^2}{2}u_{tt}(t_j)+\frac{1}{2}\int_{t_j}^{t_{j-1}}(t_{j-1}-s)^2u_{ttt}(s)\ ds.$$

and

$$u(t_{j+1}) = u(t_j) + \Delta t\ u_t(t_j) + \frac{\Delta t^2}{2} u_{tt}(t_j) + \frac{1}{2} \int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 u_{ttt}(s)\ ds.$$

So that

$$\begin{aligned} \Delta_t^2 u(t_j) &= u_{tt}(t_j) + \frac{1}{2\Delta t^2} \Big(\int_{t_j}^{t_{j-1}} (t_{j-1} - s)^2 u_{ttt}(s)\ ds + \int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 u_{ttt}(s)\ ds \Big) \\ &= u_{tt}(t_j) + \frac{1}{2\Delta t^2} \Big(-\int_{-\Delta t}^{0} (\Delta t + t)^2 u_{ttt}(t + t_j)\ dt + \int_{0}^{\Delta t} (\Delta t - t)^2 u_{ttt}(t + t_j)\ dt \Big). \end{aligned} \tag{3.6.69}$$

Thus

$$\begin{aligned} (I - R_h)\Delta_t^2 u(t_j) = (I - R_h) u_{tt}(t_j) + \frac{1}{2\Delta t^2} \Big(&- \int_{-\Delta t}^{0} (\Delta t + t)^2 (I - R_h) u_{ttt}(t + t_j)\ dt \\ &+ \int_{0}^{\Delta t} (\Delta t - t)^2 (I - R_h) u_{ttt}(t + t_j)\ dt \Big). \end{aligned}$$

So that

$$\begin{aligned} \|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} &\le \|(I - R_h) u_{tt}(t_j)\|_{0,\Omega} + \frac{1}{2} \frac{\sqrt{\Delta t}}{\sqrt{5}} \Big[\Big(\int_{-\Delta t}^{0} \|(I - R_h) u_{ttt}(t + t_j)\|_{0,\Omega}^2\ dt \Big)^{\frac{1}{2}} \\ &\qquad + \Big(\int_{0}^{\Delta t} \|(I - R_h) u_{ttt}(t + t_j)\|_{0,\Omega}^2\ dt \Big)^{\frac{1}{2}} \Big] \\ &\le \|(I - R_h) u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{2}}{2} \frac{\sqrt{\Delta t}}{\sqrt{5}} \Big(\int_{-\Delta t}^{\Delta t} \|(I - R_h) u_{ttt}(t + t_j)\|_{0,\Omega}^2\ dt \Big)^{\frac{1}{2}} \\ &= \|(I - R_h) u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{\Delta t}}{\sqrt{10}} \Big(\int_{-\Delta t + t_j}^{\Delta t + t_j} \|(I - R_h) u_{ttt}(s)\|_{0,\Omega}^2\ ds \Big)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} \le \|(I - R_h) u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{\Delta t}}{\sqrt{10}} \Big(\int_{-\Delta t + t_j}^{\Delta t + t_j} \|(I - R_h) u_{ttt}(s)\|_{0,\Omega}^2\ ds \Big)^{\frac{1}{2}}.$$

From the estimate (3.5.22) for the elliptic projection we obtain:

$$\|(I - R_h) u_{tt}(t_j)\|_{0,\Omega} \lesssim h \Big[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} \Big],$$

and in view of remark (3.5.8)

$$\begin{aligned} \Delta t \int_{-\Delta t + t_j}^{\Delta t + t_j} \|(I - R_h) u_{ttt}(s)\|_{0,\Omega}^2\ ds &\lesssim (\Delta t) h^2 \Big[\int_{-\Delta t + t_j}^{\Delta t + t_j} \Big(|u_{ttt}(s)|_{1,1;\phi,\Omega}^2 + |u_{ttt}(s)|_{1,\Omega}^2 \Big)\ ds \Big] \\ &\lesssim \Delta t^2 h^2 \Big[|u_{ttt}|_{L^\infty(H_\phi^{1,1})}^2 + |u_{ttt}|_{L^\infty(H^1)}^2 \Big]. \end{aligned}$$

Thus

$$\sqrt{\Delta t}\left(\int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I-R_h)u_{ttt}(s)\|_{0,\Omega}^2\ ds\right)^{\frac{1}{2}} \lesssim h\Big[|u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)}\Big].$$

Combining these last inequalities, we get

$$\begin{aligned}\Delta t\sum_{j=1}^{N}\|(I-R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} &\lesssim h\Big[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} \\ &\qquad + |u_{ttt}|_{L^\infty(H^1)}\Big] \times \sum_{j=1}^{N}\Delta t \\ &\lesssim h\Big[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)}\Big].\end{aligned}$$

Now, for the term $(u_{tt})^{j,\frac{1}{4}} - \Delta_t^2 u(t_j)$, we can write

$$(u_{tt})^{j,\frac{1}{4}} - \Delta_t^2 u(t_j) = (u_{tt})^{j,\frac{1}{4}} - u_{tt}(t_j) + u_{tt}(t_j) - \Delta_t^2 u(t_j). \tag{3.6.70}$$

Using once again Taylor expansion we get

$$u_{tt}(t_{j-1}) = u_{tt}(t_j) - \Delta t\ u_{ttt}(t_j) + \int_{t_j}^{t_{j-1}} (t_{j-1}-s)u_{tttt}(s)\ ds.$$

and

$$u_{tt}(t_{j+1}) = u_{tt}(t_j) + \Delta t\ u_{ttt}(t_j) + \int_{t_j}^{t_{j+1}} (t_{j+1}-s)u_{tttt}(s)\ ds.$$

Together we get

$$u_{tt}(t_{j+1}) + u_{tt}(t_{j-1}) = 2u_{tt}(t_j) + \int_{t_j}^{t_{j+1}} (t_{j+1}-s)u_{tttt}(s)\ ds + \int_{t_j}^{t_{j-1}} (t_{j-1}-s)u_{tttt}(s)\ ds.$$

Thus

$$\begin{aligned}(u_{tt})^{j,\frac{1}{4}} - u_{tt}(t_j) &= \frac{1}{4}\left(\int_{t_j}^{t_{j+1}} (t_{j+1}-s)u_{tttt}(s)\ ds + \int_{t_j}^{t_{j-1}} (t_{j-1}-s)u_{tttt}(s)\ ds\right) \\ &= \frac{1}{4}\left(\int_0^{\Delta t} (\Delta t - t)u_{tttt}(t_j+t)\ dt + \int_{-\Delta t}^{0} (\Delta t + t)u_{tttt}(t_j+t)\ dt\right).\end{aligned}$$

Hence

$$\begin{aligned}\|(u_{tt})^{j,\frac{1}{4}} - u_{tt}(t_j)\|_{0,\Omega} &\le \frac{1}{4}\frac{\Delta t^{\frac{3}{2}}}{\sqrt{3}}\left[\left(\int_{-\Delta t}^{0} \|u_{tttt}(t+t_j)\|_{0,\Omega}^2\ dt\right)^{\frac{1}{2}} + \left(\int_0^{\Delta t} \|u_{tttt}(t+t_j\|_{0,\Omega}^2\ dt\right)^{\frac{1}{2}}\right] \\ &\lesssim \Delta t^{\frac{3}{2}}\left(\int_{-\Delta t}^{\Delta t} \|u_{tttt}(t+t_j)\|_{0,\Omega}^2\ dt\right)^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\|(u_{tt})^{j,\frac{1}{4}} - u_{tt}(t_j)\|_{0,\Omega} \lesssim \Delta t^{\frac{3}{2}} \left(\int_{-\Delta t + t_j}^{\Delta t + t_j} \|u_{tttt}(s)\|_{0,\Omega}^2 \, ds \right)^{\frac{1}{2}}. \tag{3.6.71}$$

So that

$$\begin{aligned} \Delta t \sum_{j=1}^{N} \|(u_{tt})^{j,\frac{1}{4}} - u_{tt}(t_j)\|_{0,\Omega} &\lesssim \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)} \sum_{j=1}^{N} \Delta t \\ &\lesssim \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}. \end{aligned} \tag{3.6.72}$$

From (3.6.38) we get an estimation of the last two terms in the right-hand side of (3.6.70) as

$$\Delta t \sum_{j=1}^{N} \|\Delta_t^2 u(t_j) - u_{tt}(t_j)\|_{0,\Omega} \lesssim \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}. \tag{3.6.73}$$

Finally (3.6.73), (3.6.72) and triangle inequality in (3.6.68) yield

$$\begin{aligned} ||\varepsilon_h||_{L^\infty(L^2)} + ||r_h||_{L^\infty(L^2)} \lesssim h\Big[&|u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)} \Big] \\ &+ \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}. \end{aligned}$$

By the triangle inequality and taking the supremum on t in (3.5.15) we then find

$$\begin{aligned} ||\sigma - \sigma_h||_{L^\infty(L^2)} + ||p - p_h||_{L^\infty(L^2)} \lesssim h\Big[&|u|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} \\ &+ |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)} \Big] + \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}. \end{aligned}$$

By the inf-sup and the triangle inequality, we get

$$\begin{aligned} \|\omega(t_n) - \omega_h^n\|_{0,\Omega} + \|P_h^0 u(t_n) - u_h^n\|_{0,\Omega} \lesssim \Big[&\|\sigma(t_n) - \sigma_h^n\|_{0,\Omega} + \|p(t_n) - p_h^n\|_{0,\Omega} \\ &+ \|P_h^1 \omega(t_n) - \omega(t_n)\|_{0,\Omega} \Big]. \end{aligned}$$

Taking the supremum on n we get

$$\begin{aligned} ||\omega - \omega_h||_{L^\infty(L^2)} + ||P_h^0 u - u_h||_{L^\infty(L^2)} \lesssim \Big[&||\sigma - \sigma||_{L^\infty(L^2)} + ||p - p_h||_{L^\infty(L^2)} \\ &+ ||P_h^1 \omega - \omega||_{L^\infty(L^2)} \Big]. \end{aligned}$$

By the propriety of the projection operator (3.5.10) we hence find

$$\begin{aligned} ||\omega - \omega_h||_{L^\infty(L^2)} + ||P_h^0 u - u_h||_{L^\infty(L^2)} \lesssim h\Big[&|u|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} \\ &+ |u_{ttt}|_{L^\infty(H^1)} \Big] + \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}. \end{aligned}$$

The triangle inequality and the propriety of the projection operator (1.47) p.27 of [36], give

$$||u-u_h||_{L^\infty(L^2)} \lesssim h\Big[|u|_{L^\infty(H^{1,1}_\phi)} + |u|_{L^\infty(H^1)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)}\Big] + \Delta t^2 ||u_{tttt}||_{L^\infty(L^2)}.$$

■

3.7 Conclusion

We have constructed and analysed a finite element method for approximating the elastodynamic system using dual mixed formulation for spacial discretization. Our formulation requires less regularity on displacement than standard one.

Optimal order L^∞-in-time L^2-in-space a priori error estimates are derived and a quadratic convergence rate in time for the fully discretized scheme has been established for both the explicit and the implicit numerical schemes.

3.8 Implementation and numerical results

In this section win view of testing our theoretical analysis, we first introduce the so called "Hybrid formulation" [8, 15, 33] for solving the system of the mixed elastodynamic problems 3.6.2 -3.6.4 and 3.6.39 -3.6.40 . The numerical results are presented on a L-shaped domain. Given $f : [0,T]\times\Omega \longmapsto \mathbb{R}^2$ and a surface force density $g : [0,T]\times\Gamma_N \longmapsto \mathbb{R}^2$ and the initial conditions on displacements and velocities u_0 and u_1, the displacement field $u = (u_1, u_2)$ satisfies the following equations :

$$\left\{\begin{array}{ccccc} u_{tt} - div\,\sigma_s(u) & = & f & \text{in} & [0,T]\times\Omega, \\ u & = & 0 & \text{on} & [0,T]\times\Gamma_D, \\ \sigma_s(u).n & = & g & \text{on} & [0,T]\times\Gamma_N, \\ u(0,.) & = & u_0 & & \text{in} \quad \Omega, \\ u_t(0,.) & = & u_1 & & \text{in} \quad \Omega, \end{array}\right. \tag{3.8.1}$$

3.8.1 Explicit-in-time scheme

Hybrid formulation

We first introduce the enlarged space $\widetilde{\Sigma}_h$ (with respect to $\Sigma_{h,0}$) by suppressing the requirement for its elements to have continuous normal component at the interfaces of the triangulation $\mathcal{T}_h$:

$$\widetilde{\Sigma}_h := \{(\tau_h, q_h) \in L^{2\times 2}(\Omega)\times L^2(\Omega);\ \forall T \in \mathcal{T}_h :\ q_{h|T} \in \mathbb{P}_1(T) \ \text{ and } \\ (\tau_h - q_h\delta)_{|T} \in [\mathbb{P}_1(K)]^{2\times 2} \oplus [\mathbb{R}\,Curl\ b_T]^2\}.$$

We introduce the following hybrid formulation: Find $(\widetilde{\sigma}_h^{n+1}, \widetilde{p}_h^{n+1}, \lambda_h^{n+1}) \in \widetilde{\Sigma}_h \times \Lambda_h$ and $(\widetilde{u}_h^{n+1}, \widetilde{\omega}_h^{n+1}) \in V_h \times W_h$ such that

$$\widetilde{u}_h^0 = \hat{u}_h(0),\ \widetilde{u}_h^{-1} = \hat{u}_h(-\Delta t) \simeq \hat{u}_h(0) - \Delta t\ \hat{u}_{h,t}(0) + \frac{\Delta t^2}{2}\hat{u}_{h,tt}(0),$$

$$\begin{cases} \dfrac{1}{2\mu}(\widetilde{\sigma}_h^{n+1}, \tau_h) \ + \ \dfrac{1}{\lambda}(\widetilde{p}_h^{n+1}, q_h) + \displaystyle\sum_{K\in\mathcal{T}_h}\int_K div\ (\tau_h - q_h\delta).\widetilde{u}_h^{n+1}\ dx \ + \ (as\ (\tau_h), \widetilde{\omega}_h^{n+1}) \\ -\displaystyle\sum_{K\in\mathcal{T}_h}\int_{\partial K}\lambda_h^{n+1}(\tau_h - q_h\delta).n_K\ ds = 0, \quad \forall\ (\tau_h, q_h) \in \widetilde{\Sigma}_h, \quad n \geq -1, \\ (\Delta_t^2\widetilde{u}_h^n, v_h) - \displaystyle\sum_{K\in\mathcal{T}_h}\int_K div\ (\widetilde{\sigma}_h^n - \widetilde{p}_h^n\delta).v_h\ dx \ - \ (as\ (\widetilde{\sigma}_h^{n+1}), \theta_h) - (f^n, v_h) = 0, \\ \quad \forall\ (v_h, \theta_h) \in V_h \times W_h, \quad n \geq 0, \\ \displaystyle\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mu_h(\widetilde{\sigma}_h^{n+1} - \widetilde{p}_h^{n+1}\delta).n_K\ ds = \sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Gamma_N}\mu_h.g^{n+1}\ ds, \quad \forall\ \mu_h \in \Lambda_h, \quad n \geq -1, \end{cases} \tag{3.8.2}$$

where

$$\Lambda_h := \{\mu_h \in [L^2(\mathcal{E}_h)]^2;\ \mu_{h|e} \in [\mathbb{P}_1(e)]^2\ \forall e \in \mathcal{E}_h \text{ and } \mu_{h|e} = 0\ \forall e \in \Gamma_D\}.$$

$\mathcal{E}_h$ denotes the set of all edges in $\mathcal{T}_h$ and $V_h \times W_h$ is defined in (3.5.2). Let us note that the hybrid formulation (3.8.2) is equivalent to the formulation (3.6.4) in the sense that $\widetilde{\sigma}_h^n = \sigma_h^n$, $\widetilde{p}_h^n = p_h^n$, $\widetilde{u}_h^n = u_h^n$ and $\widetilde{\omega}_h^n = \omega_h^n$. Taking as a test function $(v_h, 0)$ in the third equation of system (3.8.2) we get explicitly u_h^{n+1}

$$u_{h|K}^{n+1} = \Delta t^2\big[div\ (\sigma_{h|K}^n - \sigma_{h|K}^n\delta) + P_{h|K}^0 f^n\big] + 2u_{h|K}^n - u_{h|K}^{n-1}, \quad n \geq 0.$$

Still noting by σ_h^{n+1}, p_h^{n+1}, u_h^{n+1}, ω_h^{n+1} and λ_h^{n+1} the vectors of the degrees of freedom of these same unknowns, the algebraic equations generated by (3.8.2) takes the following form

$$\begin{cases} A\sigma_h^{n+1} - E^T\lambda_h^{n+1} + H^T\omega_h^{n+1} = -F_1^{n+1}, \\ Pp_h^{n+1} + G^T\lambda_h^{n+1} = F_2^{n+1}, \\ H\sigma_h^{n+1} = 0, \\ E\sigma_h^{n+1} - Gp_h^{n+1} = F_3^{n+1}, \end{cases} \tag{3.8.3}$$

where $A,\ E,\ H,\ P$ are the corresponding matrices of the bilinear forms of the different terms in system (3.8.2), and $F_1^{n+1},\ F_2^{n+1}$ are vectors at the $n+1$ time step obtained by replacing the variable u_h^{n+1} in the first equation of system (3.8.2) by its value obtained from the second equation and putting these terms in the right-hand side. Finally F_3^{n+1} correspond to the traction on the Neumann boundary Γ_N at the $n+1$ time step. In the system (3.8.3), we start by eliminating σ_h^{n+1} and p_h^{n+1} and after we eliminate ω_h^{n+1}. These eliminations are made element by element. This procedure done, we arrive at the following system:

$$R\lambda_h^{n+1} = F^{n+1}, \qquad (3.8.4)$$

where

$$R = EA^{-1}E^T - EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}E^T + GP^{-1}G^T,$$

and

$$F^{n+1} = EA^{-1}F_1^{n+1} + GP^{-1}F_2^{n+1} - EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}F_1^{n+1} + F_3^{n+1}.$$

The matrix R is symmetric and positive definite.

Numerical test

We now present some numerical results on a test problem in the L-shaped domain $\Omega =]-1,1[^2\backslash[0,1[\times]-1,0]$ which models one singularity arising at the re-entrant corner . The numerical tests are performed with $T = 1s$. Using polar coordinates (r,θ), $0 \leq \theta \leq \omega := \frac{3\pi}{2}$, which are centered at the re-entrant corner, the analytical solution is

$$u(r,\theta) = e^{-t} r^{\alpha} \overrightarrow{\phi}_{\alpha}(\theta)$$

where

$$\overrightarrow{\phi}_{\alpha}(\theta)_1 = C_1(\rho+\tau)\{\cos(\alpha-2)\theta - \cos(\alpha\theta)\} + C_2((\rho+\tau)sin(\alpha-2)\theta + (\tau-3\rho)sin(\alpha\theta)),$$

$$\overrightarrow{\phi}_{\alpha}(\theta)_2 = C_1(-(\rho+\tau)\sin(\alpha-2)\theta + (3\rho-\tau)\sin(\alpha\theta)) + C_2(\rho+\tau)\{\cos(\alpha-2)\theta - \cos(\alpha\theta)\}.$$

The parameters are

$$C_1 = (\rho+\tau)\sin(\alpha-2)\omega - (3\tau-\rho)\sin(\alpha\omega),$$

$$C_2 = (\rho+\tau)\{\cos(\alpha\omega) - \cos(\alpha-2)\omega\},$$

$$\rho = \frac{\lambda+\mu}{\mu}(\alpha-1) - 2, \quad \tau = \frac{\lambda+\mu}{\mu}(\alpha+1) + 2,$$

where $\alpha = 0.54872335366$ is the smallest strictly positive solution of the transcendental equation (1.3.1) for $\omega = \frac{3\pi}{2}$, $\lambda = 1000$, $\mu = \lambda/49$. We fix $\Delta t = 10^{-5}s$ and $N = \frac{T}{\Delta t} = 10^5$. All numerical results will be presented at the final time $T = 1s$ ($N = 10^5$). The initial conditions u_h^0 and u_h^{-1} are chosen as the elliptic projection of $u(0)$ and $u(-\Delta t)$ respectively as follows $u_h^0 = \hat{u}_h(0)$, $u_h^{-1} = \hat{u}_h(-\Delta t)$. We use two kinds of meshes. The first one (uniform) is obtained by dividing the intervals $[0,1]$ and $[-1,0]$ into n subintervals of

Table 3.1: Convergence results when using uniform meshes at $T = 1s$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
2.828427e-001	7.379843e-002	3.733002e-001	6.127325e-002	1.263216e-001
1.414214e-001	5.071551e-002	2.513737e-001	3.054023e-002	8.408249e-002
9.428090e-002	4.070448e-002	2.002361e-001	2.027724e-002	6.671641e-002
7.071068e-002	3.480387e-002	1.706050e-001	1.518150e-002	5.674623e-002
5.656854e-002	3.081811e-002	1.507419e-001	1.212897e-002	5.008866e-002

length $\frac{1}{n}$, and then each square is divided into triangles (see figure 3.1 where we have chosen n=10). The second kind of meshes (refined) is obtained from the first one by refinement near $(0,0)$ according to Raugel's procedure [34]. Namely, Ω is divided into six big triangles; on the three ones which do not contain $(0,0)$, a uniform mesh is used; each big triangle containing $(0,0)$ is divided according to the ratios $(\frac{i}{n})^n$, $1 \leq i \leq n$, where $\beta \geq \frac{1}{(1-\alpha)}$ along the sides which end at $(0,0)$ and finally divide uniformly each of these strips (see figure 3.2 where we have chosen n=10 and $\beta = 1.8$). We then represent the variations of the errors

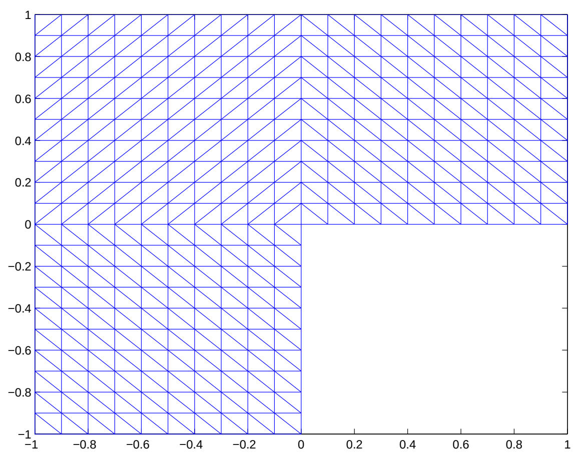


Figure 3.1: Uniform meshes

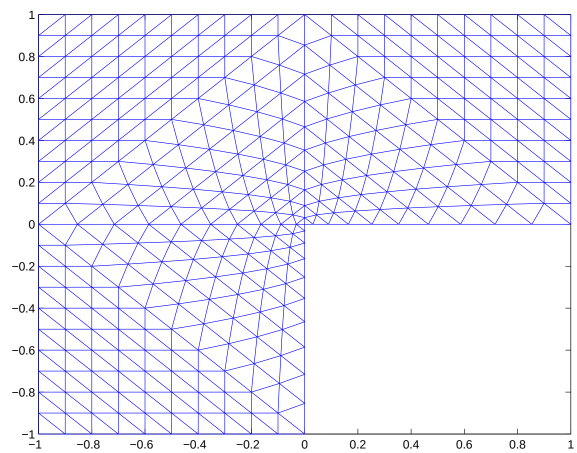


Figure 3.2: Refined meshes

$\|\sigma_h^N - \sigma(T)\|_{0,\Omega}$, $\|p_h^N - p(T)\|_{0,\Omega}$, $\|u_h^N - u(T)\|_{0,\Omega}$ and $\|\omega_h^N - \omega(T)\|_{0,\Omega}$, with respect to the mesh size h, in figure 3.3 and figure 3.4. A double logarithmic scale was used such that the slope of the curves yields the order of convergence $O(h)$ for refinement meshes (see figure 3.4) according to the theoretical results, and $O(h^{\frac{2}{3}})$ for uniforming meshes (see figure 3.4) due to the singular behavior of the solution. In Table 3.1 and Table 3.2 we summarize the results on the errors for both the uniform and refined meshes.

Let us mention that in order to fit the complex geometry of the boundary the mesh may be refined according to the rules among R_1 - R_3, and so contain element of very small size, which imply, because of the CFL stability condition, the use of a very small time step. Explicit in time scheme is thus more appropriate when we are interested by the behavior of the wave in the neighborhoods of the initial conditions.

Table 3.2: Convergence results when using refined meshes at $T = 1s$

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
3.307907e-001	4.706629e-002	1.943443e-001	5.413231e-002	6.239506e-002
1.727505e-001	2.408332e-002	9.796571e-002	2.686243e-002	3.111419e-002
1.167855e-001	1.634564e-002	6.628362e-002	1.806031e-002	2.100029e-002
8.819391e-002	1.219502e-002	4.939540e-002	1.340491e-002	1.563465e-002
7.084489e-002	9.788028e-003	3.962408e-002	1.072136e-002	1.253589e-002

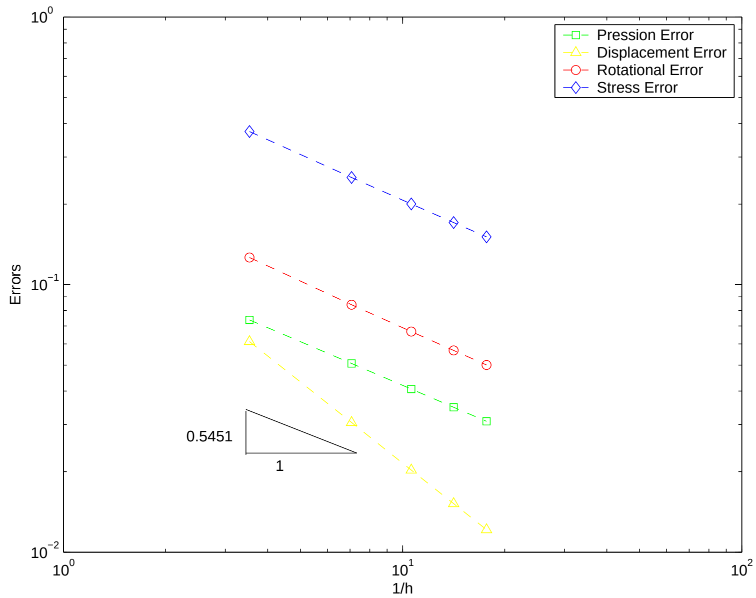


Figure 3.3: Errors as a function of $1/h$ for uniform meshes.

3.8.2 Implicite-in-time scheme

Hybrid formulation

The hybrid formulation corresponding to the mixed formulation (3.6.40) is of the following:

Find $(\widetilde{\sigma}_h^{n+1}, \widetilde{p}_h^{n+1}, \lambda_h^{n+1}) \in \widetilde{\Sigma}_h \times \Lambda_h$ and $(\widetilde{u}_h^{n+1}, \widetilde{\omega}_h^{n+1}) \in V_h \times W_h$ such that

$$\widetilde{u}_h^0 = \hat{u}_h(0), \;\; \widetilde{u}_h^1 = \hat{u}_h(\Delta t) \simeq \hat{u}_h(0) + \Delta t \;\hat{u}_{h,t}(0) + \frac{\Delta t^2}{2}\hat{u}_{h,tt}(0),$$

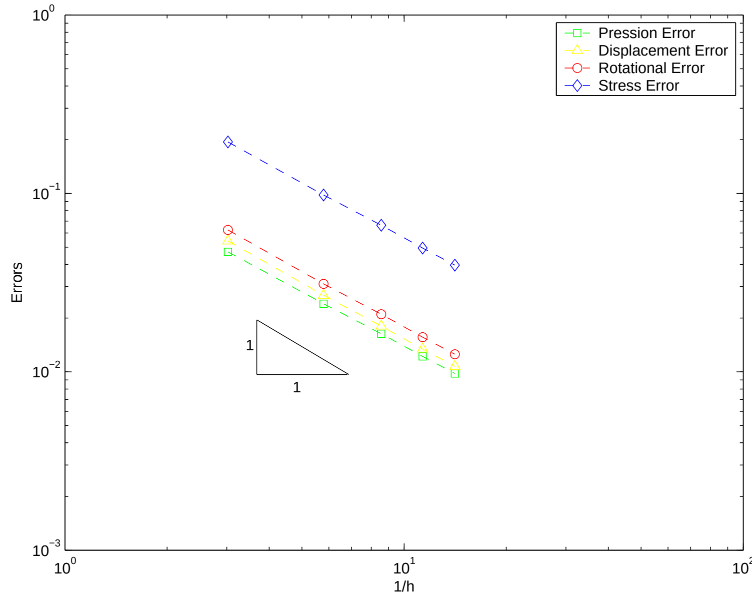


Figure 3.4: Errors as a function of $1/h$ for refined meshes.

$$
\left\{
\begin{array}{l}
\dfrac{1}{2\mu}(\widetilde{\sigma}_h^{n+1}, \tau_h) \;+\; \dfrac{1}{\lambda}(\widetilde{p}_h^{n+1}, q_h) + \displaystyle\sum_{K\in\mathcal{T}_h}\int_K div\,(\tau_h - q_h\delta).\widetilde{u}_h^{n+1}\,dx \;+\; (as\,(\tau_h), \widetilde{\omega}_h^{n+1}) \\[2ex]
-\displaystyle\sum_{K\in\mathcal{T}_h}\int_{\partial K}\lambda_h^{n+1}(\tau_h - q_h\delta).n_K\,ds = 0, \quad \forall\,(\tau_h, q_h)\in\widetilde{\Sigma}_h, \quad n\geq -1, \\[2ex]
(as\,(\widetilde{\sigma}_h^{n+1}), \theta_h) = 0, \quad \forall\,\theta_h\in W_h, \quad n\geq -1, \\[2ex]
(\Delta_t^2\widetilde{u}_h^n, v_h) - \displaystyle\sum_{K\in\mathcal{T}_h}\int_K div\,(\widetilde{\sigma}_h^{n,\frac{1}{4}} - \widetilde{p}_h^{n,\frac{1}{4}}\delta).v_h\,dx - (f^{n,\frac{1}{4}}, v_h) = 0, \\[2ex]
\quad\forall\,v_h\in V_h, \quad n\geq 1, \\[2ex]
\displaystyle\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mu_h(\widetilde{\sigma}_h^{n+1} - \widetilde{p}_h^{n+1}\delta).n_K\,ds = \sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Gamma_N}\mu_h.g^{n+1}\,ds, \quad \forall\,\mu_h\in\Lambda_h, \quad n\geq -1.
\end{array}
\right.
\tag{3.8.5}
$$

The corresponding algebraic system takes the following form

$$\begin{cases} A\sigma_h^{n+1} + C^T u_h^{n+1} - E^T \lambda_h^{n+1} + H^T \omega_h^{n+1} = 0, \\ Pp_h^{n+1} - B^T u_h^{n+1} + G^T \lambda_h^{n+1} = 0, \\ H\sigma_h^{n+1} = 0, \\ Mu_h^{n+1} - C\sigma_h^{n+1} + Bp_h^{n+1} = F^{n+1}, \\ E\sigma_h^{n+1} - Gp_h^{n+1} = T^{n+1}, \end{cases} \tag{3.8.6}$$

where $A,\ E,\ H,\ P,\ M$ are the corresponding matrices of the bilinear forms of the different terms appearing in system (3.8.5), and F^{n+1} is the second member vector at the $n+1$ time step obtained by putting, in the right-hand side of the fourth equation of system (3.8.5), the massic force term and all terms former to the time step n+1. Thus we get the explicit form of F^{n+1}:

$$F_{|K}^{n+1} = 4P_{h|K}^0 f^{n,\frac{1}{4}} + div\ (2(\sigma_{h|K}^n - p_{h|K}^n\delta) + (\sigma_{h|K}^{n-1} - p_{h|K}^{n-1}\delta)) + \frac{4}{\Delta t^2}(2u_{h|K}^n - u_{h|K}^{n-1}).$$

Finally T^{n+1} corresponds to the traction on the Neumann boundary Γ_N at the $n+1$ time step. The quantity σ_h^{n+1}, p_h^{n+1}, ω_h^{n+1} and λ_h^{n+1} for $n=-1$ and $n=0$, needed to start the implicit scheme (3.8.6), are obtained by resolving the following system at $n=-1$ and $n=0$:

$$\begin{cases} A\sigma_h^{n+1} - E^T \lambda_h^{n+1} + H^T \omega_h^{n+1} = -F_1^{n+1}, \\ Pp_h^{n+1} + G^T \lambda_h^{n+1} = F_2^{n+1}, \\ H\sigma_h^{n+1} = 0, \\ E\sigma_h^{n+1} - Gp_h^{n+1} = T^{n+1}, \end{cases} \tag{3.8.7}$$

where F_1^{n+1}, F_2^{n+1} are second members vector at the $n+1$ time step obtained by replacing the variable u_h^{n+1} by its initial value in the first and second equations of system (3.8.6) and putting these terms in the right-hand side. In system (3.8.6), we start by eliminating σ_h^{n+1} and p_h^{n+1} and after we eliminate firstly ω_h^{n+1} and secondly u_h^{n+1}. These eliminations are made element by element. After this procedure, we find the following system:

$$R\lambda_h^{n+1} = \mathcal{F}^{n+1}, \tag{3.8.8}$$

where

$$\begin{aligned} R = {} & GP^{-1}G^T + EA^{-1}E^T - EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}E^T - \\ & (EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}C^T - EA^{-1}C^T + GP^{-1}B^T)(M + CA^{-1}C^T \\ & + BP^{-1}B^T - CA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}C^T)^{-1}(CA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}E^T \\ & - CA^{-1}E^T - BP^{-1}G^T), \end{aligned}$$

and

$$\mathcal{F}^{n+1} = T^{n+1} - (EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}C^T - EA^{-1}C^T - GP^{-1}B^T)(M + CA^{-1}C^T + BP^{-1}B^T - CA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}C^T)^{-1}F^{n+1}.$$

The matrix R is symmetric and positive definite.

Numerical test

We represent the variations of the errors $\|\sigma_h^N - \sigma(T)\|_{0,\Omega}$, $\|p_h^N - p(T)\|_{0,\Omega}$, $\|u_h^N - u(T)\|_{0,\Omega}$ and $\|\omega_h^N - \omega(T)\|_{0,\Omega}$, with respect to the mesh size h, in figure 3.5 and fig 3.6. The time step Δt is selected equally to the mesh size h. A double logarithmic scale was used such that the slope of the curves yields the order of convergence $O(h)$ for refinement meshes (see figure 3.6) according to the theoretical results, and $O(h^{\frac{2}{3}})$ for uniforming meshes (see figure 3.5) due to the singular behavior of the solutions. In Table 3.4 and Table 3.5 we summarize the results on the errors in both the uniform and refined meshes. Let us mention that the implicit in time scheme possesses no limitation on the time step and convergence results seem to be better than ones in the case of explicit in time scheme. That thus suggests that the implicit in time scheme is preferable to the explicit in time scheme.

Table 3.3: Convergence results when using uniform meshes at $T = 1s$ with $\Delta t = h$

h=Δt	Pressure errors	Strain errors	Displacement errors	Rotational errors
2.828427e-001	4.950115e-002	2.379090e-001	3.944810e-002	7.684252e-002
1.414214e-001	3.416687e-002	1.481635e-001	1.794047e-002	4.790442e-002
9.428090e-002	2.746780e-002	1.223985e-001	1.223487e-002	3.985001e-002
7.071068e-002	2.288484e-002	1.021814e-001	8.927111e-003	3.337248e-002
5.656854e-002	2.022493e-002	9.194350e-002	7.247770e-003	3.013416e-002

Table 3.4: Convergence results when using refined meshes at $T = 1s$ with $\Delta t = h$

h=Δt	Pressure errors	Strain errors	Displacement errors	Rotational errors
3.307907e-001	2.957709e-002	1.164499e-001	3.349528e-002	3.736893e-002
1.727505e-001	1.574251e-002	6.344335e-002	1.751535e-002	2.017349e-002
1.167855e-001	1.017447e-002	4.111925e-002	1.124376e-002	1.304880e-002
8.819391e-002	7.523381e-003	3.039981e-002	8.271745e-003	9.635101e-003
7.084489e-002	5.969955e-003	2.412921e-002	6.540737e-003	7.641664e-003

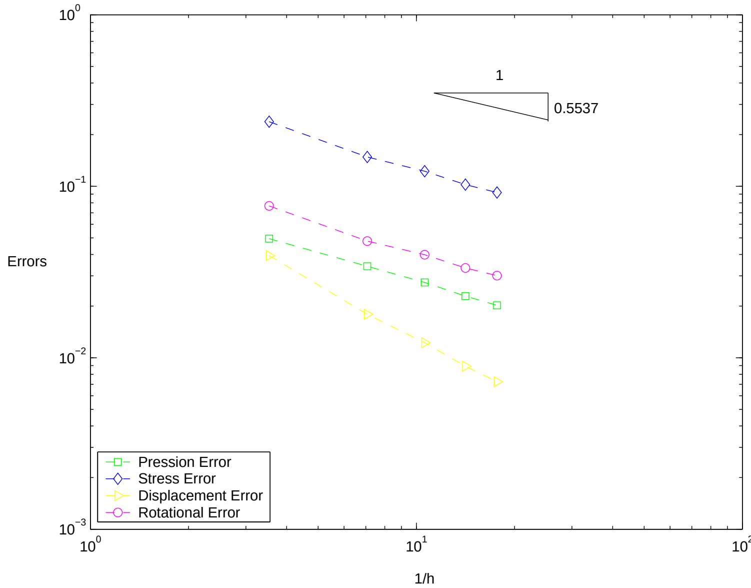


Figure 3.5: Errors as a function of $1/h$ for uniform meshes with time step $\Delta t = h$.

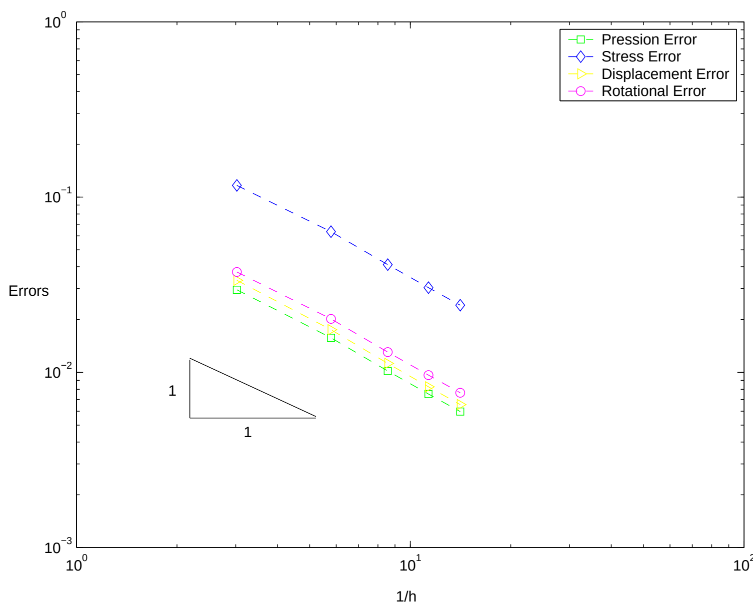


Figure 3.6: Errors as a function of $1/h$ for refined meshes with time step $\Delta t = h$.

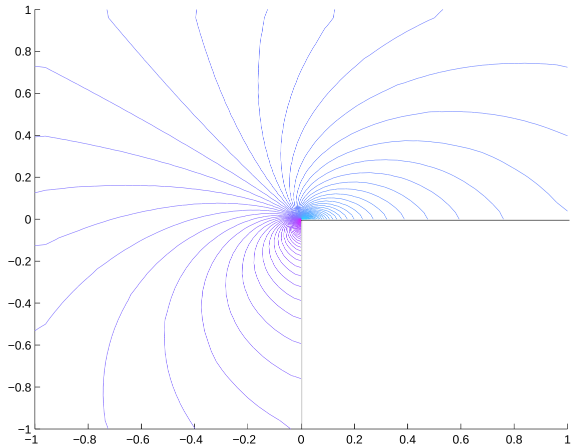


Figure 3.7: Streamlines of the Pressure

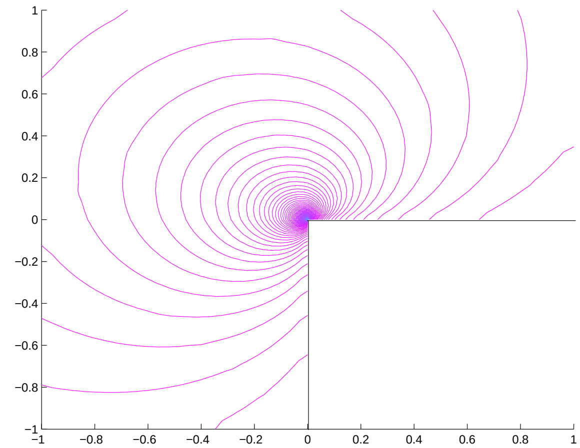


Figure 3.8: Streamlines of the Rotational

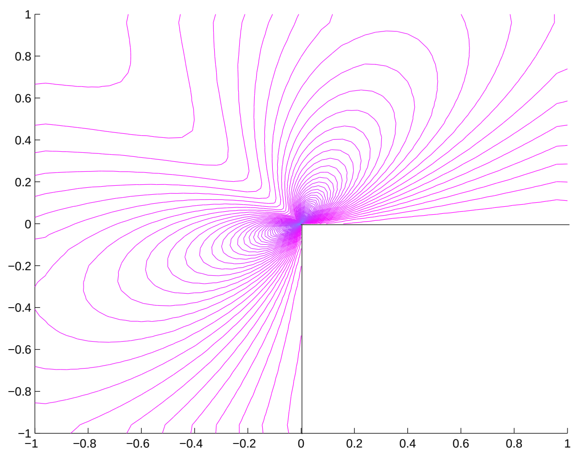


Figure 3.9: Streamlines of the Strain in the x direction

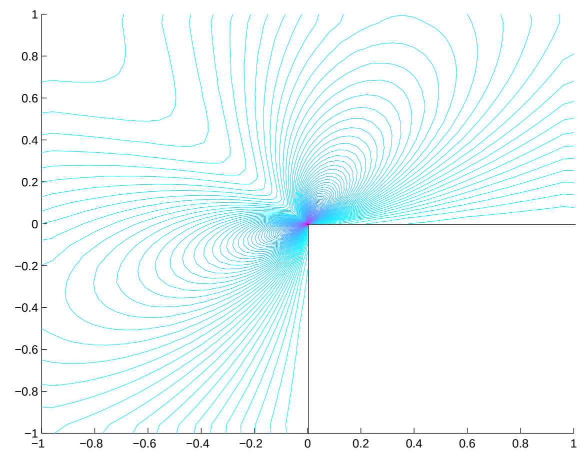


Figure 3.10: Streamlines of the Strain in the y direction

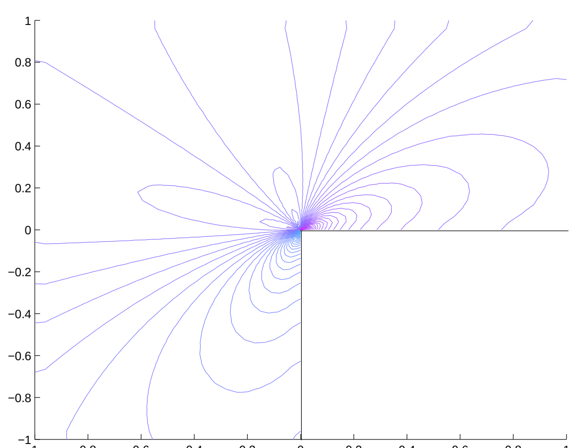


Figure 3.11: Streamlines of $\sigma_{1,2}$

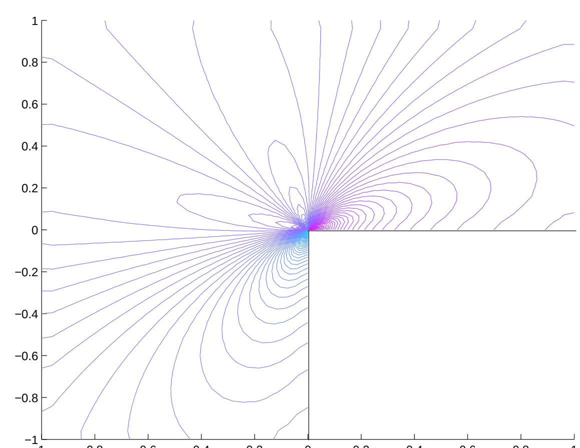


Figure 3.12: Streamlines of $\sigma_{2,1}$

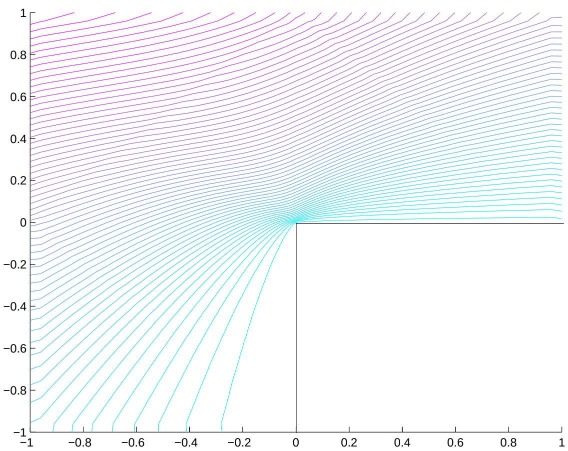


Figure 3.13: Streamlines of the Displacement in the x direction

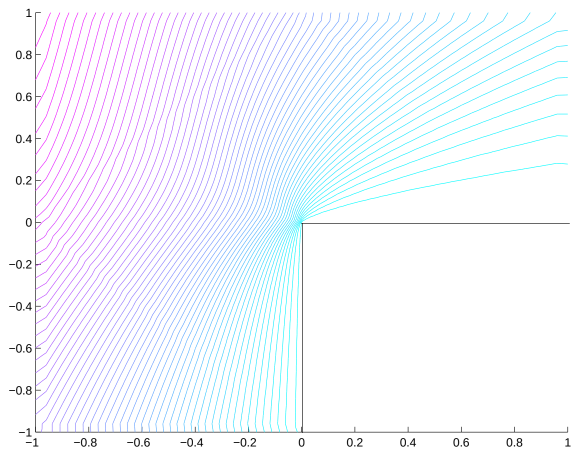


Figure 3.14: Streamlines of the Displacement in the y direction

4

Conclusions and perspectives

In this work, we study the refinement of grids for the dual mixed finite element method for two types of problems: the first one concerns the linear elasticity problem and the second one the linear elastodynamic problem.

For these two types of problems and in nonregular domains, the mixed finite element methods analyzed until now relate to the primal mixed methods. Here, we have considered a dual mixed formulation with reduced symmetry whose unknown factors are the tensor of deformations, the pressure, the displacement and the rotational of displacement [24]. This formulation is robust in the sense that it overcomes the locking phenomenon which appears when the material is close to the incompressible state i.e. for λ very large [7]. Because of geometrical singularities and with the aim of restoring the optimal convergence rate of the method, we impose a priori some refinement rules of the grids in order to recapture optimal order of convergence of the interpolates. The main ingredient is the knowledge of the singular behavior of the solution [27] in order to determine the real regularity of the solution in terms of weighted Sobolev spaces. The finite elements considered are of the BDM_1 family stabilized by the bubbles [3, 8].

For the dual mixed finite element method for linear elasticity problem , a new a posteriori error estimator is introduced and analyzed [20]. It is shown that this error estimator is reliable and efficient for simply-connected domains and also for multiply-connected domains. The lower and upper error bounds obtained are uniform with respect to the Lamé coefficient λ (thus avoiding the so-called locking effect). The estimator allows an adaptive finite element scheme which refines a given grid only in regions where the error is relatively large and thus restoring the optimal convergence rate with a lower cost of calculations. Reliability, efficiency and robustness of our estimator have been corroborated by several numerical tests. A strategy based on the "red-green-blue refinement" was implemented successfully [38]. Finally, the technique developed to establish this estimator can be extended to the three-dimensional case.

For the linear elastodynamic problem, we have constructed and analysed a finite element method using dual mixed formulation for spacial discretization and the explicit or implicit Newmark scheme for the time discretization [19]. Our formulation requires less regularity on displacement than standard one.

Optimal order L^∞-in-time L^2-in-space a priori error estimates are derived and a quadratic convergence rate in time for the fully discretized scheme has been established for both the explicit and the implicit numerical schemes. From this point of view, the complexity of certain physical models require an effective procedure based on the mixed finite element method. On this I aim primarily two directives of research:

- ▷ **Application of the mixed method to the mortar finite element:** the interest for the mortar method has increased considerably. Indeed mortar finite element, introduced Bernardi, Maday and Patera like a domain decomposition method, offers the possibility of working with completely independent discretizations on the subdomains partitioning the domain Ω without overlapping. This is very effective if a strategy of mesh-adaptation is more requested on a particular subdomain. An a posteriori error analysis is thus necessary to work out reliable and effective errors indicators. I intend to extend my results already establish about a posteriori error analysis for the mixed finite elements method for the linear elasticity to the finite element mixed mortar. Possibly a parallel implementation of the code by using the technique MPI (Passing Message Interfaces) could be done.

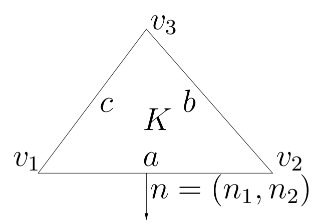
- ▷ **Application of the mixed method to the problem of propagation of waves:**

  One taking as a starting point my former work on modeling by mixed finite elements the wave propagation in homogeneous, isotropic and elastic mediums, and works of Bécache, Tsogka and Joly, I intend to extend this study to the time dependent elastic wave propagation in complex media such as heterogeneous anisotropic media of complex geometry. The use of the fictitious domain method for the resolution of the elastic diffraction of wave by cracks, could be adopted. I then hope to establish dynamic a posteriori error estimators for the automatic adaptation of grids.

Annex

1 Basis functions

Let $K \in \mathcal{T}_h$ be a triangle with vertices $v_i = (v_{x_i}, v_{y_i})$ $i = 1, \cdots, 3$. We denote by $g = (g_1, g_2) = \dfrac{v_1 + v_2 + v_3}{3}$ the center of mass. By (x_i, y_i) $i = 1, \cdots, 3$ we denote the midpoints of the three edges and by $l_1,\ l_2\ l_3$ the length of these three edges of the triangle K (see Figure below)



Let us recall the finite dimensional subspace $\widetilde{\Sigma}_h$:

$$\widetilde{\Sigma}_h = S_h \times Q_h := \{(\tau_h, q_h) \in [L^2(\Omega)]^{2\times 2} \times L^2(\Omega);\ \forall T \in \mathcal{T}_h :\ q_{h|T} \in \mathbb{P}_1(T) \ \text{ and } (\tau_h - q_h\delta)_{|T} \in [\mathbb{P}_1(K)]^{2\times 2} \oplus [\mathbb{R}\, Curl\ b_T]^2\},$$

We chose $\phi_1, \cdots, \phi_{14}$ as a basis of S_h:

S_h	$\phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \phi_2 = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$
	$\phi_3 = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \phi_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
	$\phi_5 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ $\quad \phi_6 = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$
	$\phi_7 = \begin{pmatrix} \frac{\partial b}{\partial y} & -\frac{\partial b}{\partial x} \\ 0 & 0 \end{pmatrix}$ $\quad \phi_8 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
	$\phi_9 = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$ $\quad \phi_{10} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$
	$\phi_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \phi_{12} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$
	$\phi_{13} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$ $\quad \phi_{14} = \begin{pmatrix} 0 & 0 \\ \frac{\partial b}{\partial y} & -\frac{\partial b}{\partial x} \end{pmatrix}$

The basis functions of Q_h are defined by φ_1, φ_2, φ_3. The basis functions of the rotations in W_h are the same ones as for Q_h . The basis functions of the displacements in V_h are denoted α_1 and α_2. Finally the basis functions of the Lagrange multipliers in Λ_h are denoted by $\gamma_1, \cdots, \gamma_4$:

V_h	$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	
W_h	$\varphi_1 = 1$	$\varphi_2 = x$	$\varphi_3 = y$
Λ_h	$\gamma_1 = (2-3t)\alpha_1$	$\gamma_2 = (2-3t)\alpha_2$	
	$\gamma_3 = (3t-1)\alpha_1$	$\gamma_4 = (3t-1)\alpha_2$	

2 Local stiffness matrices

The components of the resulting local stiffness matrix A^K are shown in

$$A^K = |K| \begin{pmatrix} \alpha & \beta & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \sigma & \upsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & \upsilon & \nu & 0 & 0 & 0 & \eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \beta & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & \sigma & \upsilon & -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & \upsilon & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & -\eta & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \beta & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & \sigma & \upsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & \upsilon & \nu & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \beta & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & \sigma & \upsilon & -\eta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & \upsilon & \nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta & 0 & -\eta & 0 & \rho \end{pmatrix},$$

where the remaining coefficients in the above matrix are denoted by

α	β	γ	σ	ν	υ	ρ	η
$\frac{1}{2\mu}$	$\frac{g_2}{2\mu}$	$\frac{g_2}{2\mu}$	$\frac{1}{6\mu\sum_{i=1}^3 x_i^2}$	$\frac{1}{6\mu\sum_{i=1}^3 y_i^2}$	$\frac{1}{6\mu\sum_{i=1}^3 x_i y_i}$	$\frac{l_1^2+l_2^2+l_3^2}{1440\mu\lvert K\rvert}$	$-\frac{1}{120\mu}$

The bilinear form $B_K(p_K, v_K)$ results into matrix B^K, with components

$$B^K = |K| \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The bilinear form $C_K(\sigma_K, v_K)$ results into matrix C^K, with components

$$C^K = |K| \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

The bilinear form $H_K(\sigma_K, \theta_K)$ results into matrix H^K, with components

$$H^K = |K| \begin{pmatrix} 0 & 0 & 0 & -1 & h_1 & h_2 & 0 & 1 & -h_1 & -h_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_1 & h_3 & h_4 & h_6 & -h_1 & -h_3 & -h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & h_4 & h_5 & 0 & -h_2 & -h_4 & -h_5 & 0 & 0 & 0 & h_6 \end{pmatrix},$$

while the remaining coefficients are

h_1	h_2	h_3	h_4	h_5	h_6
$-g_1$	$-g_2$	$-\sum_{i=1}^{3} x_i^2$	$\sum_{i=1}^{3} x_i y_i$	$-\sum_{i=1}^{3} y_i^2$	$-\frac{1}{60}$

The bilinear form $P_K(p_K,q_K) := \dfrac{1}{\lambda}\displaystyle\int_K p_K q_K\ dx$ results into matrix P^K, with components

$$P^K = |K| \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{pmatrix},$$

while the remaining coefficients are

p_1	p_2	p_3	p_4	p_5	p_6
$\frac{1}{\lambda}$	$\frac{g_1}{\lambda}$	$\frac{g_2}{\lambda}$	$\frac{\sum_{i=1}^3 x_i^2}{3\lambda}$	$\frac{\sum_{i=1}^3 x_i y_i}{3\lambda}$	$\frac{\sum_{i=1}^3 y_i^2}{3\lambda}$

For an edge $a \in \partial K$, the bilinear forms $E_K^a(\sigma_K,\mu_a) := \int_a (\sigma_K.n_K)\mu_a\ ds$ and $G_K^a(p_K,\mu_a) := \int_a p_K n_K.\mu_a\ ds$ result into two matrices E_K^a and G_K^a respectively with components

$$E_K^a = \frac{l}{2}\begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} & 0 \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} & 0 \end{pmatrix},$$

while the remaining coefficients are

E_{11}	E_{12}	E_{13}	E_{14}	E_{15}	E_{16}	E_{31}	E_{32}	E_{33}	E_{34}	E_{35}	E_{36}
n_1	$n_1 v_{x_1}$	$n_1 v_{y_1}$	n_2	$n_2 v_{x_1}$	$n_2 v_{y_1}$	n_1	$n_1 v_{x_2}$	$n_1 v_{y_2}$	n_2	$n_2 v_{x_2}$	$n_2 v_{y_2}$

and

$$G_K^a = \frac{l}{2}\begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \\ G_{41} & G_{42} & G_{43} \end{pmatrix},$$

while the remaining coefficients are

G_{11}	G_{12}	G_{13}	G_{21}	G_{22}	G_{23}	G_{31}	G_{32}	G_{33}	G_{41}	G_{42}	G_{43}
n_1	$n_1 v_{x_1}$	$n_1 v_{y_1}$	n_2	$n_2 v_{x_1}$	$n_2 v_{y_1}$	n_1	$n_1 v_{x_2}$	$n_1 v_{y_2}$	n_2	$n_2 v_{x_2}$	$n_2 v_{y_2}$

where $v_1 = (v_{x_1}, v_{y_1})$, $v_2 = (v_{x_2}, v_{y_2})$, l and (n_1,n_2) denote vertices , length and the outward normal vector along the edge a.

3 Notes on the elimination procedure

In the elimination procedure, which is made element by element, one must show that all matrices corresponding to the eliminations are at least invertible. Firstly let us note that the matrix A is bloc diagonal, symmetric and positive definite. For the matrix $CA^{-1}C^T$

it is easy to see that it is also bloc diagonal, symmetric and positive definite. Now let us show that the matrix W defined by

$$W = HA^{-1}C^T(CA^{-1}C^T)^{-1}CA^{-1}H^T - HA^{-1}H^T,$$

is symmetric and positive definite. Indeed, it is clear that W is symmetric. It remains to prove that it is positive definite. Denoting by $(.,.)$ the $\mathbb{R}^2$-scalar product, $S = HA^{-1}H^T$, $L = CA^{-1}H^T$ and $K = CA^{-1}C^T$, the matrix W may be rewritten:

$$W = S - L^T K^{-1} L.$$

$$(W\lambda, \lambda) = (S\lambda, \lambda) - (K^{-1}L\lambda, L\lambda).$$

Let $u = -K^{-1}L\lambda$. Then $L\lambda = -Ku$ and

$$\begin{aligned}(W\lambda, \lambda) &= (S\lambda, \lambda) - (Ku, u)\\ &= (S\lambda, \lambda) + (Ku, u) - 2(Ku, u)\\ &= (Ku, u) + 2(L\lambda, u) + (S\lambda, \lambda)\\ &= \left(\begin{pmatrix} K & L \\ L^T & S \end{pmatrix}\begin{pmatrix} u \\ \lambda \end{pmatrix}, \begin{pmatrix} u \\ \lambda \end{pmatrix}\right)\end{aligned}$$

We are thus brought back to show that the matrix

$$M := \begin{pmatrix} K & L \\ L^T & S \end{pmatrix} = \begin{pmatrix} CA^{-1}C^T & CA^{-1}H^T \\ HA^{-1}C^T & HA^{-1}H^T \end{pmatrix}.$$

is positive definite.

$$\begin{aligned}\left(M\begin{pmatrix} u \\ \lambda \end{pmatrix}, \begin{pmatrix} u \\ \lambda \end{pmatrix}\right) &= \left(\begin{pmatrix} CA^{-1}C^T & CA^{-1}H^T \\ HA^{-1}C^T & HA^{-1}H^T \end{pmatrix}\begin{pmatrix} u \\ \lambda \end{pmatrix}, \begin{pmatrix} u \\ \lambda \end{pmatrix}\right)\\ &= \big(CA^{-1}C^T u, u\big) + \big(CA^{-1}H^T\lambda, u\big) + \big(HA^{-1}C^T u, \lambda\big) + \big(HA^{-1}H^T, \lambda\big)\end{aligned}$$

Let $v = C^T u$ and $\nu = H^T\lambda$, then

$$\begin{aligned}\left(M\begin{pmatrix} u \\ \lambda \end{pmatrix}, \begin{pmatrix} u \\ \lambda \end{pmatrix}\right) &= \big(A^{-1}v, v\big) + \big(A^{-1}\nu, v\big) + \big(A^{-1}v, \nu\big) + \big(A^{-1}\nu, \nu\big)\\ &= \big(A^{-1}(v+\nu), (v+\nu)\big)\\ &\geq c||v+\nu||^2\end{aligned}$$

This prove that M is semi-positive definite. For proving that M is a positive definite matrix, it suffice to show that it is invertible. By standard Theorem of linear algebra this is equivalent to prove that the following form is injective

$$\begin{aligned} T: \quad \mathbb{R}^2 \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \times \mathbb{R}^3 \\ (\eta, \xi) &\longmapsto \begin{cases} CA^{-1}C^T\eta + CA^{-1}H^T\xi \\ \\ HA^{-1}C^T\eta + HA^{-1}H^T\xi \end{cases}\end{aligned}$$

Let thus $(\eta, \xi) = \big((\eta_1, \eta_2, \eta_3), (\xi_1, \xi_2)\big) \in \mathbb{R}^2 \times \mathbb{R}^3$ such that

$$\begin{cases} CA^{-1}C^T\eta + CA^{-1}H^T\xi = 0 \\ HA^{-1}C^T\eta + HA^{-1}H^T\xi = 0 \end{cases}$$

Thus, we have

$$\begin{cases} \big(\eta, CA^{-1}C^T\eta + CA^{-1}H^T\xi\big) = 0 \\ \big(\xi, HA^{-1}C^T\eta + HA^{-1}H^T\xi\big) = 0 \end{cases}$$

Hence

$$\begin{cases} \big(A^{-1}C^T\eta, C^T\eta + H^T\xi\big) = 0 \\ \big(A^{-1}H^T\eta, C^T\eta + H^T\xi\big) = 0 \end{cases}$$

Summing the two equations of this system to get

$$\big(A^{-1}(C^T\eta + H^T\xi), C^T\eta + H^T\xi\big) = 0.$$

But the matrix A is symmetric and positive definite, thus

$$C^T\eta + H^T\xi = 0.$$

Using the explicit forms of the bloc diagonal matrices C and H, we easily see that $\eta_1 = \eta_2 = 0$ and $\xi_1 = \xi_2 = \xi_3 = 0$. This proves that the form T is injective and thus the matrix M is invertible. Finally, we have proved that the matrix M is positive definite and thus the matrix W is also positive definite.

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Résumé

Dans ce travail, nous étudions le raffinement de maillage pour des méthodes d'éléments finis mixtes duales pour deux types de problèmes : le premier concerne le problème de l'élasticité linéaire et le second problème celui de l'élastodynamique.

Pour ces deux types de problèmes et dans des domaines non réguliers, les méthodes d'éléments finis mixtes analysées jusqu'à présent, sont celles qui concernent des méthodes mixtes "classiques". Ici, nous analysons la formulation mixte duale pour les deux problèmes de l'élasticité linéaire et de l'élastodynamique. Pour le problème d'élasticité, nous sommes concernés premièrement par une analyse a priori d'erreur en utilisant l'approximation par l'élément fini BDM_1 stabilisé. Afin de dériver une estimation a priori optimales d'erreur, nous établissons des règles de raffinement de maillage. Ensuite, nous faisons une analyse d'erreur à posteriori sur un domaine simplement ou multiplement connexe. En fait nous établissons un estimateur résiduel fiable et efficace. Cet estimateur est alors utilisé dans un algorithme adaptatif pour le raffinement automatique de maillage. Pour le problème de l'élastodynamique, nous faisons une analyse a priori d'erreur en utilisant le même élément fini que pour le problème d'élasticité, en utilisant une formulation mixte duale pour la discrétisation des variables spatiales. Pour la discrétisation en temps nous étudions les deux schémas de Newmark explicite et implicite. Par des règles de raffinement de maillage appropriées, nous dérivons des estimées d'erreur optimales pour les deux schémas numérique.

Mots-clés: MEF duale mixte, Espaces de Sobolev, Estimations d'erreur à priori, Estimations d'erreur à posteriori, Problème de l'élasticité, Problème de l'élastodynamique, Décomposition de Helmholtz , Schémas de Newmark , Multiplicateurs de Lagrange, Formulation Hybride.

Abstract

In this work, we study the refinement of grids for the dual mixed finite element method for two types of problems: the first one concerns the linear elasticity problem and the second one the linear elastodynamic problem.

For these two types of problems and in nonregular domains, the mixed finite element methods analyzed until present relate to the primal mixed methods. Here, we analyze the dual mixed formulation for both linear elasticity and linear elastodynamic problems. For the elasticity problem, we are concerned firstly by an a priori error analysis when using finite element approximation by stabilized BDM_1 element. Then, we make an a posteriori error analysis for the dual mixed finite element method for both a simply and a multiply connected domain. In fact we establish a residue based reliable and efficient error estimator for the dual mixed finite element method. This estimator is then used in an adaptive algorithm for automatic mesh refinement. For the elastodynamic problem, we make an a priori error analysis when using the same finite element as for the elasticity problem, using a dual mixed formulation for the discretization in the spatial variables and the explicit or implicit Newmark scheme for the discretization in time. By adequate refinement rules on the regular family of triangulations we derive optimal a priori error estimates for the explicit-in-time and implicit-in-time numerical schemes.

Keywords: Dual mixed FEM, Sobolev spaces, a priori error estimation, a posteriori error estimates, elasticity problem, elastodynamic problem, Helmholtz decomposition, Newmark scheme, Lagrange multiplier, hybrid formulation.