Singular structures in some variational problem

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Singular Structures in some Variational Problems.

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Introduction

Cette thèse représente la synthèse des résultats obtenus ces dernières quatre années de recherche et étude sous la direction et l'enseignement de M. Sergio Solimini et de M. Jean-Michel Morel. Dans cette thèse on aborde quelques problèmes d'analyse non linéaire et de calcul des variations qui donnent origine à des structures singulières. Notre but est celui de démontrer comment dans certains cas ces structures singulières sont un obstacle à surmonter pour obtenir des résultats d'existence ou multiplicité et dans d'autres cas les singularités présentes dans les solutions sont le vrai objet d'étude et justifient le choix de la fonctionnelle introduite. Ce travail se compose de deux parties: la première concerne les résultats sur une classe de problèmes qui ont été affrontés par des chercheurs en analyse non linéaire dans les dernières vingt années, sur l'existence et la multiplicité de solutions de équations elliptiques, obtenus malgré un manque de compacité dû à des phénomènes de concentration; la deuxième concerne une catégorie de problèmes d'irrigation où les trajectoires suivies par les particules de fluide donnent naissance à un ensemble monodimensionnel et qui peut être située dans la théorie du transport.

En particulier, dans les premiers deux chapitres de cette thèse on a à faire avec le suivant problème de croissance critique

$$\text{(CP)} \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u & \text{en } \Omega \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où Ω est un sous-ensemble ouvert et régulier de $I\!R^N$ ($N \geq 3$), $2^* = \frac{2N}{N-2}$ est l'exposant critique de Sobolev pour l'immersion de $H_0^1(\Omega)$ dans $L^p(\Omega)$, et $\lambda > 0$. Pour ce problème on obtient une infinité de solutions quand $N \geq 7$ et seulement un résultat de multiplicité finie qui laisse ouvert le problème sur l'existence de solutions en nombre infini quand $N \geq 4$. Plus précisément, on trouve au moins $\frac{N}{2} + 1$ (couples de) solutions (ou encore $N+1$ si λ est opportunément près de zéro) et $\lambda \in]0, \lambda_1[$, étant λ_1 la première auto-valeur de $-\Delta$ défini sur $H_0^1(\Omega)$.

Le manque de compacité dans le problème (CP) est dû à la concentration de séquences de Palais-Smale de la relative fonctionnelle énergie à certains niveaux, comme décrit avec des arguments de concentration-compacité dûs à P.L. Lions (voir [22]) ou comme dans le théorème de compacitè de Struwe (voir [25] et [26]).

Le troisième chapitre est consacré au problème

$$\text{(P)} \qquad \begin{cases} -\Delta u + a(x)u = |u|^{p-2}u & \text{en } I\!R^N \\ u \in H^1(I\!R^N) , \end{cases}$$

où $N \geq 2$, $p > 2$, $p < 2^*$ (quand $N > 2$), et la potentielle $a(x)$ est une fonction continue, positive dans $I\!R^N$, excepté au maximum un ensemble borné, qui vérifie des opportunes hypothèses de décroissance mais à laquelle on ne demande aucune symétrie. Le manque de compacité dans le problème (P) est structurellement similaire, dans le sens de la concentration-compacité, à celui du problème (CP). Ce qui arrive dans ce cas est que le manque de compacité est dû au fond au fait que le domaine est illimité, ce qui permet la création de quelques masses qui vont à l'infini alors que l'on calcule le limite d'une séquence de Palais-Smale, comme décrit dans un théorème de compacité de V. Benci et G. Cerami (voir [6]).

Dans les deux derniers chapitres (deuxième Partie) on suit l'approche utilisée in [32] où les auteurs ont introduit une fonctionnelle de coût pour modéliser les structures branchées comme les arbres, les appareils radicaux et cardiovasculaires pour l'étude de *structures branchées* définie sur un espace de recherche très ample fait d'élément que les auteurs appellent "irrigation patterns". Cette approche est très différente de celle utilisée par M. Qinglan Xia en [35]. En fait pour Xia le réseau est un graphe avec une quantité nombrable de sommets qui satisfait la Lois de Kirchhoff, en plus quand ces graphes G sont finis l'auteur considère, pour $\alpha \in]0,\, 1[$, une fonctionnelle énergie qui est formellement la même de celle considérée en [32]

$$E^\alpha(G) = \sum_{e \text{ arêtes de } G} w(e)^\alpha \text{long}(e) ,$$

où $w(e)$ représente la quantité de fluide transportée le long de l'arête e. Le fait que le paramètre α est plus petit que 1 implique que les bifurcations ne sont pas convenables du point de vue de l'économie d'énergie en établissant une lois de Poiseuille qualitative selon laquelle la résistance d'un tuyau augmente quand il se restreint.

Un système d'irrigation au contraire n'est pas défini comme un "embedded" graphe, mais comme un ensemble mesurable de voies (fibres) qui partent d'une unique source et s'arrêtent en quelques points de $I\!R^N$ sur lesquels ce que les auteurs appellent mesure d'irrigation est concentrée. L'espace de recherche d'une solution est plus grand car on permet a priori des arbres qui s'éparpillent (spreading trees) pour lesquels on pourrait avoir des ensembles de particules de mesure positive qui pourraient suivre leur chemin sans suivre une branche (voir fig. 1). Le formalisme de [32] considère les chemins qui partent d'une seule source et qui peuvent être interprétés soit comme la trajectoire en $I\!R^N$ d'une particule de fluide soit comme une fibre d'un arbre. Ces chemins en nombre infini (en principe un pour chaque particule) qui dépendent d'un paramètre t sont appelles "fibres" et notés $\chi(p,\, t)$ où t représente le temps où la particule p, qui appartient à un espace de probabilité abstraite Ω, rejoint la position $\chi(p,\, t) \in I\!R^N$. Tout ensemble mesurable de fibres χ, c'est-à-dire tout structure d'irrigation, induit sur $I\!R^N$ une mesure d'irrigation μ_χ qui donne à tout ensemble de Borel A de $I\!R^N$ la quantité, in Ω, de l'ensemble des particules qui s'arrêtent en A (c'est-à-dire qui sont absorbés en l'ensemble A). Toujours en [32] les auteurs ont proposée une fonctionnelle coût I_α définie sur l'ensemble de toutes les structures d'irrigation en posant

$$I_\alpha(\chi) = \int_{I\!R_+} \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp\, dt\ .$$

Cette fonctionnelle, formellement semblable à E^α, considère, à tout instant, seulement les points $p \in M_t(\chi)$ c'est-à-dire les points qui sont encore en mouvement au temps t et, en prenant la puissance $\alpha - 1 < 0$ de la quantité de l'ensemble $[p]_t$ des particules qui ont coulé ensemble jusqu'au temps t, pénalise la ramification en produisant une structure qui a l'aspecte de la figure 2.

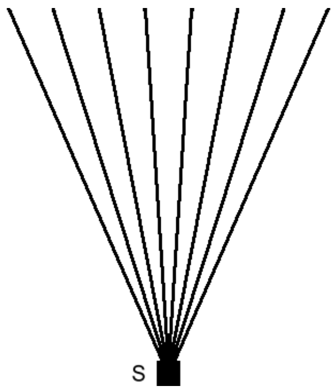

fig. 1

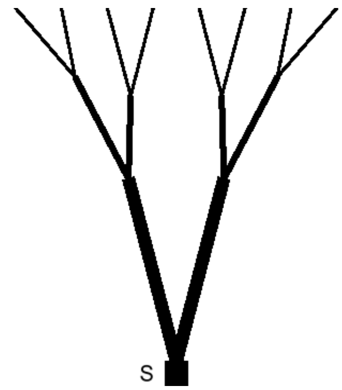

fig. 2

Les propriétés dérivées dans le Chapitre 4 pour les structures optimales sont des "propriétés élémentaires" dans le sens qu'elles ne sont pas intéressées aux points finaux de ces structures, où les propriétés d'autosimilarité présumables devraient se vérifier. Cette étude préliminaire trouve déjà une application dans le Chapitre 5 où on pose le problème de déterminer si une mesure donnée est irrigable ou pas. La réponse à cette question montre clairement, en particulier, qu'il y a des cas où les mesures de probabilité sont irrigables par une preuve différente du résultat en [35] dans un contexte très proche. On analyse une question plus générale qui consiste à caractériser, pour une valeur donnée de l'exposant, quelle mesures de probabilités sont irrigables ou pas. Dans ce but on donne une notion de dimension d'irrigabilité d'une mesure et on démontre des bornes supérieures et inférieures en fonction de la dimension minimale de Hausdorff ou respectivement de Minkowski d'un ensemble sur lequel la mesure est concentrée. On introduit ensuite une notion de dimension de résolution d'une mesure basée sur ses approximations discrètes et on étudie sa relation avec la dimension d'irrigation.

Un résumé plus détaillé des deux parties de cette thèse suit.

Part I - Résultats de multiplicité pour quelques problèmes elliptiques sans compacité

Avant les années 80 la théorie des équations elliptiques a été développée surtout sous l'hypothèse de croissance sous-critique. En particulier le problème modèle

$$-\Delta u = |u|^{p-2}u \,, \tag{0.0.1}$$

avec la condition de Dirichlet sur la frontière, a été étudié pour $p < 2^* = \frac{2N}{N-2}$ en dimension $N > 2$.

Dans le cas de croissance critique $p = 2^*$, étant l'injection de $H_0^1(\Omega)$ en L^p non compact, l'opérateur de Nemitsky $|u|^{p-2}u$ n'est pas compact et ainsi l'un d'eux ne peut pas avoir de solution non banale en (P) en utilisant la plupart des techniques variationales standard. En plus Pohozaev, dans sa célèbre étude [22], a établi une identité qui, appliquée aur problème (P) pour un domaine étoilé, a démontré la non-existence d'une solution non banale dans le cas de croissance critique. D'autre part, l'exposant critique $p = 2^*$ est le seul qui rend l'énergie fonctionnelle $I_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}$ homogène par rapport aux changements d'échelle. Dans leur travail de pionnier [9], Brezis et Nirenberg ont eu l'idée d'introduire un terme d'ordre plus bas, c'est-à-dire un terme sous-critique, en particulier le terme linéaire λu, qui ne change pas la croissance de l'énergie fonctionnelle mais rompt son homogéneité qui pénalise les concentrations, même si de façon infinitésimale. En alternative à l'identité de Pohozaev, qui garantit qu'en général (pur un domaine étoilé) (CP) n'a pas de solution non banale pour $\lambda \leq 0$, en [9] Brezis and Nirenberg donne le résultat de l'existence pour $0 < \lambda < \lambda_1$ (λ opportunément plus grand que zéro en dimension trois). Par conséquent, le résultat a été étendu en [10] to $\lambda > \lambda_1$. Pour cette raison, dans le cas de croissance critique, la rupture de l'homogénéité de l'énergie fonctionnelle peut donner le résultat de l'existence pour un problème elliptique approprié. En particulier, le changement de déviation de l'homogénéité de la fonctionnelle énergie (c'est-à-dire changeant de $\lambda < 0$ à $\lambda > 0$ dans le cas du problème (CP)) fait passer de résultats de non existence à des résultats d'existence. Le problème d'avoir des résultats de multiplicité pour le problème (CP) a été soulevé dans cette période et les techniques, alternatives à l'emploi du Théorème de Rellich, ont été introduites pour trouver une solution comme la limite des séquences de Palais Smale (ou, plus brièvement, séquence PS) pour l'énergie fonctionnelle

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} .$$

En particulier, les travaux sur la concentration-compacité comme [20], [25] and [24] donnent une analyse précise de l'obstruction à la compacité, en mettant en évidence des *mauvais niveaux* où la condition PS tombe. Dans le Chapitre 1 la question sur l'existence de infinies solutions au Problème (CP), pour tout domaine borné régulier $\Omega \subset \mathbb{R}^N$ dans le cas $N \geq 7$, obtient une réponse affirmative.

Nous avons rencontré des problèmes semblables pour trouver des résultats d'existence et de multiplicité pour des problèmes elliptiques à croissance sous-critique sur tout le domaine comme le Problème (P). Dans le cas où le coefficient $a(x)$ n'est pas symétrique le manque de compacité dérive de la naissance de quelques masses qui sont concentrées autour de quelques points qui vont à l'infini.

Dans le Chapitre 3 on pose le problème d'avoir des résultats d'existence et de multiplicité au Problème (P) et on a trouvé des solutions en nombre infini sans demander de symétrie pour $a(x)$. On a seulement adapté les mêmes techniques utilisées pour avoir une infinité de solutions au Problème (CP).

Chapitre 1 *Concentrations estimées et solutions multiples aux problèmes elliptiques à croissance critique*[1]

Les premiers résultats de multiplicité des olutions au Problème (CP) ont été obtenus pour certains domaines symétriques, voir [19], en particulier les auteurs réduisent le résultat de multiplicité à celui de l'existence en coupant le domaine en un nombre fini de sous-domaines de même grandeur et puis en collant les solutions trouvées sur chaque tranche (l'existence de solutions en nombre infini vient de la possibilité de choisir arbitrairement le nombre de tranches).

Ce type d'approche ne peut pas être appliqué quand on travaille dans un domaine de forme régulière. Pour les domaines symétriques aussi cela peut échouer quand on recherché des résultats de multiplicité d'un type particulier, par exemple les solutions radiales quand le domaine est une boule. Le cas radial permet en fait de donner résultats d'existence et de non existence complémentaires en correspondance précise. Si $N \geq 7$ et Ω est une boule, alors pour tout $\lambda > 0$ (CP) a des solutions radiales en nombre infini qui changent de signe, voir [23] and [12], si $4 \leq N \leq 6$ il y a une constante $\lambda^* > 0$ de sorte que (CP) n'a pas de solution qui change de signe si $\lambda \in]0, \lambda^*[$ (voir [1], [2]). Ainsi la condition $N \geq 7$ dans le résultat d'existence précédent ne peut pas être éliminée.

Dan ce chapitre la question de l'existence de solutions en nombre infini au Problème (CP), pour chaque domaine régulier $\Omega \subset I\!R^N$ dans le cas $N \geq 7$, obtient une réponse affirmative. La nouvelle idée qui nous a permis de prouver l'existence d'une infinité de solutions au Problème (CP) a été celle de changer le concept de "quasi solution" et de prouver la compacité pour ce nouvel objet. En fait le concept habituel de quasi solution dépendant de la norme H^{-1} de la dérivée de Fréchet de l'énergie, qui conduit à la notion bien connue de séquences PS, ne permet pas de déduire des résultats de compacité dans ce contexte. Les séquences PS ont été remplacées par des séquences de solutions de problèmes approximatifs (appelées "séquences équilibrées" ("balanced sequences" en anglais)). La circonstance que u_n est une solution d'un problème autonome du même type que le Problème (CP) nous permet d'établir une inégalité locale de type Pohozaev qui, clairement, tient compte de la modification de l'énergie I_n (relative au problème approximatif n-sime) relativement aux concentrations. On prouve en premier lieu que même pour les suites équilibrées limitées que l'on suppose non compactes il y a le même phénomène de concentration qui se vérifie pour les séquences PS non compactes. En travaillant avec les séquences non compactes équilibrées et en changeant leur paramétrer de concentration on peut produire une modification locale de la fonctionnelle approximative qui est du même ordre de la fonction en contradiction avec le fait que les éléments d'une suite équilibrée sont des points critiques. La variation de la fonctionnelle sous cette modification locale a été évaluée par une inégalité locale de type Pohozaev et une estimation *a priori* uniforme

[1] G. Devillanova & S. Solimini *Concentrations estimates and multiple solutions to elliptic problems at critical growth* Advances in Differential Equations, 7 (2002), 1257-1280.

de décroissance sur les termes d'une suite équilibrée limitée et de leurs dérivés qui, portées dans cette inégalité, engendrent formellement la contradiction.

Ce résultat de compacité peut être employé aussi dans le cas radial, en produisant l'existence d'une infinité de solutions radiales. Par les résultats en [1] et [2], où l'unicité d'une solution radiale non banale est affirmée pour $N < 7$, on déduit que l'argument de compacité est faux pour les dimensions basses. Cette circonstance ne signifie pas que l'existence de solutions en nombre infini peut être démontrée avec différent instruments comme il arrive, par exemple, dans le cas de domaines symétrique (voir [19]).

La dimension sept n'a pas de sens particulier comme il arrive, par exemple, avec les problèmes de surface minimale, mais elle dépend de la linéarité du terme λu de perturbation sous-critique. Un terme non linéaire sous-critique demanderait une limite différente dans la dimension. Remarquons que $N \geq 4$ est suffisant pour l'existence d'une solution non-banale pour chaque $\lambda > 0$ mais trois dimensions de plus sont nécessaires pour les résultats de multiplicité. La raison est due au fait qu'en partant de la dimension 4, l'avantage dû au terme quadratique $-\lambda \int_\Omega u^2$ de la fonctionnelle I_λ est plus grand que le coût d'un terme de troncature qui porte une fonction de Talenti à zéro dans les bornes de Ω. D'autre part, si l'on veut faire un test similaire pour le problème de la multiplicité, on doit couper la fonction pour atteindre une valeur de signe opposé. Ainsi le coût de la troncature est considérablement plus élevé et trois dimensions de plus sont nécessaires pour rejoindre une estimation appropriée.

Chapitre 2 *Un Résultat de Multiplicité pour des Equations Elliptiques à Croissance Critique en Dimension Basse* [2]

On a abordé le problème(CP) en dimension basse $N \geq 4$ et on a montré, en travaillant sur la double contrainte naturelle

$$(0.0.2) \quad U = \left\{ u \in H_0^1(\Omega) \mid u^\pm \neq 0,\ (\nabla I_\lambda(u), u^\pm) = 0 \text{ , pour les deux} \pm \text{signes} \right\} ,$$

que, pour $\lambda \in]0, \lambda_1[$ le problème (CP) a au moins $\frac{N}{2} + 1$ (paires de) solutions ($N + 1$ for λ assez proche en 0), améliorant ainsi le résultat de [11] obtenu, en travaillant sur la contrainte U, en cherchant des solutions qui changent de signe tout près de la frontière. L'idée est de comparer le problème dans un domaine général avec le problème dans une boule où on peut employer la symétrie. Le phénomène de concentration sous le seuil $\frac{2}{N} S^{\frac{N}{2}}$ (où S est la constante de Sobolev) est dû seulement à une unique fonction de Talenti et pour cette raison si $c < \frac{2}{N} S^{\frac{N}{2}}$ est un niveau de min-max, et donc au moins un entre c and $c - \frac{1}{N} S^{\frac{N}{2}}$ est critique. En plus, on peut trouver $N + 1$ niveaux de min-max sous le seuil $\frac{2}{N} S^{\frac{N}{2}}$, auquel on doit ajouter le niveau de l'état fondamental (ground state en anglais).

[2] G. Devillanova & S. Solimini *A Multiplicity Result for Elliptic Equations at Critical Growth in Low Dimension* Comm. in Contemporary Math., Vol. 5 N. 2(April 2003), 171-177.

On peut dire aussi que si quelques-uns de ces $N+1$ niveaux de min-max concident le problème admet des solutions en nombre infini, même si on ne sait pas si un tel niveau est critique ou pas (on ne sait pas si l'on doit le diminuer de la quantité $\frac{1}{N}S^{\frac{N}{2}}$). Dans ce cas, nous avons $\frac{N+2}{2}$ paires de solutions.

Ce résultat a été récemment étendu par [14] dans le cas $\lambda \geq \lambda_1$. En tout cas, il n'y a pas de raison pour supposer que ce résultat peut être optimal et le problème de démontrer l'existence de solutions en nombre infini, ou même si on donne une estimation optimale sur le nombre de solutions, reste, comme on sait, largement ouvert pour $N \leq 6$.

Chapitre 3 *États bornés infinis pour quelques équations non linéaires de champs scalaire* [3]

Finalement, on aborde le Problème (P) qui est similaire aux problèmes qui naturellement apparaissent dans différentes branches de Physique Mathématique, en fait les solutions à (P) peuvent être vues comme des ondes solitaires (états stationnaires) dans des équations non linéaires du type de Klein-Gordon ou de Schrödinger. En plus, ils présentent des difficultés mathématiques spécifiques qui sont un défi pour les chercheurs. Les solutions au Problème (P) peuvent être cherchées comme points critiques de l'énergie fonctionnelle $I : H^1(I\!R^N) \to I\!R$ defined by

$$I(u) = \frac{1}{2}\int_{I\!R^N} (|\nabla u|^2 + a(x)u^2)dx - \frac{1}{p}\int_{I\!R^N} |u|^p dx \,. \tag{0.0.3}$$

Les méthodes habituelles de calcul des variations, qui permettent de démontrer l'existence de solutions en nombre infini de (P) dans un domaine limité, ne peuvent pas être appliquées telles quelles en I. En fait, l'injection $j : H^1(I\!R^N) \to L^p(I\!R^N)$ est continue mais non compacte, pour cette raison la condition de base de Palais-Smale n'est pas satisfaite par I à tous les niveaux d'énergie. Dans ce cas aussi il y a des résultats partiels quand $a(x)$ a quelques propriétés de symétrie. En effet, les premiers résultats connus ont été obtenus en considérant $a(x) = a(|x|)$ ou encore $a(x) = a_\infty \in I\!R^+ \setminus \{0\}$ (see [21], [8], [15], [16], [24], [7]). Dans ce cas, la restriction de I à $H^1_r(I\!R^N)$, le sous-espace de $H^1(I\!R^N)$ qui consiste de fonction symétriques sphériques, rétablit la compacité, parce que l'injection de $H^1_r(I\!R^N)$ dans $L^p(I\!R^N)$ est compacte. Ainsi, l'existence d'une solution positive de (P) peut être montrée soit en utilisant le Théorème du Col soit par minimisation sur une contrainte naturelle, alors que l'existence de solutions en nombre infini se fait par des arguments standard de minimax. En plus il est bien de rappeler que, même sous l'hypothèse $a(x) = a(|x|)$, on peut trouver aussi l'existence de solutions en nombre infini non radiales qui changent de signe, en brisant la symétrie radiale de l'équation (see [5] et références ici).

[3] G. Cerami, G. Devillanova and S. Solimini *Infinitely many bound states for some non linear scalar field equation* Calc. of Var. and PDE's, Vol. 23 N. 2, 139-168.

Quand $a(x)$ ne jouit d'aucune propriété de symétrie, le problème devient plus difficile et même démontrer l'existence d'une solution positive n'est pas une affaire banale. Cette situation demande une compréhension plus profonde de la nature des obstructions à la compacité et l'emploi d'instruments plus subtils. La plupart des recherches se sont intéressées au cas

$$\lim_{|x|\to+\infty} a(x) = a_\infty > 0 \tag{0.0.4}$$

ainsi que (P) peut être rapporté au "problème à l'infini"

$$(P_\infty) \qquad -\Delta u + a_\infty u = |u|^{p-2}u \qquad \text{in } I\!R^N .$$

On a donné une première réponse à la question de l'existence en démontrant que, dans quelques cas, étant vraies quelques solutions qui apportent des inégalités entre les solutions de (P) et (P_∞), on peut appliquer le principe de concentration-compacités et on peu résoudre (P) par minimisation, voir [20]. C'est le cas, par exemple, quand $a(x)$ est une fonction continue qui a en plus (0.0.10) et quelques hypothèses de décroissance, satisfait

$$0 < \delta_1 \le a(x) \le a_\infty \qquad \forall x \in I\!R^N . \tag{0.0.5}$$

Par conséquent, une analyse attentive du comportement des séquences de Palais-Smale (voir [6], [3]) a permis d'affirmer que la compacité peut être perdue (dans le sens qu'une PS-séquence ne converge pas vers un point critique) si et seulement si cette séquence se brise en un nombre fini de solutions de (P_∞) qui sont *centrés* en des points qui vont à l'infini. Comme conséquence, il a été possible d'estimer les niveaux d'énergie où la condition PS échoue en termes d'énergie de ces masses et de mieux faire face à quelques questions d'existence et de multiplicité pour (P). En fait, l'existence d'une solution positive à (P) a été démontrée (voir [3]) même quand une solution de *ground state* ne peut pas exister, c'est-à-dire, par exemple, quand, en plus (0.0.10) et quelques opportunes hypothèses de décroissance, la potentielle satisfait la condition $a(x) > a_\infty$ pour tout $x \in I\!R^N$; en plus, dans les conditions (0.0.10), (0.0.11) et une opportune hypothèse de décroissance à l'infini, on a démontré l'existence d'une solution qui change de signe en plus de la solution positive (voir [33]). En suivant [13] on pose la fonction a pour satisfaire les conditions suivantes.

(a_1) $a \in \mathcal{C}^1(I\!R^N, I\!R)$;

(a_2) $\liminf_{|x|\to+\infty} a(x) = a_\infty > 0$;

(a_3) $\dfrac{\partial a}{\partial \vec{x}}(x) e^{\alpha|x|} \underset{|x|\to+\infty}{\longrightarrow} +\infty$, $\forall \alpha > 0$, où $\forall x \in I\!R^N \setminus \{0\}$, $\vec{x} = \frac{x}{|x|}$;

(a_4) il y a une constante $\overline{c} > 1$ telle que

$$|\nabla_{\tau_x} a(x)| \leq \overline{c}\, \frac{\partial a}{\partial \vec{x}}(x) \qquad \forall x \in I\!R^N : |x| > \overline{c},$$

où $\nabla_{\tau_x} a(x)$ dénote la composante de $\nabla a(x)$ dans l'hyperplan orthogonal au $\vec{x}$ et qui contient x.

Les hypothèses $(a_1) - (a_2)$ permettent que le terme super-quadratique de la fonctionnelle soit borné en termes de la partie quadratique et permettent la démonstration d'un théorème de décomposition qui est semblable à celui de [6]. La régularité demandée en (a_1) est plus forte que celle dont on a besoin dans ce but ($a^- \in L^{\frac{N}{2}}(I\!R^N)$ serait plus que suffisant) mais elle est demandée pour les autres hypothèses. La condition (a_3) est équivalente à la borne $N \geq 7$ pour le problème (CP). La démonstration des résultats de multiplicité sans cette hypothèse demanderait probablement des arguments différents. On peut démontrer aussi des résultats d'unicité dans quelques cas particuliers, comme il arrive pour le Problème (CP) dans l'hypothèse de symétrie radiale. Au contraire (a_4) a été utilisé seulement dans le but de passer d'une intégrale de la fonction $\frac{\partial a}{\partial \vec{t}}$ à une intégrale de $\frac{\partial a}{\partial \vec{x}}$ et pour démontrer qu'il n'est pas admis que les masses fuient. Elle peut être certainement affaiblie, le problème est de savoir si quelque hypothèse de ce type est nécessaire pour le résultat de multiplicité ou quelle est la condition minimale est ouvert.

De la même façon que pour le problème (CP) on change la notion de "quasi solution" et on démontre la compacité pour ce nouvel objet. En fait on abandonne la notion de séquences PS et adopte la notion de "séquences contrôlées " ("controlled sequences" en anglais) parce que le rôle du problème limite est joué par l'inégalité elliptique $-\Delta u + a_\infty u \leq |u|^{p-2}u$.

D'autre part si on pose qu'une séquence limitée contrôlée n'est pas compacte, on démontre qu'elle se "brise" en quelques masses qui vont à l'infini et, en changeant le vecteur de translation, qui correspond à une des masses qui vont à l'infini de façon plus lente, on peut produire une modification locale de la fonctionnelle approximative qui a le même ordre de la variation de la fonction en obtenant ainsi une contradiction. De cette contradiction on tire que toute les séquences limitées contrôlées, au maximum en passant à une sous séquence, doivent converger vers une solution non banale du problème. Comme fait dans le cas du Problème (CP), ici aussi la variation de la fonctionnelle sous une telle modification locale a été analysée avec des instruments d'une opportune inégalité locale du type Pohozaev et quelques estimées *a priori* uniformes de décroissance dans les termes d'une séquence limitée contrôlée et leurs dérivées qui, transportées dans cette inégalité, produisent clairement la contradiction.

Part II - Des propriétés sur les structures de irrigation et le mesures irrigables

Ces dernières 20 années on a considéré plusieurs problèmes de Calcul des Variations (Calculus of Variations) impliqués dans la modélisation de structures singulières. Entre autres on peut mentionner les problème avec des discontinuités libres, introduites au début pour la segmentation de l'image. Le but était de déterminer l'ensemble des points de la discontinuité d'une fonction donnée, c'est-à-dire d'une image numérique donnée.

La solution qui minimise la fonctionnelle, introduite au début par Mumford et Shah en [34], est pour cette raison irrégulière et ce manque de régularité n'est pas un point faible de la théorie mais l'ensemble de la discontinuité contient, au contraire, l'information principale qu'on cherche. Il y a beaucoup d'autres problèmes d'interfaces et de frontière libres où le manqué de régularité est essentiel parce qu'elles ont été introduites dans le but d'uniformiser les phenomenes qu'on ne peut décrire que par des structures singulières.

Dans ce grand contexte on peut aussi inclure quelques problèmes de transport où la densité de transport devient concentrée dans un ensemble uni-dimensionnel, comme les problèmes de transport avec un ensemble de Dirichlet ou les problèmes d'irrigation. Les derniers viennent de l'observation que le but de beaucoup de systèmes naturels de flux est celui d'irriguer un volume fini à partir d'une source. Dans les travaux concernant les réseaux de drainage, plantes, arbres, systèmes de racines, systèmes bronchiques et systèmes cardiovasculaires (voir [36], [37] et [38]), le même système de réseau est dessinée d'après des principes, c'est-à-dire axiomes, surtout les suivants, voir par exemple [36] et les références qu'il contient:

- le réseau irrigue un volume entier d'un organisme et la structure qui remplit l'espace doit avoir un branchement hiérarchique;
- les réseaux biologiques ont évolué pour minimiser la dissipation d'énergie;
- la mesure des branches finales du réseau est une unité invariante de grandeur;
- l'égalité de la fourniture de flux par le système du réseau.

En plus, on utilise d'habitude une autre hypothèse de base, c'est-à-dire que le réseau est une structure arborescente faite à chaque échelle de tubes d'une certaine uniforme longueur, rayons et avec un nombre donné de branches. Le résultat est que le réseau a une structure presque fractale avec des propriétés auto-similaires (voir [36], [37] et [38]). Le point faible de ce raisonnement heuristique et qu'on "suppose" l'existence d'un réseau qui accomplit son travail et que ce réseau est une structure arborée. En plus, il n'y a pas de théorie générale basée sur des lois fondamentales d'où on peut déduire le comportement fractal.

Dans le travail de pionnier de Maddalena, Morel and Solimini (voir [32]) les auteurs ont uniformisé "la forme" d'un arbre seulement en minimisant un certain type d'énergie fonctionnelle. De cette façon, ils ont posé sur des bases mathématiques le susdites lois empiriques hypothèses dans l'étude de systèmes d'irrigation et drainage en laissant à quelqu'un de les démontrer comme théorèmes ou de les laisser comme conjectures mathématiques claires. L'étude n'est pas, pourtant, le premier essai mathématique et on sait que d'autres travaux affrontent le problème de l'existence, notamment [28].

Dans [28] Caselles and Morel abordent le the problème de trouver un volume maximal irrigué avec un coût minimal. Ils fixent un domaine ouvert Ω et un point S en lui, après ils donnent la notion d'ensembles irrigable en disant qu'un ensemble compact $K \subset \Omega$ est irrigable si le complémentaire $U = \Omega \setminus K$, appelé réseau d'irrigation, est connexe et contient S. Puis ils introduisent un " profil d'accessibilité ", c'est-à-dire une fonction positive, croissante définie en $\mathbb{R}_+$ qui est nulle en zéro pour définir les points irrigables d'un ensemble irrigable compact K. Ils appellent un point $x \in K$ f*-irrigable* s'il y a un chemin $x(s)$ paramétré sur $[0, L]$ tel que $x(0) = x$ and $x(L) = S$ et pour tout $s \in [0, L]$ la boule du centre $x(s)$ avec rayon $f(s)$ se trouve dans le réseau d'irrigation U. En d'autres mots ils disent qu'un point x d'un ensemble irrigable K est f-irrigable s'il y a une voie dans le réseau d'irrigation qui relie x à S. Le profil f est une fonction croissante et donc cette voie devient de plus en plus petite en se rapprochant de x mais le rapport de rapetissement reste uniformément borné inférieurement. Appelant ensembles f-irrigables les sous-ensembles d'un ensemble irrigable K qui consistent de points f-irrigables, les auteurs ont démontré l'existence d'un ensemble f-irrigable avec un volume maximal positif sous l'hypothèse de sous-linéarité de f (comme, par exemple, le profil $f(s) = s^{\alpha}$ avec $0 < \alpha < 1$). Puis ils associent à toute voie d'accessibilité $x(s)$ pour atteindre x un coût qu'on suppose semi-continue inférieurement par rapport à la convergence uniforme de chemins en assignant de cette façon à tout point $x \in K$ un coût minimal d'accessibilité $c_K(x)$. Pour cela en appelant le coût d'irrigation de K la fonctionnelle $c(K) = \int_K c_K(x)dx$ ils trouvent un volume maximal irrigué K avec un coût minimal.

Ces problèmes récents sont souvent insérés dans la littérature du problème de transport de Monge même si le problème qu'ils abordent est radicalement différent de celui proposé par Monge en [33]. En effet, dans le modèle de Monge-Kantorovitch le coût d'une unique particule de fluide n'est pas influencé par les interactions avec la partie restante du fluide ou par leur mouvement alors que, dans ce contexte, on n'est pas tellement intéressé à connaître la destination finale d'une unique particule de fluide (le problème "qui va où ") comme dans la forme de l'ensemble des trajectoires, en sachant, en particulier, que si les particules bougent ensemble elle donnent origine à un grand flot et si elles bougent "solitaires" elles donnent origine à beaucoup de petits flots.

Un *irrigation pattern* avec source en un point $S \in \mathbb{R}^N$ est une application

$$\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$$

telle que:

C1) pour presque tout *point matériel* $p \in \Omega$, la "fibre" $\chi_p : t \mapsto \chi(p,t)$ est une application Lipschitz continue avec une constante de Lipschitz plus petite que ou égale à un;

C2) pour presque tout $p \in \Omega$: $\chi_p(0) = S$;

où $(\Omega, \ |\cdot|)$ est un espace de probabilité non atomique qui est interprété comme la configuration de référence d'un corps fluide. On peut penser que Ω joue le rôle d'un section d'un tronc d'arbre. Ce tronc est imaginé comme un ensemble de fibres qui peuvent se diviser en branches.

$\mathbf{P}_S(\Omega)$ représente l'ensemble de tous les *irrigation patterns* de Ω avec source en $S \in I\!R^N$.

Tout $\chi \in \mathbf{P}_S(\Omega)$, à tout instant , définit une relation d'équivalence $\simeq_t$ sur Ω en mettant en relation au temps t deux points p et $q \in \Omega$ si les deux fibres χ_p et χ_q coincident en $[0,\, t]$. Ainsi tout *irrigation pattern* à tout temps t divise Ω en classes d'équivalence appelées χ-*vessels*. Pour tout $p \in \Omega$, $[p]_t$ représente le χ-*vessel* au temps t qui contient p, alors que pour tout $t \geq 0$ $\mathcal{V}_t(\chi)$ représente l'ensemble de tous les χ-*vessels* au temps t.

La fonction $\sigma_\chi : \Omega \to I\!R_+$ définie, $\forall p \in \Omega$, comme la borne inférieure de l'ensemble $\{t \in I\!R_+ \mid \chi_p(\cdot)$ est constante en $[t, +\infty[\}$, donne le temps d'absorption d'un point et est appelée fonction de *stopping* ou d'*absorption* pour χ. L'ensemble

$$M_t(\chi) = \{p \in \Omega \mid \sigma_\chi(p) > t\}$$

est l'ensemble des points qui, au temps t, ne se sont pas encore arrêtés.

La *function d'irrigation* $i_\chi : A_\chi \to I\!R^N$, definie en posant $\forall p \in A_\chi : i_\chi(p) = \chi(p, \sigma_\chi(p))$ donne, à tout point absorbé, la position d'absorption.

La fonction i_χ induit sur $I\!R^N$ la mesure image (push-forward) μ_χ définie par la formule

$$\mu_\chi(A) = |i_\chi^{-1}(A)| \ ,$$

pour tout ensemble de Borel $A \subset I\!R^N$. On se réfère à μ_χ comme à l'*irrigation measure* produit par la structure χ.

Pour un exposant fixe $\alpha \in]0,1[$, on introduit le coût fonctionnel I_α as in [32], défini dans l'ensemble $\mathbf{P}_S(\Omega)$ de toutes les structures d'irrigation χ, par la formule suivante

$$I_\alpha(\chi) = \int_{I\!R_+} \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp\, dt \ .$$

Cette fonctionnelle ne prend en condsidération que les points en mouvement. Comme $0 < \alpha < 1$, la fonctionelle pénalise les points mobiles dans des récipients minces, c'est-à-dire la fonctionnelle pénalise l'embranchement. Donée une mesure de probabilité $\overline{\mu}$, on dit que c'est α-irrigable, dans le sens de [32], s'il y a une structure d'irrigation χ de coût fini tal que $\mu_\chi = \overline{\mu}$.

Le Problème de Dirichlet qui consiste à minimiser en $P_S(\Omega)$ la fonctionnelle $I_\alpha(\chi) + J(\mu_\chi)$, où J est défini en posant

$$J(\mu) = \begin{cases} 0 & \text{si } \mu = \overline{\mu} \\ +\infty & \text{otherwise} \end{cases} \tag{0.0.6}$$

est analysé. Le but de la fonctionnelle I est de contraindre les fibres àrester ensemble en pénalisant, de cette façon, leur embranchement. La nécessité de maintenir la fonctionnelle à un bas niveau entre en compétition avec la condition de contour exprimée par la présence de la fonctionnelle J qui, d'autre part, contraint les fibres à se diviser en deux vu que le fluide qu'elles transportent doit attendre une mesure donnée distribuée dans un volume. Le résultat de cette compétition est que les fibres ont avantage à se tenir ensemble le plus longtemps possible et puis qu'elles se divisent en s'approchant des points finaux, en mettant en évidence la structure branchée. Tout minimum χ de $I_\alpha(\chi) + J(\mu_\chi)$ est appelé *une structure d'irrigation optimale ("optimal irrigation pattern" en anglais) pour* $\overline{\mu}$ et $I_\alpha(\chi)$ est appelé *(irrigation) coût* de la mesure de probabilité $\overline{\mu}$. En bref on dit que χ is *une structure optimale* s'il est optimal pour sa mesure d'irrigation μ_χ.

Une fonctionnelle très semblable a celle utilisée en [32] a été proposé en [35] seulement dans le but de décrire une variante du problèm de transport de Monge-Kantorovitch. Dans cette approche, soit le début que la configuration du cible sont décrits avec des instrument d'approximation avec de mesures atomiques finies mais la configuration de début n'est pas nécessairement un point source comme supposé en [32]. Dans [32] et dans [35] les auteurs supposent que le mouvement d'une particule est influencé par d'autres particules dans le sens que c'est moins cher que deux ou plus particules bougent ensemble plutôt qu'elles coulent toutes seules. Ainsi les deux différentes approches mènent à une fonctionnelle très similaire. Dans des articles plus récents (voir par exemple [27] et [34]), ces questions sont vues comme des problèmes particuliers d'évolution de mesures en Espaces de Wasserstein.

En [32] les auteurs ont abordé the problème en supposant que la morphologie des arbres dérive par un essai de nature pour réduire une opportune fonctionnelle d'énergie. De cette façon, il n'ont pas explicitement considéré les lois hydrodynamiques, comme, par exemple, la lois de Kirchhoff pour laisser l'énergie constant ou la lois de Poiseuille pour quantifier la résistance qu'un fluide rencontre pendant qu'il coule dans un réseau de tubes, laissant aux études successives la raison de cette fonctionnelle, ou de quelques variantes proches avec un comportement similaire, à partir de lois hydrodynamiques.

Comme dans les problèmes classiques de Calcul des Variation (Calculus of Variations) la minimisation des fonctionnelle convexes amène à une solution régulière, en [32] les auteurs introduisent une fonctionnelle avec des termes concaves qui rendent les concentrations convenables et aènent à la présence de singularités dans les solutions.

Le but de la fonctionnelle d'irrigation est d'obliger les fibres de rester ensemble en pénalisant, de cette façon, leur branchement. La nécessité de maintenir la fonctionnelle basse se heurte à une condition de frontière qui, d'autre part, contraint les fibres à se diviser en deux, en établissant que le fluide qu'elles transportent doit atteindre une mesure donnée répandue dans un volume. Le résultat de cette compétition est que les fibres ont avantage à rester ensemble le plus longtemps possible et puis en se divisant en branches pendant qu'elle se rapprochent aux points terminaux, en donnant naissance à une structure branchée.

Chapitre 4 *Propriétés élémentaires de structures d'irrigation optimales* [4]

L'existence de minima, démontré en [32], pose un problème de régularité qui devrait mener à la preuve de la structure arborée et des propriétés de self similarité.

Comme premier pas dans cette direction, dans ce chapitre on commence cette étude par l'identification de quelques propriétés élémentaires géométriques, dont jouissent les structures d'irrigation optimales. Les propriétés dérivées pour des structures optimales sont des "propriétés élémentaires" dans le sens qu'elles ne sont pas impliquées dans la régularité aux points terminaux de ces structures, où les présumables propriétés de self similarité devraient se trouver. Parmi ces propriétés, on rappelle la notion de *structure simple* caractérisée par les propriétés suivantes:

- pour p.p.t. point $p \in \Omega$ la fibre ouverte χ_p est une courbe simple en $[0, \sigma_\chi(p)[$;

- for a.e. paires de points p et q de Ω $\chi_p(t) \neq \chi_q(s)$ pour tous $s,\, t > s_\chi(p,q)$, $t < \sigma_\chi(p)$, $s < \sigma_\chi(q)$, où $\sigma_\chi(q) = \inf\{t \in I\!R_+ \mid \chi_p(t) \neq \chi_q(t)\}$ est le *temps de separation* de les points p et q;

exprimée en termes de fibres ou également par la notion de courbes de flux qui, plus clairement, représente les trajectoires suivies par le flux. Après l'introduction de la notion de branche et du coût d'une branche, on a un théorème qui clairement nous permet de *tailler un arbre* en sorte que la somme du coût des branches taillées soit arbitrairement petite (voir [29, Pruning Theorem] pour la formulation précise). L'ensemble F_χ, appelé zone de flux ("flow zone" en anglais), qui contient les trajectoires des flux, peut être considéré comme le support d'une structure χ et en un certain sens le caractérise. Sous quelques hypothèses sur χ (c'est-à-dire χ étant non-spread), F_χ est l'union énumérable des supports de courbes rectifiables. Quand on suppose que χ est une structure simple, alors trois structures sont induites sur le support F_χ: un ordre partiel, une fonction "temps" et une fonction "quantité". Ces trois structures caractérisent la structure d'irrigation. En réalité on démontre qu'on peut établir ce type de structures dans un " ensemble branché " sans faire aucune référence à une structure. Ces structures, sous quelques hypothèses, identifient une classe de structures simples et bien paramétrées (un histogramme) par laquelle les trois structures s'accordent avec celle dérivées d'une structure d'irrigation.

[4] G. Devillanova, S. Solimini, *Elementary properties of optimal irrigation patterns* to appear

Pour cela le α-coût d'une structure non-spread simple et bien paramétrée peut être évalué par un intégral sur son support.

Cette étude préliminaire a le but de donner quelques instruments de base qui seront utiles dans l'approche des sus-dits problèmes de régularité et elle trouve déjà une application dans le Chapitre 5, où ces propriétés sont utilisées pour analyser l'irrigabilité d'une mesure donnée.

Chapitre 5 *Sur la dimension d'une mesure irrigable* [5]

Dans ce dernier chapitre on explore si une mesure est irrigable en termes d'un ensemble dans lequel elle se concentre, comme suggéré par le fait qu'une mesure répandue dans un ensemble de haute dimension contraint les fibres à un embranchement plus fréquent et pour cette raison la structure à augmenter son coût. Dans ce but, on introduit la notion de dimension d'irrigation ("irrigability dimension") $d(\mu)$ d'une mesure de probabilité μ et puis on exprime le problème posé plus avant en termes de donner quelques estimations sur la dimension d'irrigation d'une mesure positive donnée qu'on suppose être Borel régulière, avec un support borné et une masse finie (par normalisation on suppose qu'elle est une mesure de probabilité). On montre, avec quelques exemples, que l'idée intuitive et discutable que la dimension d'irrigation d'une mesure concide avec la dimension de Hausdorff de son support est fausse, malgré le fait que les deux valeurs expriment combien la mesure est répandue. D'autre part, on donne quelques bornes par le haut et par le bas pour la dimension d'irrigation $d(\mu)$ d'une mesure de probabilité μ à travers la dimension minimale de Hausdorff et respectivement de Minkowski d'un ensemble où la mesure est concentrée.

Ce résultat sera démontré avec différentes approches. En fait, on le démontre directement, avec quelques informations plus significatives et en introduisant quelques instruments qui seront aussi utilisés dans d'autres parties du chapitre mais on pourra le déduire par une estimation plus approfondie de $d(\mu)$ qui demandera l'introduction de nouvelles notions. Plus précisément, il faudra la notion de *dimension de résolution* d'une mesure qui, intéressée par un index, exprime la possibilité de décrire la mesure par des approximations discrètes. Quand la mesure est opportunément régulière, la valeur de la dimension de résolution ne dépend pas de l'index, alors que pour une mesure générique, comme on expliquera avec quelques exemples, la dimension de résolution est "au dehors du focus" dans le sens que des index différents donnent des valeurs différentes. On montre que, en tout cas, il est toujours possible de trouver un index, opportunément caractérisé, qui donne une dimension de résolution qui concide avec la dimension d'irrigation.

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Introduction

This thesis represents the synthesis of the results obtained during these last four years of research and study under the guide and teachings of professor S. Solimini and professor J.-M. Morel. In this thesis we approach some problems of Nonlinear Analysis and of Calculus of Variations which give rise to "singular" structures. Our purpose is to show how in some cases these singular structures are an obstacle to overcome to get existence or multiplicity results and in other cases the singularities in the solutions are the object of the study and justify the choice of the functional introduced. The work is divided into two parts: the first one concerns results on a class of problems, which have been faced by researchers in nonlinear analysis in the last twenty years, about the existence and the multiplicity of solutions of elliptic equations, obtained in spite of a lack of compactness due to some concentration phenomena; the second one regards a certain category of irrigation problems in which the trajectories followed by fluid particles give rise to a one-dimensional set and which can be set in the transport theory.

In particular, in the first two chapters of this thesis we deal with the following critical growth problem

$$\text{(CP)} \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, and $\lambda > 0$. For this problem we get infinitely many solutions for $N \geq 7$ and only a finite multiplicity result which leaves the question about the infinite multiplicity still open when $N \geq 4$. More precisely, we find at least $\frac{N}{2} + 1$ (pairs of) solutions (or even $N + 1$ if λ is suitably close to zero) and $\lambda \in]0, \lambda_1[$, λ_1 being the first eigenvalue of $-\Delta$ defined on $H_0^1(\Omega)$.

The lack of compactness in problem (CP) is due to the "concentration" of Palais Smale sequences for the relative energy functional at some levels, as described trough arguments of concentration-compactness due to P.L. Lions (see [22]) or as in Struwe Compactness Theorem (see [25] and [26]).

The third chapter is devoted to the problem

$$\text{(P)} \qquad \begin{cases} -\Delta u + a(x)u = |u|^{p-2}u & \text{in } I\!R^N \\ u \in H^1(I\!R^N), \end{cases}$$

where $N \geq 2, p > 2, p < 2^*$ (when $N > 2$), and the potential $a(x)$ is a continuous function, positive in $I\!R^N$, except at most a bounded set, verifying suitable decay assumption but not required to possess any symmetry property. The lack of compactness in problem (P) is structurally similar, in the spirit of Concentration-Compactness, to that in problem (CP). What happens in this case is that the lack of compactness is fundamentally due to the domain being unbounded, which allows the "escaping" to infinity of masses while taking the limit of a Palais Smale sequence, as described in a compactness theorem of V. Benci and G. Cerami (see [6]).

In the last two chapters (Part II) we follow the approach in [32] where the authors have introduced a cost functional aiming at modeling ramified structures, such as trees, root systems, lungs and cardiovascular systems for the study of the *ramified structures* defined on very large search space consisting in what the authors call irrigation patterns. This approach is very different from that used by Qinglan Xia in [35]. Actually for Xia the network is an embedded graph with a countable number of vertices which satisfy the Kirchhoff Law, moreover when such graphs G are finite the author considers, for $\alpha \in]0, 1[$, an energy functional which is formally the same as the one considered in [32]

$$E^{\alpha}(G) = \sum_{e \text{ edge of } G} w(e)^{\alpha} \text{length}(e) ,$$

where $w(e)$ represents the amount of fluid carried along the edge e. The parameter α being less than 1 implies that byforcations are not convenient from the energy saving viewpoint stating in this way a qualitative Poiseuille Law according to which the resistance of a tube increases when it gets thinner.

An irrigation system on the contrary is not defined as an embedded graph, but as measurable set of paths (fibers) starting from a unique source and stopping in some points in $I\!R^N$ on which what the authors call irrigation measure is concentrated. The search space for a solution is larger since it allows a priori spreading trees for which could exist sets of positive measure of particles which could go its way without following a branch (see fig. 1). The formalism in [32] considers paths starting from the source and can be interpreted either as the trajectory in $I\!R^N$ of a fluid particle or as a fiber of a tree. These infinitely many paths (in principle one for each particle) depending on a parameter t are called "fibers" and denoted by $\chi(p, t)$ where t represents the time in which the particle p, belonging to an abstract probability space Ω, gets the position $\chi(p, t) \in I\!R^N$. Any measurable set of fibers χ, i.e. any irrigation pattern, induces on $I\!R^N$ an "irrigation measure" μ_χ which gives to any Borel set A of $I\!R^N$ the amount, in Ω, of the set of the particles which stop in A (i.e. which are absorbed in the set A). Always in [32] the authors proposed a cost functional I_α defined on the set of all the irrigation patterns by setting

$$I_\alpha(\chi) = \int_{I\!R_+} \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp\, dt .$$

This functional, formally similar to E^α, takes into account, time by time, only the points $p \in M_t(\chi)$ i.e. the points which are still moving at the time t and, taking the power $\alpha - 1 < 0$ of the amount of the set $[p]_t$ of particles which have flown in a solid way up to the time t, penalizes the branching leading to a structure shaped as in fig. 2.

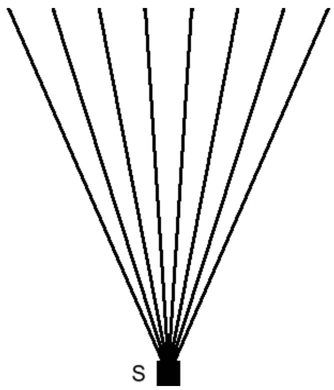

fig. 1

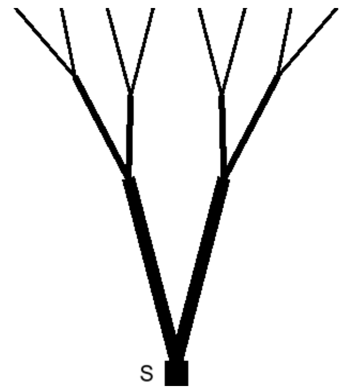

fig. 2

The properties derived in Chapter 4 for optimal patterns (an irrigation pattern χ is an optimal pattern if it minimizes the cost among all the irrigation patterns which have the same irrigation measure μ_χ) are "elementary properties" in the sense that they are not concerned with the regularity at the ending points of these structures, where the presumable selfsimilarity properties should take place. This preliminary study already finds an application in Chapter 5 where the problem of determining if a given measure is irrigable or not is addressed (a probability measure μ is "α-irrigable" if there exists an irrigation pattern χ such that $I_\alpha(\chi) < \infty$ and $\mu_\chi = \mu$). The answer to this question clearly shows, in particular, what are the cases in which all the probability measures will turn out to be irrigable, giving in this way a different proof of a result in [35] in a very close setting. We investigate a more general question consisting in characterizing, for a given value of the index, what probability measures are irrigable or not. To this aim a notion of irrigability dimension of a measure is given and lower and upper bounds are proved in terms of the minimal Hausdorff and respectively Minkowski dimension of a set on which the measure is concentrated. A notion of resolution dimension of a measure based on its discrete approximations is also introduced and its relation with the irrigation dimension is studied.

A more detailed abstract of the two parts of this thesis follows.

Part I - Multiplicity results for some elliptic problems with lack of compactness

Before the eighties the elliptic equations theory was mainly developed under subcritical growth hypothesis. In particular the model problem

$$-\Delta u = |u|^{p-2}u \ , \tag{0.0.7}$$

with boundary Dirichlet condition, was studied for $p < 2^* = \frac{2N}{N-2}$ in dimension $N > 2$.

In the case of critical growth $p = 2^*$, being the embedding of $H_0^1(\Omega)$ into L^p not compact, the Nemitsky operator $|u|^{p-2}u$ is not compact and so one can not get a nontrivial solution to (P) by using the most standard variational techniques. Moreover Pohozaev, in his celebrated paper [22], stated an identity which, applied to problem (P) for a star-shaped domain, proved the nonexistence of nontrivial solution in the case of critical growth. On the other hand, the critical exponent $p = 2^*$ is the only one which makes the energy functional $I_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}$ homogeneous with respect to scalings. In their pioneering work [9], Brezis and Nirenberg had the idea to introduce a lower order term, i.e. a subcritical term, in particular the linear one λu, which does not change the growth of the energy functional but breaks its homogeneity penalizing, even if in an infinitesimal way, concentrations. In alternative to Pohozaev identity, which guaranteed that in general (for a star-shaped domain) (CP) has no nontrivial solution for $\lambda \leq 0$, in [9] Brezis and Nirenberg get the existence result for $0 < \lambda < \lambda_1$ (λ suitably bigger that zero in dimension three). Subsequently, the result has been extended in [10] to $\lambda > \lambda_1$. Therefore, in the case of critical growth, the breaking of the homogeneity of the energy functional can give the existence result for a suitable elliptic problem. In particular, by changing the deviation from the homogeneity of the energy functional (i.e. changing from $\lambda < 0$ to $\lambda > 0$ in the case of problem (CP)) one switches from nonexistence to existence results. The problem of getting multiplicity results for problem (CP) was raised in that period and techniques, alternative to the use of the Rellich Theorem, where introduced to find solution as the limit of a Palais Smale sequence (or, more briefly, PS sequence) for the energy functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} .$$

In particular, the works on concentration-compactness such as [20], [25] and [24] gave a precise analysis of the obstruction to compactness, pointing out the *bad levels* at which the PS condition fails. In Chapter 1 the question about existence of infinitely many solutions to Problem (CP), for any bounded smooth domain $\Omega \subset \mathbb{R}^N$ in the case $N \geq 7$, is affirmatively answered.

Analogous problems are met in finding existence and multiplicity results for elliptic problems at subcritical growth on the whole domain such as Problem (P). In the case in which the coefficient $a(x)$ is not symmetric the lack of compactness comes out by giving rise to some masses which are concentrating around some points which escape to infinity. In Chapter 3 the problem of getting existence and multiplicity results to Problem (P) is addressed and infinitely many solutions are found without asking symmetry for $a(x)$. We just fit the same techniques employed to get infinitely many solutions to Problem (CP).

Chapter 1 *Concentrations estimates and multiple solutions to elliptic problems at critical growth* [1]

The first multiplicity results on the number of solutions to Problem (CP) where obtained for suitably symmetric domains, see [19], in particular the authors essentially reduced the multiplicity result to the existence one by slicing the domain into a finite number of equal subdomains and then glueing the solutions found on each slice (the existence of infinitely many solutions follows from the possibility of arbitrarily choosing the number of the slices).

This kind of approach can not be applied when one has to work in a general shape domain. Even for symmetric domains it may fail when one searches for multiplicity results of solutions of a particular type, for instance the radial solutions when the domain is a ball. The radial case actually allows to give complementary existence and non existence results in sharp correspondence. If $N \geq 7$ and Ω is a ball, then for any $\lambda > 0$ (CP) has infinitely many changing sign radial solutions, see [23] and [12], if $4 \leq N \leq 6$ there exists a constant $\lambda^* > 0$ such that (CP) has no changing sign radial solution if $\lambda \in]0, \lambda^*[$ (see [1], [2]). So the bound $N \geq 7$ in the previous existence result cannot be removed.

In this chapter the question about existence of infinitely many solutions to Problem (CP), for any bounded smooth domain $\Omega \subset I\!R^N$ in the case $N \geq 7$, is affirmatively answered. The new idea, which has allowed us to prove the existence of infinitely many solutions to Problem (CP) has been that of changing the concept of "almost solution" and prove compactness for this new object. Actually the usual concept of almost solution in terms of the H^{-1}-norm of the Fréchet derivative of the energy functional, which leads to the well known notion of PS sequences does not allow us to deduce any compactness results in such a context. The PS sequences have been substituted in favor of sequences of solutions of approximating problems (called "balanced sequences"). Every term u_n of a balanced sequence is a solution in $H^1_0(\Omega)$ to the problem $-\Delta u = |u|^{p_n-2}u + \lambda u$ where the corresponding sequence of subcritical exponents p_n converges to 2^*. The circumstance that u_n is a solution of an autonomous problem of the same type of Problem (CP) allows us to establish a local Pohozaev inequality which, roughly speaking, takes into account the modification of the energy functional I_n (which is relative to the n-th approximating problem) with respect to concentrations. We have firstly proved that also for bounded balanced sequences which are supposed to be noncompact there is the same concentration phenomena which occurs for noncompact PS sequences. By working with noncompact balanced sequences and by perturbing their concentration parameter we are then able to produce a local modification of the approximating functional which is of the same order as the function in contradiction to the fact that the elements of a balanced sequence are critical points. The variation of the functional under such a local modification has been

[1] G. Devillanova & S. Solimini *Concentrations estimates and multiple solutions to elliptic problems at critical growth.* Advances in Differential Equations, 7 (2002), 1257-1280

evaluated by a local Pohozaev inequality and some *a priori* uniform decay estimates on the terms of a bounded balanced sequence and of their derivatives which, carried in such inequality, formally produce the contradiction.

This compactness result can be employed also in the radial case, producing the existence of infinitely many radial solutions. By the results in[1] and [2], in which the uniqueness of a nontrivial radial solution is stated for $N < 7$, we deduce that the compactness argument is false for lower dimensions. This circumstance does not mean that the existence of infinitely many solutions cannot be proved through different tools as it happens, for instance, in the case of symmetric domains (see [19]).

The dimension seven has no particular meaning as happens, for instance, with the minimal surface problems, but depends on the linearity of the subcritical perturbation term λu. A nonlinear subcritical term would require a different bound on the dimension. Let us point out that $N \geq 4$ is enough for the existence of a nontrivial solution for every $\lambda > 0$ but three more dimensions are needed for multiplicity results. The reason is due to the fact that starting on dimension 4, the gain due to the quadratic term $-\lambda \int_\Omega u^2$ in the functional I_λ is more consistent than the cost of a cut-off term which brings a Talenti function to zero within the boundary of Ω. On the other hand, if one wants to perform a similar test for the multiplicity problem, one must cut-off the function in order to reach a value of opposite sign. So the cost of the cut-off is considerably more expensive and three more dimensions are needed to reach the appropriate estimate.

Chapter 2 *A Multiplicity Result for Elliptic Equations at Critical Growth in Low Dimension* [2]

We have approached problem (CP) in low dimension $N \geq 4$ and we have shown, working on the double natural constraint

$$U = \left\{ u \in H_0^1(\Omega) \mid u^\pm \neq 0,\ (\nabla I_\lambda(u), u^\pm) = 0 \text{ , for both} \pm \text{signs} \right\} , \tag{0.0.8}$$

that, for $\lambda \in]0, \lambda_1[$ problem (CP) has at least $\frac{N}{2} + 1$ (pairs of) solutions ($N+1$ for λ close enough to 0) improving thus the result in [11] which has been obtained, working on the constraint U, by searching for solutions which change sign near the boundary. The idea is that of making a comparison between the problem in a general domain with the problem in a ball where one can use symmetry. The concentration phenomena below the threshold $\frac{2}{N}S^{\frac{N}{2}}$ (being S the so called Sobolev constant) is due only to a unique Talenti function and therefore if $c < \frac{2}{N}S^{\frac{N}{2}}$ is a min-max level, then at least one between c and $c - \frac{1}{N}S^{\frac{N}{2}}$ is critical. Furthermore, we are able to find $N + 1$ min-max levels under the threshold $\frac{2}{N}S^{\frac{N}{2}}$, to which one must add the ground state level. We can also say that if some of

[2] G. Devillanova & S. Solimini *A Multiplicity Result for Elliptic Equations at Critical Growth in Low Dimension* Comm. in Contemporary Math., Vol. 5 N. 2(April 2003), 171-177.

these $N+1$ min-max levels coincide the problem admits infinitely many solutions, even if we do not know if such a level is critical or not (we do not know if we must decrease it of the amount $\frac{1}{N}S^{\frac{N}{2}}$). In this way, we get at least $\frac{N+2}{2}$ pairs of solutions.

This result has been recently extended in [14] to the case $\lambda \geq \lambda_1$. In any case, there is no reason to suppose that such a result may be optimal and the problem of proving the existence of infinitely many solutions, or even of getting an optimal estimate on the number of solutions, remain, as far as we know, largely open for $N \leq 6$.

Chapter 3 *Infinitely many bound states for some non linear scalar field equation* [3]

Finally, we have approached Problem (P) which is similar to those problems which naturally arise in various branches of Mathematical Physics; actually the solutions to (P) can be seen as solitary waves (stationary states) in nonlinear equations of the Klein-Gordon or Schrödinger types. Moreover, they present specific mathematical difficulties that make them challenging to the researchers. The solutions to Problem (P) can be searched as critical points of the energy functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2)dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx \,. \tag{0.0.9}$$

The usual variational methods, that allow to prove the existence of infinitely many solutions to (P) in a bounded domain, cannot be straightly applied to I. Indeed, the embedding $j : H^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is continuous but not compact, therefore the basic Palais-Smale condition is not satisfied by I at all energy levels. Also in this case there are some partial results when $a(x)$ enjoys some symmetry. Indeed, the first known results have been obtained considering $a(x) = a(|x|)$ or even $a(x) = a_\infty \in \mathbb{R}^+ \setminus \{0\}$ (see [21], [8], [15], [16], [24], [7]). In this case, the restriction of I to $H^1_r(\mathbb{R}^N)$, the subspace of $H^1(\mathbb{R}^N)$ consisting of spherically symmetric functions, restores compactness, because the embedding of $H^1_r(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ is compact. So, the existence of a positive solution to (P) can be proved either by using Mountain Pass Theorem or by minimization on a natural constraint, while the existence of infinitely many solutions follows by standard minimax arguments. Moreover it is worth recalling that, still under the assumption $a(x) = a(|x|)$, one can also find the existence of infinitely many nonradial changing sign solutions, breaking the radial symmetry of the equation (see [5] and reference therein).

When $a(x)$ does not enjoy any symmetry property, the problem becomes more difficult and even proving the existence of one positive solution is not a trivial matter. This situation requires a deeper understanding of the nature of the obstructions to the compactness

[3] G. Cerami, G. Devillanova and S. Solimini *Infinitely many bound states for some non linear scalar field equation* Calc. of Var. and PDE's., Vol. 23 N. 2, 139-168

and the use of more subtle tools. Most of the researches have been concerned with the case

$$\lim_{|x|\to+\infty} a(x) = a_\infty > 0 \tag{0.0.10}$$

so that (P) can be related to the "problem at infinity"

$$-\Delta u + a_\infty u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N . \tag{P_∞}$$

A first answer to the existence question has been given proving that, in some cases, being true some inequalities relating to solutions to (P) and to (P_∞), the concentration-compactness principle can be applied and (P) can be solved by minimization, see [20]. This is the case, for instance, when $a(x)$ is a continuous function that, besides (0.0.10) and some decay assumptions, satisfies

$$0 < \delta_1 \le a(x) \le a_\infty \qquad \forall x \in \mathbb{R}^N . \tag{0.0.11}$$

Subsequently, a careful analysis of the behavior of the Palais-Smale sequences (see [6], [3]) has allowed to state that the compactness can be lost (in the sense that a PS-sequence does not converge to a critical point) if and only if such a sequence breaks into a finite number of solutions to (P_∞) which are *centered* at points which go to infinity. As a consequence, it has been possible to give an estimate of the energy levels in which the PS condition fails in terms of the energy of such masses and to face better some existence and multiplicity questions for (P). Indeed, the existence of a positive solution to (P) has been proved (see [3]) even when a ground state solution cannot exist, that is, for instance, when, besides (0.0.10) and suitably decay assumptions, the potential satisfies the condition $a(x) > a_\infty$ for all $x \in \mathbb{R}^N$; moreover, under conditions (0.0.10), (0.0.11) and a suitable decay at infinity, it has been shown the existence of a changing sign solution in addition to the positive one (see [33]). Following [13] we assume the function a to satisfy the following conditions.

- (a_1) $a \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$;
- (a_2) $\liminf_{|x|\to+\infty} a(x) = a_\infty > 0$;
- (a_3) $\dfrac{\partial a}{\partial \vec{x}}(x) e^{\alpha|x|} \underset{|x|\to+\infty}{\longrightarrow} +\infty$, $\forall \alpha > 0$, where $\forall x \in \mathbb{R}^N \setminus \{0\}$, $\vec{x} = \frac{x}{|x|}$;
- (a_4) there exists a constant $\bar{c} > 1$ such that
$$|\nabla_{\tau_x} a(x)| \le \bar{c}\, \frac{\partial a}{\partial \vec{x}}(x) \qquad \forall x \in \mathbb{R}^N : |x| > \bar{c},$$
where $\nabla_{\tau_x} a(x)$ denotes the component of $\nabla a(x)$ lying in the hyperplane orthogonal to $\vec{x}$ and containing x.

The assumptions $(a_1)-(a_2)$ let the superquadratic term of the functional be bounded in terms of the quadratic part and allow the proof of a decomposition Theorem which is similar to that in [6]. The smoothness asked in (a_1) is more than what we need to this aim ($a^- \in L^{\frac{N}{2}}(I\!R^N)$ would be more than enough) but it is required for the other assumptions. Condition (a_3) is the equivalent of the bound $N \geq 7$ for (CP). The proof of the multiplicity results without this assumption would probably require completely different arguments. One can possibly show even uniqueness results in some particular cases, as happens for (CP) with the radial symmetry. On the contrary (a_4) has been only used to the aim of passing from an integral of the function $\frac{\partial a}{\partial t}$ to an integral of $\frac{\partial a}{\partial \bar{x}}$ and to prove that escaping of the masses is avoided. It can certainly be weakened; the question if some assumption of this kind is necessary for the multiplicity result or what is the minimal condition is open.

Analogously to problem (CP) we change the notion of "almost solution" and prove compactness for this new object. Actually we abandon the notion of PS sequences and adopt the notion of "controlled sequences", being the role of the limit problem played by the elliptic inequality $-\Delta u + a_\infty u \leq |u|^{p-2}u$.

On the other hand if one assumes that a bounded controlled sequence is noncompact we prove that it is "breaking" into some masses which are escaping to infinity and by perturbing the translation vector, which corresponds to one of the masses which runs toward infinity in the slowest way, we are then able to produce a local modification of the approximating functional which has the same order of the variation of the function getting in this way a contradiction. ¿From this contradiction we get that any bounded controlled sequence, at most passing to a subsequence, must converge to a nontrivial solution to the problem. As done in the case of Problem (CP), here also the variation of the functional under such a local modification has been evaluated by means of a suitable local Pohozaev inequality and some *a priori* uniform decay estimates on the terms of a bounded controlled sequence and of their derivatives which, carried in such inequality, formally produce the contradiction.

Part II - Some properties of irrigation patterns and irrigable measures

In the last twenty years several problems of Calculus of Variations concerned with modeling singular structures have been considered. Among them we can mention the problems with free discontinuities, initially introduced for image segmentation. The goal was to determine the set of the discontinuity points of a given function, i.e. of a given digital image. The solution which minimize the functional, firstly introduced by Mumford and Shah in [34], is therefore not regular and this lack of regularity is not a weak point of the theory but the discontinuity set contains, on the contrary, the main information which one is looking for. There are many other interface and free boundary problems in which the lack of regularity is essential because they have been introduced to the aim of modeling phenomena which can only be described by singular structures.

In this large context, we can also include some transport problems in which the transport density gets concentrated in a one-dimensional set, as the transport problems with a Dirichlet set or the irrigation problems. The last ones come from observing that the function of many natural flow systems is to connect by a fluid a finite size volume to a source. In works dealing with drainage networks, plants, trees, root systems, bronchial systems and cardiovascular systems (see [36], [37] and [38]), the network system itself is designed according to some principles, i.e. axioms which mainly are the following ones, see for instance [36] and the references therein:

- the network supplies an entire volume of an organism and a space filling hierarchical branching pattern is required;
- the biological networks have evolved to minimize energy dissipation;
- the size of the final branches of the network is a size-invariant unit;
- the equality of flow supply through the network system.

Moreover, another basic assumption is usually used, namely that the network is a branched tree structure made at each scale of tubes of a certain uniform length, radius and with a given branching number. The result is that the network has a fractal like structure with selfsimilar properties (see [36], [37] and [38]). The weak point in this euristic reasoning is that one "assumes" the existence of a network which does the task and that this network is a tree-like structure. Moreover, there was no general theory based on fundamental laws from which to deduce the fractal behavior.

In the pioneering work of Maddalena, Morel and Solimini (see [32]) the authors have modeled the "shape" of a tree only by minimizing a certain type of energy functional. In this way, they have put on a mathematical basis the above listed empirical laws assumed in the study of irrigation and draining systems letting someone prove them as theorems or state them as clear mathematical conjectures. That paper is not, however, the first mathematical attempt and we know other works addressing the existence problem, namely [28].

In [28] Caselles and Morel address the problem of finding a maximal irrigated volume with a minimal cost. They fix an open domain Ω and a point S in it, then they give the notion of irrigable sets by saying that a compact set $K \subset \Omega$ is irrigable if the complementary $U = \Omega \setminus K$, called irrigation network, is connected and contains S. Then they introduce an "accessibility profile", i.e. a positive, increasing function defined on $\mathbb{R}_+$ which is null in zero to define the irrigable points of an irrigable compact set K. They call a point $x \in K$ *f-irrigable* if there exists an (accessibility) path $x(s)$ parameterized on $[0, L]$ such that $x(0) = x$ and $x(L) = S$ and for every $s \in [0, L]$ the ball of center

$x(s)$ with radius $f(s)$ is contained in the irrigation network U. In other terms they say that a point x of an irrigable set K is f-irrigable if there exists a thick path inside the irrigation network which connects x to S. Being the profile f an increasing function, this path becomes thinner and thinner while approaching x but the thinning rate remains uniformly bounded from below. Calling f-irrigable sets the subsets of an irrigable set K consisting of f-irrigable points, the authors prove the existence of an f-irrigable set with maximal positive volume under superlinearity assumption on f (such as, for instance, the profile $f(s) = s^\alpha$ with $0 < \alpha < 1$). Then they associate to every accessibility path $x(s)$ to reach x a cost which is supposed to be lower semicontinuous with respect to the uniform convergence of paths allocating in this way at any point $x \in K$ a minimal accessibility cost $c_K(x)$. Therefore calling irrigation cost of K the functional $c(K) = \int_K c_K(x)dx$ they find a maximal irrigated volume K with minimal cost.

These recent problems are often included in the literature of the Monge transport problem even if the problem they approach is radically distinct from that proposed by Monge in [33]. Indeed, in the Monge-Kantorovitch model the cost of a single fluid particle is not influenced by interactions with the remaining part of the fluid or by their motion while, in this context, one is not so much interested in knowing the final destination of a single fluid particle (the so called "who goes where" problem) as in the shape of the set of the trajectories, knowing, in particular, if particles move together giving rise to a big river or move "alone" giving rise to many little rivers. An irrigation pattern with source point $S \in I\!R^N$ is a measurable mapping

$$\chi : \Omega \times I\!R_+ \to I\!R^N$$

such that:

C1) For a.e. *material point* $p \in \Omega$, the "fiber" $\chi_p : t \mapsto \chi(p,t)$ is a Lipschitz continuous map with a Lipschitz constant less than or equal to one;

C2) For a.e. $p \in \Omega$: $\chi_p(0) = S$;

where $(\Omega, \ |\cdot|)$ is a nonatomic probability space which is interpretated as the reference configuration of a fluid material body. We can think Ω playing the role of the trunk section of a tree, this trunk being thought as a set of fibers which can bifurcate into branches. $\mathbf{P}_S(\Omega)$ denotes the set of all irrigation patterns of Ω with source at a given point $S \in I\!R^N$.

Every $\chi \in \mathbf{P}_S(\Omega)$, time by time, defines an equivalence relation $\simeq_t$ on Ω by relating two points p and $q \in \Omega$ at the time t if the two fibers χ_p and χ_q coincide on $[0,\ t]$. So every irrigation pattern at every time t divides Ω into equivalence classes which are called χ-vessels. For any $p \in \Omega$ $[p]_t$ denotes the χ-vessel at time t which contains p, while for any $t \geq 0$ $\mathcal{V}_t(\chi)$ denotes the set of all the χ-vessels at time t.

The function $\sigma_\chi : \Omega \to I\!R_+$ defined by setting for a.e. $p \in \Omega$ $\sigma_\chi(p)$ as the infimum of the set $\{t \in I\!R_+ \mid \chi_p(\cdot)$ is constant on $[t, +\infty[\}$, gives the absorption time of a point and is called *stopping* or *absorption* function for χ. The set

$$M_t(\chi) = \{p \in \Omega \mid \sigma_\chi(p) > t\}$$

is the set of the points that, at time t, are still moving.

The *irrigation function* $i_\chi : A_\chi \to I\!R^N$, defined by setting $\forall p \in A_\chi \ : \ i_\chi(p) = \chi(p, \sigma_\chi(p))$ gives, point by point, the absorption position of the absorbed points.

The function i_χ induces on $I\!R^N$ the image (push-forward) measure μ_χ defined by the formula

$$\mu_\chi(A) = |i_\chi^{-1}(A)| \ ,$$

for any Borel set $A \subset I\!R^N$. We refer to μ_χ as to the *irrigation measure* induced by the pattern χ.

For a fixed exponent $\alpha \in]0,1[$, we introduce the functional cost I_α as in [32], defined on the set $\mathbf{P}_S(\Omega)$ of all the irrigation patterns χ, by the following formula

$$I_\alpha(\chi) = \int_{I\!R_+} \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp \, dt \ .$$

This functional takes into account only the moving points moreover, being $0 < \alpha < 1$, the functional penalizes the points which move into thin vessels, i.e. the functional penalizes the branching. Given a probability measure $\overline{\mu}$, we say that it is α-irrigable, in the sense of [32], if there exists an irrigation pattern χ of finite cost such that $\mu_\chi = \overline{\mu}$.

The Dirichlet Problem which consists in minimizing on $P_S(\Omega)$ the functional $I_\alpha(\chi) + J(\mu_\chi)$, where J is defined by setting

$$J(\mu) = \begin{cases} 0 & \text{if } \mu = \overline{\mu} \\ +\infty & \text{otherwise} \end{cases} \tag{0.0.12}$$

is approached. The aim of the functional I_α is to force the fibers to keep themselves together penalizing, in this way, their branching. The necessity of keeping the functional low competes with the boundary condition expressed by the presence of the functional J which, on the other hand, forces the fibers to bifurcate prescribing that the fluid they carry must reach a given measure spread out on a volume. The result of this competition is that the fibers take advantage in keeping themselves together as long as possible and then branching while approaching the terminal points, giving rise to the ramified structure. Any minimum χ of $I_\alpha + J$ is called *an optimal irrigation pattern for* $\overline{\mu}$ and $I_\alpha(\chi)$ is called *(irrigation) cost* of the probability measure $\overline{\mu}$. We briefly say that χ is *an optimal pattern* if it is optimal for its irrigation measure μ_χ.

A very similar functional to I_α, introduced in [32], has been proposed in [35] only to the aim of describing a variant of Monge-Kantorovitch transport problems. In this approach, both the starting and the target configuration are described by means of approximation with finite atomic measures but the starting configuration is not necessarily a source point as assumed in [32]. In [32] and in [35] the authors take the assumption that the motion of a particle is influenced by other particles in the sense that it is cheaper that two or more particles move together rather than flow lonely. So the two distinct approaches lead to a very similar functional. In more recent papers (see for instance [27] and [34]), these questions are addressed as particular measures evolutions problems in Wasserstein Spaces.

In [32] the authors approached the problem assuming that the morphology of trees derives by the attempt of nature to reduce a suitable energy functional. In this way, they have not explicitely taken into account hydrodynamic laws, such as, for instance, Kirchhoff Law, to keep energy constant or Poiseuille law to quantify the resistance a fluid encounters while flowing into a tube network, leaving to subsequent studies the motivation of such a functional, or of some close variants with a similar behaviour, from hydrodynamic laws. As in classical problems of Calculus of Variations the minimization of convex functionals leads to regular solutions, in [32] the authors introduce a functional with concave terms which make concentrations convenient and lead to the presence of singularities in the solutions. The aim of the irrigation functional is to force the fibers to keep themselves together penalizing, in this way, their branching. The necessity of keeping the functional low competes with a boundary condition which, on the other hand, forces the fibers to bifurcate, prescribing that the fluid they carry must reach a given measure spread out on a volume. The result of this competition is that the fibers take advantage in keeping themselves together as long as possible and then in branching while approaching the terminal points, giving rise to the ramified structure.

Chapter 4 *Elementary properties of optimal irrigation patterns*[4]

The existence of minima, proved in [32], raises regularity problem which should hopefully lead to a proof of the tree-like structure and the selfsimilarity properties.

As a first step in this direction, in this chapter we begin this study by identifying some geometric, elementary properties which are enjoyed by optimal irrigation patterns. The properties derived for optimal patterns are "elementary properties" in the sense that they are not concerned with the regularity at the ending points of these structures, where the presumable selfsimilarity properties should take place. Among these properties, we recall the notion of *simple pattern* characterized by the following properties:

- for a.e. point $p \in \Omega$ the open fiber χ_p is a simple curve on $[0, \sigma_\chi(p)[$;
- for a.e. pair of points p and q of Ω $\chi_p(t) \neq \chi_q(s)$ for all s, $t > s_\chi(p,q)$, $t < \sigma_\chi(p)$, $s < \sigma_\chi(q)$, where $s_\chi(p, q) = \inf\{t \in I\!R_+ \mid \chi_p(t) \neq \chi_q(t))\}$ is the *separation time* of the two points;

[4] G. Devillanova, S. Solimini, *Elementary properties of optimal irrigation patterns*, to appear.

expressed in terms of fibers or equivalently by the notion of flow-curves which, roughly speaking, represent the trajectories followed by flows. After introducing the notion of branch and the cost of a branch, we give a theorem which roughly speaking allows us to *prune a tree* so that the amount of the cost of the pruned branches is arbitrarily small (see Theorem 4.7.1 (Pruning Theorem) for the precise statement). The set F_χ, called flow zone, which contains the trajectories of the flows, can be considered as the support of a pattern χ and in a certain sense characterizes it. Under some hypotheses on χ (namely χ being non-spread), F_χ is the countable union of supports of rectifiable curves. When χ is also supposed to be a simple pattern, then three structures are induced on the support F_χ: a partial order, a "time" function and a "quantity" function. These three structures characterize the pattern. Actually we prove that one can prescribe such a kind of structures on a "branching set" without making any reference to a pattern. These structures, under some assumptions, identify a class of simple and well parameterized patterns (an histogram) for which the three structures agree with those derived from the pattern. Moreover the α-cost of a non-spread simple and well parameterized pattern can be evaluated by an integral on its support.

This preliminary study has the aim of giving some basic tools which will be hopefully useful for the approach to the above mentioned regularity problems and already finds an application in Chapter 5, where these properties are used in order to discuss the irrigability of a given measure.

Chapter 5 *On the dimension of an irrigable measure* [5]

In this last chapter we are investigating the irrigability of a measure in the terms of the set on which it concentrates, as suggested by the fact that a measure spread out on a set of high dimension forces the fibers to a more frequent branching and therefore the pattern to increase its cost. To this aim, we have introduced the notion of irrigability dimension $d(\mu)$ of a probability measure μ and then we have expressed the above stated problem in the terms of giving some estimates on the irrigability dimension of a given positive measure which is always supposed to be Borel regular, with a bounded support and a finite mass (by normalization we suppose it to be a probability measure). We have shown, with some examples, that the intuitive and conjecturable idea that the irrigability dimension of a measure coincides with the Hausdorff dimension of its support is false, in spite of the fact that both the two values express how much the measure is spread out. On the other hand, we have given some lower and upper bounds for the irrigability dimension $d(\mu)$ of a probability measure μ by means of the minimal Hausdorff and respectively Minkowski dimension of a set on which the measure is concentrated.

This result will be overproved. Indeed, we prove it directly, getting some further meaningful information and introducing some tools which will be also used in other parts

[5] G. Devillanova, S. Solimini, *On the dimension of an irrigable measure*, to appear.

of the chapter but we shall be also able to deduce it from a deeper estimate of $d(\mu)$ which will need the introduction of new notions. More precisely, it will need the notion of *resolution dimension* of a measure which, affected by an index, expresses the possibility to describe the measure by means of discrete approximations. When the measure is suitably regular, the value of the resolution dimension does not depend on the index, while for a generic measure, as will be explained by some examples, the resolution dimension is "out of focus" in the sense that different indexes give different values. We have shown that, in any case, it is always possible to find an index, suitably characterized, which gives a resolution dimension which coincides with the irrigability dimension.

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Contents

Part I

Multiplicity results for some elliptic problems with lack of compactness

Chapter 1

Concentration estimates and multiple solutions to elliptic problems at critical growth[1]

In this paper we consider the problem $-\Delta u = |u|^{2^*-2}u + \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is an open regular bounded subset of $\mathbb{R}^N$ $(N \geq 3)$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\lambda > 0$. Our main result asserts that, if $N \geq 7$, the problem has infinitely many solutions and, from the point of view of the compactness arguments employed here, the restriction on the dimension N cannot be weakened.

Introduction

This paper deals with the critical growth problem

$$\text{(CP)} \qquad \begin{cases} -\Delta u &= |u|^{2^*-2}u + \lambda u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open regular subset of $\mathbb{R}^N$ $(N \geq 3)$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, and $\lambda > 0$. Several people have got involved with this problem and the main known results have been collected in the introduction to [2]; adding theorems 1 and 2 of a subsequent paper [3] we complete the "state of art" obtaining the following list of main results.

1. If $\lambda \leq 0$, Pohozaev Identity (see [12]) allows us to say that Problem (CP) has, in general (for a star-shaped Ω), no nontrivial solution.

[1]G. Devillanova & S. Solimini *Concentrations estimates and multiple solutions to elliptic problems at critical growth.* Advances in Differential Equations, 7 (2002), 1257-1280

2. There exists a constant $\lambda^* \in [0, \lambda_1[$ such that (CP) has a positive solution if $\lambda \in]\lambda^*, \lambda_1[$, where λ_1 is the first eigenvalue of $-\Delta$ defined on $H_0^1(\Omega)$. Moreover if $N \geq 4$, $\lambda^* = 0$ (see Brezis-Nirenberg [6]). In the three-dimensional case and when Ω is a ball then $\lambda^* = \frac{\lambda_1}{4}$. Moreover, by using also in this case a suitable version of Pohozaev Identity we known that, for $\lambda \in]0, \lambda^*[$, (CP) has no radial solution (see [6]) but it is still unknown if there exist non radial solutions to (CP) (changing sign).

3. If $N \geq 4$ and Ω is a ball, then for any $\lambda > 0$ (CP) has infinitely many solutions changing their sign (which cannot be all radial, as shown in [2]) which are built using the particular symmetry of the domain Ω (see Fortunato-Jannelli [8]).

4. If $N \geq 7$ and Ω is a ball, then for each $\lambda > 0$ Problem (CP) has infinitely many changing sign radial solutions (the so called "nodal solutions"): see Solimini [15] and a previous paper of Cerami-Solimini-Struwe [7] where it is also shown that for $N \geq 6$ (CP) has at least two (pairs of) solutions on any smooth bounded domain.

5. When $4 \leq N \leq 6$ and Ω is a ball there exists a constant $\lambda^* > 0$ such that (CP) has no radial solutions which change sign if $\lambda \in]0, \lambda^*[$. So the bound $N \geq 7$ in the previous result cannot be removed, see Atkinson, Brezis and Peletier [3].

A natural question, which seems to be still open, is whether (CP) has infinitely many solutions on every bounded smooth domain. The above mentioned results suggest that the lower bound $N \geq 7$ on the dimension should be considered as a natural assumption, since it is necessary in the radial case. The main difficulty in solving Problem (CP) is the existence of noncompact Palais-Smale sequences (*PS* sequences), whose definition will be recalled in next section, of the corresponding functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*},$$

defined on the Hilbert space $H_0^1(\Omega)$. The behavior of noncompact *PS* sequences has been studied in [16] which, roughly speaking, guarantees the existence of a subsequence approximated by its weak limit plus terms which tend to concentrate around a finite number of points, see Theorem 1.1.3 below. This theorem allows a precise description of the behavior of noncompact *PS* sequences at every level of the functional I_λ and suggests the way to look for "good levels" in order to get compactness. On the contrary, we shall give an answer to the question about multiplicity of solutions by introducing some compactness techniques which show as, in dimension $N \geq 7$, every min-max admissible class produces precompact *PS* sequences. This will follow as a consequence of a uniform bound theorem stated for bounded sets U of solutions to

$$\text{(SP)} \qquad \begin{cases} -\Delta u = |u|^{p-2}u + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with p varying in $[2, 2^*]$. The precise results will be stated in the next section, where the characterization of PS sequences will also be recalled and some related terminology will be introduced. Firstly, we shall need to establish suitable *a priori* estimates on some norms of the functions in U, which will be proved in Section 1.2 and which will be employed to the aim of finding sharper estimates on the functions and on their derivatives, see sections 1.3 and 1.4 respectively. Subsequently, in Section 1.5, we establish a local Pohozaev Identity which allows, in Section 1.6, the proof of the uniform bound theorem. Finally, in Section 1.7, we show how this technique allows to apply classical min-max arguments to (CP) and to prove, in this way, the existence of infinitely many solutions.

1.1 Statement of the results and notation

The main results in this paper are the following ones.

Theorem 1.1.1 (Uniform bound through concentration estimates) *Let $N \geq 7$ and U be a bounded set in $H_0^1(\Omega)$ whose elements are solutions, for a fixed $\lambda > 0$, to problems (SP), for p varying in $[2, 2^*]$. Then U is uniformly bounded, i.e. there exists a constant $C > 0$ such that*

$$\sup_{u \in U} \sup_{x \in \Omega} |u(x)| \leq C.$$

Theorem 1.1.2 (Infinitely many solutions to (CP) in large dimension) *If $N \geq 7$, then problem (CP) admits infinitely many solutions.*

Analogous multiplicity results, like the existence of infinitely many radial solutions to (CP) when Ω is a ball, can be obtained from Theorem 1.1.1 in the same way as Theorem 1.1.2. Therefore the uniqueness result in [3, Theorem A] allows us to deduce the following remark.

Remark 1.1.1 *The restriction $N \geq 7$ in Theorem 1.1.1 cannot be removed. Indeed, the theorem is false for $N \leq 6$.*

On the other side, we do not know if Theorem 1.1.2 can still hold or not for $N \leq 6$. We have already pointed out that the statement still holds true for $N \geq 4$ if Ω is a ball and that for $N = 6$ and $0 < \lambda < \lambda_1$ one still has multiplicity of solutions for every bounded smooth domain. Any question about sharper results or extensions to lower dimensions on general domains seems to be open.

Now we shall introduce some notation and terminology we shall use during this note. Given $\sigma > 0$ and $\bar{x} \in I\!R^N$, let us consider the following scaled function

$$\rho(u) = u_\sigma : x \mapsto \sigma^{\frac{N}{2^*}} u(\bar{x} + \sigma(x - \bar{x})).$$

This scaling operation ρ keeps constant the norms $\| \nabla u_\sigma \|_{L^2}$ and $\| u_\sigma \|_{L^{2^*}}$ and is determined by the *center* or *concentration point* $\bar{x}$ and the *modulus* σ. For every real number c we shall say that a sequence $(u_n)_{n\in\mathbb{N}}$ is a *PS* sequence for the functional I_λ at level c if the following two conditions hold:

1. $I_\lambda(u_n) \to c$;
2. $\nabla I_\lambda(u_n) \to 0$ strongly in H^{-1} where ∇I_λ is the Fréchet derivative of I_λ .

We shall briefly say that $(u_n)_{n\in\mathbb{N}}$ is a *PS* sequence if there exists a level $c \in \mathbb{R}$ such that $(u_n)_{n\in\mathbb{N}}$ is a *PS* sequence at level c. In order to produce estimates on the values of solutions u to (SP), we observe that $v = |u|$ (extended by zero out of Ω) solves

$$\text{(EI)} \qquad -\Delta v \leq bv^{2^*-1} + A,$$

where b is any coefficient bigger than one and $A = -\inf(bs^{2^*-1} - s^{p-1} - \lambda s)$ (taken for $1 \leq p \leq 2^*$, $s > 0$) is a constant which does not depend on u. Since b can be trivially normalized, we shall always take $b = 1$ in (EI). So the estimates in these next two sections will be derived for solutions to (EI) in $H^1(\mathbb{R}^N)$ and this will make us free from caring about the sign of u or taking into account the domain Ω.

Definition 1.1.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a given sequence. We shall say that $(u_n)_{n\in\mathbb{N}}$ is:*

- *a* controlled sequence *if each u_n is a solution to (EI);*
- *a* balanced sequence *if each u_n solves (SP) for some $p \in [2, 2^*]$.*

Remark 1.1.2 *As we have already pointed out, every solution to (SP) (under a null extension out of Ω) is solution also to (EI). Therefore every balanced sequence is a controlled sequence. On the other side, when we shall deal with controlled sequences, we shall assume that they are positive since we can always replace them with their absolute values, and that $\Omega = \mathbb{R}^N$.*

Let us recall the main result in Struwe (see [16], [17] and, for some terminology used below, [15]).

Theorem 1.1.3 *Let $(u_n)_{n\in\mathbb{N}}$ be a noncompact* PS *sequence. Then, by replacing $(u_n)_{n\in\mathbb{N}}$ with a suitable subsequence, there exists a finite number k, depending on a bound M on $\|u_n\|_{H_0^1}$ (namely $k \leq MS^{\frac{N}{2}}$, where S is the so called Sobolev constant), of global solutions φ_i to (CP) in $H^1(\mathbb{R}^N)$ with $\lambda = 0$ with corresponding k sequences of mutually diverging scalings $(\rho_n^i)_{n\in\mathbb{N}}$ with respective concentration points x_n^i and diverging moduli σ_n^i (i.e. $\lim_{n\to+\infty} \sigma_n^i = +\infty$) such that*

$$(1.1.1) \qquad u_n - \sum_{i=1}^{k} \rho_n^i(\varphi_i) \to u_\infty \qquad \text{in } H_0^1(\Omega) \ ,$$

where u_∞, weak limit of the sequence, solves (CP).

We shall call *concentrating sequence* any bounded sequence which satisfies a weaker case of the property in the above thesis. More precisely, we shall say that $(u_n)_{n\in\mathbb{N}}$ is a *concentrating sequence* if the limit (1.1.1) holds in the L^{2^*} strong topology with $1 \le k \le MS^{-\frac{N}{2}}$, u_∞ solution to (CP) and φ_i multiple of a global solution by a constant $\alpha_i \ge 1$. So Theorem 1.1.3 says, in particular, that from any noncompact *PS* sequence we can extract a concentrating sequence. Given any concentrating sequence we shall also consider the scalings ρ_n^i and the limit functions φ_i (which are not uniquely determined by Theorem 1.1.3) as also given. In what follows, we shall call $(\rho_n)_{n\in\mathbb{N}}$ one of the "basic scaling sequences" ρ_n^i which corresponds to a function φ_i which concentrates in $x_n = x_n^i$ in the slowest way (for which the respective $\sigma_n = \sigma_n^i$ is the lowest order infinite). So σ_n and x_n will be also considered to be given, once we have fixed any concentrating sequence. Actually, we may find two different sequences of scalings with the modulus of the same order: in this last case we shall choose arbitrarily one of them. We shall now consider, for any n, $k+1$ concentric annuli centered in x_n and of width $7\sigma_n^{-1/2}$. Among them, for every $n \in \mathbb{N}$, we can find at least one annulus without concentration points. Being $k \le MS^{-\frac{N}{2}}$ independent by n, this procedure allows, passing to a subsequence, to choose a constant $\overline{C}$, which does not depend on n, such that $1 \le \overline{C} \le 7k+1 \le 7MS^{-\frac{N}{2}}+1$, such that the $\sigma_n^{-1/2}$-neighborhood of the annulus $\mathcal{A}_n^1 = B_{(\overline{C}+5)\sigma_n^{-1/2}}(x_n) \setminus \overline{B}_{\overline{C}\sigma_n^{-1/2}}(x_n)$ doesn't contain any concentration point for every $n \in \mathbb{N}$. *We add this last requirement to the definition of concentrating sequence*, so the sequence of the annuli $\mathcal{A}_n^1$ and of their thinner subsets $\mathcal{A}_n^2 = B_{(\overline{C}+4)\sigma_n^{-1/2}}(x_n) \setminus \overline{B}_{(\overline{C}+1)\sigma_n^{-1/2}}(x_n)$ and $\mathcal{A}_n^3 = B_{(\overline{C}+3)\sigma_n^{-1/2}}(x_n) \setminus \overline{B}_{(\overline{C}+2)\sigma_n^{-1/2}}(x_n)$ will be also considered to be fixed in correspondence of any given concentrating sequence. We shall refer to these terms as to *safe regions* of the sequence and they are the sets on which the local uniform bounds will be established.

1.2 Integral estimates for controlled concentrating sequences

Definition 1.2.1 *Let $p_1, p_2 \in]2, +\infty[$ be real numbers such that $p_2 < 2^* < p_1$, $\alpha > 0$ and $\sigma > 0$. We consider an inequalities system*

$$\text{(1.2.1)} \qquad \begin{cases} \|u_1\|_{p_1} & \le & \alpha \\ \|u_2\|_{p_2} & \le & \alpha\,\sigma^{\frac{N}{2^*}-\frac{N}{p_2}} \end{cases}$$

which will let us introduce a norm depending on p_1, p_2 and σ, by setting

$$\| u \|_{p_1,p_2,\sigma} = inf\,\{\alpha > 0 \mid \exists u_1,\, u_2 \text{ such that } (1.2.1) \text{ is satisfied and } |u| \le u_1 + u_2\}\ .$$

The above norm will be briefly denoted by $\| u \|_\sigma$ when p_1 and p_2 can be supposed to be given.

Remark 1.2.1 *Let $p_1, p_2 \in]2, +\infty[$ real numbers such that $p_2 < 2^* < p_1$ and $\sigma > 0$, then, by the very definition, for any function u we get*

$$\|u\|_\sigma \leq \|u\|_{p_1} , \qquad \|u\|_\sigma \leq \|u\|_{p_2} \sigma^{\frac{N}{p_2} - \frac{N}{2^*}} .$$

The goal of this section is the following Brezis-Kato type regularity result (see [5, Theorem 2.3]):

Proposition 1.2.1 *Let $(u_n)_{n \in \mathbb{N}}$ be a controlled concentrating sequence, then for any p_1, $p_2 \in]\frac{2^*}{2}, +\infty[$, $p_2 < 2^* < p_1$ there exists a constant $C(p_1, p_2)$ depending on the sequence and on the exponents p_1 and p_2, such that for any $n \in \mathbb{N}$*

$$\| u_n \|_{\sigma_n} \leq C .$$

To this aim, we shall state three preliminary lemmas: a continuity lemma, a bootstrap lemma and the relative initialization lemma.

Lemma 1.2.1 *Let u and $v \in H^1(\mathbb{R}^N)$ and $a \in L^{\frac{N}{2}}(\mathbb{R}^N)$ be three positive functions such that*

$$-\Delta u \leq a(x) v .$$

Then for each $p_1, p_2 \in]2, +\infty[$ there exists a constant $C(N, p_1, p_2)$, depending on the dimension N and on the exponents p_1 and p_2, such that for any $\sigma > 0$

$$\| u \|_\sigma \leq C(N, p_1, p_2) \| a \|_{\frac{N}{2}} \|v\|_\sigma .$$

PROOF. Let u, v be as in the statement of the lemma and let fix $\sigma > 0$ and $\varepsilon > 0$. Let $v \leq v_1 + v_2$ such that v_1 and v_2 satisfy (1.2.1) for $\alpha = \| v \|_{p_1, p_2, \sigma} + \varepsilon$. Let us consider, for $i = 1$, 2, the solution $u_i \in H^1(\mathbb{R}^N)$ to $-\Delta u_i = a v_i$. Then

$$\| u_i \|_{p_i} \leq C(N, p_i) \| a \|_{\frac{N}{2}} \| v_i \|_{p_i}$$

and, being $-\Delta u_1 - \Delta u_2 = a v_1 + a v_2 \geq a v \geq -\Delta u$, by the maximum principle we have $u \leq u_1 + u_2$. Since the functions u_i satisfy (1.2.1) with $\alpha = C(N, p_i) \|a\|_{\frac{N}{2}} (\|v\|_\sigma + \varepsilon)$, by the arbitrariness of ε we get the thesis. ∎

The bootstrap argument relies in the use of the following lemma.

Lemma 1.2.2 *Let $p_1, p_2 \in]\frac{N+2}{N-2}, \frac{N}{2} \frac{N+2}{N-2}[$ such that $p_2 < 2^* < p_1$ and let q_i be defined, for $i = 1$, 2, by*

$$\frac{1}{q_i} = \frac{N+2}{N-2} \frac{1}{p_i} - \frac{2}{N} . \tag{1.2.2}$$

If u and v are two positive functions whose support is contained in a bounded set Ω and such that

$$-\Delta u \leq v^{2^*-1} + A ,$$

then there exists a constant $C(N, p_1, p_2, \Omega)$ such that for any $\sigma > 0$:

$$\| u \|_{q_1,q_2,\sigma} \leq C(N, p_1, p_2, \Omega) \left((\| v \|_{p_1,p_2,\sigma})^{\frac{N+2}{N-2}} + 1 \right). \tag{1.2.3}$$

PROOF. By proceeding as in the previous lemma, we consider $v = v_1 + v_2$ where the functions v_i satisfy (1.2.1) for $\alpha = \| v \|_{p_1,p_2,\sigma} + \varepsilon$ and ε is a real strictly positive number arbitrarily small. Let u_1 and u_2 be two functions in $H_0^1(\Omega)$ such that

$$-\Delta u_1 = 2^{\frac{4}{N-2}} v_1^{\frac{N+2}{N-2}} + A$$

$$-\Delta u_2 = 2^{\frac{4}{N-2}} v_2^{\frac{N+2}{N-2}}.$$

Since

$$-\Delta u \leq v^{\frac{N+2}{N-2}} + A \leq 2^{\frac{N+2}{N-2}-1} v_1^{\frac{N+2}{N-2}} + A + 2^{\frac{N+2}{N-2}-1} v_2^{\frac{N+2}{N-2}} = -\Delta u_1 - \Delta u_2 ,$$

by the maximum principle $u \leq u_1 + u_2$ follows . Hence, we have to estimate $\| u_1 \|_{q_1}$ and $\| u_2 \|_{q_2}$. We have, using (1.2.2) and being $\frac{N+2}{N-2} < p_i < \frac{N}{2}\frac{N+2}{N-2}$,

$$\begin{aligned} \| u_1 \|_{q_1} &\leq C(N, p_1) \| v_1^{\frac{N+2}{N-2}} + A \|_{L^{p_1 \frac{N-2}{N+2}}} \leq C(N, p_1) \left(\| v_1 \|_{p_1}^{\frac{N+2}{N-2}} + A |\Omega|^{\frac{1}{p_1}\frac{N+2}{N-2}} \right) \\ &\leq C(N, p_1, \Omega) \left((\| v \|_{p_1,p_2,\sigma} + \varepsilon)^{\frac{N+2}{N-2}} + 1 \right). \end{aligned}$$

Analogously, if we use the equality

$$\frac{N}{2^*} - \frac{N}{q_2} = \left(\frac{N}{2^*} - \frac{N}{p_2} \right) \frac{N+2}{N-2} ,$$

we get

$$\begin{aligned} \| u_2 \|_{q_2} &\leq C(N, p_2) (\| v_2 \|_{p_2})^{\frac{N+2}{N-2}} \leq C(N, p_2) \left[(\| v \|_{p_1,p_2,\sigma} + \varepsilon) \sigma^{\frac{N}{2^*} - \frac{N}{p_2}} \right]^{\frac{N+2}{N-2}} \\ &= C(N, p_2) (\| v \|_{p_1,p_2,\sigma} + \varepsilon)^{\frac{N+2}{N-2}} \sigma^{(\frac{N}{2^*} - \frac{N}{p_2})\frac{N+2}{N-2}} \\ &= C(N, p_2) (\| v \|_{p_1,p_2,\sigma} + \varepsilon)^{\frac{N+2}{N-2}} \sigma^{\frac{N}{2^*} - \frac{N}{q_2}} . \end{aligned}$$

So u_1 and u_2 solve (1.2.1) for $C = C(N, p_1, p_2, \Omega) \left((\| v \|_{p_1,p_2,\sigma} + \varepsilon)^{\frac{N+2}{N-2}} + 1 \right)$; this concludes the proof by the arbitrary choice of ε. ∎

Now we need to initialize the exponents through the following lemma.

Lemma 1.2.3 *Let $(u_n)_{n \in I\!N}$ be a controlled concentrating sequence then there exists a constant C and exponents p_1, $p_2 \in]\frac{2^*}{2}, +\infty[$, $p_2 < 2^* < p_1$, such that for any $n \in I\!N$*

$$\| u_n \|_{\sigma_n} \leq C . \tag{1.2.4}$$

PROOF. This proof will follow a Brezis-Kato type argument (see [5]) in order to get free from an infinitesimal term which is the only real obstacle to our estimates. For any $n \in I\!N$, we can consider, using a homogeneous notation, $u_n = u_n^0 + u_n^1 + u_n^2$, where

- u_n^1 stands for the weak limit u_∞;
- u_n^2 stands for the sum of rescaled functions φ_i, $u_n^2 = \sum_{i=1}^k \rho_n^i(\varphi_i)$;
- $u_n^0 = u_n - u_n^2 - u_\infty$ is, by definition of concentrating sequence, an infinitesimal term in L^{2^*}-norm.

We shall overcome the difficulty due to the presence of u_n^0 by taking advantage of the assumption that we are dealing with a controlled concentrating sequence. Let u be one of the terms u_n, $u_i = u_n^i$ and $a_i = \max(1, 3^{\frac{6-N}{N-2}}) u_i^{\frac{4}{N-2}}$ for $i = 1,\ 2,\ 3$, and $\sigma = \sigma_n$. The infinitesimal character of u_n^0 shall allow us to consider a_0 as small as we want in the $L^{\frac{N}{2}}$ norm ((1.2.4) is easily checked on a finite number of terms, see [5] and [9]). Being

$$a = u^{2^*-2} \leq \max(1, 3^{\frac{6-N}{N-2}}) \left(|u_0|^{\frac{4}{N-2}} + u_1^{\frac{4}{N-2}} + u_2^{\frac{4}{N-2}} \right) ,$$

we can consider u as a solution to $-\Delta u \leq (a_0 + a_1 + a_2)u + A$, so by the monotonicity of the Green operator $\mathcal{G}$ ($\mathcal{G} : H^{-1}(\Omega) \to H_0^1(\Omega)$ denotes the inverse operator of $-\Delta$) we have

$$u \leq \mathcal{G}(a_0 u) + \mathcal{G}(a_1 u + A) + \mathcal{G}(a_2 u) . \tag{1.2.5}$$

Since Ω is a bounded set and $a_1 \in L^\infty$, we get that $\mathcal{G}(a_1 u + A)$ is bounded in $W^{2,2^*} \hookrightarrow L^{p_1}$, for any p_1 such that

$$\frac{1}{p_1} \geq \frac{1}{2^*} - \frac{2}{N} = \frac{N-6}{2N}$$

and so (see Remark 1.2.1)

$$\|\mathcal{G}(a_1 u + A)\|_\sigma \leq \|\mathcal{G}(a_1 u + A)\|_{p_1} \leq C. \tag{1.2.6}$$

Now let $2^{*\prime} < p_2 < 2^*$ be given. We consider the index r such that

$$\frac{1}{p_2} = \frac{1}{r} + \frac{1}{2^*} - \frac{2}{N} ,$$

from $p_2 > 2^{*\prime}$ we get $r > \frac{N}{4}$. The decay speed of the solution $\varphi = \varphi_i$ (see [9]) allows us to say that $a_2 \in L^r$ and, if we want to estimate the L^r-norm of a_2, we just have to take into account the less concentrated term, namely $\rho_n(\varphi)$, as follows from $r < \frac{N}{2}$, which is in turn a consequence of $p_2 < 2^*$. By easy computations we have

$$\|a_2\|_{L^r} \leq C\sigma^{2-\frac{N}{r}} ,$$

which, taking into account that $2 - \frac{N}{r} = \frac{N}{2^*} - \frac{N}{p_2}$, implies

$$\|\mathcal{G}(a_2 u)\|_{p_2} \leq C\|a_2\|_{L^r}\|u\|_{L^{2^*}} \leq C\sigma^{\frac{N}{2^*}-\frac{N}{p_2}} , \tag{1.2.7}$$

therefore, from Remark 1.2.1

$$\|\mathcal{G}(a_2 u)\|_{\sigma} \leq \sigma^{\frac{N}{p_2}-\frac{N}{2^*}}\|\mathcal{G}(a_2 u)\|_{p_2} \leq C . \tag{1.2.8}$$

Now we point out that with the above choice for p_1 and p_2 we get

$$\|\mathcal{G}(a_0 u)\|_{\sigma} \leq \frac{1}{2}\|u\|_{\sigma} . \tag{1.2.9}$$

Indeed, by Lemma 1.2.1, we get

$$\|\mathcal{G}(a_0 u)\|_{\sigma} \leq C\|a_0\|_{\frac{N}{2}}\|u\|_{\sigma} \leq \frac{1}{2}\|u\|_{\sigma} , \tag{1.2.10}$$

under a suitable choice of the bound on the norm of a_0. So by (1.2.5), (1.2.9) and the triangular inequality, we finally obtain

$$\|u\|_{\sigma} \leq 2\|\mathcal{G}(a_1 u + A)\|_{\sigma} + 2\|\mathcal{G}(a_2 u)\|_{\sigma} ,$$

which, combined with (1.2.6) and (1.2.8), gives the thesis. ∎

Proof of Proposition 1.2.1. Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence. By applying the initialization Lemma 1.2.3, we can find a constant $C > 0$ and two exponents, p_1 and $p_2 \in]\frac{N+2}{N-2}, \frac{N}{2}\frac{N+2}{N-2}[$, $p_2 < 2^* < p_1$ such that (1.2.4) holds. Using the bootstrap Lemma 1.2.2 we can repeatedly enlarge the interval $]p_2, p_1[$ to $]q_2, q_1[$, where the exponents q_i are given by (1.2.2), obtaining (1.2.3). This procedure allows us to manage, in a finite number of steps, every exponent p_1, $p_2 \in]\frac{2^*}{2}, +\infty[$. ∎

1.3 Local uniform bounds on controlled concentrating sequences

In this section we shall establish a local uniform bound on the terms of a controlled concentrating sequence on the safe regions $\mathcal{A}_n^2$.

Proposition 1.3.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence. Then there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and for any $x \in \mathcal{A}_n^2$:*

$$u_n(x) \leq C.$$

The proof is a simple variant of the argument used in [14] and in [9] and shall require some preliminary steps. We begin by establishing a weaker estimate.

Proposition 1.3.2 *Let $(u_n)_{n\in I\!N}$ a controlled concentrating sequence. Then there exists a constant $C > 0$ such that for any $n \in I\!N$ and for any $x \in \mathcal{A}_n^1$:*

$$u_n(x) \leq C\sigma_n^{\frac{N-2}{4}} .$$

PROOF. We shall proceed by contradiction: let $(y_n)_{n\in I\!N}$ be a sequence such that $y_n \in \mathcal{A}_n^1$ for any $n \in I\!N$ and

$$\lim_{n\to+\infty} u_n(y_n)\sigma_n^{\frac{2-N}{4}} = +\infty , \tag{1.3.1}$$

and let us scale the functions u_n in such a way to carry the point y_n in the origin and normalize the value of the functions. The required scaling sends u_n in $\tilde{u}_n$ defined as

$$\tilde{u}_n(x) = \rho_n^{\frac{N}{2^*}} u_n(\rho_n x + y_n) ,$$

where

$$\rho_n = (u_n(y_n))^{\frac{2}{2-N}} = (u_n(y_n))^{-\frac{2^*}{N}} ,$$

so that $\tilde{u}_n(0) = 1$. Note that, using (1.3.1), we have:

$$\lim_{n\to+\infty} \frac{\rho_n}{\sigma_n^{-1/2}} = 0 . \tag{1.3.2}$$

Therefore, since $y_n \in \mathcal{A}_n^1$, there is no concentration point which approximates y_n at a distance less or equal to $\sigma_n^{-1/2}$ and so of the order of ρ_n, we can deduce that $\tilde{u}_n \rightharpoonup \tilde{u} = 0$. The contradiction will be archived by showing that we can choose the points y_n in such a way to have $\tilde{u} \neq 0$. This shall possibly force us to work on a $\varepsilon\sigma_n^{-1/2}$-neighborhood of $\mathcal{A}_n^1$, but this change will obviously not make any relevant difference in the above argument. The choice will consist in forcing the property

$$\tilde{u}_n(y) \leq 2 \quad (= 2\tilde{u}_n(0)) \quad \forall y \in B_\rho(0) \tag{1.3.3}$$

for some given $\rho > 0$. Then by using that $\tilde{u}_n$ still satisfies (EI) and by estimating the variation of the mean value of u_n, we have for $0 < r \leq \rho$

$$\begin{aligned} \fint_{\partial B_r} \tilde{u}_n &= \tilde{u}_n(0) + \int_0^r \frac{1}{Nb_N t^{N-1}} \left(\int_{B_t} \Delta\tilde{u}_n \right) dt \\ &\geq 1 - C\int_0^r \frac{1}{t^{N-1}} \int_{B_t} (2^{2^*-1} + A) dt = 1 - Cr^2 \geq \frac{1}{2} , \end{aligned}$$

where b_N stands for the $N-1$ dimensional measure of the unit sphere in $I\!R^N$, provided we choose r conveniently small. So the weak limit $\tilde{u}$ can't be zero. Therefore we only have to prove (1.3.3). To this aim, let us fix $\rho > 0$ and assume that, for a given $n \in I\!N$, y_n does not satisfy (1.3.3). Then we must fire y_n and look for a better point to hire for the same job. Since (1.3.3) is false, we can find $z_n \in B_\rho(0)$ such that

$$\tilde{u}_n(z_n) = \rho_n^{\frac{N-2}{2}} u_n(\rho_n z_n + y_n) \geq 2 \ . \tag{1.3.4}$$

The first candidate to replace y_n is

$$y_n^{(1)} = \rho_n z_n + y_n$$

which leads us to replace ρ_n by

$$\rho_n^{(1)} = \left[u_n\left(y_n^{(1)}\right)\right]^{\frac{2}{2-N}} \leq 2^{\frac{2}{2-N}} \rho_n \ . \tag{1.3.5}$$

We can be sure that $y_n^{(1)}$ is at least as good as y_n to let (1.3.1) hold since (1.3.4) implies that

$$u_n\left(y_n^{(1)}\right) \geq 2 u_n(y_n) \ . \tag{1.3.6}$$

Moreover, being $z_n \in B_\rho(0)$, we get

$$|y_n^{(1)} - y_n| = |z_n \rho_n| \leq \rho \rho_n \ . \tag{1.3.7}$$

We can define $\tilde{u}_n$ as before by substituting y_n and ρ_n with $y_n^{(1)}$ and $\rho_n^{(1)}$ respectively. If this new $\tilde{u}_n$ satisfies (1.3.3) we do not have to look for other choices. Otherwise, we repeat the same argument and we choose a second candidate $y_n^{(2)}$ by arguing in the same way. For any fixed $n \in I\!N$, we proceed recoursively finding a sequence $y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k)}, \ldots$ as far as we don't find a successful choice, which lets us claim (1.3.3). We can easily see that this process cannot go on indefinitely. Indeed (1.3.5) becomes in the general case, for $i > 0$

$$\rho_n^{(i+1)} \leq 2^{\frac{2}{2-N}} \rho_n^{(i)}$$

and (1.3.7)

$$|y_n^{(i+1)} - y_n^{(i)}| \leq \rho \rho_n^{(i)} \ .$$

Then one easily sees, by taking the sum of a geometric sequence, that $y_n^{(i)}$ converges to a point $y_n^{(\infty)}$ as $i \to +\infty$ but, by construction, we have $u_n(y_n^{(i)}) \to +\infty$, in contradiction to the smoothness of u_n. Finally, for every $i > 0$, we have

$$|y_n^{(i)} - y_n| \leq \rho \rho_n \sum_{j=0}^{+\infty} 2^{\frac{2}{2-N} j} < \varepsilon \sigma_n^{-1/2} \ ,$$

for n large. So all the points $y_n^{(i)}$ are in the $\varepsilon \sigma_n^{-1/2}$-neighborhood of $\mathcal{A}_n^1$ and so can be used to replace y_n. ∎

Proposition 1.3.3 *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence, then there exists a constant $C>0$ such that for any $n\in\mathbb{N}$ and for any $r\in\left[\overline{C}\sigma_n^{-1/2},\ (\overline{C}+5)\sigma_n^{-1/2}\right]$:*

$$\fint_{\partial B_r(x_n)} u_n \le C\ .$$

PROOF. By continuity, being $(u_n)_{n\in\mathbb{N}}$ bounded in $L^{2^*}\subset L^1$, we can suppose

$$\int_{B_1(x_n)} u_n \le C$$

with a constant C independent from n. So, for any $n\in\mathbb{N}$, there exist $r_n\in[\frac{1}{2},1]$, such that

$$\fint_{\partial B_{r_n}(x_n)} u_n = C\ .$$

We are going to use Proposition 1.2.1 for $p_1=N\frac{N+2}{N-2}$ and $p_2=\frac{N+2}{N-2}$, so, for any $n\in\mathbb{N}$, we choose $u_1=u_{1,n}$ and $u_2=u_{2,n}$ such that (1.2.1) is satisfied for $\sigma=\sigma_n$ and with a constant α that does not depend on n. Estimating the spherical mean variation from r_n to r and taking into account that $(\overline{C}+5)\sigma_n^{-1/2}<\frac{1}{2}$, i.e. $r<r_n$ for n large, we find:

$$\begin{aligned}
\fint_{\partial B_r(x_n)} u_n &= C+\int_{r_n}^{r}\frac{d}{dt}\fint_{\partial B_t(x_n)} u_n dt = C+\int_r^{r_n}\frac{1}{Nb_Nt^{N-1}}\int_{B_t(x_n)} -\Delta u_n dt\\
&\le C+\int_{\overline{C}\sigma_n{}^{-1/2}}^{1}\frac{1}{Nb_Nt^{N-1}}\int_{B_t(x_n)}(u_n{}^{2^*-1}+A)dt\\
&\le C+\int_{\overline{C}\sigma_n^{-1/2}}^{1}2^{\frac{4}{N-2}}\frac{1}{Nb_Nt^{N-1}}\int_{B_t(x_n)}u_{1,n}{}^{\frac{N+2}{N-2}}dt\\
&\quad+\int_{\overline{C}\sigma_n^{-1/2}}^{1}2^{\frac{4}{N-2}}\frac{1}{Nb_Nt^{N-1}}\int_{B_t(x_n)}u_{2,n}{}^{\frac{N+2}{N-2}}dt+\frac{A}{N}\int_0^1 tdt\\
&= C+\frac{2^{\frac{4}{N-2}}}{Nb_N}(A_1+A_2)+\frac{A}{2N}\ ,
\end{aligned}$$

where for $i=1,2$

$$A_i=\int_{\overline{C}\sigma_n^{-1/2}}^{1}\frac{1}{t^{N-1}}\int_{B_t(x_n)}u_{i,n}{}^{\frac{N+2}{N-2}}dt\ .$$

Being $u_{1,n}\in L^{N\frac{N+2}{N-2}}$, by Hölder inequality we get

$$A_1\le C\int_0^1\frac{1}{t^{N-1}}(t^N)^{1-\frac{1}{N}}\|u_{1,n}\|_{L^{N\frac{N+2}{N-2}}}^{\frac{N+2}{N-2}}dt\le C\alpha\le C\ .$$

On the other side, being $u_{2,n}\in L^{\frac{N+2}{N-2}}$, i.e. $u_{2,n}^{\frac{N+2}{N-2}}\in L^1$ we have

$$A_2\le\int_{\overline{C}\sigma_n^{-1/2}}^{1}\frac{1}{t^{N-1}}\left[\alpha\sigma_n^{\left(\frac{N}{2^*}-N\frac{N-2}{N+2}\right)}\right]^{\frac{N+2}{N-2}}dt=\alpha^{\frac{N+2}{N-2}}\sigma_n^{\frac{2-N}{2}}\int_{\overline{C}\sigma_n^{-1/2}}^{1}\frac{1}{t^{N-1}}dt\le C\ ,$$

and this concludes the proof. ∎

¿From Proposition 1.3.3 we see, by integrating with respect to r, that

$$\fint_{\mathcal{A}_n^1} u_n \le C \,. \tag{1.3.8}$$

Since, $\forall x \in \mathcal{A}_n^2 \; : \; B_{\sigma_n^{-1/2}(x)} \subset \mathcal{A}_n^1$ and the measure of the two sets are of the same order, from (1.3.8) we deduce that

$$\forall x \in \mathcal{A}_n^2 \; : \; \fint_{B_{\sigma_n^{-1/2}(x)}} u_n \le C \,. \tag{1.3.9}$$

Since

$$u_n(x) = \lim_{\rho \to 0} \fint_{B_\rho(x)} u_n \,,$$

Proposition 1.3.1 follows from (1.3.9) if we estimate the variation of $\fint_{B_\rho(x)} u_n$ for $0 \le \rho \le \sigma_n^{-1/2}$.

Proof Proposition 1.3.1. Let us fix an index $n \in I\!N$ and a point $x \in \mathcal{A}_n^2$. If $u_n(x) \le 2 \fint_{B_{\sigma_n^{-1/2}(x)}} u_n$, by (1.3.9) we have done. Otherwise, setting for any $\rho > 0$

$$m(\rho) = \fint_{\partial B_\rho(x)} u_n \quad \text{and} \quad m(0) = u_n(x) \,,$$

we deduce that

$$\exists \bar\rho \le \sigma_n^{-1/2} \text{ such that } m(\bar\rho) \le \frac{1}{2} m(0) = \frac{1}{2} u_n(x) \,.$$

Then we take ρ_1 and $\rho_2 \in [0, \bar\rho]$ such that $m(\rho)$ attains its maximum in ρ_1, and ρ_2 is the least value of $\rho \ge \rho_1$ such that $m(\rho) \le \frac{1}{2} m(\rho_1)$.

Being u_n solution to (EI), and $B_{\rho_2}(x) \subset \mathcal{A}_n^1$, we have on such a set, by Proposition 1.3.2, $u_n^{\frac{4}{N-2}} \le C\sigma_n$. So we find, for n sufficiently large,

$$\begin{aligned}
\frac{1}{2} m(\rho_1) &= \int_{\rho_2}^{\rho_1} \left(\frac{d}{d\rho} \fint_{\partial B_\rho(x)} u_n \right) d\rho = \int_{\rho_1}^{\rho_2} \frac{1}{N b_N \rho^{N-1}} \int_{B_\rho(x)} -\Delta u_n d\rho \\
&\le \int_{\rho_1}^{\rho_2} \frac{1}{N b_N \rho^{N-1}} \int_{B_\rho(x)} \left(u_n^{\frac{4}{N-2}} u_n + A \right) d\rho \\
&\le \frac{1}{N b_N} \int_{\rho_1}^{\rho_2} \frac{1}{\rho^{N-1}} \left(\left(\sup_{B_\rho(x)} u_n^{\frac{4}{N-2}} \right) \left(\int_{B_\rho(x)} u_n \right) + A b_N \rho^N \right) d\rho \\
&\le C \int_{\rho_1}^{\rho_2} \frac{1}{\rho^{N-1}} \left(\sigma_n \int_{B_\rho(x)} u_n + A\rho^N \right) d\rho \\
&\le C(m(\rho_1)\sigma_n + A) \int_{\rho_1}^{\rho_2} \rho d\rho \le C m(\rho_1) \sigma_n (\rho_2^2 - \rho_1^2) \,,
\end{aligned}$$

therefore $(\rho_2^2 - \rho_1^2) > C\sigma_n^{-1}$ and so $\rho_2 - \rho_1 > C\sigma_n^{-1/2}$. Denoting by $\mathcal{A}$ the annulus centered in x of radii ρ_1 and ρ_2 we have that measure of $\mathcal{A}$ is of the order of $\sigma_n^{-\frac{N}{2}}$, i.e. of the same order of $\mathcal{A}_n^1$ and so as in (1.3.9) we have

$$\fint_{\mathcal{A}} u_n \leq C.$$

On the other hand:

$$\fint_{\mathcal{A}} u_n \geq m(\rho_2) = \frac{1}{2} m(\rho_1)$$

and so

$$u_n(x) = m(0) \leq m(\rho_1) \leq C.$$

∎

1.4 Gradient estimates

In this section we shall give a integral bound for the derivatives of every term u_n of a controlled concentrating sequence in its safe regions $\mathcal{A}_n^3$ which shall be used, jointly to a local Pohozaev Identity, to prove the nonexistence of balanced concentrating sequences. One can easily guess that, since u_n and Δu_n are uniformly bounded on $\mathcal{A}_n^2$ and the width of $\mathcal{A}_n^2$ is of the order of $\sigma_n^{-1/2}$, ∇u_n can be expected to be of the order of $\sigma_n^{1/2}$. Such estimate can be very easily proved in a rigorous way in an integral form on the smaller annulus $\mathcal{A}_n^3$, by a Caccioppoli-type inequality.

Proposition 1.4.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence. Then there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$:*

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 \leq C\sigma_n^{\frac{2-N}{2}}. \tag{1.4.1}$$

PROOF. Let us fix $n \in \mathbb{N}$ and consider $\varphi_n : \mathbb{R}^N \to [0,1]$ a smooth positive mollifier radially symmetric around x_n such that

1) $\varphi_n = 1$ on $\mathcal{A}_n^3$;

2) $\varphi_n = 0$ out of $\mathcal{A}_n^2$;

3) $\Delta\varphi_n \leq C\sigma_n$.

By 2) we have $\varphi_n = 0$ and $\nabla\varphi_n = 0$ on $\partial\mathcal{A}_n^2$, and so, integrating by parts, by 1) we get

$$\begin{aligned} \int_{I\!R^N} -\Delta u_n u_n \varphi_n &= \int_{\mathcal{A}_n^2} |\nabla u_n|^2 \varphi_n + \int_{\mathcal{A}_n^2} \nabla u_n \cdot \nabla \varphi_n \, u_n \\ &\geq \int_{\mathcal{A}_n^3} |\nabla u_n|^2 + \int_{\mathcal{A}_n^2} \nabla \left(\frac{1}{2} u_n^2\right) \cdot \nabla \varphi_n \\ &= \int_{\mathcal{A}_n^3} |\nabla u_n|^2 - \frac{1}{2} \int_{\mathcal{A}_n^2} \Delta \varphi_n \, u_n^2 \, . \end{aligned}$$

Therefore, being u_n solution to (EI), by Proposition 1.3.1 and 3) we have:

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 \leq \int_{\mathcal{A}_n^2} \left(|u_n|^{2^*} + A u_n\right) \varphi_n + \frac{1}{2} \int_{\mathcal{A}_n^2} \Delta \varphi_n \, u_n^2 \leq C(1+\sigma_n) |\mathcal{A}_n^2| \, . \tag{1.4.2}$$

Since $\sigma_n \geq 1$ for n large one has (1.4.1). ∎

Corollary 1.4.1 *For any $n \in I\!N$ there exists $t_n \in [\overline{C}+2, \overline{C}+3]$ such that, denoting by $B_n = B(x_n, t_n \sigma_n^{-1/2})$:*

$$\int_{\partial B_n} |\nabla u_n|^2 \leq C \sigma_n^{\frac{3-N}{2}} \, , \tag{1.4.3}$$

where C is the constant in the above proposition.

1.5 Local Pohozaev Identity

In the next section, we shall test the presence of concentrations which would prevent us to find solutions to (CP) as limits of a balanced concentrating sequence. To this aim, we shall evaluate the infinitesimal variation of the functional corresponding to (SP) under a scaling of a concentrated part of u_n. Such a variation must be null because we are dealing with a balanced sequence. This condition is equivalent to the well-known Pohozaev Identity which we must establish in a local form (namely without using boundary conditions) since it shall be tested on a small concentrated part of the functions u_n. We fix a general open smooth set B in $I\!R^N$ and shall consider, more in general, a semilinear elliptic equation of the form

$$-\Delta u = g(u) \, . \tag{1.5.1}$$

Let u be a smooth solution to (1.5.1) on a smooth domain B. Multiplying by u and integrating by parts we get

$$\int_B |\nabla u|^2 = \int_B g(u) u + \int_{\partial B} (\nabla u \cdot \vec{n}) u \, , \tag{1.5.2}$$

where $\vec{n}$ is the outward normal to ∂B. Multiplying (1.5.1) for $\nabla u \cdot x$, since

$$\nabla \cdot ((\nabla u \cdot x) \nabla u) = \Delta u (\nabla u \cdot x) + (\nabla(\nabla u \cdot x)) \cdot \nabla u \, ,$$

using the Divergence Theorem and by integrating by parts we get

$$\begin{aligned} \int_B -\Delta u(\nabla u\cdot x) &= -\int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \int_B \nabla u\cdot(\nabla^2 u\cdot x + I\cdot\nabla u) \\ &= -\int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \int_B \nabla\left(\frac{1}{2}|\nabla u|^2\right)\cdot x + \int_B |\nabla u|^2 \\ &= -\int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \frac{1}{2}\int_{\partial B}|\nabla u|^2(x\cdot\vec{n}) + \frac{2-N}{2}\int_B|\nabla u|^2. \end{aligned} \tag{1.5.3}$$

On the other side, calling $G(u)$ a primitive of the function $g(u)$, integrating by parts we get:

$$\int_B g(u)(\nabla u\cdot x) = \int_B \nabla G(u)\cdot x = \int_{\partial B} G(u)(x\cdot\vec{n}) - N\int_B G(u)\,. \tag{1.5.4}$$

Combining (1.5.3) with (1.5.4) we obtain

$$\begin{aligned} \frac{N}{2^*}\int_B|\nabla u|^2 &= N\int_B G(u) - \int_{\partial B} G(u)(x\cdot\vec{n}) \\ &\quad -\int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \frac{1}{2}\int_{\partial B}|\nabla u|^2(x\cdot\vec{n})\,. \end{aligned} \tag{1.5.5}$$

Multiplying (1.5.2) for $-\frac{N}{2^*}$ and summing (1.5.5), we have

$$\begin{aligned} N\int_B G(u) - \frac{N}{2^*}\int_B g(u)u &= \int_{\partial B} G(u)(x\cdot\vec{n}) + \int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) \\ &\quad -\frac{1}{2}\int_{\partial B}|\nabla u|^2(x\cdot\vec{n}) + \frac{N}{2^*}\int_{\partial B}(\nabla u\cdot\vec{n})u \end{aligned} \tag{1.5.6}$$

which becomes in our case (i.e. $g(u) = |u|^{p-2}u + \lambda u$)

$$\begin{aligned} \left(\frac{N}{p} - \frac{N}{2^*}\right)\int_B|u|^p + \lambda\int_B|u|^2 &= \frac{1}{p}\int_{\partial B}|u|^p(x\cdot\vec{n}) + \frac{\lambda}{2}\int_{\partial B}|u|^2(x\cdot\vec{n}) \\ &\quad + \int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) - \frac{1}{2}\int_{\partial B}|\nabla u|^2(x\cdot\vec{n}) \\ &\quad + \frac{N}{2^*}\int_{\partial B}(\nabla u\cdot\vec{n})u\,. \end{aligned} \tag{1.5.7}$$

By a translation, we can move the origin to any fixed point $x_0 \in I\!R^N$ and we can forget, being $p < 2^*$, the positive term $\left(\frac{N}{p} - \frac{N}{2^*}\right)\int_B |u|^p$, in order to obtain the following *"Pohozaev-type"* inequality:

$$\begin{aligned} \lambda\int_B|u|^2 &\le \frac{1}{p}\int_{\partial B}|u|^p((x-x_0)\cdot\vec{n}) + \frac{\lambda}{2}\int_{\partial B}|u|^2((x-x_0)\cdot\vec{n}) \\ &\quad + \int_{\partial B}(\nabla u\cdot(x-x_0))(\nabla u\cdot\vec{n}) \\ &\quad -\frac{1}{2}\int_{\partial B}|\nabla u|^2((x-x_0)\cdot\vec{n}) + \frac{N}{2^*}\int_{\partial B}(\nabla u\cdot\vec{n})u\,. \end{aligned} \tag{1.5.8}$$

1.6 Concentration estimates

In this section we shall use the local Pohozaev Identity to prove that concentrations are not possible for balanced sequences in dimension $N \geq 7$.

Lemma 1.6.1 *If $N \geq 7$ no concentrating sequence can be balanced.*

PROOF. Let a concentrating sequence $(u_n)_{n\in\mathbb{N}}$ be given and assume by contradiction that it is balanced. Let us fix $n \in \mathbb{N}$, we shall use (1.5.8) on $B_n = B(x_n, t_n\sigma_n^{-1/2}) \cap \Omega$, where t_n is the same as in Corollary 1.4.1, and we shall split $\partial B_n = \partial_i B_n \cup \partial_e B_n$ where $\partial_e B_n$, (empty in the case in which the concentration point x_n of the basic rescaled function φ is sufficiently far from $\partial\Omega$) is $\partial\Omega \cap \overline{B}_n$. When $\partial_e B_n = \emptyset$, to the aim of applying (1.5.8), we shall take x_0 equal to the concentration point x_n. Otherwise we shall take x_0 out of Ω such that $d(x_0, x_n) \leq 2t_n\sigma_n^{-1/2}$ and

$$\forall x \in \partial_e B_n \ : \ \vec{n}\cdot(x - x_0) < 0 \ , \tag{1.6.1}$$

where $\vec{n}$ is the outward normal to ∂B_n (roughly speaking x_0 is the "symmetric" of x_n with respect to $\partial\Omega$). We want to show that (1.5.8) cannot hold true, in contradiction to the assumption that the sequence is balanced. To this aim, we must give a lower bound to the left hand side of (1.5.8) and a smaller upper bound to the right hand side. In the first case, we shall restrict the integral on the ball $B'_n = B(x_n, \sigma_n^{-1})$, which is contained in Ω for n large, and we shall make use of the decomposition $u_n = u_n^0 + u_n^1 + u_n^2$ introduced in the proof of Lemma 1.2.3. So we have

$$\int_{B_n\cap\Omega} (u_n)^2 \geq \int_{B'_n} (u_n)^2 \geq \frac{1}{2}\int_{B'_n} (u_n^2)^2 - 2\int_{B'_n} (u_n^0)^2 - 2\int_{B'_n} (u_n^1)^2 \ . \tag{1.6.2}$$

Now $\int_{B'_n} (u_n^2)^2$ is of the same order of $\int_{B'_n} (\rho_n(\varphi))^2$, namely of the order of σ_n^{-2} because φ corresponds to the less concentrated global solution. Moreover

$$\int_{B'_n} (u_n^1)^2 \leq \|u_\infty\| \, |B'_n| \leq C\sigma_n^{-N}$$

and

$$\int_{B'_n} (u_n^0)^2 \leq \|(u_n^0)^2\|_{\frac{2^*}{2}} |B'_n|^{1-\frac{2}{2^*}} \leq \|u_n^0\|_{2^*}^2 \sigma_n^{-2} \ .$$

Since $\|u_n^0\|_{2^*} \to 0$, by (1.6.2), we see that the left hand side of (1.5.8) has a lower bound of the form $C\sigma_n^{-2}$, for a suitable constant C. Passing to the right hand side, we firstly evaluate the possible contributions of $\partial_e B_n$. On such set only two of the integrals must be taken into account because we have $u_n = 0$ on $\partial_e B_n \subset \partial\Omega$. For the same reason, ∇u_n has the direction of $\vec{n}$ and so the whole sum, from (1.6.1), can be written as

$$\frac{1}{2}\int_{\partial_e B_n} |\nabla u_n|^2 (x - x_0)\cdot\vec{n} \leq 0 \ .$$

So we can focus our attention to the integrals extended to $\partial_i B_n$. Hence from Proposition 1.3.1, we get

$$\frac{\lambda}{2}\int_{\partial_i B_n} |u_n|^2((x-x_0)\cdot\vec{n}) + \frac{1}{p}\int_{\partial_i B_n} |u_n|^p((x-x_0)\cdot\vec{n}) \leq C\int_{\partial_i B_n}((x-x_0)\cdot\vec{n}) \leq C\sigma_n^{-\frac{N}{2}},$$

and from Corollary 1.4.1 and our choice of B_n

$$\int_{\partial_i B_n} |\nabla u_n|^2|x-x_0| \leq C\sigma_n^{\frac{2-N}{2}}.$$

Finally, from both Proposition 1.3.1 and Corollary 1.4.1, by Hölder Inequality

$$\int_{\partial_i B_n}(\nabla u_n\cdot\vec{n})u_n \leq \left(\int_{\partial_i B_n}|\nabla u_n|^2\right)^{1/2}\left(\int_{\partial_i B_n}|u_n|^2\right)^{1/2} \leq C\sigma_n^{\frac{2-N}{2}}.$$

Combining these estimates, we see that the right hand side of (1.5.8) is therefore bounded by $C\sigma_n^{\frac{2-N}{2}}$. So (1.5.8) requires

$$\lambda\sigma_n^{-2} \leq C\sigma_n^{\frac{2-N}{2}}, \tag{1.6.3}$$

which is clearly false for n large. ∎

The utility of the previous lemma is guaranteed by the next statement, which can be seen as a variant of Theorem 1.1.3 and which allows us to say that from a noncompact balanced sequence $(u_n)_{n\in\mathbb{N}}$ we can always extract a concentrating sequence, even if we do not know if $(u_n)_{n\in\mathbb{N}}$ is a PS sequence.

Lemma 1.6.2 *Let $(u_n)_{n\in\mathbb{N}}$ a noncompact bounded balanced sequence in $H_0^1(\Omega)$. Then from $(u_n)_{n\in\mathbb{N}}$ we can extract a concentrating subsequence.*

PROOF. We can assume (by passing to a subsequence) that $(u_n)_{n\in\mathbb{N}}$ has no converging subsequence. Under a null extension of u_n to the whole of $\mathbb{R}^N$, we can use the structure theorem for bounded sequences in [15], according to which every term of the sequence can be approximated by a sum in $H^1(\mathbb{R}^N)$ of the scaled "restored scale limits" of the sequence itself. What we still need to know is how to quantify the number of such limits and to qualify them as multiple of global solutions. To this aim, we shall prove that:

a) the weak limit u_∞ of the sequence solves (CP);

b) any restored scale limit φ_i of the sequence is a solution to the limit critical problem on $\mathbb{R}^N$ multiplied by a constant $\alpha_i \geq 1$.

For any $n \in \mathbb{N}$ we call p_n the exponent p such that u_n is solution to (SP). Now, being $(u_n)_{n\in\mathbb{N}}$ a bounded sequence, by reflexivity of $H_0^1(\Omega)$, using Rellich Theorem, we can pass to a subsequence such that

- $p_n \to \bar{p} \le 2^*$;
- $u_n \rightharpoonup u_\infty$ weakly in $H_0^1(\Omega)$;
- $u_n \to u_\infty$ a.e.

Therefore

$$|u_n|^{p_n-2}u_n + \lambda u_n \to |u_\infty|^{\bar{p}-2}u_\infty + \lambda u_\infty \quad \text{a.e.} \quad , \tag{1.6.4}$$

moreover, by linearity, we can say that $-\Delta u_n \rightharpoonup -\Delta u_\infty$. Being u_n a solution to (SP) the two limits must coincide i.e. $-\Delta u_\infty = |u_\infty|^{\bar{p}-2}u_\infty + \lambda u_\infty$. Now $\bar{p} = 2^*$, otherwise we would use the compact Sobolev immersion of $H_0^1(\Omega)$ in $L^{\bar{p}}(\Omega)$ obtaining a strongly converging subsequence and this is a contradiction to our hypotheses; therefore a) is proved. Let $\varphi = \lim_{n\to+\infty} \rho_n(u_n)$ in the weak H^1-topology be any restored scale limit (see [15]), where $(\rho_n)_{n\in\mathbb{N}}$ is any diverging sequence of scalings each one of modulus $\nu_n > 0$. By easy calculations, being u_n solution to (SP) we get that

$$-\Delta\rho_n(u_n) = (\nu_n)^{2-\frac{N}{2^*}(p_n-2)}|\rho_n(u_n)|^{p_n-2}\rho_n(u_n) + \lambda\nu_n^2\rho_n(u_n) \; . \tag{1.6.5}$$

Now it is obvious that, being Ω a fixed bounded domain, the only way to get non zero weak limits is to have $(\rho_n)_{n\in\mathbb{N}}$ diverging by vanishing, i.e. $\lim_{n\to+\infty}\nu_n = 0$. Passing to a subsequence, we can assume that

$$\nu_n^{2-\frac{N}{2^*}(p_n-2)} = \nu_n^{N\left(1-\frac{p_n}{2^*}\right)} \to \mu \le 1$$

and $\mu > 0$ because $\varphi \ne 0$. Using the linearity of the operator $-\Delta$ and Rellich Theorem we can pass to the limit in (1.6.5) obtaining $-\Delta\varphi = \mu|\varphi|^{2^*-2}\varphi$. So, $\mu^{\frac{N-2}{4}}\varphi$ solves (CP) in $H^1(\mathbb{R}^N)$ for $\lambda = 0$ and, since $\mu^{\frac{N-2}{4}} \le 1$, φ is as required by the definition of concentrating sequence. Statement b) implies, in particular, that $\|\varphi_i\|_{H^1} \ge S^{\frac{N}{2}}$ and so gives the bound on k required by the definition of concentrating sequence. ∎

Proof of Theorem 1.1.1. Let us suppose, by contradiction, that there exists a bounded balanced sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\sup_{n\in\mathbb{N}}\sup_{x\in\Omega}|u_n(x)| = +\infty.$$

A standard regularity argument [5] shows that u_n cannot be compact in H^1, so by Lemma 1.6.2 it has a balanced concentrating subsequence and this is excluded by Lemma 1.6.1.

∎

1.7 Multiple solutions to the critical problem

We can now give the proof of Theorem 1.1.2 stated in Section 1.1. Let us choose a sequence $(p_n)_{n\in I\!N}$ in $]2,2^*[$ such that $p_n \to 2^*$ and the functionals

$$I_\lambda^n(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\lambda}{2}\int_\Omega |v|^2 - \frac{1}{p_n}\int_\Omega |v|^{p_n} ,$$

whose critical points are solutions to problems (SP) for $p = p_n$. We shall find, in a usual way, an infinite number of critical levels $c_n^{(k)}$ of the subcritical functionals I_λ^n obtained by means of min-max levels on a k dimensional min-max class of compact sets Γ_k on the constraint

$$V = \{u \in H_0^1(\omega) \mid \int |\nabla u|^2 - \lambda \int |u|^2 = 1\} ,$$

which does not depend on n. Precisely we call Γ_k the set of compact subset of V which have Krasnoselskii genus greater than k for all k such that $\lambda < \lambda_k$. For fixed n, $k \in I\!N$ we set

$$\overline{c}_k = \inf_{A\in\Gamma_k} \sup_{v\in A} I_\lambda(v)$$

and

$$\overline{c}_n^{(k)} = \inf_{A\in\Gamma_k} \sup_{v\in A} I_k^n(v) .$$

It is easy to show that given $\overline{u}_n^{(k)} \in V$ such that $I_\lambda^n(\overline{u}_n^{(k)}) = \overline{c}_n^{(k)}$ then $\alpha_n^{(k)}\overline{u}_n^{(k)}$, with

$$\alpha_n^{(k)} = \left[\left(\frac{1}{2} - \overline{c}_n^{(k)}\right) p_n\right]^{\frac{1}{2-p_n}} ,$$

is a solution to (SP) at level

$$c_n^{(k)} = \left[\left(\frac{1}{2} - \overline{c}_n^{(k)}\right) p_n\right]^{\frac{2}{2-p_n}} \left(\frac{1}{2} - \frac{1}{p_n}\right) . \tag{1.7.1}$$

Analogously we shall call

$$c_k = \left[\left(\frac{1}{2} - \overline{c}_k\right) 2^*\right]^{\frac{2}{2-2^*}} \left(\frac{1}{2} - \frac{1}{2^*}\right) . \tag{1.7.2}$$

The proof will follow from the following lemmas.

Lemma 1.7.1 $\lim_{n\to+\infty} c_n^{(k)} = c_k$ *for any* $k \in I\!N$.

Lemma 1.7.2 $\lim_{k\to+\infty} c_k = +\infty$.

PROOF OF THEOREM 1.1.2. Fixed $k \in \mathbb{N}$, we take, for any $n \in \mathbb{N}$, $u_n = u_n^{(k)}$ a critical point at level $c_n^{(k)}$ for the functional I_λ^n. First we use Lemma 1.7.1 and [1, Lemma 3.6] to have $(u_n^{(k)})_{n\in\mathbb{N}}$ bounded in H_0^1. The sequence $(u_n)_{n\in\mathbb{N}}$, by Theorem 1.1.1 is then uniformly bounded, so by standard compactness arguments we can find a converging subsequence to a solution $u^{(k)}$ to (CP) at level c_k, as follows from Lemma 1.7.1. By Lemma 1.7.2 we have infinitely many distinct values of c_k for $k \in \mathbb{N}$ and so the proof is concluded. ∎

Now we give the proofs of lemmas 1.7.1 and 1.7.2 which conclude the paper.

PROOF OF LEMMA 1.7.1. Let us fix $k \in \mathbb{N}$ and $A \in \Gamma_k$, then for any $u \in A$: $I_\lambda^n(u) \to I_\lambda(u)$. Being A compact and the functionals equicontinuous,

$$\sup_{u\in A} I_\lambda^n(u) \to \sup_{u\in A} I_\lambda(u) \ .$$

Then $\limsup_{n\in\mathbb{N}} \overline{c}_n^{(k)} \le \sup_{u\in A} I_\lambda(u)$ and, being A an arbitrary set in Γ_k, we get

$$\limsup_{n\to+\infty} \overline{c}_n^{(k)} \le \overline{c}_k \ .$$

By the very definition, see (1.7.1) and (1.7.2), we get

$$\limsup_{n\to+\infty} c_n^{(k)} \le c_k \ .$$

Since for $s > 0$ the function $h(s) = \frac{1}{p_n} s^{p_n} - \frac{1}{2^*} s^{2^*}$ gets its maximum value in $s = 1$ we have $h(s) \le \frac{1}{p_n} - \frac{1}{2^*}$ for all $s > 0$. Therefore for every $u \in H_0^1$:

$$I_\lambda(u) \le I_\lambda^n(u) + \left(\frac{1}{p_n} - \frac{1}{2^*}\right) |\Omega| \ ,$$

so, for any $k \in \mathbb{N}$,

$$\overline{c}_k \le \liminf_{n\to+\infty} \overline{c}_n^{(k)}$$

and by (1.7.1) and (1.7.2) we get

$$c_k \le \liminf_{n\to+\infty} c_n^{(k)}$$

and the thesis.

∎

PROOF OF LEMMA 1.7.2. We want to remark that such a result is not based on compactness properties because, if we prove the statement for $\lambda > 0$, i.e. when we have compactness, the statement itself is obviously true when $\lambda \le 0$ and compactness fails. Moreover this lemma is also true in lower dimension, as we can see, in the same way, by adding a suitably big subcritical term. On the other side, the use of compactness techniques takes advantage of the previous results in this paper. Let us suppose, by contradiction, that the sequence $(c_k)_{k\in I\!N}$ is bounded, hence it converges to a real number c. For any $k \in I\!N$ by Lemma 1.7.1 there exists $n_k > k$ such that $|c^{(k)}_{n_k} - c_k| < \frac{1}{k}$; hence

$$\lim_{k\to+\infty} c^{(k)}_{n_k} = \lim_{k\to+\infty} c_k = c \tag{1.7.3}$$

and the sequence $(n_k)_{k\in I\!N}$ is diverging, i.e.

$$\lim_{k\to+\infty} n_k = +\infty.$$

Let u_{n_k} be a solution of (SP) at level $c^{(k)}_{n_k}$. Using the Morse Index estimates on min-max points (see [4] and [10]), we can select the sequence $(u_{n_k})_{k\in I\!N}$ such that every u_{n_k} has an augmented Morse index greater or equal to n_k. By our assumptions, we can claim that the sequence $(u_{n_k})_{k\in I\!N}$ is bounded in $H^1_0(\Omega)$. Indeed, being u_{n_k} a solution to (SP) we have

$$I^{n_k}_\lambda(u_{n_k}) = \left(\frac{1}{2} - \frac{1}{p_{n_k}}\right)\int_\Omega |u_{n_k}|^{p_{n_k}} \to c\ , \tag{1.7.4}$$

which gives the boundedness of $-\Delta u_{n_k} - \lambda u_{n_k}$ in H^{-1} and, in turn, the boundedness of u_{n_k} on H^1_0 (this is obvious when λ is not an eigenvalue and, in the general case, a blow up of the component of u_{n_k} in the eigenspace is clearly excluded by (1.7.4)). So the sequence $(u_{n_k})_{k\in I\!N}$ is uniformly bounded by Theorem 1.1.1 and therefore the Morse Index of u_{n_k} must keep bounded in contradiction to our construction. ∎

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Chapter 2

A multiplicity result for elliptic equations at critical growth in low dimension[2]

We consider the problem $-\Delta u = |u|^{2^*-2}u + \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is an open regular subset of $I\!R^N$ $(N \geq 3)$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and λ is a constant in $]0, \lambda_1[$ and λ_1 is the first eigenvalue of $-\Delta$. In this paper we show that, when $N \geq 4$, the problem has at least $\frac{N}{2} + 1$ (pairs of) solutions, improving a result obtained in [4] for $N \geq 6$.

Introduction

In this note, we shall deal with the problem

$$(P) \qquad \begin{cases} -\Delta u &= |u|^{2^*-2}u + \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases},$$

where Ω is an open regular subset of $I\!R^N$ $(N \geq 3)$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, and $\lambda \in]0, \lambda_1[$ is a constant, where λ_1 is the first eigenvalue of $-\Delta$. Brezis and Nirenberg in a celebrated paper [2] found a positive solution to (P) at a level $c_0 < \frac{1}{N} S^{\frac{N}{2}}$, where S denotes the so called Sobolev constant,

$$S = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{2^*}\right)^{\frac{2}{2^*}}} = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\| \nabla u \|_2^2}{\| u \|_{2^*}^2}.$$

Our aim consists in showing that, for $N \geq 4$, (P) has at least $\overline{m}$ (pairs of) solutions where $\frac{N}{2} + 1 \leq \overline{m} \in I\!N$. We are really concerned with low values of N, since we have recently proved

[2] G. Devillanova & S. Solimini *A Multiplicity Result for Elliptic Equations at Critical Growth in Low Dimension* Comm. in Contemporary Math., Vol. 5 N. 2(April 2003), 171-177.

in [5] that (P) actually has infinitely many critical levels in dimension $N \geq 7$, thanks to some compactness properties which can be shown to be false if $N \leq 6$. Nevertheless, if $N = 6$, a result proved in [4] shows that (P) has at least two (pairs of) solutions, corresponding to a ground state level c_0 (the one considered in [2]) and to a different level $c_1 > c_0$, obtained as the minimum of the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} , \tag{2.0.1}$$

on the "double" natural constraint

$$U = \left\{u \in H_0^1(\Omega) \mid u^\pm \neq 0,\ (\nabla I_\lambda(u), u^\pm) = 0 \text{ , for both} \pm \text{signs}\right\} . \tag{2.0.2}$$

In problems involving critical growth, the main difficulty is in the "lack of compactness" of the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ and this makes the Palais-Smale condition do not hold "globally" with respect to the functional I_λ defined on $H_0^1(\Omega)$, whose critical points are solutions to (P). More precisely, the $[P.S.]_c$ condition is false at some levels $c \in \mathbb{R}$, namely there exists some $c \in \mathbb{R}$ and a $P.S.$ sequence at level c, defined as a sequence $(u_n)_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ verifying

(a) $I_\lambda(u_n) \to c$,

(b) $\nabla I_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$ strongly ,

without any strongly converging subsequence.

The existence of a critical point at level $c_1 > c_0$ is proved in [4] thanks to the estimate

$$c_1 < c_0 + \frac{1}{N}S^{\frac{N}{2}} . \tag{2.0.3}$$

The proof of such inequality is the step which requires the lower bound on the dimension and which we are not able to extend to our case. Nevertheless, we shall show in this note that one can also build some different min-max classes which will provide the above mentioned multiplicity result.

Stronger results can be proved in suitably symmetric cases, see [6], but their extension to general domains does not look to be trivial, as one can appreciate by looking to the corresponding radial problem. Indeed, in such a case, we even need the stronger condition $N \geq 7$ in order to get (2.0.3) and moreover, when $4 \leq N \leq 6$, in [1] the existence of a constant $\lambda^* \in]0, \lambda_1[$ such that (P) have not radial changing sign solutions for $\lambda \in]0, \lambda^*[$ is proved.

After few preliminary lemmas, which will be archived in next section, in Section 2.2 we shall prove the following theorem.

Theorem 2.0.1 *Let $N \geq 4$, then (P) has at least $\frac{N}{2}+1$ distinct (pairs of) solutions $\forall\lambda \in]0, \lambda_1[$.*

In particular we get at least three pairs of solutions for $N = 4$ and four pairs of solutions for $N = 5$ and $N = 6$, improving also in this last case the analogous result in [4], see also [3].

The above estimate can be further on improved if we restrict the parameter λ in a smaller neighbourhood of 0. Indeed, in section 2.2 we shall also prove the following statement.

Theorem 2.0.2 *Let $N \geq 4$, then there exists a positive number $\overline{\lambda} \in]0, \lambda_1[$ such that problem (P) has at least $N + 1$ distinct (pairs of)solutions for every $\lambda \in]0, \overline{\lambda}[$.*

2.1 Variational approach and preliminary lemmas

We shall need some preliminary lemmas which will allow us to use a suitable variational approach and to build *P.S.* sequences which must have a nonzero weak limit, by taking advantage of the behavior of non compact *P.S.* sequences described in [8].

We start by observing that the level c_0 of the solution u_0 found in [2] can be obtained as

$$c_0 = \inf_{u \in U_0} I_\lambda(u) ,$$

where

$$U_0 = \{u \in H_0^1(\Omega) \mid u \neq 0, (\nabla I_\lambda(u), u) = 0\} .$$

Analogously, we can use as in [4] the constraint defined in (2.0.2) and set

$$c_1 = \inf_{u \in U} I_\lambda(u) . \tag{2.1.1}$$

It is easy to prove that U is a regular constraint (see Lemma 2.1.1 below) and this allow us to state the following definition.

Definition 2.1.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H_0^1(\Omega)$, we shall say that $(u_n)_{n\in\mathbb{N}}$ is a constrained P.S. sequence in U at level c if every element u_n belongs to U, the sequence $(I_\lambda(u_n))_{n\in\mathbb{N}}$ converges to c and the tangential component to U in u_n of $\nabla I_\lambda(u_n)$ is infinitesimal in $H^{-1}(\Omega)$.*

It is well known, and it can be easily proved, that U is a natural constraint, in the sense that the infinitesimal character of the tangential part of $\nabla I_\lambda(u_n)$ forces the whole $\nabla I_\lambda(u_n)$ to be infinitesimal, as stated in the following lemma.

Lemma 2.1.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a constrained P.S. sequence in U, then $(u_n)_{n\in\mathbb{N}}$ is a P.S. sequence i.e. $\nabla I_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$.*

PROOF. By the very definition we get

$$I_\lambda(u_n^\pm) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\int |\nabla u_n^\pm|^2 - \lambda \int u_n^{\pm 2}\right) \leq C , \tag{2.1.2}$$

where C is a constant which does not depend on n, which gives a bound to $(u_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ since $\lambda < \lambda_1$. Taking into account that $u_n \in U$ and that the tangential component of $\nabla I_\lambda(u_n)$ is infinitesimal, by the definition of U we can deduce that also the normal part of $\nabla I_\lambda(u_n)$, multiplied by $u_n^\pm$ tends to zero. Namely, we can find a sequence of Lagrange multipliers $(\mu_n)_{n\in\mathbb{N}}$ such that the normal part of $\nabla I_\lambda(u_n)$ in u_n to U is given by $\mu_n(-2\Delta u_n - 2\lambda u_n - 2^* u_n^{2^*-1})$ and

$$\mu_n \int \left((-2\Delta u_n - 2\lambda u_n - 2^* u_n^{2^*-1}) u_n^\pm\right) \to 0 .$$

Therefore, by using once more that $u_n \in U$, we get

$$\mu_n(2 - 2^*) \int (u_n^\pm)^{2^*} \to 0 .$$

Since the sequence $(\|u_n^\pm\|_{2^*})_{n\in\mathbb{N}}$ is bounded from below we get $\mu_n \to 0$ and therefore the normal part of $\nabla I_\lambda(u_n)$ tends to zero in $H^{-1}(\Omega)$. ∎

We shall state a lemma which will be used as a compactness type property for constrained *P.S.* sequences in U at levels $c < \frac{2}{N} S^{\frac{N}{2}}$.

Lemma 2.1.2 *Let $(u_n)_{n\in\mathbb{N}}$ be a constrained P.S. sequence in U at level $c < \frac{2}{N}S^{\frac{N}{2}}$, then the sequence cannot converge weakly to zero.*

PROOF. Being $(u_n)_{n\in\mathbb{N}}$ a constrained *P.S.* sequence in U and $\lambda \in]0, \lambda_1[$ we have $\|\nabla u_n^\pm\| \geq S^{\frac{N}{4}}$ therefore an eventual strong limit of the two sequences $(u_n^\pm)_{n\in\mathbb{N}}$ cannot be zero. We have to extend this claim to the case of a weak limit, which brings a weaker information when the sequence is not compact. We know, thanks to [8, Proposition 2.1], that the "bad" levels c for noncompact *P.S.* sequences in $]0, \frac{2}{N}S^{\frac{N}{2}}[$ are $c = c' + \frac{1}{N}S^{\frac{N}{2}}$ with c' critical level for I_λ and the weak limit of the sequence is a solution u to (P) at level c'. The obstruction to the compactness of $(u_n)_{n\in\mathbb{N}}$ is given by scaled copies of a global solution (i.e. a solution to (P) in $\mathbb{R}^N$ for $\lambda = 0$) which has a constant sign and disappears if we restrict ourselves to $u_n^\pm$ for one of the $+$ or $-$ signs. So, in one of the two cases, we have $u_n^\pm \to u^\pm$ strongly and so $u \neq 0$, according to the assertion in the beginning of the proof, see also [3] and [4]. Therefore or c or $c - \frac{1}{N}S^{\frac{N}{2}}$ is a non null critical level which corresponds to the strong or respectively to the weak limit of the sequence $(u_n)_{n\in\mathbb{N}}$. ∎

We shall introduce some usual min-max classes; for $k \in \mathbb{N} \setminus \{0\}$ let us set

$$\Gamma_k = \{A \subset U \mid A \text{ is compact, } A = -A,\ \gamma(A) \geq k\}\ ,$$

where $\gamma(A)$ is the Krasnoselskii genus of the set A and $\forall k > 0$

$$c_k = \inf_{A\in\Gamma_k} \sup_{u\in A} I_\lambda(u) = \inf_{\Gamma_k} \sup I_\lambda\ .$$

We point out that for $k = 1$ we obtain (2.1.1) because Γ_1 gives the set of all compact symmetric and nonempty subsets of U.

Lemma 2.1.3 *For all $k \in \{1, \ldots, N+1\}$ we have*

$$2c_0 \leq c_k < \frac{2}{N}S^{\frac{N}{2}}\ .$$

PROOF. The finite sequence of sets $(\Gamma_k)_{k\in\{1,\ldots,N+1\}}$ is decreasing and therefore we must only prove that $2c_0 < c_1$ and $c_{N+1} < \frac{2}{N}S^{\frac{N}{2}}$. The first inequality is obvious since, for all $u \in U$: $u^+, u^- \in U_0$ and $I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-) \geq 2c_0$. We start by determining a constant $\bar{c} < \frac{1}{N}S^{\frac{N}{2}}$ such that $c_N < 2\bar{c}$, namely we can find a set $A \in \Gamma_N$ such that $\sup_A I_\lambda \leq 2\bar{c} < \frac{2}{N}S^{\frac{N}{2}}$. Let $B = B_R$ be a ball with radius $R > 0$ contained in Ω which, in order to use a simpler notation, will be assumed, without any restriction, to be centered in the origin. For any $\nu \in S^{N-1} = \partial B$, we consider two balls B_1^ν and B_2^ν of radius $\frac{R}{2}$ contained in B and tangent to ∂B respectively in ν and in $-\nu$. Let us consider the function $\varphi : \partial B \to U$ such that for any $\nu \in \partial B$, $\varphi(\nu) = u^\nu : \Omega \to \mathbb{R}$ where $(u^\nu)^+$ and $(u^\nu)^-$ are respectively the positive solutions $\overline{u}_1$ and $\overline{u}_2$ found in [2] on B_1^ν and B_2^ν and let us call $\overline{c} = I_\lambda((u^\nu)^+) = I_\lambda((u^\nu)^-)$. In this way we map ∂B into a set $A \subset U$ in an odd continuous way, therefore $\gamma(A) \geq \gamma(\partial B) = N$. Moreover $\forall \nu \in \partial B$: $I_\lambda(u^\nu) = I_\lambda((u^\nu)^+) + I_\lambda((u^\nu)^-) = 2\overline{c} < \frac{2}{N} S^{\frac{N}{2}}$.

¿From the bound $c_N < 2\overline{c}$ we shall now pass to show that c_{N+1} is below $\frac{2}{N} S^{\frac{N}{2}}$. Actually we shall find in this new case the bound $c_{N+1} \leq \overline{c} + \frac{1}{N} S^{\frac{N}{2}}$, by extending the above introduced map φ from ∂B to a $N+1$ dimensional sphere. We shall find more convenient now not to take $\overline{c}$ as the infimum but as the test level considered in [2] by using as $\overline{u}_1$ and $\overline{u}_2$ scaled copies of the minimal global solution multiplied by a cut off coefficient. This choice is convenient because allows us to trivially deduce that $\overline{u}_1$ can be modified continuously to a function $\overline{u}_1^s$, which has a support on the ball $B^\nu(s)$ concentric with B_1^ν and radius s, belongs to U_0 and keeps the property $I_\lambda(\overline{u}_1^s) < \frac{1}{N} S^{\frac{N}{2}}$. Of course $I_\lambda(\overline{u}_1^s) \to \frac{1}{N} S^{\frac{N}{2}}$ as $s \to 0$.

We shall first extend φ to B, namely we shall find a continuous homotopy with values in U from φ to a constant map. This homotopy will be performed in two steps.

Firstly we shall shrink the radius of B_1^ν to a conveniently small radius ρ by taking

$$H_1(s,\nu) = \overline{u}_1^s - \overline{u}_2$$

with $s \in \left[0, \frac{R}{2}\right]$. Then we shall translate the balls $B^\nu(\rho)$ and B_2^ν bringing the two centers on the origin, obtaining the homotopy

$$H_2(t,\nu) := x \mapsto \overline{u}_1^\rho \left(x + t\frac{\nu}{2}\right) - \overline{u}_2 \left(x - t\frac{\nu}{2}\right)$$

for $t \in [0,1]$.

The very different scales of $\overline{u}_1^\rho$ and $\overline{u}_2$ make infinitesimal the mixed terms in the evaluation of I_λ, so we have for all $t \in [0,1]$

$$I_\lambda(H_2(t,\nu)) = I_\lambda(\overline{u}_1^\rho) + I_\lambda(\overline{u}_2) + \varepsilon_1(\rho) = \overline{c} + \frac{1}{N} S^{\frac{N}{2}} + \varepsilon_2(\rho)\ , \tag{2.1.3}$$

where for $i = 1, 2$, $\varepsilon_i(\rho) \to 0$ when $\rho \to 0$. The definition of H_2 should still be changed by multiplying the positive and the negative part of $H_2(t,\nu)$ by coefficients $\alpha_\pm(t)$ in such a way to provide that $H_2(t,\nu) \in U$ for all $t \in [0,1]$. Since the function $\alpha_\pm(t)$ are clearly continuous and $\alpha_\pm(t) \to 1$, as $\rho \to 0$, as we can see by arguing as in (2.1.3), we still keep (2.1.3) after this small correction. $H_2(1,\nu)$ gives the same function whatever is $\nu \in \partial B$, so the homotopy connects φ to a constant map.

We have in this way an extension of φ to B and so to a hemisphere in dimension $N+1$. By an odd extension we define φ on the whole sphere.

Then the set of the functions obtained via this transformation is a compact symmetric set contained in U whose genus is greater or equal to $N+1$, moreover on this set the functional I_λ is bounded by $\overline{c} + \frac{1}{N} S^{\frac{N}{2}} + \varepsilon_2(\rho) < \frac{2}{N} S^{\frac{N}{2}}$ and the thesis follows. ∎

We shall now observe that if two different levels c_i coincide, then (P) must necessarily have infinitely many solutions, even if the $[P.S.]_{c_i}$ condition fails. Note that this lack of compactness make us unable even to say that such a level must be critical, however Lemma 2.1.2 permits to conclude that the number of solutions at a possibly lower level is necessarily infinite.

Lemma 2.1.4 *If there exist $i, j \in \{1,\ldots,N+1\}$, $i \neq j$, such that $c_i = c_j$, then (P) has infinitely many solutions.*

PROOF. We can reduce ourselves to the case in which $j = i+1$ with $i \in \{1,\ldots,N\}$. We shall prove that, in this hypothesis, we can find a solution to (P) which is orthogonal to every given test function v. Let $(A_n)_{n\in\mathbb{N}} \in \Gamma_{i+1}^{\mathbb{N}}$ a minimizing sequence i.e. $\limsup_n (\sup_{A_n} I_\lambda) = c_{i+1}$. Let us fix a test function v and consider for every $n \in \mathbb{N}$

$$A'_n = \{u \in A_n \mid \int uv = 0\} .$$

The sequence $(A'_n)_{n\in\mathbb{N}}$ is, by construction, a sequence in Γ_i and being

$$\limsup_n \left(\sup_{A'_n} I_\lambda\right) \leq \limsup_n \left(\sup_{A_n} I_\lambda\right) = c_{i+1} = c_i ,$$

it is a minimizing sequence. Let $(u_n)_{n\in\mathbb{N}}$ be a constrained $P.S.$ sequence at level c_i close to the sequence $(A'_n)_{n\in\mathbb{N}}$ (i.e. $\lim_n d(u_n, A_n) = 0$), then by Lemma 2.1.2 its weak limit $\overline{u}$ is a nontrivial solution to (P) which is orthogonal to v. If (P) has a finite number of solutions we can built a test function whose scalar product with every nontrivial solution is not null and we find a contradiction. ∎

2.2 Proof of the Theorems

PROOF OF THEOREM 2.0.1. Taking into account Lemma 2.1.4, we can suppose that $\forall i, j \in \{1,\ldots,N+1\} : c_i \neq c_j$. By Lemma 2.1.2, we know that for every $i \in \{1,\ldots,N+1\}$ we have the following alternative: or c_i or $c'_i = c_i - \frac{1}{N}S^{\frac{N}{2}}$ is a nonzero critical level. Taking into account that

$$0 < c_0 < c_1 < \ldots < c_{N+1} < \frac{2}{N}S^{\frac{N}{2}} ,$$

we can deduce that at most two different values c_i can determine the same solution, so we get at least $\frac{N+2}{2} = \frac{N}{2} + 1$ solutions to (P). More precisely, when N is even we have at least $\frac{N}{2}+1$ pairs of solutions to (P) while if N is odd we get at least $\frac{N+1}{2}+1$ pairs of solutions to (P). ∎

PROOF OF THEOREM 2.0.2. The proof is obvious taking into account that $\exists\overline{\lambda} \in]0, \lambda_1[$ such that for all $\lambda \in]0, \overline{\lambda}[\, : \, 2c_0 > \frac{1}{N}S^{\frac{N}{2}}$. This last property implies that $\forall i \in \{2,\ldots,N+1\}$ $c_i - \frac{1}{N}S^{\frac{N}{2}} < c_1$, so the only critical value which can be given by two different min-max approaches of the type considered above is c_0. ∎

Remark 2.2.1 *We conclude this short note pointing out that in our approach we have lost the initial variational characterization of the solutions and this does not allow us to be sure to find a changing sign solution as in [4].*

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Chapter 3

Infinitely many bound states for some nonlinear scalar field equations[3]

In this paper we consider the problem $-\Delta u + a(x)u = |u|^{p-2}u$ in $I\!R^N$, where $p > 2$ and $p < 2^* = \frac{2N}{N-2}$ if $N > 2$. Assuming that the potential $a(x)$ is a regular function such that $\liminf_{|x|\to+\infty} a(x) = a_\infty > 0$ and that verifies suitable decay assumptions, but not requiring any symmetry property on it, we prove that the problem has infinitely many solutions.

3.1 Introduction and statement of the results

In this paper we are concerned with the existence of multiple solutions to

$$\text{(P)} \qquad \begin{cases} -\Delta u + a(x)u &= |u|^{p-2}u \qquad \text{in } I\!R^N \\ u &\in H^1(I\!R^N)\,, \end{cases}$$

where $N \geq 2$, $p > 2$ and $p < \frac{2N}{N-2}$ when $N > 2$, and the potential $a(x)$ is a continuous function, positive in $I\!R^N$, except at most a bounded set, verifying suitable decay assumptions, but not required to possess any symmetry property.

During the past years there has been a considerable interest in problems like (P) due essentially to two reasons: such problems arise naturally in various branches of Mathematical Physics, indeed the solutions of (P) can be seen as solitary waves (stationary states) in nonlinear equations of the Klein - Gordon or Schrödinger type, and, on the other hand, they present specific mathematical difficulties that make them challenging to the researchers.

Problem (P) has a variational structure: its solutions can be searched as critical points of

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the energy functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2)dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx \; . \tag{3.1.1}$$

However, the usual variational methods, that allow to prove the existence of infinitely many solutions to (P) in a bounded domain, cannot be applied straightly to I. Indeed, the embedding $j : H^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is continuous, but not compact, therefore the basic Palais - Smale condition is not satisfied by I in all the energy levels. This difficulty can be avoided when $a(x)$ enjoys of some symmetry. Indeed the first known results have been obtained considering either $a(x) = a_\infty \in \mathbb{R}^+ \setminus \{0\}$, or $a(x) = a(|x|)$ (see [18], [7], [8], [9], [23], [6]). In this case the restriction of I to $H^1_r(\mathbb{R}^N)$, the subspace of $H^1(\mathbb{R}^N)$ consisting of spherically symmetric functions, restores compactness, because the embedding of $H^1_r(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ is compact. So, the existence of a positive solution to (P) can be proved either by using Mountain Pass Theorem or by minimization on a natural constraint, while the existence of infinitely many solutions follows applying standard minimax arguments. Moreover it is worth recalling that, still under the assumption $a(x) = a(|x|)$, one can also find the existence of infinitely many nonradial solutions, which change sign, breaking the radial symmetry of the equation (see [4] and reference therein).

The question becomes more difficult when $a(x)$ has not symmetry properties, then even proving the existence of one positive solution is not a simple matter. To handle this situation a deeper understanding of the nature of the obstructions to the compactness and subtle tools are needed. Most of the researches have been concerned with the case

$$\lim_{|x|\to+\infty} a(x) = a_\infty > 0 \tag{3.1.2}$$

so that (P) can be related to the "problem at infinity"

$$-\Delta u + a_\infty u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N \; . \tag{P_∞}$$

A first answer to the existence question has been given, proving that, in some cases, being true some inequalities relating (P) and (P_∞), the concentration-compactness principle can be applied and (P) can be solved by minimization [15]. This is the case, for istance, when $a(x)$ is a continuous function that, besides (3.1.2) and decay assumptions, satisfies

$$0 < \delta_1 \le a(x) \le a_\infty \qquad \forall x \in \mathbb{R}^N \; . \tag{3.1.3}$$

Subsequently a careful analysis of the behaviour of the Palais-Smale sequences (see [5], [2]) has allowed to state that the compactness can be loosen (in the sense that a PS-sequence does not converge to a critical point) if and only if such a sequence breaks into a finite number of solutions to (P_∞) which are "centered" at points whose inter distances go to infinity. As a consequence, it has been possible to give an estimate of the energy levels in which the PS condition fails in terms of the energy of such masses and to face better some existence and multiplicity questions for (P). In fact, the existence of a positive solution to (P) has been proved (see [2]) even when a ground state solution cannot exist, that is, for istance, when, besides (3.1.2) and suitably decay assumptions, the potential satisfies the condition $a(x) > a_\infty$ $\forall x \in I\!R^N$; moreover, under the conditions (3.1.2), (3.1.3) and of a suitable decay at infinity, it has been shown the existence of a changing sign solution in addition to the positive one (see [16]).

To conclude this brief review of known results, let us mention that there is some other work involving the use of variational methods to treat standing waves of nonlinear Schrödinger equations. Some of these papers mainly deal with the existence of solutions for (P) using mountain pass and comparison arguments. See e.g. [20], [12] as well as their bibliographies. In particular, we point out that in [20] the existence of a positive and a negative solution is proved, provided

$$\text{(3.1.4)} \qquad i) \quad \inf_{I\!R^N} a(x) > 0 \ ; \qquad ii) \quad lim_{|x|\to+\infty} a(x) = +\infty.$$

while in [3] the existence of a third changing sign solution is shown.

Some other papers discuss cases in which the potential $a(x)$ possesses nondegenerate critical points and depends on a parameter, i.e. it appears like $a_h(x) := a(hx)$, and contain results of multiplicity of positive solutions under restriction on the size of h (see [13], [19], [1], and for $a(x)$ of a special form [17]). Finally we remind that, under assumptions of periodicity on a, (P) has been shown to posses infinitely many solutions [10].

As far as we know the question of the existence of infinitely many solutions to (P) , without symmetry or periodicity assumptions on the potential $a(x)$, is largely open: the result we present here is a contribution to the settlement of it.

Let us now state the hypotheses on the function a that will be used and our main result.

(a_1) $a \in C^1(I\!R^N, I\!R)$

(a_2) $\liminf_{|x|\to+\infty} a(x) = a_\infty > 0$

(a_3) $\dfrac{\partial a}{\partial \vec{x}}(x)e^{\alpha|x|} \underset{|x|\to+\infty}{\longrightarrow} +\infty \, , \forall \alpha > 0$
where for all $x \in I\!R^N \setminus \{0\}$, $\vec{x} = \frac{x}{|x|}$

(a_4) there exists a constant $\bar{c} > 1$ such that

$$|\nabla_{\tau_x} a(x)| \le \bar{c}\, \frac{\partial a}{\partial \vec{x}}(x) \qquad \forall x \in I\!R^N : |x| > \bar{c}$$

$\nabla_{\tau_x} a(x)$ denoting the component of the gradient of a at x, in the hyperplane orthogonal to $\vec{x}$ and containing x.

Theorem 3.1.1 *If the potential $a(x)$ satisfies the assumptions $(a_1) - (a_4)$, then problem* (P) *has infinitely many solutions.*

Let us remark that we need to assume neither the existence of $\lim_{|x|\to+\infty} a(x)$ nor $a(x) \ge \delta > 0$ in all $I\!R^N$. Assumptions $(a_2) - (a_3)$ imply only that $a(x)$ is bounded from below at infinity by a positive constant and that, for large values of x, $a(x)$ is increasing in a reasonable way in the direction $\vec{x}$. The assumption (a_4) implies a kind of stability of the value of $\frac{\partial a}{\partial \vec{x}}(x)$ with respect to small perturbations of the direction.

The ingredients of the proof are a quite natural approach to (P) by approximation combined with compactness techniques and estimates, in the spirit of what already done in [11] to prove the existence of infinitely many solutions for problems having critical growth in bounded domains. Indeed, let us consider a sequence of balls in $I\!R^N$, $B_{\rho_n}(0) = \{x \in I\!R^N : |x| < \rho_n\}$, $\rho_n \underset{n\to+\infty}{\longrightarrow} +\infty$, and the related problems approaching (P)

$$(\mathrm{P_n}) \qquad \begin{cases} -\Delta u + a(x)u = |u|^{p-2}u & \text{in } B_{\rho_n}(0) \\ u = 0 & \text{on } \partial B_{\rho_n}(0) \end{cases} .$$

Since it is possible to prove that $(\mathrm{P_n})$ possesses infinitely many solutions, obtained constructing infinitely many critical levels for the related functionals as minimax on suitable classes of functions, it is a natural idea considering sequences $\{u_n\}$, consisting of solutions u_n to $(\mathrm{P_n})$, corresponding to minimax classes of the same type, and then trying to pass to the limit.

Clearly such argument, by itself, is not sufficient, because, "a priori", such a sequences are not necessarily precompact. Hence some additional tool is needed to control the situation. This is just a local Pohozaev type inequality that, together with some uniform decay estimates and integral bounds on any bounded sequence of solutions to $(\mathrm{P_n})$, allows to conclude that, in our assumptions, the loss of compactness due to translations cannot occur.

The paper is organized as follows: in section 3.2 some notation is introduced, useful facts and preliminary results are stated, section 3.3 and 3.4 are devoted to the compactness question settlement, section 3.5 contains the proof of Theorem 1.1.

Remark 3.1.1 *It is worth pointing out that, if in* (P) *we replace* $I\!R^N$ *by* $I\!R^N \setminus \bar{\Omega}$ *where* Ω *is any bounded smooth open set in* $I\!R^N$*, Theorem 1.1 is still true, because the arguments we shall develop still hold after very simple modification.*

3.2 Notation, preliminary remarks and results, useful tools

Throughout the paper we make use of the following notations

- $L^p(\Omega)$, $1 \le p \le +\infty$, $\Omega \subseteq I\!R^N$ denotes a Lebesgue space, the norm in $L^p(\Omega)$ is denoted by $|\cdot|_{p,\Omega}$, when Ω is a proper subset of $I\!R^N$, by $|\cdot|_p$ when $\Omega = I\!R^N$.
- $H^1_0(\Omega)$, $\Omega \subset I\!R^N$, and $H^1(I\!R^N)$ denote the Sobolev spaces obtained as closure of $\mathcal{C}^\infty_0(\Omega)$, $\mathcal{C}^\infty_0(I\!R^N)$ respectively, with respect to the norms

$$||u||_\Omega = \left[\int_\Omega (|\nabla u|^2 + u^2)dx\right]^{\frac{1}{2}}$$

$$||u|| = \left[\int_{I\!R^N} (|\nabla u|^2 + u^2)dx\right]^{\frac{1}{2}} .$$

- $H^{-1}(\Omega)$, $\Omega \subset \mathbb{R}^N$, and $H^{-1}(\mathbb{R}^N)$ denote the dual spaces of $H_0^1(\Omega)$ and $H^1(\mathbb{R}^N)$ respectively.
- If $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N$, we denote also by u its extension to $\mathbb{R}^N$ made setting $u \equiv 0$ outside Ω.

We consider some inequalities, related to problem (P), that will be very useful in producing estimates on the solutions to problems approximating (P):

$$\text{(EI)} \qquad \begin{cases} -\Delta u + a(x)u & \leq & u^{p-1} & \text{in } \mathbb{R}^N \\ u & \geq & 0 & \text{in } \mathbb{R}^N \\ u & \in & H^1(\mathbb{R}^N) & \end{cases},$$

$$(\text{EI}_\infty) \qquad \begin{cases} -\Delta u + a_\infty u & \leq & u^{p-1} & \text{in } \mathbb{R}^N \\ u & \geq & 0 & \text{in } \mathbb{R}^N \\ u & \in & H^1(\mathbb{R}^N) & \end{cases}.$$

We remark that if u weakly solves

$$(\text{P}_\Omega) \qquad \begin{cases} -\Delta u + a(x)u & = & |u|^{p-2}u & \text{in } \Omega \\ u & \in & H_0^1(\Omega) & \end{cases},$$

$\Omega \subseteq \mathbb{R}^N$, then $|u|$, eventually extended by 0 out of Ω, weakly solves (EI).

Definition 3.2.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions. We say that $(u_n)_{n\in\mathbb{N}}$ is a:*

- balanced sequence *if, for each n, u_n is a nontrivial weak solution to* $(\text{P}_{\text{B}_{\rho_n}(0)})$ *where $(\rho_n)_{n\in\mathbb{N}}$, $\rho_n \in \mathbb{R}^+$, is any sequence so that $\rho_n \to +\infty$.*
- controlled sequence *if, for each n, u_n is a nontrivial weak solution to* (EI).

Remark 3.2.1 *It is worth pointing out that to any balanced sequence $(u_n)_{n\in\mathbb{N}}$ there corresponds a controlled sequence $(v_n)_{n\in\mathbb{N}}$, where $v_n = |u_n|$ in $B_{\rho_n}(0)$ and $v_n = 0$ in $\mathbb{R}^N \setminus B_{\rho_n}(0)$.*

We also observe that it is easy to see that given a balanced sequence $(u_n)_{n\in\mathbb{N}}$ and a sequence $(t_n)_{n\in\mathbb{N}}$, $t_n \in \mathbb{R}^N$, $|t_n| \underset{n\to+\infty}{\longrightarrow} +\infty$, if

$$u(\cdot) := \lim_{n\to+\infty} u_n(\cdot - t_n) \ a.e. \ in \ \mathbb{R}^N ,$$

then $|u|$ weakly solves (EI_∞).

We recall that a sequence $(u_n)_{n\in I\!N}$, $u_n \in H^1(I\!R^N)$, is called a Palais-Smale (briefly PS) sequence for the functional I, if there exists a level $c \in I\!R$ such that $I(u_n) \underset{n\to+\infty}{\longrightarrow} c$ and $dI(u_n) \underset{n\to+\infty}{\longrightarrow} 0$ strongly in $H^{-1}(I\!R^N)$. As mentioned in section 3.1, a basic tool to face problems in unbounded domains has been the analysis of the PS - sequences behaviour and the information that, when (3.1.2) is satisfied, a noncompact PS-sequence differs from its weak limit by one or more sequences that, after suitable translation, go to a solution of (P_∞) (see [5] and [2]).

Here, since our aim is finding solutions to (P) as limit of balanced sequences, we need to know how a noncompact bounded balanced sequence can look like. Moreover, since, instead of (3.1.2), we have the more general assumption (a_2), we cannot say there is a limit equation corresponding to (P). Nevertheless, taking into account Remark 3.2.1, it is not difficult to understand that in our case the role of the limit problem can be played by $(\mathrm{EI})_\infty$.

The following lemma gives the necessary information that the set of solutions to (EI_∞) is bounded from below.

Lemma 3.2.1 *There exists a positive constant $C_0 > 0$ such that for any nontrivial solution φ to $(\mathrm{EI})_\infty$:*

(3.2.1) $$|\varphi|_p \geq C_0$$

holds.

Proof. Let φ be a nontrivial solution to $(\mathrm{EI})_\infty$, then φ satisfies

$$|\nabla\varphi|_2^2 + a_\infty|\varphi|_2^2 \leq |\varphi|_p^p .$$

By using Sobolev embedding theorem and by interpolating L^p norm (taking into account that $2 < p < 2^*$) we have

$$S|\varphi|_{2^*}^2 + a_\infty|\varphi|_2^2 \leq |\varphi|_p^p \leq (|\varphi|_{2^*}^\alpha |\varphi|_2^{1-\alpha})^p$$

where S denotes the best Sobolev constant and $\alpha \in (0,1)$ is such that $\frac{\alpha}{2^*} + \frac{1-\alpha}{2} = \frac{1}{p}$.

By applying Young Inequality we get

$$\begin{aligned}|\varphi|_{2^*}^\alpha|\varphi|_2^{1-\alpha} &\leq \alpha|\varphi|_{2^*} + (1-\alpha)|\varphi|_2 \leq k_1\left[\left(\sqrt{S}|\varphi|_{2^*} + \sqrt{a_\infty}|\varphi|_2\right)^2\right]^{\frac{1}{2}} \\ &\leq k_1 2^{\frac{1}{2}}\left(S|\varphi|_{2^*}^2 + a_\infty|\varphi|_2^2\right)^{\frac{1}{2}}\end{aligned}$$

where k_1 is chosen so that $k_1 \geq \max\left(\frac{\alpha}{\sqrt{S}}, \frac{1-\alpha}{\sqrt{a_\infty}}\right)$.

Hence

$$(S|\varphi|_{2^*}^2 + a_\infty|\varphi|_2^2)^{\frac{p}{2}-1} \geq \frac{1}{k_1 2^{\frac{p}{2}}}$$

so we deduce, as desired, $|\varphi|_p \geq C_0 > 0$ where C_0 is a costant not depending on φ. ∎

Taking advantage of Lemma 3.2.1 and by using either arguments analogous to those of [5] or the results contained in [21] and in [15], it is possible to prove the following proposition that provides the desired picture of the balanced sequences behaviour.

Proposition 3.2.1 *Let $a(x)$ satisfy (a_1) - (a_2). Let $(u_n)_{n\in\mathbb{N}}$ be a noncompact balanced sequence bounded in $H^1(\mathbb{R}^N)$. Then, there exists a subsequence (still denoted by u_n) for which the following holds: there exist an integer $k > 0$, nontrivial solutions to* (EI_∞) φ_i, $1 \le i \le k$, *sequences* $(t_n^i)_{n\in\mathbb{N}}$, $1 \le i \le k$, $t_n^i \in \operatorname{supp}(u_n)$, *such that*

$$(3.2.2)\qquad \begin{gathered} |u_n| - \textstyle\sum_{i=1}^k \varphi_i(\cdot - t_n^i) \to |u_0| \text{ in } H^1(\mathbb{R}^N) \\ |t_n^i| \underset{n\to+\infty}{\longrightarrow} +\infty \qquad |t_n^i - t_n^j| \underset{n\to+\infty}{\longrightarrow} +\infty \qquad 1 \le i \ne j \le k \end{gathered}$$

u_0 being the weak limit of $(u_n)_{n\in\mathbb{N}}$.

For the reader's convenience, a self consistent proof of Proposition 3.2.1 is given in the Appendix.

Definition 3.2.2 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions, $u_n \in H^1(\mathbb{R}^N)$,* **bounded** *in $H^1(\mathbb{R}^N)$. We say that $(u_n)_{n\in\mathbb{N}}$ is a:*

- **broken controlled sequence** *if there exist an integer $k > 0$, nontrivial solutions to* (EI_∞) φ_i, $1 \le i \le k$, *sequences* $(t_n^i)_{n\in\mathbb{N}}$, $1 \le i \le k$, $t_n^i \in \operatorname{supp}(u_n)$, *such that, up to a subsequence,*

$$(3.2.3)\qquad \begin{gathered} u_n - \textstyle\sum_{i=1}^k \varphi_i(\cdot - t_n^i) \to u_0 \text{ in } H^1(\mathbb{R}^N) \\ |t_n^i| \underset{n\to+\infty}{\longrightarrow} +\infty \qquad |t_n^i - t_n^j| \underset{n\to+\infty}{\longrightarrow} +\infty \qquad 1 \le i \ne j \le k \end{gathered}$$

  u_0 being the weak limit of $(u_n)_{n\in\mathbb{N}}$.

- **broken balanced sequence** *if there exist an integer $k > 0$, nontrivial solutions to* (EI_∞) φ_i, $1 \le i \le k$, *sequences* $(t_n^i)_{n\in\mathbb{N}}$, $1 \le i \le k$, *such that, up to a subsequence,*

$$(3.2.4)\qquad \begin{gathered} |u_n| - \textstyle\sum_{i=1}^k \varphi_i(\cdot - t_n^i) \to |u_0| \text{ in } H^1(\mathbb{R}^N) \\ |t_n^i| \underset{n\to+\infty}{\longrightarrow} +\infty\ ,\ |t_n^i - t_n^j| \underset{n\to+\infty}{\longrightarrow} +\infty \qquad 1 \le i \ne j \le k \end{gathered}$$

  u_0 being the weak limit of $(u_n)_{n\in\mathbb{N}}$.

In what follows, given any broken sequence, controlled or balanced, $(u_n)_{n\in\mathbb{N}}$, we assume as given, also, the functions φ_i and the translations vectors t_n^i (even if they are not uniquely determined) that appear, respectively, in the relations (3.2.3) and (3.2.4). Moreover, by replacing the sequences $(t_n^i)_{n\in\mathbb{N}}$ with suitable subsequences, we can suppose them ordered term by term, so it makes sense denoting by $(t_n)_{n\in\mathbb{N}}$ the *smallest sequence*, that is the one for which

$$(3.2.5)\qquad |t_n| \le t_n^i\ ,\ \forall i\ :\ 1 \le i \le k\ ,\ \forall n \in \mathbb{N}\ .$$

In order to associate to any broken sequence some suitable regions of the space, we recall the following

Definition 3.2.3 *Let $A \subset I\!R^N$ be a subset of $I\!R^N$ and $v \in I\!R^N$ a point $v \notin A$. We call* cone of vertex v generated by A *the smallest set containing A and positively homogenous with respect to the vertex v, i.e. the set*

$$\{w \in I\!R^N \;:\; w = v + \lambda(x - v),\; x \in A,\; \lambda \in I\!R^+\}\,.$$

Let us now consider a broken sequence (either controlled or balanced) $(u_n)_{n\in I\!N}$, let $(t_n)_{n\in I\!N}$ be the respective smallest sequence (for which (3.2.5) holds) of those appearing in (3.2.3) or (3.2.4) respectively, and let us define some sequences of subsets of $I\!R^N$ that are related to $(t_n)_{n\in I\!N}$.

First of all, we construct a sequence of cones $\mathcal{C}_n$, having, for all n, vertex $\frac{t_n}{2}$ and generated by a ball $B_{R_n}(t_n)$. We begin taking the cone $\mathcal{C}_{1,n}$ having as vertex $\frac{t_n}{2}$ and generated by the ball $B_{1,n} := B_{r_n}(t_n)$, where

$$r_n = \frac{\hat\gamma}{k}\frac{|t_n|}{2} \qquad \text{with } 0 < \hat\gamma < \min\left(\frac{1}{5}, \frac{1}{4(\bar c + 1)}\right)\,, \tag{3.2.6}$$

$\bar c$ being the constant appearing in (a_4).

If $\partial\mathcal{C}_{1,n} \cap B_{\frac{r_n}{2}}(t_n^i) = \emptyset$ for all $t_n^i \neq t_n$, $1 \le i \le k$, we set $\mathcal{C}_n = \mathcal{C}_{1,n}$ and $R_n = r_n$. Otherwise we consider the larger cone $\mathcal{C}_{2,n}$ having vertex $\frac{t_n}{2}$ and generated by $B_{2r_n}(t_n)$. Then, taking into account that $|t_n| \le |t_n^i|$, $1 \le i \le k$, for any index i for which $\partial\mathcal{C}_{1,n} \cap B_{\frac{r_n}{2}}(t_n^i) \neq \emptyset$, we have $B_{\frac{r_n}{2}}(t_n^i) \subset \mathcal{C}_{2,n}$, and we set $\mathcal{C}_n = \mathcal{C}_{2,n}$ if $\partial\mathcal{C}_{2,n}$ does not touch any of the other balls $B_{\frac{r_n}{2}}(t_n^i)$, $t_n^i \neq t_n$. Otherwise we pass to the cone $\mathcal{C}_{3,n}$, having vertex $\frac{t_n}{2}$, generated by $B_{3r_n}(t_n)$ that surely contains the balls, of radius $\frac{r_n}{2}$ centered at the points t_n^i, touching $\partial\mathcal{C}_{2,n}$.

We iterate this arguments and, after at most k steps, we associate to t_n a cone $\mathcal{C}_n$, having vertex $\frac{t_n}{2}$, generated by a ball $B_{R_n}(t_n)$, with $\frac{\hat\gamma}{k}\frac{|t_n|}{2} = r_n \le R_n \le kr_n = \hat\gamma\frac{|t_n|}{2}$, and having the property that $\partial\mathcal{C}_n \cap B_{\frac{r_n}{2}}(t_n^i) = \emptyset$, for any index i, $1 \le i \le k$, such that $t_n^i \neq t_n$.

Remark 3.2.2 *Denoting by θ_n the "width angle" of the cone $\mathcal{C}_n$, we emphasize that, since $R_n = \frac{|t_n|}{2}\tan\theta_n$,*

$$0 < \frac{\hat\gamma}{k} \le \tan\theta_n \le \hat\gamma < \min\left(\frac{1}{5},\ \frac{1}{4(\bar c + 1)}\right)\,.$$

For any $s \in I\!R$ and for all $n \in I\!N$, we consider the cones

$$\mathcal{C}_{s,n} = \mathcal{C}_n - s\frac{t_n}{|t_n|} \tag{3.2.7}$$

and, for all $n \in I\!N$, the regions around the boundary of $\mathcal{C}_n$

$$\mathcal{S}_{2s,n} = \mathcal{C}_{s,n} \setminus \mathcal{C}_{-s,n}\,. \tag{3.2.8}$$

Lastly we set

$$\mathcal{S}_n = I\!R^N \setminus \bigcup_{i=0}^{k} B_{\frac{r_n}{2}}(t_n^i) \tag{3.2.9}$$

where, for all $n \in I\!N$, $t_n^0 = 0$, r_n and t_n^i, $1 \le i \le k$, are as above.

3.3 Some estimates for controlled sequences

The purpose of this section is to establish some decay estimates and integral bounds, concerning bounded controlled sequences, that are contained in the following propositions:

Proposition 3.3.1 *Let $a(x)$ satisfy (a_1) - (a_2). Let $(u_n)_{n\in I\!N}$ be a broken controlled sequence. Then for any constant $\alpha \in (0, \sqrt{a_\infty})$ there exists a constant $c_\alpha > 0$ such that for n large enough*

(3.3.1) $$u_n(x) \le c_\alpha e^{-\alpha\sigma_n(x)} \qquad \forall x \in \mathcal{S}_n$$

where σ_n is defined by

(3.3.2) $$\sigma_n(x) := \inf_{0\le i\le k} |x - t_n^i| \qquad x \in I\!R^N$$

and $\mathcal{S}_n$ and t_n^i, are as in (3.2.9).

Proposition 3.3.2 *Let $a(x)$, $(u_n)_{n\in I\!N}$, $\mathcal{S}_n$, be as in Proposition 3.3.1. Then, for all $p \ge 2$, there exist constants $\tilde{\alpha} > 0$ and $\tilde{c} > 0$ such that for n large enough*

(3.3.3) $$\int_{\mathcal{S}_n} (u_n)^p dx \le \tilde{c}\, e^{-\tilde{\alpha}|t_n|}$$

$(t_n)_{n\in I\!N}$ being the smallest sequence appearing in (3.2.3).

Proposition 3.3.3 *Let $a(x)$ and $(u_n)_{n\in I\!N}$ be as in Proposition 3.3.1. Then there exist constants $\alpha^* > 0$, $c^* > 0$ and a sequence $(s_n)_{n\in I\!N}$, $s_n \in (-\frac{1}{2}, \frac{1}{2})$ such that for all $n \in I\!N$*

(3.3.4) $$\int_{\partial\mathcal{C}_{s_n,n}} |\nabla u_n|^2 dx \le c^*\, e^{-\alpha^*|t_n|}$$

where, for all n, $\mathcal{C}_{s_n,n}$ is as defined in (3.2.7).

The proof of Proposition 3.3.1 is carried out through some estimates, on bounded controlled sequences, proved in a slightly general setting. In order to do this we introduce the following definitions.

Definition 3.3.1 *Given a sequence of functions $(u_n)_{n\in I\!N}$, $u_n \in H^1(I\!R^N)$, and a sequence $(x_n)_{n\in I\!N}$, $x_n \in I\!R^N$, we say that $(x_n)_{n\in I\!N}$ is a sequence of* drift points *for $(u_n)_{n\in I\!N}$ if*

(3.3.5) $$u_n(\cdot - x_n) \rightharpoonup 0 \text{ weakly in } H^1(I\!R^N)\,.$$

Definition 3.3.2 *Given a sequence of functions $(u_n)_{n\in I\!N}$, $u_n \in H^1(I\!R^N)$, and a drift points sequence $(x_n)_{n\in I\!N}$ for $(u_n)_{n\in I\!N}$, we say that a diverging sequence of real numbers $(\delta_n)_{n\in I\!N}$ is a sequence of* drift distances *for $(x_n)_{n\in I\!N}$ with respect to $(u_n)_{n\in I\!N}$ if any sequence $(y_n)_{n\in I\!N}$, $y_n \in I\!R^N$, such that $d(x_n, y_n) \le h\delta_n$, for some constant $h \in [0, 1]$ independent of n, is a drift points sequence for $(u_n)_{n\in I\!N}$.*

Remark 3.3.1 *We observe that, if $(u_n)_{n\in\mathbb{N}}$ is a broken sequence, any sequence of points $(x_n)_{n\in\mathbb{N}}$, such that $x_n \in \mathcal{S}_n$ for all n, is a drift points sequence for $(u_n)_{n\in\mathbb{N}}$ and $\delta_n = \sigma_n(x_n)$ (where σ_n is defined in (3.3.2)) is a drift distances sequence for $(x_n)_{n\in\mathbb{N}}$.*

Moreover, we remark that, given a drift points sequence $(x_n)_{n\in\mathbb{N}}$ for a sequence of functions $(u_n)_{n\in\mathbb{N}}$, a sequence $(\delta_n)_{n\in\mathbb{N}}$, as in Definition 3.3.2, is not uniquely determined. For istance, given a broken sequence $(u_n)_{n\in\mathbb{N}}$, any sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \partial\mathcal{C}_n$ is a drift points sequence and considering either $\delta_n = \frac{r_n}{2}$ (r_n as defined in (3.2.6)) either $\hat{\delta}_n = |t_n|$, $n \in \mathbb{N}$, we obtain drift distances sequences.

The first step is proving a lemma that allows to obtain an uniform upper bound on the values of the Laplacian on a controlled sequence.

Lemma 3.3.1 *Let $a(x)$ satisfy (a_1) - (a_2). Let $(u_n)_{n\in\mathbb{N}}$ be a controlled sequence bounded in $H^1(\mathbb{R}^N)$. Then $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^N)$.*

PROOF. By (a_1) - (a_2), there exist a constant $\tilde{a} \in (0, a_\infty)$ and a positive function $c(x) \in \mathcal{C}_0(\mathbb{R}^N)$ such that $a(x) \geq \tilde{a} - c(x)$, $\forall x \in \mathbb{R}^N$. Therefore u_n weakly solves

$$-\Delta u_n + \tilde{a} u_n \leq u_n^{p-1} + c(x) u_n \quad \text{in } \mathbb{R}^N$$

moreover, by the maximum principle, for any weak positive solution $v_n \in H^1(\mathbb{R}^N)$ to

$$-\Delta v + \tilde{a} v = u_n^{p-1} + c(x) u_n \quad \text{in } \mathbb{R}^N \tag{3.3.6}$$

the relation

$$u_n(x) \leq v_n(x) \quad \text{in } \mathbb{R}^N \tag{3.3.7}$$

holds.

Now, let us consider a sequence $(v_n)_{n\in\mathbb{N}}$, $v_n \in H^1(\mathbb{R}^N)$, such that, for all $n \in \mathbb{N}$, v_n solves (3.3.6). By (3.3.7), the claim follows proving that $(|v_n|_\infty)_{n\in\mathbb{N}}$ is bounded.

Since $u_n \in H^1(\mathbb{R}^N)$ and $c(x) \in \mathcal{C}_0(\mathbb{R}^N)$ we can assume $u_n^{p-1} + c(x)u_n \in L^{\frac{2^*}{p-1}}$, so by regularity results $v_n \in W^{2,\frac{2^*}{p-1}}(\mathbb{R}^N)$. Now, the space $W^{2,\frac{2^*}{p-1}}(\mathbb{R}^N)$ embeds continuously in $L^{\hat{q}}(\mathbb{R}^N)$, where $\hat{q} = \frac{N\frac{2^*}{p-1}}{N-2\frac{2^*}{p-1}}$ and, since $\frac{2^*}{p-1} > \frac{2^*}{2^*-1} = \frac{2N}{N+2} = (2^*)'$, $\hat{q} > \frac{N(2^*)'}{N-2(2^*)'} = 2^*$. Then, by (3.3.7), $u_n \in L^{\hat{q}}(\mathbb{R}^N)$, with $\frac{\hat{q}}{2^*} > 1$, and

$$|u_n|_{\hat{q}} \leq |v_n|_{\hat{q}} \leq k_1 |u_n^{p-1} + c(x)u_n|_{\frac{2^*}{p-1}} < k_2 \,.$$

By iterating the same argument, we gradually increase the regularity properties of u_n and v_n, obtaining also uniform bounds to the norms in the respective spaces. After a finite number of steps we obtain $v_n \in W^{2,\tilde{q}}(\mathbb{R}^N)$ with $\tilde{q} > \frac{N}{2}$ and $\|v_n\|_{W^{2,\tilde{q}}} < k_3$, k_3 not depending on n.

Then Sobolev embedding theorem gives $v_n \in \mathcal{C}^{0,\mu}(\mathbb{R}^N)$ for some $\mu \in (0,\,1)$, and $\|v_n\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^N)} < k_4$.

This last relation with the L^2 sommability allows to obtain an L^∞ uniform bound on $(v_n)_{n\in\mathbb{N}}$ and, in turn, on $(u_n)_{n\in\mathbb{N}}$ as desired. ∎

Corollary 3.3.1 *Let $(u_n)_{n\in\mathbb{N}}$ and $a(x)$ be as in Lemma 3.3.1. Then there exists a constant $c_1 > 0$ such that for all $n \in \mathbb{N}$, the relation*

$$-\Delta u_n \leq c_1 \tag{3.3.8}$$

weakly holds.

The following lemma guarantees that the values that a controlled bounded sequence takes around the drift points x_n of a sequence are small.

Lemma 3.3.2 *Let $a(x)$ and $(u_n)_{n\in\mathbb{N}}$ be as in Lemma 3.3.1. Let $(x_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ be respectively a drift points sequence for $(u_n)_{n\in\mathbb{N}}$ and a drift distances sequence for $(x_n)_{n\in\mathbb{N}}$. Then for all $h \in (0,\, 1)$*

$$\lim_{n\to+\infty} \sup_{B_{h\delta_n}(x_n)} u_n(x) = 0\,. \tag{3.3.9}$$

PROOF.
We argue by contradiction and we assume that there exist real numbers $h \in (0,\, 1)$, $\eta > 0$ and a sequence $(y_n)_{n\in\mathbb{N}}$, $y_n \in B_{h\delta_n}(x_n)$ such that, for large n,

$$u_n(y_n) > \left(\sup_{B_{h\delta_n}(x_n)} u_n(x)\right) - \frac{1}{n} > \eta\,.$$

The above relation, combined with (3.3.8), allows to conclude that for large n and ρ small enough

$$⨍_{B_\rho(y_n)} u_n dx := \frac{1}{|B_\rho(y_n)|}\int_{B_\rho(y_n)} u_n dx > \frac{\eta}{2}\,,$$

where $|B_\rho(y_n)|$ denotes the Lebesgue N-dimensional measure of $B_\rho(y_n)$.

Hence $u_n(\cdot - y_n) \rightharpoonup v \neq 0$ as $n \to +\infty$. This is impossible because, by the choice of δ_n, h and y_n and by Definition 3.3.1 $u_n(\cdot - y_n) \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. ∎

Next lemma contains the key estimate for proving Proposition 3.3.1.

Lemma 3.3.3 *Let $a(x)$, $(u_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, $(\delta_n)_{n\in\mathbb{N}}$ be as in Lemma 3.3.2 and, moreover, let $(x_n)_{n\in\mathbb{N}}$ be supposed diverging. Then, for all $\alpha \in (0,\, \sqrt{a_\infty})$ there exists a constant $\hat{c}_\alpha > 0$ such that for all n*

$$u_n(x_n) \leq \hat{c}_\alpha e^{-\alpha\delta_n}\,. \tag{3.3.10}$$

PROOF. Let $\alpha \in \mathbb{R}$, $0 < \alpha < \sqrt{a_\infty}$ be fixed, and let us choose $h \in \left(\frac{\alpha}{\sqrt{a_\infty}},\, 1\right)$ and $\overline{\alpha} \in (\alpha,\, \sqrt{a_\infty}h)$.

Then, by using Lemma 3.3.2, we obtain that, for any n large enough, u_n weakly satisfies

$$\Delta u_n \geq a(x)u_n - u_n^{p-1} > \overline{\alpha}^2 h^{-2} u_n \geq 0 \qquad \text{in } B_{h\delta_n}(x_n)\,. \tag{3.3.11}$$

Thus, since $h\delta_n > 1$ for large n, we have

$$u_n(x_n) \le \int_{\partial B_r(x_n)} u_n d\sigma \qquad \forall r : \quad 0 < r \le 1 \tag{3.3.12}$$

and we deduce

$$u_n(x_n) \le \int_0^1 \left[\int_{\partial B_r(x_n)} u_n d\sigma \right] dr = \int_{B_1(x_n)} u_n dx \; .$$

So, in order to obtain (3.3.10) for large n, it is enough to show that a constant $\overline{c}_\alpha > 0$ exists such that

$$\int_{B_1(x_n)} u_n dx \le \overline{c}_\alpha e^{-\alpha\delta_n} \; . \tag{3.3.13}$$

To do this, let us consider the functions

$$v_n(\rho) := \int_{B_\rho(x_n)} u_n dx \qquad , \qquad w_n(\rho) := \frac{(h\delta_n)^N \omega_N}{e^{\overline{\alpha}\delta_n}} e^{\overline{\alpha}\frac{\rho}{h}}$$

where ω_N is the Lebesgue measure of the unitary ball in $\mathbb{R}^N$, and let us remark that $v_n(1)$ is just the left hand side of (3.3.13), while

$$w_n(1) = \frac{(h\delta_n)^N \omega_N}{e^{\overline{\alpha}\delta_n}} e^{\frac{\overline{\alpha}}{h}} \le \; h\omega_N e^{\sqrt{a_\infty}} \frac{\delta_n^N}{e^{\overline{\alpha}\delta_n}} \le \; \overline{c}_\alpha e^{-\alpha\delta_n}$$

for n large enough. So (3.3.10) follows, by proving $v_n(1) \le w_n(1)$ and taking into account that for any finite set $u_n(x_n)$, $n < \overline{n}$, (3.3.10) is obviously true for a suitable choice of the constant $\hat{c}_\alpha$.

Let us then show that for n large

$$v_n(\rho) \le w_n(\rho) \qquad \forall \rho \in [0, \, h\delta_n] \; . \tag{3.3.14}$$

First, let us observe that

$$v_n(0) \le w_n(0) \qquad \forall n \in \mathbb{N}$$

and that, for n large, by Lemma 3.3.2,

$$v_n(h\delta_n) \le |B_{h\delta_n}(x_n)| \sup_{B_{h\delta_n}(x_n)} u_n(x) \le \omega_N (h\delta_n)^N = w_n(h\delta_n) \; .$$

Now, if for some point in $[0, h\delta_n]$ (3.3.14) were false, then the function $(v_n - w_n)(\rho)$ should have a maximum point, $\overline{\rho}_n \in (0, \, h\delta_n)$, for which $(v_n - w_n)(\overline{\rho}_n) > 0$ and, of course, $v''(\overline{\rho}_n) - w''(\overline{\rho}_n) \le 0$. Let us show that this is impossible. Indeed, since

$$v_n(\rho) = \int_{B_\rho(x_n)} u_n(x) dx = \int_0^\rho \left[\int_{\partial B_r(x_n)} u_n d\sigma \right] dr \; ,$$

we have

$$v_n'(\rho) = \frac{d}{d\rho} v_n(\rho) = \int_{\partial B_\rho(x_n)} u_n d\sigma$$

moreover

$$\fint_{\partial B_\rho(x_n)} u_n d\sigma = \frac{1}{N\omega_N \rho^{N-1}} \int_{\partial B_\rho(x_n)} u_n d\sigma = \frac{v_n'(\rho)}{N\omega_N \rho^{N-1}}$$

and, by using divergence theorem,

$$\begin{aligned} \fint_{\partial B_\rho(x_n)} u_n d\sigma &= \int_0^\rho \frac{1}{N\omega_N r^{N-1}} \left[\int_{\partial B_r(x_n)} \frac{\partial}{\partial \nu} u_n d\sigma \right] dr \\ &= \int_0^\rho \frac{1}{N\omega_N r^{N-1}} \left[\int_{B_r(x_n)} \Delta u_n dx \right] dr \, . \end{aligned}$$

So

$$\frac{d}{d\rho} \frac{v_n'(\rho)}{N\omega_N \rho^{N-1}} = \frac{1}{N\omega_N \rho^{N-1}} \int_{B_\rho(x_n)} \Delta u_n dx \, ,$$

from which, using (3.3.11), we obtain

$$\frac{v_n''(\rho)}{\rho^{N-1}} + (1-N)\frac{v_n'(\rho)}{\rho^N} = \frac{d}{d\rho}\left(\frac{v_n'(\rho)}{\rho^{N-1}}\right) = \frac{1}{\rho^{N-1}} \int_{B_\rho(x_n)} \Delta u_n dx \geq \frac{\overline{\alpha}^2 h^{-2}}{\rho^{N-1}} v_n(\rho) \, .$$

Hence, taking into account that $v_n'(\rho) > 0$ and $N > 1$

$$v_n''(\rho) \geq \overline{\alpha}^2 h^{-2} v_n(\rho) \qquad \forall \rho \in (0,\, h\delta_n) \tag{3.3.15}$$

follows.

Let now $\overline{\rho}_n \in (0,\, h\delta_n)$ be a maximum point for $(v_n - w_n)(\rho)$ for which $(v_n - w_n)(\overline{\rho}_n) > 0$ then by (3.3.15) we get

$$v_n''(\overline{\rho}_n) - w_n''(\overline{\rho}_n) \geq \overline{\alpha}^2 h^{-2}(v_n(\overline{\rho}_n) - w_n(\overline{\rho}_n)) > 0 \, ,$$

and we are in contradiction. ∎

Proof of Proposition 3.3.1. Arguing by contradiction, we assume that there is $\alpha \in (0,\, \sqrt{a_\infty})$ such that for all $q \in I\!N$ there exist $n_q \in I\!N$ and $x_q \in \mathcal{S}_{n_q}$ for which

$$u_{n_q}(x_q) > q e^{-\alpha \sigma_{n_q}(x_q)} \, .$$

This is impossible because it contradicts Lemma 3.3.3 whose assumptions are fulfilled by α, $(u_{n_q})_{q \in I\!N}$, x_q and $\delta_q = \sigma_{n_q}(x_q)$. ∎

Proof of Proposition 3.3.2. By using Proposition 3.3.1, we deduce that, for n large enough and $\alpha \in (0,\, \sqrt{a_\infty})$,

$$\begin{aligned} \int_{\mathcal{S}_n} (u_n)^p dx &\leq c_\alpha \int_{\mathcal{S}_n} e^{-\alpha p \sigma_n(x)} dx \leq c_\alpha \sum_{i=0}^{k} \int_{\mathcal{S}_n} e^{-\alpha p |x - t_n^i|} dx \\ &\leq c_\alpha k \int_{\frac{\hat{\gamma}}{4k}|t_n|}^{+\infty} e^{-\alpha p t} t^{N-1} dt \leq \tilde{c} e^{-\tilde{\alpha}|t_n|} \, . \end{aligned}$$

∎

The proof of Proposition 3.3.3 is based on the following lemma that is proved by using a Caccioppoli type argument.

Lemma 3.3.4 *Let $a(x)$ satisfy (a_1) - (a_2). Let $(u_n)_{n\in I\!N}$ be a broken controlled sequence . Then there exist constants $\alpha_* > 0$ and $c_* > 0$ such that for all $n \in I\!N$*

$$\int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 dx \leq c_* e^{-\alpha_* |t_n|} \tag{3.3.16}$$

where $\mathcal{S}_{1,n}$ is as defined in (3.2.8).

PROOF. For any fixed $n \in I\!N$, let $\varphi_n \in \mathcal{C}^\infty(I\!R^N, [0, 1])$ be a function fulfilling the following conditions:

$$\begin{cases} i) & \varphi_n = 1 \text{ on } \mathcal{S}_{1,n} \\ ii) & \text{supp}(\varphi_n) \subset \mathcal{S}_{2,n} \\ iii) & \Delta\varphi_n \leq C \qquad C \in I\!R\,. \end{cases} \tag{3.3.17}$$

Since u_n weakly solves (EI) and $\varphi_n = 0$ in $I\!R^N \setminus \mathcal{S}_{2,n}$ we have

$$\begin{aligned} \int_{\mathcal{S}_{2,n}} (-\Delta u_n)(u_n\varphi_n) &= \int_{I\!R^N} (-\Delta u_n)(u_n\varphi_n) \leq \int_{I\!R^N} (u_n^p - a(x)u_n^2)\varphi_n \\ &= \int_{\mathcal{S}_{2,n}} (u_n^p - a(x)u_n^2)\varphi_n\,. \end{aligned} \tag{3.3.18}$$

On the other hand, taking into account that by (3.3.17)(ii), $\varphi_n = 0$ and $\nabla\varphi_n = 0$ on $\partial\mathcal{S}_{2,n}$ and using (3.3.17)(i) we get

$$\begin{aligned} \int_{\mathcal{S}_{2,n}} (-\Delta u_n)(u_n\varphi_n) &= \int_{\mathcal{S}_{2,n}} |\nabla u_n|^2\varphi_n + \int_{\mathcal{S}_{2,n}} (\nabla u_n \cdot \nabla\varphi_n)u_n \\ &\geq \int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 + \int_{\mathcal{S}_{2,n}} \left(\nabla\left(\frac{1}{2}u_n^2\right) \cdot \nabla\varphi_n\right) \\ &= \int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 - \frac{1}{2}\int_{\mathcal{S}_{2,n}} (\Delta\varphi_n)\, u_n^2\,. \end{aligned} \tag{3.3.19}$$

So, inserting (3.3.18) in (3.3.19), using (3.3.17)(iii) and taking into account that, if n is large enounngh, $\mathcal{S}_{2,n} \subset \mathcal{S}_n$ and $a(x) > 0$ on $\mathcal{S}_n$, we deduce for large n

$$\begin{aligned} \int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 &\leq \int_{\mathcal{S}_{2,n}} \left(u_n^p - a(x)u_n^2\right)\varphi_n + \frac{1}{2}\int_{\mathcal{S}_{2,n}} (\Delta\varphi_n)\, u_n^2 \\ &\leq \int_{\mathcal{S}_n} u_n^p\varphi_n + \frac{1}{2}\int_{\mathcal{S}_n} (\Delta\varphi_n)\, u_n^2 \\ &\leq \int_{\mathcal{S}_n} u_n^p + \frac{1}{2}C\int_{\mathcal{S}_n} u_n^2\,. \end{aligned}$$

Then, applying Proposition 3.3.2, taking also into account that for any finite set of indices (3.3.16) is true for a suitable choice of the constant c_*, we obtain the thesis. ∎

PROOF OF PROPOSITION 3.3.3. Denoting by θ_n the width angle of the cone $\mathcal{C}_n$ we have

$$\int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{\partial \mathcal{C}_{s,n}} |\nabla u_n|^2 d\sigma \right) \sin\theta_n ds .$$

So, using the integral mean value theorem and considering (3.3.16), we deduce that, for all n, $s_n \in (-\frac{1}{2}, \frac{1}{2})$ must exists so that

$$0 < \sin\theta_n \int_{\partial \mathcal{C}_{s_n,n}} |\nabla u_n|^2 d\sigma < c_* \, e^{-\alpha_* |t_n|}$$

from which (3.3.4) follows because, as a consequence of what observed in Remark 3.2.2, $\sin\theta_n \geq s > 0$, $s =$ const. independent on n. ∎

3.4 Local Pohozaev Identity and Compactness results for Balanced Sequences

The aim of this section is showing that our assumptions (a_3) and (a_4) prevent a bounded balanced sequence to be not compact.

In order to do this, we start considering not compact balanced sequences bounded in $H^1(\mathbb{R}^N)$ and stating basic relations they must satisfy.

In the following Lemmas 3.4.1, 3.4.2, 3.4.3 we deal with balanced sequences $(u_n)_{n\in\mathbb{N}}$, (consisting $\forall n$ of solutions u_n to $(\mathrm{P}_{\mathrm{B}_{\rho_n}(0)})$ with $\rho_n \to +\infty$) that are bounded in $H^1(\mathbb{R}^N)$ and not compact. To such sequences there correspond, in view of Remark 3.2.1 and Proposition 3.2.1, bounded broken controlled sequences $(|u_n|)_{n\in\mathbb{N}}$, to which Propositions 3.3.1, 3.3.2 and 3.3.3 apply. Then, it makes sense setting, for all $n \in \mathbb{N}$,

$$\mathcal{D}_n := \tilde{\mathcal{C}}_n \cap B_{\rho_n}(0) ,$$

where $\tilde{\mathcal{C}}_n$ denotes the cone $\mathcal{C}_{s_n,n}$, whose existence is stated in Proposition 3.3.3. We remark that, for large n, $\frac{t_n}{2} \in \mathcal{D}_n$; in fact, even if $\rho_n < |t_n|$, for all n, $|t_n| - \rho_n \leq C$ for some constant C, otherwise $u_n(\cdot - t_n) \underset{n\to+\infty}{\rightharpoonup} 0$ contradicting the choice of t_n. Moreover we remark that $\partial\mathcal{D}_n$ consists of an "internal part"

$$(\partial\mathcal{D}_n)_i := \partial\tilde{\mathcal{C}}_n \cap B_{\rho_n}(0)$$

and an "external" one

$$(\partial\mathcal{D}_n)_e := \tilde{\mathcal{C}}_n \cap \partial B_{\rho_n}(0) .$$

We recall, also, that $\forall y \in \mathbb{R}^N \setminus \{0\}$, $\vec{y}$ denotes the unitary vector $\frac{y}{|y|}$.

In next Lemma 3.4.1 we take a not compact bounded balanced sequence $(u_n)_{n\in\mathbb{N}}$ and we evaluate the infinitesimal variation of the functional I under translation, along the direction $\vec{t_n}$, of that part of the function u_n that is contained in the cone $\tilde{\mathcal{C}}_n$. Since $(u_n)_{n\in\mathbb{N}}$ is balanced, such a variation must be zero, then we are led to an identity of the well known Pohozaev type.

Lemma 3.4.1 *Let $a(x)$ satisfy (a_1) - (a_2). Let $(u_n)_{n\in \mathbb{N}}$ be a noncompact balanced sequence. Then the identity*

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{D}_n} u_n^2(\nabla a(x)\cdot \vec{t_n})dx &= \frac{1}{2}\int_{\partial\mathcal{D}_n}(|\nabla u_n|^2 + a(x)u_n^2)(\nu_n\cdot \vec{t_n}) \\ &- \int_{\partial\mathcal{D}_n}(\nabla u_n\cdot \nu_n)(\nabla u_n\cdot \vec{t_n}) - \frac{1}{p}\int_{\partial\mathcal{D}_n}|u_n|^p(\nu_n\cdot\vec{t_n})\ ,\end{aligned} \tag{3.4.1}$$

where ν_n is the outward normal to $\partial\mathcal{D}_n$, holds.

Proof. Since $\{u_n\}$ is a balanced sequence we have

$$\int_{\mathcal{D}_n}(-\Delta u_n + a(x)u_n - |u_n|^{p-2}u_n)\ (\nabla u_n\cdot\vec{t_n})dx = 0\ . \tag{3.4.2}$$

Now integrating by parts, we obtain

$$\int_{\mathcal{D}_n} -\Delta u_n\ (\nabla u_n\cdot\vec{t_n})dx = \int_{\mathcal{D}_n}(\nabla u_n\cdot\nabla(\nabla u_n\cdot\vec{t_n}))dx - \int_{\partial\mathcal{D}_n}(\nabla u_n\cdot\nu_n)\ (\nabla u_n\cdot\vec{t_n})d\sigma\ .$$

Then, taking into account that $\vec{t_n}$ does not depend on x, again using divergence theorem, we get

$$\begin{aligned}\int_{\mathcal{D}_n}(\nabla u_n\cdot\nabla(\nabla u_n\cdot\vec{t_n}))dx &= \int_{\mathcal{D}_n}(\nabla u_n\cdot(\nabla^2 u_n\cdot\vec{t_n}))dx \\ &= \frac{1}{2}\int_{\mathcal{D}_n}(\nabla|\nabla u_n|^2\cdot\vec{t_n})dx = \frac{1}{2}\int_{\partial\mathcal{D}_n}|\nabla u_n|^2(\vec{t_n}\cdot\nu_n)d\sigma\end{aligned}$$

and, then,

$$\begin{aligned}\int_{\mathcal{D}_n} -\Delta u_n\ (\nabla u_n\cdot\vec{t_n})dx &= \frac{1}{2}\int_{\partial\mathcal{D}_n}(|\nabla u_n|^2(\vec{t_n}\cdot\nu_n)d\sigma \\ &- \int_{\partial\mathcal{D}_n}(\nabla u_n\cdot\nu_n)\ (\nabla u_n\cdot\vec{t_n})d\sigma\ .\end{aligned} \tag{3.4.3}$$

Analogously we deduce

$$\begin{aligned}\int_{\mathcal{D}_n} a(x)u_n\ (\nabla u_n\cdot\vec{t_n})dx &= \frac{1}{2}\int_{\mathcal{D}_n} a(x)(\nabla|u_n|^2\cdot\vec{t_n})dx \\ &= -\frac{1}{2}\int_{\mathcal{D}_n} u_n^2(\nabla a(x)\cdot\vec{t_n})dx \\ &+ \frac{1}{2}\int_{\partial\mathcal{D}_n} a(x)u_n^2(\nu_n\cdot\vec{t_n})d\sigma\end{aligned} \tag{3.4.4}$$

and

$$\int_{\mathcal{D}_n}|u_n|^{p-2}u_n(\nabla u_n\cdot\vec{t_n})dx = \frac{1}{p}\int_{\mathcal{D}_n}(\nabla|u_n|^p\cdot\vec{t_n})dx = \frac{1}{p}\int_{\partial\mathcal{D}_n}|u_n|^p(\nu_n\cdot\vec{t_n})d\sigma \tag{3.4.5}$$

Combining (3.4.2), (3.4.3), (3.4.4) and (3.4.5) we obtain (3.4.1). ∎

Lemma 3.4.2 *Let $a(x)$ satisfy (a_1), (a_2), (a_4). Let $(u_n)_{n\in\mathbb{N}}$ be a noncompact balanced sequence. Then, for large n, the inequality*

$$\int_{\mathcal{D}_n} (\nabla a(x)\cdot \vec{t}_n) u_n^2 dx \geq \frac{1}{2}\int_{\mathcal{D}_n} \frac{\partial a}{\partial \vec{x}}(x) u_n^2 dx \tag{3.4.6}$$

holds.

PROOF.

Denoting by $(\vec{t}_n)_{\tau_x}$ the component of $\vec{t}_n$ lying in the space orthogonal to $\vec{x}$ and containing x, using (a_4), we get, for large n

$$\begin{aligned}(\nabla a(x)\cdot \vec{t}_n) &= (\nabla a(x)\cdot \vec{x})(\vec{t}_n\cdot\vec{x}) + (\nabla_{\tau_x} a(x)\cdot (\vec{t}_n)_{\tau_x})\\ &\geq \frac{\partial a}{\partial \vec{x}}(x)(\vec{t}_n\cdot\vec{x}) - \bar{c}\frac{\partial a}{\partial\vec{x}}(x)|(\vec{t}_n)_{\tau_x}|\\ &= \frac{\partial a}{\partial \vec{x}}(x)\left[(\vec{t}_n\cdot\vec{x}) - \bar{c}|(\vec{t}_n)_{\tau_x}|\right].\end{aligned}$$

In order to evaluate $\left[(\vec{t}_n\cdot\vec{x}) - \bar{c}|(\vec{t}_n)_{\tau_x}|\right]$, let us first suppose $x\in B_{2R_n}(t_n)$, so that $|x-t_n| < 2R_n < \hat{\gamma}|t_n|$ then we have

$$(\vec{t}_n\cdot\vec{x}) = \left(\frac{t_n}{|t_n|}\cdot\frac{t_n + x - t_n}{|x|}\right) \geq \frac{|t_n| - |x-t_n|}{|x|} \geq \frac{|t_n|-|x-t_n|}{|t_n|+|x-t_n|} \geq \frac{1-\hat{\gamma}}{1+\hat{\gamma}} \tag{3.4.7}$$

and, since

$$\vec{t}_n = \frac{x}{|t_n|} + \frac{t_n - x}{|t_n|},$$

$$|(\vec{t}_n)_{\tau_x}| \leq \frac{|t_n - x|}{|x|} < \hat{\gamma}. \tag{3.4.8}$$

On the other hand, we can assert that, by homothety, (3.4.7) and (3.4.8) are also true for all x belonging to the cone $\mathcal{K}$ having as vertex the origin and generated by $B_{2R_n}(t_n)$. Then, in particular, (3.4.7) and (3.4.8) are true for all $x\in\mathcal{D}_n$, being $\mathcal{D}_n\subset\tilde{\mathcal{C}}_n\subset\mathcal{K}$.

Thus (3.4.6) follows because we have, by the choice of $\hat{\gamma}$, $\frac{3}{4}\frac{1-\hat{\gamma}}{1+\hat{\gamma}} > \frac{1}{2}$ and $\frac{1-\hat{\gamma}}{1+\hat{\gamma}} - 4\bar{c}\hat{\gamma} > 0$.

∎

Lemma 3.4.3 *Let $a(x)$ and $(u_n)_{n\in\mathbb{N}}$ be as in Lemma 3.4.2. Then the inequality*

$$\begin{aligned}\frac{1}{4}\int_{\mathcal{D}_n}\frac{\partial a}{\partial\vec{x}}(x)u_n^2(x) &\leq \frac{1}{2}\int_{(\partial\mathcal{D}_n)_i}(|\nabla u_n|^2 + a(x)u_n^2)(\nu_n\cdot\vec{t}_n)\\ &- \int_{(\partial\mathcal{D}_n)_i}(\nabla u_n\cdot\nu_n)\,(\nabla u_n\cdot\vec{t}_n) - \frac{1}{p}\int_{(\partial\mathcal{D}_n)_i}|u_n|^p(\nu_n\cdot\vec{t}_n)\end{aligned} \tag{3.4.9}$$

holds.

PROOF. Combining (3.4.1) and (3.4.6) we obtain

$$\begin{aligned} \frac{1}{4}\int_{\mathcal{D}_n} \frac{\partial a}{\partial \vec{x}} u_n^2 &\leq \frac{1}{2}\int_{\partial \mathcal{D}_n} (|\nabla u_n|^2 + a(x)u_n^2)(\nu_n \cdot \vec{t}_n) \\ &- \int_{\partial \mathcal{D}_n} (\nabla u_n \cdot \nu_n)\,(\nabla u_n \cdot \vec{t}_n) - \frac{1}{p}\int_{\partial \mathcal{D}_n} |u_n|^p (\nu_n \cdot \vec{t}_n)\,. \end{aligned} \tag{3.4.10}$$

Now, for all n, u_n solves $(\mathrm{P}_{\mathrm{B}_{\rho_n}(0)})$, $u_n = 0$ on $\partial B_{\rho_n}(0) \supset (\partial \mathcal{D}_n)_e$, so ∇u_n and ν_n have the same direction, moreover on $(\partial \mathcal{D}_n)_e$ it is $(\nu_n \cdot \vec{t}_n) \geq 0$, thus we deduce

$$\int_{(\partial \mathcal{D}_n)_e} a(x)u_n^2(\nu_n \cdot \vec{t}_n) = 0 = \int_{(\partial \mathcal{D}_n)_e} |u_n|^p(\nu_n \cdot \vec{t}_n) \tag{3.4.11}$$

and

$$\begin{aligned} \frac{1}{2}\int_{(\partial \mathcal{D}_n)_e} |\nabla u_n|^2(\nu_n \cdot \vec{t}_n) &- \int_{(\partial \mathcal{D}_n)_e} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \vec{t}_n) \\ &= \frac{1}{2}\int_{(\partial \mathcal{D}_n)_e} |\nabla u_n|^2(\nu_n \cdot \vec{t}_n) \\ &- \int_{(\partial \mathcal{D}_n)_e} (\nabla u_n \cdot \theta \nabla u_n)(\frac{1}{\theta}\nu_n \cdot \vec{t}_n) \\ &= -\frac{1}{2}\int_{(\partial \mathcal{D}_n)_e} |\nabla u_n|^2(\nu_n \cdot \vec{t}_n) \leq 0\,. \end{aligned} \tag{3.4.12}$$

Hence (3.4.9) follows inserting (3.4.11) and (3.4.12) in (3.4.10). ∎

Proposition 3.4.1 *Let $a(x)$ satisfy (a_1), (a_2), (a_3), (a_4). Let $(u_n)_{n\in \mathbb{N}}$ be a balanced sequence bounded in $H^1(\mathbb{R}^N)$ then $(u_n)_{n\in\mathbb{N}}$ is relatively compact.*

PROOF. We argue by contradiction and we assume that $(u_n)_{n\in\mathbb{N}}$ is not compact. Then, by Proposition 3.2.1, up to a subsequence, it is broken and, by Lemma 3.4.3, the inequality (3.4.9) must be true.

Let us consider, for n large, the right hand side of (3.4.9). First of all, let us observe that, by (a_2), $a(x) \geq 0$ for all $x \in (\partial \mathcal{D}_n)_i$ so, taking into account that $(\nu_n \cdot \vec{t}_n) \leq 0$ on $(\partial \mathcal{D}_n)_i$, we have

$$\int_{(\partial \mathcal{D}_n)_i} \left(|\nabla u_n|^2 + a(x)u_n^2\right)(\nu_n \cdot \vec{t}_n) \leq 0\,. \tag{3.4.13}$$

Moreover, by using Proposition 3.3.3, we deduce

$$\begin{aligned} -\int_{(\partial \mathcal{D}_n)_i} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \vec{t}_n)d\sigma &\leq \int_{(\partial \mathcal{D}_n)_i} |\nabla u_n|^2 d\sigma \\ &\leq \int_{\partial \tilde{\mathcal{C}}_n} |\nabla u_n|^2 d\sigma \leq c^* e^{-\alpha^*|t_n|}\,. \end{aligned} \tag{3.4.14}$$

Let us now show that there exist constants $\alpha' > 0$ and $c' > 0$, independent on n, so that

$$-\int_{(\partial \mathcal{D}_n)_i} |u_n|^p(\nu_n \cdot \vec{t}_n)d\sigma \leq \int_{(\partial \mathcal{D}_n)_i} |u_n|^p d\sigma \leq c' e^{-\alpha'|t_n|}\,. \tag{3.4.15}$$

Since $(\partial\mathcal{D}_n)_i \subset \partial\tilde{\mathcal{C}}_n$ and, for large n, $\partial\tilde{\mathcal{C}}_n \subset \mathcal{S}_n$, using Proposition 3.3.1, we infer,

$$
\begin{aligned}
-\int_{(\partial\mathcal{D}_n)_i} |u_n|^p d\sigma &\le \int_{\partial\tilde{\mathcal{C}}_n} |u_n|^p d\sigma \\
&\le c_\alpha \int_{\partial\tilde{\mathcal{C}}_n} e^{-\alpha\sigma_n(x)p} d\sigma \le c_\alpha \sum_{i=1}^{k} \int_{\partial\tilde{\mathcal{C}}_n} e^{-\alpha p|x-t_n^i|} d\sigma \ ,
\end{aligned}
\tag{3.4.16}
$$

$\alpha \in (0, \sqrt{a_\infty})$, $c_\alpha > 0$.

Setting, for $h \ge 1$ and $i = 0, 1, \dots, k$

$$
A_{h,i} := \left\{ x \in \partial\tilde{\mathcal{C}}_n \ : \ 2^{h-1}\frac{r_n}{2} < |x - t_n^i| < 2^h \frac{r_n}{2} \right\}
\tag{3.4.17}
$$

and denoting by $|A_{h,i}|$ the $(N-1)$ - dimensional (Hausdorff) measure of $A_{h,i}$, we have for $i = 0, 1, \dots, k$

$$
|A_{h,i}| \le C \left[2^h \frac{r_n}{2} \right]^{N-1} \qquad C \in I\!R
\tag{3.4.18}
$$

because it is not difficult to understand that, $\forall h$, $|A_{h,i}|$ can be estimated by the surface of the cylinder having height and basis diameter measure equal to $\frac{r_n}{2}2^h$.

Thus, in view of (3.4.17) and (3.4.18), we deduce

$$
\int_{\partial\tilde{\mathcal{C}}_n} e^{-\alpha p|x-t_n^i|} d\sigma \le \sum_{h=1}^{\infty} \int_{A_{h,i}} e^{-\alpha p 2^{h-1}\frac{r_n}{2}} d\sigma \le C \sum_{h=1}^{\infty} e^{-\alpha p 2^{h-1}\frac{r_n}{2}} \left[2^h \frac{r_n}{2} \right]^{N-1}
\tag{3.4.19}
$$

hence, inserting (3.4.19) in (3.4.16), we obtain as desired,

$$
\int_{(\partial\mathcal{D}_n)_i} |u_n|^p d\sigma \le c'_\alpha k r_n^{N-1} e^{-\alpha p\frac{r_n}{2}} \sum_{h=0}^{\infty} e^{-\alpha p 2^h} 2^{h(N-1)} \le c' e^{-\alpha'|t_n|} \ .
\tag{3.4.20}
$$

On the other hand, denoting by $\tilde{\rho}_n := \max\{\rho_n, |t_n|\}$ and by

$$
\tilde{\mathcal{D}}_n := \tilde{\mathcal{C}}_n \cap B_{\tilde{\rho}_n}(0)
$$

we have, for large n,

$$
\int_{\mathcal{D}_n} \frac{\partial a}{\partial \vec{x}}(x) u_n^2 dx \ge \inf_{\mathcal{D}_n} \left(\frac{\partial a}{\partial \vec{x}}(x) \right) \int_{\mathcal{D}_n} u_n^2 dx \ge \overline{C} \inf_{\tilde{\mathcal{D}}_n} \left(\frac{\partial a}{\partial \vec{x}}(x) \right) \int_{\tilde{\mathcal{D}}_n} u_n^2 dx \ ,
\tag{3.4.21}
$$

$\overline{C} > 0$ constant, because, as remarked at the beginning of the section, $(|\ |t_n| - \rho_n|)_{n\in I\!N}$ is bounded from above. Moreover, in view of Proposition 3.2.1 and of the choice of t_n, we infer

$$
\liminf_{n\to+\infty} \int_{\tilde{\mathcal{D}}_n} u_n^2 dx \ge \lambda > 0, \qquad \lambda = \text{const} \ .
\tag{3.4.22}
$$

Then, combining (3.4.9) with (3.4.13), (3.4.14), (3.4.15), (3.4.21) and (3.4.22), we obtain

$$
\frac{1}{4}\lambda\overline{C} \inf_{\tilde{\mathcal{D}}_n} \left(\frac{\partial a}{\partial \vec{x}}(x) \right) \le c^* e^{-\alpha^*|t_n|} + \frac{c'}{p} e^{-\alpha'|t_n|} \le \overline{c} e^{-\overline{\alpha}|t_n|}
$$

$\overline{\alpha} = \min(\alpha^*, \alpha')$, and this is impossible by (a_3).

∎

3.5 Multiplicity of solutions

This section is devoted to the proof of the existence of infinitely many solutions to problem (P).

Let us fix a sequence $(\rho_n)_{n\in I\!N}$, $\rho_n \in I\!R^+$ such that $\rho_n \underset{n\to+\infty}{\longrightarrow} +\infty$ and consider the problems

$$(\mathrm{P_n}) \qquad \begin{cases} -\Delta u + a(x)u = |u|^{p-2}u & \text{in } B_{\rho_n}(0) \\ u = 0 & \text{on } \partial B_{\rho_n}(0) \end{cases}$$

approximating (P).

Our first step is proving that, for all $n \in I\!N$, (P_n) possesses infinitely many solutions. In order to do this, we consider, $\forall n \in I\!N$, the homogeneus functional

$$J_n : H_0^1(B_{\rho_n}(0)) \setminus \{0\} \to I\!R$$

defined by

$$J_n(u) = \frac{\displaystyle\int_{B_{\rho_n}(0)} (|\nabla u|^2 + a(x)u^2)dx}{\left(\displaystyle\int_{B_{\rho_n}(0)} |u|^p dx\right)^{\frac{2}{p}}}$$

and we look for critical points of it constrained on

$$\Sigma_n := \{u \in H_0^1(B_{\rho_n}(0))\ ; \|u\|_{B_{\rho_n}(0)} = 1\}\ .$$

We remark that, by the homogeneity of J_n, $J_n(t\,u) = J_n(u)$, $\forall u \in \Sigma_n$, $\forall t \in I\!R^+ \setminus \{0\}$, hence, $\forall u \in \Sigma_n$, $(\nabla J_n(u)\,,\, u)_{H_0^1} = 0$. So a critical point of J_n constrained on Σ_n is actually a free critical point of J_n. Moreover, it is easy to verify that, to any $\overline{u} \in \Sigma_n$ for which $J_n(\overline{u}) > 0$ and $\nabla J_n(\overline{u}) = 0$, there corresponds, unique $\overline{t} \in I\!R^+ \setminus \{0\}$ such that $\overline{t}\overline{u}$ belongs to the set

$$\mathcal{N}_n := \{u \in H_0^1(B_{\rho_n}(0))\ :\ u \neq 0, \int_{B_{\rho_n}(0)} |\nabla u|^2 + a(x)u^2 = \int_{B_{\rho_n}(0)} |u|^p\}$$

and, clearly, $\nabla J_n(\overline{t}\overline{u}) = 0$. Furthermore, a direct computation shows that $u \in H_0^1(B_{\rho_n}(0))$ solves (P_n), i.e. is a critical point of the free functional

$$I_n(u) = \frac{1}{2}\int_{B_{\rho_n}(0)} (|\nabla u|^2 + a(x)u^2)dx - \frac{1}{p}\int_{B_{\rho_n}(0)} |u|^p dx\ ,$$

if and only if $\nabla J_n(u) = 0$ and $u \in \mathcal{N}_n$ (see also Lemma 3.6.2 and Remark 3.6.1 in the Appendix).

Now, let us call, for any fixed n and for all $k \in I\!N$

$$\Gamma_k^n := \{A \subset \Sigma_n\ :\ A \text{ compact}, A = -A, \gamma(A) \geq k\}\ ,$$

$\gamma(A)$ denoting the Krasnosel'skii genus of A, and let us set

$$c_k^n = \inf_{A\in\Gamma_k^n} \sup_{u\in A} J_n(u) \ . \tag{3.5.1}$$

Since the assumptions on $a(x)$ do not imply that J_n is bounded from below on Σ_n, we cannot apply straightly the well known minimax principle to conclude that the numbers $(c_k^n)_k$ defined in (3.5.1) are, for all k, critical levels for the functional J_n. Nevertheless this difficulty is overcame thanks to the following

Proposition 3.5.1 *Let $a(x)$ satisfy (a_1) - (a_2). Then there exists $\overline{k} \in I\!N$ such that*

$$c_k^n > 0 \qquad \forall k > \overline{k} \qquad \forall n \in I\!N \ . \tag{3.5.2}$$

To prove Proposition 3.5.1 we need to state beforehand the

Lemma 3.5.1 *Let $a(x)$ satisfy (a_1) - (a_2). Then for all $\beta < \min(1\,,\, a_\infty)$, there exists $\overline{k} \in I\!N$ such that for all compact sets A such that*

$$A \subset \{u \in H^1(I\!R^N) \,:\, \|u\| = 1\} =: \Sigma$$

$A = -A$, $\gamma(A) > \overline{k}$, there exists $u_A \in A$ such that

$$\int_{I\!R^N} |\nabla u_A|^2 + a(x)u_A^2 \geq \beta \|u_A\|^2 \ . \tag{3.5.3}$$

PROOF. The desired result can be easily obtained once proved the following claim: $\forall\beta < \min(1\,,\, a_\infty)$ *there exists a subspace E of $H^1(I\!R^N)$, codim$(E) < +\infty$ such that*

$$\int_{I\!R^N} (|\nabla u|^2 + a(x)u^2)dx \geq \beta\|u\|^2 \qquad \forall u \in E \ . \tag{3.5.4}$$

In fact, setting $\overline{k} = codim(E)$, by the genus properties

$$A \cap E \neq \emptyset \qquad \forall A \subset \Sigma, \text{ compact}, A = -A,\ \gamma(A) > \overline{k} \tag{3.5.5}$$

and (3.5.4)- (3.5.5) imply (3.5.3).

Let us now prove the claim. We argue by contradiction. Then, we can select a sequence $(u_n)_{n\in I\!N}$, $u_n \in H^1(I\!R^N)$ such that

$$\|u_n\| = 1, \qquad (u_n,\, u_m)_{H^1} = 0 \qquad n \neq m \tag{3.5.6}$$

$$\int_{I\!R^N} (|\nabla u_n|^2 + a(x)u_n^2)dx < \beta\|u_n\|^2 = \beta \qquad \forall n \in I\!N \ . \tag{3.5.7}$$

Indeed, setting $\forall n \in I\!N$, $E_n = \{[u_0,\, u_1, \ldots,\, u_{n-1}]\}^\perp$, $[u_0,\, u_1, \ldots,\, u_{n-1}]$ being the subspace spanned by the mutually orthogonal unitary vectors $\{u_0,\, u_1, \ldots,\, u_{n-1}\}$ since we assume false the claim, we can find a unitary vector, $u_n \in E_n$, which verifies (3.5.7).

By (3.5.6), passing eventually to a subsequence, we have $u_n \rightharpoonup 0$.

On the other hand, by (a_1) - (a_2), choosing a constant $\hat{a} \in (\beta,\, a_\infty)$, there exists a function $c(x) \in \mathcal{C}_0(I\!R^N)$ such that $a(x) \geq \hat{a} + c(x)\ \forall x \in I\!R^N$. Hence

$$\beta > \int_{I\!R^N} (|\nabla u_n|^2 + a(x)u_n^2)dx \geq \int_{I\!R^N} (|\nabla u_n|^2 + \hat{a}u_n^2) + \int_{I\!R^N} c(x)u_n^2 \geq \min(1,\, \hat{a})\|u_n\|^2 + \int_{I\!R^N} c(x)u_n^2\ ,$$

so, taking into account that $\int_{I\!R^N} c(x)u_n^2 \underset{n\to+\infty}{\longrightarrow} 0$, we obtain $\beta \geq \min(1, \hat{a})$ contradicting the choice of β and $\hat{a}$. ∎

Proof of Proposition 3.5.1.

Setting

$$J(u) = \frac{\int_{I\!R^N} (|\nabla u|^2 + a(x)u^2)dx}{\left(\int_{I\!R^N} |u|^p dx\right)^{\frac{2}{p}}} \qquad \forall u \in H^1(I\!R^N) \tag{3.5.8}$$

and

$$c_k = \inf_{A\in\Gamma_k} \sup_{u\in A} J(u)\ ,$$

where

$$\Gamma_k := \{A \subset \Sigma : A \textit{ compact}, A = -A, \gamma(A) \geq k\}\ ,$$

we have, by Lemma 3.5.1,

$$c_k > 0 \qquad \forall k > \overline{k}\ . \tag{3.5.9}$$

Moreover, for any fixed k

$$\Gamma_k^n \subset \Gamma_k \qquad \forall n \in I\!N$$

and, obviously, $J(u) = J_n(u)$, for all $u \in A \in \Gamma_k^n$, thus

$$c_k \leq c_k^n \qquad \forall n \in I\!N, \qquad \forall k \in I\!N \tag{3.5.10}$$

and (3.5.9) - (3.5.10) imply (3.5.2). ∎

We are now ready to give the

Proof of Theorem 3.1.1.

In view of (3.5.2), by using well known results of minimax theory (see for istance [24]) we can assert that, for any fixed $n \in I\!N$, the numbers c_k^n, defined in (3.5.1), are, for all $k > \overline{k}$, critical values of the functionals J_n. Moreover for any fixed $k > \overline{k}$

$$c_k^n \geq c_k^{n+1}\ . \tag{3.5.11}$$

Then, it is easy to verify that the numbers b_k^n defined, for all $n \in I\!N$, for all $k > \overline{k}$, by

$$b_k^n := \left(\frac{1}{2} - \frac{1}{p}\right)(c_k^n)^{\frac{p}{p-2}}$$

are critical values for the functional I_n and that, for any fixed $k > \overline{k}$, by (3.5.11) (3.5.9) (3.5.10), the sequence $(b_k^n)_{n\in I\!N}$ is a decreasing sequence bounded from below by $\left(\frac{1}{2} - \frac{1}{p}\right)(c_k)^{\frac{p}{p-2}} > 0$.

Set now

$$b_k := \lim_{n\to+\infty} b_k^n \qquad \forall k > \overline{k}\,.$$

Clearly $b_k > 0$, we claim that

$$\lim_{k\to+\infty} b_k = +\infty\,. \tag{3.5.12}$$

Once proved (3.5.12), it is easy to conclude the proof. Indeed, for any fixed $k > \overline{k}$ we can construct a balanced sequence $(v_k^n)_{n\in I\!N}$, taking, for all $n \in I\!N$, a critical point v_k^n of I_n at level b_k^n. Since $I_n(v_k^n) \underset{n\to+\infty}{\longrightarrow} b_k$ and $\nabla I_n(v_k^n) = 0$, it is a standard matter to derive that $(\|v_k^n\|)_{n\in I\!N}$ is bounded; then, by Proposition 3.4.1, $(v_k^n)_{n\in I\!N}$ is relatively compact and strongly converges (up to a subsequence) in $H^1(I\!R^N)$ to a solution v_k of problem (P) such that $I(v_k) = b_k$. By (3.5.12) we have infinitely many distinct values of b_k, so the conclusion follows.

Let us now show that (3.5.12) holds.

We argue by contradiction and we assume

$$\lim_{k\to+\infty} b_k = \overline{b} \in I\!R\,.$$

Then, there exists a $\hat{k} \geq \overline{k}$ such that for all $k > \hat{k}$ there exists n_k for which

$$b_k^n < \overline{b} + 1 \qquad \forall n \geq n_k\,. \tag{3.5.13}$$

Then, by using Morse index estimates on min-max critical points (see [22], Lemma 3.6.2 and Remark 3.6.1 in the Appendix) we can select a sequence $(w_k)_k$, $w_k \in H_0^1(B_{\rho_{n_k}}(0))$, $k \in I\!N$, $k > \hat{k}$, such that for all k

$$\begin{array}{lll} a) & I_{n_k}(w_k) & = b_k^{n_k} \\ b) & \nabla I_{n_k}(w_k) & = 0 \end{array} \tag{3.5.14}$$

and

$$i_M(w_k) \geq k \tag{3.5.15}$$

$i_M(w_k)$ denoting the augmented Morse index of w_k (see Definition 3.6.1 in the Appendix).

$(w_k)_k$ is, by construction, a balanced sequence and using (3.5.13) - (3.5.14), it is not difficult to deduce that $(\|w_k\|)_k$ is bounded. Hence, Proposition 3.4.1 apply to $(w_k)_k$ and, passing eventually to a subsequence, still denoted by $(w_k)_k$, $w_k \underset{k\to+\infty}{\longrightarrow} \overline{w}$ strongly in $H^1(I\!R^N)$ and $\overline{w}$ is a solution to (P).

Hence, taking into account that the augmented Morse index of the critical points of I is well defined and finite (see Lemma 3.6.1 in the Appendix), we can assert that a finite dimensional subspace $M \subset H^1(\mathbb{R}^N)$ and a constant $\varepsilon > 0$ must exist so that

$$(\nabla^2 I(\overline{w})z, z) \geq \varepsilon \|z\|^2 \qquad \forall z \in M^\perp \,, \tag{3.5.16}$$

moreover, the strong convergence of $(w_k)_k$ to $\overline{w}$ imply, for k large enough,

$$\|\nabla^2 I(w_k) - \nabla^2 I(\overline{w})\| < \frac{\varepsilon}{2} \,. \tag{3.5.17}$$

On the other hand, by (3.5.15), for all k large enough, we can find $z_k \in M^\perp$ so that

$$(\nabla^2 I_{n_k}(w_k) z_k, z_k) \leq 0 \,. \tag{3.5.18}$$

Then, we deduce from (3.5.17) and (3.5.18)

$$\begin{aligned}(\nabla^2 I(\overline{w}) z_k, z_k) &\leq (\nabla^2 I(\overline{w}) z_k - \nabla^2 I_{n_k}(w_k) z_k, z_k) + (\nabla^2 I_{n_k}(w_k) z_k, z_k) \\ &\leq \|\nabla^2 I(\overline{w}) - \nabla^2 I_{n_k}(w_k)\| \, \|z_k\|^2 = \|\nabla^2 I(\overline{w}) - \nabla^2 I(w_k)\| \, \|z_k\|^2 < \frac{\varepsilon}{2} \, \|z_k\|^2\end{aligned}$$

contradicting (3.5.16) and completing the argument. ∎

3.6 Appendix

PROOF OF PROPOSITION 3.2.1.

Let $(u_n)_{n \in \mathbb{N}}$ be a non compact balanced sequence bounded in $H^1(\mathbb{R}^N)$.

Being $(\|u_n\|)_{n \in \mathbb{N}}$ bounded, there exists $u_0 \in H^1(\mathbb{R}^N)$ such that, up to a subseqence, still denoted by u_n,

(A.1) $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$

(A.2) $u_n \to u_0$ strongly in $L^q_{loc}(\mathbb{R}^N)$ $\quad \forall q < \frac{2N}{N-2} =: 2^*$

(A.3) $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$.

Since u_n is not compact, we have also

$$u_n - u_0 \not\to 0 \text{ strongly in } H^1(\mathbb{R}^N) \tag{A.4}$$

so there exists $\tau \in \mathbb{R}$, $\tau > 0$ such that

$$\|u_n - u_0\| \geq \tau > 0 \qquad \forall n \in \mathbb{N} \,. \tag{A.5}$$

Let us decompose $I\!R^N$ into N-dimensional unitary hypercubes, Q, with vertices having integer coordinates and put

$$d_n = \sup_Q \int_Q |u_n - u_0|^2 dx \ .$$

We claim there exists $\eta \in I\!R$, $\eta > 0$ such that

$$(A.6) \qquad d_n \geq \eta > 0 \qquad \forall n \in I\!N \ .$$

Arguing by contradiction, we assume (A.6) false. Therefore, up to a subsequence, $d_n \underset{n\to+\infty}{\longrightarrow} 0$. Now, by Hölder's inequality, we have, for all q, r : $2 < q < r < 2^*$

$$\int_{I\!R^N} |u_n - u_0|^q dx \ = \ \sum_Q \int_Q |u_n - u_0|^q dx \leq \sum_Q \left(\int_Q |u_n - u_0|^2\right)^\alpha \left(\int_Q |u_n - u_0|^r\right)^\beta$$

where $\alpha = \frac{r-q}{r-2}$, $\beta = \frac{q-2}{r-2}$.

Thus by Sobolev's inequalities we deduce

$$\int_{I\!R^N} |u_n - u_0|^q dx \leq c_1 d_n^\alpha \sum_Q \left[\int_Q (|\nabla(u_n - u_0)|^2 + |u_n - u_0|^2)\right]^{\frac{r\beta}{2}} = c_1 d_n^\alpha \|u_n - u_0\|^{r\,\beta}$$

if $\frac{r\beta}{2} > 1$, with $c_1 \in I\!R^+ \setminus \{0\}$.

Since $\dfrac{r\beta}{2} \underset{r\to q}{\longrightarrow} \dfrac{q}{2} > 1$, we obtain for all $q \in (2,\, 2^*)$

$$(A.7) \qquad \int_{I\!R^N} |u_n - u_0|^q dx \to 0$$

On the other hand, since u_n is balanced,

$$\int_{I\!R^N} \left[|\nabla(u_n - u_0)|^2 + a(x)(u_n - u_0)^2\right] dx = \int_{I\!R^N} |u_n - u_0|^p dx$$

and, by (a_1) - (a_2), there exist a constant $\tilde{a} \in (0,\, a_\infty)$ and a positive function $c(x) \in \mathcal{C}_0(I\!R^N)$ such that $a(x) \geq \tilde{a} - c(x)$, $\forall x \in I\!R^N$. Therefore there exists $c_2 \in I\!R^+ \setminus \{0\}$ such that

$$c_2 \|u_n - u_0\|^2 + o(1) \leq |u_n - u_0|_p^p$$

that, in view of (A.5), contradicts (A.7) and gives (A.6).

Being true (A.6), there exists a sequence of hypercubes $Q_n \subset I\!R^N$ such that

$$(A.8) \qquad \int_{Q_n} |u_n - u_0|^2 dx > \frac{\eta}{2} > 0 \ .$$

Let us denote, for all n, by t_n the center of Q_n and by

$$\tilde{u}_n(\cdot) = (u_n - u_0)(\cdot - t_n) \ .$$

We may assume, up to a subsequence, that, as $n \to +\infty$, $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^N)$, $\tilde{u}_n \to \tilde{u}$ in $L^2_{loc}(\mathbb{R}^N)$ and $\tilde{u}_n(x) \to \tilde{u}(x)$ a.e. in $\mathbb{R}^N$. Denoting by $\tilde{Q}$, the N-dimensional unitary hypercube centered at the origin, (A.8) gives

$$\int_{\tilde{Q}} |\tilde{u}_n|^2 dx > \frac{\eta}{2} > 0 \, ,$$

so, it follows from the Rellich Theorem

$$\int_{\tilde{Q}} |\tilde{u}|^2 dx \geq \frac{\eta}{2} > 0$$

and $\tilde{u} \neq 0$. But $u_n - u_0 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, hence $|t_n| \underset{n\to+\infty}{\longrightarrow} +\infty$.

Note that

$$\tilde{u} = \lim_{n\to+\infty} \tilde{u}_n = \lim_{n\to+\infty} (u_n - u_0)(\cdot - t_n) = \lim_{n\to+\infty} u_n(\cdot - t_n)$$

weakly, therefore, according to Remark 3.2.1, $|\tilde{u}|$ solves (EI_∞).

Moreover, since $(\tilde{u}_n)_{n\in\mathbb{N}}$ and $(\tilde{u}_n - \tilde{u})_{n\in\mathbb{N}}$ are weakly orthogonal we have

(A.9)

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} |\nabla(u_n - u_0)|^2 dx & = \displaystyle\int_{\mathbb{R}^N} |\nabla\tilde{u}_n|^2 dx = \int_{\mathbb{R}^N} |\nabla(\tilde{u}_n - \tilde{u})|^2 dx + \int_{\mathbb{R}^N} |\nabla\tilde{u}|^2 dx + o(1) \\ \displaystyle\int_{\mathbb{R}^N} |u_n - u_0|^2 dx & = \displaystyle\int_{\mathbb{R}^N} |\tilde{u}_n|^2 dx = \int_{\mathbb{R}^N} |\tilde{u}_n - \tilde{u}|^2 dx + \int_{\mathbb{R}^N} |\tilde{u}|^2 dx + o(1) \, . \end{cases}$$

Iterating the above argument and observing that the iteration procedure has to stop in a finite number of steps because of (A.9) and Lemma 3.2.1, we obtain

$$u_n - \sum_{i=1}^k \varphi_i(\cdot - t_n^i) \to u_0 \qquad \text{in } H^1(\mathbb{R}^N)$$

$$|t_n^i| \underset{n\to+\infty}{\longrightarrow} +\infty \qquad\qquad |t_n^i - t_n^j| \underset{n\to+\infty}{\longrightarrow} +\infty \qquad 1 \leq i \neq j \leq k$$

where $|\varphi_i|$ are solutions of (EI_∞).

In order to prove (3.2.2) let us observe that for any given $\varepsilon > 0$ there exist sets $B_0, B_1, \dots, B_k$ such that

$$\int_{\mathbb{R}^N \setminus B_0} |\nabla u_0|^2 < \varepsilon \qquad\qquad \int_{\mathbb{R}^N \setminus B_i} |\nabla\varphi_i|^2 < \varepsilon \qquad\qquad 1 \leq i \leq k \, .$$

So for $n \in \mathbb{N}$ large enough, denoting by H a positive real constant, we have:

$$\int_{\mathbb{R}^N} |\nabla(u_n - (u_0 + \sum_{i=1}^k \varphi_i(\,\cdot - t_n^i)))|^2 dx =$$

$$\int_{B_0 \cup \left(\bigcup_{i=1}^k (B_i + t_n^i)\right)} |\nabla(u_n - (u_0 + \sum_{i=1}^k \varphi_i(\,\cdot - t_n^i)))|^2 dx +$$

$$+\int_{I\!R^N\setminus\left(B_0\cup\left(\bigcup_{i=1}^k(B_i+t_n^i)\right)\right)}|\nabla(u_n-(u_0+\sum_{i=1}^k\varphi_i(\cdot-t_n^i)))|^2dx=$$

$$=\int_{B_0}|\nabla(u_n-u_0)|^2dx+\sum_{i=1}^k\int_{B_i+t_n^i}|\nabla(u_n-\varphi_i(\cdot-t_n^i))|^2dx+H\varepsilon=$$

$$=\int_{B_0}|\nabla(|u_n|-|u_0|)^2dx+\sum_{i=1}^k\int_{B_i}|\nabla(|u_n(\cdot+t_n^i)|-|\varphi_i|)|^2dx+H\varepsilon=$$

$$=\int_{B_0\cup\left(\bigcup_{i=1}^k(B_i+t_n^i)\right)}|\nabla(|u_n|-(|u_0|+\sum_{i=1}^k|\varphi_i(\cdot-t_n^i)|))|^2dx+H\varepsilon=$$

$$=\int_{I\!R^N}|\nabla(|u_n|-(|u_0|+\sum_{i=1}^k|\varphi_i(\cdot-t_n^i)|))|^2.$$

∎

Let us recall the following definition (see [22]).

Definition 3.6.1 *We call* augmented Morse index *of a critical point u for a functional I, the number (possibly $+\infty$) of the eigenvalues of $\nabla^2 I(u)$ less or equal than zero.*

Lemma 3.6.1 *Let $a(x)$ satisfy (a_1) - (a_2). Let $w\in H^1(I\!R^N)$ be a critical point of I. Then the augmented Morse index of w is well defined and finite.*

PROOF. By (a_1) - (a_2), choosing a constant $\tilde a\in(0,a_\infty)$, there exists a function $\tilde c(x)\in\mathcal{C}_0(I\!R^N)$ such that

$$a(x)-(p-1)|w|^{p-2}>\tilde a+\tilde c(x)\ .$$

Arguing by contradiction, we now assume that the linear operator

$$\tilde L_w v:=-\Delta v+a(x)v-(p-1)|w|^{p-2}v$$

possesses infinitely many eigenfunctions v_n so that

$$\|v_n\|=1\qquad (v_n,\,v_m)=0\quad\text{for } n\neq m \tag{3.6.8}$$

and

$$<\tilde L_w v_n,v_n>\le\beta\|v_n\|^2$$

for some $\beta\in I\!R:\ 0<\beta<\min(1,\,\tilde a)$.
Then

$$\min(1,\tilde a)\,\|v_n\|^2+<\tilde c(x)v_n,\,v_n>\le\beta\|v_n\|^2$$

so

$$< \tilde{c}(x)v_n, v_n > \leq (\beta - \min(1, \tilde{a})) < 0 .$$

On the other hand $v_n \rightharpoonup 0$, thus $< \tilde{c}(x)v_n, v_n > \underset{n\to+\infty}{\longrightarrow} 0$ and we are in contradiction. ∎

Lemma 3.6.2 *A function $\overline{u} \in H^1(I\!R^N)$ is a critical point for the functional I if and only if it is a critical point for the functional J and belongs to the mainfold*

$$\mathcal{N} := \left\{ u \in H^1(I\!R^N) \ : \ u \neq 0, \ \int_{I\!R^N} (|\nabla u|^2 + a(x)u^2)dx = \int_{I\!R^N} u^p dx \right\} .$$

Moreover the augmented Morse Index of $\overline{u}$ as critical point of I is greather o equal than the augmented Morse index of $\overline{u}$ as critical point of J.

Proof. Since

$$\nabla J(\overline{u}) = 2\left(\int_{I\!R^N} |\overline{u}|^p dx\right)^{-\frac{2}{p}} \left[-\Delta\overline{u} + a(x)\overline{u} - \frac{\int_{I\!R^N} (|\nabla\overline{u}|^2 + a(x)\overline{u}^2)dx}{\int_{I\!R^N} |\overline{u}|^p dx} |\overline{u}|^{p-2}\overline{u} \right] ,$$

the first part of the claim follows straightly. In order to get the thesis it is enough to show that if $\overline{u}$ is a critical point of I (and then of J)

$$\text{(3.6.9)} \quad \int_{I\!R^N} [\nabla^2 J(\overline{u})][v]v dx \geq 2\left(\int_{I\!R^N} |\overline{u}|^p dx\right)^{-\frac{2}{p}} \int_{I\!R^N} [\nabla^2 I(\overline{u})][v]v dx \quad \forall v \in H^1(I\!R^N) .$$

Indeed, observing that

$$\left[-\Delta\overline{u} + a(x)\overline{u} - \frac{\int_{I\!R^N} (|\nabla\overline{u}|^2 + a(x)\overline{u}^2)dx}{\int_{I\!R^N} |\overline{u}|^p dx} |\overline{u}|^{p-2}\overline{u} \right] \int_{I\!R^N} \left[\nabla \left(\int_{I\!R^N} |\overline{u}|^p dx\right)^{-\frac{2}{p}} \right] v = 0$$

we have

$$\begin{aligned} [\nabla^2 J(\overline{u})][v] &= 2\left(\int_{I\!R^N} |\overline{u}|^p dx\right)^{-\frac{2}{p}} \left\{ -\Delta v + a(x)v - (p-1)|\overline{u}|^{p-2} v \frac{\int_{I\!R^N} (|\nabla\overline{u}|^2 + a(x)\overline{u}^2)dx}{\int_{I\!R^N} |\overline{u}|^p dx} \right. \\ &- \frac{2}{\int_{I\!R^N} |\overline{u}|^p dx} \left[\int_{I\!R^N} (-\Delta\overline{u} + a(x)\overline{u})v \right] |\overline{u}|^{p-2}\overline{u} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\int_{I\!R^N}(|\nabla\overline{u}|^2+a(x)\overline{u}^2)dx}{\left(\int_{I\!R^N}|\overline{u}|^p dx\right)^2}\left(\int_{I\!R^N}p|\overline{u}|^{p-2}\overline{u}v\right)|\overline{u}|^{p-2}\overline{u}\Bigg\} \\
&= \quad 2\left(\int_{I\!R^N}|\overline{u}|^p dx\right)^{-\frac{2}{p}}\Big\{-\Delta v+a(x)v-(p-1)|\overline{u}|^{p-2}v \\
&- \quad \frac{2}{\int_{I\!R^N}|\overline{u}|^p dx}\left[\int_{I\!R^N}|\overline{u}|^{p-2}\overline{u}v dx\right]|\overline{u}|^{p-2}\overline{u} \\
&+ \quad \frac{p}{\int_{I\!R^N}|\overline{u}|^p dx}\left(\int_{I\!R^N}|\overline{u}|^{p-2}\overline{u}v dx\right)|\overline{u}|^{p-2}\overline{u}\Bigg\} \\
&= \quad 2\left(\int_{I\!R^N}|\overline{u}|^p dx\right)^{-\frac{2}{p}}\left\{[\nabla^2 I(\overline{u})][v]+\frac{p-2}{\int_{I\!R^N}|\overline{u}|^p dx}\left(\int_{I\!R^N}|\overline{u}|^{p-2}\overline{u}v dx\right)|\overline{u}|^{p-2}\overline{u}\right\}.
\end{aligned}
$$

So, being $p \geq 2$, (3.6.9) follows. ∎

Remark 3.6.1 *We point out that the same conclusions of Lemma 3.6.2 hold if* $\overline{u} \in H^1(B_{\rho_n}(0))$ *is a critical point for the functional* I_n*, because the arguments of Lemma 3.6.2 can be repeated just replacing* $I\!R^N$*,* I*,* J *,* $\mathcal{N}$ *respectively with* $B_{\rho_n}(0)$*,* I_n*,* J_n *,* $\mathcal{N}_n$*.*

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Part II

Some properties of irrigation patterns and irrigable measures

Chapter 4

Elementary properties of optimal irrigation patterns[4]

In this paper we follow the approach in [3] for the study of the *ramified structures* and we identify some geometrical properties enjoyed by optimal irrigation patterns. These properties are “elementary” in the sense that they are not concerned with the regularity at the ending points of such structures, where the presumable selfsimilarity properties should take place. This preliminary study already finds an application in [2], where it is used in order to discuss the irrigability of a given measure.

4.1 Introduction

In many works, see for instance [4], [6], [7], [8] and [9], irrigation and draining systems, trees and roots, lungs and cardiovascular systems are described trough empirical observations assuming some selfsimilarity laws which lead to their fractal structures.

In [3] the authors, introducing the notion of *irrigation pattern*, propose a model, similar to that presented in [5] for the study of some transport problems, to the aim of setting on a mathematical basis the question and to investigate the causes which justify the ramified structures. They assume that the ramified structure is the result of a compromise between the necessity of keeping together the fibers (to reduce a cost) and the necessity of reaching a measure spread out on a large set. Thanks to [3], one can reformulate all the empirical observations which supply this recent literature in terms of precise mathematical conjectures or prove them as theorems. As a first step in this direction, in this paper we begin this study by identifying some geometrical, elementary properties which are enjoyed by optimal irrigation patterns, i.e. by those patterns which are solutions to a variational problem with Dirichlet boundary conditions. These properties are “elementary properties” in the sense that they are not concerned with the regularity at the ending points of these structures, where the presumable selfsimilarity properties should take place. This preliminary study has the aim of giving some

[4]G. Devillanova, S. Solimini, *Elementary properties of optimal irrigation patterns*, to appear.

basic tools which will be hopefully useful for the approach to the above mentioned problems and already finds an application in [2], where these properties are used in order to discuss the irrigability of a given measure.

4.2 Fundamental notions and notation

In this section we recall some notions and notation introduced for the first time in [3] and at the same time we introduce some new terminology.

Let $(\Omega,\ |\cdot|)$ be a nonatomic probability space which we interpret as the reference configuration of a fluid material body. We can think Ω playing the role of the trunk section of a tree, this trunk being thought as a set of fibers which can bifurcate into branches. A *set of fibers of* Ω *with source point* $S \in \mathbb{R}^N$ is a mapping

$$\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$$

such that:

C1) For a.e. *material point* $p \in \Omega$, $\chi_p : t \mapsto \chi(p,t)$ is a Lipschitz continuous map with a Lipschitz constant less than or equal to one.

C2) For a.e. $p \in \Omega$: $\chi_p(0) = S$.

The condition $|\Omega| = 1$ is of course assumed by normalization in order to simplify the exposition. In some cases this normalization will be impossible (we can, for instance, work with two different spaces and assume an inclusion), then we shall consider all the notions trivially extended to the case $|\Omega| < +\infty$. We shall consider the source point $S \in \mathbb{R}^N$ as given and we shall denote by $\mathbf{C}_S(\Omega)$ and $\mathbf{P}_S(\Omega)$ the set of all the sets of fibers of Ω and respectively the set of all the measurable sets of fibers of Ω and we shall call the elements of $\mathbf{P}_S(\Omega)$ *irrigation patterns*.

When the pattern χ is the constant map of constant value S we shall say that χ is *a trivial pattern*. When we shall deal with subsets $\Omega' \subset \Omega$ we shall use $\chi_{|\Omega'}$ instead of $\chi_{|\Omega' \times \mathbb{R}_+}$ to denote the restriction of χ to $\Omega' \times \mathbb{R}_+$ and we shall call $\chi_{|\Omega'}$ the *subpattern* of χ defined on Ω'.

Let $(\Omega_1,\ |\cdot|_1)$ and $(\Omega_2,\ |\cdot|_2)$ be two disjoint probability spaces, let $S \in \mathbb{R}^N$ and let $\chi_1 \in \mathbf{P}_S(\Omega_1)$ and $\chi_2 \in \mathbf{P}_S(\Omega_2)$ be two irrigation patterns with the same source S. Let us consider the set $\Omega = \Omega_1 \cup \Omega_2$ endowed with the finite measure defined by setting, for all $A \subset \Omega$, $|A| = |A \cap \Omega_1|_1 + |A \cap \Omega_2|_2$. Then we can consider χ_1 and χ_2 as subpatterns of a pattern $\chi \in \mathbf{P}_S(\Omega)$ defined by setting for a.e. $p \in \Omega$ and for all $t \in \mathbb{R}_+$

$$\chi(p,\ t) = \begin{cases} \chi_1(p,\ t) & \text{if } p \in \Omega_1 \\ \chi_2(p,\ t) & \text{if } p \in \Omega_2\ . \end{cases}$$

The above defined pattern will be called *bunch* of the patterns χ_1 and χ_2. It is clear that the definition of bunch can be extended to a finite number of patterns defined on disjoint probability spaces and also, with little changes, to a sequence of patterns, as in the following definition.

Definition 4.2.1 *Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence of irrigation patterns $\chi_n : \Omega_n \times \mathbb{R}_+ \to \mathbb{R}^N$ all with the same source $S \in \mathbb{R}^N$, where $(\Omega_n, |\cdot|_n)_{n\in\mathbb{N}}$ is a sequence of measurable spaces such that $\sum_n |\Omega_n|_n < +\infty$. Let Ω be the disjoint union of the sequence $(\Omega_n)_{n\in\mathbb{N}}$ endowed with the probability measure $|\cdot|$ defined by setting, $\forall A \subset \Omega$, $|A| = \sum_n |A \cap \Omega_n|_n$. The function $\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$ defined by setting for a.e. $p \in \Omega$ and for all $t \geq 0$, $\chi(p, t) = \chi_n(p, t)$ if $p \in \Omega_n$ will be called* bunch *of the sequence of patterns $(\chi_n)_{n\in\mathbb{N}}$.*

We recall that every set of fibers of Ω, time by time, defines an equivalence relation $\simeq_t$ on Ω by relating two points p and $q \in \Omega$ at the time t if χ_p and χ_q coincide on $[0, t]$. So every set of fibers at every time t divides Ω into equivalence classes which we shall call χ-vessels. For any $p \in \Omega$, we shall denote by $[p]_t$ the χ-vessel at time t which contains p while for any $t \geq 0$ we shall denote by $\mathcal{V}_t(\chi)$ the set of all the χ-vessels at time t. Then the following lemma trivially follows.

Lemma 4.2.1 *Let χ be an irrigation pattern. Then for all $0 \leq t_1 \leq t_2$ and for all $V_{t_1} \in \mathcal{V}_{t_1}(\chi)$ and $V_{t_2} \in \mathcal{V}_{t_2}(\chi)$ we have the following two alternatives:*

1. $V_{t_2} \subset V_{t_1}$
2. $V_{t_2} \cap V_{t_1} = \emptyset$.

Definition 4.2.2 *Let, for $i \in \{1, 2\}$, $\gamma_i : [0, T_i[\to \mathbb{R}^N$ be a curve in $\mathbb{R}^N$. Let*

$$X = \{t < \min\{T_1, T_2\} \mid \gamma_1(t) \neq \gamma_2(t))\}$$

be a nonempty set, then we shall call

$$s(\gamma_1, \gamma_2) = \inf X$$

the separation time *of the two curves γ_1 and γ_2.*

Definition 4.2.3 *Let χ be an irrigation pattern, then if one takes $\gamma_1 = \chi_p$ and $\gamma_2 = \chi_q$ for some points p and $q \in \Omega$ with $p \not\simeq_t q$ at some time $t > 0$ we shall call $s(\gamma_1, \gamma_2)$* the separation time *of the two points p and q and we shall denote it by $s_\chi(p, q)$.*

We can introduce for any time $t > 0$ a more restrictive equivalence relation among the points of Ω as stated in the following definition.

Definition 4.2.4 *Let χ be an irrigation pattern, $p \in \Omega$ and $t \geq 0$. We shall say that two points $p, q \in \Omega$ are strictly equivalent at the time t, and we shall write $p \simeq_t^s q$, if there exists $\varepsilon > 0$ such that $p \simeq_{t+\varepsilon} q$. We shall call* strict equivalence class *of p at the time t, or equivalently* strict vessel *of the point p at the time t, the following set*

$$[p]_t^s = \{q \in \Omega \mid p \simeq_t^s q\}$$

and we shall denote by $\mathcal{V}_t^s(\chi)$ the set of the strict vessels at time t.

Remark 4.2.1 *Let χ be an irrigation pattern, $p \in \Omega$ and $t \geq 0$. Then the* strict equivalence class *of p at the time t coincides with the union of the equivalence classes $[p]_{t'}$ of p at times $t' > t$, i.e.*

$$[p]^s_t = \bigcup_{t'>t} [p]_{t'} = \bigcup_{t'>t} [p]^s_{t'} \ . \tag{4.2.1}$$

Remark 4.2.2 *For a.e. p, $q \in \Omega$ and for all $t \geq 0$:*

- $p \simeq_t q$ *for all* $t \leq s_\chi(p,q)$
- $p \simeq^s_t q$ *for all* $t < s_\chi(p,q)$.

In the following we assume $\chi \in \mathbf{C}_S(\Omega)$. We introduce the following function $\sigma_\chi : \Omega \to I\!R_+$ which gives the absorption time of a point defined as follows

$$\forall p \in \Omega \ : \quad \sigma_\chi(p) = \inf\{t \in I\!R_+ \mid \chi_p(\cdot) \text{ is constant on } [t, +\infty[\} \ ,$$

which will be called *stopping* or *absorption function* for χ.

Definition 4.2.5 *We shall say that a point $p \in \Omega$ is absorbed, according to χ, when $\sigma_\chi(p) < +\infty$. A point $p \in \Omega$ is absorbed at the time t if $\sigma_\chi(p) \leq t$. Analogously, we shall say that a set $X \subset \Omega$ is an* absorbed set *at the time t if $\sigma_\chi(p) \leq t$ for a.e. $p \in X$, in particular when the set X is a χ-vessel we shall say that X is an absorbed χ-vessel.*

We shall denote by $A_t(\chi)$ the set of the points of Ω which are absorbed at the time t and by $A_\chi = \bigcup_{t>0} A_t(\chi)$ the set of the absorbed points. On the contrary, the set

$$M_t(\chi) = \{p \in \Omega \mid \sigma_\chi(p) > t\} = \Omega \setminus A_t(\chi)$$

is the set of the points that, at time t, are still moving.

Definition 4.2.6 *We shall call χ-*flow *at time t any χ-vessel which is not absorbed and we shall denote by $\mathcal{F}_t(\chi)$ the set of the χ-flows at time t and by $F_t(\chi)$ the union of all the χ-flows at time t.*

Definition 4.2.7 *Let $I \subset I\!R_+$. We shall say that the one parameter family of sets $V_t = (V_t)_{t \in I}$ is a χ-vessel evolution if*

- V_t *is decreasing under inclusion,*
- $V_t \in \mathcal{V}_t(\chi)$ *for every* $t \in I$.

*In particular, when for all t $V_t \in \mathcal{F}_t(\chi)$, the χ-vessel evolution $V_t = (V_t)_{t \in I}$ will be called χ-*flow evolution.

Definition 4.2.8 *At any time $t > 0$, the set $S_t(\chi) = M_t(\chi) \setminus F_t(\chi)$ will be called* spread flow *at time t. We shall say that the pattern χ is a* non-spread irrigation pattern *if $|S_t(\chi)| = 0$ for every $t \in I\!R_+$.*

Definition 4.2.9 *For every set of fibers $\chi \in \mathbf{C}_S(\Omega)$ we introduce the* irrigation function

$$i_\chi : A_\chi \to I\!R^N ,$$

defined by setting

$$\forall p \in A_\chi \qquad i_\chi(p) = \chi(p, \sigma_\chi(p)) ,$$

which gives, point by point, the absorption position of the absorbed points.

In the case in which we deal with an irrigation pattern $\chi \in \mathbf{P}_S(\Omega)$, the absorption time function σ_χ and, for all $t \geq 0$, the set $A_t(\chi)$ of the absorbed points at time t are both measurable (see [3]). We remark that $i_\chi(p) = \lim_{t\to\infty} \chi(p,t)$ and so also $i_\chi : A_\chi \to I\!R^N$ is a measurable function, as a pointwise limit of a sequence of measurable functions, when $\chi \in \mathbf{P}_S(\Omega)$.

Definition 4.2.10 *The image (push-forward) measure μ_χ determined by the irrigation function i_χ, defined by setting*

$$\mu_\chi(A) = |i_\chi^{-1}(A)| ,$$

for any Borel set $A \subset I\!R^N$, will be called irrigation measure *induced by the pattern χ.*

4.3 Flow curves and Dispersion

Definition 4.3.1 *Let χ be an irrigation pattern and $T > 0$. We shall call* flow curve *parameterized on $[0, T[$ any measurable function $\gamma : [0, T[\to I\!R^N$ for which there exists a (unique) χ-flow evolution $(V_t)_{0\leq t<T}$ such that*

$$\forall t \in [0, T[\quad : \quad \gamma(t) = \chi(p, t) = \chi_p(t) \;\; \textit{for } p \in V_t . \tag{4.3.1}$$

In such a case we shall say that $(V_t)_{0\leq t<T}$ is the *χ-flow evolution* relative to the flow curve γ.

Definition 4.3.2 *Let χ be an irrigation pattern and γ be a flow curve. We shall say that γ is a* maximal *flow curve if it is not the restriction of any other flow curve.*

Remark 4.3.1 *Let χ be a non-spread pattern and let γ be a flow curve with the relative χ-flow evolution $(V_t)_{0\leq t<T}$. Then, γ is maximal if and only if $V_T = \bigcap_{0\leq t<T} V_t$ is not a flow at time T. Indeed, by continuity, the assumption $V_T \in \mathcal{F}_T$ would state the existence of a positive measure subset of V_T consisting of points which are not absorbed at a time $\overline{s} > T$, so, χ being non-spread, V_T would contain a flow at the time $\overline{s}$.*

Definition 4.3.3 *Let χ be an irrigation pattern and let $\gamma : [0, T[\to I\!R^N$ be a flow curve of χ. Let $p \in \Omega$. We shall say that* the point p follows γ *if*

1. *$\sigma_\chi(p) < T$ and*
2. *$\gamma = \chi_p$ on $[0, \sigma_\chi(p)]$.*

and we shall set $D_\gamma = \{p \in \Omega \mid p \text{ follows } \gamma\}$ and $\tau_\gamma = \sup_{D_\gamma} \sigma_\chi$ (modulo a negligible set).

Definition 4.3.4 *Let χ be an irrigation pattern. For any point $p \in \Omega$ we shall call open fiber of the point p the restriction of the χ-fiber χ_p to the set $[0, \sigma_\chi(p)[$.*

Remark 4.3.2 *Let χ be an irrigation pattern and let $\gamma : [0, T[\to \mathbb{R}^N$ be a nonmaximal flow curve. Then γ can be seen as a set of χ fibers of positive measure. Indeed, by Definition 4.3.1 there exists a χ-flow evolution $(V_t)_{0\leq t<T}$ such that $\gamma = \chi_p$ on $[0, T[$ for all $p \in V_T$.*

To get the counterpart of the above remark we must add the hypotheses that the pattern is non-spread.

Remark 4.3.3 *Let χ be a non-spread irrigation pattern, then almost every open fiber is a flow curve. Indeed, by [3, Proposition 1.5], almost every point p belongs to a flow at any time $t < \sigma_\chi(p)$.*

The two above remarks suggest the idea that flow curves and open fibers are very similar objects and therefore they share many properties. Let us begin by pointing out with the following examples that this is not the case of every general property.

Example 4.3.1 *There exists an irrigation pattern χ such that any χ fiber has a length strictly smaller than 1 while all its maximal flow curves have length equal to 1.*

PROOF. Let us consider the pattern χ defined on $\Omega = [0, 1[$ by setting for all $p \in [0, 1[$ and for all $t \in \mathbb{R}_+$, $\chi(p, t) = \min\{p, t\}$. The pattern χ gives rise to a unique maximal flow curve whose support is the whole of the segment $[0, 1[$ and therefore it has a length equal to 1, while any χ fiber has a length strictly smaller than 1. ∎

Example 4.3.2 *There exists an irrigation pattern χ such that any χ fiber has a length equal to 1 and any nonmaximal flow curve has length strictly smaller than 1.*

PROOF. Let us consider the pattern χ defined on $\Omega = [0, 1[$ by setting for all $p \in [0, 1[$ and for all $t \in \mathbb{R}_+$, $\chi(p, t) = \min\{1, t\}$. For any $p \in \Omega$ the fiber χ_p has a support which is the segment $[0, 1]$ and therefore has a length equal to 1. On the contrary, any nonmaximal flow curve has a length strictly smaller than 1. ∎

The above stated examples lead us to give the following definition.

Definition 4.3.5 *Let P be a property enjoyed by curves. We shall say that P is an* inductive property *if the following equivalence holds true: P is satisfied by a curve parameterized on $[0, T[$ if and only if P is satisfied by all the restrictions to $[0, s[$ for $s < T$.*

Example 4.3.3 *The property of being without self intersections is an inductive property.*

Example 4.3.4 *The property of having length strictly smaller than 1 is not an inductive property.*

The following proposition will explain how the introduction of flow curves gives the advantage of studying the inductive properties of the fibers without caring of negligible sets.

Proposition 4.3.1 *Let χ be an irrigation pattern and let P be an inductive property. If almost all the open χ-fibers satisfy P, then P is also satisfied by any flow curve. Moreover, if χ is a non-spread pattern, if all the flow curves satisfy P, then P is satisfied by almost every open χ-fiber.*

PROOF. Let us assume that the inductive property P is enjoyed by almost any open χ-fiber then, being, by Remark 4.3.2, a nonmaximal flow curve coincident with a set of open χ fibers of positive measure, P is also satisfied by nonmaximal flow curves and therefore, by induction, by any flow curve. Now let us assume that χ is non-spread. By Remark 4.3.3 almost every open χ-fiber is a flow curve. So if P is a property which is satisfied by all the flow curves then it is also satisfied by almost every fiber. ∎

Definition 4.3.6 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern. We shall say that*

$$F_\chi = \{x \in I\!R^N \mid \exists t > 0,\ \exists A \in \mathcal{F}_t(\chi)\ s.t.\ x = \chi(p,t),\ p \in A\}$$

is the flow zone of χ.

Definition 4.3.7 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, then the set*

$$D_\chi = \{p \in \Omega \mid p \in F_{\sigma_\chi(p)}(\chi)\} \tag{4.3.2}$$

will be called dispersion *of the pattern χ. Moreover we shall say that*

- χ has a complete dispersion *or, equivalently,* χ is totally dispersed *if* $|\Omega \setminus D_\chi| = 0$
- χ *is* a pattern with dispersion *if* $|D_\chi| > 0$
- χ *is* a pattern without dispersion *if* $|D_\chi| = 0$.

Hence, when a pattern χ has a complete dispersion, every point is absorbed just because it stops its motion while it still belongs to a flow.

Remark 4.3.4 *By Definition 4.3.3 and Remark 4.3.1, if χ is non-spread $D_\chi = \bigcup D_\gamma$ where the union is extended to the (nonmaximal) flow curves γ.*

Remark 4.3.5 *Let χ be an irrigation pattern. Then the irrigation function sends the dispersion D_χ in the flow zone F_χ, i.e.*

$$i_\chi(D_\chi) \subset F_\chi\ . \tag{4.3.3}$$

As a consequence, by the definition of irrigation measure induced by χ, we have

$$|D_\chi| \leq \mu_\chi(F_\chi)\ . \tag{4.3.4}$$

Therefore any irrigation pattern χ such that $\mu_\chi(F_\chi) = 0$ is a pattern without dispersion.

Lemma 4.3.1 *Let χ be given. Then there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of flow curves such that for any nonmaximal flow curve γ there exists n such that γ is a restriction of γ_n.*

PROOF. Let $T > 0$ and let γ be a nonmaximal flow curve defined on $[0, T[$. Being γ nonmaximal there exist $\overline{T} > T$ and a flow curve $\overline{\gamma}$ defined on $[0, \overline{T}[$ such that $\gamma = \overline{\gamma}_{|[0,T[}$. Let us fix $s \in \mathbb{Q}$ such that $T < s < \overline{T}$, then $\gamma_s = \overline{\gamma}_{|[0,\,s[}$ is a nonmaximal flow curve defined on $[0,\ s[$ with $s \in \mathbb{Q}$. Since there is an injective map between the nonmaximal flow curves defined on $[0,\ s[$ and the flows in $\mathcal{F}_s(\chi)$ and since $\mathcal{F}_s(\chi)$ and $\mathbb{Q}$ are countable sets, the thesis follows. ∎
By Remark 4.3.4 we have the following corollary.

Corollary 4.3.1 *Let χ be a non-spread irrigation pattern. Then*

$$D_\chi = \bigcup_{n\in\mathbb{N}} D_{\gamma_n}\ ,$$

where the sequence $(\gamma_n)_{n\in\mathbb{N}}$ of nonmaximal flow curves is provided by the above lemma.

Corollary 4.3.2 *Let χ be a non-spread irrigation pattern. Then F_χ is the countable union of the support of a sequence of flow curves (which can also be supposed to be maximal or nonmaximal).*

PROOF. Let $x \in F_\chi$, then by definition there exists $V \in \mathcal{F}_t(\chi)$ such that $x = \chi(p,\ t)$ for all $p \in V$. Let us fix $p \in V$, then x belongs to the support of the open fiber χ_p and so by, Remark 4.3.3, to the support of a (nonmaximal) flow curve. Then the thesis follows by applying Lemma 4.3.1. ∎

Corollary 4.3.3 *For any pattern χ the flow zone F_χ is a Borel set and $d(F_\chi) = 1$.*

Lemma 4.3.2 *Let χ be an irrigation pattern. Then $\chi_{|(\Omega\setminus D_\chi)}$ is a subpattern of χ without dispersion.*

PROOF. Let us set $\chi' = \chi_{|(\Omega\setminus D_\chi)}$ and $p \in \Omega \setminus D_\chi$. Then if $t = \sigma_\chi(p) = \sigma_{\chi'}(p)$, $p \notin F_t(\chi) \supset F_t(\chi')$. ∎

On the contrary, we cannot say that the complementary restriction $\chi_{|D_\chi}$ has in general a complete dispersion, as shown by the following example.

Example 4.3.5 *There exists some irrigation pattern χ such that $|D_\chi| > 0$ and $\chi_{|D_\chi}$ is without dispersion.*

PROOF. Let $\Omega = [0,\ 1]$ and $\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be defined by setting

$$\chi(p,\ t) = \begin{cases} \min\left\{t,\ \frac{1}{2}\right\} & \text{if } p \le \frac{1}{2} \\ \min\{t,\ 1\} & \text{if } p > \frac{1}{2}\ . \end{cases}$$

It is easy to see that

1) $\mu_\chi = \frac{1}{2}\delta_{\frac{1}{2}} + \frac{1}{2}\delta_1$, where $\delta_{\frac{1}{2}}$ and δ_1 are Dirac masses respectively centered in $\frac{1}{2}$ and 1;

2) $D_\chi = \left[0, \frac{1}{2}\right]$ and, by consequence, $\chi_{|D_\chi}$ is without dispersion.

∎

In the above example we have considered a case of a discrete irrigation measure. It is easy to see that this is the only obstruction to the possibility of splitting a pattern χ as the bunch of a pattern without dispersion and a totally dispersed one.

Lemma 4.3.3 *Let χ be an irrigation pattern such that $\mu_\chi(X) = 0$ if X is a finite set. Then $\chi_{|D_\chi}$ is a subpattern of χ with a complete dispersion.*

PROOF. Let us set $\chi' = \chi_{|D_\chi}$, we have to prove that $|D_\chi \setminus D_{\chi'}| = 0$. Thanks to the decomposition of D_χ stated in Corollary 4.3.1, we just have to prove that $|D_\gamma \setminus D_{\chi'}| = 0$ for any flow curve γ. Given γ, we shall split D_γ in the sets $A_\gamma = \{q \in D_\gamma \mid \sigma_\chi(q) = \tau_\gamma\}$ and $B_\gamma = \{q \in D_\gamma \mid \sigma_\chi(q) < \tau_\gamma\}$. We know that $|A_\gamma| = 0$ because otherwise, if $p \in A_\gamma$, $i_\chi(p)$ would be the center of a Dirac mass in μ_χ, in contradiction to our assumptions. Let $p \in B_\gamma$, then there exists a set of positive measure of points $q \in D_\gamma$ such that $\sigma_\chi(p) < \sigma_\chi(q)$, by consequence $p \in F_{\sigma_{\chi'(p)}}(\chi')$. So for a.e. point $p \in D_\gamma$ we have $p \in B_\gamma \subset D_{\chi'}$. ∎

As a consequence of lemmas 4.3.2 and 4.3.3 we get the following corollaries.

Corollary 4.3.4 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, then we can consider it as the bunch of three irrigation patterns χ_1, χ_2 and χ_3 where χ_1 has a discrete (i.e. with a countable support) irrigation measure μ_{χ_1}, χ_2 has a complete dispersion and χ_3 has no dispersion.*

Corollary 4.3.5 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, then we can consider it as the bunch of a sequence of patterns $(\chi_n)_{n \in \mathbb{N}}$ where χ_0 is totally dispersed while for all $n \geq 1$ χ_n is without dispersion.*

4.4 Reduction formula and fiber cost

Given $\chi \in P_S(\Omega)$ and $0 < \alpha < 1$, we shall make use of the function $\varphi_\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\varphi_\chi(p, t) = |[p]_t|^{\alpha - 1} \mathbb{1}_{M_t(\chi)}(p) \,. \tag{4.4.1}$$

Remark 4.4.1 *Let $\chi \in P_S(\Omega)$ and $\Omega' \subset \Omega$. If $\chi' = \chi_{|\Omega'}$ we have*

$$\varphi_{\chi'} \geq \varphi_\chi \text{ on } \Omega' \times \mathbb{R}_+ \,.$$

By integrating with respect to one of the two variables we obtain the following two functions, respectively defined on $\mathbb{R}_+$ and Ω

$$c_\chi(t) = \int_\Omega \varphi_\chi(p, t)dp = \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp \tag{4.4.2}$$

and

$$\bar{c}_\chi(p) = \int_0^{+\infty} \varphi_\chi(p,t)dt = \int_0^{\sigma_\chi(p)} |[p]_t|^{\alpha-1}dt \,. \tag{4.4.3}$$

The function c_χ, already introduced in [3], gives the α-cost of χ at a time $t \in I\!R_+$. On the other hand, the function $\bar{c}_\chi$ gives, for every point $p \in \Omega$, the α-cost of the fiber χ_p. Being φ_χ a positive measurable function, the α-cost

$$I_\alpha(\chi) = \int_{\Omega\times I\!R_+} \varphi_\chi(p,\, t)dpdt$$

of the pattern χ can be either obtained, as in [3], by

$$I_\alpha(\chi) = \int_0^{+\infty} c_\chi(t)dt \tag{4.4.4}$$

or by

$$I_\alpha(\chi) = \int_\Omega \bar{c}_\chi(p)dp \,. \tag{4.4.5}$$

Remark 4.4.2 *Let χ be an irrigation pattern of finite cost, then a.e. point $p \in \Omega$ is absorbed, i.e. for a.e. $p \in \Omega$ $\sigma_\chi(p) < +\infty$.*

Given a probability measure $\bar{\mu}$, in the case in which there exists an irrigation pattern χ of finite cost such that $\mu_\chi = \bar{\mu}$ we shall say that $\bar{\mu}$ is an irrigable measure and that χ irrigates $\bar{\mu}$. Then we shall consider the Dirichlet problem which consists in minimizing on $P_S(\Omega)$ the functional $I_\alpha(\chi) + J(\mu_\chi)$, where J is defined by setting

$$J(\mu) = \begin{cases} 0 & \text{if } \mu = \bar{\mu} \\ +\infty & \text{otherwise}\,. \end{cases} \tag{4.4.6}$$

Definition 4.4.1 *Let $\bar{\mu}$ be a given probability measure. The Dirichlet Problem (4.4.6) admits a minimum if and only if $\bar{\mu}$ is irrigable, see [3, Section 9]. Any minimum χ of $I_\alpha + J$ will be called* an optimal irrigation pattern for *$\bar{\mu}$ and $I_\alpha(\chi)$ will be called* (irrigation) cost *of the probability measure $\bar{\mu}$. We shall say that χ is* an optimal pattern *if it is optimal for its irrigation measure μ_χ.*

Lemma 4.4.1 *Let χ be the bunch of a sequence of patterns $\chi_n : \Omega_n \times I\!R_+ \to I\!R^N$, where $(\Omega_n, |\cdot|_n)_{n\in I\!N}$ is a sequence of measurable spaces such that $\sum_n |\Omega_n|_n < +\infty$. Then*

$$I_\alpha(\chi) \le \sum_{n\in I\!N} I_\alpha(\chi_n) \,.$$

PROOF. For any $n \in I\!N$ and $p \in \Omega_n$ let $[p]_t^n$ be the χ_n-vessel at time t which contains p, then $[p]_t^n \subset [p]_t$. Therefore, being $\sigma_{\chi_n}(p) = \sigma_\chi(p)$ for all $p \in \Omega_n$

$$\bar{c}_{\chi_n}(p) = \int_0^{\sigma_{\chi_n}(p)} |[p]_t^n|^{\alpha-1}dt \ge \int_0^{\sigma_\chi(p)} |[p]_t|^{\alpha-1}dt = \bar{c}_\chi(p) \,.$$

Therefore, by (4.4.5),

$$I_\alpha(\chi) = \int_\Omega \bar{c}_\chi(p)dp \le \sum_n \int_{\Omega_n} \bar{c}_{\chi_n}(p)dp = \sum_n I_\alpha(\chi_n) \,.$$

∎

4.5 Good parameterization of a pattern

Let χ be an irrigation pattern, then by definition for a.e. $p \in \Omega$ the fiber χ_p is a Lipschitz continuous map with a Lipschitz constant less than or equal to one. Therefore, by Rademacher Theorem, for a.e. $p \in \Omega$ we know that for a.e. $t \in I\!R_+$ the derivative $\partial_t \chi(p, t)$ exists and $|\partial_t \chi(p, t)| \leq 1$.

Definition 4.5.1 *Let χ be an irrigation pattern. We shall say that χ is well parameterized if for a.e. $p \in \Omega$ and for a.e. $t \geq 0$, $t < \sigma_\chi(p)$, we have*

$$\left| \frac{\partial \chi_p}{\partial t}(t) \right| = 1 . \tag{4.5.1}$$

Remark 4.5.1 *When χ is a well parameterized pattern, for a.e. $p \in \Omega$, $\chi(p, \cdot)$ is parameterized with respect to the length.*

Remark 4.5.2 *The property of a curve of being parameterized with respect to the length is an inductive property. Therefore, by Proposition 4.3.1, a non-spread pattern χ is well parameterized if and only if any flow curve is parameterized with respect to the length.*

In the case of a general χ, we can consider the following function defined on $I\!R_+$ by setting for all $t \geq 0$

$$\eta_p(t) = \int_0^t \left| \frac{\partial \chi_p}{\partial s}(s) \right| ds , \tag{4.5.2}$$

which turns out to be a 1-Lipschitz function.

Remark 4.5.3 *For a.e. $p \in \Omega$, η_p is the identity function on $[0, \sigma_\chi(p)]$ if and only if χ is well parameterized.*

Remark 4.5.4 *Let us fix $p \in \Omega$ and $t \geq 0$, then, for all $q \in [p]_t$, being $\chi(p, \cdot) = \chi(q, \cdot)$ on $[0, t]$, we have $\eta_p(t) = \eta_q(t)$.*

Remark 4.5.5 *Let $\chi \in P_S(\Omega)$. Let us fix $p \in \Omega$ and $t \geq 0$ such that $t < \sigma_\chi(p)$. Then $\eta_p(t) < \eta_p(\sigma_\chi(p))$. Indeed, being $t < \sigma_\chi(p)$, the fiber $\chi(p, \cdot)$ can not be constant on $[t, \sigma_\chi(p)]$.*

Definition 4.5.2 *Let χ be an irrigation pattern. We shall say that $\overline{\chi} : \Omega \times I\!R_+ \to I\!R^N$ is the good parameterization of χ if, for a.e. $p \in \Omega$ and for all $t \in I\!R_+$,*

$$\chi(p, t) = \overline{\chi}(p, \eta_p(t)) , \tag{4.5.3}$$

and

$$\overline{\chi}(p, t) = const \qquad \forall t \geq \eta_p(\sigma_\chi(p)) .$$

It is easy to see that, for any given irrigation pattern χ, $\overline{\chi}$ exists and it is determined by (4.5.3), that $\overline{\chi}$ is a well parameterized pattern and that $\chi = \overline{\chi}$ if and only if χ is well parameterized. In the remaining part of this section we shall consider the patterns χ and $\overline{\chi}$ as given. Let $\overline{[p]}_t$ be the $\overline{\chi}$-vessel at time t which contains the point p.

Lemma 4.5.1 *The following properties hold true*

(4.5.4) $$\text{for a.e. } p \in \Omega \text{ and for all } t \geq 0 \quad : \quad [p]_t \subset \overline{[p]}_{\eta_p(t)} \ ,$$

(4.5.5) $$\text{for a.e. } p \in A_\chi \quad : \quad \sigma_{\overline{\chi}}(p) = \eta_p(\sigma_\chi(p)) \ .$$

PROOF. The proof is a straightforward application of the definition. Indeed, the thesis follows from (4.5.3) with the help of remarks 4.5.4 and 4.5.5. ∎

Corollary 4.5.1 *For $p \in \Omega$ and for $t < \sigma_\chi(p)$ if $[p]_t$ is a χ-flow at the time t then $\overline{[p]}_{\eta_p(t)}$ is a $\overline{\chi}$-flow at the time $\eta_p(t)$.*

Corollary 4.5.2 *Being, for all $p \in \Omega$, η_p a 1-Lipschitz function, from (4.5.5) we get $\sigma_{\overline{\chi}} \leq \sigma_\chi$ and consequently for all $t \geq 0$*

(4.5.6) $$A_t(\chi) \subset A_t(\overline{\chi}) \ .$$

Corollary 4.5.3 *$A_\chi \subset A_{\overline{\chi}}$ and $(i_{\overline{\chi}})_{|A_\chi} = i_\chi$.*

PROOF. By (4.5.6) $A_\chi \subset A_{\overline{\chi}}$. The remaining part of the thesis is only an application of the definition of the irrigation function, indeed for a.e. $p \in A_\chi$ we have by (4.5.5) and (4.5.3)

$$i_{\overline{\chi}}(p) = \overline{\chi}(p, \sigma_{\overline{\chi}}(p)) = \overline{\chi}(p, \eta_p(\sigma_\chi(p))) = \chi(p, \sigma_\chi(p)) = i_\chi(p) \ .$$

∎

Corollary 4.5.4 *When the pattern χ has a finite cost we have $A_\chi = \Omega$ and consequently*

(4.5.7) $$i_{\overline{\chi}} = i_\chi \qquad \text{and} \qquad \mu_{\overline{\chi}} = \mu_\chi \ .$$

Remark 4.5.6 *For all $t \geq 0$ and for a.e. $p \in \Omega$*

$$t < \sigma_{\overline{\chi}}(p) \qquad \Rightarrow \qquad \exists s < \sigma_\chi(p) \text{ such that } t = \eta_p(s) \ .$$

PROOF. Being η_p a continuous function such that $\eta_p(0) = 0$ and, by (4.5.5), $\eta_p(\sigma_\chi(p)) = \sigma_{\overline{\chi}}(p) > t > 0$ the existence of s follows by continuity. ∎

Lemma 4.5.2 *Let χ be a non-spread irrigation pattern. Then the good parameterization $\overline{\chi}$ of χ is also non-spread.*

PROOF. Let us fix $p \in \Omega$ and $t < \sigma_{\overline{\chi}}(p)$, then by Remark 4.5.6 we get the existence of a time $s < \sigma_\chi(p)$ such that $t = \eta_p(s)$. So, being χ a non spread pattern, we have for a.e. p that $[p]_s$ is a χ-flow at the time t and, by Corollary 4.5.1, that $\overline{[p]}_{\eta_p(s)}$ is a $\overline{\chi}$- flow at the time $\eta_p(s) = t$. ∎

Lemma 4.5.3 *For a.e. $p \in \Omega$*

(4.5.8)
$$\overline{c}_{\overline{\chi}}(p) \leq \overline{c}_{\chi}(p) \; .$$

PROOF. By (4.5.4) and (4.5.5), being $\eta_p' \leq 1$ for a.e. $p \in \Omega$,

$$\begin{aligned}
\overline{c}_{\chi}(p) &= \int_0^{\sigma_\chi(p)} |[p]_t|^{\alpha-1} dt \geq \int_0^{\sigma_\chi(p)} |\overline{[p]}_{\eta_p(t)}|^{\alpha-1} dt \\
&\geq \int_0^{\sigma_\chi(p)} |\overline{[p]}_{\eta_p(t)}|^{\alpha-1} \eta_p'(t) dt = \int_0^{\eta_p(\sigma_\chi(p))} |\overline{[p]}_t|^{\alpha-1} dt \\
&= \int_0^{\sigma_{\overline{\chi}}(p)} |\overline{[p]}_t|^{\alpha-1} dt = \overline{c}_{\overline{\chi}}(p) \; .
\end{aligned}$$

∎

Remark 4.5.7 *The proof of Lemma 4.5.3 shows that if (4.5.8) is an equality $\eta_p' = 1$ a.e. in $I\!R_+$. So (4.5.8) is an equality for a.e. p if and only if the pattern χ is well parameterized.*

Corollary 4.5.5 *$I_\alpha(\overline{\chi}) \leq I_\alpha(\chi)$ and the equality holds if and only if χ is well parameterized.*

Remark 4.5.8 *By corollaries 4.5.4 and 4.5.5 any optimal irrigation pattern is a well parameterized pattern.*

Definition 4.5.3 *Let $\chi \in P_S(\Omega)$. We shall say that χ is a* partitioned pattern *if for all $t \in I\!R_+$ and for all p, $q \in \Omega$ such that $p \simeq_t q$*

$$p \in A_t(\chi) \text{ and } q \in M_t(\chi) \quad \Rightarrow \quad \sigma_\chi(p) = t \; .$$

Remark 4.5.9 *Let χ be an irrigation pattern such that for any $p \in \Omega$ the fiber χ_p is not constant on any interval $[a, b]$ such that $0 \leq a < b \leq \sigma_\chi(p)$. Then χ is a partitioned pattern. In particular, a well parameterized pattern is partitioned.*

Lemma 4.5.4 *Let $\chi \in P_S(\Omega)$ be a partitioned non-spread irrigation pattern. Then for a.e. $t \in I\!R_+$*

(4.5.9)
$$c_\chi(t) = \sum_{A \in \mathcal{F}_t(\chi)} |A|^\alpha \; .$$

PROOF. We shall prove a little bit more, actually we shall prove that (4.5.9) holds true with the exception of a countable set of times, which are the values of t for which the level set $\{q \in \Omega \mid \sigma_\chi(q) = t\}$ has a positive measure. Indeed, for the other values of $t \in I\!R_+$ if $A \in \mathcal{F}_t(\chi)$ then, for a.e. $p \in A$, $\sigma_\chi(p) \neq t$ and therefore, being χ partitioned, p cannot be absorbed. So for all $A \in \mathcal{F}_t(\chi)$, $A \subset M_t(\chi)$ and, being χ non-spread, $M_t(\chi)$ is the countable union of the flows in $\mathcal{F}_t(\chi)$. Therefore we have

$$c_\chi(t) = \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp = \int_{\bigcup_{A\in\mathcal{F}_t(\chi)}} |[p]_t|^{\alpha-1} dp = \sum_{A\in\mathcal{F}_t(\chi)} \int_A |A|^{\alpha-1} dp = \sum_{A\in\mathcal{F}_t(\chi)} |A|^\alpha .$$

∎

Remark 4.5.10 *The hypothesis that χ is a non-spread pattern can not be removed. Indeed, if χ is totally spread then it has no flows and so, for all t, $\sum_{A\in\mathcal{F}_t(\chi)} |A|^\alpha = 0$ while, by [3, Proposition 2.1], $I_\alpha(\chi) = +\infty$.*

By [3, Proposition 2.1] we know that a pattern of finite cost is non-spread, therefore Lemma 4.5.4 admits the following corollary.

Corollary 4.5.6 *Let χ be a partitioned pattern of finite cost. Then for a.e. $t \in I\!R_+$ (4.5.9) holds true. Therefore*

$$I_\alpha(\chi) = \int_0^{+\infty} \sum_{A\in\mathcal{F}_t(\chi)} |A|^\alpha \, dt . \tag{4.5.10}$$

Corollary 4.5.7 *Let χ be a partitioned pattern of finite cost, then for any subpattern $\chi' = \chi_{|\Omega'}$ of χ and for a.e. $t \in I\!R_+$ we have $c_{\chi'}(t) \leq c_\chi(t)$. Moreover the inequality is strict unless Ω' contains $F_t(\chi)$.*

Proof. The thesis easily follows from (4.5.9) since the flows of χ' are the traces of the flows of χ on Ω'. ∎

Corollary 4.5.8 *Let χ be a partitioned pattern, then, for any subpattern χ' of χ, $I_\alpha(\chi') \leq I_\alpha(\chi)$.*

In Corollary 4.5.6, which, in some sense, states a "monotonicity" property of the cost functional I_α, the hypothesis that χ is partitioned can not be removed, as shown by the following example.

Example 4.5.1 *There exists a pattern χ (which is not partitioned) and a subpattern χ' of χ such that $I_\alpha(\chi) < I_\alpha(\chi')$*

Proof. Fixed $\varepsilon < T \in I\!R_+$, let $\Omega = [0, 1]$, $S = 0 \in I\!R$ and for all $p \in \Omega$ let χ_p be defined by

$$\frac{\partial\chi}{\partial t}(p, t) = \begin{cases} 1 & \text{if } t \in [T, T+\varepsilon] \text{ and } p \in \left[0, \frac{1}{2}\right] \\ 0 & \text{otherwise} \end{cases} ,$$

and let $\chi' = \chi_{|[0, \frac{1}{2}]}$. An easy computation shows that $I_\alpha(\chi) = \frac{1}{2}T + \frac{1}{2^\alpha}\varepsilon >$ while $I_\alpha(\chi') = \frac{1}{2^\alpha}(T+\varepsilon) > I_\alpha(\chi)$. ∎

Lemma 4.5.5 *If χ is non-spread then the flow curves of $\overline{\chi}$ coincide with the flow curves of χ parameterized with respect to the length.*

PROOF. It is a not completely straightforward but simple consequence of Proposition 4.3.1. The details are left to the reader. ∎

Corollary 4.5.9 *If χ is non-spread then*

$$F_\chi = F_{\overline{\chi}} .$$

4.6 Simple patterns, branches and cost of a branch

Definition 4.6.1 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω. We shall say that χ is a* simple pattern *if:*

- *for a.e. point $p \in \Omega$ the open fiber χ_p is a simple curve on $[0, \sigma_\chi(p)[$;*
- *for a.e. pair of points p and q of Ω $\chi_p(t) \neq \chi_q(s)$ for all $s, t > s_\chi(p,q)$, $t < \sigma_\chi(p)$, $s < \sigma_\chi(q)$.*

Remark 4.6.1 *By Remark 4.5.9, any simple irrigation pattern is a partitioned pattern.*

Remark 4.6.2 *Let χ be a simple irrigation pattern, then any subpattern χ' of χ is simple too.*

By Proposition 4.3.1 the above stated definition can be formulated as in the following remark by means of flow curves, since the property of being a simple curve is an inductive property.

Remark 4.6.3 *Let χ be a simple irrigation pattern. Then*

1) any flow curve γ is a simple curve;

2) for any pair (γ_1, γ_2) of flow curves we have $\gamma_1(t) \neq \gamma_2(s)$ for all $s, t > s(\gamma_1, \gamma_2)$.

Moreover, if a non-spread pattern χ satisfies 1) and 2) then it is a simple pattern.

Remark 4.6.4 *A non spread irrigation pattern χ is simple if and only if*

$$\forall x \in F_\chi \ \exists | t \geq 0 \ \exists | [p]_t \in \mathcal{F}_t(\chi) \ s.t. \ x = \chi(p, t) .$$

Definition 4.6.2 *Let χ be an irrigation pattern. For any pair $(p, t) \in \Omega \times I\!R_+$ the function $\chi_{(p,t)} : [p]_t \times I\!R_+ \to I\!R^N$, defined, for all $(q, s) \in [p]_t \times I\!R_+$, by*

$$\chi_{(p,t)}(q, s) = \chi(q, s + t) ,$$

is the branch of χ *starting from the vessel $[p]_t$ at time t.*

Remark 4.6.5 *If for the pair $(p, t) \in \Omega \times I\!R_+$ the vessel $[p]_t$ is not a flow, for a.e. $q \in [p]_t$ and for all $s \geq 0$ we would have $\chi(q, s+t) = \chi(p, \sigma_\chi(p))$ and therefore the branch of χ starting from $[p]_t$ at time t would be trivial. This is the reason for which the notion of branch starting from a vessels $[p]_t$ is meaningful when $[p]_t$ is actually a flow.*

Definition 4.6.3 *Let χ be a simple irrigation pattern. For any pair $(p, t) \in \Omega \times I\!R_+$ the pattern $\chi^s_{(p,t)} = \chi_{(p,t)|[p]^s_t}$ is* the single branch *of χ starting from the strict vessel $[p]^s_t$ at time t.*

Where $[p]_t \neq [p]^s_t$ we shall have branches which are not "single" and, in order to point out that the point $\chi(p, t)$ give rise to more than one single branch, we shall call $\chi_{(p,t)}$ *multiple branch.* Remark 4.6.4 allows us to give, for a simple pattern χ, the definition of branch of χ which starts from a point $x \in F_\chi$, according to Definition 4.6.2 applied to any pair (p, t) such that $x = \chi(p, t)$. The following definition is the analogous of Definition 4.6.2 for a point $x \in F_\chi$.

Definition 4.6.4 *Let χ be a simple irrigation pattern. For any $x \in F_\chi$ we shall call* branch *of χ with source point x the branch of χ starting from the flow $[p]_t$ at the time t where $t \geq 0$ and $[p]_t$ are determined by the condition $\chi(p, t) = x$, see Remark 4.6.4.*

Remark 4.6.6 *Let $x \in F_\chi$, $x = \chi(p, t)$, being, in general, $[p]_t \neq [p]^s_t$ we have not a single branch but a* multiple branch *with source point x at time t. More precisely, the multiple branch of χ starting from $x \in F_\chi$ can be seen as the bunch of the single branches which start from x.*

Definition 4.6.5 *Let χ be a simple irrigation pattern with source point S. For any branch χ' of χ starting from $x = \chi(p, t) \in F_\chi$, we shall call* pattern of χ "stumped" of the branch χ', *the restriction of χ to $(\Omega \setminus [p]_t)$ and we shall denote it by $\chi \setminus \chi'$.*

Lemma 4.6.1 *Let χ be a simple pattern and let χ' be a branch of χ. Then*

$$\mu_\chi = \mu_{\chi'} + \mu_{\chi\setminus\chi'} \tag{4.6.1}$$

and

$$I_\alpha(\chi) \leq I_\alpha(\chi') + I_\alpha(\chi \setminus \chi') \,. \tag{4.6.2}$$

PROOF. Equality (4.6.1) is obvious. To prove (4.6.2) we have only to remark that, if the branch χ' starts from the point $x = \chi(p, t)$, for a.e. $s \geq t$ we have

$$c_\chi(s) = c_{\chi'}(s-t) + c_{\chi\setminus\chi'}(s);.$$

Then the thesis follows by Corollary 4.5.7. ∎

Definition 4.6.6 *Given a pattern χ and a χ-vessel $V = [p]_t$ at a time t, we shall call* cost of the vessel V *and we shall denote it by $I_\alpha(V, t)$ or more briefly, when no ambiguity is possible, by $I_\alpha(V)$ the cost $I_\alpha(\chi')$ of the branch χ' which starts from the vessel $[p]_t$ at the time t.*

Remark 4.6.7 *Let χ be an irrigation pattern, then for all $t \geq 0$*

$$\sum_{V \in \mathcal{V}_t(\chi)} I_\alpha(V) = \int_t^{+\infty} c_\chi(s)ds \,. \tag{4.6.3}$$

The analogous property also holds true for strict vessels.

Lemma 4.6.2 *Let χ be an optimal irrigation pattern. Then any flow curve is a simple curve.*

PROOF. Let us assume, by contradiction, the existence of a flow curve γ defined on $[0, T[$ and of two real positive numbers $0 < t_1 < t_2 < T$ such that

$$\gamma(t_1) = \gamma(t_2) \tag{4.6.4}$$

Let $(V_t)_{0 \leq t < T}$ be the flow evolution relative to the flow curve γ. Let us define $\chi' : \Omega \times I\!R_+ \to I\!R^N$ by setting for all $p \in \Omega$ and $t \in I\!R_+$

$$\chi'(p, t) = \begin{cases} \chi(p, t) & \text{if } t \leq t_1 \quad \text{or} \quad p \notin V_{t_2} \\ \chi(p, t + t_2 - t_1) & \text{if } t > t_1 \quad \text{and} \quad p \in V_{t_2} \,. \end{cases} \tag{4.6.5}$$

It is easy to show that $\chi' \in P_S(\Omega)$ and $\mu_{\chi'} = \mu_\chi$. The contradiction will follow if we show that $I_\alpha(\chi') < I_\alpha(\chi)$. In estimating $I_\alpha(\chi)$ and $I_\alpha(\chi')$ we shall make use of two distinct partitions of $\Omega \times I\!R_+$. More precisely, we shall split

$$I_\alpha(\chi) = \sum_{i=1}^{4} \int_{E_i} \varphi_\chi(p, t)dp\,dt \,, \tag{4.6.6}$$

where the partition E_1, E_2, E_3, E_4 of the cylinder $\Omega \times I\!R_+$ is given by

$$E_1 = \Omega \times [0, t_1] \,; \tag{4.6.7}$$

$$E_2 = V_{t_2} \times [t_2, +\infty[\,; \tag{4.6.8}$$

$$E_3 = (\Omega \times [t_2, +\infty[) \setminus E_2 = (\Omega \setminus V_{t_2}) \times [t_2, +\infty[\,; \tag{4.6.9}$$

$$E_4 = \Omega \times [t_1, t_2] \,. \tag{4.6.10}$$

Analogously

$$I_\alpha(\chi') = \sum_{i=1}^{4} \int_{E'_i} \varphi_{\chi'}(p, t)dp\,dt \,, \tag{4.6.11}$$

where

$$E'_1 = E_1 \,; \tag{4.6.12}$$

$$E'_2 = V_{t_2} \times [t_1, +\infty[\,; \tag{4.6.13}$$

$$E'_3 = E_3 \,; \tag{4.6.14}$$

$$E'_4 = E_4 \setminus E'_2 \,. \tag{4.6.15}$$

By (4.6.5) for all $(p,\, t) \in E_1$ we have $\varphi_{\chi'}(p,\, t) = \varphi_\chi(p,\, t)$, while for all $(p,\, t) \in E_3$ we have $\varphi_{\chi'} \le \varphi_\chi$ so that

$$\int_{E'_i} \varphi_{\chi'}(p,\, t)dp\, dt \le \int_{E_i} \varphi_\chi(p,\, t)dp\, dt \qquad \text{for } i \in \{1,\, 3\}\ . \tag{4.6.16}$$

Moreover, by (4.6.15) being χ a well parameterized pattern, as stated in Remark 4.5.8, by Corollary 4.5.7 and by Remark 4.4.1, being $\chi_{|\Omega\setminus V_{t_2}} = \chi'_{|\Omega\setminus V_{t_2}}$, we find

$$\begin{aligned}\int_{E'_4} \varphi_{\chi'}(p,\, t)dp\, dt \;&\le\; \int_{E_4} \varphi_{\chi'}(p,\, t)dp\, dt \le \int_{E_4} \varphi_{\chi'_{|\Omega\setminus V_{t_2}}}(p,\, t)dp\, dt = \int_{E_4} \varphi_{\chi_{|\Omega\setminus V_{t_2}}}(p,\, t)dp\, dt \\ &=\; \int_{t_1}^{t_2} c_{\chi_{|\Omega\setminus V_{t_2}}}(t)dt < \int_{t_1}^{t_2} c_\chi(t)dt = \int_{E_4} \varphi_\chi(p,\, t)dp\, dt\ .\end{aligned}$$

We still need to show that

$$\int_{E'_2} \varphi_{\chi'}(p,\, t)dp\, dt \le \int_{E_2} \varphi_\chi(p,\, t)dp\, dt\ . \tag{4.6.17}$$

Actually, for $(p,\, t) \in E'_2$ we have by (4.6.13) that $p \in V_{t_2}$ and $t \ge t_1$. Then it is easy to see that

1. $[p]_{t+t_2-t_1} \subset [p]'_t$ where $[p]'_t$ is the χ'-vessel of the point p at time t;
2. $\sigma_{\chi'}(p) \le \max\{t_1,\, \sigma_\chi(p) + t_1 - t_2\}$.

As a consequence, being $\alpha - 1 < 0$, we get that

$$\varphi_{\chi'}(p,\, t) = |[p]'_t|^{\alpha-1}\mathbb{1}_{M_t(\chi')}(p) \le |[p]_{t+t_2-t_1}|^{\alpha-1}\mathbb{1}_{M_{t+t_2-t_1}(\chi')}(p) = \varphi_\chi(p,\, t+t_2-t_1)\ ,$$

from which (4.6.17) easily follows taking into account that $(p,\, t) \in E'_2$ gives $(p,\, t+t_2-t_1) \in E_2$. ∎

Lemma 4.6.3 *Let $\chi \in P_S(\Omega)$ be an optimal irrigation pattern. Then for any pair $(\gamma_1,\, \gamma_2)$ of flow curves of χ we have*

$$\gamma_1(t) \ne \gamma_2(s) \qquad \forall t,\, s > s(\gamma_1,\, \gamma_2)\ . \tag{4.6.18}$$

Proof. Let us assume by contradiction that there exist a pair of flow curves $(\gamma_1,\, \gamma_2)$ and real positive numbers $t_1, t_2 > t_0 = s(\gamma_1,\, \gamma_2)$ such that $\gamma_1(t_1) = \gamma_2(t_2)$. Now let us call $(V_t)_{0\le t<T_1}$ and $(W_t)_{0\le t<T_2}$ the χ-flow evolutions relative to γ_1 and to γ_2 respectively. Let us call $a = |V_{t_1}|$ and $b = |W_{t_2}|$. Let us define the following pattern by setting for all $p \in \Omega$ and $t \ge 0$

$$\chi_1(p,\, t) \;=\; \begin{cases} \chi(p,\, t) & \text{if } p \notin W_{t_2} \\ \gamma_1(t) & \text{if } t \in [0,\, t_1] \text{ and } p \in W_{t_2} \\ \chi(p,\, t + (t_2 - t_1)) & \text{if } t \ge t_1 \text{ and } p \in W_{t_2}\ . \end{cases} \tag{4.6.19}$$

Since the absorption position of a point $p \in \Omega$ is the value of $\chi(p,\, t)$ for a large t, it is easy to see that

$$i_{\chi_1} = i_\chi \tag{4.6.20}$$

and by consequence

$$\mu_{\chi_1} = \mu_\chi \,. \tag{4.6.21}$$

Let $\delta_1 = I_\alpha(\chi_1) - I_\alpha(\chi)$. One easily cecks that

$$\delta_1 \le \int_{t_0}^{t_1} ((|V_t| + b)^\alpha - |V_t|^\alpha)dt + \int_{t_0}^{t_2} ((|W_t| - b)^\alpha - |W_t|^\alpha)dt \,. \tag{4.6.22}$$

In a symmetric way, we set

$$\chi_2(p,\, t) \;=\; \begin{cases} \chi(p,\, t) & \text{if } p \notin V_{t_1} \\ \gamma_2(t) & \text{if } t \in [0,\, t_2] \text{ and } p \in V_{t_1} \\ \chi(p,\, t + (t_1 - t_2)) & \text{if } t \ge t_2 \text{ and } p \in V_{t_1} \,. \end{cases} \tag{4.6.23}$$

Analogously as before, one can deduce that

$$i_{\chi_2} = i_\chi \tag{4.6.24}$$

and by consequence

$$\mu_{\chi_2} = \mu_\chi \,. \tag{4.6.25}$$

Symmetrically, the difference $\delta_2 = I_\alpha(\chi_2) - I_\alpha(\chi)$ can be estimated by

$$\delta_2 \le \int_{t_0}^{t_1} ((|V_t| - a)^\alpha - |V_t|^\alpha)dt + \int_{t_0}^{t_2} ((|W_t| + a)^\alpha - |W_t|^\alpha)dt \,. \tag{4.6.26}$$

The contradiction will follow from the inequality

$$a\delta_1 + b\delta_2 < 0 \,, \tag{4.6.27}$$

which we are going to prove and which shows that χ is not optimal. Indeed, being $0 < \alpha < 1$, the concavity of the function x^α gives

$$\begin{aligned} \frac{\delta_1}{b} + \frac{\delta_2}{a} \;&\le\; \int_{t_0}^{t_1} \left[\frac{(|V_t| + b)^\alpha - |V_t|^\alpha}{b} - \frac{(|V_t|)^\alpha - (|V_t| - a)^\alpha}{a} \right] dt \\ &+\; \int_{t_0}^{t_2} \left[\frac{(|W_t| + a)^\alpha - |W_t|^\alpha}{a} - \frac{|W_t|^\alpha - (|W_t| - b)^\alpha}{b} \right] dt < 0 \,. \end{aligned}$$

∎

From lemmas 4.6.2 and 4.6.3 and Remark 4.6.3 the following theorem easily follows.

Theorem 4.6.1 *Let χ be an optimal irrigation pattern. Then χ is a simple pattern.*

4.7 Pruning Theorem

We shall devote this section to the proof of the following theorem.

Theorem 4.7.1 (Pruning Theorem) *Let $\varepsilon > 0$ and $\chi \in P_S(\Omega)$ be an irrigation pattern of finite cost without dispersion. Then there exists a finite number k of points $x_i \in F_\chi$ such that, denoting by χ_i the branchs of χ with source point x_i, we have*

$$\sum_{i=1}^{k} I_\alpha(\chi_i) < \varepsilon \tag{4.7.1}$$

$$(\mu_\chi - \sum_{i=1}^{k} \mu_{\chi_i})(I\!R^N) < \varepsilon \, . \tag{4.7.2}$$

We shall need to introduce some technical tools to be essentially used in this section. The following definition quantifies the short life-flows i.e. the flows such that most of their points are going to become absorbed in a short time.

Definition 4.7.1 *Let $\chi \in P_S(\Omega)$, $t > 0$ and $A \in \mathcal{F}_t(\chi)$. For fixed $\varepsilon > 0$ and $\delta > 0$ we shall say that a flow A at time t is a $(\varepsilon,\, \delta)$-flow if*

$$|\{p \in A \mid \sigma_\chi(p) > t + \delta\}| < \varepsilon \, . \tag{4.7.3}$$

In the following, we shall denote by $F_{\varepsilon,\delta}$ the set of all the $(\varepsilon,\, \delta)$-flows of the pattern χ at some time. For all $A \in F_{\varepsilon,\delta}$ there exists therefore by definition $t_A > 0$ such that A is an $(\varepsilon,\, \delta)$-flow at the time t_A.

Lemma 4.7.1 *Let χ be an irrigation pattern of finite cost and without dispersion. Then for all $\varepsilon > 0$ and $\delta > 0$, for a.e. $p \in \Omega$, there exists $A \in F_{\varepsilon,\delta}$ such that $p \in A$.*

Proof. We shall prove that the set of the points p for which the thesis is false is a negligible set. By [3, Proposition 2.1] the pattern χ is a non-spread pattern and therefore, by [3, Proposition 1.5], for a.e. p, if $t < \sigma_\chi(p)$, $\exists A \in \mathcal{F}_t(\chi)$ such that $p \in A$. So, let us fix $p \in \Omega$, $\varepsilon > 0$ and $\delta > 0$ and let us assume that

$$\forall t < \sigma_\chi(p),\ \forall A \in F_t(\chi),\ \text{such that } p \in A \ : \quad |\{q \in A \mid \sigma_\chi(q) > t + \delta\}| \geq \varepsilon \, . \tag{4.7.4}$$

By Remark 4.4.2 we know that, being χ a pattern of finite cost, for a.e. $p \in \Omega$, $\sigma_\chi(p) < +\infty$. Let $(t_n)_{n\in I\!N}$ be an increasing sequence of real positive numbers such that $\lim_{n\to+\infty} t_n = \sigma_\chi(p)$. So, for every $n \in I\!N$ let $A_n \in \mathcal{F}_{t_n}(\chi)$ be the flow at time t_n such that $p \in A_n$. Setting $X_n = \{q \in A_n \mid \sigma_\chi(q) > t_n + \delta\}$ we get, by (4.7.4), $|X_n| \geq \varepsilon$. Being the sequences $(A_n)_{n\in I\!N}$ and $(t_n)_{n\in I\!N}$ respectively decreasing and increasing, we get that $(X_n)_{n\in I\!N}$ is a decreasing sequence. So, if $X = \bigcap_{n\in I\!N} X_n$, $|X| \geq \varepsilon$. By construction X is made of points which belong to the same χ-vessel up to time $\sigma_\chi(p)$ and which are not absorbed before the time $\sigma_\chi(p) + \delta$. So X is contained in a flow at the time $\sigma_\chi(p)$ and, since $X \subset [p]_{\sigma_\chi(p)}$, $p \in D_\chi$, which is a negligible set being χ a pattern without dispersion. ∎

Lemma 4.7.2 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of finite cost without dispersion, $\mathcal{F}$ a set of χ-flows, $\overline{A} \in \mathcal{F}$, $\eta > 0$ such that*

$$\forall A \in \mathcal{F} \text{ such that } \overline{A} \subset A \ : \ |A \setminus \overline{A}| \le \eta \,. \tag{4.7.5}$$

Then there exists a measurable set T such that $|T| \le \eta$ and for all $A \in \mathcal{F}$, $\overline{A} \subset A$ we have $A \setminus \overline{A} \subset T$.

Proof. Let us set

$$D = \{t \in I\!R_+ \mid \exists A \in \mathcal{F} \text{ flow at the time } t, \text{ s.t. } \overline{A} \subset A\} \qquad \text{and} \qquad \bar{t} = \inf D \,. \tag{4.7.6}$$

Let us call A' the strict flow at time $\bar{t}$ which contains $\overline{A}$ and $T = A' \setminus \overline{A}$. Let $(t_n)_{n \in I\!N}$ be a sequence of elements of D such that $t_n \to \bar{t}$ for $n \to +\infty$. For any $n \in I\!N$ let us call A_n the flow at the time t_n such that $\overline{A} \subset A_n$. Being $t_n \in D$, we deduce that $A_n \in \mathcal{F}$ and $A_n \subset A'$. Therefore, by (4.7.5), $|A_n \setminus \overline{A}| \le \eta$ for all $n \in I\!N$. Then $|T| = |A' \setminus \overline{A}| = \sup_n |A_n \setminus \overline{A}| \le \eta$. For all $t \in D$ let $A \in \mathcal{F}$ be a flow at the time t which contains $\overline{A}$, we have $A \setminus \overline{A} \subset A' \setminus \overline{A} = T$. ∎

Lemma 4.7.3 *Let χ be an irrigation pattern of finite cost without dispersion, then for any given ε, δ and $\eta > 0$ there exists a finite set $\mathcal{A} \subset F_{\varepsilon,\delta}$ consisting of disjoint flows such that*

$$|\Omega \setminus \bigcup_{A \in \mathcal{A}} A| \le \eta \,. \tag{4.7.7}$$

Proof. Let us fix ε, δ and $\eta > 0$. Let us set $F_0 = F_{\varepsilon,\delta}$ and $s_1 = \sup_{A \in F_0} |A|$. Let us fix $A_1 \in F_0$ such that $|A_1| > s_1 - \frac{\eta}{2}$. By Lemma 4.7.2 applied to $\mathcal{F} = F_0$ we can find a measurable subset $T_1 \subset \Omega$, $|T_1| \le \frac{\eta}{2}$ such that for all $A \in F_{\varepsilon,\delta}$, $A_1 \subset A$, we have $A \setminus A_1 \subset T_1$. We shall recoursively procede to define other sets A_i. For all $k \in I\!N$, $k \ge 1$ let us assume to have already defined A_1, A_2, $\ldots$, A_k, and let us set $F^k = \{A \in F_{\varepsilon,\delta} \mid A \cap (\cup_{i=1}^k A_i) = \emptyset\}$. In the first part of the construction we shall assume that $\forall k \in I\!N$, $k \ge 1$, $F^k \ne \emptyset$, so we can set

$$s_k = \sup_{A \in F^{k-1}} |A| \,. \tag{4.7.8}$$

We chose $A_k \in F^{k-1}$ such that

$$|A_k| > s_k - \frac{\eta}{2^k} \,. \tag{4.7.9}$$

By Lemma 4.7.2 applied to $\mathcal{F} = F^{k-1}$ we can find a subset T_k of Ω, $|T_k| \le \frac{\eta}{2^k}$ such that for all $A \in F^{k-1}$, $A_k \subset A$ we have $A \setminus A_k \subset T_k$. If, on the contrary, we reach a k such that $F^k = \emptyset$, we put $s_n = 0$, $A_n = T_n = \emptyset$ for $n > k$. Let us remark that by construction $(|A_n|)_{n \in I\!N}$ is an infinitesimal sequence, so (4.7.9) gives

$$\lim_{n \to +\infty} s_n = 0 \,. \tag{4.7.10}$$

Setting $T = \bigcup_{n=1}^{\infty} T_n$ we have

$$|T| \le \sum_{n=1}^{\infty} |T_n| < \sum_{n=1}^{\infty} \frac{\eta}{2^n} = \eta \,. \tag{4.7.11}$$

We claim that

$$|(\Omega \setminus \bigcup_{i=1}^{\infty} A_i) \setminus T| = 0 \,. \tag{4.7.12}$$

Indeed, otherwise by Lemma 4.7.1 we could find $A \in F_{\varepsilon,\delta}$ such that $((\Omega \setminus \bigcup_{i=1}^{\infty} A_i) \setminus T) \cap A \neq \emptyset$ and so, in particular,

$$A \not\subset \bigcup_{i=1}^{\infty} A_i \,. \tag{4.7.13}$$

By (4.7.10) we can find $k \in I\!N$ such that $s_{k+1} < |A|$, so by construction (see (4.7.8)) $A \notin F^k$, therefore $A \cap (\bigcup_{i=1}^{k} A_i) \neq \emptyset$ and so, by (4.7.13) and Lemma 4.2.1, $A_i \subset A$ for some $i \le k$. Let h be the minimum of such values of i. By construction $A \in F^{h-1}$, $A_h \subset A$ and so $A \setminus A_h \subset T_h$, hence $A \subset A_h \cup T$, in contradiction to the choice of A. Therefore by (4.7.12), $|\Omega \setminus \bigcup_{i=1}^{\infty} A_i| \le |T| < \eta$. So for some $n \in I\!N$, $|\Omega \setminus \bigcup_{i=1}^{n} A_i| < \eta$. ∎

Lemma 4.7.4 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of finite cost and without dispersion, then given $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $\mathcal{A}$ consisting of disjoint flows in $F_{\delta^2,\delta}$*

$$\sum_{A \in \mathcal{A}} I_\alpha(A) \le \varepsilon \,. \tag{4.7.14}$$

Proof. Let $\varepsilon > 0$ be given, then by the absolute continuity of the integral we can find $\delta > 0$ such that the following properties hold true

$$\text{for any } A \subset \Omega \times I\!R_+ \text{ s.t. } |A| < \delta \quad : \quad \int_A \varphi_\chi(p,t)\,dpdt < \frac{\varepsilon}{3} \,; \tag{4.7.15}$$

$$\text{for any } B \subset \Omega \text{ s.t. } |B| < \delta \quad : \quad \int_{B \times I\!R_+} \varphi_\chi(p,t)\,dpdt < \frac{\varepsilon}{3} \,; \tag{4.7.16}$$

$$\int_{\{\varphi_\chi \ge \delta^{\alpha-1}\}} \varphi_\chi(p,t)\,dpdt < \frac{\varepsilon}{3} \,. \tag{4.7.17}$$

Let us fix a set $\mathcal{A}$ of disjoint elements of $F_{\delta^2,\delta}$. Then we can split $\mathcal{A}$ in the following two sets

$$\mathcal{A}_1 = \{A \in \mathcal{A} \mid |A| \ge \delta\} \quad \text{and} \quad \mathcal{A}_2 = \{A \in \mathcal{A} \mid |A| < \delta\} \,.$$

Let us evaluate the contribution given to the left-hand side of (4.7.14) by $\mathcal{A}_1$. To this aim, we shall split any flow $A \in \mathcal{A}_1$ in $A' = \{q \in A \mid \sigma_\chi(q) > t_A + \delta\}$ and $A'' = \{q \in A \mid \sigma_\chi(q) \le t_A + \delta\}$, so that

$$\sum_{A \in \mathcal{A}_1} I_\alpha(A) = \int_{\bigcup_{A \in \mathcal{A}_1} A' \times [t_A, +\infty[} \varphi_\chi(p,t)\,dpdt + \int_{\bigcup_{A \in \mathcal{A}_1} A'' \times [t_A, +\infty[} \varphi_\chi(p,t)\,dpdt \,. \tag{4.7.18}$$

Taking into account that, by the definition of $\mathcal{A}_1$, we have $\mathrm{card}(\mathcal{A}_1) \le \delta^{-1}$ and that, by the definition of $F_{\delta^2,\delta}$ flow, $|A'| \le \delta^2$ we have

$$| \bigcup_{A\in\mathcal{A}_1} A'| \le \mathrm{card}(\mathcal{A}_1)\delta^2 \le \delta \; ,$$

and therefore, by (4.7.16),

$$\int_{\bigcup_{A\in\mathcal{A}_1} A'\times[t_A,\, +\infty[} \varphi_\chi(p,\, t)dpdt < \frac{\varepsilon}{3} \; . \tag{4.7.19}$$

On the other hand, being the elements of $\mathcal{A}_1$ disjoint sets and $|\Omega| = 1$,

$$| \bigcup_{A\in\mathcal{A}_1} A'' \times [t_A,\, t_A + \delta[| \le \delta \sum_{A\in\mathcal{A}_1} |A''| \le \delta \; .$$

Therefore from (4.7.15)

$$\int_{\bigcup_{A\in\mathcal{A}_1} A''\times[t_A,\, +\infty[} \varphi_\chi(p,\, t)dpdt = \int_{\bigcup_{A\in\mathcal{A}_1} A''\times[t_A,\, t_A+\delta[} \varphi_\chi(p,\, t)dpdt < \frac{\varepsilon}{3} \; . \tag{4.7.20}$$

By combining (4.7.18) with (4.7.19) and (4.7.20) we get

$$\sum_{A\in\mathcal{A}_1} I_\alpha(A) < \frac{2}{3}\varepsilon \; . \tag{4.7.21}$$

Finally, if $A \in \mathcal{A}_2$ and $(p,\, t) \in A \times [t_A,\, +\infty[$ we have $\varphi_\chi(p,\, t) = |[p]_t|^{\alpha-1} \ge |[p]_{t_A}|^{\alpha-1} \ge \delta^{\alpha-1}$ or $\varphi_\chi(p,\, t) = 0$ if p is absorbed at the time t. So by (4.7.17)

$$\sum_{A\in\mathcal{A}_2} I_\alpha(A) = \int_{\bigcup_{A\in\mathcal{A}_2} A\times[t_A,\, +\infty[} \varphi_\chi(p,\, t)dpdt \le \int_{\{\varphi_\chi \ge \delta^{\alpha-1}\}} \varphi_\chi(p,\, t)dpdt < \frac{\varepsilon}{3} \; . \tag{4.7.22}$$

The thesis trivially follows from (4.7.21) and (4.7.22). ∎

Proof of Theorem 4.7.1. Let us fix $\varepsilon > 0$ and consequently $\delta > 0$ as in the thesis of Lemma 4.7.4. By Lemma 4.7.3 we can find a finite set $\mathcal{A} = \{A_1,\, A_2,\, \ldots,\, A_k\}$ of disjoint elements of $F_{\delta^2,\delta}$ such that $|\Omega \setminus \bigcup_{i=1}^k A_i| \le \varepsilon$. The thesis follows from Lemma 4.7.4, see also Definition 4.6.6, taking, for all $i \in \{1,\, 2, \ldots,\, k\}$, χ_i as the branch which starts from $x_i = \chi(p, t_{A_i})$ for $p \in A_i$. ∎

4.8 Rearranged patterns

Definition 4.8.1 *Let χ be an irrigation pattern with source point S. We define $\tilde{\chi} : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ by setting, for a.e. $p \in \Omega$, and for all $t \ge 0$*

$$\tilde{\chi}(p,\, t) = \eta_p(t) \; , \tag{4.8.1}$$

where η_p is defined by (4.5.2). We shall call $\tilde{\chi}$ the stretched pattern *of χ on $\mathbb{R}_+$.*

Remark 4.8.1 *If $\chi \in P_S(\Omega)$, being, for a.e. $p \in \Omega$, η_p a 1-Lipschitz function, we have $\tilde{\chi} \in P_0(\Omega)$. Moreover, if χ is a well parametrized pattern then, for a.e. $p \in \Omega$, $\tilde{\chi}(p, \cdot)$ is the identity function on $[0, \sigma_\chi(p)]$ see Remark 4.5.3.*

For every $p \in \Omega$ and for all $t \in \mathbb{R}_+$ we shall denote by $\tilde{[p]}_t$ the $\tilde{\chi}$-vessel at time t which contains the point p;

Lemma 4.8.1 *Let χ be an irrigation pattern and let $\tilde{\chi}$ be the stretched pattern of χ. Then the following properties hold true*

$$\text{for a.e. } p \in \Omega \text{ and for all } t \geq 0 \quad : \quad [p]_t \subset \tilde{[p]}_t \, , \tag{4.8.2}$$

$$\text{for a.e. } p \in \Omega \quad : \quad \sigma_{\tilde{\chi}}(p) = \sigma_\chi(p) \, . \tag{4.8.3}$$

PROOF. The first part of the thesis is an easy consequence of Remark 4.5.4. To prove the second part let us remark that, by definition, for a.e. $p \in \Omega$ we have $\sigma_{\tilde{\chi}}(p) \leq \sigma_\chi(p)$. The reverse inequality follows from Remark 4.5.5 ∎

Corollary 4.8.1 *Let χ be an irrigation pattern and let $\tilde{\chi}$ be the stretched pattern of χ. Then*

$$\text{for all } t \geq 0 \quad : \quad M_t(\tilde{\chi}) = M_t(\chi) \quad , \quad A_t(\tilde{\chi}) = A_t(\chi) \quad \text{and} \quad \varphi_{\tilde{\chi}} \leq \varphi_\chi \, . \tag{4.8.4}$$

Corollary 4.8.2 *Let χ be an irrigation pattern and let $\tilde{\chi}$ be the stretched pattern of χ. Then*

$$I_\alpha(\tilde{\chi}) \leq I_\alpha(\chi) \, . \tag{4.8.5}$$

For any $p \geq 1$ we recall the definition of Kantorovitch-Wasserstein distance of index p.

Definition 4.8.2 *Let $p \geq 1$ and let μ, ν be two probability measures. We define the Kantorovitch-Wasserstein distance of index p between μ and ν by*

$$d_p(\mu, \nu) = \left(\min_\sigma \int_{\Omega \times \Omega} |x - y|^p d\sigma \right)^{\frac{1}{p}} \, ,$$

where the minimum is taken on all the transport plans σ which lead μ to ν, i.e. measures on $\Omega \times \Omega$ such that their push forward measures by the first and the second projection on Ω respectively are μ and ν ($\pi_{1\#}\sigma = \mu$ and $\pi_{2\#}\sigma = \nu$) (see [1] for more details).

Taking into account that $\tilde{\chi}$ maximizes the distance of the absorbed points from the source one can easily prove the following lemma.

Lemma 4.8.2 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern and let $\tilde{\chi}$ be the stretched pattern of χ. Then we have*

$$\forall q \in \Omega : \quad |i_\chi(q) - S| \leq |i_{\tilde{\chi}}(q) - 0| \, . \tag{4.8.6}$$

PROOF. Given $q \in \Omega$, we have

$$|i_\chi(q) - S| \leq \int_0^{\sigma_\chi(q)} |\frac{\partial \chi}{\partial t}(q,\, t)|dt = \eta_q(\sigma_\chi(q)) = \tilde{\chi}(q,\, \sigma_{\tilde{\chi}}(q)) = i_{\tilde{\chi}}(q) = |i_{\tilde{\chi}}(q) - 0|\ .$$

∎

Corollary 4.8.3 *Let χ be an irrigation pattern and let $\tilde{\chi}$ be the stretched pattern of χ. Then for all $p \geq 1$ we have*

(4.8.7) $$d_p(\mu_\chi,\, \delta_S) \leq d_p(\mu_{\tilde{\chi}},\, \delta_0)\ .$$

Lemma 4.8.3 *Let χ be an irrigation pattern and let $\tilde{\chi}$ be the stretched pattern of χ. Then for all $p \geq 1$ we have*

(4.8.8) $$d_p(\mu_{\tilde{\chi}},\, \delta_0) \leq I_{\frac{1}{p}}(\tilde{\chi})\ .$$

PROOF. Let us fix $p \geq 1$. We must prove that the Kantorovitch-Wasserstein distance of index p

$$d = d_p(\mu_{\tilde{\chi}}, \delta_0) = \left(\int_0^{+\infty} s^p d\mu_{\tilde{\chi}}(s)\right)^{\frac{1}{p}}$$

of $\mu_{\tilde{\chi}}$ from the source point 0 of $\tilde{\chi}$ is less or equal to the cost

$$c = I_{\frac{1}{p}}(\tilde{\chi}) = \int_0^{+\infty} \left(\int_t^{+\infty} d\mu_{\tilde{\chi}}(s)\right)^{\frac{1}{p}} dt\ .$$

We can assume without any restriction that μ_χ is a discrete measure and then recover the general case by weak continuity. So let $\mu_\chi = \sum_{i=1}^k m_i \delta_{s_i}$, where $s_0 = 0 < s_1 < \ldots < s_k$. Then

$$d = \left(\sum_{j=1}^k m_j s_j^p\right)^{\frac{1}{p}} \quad \text{and} \quad c = \sum_{j=1}^k \left(\sum_{i=j}^k m_i\right)^{\frac{1}{p}} (s_j - s_{j-1})\ .$$

We shall prove that $d \leq c$ by induction on k. Let us assume the statement true for $k-1$ masses and let us estimate the variation of the distance d and of the cost c with respect to the position s_k of the last mass m_k. We get

$$\frac{\partial}{\partial s_k} d = \frac{1}{p}\left(\sum_{j=1}^k m_j\, s_j^p\right)^{\frac{1}{p}-1} p\, m_k\, s_k^{p-1} = \left(\frac{\sum_{j=1}^k m_j s_j^p}{m_k s_k^p}\right)^{\frac{1}{p}-1} m_k^{\frac{1}{p}} \leq m_k^{\frac{1}{p}} = \frac{\partial}{\partial s_k} c\ .$$

Therefore it is sufficient to prove that the inequality $d \leq c$ is true when the distance $s_k - s_{k-1}$ from the masses m_k and m_{k-1} is zero. In such a case, the number of the masses reduces to $k-1$ and the inequality is given by the induction assumption. ∎

By combining in order Corollary 4.8.3, Lemma 4.8.3 and Corollary 4.8.2 we get the following theorem.

Theorem 4.8.1 *Let χ be an irrigation pattern. Then for all $p \geq 1$ we have*

$$d_p(\mu_\chi, \delta_S) \leq I_{\frac{1}{p}}(\chi)\ . \tag{4.8.9}$$

Definition 4.8.3 *Let χ be an irrigation pattern with source point S and $\sigma : \Omega \to \mathbb{R}_+$ be a measurable function. Let us consider the function $\hat{\chi} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$ defined by setting for a.e. $p \in \Omega$ and for all $t \geq 0$*

$$\hat{\chi}(p, t) = \begin{cases} \chi(p, t) & \text{if } t < \sigma(p) \\ \chi(p, \sigma(p)) & \text{if } t \geq \sigma(p)\ . \end{cases} \tag{4.8.10}$$

Being $\chi \in P_S(\Omega)$ it is easy to check that $\hat{\chi} \in P_S(\Omega)$. We shall say that $\hat{\chi}$ is the "σ-forced absorption pattern" of χ.

Remark 4.8.2 *Let $\Omega' \subset \Omega$. By Definition 4.8.3 we can say that the bunch of a subpattern $\chi' = \chi_{|\Omega'}$ and a trivial pattern gives a σ-forced absorption pattern of χ trough the function $\sigma = \sigma_\chi \cdot \mathbf{I}_{\Omega'}$. Equivalently, we can say that any subpattern χ', modulo the bunching with a trivial pattern, is the σ-forced absorption pattern of χ trough a characteristic function.*

Therefore, in general, all what we shall say about σ-forced absorption patterns of χ still holds true for subpatterns of χ modulo the bunching with a trivial pattern.

Remark 4.8.3 *Let χ be a well parameterized (or simple) pattern. Then for any measurable function $\sigma : \Omega \to \mathbb{R}_+$ the σ forced absorption pattern of χ is a well parameterized (or simple) pattern.*

Remark 4.8.4 *Both the assumptions on χ of being a well parameterized or a simple pattern guarantee that for any measurable function $\sigma : \Omega \to \mathbb{R}_+$ the σ forced absorption pattern of χ is a partitioned pattern. However, the property of a pattern of being partitioned is not inherited, in general, by its forced absorption patterns as shown by the following example.*

Example 4.8.1 *Let us consider $\Omega = [0, 1]$, $S = 0$ and the pattern χ defined by setting for all $p \in [0, 1]$ and for all $t \in \mathbb{R}_+$*

$$\chi(p, t) = \begin{cases} t & \text{if } t < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{2} \\ \min\{t-1, 1\} & \text{if } t > \frac{3}{2}\ . \end{cases} \tag{4.8.11}$$

For any $t \geq 0$ and for all $p \in \Omega$, $[p]_t = \Omega$ and so χ is a partitioned pattern. Let us take a subset $\Omega' \subset \Omega$ and $\sigma = 2 - \mathbb{I}_{\Omega'}$. Then the σ-forced absorption pattern $\hat{\chi}$ of χ is not a partitioned pattern. Indeed, let us take $p \in \Omega'$ and $q \in \Omega \setminus \Omega'$, then we have, for $t = 1$, $\hat{\chi}_p = \hat{\chi}_q$ on $[0, t]$, $p \in A_t(\hat{\chi})$, $q \in M_t(\hat{\chi})$ but $\sigma_{\hat{\chi}}(p) = \frac{1}{2} < t$.

Lemma 4.8.4 *Let χ and σ be given and let $\hat{\chi}$ be the σ-forced absorption pattern of χ. Then*

$$\sigma_{\hat{\chi}} \leq \inf\{\sigma_\chi, \sigma\}\ . \tag{4.8.12}$$

Moreover if χ, and therefore by Remark 4.8.3 $\hat{\chi}$, is well parameterized or simple then

$$\sigma_{\hat{\chi}} = \inf\{\sigma_\chi, \sigma\}\ . \tag{4.8.13}$$

PROOF. Inequality (4.8.12) follows from (4.8.10). Moreover, if the inequality is strict, $\hat{\chi}(p, \cdot)$ is constant on $[\sigma_{\hat{\chi}}, \inf\{\sigma_\chi, \sigma\}]$ for a set of positive measure of points $p \in \Omega$. This case is avoided when χ is well parameterized or simple. ∎

Corollary 4.8.4 *Let χ and σ be given and let $\hat{\chi}$ be the σ-forced absorption pattern of χ. Then*

$$M_t(\hat{\chi}) \subset M_t(\chi) \, . \tag{4.8.14}$$

Lemma 4.8.5 *Let χ be a well parameterized pattern and let $\sigma : \Omega \to I\!R_+$ be a measurable function. Let $\hat{\chi}$ be the σ-forced absorption pattern of χ. Then for any $t \in I\!R_+$ and for any $A \in \mathcal{F}_t(\hat{\chi})$ there exists $A' \in \mathcal{F}_t(\chi)$ such that $A \subset A'$.*

PROOF. Let us fix $t \in I\!R_+$ and $A \in \mathcal{F}_t(\hat{\chi})$. By definition of $\mathcal{F}_t(\hat{\chi})$, there exists $B \subset A$, $|B| > 0$, such that, for all $p \in B$, $\sigma_{\hat{\chi}}(p) > t$. Let us fix $p \in B$, then $A = [\hat{p}]_t$. Moreover, by (4.8.12), for any $q \in B$ we have $\sigma(q) \geq \sigma_{\hat{\chi}}(q) > t$ and therefore, by Definition 4.8.3, $\hat{\chi}_q = \chi_q$ on $[0, t]$. By consequence we have $B \subset [p]_t$ and, by(4.8.12), $B \subset M_t(\chi)$. Therefore, set $A' = [p]_t$, we get that A' is a χ-flow at the time t. Now we must prove that also $A \setminus B \subset [p]_t$. This is the step which requires $\hat{\chi}$ to be partitioned, as guaranteed by Remark 4.8.4. Indeed, for any $q \in A \setminus B \subset [\hat{p}]_t$ we have $\hat{\chi}_q = \hat{\chi}_p$ on $[0, t]$ and therefore $\sigma_{\hat{\chi}}(q) = t$. By (4.8.12) it follows that $\sigma(q) \geq \sigma_{\hat{\chi}}(q) \geq t$ for all $q \in A \setminus B$. By consequence, for any $q \in A \setminus B$, $\hat{\chi}_q = \chi_q$ on $[0, t[$ and, by continuity, on $[0, t]$. Being, for all $q \in A \setminus B$, $q \in [\hat{p}]_t$ and $\hat{\chi}_q = \chi_q$ on $[0, t]$, we have $A \setminus B \subset [p]_t$. ∎

As shown by the following example the hypotheses of the good parameterization of the pattern χ in Lemma 4.8.5, which can be replaced by χ being simple, can not be completely removed.

Example 4.8.2 *Let us consider $\Omega = [0, 1]$ endowed with the Lebesgue measure. Let us set*

$$\chi(p, t) \;=\; \begin{cases} \left(t - \frac{2}{3}\right)^+ & \text{if } p \in \left[0, \frac{1}{2}\right[\\ \left(t - \frac{1}{3}\right)^+ & \text{if } p \in \left[\frac{1}{2}, 1\right] \end{cases} , \tag{4.8.15}$$

and $\sigma = \mathbf{I}_{[0, \frac{1}{2}[}$. Let $\hat{\chi}$ be the σ-forced absorption patter of χ. Then set $p = \frac{1}{2}$ and $t = \frac{1}{2}$ we have that $[p]_t = \left[\frac{1}{2}, 1\right] \in \mathcal{F}_t(\chi)$, $[\hat{p}]_t = [0, 1] \in \mathcal{F}_t(\hat{\chi})$ and $[\hat{p}]_t \not\subset [p]_t$.

Remark 4.8.5 *In other terms, the above lemma states that, for a.e. $p \in \Omega$ and for all $t \geq 0$, if $[\hat{p}]_t$ is a $\hat{\chi}$-flow at the time t then*

$$[\hat{p}]_t \subset [p]_t \tag{4.8.16}$$

and $[p]_t$ is a χ-flow at the time t.

Lemma 4.8.6 *Let χ be a non-spread irrigation pattern, then for any measurable function $\sigma : \Omega \to I\!R_+$ the σ-forced absorption pattern of χ is also non-spread.*

PROOF. We just remark that for any $t > 0$ the function σ must be strictly greater than t on the spread flow $S_t(\hat{\chi})$. So, by Definition 4.8.3, $S_t(\hat{\chi})$ is actually contained in the spread flow $S_t(\chi)$. Being χ a non-spread pattern $S_t(\chi)$, and therefore $S_t(\hat{\chi})$, must be a negligible set. ∎

By Remark 4.8.2, Lemma 4.8.6 admits the following corollary.

Corollary 4.8.5 *Let χ be a non-spread irrigation pattern, then any subpattern χ' of χ is also non-spread.*

Corollary 4.8.6 *Let χ be a well parameterized irrigation pattern and let σ be a measurable function. Let $\hat{\chi}$ be the σ-forced absorption pattern of χ. Then $I_\alpha(\hat{\chi}) \leq I_\alpha(\chi)$.*

PROOF. We can assume, without any restriction, that χ is a pattern with a finite cost and therefore a non-spread pattern. By Lemma 4.8.6 $\hat{\chi}$ is also a non spread pattern, then the thesis easily follows by Corollary 4.5.6. ∎

Definition 4.8.4 *Let χ be an irrigation pattern with source point S. We shall call "rearranged pattern" of χ the function $\chi^* : \Omega \times I\!R_+ \to I\!R_+$ s.t., for a.e. $p \in \Omega$ and for all $t \geq 0$*

$$\chi^*(p,\, t) = \min\{t,\, |S - i_\chi(p)|\}\ . \tag{4.8.17}$$

Remark 4.8.6 *By the definition of χ^* it follows that $\chi^* \in P_0(\Omega)$ and*

$$\forall p \in \Omega \quad : \quad |0 - i_{\chi^*}(p)| = i_{\chi^*}(p) = |S - i_\chi(p)| = \sigma_{\chi^*}(p)\ .$$

Therefore μ_{χ^*} is the image measure of μ_χ trough the function $\psi : p \mapsto |S - i_\chi(p)|$.

Corollary 4.8.7 *Let χ be an irrigation pattern, then for any $p \geq 1$*

$$d_p(\mu_{\chi^*}, \delta_0) = d_p(\mu_\chi, \delta_S)\ . \tag{4.8.18}$$

Remark 4.8.7 *We can obtain the rearrangement χ^* of an irrigation pattern χ by*

1. *the good parameterization $\overline{\chi}$ (see Definition 4.5.2);*
2. *the stretched pattern $\tilde{\chi}$ of $\overline{\chi}$ on $I\!R_+$ (see Definition 4.8.1);*
3. *the σ-forced absorption pattern of $\tilde{\chi}$ (see Definition 4.8.3) where the function σ which forces the absorption is defined by setting, for all $p \in \Omega$, $\sigma(p) = |S - i_\chi(p)|$.*

Lemma 4.8.7 *Let χ be an irrigation pattern then for any $\alpha \in]0,\, 1[$:*

$$I_\alpha(\chi^*) \leq I_\alpha(\chi)\ .$$

PROOF. The proof easily follows from Remark 4.8.7. Indeed, we get the thesis by applying in sequence corollaries 4.8.6, 4.8.2 and 4.5.5. ∎

4.9 Support of a simple pattern

The set F_χ which contains the trajectories of the flows can be considered as the support of a pattern χ. Under some regularity conditions, it can also be considered to be equipped with some structure. By Corollary 4.3.2, if χ is non-spread, F_χ is the countable union of supports of rectifiable curves.

Definition 4.9.1 *We shall say that the sequence of open simple curves $(\gamma_n)_{n\in I\!N}$, each one with left extreme in $S \in I\!R^N$ and defined in $[0, T_n[$, satisfies the "separation property" if*

$$\forall m \neq n \quad \gamma_m(s) \neq \gamma_n(t) \quad \forall s, t > s(\gamma_m, \gamma_n)\,. \tag{4.9.1}$$

Definition 4.9.2 *Let $F \subset I\!R^N$ and $S \in I\!R^N$. We shall say that the set F is a "branching set" with respect to S if for all $x \in F$ there exists a unique simple curve which joins x with S.*

Remark 4.9.1 *If $(\gamma_n)_{n\in I\!N}$ satisfies the separation property, then $F = \bigcup_{n\in I\!N} \mathrm{supp}(\gamma_n)$ is a branching set with respect to S.*

Remark 4.9.2 *Let $\chi \in P_S(\Omega)$ be a simple non-spread pattern. Then, by Corollary 4.3.2 and Remark 4.6.3, the support of χ is the countable union of a sequence $(\gamma_n)_{n\in I\!N}$ of flow curves which satisfies the separation property. In particular, the support of χ is a branching set with respect to S.*

When χ is a non-spread simple pattern three structures are induced on the support F_χ: a partial order, a "time" function and a "quantity" function as will be specified in the following definitions.

Definition 4.9.3 *Let $\chi \in P_S(\Omega)$ be a simple non-spread pattern. Then for any $x \in F_\chi$, $x = \chi(p, t)$ we shall refer to t, which, being χ simple, is determined by x (see Remark 4.6.4), as to the "time" of the point x and we shall denote it by t_x.*

Remark 4.9.3 *For all $x \in F_\chi$ the time t_x represents the "geodetical" distance of x from S along the unique simple curve which joins x to S if and only if χ is well parameterized.*

Definition 4.9.4 *Let $\chi \in P_S(\Omega)$ be a simple non-spread pattern. Let $x = \chi(p, s)$ and $y = \chi(q,t)$ be two points of the support F_χ. We shall say that x precedes y on F_χ and we shall write $x \preceq_\chi y$ if $[q]_t \subset [p]_s$ and $s \leq t$.*

Definition 4.9.5 *Let χ be a simple pattern. The function P_χ defined on F_χ by setting for all $x \in F_\chi$, $x = \chi(p, t)$, $P_\chi(x) = |[p]_t|$ will be called* quantity function *of χ.*

The following lemma can be trivially proved.

Lemma 4.9.1 *Let $\chi \in P_S(\Omega)$ be a simple non-spread pattern. Then for all $x \in F_\chi$*

$$\sum_{i=1}^{n} P_\chi(x_i) \leq P_\chi(x) \tag{4.9.2}$$

holds true for any finite sequence $x_1, x_2, \ldots, x_n$ of non-comparable points of F_χ such that $x \preceq_\chi x_i$ $\forall i \in \{1, \ldots, n\}$.

Remark 4.9.4 *Let $\chi \in P_S(\Omega)$ be a simple non-spread pattern. Then F_χ, equipped with the partial order $\preceq_\chi$, the time function and the quantity function, is invariant modulo equivalence.*

One can wonder if any given branching set can be seen as the support of an equivalence class of patterns. Let $S \in \mathbb{R}^N$ be a given point and let $F \subset \mathbb{R}^N$ be a branching set with respect to S. Then we can define on F a partial order as follows.

Definition 4.9.6 *We shall say that $x \preceq y$ if the simple curve which joins y to S passes through the point x. The partial order $\preceq$ will be called* branching order.

A time function can also be assigned on F or it can be automatically defined by the length in such a way to lead to a well parameterized pattern.

Definition 4.9.7 *For any $x \in F$, we shall call "time" of the point x the parameter t_x which represents the length of the unique arc of curve which joins x to S. For any $t \geq 0$ we set $\mathcal{F}_t = \{x \in F \mid t_x = t\}$.*

Definition 4.9.8 *Let F be a branching set and P be a measurable positive function defined on F. We shall say that P satisfies the* superadditivity property *on F if for all $x \in F$*

$$\sum_{i=1}^{n} P(x_i) \leq P(x) \tag{4.9.3}$$

holds true for any finite sequence $x_1, x_2, \ldots, x_n$ of non-comparable points of F such that $x \preceq x_i$ $\forall i \in \{1, \ldots, n\}$.

If we take as the parameter t the values of the time function, as the flows at a time t the points $x \in F$ such that $t_x = t$ and as their measure the values of the quantity function, the same argument already used to prove [3, Theorem 7.1] can be employed to prove the following theorem. The proof follows the same framework as in [3, Section 7], the easy changes are left to the reader.

Theorem 4.9.1 *Let $S \in \mathbb{R}^N$ and let F be a branching set with respect to S. Let $P : F \to \mathbb{R}_+$ be a measurable positive function which satisfies the superadditivity property. Then there exists a simple and well parameterized pattern χ, with source point S, such that $F = F_\chi$ and P is the quantity function P_χ of the pattern.*

The α-cost of a non-spread simple well parameterized pattern can be evaluated by an integral on its support. This property is a particular case of the following lemma.

Lemma 4.9.2 *Let χ be a non-spread simple and well parameterized pattern. Then, if $\mathcal{F}_t$ is used as in Definition 4.9.7 for $F = F_\chi$, for any measurable function $f : F_\chi \to I\!R_+$ we have*

$$\int_{F_\chi} f(x)d_{\mathcal{H}_1}(x) = \int_0^{+\infty} \sum_{x\in\mathcal{F}_t(\chi)} f(x)dt\ . \tag{4.9.4}$$

PROOF. By Corollary 4.3.2, we can find a sequence $(\gamma_n)_{n\in I\!N}$ of simple flow curves parameterized with respect to the length such that $F_\chi = \bigcup_{n\in I\!N} \operatorname{supp}(\gamma_n)$. Let us assume f positive and, for all $k \geq 1$, let us set $F^k = \bigcup_{i=1}^k \operatorname{supp}(\gamma_i)$ and $\mathcal{F}_t^k = \mathcal{F}_t \cap F^k$. Let us remark that, being χ well parameterized, for all $k \geq 1$ we have

$$\mathcal{F}_t^k = \{x \in F^k \mid t_x = t\} = \{\gamma_i(t) \mid 1 \leq i \leq k\}\ . \tag{4.9.5}$$

First we shall prove that (4.9.4) holds true in the case in which F_χ and $\mathcal{F}_t(\chi)$ are respectively replaced by F^k and $\mathcal{F}_t^k$ for some $k \geq 1$. We shall proceed by induction on k. The case $k = 1$ is obvious. Let us assume that (4.9.4) holds true for F^k and let us prove it for F^{k+1}. We can obviously assume that γ_{k+1} has a support which is not completely contained in F^k. Let us call $\bar{t}$ the biggest separation time among γ_{k+1} and the other curves γ_i, $i \leq k$. Then

$$\int_{F^{k+1}} f(x)d_{\mathcal{H}_1}(x) = \int_{F^k} f(x)d_{\mathcal{H}_1}(x) + \int_{\{\gamma_{k+1}(s)\,|\,s\geq\bar{t}\}} f(x)d_{\mathcal{H}_1}(x)\ ,$$

which by the induction hypotheses gives

$$\int_{F^{k+1}} f(x)d_{\mathcal{H}_1}(x) = \int_0^{+\infty} \sum_{x\in\mathcal{F}_t^k} f(x)dt + \int_{\bar{t}}^{+\infty} f(\gamma_{k+1}(t))dt\ . \tag{4.9.6}$$

Taking into account (4.9.5), we have that, for $t < \bar{t}$, $\mathcal{F}_t^{k+1} = \mathcal{F}_t^k$, while, for $t \geq \bar{t}$, $\mathcal{F}_t^{k+1} = \mathcal{F}_t^k \cup \{\gamma_{k+1}(t)\}$, so by (4.9.6) we get

$$\int_{F^{k+1}} f(x)d_{\mathcal{H}_1}(x) = \int_0^{+\infty} \sum_{x\in\mathcal{F}_t^{k+1}} f(x)dt\ . \tag{4.9.7}$$

Being f a positive measurable function, by (4.9.7) we have by monotone convergence

$$\int_{F_\chi} f(x)d_{\mathcal{H}_1}(x) = \sup_n \int_{F^n} f(x)d_{\mathcal{H}_1}(x) = \sup_n \int_0^{+\infty} \sum_{x\in\mathcal{F}_t^n} f(x)dt = \int_0^{+\infty} \sum_{x\in\mathcal{F}_t(\chi)} f(x)dt\ .$$

∎

Theorem 4.9.2 *Let χ be a non-spread simple and well parameterized pattern. Then for any* $\alpha \in]0, 1[$

$$I_\alpha(\chi) = \int_{F_\chi} |P_\chi(x)|^\alpha d_{\mathcal{H}_1}(x)\,. \tag{4.9.8}$$

PROOF. Being χ a well parameterized pattern, one can apply Corollary 4.5.6 so that $I_\alpha(\chi) = \int_0^{+\infty} \sum_{A\in\mathcal{F}_t(\chi)} |A|^\alpha\, dt$. So, the thesis follows by applying Lemma 4.9.2 to the function $f(x) = |P_\chi(x)|^\alpha$. ∎

4.10 Index of the main notation in order of appearance

- $(\Omega, |\cdot|)$ a nonatomic probability space
- $\chi : \Omega \times I\!R^+ \to I\!R^N$ = set of fibers
- $\chi(p,t) \in I\!R^N$ position of the point $p \in \Omega$ at the time t
- $\chi_p = t \mapsto \chi(p,t)$ = fiber of p
- $\mathbf{C}_S(\Omega)$ = set of sets of fibers of Ω
- $\mathbf{P}_S(\Omega)$ = set of all irrigation patterns, i.e. the set of all the measurable sets of fibers of Ω
- $\chi_{|\Omega'}$ = subpattern of χ defined on $\Omega' \subset \Omega$
- $[p]_t$ = equivalence class of p under the equivalence $p \simeq_t q$ if $\chi_p(s) = \chi_q(s)$ for all $s \in [0,t]$
- *χ-vessels* = class of equivalence at time t under $\simeq_t$
- $\mathcal{V}_t(\chi) = \Omega/\simeq_t$ = set of χ-vessels at time t
- $s(\gamma_1, \gamma_2)$ = separation time of the two curves γ_1 and γ_2, see Definition 4.2.2
- $s_\chi(p, q)$ = separation time of the two points p and q, see Definition 4.2.3
- $[p]_t^s$ = equivalence class of p under the equivalence $p \simeq_t^s q$ if there exists $\varepsilon > 0$ s.t. $p \simeq_{t+\varepsilon} q$
- *strict χ-vessels* = class of equivalence at time t under $\simeq_t^s$
- $\mathcal{V}_t^s(\chi) = \Omega/\simeq_t^s$ = set of the strict χ-vessels at time t
- $\sigma_\chi(p) = \inf\{t \in I\!R_+ \mid \chi_p(s)$ is constant on $[t, +\infty[\}$: absorption (stopping) time of p, p is absorbed at time t if $\sigma_\chi(p) \le t$
- $X \subset \Omega$ is an absorbed set at time t if $\sigma_\chi(p) \le t$ for a.e. $p \in X$, see Definition 4.2.5

- $A_t(\chi)$ = set of the points of Ω which are absorbed at time t
- $A_\chi = \bigcup_{t>0} A_t(\chi)$ = set of the absorbed points
- $M_t(\chi) = \Omega \setminus A_t(\chi)$ = set of the points of Ω that at time t are still moving
- χ-flow = non absorbed χ-vessel (has positive measure in Ω), see Definition 4.2.6
- $\mathcal{F}_t(\chi)$ = set of χ-flows at time t
- $F_t(\chi) = \bigcup_{A\in\mathcal{F}_t(\chi)} A$ = union of the χ-flows at time t
- $S_t(\chi) = M_t(\chi) \setminus F_t(\chi)$= spread flow at time t, see Definition 4.2.8
- $i_\chi : A_\chi \to I\!R^N$ = irrigation function defined by $i_\chi(p) = \chi(p, \sigma_\chi(p))$, see Definition 4.2.9
- μ_χ = irrigation measure induced by the pattern χ by setting $\mu_\chi(A) = |i_\chi^{-1}(A)|$ for any Borel set $A \subset I\!R^N$, see Definition 4.2.10
- $D_\gamma = \{p \in \Omega \mid p \text{ follows } \gamma\}$, see Definition 4.3.3
- $\tau_\gamma = \sup_{D_\gamma} \sigma_\chi$, see Definition 4.3.3
- $F_\chi = \{x \in I\!R^N \mid \exists t > 0, \exists A \in \mathcal{F}_t(\chi) \text{ s.t. } x = \chi(p, t), p \in A\}$ = flow zone or support of χ, see Definition 4.3.6
- $D_\chi = \{p \in \Omega \mid p \in \mathcal{F}_{\sigma_\chi(p)}(\chi)\}$ = dispersion of the pattern χ, see Definition 4.3.7
- $\varphi_\chi(p, t) = |[p]_t|^{\alpha-1}\mathbf{1}_{M_t(\chi)}(p)$
- $c_\chi(t) = \int_\Omega \varphi_\chi(p, t)dp = \int_{M_t(\chi)} |[p]_t|^{\alpha-1}dp$
- $\bar{c}_\chi(p) = \int_0^{+\infty} \varphi_\chi(p, t)dt = \int_0^{\sigma_\chi(p)} |[p]_t|^{\alpha-1}dt$
- $I_\alpha(\chi) = \int_{\Omega\times I\!R_+} \varphi_\chi(p, t)dp\,dt$ = cost of the pattern χ
- $\eta_p(t) = \int_o^t \left|\frac{\partial \chi_p}{\partial s}(s)\right| ds$
- $\overline{\chi}$ = the good parameterization of χ, see Definition 4.5.2
- $\overline{[p]}_t$ = $\overline{\chi}$-vessel at the time t which contains the point p
- $\chi' : [p]_t \times I\!R_+ \to I\!R^N$ = branch of χ starting from $\chi(p, t)$ defined by setting $\chi'(q, \cdot) = \chi(q, \cdot + t)$, see Definition 4.6.2.
- $\chi \setminus \chi'$ = pattern χ stumped of the branch χ'

- $I_\alpha(V,t) = I_\alpha(V)$ = cost of the vessel V at time t
- $F_{\varepsilon,\delta}$ the set of all the $(\varepsilon,\ \delta)$-flows of the pattern χ at some time, see Definition 4.7.1
- $\tilde{\chi}$ = the stretched pattern of χ on $\mathbb{R}_+$, see Definition 4.8.1
- $[\tilde{p}]_t = \tilde{\chi}$-vessel at the time t which contains the point p
- $d_p(\mu,\ \nu) = \left(\min_\sigma \int_{\Omega\times\Omega} |x-y|^p d\sigma\right)^{\frac{1}{p}}$ = Kantorovitch-Wasserstein distance of index p between μ and ν, see Definition 4.8.2
- $\hat{\chi}$ = the σ forced absorption pattern of χ trough a measurable function σ, see Definition 4.8.3
- $[\hat{p}]_t = \hat{\chi}$-vessel at the time t which contains the point p
- χ^* = the rearranged pattern of χ, see Definition 4.8.4
- P_χ= quantity function of the pattern χ, see Definition 4.9.5

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Chapter 5

On the dimension of an irrigable measure[5]

In this paper the problem of determining if a given measure is irrigable, in the sense of [4], or not is addressed. A notion of irrigability dimension of a measure is given and lower and upper bounds are proved in terms of the minimal Hausdorff and respectively Minkowski dimension of a set on which the measure is concentrated.

A notion of resolution dimension of a measure based on its discrete approximations is also introduced and its relation with the irrigation dimension is studied.

Introduction

In the paper [4] the authors have introduced a cost functional to the aim of modeling ramified structures, such as trees, root systems, lungs and cardiovascular systems. A very similar functional (even if the variable employed has a different form) has been introduced in [10]. The aim of the functional is to force the fibers to keep themselves together penalizing, in this way, their branching. The necessity of keeping the functional low competes with a boundary condition which, on the other hand, forces the fibers to bifurcate prescribing that the fluid they carry must reach a given measure spread out on a volume. The result of this competition is that the fibers take advantage in keeping themselves together as long as possible and then branching, always into a finite number of branches, while approaching the terminal points, giving rise to the ramified structure. In [10] the problem consisting in determining the cases, depending on an index, in which all the probability measures can be reached by a system of fibers (an irrigation pattern) of finite cost, i.e. are irrigable measures, is formulated and solved in a very close setting.

In this work we shall investigate a more general question consisting in characterizing, for a given value of the index, what probability measures are irrigable or not. The answer to this question will clearly show, in particular, what are the cases in which all the probability measures will turn out to be irrigable, giving in this way a different proof of the already mentioned result

[5] G. Devillanova, S. Solimini, *On the dimension of an irrigable measure*, to appear.

in [10]. The fact that a measure spread on a set of high dimension forces the fibers to a more frequent branching, and therefore needs a higher cost, seems to suggest that the higher it is the dimension on which a measure is spread the more difficult it becomes to irrigate it. For a better formalization, we introduce the notion of irrigability dimension of a measure and then we equivalently express the above stated problem in terms of giving some estimates on the irrigability dimension of a given positive measure which is always supposed to be Borel regular, with a bounded support and a finite mass (by normalization we shall suppose it to be a probability measure). We shall show, with some examples, that the intuitive and conjecturable idea that the irrigability dimension of a measure coincides with the Hausdorff dimension of its support is groundless in spite of the fact that both the two values express how much the measure is spread out. On the other hand, we shall give some lower and upper bounds for the irrigability dimension $d(\mu)$ of a probability measure μ by means of the minimal Hausdorff and respectively Minkowski dimension of a set on which the measure is concentrated.

This result will be overproved. Indeed, we shall prove it directly, getting some further meaningful information and introducing some tools which will be also used in other parts of the paper but we shall also be able to deduce it from a deeper estimate of $d(\mu)$ which will need the introduction of new notions. More precisely, it will need the notion of *resolution dimension* of a measure which, affected by an index, expresses the possibility to describe the measure by means of discrete approximations. When the measure is suitably regular, the value of the resolution dimension does not depend on the index, while for a generic measure, as will be explained by some examples, the resolution dimension is "out of focus" in the sense that different indexes give different values. We shall show that, in any case, it is always possible to find an index, suitably characterized, which gives a resolution dimension which coincides with the irrigability dimension.

The paper is organized as follows: In Section 5.1 we shall introduce the notion of irrigability dimension and we shall state the main results which do not make use of the notion of resolution dimension of a measure. Sections 5.2 and 5.3 are respectively dedicated to the lower and upper estimate given for $d(\mu)$ by means of the minimum among the Hausdorff and the Minkowski dimension of the sets on which the measure is concentrated. In Section 5.4 remarks and examples, mainly based on the compactness results stated in [4], which show that the estimates are, in a certain sense, sharp are collected. In Section 5.5 we shall introduce the notion of resolution dimension of a measure and we shall state some fundamental properties. The proof of the irrigability and nonirrigability results which can be deduced from conditions on the resolution dimension will be respectively shown in sections 5.6 and 5.7. In Section 5.8 we shall show how the irrigability dimension of a measure can be seen as a resolution dimension with respect to some index $p \geq 1$ and how to chose such a suitable value of p. Then we shall give another proof of the main result in Section 5.9 (Theorem 5.1.1).

Since we are dealing with notions introduced for the first time in [4] and [3], which will be used without any explanation, in order to help the reader we have gathered up in Appendix A the notation and the results in [4] and [3] which are essential for the understanding of this paper. In Appendix B we give the proof of the propositions stated in Section 5.5 with some examples which justify the required assumptions. Finally, in Appendix C we give the index of the main notation.

5.1 Dimensions of a measure and irrigability results

We just recall the definition of irrigation pattern while, as said in the introduction, we have gathered up in Appendix A the notation and the results in [4] and [3] which will help the reader for the understanding of this paper.

Let $(\Omega, |\cdot|)$ be a nonatomic probability space which we interpret as the reference configuration of a fluid material body. We can also interpret it as the trunk section of a tree, this trunk being thought of as a set of fibers which can bifurcate into branches. A *set of fibers of* Ω *with source point* $S \in I\!R^N$ is a mapping

$$\chi : \Omega \times I\!R_+ \to I\!R^N$$

such that:

C1) For a.e. *material point* $p \in \Omega$, $\chi_p(t) : t \mapsto \chi(p,t)$ is a Lipschitz continuous map with a Lipschitz constant less than or equal to one.

C2) For a.e. $p \in \Omega$: $\chi_p(0) = S$.

The condition $|\Omega| = 1$ is of course assumed by normalization in order to simplify the exposition. In some cases this normalization will be impossible (we can, for instance, work with two different spaces and assume an inclusion), then we shall consider all the notions trivially extended to the case $|\Omega| < +\infty$. We shall consider the source point $S \in I\!R^N$ as given and we shall denote by $\mathbf{C}_S(\Omega)$ and $\mathbf{P}_S(\Omega)$ the set of all the set of fibers of Ω and respectively the set of all the measurable sets of fibers of Ω and we shall call the elements of $\mathbf{P}_S(\Omega)$ *irrigation patterns.*

We shall introduce some definitions which will be used to formalize the irrigability problem.

Definition 5.1.1 *For a fixed real number* $\alpha \in]0, 1[$ *we shall call* critical dimension *of the exponent* α *the constant* $d_\alpha = \frac{1}{1-\alpha} = \left(\frac{1}{\alpha}\right)' > 1$.

Definition 5.1.2 *Let* $\alpha \in]0, 1[$ *be given and let* μ *be a probability measure on* $I\!R^N$. *We shall say that* μ *is an irrigable measure with respect to* α *(or that* μ *is* α*-irrigable) if there exists a pattern* $\chi \in \mathbf{P}_S(\Omega)$ *of finite cost* $I_\alpha(\chi) < +\infty$ *such that* $\mu_\chi = \mu$.

It is clear that two approaches are possible and equivalent: one can fix a constant $\alpha \in]0, 1[$ and investigate the irrigable measures with respect to this constant or fix a measure μ and find out the constants $\alpha \in]0, 1[$ with respect to which μ is irrigable. This second point of view leads us to introduce the following definition.

Definition 5.1.3 *Let* μ *be a positive Borel measure on* $I\!R^N$, *then we shall call* irrigability dimension of μ *the number*

$$d(\mu) = \inf\{d_\alpha \,|\, \mu \text{ is irrigable with respect to } \alpha\} .$$

Remark 5.1.1 *For any probability measure* μ*, by definition, the irrigability dimension* $d(\mu)$ *of* μ *is greater or equal to* 1.

Remark 5.1.2 *If μ is an irrigable measure with respect to α, then μ is also irrigable with respect to every constant $\beta \in [\alpha, 1[$. Indeed, let $\chi \in P_S(\Omega)$ such that $I_\alpha(\chi) < +\infty$ and $\mu_\chi = \mu$, then for all $\beta \geq \alpha$, $I_\beta(\chi) \leq I_\alpha(\chi)$.*

Remark 5.1.3 *By the definition of $d(\mu)$ and by Remark 5.1.2 it follows that for a given $\alpha \in]0, 1[$ and for a given measure μ:*

1. *if $d(\mu) < d_\alpha$ then μ is α-irrigable;*
2. *if $d(\mu) > d_\alpha$ then μ is not α-irrigable.*

As we shall show in Section 5.4, both cases can occur when $d_\alpha = d(\mu)$, see examples 5.4.4 and 5.4.5.

The aim of the first part of this paper is to give operative estimates of $d(\mu)$ in terms of geometrical properties of the measure μ. So we introduce the following two definitions.

Definition 5.1.4 *We shall say that a positive Borel measure μ on $\mathbb{R}^N$ is* concentrated on *a Borel set B if $\mu(\mathbb{R}^N \setminus B) = 0$ and we shall call* concentration dimension *of μ the smallest Hausdorff dimension $d(B)$ of a set B on which μ is concentrated i.e. the number*

$$d_c(\mu) = inf\{d(B) \mid \mu \text{ is concentrated on } B\} .$$

Definition 5.1.5 *We shall denote by* supp(μ) *the support of μ in the sense of distributions and shall call* support dimension *of μ, $d_s(\mu)$, its Hausdorff dimension.*

Remark 5.1.4 *The support of a measure can be characterized as the smallest closed set on which μ is concentrated and the existence of such a set* a priori *follows by the separability of $\mathbb{R}^N$, precisely by the Lindelöf property. While, as stated above, the existence of the smallest closed set on which μ is concentrated is granted, it is clear that the smallest set on which μ is concentrated, in general, does not exist. This is the reason for which the infimum is taken in Definition 5.1.4, even if a set B of minimal dimension on which μ is concentrated can always be fixed. Moreover being* supp(μ) *a set on which μ is concentrated, it follows that*

$$d_c(\mu) \leq d_s(\mu) .$$

These two geometrical dimensions are not sufficient to study the irrigability of a measure, as we shall show later in examples 5.4.1 and 5.4.3.

Definition 5.1.6 *Let $X \subset \mathbb{R}^N$ be a bounded set. We shall call Minkowski dimension of the set X (see [8]) the constant*

(5.1.1) $$d_M(X) = N - \liminf_{\delta \to 0} \log_\delta |N_\delta(X)|$$

where, for all $\delta > 0$,

$$N_\delta(X) = \{y \in \mathbb{R}^N | d(y, X) < \delta\} .$$

It is useful to remark that

(5.1.2) $$0 \leq d_M(X) \leq N \qquad \forall X \neq \emptyset\,.$$

Moreover the Minkowski dimension of a set $X \subset I\!R^N$ can be characterized by the following two properties:

(5.1.3) $$\forall \beta < d_M(X) \quad \limsup_{\delta \to 0} |N_\delta(X)| \delta^{\beta - N} = +\infty$$

and

(5.1.4) $$\forall \beta > d_M(X) \quad \lim_{\delta \to 0} |N_\delta(X)| \delta^{\beta - N} = 0\,.$$

Lemma 5.1.1 *Let $X \subset I\!R^N$ and $\beta > d_M(X)$. Then we can cover X by using $\delta^{-\beta}$ balls of radius δ for all δ sufficiently small.*

PROOF. Being $\beta > d_M(X)$, we have, by (5.1.4), that for all $C > 0$ and for $\delta > 0$ sufficiently small
$$|N_{\frac{\delta}{2}}(X)| \leq C\delta^{N-\beta}\,.$$
We consider any family of disjoint balls $(B_i)_{i \in I}$ of radius $\frac{\delta}{2}$ contained in $N_{\frac{\delta}{2}}(X)$. We know that, being, for all $i \in I$, $|B_i| = b_N \left(\frac{\delta}{2}\right)^N$ (b_N stands for the measure of the unitary ball of $I\!R^N$), $\mathrm{card}(I) b_N \left(\frac{\delta}{2}\right)^N \leq |N_{\frac{\delta}{2}}(X)| \leq C\delta^{N-\beta}$ so, taking as C the constant $\frac{b_N}{2^N}$,

(5.1.5) $$\mathrm{card}(I) \leq C \frac{2^N}{b_N} \delta^{-\beta} = \delta^{-\beta}\,.$$

We have shown that the number of elements of any family consisting of disjoint balls contained in $N_{\frac{\delta}{2}}(X)$ is bounded by $\delta^{-\beta}$. This allows us to find a family of such balls which is maximal by inclusion. The corresponding family of balls with the same centers but with double radius, by maximality, turns out to be a covering of X. Inequality (5.1.5) gives the thesis. ∎

Lemma 5.1.2 *Let $X \subset I\!R^N$ and $\beta < d_M(X)$. It is not possible to find a constant $C > 0$ such that one can cover X with only $C\delta^{-\beta}$ balls of radius δ for all δ sufficiently small.*

PROOF. We shall proceed by contradiction assuming that there exists a constant $C > 0$ such that for δ sufficiently small it is possible to cover X using $C\delta^{-\beta}$ balls of radius δ. It is useful to remark that doubling the radius of these balls we get a covering of $N_\delta(X)$, so we have
$$|N_\delta(X)| \leq C\delta^{-\beta} b_N (2\delta)^N \leq cost\ \delta^{N-\beta}\,,$$
which gives $\beta \geq d_M(X)$ by (5.1.3). ∎

Remark 5.1.5 *Collecting the last two lemmas, we can say that for a set $X \subset I\!R^N$*

(5.1.6) $$d_M(X) = \inf\{\beta \geq 0 | X \text{ can be covered by } C_\beta \delta^{-\beta} \text{ balls of radius } \delta \text{for all } \delta \leq 1\}\,.$$

Definition 5.1.7 *Let μ be a probability measure, we shall use the notation*

$$d_M(\mu) = \inf\{d_M(X) \mid \mu \text{ is concentrated on } X\} \tag{5.1.7}$$

and we shall call it Minkowski dimension of *μ.*

Remark 5.1.6 *For any subset X of $I\!R^N$ the Minkowski dimensions of X and of its closure $\overline{X}$ are the same. Therefore*

$$d_M(\mu) = d_M(\text{supp}(\mu)) .$$

Moreover the Hausdorff dimension $d(X)$ of a set X is less or equal to $d_M(X)$. So for any probability measure μ

$$d_s(\mu) \leq d_M(\mu) . \tag{5.1.8}$$

Remark 5.1.7 *Let μ be a probability measure, then collecting Remark 5.1.4 and (5.1.8) we have that the following inequalities hold for $d_s(\mu)$*

$$d_c(\mu) \leq d_s(\mu) \leq d_M(\mu) . \tag{5.1.9}$$

A similar estimate is enjoyed by $d(\mu)$. Indeed, we shall prove the following statement.

Theorem 5.1.1 (Lower and Upper bound on $d(\mu)$) *Let μ be a probability measure then the following bounds hold for $d(\mu)$*

$$d_c(\mu) \leq d(\mu) \leq \max\{d_M(\mu), 1\} . \tag{5.1.10}$$

The first inequality in (5.1.10) is a straightforward consequence of a deeper and more precise result stated in the following theorem, whose proof is in Section 5.2.

Theorem 5.1.2 *Let $\alpha \in]0, 1[$ and let μ be an α-irrigable probability measure, then μ is concentrated on a d_α-negligible set, in particular,*

$$d_c(\mu) \leq d_\alpha . \tag{5.1.11}$$

Theorems 5.1.1 and 5.1.2 widely answer the question considered in [10] about the values of α which make every measure of bounded support irrigable. Indeed, we can deduce the following corollaries.

Corollary 5.1.1 *Let $\alpha \in]0, 1[$, $\alpha > \frac{1}{N'}$. Then any probability measure μ with a bounded support is α-irrigable.*

PROOF. Remarking that $\alpha > \frac{1}{N'}$ is equivalent to $d_\alpha = \left(\frac{1}{\alpha}\right)' > N$, combining (5.1.2) with (5.1.10), we have, for every μ,

$$d_\alpha > N \geq \max\{d_M(\mu), 1\} \geq d(\mu) ,$$

so every probability measure μ with a bounded support is α-irrigable by Remark 5.1.3,1. ∎

Corollary 5.1.2 *Let $\alpha \in]0, 1[$ be such that any probability measure μ with a bounded support is α-irrigable, then $\alpha > \frac{1}{N'}$.*

PROOF. ¿From Theorem 5.1.2 we have that any probability measure μ with a bounded support is concentrated on a d_α-negligible set. So, $N < d_\alpha$, namely $\alpha > \frac{1}{N'}$. ∎

In spite of inequalities (5.1.10) and (5.1.9) it is not possible to establish some general inequality between $d(\mu)$ and $d_s(\mu)$, as shown in Section 5.4 by examples 5.4.1 and 5.4.3.

By the following lemmas we shall make the estimates on the dimension $d(\mu)$ more precise in the case in which the probability measure μ enjoys some regularity properties.

Definition 5.1.8 *Let μ be a probability measure and $\beta \geq 0$. We shall say that μ is Ahlfors regular in dimension β if*

$$\text{(AR)} \qquad \exists C_1,\, C_2 > 0 \ s.t.\ \forall r \in [0,\, 1],\ \forall x \in \operatorname{supp}(\mu): \quad C_1 r^\beta \leq \mu(B(x,r)) \leq C_2 r^\beta\ .$$

We shall separately consider the two bounds in (AR). So for a probability measure μ and a real number $\beta \geq 0$ we shall consider the two conditions

$$\text{(LAR)} \qquad \exists C > 0 \text{ s.t. } \forall r \in [0,\, 1],\ \forall x \in \operatorname{supp}(\mu): \quad C r^\beta \leq \mu(B(x,r))\ .$$

and

$$\text{(UAR)} \qquad \exists C > 0 \text{ s.t. } \forall r \in [0,\, 1],\ \forall x \in \operatorname{supp}(\mu): \quad \mu(B(x,r)) \leq C r^\beta\ .$$

In (UAR) the restriction $x \in \operatorname{supp}(\mu)$ can be removed, this could make the value of C_2 increase at most of 2^β. It is useful to recall the following definition.

Definition 5.1.9 *A probability measure $\nu : I\!R^N \to I\!R_+$* satisfies the uniform density property (in short u.d.p.) in dimension $\beta \geq 0$ *on a set M if*

$$\exists C_1 > 0 \ s.t.\ \forall x \in M,\ \forall r \in [0,\, 1]: \ C_1 r^\beta \leq \nu(B(x,r))\ .$$

Lemma 5.1.3 *Let ν be a probability measure which satisfies the u.d.p. in dimension $\beta \geq 0$ on a subset B. Then*

$$d_M(B) \leq \beta\ . \qquad (5.1.12)$$

PROOF. Let us fix $\delta > 0$ and let us consider any family $(B_i)_{i\in I}$ of disjoint balls of radius $\frac{\delta}{2}$ with centers on B. By hypotheses, $\nu(B_i) \geq C2^{-\beta}\delta^\beta$ and $\nu(B) \leq 1$, therefore $\operatorname{card}(I) \leq 2^\beta C^{-1}\delta^{-\beta}$. So we can consider a family $(B_i)_{i\in I}$ as above maximal by inclusion. The maximality of $(B_i)_{i\in I}$ guarantees that, for any other point $x \in B$, $d(x, \bigcup_{i\in I} B_i) < \frac{\delta}{2}$ holds. Therefore the family $(\tilde{B}_i)_{i\in I}$ which is obtained by doubling the radius of the balls B_i is a covering of B. So we have proved that B can be covered by const$\delta^{-\beta}$ balls of radius δ arbitrarily small and so by Remark 5.1.5 $d_M(B) \leq \beta$. ∎

Corollary 5.1.3 *Let μ be a probability measure. Let $\beta \geq 0$ such that μ satisfies (LAR) (i.e. μ satisfies the uniform density property in dimension β on $\operatorname{supp}(\mu)$). Then*

$$d_M(\mu) \leq \beta \,. \tag{5.1.13}$$

Remark 5.1.8 *The thesis of Corollary 5.1.3 still holds true if one assumes the existence of a probability measure ν which satisfies the uniform density property in dimension β on a set B on which μ is concentrated.*

Lemma 5.1.4 *Let μ be a probability measure concentrated on a set $A \subset \mathbb{R}^N$. Let $\beta \geq 0$ such that μ satisfies (UAR). Then*

$$\mathcal{H}^\beta(A) > 0. \tag{5.1.14}$$

PROOF. Let $(X_i)_{i\in I}$ be any countable covering of A. Every X_i is contained in a ball B_i with a radius equal to $\operatorname{diam}(X_i)$. So, by (UAR)

$$1 = \mu(\mathbb{R}^N) = \mu(A) \leq \sum_{i\in I} \mu(B_i) \leq C \sum_{i\in I} \operatorname{diam}(X_i)^\beta \,,$$

from which we have

$$\sum_{i\in I} \operatorname{diam}(X_i)^\beta \geq C^{-1} > 0 \,.$$

∎

Corollary 5.1.4 *Let μ be a probability measure. Let $\beta \geq 0$ such that μ satisfies (UAR). Then*

$$d_c(\mu) \geq \beta \,. \tag{5.1.15}$$

Corollary 5.1.5 *Let μ be an Ahlfors regular probability measure in dimension $\beta \geq 1$. By Corollary 5.1.3 and Corollary 5.1.4, being $\beta = \max\{\beta, 1\} \geq \max\{d_M(\mu), 1\}$, the lower and upper bounds stated in Remark 5.1.7 and Theorem 5.1.1 for $d_s(\mu)$ and $d(\mu)$ respectively, give*

$$d_c(\mu) = d_s(\mu) = d(\mu) = d_M(\mu) = \beta \,.$$

This guarantees that, in the case of an Ahlfors regular probability measure, all the geometrical dimensions $d_c(\mu)$, $d_s(\mu)$ and $d_M(\mu)$ and the irrigability dimension $d(\mu)$ are equal to the Ahlfors dimension β.

Corollary 5.1.6 *An Ahlfors regular probability measure μ of dimension $\beta \geq 1$, is α-irrigable for all $\alpha \in]0, 1[$ s.t. $d_\alpha > \beta$ i.e. for all $\alpha \in]\frac{1}{\beta'}, 1[$ and is not irrigable for all $\alpha \in]0, 1[$ s.t. $d_\alpha \leq \beta$ i.e. for all $\alpha \in]0, \frac{1}{\beta'}]$.*

PROOF. Let $\alpha \in]0, 1[$. If $d_\alpha \neq \beta = d(\mu)$, the thesis follows from Remark 5.1.3. Moreover, when $d_\alpha = \beta$, by Theorem 5.1.2 it is clear that an Ahlfors regular probability measure of dimension $\beta = d_\alpha$, is not α-irrigable. Indeed by Lemma 5.1.4 it cannot be concentrated on a d_α-negligible set. ∎

We shall make use of this last argument when in Section 5.4 we shall show that, in general, $d(\mu) = \inf\{d_\alpha \mid \mu \text{ is } \alpha\text{-irrigable}\}$ is not a minimum, see Example 5.4.4.

5.2 Lower bound on $d(\mu)$

This section is devoted to the proof of Theorem 5.1.2 from which $d_c(\mu) \leq d(\mu)$ trivially follows.

Lemma 5.2.1 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω and $r > 0$, then*

$$\mu_\chi(I\!R^N \setminus B_r(S)) \leq \left(\frac{I_\alpha(\chi)}{r}\right)^{\frac{1}{\alpha}} . \tag{5.2.1}$$

PROOF. Taking into account that the less expensive way to carry some part of the fluid out of $B_r(S)$ is to move it in a unique tube in the radial direction and to leave the other part in the source point, we have

$$[\mu_\chi(I\!R^N \setminus B_r(S))]^\alpha r \leq I_\alpha(\chi) ,$$

from which the thesis follows. ∎

Corollary 5.2.1 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω. If $r \geq (I_\alpha(\chi))^{1-\alpha}$, then*

$$\mu_\chi(I\!R^N \setminus B_r(S)) \leq I_\alpha(\chi) . \tag{5.2.2}$$

In [3] the following lemma has been proved.

Lemma 5.2.2 *Let $\chi \in P_S(\Omega)$ be a simple irrigation pattern of Ω without dispersion and $\varepsilon > 0$, then there exists a finite number $k \in I\!N$ of points $x_i \in F_\chi$ such that, denoting by χ_i the branch of χ with source point x_i,*

$$\sum_{i=1}^{k} I_\alpha(\chi_i) < \varepsilon \tag{5.2.3}$$

$$(\mu_\chi - \sum_{i=1}^{k} \mu_{\chi_i})(I\!R^N) < \varepsilon . \tag{5.2.4}$$

Lemma 5.2.3 *Let $\chi \in P_S(\Omega)$ be a simple irrigation pattern of Ω without dispersion and $\varepsilon > 0$, then $\exists A \subset I\!R^N$ such that*

1. *A can be covered by a finite number of balls $B_i = B_{r_i}(x_i)$, s.t. $\sum_i (r_i)^{d_\alpha} < \varepsilon$;*
2. *$\mu_\chi(I\!R^N \setminus A) \leq \varepsilon$.*

PROOF. By Lemma 5.2.2 we can find a finite number $k \in I\!N$ of points $x_i \in I\!R^N$ such that, by denoting by χ_i the branch of χ starting from x_i and by $\varepsilon_i = I_\alpha(\chi_i)$, we have

$$\sum_{i=1}^{k} \varepsilon_i < \varepsilon . \tag{5.2.5}$$

Calling, for all $i \in \{1, \dots, k\}$, as suggested by Corollary 5.2.1, $r_i = (I_\alpha(\chi_i))^{1-\alpha} = (\varepsilon_i)^{1-\alpha}$ we have,

$$\sum_i r_i^{d_\alpha} = \sum_i \varepsilon_i < \varepsilon .$$

Moreover, from (5.2.2), we can deduce that

$$\mu_{\chi_i}(I\!R^N \setminus B_{r_i}(x_i)) \leq \varepsilon_i .$$

Applying (5.2.4), (5.2.3) and (5.2.5) we get, for $A = \bigcup_{i=1}^k B_{r_i}(x_i)$, that

$$\mu_\chi(I\!R^N \setminus A) \leq (\mu_\chi - \sum_{i=1}^k \mu_{\chi_i})(I\!R^N) + \sum_{i=1}^k \mu_{\chi_i}(I\!R^N \setminus B_{r_i}(x_i)) < \varepsilon + \sum_{i=1}^k \varepsilon_i < 2\varepsilon .$$

Replacing ε by $\frac{\varepsilon}{2}$ we complete the proof. ∎

Proof of Theorem 5.1.2. By hypotheses there exists an irrigation pattern $\chi \in P_S(\Omega)$ of finite cost $I_\alpha(\chi) < +\infty$, such that $\mu = \mu_\chi$. By Lemma 5.9.2 we know that $d(F_\chi) = 1 < d_\alpha$, therefore we can reduce ourselves, as Remark 5.9.3 suggests, to a pattern χ without dispersion. Moreover, if one considers a pattern which is optimal with respect to the cost functional, the pattern can also be supposed to be simple (see Definition 5.9.7), see Lemma 4.6.2.

So, for every $n \in I\!N$, we can apply Lemma 5.2.3 to the pattern χ and to $\varepsilon = 2^{-n} > 0$. Therefore for all $n \in I\!N$ there exists $A_n \subset I\!R^N$ which satisfies 1) and 2) of Lemma 5.2.3 for $\varepsilon = 2^{-n}$. For a fixed $h \in I\!N$ we shall denote by $D_h = \bigcap_{n>h} A_n$

Then

$$\mu_\chi(I\!R^N \setminus D_h) = \mu_\chi(\bigcup_{n>h} I\!R^N \setminus A_n) \leq \sum_{n>h} \mu_\chi(I\!R^N \setminus A_n) \leq \sum_{n>h} \frac{1}{2^n} = \frac{1}{2^h} . \tag{5.2.6}$$

Moreover, being $D_h \subset A_n$ for all $n > h$, by Lemma 5.2.3,1), D_h is covered by a finite number k of balls of radius r_i verifying $\sum_{i=1}^k r_i^{d_\alpha} < 2^{-n}$, from which $\mathcal{H}^{d_\alpha}(D_h) = 0$ follows by the definition of Hausdorff outer measure. For all $i \in I\!N$

$$\mu_\chi(I\!R^N \setminus \bigcup_{h \in I\!N} D_h) \leq \mu_\chi(I\!R^N \setminus D_i) \leq \frac{1}{2^i} ,$$

therefore we have

$$\mu_\chi(I\!R^N \setminus \bigcup_{h \in I\!N} D_h) = 0$$

and so μ is concentrated on $\bigcup_{h \in I\!N} D_h$. Since, for all $h \in I\!N$, $\mathcal{H}^{d_\alpha}(D_h) = 0$ we get that μ is concentrated on a d_α-negligible set. ∎

Proof of Theorem 5.1.1 (lower bound $d_c(\mu) \leq d(\mu)$). By Theorem 5.1.2 we have proved in particular that, for every $\alpha \in]0, 1[$, if μ is α-irrigable then $d_c(\mu) \leq d_\alpha$. By the definition of $d(\mu)$, taking the infimum on d_α in the above inequality, the thesis follows. ∎

5.3 Upper bound on $d(\mu)$

The main goal of this section is the proof of the following theorem, from which the upper bound on $d(\mu)$ stated in Theorem 5.1.1 easily follows.

Theorem 5.3.1 *Let μ be a probability measure and $\alpha \in]0,1[$, then μ is α-irrigable provided $d_M(\mu) < d_\alpha$.*

To this aim, we need to introduce some definitions and to establish some preliminary lemmas.

Definition 5.3.1 *Let $I = \{1, 2, \ldots, n\} \subset I\!N$ be a finite set of indexes. We shall say that $(P_i, \gamma_i)_{i\in I}$ is a* hierarchy of collectors *if*

- *$\forall i \in I$: P_i is a finite subset of $I\!R^N$ with k_i elements x_j^i, $1 \leq j \leq k_i$;*
- *$\forall i \in I, i \neq n$, γ_i maps P_i in P_{i+1} while γ_n is a map on P_n of constant value S (which is the "head" of the hierarchy and will be the source S in the applications).*

In the following we shall call each map γ_i the "dependence" map of the points $x_j^i \in P_i$ from those $x_j^{i+1} \in P_{i+1}$.

Remark 5.3.1 *For a given hierarchy of collectors $(P_i, \gamma_i)_{i\in I}$, every time we fix a point $x = x_j^1 \in P_1$, we find, using the dependence maps, a chain of points $\{x, \gamma_1(x), \gamma_2(\gamma_1(x)), \ldots, S\}$ which allows us to reach the source S "in a hierarchical way". We can consider the elements of such a chain as the vertices of a polygonal which runs with unitary speed. Reversing the time, we get a path which starts from the source S and arrives in x. We shall call by $g_x : I\!R_+ \to I\!R^N$ this path, parameterized in the whole of $I\!R_+$, by considering it constant after reaching x.*

In what follows, let us set, $\forall x \in P_1$, $\gamma^1(x) = \gamma_1(x)$, $\gamma^2(x) = \gamma_2(\gamma^1(x))$ and recoursively

$$\gamma^j(x) = \gamma_j(\gamma^{j-1}(x)) \in P_{j+1} \,.$$

For a given hierarchy of collectors $(P_i, \gamma_i)_{i\in I}$, we shall deal with a probability measure $\overline{\mu}_1$ concentrated on P_1, namely $\overline{\mu}_1 = \sum_{i=1}^{k_1} m_j^1 \delta_{x_j^1}$ is the sum of a finite number of Dirac masses centered on the points x_j^1 of P_1.

Being Ω a non atomic probability space, by Lyapunov Theorem, we can split Ω into $k_1 (= \mathrm{card}(P_1))$ sets Ω_j such that $|\Omega_j| = m_j^1$, i.e. we can split Ω into k_1 sets whose measures are just the masses m_j^1 we find in the points $(x_j^1)_{j\in k_1}$ at the base of the hierarchy.

Definition 5.3.2 *Let $(P_i, \gamma_i)_{i\in I}$ be a hierarchy of collectors and $\overline{\mu}_1 = \sum_{j=1}^{k_1} m_j^1 \delta_{x_j^1}$ a probability measure concentrated on the base P_1 of the hierarchy.*

We shall say that $\chi : \Omega \times I\!R_+ \to I\!R^N$ is a distribution pattern relative to $\overline{\mu}_1$ and to the hierarchy $(P_i, \gamma_i)_{i\in I}$ *if $\forall p \in \Omega$, and $\forall t \geq 0$:*

$$\chi(p,t) = g_{x_j^1}(t) \qquad \textit{for } p \in \Omega_j \,,$$

where the paths $g_{x_j^1}$ and the partition $(\Omega_j)_{1\leq j\leq k_1}$ are as above.

Remark 5.3.2 *Let $(P_i, \gamma_i)_{i\in I}$ be a hierarchy of collectors and $\overline{\mu}_1 = \sum_{j=1}^{k_1} m_j^1 \delta_{x_j^1}$ a probability measure concentrated on the base P_1 of the hierarchy. Let χ be a distribution pattern relative to $\overline{\mu}_1$ and to the hierarchy $(P_i, \gamma_i)_{i\in I}$. Then, by construction, being $i_\chi(\Omega_j) = \{x_j^1\}$, we have*

$$\mu_\chi = \sum_{j=1}^{k_1} m_j^1 \delta_{x_j^1} = \overline{\mu}_1 \ .$$

Definition 5.3.3 *Let $(P_i, \gamma_i)_{1\le i\le n}$ be a hierarchy of collectors. For any discrete probability measure $\overline{\mu}_1$ concentrated on P_1 we shall recoursively call for all $i \in \{2, \ldots, n\}$, $\overline{\mu}_i$ the image measure of $\overline{\mu}_{i-1}$ through the function γ_{i-1}.*

Each one of these measures can be considered as a discrete measure defined on the whole of the space and concentrated on P_i.

Lemma 5.3.1 *Let $(P_i, \gamma_i)_{i\in I}$ be a hierarchy of collectors and $\overline{\mu}_1$ be a discrete probability measure concentrated on the base level P_1 of the hierarchy. Let χ be a distribution pattern relative to $\overline{\mu}_1$ and to the hierarchy, then*

$$I_\alpha(\chi) = \sum_{i\in I} \sum_{x\in P_i} (m_i(x))^\alpha |x - \gamma_i(x)| \ , \tag{5.3.1}$$

where, for all $i \in I$, and for all $x \in P_i$

$$m_i(x) = \overline{\mu}_i(\{x\}) \ .$$

Proof. We shall proceed by induction on $n = \text{card}(I)$. The thesis is obvious in the case $n = 1$. Let us suppose that the statement is true for $\text{card}(I) = n - 1$ and let us prove that the statement is also true for $\text{card}(I) = n$. Let us remark that each one of the $k_n = \text{card}(P_n)$ elements of the last level set P_n can be seen as the head of a hierarchy of $n - 1$ levels, given by the sets $P_i(x)$, where for all $1 \le i \le n - 1$:

$$P_i(x) = \{y \in P_i \mid \gamma_{n-1}(\gamma_{n-2}(\ldots(\gamma_i(y)))) = x\} \ .$$

and by the suitable restrictions of the maps γ_i, $i = 1, \ldots, n - 1$.

Therefore we can apply the induction hypotheses to the k_n branches $\overline{\chi}_x$ of χ which start from the point $x \in P_n$. So each one of these patterns has a cost which can be estimated by

$$I_\alpha(\overline{\chi}_x) = \sum_{i=1}^{n-1} \sum_{y\in P_i(x)} (m_i(y))^\alpha |x - \gamma_i(y)| \ .$$

To bring $\overline{\chi}_x$ back to the source S obtaining the pattern χ_x, restriction of χ to $\bigcup_{x_j^1\in P_1(x)} \Omega_j$, we must add to $I_\alpha(\overline{\chi}_x)$ the cost necessary for the connection of x to the source S. Therefore we have

$$I_\alpha(\chi_x) = (m_n(x))^\alpha |x - S| + I_\alpha(\overline{\chi}_x) = (m_n(x))^\alpha |x - \gamma_n(x)| + \sum_{i=1}^{n-1} \sum_{y \in P_i(x)} (m_i(y))^\alpha |x - \gamma_i(y)| \, .$$

Since the whole χ can be regarded as a multiple branch starting from the source S which has the patterns χ_x as the corresponding single branches, by additivity we have

$$\begin{aligned} I_\alpha(\chi) &= \sum_{x \in P_n} I_\alpha(\chi_x) = \sum_{x \in P_n} (m_n(x))^\alpha |x - \gamma_n(x)| + \sum_{x \in P_n} \sum_{i=1}^{n-1} \sum_{y \in P_i(x)} (m_i(y))^\alpha |x - \gamma_i(y)| \\ &= \sum_{i=1}^{n} \sum_{x \in P_i} (m_i(x))^\alpha |x - \gamma_i(x)| \, . \end{aligned}$$

∎

Lemma 5.3.1 admits the following corollary.

Corollary 5.3.1 *Let $(P_i, \gamma_i)_{i \in I}$ be a hierarchy of collectors and $\overline{\mu}_1$ be a probability measure concentrated on the base level P_1 of the hierarchy. Let χ be a distribution pattern relative to $\overline{\mu}_1$ and to the hierarchy, then*

$$I_\alpha(\chi) \le \sum_{i \in I} k_i^{1-\alpha} l_i \, ,$$

where for all $i \in \{1, \dots, n\}$

$$l_i = \max_{x \in P_i} |x - \gamma_i(x)| \, .$$

PROOF. The thesis follows because for all $i \in I$:

$$\sum_{x \in P_i} (m_i(x))^\alpha \le (k_i)^{1-\alpha} \, .$$

Indeed, by Hölder inequality, being, for all $i \in I$, $\sum_{x \in P_i} m_i(x) = 1$, we have:

$$\sum_{x \in P_i} (m_i(x))^\alpha \le \left(\sum_{x \in P_i} m_i(x) \right)^\alpha \left(\sum_{x \in P_i} 1 \right)^{1-\alpha} = k_i^{1-\alpha} \, .$$

∎

PROOF OF THEOREM 5.3.1. If $d_M(\mu) < d_\alpha$, we can fix a constant β such that $d_M(\mu) < \beta < d_\alpha$.

Given $n \in \mathbb{N}$, $n \ge 1$, let us consider a covering of $\text{supp}(\mu)$ consisting of balls with radius 2^{-n}. Let us call X_n the set made of the centers of such balls and let us set $X_0 = \{S\}$. We introduce for $n \ge 1$ the map $\varphi_n : X_n \to X_{n-1}$ which chooses, for every point $x \in X_n$, one of the closest points $\varphi_n(x) \in X_{n-1}$. It is easy to see that for $n \geq 2$ (and for $n \geq 1$, with a suitable choice of S and a normalization of the diameter of the support of μ)

$$\forall x \in X_n \quad : \quad d(x, \varphi_n(x)) \leq 3\, 2^{-n}\ . \tag{5.3.2}$$

Moreover, by Lemma 5.1.1, being $d_M(\mu) < \beta$, we can choose X_n and a constant $C > 0$ so that

$$\mathrm{card}(X_n) \leq C\,(2^{-n})^{-\beta} = C\,2^{n\beta}\ . \tag{5.3.3}$$

Let us now put a total order on X_n. On each center $x \in X_n$ we shall put the mass

$$m_x^n = \mu(B_{2^{-n}}(x) \setminus \bigcup_{y<x} B_{2^{-n}}(y))\ .$$

In this way we get a probability measure $\mu_n = \sum_{x \in X_n} m_x^n \delta_x$ such that $\mu_n \rightharpoonup \mu$. Now, for a fixed $n \in I\!N$, all $1 \leq i \leq n$, let us call $P_i = X_{n-i+1}$ and $\gamma_i = \varphi_{n-i+1}$. By (5.3.3) we have:

$$\forall i \in \{1, \ldots, n\}: \qquad k_i = \mathrm{card}(P_i) = \mathrm{card}(X_{n-i+1}) \leq C\,(2^{-(n-i+1)})^{-\beta}\ , \tag{5.3.4}$$

while, by (5.3.2),

$$\forall i \in \{1, \ldots, n\}: \quad l_i = \max_{x \in P_i} |x - \gamma_i(x)| = \max_{x \in X_{n-i+1}} |x - \varphi_{n-i+1}(x)| \leq 3(2^{-(n-i+1)})\ . \tag{5.3.5}$$

If we denote by χ_n a distribution pattern relative to the hierarchy of collectors $(P_i, \gamma_i)_{1 \leq i \leq n}$ and to $\overline{\mu}_1 = \mu_n$, by Corollary 5.3.1, using also (5.3.4) and (5.3.5), we have

$$\begin{aligned} I_\alpha(\chi_n) &\leq \sum_{i=1}^n (k_i)^{1-\alpha} l_i \leq C^{1-\alpha} \sum_{i=1}^n [(2^{-(n-i+1)})^{-\beta}]^{1-\alpha} 3(2^{-(n-i+1)}) \\ &= 3C^{1-\alpha} \sum_{i=1}^n 2^{-(n-i+1)(-\beta(1-\alpha)+1)} = 3C^{1-\alpha} \sum_{j=1}^n 2^{-jb} \leq \frac{3C^{1-\alpha}}{2^b - 1}\ , \end{aligned}$$

where, being $\beta < d_\alpha$, is

$$b = -\beta(1-\alpha) + 1 > 0\ .$$

The independence on n of the above bound allows us to build a sequence of patterns $(\chi_n)_{n \in I\!N}$ to which we can apply the compactness theorem [4, Theorem 8.1] and to get, in this way, the existence of a limit pattern χ of finite cost such that $\mu_\chi = \mu$. ∎

It is worth remarking that the measure μ_n taken in the proof of Theorem 5.3.1 could be replaced by any probability measure centered on the points of X_n such that the Kantorovitch-Wasserstein distance between μ_n and μ (see Definition 5.5.3) is less or equal to 2^{-n}.

Proof of Theorem 5.1.1 (upper bound $d(\mu) \leq \max\{d_M(\mu), 1\}$). Arguing by contradiction, let us suppose $d(\mu) > \max\{d_M(\mu), 1\}$. Then there exists a constant $\alpha \in]0, 1[$ such that $d_M(\mu) < d_\alpha < d(\mu)$. ¿From one side $d_M(\mu) < d_\alpha$, so we have from Theorem 5.3.1 that μ is α-irrigable; on the other side $d_\alpha < d(\mu)$, so we get from Remark 5.1.3 (2) that μ cannot be α-irrigable. ∎

5.4 Remarks and examples

Definition 5.4.1 *Let $\alpha \in]0,1[$ and let μ be a finite measure on $\mathbb{R}^N$. We shall call α-cost of the measure μ the value of the functional I_α on the optimal patterns χ which irrigate the measure μ.*

Lemma 5.4.1 *Let $\alpha \in]0,1[$, ν and μ be two finite measure on $\mathbb{R}^N$ such that $\nu \le \mu$. If μ is α-irrigable then also ν is α-irrigable, moreover the α-cost to irrigate ν is less expensive that the α-cost for μ.*

PROOF. For any $n \in \mathbb{N}$, let us consider a countable borel partition $\mathcal{A}_n = (A_i^n)_{i\in I}$ of $\mathbb{R}^N$ made of sets of diameter less or equal to $\frac{1}{n}$ for all $i \in I$. By hypotheses there exists an irrigation pattern χ, defined on $\Omega \times \mathbb{R}_+$, where Ω is a probability space, s.t. $I_\alpha(\chi) < +\infty$ and $\mu_\chi = \mu$. Let us call, for all i, $\Omega_{i,n} = i_\chi^{-1}(A_i^n)$. By construction we get

$$|\Omega_{i,n}| = |i_\chi^{-1}(A_i^n)| = \mu_\chi(A_i^n) = \mu(A_i^n) \ge \nu(A_i^n) \ .$$

Therefore, being any $\Omega_{i,n}$ a non atomic set, by Lyapunov Theorem, $\Omega_{i,n}$ admits a subset $\Omega'_{i,n}$ such that $|\Omega'_{i,n}| = \nu(A_i^n)$. Let us consider $\Omega'_n = \bigcup_i \Omega'_{i,n}$ and let us denote $\chi_n = \chi_{|\Omega'_n}$. By construction $\mu_{\chi_n} \rightharpoonup \nu$ and

$$I_\alpha(\chi_n) \le I_\alpha(\chi) < +\infty \ . \tag{5.4.1}$$

Therefore, by compactness, we get a limit pattern $\overline{\chi}$ such that $\mu_{\overline{\chi}} = \nu$. Moreover, being I_α a lower semicontinuous functional, (5.4.1) gives $I_\alpha(\overline{\chi}) \le \liminf_{n\to+\infty} I_\alpha(\chi_n) \le I_\alpha(\chi)$. ∎

The following corollaries easily follow.

Corollary 5.4.1 *Let $\alpha \in]0,1[$, $c \in \mathbb{R}$ and ν and μ two finite Radon measures on $\mathbb{R}^N$ such that $\nu \le c\mu$. Then*

$$d(\nu) \le d(\mu) \ .$$

Corollary 5.4.2 *Let μ and ν be finite Radon measures such that $c_1\mu \le \nu \le c_2\mu$ for some positive constants c_1, c_2. Then we have*

$$d(\nu) = d(\mu) \ .$$

Remark 5.4.1 *The pattern $\overline{\chi}$, found in the proof of Lemma 5.4.1, is the limit pattern, modulo equivalence, of a sequence of subpatterns of χ but it is not a subpattern in general. So one could wonder if it is always possible to find a subpattern of χ which irrigates ν.*

The answer to this question is negative. For instance, one can consider $\Omega = [0, 1]$ and, for a.e. $p \in [0, 1]$ and for all $t \geq 0$,

$$\chi(p, t) = \min(p, t) \ . \tag{5.4.2}$$

It is clear that χ irrigates the Lebesgue measure μ_L on $[0, 1]$. On the other side, it is not possible to find a subpattern of χ which can irrigate $\nu = \frac{1}{2}\mu_L$. Indeed, in such a case, one should find a subset $A \subset [0, 1]$ of density $\frac{1}{2}$ everywhere and this is not possible. A pattern $\overline{\chi} : \left[0, \frac{1}{2}\right] \times I\!R_+ \to I\!R$, provided by the proof of Lemma 5.4.1 is, for instance,

$$\chi(p, t) = \min(2p, t) \ , \tag{5.4.3}$$

which irrigates $\frac{1}{2}\mu_L$.

The simple idea that the irrigability of a probability measure μ depends only on the dimension of the support is false. Indeed, $d_s(\mu)$ and $d(\mu)$ are not comparable in general, even if the dimensions $d_c(\mu)$ and $d_M(\mu)$ which respectively give a lower and an upper bound on $d_s(\mu)$ are also bounds for $d(\mu)$, as stated in Remark 5.1.7 and Theorem 5.1.1.

It is easy to see that, in general, $d_s(\mu) \not\leq d(\mu)$, as stated in the following example.

Example 5.4.1 *There exist probability measures μ such that $d_s(\mu) = N$ (maximum possible value) and $d(\mu) = 1$ (minimum possible value) i.e. which are α-irrigable for all $\alpha \in]0, 1[$.*

Proof. Let us call B the unit ball of $I\!R^N$, $S = 0$ and let $\tilde{B} = \{x_1, x_2, \dots, x_n, \dots\}$ be the countable set consisting in the points of B with rational coordinates.

Let us consider $\mu = \sum_{n\geq 1} \left(\frac{1}{2}\right)^n \delta_n$ where, $\forall n \in I\!N$, δ_n is a Dirac mass centered in x_n. By construction, $d_s(\mu) = N$. Moreover we shall prove that μ is α - irrigable for all $\alpha \in]0, 1[$, i.e. $d(\mu) = 1$. Let χ be the pattern which at unitary speed carries from S in the n-th point of $\tilde{B}$ the mass $\frac{1}{2^n}$. Then for any $\alpha \in]0, 1[$ we have: $I_\alpha(\chi) \leq \sum_{n\geq 1} \left(\frac{1}{2}\right)^{\alpha n} = \frac{1}{2^\alpha - 1} < +\infty$. Therefore, being by construction $\mu_\chi = \mu$, μ is α-irrigable. ∎

In order to show that also the converse inequality is, in general, not true, we shall point out the following property.

Proposition 5.4.1 *Let $\alpha \in]0, 1[$ and μ be a probability measure which is not α-irrigable. Then for all $n \in I\!N$ it is possible to find a discrete approximation $\tilde{\mu}$ of μ, of sufficiently high resolution (see Definition 5.5.1), such that any pattern $\tilde{\chi}$ which irrigates $\tilde{\mu}$ has a cost $I_\alpha(\tilde{\chi}) \geq n$.*

Proof. Assume by contradiction that we can find a sequence $(\tilde{\mu}_n)_{n\in I\!N}$ of discrete approximations of μ weakly converging to μ and a sequence $(\tilde{\chi}_n)_{n\in I\!N}$ of patterns, where, $\forall n \in I\!N$, $\tilde{\chi}_n$ irrigates $\tilde{\mu}_n$, such that, $\forall n \in I\!N$: $I_\alpha(\tilde{\chi}_n) < c$. Then we could apply the compactness theorem [4, Theorem 8.1] obtaining a limit pattern $\overline{\chi}$ of finite cost which irrigates μ. ∎

Example 5.4.2 *There exists a probability measure μ with a countable support which is not α-irrigable for $\alpha = \frac{1}{N'}$.*

PROOF. Let B be the unit ball of $\mathbb{R}^N$, let μ_L be the normalized Lebesgue measure on B and $\alpha = \frac{1}{N'}$. Being, by Theorem 5.1.2, μ_L not α-irrigable, by Proposition 5.4.1, we can consider a discretization μ_1 of μ_L such that for any pattern $\tilde{\chi}_1$ which irrigates μ_1: $I_\alpha(\chi_1) \geq 1$. Analogously, let μ_2 be a discretization of $\frac{1}{2}\mu_L$ distributed on $\frac{1}{2}B$ such that for any pattern χ_2 which irrigates μ_2: $I_\alpha(\chi_2) \geq 2$. Recoursively, for any $n \in \mathbb{N}$ let μ_n be a discretization of $\frac{1}{2^n}\mu_L$ restricted to $\frac{1}{n}B$ such that for any pattern χ_n which irrigates it,

$$I_\alpha(\chi_n) \geq n \tag{5.4.4}$$

holds true. Let $\overline{\mu} = \sum_{n\geq 1}\mu_n$ and let us remark that $\operatorname{supp}(\overline{\mu}) = \bigcup_{n\geq 1}\operatorname{supp}(\mu_n) \cup \{0\}$ and therefore, being for all $n \geq 1$ $\operatorname{supp}(\mu_n)$ a finite set, $\operatorname{supp}(\overline{\mu})$ is countable.

Let us show that $\overline{\mu}$ is not α-irrigable. Indeed, the α-irrigability of $\overline{\mu}$ would imply, by Lemma 5.4.1 (being $\mu_n \leq \overline{\mu}$ for all $n \geq 1$), that any μ_n is α-irrigable with a bounded cost and this is in contradiction with (5.4.4). ∎

A measure as in the above statement satisfies, in particular, the condition in the following one and shows that, in general, $d(\mu) \not\leq d_s(\mu)$.

Example 5.4.3 *There exist probability measures μ such that $d_s(\mu) = 0$ (minimum possible value) and $d(\mu) = N$ (maximum possible value).*

We have stated in Section 5.1 that the information that, for a probability measure μ and a real number $\alpha \in]0,1[$, the critical dimension d_α coincides with the irrigability dimension $d(\mu)$, (i.e. $\alpha = \frac{1}{(d(\mu))'}$) does not allow to decide whether the measure is irrigable or not. Examples 5.4.4 and 5.4.5 will motivate this claim.

Example 5.4.4 *Let μ be an Ahlfors probability measure in dimension $\beta \geq 0$. Then μ is not α-irrigable if $d_\alpha = \beta = d(\mu)$.*

PROOF. The thesis follows from Corollary 5.1.6. ∎

Remark 5.4.2 *One has Ahlfors regular measures for every dimension $\beta < N$. Indeed, let $\mathcal{C}$ be a selfsimilar (Cantor) set of $\mathbb{R}^N$ with dimension $\beta > 0$. Let us call $\mathcal{H}^\beta_{\lfloor \mathcal{C}}$ the Hausdorff measure* distributed *on $\mathcal{C}$, i.e. the measure on $\mathbb{R}^N$ defined setting $\forall X \subset \mathbb{R}^N$*

$$\mathcal{H}^\beta_{\lfloor \mathcal{C}}(X) = \mathcal{H}^\beta(X \cap \mathcal{C})\,.$$

Then $\mathcal{H}^\beta_{\lfloor \mathcal{C}}$ is Ahlfors regular with dimension β.

Example 5.4.5 *There exist some measures μ for which $d(\mu)$ is a minimum, i.e. there exist some measures μ and some exponents $\alpha \in]0,1[$ such that $d(\mu) = d_\alpha$ and μ is α-irrigable.*

PROOF. Indeed, let us fix $\alpha \in]0, 1[$ and let $(\mathcal{C}_n)_{n\in I\!N}$ be a sequence of self similar (Cantor) sets in $I\!R^N$ with dimension d_{α_n} where $(\alpha_n)_{n\in I\!N}$ is a sequence converging to α from below, by Corollary 5.1.5 and Remark 5.4.2 we know that $d(\mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}) = d_{\alpha_n} < d_\alpha$ and, by Remark 5.1.3 (1), we get that $\mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}$ is α-irrigable. Let us consider a suitable sequence $(\varepsilon_n)_{n\in I\!N}$ of positive real numbers, sufficiently small to allow us to consider $\mu = \sum_{n\in I\!N} \varepsilon_n \mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}$. We know, by Corollary 5.4.2 that also $\varepsilon_n \mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}$ are irrigable and we call χ_n an irrigation pattern which irrigates $\varepsilon_n \mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}$ (i.e. such that $I_\alpha(\chi_n) < +\infty$ and $\mu_{\chi_n} = \varepsilon_n \mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}$). Under the choice of a sufficiently infinitesimal sequence of coefficients $(\varepsilon_n)_{n\in I\!N}$, we have $\sum_{n\in I\!N} I_\alpha(\chi_n) < +\infty$.

Now let us consider the bunch χ of the sequence of patterns χ_n (see (5.9.1)) so that by Remark 5.9.1 we have

$$\mu_\chi = \mu \; . \tag{5.4.5}$$

and

$$I_\alpha(\chi) \le \sum_{n\in I\!N} I_\alpha(\chi_n) < +\infty \; . \tag{5.4.6}$$

Equality (5.4.5) and inequality (5.4.6) give the α - irrigability of μ and therefore $d(\mu) \le d_\alpha$. Moreover $d(\mu) \ge d_\alpha$. Indeed being, for all $n \in I\!N$, $\mu \ge \varepsilon_n \mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}$ we get, by Corollary 5.4.1 and Remark 5.4.2 that $d(\mu) \ge d(\varepsilon_n \mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}) = d(\mathcal{H}^{d_{\alpha_n}}_{\lfloor \mathcal{C}_n}) = d_{\alpha_n} \to d_\alpha$. ∎

5.5 Discretizations and resolution dimensions of a measure

Definition 5.5.1 *We shall say that a measure μ is a* discrete measure *if*

$$\text{card}(\text{supp}(\mu)) < \infty$$

and we shall call card(supp(μ)) "resolution" *of μ.*

Definition 5.5.2 *For every $n \in I\!N$ we shall denote by D_n the set containing all the discrete probability measures whose resolution is less or equal to n. Equivalently, D_n is the set of all the convex combinations of n Dirac masses.*

For any $p \ge 1$ we recall the definition of Kantorovitch-Wasserstein distance of index p.

Definition 5.5.3 *Let $p \ge 1$ and let μ, ν be two probability measures. We define the Kantorovitch-Wasserstein distance of index p between μ and ν by*

$$d_p(\mu, \nu) = \left(\min_\sigma \int_{\Omega\times\Omega} |x-y|^p d\sigma \right)^{\frac{1}{p}} ,$$

where the minimum is taken on all the transport plans σ which lead μ to ν, i.e. measures on $\Omega \times \Omega$ such that their push forward measures by the first and the second projection on Ω respectively are μ and ν ($\pi_{1\#}\sigma = \mu$ and $\pi_{2\#}\sigma = \nu$) (see [1] for more details).

Definition 5.5.4 *Let μ be a probability measure. For every $n \in I\!N$, given $p \geq 1$, we shall denote by μ_n (or, when necessary, by μ_n^p) one of the elements of D_n of minimal distance with respect to the Kantorovitch-Wasserstein distance of index p from μ. We shall refer to μ_n as to* a discretization of resolution n of μ *(with respect to the index p).*

Proposition 5.5.1 *Let $1 \leq p \leq q$ and let μ, ν be two probability measures. Then*

$$d_p(\mu, \nu) \leq d_q(\mu, \nu) \tag{5.5.1}$$

and

$$d_q(\mu, \nu) \leq d^{1-\frac{p}{q}} (d_p(\mu, \nu))^{\frac{p}{q}} \ , \tag{5.5.2}$$

where the constant d is the diameter of $\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu)$.

PROOF. Let τ be an optimal transport plan from μ to ν with respect to the Kantorovitch-Wasserstein distance of index q. Then, by Hölder inequality,

$$[d_p(\mu, \nu)]^p \leq \int_{\Omega\times\Omega} |x-y|^p d\tau \leq \left(\int_{\Omega\times\Omega} |x-y|^q d\tau\right)^{\frac{p}{q}} \left(\int_{\Omega\times\Omega} d\tau\right)^{1-\frac{p}{q}} = [d_q(\mu, \nu)]^p \ .$$

To prove (5.5.2) we shall consider an optimal transport plan τ from μ to ν with respect to the p distance. Let us call $d = \operatorname{diam}(\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu))$, then

$$[d_q(\mu, \nu)]^q \leq \int_{\Omega\times\Omega} |x-y|^q d\tau \leq d^{q-p} \int_{\Omega\times\Omega} |x-y|^p d\tau = d^{q-p}[d_p(\mu, \nu)]^p \ ,$$

from which the thesis follows. ∎

In the following, for any $n \in I\!N$ and $p \geq 1$, we shall use the Kantorovitch-Wasserstein distance of index p of μ from D_n

$$\delta_n^p = d_p(\mu, \mu_n) = d_p(\mu, D_n) \ , \tag{5.5.3}$$

to test "how good" a discretization of resolution n can be. When we shall deal with more than one measure we shall use the more detailed notation $\delta_n^p(\mu) = d_p(\mu, D_n)$.

In [3] the following proposition, which gives a relation between the cost of an irrigation pattern χ and the Kantorovitch-Wasserstein distance $\delta_1^{\frac{1}{\alpha}}(\mu_\chi)$ of the irrigation measure μ_χ from a Dirac delta, has been proved.

Proposition 5.5.2 *Let χ be an irrigation pattern, with a source point S, then*

$$\delta_1^{\frac{1}{\alpha}}(\mu_\chi) \leq d_{\frac{1}{\alpha}}(\mu_\chi, \delta_S) \leq I_\alpha(\chi) \ .$$

Remark 5.5.1 *Let $1 \leq p \leq q$, $n \in I\!N$, $n \geq 1$ and let μ be a probability measure. Then from (5.5.1) and (5.5.2), applied for $\nu = \mu_n^r$ and $\nu = \mu_n^q$, we get*

$$\delta_n^p \leq \delta_n^q \tag{5.5.4}$$

and

$$\delta_n^q \leq d^{1-\frac{p}{q}} (\delta_n^p)^{\frac{p}{q}} \ , \tag{5.5.5}$$

where the constant d is the diameter of $\operatorname{supp}(\mu)$.

It is clear that, increasing the number n, the discretizations μ_n become more accurate, therefore it is rather natural to make some decay hypothesis on δ^p_n.

Definition 5.5.5 *Let μ be a probability measure and $p \geq 1$, then we shall call* resolution dimension *of μ of index p the constant $d^p_r(\mu)$ defined as follows*

$$d^p_r(\mu) = \left(-\limsup_{n\to+\infty} \log_n \ \delta^p_n\right)^{-1} . \tag{5.5.6}$$

Remark 5.5.2 *Let $0 < a < (d^p_r(\mu))^{-1}$, then there exists $\overline{n}$ such that*

$$\forall n \geq \overline{n} \, : \, \delta^p_n \leq n^{-a} .$$

Conversely, if $a > (d^p_r(\mu))^{-1}$ then for any $C > 0$ we have

$$\delta^p_n > n^{-a}$$

for arbitrarily large values of $n \in I\!N$.

Proposition 5.5.1 allows us to state the corresponding properties of $d^p_r(\mu)$ in terms of the index p.

Proposition 5.5.3 *Let $1 \leq p \leq q$ and μ a probability measure, then*

$$d^p_r(\mu) \leq d^q_r(\mu) \tag{5.5.7}$$

and

$$d^q_r(\mu) \leq \frac{q}{p} d^p_r(\mu) . \tag{5.5.8}$$

PROOF. Taking into account (5.5.6), both inequalities easily follow from (5.5.4) and (5.5.5). ∎

Remark 5.5.3 *It is useful to remark that, by (5.5.7) and (5.5.8), $d^p_r(\mu)$ changes with continuity with respect to the index p. Moreover, if there exists a index $p \geq 1$ for which $d^p_r(\mu) = 0$, then for all $q < +\infty$ $d^q_r(\mu) = 0$. This means that, in such a case, we can not change the resolution dimension of μ acting on the index p as far as it is finite.*

In Appendix B we shall prove the following propositions.

Proposition 5.5.4 *Let μ be a probability measure, then*

$$d^\infty_r(\mu) = d_M(\mu) .$$

Proposition 5.5.5 *Let μ be a probability measure. Then*

$$d_c(\mu) \leq d^1_r(\mu) \leq d^p_r(\mu) \qquad \forall p \geq 1 .$$

Remark 5.5.4 *Since in the case $p = +\infty$ the dimension $d_r^p(\mu)$ agrees with $d_M(\mu)$ we shall use the notation d_r^∞ in order to denote a weaker case, according to Proposition 5.5.3, of the dimension of index $+\infty$, defined as*

$$d_r^\infty(\mu) = \sup_{p \geq 1} d_r^p(\mu) \; .$$

Example 5.10.1 will show how for $p = +\infty$ the "strong" and the "weak" dimensions $d_M(\mu)$ and $d_r^\infty(\mu)$ are, in general, distinct.

By the following lemmas we shall estimate the resolution dimension $d_r^p(\mu)$ in the case in which the probability measure μ enjoys some regularity properties, beginning by considering a probability measure which satisfies the lower Ahlfors regularity (LAR). As a consequence of Corollary 5.1.3, taking into account that, by Proposition 5.5.4, $d_M(\mu) = d_r^\infty(\mu)$ we have the following corollary.

Corollary 5.5.1 *Let μ be a probability measure which satisfies (LAR) in dimension $\beta \geq 0$. Then*

$$d_r^p(\mu) \leq \beta \qquad \forall p \geq 1 \; . \tag{5.5.9}$$

PROOF. Indeed, by Corollary 5.1.3, we have $d_r^\infty(\mu) = \sup_{p \geq 1} d_r^p(\mu) \leq d_M(\mu) \leq \beta$. ∎

In the case in which a probability measure μ satisfies the upper Ahlfors regularity (UAR), by Corollary 5.1.4 Proposition 5.5.5 admits the following corollary.

Corollary 5.5.2 *Let μ be a probability measure such that (UAR) holds true. Then*

$$\beta \leq d_r^1(\mu) \; . \tag{5.5.10}$$

From Corollary 5.5.1 and Corollary 5.5.2 we easily get the following proposition.

Proposition 5.5.6 *Let μ be an Ahlfors regular probability measure of dimension $\beta \geq 0$. Then*

$$d_r^p(\mu) = \beta \qquad \forall p \geq 1 \; . \tag{5.5.11}$$

Remark 5.5.5 *So, when the measure is Ahlfors regular, the value of the resolution dimensions does not depend on the index, while for a generic measure, as it will be shown in the Appendix B by some examples, the resolution dimension is "out of focus" in the sense that different indexes give different values. We shall show that, in any case, it is always possible to find an index, suitably characterized, which gives a resolution dimension which coincides with the irrigability dimension of the measure (see Theorem 5.8.1 below).*

5.6 Irrigability results via resolution dimension

In this section we shall establish some irrigability estimates, in particular we shall prove the following proposition.

Proposition 5.6.1 *Let μ be a probability measure and $p \geq 1$. If $p' > d_r^p(\mu)$, then μ is α-irrigable, with $\alpha = \frac{1}{p}$.*

The idea of the proof consists in fixing $n \in I\!N$ and in using the distribution pattern introduced in Section 5.2, induced by a hierarchy related to the measures μ_h, in order to irrigate μ_n. An estimate of the cost and a passage to the limit will then ensure the irrigability of μ.

More precisely, let us fix an integer $k > 1$ (we shall assume that k is large enough to have the estimate

$$2k^{-a} < 1 \,, \tag{5.6.1}$$

where we choose a number a such that $0 < 1 - \alpha = \frac{1}{p'} < a < (d_r^p(\mu))^{-1}$, such a bound will be useful later on). Then we shall denote by X_h the support of μ_{k^h} for any $h \leq n$. Let $X_{h+1} = \{x_1, x_2, \ldots, x_{k^{h+1}}\}$ and let $m_i = \mu_{k^{h+1}}(\{x_i\})$. We would like to have a map $\varphi_h : X_{h+1} \to X_h$ such that $d_p(\mu_{k^h}, \mu_{k^{h+1}}) = \left(\sum_{i=1}^{k^{h+1}} m_i l_i^p\right)^{\frac{1}{p}}$, where $l_i = |x_i - \varphi_h(x_i)|$. However such a formula would require the Kantorowich-Wassernstain distance between $\mu_{k^{h+1}}$ and μ_{k^h} to be achieved by a transport map, while the discrete nature of $\mu_{k^{h+1}}$ guarantees only the existence of an optimal transport plan, see [1]. Roughly speaking, if we want to carry in an optimal way the masses m_i, given on X_{h+1}, to the set X_h in order to reconstruct μ_{k^h}, we cannot bring each m_i to a unique point $\varphi_h(x_i)$ but we must split it in several parts and bring each piece to a different point. To avoid this problem, we shall replace the measures μ_{k^h} by other measures $\tilde{\mu}_{k^h}$, recoursively defined taking $\tilde{\mu}_{k^n} = \mu_{k^n}$ and proceeding backward by choosing, for $h < n$, a measure $\tilde{\mu}_{k^h}$ which gives an optimal approximation of $\tilde{\mu}_{k^{h+1}}$ on D_{k^h}.

For $h \leq n$, let $\tilde{X}_h = \mathrm{supp}(\tilde{\mu}_{k^h})$, $\tilde{X}_h = \{x_1^h, x_2^h, \ldots, x_{k^h}^h\}$ and $m_i^h = \tilde{\mu}_{k^h}(\{x_i^h\})$. Now, for $h < n$, it is not difficult to choose $\tilde{\mu}_{k^h}$ in such a way that an optimal transport plan for the Kantorowich-Wassernstain distance of index p between $\tilde{\mu}_{k^{h+1}}$ and $\tilde{\mu}_{k^h}$ can be induced by a transport map $\varphi_h : \tilde{X}_{h+1} \to \tilde{X}_h$.

Indeed, if an optimal transport plan splits a mass m_i^{h+1} bringing each piece to a different point of $\tilde{X}_h$, we just need to modify the values of the masses m_j^h in the points of $\tilde{X}_h$ of minimal distance from x_i^{h+1} in such a way to let the mass m_i^{h+1} fully carried to one of such points, arbitrarily chosen, which we shall chose as $\varphi_h(x_i^{h+1})$. Such a modification does not affect the minimality of $\tilde{\mu}_{k^n}$.

Therefore, by letting

$$\tilde{l}_i^{h+1} = |x_i^{h+1} - \varphi_h(x_i^{h+1})| \,,$$

we can be sure that

$$d_p(\tilde{\mu}_{k^h}, \tilde{\mu}_{k^{h+1}}) = \left(\sum_{i=1}^{k^{h+1}} m_i^{h+1} (\tilde{l}_i^{h+1})^p\right)^{\frac{1}{p}} . \tag{5.6.2}$$

In the following, for $p \geq 1$ and for all $h \in \{0, 1, \ldots, n\}$, we shall set

$$\tilde{\delta}_{k^h}^p = d_p(\mu, \tilde{\mu}_{k^h}) \,,$$

beside the already introduced notation

$$d^p_{k^h} = d_p(\mu,\, \mu_{k^h})\ .$$

The following metric lemmas, which are only based on the optimality of $\tilde{\mu}_{k^h}$ asked in the recursive choice, will provide an estimate of the left hand side of (5.6.2).

Lemma 5.6.1 *For all $p \geq 1$*

$$\forall h < n: \quad \tilde{\delta}^p_{k^h} \leq \sum_{i=0}^{n-h} 2^i \delta^p_{k^{h+i}}\ . \tag{5.6.3}$$

Proof. In the following we will forget the index p. The result easily follows from an iteration of the following inequality

$$\begin{aligned}\forall h < n: \quad \tilde{\delta}_{k^h} &\leq d(\tilde{\mu}_{k^h}, \tilde{\mu}_{k^{h+1}}) + d(\tilde{\mu}_{k^{h+1}}, \mu) \leq d(\mu_{k^h}, \tilde{\mu}_{k^{h+1}}) + \tilde{\delta}_{k^{h+1}} \\ &\leq d(\mu_{k^h}, \mu) + d(\mu, \tilde{\mu}_{k^{h+1}}) + \tilde{\delta}_{k^{h+1}} = \delta_{k^h} + 2\tilde{\delta}_{k^{h+1}}\ ,\end{aligned} \tag{5.6.4}$$

which just uses the triangular inequality and the optimality of $\tilde{\mu}_{k^h}$. Then (5.6.3) follows by induction on $n-h$. It holds true for $h=n$ (being $\tilde{\mu}_{k^n} = \mu_{k^n}$) then, if we assume it true if h is replaced by $h+1$, taking into account (5.6.4) we have

$$\tilde{\delta}_{k^h} \leq \delta_{k^h} + 2\tilde{\delta}_{k^{h+1}} \leq 2 \sum_{i=0}^{n-h-1} 2^i \delta_{k^{h+i+1}} + \delta^h_k = \sum_{i=0}^{n-h-1} 2^{i+1} \delta_{k^{h+i+1}} + \delta_{k^h} = \sum_{i=0}^{n-h} 2^i \delta_{k^{h+i}}\ .$$

∎

Lemma 5.6.2 *Given $p \geq 1$, $a < (d^p_r(\mu))^{-1}$. Then $\exists C > 0$ such that*

$$\forall h \leq n: \quad \tilde{\delta}^p_{k^h} \leq C(k^h)^{-a}\ . \tag{5.6.5}$$

Proof. By Lemma 5.6.1 and Remark 5.5.2, we have

$$\begin{aligned}\delta^p_{k^h} &\leq \sum_{i=0}^{n-h} 2^i \delta^p_{k^{h+i}} \leq C \sum_{i=0}^{n-h} 2^i (k^{h+i})^{-a} = C(k^h)^{-a} \sum_{i=0}^{n-h} (2k^{-a})^i \\ &< C(k^h)^{-a} \sum_{i=0}^{+\infty} (2k^{-a})^i = C(k^h)^{-a}\ ,\end{aligned}$$

where for the last inequality we have used (5.6.1). ∎

By the triangular inequality and (5.6.5) we get the following corollary.

Corollary 5.6.1 *Let $p \geq 1$ and let $a < (d_r^p(\mu))^{-1}$. Then $\exists C > 0$ such that*

$$\forall h < n: \quad d_p(\tilde{\mu}_{k^h}, \tilde{\mu}_{k^{h+1}}) \leq C(k^h)^{-a} . \tag{5.6.6}$$

Combining (5.6.2) and (5.6.6) we have

$$\left(\sum_{i=1}^{k^{h+1}} m_i^{h+1} (\tilde{l}_i^{h+1})^p \right)^{\frac{1}{p}} \leq C(k^h)^{-a} . \tag{5.6.7}$$

Now let us call χ_n a distribution pattern relative to the hierarchy $((P_i, \gamma_i))_{i \in I}$, $I = \{1, \ldots, n\}$, (where, for $1 \leq i \leq n$, $P_i = \tilde{X}_{n-i+1}$ and, for $1 \leq i < n$, $\gamma_i = \varphi_{n-i}$ while γ_n is the constant map on P_n of a constant value given by the source S) and to the measure $\overline{\mu}_1 = \mu_{k^n}$ defined on the basic level $P_1 = \tilde{X}_n$. Lemma 5.3.1 allows us to prove Proposition 5.6.1.

Proof of Proposition 5.6.1. Since $p' > d_r^p(\mu)$, we can fix a such that $0 < 1 - \alpha = \frac{1}{p'} < a < (d_r^p(\mu))^{-1}$ and $k \in I\!N$ satisfying (5.6.1). Given $n \in I\!N$, we get the existence of a sequence $(\tilde{\mu}_{k^h})_{0 \leq h \leq n}$ such that Corollary 5.6.1 holds true. The cost $I_\alpha(\chi_n)$ needed to irrigate μ_{k^n} can be estimated with the use of (5.3.1) in Lemma 5.3.1. Taking into account (5.6.7) and calling $b = a - 1 + \alpha > 0$ we get by Hölder Inequality

$$\begin{aligned} I_\alpha(\chi_n) &= \sum_{i=1}^{n} \sum_{j=1}^{k^{n-i+1}} (m_j^{n-i+1})^\alpha |x_j^{n-i+1} - \gamma_i(x_j^{n-i+1})| = \sum_{i=1}^{n} \sum_{j=1}^{k^{n-i+1}} (m_j^{n-i+1})^\alpha \tilde{l}_j^{n-i+1} \\ &\leq \sum_{i=1}^{n} \left[\sum_{j=1}^{k^{n-i+1}} m_j^{n-i+1} (\tilde{l}_j^{n-i+1})^{\frac{1}{\alpha}} \right]^\alpha (k^{n-i+1})^{1-\alpha} \\ &\leq Ck^a \sum_{i=1}^{n} (k^{n-i+1})^{-(a-1+\alpha)} < Ck^a \sum_{i=1}^{+\infty} (k^{-b})^i < +\infty . \end{aligned}$$

Therefore we get a bound on the cost to irrigate μ_{k^n} which does not depend on n. Therefore there exist a constant $\tilde{C} = Ck^a > 0$ such that we can irrigate every discretization spending at most $\tilde{C} \sum_{i=1}^{+\infty} (k^{-b})^i$, where $b = a - 1 + \alpha > 0$. So, by the compactness theorem [4, Theorem 8.1], we have, passing to a subsequence, a limit pattern modulo equivalence χ such that $I_\alpha(\chi) < +\infty$. By construction, its irrigation measure μ_χ is just the measure μ which is therefore α-irrigable. ∎

Taking into account that $d_\alpha = \left(\frac{1}{\alpha}\right)'$, we can also restate Proposition 5.6.1 in the following way.

Proposition 5.6.2 *Let μ be a probability measure for which there exists a constant $\alpha \in]0, 1[$ such that*

$$d_r^{\frac{1}{\alpha}}(\mu) < d_\alpha .$$

Then μ is α-irrigable.

Corollary 5.6.2 *Let μ be a probability measure and let $p \geq 1$ be a solution to*

$$d_r^p(\mu) \leq p' \,. \tag{5.6.8}$$

Then

$$d(\mu) \leq p' \,. \tag{5.6.9}$$

PROOF. Let $\alpha \in]0,1[$ be such that $d_\alpha > p'$. From $d_\alpha > p'$ we get $\frac{1}{\alpha} < p$, therefore, applying (5.5.7) and taking into account (5.6.8), we have $d_r^{\frac{1}{\alpha}}(\mu) \leq d_r^p(\mu) \leq p' < d_\alpha$. So, by Proposition 5.6.2, we know that μ is α-irrigable, therefore $d(\mu) \leq d_\alpha$. Letting $d_\alpha \to p'$ we get the thesis. ∎

5.7 Nonirrigability results via resolution dimension

The aim of this section is to prove Proposition 5.7.1, or equivalently Proposition 5.7.2, which gives the counterpart of the results stated in Proposition 5.6.1 (and respectively of Proposition 5.6.2).

Proposition 5.7.1 *Let μ be a probability measure and $p \geq 1$. If $p' < d_r^p(\mu)$, then μ is not α-irrigable, with $\alpha = \frac{1}{p}$.*

Taking into account that $d_\alpha = \left(\frac{1}{\alpha}\right)'$, Proposition 5.7.1 can be also restated in the following way.

Proposition 5.7.2 *Let μ be a probability measure for which there exists a constant $\alpha \in]0,1[$ such that*

$$d_r^{\frac{1}{\alpha}}(\mu) > d_\alpha \,.$$

Then μ is not α-irrigable.

Proposition 5.7.2 admits the following corollary whose proof is similar to the proof of Corollary 5.6.2.

Corollary 5.7.1 *Let μ be a probability measure and let $p \geq 1$ be a solution to*

$$d_r^p(\mu) \geq p' \,. \tag{5.7.1}$$

Then

$$d(\mu) \geq p' \,. \tag{5.7.2}$$

The proof of Proposition 5.7.1 is based on the semicontinuity properties of two functions which we are going to introduce and which, time by time, give the maximum cost of the single or multiple branches (see Definition 5.9.9).

Definition 5.7.1 *Let χ be an irrigation pattern, then we shall consider the following two functions W_χ and S_χ defined on $I\!R_+$ by setting, for all $t \geq 0$,*

$$W_\chi(t) = \max\{I_\alpha(V) \mid V \in \mathcal{V}_t(\chi)\} \tag{5.7.3}$$

$$S_\chi(t) = \max\{I_\alpha(V) \mid V \in \mathcal{V}_t^s(\chi)\} \,. \tag{5.7.4}$$

We state some properties enjoyed by the above defined functions.

Proposition 5.7.3 *Let χ be an irrigation pattern of finite cost. Then*

$$\forall t \geq 0 : \qquad S_\chi(t) \leq W_\chi(t) \tag{5.7.5}$$

and

$$\forall t_1 < t_2 : \qquad S_\chi(t_1) \geq W_\chi(t_2) \,. \tag{5.7.6}$$

So, in particular, S_χ and W_χ are decreasing functions.

Proof. The proof of the statement relies on the definition of strict equivalence relation (see Definition 5.9.4). Indeed, a strict vessel at time t contains a multiple vessel at a bigger time. ∎

Proposition 5.7.4 *Let χ be an irrigation pattern of finite cost. Then S_χ is a lower semicontinuous function.*

Proof. In view of Proposition 5.7.3, the statement is equivalent to the right continuity of S_χ.

Let $(t_n)_{n\in I\!N}$ be a decreasing sequence of real positive numbers such that $\lim_{n\to+\infty} t_n = \bar{t} \in I\!R_+$. We shall prove that $\lim_{n\to+\infty} S_\chi(t_n) = S_\chi(\bar{t})$. Let $\overline{V} \in \mathcal{V}_{\bar{t}}^s(\chi)$ such that $S_\chi(\bar{t}) = I_\alpha(\overline{V})$ and fix $\bar{p} \in \overline{V}$. Let us consider $V_n = [\bar{p}]_{t_n}^s \in \mathcal{V}_{t_n}^s(\chi)$. The sequence $(V_n)_{n\in I\!N}$ is monotone increasing under inclusion, moreover $\bigcup_{n\in I\!N} V_n = \overline{V}$.

Let us set, for any $n \in I\!N$, $A_n = V_n \times [t_n, +\infty[$, then using Remark 5.9.11 we have

$$S_\chi(\bar{t}) = I_\alpha(\overline{V}) = I_\alpha(\bigcup_{n\in I\!N} V_{t_n}, \bar{t}) = \nu(\bigcup_{n\in I\!N} A_n) = \lim_{n\to+\infty} \nu(A_n) = \lim_{n\to+\infty} I_\alpha(V_n, t_n) \leq \lim_{n\to+\infty} S_\chi(t_n) \,.$$

∎

Proposition 5.7.5 *Let χ be an irrigation pattern of finite cost. Then W_χ is an upper semicontinuous function.*

Proof. In view of Proposition 5.7.3, the statement is equivalent to the left continuity of W_χ. Let $(t_n)_{n\in I\!N}$ be an increasing sequence of real positive numbers less or equal to $\bar{t}$ such that $\lim_{n\to+\infty} t_n = \bar{t} \in I\!R_+$.

Let us consider, for any $n \in I\!N$, V_n a vessel at time t_n such that

$$I_\alpha(V_n) = W_\chi(t_n) .$$

Let us set, for any $n \in I\!N$, $A_n = V_n \times [t_n, +\infty[$, then using Remark 5.9.11 we have

$$\nu(A_n) = I_\alpha(V_n, t_n) \geq I_\alpha(\overline{V}, \overline{t}) . \tag{5.7.7}$$

We can assume that $\lim_{n\to+\infty} \nu(A_n) = \lim_{n\to+\infty} W_\chi(t_n) > 0$ (otherwise we have nothing to prove), therefore $(A_n)_{n\in I\!N}$ admits a subsequence, still denoted by $(A_n)_{n\in I\!N}$, with a nonempty intersection.

By using Lemma 5.9.1 we get that the sequence of the vessels V_n is decreasing. Let us set $V = \bigcap_{n\in I\!N} V_n$. By construction, we have that V is a vessel at time $\overline{t}$. Consequently, being $(t_n)_{n\in I\!N}$ increasing, it follows that $(A_n)_{n\in I\!N}$, is decreasing. Therefore, by (5.7.7), we have $I_\alpha(V, \overline{t}) = \nu(\bigcap_{n\in I\!N} A_n) = \lim_{n\to+\infty} \nu(A_n) = \lim_{n\to+\infty} I_\alpha(V_n, t_n) = \lim_{n\to+\infty} W_\chi(t_n)$ and so

$$\lim_{n\to+\infty} W_\chi(t_n) = I_\alpha(V) \leq W_\chi(\overline{t}) .$$

∎

Propositions 5.7.4 and 5.7.5 allow us to give the following proposition.

Proposition 5.7.6 *Let χ be an irrigation pattern, with a finite cost $I_\alpha(\chi) < +\infty$. Let $0 < a < I_\alpha(\chi)$. Then there exists a multiple branch χ' of χ such that $I_\alpha(\chi') \geq a$ which is the bunch of single branches $(\chi'_s)_j$ such that $I_\alpha((\chi'_s)_j) \leq a$ for all j.*

PROOF. Let $\overline{t} = \sup\{t \,|\, W_\chi(t) \geq a\}$. Using Proposition 5.7.5 we get the existence of a vessel $V \in \mathcal{V}_{\overline{t}}(\chi)$ such that $I_\alpha(V) \geq a$. On the other hand, being $S_\chi(t) \leq W_\chi(t)$, by Proposition 5.7.4 we know that can not exist any vessel $V^s \in \mathcal{V}^s_{\overline{t}}(\chi)$ such that $I_\alpha(V^s) > a$. Indeed, on the contrary, by using the right continuity of S_χ, we would get a time $t > \overline{t}$ such that $W_\chi(t) \geq S_\chi(t) \geq a$, in contradiction to the maximality of $\overline{t}$. ∎

Theorem 5.7.1 *Let $\chi : \Omega \times [0, +\infty[\to I\!R^N$ be an irrigation pattern with a finite cost $c = I_\alpha(\chi)$. Then for $n \geq 1$ there exist n source points for a finite number of patterns χ_i such that*

1. $\forall i \; I_\alpha(\chi_i) \leq \frac{c}{n}$
2. $\mu_\chi = \sum_i \mu_{\chi_i}$.

PROOF. Let us fix $n \in I\!N$, $n \geq 1$, and let us apply Proposition 5.7.6 to the constant $a = \frac{c}{n} < c$, where $c = I_\alpha(\chi) < +\infty$. Then there exists $t_n \geq 0$ and a multiple branch χ' of χ such that $I_\alpha(\chi') \geq \frac{c}{n}$ and χ' is the bunch of single branches whose cost is less or equal to $\frac{c}{n}$. We can regard χ' as the union of a finite number of (not necessarily single) branches with a cost less or equal to $\frac{c}{n}$. If we consider the pattern $\chi \setminus \chi'$ of χ stumped of the branch χ' (see Definition 5.9.8), according to Lemma 5.9.3, we have that $I_\alpha(\chi \setminus \chi') \leq I_\alpha(\chi) - I_\alpha(\chi') \leq c - \frac{c}{n}$.

Let us apply Proposition 5.7.6 with the same constant $\frac{c}{n}$ to the stumped pattern $\chi \setminus \chi'$ and proceed recoursively in this way. At every step, the cost of the iteratively stumped pattern loses at least $\frac{c}{n}$. So we can do at most $n-1$ stumps of this kind. At the end of this procedure we get at most n sources, which are the cut point in the stumping procedure, which globally give rise to a finite number of patterns each one with a cost which is less or equal to $\frac{c}{n}$. The second part of the statement is easily obtained by iterating (5.9.10). ∎

Proposition 5.7.7 *Let μ be a probability measure which is α-irrigable. Then $\exists C > 0$ such that for all $n \in \mathbb{N}$*

$$\delta_n^{\frac{1}{\alpha}} \leq C n^{-(1-\alpha)} . \tag{5.7.8}$$

PROOF. Let χ be an irrigation pattern such that $I_\alpha(\chi) < +\infty$ and $\mu_\chi = \mu$. Let us apply the decomposition Theorem 5.7.1 to the pattern χ, so we get, for a fixed $n \in \mathbb{N}$, n source points S_i and a finite number of subpatterns χ_i verifying 1. and 2. of the thesis of the theorem.

Let $\mu_n = \sum_{i=1}^n \mu_{\chi_i}(\mathbb{R}^N)\delta_{S_i} \in \mathcal{D}_n$. By Proposition 5.5.2 we have

$$\begin{aligned}\left[\delta_n^{\frac{1}{\alpha}}\right]^{\frac{1}{\alpha}} &\leq \left[d_{\frac{1}{\alpha}}(\mu, \mu_n)\right]^{\frac{1}{\alpha}} \leq \left[d_{\frac{1}{\alpha}}\Big(\sum_i \mu_{\chi_i}, \sum_i \mu_{\chi_i}(\mathbb{R}^N)\delta_{S_i}\Big)\right]^{\frac{1}{\alpha}} \\ &\leq \sum_i (I_\alpha(\chi_i))^{\frac{1}{\alpha}} \leq \sup_i (I_\alpha(\chi_i))^{\frac{1}{\alpha}-1} \sum_i I_\alpha(\chi_i) \leq \left(\frac{c}{n}\right)^{\frac{1}{\alpha}-1} c ,\end{aligned}$$

from which we get

$$\delta_n^{\frac{1}{\alpha}} \leq C n^{-(1-\alpha)} .$$

∎

PROOF OF PROPOSITION 5.7.2. Assuming, by contradiction that μ is α-irrigable, by (5.7.8) it would follow, by Remark 5.5.2, $(1-\alpha) \leq \left(d_r^{\frac{1}{\alpha}}\right)^{-1}$ and so $d_r^{\frac{1}{\alpha}} \leq d_\alpha$, in contradiction to our assumptions. ∎

5.8 The irrigability dimension as a resolution dimension

In this section we show that the irrigability dimension of a measure can be seen as a resolution dimension with respect to an appropriate choice of the index p.

We shall prove the following theorem.

Theorem 5.8.1 *Let μ be a probability measure. Then*

a) if $d_r^\infty(\mu) \leq 1$ then $d(\mu) = 1$;

b) if $d_r^\infty(\mu) > 1$ then $\exists p \geq N'$ such that $d(\mu) = d_r^p(\mu)$.

Moreover, an exponent p for which the above inequality holds true is the unique solution of the equation

$$d_r^p(\mu) = p' \; . \tag{5.8.1}$$

PROOF. In the case a) we have that for all $p \geq 1$,

$$d_r^p(\mu) \leq d_r^\infty(\mu) \leq 1 \leq p' \; ,$$

so by applying Corollary 5.6.2 we have that $d(\mu) \leq p'$. The thesis follows taking the limit for $p \to +\infty$.

For the proof of the remaining part of the statement, it is sufficient to show that equation (5.8.1) admits a (unique) solution $p \geq N'$. Indeed, by applying Corollaries 5.6.2 and 5.7.1 one gets $d(\mu) = p'$ and therefore b) follows.

We can get a solution to (5.8.1) since by means of Proposition 5.5.3, the map $p \mapsto d_r^p(\mu)$ is a continuous map and by Proposition 5.5.4, being $d_r^\infty(\mu) \leq d_M(\mu) \leq N$, for $p_1 = N'$ we have $d_r^{p_1}(\mu) \leq p_1'$. On the other side, by (5.5.7) we get $d_r^\infty(\mu) = \lim_{p\to+\infty} d_r^p(\mu) > 1 = \lim_{p\to+\infty} p'$. So for p_2 large enough we have $d_r^{(p_2)}(\mu) \geq p_2'$. Moreover equation (5.8.1) admits a unique solution because the map $p \mapsto d_r^p(\mu)$ is increasing and the map $p \mapsto p'$ is strictly decreasing. ∎

Remark 5.8.1 *The uniqueness of the solution to (5.8.1) does not guarantee in any way the uniqueness of the exponent p for which $d_r^p(\mu) = d(\mu)$. Indeed, by Proposition 5.5.6 for a measure μ which is Ahlfors regular in dimension $\beta = d(\mu)$ any exponent $p \geq 1$ gives $d_r^p(\mu) = d(\mu)$.*

As we have said in the introduction, Theorem 5.1.1 can be deduced from the previous result.

ALTERNATIVE PROOF OF THEOREM 5.1.1. If $d_r^\infty(\mu) \leq 1$, by Theorem 5.8.1 we have $d(\mu) = 1$ and so

$$d_c(\mu) \leq d_r^\infty(\mu) = 1 = d(\mu) \leq \max\{1, \, d_M(\mu)\} \; .$$

On the other hand, by propositions 5.5.4 and 5.5.5, we have for a suitable p

$$d_c(\mu) \leq d_r^p(\mu) = d(\mu) \leq d_M(\mu) = \max\{1, \, d_M(\mu)\} \; .$$

∎

5.9 Appendix A - Fundamental notions, remarks and notation

In this appendix we shall introduce some terminology which has been used in this paper, in particular we shall recall the same notation as in [4], introducing the notion of irrigable measure and referring to that paper for more details. Then we shall give some new definitions and useful tools.

Let $(\Omega, |\cdot|)$ be a nonatomic probability space and $\chi : \Omega \times I\!R_+ \to I\!R^N$ an irrigation pattern, as defined in Section 5.1. When we shall deal with subsets $\Omega' \subset \Omega$ we shall use $\chi_{|\Omega'}$ instead of $\chi_{|\Omega' \times I\!R_+}$ to denote the restriction of χ to $\Omega' \times I\!R_+$ and we shall call $\chi_{|\Omega'}$ the *subpattern* of χ defined on Ω'.

Let be $(\Omega_1, |\cdot|_1)$ and $(\Omega_2, |\cdot|_2)$ two disjoint probability spaces, let $S \in I\!R^N$ and let $\chi_1 \in \mathbf{P}_S(\Omega_1)$ and $\chi_2 \in \mathbf{P}_S(\Omega_2)$ be two irrigation pattern with the same source S. Let us consider the set $\Omega = \Omega_1 \cup \Omega_2$ endowed with the finite measure defined by setting for all $A \subset \Omega$, $|A| = |A \cap \Omega_1|_1 + |A \cap \Omega_2|_2$. Then we can consider χ_1 and χ_2 as subpatterns of a pattern $\chi \in \mathbf{P}_S(\Omega)$ defined by setting for a.e. $p \in \Omega$ and for all $t \in I\!R_+$

$$\chi(p,\, t) = \begin{cases} \chi_1(p,\, t) & \text{if } p \in \Omega_1 \\ \chi_2(p,\, t) & \text{if } p \in \Omega_2\,. \end{cases} \tag{5.9.1}$$

The above defined pattern will be called *bunch* of the patterns χ_1 and χ_2. It is clear that the definition of bunch of patterns can be extended to a sequence of patterns, see [3].

We recall that every set of fibers of Ω, time by time, defines an equivalence relation $\simeq_t$ on Ω by relating two points p and $q \in \Omega$ at the time t if χ_p and χ_q coincide on $[0,\, t]$. So every set of fibers at every time t divides Ω into equivalence classes which we shall call χ-vessels. For any $p \in \Omega$, we shall denote by $[p]_t$ the χ-vessel at time t which contains p, while for any $t \geq 0$ we shall denote by $\mathcal{V}_t(\chi)$ the set of all the χ-vessels at time t. The following lemma can be trivially proved, see [4].

Lemma 5.9.1 *Let χ be an irrigation pattern. Then for all $0 \leq t_1 \leq t_2$ and for all $V_{t_1} \in \mathcal{V}_{t_1}(\chi)$ and $V_{t_2} \in \mathcal{V}_{t_2}(\chi)$ we have the following two alternatives:*

1. $V_{t_2} \subset V_{t_1}$
2. $V_{t_2} \cap V_{t_1} = \emptyset$.

For a set of fibers $\chi \in \mathbf{C}_S(\Omega)$, we introduce the following function $\sigma_\chi : \Omega \to I\!R_+$ which gives the absorption time of a point defined as follows

$$\forall p \in \Omega \ : \quad \sigma_\chi(p) = \inf\{t \in I\!R_+ \mid \chi_p(\cdot) \text{ is constant on } [t, +\infty[\} \ ,$$

which will be called *stopping or absorption function for χ.*

We shall say that a point $p \in \Omega$ is absorbed when $\sigma_\chi(p) < +\infty$. A point $p \in \Omega$ is absorbed at the time t if $\sigma_\chi(p) \leq t$. Analogously we shall say that a set $X \subset \Omega$ is an absorbed set at time t if $\sigma_\chi(p) \leq t$ for a.e. $p \in X$, in particular when the set X is a χ-vessel we shall say that X is an absorbed χ-vessel. We shall denote by $A_t(\chi)$ the set of the points of Ω which are absorbed at time t, and by $A_\chi = \bigcup_{t>0} A_t(\chi)$ the set of the absorbed points. On the contrary, the set

$$M_t(\chi) = \{p \in \Omega \mid \sigma_\chi(p) > t\} = \Omega \setminus A_t(\chi)$$

is the set of the points that, at time t, are still moving. We shall call *χ-flow* at time t any not absorbed χ-vessel, and we shall denote by $\mathcal{F}_t(\chi)$ the set of the χ-flows at time t and by $F_t(\chi)$ the union of all the χ-flows at time t.

For every pattern $\chi \in \mathbf{C}_S(\Omega)$ we introduce the *irrigation function*

$$i_\chi : A_\chi \to I\!R^N ,$$

defined by setting

$$\forall p \in A_\chi \ : \qquad i_\chi(p) = \chi(p, \sigma_\chi(p))$$

and giving, point by point, the absorption position of the absorbed points.

In the case in which we deal with an irrigation pattern $\chi \in \mathbf{P}_S(\Omega)$, the absorption time function σ_χ, and, for all $t \geq 0$, the vessels and the set $A_t(\chi)$ of the absorbed points at time t are both measurable (see [4]).

We remark that $i_\chi(p) = \lim_{t\to\infty} \chi(p,t)$ and so also $i_\chi : A_\chi \to I\!R^N$ is a measurable function, as a pointwise limit of a sequence of measurable functions, when $\chi \in \mathbf{P}_S(\Omega)$.

The function i_χ induces on $I\!R^N$ the image (push-forward) measure μ_χ defined by the formula

$$\mu_\chi(A) = |i_\chi^{-1}(A)| ,$$

for any Borel set $A \subset I\!R^N$. We shall refer to μ_χ as to the *irrigation measure* induced by the pattern χ.

For a fixed cost exponent $\alpha \in]0,1[$, we introduce the functional cost I_α in [4], defined on the set $\mathbf{P}_S(\Omega)$ of the irrigation patterns χ, by the following formula

$$I_\alpha(\chi) = \int_{I\!R_+} c_\chi(t)dt ,$$

where

(5.9.2) $$c_\chi(t) = \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp$$

is the relative density cost function.

Remark 5.9.1 *Let $(\chi_n)_{n\in I\!N}$ be a sequence of patterns $\chi_n : \Omega_n \times I\!R_+ \to I\!R^N$ all with the same source $S \in I\!R^N$, let χ be the bunch of the sequence $(\chi_n)_{n\in I\!N}$. Then it is easy to show that for any $\alpha \in]0, 1[$ we have*

(5.9.3) $$I_\alpha(\chi) \leq \sum_n I_\alpha(\chi_n)$$

and

(5.9.4) $$\mu_\chi = \sum_n \mu_{\chi_n} .$$

We introduce some more definitions.

Definition 5.9.1 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, we will say that*

$$F_\chi = \{x \in I\!R^N \mid \exists t > 0,\ \exists A \in \mathcal{F}_t(\chi) \text{ s.t. } x = \chi(p,t),\ p \in A\}$$

is the flow zone of χ.

The following lemma has been proved in [3].

Lemma 5.9.2 *For any pattern χ of finite cost, F_χ is a Borel set and $d(F_\chi) = 1$.*

Definition 5.9.2 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, then the set*

$$D_\chi = \{p \in \Omega \mid p \in F_{\sigma_\chi}(p)\} \tag{5.9.5}$$

will be called dispersion *of the pattern χ. Moreover we shall say that*

- χ has a complete dispersion *(or equivalently χ is totally dispersed) if* $|\Omega \setminus D_\chi| = 0$
- χ *is* a pattern with dispersion *if* $|D_\chi| > 0$
- χ *is* a pattern without dispersion *if* $|D_\chi| = 0$.

Remark 5.9.2 *Let χ be an irrigation pattern. Then the irrigation function sends the dispersion D_χ in the flow zone F_χ, i.e.*

$$i_\chi(D_\chi) \subset F_\chi\ . \tag{5.9.6}$$

As a consequence, by the definition of irrigation measure induced by χ, we have

$$|D_\chi| \le \mu_\chi(F_\chi)\ . \tag{5.9.7}$$

Therefore to get a pattern without dispersion it is sufficient to check that $\mu_\chi(F_\chi) = 0$.

Hence, when a pattern χ has a complete dispersion, every point is absorbed just because it stops its motion while it still belongs to a flow.

Remark 5.9.3 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, then the subpattern of χ restricted to $\Omega \setminus D_\chi$ is a pattern without dispersion.*

Definition 5.9.3 *Let χ be an irrigation pattern, p, $q \in \Omega$ and $t \ge 0$. We shall introduce the separation time $s_\chi(p,q)$ of the two points p and q defined as*

$$s_\chi(p,q) = \inf\{t \ge 0 \mid \chi(p,t) \ne \chi(q,t)\} \tag{5.9.8}$$

Definition 5.9.4 *Let χ be an irrigation pattern, $p \in \Omega$ and $t \geq 0$. We shall say that two points $p, q \in \Omega$ are strictly equivalent at time t, and we shall write $p \simeq_t^s q$, if there exists $\varepsilon > 0$ such that $p \simeq_{t+\varepsilon} q$. We shall call $[p]_t^s$* strict equivalence class *defined by p at time t or equivalently* strict vessel *of the point p at time t, the following set*

$$[p]_t^s = \{q \in \Omega \,|\, p \simeq_t^s q\}$$

and we shall denote by $\mathcal{V}_t^s(\chi)$ the set of the strict vessels at time t according to the pattern χ.

Remark 5.9.4 *Let χ be an irrigation pattern, $p \in \Omega$ and $t \geq 0$. Then the* strict equivalence class *defined by p at time t coincides with the union of the equivalence classes $[p]_{t'}$ defined by p at times $t' > t$, i.e.*

$$[p]_t^s = \bigcup_{t'>t} [p]_{t'} = \bigcup_{t'>t} [p]_{t'}^s \,. \tag{5.9.9}$$

Remark 5.9.5 *For a.e. $p, q \in \Omega$ and for all $t \geq 0$:*

- *$p \simeq_t q$ for all $t \leq s_\chi(p, q)$*
- *$p \simeq_t^s q$ for all $t < s_\chi(p, q)$.*

Definition 5.9.5 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω. For any pair $(p, t) \in \Omega \times I\!R_+$ the function $\chi_{(p,t)} : [p]_t \times I\!R_+ \to I\!R^N$, defined, for all $(q, s) \in [p]_t \times I\!R_+$, by $\chi_{(p,t)}(q, s) = \chi(q, s+t)$ is the* branch *of χ starting from $\chi(p, t)$.*

Remark 5.9.6 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω. Then for any $(p, t) \in \Omega \times I\!R_+$ the branch of χ starting from $\chi(p, t)$ does not depends on p but only on the χ-vessel $[p]_t$. Moreover to get nontrivial (constant) branches one must require the vessel $[p]_t$ to be a flow.*

Definition 5.9.6 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω. For any pair $(p, t) \in \Omega \times I\!R_+$ the function $\chi'_{(p,t)} : [p]_t \times I\!R_+ \to I\!R^N$, defined, for all $(q, s) \in [p]_t^s \times I\!R_+$, by $\chi'_{(p,t)}(q, s) = \chi(q, s+t)$, where $[p]_t^s$ is the strict χ-vessel of p at time t, is the* single branch *of χ starting from $\chi(p, t)$.*

Where $[p]_t \neq [p]_t^s$ we shall have branches which are not single ones and, in order to point out that the point $\chi(p, t)$ give rise to more than one single branch, we shall call $[p]_t$ *multiple branch.*

We introduce the notion of a simple pattern which will allow us to extend Definitions 5.9.5 and 5.9.6 to any point $x \in F_\chi$.

Definition 5.9.7 *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, we will say that χ is a* simple pattern if:

- for a.e. point $p \in \Omega$ the χ-fiber of the point p, i.e. the function $\chi_p : I\!R_+ \to I\!R^N$ is a simple curve up to the stopping time $\sigma_\chi(p)$, i.e. once restricted to the interval $[0, \sigma_\chi(p)]$
- for a.e. pair of points p and q of Ω: $\chi_p(s) \neq \chi_q(t)$ for all $s, t > s_\chi(p, q)$.

It is easy to show that any subpattern of a simple pattern is simple too. Moreover, when we deal with an irrigable measure μ, it is always possible to irrigate μ by means of a simple pattern, see Lemma 4.6.2. Lemma 5.9.2 and remarks 5.9.2 and 5.9.3 allow us to assume that the restriction of μ out of a 1-dimensional set can be irrigated by a simple pattern without dispersion.

Remark 5.9.7 *It is worth to remark, see Lemma 4.6.2, that one can say that a pattern χ is simple if*

$$\forall x \in F_\chi, \exists | t \geq 0, \exists | V = [p]_t \in \mathcal{F}_\chi(t) \text{ s.t. } x = \chi(p, t) .$$

The above remark allows us to give, for simple patterns χ, the definition of branch of χ which starts from a point $x \in F_\chi$, according to definitions 5.9.5 and 5.9.6, applied to any pair (p, t) such that $x = \chi(p, t)$.

Definition 5.9.8 *Let χ be a simple irrigation pattern with source point S. For any branch χ' from $x = \chi(p, t)$ of χ, we shall call* pattern χ "stumped" of the branch χ', *the restriction of χ to $\Omega \setminus [p]_t$ and we shall denote it by $\chi \setminus \chi'$.*

The following lemma can be trivially proved.

Lemma 5.9.3 *Let χ be an irrigation pattern and let χ' be a branch of χ. Then*

$$\mu_\chi = \mu_{\chi'} + \mu_{\chi \setminus \chi'} \tag{5.9.10}$$

moreover

$$I_\alpha(\chi) \geq I_\alpha(\chi') + I_\alpha(\chi \setminus \chi'). \tag{5.9.11}$$

Definition 5.9.9 *Given a pattern χ and a vessel $V = [p]_t$ at a time t, we shall call* cost of the vessel V and we shall denote it by $I_\alpha(V, t)$ or, when there is no doubt about the time at which one refers, by $I_\alpha(V)$ the cost $I_\alpha(\chi')$, where χ' is the branch of χ which starts from $\chi(p, t)$.

Remark 5.9.8 *Let χ be an irrigation pattern, then for all $t \geq 0$*

$$\sum_{V \in \mathcal{V}_t(\chi)} I_\alpha(V) = \int_t^{+\infty} c_\chi(s) ds. \tag{5.9.12}$$

The analogous property also holds true for the strict vessels.

We recall the definition of χ-vessel evolution introduced in [4]

Definition 5.9.10 *Let $I \subset \mathbb{R}_+$. We shall say that the one-parameter family of sets $V_t = (V_t)_{t \in I}$ is a χ-vessel evolution if:*

- *it is decreasing under inclusion*
- *$V_t \in \mathcal{V}_t(\chi)$ for every $t \in I$.*

Remark 5.9.9 *Let V_t be a χ-vessel evolution, then the family $(I_\alpha(V_t))_{t\in\mathbb{R}_+}$ which, time by time, gives the cost of the vessel V_t, is decreasing but it is, in general, not continuous because of the possible "multiple branching" of the pattern χ at some time.*

The function $\varphi_\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by setting for a.e. $p \in \Omega$ and for all $t \in \mathbb{R}_+$

$$\varphi_\chi(p,\, t) = |[p]_t|^{\alpha-1}\mathbb{1}_{M_t(\chi)}(p)\ , \tag{5.9.13}$$

so that $c_\chi(t) = \int_\Omega \varphi_\chi(p,\, t)dp$.

Definition 5.9.11 *Let χ be an irrigation pattern. Then χ induces on $\Omega\times\mathbb{R}_+$ a positive measure ν defined in the following way:*

$$\nu(A) = \int_A |[p]_t|^{\alpha-1}\mathbb{1}_{M_t}(p)dp\,dt = \int_A \varphi_\chi(p,\, t)dp\,dt\ .$$

Remark 5.9.10 *Let χ be an irrigation pattern. Then*

$$\int_{\mathbb{R}_+} c_\chi(t)dt = \nu(\Omega\times\mathbb{R}_+) = I_\alpha(\chi)\ .$$

Remark 5.9.11 *Let χ be an irrigation pattern, then for all $t \geq 0$ and for all vessel $V_t \in \mathcal{V}_t(\chi)$ at time t*

$$I_\alpha(V_t) = \nu(V_t \times [t,\, +\infty[)\ .$$

5.10 Appendix B

This second appendix is devoted to the proof of some tools and propositions stated and used in Section 5.5.

Lemma 5.10.1 *Let μ be a probability measure. Then for all $d' > d^1_r(\mu)$ and for all $\varepsilon > 0$ $\exists A_\varepsilon \subset \mathbb{R}^N$ such that*

$$\mu(\mathbb{R}^N \setminus A_\varepsilon) < \varepsilon \text{ and } d_M(A_\varepsilon) \leq d'\ . \tag{5.10.1}$$

Proof. Let us call $d = d^1_r(\mu)$ and let us fix $d' > d$ and $\varepsilon > 0$. For any $k \in \mathbb{N}$ let us set $\varepsilon_k = 2^{-k}\varepsilon$. Fix $k \in \mathbb{N}$, since $\lim_{n\to+\infty} d_1(\mu,\, D_n) = 0$ there exists $n_k \in \mathbb{N}$ and $\mu_{n_k} \in D_{n_k}$ such that

$$d_1(\mu,\, \mu_{n_k}) < \varepsilon_k\ . \tag{5.10.2}$$

Let $\{x_1,\, x_2,\, \ldots,\, x_{n_k}\} = \mathrm{supp}(\mu_{n_k})$ and $U_k = \bigcup_{i=1}^{n_k} B(x_i,\, \varepsilon_k^{\frac{d}{d'}})$. By (5.10.2)

$$\mu(\mathbb{R}^N \setminus U_k) < \varepsilon_k^{1-\frac{d}{d'}}\ . \tag{5.10.3}$$

Let us call $A_\varepsilon = \bigcap_{k\in\mathbb{N}} U_k$. Then, by (5.10.3), being $d < d'$, we have

$$\mu(\mathbb{R}^N \setminus A_\varepsilon) \leq \sum_{k\in\mathbb{N}} \mu(\mathbb{R}^N \setminus U_k) < \sum_{k\in\mathbb{N}} \varepsilon_k^{1-\frac{d}{d'}} = \varepsilon^{1-\frac{d}{d'}} \sum_{k\in\mathbb{N}} (2^{-k})^{1-\frac{d}{d'}} = \frac{\varepsilon^{1-\frac{d}{d'}}}{1-\left(\frac{1}{2}\right)^{1-\frac{d}{d'}}}\ .$$

Replacing ε by $\left(\left(1-\left(\frac{1}{2}\right)^{\frac{d'-d}{d'}}\right)\varepsilon\right)^{\frac{d'}{d'-d}}$ we get the first inequality in (5.10.1).

Let us remark that, for all $k \in I\!N$ large enough, we can fix n_k in order to have

$$n_k < \varepsilon_k^{-d'} . \tag{5.10.4}$$

Indeed, being $d < d'$, by (5.5.6), we have, for large enough n, $\log_n \delta_n^1 < -\frac{1}{d'}$ i.e. $\delta_n^1 < n^{-\frac{1}{d'}}$, namely

$$n < (\delta_n^1)^{-d'} . \tag{5.10.5}$$

Then set $n_k = \min\{n \mid \delta_n^1 \leq \varepsilon_k\}$ we have $\delta_{n_k-1}^1 > \varepsilon_k$ and so, by (5.10.5), $n_k \leq \varepsilon_k^{-d'} + 1$, from which (5.10.4) follows by the arbitrariness of d'.

Now let us consider $k = \min\{h \in I\!N \mid \varepsilon_h^{\frac{d}{d'}} \leq \delta\}$, where $\delta > 0$ is a fixed real positive number small enough to have $k \geq 1$ ($\delta < \varepsilon^{\frac{d}{d'}}$ is enough), then

$$\delta < \varepsilon_{k-1}^{\frac{d}{d'}} = 2^{\frac{d}{d'}} \varepsilon_k^{\frac{d}{d'}} . \tag{5.10.6}$$

Being $A_\varepsilon \subset U_k$, A_ε can be covered by using n_k balls with a radius $\varepsilon_k^{\frac{d}{d'}} \leq \delta$. Combining (5.10.4) with (5.10.6) we have

$$n_k \leq 2^{d'} \delta^{-\frac{(d')^2}{d}} .$$

This last inequality gives, by Lemma 5.1.2, $d_M(A_\varepsilon) \leq \frac{(d')^2}{d}$. By the arbitrariness of d' the thesis follows. ∎

Proof of Proposition 5.5.5. We shall prove that, for any $d' > d_r^1(\mu)$ we have $d_c(\mu) \leq d'$. Let $d' > d_r^1(\mu)$, then by applying Lemma 5.10.1 to $\varepsilon = \frac{1}{n}$ we get the existence of a sequence of sets $(A_n)_{n\in I\!N}$ such that

$$\mu(I\!R^N \setminus \bigcup_{n=1}^{+\infty} A_n) = 0 \tag{5.10.7}$$

and

$$d(\bigcup_{n=1}^{+\infty} A_n) \leq \sup_{n\in I\!N}(d(A_n)) \leq \sup_{n\in I\!N}(d_M(A_n)) \leq d' . \tag{5.10.8}$$

By (5.10.7) and (5.10.8) we get that $d_c(\mu) \leq d'$ and therefore the thesis. ∎

We shall now prove Proposition 5.5.4 which states that the resolution dimension of index $p = +\infty$ coincides with the Minkowski dimension d_M introduced in Definition 5.1.7. This circumstance explains why we have used the notation $d_r^\infty(\mu)$ to refer to $\sup_{p\geq 1} d_r^p(\mu)$ which is, according to Proposition 5.5.3, a weaker option see Remark 5.5.4. However in the following part of this appendix, to the aim of proving the equivalence with the Minkowski dimension, we shall need to use the notation d_r^∞ according to Definition 5.5.5, taking $p = +\infty$ in (5.5.6).

By the definition of Kantorowich-Wasserstein distance we can easily deduce the following remark.

Remark 5.10.1 *Let μ be a probability measure, then for all $n \in \mathbb{N}$ δ_n^∞ is the infimum of the numbers $\delta > 0$ such that there exists a δ-net of* $\mathrm{supp}(\mu)$ *of cardinality n.*

Proof of Proposition 5.5.4. Let $\beta > d_M(\mu)$. Given $n \in \mathbb{N}$ large enough, let $\delta = n^{-\frac{1}{\beta}}$. By Lemma 5.1.1 we can cover $\mathrm{supp}(\mu)$ by using $\delta^{-\beta} = n$ balls of radius δ and so, by Remark 5.10.1, $\delta_n^\infty \leq \delta$. Therefore $\limsup_{n\to+\infty} \log_n \delta_n^\infty \leq -\frac{1}{\beta}$, namely $d_r^\infty \leq \beta$. By the arbitrariness of β, $d_r^\infty \leq d_M$ follows. Conversely, let $\beta > d_r^\infty$, namely $-\frac{1}{\beta} > \limsup_{n\to+\infty} \log_n \delta_n^\infty$. Then $\delta_n^\infty < n^{-\frac{1}{\beta}}$, namely $n < (\delta_n^\infty)^{-\beta}$ for n large enough and, by Remark 5.10.1, $\mathrm{supp}(\mu)$ has a δ_n^∞-net of cardinality n. Since $n < (\delta_n^\infty)^{-\beta}$, we can deduce from Lemma 5.1.2 that $d_M \leq \beta$ and, by the arbitrariness of β, that $d_M \leq d_r^\infty$. ∎

We know by Proposition 5.5.3 that $\sup_{p\geq 1} d_r^p(\mu) \leq d_r^\infty(\mu) = d_M(\mu)$. We shall show by the following example that, in general, the two values are different, so the weaker definition of d_r^∞ gives a different dimension.

Example 5.10.1 *There exist probability measures μ such that $d_M(\mu) = N$ and $\sup_{p\geq 1} d_r^p(\mu) = 0$.*

Proof. Let $\mu = \sum_{n=1}^\infty m_n \delta_{x_n}$ where, for all $n \in \mathbb{N}$, $x_n \in \mathbb{Q}^N \cap B_{\frac{1}{2}}(S)$ and $m_n = ce^{-n} > 0$, where the constant $c > 0$ is a normalization constant which allows $\sum_{n\geq 1} m_n = 1$. By construction $d_M(\mu) = d_M(\mathrm{supp}(\mu)) = d_M(B_{\frac{1}{2}}(S)) = N$. For any $p \geq 1$ we shall bound δ_n^p by considering as an element of $\mathcal{D}_n$ the sum of the first n masses of μ.

So

$$\delta_n^p \leq d_p(\mu, \sum_{k=1}^n m_k \delta_{x_k}) \leq \left(\sum_{k=n+1}^\infty m_k\right)^{\frac{1}{p}} \leq \left(\int_n^\infty ce^{-x}dx\right)^{\frac{1}{p}} = (ce^{-n})^{\frac{1}{p}} .$$

Therefore

$$\log_n(\delta_n^p) \leq \frac{1}{p}\log_n c - \frac{n}{p}\log_n e = \frac{1}{p}\log_n c - \frac{n}{p\log n} \to -\infty ,$$

which, taking into account (5.5.6) easily leads to $d_r^p(\mu) = 0$ for all $p \geq 1$. ∎

In the following example we shall show a probability measure (which cannot be Ahlfors regular, see Proposition 5.5.6) for which there is a real the dependence of $d_r^p(\mu)$ on the index $p \geq 1$. In particular, we shall show that $\frac{q}{p}$ is the best possible constant in (5.5.8) while any Ahlfors regular μ shows the optimality of (5.5.7).

Example 5.10.2 *For any $p < q$, the constant $\frac{q}{p}$ in (5.5.8) of Proposition 5.5.3 cannot be improved.*

PROOF. Let $N = 1$, $S = 0$ and let us consider a $\mu = \sum_{i\in I\!N} m_i \delta_{x_i}$ where, for all $i \in I\!N$, δ_i denotes the Dirac mass concentrated in a point $x_i \in I\!R$. Let us fix two exponents γ, $\beta > 1$ and take $\forall i \in I\!N$,

$$r_i = x_{i+1} - x_i = \operatorname{dist}(x_i, \{x_j \mid j \neq i\}) = i^{-\gamma} \tag{5.10.9}$$

and

$$m_i = i^{-\beta} . \tag{5.10.10}$$

By using the triangular inequality, we get that

$$\forall x \in I\!R \ \exists \text{ at most one } i \in I\!N \text{ s.t. } |x - x_i| \leq \frac{r_i}{2} . \tag{5.10.11}$$

For a given $p \geq 1$ and for a fixed $n \in I\!N$ let us evaluate δ_n^p. Let us set $\mu_n = \sum_{i=1}^n m_i' \delta_{x_i}$ where for all $i < n$, $m_i' = m_i$, while $m_n' = \sum_{j\geq n} m_j$. Then we can bound δ_n^p by $C\, n^{1-\gamma-\frac{\beta-1}{p}}$. Indeed, by (5.10.10) and (5.10.9)

$$\begin{aligned} \delta_n^p \leq d_p(\mu, \mu_n) &\leq \left[\left(\sum_{i=n+1}^{\infty} m_i \right) \left(\sum_{i=n+1}^{\infty} r_i \right)^p \right]^{\frac{1}{p}} \\ &= \left(\sum_{i=n+1}^{\infty} i^{-\beta} \right)^{\frac{1}{p}} \sum_{i=n+1}^{\infty} i^{-\gamma} \leq C\, n^{1-\gamma-\frac{\beta-1}{p}} . \end{aligned} \tag{5.10.12}$$

Fix now an arbitrary discretization $\mu_n \in D_n$ of μ. By (5.10.11) for any point x of $\operatorname{supp}(\mu_n)$ we find at most a point x_i such that $|x - x_i| \leq \frac{r_i}{2}$. So we can find at most n points x_i which are at a distance less or equal to $\frac{r_i}{2}$ from $\operatorname{supp}(\mu_n)$. We can assume, being $(m_i)_{i\in I\!N}$ and $(r_i)_{i\in I\!N}$ decreasing sequences, that such points x_i are the first n ones. Therefore we have

$$\delta_n^p = d_p(\mu, \mu_n) \geq \left(\sum_{i=n+1}^{\infty} m_i \left(\frac{r_i}{2} \right)^p \right)^{\frac{1}{p}} \geq c \left(\sum_{i=n+1}^{\infty} i^{-\beta} i^{-\gamma\, p} \right)^{\frac{1}{p}} \geq c\, n^{-\gamma-\frac{\beta-1}{p}} . \tag{5.10.13}$$

¿From (5.10.13) and (5.10.12) we have the existence of two positive constants, c and C such that for all $p \geq 1$ and $n \in I\!N$:

$$c\, n^{-\gamma-\frac{\beta-1}{p}} \leq \delta_n^p \leq C\, n^{1-\gamma-\frac{\beta-1}{p}} . \tag{5.10.14}$$

Taking the $\log_n$ and then taking the lim sup of the three members of (5.10.14) as in (5.5.6), we have for all $p \geq 1$

$$\frac{p}{p\gamma + \beta - 1} \leq d_r^p(\mu) \leq \frac{p}{p(\gamma - 1) + \beta - 1} \tag{5.10.15}$$

and therefore

$$\frac{q}{p} \frac{p(\gamma - 1) + \beta - 1}{q\gamma + \beta - 1} \leq \frac{d_r^q(\mu)}{d_r^p(\mu)} \leq \frac{q}{p} \frac{p\gamma + \beta - 1}{q(\gamma - 1) + \beta - 1} . \tag{5.10.16}$$

Taking into account that, for a fixed value of $\gamma > 1$, both the bounds in (5.10.16) go to $\frac{q}{p}$ as $\beta \to +\infty$ we conclude the proof. ∎

5.11 Appendix C - Index of the main notation

The order first follows the exposition in Appendix A and then the exposition of the paper:

- $(\Omega, |\cdot|)$ a nonatomic probability space
- $\chi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^N$ = set of fibers
- $\chi(p,t) \in \mathbb{R}^N$ position of the point $p \in \Omega$ at the time t
- $\chi_p = t \mapsto \chi(p,t)$ = fiber of p
- $\mathbf{C}_S(\Omega)$ = set of sets of fibers of Ω
- $\mathbf{P}_S(\Omega)$ = set of all irrigation patterns, i.e. the set of all the measurable sets of fibers of Ω
- $[p]_t$ = equivalence class of p under the equivalence $p \simeq_t q$ if $\chi_p(s) = \chi_q(s)$ for all $s \in [0,t]$
- χ-*vessels* = class of equivalence at time t under $\simeq_t$
- $\mathcal{V}_t(\chi) = \Omega/\simeq_t$ = set of χ-vessels at time t
- $\sigma_\chi(p) = \inf\{t \in \mathbb{R}_+ \mid \chi_p(s)$ is constant on $[t, +\infty[\}$: absorption (stopping) time of p, p is absorbed at time t if $\sigma_\chi(p) \leq t$
- $X \subset \Omega$ is an absorbed set at time t if $\sigma_\chi(p) \leq t$ for a.e. $p \in X$
- χ-flow = non absorbed χ-vessel (has positive measure in Ω)
- $\mathcal{F}_t(\chi)$ = set of χ-flows at time t
- $A_t(\chi)$ = set of the points of Ω which are absorbed at time t
- $A_\chi = \bigcup_{t>0} A_t(\chi)$ = set of the absorbed points
- $M_t(\chi) = \Omega \setminus A_t(\chi)$ = set of the points of Ω that at time t are still moving
- $F_t(\chi) = \bigcup_{A \in \mathcal{F}_t(\chi)} A$ = union of the χ-flows at time t
- $i_\chi : A_\chi \to \mathbb{R}^N$ = irrigation function defined by $i_\chi(p) = \chi(p, \sigma_\chi(p))$
- μ_χ = irrigation measure induced by the pattern χ by setting $\mu_\chi(A) = |i_\chi^{-1}(A)|$ for any Borel set $A \subset \mathbb{R}^N$
- $c_\chi(t) = \displaystyle\int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp$ = density cost function see (5.9.2)
- $I_\alpha(\chi) = \displaystyle\int_{\mathbb{R}_+} c_\chi(t)dt$ = cost of the pattern χ
- $F_\chi = \{x \in \mathbb{R}^N \mid \exists t > 0, \exists A \in \mathcal{F}_t(\chi) \text{ s.t. } x = \chi(p,t), p \in A\}$ = flow zone of χ, see Definition 5.9.1

- $D_\chi = \{p \in \Omega \mid p \in \mathcal{F}_{\sigma_\chi(p)}(\chi)\}$ = dispersion of the pattern χ, see Definition 5.9.2
- $s_\chi(p, q) = \inf\{t \geq 0 \mid \chi(p, t) \neq \chi(q, t)\}$ = separation time of the two points p and q, see Definition 5.9.3
- $[p]_t^s$ = equivalence class of p under the equivalence $p \simeq_t^s q$ if there exists $\varepsilon > 0$ s.t. $p \simeq_{t+\varepsilon} q$, see Definition 5.9.4
- *strict χ-vessels* = class of equivalence at time t under $\simeq_t^s$, see Definition 5.9.4
- $\mathcal{V}_t^s(\chi) = \Omega/\simeq_t^s$ = set of the strict χ-vessels at time t, see Definition 5.9.4
- $\chi' : [p]_t \times I\!R_+ \to I\!R^N$ = branch of χ starting from $\chi(p, t)$ defined by setting $\chi'(q, \cdot) = \chi(q, \cdot + t)$, see Definition 5.9.5
- $\chi \setminus \chi'$ = pattern χ stumped of the branch χ', see Definition 5.9.8
- $I_\alpha(V, t) = I_\alpha(V)$ = cost of the vessel V at time t, see Definition 5.9.9
- $\varphi_\chi(p, t) = |[p]_t|^{\alpha-1}\mathbf{1}_{M_t(\chi)}(p)$, see (5.9.13)
- $\nu(A) = \int_A \varphi_\chi(p, t)dp\,dt$, see Definition 5.9.11
- $d_\alpha = \frac{1}{1-\alpha} = \left(\frac{1}{\alpha}\right)'$ = critical dimension of the exponent α, see Definition 5.1.1
- $d(\mu) = \inf\{d_\alpha \mid \mu \text{ is irrigable with respect to} \alpha\}$ = irrigability dimension of μ, see Definition 5.1.3
- $d(B)$ = Hausdorff dimension of the set B
- $d_c(\mu) = \inf\{d(B) \mid \mu \text{ is concentrated on } B\}$, see Definition 5.1.4
- $d_s(\mu)$ = Hausdorff dimension of the supp(μ), see Definition 5.1.5
- $d_M(X)$ = Minkowski dimension of the set X, see Definition 5.1.6
- $N_\delta(X) = \{y \in I\!R^N \mid d(y, X) < \delta\}$, see Definition 5.1.6
- $d_M(\mu) = \inf\{d_M(X) \mid \mu \text{ is concentrated on } X\}$ = Minkowski dimension of μ or equivalently strong resolution dimension of index $+\infty$, see Definition 5.1.7, Definition 5.5.5 and Proposition 5.5.4
- *resolution* of μ = card(supp(μ)) $< \infty$, see Definition 5.5.1
- D_n = set of all the convex combinations of n Dirac masses, see Definition 5.5.2
- $d_p(\mu, \nu) = \left(\min_\sigma \int_{\Omega\times\Omega} |x-y|^p d\sigma\right)^{\frac{1}{p}}$ = Kantorovitch-Wasserstein distance of index p between μ and ν, see Definition 5.5.3
- $\delta_n^p = d_p(\mu, D_n)$, see (5.5.3)

- $d_r^p(\mu) = \left(-\limsup_{n\to+\infty} \log_n (\delta_n^p)\right)^{-1}$ = resolution dimension of μ of index p, see Definition 5.5.5
- $d_r^\infty(\mu) = \sup_{p\geq 1} d_r^p(\mu)$ = weak resolution dimension of index $+\infty$, see Remark 5.5.4
- $W_\chi(t) = \max\{I_\alpha(V) \mid V \in \mathcal{V}_t(\chi)\}$, see (5.7.3)
- $S_\chi(t) = \max\{I_\alpha(V) \mid V \in \mathcal{V}_t^s(\chi)\}$, see (5.7.4).

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