Two applications of positivity to the classification theory of complex projective varieties

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TWO APPLICATIONS OF POSITIVITY TO THE CLASSIFICATION THEORY OF COMPLEX PROJECTIVE VARIETIES

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Au vu des rapports de Christophe Mourougane et Jaroslaw Wiśniewski

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Contents

Zwei Anwendungen von Positivität in der Klassifikationstheorie komplexer projektiver Mannigfaltigkeiten

Das Ziel dieser Arbeit ist die Untersuchung zweier sehr natürlicher Fragestellungen aus der komplexen algebraischen Geometrie.

Beim ersten Problem geht es darum ob die universelle Überlagerung einer kompakten Kählermannigfaltigkeit mit spaltendem Tangentialbündel ein Produkt von Mannigfaltigkeiten ist. Wir werden eine Strukturtheorie für Mannigfaltigkeiten mit spaltendem Tangentialbündel entwickeln und überdeckende Familien von rationalen Kurven benutzen um die Existenz von Faserraumstrukturen zeigen. Eine genaue Diskussion der Faserraumstruktur erlaubt es dann die gestellte Frage für mehrere Klassen von Mannigfaltigkeiten positiv zu beantworten.

Beim zweiten Problem fragen wir ob die Positivität eines Geradenbündels die Positivität der direkten Bildgarbe des adjungierten Geradenbündel unter einer flachen projektiven Abbildung impliziert. Die Antwort auf diese Frage hängt von der Positivität des Geradenbündels und dessen Zusammenhang mit der Geometrie der Abbildung ab. Wir zeigen, dass unter Bedingungen die typischerweise in der Klassifikationstheorie projektiver Varietäten auftreten, die Antwort positiv ist.

Obwohl die beiden Probleme vollkommen unabhängig sind, sind sie durch die zur Lösung verwendeten Methoden verbunden: Wir benutzen die Positivität kohärenter Garben und Klassifikationstheorie um die Existenz und Eigenschaften von Faserraumstrukturen zu studieren. Wir geben jetzt eine Zusammenfassung der wichtigsten Ergebnisse der Arbeit, die Einleitungen der Teile I und II geben genauere Informationen zu den verwendeten Methoden und offenen Fragen.

Teil I: Kählermannigfaltigkeiten mit gespaltenem Tangentialbündel

Eine häufig verwendete Strategie in der algebraischen Geometrie ist Eigenschaften einer Mannigfaltigkeit aus Eigenschaften des Tangentialbündels abzuleiten. Das Tangentialbündel ist häufig einfacher zu verstehen, da es als eine linearisierte Version der Mannigfaltigkeit angesehen werden kann. Wenn eine Mannigfaltigkeit ein Produkt von zwei Mannigfaltigkeiten ist, dann ist das Tangentialbündel eine direkte Summe von Vektorbündel. Im Folgenden wollen wir fragen, ob es möglich ist von der Spaltung des Tangentialbündels auf eine Produktstruktur der Mannigfaltigkeit zu schließen. Etwas genauer gesprochen soll folgende Vermutung betrachtet werden.

Vermutung 1. *(A. Beauville) Sei X eine kompakte Kählermannigfaltigkeit so dass $T_X = V_1 \oplus V_2$, wobei V_1 und V_2 Vektorbündel sind. Sei $\mu : \tilde{X} \to X$ die* *universelle Überlagerung von X. Dann ist $\tilde{X} \simeq X_1 \times X_2$ und $p_{X_j}^* T_{X_j} \simeq \mu^* V_j$. Falls außerdem das Unterbündel V_j integrabel ist, dann gibt es einen Automorphismus von $\tilde{X}$ so dass wir eine Identität $\mu^* V_j = p_{X_j}^* T_{X_j}$ von Unterbündeln des Tangentialbündels haben.*

Diese Vermutung wurde zuvor von Beauville [Bea00], Druel [Dru00], Campana-Peternell [CP02] und zuletzt von Brunella-Pereira-Touzet [BPT04] studiert. Der letztgenannte Artikel verallgemeinert die meisten vorher bekannten Ergebnisse, das wichtigste Ergebnis ist das

Theorem. [BPT04, Thm.1] *Sei X eine kompakte Kählermannigfaltigkeit. Wenn das Tangentialbündel in $T_X = V_1 \oplus V_2$ spaltet wobei $V_2 \subset T_X$ ein Unterbündel vom Rang $\dim X - 1$ ist, dann gibt es zwei Fälle:*

1.) Falls V_2 nicht integrabel ist, ist V_1 tangential zu den Fasern eines $\mathbb{P}^1$-Bündels.

2.) Falls V_2 integrabel ist, ist Vermutung 1 wahr.

Das Theorem stellt eine überraschende Verbindung zwischen der Existenz rationaler Kurven entlang der Blätterung V_1 und der Integrabilität des komplementären Faktors V_2 her. Dies weist darauf hin, dass unigeregelte Mannigfaltigkeiten eine besondere Rolle bei der Beantwortung der Vermutung spielen werden.

Definition. *Eine kompakte Kählermannigfaltigkeit X ist unigeregelt falls es eine überdeckende Familie von rationalen Kurven auf X gibt. Sie ist rational zusammenhängend falls zwei allgemeine Punkte durch eine rationale Kurve verbunden werden können.*

Ein tiefer Satz von Campana [Cam04b, Cam81] zeigt dass es auf einer unigeregelten kompakten Kählermannigfaltigkeit X immer eine meromorphe Faserung $\phi : X \dashrightarrow Y$ auf eine normale Varietät Y gibt bei der die allgemeine Faser rational zusammenhängend und die Basis Y nicht unigeregelt ist (siehe auch [GHS03]). Im projektiven Fall können wir die Aussage des obigen Theorems zur Integrabilität auf eine Spaltung in Vektorbündel von beliebigem Rang verallgemeinern.

Theorem. *Sei X eine projektive Mannigfaltigkeit mit gespaltenem Tangentialbündel $T_X = V_1 \oplus V_2$. Es sei angenommen, dass für die allgemeine Faser F des rationalen Quotienten $T_F \subset V_2|_F$ gilt. Dann ist V_2 integrabel und $\det V_1^*$ ist pseudoeffektiv.*

Insbesondere gilt: Ist X nicht unigeregelt, dann sind V_1 und V_2 integrabel.

Da das Tangentialbündel einer rational zusammenhängenden Mannigfaltigkeit sehr starke Positivitätseigenschaften hat, erscheint es vernünftig unsere Untersuchung mit dieser Klasse von Mannigfaltigkeiten zu beginnen. Als erstes wichtiges Ergebnis erhalten wir das folgende

Theorem. *Sei X eine rational zusammenhängende Mannigfaltigkeit so dass $T_X = V_1 \oplus V_2$. Wenn V_1 oder V_2 integrabel ist, dann sind V_1 und V_2 integrabel; in diesem Fall ist Vermutung 1 wahr.*

Dieses Theorem verallgemeinert einen Satz von Campana and Peternell [CP02] für Fanomannigfaltigkeiten deren Dimension kleiner gleich fünf ist.

Der nächste Schritt ist die folgende Beobachtung: Sei X eine unigeregelte Mannigfaltigkeit X so dass $T_X = V_1 \oplus V_2$, und sei $\psi : X \dashrightarrow Y$ die rationale Quotientenabbildung, dann gilt für die allgemeine ψ-Faser F

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

Da die allgemeine ψ-Faser rational zusammenhängend ist, zeigt diese Beobachtung, dass das obige Theorem auch bei der Betrachtung der viel größeren Klasse der unigeregelten Mannigfaltigkeiten nützlich sein wird. Ein wichtiges Zwischenergebnis ist das

Theorem. *Sei X eine unigeregelte kompakte Kählermannigfaltigkeit so dass $T_X = V_1 \oplus V_2$ und rg $V_1 = 2$. Sei F eine allgemeine Faser der rationalen Quotientenabbildungen, dann gilt*

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

Es gibt dann drei Fälle:

1.) $T_F \cap V_1|_F = V_1|_F$. Falls $T_F \cap V_1|_F$ integrabel ist, hat die Mannigfaltigkeit X die Struktur eines analytischen Faserbündels $X \to Y$ so dass $T_{X/Y} = V_1$. Falls außerdem V_2 integrabel ist, ist die Vermutung 1 für X wahr.

2.) $T_F \cap V_1|_F$ ist ein Geradenbündel. Dann gibt es eine equidimensionale Abbildung $\phi : X \to Y$ so dass für die allgemeine ϕ-Faser M die Inklusion $T_M \subset V_1|_M$ gilt. Falls die Abbildung ϕ flach ist und V_2 integrabel ist, ist die Vermutung 1 für X wahr.

3.) $T_F \subset V_2|_F$.

Im projektiven Fall kann die Analyse der einzelnen Fälle noch verfeinert werden so dass wir eine Ergebnis erhalten, das analog ist zum Theorem von Brunella, Pereira und Touzet.

Theorem. *Sei X eine unigeregelte projektive Mannigfaltigkeit so dass $T_X = V_1 \oplus V_2$ und rg $V_1 = 2$. Sei F eine allgemeine Faser der rationalen Quotientenabbildungen, dann gilt eine der folgenden Aussagen.*

1.) $T_F \cap V_1|_F \neq 0$. Wenn V_1 und V_2 integrabel sind, ist Vermutung 1 wahr.

2.) $T_F \cap V_1|_F = 0$. Dann ist V_2 integrabel und $\det V_1^*$ ist pseudoeffektiv.

Eines der wichtigsten Ergebnisse des Artikels [Hör05] ist ein Korollar dieses Satzes.

Korollar. *[Hör05, Thm.1.5] Sei X eine projektive unigeregelte vierdimensionale Mannigfaltigkeit so dass $T_X = V_1 \oplus V_2$ und rg $V_1 =$ rg $V_2 = 2$. Wenn V_1 und V_2 integrabel sind, ist Vermutung 1 wahr.* □

Teil II: Direkte Bildgarben adjungierter Geradenbündel

Eines der grundlegenden Probleme bei der Betrachtung einer Faserung $\phi : X \to Y$, das heißt eines Morphismus mit zusammenhängenden Fasern zwischen projektiven normalen Varietäten, ist es eine Verbindung zwischen den dualisierenden Garben des Totalraums X und der Basis Y herzustellen. Es gibt zwei Gründe warum man dieses Problem stets mit einer Untersuchung der direkten Bildgarbe $\phi_* \omega_{X/Y}$ der relativen dualisierenden Garbe $\omega_{X/Y} = \omega_X \otimes \phi^* \omega_Y^*$ beginnen sollte: Erstens ist die Einschränkung von $\omega_{X/Y}$ auf eine allgemeine Faser F die dualisierende Garbe der Faser F. Daher ist der Halm von $\phi_* \omega_{X/Y}$ in einem allgemeinen Punkt kanonisch isomorph zum Raum der globalen Schnitte $H^0(F, \omega_F)$, welcher als ein Maßfür die Positivität von ω_X in der Umgebung der Faser betrachtet werden kann. Zweitens enthält die globale Struktur von $\phi_* \omega_{X/Y}$ Information über die Variation der Positivität zwischen den Fasern. Etwas wage gesprochen ist die Positivität von $\phi_* \omega_{X/Y}$ die Positivität von ω_X modulo der Positivität entlang der Fasern. Da Y der Parameterraum der Fasern ist sollte die Positivität von ω_Y dieser „Quotientenpositivität" entsprechen. In seinen bedeutenden Arbeiten [Vie82, Vie83] hat Eckart Viehweg den Begriff der schwachen Positivität eingeführt, der für die Untersuchung direkter Bildgarben besonders gut geeignet ist.

Definition. *Sei X eine quasi-projektive Varietät. Eine torsionsfreie kohärente Garbe $\mathcal{F}$ ist schwach positiv wenn es ein amples Geradenbündel H gibt so dass es für jede natürliche Zahl $\alpha \in \mathbb{N}$ ein $\beta \in \mathbb{N}$ gibt so dass $(\mathrm{Sym}^{\beta\alpha} \mathcal{F})^{**} \otimes H^\beta$ in einem allgemeinem Punkt von globalen Schnitten erzeugt wird.*

Eines der wichtigsten Ergebnisse in Viehweg's Arbeiten ist das

Theorem. *[Vie82] Sei $\phi : X \to Y$ eine Faserung zwischen projektiven Mannigfaltigkeiten. Dann ist für jedes $m \in \mathbb{N}$die direkte Bildgarbe $\phi_*(\omega_{X/Y}^{\otimes m})$ schwach positiv.*

Für Anwendungen, zum Beispiel im Zusammenhang mit Modulräumen polarisierter Mannigfaltigkeiten (vergleiche [Vie95]), ist es wichtige eine allgemeinere Situation zu betrachten: Gegeben sei eine Faserung $\phi : X \to Y$, und ein Geradenbündel L auf X, was kann man über die Positivität der direkten Bildgarbe $\phi_*(L \otimes \omega_{X/Y})$ aussagen? Es ist sofort einsichtig dass es keinen Sinn macht solch eine Frage für ein Geradenbündel L zu stellen, dass nicht selbst in einem gewissen Sinn positiv (z.B. ample, nef, weakly positive,...) ist. Außerdem ist es notwendig Einschränkungen bezüglich der Geometrie der Faserung $\phi : X \to Y$ zu formulieren, zum Beispiel (möglichst schwache) Bedingungen bezüglich der Singularitäten der Varietät X. Aufbauend auf den fundamentalen Arbeiten von Kollár [Kol86] und Viehweg [Vie82, Vie83] werden wir eine Strategie verfeinern, die C. Mourougane in seiner Dissertation verwendet hat um die Positivität direkter Bildgarben zu zeigen.

Theorem. *[Mou97, Thm.1] Sei $\phi : X \to Y$ eine glatte Faserung zwischen projektiven Mannigfaltigkeiten und sei L eine Geradenbündel auf X das nef und ϕ-big ist. Dann ist die direkte Bildgarbe $\phi_*(L \otimes \omega_{X/Y})$ lokal frei und nef.*

Das Ziel dieser Arbeit ist dann sein Ergebnis in verschiedene Richtungen zu verallgemeinern. In erster Linie geht es darum ein analoges Ergebnis für Faserungen zu zeigen die flach, aber nicht notwendigerweise glatt sind. Zweitens sollte ein solcher Satz auch für singuläre Varietäten gelten. Drittens möchte man die Voraussetzung hinsichtlich der Positivität von L verändern oder abschwächen. Insbesondere wird man dann auf Situationen treffen in denen die direkte Bildgarbe $\phi_*(L \otimes \omega_{X/Y})$ nicht lokal frei ist. Wir werden diese Ziele unter einer Vielzahl von unterschiedlichen Bedingungen an die Positivität des Geradenbündels und die geometrische Situation realisieren.

Theorem. *Sei X eine normale $\mathbb{Q}$-Gorensteinvarietät mit höchstens kanonischen Singularitäten, und sei Y eine normale $\mathbb{Q}$-Gorensteinvarietät. Sei $\phi : X \to Y$ eine flache Faserung und sei L ein Geradenbündel auf X das nef und ϕ-big ist. Dann ist $\phi_*(L \otimes \omega_{X/Y})$ schwach positiv.*

Das zweite Ergebnis sollte für viele Anwendungen nützlich sein, es verallgemeinert insbesondere den klassischen Fall der direkten Bildgarbe $\phi_*\omega_{X/Y}$.

Theorem. *Sei X eine normale $\mathbb{Q}$-Gorensteinvarietät mit höchstens kanonischen Singularitäten, und sei Y eine normale $\mathbb{Q}$-Gorensteinvarietät. Sei $\phi : X \to Y$ eine flache Faserung und sei L ein semiamples Geradenbündel auf X. Dann ist $\phi_*(L \otimes \omega_{X/Y})$ schwach positiv.*

Eine Verallgemeinerung des letzten Ergebnisses für Geradenbündel L mit nicht-negativer Kodairadimension, d.h. ein multiples von L hat globale Schnite, ist nicht ohne weiteres möglich. Sei $N \in \mathbb{N}$ eine hinreichende hohe und teilbare natürliche Zahl so dass das Linearsystem $|L^{\otimes N}|$ eine rationale Abbildung $\phi : X \dashrightarrow Y$ auf eine normale Varietät Y induziert. Wenn L nicht semiampel ist, kann diese Abbildung kein Morphismus sein, aber wir können durch eine Aufblasung $\mu : X' \to X$ den unbestimmten Ort auflösen. Dann gilt

$$\mu^* L^{\otimes N} \otimes \mathcal{O}_{X'}(-D) \simeq M,$$

wobei D ein effektiver Divisor und M ein semiamples Geradenbündel ist. Grob gesprochen misst der Divisor D den Abstand von L von der Eigenschaft semiampel zu sein (genauer genommen von der Eigenschaft nef und abundant zu sein). Die grundlegende Idee der Theorie asymptotischer Multiplierideals ist L eine Idealgarbe $\mathcal{J}(||L||)$ zuzuordenen die diesen Abstand repräsentiert. Den Ort auf X der durch diese Idealgarbe definiert wird nennt man den Koträger der Idealgarbe und typischerweise ist dies der Ort auf dem das Geradenbündel L nicht nef ist. Dies bringt uns zu unserem nächsten Ergebnis.

Theorem. *Sei $\phi : X \to Y$ eine flache Faserung zwischen projektiven Mannigfaltigkeiten und sei L ein Geradenbündel auf X dessen Kodairadimension nicht negativ ist. Es sei $\mathcal{J}(X, ||L||)$ das asymptotische Multiplierideal von L. Wenn der Koträger von $\mathcal{J}(X, ||L||)$ nicht surjektiv auf Y abgebildet wird, ist die direkte Bildgarbe $\phi_*(L \otimes \omega_{X/Y})$ schwach positiv.*

Wir zeigen durch eine Reihe von Beispielen und Gegenbeispielen, dass diese Ergebnisse optimal sind. Sei $Z \subset Y$ eine Untervarietät, dann ist die (schwache) Positivität der direkten Bildgarbe auf Z in den folgenden Situation im Allgemeinen nicht gewährleistet.

1.) Die allgemeine Faser über Z ist nicht reduziert.

2.) Das Urbild von Z hat viele irrationale Singularitäten.

3.) Der Koträger des Multiplierideals wird surjektiv auf Z abgebildet.

Deux applications de la positivité à l'étude des variétés projectives complexes

Dans cette thèse, nous étudions deux problèmes très naturels en géométrie algébrique complexe.

La première question étudiée est de savoir si le revêtement universel d'une variété kählérienne lisse compacte avec un fibré tangent décomposé est un produit de deux variétés. A l'aide des familles couvrantes de courbes rationnelles - lorsqu'elles existent - nous montrons que les variétés avec un fibré tangent décomposé possèdent une structure d'espace fibré, que nous étudions ensuite de façon systématique. Ceci nous permet de donner une réponse affirmative à la question initiale pour plusieurs nouvelles classes de variétés.

La deuxième question étudiée est de savoir si la positivité d'un fibré en droites implique la positivité de l'image directe, par un morphisme projectif et plat, du fibré en droites adjoint. La réponse à cette question dépend de la positivité du fibré en droites et de ses liens avec la géométrie du morphisme considéré. Nous montrons que la réponse à la question est positive sous des conditions apparaissant naturellement dans les problèmes de classification des variétés projectives complexes.

Bien que les deux problèmes soient indépendants, les méthodes utilisées sont assez proches : nous utilisons la positivité des faisceaux cohérents et les outils de la classification des variétés complexes pour obtenir l'existence de structures d'espace fibré et pour en étudier leurs propriétés. Donnons maintenant un résumé des résultats principaux, les introductions des parties I et II fournissent des renseignements plus précis sur les méthodes employées et les problèmes encore ouverts.

Première partie : Variétés kählériennes avec un fibré tangent décomposé

Une stratégie standard en géométrie algébrique est d'obtenir des informations sur la structure d'une variété lisse à partir d'informations sur son fibré tangent. Ce dernier est souvent plus facile à manier puisqu'il peut être considéré comme une version linéarisée de la variété. Si une variété est un produit, son fibré tangent est la somme de deux fibrés vectoriels et on peut se demander si l'implication inverse est vraie. Plus précisément nous allons étudier la conjecture suivante.

Conjecture 1. *(A. Beauville) Soit X une variété kählérienne lisse compacte telle que $T_X = V_1 \oplus V_2$, où V_1 et V_2 sont des fibrés vectoriels holomorphes. Soit $\mu : \tilde{X} \to X$ le revêtement universel de X. Alors $\tilde{X} \simeq X_1 \times X_2$ et $p_{X_j}^* T_{X_j} \simeq \mu^* V_j$. Si de plus nous supposons que V_j est intégrable, alors il existe un automorphisme de $\tilde{X}$ tel que $\mu^* V_j = p_{X_j}^* T_{X_j}$.*

Cette conjecture a été étudiée par Beauville [Bea00], Druel [Dru00], Campana-Peternell [CP02] et récemment par Brunella-Pereira-Touzet [BPT04]. Ce dernier papier généralise la plupart des résultats précédents, son résultat principal étant le

Théorème. [BPT04, Thm.1] *Soit X une variété kählérienne lisse compacte. Supposons que le fibré tangent de X se décompose en $T_X = V_1 \oplus V_2$, où $V_2 \subset T_X$ est un sous-fibré vectoriel de rang* $\dim X - 1$*. Alors il y a deux cas :*

1.) si V_2 n'est pas intégrable, V_1 est tangent aux fibres d'un fibré en $\mathbb{P}^1$.

2.) si V_2 est intégrable, la conjecture 1 est vraie.

Ce théorème établit un lien surprenant entre l'existence de courbes rationnelles le long du feuilletage V_1 et l'intégrabilité du supplémentaire V_2. Il est donc probable que les variétés uniréglées vont jouer un rôle particulier dans la résolution de cette conjecture.

Définition. *Une variété kählérienne lisse compacte X est uniréglée s'il existe une famille couvrante de courbes rationnelles sur X. La variété est rationnellement connexe si pour deux points généraux il existe une courbe rationnelle qui relie ces deux points.*

Un résultat important dû à Campana [Cam81, Cam04b] montre qu'une variété kählérienne lisse compacte uniréglée X admet une fibration méromorphe $\phi : X \dashrightarrow Y$ sur une variété normale Y telle que la fibre générale est rationnellement connexe et la variété Y n'est pas uniréglée (cf. [GHS03]). Dans le cas projectif, nous allons généraliser le résultat d'intégrabilité pour une décomposition du fibré tangent dont le rang est arbitraire.

Théorème. *Soit X une variété projective avec un fibré tangent décomposé $T_X = V_1 \oplus V_2$. Supposons que la fibre générale F du quotient rationnel satisfait $T_F \subset V_2|_F$. Alors V_2 est intégrable et* $\det V_1^*$ *est pseudo-effectif.*

En particulier si X n'est pas uniréglée, les sous-fibrés V_1 et V_2 sont intégrables.

Comme le fibré tangent d'une variété rationnellement connexe a des propriétés de positivité très fortes, il est raisonnable de commencer l'étude de la conjecture avec cette classe de variétés. Notre premier résultat principal est le

Théorème. *Soit X une variété rationnellement connexe tel que $T_X = V_1 \oplus V_2$. Si V_1 ou V_2 est intégrable, alors V_1 et V_2 sont intégrables ; de plus la conjecture 1 est vraie.*

Ce théorème généralise un résultat de Campana et Peternell [CP02] pour les variétés de Fano de dimension au plus 5.

L'étape suivante est de démontrer que si X est une variété uniréglée telle que $T_X = V_1 \oplus V_2$ et si $\psi : X \dashrightarrow Y$ est le quotient rationnel, alors la ψ-fibre générale F satisfait

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

Puisque la ψ-fibre générale est rationnellement connexe, ceci montre que le théorème précédent est très utile pour traiter la classe beaucoup plus large des variétés uniréglées. Un résultat technique important est le

Théorème. *Soit X une variété kählérienne lisse compacte uniréglée tel que $T_X = V_1 \oplus V_2$ et $\mathrm{rg} V_1 = 2$. Soit F une fibre générale du quotient rationnel, alors*

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

Il y a donc trois possibilités :

1.) $T_F \cap V_1|_F = V_1|_F$. Supposons que $T_F \cap V_1|_F$ est intégrable. Alors la variété X possède une structure de fibré analytique $X \rightarrow Y$ tel que $T_{X/Y} = V_1$. Si de plus V_2 est intégrable, la conjecture 1 est vraie pour X.

2.) $T_F \cap V_1|_F$ est un fibré en droites. Alors il existe un morphisme équidimensionnel $\phi : X \rightarrow Y$ tel que la ϕ-fibre générale M satisfait $T_M \subset V_1|_M$. Si l'application ϕ est plate et si V_2 est intégrable, la conjecture 1 est vraie pour X.

3.) $T_F \subset V_2|_F$.

Dans le cas projectif, il est possible de préciser cette analyse et d'obtenir un énoncé analogue au théorème de Brunella, Pereira et Touzet.

Théorème. *Soit X une variété uniréglée projective telle que $T_X = V_1 \oplus V_2$ et $\mathrm{rg} V_1 = 2$. Soit F la fibre générale du quotient rationnel, alors l'un des deux cas suivants se produit.*

1.) $T_F \cap V_1|_F \neq 0$. Si V_1 et V_2 sont intégrables, la conjecture 1 est vraie.

2.) $T_F \cap V_1|_F = 0$. Dans ce cas V_2 est intégrable et $\det V_1^*$ est pseudo-effectif.

On obtient comme corollaire de ce théorème un des résultats principaux de l'article [Hör05].

Corollaire. *[Hör05, Thm.1.5] Soit X une variété projective uniréglée de dimension 4 telle que $T_X = V_1 \oplus V_2$ et $\mathrm{rg} V_1 = \mathrm{rg} V_2 = 2$. Si V_1 et V_2 sont intégrables, la conjecture 1 est vraie.*

Seconde partie : Images directes des fibrés en droites adjoints

Etant donnée une fibration $\phi : X \to Y$, c'est-à-dire un morphisme à fibres connexes entre des variétés projectives normales, il s'agit d'un problème naturel et fondamental que d'essayer d'établir un lien entre les propriétés de positivité des faisceaux dualisants de l'espace total X et de la base Y. Il y deux raisons pour lesquelles cette analyse devrait commencer avec l'étude de l'image directe $\phi_* \omega_{X/Y}$ du faisceau dualisant relatif $\omega_{X/Y} = \omega_X \otimes \phi^* \omega_Y^*$: premièrement, la restriction de $\omega_{X/Y}$ à une fibre générale F est le faisceau dualisant de la fibre F. Le germe de $\phi_* \omega_{X/Y}$ en un point général est donc canoniquement isomorphe à l'espace vectoriel $H^0(F, \omega_F)$, que l'on peut voir comme une mesure de la positivité de ω_X autour de cette fibre. Deuxièmement, la structure globale de $\phi_* \omega_{X/Y}$ donne des informations sur la variation de la positivité entre les fibres, donc sur la positivité de ω_X après avoir pris le quotient par la positivité le long des fibres (nous allons préciser cet énoncé un peu vague dans la suite). Puisque Y est l'espace paramétrant les fibres, la positivité de ω_Y devrait donc refléter cette „positivité du quotient". Dans ses papiers fondamentaux [Vie82, Vie83], Eckart Viehweg a introduit la notion de positivité faible.

Définition. *Soit X une variété quasi-projective. Un faisceau cohérent sans torsion $\mathcal{F}$ est faiblement positif s'il existe un fibré en droites ample H tel que pour tout entier positif $\alpha \in \mathbb{N}$ il existe $\beta \in \mathbb{N}$ tel que $(\mathrm{Sym}^{\beta\alpha} \mathcal{F})^{**} \otimes H^\beta$ est engendré par ses sections globales au point général de X.*

Un des résultats principaux des articles de Viehweg est le

Théorème. *[Vie82] Soit $\phi : X \to Y$ une fibration entre des variétés projectives. Alors pour tout $m \in \mathbb{N}$, le faisceau image directe $\phi_*(\omega_{X/Y}^{\otimes m})$ est faiblement positif.*

Pour des applications, par exemple dans le contexte des espaces de modules des variétés polarisées (cf. [Vie95]), il est important d'étudier un problème plus général : étant donnés une fibration $\phi : X \to Y$ et un fibré en droites L sur X, on peut s'intéresser à la positivité de l'image directe $\phi_*(L \otimes \omega_{X/Y})$. Un moment de réflexion va convaincre le lecteur qu'on ne peut pas espérer obtenir un résultat positif si on ne suppose pas que le fibré en droites L est lui-même positif en un certain sens (par exemple ample, nef, faiblement positif,...). De plus, il est nécessaire d'imposer des restrictions sur la géométrie de la fibration $\phi : X \to Y$, par exemple sur les singularités de la variété X. En utilisant les papiers importants de Kollár [Kol86] et Viehweg [Vie82, Vie83], nous allons adapter une stratégie utilisée par C. Mourougane dans sa thèse pour démontrer la positivité des faisceaux image directe.

Théorème. *[Mou97, Thm.1] Soit $\phi : X \to Y$ une fibration lisse entre des variétés projectives lisses et soit L un fibré en droites nef et ϕ-big sur X. Alors $\phi_*(L \otimes \omega_{X/Y})$ est localement libre et nef.*

Le but de notre travail est de généraliser ce résultat dans des directions différentes. La première, et la plus importante, est de montrer un résultat analogue pour une fibration qui est plate, mais pas nécessairement lisse. Deuxièmement, nous faisons ceci pour une fibration entre des variétés projectives qui ne sont pas forcément lisses. Troisièmement, nous affaiblissons ou changeons la condition sur la positivité de L. En particulier, nous rencontrons des situations où $\phi_*(L \otimes \omega_{X/Y})$ n'est pas localement libre. Nous réalisons ce programme sous des conditions diverses sur la positivité du fibré en droites et dans différents contextes géométriques.

Théorème. *Soit X une variété normale $\mathbb{Q}$-Gorenstein avec au plus des singularités canoniques et soit Y une variété normale $\mathbb{Q}$-Gorenstein. Soit $\phi : X \to Y$ une fibration plate et soit L un fibré en droites nef et ϕ-big sur X. Alors $\phi_*(L \otimes \omega_{X/Y})$ est faiblement positif.*

Le deuxième résultat devrait être utile pour beaucoup d'applications, en particulier il généralise le cas classique du faisceau image directe $\phi_* \omega_{X/Y}$.

Théorème. *Soit X une variété normale $\mathbb{Q}$-Gorenstein avec au plus des singularités canoniques et soit Y une variété normale $\mathbb{Q}$-Gorenstein. Soit $\phi : X \to Y$ une fibration plate et soit L un fibré en droites semiample sur X. Alors $\phi_*(L \otimes \omega_{X/Y})$ est faiblement positif.*

Pour démontrer un énoncé analogue pour un fibré en droites L dont la dimension de Kodaira est non-négative, - c'est-à-dire dont un multiple a des sections globales - il faut être plus prudent. Soit $N \in \mathbb{N}$ un entier suffisamment grand et divisible tel que le systeme linéaire $|L^{\otimes N}|$ induit une application rationnelle $\phi : X \dashrightarrow Y$ sur une variété normale Y. Si L n'est pas semiample, cette application n'est pas un morphisme, mais il est possible de résoudre les points d'indétermination en éclatant $\mu : X' \to X$. Alors

$$\mu^* L^{\otimes N} \otimes \mathcal{O}_{X'}(-D) \simeq M,$$

où D est un diviseur effectif et M est semiample. Moralement le diviseur D décrit le défaut de L à être semiample (plus précisement le défaut de L à être nef et abondant). L'idée centrale de la théorie des idéaux multiplicateurs asymptotiques est qu'on peut associer à L un faisceau d'idéaux $\mathcal{I}(X, ||L||)$ qui mesure ce défaut. Le lieu sur X défini par ce faisceau d'idéaux est appelé le cosupport du faisceau d'idéaux et typiquement c'est le lieu où L n'est pas nef. Ceci nous amène à notre dernier résultat.

Théorème. *Soit $\phi : X \to Y$ une fibration plate entre des variétés projectives lisses et soit L un fibré en droites de dimension de Kodaira non-négative sur X. Notons $\mathcal{I}(X, ||L||)$ le faisceau d'idéaux multiplicateurs asymptotiques associé à L. Si la restriction de ϕ au cosupport de $\mathcal{I}(X, ||L||)$ n'est pas surjective sur Y, alors le faisceau image directe $\phi_*(L \otimes \omega_{X/Y})$ est faiblement positif.*

Une série d'exemples et contre-exemples montre que ces résultats sont optimaux. Etant donné un certain lieu $Z \subset Y$, la positivité sur Z du faisceau image directe n'est pas assurée dans les situations suivantes :

1.) La fibre générale sur Z n'est pas réduite.

2.) La préimage de Z a beaucoup de singularités irrationnelles.

3.) Le cosupport du faisceau d'idéaux multiplicateurs asymptotiques se surjecte sur Z.

Two applications of positivity to the classification theory of complex projective varieties

The subject of this thesis is to investigate two very natural questions in complex algebraic geometry.

The first question asks if the universal covering of a compact Kähler manifold with a split tangent bundle is a product of two manifolds. We will establish a structure theory for manifolds with a split tangent bundle and use covering families of rational curves to show the existence of a fibre space structure. A discussion of the fibre space structure allows to give an affirmative answer to the question for several classes of manifolds.

The second question asks if the positivity of a line bundle implies the positivity of the direct image of the adjoint line bundle under a flat projective morphism. We will see that the answer to this question depends on the positivity of the line bundle and its relation to the geometry of the morphism. We will show that under conditions that are typical for problems in classification theory of projective varieties, the answer is to the affirmative.

Although the two problems are completely independent, the methods involved are rather similar: we use positivity of coherent sheaves and classification theory to discuss the existence and properties of certain fibre spaces structures. We will now give an overview of the results, the introductions of part I and II will give more precise informations on the method and open problems.

Part I: Kähler manifolds with split tangent bundle

One of the basic strategies in algebraic geometry is to deduce properties of a manifold from properties of its tangent bundle which can be seen as a linearized version of the manifold. If a manifold is a product of two manifolds, the tangent bundle has the property of being a direct sum of vector bundles and we ask if there exists an inverse statement. More precisely we have the following conjecture.

Conjecture 1. *(A. Beauville) Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$, where V_1 and V_2 are vector bundles. Let $\mu : \tilde{X} \to X$ be the universal covering of X. Then $\tilde{X} \simeq X_1 \times X_2$, where $p_{X_j}^* T_{X_j} \simeq \mu^* V_j$. If moreover V_j is integrable, then there exists an automorphism of $\tilde{X}$ such that we have an identity of subbundles of the tangent bundle $\mu^* V_j = p_{X_j}^* T_{X_j}$.*

The conjecture has been studied before by Beauville [Bea00], Druel [Dru00], Campana-Peternell [CP02] and recently by Brunella-Pereira-Touzet [BPT04]. The last paper contains most of the preceeding results, its main result is the

Theorem. [BPT04, Thm.1] *Let X be a compact Kähler manifold. Suppose that its tangent bundle splits as $T_X = V_1 \oplus V_2$, where $V_2 \subset T_X$ is a subbundle of rank* $\dim X - 1$. *Then there are two cases:*

1.) if V_2 is not integrable, then V_1 is tangent to the fibres of a $\mathbb{P}^1$-bundle.

2.) if V_2 is integrable, then conjecture 1 holds.

The theorem establishes a surprising link between the existence of rational curves along the foliation V_1 and the integrability of the complement V_2. This suggest that uniruled manifolds will play a distinguished role in the solution of the conjecture.

Definition. *A compact Kähler manifold X is uniruled if there exists a covering family of rational curves. It is rationally connected if for two general points there exists a rational curve through these two points.*

A deep result of Campana [Cam81, Cam04b] shows that a uniruled compact Kähler manifold X admits a meromorphic fibration $\phi : X \dashrightarrow Y$ to a normal variety Y such that the general fibre is rationally connected and the variety Y is not uniruled (see also [GHS03]). In the projective case we obtain a generalisation of the integrability result to a splitting in vector bundles of arbitrary rank.

Theorem. *Let X be a projective manifold with split tangent bundle $T_X = V_1 \oplus V_2$. Suppose that a general fibre of the rational quotient F satisfies $T_F \subset V_2|_F$. Then V_2 is integrable and* $\det V_1^*$ *is pseudo-effective.*

In particular if X is not uniruled, then V_1 and V_2 are integrable.

Since the tangent bundle of a rationally connected manifold has very strong positivity properties, it is reasonable to start the investigation of the conjecture with this class of manifolds. As a first main result, we show the following

Theorem. *Let X be a rationally connected manifold such that $T_X = V_1 \oplus V_2$. If V_1 or V_2 is integrable, then V_1 and V_2 are integrable; furthermore conjecture 1 holds.*

This generalizes a result due to Campana and Peternell [CP02] for Fano manifolds of dimension at most 5.

As a next step we show that if X is a uniruled manifold such that $T_X = V_1 \oplus V_2$ and $\psi : X \dashrightarrow Y$ is the rational quotient map, then the general ψ-fibre F satisfies

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

Since the general ψ-fibre is rationally connected, this shows that the preceeding theorem is very useful to treat the much larger class of uniruled manifolds. An important intermediate result is the

Theorem. *Let X be a uniruled compact Kähler manifold such that $T_X = V_1 \oplus V_2$ and $\mathrm{rk} V_1 = 2$. Let F be a general fibre of the rational quotient map, then*

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

Furthermore there are three possibilities:

1.) $T_F \cap V_1|_F = V_1|_F$. Suppose that $T_F \cap V_1|_F$ is integrable. Then the manifold X admits the structure of an analytic fibre bundle $X \to Y$ such that $T_{X/Y} = V_1$. If moreover V_2 is integrable, then conjecture 1 holds for X.

2.) $T_F \cap V_1|_F$ is a line bundle. There exists an equidimensional map $\phi : X \to Y$ such that the general ϕ-fibre M satisfies $T_M \subset V_1|_M$. If the map ϕ is flat and V_2 is integrable, then conjecture 1 holds for X.

3.) $T_F \subset V_2|_F$.

In the projective case we can go further and obtain a statement that is analogous to the theorem of Brunella, Pereira, and Touzet.

Theorem. *Let X be a uniruled projective manifold such that $T_X = V_1 \oplus V_2$ and* $\mathrm{rk} V_1 = 2$*. Let F be a general fibre of the rational quotient map, then one of the following holds.*

1.) $T_F \cap V_1|_F \neq 0$. If V_1 and V_2 are integrable, conjecture 1 holds.

2.) $T_F \cap V_1|_F = 0$. Then V_2 is integrable and $\det V_1^*$ *is pseudo-effective.*

We recover as a corollary one of the main results of [Hör05].

Corollary. *[Hör05, Thm.1.5] Let X be a projective uniruled fourfold such that $T_X = V_1 \oplus V_2$ where* $\mathrm{rk} V_1 = \mathrm{rk} V_2 = 2$*. If V_1 and V_2 are integrable, then conjecture 1 holds.* □

Part II: Direct images of adjoint line bundles

Given a fibration $\phi : X \to Y$, i.e. a morphism with connected fibres between projective normal varieties, it is a natural und fundamental problem to try to relate positivity properties of the dualising sheaves of the total space X and the base Y. There are two reasons why such an analysis should start with an investigation of the direct image $\phi_* \omega_{X/Y}$ of the relative dualising sheaf $\omega_{X/Y} = \omega_X \otimes \phi^* \omega_Y^*$: firstly, the restriction of $\omega_{X/Y}$ to a general fibre F is the dualising sheaf of the fibre F. Therefore the germ of $\phi_* \omega_{X/Y}$ in a general point is the space of global sections $H^0(F, \omega_F)$ which can be interpreted as a measure of the positivity of ω_X around this fibre. Secondly, the global structure of $\phi_* \omega_{X/Y}$ gives some information about the variation of the positivity between the fibres, so in some very vague sense the positivity of ω_X after taking the quotient by the positivity along the fibres. Since Y is the parameter space of the fibres, the positivity of ω_Y should reflect this „quotient positivity“. In his landmark papers [Vie82, Vie83], Eckart Viehweg introduced the notion of weak positivity.

Definition. *Let X be a quasi-projective variety. A torsion-free coherent sheaf $\mathcal{F}$ is weakly positive if there exists an ample line bundle H such that for every natural number $\alpha \in \mathbb{N}$ there exists some $\beta \in \mathbb{N}$ such that $(\mathrm{Sym}^{\beta\alpha} \mathcal{F})^{**} \otimes H^{\beta}$ is generated in the general point.*

One of the main results in Viehweg's papers is the

Theorem. *[Vie82] Let $\phi : X \to Y$ be a fibration between projective manifolds. Then for all $m \in \mathbb{N}$, the direct image sheaf $\phi_*(\omega_{X/Y}^{\otimes m})$ is weakly positive.*

For applications, for example in the context of moduli spaces for polarized manifolds (compare [Vie95]), it is important to study a more general setting: given a fibration $\phi : X \to Y$, and a line bundle L on X, one can ask for the positivity of the direct image $\phi_*(L \otimes \omega_{X/Y})$. A moment of reflection will convince the reader that it is hopeless to ask such a question for a line bundle L that is not itself positive in some sense (e.g. ample, nef, weakly positive,...). Furthermore it is necessary to put some restrictions on the geometry of the fibration $\phi : X \to Y$, for example some mild conditions on the singularities of the variety X. Building up on the important papers of Kollár [Kol86] and Viehweg [Vie82, Vie83], we will refine a strategy used by C. Mourougane in his thesis to show the positivity of direct image sheaves.

Theorem. *[Mou97, Thm.1] Let $\phi : X \to Y$ be a smooth fibration between projective manifolds, and let L be a nef and ϕ-big line bundle on X. Then $\phi_*(L \otimes \omega_{X/Y})$ is locally free and nef.*

The aim of this work is to generalise his result in different directions. First and foremost is to show an analogous result for a fibration that is flat, but not necessarily smooth. Secondly we would like to do this for a fibration between projective varieties that are not smooth. Thirdly we would like to weaken or change the positivity hypothesis on L. In particular we might encounter situations where $\phi_*(L \otimes \omega_{X/Y})$ is not locally free. We will do this under a variety of conditions on the positivity of line bundles and geometric settings.

Theorem. *Let X be a normal $\mathbb{Q}$-Gorenstein variety with at most canonical singularities, and let Y be a normal $\mathbb{Q}$-Gorenstein variety. Let $\phi : X \to Y$ be a flat fibration, and let L be a nef and ϕ-big line bundle on X. Then $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

The second result should be useful for a lot of applications, in particular it contains the classical case of the direct image sheaf $\phi_*\omega_{X/Y}$.

Theorem. *Let $\phi : X \to Y$ be a flat Cohen-Macaulay fibration from a projective*

$\mathbb{Q}$-Gorenstein variety X with at most canonical singularities to a normal projective $\mathbb{Q}$-Gorenstein variety Y. Let L be a semiample line bundle over X, then $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.

If we want to show such a statement for a line bundle L with non-negative Kodaira dimension, i.e. such that some multiple has global sections, we have to be more careful. Let $N \in \mathbb{N}$ be a sufficiently high and divisible integer such that the linear system $|L^{\otimes N}|$ induces a rational map $\phi : X \dashrightarrow Y$ on a normal variety Y. If L is not semiample, this map will never be a morphism, but we can resolve the indeterminacies by blowing-up $\mu : X' \to X$. Then

$$\mu^* L^{\otimes N} \otimes \mathcal{O}_{X'}(-D) \simeq M,$$

where D is an effective divisor and M is semiample. Morally speaking the divisor D describes the distance of L from being semiample (more precisely from being nef and abundant). The basic idea of the asymptotic multiplier ideal theory is that we can associate to L an ideal sheaf $\mathcal{I}(||L||)$ that represents this distance. The locus on X defined by this ideal sheaf is then called the cosupport of the ideal sheaf and is typically the locus where L fails to be nef. This leads us to our third main result.

Theorem. *Let $\phi : X \to Y$ be a flat fibration between projective manifolds, and let L be a line bundle of non-negative Kodaira dimension over X. Denote by $\mathcal{I}(X, ||L||)$ the asymptotic multiplier ideal of L. If the cosupport of $\mathcal{I}(X, ||L||)$ does not project onto Y, then the direct image sheaf $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

A series of examples and counterexamples shows the optimality of these results. Given a certain locus $Z \subset Y$, the positivity of the direct image sheaf on Z can not be guaranteed in the following situations:

1.) The general fibre over Z is not reduced.

2.) The preimage of Z has many irrational singularities.

3.) The cosupport of the multiplier ideal surjects onto Z.

Part I

Kähler manifolds with split tangent bundle

Chapter 1

Introduction to Part I

1.1 Main results

Differentiable manifolds with split tangent bundle are a classical object of study in differential geometry. The most important result in this context is de Rham's theorem.

1.1.1 Theorem. *[KN63, IV, Thm.6.1] Let X be a complete Riemannian manifold such that $T_X = V_1 \oplus V_2$. Suppose that this decomposition is invariant under the linear holonomy group. Let $\mu : \tilde{X} \to X$ be the universal covering of X. Then $\tilde{X} \simeq X_1 \times X_2$ and there exists an automorphism of $\tilde{X}$ such that we have an identity of subbundles of the tangent bundle $\mu^* V_j = p_{X_j}^* T_{X_j}$* [1]*.*

An analogous result holds in the analytic category for Kähler manifolds. Since a splitting of a vector bundle in the analytic category is a much stronger property than in the setting of real differential geometry, one might hope to get the same result for compact complex manifolds without making a hypothesis about invariance under holonomy. The most well-known example of such a statement is the decomposition theorem of Beauville 6.1.4 which states that a projective manifold with trivial canonical class decomposes in a product according to its holonomy. For general compact Kähler manifolds we have the following conjecture.

1.1.2 Conjecture. *(A. Beauville) Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$, where V_1 and V_2 are vector bundles. Let $\mu : \tilde{X} \to X$ be the universal covering of X. Then $\tilde{X} \simeq X_1 \times X_2$, where $p_{X_j}^* T_{X_j} \simeq \mu^* V_j$. If moreover V_j is integrable, then there exists an automorphism of $\tilde{X}$ such that we have an identity of subbundles of the tangent bundle $\mu^* V_j = p_{X_j}^* T_{X_j}$.*

[1] We adapt the convention that for a product $X_1 \times X_2$, the vector bundle $p_{X_j}^* T_{X_j}$ is embedded in $T_{X_1 \times X_2}$ as the relative tangent bundle of the projection $X_1 \times X_2 \to X_j$.

The Kähler hypothesis is needed, since Beauville has shown in [Bea00] that there are Hopf surfaces with split tangent bundle whose universal covering is not a product. The conjecture has been studied before by Beauville [Bea00], Druel [Dru00], Campana-Peternell [CP02] and recently by Brunella-Pereira-Touzet [BPT04]. The last paper contains most of the preceeding results, its main result is the

1.1.3 Theorem. *[BPT04, Thm.1] Let X be a compact Kähler manifold. Suppose that its tangent bundle splits as $T_X = V_1 \oplus V_2$, where $V_2 \subset T_X$ is a subbundle of rank* $\dim X - 1$. *Then there are two cases:*

1.) if V_2 is not integrable, then V_1 is tangent to the fibres of a $\mathbb{P}^1$-bundle.

2.) if V_2 is integrable, then conjecture 1.1.2 holds.

Brunella, Pereira and Touzet use analytic techniques for codimension 1 foliations to establish this result in a surprisingly short paper. In this thesis, we want to take a different point of view which combines techniques from the classification theory of compact Kähler manifolds and foliation theory. Our general approach is not limited to a splitting in direct factors with a certain rank, but for geometric reasons it will often be necessary to limit ourselves to the case where one of the direct factors has rank 2. Although conjecture 1.1.2 remains our main objective, we will be interested in a larger spectrum of questions.

1.1.4 Question. Let X be a compact Kähler manifold with split tangent bundle $T_X = V_1 \oplus V_2$. What can we say about the integrability of the direct factors V_1 and V_2 ?

We will see that the answer to this question is closely related to the uniruledness of the manifold.

1.1.5 Definition. *A compact Kähler manifold X is uniruled if there exists a covering family of rational curves. It is rationally connected if for two general points there exists a rational curve through these two points.*

A deep result of Campana [Cam81, Cam04b] shows that a uniruled compact Kähler manifold X admits a meromorphic fibration $\phi : X \dashrightarrow Y$ to a normal variety Y such that the general fibre is rationally connected and the variety Y is not uniruled (see also [GHS03]). This map is not holomorphic in general, but almost holomorphic, that is the image of the indeterminate locus does not cover Y. Furthermore it is unique up to meromorphic equivalence of fibrations, so we are entitled to call it *the* rational quotient of X[2].

Theorem 1.1.3 establishes a surprising link between the existence of rational curves along the foliation V_1 and the integrability of the complement V_2. We obtain an analogous result for projective manifolds.

[2] Chapter 4 gives a more detailed introduction to uniruled manifolds and the rational quotient.

2.2.1 Theorem. *Let X be a projective manifold with split tangent bundle $T_X = V_1 \oplus V_2$. Suppose that a general fibre of the rational quotient F satisfies $T_F \subset V_2|_F$. Then V_2 is integrable and $\det V_1^*$ is pseudo-effective.*

In particular if X is not uniruled, then V_1 and V_2 are integrable.

This result should also hold for compact Kähler manifolds that are not uniruled, but the proof of our result relies on the deep results from [BDPP04] which are difficult to generalise. There are counterexamples to the integrability of the direct factors in the uniruled case (cf. example 2.2.3), but these examples are not rationally connected. Therefore we conjecture

1.1.6 Conjecture. *Let X be a projective manifold with split tangent bundle $T_X = V_1 \oplus V_2$. If X is rationally connected, then V_1 or V_2 is integrable.*

Lemma 2.2.6 will provide some evidence for this conjecture, in particular we show that it holds if X has dimension at most 4.

A second line of investigation is to study compact Kähler manifold X with split tangent bundle $T_X = V_1 \oplus V_2$ that admit a fibre space structure. It is clear that this is a hopeless task if the fibre space structure has no relation with the decomposition of the tangent bundle, so we are interested in fibrations such that for a general fibre F, we have

$$T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F).$$

We then say that the fibration satisfies the ungeneric position property and study this property extensively in chapter 3.

1.1.7 Question. Let X be a compact Kähler manifold with split tangent bundle $T_X = V_1 \oplus V_2$. Which fibrations satisfy the ungeneric position property ? Given such a fibration, what can we say about the global structure of such a fibration ?

The first question is relatively easy: all the maps that reflect some positivity property of the tangent bundle, cotangent bundle or (anti-)canonical divisor should satisfy the ungeneric position property. We show this for rational quotient maps (corollary 4.4.4), Mori fibre spaces (lemma 4.3.3) and Albanese maps (proposition 3.2.8). Furthermore we obtain a similar property for some Iitaka fibrations (proposition 6.1.6).

The second question is a much more difficult task, since a priori the degenerate fibres can be very bad. Still there is a hope that the fibres are not worse than in the situation of a fibration $\phi = \phi_1 \circ p_{X_1} : X_1 \times X_2 \to X_1 \to Y_1$ where $\phi_1 : X_1 \to Y_1$ is a fibration of the first factor. The fibres then have a (local) product structure which gives restrictions on the singularities of the fibre. We illustrate this principle in section 3.3 in a non-trivial case. Although multiple and higher-dimensional fibres make this problem rather arduous, it is also particularly interesting. In fact it should be seen as a test case for studying the relation between the fibre space structure and the foliated structure of a manifold. This is an important tool in the classification theory of foliations (cf. Brunella's excellent survey [Bru00] on the state of the art for surfaces).

Last but not least we turn our attention to uniruled manifolds. Here our techniques can finally develop their full power, since some of the technical obstacles related to the existence of multiple fibres will disappear (for an example compare proposition 2.3.8 with corollary 4.1.2). As a first main result, we show

4.2.4 Theorem. *Let X be a rationally connected manifold such that $T_X = V_1 \oplus V_2$. If V_1 or V_2 is integrable, then V_1 and V_2 are integrable; furthermore conjecture 1.1.2 holds.*

This statement is essentially based on a theorem of Bogomolov and McQuillan [BM01, KSCT05] on the algebraicity and rational connectedness of leaves of certain foliations. The study of this subject, the so-called foliated Mori theory, was initiated by Miyaoka [Miy87, Miy88] and his characterisation of non-uniruled manifolds in terms of some weak positivity property of the cotangent bundle.

Note that an affirmative answer to conjecture 1.1.6 would imply conjecture 1.1.2 for rationally connected manifolds. We then move from rationally connected manifolds to uniruled manifolds, a first corollary of the theorem is

4.4.6 Corollary. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Suppose that the rational quotient map $\phi : X \dashrightarrow Y$ is a map on a curve Y. If V_1 or V_2 is integrable, then V_1 and V_2 are integrable; furthermore conjecture 1.1.2 holds for X.*

Once we have treated these classical fibrations, we move on to the fibrations obtained by Mori theory. These are defined as fibrations $X \to Y$ such that the anticanonical divisor $-K_X$ is ample on all the fibres and the relative Picard number $\rho(X) - \rho(Y)$ equals one. Mori fibre spaces satisfy a very particular case of the ungeneric position property which allows us to show numerous structure results in section 4.3. One interesting result in this context is

4.3.8 Theorem. *Let X be a projective manifold with $T_X = V_1 \oplus V_2$. If X admits an elementary Mori contraction on a surface, then V_1 or V_2 are integrable. If furthermore both V_1 and V_2 are integrable, then conjecture 1.1.2 holds.*

Since Mori theory for compact Kähler manifolds is far from being complete, it is not possible to use this strategy to obtain results in the non-projective case. We therefore study this problem in section 4.4 by replacing Mori contractions with the rational quotient map. An important intermediate result is the

4.4.11 Theorem. *Let X be a uniruled compact Kähler manifold such that $T_X = V_1 \oplus V_2$ and $\mathrm{rk} V_1 = 2$. Let F be a general fibre of the rational quotient map, then*

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

Furthermore there are three possibilities:

1.) $T_F \cap V_1|_F = V_1|_F$. *Suppose that* V_1 *is integrable or conjecture 1.1.6 holds for* F*, that is* $T_F \cap V_1|_F$ *or* $T_F \cap V_2|_F$ *is integrable. Then the manifold* X *admits the structure of an analytic fibre bundle* $X \to Y$ *such that* $T_{X/Y} = V_1$. *If* V_2 *is integrable, then conjecture 1.1.2 holds for* X.

2.) $T_F \cap V_1|_F$ *is a line bundle. There exists an equidimensional map* $\phi : X \to Y$ *such that the general* ϕ*-fibre* M *satisfies* $T_M \subset V_1|_M$. *If the map* ϕ *is flat and* V_2 *is integrable, then conjecture 1.1.2 holds for* X.

3.) $T_F \subset V_2|_F$.

In the projective case we can go further and obtain a more complete statement.

4.4.12 Theorem. *Let* X *be a uniruled projective manifold such that* $T_X = V_1 \oplus V_2$ *and* $\mathrm{rk} V_1 = 2$. *Let* F *be a general fibre of the rational quotient map, then one of the following holds.*

1.) $T_F \cap V_1|_F \neq 0$. *If* V_1 *and* V_2 *are integrable, then conjecture 1.1.2 holds.*

2.) $T_F \cap V_1|_F = 0$. *Then* V_2 *is integrable and* $\det V_1^*$ *is pseudo-effective.*

We recover as a corollary one of the main results of [Hör05].

1.1.8 Corollary. *[Hör05, Thm.1.5] Let* X *be a projective uniruled fourfold such that* $T_X = V_1 \oplus V_2$ *where* $\mathrm{rk} V_1 = \mathrm{rk} V_2 = 2$. *If* V_1 *and* V_2 *are integrable, then conjecture 1.1.2 holds.* □

1.2 Leitfaden

Chapter 2 contains a detailed introduction to foliations on complex manifolds, including the very important classical theorems on the stability of foliations on Kähler manifolds. Section 2.2 contains our new results on the integrability of the direct factors of a split tangent bundle.

Chapter 3 introduces the notion of ungeneric positin and is of a technical nature. The reader that is interested in the main statements should skip this chapter and only come back to the results when they are applied in the geometric context.

Chapter 4 is the core of this first part. It contains the proofs of the main results and shows that the ungeneric position approach works very well for uniruled manifolds. A reader that is familiar with the basic of the theory of foliations and classification theory can focus on this chapter.

Chapter 5 is an implementation of the minimal model program for fourfolds with split tangent bundle. We show that in dimension 4, it is sufficient to discuss conjecture 1.1.2 for Mori fibre spaces and smooth minimal models.

Chapter 6 shows that our approach is not limited to uniruled manifolds. We do not make much progress on conjecture 1.1.2, but we will obtain a global vision of the structure of manifolds with split tangent bundle. This chapter is mainly for those that are interested in tackling conjecture 1.1.2 themselves.

1.3 Notational conventions

The nonsingular locus of a complex variety X will be denoted by X_{reg}, the singular locus X_{sing}.

Let $X_1 \times X_2$ be a product of manifolds. Then we denote by $p^*_{X_1}T_{X_1}$ the subbundle $T_{X_1\times X_2/X_2} \subset T_{X_1\times X_2}$ (and not only some abstract vector bundle isomorphic to $T_{X_1\times X_2/X_2}$), where the relative tangent bundle is defined by the canonical projection. Analogously, we denote by $p^*_{X_2}T_{X_2}$ the subbundle $T_{X_1\times X_2/X_1} \subset T_{X_1\times X_2}$.

I have tried to structure the more involved proofs by separating them into several steps. For the notation of these steps, I follow Hartshorne's book [Har77] and write

Step 1. p_X is finite.

to say what will be done in this step. Some proofs will be preceeded by an

Idea of the proof..

This is meant to give an intuition why the result should be true. The statements will be vague and non-mathematical, but hopefully interesting for the reader.

A remark on the generality of statements. The natural setting for our problem is the category of complex analytic varieties. For this reason we will give all statements in this context although some results certainly hold for complex analytic spaces or even more general objects.

A complex variety is an irreducible and reduced complex analytic space of finite dimension. Topological notions refer to the analytic topology if not mentioned otherwise.

The text assumes knowledge of the basics of algebraic and analytic geometry, as presented in [Har77] and [KK83]. Furthermore we use a plethora of results from classification theory of projective manifolds which can be found in [Deb01].

Chapter 2

Holomorphic foliations

In this chapter we introduce some basic tools from the theory of foliations. Although all the material presented is fairly standard, it is probably not so familiar to algebraic geometers. For this reason the exposition is relatively large, more details and proofs can be found in the book by Camacho and Lins Neto [CLN85].

2.1 Definitions

2.1.1 Definition. *[CLN85, II,§3, Defn.] Let X be a complex manifold. A subbundle $V \subset T_X$ of rank* $\dim X - k$ *is integrable if there exists a collection of pairs $(U_i, f_i)_{i\in I}$, where U_i is an open subset of X and $f_i : U_i \to \mathbb{D}^k$ is a submersion such that $T_{U_i/\mathbb{D}^k} = V|_{U_i}$, and such that the collection satisfies:*

1.) $\cup_{i\in I} U_i = X$

2.) if $U_i \cap U_j \neq \emptyset$, there exists a local biholomorphism g_{ij} of $\mathbb{D}^k$ such that $f_i = g_{ij} \circ f_j$ on $U_i \cap U_j$.

A maximal collection of such pairs $(U_i, f_i)_{i\in I}$ is called the foliation induced by V and the f_i's are called the distinguished maps of V.

Notation. If $V \subset T_X$ is an integrable subbundle, we also denote by V the foliation induced by it. Confusion will not arise.

Remarks and definitions. The level sets of the distinguished maps $f_i : U_i \to \mathbb{D}^k$ are called the plaques of the foliation. It is clear by definition that the plaques are locally closed subsets of X. The foliation V induces an equivalence relation on X, two points being equivalent if and only if they can be connected by chains of smooth (open) curves C_i such that $T_{C_i} \subset V|_{C_i}$. An equivalence class is called a leaf of the foliation. Let $\mathfrak{V}$ be a leaf of the foliation V and endow $\mathfrak{V}$ with the smallest topology such that the plaques contained in $\mathfrak{V}$ are open. Then $\mathfrak{V}$ admits the structure of a complex manifold such that the inclusion map $i : \mathfrak{V} \to X$ is an injective immersion [CLN85, II,§2, Thm. 1]. Note that the inclusion is in general *not* an embedding, so the leaf is *not* a closed submanifold of X. This is due to the fact that the (so-called intrinsic or fine) topology on $\mathfrak{V}$ defined by the plaques is finer than the topology induced by the topology of X. If $\mathfrak{V}$ is compact for the fine topology, the inclusion $\mathfrak{V} \subset X$ is proper, injective and immersive, so it is an embedding. In particular the fine topology coincides with the topology induced by the topology of X. In this case we say that $\mathfrak{V}$ is a compact leaf. One of the major issues in this thesis will be to search (and find) foliations with compact leaves.

Let $X \to X/V$ be the quotient map associated to the equivalence relation induced by the foliation V, i.e. the set-theoretical map such that the fibres are the leaves of the foliation. If one puts the quotient topology on the set X/V this map is open and continuous, but in general the topology of X/V is very complicated, possibly non-Hausdorff (cf. [CLN85, ch. III]).

A subset $X^* \subset X$ is saturated (or V-saturated) if every leaf of the foliation is either contained in X^* or disjoint from it. We will say that the general leaf of a foliation is compact if there exists a non-empty saturated open subset $X^* \subset X$ such that every leaf contained in X^* is compact. We come to the first fundamental result of foliation theory.

2.1.2 Theorem. *(Frobenius theorem) Let X be a complex manifold, and let $V \subset T_X$ be a subbundle. Then V is integrable if and only if it is involutive, that is the restriction of the Lie bracket*

$$[.,.] \ : \ T_X \times T_X \ \to \ T_X$$

to $V \times V$ has its image in V.

Remark. Note that integrability is a property of a subbundle $V \subset T_X$ and not of the abstract vector bundle V. In fact, for the same abstract vector bundle V there may exist different embeddings $V \hookrightarrow T_X$, some of them such that the image is integrable but not for the others. For an example consider example 2.2.3.

Examples.

1.) Let $\phi : X \to Y$ be a smooth map between complex manifolds, then $T_{X/Y} \subset T_X$ is integrable

2.) Let $v \in H^0(X, T_X)$ be a non-vanishing vector field on a complex manifold. The image of the corresponding morphism $\mathcal{O}_X \to T_X$ is an integrable subbundle.

3.) Let $V_1 \subset T_X$ and $V_2 \subset T_X$ two integrable subbundles, then $V_1 \cap V_2 \subset T_X$ is an integrable subbundle.

The Frobenius theorem implies some elementary properties of foliations.

2.1.3 Corollary. *Let X be a complex manifold, and let $V \subset T_X$ be a subbundle.*

1.) If $H^0(X, \mathcal{H}om(\wedge^2 V, T_X/V)) = 0$, then the subbundle V is integrable.

2.) If there exists a covering family of subvarieties $(Z_s)_{s\in S}$ of X such that a general member of the family satisfies $H^0(Z_s, \mathcal{H}om(\wedge^2 V, T_X/V)|_{Z_s}) = 0$, then the subbundle V is integrable.

3.) Let $X^* \subset X$ be a non-empty Zariski open set. Then V is integrable if and only if $V|_{X^*}$ is integrable.

4.) Let $X \dashrightarrow Y$ be a birational map to a complex manifold and let $W \subset T_Y$ be a subbundle such that W coincides with V on the locus where the map is an isomorphism. Then W is integrable if and only if V is integrable.

Proof. The Lie bracket

$$[.,.] : V \times V \rightarrow T_X$$

is a bilinear antisymmetric mapping that is not $\mathcal{O}_X$-linear but induces an $\mathcal{O}_X$-linear map $\alpha : \wedge^2 V \to T_X/V$. By the Frobenius theorem 2.1.2 this map is zero if and only if V is integrable.

1) Since $\alpha \in H^0(X, \mathcal{H}om(\wedge^2 V, T_X/V))$, we can conclude.

2) The morphism α is a morphism between vector bundles, so it is zero if the restrictions $\alpha|_{Z_s} \in H^0(Z_s, \mathcal{H}om(\wedge^2 V, T_X/V)|_{Z_s})$ are zero.

3) If $V|_{X^*}$ is integrable, the morphism $\alpha|_{X^*}$ is zero. Since X^* is dense, α is zero, so V is integrable. The other implication is trivial.

4) Follows from 3). □

Remark. Corollary 2.1.3,3 can be shortly stated as „integrability is a generic property". This will often allow us to make some extra assumptions if we want to prove the integrability.

2.2 Two integrability results

We show theorem 2.2.1 and give a counterexample in the uniruled case. This shows that the distinction between the uniruled and the non-uniruled case is appropriate to the nature of the problem. In the uniruled case we show an integrability result for a special case (lemma 2.2.6).

2.2.1 Theorem. *Let X be a projective manifold with split tangent bundle $T_X = V_1 \oplus V_2$. Suppose that a general fibre of the rational quotient F satisfies $T_F \subset V_2|_F$. Then V_2 is integrable and* $\det V_1^*$ *is pseudo-effective.*

In particular if X is not uniruled, then V_1 and V_2 are integrable.

Proof.

Step 1. Suppose that $L := \det V_1^*$ is pseudoeffective. Since V_1^* is a direct factor of Ω_X, the vector bundle $\det V_1 \otimes \wedge^{\mathrm{rk} V_1} \Omega_X$ has a trivial direct factor. If $\theta \in H^0(X, L^{-1} \otimes \wedge^{\mathrm{rk} V_1} \Omega_X)$ is the associated nowhere-vanishing $\det V_1$ -valued form, and ζ a germ of any vector field, a local computation shows that $i_\zeta \theta = 0$

if and only if ζ is in V_2. An integrability criterion by Demailly [Dem02, Thm.] shows that V_2 is integrable.

Step 2. $\det V_1^*$ *is pseudoeffective* Suppose that $\det V_1^*$ is not pseudoeffective, then by [BDPP04] there exists a birational morphism $\phi : X' \to X$ and a general intersection curve $C := D_1 \cap \ldots \cap D_{\dim X-1}$ of very ample divisors $D_1, \ldots, D_{\dim X-1}$ where $D_i \in |m_i H|$ for some ample divisor H such that $\phi^* \det V_1^* \cdot L < 0$. Let

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = \phi^* V_1$$

be the Harder-Narasimham filtration with respect to the polarisation H, i.e. the graded pieces E_{i+1}/E_i are semistable with respect to H. Since $m_1, \ldots, m_{\dim X-1}$ can be arbitrarily high, we can suppose that the filtration commutes with restriction to C. Furthermore since C is general and E_1 a reflexive sheaf, the curve C is contained in the locus where E_1 is locally free. Since

$$\mu(E_1|_C) \geq \mu(\phi^* V_1|_C) = \frac{\deg_C V_1}{\mathrm{rk} \phi^* V_1} > 0$$

and $E_1|_C$ is semistable, it is ample by [Laz04a, p.62]. By corollary 2.2.2 below this implies that E_1 is vertical with respect to the rational quotient, that is a general fibre F of the rational quotient satisfies $E_1|_F \cap T_F = E_1|_F$. It follows that the intersection $T_F \cap V_1|_F$ is not empty. $\square$

2.2.2 Corollary. *[KSCT05, Cor.1.5] Let X be a projective manifold, and let $C \subset X$ be a general complete intersection curve. Assume that the restriction $T_X|_C$ contains an ample locally free subsheaf $\mathcal{F}_C$. Then $\mathcal{F}_C$ is vertical with respect to the rational quotient of X.*

Remark. The integrability lemma is optimal, in the sense that there is the following counterexample to the integrability in the uniruled case.

2.2.3 Example. (Beauville) Let A be an abelian surface and u_1, u_2 be linearly independent vector fields on A. Let z_1, z_2 be nonzero vector fields on $\mathbb{P}^1$ such that $[z_1, z_2] \neq 0$. Then $v_1 := p_A^*(u_1) + p_{\mathbb{P}_1}^*(z_1)$ and $v_2 := p_A^*(u_2) + p_{\mathbb{P}_1}^*(z_2)$ are everywhere nonzero vector fields on $X := A \times \mathbb{P}^1$. The subbundle $V := \mathcal{O}_X v_1 \oplus \mathcal{O}_X v_2 \subset T_X$ is not integrable and $T_X = V \oplus p_{\mathbb{P}^1}^* T_{\mathbb{P}^1}$.

In view of this example it seems reasonable to ask whether all the counterexamples to the integrability of the direct factors arise in this manner.

2.2.4 Question. Let X be a compact Kähler manifold such that $T_X \simeq V_1 \oplus V_2$. Are there embeddings $V_1 \subset T_X$ and $V_2 \subset T_X$ such that $T_X = V_1 \oplus V_2$ and V_1 and V_2 are integrable ?

I am rather optimistic that this question has a positive answer. In fact if a vector bundle V admits an integrable embedding $V \subset T_X$, certain characteristic classes vanish. In our situation, this necessary condition is always satisfied. This is due to a fundamental result on split tangent bundles which is a translation of a theorem of Baum and Bott [BB70] to the compact Kähler situation.

2.2.5 Lemma. *[CP02, Lemma 0.4] Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Then we have*

$$c_j(V_i) \in H^j(X, \bigwedge^j V_i^*) \subset H^j(X, \Omega^j) \qquad \forall\ j = 1, \ldots, \mathrm{rk} V_i$$

and $i = 1, 2$.

We give a first application of this important lemma.

2.2.6 Lemma. *Let X be a uniruled compact Kähler manifold such that $T_X = \oplus_{j=1}^k V_j$, where for all $j = 1, \ldots, k$ we have $\mathrm{rk} V_j \leq 2$. Then one of the direct factors is integrable.*

In particular if $\dim X \leq 4$, then one of the direct factors is integrable.

Proof. The statement is trivial if one direct factor has rank 1, so we suppose that all the direct factors have rank 2. Let $f : \mathbb{P}^1 \to X$ be a minimal rational curve on X, then

$$\bigoplus_{j=1}^k f^* V_j = f^* T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus b},$$

We may suppose up to renumbering that $f^* V_1 \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ where $c = 0$ or 1. It follows that for $i \geq 2$, we have $f^* V_i \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$ or $f^* V_i \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ or $f^* V_i \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, in particular

$$H^1(\mathbb{P}^1, f^* V_i^*) = 0 \qquad \forall\ i \geq 2.$$

By lemma 2.2.5, we have $c_1(V_i) \in H^1(X, V_i^*)$, so $c_1(f^* V_i) \in H^1(\mathbb{P}^1, f^* V_i^*)$ is zero for $i \geq 2$. It follows that $f^* \det V_i \simeq \mathcal{O}$, since $f^* V_i$ is nef this implies $f^* V_i \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ for $i \geq 2$. In particular $a + 1 \leq \mathrm{rk} V_1 = 2$. Since $\det f^* V_1$ is ample we obtain

$$H^0(f(\mathbb{P}^1), (\det V_1^* \otimes \bigoplus_{i \geq 2} V_i)|_{f(\mathbb{P}^1)}) \subset H^0(\mathbb{P}^1, f^* \det V_1^* \otimes \bigoplus_{i \geq 2} f^* V_i) = 0.$$

Since the minimal rational curves form a covering family of X, corollary 2.1.3,2 implies the integrability of V_1. $\square$

2.3 Classical results

We continue the introduction to foliations with a series of classical considerations.

For applications it is convenient to have a theory of integrable subsheaves on varieties that are not necessarily smooth. Although we will use these more general notions only in some examples, we include the definitions for completeness' sake.

2.3.1 Definition. *Let X be a complex variety and let $\mathcal{S} \subset V$ be a subsheaf of a vector bundle. The saturation $\overline{\mathcal{S}}$ of $\mathcal{S}$ in V is the kernel of the map $V \to (V/\mathcal{S})/\operatorname{Tor}(V/\mathcal{S})$.*

A subsheaf is saturated if it equals its saturation.

Remark. By [Har80, Prop.1.1] the saturation of $\mathcal{S}$ in V is a reflexive sheaf, so $\overline{\mathcal{S}} \simeq \overline{\mathcal{S}}^{**}$. If furthermore X is normal, the saturation is locally free in codimension 1, i.e. there exists a subvariety $Z \subset X$ such that $\operatorname{codim}_X Z \geq 2$ and $\overline{\mathcal{S}}|_{X \setminus Z}$ is locally free.

2.3.2 Definition. *Let X be a complex variety, and let $\Omega_X \to Q \to 0$ be a quotient of the cotangent sheaf. Q defines a foliation if there exists a non-empty Zariski open subset $X^* \subset X_{\text{reg}}$ such that $Q^*|_{X^*} \subset T_{X^*}$ is an integrable subbundle.*

Example. Let $\phi : X \to Y$ be a fibration between complex manifolds X and Y. Let $L \subset T_Y$ be an integrable subbundle of rank $\dim Y - k$. The natural morphism $T_X \to \phi^* T_Y$ induces a generically surjective map $T_X \to \phi^*(T_Y/L)$ and we denote by Q the image. Then Q defines a foliation of rank $\dim X - k$ on X and we say that it is obtained as the preimage of the foliation L. One can see this in the following way: since integrability is a generic property, we may suppose that ϕ is smooth. Then $T_X \to \phi^* Q$ is a quotient bundle and we have to show that the $V := \ker(T_X \to \phi^* Q)$ is integrable. Let $x \in X$ be an arbitrary point and let $f_y : U_y \to \mathbb{D}^k$ be a distinguished map for the foliation L in the point $y = \phi(x)$. Then $f_y \circ \phi|_{\phi^{-1}(U_y)} : \phi^{-1}(U_y) \to \mathbb{D}^k$ is a distinguished map for V.

As the name says, the idea of the preimage of the foliation L is to take the preimages of each leaf and make this into a foliation. This is exactly what happens if ϕ is smooth. If this is not the case, we must be more careful, in particular higher-dimensional fibres might not be contained in the leaves of V.

Given a smooth fibration $\phi : X \to Y$ between complex manifolds and an integrable subbundle $V \subset T_X$, one might ask if V is a preimage of a foliation on Y. A necessary condition is certainly that for all fibres $T_F \subset V|_F$, but this condition is far from being sufficient.

2.3.3 Lemma. *Let $\phi : X \to Y$ be a smooth fibration between complex manifolds. Let $V \subset T_X$ be a subbundle such that for all fibres $T_F \subset V|_F$. Let $W \subset \phi^* T_Y$ be the image of the canonical map $V \to T_X \to \phi^* T_Y$ and suppose that $W|_F$ is trivial for all fibres. Then there exists a subbundle $L \subset T_Y$ such that $V = \ker(T_X \to \phi^*(T_Y/L))$. Furthermore L is integrable if and only if V is integrable*

Proof. Since ϕ is smooth and $T_F \subset V|_F$ for all fibres, the map $V \to T_X \to \phi^* T_Y$ has constant rank, so W is a vector bundle. Since W is trivial on all the fibres and ϕ is proper, the inclusion $W \subset \phi^* T_Y$ pushes down to an inclusion $L := \phi_* W \subset T_Y$ where L is a vector bundle of rank $\operatorname{rk} W$. By construction $V = \ker(T_X \to \phi^*(T_Y/L))$. We have seen before that V is integrable if L is integrable, the other implication can be seen as follows: for $y \in Y$, take an $x \in \phi^{-1}(y)$. Let W_x be an open coordinate neighborhood of x that admits a distinguished map $f : W_x \to \mathbb{D}^k$ for V and such that $\phi|_{W_x} : W_x \to \phi(W_x)$ can be written as

$$(z_1, \ldots, z_{\dim X}) \to (z_1, \to z_{\dim Y})$$

Take a section s of $\phi|_{W_x}$, then $f \circ s : \phi(W_x) \to \mathbb{D}^k$ defines a distinguished map of L in a neighborhood of y. □

Remark. We will see in corollary 3.2.5 an application of this apparently technical lemma.

We have said before that the leaves of a foliation are in general not closed submanifolds. The stability problems for foliations asks two things: given a foliation with one compact leaf, is the general leaf (or even all leaves) compact ? Given a foliation such that all leaves are compact, are the leaves the level sets of a proper map ? In general, that is on complex manifolds and without any extra hypothesis, the answer to these questions is negative, but the works of Reeb and Holmann give positive answers in a lot of interesting situations.

The natural setting for their statements makes use of the holonomy group of a leaf. Since we will not use holonomy groups, we give only an informal description and refer to [CLN85, IV., §1] for a detailed introduction: let X be a complex manifold and let $V \subset T_X$ be an integrable subbundle of rank $\dim X - k$. Let F be a compact leaf and let $x \in F$ be a point. Let furthermore $\mathbb{D}^k$ be a small disc that is transverse to F and intersects the leaf only in the point x. Let $G(\mathbb{D}^k, x)$ be the group of germs of local homeomorphisms of $\mathbb{D}^k$ keeping fixed the point x. Then one can define a map

$$\Phi : \pi_1(F, x) \to G(\mathbb{D}^k, x)$$

by „transporting a point of $\mathbb{D}^k$ along the plaques of the foliation over a given path“. We denote by $\mathrm{Hol}(F, x)$ the image of this morphism and call it the holonomy group of F at x. If $y \in F$ is a second point, the groups $\mathrm{Hol}(F, x)$ and $\mathrm{Hol}(F, y)$ are isomorphic, so we can speak of the holonomy group of the leaf F.

2.3.4 Theorem. *[CLN85, V.,§4,Thm.3] (Reeb's local stability theorem) Let V be a foliation on a complex manifold, and let F be a compact leaf with finite holonomy group. Then there exists a V-saturated neighbourhood U of F such that the leaves contained in U are compact with finite holonomy groups.*

We will be most interested in the case where the compact leaf has a finite fundamental group. In this case we have a more precise statement.

2.3.5 Corollary. *[CLN85, V.,§4,Cor.] Let V be a foliation on a complex manifold, and let F be a compact leaf with finite fundamental group $\pi_1(F)$. Then there exists a V-saturated neighbourhood U of F such that the leaves contained in U are compact with finite fundamental groups.*

While the local stability theorem holds on arbitrary manifolds, it is not true in general that if a foliation has a compact leaf with finite fundamental group, then all the leaves are compact with finite fundamental group. The global stability theorem states that this holds on compact Kähler manifolds.

2.3.6 Theorem. *[Hol80],[Per01] (Global stability theorem) Let V be a holomorphic foliation on a compact Kähler manifold. If V has a compact leaf with finite holonomy group then every leaf of V is compact with finite holonomy group.*

2.3.7 Definition. *A morphism $X \to Y$ is almost smooth if it is equidimensional and the set-theoretical fibres are smooth.*

2.3.8 Proposition. *Let X be a compact Kähler manifold, and let $V \subset T_X$ be an integrable subbundle. Assume that the general V-leaf is compact. Then every V-leaf of the foliation is compact and there exists an almost smooth map $\phi : X \to Y := X/V$ such that the set-theoretical fibres are V-leaves.*

Proof. The general leaf is compact, so there exists a leaf with finite holonomy group [Hol80]. The compactness of the leaves follows from the global stability theorem 2.3.6. Holmann [Hol80] has shown that in this case the leaf space $Y := X/V$ admits the structure of an analytic space such that the projection is almost smooth. □

2.3.9 Corollary. *Let X be a compact Kähler manifold and $V \subset T_X$ a subbundle. Suppose that there exists a fibration $\phi' : X \to Y'$ such that a general fibre F satisfies $T_F = V|_F$. Then V is integrable with compact leaves. Then the almost smooth map $\phi : X \to Y$ from proposition 2.3.8 is a factorisation of ϕ, i.e. there exists a birational morphism $g : Y \to Y'$ such that $g \circ \phi = \phi'$. In particular if ϕ is equidimensional, then it is almost smooth.* □

Proof. The only non-trivial statement is the existence of the morphism $g : Y \to Y'$: by construction ϕ' contracts the general fibre of ϕ. Since all the ϕ-fibres are multiples of the same homology class, all the ϕ-fibres are contracted by ϕ'. The existence of g follows from the rigidity lemma [Deb01, Lemma 1.15] (the proof works in the analytic category). □

2.4 Around the Ehresmann theorem

The classical Ehresmann theorem gives a sufficient condition for a manifold to have a universal covering that is a product. We state this theorem in a slightly more general version than usual and add some rather technical corollaries that we will need later.

2.4.1 Theorem. *[CLN85, V.,§2,Prop.1 and Thm.3] (Ehresmann theorem) Let $\phi : X \to Y$ be a submersion of complex manifolds with an integrable connection, i.e., an integrable subbundle $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$. Suppose furthermore one of the following:*

1.) ϕ is proper.

2.) the restriction of ϕ to every V-leaf is a (not necessarily finite) étale map.

Then $\phi : X \to Y$ is an analytic fibre bundle with typical fibre F . More precisely, if $\tilde{Y} \to Y$ is the universal cover, there is a representation $\rho : \pi_1(Y) \to \mathrm{Aut}(F)$ such that X is isomorphic to $(\tilde{Y} \times F)/\pi_1(Y)$. Denote by $\tilde{F} \to F$ the universal cover of F, then the map $\mu : \tilde{Y} \times \tilde{F} \to \tilde{Y} \times F \to (\tilde{Y} \times F)/\pi_1(Y) \simeq X$ is the universal cover $\tilde{X}$ of X. There exists an automorphism of $\tilde{X} \simeq \tilde{Y} \times \tilde{F}$ such that we have an identity of subbundles $\mu^* V = p_{\tilde{Y}}^* T_{\tilde{Y}}$ and $\mu^* T_{X/Y} = p_{\tilde{F}}^* T_{\tilde{F}}$.

2.4.2 Lemma. *Let X be a complex manifold that admits a proper submersion $\phi : X \to Y$ on a complex manifold Y. Suppose furthermore that ϕ admits a connection, i.e. a vector bundle $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$. Then ϕ is an analytic fibre bundle.*

Proof. In general V is not integrable, but if $C \subset Y$ is a smooth (open) curve, the restriction $\phi|_{\phi^{-1}(C)} : \phi^{-1}(C) \to C$ is a smooth map over a curve and $V \cap T_{\phi^{-1}(C)}$ is a rank 1 bundle that provides a connection (cf. the proof of lemma 4.2.3 for details). Since the connection has rank 1 it is integrable, so $\phi|_{\phi^{-1}(C)}$ is an analytic bundle. In particular its fibres are isomorphic. Since we can connect any two points in Y by a chain of smooth curves, this shows that all fibres are isomorphic complex manifolds. By the Grauert-Fischer theorem [FG65] this shows that ϕ is a fibre bundle. □

The Ehresmann theorem implies a corollary of proposition 2.3.8

2.4.3 Corollary. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Suppose that V_1 is integrable with general leaf compact. Then there exists an almost smooth map $X \to Y := X/V_1$ such that the set-theoretical fibres are V_1-leaves. The V_1-leaves have the same universal covering.*

Proof. Proposition 2.3.8 yields an almost smooth map $\phi : X \to Y$ on the leaf space $Y := X/V$.

Let $y \in Y$ be an arbitrary point. By [Mol88, Prop.3.7.] there exists a neighbourhood U of y which is isomorphic to $\mathbb{D}^k/G$ where $\mathbb{D}^k$ is the $k := \dim Y$-dimensional unit disc and G is the holonomy group of the leaf $\phi^{-1}(y)_{\mathrm{red}}$. Denote now by $q : \mathbb{D}^k \to U$ the quotient map and by X_U the normalisation of $\phi^{-1}(U) \times_U \mathbb{D}^k$. Let $\phi' : X_U \to \mathbb{D}^k$ be the induced map and $q' : X_U \to \phi^{-1}(U)$ the induced étale covering. Then ϕ' is a submersion and the ϕ'-fibres are the $q'^* V_1$-leaves. Since q' is étale, we have

$$T_{X_U} = q'^* T_X = q'^* V_1 \oplus q'^* V_2,$$

so $q'^* V_2$ is a connection on the submersion ϕ'. By lemma 2.4.2 all the ϕ'-fibres are isomorphic, so all the V_1-leaves that are set-theoretical fibres of $\phi^{-1}(U) \to U$ have the same universal covering. We conclude via connectedness of Y. □

The next lemma is a technical generalisation of the Ehresmann theorem, its usefulness will become apparent later.

2.4.4 Lemma. *Let $\phi : X \to Y_1 \times Y_2$ be a proper surjective map from a complex manifold X on a product of (not necessarily compact) complex manifolds such that the morphism $q := p_{Y_2} \circ \phi : X \to Y_2$ is a submersion that admits an integrable connection $V \subset T_X$. Suppose that for every V-leaf $\mathfrak{V}$, there exists a $y_1 \in Y_1$ such that $\phi(\mathfrak{V}) = y_1 \times Y_2$. Then the restriction of q to every V-leaf is an étale covering.*

Idea of the proof. It is sufficient to show that the restriction of ϕ to every V-leaf $\mathfrak{V}$ is an étale covering. If we consider the map $\phi|_{\phi^{-1}(\phi(\mathfrak{V}))} : \phi^{-1}(\phi(\mathfrak{V})) \to \phi(\mathfrak{V})$, it is a proper submersion with a smooth base and integrable connection $V|_{\phi^{-1}(\phi(\mathfrak{V}))}$, but the total space might be singular. Therefore we have to rephrase the proof of [CLN85, V,§2,Prop.1] for this situation.

Proof. In this proof all fibres and intersections are set-theoretical.

Let $\mathfrak{V}$ be a V-leaf, and let $y_1 \in Y_1$ such that $\phi(\mathfrak{V}) = y_1 \times Y_2$. Since $p_{Y_2}|_{y_1 \times Y_2} : y_1 \times Y_2 \to Y_2$ is an isomorphism, it is sufficient to show that $\phi|_{\mathfrak{V}} : \mathfrak{V} \to y_1 \times Y_2$ is an étale map. Furthermore it is sufficient to show that for $y_1 \times y_2 \in y_1 \times Y_2$, there exists a disc $D \subset y_1 \times Y_2$ such that for $y \in D$, the fibre $\phi^{-1}(y)$ cuts each leaf of the restricted foliation $V|_{\phi^{-1}(D)}$ exactly in one point. Granting this for the moment, we show how this implies the result. The connected components of $\mathfrak{V} \cap \phi^{-1}(D)$ are leaves of $V|_{\phi^{-1}(D)}$ Let $\mathfrak{V}'$ be such a connected component. Since for $y \in D$, the intersection $\mathfrak{V}' \cap \phi^{-1}(y)$ is exactly one point, the restricted morphism $\phi|_{\mathfrak{V}'} : \mathfrak{V}' \to D$ is one-to-one and onto, so it is a biholomorphism. This shows that $\phi|_{\mathfrak{V} \cap \phi^{-1}(D)} : \mathfrak{V} \cap \phi^{-1}(D) \to D$ is a trivialisation of $\phi|_{\mathfrak{V}}$.

Let us now show the claim. Set $k := \mathrm{rk} V$ and $n := \dim X$, and set $Z := \phi^{-1}(y_1 \times Y_2)$. Since every V-leaf is sent on some $b \times Y_2$, the complex space Z is V-saturated. In particular if $\mathfrak{V} \subset Z$ is leaf, the restriction of a distinguished map $f_i : W_i \to \mathbb{D}^{n-k}$ to Z which we denote by $f_i|_{W_i \cap Z} : W_i \cap Z \to \mathbb{D}^{n-k}$, is a distinguished map for the foliation $V|_Z$ and a plaque of f_i is contained in $\mathfrak{V}$ if and only if it is a plaque of $f_i|_{W_i \cap Z}$.

Step 1. The local situation. Let $x \in \phi^{-1}(y_1 \times y_2)$ be a point. Since q is a submersion with integrable connection V there exists coordinate neighbourhood $x \in W'_x \subset X$ with local coordinates $z_1, \ldots, z_k, z_{k+1}, \ldots, z_n$ and a coordinate neighbourhood $y_2 \in U_x \subset Y_2$ with coordinate $w_1, \ldots, w_k$ such that $q(W'_x) = U_x$ and $q|_{W'_x} : W_x \to U_x$ is given in these coordinates by

$$(z_1, \ldots, z_n) \to (z_1, \ldots, z_k).$$

Furthermore there exists a distinguished map $f_x : W'_x \to \mathbb{D}^{n-k}$ given in these coordinates by

$$(z_1, \ldots, z_n) \to (z_{k+1}, \ldots, z_n).$$

Since $x \in \phi^{-1}(y_1 \times y_2)$ and ϕ is equidimensional over a smooth base, so open, $\phi(W'_x)$ is a neighbourhood of $y_1 \times y_2$ in $Y_1 \times Y_2$. Since $p_{Y_2}|_{y_1 \times U_x} : y_1 \times U_x \to U_x$ is an isomorphism we can suppose that up to restricting U_x and W'_x a bit that

$$\phi(W'_x) \cap (y_1 \times Y_2) = y_1 \times U_x.$$

Set $W_x := W'_x \cap Z$, then $\phi|_Z(W_x) = y_1 \times U_x$. It then follows from this local description that $\phi|_{W_x} : W_x \to y_1 \times U_x$ has the property that for $y \in y_1 \times U_x$ the fibre $\phi^{-1}(y)$ intersects each plaque of the distinguished map $f_x|_{W_x} : W_x \to \mathbb{D}^k$ in exactly one point.

Step 2. Using the properness. Since the fibre $\phi^{-1}(y_1 \times y_2)$ is compact, we can take a finite cover of the fibre by $W_i := W_{x_i}$ where $i = 1, \ldots, l$ and W_{x_i} is as in step 1. For each $i \in \{1, \ldots, l\}$, the image $\phi(W_i)$ is a neighbourhood of $y_1 \times y_2 \in y_1 \times Y_2$. Let $D \subset \cap_{i=1}^{l} \phi(W_i)$ be a disc that contains $y_1 \times y_2$. If $\mathfrak{V}'$ is a leaf of $V|_{\phi^{-1}(D)}$, it is contained in some plaque P of W_i for some $i \in \{1, \ldots, l\}$. Since the plaques intersect each fibre at most in one point, $\phi|_{\mathfrak{V}'} : \mathfrak{V}' \to D$ is injective. The equality $P \cap \phi^{-1}(D) = \mathfrak{V}'$ then implies that

$$\phi(\mathfrak{V}') = \phi(P \cap \phi^{-1}(D)) = \phi(P) \cap D = D,$$

so $\phi|_{\mathfrak{V}'} : \mathfrak{V}' \to D$ is surjective. So $\mathfrak{V}'$ intersects each fibre exactly in one point.
$\square$

Chapter 3

Ungeneric position

In this chapter we develop the concept of ungeneric position. Since this concept is at the heart of our approach to the problem, we will develop it very systematically and in a more general context than we will actually need. This large exposition allows us to show without too much effort the following proposition which will be a cornerstone of chapter 4.

3.0.5 Proposition. *(Proof on page 49) Let X be a Kähler manifold such that $T_X = V_1 \oplus V_2$. Let $\phi : X \to Y$ be a fibration such that the general fibre is rationally connected. Then ϕ satisfies the ungeneric position property, that is the general fibre F satisfies*

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

3.1 Definition and elementary properties

We introduce the notion of a split vector bundle and a subsheaf of a split vector bundle that is in ungeneric position (definition 3.1.5). We show some useful elementary properties which are mere generalizations of well-known properties of direct sums of vector spaces. We finish the paragraph with a sufficient condition for ungeneric position (lemma 3.1.13) that does not have an equivalent in the category of vector spaces. This observation will be the cornerstone of the geometric applications in the following chapters.

3.1.1 Definition. *Let X be a complex variety, and let V be a vector bundle over X. Then V is said to be splitting if there exists an exact sequence of vector bundles*

$$0 \to V_1 \xrightarrow{i_1} V \xrightarrow{q_2} V_2 \to 0,$$

and a morphism $i_2 : V_2 \to V$ such that $id_{V_2} = q_2 \circ i_2$ and $0 < \mathrm{rk} V_2 < \mathrm{rk} V$. We then say that the exact sequence splits and write $V \simeq V_1 \oplus V_2$.

A split vector bundle $V = V_1 \oplus V_2$ is a collection of vector bundles V, V_1, V_2 and morphisms $i_j : V_j \to V$ that fit in a split exact sequence as above.

In general a splitting vector bundle does not determine a split vector bundle, the next lemma measures the difference between the two objects.

3.1.2 Lemma. *Let X be a compact complex variety and V, V_1, V_2 vector bundles on X such that $V \simeq V_1 \oplus V_2$. Then every exact sequence*

$$0 \to V_1 \xrightarrow{i_1} V \xrightarrow{q_2} V_2 \to 0$$

splits, independently of i_1. Then the set of splitting maps $i_2 : V_2 \to V$ is parametrized by $H^0(X, \mathcal{H}om(V_2, V_1))$. In particular the splitting map is unique if and only if $H^0(X, \mathcal{H}om(V_2, V_1)) = 0$.

Proof. The exact sequence of vector bundles induces a long exact sequence

$$0 \to \mathrm{Hom}(V_2, V_1) \to \mathrm{Hom}(V_2, V) \xrightarrow{\phi} \mathrm{Hom}(V_2, V_2) \to \mathrm{Ext}^1(V_2, V_1) \to \dots.$$

By definition the set of splitting maps is the affine space $\phi^{-1}(id_{V_2})$. If it is non-empty it is isomorphic to $\ker \phi = \mathrm{Hom}(V_2, V_1)$, so we are left to show the surjectivity of ϕ: since $V \simeq V_1 \oplus V_2$ and X is compact, we have

$$h^0(X, \mathcal{H}om(V_2, V_1)) + h^0(X, \mathcal{H}om(V_2, V_2)) = h^0(X, \mathcal{H}om(V_2, V)),$$

so $\dim(\mathrm{im}\phi) = \dim \mathrm{Hom}(V_2, V_2)$. $\square$

3.1.3 Example. Let $X = Y \times Z$ be a product of compact Kähler manifolds such that $b_1(X) = 0$. Then the splitting $T_X \simeq p_Y^* T_Y \oplus p_Z^* T_Z$ is unique, i.e. there exists exactly one couple of morphisms $i_1 : p_Y^* T_Y \to T_X$ and $i_2 : p_Z^* T_Z \to T_X$ that fit in a split exact sequence. Indeed $b_1(X) = b_1(Y) + b_1(Z) = 0$ implies by the Hodge decomposition that $h^{0,1}(Y) = h^{0,1}(Z) = 0$. Let $\alpha \in H^0(X, p_Y^* \Omega_Y \otimes p_Z^* T_Z)$, then for all $z \in Z$, we have

$$\alpha|_{Y \times z} \in H^0(Y \times z, (p_Y^* \Omega_Y \otimes p_Z^* T_Z)|_{Y \times z}) \simeq H^0(Y, p_Y^* \Omega_Y \otimes \mathcal{O}_Y^{\oplus \dim Z}) = 0.$$

Hence $\alpha = 0$, so $H^0(X, p_Y^* \Omega_Y \otimes p_Z^* T_Z) = 0$. By a similar argument we have $H^0(X, p_Y^* T_Y \otimes p_Z^* \Omega_Z) = 0$ and we conclude with lemma 3.1.2.

3.1.4 Example. Let $X = E_1 \times E_2$ be a product of elliptic curves, then $T_X \simeq \mathcal{O}_X^{\oplus 2}$. We define $V_1 = T_{X/E_1}$ and $V_2 = T_{X/E_2}$, then clearly $T_X = V_1 \oplus V_2$. Since $h^0(X, \mathcal{H}om(V_1, V_2)) = h^0(X, \mathcal{O}_X) = 1$, this splitting is not unique. Keeping the embedding $V_1 \to T_X$ fixed, if we choose a generic embedding $V_2 \to T_X$ with image V_2', we still have $T_X = V_1 \oplus V_2'$, but the geometry of the splitting has changed significantly: while the V_2-leaves are the fibres of the projection on E_2, the V_2'-leaves are entire non-compact curves on X.

The two examples show that if we want to deduce *geometric* properties from the existence of a splitting tangent bundle, then we should better fix the embeddings and consider the split tangent bundle instead. Doing this it might happen that we choose a „bad embedding“ like the splitting $T_X = V_1 \oplus V_2'$ in the second example, while in the case of the first example we are sure to choose the right embedding, i.e. the one corresponding to the product structure of the variety. This problem should be kept in mind and getting along with it is the main motivation for our approach, although this will not be apparent immediately.

Given a split vector bundle $V = V_1 \oplus V_2$ and a subsheaf of V, we would like to relate their properties. In a geometric situation these are typically positivity properties or information on the compactness or the structure of leaves of some foliation. The following definition which is the central notion of our approach is quite restrictive and yields many geometric consequences. It is thus even more surprising that as we will see in chapter 3.2, it holds in almost every geometric situation where this is reasonable.

3.1.5 Definition. *Let X be a complex variety, and let $V = V_1 \oplus V_2$ be a split vector bundle. Let $i : \mathcal{S} \to V$ be a generically injective morphism of sheaves and let $X^* \subset X$ be the maximal open locus where $i(\mathcal{S})$ is a subbundle of V. Then $\mathcal{S}$ is in ungeneric position (with respect to the splitting) if*

$$i(\mathcal{S})|_{X^*} = (i(\mathcal{S}) \cap V_1)|_{X^*} \oplus (i(\mathcal{S}) \cap V_2)|_{X^*}.$$

We introduce some technical notations which will be the main tool for our analysis of ungeneric position statements.

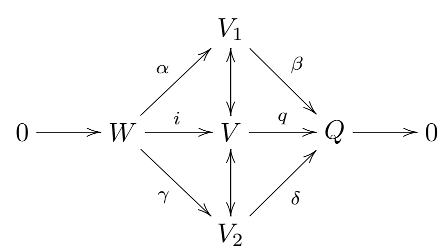
3.1.6 Notation. *Let X be a complex variety, and let $V = V_1 \oplus V_2$ be a split vector bundle. Let W be a subbundle of V, then we denote the restriction of q_1 and q_2 to W as*

$$\alpha := q_1|_W : W \to V_1 \qquad \gamma := q_2|_W : W \to V_2.$$

Let furthermore $q : V \to Q := V/W$ be the quotient map, then we define the restriction of q to the subbundles V_1 and V_2 as

$$\beta := q|_{V_1} : V_1 \to Q \qquad \delta := q|_{V_2} : V_2 \to Q.$$

We obtain the following (in almost all cases **not commutative***) diagram, which we call a basic diagram:*



The following example present the two extreme cases of ungeneric position: one direct factor is contained in the subsheaf, resp. the subsheaf is contained in one of the direct factors. These cases will be highly relevant in section 4.3 and section 3.3 respectively.

3.1.7 Example. Suppose that $W \subset V_1$, then α is an embedding and $\gamma = 0$, so we have ungeneric position.

3.1.8 Example. Suppose that $V_1 \subset W$, then γ has rank equal to $\mathrm{rk}W - \mathrm{rk}V_1$, while α has rank at most $\mathrm{rk}V_1$. Hence $\mathrm{rk}W \leq \mathrm{rk}\alpha + \mathrm{rk}\gamma \leq \mathrm{rk}W$, so we have ungeneric position.

3.1.9 Lemma. *(Characterisation of ungeneric position) Let X be a complex variety, and let $V = V_1 \oplus V_2$ be a split vector bundle. Let W be a subbundle of $V = V_1 \oplus V_2$. Then we have*

$$\mathrm{rk}\alpha + \mathrm{rk}\gamma \geq \mathrm{rk}W \tag{3.1}$$

and equality holds if and only if W is in ungeneric position.

Proof. a) We deduce from the exact sequences of sheaves

$$\begin{aligned} 0 \to W \cap V_2 &\to W &\to \mathrm{im}\alpha \to 0, \\ 0 \to W \cap V_1 &\to W &\to \mathrm{im}\gamma \to 0, \end{aligned}$$

that $\mathrm{rk}\alpha = \mathrm{rk}W - \mathrm{rk}(W \cap V_2)$ and $\mathrm{rk}\gamma = \mathrm{rk}W - \mathrm{rk}(W \cap V_1)$.

We have an injection $(W \cap V_1) \oplus (W \cap V_2) \hookrightarrow W$, so $\mathrm{rk}(W \cap V_1) + \mathrm{rk}(W \cap V_2) \leq \mathrm{rk}W$. This implies inequality 3.1, since

$$\mathrm{rk}\alpha + \mathrm{rk}\gamma = 2\mathrm{rk}W - (\mathrm{rk}(W \cap V_1) + \mathrm{rk}(W \cap V_2)) \geq \mathrm{rk}W.$$

b) Suppose now that equality holds. Since α and γ are morphisms between vector bundles, the rank in every point $x \in X$ is inferior or equal to the rank. So by assumption

$$\mathrm{rk}\alpha_x + \mathrm{rk}\gamma_x \leq \mathrm{rk}W.$$

On the other hand, we know that $W_x \subset V_x$ is a k-dimensional subspace, so $\mathrm{rk}\alpha_x + \mathrm{rk}\gamma_x \geq \mathrm{rk}W$. By the lower semicontinuity of the rank this shows that α and γ have constant rank in every point, in particular $\ker\gamma = W \cap V_1$ and $\ker\alpha = W \cap V_2$ are subbundles of W such that

$$W = (W \cap V_1) \oplus (W \cap V_2).$$

This shows the only if part of the assertion, the if part is trivial. □

3.1.10 Corollary. *Let X be a complex variety, and let $V = V_1 \oplus V_2$ be a split vector bundle. Let $i : \mathcal{S} \to V$ be a generically injective morphism of sheaves and let $X^* \subset X$ be any open dense set where $i(\mathcal{S})$ is is a subbundle of V. Then $\mathcal{S}$ is in ungeneric position if and only if*

$$i(\mathcal{S})|_{X^*} = (i(\mathcal{S}) \cap V_1)|_{X^*} \oplus (i(\mathcal{S}) \cap V_2)|_{X^*}.$$

Proof. Let X' be the maximal open locus where $i(\mathcal{S})$ is is a subbundle of V. Then $X^* \subset X'$, so the only if part is trivial. For the if part consider the morphisms $\alpha = q_1 \circ i : \mathcal{S} \to V_1$ and $\gamma = q_2 \circ i : \mathcal{S} \to V_2$. By lemma 3.1.9 applied to X^* we have $\mathrm{rk}\alpha|_{X^*} + \mathrm{rk}\gamma|_{X^*} = \mathrm{rk}\mathcal{S}$. Hence $\mathrm{rk}\alpha|_{X'} + \mathrm{rk}\gamma|_{X'} = \mathrm{rk}\mathcal{S}$, so a second application of lemma 3.1.9 yields the result. □

Remark. The corollary shows that we could have given an apparently less restrictive, but in fact equivalent definition of ungeneric position. In the following we will always verify ungeneric position on *some* non-empty open subset of X, not necessarily a maximal one.

3.1.11 Corollary. *Let X be a complex variety, and let $V = V_1 \oplus V_2$ be a split vector bundle. Let S be an irreducible complex space parametrizing a family of subvarieties $Z_s \subset X$ such that the canonical projection from the graph $\Gamma \subset S \times X$ on X is dominant. Let W be a subbundle of V. Then W is in ungeneric position if and only if $W|_{Z_s} \subset V|_{Z_s} = V_1|_{Z_s} \oplus V_2|_{Z_s}$ is in ungeneric position for every generic $s \in S$.*

Proof. We consider the morphisms $\alpha = q_1|_W : W \to V_1$ and $\gamma = q_2|_W : W \to V_2$. Then we see by lemma 3.1.9 that W is in ungeneric position if and only if $\mathrm{rk}\alpha + \mathrm{rk}\gamma = \mathrm{rk}W$. This holds if and only if $\mathrm{rk}\alpha|_{Z_s} + \mathrm{rk}\gamma|_{Z_s} = \mathrm{rk}W|_{Z_s}$ for all $s \in S$, which holds if and only if $W|_{Z_s}$ is in ungeneric position. □

3.1.12 Lemma. *(Ungeneric position and duality) Let X be a complex variety, and let $V = V_1 \oplus V_2$ be a split vector bundle. Let furthermore $V^* = V_1^* \oplus V_2^*$ be the splitting induced on the dual bundle. For W a subbundle of V, denote $q : V \to Q$ the quotient map and let $Q^* \overset{q^*}{\hookrightarrow} V^*$ be the corresponding subbundle of V^*. Then W is in ungeneric position if and only if Q^* is in ungeneric position.*

Proof. We use the notation 3.1.6. It is elementary to see that the dual map β^* (resp. δ^*) is the restriction of the projection on V_1^* (resp. V_2^*) along V_2^* (resp. V_1^*) to Q^*. So by lemma 3.1.9 we know that Q^* is in ungeneric position if and only if $\mathrm{rk}\beta^* + \mathrm{rk}\delta^* = \mathrm{rk}Q^*$.

Suppose now that W is in ungeneric position, i.e.

$$W = (V_1 \cap W) \oplus (V_2 \cap W).$$

Since $\ker \beta = (V_1 \cap W)$ and $\ker \delta = (V_2 \cap W)$, this implies

$$\mathrm{rk}\beta + \mathrm{rk}\delta = \mathrm{rk}V_1 - \mathrm{rk}(V_1 \cap W) + \mathrm{rk}V_2 - \mathrm{rk}(V_2 \cap W) = \mathrm{rk}V - \mathrm{rk}W = \mathrm{rk}Q.$$

By dualizing, we obtain $\mathrm{rk}\beta^* + \mathrm{rk}\delta^* = \mathrm{rk}Q = \mathrm{rk}Q^*$. The proof of the other implication is completely analogous. □

The following lemma gives a sufficient condition for ungeneric position.

3.1.13 Lemma. *(Ungeneric position lemma) Let X be a complex variety, and let $V = V_1 \oplus V_2$ be a split vector bundle. Let W be a subbundle of V, and let $Q := V/W$ be the quotient. If*

$$H^0(X, \mathcal{H}om(W, Q)) = 0,$$

then W is in ungeneric position.

Proof. We use the notation 3.1.6. The morphisms of sheaves $\beta \circ \alpha$ and $\delta \circ \gamma$ are elements of $H^0(X, \mathcal{H}om(W, Q))$, so they are zero. This implies

$$\mathrm{rk}\alpha \le \mathrm{rk}(\ker \beta) = \mathrm{rk}V_1 - \mathrm{rk}(\mathrm{im}\beta)$$

and

$$\mathrm{rk}\gamma \le \mathrm{rk}(\ker \delta) = \mathrm{rk}V_2 - \mathrm{rk}(\mathrm{im}\delta).$$

Applying formula 3.1 to the subbundle $Q^* \subset V_1^* \oplus V_2^*$, we have

$$\mathrm{rk}(\mathrm{im}\beta) + \mathrm{rk}(\mathrm{im}\delta) = \mathrm{rk}(\mathrm{im}\beta^*) + \mathrm{rk}(\mathrm{im}\delta^*) \ge \mathrm{rk}Q^* = \mathrm{rk}V - \mathrm{rk}W.$$

Applying formula 3.1 to the subbundle $W \subset V_1 \oplus V_2$ and using the preceeding inequalities, we obtain:

$$\begin{aligned} \mathrm{rk}W &\le \mathrm{rk}\alpha + \mathrm{rk}\gamma \\ &\le \mathrm{rk}V_1 + \mathrm{rk}V_2 - \mathrm{rk}(\mathrm{im}\beta) - \mathrm{rk}(\mathrm{im}\delta) \\ &\le \mathrm{rk}V - (\mathrm{rk}V - \mathrm{rk}W) = \mathrm{rk}W. \end{aligned}$$

We conclude with lemma 3.1.9. □

Remark. The reader will have noticed that the condition $H^0(X, \mathcal{H}om(W, Q)) = 0$ looks similar to the condition assuring the uniqueness of the splitting map in lemma 3.1.2. The same condition was used in [Bea00][4.4.] to show a more special result, it also appears in other contexts involving split exact sequences, see for example [Jah05]. Morally speaking it rigidifies the situation and excludes cases like the example 3.1.4.

3.2 Ungeneric position in a geometric context

We specify the situation to the geometric context, namely to the case where the tangent bundle T_X of a complex manifold X splits. Potential candidates for subsheaves of T_X that are in ungeneric position are relative tangent sheaves $T_{X/Y}$ induced by some morphism $\phi : X \to Y$. In this case we obtain an induced splitting of the tangent bundle of a general fibre (lemma 3.2.2), and, under appropriate conditions, of the tangent bundle of the base space (lemma 3.2.4). As a first positive result we show ungeneric position for Albanese maps (proposition 3.2.8).

For the following fundamental definition, recall that a morphism of complex varieties $\phi : X \to Y$ induces an exact sequence of sheaves

$$\phi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

If X is smooth, the dual exact sequence yields an injective morphism from the relative tangent sheaf $T_{X/Y} := \mathcal{H}om(\Omega_{X/Y}, \mathcal{O}_X)$ to T_X, so it makes sense to ask if it is in ungeneric position.

3.2.1 Definition. *Let X be a complex manifold such that $T_X = V_1 \oplus V_2$, and let $\phi : X \to Y$ be a fibration. The morphism ϕ satisfies the ungeneric position property if $T_{X/Y}$ is in ungeneric position.*

We characterize ungeneric position of a fibration via duality and in terms of the general fibres of the morphism.

3.2.2 Lemma. *Let X be a complex manifold such that $T_X = V_1 \oplus V_2$, and let $\phi : X \to Y$ be a fibration.*

1. The morphism ϕ satisfies the ungeneric position property if and only if $\phi^*\Omega_Y$ is in ungeneric position in $\Omega_X = V_1^* \oplus V_2^*$.

2. The morphism ϕ satisfies the ungeneric position property if and only if for every smooth fibre $F = \phi^{-1}(y)$ we have

$$T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F).$$

Proof. By corollary 3.1.10 it is sufficient to prove the claims for some non-empty Zariski open set $X^* := \phi^{-1}(Y^*)$ such that $\phi|_{X^*} : X^* \to Y^*$ is a smooth morphism, so we suppose without loss of generality that ϕ is smooth. Then Y is smooth and we have an exact sequence of vector bundles

$$0 \to \phi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

The first assertion now follows from lemma 3.1.12, the second from corollary 3.1.11. □

Notation. Let $\phi : X \to Y$ be a fibration between complex manifolds. The canonical map $\phi^*\Omega_Y \to \Omega_X$ induces a a generically surjective sheaf homomorphism $T\phi : T_X \to \phi^*T_Y$. In particular for $\mathcal{S} \subset T_X$ a quasicoherent subsheaf, we obtain a quasicoherent subsheaf $\phi_*(T\phi(\mathcal{S})) \subset T_Y$. For a point $y \in Y$ and a point $x \in \phi^{-1}(y)$ we denote $(T\phi)_x : T_{X,x} \to T_{Y,y}$ the tangent map between the vector spaces $T_{X,x}$ and $T_{Y,y}$.

3.2.3 Definition. *Let $\phi : X \to Y$ be a fibration between complex manifolds. Suppose that there exists a non-empty Zariski open set $Y^* \subset Y$ such that $Y \setminus Y^*$ has codimension at least 2, and such that for $y \in Y^*$, there exists a point $x \in \phi^{-1}(y)$ such that*

$$\mathrm{rk}((T\phi)_x : T_{X,x} \to T_{Y,y}) = \dim Y.$$

Then we say that ϕ has generically reduced fibres in codimension 1.

3.2.4 Lemma. *Let X be a complex manifold such that $T_X = V_1 \oplus V_2$. Let $\phi : X \to Y$ be a fibration on a complex manifold Y that satisfies the ungeneric position property and has generically reduced fibres in codimension 1 (cf. definition 3.2.3). Then for $j = 1, 2$, the reflexive sheaf $W_j := (\phi_*(T\phi(V_j)))^{**} \subset T_Y$ is a subbundle of T_Y and*

$$T_Y = W_1 \oplus W_2.$$

Proof. *Step 1. Suppose that ϕ is smooth.* Then the map $T\phi : T_X \to \phi^*T_Y$ is surjective. Its restriction to the subbundle V_j, denoted by $q_j : V_j \to \phi^*T_Y$ is a morphism of sheaves. Since $T_{X/Y} = \ker(T\phi)$, we have $\ker q_j = T_{X/Y} \cap V_j$, so the rank of q_j at a general point is $\mathrm{rk}V_j - \mathrm{rk}(T_F \cap V_j|_F)$. By hypothesis this implies $\mathrm{rk} q_1 + \mathrm{rk} q_2 = \dim Y$. Since $T\phi = q_1 \oplus q_2$ and $T\phi$ has rank equal to $\dim Y$ at every point, this implies that q_j is a morphism of vector bundles and $\mathrm{im}(q_1) \oplus \mathrm{im}(q_2) = \phi^* T_Y$. We verify that this induces a splitting of T_Y: for every fibre F, we have

$$\mathrm{im}(q_1)|_F \oplus \mathrm{im}(q_2)|_F = \phi^* T_Y|_F \simeq \mathcal{O}_F^{\oplus \dim Y},$$

so it is elementary to see that $\mathrm{im}(q_j)|_F$ is trivial. This implies $\mathrm{im}(q_j) = \phi^* E_j$ where E_j is a vector bundle on Y, so the splitting pushes down to Y.

Step 2. We show the general case. Let $W_j := (\phi_*(T\phi(V_j)))^{**} \subset T_Y$ be the saturation of $\phi_*(T\phi(V_j)) \subset T_Y$, then W_j is a reflexive sheaf, so the locus $Y' \subset Y$ where W_1 and W_2 are locally free satisfies $\mathrm{codim}_Y(Y \setminus Y') \geq 2$. Apply the first step to the ϕ-smooth locus to see that $\mathrm{rk} W_1 + \mathrm{rk} W_2 = \dim Y$.

For $y \in Y' \cap Y^*$, let $x \in \phi^{-1}(y)$ be such that the rank of $(T\phi)_x : T_{X,x} \to T_{Y,y}$ is maximal. Denote by $V_{j,x} \subset T_{X,x}$ and $W_{j,y} \subset T_{Y,y}$ the subspaces induced by the subbundles V_j and W_j. Since $(T\phi)_x(V_{j,x}) \subset W_{j,y}$, we have

$$T_{Y,y} = \mathrm{im}(T\phi)_x \subset (T\phi)_x(V_{1,x}) + (T\phi)_x(V_{2,x}) \subset W_{1,y} + W_{2,y} \subset T_{Y,y}.$$

Since $\mathrm{rk} W_1 + \mathrm{rk} W_2 = \dim Y$ this implies that $W_{1,y} \oplus W_{2,y} = T_{Y,y}$, hence $T_{Y' \cap Y^*} = W_1|_{Y' \cap Y^*} \oplus W_2|_{Y' \cap Y^*}$. Since $Y \setminus (Y^* \cap Y')$ has codimension 2 and Y is smooth, we have

$$T_Y = W_1 \oplus W_2.$$

In particular the sheaves W_j are locally free. □

3.2.5 Corollary. *In the situation of lemma 3.2.4, suppose furthermore that a general fibre F satisfies $T_F \subset V_1|_F$. Then W_1 is integrable if and only if V_1 is integrable. In particular V_1 is integrable if* $\mathrm{rk} V_1 \leq \dim F + 1$.

Proof. Integrability is a generic property (lemma 2.1.3), so we suppose without loss of generality that ϕ is smooth. Then the image of $q_1 : V_1 \to T_X \to \phi^* T_X$ is a vector bundle of rank $\mathrm{rk} V_1 - \dim F$ and we have seen in the proof of lemma 3.2.4 that $\mathrm{im}(q_1)|_F$ is trivial. The first statement now follows from lemma 2.3.3. The second statement is trivial, since then W_1 has rank at most 1, so it is integrable. □

We come to the critical technical lemma that links the ungeneric position property and conjecture 1.1.2.

3.2.6 Lemma. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Let $\phi : X \to Y$ be an equidimensional fibration on a compact Kähler manifold such that the general fibre F satisfies $T_F \subset V_1|_F$ and* $\mathrm{rk} V_1 = \dim F + 1$. *Suppose that ϕ has generically reduced fibres in codimension 1. If V_1 and V_2 are integrable, then conjecture 1.1.2 holds for X.*

Proof. Since $T_F \subset V_1|_F$, a general fibre is contained in a V_1-leaf. Since leaves are locally closed subset of X and ϕ is equidimensional, this property is closed, so all the fibres are contained in V_1-leaves. The morphism ϕ satisfies the ungeneric position property and has generically reduced fibres in codimension 1, so by lemma 3.2.4 we have a splitting

$$T_Y = (\phi_* T\phi(V_1))^{**} \oplus (\phi_* T\phi(V_2))^{**}.$$

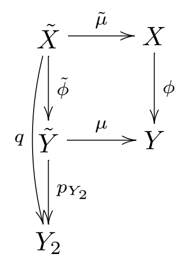
For $j = 1$ and 2, set $W_j := (\phi_* T\phi(V_j))^{**}$. Since $\mathrm{rk} W_1 = 1$, we can apply theorem 1.1.3 to Y. There are two cases.

Case 1. The variety Y is a $\mathbb{P}^1$-bundle $\psi : Y \rightarrow Z$ such that $T_{Y/Z} = W_1$. Then $\psi \circ \phi : X \rightarrow Z$ is a an almost smooth map such that the set-theoretical fibres are V_1-leaves.

Suppose that $\psi \circ \phi$ has a fibre $(\psi \circ \phi)^{-1}(z_0)$ that is not reduced. Let $x \in (\psi \circ \phi)^{-1}(z_0)$, and let D be a small disc contained in the V_2-leaf through x. We suppose without loss of generality that the disc intersects the fibre only in x. The restricted morphism $\psi \circ \phi|_D : D \rightarrow \psi \circ \phi(D)$ is then finite and ramifies in x. Since $\psi \circ \phi(D)$ is smooth, purity of branch implies that $\psi \circ \phi|_D$ is ramified in codimension 1. Since the set-theoretical fibres of $\psi \circ \phi$ are V_1-leaves, this implies that there exists a divisor $E \subset Z$ such that for $z \in E$, the homology class of the fibre $(\psi \circ \phi)^{-1}(z)$ is a multiple of the class of the V_1-leaf, so not reduced in any point. Since ψ is smooth, this implies that for every $y \in \psi^{-1}(E)$, the fibre $\phi^{-1}(y)$ is not reduced in any point. But this implies that ϕ does not have generically reduced fibres in codimension 1, a contradiction.

So $\psi \circ \phi$ is a smooth map. By hypothesis the connection V_2 is integrable and we conclude with the Ehresmann theorem 2.4.1.

Case 2. The bundle $W_2 := (\phi_*(T\phi(V_2)))^{**}$ is integrable, and the universal covering $\mu : \tilde{Y} \rightarrow Y$ satisfies $\tilde{Y} \simeq Y_1 \times Y_2$ such that $\mu^* W_1 = p_{Y_1}^* T_{Y_1}$ and $\mu^* W_2 = p_{Y_2}^* T_{Y_2}$. Furthermore we have a commutative diagram



where $\tilde{\mu} : \tilde{X} := X \times_Y \tilde{Y} \rightarrow X$ is étale. Then q is an almost smooth map such that the set-theoretical fibres are $\tilde{\mu}^* V_1$-leaves. Since $\tilde{\phi}$ is obtained as the pull-back of ϕ via an étale covering, it has generically reduced fibres in codimension 1. We use the same argument as in the first case to show that q is smooth with connection $\tilde{\mu}^* V_2$. Since

$$(\tilde{\phi}_*(T\tilde{\phi}\tilde{\mu}^* V_2))^{**} = \mu^* W_2 = p_{Y_2}^* T_{Y_2},$$

there exists for every $\tilde{\mu}^* V_2$ leaf $\mathfrak{V}_2$ a $y_1 \in Y_1$ such that $\tilde{\phi}(\mathfrak{V}_2) = y_1 \times Y_2$. By lemma 2.4.4 the restriction of q to a $\tilde{\mu}^* V_2$ leaf is an étale covering, so the Ehresmann theorem 2.4.1 applies. $\square$

Remark. The condition $\dim F+1 = \mathrm{rk}V_1$ is only necessary to apply theorem 1.1.3, so a positive answer to conjecture 1.1.2 for a splitting $T_X = V_1 \oplus V_2$ where $\mathrm{rk}V_1 = k$ opens the way for (partial) results in the case where $\mathrm{rk}V_1 = k+1$. This inductive approach to the conjecture has its limits in the fact that some of the varieties considered will not admit any fibre space structure at all. I hope that the chapters 6 and 4 will convince the reader that the strategy nevertheless works in many interesting cases.

We can now state the ungeneric position lemma 3.1.13 for fibrations. The statement is slightly technical, but is needed in this generality to be useful both for Iitaka fibrations (cf. section 6.1) and fibrations with rationally connected fibres (cf. chapter 4).

3.2.7 Lemma. *(Ungeneric position lemma, fibrewise version) Let X be a Kähler manifold such that $T_X = V_1 \oplus V_2$. Let $\phi : X \rightarrow Y$ be a fibration such that a general fibre F satisfies the following property: the compact Kähler manifold F admits a morphism $\psi : F \rightarrow Z$ such that $\psi_* \mathcal{O}_F = \mathcal{O}_Z$, and $0 \leq \dim Z < \dim F$, and $b_1(G) = 0$ for a general ψ-fibre G.*

Then $T_G \subset T_X|_G$ is in ungeneric position. In particular if $\dim Z = 0$, then the morphism ϕ satisfies the ungeneric position property.

Proof. Since F is a general fibre of ϕ, we have $N_{F/X} \simeq \mathcal{O}_F^{\dim Y}$ and analogously $N_{G/F} \simeq \mathcal{O}_G^{\dim Z}$. Furthermore we have an exact sequence of vector bundles on G

$$0 \rightarrow N_{G/F} \rightarrow N_{G/X} \rightarrow N_{F/X}|_G \rightarrow 0.$$

The manifold G is compact Kähler, so by hypothesis and Hodge duality $h^1(G, \mathcal{O}_G) = h^0(G, \Omega_G) = 0$. Hence the exact sequence splits by the h^1-criterion, so $N_{G/X} \simeq \mathcal{O}_G^{\dim Y + \dim Z}$. Now consider the tangent sequence for $G \subset X$

$$0 \rightarrow T_G \rightarrow T_X|_G \rightarrow N_{G/X} \rightarrow 0.$$

We have

$$\mathcal{H}om(T_G, N_{G/X}) \simeq \Omega_G^{\oplus \dim Y + \dim Z},$$

and as seen $h^0(G, \Omega_G) = 0$. We conclude with the ungeneric position lemma 3.1.13. □

Remark. The Kähler condition on X could be replaced by $h^1(G, \mathcal{O}_G) = h^0(G, \Omega_G)$ for a general G, but ungeneric position statements in a non-Kähler context are hardly useful: the idea of the theory is to extend properties of the general fibres to all the fibres via holomorphic foliations. Since properties like the global stability theorem 2.3.6 are false in the non-Kähler setting, this approach works only in the Kähler category. As a direct application of the ungeneric position lemma, we can now show the central result of this chapter: the ungeneric position property for fibrations with rationally connected general fibre.

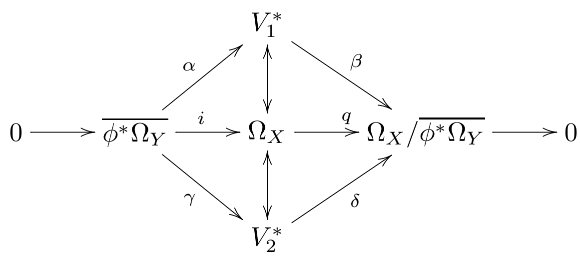
Proof of proposition 3.0.5. The general fibre F is rationally connected, so simply connected. Hence $b_1(F) = 0$ and we apply lemma 3.2.7. □

The Albanese maps are the second large class of maps for which we show the ungeneric position property.

3.2.8 Proposition. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Let $\alpha_X : X \to Alb(X)$ be the Albanese map and denote $\phi : X \to Y$ its Stein factorisation. Then ϕ satisfies the ungeneric position property.*

Idea of the proof. The global holomorphic 1-forms on X are obtained as a pull-back from Y and the image of $\phi^*\Omega_Y$ in Ω_X is the sheaf generated by the global 1-forms. If it is not in ungeneric position, the kernel of the projection maps on V_1^* and V_2^* is „small“, so V_1^* and V_2^* have „many“ global sections. Since $\Omega_Y = V_1^* \oplus V_2^*$, this yields a contradiction.

Proof. The canonical morphism $\phi^*\Omega_Y \to \Omega_X$ is generically injective and we denote $\overline{\phi^*\Omega_Y}$ its image. Since Y is finite over a torus, the sheaf $\overline{\phi^*\Omega_Y}$ is generically generated by its global sections and by definition of the Albanese map $q(X) = h^0(X, \overline{\phi^*\Omega_Y})$. We consider the basic diagram (cf. notation 3.1.6)



We argue by contradiction and suppose that we don't have ungeneric position. We have on the smooth ϕ-locus $T_{X/Y} = \Omega_X / \overline{\phi^*\Omega_Y}$, so this implies by lemma 3.1.9 that $\mathrm{rk}\beta^* + \mathrm{rk}\delta^* > \mathrm{rk}(\Omega_X / \overline{\phi^*\Omega_Y})$. By an elementary computation this is equivalent to $\mathrm{rk}(\ker\beta) + \mathrm{rk}(\ker\delta) < \mathrm{rk}(\overline{\phi^*\Omega_Y})$.

Since $\ker\beta = \overline{\phi^*\Omega_Y} \cap V_1^*$ and $\ker\delta = \overline{\phi^*\Omega_Y} \cap V_2^*$ this implies that $\mathcal{S} := (\overline{\phi^*\Omega_Y} \cap V_1^*) \oplus (\overline{\phi^*\Omega_Y} \cap V_2^*)$ is a subsheaf of $\overline{\phi^*\Omega_Y}$ such that $\mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\overline{\phi^*\Omega_Y})$. Hence $h^0(X, \mathcal{S}) < h^0(X, \overline{\phi^*\Omega_Y})$, since otherwise $\overline{\phi^*\Omega_Y}$ would not be generically generated by its global sections. Since

$$h^0(X, \overline{\phi^*\Omega_Y} \cap V_1^*) + h^0(X, \overline{\phi^*\Omega_Y} \cap V_2^*) = h^0(X, \mathcal{S}),$$

we have

$$h^0(X, \overline{\phi^*\Omega_Y} \cap V_1^*) + h^0(X, \overline{\phi^*\Omega_Y} \cap V_2^*) < q(X). \qquad (*)$$

The exact sequence

$$0 \to \overline{\phi^*\Omega_Y} \cap V_2^* \quad \to \overline{\phi^*\Omega_Y} \quad \to \mathrm{im}\alpha \to 0$$

induces a long exact sequence

$$0 \to H^0(X, \overline{\phi^*\Omega_Y} \cap V_2^*) \to H^0(X, \overline{\phi^*\Omega_Y}) \stackrel{\alpha(X)}{\to} H^0(X, \mathrm{im}\alpha) \to \ldots.$$

Since $\mathrm{im}\alpha \subset V_1^*$ a dimension count gives

$$h^0(X, V_1^*) \geq h^0(X, \mathrm{im}\alpha) \geq q(X) - h^0(X, \overline{\phi^*\Omega_Y} \cap V_2^*).$$

We obtain analogously that

$$h^0(X, V_2^*) \geq h^0(X, \mathrm{im}\gamma) \geq q(X) - h^0(X, \overline{\phi^*\Omega_Y} \cap V_1^*).$$

Using the inequality (*) it follows that

$$\begin{aligned} q(X) &= h^0(X, V_1^*) + h^0(X, V_2^*) \\ &\geq 2q(X) - h^0(X, \overline{\phi^*\Omega_Y} \cap V_1^*) - h^0(X, \overline{\phi^*\Omega_Y} \cap V_2^*) > q(X), \end{aligned}$$

a contradiction. □

For a fibration over a curve we obtain a slightly stronger result.

3.2.9 Lemma. *Let X be a compact complex manifold such that $T_X = V_1 \oplus V_2$. Let $\phi : X \to C$ be a fibration on a smooth curve such that $q(X) < 2g(C)$. Then ϕ satisfies the ungeneric position property.*

Proof. We use the notation of the basic diagram 3.1.6. We want to show that $\phi^*\Omega_C$ is in ungeneric position and argue by contradiction. Since $\phi^*\Omega_C$ has rank 1 this is equivalent by lemma 3.1.9 to supposing that the morphisms $\alpha : \phi^*\Omega_C \to V_1^*$ and $\gamma : \phi^*\Omega_C \to V_2^*$ have both rank different from zero. Since $\phi^*\Omega_C$ is torsion-free, they are injective morphisms of sheaves. In particular we have injections of global sections

$$\begin{aligned} H^0(X, \phi^*\Omega_C) &\hookrightarrow H^0(X, V_1^*) \\ H^0(X, \phi^*\Omega_C) &\hookrightarrow H^0(X, V_2^*). \end{aligned}$$

Since $q(X) = h^0(X, V_1^*) + h^0(X, V_2^*)$, this implies $q(X) \geq 2g(C)$, a contradiction. □

Remark. Example 3.1.4 shows that for $q(X) = 2g(C)$ we can't expect ϕ to satisfy the ungeneric position property. Note nevertheless that for reasons that will become apparent in section 6.3 this example does not generalise to a product of two curves of higher genus.

3.3 An example

In this chapter we wish to illustrate how to use the ungeneric position property to give detailed information about the manifold X. We will obtain informations both on the structure of the fibration $\phi : X \to C$ (theorem 3.3.3), the sheaves V_1, V_2 (ibid.) and $T_{X/C}$ (lemma 3.3.2). The goal is to give a generalisation of the following (trivial) statement.

3.3.1 Proposition. *Let S be a complex surface, and let Y be a complex manifold of positive dimension. Suppose furthermore that S admits a fibration $f : S \to C$ on a smooth curve C. Let $\phi := f \circ p_S : X := S \times Y \to C$ be the induced fibration and let $F = \phi^{-1}(t)$ be a geometric fibre. Denote by F' the normalisation of F, then F' is a disjoint union $C_1 \times Y \sqcup \ldots \sqcup C_k \times Y$, where $C_1, \ldots, C_k$ are smooth curves. In particular the normalisation of an irreducible component of a set-theoretical fibre is smooth and has a natural product structure.*

Proof. By construction of ϕ, we have $F_0 = f^{-1}(t) \times Y$. Since $f^{-1}(t)$ is a finite union of irreducible curves, the claim follows. □

Remark. In the statement of the proposition we used the slightly abusive notion of normalisation for a not necessarily irreducible variety. For a reducible reduced variety, the normalisation is the disjoint union of the normalisations of the irreducible components.

We will now replace the global product structure of X by a splitting of the tangent bundle $T_X = V_1 \oplus V_2$, the construction of ϕ via a projection map will be replaced by the ungeneric position property $V_1|_F \subset T_F$ for F a general fibre (cf. example 3.1.8).

3.3.2 Lemma. *Let X be a complex manifold such that $T_X = V_1 \oplus V_2$, where* $\mathrm{rk} V_2 = 2$. *Let $\phi : X \to C$ be a fibration on a smooth curve such that the general fibre satisfies $V_1|_F \subset T_F$. Then $T_{X/C}$ is locally free and splits as*

$$T_{X/C} = V_1 \oplus L,$$

where L is a line bundle.

Remark. A fibration is a smooth map if and only if $\Omega_{X/C}$ is locally free, but $T_{X/C}$ is locally free does not imply that ϕ is smooth. Indeed this is false in general, just consider the fibration from proposition 3.3.1. Since $T_{X/C}$ is obtained by dualizing the non-locally free sheaf $\Omega_{X/C}$, some information is lost.

Proof. By lemma 3.2.2 and duality the condition $V_1|_F \subset T_F$ is equivalent to supposing that $\phi^*\Omega_C \hookrightarrow \Omega_X$ has its image in V_2^*. Consider the exact sequence

$$0 \to \phi^*\Omega_C \to \Omega_X = V_1^* \oplus V_2^* \to \Omega_{X/C} \to 0.$$

Since $\phi^*\Omega_C \hookrightarrow V_2^*$, we have $\Omega_{X/C} = V_1^* \oplus (V_2^*/\phi^*\Omega_C)$, so it is sufficient to show that $L := \mathcal{H}om((V_2^*/\phi^*\Omega_C), \mathcal{O}_X)$ is a locally free sheaf of rank 1. The sheaf $V_2^*/\phi^*\Omega_C$ has rank 1, so its dual L is a reflexive sheaf of rank 1. Since X is smooth, the sheaf L is locally free. □

3.3.3 Theorem. *Let X be a complex manifold such that $T_X = V_1 \oplus V_2$, where V_1 and V_2 are integrable subbundles and* $\mathrm{rk} V_2 = 2$. *Let $\phi : X \to C$ be a fibration on a smooth curve C such that the general fibre satisfies $V_1|_F \subset T_F$. Then a V_1-leaf is contained in a unique fibre.*

Let $F = \phi^{-1}(c)$ be a set-theoretical fibre and denote by $\nu : F' \to F$ the normalisation of F. Then F' is a disjoint union $F_1 \sqcup \ldots \sqcup F_k$ of manifolds such that for $j = 1, \ldots, k$

$$T_{F_j} = (\nu^* V_1)_{F_j} \oplus L_j$$

where L_j is a line bundle. In particular the normalisation of an irreducible component of a fibre is smooth and has a split tangent bundle.

If furthermore X is a Kähler manifold, then the V_1-leaves contained in the same ϕ-fibre have the same universal covering.

Proof. The proof has several steps, for every step we will first announce what we will show and then prove the claim.

Step 1. Every V_1-leaf is contained in a unique fibre. By hypothesis a general fibre F satisfies $V_1|_F \subset T_F$. This shows that a V_1-leaf that meets a general fibre is contained in it. Since the special fibres are isolated, this remains true for the special fibres.

For the rest of the proof, let $F := \phi^{-1}(c)_{\text{red}}$ be a geometric fibre and denote by $\nu : F' \to F$ the normalisation of F.

Step 2. The normalisation F' is smooth. Fix a point $x \in F$. Then there exists a neighbourhood $U \subset X$ of x and a submersion $p : U \to \mathbb{D}^2$ such that $V_1|_U = T_{U/\mathbb{D}^2}$ and $U \simeq \mathbb{D}^{\text{rk}V_1} \times \mathbb{D}^2$. We define a section $s : \mathbb{D}^2 \to \mathbb{D}^{\text{rk}V_1} \times \mathbb{D}^2$ of p by $w \mapsto (w, 0)$. Since the V_1-leaves are contained in the ϕ-fibres, the map $f := \phi \circ s : \mathbb{D}^2 \to \phi(U)$ gives a factorisation $\phi = f \circ p$. This implies that

$$Y := (p(F \cap U))_{\text{red}} = (f^{-1}(c))_{\text{red}}$$

is an analytic subset of dimension 1 and

$$F \cap U = (\phi^{-1}(c))_{\text{red}} \cap U = p^{-1}((f^{-1}(c))_{\text{red}}) \cap U \simeq \mathbb{D}^{\text{rk}V_1} \times Y.$$

This description of the local structure of F shows two things.

a) The normalisation of $F \cap U$ is isomorphic to the product $\mathbb{D}^{\text{rk}V} \times Y'$, where Y' is the normalisation of the curve Y. Since Y' is smooth and smoothness is a local property this concludes the proof.

b) The singular locus of F is a union of V_1-leaves. Indeed we have $x \in F_{\text{sing}}$ if and only $p(x) \in Y_{sing}$ if and only if $p^{-1}(p(x)) \subset F_{\text{sing}}$ and the fibres of p are contained in V_1-leaves.

Step 3. Every irreducible component of F' has a split tangent bundle $(\nu^* V_1)_{F_j} \oplus L_j$. In order to simplify the notation we suppose without loss of generality that F is irreducible and show that F' has a split tangent bundle. Fix a point $x \in F$. Then there exists a neighbourhood $U \subset X$ of x that admits submersions $p : U \to \mathbb{D}^2$ and $q : U \to \mathbb{D}^{\text{rk}V_1}$ satisfying $V_1|_U = T_{U/\mathbb{D}^2}$ and $V_2|_U = T_{U/\mathbb{D}^2}$. Up to restricting a bit further this induces an isomorphism $U \simeq \mathbb{D}^{\text{rk}V_1} \times \mathbb{D}^2$. We have seen in step 2 that under this isomorphism $F \cap U \simeq \mathbb{D}^{\text{rk}V_1} \times Y$ where Y is an analytic curve. So by construction the restrictions $p|_{F\cap U} : F \cap U \to Y$ and $q|_{F\cap U} : F \cap U \to \mathbb{D}^{\text{rk}V_1}$ identify to the projections $p_Y : \mathbb{D}^{\text{rk}V_1} \times Y \to Y$ and $p_{\mathbb{D}^{\text{rk}V_1}} : \mathbb{D}^{\text{rk}V_1} \times Y \to \mathbb{D}^{\text{rk}V_1}$. It follows that these maps induce projections $p_{Y'} : \mathbb{D}^{\text{rk}V_1} \times Y' \to Y'$ and $p_{\mathbb{D}^{\text{rk}V_1}} : \mathbb{D}^{\text{rk}V_1} \times Y' \to \mathbb{D}^{\text{rk}V_1}$ where Y' is the normalisation of Y. Since the normalisation of $F \cap U$ is isomorphic to $\mathbb{D}^{\text{rk}V_1} \times Y'$, this induces a splitting of $T_{\nu^{-1}(F\cap U)}$. Furthermore $\nu|_{F\cap U}$ identifies under this isomorphism to $\text{id}_{\mathbb{D}^{\text{rk}V_1}} \times \mu$, where $\mu : Y' \to Y$ is the normalisation of Y. This gives a natural embedding of $\nu|_{F\cap U}^* V_1$ as a subbundle of $T_{\nu^{-1}(F\cap U)}$ and the image identifies under the isomorphism $\nu^{-1}(F \cap U) \simeq \mathbb{D}^{\text{rk}V_1} \times Y'$ to $T_{(\mathbb{D}^{\text{rk}V_1} \times Y')/Y'}$. These local maps can be glued since they are induced by the projection maps of the foliations V_1 and V_2 which glue together.

Step 4. If X is Kähler, then the V_1-leaves contained in a fibre have the same universal covering. Let F_j be a connected component of the normalisation F'.

By the preceeding steps F_j is smooth and has a splitting tangent bundle

$$T_{F_j} = \nu^* V_1|_{F_j} \oplus L_i,$$

the second direct factor being a line bundle. Since F_i is compact Kähler and $\nu^* V_1|_{F_j}$ integrable (the morphism ν is generically an isomorphism), we know by [BPT04][Thm.0.1] that the $(\nu^* V_1)|_{F_j}$-leaves have the same universal covering. By b) of step 2 the singular locus of $\nu(F_j)$ is a union of V_1-leaves. It follows that every V_1-leaf $\mathcal{V}$ is isomorphic to a $\nu^* V_1$-leaf $\mathcal{V}'$, in particular they have the same universal covering. This shows that all the V_1-leaves in $\nu(F_j)$ have the same universal covering. If an irreducible component $\nu(F_j)$ intersects another irreducible component $\nu(F_i)$, the intersection is contained in the singular locus of the fibre F. Since F_{sing} is a union of V_1-leaves, this shows that there exists a leaf that is contained in $\nu(F_j)$ and $\nu(F_i)$. By connectedness of the fibre all the V_1-leaves have the same universal covering. $\square$

Chapter 4

Uniruled manifolds

This chapter contains our most important structure results about manifolds with split tangent bundle. We have seen that a map with rationally connected fibres always satisfies the ungeneric position property (cf. proposition 3.0.5) and we will show now that elementary Mori fibre spaces even satisfy a stronger ungeneric position property (cf. lemma 4.3.3).

Our first goal is to understand rationally connected manifolds with split tangent bundle and we will show conjecture 1.1.2 for this class of manifolds modulo an integrability condition (theorem 4.2.4). In section 4.3 we discuss elementary Mori contractions of fibre type and show various structure results. This encourages us to tackle the general case in section 4.4 via the rational quotient. The global strategy is to reduce the proof of conjecture 1.1.2 to proving it for non-uniruled manifolds by contracting the covering families of rational curves. We are still far away from this situation, but theorem 4.4.11 is certainly an important step in this direction.

4.1 Ungeneric position revisited

4.1.1 Definition. *A compact Kähler manifold X is uniruled if there exists a covering family of rational curves. It is rationally connected if for two general points there exists a rational curve through these two points.*

Remark. A rationally connected manifold is by definition algebraically connected, so it is Moishezon [Cam81]. Smooth Moishezon compact Kähler manifolds are projective, so rationally connected manifolds are projective. Fano manifolds are rationally connected ([Cam92, KMM92]).

Rationally connected manifolds have several properties that are interesting for us (cf. [Deb01, ch. 4] for an introduction): they are simply connected, so Reeb's local stability theorem 2.3.5 applies to a rationally connected compact leaf. If $X \to Y$ is an étale cover such that X is rationally connected, Y is rationally connected and the étale cover is an isomorphism. Furthermore if $\phi : X \to Y$ is a smooth projective morphism between complex manifolds, one fibre is rationally connected if and only if all the fibres are rationally connected (deformation invariance of rational connectedness ([Kol96, Thm.3.11]).

Proposition 2.3.8 says that if the general leaf of a foliation is compact, there exists an almost smooth map (cf. definition 2.3.7) such that the set-theoretical fibres are leaves. The next corollary shows that one rationally connected compact leaf yields the existence of a smooth map such that the scheme-theoretical fibres are leaves.

4.1.2 Corollary. *Let X be a compact Kähler manifold, and let $V \subset T_X$ be an integrable subbundle such that one leaf is compact and rationally connected. Then there exists a submersion $X \to Y$ on a smooth variety Y such that $T_{X/Y} = V$.*

Remark. Kebekus, Solá and Toma obtained independently similar results ([KSCT05, Thm.28]) for singular foliations on projective manifolds.

Proof. A rationally connected leaf is simply connected, so by Reeb's local stability theorem (cf. [CLN85, IV, §3, Thm.3]), the general V-leaf is compact. Proposition 2.3.8 yields an almost smooth map $g : X \to Y$ on the leaf space $Y := X/V$.

Let $y \in Y$ be an arbitrary point. By [Mol88, Prop.3.7.] there exists a neighbourhood U of y which is isomorphic to $\mathbb{D}^k/G$ where $\mathbb{D}^k$ is the $k := \dim Y$-dimensional unit disc and G is a finite group. More precisely G is the image of a representation of the fundamental group $\pi_1(g^{-1}(y)_{\text{red}})$. Denote now by $q : \mathbb{D}^k \to U$ the quotient map and by X_U the normalisation of $g^{-1}(U) \times_U \mathbb{D}^k$. Let $g' : X_U \to \mathbb{D}^k$ be the induced map and $q' : X_U \to g^{-1}(U)$ the induced étale covering. Then g' is a submersion and the g'-fibres are the q'^*V-leaves.

A V-leaf is rationally connected if and only if the corresponding q'^*V-leaf is rationally connected. Furthermore one g'-fibre is rationally connected if and only if all g'-fibres are rationally connected (by deformation invariance of rational connectedness, cf. [Kol96, Thm.3.11]). Now by hypothesis there exists one rationally connected V-leaf, so by connectedness of Y all V-leaves are rationally connected. Since rationally connected varieties are simply connected, the group G is trivial. It follows that $U \simeq \mathbb{D}^k/G = \mathbb{D}^k$ is smooth and the quotient map $q : \mathbb{D}^k \to U$ is an isomorphism. Hence $q' : X_U \to g^{-1}(U)$ is also an isomorphism, so g' and $g|_{g^{-1}(U)}$ identify under this isomorphism. In particular $g|_{g^{-1}(U)}$ is a submersion. Since y is arbitrary, this shows the claim. □

The striking difference between the corollary and proposition 2.3.8 is the absence of multiple fibres. This is a well-known phenomen for fibrations with rationally connected general fibre which we precise in the next lemma.

4.1.3 Lemma. *Let $\phi : X \to Y$ a fibration between complex manifolds such that the general fibre is rationally connected. Suppose one of the following:*

1.) X and Y are projective.

2.) $\dim X - \dim Y = 1$.

Then ϕ has generically reduced fibres in codimension 1 (cf. definition 3.2.3). Furthermore $\pi_1(X) \simeq \pi_1(Y)$.

Proof. For the first statement we argue by contradiction and suppose that the non-reduced ϕ-locus $N \subset Y$ has a codimension 1 component.

1st case. X and Y are projective. Let C be an intersection of general hyperplane sections $H_1, \ldots, H_{\dim Y-1}$, then C and $\phi^{-1}(C)$ are smooth and the restriction $\phi|_{\phi^{-1}(C)} : \phi^{-1}(C) \rightarrow C$ has at least one fibre such that all the components are not generically reduced. In particular $\phi|_{\phi^{-1}(C)}$ does not admit a section. Since the general fibre is rationally connected, this contradicts the Graber-Harris-Starr theorem [GHS03].

2nd case. $\dim X - \dim Y = 1$. There exists an (open) curve $\mathbb{D} \subset Y$ such that $\mathbb{D}$ intersects N in a point y and $S := \phi^{-1}(\mathbb{D})$ is a smooth surface. Set $\phi^{-1}(y) = \sum_{i=1}^{k} a_i C_i$ where $a_i > 1$ and the $C_i's$ are the reductions of the irreducible components of the fibre. We proceed by induction on k.

If $k = 1$, the curve C_1 has arithmetic genus 0, so it is a smooth rational curve. The restriction of the normal bundle to the set-theoretical fibre $\mathcal{O}_{C_1}(C_1)$ is a non-trivial torsion line bundle ([BPVdV84, III, Lemma 8.3]). Since C_1 is simply connected, we obtain a contradiction.

If $k > 1$ Zariski's lemma [BPVdV84, III, Lemma 8.2] implies that $C_i^2 < 0$ for every irreducible component C_i. Since the general fibre F is rational, we have

$$-2 = K_S \cdot F = K_S \cdot \sum_{i=1}^{k} a_i C_i.$$

So there exists a component C_i such that $K_S \cdot C_i < 0$. By the adjunction formula [BPVdV84, p.68] this implies $K_S \cdot C_i = -1$ and $C_i^2 = -1$, so C_i can be contracted by a birational map $S \rightarrow S'$ on a smooth surface S'. Let $\phi' : S' \rightarrow \mathbb{D}$ be the induced morphism, then $\phi'^{-1}(y)$ has $k-1$ irreducible components, none of them being reduced. We apply the induction hypothesis.

For the second statement, a result due to Nori [Nor83, Lemma 1.5] says that for a fibration with generically reduced fibres in codimension 1 there exists an exact sequence

$$\pi_1(F) \rightarrow \pi_1(X) \rightarrow \pi_1(Y) \rightarrow 1,$$

where F is a set-theoretical fibre. Since the general fibre F is simply connected, we conclude. $\square$

Lemma 3.2.4 is a tool that can help us to solve conjecture 1.1.2 at least partially by induction on the dimension. The following corollary shows that we can forget about the technical conditions in lemma 3.2.4 if the general fibre is rationally connected.

4.1.4 Corollary. *Let X be a complex manifold such that $T_X = V_1 \oplus V_2$. Let $\phi : X \rightarrow Y$ be a fibration on a complex manifold Y such that the general fibre is rationally connected. Suppose one of the following:*

1.) X and Y are projective.

2.) $\dim X - \dim Y = 1$.

Then the tangent bundle T_Y splits as

$$(\phi_* T\phi(V_1))^{**} \oplus (\phi_* T\phi(V_2))^{**} = T_Y.$$

Proof. By proposition 3.0.5 the fibration satisfies the ungeneric position property. Lemma 4.1.3 shows that ϕ has generically reduced fibres in codimension 1, so we conclude with lemma 3.2.4. □

4.2 Rationally connected manifolds

In this section we show theorem 4.2.4. The strategy is to construct in a first step a fibre space structure on X and to show in a second step that this fibre space structure comes from a product structure on X. The main technical ingredient is the following deep theorem by Bogomolov and McQuillan on the algebraicity of leaves.

4.2.1 Theorem. *[BM01, Thm.0.1.a)], [KSCT05, Thm.1.1] Let X be a projective manifold, and let $V \subset T_X$ be an integrable subbundle. Let $f : C \to X$ be a curve on X such that f^*V is ample. Then all V-leaves meeting $f(C)$ are rationally connected closed submanifolds of X.*

4.2.2 Lemma. *Let X be a projective rationally connected manifold such that $T_X = V_1 \oplus V_2$ and $V_1 \subset T_X$ is an integrable subbundle. Then there exists a submersion $\phi : X \to Y$ such that $V_1 = T_{X/Y}$.*

Proof. Since X is rationally connected, there exists a (very free) rational curve $f : \mathbb{P}^1 \to C$ on X. Since f^*T_X is ample, its quotient f^*V_1 is ample. By theorem 4.2.1 this implies that the V_1-leaves passing through a point $x \in C$ are rationally connected subvarieties of X. Now apply corollary 4.1.2. □

We come now to a second property which is proper to rationally connected manifolds: we have seen in example 2.2.3 that if there is a holomorphic connection on a submersion, it is not necessarily integrable. The next lemma shows that if the base of the submersion is rationally connected, the connection is integrable.

4.2.3 Lemma. *Let X be a projective manifold that admits a submersion $\phi : X \to Y$ on a variety Y. Suppose furthermore that ϕ admits a connection, i.e. a vector bundle $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$. If Y is rationally connected or a uniruled surface, then V is integrable and $X \simeq Y \times F$ where F is a general fibre.*

Proof. If $\dim Y = \mathrm{rk} V = 1$ the result is trivial, so we suppose that $\dim Y > 1$. We will construct a covering family of curves on X such that the general member C' is a smooth rational curve $f' : \mathbb{P}^1 \to X$ such that $f'^*T_{X/Y}$ is trivial and $f'^* \wedge^2 V$ is ample. Then $f'^*\mathcal{H}om(\wedge^2 V, T_X/V)$ is antiample, so clearly

$$H^0(C', \mathcal{H}om(\wedge^2 V, (T_X/V))|_{C'}) = H^0(\mathbb{P}^1, f'^*\mathcal{H}om(\wedge^2 V, (T_X/V))) = 0.$$

This shows the integrability of V (see corollary 2.1.3,2). Since ϕ admits an integrable connection, it follows from the Ehresmann theorem that the structure of an analytic bundle on X arises as a representation of the fundamental group of Y. Since Y is simply connected, this implies $X \simeq Y \times F$, where F is a ϕ-fibre.

We come to the construction of the family. If Y is rationally connected of dimension at least 3, there exists a covering family of curves on Y such that the general member is a very free *smooth* rational curve $f : \mathbb{P}^1 \to C$ (cf. [Kol96, Thm.3.14]). In particular $f^* \wedge^2 T_Y$ is ample. If Y is a uniruled surface its relative minimal model is a ruled surface or $\mathbb{P}^2$. Taking either the preimages of the fibres of the ruling or the lines in $\mathbb{P}^2$, we obtain a covering family of rational curves such that the general member is smooth and $f^*T_Y \simeq \mathcal{O}(2) \oplus \mathcal{O}$ or $f^*T_Y \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)$. In both cases $f^* \wedge^2 T_Y$ is ample.

Fix now a general member $f : \mathbb{P}^1 \to Y$ of the covering family and denote by C the image. Since C is smooth and ϕ is a submersion, the fibre product $Z := X \times_Y \mathbb{P}^1$ is smooth and admits a submersion $\tilde{\phi} : Z \to \mathbb{P}^1$. Denote by $\mu : Z \to X$ the natural projection, then $\mu^* T_{X/Y} = T_{Z/\mathbb{P}^1}$ and $T_Z \subset \mu^* T_X$ is a subbundle. We then consider the sequence of sheaf homomorphisms on Z

$$T_{Z/\mathbb{P}^1} \hookrightarrow T_Z \hookrightarrow \mu^* T_X \to \mu^* T_{X/Y}.$$

The first two maps are the canonical embeddings, while the last one is the projection along $\mu^* V$. Since $T_{Z/\mathbb{P}^1} \simeq \mu^* T_{X/Y}$ the map $T_Z \to \mu^* T_{X/Y}$ has maximal rank in every point. It follows that $L := \mu^* V \cap T_Z = \ker(T_Z \to \mu^* T_{X/Y})$ is a rank 1 subbundle of T_Z such that

$$T_Z := L \oplus T_{Z/\mathbb{P}^1}$$

Since L has rank 1 it is integrable. This shows that $\tilde{\phi} : Z \to \mathbb{P}^1$ admits an integrable connection, so by the Ehresmann theorem $Z \simeq \mathbb{P}^1 \times F$ where F is a fibre. It follows that for any $a \in F$, we obtain a rational curve $\mu' : \mathbb{P}^1 \times \{a\} \to C'$ on X such that $\mu'^* T_{X/Y}$ is trivial and $\mu'^* \wedge^2 V = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$. Since the rational curves we use cover a dense open subset on Y, the constructed curves cover a dense open subset on X. $\square$

We come now to the first main result.

4.2.4 Theorem. *Let X be a rationally connected manifold such that $T_X = V_1 \oplus V_2$. If V_1 or V_2 is integrable, then V_1 and V_2 are integrable; furthermore conjecture 1.1.2 holds.*

Proof. Lemma 4.2.2 yields the existence of a submersion $\phi : X \to Y$ with connection one of the direct factors V_j. Since X is rationally connected, the manifold Y is rationally connected, so lemma 4.2.3 applies. $\square$

It is very well possible that the integrability hypothesis in theorem 4.2.4 is superfluous. A first step in this direction is the next corollary.

4.2.5 Corollary. *Let X be a projective rationally connected manifold such that $T_X = \oplus_{j=1}^k V_j$, where $\mathrm{rk} V_j \leq 2$ for all $j = 1, \ldots, k$. Then conjecture 1.1.2 holds.*

In particular the conjecture holds for rationally connected manifolds of dimension up to 4.

Proof. By lemma 2.2.6 one of the direct factors is integrable, so up to renumbering V_k is integrable. Then by theorem 4.2.4 we have $X \simeq X' \times X_k$ such that $V_k = p_{X_k}^* T_{X_k}$ and $\oplus_{j=1}^{k-1} V_j = p_{X'}^* T_{X'}$. The manifold X' is rationally connected with split tangent bundle $\oplus_{j=1}^{k-1} (p_{X'})_* V_j$, so we can conclude by induction on k. $\square$

Remark. Campana and Peternell have shown the conjecture for Fano manifolds of dimension up to 5 ([CP02]), but their results are based on classification theory and it seems hard to generalize them to higher dimension.

4.3 Mori fibre spaces

Elementary Mori contractions of fibre type (Mori fibre spaces) are an important tool in the classification theory of uniruled manifolds. Since the relative Picard number of such a fibre space is one, we have much better control over the singular fibres than usual. This will allow us to deduce properties of the whole fibre space from a property of the general fibre (the ungeneric position property). Once we have established these results we can deduce various corollaries about the universal covering of X (corollaries 4.3.5, 4.3.7, 4.3.9).

We introduce the standard terminology of Mori theory. A Mori contraction of a projective manifold X is a morphism with connected fibres $\phi : X \to Y$ to a normal variety Y such that the anticanonical bundle $-K_X$ is ϕ-ample. We say that the contraction is elementary if $\rho(X/Y) := \rho(X) - \rho(Y) = 1$. The contraction is said to be of fibre type if $\dim Y < \dim X$, otherwise it is birational.

4.3.1 Proposition. *[CP02, Prop.1.8.] Let X be a projective manifold such that $T_X = V_1 \oplus V_2$. Let $\phi : X \to Y$ be an elementary extremal contraction.*

1.) If $L_j = \det V_j$ is not ϕ-trivial, then $\dim F \leq \mathrm{rk} V_j$ for every ϕ-fibre F.

2.) $\dim F \leq \max(\mathrm{rk} V_1, \mathrm{rk} V_2)$.

3.) Suppose some fibre of ϕ contains a rational curve $f : \mathbb{P}^1 \to X$ such that

$$f^* T_X = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \bigoplus_{i=2}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i) \qquad \text{with} \quad a_i \leq 1 \qquad \forall\ i > 1. \tag{4.1}$$

Then up to renumbering L_1 is ϕ-ample and L_2 is ϕ-trivial. In this case we have $\mathcal{O}_{\mathbb{P}^1}(2) \subset f^* V_1$.

Remark. If $\phi : X \to Y$ is an elementary contraction of fibre type, a general fibre F always contains a rational curve of splitting type (4.1). Indeed F is a Fano manifold of positive dimension, so by [Deb01, Ex.4.8.3] there exists a rational curve $f : \mathbb{P}^1 \to F \subset X$ such that $f^* T_F$ is nef and has splitting type (4.1). Conclude with lemma 4.3.2.

4.3.2 Lemma. *Let $\phi : X \to Y$ be a morphism from a projective manifold X to a normal variety Y, and let Z be a fibre such that its scheme-theoretic structure is reduced. Let F be an irreducible component of Z that is smooth. Let $f' : \mathbb{P}^1 \to F$ be a rational curve $C := f'(\mathbb{P}^1)$ such that f'^*T_F is nef. Then there exists a deformation $f : \mathbb{P}^1 \to F$ of C in F, such that f^*T_F is nef and*

$$f^*T_X \simeq f^*T_F \oplus f^*N_{F/X}. \tag{4.2}$$

Proof. Since Z is reduced, the canonical morphism

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \otimes \mathcal{O}_F \to \mathcal{I}_F/\mathcal{I}_F^2$$

is generically surjective. Since $\mathcal{I}_Z/\mathcal{I}_Z^2$ is globally generated, $\mathcal{I}_F/\mathcal{I}_F^2 = N^*_{F/X}$ is generically generated. Since F is smooth, the deformations of f' cover F ([Deb01, Prop.4.8]). So for a general deformation $f : \mathbb{P}^1 \to F$ the restriction of the bundle $N^*_{F/X}$ to $f(\mathbb{P}^1)$ is generically generated and f^*T_F is nef. It follows that $f^*(N^*_{F/X} \otimes T_F)$ is nef, so $h^1(\mathbb{P}^1, f^*(N^*_{F/X} \otimes T_F)) = 0$. Hence the exact sequence

$$0 \to f^*T_F \to f^*T_X \to f^*N_{F/X} \to 0$$

splits. □

We can now proof the central ungeneric position result for Mori fibre spaces. It shows that an elementary contraction of fibre type not only satisfies the ungeneric position property (this is clear from proposition 3.0.5), but even that T_F is contained in one of the direct factors $V_1|_F$ or $V_2|_F$. This is the special case of ungeneric position that we saw already in example 3.1.7.

4.3.3 Lemma. *Let X be a projective manifold with $T_X = V_1 \oplus V_2$. Let $\phi : X \to Y$ be an elementary extremal contraction of fibre type. Let F be a general fibre, then we have after possible renumbering $T_F \subset V_1|_F$.*

If furthermore $\dim X - \dim Y = \mathrm{rk} V_1$ *or* $\dim X - \dim Y + 1 = \mathrm{rk} V_1$*, then V_1 is integrable.*

Proof. The general fibre F is a Fano manifold, so there exists a covering family such that the general member $f : \mathbb{P}^1 \to F$ is a very free rational curve in F, that is

$$f^*T_F = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \qquad \text{with} \quad a_i \geq 1 \qquad \forall\ i \geq 1.$$

We use lemma 4.3.2 to obtain

$$f^*T_X = f^*(T_F) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim Y} = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim Y}.$$

By proposition 4.3.1 we may assume after possible renumbering that $\det V_1$ is ϕ-ample and $\det V_2$ is ϕ-trivial. This implies $f^*V_2 = \mathcal{O}_{\mathbb{P}^1}^{\oplus \mathrm{rk} V_2}$. We denote by $\delta : T_F \to V_2|_F$ the projection on $V_2|_F$ along $V_1|_F$. Since f^*T_F is ample and f^*V_2 is trivial, the restriction of δ to $f(\mathbb{P}^1)$ is zero. Since the very free curves cover a dense set in F we see that δ is zero. This is equivalent to $T_F \subset V_1|_F$.

Since integrability is a generic property (cf. corollary 2.1.3), we suppose without loss of generality that ϕ and Y are smooth. The integrability follows from corollary 3.2.5. □

We now come to the core of this chapter where we fully exploit the ungeneric position property shown in lemma 4.3.3. The idea of the structure results is as follows: let $X = X_1 \times X_2$ be a product of projective manifolds and suppose that $X_1 \to Y_1$ is an elementary Mori fibre space. Then $X = X_1 \times X_2 \to Y_1 \times X_2 =: Y$ is an elementary fibre space that has „the same fibres". In particular if $\dim Y_1 = 0$, the fibration is a fibre bundle and if $\dim Y_1 = 1$, it is flat. Analogously if X has a split tangent bundle $T_X = V_1 \oplus V_2$ and is an elementary Mori fibre space $X \to Y$ such that the general fibre F satisfies $T_F \subset V_1|_F$, then $\mathrm{rk} V_1 - \mathrm{rk} T_F$ should be seen as the „essential dimension" of the base Y. In particular if this difference equals zero, the fibre space is a fibre bundle (proposition 4.3.4), and if this difference equals 1, the fibration is equidimensional (lemma 4.3.6).

4.3.4 Proposition. *Let X be a projective manifold with $T_X = V_1 \oplus V_2$. Let $\phi : X \to Y$ be an elementary extremal contraction of fibre type. Suppose that a general fibre F satisfies $T_F = V_1|_F$. Then ϕ is an analytic fibre bundle such that $T_{X/Y} = V_1$. If V_2 is integrable, conjecture 1.1.2 holds for X.*

Proof. By proposition 4.3.1 all fibres have dimension at most $\mathrm{rk} V_1$, so ϕ is equidimensional. By corollary 2.3.9 this implies that ϕ is almost smooth and identifies to the map on the leaf space $X \to X/V_1$. Since the general fibre is a Fano manifold and a V_1-leaf, corollary 4.1.2 implies that the map on the leaf space is smooth, so ϕ is even smooth. The vector bundle V_2 is a holomorphic connection, so by lemma 2.4.2 we know that ϕ is a fibre bundle.

If furthermore V_2 is integrable, the Ehresmann theorem 2.4.1 shows that conjecture 1.1.2 holds for X. □

4.3.5 Corollary. *Let X be a projective manifold with $T_X = V_1 \oplus V_2$. Let $\phi : X \to C$ be an elementary contraction to a smooth curve C. Then ϕ is an analytic fibre bundle and conjecture 1.1.2 holds for X.*

Proof. By lemma 4.3.3, we have after possible renumbering, $T_F \subset V_1|_F$. Since $\dim X - 1 = \dim F \leq \mathrm{rk} V_1 \leq \dim X - 1$, we even have $T_F = V_1|_F$. Apply the first part of proposition 4.3.4 to obtain the structure result on ϕ. The bundle V_2 has rank 1, so the second part of proposition 4.3.4 applies. □

Remark. The corollary is a generalisation of [CP02][Thm.2.9.] which gives the same statement for an elementary contraction of a threefold on a curve. Since the decomposition $T_X = V_1 \oplus V_2$ has one direct factor of rank 1, the decomposition of the universal covering could also be obtained by [BPT04, Thm. 1].

4.3.6 Lemma. *Let X be a projective manifold with $T_X = V_1 \oplus V_2$, let $\phi : X \to Y$ be an elementary contraction of fibre type on a normal variety Y. Suppose that a general fibre F satisfies $T_F \subset V_1|_F$ and $\dim F + 1 = \mathrm{rk} V_1$. Then ϕ is equidimensional.*

Idea of the proof. Suppose that there exists a higher-dimensional fibre F. There are two cases: either F is a V_1-leaf, but this is impossible since the „leaf deforms in all the directions, but the higher-dimensional fibre does not". Or the V_1-leaves intersect with F in subvarieties of dimension $\dim X - \dim Y$. This allows us to construct a factorization of ϕ. Thus the relative Picard number is not 1.

Proof. By lemma 4.3.3 we know that V_1 is integrable.

Let $Y^* \subset Y$ be the ϕ-smooth locus. We embed Y^* in the Chow scheme $\mathcal{C}(X)$ and denote by $\bar{Y}$ its closure, endowed with the reduced structure. Let $\Gamma \subset \bar{Y} \times X$ be the reduction of the graph over $\bar{Y}$, and denote by $p_{\bar{Y}} : \Gamma \to \bar{Y}$ and $p_X : \Gamma \to X$ the restrictions of the natural projections to the graph. Since every $p_{\bar{Y}}$-fibre is contracted by ϕ (this depends only on the homology class), a standard rigidity result [Deb01][Lemma1.15] implies the existence of a factorization $g : \bar{Y} \to Y$, so that we obtain a commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{p_X} & X \\ {\scriptstyle p_{\bar{Y}}}\downarrow & & \downarrow{\scriptstyle \phi} \\ \bar{Y} & \xrightarrow{g} & Y \end{array}$$

Step 1. We show that p_X is bijective. In order to show that p_X is bijective we first show that the intersection of any V_1-leaf and a ϕ-fibre is a subvariety of dimension $k := \dim X - \dim Y$.

a) The intersection does not have dimension $k+1$.

Suppose the contrary, i.e. let V_1^x be a V_1-leaf passing through a point x, such that $\phi^{-1}(\phi(x))$ contains V_1^x (as a set). Since every irreducible component of the fibre has dimension at most $k+1 = \mathrm{rk} V_1$, this implies that the leaf V_1^x is an irreducible component $F \subset \phi^{-1}(\phi(x))$. This shows that F_{red} is smooth and V_1^x is compact. Since $T_{F_{red}} = V_1|_{F_{red}}$ and $\det V_1$ is ϕ-ample, we see that V_1^x is a Fano variety. Corollary 4.1.2 shows that there exists a submersion $f : X \to Z$ such that all fibres are V_1-leaves. Since ϕ contracts F, we can apply the rigidity lemma [Deb01][Lemma 1.15] to obtain a dominant factorization $g : Z \dashrightarrow Y$. But this is impossible, since $\dim X - k - 1 = \dim Z < \dim Y = \dim X - k$.

b) The intersection has dimension at least k.

For $y \in \bar{Y}$, denote by Γ_y the fibres of $p_{\bar{Y}}$. Let $x \in X$ be an arbitrary point, then there exists an Γ_y such that $x \in p_X(\Gamma_y)$. The scheme $p_X(\Gamma_y)$ is contained set-theoretically in the fibre $\phi^{-1}(\phi(x))$. Consider now the foliation induced by $p_X^* V_1$ on $\Gamma \subset \bar{Y} \times X$. Since a general $p_{\bar{Y}}$-fibre is contained in a $p_X^* V_1$-leaf and this is a closed condition, every fibre Γ_y is contained in a $p_X^* V_1$-leaf. Hence $p_X(\Gamma_y)$ is contained in the V_1-leaf through x. Since the restriction of p_X to Γ_y is injective, the scheme $p_X(\Gamma_y)$ has dimension k.

c) The map p_X is bijective.

It is sufficient to show that for an arbitary $x \in X$, there exists exactly one ϕ-fibre F_x and one V_1-leaf V_1^x passing through x. Steps a) and b) show that for any $z \in p_X^{-1}(x)$ and $y = p_{\bar{Y}}(z)$, we have $p_X(\Gamma_y) \subset F_x \cap V_1^x$ (set-theoretically). Since $F_x \cap V_1^x$ has dimension equal to Γ_y, there exist at most finitely many

$p_X(\Gamma_y)$ contained in $F_x \cap V_1^x$, in particular p_X is finite. Since p_X is birational and X is smooth, the fibres are connected, so p_X is bijective.

Step 2. We show that ϕ is equidimensional. Since X is smooth and Γ reduced, p_X bijective implies that p_X is an isomorphism. We identify Γ to X via the isomorphism p_X. The factorization map g is birational and we want to show that it is an isomorphism. Now $\rho(X) - \rho(Y) = 1$ implies that $\rho(Y) = \rho(\bar{Y})$. Since g is birational and Y normal, this implies that it is an isomorphism. Since $p_{\bar{Y}}$ is equidimensional this shows that ϕ is equidimensional. □

Remark. It is tempting to conjecture that ϕ is even flat, but there is a technical obstacle: In step 1, c) we argue that if there exists a subvariety $\Delta \subset \mathcal{C}(X)$ parametrizing cycles such that the support has dimension k, and the image of the family $p_X(p_{\bar{Y}}^{-1}(\Delta)$ has dimension k, then Δ has dimension 0. The same argument does not work for the Hilbert scheme, since we may very well have a positive-dimensional family of schemes having the same support, but different scheme structure. We will come back to this problem in section 4.4.

We give two applications of lemma 4.3.6 where flatness is guaranteed by classification results.

4.3.7 Corollary. *Let X be a projective manifold with $T_X = V_1 \oplus V_2$. Let $\phi : X \to S$ be an elementary contraction of fibre type to a surface S. There are two cases:*

1st case. ϕ is an analytic fibre bundle such that up to renumbering $T_{X/Y} = V_1$. If V_1 and V_2 are integrable, then conjecture 1.1.2 holds.

2nd case. ϕ is a flat map such that a general fibre F satisfies up to renumbering $T_F \subsetneq V_1|_F$. Then conjecture 1.1.2 holds.

In both cases V_1 is integrable. In particular if V_1 and V_2 are integrable, then conjecture 1.1.2 holds.

This corollary almost immediately implies the following

4.3.8 Theorem. *Let X be a projective manifold with $T_X = V_1 \oplus V_2$. If X admits an elementary Mori contraction on a surface, then either V_1 or V_2 is integrable. If furthermore both V_1 and V_2 are integrable, then conjecture 1.1.2 holds.*

Proof of corollary 4.3.7 and theorem 4.3.8. By 4.3.3 we have, after possible renumbering, $T_F \subset V_1|_F$. Since

$$\dim X - 2 = \dim F \leq \mathrm{rk} V_1 \leq \dim X - 1,$$

there are two cases.

1st case. $\dim F = \mathrm{rk} V_1$. Apply proposition corollay 4.3.4.

2nd case. $\dim F + 1 = \mathrm{rk} V_1$. Lemma 4.3.6 shows that ϕ is equidimensional. Since Y is a surface and ϕ elementary, Y is smooth. Hence equdimensionality implies flatness. Lemma 4.1.3 shows that ϕ has generically reduced fibres in codimension 1. The rank 1 bundle V_2 is integrable, so we conclude with lemma 3.2.6.

In both cases the integrability of V_1 follows from corollary 3.2.5. From what preceeds it is clear that corollary 4.3.7 implies theorem 4.3.8. □

4.3.9 Corollary. *Suppose that X is a projective manifold with split tangent bundle $T_X = V_1 \oplus V_2$ and admits an elementary Mori contraction $\phi : X \to Y$ such that $\dim Y = \dim X - 1$. Then a general fibre F satisfies, after possible renumbering, $T_F \subset V_1|_F$ and we suppose furthermore that $\mathrm{rk} V_1 = 2$. If V_2 is integrable, then conjecture 1.1.2 holds.*

Furthermore X has the structure of a $\mathbb{P}^1$-bundle or conic bundle. The map ϕ induces a splitting $T_Y = (\phi_*(T\phi(V_1)))^{**} \oplus (\phi_*(T\phi(V_2)))^{**}$. If F is a singular fibre, then F is isomorphic to two smooth rational curves intersecting transversally in one point. Furthermore the ϕ-singular locus $\Delta \subset Y$ is smooth and satisfies

$$T_\Delta = (\phi_*(T\phi(V_2)))^{**}|_\Delta.$$

Proof. By lemma 4.3.6 the map ϕ is equidimensional of dimension 1. Since ϕ is elementary, Ando's result implies that ϕ is a $\mathbb{P}^1$-bundle or conic bundle. In particular it is flat, so Y is smooth. By 4.1.4 we have a splitting

$$T_Y = (\phi_*(T\phi(V_1)))^{**} \oplus (\phi_*(T\phi(V_2)))^{**}.$$

Since V_1 is integrable by lemma 4.3.3 and V_2 by hypothesis, lemma 3.2.6 applies.

Let $D \subset X$ be an irreducible effective divisor such that $\phi(D) \subset \Delta$. The general fibre of $D \to \phi(D)$ is a union of two smooth rational curves intersecting transversally in one point. Let C be an irreducible component of a general singular fibre, then

$$T_X|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim X - 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

By proposition 4.3.1 we know that $\det V_1$ is ϕ-ample and $\det V_2$ is ϕ-trivial, so $T_C \subset V_1|_C$ and $V_2|_C \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim X - 2}$. The canonical map $\gamma : V_2|_D \to N_{D/X}$ is zero since its restriction to every component $C \subset D$ of a fibre is given by $\gamma|_C : \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim X - 2} \to \mathcal{O}_{\mathbb{P}^1}(-1)$, which is the zero map. Hence we obtain a splitting of the tangent bundle of the nonsingular locus $D_{\mathrm{reg}} \subset D$ as

$$T_{D_{\mathrm{reg}}} = (V_1|_{D_{\mathrm{reg}}} \cap T_{D_{\mathrm{reg}}}) \oplus V_2|_{D_{\mathrm{reg}}}.$$

The inclusion $T_C \subset V_1|_C$ implies $T_{D_{\mathrm{reg}}/\phi(D_{\mathrm{reg}})} = V_1|_{D_{\mathrm{reg}}} \cap T_{D_{\mathrm{reg}}}$. So if $y \in \phi(D)$ is a smooth point, we have

$$T\phi(D)_y = W_{2,y}.$$

This shows that $\phi(D)$ coincides with a W_2-leaf is a general point of $\phi(D)$. Since leaves are locally closed, this shows that $\phi(D)$ is a W_2-leaf, so it is smooth. Since leaves don't intersect this shows that Δ is smooth. By [Sar82, Prop.1.8] this implies that there are no double lines as fibres. □

Remark. Since elementary contractions $\phi : S \to C$ from a projective surface to a curve are always $\mathbb{P}^1$-bundles, one might expect that in the situation above there are no singular fibres at all. The following counterexample, for which we thank M. Brunella, shows that this is not true, thereby correcting and completing [CP02, Thm.2.8].

Let $S' := \mathbb{P}^1 \times \mathbb{P}^1$, we identify $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and choose coordinates (z, w) on S'. The following map is an involution on $S'' := S' \setminus \{(0,0), (\infty, \infty)\}$.

$$\phi' : S'' \to S'' \quad (z, w) \mapsto (z, \frac{z}{w})$$

If $C_z := \{z\} \times \mathbb{P}^1$ is a fibre of the projection pr_1 on the first factor, then $\phi'|_{C_z}$ is the involution on $\mathbb{P}^1$ with fixed points $\sqrt{z}$ and $-\sqrt{z}$. Blow up S' in $(0,0)$ and (∞, ∞) to resolve the indeterminacies of ϕ'. If we denote by $\mu : S \to S'$ the blow-up, ϕ' lifts to an involution ϕ of S. The total transform of $\{0\} \times \mathbb{P}^1$ has two irreducible components E_1 and E_2 such that $\phi(E_1) = E_2$ and $\phi(E_2) = E_1$ (an analogous statement holds for the total transform of $\{\infty\} \times \mathbb{P}^1$).

Let $T := \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ be an elliptic curve and $\psi : T \to T \quad t \mapsto t + \frac{1}{2}$. Then $T' \simeq T/\{\mathrm{id}_T, \psi\}$ is an elliptic curve. Define $X := (S \times T)/\{\mathrm{id}_S \times \mathrm{id}_T, \phi \times \psi\}$, then X is a smooth projective variety with split tangent bundle. The map $(pr_1 \circ \mu) \times \mathrm{id}_T$ induces a morphism

$$f : X \to \mathbb{P}^1 \times T'.$$

Let us show that f is an elementary contraction. Let $F_0' = E_1' + E_2'$ be a singular fibre and let $F_0 = E_1 + E_2$ be its lifting to $S \times T$. Then $F_0 \subset S \times \{t\}$ for some $t \in T$ and clearly

$$E_1 = (\phi \times \psi)(\tilde{E})$$

where $\tilde{E}$ is the copy of E_2 in $S \times \{t + \frac{1}{2}\}$. Since $\tilde{E}$ is a deformation of E_2 in $S \times T$, this shows that E_2' is a deformation of E_1' in X, hence $E_1' = E_2'$ in $N_1(X)$. The special fibre F_0 is homologous to a general fibre F, so

$$F = F_0 = E_1' + E_2' = 2E_1'$$

in $N_1(X)$. This shows that f is the elementary contraction of the ray generated by E_1'.

4.4 The rational quotient

We turn now to the case of compact Kähler manifolds. Since we can't use Mori contractions, we have to work with the rational quotient map. This makes our task more difficult, since this is only a rational map, so we will have less control over the degenerations of rational curves. In the projective case, an auxiliary result (proposition 4.4.10) will allows us to see that the situation is slightly better than what can a priori expect.

4.4.1 Definition. *An almost holomorphic fibration is an almost holomorphic map $\phi : X \dashrightarrow Y$, i.e. a map a meromorphic map such that the image of the indeterminacy locus does not surject onto the base, such that for some non-empty Zariski open subset $Y^* \subset Y$, the induced holomorphic map $\phi|_{\phi^{-1}(Y^*)} : \phi^{-1}(Y^*) \to Y^*$ is a fibration.*

A deep result of Campana shows that a uniruled compact Kähler manifold X admits an almost holomorphic fibration $\phi : X \dashrightarrow Y$ to a normal variety Y such that the general fibre is rationally connected and the variety Y is not uniruled (for the existence of so-called geometric quotients consider [Cam04b], a very readable introduction to the rational quotient can be found in [Deb01], the non-uniruledness of the base is a consequence of the Graber-Harris-Starr result [GHS03]). This map is unique up to meromorphic equivalence of fibrations, so we are entitled to call it *the* rational quotient of X. Since the rational quotient is fundamental for the study of uniruled manifolds we have to generalise slightly the definition of the ungeneric position property.

4.4.2 Definition. *Let X be a complex manifold such that $T_X = V_1 \oplus V_2$. An almost holomorphic fibration $\phi : X \dashrightarrow Y$ satisfies the ungeneric position property if for some non-empty Zariski open subset $Y^* \subset Y$, the induced holomorphic fibration $\phi|_{\phi^{-1}(Y^*)} : \phi^{-1}(Y^*) \to Y^*$ satisfies this property.*

4.4.3 Lemma. *In the situation of definition 4.4.2, the definition does not depend on the set Y^*.*

Proof. By lemma 3.2.2 the map $\phi|_{\phi^{-1}(Y^*)} : \phi^{-1}(Y^*) \to Y^*$ satisfies the ungeneric position property if and only if a general fibre F satisfies

$$T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F).$$

Since F is general this property does not depend on the choice of Y^*. $\square$

The ungeneric position of the rational quotient is now an immediate corollary of lemma 3.2.7.

4.4.4 Corollary. *Let X be a uniruled compact Kähler manifold with split tangent bundle $T_X = V_1 \oplus V_2$, and let $\phi : X \dashrightarrow Y$ be an almost holomorphic fibration such that the general fibre is rationally connected (e.g. ϕ is the rational quotient map). Then ϕ satisfies the ungeneric position property.*

Proof. Let $Y^* \subset Y$ be a non-empty Zariski open subset such that $\phi|_{\phi^{-1}(Y^*)} : \phi^{-1}(Y^*) \dashrightarrow Y^*$ is a holomorphic map. The general fibre F of this map is rationally connected, so proposition 3.0.5 applies. $\square$

We start our investigation of the rational quotient with the easiest case possible, this should be seen as an analogoue to proposition 4.3.4.

4.4.5 Proposition. *Let X be a uniruled compact Kähler manifold with split tangent bundle $T_X = V_1 \oplus V_2$, and let $\phi : X \dashrightarrow Y$ be the rational quotient map. Suppose that a general fibre F satisfies $T_F \cap V_1|_F = V_1|_F$. If V_1 or V_2 are integrable or conjecture 1.1.6 holds for F, the manifold X has the structure of an analytic fibre bundle $X \to Z$ such that $T_{X/Z} = V_1$. If V_1 and V_2 are integrable, then conjecture 1.1.2 holds for X.*

Proof. By hypothesis and corollary 4.4.4, we have

$$T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F) = V_1|_F \oplus (V_2|_F \cap T_F).$$

Since V_1 or V_2 is integrable or conjecture 1.1.6 holds for F, one of the bundles $V_1|_F$ or $V_2|_F \cap T_F$ is integrable. By theorem 4.2.4, the manifold F is a product $F_1 \times F_2$ and

$$T_F = V_1|_F \oplus (V_2|_F \cap T_F) = p^*_{F_1} T_{F_1} \oplus p^*_{F_2} T_{F_2}.$$

In particular $V_1|_F$ is integrable. Since F is general, V_1 is integrable. Furthermore V_1 has a rationally connected leaf, so by corollary 4.1.2 there exists a submersion $X \to Z$ such that $T_{X/Z} = V_1$

If V_1 and V_2 are integrable, the Ehresmann theorem 2.4.1 applies to $X \to Z$. □

This proposition together with theorem 4.2.4 has an interesting corollary.

4.4.6 Corollary. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Suppose that the rational quotient map $\phi : X \dashrightarrow Y$ is a map on a curve Y. If V_1 or V_2 is integrable, then V_1 and V_2 are integrable; furthermore conjecture 1.1.2 holds for X.*

Proof. By corollary 4.4.4 the rational quotient map satisfies the ungeneric position property, so the general fibre F satisfies

$$T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F).$$

Since V_1 or V_2 is integrable, $V_1|_F \cap T_F$ or $V_2|_F \cap T_F$ is integrable. Hence theorem 4.2.4 applies to F, so the manifold F is a product $F_1 \times F_2$ and

$$T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F) = p^*_{F_1} T_{F_1} \oplus p^*_{F_2} T_{F_2}.$$

Since $\dim Y = 1$, we have $V_1|_F \cap T_F = V_1|_F$ or $V_2|_F \cap T_F = V_2|_F$. Up to renumbering we can suppose that $V_2|_F \cap T_F = V_2|_F$. Since F is general, V_2 is integrable. Since F is a product of rationally connected manifolds, there exists a covering family of rational curves such that for a general member $f : \mathbb{P}^1 \to F$ the pull-back $f^* V_2|_F$ is trivial and $f^*(V_1|_F \cap T_F)$ is ample (the rational curve is very free in the factor F_1). Since $\mathrm{rk}(V_1|_F \cap T_F) = \mathrm{rk} V_1 - 1$, this implies that $f^* \bigwedge^2 V_1$ is ample. Hence

$$H^0(f(\mathbb{P}^1), \mathcal{H}om(\bigwedge^2 V_1, V_2)|_{f(\mathbb{P}^1)}) \subset H^0(\mathbb{P}^1, f^*(\bigwedge^2 V_1^* \otimes V_2)) = 0,$$

so $H^0(F, \mathcal{H}om(\bigwedge^2 V_1, V_2)|_F) = 0$. Since F is a general fibre, corollary 2.1.3 implies the integrability of V_1. Conjecture 1.1.2 follows from proposition 4.4.5. □

Remark. A positive answer to conjecture 1.1.6 would make it possible to state the proposition 4.4.5 and corollary 4.4.6 with a weaker hypothesis. For example, since this conjecture holds in dimension up to 4 by lemma 2.2.6, we have

4.4.7 Corollary. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Suppose that the rational quotient map $\phi : X \dashrightarrow Y$ is a map on a curve Y and $\dim X \leq 5$. Then Conjecture 1.1.2 holds for X.* □

After these easy corollaries, we come to some more serious business: what can we do if the general fibre F of the rational quotient satisfies $T_F \cap V_j|_F \subsetneq V_j|_F$ for $j = 1$ and 2 ? We have seen in section 4.3 that even if we start with a *holomorphic* map such that the general fibre satisfies $T_F \subset V_1|_F$, it takes some work to say something about the singular fibres (remember the proof of lemma 4.3.6). We therefore have to reduce our ambitions a bit and start by treating the special case, where $\mathrm{rk} V_1 = 2$ and $\mathrm{rk}(V_1|_F \cap T_F) = 1$. In this situation a variation of the proof of lemma 4.3.6 yields a satisfactory result.

4.4.8 Proposition. *Let X be a uniruled compact Kähler manifold with split tangent bundle $T_X = V_1 \oplus V_2$ where $\mathrm{rk} V_1 = 2$. Suppose that the general fibre F of the rational quotient map satisfies $\mathrm{rk}(T_F \cap V_1|_F) = 1$. Then there exists an equidimensional map $\phi : X \to Y$ on a compact Kähler variety such that the general fibre is a rational curve C that satisfies $T_C \subset V_1|_C$.*

Proof. By corollary 4.4.4 the rational quotient map satisfies the ungeneric position property, so the general fibre F satisfies

$$T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F).$$

The line bundle $T_F \cap V_1|_F$ is integrable, so theorem 4.2.4 implies F is a product $F_1 \times F_2$ such that $T_F \cap V_1|_F = p_{F_1}^* T_{F_1}$. Since $\mathrm{rk}(T_F \cap V_1|_F) = 1$, the variety F_1 is a smooth rational curve such that $T_F|_{F_1} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\dim F - 1}$. Since F is a general fibre, this implies that there exists a dominant family of smooth rational curves on X such that a general member $C := F_1$ satisfies

$$T_X|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\dim X - 1}$$

and $T_C \subset V_1|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$. Since $h^0(C, N_{C/X}) = \dim X - 1$ and $h^1(C, N_{C/X}) = 0$, the corresponding open subvariety of the cycle space $\mathcal{C}(X)$ is smooth of dimension $\dim X - 1$. We denote by Y its closure in $\mathcal{C}(X)$ and endow it with the reduced structure. Denote by $\Gamma \subset Y \times X$ the reduction of the graph over Y. Denote furthermore by $p_X : \Gamma \to X$ and $p_Y : \Gamma \to Y$ the restrictions of the projections to the graph.

Step 1. We show that p_X is finite. We argue by contradiction, then by the analytic version of Zariski's main theorem there are fibres of positive dimension. Let $x \in X$ be a point such that $p_X^{-1}(x)$ has a component of positive dimension. Let $\Delta \subset p_Y(p_X^{-1}(x))$ be an irreducible component of dimension $k > 0$. Then $\Gamma_\Delta := p_Y^{-1}(\Delta)$ has dimension $k+1$. Consider now the foliation induced by $p_X^* V_1$ on $\Gamma \subset Y \times X$. Since a general p_Y-fibre is contained in a $p_X^* V_1$-leaf and this is a closed condition, every fibre $p_Y^{-1}(y)$ is contained in a $p_X^* V_1$-leaf. So for $y \in \Delta$, the set $p_X(p_Y^{-1}(y))$ is contained in $\mathfrak{V}_1^x$, the V_1-leaf through x. It follows that $S := p_X(p_Y^{-1}(\Delta))$ is contained set-theoretically in $\mathfrak{V}_1^x$. Since p_X is injective on the fibres of p_Y, and $p_Y^{-1}(\Delta)$ has dimension $k + 1 \geq 2$, the subvariety S has dimension at least 2. Since $\mathrm{rk} V_1 = 2$, it has dimension 2 and $S = \mathfrak{V}_1^x$ (at least set-theoretically). So $\mathfrak{V}_1^x$ is a compact leaf and is covered by a family of rational cycles that intersects in the point x. Hence $\mathfrak{V}_1^x$ is rationally connected, so by corollary 4.1.2 there exists a submersion $\psi : X \to Z$ such that $T_{X/Z} = V_1$ and the fibres are rationally connected. By the universal property of the rational quotient the general ψ-fibre is contracted by rational quotient map. This implies $\mathrm{rk}(T_F \cap V_1|_F) = \mathrm{rk}V_1$, a contradiction.

Step 2. Construction of ϕ. Since p_X is birational by construction and finite, it is bijective by the analytic version of Zariski's main theorem. Since X is smooth and Γ reduced this shows that p_X is an isomorphism. Since p_Y is equidimensional, $\phi := p_Y \circ p_X^{-1} : X \to Y$ is equidimensional. □

4.4.9 Corollary. *In the situation of proposition 4.4.8 suppose that ϕ is flat or, equivalently, that Y is smooth. If V_2 is integrable, then conjecture 1.1.2 holds for X.*

Remark. The image of a smooth variety by a flat map is smooth and an equidimensional fibration between smooth varieties is flat, so the two conditions are equivalent.

Proof. Since $\dim X - \dim Y = 1$, the fibration ϕ has generically reduced fibres in codimension 1 by lemma 4.1.3. Since V_1 is integrable by corollary 3.2.5 and V_2 is integrable by hypothesis, lemma 3.2.6 shows that the conjecture holds. □

4.4.10 Proposition. *Let X be a projective manifold, and let $\phi : X \to Y$ be an equidimensional fibration of relative dimension 1 on a normal variety Y such that the general fibre F is a rational curve. Then ϕ is flat.*

Proof. Since X is Cohen-Macaulay and ϕ is equidimensional, it is sufficient to show that Y is smooth. We argue by induction on the relative Picard number $\rho(X/Y)$.

If $\rho(X/Y) = 1$, the anticanonical divisor $-K_X$ is ϕ-ample and the contraction is elementary, so by Ando's theorem ϕ is a $\mathbb{P}^1$-bundle or a conic bundle. In both cases ϕ is flat, so Y is smooth by [Mat89, Thm. 23.7].

Suppose now that $\rho(X/Y) > 1$. Since $-K_X \cdot F = 2$, the canonical divisor is not ϕ-nef. It follows from the relative contraction theorem [KMM87, Thm.4-1-1] that there exists an elementary contraction $\psi : X \to X'$ that is a Y-morphism, i.e. there exists a morphism $\phi' : X' \to Y$ such that $\phi = \phi' \circ \psi$. Since ϕ is equidimensional of relative dimension 1, it follows that all the ψ-fibres have dimension at most 1. Thus ψ is of fibre type of relative dimension 1 or of birational type.

Step 1. ψ is not of fibre type. We argue by contradiction and suppose that $\dim X = \dim X' + 1$. Then $\dim X' = \dim Y$, so ϕ' is a birational morphism. Since $\rho(X'/Y) = \rho(X/Y) - 1 > 0$, the map ϕ' is not an isomorphism, so there exists a fibre $\phi'^{-1}(y)$ of positive dimension. Since ψ is of fibre type, $\phi^{-1}(y) = \psi^{-1}(\phi'^{-1}(y))$ has dimension at least 2, a contradiction.

Step 2. Description of ψ. We know that ψ is a birational contraction such that all the fibres have dimension at most 1. Denote by E the exceptional locus.

By Wiśniewski's inequality [Wiś91] we have $\dim E + \dim G \geq \dim X$ for every positive-dimensional fibre G [Deb01, p.156]. Since all the fibres have dimension at most 1, it follows that the contraction is divisorial and all the fibres have dimension at most 1. By Ando's theorem we know that X' is smooth and X is the blow-up of X' along a smooth submanifold of codimension 2. Now $\rho(X'/Y) = \rho(X/Y) - 1$ and ϕ' is equidimensional of relative dimension 1 over Y, so by the induction hypothesis ϕ' is flat. Hence Y is smooth. $\square$

Remark. Let us note where the proof of proposition 4.4.10 uses the projectiveness of X or more precisely let us mention the statements we would have to show to generalise the argument to the compact Kähler case. In the Kähler case, an elementary contraction is a morphism with connected fibres $\phi : X \to Y$ from a compact Kähler manifold X to a normal Kähler variety such that $-K_X$ is ϕ-ample and $\rho(X/Y) = 1$.

1.) Let $\phi : X \to Y$ be an equidimensional fibration such that the general fibre is a rational curve. There exists an elementary contraction $\psi : X \to X'$ that is a Y-morphism.

2.) Let $\phi : X \to Y$ be an elementary contraction of fibre type of relative dimension 1. If ϕ is equidimensional, it is a $\mathbb{P}^1$-bundle or a conic bundle.

3.) Let $\phi : X \to Y$ be an elementary contraction of birational type such that all the fibres have dimension at most 1. Then Y is smooth and X is the blow-up of Y along a smooth submanifold of codimension 2.

We have now established all the tools that are necessary for the proofs of the two central theorems of this first part of the thesis.

4.4.11 Theorem. *Let X be a uniruled compact Kähler manifold such that $T_X = V_1 \oplus V_2$ and $\mathrm{rk} V_1 = 2$. Let F be a general fibre of the rational quotient map, then*

$$T_F = (T_F \cap V_1|_F) \oplus (T_F \cap V_2|_F).$$

Furthermore there are three possibilities:

1.) $T_F \cap V_1|_F = V_1|_F$. Suppose that V_1 is integrable or conjecture 1.1.6 holds for F, that is $T_F \cap V_1|_F$ or $T_F \cap V_2|_F$ is integrable. Then the manifold X admits the structure of an analytic fibre bundle $X \to Y$ such that $T_{X/Y} = V_1$. If V_2 is integrable, then conjecture 1.1.2 holds for X.

2.) $T_F \cap V_1|_F$ is a line bundle. There exists an equidimensional map $\phi : X \to Y$ such that the general ϕ-fibre M satisfies $T_M \subset V_1|_M$. If the map ϕ is flat and V_2 is integrable, then conjecture 1.1.2 holds for X.

3.) $T_F \subset V_2|_F$.

Proof. By corollary 4.4.4, the rational quotient map satisfies the ungeneric position property, so the general fibre F satisfies

$$T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F).$$

If $V_1|_F \cap T_F = V_1|_F$, proposition 4.4.5 shows that we are in the first case of the statement.

If $0 \subsetneq V_1|_F \cap T_F \subsetneq T_F$, the intersection has rank 1. Proposition 4.4.8 and corollary 4.4.9 show that we are in the second case of the statement.

If $V_1|_F \cap T_F = 0$, clearly $T_F = T_F \cap V_2|_F \subset V_2|_F$. □

4.4.12 Theorem. *Let X be a uniruled projective manifold such that $T_X = V_1 \oplus V_2$ and $\mathrm{rk} V_1 = 2$. Let F be a general fibre of the rational quotient map, then one of the following holds.*

1.) $T_F \cap V_1|_F \neq 0$. If V_1 and V_2 are integrable, then conjecture 1.1.2 holds.

2.) $T_F \cap V_1|_F = 0$. Then V_2 is integrable.

Proof. If $T_F \cap V_1|_F = V_1|_F$ we are in the first case of theorem 4.4.11. Since V_1 and V_2 are integrable, the conjecture holds by the same theorem.

If $0 \subsetneq T_F \cap V_1|_F \subsetneq T_F$, we are in the second case of theorem 4.4.11, so there exists an equidimensional fibration $X \to Y$ such that the general fibre is a rational curve. Since X is projective the map is flat by proposition 4.4.10, so we conclude with theorem 4.4.11.

If $0 = V_1|_F \cap T_F$, we are in the third case of theorem 4.4.11. Since X is projective, the integrability of V_2 and the pseudo-effectiveness of $\det V_1^*$ follow from theorem 2.2.1.

□

Chapter 5

Birational contractions in dimension 4

The content of this chapter is very different from the other ones, since we will not discuss fibre spaces but birational morphisms. In particular the general fibres will be points, so we can't hope to get any information from them. Instead we have to analyse directly the special fibres which is a much more arduous task. We will rely on classification relations for fourfolds to give a description of elementary contractions of birational type, thus completing the first part minimal model program for manifolds with split tangent bundle in dimension 4 (cf. proposition 5.1.2). The second part would be to show the abundance conjecture for these manifolds (cf. section 6.1), but this is out of the scope of our methods.

5.1 Birational geometry

Notation. Let $\phi : X \to Y$ be an elementary contraction of birational type. If the exceptional locus E of ϕ is irreducible, let $k := \dim E$ and $l := \dim \phi(E)$. The birational contraction is then said to be of type (k, l).

5.1.1 Proposition. *Let X be projective fourfold such that $T_X = V_1 \oplus V_2$ with $\mathrm{rk} V_1 = \mathrm{rk} V_2 = 2$, and let $\phi : X \to Y$ be an elementary contraction of birational type. Then Y is smooth and ϕ is the blow-up of a smooth 2-dimensional subvariety of the manifold Y. Furthermore the tangent bundle of Y splits as*

$$T_Y = (\phi_*(T\phi(V_1)))^{**} \oplus (\phi_*(T\phi(V_2)))^{**}.$$

Denote by $W_j := (\phi_*(T\phi(V_j)))^{**}$. If the universal covering $\mu : \tilde{Y} \to Y$ splits as $\tilde{Y} \simeq Y_1 \times Y_2$ such that $\mu^* W_j = p_{Y_j}^* T_{Y_j}$, then the analogous statement holds for X.

Proof. Let E be the exceptional locus of the contraction and F an irreducible component of a non-trivial fibre, then by proposition 4.3.1 we have

$\dim F \leq 2$. In particular the contraction can't be of type $(3,0)$. Contractions of type $(3,1)$ have been classified by Takagi [Tak99, p.316], the map $\phi|_E : E \to \phi(E)$ is a $\mathbb{P}^2$-bundle or quadric bundle, so the general fibre is reduced and isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a quadric cone. If ϕ is of type $(3,2)$, a fibre of dimension 2 is isolated, so F is reduced and either $\mathbb{P}^2$, $\mathbb{P}^2 \cup \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a quadric cone [AW97, Thm.4.6]. Elementary fourfold contractions such that the exceptional locus is not an irreducible divisor have been classified by Kawamata [Kaw89, Thm.1.1], the positive dimensional fibres are reduced and isomorphic to $\mathbb{P}^2$.

We will exclude case by case the existence of 2-dimensional fibres. The strategy is to choose appropriately rational curves $f : \mathbb{P}^1 \to F$ such that f^*T_F is nef so that we can use equation (4.2) to compute f^*T_X.

Case 1. $F \simeq \mathbb{P}^2$ or $F \simeq \mathbb{P}^2 \cup \mathbb{P}^2$.

If $F \simeq \mathbb{P}^2$, let $f : \mathbb{P}^1 \to F$ be a line $C = f(\mathbb{P}^1)$, then $f^*N^*_{F/X}$ is nef, so by equation (4.2)

$$f^*T_X = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b),$$

where $a \geq 0$, $b \geq 0$ and $a + b > 0$ (otherwise the deformations of C would cover X). Since the curve is of splitting type (4.1), we have up to renumbering $\mathcal{O}_{\mathbb{P}^1}(2) \subset f^*V_1$. Since C is a line in projective space it deforms keeping a point fixed. This shows that $\mathcal{O}_{\mathbb{P}^1}(2) \subset f^*V_1$ implies $T_{\mathbb{P}^2} = V_1|_{\mathbb{P}^2}$. In particular $f^*V_1 = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, so $f^*V_2 = \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$. This implies that $f^* \det V_2 = \mathcal{O}_{\mathbb{P}^1}(-a-b)$ is not ϕ-trivial, a contradiction to proposition 4.3.1. The same argument works in the case $F \simeq \mathbb{P}^2 \cup \mathbb{P}^2$.

Case 2. $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Choose a ruling line $f : \mathbb{P}^1 \to F$ with image $C = f(\mathbb{P}^1)$, then

$$f^*T_X = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$$

where $a \geq 0$ and $b \geq 0$ and $a + b > 0$. Up to renumbering, this implies $\mathcal{O}_{\mathbb{P}^1}(2) \subset f^*V_1$. By proposition 4.3.1 we see that $\det V_1$ is ϕ-ample and $\det V_2$ is ϕ-trivial. Now choose a line $f' : \mathbb{P}^1 \to F$ from the second ruling that is transversal to the first one, then

$$f'^*T_X = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$$

with coefficients as above. Then again $\mathcal{O}_{\mathbb{P}^1}(2) \subset f'^*V_1$, since otherwise $f'^*(c_1(V_1)) = 0$ (for details cf. the proof of [CP02, Lemma 1.3]), which would contradict the ϕ-ampleness of $\det V_1$. Since the lines are transversal, we obtain

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} = f^*T_F = f^*V_1.$$

As in the first case we obtain $f^*V_2 = \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$. Clearly $f^* \det V_2 = \mathcal{O}_{\mathbb{P}^1}(-a-b)$ is not trivial, a contradiction.

Case 3. F is a quadric cone.

Let $f : \mathbb{P}^1 \to F$ be a line passing through the vertex x of the cone F. Campana and Peternell have shown in [CP02, Thm.3.6] that $C = f(\mathbb{P}^1)$ is of splitting type (4.1), so up to renumbering, we have $\mathcal{O}_{\mathbb{P}^1}(2) \subset f^*V_1$. In particular $T_{C,x} \subset V_{1,x}$ for every such line. Since x is a singularity of F and the vector spaces $T_{C,x}$ generate the Zariski tangent space of F in x, they generate a subspace of dimension at least 3 in $V_{1,x}$. This contradicts $\mathrm{rk} V_1 = 2$.

We summarize: the morphism ϕ is a birational contraction of divisorial type such that all non-trivial fibres are of pure dimension 1. So the structure of ϕ follows from a result by Ando [And85, Thm.2.1].

Let F be a positive dimensional fibre, then $T_X|_F \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Up to renumbering $\det V_1$ is ϕ-ample and $\det V_2$ is ϕ-trivial, so $T_F \subset V_1|_F$ and $V_2|_F \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. The splitting of the tangent bundle T_Y is a consequence of lemma 3.2.4. Let E be the exceptional divisor, then the canonical map $\gamma : V_2|_E \to N_{E/X}$ is zero since its restriction to the positive dimensional fibres is given by $\gamma|_F : \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(-1)$, which is the zero map. It follows easily that $T_E = (V_1|_E \cap T_E) \oplus V_2|_E$ and $T_F \subset V_1|_F$ implies $T_{E/\phi(E)} = V_1|_E \cap T_E$, so we have $T_{\phi(E)} = W_2|_{\phi(E)}$.

Let now $\mu : \tilde{Y} \to Y$ be the universal covering map and suppose that $\tilde{Y} \simeq Y_1 \times Y_2$ and $\mu^* W_j = p_{Y_j}^* T_{Y_j}$. Note that this implies the integrability of W_1 and W_2. Since $\pi_1(X) = \pi_1(Y)$, the pull-back $\tilde{X} = X \times_Y \tilde{Y}$ is the universal covering of X.

We have seen that ϕ is the blow-up of Y along the W_2-leaf $\phi(E)$. It follows that $\tilde{\phi}$ is the blow-up of $\tilde{Y}$ along a $\mu^* W_2 = p_{Y_2}^* T_{Y_2}$-leaf. So there exists a $y \in Y$ such that

$$\tilde{X} \simeq \mathrm{Bl}_{y \times Y_2} \tilde{Y} \simeq \mathrm{Bl}_y Y_1 \times Y_2 =: X_1 \times X_2,$$

where $\mathrm{Bl}_A B$ denotes the blow-up of a manifold B along a submanifold A. Since

$$\tilde{\mu}^* V_j = \tilde{\phi}^* \circ \mu^* W_j = \tilde{\phi}^* p_{Y_j}^* T_{Y_j},$$

we have $\tilde{\mu}^* V_1 = p_{X_1}^* T_{X_1}$ and $\tilde{\mu}^* V_2 = p_{X_2}^* T_{X_2}$. $\square$

5.1.2 Proposition. *Let X be a projective fourfold such that $T_X = V_1 \oplus V_2$ where $\mathrm{rk} V_1 = \mathrm{rk} V_2 = 2$. There exists a birational morphism $\phi : X \to Y$ to a smooth projective fourfold such that the tangent bundle of Y splits as*

$$T_Y = (\phi_*(T\phi(V_1)))^{**} \oplus (\phi_*(T\phi(V_2)))^{**}.$$

and Y satisfies one of the following:

1.) The canonical divisor K_Y is nef.

2.) The variety Y admits an elementary contraction of fibre type.

If V_1 and V_2 are integrable, conjecture 1.1.2 holds for X if it holds for Y.

Proof. If the canonical divisor of X is nef or X admits an elementary contraction of fibre type, there is nothing to show. So suppose that none of the two holds, then X admits an elementary contraction of birational type $\psi : X \to X_1$. Proposition 5.1.1 shows that X_1 is smooth and

$$T_{X_1} = (\psi_*(T\psi(V_1)))^{**} \oplus (\psi_*(T\psi(V_2)))^{**}.$$

If V_1 and V_2 are integrable, the vector bundles $(\psi_*(T\psi(V_1)))^{**}$ and $(\psi_*(T\psi(V_2)))^{**}$ are integrable by corollary 2.1.3,4. So if conjecture 1.1.2 holds for X_1, it holds for X by proposition 5.1.1. If the canonical divisor of X_1 is nef or X_1 admits an elementary contraction of fibre type, we set $Y := X_1$ and the program stops, otherwise we have another contraction of birational type $X_1 \to X_2$ and proceed inductively. Since

$$\rho(X) > \rho(X_1) > \rho(X_2) > \ldots \geq 1,$$

the program stops after finitely many steps. □

Chapter 6

Non-uniruled manifolds

It has been shown in theorem 2.2.1 that for projective manifolds the direct factors of a split tangent bundle are always integrable. This suggests that (analytic) foliation theory is the best approach to conjecture 1.1.2 in this case. In this chapter we take a different stance and show how ungeneric position properties yield rich structure results for non-uniruled manifolds with split tangent bundle. A positive answer to conjecture 1.1.2 would not directly imply these results, this makes them interesting in their own right.

6.1 Iitaka fibrations

We have shown in proposition 3.2.8 that the Albanese map always satisfies the ungeneric position property. We now turn our attention to another important fibration in classfication theory and ask if the Iitaka fibration also satisfies the ungeneric position property. We will obtain a partial positive answer (proposition 6.1.6), but the general case remains open.

The critical reader may point out that the existence of the Iitaka fibration as a holomorphic map is a rather restrictive condition on X. This is certainly true, but we have shown in proposition 5.1.2 that in order to show the conjecture 1.1.2 dimension 4, it is sufficient to do this for Mori fibre spaces and minimal varieties. If the abundance conjecture holds in dimension 4 (which in the projective case is widely believed to be true), minimal fourfolds admit a holomorphic Iitaka fibration. This explains our interest in this class of fibrations although the picture is far from being complete.

6.1.1 Definition. *Let X be a compact Kähler manifold, and let ω_X be the canonical bundle. Suppose that ω_X is semiample, i.e. for some $m \in \mathbb{N}$, the sheaf $\omega_{\otimes m}$ is globally generated, the fibration $\phi : X \to Y$ induced by ω_X is then called the Iitaka fibration.*

Remark. Let F be a general fibre of the Iitaka fibration, then $\omega_F \simeq \omega_X|_F$, so there exists a positive integer m such that $\omega_F^{\otimes m} \simeq \mathcal{O}_F$. The Bogomolov-

Beauville decomposition theorem then gives precise information on the structure of F. Before we can state the theorem we recall two definitions.

6.1.2 Definition. *A compact Kähler manifold X is a Calabi-Yau manifold if it is simply connected, the canonical bundle ω_X is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.*

6.1.3 Definition. *[Bea96, Prop.5.4] A symplectic structure on a complex manifold X is a holomorphic 2-form ω that is non-degenerate in every point. A simply connected compact Kähler manifold is symplectic if it admits a symplectic structure. It is irreducible symplectic if the symplectic structure is unique up to a scalar.*

Remark. An irreducible symplectic manifold X satisfies

$$H^0(X, \wedge^{2k+1}\Omega_X) = 0 \qquad \forall\ k \geq 0,$$
$$H^0(X, \wedge^{2k}\Omega_X) \simeq \mathbb{C}[\omega]^k \qquad \forall\ k > 0,$$

where $[\omega]$ is the cohomology class of a symplectic form.

6.1.4 Theorem. *[Bea96, Thm.2] Let X be a compact Kähler manifold such that $c_1(\omega_X) = 0$ in $H^2(X, \mathbb{R})$. Then there exists a finite étale covering $\mu : X' \rightarrow X$ such that*

$$X' \simeq A \times Y_1 \times \ldots \times Y_k \times Z_1 \times \ldots \times Z_l$$

where A is a complex torus, the varieties Y_i are Calabi-Yau manifolds and the Z_j irreducible symplectic manifolds (we may have $\dim A = 0$ or $k = 0$ or $l = 0$). The decomposition is unique up to order.

6.1.5 Lemma. *Let X be a Calabi-Yau or irreducible symplectic manifold. Then X does not admit a split tangent bundle.*

Proof. We argue by contradiction and suppose that $T_X = V_1 \oplus V_2$. Every compact Kähler manifold with numerically trivial canonical bundle is a Kähler-Einstein manifold, so by [Bea00] the universal covering of X decomposes into a product. Since X is simply connected, this means that we have a decomposition

$$X \simeq X_1 \times X_2,$$

where $\dim X > \dim X_1 > 0$. Since $\mathcal{O}_X \simeq \omega_X \simeq p_{X_1}^* \omega_{X_1} \otimes p_{X_2}^* \omega_{X_2}$, the manifolds X_1 and X_2 have trivial canonical line bundle. This contradicts the uniqueness of the decomposition in the decomposition theorem 6.1.4. □

The next proposition is the central ungeneric position result in this chapter; it is a first application of lemma 3.2.7.

6.1.6 Proposition. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Suppose that ω_X is semiample and $0 < \kappa(X) < \dim X$. Let $\phi : X \rightarrow Y$ be the Iitaka fibration. Furthermore let*

$$F' \simeq A \times Y_1 \times \ldots \times Y_k \times Z_1 \times \ldots \times Z_l$$

be the Beauville decomposition $\mu : F' \to F$ of a general fibre F. Then for every $r \in \{1, \ldots, k\}$ (resp. for $s \in \{1, \ldots, l\}$), we have

$$p_{Y_r}^* T_{Y_r} \subset \mu^* V_j$$

(resp. $p_{Z_s}^* T_{Z_s} \subset \mu^* V_j$) for $j = 1$ or 2.

Proof. Fix $r \in \{1, \ldots, k\}$ (resp. $s \in \{1, \ldots, l\}$), then $F' \simeq U \times V$ where $V = Y_r$ (resp. $V = Z_s$). Since V is a Calabi-Yau manifold (resp. an irreducible symplectic manifold) the tangent bundle T_V does not split (lemma 6.1.5) and $b_1(V) = 0$. Furthermore the action of $\Gamma := \pi_1(F)$ on the direct factors of the Beauville decomposition is diagonal, so the projection $p_U : F' \to U$ descends to a fibration $\psi : F \simeq F'/\Gamma \to U/\Gamma$. Let G be a general ψ-fibre, then we can restrict the covering μ to a p_U-fibre $\{u\} \times V \simeq V$ to obtain an étale covering $\mu|_V : V \to G$, so $b_1(G) = 0$. We apply lemma 3.2.7 and obtain

$$T_G = (V_1|_G \cap T_G) \oplus (V_2|_G \cap T_G).$$

Since $\mu|_V : V \to G$ is étale, this implies

$$T_V = (\mu|_V^* V_1 \cap T_V) \oplus (\mu|_V^* V_2 \cap T_V).$$

Since the tangent bundle T_V does not split, one of the intersections $\mu|_V^* V_j \cap T_V$ equals zero. It follows that up to renumbering $T_V \subset \mu|_V^* V_1 \cap T_V$. Since G is a general fibre, this implies $T_{F'/U} = p_V^* T_V \subset \mu^* V_1$. □

The preceeding proposition shows that if the general fibre of the Iitaka fibration is not covered by a torus, then the non-abelian factors of the Beauville decomposition satisfy a very strong ungeneric position property. In particular we have the following

6.1.7 Corollary. *In the situation of proposition 6.1.6, suppose that for some $r \in \{1, \ldots, k\}$ (resp. for $s \in \{1, \ldots, l\}$), the inclusion $p_{Y_r}^* T_{Y_r} \subset \mu^* V_j$ (resp. $p_{Z_s}^* T_{Z_s} \subset \mu^* V_j$) is an equality. Then there exists an almost smooth map $\psi : X \to M$ such that the geometric fibres are V_j-leaves. All the V_j-leaves have the same uniformisation.*

Proof. Up to renumbering we may suppose that $\mu^* V_1 = p_{Y_1}^* T_{Y_1}$. This shows the general $V_1|_F$-leaf is compact. Since F is general, this implies that the general V_1-leaf is compact. The statement follows from 2.4.3. □

Still we can say nothing about the general case remains, so we formulate this as an open

6.1.8 Problem. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Suppose that ω_X is semiample and $0 < \kappa(X) < \dim X$. Let $\phi : X \to Y$ be the Iitaka fibration. Does ϕ satisfy the ungeneric position property ?*

It is clear that our techniques can't be used to solve this problem, since a torus does not have first Betti number equal to zero and the tangent bundle always splits. In order to give an affirmative answer it is probably necessary to look not only at the general fibre, but to use the global construction of the Iitaka fibration, just as we did for the Albanese map.

6.2 An example

We will now focus our attention on the description of the Iitaka fibration of a fourfold with Kodaira dimension 1. In this case the general fibre is covered by a Calabi-Yau threefold or a product of a K3 surface and an elliptic curve or a torus. The first case is easy and treated in corollary 6.2.1, the other two cases are more difficult. We will have to make more precise the results from the example introduced in section 3.3 for the special case of an Iitaka fibration to obtain a good description of the structure of the fibration.

6.2.1 Corollary. *Let X be a compact Kähler fourfold such that $T_X = V_1 \oplus V_2$. Suppose that ω_X is semiample and $\kappa(X) = 1$. Let $\phi : X \to Y$ be the Iitaka fibration and suppose that the general fibre F is Calabi-Yau. Then, up to renumbering,* $\mathrm{rk} V_1 = 3$*, and the set-theoretical fibres are V_1-leaves. In particular ϕ is almost smooth.*

Proof. Let $\mu : F' \to F$ be the Beauville decomposition, then by proposition 6.1.6, we have up to renumbering $T_{F'} \subset \mu^* V_1$. Since $\mathrm{rk} V_1 \leq 3$ the inclusion is an equality, so the generic V_1-leaves are the ϕ-fibres. The corollary 6.1.7 yields a map $\psi : X \to C'$ such that the geometric fibres are the V_1-leaves and there exists a birational map $C' \to C$ that factors ϕ through ψ. A birational map between smooth curves is an isomorphism, so ϕ identifies to ψ.$\square$

6.2.2 Proposition. *Let X be a compact Kähler manifold with split tangent bundle $T_X = V_1 \oplus V_2$ such that* $\mathrm{rk} V_2 = 2$*. Suppose that $\kappa(X) = 1$ and ω_X is semiample, and let $f : X \to C$ be the Iitaka fibration. Suppose that $f^* \Omega_C \subset V_2^*$. Let F be a set-theoretical fibre that is not smooth and denote by $\nu : F' \to F$ the normalisation of F. Let F_j be an irreducible component of F', then F_j is uniruled and admits the structure of a $\mathbb{P}^1$-bundle $\phi : F_j \to Y_j$ such that $\phi^* T_{Y_j} \simeq (\nu^* V_1)_{F_j}$. Furthermore the canonical bundle of Y_j is numerically trivial.*

Proof. Let $m \in \mathbb{N}$ be a natural number such that $\omega_X^{\otimes m}$ is globally generated. By the Baum-Bott theorem (cf. lemma 2.2.5) we have $c_1(V_j^*) \in H^1(X, V_j^*)$, so the decomposition $mc_1(\omega_X) = mc_1(V_1^*) + mc_1(V_2^*)$ is the unique decomposition of $mc_1(\omega_X)$ induced by

$$H^1(X, \Omega_X) = H^1(X, V_1^*) \oplus H^1(X, V_2^*).$$

Since $\omega_X^{\otimes m} \simeq f^* A$ with A a line bundle on C, we have $mc_1(\omega_X) \in H^1(X, f^* \Omega_C)$. The inclusion $f^* \Omega_C \subset V_2^*$ implies that the canonical morphism

$$H^1(X, f^* \Omega_C) \to H^1(X, \Omega_X) \to H^1(X, V_1^*)$$

is zero. It follows that $mc_1(V_1^*) = 0$, so $\det V_1^*$ is numerically trivial.

Let F be a set-theoretical fibre that is not smooth and let F_j be an irreducible component of the normalisation of F. We suppose without loss of generality that $j = 1$ and denote by $\nu : F_1 \to F$ the restriction of the normalisation to F_1.

By theorem 3.3.3 we know that F_1 is smooth and $T_{F_1} = (\nu^* V_1)|_{F_1} \oplus L_1$ where L_1 is a line bundle. The fibre F being an effective divisor, we can write

$$F = \nu(F_1) + F',$$

where F' is an effective divisor or $\nu(F_1)$ is singular. By the adjunction formula

$$\omega_{\nu(F_1)} \simeq \omega_X(F_1) \otimes \mathcal{O}_{\nu(F_1)} \simeq \omega_X(F - F') \otimes \mathcal{O}_{\nu(F_1)}$$

and by subadjunction [Rei94, Prop.2.4] we have

$$\omega_{F_1} \simeq \nu^* \omega_{\nu(F_1)} \otimes \mathcal{O}_{F_1}(-E) \simeq \nu^*(\omega_X(F - F') \otimes \mathcal{O}_{\nu(F_1)}) \otimes \mathcal{O}_{F_1}(-E)$$

where E is an effective divisor defined by the pull-back of the conductor ideal. The fibre F is not smooth, so $F - F'$ is not zero (this means that F has at least two irreducible components) or E is not zero. It follows that the divisor corresponding to $\nu^* \mathcal{O}_{\nu(F_1)}(F - F') \otimes \mathcal{O}_{F_1}(-E)$ is strictly antieffective. Since $\nu^*(\omega_X|_{\nu(F_1)})$ is numerically trivial this implies that ω_{F_1} is not pseudoeffective. The decomposition $T_{F_1} = (\nu^* V_1)|_{F_1} \oplus L_1$ yields a decomposition $c_1(\omega_{F_1}) = c_1(\nu^* V_1) + c_1(L_1)$. We have seen that $\det V_1$ is numerically trivial, so $c_1(\nu^* V_1) \equiv 0$. Hence L_1 is not pseudo-effective. By [BPT04] this shows that F_1 admits the structure of a $\mathbb{P}^1$-bundle $\phi : F_1 \to Y_1$ such that $T_{F_1/Y_1} = L_1$. Hence $\phi^* T_{Y_j} \simeq (\nu^* V_1)_{F_j}$, the numerical triviality of $\det V_1$ implies that Y_1 has a numerically trivial canonical bundle. □

6.2.3 Corollary. *In the situation of proposition 6.2.2, suppose that* $\dim X = 4$*. Then all the* V_1*-leaves have the same universal covering.*

Proof. By theorem 3.3.3 we know that all the V_1-leaves contained in the same fibre have the same universal covering, so we only have to show that the universal covering does not change with the fibre.

Suppose first that the general ϕ-fibre F is not covered by a torus. Since $\dim F = 3$ and the tangent bundle splits as $T_F = V_1|_F \oplus (V_2|_F \cap T_F)$, the fibre is covered by a product $\mu : S \times E \to F$ where S is a K3 surface and E an elliptic curve. By 6.1.6 we have $\mu^* V_1 = p_S^* T_S$. We conclude with corollary 6.1.7.

Suppose that the general ϕ-fibre F is covered by a torus $\mu : A \to F$. Since the tangent bundle splits as $T_F = V_1|_F \oplus (V_2|_F \cap T_F)$, the direct factor $\mu^* V_1 \subset T_A$ is trivial and the V_1-leaves in F are covered by $\mathbb{C}^2$. We argue now by contradiction and suppose that there exists a set-theoretical fibre F such that the V_1-leaves in F are not covered by $\mathbb{C}^2$. Then F is not smooth (if the fibre is multiple we can perform a local base change $C' \subset C$ such that the corresponding fibre of $X' := (X \times_C C')_{\text{norm}}$ is smooth). Let F_1 be an irreducible component of the normalisation of F and denote by $\nu : F_1 \to F$ the restriction of the normalisation to F_1. By proposition 6.2.2 the variety F_1 is a $\mathbb{P}^1$-bundle $\phi : F_1 \to Y_1$ such that $\phi^* T_{Y_1} = (\nu^* V_1)_{F_1}$ and Y_1 is smooth with trivial canonical bundle. By the Ehresmann theorem 2.4.1, the $\nu^* V_1$-leaves and Y_1 have the same universal covering. Since the $\nu^* V_1$-leaves are not uniformised by $\mathbb{C}^2$, the surface Y_1 is not abelian or hyperelliptic. So it is a K3 or Enriques surfaces, in particular the

$(\nu^* V_1)_{F_1}$-leaves are uniformised by K3 or Enriques surfaces. This shows that the V_1-leaves in $\nu(F_1)$ are compact and have finite fundamental group. By Reeb's local stability theorem the general V_1-leaf has finite fundamental group. This contradicts the fact that the general V_1-leaf is uniformised by $\mathbb{C}^2$. $\square$

Remark. In the situation of proposition 6.2.2 all the set-theoretical fibres are smooth or there exists a compact V_1-leaf. Indeed if there exists a set-theoretical fibre F that is not smooth, the singular locus of F is a union of V_1-leaves, which necessarily are compact. We could not figure out if in this case the holonomy representation of the fundamental group has finite image. By Reeb's local stability theorem and global stability this would imply that all the V_1-leaves are compact, which is of course much better than the preceeding result.

6.3 Irregular varieties

In this section we give a refinement of proposition 3.2.8. In fact the ungeneric position property $T_F = (V_1|_F \cap T_F) \oplus (V_2|_F \cap T_F)$ is often too weak to obtain precise structure results, so one tries to produce fibrations such that $T_F = V_1|_F \oplus (V_2|_F \cap T_F)$. For irregular varieties this is sometimes possible by the generalized Castelnuovo-de Francis theorem.

6.3.1 Definition. *Let X be a compact Kähler manifold. Then X is said to be irregular if $q(X) := h^1(X, \mathcal{O}_X) > 0$. Let X be a compact Kähler manifold, and let $\alpha : X \to Alb(X)$ be the Albanese map. The Albanese dimension $a(X)$ is the dimension of $\alpha(X)$. The manifold X is Albanese general if it has maximal Albanese dimension $\dim X = a(X)$ and $q(X) > \dim X$.*

A normal variety X is Albanese general if a desingularisation of X is Albanese general.

Remark. It is easy to see that a compact Kähler manifold X is Albanese general if and only the cotangent bundle Ω_X is generically generated by its global sections and $q(X) > \dim X$.

If $X' \to X$ is a desingularisation of a normal variety that is Albanese general, the manifold X' is Albanese general. To see this, let $X'' \to X$ be a desingularisation such that X'' is Albanese general, and let X''' a desingularisation dominating X' and X''. By what preceeds it is clear that X''' is Albanese general. Since X' is smooth it has only rational singularities, so $h^1(X', \mathcal{O}_{X'}) = h^1(X''', \mathcal{O}_{X'''})$. Since $\Omega_{X'''}$ is generically generated by global sections this holds for $\Omega_{X'}$. This shows that X' is Albanese general.

6.3.2 Theorem. *(Generalised Castelnuovo-de Francis or Catanese's theorem) [Cat91, Thm.1.14] Let X be a compact Kähler manifold and let $U \subset H^0(X, \Omega_X)$ be a $(k+1)$-dimensional strict k-wedge subspace, i.e. a subspace with a basis given by forms $\omega_1, \ldots, \omega_{k+1}$, and such that $\omega_1 \wedge \ldots \wedge \omega_{k+1} = 0$ while $\wedge^k(U)$ embeds into $H^0(X, \Omega_X^k)$ under the natural homomorphism.*

Then there exists a holomorphic map $\phi : X \to Y$ to a k-dimensional normal variety such that $U \subset \phi^*(H^0(Y, \Omega_Y))$. The variety Y is Albanese general.

6.3.3 Corollary. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. If $H^0(X, V_2^*)$ contains a strict k-wedge subspace, then there exists a fibration $\phi : X \to Y$ such that*

$$\phi^*\Omega_Y/(\mathrm{Tor}\,\phi^*\Omega_Y) \subset V_2^*$$

In particular ϕ satisfies the ungeneric position property.

Proof. Denote by $U \subset H^0(X, V_2^*)$ a strict k-wedge subspace of dimension $k+1$, then theorem 6.3.2 implies the existence of a holomorphic map $\phi : X \to Y$ such that Y has dimension k and $U \subset \phi^* H^0(Y, \Omega_Y)$. Since U is a strict k-wedge subspace, it generates a subsheaf $\mathcal{S} \subset \phi^*\Omega_Y$ of rank $k = \dim Y$. In particular there exists a non-empty Zariski open subset $X^* \subset X$ such that $\mathcal{S}|_{X^*} = \phi^*\Omega_Y|_{X^*}$. Since $U \subset H^0(X, V_2^*)$, we have $\mathcal{S} \subset V_2^*$. It follows that the restriction of $\gamma : \phi^*\Omega_Y \to V_2^*$ has rank k on the dense set X^*. This implies the result. □

Example. Let X be a surface with split tangent bundle $T_X = V_1 \oplus V_2$. If $h^0(X, V_1^*) > 1$ and $h^0(X, V_2^*) > 1$, the splitting of T_X is unique. Since V_2 has rank 1 and $h^0(X, V_2^*) > 1$, it follows by corollary 6.3.3 that there exists a fibration $\phi_2 : X \to C_2$ such that a general fibre satisfies $T_F = V_1|_F$. Since $h^0(X, V_1^*) > 1$, the general fibre has higher genus. It follows that

$$H^0(F, \mathcal{H}om(V_2|_F, V_1|_F)) \simeq H^0(F, T_F) = 0,$$

so $h^0(X, \mathcal{H}om(V_2, V_1)) = 0$. An analogous argument shows that $h^0(X, \mathcal{H}om(V_1, V_2)) = 0$, so we conclude with lemma 3.1.2. In particular example 3.1.4 does not generalize to a product of higher genus curves.

The next lemma gives a condition for the existence of a strict k-wedge that is easy to verify.

6.3.4 Lemma. *([Cat91, Lemma 2.5]) Let U be a vector subspace of $H^0(X, \Omega_X)$. Then U is said to be a k-wedge subspace if $\Lambda^k U$, the natural image of $\Lambda^k(U)$ into $H^0(X, \Omega_X^k)$ is nonzero, whereas the dimension of U is at least $k+1$ and $\Lambda^{k+1}U = 0$.*

Let U be a k-wedge subspace: then there is an integer $k' \leq k$ and $U' \subset U$ such that U' is a strict k'-wedge subspace.

We illustrate the technique with an example.

6.3.5 Proposition. *Let X be a non-uniruled compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Suppose that $\mathrm{rk} V_2 = 2$ and $h^0(X, V_2^*) > 2$. Then there exists a fibration $\phi : X \to Y$ such that one of the following holds:*

1.) $\dim Y = 1$. *The ϕ-fibres are V_1-saturated and all the V_1-leaves contained in the same fibre have the same universal covering.*

2.) $\dim Y = 2$. *The ϕ-fibres are V_1-leaves. All the V_1-leaves have the same universal covering.*

Proof.

Since $h^0(X, V_2^*) > \mathrm{rk} V_2^*$ one sees easily that $H^0(X, V_2^*) \subset H^0(X, \Omega_X)$ is a 2- or 1-wedge subspace in the sense of lemma 6.3.4. By the same lemma $H^0(X, V_2^*)$ contains a strict k-wedge subspace with $k = 1$ or 2.

If $k = 1$, corollary 6.3.3 implies the existence of a fibration $\phi : X \to Y$ on a smooth projective curve Y such that $\phi^* \Omega_Y \subset V_2^*$. The statement follows from theorem 3.3.3.

If $k = 2$, corollary 6.3.3 implies the existence of a fibration $\phi' : X \to Y'$ such that $\phi'^* \Omega_{Y'} / (\mathrm{Tor}\, \phi'^* \Omega_{Y'}) \subset V_2^*$. Since $\dim Y' = \mathrm{rk} V_2^*$, we have equality on the preimage $\phi'^{-1}(Y'^*)$ of the ϕ'-smooth-locus $Y'^* \subset Y'$. By duality this implies that a general fibre F satisfies $T_F = V_1|_F$. The existence of ϕ and the universal covering of the V_1-leaves then follows from corollary 2.4.3. □

In order to move further on in this direction, one would have to study the cases where $0 < h^0(X, V_2^*) \leq \mathrm{rk} V_2$. Some cases will be covered by the generalized Castelnuovo-de Francis theorem but others not. In these cases it is interesting to analyse the structure of the image of the Albanese map $\alpha(X)$ (or the base of the Stein factorisation of the Albanese map) via the results due to Ueno [Uen75, Thm.10.9] and Kawamata [Kaw81, Thm.13]. The following example gives an idea of one of the tricky cases that might arise in this study.

6.3.6 Example. Let X_1 be an elliptic surface of Kodaira dimension one such that $h^0(X_1, T_{X_1}) > 0$ and the Albanese map $X_1 \to \mathrm{Alb}(X_1)$ is a ramified covering (such surfaces can be constructed as an étale quotient of a product $E \times C$ where E is an elliptic and C a higher genus curve). Set $X_2 := X_1$. The Albanese map of $X = X_1 \times X_2$ is then a ramified covering. Furthermore

$$H^0(X_1 \times X_2, \mathcal{H}om(p_{X_1}^* \Omega_{X_1}, p_{X_2}^* \Omega_{X_2})) = H^0(X_1 \times X_2, p_{X_1}^* T_{X_1} \otimes p_{X_2}^* \Omega_{X_2}) = H^0(X_1, T_{X_1}^{\oplus 2})$$

is not zero, so lemma 3.1.2 shows that there are „a lot of“ embeddings $p_{X_1}^* \Omega_{X_1} \hookrightarrow \Omega_X$ with image V_1^* such that

$$V_1^* \oplus p_{X_2}^* \Omega_{X_2} = \Omega_X.$$

For a general choice of V_1^*, the leaves of $V_2 = \ker(\Omega_X \to V_1^*)$ will not be compact. By symmetry the same can be done for the leaves of V_1. In total we have constructed a product $X = X_1 \times X_2$ with a split tangent bundle $T_X = V_1 \oplus V_2$ such that $V_j \simeq p_{X_j}^* T_{X_j}$, but the V_j-leaves are not compact. It should be difficult to recover the product structure of X if we only have $T_X = V_1 \oplus V_2$.

The example shows the main flaw of our approach: if the positivity properties of the direct factors of the splitting are not strong enough, there is no reason for the leaves to be compact/algebraic. Thus algebro-geometric techniques will probably fail. For uniruled manifolds the situation is totally different and we very often encounter foliations with compact leaves.

Part II

Direct images of adjoint line bundles

Chapter 7

Introduction to Part II

7.1 Main results

Given a fibration $\phi : X \to Y$, that is a projective morphism with connected fibres, it is a natural und fundamental problem to try to relate positivity properties of the total space X, the base Y and the general fibre F. The most important problem in this context is Iitaka's $C_{n,m}$-conjecture that predicts the subadditivity of the Kodaira dimension. More precisely if we denote by $\kappa(.)$ the Kodaira dimension of a projective manifold, the $C_{n,m}$-conjecture states that

$$\kappa(X) \geq \kappa(Y) + \kappa(F).$$

It turns out that one of the main issues in the proof of this conjecture is a discussion of the positivity of the direct image sheaves $\phi_*(\omega_{X/Y}^{\otimes m})$ where m is a sufficiently large integer. Building up on one of the landmark papers due to Kawamata [Kaw81, Kaw82], this question has been settled by Viehweg in a series of great articles [Vie82, Vie83]. He introduced the notion of weak positivity that is particularly well-adapted to this kind of questions: a torsion-free coherent sheaf $\mathcal{F}$ is weakly positive if there exists an ample line bundle H such that for every positive integer $\alpha \in \mathbb{N}$ there exists some $\beta \in \mathbb{N}$ such that $(\mathrm{Sym}^{\beta\alpha} \mathcal{F})^{**} \otimes H^{\beta}$ is generated in the general point. One of his main results is the

7.1.1 Theorem. *[Vie82] Let $\phi : X \to Y$ be a fibration between projective manifolds. Then for all $m \in \mathbb{N}$, the direct image sheaf $\phi_*(\omega_{X/Y}^{\otimes m})$ is weakly positive.*

For further applications, for example in the context of moduli spaces for polarized manifolds (compare [Vie95]), it is important to produce more general versions of this theorem. Given a fibration $\phi : X \to Y$ and a line bundle L on X, one can ask for the positivity of the direct image $\phi_*(L \otimes \omega_{X/Y})$. A moment of reflection will convince the reader that it is hopeless to ask such a question for a line bundle L that is not itself positive in some sense (e.g.

ample, nef, weakly positive,...). In this situation vanishing theorems for adjoint line bundles $L \otimes \omega_X$ replace the Hodge-theoretic approach that was the starting point of Kawamata's results [1]. An example for a result in this context of adjoint line bundles is a theorem that was probably known by specialists and written down by C. Mourougane in his thesis.

7.1.2 Theorem. *[Mou97, Thm.1] Let $\phi : X \to Y$ be a smooth fibration between projective manifolds, and let L be a nef and ϕ-big line bundle on X. Then $\phi_*(L \otimes \omega_{X/Y})$ is locally free and nef.*

The aim of this work is to generalise this result in different directions. First and foremost is to show an analogous result for a fibration that is flat, but not necessarily smooth. Secondly we would like to do this for a fibration between projective varieties that are not smooth. Note that this immediately raises the question about the existence of a relative dualising sheaf $\omega_{X/Y}$. Thirdly we would like to weaken or change the positivity hypothesis on L. In particular we might encounter situations where $\phi_*(L \otimes \omega_{X/Y})$ is not locally free. We summarize this in the following

7.1.3 Problem. *(The direct image problem) Let $\phi : X \to Y$ be a flat morphism for X a projective Cohen-Macaulay scheme over $\mathbb{C}$ and let L be a line bundle over X that satisfies certain positivity properties (e.g. nef and ϕ-big). Denote by $\omega_{X/Y}$ the relative dualising sheaf and $L \otimes \omega_{X/Y}$ the adjoint sheaf. What can we say about the positivity properties of the direct image sheaf $\phi_*(L \otimes \omega_{X/Y})$?*

It is clear that one can't expect to obtain weak positivity of the direct image sheaf in every imaginable situation, in particular strange things are bound to happen for non-reduced schemes or varieties with very bad singularities. Nevertheless I think that it is preferable to start with this general setting and then try to figure out the obstructions to giving a positive answer. We will realize this program with a series of counterexamples that show the optimality of the results.

One of the first observations is that the proof of Mourougane's theorem can be easily adapted as long as we suppose the varieties to be Gorenstein with at most canonical singularities. This is no longer true if we consider $\mathbb{Q}$-Gorenstein varieties with at most canonical singularities, which is the natural class of varieties for problems in Mori theory. As a general rule of (mathematical) life, you have to work much harder if you pass from Gorenstein to $\mathbb{Q}$-Gorenstein varieties and so it is here. The strategy remains the same, but every step comes with an extra twist due to the fact that the relative dualising sheaf is not locally free. We end up with the rather technical, but very useful lemma 10.2.4 that reduces the direct image problem 7.1.3 to the following, probably easier

[1] Using cyclic coverings, one could still use Kawamata's approach. Nevertheless this would force us to restrict ourselves to line bundles L such that some multiple has sections, so we prefer the approach via vanishing theorems.

7.1.4 Problem. *(The extension problem) Let $\phi : X \to Y$ be a fibration from a projective manifold to a normal projective variety Y. Fix a very ample line bundle H on Y. Let L be a line bundle on X that satisfies certain positivity properties (e.g. nef and ϕ-big). Denote by ω_X the canonical sheaf, and let $L \otimes \omega_X$ be the adjoint sheaf. Is the coherent sheaf $\phi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1}$ generically generated by global sections ?*

An application of the relative Kawamata-Viehweg vanishing theorem allows us to give an affirmative answer to this problem for a line bundle that is nef and relatively big. This implies our first main result.

10.3.2 Theorem. *(Nef and relatively big version) Let X be a normal $\mathbb{Q}$-Gorenstein variety with at most canonical singularities, and let Y be a normal $\mathbb{Q}$-Gorenstein variety. Let $\phi : X \to Y$ be a flat fibration, and let L be a nef and ϕ-big line bundle on X. Then $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

If ϕ is generically smooth, then $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive on the ϕ-smooth locus.

Problem 7.1.4 gets much more delicate if we do not suppose that the line bundle is relatively big. In this case there is no analogue of the Kawamata-Viehweg theorem that would allow to vanish the cohomology on the total space X. Instead one has to show a vanishing result for the cohomology of some direct image sheaf on Y which is a much harder task. The key result in this direction is due to Kollár [Kol86] and shows that for any fibration $\phi : X \to Y$, and any ample line bundle A on Y, we have

$$H^i(Y, \phi_*\omega_X \otimes A) = 0 \qquad \forall i > 0.$$

In section 9.4 we will establish twisted versions of this theorem for the adjoint line bundle. This allows us to treat the case where L is semiample.

10.3.2 Theorem. *(Semiample version) Let X be a normal $\mathbb{Q}$-Gorenstein variety with at most canonical singularities, and let Y be a normal $\mathbb{Q}$-Gorenstein variety. Let $\phi : X \to Y$ be a flat fibration, and let L be a semiample line bundle on X. Then $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

If ϕ is generically smooth, then $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive on the ϕ-smooth locus.

One should note that a considerable amount of work has been done on both the direct image problem 7.1.3 and the extension problem 7.1.4. In particular Viehweg's book on moduli spaces [Vie95] contains a considerable amount of very precise results under various conditions. Unfortunately these statements come along with some technical conditions and are not easy to apply. To my best knowledge this is the first time that these theorems are stated and shown in this way.

The theorems 10.3.2 and 10.3.2 can nevertheless not be fully satisfactory, since L semiample is a condition on the positivity of L in *every* point of X, while $\phi_*(L \otimes \omega_{X/Y})$ weakly positive is a positivity property in a *general* point.

Counterexamples show that it is not possible to obtain that $\phi_*(L\otimes\omega_{X/Y})$ is nef, so we try to weaken the hypothesis. One possible generalisation is to consider line bundles with non-negative Kodaira dimension, but an easy counterexample (proposition 11.5.1) shows that this property is not strong enough to obtain weak positivity. Therefore we have to introduce the asymptotic multiplier ideal of the line bundle.

Suppose that L has Kodaira dimension d, and let $N \in \mathbb{N}$ be a sufficiently high and divisible integer such that the linear system $|L^{\otimes N}|$ induces a rational map $\phi : X \dashrightarrow Y$ on a normal variety Y of dimension d. If L is not semiample, this map will never be a morphism, but we can resolve the indeterminacies by blowing-up $\mu : X' \to X$. Then

$$\mu^* L^{\otimes N} \otimes \mathcal{O}_{X'}(-D) \simeq M,$$

where D is an effective divisor and M is semiample. Then we can define

$$\mathcal{I}(\frac{1}{N}|L^{\otimes N}|) := \mu_* \mathcal{O}_{X'}(K_{X'/X} - [\frac{1}{N}D]),$$

and the asymptotic multiplier sheaf $\mathcal{I}(||L||)$ is the unique maximal ideal sheaf obtained in this way (cf. section 9.3 for details). The locus on X defined by this ideal sheaf is then called the cosupport of the ideal sheaf and is typically the locus where L fails to be nef. This leads us to our third main result.

10.3.6 Theorem. *Let $\phi : X \to Y$ be a flat fibration between projective manifolds, and let L be a line bundle of non-negative Kodaira dimension over X. Denote by $\mathcal{I}(X, ||L||)$ the asymptotic multiplier ideal of L. If the cosupport of $\mathcal{I}(X, ||L||)$ does not project onto Y, then the direct image sheaf $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

The reader that is familiar with the papers of Viehweg (e.g. [Vie82, Vie83, Vie95, Vie06]) will notice that our results and techniques owes a lot to his imposing work. Theorem 10.3.6 has a partial overlap with theorem 4.13 in [Cam04a] and it is quite interesting to compare the (very different) techniques.

The idea of our work is to show that these various results are in fact based on the same strategy and to show where the conditions on the geometry of the fibration and the positivity of the line bundle actually enter the game. I hope furthermore that the results are stated in a generality that is useful for the working mathematician who does not have the time to learn how the positivity of direct image sheaves can be shown. It would be very interesting to extend this type of result to the case of a pseudo-effective line bundle. At the moment this case is not accessible since we do not have any vanishing theorem nor the appropriate extension theorem for sections. Nevertheless the following statement should be true.

7.1.5 Conjecture. *Let $\phi : X \to Y$ be a flat fibration between projective manifolds, and let L be a pseudo-effective line bundle on X. Denote by $\mathcal{I}(L)$ the multiplier ideal associated to some positive singular metric h on L. If for a* *general fibre F, the restriction of $\mathcal{I}(L)$ to F induces the trivial ideal sheaf, then the direct image sheaf $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

7.2 The global strategy

Since the steps in our proofs are rather technical, we will try to explain how to show the positivity of direct images sheaves. Therefore we will first give a summary of the proof of Mourougane's theorem and then explain how we must adapt the strategy to our setting.

7.2.1 Theorem. *[Mou97, Thm.1] Let $\phi : X \rightarrow Y$ be a smooth fibration between projective manifolds, and let L be a nef and ϕ-big line bundle on X. Then $E := \phi_*(L \otimes \omega_{X/Y})$ is locally free and nef.*

Outline of the proof of Mourougane's theorem.

Step 1. E is locally free. By the relative Kawamata-Viehweg theorem there are no higher direct images and we conclude via flatness.

Step 2. The fibre space trick. Let $X^s := X \times_Y \ldots \times_Y X$ be the s-times fibred product and denotes by $\phi^s : X^s \rightarrow Y$ the induced map to Y and by $\pi^i : X^s \rightarrow X$ the projection on the i-th factor. Then X^s is smooth, ϕ^s is a smooth fibration and $L_s := \otimes_{i=1}^s (\pi^i)^* L$ is nef and ϕ^s-big. A combination of flat base change and projection formula arguments shows that

$$E^{\otimes s} \simeq \phi^s_*(L_s \otimes \omega_{X^s/Y}).$$

Step 3. An extension property for nef and relatively big line bundles. An application of the Kawamata-Viehweg vanishing theorem combined with an argument via Castelnuovo-Mumford regularity shows the following: let $\psi : A \rightarrow B$ be a smooth fibration between projective manifolds and M be a nef and ψ-big line bundle on A. Let H be a very ample line bundle on B, then

$$\psi_*(M \otimes \omega_A) \otimes H^{\dim Y+1}$$

is globally generated. This means that a global section of the restriction of $M \otimes \omega_A$ to an arbitrary fibre extends to a global section of $M \otimes \omega_A \otimes \psi^* H^{\dim Y+1}$.

Step 4. The vector bundle E is nef. Let H be a very ample line bundle on Y. Apply step 3 to the smooth fibrations $\phi^s : X^s \rightarrow Y$ to see that

$$E^{\otimes s} \otimes \omega_Y \otimes H^{\dim Y+1} \simeq \phi^s_*(L_s \otimes \omega_{X^s}) \otimes H^{\dim Y+1}$$

is globally generated for all $s > 0$. This shows that E is nef. □

Here comes the list of ideas that we have to realize to adapt this proof to our situation.

Idea 1. Skip step 1. It is clear that for L an arbitrary line bundle, the direct image sheaf $\phi_*(L \otimes \omega_{X/Y})$ is not locally free. Relative duality implies that it is reflexive, this must be enough. Note that this means that we can't use the projection formula.

Idea 2. Try to understand the fibre products X^s. The main setback in our situation is that the fibre products X^s can acquire some really bad singularities. Flatness implies that they are Cohen-Macaulay and under some conditions even normal. The most important point for us is to find a non-empty Zariski open subset $Y^* \subset Y$, such that $(\phi^s)^{-1}(Y^*) \subset X^s$ has at most rational singularities.

Idea 3. Isomorphisms become morphisms. In our case, a formula like

$$(\phi_*(L \otimes \omega_{X/Y}))^{\otimes s} \simeq \phi^s_*(L_s \otimes \omega_{X^s/Y})$$

does not necessary hold, since the the tensor product of a reflexive sheaf is not necessarily reflexive. A simple formula like

$$\phi^s_*(L_s \otimes \omega_{X^s/Y}) \otimes \omega_Y \simeq \phi^s_*(L_s \otimes \omega_{X^s})$$

might be false if Y is not Gorenstein. We will take care of this problem by showing these isomorphisms in codimension 1, then extend to a morphism via reflexiveness.

Idea 4. Show extension properties. So far we haven't made use of the positivity of L. The positivity of L only enters the picture in step 3, which we generalise for a variety of different conditions on the positivity of L.

We will carry out the first three points in the sections 10.1 and 10.2, the extension of sections will then be treated in section 10.3.

7.3 Leitfaden

Chapter 8 is a collection of definitions and results that are classical, but maybe not known to everyone. Its introductory style addresses in the first place the young researcher wanting to learn about singular varieties or the gymnastics of duality theory, the experienced geometer has certainly seen everything before.

Chapter 9 starts with a comprehensive introduction to the positivity theory of coherent sheaves. Section 9.3 contains some important preparatory work, but will only be used much later. Section 9.5 is completely independent of the rest, it fills the gap left by the fact that we only treat morphisms with connected fibres. Section 9.4 is the most interesting and should give the reader a first idea how to show weak positivity.

Chapter 10 is the core chapter of this second part. It contains the technical constructions indicated in the global strategy. Section 10.3 contains the (surprisingly short) proofs of the main theorems. A reader that is interested in showing a direct image result himself, should read the reduction lemma 10.2.4 which is the fruit of our technical preparation

Chapter 11 is the canonical closure of the preceeding chapter. A reader that wants to get a better understanding of the conditions in our results, will hopefully find these examples enlightening.

7.4 Notation

We work over the complex field $\mathbb{C}$. If not mentioned otherwise, all the topological notions refer to the Zariski topology. In particular sheaves are defined with respect to the Zariski topology. Schemes will always be supposed to be quasi-projective over $\mathbb{C}$. A variety is an integral scheme of finite type over $\mathbb{C}$. A divisor on a normal variety is a Weil divisor. We will identify locally free sheaves and vector bundles. By a point on a variety we will always mean a closed point, the fibres of a morphism are also the fibres over a closed point of the base.

Let X be a scheme, and let $\mathcal{S}$ be a coherent sheaf on X. If $X^* \subset X$ is an open subscheme that is endowed with the canonical subscheme structure, we denote by $\mathcal{S}|_{X^*}$ the restriction to X^*. If $Z \subset X$ is a closed subscheme we denote by

$$\mathcal{S} \otimes \mathcal{O}_Z := \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$$

the restriction to Z.

Let $\phi : X \to Y$ be a morphism of schemes, and let $y \in Y$ be a closed point. Then we denote by $X_y := \phi^{-1}(y) := X \times_Y y$ the (scheme-theoretical) fibre. More generally if $Z \subset Y$ is a subscheme, we denote by $X_Z := X \times_Y Z$ the fibre via ϕ.

We will say that a certain property on X holds over a general point of Y if there exists a non-empty Zariski open subset $Y^* \subset Y$ such that the property holds on $\phi^{-1}(Y^*)$.

Chapter 8

Recalling the basics

The results in this chapter are classical and well-known. For completeness of reference and for the convenience of the reader we will explain some of the arguments that will be used frequently in the following.

8.1 Reflexive sheaves

The category of locally free sheaves, although very convenient, is too restricted for our purposes. In particular if we work with non-Gorenstein varieties (cf. section 8.2 for complements on the terminology for singular varieties), the dualising sheaf is not locally free. Reflexive sheaves are the appropriate framework to fill this gap. Our exposition follows closely [Har80, Ch. 1].

8.1.1 Definition. *Let X be an integral scheme and $\mathcal{S}$ a coherent sheaf on X. We define*

$$\mathcal{S}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{S}, \mathcal{O}_X)$$

to be the dual of $\mathcal{S}$ and $\mathcal{S}^{**} := (\mathcal{S}^*)^*$ to be the bidual of $\mathcal{S}$. A sheaf $\mathcal{S}$ is torsion-free if the natural map

$$\mathcal{S} \to \mathcal{S}^{**}$$

is injective. It is reflexive if it is an isomorphism.

Remark. Let $\operatorname{Tor} \mathcal{S} \subset \mathcal{S}$ be the torsion subsheaf of a coherent sheaf $\mathcal{S}$ (cf. definition in [Kob87, p.159]). It is elementary to see that $\mathcal{S}$ is torsion-free if and only if $\operatorname{Tor} \mathcal{S} = 0$ (ibid).

8.1.2 Corollary. *Let X be an integral scheme, and let $\mathcal{F}$ and $\mathcal{S}$ be coherent sheaves on X. If $\mathcal{S}$ is torsion-free, then $\mathcal{H}om(\mathcal{F}, \mathcal{S})$ is torsion-free.*

Proof. The statements are local, so we can suppose that $X = \operatorname{Spec} A$ for some integral noetherian ring A and consider F and S the noetherian A-modules that induce $\mathcal{F}$ and $\mathcal{S}$.

For the first statement, let $\phi : F \to S$ be a morphism such that there exists an $a \in A$ such that $a \otimes \phi : F \to S, x \mapsto a \cdot \phi(x)$ is the zero map. Since S is torsion-free, it follows that $\phi(x)$ is zero for all $x \in F$, so ϕ is the zero map. □

By upper semicontinuity [Har77, II,Exerc.5.8] there exists a non-empty open subset $X^* \subset X$ such that $\mathcal{S}|_{X^*}$ is locally free. It is important to give a lower bound on the codimension of $X \setminus X^*$.

8.1.3 Notation. *Let X be an integral scheme, and let $\mathcal{S}$ be a coherent sheaf on X. We say that $\mathcal{S}$ is locally free in codimension k if there exists a closed subvariety $Z \subset X$ such that $\operatorname{codim}_X Z \geq k+1$ and $\mathcal{S}|_{X \setminus Z}$ is locally free.*

Remark. It is well-known that on a smooth variety a torsion-free sheaf is locally free in codimension 1 (cf. [Kob87, Cor.5.15]) and a reflexive sheaf is locally free in codimension 2 (cf. [Har80, Cor.1.4]). Since a normal variety is regular in codimension 1, this implies immediately

8.1.4 Proposition. *Let X be a normal variety and $\mathcal{S}$ a torsion-free sheaf on X. Then $\mathcal{S}$ is locally free in codimension 1.*

8.1.5 Proposition. *[Har80, Prop.1.1,Cor.1.2] A coherent sheaf $\mathcal{E}$ on an integral scheme X is reflexive if and only if (at least locally) it can be included in an exact sequence*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

where $\mathcal{F}$ is locally free and $\mathcal{G}$ is torsion-free. In particular the dual of any coherent sheaf is reflexive.

8.1.6 Definition. *A coherent sheaf $\mathcal{S}$ on an irreducible scheme X is normal if for every open set $X^* \subset X$ and every closed subset $Z \subset X^*$ such that $\operatorname{codim}_{(X^*)} Z \geq 2$, the restriction map*

$$\mathcal{S}(X^*) \to \mathcal{S}(X^* \setminus Z)$$

is bijective.

The next lemma gives useful characterisation of reflexive sheaves on a normal variety.

8.1.7 Lemma. *[Har80] Let $\mathcal{S}$ be a coherent sheaf on a normal variety X. The following conditions are equivalent:*

1.) $\mathcal{S}$ is reflexive;

2.) $\mathcal{S}$ is torsion-free and normal;

3.) $\mathcal{S}$ is torsion-free, and for each open set $X^* \subset X$ and each closed subset $Z \subset X^*$ such that $\operatorname{codim}_{(X^*)} Z \geq 2$, we have $\mathcal{S}|_{X^*} \simeq j_* \mathcal{S}|_{X^* \setminus Z}$, where $j : X^* \setminus Z \to X^*$ is the inclusion map.

In particular if $\mathcal{S}$ *and* $\mathcal{F}$ *are reflexive sheaves on* X *and there exists a closed set* $Z \subset X$ *such that* $\operatorname{codim}_X Z \geq 2$ *and* $\mathcal{S}|_{X \setminus Z} \simeq \mathcal{F}|_{X \setminus Z}$, *then* $\mathcal{S} \simeq \mathcal{F}$.

8.1.8 Corollary. *Let* X *be a normal variety, and let* $\mathcal{F}$ *and* $\mathcal{S}$ *be coherent sheaves on* X. *If* $\mathcal{S}$ *is reflexive, then* $\mathcal{H}om(\mathcal{F}, \mathcal{S})$ *is reflexive.*

In particular if there exists a closed set $Z \subset X$ *such that* $\operatorname{codim}_X Z \geq 2$ *and a morphism* $\mathcal{F}|_{X \setminus Z} \to \mathcal{S}|_{X \setminus Z}$, *then the morphism extends to a unique morphism* $\mathcal{F} \to \mathcal{S}$ *on* X.

Remark. The same statement for X smooth is Prop. 5.23 from [Kob87], the proof goes through without changes. For the readers convenience we nevertheless write it down here.

Proof. We admit for the moment the following claim: *let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of coherent sheaves where $\mathcal{F}$ *is reflexive and* $\mathcal{G}$ *is torsion-free. Then* $\mathcal{E}$ *is reflexive.*

Since $\mathcal{S}$ is reflexive, there exists (at least locally) an exact sequence

$$0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0,$$

where $\mathcal{T}$ is locally free and $\mathcal{U}$ torsion-free. By left-exactness of the $\mathcal{H}om(\mathcal{F}, .)$-functor we have an exact sequence

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{S}) \to \mathcal{H}om(\mathcal{F}, \mathcal{T}) \to \mathcal{H}om(\mathcal{F}, \mathcal{U}).$$

Since $\mathcal{T}$ is locally free, $\mathcal{H}om(\mathcal{F}, \mathcal{T})$ is reflexive. Since $\mathcal{H}om(\mathcal{F}, \mathcal{U})$ is torsion-free by corollary 8.1.2, the image $\mathcal{Q}$ of the morphism $\mathcal{H}om(\mathcal{F}, \mathcal{T}) \to \mathcal{H}om(\mathcal{F}, \mathcal{U})$ is torsion-free. So we have an exact sequence

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{S}) \to \mathcal{H}om(\mathcal{F}, \mathcal{T}) \to \mathcal{Q} \to 0$$

where the middle sheaf is reflexive and the right hand sheaf is torsion-free, so we conclude with the claim.

Proof of the claim: Since $\mathcal{F}$ is torsion-free, $\mathcal{E}$ is torsion-free, so it suffices to show that $\mathcal{E}$ is normal. Let $X^* \subset X$ be an open subset and $Z \subset X^*$ be a closed set such that $\operatorname{codim}_{(X^*)} Z \geq 2$ Then we have a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(X^*, \mathcal{E}) & \longrightarrow & \Gamma(X^*, \mathcal{F}) & \longrightarrow & \Gamma(X^*, \mathcal{G}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(X^* \setminus Z, \mathcal{E}) & \longrightarrow & \Gamma(X^* \setminus Z, \mathcal{F}) & \longrightarrow & \Gamma(X^* \setminus Z, \mathcal{G}) \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

where all vertical and horizontal sequences are exact. A simple diagram chase shows that the restriction map $\Gamma(X^*, \mathcal{E}) \to \Gamma(X^* \setminus Z, \mathcal{E})$ is surjective. □

8.1.9 Proposition. *[Har80, Cor.1.7, Prop.1.8] Let $\phi : X \to Y$ be a morphism of normal varieties.*

If ϕ is equidimensional and dominant, and $\mathcal{S}$ is a reflexive sheaf on X, the direct image $\phi_*\mathcal{S}$ is reflexive.

If ϕ is flat and $\mathcal{S}$ is a reflexive sheaf on Y, the pull-back $\phi^*\mathcal{S}$ is reflexive.

It is a well-known and basic fact [Har77, II, Cor.6.16.] that on a smooth variety there is a bijection between classes of Weil divisors and isomorphism classes of invertible sheaves. On a normal variety this is no longer true since a Weil divisor is no longer necessarily Cartier. Nevertheless if we denote by X_{reg} the nonsingular locus of X, then by [Har77, II, Prop.6.5] we can identify the divisor class groups

$$Cl(X) = Cl(X_{\text{reg}}).$$

Thus, given a Weil divisor D, we can associate a coherent sheaf $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D) := j_*\mathcal{O}_{X_{\text{reg}}}(D|_{X_{\text{reg}}}),$$

where $j : X_{\text{reg}} \to X$ is the inclusion. By lemma 8.1.7 the sheaf $\mathcal{O}_X(D)$ is reflexive. On the other hand given a reflexive sheaf $\mathcal{F}$ of rank 1 on X, there exists an open subset $X^* \subset X$ such that $\operatorname{codim}_X(X \setminus X^*) \geq 2$ and $\mathcal{F} \otimes \mathcal{O}_{X^*}$ is locally free [Har80, Cor.1.4]. So there exists a Weil divisor D' on X^* such that $\mathcal{F} \otimes \mathcal{O}_{X^*} = \mathcal{O}_{X^*}(D')$, so $Cl(X) = Cl(X^*)$ (cf. [Har77, II, Prop.6.5]) implies that in fact D' can be seen as a Weil divisor on X. Therefore we have $\mathcal{O}_X(D') \simeq \mathcal{F}$ by lemma 8.1.7. This shows that on a normal variety we have a bijection between divisor classes and reflexive rank 1 sheaves modulo multiplication by non-vanishing functions. Unfortunately this bijection is not an isomorphism $\mathbb{Z}$-modules, since the class of reflexive sheaves is not closed under the tensor product. In particular

$$\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \simeq \mathcal{O}_X(D_1 + D_2)$$

does not hold in general, we have to take the bidual on the left hand side to obtain an isomorphism. We sum up these observations in the following

8.1.10 Proposition. *[Rei80, App., Thm. 3] Let X be a normal variety, then the correspondence*

$$\delta : D \mapsto \mathcal{O}_X(D)$$

where D is a Weil divisor induces a bijection

$$\delta : Cl(X) \to \{\textit{reflexive rank 1 sheaves}\} / H^0(X, \mathcal{O}_X^*).$$

This bijection becomes an isomorphism of $\mathbb{Z}$-modules if one sets

$$\delta(D_1 + D_2) := (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{**}$$

and

$$\delta(kD) := \mathcal{O}_X(kD)$$

for any divisor D, D_1, D_2 and $k \in \mathbb{Z}$.

8.2 Singularities

We start with an introduction to the zoology of singular varieties. All these notions are well-known to the experienced researcher, but often confusing for the young algebraic geometer. Therefore we will build up the notions as systematically as possible, but not give any proof.

8.2.1 Definition. *Let X be a scheme. We denote by X_{reg} the open subscheme such that for $p \in X_{\mathrm{reg}}$, the local ring $\mathcal{O}_{X,p}$ is regular. We denote by $X_{\mathrm{sing}} = X \setminus X_{\mathrm{reg}}$ the singular locus of X.*

If X is irreducible and $\operatorname{codim}_X(X \setminus X_{\mathrm{reg}}) \geq k$, we say that X is regular in codimension $k-1$.

8.2.2 Definition. *An integral scheme X is normal if all the local rings $\mathcal{O}_{X,p}$ are integrally closed domains, that is integrally closed in their field of fractions.*

Remark. It is possible to define normality for Noetherian rings that are not necessarily integral (cf. [Mat89, p.64]) and thus for schemes that are not integral, but we will not need this more general notion.

8.2.3 Definition. *[Mat89, p.134] A scheme is Cohen-Macaulay if all the local rings $\mathcal{O}_{X,p}$ are Cohen-Macaulay rings in the sense of commutative algebra, that is*

$$\operatorname{Depth} \mathcal{O}_{X,p} = \dim \mathcal{O}_{X,p}.$$

Here the depth of a local is the maximal length of a regular sequence in $\mathcal{O}_{X,p}$: a sequence $x_1, \ldots, x_r$ is regular if x_1 is not a zero divisor in $\mathcal{O}_{X,p}$ and for all $i \in \{2, \ldots, r\}$, the image of x_i in $A/(x_1, \ldots, x_{i-1})$ is not a zero divisor. The dimension $\dim \mathcal{O}_{X,p}$ is the Krull dimension of the local ring, that is the maximal length of sequences of distinct prime ideals $p_i \subset \mathcal{O}_{X,p}$ such that

$$p_0 \subset p_1 \subset \ldots \subset p_r \subset \mathcal{O}_{X,p}$$

Example. Any local complete intersection in a manifold is Cohen-Macaulay [Har77, II, Prop.8.23].

Serre's criterion R1 equivalent S2 gives us a convenient way to check if an integral scheme is normal.

8.2.4 Theorem. *[Har77, II,Thm.8.22A] Let X be an integral Cohen-Macaulay scheme. Then X is normal if and only if it is regular in codimension 1.*

The Cohen-Macaulay condition will be very useful for us, since it assures the existence of a relative dualising sheaf such that the natural dualising morphisms are isomorphisms (cf. section 8.4). For some arguments it is nevertheless necessary to suppose the stronger Gorenstein property. We will see that for all the questions we are interested in, it makes no difference to work with smooth or the more general Gorenstein varieties.

8.2.5 Definition. *An n-dimensional scheme X is Gorenstein if it is Cohen-Macaulay and for every local ring $\mathfrak{m}_p \subset \mathcal{O}_{X,p}$, we have*

$$Ext^n_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/\mathfrak{m}_p, \mathcal{O}_{X,p}) \simeq \mathcal{O}_{X,p}/\mathfrak{m}_p.$$

Remark. The definition above is the most general, but not very useful for our purpose. Since every projective scheme admits a dualising sheaf in the sense of [Har77, III, 7], we can talk of a dualising sheaf on a quasi-projective scheme by taking the restriction of the dualising sheaf of some compactification. Thus we can express the Gorenstein property in terms of the dualising sheaf.

8.2.6 Theorem. *[BH93, Thm.3.3.7] A scheme of pure dimension n is Gorenstein if and only if it is Cohen-Macaulay and the dualising sheaf ω_X is invertible.*

Example. Any effective Weil divisor D in a manifold X is Gorenstein. Indeed by adjunction we have $\omega_D \simeq \omega_X \otimes \mathcal{O}_D(D)$.

In the context of classification theory it is not sufficient to work with Gorenstein varieties, since Mori's program produces singularities such that the local rings are Cohen-Macaulay but not Gorenstein. Still this category is smaller than the category of Cohen-Macaulay varieties, since a multiple of the canonical sheaf is invertible.

8.2.7 Definition. *Let X be a normal variety, and let D be a divisor such that there exists a natural number $m \in \mathbb{N}^*$ such that $\mathcal{O}_X(mD)$ is locally free (or equivalently that mD is a Cartier divisor). Then D is said to be a $\mathbb{Q}$-Cartier divisor and the smallest natural number i such that $\mathcal{O}_X(iD)$ is locally free is called the index of D.*

Recall now that for a normal variety X, we can define the canonical divisor by extending the canonical divisor of the nonsingular locus to X. This yields a coherent sheaf $\mathcal{O}_X(K_X)$ and if X is Cohen-Macaulay this sheaf is isomorphic to the dualising sheaf ω_X (cf. e.g. [Rei80, App., Prop. 6]).

8.2.8 Definition. *A normal variety X is $\mathbb{Q}$-Gorenstein of index i if it is Cohen-Macaulay and the canonical divisor K_X is a $\mathbb{Q}$-Cartier divisor of index i.*

Example. The quotient of $\mathbb{C}^3$ by the $\mathbb{Z}/2\mathbb{Z}$-action $(x, y, z) \to (-x, -y, -z)$ has a $\mathbb{Q}$-Gorenstein singularity of index 2 in the point corresponding to the origin $(0, 0, 0)$.

Remark. In the literature there exists also a weaker definition of $\mathbb{Q}$-Gorenstein: a normal variety X is i-Gorenstein if the canonical divisor K_X is a $\mathbb{Q}$-Cartier divisor of index i. We will not work with this definition, since we have to work with dualising sheaves $\omega_X \simeq \mathcal{O}_X(K_X)$ and not mere canonical sheaves.

8.2.9 Definition. *A normal variety X has at most rational singularities if X is Cohen-Macaulay and there exists a desingularisation $d : X' \to X$ such that*

$$d_*\omega_{X'} \simeq \omega_X.$$

Remark. Note that if X has rational singularities, then every desingularisation satisfies the condition from the definition. By [KKMSD73] and [Elk81] the definition above is equivalent to asking that there exists a desingularisation $d : X' \to X$ such that $d_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ and

$$R^i d_* \mathcal{O}_{X'} = 0 \qquad \forall\ i > 0.$$

The reader might be astonished that the definition of rational singularities does not make any statement about the higher direct images $R^i d_* \omega_{X'}$. In fact by the Grauert-Riemenschneider vanishing theorem [GR70], we have

$$R^i d_* \omega_{X'} = 0 \qquad \forall\ i > 0,$$

independently of the singularities of X.

It is clear from the definition that the subset $\mathrm{Irr}(X) \subset X$ where X has non-rational singularities is closed and we will call it the irrational locus of X.

The most important class of singularities in classification theory is that of canonical singularities. These singularities arise naturally as the singularities of the so-called canonical model of a variety of general type.

8.2.10 Definition. *A normal $\mathbb{Q}$-Gorenstein variety X of index i has canonical singularities if there exists a desingularisation $d : X' \to X$ such that*

$$iK_{X'} = d^*(iK_X) + F,$$

where F is an effective divisor such that its support is contained in the d-exceptional locus.

The variety X has terminal singularities if furthermore for every d-exceptional prime divisor E, we have $E \leq F$, that is $E - F$ is an effective divisor.

Example. The surface

$$\{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$$

has a canonical singularity in the origin (more precisely it is a rational double point of type E8, see also [BPVdV84]).

Remark. Since iK_X is a Cartier divisor, the pull-back $d^*(iK_X)$ is the sheaf-theoretical pull-back. It is not very hard to see that a normal Gorenstein variety has canonical singularities if and only if it has rational singularities. Furthermore we have the

8.2.11 Theorem. *[Rei87, p. 363] Canonical singularities are rational singularities.*

8.3 Flat morphisms

We introduce a variety of technical conditions that we will impose on the maps that we will consider. The most important result in this chapter is theorem 8.3.7 which is classical, but apparently not so well-known.

8.3.1 Definition. *A projective morphism $\phi : X \to Y$ from a scheme X to a normal variety Y is a fibration if it is surjective and the generic fibre is irreducible.*

Remark. Since Y is supposed to be normal, every fibre of a fibration is connected. Since we work over a field of characteristic 0 this is equivalent to $\phi_* \mathcal{O}_X \simeq \mathcal{O}_Y$.

8.3.2 Definition. *A fibration $\phi : X \to Y$ is generically smooth if the singular locus of X does not surject on Y.*

Remark. A fibration is generically smooth if and only if there exists a non-empty Zariski open subset $Y^* \subset Y$ such that for all $y \in Y^*$, the fibre X_y is a smooth variety. Indeed, if this is the case up to making Y^* a bit smaller we may suppose that Y^* is smooth. Then $\phi^{-1}(Y^*) \subset X$ is smooth, so $\phi(X_{\mathrm{sing}})$ does not meet Y^*.

For the other implication suppose that ϕ is generically smooth, then set $Y' := Y \setminus (\phi(X_{\mathrm{sing}}) \cup Y_{\mathrm{sing}})$ and $X' := \phi^{-1}(Y^*)$. The restricted morphism $\phi|_{X'} : X' \to Y'$ is a morphism between manifolds, so the generic smoothness theorem applies.

8.3.3 Definition. *A fibration has generically reduced (resp. Gorenstein) fibres in codimension 1, if there exists a Zariski open subset $Y^* \subset Y$ such that*

$$\operatorname{codim}_Y(Y \setminus Y^*) \geq 2$$

and for all $y \in Y^*$, the fibre X_y is generically reduced (resp. Gorenstein).

Remark. Note that the definition of a fibration with generically reduced fibres in codimension 1 is *not* the same as in the first part of this thesis. If a fibre X_y is reducible, we want *all* the irreducible components to be generically reduced and not only one.

8.3.4 Definition. *Let $\phi : X \to Y$ be a generically smooth fibration. The ϕ-smooth locus is the maximal open subset $Y^* \subset Y$ such that $\phi^{-1}(Y^*)$ is smooth and such that for every $y \in Y^*$ the fibre X_y is smooth and of dimension* $\dim X - \dim Y$.

The ϕ-singular locus Δ is the complement $Y \setminus Y^*$.

8.3.5 Definition. *A morphism of schemes $\phi : X \to Y$ is flat if for every $p \in X$, the local ring $\mathcal{O}_{X,p}$ is flat over $\mathcal{O}_{Y,\phi(p)}$*

Remark. If $\phi : X \to Y$ is a morphism of schemes and $y \in Y$ a point, we denote by $\phi^{-1}(\mathcal{I}_y)$ the ideal sheaf of the fibre X_y as defined in [Har77, p.163], so $\phi^{-1}(\mathcal{I}_y)$ is the image of $\phi^* \mathcal{I}_y$ in $\mathcal{O}_X$ [Har77, II, Caution 7.12.2]. If ϕ is flat in a neighborhood of the fibre X_y, the pull-back of $\mathcal{I}_y \to \mathcal{O}_Y$ to X, namely the morphism $\phi^* \mathcal{I}_y \to \mathcal{O}_X$ is injective. Thus we can and will identify $\phi^* \mathcal{I}_y$ and $\phi^{-1}(\mathcal{I}_y)$.

8.3.6 Definition. *A fibration $\phi : X \to Y$ is a flat Cohen-Macaulay (resp. Gorenstein) fibration if it is flat and X is an irreducible Cohen-Macaulay (resp. Gorenstein) scheme.*

The following theorem shows that the Cohen-Macaulay (resp. Gorenstein) condition is well-behaved under flat maps.

8.3.7 Theorem. *[Mat89, Cor.23.3,Thm.23.4] Let $\phi : X \to Y$ be a flat morphism. Then X is Cohen-Macaulay (resp. Gorenstein) if and only if Y and every ϕ-fibre is Cohen-Macaulay (resp. Gorenstein).*

Remark. A heuristic reason why this theorem should be true is as follows: if $y \in Y$ is a closed point of a Cohen-Macaulay scheme, there exists a subscheme $Z \subset Y$ that is a local complete intersection and is supported in y. It is then clear that the fibre X_Z is a local complete intersection. But of course X_Z is „a multiple“ of the fibre X_y, so it seems reasonable that X_y is Cohen-Macaulay if X is Cohen-Macaulay.

A similar property should hold in the $\mathbb{Q}$-Gorenstein case, but I don't know a reference or proof for this fact.

8.4 Coherent sheaves and duality theory

We recall the basics of duality theory, for proofs we refer to [Kle80]. The crucial result that we will apply frequently in the following is the technical corollary 8.4.5. For the whole section, we fix the notation: let $\phi : X \to Y$ and $\psi : Y' \to Y$ be morphisms of schemes, then we have the base change diagram for $X' := X \times_Y Y'$:

$$\begin{array}{ccc} X' & \xrightarrow{\psi'} & X \\ \downarrow{\scriptstyle \phi'} & \searrow^{\eta} & \downarrow{\scriptstyle \phi} \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

8.4.1 Definition. *[Kle80, Def.10] Let $\phi : X \to Y$ be a flat projective morphism of schemes. Denote by $\mathcal{E}xt^m_\phi$ the m-th derived functor of $\phi_*\mathcal{H}om_X$. We say that relative duality holds for the morphism ϕ if there exists a quasi-coherent sheaf $\omega_{X/Y}$ such that the natural map*

$$D^m : \mathcal{E}xt^m_\phi(\mathcal{S}, \omega_{X/Y} \otimes \phi^*\mathcal{T}) \to \mathcal{H}om_Y(R^{r-m}\phi_*\mathcal{S}, \mathcal{T})$$

is an isomorphism for all $0 \leq m \leq r := \dim X - \dim Y$, and $\mathcal{S}$ a quasi-coherent sheaf on X, and $\mathcal{T}$ a quasi-coherent sheaf on Y. In this case we call $\omega_{X/Y}$ the relative dualising sheaf.

8.4.2 Theorem. *[Kle80, Thm.21], [Kle80, Prop.9] Let $\phi : X \to Y$ be a flat projective morphism of schemes. Then relative duality holds if and only if all the fibres are Cohen-Macaulay. In this case the relative dualising sheaf $\omega_{X/Y}$ is* *flat over Y. Let $\psi : Y' \to Y$ be any (not necessarily flat) base change, then full duality holds for the flat morphism $\phi' : X' \to Y'$ and*

$$\omega_{X'/Y'} \simeq (\psi')^* \omega_{X/Y}. \tag{8.1}$$

8.4.3 Theorem. *([Kle80, p.58], see also theorem 8.3.7) Let $\phi : X \to Y$ be a flat projective morphism. If X is Cohen-Macaulay and Y is Gorenstein, relative duality holds and*

$$\omega_{X/Y} \simeq \omega_X \otimes \phi^* \omega_Y^*. \tag{8.2}$$

This allows to slightly extend the definition of the relative dualising sheaf to the non-flat case.

8.4.4 Definition. *Let $\phi : X \to Y$ be a fibration such that X is irreducible Cohen-Macaulay and Y is Gorenstein. Then we set $\omega_{X/Y} := \omega_X \otimes \phi^* \omega_Y^*$ and call it the relative dualising sheaf of ϕ.*

8.4.5 Corollary. *Let $\phi : X \to Y$ and $\psi : Y' \to Y$ be flat Cohen-Macaulay fibrations. Then η is a flat Cohen-Macaulay fibration. Furthermore we have*

$$\omega_{X'/Y} = (\phi')^* \omega_{Y'/Y} \otimes (\psi')^* \omega_{X/Y}. \tag{8.3}$$

If X and Y' are Gorenstein, X' is Gorenstein. If X and Y' are integral, X' is integral.

Proof.

Step 1. The Cohen-Macaulay case. Since X (resp. Y') is Cohen-Macaulay, the fibres of the morphism ϕ (resp ψ) are Cohen-Macaulay by theorem 8.3.7, so relative duality holds by theorem 8.4.2. So all the fibres of the induced morphism $\psi' : X' \to X$ are Cohen-Macaulay. Since X is Cohen-Macaulay, this implies by theorem 8.3.7 that X' is Cohen-Macaulay. Since the general fibres of ϕ and ψ are irreducible, this holds for the general fibre of η. By [Cam04b, Lemma 1.10] this shows that there exists an open subset $Y^* \subset Y$ such that $\eta^{-1}(Y^*)$ is irreducible. Since η is flat, it is an open mapping, so $\eta^{-1}(Y^*)$ is dense in X'. This shows the irreducibility of X'. By [Kle80, p.58] we have

$$\omega_{X'/Y} = \omega_{X'/X} \otimes (\psi')^* \omega_{X/Y},$$

so formula (8.1) implies formula (8.3).

Step 2. The Gorenstein case. Since Y' is Gorenstein, we know by formula 8.2 that $\omega_{X'/X} = (\phi')^* \omega_{Y'/Y}$ is locally free. Using the same formula we see that in this case the fibres of ψ' are even Gorenstein. Since X is Gorenstein, this implies by theorem 8.4.3 that X' is Gorenstein.

Step 3. Integrality. Since X and Y' are integral, they admit non-empty open subsets $X^* \subset X$ and $Y^* \subset Y$ that are smooth. By generic smoothness applied to the induced morphisms $X^* \to Y$ and $Y^* \to Y$ we can suppose up to restricting a bit further that they are smooth over a smooth base. Therefore $X^* \times_Y Y^*$ is smooth and dense in X', in particular X' is generically reduced. Since X' is flat over the integral scheme Y, it follows from [Laz04a, p.246] that X' is reduced. □

We conclude with an absolutely trivial, but nevertheless crucial remark.

8.4.6 Corollary. *Let $\phi : X \to Y$ be a flat Cohen-Macaulay fibration, and let $\mathcal{S}$ be a locally free sheaf on X. Then*

$$\phi_*(\mathcal{S} \otimes \omega_{X/Y})$$

is reflexive.

Proof. Since $\mathcal{S}$ is locally free, we have

$$\mathcal{S} \otimes \omega_{X/Y} \simeq \mathcal{H}om_X(\mathcal{S}^*, \omega_{X/Y})$$

so relative duality implies

$$\phi_*(\mathcal{S} \otimes \omega_{X/Y}) \simeq \phi_* \mathcal{H}om_X(\mathcal{S}^*, \omega_{X/Y}) \simeq \mathcal{H}om_Y(R^{\dim X - \dim Y} \phi_* \mathcal{S}^*, \mathcal{O}_Y).$$

The dual sheaf of a coherent sheaf is reflexive. □

Chapter 9

Positivity notions

9.1 Positivity of locally free sheaves

We recall some of the basic definitions on positivity of locally free sheaves, for details and complements we refer to [Laz04a], [Laz04b].

9.1.1 Definition. *Let X be a projective scheme, and let L be a line bundle on X. Then L is nef if*

$$c_1(L) \cdot C \geq 0$$

for every complete curve $C \subset X$. Let E be a vector bundle over X, and denote by $\pi : \mathbb{P}(E) \to X$ the projectivisation of E. Then E is nef, if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef.

Let $\phi : X \to S$ be a projective morphism, and let L be a line bundle on X. Then L is ϕ-nef if for some $k > 0$, the bundle $L \otimes \phi^* \mathcal{O}_S(k)$ is nef. Let E be a vector bundle over X and denote by $\pi : \mathbb{P}(E) \to X$ the projectivisation of E. Then E is ϕ-nef, if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is $\phi \circ \pi$-nef.

Remark. We will say that a vector bundle is antinef if the dual bundle is nef.

9.1.2 Theorem. *(Barton-Kleiman criterion, [Laz04b, Prop.6.1.18]) Let X be a projective variety and let E be a vector bundle on X. Then E is nef if and only if the following holds: for any smooth projective curve C and any finite morphism $\nu : C \to X$ and any quotient line bundle $\nu^* E \to Q$ of $\nu^* E$, we have*

$$\deg_C Q \geq 0.$$

9.1.3 Lemma. *Let C be a smooth projective curve, and let $\mathcal{E} \to \mathcal{F}$ be a generically surjective morphism between vector bundles. If $\mathcal{E}$ is nef, $\mathcal{F}$ is nef. Let $\mathcal{E} \to \mathcal{F}$ be an injective morphism between vector bundles. If $\mathcal{F}$ is antinef, $\mathcal{F}$ is antinef.*

Proof. The first statement is [Laz04b, Ex.6.4.17], the second statement follows from the first by dualizing. □

9.1.4 Definition. *Let X be a projective irreducible scheme, and let L be a line bundle over X. The line bundle has Kodaira dimension at least m if*

$$\limsup_{k\to\infty} \frac{h^0(X,(L)^{\otimes k})}{k^m} > 0.$$

In this case we write

$$\kappa(X,L) = \kappa(L) \geq m.$$

If $h^0(X,(L)^{\otimes k}) = 0$ for all $k \in \mathbb{N}$, we set

$$\kappa(X,L) = \kappa(L) := -\infty.$$

If $\kappa(L) \geq 0$ we say that L has non-negative Kodaira dimension. The line bundle L is said to be big, if its Kodaira dimension is equal to the dimension of X.

9.1.5 Definition. *Let $\phi : X \to S$ be a projective morphism and let L be a line bundle on X. Then L is ϕ-big if for some $k > 0$, the bundle $L \otimes \phi^* \mathcal{O}_S(k)$ is big.*

Remark. An alternative definition would be to ask that the restriction of L to a general fibre F is big, i.e. we have

$$\limsup_{k\to\infty} \frac{h^0(F,(L\otimes \mathcal{O}_F)^{\otimes k})}{k^{\dim F}} > 0.$$

If in the situation of the definition we suppose that L is ϕ-nef, then L is ϕ-big if and only if

$$c_1(L)^{\dim F} \cdot F > 0.$$

9.1.6 Definition. *Let X be a projective manifold, and L be a line bundle over X. Then L is pseudo-effective if the first Chern class $c_1(L)$ is contained in the closure of the cone of the first Chern classes of big line bundles in $NS(X) \otimes \mathbb{R}$.*

Pseudo-effective line bundles have been characterized by Boucksom-Demailly-Paun-Peternell.

9.1.7 Theorem. *[BDPP04, Thm.0.2] A line bundle L on a projective manifold X is pseudo-effective if and only if $L \cdot C \geq 0$ for all irreducible curves C which move in a family covering X.*

9.2 Positivity of coherent sheaves

Since direct images sheaves are in general not locally free, it is necessary to find more general positivity notions than those established in section 9.1. This can be done in two different ways: try to use the same definition in the more general context or introduce a new, weaker definition for the general setting. The first way leads to the notion of ample and nef sheaves as defined in [AT82] and [Kub70], but it turns out that for our problem the notion of weak positivity in the sense of Viehweg is much more natural. Furthermore we introduce the even weaker notion of generic nefness in the sense of Miyaoka, this allows us to say a word about general fibrations.

9.2.1 Notation. *[Vie83, Defn.1.1] Let X be a normal variety and let $\mathcal{F}$ be a torsion-free sheaf on X. Let $i : X^* \to X$ be the largest open subvariety such that $\mathcal{F} \otimes \mathcal{O}_{X^*}$ is locally free. For all $m \in \mathbb{N}$, denote by $\operatorname{Sym}^m(\mathcal{F} \otimes \mathcal{O}_{X^*})$ (resp. $(\mathcal{F} \otimes \mathcal{O}_{X^*})^{\otimes m}$) the m-th symmetric product (resp. the m-fold tensor product). Then we define $S^{[m]}\mathcal{F} := i_* \operatorname{Sym}^m(\mathcal{F} \otimes \mathcal{O}_{X^*})$ and $\mathcal{F}^{[\otimes m]} := i_*(\mathcal{F} \otimes \mathcal{O}_{X^*})^{\otimes m}$, in particular these sheaves are reflexive by lemma 8.1.7.*

Remark. Since $\mathcal{F}$ is supposed to be torsion-free, the set $X \setminus X^*$ has codimension at least 2. It follows from lemma 8.1.7 that

$$S^{[m]}\mathcal{F} \simeq ((\operatorname{Sym})^m\mathcal{F})^{**} \qquad \text{and} \qquad \mathcal{F}^{[\otimes m]} \simeq (\mathcal{F}^{\otimes m})^{**}.$$

This implies that

$$(\mathcal{F}^{[\otimes a]})^{[\otimes b]} \simeq \mathcal{F}^{[\otimes ab]} \qquad \forall\, a, b \in \mathbb{N}.$$

9.2.2 Definition. *[Vie83, Defn.1.2] Let X be a normal variety, and let $X^* \subset X$ be a non-empty open subset. We say that a coherent sheaf $\mathcal{F}$ is weakly positive (in the sense of Viehweg) over X^* if for every ample line bundle H on X and every $\alpha \in \mathbb{N}$ there exists some $\beta \in \mathbb{N}$ such that $S^{[\beta\cdot\alpha]}\mathcal{F} \otimes H^{\otimes\beta}$ is globally generated over X^*, that is the evaluation map of sections on X*

$$H^0(X, S^{[\beta\cdot\alpha]}\mathcal{F} \otimes H^{\otimes\beta}) \otimes \mathcal{O}_X \to S^{[\beta\cdot\alpha]}\mathcal{F} \otimes H^{\otimes\beta}$$

is surjective over X^*. The sheaf $\mathcal{F}$ is weakly positive if there exists some non-empty open subset $X^* \subset X$ such that $\mathcal{F}$ is weakly positive over X^*

9.2.3 Lemma. *[Vie83, Remark 1.3] Let X be a normal variety, and let $\mathcal{F}$ be a torsion-free coherent sheaf over X. Then $\mathcal{F}$ is weakly positive over some open subset $X^* \subset X$ if and only if there exists a not necessarily ample line bundle L such that for every $\alpha \in \mathbb{N}$ there exists some $\beta \in \mathbb{N}$ such that $S^{[\beta\cdot\alpha]}\mathcal{F} \otimes L^{\otimes\beta}$ is globally generated over X^*.*

9.2.4 Lemma. *[Vie83, Lemma 1.4,3)] Let X be a normal variety, and let $\mathcal{F}$ be a torsion-free coherent sheaf over X. Then $\mathcal{F}$ is weakly positive over $X^* \subset X$ if and only if for some $m \in \mathbb{N}$, the tensor product $\mathcal{F}^{[\otimes m]}$ is weakly positive over $X^* \subset X$.*

The next lemma relates the notions of nefness and weak positivity. Morally speaking, it shows that a weakly positive sheaf is nef with the exception of some proper subscheme locus that contains curves where the sheaf fails to be nef.

9.2.5 Lemma. *[Vie82, Lemma 1.10] Let X be a normal quasi-projective variety, and let $\mathcal{F}$ be a locally free sheaf over X. Suppose that $\mathcal{F}$ is weakly positive over some open subset $X^* \subset X$. Then for every projective curve $C \subset X$ such that $C \cap X^* \neq \varnothing$, the restriction $\mathcal{F} \otimes \mathcal{O}_C$ is nef.*

9.2.6 Corollary. *Let X be a normal projective variety, and let $\mathcal{F}$ be a coherent sheaf over X. Suppose that $\mathcal{F}$ is weakly positive over $X^* \subset X$. Let C be a projective curve such that $C \cap X^* \neq \varnothing$ and $\mathcal{F}$ is locally free in a neighborhood of C. Then the restriction $\mathcal{F} \otimes \mathcal{O}_C$ is nef.*

We conclude this section with the notion of generic nefness to which we will come back only in section 10.4.

9.2.7 Definition. *Let X be a normal projective variety of dimension n, and let $\mathcal{F}$ be a torsion-free sheaf on X. Then $\mathcal{F}$ is generically nef (in the sense of Miyaoka) if for a collection of ample line bundles $H_1, \ldots, H_{n-1}$ and sufficiently high integers $m_1, \ldots, m_{n-1}$ the intersection*

$$C := D_1 \cap \ldots \cap D_{n-1}$$

of general elements $D_j \in |m_j H_j|$ satisfies $\mathcal{F} \otimes \mathcal{O}_C$ is nef.

Remark. A torsion-free sheaf $\mathcal{F}$ on a normal variety X is locally free in codimension 1, so the locus $Z \subset X$ such that for $x \in Z$, the stalk $\mathcal{F}_x$ has rank strictly higher than rk $\mathcal{F}$, has codimension at least 2. In particular a curve C that is obtained as an intersection of general hyperplane sections does not meet Z, so $\mathcal{F} \otimes \mathcal{O}_C$ is locally free. This shows that the definition makes sense. Furthermore we have the easy

9.2.8 Lemma. *Let X be a normal projective variety, and let $\mathcal{F}$ be a torsion-free sheaf on X. If $\mathcal{F}$ is weakly positive, it is generically nef.*

Proof. Let $X^* \subset X$ be a non-empty Zariski-open subset such that $\mathcal{F}$ is weakly positive over X^*. A curve C that is obtained as the intersection of general hyperplane sections satisfies $C \cap X^* \neq \varnothing$. By lemma 9.2.6, the vector bundle $\mathcal{F} \otimes \mathcal{O}_C$ is nef. □

Remark. Note that generic nefness in the sense of Miyaoka is a much weaker property than weak positivity in the sense of Viehweg. In fact if $\mathcal{F}$ is generically nef, there might exist a *covering* family of curves such that the restriction of $\mathcal{F}$ to the general member of the family is not nef !

9.3 Multiplier ideals

We will now introduce the notion of a multiplier ideal of a linear series and the asymptotic multiplier of a line bundle. For simplicity's sake we will restrict our considerations to smooth varieties. It might seem redundant to work with two different types of multiplier ideals, but while the multiplier ideal of a linear series is an object that can be manipulated rather easily, it is actually not the right object if one is interested in the positivity of the line bundle. Therefore we will try to make our statements always in terms of the asymptotic multiplier ideal of the line bundle while we use the multiplier ideal of an appropriately chosen linear series to prove the statements.

In the whole section, a $\mathbb{Q}$-divisor will always be understood to be a $\mathbb{Q}$-Weil divisor, that is a formal finite sum

$$\sum a_i D_i$$

where $a_i \in \mathbb{Q}$ and D_i are prime divisors on X. Furthermore for a birational morphism $\mu : X' \to X$ between quasi-projective manifolds we will denote by $K_{X'/X}$ the relative canonical Cartier divisor defined by the vanishing of the determinant of the derivative $\det d\mu$ (in contrast to the relative dualising bundle $\omega_{X'/X}$ which is only defined up to isomorphism by $\omega_{X'} \otimes \mu^* \omega_X^*$).

9.3.1 Definition. *[Laz04b, Defn.9.1.11] Let X be a quasi-projective manifold, and let L be a line bundle on X. Let $V \subset H^0(X, L)$ be a non-zero finite-dimensional subspace. A log-resolution of the linear system $|V|$ is a birational morphism $\mu : X' \to X$ such that X' is smooth and*

$$\mu^*|V| = |W| + D,$$

where $D + exc(\mu)$ is a SNC divisor and

$$W \subset H^0(X, \mu^* L \otimes \mathcal{O}_{X'}(-D))$$

defines a free linear system. Let $K_{X'/X}$ be the relative canonical divisor, then for any rational number $c > 0$, the multiplier ideal $\mathcal{I}(c \cdot |V|)$ corresponding to c and $|V|$ is the ideal sheaf on X defined by

$$\mathcal{I}(c \cdot |V|) := \mathcal{I}(X, c \cdot |V|) := \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cD]),$$

where $[.]$ is the round-down of the $\mathbb{Q}$-divisor.

Remark. It follows from Hironaka's theorem that every linear series as in the definition actually has a log-resolution. By a theorem of Esnault and Viehweg [Laz04b, Thm.9.2.18], the definition of the multiplier ideal does not depend on the log-resolution. By convention, we define that the multiplier ideal of an empty linear series is the zero ideal.

The next theorem, called the local vanishing theorem establishes one of the fundamental local properties of log-resolutions.

9.3.2 Theorem. *Let X be a quasi-projective manifold, and let L be a line bundle on X. Let $\mu : X' \to X$ be a log-resolution of a finite-dimensional linear series $V \subset H^0(X, L)$, that is*

$$\mu^*|V| = |W| + D,$$

where $|W|$ is a free linear system and D an effective divisor such that $D+exc(\mu)$ is a SNC divisor. Then for all $i > 0$ and $c > 0$

$$R^i\mu_*\mathcal{O}_{X'}(K_{X'/X} - [cD]) = 0.$$

In what follows we will frequently consider the restriction of a linear series and the multiplier ideal to some submanifold. The restriction theorems tell us how these two different types of restrictions are related.

9.3.3 Definition. *[Laz04b, Ex.9.5.5] Let X be a quasi-projective manifold, and let L be a line bundle on X. Let $V \subset H^0(X, L)$ be a non-zero finite-dimensional linear series, and let $B := Bs(|V|)$ be its base locus. Let Y be a submanifold of X that is not contained in B. Then we define the trace linear series by*

$$|V|_Y := \operatorname{im}\left(V \to H^0(Y, L)\right).$$

and for $c > 0$ we denote by

$$\mathcal{J}(Y, c \cdot |V|_Y)$$

the multiplier ideal of the restricted linear series. If $V = H^0(X, L)$, we simply write

$$\mathcal{J}(Y, c \cdot |L|_Y) := \mathcal{J}(Y, c \cdot |V|_Y).$$

Remark. The hypothesis that Y is not contained in the stable base locus of the graded linear series assures that the trace linear series is not trivial.

9.3.4 Theorem. *(Restriction theorem [Laz04b, thm.9.5.1, Ex.9.5.5]) Let X be a quasi-projective manifold, and let L be a line bundle on X. Let $V \subset H^0(X, L)$ be a non-zero finite-dimensional linear series, and let $B := Bs(|V|)$ be its base locus. Let Y be a submanifold of X that is not contained in B. For $c > 0$, denote by $\mathcal{J}(X, c\cdot|V|)_Y$ the image of $\mathcal{J}(X, c\cdot|V|)$ under the restriction morphism $\mathcal{O}_X \to \mathcal{O}_Y$. Then*

$$\mathcal{J}(Y, c \cdot |V|_Y) \subset \mathcal{J}(X, c \cdot |V|)_Y.$$

In view of the restriction theorem it is natural to ask if there are conditions that the inclusion of the ideal sheaves is in fact an equality. The generic restriction theorem states that for a sufficiently general submanifold, this is indeed the case.

9.3.5 Theorem. *(Generic restriction theorem [Laz04b, Thm.9.5.35, Ex.9.5.37]) Let X be a quasi-projective manifold, and let L be a line bundle on X. Let $V \subset H^0(X, L)$ be a non-zero finite-dimensional linear series. Let $\phi : X \to Y$ be a smooth morphism to a quasi-projective manifold Y, and denote by X_y its fibres. Then there exists a non-empty Zariski open subset $Y^* \subset Y$ such that for $y \in Y^*$ we have*

$$\mathcal{J}(X_y, c \cdot |V|_{X_y}) = \mathcal{J}(X, c \cdot |V|)_{X_y}$$

for all $c > 0$.

Before we come to the main statement, let us recall an innocuous, but crucial property.

9.3.6 Lemma. *[Laz04b, Prop.9.5.22] Let $X_1 \times X_2$ be a product of quasi-projective manifolds and denote by $p_i : X_1 \times X_2 \to X_i$ the projection on the i-th factor. Let V_1 (resp. V_2) be a non-empty finite-dimensional linear system on X_1 (resp. X_2) associated to a line bundle L_1 (resp. L_2). Denote by $\pi_1^* V_1 \otimes \pi_2^* V_2$ the linear system on $X_1 \times X_2$ associated to $\pi_1^* L_1 \otimes \pi_2^* L_2$, that is*

$$\pi_1^* V_1 \otimes \pi_2^* V_2 \subset H^0(X_1 \times X_2, \pi_1^* L_1 \otimes \pi_2^* L_2).$$

Then we have

$$\mathcal{I}(X_1 \times X_2, |\pi_1^* V_1 \otimes \pi_2^* V_2|) = \pi_1^{-1}(\mathcal{I}(X_1, |V_1|)) \cdot \pi_2^{-1}(\mathcal{I}(X_2, |V_2|)).$$

The next proposition shows how to generalise the lemma to the relative situation, i.e. the situation of a smooth fibre product over some base manifold. It is clear from examples that we can't hope to have exactly the same result as in the lemma, but the property will still hold on a general fibre. Let us first introduce a notation.

9.3.7 Definition. *Let X be a projective variety and let $\mathcal{I}$ be an ideal sheaf on X. The cosupport of $\mathcal{I}$ is the support of the sheaf $\mathcal{O}_X/\mathcal{I}$. Let $\phi : X \to Y$ be a surjective morphism, then we say that cosupport Z of $\mathcal{I}$ does not project onto Y if $\phi(Z) \subsetneq Y$.*

9.3.8 Proposition. *Let $\phi : X \to Y$ be a smooth surjective morphism between quasi-projective manifolds. Let L be a line bundle on X, and let V be a non-empty finite-dimensional linear system on X attached to L. For $s \in \mathbb{N}$, let*

$$X^s := X \times_Y \ldots \times_Y X$$

be the s-fold fibered product and denote by $\pi^i : X^s \to X$ the projection on the i-th factor. Let $\phi^s : X^s \to Y$ be the induced morphism on Y and denote by X^s_y the fibre over a point y. Let furthermore V_s be the linear sytem attached to $\otimes_{i=1}^s (\pi^i)^* L$ defined by

$$V_s := \otimes_{i=1}^k (\pi^i)^* V.$$

Fix a $c > 0$ and suppose that the cosupport of $\mathcal{I}(X, c \cdot |V|)$ does not project onto Y. Then there exists a non-empty Zariski open subset $Y^* \subset Y$ such that for all $s \in \mathbb{N}$ and for all $y \in Y^*$ we have

$$\mathcal{I}(X^s, c \cdot |V_s|)_{X^s_y} = \mathcal{I}(X^s_y, c \cdot |V_s|_{X^s_y}) = \mathcal{O}_{X^s_y}.$$

Proof. Denote by Z the cosupport of $\mathcal{I}(X, c \cdot |V|)$. By the generic restriction theorem 9.3.5 there exists a Zariski open subset $Y^* \subset Y$ such that

$$\phi(Z) \cap Y^* = \emptyset$$

and for $c > 0$ we have

$$\mathcal{I}(X_y, c \cdot |V|_{X_y}) = \mathcal{I}(X, c \cdot |V|)_{X_y}$$

for all $y \in Y^*$. Hence

$$\mathcal{I}(X_y, c \cdot |V|_{X_y}) = \mathcal{O}_{X_y}$$

for all $y \in Y^*$. By the restriction theorem 9.3.4 for any $s \in \mathbb{N}$ and $y \in Y^*$, the multiplier ideal $\mathcal{I}(X_y^s, c \cdot |V_s|_{X_y^s})$ satisfies

$$\mathcal{I}(X_y^s, c \cdot |V_s|_{X_y^s}) \subset \mathcal{I}(X_y^s, c \cdot |V_s|)_{X_y^s}.$$

If we show that $\mathcal{I}(X_y^s, c \cdot |V_s|_{X_y^s}) = \mathcal{O}_{X_y^s}$ for all $y \in Y^*$, then this inclusion is clearly an equality and the claim follows.

Since X_y^s is isomorphic to the s-fold product $X_y \times \ldots \times X_y$ we have an isomorphism

$$H^0(X_y^s, \otimes_{i=1}^s (\pi^i)^* L)) \simeq \otimes_{i=1}^s (\pi^i)^* H^0(X_y, L)$$

such that the image of the morphism

$$V_s \to H^0(X_y^s, \otimes_{i=1}^s (\pi^i)^* L)$$

identifies to

$$\otimes_{i=1}^s (\pi^i)^* [\mathrm{im}(V \to H^0(X_y, L))].$$

This shows that

$$|V_s|_{X_y^s} = | \otimes_{i=1}^s (\pi^i)^* |V|_{X_y}|.$$

Therefore by lemma 9.3.6

$$\mathcal{I}(X_y^s, c \cdot |V_s|_{X_y^s}) = (\pi^1|_{X_y^s})^{-1}(\mathcal{I}(X_y, c \cdot |V|_{X_y})) \cdot \ldots \cdot (\pi^s|_{X_y^s})^{-1}(\mathcal{I}(X_y, c \cdot |V|_{X_y})).$$

Since we have $\mathcal{I}(X_y, c \cdot |V|_{X_y}) = \mathcal{O}_{X_y}$ for $y \in Y^*$ we obtain

$$\mathcal{I}(X_y^s, c \cdot |V_s|_{X_y^s}) = \mathcal{O}_{X_y^s}.$$

This concludes the proof. □

Remark. Note that we have a (in general strict) inclusion

$$V_s \subset H^0(X^s, \otimes_{i=1}^s (\pi^i)^* L),$$

so the preceeding result shows in particular that

$$\mathcal{I}(X_y^s, c \cdot | \otimes_{i=1}^s (\pi^i)^* L)|_{X_y^s}) = \mathcal{O}_{X_y^s}.$$

Let L be a line bundle of Kodaira dimension d on a projective manifold X. Let $N \in \mathbb{N}$ be such that the linear system $|L^{\otimes N}|$ induces a rational map $\phi : X \dashrightarrow Y$ on a variety of dimension d. If L is not semiample, this map will never be a morphism, but we can resolve the indeterminacies by blowing-up $\mu : X' \to X$. Then

$$\mu^* L^{\otimes N} \otimes \mathcal{O}_{X'}(-D) \simeq M,$$

where D is an effective divisor and M is semiample. Morally speaking the divisor D describes the distance of L from being semiample (more precisely from being nef and abundant, cf. also corollary 10.3.7). The basic idea of the asymptotic multiplier ideal theory is that if L is a line bundle on a projective manifold and L is not semi-ample, then there should exist an ideal sheaf on X that represents this distance. One could of course just use the multiplier ideal of the linear system $|L^{\otimes N}|$, but that would mean that we always have to drag along the integer N which is not canonically chosen.

Although asymptotic multiplier ideals as the multiplier ideals of a linear series can be defined without problems on quasi-projective manifolds we will restrict ourselves from now on to projective manifolds, since we will use the asymptotic multiplier ideals only in this setting.

9.3.9 Definition. *Let X be a projective manifold, and let L be a line bundle on X. A graded linear series $V_\bullet = \{V_k\}_{k\in\mathbb{N}}$ is a set of subspaces*

$$V_k \subset H^0(X, L^{\otimes k}),$$

which are not all empty and such that

$$V_k \cdot V_l \subset V_{k+l} \qquad \forall\ k, l > 0,$$

where $V_k \cdot V_l$ denotes the image of $V_m \otimes V_l$ under the homomorphism

$$H^0(X, L^{\otimes k}) \otimes H^0(X, L^{\otimes l}) \to H^0(X, L^{\otimes k+l})$$

determined by multiplication.

Remark. Note that since we want the collection of V_k not to consist of empty subspaces only, we necessarily have $\kappa(L) \geq 0$. Furthermore it follows from the multiplication property that there are in fact infinitely many V_k that are non-zero.

Given a graded linear series $V_\bullet$, one checks easily [Laz04b, p.276-281] that

$$\mathcal{I}(X, \frac{1}{k}|V_k|) \subset \mathcal{I}(X, \frac{1}{kl}|V_{kl}|)$$

for every $k, l \in \mathbb{N}$. Furthermore it follows from the Noetherian property that the family of ideals that the family of ideals

$$\left\{\mathcal{I}(X, \frac{1}{k}|V_k|)\right\}_{k\geq 1}$$

has a unique maximal element (recall that the multiplier ideal of an empty linear series is the zero ideal sheaf). This brings us to the central definition of this section.

9.3.10 Definition. *Let X be a projective manifold, and let L be a line bundle on X of non-negative Kodaira dimension. Let $V_\bullet = \{V_k\}_{k\in\mathbb{N}}$ be a graded linear series attached to L, then the asymptotic multiplier ideal*

$$\mathcal{I}(X, ||V_\bullet||) = \mathcal{I}(||V_\bullet||)$$

is the unique maximal element among the ideals $\mathcal{I}(X, \frac{1}{k}|V_k|)$. If for all $k \in \mathbb{N}$ we have

$$V_k = H^0(X, L^{\otimes k}),$$

we write $||L|| := V_\bullet$ and

$$\mathcal{I}(||L||) := \mathcal{I}(||V_\bullet||).$$

For our means it is essential to understand the behaviour of multiplier ideals of line bundles under such basic operations as pull-backs or restrictions to submanifolds. In particular the restriction poses serious difficulties since there are several a-priori reasonable possibilities. Since the positivity of a line bundle can change a lot after restricting to some submanifold, it is not a good idea to work with the restriction of the line bundle. Instead one has to consider the restricted linear series.

9.3.11 Definition. *[Laz04b, Ex.11.2.2] Let X be a projective manifold, and let L be a line bundle of non-negative Kodaira dimension on X. Let $||L||$ be the complete graded linear series attached to L, and let*

$$B(||L||) := \cap_{k\geq 1} Bs(|L^{\otimes k}|)$$

be the stable base set of the graded linear series[1]. Let Y be a submanifold of X that is not contained in $B(||L||)$. Then we define the trace graded linear series $||L||_Y := Tr_Y(||L||)_\bullet$ by

$$Tr_Y(||L||)_k := \operatorname{im}\left(H^0(X, L^{\otimes k}) \to H^0(Y, L^{\otimes k})\right).$$

We define the multiplier of the restricted linear series by

$$\mathcal{I}(Y, ||L||_Y) := \mathcal{I}(Y, ||Tr_Y(||L||)_\bullet||).$$

Remark. The hypothesis that Y is not contained in the stable base locus of the graded linear series assures that the trace graded linear series is not trivial. As in the case of the multiplier ideal of a linear series we have a restriction theorem.

9.3.12 Theorem. *(Restriction theorem for asymptotic multiplier ideals [Laz04b, Ex.11.2.2]) Let X be a projective manifold, and let L be a line bundle of non-negative Kodaira dimension on X. Let $||L||$ be a the complete graded linear series attached to L, let Y be a submanifold of X that is not contained in the stable base locus of $||L||$. Denote by $\mathcal{I}(X, ||L||)_Y$ the image of $\mathcal{I}(X, ||L||)$ under the restriction morphism $\mathcal{O}_X \to \mathcal{O}_Y$. Then*

$$\mathcal{I}(Y, ||L||_Y) \subset \mathcal{I}(X, ||L||)_Y.$$

[1]Note that we do not have any particular scheme structure on $B(||L||)$.

The question of an analogue of the generic restriction theorem 9.3.5 is slightly more delicate. In fact given a projective morphism $\phi : X \to Y$ from a projective manifold to some projective variety and L a line bundle on X, the subset $Y^* \subset Y$ such that for $y \in Y^*$, we have

$$\mathcal{I}(X_y, ||L||_{X_y}) = \mathcal{I}(X, ||L||)_{X_y}$$

is Zariski dense [Laz04b, Thm.11.2.8, Rem.11.2.9], but in general not open. However if the cosupport of the asymptotic ideal does not project onto the base we can show the existence of such an open set.

9.3.13 Corollary. *Let $\phi : X \to Y$ be a morphism with connected fibres from a projective manifold X to a normal variety Y, and let L be a line of non-negative Kodaira dimension on X. Let $\mathcal{I}(X, ||L||)$ be the asymptotic multiplier ideal of L, and suppose that the cosupport of $\mathcal{I}(X, ||L||)$ does not project onto Y. Then there exists a non-empty Zariski open subset $Y^* \subset Y$ such that for $y \in Y^*$, the fibre X_y is smooth and*

$$\mathcal{I}(X_y, ||L||_{X_y}) = \mathcal{I}(X, ||L||)_{X_y} \simeq \mathcal{O}_{X_y}.$$

Proof. Let $N \in \mathbb{N}$ be such that

$$\mathcal{I}(\frac{1}{N} \cdot |L^{\otimes N}|) = \mathcal{I}(||L||),$$

and denote by Z its cosupport. Since L has non-negative Kodaira dimension, the base locus of $|L^{\otimes N}|$ is a proper subset of X. Therefore by hypothesis and the generic smoothness theorem there exists a non-empty Zariski open subset $Y' \subset Y$ such that

$$\phi(Z) \cap Y' = \emptyset,$$

and for $y \in Y'$ the fibre X_y is smooth and not contained in the base locus of $|L^{\otimes N}|$[2]. Therefore it makes sense to speak of the restricted linear series $|L^{\otimes N}|_{X_y}$. Set $X' := \phi^{-1}(Y')$, then the generic restriction theorem 9.3.5 applied to the smooth morphism $\phi|_{X'} : X' \to Y'$ and to the linear subsystem of $|L|_{X'}|$ defined by $|L|$ yields the existence of a non-empty Zariski open subset $Y^* \subset Y'$ such that

$$\mathcal{I}(X_y, \frac{1}{N} \cdot |L^{\otimes N}|_{X_y}) = \mathcal{I}(X, \frac{1}{N} \cdot |L^{\otimes N}|)_{X_y}$$

for all $y \in Y^*$. Since $\phi(Z) \cap Y' = \emptyset$, this implies that

$$\mathcal{I}(X_y, \frac{1}{N} \cdot |L^{\otimes N}|_{X_y}) \simeq \mathcal{O}_{X_y}$$

for all $y \in Y^*$, so we are left to show that

$$\mathcal{I}(X_y, \frac{1}{N} \cdot |L^{\otimes N}|_{X_y}) = \mathcal{I}(X_y, ||L||_{X_y})$$

[2] Since the inclusion $Z \subset Bs(|L^{\otimes N}|)$ might be strict, it is nevertheless possible that L has base points on X_y.

for all $y \in Y^*$. Recall that $\mathcal{I}(X_y, ||L||_{X_y})$ is defined as the unique maximal element of

$$\left\{ \mathcal{I}(X_y, \frac{1}{k} \cdot |L^{\otimes k}|_{X_y}) \right\}_{k \geq 1}.$$

Since all these sheaves are ideal sheaves, so contained in $\mathcal{O}_{X_y}$, it is clear that $\mathcal{I}(X_y, \frac{1}{N} \cdot |L^{\otimes N}|_{X_y}) = \mathcal{O}_{X_y}$ is maximal. □

It will be our goal in section 10.3 to prove an analogue of proposition 9.3.8 for a asymptotic multiplier ideals and a flat morphism between projective manifolds. This of course a bit more complicated since the fibre products involved can be very singular. Therefore we have to acquire some additional knowledge and postpone this to proposition 10.3.5. Nevertheless corollary 9.3.13 gives a good impression of what we will have to do: since for a general fibre X_y the asymptotic multiplier ideal of the restricted graded linear series is trivial, we can use the fact the fibres X_y^s of the fibre products $X^s \to Y$ are products to lift this property. The restriction theorem 9.3.12 then allows to conclude.

9.4 Vanishing theorems

Vanishing theorems play a central role for the positivity of direct images sheaves. In fact every time that we can show that for some fibration $\phi : X \to Y$ and a line bundle L (with some positivity property) on X and an ample line bundle A on Y, we have

$$(*) \qquad H^i(Y, \phi_*(L \otimes \omega_X) \otimes H) = 0 \qquad \forall\ i > 0,$$

we have a good chance of showing the weak positivity of $\phi_*(L \otimes \omega_{X/Y}$. The main reason why this works is Castelnuovo-Mumford regularity.

9.4.1 Theorem. *(Castelnuovo-Mumford regularity, [Laz04a, Thm.1.8.5]) Let X be a projective variety and H an ample line bundle that is generated by global sections. Let $\mathcal{S}$ be a coherent sheaf on X, and let m be a natural number such that*

$$H^i(X, \mathcal{S} \otimes H^{m-i}) = 0 \qquad \forall\ i > 0$$

Then $\mathcal{S} \otimes H^m$ is generated by its global sections.

There are two ways of showing a vanishing statement of type $(*)$. The first way which is in some sort the brute force method is to show the vanishing of

$$H^i(X, L \otimes \omega_X \otimes \phi^* H) = 0 \qquad \forall\ i > 0,$$

and to make a spectral sequence argument to obtain the vanishing statement $(*)$. This method only works if L is sufficiently positive on the general fibre of the fibration. The fundamental tool in this case is the relative Kawamata-Viehweg vanishing theorem 9.4.3 below. The second method is much more refined and aims at showing directly the vanishing $(*)$. This approach is based on a famous theorem due to Kollár.

9.4.2 Theorem. *[Kol86, Thm.2.1] Let X be a projective manifold, and let $\phi : X \rightarrow Y$ be a fibration on a normal variety Y. Then for all $j \geq 0$ the coherent sheaf*

$$R^j \phi_* \omega_X$$

is torsion-free. Let furthermore H be an ample line bundle Y, then for all $j \geq 0$ we have

$$H^i(Y, R^j \phi_* \omega_X \otimes H) = 0 \qquad \forall\ i > 0.$$

We will show a number of twisted versions of this theorem (theorems 9.4.5 and theorem 9.4.8). These results are not really new and essentially due to Viehweg, but we will state them in a form that is convenient for our purpose and indicate how to prove them based on what can be found in the literature.

9.4.3 Theorem. *(Relative Kawamata-Viehweg vanishing theorem, [BS95, Lemma2.2.5]) Let X be an normal quasi-projective variety and let $\phi : X \rightarrow Y$ be a projective morphism to a quasi-projective variety. Let L be a ϕ-nef and ϕ-big line bundle. Then*

$$R^i \phi_*(L \otimes \omega_X) = 0 \qquad \forall i > \max_{y \in \phi(\mathrm{Irr}(X))} \dim(\phi^{-1}(y) \cap \mathrm{Irr}(X)).$$

9.4.4 Corollary. *In the situation above, suppose that X has at most rational singularities and ϕ is a flat Cohen-Macaulay fibration on a Gorenstein variety Y. Then $\phi_*(L \otimes \omega_{X/Y})$ is locally free.*

Remark. It is a bit unfortunate that we have to suppose that the base is Gorenstein. This hypothesis is necessary to be able to switch back and forth between $\omega_X \otimes \phi^* \omega_Y^*$ and $\omega_{X/Y}$. A way to work around this problem would be to show a relative Kawamata-Viehweg theorem that involves the relative dualising sheaf $\omega_{X/Y}$ and not the canonical sheaf ω_X.

Proof. Since ϕ is a flat Cohen-Macaulay fibration, the relative dualising sheaf $\omega_{X/Y}$ exists and is flat over Y by theorem 8.4.2. Since Y is Gorenstein, we have

$$\omega_X \simeq \omega_{X/Y} \otimes \phi^* \omega_Y$$

by theorem 8.4.3, so ω_X is flat over Y. By the relative Kawamata-Viehweg vanishing theorem the higher direct images $R^i \phi_*(L \otimes \omega_X)$ vanish, so $\phi_*(L \otimes \omega_X)$ is locally free ([Mum70, Cor.2,p. 50], see also [Har77, III,Thm.12.11]). Hence $\phi_*(L \otimes \omega_{X/Y}) \simeq \phi_*(L \otimes \omega_X) \otimes \omega_Y^*$ is locally free. □

For a semiample line bundle it is no longer possible to vanish all the cohomology of the total space and the higher direct images, but it is still possible to show a vanishing result on the base.

9.4.5 Theorem. *[Vie95, Cor. 2.35,Cor. 2.37] Let X be a projective Cohen-Macaulay variety with at most rational singularities, and let L be a semiample line bundle on X. Let $\phi : X \rightarrow Y$ be a fibration on a normal variety Y, then for all $j \geq 0$ the coherent sheaf*

$$R^j \phi_*(L \otimes \omega_X)$$

is torsion-free. Let furthermore H be an ample line bundle on Y, then

$$H^i(Y, \phi_*(L \otimes \omega_X) \otimes H) = 0 \qquad \forall\ i > 0.$$

If furthermore H is very ample, the sheaf

$$\phi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y + 1}$$

is generated by global sections.

The theorem is a twisted version of Kollár's theorem 9.4.2 and in fact can be derived from it. Since the technique is quite interesting we show how to prove the vanishing of the first cohomology group.

9.4.6 Lemma. *Let $\phi : X \rightarrow Y$ be a fibration from a projective Cohen-Macaulay variety with at most rational singularities X to a normal variety Y. Let L be a semi-ample line bundle over X. Let H be an ample line bundle on C, then*

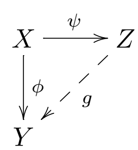
$$H^1(Y, \phi_*(L \otimes \omega_X) \otimes H) = 0$$

Proof. *Step 1. Reduction to the case X smooth.* Let $d : X' \rightarrow X$ be a desingularisation, then $d_* \omega_{X'} \simeq \omega_X$ and $R^i d_* \omega_{X'} = 0$ for $i > 0$, so

$$(\phi \circ d)_*(d^* L \otimes \omega_{X'}) \otimes H \simeq \phi_*(L \otimes \omega_X) \otimes H.$$

Hence it is sufficient to show the vanishing for $(\phi \circ d)_*(d^* L \otimes \omega_{X'}) \otimes H$. Since $d^* L$ is semiample, we are reduced to the case X smooth.

Step 2. Suppose that $L \otimes \phi^* H$ is globally generated. Since $L \otimes \phi^* H$ is globally generated, there exists a fibration $\psi : X \rightarrow Z$ on a normal variety Z such that $L \otimes \phi^* H \simeq \psi^* N$, with N ample on Z. Let us show that ψ is a Y-morphism, that is there exists a morphism $g : Z \rightarrow Y$ such that $\phi = g \circ \psi$. By the rigidity lemma [Deb01, Lemma 1.15] it is sufficient to show that ϕ contracts every fibre of ψ.



We argue by contradiction, so let F be a ψ-fibre such that there exists a projective curve $C \subset F$ such that $\phi(C)$ is a curve. Since L is semiample, so nef the projection formula yields

$$c_1(\psi^* N) \cdot C = c_1(L \otimes \phi^* H) \cdot C \geq c_1(H) \cdot \phi_* C > 0.$$

But C is contracted by ψ, a contradiction.

It is now clear that

$$g_*(\psi_* \omega_X \otimes N) \simeq \phi_*(\omega_X \otimes L) \otimes H,$$

so the Leray spectral sequence applied to g yields an injective map

$$H^1(Y, \phi_*(\omega_X \otimes L) \otimes H) \hookrightarrow H^1(Z, N \otimes \psi_* \omega_X).$$

Yet by Kollár's theorem 9.4.2 we have

$$H^j(Y, N \otimes \psi_* \omega_X) = 0 \qquad \forall\ j > 0.$$

Step 3. Reduction to the globally generated case (we follow Kollár [Kol86]). Since L and $\phi^* H$ are semiample, the line bundle $L \otimes \phi^* H$ is semiample, so there exists a $m \in \mathbb{N}$ such that $(L \otimes \phi^* H)^{\otimes m}$ is globally generated.

We argue by induction on the dimension k of the base locus B of the linear system $|L \otimes \phi^* H|$, the case $k = -\infty$ has been treated in step 2. Suppose now that we have shown the statement if the base locus has dimension at most $k-1$, and suppose that the components of B have dimension at most k. Let $A \subset B$ be a component of dimension k and let $D \in |(L \otimes \phi^* H)^{\otimes m}|$ be a smooth divisor such that $A \not\subset D$. Let $\mu : X' \rightarrow X$ be the m-sheeted cyclic cover ramifying along D. Then by [Laz04a, Rem.4.1.7] we have

$$\mu_* \mathcal{O}_{X'} \simeq (\mu_* \omega_{X'/X})^* \simeq \bigoplus_{j=0}^{m-1} (L \otimes \phi^* H)^{-j}.$$

Hence

$$\begin{aligned} (\phi \circ \mu)_* (\mu^* (L \otimes \phi^* H) \otimes \omega_{X'}) &\simeq \phi_* (\mu_* \omega_{X'/X} \otimes (L \otimes \phi^* H) \otimes \omega_X) \\ &\simeq \bigoplus_{j=0}^{m-1} \phi_* ((L \otimes \phi^* H)^{1+j} \otimes \omega_X), \end{aligned}$$

so

$$H^1(Y, \phi_*(L \otimes \omega_X) \otimes H) \subset H^1(Y, (\phi \circ \mu)_* (\mu^* L \otimes \omega_{X'}) \otimes H).$$

In particular it is sufficient to show the vanishing on the right hand side. Since $\frac{1}{m} \mu^* D$ is a section of $\mu^* L$ that does not vanish on $\mu^{-1}(A)$, the base locus of $\mu^* L$ has at least one k-dimensional component less than $\mu^{-1}(B)$. By an induction on the number of k-dimensional irreducible components of the base locus, one sees that after taking a finite number of coverings we may suppose that the base locus of $\mu^* L$ has no k-dimensional components at all. Since $\mu^* L$ is semiample we conclude by the induction hypothesis. □

It is possible to improve theorem 9.4.5 and lemma 9.4.6 significantly by introducing multiplier ideals. In fact these results should also hold for line bundles with non-negative Kodaira dimension if we take the base locus into account in some appropriate way. This is exactly what can be done by the asymptotic multiplier ideal of the line bundle. Let us start with an immediate corollary to the local vanishing theorem.

9.4.7 Corollary. *Let X be a projective manifold, and let $\phi : X \rightarrow Y$ be a fibration on a normal variety. Let L be a line bundle on X, and let $\mu : X' \rightarrow X$ be a log-resolution of the linear series $|L|$, that is*

$$\mu^* |L| = |M| + D,$$

where $|M|$ is a free linear system and D an effective divisor such that $D+exc(\mu)$ is a SNC divisor. Then for all $i \geq 0$ and $c > 0$

$$R^i(\phi \circ \mu)_*(\mu^* L \otimes \mathcal{O}_{X'}(-[cD]) \otimes \omega_{X'}) \simeq R^i\phi_*(L \otimes \mathcal{J}(c \cdot |L|) \otimes \omega_X).$$

Proof. By the local vanishing theorem 9.3.2 and the projection formula, we have

$$R^j\mu_*(\mu^*(L \otimes \omega_X) \otimes \mathcal{O}_{X'}(K_{X'/X} - [cF])) = 0$$

for all $j > 0$. Therefore the spectral sequence

$$R^p\phi_* R^q\mu_*(\mu^*(L\otimes\omega_X)\otimes\mathcal{O}_{X'}(K_{X'/X}-[cF])) \Rightarrow R^{p+q}(\phi\circ\mu)_*(\mu^*(L\otimes\omega_X)\otimes\mathcal{O}_{X'}(K_{X'/X}-[cF]))$$

simplifies to

$$R^i\phi_*\mu_*(\mu^*(L\otimes\omega_X)\otimes\mathcal{O}_{X'}(K_{X'/X}-[cF])) \simeq R^i(\phi\circ\mu)_*(\mu^*(L\otimes\omega_X)\otimes\mathcal{O}_{X'}(K_{X'/X}-[cF]))$$

for all $i \geq 0$. Therefore the spectral sequence degenerates, since

$$\mu^*(L \otimes \omega_X) \otimes \mathcal{O}_{X'}(K_{X'/X} - [cF]) \simeq \mu^* L \otimes \mathcal{O}_{X'}(-[cF]) \otimes \omega_{X'},$$

we conclude with the projection formula. □

9.4.8 Theorem. *Let X be a projective manifold and let L be a line bundle of non-negative Kodaira dimension on X. Denote by $\mathcal{J}(||L||)$ the asymptotic multiplier ideal sheaf of L. Let $\phi : X \to Y$ be a fibration on a normal variety Y, then for all $j \geq 0$ the coherent sheaf*

$$R^j\phi_*(L \otimes \mathcal{J}(||L||) \otimes \omega_X)$$

is torsion-free. Let furthermore H be an ample line bundle Y, then

$$H^i(Y, \phi_*(L \otimes \mathcal{J}(||L||) \otimes \omega_X) \otimes H) = 0 \qquad \forall\ i > 0.$$

If furthermore H is very ample, the sheaf

$$\phi_*(L \otimes \mathcal{J}(||L||) \otimes \omega_X) \otimes H^{\otimes \dim Y+1}$$

is generated by global sections.

Remark. This result, although not stated like this, is essentially a corollary of [Vie95, Thm.2.33, Cor.2.34] and we only show how to prove it modulo these results, following the ideas of [Vie95, Cor 2.37] and [Kol86, p.18].

Proof. Fix $N \in \mathbb{N}$ such that the multiplier ideal of the linear system $|NL|$ can be used to compute the asymptotic multiplier ideal, that is

$$\mathcal{J}(\frac{1}{N}|NL|) = \mathcal{J}(||L||).$$

Let $\mu : X' \to X$ be a log resolution of the linear sytem $|NL|$, that is

$$\mu^*|NL| = |M| + D,$$

where M is a free linear system and D is an effective divisor such that $D+exc(\mu)$ has SNC support. Then

$$\mathcal{I}(||L||) = \mathcal{I}(\frac{1}{N}|NL|) = \mu_*(\mathcal{O}_{X'}(K_{X'/X} - [\frac{1}{N}D])),$$

so corollary 9.4.7 implies

$$R^i(\phi \circ \mu)_*(\mu^* L \otimes \mathcal{O}_{X'}(-[\frac{1}{N}D]) \otimes \omega_{X'}) \simeq R^i \phi_*[L \otimes \mathcal{I}(||L||) \otimes \omega_X]$$

for all $i \geq 0$. It follows that up to replacing X by X' and L by $\mu^* L$, we may suppose without loss of generality that

$$|NL| = |M| + D,$$

where M is a free linear system on X and D is an effective SNC divisor on X. In particular we can suppose that

$$\mathcal{I}(||L||) = \mathcal{I}(\frac{1}{N}|NL|) = \mathcal{O}_X(-[\frac{1}{N}D]).$$

The torsion-freeness of the higher direct image sheaves is then exactly [Vie95, Cor.2.34]. To lighten the notation we set

Step 1. Construction of the exact sequence. Let $\gamma \in \mathbb{N}$ be such that $H^{\otimes \gamma}$ is very ample and

$$H^i(Y, \phi_*(L \otimes \mathcal{I}(||L||) \otimes \omega_X) \otimes H^{\otimes \gamma+1}) = 0$$

for all $i \geq 1$. Fix a general element $B \in |H^{\otimes \gamma}|$ such that B is a normal variety, the preimage $A := \phi^{-1}(B) \in |\phi^* H^{\otimes \gamma}|$ is a manifold, the manifold B is not in the base locus of $|L^{\otimes N}|$ and

$$\mathcal{I}(A, \frac{1}{N} \cdot |L^{\otimes N}|_A) = \mathcal{I}(X, \frac{1}{N} \cdot |L^{\otimes N}|)_A = \mathcal{O}_A(-[\frac{1}{N}D]).$$

Denote by $\phi|_A : A \rightarrow B$ the restriction of ϕ to A. Then we have an exact sequence

$$\begin{aligned} 0 \quad &\rightarrow \quad L \otimes \mathcal{O}_X(-[\frac{1}{N}D]) \otimes \omega_X \otimes \phi^* H \\ &\rightarrow \quad L \otimes \mathcal{O}_X(-[\frac{1}{N}D]) \otimes \omega_X \otimes \phi^* H^{\otimes \gamma+1} \\ &\rightarrow \quad L|_A \otimes \mathcal{O}_A(-[\frac{1}{N}D]) \otimes \omega_A \otimes \phi|_A^* H \rightarrow 0 \end{aligned}$$

We want to show that this sequence pushes down to an exact sequence

$$\begin{aligned} (*) \qquad 0 \quad &\rightarrow \quad \phi_*(L \otimes \mathcal{O}_X(-[\frac{1}{N}D]) \otimes \omega_X) \otimes H \\ &\rightarrow \quad \phi_*(L \otimes \mathcal{O}_X(-[\frac{1}{N}D]) \otimes \omega_X) \otimes H^{\otimes \gamma+1} \\ &\rightarrow \quad (\phi|_A)_*(L|_A \otimes \mathcal{O}_A(-[\frac{1}{N}D]) \otimes \omega_A) \otimes H|_B \rightarrow 0, \end{aligned}$$

so we have to show that the morphism

$$\tau : R^1\phi_*(L\otimes \mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X)\otimes H \to R^1\phi_*(L\otimes \mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X)\otimes H^{\otimes\gamma+1}$$

is injective. By the projection formula

$$R^1\phi_*(L\otimes\mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X\otimes\phi^*H^{\otimes\gamma+1}) \simeq R^1\phi_*(L\otimes\mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X\otimes\phi^*H)\otimes H^{\otimes\gamma}$$

and the morphism τ identifies to the morphism

$$[\mathcal{O}_Y \to H^{\otimes\gamma}]\otimes R^1\phi_*(L\otimes\mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X),$$

where $\mathcal{O}_Y \to H^{\otimes\gamma}$ is the section corresponding to the divisor B. This evaluation morphism is an injective morphism of sheaves, since A is general and $R^1\phi_*(L\otimes \mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X)$ is torsion-free, the tensored morphism is still injective.

Step 2. Vanishing of the first cohomology. It follows from the long exact cohomology sequence associated to the exact sequence $(*)$ that

$$H^1(Y,\phi_*(L\otimes\mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X)\otimes H) = 0$$

if and only if the morphism

$$H^0(Y,\phi_*(L\otimes\mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X)\otimes H^{\otimes\gamma+1}) \to H^0(B,(\phi|_A)_*(L|_A\otimes\mathcal{O}_A(-[\frac{1}{N}D])\otimes\omega_A)\otimes H|_B)$$

is surjective. This is of course equivalent to showing that

$$(**) \qquad H^0(X,L\otimes\mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X\otimes\phi^*H^{\otimes\gamma+1}) \to H^0(A,L|_A\otimes\mathcal{O}_A(-[\frac{1}{N}D])\otimes\omega_A\otimes\phi|_A^*H)$$

is surjective. By construction, the line bundle $(L\otimes\phi^*H)^{\otimes N}\otimes\mathcal{O}_X(-D)$ is semiample. The divisor A is an element of $|\phi^*H^{\otimes\gamma}|$, so for $\nu\in\mathbb{N}$ such that $N\cdot\nu>\gamma$, the line bundle $\phi^*H^{\otimes(N\cdot\nu)}\otimes\mathcal{O}_X(-A)$ has global sections. Since $L^{\otimes N}\otimes\mathcal{O}_X(-D)$ is semiample, it is then clear that for ν sufficiently high, we have

$$H^0(X,((L\otimes\phi^*H)^{\otimes N}\otimes\mathcal{O}_X(-D))^{\otimes\nu}\otimes\mathcal{O}_X(-A))\neq 0.$$

Thus the surjectivity of $(**)$ follows from [Vie95, Thm.2.33].

Step 3. Vanishing of the higher cohomology. We will proceed by induction on the dimension of Y, the case $\dim Y = 1$ is covered by step 2. By the long exact cohomology sequence associated to the exact sequence $(*)$ it follows by the induction hypothesis that

$$H^i(Y,\phi_*(L\otimes\mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X)\otimes H) \simeq H^i(Y,\phi_*(L\otimes\mathcal{O}_X(-[\frac{1}{N}D])\otimes\omega_X)\otimes H^{\otimes\gamma+1})$$

for all $i\geq 2$. The second term is zero by construction of γ, so we obtain the stated vanishing.

The global generation statement is an immediate corollary of the stated vanishing and Castelnuovo-Mumford regularity (theorem 9.4.1). □

9.5 Finite flat morphisms

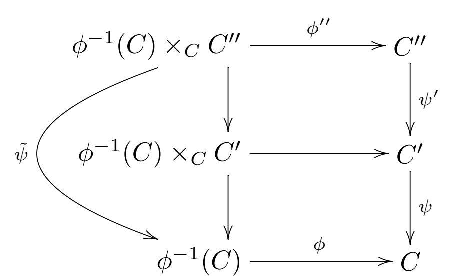
We show a positivity result for direct image sheaves of finite flat morphisms, which is an easy generalisation of a result by Lazarsfeld [PS00, Appendix]. It is interesting to study the proof since it points out some of the difficulties that we will encounter later in the study of fibrations. The finiteness of the morphism $\phi : X \to Y$ allows us to do things that won't work for more complicated maps: if $C \subset Y$ is an *arbitrary* curve that is not contained in the branch locus, the fibre $C \times_Y X$ is a union of reduced curves and up to normalising, we understand its geometry very well. For a fibration this would only be possible for *sufficiently general* curves, so weak positivity can't be shown this way. A second advantage of finite morphisms is that the relation between the sections of a sheaf $\mathcal{F}$ and $\phi_*\mathcal{F}$ is much „closer". In fact, if $y \in Y$ is not in the branch locus and $U \subset Y$ is a small analytic neighborhood of y, the module $\Gamma(U, \phi_*(\mathcal{F}))$ is a direct sum $\oplus_x \Gamma(V_x, \mathcal{F})$ where the sum goes over the points of the fibre $\phi^{-1}(x)$ and V_x is the connected component of $\phi^{-1}(U)$ that contains x. This can't happen for a morphism with connected fibres and we will see this idea again in the proof.

Notation. For a finite morphism $\phi : X \to Y$ it is standard to call the ϕ-singular locus the branch locus.

9.5.1 Proposition. *Let $\phi : X \to Y$ be a finite flat morphism from a reduced projective Cohen-Macaulay scheme X to a projective manifold Y. Let V be a nef vector bundle on X and $C \subset Y$ a curve that is not contained in the branch locus. Then the restriction $\phi_*(V \otimes \omega_{X/Y}) \otimes \mathcal{O}_C$ is nef.*

Proof. The coherent sheaf $V \otimes \omega_{X/Y}$ is flat over Y by theorem 8.4.2, since there are no higher direct images, the flatness of ϕ implies that $\phi_*(V \otimes \omega_{X/Y})$ is locally free.

Step 1. Reduction to the case where Y is a curve. Let $\psi : C' \to C$ be the normalisation, and let $\phi' : \phi^{-1}(C) \times_C C' \to C'$ be the induced morphism. Since C is not contained in the branch locus, the scheme $\phi^{-1}(C) \times_C C'$ is generically reduced and flat over the smooth curve C', so it is reduced by [Laz04a, p.246]. By lemma 9.5.2, there exists a finite covering $\psi' : C'' \to C'$ such that the restriction of the induced morphism $\phi'' : \phi^{-1}(C) \times_C C'' \to C''$ to any irreducible component is an isomorphism. We repeat the same argument to see that $\phi^{-1}(C) \times_C C''$ is reduced and Cohen-Macaulay.



Denote by $\tilde{\psi} : \phi^{-1}(C) \times_C C'' \to \phi^{-1}(C)$ the canonical map on the first factor, then [Kle80, Lemma3,iii] implies

$$(\psi \circ \psi')^*(\phi_* V \otimes \omega_{X/Y}) \simeq \phi''_* (\tilde{\psi})^*(V \otimes \omega_{X/Y}).$$

Since $(\psi \circ \psi')^*(\phi_* V \otimes \omega_{X/Y})$ is antinef if and only if $\phi_*(V \otimes \omega_{X/Y}) \otimes \mathcal{O}_C$ is antinef, we are reduced to consider ϕ''. To simplify notation, we set $C'' = Y$ and $\phi^{-1}(C) \times_C C'' = X$ and $\phi'' = \phi$.

Step 2. Comparing direct images. We have reduced to the case where $\phi : X \to Y$ is a morphism to a smooth curve such that the restriction to an irreducible component of X is an isomorphism. Relative duality implies

$$\phi_*(V \otimes \omega_{X/Y}) \simeq (\phi_* V^*)^{**},$$

so it is sufficient to show that $\phi_* V^*$ is antinef. Denote by $X = Y_1 \cup \ldots \cup Y_s$ the irreducible components, then for all $i \in 1, \ldots, s$, the restriction $V^* \otimes \mathcal{O}_{Y_i}$ is antinef, so

$$\oplus_{i=1}^s (\phi|_{Y_i})_* (V^* \otimes \mathcal{O}_{Y_i})$$

is an antinef vector bundle on Y. For all Zariski open sets $Y^* \subset Y$ we have a natural inclusion

$$H^0(\phi^{-1}(Y^*), V^*) \to \oplus_{i=1}^s H^0(\phi^{-1}(Y^*) \cap Y_i, V^* \otimes \mathcal{O}_{Y_i}).$$

These inclusions induce a natural map

$$\phi_* V^* \to \oplus_{i=1}^s (\phi|_{Y_i})_* (V^* \otimes \mathcal{O}_{Y_i}),$$

which is easily seen to be an isomorphism on the complement of the branch locus. We conclude by lemma 9.1.3. □

9.5.2 Lemma. *[PS00, App.Lemma C] Let $f : X \to Y$ be a branched covering of curves such that Y is smooth and irreducible and X reduced. Then there exists a finite covering $p : Y' \to Y$ of smooth irreducible curves such that $Y' := X \times_Y Y'$ is a reduced curve and the restriction of the induced map $f' : X' \to Y'$ to an irreducible component of X' is an isomorphism.*

Chapter 10

Positivity of direct images sheaves

This chapter contains the core of our considerations. The goal is to give sufficient conditions such that the problem 7.1.3 reduces to the more tractable problem 7.1.4. This goal is achieved with sufficient generality with the reduction lemma 10.2.4 which is our main technical result and really the heart of the whole direct image problem. The statement is quite technical but the idea is as follows: to show the weak positivity of $\phi_*(L \otimes \omega_{X/Y})$ it is sufficient to show that for some very ample line bundle H and all $s \in \mathbb{N}$, the sheaves

$$[\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]} \otimes \omega_Y \otimes H^{\otimes \dim Y + 1}$$

are globally generated on some non-empty open set. This could be simpler, since we „added a lot of positivity“ to the sheaf. On the other hand it is very difficult to say something about the tensor products $[\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]}$. The so-called fibre product trick will allow us to get around this problem by showing that up to correction terms the sheaf $\phi_*(L \otimes \omega_{X/Y})^{[\otimes s]}$ can be realized as a direct image sheaf of some associated fibration $X^{(s)} \to Y$.

Once we have proven the reduction lemma 10.2.4, we can earn the fruits of our labour: the sections 10.3 and 10.4 contain the (relatively simple) proofs of our main results and some corollaries. For the reader's convenience, let us recall the global strategy as explained in more detail in section 7.2 of the introduction.

Goal 1. Do the fibre product trick in a singular setting.

Given a flat fibration $\phi : X \to Y$, we want to understand the geometry of the s-times fibred product

$$X^s := X \times_Y \ldots \times_Y X$$

and the induced morphism $\phi^s : X^s \to Y$. This means that we have to show two things: for every $s \in \mathbb{N}$, there is a line bundle L_s on X^s such that

$$[\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]} \simeq \phi^s_*(L_s \otimes \omega_{X^s/Y}),$$

furthermore there exists a non-empty open set $Y^* \subset Y$ such that for every $s \in \mathbb{N}$, the preimage $(\phi^s)^{-1}(Y^*)$ has at most rational singularities.

Goal 2. Show the generic generation by global sections.

Let i be the index of the $\mathbb{Q}$-Gorenstein variety Y. Under appropriate hypothesis on L, we want to show that for H a very ample line bundle on Y, there exists a non-empty open set $\tilde{Y} \subset Y$ such that for every $s \in \mathbb{N}$, the sheaf

$$[\phi_*(L \otimes \omega_{X/Y})]^{[\otimes si]} \otimes \mathcal{O}(-iK_Y) \otimes H^{\otimes i(\dim Y+1)}$$

is generated by global sections on $\tilde{Y}$. In particular we have to assure that the set $\tilde{Y}$ is independent of s.

10.1 Fibre products

Give a flat fibration $\phi : X \to Y$, we will now discuss the geometry of the fibre products $X^s := X \times_Y \ldots \times_Y X$. A priori the singularities of these varieties could be arbitrarily bad, but if X is $\mathbb{Q}$-Gorenstein with at most canonical singularities, X^s is $\mathbb{Q}$-Gorenstein with at most canonical singularities over a general point of Y (lemma 10.1.3). Furthermore we will show how the tensor products $[\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]}$ can be expressed in a more geometric way via the fibration $X^s \to Y$ (lemma 10.1.7).

10.1.1 Construction. *Let $\phi : X \to Y$ be a flat Cohen-Macaulay fibration from a Cohen-Macaulay variety X to a normal variety Y, and let L be a line bundle over X. Let furthermore*

$$X^s := X \times_Y \ldots \times_Y X$$

be the s-times fibered product taken with respect to the map ϕ. Denote $\pi^s : X^s \to X$ the projection on the s-th factor and $\phi^s : X^s \to Y$ and $\pi' : X^s \to X^{s-1}$ the induced maps. We sum up this in the commutative diagram

$$\begin{array}{ccc} X^s & \xrightarrow{\pi^s} & X \\ \downarrow{\scriptstyle \pi'} & \searrow^{\phi^s} & \downarrow{\scriptstyle \phi} \\ X^{s-1} & \xrightarrow{\phi^{s-1}} & Y \end{array}$$

Corollary 8.4.5 shows that ϕ^s is a flat Cohen-Macaulay fibration and

$$\omega_{X^s/Y} = (\pi^s)^*\omega_{X/Y} \otimes (\pi')^*\omega_{X^{s-1}/Y}.$$

By the same corollary, X^s is integral if X is integral. We define inductively for $s > 1$

$$L_s := (\pi^s)^*L \otimes (\pi')^*L_{s-1},$$

where $L_1 := L$.

10.1.2 Lemma. *In the situation of construction 10.1.1, suppose that X is normal and that ϕ has generically reduced fibres in codimension 1. Then X^s is normal.*

Proof. Since X^s is Cohen-Macaulay, by theorem 8.2.4 it is sufficient to show that it is nonsingular in codimension 1. Since X^s is equidimensional over Y and Y is nonsingular in codimension 1, we may suppose without loss of generality that Y is smooth. Since ϕ has generically reduced fibres in codimension 1, we may furthermore suppose that all the fibres are generically reduced. Note first that the generalised Seidenberg theorem [BS95, Thm.1.7.1] implies that the general fibre of ϕ is normal, hence nonsingular in codimension 1. Set

$$Z := \{x \in X_{\text{reg}} \mid \text{rk}((T\phi)_x : T_{X,x} \to T_{Y,\phi(x)}) < \dim Y\},$$

then $x \in Z$ if and only if the fibre $\phi^{-1}(\phi(x))$ is singular in x and $x \notin X_{\text{sing}}$. The general fibre is normal, so nonsingular in codimension 1. Set $X^* := X_{\text{reg}} \setminus Z$, then it follows that

$$\text{codim}_X(X \setminus X^*) \geq 2$$

and $\phi|_{X^*} : X^* \to Y$ is smooth. Since ϕ is equidimensional, it follows inductively that the s-times fibered product

$$(X^*)^s := X^* \times_Y \ldots \times_Y X^*$$

is smooth and

$$\text{codim}_{X^s}(X^s \setminus (X^*)^s) \geq 2.$$

This concludes the proof. □

10.1.3 Lemma. *In the situation of construction 10.1.1, suppose that X is a normal $\mathbb{Q}$-Gorenstein variety of index i with at most canonical singularities. Then there exists a non-empty open set $Y^* \subset Y$ such that for all $s > 0$ and $X^* := \phi^{-1}(Y^*)$, the s-times fibered product*

$$X^* \times_{Y^*} \ldots \times_{Y^*} X^*$$

is a normal $\mathbb{Q}$-Gorenstein variety of index i with at most canonical singularities.

Proof. Let $d : \tilde{X} \to X$ be a desingularisation of X, then $\phi \circ d : \tilde{X} \to Y$ is generically smooth. We choose $Y^* \subset Y$ such that over Y^*, the morphism $\phi \circ d$ is a smooth morphism of quasi-projective manifolds and ϕ has generically reduced fibres in codimension 1. Since the claim is local on the base and to simplify the notation, we will suppose without loss of generality that $Y^* = Y$. Then the s-times fibered product

$$(\tilde{X})^s := \tilde{X} \times_Y \ldots \times_Y \tilde{X}$$

is smooth and smooth over Y. We denote by $\tilde{\pi}^j : \tilde{X}^s \to \tilde{X}$ the projection on the j-th factor. Since ϕ has generically reduced fibres in codimension 1, the fibre product X^s is normal by lemma 10.1.2 and we denote by $\pi^j : X^s \to X$ the projection on the j-th factor. Denote by $d^s : \tilde{X}^s \to X^s$ the birational morphism induced by d, then we have a commutative diagram

$$\begin{array}{ccc} (\tilde{X})^s & \xrightarrow{\tilde{\pi}^j} & \tilde{X} \\ \Big\downarrow{\scriptstyle d^s} & & \Big\downarrow{\scriptstyle d} \\ X^s & \xrightarrow{\pi^j} & X \\ & & \Big\downarrow{\scriptstyle \phi} \\ & & Y. \end{array}$$

We want to show that the canonical divisor K_{X^s} is $\mathbb{Q}$-Cartier of index i and that

$$(*) \qquad iK_{\tilde{X}^s} = (d^s)^*(iK_{X^s}) + F,$$

where F is effective and supported in the d^s-exceptional locus. Since Y is smooth, we can define relative canonical divisors

$$K_{\tilde{X}^s/Y} := K_{\tilde{X}^s} - (\phi^s \circ d)^* K_Y, \quad K_{X^s/Y} := K_{X^s} - (\phi^s) K_Y,$$
$$K_{\tilde{X}/Y} = K_{\tilde{X}} - (\phi \circ d)^* K_Y, \qquad K_{X/Y} = K_X - \phi^* K_Y$$

and it is clear from the definition that is sufficient to show $(*)$ for the relative canonical divisors. Since $\phi \circ d : \tilde{X} \to Y$ is smooth, we clearly have

$$\omega_{\tilde{X}^s/Y}^{\otimes i} \simeq \otimes_{j=1}^s ((\tilde{\pi})^j)^* \omega_{\tilde{X}/Y}^{\otimes i}$$

and thus an equality of divisors

$$iK_{\tilde{X}^s/Y} \equiv \sum_{j=1}^s ((\tilde{\pi})^j)^* (iK_{\tilde{X}/Y}).$$

Since $\phi : X \to Y$ is a flat Cohen-Macaulay fibration, we have by corollary 8.4.5

$$\omega_{X^s/Y} \simeq \otimes_{j=1}^s (\pi^j)^* \omega_{X/Y}.$$

By the proof of lemma 10.1.2 the s-times fibered product X^s_{reg} is Gorenstein and $\mathrm{codim}_{X^s}(X^s \setminus X^s_{\mathrm{reg}}) \geq 2$. It follows that

$$\mathcal{O}_{X^s_{\mathrm{reg}}}(iK_{X^s/Y}) \simeq \omega_{X^s/Y}^{\otimes i} \otimes \mathcal{O}_{X^s_{\mathrm{reg}}} \simeq \otimes_{j=1}^s (\pi^j)^* \omega_{X/Y}^{\otimes i} \otimes \mathcal{O}_{X^s_{\mathrm{reg}}} \simeq \otimes_{j=1}^s (\pi^j)^* \mathcal{O}_{X^s_{\mathrm{reg}}}(iK_{X/Y})$$

and since this isomorphism holds in codimension 1, it extends to a global isomorphism of the biduals

$$\mathcal{O}_{X^s}(iK_{X^s/Y}) \simeq (\otimes_{j=1}^s (\pi^j)^* \mathcal{O}_X(iK_{X/Y}))^{**}.$$

Yet $K_{X/Y}$ has index i, so $\mathcal{O}_X(iK_{X/Y})$ is locally free. This shows that X^s is $\mathbb{Q}$-Gorenstein of index i and we can write

$$iK_{X^s/Y} \equiv \sum_{j=1}^{s} (\pi^j)^*(iK_{X/Y}).$$

Since X has canonical singularities, we have

$$iK_{\tilde{X}/Y} = d^*(iK_{X/Y}) + E,$$

where E is effective with support in the d-exceptional locus. By what preceeds it follows that

$$\begin{aligned} iK_{\tilde{X}^s/Y} &= \sum_{j=1}^{s}((\tilde{\pi})^j)^*(iK_{\tilde{X}/Y}) = \sum_{j=1}^{s}((\tilde{\pi})^j)^*(d^*(iK_{X/Y}) + E) \\ &= \sum_{j=1}^{s}(d \circ ((\tilde{\pi})^j))^*(iK_{X/Y}) + \sum_{j=1}^{s}((\tilde{\pi})^j)^*E = (d^s)^*(iK_{X^s/Y}) + \sum_{j=1}^{s}((\tilde{\pi})^j)^*E. \end{aligned}$$

Since E is contained in the d-exceptional locus, the effective divisor $\sum_{j=1}^{s}((\tilde{\pi})^j)^*E$ is contained in the d^s-exceptional locus. □

Remark. It would be nice to improve this statement in the following way: suppose furthermore that ϕ has generically reduced fibres in codimension 1, then Y^* may be chosen such that $\operatorname{codim}_Y(Y \setminus Y^*) \geq 2$. If one goes through the proof above, one is tempted to take Y^* such that $\tilde{X}$ is flat over Y^* and the $\phi \circ d$-singular locus intersects Y^* in a smooth divisor. Nevertheless there is a problem, since ϕ having generically reduced fibres in codimension 1 does **not** imply this property for $\phi \circ d$. This makes it hard to say something about $(X^*)^s$.

Recent work by Viehweg [Vie06] suggests a way out of this dilemma, but it is still necessary to make a base change $Y' \to Y$. Since it is not clear how these deep results work in our context, we have to refrain from using these tools and only mention a classical known result in this direction.

10.1.4 Lemma. *(compare [Kol86, Lemma 3.4]) In the situation of construction 10.1.1, suppose that $\phi : X \to Y$ is a flat fibration between quasi-projective manifolds and denote by Δ the ϕ-singular locus. Suppose that there exists an open subset $Y^* \subset Y$ such that*

1.) the intersection $\Delta \cap Y^*$ is a smooth divisor,

2.) the preimage $\phi^{-1}(\Delta \cap Y^*)$ is a simple normal crossings divisor,

3.) and $\operatorname{codim}_Y(Y \setminus Y^*) \geq 2$.

Then $(\phi^s)^{-1}(Y^*) \subset X^s$ is normal with at most rational singularities.

Proof. The conditions 2 and 3 imply that ϕ has generically reduced fibres in codimension 1, so lemma 10.1.2 shows that X^s is smooth. By [Vie83, Lemma 3.6] the singularities of $(\phi^s)^{-1}(Y^*)$ are rational. □

We turn now to the most crucial question of construction 10.1.1: what is the relation between $\phi_*(L\otimes\omega_{X/Y})$ and $\phi_*^s(L_s\otimes\omega_{X^s/Y})$? The ideal relation would of course be

$$(\phi_*(L\otimes\omega_{X/Y})^{\otimes s})\simeq\phi_*^s(L_s\otimes\omega_{X^s/Y}),$$

but this is not true in general. Nevertheless we will, as a demonstration of the general principle, show a case where this is true and then deduce the more technical cases.

10.1.5 Lemma. *(Base change, Gorenstein case) In the situation of construction 10.1.1, suppose that $\phi_*(L\otimes\omega_{X/Y})$ is locally free and ϕ has Gorenstein fibres. Then*

$$\phi_*^s(L_s\otimes\omega_{X^s/Y})\simeq[\phi_*(L\otimes\omega_{X/Y})]^{\otimes s}.$$

Proof. We proceed by induction on s, the case $s=1$ is clear. By construction, we have

$$L_s\otimes\omega_{X^s/Y}\simeq(\pi^s)^*(L\otimes\omega_{X/Y})\otimes(\pi')^*(L_{s-1}\otimes\omega_{X^{s-1}/Y}).$$

Since ϕ has Gorenstein fibres, $\omega_{X/Y}$ is locally free by [Kle80, p.58]. Hence $L\otimes\omega_{X/Y}$ is locally free and the projection formula implies

$$\phi_*^s(L_s\otimes\omega_{X^s/Y})\simeq\phi_*(\pi_*^s(\pi'^*(L_{s-1}\otimes\omega_{X^{s-1}/Y})))\otimes L\otimes\omega_{X/Y}).$$

Since ϕ is flat, we can apply flat base change to obtain

$$\pi_*^s(\pi'^*(L_{s-1}\otimes\omega_{X^{s-1}/Y}))\simeq\phi^*(\phi_*^{s-1}(L_{s-1}\otimes\omega_{X^{s-1}/Y})).$$

By the induction hypothesis

$$\phi_*^{s-1}(L_{s-1}\otimes\omega_{X^{s-1}/Y})\simeq[\phi_*(L\otimes\omega_{X/Y})]^{\otimes s-1}$$

is locally free, so we can apply the projection formula a second time to see that

$$\begin{aligned}\phi_*(L\otimes\omega_{X/Y}\otimes\pi_*^s(\pi'^*(L_{s-1}\otimes\omega_{X^{s-1}/Y}))) &\simeq \phi_*(L\otimes\omega_{X/Y}\otimes\phi^*(\phi_*^{s-1}(L_{s-1}\otimes\omega_{X^{s-1}/Y})))\\ &\simeq \phi_*(L\otimes\omega_{X/Y})\otimes[\phi_*(L\otimes\omega_{X/Y})]^{\otimes s-1},\end{aligned}$$

this concludes the proof. □

10.1.6 Lemma. *(Base change, Cohen-Macaulay case) In the situation of construction 10.1.1, suppose that $\phi_*(L\otimes\omega_{X/Y})$ is locally free. Suppose furthermore that X is normal and Y is Gorenstein. Then*

$$\phi_*^s(L_s\otimes\omega_{X^s/Y})\simeq[\phi_*(L\otimes\omega_{X/Y})]^{\otimes s}.$$

Proof. We proceed by induction on s, the case $s=1$ is clear. Since ϕ is flat, we can apply flat base change to obtain

$$\pi_*^s(\pi'^*(L_{s-1}\otimes\omega_{X^{s-1}/Y}))\simeq\phi^*(\phi_*^{s-1}(L_{s-1}\otimes\omega_{X^{s-1}/Y})).$$

By the induction hypothesis this sheaf is locally free. Since Y is Gorenstein, the relative dualising sheaf satisfies

$$\omega_{X/Y} \simeq \omega_X \otimes \omega_Y^*$$

by theorem 8.4.3, so it is locally free on X_{reg}. By construction, we have

$$L_s \otimes \omega_{X^s/Y} \simeq (\pi^s)^*(L \otimes \omega_{X/Y}) \otimes (\pi')^*(L_{s-1} \otimes \omega_{X^{s-1}/Y}),$$

so the projection formula implies

$$(*) \qquad \pi^s_*(L_s \otimes \omega_{X^s/Y}) \otimes \mathcal{O}_{X_{\text{reg}}} \simeq L \otimes \omega_{X/Y} \otimes \pi^s_*((\pi')^*(L_{s-1} \otimes \omega_{X^{s-1}/Y})) \otimes \mathcal{O}_{X_{\text{reg}}}.$$

Since X is normal, the relative dualising sheaf

$$\omega_{X/Y} \simeq \omega_X \otimes \omega_Y^* \simeq \mathcal{O}_X(K_X) \otimes \omega_Y^*$$

is reflexive, so

$$L \otimes \omega_{X/Y} \otimes \pi^s_*(\pi'^*(L_{s-1} \otimes \omega_{X^{s-1}/Y}))$$

is reflexive. Since π^s is a flat Cohen-Macaulay fibration, the sheaf

$$\pi^s_*(L_s \otimes \omega_{X^s/Y})$$

is reflexive by corollary 8.4.6. Since these sheaves are isomorphic in codimension 1 by $(*)$ and reflexive, we have an isomorphism

$$\pi^s_*(L_s \otimes \omega_{X^s/Y}) \simeq L \otimes \omega_{X/Y} \otimes \phi^*(\phi^{s-1}_*(L_{s-1} \otimes \omega_{X^{s-1}/Y})).$$

The projection formula implies that

$$\phi^s_*(L_s \otimes \omega_{X^s/Y}) \simeq \phi_*(L \otimes \omega_{X/Y}) \otimes [\phi^{s-1}_*(L_{s-1} \otimes \omega_{X^{s-1}/Y})]^{\otimes s-1}.$$

By the induction hypothesis, this concludes the proof. □

In view of the preceeding lemmata, it is not hard to guess how to proceed in the general case: establish an isomorphism on a subset $Y' \subset Y$ such that $Y \setminus Y'$ has codimension at least 2, then extend by taking biduals. Although taking biduals comes along with a certain loss of information, it seems to be inevitable and allows us to weaken considerable the initial conditions.

10.1.7 Lemma. *(Base change, the general case) In the situation of construction 10.1.1, suppose one of the following:*

(H1) ϕ has Gorenstein fibres in codimension 1.

(H2) X is a normal variety.

Then $\phi^s_*(L_s \otimes \omega_{X^s/Y})$ is reflexive and

$$\begin{aligned} \phi^s_*(L_s \otimes \omega_{X^s/Y}) &\simeq [\phi_*(L \otimes \omega_X) \otimes \mathcal{O}_Y(-K_Y)]^{[\otimes s]} \qquad (10.1) \\ &\simeq [\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]}. \end{aligned}$$

Denote by $Y^* \subset Y$ the locus where $\phi_*(L \otimes \omega_X)$ is locally free, then

$$\phi^s_*(L_s \otimes \omega_{X^s/Y}) \otimes \mathcal{O}_{Y^*} \simeq [\phi_*(L \otimes \omega_X)]^{\otimes s} \otimes \mathcal{O}_{Y^*}(-sK_{Y^*}) \qquad (10.2)$$

Proof. Since $\phi^s : X^s \to Y$ is a flat Cohen-Macaulay fibration, the sheaf $\phi^s_*(L_s \otimes \omega_{X^s/Y})$ is reflexive by corollary 8.4.6. Furthermore it is locally free in codimension 1 by proposition 8.1.4. Since Y is normal, it is smooth in codimension 1. Thus lemma 10.1.5 (resp. lemma 10.1.6) shows that there exists an open subset $Y' \subset Y$ such that

$$\operatorname{codim}_Y(Y \setminus Y') \geq 2$$

and

$$\phi^s_*(L_s \otimes \omega_{X^s/Y}) \otimes \mathcal{O}_{Y'} \simeq [\phi_*(L \otimes \omega_{X/Y})]^{\otimes s} \otimes \mathcal{O}_{Y'} \simeq [\phi_*(L \otimes \omega_X) \otimes \mathcal{O}_{Y'}(-K_{Y'})]^{\otimes s}.$$

Since the bidual $[\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]}$ is reflexive, we obtain the isomorphism (10.1). The isomorphism (10.2) is then a direct consequence of the fact that $\mathcal{O}_{Y^*}(-sK_{Y^*})$ is reflexive, so $[\phi_*(L \otimes \omega_X)]^{\otimes s} \otimes \mathcal{O}_{Y^*}(-sK_{Y^*})$ is reflexive. □

10.2 Desingularisation

In the preceeding section we have seen that in some cases the fibre product X^s has rational singularities over a general point of Y, but it has also become clear that we can almost never expect the fibre products to be globally normal with at most rational singularities. This is somehow annoying since vanishing theorems tend to fail for varieties with irrational singularities. We therefore have to take a desingularisation $X^{(s)} \to X^s$ of the fibre product, so the total space will be smooth. This comes at a certain price, since the new map $X^{(s)} \to Y$ is no longer flat. In lemma 10.2.3 we construct the crucial morphism that allows us to deduce positivity properties of $\phi_*(L \otimes \omega_{X/Y})$ from positivity properties of a direct image sheaf of the fibration $X^{(s)} \to Y$. The reduction lemma 10.2.4 summarizes all these technicalities in a compact statement.

10.2.1 Construction. *Suppose that we are in the situation of construction 10.1.1, Let $\nu : (X^s)' \to X^s$ be the composition of the reduction $(X^s)_{red} \to X^s$ and normalisation $(X^s)' \to (X^s)_{red}$, and suppose that ν is a finite morphism (this holds if X is projective [Har77, III, Ex.11.3]). By relative duality for finite morphisms (cf. [Har77, III, ex.6.10., ex.7.2.]), there exists a natural morphism*

$$\nu_*\omega_{(X^s)'} \to \omega_{X^s}.$$

Let $d : X^{(s)} \to (X^s)'$ be a desingularisation, then we have a morphism

$$d_*\omega_{X^{(s)}} \to \omega_{(X^s)'}.$$

Pushing the morphism down on X^s, we obtain a morphism

$$(\nu \circ d)_*(\omega_{X^{(s)}}) \to \nu_*\omega_{(X^s)'} \to \omega_{X^s}, \tag{10.3}$$

which by definition 8.2.9 is an isomorphism on the locus where X^s is normal with at most rational singularities. By the projection formula, the morphism

(10.3) induces a morphism $(\nu \circ d)_*(\omega_{X^{(s)}} \otimes (\nu \circ d)^* L_s) \to L_s \otimes \omega_{X^s}$. Set $\phi^{(s)} := \phi^s \circ \nu \circ d : X^{(s)} \to Y$. Pushing down via ϕ_s we obtain a morphism

$$\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \to \phi_*^s(L_s \otimes \omega_{X^s}), \tag{10.4}$$

which is an isomorphism on the largest open set $Y^* \subset Y$ such that $(\phi^s)^{-1}(Y^*)$ is normal with at most rational singularities.

Remark. Since $\phi^{(s)}$ is not necessarily equidimensional, the direct image sheaf $\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}})$ is torsion-free, but not necessarily reflexive. So even if the morphism (10.4) is an isomorphism in codimension 1, it is not necessarily an isomorphism on Y. This is a significant difference to the case $L \simeq \mathcal{O}_X$, where we have the following result due to Kawamata:

10.2.2 Theorem. *[Kaw81, Thm. 5] Let $\phi : X \to Y$ be a fibration between projective manifolds and let $\Delta \subset Y$ be the ϕ-singular locus. If Δ is a normal crossing divisor, the direct image sheaf $\phi_* \omega_X$ is locally free.*

The reader will have noticed that while in section 10.1 we always worked with the relative dualising sheaf $\omega_{X/Y}$ (or $\omega_{X^s/Y}$), we do now work with the canonical sheaf ω_{X^s}. The reason for this change is that $\phi^{(s)}$ is not necessarily flat, so the relative dualising sheaf does not exist in general. Since the base Y is not necessarily Gorenstein, we can't use the definition 8.4.4 either. Since vanishing theorems are always stated for the canonical sheaf, this defect in the theory does not pose any problem from this point of view. The real problem arises from the fact that morphism (10.4) goes to $\phi_*^s(L_s \otimes \omega_{X^s})$ and not to $\phi_*^s(L_s \otimes \omega_{X^s/Y})$, so formula (10.1) does not apply. With some work it is possible to see that there exists a morphism

$$\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \to \{[\phi_*(L \otimes \omega_{X/Y})]^{\otimes s} \otimes \omega_Y\}^{**},$$

but this is still not exactly what we want. The next lemma shows how to fix this problem if we suppose Y to be $\mathbb{Q}$-Gorenstein.

10.2.3 Lemma. *In the situation of construction 10.2.1 suppose that the hypothesis H1 or the hypothesis H2 of lemma 10.1.7 holds. Suppose furthermore that Y is $\mathbb{Q}$-Gorenstein of index i. Fix $s \in \mathbb{N}$, then there exists a natural map*

$$\tau^s : [\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}})]^{\otimes i} \to [\phi_*(L \otimes \omega_{X/Y})]^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y), \tag{10.5}$$

which is an isomorphism on the largest open set $Y^* \subset Y$ such that

1.) $(\phi^s)^{-1}(Y^*)$ is normal with at most rational singularities,

2.) and Y^* is Gorenstein.

Remark. Note that the set Y^* might a priori depend on s. In the cases we are interested in, we will always be able to find a Y^* that works for all every s (recall that in general an infinite intersection of open sets is not open).

Proof. We have a global morphism

$$\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \to \phi_*^s(L_s \otimes \omega_{X^s}),$$

which is an isomorphism on the largest open subset $Y^* \subset Y$ such that $(\phi^s)^{-1}(Y^*)$ is normal with at most rational singularities. Hence the induced morphism

$$[\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}})]^{\otimes i} \to [\phi_*^s(L_s \otimes \omega_{X^s})]^{\otimes i}$$

is an isomorphism on Y^* and we are left to show the existence of a morphism

$$[\phi_*^s(L_s \otimes \omega_{X^s})]^{\otimes i} \to [\phi_*(L \otimes \omega_{X/Y})]^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y),$$

that is an isomorphism on the Gorenstein locus $\tilde{Y} \subset Y$. Since Y is normal, so regular in codimension 1, it is clear that

$$\operatorname{codim}_Y(Y \setminus \tilde{Y}) \geq 2.$$

If we set $(\tilde{X})^s := (\phi^s)^{-1}(\tilde{Y})$, we have

$$\omega_{(\tilde{X})^s/\tilde{Y}} \simeq \omega_{(\tilde{X})^s} \otimes (\phi^s)^* \omega_{\tilde{Y}}^* \simeq \omega_{(\tilde{X})^s} \otimes (\phi^s)^* \mathcal{O}_{\tilde{Y}}(-K_{\tilde{Y}})$$

by theorem 8.4.3. We have an isomorphism

$$[\phi_*^s(L_s \otimes \omega_{X^s})]^{\otimes i} \otimes \mathcal{O}_{\tilde{Y}} \simeq [\phi_*^s(L_s \otimes \omega_{X^s/Y})]^{\otimes i} \otimes \mathcal{O}_Y(iK_Y) \otimes \mathcal{O}_{\tilde{Y}}$$

on the Gorenstein locus. Since the canonical divisor K_Y is $\mathbb{Q}$-Cartier of index i, the reflexive sheaf $\mathcal{O}_Y(iK_Y)$ is locally free. It follows that

$$[\phi_*^s(L_s \otimes \omega_{X^s/Y})]^{\otimes i} \otimes \mathcal{O}_Y(iK_Y) \simeq [\phi_*(L \otimes \omega_{X/Y})]^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y)$$

is reflexive, so the isomorphism on $\tilde{Y}$ extends to a morphism on Y by reflexiveness of the $\mathcal{H}om$-sheaf (corollary 8.1.8). □

10.2.4 Lemma. *(Reduction lemma 1) Let $\phi : X \to Y$ be a flat Cohen-Macaulay fibration on Y a $\mathbb{Q}$-Gorenstein variety of index i. Suppose that X is $\mathbb{Q}$-Gorenstein and has at most canonical singularities. Then there exists a maximal non-empty Zariski open subset $Y^* \subset Y$ such that for all $s \in \mathbb{N}$, the fibre product $(\phi^s)^{-1}(Y^*)$ has at most canonical singularities and Y^* is Gorenstein. If ϕ is generically smooth, then Y^* contains the ϕ-smooth locus.*

Let L be a line bundle on X, and set $\mathcal{E} := \phi_*(L \otimes \omega_{X/Y})$. Suppose furthermore that there exists a line bundle H on Y and a non-empty Zariski open subset $Y' \subset Y$ such that for all $s \in \mathbb{N}$, the sheaf

$$\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y + 1}$$

is generated on Y' by global sections on Y. Then $\mathcal{E}$ is weakly positive on $Y^* \cap Y'$.

Remark. It may appear to be pointless to make a statement with two different sets Y^* and Y', since in the end we show a property on their intersection. This distinction is motivated by the fact that Y^* is completely determined by the geometry of the fibration $\phi : X \to Y$ and has nothing to do with the line bundle L. The set Y' depends a lot on L or more precisely on the positivity of L. We will detail this in section 10.3.

Proof. By lemma 10.1.3 there exists a non-empty Zariski-open subset $Y^* \subset Y$ such that for all $s \in \mathbb{N}$, the fibre product $(\phi^s)^{-1}(Y^*)$ has at most canonical singularities. This is an open property and so is the Gorenstein property, so we can choose Y^* maximal with this property such that Y^* is Gorenstein. If ϕ is smooth, it is clear that ϕ^s is smooth over the ϕ-smooth locus, so Y^* contains the ϕ-smooth locus.

Since X is normal, it satisfies the hypothesis H2 from lemma 10.1.7, so lemma 10.2.3 implies the existence of a morphism

$$\tau^s : [\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}})]^{\otimes i} \to \mathcal{E}^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y)$$

that is an isomorphism on Y^*. Since H is locally free, the induced morphism

$$[\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}]^{\otimes i} \to \mathcal{E}^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y) \otimes H^{\otimes i(\dim Y+1)}$$

is still an isomorphism on Y^*. Since the left hand sheaf is generated by global sections on Y', it follows that

$$\mathcal{E}^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y) \otimes H^{\otimes i(\dim Y+1)} \simeq (\mathcal{E}^{[\otimes i]})^{[\otimes s]} \otimes \mathcal{O}_Y(iK_Y) \otimes H^{\otimes i(\dim Y+1)}$$

is generated on $Y^* \cap Y'$ by global sections. By definition this implies that $\mathcal{E}^{[\otimes i]}$ is weakly positive on $Y^* \cap Y'$, so $\mathcal{E}$ is weakly positive on $Y^* \cap Y'$ by lemma 9.2.4. $\square$

We give a variation of the reduction lemma for non-normal varieties. Unfortunately we have to put quite heavy restrictions on the fibration ϕ, but the lemma might still be interesting for flat degenerations of manifolds.

10.2.5 Lemma. *(Reduction lemma 2) Let $\phi : X \to Y$ be a flat Cohen-Macaulay fibration on Y a $\mathbb{Q}$-Gorenstein variety of index i. Suppose that ϕ is generically smooth and has Gorenstein fibres in codimension 1.*

Let L be a line bundle on X, and set $\mathcal{E} := \phi_*(L \otimes \omega_{X/Y})$. Suppose furthermore that there exists a line bundle H on Y and a non-empty Zariski open subset $Y' \subset Y$ such that for all $s \in \mathbb{N}$, the sheaf

$$\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}$$

is generated by global sections on Y'. Then $\mathcal{E}$ is weakly positive on the intersection of Y' with the ϕ-smooth locus.

Proof. Since ϕ is generically smooth there exists a non-empty Zariski-open subset $Y^* \subset Y$ such that $\phi|_{\phi^{-1}(Y^*)} : \phi^{-1}(Y^*) \to Y^*$ is a smooth projective morphism of quasi-projective manifolds. It is then clear that $(\phi^s)^{-1}(Y^*)$ is smooth and Y^* is Gorenstein.

Since ϕ has Gorenstein fibres in codimension 1, it satisfies the hypothesis H1 from lemma 10.1.7, so lemma 10.2.3 implies the existence of a morphism

$$\tau^s : [\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}})]^{\otimes i} \to \mathcal{E}^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y)$$

that is an isomorphism on Y^*. Since H is locally free, the induced morphism

$$[\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}]^{\otimes i} \to \mathcal{E}^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y) \otimes H^{\otimes i(\dim Y+1)}$$

is still an isomorphism on Y^*. Since the left hand sheaf is generated by global sections on Y', it follows that

$$\mathcal{E}^{[\otimes si]} \otimes \mathcal{O}_Y(iK_Y) \otimes H^{\otimes i(\dim Y+1)} \simeq (\mathcal{E}^{[\otimes i]})^{[\otimes s]} \otimes \mathcal{O}_Y(iK_Y) \otimes H^{\otimes i(\dim Y+1)}$$

is generated on $Y^* \cap Y'$ by global sections. By definition this implies that $\mathcal{E}^{[\otimes i]}$ is weakly positive on $Y^* \cap Y'$, so $\mathcal{E}$ is weakly positive on $Y^* \cap Y'$ by lemma 9.2.4. □

Note added in proof. C. Mourougane pointed out that it should be possible to establish the results above for varieties with rational singularities. In fact, after deposing this thesis I continued to work in this direction, also in view of a generalisation to fibrations that are not flat. The rest of the paragraph is devoted to a complement which shows that both generalisation do work. Note nevertheless that there is a small drawback since we do not know how to define the dualising sheaf if the morphism is not flat on the base not Gorenstein. Therefore we will restrict ourselves to the case where the base is Gorenstein and use definition 8.4.4 to define the dualising sheaf. It is clear that these results allow to generalise our main statements, but for lack of time I prefer not to change the exposition. These results will be included in the forthcoming article [Hör06].

10.2.6 Construction. *Let $\phi : X \to Y$ be be a fibration from a projective manifold X onto a Gorenstein variety Y. For every $s \in \mathbb{N}$, denote by X^s the unique component of the fibre product*

$$X \times_Y \ldots \times_Y X,$$

that dominates Y. Denote by $\phi^s : X^s \to Y$ the restriction of the natural morphism $X \times_Y \ldots \times_Y X \to Y$ to X^s.

Let $\nu : (X^s)' \to X^s$ be the composition of the reduction $(X^s)_{red} \to X^s$ and normalisation $(X^s)' \to (X^s)_{red}$, then ν is a finite morphism. Denote by $\tilde{Y}$ the ϕ-flat locus and set for all $s > 0$

$$\tilde{X}^s := \phi^{-1}(\tilde{Y}) \times_{\tilde{Y}} \ldots \times_{\tilde{Y}} \phi^{-1}(\tilde{Y}).$$

Set furthermore $(\tilde{X}^s)' := \nu^{-1}(\tilde{X}^s)$, then $(\tilde{X}^s)'$ is normal and the restriction of ν to $(\tilde{X}^s)'$ is a finite morphism. Thus the morphism $\phi|_{\phi^{-1}(\tilde{Y})} : \phi^{-1}(\tilde{Y}) \to \tilde{Y}$

satisfies the conditions of the constructions 10.1.1 and 10.2.1. Let $d : X^{(s)} \to (X^s)'$ *be a desingularisation, then the restriction of* d *to* $\tilde{X}^{(s)} := d^{-1}((\tilde{X}^s)')$ *is a desingularisation.*

Furthermore denote by $\phi^{(s)} : X^{(s)} \to Y$ *the morphism given by* $\phi^s \circ \nu \circ d$*.*

10.2.7 Lemma. *Suppose that we are in the situation of construction 10.2.6, e.g. let* $\phi : X \to Y$ *be be a fibration from a projective manifold* X *onto a Gorenstein variety* Y*. Denote by* L_s *the restriction of* $\otimes_{j=1}^{s}(\pi^j)^* L$ *to* X^s*, where*

$$\pi^j : X \times_Y \ldots \times_Y X \to X$$

is the natural projection on the j*-th factor. Fix* $s \in \mathbb{N}$*, then there exists a natural map*

$$\tau^s : [\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}})] \to [\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]} \otimes \mathcal{O}_Y(K_Y), \qquad (10.6)$$

which is an isomorphism on the largest open set $Y^* \subset Y$ *such that* $(\phi^s)^{-1}(Y^*)$ *is normal with at most rational singularities. In particular, for all* $s \in \mathbb{N}$*, it is an isomorphism on the* ϕ*-smooth locus.*

Remark. The general problem in treating a situation where the morphism ϕ is not flat, comes from the fact that the fibre products $X \times_Y \ldots \times_Y X$ may be „wild“. In particular they are not necessarily Cohen-Macaulay, so it does not make any sense to speak of a dualizing sheaf. Nevertheless after normalizing and desingularisation we certainly end up with a projective manifold $X^{(s)}$ and the trick is now to obtain the morphism 10.6 *without* having a morphism in the middle coming from the dualising sheaf of the fibre product. Well, this can be done by constructing such a morphism on the flat locus and extend it to the whole of Y by taking biduals.

Proof. We use the notation of construction 10.2.6, let $\tilde{Y}$ be the ϕ-flat locus, then the morphism $\phi|_{\phi^{-1}(\tilde{Y})} : \phi^{-1}(\tilde{Y}) \to \tilde{Y}$ satisfies the conditions of the constructions 10.1.1 and 10.2.1. Furthermore it satisfies the condition $(H2)$ of lemma 10.1.7. To simplify the notation, we will denote the restrictions of ν (resp. d and $\phi^{(s)}$) to $(\tilde{X}^s)'$ (resp. $\tilde{X}^{(s)}$) by the same letter. Then lemma 10.2.3 applies and yields for all $s \in \mathbb{N}$ the existence of a natural morphism

$$\tau^s|_{\tilde{Y}} : [\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{\tilde{X}^{(s)}} \otimes \mathcal{O}_{\tilde{X}^{(s)}})] \to [\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]} \otimes \mathcal{O}_Y(K_Y) \otimes \mathcal{O}_{\tilde{Y}},$$

which is an isomorphism on the largest open set $Y^* \subset \tilde{Y}$ such that $(\phi^s)^{-1}(Y^*)$ is normal with at most rational singularities, Since ϕ is generically smooth, it is clear that this locus is not empty and contains the ϕ-smooth locus.

Since X is Cohen-Macaulay, the ϕ-flat locus $\tilde{Y}$ is exactly the locus where the fibres of ϕ have dimension $\dim X - \dim Y$ [Har77, III, Ex.10.9]. Therefore it is easy to see that

$$\mathrm{codim}_Y(Y \setminus \tilde{Y}) \geq 2.$$

Since $\omega_{\tilde{X}^{(s)}} \simeq \omega_{X^{(s)}} \otimes \mathcal{O}_{\tilde{X}^{(s)}}$ and $[\phi_*(L \otimes \omega_{X/Y})]^{[\otimes s]} \otimes \mathcal{O}_Y(K_Y)$ is reflexive (recall that Y is Gorenstein), the morphism $\tau^s|_{\tilde{Y}}$ extends to the stated morphism τ^s by reflexiveness of the $\mathcal{H}om$-sheaf. $\square$

10.2.8 Lemma. *(Reduction lemma - non flat case) Let $\phi : X \to Y$ be a fibration from a projective manifold onto a Gorenstein variety Y. Let L be a line bundle on X, and set $\mathcal{E} := \phi_*(L \otimes \omega_{X/Y})$.*

Suppose furthermore that there exists a line bundle H on Y and a non-empty Zariski open subset $Y' \subset Y$ such that for all $s \in \mathbb{N}$, the sheaf

$$\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y + 1}$$

is on Y' generated by global sections on Y. Then $\mathcal{E}$ is weakly positive on the intersection of Y' with the ϕ-smooth locus.

Important remark. Note that this reduction lemma is in fact not a statement about fibrations from a projective manifold X to a Gorenstein variety, but from a variety with at most rational singularities X to a Gorenstein variety. Indeed, let $q : X' \to X$ a desingularisation, then $q_* \omega_{X'} \simeq \omega_X$ and the projection formula implies

$$(\phi \circ q)_*(q^* L \otimes \omega_{X'/Y}) \simeq \phi_*(L \otimes \omega_{X/Y}).$$

Therefore $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive if and only $(\phi \circ q)_*(q^* L \otimes \omega_{X'/Y})$ is weakly positive.

Proof. We are in the situation of construction 10.2.6, so lemma 10.2.7 implies the existence of a morphism

$$\tau^s : [\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}})] \to \mathcal{E}^{[\otimes s]} \otimes \mathcal{O}_Y(K_Y)$$

that is an isomorphism on the ϕ-smooth locus Y^*. Since H is locally free, the induced morphism

$$\phi_*^{(s)}((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y + 1} \to \mathcal{E}^{[\otimes s]} \otimes \mathcal{O}_Y(K_Y) \otimes H^{\otimes \dim Y + 1}$$

is still an isomorphism on Y^*. Since the left hand sheaf is generated by global sections on Y', it follows that

$$\mathcal{E}^{[\otimes s]} \otimes \mathcal{O}_Y(K_Y) \otimes H^{\otimes \dim Y + 1} \simeq \mathcal{E}^{[\otimes s]} \otimes \mathcal{O}_Y(K_Y) \otimes H^{\otimes \dim Y + 1}$$

is generated by global sections on $Y^* \cap Y'$. By definition this implies that $\mathcal{E}$ is weakly positive on $Y^* \cap Y'$. $\square$

10.3 Extension of sections

The difficult part is done. We will now specify the condition on L and check that the line bundles $(\nu \circ d)^* L_s$ satisfy the same condition. Then we can apply vanishing theorems and Castelnuovo-Mumford regularity to check the condition in the reduction lemma 10.2.4. Note that while so far all our considerations were local on the base Y and worked in the quasi-projective setting, we will now assume X and Y to be projective.

Proof of theorem 10.3.2.

The weak positivity of the direct image sheaf $\phi_*(L \otimes \omega_{X/Y})$ for L a nef and ϕ-big line bundle is the subject of Mourougane's theorem and it was also the starting point of our investigation. Although it turns out to be the easiest case we will explain every step in detail to give a good idea of the method. Once this preparatory work is done, we can show theorem 10.3.2 which settles at once the case of a nef and relatively big and the case of a semiample line bundle.

10.3.1 Lemma. *Let $\phi : X \to Y$ be a fibration from a projective manifold X to a normal projective variety Y. Let L be a nef and ϕ-big line bundle on X, and let H be a very ample line bundle on Y. Then the coherent sheaf*

$$\phi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1}$$

is generated by global sections.

Proof. By Castelnuovo-Mumford regularity (theorem 9.4.1) it is sufficient to show that

$$H^j(Y, \phi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1-j}) = 0 \qquad \forall \ \dim Y \geq j > 0.$$

Fix a $j \in \{1, \ldots, \dim Y\}$. By the relative Kawamata-Viehweg vanishing theorem, we have

$$R^i\phi_*(L \otimes \omega_X) = 0 \qquad \forall i > 0,$$

so by a degenerate case of the Leray spectral sequence

$$H^k(Y, \phi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1-j}) \simeq H^k(X, L \otimes \omega_X \otimes \phi^* H^{\otimes \dim Y+1-j}) \qquad \forall \ k \geq 0.$$

Since H is ample and L is nef and ϕ-big, the line bundle $M := \phi^* H^{\otimes \dim Y+1-j} \otimes L$ is nef. Furthermore we have

$$c_1(M)^{\dim X} = \sum_{i=0}^{\dim Y} c_1(\phi^* H^{\otimes \dim Y+1-j})^i \cdot c_1(L)^{\dim X - i}.$$

Since L is nef and H ample on Y, every term in this sum is nonnegative. Since L is ϕ-big, the last term is strictly positive, so M is nef and big. The standard Kawamata-Viehweg theorem yields

$$H^k(X, L \otimes \omega_X \otimes \phi^* H^{\otimes \dim Y+1-j}) = 0 \qquad \forall \ k \geq 1.$$

□

10.3.2 Theorem. *Let $\phi : X \to Y$ be a flat Cohen-Macaulay fibration from a projective $\mathbb{Q}$-Gorenstein variety X with at most canonical singularities to a normal projective $\mathbb{Q}$-Gorenstein variety Y. Let L be a nef and ϕ-big (resp. semiample) line bundle over X, then $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

If ϕ is generically smooth, then $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive on the ϕ-smooth locus.

Remark. If Y is Gorenstein, the direct image sheaf $\phi_*(L\otimes\omega_{X/Y})$ is locally free by corollary 9.4.4.

Proof. We use the notation of the constructions 10.1.1 and 10.2.1. Since L is nef and ϕ-big (resp. semiample), the bundle L_s is nef and ϕ^s-big (resp. semiample). Since the restriction of $\nu\circ d : X^{(s)}\to X^s$ to a general $\phi^{(s)}$-fibre is a birational morphism, the pull-back $(\nu\circ d)^*L_s$ is nef and $\phi^{(s)}$-big (resp. semiample). Let H be a very ample line bundle on Y, then the coherent sheaf

$$\phi^{(s)}_*((\nu\circ d)^*L_s\otimes\omega_{X^{(s)}})\otimes H^{\otimes \dim Y+1}$$

is globally generated for all $s>0$ by lemma 10.3.1 (resp. by theorem 9.4.5), so in the notation of the reduction lemma 10.2.4 we have $Y'=Y$. By the same lemma there exists a Zariski open set $Y^*\subset Y$ such that $\phi_*(L\otimes\omega_{X/Y})$ is weakly positive over Y^* and Y^* contains the ϕ-smooth locus. □

10.3.3 Remark. The proof shows that $\phi_*(L\otimes\omega_{X/Y})$ is weakly positive over the maximal Zariski open subset $Y^*\subset Y$ such that for all $s>0$ the fibre product $(\phi^s)^{-1}(Y^*)$ is normal with at most rational singularities.

The theorem has an immediate corollary for a morphism to a curve.

10.3.4 Corollary. *Let $\phi : X\to C$ be a morphism from a normal projective Cohen-Macaulay variety onto a smooth projective curve C such that the irrational locus of X does not project onto C and let L be a nef and ϕ-big line bundle. Then $\phi_*(L\otimes\omega_{X/C})$ is a nef vector bundle.*

Proof. *Step 1. Suppose that ϕ is a fibration.* Since C is smooth, we have $\omega_{X/C}=\omega_X\otimes\phi^*\omega_C^*$. Since $L\otimes\omega_{X/C}$ is torsion-free, the direct image sheaf $\phi_*(L\otimes\omega_{X/C})$ is torsion-free on a smooth curve, so locally free. Let $d : X'\to X$ be a resolution of singularities, then the morphism $d_*\omega_{X'}\to\omega_X$ is an isomorphism on the complement of the irrational locus. Since the irrational locus does not project on C, the induced morphism

$$(\phi\circ d)_*(d^*L\otimes\omega_{X'/C})\to\phi_*(L\otimes\omega_{X/C})$$

is generically an isomorphism.

Since C is a smooth curve, the morphism $\phi\circ d : X'\to C$ is a flat Cohen-Macaulay fibration between projective manifolds. It follows that $(\phi\circ d)_*(d^*L\otimes\omega_{X'/C})$ is nef by theorem 10.3.2, so $\phi_*(L\otimes\omega_{X/C})$ is nef by lemma 9.1.3.

Step 2. The general case. Let $\phi' : X\to C'$ be the Stein factorisation of ϕ, and let $\nu : C'\to C$ be the induced finite morphism. Then ϕ' is a fibration, so by the first step $\phi'_*(L\otimes\omega_{X/C'})$ is a nef vector bundle. Since C and C' are smooth, we have

$$\phi_*(L\otimes\omega_{X/C})\simeq\nu_*(\phi'_*(L\otimes\omega_{X/C'})\otimes\omega_{C'/C})$$

which is nef by proposition 9.5.1. □

Line bundles with non-negative Kodaira dimension.

We have seen in the last paragraph that the semiample case of theorem 10.3.2 is a rather straightforward consequence of theorem 9.4.5. Since theorem 9.4.8 generalizes this result, one should not expect too much difficulties in proving theorem 10.3.6 along the same lines. Yet there is one issue that has to be treated carefully: the asymptotic multiplier ideal sheaf $\mathcal{J}_s$ of the line bundle $(\nu \circ d)^* L_s$ is of course not independent of s, i.e. it is not just something like

$$(\nu \circ d)^{-1}(\otimes_{i=1}^s (\pi^i)^{-1}(\mathcal{J})),$$

where $\mathcal{J}$ is the asymptotic multiplier ideal sheaf of L. Therefore it is a priori not clear if the cosupports Z_s of $\mathcal{J}_s$ projects onto Y. If this is not the case, it still might happen that the union of the images

$$\cup_{s \geq 1} \phi^{(s)}(Z_s)$$

is not contained in a finite union of strict subvarieties of Y. In that case it would be impossible to show weak positivity of the direct image sheaf. The next proposition shows that none of these accidents can actually occur, the proof of theorem 10.3.6 then follows the same strategy that we used before.

10.3.5 Proposition. *Let $\phi : X \rightarrow Y$ be a flat fibration between projective manifolds, and let L be a line bundle with non-negative Kodaira dimension on X. Let $\mathcal{J}(X, ||L||)$ be the asymptotic multiplier ideal of L and suppose that the cosupport of $\mathcal{J}(X, ||L||)$ does not project onto Y. Using the notation of the constructions 10.1.1 and 10.2.1, let $(\nu \circ d)^* L_s$ be the corresponding line bundle on the projective manifold $X^{(s)}$ and denote by $\phi^{(s)} : X^{(s)} \rightarrow Y$ the induced morphism. Then there exists a non-empty Zariski open subset $Y^* \subset Y$ such that for all $s \in \mathbb{N}$, the cosupport Z_s of $\mathcal{J}(X^s, ||(\nu \circ d)^* L_s||)$ satisfies*

$$\phi^{(s)}(Z_s) \cap Y^* = \emptyset.$$

Proof. Fix $N \in \mathbb{N}$ such that

$$\mathcal{J}(X, \frac{1}{N}|L^{\otimes N}|) = \mathcal{J}(X, ||L||),$$

in particular the cosupport of $\mathcal{J}(X, \frac{1}{N}|L^{\otimes N}|)$ does not project onto the base. Since $L_s = \otimes_{i=1}^s (\pi^i)^* L$, we have a natural map

$$(\nu \circ d)^* \left[\otimes_{i=1}^s (\pi^i)^* H^0(X, L^{\otimes N}) \right] \rightarrow H^0(X^{(s)}, ((\nu \circ d)^* L_s)^{\otimes N}).$$

Therefore

$$\mathcal{J}(X^{(s)}, \frac{1}{N}(\nu \circ d)^* \left[\otimes_{i=1}^s (\pi^i)^* H^0(X, L^{\otimes N}) \right]) \subset \mathcal{J}(X^{(s)}, \frac{1}{N}((\nu \circ d)^* L_s)^{\otimes N}).$$

Since $\mathcal{J}(X^s, ||(\nu \circ d)^* L_s||)$ is the unique maximal element of

$$\left\{ \mathcal{J}(X^{(s)}, \frac{1}{k}((\nu \circ d)^* L_s)^{\otimes k}). \right\}_{k \geq 1},$$

this implies that it is sufficient to show the following claim: for $s \in \mathbb{N}$ denote by Z'_s the cosupport of $\mathcal{I}(X^{(s)}, \frac{1}{N}(\nu \circ d)^* \left[\otimes_{i=1}^s (\pi^i)^* H^0(X, L^{\otimes N})\right])$, then there exists a non-empty Zariski open subset $Y^* \subset Y$ such that

$$\phi^{(s)}(Z'_s) \cap Y^* = \emptyset$$

for all $s \in \mathbb{N}$.

Let $\tilde{Y} \subset Y$ be the ϕ-smooth locus, and set $\tilde{X} := \phi^{-1}(\tilde{Y})$. Denote by $\tilde{X}^s$ the s-fold fibre product

$$\tilde{X} \times_{\tilde{Y}} \ldots \times_{\tilde{Y}} \tilde{X}.$$

Since the cosupport of $\mathcal{I}(X, \frac{1}{N}|L^{\otimes N}|)$ does not project onto the base, the restriction of the global sections $H^0(X, L^{\otimes N})$ to X^* yield a non-empty finite-dimensional linear system V attachted to $L|_{\tilde{X}}^{\otimes N}$ such that the cosupport of $\mathcal{I}(\tilde{X}, \frac{1}{N}|V|)$ does not project onto $\tilde{Y}$. Since $\tilde{X}^s$ is smooth the restriction of $\nu \circ d$ to $(\nu \circ d)^{-1}(\tilde{X}^s)$ is an isomorphism, so it is clear $\otimes_{i=1}^s (\pi^i|_{\tilde{X}^s})^* V$ is the restriction of $(\nu \circ d)^* \left[\otimes_{i=1}^s (\pi^i)^* H^0(X, L^{\otimes N})\right]$ to $\tilde{X}^s$. Therefore

$$(*) \qquad \mathcal{I}(X^{(s)}, (\nu \circ d)^* \left[\otimes_{i=1}^s (\pi^i)^* H^0(X, L^{\otimes N})\right]) \otimes \mathcal{O}_{\tilde{X}^s} = \mathcal{I}(\tilde{X}^s, \otimes_{i=1}^s (\pi^i|_{\tilde{X}^s})^* V)$$

Yet by corollary 9.3.13 there exists a non-empty Zariski open subset $Y^* \subset \tilde{Y}$ such that for all $s \in \mathbb{N}$ and all $y \in Y^*$ we have

$$\mathcal{I}(\tilde{X}^s, \otimes_{i=1}^s (\pi^i|_{\tilde{X}^s})^* V)_{X_y^s} = \mathcal{O}_{X_y^s}.$$

By the equality (*) this shows that

$$\phi^{(s)}(Z'_s) \cap Y^* = \emptyset$$

and concludes the proof of the claim. □

10.3.6 Theorem. *Let $\phi : X \to Y$ be a flat fibration between projective manifolds, and let L be a line bundle of non-negative Kodaira dimension over X. Denote by $\mathcal{I}(X, ||L||)$ the asymptotic multiplier ideal of L. If the cosupport of $\mathcal{I}(X, ||L||)$ does not project onto Y, the direct image sheaf $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

Remark. Note that it is not possible to make a precise statement about the locus $Y^* \subset Y$ where the direct image sheaf is actually weakly positive. In fact even if we suppose ϕ to be smooth and denote by Z the cosupport of $\mathcal{I}(X, ||L||)$, it is not clear that the direct image sheaf is weakly positive on $Y \setminus \phi(Z)$. This is due to the fact that the „good locus Y' “ from the reduction lemma 10.2.4 will be the locus where the multiplier ideal of the restricted graded linear series is trivial. This locus will in general be strictly contained in $Y \setminus \phi(Z)$.

Proof. We use the notation of the constructions 10.1.1 and 10.2.1. Since L has non-negative Kodaira dimension, the bundle $(\nu \circ d)^* L_s$ has non-negative Kodaira dimension and we denote by $\mathcal{I}_s := \mathcal{I}(X^{(s)}, ||(\nu \circ d)^* L_s||)$ its asymptotic multiplier ideal. It follows by theorem 9.4.8 that for every very ample line bundle H on Y, the coherent sheaf

$$\phi^{(s)}_*((\nu \circ d)^* L_s \otimes \mathcal{I}_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}$$

is globally generated. Denote by Z_s the cosupport of $\mathcal{I}_s$, then it is clear that

$$\phi^{(s)}_*((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}$$

is globally generated on $Y \setminus \phi^{(s)}(Z_s)$.

Since the cosupport of $\mathcal{I}(X, ||L||)$ does not project onto Y, proposition 10.3.5 implies the existence of a non-empty Zariski open set $Y' \subset Y$ such that

$$\phi^{(s)}(Z_s) \cap Y' = \emptyset.$$

for all $s > 0$. It follows that for all $s > 0$

$$\phi^{(s)}_*((\nu \circ d)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}$$

is globally generated on Y'. Conclude with the reduction lemma 10.2.4. □

10.3.7 Corollary. *Let $\phi : X \to Y$ be a flat fibration between projective manifolds, and let L be a nef and abundant line bundle over X. Then the direct image sheaf $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

Proof. By a result of Russo [Laz04b, Rem.11.2.20], the asymptotic multiplier ideal sheaf of L is trivial, so the hypothesis of theorem 10.3.6 is satisfied. □

10.4 Fibrations that are not flat

Let $\phi : X \to Y$ be a morphism between projective manifolds, and let L be a line bundle on X that satisfies some positivity property. While the non-flatness of ϕ makes our strategy to show weak positivity of $\phi_*(L \otimes \omega_{X/Y})$ more complicated, it is easy to show generic nefness (cf. definition 9.2.7) if one can do this for the case where Y is a curve. More precisely we have the following meta-theorem.

10.4.1 Theorem. *Let $\phi : X \to Y$ be a fibration between projective manifolds. Let L be a line bundle on X that satisfies a property (P). Suppose that the following two conditions hold:*

1.) *Let A be a projective manifold, and let M be a line bundle on A. Let V be a free linear system of projective dimension at least 1 on A, and let $D \subset A$ be a general element of this linear sytem. If M satisfies (P), the restriction $M|_D$ satisfies (P).*

2.) *Let $\psi : A \to B$ be a fibration from a projective manifold A to a smooth curve B. Let M be a line bundle that satisfies (P). Then $\psi_*(L \otimes \omega_{A/B})$ is nef.*

Then $\phi_*(L \otimes \omega_{X/Y})$ is generically nef.

Proof. Set $n := \dim Y$. We proceed by induction on n.

The case $n = 1$ is clear, this is condition 2.

For the step $n - 1 \to n$, let $H_1, \ldots, H_{n-1}$ be very ample line bundles on Y and let $D_1 \in |H_1|$ be a general element such that D_1 and $\phi^{-1}(D_1)$ are smooth and $L|_{\phi^{-1}(D_1)}$ has the property P (this is possible by Bertini's theorem and property 1). If we choose now $D_j \in |H_j|$ general elements for $j \geq 2$ such that $D_j|_{D_1} \in |H_j|_{D_1}$ are also general elements, then

$$C := D_1 \cap \ldots \cap D_{n-1}$$

is a very general complete intersection curve in D_1. By the induction hypothesis the restriction of

$$(\phi|_{\phi^{-1}(D_1)})_*(L|_{\phi^{-1}(D_1)} \otimes \omega_{\phi^{-1}(D_1)/D_1})$$

to C is nef. Since this is the same as the restriction of $\phi_*(L \otimes \omega_{X/Y})$ to C, we conclude. $\square$

Remark. The theorem shows that generic nefness essentially follows from the nefness of a direct image sheaf for a fibration to a smooth curve. Such a map is necessarily flat, so our considerations about flat morphisms contribute to the study of this more general case. Roughly speaking, a weak positivity result for flat morphisms gives a generic nefness result for arbitrary morphisms for free. Unfortunately generic nefness is in general too weak to be useful in applications.

Chapter 11

Examples and counterexamples

This chapter is devoted to examples and counterexamples that show the limits of the theory. In particular we try to indicate which properties a fibre X_y should *not* have if we want $\phi_*(L \otimes \omega_{X/Y})$ to be weakly positive over a Zariski neighborhood of y.

11.1 Conic bundles

11.1.1 Definition. *Let X be a normal quasi-projective $\mathbb{Q}$-Gorenstein variety with at most canonical singularities. We say that X is a conic bundle if it admits a flat Cohen-Macaulay fibration $\phi : X \to Y$ such that there exists a vector bundle E of rank 3 on Y and an embedding $X \hookrightarrow \mathbb{P}(E)$ over Y identifying every fibre X_y is isomorphic to a conic in $\mathbb{P}(E_y)$.*

Remark. Since the fibres of a conic bundle are Gorenstein, it follows from general arguments that Y is also $\mathbb{Q}$-Gorenstein.

11.1.2 Proposition. *Let $\phi : X \to Y$ be a conic bundle such that X is non-singular in codimension 2 (e.g. X has at most terminal singularities). Let L be a nef line bundle on X, then there exists a Zariski open subset $Y^* \subset Y$ such that*

$$\operatorname{codim}_Y(Y \setminus Y^*) \geq 2$$

and $\phi_*(L \otimes \omega_{X/Y})$ is weakly positive on Y^*.

Remark. If Y is a surface, the result implies that for *any* curve $C \subset Y$, the restriction $\phi_*(L \otimes \omega_{X/Y}) \otimes \mathcal{O}_C$ is a nef vector bundle (modulo torsion).

Proof. The general ϕ-fibre is a normal conic, so it is a smooth rational curve. In particular ω_X is locally free over a general point of Y. If $\phi_*(L \otimes \omega_{X/Y}) = 0$ there is nothing to show, so we suppose without loss of generality that $h^0(F, L\otimes\omega_X/Y\otimes\mathcal{O}_F) > 0$. Since $\omega_X\otimes\mathcal{O}_F\simeq\omega_F$ is anti-ample, this shows that L is ample on the general fibre, so L is ϕ-big. By theorem 10.3.2 and remark 10.3.3 it follows that $\phi_*(L\otimes\omega_{X/Y})$ is weakly positive over the maximal Zariski open subset $Y^*\subset Y$ such that (in the notation of the reduction lemma 10.2.4), the fibre product $(\phi^s)^{-1}(Y^*)$ is normal with at most rational singularities. We are left to show that $Y\setminus Y^*$ has codimension at least 2.

Since ϕ is flat, we have $Y_{\text{sing}}\subset\phi(X_{\text{sing}})$. Set $Y' := Y\setminus\phi(X_{\text{sing}})$ and $X' := \phi^{-1}(Y')$, then the induced morphism $\phi|_{X'} : X'\to Y'$ is a conic bundle between smooth manifolds. By hypothesis the singular locus of X has dimension at most $\dim X-3$, so $\phi(X_{\text{sing}})$ has dimension at most $\dim X-3=\dim Y-2$, hence $\text{codim}_Y(Y\setminus Y')\geq 2$. Let $\Delta\subset Y'$ be the $\phi|_{X'}$-singular locus, then by [Sar82, Prop.1.8,5] the set $\Delta_1\subset\Delta$ such that for $y\in\Delta_1$, the fibre X_y is a double line, is contained in the singular locus of Δ. In particular for $\tilde{Y} := Y'\setminus\Delta_{\text{sing}}$ and $\tilde{X} := \phi^{-1}(\tilde{Y})$, the induced morphism $\phi|_{\tilde{X}} : \tilde{X}\to\tilde{Y}$ is a conic bundle between smooth manifolds such that the $\phi|_{\tilde{X}}$-singular locus $\tilde{\Delta}$ is smooth and the singular fibres are two smooth rational curves intersecting transversally in one point. It follows that $\phi^{-1}(\tilde{\Delta})$ is a simple normal crossings divisor, since (at least locally) it is a union of two $\mathbb{P}^1$-bundles intersecting transversally. Furthermore $\text{codim}_Y(Y\setminus\tilde{Y})\geq 2$.

Using the notation of construction 10.1.1 this implies by lemma 10.1.4 that for all $s\in\mathbb{N}$, the fibre product $(\phi^s)^{-1}(\tilde{Y})$ is normal with at most rational singularities. In the notation of the reduction lemma 10.2.4 this shows that $\tilde{Y}\subset Y^*$, in particular $Y\setminus Y^*$ has codimension at least 2. $\square$

In view of this last result it is tempting to conjecture that if $\phi : X\to Y$ is a conic bundle and L be a nef line bundle over X, then $\phi_*(L\otimes\omega_{X/Y})$ is nef, at least if X is smooth. We will show in section 11.3 that this is not true and that our result cannot be improved.

11.2 Direct images and non-vanishing

Vanishing theorems play a prominent role in our approach to positivity of direct image sheaves. One can ask an „inverse" question: let $X\to Y$ be a flat Cohen-Macaulay fibration and let L be a nef and ϕ-ample line bundle. Suppose that $\phi_*(L\otimes\omega_{X/Y})$ is weakly positive. Can we say something about the cohomology groups

$$H^i(X, (L\otimes\phi^*H)\otimes\omega_X),$$

where H is ample on Y ? If the Kodaira vanishing theorem holds for X, this question is trivial since $L\otimes\phi^*H$ is ample. We will now give a construction where the weak positivity of $\phi_*(L\otimes\omega_{X/Y})$ is equivalent to the vanishing of certain of these cohomology groups. This will enable us later to give counterexamples to the weak positivity of direct image sheaves.

11.2.1 Proposition. *Let $\phi : X\to\mathbb{P}^1$ be a flat Gorenstein fibration where X is an irreducible projective Gorenstein scheme (not necessarily reduced, nor* *normal or some restrictions on the singularities). Let L be a line bundle such that*

$$R^i\phi_*(L \otimes \omega_X) = 0 \qquad \forall\ i > 0.$$

Then $\phi_*(L \otimes \omega_{X/\mathbb{P}^1})$ *is locally free and it is nef if and only if*

$$h^1(X, L \otimes \omega_X \otimes \phi^*(\omega^*_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))) = h^1(X, L \otimes \omega_X \otimes \phi^*(\omega^*_{\mathbb{P}^1})).$$

Proof. Since ϕ is flat the vanishing of the higher direct images implies by [Har77, III,Thm.12.11]

$$H^i(F, L \otimes \mathcal{O}_F \otimes \omega_F) = 0 \qquad \forall\ i > 0$$

for every fibre F, so

$$h^0(F, L \otimes \mathcal{O}_F \otimes \omega_F) = \chi(F, L \otimes \mathcal{O}_F \otimes \omega_F)$$

is constant by [Har77, III,Thm.9.9]. It follows from the Bildgarbensatz that $\phi_*(L \otimes \omega_{X/C})$ is a locally free sheaf. Furthermore, since ϕ is flat, we have $\mathcal{I}_F = \phi^*\mathcal{O}_{\mathbb{P}^1}(-1)$ for any fibre $F = \phi^{-1}(y)$. So we have an exact sequence

$$0 \to L \otimes \omega_X \otimes \phi^*(\omega^*_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \to L \otimes \omega_X \otimes \phi^*(\omega^*_{\mathbb{P}^1}) \to L \otimes \mathcal{O}_F \otimes \omega_F \to 0$$

Since $h^1(F, L \otimes \mathcal{O}_F \otimes \omega_F) = 0$, the associated long exact cohomology sequence shows that

$$h^1(X, L \otimes \omega_X \otimes \phi^*(\omega^*_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))) = h^1(X, L \otimes \omega_X \otimes \phi^*(\omega^*_{\mathbb{P}^1}))$$

if and only if the morphism

$$H^0(X, L \otimes \omega_X \otimes \phi^*(\omega^*_{\mathbb{P}^1})) \to H^0(F, L \otimes \mathcal{O}_F \otimes \omega_F)$$

is surjective. Since

$$H^0(X, L \otimes \omega_X \otimes \phi^*(\omega^*_{\mathbb{P}^1})) \simeq H^0(\mathbb{P}^1, \phi_*(L \otimes \omega_{X/C}))$$

and

$$(\phi_*(L \otimes \omega_{X/C}))_y \simeq H^0(F, L \otimes \mathcal{O}_F \otimes \omega_F),$$

this morphism is surjective if and only if $\phi_*(L \otimes \omega_{X/C})$ is generated in y by its global sections. A vector bundle on $\mathbb{P}^1$ is nef if and only if it is generated by its global sections. $\square$

Remark. If L is nef and ϕ-ample, the bundles $L \otimes \phi^*(\omega^*_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ and $L \otimes \phi^*(\omega^*_{\mathbb{P}^1})$ are ample. So if the Kodaira vanishing theorems holds, both cohomology groups are zero, so $\phi_*(L \otimes \omega_{X/C})$ is nef. In section 11.3 and 11.4 we will construct examples where the equality does not hold.

11.3 Multiple fibres and a conic bundle

11.3.1 Proposition. *Set* $V := \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2a)$ *where* $a \geq 1$ *and denote by* $\phi : P := \mathbb{P}(V) \to \mathbb{P}^1$ *the projection map. Let* X_{red} *be the divisor defined by the quotient vector bundle*

$$V \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$$

and denote by X *the divisor* $2X_{\text{red}}$*. For* $b \geq 1$*, set* $L := \phi^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_P(b)$*. Then* $L \otimes \mathcal{O}_X$ *is an ample line bundle, but for* b *sufficiently large, the direct image sheaf*

$$(\phi|_X)_*(L \otimes \mathcal{O}_X \otimes \omega_{X/\mathbb{P}^1})$$

is not nef.

Proof. For $y \in \mathbb{P}^1$ arbitrary, we have

$$L \otimes \omega_X \otimes \mathcal{O}_{\phi|_X^{-1}(y)} \simeq \mathcal{O}_P(b) \otimes \mathcal{O}_{\phi|_X^{-1}(y)} \otimes \omega_{\phi|_X^{-1}(y)}.$$

Since $\mathcal{O}_P(b)$ is relatively ample, Serre vanishing implies that for b large we have

$$H^i(\phi|_X^{-1}(y), \mathcal{O}_P(b) \otimes \mathcal{O}_{\phi|_X^{-1}(y)} \otimes \omega_{\phi|_X^{-1}(y)}) = 0 \qquad \forall\, i > 0.$$

Since $\phi|_X$ is flat, this implies by [Har77, III, Ex.11.8] that

$$R^i(\phi|_X)_*(L \otimes \mathcal{O}_X \otimes \omega_X) = 0 \qquad \forall\, i > 0.$$

The statement follows from proposition 11.2.1 if we show that

$$(*) \qquad h^1(X, L \otimes \omega_X \otimes \phi^*(\omega_{\mathbb{P}^1}^* \otimes \mathcal{O}_{\mathbb{P}^1}(-1))) > h^1(X, L \otimes \omega_X \otimes \phi^*\omega_{\mathbb{P}^1}^*).$$

Step 1. Reduction to X_{red}*.* Consider the exact sequence

$$0 \to L \otimes \omega_{P/\mathbb{P}^1} \to L(X) \otimes \omega_{P/\mathbb{P}^1} \to L \otimes \mathcal{O}_X \otimes \omega_{X/\mathbb{P}^1} \to 0.$$

Since L is ample, $L \otimes \phi^*\omega_{\mathbb{P}^1}^*$ is ample, so Kodaira vanishing on P and the associated long exact sequence imply that

$$h^i(P, L(X) \otimes \omega_{P/\mathbb{P}^1}) = h^i(X, L \otimes \mathcal{O}_X \otimes \omega_{X/\mathbb{P}^1}) \qquad \forall\, i > 0.$$

Tensor the exact sequence with $\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)$, then $L \otimes \phi^*(\omega_{\mathbb{P}^1}^* \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ is ample, so the same argument shows that

$$h^i(P, L(X) \otimes \omega_{P/\mathbb{P}^1} \otimes \phi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = h^i(X, L \otimes \phi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_X \otimes \omega_{X/\mathbb{P}^1}).$$

for all $i > 0$. Consider now the exact sequence

$$0 \to L(X_{\text{red}}) \otimes \omega_{P/\mathbb{P}^1} \to L(X) \otimes \omega_{P/\mathbb{P}^1} \to (L(X_{\text{red}}) \otimes \mathcal{O}_{X_{\text{red}}}) \otimes \omega_{X_{\text{red}}/\mathbb{P}^1} \to 0.$$

Since L is ample, $L \otimes \phi^*\omega_{\mathbb{P}^1}^*$ is ample. The divisor X_{red} is smooth, so Norimatsu vanishing (cf. theorem 11.3.2 below) and the associated long exact sequence imply that

$$h^i(P, L(X) \otimes \omega_{P/\mathbb{P}^1}) = h^i(X_{\text{red}}, (L(X_{\text{red}}) \otimes \mathcal{O}_{X_{\text{red}}}) \otimes \omega_{X_{\text{red}}/\mathbb{P}^1}) \qquad \forall\, i > 0.$$

Tensor the exact sequence with $\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)$, then $L\otimes\phi^*(\omega^*_{\mathbb{P}^1}\otimes\mathcal{O}_{\mathbb{P}^1}(-1)$ is ample, so the same argument shows that

$$h^i(P, L(X)\otimes\omega_{P/\mathbb{P}^1}\otimes\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = h^i(X_{\text{red}}, (L(X_{\text{red}})\otimes\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\mathcal{O}_{X_{\text{red}}})\otimes\omega_{X_{\text{red}}/\mathbb{P}^1})$$

for all $i > 0$. As an intermediate result we obtain that

$$h^i(X_{\text{red}}, (L(X_{\text{red}})\otimes\mathcal{O}_{X_{\text{red}}})\otimes\omega_{X_{\text{red}}/\mathbb{P}^1}) = h^i(X, L\otimes\mathcal{O}_X\otimes\omega_{X/\mathbb{P}^1}) \qquad \forall\ i > 0$$

and

$$h^i(X_{\text{red}}, (L(X_{\text{red}})\otimes\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\mathcal{O}_{X_{\text{red}}})\otimes\omega_{X_{\text{red}}/\mathbb{P}^1}) = h^i(X, L\otimes\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\mathcal{O}_X\otimes\omega_{X/\mathbb{P}^1})$$

for all $i > 0$. Since $\mathcal{O}_{X_{\text{red}}}(X_{\text{red}}) \simeq N_{X_{\text{red}}/P}$, by $(*)$ we are left to show that

$$(**)\qquad h^1(X_{\text{red}}, (L\otimes\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\mathcal{O}_{X_{\text{red}}})\otimes N_{X_{\text{red}}/P}\otimes\omega_{X_{\text{red}}/\mathbb{P}^1}) > h^1(X_{\text{red}}, (L\otimes\mathcal{O}_{X_{\text{red}}})\otimes N_{X_{\text{red}}/P}\otimes\omega_{X_{\text{red}}/\mathbb{P}^1}).$$

Step 2. The computation on X_{red}. The divisor X_{red} is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$ and we will identify the restriction of ϕ to X_{red} to the projection on the first factor. Using the usual notation for line bundles on $\mathbb{P}^1\times\mathbb{P}^1$ and the fact that $\mathcal{O}_P(1)|_{X_{\text{red}}} \simeq \mathcal{O}_{X_{\text{red}}}(1,1)$, we have

$$L\otimes\mathcal{O}_{X_{\text{red}}} \simeq \phi^*\mathcal{O}_{\mathbb{P}^1}(a)\otimes\mathcal{O}_P(b)\otimes\mathcal{O}_{X_{\text{red}}} \simeq \mathcal{O}_{X_{\text{red}}}(a,b).$$

Since $\omega_P \simeq \phi^*\mathcal{O}_{\mathbb{P}^1}(2a-2)\otimes\mathcal{O}_P(-3)$ and $\omega_{X_{\text{red}}} \simeq \mathcal{O}_{X_{\text{red}}}(-2,-2)$, we have

$$N_{X_{\text{red}}/P} \simeq \omega^*_P\otimes\mathcal{O}_{X_{\text{red}}}\otimes\omega_{X_{\text{red}}} \simeq \mathcal{O}_{X_{\text{red}}}(-2a,1).$$

This implies

$$\begin{aligned}(L\otimes\mathcal{O}_{X_{\text{red}}})\otimes N_{X_{\text{red}}/P}\otimes\omega_{X_{\text{red}}/\mathbb{P}^1} &\simeq \mathcal{O}_{X_{\text{red}}}(a,b)\otimes\mathcal{O}_{X_{\text{red}}}(-2a,1)\otimes\mathcal{O}_{X_{\text{red}}}(0,-2)\\ &\simeq \mathcal{O}_{X_{\text{red}}}(-a,b-1)\end{aligned}$$

Since $b \geq 1$, this implies that

$$(\phi|_{X_{\text{red}}})_*((L\otimes\mathcal{O}_{X_{\text{red}}})\otimes N_{X_{\text{red}}/P}\otimes\omega_{X_{\text{red}}/\mathbb{P}^1}) \simeq \mathcal{O}_{\mathbb{P}^1}(-a)^{\oplus b},$$

furthermore there are no higher direct image, so

$$h^1(X_{\text{red}}, (L\otimes\mathcal{O}_{X_{\text{red}}})\otimes N_{X_{\text{red}}/P}\otimes\omega_{X_{\text{red}}/\mathbb{P}^1}) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a)^{\oplus b}).$$

By the same argument we obtain that

$$h^1(X_{\text{red}}, (L\otimes\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\mathcal{O}_{X_{\text{red}}})\otimes N_{X_{\text{red}}/P}\otimes\omega_{X_{\text{red}}/\mathbb{P}^1}) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a-1)^{\oplus b}).$$

Since $a \geq 1$, we have

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a-1)^{\oplus b}) > h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a)^{\oplus b}).$$

This concludes the proof by $(**)$. □

11.3.2 Theorem. *(Norimatsu vanishing theorem, [Nor78]) Let X be a projective manifold and D a simple normal crossings divisor on X. Let L be an ample line bundle over X, then*

$$H^q(X, \omega_X(D) \otimes L) = 0 \qquad \forall\ q > 0$$

Proposition 11.3.1 shows that the direct image sheaf can have surprising properties if the total space is not reduced. We will now show that even if we start with a flat fibration between projective manifolds, such a situation may appear if we consider a curve that is contained in the ϕ-singular locus. In particular it is in general not possible to obtain a better statement than weak positivity.

11.3.3 Proposition. *Let Y be a projective manifold and suppose that there exists a line bundle M on Y that has the following properties.*

1.) *The linear system $|2M|$ contains two smooth divisors D_1 and D_2 such that $D_1 + D_2$ is a simple normal crossings divisor.*

2.) *The non-nef locus of M is contained in $D_1 \cap D_2$ and there exists a smooth rational curve $C \subset D_1 \cap D_2$ such that $M|_C \simeq \mathcal{O}_C(-2a)$ with $a > 0$.*

3.) *There exist an ample line bundle N on Y such that*

$$N|_{D_1 \cap D_2} \equiv_{num} M^*|_{D_1 \cap D_2}.$$

Then there exists a smooth projective manifold X that admits a conic bundle structure $\phi : X \to Y$ and an ample line bundle L on X such that

$$\phi_*(L \otimes \omega_{X/Y})$$

is not nef on C.

Remark. For the construction of the conic bundle we will use the descriptions from [Bea77] and [Sar82]. At the moment I do not know an explicit example that satisfies all the conditions from the proposition, although I don't see a reason why it should not exist. If we weaken the condition 1) and demand only that the divisors D_1 and D_2 are integral, the conclusion still holds but X might be only a normal Gorenstein variety. Under this weaker hypothesis I can give an explicit example: let Y be a smooth Fano fourfold that admits a small contraction $\mu : Y \to Y'$ that contracts exactly a subvariety $\mathbb{P}^2 \subset Y$. Let H be a sufficiently ample divisor on Y' such that $M := \omega_Y^{\otimes 2} \otimes \phi^* H$ is base-point free in the complement of $\mathbb{P}^2$. Then by Bertini's theorem the linear system $|M^{\otimes 2}|$ contains divisors that are irreducible and smooth in the complement of $\mathbb{P}^2$, so the weakened condition 1) is satisfied. Furthermore for every line $C \subset \mathbb{P}^2$, we have $M|_C \simeq \mathcal{O}_C(-2)$, so condition 2) is satisfied. Set $N := \omega_Y^{\otimes -2}$, then N satisfies condition 3).

Proof. Set $V := M^{\oplus 2} \oplus \mathcal{O}_Y$. An everywhere non-zero section of

$$H^0(Y, Sym^2(M^{\oplus 2} \oplus \mathcal{O}_Y)).$$

determines a conic bundle embedded in $\mathbb{P}(V)$. For $i = 1$ and 2, let s_i be the section of $M^{\otimes 2}$ associated to the divisor D_i. Then

$$s := (s_1, s_2, 1) \in H^0(Y, M^{\otimes 2} \oplus M^{\otimes 2} \oplus \mathcal{O}_Y^{\otimes 2}) \subset H^0(Y, Sym^2 V).$$

is non-zero everywhere. Let $\phi : X \to Y$ be the associated conic bundle, then the ϕ-singular locus equals $D_1 \cup D_2$, in particular it is a normal crossings divisor. Furthermore the associated quadratic form has rank equal to two exactly on $(D_1 \cup D_2) \setminus (D_1 \cap D_2)$ and rank equal to one exactly on $D_1 \cap D_2$. Hence X is smooth by [Sar82][Prop.1.8,5].

Identify $\mathbb{P}(V)$ and $\mathbb{P}(V \otimes N)$ via the canonical isomorphism and embed X in $\mathbb{P}(V \otimes N)$ via this isomorphism. By hypothesis 3), the vector bundle $V \otimes N$ is nef. Hence for all $b \geq 1$, the line bundle $\mathcal{O}_{\mathbb{P}(V \otimes N)}(b)$ is nef and relatively ample on $\mathbb{P}(V \otimes N) \to Y$, so $L := \phi^* N \otimes \mathcal{O}_{\mathbb{P}(V \otimes N)}(b) \otimes \mathcal{O}_X$ is an ample line bundle on X

Consider now the smooth rational curve C. By hypothesis 2) and 3), we have $M|_C \simeq \mathcal{O}_{\mathbb{P}^1}(-2a)$ for some $a > 0$ and $N|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2a)$, hence

$$(V \otimes N)|_C \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2a).$$

Furthermore the scheme-theoretic intersection $X_C := X \cap \mathbb{P}((V \otimes N)|_C)$ is by construction equal to $2D \subset \mathbb{P}((V \otimes N)|_C)$, where D is the divisor associated to the quotient bundle

$$(V \otimes N)|_C \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}.$$

Since $N|_C \simeq M^*|_C$, we have

$$L|_C \simeq \phi^* \mathcal{O}_{\mathbb{P}^1}(2a) \otimes \mathcal{O}_{\mathbb{P}((V \otimes N)|_C)}(b).$$

It follows from proposition 11.3.1 that

$$\phi_*(L \otimes \omega_{X/Y})|_C \simeq (\phi|_{X_C})_*(L \otimes \mathcal{O}_{X_C} \otimes \omega_{X_C/C})$$

is not nef. $\square$

11.4 Non-rational singularities

In corollary 10.3.4 we have shown that for a fibration $X \to C$ over a smooth curve and a line bundle that is nef and relatively big, the direct image sheaf is nef, *if the irrational locus of X does not project onto C*. We will give an example to show that this last condition is crucial. This example should also be seen as an indication that in general the direct image sheaves will not be nef. If $\phi : X \to Y$ is a flat fibration between projective manifolds and $C \subset Y$ is a smooth projective curve that is contained in the ϕ-singular locus, the preimage $\phi^{-1}(C)$ might have many irrational singularities.

11.4.1 Proposition. *(We follow closely [BS95, Ex.2.2.10]) Set* $V := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$ *and denote by* $\phi : P := \mathbb{P}(V) \to \mathbb{P}^1$ *the projection map. For* $d \geq 4$*, let* X *be a general member of the linear system* $|d(\mathcal{O}_P(1) \otimes \phi^*\mathcal{O}(-1))|$*. Then the Stein factorisation of the restriction* $\phi|_X : X \to C$ *is a flat morphism of normal Gorenstein varieties such that the general fibre is a cone in* $\mathbb{P}^3$ *over a curve of degree* d*. Set* $L := \mathcal{O}_P(1)$*, then* L *is nef and* ϕ*-ample and the direct image sheaf* $(\phi|_X)_*(L \otimes \omega_{X/\mathbb{P}^1})$ *is not nef.*

Proof. If we identify P and $\mathbb{P}(V \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ via the canonical isomorphism, the line bundle $\mathcal{O}_P(1) \otimes \phi^*\mathcal{O}(-1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(V\otimes\mathcal{O}_{\mathbb{P}^1}(-1))}(1)$. Since

$$V \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$$

it follows that for $d \geq 1$, the base locus of the linear system $|d(\mathcal{O}_P(1) \otimes \phi^*\mathcal{O}(-1))|$ is exactly the ϕ-section induced by the quotient line bundle

$$V \to \mathcal{O}_{\mathbb{P}^1}.$$

It follows from Bertini's theorem that a general member X of the linear system is nonsingular in the complement of the base locus, therefore X is nonsingular in codimension 1. Since it is a divisor in P, it is Gorenstein, so it is even normal. Since X has no embedded points and $\mathbb{P}^1$ is a smooth curve it is clear that $\phi|_X$ is flat.

We want to show that the direct image sheaf $(\phi|_X)_*(L \otimes \omega_{X/\mathbb{P}^1})$ is not nef. Note first that since X is singular at most along a ϕ-section, it has at most finitely many irrational singularities in every fibre. Therefore the relative Kawamata-Viehweg vanishing theorem 9.4.3 applies to X, so

$$R^i(\phi|_X)_*(L \otimes \omega_X) = 0 \qquad \forall\ i > 0.$$

Consider the exact sequence

$$0 \to L \otimes \omega_{P/\mathbb{P}^1} \to L(X) \otimes \omega_{P/\mathbb{P}^1} \to (L \otimes \mathcal{O}_X) \otimes \omega_{X/\mathbb{P}^1} \to 0$$

Since $L \otimes \phi^*\omega_{\mathbb{P}^1}^*$ is ample, the Kodaira vanishing theorem on P and the associated long exact cohomology sequence imply

$$h^1(P, L(X) \otimes \omega_{P/\mathbb{P}^1}) = h^1(X, (L \otimes \mathcal{O}_X) \otimes \omega_{X/\mathbb{P}^1}).$$

If we tensor the exact sequence with $\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)$, the line bundle $L \otimes \phi^*(\omega_{\mathbb{P}^1}^* \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ is ample, so we can apply the same argument to obtain

$$h^1(P, L(X) \otimes \omega_{P/\mathbb{P}^1} \otimes \phi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = h^1(X, (L \otimes \mathcal{O}_X) \otimes \omega_{X/\mathbb{P}^1} \otimes \phi^*\mathcal{O}_{\mathbb{P}^1}(-1)).$$

We have $\omega_P = \phi^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_P(-4)$, so

$$L(X) \otimes \omega_{P/\mathbb{P}^1} = \phi^*\mathcal{O}_{\mathbb{P}^1}(3-d) \otimes \mathcal{O}_P(d-3).$$

Since $d \geq 4$, the higher direct images of $\phi^*\mathcal{O}_{\mathbb{P}^1}(3-d) \otimes \mathcal{O}_P(d-3)$ vanish, so

$$h^1(P, L(X) \otimes \omega_{P/\mathbb{P}^1}) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3-d) \otimes (\mathrm{Sym})^{d-3}V).$$

In the same way we obain

$$h^1(P, L(X) \otimes \omega_{P/\mathbb{P}^1} \otimes \phi^* \mathcal{O}_{\mathbb{P}^1}(-1)) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-d) \otimes (\mathrm{Sym})^{d-3} V).$$

An easy computation shows that for $d \geq 4$, we have

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-d) \otimes (\mathrm{Sym})^{d-3} V) > h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3-d) \otimes (\mathrm{Sym})^{d-3} V),$$

so

$$h^1(X, (L \otimes \mathcal{O}_X) \otimes \omega_{X/\mathbb{P}^1} \otimes \phi^* \mathcal{O}_{\mathbb{P}^1}(-1)) > h^1(X, (L \otimes \mathcal{O}_X) \otimes \omega_{X/\mathbb{P}^1}).$$

Conclude with proposition 11.2.1. □

11.5 Large multiplier ideals

The positivity of the direct image sheaf for a line bundle that is pseudo-effective but not nef comes with an extra difficulty. It is not sufficient to discuss the geometry of the fibration $X \to Y$, but also its relation with the geometry of the non-nef locus of the line bundle, represented by some multiplier ideal. In the next example we will take a line bundle L such that the base locus of its linear system surjects on the base. This will be enough to obtain direct image sheaves that are not weakly positive. Although we do not compute the multiplier ideal explicitly, it is then clear from theorem 10.3.6 that the cosupport of the multiplier ideal also surjects on the base.

11.5.1 Proposition. *Set $V := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$ and denote by $\phi : P := \mathbb{P}(V) \to \mathbb{P}^1$ the projection map. Set $L := \mathcal{O}_P(4) \otimes \phi^* \mathcal{O}_{\mathbb{P}^1}(-4)$, then for any $k \geq 1$, the linear system $|kL|$ is not empty, but the direct image sheaf $\phi_*(L^{\otimes k} \otimes \omega_{P/\mathbb{P}^1})$ is not nef.*

Proof. For $k \geq 1$, we have

$$\phi_* L^{\otimes k} \simeq \mathcal{O}_{\mathbb{P}^1}(-4k) \otimes (\mathrm{Sym})^{4k} V.$$

Since $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$, the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-4k) \otimes (\mathrm{Sym})^{4k} V$ has at least one direct factor isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. Therefore

$$h^0(P, L^{\otimes k}) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-4k) \otimes (\mathrm{Sym})^{4k} V) > 0,$$

so the linear system is not empty. Since $\omega_{P/\mathbb{P}^1} \simeq \mathcal{O}_P(-4) \otimes \phi^* \mathcal{O}_{\mathbb{P}^1}(3)$, we have

$$L^{\otimes k} \otimes \omega_{P/\mathbb{P}^1} \simeq \mathcal{O}_P(4k-4) \otimes \phi^* \mathcal{O}_{\mathbb{P}^1}(3-4k).$$

Hence

$$\phi_*(L^{\otimes k} \otimes \omega_{P/\mathbb{P}^1}) \simeq (\mathrm{Sym})^{4k-4} V \otimes \mathcal{O}_{\mathbb{P}^1}(3-4k)$$

has at least one direct factor isomorphism to $\mathcal{O}_{\mathbb{P}^1}(3-4k)$, so it is not nef. For $k = 1$, we even have $\phi_*(L \otimes \omega_{P/\mathbb{P}^1}) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, so the direct image sheaf is antiample ! □

Bibliography

[And85] Tetsuya Ando. On extremal rays of the higher-dimensional varieties. *Invent. Math.*, 81(2):347–357, 1985.

[AT82] Vincenzo Ancona and Giuseppe Tomassini. *Modifications analytiques*, volume 943 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[AW97] Marco Andreatta and Jarosław A. Wiśniewski. A view on contractions of higher-dimensional varieties. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 153–183. Amer. Math. Soc., Providence, RI, 1997.

[BB70] Paul F. Baum and Raoul Bott. On the zeros of meromorphic vector-fields. In *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*, pages 29–47. Springer, New York, 1970.

[BDPP04] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Paun, and Thomas Peternell. The pseudo-effective cone of a compact kähler manifold and varieties of negative Kodaira dimension. *arxiv preprint*, 0405285, 2004.

[Bea77] Arnaud Beauville. Variétés de Prym et jacobiennes intermédiaires. *Ann. Sci. École Norm. Sup. (4)*, 10(3):309–391, 1977.

[Bea96] Arnaud Beauville. *Complex algebraic surfaces*, volume 34 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid.

[Bea00] Arnaud Beauville. Complex manifolds with split tangent bundle. In *Complex analysis and algebraic geometry*, pages 61–70. de Gruyter, Berlin, 2000.

[BH93] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[BM01] Fedor A. Bogomolov and Michael L. McQuillan. Rational curves on foliated varieties. *Prépublications de l'IHES*, pages 1–29, February 2001.

[BPT04] Marco Brunella, Jorge V. Pereira, and Frédéric Touzet. Kähler manifolds with split tangent bundle. *preprint*, 2004.

[BPVdV84] W. Barth, C. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[Bru00] Marco Brunella. *Birational geometry of foliations.* Monografías de Matemática. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000. Available electronically at http://www.impa.br/Publicacoes/Monografias/Abstracts/brunella.ps.

[BS95] Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *de Gruyter Expositions in Mathematics.* Walter de Gruyter & Co., Berlin, 1995.

[Cam81] F. Campana. Coréduction algébrique d'un espace analytique faiblement kählerien compact. *Invent. Math. 63*, pages 187–223, 1981.

[Cam92] F. Campana. Connexité rationnelle des variétés de Fano. *Ann. Sci. École Norm. Sup. (4)*, 25(5):539–545, 1992.

[Cam04a] Frédéric Campana. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)*, 54(3):499–630, 2004.

[Cam04b] Frédéric Campana. Orbifolds, special varieties and classification theory: an appendix. *Ann. Inst. Fourier (Grenoble)*, 54(3):631–665, 2004.

[Cat91] F. Catanese. Moduli and classification of irregular kaehler manifolds. *Inventiones mathematicae*, 104:263–289, 1991.

[CLN85] César Camacho and Alcides Lins Neto. *Geometric theory of foliations.* Birkhäuser Boston Inc., Boston, MA, 1985. Translated from the Portuguese by Sue E. Goodman.

[CP02] Frédéric Campana and Thomas Peternell. Projective manifolds with splitting tangent bundle. I. *Math. Z.*, 241(3):613–637, 2002.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[Dem02] Jean-Pierre Demailly. On the Frobenius integrability of certain holomorphic p-forms. In *Complex geometry (Göttingen, 2000)*, pages 93–98. Springer, Berlin, 2002.

[Dru00] Stéphane Druel. Variétés algébriques dont le fibré tangent est totalement décomposé. *J. Reine Angew. Math.*, 522:161–171, 2000.

[Elk81] Renée Elkik. Rationalité des singularités canoniques. *Invent. Math.*, 64(1):1–6, 1981.

[FG65] Wolfgang Fischer and Hans Grauert. Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1965:89–94, 1965.

[GHS03] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67 (electronic), 2003.

[GR70] Hans Grauert and Oswald Riemenschneider. Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen. *Invent. Math.*, 11:263–292, 1970.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Har80] Robin Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.

[Hol80] Harald Holmann. On the stability of holomorphic foliations. In *Analytic functions, Kozubnik 1979 (Proc. Seventh Conf., Kozubnik, 1979)*, volume 798 of *Lecture Notes in Math.*, pages 192–202. Springer, Berlin, 1980.

[Hör05] Andreas Höring. Uniruled varieties with split tangent bundle. *to appear in Math. Zeit.*, 2005.

[Hör06] Andreas Höring. Direct images of adjoint line bundles. *article in preparation*, 2006.

[Jah05] Priska Jahnke. Submanifolds with splitting tangent sequence. *Math. Z.*, 251(3):491–507, 2005.

[Kaw81] Yujiro Kawamata. Characterization of abelian varieties. *Compositio Math.*, 43(2):253–276, 1981.

[Kaw82] Yujiro Kawamata. Kodaira dimension of algebraic fiber spaces over curves. *Invent. Math.*, 66(1):57–71, 1982.

[Kaw89] Yujiro Kawamata. Small contractions of four-dimensional algebraic manifolds. *Math. Ann.*, 284(4):595–600, 1989.

[KK83] Ludger Kaup and Burchard Kaup. *Holomorphic functions of several variables*, volume 3 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1983. An introduction to the fundamental theory, With the assistance of Gottfried Barthel, Translated from the German by Michael Bridgland.

[KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Springer-Verlag, Berlin, 1973. Lecture Notes in Mathematics, Vol. 339.

[Kle80] Steven L. Kleiman. Relative duality for quasicoherent sheaves. *Compositio Math.*, 41(1):39–60, 1980.

[KMM87] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 283–360. North-Holland, Amsterdam, 1987.

[KMM92] Janos Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. *J. Diff. Geom. 36*, pages 765–769, 1992.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963.

[Kob87] S. Kobayashi. *Differential geometry of complex vector bundles.* Iwanami Shoten and Princeton University Press, Princeton, 1987.

[Kol86] János Kollár. Higher direct images of dualizing sheaves. I. *Ann. of Math. (2)*, 123(1):11–42, 1986.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 1996.

[KSCT05] Stefan Kebekus, Luis Solá Conde, and Matei Toma. Rationally connected foliations after Bogomolov and McQuillan. *arXiv preprint*, 0505222, 2005.

[Kub70] Kazuji Kubota. Ample sheaves. *J. Fac. Sci. Univ. Tokyo Sect. I A Math.*, 17:421–430, 1970.

[Laz04a] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Laz04b] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[Miy87] Yoichi Miyaoka. Deformations of a morphism along a foliation and applications. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 245–268. Amer. Math. Soc., Providence, RI, 1987.

[Miy88] Yoichi Miyaoka. On the Kodaira dimension of minimal threefolds. *Math. Ann.*, 281(2):325–332, 1988.

[Mol88] Pierre Molino. *Riemannian foliations*, volume 73 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1988. Translated from the French by Grant Cairns.

[Mou97] Christophe Mourougane. Images directes de fibrés en droites adjoints. *Publ. Res. Inst. Math. Sci.*, 33(6):893–916, 1997.

[Mum70] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[Nor78] Yoshiki Norimatsu. Kodaira vanishing theorem and Chern classes for ∂-manifolds. *Proc. Japan Acad. Ser. A Math. Sci.*, 54(4):107–108, 1978.

[Nor83] Madhav V. Nori. Zariski's conjecture and related problems. *Ann. Sci. École Norm. Sup. (4)*, 16(2):305–344, 1983.

[Per01] Jorge V. Pereira. Global stability for holomorphic foliations on Kaehler manifolds. *Qual. Theory Dyn. Syst.*, 2(2):381–384, 2001.

[PS00] Thomas Peternell and Andrew J. Sommese. Ample vector bundles and branched coverings. *Comm. Algebra*, 28(12):5573–5599, 2000. With an appendix by Robert Lazarsfeld, Special issue in honor of Robin Hartshorne.

[Rei80] Miles Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[Rei87] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[Rei94] Miles Reid. Nonnormal del Pezzo surfaces. *Publ. Res. Inst. Math. Sci.*, 30(5):695–727, 1994.

[Sar82] Victor G. Sarkisov. On conic bundle structures. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(2):371–408, 432, 1982.

[Tak99] Hiromichi Takagi. Classification of extremal contractions from smooth fourfolds of $(3,1)$-type. *Proc. Amer. Math. Soc.*, 127(2):315–321, 1999.

[Uen75] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces*. Springer-Verlag, Berlin, 1975. Notes written in collaboration with P. Cherenack, Lecture Notes in Mathematics, Vol. 439.

[Vie82] Eckart Viehweg. Die Additivität der Kodaira Dimension für projektive Faserräume über Varietäten des allgemeinen Typs. *J. Reine Angew. Math.*, 330:132–142, 1982.

[Vie83] Eckart Viehweg. Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 329–353. North-Holland, Amsterdam, 1983.

[Vie95] Eckart Viehweg. *Quasi-projective moduli for polarized manifolds*. Springer Verlag, 1995.

[Vie06] Eckart Viehweg. Compactifications of smooth families and of moduli spaces of polarized manifolds. *arXiv preprint*, 0605093, 2006.

[Wiś91] Jarosław A. Wiśniewski. On contractions of extremal rays of Fano manifolds. *J. Reine Angew. Math.*, 417:141–157, 1991.

Summary

The subject of this thesis is to investigate two very natural questions in complex algebraic geometry.

The first question asks if the universal covering of a compact Kähler manifold with a split tangent bundle is a product of two manifolds. We will establish a structure theory for manifolds with a split tangent bundle and use covering families of rational curves to show the existence of a fibre space structure. A discussion of the fibre space structure allows to give an affirmative answer to the question for several classes of manifolds.

The second question asks if the positivity of a line bundle implies the positivity of the direct image of the adjoint line bundle under a flat projective morphism. We will see that the answer to this question depends on the positivity of the line bundle and its relation to the geometry of the morphism. We will show under a variety of conditions that the answer is to the affirmative.

Résumé

Dans cette thèse, nous étudions deux problèmes très naturels en géométrie algébrique complexe.

La première question étudiée est de savoir si le revêtement universel d'une variété kählérienne lisse compacte avec un fibré tangent décomposé est un produit de deux variétés. A l'aide des familles couvrantes de courbes rationnelles, nous montrons que certaines variétés avec un fibré tangent décomposé possèdent une structure d'espace fibré. Une étude systématique nous permet de donner une réponse affirmative à la question pour plusieurs classes de variétés.

La deuxième question étudiée est de savoir si la positivité d'un fibré en droites implique la positivité de l'image directe, par un morphisme projectif et plat, du fibré en droites adjoint. La réponse à cette question dépend de la positivité du fibré en droites et de ses liens avec la géométrie du morphisme considéré. Nous donnons une réponse positive à la question sous de faibles conditions géométriques.

Key words: holomorphic foliations, split vector bundle, universal covering, rational curves, Mori theory, rationally connected variety, uniruled variety, direct images of sheaves, positivity of coherent sheaves, fibre space

Mots clés : feuilletages holomorphes, fibré vectoriel décomposé, revêtement universel, courbes rationnelles, théorie de Mori, variété rationnellement connexe, variété uniréglée, images directes de faisceaux, positivité des faisceaux cohérents, espace fibré

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