Optimal transport and irrigation

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Introduction générale

En observant les images de la figure 1, on constate que les structures branchues sont présentes dans de nombreux systèmes naturels et artificiels : poumons, réseau des veines et artères, arbres, nervures des feuilles, réseaux de drainage et d'irrigation, réseau électrique et de télécommunication... Sans vouloir à tout prix trouver un principe unique caché derrière cette diversité, on peut toutefois chercher à modéliser la géométrie et les fonctions de ces systèmes, et se demander si leur structure et leurs propriétés peuvent être reliés à des principes d'optimisation simples. Le paradigme qui soutient cette volonté est classique : la nature étant bien faite, les systèmes qu'elle propose sont efficaces pour la tâche qu'ils réalisent. Quelle est cette tâche dans le cas présent? On constate que les systèmes branchus précédemment mentionnés partagent certaines propriétés qui nous incitent à les dénommer désormais systèmes irrigants :

- ces systèmes acheminent un fluide (ou un signal) d'une source vers un but : coeur vers tout le corps humain pour les artères (et trajet inverse pour les veines) ; tout un bassin de rivière vers la mer ; de la tige vers les cellules de la feuille ou inversement (xylème et phylème).
- les points terminaux constituent tout un volume : les capillaires sanguins irriguent "tous" les points du corps humains ; les bronchioles amènent de l'air en presque tout point des poumons ; le bassin d'une rivière occupe toute une surface...
- ces systèmes assurent une égalité de distribution (même débit aux points terminaux des poumons, des veines) ou bien une distribution imposée (pluviométrie moyenne inhomogène sur tout un bassin de rivière).

C'est l'objet de cette thèse de proposer et d'étudier une formulation variationelle de ce qu'on appellera le problème d'irrigation (et ses variantes). Nous considérons dans cette introduction les cinq points principaux de l'étude : modélisation, existence, régularité, équivalence entre les modèles, simulations numériques. Cette thèse apporte des contributions à chacun de ces cinq points.

MODELISATION

C'est le système irrigant que l'on souhaite modéliser. Il s'agit d'en retenir les propriétés essentielles : structure géométrique et flots/capacité en chaque point de la structure. Beaucoup de propositions ont été faites pour modéliser ce type d'objet, on distinguera les modèles à tubes "épais" et les modèles de transport de masse.

Les modèles à tube épais

Les articles de Brown, West et Enquist [31] et [32] utilisent un tel modèle. Comme illustré par la figure 1.1 du chapitre 1, ces auteurs considèrent un ensemble de tubes qu'ils regroupent par génération. Ni

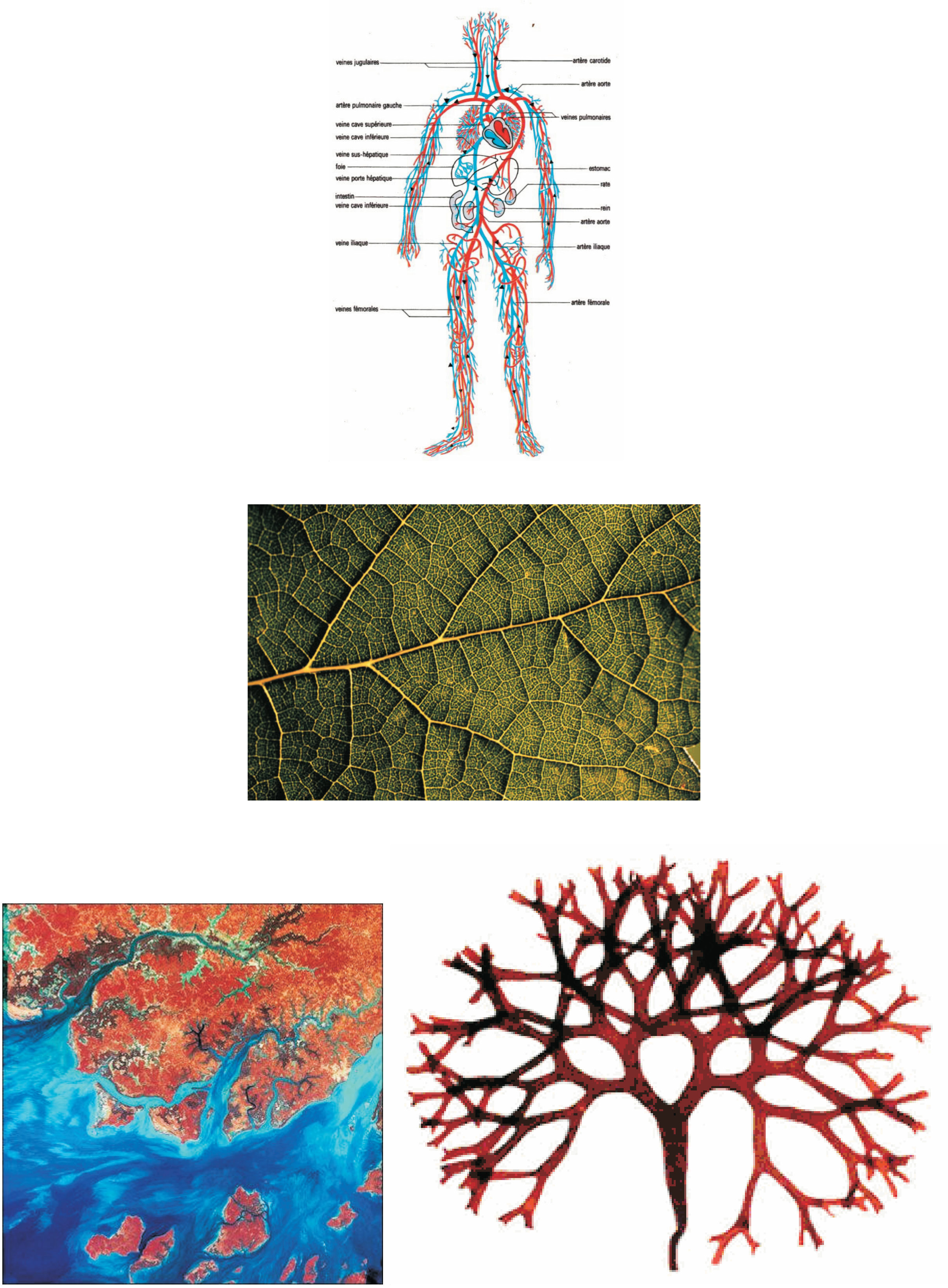

Figure 1: Le réseau des veines et artères. Les nervures d'une feuille. Une rivière vue du ciel. Une variété d'algue rouge.

la manière dont les tubes sont connectés entre deux générations, ni le plongement des tubes dans l'espace ambiant ne sont explicités. Seules sont retenues les sections et longueurs des tubes. L'objectif que se donnent les auteurs en considérant de tels modèles est de déduire des lois d'échelles à partir de certains axiomes : la structure minimise une énergie, la structure a des points terminaux décrivant un volume, le nombre de branchements est constant à chaque génération... Notons que les résultats obtenus par ces auteurs sont critiqués par Dodds, Rothman et Weitz [15] à juste raison. Il est intéressant de mentionner que ce type de modèle a également été utilisé par Sapoval, Filoche, Mauroy et Weibel dans l'article [23]. Bernard Sapoval et son équipe montrent que des poumons qui seraient "optimaux" dans le sens où ils prennent le moins de place possible tout en nécessitant un effort d'inspiration tolérable, seraient dangereux, i.e. qu'une petite diminution de la section des tubes entraînerait un effort d'inspiration trop important.

Nous présentons dans le chapitre 1 un autre type de modèle à tube épais, celui proposé par Caselles et Morel dans [12]. Le système irrigant est tout simplement un ouvert U de l'espace ambiant (cf figure 1.1). Les auteurs introduisent une notion de profil pour contrôler la vitesse de décroissance du rayon des tubes. On dit alors qu'un point est accessible/irrigué selon un profil f donné (voir la définition 1.1.1 et la figure 1.2) si ce point se trouve à l'extrémité d'un tube contenu dans U et de profil f. La question abordée dans [12] est de trouver des conditions nécessaires et suffisantes sur un profil pour que des systèmes irrigant tout un volume puissent exister.

L'article [12] poursuit en montrant que ce type de modèle à tubes épais permet d'introduire une définition naturelle de l'égalité de distribution. Soit $U \subseteq \Omega$ (où Ω est un ouvert de $\mathbb{R}^N$ tel que $|\partial U| > 0$). Un ensemble U permet l'égalité de distribution s'il existe un champ de vecteurs borné v dans Ω, nul en dehors de U, et une mesure source μ dont le support est dans U tels que $-div\, v = -\mu + \chi_{\partial U}$ où $\chi_{\partial U}$ est la mesure de Lebesgue restreinte à l'ensemble irrigué ∂U(cf section 1.2).

Les modèles de transport de mesure

Une autre approche de la modélisation des systèmes irrigants consiste à ne retenir que le squelette de la géométrie formée par les tubes, ainsi que les flots/capacités associés à ces tubes. L'objet qui vient immédiatement à l'esprit pour modéliser ce type de structure est le graphe orienté à poids (vérifiant les lois de Kirchhoff). Un tel objet permet en effet de décrire comment un flot initial se scinde et se répartit entre tous les points terminaux. Le problème principal lié aux graphes finis est qu'ils ne permettent pas d'appréhender des structures irrigant des volumes. Il s'agit alors de plonger les graphes finis dans un espace plus important, si possible avec une propriété de compacité. L'espace que propose Xia dans [35] est celui des mesures de Radon vectorielles ou des 1-courants ; on dit alors que G transporte μ^+ vers μ^- si le bord de G est $\mu^- - \mu^+$. Dans [22], Maddalena, Morel et Solimini introduisent une description Lagrangienne en décrivant la structure irrigante par une application $\chi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^N$ appelée *pattern*. L'ensemble Ω est l'ensemble des particules et chaque fibre $\chi(\omega, \cdot)$ indique le chemin suivi par la particule ω. Les graphes orientés finis n'ayant qu'une seule source peuvent facilement s'écrire comme *pattern* et la mesure irriguée par un *pattern* χ est simplement la mesure image de Ω par $\chi(\cdot, \infty)$. Une autre possibilité proposée par Brancolini, Buttazzo et Santambrogio est de considérer un transport comme un chemin γ sur l'espace des mesures [7], où γ est tel que $\gamma(0) = \mu^+$ et $\gamma(1) = \mu^-$. L'objet que nous avons choisi d'introduire pour modéliser les systèmes irrigants se veut être la généralisation des *patterns* et consiste simplement en l'ensemble des mesures de probabilité sur l'espace des courbes 1-Lipschitzienne (cf figure 2). De la même manière que pour les *patterns*, on peut associer canoniquement une mesure irriguée à un plan d'acheminement. Les plans d'acheminement offrent cependant plus de souplesse puisque l'on peut également y associer une mesure source ainsi qu'un plan de transfert de la masse transportée.

Que souhaite-t-on optimiser?

Autrement dit, quel coût souhaite-t-on qu'une structure irrigante optimise? On va distinguer deux types de coût : le coût de fabrication ; le coût du transport le long de la structure. Donnons deux exemples.

Imaginons une structure faite d'un seul type de tube. Le coût de cette structure est obtenu en multipliant la longueur totale du réseau par le coût au linéaire d'un tube. Si le prix du linéaire est proportionnel à la quantité de matière d'un tube, un tube de section S et de longueur L aura une quantité de matière de l'ordre de $L\sqrt{S}$, de sorte que le coût d'un tube par unité de longueur sera de l'ordre de $\sqrt{S}$. Il vaut donc mieux du point de vue de la quantité de matière un tube de section S que deux tubes de section $\frac{S}{2}$. Pour une structure de type graphe, la fonctionnelle que l'on va chercher à optimiser est donc de la forme $\sum_e f(c_e)l(e)$ où l'on somme sur toutes les arêtes de la structure et c_e désigne la capacité de l'arête e, $l(e)$ la longueur de l'arête e et $f(c)$ est le coût d'une arête de capacité c par unité de longueur. C'est ce type de coût que Gilbert [18] a utilisé en 1967 pour optimiser des réseaux de télécommunication.

Imaginons, à l'instar du problème de Monge-Kantorovitch, que l'on souhaite transporter deux tas de sable de 1kg vers une configuration faite d'un seul tas de 2kg situé à 100m. On peut soit transporter les deux tas de 1kg séparément ou bien les amener en un point commun puis les transporter ensemble. La fonctionnelle qui décrit ce coût prend la même forme que dans l'exemple précédent : $\sum_e f(c_e)l(e)$ où c_e désigne la masse/flot transportée le long de l'arête e, et $f(c)$ désigne ce qu'il en coûte de transporter c par unité de longueur. Pour coder le fait que l'on encourage la masse à se grouper pour être transportée, on prend une fonction f concave de sorte que $f(a+b) \leq f(a) + f(b)$.

Dans les deux cas, nous sommes amenés à considérer la fonctionnelles de type $\sum_e f(c_e)l(e)$ qui a été introduite pour la première fois par Gilbert [18].

Sous quelle contrainte?

Quand on modélise un système irrigant à l'aide du transport de mesure, la contrainte que l'on impose à la structure est constituée de sa mesure irrigante et de sa mesure irriguée, i.e. on cherche à optimiser le coût d'une structure, ses mesures irrigantes et irriguées étant prescrites. Notons que dans le cas des plans d'acheminement, on peut également imposer le plan de transfert. En effet, la description lagrangienne permet de garder la trace de la trajectoire précise de chaque particule et rien n'empêche de prescrire le plan de transfert puisqu'il reste alors tout un éventail de possibilités pour la structure réalisant ce plan de transfert. Bien entendu, cette contrainte n'aurait aucun sens dans le cas du problème de Monge-Kantorovitch puisque c'est précisément parmi l'ensemble des plans de transfert que l'on cherche le transport optimal. L'objet plan d'acheminement (*traffic plan*) permet donc d'introduire un nouveau problème que l'on nommera le problème de "qui va où" (*"who goes where"*). Imposer la contrainte du plan de transfert pourrait par exemple permettre de modéliser la structure des transport urbains où le plan de transfert logements vers lieux de travail est prescrit (cf les travaux de Buttazzo et Stepanov [10], [9] et [8] pour une autre approche).

EXISTENCE

Le modèle épais

On s'intéresse à l'ensemble des points accessibles et, en particulier, on se demande sous quelle condition de profil il est possible d'irriguer un ensemble de mesure non nulle. Le corollaire 1.1.3 montre

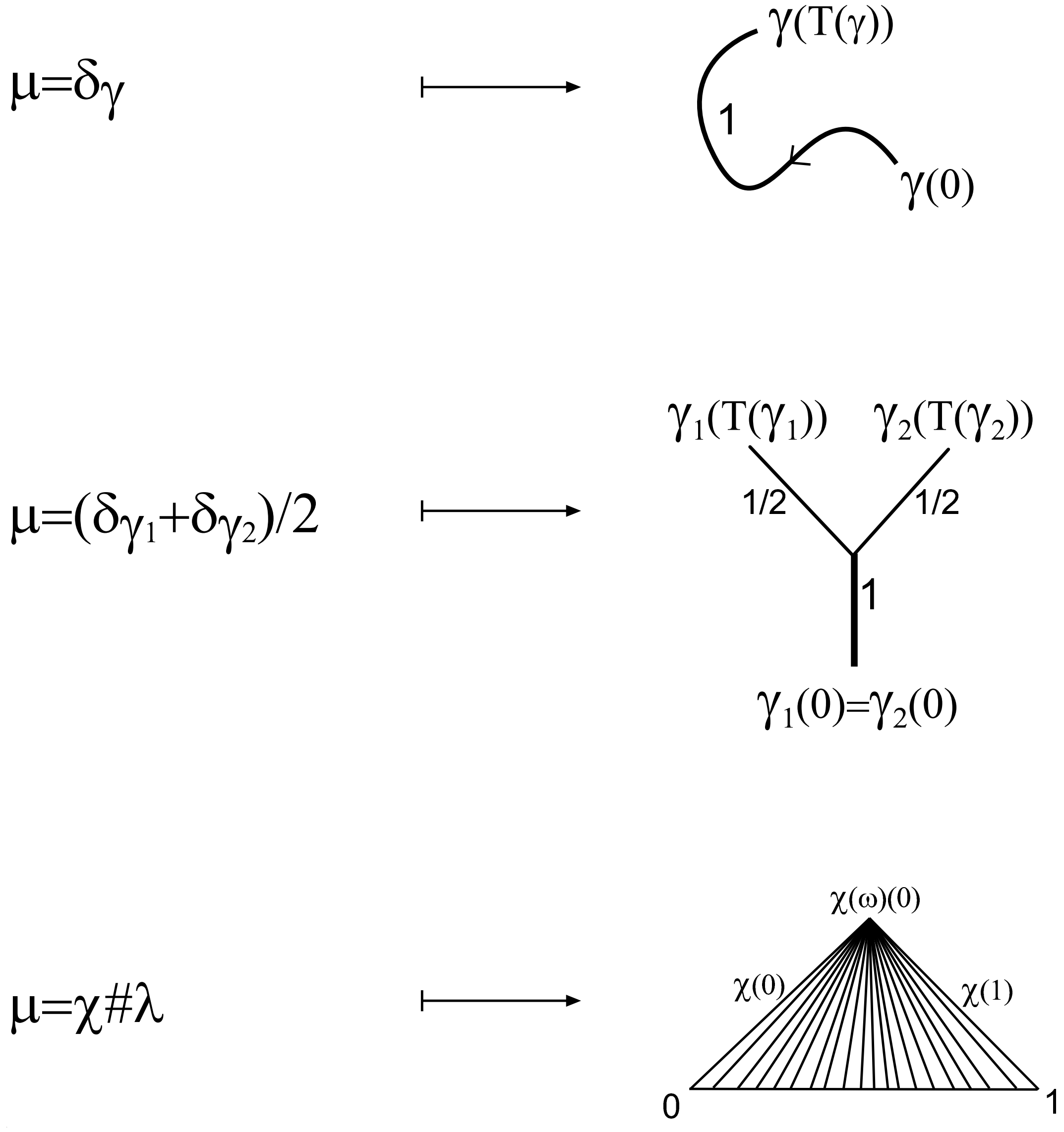

Figure 2: Trois plans d'acheminement (*traffic plans*) : une masse de Dirac en γ, un arbre avec une bifurcation, un arbre irrigant la mesure de Lebesgue sur le segment $[0,1] \times \{0\}$ du plan. Dans le cas du dernier exemple, à $\omega \in [0,1]$ correspond $\chi(\omega) \in K$, le chemin paramétré par sa longueur reliant une masse de Dirac situé en $(1/2, 1)$ au point $(\omega, 0)$.

qu'une telle structure n'est possible que si le profil vérifie

$$\limsup_{r\to 0+} f(r)/r = 0.$$

Réciproquement le lemme 1.1.5 donne une condition suffisante sur le profil, en construisant une structure inspirée du tapis de Sierpinsky. Cette condition suffisante est vérifiée par des profils de type $f(s) = s^p$ pour $p > 1$ et $f(s) = \frac{s}{|\log s|^\beta}$ pour $\beta > 1$ (cf lemme 1.1.6).

En considérant l'exemple inspiré du cube de Sierpinsky, on construit une suite de champs de vecteurs satisfaisant l'équation sur la structure obtenue à la n-ème itération de la construction du cube. On se sert de cette suite pour obtenir l'existence du champ de vecteurs recherché sur la structure générale.

Les deux résultats que l'on vient de mentionner montrent qu'il n'y a pas d'obstruction à l'existence de systèmes irrigant un volume et assurant une égalité de distribution. Toutefois, si l'on tient en plus compte de la résistance du réseau, une obstruction énergétique survient. L'objet du chapitre 3 est principalement de montrer l'obstruction suivante aux modèles infinitésimaux précedemment introduits : il ne peut y avoir de structure irrigant un volume fini à énergie finie. La principale conséquence est donc que les tubes s'arrêtent nécessairement à une certaine échelle.

Le modèle transport de masse

Une fois que les fonctionnelles à optimiser sont bien définies, on s'intéresse au résultat d'existence d'une structure optimale transportant une mesure μ^+ vers μ^- (en imposant de plus un plan de transfert π dans le cas du problème "qui va où"). Les différents modèles emploient tous la méthode directe qui consiste à montrer que la fonctionnelle est semicontinue inférieurement tout en montrant que l'on peut extraire une suite convergente d'une suite minimisante. Xia [35] obtient aisément ce résultat d'existence d'une structure optimale pour le problème d'irrigation puisque la fonctionnelle coût qu'il utilise est semicontinue inférieurement par définition.

Le modèle des plans d'acheminement est la généralisation naturelle des *patterns* [22] et les fonctionnelles de coût sont presque identiques. La preuve d'existence d'une structure optimale pour le problème d'irrigation présentée dans le chapitre 4 est donc très similaire à celle obtenue pour les *patterns* [22]. Notons toutefois que les plans d'acheminement permettent d'obtenir le résultat d'existence également sous la contrainte du plan de transfert, répondant ainsi au problème de l'existence du problème "qui va où".

Une fois l'existence d'une structure optimale démontrée, il s'agit de démontrer qu'il existe une telle structure à coût fini. Pour ce faire, on estime le coût pour irriguer une approximation dyadique de la mesure irriguée (cf [35]). Les estimations obtenues en considérant les approximations dyadiques permettent alors de conclure à l'existence d'une structure à coût fini dans le cas où $\alpha > 1 - \frac{1}{N}$, où N est la dimension de l'espace ambiant. Comme cela est montré dans le chapitre 5, on peut adapter ce type d'argument au cas des plans d'acheminement et montrer également qu'il existe une structure à coût finie sous la contrainte d'un plan de transfert.

Mentionnons que De Villanova et Solimini donnent dans [29] des conditions très précises sous lesquelles une mesure donnée peut être irriguée à coût fini.

Variantes

La plupart des systèmes irrigants naturels considérés évitent les variations d'angle importantes. Comme on peut le lire dans le manuel d'hydraulique [11], des angles dans un réseau de tubes entraînent des chutes de pression et des turbulences, de telle sorte que l'on fait tout pour les éviter au maximum. Ces considérations nous font nous intéresser à l'existence de structures à coût fini ayant des variations d'angles finies. On montre qu'en trois dimensions et dimensions supérieures (cf section 5.5), il existe des arbres irrigant un volume tout en maintenant une variation d'angle fini le long des chemins.

STABILITE ET REGULARITE

En suivant le travail de Xia [35], le résultat d'existence d'un plan d'acheminement optimal entre deux mesures permet d'introduire une distance sur les probabilités analogue à la distance de Wasserstein. L'article [35] contient de plus une preuve du résultat de stabilité suivant: la limite d'une suite convergente de plans d'acheminement optimaux est un plan d'acheminement optimal (cf corollaire 5.3.3).

En ce qui concerne la régularité, très peu de résultats sont pour l'instant acquis. Un résultat satisfaisant serait bien sûr cet énoncé : soit x un point du support de la structure, et $B(x, r)$ une boule n'intersectant pas le support des mesures irrigante et irriguée ; alors le support de la structure dans $B(x, r)$ est un graphe fini. C'est une version un peu affaiblie de cet énoncé qui est annoncée dans [36]. L'article comporte toutefois une erreur et plusieurs imprécisions si bien que l'énoncé reste actuellement une conjecture. La stratégie employée dans [36] est dans un premier temps d'effectuer un blow-up, puis de montrer par des estimations utilisant l'optimalité que la structure coïncide avec le blow-up dans un voisinage assez petit de x. L'existence du blow-up est correcte, mais l'usage qui en est fait est erroné.

Une classe de résultats de régularité très utile sont les lemmes de "nettoyage" qui réduisent l'éventail de ce à quoi peut ressembler un plan d'acheminement optimal. Mentionnons la preuve élégante d'un résultat de non présence de boucle dans un plan d'acheminement optimal proposée par De Villanova et Solimini [28] (cf lemme 6.2.4). Le premier lemme montrant qu'il n'y a pas de circuit à flux minoré par un $c > 0$ dans un transport path optimal a été donné par Xia [35] et est redémontré dans le lemme 6.2.5 (dans le cadre plan d'acheminement). Notons toutefois que ce résultat n'est valable que dans le cas du problème d'irrigation et ne conserve pas la contrainte d'un plan de transfert. Il ne peut donc s'appliquer dans le cas du problème "qui va où". La proposition 6.2.7 montre quant à elle qu'il ne peut y avoir de boucle en toute généralité. Ce résultat apporte un surcroît d'information par rapport au lemme 6.2.5. En effet, la proposition 6.2.7 ne requiert pas que le flux soit minoré par un $c > 0$ le long de la boucle.

L'absence de boucle est cruciale pour démontrer la régularité d'un optimum dans le cas d'un transport entre deux mesures atomiques (cf proposition 6.3.3). Notons que ce résultat n'est pas montré dans [35]. En effet, Xia définit le coût E^α sur les graphes finis puis l'étend par relaxation sur l'espace formé par les limites de graphes finis. Rien n'assure alors que les graphes finis "demeurent" optimaux pour transporter des mesures atomiques.

La régularité étant démontrée dans le cas du transport entre masses atomiques, on peut s'intéresser à la structure des embranchements. Les contraintes d'angles (cf 7.1.2) associées à d'autres arguments permettent de montrer qu'en deux dimensions, et pour $\alpha \leq \frac{1}{2}$, le seul type d'embranchement possible pour un plan d'acheminement optimal est l'embranchement de type Y.

EQUIVALENCE

Les résultats de régularité obtenus précedemment assurent que dans le cas du transport entre mesures atomiques, un plan d'acheminement optimal a la structure d'un graphe fini. Ce résultat suffit à montrer l'équivalence entre la formulation du problème de Gilbert-Steiner (cf théorème 6.4.2) et le problème de l'irrigation. Le fait qu'un optimal n'ait pas de boucle générale de masse permet par ailleurs d'identifier les modèles de plan d'acheminement et de *patterns* dans le cas où la source est réduite à une seule masse de Dirac (cf théorème 6.4.1). Enfin, l'équivalence entre le problème d'irrigation et le modèle de Xia est démontrée dans le chapitre 4.

SIMULATIONS NUMERIQUES

Un algorithme ayant pour ambition d'approcher un optimum global est présenté dans l'article de Xia [35]. L'algorithme consiste à résoudre une tour de problèmes simples obtenus par approximation dyadique. Décrivons-en rapidement le fonctionnement : soit μ^- la mesure que l'on souhaite irriguer à partir d'une source S. Une subdivision dyadique de l'espace permet d'approcher la mesure μ^- par deux masses de Dirac. On est alors ramené au problème très simple de trouver la structure optimale transportant une masse de Dirac vers deux masses de Dirac en S_1 et S_2. Une fois cette structure trouvée on applique à nouveau cette procédure pour transporter S_1 et S_2 vers le raffinement dyadique suivant de μ^-. Une fois que la structure globale est obtenue, Xia optimise la position des points de bifurcation. En deux mots, cet algorithme consiste en une stratégie multiéchelle utilisant des approximations à deux masses de Dirac. Comme on le montre dans le chapitre 8, cet algorithme ne peut que très rarement trouver l'optimum global. La raison principale en est que l'arbre ainsi obtenu a une topologie dyadique imposée. Or, le problème d'optimisation posé par l'irrigation se scinde en deux sous-problèmes. D'une part l'optimisation de la topologie de la structure ; d'autre part l'optimisation des points de bifurcations (pour une structure de topologie donnée).

Comme cela a été mentionné précedemment, l'algorithme de Xia n'explore qu'une seule topologie. Par ailleurs, l'optimisation de la position des points de bifurcation qu'il propose converge lentement et reste approchée. Pourtant l'article [18] de Gilbert décrit une construction à la règle et au compas récursive permettant d'obtenir la position exacte des points de bifurcation d'une structure optimale de topologie prescrite. Cette construction est décrite dans la section 8.2.

En utilisant cette construction récursive, nous pouvons alors très rapidement obtenir le coût d'une structure optimale de topologie prescrite. La recherche exhaustive à travers toutes les topologies est alors envisageable pour des mesures n'ayant pas plus de 6 masses de Dirac (au delà, le nombre de topologies à explorer est trop important). Dans le cas où les masses de Dirac sont alignées, le nombre des topologies qui méritent d'être prises en compte est considérablement réduit. L'exploration exhaustive est alors possible pour une dizaine de masses de Dirac.

Afin d'éviter les recherches exhaustives quand le nombre de masses de Dirac est trop important, on adopte une approche multiéchelle. Celle-ci consiste à approcher le problème par k masses de Dirac où k est le nombre de masses pour lequel l'exploration exhaustive reste possible. Ce sous-problème fournit une structure optimale dont la première bifurcation S' permet de scinder la mesure irriguée en deux mesures μ_1 et μ_2. On applique alors la stratégie multiéchelle aux problèmes de transporter S' vers μ_1 et S' vers μ_2. Cette approche multiéchelle permet d'obtenir dans un temps raisonnable une topologie efficace de transport. Celle-ci peut alors être affinée par perturbation (cf figure 8.15).

Chapitre 1 : *Irrigation géométrique et EDP.*

Ce chapitre présente les résultats obtenus dans [12] par Caselles et Morel qui explorent deux facettes de l'irrigation : l'irrigation de volume et l'égalité de distribution. Il s'agit d'abord de préciser ce que l'on souhaite entendre par réseau irrigant un volume. En considérant le cas des poumons par exemple, on est amené à considérer qu'un réseau irrigue un volume si ses points terminaux forment un ensemble de mesure positive. Cette approche géométrique permet de donner une première obstruction. En effet, on montre que le rayon des tubes d'une structure arborescente irrigant un volume doit nécessairement décroître plus que linéairement. Le problème de l'égalité de distribution peut également être motivé par l'étude des poumons. En effet, lorsque l'on respire, il est souhaitable que la structure des poumons soit telle que chaque bronchiole terminale reçoive de l'air à la même pression. Nous formalisons dans ce chapitre l'égalité de distribution par l'intermédiaire d'une EDP et donnons un exemple de structure irrigant un volume tout en permettant l'égalité de distribution.

Chapitre 2 : *L'irrigation vue comme transport de mesure.*

Les modèles de transport de mesure permettent de définir de manière satisfaisante à la fois l'irrigation de volume et l'égalité de distribution. En effet, si l'on considère un transport de μ^+ vers μ^-, on dira qu'un volume est irrigué si le support de μ^- est de mesure positive. On dira qu'il y a égalité de distribution si μ^- est la mesure de Lebesgue sur un ensemble K. Ce chapitre fait la synthèse des différents modèles basés sur le transport de mesure et proposés jusqu'alors.

Chapitre 3 : *Une obstruction énergétique à l'irrigation des volumes.*[1]

Le résultat principal de cet article est un résultat de non existence : si l'on considère que la loi de Poiseuille est satisfaite même aux plus petites échelles, alors une structure arborescente ne peut à la fois irriguer un volume et causer une dissipation d'énergie finie.

Chapitre 4 : *Le modèle de plan d'acheminement.*[2]

Ce chapitre décrit en détail le modèle de plan d'acheminement. On y démontre la semicontinuité inférieure du coût et l'existence de plans d'acheminement optimaux dans le cas du problème d'irrigation et du problème qui-va-où.

Chapitre 5 : *Irrigation à coût fini et questions de stabilité.*

Pour un $\alpha \geq 1 - \frac{1}{N}$ où N est la dimension de l'espace ambiant, on montre que le coût de transport entre deux mesures est fini, que ce soit pour le problème de l'irrigation ou le problème qui-va-où. Toujours pour $\alpha \geq 1 - \frac{1}{N}$, on montre ce résultat de stabilité : la limite d'une suite de structures optimales est optimale.

Chapitre 6 : *Régularité et structure des branchements d'un optimum.*

On montre dans un premier temps que des structures optimales n'ont pas de boucles ou pas de circuit (suivant si l'on considère le problème "qui va où" ou le problème d'irrigation). Cette très forte contrainte permet de montrer la régularité dans le cas du transport entre deux mesures atomiques. On étudie ensuite

[1]M. Bernot, V. Caselles and J.-M. Morel, *Are there infinite irrigation trees?*, Journal of Mathematical Fluid Mechanics, Vol. 7, 2005.

[2]M. Bernot, V. Caselles and J.-M. Morel, *Traffic plans* Publicacions Matemàtiques Vol. 49, Núm. 2, pp. 417-451, 2005

quels sont les branchements possibles en un point de bifurcation. Dans le cas où $\alpha \leq \frac{1}{2}$ et en dimension 2, les seuls branchements possibles sont en Y.

Chapitre 7 : *Exemples d'irrigation optimale.*

Cette partie d'exemples étudie complètement la structure optimale du transport d'une masse de Dirac vers deux masses de Dirac. On s'intéresse ensuite à la structure d'un optimum pour l'irrigation de la mesure de Lebesgue portée par un segment. On étudie alors une classe assez différente d'exemples, i.e. on se demande si une structure de coût fini, irrigant un volume, et telle que la variation totale de l'angle le long des fibres reste finie peut exister. La réponse est oui en toute dimension. La réponse demeure oui en dimension supérieure à trois si l'on exige en plus d'avoir une vraie structure d'arbre dans le sens où les fibres ne s'entrecoupent pas.

Chapitre 8 : *Algorithmes de recherche des optima locaux et globaux.*

Ce chapitre présente une méthode de construction des emplacements optimaux des points de bifurcations d'une structure à topologie donnée. Les heuristiques multiéchelles et de perturbation topologique permettent quant à elles d'obtenir des topologies efficaces en un temps raisonnable.

Chapter 1

Irrigation: the geometric and PDE framework

Introduction

In many natural or artificial flow systems, a fluid flow network succeeds both in connecting every point of a volume to a source, and in ensuring equality of supply (in the sense that the tips of a network receive roughly the same flow). Examples are the blood vessels, the bronchial tree and many irrigation and draining systems. The aim of this chapter is twofold ; to propose a definition of irrigating systems i.e. ”structure irrigating a volume from a source”, and to introduce a PDE model to define the equality of supply condition.

In the articles [31], [32] and [33], irrigating systems are viewed as homogeneous trees made of tubes (see figure 1.1) in the sense that bifurcation ratio, scaling of the length and scaling of the section are associated to each level of the tree. The problem of such a model is that it considers sets of tubes separating them by generation, but does not take into account the set of all tubes as a whole so that it avoids the question of the embedding of that tree in the real 2 or 3 dimension space.

In the first section of this chapter we present a much more general approach due to Caselles and Morel [12], where the irrigating system is only supposed to be an open connected set. A point on the boundary is said to be accessible or irrigable for some profile f if Ω is not too much ”narrow” in the neighborhood of that point (see definition 1.1.1 and figure 1.2). The main question that is asked in this purely geometric framework is whether or not an open set can irrigate a set with positive measure. Proposition 1.1.2 gives a geometrical obstruction to irrigability (for instance, a profile of an irrigating set cannot be linear). In subsection 1.1.2, we show the construction (given [12]) of a ”Sierpinsky gasket” like irrigating tree for many different profiles.

In the second section of this chapter, we define the equality of supply requirement through a suitable PDE, as it is proposed in [12]. Caselles and Morel say that $U \subseteq \Omega$ (where Ω is an open set in $\mathbb{R}^N$ with Lipschitz boundary, and $|\partial U| > 0$) permits the equality of supply, if there is a bounded velocity vector field v in Ω such that $-div\, v = -\mu + \chi_{\partial U}$ where $\chi_{\partial U}$ is Lebesgue measure restricted to the irrigated set ∂U, and $v = 0$ outside U. We shall then give an example of a set permitting equality of supply in 2 dimensions, and such that U irrigates a set with positive measure. Of course, if U permits an equality of supply flow, then U cannot be any set, but a useful description of those sets is lacking (even if integrating the PDE against characteristic functions of rectifiable sets in $\mathbb{R}^N$ gives necessary and sufficient conditions for its existence).

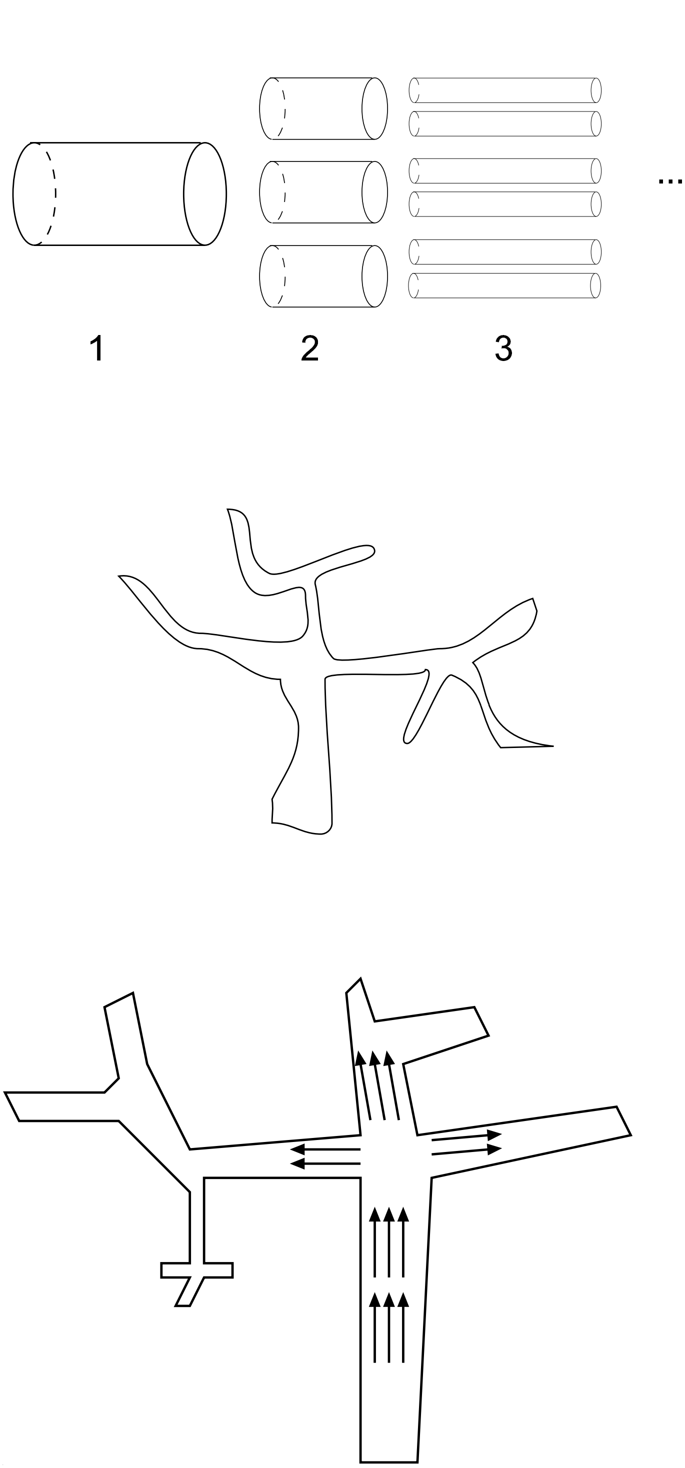

Figure 1.1: The homogeneous tree model, the irrigating set model and the PDE framework for the equality of supply. The homogeneous tree model only considers very simple trees made of tubes without asking the question of the embedding of the tree in space or the question of the real flow in it. The irrigating set model permits to study geometric obstructions, i.e. what kind of profile on the section of tubes allows to irrigate a set with positive measure. The PDE framework permits to define precisely the equality of supply, i.e. is it possible for a fluid to flow in X from a source to the boundary of X.

1.1 A geometric model for irrigability

In this section, a model of irrigating set based on irrigation at the boundary is considered. Let X be an open set in $\mathbb{R}^N$ and $S \in X$ a source point. A point $x \in \partial X$ is said to be accessible if there is a path connecting S to x, so that a tube of prescribed profile along the path is contained in X. We call irrigated set the set of accessible points. Conditions are given on the profile so that it prevents a set from irrigating a set of positive measure. Examples of irrigating sets with bounded mean lengths along accessible paths is given in lemma 1.1.5 and lemma 1.1.6.

1.1.1 Accessible points

We denote by $B(x,r)$ the open ball of center $x \in \mathbb{R}^N$ and radius $r > 0$.

Let $f : [0,\infty) \to [0,\infty)$ be an increasing continuous function such that $f(0) = 0$.

Definition 1.1.1 *Let X be an open set in $\mathbb{R}^N$, $S \in X$. We say that $x \in \partial X$ is accessible from S with profile given by f if there is a curve $\gamma : [0, L(\gamma)] \to \mathbb{R}^N$ parameterized by its arc length such that $\gamma(0) = x$ and $\gamma(L(\gamma)) = S$,*

$$B(\gamma(s), f(s)) = (\gamma(s) + B(0, f(s))) \subset X \tag{1.1}$$

for all $s \in (0, L(\gamma)]$ (see figure 1.2).

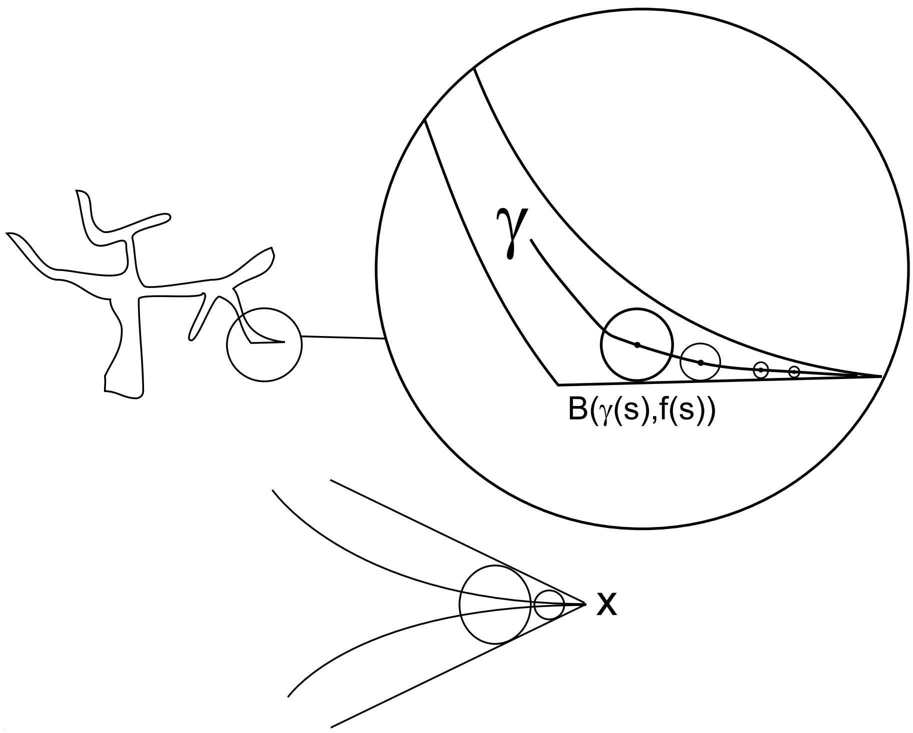

Figure 1.2: A point on the boundary of X is said to be accessible from S with profile f if there is a path γ such that balls centered on $\gamma(s)$ with radii $f(s)$ lie within X. On the figure at bottom, x is not accessible with a linear profile $f(r) = kr$ because of the cusp.

If $E \subset \mathbb{R}^N$ is Lebesgue-measurable and $x \in \mathbb{R}^N$, the upper and lower densities of x in E are defined by

$$\overline{d}(E,x) := \limsup_{\rho \to 0^+} \frac{|E \cap B(x,\rho)|}{|B(x,\rho)|}$$

$$\underline{d}(E,x) := \liminf_{\rho \to 0^+} \frac{|E \cap B(x,\rho)|}{|B(x,\rho)|}.$$

When the upper and lower limits are equal, we denote their common value by $d(E,x)$ and we call it the density of E at x. By Lebesgue density theorem [25], both densities are equal to 1 at almost every point of E.

Proposition 1.1.2 *Let $x \in \partial X$ be irrigable from S with profile f. Assume that $d(\mathbb{R}^N \setminus X, x) = 1$. Then $\limsup_{r \to 0+} f(r)/r = 0$. As a consequence $\int_0^R \frac{1}{f(r)}\, dr = \infty$, $R > 0$.*

Proof: Let γ be a curve of accessibility to x, and $r < L(\gamma)$. Then, since γ is parameterized by its arc length, we have $\gamma(\frac{r}{2}) \in \overline{B(x, \frac{r}{2})}$. As a consequence, $B(\gamma(\frac{r}{2}), \frac{r}{2}) \subset B(x,r)$.

If $f(\frac{r}{2}) < \frac{r}{2}$, then $B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \subset B(x,r)$, so that $B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \cap B(x,r) = B(\gamma(\frac{r}{2}), f(\frac{r}{2}))$. If $f(\frac{r}{2}) \geq \frac{r}{2}$, then $B(\gamma(\frac{r}{2}), \frac{r}{2}) \subset B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \cap B(x,r)$. By definition of accessibility, we have $B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \subset X$, hence

$$\frac{|(X) \cap B(x,r)|}{|B(x,r)|} \geq \frac{|B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \cap B(x,r)|}{|B(x,r)|} \geq \frac{min(\frac{r}{2}, f(\frac{r}{2}))^N}{r^N}$$

Taking the limsup, the inequality yields $\overline{d}(X,x) \geq \frac{1}{2^N} min(\limsup_{r \to 0+} f(r)/r, 1)^N$. Then, $d(\mathbb{R}^N \setminus X, x) = 1$ implies that $\limsup_{r \to 0+} f(r)/r = 0$.

Finally, observe that, for some $R > 0$, $\frac{f(r)}{r} < 1$ for all $r < R$; otherwise we would have $\limsup_{r \to 0+} f(r)/r \geq 1$. It follows that $\frac{1}{r} < \frac{1}{f(r)}$ for all $r < R$, and thus $\int_0^R \frac{1}{f(r)}\, dr = \infty$. □

Corollary 1.1.3 *If X irrigates a set of positive measure, then the profile f is such that*

$$\limsup_{r \to 0+} f(r)/r = 0.$$

Proof: Let us denote A the set of accessible points. By Lebesgue density theorem [25], $d(A,x) = 1$ at almost every point of A. Since $A \cap X = \emptyset$ we have $d(\mathbb{R}^N \setminus X, x) = 1$ and proposition 1.1.2 asserts that the profile f is such that $\limsup_{r \to 0+} f(r)/r = 0$. □

Remark 1.1.4 *Let us consider a linear profile, i.e. f(r)=ar (see figure 1.3). Corollary 1.1.3 states that it is not possible for a set to irrigate a set of positive measure with the profile f.*

1.1.2 An example of irrigated set with positive measure in 2D

In this section, sufficient conditions are given on the profile of a particular 2D tree so that it permits to irrigate a set of positive measure.

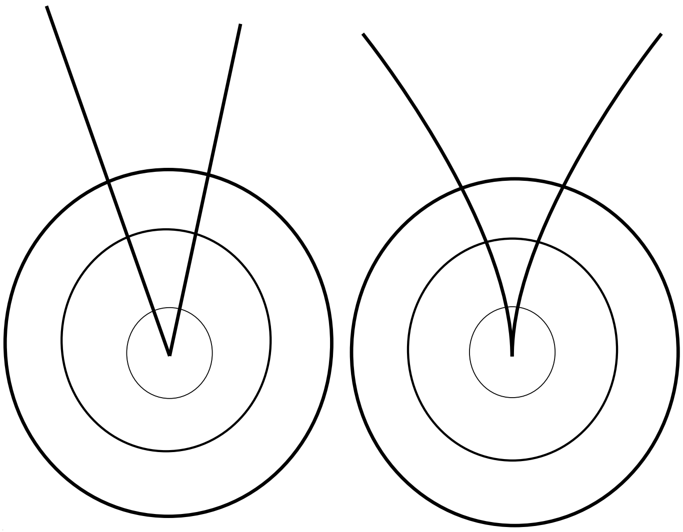

Figure 1.3: The geometric content of proposition 1.1.2 is very natural. Indeed, if a point is accessible from $S \in X$ and has density 1 in $\mathbb{R}^N \setminus X$, then the ratio of the area of the profile upon the area of the ball has to go to 0 since the profile is contained in X. The ratio for the linear profile (on the left hand side) is constant so that there is no tree irrigating a set with positive measure with such a profile. The profile on the right hand side is such that $\limsup_{r\to 0+} f(r)/r = 0$.

Lemma 1.1.5 *Let $\ell > 0$. Assume that*

$$\sum_{n=1}^{\infty} 2^n f(\frac{\ell}{2^n}) < \infty. \tag{1.2}$$

Then there is an open bounded subset X of $\mathbb{R}^2$ whose boundary ∂X is of positive measure and accessible with profile f. In addition, accessibility paths can be taken with bounded lengths.

Proof:

We shall construct a Sierpinski carpet which is irrigable and of positive measure as illustrated on figure 1.4. Let $\ell_0 = \ell$. Take the square $\Omega = [-\frac{\ell}{2}, \frac{\ell}{2}]^2$ and take out the cross $X_0^1 = (-\frac{\ell_0}{2}, \frac{\ell_0}{2}) \times (-\frac{\delta_1}{2}, \frac{\delta_1}{2}) \cup (-\frac{\delta_1}{2}, \frac{\delta_1}{2}) \times (-\frac{\ell_0}{2}, \frac{\ell_0}{2})$ with $\delta_1 < \ell_0$. This cross will be the step 0 cross. We shall say that the cross has length ℓ_0 and width δ_1. The square $(-\frac{\delta_1}{2}, \frac{\delta_1}{2}) \times (-\frac{\delta_1}{2}, \frac{\delta_1}{2})$ will be called the center of the cross. There are four squares remaining in $\Omega \setminus X_0^1$ of lateral size $\ell_1 = \frac{\ell_0 - \delta_1}{2}$. Consider in each of those squares a cross of length ℓ_1 and width $\delta_2 (< \ell_1)$. We call these crosses the step 1 crosses and denote them by X_1^j, $j = 1, ..., 4$. We continue iteratively in this way, thus, at step n we have 4^n crosses X_n^j, $j = 1, ..., 4^n$, and each of them has length $\ell_n = \frac{\ell_{n-1} - \delta_n}{2}$ and width $\delta_{n+1} (< \ell_{n-1})$. Observe that $|X_n^j| = 2\ell_n \delta_{n+1} - \delta_{n+1}^2$. At step n the projections of the squares at the center of all crosses onto the x-axis are a finite number of intervals whose total length is $\sum_{j=1}^{n+1} 2^{j-1}\delta_j$. Thus our constraint on δ_n is

$$\sum_{j=1}^{\infty} 2^{j-1}\delta_j \le \ell. \tag{1.3}$$

Let $X = \cup_{n=0}^{\infty} \cup_{j=1}^{4^n} X_n^j$. Then

$$|X| = \sum_{n=0}^{\infty} 4^n (2\ell_n \delta_{n+1} - \delta_{n+1}^2)$$

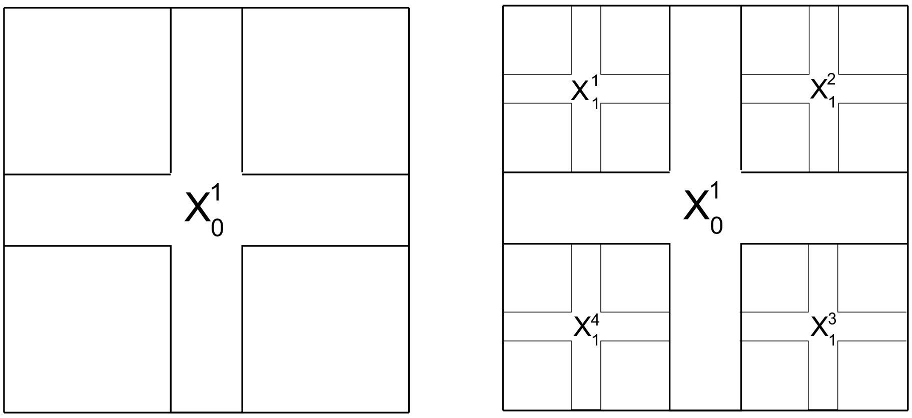

Figure 1.4: The irrigating set is constructed iteratively as a union of cross of controlled thickness. The first step consists of the cross X_0^1. Then we consider the set made of X_0^1 and the four additional crosses of the next step. For a suitable choice of the thickness of crosses, the set of points that do not lie in the countable union of crosses is of positive measure and all of them are accessible for a particular profile.

Let us introduce the parameters $t_j = \frac{2^{j-1}\delta_j}{\ell}$, $j = 1, 2, ...$ which represents the proportion of the interval $(-\ell, \ell)$ covered by the projections of the squares at the center of the crosses constructed at step $j-1$. In terms of t_j the constraint (1.3) becomes

$$t := \sum_{j=1}^{\infty} t_j \leq 1. \tag{1.4}$$

Observe that

$$\ell_n = \frac{\ell}{2^n} - \frac{1}{2^n} \sum_{j=1}^{n} 2^{j-1}\delta_j,$$

where this equality holds for all $n \geq 0$ if we understand that the sum at the right hand side is equal to zero when $n = 0$. Thus,

$$|X| = 2\sum_{n=0}^{\infty} 2^n \delta_{n+1}(\ell - \sum_{j=1}^{n} 2^{j-1}\delta_j) - \sum_{n=0}^{\infty} 4^n \delta_{n+1}^2$$

which we may write in terms of t_n as

$$\begin{aligned} |X| &= 2\ell^2 \sum_{n=0}^{\infty} t_{n+1}(1 - \sum_{j=1}^{n} t_j) - \ell^2 \sum_{n=0}^{\infty} t_{n+1}^2 \\ &= 2\ell^2 \sum_{n=0}^{\infty} t_{n+1} - \ell^2 \left(2\sum_{n=0}^{\infty} t_{n+1} \sum_{j=1}^{n} t_j + \sum_{n=0}^{\infty} t_{n+1}^2 \right) \\ &= 2\ell^2 t - \ell^2 t^2 = \ell^2(2t - t^2). \end{aligned}$$

We conclude that $|X| < \ell^2$ if and only if we have the strict inequality in (1.4), or equivalently, in (1.3). In this case $K = \Omega \setminus X$ is of positive measure. Let us prove that K is accessible with profile f by properly choosing the values of δ_k. This will imply, in particular, that $K = \partial X$.

Given a point $p \in K$, there is a sequence of arms of crosses joining p to the center of Ω. Let γ be the curve formed by the segments going through the centers of these arms. The worst case would happen if the arc consists of segments, called $s_0, s_1, s_2, s_3, ..., s_{2n-2}, s_{2n-1}, ...$ of lengths

$$\frac{\ell_1}{2} + \frac{\delta_1}{2}, \frac{\ell_1}{2} + \frac{\delta_1}{2}, \frac{\ell_2}{2} + \frac{\delta_2}{2}, \frac{\ell_2}{2} + \frac{\delta_2}{2}, ..., \frac{\ell_n}{2} + \frac{\delta_n}{2}, \frac{\ell_n}{2} + \frac{\delta_n}{2}, ...$$

Observe that the total length is less than 2ℓ. We consider the s_0 as part of the step 0 cross, s_1, s_2 as part of the crosses constructed at step 1, etc. Let m_k be the length of segment s_k. If $s \in [\sum_{k=n+1}^{\infty} m_k, \sum_{k=n}^{\infty} m_k]$ we are just describing segment s_n. If n is even (odd) we are in a cross of type $n/2$ (resp., $\frac{n+1}{2}$). Suppose that $n = 2p$, $p = 0, 1, ...$, and $s \in [\sum_{k=n+1}^{\infty} m_k, \sum_{k=n}^{\infty} m_k]$. Since

$$\begin{aligned}
\sum_{k=n}^{\infty} m_k &= \sum_{j=p+1}^{\infty} (\ell_j + \delta_j) = \sum_{j=p+1}^{\infty} \frac{\ell}{2^j} + \sum_{j=p+1}^{\infty} (\delta_j - \frac{1}{2^j} \sum_{i=1}^{j} 2^{i-1} \delta_i) \\
&\leq \frac{\ell}{2^p} + \sum_{j=p+1}^{\infty} (\delta_j - \frac{\delta_j}{2}) = \frac{\ell}{2^p} + \frac{1}{2} \sum_{j=p+1}^{\infty} \delta_j \\
&= \frac{\ell}{2^p} + \ell \sum_{j=p+1}^{\infty} \frac{t_j}{2^j} = \frac{\ell}{2^p} + \frac{\ell}{2^p} = \frac{\ell}{2^{p-1}},
\end{aligned}$$

f is increasing and we are in a cross constructed at step p whose width is δ_{p+1}. Thus, (1.1) will be satisfied if we have the inequality

$$f(\frac{\ell}{2^{p-1}}) \leq \frac{\delta_{p+1}}{2}. \tag{1.5}$$

In the same way, if $n = 2p - 1$, $p = 1, 2, ...$ and $s \in [\sum_{k=n+1}^{\infty} m_k, \sum_{k=n}^{\infty} m_k]$, (1.1) will be satisfied if the inequality (1.5) holds. By our assumption on f, (1.5) will be satisfied with a proper choice of δ_k which has to satisfy the constraint (1.3) with a strict inequality sign to guarantee that K is of positive measure. This ends the proof that K is irrigable and the length of accessibility curves is less than 2ℓ. $\square$

Remark 1.1.6 *The function $f(s) = s^p$ satisfies (1.2) if and only if $p > 1$. The function $f(s) = \frac{s}{|logs|^{\beta}}$ satisfies (1.2) if and only if $\beta > 1$. All these profiles combined with lemma 1.1.5 give a whole bunch of sets irrigating a set with positive measure.*

1.2 The equality of supply flow problem

Let Ω be a bounded set in $\mathbb{R}^N$ with Lipschitz boundary (we may also take $\Omega = \mathbb{R}^N$). Let U be an open bounded set such that $U \subset\subset \Omega$ and $|\partial U| > 0$.

Definition 1.2.1 *We say that the open set U permits an equality of supply flow if there is a positive measure μ with support $supp(\mu) \subset\subset U$ with mass $\int_\Omega \mu = |\partial U|$ and a vector field $v \in L^\infty(\Omega)$ such that*

$$-div\, v = -\mu + \chi_{\partial U} \quad \text{in } \Omega \tag{1.6}$$

$$v = 0 \quad \text{outside } U. \tag{1.7}$$

The measure μ will be called the source measure.

Remark 1.2.2 *Notice that condition (1.7) implies that $v \cdot \nu = 0$ on the points of ∂U where ∂U is described by a regular manifold.*

We are interested in studying conditions on the structure of the open set U which guarantee the existence of a flow with equality of supply. We shall consider open sets such that $U = \cup_n U_n$, where U_n are open sets in $\mathbb{R}^N$ with Lipschitz continuous boundary. The idea is then to solve a slightly modified problem on each U_n and to use the sequence of vector fields v_n thus obtained to solve the problem for U. In the case of Lipschitz boundary sets, Proposition 1.2.4 below gives a simple criterion to ensure the existence of a vector field of prescribed divergence. This criterion is then used to show that the 2D tree introduced in section 1.1.2 permits an equality of supply flow.

Let us first consider the solvability of (1.6) in Lipschitz domains. We recall the following result which was proved in $\mathbb{R}^N$ by Bellettini, Caselles and Novaga in [4].

Proposition 1.2.3 *[4] Let W be a bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $f \in L^2(W) \cap L^N(W)$. Then the function u is a solution of*

$$\min_{w \in L^2(W) \cap BV(W)} \int_W |Dw| + \frac{1}{2}\int_W (w-f)^2\, dx \tag{1.8}$$

if and only if there is a vector field $v \in L^\infty(W, \mathbb{R}^N)$ with $\| v \|_\infty \leq 1$ such that $\int_W (v, Du) = \int_W |Du|$ and

$$\begin{aligned} u - div\, v = f \quad \text{in } W \\ v \cdot \nu = 0 \quad \text{in } \partial W. \end{aligned} \tag{1.9}$$

The following result is an easy consequence of Proposition 1.2.3.

Proposition 1.2.4 *Let W be a bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $f \in L^2(W) \cap L^N(W)$. Then there is a vector field $v \in L^\infty(W, \mathbb{R}^N)$ with $\| v \|_\infty \leq C$ such that*

$$\begin{aligned} -div\, v = f \quad \text{in } W \\ v \cdot \nu = 0 \quad \text{in } \partial W \end{aligned} \tag{1.10}$$

if and only if

$$\int_W f = 0 \tag{1.11}$$

and

$$\Big| \int_W fw \Big| \leq C \int_W |Dw| \quad \text{for all } w \in BV(W). \tag{1.12}$$

Proof: By changing v into $\frac{v}{C}$ we may assume that $C = 1$. The vector field $v \in L^\infty(W, \mathbb{R}^N)$ with $\| v \|_\infty \leq 1$ is a solution of (1.10) if and only if $\int_W f = 0$ and the function $u = 0$ is a solution of (1.9). Under the assumption that $\int_W f = 0$, $u = 0$ is a solution of (1.9) if and only if

$$\int_W |Dw| + \frac{1}{2}\int_W (w-f)^2\, dx \geq \frac{1}{2}\int_W f^2\, dx \quad \forall w \in L^2(W) \cap BV(W). \tag{1.13}$$

Replacing w by ϵw (where $\epsilon > 0$), expanding the L^2-norm, dividing by $\epsilon > 0$, and letting $\epsilon \to 0+$, we have

$$\Big| \int_W f(x) w(x)\, dx \Big| \leq \int_W |Dw| \quad \forall w \in L^2(W) \cap BV(W). \tag{1.14}$$

Since (1.14) implies (1.13), we have that (1.13) and (1.14) are equivalent. □

Before proceeding, let us recall some results about BV functions and traces. In the following we note $\mathcal{H}^N$ for the $N-$dimensional Hausdorff measure, and $[u > t] = \{y \mid u(y) > t\}$. Let $u \in BV(U)$. We define

$$u^+(x) = \inf\{t : d([u > t], x) = 0\}$$

$$u^-(x) = \sup\{t : d([u < t], x) = 0\}.$$

It is useful to introduce u^* defined by the formula $u^*(x) := \dfrac{u^+(x) + u^-(x)}{2}$. For a suitable mollifier, $u^*(x) = \lim(\rho_n * u)(x)$ for almost every x ([17], p. 216, Corollary 1), relatively to the $\mathcal{H}^{N-1}$ measure. This property is of particular interest since it permits to define the following linear form on $BV(U)$

$$\delta_X(w) = \int_X w^* \, d\mathcal{H}^{N-1} \quad w \in BV(U).$$

for any $X \subset U$ which is $\mathcal{H}^{N-1}$ rectifiable with $\mathcal{H}^{N-1}(X) < \infty$. If f is a linear form on $BV(U)$, we shall write indiscriminately $\int_U fw$ or $f(w)$.

1.2.1 A solution to the equality of supply for the 2D tree described in Section 1.1.2

A convenient iterative description of the tree

Let $X_n = \cup_{k=1}^n \cup_{j=1}^{4^n} X_k^j$. The centers of the 4^n crosses X_n^j will be called the sink boxes of the draining network X_n while the center of X_0^1 will be called source box of X_n. At each sink box we place two segments joined in the form of $'V'$ whose total length $\sqrt{2}\delta_{n+1}$, which coincides with the length of the diagonal of the box, and at the source box we place a $'V'$ of length $\sqrt{2}\delta_1$. Observe that, if $f(s) = s^2$, then $\frac{\delta_{p+1}}{\delta_p} = \frac{1}{4}$. Hence $4^n\delta_{n+1} = \delta_1$. Observe also that, if $f(s) = s^p$, then $4^n\delta_{n+1} \geq \delta_1$ if and only if $p \leq 2$.

In the sequel we shall consider the case $f(s) = s^2$. Let us describe precisely how we shall connect the sink segments to the source segment. We shall describe with detail the first step of the construction. Let us first describe the position of the sinks inside its box. Let us consider a box which we normalize to be $(0, 1)^2$. We shall use as a sink two segments of length $\frac{\sqrt{2}}{2}$ joined by its end point in the form of $'V'$, or an inverted $'V'$, and forming an angle of $\frac{\pi}{2}$, and we center it in the sink box. Similarly we construct the source in a form of $'V'$ located at the source box. Let us describe how to connect the sinks and the source in the draining network X_1. At the two upper sink boxes we put inverted $'V'$ sinks of length $\sqrt{2}\delta_2$, while we will place $'V'$'s of the same size at the two lower sink boxes. To connect the sink at the upper left cross, say X_1^1, to the source we shall use first the descending arm of X_1^1 until we reach the center of the left arm of the cross X_0^1. This will be called a re-directing station of the draining network. There we place a re-directing join made of two segments in the form of a $' <'$. Each segment of the $' <'$ re-directing station has length $\sqrt{2}\delta_2$. The re-directing station collects the flux from the upper and lower left crosses and redirects it towards the source. This disposition permits to collect the flow from the two arms of the type 1 crosses which are incoming into the left arm of the cross X_0^1 and redirect it to its center. Observe that the sum of the lengths of the 4 sinks equals the length of the $'V'$ source. Thus, each segment of each $'V'$ sink is mapped into a segment of equal length of the source. We observe that by a similar construction we may place a $'V'$ segment at each sink box of the n-th draining network X_n and $'V'$ segments in the corresponding redirecting stations in such a way that each segment of each of the $'V'$ sinks is mapped to a segment of equal length of the source. For latter use, let us fix some notation. For each n, we divide the square Ω into a family $\mathcal{C}_n$ made of 2^{2n} squares whose side has length

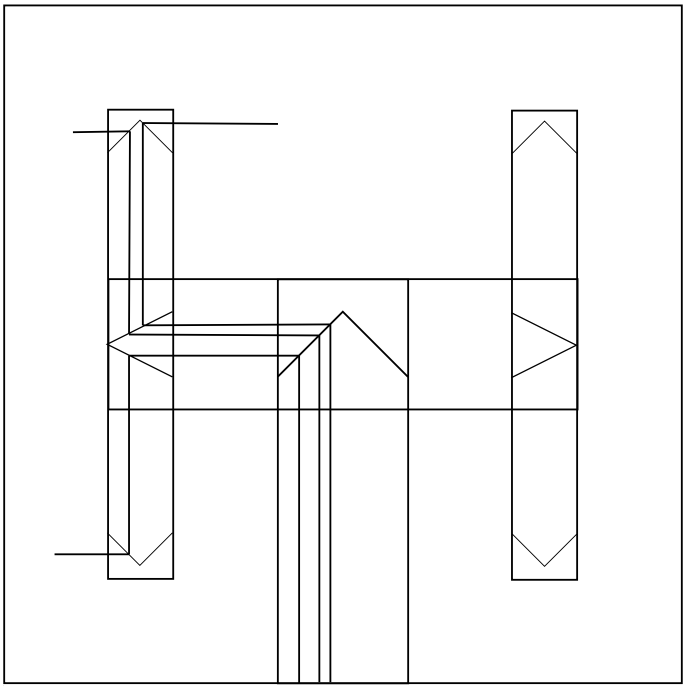

Figure 1.5: An irrigating tree in $\mathbb{R}^2$ with its re-directing stations.

$\alpha_n = \frac{\ell}{2^n}$. For each $Q \in \mathcal{C}_n$ there is a cross X_n^j of X_n inside Q in which we place a sink V_Q in the form of a $'V'$, or an inverted $'V'$. The length of V_Q is $\sqrt{2}\delta_{n+1}$. Let us call Δ_Q the box where the sink V_Q is located. Let V_0 be the source. Let $Q \in \mathcal{C}_n$. Let W_Q be segment of the source V_0 corresponding to V_Q which is of equal length. When we go from X_n to X_{n+1} the source is unchanged, the sinks are now at the centers of the crosses X_{n+1}^j, $j = 1, ..., 4^{n+1}$, while the sinks of the previous stages are now transformed into re-directing stations. We have just to add some re-directing stations to connect the new sinks $V_{Q'}$, $Q' \in \mathcal{C}_{n+1}$, to the previous ones V_Q, $Q \in \mathcal{C}_n$, converted now into re-directing stations. These new re-directing stations will be placed in the same way we did for the re-directing stations connecting V_Q, $Q \in \mathcal{C}_1$, to V_0. We are now in position to prove that a bounded vector field may be constructed in X_n sending the flux from the source to the sinks, and to prove that the bound on the supremum of the norm of the vector field is independent of n.

The solution in U_n

Let us now write U and U_n, instead of X and X_n, respectively. Consider Ω to be a box such that $U \subset\subset \Omega$. Our purpose is to construct a vector field $v \in L^\infty(\Omega)$ and a positive measure μ supported on the source V_0 such that

$$-div\, v = -\mu + \chi_{\partial U} \quad \text{in } \Omega \tag{1.15}$$

and $v = 0$ in $\Omega \setminus U$. Observe that integrating (1.15) in Ω and using that $v \cdot \nu = 0$ in $\partial\Omega$ we deduce that $\mu(V_0) = |\partial U|$. To prove the existence of v we shall proceed by constructing vector fields $v_n \in L^\infty(U_n)$, and measures μ_n on V_0, such that

$$-div\, v_n = -\mu_n + f_n \quad \text{in } U_n, \tag{1.16}$$

$$v_n \cdot \nu = 0 \quad \text{in } \partial U_n, \tag{1.17}$$

where f_n is a sequence of functions converging to $\chi_{\partial U}$ weakly as measures. The sequence f_n will be chosen such that v_n is bounded independently of n.

Construction of f_n

Let

$$f_n = \sum_{Q \in \mathcal{C}_n} |Q \cap \partial U| \frac{\delta_{V_Q}}{\mathcal{H}^1(V_Q)}.$$

Observe that, in particular, we have that

$$\int_\Omega f_n = |\partial U|.$$

Let $k \leq n$, $Q' \in \mathcal{C}_k$, $Q \in \mathcal{C}_n$. Observe that either $V_Q \subseteq Q'$ or $V_Q \cap Q' = \emptyset$. Then

$$\int_\Omega f_n \chi_{Q'} = \sum_{Q \in \mathcal{C}_n} |Q \cap \partial U| \int_\Omega \frac{\delta_{V_Q}}{\mathcal{H}^1(V_Q)} \chi_{Q'} = \sum_{Q \in \mathcal{C}_n, Q \subseteq Q'} |Q \cap \partial U| = |Q' \cap \partial U|.$$

Thus

$$\int_\Omega f_n \chi_{Q'} \to |Q' \cap \partial U| \quad \text{for all } Q' \in \cup_k \mathcal{C}_k.$$

This implies that

$$\int_\Omega f_n \varphi \to \int_\Omega \chi_{\partial U} \varphi \quad \text{for all } \varphi \in C(\overline{\Omega}).$$

Construction of μ_n

In the notation introduced before, let W_Q, $Q \in \mathcal{C}_n$, be the segment of the source V_0 corresponding to V_Q. Recall that $\mathcal{H}^1(V_Q) = \mathcal{H}^1(W_Q)$. Let

$$\mu_n = \sum_{Q \in \mathcal{C}_n} |Q \cap \partial U| \frac{\delta_{W_Q}}{\mathcal{H}^1(W_Q)}.$$

By extracting a subsequence, if necessary, we may assume that $\mu_n \rightharpoonup \mu$ where μ is a positive measure with support in V_0 such that $\int_{V_0} \mu = |\partial U|$.

In order to apply Proposition 1.2.4, conditions (1.11) and (1.12) are to be verified. This will prove the existence of a vector field $v_n \in L^\infty(U_n)$ satisfying (1.16), (1.17) with an L^∞ bound depending on the constant C appearing in (1.12). First, we observe that $\int_{U_n} f_n = \int_{U_n} \mu_n$. Next, let $w \in BV(U_n)$. We evaluate

$$\int_{U_n} (\mu_n - f_n) w = \sum_{Q \in \mathcal{C}_n} \frac{|Q \cap \partial U|}{\mathcal{H}^1(V_Q)} \Big(\int_{W_Q} w^* \, d\mathcal{H}^1 - \int_{V_Q} w^* \, d\mathcal{H}^1 \Big)$$

Since for each $Q \in \mathcal{C}_n$ we have

$$\frac{|Q \cap \partial U|}{\mathcal{H}^1(V_Q)} \leq \frac{|Q|}{\mathcal{H}^1(V_Q)} = \frac{|Q|}{\sqrt{2}\delta_{n+1}} = \frac{\ell^2/4^n}{\sqrt{2}\delta_1/4^n} = \frac{\ell^2}{\sqrt{2}\delta_1}, \tag{1.18}$$

and

$$\sum_{Q \in \mathcal{C}_n} \Big| \int_{W_Q} w^* \, d\mathcal{H}^1 - \int_{V_Q} w^* \, d\mathcal{H}^1 \Big| \leq C \int_{U_n} |Dw| \tag{1.19}$$

where the constant C does not depend on n, we have

$$\Big|\int_{U_n}(\mu_n - f_n)w\Big| \le C\frac{\ell^2}{\sqrt{2}\delta_1}\int_{U_n}|Dw| \tag{1.20}$$

Having stated Proposition 1.2.4 for functions and not for measures, we have to regularize f_n and μ_n. Let $\rho \in C_0^\infty(\mathbb{R}^N)$ be such that $\rho \ge 0$, $supp(\rho) \subseteq B(0,1)$, $\int_{\mathbb{R}^N}\rho(x)\,dx = 1$, and let $\rho_\epsilon(x) = \epsilon^{-N}\rho(\frac{x}{\epsilon})$. We choose $\epsilon = \epsilon_n$ such that the support of $\rho_n(x) = \rho_{\epsilon_n}(x)$ is contained in a ball of radius strictly less than the distance from the support of f_n to the boundary of U_n. Observe that $(\rho_n * f_n)|_{U_n} = \rho_n * (f_n|_{U_n})$ and the same property also holds for μ_n. The functions $(\rho_n * \mu_n - \rho_n * f_n)|_{U_n}$ satisfy (1.11) and (1.12) with the same constant C than $\mu_n - f_n$.

Let $v_n \in L^\infty(U_n, \mathbb{R}^N)$ be the solution of (1.16), (1.17) in U_n corresponding to $(\rho_n * \mu_n - \rho_n * f_n)|_{U_n}$. To extend v_n, we use the following Lemma.

Lemma 1.2.5 *Let W, W_1, W_2 be two open bounded sets with Lipschitz boundary. Assume that W_1 and W_2 have a common boundary F and $W = W_1 \cup W_2 \cup F$. Let $g_i \in L^N(W_i)$, $i = 1, 2$. Suppose that for each $i = 1, 2$, there are vector fields z_i satisfying*

$$\begin{aligned} -div\, z_i &= g_i \quad \text{in } W_i \\ z_i \cdot \nu &= 0 \quad \text{in } \partial W_i. \end{aligned} \tag{1.21}$$

Let $g = g_1\chi_{W_1} + g_2\chi_{W_2}$, $z = z_1\chi_{W_1} + z_2\chi_{W_2}$. Then

$$\begin{aligned} -div\, z &= g \quad \text{in } W \\ z \cdot \nu &= 0 \quad \text{in } \partial W. \end{aligned} \tag{1.22}$$

By setting $v_n = 0$ in $W_2 = \Omega \setminus \overline{U_n}$, $W_1 = U_n$ and $F = \partial U_n$, applying Lemma 1.2.5, we may extend v_n to $\Omega \setminus U_n$.

Proposition 1.2.6 *The 2D irrigating tree defined in Section 1.1.2 with profile $f(s) = s^p$, $p \le 2$, permits an equality of supply flow.*

Proof: Let us consider the sequence of vector fields v_n obtained from the previous construction. Let us prove that we may extract a subsequence from v_n which permits to solve (1.6), (1.7). Let us observe first that $\rho_n * \mu_n - \rho_n * f_n \rightharpoonup \mu - \chi_{\partial U}$ in Ω. Indeed, $\rho_n * \varphi \to \varphi$ uniformly for each $\varphi \in C_c(\Omega)$, and $\mu_n - f_n \rightharpoonup \mu - \chi_{\partial U}$ in Ω. Thus we have that $\int_\Omega(\rho_n * \mu_n - \rho_n * f_n)\varphi = \int_\Omega(\mu_n - f_n)\rho_n * \varphi \to \int_\Omega(\mu - f)\varphi$ when $n \to \infty$ for all $\varphi \in C_c(\Omega)$. Since $\mu - f$ does not charge $\partial\Omega$, we deduce that $\int_\Omega(\mu_n - f_n)\varphi \to \int_\Omega(\mu - f)\varphi$ as $n \to \infty$ for all $\varphi \in C(\overline{\Omega})$. By extracting a subsequence, if necessary, we may assume that v_n converges weakly* in $L^\infty(\Omega, \mathbb{R}^N)$ to a bounded vector field v such that $v = 0$ in $\Omega \setminus U$. In addition we have

$$-div\, v = -\mu + \chi_{\partial U} \quad \text{in } \Omega. \tag{1.23}$$

□

Remark 1.2.7 *In fact, it is possible to give directly a vector field which answers the problem. The vector field is the one that appears in figure 1.5. Let x be a point in U, there is a n such that $x \in U_n$. Then, if x is on a path between the source and a sink, $v(x)$ is set to be the unit vector colinear to the path, otherwise $v(x) = 0$.*

Chapter 2

Measure transportation models

Introduction

In the previous chapter, we studied some geometrical obstruction to the existence of irrigating systems. In this chapter we shall no longer consider systems with ”thick” tubes but rather an idealized structure which will consist only of the skeleton of the structure. The information we want to keep trace of is the way mass is transported from the sources to the tips. To do this we shall consider different formulations of mass transportation problems. The first mathematical transportation problem was formalized by Monge, then given a relaxed formulation by Kantorovitch ([24],[19]). The problem he considered was the one of moving a pile of sand from a place to another with the less possible work. In the Monge-Kantorovitch framework, μ^+ and μ^- are measures on $\mathbb{R}^N$, and to transport μ^+ onto μ^- means to tell where the mass of μ^+ is sent, i.e. to give a measure π on $\mathbb{R}^N \times \mathbb{R}^N$ where $\pi(A \times B)$ represents the amount of mass going from A to B. This measure π is called a transference plan. To evaluate the efficiency of a transference plan, we consider the cost function $c : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ where $c(x, y)$ is the cost of transporting a unit mass from x to y. The cost associated with a transference plan is $\int_{\mathbb{R}^N \times \mathbb{R}^N} c(x, y) d\pi(x, y)$. The minimization of this functional is the Monge-Kantorovitch problem.

If we see μ^+ and μ^- as supply (factories) and demand (clients) measures, the Monge-Kantorovitch framework is well adapted to model the way the clients should be delivered when the roads already exist. This problem is sometimes named the transport problem or the Hitchcock problem in the linear programming literature. We also consider measure transportation framework as an alternative and convenient way to formalize irrigation of volume and equality of supply (see Chapter 1). Indeed, let us take $\mu^+ = \delta_S$ and μ^- some measure of $\mathbb{R}^N$. In such a context, for a source to irrigate a volume means that we consider a transport from μ^+ to a measure μ^- with a support of positive measure. To have the equality of supply would be translated by the fact that μ^- is Lebesgue measure on some set K.

As an example, consider the cost function $c(x, y) = |x - y|^2$, and the supply and demand measures $\mu^+ = \delta_x$ and $\mu^- = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$. The minimizer π is the measure on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\pi(\{x\} \times \{y_1\}) = \frac{1}{2}$ and $\pi(\{x\} \times \{y_2\}) = \frac{1}{2}$. The actual transportation, for the real problem of transporting sand, is achieved along geodesics between x, y_1 and y_2 as represented in Figure 2.1.

In the Monge-Kantorovitch framework, the transport structure along which the mass would be really transported is all made of geodesics between starting and ending points, it is given by the transference plan. We stress the fact that the structure plays no role in the cost functional, in the sense that the structure depends completely on the transference plan. This is why the cost functional has to be adapted if we want to apply this framework to the irrigation problem or to some particular supply-demand problems. Indeed,

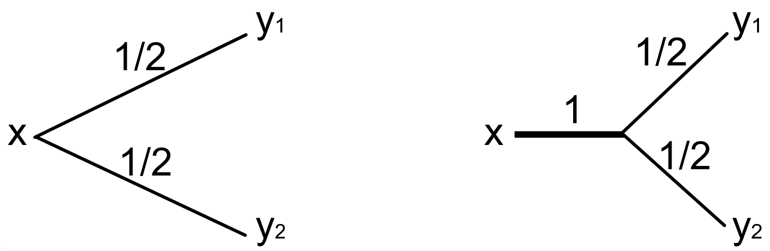

Figure 2.1: The transport from δ_x to $\frac{1}{2}(\delta_{y_1} + \delta_{y_2})$. Monge-Kantorovich versus Q. Xia's solution.

in the case of a supply-demand problem where the structure is still to be built, it could be preferable to incorporate the cost of the construction of this structure into the total cost (finding a compromise between construction cost and efficiency of the structure). A second motivation for taking into account the structure is that it is in some cases preferable for the mass to be transported in a grouped way: concerning the sand example, it is better for instance to use trucks, wheelbarrows and buckets rather than just a shovel. In a fluid mechanics context, Poiseuille's law states that the resistance of a tube increases when a tube gets thinner in such a way that it is preferable to have a tube of section S rather than two tubes of section $S/2$. This is also an invitation to group the mass/the flow in the case of the irrigation problem, as it is illustrated by the structure of the lungs.

This chapter is dedicated to survey briefly the different mathematical objects that have been proposed to model efficient transport structures.

2.1 The Gilbert-Steiner problem [18]

The Steiner problem consists in minimizing the total length of a network connecting a given set of points. It is a good model to penalize the cost of the construction of a homogeneous transport structure. However, this cost is not realistic since it does not discriminate the cost of high or low capacity edges (a road has not the same cost as a highway). The first model taking into account capacities of edges was proposed by Gilbert [18] in the case of communication networks. This author models the network as a graph such that each edge e is associated with a capacity c_e. Let $f(c)$ denote the cost per unit length of an edge with capacity c. It is assumed that $f(c)$ is subadditive and increasing, i.e., $f(a) + f(b) \geq f(a + b) \geq \max(f(a), f(b))$. In this context, the cost of a graph G is $C(G) = \sum_e f(c_e)l(e)$ where the sum is taken over all the edges of the graph and $l(e)$ is the length of e. Gilbert then considers the problem of minimizing this cost over all networks supporting a given set of flows between prescribed terminals. The subadditivity of the cost f translates the fact that it is more advantageous to construct an edge with capacity c rather than two edges of capacity $c/2$. Let us mention that Gilbert's model was also used in the study of optimal pipeline or drainage networks ([6],[20]).

2.2 Xia's model of transport paths [35]

Though Gilbert clearly proposed this functional to optimize the construction cost of a network, the subadditivity of f could also be interpreted as a way to encourage the mass to move in a grouped way. This is the object of articles [35] and [22] to use Gilbert-Steiner cost in a more general continuous

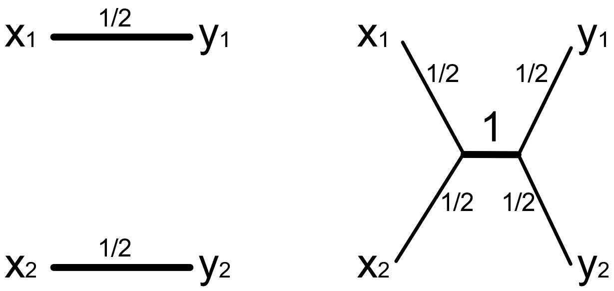

Figure 2.2: Irrigation problem minimizer versus traffic problem minimizer in the case $\alpha = 0$.

framework where the supply and demand measures are not constrained to be atomic. Let us now detail the approaches of [35] and [22].

Xia models the transportation network as an embedded graph with a countable number of vertices and satisfying Kirchhoff's law. This author starts with finite atomic measures a and b and defines a "path" from a to b as a flow on a finite embedded graph whose end vertices end up on a or b. He denotes by e the (straight) edges of the graph, by $w(e)$ the flow in the edge e, and by $\vec{e}$ the unit vector in the direction of e. He denotes by $[[e]] = \mathcal{H}^1_{|e}\,\vec{e}$ the vectorial measure obtained as the product of the Hausdorff measure restricted to e and of the vector $\vec{e}$. Then the embedded path from a to b can be written as the vectorial measure

$$G = \sum_e w(e)[[e]].$$

The Kirchhoff law (K) is simply expressed as

$$div(G) = a - b,$$

where a and b are the supply and the demand measures. The cost functional is defined as in the functional of Gilbert-Steiner [18]:

$$M^\alpha(G) = \sum_e w(e)^\alpha \text{length}(e),$$

where $\alpha \in [0, 1]$. Notice that this cost corresponds to a cost per unit length of $w(e)^\alpha$ for each edge e. It is subadditive because of the concavity of $f(x) = x^\alpha$. Then, Xia proceeds to define transport paths between probability measures more general than finite graphs. He says that a vector measure T is a transport path between μ^+ and μ^- if there are sequences of atomic measures a_i and b_i and paths G_i connecting a_i to b_i such that a_i and b_i converge weakly to μ^+ and μ^- and $G_i \to T$ weakly in the sense of vector measures. This implies $div(T) = \mu^+ - \mu^-$ in the distribution sense. The energy of any such path is defined by relaxation as

$$M^\alpha(T) := \inf \liminf_{i\to\infty} M^\alpha(G_i),$$

where the infimum is taken over the set of all possible approximating graph sequences $a_i,\ b_i,\ G_i$ of T. As a simple example, the minimizer of M^α with $\mu^+ = \delta_x$ and $\mu^- = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$ is represented in Figure 2.1. Let us consider another example that will illustrate the difference with the traffic plan approach: take $\mu^+ = \frac{1}{2}(\delta_{x_1} + \delta_{x_2})$ and $\mu^- = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$. The locations of x_1, x_2, y_1 and y_2 and the minimizer are represented in Figure 2.2.

2.3 The pattern model [22]

The article [22] describes a (Lagrangian) formulation quite related to the transport paths proposed by Xia.The pattern model describes an irrigation system as a (usually uncountable) set of paths or “fibers” starting from a point source S and arriving at every point of the support of the irrigated measure. The fibers represent either the trajectory in $\mathbb{R}^d$ of a fluid particle, or a fiber of a tree. Each fiber is parameterized as $\chi(\omega, l) \in \mathbb{R}^d$, where l is time (or length along the fiber) and ω denotes a particle, belonging to an abstract probability space Ω. A stopping time (or length) $\sigma_\chi(\omega)$ is associated with each fiber. This permits to define the irrigation measure as a density measure of the fibers stopping in any given volume. Let us denote $T(\omega) := \chi(\omega, \sigma_\chi(\omega))$; the amount of fluid irrigating a Borel set A is the measure of $T^{-1}(A)$ in Ω. The authors define χ-vessels, or branches, as equivalence classes by the equivalence relation $\omega \simeq_l \omega'$ if $\chi(\omega, s)$ and $\chi(\omega', s)$ coincide up to time l. The cost of a pattern is defined as

$$E(\chi) = \int_\Omega \int_{\mathbb{R}^+} |[\chi(\omega,t)]_\chi|^{\alpha-1} |\dot\chi(\omega,t)| dt d\omega,$$

where $|[\chi(\omega,t)]_\chi|$ is the measure of the equivalence class of ω at time t, and $\alpha \in [0,1]$.

If we consider the simplest example of transportation with two Dirac masses as a demand (see Figure 2.1), Maddalena-Morel-Solimini’s solution coincides with the Xia’s one displayed in Figure 2.1. In this case the solution is given by the set of fibers $\chi : [0,1] \times [0,\infty) \to \mathbb{R}^2$, where $\chi(p,t)$ is either the path from x to y_1 (if $p \in [0,1/2]$), or the path from x to y_2 (if $p \in (1/2,1]$). More details about this model are given in Chapter 3.

2.4 Path functionals over Wasserstein spaces [7]

Quoting [7], the idea of the path functionals approach is that ”during the interpolation between the starting configuration (a probability measure) and the terminal one, the condition of keeping the mass together can be expressed by the requirement of passing through measures concentrated on discrete sets”. Let us consider $\mathcal{W}_p(\Omega)$ the space of probability measures with Wasserstein distance W_p. Given a source or initial measure μ_0 and a target or final measure μ_1, the object that realizes the transport is a continuous path $\gamma : [0,1] \to \mathcal{W}_p(\Omega)$ such that $\gamma(0) = \mu^+$ and $\gamma(1) = \mu^-$ and the goal is to minimize a suitable cost $\mathcal{J}(\gamma)$. To code for the fact that it is cheaper to transport the mass in a grouped way, the authors make paths through atomic and concentrated measures cheaper with the functional

$$\mathcal{J}(\gamma) = \int_0^1 J(\gamma(t)) |\gamma'(t)| dt,$$

where $|\gamma'|$ is the metric derivative in the Wassertstein space $\mathcal{W}_p(\Omega)$ and

$$J(\mu) = \begin{cases} \sum_k (a_k)^\alpha & \text{if } \mu = \sum_k a_k \delta_{x_k} \\ +\infty & \text{otherwise,} \end{cases}$$

with $\alpha < 1$. Such a functional is indeed such that the path has to go through (possibly infinite) atomic measure for the cost to be finite.

2.5 Optimal urban transportation networks [10],[9] and [8]

In [10], [9] and [8], a variation of the Monge-Kantorovitch problem has been proposed to model urban transportation network. In [10], a transportation network is modelled as a connected closed set Σ.

The users can either walk or join and use Σ. Thus, the cost for going from x to y is $d_\Sigma(x,y) := d(x,y) \wedge (dist(x,\Sigma) + dist(y,\Sigma))$, i.e. the minimum between the Euclidian (walking) distance $d(x,y)$ and the sum of distances from x and y to the network. Notice that the distance d_Σ describes how the Euclidian distance is twisted by the network. Given a population density μ^+ and a density of workplaces μ^-, the cost of this transportation network is given by the Monge-Kantorovitch distance between μ^+ and μ^- (in $\mathbb{R}^N$ equipped with the twisted distance $d_\Sigma(x,y)$). The authors of [10] then consider optimal transportation networks, i.e. transportation networks Σ with minimal cost among all possible Σ with length less than a prescribed length L, and study their qualitative topological and geometrical properties.

2.6 The traffic plan model (see Chapter 4)

We define a traffic plan as a measure on the set of all possible paths. Thus the traffic plan model is a straightforward generalization of patterns, since rather considering a particular parameterization $\chi(\omega)$ of fibers, we only keep the information given by $\chi\#\lambda$, i.e. the measure on paths induced by χ. Figure 2.3 shows three examples of traffic plans: a Dirac mass on a finite length path γ (which means that a unit mass is transported from $\gamma(0)$ to $\gamma(L)$), a traffic plan with ”Y” shape, and a traffic plan transporting a Dirac mass to the Lebesgue measure on a segment of the plane. In the same way as for the ”Y” shape, a weighted graph can easily be modelled by an atomic measure on the space of paths in the graph.

This very handy object generalizes finite graphs and can allow more general structure as can be seen on figure 2.3. In addition, this Lagrangian formalism is such that we can associate canonically a transference plan, an irrigating measure, and irrigated measures to any traffic plan. We denote by $|x|_\mu$ the multiplicity at a point x that will be the analogous of the capacity of an edge. It is the measure of paths going through x. The cost of the structure can then be written very similarly to the cost of patterns:

$$E(\mu) = \int_K \int_{\mathbb{R}^+} |\gamma(t)|_\mu^{\alpha-1} |\dot{\gamma}(t)| dt d\mu(\gamma),$$

where K is the set of 1-Lipschitz paths. We shall see further that it is the exact analogous of Gilbert-Steiner and Xia cost.

2.7 The irrigation problem versus the ”who goes where” problem

The ”who goes where” problem.

The irrigation problem consists in optimizing some cost on the set of all structures transporting μ^+ to μ^-. In contrast, the ”who goes where” problem consists in optimizing some cost on the set of all structures with prescribed transference plan. In the Monge-Kantorovitch framework, it would be absurd to consider the ”who goes where” problem since the ambient space of transports is precisely the set of transference plans. However, in the other models we presented, the structure and the transference plan are in some way dissociated. In case that we incorporate a transference plan constraint, that is to say, a ”who is going where” set of constraints, we call this generalization the traffic problem and its solution a traffic plan. This problem was briefly addressed by Xia in [35], but its solution is not satisfactory, to the best of our knowledge as we shall detail in the next paragraph. In order to understand the discussion, it is good to consider the very basic problem where $\mu^+ = \delta_{x_1} + \delta_{x_2}$ and $\mu^- = \delta_{y_1} + \delta_{y_2}$ as in Figure 2.2, i.e. $d(x_1, y_1) = d(x_2, y_2)$ is smaller than $d(y_1, y_2) = d(x_1, x_2)$. From the irrigation problem viewpoint, the solution is the same as the Monge-Kantorovitch one since it is not efficient to group the mass of μ^+

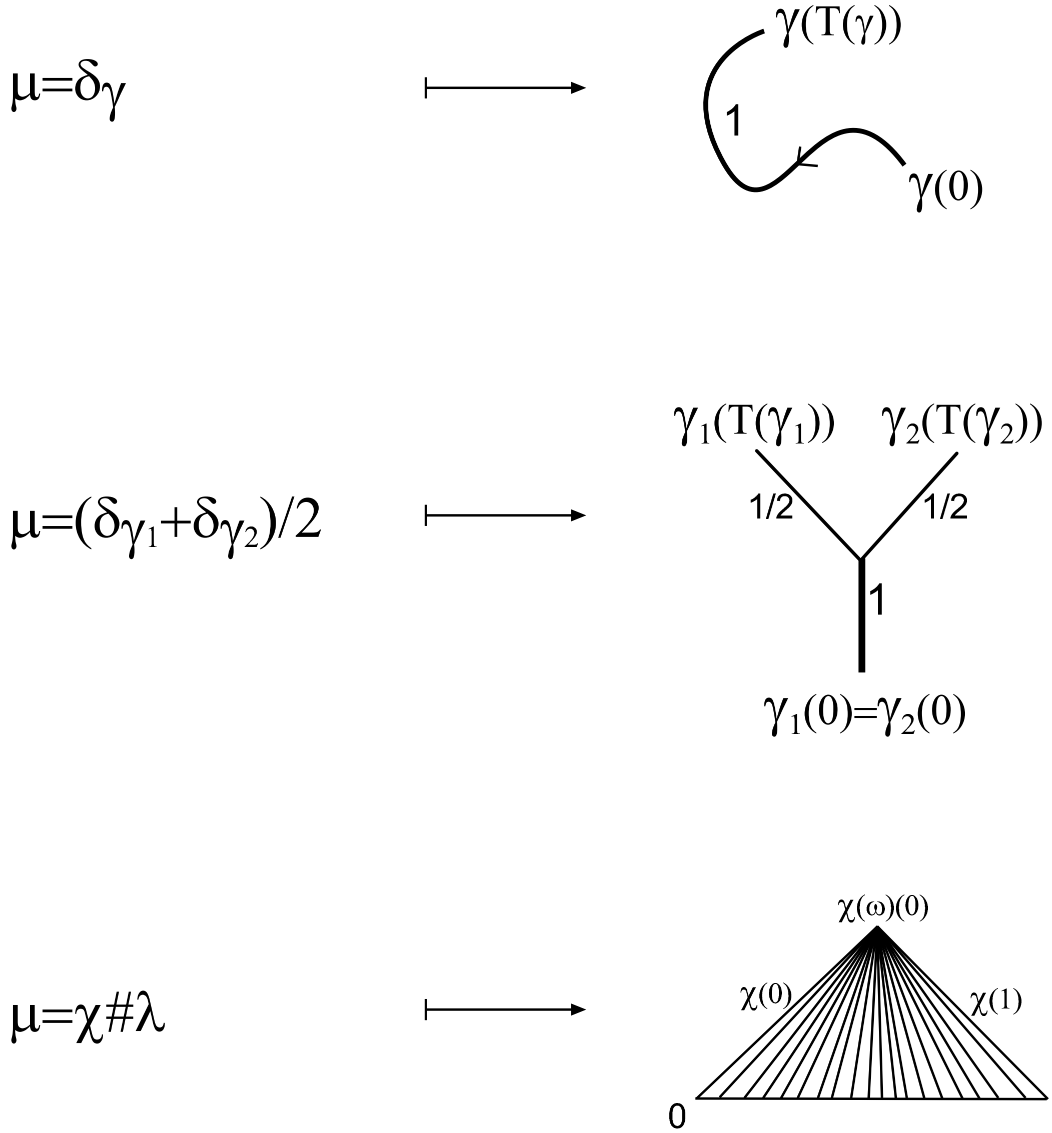

Figure 2.3: Three traffic plans and their associated embedding: a Dirac measure on γ, a tree with one bifurcation, a spread tree irrigating Lebesgue's measure on the segment $[0,1]\times\{0\}$ of the plane. Let us detail this last example. In that case, to $\omega\in[0,1]$ correspond $\chi(\omega)\in K$, the path parameterized by its length from the Dirac mass located at $(1/2,1)$, to the point $(\omega,0)$.

together. It is not if instead we want to find the best transportation network with the ”who goes where” constraint that all the mass in x_1 is sent onto y_2, and all the mass in x_2 onto y_1. The solution of the traffic problem versus the solution of the irrigation problem is displayed in Figure 2.2.

A traffic plan as a compatible pair of a transport path and a transference plan.

As mentioned in the previous paragraph, a graph approach modelling the traffic (or mailing) problem was presented in section 7 of [35]. To express the transference plan constraint, Xia considers what he calls ”compatible pairs” of a transport path and a transference plan. A piecewise rectilinear curve γ can be viewed as a graph with starting and ending points denoted by γ_i^- and γ_i^+. Given an atomic transference plan π, a transport path (a weighed finite graph in that case) is said to be compatible with π if it can be decomposed as a sum of curves γ_i with weight w_i so that $\pi(\gamma_i^-, \gamma_i^+) = w_i$. Notice that the notion of traffic plan is a convenient way to handle such compatible pairs. Indeed, the traffic plan $\sum_i w_i \delta_{\gamma_i}$ contains both the transference plan and the transport path information and is such that they are automatically ”compatible”. Xia then extends this compatibility definition to more general, non atomic, irrigating and irrigated measures. A transport path T and a transference π from μ^+ to μ^- are said to be compatible if

- there exist atomic measures a_i and b_i such that $a_i \rightharpoonup \mu^+$ and $b_i \rightharpoonup \mu^-$
- there exists a compatible pair (G_i, π_i) of transport path and transference plan from a_i to b_i such that $G_i \rightharpoonup T$ and $\pi_i \rightharpoonup \pi$.

We were not able to find a way to make this definition consistent with the discrete case. Indeed, a pair of a transport path with a transference plan can be both at a time compatible with respect to this last general definition but not compatible with respect to the atomic case definition.

To prove that, let us consider $\mu^+ = \mu^- = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$. Let T be the null transport path i.e the one associated with an empty graph. It is such that $div(T) = \mu^+ - \mu^- = 0$ so that T is a transport path from μ^+ to μ^-. Let π be the transference plan such that $\pi(x, y) = \frac{1}{2}$ and $\pi(y, x) = \frac{1}{2}$. This means that the mass in x and the mass in y are swapped by π. Thus defined, T and π form a compatible pair with respect to the general definition. Indeed, take G_i the graph made of and edge (x, y) with weight $\frac{1}{2}$ and of an edge (y_i, x_i) with weight $\frac{1}{2}$, parallel to (x, y) where y_i and x_i are getting closer and closer of x and y (see figure 2.4). Then G_i is weakly converging to $T = 0$. Let us define $a_i = \frac{1}{2}\delta_x + \frac{1}{2}\delta_{y_i}$ and $b_i = \frac{1}{2}\delta_y + \frac{1}{2}\delta_{x_i}$ so that a_i and b_i are weakly converging to μ^+ and μ^- . Finally, let π_i be the transference plan such that $\pi_i(x, y) = \frac{1}{2}$ and $\pi_i(y_i, x_i) = \frac{1}{2}$ so that π_i is weakly converging to π (see figure 2.5). Since G_i and π_i form a compatible pair, it follows that T and π are compatible. However, considered as a pair of a transport path and transference plan irrigating atomic measures, they are no more compatible with respect to the atomic case definition. This proves that the general definition of a compatible pair does not fit with what Xia wants a compatible pair in the atomic case to be.

Thus, it seems to us that the traffic plan object is a more convenient way to handle the transference plan constraint since it conveys both at a time the transport path and the transference plan information.

2.8 Comparison of models

The table 2.8 presents a synthesis of different objects that were proposed to model irrigation type problems.

Cost of patterns versus cost of transport paths.

Name	Object	Functional	Constraints	Comments	Article
Monge-Kantorovitch problem	Transference plan π	$\int_{\mathbb{R}^N \times \mathbb{R}^N} c(x,y) d\pi(x,y)$	marginals of π are ν^+ and ν^-	Does not take into account the cost of the network	[24, 19]
Gilbert-Steiner problem	Finite weighed directed graph G	$E(G) = \sum_e f(c(e))\text{length}(e)$, where $c(e)$ is the capacity of an edge and $f(c)$ is the cost per unit length of an edge with capacity c	Compatibility of capacities at nodes	Finite graph framework	[18]
Transport path	1-current or Radon vectorial measure T	$M^\alpha(G) = \sum_e w(e)^\alpha \text{length}(e)$ for finite graphs extended by relaxation on a subclass of Radon vectorial measures	$\partial T = \nu^+ - \nu^-$	The relaxation process makes unclear the identification of transport paths	[35]
Path in Wasserstein spaces	path $\gamma(t)$ in a Wasserstein spaces of probabilities	$\mathcal{J}(\gamma) = \int_0^1 J(\gamma(t))\|\gamma'(t)\|dt$ where J is encouraging the path to go through atomic measures	$\gamma(0) = \nu^+$ and $\gamma(1) = \nu^-$	Different optima than other models	[7]
Pattern	$\chi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^N$ such that $\chi(\omega, \cdot)$ is a 1-Lipschitz curve describing the trajectory of the particle ω	$E(\chi) = \int_\Omega \int_{\mathbb{R}^+} \|[\chi(\omega,t)]_\chi\|^{\alpha-1} \|\dot{\chi}(\omega,t)\| dt d\omega$	$\chi(\omega, 0) = S$ the source point and $i_\chi = \mu^-$	The Lagrangian approach permits to follow each particle but handles only one source	[22]
Traffic plan	Measure μ on the set of 1-Lipschitz curves	$E(\mu) = \int_K \int_{\mathbb{R}^+} \|\gamma(t)\|_\mu^{\alpha-1} \|\dot{\gamma}(t)\| dt d\mu(\gamma)$ where $\|x\|_\mu = \mu(\{\gamma : x \in \gamma(\mathbb{R})\})$.	$\pi_\mu = \pi$ or $\mu^+ = \nu^+$ and $\mu^- = \nu^-$	Permits to handle a ”who goes where” set of constraint	[5]

Table 2.1: Comparison of different models.

Figure 2.4: On the left hand side: the transport path $G_i = \frac{1}{2}[[e]] + \frac{1}{2}[[e_i]]$ where $[[e]]$ is the vector measure $\mathcal{H}^1\lfloor_e \mathbf{e}$ with $\mathbf{e}$ the unit directional vector of the edge e. On the right hand side: the weak limit of G_i is the null transport path.

Figure 2.5: On the left hand side: the transference plan π_i is such that $\pi_i(x,y) = \pi_i(y_i,x_i) = \frac{1}{2}$. On the right hand side: the limit of π_i is the transference plan π such that $\pi(x,y) = \pi(y,x) = \frac{1}{2}$.

Let us first mention that the cost functional defined in [22] is slightly different from the energy proposed in [35]. Indeed, both functionals coincide on trees, and [22] only handles such tree like objects by definition of *patterns*. To see why the two costs are different, let us consider $\mu^- = \frac{2}{5}\delta_{y_1} + \frac{2}{5}\delta_{y_2} + \frac{1}{5}\delta_{y_3}$ and $\mu^+ = \delta_x$. The left-hand side of Figure 2.6 shows that once two fibers get separated, they are considered to be separated until the end, even if they coincide geometrically afterwards. Thus, the cost of the segment part of the graph irrigating y_3 is $2l(1/10)^\alpha$ on the left-hand side of Figure 2.6 and $l(1/5)^\alpha$ on the right-hand side. Now, this difference does not matter, as it is easily shown [18], [35] that optimal networks are loop free (due to the concavity of x^α for $\alpha \in [0,1[$).

The path functional model.

The path functionals model seems to be quite different of other approaches. Let consider an example to illustrate that difference. Let γ be a path made of two Dirac masses, one is moving on a distance 1 at speed 1 and the other one is still. By definition we have $J(\gamma) = 2$ and $\mathcal{J}(\gamma) = 2$. That is to say, the still Dirac mass contributes to the global cost because some other part of the mass is moving. This model may get closer of the one we shall consider if we were able to define a cost where only the moving mass contributes to the cost.

Equivalence results.

Because of regularity results that are to be proven in Chapter 6, we can state equivalence between some of the presented models.

Theorem 2.8.1 *Traffic plans and patterns ([22]) are equivalent with respect to the irrigation problem when μ^+ consists of a single Dirac mass.*

Proof: The small difference between the traffic plan model and the pattern model is the definition of the multiplicity. In the pattern model, when two fibers coincide for time $[0,T]$ then separate, there are viewed as being separated for the remaining time even if the fibers happen to coincide again geometrically. This is due to the fact that multiplicity of the fiber ω at time t is the measure of all equivalent fibers (i.e. fibers coinciding with ω during time $[0,t]$). Let μ^+ being a single Dirac mass at a source point S and μ an optimal traffic plan for the irrigation problem. Proposition 6.2.7 asserts that a parameterization χ of μ

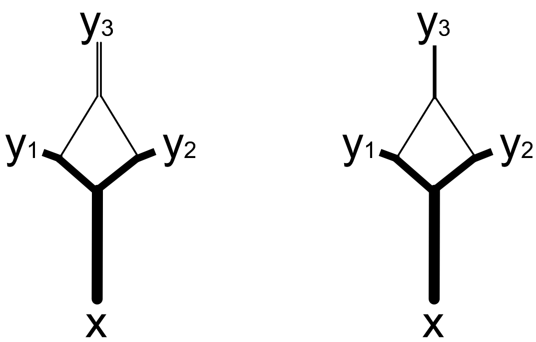

Figure 2.6: Maddalena-Morel-Solimini's versus Xia's model of the irrigation problem with $\mu^+ = \delta_x$ and $\mu^- = \frac{2}{5}\delta_{y_1} + \frac{2}{5}\delta_{y_2} + \frac{1}{5}\delta_{y_3}$. The two geometric objects are the same but on the left-hand side, once fibers separate, they are considered to be separated until they stop. This difference, however, is irrelevant for optimal networks, which are loop free (because of the concavity of $x \mapsto x^\alpha$).

has a tree structure, so that the definition of multiplicity in the traffic plan framework coincide with the one of patterns. Since the cost of tree structures are identical, the models are then equivalent. □

Theorem 2.8.2 *The irrigation problem for traffic plans when μ^+ and μ^- are atomic measures and the Gilbert-Steiner problem are equivalent*

Proof: Let μ^+ and μ^- be atomic measures and μ an optimal traffic plan for the irrigation problem. Proposition 6.3.3 asserts that μ has a graph structure so that the E^α cost is the same than the Gilbert-Steiner problem cost for $f(c) = c^\alpha$. Thus, both problems give same optima. □

Chapter 3

Physical irrigation

Introduction

As was mentioned earlier, in many natural or artificial flow systems, a fluid flow network succeeds in irrigating every point of a volume from a source. Examples are the blood vessels, the bronchial tree and many irrigation and draining systems. Such systems have raised recently a lot of interest and some attempts have been made to formalize their description, as a finite tree of tubes, and their scaling laws [31], [32] or alternately as an open set along with a profile constraint (see Chapter 1). Alternatively, several mathematical models [12], [35], [22] (see Chapter 2) propose an idealization of these irrigation trees, where only the skeleton structure of the network is preserved along with the mass transportation information. There is no geometric obstruction for irrigating systems to exist (see Chapter 1). As we show, there may instead be an energetic obstruction. Under Poiseuille law $R(s) = s^{-2}$ for the resistance of tubes with section s, the dissipated power of a volume irrigating tree cannot be finite. In other terms, infinite irrigation trees seem to be impossible from the fluid mechanics viewpoint. This also implies that the usual principle analysis performed for the biological models needs not to impose a minimal size for the tubes of an irrigating tree ; the existence of the minimal size can be proven from the only two obvious conditions for such irrigation trees, namely the Kirchhoff and Poiseuille laws.

3.1 Irrigation networks made of tubes

The function of many natural or artificial irrigation or drainage systems is to connect by a fluid flow a finite size volume to a source. This happens, e.g., with the lungs [27] or with the blood circulation. A space filling hierarchical branching pattern is obviously required and observed. The resulting irrigation circuitry is a tree of tubes branching from a source and going as close as possible to any point of the irrigated volume. The following principles have been proposed to characterize such irrigation patterns:

(SF) Space filling requirement: The network supplies uniformly an entire volume of the organism.

(K) Kirchhoff law at branching (conservation of fluid mass).

(W) Energy minimization: the biological networks have evolved to minimize energy dissipation.

(MSU) Minimal size unit: the size of the final branches of the network is lower bounded and the lower bound does not depend on the global size of the structure.

These principles are considered basic principles in all presentations of irrigation circuits [31], [32], [33], [3]. In the case of trees and plants, the energy criterion must be related to the mechanical stability of the trunk and branches in response to wind and gravity. In the case of irrigation or drainage networks, the energy criterion aims at a reduction of the overall resistance of the system, or, equivalently, to a minimization of the dissipated power.

In the mentioned papers, several additional assumptions are usually made to derive conclusions from this set of principles, namely

(H) Homogeneous tree: The irrigation system is assumed to be a tree made of tubes, fully homogeneous in lengths and sections.

Let us describe in some detail this homogeneous framework and its consequences. We denote by $k \in \mathbb{N}$, $k \leq k_{max}(\leq \infty)$ the branching level in the tree. The tubes at the final level k_{max} will be called the capillaries. There is a single tube at level 0, and N_k tubes at level k. By (H), at each level k all tubes (which we shall refer to as k-tubes) are equal and are described by the same parameters: $l_k,\ r_k,\ f_k$, namely the common value of their length, radius and flow. We shall also use the variable $s_k = r_k^2$ which is proportional to the area of the constant section of the tube. With these variables, the power dissipated by the irrigation network is expressed as

$$W = \sum_{k=1}^{k_{max}} N_k \, l_k \, s_k^{-\beta} f_k^2. \tag{3.1}$$

Although we treat β as a free parameter, Poiseuille's law (see the appendix) states that for all Newtonian fluids in laminar mode, $\beta = 2$. The homogeneity of the irrigation tree can be rendered still more specific by imposing the realistic

(CB) Constant branching: $\frac{N_{k+1}}{N_k} = \nu = constant.$

The space filling requirement can be formalized in a rough way by stating that the k-th tube irrigates a volume proportional to l_k^3. This is a possible interpretation of (SF) which we shall call $(SF1)$. So we can summarize as a set of equations the constraints usually proposed for homogeneous trees

(H) Homogeneous tree with unknown $k,\ l_k,\ r_k,\ f_k, k \leq k_{max}$.

(K) Kirchhoff $N_k f_k = constant.$

(SF1) Space filling $N_k l_k^3 = constant.$

(MSU) Minimal size capillaries: $k_{max} < \infty$.

(CB) Constant branching (optional), $\frac{N_{k+1}}{N_k} = \nu = constant.$

The aim of this set of assumptions was in [31], [32], [33] to prove that the network has a fractal-like structure with self-similar properties. In the mentioned papers, it is claimed that the minimization of the energy (3.1) with prescribed volume $\sum_k N_k l_k r_k^2 = V$ leads to self-similar properties, namely constant ratios $\frac{l_{k+1}}{l_k} = constant$, $\frac{r_{k+1}}{r_k} = constant$, so that also $\frac{l_k}{r_k} = constant$, namely the tubes have a scale invariant shape and all quantities scale as powers of n. Actually, such results were not proven, the main focus of the mentioned papers being rather to discuss scaling laws in animal metabolism. A mathematically more comprehensive study of the consequences of the above mentioned axioms is given in [15] where the correct consequences are drawn. We shall recall these results and extend their techniques in section 3.3.

The above axiomatic of irrigation systems is simple and efficient enough, but has some weak points. There is no mention of the tube non-intersection constraint. From that point of view, the homogeneity assumption is probably not quite realistic. Also, the space filling assumption does not take into account the volume occupied by the network itself. In short, the realistic embedding of the circuit in a volume is not directly considered and the Lagrangian calculus involved in the mentioned papers is done as though all lengths, radii and even branching numbers could move freely. This is certainly not the case for a realistic embedded circuit. Thus, it would be good to get rid of the homogeneity assumption (H) which clearly should be derived as a property from the first four principles. Also, the question arises of whether the four basic principles (SF), (K), (W) and (MSU) are redundant or not. One of the main outcomes of our discussion here will be to eliminate (MSU), that is, the minimal size constraint for the capillaries. The (MSU) assumption, essential in the above mentioned physical models, was simply written as $k_{max} < \infty$ and forbids infinite branching. It also actually excludes a volume direct irrigation and only permits any point of a volume to be "close enough" to a capillary.

There is, however, no geometric obstruction for the existence of infinite trees irrigating a positive volume K in a strong sense, namely with a branch of the tree (a sequence of tubes) arriving at every point of K. Such tube trees can be constructed by rather explicit rules as in Chapter 1 ; they can satisfy the Kirchhoff law and can even have the fluid speed decrease and be null at the tips of capillaries. Such constructions can be found (e.g.) in [3], [26] and [12]. See Figure 3.1 for an example.

We shall prove that the only obstruction to infinite trees is the infinite resistance of such circuits. We assume without loss of generality that Poiseuille law holds throughout the circuit: it is generally acknowledged that this law is valid in all biological circuits, at least for the smaller tubes [23]. We shall prove:

Theorem 3.1.1 *Let $\beta \geq 2$. Then $W = +\infty$ for any set of tubes obeying (K) and (W) and irrigating a positive volume.*

(See Theorem 3.6.4 for a more precise statement.)

This result may invalidate the infinitesimal models, admitting infinite branching, proposed in several recent mathematical works [12], [35], [22]. Now, as we shall see, the tools developed in the mentioned paper turn out to be quite handy to perform the present axiomatic discussion. And, of course, nothing hinders the consideration of other resistance laws than Poiseuille law for other human built transportation circuits. Poiseuille law states that for fluids, the resistance $R(s) = Cs^{-2}$ of a tube with section s scales as the inverse second power of s. If we instead consider $R(s) = Cs^{-\beta}$, then infinitesimal circuits are possible. The power $\beta = 2$ is the limit exponent.

Two of the mentioned mathematical models, [35] and [22], do not involve the radius of the tubes. They instead express a "cost" of the flow directly as

$$\tilde{W} = \sum_{i \in I} l_i f_i^{\alpha}$$

where $0 < \alpha < 1$ and I denotes the countable set of all tubes. There is, however, a way to relate this expression of the cost to the energy W, at least for optimal and homogeneous circuits.

Proposition 3.1.2 *Let us consider an irrigation network which optimizes the dissipated power W given by (3.1) under the constraints of fixed volume V and prescribed lengths of tubes l_i and flows f_i. Then*

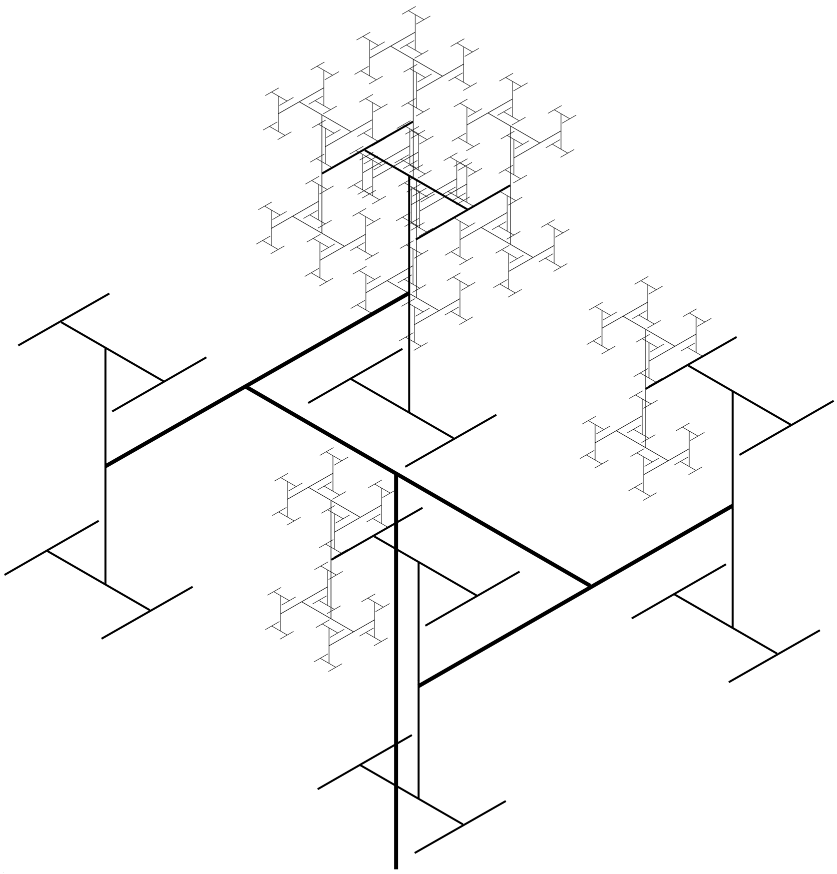

Figure 3.1: An irrigating tree

$s_i = C_1 f_i^{2/(\beta+1)}$, and $W = C_2 \sum l_i f_i^{2/(\beta+1)}$ for some constants $C_1, C_2 > 0$. (Poiseuille law corresponds to the case $\beta = 2$ in dimension 3, and in this case $s_i = C f_i^{2/3}$ for some constant $C > 0$).

The proof of this proposition is easy and can be done along the lines of the proof of Proposition 3.3.1 in Section 3.3. The model equivalence thus obtained is not quite satisfactory: we are not a priori allowed to move freely the radii in an optimal embedded circuit, since we do not take into account the fact that the tubes should not intersect. Let us concede anyway some validity to the model equivalence thus indicated. Then we see that there is no contradiction with the existence results in [35] and [22]. Indeed, these authors assume (in dimension 3) $\alpha > \frac{2}{3}$ which corresponds to $\beta < 2$, and we prove that $\beta \geq 2$ is not compatible with Poiseuille law.

3.2 Mathematical, infinitesimal approaches

Let us give some details on the existing mathematical formalizations of the problem, since we shall use some of them. The model proposed in [12] is directly compatible, but more general than the above tube model. It directly considers the problem of finding a maximal irrigated volume with minimal cost. Let D be an open domain of $\mathbb{R}^d$ (of course, $d = 2$ or 3). A point source $S \in D$ is fixed. Say that a compact set $E \subset D$ is *irrigable* if the complementary open set $U = D \setminus E$ is connected and contains S. U is called the irrigation network and is nothing but an open set at this point. Caselles and Morel then fix an "accessibility profile", namely a function $f(s) : \mathbb{R}_+ \to \mathbb{R}_+$, increasing and such that $f(0) = 0$. A point $x \in E$ is said f-irrigable if there is a path $x(s)$ such that $x(0) = x$, $x(L) = S$, and for every $s \in [0, L]$, $B(x(s), f(s)) \subset U$, where $B(x, r)$ denotes the ball with center x and radius r. Such paths exist in the physical tube model as a branch of the irrigation tree. In other terms, there is a thick path inside U leading to x. This path becomes thinner when approaching the irrigated point x, but with a thinning rate uniformly bounded from below. The authors show first that if f slightly super-linear at 0 (e.g. $f(s) = s^\alpha$, $0 < \alpha < 1$) then the problem of irrigating a maximal positive volume is well posed. Namely: there exists K with maximal volume among all f-irrigable sets.

From this paper, we shall retain the following result which will be a main ingredient here.

Proposition 3.2.1 *Let $x \in E$ be irrigable from S with profile f. Assume that x is a Lebesgue point of E. Then $\limsup_{s \to 0+} f(s)/s = 0$. As a consequence $\int_0^R \frac{1}{f(s)}\, dr = \infty$ for all $R > 0$.*

Almost every point of a measurable set is a Lebesgue point, and this yields a generic constraint on accessibility profiles, not taken into account in finite models, but handy in infinitesimal ones. In the terminology of homogeneous trees of tubes, this constraint yields under (H)

$$\sum_k \frac{l_k}{r_k} = +\infty. \tag{3.2}$$

Our plan is as follows: Section 3.3 is devoted to the classical physical tube models, the derivation of scaling laws and the proof of our result in the homogeneous case (hypothesis (H)). Section 3.4 gives all elements we need from [22] to perform integration on the set of fibers. Section 3.5 constructs from any embedded set of tubes a set of fibers $\chi(\omega, l)$. Section 3.6 proves the main result. Three small appendices are devoted, for a sake of completeness, to a proof of Proposition 3.2.1, the optimality of circular section for tubes and the derivation of Poiseuille law in tubes.

3.3 Dissipated power in a homogeneous network of tubes

In this section, we take the standard notation given in the introduction. We consider a homogeneous irrigation network as a set of tubes which are organized as a hierarchical branching system from level 0 up to a final level $k_{max}(\leq \infty)$. There is a single tube at level 0, and N_k tubes at level k. At each level k, all tubes (which we shall refer to as k-tubes) are equal and are described by their length l_k, radius r_k, and flow f_k We set $s_k = r_k^2$ which is proportional to the area of the constant section of the tube. With these variables, the power dissipated by the irrigation network is expressed as

$$W = \sum_{k=1}^{k_{max}} N_k l_k s_k^{-\beta} f_k^2 \tag{3.3}$$

for some $\beta > 0$ (Poiseuille law corresponds to $\beta = 2$). As proposed in [31], [32], [33], if we prescribe the volume occupied by the irrigation network, physical networks are designed to minimize the dissipated power W, and satisfy the following assumptions:

(K) Kirchhoff's law: the fluid is conserved as it flows through the system, that is, $N_k f_k = N_{k+1} f_{k+1}$ for each k. In other words, Kirchhoff's law holds in the network.

$(SF1)$ Space filling requirement: at each level k the volume supplied by the set of k-tubes is independent of k and is approximately given by the sum of the volumes of N_k spheres of diameter $l_k/2$. This total volume is $N_k l_k^3$ and we assume that this quantity is a constant.

For a homogeneous irrigation network satisfying (K) and $(SF1)$, there are constants $C, C' > 0$ such that

$$f_k = \frac{C'}{N_k}, \quad l_k = C N_k^{-1/3}$$

and the dissipated power may be written as

$$W(s_k) = C'^2 C \sum_{k=1}^{k_{max}} N_k^{-4/3} s_k^{-\beta}.$$

In the same way the volume $V = \sum_{k=1}^{k_{max}} N_k l_k s_k$ can be written as

$$V(s_k) = C \sum_{k=1}^{k_{max}} N_k^{2/3} s_k.$$

We shall consider the geometry of the network as given, i.e. the values N_k are prescribed, hence the dissipated power is only a function of the variables s_k. Under the constraint of given volume, we consider an optimal irrigation network as a minimizer of the dissipated power W.

Proposition 3.3.1 *Assume that* $\beta \geq 2$. *Under the assumptions* (K) *and* $(SF1)$*, an optimal homogeneous irrigation network with prescribed volume satisfies* $k_{max} < \infty$ *and* $N_k r_k^{\beta+1} = constant$.

We observe that if $k_{max} < \infty$, then the relation $N_k r_k^{\beta+1} = constant$ does not require to assume that $\beta \geq 2$.

In particular, accepting that assumption (K) is a sound one, this proposition proves that the assumption $(SF1)$ cannot be fulfilled if we want to consider infinite trees. If we accept it, we have to assume that capillaries cannot be infinitely thin.

Proof. Assume first that $k_{max} < \infty$. For simplicity, let us assume that $C'^2 C = 1$. Then, by Lagrange multiplier's Theorem, there is a constant $\lambda \in \mathbb{R}$ such that $\frac{\partial W}{\partial s_k} = \lambda \frac{\partial V}{\partial s_k}$, that is,

$$-\beta N_k^{-4/3} s_k^{-(\beta+1)} = \lambda C N_k^{2/3}.$$

Hence $N_k^2 s_k^{\beta+1} = -\frac{\beta}{\lambda C}$, and therefore $N_k r_k^{\beta+1} = constant$.

Assume that $k_{max} = \infty$, and there exists a homogeneous irrigation network with specified volume $V = V_0 < \infty$ and finite dissipated power. Then the dissipated power has a minimum in the set $\mathcal{S} = \{(s_k)_{k=1}^{\infty} : s_k > 0, C \sum_{k=1}^{\infty} N_k^{2/3} s_k = V_0\}$. Indeed, since the infimum of W in $\mathcal{S}$ is finite, let $\{(s_k(n))_k\}_n$ be a minimizing sequence of elements in $\mathcal{S}$. By extracting a subsequence, if necessary, we may assume that $s_k(n) \to s_k$ as $n \to \infty$ for all k. If $s_k = 0$ for some k, then $W(s_k(n)) \geq N_k^{-4/3}(s_k(n))^{-\beta} \to \infty$ as $n \to \infty$, a contradiction with the fact that $(s_k(n))_k$ is a minimizing sequence. Hence $s_k > 0$ for all k. Now, for each $p \geq 1$, we have

$$C \sum_{k=1}^{p} N_k^{2/3} s_k \leq \lim_n C \sum_{k=1}^{p} N_k^{2/3} s_k(n) \leq \lim_n C \sum_{k=1}^{\infty} N_k^{2/3} s_k(n) = V_0.$$

Thus $M := C \sum_{k=1}^{\infty} N_k^{2/3} s_k \leq V_0$. If $M < V_0$, we define $S_k = \frac{V_0}{M} s_k$ and we have $V(S_k) = V_0$. Now,

$$\begin{aligned} \sum_{k=1}^{p} N_k^{-4/3} S_k^{-\beta} &= \left(\frac{M}{V_0}\right)^{\beta} \sum_{k=1}^{p} N_k^{-4/3} s_k^{-\beta} = \left(\frac{M}{V_0}\right)^{\beta} \lim_n \sum_{k=1}^{p} N_k^{-4/3} (s_k(n))^{-\beta} \\ &\leq \left(\frac{M}{V_0}\right)^{\beta} \lim_n \sum_{k=1}^{\infty} N_k^{-4/3} (s_k(n))^{-\beta} = \left(\frac{M}{V_0}\right)^{\beta} \inf_{\mathcal{S}} W. \end{aligned}$$

In particular, we deduce that $\inf_{\mathcal{S}} W > 0$, and

$$W(S_k) \leq \left(\frac{M}{V_0}\right)^{\beta} \inf_{\mathcal{S}} W < \inf_{\mathcal{S}} W.$$

This contradiction proves that $M = V_0$, hence $(s_k) \in \mathcal{S}$, and $(s_k)_k$ is a minimum of W in $\mathcal{S}$. Let us denote $\vec{s} = (s_k)_k$, and for each $p \geq 1$, $-s_p N_p^{2/3} < \epsilon < s_{p+1} N_{p+1}^{2/3}$, $\vec{s}_\epsilon^{\,p} = (s_1, \ldots, s_p + \frac{\epsilon}{N_p^{2/3}}, s_{p+1} - \frac{\epsilon}{N_{p+1}^{2/3}}, s_{p+2}, \ldots)$. Then computing

$$\lim_{\epsilon \to 0+} \frac{W(\vec{s}_\epsilon) - W(\vec{s})}{\epsilon} = 0,$$

we obtain that

$$N_p^2 s_p^{\beta+1} = N_{p+1}^2 s_{p+1}^{\beta+1}.$$

Since this holds for all p, we obtain that $N_k^2 s_k^{\beta+1} = constant$, hence also $N_k^{2/3} s_k^{(\beta+1)/3} = constant$. Using that $(\beta+1)/3 \geq 1$, we have

$$N_k^{2/3} s_k^{(\beta+1)/3} \leq N_k^{2/3} s_k,$$

when $s_k < 1$. We conclude that $V(s_k) = \infty$. Notice that we also have $W(s_k) = C \sum_{k=1}^{\infty} s_k^{(2-\beta)/3} = \infty$ since $\beta \geq 2$ and $s_k \to 0$ as $k \to \infty$. □

Remark 3.3.2 Observe that no relation of the type $N_{k+1}/N_k = constant$ follows for optimal irrigation trees under the assumption (K) and $(SF1)$ as suggested in [31], [32], [33]. This fact has also been observed in [15]. Now, if we add the assumption of constant branching

(CB) $\frac{N_{k+1}}{N_k} = \nu$,

we obtain the relations (written modulo multiplicative constants) $N_k = \nu^k$, $s_k = \nu^{-2k/(\beta+1)}$, $W = \sum_k \nu^{(\frac{2\beta}{\beta+1}-\frac{4}{3})k}$, $V = \sum_k \nu^{(\frac{2}{3}-\frac{2}{\beta+1})k}$ and both quantities are infinite if $k_{max} = \infty$, and $\beta \geq 2$.

We shall replace the space filling assumption $(SF1)$ by a different assumption which is related to the existence of a positive volume irrigated by the network. Indeed, we assume

$(SF2)$ $k_{max} = \infty$ and $\sum_{k=1}^{\infty} \frac{l_k}{r_k} = \infty$.

This implies that the length of the tubes cannot be too small compared to its radius. Our analysis in section 1.1 will prove that this assumption holds for networks irrigating a positive volume.

Proposition 3.3.3 *Assume that $\beta \geq 2$. Under the assumptions (K) and $(SF2)$, both V and W cannot be finite.*

Proof. Recall that $V = \sum_{k=1}^{\infty} N_k l_k s_k$. Since $N_k f_k = C$ for some constant $C > 0$, we may write $W = C^2 \sum_{k=1}^{\infty} \frac{l_k s_k^{-\beta}}{N_k}$, using Cauchy-Schwarz inequality, we have

$$\sqrt{W}\sqrt{V} \geq C \sum_{k=1}^{\infty} \sqrt{l_k^2 s_k^{1-\beta}} = C \sum_{k=1}^{\infty} \frac{l_k}{r_k^{\beta-1}} = \infty,$$

and the conclusion follows. □

Remark 3.3.4 The proof of Proposition 3.3.3 can also be done using Lagrange multiplier's theorem as we did in the proof of Proposition 3.3.1.

3.4 A model of abstract tree

The purpose of this section is to recall the formalization defining "set of fibers" in the sense of [22]. In Section 3.5, we shall make the link with irrigation trees. To do this, we shall describe the tree made of tubes as a tree of segments, each segment being the medial axis of a tube. We shall also keep the flow information inside each tube. These informations are enough, as we shall see, to associate with the concrete tree an abstract "set of fibers". The reason for making this association will become clear in Section 3.6: we wish to compute the volume or the dissipated power by integrating along the sections of the irrigation tree. These computations are facilitated by the "set of fibers" formalism.

Let us recall the main concepts introduced in [22]. Let $(\Omega, \ |.|)$ be a probability space which we interpret as the reference configuration of a fluid incompressible material body. We can also interpret it as the trunk section of a tree, this trunk being thought of as a set of fibers (which can bifurcate into branches). A *set of fibers of Ω with source point $S \in \mathbb{R}^d$* is a mapping

$$\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$$

such that:

C1) For a.e. *material point* $p \in \Omega$, $\chi_p : t \mapsto \chi(p,t)$ is Lipschitz continuous with Lipschitz constant less than or equal to one.

C2) For a.e. $p \in \Omega$: $\chi_p(0) = S$.

We shall consider the source point $S \in \mathbb{R}^d$ as given and we will denote by $\mathbf{C}_S(\Omega)$ the set of possible $\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$.

Definition 3.4.1 *[22] Given $l \in \mathbb{R}_+$, we shall say that two points $p,\ q \in \Omega$ belong to the same χ-vessel of value l and we will write $p \simeq_l q$ if*

$$\chi_p(s) = \chi_q(s) \text{ for all } s \in [0, l].$$

For every $l \in \mathbb{R}_+$, the equivalence relation $\simeq_l$ induces a decomposition of Ω into equivalence classes X. We will call χ-*vessels* such classes.

Definition 3.4.2 *[22] Given $p \in \Omega$ and $l \in \mathbb{R}_+$, the equivalence class of $\simeq_l$ which contains p and which will be denoted by $[p]_t$ will be named χ-vessel of the point p at l.*

Given $\chi \in \mathbf{C}_S(\Omega)$ and $l > 0$, we shall denote by $\Omega_l(\chi)$ the set of all the χ-vessels at the value l, that is

$$\Omega_t(\chi) := \Omega / \simeq_l.$$

The decomposition of Ω induced by $\simeq_l$ can be viewed as dividing the body in parts which are mapped, through χ, into tube-like regions of $\mathbb{R}^d$ which we identify with rectifiable curves. Since we control only the total amount of fluid carried by these regions, we describe them by giving their axial curves. Thus, at each l a set of fibers χ can be regarded as a set of curves, obtained by varying $[p]_l$. Indeed, by Definition 3.4.1, on the interval $[0, l]$, χ_p coincides with any other function χ_q for q varying in the set $[p]_l$. A set of fibers can also be interpreted as modelling a tree, in which case the χ-vessels represent the branches.

Definition 3.4.3 *[22] Let $\chi \in \mathbf{C}_S(\Omega)$ be given. The function $\sigma_\chi : \Omega \to \mathbb{R}_+$ defined by*

$$\sigma_\chi(p) := \inf\{l \in \mathbb{R}_+ \mid \chi_p(s) \text{ is constant on } [l, +\infty[\}$$

will be called absorption time. We shall say that a point $p \in \Omega$ is absorbed when $\sigma_\chi(p) < +\infty$. A point $p \in \Omega$ is absorbed at time l if $\sigma_\chi(p) \le l$. We denote $A_l(\chi)$ the set of absorbed points at time l, and A_χ the set of absorbed points at some time. In the following, we shall only consider patterns such that almost all points are absorbed.

Lemma 3.4.4 *[22] Let $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be such that $f(\cdot, l)$ is measurable for l in a dense subset $D \subset \mathbb{R}_+$ and $f(p, \cdot)$ is continuous for a.e. $p \in \Omega$. Then f is a measurable mapping.*

Theorem 3.4.5 *[22] For every set of fiber $\chi \in \mathbf{C}_S(\Omega)$ the following statements are equivalent.*

1. *χ is measurable.*
2. *$\chi(\cdot, l)$ is measurable for every l in a dense subset $D \subset \mathbb{R}_+$.*
3. *$\chi(\cdot, l)$ is measurable for every $l \in \mathbb{R}_+$.*

In the following we only consider measurable sets of fibers.

Proposition 3.4.6 *[22] For every $\chi \in \mathbf{P}_S(\Omega)$, the absorption function σ_χ is a measurable mapping.*

Let $\chi \in \mathbf{P}_S(\Omega)$. We introduce the *irrigation function* defined on the set A_χ of the absorbed points:

$$i_\chi(p) = \chi(p, \sigma_\chi(p))$$

We have $i_\chi(p) = \lim_{t\to\infty} \chi(p, l)$ and so $i_\chi : A_\chi \to \mathbb{R}^d$ is a measurable function, as a pointwise limit of a sequence of measurable functions. The function i_χ induces the image (push-forward) measure μ_χ defined by the formula

$$\mu_\chi(A) := |i_\chi^{-1}(A)|$$

for any Borel set $A \subset \mathbb{R}^d$. We shall refer to μ_χ as to the *irrigation measure* induced by χ.

3.5 The set of fibers associated to the skeleton of a tree of tubes

Our purpose in this section is to obtain an abstract description of a physically realized tree. We first introduce the skeleton of a tree of tubes which is a depurate description of an embedded tree (the tree being viewed as a set of tubes). The skeleton description of a tree permits to associate a set of fibers to it, in the sense of [22] as described in the section 3.4. Integration of functions which are constants on any tube is then allowed and made easier with this formalism.

Since notation here is necessarily a bit cumbersome, we refer to Figure 3.2.

3.5.1 Embedded irrigation tree through its skeleton

Definition 3.5.1 *Let $\{[x_n^k, y_n^k] \mid n \geq 1, k \in [1, N(n)]\}$ be a family of segments in $\mathbb{R}^d$ such that $]x_n^k, y_n^k]$ are disjoint. We shall say that the set $S = \cup_{n=1}^\infty \cup_{k=1}^{N(n)} [x_n^k, y_n^k]$ is a skeleton if there are increasing surjective functions $\phi_n : [1, N(n)] \to [1, N(n-1)]$ such that $x_n^k = y_{n-1}^{\phi_n(k)}$.*

The number $N(n)$ will be called the number of branches at generation n. The segment $[x_n^k, y_n^k]$ will be called the (n,k) tube. We will consider skeletons such that $N(1) = 1$.

The set $K_N = \cup_{n=1}^N \cup_{k=1}^{N(n)} [x_n^k, y_n^k]$ will be called the partial tree at generation N of the skeleton.

We shall consider skeletons with a flow attached to each tube so that Kirchhoff's law is satisfied at each bifurcation.

Definition 3.5.2 *Let S be a skeleton. We say that S is a skeleton with a flow F if the family $F = \{f_n^k \mid n \in \mathbb{N},\ k \in [1, N(n)]\}$ is such that $\max_k f_n^k \to 0$ as $n \to \infty$ and satisfies Kirchhoff's law, i.e.,*

$$\sum_{l \in \phi_{n+1}^{-1}(k)} f_{n+1}^l = f_n^k.$$

We shall say that f_1^1 is the total flow on S. In the sequel we normalize the total flow $f_1^1 = 1$.

We associate to a skeleton with a flow the family $R = \{r_n^k \mid n \in \mathbb{N},\ k \in [1, N(n)]\}$, where r_n^k represents the radius of the (n,k) tube. We shall assume that $\sup_{n,k} r_n^k < \infty$.

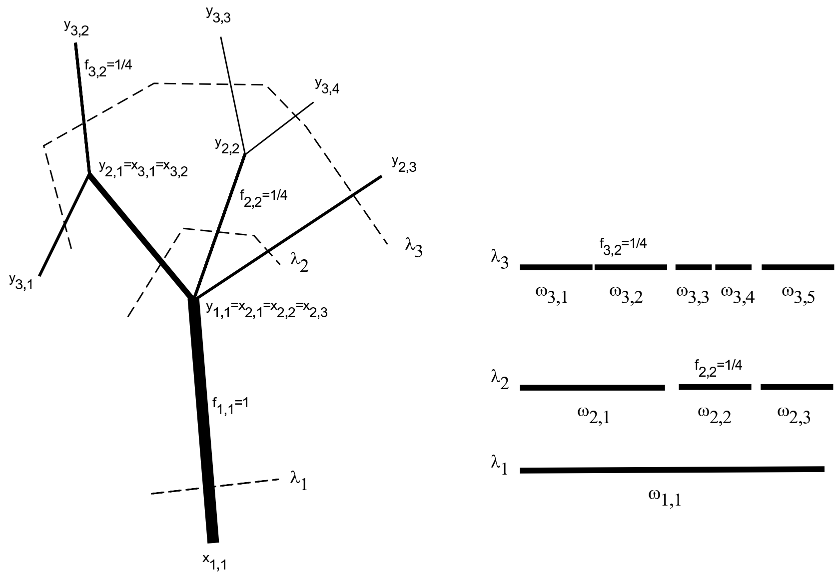

Figure 3.2: Skeleton of a tree of tubes

3.5.2 Correspondence between a skeleton and a filtration of [0,1)

The idea is to associate to each tube of generation n of the tree some interval $\omega_n^k \subset [0,1]$, so that σ-algebras $\mathcal{A}_n$ generated by the finite number of sets ω_n^k form a filtration. A point of $[0,1)$ will then correspond to a path in the tree. This construction follows [22].

Proposition 3.5.3 *Let S be a skeleton with a flow F. We assume that the total flow is* 1. *Then, there is a family ω_n^k such that $|\omega_n^k| = f_n^k$ for all $k \in [1, N(n)]$ and the family of intervals $\{\omega_n^k : k \in [1, N(n)]\}$ forms a partition of $\Omega = [0,1)$. Moreover, the σ-algebras $\mathcal{A}_n$ generated by $\{\omega_n^k \mid k \in [1, N(n)]\}$ form a filtration and the σ-algebra $\mathcal{A}$ generated by $\bigcup_n \mathcal{A}_n$ coincides with the σ-algebra of Borel sets of* $[0,1)$.

Proof. Let $\omega_1^1 = [0,1)$. Suppose that ω_n^k are defined for all $k \in [1, N(n)]$. We have to define ω_{n+1}^l for all $l \in [1, N(n+1)]$. Let $k \in [1, N(n)]$. Then, if $\omega_n^k = [a,b)$, for all $r \in \phi_{n+1}^{-1}(k) = [l_k+1, l_{k+1}]$, we define

$$\omega_{n+1}^r = [\ a + \sum_{i=l_k+1}^{r-1} f_n^i\ ,\ a + \sum_{i=l_k+1}^{r} f_n^i\ )$$

From the definition, $|\omega_{n+1}^r| = f_{n+1}^r$, and ω_{n+1}^r are intervals of the form $[c,d)$ forming partition of ω_n^k because of Kirchhoff's law. Repeating the same construction for all $k \in [1, N(n)]$ we obtain the family $\{\omega_{n+1}^l\}$.

By construction, the σ-algebras $\mathcal{A}_n$ generated by $\{\omega_n^k \mid k \in [1, N(n)]\}$ form a filtration. Since $\max_k |\omega_n^k| = \max_k f_n^k$ converges towards 0 when n goes to infinity, the σ-algebra $\mathcal{A}$ generated by $\bigcup_n \mathcal{A}_n$ coincides with the σ-algebra of Borel subsets of $[0,1)$. Moreover, if $\omega \in \Omega = [0,1)$ there is a unique decreasing family of intervals $\{\omega_n^{k(n)}, n \geq 1\}$ such that $\omega = \bigcap_n \omega_n^{k(n)}$, or, in other words, paths of the tree are in a one-to-one correspondence with points of $[0,1)$. Note that $\mathcal{A}_1 = \{\omega_1^1\}$ with $\omega_1^1 = [0,1)$. □

3.5.3 Construction of the set of fibers associated to the skeleton. The equality of supply.

Let S be a skeleton with a flow, and let $l_n^k = |x_n^k - y_n^k|$ be the length of the (n,k) tube. By definition of skeletons, there is a unique path from x_n^k to the source x_1^1, that is to say, given ω_n^k, there is a unique family of intervals $\omega_i^{k(i)}$ such that $\omega_n^k \subset \omega_i^{k(i)}$ for all $i \le n$. Notice that $\omega_n^{k(n)} = \omega_n^k$. We shall denote by L_n^k the sum of lengths corresponding to the tubes $\{\omega_i^{k(i)} : i \le n\}$, i.e., $L_n^k = \sum_{i=1}^n l_i^{k(i)}$. We also set $L_n^{k*} = \sum_{i=1}^{n-1} l_i^{k(i)}$. More generally, for all $\omega \in \Omega$, there exists a unique sequence $k(n)$ such that $\omega = \cap_n \omega_n^{k(n)}$. We define $L(\omega) = \sum_n l_n^{k(n)} \in \mathbb{R} \cup \{\infty\}$ to be the length of the path ω.

Proposition 3.5.4 *Let S be a skeleton with a flow. Let us define by recursively*

$$\chi_1(\omega,l) = \begin{cases} x_1^1 + l\frac{y_1^1 - x_1^1}{|y_1^1 - x_1^1|} & \text{if } l \le l_1^1 \\ y_1^1 & \text{if } l > l_1^1 \end{cases}$$

and, for $n \ge 2$, $\omega \in \omega_n^k$, let

$$\chi_n(\omega,l) = \begin{cases} \chi_{n-1}(\omega,l) & \text{if } l \le L_n^{k*} \\ x_n^k + (l - L_n^{k*})\frac{y_n^k - x_n^k}{|y_n^k - x_n^k|} & \text{if } l \in [L_n^{k*}, L_n^k] \\ y_n^k & \text{if } l > L_n^k \end{cases}$$

Then the pointwise limit $\chi(\omega,l) := \lim_n \chi_n(\omega,l)$ exists for any $(\omega,l) \in [0,1]\times\mathbb{R}^+$, and it is measurable. Hence χ is a measurable set of fibers in the sense [22].

Proof. Let us prove that χ_n is $\mathcal{A}_n \times \mathcal{B}(\mathbb{R}^+)$ measurable. First, since for any given l, $\chi_n(\cdot,l)$ is constant on every interval ω_n^k, the inverse image of any subset of $\mathbb{R}^d$ is a finite union of intervals ω_n^k, hence it is in $\mathcal{A}_n$. Thus $\chi_n(\cdot,l)$ is measurable for any l. Moreover, since for any $\omega \in \Omega$, $\chi_n(\omega,\cdot)$ is 1-Lipschitz, by Lemma 3.4.4, we obtain that χ_n is measurable, hence it is a set of fibers.

Let us prove that the pointwise limit $\chi(\omega,l) = \lim_n \chi_n(\omega,l)$ exists for any $(\omega,l) \in [0,1] \times \mathbb{R}^+$. If $l < L(\omega)$, and $\omega = \cap_m \omega_m^{k(m)}$, then there is an integer n such that $l \in (L_n^{k(n)*}, L_n^{k(n)}]$. The sequence $\{\chi_i(\omega,l)\}_{i\ge n}$ is constant, hence it is convergent. If $l \ge L(\omega)$, then $\chi_n(\omega,l)$ is a Cauchy sequence. Indeed,

$$|\chi_n(\omega,l) - \chi_m(\omega,l)| = |\chi_n(\omega,L_n^{k(n)}) - \chi_m(\omega,L_m^{k(m)})| = |y_m^{k(m)} - y_n^{k(n)}|$$

and the conclusion follows from the fact that $y_n^{k(n)}$ is a Cauchy sequence ; this being so because $L(\omega) \le l < \infty$. We conclude that χ is Borel measurable, being a pointwise limit of Borel measurable functions. □

Let χ_n and χ be the set of fibers associated to the skeleton S constructed in Proposition 3.5.4. Let us define the functions

$$r_{\chi_n}(\omega,l) = \sum_{\{(m,k):m\le n,\, k\le N(m)\}} \mathbb{1}_{\omega_m^k}(\omega)\mathbb{1}_{(L_m^{k*},L_m^k]}(l) r_m^k,$$

$$f_{\chi_n}(\omega,l) = \sum_{\{(m,k):m\le n,\, k\le N(m)\}} \mathbb{1}_{\omega_m^k}(\omega)\mathbb{1}_{(L_m^{k*},L_m^k]}(l) f_m^k,$$

for $(\omega,l) \in [0,1] \times \mathbb{R}^+$. Observe that

$$r_{\chi_n}(\omega,l) = r_{\chi_{n-1}}(\omega,l) \quad \text{if } \omega \in \omega_n^k,\, l \le L_n^{k*},$$

and similarly for $f_{\chi_n}(\omega, l)$. Thus if $\omega = \cap_m \omega_m^{k(m)}$, and $l < L(\omega)$, there is an integer n such that $l \in (L_n^{k(n)*}, L_n^{k(n)}]$, and $r_{\chi_i}(\omega, l) = r_n^{k(n)}$, $f_{\chi_i}(\omega, l) = f_n^{k(n)}$ for all $i \geq n$. Thus the pointwise limits

$$r_\chi(\omega, l) = \lim_n r_{\chi_n}(\omega, l),$$

$$f_\chi(\omega, l) = \lim_n f_{\chi_n}(\omega, l)$$

exist for any $(\omega, l) \in [0,1] \times \mathbb{R}^+$ such that $l < L(\omega)$. If $l \geq L(\omega)$, then $r_{\chi_n}(\omega, l) = 0$, $f_{\chi_n}(\omega, l) = 0$, and we may define

$$r_\chi(\omega, l) = 0,$$

$$f_\chi(\omega, l) = 0.$$

Observe that the functions r_{χ_n}, f_{χ_n} are measurable, and, hence, r_χ, f_χ are also measurable.

The function $L(\omega)$ can be seen as an absorption length since it may be written as

$$L(\omega) = \inf\{l \in \mathbb{R}_+ \mid \chi(\omega, l) \text{ is constant on } [l, +\infty)\}.$$

Then, by Proposition 3.4.6, it is also Lebesgue measurable. As in [22] and Section 3.4, we define the irrigation measure μ by $\mu(A) = |T^{-1}(A)|$, where $T : \omega \to \chi(\omega, L(\omega))$.

Definition 3.5.5 *Let S be a skeleton with a flow. We shall say that S satisfies weak equality of supply if its associated set of fibers defines an image measure μ such that $\mu = f(x)\lambda$ where λ is the Lebesgue measure in $\mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d)$, $f \geq 0$, $f \neq 0$.*

We say that S satisfies the equality of supply if $f = c\mathbb{1}_K$ where K is some set of positive measure (we denote by $\mathbb{1}_A$ the characteristic function of a set A). In the general case, we shall denote by $K := \{x \in \mathbb{R}^d : f(x) > 0\}$.

Remark 3.5.6 The set K can be taken as being a subset of $T(\Omega)$, indeed

$$\int_{K \setminus T(\Omega)} f(x) d\lambda = \mu(K \setminus T(\Omega)) = |T^{-1}(K \setminus T(\Omega))| = |\emptyset| = 0.$$

Since $f > 0$ on K, we have that $K \subset T(\Omega)$ almost everywhere and we may write $\mu = f(x)\mathbb{1}_{K \cap T(\Omega)}\lambda$. Thus, replacing K by $K \cap T(\Omega)$ if necessary, we may assume that $K \subset T(\Omega)$.

The aim of the above construction is to be able to reformulate the energy and the volume of the tube network as Lebesgue integrals of adequate functions defined on the set Ω of paths, as we shall see in the next section.

3.6 Source to volume transfer energy

There are technical difficulties if one wants to make calculations on a tree. For instance, if one wants to write the volume of a tree as an integral, either one writes it as an integral of the total sections over all branches, from the source to the tips; or we write it as an integral over $[0, 1) \times \mathbb{R}^+$, i.e., an integral along the paths of the tree. The construction of the last section will enable us to follow the latter approach. In what follows, we introduce the volume and the dissipated power of a skeleton with a flow. It is to be mentioned that these definitions only intend to be of the same order as the exact volume and dissipated power of an associated embedded tree. Indeed, due to the fact that we assimilate the thick tree with its skeleton, we neglect the influence of the real structure of junctions at bifurcations.

Definition 3.6.1 *Let S be a skeleton with a flow. Let l_n^k, r_n^k, f_n^k be the length, radius, and flow, respectively, of the (n,k) tubes. We define the volume of the tree associated to S by $V = \sum_{n=1}^{\infty}\sum_{k=1}^{N(n)} l_n^k s_n^k$, and its dissipated power associated with a resistance law $R(s)$ by $W = \sum_{n=1}^{\infty}\sum_{k=1}^{N(n)} l_n^k R(s_n^k)(f_n^k)^2$, where $s_n^k = (r_n^k)^{d-1}$ (the quantities are taken modulo constants).*

To prepare the proof of Theorem 3.6.4, it will be convenient to write them as double integrals over l and ω as follows.

Proposition 3.6.2 *We may express the volume and the dissipated power of the tree by the formulas $V = \int_0^\infty \int_0^1 Q_1(\omega,l) dl d\omega$ where $Q_1(\omega,l) = \frac{r_\chi(\omega,l)^{d-1}}{f_\chi(\omega,l)}$ for $l \le L(\omega)$ and $Q_1(\omega,l) = 0$ for $l > L(\omega)$, and $W = \int_0^\infty \int_0^1 Q_2(\omega,l) dl d\omega$ where $Q_2(\omega,l) = f_\chi(\omega,l) R(s_\chi(\omega,l))$, where $s_\chi(\omega,l) = r_\chi(\omega,l)^{d-1}$, for $l \le L(\omega)$ and $Q_2(\omega,l) = 0$ for $l > L(\omega)$.*

Proof. Let us define
$$\frac{r_{\chi_n}(\omega,l)^{d-1}}{f_{\chi_n}(\omega,l)} = 0$$
when both terms are 0. Then it is easy to check that
$$\frac{r_{\chi_n}(\omega,l)^{d-1}}{f_{\chi_n}(\omega,l)} = \sum_{\{(m,k): m\le n,\, k\le N(m)\}} \mathbb{1}_{\omega_m^k}(\omega) \mathbb{1}_{(L_m^{k*}, L_m^k]}(l) \frac{(r_n^k)^{d-1}}{f_n^k} \quad (\omega,l) \in [0,1]\times\mathbb{R}^+,$$
and
$$\int_0^\infty \int_0^1 \frac{r_{\chi_n}(\omega,l)^{d-1}}{f_{\chi_n}(\omega,l)} \, dl \, d\omega = \sum_{m=1}^{n} \sum_{k=1}^{N(m)} l_m^k (r_m^k)^{d-1} \tag{3.4}$$
for each $n \ge 1$. Since $\frac{r_{\chi_n}(\omega,l)^{d-1}}{f_{\chi_n}(\omega,l)} \uparrow Q_1(\omega,l)$ pointwise as $n \to \infty$, letting $n \to \infty$ in (3.4) we deduce that
$$V = \int_0^\infty \int_0^1 Q_1(\omega,l) dl d\omega.$$
In a similar way, we prove that $W = \int_0^\infty \int_0^1 Q_2(\omega,l) dl d\omega$. □

Definition 3.6.3 *We shall say that S is a skeleton with almost surely finite paths if $L(\omega) < \infty$ for almost every $\omega \in \Omega$.*

Theorem 3.6.4 *Let $0 < \alpha \le 1 - \frac{1}{d}$. Let us assume that the resistivity function is $R(s) = s^{(\alpha-2)/\alpha}$. Let S be a skeleton with a flow which has almost surely finite paths and satisfies weak equality of supply. Then, V and W cannot be simultaneously finite.*

Proof. Observe that we have $Q_1 Q_2 = r_\chi(\omega,l)^{2(d-1)(1-\alpha^{-1})}$. Hence $Q_1 Q_2 \ge \frac{c^2}{r_\chi(\omega,l)^2}$ when $r_\chi(\omega,l) > 1$ where $c = (\sup_{\omega,l} r_\chi(\omega,l))^{(d-1)(1-\alpha^{-1})}$. By Cauchy-Schwarz inequality we have
$$\sqrt{V}\sqrt{W} \ge \int_0^\infty \int_0^1 \sqrt{Q}_1 \sqrt{Q}_2 = \int_0^\infty \int_0^1 r_\chi(\omega,l)^{(d-1)(1-\alpha^{-1})} d\omega dl \ge c \int_0^\infty \int_0^1 \frac{1}{r_\chi(\omega,l)} d\omega dl.$$
Let K be the set where $f > 0$, f being the function such that $\mu = f\lambda$, where μ is the irrigation measure defined by the set of fibers associated to the skeleton (see Definition 3.5.5). Let us decompose K as $K = A \cup R$, where A are the points of Lebesgue density 1 of K, R has zero Lebesgue measure and,

because of weak equality of supply, it is also of μ measure zero. Then $|T^{-1}(R)| = 0$, so that $|T^{-1}(A)|$ is of non zero measure. By Proposition 3.2.1 in section 1.1, the profile of an irrigating branch must decrease faster than linearly and $\int_0^\infty \frac{1}{r_\chi(\omega,l)} dl$ is infinite for all ω such that $T(\omega) = x \in A$. Then it turns out that

$$\int_0^1 \int_0^\infty \frac{1}{r_\chi(\omega,l)} dl d\omega \geq \int_{T^{-1}(A)} \int_0^\infty \frac{1}{r_\chi(\omega,l)} dl d\omega \geq \infty.$$

We conclude that

$$\sqrt{V}\sqrt{W} \geq \int_0^1 \int_0^\infty \frac{1}{r_\chi(\omega,l)} dl d\omega = \infty.$$

□

Thus, the exponent $\alpha = 1 - \frac{1}{d}$ is critical relatively to the fact that a tree cannot irrigate a volume at finite cost. This result is consistent with the results presented in Chapter 5.

Chapter 4

The traffic plan model

Introduction

A traffic plan is a measure on the set of paths. As it is possible to see on figure 2.3, this object can describe a great variety of structures. We can associate to a traffic plan a canonical transference plan π_μ along with its marginals μ^+ and μ^-. Thus, as was mentioned in chapter 2, traffic plan can model both the irrigation problem and the who goes where problem where the whole transference plan is prescribed. Let us now give the plan of the present chapter. In Section 4.1, we define traffic plans and transference plans. In Section 4.2, we model probability measures in a Lagrangian way as sets of particles indexed by $[0, 1]$. In Section 4.3, we prove semicontinuity results, and sequential compactness properties of traffic plans. Section 4.4 is devoted to the proof of existence of minimizers of the Monge-Kantorovitch problem within our framework. In Section 4.5, we prove the existence of a minimizer for both the irrigation and the traffic problems. This result in particular retrieves the existence results of [22] and [35] in a more general setting.

4.1 Traffic plans with prescribed transference plans

Let $X \subset \mathbb{R}^N$ be a compact set.

Definition 4.1.1 *Let us denote by K the set of 1-Lipschitz maps $\gamma : \mathbb{R}^+ \to X$ endowed with the distance*

$$d(\gamma, \gamma') := \sup_{k \in \mathbb{N}^*} \frac{1}{k} ||\gamma - \gamma'||_{L^\infty([0,k])}.$$

From now on, we consider $\mathcal{B}$, the Borel σ-algebra on K.

Definition 4.1.2 *Let $\gamma \in K$. We define its stopping time as*

$$T(\gamma) := \inf\{t : \gamma \textit{ constant on } [t, \infty[\}.$$

Remark 4.1.3 Observe that the stopping time $T : K \to \bar{\mathbb{R}}$ is measurable. Indeed, using lemma 4.3.5 below, T is lower semicontinuous. This means that $T^{-1}(]A, +\infty])$ is open, then measurable. Thus, T is measurable.

Lemma 4.1.4 *The metric space (K, d) is compact.*

Proof: The space K is complete and the totally boundedness is a straightforward consequence of Ascoli-Arzela's Theorem.

Definition 4.1.5 *We define a traffic plan μ as a probability measure on $(K, \mathcal{B})$ such that*

$$\int_K T(\gamma) d\mu(\gamma) < \infty. \tag{4.1}$$

We denote by $TP(X)$ the set of all traffic plans in X. We denote by $TP_C(X)$ the set of traffic plans μ such that $\int_K T(\gamma)d\mu(\gamma) \leq C$. We shall omit the mention of X in the following.

Remark 4.1.6 This definition is realistic for a traffic plan, as $T(\gamma)$ represents a transportation time and we don't want the average transportation time to be infinite! Observe that (4.1) implies that $T(\gamma) < \infty$, $\mu-$almost everywhere.

Definition 4.1.7 *With any traffic plan μ is associated a transference plan, that is to say a probability measure on $X \times X$ that we denote by π_μ and define by*

$$< \pi_\mu, \phi >:= \int_K \phi(\gamma(0), \gamma(T(\gamma)))d\mu(\gamma),$$

where $\phi \in C(X \times X, \mathbb{R})$. In an informal way, $\pi_\mu(A \times B)$ is the mass carried from A to B by means of the traffic plan μ. We denote by $TP(\pi)$ the set of traffic plans μ such that $\pi_\mu = \pi$. This is the set of traffic plans with prescribed transference plan π.

Definition 4.1.8 *If μ is a traffic plan, we define its irrigating and irrigated measure by*

$$< \mu^+, \phi_1 >:=< \pi_\mu, \phi_1 \otimes \mathbb{1}_X > \quad \text{and} \quad < \mu^-, \phi_2 >:=< \pi_\mu, \mathbb{1}_X \otimes \phi_2 > \quad \phi_1, \phi_2 \in C(X).$$

We denote by $TP(\nu^+, \nu^-)$ the set of traffic plans μ such that $\mu^+ = \nu^+$ and $\mu^- = \nu^-$.

4.2 Parameterization of a probability measure on a totally bounded metric space

The aim of this section is to show that we can associate with any probability measure a system of "elementary particles" such that $\mu_n \rightharpoonup \mu$ becomes "almost every elementary particle of μ_n tends to an elementary particle of μ". In an abstract setting, we assume in this section that (K, d) is a totally bounded metric space equipped with the $\sigma-$algebra of its Borel sets. The results in this section are well-known [16], the main aim being to prove the Skorohod (or Skorokhod in other textbooks) representation theorem, i.e. theorem 4.2.8. The results of this section will be applied to traffic plans but it is convenient to develop them in a more general setting.

Definition 4.2.1 *Let μ be a probability measure on K. We call parameterization of μ a measurable application $\chi : \omega \in [0, 1] \to K$ such that $\mu = \chi\#\lambda$ where λ is the Lebesgue measure on $[0, 1]$. That is to say $\mu(A) = \lambda(\chi^{-1}(A))$. Observe that if $\phi : K \to \mathbb{R}^+$ is a $\mu-$measurable function, then $\int_K \phi(\gamma)d\mu(\gamma) = \int_\Omega \phi(\chi(\omega))d\omega$ ([2], Def. 1.70, p. 32).*

Remark 4.2.2 As an illustrative example, if $K = [-1, 1]$, the Dirac mass at 0 is parameterized by the null constant application on $[0, 1]$. In the same way, an atomic measure $\sum_1^n a_i \delta_{x_i}$ can be parameterized by the piecewise constant function $\chi(\omega) = x_1$ on $[0, a_1]$, $\chi(\omega) = x_2$ on $]a_1, a_2]$ and so on.

Remark 4.2.3 Recall that the function $\chi : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^N$ is called a Carathéodory function if $\chi(\omega, t)$ is a continuous function of t for almost every $\omega \in [0,1]$ and is measurable in ω for every $t \in \mathbb{R}^+$. As it is well-known, Carathéodory functions are measurable as functions of (ω, t) [13]. As a function of (ω, t), the parameterization χ defined in Definition 4.2.1 is a Carathédory function. Observe that, as a consequence of Proposition 4.3.1, both concepts coincide for functions $\chi : [0,1] \to K$.

In lemma 4.2.5, we shall construct a filtration on K of a special kind which gives us a parameterization of μ (see lemma 4.2.6). For that, we first prove that we can construct a filtration on K whose sets have a specified diameter. Then, in lemma 4.2.5, we prove that we can adapt the filtration so that μ does not charge the boundaries of its elements.

Lemma 4.2.4 *There exists a filtration of K made of finite partitions $\mathcal{F}_l = \{F_j^l : 1 \le j \le J_l\}$, where $J_l \in \mathbb{N}^*$, such that the diameters of the sets F_j^l are less than 2^{-l}.*

Proof: We construct this filtration recursively. In order to construct $\mathcal{F}_1$, we cover K with a finite number of balls of radii $1/4$. Let us denote by B_i, where $1 \le i \le n$, the intersection of these balls with K. Let us find a partition of $K = \cup_i B_i$ with at most n elements. To do this, we denote $\tilde{F}_1^1 := B_1$ and, in a recursive way, we define $\tilde{F}_{i+1}^1 := B_{i+1} \setminus \cup_{j\le i} B_j$. If any of the $\tilde{F}_i^1$ is empty, we do not take it into account, so that we obtain a family of non empty elements F_i^1 where $i \le J_1$. Since the F_i^1 are totally bounded, we can iterate the above process by covering them with balls of radius $1/8$. Proceeding iteratively we construct the desired filtration. □

Lemma 4.2.5 *Let μ be a probability measure on K. There exists a filtration made of finite partitions $\mathcal{F}_l = \{F_j^l : 1 \le j \le J_l\}$, $J_l \in \mathbb{N}^*$, such that the diameters of F_j^l are less than 2^{-l+1} and $\mu(\partial F_j^l) = 0$ for all l and $j \le J_l$.*

Proof: To obtain this filtration, we slightly modify the construction of lemma 4.2.4. We only need to request in addition that $\mu(\partial F_j^l) = 0$ for all l and $j \in J_l$. For that, it is enough to perturb the radii $r_l = 2^{-l}(1+\epsilon_l)$, with $\epsilon_l \le 1$ so that μ does not charge the boundaries of the balls with radius r_l used to construct $\mathcal{F}_l$. □

The filtration obtained in lemma 4.2.5 allows us to define a parameterization of μ. The idea is to group together the ω's whose images are close.

Lemma 4.2.6 *Let μ be a probability measure on K and $\mathcal{F}$ be the filtration constructed in lemma 4.2.5. There exists a parameterization χ of μ such that for all l, the sets*

$$\Omega_{j,l} = \{\omega : \chi(\omega) \in F_j^l\}$$

are intervals ordered in an increasing way with j.

Proof: We construct χ by successive approximations χ_n using the filtration of lemma 4.2.5.

Step 1: Definition of χ_n. Let $t_0^n := 0$ and $t_j^n := \sum_{i\le j} \mu(F_i^n)$ where $1 \le j \le J_n$. The application χ_n is defined as a piecewise constant function sending each interval $[t_{j-1}^n, t_j^n[$ onto an arbitrary element of F_j^n. By construction, $\Omega_{j,l} := \{\omega : \chi_n(\omega) \in F_j^l\} = [t_{j-1}^l, t_j^l[$ for all $j \le J_l$. We notice that the intervals $[t_{j-1}^l, t_j^l[$ where $1 \le j \le J_l$, are intervals ordered in an increasing way when j goes from 1 to J_l, so that their union is $[0,1[$. Notice also that $\mu(F_j^l) = \lambda(\Omega_{j,l})$.

Step 2: The sequence $\chi_n(\omega)$ converges for all ω. Let us prove that χ_n is a Cauchy sequence. Let us first observe that $\chi_n(\Omega_j^m) \subset F_j^m$ for any $n \geq m.$ Indeed, let us fix m and $n \geq m.$ By the definition of filtration, Ω_j^m is the union of Ω_k^n where k describes the set of indices such that $F_k^n \subset F_j^m$. Thus, χ_n sends every element of Ω_k^n to an element of $F_k^n \subset F_j^m$. A fortiori, the image of Ω_j^m under χ_n is in F_j^m. Now, since the sets F_j^m have diameter less than 2^{-m}, we deduce that $d(\chi_n(\omega), \chi_m(\omega)) < 2^{-m}$ for all $m \leq n$. Thus, $\chi_n(\omega)$ is a Cauchy sequence.

Let χ be the pointwise limit of χ_n. Observe that χ is measurable as a pointwise limit of measurable functions.

Step 3: The measure $\chi\#\lambda$ is exactly μ. We have to show that $\chi\#\lambda(F_j^l) = \mu(F_j^l)$ for all (j,l). The measures μ and $\chi\#\lambda$ will then be equal on the sets F_j^l which form a Π-system. Then the extension Theorem of Π-systems (lemma 1.6, p.19 [34]) shows that $\mu = \chi\#\lambda$ on the σ-algebra generated by this Π-system, that is, on the σ-algebra of Borel sets of K.

Let us fix l, $j \leq J_l$, and let us define

$$G_p := \{\gamma \in F_j^l : d(\gamma, \partial F_j^l) \geq 1/p\}.$$

This is a non decreasing sequence of sets such that $\cup_p G_p = F_j^l \setminus \partial F_j^l$. Fix $\epsilon > 0$. For a sufficiently large p, we have

$$\mu(G_p) \geq \mu(F_j^l) - \epsilon. \tag{4.2}$$

Now, consider an l' such that $2^{-l'} < \frac{1}{2p}$. For any $y \in G_p$, there exists k so that $y \in F_k^{l'}$. Since the diameter of $F_k^{l'}$ is less than $\frac{1}{2p}$, $F_k^{l'} \subset G_{2p}$ so that $\bar{F}_k^{l'} \subset F_j^l$. For $n \geq l'$, the construction of χ_n ensures that $\chi_n(\Omega_k^{l'}) \subset F_k^{l'}$. Since χ is the pointwise limit of χ_n,

$$\chi(\Omega_k^{l'}) \subset \bar{F}_k^{l'} \subset F_j^l. \tag{4.3}$$

We obtain a covering of G_p with sets of the form $F_k^{l'}$ satisfying (4.3), and, using (4.2), we have $\chi\#\lambda(F_j^l) \geq \mu(F_j^l) - \epsilon$. This being true for all $\epsilon > 0$, we deduce that $\chi\#\lambda(F_j^l) \geq \mu(F_j^l)$. Since these sets form a partition for $1 \leq j \leq J_l$, and $\chi\#\lambda$ is a probability measure, the inequality is indeed an equality, that is: $\chi\#\lambda(F_j^l) = \mu(F_j^l)$. As a consequence, we have $\chi^{-1}(F_j^l) = \Omega_{j,l}$ modulo a null set. □

Definition 4.2.7 *Let $(\mu_n)_n$ and μ be probability measures on (K,d). We say that μ_n tends to μ "pointwise" whenever there exist parameterizations χ_n and χ of μ_n and of μ, respectively, such that $d(\chi_n(\omega), \chi(\omega)) \to 0$ almost everywhere in $[0,1]$.*

Theorem 4.2.8 *Let $(\mu_n)_n$ be a sequence of probability measures on (K,d). Then μ_n weakly-* converges to μ if and only if μ_n to μ tends to μ "pointwise".*

Proof: Assume that μ_n converges to μ "pointwise", and let χ_n, χ denote the parameterizations of μ_n and μ, respectively. Since $\chi_n(\omega)$ converges to $\chi(\omega)$ for almost every ω, using Lebesgue's theorem, for all $\phi \in C(K)$, we have

$$\begin{aligned} <\mu_n, \phi> &= \int_K \phi(\gamma) d\mu_n(\gamma) = \int_{[0,1]} \phi(\chi_n(\omega)) d\omega \\ &\to \int_{[0,1]} \phi(\chi(\omega)) d\omega = \int_K \phi(\gamma) d\mu(\gamma) = <\mu, \phi> . \end{aligned}$$

Conversely, let μ_n be weakly-* converging to μ. Let us consider the filtration associated with μ constructed in lemma 4.2.5. Since $\mu(\partial F_j^l) = 0$, we deduce that $\mu_n(F_j^l)$ converges to $\mu(F_j^l)$. Next, applying lemma 4.2.6 to measures μ_n and μ, we get applications χ_n and χ such that $\chi_n\#\lambda = \mu_n$ and $\chi\#\lambda = \mu$. The fact that $\mu_n(F_j^l)$ converges to $\mu(F_j^l)$ implies that $\lambda(\Omega_{j,l}^n)$ converges to $\lambda(\Omega_{j,l})$, where $\Omega_{j,l}^n := \{\omega : \chi_n(\omega) \in F_j^l\}$ and $\Omega_{j,l} := \{\omega : \chi(\omega) \in F_j^l\}$. This convergence of measures implies the convergence of intervals $\Omega_{j,l}^n$ to some intervals $\Omega_{j,l}$, ordered in an increasing way with j.

We are now in a position to prove that for almost all ω the sequence $\chi_n(\omega)$ converges to $\chi(\omega)$. Notice that for almost all ω and for any $l \in \mathbb{N}$, there exists a $j \leq J_l$ such that ω is in the interior of $\Omega_{j,l}$. Indeed, there is a finite number of such intervals at each rank of the filtration, and, thus, the set of its endpoints is countable, hence of measure zero. Thus, for n large enough, we have that $\omega \in \Omega_{j,l}^n$, i.e., $\chi_n(\omega) \in F_j^l$. This yields $d(\chi_n(\omega), \chi(\omega)) < 2^{-l}$. □

4.3 Stability properties of traffic plans

From now on, we will denote $|A| := \lambda(A)$, the Lebesgue measure of a measurable set $A \subset [0,1]$. Throughout this section, (K, d) is the compact metric space of Definition 4.1.1. According to lemma 4.2.6, we can associate with a traffic plan μ a parameterization $\chi : \Omega \to K$. We set $\chi(\omega, t) := \chi(\omega)(t)$. It is easy to check that χ is a measurable function from $\Omega \times \mathbb{R}^+ \to X$. This is true, since χ is a Carathéodory function (see Remark 4.2.3). Moreover, if a function $\chi : [0,1] \to K$ is measurable as a function of (ω, t), then it is measurable as a function from $[0,1]$ to (K, d). Since this is a simple argument, we include it here for the sake of completeness.

Proposition 4.3.1 *The application $\chi : \Omega \times \mathbb{R}^+ \to X$ is measurable if and only if the application $\omega \in [0,1] \mapsto \chi(\omega, \cdot) \in K$ is measurable.*

Proof: Let $\chi : \Omega \times \mathbb{R}^+ \to X$ be a measurable function. Observe that

$$\begin{aligned} \chi^{-1}(B(\gamma, r)) &= \{\omega : d(\chi(\omega), \gamma) \leq r\} \\ &= \{\omega : \forall k, \ \frac{||\chi(\omega) - \gamma||_{L^\infty([0,k])}}{k} \leq r\} \\ &= \cap_k \{\omega : ||\chi(\omega) - \gamma||_{L^\infty([0,k])} \leq kr\} \\ &= \cap_k \cap_{t \in \mathbb{Q} \cap [0,k]} \{\omega : |\chi(\omega)(t) - \gamma(t)| \leq kr\} \end{aligned}$$

This last expression is a countable intersection of measurable sets since the maps $\tilde{\chi} : \omega \mapsto \tilde{\chi}(\omega, t)$ are measurable for any $t \in [0,1]$. □

This shows that if $\chi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^N$ is measurable, we can define its associated traffic plan $\mu := \chi\#\lambda$. Of course, as we can deduce from the preceding section, a traffic plan can have many different parameterizations.

Definition 4.3.2 *Let μ_n be a sequence of traffic plans. We shall say that μ_n converges to a traffic plan μ if one of the equivalent relations is satisfied:*

$$\mu_n \rightharpoonup \mu,$$

$$\chi_n(\omega) \to \chi(\omega) \text{ in } K \text{ for almost all } \omega \in \Omega,$$

where μ_n and μ are parameterized using a common filtration constructed as in lemma 4.2.5, such that $\mu_n(\partial F_j^l) = \mu(\partial F_j^l) = 0$ for any j, l.

Remark 4.3.3 An immediate adaptation of lemma 4.2.5 permits to use the same filtration to construct the parameterizations of all measures μ_n and μ.

4.3.1 Lower semicontinuity of length, stopping time, averaged length and averaged stopping time

Lemma 4.3.4 *Let μ_n be a sequence of probability measures on a compact metric space K and such that μ_n weakly converges to μ. Let $\gamma \mapsto f(\gamma)$ be a lower semicontinuous function on K. Then,*

$$\int_K f(\gamma)d\mu(\gamma) \leq \liminf \int_K f(\gamma)d\mu_n(\gamma).$$

Proof: This is a straightforward application of the fact that any lower semicontinuous function f on a metric compact space is the increasing limit of a sequence of continuous functions ([2], lemma 1.61, p. 27), and the monotone convergence theorem. □

Lemma 4.3.5 *Let $L(\gamma)$ denote the length of $\gamma \in K$. If the sequence $\gamma_n \in K$ converges to γ for the metric d, then*

$$T(\gamma) \leq \liminf T(\gamma_n),$$

and

$$L(\gamma) \leq \liminf L(\gamma_n).$$

Proof: For all $t \geq s > \liminf T(\gamma_n)$, there exists an increasing sequence of indices n_k going to infinity such that $T(\gamma_{n_k}) < s \leq t$. This ensures that $\gamma_{n_k}(t) = \gamma_{n_k}(s)$. Considering the limit of this equality, we obtain $\gamma(t) = \gamma(s)$. Then γ is constant on $]\liminf T(\gamma_n), +\infty[$, so that $T(\gamma) \leq \liminf T(\gamma_n)$. The lower semicontinuity of the length functional is well-known and we shall omit the details. □

Lemma 4.3.6 *If a sequence of traffic plans μ_n converges to μ, then*

$$\int_K T(\gamma)d\mu(\gamma) \leq \liminf \int_K T(\gamma)d\mu_n(\gamma)$$

and

$$\int_K L(\gamma)d\mu(\gamma) \leq \liminf \int_K L(\gamma)d\mu_n(\gamma).$$

Proof: Because of lemma 4.3.5, the applications $\gamma \mapsto T(\gamma)$ and $\gamma \mapsto L(\gamma)$ are lower semicontinuous. The desired inequalities then directly come from lemma 4.3.4. □

4.3.2 Multiplicity of a traffic plan and its upper semicontinuity

Definition 4.3.7 *Let μ be a traffic plan. We call multiplicity of μ at a point $x \in \mathbb{R}^N$ the number*

$$|x|_\mu := \mu(\{\gamma : \exists t, \gamma(t) = x\}).$$

If χ is a parameterization of μ, then we define the path class of $x \in \mathbb{R}^N$ as the set

$$[x]_\chi := \{\omega : \exists t, \chi(\omega, t) = x\}.$$

Since $\chi\#\lambda = \mu$, we have that $|[x]_\chi| = |x|_\mu$.

Remark 4.3.8 The multiplicity is well defined since the set $\{\gamma : \exists t, \gamma(t) = x\}$ is a Borel set of K. Indeed, $\{\gamma : \exists t, \gamma(t) = x\} = \cup_n \{\gamma : \exists t \leq n, \gamma(t) = x\}$ is a union of closed sets in K.

Proposition 4.3.9 *(lemma 6.2, [22]) Let χ_n be a sequence of parameterizations of traffic plans converging to χ. Suppose further that there is $C > 0$ such that $\int_\Omega T(\chi_n(\omega))d\omega \leq C$. Then, for almost all ω,*

$$\limsup |[\chi_n(\omega,t)]_{\chi_n}| \leq |[\chi(\omega,t)]_\chi|.$$

Proof: Set $\epsilon = C/M$. By Markov's inequality,

$$|\{\omega : T(\chi_n(\omega)) > M\}| \leq \frac{C}{M} = \epsilon.$$

Let us define an approximate multiplicity by

$$[\chi(\omega,t)]^\epsilon_\chi := \{\omega' \in [\chi(\omega,t)]_\chi : T(\chi(\omega')) \leq M\}.$$

Next, let us take an element ω' in $\cap_k \cup_{n>k} [\chi_n(\omega,t)]^\epsilon_{\chi_n}$. This means that there exists a sequence of indices n_i which goes to infinity, and times $s_i \leq T(\chi_{n_i}(\omega)) \leq M$ such that $\chi_{n_i}(\omega', s_i) = \chi_{n_i}(\omega,t)$. Since s_i is bounded, it is possible to extract $s_i \to s$ and because of uniform convergence of $\chi_{n_i}(\omega', \cdot)$ on $[0, M]$, we obtain $\chi(\omega', s) = \chi(\omega, t)$, hence $\omega' \in [\chi(\omega,t)]_\chi$. This shows that $\cap_k \cup_{n>k} [\chi_n(\omega,t)]^\epsilon_{\chi_n} \subset [\chi(\omega,t)]_\chi$, so that

$$\limsup |[\chi_n(\omega,t)]^\epsilon_{\chi_n}| \leq |[\chi(\omega,t)]_\chi|.$$

Thus,

$$\limsup |[\chi_n(\omega,t)]_{\chi_n}| - \epsilon \leq |[\chi(\omega,t)]_\chi|.$$

□

We prove another kind of upper semicontinuity which will be useful to prove Corollary 4.3.11.

Lemma 4.3.10 *Let χ be a parametrization of a traffic plan μ. Then, the function $\phi : x \mapsto |[x]_\chi|$ is upper semicontinuous.*

Proof: Let us show that for each x such that $|[x]_\chi| < r$, there is a ball $B(x,\epsilon)$ such that for all y in $B(x,\epsilon)$, $|[y]_\chi| < r$. This will prove that $\phi^{-1}([0,r[)$ is an open set, and therefore that ϕ is upper semicontinuous. Suppose that it is not the case. Then, for each ball $B_n := B(x, 1/n)$, there is a $y_n \in B_n$ so that $|[y_n]_\chi| \geq r$. Notice that y_n tends to x when n goes to infinity. Let us consider

$$\tilde{\Omega} := \cap_n \cup_{m \geq n} [y_m]_\chi.$$

Then, modulo a null set, $\tilde{\Omega} \subset [x]_\chi$. Indeed, for almost every ω, $T(\chi(\omega)) < \infty$. For such an ω in $\tilde{\Omega}$, this means that for all n, there is an $m \geq n$ such that $\omega \in [y_m]_\chi$, that is, there is a t_m such that $\chi(\omega, t_m) = y_m$. Since $T(\chi(\omega)) < \infty$, the sequence $(t_m)_m$ can be supposed to be bounded. Thus, it is possible to extract a convergent subsequence $t_m \to t$ such that $\chi(\omega,t) = x$, i.e., $\omega \in [x]_\chi$. Thus $|\tilde{\Omega}| \leq |[x]_\chi| < r$ and $|\tilde{\Omega}| = \lim_n |\cup_{m\geq n} [y_n]_\chi| \geq r$. This contradicts our initial assumption. □

Corollary 4.3.11 *Let χ be a parametrization of a traffic plan μ. The function $(\omega,t) \mapsto |[\chi(\omega,t)]_\chi|$ is measurable.*

Proof: This a consequence of the measurability of $x \mapsto |[x]_\chi|$ (lemma 4.3.10). Indeed, we have

$$\begin{aligned}\{(\omega,t) : |[\chi(\omega,t)]_\chi| < r\} &= \{(\omega,t) : \chi(\omega,t) = x \text{ and } |[x]_\chi| < r\} \\ &= \chi^{-1}(\{x : |[x]_\chi| < r\}).\end{aligned}$$

□

4.3.3 Sequential compactness of traffic plans

Theorem 4.3.12 *If $(\mu_n)_n$ is a sequence of TP_C such that $\mu_n \rightharpoonup \mu$, then $\pi_{\mu_n} \rightharpoonup \pi_\mu$. Hence, given a sequence $(\mu_n)_n$ of TP_C, it is possible to extract a convergent subsequence such that π_{μ_n} converges.*

Proof: Set $\epsilon = C/M$. By Markov's inequality, we have $\mu_n(K \setminus K_\epsilon) \le \frac{C}{M} = \epsilon$ where $K_\epsilon := \{\gamma : T(\gamma) \le M\}$. Because of lemma 4.3.6, we also have that $\int_K T(\gamma) d\mu(\gamma) \le C$, and, thus, $\mu(K \setminus K_\epsilon) < \epsilon$. Let $\phi \in C(X \times X, \mathbb{R})$. Since, by definition of the distance on K, the map $\gamma \mapsto \phi(\gamma(0), \gamma(M))$ is continuous from K to $\mathbb{R}$, then, by definition of the transference plan associated with a traffic plan, we have

$$\begin{aligned}\limsup_n < \pi_{\mu_n}, \phi > &\le \limsup_n (\int_{K_\epsilon} \phi(\gamma(0), \gamma(T(\gamma))) d\mu_n(\gamma) + \epsilon ||\phi||_\infty) \\ &= \limsup_n \int_{K_\epsilon} \phi(\gamma(0), \gamma(M)) d\mu_n(\gamma) + \epsilon ||\phi||_\infty \\ &\le \limsup_n \int_K \phi(\gamma(0), \gamma(M)) d\mu_n(\gamma) + 2\epsilon ||\phi||_\infty \\ &= \int_K \phi(\gamma(0), \gamma(M)) d\mu(\gamma) + 2\epsilon ||\phi||_\infty \\ &\le \int_K \phi(\gamma(0), \gamma(T(\gamma))) d\mu(\gamma) + 4\epsilon ||\phi||_\infty \\ &= < \pi_\mu, \phi > + 4\epsilon ||\phi||_\infty.\end{aligned}$$

In the same way,

$$\liminf_n < \pi_{\mu_n}, \phi > \ge < \pi_\mu, \phi > - 4\epsilon ||\phi||_\infty.$$

□

Corollary 4.3.13 *Let π be a probability measure on $X \times X$. There exists a traffic plan μ such that $\pi_\mu = \pi$.*

Proof: Let us first prove this property in the case of finite atomic measures π. Let $(a_i)_{i=1}^k$ and $(b_j)_{j=1}^l$ the elements of the support of the two marginals of π. Let us denote by $\pi_{i,j}$ the values $\pi(\{a_i\} \times \{b_j\})$. We now define $\gamma_{i,j} \in K$, the segment joining a_i to b_j, i.e. $\gamma_{i,j}(0) = a_i$, for $t \in]0, |a_i - b_j|]$,

$$\gamma_{i,j}(t) := \frac{t}{|a_i - b_j|} b_j + \frac{1-t}{|a_i - b_j|} a_i$$

and $\gamma_{i,j}$ is constant on $[|a_i - b_j|, \infty[$. The traffic plan $\mu := \sum_{i,j} \pi_{i,j} \delta_{\gamma_{i,j}}$ is such that $\pi_\mu = \pi$ by construction.

Let us now consider a general transference plan π and a sequence of atomic measures π_n such that $\pi_n \rightharpoonup \pi$. The first part of the proof tells that there are traffic plans μ_n such that $\pi_{\mu_n} = \pi_n$. By theorem 4.3.12, we can extract a converging subsequence from $(\mu_n)_n$ such that μ_n converges to μ with $\pi_{\mu_n} \rightharpoonup \pi_\mu$. Thus, the traffic plan μ is such that $\pi_\mu = \pi$. □

4.4 The Monge-Kantorovitch problem

For a sake of completeness, we show that the traffic plan formalism is adapted to solve the Monge-Kantorovitch problem. Of course, no result is new here.

Definition 4.4.1 *We call cost of a traffic plan a functional*

$$I(\mu) = \int_K c(\gamma(0), \gamma(T(\gamma)))d\mu(\gamma),$$

where c is a bounded non-negative lower semicontinuous function which informally represents the cost for transporting a unit of mass from x to y.

Let us notice that $I(\mu) = \int_{X\times X} c(x,y)d\pi_\mu(x,y)$ where π_μ is the transference plan associated to the traffic plan μ. Given two measures ν^+ and ν^-, the Monge-Kantorovitch problem consists in minimizing $\int_{X\times X} c(x,y)d\pi(x,y)$ under prescribed marginal measures ν^+ and ν^-. By corollary 4.3.13, any transference plan can be obtained (in a not unique way) as the transference plan π_μ associated to a traffic plan μ. Thus, the problem of minimizing $I(\mu)$ under prescribed marginal measures ν^+ and ν^- is equivalent to the Monge-Kantorovitch problem. The existence of an optimal transference plan is given by standard lower semicontinuity argument and compactness. The next two propositions uses the same strategy at the level of traffic plans.

Proposition 4.4.2 *If $(\mu_n)_n$ and μ are traffic plans such that $\mu_n \rightharpoonup \mu$, then*

$$I(\mu) \leq \liminf I(\mu_n).$$

Proof: The application $\gamma \mapsto c(\gamma(0), \gamma(M))$ is lower semicontinuous because of the lower semicontinuity of c. Then lemma 4.3.4 asserts that

$$\liminf \int_K c(\gamma(0), \gamma(M))d\mu_n(\gamma) \geq \int_K c(\gamma(0), \gamma(M))d\mu(\gamma).$$

Set $\epsilon = C/M$. By Markov's inequality, $\mu_n(K \setminus K_\epsilon) \leq \frac{C}{M} = \epsilon$ where

$$K_\epsilon := \{\gamma : T(\gamma) \leq M\}.$$

For such an M, we have

$$\int_K c(\gamma(0), \gamma(M))d\mu_n(\gamma) \leq I(\mu_n) + \epsilon||c||_\infty$$

and

$$\int_K c(\gamma(0), \gamma(M))d\mu(\gamma) \geq I(\mu) - \epsilon||c||_\infty,$$

so that

$$I(\mu_n) + \epsilon||c||_\infty \geq I(\mu) - \epsilon||c||_\infty.$$

□

Proposition 4.4.3 *The problem of minimizing $I(\mu)$, with $\mu \in TP_C(\nu^+, \nu^-)$ admits a solution.*

Proof: Let μ_n be a minimizing sequence. Because of Theorem 4.3.12, there exists a subsequence such that $\mu_n \rightharpoonup \mu$ and $\pi_{\mu_n} \rightharpoonup \pi_\mu$. In particular, we have $\mu_n^+ \rightharpoonup \mu^+$ and $\mu_n^- \rightharpoonup \mu^-$. Since $\mu_n^+ = \nu^+$ and $\mu_n^- = \nu^-$ for all n, μ is a traffic plan satisfying the constraints and such that $I(\mu) \leq \liminf I(\mu_n)$. Since μ_n is a minimizing sequence, μ is a minimizer of I under the constraints of irrigating and irrigated measures. □

4.5 Irrigation and traffic models

In this section, the cost functional we consider is taken from two irrigation models proposed in [35] and [22]. As in these models, we prove that the functional admits a minimizer under the constraint of prescribed irrigating and irrigated measures. In addition, our model permits to handle a prescribed transference plan constraint. We prove the existence of minimizing traffic plans with this new constraint. So we move from an irrigation model to a traffic model. The first three subsections are devoted to the proof of the existence of minimizers of the energy functional under the two different sets of constraints. In the other two subsections, we show that there exists a minimizer of the energy with simple paths. A change of variable formula permits us to prove that the energy functional coincides with Q. Xia's one ([35, 36]) on traffic plans with simple paths.

4.5.1 Energy of a traffic plan and existence of a minimizer

We use the convention that $0^{\alpha-1} = \infty$ with $\alpha \in [0, 1)$.

Definition 4.5.1 *Let $\alpha \in [0, 1]$. We call energy of a traffic plan the functional*

$$E(\mu) = \int_\Omega \int_{\mathbb{R}^+} |[\chi(\omega, t)]_\chi|^{\alpha-1} |\dot{\chi}(\omega, t)| dt d\omega, \tag{4.4}$$

where χ is a parameterization of μ.

Remark 4.5.2 This energy will be proved to be a reformulation of the one used in [35] (see Proposition 4.6.6).

Remark 4.5.3 The application $(\omega, t) \mapsto |[\chi(\omega, t)]_\chi|$ was shown to be measurable in Corollary 4.3.11. Let us denote $|\dot{\chi}(\omega, t)|_{sup} := \limsup_{s\to t} |\frac{\chi(\omega,t)-\chi(\omega,s)}{t-s}|$ and $|\dot{\chi}(\omega, t)|_{inf} := \liminf_{s\to t} |\frac{\chi(\omega,t)-\chi(\omega,s)}{t-s}|$. Both applications $(\omega, t) \mapsto |\dot{\chi}(\omega, t)|_{sup}$ and $(\omega, t) \mapsto |\dot{\chi}(\omega, t)|_{inf}$ are measurable since they can be interpreted as a pointwise limit of measurable functions. For almost every ω and for almost every t, $|\dot{\chi}(\omega, t)|_{inf} = |\dot{\chi}(\omega, t)|_{sup}$ since $\chi(\omega, \cdot)$ is 1-Lipschitz. Thus, the set C where $|\dot{\chi}(\omega, t)|$ is well defined is measurable. If $|\dot{\chi}|$ is extended by 0 on $\Omega \times \mathbb{R} \setminus C$ (which is of null measure), the function thus defined is measurable.

Remark 4.5.4 The energy of a traffic plan could also be written

$$E(\mu) = \int_K \int_{\mathbb{R}^+} |\gamma(t)|_\mu^{\alpha-1} |\dot{\gamma}(t)| dt d\mu(\gamma).$$

The traffic problem is the following: given two measures ν^+ and ν^-, and a transference plan π between those measures, we look for minimizers of E with this prescribed transference plan. The irrigation problem is the less constrained case where we specify globally the supply and the demand. This latter case is essentially the same as in [35].

Lemma 4.5.5 *Let μ be a traffic plan. Then, we have*

$$E(\mu) \geq \int_K L(\gamma) d\mu(\gamma).$$

Proof: As the multiplicity at a point x is always less than 1, we have $|x|_\mu^{\alpha-1} \geq 1$ and then

$$E(\mu) \geq \int_K \int_{\mathbb{R}^+} |\dot\gamma(t)| dt d\mu(\gamma) = \int_K L(\gamma) d\mu(\gamma).$$

□

4.5.2 Normalization of a traffic plan

Lemma 4.5.6 *Let $\chi : [0,1] \to K$ be a parameterization of the traffic plan μ. We define $\tilde\chi(\omega)$ the arc-length reparameterization of $\chi(\omega)$ in the usual way. Let*

$$S(\omega,t) = \int_0^t |\dot\chi(\omega,r)|\, dr,$$

and let

$$T(\omega,s) = \inf\{t \in [0,\infty) :\, S(\omega,t) = s\}.$$

Let $\tilde\chi(\omega,s) = \chi(\omega, T(\omega,s))$. Then $\tilde\chi(\omega) \in K$ is Lebesgue measurable and for all $\omega \in [0,1]$, $\tilde\chi(\omega)$ is the arc-length reparameterization of $\chi(\omega)$.

Proof: The map $\tilde\chi$ is the composition of the maps $(I,T) : [0,1] \times [0,\infty) \to [0,1] \times [0,\infty)$ and $\chi : [0,1] \times [0,\infty) \to \mathbb{R}^N$. The measurability of $\tilde\chi$ will be a consequence of the measurability of (I,T) and χ, and the fact that $(I,T)^{-1}(N)$ is a null set in $[0,1] \times [0,\infty)$ for any null set N in $[0,1] \times [0,\infty)$.

Let us prove first that (I,T) is measurable. It suffices to prove that the function $T : [0,1] \times [0,\infty) \to \mathbb{R}$ is measurable. For that it will be sufficient to prove that $T^{-1}((-\infty,\lambda])$ is measurable for any $\lambda \in \mathbb{R}$. Let $\{t_m\}_m$ be a dense sequence in $[0,\infty)$. Using that T is non decreasing and lower semicontinuous in s we may write

$$T^{-1}((-\infty,\lambda]) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{\omega \in [0,1] :\, T(\omega,t_m) \leq \lambda\} \times [0, t_m + \frac{1}{n}].$$

Since $\{\omega \in [0,1] :\, T(\omega,t_m) \leq \lambda\} = \{\omega \in [0,1] :\, S(\omega,\lambda) \geq t_m\}$ is measurable, we deduce that $T^{-1}((-\infty,\lambda])$ is measurable.

Now, let N be a null set in $[0,1] \times [0,\infty)$ and let B be a Borel set containing N (of total measure less than ϵ). Observe that $F(\omega,s) := \mathbb{1}_B(\omega, T(\omega,s))$ is a measurable map. Now, for a.e. fixed value of each $\omega \in [0,1]$, we have

$$\int_0^\infty F(\omega,s) ds = \int_0^\infty \mathbb{1}_B(\omega,t) S_t(\omega,t) dt \leq \int_0^\infty \mathbb{1}_B(\omega,t) dt,$$

the last inequality being true since $S_t(\omega,t) \leq 1$. Integrating with respect to $\omega \in [0,1]$, and observing that both F and $\mathbb{1}_B$ are measurable in $[0,1] \times [0,\infty)$, we have

$$|(I,T)^{-1}(B)| \;=\; \int_0^1 \int_0^\infty \mathbb{1}_B(\omega, T(\omega,s)) ds d\omega \leq \int_0^1 \int_0^\infty \mathbb{1}_B(\omega,t) dt d\omega \leq \epsilon.$$

We deduce that $(I,T)^{-1}(N)$ is a null set. □

Definition 4.5.7 *We say that $\tilde\mu$ is a normalization of a traffic plan μ if for some parameterization χ of μ, $\tilde\chi \# \lambda = \tilde\mu$, where $\tilde\chi(\omega)$ is the arc-length reparameterization of $\chi(\omega)$ defined in lemma 4.5.6. Observe that $E(\tilde\mu) = E(\mu)$.*

Remark 4.5.8 Due to the fact that $\{\gamma \in K : |\dot\gamma| = 1\}$ is not closed under the distance d, it is not true that $\mu_n \rightharpoonup \mu$ implies $\tilde\mu_n \rightharpoonup \tilde\mu$.

4.5.3 Existence of a minimizer

Proposition 4.5.9 *If $(\mu_n)_n$ is a normalized sequence in TP_C, and μ is a traffic plan such that $\mu_n \rightharpoonup \mu$, then*

$$E(\mu) \leq \liminf E(\mu_n).$$

Proof: Let χ_n, χ' be parameterizations of μ_n and μ, respectively, such that $\chi_n(\omega) \to \chi'(\omega)$ converges in (K, d) for almost every $\omega \in [0,1]$. Because of the upper semicontinuity of multiplicity which was proved in Proposition 4.3.9 and the lower semicontinuity of $L(\gamma)$, we have

$$\begin{aligned}
\liminf_n E(\mu_n) &= \liminf_n \int_\Omega \int_0^{L(\chi_n(\omega))} |[\chi_n(\omega,t)]_{\chi_n}|^{\alpha-1} dt d\omega \\
&\geq \int_\Omega \int_0^{L(\chi'(\omega))} |[\chi'(\omega,t)]_{\chi'}|^{\alpha-1} dt d\omega \\
&\geq \int_\Omega \int_0^{L(\chi'(\omega))} |[\chi'(\omega,t)]_{\chi'}|^{\alpha-1} |\dot{\chi}'(\omega,t)| dt d\omega \\
&= E(\chi') = E(\mu).
\end{aligned}$$

□

Proposition 4.5.10 *The problem of minimizing $E(\mu)$ in $TP(\nu^+, \nu^-)$ admits a solution.*

Proof: In the case $\inf_{TP(\nu^+,\nu^-)} E(\mu) = \infty$, there is nothing to prove. Otherwise, there is some $C < \infty$ such that $\inf_{TP(\nu^+,\nu^-)} E(\mu) \leq C$. Because of lemma 4.5.5, $\inf_{TP(\nu^+,\nu^-)} E(\mu) = \inf_{TP_C(\nu^+,\nu^-)} E(\mu)$ so that we can consider a minimizing sequence $(\mu_n)_n$ in $TP_C(\nu^+, \nu^-)$. Since $E(\mu_n) = E(\tilde{\mu}_n)$, without loss of generality, we can take μ_n as being normalized. Because of Theorem 4.3.12, it is possible to extract a converging subsequence such that $\mu_n \rightharpoonup \mu$, $\nu^+_{\mu_n} \rightharpoonup \nu^+_\mu$, and $\nu^-_{\mu_n} \rightharpoonup \nu^-_\mu$. Since $\nu^+_{\mu_n} = \nu^+$ for all n, and $\nu^-_{\mu_n} = \nu^-$, μ is a traffic plan satisfying the constraints and $E(\mu) \leq \liminf E(\mu_n)$. Since μ_n is a minimizing sequence, μ is a minimizer of E under the constraint of the prescribed irrigating and irrigated measures. □

Proposition 4.5.11 *The problem of minimizing $E(\mu)$ in $TP(\pi)$ admits a solution.*

Proof: As in the proof of Proposition 4.5.10, we can consider a minimizing sequence $(\mu_n)_n$ in $TP_C(\pi)$, where C is such that $\inf_{TP(\pi)} E(\mu) \leq C$. Since $E(\mu_n) = E(\tilde{\mu}_n)$, without loss of generality, we can take μ_n as being normalized. Because of Theorem 4.3.12, it is possible to extract a subsequence, which we denote again by μ_n, such that $\mu_n \rightharpoonup \mu$ and $\pi_{\mu_n} \rightharpoonup \pi_\mu$. Since $\pi_{\mu_n} = \pi$ for all n, μ is a traffic plan satisfying the constraints and such that $E(\mu) \leq \liminf E(\mu_n)$. Since μ_n is a minimizing sequence, μ is a minimizer of E under the constraint of the prescribed transference plan. □

4.6 Simple paths traffic plan

Definition 4.6.1 **Simple paths traffic plan** *A traffic plan μ is said to be with simple paths if there is a parameterization χ of μ such that for almost all $\omega \in [0,1]$, the element $\chi(\omega)$ of K is injective on* $[0, T(\chi(\omega))]$.

Definition 4.6.2 Support *Let μ be a traffic plan. The support of μ is defined as $S_\mu := \{x : [x]_\mu > 0\}$.*

Proposition 4.6.3 *Let μ be a traffic plan such that $E(\mu) < \infty$. There exists a traffic plan with simple paths $\tilde{\mu}$ so that $S_{\tilde{\mu}} \subset S_\mu$ and $\pi_{\tilde{\mu}} = \pi_\mu$.*

Proof: Since the geometric embedding and the transference plans are invariant under normalization of a traffic plan μ, we can suppose μ to be normalized. Let χ be a parameterization of μ. Because of lemma 4.5.5, $L(\chi(\omega)) < \infty$ for almost all $\omega \in \Omega$. For these ω, we reparameterize the path $\chi(\omega)$, so that we suppress loops. To do so, we introduce the set

$$X_\omega = \{x \in \chi(\omega, \mathbb{R}^+) | \# \chi(\omega, \cdot)^{-1}(x) \cap [0, L(\chi(\omega)] > 1\},$$

which is empty if and only if $\chi(\omega)$ is injective.

Step 1: Existence of a maximal set of injectivity. We shall call a set of injectivity , a set

$$A_\omega = \bigcup_{x \in X_\omega} [t_x^-, t_x^+[$$

such that $\chi(\omega)$ is injective on $[0, L(\chi(\omega))] \setminus A_\omega$, where t_x^- and t_x^+ are elements of $\chi(\omega, \cdot)^{-1}(x)$.

Let us use an iterative process to construct such a set. Let us consider first the set $T_\omega^0 = [0, L(\chi(\omega))]$. If $\chi(\omega)$ is injective on T_ω^0, then the empty set is a set of injectivity. Otherwise, we consider one of the largest interval $[t_1^-, t_1^+[$ where t_1^- and t_1^+ are in $T_\omega^0 \cap \chi(\omega, \cdot)^{-1}(x)$ with x in X_ω. Such an interval exists since $[0, L(\chi(\omega))]$ is bounded. We then set $T_\omega^1 = T_\omega^0 \setminus [t_1^-, t_1^+[$. Continuing this process iteratively, we obtain a decreasing sequence of sets

$$T_\omega^n = T_\omega^{n-1} \setminus [t_n^-, t_n^+[,$$

where $t_n^-, t_n^+ \in T_\omega^{n-1} \cap \chi(\omega, \cdot)^{-1}(x)$ and $x \in X_\omega$. The process stops whenever $\cup_{k=1}^n [t_k^-, t_k^+[$ is a set of injectivity. If the process never ends, the set $\cup_{k=1}^\infty [t_k^-, t_k^+[$ is a set of injectivity. Indeed, let us assume that $s_1, s_2 \in [0, L(\omega)] \setminus \cup_k [t_k^-, t_k^+[$ are such that $\chi(\omega, s_1) = \chi(\omega, s_2)$. Then, by construction,

$$\infty > L(\chi(\omega)) \geq \sum_n |t_n^+ - t_n^-| \geq \sum_n |s_1 - s_2|,$$

thus $s_1 = s_2$. We shall denote by T_ω the set $[0, L(\omega)] \setminus \cup_k [t_k^-, t_k^+[$.

Step 2: Definition of the reparameterization. The set T_ω is a set of time parameters describing an injective subpath of $\chi(\omega)$. Let us consider the non-decreasing continuous function

$$S_\omega(u) = \int_0^u \mathbb{1}_{T_\omega}(s) ds$$

and let us define $\tau_\omega(t) := \inf\{u \in [0, \infty) : S_\omega(u) = t\}$. Then, $\tau_\omega(t)$ is such that $|T_\omega \cap [0; \tau_\omega(t)]| = t$.

Let us observe that the map $\tau_\omega(t)$ is measurable as a function of (ω, t). Let $\{t_m\}$ be a dense sequence in $[0, \infty)$. Following the proof of lemma 4.5.6, since $\tau_\omega(t)$ is non-decreasing, lower semicontinuous, and

$$\{\omega \in [0,1] : \tau_\omega(t_m) \leq \lambda\} = \{\omega \in [0,1] : S_\omega(\lambda) \geq t_m\}$$

it suffices to prove that the sets $\{\omega \in [0,1] : S_\omega(\lambda) \geq t_m\}$ are measurable for any $\lambda \geq 0$. For that, it is sufficient to prove that the sets

$$\begin{aligned} \mathcal{S} = \{\omega \in [0,1] : S_\omega(\lambda) \leq t_m\} &= \{\omega \in [0,1] : |T_\omega \cap [0, \lambda]| \leq t_m\} \\ &= \{\omega \in [0,1] : |T_\omega^c \cap [0, \lambda]| \geq \lambda - t_m\} \end{aligned}$$

are measurable for any $\lambda \geq 0$. Let

$$T_{\omega,p} = [0, L(\omega)] \setminus \cup_{\{k: t_k^+ - t_k^- \geq \frac{1}{p}\}} [t_k^-, t_k^+[$$

and observe that $\cap_p T_{\omega,p} = T_\omega$ Let us prove that for any $p \geq 1$, the set

$$\mathcal{S}_p := \{\omega \in [0,1] : |T^c_{\omega,p} \cap [0,\lambda]| \geq \lambda - t_m\}$$

is measurable. Recall that, since $\chi : [0,1] \to K$ is measurable, for each $j \in \mathbb{N}$, there is a compact set $B_j \subseteq [0,1]$ such that $\chi : B_j \to K$ is continuous [14]. Let us prove that for any $j \in \mathbb{N}$ the set

$$\mathcal{S}_{p,j} := \{\omega \in [0,1] : |T^c_{\omega,p} \cap [0,\lambda]| \geq \lambda - t_m\} \cap B_j$$

is closed, hence, a Borel set. Let $\omega_i \in \mathcal{S}_{p,j}$, $\omega_i \to \omega$. Then, for each of the curves $\chi(\omega_i)$, the sum of the lengths of the loops of length $\geq \frac{1}{p}$ is $\geq \lambda - t_m$. Letting $i \to \infty$, we deduce that the sum of the lengths of the loops of $\chi(\omega)$ of length $\geq \frac{1}{p}$ is also $\geq \lambda - t_m$. In other words, $\omega \in \mathcal{S}_{p,j}$. Since $\mathcal{S}_p = \cup_j \mathcal{S}_{p,j} \cup N$ where N is a null set, we deduce that $\mathcal{S}_p$ is a measurable set. Now, since $\cup_p T^c_{\omega,p} = T^c_\omega$, we have that

$$\begin{aligned} \{\omega \in [0,1] : |T^c_\omega \cap [0,\lambda]| \geq \lambda - t_m\} &= \{\omega \in [0,1] : \sup_p |T^c_{\omega,p} \cap [0,\lambda]| \geq \lambda - t_m\} \\ &= \cap_j \cup_k \{\omega \in [0,1] : |T^c_{\omega,k} \cap [0,\lambda]| \geq \lambda - t_m - \frac{1}{j}\}. \end{aligned}$$

Hence $\mathcal{S}$ is measurable. We conclude that $\tau_\omega(t)$ is measurable as a function of (ω, t).

We reparameterize the paths $\chi(\omega, s)$ by $\tilde{\chi}(\omega, t) := \chi(\omega, \tau_\omega(t))$. As in lemma 4.5.6, to prove that the application $\tilde{\chi}(\omega, t)$ is measurable it suffices to prove that $(I, \tau)^{-1}(N)$ is a null set for any null set $N \subseteq [0,1] \times [0, \infty)$. As in the proof of lemma 4.5.6, let B be a Borel set containing N (of total measure less than ϵ). Observe that $G(\omega, s) := \mathbb{1}_B(\omega, \tau_\omega(s))$ is a measurable map. Now, for a.e. fixed value of each $\omega \in [0,1]$, we have

$$\int_0^\infty G(\omega, s) ds = \int_0^\infty \mathbb{1}_B(\omega, u) S'_\omega(u) du \leq \int_0^\infty \mathbb{1}_B(\omega, u) du,$$

the last inequality being true since $S'_\omega(u) \leq 1$. Integrating with respect to $\omega \in [0,1]$, and observing that both G and $\mathbb{1}_B$ are measurable in $[0,1] \times [0, \infty)$, we have

$$|(I, \tau)^{-1}(B)| = \int_0^1 \int_0^\infty \mathbb{1}_B(\omega, \tau_\omega(s)) ds d\omega \leq \int_0^1 \int_0^\infty \mathbb{1}_B(\omega, u) du d\omega \leq \epsilon.$$

We deduce that $(I, \tau)^{-1}(N)$ is a null set. We conclude that $\tilde{\chi}$ is measurable. We can then define $\tilde{\mu} := \tilde{\chi} \# \lambda$.

Step 3: The traffic plan $\tilde{\mu}$ is with simple paths. Indeed, if there is an ω such that $\tilde{\chi}(\omega)$ is not injective, there are t_1 and t_2 such that $y = \tilde{\chi}(\omega, t_1) = \tilde{\chi}(\omega, t_2)$ with $t_1 \neq t_2$. Then, since τ_ω is increasing, $\tau_\omega(t_1) \neq \tau_\omega(t_2)$. Thus $\#\chi_\omega^{-1}(y) > 1$ so by definition of A_ω one of these two elements has to be in A_ω. But this is not possible since the image of τ_ω is disjoint from A_ω. Thus, $\tilde{\chi}$ is with simple paths. By definition of $\tilde{\chi}$, $\pi_{\tilde{\mu}} = \pi_\mu$ and $S_{\tilde{\mu}} \subset S_\mu$. □

4.6.1 A change of variable formula

Let μ be a traffic plan and χ a parameterization of μ. It will be called non-trivial if $L(\chi(\omega)) > 0$ on a set of positive measure in $\Omega := [0,1]$. Since we can eliminate the paths whose length is null, without loss of generality we shall assume that for non-trivial traffic plans we have $L(\chi(\omega)) > 0$ a.e.. First, we prove that the geometric embedding of a non-trivial traffic plan with finite energy can be covered by a countable set of paths. This permits us to compare our energy with the formulation given by Q. Xia [35, 36]. For a sake of simplicity, we shall denote in the sequel $[x]$ instead of $[x]_\chi$.

Lemma 4.6.4 *Let μ be a non-trivial traffic plan with finite energy and χ a parameterization of μ. There exists a sequence $(\omega_j)_j$ such that*

$$|[x]_\chi| = 0 \quad \mathcal{H}^1 - \text{a.e., for} \quad x \in \mathbb{R} \setminus \cup_{j=1}^\infty \mathrm{Im}\,\chi(\omega_j). \tag{4.5}$$

Proof: Let us first prove that we may cover the set

$$D := \{(\omega,t) \in \Omega \times [0,\infty) : 0 < t < L(\chi(\omega))\}$$

with a countable number of sets of the form $D_\omega = \{(\tilde\omega,t) \in D : \chi(\tilde\omega,t) \in \mathrm{Im}\chi(\omega)\}$. Since $E(\mu)$ is finite and χ is non-trivial, then for almost all $(\omega,t) \in D$, $|[\chi(\omega,t)]| > 0$. For each $\omega \in \Omega$, let

$$D^1_\omega := \{(\tilde\omega,t) : \chi(\tilde\omega,t) \in \mathrm{Im}\chi(\omega)\}.$$

Observe that

$$\begin{aligned}\int_\Omega |D^1_\omega|\,d\omega &= \int_\Omega |\{(\tilde\omega,t) : \chi(\tilde\omega,t) \in \mathrm{Im}\chi(\omega)\}|\,d\omega = \int_\Omega\int_0^\infty\int_\Omega \mathbb{1}_{\mathrm{Im}\chi(\omega)}(\chi(\tilde\omega,t))\,d\tilde\omega\,dt\,d\omega \\ &= \int_\Omega\int_0^\infty\int_\Omega \mathbb{1}_{\mathrm{Im}\chi(\omega)}(\chi(\tilde\omega,t))\,d\omega\,dt\,d\tilde\omega = \int_\Omega\int_0^\infty |[\chi(\tilde\omega,t)]|\,dt\,d\tilde\omega > 0.\end{aligned}$$

Hence $d^1 := \sup_\omega |D^1_\omega| > 0$. Let us choose $\omega_1 \in \Omega$ such that

$$|D^1_{\omega_1}| \geq \frac{d^1}{2} > 0.$$

Either D_{ω_1} covers all D, or

$$|D^1_{\omega_1}| < \int_\Omega\int_0^{L(\chi(\omega))} dt\,d\omega.$$

Proceeding iteratively in this way, and assuming that

$$\sum_{j=1}^{k-1} |D^j_{\omega_j}| < \int_\Omega\int_0^{L(\chi(\omega))} dt\,d\omega,$$

we define

$$D^k_\omega := \{(\tilde\omega,t) : \chi(\tilde\omega,t) \in \mathrm{Im}\,\chi(\omega) \setminus \cup_{j=1}^{k-1}\mathrm{Im}\,\chi(\omega_j)\}$$

and we may check that

$$\int_\Omega |D^k_\omega|\,d\omega = \int_{(\cup_{j=1}^{k-1} D^j_{\omega_j})^c} |[\chi(\tilde\omega,t)]|\,dt\,d\tilde\omega > 0,$$

which implies that $d^k := \max_\omega |D^k_\omega| > 0$. Then we choose $\omega_k \in \Omega$ such that

$$|D^k_{\omega_k}| \geq \frac{d^k}{2} > 0.$$

Either this construction ends in a finite number of steps k and we obtain that

$$\text{a.e. } \omega \in \Omega \quad \mathrm{Im}\,\chi(\omega) \subseteq \cup_{j=1}^k \mathrm{Im}\,\chi(\omega_j),$$

or we have an infinite number of sets $D^j_{\omega_j}$ and we have

$$\text{a.e. } \omega \in \Omega \quad \mathrm{Im}\,\chi(\omega) \subseteq \cup_{j=1}^\infty \mathrm{Im}\,\chi(\omega_j). \tag{4.6}$$

Indeed, if (4.6) does not hold then

$$\sum_{j=1}^\infty |D^j_{\omega_j}| < \int_\Omega \int_0^{L(\chi(\omega))} dt\, d\omega.$$

In particular, we have $d^j \leq 2|D^j_{\omega_j}| \to 0$ as $j \to \infty$, hence

$$\sup_{\omega\in\Omega} |D^j_\omega| \to 0 \quad \text{as } j \to \infty. \tag{4.7}$$

Since

$$\int_\Omega |D^j_\omega|\, d\omega = \int_{(\cup_{i=1}^{j-1} D^i_{\omega_i})^c} |[\chi(\tilde\omega, t)]|\, dt\, d\tilde\omega \geq \int\int_{(\cup_{i=1}^{\infty} D^i_{\omega_i})^c} |[\chi(\tilde\omega, t)]|\, dt\, d\tilde\omega > 0,$$

we obtain a contradiction since the left-hand side tends to 0 as $j \to \infty$ while the right-hand side is a positive constant. We have proved that $\cup_{j=1}^\infty D^j_{\omega_j}$ covers D (modulo a null set), and, therefore (4.6) holds.

To prove that (4.5) holds, assume on the contrary that there exists a set C such that $\mathcal{H}^1(C) > 0$,

$$C \cap (\cup_{i=1}^\infty \mathrm{Im}\,\chi(\omega_i)) = \emptyset, \tag{4.8}$$

and such that $|[x]| > 0$ for all $x \in C$. Then

$$\begin{aligned} 0 \;&<\; \int_C |[x]|\, d\mathcal{H}^1(x) = \int_C \int_\Omega \mathbb{1}_{[x]}(\omega)\, d\omega\, d\mathcal{H}^1(x) \\ &=\; \int_\Omega \int_C \mathbb{1}_{[x]}(\omega)\, d\mathcal{H}^1(x)\, d\omega = \int_\Omega \mathcal{H}^1(C \cap \mathrm{Im}\,\chi(\omega))\, d\omega. \end{aligned}$$

This implies that there exists a subset Ω_C of Ω such that $\mathcal{H}^1(C \cap \mathrm{Im}\,\chi(\omega)) > 0$ for any $\omega \in \Omega_C$, hence for any $\omega \in \Omega_C$ the set $I_\omega := \{t \in [0,\infty) : \chi(\omega, t) \in C\}$ is of positive measure. Since

$$\{(\omega, t) : \omega \in \Omega_C,\, t \in I_\omega\} \subseteq \{(\omega, t) : \chi(\omega, t) \in C\},$$

we conclude that $|\{(\omega, t) : \chi(\omega, t) \in C\}| > 0$. This contradicts (4.8). The lemma follows. □

Definition 4.6.5 *Let μ be a traffic plan and χ a parameterization of μ. For each $\omega \in \Omega$, we define*

$$\mathcal{D}^\chi(\omega) = \{x \in \mathbb{R}^N :\ x \textit{ is a double point of } \chi(\omega)\}.$$

We say that χ has simple paths if $\mathcal{H}^1(\mathcal{D}^\chi(\omega)) = 0$ for almost every $\omega \in \Omega$.

Assume that for a given $\omega \in \Omega$, $\chi(\omega)$ is parameterized by arc-length. Let

$$\mathcal{D}_\chi(\omega) = \{t \in [0, \infty) : \exists s < t, \ \chi(\omega, t) = \chi(\omega, s)\}.$$

Observe that $\mathcal{H}^1(\mathcal{D}^\chi(\omega)) = 0$ if and only if $|\mathcal{D}_\chi(\omega)| = 0$. Thus, if χ is normalized, χ has simple paths if and only if $|\mathcal{D}_\chi(\omega)| = 0$ for almost every $\omega \in \Omega$.

Our purpose is to prove the following change of variable formula. Notice that, in the case of a graph with the structure of a tree, the right-hand side of the identity (4.10) takes the form ($\sum_e w(e)^\alpha l(e)$), so that our framework generalizes [35].

Proposition 4.6.6 *Let χ be a parameterization of a nontrivial traffic plan μ with finite energy. Then, we have*

$$E(\mu) = \int_\Omega \int_0^\infty |[\chi(\omega, t)]|^{\alpha-1} |\dot\chi(\omega, t)| \, dt \, d\omega \geq \int_{\mathbb{R}^N} |[x]_\chi|^\alpha \, d\mathcal{H}^1(x). \tag{4.9}$$

If we assume, in addition, that χ has simple paths, we have

$$E(\mu) = \int_\Omega \int_0^\infty |[\chi(\omega, t)]|^{\alpha-1} |\dot\chi(\omega, t)| \, dt \, d\omega = \int_{\mathbb{R}^N} |[x]_\chi|^\alpha \, d\mathcal{H}^1(x). \tag{4.10}$$

Proof: Since the reparameterization $\tilde\chi$ of χ is measurable (lemma 4.5.6), and since $[x]_\chi = [x]_{\tilde\chi}$ for all $x \in \mathbb{R}^N$, we may assume that $|\dot\chi(\omega, t)| = 1$ for almost all $\omega \in \Omega$, a.e. $t \in [0, L(\chi(\omega))[$. Let us consider the sequence $(\omega_j)_j$ constructed in lemma 4.6.4. We denote by D the set

$$D := \{(\omega, t) \in \Omega \times [0, \infty) : 0 \leq t < L(\chi(\omega))\}.$$

Let us prove first that

$$\int_{D_{\omega_1}} |[\chi(\omega, t)]|^{\alpha-1} \, d\omega \, dt = \int_{\mathrm{Im}\, \chi(\omega_1)} |[x]|^\alpha \, d\mathcal{H}^1(x),$$

where D_{ω_1} is the set

$$D_{\omega_1} = \{(\tilde\omega, t) \in D : \chi(\tilde\omega, t) \in \mathrm{Im}\chi(\omega_1)\}.$$

Let us define

$$\Omega_{\omega_1} := \{\omega \in \Omega : \mathrm{Im}\, \chi(\omega) \cap \mathrm{Im}\, \chi(\omega_1) \neq \emptyset\},$$

$$I_\omega = \{t < L(\chi(\omega)) : \chi(\omega, t) \in \mathrm{Im}\, \chi(\omega_1)\},$$

and

$$I'_\omega := \{t \in \mathbb{R}^+ \setminus \mathcal{D}_\chi(\omega) : \chi(\omega, t) \in \mathrm{Im}\, \chi(\omega_1)\}.$$

Notice that

$$D_{\omega_1} = \cup_\omega \{\omega\} \times I_\omega.$$

Let t be in I'_ω. Since $\chi(\omega, t) \in \mathrm{Im}\, \chi(\omega_1)$ and because of the definition of $\mathcal{D}_\chi(\omega_1)$, there is a unique $s = \varphi(t) \in \mathbb{R}^+ \setminus \mathcal{D}_\chi(\omega_1)$ such that $\chi(\omega_1, s) = \chi(\omega, t)$. Let I^*_ω be the set

$$I^*_\omega = \varphi(I'_\omega) = \{s \in \mathbb{R}^+ \setminus \mathcal{D}_\chi(\omega_1) : \chi(\omega_1, s) \in \mathrm{Im}\, \chi(\omega)\}.$$

Then I^*_ω is a Borel set of the same one-dimensional Lebesgue measure as I'_ω. As in the proof of lemma 4.6.3, to prove the measurability of the set

$$Q = \{(\omega, s) : \omega \in \Omega_{\omega_1}, \ \chi(\omega_1, s) \in \mathrm{Im}\, \chi(\omega)\},$$

we recall that for each $\epsilon > 0$, there is a compact set $B_\epsilon \subseteq [0,1]$ such that $\chi : B_\epsilon \to K$ is continuous [14]. Now, one can easily check that $Q \cap B_\epsilon$ is a closed set. We deduce that Q is measurable. Since

$$\{(\omega,s) : \omega \in \Omega_{\omega_1},\, \chi(\omega_1,s) \in \operatorname{Im}\chi(\omega) \setminus \mathcal{D}^\chi(\omega_1)\} = Q \cap \{(\omega,s) : \omega \in \Omega_{\omega_1}, s \notin \mathcal{D}_\chi(\omega_1)\}$$

we deduce that the set

$$\{(\omega,s) : \omega \in \Omega_{\omega_1},\, \chi(\omega_1,s) \in \operatorname{Im}\chi(\omega) \setminus \mathcal{D}^\chi(\omega_1)\}$$

is measurable. Finally observe that $\mathbb{1}_{I^*_\omega}(s) = 1$ if and only if $\omega \in [\chi(\omega_1,s)]$ and $s \notin \mathcal{D}_\chi(\omega_1)$. Thus, we have

$$\int_{\Omega_{\omega_1}} \mathbb{1}_{I^*_\omega}(s)\, d\omega = |[\chi(\omega_1,s)]| \mathbb{1}_{\mathbb{R}^+ \setminus \mathcal{D}_\chi(\omega_1)}.$$

Then, we have

$$\begin{aligned}
\int_{D_{\omega_1}} |[\chi(\omega,t)]|^{\alpha-1}\, d\omega\, dt &= \int_{\Omega_{\omega_1}} \int_{I_\omega} |[\chi(\omega,t)]|^{\alpha-1}\, dt\, d\omega \\
&\geq \int_{\Omega_{\omega_1}} \int_{I'_\omega} |[\chi(\omega,t)]|^{\alpha-1}\, dt\, d\omega \\
&= \int_{\Omega_{\omega_1}} \int_{I^*_\omega} |[\chi(\omega_1,s)]|^{\alpha-1}\, ds\, d\omega \\
&= \int_{\Omega_{\omega_1}} \int_0^\infty \mathbb{1}_{I^*_\omega}(s) |[\chi(\omega_1,s)]|^{\alpha-1}\, ds\, d\omega \\
&= \int_0^\infty \int_{\Omega_{\omega_1}} \mathbb{1}_{I^*_\omega}(s) |[\chi(\omega_1,s)]|^{\alpha-1}\, d\omega\, ds \\
&= \int_0^\infty |[\chi(\omega_1,s)]|^{\alpha-1} \int_{\Omega_{\omega_1}} \mathbb{1}_{I^*_\omega}(s)\, d\omega\, ds \\
&= \int_{[0,\infty)\setminus\mathcal{D}_\chi(\omega_1)} |[\chi(\omega_1,s)]|^{\alpha}\, ds = \int_{\operatorname{Im}\chi(\omega_1)} |[x]|^\alpha\, d\mathcal{H}^1(x).
\end{aligned}$$

Notice that in the case μ has simple paths, modulo a null set we have the identity

$$I_\omega = I'_\omega.$$

This proves that for a traffic plan with simple paths,

$$\int_{D_{\omega_1}} |[\chi(\omega,t)]|^{\alpha-1}\, d\omega\, dt = \int_{\operatorname{Im}\chi(\omega_1)} |[x]|^\alpha\, d\mathcal{H}^1(x).$$

We may reproduce iteratively the same argument for the arcs forming $\operatorname{Im}\chi(\omega_k) \setminus \cup_{j=1}^{k-1} \operatorname{Im}\chi(\omega_j)$ to obtain

$$\int_{\cup_{j=1}^k D^j_{\omega_j}} |[\chi(\omega,t)]|^{\alpha-1}\, d\omega\, dt \geq \int_{\cup_{j=1}^k \operatorname{Im}\chi(\omega_j)} |[x]|^\alpha\, d\mathcal{H}^1(x).$$

Notice that there is equality in the case μ has simple paths. Letting $k \to \infty$, and using that $\cup_{j=1}^\infty D^j_{\omega_j}$ is a covering (modulo a null set) of

$$D = \{(\omega,t) \in \Omega \times [0,\infty) : 0 \leq t < L(\chi(\omega))\},$$

we obtain

$$\int_\Omega \int_0^{L(\chi(\omega))} |[\chi(\omega,t)]|^{\alpha-1}\, dt\, d\omega \geq \int_{\cup_{j=1}^\infty \mathrm{Im}\,\chi(\omega_j)} |[x]|^\alpha \, d\mathcal{H}^1(x),$$

and

$$\int_\Omega \int_0^{L(\chi(\omega))} |[\chi(\omega,t)]|^{\alpha-1}\, dt\, d\omega = \int_{\cup_{j=1}^\infty \mathrm{Im}\,\chi(\omega_j)} |[x]|^\alpha \, d\mathcal{H}^1(x)$$

if μ has simple paths. The proposition follows by using lemma 4.6.4. □

Let us denote

$$E_x(\mu) = \int_{\mathbb{R}^N} |[x]_\mu|^\alpha \, d\mathcal{H}^1(x)$$

Proposition 4.6.7 *The minimum of E on the set of traffic plans is attained at a traffic plan with simple paths. Moreover $\inf E = \inf E_x$ where both infima can be taken with respect to the set of all traffic plans or the set of traffic plans with simple paths.*

Proof: We observe that if μ is a traffic plan and $\tilde{\mu}$ its associated traffic plan with simple paths constructed in Proposition 4.6.3, we have $E(\tilde{\mu}) \leq E(\mu)$. To prove it, we observe that eliminating loops can only decrease the multiplicity, hence $E_x(\mu) \geq E_x(\tilde{\mu})$. Now, by Proposition 4.6.6, we have

$$E(\mu) \geq E_x(\mu) \geq E_x(\tilde{\mu}) = E(\tilde{\mu}).$$

Our assertions are a simple consequence of Proposition 4.6.3 and this inequality. □

Chapter 5

Irrigation at finite cost, stability.

Introduction

We proved in the previous chapter the existence of a traffic plan μ minimizing $E^\alpha(\mu)$ where $\mu \in TP(\mu^+, \mu^-)$. We prove in that section that for $\alpha > 1 - \frac{1}{N}$ where N is the dimension of the ambient space, the optimal cost is finite. To do this we introduce the pseudo-distance d^α (the pseudo is here to stress that d^α is not always finite) between measures of $\mathbb{R}^N$ and construct a chain of traffic plans transporting μ_i to μ_{i+1} where the sequence μ_i is the sequence of dyadic approximation of μ^-. The d^α pseudo-distance between μ_i and μ_{i+1} is easy to estimate so that we get a bound on $E^\alpha(\mu)$ where μ is the concatenation of traffic plans obtained by transporting μ_i to μ_{i+1} when $i \in \mathbb{N}$. As a consequence, this bound is also a bound on the cost of an optimal structure. The d^α pseudo-distance allows also to look closer at the stability problem. Indeed, we prove in lemma 5.3.2 that $d^\alpha(\nu_n, \nu) \to 0$ when ν_n is a sequence of probability measures on the compact $X \subset \mathbb{R}^N$ weakly converging to ν. Finally we investigate the existence of structures at finite cost adding a constraint on the angle variation. The question we ask is: can we irrigate a measure with a support of positive measure in such a way that the total variation of the angles along fibers is bounded.

5.1 Preliminaries

5.1.1 Concatenation of a chain of traffic plans

Lemma 5.1.1 *Let $\mu \in TP(\mu^+, \mu^-)$ and $\nu \in TP(\nu^+, \nu^-)$ such that $\mu^- = \nu^+$. There is a traffic plan $\sigma \in TP(\mu^+, \nu^-)$ such that $E^\alpha(\sigma) \le E^\alpha(\mu) + E^\alpha(\nu)$.*

Proof : Let χ and ξ be parameterizations of μ and ν. Let us denote $f(\omega) := \chi(\omega, \infty)$ and $g(\omega) := \xi(\omega, 0)$. By definition, $\mu^- = \nu^+$ means that $f\#\lambda = g\#\lambda$. Thus, there is a measure preserving application ψ such that $f(\omega) = g(\psi(\omega))$ for almost all ω. The gluing of the fiber $\chi(\omega)$ with the fiber $\xi(\psi(\omega))$ is thus well defined and we denote $\tilde{\chi}$ the parameterization

$$\tilde{\chi}(\omega, t) = \begin{cases} \chi(\omega, t) & \text{if } t \le T_\chi(\omega), \text{ where } T_\chi(\omega) \text{ is the stopping time of the fiber } \omega \\ \xi(\psi(\omega), t - T_\chi(\omega)) & \text{if } t > T_\chi(\omega). \end{cases}$$

The traffic plan $\sigma := \tilde{\chi}\#\lambda$ is such that $|[x]_\sigma| \leq |[x]_\chi| + |[x]_{\tilde{\xi}}|$. Thus, we have

$$\begin{aligned} E^\alpha(\sigma) &= \int_{x\in\mathbb{R}^N} |[x]_{\tilde{\chi}}|^\alpha d\mathcal{H}^1(x) \\ &\leq \int_{x\in\mathbb{R}^N} (|[x]_\chi| + |[x]_{\tilde{\xi}}|)^\alpha d\mathcal{H}^1(x) \\ &\leq \int_{x\in\mathbb{R}^N} (|[x]_\chi|^\alpha + |[x]_{\tilde{\xi}}|^\alpha) d\mathcal{H}^1(x) \\ &= E^\alpha(\chi) + E^\alpha(\xi). \end{aligned}$$

□

5.1.2 The d^α pseudo-distance

Definition 5.1.2 *Let μ^+ and μ^- be two probability measures. We denote*

$$d^\alpha(\mu^+, \mu^-) = \inf_{\mu \in TP(\mu^+, \mu-)} E^\alpha(\mu).$$

Lemma 5.1.3 *Let us denote W_1 the Wasserstein distance of order 1 and let μ^+ and μ^- be two probability measures. We have $W_1(\mu^+, \mu^-) \leq d^\alpha(\mu^+, \mu^-)$ for all $\alpha \in [0, 1]$.*

Proof : Indeed,

$$d^\alpha(\mu^+, \mu^-) := \inf \int_\Omega \int_t |[\chi(\omega, t)]_\chi|^{\alpha-1} |\dot{\chi}(\omega, t)| d\omega dt,$$

where the infimum is taken over all parameterizations transporting μ^+ to μ^-. Thus,

$$d^1(\mu^+, \mu^-) := \inf \int_\Omega \int_t |\dot{\chi}(\omega, t)| d\omega dt,$$

is precisely $W_1(\mu^+, \mu^-)$ and the inequality obviously comes from $|[\chi(\omega, t)]_\chi|^{\alpha-1} \geq 1$. □

Proposition 5.1.4 *d^α is a pseudo-distance on the space of probability measures on X.*

Proof : Because of lemma 5.1.3, we have $d^\alpha(\nu_1, \nu_2) = 0$ if and only if $\nu_1 = \nu_2$. Next, the triangular inequality is easily proved as follow : let μ and ν be optimal traffic plans respectively from a to b and from b to c. By definition of d^α, we have

$$d^\alpha(a, c) \leq E^\alpha(\sigma),$$

where σ is the concatenation defined by lemma 5.1.1. Thus

$$d^\alpha(a, c) \leq E^\alpha(\mu) + E^\alpha(\nu) = d^\alpha(a, b) + d^\alpha(b, c).$$

□

5.1.3 Dyadic approximation of a measure

Let C be a cube with edge length d and center c. Let ν be a probability measure on the compact X where $X \subset C$. We may approximate ν by atomic measures in $\mathcal{A}_\Lambda(X)$ as follow. For each i, let

$$\mathcal{C}_i := \{C_i^h : h \in \mathbb{Z}^N \cap [0, 2^i)^N\}$$

be a partition of C into cubes of edge length $\frac{d}{2^i}$. Now, for each $h \in \mathbb{Z}^N \cap [0, 2^i)^N$, let c_i^h be the center of C_i^h and $m_i^h = \nu(C_i^h)$ be the μ mass of the cube C_i^h. We define the atomic measure

$$A_i(\nu) = \sum_{h \in \mathbb{Z}^N \cap [0,2^i)^N} m_i^h \delta_{c_i^h},$$

which is classically weakly converging to μ.

Lemma 5.1.5 *The atomic measure $A_i(\nu)$ weakly converges to ν. We call $A_i(\nu)$ the dyadic approximation of ν.*

5.2 Existence of a finite cost traffic plan

Lemma 5.2.1 *The maximum of $f : (x_1, ..., x_n) \mapsto \sum x_i^\alpha$ under the constraint $\sum x_i = 1$ is $n^{1-\alpha}$.*

Proof : Because of the concavity of $x \mapsto x^\alpha$, we have $\frac{1}{n} \sum x_i^\alpha \leq (\frac{\sum x_i}{n})^\alpha$. Thus the maximum of f is lower than $n(\frac{1}{n})^\alpha$. This value is attained for $x_i = \frac{1}{n}$ for all i. □

Lemma 5.2.2 *Let ν be a probability measure on a compact set X, where X is include in a cube C of edge length L. Then,*

$$d^\alpha(A_i(\nu), A_{i+1}(\nu)) \leq \frac{\sqrt{N}L}{2} 2^{i(N(1-\alpha)-1)}.$$

Proof : The atomic measure $A_i(\nu)$ is made of 2^{iN} Dirac masses at the centers of the cubes C_i^h. We consider the traffic plan μ obtained as the sum of μ_h, where μ_h is a traffic plan transporting $m_i^h \delta_{c_i^h}$ on $A_{i+1}(\nu)|_{C_i^h}$, for all of the 2^{iN} values of h. Let us denote $A_{i+1}(\nu)|_{C_i^h} = \sum_{k=1}^{2^N} m_k \delta_{x_k}$, where $\sum_{k=1}^{2^N} m_k = m_i^h$ by definition of $A_i(\nu)$. We choose μ_n as being the Monge-Kantorovitch transport i.e the traffic plan made of weighted directed segments $(c_i^h x_k, m_k)$, as illustrated on Figure 5.1. The cost of μ_h is such that

$$\begin{aligned} E^\alpha(\mu_h) &= \sum_k (m_k)^\alpha |c_i^h x_k| \\ &\leq \sum_k (m_k)^\alpha \frac{\sqrt{N}L}{2^{i+1}} \end{aligned}$$

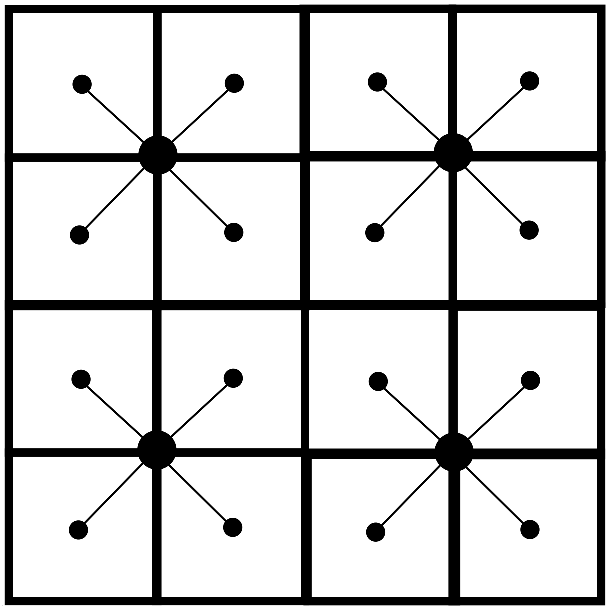

Figure 5.1: To transport $A_i(\nu)$ to $A_{i+1}(\nu)$, we simply transport straightforward all the mass at the center of a cube with edge length $\frac{L}{2^i}$ to the centers of its subcubes with edge length $\frac{L}{2^{i+1}}$.

Thus,

$$\begin{aligned} E^\alpha(\mu) &= \sum_h E^\alpha(\mu_h) \\ &\leq \sum_h \sum_k (m_k)^\alpha \frac{\sqrt{N}L}{2^{i+1}} \\ &\leq \frac{\sqrt{N}L}{2^{i+1}} 2^{iN} (\frac{1}{2^{iN}})^\alpha \quad \text{because of lemma 5.2.1} \\ &\leq \frac{\sqrt{N}L}{2} 2^{i(N(1-\alpha)-1)} \end{aligned}$$

□

Proposition 5.2.3 *Let $\alpha \in (1 - \frac{1}{N}, 1]$. Let ν be a probability measure with support in a cube centered at c and of edge length L. We have*

$$d^\alpha(A_n(\nu), \nu) \leq \frac{2^{n(N(1-\alpha)-1)}}{2^{1-N(1-\alpha)} - 1} \frac{\sqrt{N}L}{2}.$$

In particular, $d^\alpha(A_n(\nu), \nu) \to 0$ uniformly for all ν when $n \to \infty$

Proof : Let $A_i(\nu)$ be the dyadic approximation of ν. Lemma 5.2.2 combined with lemma 5.1.1 permits to iteratively construct a traffic plan μ_i from $A_n(\nu)$ to $A_i(\nu)$ with $i > n$. By construction this traffic plan converges to a traffic plan μ such that

$$E^\alpha(\mu) \leq \sum_{j=n}^{\infty} d^\alpha(A_j(\nu), A_{j+1}(\nu)).$$

Since μ is irrigating the measure ν, we have

$$\begin{aligned} d^\alpha(A_n(\nu),\nu) &\leq E^\alpha(\mu) \\ &\leq \sum_{j=n}^{\infty} d^\alpha(A_j(\nu),A_{j+1}(\nu)) \\ &\leq \frac{\sqrt{N}L}{2}\sum_{j=n}^{\infty} 2^{j(N(1-\alpha)-1)} \\ &= \frac{2^{n(N(1-\alpha)-1)}}{2^{1-N(1-\alpha)}-1}\frac{\sqrt{N}L}{2} \quad \text{since } \alpha > 1-\frac{1}{N}. \end{aligned}$$

Thus $d^\alpha(A_n(\nu),\nu) \to 0$ uniformly for all ν when $n \to \infty$. □

Since $A_0(\nu) = \delta_c$, we obtain directly from the previous proposition applied with $n = 0$, the following uniform bound on the energy required to irrigate a measure.

Corollary 5.2.4 *Let $\alpha \in (1-\frac{1}{N},1]$ and $\nu \in \mathcal{M}_1(X)$, where X is of diameter L. There exists $\mu \in TP(\delta_c,\nu)$ such that*

$$E(\mu) \leq \frac{1}{2^{1-N(1-\alpha)}-1}\frac{\sqrt{N}L}{2}.$$

Remark 5.2.5 *In the case we transport a measure with mass Λ, the uniform bound obtained in corollary 5.2.4 scales as Λ^α and we have*

$$d^\alpha(\delta_c,\nu) \leq \frac{1}{2^{1-N(1-\alpha)}-1}\frac{\sqrt{N}L}{2}\Lambda^\alpha.$$

Finally, combining a transport from μ^+ to δ_c with a transport from δ_c to μ^-, it is possible to obtain any transference plan, so that the who goes where problem has a solution at finite cost in the case $\alpha > 1-\frac{1}{N}$.

Corollary 5.2.6 *Let $\alpha \in (1-\frac{1}{N},1]$. Let μ^+ and μ^- in $\mathcal{M}_1(X)$ and π a prescribed transference plan with marginals μ^+ and μ^-. There exists $\mu \in TP(\pi)$ such that*

$$E(\mu) \leq \frac{1}{2^{1-N(1-\alpha)}-1}\sqrt{N}L.$$

Proof : Indeed, we can find a traffic plan μ transporting μ^+ to δ_c and a traffic plan ν transporting δ_c to μ^- with costs $E^\alpha(\mu)$ and $E^\alpha(\nu)$ inferior to $\frac{1}{2^{1-N(1-\alpha)}-1}\frac{\sqrt{N}L}{2}$. Since all fibers of μ terminates at c, it is possible to glue fibers of μ with fibers of ν so that we obtain a traffic plan $\tilde{\mu}$ with a transference plan $\pi_{\tilde{\mu}}$ that can be any transference plan with marginals μ^+ and μ^-. Since $|[x]_{\tilde{\mu}}| \leq |[x]_\mu| + |[x]_\nu|$, we have

$$E^\alpha(\tilde{\mu}) \leq E^\alpha(\mu) + E^\alpha(\nu) \leq \frac{1}{2^{1-N(1-\alpha)}-1}\sqrt{N}L.$$

□

Remark 5.2.7 *In the case $\alpha \in (1-\frac{1}{N},1]$, it is not clear wether or not d^α and W_1 are equivalent distances, i.e. does there exist a constant C depending on α and N such that $d^\alpha \leq W_1$. An answer to this question raised by Cedric Villani would make clearer the relation between Monge-Kantorovitch and the irrigation problem.*

Remark 5.2.8 *The work of De Villanova and Solimini [29] refine widely the result of corollary 5.2.4. They call* **irrigable for the exponent** α *a probability measure* ν *such that there exists a traffic plan* $\mu \in TP(\delta_S, \nu)$ *with finite energy* $E^\alpha(\mu) < \infty$*. The article [29] then gives precise condition for a measure to be irrigable. In particular let us mention*

Theorem 5.2.9 *If* ν *is irrigable for the exponent* α*, then* ν *is concentrated on a* $\frac{1}{1-\alpha}$ *negligible set (in the sense of Hausdorff measure).*

5.3 Stability results

In this section we partially answer to the stability question, i.e. "is the limit of a sequence of optimal traffic plans optimal?". The property of the d^α pseudo-distance in the case $\alpha \in (1 - \frac{1}{N}, 1]$ permits to answer by a yes as stated by corollary 5.3.3. However, in the case $\alpha \leq 1 - \frac{1}{N}$ this stability is conjectural.

Lemma 5.3.1 *Let* $\alpha \in (1 - \frac{1}{N}, 1]$*. If* ν_n *is a sequence of probability measures on the compact* $X \subset \mathbb{R}^N$ *weakly converging to* ν*, then* $d^\alpha(A_k(\nu_n), A_k(\nu)) \to 0$ *when* $n \to \infty$*.*

Proof : The weak convergence of ν_n to ν applied to characteristic functions of the cubes C_k^h implies that $m_k^h(\nu_n) \to m_k^h(\nu)$ when $n \to \infty$, where $m_k^h(\nu)$ is the mass of ν contained in the cube C_k^h. Thus, for any $\epsilon > 0$, for n large enough we have

$$\sum_h |m_k^h(\nu_n) - m_k^h(\nu)| < \epsilon.$$

Let us denote $a^h := \min(m_k^h(\nu_n), m_k^h(\nu))$ and $\gamma_h(t) = c_k^h$ for all $t \in \mathbb{R}$. Let us consider the traffic plan

$$\mu = \sum_h a^h \delta_{\gamma_h} + \tilde{\mu},$$

where $\tilde{\mu}$ transports $\sum_h (m_k^h(\nu_n) - a_n)\delta_{c_k^h}$ on $\sum_h (m_k^h(\nu) - a_n)\delta_{c_k^h}$. Notice that the first term of μ consists of a "still" transport, i.e. the irrigating mass that is already at a position to be irrigated does not move. The total mass of $\sum_h (m_k^h(\nu_n) - a_n)\delta_{c_k^h}$ is such that

$$\sum_h (m_k^h(\nu_n) - a_n) \leq \sum_h |m_k^h(\nu_n) - m_k^h(\nu)| \leq \epsilon.$$

Thus, corollary 5.2.4 asserts that $\tilde{\mu}$ can be chosen with a cost inferior to $C\epsilon^\alpha L$ where L is the diameter of X and C a constant depending on N and α. The "still" component of μ does not contribute to its cost, so that we have $E^\alpha(\mu) \leq C\epsilon^\alpha L$. Thus, for any $\epsilon > 0$, for n large enough, we have $d^\alpha(A_k(\nu_n), A_k(\nu)) \leq C\epsilon^\alpha L$. □

Lemma 5.3.2 *Let* $\alpha \in (1 - \frac{1}{N}, 1]$*. If* ν_n *is a sequence of probability measures on the compact* $X \subset \mathbb{R}^N$ *weakly converging to* ν*, then* $d^\alpha(\nu_n, \nu) \to 0$ *when* $n \to \infty$*.*

Proof : Let us fix $\epsilon > 0$. Proposition 5.2.3 applied to ν_n and ν asserts that for k large enough, $d^\alpha(A_k(\nu_n), \nu_n) < \epsilon$ for all n and $d^\alpha(A_k(\nu), \nu) < \epsilon$. Thus

$$\begin{aligned} d^\alpha(\nu_n, \nu) &\leq d^\alpha(\nu_n, A_k(\nu_n)) + d^\alpha(A_k(\nu_n), A_k(\nu)) + d^\alpha(A_k(\nu), \nu) \\ &\leq 2\epsilon + d^\alpha(A_k(\nu_n), A_k(\nu)). \end{aligned}$$

Since ν_n weakly converges to ν, lemma 5.3.1 asserts that for n large enough, $d^\alpha(A_k(\nu_n), A_k(\nu)) < \epsilon$. Thus, for n large enough, $d^\alpha(\nu_n, \nu) < 3\epsilon$ and the result follows. □

Corollary 5.3.3 *Let $\alpha \in (1-\frac{1}{N},1]$. If μ_n is a sequence of optimal traffic plans for the irrigation problem and μ_n is converging to μ, then μ is optimal.*

Proof : Because of the lower semicontinuity of E^α, we have

$$\begin{aligned} E^\alpha(\mu) &\leq \liminf E^\alpha(\mu_n) = \liminf d^\alpha(\mu_n^+,\mu_n^-) \\ &\leq \liminf d^\alpha(\mu_n^+,\mu^+) + d^\alpha(\mu^+,\mu^-) + d^\alpha(\mu^-,\mu_n^-) \\ &\leq d^\alpha(\mu^+,\mu^-) \text{ since } \mu_n^+ \to \mu^+ \text{ and } \mu_n^+ \to \mu^+. \end{aligned}$$

Thus, μ is optimal. □

Remark 5.3.4 *In the case $\alpha < 1-\frac{1}{N}$, the stability of optimal traffic plans remains an open question. Of course, only the case when μ_n is a sequence of optimal traffic plans with $E^\alpha(\mu_n) < \infty$ is of interest. Is a limit of μ_n still optimal? The stability in the case of the who goes where problem is also an open problem.*

5.4 The topology induced by the d^α pseudo-distance

Proposition 5.4.1 *If $\alpha \in (1-\frac{1}{N},1]$, d^α metrizes the weak * topology of probability measures $\mathcal{M}_1(X)$.*

Proof : Indeed, lemma 5.3.2 asserts that if ν_n weakly converges to ν then $d^\alpha(\nu_n,\nu) \to 0$. Conversely, if $d^\alpha(\nu_n,\nu) \to 0$, then lemma 5.1.3 asserts that $W_1(\nu_n,\nu) \to 0$, so that ν_n weakly converges to ν. □

Remark 5.4.2 *If $\alpha \leq 1-\frac{1}{N}$, then it is no longer true that ν_n weakly converges to ν implies $d^\alpha(\nu_n,\nu) \to 0$. Indeed, let us consider $\nu_n := \frac{1}{v_n}\mathbb{1}_{B(0,\frac{1}{n})}$, where v_n is the volume of a ball with radius $\frac{1}{n}$. Indeed, we have $\nu_n \rightharpoonup \delta_0$, but due to theorem 5.2.9, $d^\alpha(\nu_n,\delta_0) = \infty$ in the case $\alpha \leq 1-\frac{1}{N}$.*

5.5 The total variation of the angle question

Real irrigating systems (blood vessels, pipe networks) seem to avoid big variation of angles since it can cause turbulence and pressure drop [11]. At the same time, these systems manage to irrigate many points: the whole human body in the case of the blood system or many users in a city. A natural question is then the following: are there traffic plans both at a time irrigating a set with positive measure and such that the angle variation along fibers is bounded?

Proposition 5.5.1 *Let $\alpha \in]1-\frac{1}{N-1},1]$. There is a traffic plan of finite cost in $\mathbb{R}^N$ irrigating a measure which support is of codimension 1 and such that the total angle variation along fibers is bounded.*

Proof: Let us first describe the case of dimension 2.

For a sake of convenience, we shall not define a parameterization of the traffic plan but rather define the underlying infinite directed weighed graph.

We denote by "level of an edge e", the number of edges from the source to e. In this tree, all edges of a same level will have the same length. For the angle variation to be as readable as possible, we shall consider a tree made of vertical edges of length d_i at all even level $2i$. Let us denote l_i the length of edges of level $2i+1$. Let us denote α_i, the angle of a level $2i+1$ branch with the vertical. We choose

l_i such that the vertical projection of an edge of level $2i+1$ is of length $\frac{1}{2^{i+1}}$. By definition we have $\sin(\alpha_i)l_i = \frac{1}{2^{i+1}}$. Since levels $2i$ and $2i+1$ are made of 2^i segments with weights $\frac{1}{2^i}$, the total cost of this tree is $\sum_i (l_i + d_i)2^{i(1-\alpha)}$. Thus, the total cost and angle variation of fibers is finite if and only if the series $\sum_i (l_i + d_i)2^{i(1-\alpha)}$ and $\sum_i \alpha_i$ are convergent. Let us take $d_i = l_i$. We notice that when $l_i 2^{i+1} \to +\infty$,

$$\alpha_i = \arcsin(\frac{1}{l_i 2^{i+1}}) \sim \frac{1}{l_i 2^{i+1}}.$$

Thus, the convergence of both series is equivalent to the convergence of $\sum_i l_i 2^{i(1-\alpha)}$ and $\sum_i \frac{1}{l_i 2^i}$. Numerous choices can fit these requirements, for instance $l_i = 2^{-\beta i}$ makes the series convergent if and only if $\beta < 1$ and $\beta > 1 - \alpha$.

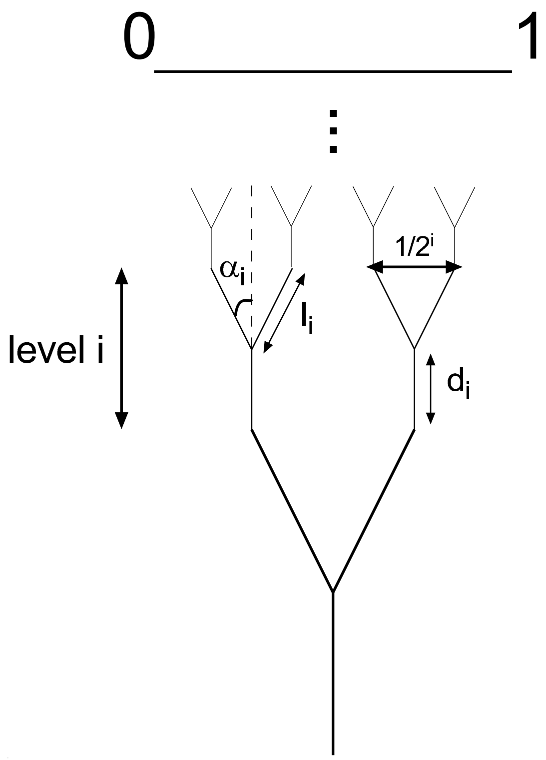

Figure 5.2: Finite cost traffic plan irrigating the Lebesgue segment and such that the total angle variation along fibers is bounded.

Let us generalize this example to any dimension. The main feature of the tree in 2 dimensions is that the graph made of the $2i$ first levels irrigates the dyadic approximation of the Lebesgue measure of the segment. Now we shall consider a tree in dimension N, such that the graph stopped at level $2i$ irrigates the dyadic approximation of the Lebesgue measure on the hypercube of dimension $N-1$. Let us describe the nodes of this tree: at level $2i$, the nodes are located at dyadic coordinates on a hybercube of dimension $N-1$ which lies in the plane $z_N = h_N$. To describe the positions of the nodes at level i, it is convenient to enumerate all 2^{N-1} subcubes of an hypercube by $(v_i)_{i=1}^{2^{N-1}}$, where v_i describes all the elements $(\pm1, ..., \pm1) \in \mathbb{R}^{N-1}$ (see figure 5.3). Indeed, a sequence $k_i \in [1, 2^{N-1}]$ of elements of type v_i can describe all dyadic nodes in this way: k_1 codes for the fact that the node is in the cube subcube C_{k_1} of C defined by v_{k_1}, k_2 says that the node is in the v_{k_2} subcube of C_{k_1}... Roughly speaking, the sequence k_i tells which direction to take at each bifurcation, i.e the node at level n described by the sequence (k_i) has coordinates $\sum_{i=1}^{n} \frac{\sqrt{N-1}v_{k_i}}{2^i} + (0, ..., 0, h_n)$, where h_n stands for the height of the hyperplane containing nodes of order n. As in the 2 dimension example,

$$\alpha_i = \arcsin(\frac{\sqrt{N-1}}{l_i 2^{i+1}}) \sim \frac{\sqrt{N-1}}{l_i 2^{i+1}}.$$

The total cost is $\sum_i (l_i + d_i) 2^{(N-1)i(1-\alpha)}$. If we take $d_i = l_i$, the traffic plan has finite energy and bounded total angle variation if and only if the two series $\sum_i l_i 2^{(N-1)i(1-\alpha)}$ and $\sum_i \frac{1}{l_i 2^i}$ are convergent. This is possible for the choice $l_i = 2^{-\beta i}$ with $\beta \in](1-\alpha)(N-1), 1]$ (this interval is not empty if $\alpha \in]1 - \frac{1}{N}, 1]$). Notice that this traffic plan irrigates the Lebesgue measure on $[-1,1]^{N-1} \times \{h\}$, where $h = \lim h_i$. □

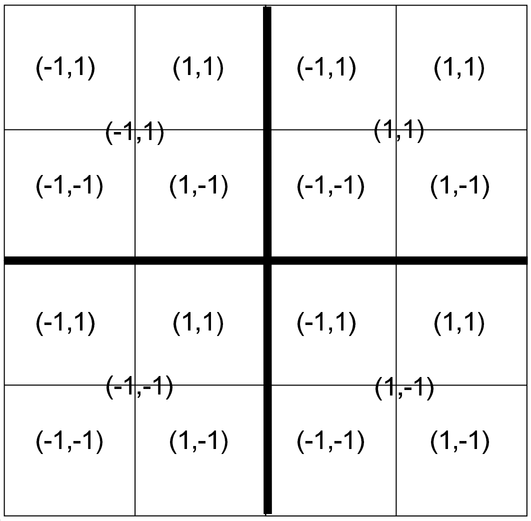

Figure 5.3: A sequence of n elements of the form $(\pm 1, ..., \pm 1)$ permits to describe all the $2^{n(N-1)}$ subcubes at level n. For instance the element $(-1,-1)$ means west-south and the sequence of elements of type $(\pm 1, \pm 1)$ permits to describe iteratively along a finer and finer mesh, all the dyadic subcubes.

Proposition 5.5.2 *Let $\alpha \in]1 - \frac{1}{N}, 1]$. There is a traffic plan in $\mathbb{R}^N$ with finite cost, transporting a Dirac mass to Lebesgue measure on a parallelepiped, and such that the total angle variation of fibers is finite.*

Proof: Indeed, such a traffic plan is obtained through a suitable projection of the traffic plan obtained in proposition 5.5.1 for the dimension $N+1$. The projection has to be such that the total angle variation is not increased. This can be done with a direction of projection having an angle with vertical direction superior to the maximal angle variation between two adjacent edges. □

Remark 5.5.3 *Notice that in general, the projection of the tree will be such that the projected structure has intersecting edges, so that it has not a "tree" structure. This raises a natural question: is it possible for a traffic plan to irrigate the Lebesgue measure on a set with positive measure, to have finite total angle variation and so that the traffic plan has a tree structure? It is obvious that it is impossible to project the 3-D tree of proposition 5.5.1 in $\mathbb{R}^2$, so that edges do not intersect. This leads to state the following conjecture. On the contrary, as proved in proposition 5.5.5, there is more room in $\mathbb{R}^3$ for edges not to intersect.*

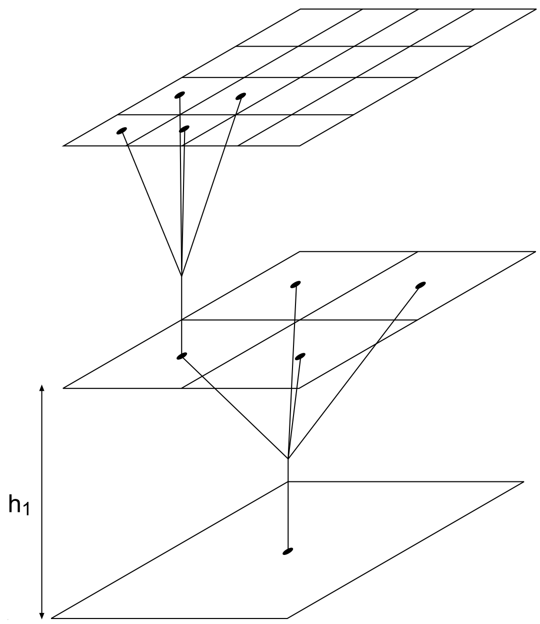

Figure 5.4: Finite cost traffic plan irrigating the Lebesgue hypercube of dimension $N-1$ and such that the total angle variation along fibers is bounded. If we stop fibers at the hyperplane of height h_i, we irrigate the dyadic approximation of level i of the Lebesgue measure on the hypercube of dimension $N-1$.

Conjecture 5.5.4 *A traffic plan with finite cost in $\mathbb{R}^2$ cannot at the same time irrigate Lebesgue measure restricted to a set with positive measure, have fibers with bounded total angle variation* **and be a tree**.

Proposition 5.5.5 *Let $\alpha \in]1-\frac{1}{N}, 1]$ and $N \geq 3$. There is a traffic plan* **with tree structure** *and finite cost, transporting a Dirac mass to Lebesgue measure on a parallelepiped, and such that the total angle variation of fibers is finite.*

Proof: Let us project the tree obtained in proposition 5.5.1 for the dimension $N+1$ on the hyperplane of dimension N, $z_{N+1} = 0$, so that we obtain a traffic plan in $\mathbb{R}^N$. We shall assimilate the space of projections on $z_{N+1} = 0$ to the hyperplane $\mathbb{R}^N \times \{1\}$. If not chosen specifically, the projection may be such that some projected edges intersect one another. Let us prove that it is possible to choose a suitable projection so that no intersection occurs (so that the resulting projected traffic plan has a tree structure).

We shall say that a projection is forbidden if it introduces a strict intersection (i.e. not at tips) between two segments of the tree. Let x, x', y, y' be four points of $\mathbb{R}^4$ and let us consider the two segments $[x, x']$ and $[y, y']$. To $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ where $v_4 \neq 0$, we associate the projection vector $\tilde{v} = \frac{v}{v_4}$. The set of forbidden projections consists of directions given by a point on $]x, x'[$ and a point of $]y, y'[$, i.e.:

$$P_f([x, x'], [y, y']) = \{\tilde{ab} | a \in]x, x'[, b \in]y, y'[\}.$$

The set $P_f([x, x'], [y, y'])$ is a submanifold of dimension 2. Indeed,

$$P_f([x, x'], [y, y']) = \{\tilde{ab} | a = \lambda x + (1-\lambda)x', b = \lambda' y + (1-\lambda')y' \text{ where } \lambda, \lambda' \in]0, 1[\},$$

so that it is described by the two parameters λ and λ'. Since the projective space of $\mathbb{R}^4$ is of dimension 3, the submanifold $P_f([x,x'],[y,y'])$ has null measure and the countable union of all forbidden projection sets associated to all couple of segments has null measure too. Thus there is a projection direction that is allowed for all couple of segments and this projection permits to obtain a traffic plan irrigating a set with positive measure, with a finite total angle variation and with a tree structure. Since the irrigated measure of the traffic plan described in proposition 5.5.1 is a cube, the projected traffic plan irrigates a parallelepiped. □

Remark 5.5.6 *Proposition 5.5.5 gives the example of a tree irrigating Lebesgue measure on the parallelepiped and such that the total angle variation along the paths of this tree is finite. Of course this tree is a mathematical object with branches of no thickness. The next question would ask if it is possible for a tree with "thick" tubes to irrigate a set with positive measure with the same angle condition. The human body seems to answer this question by a yes since blood vessels manage to irrigate the whole body with very low angle variation.*

Chapter 6

Structure and regularity of an optimum

Introduction

Let μ^+ and μ^- be measures on $\mathbb{R}^N$ and let π be a transference plan with marginals μ^+ and μ^-. We can either consider the irrigation problem which consists in optimizing $E^\alpha(\mu)$ over all $\mu \in TP(\mu^+, \mu^-)$ or the who goes where problem where we optimize $E^\alpha(\mu)$ over all $\mu \in TP(\pi)$. These two problems are quite different from a regularity point of view. Indeed, $TP(\pi)$ is much smaller than $TP(\mu^+, \mu^-)$ so that given a traffic plan μ, there are less possibilities of perturbation of μ to try to find a better competitor. Typically, the no circuit lemma 6.2.5 concerns only the irrigation problem: we suppose that μ is optimal and has a circuit, we construct μ_ϵ a perturbation of μ such that $E^\alpha(\mu_\epsilon) < E^\alpha(\mu)$ and $\mu_\epsilon \in TP(\mu^+, \mu^-)$ so that there is a contradiction and μ has no circuit. However, this perturbation μ_ϵ is not in $TP(\pi)$ so that we cannot conclude for the who goes where problem. The main proposition 6.2.7 of this chapter asserts that mass cannot split and get together again (for both irrigation and who goes where problems). This is different from the no circuit lemma since it covers who goes where problem and it does not assume a lower bound on the multiplicity along the fibers. These no loop or no circuit properties are essentials since they permit to state a regularity result when μ^+ and μ^- are atomic measures so that we can state equivalence results between models in section 6.4. The last section investigates the possible structure of branches at a bifurcation point.

6.1 Convex hull property

Definition 6.1.1 *A traffic plan μ is said to be optimal, respectively $\pi-$optimal if it is of minimal cost in $TP(\mu^+, \mu^-)$, respectively in $TP(\pi_\mu)$.*

Definition 6.1.2 (Support) *Let μ be a traffic plan. The support of μ is defined as $S_\mu := \{x : [x]_\mu > 0\}$. We will denote by S_{μ^+} the support of a measure μ^+ of $\mathbb{R}^N$.*

Lemma 6.1.3 *An optimal traffic plan μ is such that $S_\mu \subset conv(S_{\mu^-}, S_{\mu^+})$ where $conv(E)$ is the convex hull of a set E.*

Proof: Let $C := conv(S_{\mu^-}, S_{\mu^+})$ and χ be a parameterization of μ. For all $\omega \in \Omega$, let us define $\tilde{\chi}(\omega, t) = p_C(\chi(\omega, t))$ where p_C denotes the projection on the convex C. Since $\chi(\omega, 0)$ and $\chi(\omega, \infty)$

are in C, $\tilde{\chi}$ has the same transference plan as χ. Next, we have

$$\begin{aligned} E^\alpha(\tilde{\chi}) &= \int_{S_{\tilde{\chi}}} |[y]_{\tilde{\chi}}|^\alpha d\mathcal{H}^1(y) \\ &= \int_{S_{\tilde{\chi}}} (\sum_{x\in p_C^{-1}(y)\cap S_\chi} |[x]_\chi|)^\alpha d\mathcal{H}^1(y) \\ &\le \int_{S_{\tilde{\chi}}} \sum_{x\in p_C^{-1}(y)\cap S_\chi} |[x]_\chi|^\alpha d\mathcal{H}^1(y) \\ &\le \int_{S_\chi} |[x]_\chi|^\alpha d\mathcal{H}^1(x) = E^\alpha(\chi). \end{aligned}$$

The first inequality is obtained by the concavity of $x \mapsto x^\alpha$. The last inequality comes from the contraction of the length of fibers by the convex projection p_C and is a strict inequality if $\tilde{\chi} \neq \chi$. Thus $\tilde{\chi} = \chi$, by optimality of μ. □

6.2 The no-loop and no-circuit properties for an optimum

Definition 6.2.1 (Arc) *Let μ be a traffic plan and χ a parameterization of μ. Let $\gamma : [0,T] \to X$ be a curve parameterized by its arclength and $\Gamma := \gamma([0,T])$. Set $\Omega_\Gamma := \{\omega : \Gamma \subset \chi(\omega,\mathbb{R})\}$. The curve γ is said to be an arc of μ if $|\Omega_\Gamma| > 0$. Note that this definition does depend only on μ and not on the choice of the parameterization.*

Lemma 6.2.2 *Let μ be a simple path traffic plan parameterized by χ and γ an arc of μ. For any $\omega \in \Omega_\Gamma$, there are unique $t_\gamma^-, t_\gamma^+ \in \mathbb{R}$ such that $\chi(\omega)|_{[t_\gamma^-, t_\gamma^+]}$ coincides with a reparameterization of γ. Thus, we can define Ω_Γ^+ and Ω_Γ^- respectively as the ω such that $\chi(\omega, t_\gamma^-) = \gamma(0)$ and the ω such that $\chi(\omega, t_\gamma^+) = \gamma(0)$*

Proof: The parameterization χ is such that $\chi(\omega,\cdot)$ is one to one. Thus, for all $\omega \in \Omega_\Gamma$, the set $I := \{t : \chi(\omega,t) \in \Gamma\} = \chi^{-1}(\omega,\cdot)$ is closed and connected. □

Lemma 6.2.3 No both ways *Let μ be an optimal traffic plan from μ^+ to μ^-. If γ is an arc of μ then either $|\Omega_\Gamma^+| = 0$ or $|\Omega_\Gamma^-| = 0$.*

Proof: If Ω_Γ^+ and Ω_Γ^- are both non-negligible, consider two subsets of same positive measure $\Omega_1 \subset \Omega_\Gamma^+$ and $\Omega_2 \subset \Omega_\Gamma^-$ and $\phi := \Omega_1 \to \Omega_2$ bijective and measure preserving. Let us define $\tilde{\chi}$ as χ for all $\omega \notin \Omega_1 \cup \Omega_2$. For all $\omega \in \Omega_1$, we define

$$\tilde{\chi}(\omega,t) = \begin{cases} \chi(\omega,t) & \text{if } t \le t_\gamma^-(\omega) \\ \chi(\phi(\omega), t - t_\gamma^-(\omega) + t_\gamma^+(\phi(\omega))) & \text{if } t \ge t_\gamma^-(\omega) \end{cases}$$

We define $\tilde{\chi}$ in the same way on Ω_2. The traffic plan $\tilde{\mu} := \tilde{\chi}\#\lambda$ has a lower cost than μ and has the same transference plan. This is absurd so that the lemma is proved. □

Lemma 6.2.4 No splitting and grouping of mass: the case of two arcs*[28] Let μ be an optimal traffic plan from μ^+ to μ^- with a parameterization χ. If γ_1 and γ_2 are two arcs of μ such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(T_1) = \gamma_2(T_2)$, then $\gamma_1 = \gamma_2$.*

Proof: Without loss of generality, we can restrict to the following two cases: $\Omega^+_{\Gamma_1}$ and $\Omega^-_{\Gamma_2}$ are non-negligible, or $\Omega^+_{\Gamma_1}$ and $\Omega^+_{\Gamma_2}$ are non-negligible. In the first case, the traffic plan has an oriented loop that we can easily remove as in the lemma **??**. In the second case, we suppose the two arcs to be different and prove that we can decrease the energy. This proof is mainly reproduced from [28]. Let us consider the two traffic plans χ_1 and χ_2 respectively obtained sending the arc Γ_1 on Γ_2 and sending the arc Γ_2 on Γ_1. That is to say, let us define χ_1 as χ for all $\omega \notin \Omega_1 \cup \Omega_2$. For all $\omega \in \Omega_1$, we define

$$\chi_1(\omega, t) = \begin{cases} \chi(\omega, t) & \text{if } t \leq t^-_{\gamma_1}(\omega) \\ \gamma_2(t - t^-_{\gamma_1}(\omega)) & \text{if } t^-_{\gamma_1}(\omega) \leq t \leq t^-_{\gamma_1}(\omega) + T_2 \\ \chi(\omega, t - t^-_{\gamma_1}(\omega) - T_2 + t^+_{\gamma_1}(\omega)) & \text{if } t \geq t^-_{\gamma_1}(\omega) + T_2 \end{cases}$$

where $t^-_{\gamma_1}$ and $t^+_{\gamma_1}$ are defined as in lemma 6.2.2. We define χ_2 in the same way. Let us denote $m_i := |\Omega_i|$ and $m_i(t) := |[\gamma_i(t)]_m u|$. The energy difference between μ and $\mu_1 := \chi_1 \# \lambda$ is

$$\begin{aligned} \delta_1 : &= E(\mu) - E(\mu_1) \\ &= \int_0^{T_1} m_1(t)^\alpha dt + \int_0^{T_1} m_2(t)^\alpha dt - \int_0^{T_1} (m_1(t) - m_1)^\alpha dt - \int_0^{T_2} (m_2(t) + m_1)^\alpha dt \\ &= \int_0^{T_1} (m_1(t)^\alpha - (m_1(t) - m_1)^\alpha) dt + \int_0^{T_2} (m_2(t)^\alpha - (m_2(t) + m_1)^\alpha) dt. \end{aligned}$$

In the same way,

$$\delta_2 := E(\mu) - E(\mu_2) = \int_0^{T_2} (m_2(t)^\alpha - (m_2(t) - m_2)^\alpha) dt + \int_0^{T_1} (m_1(t)^\alpha - (m_1(t) + m_2)^\alpha) dt.$$

Let us now prove that $\frac{\delta_1}{m_1} + \frac{\delta_2}{m_2} < 0$. Since $m_1, m_2 > 0$, this will prove that either $\delta_1 < 0$ or $\delta_2 < 0$.

$$\begin{aligned} \frac{\delta_1}{m_1} + \frac{\delta_2}{m_2} &= \int_0^{T_1} \frac{(m_1(t) + m_2)^\alpha - m_1(t)^\alpha}{m_2} - \frac{m_1(t)^\alpha - (m_1(t) - m_1)^\alpha}{m_1} - dt \\ &+ \int_0^{T_2} \frac{(m_2(t)^\alpha - (m_2(t) - m_2)^\alpha)}{m_2} - \frac{(m_2(t) + m_1)^\alpha - m_2(t)^\alpha}{m_1} dt \\ &< 0, \end{aligned}$$

because of the concavity of $x \mapsto x^\alpha$. □

The next lemma is a restatement of [35, proposition 2.1 p.256], and the proof is strongly inspired from it. Still, in the author's point of view, it makes clearer the perturbation used to decrease the energy, in the case there is a circuit with a positive flow.

Lemma 6.2.5 No circuit made of arcs in the irrigation problem [35] *Let μ be a traffic plan from μ^+ to μ^- and $\alpha < 1$. If there are $(\gamma_i)_{i=1}^n$, arcs of μ such that $\gamma_i(T_i) = \gamma_{i+1}(0)$ for all $i \in [1, n-1]$ and $\gamma_n(T_n) = \gamma_1(0)$, then μ is not optimal for the irrigation problem.*

Proof: Thanks to lemma **??**, it is consistent to define respectively L^+ and L^- as the set of indices such that respectively $|\Omega^-_{\Gamma_i}| = 0$ and $|\Omega^+_{\Gamma_i}| = 0$. Let us consider sets $\Omega_i \subset \Omega_{\Gamma_i}$ such that $|\Omega_i| = m$ for all i. If two consecutive indices are in L^+, we can shrink arcs Γ_i and Γ_{i+1} to a single one up to a mixing between fibers of Γ_i and Γ_{i+1}. More precisely, let ω_i and ω_{i+1} be fibers in Ω_i and Ω_{i+1}. The mixing of ω_i and ω_{i+1} consists in defining

$$\tilde{\chi}(\omega_i, t) = \begin{cases} \chi(\omega_i, t) & \text{if } t \leq \min(I_{\omega_i}) \\ \chi(\omega_{i+1}, t - \min(I_{\omega_i}) + \min(I_{\omega_{i+1}}) & \text{if } t > \min(I_{\omega_i}), \end{cases}$$

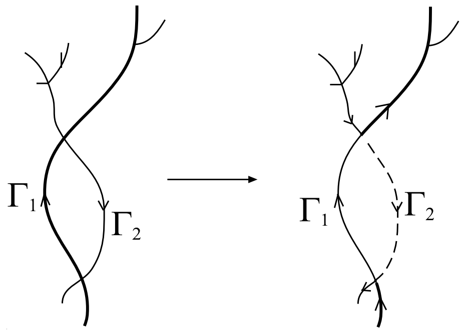

Figure 6.1: The oriented loop of the left hand side can be removed by mixing fibers of Γ_1 with those of Γ_2. Indeed, we can glue fibers of Γ_1 with the ends of fibers of Γ_2 and fibers of Γ_2 with the ends of fibers of Γ_1 as illustrated by the right hand side figure.

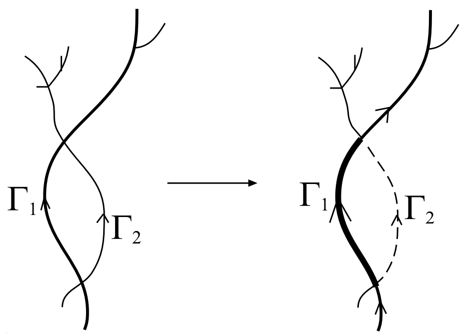

Figure 6.2: The fibers going through Γ_2 are modified between $\gamma_2(0)$ and $\gamma_2(T_2)$ so that they go through Γ_1. Lemma 6.2.4 proves that either transferring Γ_2 to Γ_1 or Γ_1 to Γ_2 decreases the cost. Notice that this transformation does not affect the transference plan.

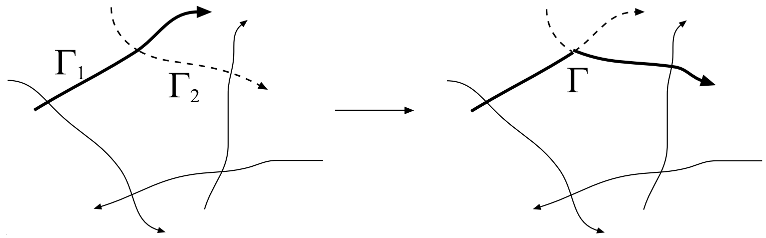

Figure 6.3: Gluing the beginning of a fiber of Γ_i with the end of a fiber of Γ_{i+1} and reciprocally permits to merge two arcs with same orientation. This mixing modifies the transference plan but not irrigating and irrigated measures.

and

$$\tilde{\chi}(\omega_{i+1}, t) = \begin{cases} \chi(\omega_{i+1}, t) & \text{if } t \leq \min(I_{\omega_{i+1}}) \\ \chi(\omega_i, t - \min(I_{\omega_{i+1}}) + \min(I_{\omega_i})) & \text{if } t > \min(I_{\omega_{i+1}}). \end{cases}$$

We define $\tilde{\chi}$ the parameterization obtained mixing ω with $\psi(\omega)$ for all $\omega \in \Omega_i$ where $\psi : \Omega_i \to \Omega_{i+1}$ is a measure preserving bijection. Notice that $\tilde{\chi}$ has the same irrigating and irrigated measures, but not the same transference plan as χ. Moreover, the swapping does not change the cost so that $E^\alpha(\tilde{\chi}) = E^\alpha(\chi)$. We have reduced the problem to the one of proving that $\tilde{\chi}$ is not optimal. It is indifferent to prove either that μ or the reversed time traffic plan obtained from μ is not optimal, thus we can assume without loss of generality that

$$\sum_{i \in L^+} \int_0^{l_i} m_i(s)^{\alpha-1} ds \leq \sum_{i \in L^-} \int_0^{l_i} m_i(s)^{\alpha-1} ds.$$

We now define χ_ϵ such that all flow along an L^+ path is increased by ϵ and all flow along an L^- paths is decreased by ϵ. This parameterization can be obtained through the convenient mixing of fibers and is such that the irrigating and irrigated measures are the same as those of χ. Let us denote $f(\epsilon) = E^\alpha(\chi_\epsilon) - E^\alpha(\chi)$. We have

$$\sum_{i \in L^+} \int_0^{l_i} (m_i(s) + \epsilon)^\alpha ds + \sum_{i \in L^-} \int_0^{l_i} (m_i(s) - \epsilon)^\alpha ds.$$

The function f is strictly concave because $\alpha < 1$. Thus

$$f'(\epsilon) < f'(0) = \sum_{i \in L^+} \int_0^{l_i} (m_i(s))^{\alpha-1} ds + \sum_{i \in L^-} \int_0^{l_i} (m_i(s))^{\alpha-1} ds \leq 0.$$

Thus, the cost of χ_ϵ is lower than the one of χ, and χ is not optimal.

□

Remark 6.2.6 *Lemma 6.2.5 proves that an optimal traffic plan* **for the irrigation problem** *has no circuit with a flow bounded below by a positive constant. This does prove that a more general circuit as the one represented on figure 6.5 is not optimal for the irrigation problem. Indeed, such a traffic plan is such that the fibers irrigate Lebesgue measure on the segment to finally group again to a Dirac mass. Of course, such a structure is far from being optimal and proposition 6.2.7 rules out such candidates through a perturbation similar to the one of lemma 6.2.4.*

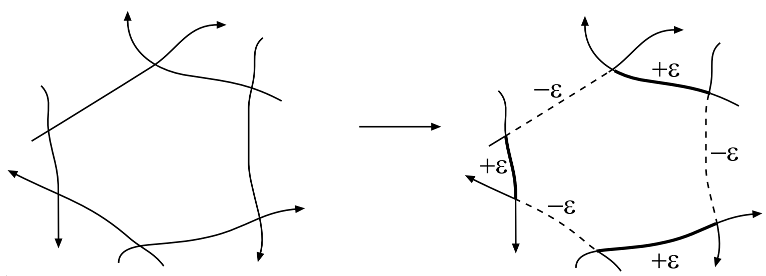

Figure 6.4: The modification of the traffic plan μ consists in transfering a multiplicity ϵ from all arcs Γ_{2i} to arcs Γ_{2i+1}. This perturbation gives a new traffic plan μ_ϵ which has a lower cost than μ and same irrigating and irrigated measures.

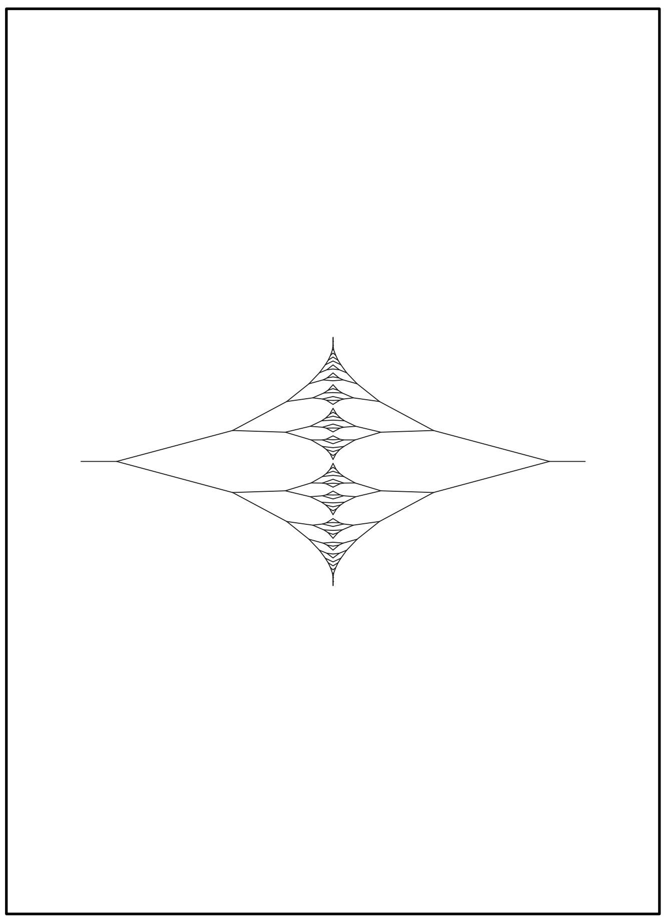

Figure 6.5: This traffic plan is obtained through the concatenation of a traffic plan transporting a Dirac mass to Lebesgue measure on a segment and a traffic plan transporting Lebesgue measure on a segment to a Dirac mass. Proposition 6.2.7 proves that such a structure is not optimal.

Proposition 6.2.7 No splitting and grouping of mass: the general case *Let μ be an optimal traffic plan with simple paths from μ^+ to μ^-, and χ a parameterization of μ. Let us denote $\Omega_x := \{\omega : x \in \chi(\omega, \mathbb{R})\}$. Let x, y be such that $\Omega_{xy} := \Omega_y \cap \Omega_x$ is of positive measure. For all $\omega \in \Omega_{xy}$, we define $t_x(\omega) := \chi(\omega)^{-1}(x)$, $t_y(\omega) := \chi(\omega)^{-1}(y)$ and I_ω the time interval between $t_x(\omega)$ and $t_y(\omega)$. For almost all $\omega_1, \omega_2 \in \Omega_{xy}$, we have $\chi(\omega_1, I_{\omega_1}) = \chi(\omega_2, I_{\omega_2})$.*

Proof: Let us first define $L_i := S_\mu \cap (\cup_{\omega \in \Omega_i} \chi(\omega, I_\omega))$ where S_μ denotes the set of points with positive multiplicity and suppose by contradiction that there are $\Omega_1, \Omega_2 \subset \Omega_{xy}$ such that $|\Omega_1|, |\Omega_2| > 0$, $|\Omega_1 \cap \Omega_2| = 0$, such that the symmetric difference $|L_1 \Delta L_2| > 0$. This means that the structure generated by fibers of Ω_1 and Ω_2 are different. Let us consider some point z in $L_1 \cup L_2$ and denote $m_i(z) := |\Omega_i \cap \Omega_z|$ for $i = 1, 2$ and $\bar{m}(z) = |\Omega_z \cap (\Omega \setminus (\Omega_1 \cup \Omega_2))|$. Notice that the multiplicity at z is $|[z]_\chi| = m_1(z) + m_2(z) + \bar{m}(z)$ for all $z \in \mathbb{R}^N$. As in lemma 6.2.4, we are going to transfer mass of L_2 through the L_1 structure. Let ρ' be the proportion of fibers of Ω_2 to be transferred to Ω_1. We take $\rho' := \rho\frac{|\Omega_1|}{|\Omega_2|} < 1$. Let $m_\rho(z) := (1+\rho)m_1(z) + (1-\rho')m_2(z) + \bar{m}(z)$. Let us prove that there exists a traffic plan μ_ρ with the same transference plan as μ such that $|[z]_{\mu_\rho}| = m_\rho(z)$. Up to a measure preserving bijection, we can suppose for the sake of convenience that $\Omega_1 = [0, |\Omega_1|]$ and $\Omega_2 =]|\Omega_1|, |\Omega_2| + |\Omega_1|]$. Let us denote $\tilde{\Omega}_1 = [0, |\Omega_1| + \rho'|\Omega_2|]$ and $\tilde{\Omega}_2 =]|\Omega_1| + \rho'|\Omega_2|, |\Omega_2| + |\Omega_1|]$. The application

$$\psi(\omega) = \begin{cases} \frac{|\Omega_1|}{|\tilde{\Omega}_1|}\omega & \text{if } \omega \in \tilde{\Omega}_1 \\ \frac{|\Omega_2|}{|\tilde{\Omega}_2|}(\omega - |\tilde{\Omega}_1|) + |\Omega_1| & \text{if } \omega \in \tilde{\Omega}_2 \\ \omega & \text{if } \omega \in \Omega \setminus (\tilde{\Omega}_1 \cup \tilde{\Omega}_2) \end{cases}$$

is an application contracting $|\tilde{\Omega}_1|$ onto $|\Omega_1|$ and dilating $|\tilde{\Omega}_2|$ onto $|\Omega_2|$. We define

$$\chi_\rho(\omega, t) = \begin{cases} \chi(\omega, t) & \text{if } t \le \min(I_\omega) \\ \chi(\psi(\omega), t - \min(I_\omega) + \min(I_{\psi(\omega)})) & \text{if } t \in [\min(I_\omega), \min(I_\omega) + |I_{\psi(\omega)}|] \\ \chi(\omega, t - (\min(I_\omega) + |I_{\psi(\omega)}|) + |I_\omega|) & \text{if } t > \min(I_\omega) + |I_{\psi(\omega)}| \end{cases}$$

which is obtained transferring uniformly mass of Ω_2 onto paths followed by fibers of Ω_1 between x and y. The traffic plan $\mu_\rho = \tilde{\chi}\#\lambda$ is by definition such that $|[z]_{\mu_\rho}| = m_\rho(z)$. Further, the transference plan of μ_ρ is the same as the one of μ since $\chi_\rho(\omega, 0) = \chi(\omega, 0)$ and $\chi_\rho(\omega, \infty) = \chi(\omega, \infty)$ for all $\omega \in [0, 1]$. Let us compare the costs of μ and μ_ρ. We define the balance of the energy as

$$f(\rho) = E^\alpha(\mu_\rho) - E^\alpha(\mu).$$

Let us denote $L := L_1 \cup L_2$. We have

$$f(\rho) = \int_L (m_\rho(z)^\alpha - |[x]_\mu|^\alpha) d\mathcal{H}^1.$$

Thus

$$f'(\rho) = \alpha \int_L m_\rho(z)^{\alpha-1}(m_1(z) - m_2(z)\frac{|\Omega_2|}{|\Omega_1|}) d\mathcal{H}^1,$$

and

$$f''(\rho) = \alpha(\alpha - 1) \int_L m_\rho(z)^{\alpha-2}(m_1(z) - m_2(z)\frac{|\Omega_2|}{|\Omega_1|})^2 d\mathcal{H}^1.$$

We then notice that on $L_1 \setminus L_2$, $m_2(z) = 0$ and $m_1(z) > 0$. Symmetrically, $m_2(z) > 0$ and $m_1(z) = 0$ on $L_2 \setminus L_1$. Thus, $(m_1(z) - m_2(z)\frac{|\Omega_2|}{|\Omega_1|}) \ne 0$ for all $z \in L_1 \Delta L_2$. Since $|L_1 \Delta L_2| > 0$ and $\alpha < 1$,

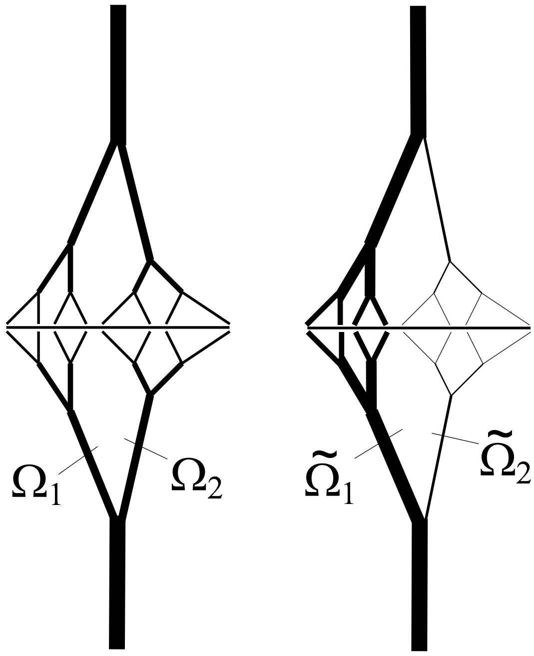

Figure 6.6: lemma 6.2.4 requires a lower bound on the multiplicity along some fibers to exclude loops in optimal traffic plans. Thus it cannot rule out structure as the one presented on this figure, where the mass is spreading on a Lebesgue measure on the segment to finally group again. The idea of the proof of proposition 6.2.7 is roughly the same as in the case of two arcs. Indeed, suppose that we have two different structures going from x to y (in the sense that the geometric support of fibers Ω_1 and Ω_2 are different). We convey some part of the mass of the second structure through the first structure or in the other way, and we prove that the resulting structure which is represented on the right is better so that we obtain a contradiction.

we obtain that $f''(\rho) < 0$. Thus $f'(\lambda) < f'(0) = \alpha \int_L (m(z)^{\alpha-1}(m_1 - m_2\frac{|\Omega_2|}{|\Omega_1|})d\mathcal{H}^1$. Without loss of generality, we can assume that $f'(0) \leq 0$, otherwise we exchange Ω_1 and Ω_2. Thus $f'(\rho) < 0$ and $f(\rho) < f(0) = 0$ for a sufficiently small ρ. This inequality contradicts the optimality of μ. □

6.3 Regularity when μ^+ and μ^- are atomic measures

Definition 6.3.1 *Let μ be a traffic plan and Γ, Γ' two arcs of μ. Let us call bifurcation point some $p \in \Gamma \cap \Gamma'$ such that $\Gamma \cup \Gamma' \setminus \{p\}$ has at least three connected components.*

Definition 6.3.2 *Let μ be a traffic plan. We say that μ has a circuit if there are arcs $(\Gamma_i)_{i=1}^n$ such that there are bifurcation points $(p_i)_{i=1}^n$ such that $p_i \in \Gamma_i \cap \Gamma_{i+1}$ for $i < n$ and $p_n \in \Gamma_n \cap \Gamma_1$.*

Proposition 6.3.3 *Let π be a transference plan such that μ^+ and μ^- are finite atomic measures. An optimum for the who goes where problem has the structure of a finite graph. An optimum for the irrigation problem is a finite tree made of segments.*

Proof: Let us denote $\mu^+ = \sum a_i\delta_{x_i}$ and $\mu^- = \sum b_j\delta_{y_j}$. Let μ be an optimum for the who goes where problem and χ a parameterization of μ. We denote $\Omega_{ij} := \{\omega : \chi(\omega, 0) = x_i$ and $\chi(\omega, \infty) = y_j\}$.

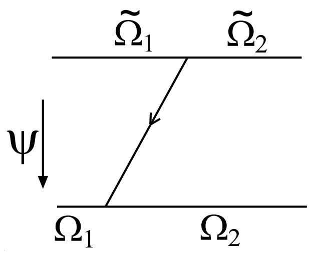

Figure 6.7: The application ψ is contracting $\tilde{\Omega}_1$ on Ω_1 and dilating $\tilde{\Omega}_2$ on Ω_2.

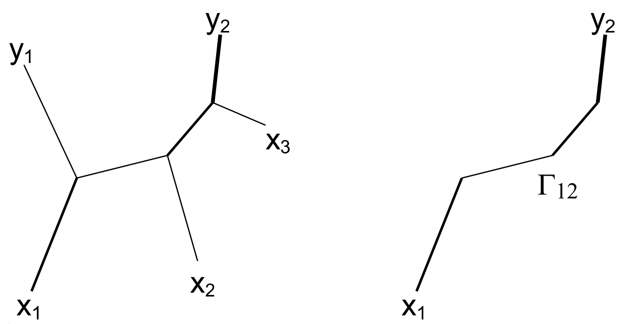

Figure 6.8: Proposition 6.2.7 asserts that fibers connecting x_i with y_j follow a single arc Γ_{ij}.

Because of proposition 6.2.7, there is an arc Γ_{ij} such that $\chi(\omega, \mathbb{R}) = \Gamma_{ij}$ for all $\omega \in \Omega_{ij}$. Thus, an optimum for the who goes where problem has the structure of a finite graph. The same argument stands for the irrigation problem. Moreover, lemma 6.2.5 permits to prove that no circuit occurs for an optimum. Thus an optimum has a tree structure. Further, since the multiplicity of points of an arc does not change between two consecutive bifurcation points, this tree is made of segments. □

Remark 6.3.4 *Neither in [35] nor in [36] does Xia investigate the question of the regularity in the atomic case. We remind the reader that Xia defines a cost on Radon vectorial measures obtained from finite graphs and then relaxes the functional to define a cost on more general Radon vectorial measures. Let us emphasize that though the initial cost is defined on finite graphs, it does not mean that the relaxation process could not bring better structure than finite graphs when μ^+ and μ^- are atomic.*

6.4 Equivalence between models

It is now time to make a stop to look at the problem of equivalence between models described in chapter 2. Indeed, the knowledge we now have on the structure of an optimum permits to conclude that optima and costs for traffic plan, patterns and Gilbert-Steiner problem are equivalent.

Theorem 6.4.1 *Traffic plans and patterns ([22]) are equivalent with respect to the irrigation problem when μ^+ consists of a single Dirac mass.*

Proof: The small difference between the traffic plan model and the pattern model is the definition of the multiplicity. In the pattern model, when two fibers coincide for time $[0, T]$ then separate, there are viewed as being separated for the remaining time even if the fibers happen to coincide again geometrically. This is due to the fact that multiplicity of the fiber ω at time t is the measure of all equivalent fibers (i.e. fibers coinciding with ω during time $[0, t]$). Let μ^+ being a single Dirac mass at a source point S and μ an optimal traffic plan for the irrigation problem. Proposition 6.2.7 asserts that a parameterization χ of μ has a tree structure, so that the definition of multiplicity in the traffic plan framework coincide with the one of patterns. Since the cost of tree structures are identical, the models are then equivalent. □

Theorem 6.4.2 *The irrigation problem for traffic plans when μ^+ and μ^- are atomic measures and the Gilbert-Steiner problem are equivalent*

Proof: Let μ^+ and μ^- be atomic measures and μ an optimal traffic plan for the irrigation problem. Proposition 6.3.3 asserts that μ has a graph structure so that the E^α cost is the same than the Gilbert-Steiner problem cost for $f(c) = c^\alpha$. Thus, both problems give same optima. □

6.5 The regularity result in [36]

In this section, we briefly survey the article [36] where Xia claims the following regularity result. Let μ^+ and μ^- be two measures of $\mathbb{R}^N$ and T an optimal transport path from μ^+ to μ^-. If x is a point on the support of T away from the support of μ^- and μ^+, then T has a finite graph structure in the neighborhood of x. Xia first proves the existence of a cone-shaped blow-up at x and then does estimates to prove that the optimal structure coincides with the blow-up in a sufficiently small neighborhood.

The second part of this proof is lacking some argument. We have mainly two criticisms relative to this article (we refer to [36] for the notations):

- Concerning lemma 4.8, the fact that $f(r)$ is decreasing because T contains no loop seems questionable. It rather seems to be a consequence of the radiality of an optimal transport from a center of the ball to the sphere. Such radiality should be proven carefully.
- Lemma 4.9 seems also to be questionable since Xia moves from a transport between positive measures to transport between an arbitrary infinite atomic measure (with positive and negative Dirac masses) and a Dirac mass centered at a ball. The justification of that change is that the boundary of the current T can be viewed either as $(\mu^+ - \mu^-) - \delta_0$ or $\mu^+ - (\mu^- + \delta_0)$. However the estimations of $M^\alpha(\Gamma_p + \lambda_q \gamma_p) - M^\alpha(\Gamma_p)$ in the proof of Lemma 4.9 strongly depends on the fact that we consider a transport from a Dirac mass to a positive measure.

Since lemma 4.9 is crucial in the proof of final theorem, we consider that the regularity claim needs another proof.

6.6 Number of branches at a bifurcation

In this section we investigate the geometry of branches at a bifurcation point of an optimal traffic plan. The optimal structure of a traffic plan from one Dirac mass to two Dirac masses is essential in all that follows. It is necessary to read section 7.1 in order to understand well the present section. Lemmas 6.6.2 and 6.6.3 give lower bound (depending on α) on the angle between two edges starting from the same point (see figure). As a consequence, we prove that it is not possible for an optimal finite traffic plan in $\mathbb{R}^2$ to have more than three edges meeting at a bifurcation point (away from μ^+ and μ^-), when $\alpha \leq \frac{1}{2}$

(proposition 6.6.4). It is still a conjecture whether or not four edges can meet at a bifurcation point in $\mathbb{R}^2$ when $1 > \alpha > \frac{1}{2}$, though numerical experiments seem to exclude this situation.

Lemma 6.6.1 *The function g defined by $g(m) = \frac{(m+1)^{2\alpha} - m^{2\alpha} - 1}{2m^\alpha}$ is nondecreasing on $]0,1]$ for $1 > \alpha > \frac{1}{2}$ and nonincreasing for $\alpha < \frac{1}{2}$. Thus,*

$$\sup_{m \in]0,1]} g(m) = \begin{cases} 2^{2\alpha-1} - 1 & \text{if } \alpha > \frac{1}{2} \\ 0 & \text{if } 1 > \alpha \leq \frac{1}{2}. \end{cases}$$

Proof: Indeed, ϕ' has the same sign as

$$(m+1)^{2\alpha} - 2(m+1)^{2\alpha-1} - m^{2\alpha} + 1$$

that we denote $\psi(\alpha)$. We notice further that $\psi(0) = \frac{m-1}{m+1} < 0$, $\psi(1/2) = \psi(1) = 0$ and that ψ is concave. Indeed,

$$\begin{aligned} \psi''(\alpha) &= 4(m+1)^{2\alpha} \ln(m+1)^2 - 8(m+1)^{2\,\alpha-1} \ln(m+1)^2 - 4m^{2\alpha} \ln(m)^2 \\ &= 4(m+1)^{2\alpha-1} \ln(m+1)^2 (m-1) - 4m^{2\alpha} \ln(m)^2 \\ &\leq 0. \end{aligned}$$

The last inequality results from the fact that $m \leq 1$. Thus, $\psi(\alpha) \geq 0$ for $1 > \alpha > \frac{1}{2}$ and ϕ' is positive so that ϕ is not decreasing. Similarly, $\psi(\alpha) \leq 0$ for $\alpha < \frac{1}{2}$ and ϕ' is negative so that ϕ is not decreasing. The monotonicity of g permits to easily calculate the supremum,

$$\sup_{m \in]0,1]} g(m) = \begin{cases} g(1) = 2^{2\alpha-1} - 1 & \text{if } 1 > \alpha > \frac{1}{2} \\ \lim_{m \to 0} g(m) = 0 & \text{if } 1 > \alpha \leq \frac{1}{2}. \end{cases}$$

□

Lemma 6.6.2 *Let $e_1 = pa_1$ and $e_2 = pa_2$ be two oriented edges of the circle $C(p,r)$. Let μ be a traffic plan made of the two edges e_1 and e_2 with masses m_1 and m_2. If μ is optimal, the angle θ between e_1 and e_2 is such that $\cos(\theta) \leq 2^{2\alpha-1} - 1$ for $1 > \alpha > \frac{1}{2}$ and $\cos(\theta) \leq 0$ for $\alpha \leq \frac{1}{2}$.*

Proof: Indeed, because of proposition 7.1.7, and lemma 6.6.1,

$$\cos(\theta) \leq \sup_{m_1, m_2 \in [0,1]} \frac{(m_1 + m_2)^{2\alpha} - m_1^{2\alpha} - m_2^{2\alpha}}{2m_1^\alpha m_2^\alpha} = \begin{cases} 2^{2\alpha-1} - 1 & \text{if } 1 > \alpha > \frac{1}{2} \\ 0 & \text{if } \alpha \leq \frac{1}{2}. \end{cases}$$

□

Lemma 6.6.3 *Let $e^+ = a^+p$ and $e^- = pa^-$ two oriented edges of the circle $C(p,r)$. Let μ be a traffic plan made of the two edges e^+ and e^- with masses m and m'. If μ is optimal, the angle θ between e^+ and e^- is such that $\cos(\theta) \leq (\frac{m}{m'} - 1)^\alpha - (\frac{m}{m'})^\alpha$. In particular, θ is strictly superior to $\frac{\pi}{2}$.*

Proof: Without loss of generality, we can suppose that $m \geq m'$. Let p_ϵ be the point on segment a^+p at a distance ϵ of p. Let us consider the traffic plan μ_ϵ made of the edges (a^+p_ϵ, m),$(p_\epsilon a^-, m')$ and $(p_\epsilon p, m - m')$. Let us denote

$$\begin{aligned} \delta(\epsilon) &= E^\alpha(\mu) - E^\alpha(\mu_\epsilon) \\ &= m^\alpha + m'^\alpha - (m^\alpha(1-\epsilon) + (m-m')^\alpha \epsilon + m'^\alpha \sqrt{1 + \epsilon^2 - 2\epsilon\cos(\theta)}). \end{aligned}$$

Since the traffic plan μ_ϵ has the same transference plan as μ and μ is optimal, $E^\alpha(\mu_\epsilon) \geq E^\alpha(\mu)$, i.e. $\delta(\epsilon) \leq 0$. Thus $\delta'(0) \leq 0$, i.e. $\cos(\theta) \leq (\frac{m}{m'} - 1)^\alpha - (\frac{m}{m'})^\alpha$. In particular, $\cos(\theta) < 0$ so that $\theta > \frac{\pi}{2}$. □

Proposition 6.6.4 *Let $\alpha \leq 1/2$ and μ be an optimal traffic plan of $\mathbb{R}^2$ with finite graph structure. A node of the graph not in the support of μ^+ and μ^- has an edge multiplicity less than or equal to 3.*

Proof: Let p be a bifurcation point with more than three edges at p. Let us consider ν, the restriction of the traffic plan μ to a small ball $B(p,r)$ such that p is the only bifurcation of ν. The traffic plan ν is optimal for the irrigation problem from ν^+ to ν^- where ν^+ and ν^- are atomic measures on the circle $C(p,r)$. Let us denote by L^- and L^+ respectively the set of edges connecting p to ν^- and ν^+. A subtraffic plan made of two edges of L^+ or two edges of L^- is optimal, otherwise ν would not be optimal. Thus because of proposition 7.1.7 the angle between two edges (e,m) and (e',m') is superior to the angle θ such that

$$\cos(\theta) = \frac{(m+m')^{2\alpha} - m^{2\alpha} - m'^{2\alpha}}{2m^\alpha m'^\alpha}.$$

In the case $\alpha \leq \frac{1}{2}$, $\cos(\theta) \leq 0$ so that the angle between e and e' is superior or equal to $\frac{\pi}{2}$. This fact in addition with lemma 6.6.3 implies that $\#L^+ \cup L^- \leq 3$. Indeed, assume that $\#L^+ \cup L^- \geq 4$, and let us extract four edges e_i from L^+ and L^-. Let us denote θ_i the four angles between the edges e_i considered in a trigonometric order. All of these angles are superior to $\frac{\pi}{2}$ and one of them is strictly superior to $\frac{\pi}{2}$ because of lemma 6.6.3. Thus, there is no room for more than three edges in $L^+ \cup L^-$. □

Remark 6.6.5 *There is a very quick and geometric argument to prove that no Ψ shape can occur for an optimal traffic plan and $\alpha \leq \frac{1}{2}$. It is illustrated by figure 6.9. The argument is the following. Let us suppose that a Ψ shape is optimal and denote p the bifurcation point. In particular the subtraffic plan made of edges pa_1 and pa_2 is optimal so that p lies within the disk D_1 defined by the equiangle circle of proposition 7.1.7. In the same way, the subtraffic plan made of edges pa_2 and pa_3 is optimal so that p lies within the disk D_2 defined by the equiangle circle. For $\alpha \leq \frac{1}{2}$, $D_1 \cap D_2 = \emptyset$ so that we obtain a contradiction.*

Conjecture 6.6.6 *Let $\frac{1}{2} < \alpha < 1$ and μ be an optimal traffic plan of $\mathbb{R}^2$ with finite graph structure. A node of the graph not in the support of μ^+ and μ^- has an edge multiplicity less than or equal to 3.*

Beginning of the proof: Let us define L^+ and L^- as in the proof of 6.6.4. Because of the minimal angle lemma, 6.6.2 both L^+ and L^- are finite. Though, it does not seem as easy as in the case $\alpha \leq \frac{1}{2}$ to reduce the cardinal of L^+ and L^-. A first step would be to prove that it is enough to consider the case $\#L^+ = \#L^- = 2$ and the case $\#L^- = 3$ and $\#L^+ = 1$. Remark 6.6.5 contains a strategy to deal with the second case. Indeed, let us consider the optimal bifurcation point p for the best Ψ shape structure and consider that this structure is globally optimum. We denote $(s,1)$ the source and $(a_i,m_i)_{i=1}^3$ the three irrigated points, such that $m_1+m_2+m_3 = 1$. Remark 6.6.5 proves that p has to be in $D_1 \cap D_2$. In addition, the first order local optimality criterion states that

$$\sum_{i=1}^{3} m_i^\alpha n_i = -n,$$

where n_i is the unit vector directed by the vector pa_i and n is the unit vector directed by ps. If we prove that $||\sum_{i=1}^3 m_i^\alpha n_i|| \neq 1$ for all $p \in D_1 \cap D_2$, the contradiction follows. Let us denote θ_1 and θ_2 respectively the angle a_1pa_2 and a_2pa_3. The disks D_1 and D_2 are the equiangle circles corresponding

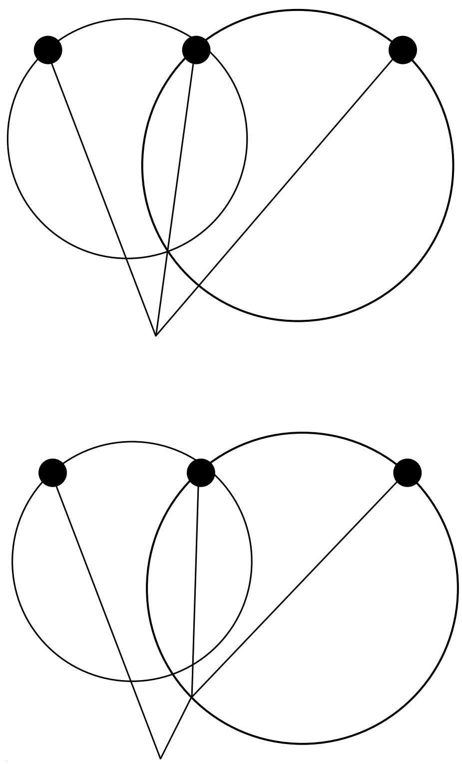

Figure 6.9: If there is a triple point outside an equiangle circle, then a ”ψ” structure can be improved as illustrated, thanks to proposition 7.1.7.

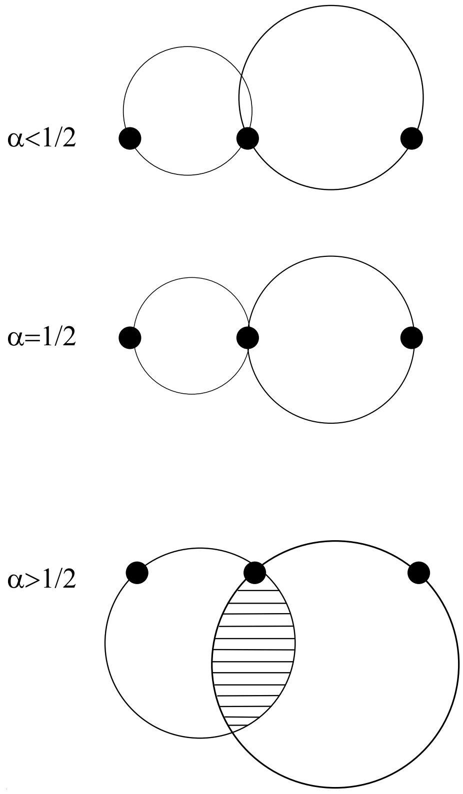

Figure 6.10: As illustrated by figure 6.9, a triple point of an optimal traffic plan has to lie within the two corresponding equiangle circles. In the case $\alpha \leq 1/2$, the intersection of these two disks is empty (both figures at the top). In the case $1 > \alpha > 1/2$, the intersection is not empty so that we cannot conclude immediately.

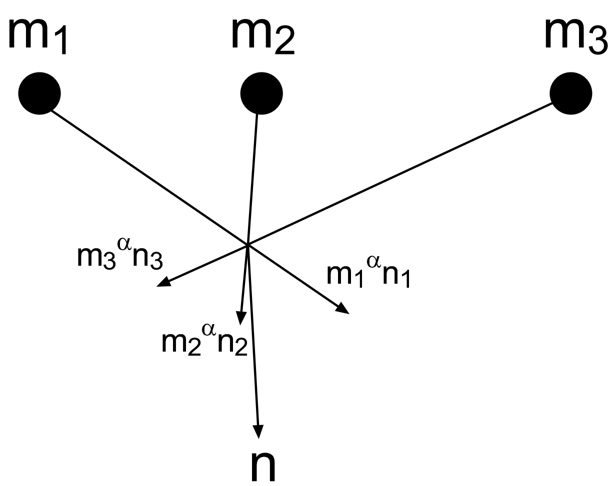

Figure 6.11: The balance equation at an optimal triple point asserts that $n := m_1^\alpha n_1 + m_2^\alpha n_2 + m_3^\alpha n_3$ has to be of norm 1. A strategy to prove that there is no triple point for an optimum is thus to prove that $||n|| < 1$ for any point in the intersection of the two equiangle disks.

respectively to $(a_1, m_1), (a_2, m_2)$ and $(a_2, m_2), (a_3, m_3)$, i.e. they are equiangle circles for the angles τ_1 and τ_2 such that

$$\cos(\tau_1) = \frac{(m_1 + m_2)^{2\alpha} - m_1^{2\alpha} - m_2^{2\alpha}}{2m_1^\alpha m_2^\alpha},$$

and

$$\cos(\tau_2) = \frac{(m_2 + m_3)^{2\alpha} - m_2^{2\alpha} - m_3^{2\alpha}}{2m_2^\alpha m_3^\alpha}.$$

Thus, the fact for p to be both at a time in D_1 and in D_2 is well expressed by the fact that $\theta_1 \geq \tau_1$ and $\theta_2 \geq \tau_2$, i.e. $\cos(\theta_1) \leq \cos(\tau_1)$ and $\cos(\theta_2) \leq \cos(\tau_2)$.

Let us evaluate $\phi(p) := ||\sum_{i=1}^3 m_i^\alpha n_i||^2$. We have

$$\begin{aligned}\phi(p) &= ||\sum_{i=1}^3 m_i^\alpha n_i||^2 \\ &= m_1^{2\alpha} + m_2^{2\alpha} + m_3^{2\alpha} + 2m_1^\alpha m_2^\alpha \cos(\theta_1) + 2m_2^\alpha m_3^\alpha \cos(\theta_2) + 2m_1^\alpha m_3^\alpha \cos(\theta_1 + \theta_2).\end{aligned}$$

If we prove that $\phi(p) < 1$ for all $p \in D_1 \cap D_2$, this proves that a Ψ shape structure cannot be globally optimal. Let us denote $a = \cos(\tau_1)$ and $b = \cos(\tau_2)$. Since $\cos(\theta_1) \leq a$ and $\cos(\theta_2) \leq b$ for $p \in D_1 \cap D_2$, it is enough to prove that

$$m_1^{2\alpha} + m_2^{2\alpha} + m_3^{2\alpha} + 2m_1^\alpha m_2^\alpha a + 2m_2^\alpha m_3^\alpha b + 2m_1^\alpha m_3^\alpha (ab - \sqrt{1-a^2}\sqrt{1-b^2}) < 1,$$

for all m_1, m_2 and m_3 such that $m_1 + m_2 + m_3$ and all $\alpha > \frac{1}{2}$. Because this expression is symmetric with respect to m_1 and m_3, we can suppose without loss of generality that $m_1 > m_3$.

Since $m_1 \geq m_3$, lemma 6.6.1 implies that $a \geq b$. Thus, $ab - \sqrt{1-a^2}\sqrt{1-b^2} \leq 2a^2 - 1$ and it is enough to prove that

$$m_1^{2\alpha} + m_2^{2\alpha} + m_3^{2\alpha} + 2m_1^\alpha m_2^\alpha a + 2m_2^\alpha m_3^\alpha b + 2m_1^\alpha m_3^\alpha (2a^2 - 1) < 1,$$

in order to prove that $\phi(p) < 1$. This expression can be simplified in

$$(1 - m_1)^{2\alpha} + (1 - m_3)^{2\alpha} - m_2^{2\alpha} + \frac{m_3^\alpha}{m_1^\alpha m_2^{2\alpha}}((m_1 + m_2)^{2\alpha} - m_1^{2\alpha} - m_2^{2\alpha})^2 - 2m_1^\alpha m_3^\alpha < 1,$$

and the fact that no Ψ shape can be optimal would then be a consequence of the following conjecture.

Conjecture 6.6.7 *For every $m_1, m_2, m_3 > 0$ such that $m_1 + m_2 + m_3 = 1$, and every $1 > \alpha > \frac{1}{2}$,* $(1 - m_1)^{2\alpha} + (1 - m_3)^{2\alpha} - m_2^{2\alpha} + \frac{m_3^\alpha}{m_1^\alpha m_2^{2\alpha}}((m_1 + m_2)^{2\alpha} - m_1^{2\alpha} - m_2^{2\alpha})^2 - 2m_1^\alpha m_3^\alpha < 1.$

Hints: This inequality seems to hold. The main argument for it is numerical: the inequality has been numerically tested on a regular mesh of 1000^3 values and was always true. The other hint is the following: let us denote $\phi(m_1, m_3, \alpha)$ the expression on the left hand side. We are interested in proving that $\phi < 1$ in the domain $D := T \times]\frac{1}{2}, 1[$ where $T := \{(x, y) : x \in]0, 1[, y < x\}$. It is easy to prove that $\phi|_{\partial D} \leq 1$. Moreover, if $m_1 = m_3 = m$, then the inequality is true. Indeed, we have

$$\phi(m, m, \alpha) = \frac{((1 - m)^{2\alpha} - m^{2\alpha})^2}{(1 - 2m)^{2\alpha}},$$

so that $\phi(m, m, \alpha) < 1$ if and only if $(1 - m)^{2\alpha} - m^{2\alpha} < (1 - 2m)^\alpha$ (since $m < \frac{1}{2}$). By concavity of $x \mapsto x^\alpha$, we have $(1 - 2m + m^2)^\alpha < (1 - 2m)^\alpha + m^{2\alpha}$ so that $\phi(m, m, \alpha) < 1$.

Chapter 7

Examples of optimal irrigation

Introduction

Because of the atomic regularity of the previous chapter, we are now in a position to investigate particular examples. In section 7.1, we shall prove that an optimal structure for the problem of irrigating two masses from one source has a tree structure and we shall describe analytically this case. This first example is very important since it gives very constraining angle conditions at bifurcation points. Further, this example is the foundation of a recursive algorithm of construction that was proposed in [18] and that we shall present in the next chapter. In section 7.2 we investigate the structure of an optimal traffic plan irrigating Lebesgue measure on the segment from one source and study if the tree gets totally spread as in the case of the Monge-Kantorovitch transport problem or if diffusion along the segment occurs.

7.1 Optimum irrigation from one source to two sinks

Let a_1, a_2, a_3 in $\mathbb{R}^N$ with $a_1 \neq a_2$, $\mu^- = m_1\delta_{a_1} + m_2\delta_{a_2}$ and $\mu^+ = m_3\delta_{a_3}$ with $m_3 = m_1 + m_2$ and $m_1, m_2 > 0$. We are looking for the optimal traffic plan from μ^- to μ^+ under the E^α cost.

Lemma 7.1.1 *In the case a_1, a_2, a_3 are aligned, an optimal traffic plan from μ^- to μ^+ has its support in the minimal segment containing a_1, a_2, a_3. Otherwise, an optimal traffic plan has its support in the triangle a_1, a_2, a_3. In addition, it is a graph with two edges or three edges.*

Proof: Because of the convex envelop property 6.1.3, the support of an optimal traffic plan from μ^- to μ^+ is in the convex envelop of a_1, a_2 and a_3. Further, proposition 6.3.3 proves that an optimal traffic plan is a graph with at most 3 edges. □

Lemma 7.1.2 *Let μ be an optimal traffic plan from μ^- to μ^+ made of three edges. With the notation of Figure 7.1, the bifurcation point a has to satisfy the following angle constraints:*

$$\cos(\theta_1) = \frac{k_1^{2\alpha} + 1 - k_2^{2\alpha}}{2k_1^\alpha} \tag{7.1}$$

$$\cos(\theta_2) = \frac{k_2^{2\alpha} + 1 - k_1^{2\alpha}}{2k_2^\alpha} \tag{7.2}$$

$$\cos(\theta_1 + \theta_2) = \frac{1 - k_2^{2\alpha} - k_1^{2\alpha}}{2k_1^\alpha k_2^\alpha}, \tag{7.3}$$

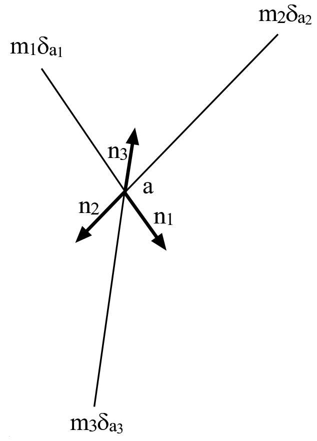

Figure 7.1: If an optimum has a Y-structure, the perturbation of the bifurcation point gives necessary condition through the cancellation of the derivative of the cost.

where $k_1 = \frac{m_1}{m_1+m_2}$, $k_2 = \frac{m_2}{m_1+m_2}$.

Proof: Because of lemma 7.1.1, it is equivalent to consider the two dimension situation. Let us consider the graph $G(a)$ made of edges (a_1a, m_1), (a_2a, m_2) and (aa_3, m_3), with $a \in \mathbb{R}^2 \setminus \{a_1, a_2, a_3\}$. The cost of this graph is

$$C(a) = m_1^\alpha ||a_1 - a|| + m_2^\alpha ||a_2 - a|| + m_3^\alpha ||a - a_3||.$$

Notice that this function is differentiable on $\mathbb{R}^2 \setminus \{a_1, a_2, a_3\}$. Thus, if $G(a)$ is an optimal path with $a \notin \{a_1, a_2, a_3\}$, we have $\frac{\partial}{\partial x} C(a) = 0$ and $\frac{\partial}{\partial y} C(a) = 0$. Let us denote respectively by (x_1, y_1), (x_2, y_2) and (x_3, y_3) the cartesian coordinates of a_1, a_2 and a_3. We have

$$\frac{\partial}{\partial x} C(a) = m_1^\alpha \frac{(x - x_1)}{||a_1 - a||} + m_2^\alpha \frac{(x - x_2)}{||a_2 - a||} + m_3^\alpha \frac{(x - x_3)}{||a_3 - a||},$$

and

$$\frac{\partial}{\partial y} C(a) = m_1^\alpha \frac{(y - y_1)}{||a_1 - a||} + m_2^\alpha \frac{(y - y_2)}{||a_2 - a||} + m_3^\alpha \frac{(y - y_3)}{||a_3 - a||}.$$

For $a \notin \{a_1, a_2, a_3\}$, let us denote by $n_i = \frac{a - a_i}{||a - a_i||}$ the unit vector from a_i to a for $i = 1, 2, 3$. The necessary condition given by the derivative of the cost function yields the balance equation

$$m_1^\alpha n_1 + m_2^\alpha n_2 + m_3^\alpha n_3 = 0. \tag{7.4}$$

Let θ_i be the angle between n_i and $-n_3$ for $i = 1, 2$ and $k_1 = \frac{m_1}{m_1+m_2}$, $k_2 = \frac{m_2}{m_1+m_2}$. Multiplying the balance equation (7.4) by n_i for $i = 1, 2, 3$ we obtain the following equalities:

$$\begin{aligned} k_1^\alpha + k_2^\alpha n_1 n_2 &= \cos(\theta_1) & (7.5)\\ k_1^\alpha n_1 n_2 + k_2^\alpha &= \cos(\theta_2) & (7.6)\\ k_1^\alpha \cos(\theta_1) + k_2^\alpha \cos(\theta_2) &= 1, & (7.7) \end{aligned}$$

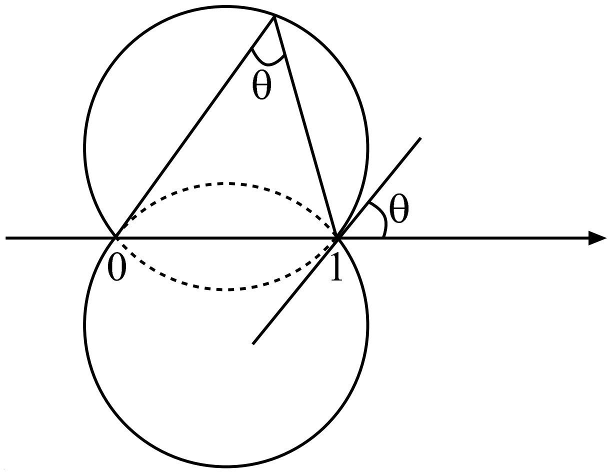

Figure 7.2: The locus of constant angle between M and two prescribed points is the union of two circle arcs.

so that the angles satisfy

$$\cos(\theta_1) = \frac{k_1^{2\alpha} + 1 - k_2^{2\alpha}}{2k_1^{\alpha}} \quad (7.8)$$

$$\cos(\theta_2) = \frac{k_2^{2\alpha} + 1 - k_1^{2\alpha}}{2k_2^{\alpha}} \quad (7.9)$$

$$\cos(\theta_1 + \theta_2) = \frac{1 - k_2^{2\alpha} - k_1^{2\alpha}}{2k_1^{\alpha} k_2^{\alpha}}. \quad (7.10)$$

This means that in the triangle $a_1 a a_3$, the angle at a is $\pi - \theta_1$ and in the triangle $a_2 a a_3$, the angle at a is $\pi - \theta_2$. □

Remark 7.1.3 *Notice that in the case* $m_1 = m_2$, $\theta_1 = \theta_2 = \arccos(2^{2\alpha-1} - 1)/2$. *If* $\alpha = \frac{1}{2}$ *the angles satisfy* $\theta_1 + \theta_2 = \frac{\pi}{2}$, $\theta_1 = \sqrt{k}_1$ *and* $\theta_2 = \sqrt{k}_2$. *Thus the bifurcation point lies on the circle of diameter* $a_1 a_2$. *If* $\alpha = 0$, *we find the* $\frac{2\pi}{3}$ *angle constraint that has to satisfy a Steiner point in the Steiner tree problem.*

Lemma 7.1.4 *Given two points* b *and* c *and an angle* θ, *the set of points* a *so that the not oriented angle* bac *is* θ *is the union of two circle arcs going through* b *and* c, *with radius* $\frac{||c-b||}{2\sin(\theta)}$.

Proof:

The set of points a so that the not oriented angle bac is θ is given by the equation

$$(b - a) \cdot (c - a) = cos(\theta)||b - a|| \cdot ||c - a||. \quad (7.11)$$

To maintain simple calculations, we can assume with a suitable rotation and scaling that $b = (0, 0)$ and $c = (1, 0)$. Let us denote by (x, y) the cartesian coordinates of a. Equation 7.11 becomes

$$x(x - 1) + y^2 = cos(\theta)\sqrt{x^2 + y^2}\sqrt{x(x - 1) + y^2}.$$

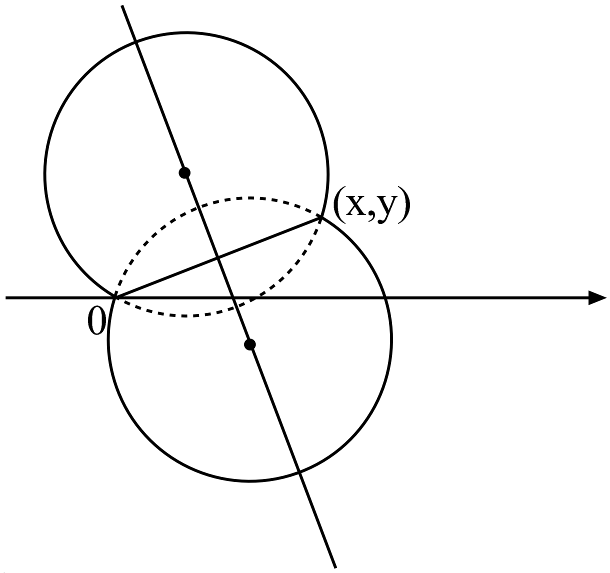

Figure 7.3: The center of these circles can be obtained through a scaling transformation.

Squaring this equation, we obtain a polynomial equation that has to satisfy $a = (x, y)$. We notice that it is the product of two circle equations:

$$(-p^2+1)[(x-1/2)^2+(y-y_c)^2-R_c^2][(x-1/2)^2+(y+y_c)^2-R_c^2]=0,$$

where $p = \cos(\theta)$, $y_c = \sqrt{\frac{p^2}{4(1-p^2)}}$ and $R_c = \sqrt{\frac{1}{4(1-p^2)}}$. Notice that if $a = (x, y)$ satisfies this equation, it is no more sufficient for the angle bac to be θ. Indeed, we squared $\cos(\theta)$, so that the angle bac is θ or $\pi - \theta$. The set of points is then a subset of the two circles and is indeed the union of two symmetric connected components of the circles from which we remove $(0, 0)$ and $(1, 0)$. Let us now give the equation of equiangle points for $b = (0, 0)$ and $c = (x', y')$. We move from $(1, 0)$ to (x', y') with a scaling of factor $||c - b||$ so that the radius of circles will be $||c - b||\sqrt{\frac{1}{4(1-p^2)}}$. To obtain the coordinates of the centers of two circles, we notice that it lies on the middle orthogonal of the segment $[bc]$ and is located at a distance $\sqrt{\frac{p^2}{4(1-p^2)}}$ of $\frac{b}{c}$. Thus, the two centers of the equiangle circles have the following coordinates: $c/2 \pm (-y', x')\sqrt{\frac{p^2}{4(1-p^2)}}$. □

Lemma 7.1.5 *Let μ be an optimal traffic plan from μ^- to μ^+ made of three edges. Let E be the equiangle circle arc associated to a_1, a_2 and θ, which is in the same half plane as a_1. Let E' be the complementary circle arc. There is a "pivot" point $p \in E'$ which does not depend on a_3 such that the bifurcation point a is the intersection of a_3p with E.*

Proof: Let us denote by p the instersection of the line a_3a with E'. The bifurcation point a has to satisfy the angle conditions given by 7.1.2, i.e. the angle paa_1 is prescribed as equal to θ_1. Since $E \cup E'$ is the only circle going through a, p and a_1, E is an equiangle circle arc for a, p and the angle θ_1 so that the point p does not depend on the source point a_3. Thus, the optimal bifurcation point is obtained as the intersection of the line a_3p with E. Let us denote c the center of the equiangle circle. The angle a_1cp is twice the angle a_1ap which is θ_1. Thus, the "pivot" point is easily constructed as the rotation of a_1 with angle $2\theta_1$. □

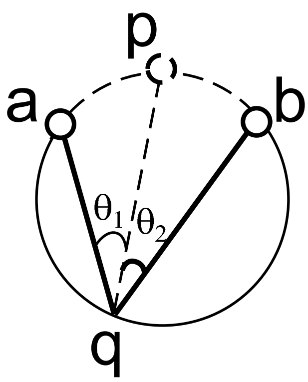

Figure 7.4: The equiangle locus associated to a, p with an angle θ_1 is supported by a circle and thus is the same circle as the equiangle locus associated to a, b with an angle $\theta_1 + \theta_2$.

Lemma 7.1.6 *Let μ be an optimal traffic plan from μ^- to μ^+ made of three edges and p the "pivot" point obtained in lemma 7.1.5. The cost of μ is $|a_3p|$.*

Proof: Indeed, it is a direct consequence of Ptolemy's theorem stating that the diagonals of a quadrilateral equals the sum of the products of the opposite sides. Let us notice first that $|a_1p| = |a_1a_2|(\frac{m_2}{m_3})^\alpha$ and $|a_2p| = |a_1a_2|(\frac{m_1}{m_3})^\alpha$. When applied to the equilateral a_1aa_2p, the theorem becomes $|a_3p||a_1a_2| = |aa_1||pa_2| + |aa_2||pa_1|$. Thus,

$$m_1^\alpha|aa_1| + m_2^\alpha|aa_2| = m_3^\alpha|a_3p|.$$

Proposition 7.1.7 *Let μ be an optimal traffic plan from μ^- to μ^+. Let p be the pivot point associated to $(a_1, m_1), (a_2, m_2)$, in the half plane not containing a_3. There are four different zones for a_3. If $a_3p \cap E = \{a\}$, either $a \in [a_3p]$ and the optimal has three edges with a the bifurcation point, or $a \notin [a_3p]$ and the optimal is made of the two edges $[a_3a_1]$ and $[a_3a_2]$. If $a_3p \cap E = \emptyset$, then either $|a_3a_1| < |a_3a_2|$ and μ is made of the two edges $[a_3a_1]$ and $[a_1a_2]$ or $|a_3a_2| < |a_3a_1|$ and μ is made of the two edges $[a_3a_2]$ and $[a_2a_1]$*

Proof: The four zones are illustrated by figure 7.6. If $a \notin [a_3p]$ or $a_3p \cap E = \emptyset$, then an optimal structure cannot have three edges because no bifurcation point is able to satisfy necessary angle conditions. The optimum thus have an "L" or "V" structure, depending on the position of the source point a_3. If $a \in [a_3p]$, the three edges graph thus obtained is optimal. □

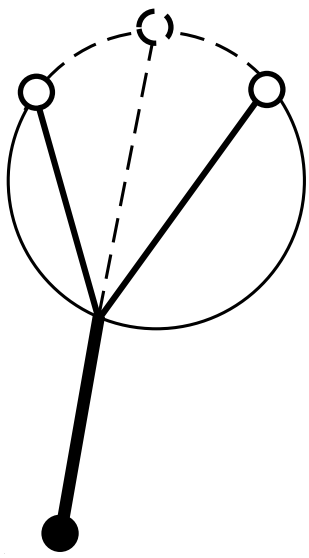

Figure 7.5: The bifurcation satisfying the angle constraints given by the balance equation is obtained as the intersection of the equiangle circle and the source to pivot point line.

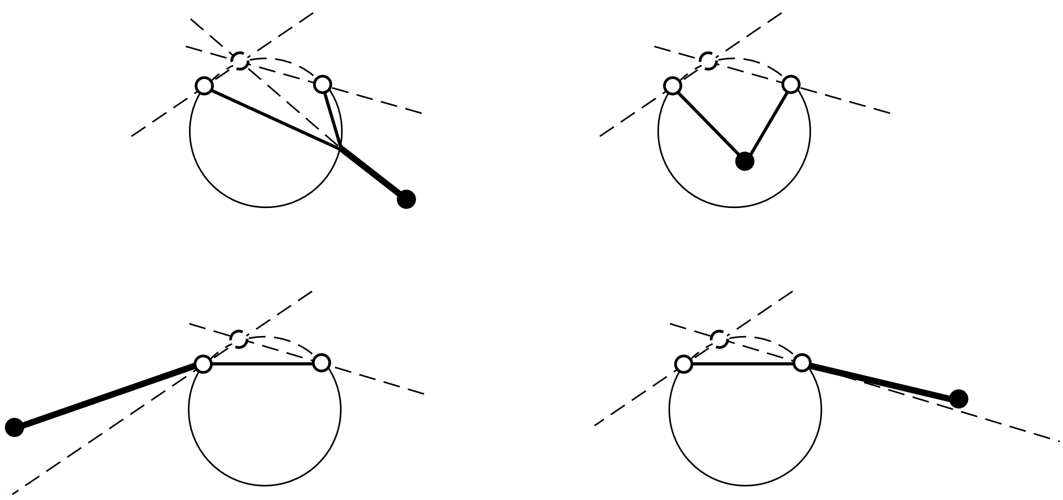

Figure 7.6: Let us sum up the process that permits to find the optimal structure from one source to two sinks. 1) Given the masses m_1 and m_2, we obtain the angle θ at an optimal bifurcation. 2) We draw the equiangle circle, the pivot point and the lines pa_1 and pa_2. 3) Depending on the position of the source point, we obtain one of the four possible configuration that are represented on this figure.

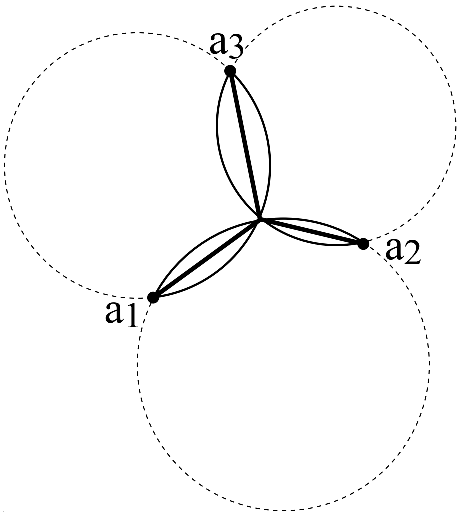

Figure 7.7: The bifurcation point of an optimal ”Y” lies on the three equiangle circles.

7.2 How to irrigate a Lebesgue segment

Let μ^+ be the Lebesgue measure on the segment $[0,1] \times \{0\}$ and $\mu^- = \delta_S$ the Dirac mass at the point S. In the following, we discuss the structure of an optimal traffic plan from μ^- to μ^+. First, we determine an optimal traffic plan in the case where $S \in \mathbb{R} \times \{0\}$. This is the case of dissipation along the path.

7.2.1 The case of a source aligned with the segment

Lemma 7.2.1 *Let $S \in \mathbb{R} \times \{0\}$ and μ be an optimal traffic plan from μ^+ to μ^-. Then μ is equivalent to $\chi \# \lambda$ where $\chi(\omega, t) = \min(\omega, t)$. If $S = (0,0)$, $E(\mu) = \frac{1}{\alpha+1}$.*

Proof: The convex envelop property 6.1.3 tells that the support of μ is in the axis of the segment. Because of the no-loop, the mass is dissipated uniformly along the fibers. Thus, $E(\mu) = \int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$. □

7.2.2 A ”T structure” is not optimal: the better Y structure proof

Let $S \notin [0,1] \times \{0\}$.

Definition 7.2.2 *Let $s \in [0,1]$ and δ_s the Dirac mass located at $(s,0)$. To every $s \in [0,1]$, we associate μ_s the traffic plan obtained as the concatenation of the optimal traffic plan from μ^- to δ_s and from δ_s to μ^+. We say that such a traffic plan has a T-structure.*

Lemma 7.2.3 *A traffic plan with T structure is not optimal.*

Proof: Let μ be the T structure associated to $s \in [0,1]$. By construction, a mass s is irrigating the segment $[0,s]$, and a mass $1-s$ is irrigating the segment $[s,1]$. We shall now prove that it is possible to find a Y structure more efficient than the T one.
Let us consider a Y configuration with ending points of coordinate $s-x$ and $s+x$ where x is to be

Figure 7.8: A $T-$structure and a Y-structure perturbation of it.

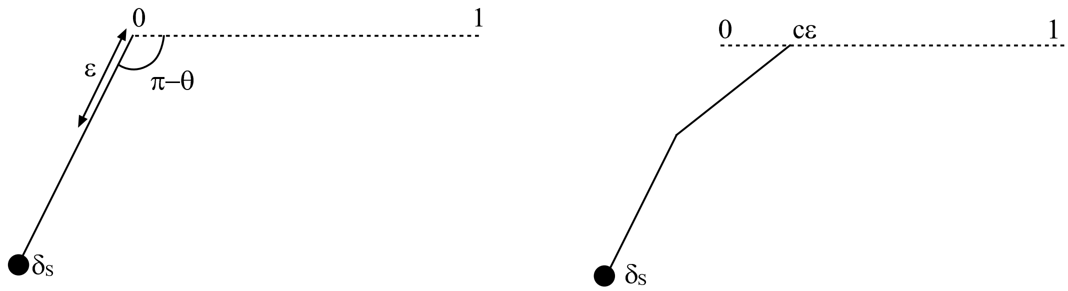

Figure 7.9: A degenerate T-structure and a perturbation of it.

determined. The bifurcation is located at a distance ϵ from s.
The cost of this Y structure can be written

$$\phi(\epsilon,x)=s^{\alpha}a+(1-s)^{\alpha}b+\frac{1}{\alpha+1}(2x^{\alpha+1}+(s-x)^{\alpha+1}+(1-s-x)^{\alpha+1})$$

where $a=\sqrt{\epsilon^2+x^2+2\epsilon x\cos(\theta)}$ and $b=\sqrt{\epsilon^2+x^2-2\epsilon x\cos(\theta)}$.
Let us define $v(\epsilon)=\phi(\epsilon,c\epsilon)$.
The cost of the modified part of the T structure is

$$u(\epsilon)=\epsilon+\frac{1}{\alpha+1}(s^{\alpha+1}+(1-s)^{\alpha+1}).$$

Notice that $v(0)=u(0)$ and $u'(0)=1$. Thus it is sufficient to show that for some suitable c, $v'(0)<1$ so that $v(\epsilon)<u(\epsilon)$ for a sufficiently small ϵ. Let us calculate the derivative of v at point 0,

$$v'(0)=\left(\sqrt{c^2+1+2c\cos(\theta)}s^{\alpha}+\sqrt{c^2+1-2c\cos(\theta)}(1-s)^{\alpha}\right)-c(s^{\alpha}+(1-s)^{\alpha}).$$

For $c=0$, $v'(0)=s^{\alpha}+(1-s)^{\alpha}$. For c near infinity, the asymptotic expansion of $v'(0)$ is

$$\begin{aligned} v'(0) &= c\left(s^{\alpha}(1+\frac{\cos(\theta)}{c}+O(\frac{1}{c^2}))+(1-s)^{\alpha}(1-\frac{\cos(\theta)}{c}+O(\frac{1}{c^2}))\right)-c(s^{\alpha}+(1-s)^{\alpha}) \\ &= \cos(\theta)(s^{\alpha}-(1-s)^{\alpha})+O(\frac{1}{c}) \end{aligned}$$

Let us suppose that $\theta\notin\pi\mathbb{Z}$ and $s\notin\{0,1\}$, then, because of the continuity of $v'(0)$ regarding c, we deduce that for a sufficiently large c, $v'(0)<1$.

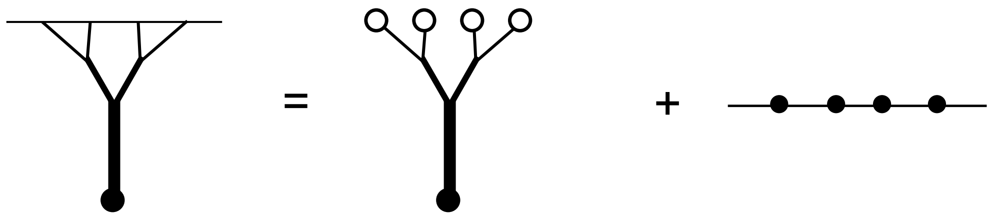

Figure 7.10: The cut at Lebesgue segment induces optimal before and after cut traffic plans.

If $s = 0$, or $s = 1$, and $\theta \notin \pi\mathbb{Z}$, let us show that the T structure is not optimal. The cost of the traffic plan defined with a junction at coordinate x is

$$\phi(\epsilon, x) = \sqrt{\epsilon^2 + x^2 + 2\epsilon x \cos(\theta)} + \frac{1}{\alpha+1}(x^{\alpha+1} + (1-x)^{\alpha+1}).$$

Let us consider $v(\epsilon) = \phi(\epsilon, c\epsilon)$. Then, $v(0) = u(0)$ and $v'(0) = \sqrt{1 + 2\cos(\theta) + c^2} - c$ so that with c at infinity, $v'(0) = \cos(\theta) + O(\frac{1}{c})$. Thus, in case $\theta \notin \pi\mathbb{Z}$, the T structure with a junction at $s = 0$ or $s = 1$ is not optimal so that we can find a better T-structure with junction in $]0, 1[$ and therefore a better Y-structure. □

7.2.3 An optimum has not finite graph + fibers along the segment structure

Proposition 7.2.4 *Let μ be an optimal traffic plan from $\mu^- = \delta_S$ to μ^+, where μ^+ is the Lebesgue measure on the unit segment. Let us denote by ν the measure obtained stopping fibers when they attain the segment. The traffic plan ν is not atomic finite.*

Proof: Let χ be a parameterization of μ and suppose that $\nu = \sum_{i=1}^{n} a_i \delta_{x_i}$. The cut at Lebesgue segment induces a traffic plan $\tilde{\mu}$ which transports μ^- to ν. Because μ is optimal, $\tilde{\mu}$ is optimal and lemma 6.3.3 proves that $\tilde{\mu}$ has a finite graph structure. Let us consider Ω_i the set of fibers going through x_i, i.e. $\Omega_i := [x_i]_\chi$. Let us denote by μ_i the measure irrigated by Ω_i. Because of the no-loop property, the Ω_i are disjoints and the support of the measures μ_i form a partition of $[0, 1]$. Thus we can consider an interval $I \subset [0, 1]$ such that $x_1 \in I$ and the restriction of the traffic plan to Ω_1 induces a traffic plan with T-structure irrigating Lebesgue measure on the interval I. It should be optimal as a restricted traffic plan but is not because of lemma 7.2.3. □

7.2.4 Can fibers move along the segment in the optimal structure?

Because of proposition 7.2.4, we know that ν is not a finite atomic measure. In the case $\alpha = 1$, the transport problem is the one of Monge-Kantorovich and then the cut of an optimum at the unit segment is the Lebesgue measure on this segment. What if $\alpha < 1$? Does it depend on the position of the source or not?

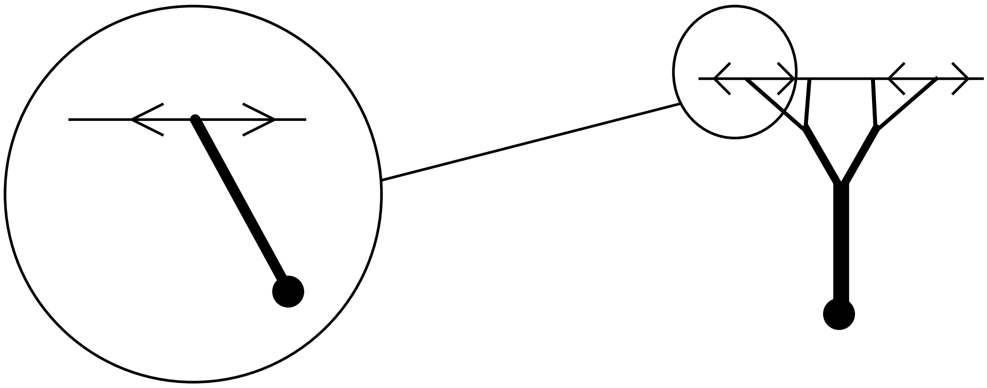

Figure 7.11: Let ν be the measure obtained stopping fibers of μ when they reach the unit segment. If ν was a finite atomic measure, then the restriction around a Dirac mass of ν would be an optimal T structure.

Conjecture 7.2.5 *Let μ be an optimal traffic plan from μ^- to μ^+ with $\alpha < 1$ and ν the cut of μ at Lebesgue segment. The measure ν is an infinite atomic measure, i.e. $\nu = \sum m_i \delta_{a_i}$, where m_i are positive and $i \in \mathbb{N}$.*

Hint: The case of $\alpha = \frac{1}{2}$ seems to be more tractable than the general case since in that case, the angle constraint formula is particularly simple, i.e. at a bifurcation point, $\cos(\theta_1) = \sqrt{k}_1$ and $\cos(\theta_2) = \sqrt{k}_2$. Very roughly speaking, there is a $\pi/2$ angle at any bifurcation of an optimal tree so that not many bifurcation can occur along a path in an optimal tree, this obliges in some way the tree not to bifurcate so that it has to dissipate.

Some numerical evidences (see chapter 8) go in the direction of the conjecture. Indeed, the optimal shape irrigating an atomic approximation of Lebesgue measure on the segment shows that as the mesh increases, the path are getting flatter and flatter so that it suggests that diffusion will occur in the end along any path.

Chapter 8

Algorithms

Introduction

Numerical experiments can be important to rule out conjectures or to gain intuition on the structure of efficient traffic plans. In the first section we present an algorithm proposed by Xia and explain why it cannot give a global optimum. We then consider the optimization problem of finding the/a best traffic plan as two separated problems: a topological optimization and an optimization of nodes. Indeed, given a topology of the structure, there generally exists a local optimum with this prescribed topology. Thus, an algorithm for this problem should both try to optimize the topology and the position of nodes of the graph. In the article [18], Gilbert presented a recursive construction with ruler and compass that permits to obtain the exact position of nodes of an optimal structure, for a general cost $\sum f(c_e)l(e)$, where f is any concave function. We present this recursive construction in section 8.2. We then give examples of exhaustive search through all possible topologies in the case where target Dirac masses are aligned. Indeed, when target Dirac masses are aligned the number of possible topologies is drastically reduced. For more than 10 target points, the combinatorial explosion requires to search through a reduced number of topologies. The multiscale approach and different type of perturbation of the topology are a good way to obtain efficient structures in a reasonable time. All the algorithms that we just spoke about are confined to the plane and to the one source to any measure problem. We explain in the last section why it is difficult to move to a ”any measure to any measure” problem and to increase dimension.

8.1 An algorithm suggested by Xia in [35]

8.1.1 Presentation of the algorithm

Let us recall the notation of the dyadic approximation of a measure presented in section 5.1.3. Let C be a cube with edge length L and center c. Let ν be a probability measure on the compact X where $X \subset C$. We may approximate ν by atomic measures in $\mathcal{A}_\Lambda(X)$ as follow. For each i, let

$$\mathcal{C}_i := \{C_i^h : h \in \mathbb{Z}^N \cap [0, 2^i)^N\}$$

be a partition of C into cubes of edge length $\frac{L}{2^i}$. Now, for each $h \in \mathbb{Z}^N \cap [0, 2^i)^N$, let c_i^h be the center of C_i^h and $m_i^h = \nu(C_i^h)$ be the μ mass of the cube C_i^h. We define the atomic measure

$$A_i(\nu) = \sum_{h \in \mathbb{Z}^N \cap [0,2^i)^N} m_i^h \delta_{c_i^h},$$

which is classically weakly converging to μ. This approach is justified by corollary 5.3.3. Indeed, the limit of sequence of optimal traffic plans is optimal.

Let μ be any probability measure in the cube $C \subset \mathbb{R}^N$ with edge length L. In section 6 of [35], Xia proposes an algorithm to compute an optimal transport path from a Dirac masses δ_p where $p \in \mathbb{R}^N$ to μ. Let H be a fixed positive real number.

1. Given an approximating depth n, let $a_n = A_n(\mu)$, be the $n-$th dyadic approximation of order n.

2. For each $h \in \mathbb{Z}^N \cap [0, 2^i)^N$, the cube C^h_{n-1} of level $n-1$ consisting in 2^N subcubes of level n. For any $x \in X \times [0, H]$, let G^h_x be the union of (the cone over $a_n \lfloor Q^h_{n-1}$ with vertex x) and the line segment $\bar{xp}$ with weight $\mu(C^h_{n-1})$. Then G^h_x is a transport path from $a_n(\mu) \lfloor C^h_{n-1}$ to $\mu(C^h_{n-1})\delta_p$. Let $q^h \in X \times [0, H]$ be the point at which $M^\alpha(G^h_x)$ achieves its minimum among all $x \in X \times [0, H]$. Let
$$a_{n-1} = \sum_{h \in \mathbb{Z}^N \cap [0,2^i)^N} \mu(C^h_{n-1})\delta_{q^h}.$$

3. For each $k = n-1, ..., 1$, repeatedly doing step 2 to get a_{k-1}. In the end, we get a transport path G_n from a_n to δ_p with finite M_α mass.

4. By using optimization from one source to two sinks, we can locally optimize the locations of the vertices of G. One may repeatedly doing upward and downward optimization until the transport path converges to a fixed graph.

5. Increase depth n to get better approximation

8.1.2 Results and criticisms

We refer the reader to [35] to see the genuine figures obtained by Xia. However, for a sake of completeness we shall represent on figure 8.1 some trees with very similar shapes. These results suggest three remarks:

- We can see on figure 8.1 and in [35] that the structure of the tree is homogeneous dyadic in the sense that at every bifurcation, the mass is split into two equal parts. This is due to the step 2 of the algorithm.

- The second remark is that the cost of trees represented in figure 8.1 are not all identical with the cost of trees in [35]. This is certainly due to the fact that the step 4 of the algorithm is not efficient in optimizing the structure. Indeed, changing the location of bifurcation points upward and downward is very costly and takes a lot of iterations to stabilize. Thus, during the optimization process, the cost decreases very slowly so that the trees obtained by Xia are in general not fully stabilized.

- The cost of the tree for $\mu^- = \lambda_{64}$ is much lower than the one obtained by Xia in [35] for another reason than the previous remark. Indeed, if one look closely at tips, the tree we obtained has a degenerate topology in the sense that the Dirac mass on the extreme left of λ_{64} is not irrigated from a bifurcation point but from a point of the support of λ_{64}.

The first remark raises the question of whether or not a ”homogeneous” dyadic structure is optimal. The answer is generally no. Indeed, two arguments prove that fact:

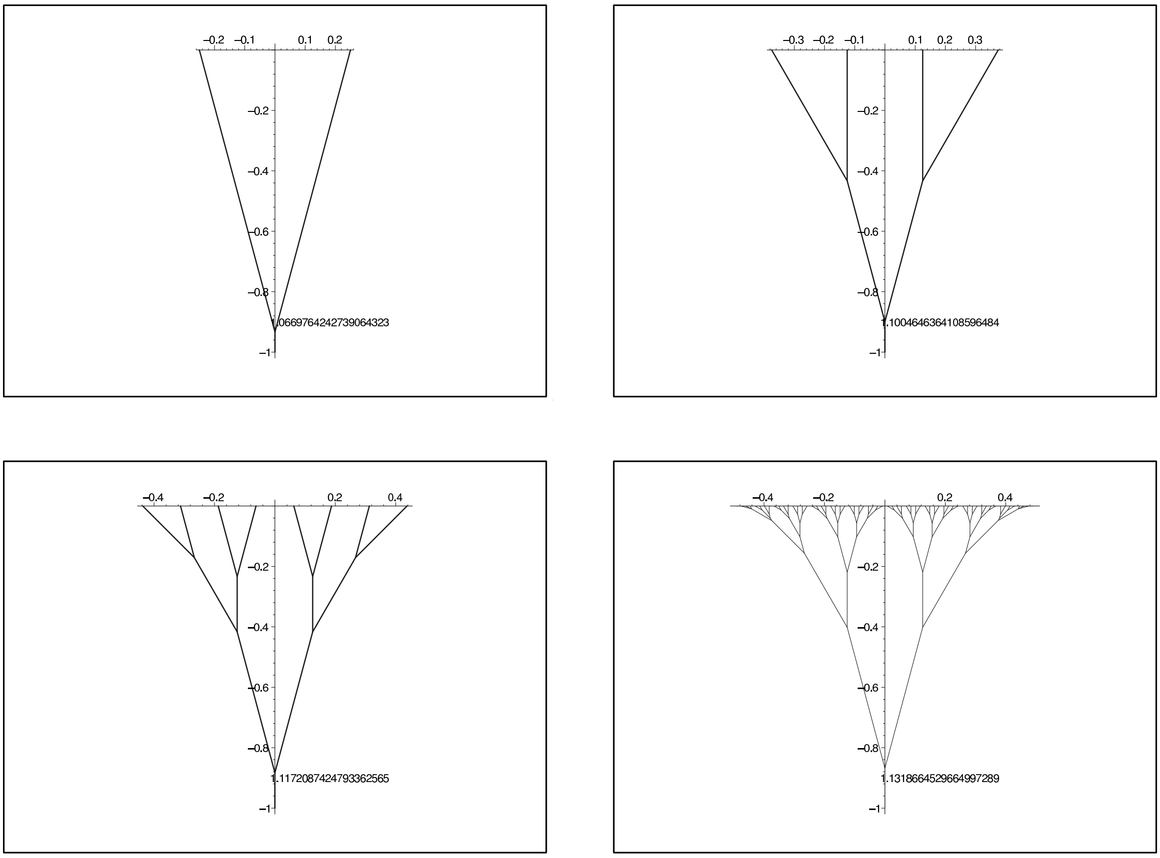

Figure 8.1: Let μ^+ be a Dirac mass at (0,-1) and $\mu^- = \lambda_n$ with $n = 2, 4, 8$ and $n = 64$ where λ_n is the dyadic approximation of level n of Lebesgue measure on the segment. From top left to bottom right, the figures represent the trees considered in [35] as being optimal for $\alpha = 0.95$.

- The numerical argument consists of the tree represented on figure 8.2. It has a better cost than the best homogeneous dyadic tree.

  Another numerical example illustrating why the dyadic structure is not always the best one is represented on figure 8.3.

- The angle argument: as stated in remark 7.1.3, when the two exit masses are equal at a bifurcation point, the angle variation is equal to $\arccos(2^{2\alpha-1}-1)/2$. Thus, if we consider the path on the left of a homogeneous dyadic tree, the angle variation after each bifurcation is $\arccos(2^{2\alpha-1}-1)/2$ (see figure 8.4). So, after n bifurcations, the path has an angle $n\arccos(2^{2\alpha-1}-1)/2$ with the vertical. Since this angle cannot exceed $\frac{\pi}{2}$, it means that diffusion has to occur if n is sufficiently large.

In the end, these three remarks can finally be formulated as criticisms of this algorithm:

- By construction, the algorithm proposed by Xia can only lead to a homogeneous dyadic tree. Such a tree is generally not optimal. The upward and downward optimization of the step 4 cannot modify the topology so that the algorithm cannot reach an optimal tree. We shall present a method to explore all possible topologies in the simple case of the irrigation of Lebesgue measure on the segment.

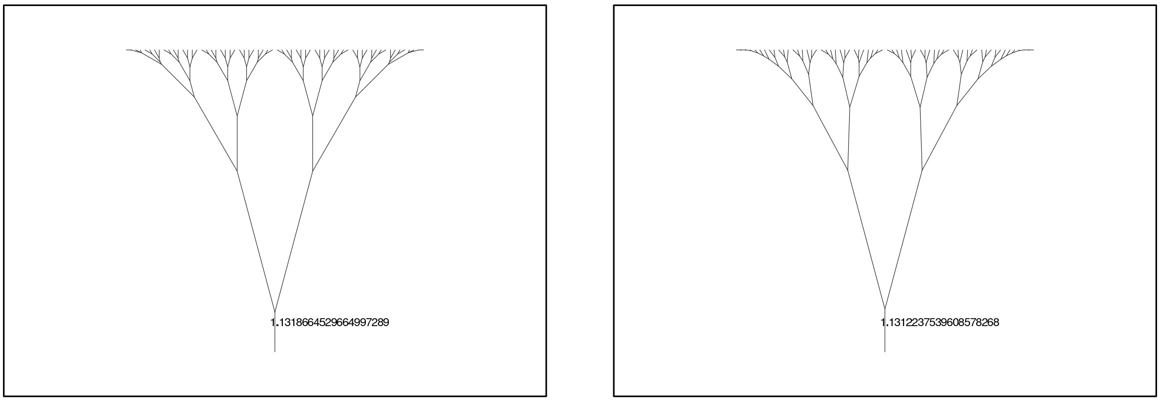

Figure 8.2: Two different topologies for the 64 problem $\alpha = 0.95$.

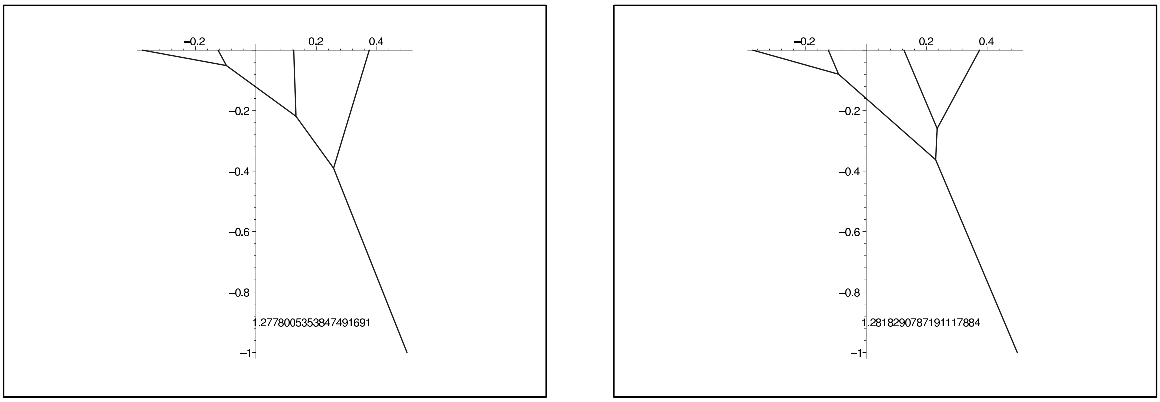

Figure 8.3: The structure on the left has a lower cost than the dyadic one represented on the right. Here is a hint to explain why: roughly speaking, the source is located on the right so that it is preferable to keep the mass as grouped as possible while it is transported from the right to the left ; each Dirac mass of the target is thus directly irrigated from the main flow.

- Step 4 consists in optimizing the location of bifurcation points upward and downward. This costs a lot of computer time since these points are numerous and since the stabilization of these location can take a while. Indeed, modifying the position of one point P obliges to change the position of all the other points to satisfy the angle condition we have at optimal bifurcations(see proposition 7.1.2). But since all the other points moved, the optimization process requires to move P again and so on. For the case $\alpha = \frac{1}{2}$, the angles at a bifurcation have to be $\frac{\pi}{2}$; due to the structure of the algorithm, this angle condition is clearly not respected by the figures of page 261 in [35]. This can be avoided by the recursive exact construction proposed in next section.

- Even looking after the best structure with the dyadic homogeneous topology, the step 4 prevents the algorithm from finding the best structure. Indeed, the step 4 is not always successful in moving from a topology to its degenerated topologies. This explains why the best dyadic tree of figure 8.1 looks different from the one in [35]. This calls for another algorithm taking into account degenerated topologies during the exhaustive search through all possible structures.

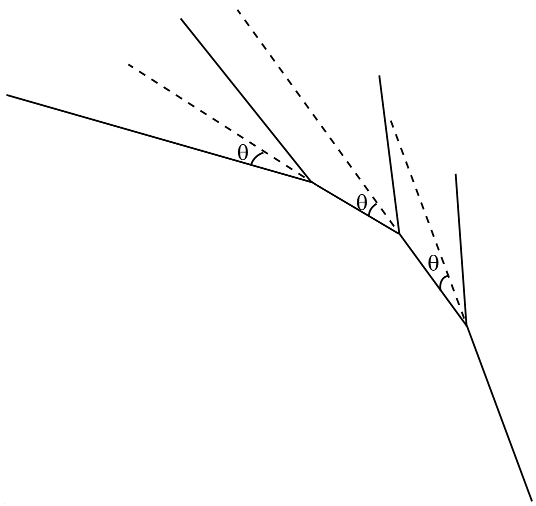

Figure 8.4: In the case of the homogeneous dyadic structure, the angle variation after each bifurcation is $\theta = \arccos(2^{2\alpha-1} - 1)/2$. This explains why there cannot be an infinite number of bifurcation when one consider the path on the extreme left of an optimal homogeneous dyadic tree.

8.2 Optimal shape of a traffic plan with given topology

The irrigation problem can be divided into two optimization problems: the optimization of the topology, and the optimization of the locations of bifurcation points. The optimization of the topology is treated in the section 8.4. Within this section, we present a recursive construction that gives the location of bifurcation points (i.e. Steiner points) of the optimal structure that has a prescribed topology. To explain this construction, we consider the simplest case of trees with full Steiner topology. We then consider the different possible degeneracies of topologies and explain how to take them into account.

8.2.1 Topology of a graph

Definition 8.2.1 *A topology $\mathcal{T}$ for a given point set $(v_i)_{i=1}^n$ of $\mathbb{R}^N$ is an undirected connected graph $G = (V, E)$ where E is the set of edges and $V = (v_i)_{i=1}^{n+m}$ is the set of vertices. The points $(v_i)_{i=n+1}^{n+m}$ which are not present in the initial point set $(v_i)_{i=1}^n$ are called Steiner points.*

Definition 8.2.2 *A finite traffic plan induces a graph structure and thus a topology. Let us denote $TP(\mu^-, \mu^+, \mathcal{T})$ the set of traffic plans with topology $\mathcal{T}$ and*

$$C(\mu^-, \mu^+, \mathcal{T}) := \inf_{TP(\mu^-,\mu^+,\mathcal{T})} E(\mu)$$

the cost of the topology $\mathcal{T}$.

Definition 8.2.3 *A Steiner topology is a topology $\mathcal{T}$ such that all vertices corresponding to Steiner point have degree 3. A full Steiner topology is such that it has $2n - 2$ vertices $(v_i)_{i=1}^{2n-2}$ and $2n - 3$ edges.*

8.2.2 A recursive construction of an optimal with full Steiner topology [18]

In this subsection, we shall assume that the optimal structure associated to a prescribed full Steiner topology is not degenerated.

Let us first recall the construction of the optimal structure in the case we transport a source S to two Dirac masses located at points a and b. Proposition 7.1.7 states that there is a pivot point P such that the only bifurcation point of the optimal structure with full Steiner topology is obtained as the intersection of the line SP with the circle abP.

In the more general case, this construction can be applied recursively as it was first described by Gilbert in [18]. Let us explain the recursive construction on a simple example. We consider μ^- as a target measure made of 4 Dirac masses a, b, c, d, and μ^+ the Dirac mass at a source point S. Let us suppose that the optimal structure has the full Steiner topology such that the first bifurcation occurs at b_1 and the first subtree irrigates a and b and the second subtree irrigates c and d. The second bifurcation at b_2 is such that one branch irrigates a and the other one b. At last, the bifurcation b_3 is such that one branch irrigates c and the other one d. This topology is in fact the simplest we can imagine and is illustrated by figure 8.6.

Let us explain why the construction of bifurcation points b_i is only a recursive way to apply the construction in the simplest "one source to 2 Dirac masses" case. Indeed, if we look for the best structure, every subtree has to be optimal for the irrigation problem it induces. That is to say, the subtree which irrigates a and b from b_1 is optimal, so is the subtree irrigating c and d from b_1. Thus, points b_2 and b_3 can be constructed thanks to pivot points p_1 and p_2 as in proposition 7.1.7. Next, the irrigation from S to b_2 and b_3 has to be optimal as a subtree of an optimal structure. As a consequence, the irrigation from S to p_1 and p_2 is also optimal. Indeed, since b_1, b_2 and p_1 are aligned, and b_1, b_3, p_2 are also aligned, the angle $p_1 b_1 p_2$ is the optimality angle so that the transport from S to p_1 and p_2 is optimal. Thus we can construct the position of b_1 through the pivot point p_3 associated to p_1 and p_2.

Let us now give the construction top to down then bottom-up.

- The prescribed topology is such that a is grouped with b and c with d. Thus we construct their associated pivot points p_1 and p_2.
- Since (to be found) bifurcation points b_2 and b_3 are then grouped, we construct the pivot point associated to p_1 and p_2.
- Since the subtree made of edges Sb_1, $b_1 p_1$ and $b_1 p_2$ is optimal, the bifurcation point b_1 is obtained as the intersection of the line Sp_3 with the circle $p_3 p_1 p_2$.
- Now that the bifurcation point b_1 is located, we obtain the bifurcation point b_2 as the intersection of the line $b_1 p_1$ with the circle $p_1 ab$. And we obtain the bifurcation point b_3 as the intersection of the line $b_1 p_2$ with the circle $p_2 cd$.

8.3 Optimal structure in the case of Lebesgue measure on the segment

8.3.1 Coding of the topology

Let $A = (a_i)_i$ be N points of the space. When the points $(a_i)_i$ are ordered on a line, it does not make sense to group first a_1 with a_3 and a_2 with a_4. No such mixing can occur in the case of an optimal structure, otherwise there would be a circuit which is impossible thanks to proposition 6.2.5. Thus, we can restrict to "parenthesis" topologies, i.e. to topologies corresponding to all the possible way to do the non-associative product $a_1 ... a_n$. We present here a convenient way to code for "parenthesis" topologies and to generate them all.

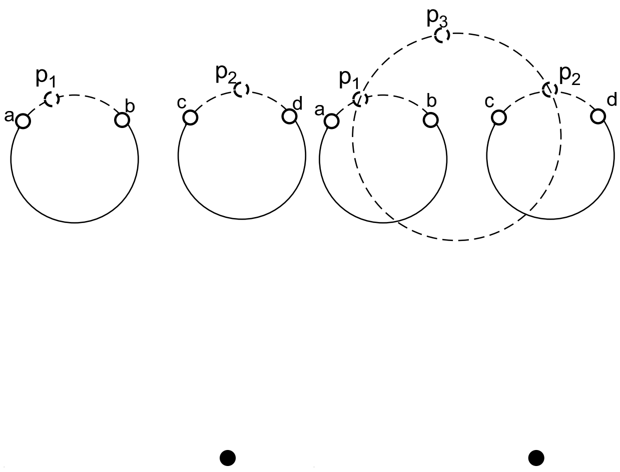

Figure 8.5: Given a topology, the pivot point permits to reduce two masses to one. Using this recursively permits to reduce the problem to the transport of a Dirac mass to a Dirac mass. This is the top-down part of the construction, i.e. the construction of the hierarchy of pivot points.

Definition 8.3.1 *All parenthesis topologies are recursively described by a list* $[t_1, t_2, .., t_{n-1}]$. *The coding works as follow:* t_1 *denotes the index of the first grouping so that we shrink* a_{t_1}, a_{t_1+1} *to a single formal point* $b1 := (a_{t_1}, a_{t_1+1})$. *Then* $[t2, ..., t_{n-1}]$ *describes the topology, of* $a_1, ..., a_{t_1-1}, b1, a_{t_1+2}, ..., a_n$. *The figure 8.7 permits to clearly understand how it works. As a matter of an example, the topology on the left of figure 8.3 is* $[1, 1, 1]$ *; the one on the right is* $[1, 2, 1]$.

Lemma 8.3.2 *The total number of topologies for* N *aligned points is the Catalan number* $\frac{1}{n}\binom{2n-2}{n-1}$.

8.3.2 Exhaustive search

Let us briefly mention that the coding of topologies is particularly adapted to the pivot point algorithm since it permits a recursive description of the topology. Thus, in the case of few Dirac masses at the target measure, it is possible to proceed to an exhaustive search through all topologies. This permits to find global optima in the case the target measure has less than 10 Dirac masses.

8.4 Heuristics for topology optimization

As it was said before, the irrigation problem can be divided into two optimization problems: the optimization of the topology, and the optimization of the locations of bifurcation points. The recursive construction presented in section 8.2 answers to the second optimization problem with an accurate construction along with an exhaustive search through all possible degeneracies of a topology. However, an exhaustive search through all topologies takes a lot of time and increasing the number of Dirac masses causes combinatorial explosion. Several heuristics can help in finding a reasonable topology within a reasonable time or in improving it. We present three of them:

- The multiscale approach permits to find efficient topologies thanks to a compromise between accuracy of the resolution of the target measure and exhaustive search.

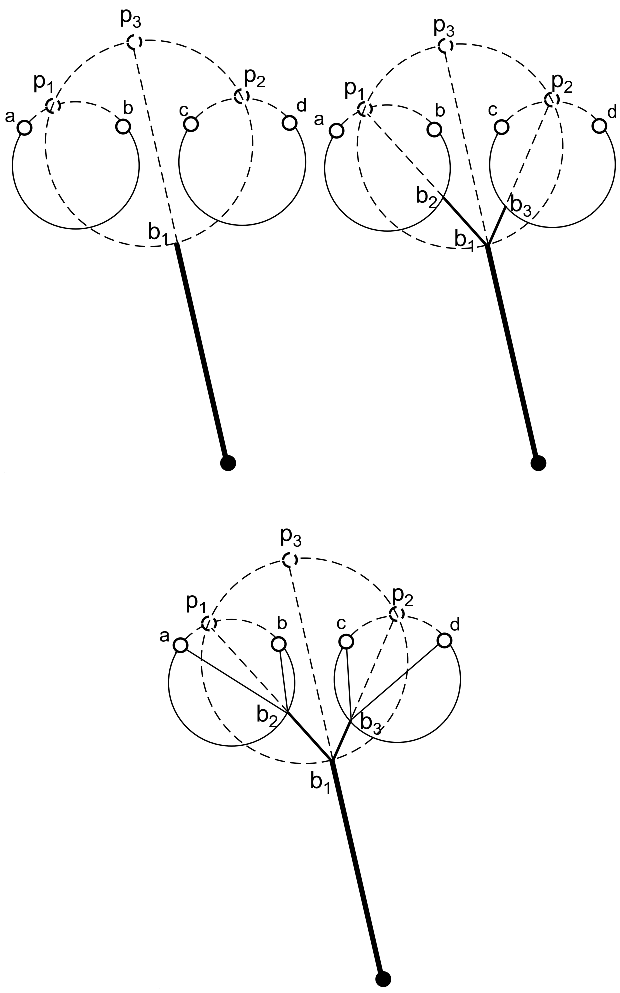

Figure 8.6: The bottom-up part of the construction: connecting the source to the last pivot point permits to find the bifurcation point which is taken as the new source point for the two induced topologies.

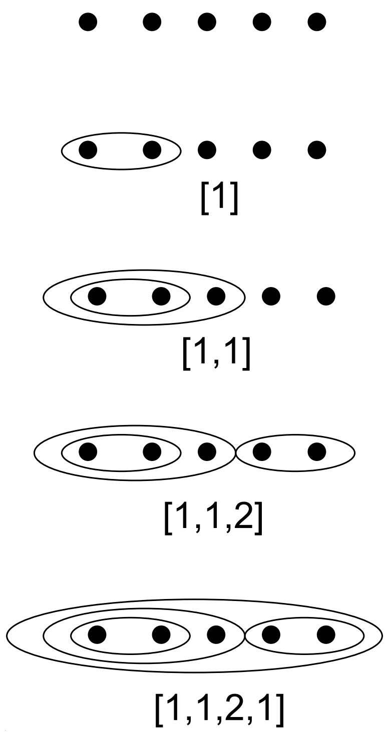

Figure 8.7: The hierarchy of grouping is coded by a chain of numbers indicating the position of the successive merging.

- The optimality of subtrees criterion looks if it possible to improve some subtrees of the global structure.

- The perturbation method permits to move from a topology to another, allowing global improvement.

8.4.1 Multiscale method

When the Dirac masses of the target measure are too numerous, the total number of possible topologies is much too big for the exhaustive exploration to take place. The multiscale approach permits to reduce the number of target points, and thus reduce the problem to a tractable one. The solution of this approximate problem gives hints on the structure of a good structure for the initial problem. These hints permit to reduce the initial problem to appropriate subtrees problems. The synthesis of all subtrees problems can then take place to obtain a reasonable (but not necessarily optimal) structure.

Let us illustrate how the multiscale approach works with $\mu^- = \lambda_{64}$ being the target measure, μ^+ the source point at (0,-1) and $\alpha = 0.95$.

The exhaustive search for an optimal structure takes less than a few minutes in the case of not more than 10 target measures.

- Best structure at a lower resolution: let us start by considering the optimal traffic plan transporting μ^- to $\mu^+ = \lambda_{10}$, we denote it by T_{10}. It is represented on figure 8.11. This tree is symmetrical and because of the symmetry of the problem we shall look for a symmetrical solution.

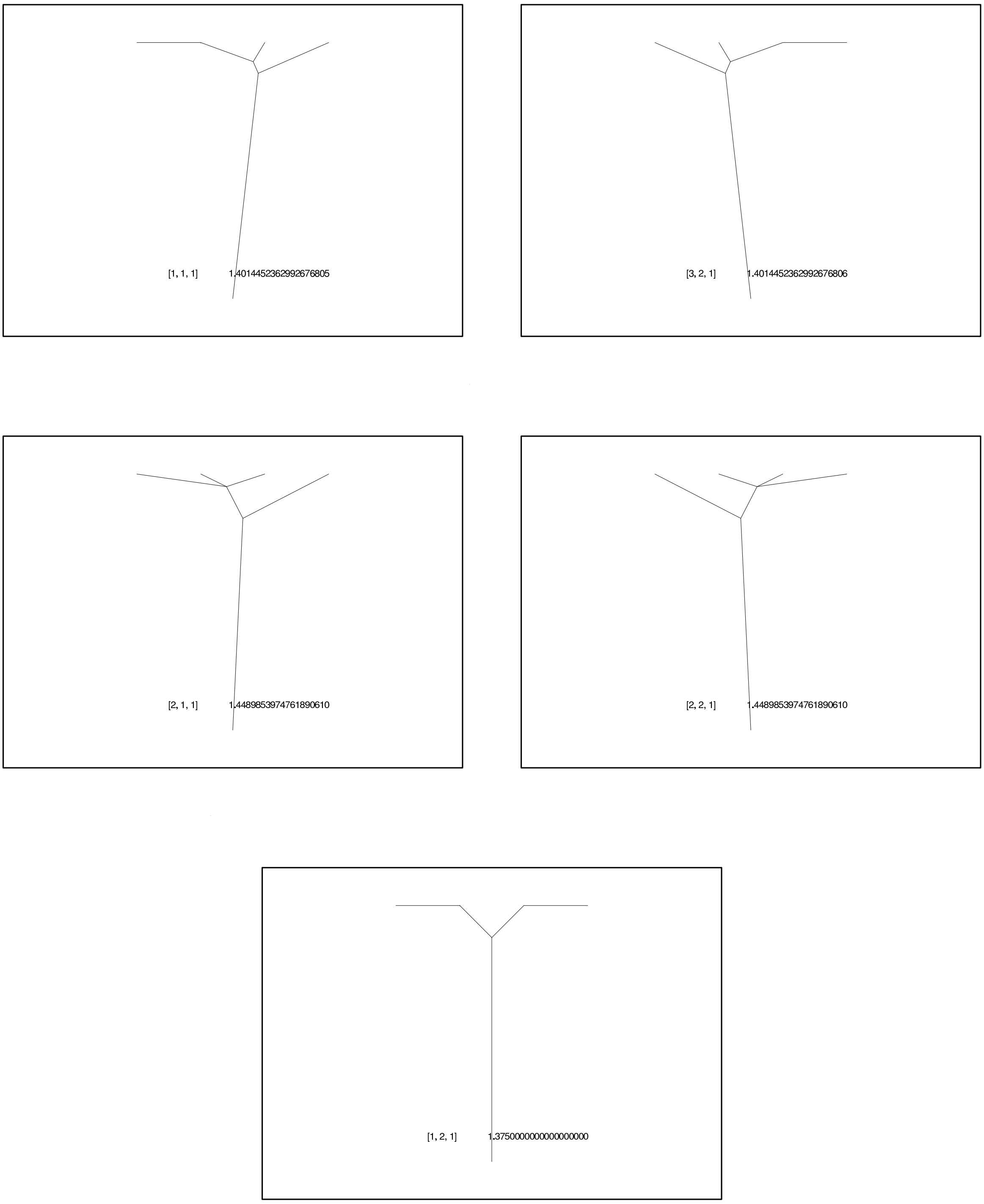

Figure 8.8: All local optima associated to each topology for $\alpha = \frac{1}{2}$.

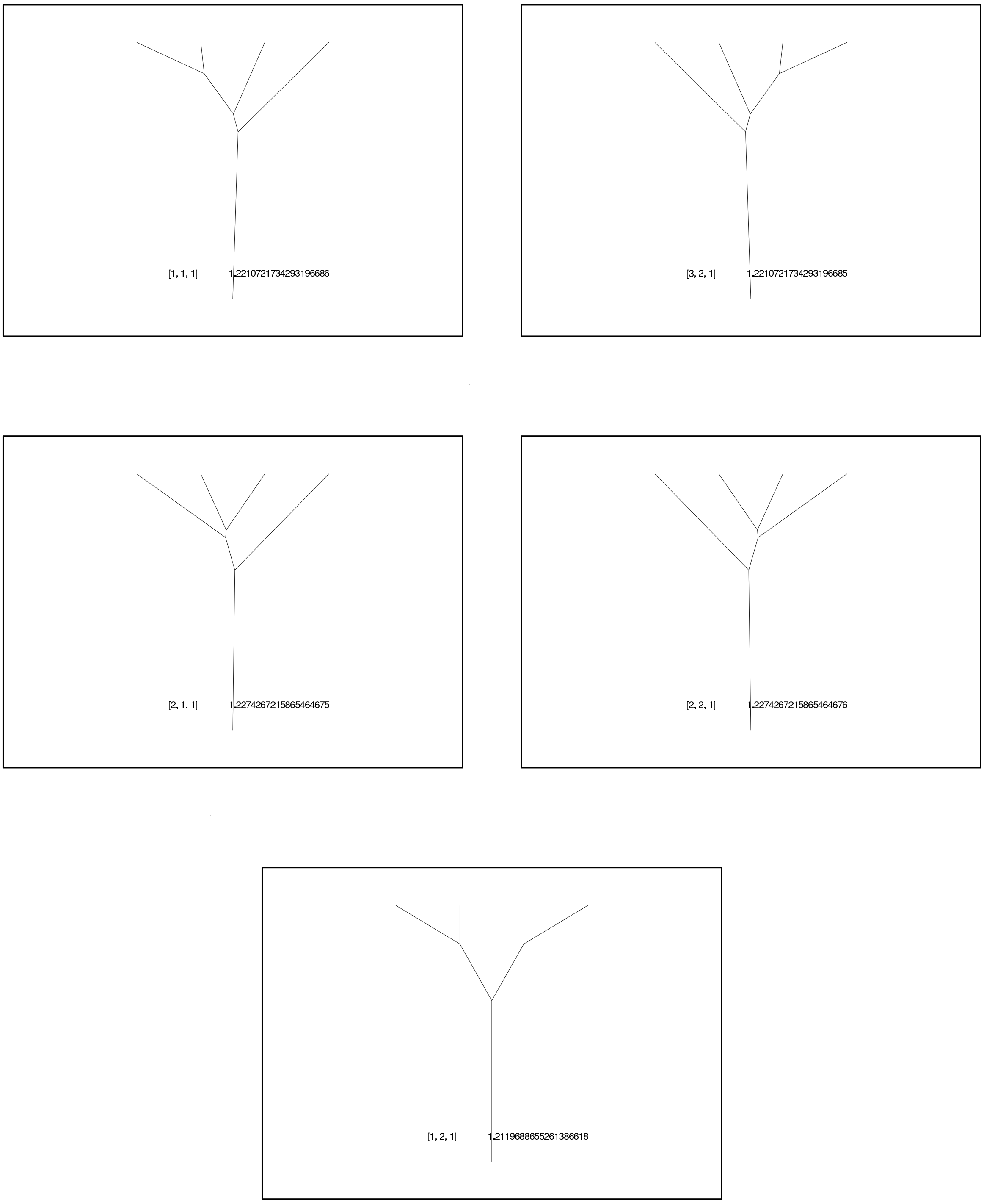

Figure 8.9: All local optima associated to each topology for $\alpha = 0.8$.

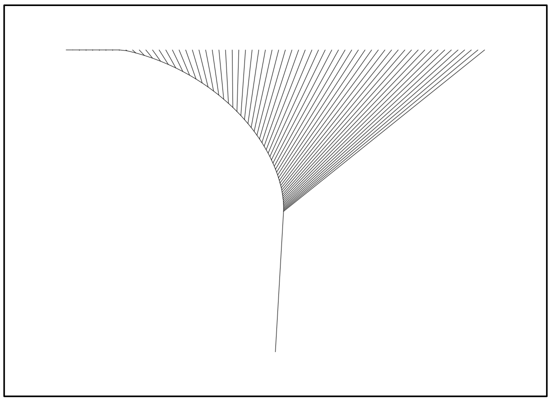

Figure 8.10: The [1,1,...] topology for the irrigation of a 64-approximation of Lebesgue measure on the segment ($\alpha = 0.95$).

- Two subtrees: we denote by P the second bifurcation point of T_{10}, it is located at (-0.123,-0.41). Two subtrees are starting from P, T_{10}^l on the left and T_{10}^r on the right.

- The range of the two subtrees: T_{10}^l irrigates target Dirac masses within $[-0.5, -0.2]$ and T_{10}^r irrigates target Dirac masses within $[-0.2, 0]$.

- Go back to the initial resolution: let μ_l^+ and μ_r^+ be respectively the sum of Dirac masses of λ_{64} located within $[-0.5, -0.2]$ and within $[-0.2, 0]$. Because of the previous point we bring back the initial problem to the one of finding efficient structures to transport P to μ_l^+ and P to μ_l^+.

- Iteration of the process: since μ_r^+ is made of 13 Dirac masses, we proceed to an exhaustive search of the optimal structure. Since μ_l^+ is made of 19 Dirac masses we apply the multiscale approach to this problem.

- Best structure at a lower resolution: We denote by ν_{10} an approximation of μ_l^+ made of 10 Dirac masses. The best traffic plan T_{10}^2 represented on figure 8.11 and 8.12 has a bifurcation point Q located at $(-0.215, -0.22)$.

- The range of the two subtrees (see figure 8.12): the two subtrees starting from Q have range $[-0.5, -0.28]$ and $[-0.28, -0.2]$. The corresponding measures at the initial resolution ν_l and ν_r are respectively made of 14 and 5 Dirac masses. The problem of finding the best structure from P to μ_l^+ thus reduces to the one of finding the best irrigation from Q to ν_l and ν_r. An exhaustive search can do this job.

- Recombination (see figure 8.13): we decomposed λ_{64} as $\lambda_{64}^+ + \lambda_{64}^-$, respectively the Dirac masses on the right and on the left. The multiscale approach made us consider λ_{64}^- as $\lambda_{64}^- = \nu_l + \nu_r + \mu_r^+$. The recombination of optimal structures from Q to ν_l and ν_r gives an efficient structure T_l from P to μ_l^+. We can then combine it with the structure T_r that transports P to μ_r^+.

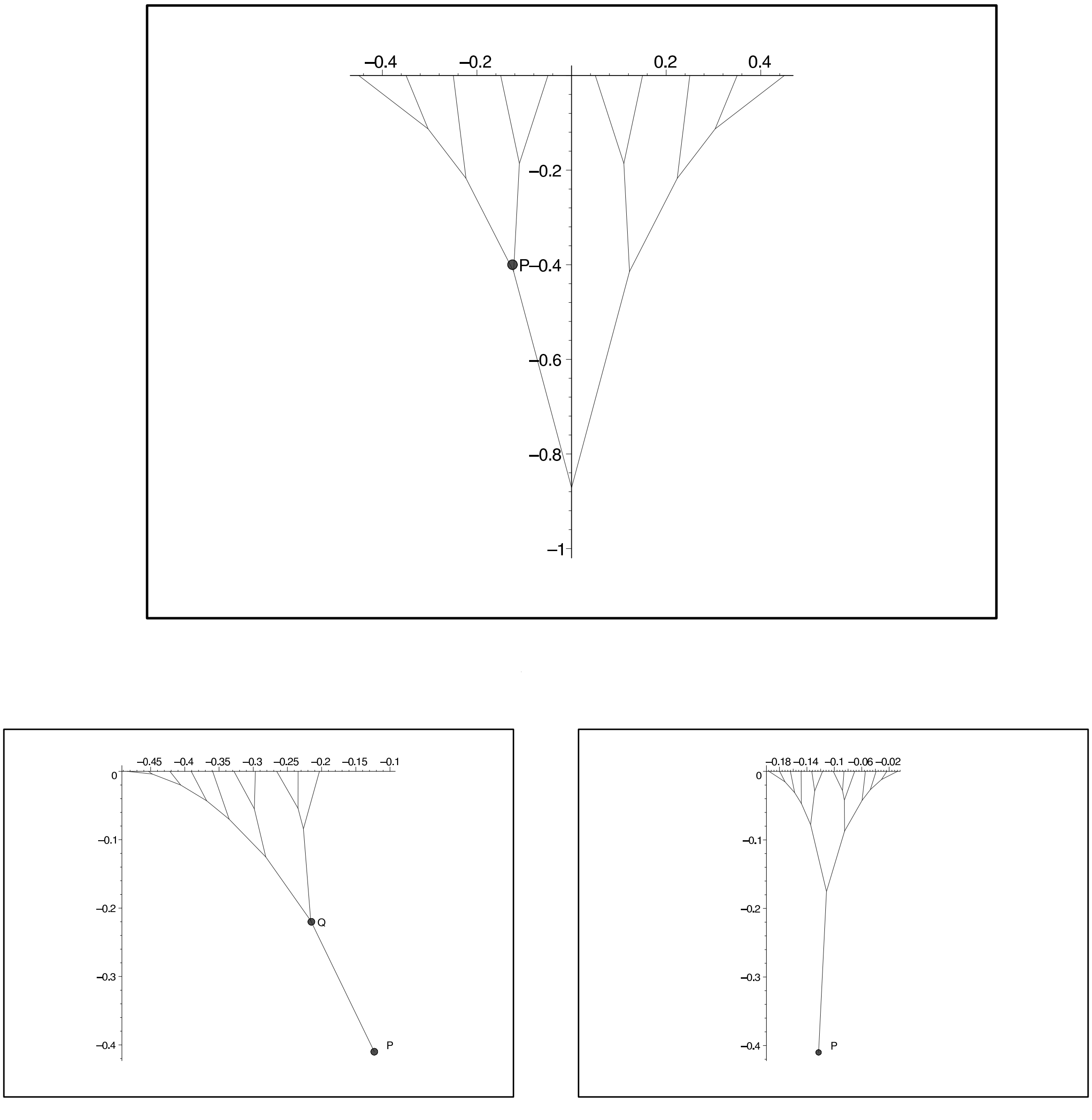

Figure 8.11: At the top, T_{10} is transporting μ^- to $\mu^+ = \lambda_{10}$. The bifurcation point P induces two subtrees. The two figures at the bottom represent these two subtrees at a better resolution so that we can continue the multiscale optimization process.

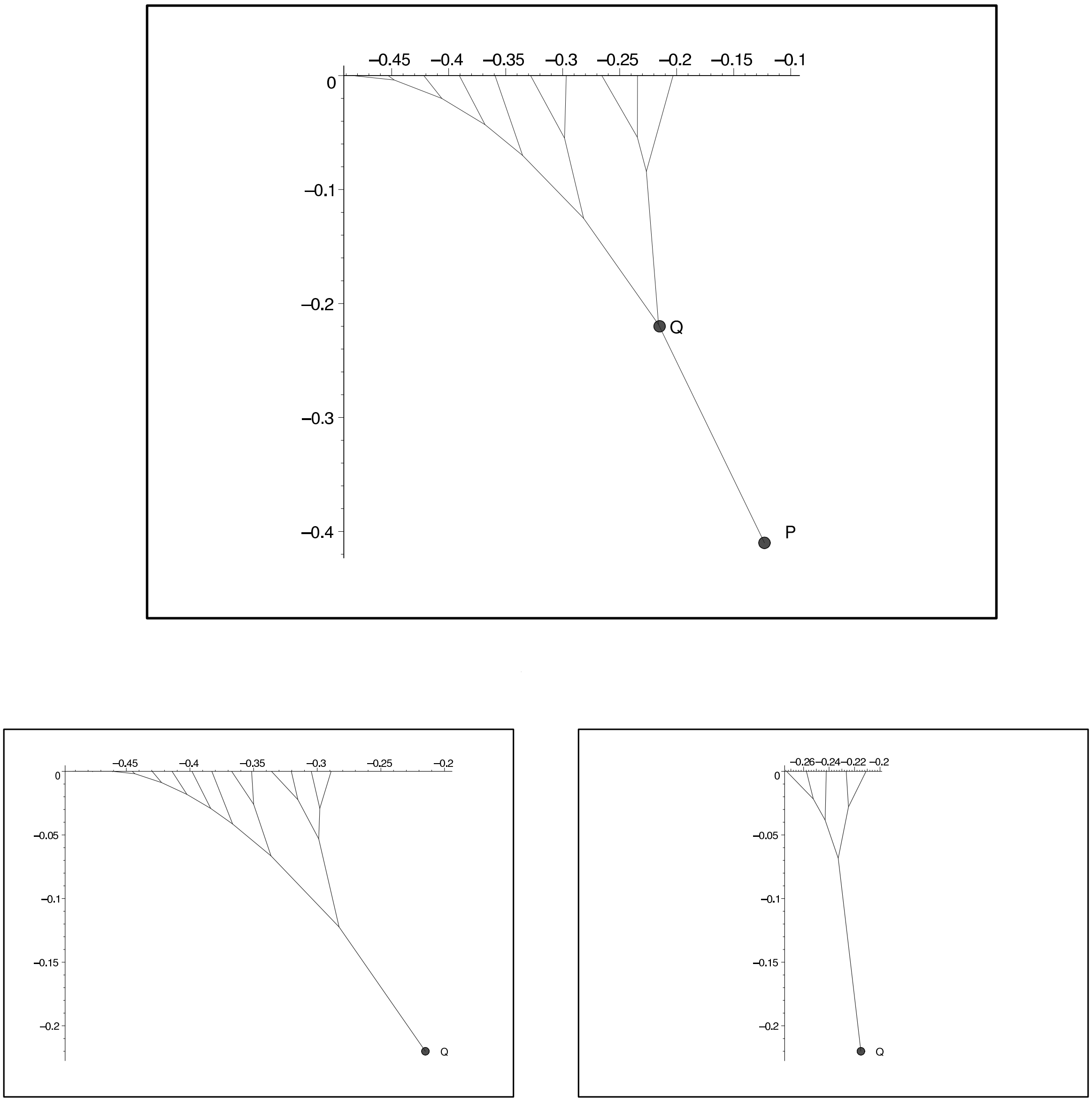

Figure 8.12: The measure ν_{10} is an approximation of μ_r^+ that is made of 10 Dirac masses. The figure at the top represents T_{10}^2, the best traffic plan irrigating ν_{10} from P. The bifurcation point Q induces two subtrees, that we look at the initial resolution. The two figures at the bottom represent these two subtrees at the initial resolution.

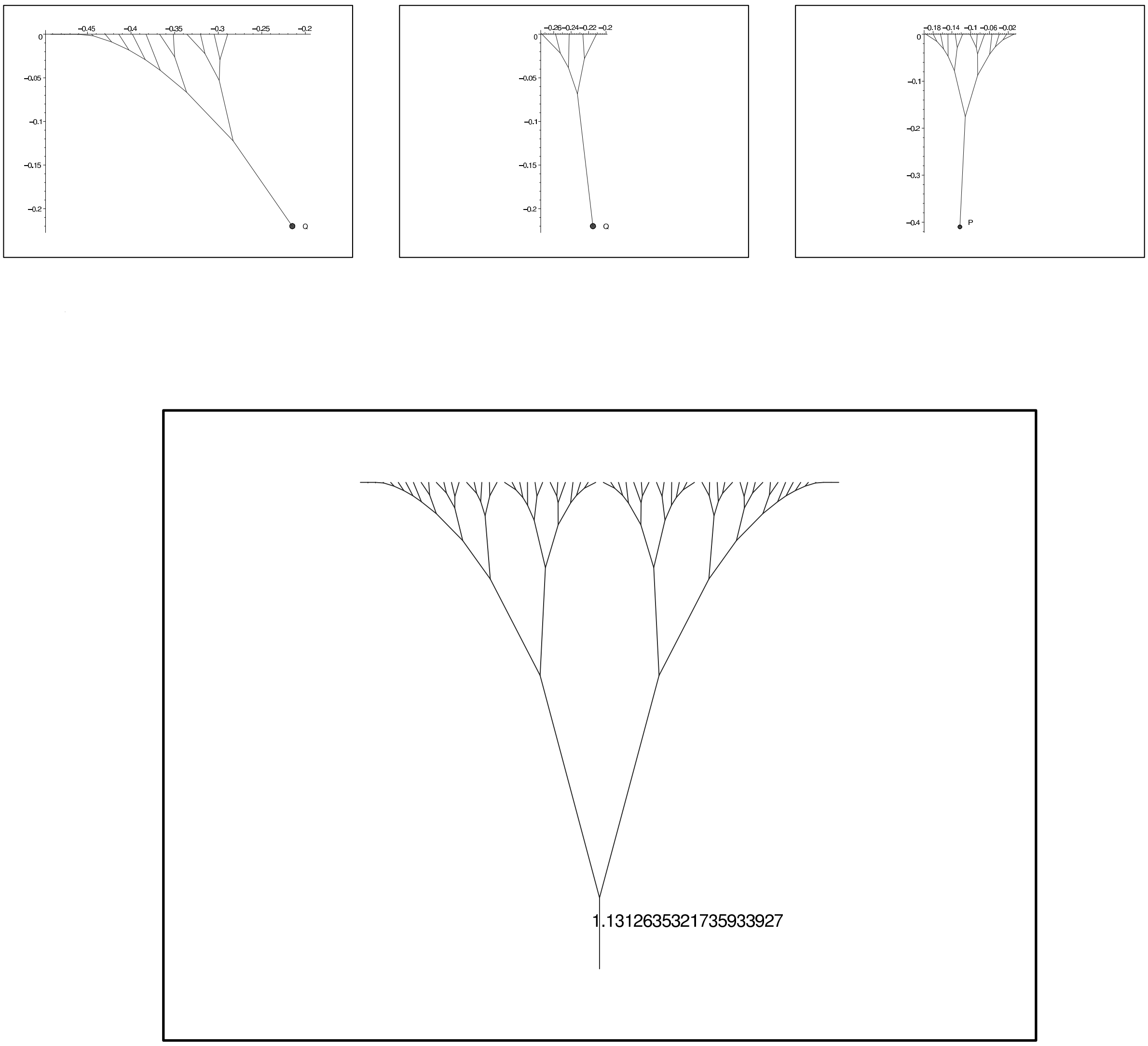

Figure 8.13: We obtained efficient structures to transport a mass at Q to ν_l and ν_r and to transport a mass at P to μ_r^+. These three structures are represented at the top. The figure at the bottom represents the combination of these three structures that gives an efficient transport from the source point $(0, -1)$ to λ_{64}. Notice that this structure is better than the dyadic homogeneous one and has a cost 1.1312635 which is very close of the optimal one 1.1312238.

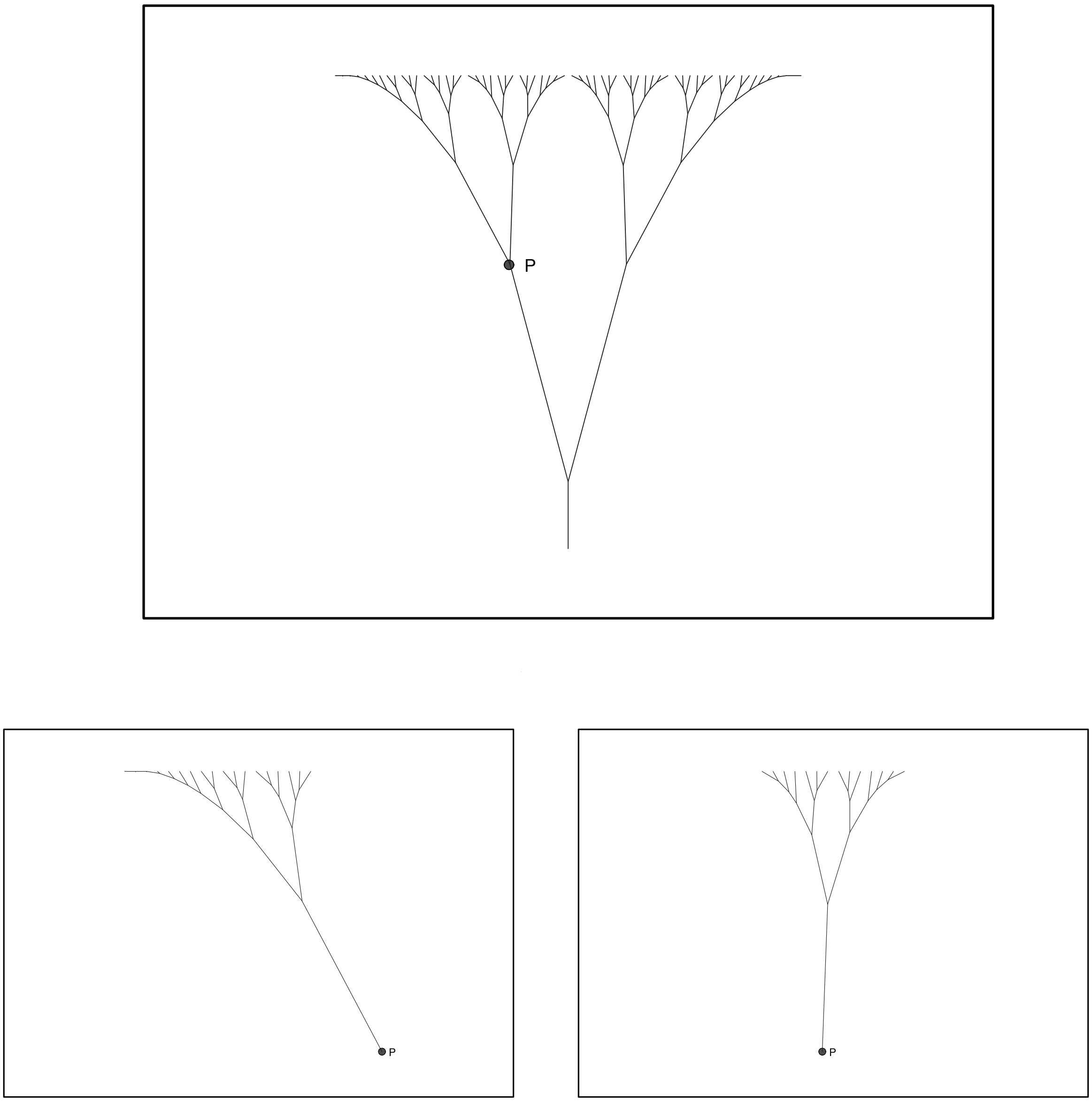

Figure 8.14: If a tree is optimal, then all of its subtrees also have to be optimal. For instance the two subtrees starting from P are optimal in this case. This tells that we can't improve the initial structure with the optimality of subtrees criterion.

8.4.2 Optimality of subtrees

Given an optimal structure T, a subtree is optimal for the problem it induces. That is to say, if we look at the two trees T_1 and T_2 (see figure 8.14) starting at a bifurcation point P of an optimal structure, these two trees have to be optimal. Indeed, if it was not the case, there would be better trees T_1' and T_2' such that a combination of T_1' and T_2' would give a better structure than T. Thus, it is possible to improve some structures, only trying to improve subparts of it. More precisely, since a target measure with 10 Dirac masses is computationally tractable, we can test all subtrees irrigating less than 10 Dirac masses in order to improve a structure.

8.4.3 Perturbation of the topology

The second heuristics that permits to improve a given structure T consists in perturbing the topology of T. That is to say, given an edge e, we can define the topological neighborhood of (T, e) the set of topologies obtained through all possible perturbation of the edge e. In the case of parenthesis topologies, we reduce these perturbations to reasonable ones (see figure 8.15).

8.5 Further

8.5.1 General measure to general measure

Let us illustrate the difficulties appearing in the case of several sources. In case the optimal structure has a point S with multiplicity 1, the structure is the union of an optimal irrigation from S to μ^+ and an optimal irrigation from S to μ^- so that the pivot point approach holds (see figure 8.16). However, if we try to find the optimal structure with the prescribed topology like the one represented on figure 8.17, then the pivot point algorithm is of no use. Indeed, as illustrated by figure 8.17, Steiner points are no longer being obtained from top to down. The point b_1 depends on the location of b_2 and the point b_2 depends on the location of b_1. This calls for another approach and another coding of topologies.

8.5.2 Three dimensions

One main difficulty is added in the case of three dimensions: it is no more possible to use a combinatoric approach, even to optimize the transportation of a Dirac mass to a measure with very few Dirac masses. Let us go back to dimension 2 to explain that. In the case of 2 dimensions, each couple of points (P_1, P_2) can be reduced either to one of the two possible pivot point, either to P_1 or P_2. An exhaustive search through all possible topologies and all possible degeneracies can then take place.

In the case of 3 dimensions, given two points (P_1, P_2) the set of possible location for the pivot point is a whole circle. Thus, even for a prescribed topology, the combinatorics is of no help and one has to use numerical approximation.

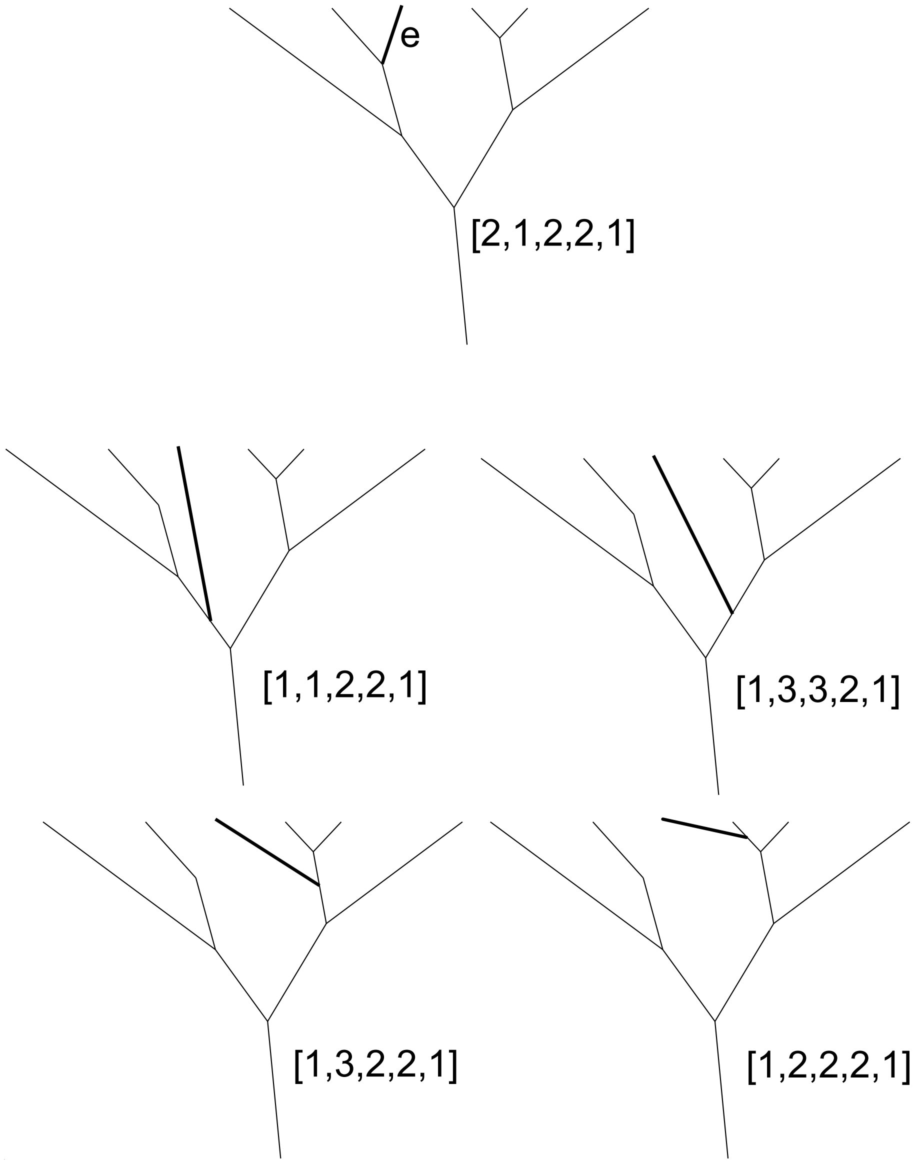

Figure 8.15: The different possible topological perturbations associated to the edge e.

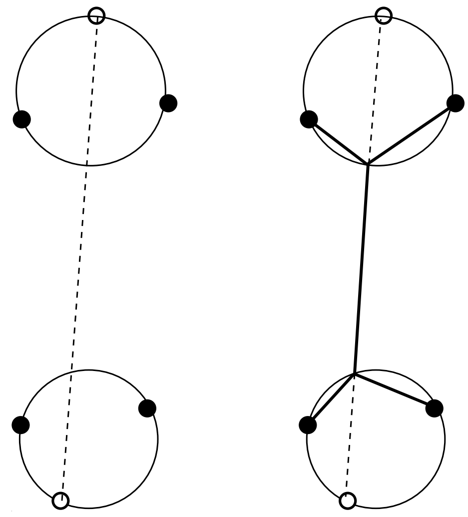

Figure 8.16: The pivot point approach can give the optimum in that case since the top-dow, bottom-up approach holds in that case. Indeed, the construction of pivot points bring back the problem to a one source one target problem. We then reconstruct the whole structure as described in section 8.2.

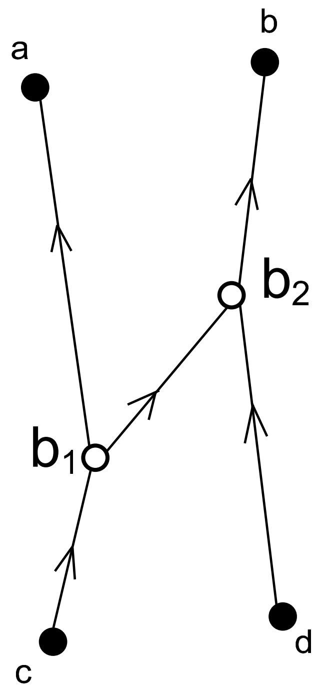

Figure 8.17: If this structure is optimal, then the pivot points are of no help in finding the location of the bifurcation points b_1 and b_2. Indeed, we need b_1 to locate b_2 and reciprocally so that a numerical algorithm seems necessary in that case.

Appendix: optimal flow and Poiseuille's law

In this appendix, we shall consider a fluid with laminar flow in a tube. We recall how Poiseuille law can be derived from Navier-Stokes equation. Next, we discuss the optimality of the circular section.

Poiseuille law

Let us consider a tube of constant circular section with a straight axis. We take (x, y, l) as coordinates in the tube, where $l \in [0; L]$ is the distance along the axis and $(x, y) \in D(0, r)$ are orthogonal cartesian coordinates.

We assume a stationary regime and that the flow is laminar, that is to say the velocity is oriented by the axis and is constant on all trajectories, so that $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$. The velocity v at a point of a tube along the z-axis is given by Navier-Stokes equation

$$- \triangle v(l)(x, y) = \frac{1}{\eta} \frac{\partial p}{\partial z}, \text{ where } \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Hence, $\frac{\partial p}{\partial z} = constant$ (where η denotes the viscosity coefficient). Thus, the gradient of pressure has the form $\frac{[p]}{L}$ where $[p]$ denotes the pressure difference at the ends of the tube, and we shall denote it by $\bigtriangledown p$. In other words, p is a linear interpolation of the initial and final pressures in the tube. We assume that the pressure is constant on the initial and ending sections of the tube, so that the pressure is constant on each section of the tube. For simplicity, let us take $\eta = 1$.

Under these hypotheses, we can calculate the velocity and the corresponding flow through the whole tube

$$v(x, y, l) = \frac{(r^2 - (x^2 + y^2))}{4} \bigtriangledown p$$

$$f = \int_{D(0,r)} v(x, y, l) = \frac{1}{4} r^4 \bigtriangledown p = r^2 v_{max}$$

The power dissipated by the steady flow is $W = fL \bigtriangledown p$. This is to be identified with $W = Lf^2R$ where by definition R stands for the resistivity of the tube. Thus we obtain $R = 4/r^4$: Poiseuille law says that the resistivity of a tube scales as the inverse fourth power of the radius.

Optimality of the circular section

What is the optimal form of the section of a tube? If we prescribe the pressure at both ends of a tube of constant section, the circular form ensures the maximal flow. We briefly present the result obtained in [30] and [1].

Let us recall the definition of the rearrangement of a set (see [21]). If $A \subset \mathbb{R}^d$, we denote by A^* the ball $B(0,r) = \{x \mid |x| < r\}$ such that $|B(0,r)| = |A|$. If $f : \mathbb{R}^d \to \mathbb{R}$ is a Borel measurable function vanishing at infinity, we define the symmetric decreasing rearrangement of f by $f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt$. It results from the definition that $|\{x \mid |f(x)| > t\}| = |\{x \mid f^*(x) > t\}|$ and $||f||_p = ||f^*||_p$.

Let u be such that $-\triangle u(x,y) = \bigtriangledown p$ in the domain Ω. Let v be such that $-\triangle v(x,y) = (\bigtriangledown p)^* = \bigtriangledown p$ in Ω^*. Then, it can be shown that $u^* \leq v$ [30]. As a consequence, the flow in a tube of section Ω is such that $\int_\Omega u = \int_{\Omega^*} u^* \leq \int_{\Omega^*} v$. Then a circular section is always more advantageous from the point of view of the flow.

In [1], the authors prove the uniqueness of the optimal form: if $\max u = \max v$, then there is x_0 such that $\Omega = x_0 + \Omega^*$ and $u = v(\cdot + x_0)$. Then, if Ω is an optimal form, we have $\int_\Omega u = \int_{\Omega^*} v$ and $u^* \leq v$, hence $\max u = \max u^* = \max v$ necessarily. Then there is x_0 such that $\Omega = x_0 + \Omega^*$, and, therefore, the circular form is the unique optimum.

Bibliography

[1] ALVINO, A., LIONS, P.-L., AND TROMBETTI, G. A remark on comparison results via symmetrisation. *Proceeding of the royal society of Edimburgh 102A* (1986), 37–48.

[2] AMBROSIO, L., FUSCO, N., AND PALLARA, D. *Functions of bounded variations and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, 2000.

[3] BEJAN, A., AND ERRERA, M. Deterministic tree networks for fluid flow: geometry for minimal flow resistance between a volume and one point. *Fractals 5(4)* (1997), 685–695.

[4] BELLETTINI, G., CASELLES, V., AND NOVAGA, M. The total variation flow in $\mathbb{R}^n$. *Journal Differential Equations 184* (2002), 475–525.

[5] BERNOT, M., CASELLES, V., AND MOREL, J.-M. Traffic plans. *Publicacions Matemàtiques 49*, 2 (2005), 417–451.

[6] BHASHKARAN, S., AND SALZBORN, F. J. M. Optimal design of gas pipeline networks. *J. Oper. Res. Society 30* (1979), 1047–1060.

[7] BRANCOLINI, A., BUTTAZZO, G., AND SANTAMBROGGIO, F. Path functionals over Wasserstein spaces. *preprint*.

[8] BUTTAZZO, G. Three optimization problems in mass transportation theory. *Preprint*.

[9] BUTTAZZO, G., AND STEPANOV, E. Minimization problems for average distance functionals. *in "Calculus of Variations : Topics from the Mathematical Heritage of Ennio De Giorgi", D. Pallara (ed.), Quaderni di Matematica, Seconda Università di Napoli*.

[10] BUTTAZZO, G., AND STEPANOV, E. Optimal transportation networks as free dirichlet regions for the monge-kantorovich problem. *Ann. Sc. Norm. Super. Pisa Cl. Sci. 5-2*, 4 (2003), 631–678.

[11] CARLIER, M. *Hydraulique générale et appliquée*. Eyrolles, 1998.

[12] CASELLES, V., AND MOREL, J.-M. *Irrigation*, vol. 51 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhaüser, 2002. Editors: F. Tomarelli and G. Dal Maso. VARMET 2001. Trieste, June, 2002.

[13] DACOROGNA, B. *Direct methods in the calculus of variations*. Springer Verlag, 1989.

[14] DIESTEL, J., AND UHL, J. *Vector measures*, vol. 15 of *Mathematical Surveys*. American Mathematical Society, Providence, R.I., 1977.

[15] DODDS, P., ROTHMAN, D., AND WEITZ, J. Re-examination of the 3/4-law of metabolism. *Journal of Theoretical Biology 209* (2001), 9–27.

[16] DUDLEY, R. *Real Analysis and Probability*. Cambridge University Press, 2002.

[17] EVANS, L., AND GARIEPY, R. *Measure theory and fine properties of functions*. CRC press, 1992.

[18] GILBERT, E. N. Minimum cost communication networks. *Bell System Tech. J. 46* (1967), 2209–2227.

[19] KANTOROVICH, L. On the transfer of masses. *Dokl. Acad. Nauk. USSR 37* (1942), 7–8.

[20] LEE, D. H. Low cost drainage networks. *Networks 6* (1976), 351–371.

[21] LIEB, E., AND LOSS, M. *Analysis*, vol. 14 of *Graduate Studies in Mathematics*. 1997.

[22] MADDALENA, F., MOREL, J.-M., AND SOLIMINI, S. A variational model of irrigation patterns. *Interfaces and Free Boundaries 5*, 4 (2003), 391–416.

[23] MAUROY, B., FILOCHE, M., WEIBEL, E., AND SAPOVAL, B. An optimal bronchial tree may be dangerous. *Nature 427* (2004), 633–636.

[24] MONGE, G. Mémoire sur la théorie des déblais et de remblais. *Histoire de l'Académie Royale des Sciences de Paris* (1781), 666–704.

[25] MORGAN, F. *Geometric measure theory. A beginner's guide.* Academic Press, 1995.

[26] NEWMAN, W., TURCOTTE, D., AND GABRIELOV, A. Fractal trees with side branching. *Fractals 5 (4)* (1997), 603–614.

[27] SAPOVAL, B. *Universalités et Fractales*, vol. 466 of *Champs*. Flammarion, 1997.

[28] SOLIMINI, S., AND DE VILLANOVA, G. Elementary properties of optimal irrigation patterns. *preprint*.

[29] SOLIMINI, S., AND DE VILLANOVA, G. On the dimension of an irrigable measure. *preprint*.

[30] TALENTI, G. Elliptic equations and rearrangements. *Ann. Scuola Norm. Sup. Pisa 3* (1976).

[31] WEST, G. The origin of universal scaling laws in biology. *Physica A 263* (1999), 104–113.

[32] WEST, G., BROWN, J., AND ENQUIST, B. A general model for the origin of allometric scaling laws in biology. *Science 276(4)* (1997), 122–126.

[33] WEST, G., BROWN, J., AND ENQUIST, B. A general model for ontogenetic growth. *Nature 413* (2001), 628–631.

[34] WILLIAMS, D. *Probability with martingales*. Cambridge University Press, 1991.

[35] XIA, Q. Optimal paths related to transport problems. *Commun. Contemp. Math. 5* (2003), 251–279.

[36] XIA, Q. Interior regularity of optimal transport paths. *Calculus of Variations and Partial Differential Equations 20*, 3 (2004), 283–299.