Étude de la stabilité des petites solutionsstationnaires pour une classe d'équations de Dirac non linéaires

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Étude de la stabilité des petites solutions stationnaires pour une classe d'équations de Dirac non linéaires

Résumé : Cette thèse est consacrée à l'étude de la stabilité de petits états stationnaires d'une équation d'évolution non linéaire issue de la mécanique quantique relativiste : l'équation de Dirac non linéaire.

Tout le long de notre étude, les équations non linéaires sont vues comme des petites perturbations non linéaires de systèmes linéaires. Une partie de cette thèse est donc consacrée à l'étude de problèmes linéaires. Nous montrons que, pour un opérateur de Dirac n'ayant pas de résonance aux seuils ni de valeur propre aux seuils, le propagateur vérifie des estimations de propagation et de dispersion. Nous en déduisons également des estimations de régularité au sens de Kato et des estimations de Strichartz.

En faisant des hypothèses *ad hoc* sur le spectre discret d'un opérateur de Dirac, nous construisons des petites variétés formées d'états stationnaires. Puis en faisant varier ces hypothèses, nous faisons apparaître des phénomènes de stabilisation et d'instabilité orbitale pour certains de ces états.

Mots clés : équations aux dérivées partielles, opérateur de Dirac, estimations de propagation, estimations de dispersion, estimations de régularité, estimations de Strichartz, équation de Dirac non linéaire, états stationnaires, stabilité, stabilité orbitale, stabilité asymptotique, directions stables.

A study of the stability of small stationary solutions for a class of nonlinear Dirac equations

Abstract: This thesis is devoted to the study of the stability of small stationary solutions of a nonlinear time dependent equation coming from relativistic quantum mechanics: the nonlinear Dirac equation.

In this study, non linear equations are viewed as small nonlinear perturbations of linear systems. A part of this thesis is hence devoted to the study of linear problems. We prove that for a Dirac operator, with no resonance at thresholds nor eigenvalue at thresholds, the propagator satisfies propagation and dispersive estimates. We also deduce smoothness estimates in the sense of Kato and Strichartz estimates.

With some *ad hoc* assumptions on the discrete spectrum of a Dirac operator, we build small manifolds of stationary states. Then with small variations on these assumptions, we can highlight some stabilization process and orbital instability phenomena for some stationary states.

Keywords: Partial Differential Equations, Dirac Operator, Propagation estimates, Dispersive estimates, Smoothness estimates, Strichartz estimates, Nonlinear Dirac equation, Stationary states, Stabilty, Orbital stability, Asymptotic stability, Stable directions.

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Nabile Boussaid

"Les deux mots les plus brefs et les plus anciens, oui et non, sont ceux qui exigent le plus de réflexion."

Pythagore de Samos (580-490).

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Le Chapitre II Partie B, *Stable directions for small nonlinear Dirac standing waves*, a été accepté pour publication dans *Communications in Mathematical Physics*.

Première partie

Introduction

Introduction et présentation des résultats

Dans ce chapitre introductif, nous présentons le problème qui nous intéresse ainsi que quelques éléments de bibliographie. Nous consacrons une bonne partie de cette présentation à quelques propriétés des objets que nous utiliserons dans notre étude. Nous donnons également une justification heuristique des hypothèses que nous ferons par la suite.

Nous présentons alors les résultats que nous avons obtenus. Dans la dernière partie, nous mentionnons quelques questions ouvertes liées à celles que nous nous sommes posées durant cette thèse.

Le problème de la stabilité

Nous étudions des problèmes de stabilité pour des solutions stationnaires d'une équation de Dirac non linéaire. Une solution stationnaire représente un état lié d'une particule. Dans la littérature scientifique, de telles solutions sont nommées *solitons*. Toutefois les physiciens préfèrent réserver ce terme aux ondes qui gardent leur forme après une collision.

Bien des travaux ont été consacrés à l'étude de l'existence de solutions stationnaires et ceci pour une grande variété d'équations. La stabilité de ces solutions reste souvent une question ouverte bien que ce soit d'importance cruciale (en particulier lors des calculs numériques ou d'expériences physiques), .

Il y a différentes définitions de stabilité qu'il convient de préciser avant toute autre chose. La première notion qui est la stabilité *orbitale*, traduit le fait que l'orbite de la perturbation d'une onde stationnaire reste près de l'onde ou d'un ensemble d'ondes stationnaires mais ne converge pas nécessairement. Une notion plus forte est celle de stabilité *asymptotique*, qui signifie que la perturbation d'une onde stationnaire tend asymptotiquement vers une onde stationnaire qui est proche de l'onde perturbée. En fait, dans beaucoup de problèmes conservatifs, la stabilité asymptotique ne peut pas être obtenue. Elle est alors remplacée par la stabilité asymptotique pour une classe restreinte de perturbations, formant ce que l'on nomme les directions *stables*.

Pour les équations de Dirac étudiées dans cette thèse, nous avons obtenu des directions stables, des résultats de stabilité asymptotique et d'instabilité orbitale. À notre connaissance, ce sont les premiers résultats de ce type pour des équations faisant intervenir l'opérateur de Dirac.

Dans ce mémoire, nous étudions le problème de la stabilité de petites ondes stationnaires de l'équation de Dirac non linéaire :

$$\mathrm{i}\partial_t\psi = (D_m + V)\psi + \nabla F(\psi) \tag{NLDE}$$

où ∇F est le gradient de la non-linéarité $F : \mathbb{C}^4 \mapsto \mathbb{R}$ pour le produit scalaire usuel de $\mathbb{R}^8$. Ici, D_m est l'opérateur de Dirac habituel, voir [Tha92], agissant sur $L^2(\mathbb{R}^3, \mathbb{C}^4)$:

$$D_m = \alpha \cdot (-\mathrm{i}\nabla) + m\beta = -\mathrm{i}\sum_{k=1}^{3} \alpha_k \partial_k + m\beta$$

où $m \in \mathbb{R}_+^*$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ et β sont des matrices hermitiennes agissant sur $\mathbb{C}^4$ satisfaisant les propriétés suivantes :

$$\begin{cases} \alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}\mathbf{1}_{\mathbb{C}^4}, & \\ \beta^2 = \mathbf{1}_{\mathbb{C}^4}, & i, k \in \{1, 2, 3\}, \\ \alpha_i\beta + \beta\alpha_i = \mathbf{0}_{\mathbb{C}^4}, & i \in \{1, 2, 3\}. \end{cases}$$

Ici, nous utilisons la représentation classique :

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{et} \quad \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}$$

$$\text{où } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{et} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{et} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dans (NLDE), V est un potentiel extérieur et $F : \mathbb{C}^4 \mapsto \mathbb{R}$ est une non-linéarité vérifiant

$$\forall(\theta, z) \in \mathbb{R} \times \mathbb{C}^4, \quad F(e^{i\theta}z) = F(z).$$

Quelques hypothèses additionnelles sur F et V seront faites dans la suite de notre étude. Les solutions stationnaires de (NLDE) sont de la forme $\psi(t, x) = e^{-\mathrm{i}Et}\phi(x)$ où ϕ est solution de l'équation

$$E\phi = (D_m + V)\phi + \nabla F(\phi). \tag{1}$$

Sous certaines hypothèses que nous préciserons par la suite, de telles solutions forment sous nos hypothèses une petite variété. Nous étudions alors la dynamique au voisinage de cette variété.

De nombreux travaux ont été consacrés à l'étude de la stabilité pour des équations de Schrödinger ou de Klein-Gordon (voir par exemple [BP95, BP92b, BP92c, BP92a, BS03, BS02, CL82, CF01, Cuc00, Cuc01, Cuc03, GSS87, GNT04, KS05, PW97, RSS05a, Sch04, SS85, SW90, SW92, SW99, SW04, SW05, Tsa03, TY02a, TY02c, TY02d, TY02b, Wed00]). Les méthodes employées pour traiter ces cas ne s'adaptent pas facilement à notre problème, puisque l'opérateur de Dirac D_m n'est pas minoré, contrairement à $-\Delta$. La positivité du Laplacien permet d'utiliser des méthodes de minimisation et de concentration-compacité pour prouver l'existence d'ondes orbitalement stables, voir par exemple Cazenave et Lions [CL82] ou plus récemment Cid et Felmer [CF01]. Dans sa revue sur des modèles non linéaires basés sur l'opérateur de Dirac, Rañada [Ran] signale que les physiciens ont d'abord affirmé que les ondes stationnaires d'équations de Dirac non linéaires ne pouvaient pas être stables. La variation seconde de la fonctionnelle d'énergie doit être, pour certains d'entre eux, définie positive pour qu'il y ait stabilité. Cependant, dans une étude très générale (mais inapplicable à notre cas), Shatah et Straus [SS85] et Grillakis, Shatah et Straus [GSS87] ont trouvé un critère de stabilité orbitale général permettant à la fonctionnelle d'énergie d'avoir une valeur propre négative simple (et un noyau de dimension un). Leur méthode ne s'applique pas directement à l'équation de Dirac mais nuance un peu les affirmations signalées par Rañada. Ceci a aussi induit une discussion intéressante au sujet de la stabilité orbitale pour l'équation de Dirac dans certains articles de physique [SV86, AS86, BSV87].

Rañada rapporte également quelques expériences numériques, qui semblent confirmer que certaines solutions stationnaires de l'équation de Dirac sont asymptotiquement stables. Dans le cas de l'équation de Schrödinger, la stabilité asymptotique a été beaucoup étudiée pendant la dernière décennie. Un résultat fondamental est celui de Soffer et Weinstein [SW90, SW92], qui est consacré à l'étude d'une petite perturbation non linéaire d'un opérateur de Schrödinger ayant une valeur propre simple. Ils ont montré que les petites ondes stationnaires perturbées se relaxent vers une onde stationnaire. Plus tard, Pillet et Wayne [PW97] ont proposé une preuve différente dans l'esprit du théorème de la variété centrale. Dans tous ces travaux, la stabilité asymptotique est une conséquence directe des estimations de propagation et de dispersion du propagateur de l'opérateur de Schrödinger. Afin de pouvoir utiliser ces estimations, il faut considérer au temps $t = 0$ des perturbations localisées *i.e.* dans L^1 ou dans L^2 pondéré par un poids croissant. Pour éviter de telles restrictions, Gustafson, Nakanishi et Tsai [GNT04] proposent d'employer des inégalités de Strichartz.

Des généralisations de ces problèmes ont été considérées par exemple par Tsai et Yau [TY02a, TY02c, TY02d, TY02b, Tsa03]. Ces auteurs considèrent le cas d'un opérateur de Schrödinger ayant deux valeurs propres simples. Un phénomène intéressant apparaît alors : si les deux valeurs propres sont suffisamment éloignées l'une de l'autre, après linéarisation autour de l'état considéré, on obtient une résonance. Tsai et Yau ont prouvé que s'il n'y a aucune résonance, la variété associée à l'état fondamental a des directions stables. Dans le cas résonnant, cette variété est asymptotiquement stable, tandis que la variété associée à l'état excité a des directions stables et instables. En cas d'instabilité, sous certaines conditions, on a la relaxation vers l'état fondamental. Pour un résultat du même type, on peut aussi consulter [SW04, SW05]. Mentionnons que Soffer et Weinstein [SW99] ont auparavant étudié un phénomène de résonance semblable dans le cas de l'équation de Klein-Gordon avec une valeur propre simple ; ils ont prouvé que si la valeur propre est assez grande, cela induit une « métastabilité ».

Un autre problème a été étudié par Cuccagna [Cuc01, Cuc03, Cuc05]. Il a considéré le cas de grandes ondes stationnaires, quand l'opérateur linéarisé a seulement une valeur propre et a obtenu la stabilité asymptotique de la variété des états fondamentaux. Tsai, Yau et Cuccagna utilisent également des estimations de propagation et de dispersion. Ils les démontrent en généralisant le travail de Yajima [Yaj95] sur les opérateurs d'ondes.

Une étude intéressante a aussi été menée par Rodnianski, Schlag et Soffer [RSS05a] qui ont prouvé la stabilité asymptotique d'un nombre arbitraire de grandes ondes stationnaires interagissant faiblement.

Schlag [Sch04] et Krieger et Schlag [KS05] ont prouvé l'existence de directions stables pour des ondes stationnaires instables. Nous précisons que certains travaux de Schlag [ES04, GS04, RSS05b] ou de Soffer [HSS99, JSS91, RSS05b] sont consacrés à la preuve des estimations de dispersion.

On peut également mentionner les travaux de Buslaev et de Perel'mann [BP95, BP92b, BP92c, BP92a], de Buslaev et Sulem [BS03, BS02] et de Weder [Wed00], pour le cas de Schrödinger en dimension un.

Dans cette thèse, nous étudions des équations de Dirac non linéaires comme perturbations d'une équation linéaire de Dirac avec un opérateur de Dirac possédant uniquement :

1. deux valeurs propres simples en l'absence de résonance sur la première valeur propre (Théorème B.6 Chapitre II) ;
2. deux valeurs propres doubles avec une condition de résonance sur la première (Théorème B.7 Chapitre III) ;
3. une valeur propre double (Théorème B.6 Chapitre IV).

1 Qu'appelle-t-on équation de Dirac ?

1.1 Les postulats de la mécanique quantique

L'équation de Dirac libre est l'équation gouvernant la dynamique d'un électron quantique relativiste. Elle vérifie les postulats de la mécanique quantique (voir Basdevant et Dalibard [BD02]) :

$$\mathrm{i}\partial_t\psi(t) = H\psi(t) \text{ (principe sur l'évolution)}$$

où ψ est dans un espace de Hilbert $\mathcal{H}$ (principe de superposition) et H est l'observable d'énergie, c'est à dire un opérateur auto-adjoint sur $\mathcal{H}$ quantifiant l'énergie (principe de quantification).

Des exemples d'observables dans $\mathcal{H} = L^2(\mathbb{R}^3)$ sont donnés par :

$$\begin{array}{ll} P & = -\mathrm{i}\nabla, \text{ observable de quantité de mouvement} \\ Q & \text{opérateur de multiplication par } x \in \mathbb{R}^3, \text{ observable de position.} \end{array}$$

Pour une fonction ψ de $L^2(\mathbb{R}^3)$ normalisée qui représente l'état d'un électron, $|\psi(x)|^2\,dx$ s'interprète comme la probabilité de présence de l'électron dans un volume dx autour de x et $\left|\widehat{\psi}(p)\right|^2 dp$ comme la probabilité que l'électron ait une quantité de mouvement dans un volume dp autour de p.

si l'on considère un électron gouverné par les principes de la mécanique newtonienne dans un champ engendré par le potentiel V, on a l'énergie

$$E = \frac{mv^2}{2} + V(x) = \frac{p^2}{2m} + V(x)$$

où m est la masse de l'électron, p le quantité de mouvement, v la vitesse et x la position. Ce qui heuristiquement, nous donne l'observable d'énergie

$$H = \frac{P^2}{2m} + V(Q) = \frac{-\Delta}{2m} + V(Q).$$

C'est le principe de correspondance

$$\begin{array}{ccc} x & \leftrightarrow & Q \\ p & \leftrightarrow & P \end{array}$$

de Schrödinger qui mène à l'équation de Schrödinger.

L'énergie d'un électron libre gouverné par les principes de la mécanique einsteinienne est donnée par la relativité restreinte :

$$E^2 = c^2p^2 + m^2c^4,$$

où c est la vitesse de la lumière que l'on prendra dans la suite égale à 1. C'est en partie cette loi qui va nous donner H l'opérateur dit de Dirac.

1.2 Qu'est ce que l'opérateur de Dirac ?

Le principe de correspondance de Schrödinger nous donne la relation

$$H^2 = -\Delta + m^2, \tag{2}$$

où H serait l'opérateur de Dirac. Ceci nous donne un bon exemple d'équation relativiste :

$$-\partial_t^2\psi(t) = -\Delta\psi(t) + m^2\psi(t),$$

l'équation de Klein-Gordon. Elle est relativiste mais elle est du second ordre en temps. Elle transgresse le postulat de la mécanique quantique sur l'évolution et la densité de charge n'a pas un signe déterminé. Ceci ne permet pas l'interpréter probabiliste de la mécanique quantique. Il faut donc réellement extraire une racine carrée de (2).

1.2.1 L'opérateur de Schrödinger relativiste

On introduit, dans un premier temps, l'opérateur de Schrödinger relativiste

$$H_{\mathrm{S}} = \sqrt{-\Delta + m^2}.$$

que l'on définit dans le domaine de Fourier comme l'opérateur de multiplication par $\sqrt{p^2 + m^2}$.

L'évolution associée à cet opérateur n'est malheureusement pas local. Considérons le front d'onde localisé en fréquence $e^{-\mathrm{i}t\sqrt{-\Delta+m^2}}\psi(P)$ avec $\psi \in \mathcal{D}(\mathbb{R}^3)$, son noyau étant un noyau de convolution. Il propage l'onde instantanément, c'est à dire avec une vitesse infiniment plus grande que celle de la lumière. Ceci transgresse le principe de causalité d'Einstein.

D'autre part, alors qu'en relativité restreinte il y a un traitement quasiment symétrique des variables d'espace et de temps, dans l'équation d'évolution associée à $\sqrt{-\Delta + m^2}$,

$$-\mathrm{i}\partial_t\psi(t) = \sqrt{-\Delta + m^2}\psi(t),$$

il apparaît une dissymétrie entre la dérivation en temps et en espace. Cela ne permet pas d'intégrer de manière relativiste un champ électromagnétique relativiste (*i.e* vérifiant les équations de Maxwell).

1.2.2 L'opérateur de Dirac

C'est pour surmonter ces difficultés que Paul Adrien Maurice Dirac propose (voir [Dir95]) en 1928, un opérateur du premier ordre en P :

$$D_m = \sum_{i=1}^{3} \alpha_i P_i + m\beta.$$

Cet opérateur, qui est en quelque sorte le linéarisé de l'opérateur de Schrodinger relativiste, doit vérifier (2) *i.e.*

$$D_m^2 = -\Delta + m^2,$$

on obtient les relations suivantes sur les coefficients :

$$\begin{cases} \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \\ \beta\alpha_i + \alpha_i\beta = 0 \\ \beta^2 = 1. \end{cases}$$

Si $m \neq 0$, comme D_m doit être autoadjoint, on doit avoir quatre matrices hermitiennes unitaires, anticommutantes les unes avec les autres et de trace nulle. La dimension minimale pour laquelle ceci est possible est 4.

On est donc sur $\mathbb{C}^4$ et Dirac fait le choix suivant :

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{et} \quad \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}$$

que l'on appelle matrices de Dirac et où

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{et} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{et} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

sont les matrices de Pauli introduites par Pauli pour tenir compte du moment magnétique intrinsèque (spin) de l'électron non relativiste (*i.e.* gouverné par l'équation de Schrödinger).

1.2.3 Quelques particularités

L'opérateur D_m est auto-adjoint sur $L^2(\mathbb{R}^3, \mathbb{C}^4)$ de domaine $H^1(\mathbb{R}^3, \mathbb{C}^4)$ et est essentiellement autoadjoint sur $\mathcal{D}(\mathbb{R}^3, \mathbb{C}^4)$ (c'est donc un cœur de D_m). Comme $L^2(\mathbb{R}^3, \mathbb{C}^4) = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$, on voit apparaître deux jeux de degrés de liberté : le signe de l'énergie et le spin.

Le spectre de D_m, qui s'obtient par transformation de Fourier :

$$\sigma(D_m) =]-\infty, -m] \cup [m, +\infty[$$

est purement absolument continu. Ce spectre n'est pas minoré, ce qui est assez gênant d'un point de vue physique mais aussi mathématique.

Point de vue mathématique Les méthodes de minimisation n'ont alors aucun sens. En particulier, il n'est pas possible d'appliquer les méthodes de concentration-compacité pour obtenir de la stabilité. Plus globalement, les méthodes variationnelles sont alors plus difficiles à mettre en oeuvre du fait qu'il y a une infinité de directions de descente.

Point de vue physique Il est assez difficile d'interpréter correctement le spectre négatif de l'opérateur libre puisqu'il serait associé à des « particules » d'énergie cinétique négative, ce qui ne semble pas avoir de sens.

De plus, plus la vitesse d'une particule du spectre d'énergie négative est élevée, plus son énergie est petite (avec une très grande valeur absolue). Dans ce cas, pour obtenir une particule au repos, il faut lui donner de l'énergie !

D'autre part, en physique, il est admis qu'une particule se relaxe naturellement sur son état de plus basse énergie. Dans le cas qui nous intéresse, les électrons de l'univers s'effondreraient tous sur l'état d'énergie « $-\infty$ ».

Pour vaincre cette nouvelle difficulté, Dirac a proposé l'hypothèse dite de « la mer de Dirac ». Selon cette hypothèse, les états d'énergie négative sont occupés par des électrons, que l'on qualifiera de virtuels puisqu'ils ne sont pas observables physiquement. Cette infinité d'électrons virtuels forme ce que l'on appelle la « mer de Dirac » ou le vide. Ainsi, si un électron d'énergie positive ne s'effondre pas, c'est parce que les électrons sont des fermions et que les fermions obéissent au principe d'exclusion de Pauli : deux fermions identiques ne peuvent pas occuper le même état. Les états d'énergies négatives étant occupés, un électron réel ne peut pas s'effondrer.

Dirac postule aussi que l'observation directe de la mer de Dirac ne nous est pas accessible et son existence peut être mise en évidence de manière indirecte uniquement.

Dirac [Dir65] pousse son raisonnement plus loin ; en excitant le vide avec un énergie $E > 2m$, on excite un électron virtuel et on obtient un électron d'énergie positive. Cet électron laisse dans le vide une lacune qui (par conservation de la charge) est de charge positive et de masse identique à celle de l'électron. D'autre part en inversant le sens du temps dans cette expérience théorique, on voit alors que cette lacune s'annihile lorsqu'elle rencontre un électron. C'est donc un antiélectron, l'antiparticule de l'électron, que l'on nomme aujourd'hui le positron.

L'observation en 1932, quatre ans après l'introduction de l'équation de Dirac, du positron par Carl Anderson [And33] finira de conforter la théorie.

2 Le problème de la stabilité linéaire

Dans la suite de notre étude, nous envisagerons toujours un système linéaire comme une petite perturbation non linéaire d'un système linéaire. Ceci explique pourquoi nous ne considérons que des petites solutions stationnaires.

Ce fait se justifie de la manière suivante :
Considérons un opérateur H ayant un spectre discret fini $\{\lambda_1, \dots, \lambda_m\}$ et$\{\phi_1, \dots, \phi_m\}$ un système orthonormal de fonctions propres associées. Pour toute solution ψ de l'équation

$$\begin{cases} -\mathrm{i}\partial_t \psi = H\psi \\ \psi(0) = \displaystyle\sum_{i=1}^{m} \alpha_i \phi_i + \phi_c \text{ où } \phi_c \in \{\phi_1, \dots, \phi_m\}^{\perp} \end{cases}$$

on a

$$\psi(t) = \sum_{i=1}^{m} \alpha_i e^{-\mathrm{i}t\lambda_i} \psi + e^{-\mathrm{i}tH} \phi_c.$$

Nous savons, par le théorème RAGE (Ruelle-Amrein-Georgescu-Enss) [AG74], qu'en moyenne de Cesaro $e^{-\mathrm{i}tH}\phi_c$ tend vers 0 dans L^2_{loc}, si par exemple l'opérateur $\mathbb{1}_E(Q)(H - \mathrm{i})^{-1}$ est compact pour tout compact $E \subset \mathbb{R}^3$. Ce fait qui est vrai pour l'opérateur de Dirac avec des potentiels assez réguliers, nous donne pour Dirac la stabilité asymptotique de la somme des espaces propres.

On peut également se restreindre aux orthogonaux de certains espaces en imposant l'annulation de certaines composantes α_i et on obtient dans ce cas des directions stables.

On tente alors de généraliser cela au problème non linéaire. Tout d'abord, il nous faut un équivalent de la somme des espaces propres. Cet objet, une variété, sera obtenu par des méthodes de bifurcation. Ce qui nous amène à faire l'hypothèse soit que chaque espace est de dimension 1, soit que la dégénérescence est due à une symétrie que possède également la non-linéarité, voir le paragraphe 2.

Toutefois, le théorème RAGE ne suffit pas à obtenir un résultat de stabilité. Il faut alors préciser comment une solution du problème linéaire orthogonale aux vecteurs propres, c'est à dire dans le spectre continu, tend vers zéro. C'est l'objet de nos deux premiers théorèmes, Théorème B.3 du chapitre II sur la propagation et Théorème B.4 chapitre II sur la dispersion.

3 Les estimations linéaires

Il s'agit ici de donner des résultats plus précis que le théorème RAGE qui nous permettent d'obtenir des résultats de stabilité non linéaire.

Dans tout ce qui suit $P_c(H)$ désigne le projecteur associé au spectre continu, c'est le projecteur orthogonal sur l'espace associé au spectre continu : $\mathcal{H}_c$ qui est l'orthogional des espaces propres.

3.1 La propagation et les estimations de régularité

La propagation La propagation met en évidence le mouvement d'une onde et ne peut donc se produire que pour des ondes non stationnaires et donc associées au spectre continu. Les estimations de propagation s'énoncent dans les espaces de Sobolev pondérés suivants :

Definition A.1 (Espace de Sobolev pondérés)**.** *Les espaces de Sobolev pondérés sont définis par*

$$H^t_\sigma(\mathbb{R}^3, \mathbb{C}^4) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3),\ \|\langle Q\rangle^\sigma \langle P\rangle^t f\|_2 < \infty \right\}$$

pour $\sigma, t \in \mathbb{R}$. On les munis de la norme

$$\|f\|_{H^t_\sigma} = \|\langle Q\rangle^\sigma \langle P\rangle^t f\|_2.$$

Si $t = 0$, on note L^2_σ au lieu de H^0_σ.

Nous avons utilisé les notations $\langle u\rangle = \sqrt{1+u^2}$, $P = -i\nabla$, et Q pour l'opérateur de multiplication par x dans $\mathbb{R}^3$.

Nous avons obtenu (voir Théorème B.3 Chapitre II Partie B) le

Théorème 1 (Propagation pour l'opérateur de Dirac à potentiel)**.** *Si $V : \mathbb{R}^3 \mapsto S_4(\mathbb{C})$ (matrices symétriques 4×4) est une fonction $\mathcal{C}^\infty$ telle qu'il existe $\rho > 5$ avec*

$$\forall \alpha \in \mathbb{N}^3,\ \exists C > 0,\ \forall x \in \mathbb{R}^3,\ |\partial^\alpha V|(x) \leq \frac{C}{\langle x\rangle^{\rho+|\alpha|}}.$$

alors

$$H := D_m + V$$

est autoadjoint de domaine $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

Si de plus H n'a ni valeurs propres aux seuils ni résonances aux seuils, on a pour tout $\sigma > 5/2$, l'estimation

$$\|e^{-\mathrm{it}H}\mathbf{P}_c(H)\|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C\langle t\rangle^{-3/2}.$$

Une résonance est dans notre cas une solution stationnaire dans $\cap_{\sigma\in(1/2,\rho-1/2)} L^2_{-\sigma} \setminus L^2$.

Un tel résultat est en fait l'équivalent pour l'opérateur de Dirac d'un résultat bien connu pour l'opérateur de Schrödinger, voir Jensen et Kato [JK79] . La preuve pour l'opérateur de Schrödinger [JK79] est basée sur l'identité suivante :

$$e^{-\mathrm{i}tH}\chi(H) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int e^{-\mathrm{i}t\lambda}\chi(\lambda)\Im\left(H-\lambda-\mathrm{i}\varepsilon\right)^{-1} d\lambda,$$

qui nous donne le propagateur comme la transformée de Fourier de la trace sur l'axe réel de la partie imaginaire de la résolvante ou encore la densité spectrale. La difficulté est alors dans l'interversion limite et intégrale. Ceci est possible sur les espaces L^2 à poids si χ est à support compact en utilisant le principe d'absorption limite :

$$\exists C>0,\ \sup_{\substack{|\Re z|>m,\\ \Im z>0}} \left\{ \left\| \langle Q\rangle^{-\sigma} (H-z)^{-\sigma} \langle Q\rangle^{-1} \right\| \right\} \leq C. \tag{LAP}$$

Pour l'opérateur de Schrödinger, le comportement de la résolvante en l'infini permet également l'interversion au voisinage de l'infini. Il suffit alors d'utiliser des résultats liant la régularité et le comportement d'une fonction à la régularité et au comportement de sa transformée de Fourier.

Malheureusement, pour l'opérateur de Dirac, on sait depuis Yamada [Yam93] que la résolvante se comporte « mal » à l'infini ; elle ne tend pas vers 0 en norme. Nous avons alors divisé l'étude en deux parties :

Au voisinage des seuils, nous utilisons la seule méthode connue pour étudier les propriétés spectrales aux seuils, celle de Jensen et Kato [JK79] par la suite améliorée par Jensen et Nenciu [JN01]. Cette étude nous donne le comportement de la résolvante et de certaines de ses dérivées aux seuils permettant ainsi d'obtenir le comportement du propagateur en temps grand pour des petites énergies.

Loin des seuils, nous avons étudié directement le propagateur en utilisant un lemme de vitesse minimale de fuite. Plus précisément, nous avons adapté à notre cas la méthode de Hunziker, Sigal et Soffer [HSS99]. Cette méthode basée sur des développements en multi-commutateurs et l'estimation de Mourre nécessite dans le cas de l'opérateur de Schrödinger une localisation dans le spectre afin d'avoir des commutateurs bornés. Dans le cas qui nous intéresse ici, le fait que l'opérateur de Dirac soit d'ordre un donne des commutateurs bornés et évite toute localisation spectrale, voir Iftimovici et Mantoiu [IM99]. Nous avons obtenu, Théorème B.8 Chapitre II Partie B, le résultat suivant :

Proposition A.2 (Propagation loin des seuils)**.** *Sous les hypothèses du Théorème 1, pour tout $\chi \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{C}^4)$ borné à support dans $\mathbb{R}\backslash]-m;m[$ et tout $\sigma\geq 0$, il existe $C>0$ tel que*

$$\|e^{-\mathrm{i}tH}\chi\left(H\right)\|_{B(L^2_\sigma,L^2_{-\sigma})} \leq C\left\langle t\right\rangle^{-\sigma}.$$

Nous notons alors que la propagation loin des seuils a lieu à des taux d'autant plus élevés que la donnée initiale est localisée.

La H-régularité Nous avons aussi obtenu, Théorème B.2 Chapitre III Partie B, un résultat proche de nos estimations de propagation et lié aux estimations de régularité au sens de Kato :

Théorème 2 (H-régularité de $\langle Q\rangle^{-1}$). *Sous les hypothèses du Théorème 1, soit $\sigma \geq 1$, pour tout $s \in \mathbb{R}$, on a les estimations*

$$\int_{\mathbb{R}} \| \langle Q\rangle^{-\sigma} e^{-\mathrm{i}tH} \mathbf{P}_c(H) \psi\|_{H^s}^2 \, dt \leq C \|\psi\|_{H^s}^2, \tag{i}$$

$$\| \int_{\mathbb{R}} e^{-\mathrm{i}tH} \mathbf{P}_c(H) \langle Q\rangle^{-\sigma} F(t) \, dt\|_{H^s} \leq C \|F\|_{L^2(\mathbb{R},H^s)}, \tag{ii}$$

$$\| \int_{s<t} \langle Q\rangle^{-\sigma} e^{-\mathrm{i}(t-s)H} \mathbf{P}_c(H) \langle Q\rangle^{-\sigma} F(s) \, ds\|_{L^2(\mathbb{R},H^s)} \leq C \|F\|_{L^2(\mathbb{R},H^s)}. \tag{iii}$$

La démonstration est similaire à celle des estimations de Strichartz. Au lieu d'utiliser les inégalités de Hardy-Littlewood-Sobolev, il faut utiliser l'estimation d'absorption limite (voir (LAP) avec $\sigma = 1$) qui est en fait équivalente à (i) pour $s = 0$.

3.2 La dispersion et les estimations de Strichartz

La dispersion La dispersion quant à elle traduit un phénomène d'étalement d'une onde associée au spectre continu. Elle s'énonce pour l'opérateur de Dirac dans les espaces de Besov :

Definition A.3 (Espace de Besov). *Pour $s \in \mathbb{R}$ et $1 \leq p, q \leq \infty$, l'espace de Besov $B_{p,q}^s(\mathbb{R}^3, \mathbb{C}^4)$ est l'ensemble des $f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ (distributions tempérées) telles que*

$$\|f\|_{B_{p,q}^s} = \left(\sum_{j\in\mathbb{N}} 2^{jsq} \|\varphi_j * f\|_p^q \right)^{\frac{1}{q}} < +\infty$$

avec $\widehat{\varphi} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ telle que $\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) = 1$ pour tout $\xi \in \mathbb{R}^3 \setminus \{0\}$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$, $j \in \mathbb{N}^*$, $\xi \in \mathbb{R}^3$ et $\widehat{\varphi_0} = 1 - \sum_{j\in\mathbb{N}^*} \widehat{\varphi}_j$.

Cet espace est naturellement muni de la norme : $f \in B_{p,q}^s(\mathbb{R}^3, \mathbb{C}^4) \mapsto \|f\|_{B_{p,q}^s}$.

Nous avons obtenu (voir Théorème B.4 Chapitre II Partie B) le

Théorème 3 (Dispersion pour l'opérateur). *Sous les hypothèses du Théorème 1, pour tous $p \in [1, 2]$, $\theta \in [0, 1]$, $s - s' \geq (2+\theta)(\frac{2}{p} - 1)$ et $q \in [1, \infty]$, il existe une constante $C > 0$ telle que l'on ait l'estimation suivante :*

$$\left\|e^{-\mathrm{i}tH} \mathbf{P}_c(H)\right\|_{B_{p,q}^s, B_{p',q}^{s'}} \leq C \left(K(t)\right)^{\frac{2}{p}-1}$$

avec $\frac{1}{p} + \frac{1}{p'} = 1$, et

$$K(t) = \begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \in (0, 1], \\ |t|^{-1-\theta/2} & \text{if } |t| \in [1, \infty). \end{cases}$$

Ce résultat dans le cas libre ($V = 0$) se déduit de ceux bien connus pour l'équation de Klein-Gordon, voir Brenner [Bre85]. Plus précisément, l'utilisation des espaces de Besov permet de se retreindre à des intégrales oscillantes de la forme :

$$I(t,x) = \int_{\mathbb{R}^3} e^{-\mathrm{i}x\cdot\xi - \mathrm{i}t\sqrt{\xi^2+m^2}}\phi(\xi)\, d\xi,$$

avec ϕ à support compact. La perte de régularité nécessaire à ces estimations ne permet pas de les généraliser facilement aux cas avec potentiel. Dans les cas où l'on peut se ramener à des équations de Klein-Gordon, comme le cas à potentiel scalaire $V(x) = f(x)\beta$, les résultats de Yajima [Yaj95] nous donnent la dispersion. Malheureusement, nous n'avons obtenu aucune généralisation de cette méthode.

À la place, nous avons utilisé une méthode inspirée de Cuccagna et Schirmer [CS01]. On généralise la méthode du cas libre en étudiant de nouvelles intégrales oscillantes où apparaissent de nouveaux termes dans l'intégrande. Ces nouveaux termes sont les fronts d'ondes perturbés par le potentiel. Ils correspondent aux vecteurs propres généralisés associés aux « valeurs propres » dans le spectre continu : les fronts d'ondes déformés.

Notons que ce théorème se reformule facilement dans les espace de Besov en utilisant les injections continues suivantes :

1. $B^s_{p,2} \subset W^{s,p} \subset B^s_{p,p}$ pour $1 < p \leq 2$,
2. $B^s_{p,p} \subset W^{s,p} \subset B^s_{p,2}$ pour $2 \leq p < \infty$,
3. $B^u_{r,q} \subset B^s_{p,q}$ si $1 \leq r \leq p \leq \infty$ et $u - n/r = s - n/p$.

On notera qu'alors nous n'obtenons pas le théorème pour $p = 1$. Pour obtenir ce cas, nous pouvons répéter la démonstration faites dans les espaces de Besov en utilisant des poids $\langle\xi\rangle^{-(s-s')}$ dans les intégrales oscillantes en lieu et place des fonctions ϕ à support compact. Nous obtenons alors une condition d'intégrabilité sur le poids qui impose l'inégalité stricte $s - s' > (2+\theta)(\frac{2}{p} - 1)$ au lieu de l'inégalité large que nous avons dans les espaces de Besov. Nous n'avons malheureusement pas pu améliorer cela.

Les estimations de Strichartz En appliquant les résultats de Keel et Tao [KT98] à nos estimations de dispersion, nous avons obtenu (voir Théorème B.4 Chapitre III Partie B) le

Théorème 4 (Estimations de Strichartz)**.** *Sous les hypothèses du Théorème 1, pour tous* $2 \leq p, q \leq \infty$, $\theta \in [0,1]$, $\beta \in [-\theta/2, \theta/2]$ *tels que* $(1 - \frac{2}{q})(1+\beta) = \frac{2}{p}$ *avec* $(p,\beta) \neq (2,0)$, *posons* $\alpha(q) = (1+\beta)(1-\frac{2}{q})$. *Alors pour* $s' - s \geq \alpha(q)$, *il existe une constante strictement positive* C *telle que*

$$\left\|e^{-\mathrm{i}tH}P_c(H)\psi\right\|_{L^p_t(\mathbb{R},B^s_{q,r}(\mathbb{R}^3,\mathbb{C}^4))} \leq C\|\psi\|_{H^{s'}(\mathbb{R}^3,\mathbb{C}^4)}, \tag{i}$$

$$\left\|\int e^{\mathrm{i}tH}P_c(H)F(t)\,dt\right\|_{H^s} \leq C\|F\|_{L^{p'}(\mathbb{R},B^{s'}_{q',r}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{ii}$$

$$\left\|\int_{s<t} e^{-\mathrm{i}(t-s)H}P_c(H)F(s)\,ds\right\|_{L^p(\mathbb{R},B^{-s}_{q,r}(\mathbb{R}^3,\mathbb{C}^4))} \leq C\|F\|_{L^{\tilde p'}(\mathbb{R},B^{\tilde s}_{\tilde q',r}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{iii}$$

pour tout $r \in [1,\infty]$, $(\tilde q, \tilde p)$ *choisi comme* (q,p) *et* $s + \tilde s \geq \alpha(q) + \alpha(\tilde q)$.

4 L'équation de Dirac non linéaire

4.1 Les états stationnaires

Selon Rañada [Ran], les solutions stationnaires de l'équation de Dirac non linéaire sont interprétées de deux façon différentes :

- soit comme les états liés d'une particule dans un espace-temps avec une métrique possédant une torsion.
- soit comme des ondes interagissant avec elles-mêmes du fait de leurs extensions spatiales.

L'existence de telles solutions n'est pas évidente. Beaucoup étudié par les physiciens, dont Soler [Sol70], ce problème n'a été mathématiquement résolu que récemment dans un certain nombre de cas, voir [ES02] pour une liste de références.

Dans notre cas, la présence du potentiel et le fait que nous nous intéressons uniquement aux petites solutions, nous permet d'utiliser la

Proposition A.4. *Soit H un opérateur autoadjoint sur $L^2(\mathbb{R}^3,\mathbb{C}^4)$ ayant une valeur propre simple λ_0 associée à un vecteur propre ϕ_0. On suppose qu'il existe un voisinage $\mathcal{O} \subset \mathbb{R}$ de λ_0 tel que pour tout $\lambda \in \mathcal{O}$ l'opérateur $(H-\lambda)^{-1}P_0$ est dans $\mathcal{B}(L^2_\sigma(\mathbb{R}^3,\mathbb{C}^4))$ pour tout $\sigma \in \mathbb{R}^+$, et dans $\mathcal{B}(H^l(\mathbb{R}^3,\mathbb{C}^4), H^{l+1}(\mathbb{R}^3,\mathbb{C}^4))$ pour tout $l \in \mathbb{N}$, où P_0 est le projecteur sur l'orthogonal de ϕ_0. Soit $F \in \mathcal{C}^{k+1}(\mathbb{C}^4,\mathbb{C}^4)$ tel que $F(z) = O(|z|^3)$.*

Alors pour tout $\sigma \in \mathbb{R}^+$, il existe Ω un voisinage de $0 \in \mathbb{C}$, une application $\mathcal{C}^k$

$$h : \Omega \mapsto \{\phi_0\}^\perp \cap H^2(\mathbb{R}^3,\mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3,\mathbb{C}^4)$$

et une application $\mathcal{C}^k$ $E : \Omega \mapsto \mathbb{R}$ telles que $S(u) = u\phi_0 + h(u)$ satisfait pour tout $u \in \Omega$,

$$HS(u) + \nabla F(S(u)) = E(u)S(u),$$

avec les propriétés suivantes

$$\begin{cases} h(e^{i\theta}u) = e^{i\theta}h(u), \quad \forall \theta \in \mathbb{R}, \\ h(u) = O(|u|^2), \\ E(u) = E(|u|), \\ E(u) = \lambda_0 + O(|u|^2). \end{cases}$$

Ce lemme est facilement démontré à partir d'un résultat de Pillet et Wayne [PW97]. L'hypothèse de simplicité de la valeur propre peut être affaiblie en présence de symétrie, voir paragraphe 2 Chapitre I Partie A plus loin, permettant ainsi une réduction au cas unidimensionnel.

4.2 La stabilité non linéaire

La question de la stabilité des grands états stationnaires d'un système de Dirac a suscité beaucoup d'intérêt dans la communauté des physiciens. Il semble, voir [Ran], que les études théoriques ont souvent conclu à l'instabilité orbitale du fait que la variation seconde de l'énergie n'est pas définie positive.

Toutefois les expériences numériques semblent mettre au jour des phénomènes bien plus subtils. Il a été observé numériquement, voir [Ran], que pour certaines non-linéarités les ondes solitaires sont stables après collision. Il a également été observé un phénomène d'attraction. Plus précisément, des ondes initialement gaussiennes, se relaxent vers des solutions stationnaires. Par ailleurs dans la même expérience, les dilatations et les contractions d'ondes stationnaires se relaxent sur d'autre ondes stationnaires en absorbant ou en émettant la quantité d'énergie nécessaire.

En ce qui nous concerne, nous avons étudié un problème plus simple en tentant de généraliser les propriétés de stabilité linéaire au cas des petites solutions. Nous avons travaillé avec l'hypothèse suivante :

Hypothèse 4.1. *La fonction $F : \mathbb{C}^4 \mapsto \mathbb{R}$ de classe $\mathcal{C}^\infty(\mathbb{R}^8, \mathbb{R})$ satisfait $F(z) = O(|z|^4)$ quand $z \to 0$. De plus, on a l'invariance de jauge :*

$$\forall z \in \mathbb{C}^4,\ \forall \theta \in \mathbb{R},\ F(e^{i\theta} z) = F(z).$$

Le cas non résonnant avec deux valeurs propres simples Nous faisons, dans ce paragraphe, l'hypothèse suivante :

Hypothèse 4.2. *L'opérateur $H := D_m + V$ a uniquement deux valeurs propres $\lambda_0 < \lambda_1$ et elles sont simples, avec ϕ_0 et ϕ_1 comme vecteurs propres associés. De plus, nous avons la condition de non résonance suivante :*

$$|\lambda_1 - \lambda_0| < \min\{|\lambda_0 + m|, |\lambda_0 - m|\}.$$

Nous obtenons alors comme dans le cas linéaire des directions stables pour chaque état propre. Plus précisément si l'espace $\mathcal{H}_c$ est l'espace associé au spectre continu,

On introduit $JH(u)$ l'opérateur linéarisé autour de l'état stationnaire $S(u)$, avec

$$H(u) = H + d^2F(S(u)) - E(u)$$

où d^2F est l'application différentielle de ∇F. On note que $H(u)$ est un opérateur $\mathbb{R}$-linéaire et non $\mathbb{C}$-linéaire. Nous travaillerons donc dans l'espace $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ au lieu de $L^2(\mathbb{R}^3, \mathbb{C}^4)$ en écrivant

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}$$

au lieu de ϕ. La multiplication par $-$i devient l'opérateur

$$J = \begin{pmatrix} 0 & -I_{\mathbb{R}^4} \\ I_{\mathbb{R}^4} & 0 \end{pmatrix}.$$

L'opérateur $JH(u)$ a un noyau géométrique de dimension 2 et deux valeurs propres simples $E_1(u)$ et $-E_1(u)$ qui sont purement imaginaire. Les espaces propres associés sont conjugués l'un à l'autre. Travaillant sur la partie réelle de la somme directe de ces espaces, on introduit une famille de bases pour ces espaces : $\left(\phi_1^1(u), \phi_1^2(u)\right)$. Le reste du spectre est le spectre essentiel. On note $\mathcal{H}_c(u)$ l'espace associé au spectre continu. Cette espace $\mathcal{H}_c(u)$ est l'orthogonal des précédents espaces par rapport au produit $(f, g) \mapsto \Re\langle f, Jg\rangle$ où $\langle\cdot,\cdot\rangle$ est le produit standard de $L^2(\mathbb{R}^3, \mathbb{C}^4)$. L'espace $\mathcal{H}_c(u)$ est isomorphe à $\mathcal{H}_c$ (l'espace associé au spectre continu de H) et l'isomorphisme est P_c, le projecteur orthogonal sur $\mathcal{H}_c$ qui est par ailleurs un opérateur borné de $H^s_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ dans lui-même, pour tout réels s et σ.

Nous avons obtenu (voir Théorème B.6 Chapitre II Partie B) le

Théorème 5 (Variété stable)**.** *Sous les hypothèses du Théorème 1, les hypothèses 4.1 et 4.2, si $s, s', \beta \in \mathbb{R}_+^*$ sont tels que $s' \geq s + 3 \geq \beta + 6$ et $\sigma > 5/2$, il existe alors $\varepsilon_0 > 0$, $R > 0$, $C > 0$, $T_0 > 0$ et une application lipschitzienne*

$$\Psi : \mathcal{S} \to \mathbb{R}^2$$

où $\mathcal{S} = \Big\{(V,\xi)\,;\ V \in B_{\mathbb{C}^2}(0,\varepsilon),\ \xi \in \mathcal{H}_c(u) \cap B_{H^{s'}_\sigma}(0,R)\Big\}$ *muni de la métrique de de* $\mathbb{C}^2 \times H^{s'}_\sigma$, *telle que* $\Psi(u,0) = 0$ *pour* $u \in B_{\mathbb{C}}(0,\varepsilon)$,

$$|\Psi(u,z)| \leq C\left(|u| + \|z\|_{H^{s'}_\sigma}\right)^2,$$

et ce qui suit est vrai. Pour toute condition initiale de la forme

$$\psi_0 = S(u_0) + z_0 + \Psi(u_0,z_0)\cdot\phi_1(u_0)$$

avec $(u_0,z_0) \in \mathcal{S}$, *on a*

(i) il existe une unique solution ψ *de* (NLDE) *de condition initiale* ψ_0, *et cette solution est dans*

$$\mathcal{C}\left(]-T_0;+\infty[, H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\right) \cap \mathcal{C}^1\left(]-T_0;+\infty[, H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right);$$

(ii) il existe $(u_\infty, z_\infty) \in \mathcal{S}$ *et* $E_\infty \in \mathbb{R}$ *avec*

$$|u_\infty - u_0| \leq C\|z_0\|^2_{H^{s'}_\sigma}, \quad |E_\infty| \leq C\|z_0\|^2_{H^{s'}_\sigma}, \quad \|z_\infty - z_0\|_{H^{s'}} \leq C\|z_0\|^2_{H^{s'}_\sigma}$$

tels que

$$\psi(t) = e^{-\mathrm{i}(tE(u_\infty)+E_\infty+r(t))}\left(S(u_\infty) + e^{JtH(u_\infty)}z_\infty + \varepsilon(t)\right),$$

où

$$\begin{cases} \|\varepsilon(t)\|_{H^{s'}} \leq \dfrac{C}{\langle t\rangle}\|z_0\|^2_{H^{s'}_\sigma} \\ \|\varepsilon(t)\|_{H^{s}_{-\sigma}} \leq \dfrac{C}{\langle t\rangle^2}\|z_0\|^2_{H^{s'}_\sigma} \\ \|\varepsilon(t)\|_{B^{\beta}_{\infty,2}} \leq \dfrac{C}{\langle t\rangle^2}\|z_0\|^2_{H^{s'}_\sigma}. \end{cases}$$

et $|r(t)|_{H^{s'}} \leq \dfrac{C}{\langle t\rangle}\|z_0\|^2_{H^{s'}_\sigma}$ *quand* $t \to +\infty$.

Nous avons, avec ce théorème, la stabilité pour un ensemble de directions tangentes à un espace de codimension un. La direction pour laquelle nous n'avons pas été capable d'obtenir de résultats est la direction associée à l'état excité. En dehors de celle-ci, nous avons une stabilisation sur la variété d'états stationnaires associés à ce que l'on a considéré comme l'état fondamental. Il est intéressant de noter que la stabilisation, qui est en t^{-2} au moins, est plus rapide que la propagation et la dispersion, qui sont en $t^{-3/2}$.

Nous avons donc une variété stable contenant la variété des états stationnaireset à l'intérieur de celle-ci, nous avons la stabilisation en temps positifs.

Nous souhaitons aussi mentionner un résultat que nous avons d'abord négligé et qui est une conséquence d'un théorème dû à Kramers. Considérons l'opérateur anti-linéaire K :

$$K\begin{pmatrix}\psi\\ \chi\end{pmatrix} = \begin{pmatrix}\sigma_2\overline{\psi}\\ \sigma_2\overline{\chi}\end{pmatrix} \text{ avec } \sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}.$$

L'opérateur D_m commute à K. Si V commute aussi à K, les valeurs propres de H sont dégénérées et les espaces propres associés sont de dimensions paires. Cette invariance bien qu'importante a été négligée dans cette première étude. Ceci nous empêche par exemple de considérer des potentiels électriques ou scalaires. Dans ce qui suit, nous introduisons l'invariance par rapport à K pour obtenir des résultats plus complets, incluant ces potentiels. Le lecteur notera que celle-ci pourrait être introduite dans cette première étude de façon tout à fait similaire au chapitre III de la partie B.

Le cas résonnant avec deux valeurs propres doubles Dans le théorème précédent, nous ne savons pas préciser le comportement d'une perturbation en dehors des directions stables. Toutefois, en faisant une hypothèse de résonance, ceci devient alors possible. Nous faisons alors l'hypothèse suivante :

Hypothèse 4.3. *Le potentiel V commute à K. L'opérateur $H := D_m + V$ a seulement deux valeurs propres $\lambda_0 < \lambda_1$ et elles sont doubles. avec $\{\phi_0, K\phi_0\}$ et $\{\phi_1, K\phi\}$ comme vecteurs propres associés.*

De plus, on a la condition de résonance

$$|\lambda_1 - \lambda_0| > \min\{|\lambda_0 + m|,\, |\lambda_0 - m|\}$$

et la règle d'or de Fermi :

$$\Gamma(\phi) = \lim_{\substack{\varepsilon\to 0,\\ \varepsilon>0}} \left\langle d^2F(\phi)\phi_1, \Im\left((H-\lambda_0) - (\lambda_1-\lambda_0) - \mathrm{i}\varepsilon\right)^{-1} P_c(H) d^2F(\phi)\phi_1 \right\rangle > 0, \tag{3}$$

pour tout vecteur propre ϕ non nul associé à λ_0.

L'application d^2F est en fait la différentielle de ∇F. Pour plus de détails sur la règle d'or de Fermi on pourra consulter [RS78].

Pour construire notre variété, nous supposerons que le problème non linéaire est lui aussi invariant par l'action de K :

$$\forall z \in \mathbb{C},\ F(Kz) = F(z).$$

On a alors :

Proposition A.5. *Pour tout $\sigma \in \mathbb{R}^+$, il existe Ω un voisinage de $0 \in \mathbb{C}^2$, une fonction de classe $\mathcal{C}^\infty$*

$$h : \Omega \mapsto \mathcal{H}_c \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4),$$

et une fonction $E \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ tels que $S((u_1, u_2)) = u_1\phi_0 + u_2 K\phi_0 + h((u_1, u_2))$ vérifie, pour tout $U = (u_1, u_2) \in \Omega$,

$$HS(U) + \nabla F(S(U)) = E(U)S(U),$$

avec $h((u_1, u_2)) = \left(\frac{u_1}{|(u_1,u_2)|} Id_{\mathbb{C}^4} + \frac{u_2}{|(u_1,u_2)|} K\right) h\left((|(u_1, u_2)|, 0)\right)$, $h(U) = O(|U|^2)$, $E(U) = E(|U|)$ et $E(U) = \lambda_0 + O(|U|^2)$.

De plus, pour tout $\alpha \in \mathbb{N}^4$, $s \in \mathbb{R}^+$ et $p, q \in [1, \infty]$, il existe $\gamma > 0$, $\varepsilon > 0$ et $C > 0$ tels que pour tout $U \in B_{\mathbb{C}^2}(0, \varepsilon)$, on a $\|e^{\gamma\langle Q\rangle} \partial^\alpha_U S(U)\|_{B^s_{p,q}} \le C\|S(U)\|_2$.

Dans ce cas (voir Théorème B.7 Chapitre III Partie B) on classifie alors complètement les directions de perturbation en directions stables et instables.

On introduit l'opérateur linéarisé $JH(U)$ autour d'un état stationnaire $S(U)$

$$H(U) = H + d^2F(S(U)) - E(U).$$

Comme l'opérateur $H(U)$ est seulement $\mathbb{R}$-linear nous travaillons dans $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ comme précédemment. L'opérateur $JH(u)$ a un noyau géométrique de dimension 4 et quatre valeurs propres doubles $E_1(U)$, $\overline{E_1}(U)$, $-E_1(U)$ et $-\overline{E_1}(U)$ avec $\Re E_1(U) > 0$. Les espaces propres associés à $E_1(U)$ et $\overline{E_1}(U)$ sont conjugués via la conjugaison complexe, ainsi comme dans le cas précédent nous travaillerons sur la partie réelles de leur somme : $\mathcal{H}^1_+(U)$. On introduit donc la base $(\xi_i(U))_{i=1,\dots,4}$ de $\mathcal{H}^1_+(U)$.

En procédant de même pour $-E_1(U)$ et $-\overline{E_1}(U)$, on introduit la partie réelle de leur somme : $\mathcal{H}^1_-(U)$, et une base : $(\xi_i(U))_{i=5,\dots,8}$.

Le reste du spectre est le spectre essentiel. On note encore $\mathcal{H}_c(U)$ l'espace associé. Ici encore, il est orthogonal aux espaces propres par rapport au produit $(f,g) \mapsto \Re\langle f, Jg\rangle$ où $\langle\cdot,\cdot\rangle$ est le produit canonique de $L^2(\mathbb{R}^3,\mathbb{C}^4)$.

On introduit $\mathcal{H}^r_c(U)$ l'intersection de $\mathcal{H}_c$ avec

$$\left\{\Im\left(H - E(U) + \lambda_1 - \lambda_0 - \frac{\mathrm{i}}{2}\Gamma(S(U))\right)^{-1} d^2F(S(U))|U|\left(\frac{u_1}{|U|}I_{\mathbb{C}^4} + \frac{u_2}{|U|}K\right)\phi_1, \quad \forall U = (u_1,u_2)\in\mathbb{C}^2\right\}^{\perp}.$$

L'espace $\mathcal{H}_c(U)$ est isomorphe à $\mathcal{H}^r_c(U)$ et cet isomorphisme est le projecteur orthogonal sur $\mathcal{H}^r_c(U)$ par rapport au produit $(f,g)\mapsto \Re\langle f, Jg\rangle$ où $\langle\cdot,\cdot\rangle$ est encore une fois le produit canonique de $L^2(\mathbb{R}^3,\mathbb{C}^4)$. Cette fois encore, ce projecteur est un opérateur borné de $H^s_\sigma(\mathbb{R}^3,\mathbb{R}^8)$ dans lui-même pour tout réels s et σ.

On a alors obtenu (voir Théorème B.7 III Partie B)

Théorème 6. *Sous les hypothèses du Théorème 1, les hypothèses 4.3 et 4.1 pour $s' > \beta + 2 > 2$ et $\sigma > 3/2$, il existe $\varepsilon > 0$ et une application continue $r : B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \mathbb{R}$ avec $r(U) = O(\Gamma(U))$, telle que pour toute condition initiale de la forme*

$$\psi_0 = S(U_0) + z_0 + A\cdot\xi(U_0)$$

avec $U_0 \in B_{\mathbb{C}^2}(0,\varepsilon)$, $z_0) \in \mathcal{H}_c(U_0)\cap B_{H^{s'}}(0, r(U_0))$ et $A\in\mathbb{R}^8$, ce qui suit est vrai.

(i) Pour l'ensemble

$$\mathcal{S} = \left\{(U,z)\,;\; U\in B_{\mathbb{C}^2}(0,\varepsilon),\; z\in\mathcal{H}_c(U)\cap B_{H^{s'}}(0,r(U))\right\}$$

muni de la métrique de de $\mathbb{C}^2\times H^{s'}$, il existe $C>0$, $\mathcal{U}$ un voisinage de $(0,0)$ de $\mathcal{S}$, une application lisse $\Psi : \mathcal{U}\mapsto\mathbb{C}^4$ de graphe $\mathcal{CM}$ avec $\Psi(\cdot,0)=0$ et $\|\Psi(0,0)\| = \|D\Psi(0,0)\| = 0$ tels que si $(U_0,z_0)\in\mathcal{U}$ et $A=\Psi(U_0,z_0)$ on aie :

(a) il existe une unique solution ψ de (NLDE) *de condition initiale ψ_0 et cette solution est dans $\mathcal{C}\left(\mathbb{R},H^{s'}\right)\cap\mathcal{C}^1\left(\mathbb{R},H^{s'-1}\right)$;*

(b) il existe $\mathcal{U}^\pm$ des voisinages de $(0,0)$ dans $\mathcal{S}$, des applications bijectives

$$(U_0;z_0)\in\mathcal{U}\mapsto(U_{\pm\infty};z_{\pm\infty})\in\mathcal{U}^\pm,$$

avec

$$|U_{\pm\infty}-U_0| \le C\|z_0\|^2_{H^{s'}},\; \|z_{\pm\infty}-z_0\|_{H^{s'}} \le C\,|U_0|\,\|z_0\|_{H^{s'}},$$

telle que pour tout $t\in\mathbb{R}$

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U(t)) + e^{JtE(U_{\pm\infty})}e^{JtH(U_{\pm\infty})}z_{\pm\infty} + \varepsilon_\pm(t)$$

avec $\dot U\in L^q(\mathbb{R}^\pm)$ pour tout $q\in[1,\infty]$, $\lim_{t\to\pm\infty} U(t) = U_{\pm\infty}$,

$$\max\left\{\|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm,H^{s'})},\; \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,H^s_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,B^\beta_{\infty,2})}\right\} \le C\,|U_0|\,\|z_0\|_{H^{s'}},$$

et

$$\lim_{t\to\pm\infty} \|\varepsilon_\pm(t)\|_{H^{s'}} = 0.$$

(ii) Pour les ensembles

$$\tilde{S_+} = \Big\{ (U,z,p)\,;\ U \in B_{\mathbb{C}^2}(0,\varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0,r(U)),$$

$$p \in B_{\mathbb{R}^8}(0,r(U)),\ \textit{avec } p_i = 0 \textit{ pour } i = 1,\dots,4 \Big\}$$

et

$$\tilde{S_-} = \Big\{ (U,z,p)\,;\ U \in B_{\mathbb{C}^2}(0,\varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0,r(U)),$$

$$p \in B_{\mathbb{R}^8}(0,r(U)),\ \textit{avec } p_i = 0 \textit{ pour } i = 5,\dots,8 \Big\}$$

munis des métriques de $\mathbb{C}^2 \times H^{s'} \times \mathbb{R}^8$ *; Il existe des voisinages ouverts* $\mathcal{W}_\pm$ *de* $(0,0,0)$ *dans* $\tilde{S}_\pm$ *et des applications lisses* $\Phi_\pm : \mathcal{W}_\pm \mapsto \mathbb{C}^4$ *avec* $\|\Phi_\pm(0,0,0)\| = \|\Phi_\pm(0,0,0)\| = 0$, $(\Phi_+(\cdot,\cdot,p))_i = p_i$ *si* $i = 5,\dots,8$, $(\Phi_-(\cdot,\cdot,p))_i = p_i$ *si* $i = 1,\dots,4$ *et*

(a) si $A \in \Phi_\pm(\mathcal{W}_\pm)$ *et* $A \notin \Psi(\mathcal{U})$ *alors il existe une unique solution* ψ *de* (NLDE) *de condition initiale* ψ_0 *et pour tout voisinage* $\mathcal{O}$ *de* $S(U_0)$ *contenant* ψ_0*, il existe* $t_\pm(\psi_0) > 0$ *tel que* ψ *est dans*

$$\mathcal{C}\left([-t_+;+\infty[, H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\right) \cap \mathcal{C}^1\left(]-t_+;+\infty[, H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right)$$

$$\Bigg(\textit{resp.}$$

$$\mathcal{C}\left(]-\infty;t_-], H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\right) \cap \mathcal{C}^1\left(]-\infty;t_-[, H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right) \Bigg),$$

et

$$\mathrm{dist}_{L^2}(\psi(t),\mathcal{CM}) = O(e^{\mp\gamma t}) \textit{ quand } t \to \pm\infty \quad \textit{et} \quad \psi(\mp t_\pm) \notin \mathcal{O}$$

où γ *est dans une boule autour de* $1/2\Gamma(S(U_0))$*, de rayon en* $O(|U_0|^6))$ *;*

(b) si $A \notin \Phi_+(\mathcal{W}^+) \cup \Phi_-(\mathcal{W}_-)$ *alors existe une unique solution* ψ *de* (NLDE) *de condition initiale* ψ_0 *et pour tout voisinage* $\mathcal{O}$ *de* $S(V_0)$ *contenant* ψ_0 *tels que* ψ *est dans*

$$\mathcal{C}\left([t_-;t_+], H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\right) \cap \mathcal{C}^1\left(]t_-;t_+[, H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right)$$

et $\psi(t_+) \notin \mathcal{O}$ *et* $\psi(t_-) \notin \mathcal{O}$*.*

Avec ce théorème, comparativement au Théorème 5, nous avons réussi à préciser la dynamique pour des perturbations associées aux états excités. Nous avons alors deux jeux de quatres directions réelles donnant deux comportements opposés. Quatres directions donnent des solutions qui en temps positif se stabilisent exponentiellement vite sur la variété centrale alors que les quatres autres directions donnent de l'instabilité orbitale. Ces informations supplémentaires nous permettent alors d'obtenir un résultat bien meilleur pour les perturbations associées au spectre continu. D'une part, cela nous permet d'utiliser nos estimations de régularité, *i.e.* le Théorème 2, et nos estimations de Strichartz, *i.e.* le Théorème 4. D'autre part, nous obtenons la stabilisation dans les deux directions du temps. Nous noterons que nous avons considérer des perturbations non localisées contrairement au Théorème 5.

Nous avons une variété centrale contenant un bout de la variété des états stationnaire-set à l'intérieur de celle-ci nous avons un phénomène de diffusion non linéaire.À l'extérieur nous avons des directions stables et instables

Le cas à une valeur propre double Dans ce cas, il n'y a plus de problème associé à la seconde valeur propre et on obtient uniquement des directions stables et un résultat de *scattering* non linéaire plus fort. Nous travaillons avec l'hypothèse suivante :

Hypothèse 4.4. *Le potentiel V commute à K et H n'a qu'une seule valeur propre λ_0 qui est double.*

Nous introduisons donc une base orthonormée de vecteurs propres donnée par $(\phi_0, K\phi_0)$. Et nous supposerons que le problème non linéaire est lui aussi invariant par l'action de K :

$$\forall z \in \mathbb{C},\ F(Kz) = F(z).$$

On introduit alors $JH(U)$ l'opérateur linéarisé au voisinage d'un état stationnaire $S(U)$ avec $H(U) = \{H + d^2F(S(U)) - E(U)\}$, où d^2F est la différentielle of ∇F. Cet opérateur a un noyau géométrique de dimension réelle 4. Le reste du spectre est le spectre essentiel et on note $\mathcal{H}_c(U)$ l'espace associé qui est aussi l'orthogonal de ce noyau pour le produit $(f, g) \mapsto \Re\,(f, \mathrm{i}g)_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$.

On introduit également l'ensemble :

$$S = \{(U, z)\,;\ U \in \Omega,\ z \in \mathcal{H}_c(U)\}\,,$$

que l'on muni de la norme de $\mathbb{C}^2 \times H^{s'}$. Nous avons obtenu, Théorème B.6 Chapitre IV Partie B, le

Théorème 7 (Stabilisation et diffusion non linéaire)**.** *Sous les hypothèses du Théorème 1, les hypothèses 4.1 et 4.4, soient $s' > \beta + 2 > 2$ et $\sigma > 3/2$, il existe un voisinage $\mathcal{V}_0$ de $(0,0)$ dans S et $C > 0$ tels que pour tout condition initiale $\psi_0 = S(U_0) + z_0$ avec $(U_0, z_0) \in S$, on a :*

(i) il existe une unique solution globale ψ et cette solution est dans la classe $\mathcal{C}\left(\mathbb{R}, H^{s'}\right) \cap \mathcal{C}^1\left(\mathbb{R}, H^{s'-1}\right)$;

(ii) il existe des bijections de classe $\mathcal{C}^\infty$

$$(U_{\pm\infty}; z_\pm) : \mathcal{V}_0 \mapsto \mathcal{V}^\pm,$$

où $\mathcal{V}^\pm$ sont des voisinages ouverts de $(0,0)$ dans S telles que

$$|U_{\pm\infty} - U_0| \le C\|z_0\|^2_{H^{s'}},\ \|z_\pm - z_0\|_{H^{s'}} \le C\,|U_0|\,\|z_0\|_{H^{s'}},$$

telles que pour tout $t \in \mathbb{R}$, $\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U(t)) + e^{-\mathrm{i}tE(U_{\pm\infty})} e^{-\mathrm{i}tH(U_{\pm\infty})} z_\pm + \varepsilon_\pm(t)$ *avec* $\dot{U} \in L^p(R^\pm)$ *pour tout* $p \in [1,\infty]$, $\lim_{t\to\pm\infty} U(t) = U_{\pm\infty}$,

$$\max\left\{ \|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^{s'})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^{s'}_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})} \right\} \le C\,|U_0|\,\|z_0\|_{H^{s'}}$$

et

$$\lim_{t\to\pm\infty} \|\varepsilon_\pm(t)\|_{H^{s'}} = 0.$$

Nous noterons que l'absence d'états excités nous permet d'espérer des résultats au moins aussi bons que ceux du théorème 6. Le fait qu'il n'y ait plus d'états excités, nous évite même de localiser nos résultats au voisinage d'un état stationnaire. On obtient donc des résultats analogues au point *(i)* du théorème 6 mais valables sur l'ensemble de la variété des états stationnaires que nous avons construite.

Nous avons donc un vrai phénomène de diffusion non linéaire.

Remarque générale sur nos théorèmes Dans tout ce qui précède, on peut formuler nos résultats de scattering en prenant pour référence H ou D_m. On peut obtenir des développements de la forme

$$\psi(t) = e^{-\mathrm{i}(\int_0^t E(U(v))\,dv} S(U_\infty) + e^{-\mathrm{i}tD_m} z_\infty + \varepsilon(t)$$

avec pour référence l'opérateur libre ou

$$\psi(t) = e^{-\mathrm{i}(\int_0^t E(U(v))\,dv} S(U_\infty) + e^{-\mathrm{i}tH} z_\infty + \varepsilon(t)$$

avec pour référence l'opérateur à potentiel et avec $z_\infty \in \mathcal{H}_c$.

Mais dans ces cas, nous avons seulement l'estimation

$$\left\| \widetilde{\xi}_\infty - \xi_0 \right\|_{H^{s'}} \le K \|\xi_0\|_{H^{s'}_\sigma}.$$

D'autre part, le Théorème 5 s'adapte facilement au cas des valeurs propres dégénérées dues à l'invariance par rapport à K et inversement les théorèmes 6 et 7 s'adaptent facilement au cas des valeurs propres simples.

4.3 Le problème de Cauchy

Dans le chapitre I de la Partie B, nous montrons que le problème de Cauchy associé à notre étude est localement bien posé. Plus précisément, si l'on a

Hypothèse 4.5. *Soit* $\rho \ge 3$, *la non-linéarité* $F : \mathbb{C}^4 \mapsto \mathbb{R}$ *est de classe* $\mathcal{C}^{[\rho]}$ *(pour la structure réelle de* $\mathbb{C}^4$*) avec* $D^\alpha F(0) = 0$, *pour* $\alpha \in \mathbb{N}^8$, $|\alpha| \le [\rho]$ *et* $D^\beta F$ *pour* $|\beta| = [\rho]$ *est hölderienne d'ordre* $\rho - [\rho]$ *(le cas* 0 *signifiant borné).*

Nous avons obtenu (voir Théorème B.2 Chapitre I Partie B) le

Théorème 8. *Si* $V \in B^s_{\infty,\infty}$ *et* F *vérifie l'hypothèse 4.5, alors pour tout* $s > 1$ *avec* $\frac{3}{2}(1 - \frac{1}{\rho}) < s < [\rho] + 1$, *tout* $\psi_0 \in H^s(\mathbb{R}^3, \mathbb{C}^4)$, $\gamma \in (2, \frac{2}{3-2s})$ *et* $\gamma \ge \rho - 1$ *il existe* $T^* = T^*(\|\psi_0\|_{H^s}) > 0$ *et une solution*

$$\psi \in \mathcal{C}((-T^*, T^*); H^s(\mathbb{R}^3, \mathbb{C}^4)) \cap \mathcal{C}^1((-T^*, T^*); L^2(\mathbb{R}^3, \mathbb{C}^4)) \\ \cap L^\gamma_{loc}((-T^*, T^*); L^\infty(\mathbb{R}^3, \mathbb{C}^4))$$

du problème de Cauchy

$$\begin{cases} \mathrm{i}\partial_t\psi &= (D_m + V(Q))\psi + \nabla F(\psi), \\ \psi(0) &= \psi_0. \end{cases} \tag{4}$$

De plus,

(i) cette solution est unique dans $L^\infty((-T,T); L^2(\mathbb{R}^3,\mathbb{C}^4)) \cap L^\gamma((-T,T); L^\infty(\mathbb{R}^3,\mathbb{C}^4))$ *pour* $T < T^*$ *;*

(ii) si $T^* \neq \infty$*, alors* $\|\psi\|_{L^\infty((-T^*,T^*);L^2(\mathbb{R}^3,\mathbb{C}^4))} + \|\psi\|_{L^\gamma((-T^*,T^*);L^\infty(\mathbb{R}^3,\mathbb{C}^4))} = \infty$ *;*

(iii) il existe $T \equiv T(\|\psi_0\|_{H^s}) < T^*(\|\psi_0\|_{H^s})$ *et un voisinage* $\mathcal{W}$ *de* ψ_0 *dans* $H^s(\mathbb{R}^3,\mathbb{C}^4)$ *tels que pour tout* $0 \leq s' < s$*, l'application* $\psi_0 \mapsto \psi(.)$ *est continue de* $\mathcal{W}$ *dans* $\mathcal{C}((-T,T); H^{s'}(\mathbb{R}^3,\mathbb{C}^4))$*.*

Ce chapitre indépendant, présente un résultat intéressant qui n'est pas nécessaire pour démontrer les autres résultats. En effet, dans chacun de nos théorèmes, nous décomposons le système en fonction du spectre de l'opérateur linéarisé autour d'un état stationnaire (état dont on étudie la stabilité). Puis nous utilisons les propriétés de cet opérateur et de son propagateur, pour construire les diverses parties de la décomposition. Ces constructions se font à l'aide du théorème du point fixe, la difficulté étant alors de choisir les ensembles sur lesquels nous allons l'appliquer .

5 Conclusion et perspectives

Dans l'étude de la stabilité des solutions stationnaires que nous avons menée, nous avons remarqué que les directions de descente de la variation seconde de l'énergie ne sont pas nécessairement des sources d'instabilité. En fait, il apparaît que les directions de descente associées au spectre continu ne sont pas à elles seules des sources d'instabilité. Nous avons d'ailleurs réussi à montrer que les propriétés propagative et dispersive associées entraînent une stabilisation. En fait, les directions instables semblent associées à des états propres d'un opérateur linéaire, qui dans le meilleur des cas donnent des modes conservatifs et dans le pire des cas donnent des modes exponentiellement instables .

Il semble ainsi que le seul biais par lequel le spectre continu participe à une instabilité soit celui des résonances qui apparaissent après une perturbation non autoadjointe d'une valeur propre plongée. A ce moment-là, il semble que le spectre continu serve d'évacuation ou d'alimentation en énergie pour une onde lorsqu'elle s'éloigne d'un état stationnaire.

C'est ce qui apparaît au travers des travaux de Tsai et Yau [TY02a, TY02c, TY02d] ou de Soffer et Weinstein [SW99, SW04, SW05]. Ainsi il convient d'essayer de préciser ce phénomène pour l'opérateur de Dirac en faisant une hypothèse de résonance sur l'état excité et non sur l'état « fondamental ». Il s'agirait alors de compléter les résultats du Théorème B.6 du Chapitre II en faisant une hypothèse supplémentaire. Toutefois l'étude dans ce cas est fondamentalement non linéaire. Les informations sur le spectre du linéarisé deviennent inutiles.

Il nous semble également très intéressant de généraliser cette étude au cas des grands solitons mis en évidence par Balabane, Cazenave, Douady et Merle [BCDM88] et Esteban et Séré [ES02]. C'est d'ailleurs le seul moyen que nous imaginons pour obtenir des résultats de stabilité pour l'équation de Dirac. Il s'agit alors essentiellement de développer deux points :

– Il faut réussir à préciser le spectre d'un opérateur de Dirac perturbé par un potentiel non autoadjoint, éventuellement avec une décroissance exponentielle. Les opérateurs

linéarisés font partie de cette classe mais ceux que nous avons envisagés ne sont que de petites perturbations d'opérateurs autoadjoints pour lesquels nous avons fait des hypothèses spectrales.
– Il faut réussir à généraliser aux opérateurs non autoadjoints les estimations de propagation et de dispersion et éventuellement celles de régularité et de Strichartz. Comme nous l'avons vu, ce sont ces propriétés du spectre continu qui nous donnent des directions stables.

Il reste alors à étudier l'équation non linéaire en la recentrant autour d'un état propre. Comme dans notre étude, il faut pouvoir suivre l'évolution de l'état propre. Pour ceci, il faut mettre une structure suffisamment régulière sur l'ensemble des états propres.

Nous aimerions aussi généraliser notre étude au cas où des dégénérescences autres que celles dues à des invariances apparaissent et cela pour un nombre arbitraire de valeurs propres. Nous souhaiterions par exemple remplacer les variétés d'états propres par des variétés centrales. Mais ici les hypothèses du théorème de la variété centrale ne sont pas vérifiées, puisqu'il n'y a pas de dichotomie exponentielle. Toutefois, si une variété centrale existe, nous pensons qu'il est possible de prouver la stabilisation vers cette variété. La seule inconnue est alors la dynamique dans cette variété, c'est à dire sur un ensemble fini de paramètres. On a alors l'espoir d'appliquer les idées de Grillakis, Shatah et Strauss [GSS87].

Un autre axe de généralisation est celui qui consiste à intégrer des résonances et des valeurs propres aux seuils. Il faudrait ainsi généraliser nos estimations de propagation et de dispersion. Cela permettrait d'envisager les cas limites que l'on a exclus : les cas où une valeur propre du linéarisé serait sur un seuil. Il faut alors réécrire la règle d'or de Fermi à la manière de Jensen et Nenciu [JN06].

Puisque nous avons abordé la question du caractère localement bien posé du problème de Cauchy, nous pouvons également nous poser la question de l'explosion en temps fini. Pour certaines équations de Schrödinger non linéaires, cela semble être une source d'instabilité pour des états stationnaires. Si l'on se réfère à Rañada [Ran], ce genre d'instabilité ne se produit pas lors des expériences numériques. Cela pourrait être relié au paradoxe de Klein, nous espérons préciser cela dans l'avenir.

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A.I

L'opérateur de Dirac à potentiel

Nous souhaitons maintenant mentionner quelques idées de la théorie spectrale qui clarifieront certaines des hypothèses faites dans la suite de ce mémoire.

Nous considérons $H := D_m + V$ avec $V : \mathbb{R}^3 \mapsto \mathcal{S}_4(\mathbb{C})$ un potentiel symétrique. Nous donnons quelques hypothèses sur V pour que H possède une réalisation autoadjointe puis nous donnons quelques précisions sur le spectre de H.

1 Caractère Autoadjoint et spectre essentiel

Pour pouvoir étudier le spectre de H, il faut montrer que cet opérateur est autoadjoint. Ce qui nous permettra d'utiliser le formalisme du calcul fonctionnel, *i.e.* de considérer des fonctions bornées de H. Le théorème de Kato-Rellich nous donne le

Théorème 9 ([Tha92], Théorème 4.2)**.** *Soit V un opérateur de multiplication par une fonction à valeur dans les matrices hermitiennes 4×4, tel que chaque composante V_{ik} est une fonction vérifiant :*

$$|V_{ik}(x)| \leq \frac{a}{2|x|} + b \text{ pour tout } x \in \mathbb{R}^3 \setminus \{0\} \text{ et } , k = 1, \ldots, 4$$

avec $0 < a < 1$ et $b > 0$. Alors l'opérateur $H = D_m + V$ est autoadjoint sur $H^1(\mathbb{R}^3, \mathbb{C}^4)$ et $\mathcal{D}(\mathbb{R}^3 \setminus \{0\})$ est un cœur.

Il est alors intéressant de noter que le théorème de Weyl [RS78, Theorem XIII.14, Corollary 1] sur la stabilité du spectre continu, nous donne le spectre essentiel de H lorsque V tend vers 0 à l'infini. On a donc

$$\sigma_{ess}(D_m + V) =]-\infty, -m] \cup [m, +\infty[.$$

Nous notons que le théorème de Weyl s'applique aussi auxperturbations non autoadjointes. Ceci nous sera très utile lors de l'étude de l'opérateur linéarisé autour d'un état stationnaire.

2 Le théorème de Kramers

Comme nous l'avons mentionné plus haut, l'opérateur de Dirac agit naturellement sur des ondes spinorielles de $\mathbb{C}^4$. Parmi ces quatre composantes, certaines nous donnent l'amplitude de l'onde selon chaque direction de spin, « haut » ou « bas ». Sous certaines conditions, la valeur du spin n'affecte pas le niveau d'énergie. C'est le cas pour l'opérateur D_m et pour certains opérateurs $D_m + V$. Ceci nous donne alors une dégénérescence des valeurs propres. C'est une conséquence du théorème de Kramers.

Afin de préciser cette propriété, nous introduisons l'opérateur anti-linéaire K :

$$K \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \sigma_2 \overline{\psi} \\ \sigma_2 \overline{\chi} \end{pmatrix} \text{ avec } \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

L'opérateur D_m commute à K si V commute aussi à K, les valeurs propres de H sont dégénérées et les espaces propres associés sont de dimensions paires. Les potentiels électriques, $V = \phi I_{\mathbb{C}^4}$ avec $\phi : \mathbb{R}^3 \to \mathbb{R}$, et scalaires, $V = \phi\beta$ avec $\phi : \mathbb{R}^3 \to \mathbb{R}$, sont les deux exemples importants de potentiels commutant à K.

3 Localisation du spectre discret

En règle générale, les valeurs propres sont toutes en dehors du spectre continu, sauf celles qui sont éventuellement aux seuils. Ceci peut se démontrer pour certains potentiels, coulombien par exemple, en utilisant le théorème du viriel. Nous souhaitons mentionner des résultats moins connus, qui nous donnent théorèmes pour une classe importante de perturbations.

Ces résultats dûs à Anne Berthier et Vladimir Georgescu nous donnent les théorèmes suivants. Nous avons, grâce au [BG87][Theorem 6], le

Théorème 10. *Soit $\lambda \in \mathbb{R} \setminus (-m; m)$ et $V = V_5 + V_\infty$ une potentiel matriciel tel que*

$$\lim_{R\to\infty} \left\| |x|^{1/2} V_5(x) \right\|_{L^5(B(0,R)^c)} = 0 \text{ et } \left\| |x|^{1/2} V_\infty(x) \right\|_{L^\infty(B(0,R_0)^c)} < \infty$$

pour un certain $R_0 > 0$.

Soit $u \in L^2(B(0,R_0)^c, \mathbb{C}^4) \cap H^1_{\text{loc}}(B(0,R_0)^c, \mathbb{C}^4)$ telle que $Hu = \lambda u$. Alors u s'annule au voisinnage de l'infini. S'il existe un ensemble $K \subset B(0,R_0)^c$ de mesure nulle tel que $B(0,R_0)^c \setminus K$ est connexe et $W \in L^5(B(0,R_0)^c \setminus K, \mathcal{M}_4(\mathbb{C}))$ alors $u = 0$ sur $B(0,R_0)^c$.

En utilsant une inégalité du type Carleman, Berthier et Georgescu obtiennent le ([BG87][Theorem A])

Théorème 11 (Propriété de prolongement unique)**.** *Soit $v : \Omega \to \mathbb{R}^+$ une fonction de $L^5_{\text{loc}}(\Omega)$. Si $\psi \in H^1_{\text{loc}}(\Omega)$ vérifie*

$$|D_0\psi(x)| \leq v(x)\, |\psi(x)| \ \text{ p.p.}$$

et $\psi(x) = 0$ sur un ouvert non vide de Ω alos $\psi \equiv 0$.

Ces deux théorèmes nous donnent donc que la seule solution de $(D_m + V)\psi = \lambda\psi$ avec $\lambda \in \mathbb{R} \setminus [-m; m]$ est nulle.

4 Estimation de Mourre et le principe d'absorption limite

Nous allons dans la suite parler d'estimations de Mourre et de principe d'absorption limite. Nous souhaitons dans ce paragraphe préciser un peu ce dont il s'agit.

Introduites par Eric Mourre [Mou81] dans les années 80, les estimations de Mourre sont liées au caractère propagatif du spectre essentiel et permettent de préciser dans des cas particuliers, les résultats de certains théorèmes comme le théorème RAGE (Ruelle-Amrein-Georgescu-Enss) [AG74].

Pour les énoncés, il faut introduire un opérateur A dit opérateur conjugué. Pour l'opérateur de Dirac, on choisit :

$$A = \frac{1}{2}\left\{ \frac{P}{D_m} \cdot Q + Q \cdot \frac{P}{D_m} \right\}.$$

Cet opérateur est essentiellement autoadjoint sur $\mathcal{D}(\mathbb{R}^3, \mathbb{C}^4)$ et son domaine contient celui de $\langle Q \rangle = \sqrt{1 + Q^2}$, voir [IM99]. On a de plus :

$$\mathrm{i}[D_m, A] = \frac{-\Delta}{-\Delta + m^2}.$$

On obtient une estimation de Mourre faible

$$\mathrm{i}[D_m, A] > 0$$

sur $L^2(\mathbb{R}^3, \mathbb{C}^4)$. Si V est $\mathcal{C}^\infty$ avec des dérivées tendant vers 0 à l'infini, alors (Lemme B.15 du chapitre II) pour tout $\lambda \in \sigma_{ess}(H)$, il existe $\varepsilon >$ tel que

$$\mathbf{1}_{|H-\lambda|\leq\varepsilon}\mathrm{i}[H, A]\mathbf{1}_{|H-\lambda|\leq\varepsilon} \geq a\mathbf{1}_{|H-\lambda|\leq\varepsilon}$$

où $a > 0$. C'est ce qu'on nomme l'estimation de Mourre locale. Les estimations de Mourre nous donnent la croissance stricte de l'observable A sur une orbite de H.

Plus généralement, on dit que l'on a une estimation de Mourre sur $J \subset \mathbb{R}$ s'il existe $a > 0$, A un opérateur autoadjoint et K un opérateur compact sur notre espace de Hilbert tels que

$$\mathbf{1}_{H\in J}\mathrm{i}[H, A]\mathbf{1}_{H\in J} \geq a\mathbf{1}_{H\in J} + K$$

Les estimations de Mourre permettent, lorqu'il n'y a pas de valeurs propres, d'établir des inéquations différentielles pour des résolvantes de « déformations » de H, voir par exemple [IM99]. Ces inéquations et certains lemmes de Gronwall (voir [ABdMG96]) donnent des principes d'absorption limite tels que

$$\forall\lambda,\ |\lambda| > m,\ \exists\varepsilon > 0,\ \exists C > 0,\ \sup_{\substack{|\Re z-\lambda|\leq\varepsilon,\\ \Im z>0}} \left\{\left\|\langle Q\rangle^{-1}(H-z)^{-1}\langle Q\rangle^{-1}\right\|\right\} \leq C. \qquad (1)$$

5 Le nombre de valeurs propres

Dans la suite de notre étude, nous allons faire des hypothèses sur le nombre et la multiplicité des valeurs propres de $H = D_m + V$. Il peut alors être intéressant d'exhiber ou de construire des exemples où ces hypothèses sont vérifiées.

Considérons un potentiel à symétrie sphérique. Dans $L^2(\mathbb{R}^3)^4$ les rotations de $SU(2)$ sont représentées par :

$$\left(e^{-\mathrm{i}\phi J\cdot n}\psi\right)(x) = e^{-\mathrm{i}\phi S\cdot n}\psi(R^{-1}x)$$

pour $\phi \in [0, 4\pi[$, R la rotation d'angle ϕ et d'axe n, $S = \frac{1}{2}\begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}$ l'opérateur de spin angulaire et $J = L + S$ le moment angulaire total avec $L = Q \wedge P$ le moment angulaire orbital. Ainsi un potentiel est à symétrie sphérique si :

$$e^{-\mathrm{i}\phi S\cdot n}V(R^{-1}x)e^{\mathrm{i}\phi S\cdot n} = V(x).$$

Des exemples importants de tels potentiels sont les potentiels électriques $\phi_{el}I_{\mathbb{C}^4}$ avec ϕ_{el} radiale et les potentiels scalaires $\phi_{sc}\beta$, avec ϕ_{sc} radiale. Considérons $V = \phi_{el}I_{\mathbb{C}^4} + \phi_{sc}\beta$ avec ϕ_{el}, ϕ_{sc} radiales.

En passant aux coordonnées sphériques (voir [Tha92][Section 4.6] et [Ynd96][Section 3.6])

$$\psi(x) \mapsto r\psi(r, \theta, \phi),$$

nous remarquons que

$$L^2(\mathbb{R}^3)^4 \cong L^2([0, \infty[, dr) \otimes L^2(S^2)^4.$$

Comme les opérateurs J^2, J_3 et $\tilde{K} = \beta(2S\cdot L + 1) = \beta(J^2 - L^2 + 1/4)$ (opérateur d'orbite de spin) n'agissent que sur les composantes de $L^2(S^2)^4$ et qu'ils commutent à H, nous pouvons décomposer $L^2(S^2)^4$ comme somme directe d'espaces stables par H, J^2, J_3 et K. Chacun de ces espaces noté $\mathcal{K}_{m_j,\kappa_j}$ est de dimension 2 et engendré par $\phi^+_{m_j,\kappa_j}$ et $\phi^-_{m_j,\kappa_j}$ où

$$\begin{aligned} J^2\phi^\pm_{m_j,\kappa_j} &= j(j+1)\phi^\pm_{m_j,\kappa_j} \text{ avec } j \in \frac{1}{2}+\mathbb{N} \\ J_3\phi^\pm_{m_j,\kappa_j} &= m_j\phi^\pm_{m_j,\kappa_j} \text{ avec } m_j = -j, -j+1, \ldots, +j \\ \tilde{K}\phi^\pm_{m_j,\kappa_j} &= \kappa_j\phi^\pm_{m_j,\kappa_j} \text{ avec } \kappa_j = -(j+1/2), +(j+1/2). \end{aligned}$$

Nous avons alors

$$L^2(S^2)^4 \cong \oplus_{j\in\frac{1}{2}+\mathbb{N}} \oplus_{m_j=-j,-j+1,\ldots,+j} \oplus_{\kappa_j=-(j+1/2),+(j+1/2)} \mathcal{K}_{m_j,\kappa_j}.$$

De plus, chacun des $L^2([0,\infty[, dr)\otimes\mathcal{K}_{m_j,\kappa_j}$ est stable par H et peut être représenté sur la base $\phi^\pm_{m_j,\kappa_j}$ par

$$h_{m_j,\kappa_j} = \begin{pmatrix} m+\phi_{sc}(r)+\phi_{el}(r) & -\partial_r+\kappa_j/r \\ \partial_r+\kappa_j/r & -m-\phi_{sc}(r)+\phi_{el}(r) \end{pmatrix}.$$

Il suffit alors de rechercher les valeurs propres de ces opérateurs pour obtenir celles de H.

Le cas d'un petit mur de potentiel électrique Considérons ici $\phi_{sc} = 0$ et $\phi_{el} = -v_0\mathbf{1}_{r<R}$ avec $v_0 > 0$. Nous avons alors deux systèmes d'équations :

-pour $r > R$:

$$\begin{cases} -\partial_r g + \frac{\kappa_j}{r}g + mf = Ef, \\ \partial_r f + \frac{\kappa_j}{r}f - mg = Eg; \end{cases}$$

-pour $r < R$:

$$\begin{cases} -\partial_r g + \frac{\kappa_j}{r}g + mf = (E+v_0)f, \\ \partial_r f + \frac{\kappa_j}{r}f - mg = (E+v_0)g. \end{cases}$$

Nous avons donc deux jeux d'équations libres. Nous imposons $|E| \le m$ et $|E+v_0| \ge m$. Pour le premier, nous obtenons

$$\begin{cases} \partial_r^2 f - \frac{\kappa_j(\kappa_j+1)}{r^2}f + (E^2-m^2)f = 0, \\ g = \frac{\partial_r+\kappa_j/r}{E+m}f. \end{cases}$$

Nous avons alors que $\frac{f(\sqrt{m^2-E^2}^{-1}r)}{\sqrt{m^2-E^2}^{-1}r}$ est obtenue à partir des solutions de l'équation de Bessel. Elle doit être de carré intégrable sur $[R,+\infty[$, nous utilisons des fonctions de Bessel de seconde espèce. On a :

$$f(r) = C\sqrt{m^2-E^2}rK_{\kappa_j-1}(\sqrt{m^2-E^2}r)$$

où

$$K_n(z) = \frac{\pi}{2}z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n\frac{e^{-z}}{z}.$$

Pour $r < R$, nous utilisons les fonctions de Bessel sphériques pour avoir l'intégrabilité en 0 :

$$f(r) = C'\sqrt{(E+v_0)^2-m^2}rJ_{\kappa_j-1}(\sqrt{(E+v_0)^2-m^2}r) \tag{2}$$

où

$$J_n(z) = z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin(z)}{z}.$$

Nous pouvons alors fixer $C = 1$, le recollement en R de g et f nous donne les valeurs de C' et E. Ce recollement n'est pas toujours possible lorsque $v_0 = 0$ par exemple ou v_0 petit. Cela va dépendre aussi de κ_j. Les relations

$$\begin{aligned} J_n'(z) &= \frac{n}{z} J_n(z) - J_{n+1}(z) \\ K_n'(z) &= \frac{n}{z} K_n(z) - K_{n+1}(z) \end{aligned}$$

lorsque R est petit et les équivalents des fonctions de Bessel au voisinage de 0 :

$$\begin{aligned} K_n(z) &\sim_0 \frac{2^{n-1}\pi^{3/2}}{\sin((n+1/2)\pi)} \frac{z^{-n-1}}{\Gamma(1/2-n)} + \frac{\pi^{3/2}}{2^{n-2}\sin((n+1/2)\pi)} \frac{z^n}{\Gamma(n+3/2)} \\ J_n(z) &\sim_0 z^n \frac{\sqrt{\pi}}{2^{n+1}\Gamma(n+3/2)} \end{aligned}$$

nous donnent, si v_0 est assez petit, un raccord possible uniquement pour $\kappa_j = +1$ (et donc pour $j = 1/2$ et $m_j = \pm/2$). Dans ce cas $D_m + V$ n'a qu'une valeur propre double. En lissant V de manière à ce qu'il commute encore à K, on peut encore supposer que V est lisse. Ceci nous donne un exemple de potentiel vérifiant les hypothèses du Théorème B.6 du chapitre IV de la partie B.

On peut aussi perturber de manière lisse la configuration précédente afin d'obtenir uniquement deux valeurs propres simples conformément aux hypothèses du Théorème B.6 du chapitre II de la partie B. Il suffit dans le cas précédent de choisir deux vecteurs propres distincts. On peut supposer que sur une petite boule fermée B, leurs premières composantes sont distinctes. On choisit alors χ une approximation lisse de $\mathbf{1}_B$ et on considère $H + V + \varepsilon W$ où V est la version lisse du potentiel obtenu dans le paragraphe précédent, ε petit et

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \chi.$$

Le cas d'un petit mur de potentiel scalaire Considérons ici $\phi_{sc} = v_0 \beta \mathbf{1}_{r<R}$ avec $v_0 > 0$ et $\phi_{el} = 0$. Dans ce cas l'étude précédente reste valable mais au lieu d'obtenir $E + v_0$ à la place de E dans les équations précédentes, on obtient $m - v_0$ à la place de m. La configuration minimale est celle de deux valeurs propres doubles et opposées. De plus elles sont proches de m et $-m$ respectivement. On se retrouve donc après avoir lissé le potentiel tout en conservant la commutation à K avec les hypothèses du Théorème B.6 du chapitre III de la partie B.

Nous espérons dans ces paragraphes avoir donné une idée de la façon de construire des potentiels qui vérifient les hypothèses de notre étude. Nous ferons aussi par la suite l'hypothèse qu'il n'y a pas de résonance aux seuils. C'est cette hypothèse qui nous intéresse maintenant.

6 Les résonances aux seuils

Une résonance sera pour nous un vecteur propre qui n'est pas dans L^2 mais dans un espace plus grand : un L^2 pondéré par un poids décroissant. De telles valeurs propres apparaissent dans notre étude des estimations de propagation. Ce sont des estimations de décroissance en temps du propagateur dans des espaces L^2 à poids.

On peut montrer que pour un potentiel borné tendant vers 0 à l'infini, une résonance ne peut pas être associée à une valeur propre dans $]-m;+m[$. En effet, la preuve du lemme B.39 du chapitre II, nous montre que de tels vecteurs propres ont une décroissance exponentielle. Les idées du paragraphe 3, nous permettent aussi de dire qu'elles ne sont pas à l'intérieur du spectre essentiel. Ainsi, pour des potentiels réguliers avec des dérivées tendant assez vite vers 0, seuls les points $+m$ et $-m$ peuvent être associés à des résonances.

On a l'équation des valeurs propres aux seuils (m ici) :

$$(D_m - m)\psi = -V\psi$$

avec le choix que l'on a fait pour l'opérateur de Dirac, on obtient :

$$\begin{cases} -\sigma \mathrm{i}\nabla\psi_2 = -V_{11}\psi_1 - V_{12}\psi_2 \\ -\sigma \mathrm{i}\nabla\psi_1 + 2m\psi_2 = -V_{21}\psi_1 - V_{22}\psi_2 \end{cases},$$

ou en passant à la transformée de Fourier

$$|\xi|\widehat{\psi_2}(\xi) \in H_s^\sigma,$$

donc si $\sigma \geq 1$, on a $\psi_2 \in H_1^s \subset L^2$.

On obtient aussi $\xi^2\widehat{\psi_1}(\xi) \in H_s^\sigma$. Donc, la valeur en $\xi = 0$ de $\xi^2\widehat{\psi_1}(\xi)$ est bien définie. Si elle est nulle alors $\psi_2 \in L^2$ comme ψ_1. Ainsi on a une vraie résonance si $\int_{\mathbb{R}^3} \Delta\psi_2 \neq 0$. Il existe donc deux fonctions $\Phi_\pm^{res}$ telles que pour toute résonance ψ, il existe $\alpha_\pm$, on a $\psi - \alpha^+\Phi_+^{res} - \alpha^-\Phi_-^{res} \in L^2$.

Ainsi les résonances quotientées par L^2 forment au plus un espace de dimension 2 pour chaque seuil. Les résonances sont portées par les composantes pour le seuil haut et les composantes basses pour le seuil bas.

D'autre part, on sait que génériquement les résonances n'existent pas. Ce résultat remonte à Rauch [Rau78][Theorem 3]. On peut montrer que pour une famille $z \in \Omega \mapsto V_z$ holomorphe de potentiels sur un ouvert $\Omega \subset \mathbb{C}$, si pour un potentiel V_{z_0} il n'y a pas de résonance, alors il n'y en pas pour tout potentiel sauf un pour un ensemble discret de paramètres z. Il est d'ailleurs possible de montrer que l'ensemble des potentiels sans résonance est un ouvert, ainsi l'ensemble des potentiels à résonance est un fermé d'intérieur vide.

Pour terminer, on signalera que l'on peut toujours trouver des potentiels à résonance. Ce fait peut s'adapter d'un exemple dû à Dolph, Mcleod et Thoe [DMT66][page 332].

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Deuxième partie

Les résultats

B.I

Le problème de Cauchy est localement bien posé

Dans ce chapitre, nous montrons que le problème de Cauchy associé à nos équations est localement bien posé.

Bien qu'il ne soit pas utile au reste de l'étude, puisque nous reconstruisons toutes nos solutions, ce résultat simple présente un intérêt propre. De plus, il permet d'introduire les espaces de Besov et les propriétés qui vont nous être utiles par la suite.

The local well posedness of the Cauchy problem

1 Setting of the problem

We study the local well posedness of the following nonlinear Dirac equation

$$\begin{aligned} \mathrm{i}\partial_t\phi &= \gamma\cdot(-\mathrm{i}\nabla\phi)+m\beta\phi+V(Q)\phi+\nabla F(\phi) \\ &= D_m\phi+V(Q)\phi+\nabla F(\phi) \\ &= H\phi+\nabla F(\phi) \end{aligned}$$

with $\phi\in\mathcal{C}^1(\mathbb{R},H^1(\mathbb{R}^3,\mathbb{C}^4))$, $(\gamma_1,\,\gamma_2,\,\gamma_3,\,\beta)$ the 4-vector of Pauli matrix, m is a positive real and V a matrix valued real function, see [Tha92] for more details about the nonlinear Dirac equation.

Units are chosen such that the speed of light and the Planck's constant equal one. The ∇ symbol means that if $\mathbb{C}^4$ is taking with its usual hermitian product, and $F:\mathbb{C}^4\mapsto\mathbb{R}$ is a differentiable map for the real structure of $\mathbb{C}^4$, we have

$$DF(v)h=\Re\langle\nabla F(v),h\rangle.$$

We work within the following

Assumption 1.1 (On the nonlinear perturbation)**.** *Let be $\rho\geq 3$. The nonlinearity $F:\mathbb{C}^4\mapsto\mathbb{R}$ is of $\mathcal{C}^{[\rho]}$ (for the real structure of $\mathbb{C}^4$) with $G^\alpha(0)=0$, for $\alpha\in\mathbb{N}^8$, $|\alpha|\leq[\rho]$ and $D^\beta F$ for $|\beta|=[\rho]$ is hölder of order $\rho-[\rho]$ (when it is zero it means bounded).*

Throughout this paper, the notation $<v>$ means $\sqrt{1+v^2}$.

2 The theorem

The local well-posedness for the free nonlinear Dirac equation ($V=0$) was proved in [EV97] and in [MNNO05] the authors proved global well-posedness for small initial data in this case. We sketch here an adapted proof.

To state our local well posedness result, we need

Definition B.1 (Besov space). *For $s\in\mathbb{R}$ and $1\leq p,q\leq\infty$, the space $B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^4)$ is the set of all $f\in L^p(\mathbb{R}^3,\mathbb{C}^4)$ such that*

$$\|f\|_{B^s_{p,q}}=\left(\sum_{j\in\mathbb{N}}2^{jsq}\|\varphi_j f\|_p^q\right)^{\frac{1}{q}}<+\infty$$

with $\widehat{\varphi}\in\mathcal{D}(\mathbb{R}^n\setminus\{0\})$ such that

$$\sum_{j\in\mathbb{Z}}\widehat{\varphi}(2^{-j}x)=1,$$

$\forall j\in\mathbb{N}^*$, $\varphi_j(x)=\varphi(2^{-j}x)$ *and* $\varphi_0=1-\sum_{j\in\mathbb{N}^*}\varphi_j$.

Remark B.1.1. *The previous norms are not well defined in case $p=\infty$ or $q=\infty$. But we easily adapt it to these cases.*

We notice that using the Proposition 13 and the Kato-Rellich theorem we have that for $V \in B^s_{\infty,\infty}$ the operator $H = D_m + V$ is a essentially self-adjoint operator with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$ and core $\mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$.

We have

Theorem B.2 (Local wellposedness)**.** *Assume $V \in B^s_{\infty,\infty}$ and F satisfies assumption 1.1, then for all $s > 1$ with $\frac{3}{2}(1 - \frac{1}{\rho}) < s < [\rho] + 1$, all $\psi_0 \in H^s(\mathbb{R}^3, \mathbb{C}^4)$ and for $\gamma \in (2, \frac{2}{3-2s})$ and $\gamma \geq \rho - 1$ there is a $T^* = T^*(\|\psi_0\|_{H^s}) > 0$ and a solution*

$$\psi \in \mathcal{C}((-T^*, T^*); H^s(\mathbb{R}^3, \mathbb{C}^4)) \cap \mathcal{C}^1((-T^*, T^*); L^2(\mathbb{R}^3, \mathbb{C}^4)) \cap L^\gamma_{loc}((-T^*, T^*); L^\infty(\mathbb{R}^3, \mathbb{C}^4))$$

of the Cauchy problem

$$\left\{ \begin{array}{rcl} \mathrm{i}\partial_t \psi & = & (D_m + V(Q))\psi + \nabla F(\psi), \\ \psi(0) & = & \psi_0. \end{array} \right. \tag{1}$$

Moreover,

(i) this solution is unique in $L^\infty((-T, T); L^2(\mathbb{R}^3, \mathbb{C}^4)) \cap L^\gamma((-T, T); L^\infty(\mathbb{R}^3, \mathbb{C}^4))$ for $T < T^*$;

(ii) if $T^* \neq \infty$, $\|\psi\|_{L^\infty((-T^*,T^*);L^2(\mathbb{R}^3,\mathbb{C}^4))} + \|\psi\|_{L^\gamma((-T^*,T^*);L^\infty(\mathbb{R}^3,\mathbb{C}^4))} = \infty$;

(iii) there is a $T \equiv T(\|\psi_0\|_{H^s}) < T^*(\|\psi_0\|_{H^s})$ and a neighborhood $\mathcal{W}$ of ψ_0 in $H^s(\mathbb{R}^3, \mathbb{C}^4)$ such that for all $0 \leq s' < s$, such that the map $\psi_0 \mapsto \psi(.)$ is continuous from $\mathcal{W}$ to $\mathcal{C}((-T, T); H^{s'}(\mathbb{R}^3, \mathbb{C}^4))$.

3 Chain rule in Besov spaces

Lemma B.3. *The following application is an equivalent norm for $B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4)$ when $s > 0$,*

$$f \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4) \mapsto \|f\|_p + \left(\int_{\mathbb{R}^3} \frac{\|\omega(l, y)(f)\|_p^q}{|y|^{3+sq}} \, dy \right)^{1/q}$$

with l an integer such that $s < l$, $\| \cdot \|_p$ is the norm of the L^p space and

$$\omega(l, y)(f) = (\Delta_y)^l f = \sum_{k=0}^{l} (-1)^k C_l^k \tau_{(2k-l)y} f,$$

this formula defines Δ_y and τ_y is the translation of vector $y \in \mathbb{R}^3$.

Proof – See [Tri78] definition 1 of 2.3.1 and theorem of 2.5.12 . □

Remark B.3.1. *In case $p = \infty$ or $q = \infty$, we can adapt easily adapt the definition of the norm.*

We introduce the

Definition B.4 (generalized Sobolev spaces). *The space* $W^{s,p}(\mathbb{R}^3, \mathbb{C}^4)$ *is the set of all* $f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ *such that*

$$\| < P >^s f\|_p < \infty.$$

We can find the following properties in [BL76]

Proposition 12. *Besov spaces are Banach spaces and*

1. $B^s_{2,2} = W^{s,2}$,
2. $B^s_{p,r} \subset B^{s'}_{p,q}$ *if* $s' < s$ *or* $s = s'$ *and* $1 \le r \le q \le \infty$,
3. $B^s_{p,2} \subset W^{s,p} \subset B^s_{p,p}$ *for* $1 < p \le 2$,
4. $B^s_{p,p} \subset W^{s,p} \subset B^s_{p,2}$ *for* $2 \le p < \infty$,
5. $B^u_{r,q} \subset B^s_{p,q}$ *if* $1 \le r \le p \le \infty$ *and* $u - n/r = s - n/p$.

with continuous embedding.

We have the

Proposition 13 (Chain Rule in Besov space 1)**.** *For* $s > 1$*, there is* $C > 0$ *such that for* $V \in B^s_{\infty,\infty}(\mathbb{R}^3, \mathcal{M}_4(\mathbb{C}))$ *and any* $\psi \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4) \cap L^\infty(\mathbb{R}^3, \mathbb{C}^4)$*, we have* $G(\psi) \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4)$ *and*

$$\|V\psi\|_{B^s_{p,q}} \le C\|V\|_{B^s_{\infty,\infty}(\mathbb{R}^3, \mathcal{M}_4(\mathbb{C}))}\|\psi\|_{B^s_{p,q}}$$

Proof – We use the identity

$$\Delta^m_y(Vf) = \sum_{k=0}^m C^k_m \tau_{-(m-k)y}\Delta^k_y(V) \times \tau_{ky}\Delta^{m-k}_y(f).$$

With help of a Minkowski inequality and then with a Hölder inequality the proposition follows. □

We also have

Proposition 14 (Chain Rule in Besov space 2)**.** *Let be* $s > 1$ *and let be* $G \in \left(B^{s'}_{\infty,\infty}\right)_{loc}(\mathbb{C}^4, \mathbb{C}^4)$ *for some* $s' \in ([s], [s] + 1)$ *such that* $G(0) = 0$ *and any* $\psi \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4) \cap L^\infty(\mathbb{R}^3, \mathbb{C}^4)$*, we have* $G(\psi) \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4)$ *and*

$$\|G(\psi)\|_{B^s_{p,q}} \le \sum_{k=1}^{[s]} \sup_{|z| \le \|\psi\|_\infty} \{\left|D^k G(z)\right|\}\|\psi\|^{k-1}_\infty \|\psi\|_{B^s_{p,q}} + \Lambda_{s,s'}(G, \|\psi\|_\infty)\|\psi\|^{s'-1}_\infty \|\psi\|_{B^s_{p,q}}$$

with

$$\Lambda_{s',s}(G, r) = \sup_{|z|, |z'| \le r} \left\{ \frac{\left|D^{[s]}G(z) - D^{[s]}G(z')\right|}{|z - z'|^{s'-[s]}} \right\}.$$

Proof – We can found a proof in [EV97] or in [Esc88]. □

Corollaire 15. *Under the previous assumptions, if moreover G is $\mathcal{C}^\rho$ then*

$$\|G(\psi)\|_{B^s_{p,q}} \leq [s]\|G\|_{\mathcal{C}^\rho}\|\psi\|_\infty^{\rho-1}\|\psi\|_{B^s_{p,q}}.$$

Proof – We notice that

$$\sup_{|z|\leq\|\psi\|_\infty}\{\left|D^kG(z)\right|\} = \|\psi\|_\infty^{\rho-k}\|D^kG\|_{\mathcal{C}^{\rho-k}},$$

and

$$\Lambda_{s,s'}(G,r) = r^{\rho-s'}\|D^{[s]}G\|_{\mathcal{C}^{\rho-[s]}}.$$

□

Remark B.4.1. *Notice that in the previous corollary, we necessarily have*

$$[s] \leq s' \leq \rho.$$

We shall also use the

Lemma B.5. *For $s \geq \frac{3}{p}$, and $u, v \in B^s_{p,q}$ then*

$$\|uv\|_{B^s_{p,q}} \leq 2^m\|u\|_{B^s_{p,q}}\|v\|_{B^s_{p,q}}.$$

Finally, the calculation that leads to the second chain rule also leads to

Lemma B.6. *For $\tau > 0$, $s > 0$, $p, q \in [1,\infty]$ we have for $u \in B^s_{p,q}$ and*

$$F \in \left(B^{s'}_{\infty,\infty}\right)_{loc}$$

for $s' \in [[s],[s]+1]$ with $F(0) = 0$, we have

$$\|\langle Q\rangle^{-\tau}u\|_{B^s_{p,q}} \leq \|u\|_\infty^{s'-1}\|\langle Q\rangle^{-\frac{\tau}{[s]+1}}u\|_{B^s_{p,q}}$$

4 Proof of Theorem B.2

For the need of the proof, we first rewrite the problem in an integral form using Duhamel's formula

$$\psi(t) = e^{-\mathrm{i}tD_m}\psi_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-u)D_m}(V\psi(u) + \nabla F(\psi(u)))\,du,$$

and we prove the statements by a fixed point argument.

Let $\gamma > 2$ such that $3/2 - 1/\gamma < s$, we study the map defined by

$$\mathcal{G}(\psi) = \{t \in \mathbb{R} \mapsto e^{-\mathrm{i}tD_m}\psi_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-u)D_m}(V\psi(u)\nabla F(\psi(u)))\,du\}$$

in

$$\mathcal{C}((-T,T);H^s(\mathbb{R}^3,\mathbb{C}^4)) \cap \mathcal{C}^1((-T,T);L^2(\mathbb{R}^3,\mathbb{C}^4)) \cap L^\gamma_{loc}((-T,T);L^\infty(\mathbb{R}^3,\mathbb{C}^4)).$$

for $T > 0$.

For $M > 0$, we define

$$X(T,M) = \{\psi \in L^\infty((-T,T); H^s(\mathbb{R}^3,\mathbb{C}^4)) \cap L^\gamma((-T,T); L^\infty(\mathbb{R}^3,\mathbb{C}^4));\ \|\psi\| \le M\}$$

where

$$\|\psi\| = \sup_{(-T,T)} \|\psi\|_{H^s} + \|\psi\|_{L^\gamma((-T,T);L^\infty(\mathbb{R}^3,\mathbb{C}^4))}.$$

Lemma B.7. *Endowed with the distance associated with the natural norm of the space $L^\infty((-T,T); L^2(\mathbb{R}^3,\mathbb{C}^4))$, $X(T,M)$ is a complete metric space.*

We prove that $\mathcal{G}$ stabilizes $X(T,M)$. We first study the H^s norm, with the Chain rule in Besov space of the Proposition 13 and Corrolary 15, we obtain

$$\begin{aligned}
&\|\mathcal{G}(\psi)(t)\|_{H^s} \\
&\le \|\psi_0\|_{H^s} + \int_0^t (\|V(Q)\psi(u) + \nabla F(\psi(u))\|_{H^s})\, du \\
&\le \|\psi_0\|_{H^s} + \int_{-T}^T (\|V(Q)\psi(u)\|_{H^s} + \|\nabla F(\psi(u))\|_{H^s})\, du \\
&\le \|\psi_0\|_{H^s} + \int_{-T}^T \|V\|_{B^s_{\infty,\infty}} \|\psi(u)\|_{H^s}\, du + \int_{-T}^T \|\psi(u))\|_\infty^{\rho-1} \|\psi(u))\|_{H^s}\, du \\
&\le \|\psi_0\|_{H^s} + \|V\|_{B^s_{\infty,\infty}} 2T \sup_{[-T,T]} \{\|\psi(u)\|_{H^s}\} \\
&\qquad + C(\nabla F) \sup_{[-T,T]} \{\|\psi(u)\|_{H^s}\} T^{1-\frac{\rho-1}{\gamma}} \left(\int_{-T}^T \|\psi(u)\|_\infty^\gamma\, du\right)^{\frac{\rho-1}{\gamma}} \\
&\le \|\psi_0\|_{H^s} + \|V\|_{B^s_{\infty,\infty}} 2TM + C(\nabla F) T^{1-\frac{\rho-1}{\gamma}} M^\rho.
\end{aligned}$$

Then we study the norm $L^\gamma([-T,T], L^\infty(\mathbb{R}^3,\mathbb{C}^4))$,

$$\begin{aligned}
\|\mathcal{G}(\psi)\|_{L^\gamma([-T,T],L^\infty)} &\le \|e^{-\mathrm{i}(\cdot)D_m}\psi_0\|_{L^\gamma([-T,T],L^\infty)} \\
&\quad + \left\|\int_0^{(\cdot)} e^{-\mathrm{i}(\cdot-u)D_m} (V(Q)\psi(u) + \nabla F(\psi(u)))\, du\right\|_{L^\gamma([-T,T],L^\infty)}
\end{aligned}$$

We need the

Proposition 16 (Strichartz-type estimates)**.** *Let $s > 1$, then for all $\gamma > 0$ such that $\frac{1}{\gamma} \in \left(\frac{3-2s}{2}, \frac{1}{2}\right)$ and all $r' > \gamma$ such that $\frac{1}{r'} \in \left(\frac{3-2s}{2}, \frac{1}{\gamma}\right)$ there is a positive constant C such that for every $T > 0$ and all $\psi_0 \in H^s(\mathbb{R}^3)$,*

$$\|e^{-\mathrm{i}tD_m}\psi_0\|_{L^\gamma([-T,T],L^\infty(\mathbb{R}^3,\mathbb{C}^4))} \le CT^{\frac{1}{\gamma}-\frac{1}{r'}} \|\psi_0\|_{H^s}.$$

Proof – It is Theorem 1.5 *(ii)* of [EV97] or Theorem B.4 of Part III. □

We write

$$
\begin{aligned}
&\|\mathcal{G}(\psi)\|_{L^\gamma([-T,T],L^\infty)}\\
\leq\ & \|\psi_0\|_{H^s} + \int_{-T}^{T} \|e^{-\mathrm{i}(\cdot-u)D_m}V(Q)\psi(u) + \nabla F(\psi(u))\|_{L^\gamma([-T,T],L^\infty)}\,du\\
\leq\ & C\|\psi_0\|_{H^s} + CT^{\frac{1}{\gamma}-\frac{1}{r'}}\|V(Q)\psi\|_{L^1([-T,T],H^s)} + CT^{\frac{1}{\gamma}-\frac{1}{r'}}\|\nabla F(\psi)\|_{L^1([-T,T],H^s)}\\
\leq\ & C\|\psi_0\|_{H^\alpha} + CT^{\frac{1}{\gamma}-\frac{1}{r'}}\|V\|_{B^s_{\infty,\infty}}2T\sup_{[-T,T]}\{\|\psi\|_{H^s}\}\\
& +C(\nabla F)CT^{\frac{1}{\gamma}-\frac{1}{r'}}\int_{[-T,T]}\|\psi\|_\infty^{\rho-1}\|\psi\|_{H^s}\,dt\\
\leq\ & C\|\psi_0\|_{H^\alpha} + CT^{\frac{1}{\gamma}-\frac{1}{r'}}\|V\|_{B^s_{\infty,\infty}}2T\sup_{[-T,T]}\{\|\psi\|_{H^s}\}\\
& +C(\nabla F)CT^{\frac{1}{\gamma}-\frac{1}{r'}}(2T)^{1-\frac{2}{\gamma}}\sup_{[-T,T]}\{\|\psi\|_{H^s}\}\|\psi\|^{\rho-1}_{L^\gamma([-T,T],L^\infty)}\\
\leq\ & C\|\psi_0\|_{H^\alpha} + CT^{\frac{1}{\gamma}-\frac{1}{r'}}\|V\|_{B^s_{\infty,\infty}}2T\sup_{[-T,T]}\{\|\psi\|_{H^s}\}\\
& +C(\nabla F)CT^{\frac{1}{\gamma}-\frac{1}{r'}}(2T)^{1-\frac{2}{\gamma}}\sup_{[-T,T]}\{\|\psi\|_{H^s}\}\|\psi\|^{\rho-1}_{L^\gamma([-T,T],L^\infty)}\\
\leq\ & C\|\psi_0\|_{H^\alpha} + 2CT^{1+\frac{1}{\gamma}-\frac{1}{r'}}\|V\|_{B^s_{\infty,\infty}}M + C(\nabla F)C2^{1-\frac{2}{\gamma}}T^{1-\frac{1}{\gamma}-\frac{1}{r'}}M^\rho
\end{aligned}
$$

So T is small enough we obtain

$$\|\mathcal{G}(\psi)\| \leq M$$

and so $\mathcal{G}$ stabilize $X(T,M)$.

We then prove the contraction property of $\mathcal{G}$ with respect to the norm

$$L^\infty((-T,T);L^2(\mathbb{R}^3,\mathbb{C}^4)).$$

We recall that $X(T,M)$ is closed for this norm. We easily obtain

$$
\begin{aligned}
&\|\mathcal{G}(\psi_1)-\mathcal{G}(\psi_2)\|_{L^\infty((-T,T);L^2(\mathbb{R}^3,\mathbb{C}^4))}\\
&\qquad\leq CT(\|V\|_\infty + C(\nabla F)M^{\rho-1})\{\|\psi_1-\psi_2\|_{L^\infty((-T,T);L^2(\mathbb{R}^3,\mathbb{C}^4))}\}
\end{aligned}
$$

So if T is small enough, we can apply the fixed point theorem.

To go on with the proof of the local well-posedness, we now treat the uniqueness problem by using

Lemma B.8. *Assume $V \in B^s_{\infty,\infty}$ and F satisfies assumption 1.1, let be $\psi_0 \in H^s(\mathbb{R}^3,\mathbb{C}^4)$ and $\gamma > \rho - 1$.*

Suppose that $\psi \in L^\infty((-T,T);H^s(\mathbb{R}^3,\mathbb{C}^4)) \cap L^\gamma((-T,T);L^\infty(\mathbb{R}^3,\mathbb{C}^4))$ is a solution of

$$\psi(t) = e^{-\mathrm{i}tD_m}\psi_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)D_m}(V\psi(s) + \nabla F(\psi(s)))\,ds.$$

Then $\psi \in \mathcal{C}([-T,T];H^s(\mathbb{R}^3,\mathbb{C}^4))$. Furthermore, if ψ_1 and ψ_2 are two solutions, then $\psi_1 = \psi_2$.

Proof – The assertion about continuity comes from the strong continuity of $W : t \mapsto e^{-\mathrm{i}tD_m}$ in L^2 and the fact that $\langle P\rangle^s$ commutes to W. It also comes from the fact that $V(Q)\psi$ and $\nabla F(\psi)$ belongs to $L^1([-T,T];H^s(\mathbb{R}^3,\mathbb{C}^4))$ since it belongs to $L^\infty([-T,T];H^s(\mathbb{R}^3,\mathbb{C}^4))$. For the uniqueness it's a well known argument of contraction that gives the results. We introduce $t_0 = \inf\{t,\, \psi_1(t) \neq \psi_2(t)\}$.

By continuity $\psi_1(t_0) = \psi_2(t_0)$, so if $t_0 < T$ then $\tau_{-t_0}\psi_1$ and $\tau_{-t_0}\psi_2$ are solution of

$$\psi(t) = e^{-\mathrm{i}tD_m}\psi_1(t_0) - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)D_m}(V\psi(s) + \nabla F(\psi(s)))\, ds,$$

on $[0, T - t_0]$.

But using the previous computation, we have

$$\begin{aligned}&\|\tau_{-t_0}\psi_1 - \tau_{-t_0}\psi_2\|_{L^\infty((0,T-t-0);L^2(\mathbb{R}^3,\mathbb{C}^4))}\\ &\qquad\leq CT(\|V\|_\infty + C(\nabla F)M^{p-1})\{\|\tau_{-t_0}\psi_1 - \tau_{-t_0}\psi_2\|_{L^\infty((-T,T);L^2(\mathbb{R}^3,\mathbb{C}^4))}\}.\end{aligned}$$

Since $CT(\|V\|_\infty + C(\nabla F)M^{p-1}) < 1$ we have $\psi_1(t) = \psi_2(t)$ for $t > t_0$ what is a contradiction, so $t_0 = T$.

□

Proof – [Proof of local well-posedness] With help of the previous lemma, we know that there exist $T^* > 0$, which is the supremum of all T for which there is a solution.

If $T^* < \infty$ then there is no solution in

$$L^\infty((-T^*,T^*);H^s(\mathbb{R}^3,\mathbb{C}^4)) \cap L^\gamma((-T^*,T^*),L^\infty(\mathbb{R}^3,\mathbb{C}^4)).$$

Otherwise, it should be in $\mathcal{C}([-T^*,T^*];H^s(\mathbb{R}^3,\mathbb{C}^4))$. So we can define $\psi(T^*) \in H^s$. Solving the Cauchy problem with this initial value, we can continue our solution beyond T^*. What contradicts the definition of T^*.

We now study the dependance towards the initial value problem. Looking to a sequence $\psi_{0,k} \in H^s$ tending to ψ_0 which satisfies $\|\psi_{0,k}\|_{H^s} \leq 2\|\psi_0\|_{H^s}$, we can solve the initial value problem for each $\psi_{0,k}$ in $X(T,M)$ for common T and M. Denoting by ψ_k each solution, with help of an already made calculus, we write

$$\|\psi - \psi_k\|_2 \leq (1 - CT^{1-\frac{\rho-1}{\gamma}})^{-1}\|e^{-\mathrm{i}tD_m}\left(\psi_{0,k} - \psi_0\right)\|_2$$

together with $\|\psi_k(t)\| \leq M$ for $t \in (-T,T)$, we obtain by interpolation that for $s' \in (0,s)$ there is $\theta \in (0,1)$ such that

$$\|\psi - \psi_k\|_{H^{s'}} \leq C(T,M)\|\psi_{0,k} - \psi_0\|_2^\theta,$$

what proves the continuity with respect to the inial value.

Finally we prove that $\psi \in \mathcal{C}^1([-T,T];L^2(\mathbb{R}^3,\mathbb{C}^4))$, it follows from a usual bootstrap argument. But we have to differentiate our integral equation.

Since $2\rho \leq \frac{6}{3-2s}$, H^s is continuously embedded in $L^{2\rho}$ and ∇F satisfies

$$\begin{aligned}\|\nabla F(\phi_1) - \nabla F(\phi_2)\|_2 &\leq \sup_{t\in[0,1]}\{\|D\nabla(F)(t\phi_1 + (1-t)\phi_2)\cdot(\phi_1 - \phi_2)\|_2\}\\ &\leq \|\,|t\phi_1 + (1-t)\phi_2|^{\rho-1}\,\|_{\frac{2\rho}{\rho-1}}\|\phi_1 - \phi_2\|_{2\rho}\\ &\leq (\|\phi_1\|_{2\rho} + \|\phi_2\|_{2\rho})^{\rho-1}\|\phi_1 - \phi_2\|_{2\rho},\end{aligned}$$

with $V \in \mathcal{B}(L^2)$, we obtain $D_m\psi + V(Q)\psi + \nabla F(\psi) \in \mathcal{C}([-T,T], L^2)$.

So we have

$$\{u \mapsto V\psi(u) + \nabla F(\psi(u))\} \in \mathcal{C}([-T,T]; L^2(\mathbb{R}^3, \mathbb{C}^4)).$$

What gives $\mathcal{G}(\psi(t))$ is differentiable with respect to t and in $[-T,T]$ and

$$\begin{cases} i\partial_t\psi &= (D_m + V)\psi + \nabla F(\psi), \\ \psi(0) &= \psi_0. \end{cases}$$

$\square$

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B.II

Directions stables pour des petites solutions stationnaires de l'équation de Dirac

Nous montrons que pour un opérateur de Dirac, n'ayant pas de résonances aux seuils ni de valeurs propres aux seuils, le propagateur vérifie des estimations de propagation et de dispersion.

Quand cet opérateur a seulement deux valeurs propres et qu'elles sont simples et suffisamment proches l'une de l'autre, nous étudions une classe d'équations de Dirac non linéaires ayant des états stationnaires. En appliquant nos estimations linéaires, nous montrons que ces solutions ont des directions stables tangentes au sous espace continu associé à notre opérateur de Dirac de référence. Ce résultat est l'analogue dans le cas de l'équation de Dirac d'un résultat de Tsai et Yau sur l'équation de Schrödinger. À notre connaissance, ce travail est la première étude mathématique du problème de la stabilité associé à une équation de Dirac non linéaire.

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Stable directions for small nonlinear Dirac standing waves

Abstract

We prove that for a Dirac operator, with no resonance at thresholds nor eigenvalue at thresholds, the propagator satisfies propagation and dispersive estimates.

When this linear operator has only two simple eigenvalues sufficiently close to each other, we study an associated class of nonlinear Dirac equations which have stationary solutions. As an application of our decay estimates, we show that these solutions have stable directions which are tangent to the subspaces associated with the continuous spectrum of the Dirac operator. This result is the analogue, in the Dirac case, of a theorem by Tsai and Yau about the Schrödinger equation. To our knowledge, the present work is the first mathematical study of the stability problem for a nonlinear Dirac equation.

Introduction

We study the stability of stationary solutions of a time-dependent nonlinear Dirac equation.

Usually, a localized stationary solution of a given time-dependent equation represents the bound state of a particle. Like Rañada [Ran], we call it a *particle-like solutions* (PLS). In the literature, the term *soliton* is also found instead of PLS, but this additionally means that the particle keeps its form after a collision. Many works have been devoted to the proof of the existence of such solutions for a large variety of equations. Although their stability is a crucial problem (in particular in numerical computation or experiment), a smaller attention has been deserved to this issue.

There are different definitions of stability. The first one is commonly called the *orbital stability*. It means that the orbit of the perturbation of a PLS stays close to the PLS or a manifold of PLS but does not necessarily converge. A stronger notion is *asymptotic stability*, which means that the perturbation of the PLS relaxes asymptotically towards a PLS which is not far from the perturbed PLS.

In fact in many conservative problems asymptotic stability does not hold. But one has asymptotic stability for a restricted class of perturbations, forming the so-called *stable manifold*.

In this paper, we deal with the problem of stability of small PLS of the following nonlinear Dirac equation:

$$\mathrm{i}\partial_t\psi = (D_m + V)\psi + \nabla F(\psi) \qquad \text{(NLDE)}$$

where ∇F is the gradient of $F : \mathbb{C}^4 \mapsto \mathbb{R}$ for the standard scalar product of $\mathbb{R}^8$. Here, D_m is the usual Dirac operator [Tha92] acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$D_m = \alpha \cdot (-\mathrm{i}\nabla) + m\beta = -\mathrm{i}\sum_{k=1}^{3} \alpha_k \partial_k + m\beta$$

where $m \in \mathbb{R}_+^*$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are $\mathbb{C}^4$ hermitian matrices satisfying the following properties:

$$\begin{cases} \alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik} \mathbf{1}_{\mathbb{C}^4}, & i, k \in \{1, 2, 3\}, \\ \alpha_i \beta + \beta \alpha_i = \mathbf{0}_{\mathbb{C}^4}, & i \in \{1, 2, 3\}, \\ \beta^2 = \mathbf{1}_{\mathbb{C}^4}. \end{cases}$$

Here we choose

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}$$

$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In (NLDE), V is the external potential field and $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a nonlinearity such that

$$\forall (\theta, z) \in \mathbb{R} \times \mathbb{C}^4, \quad F(e^{i\theta} z) = F(z).$$

Some additional assumptions on F and V will be made in the sequel. Stationary solutions (PLS) of (NLDE) take the form $\psi(t, x) = e^{-\mathrm{i}Et} \phi(x)$ where ϕ satisfies

$$E\phi = (D_m + V)\phi + \nabla F(\phi). \tag{PLSE}$$

We prove the existence of a manifold of small solutions to (PLSE), interpreted as particle-like solutions to (NLDE). Then we construct a stable manifold around this manifold. At the origin, it is tangent to the sum of the eigenspace associated with the first eigenvalue and the continuous spectral subspace of $D_m + V$. This is the analogue in the Dirac case of [TY02d, Theorem 1.1, non-resonant case]. The interpretation is that radiations (described by the continuous spectrum) do not destabilize too much the PLS manifold. To prove stabilization towards the PLS manifold, we shall need linear decay estimates associated with the continuous spectral subspace of $D_m + V$.

To our knowledge, this is the first stability result on a nonlinear Dirac equation.

The problem of stability has been extensively studied for Schrödinger and Klein-Gordon equations. The methods used to treat these cases cannot be easily adapted to our problem, due to the fact that the Dirac operator D_m is not bounded-below, contrarily to $-\Delta$. The non-negativity of the latter permits to use minimization and concentration-compactness methods to prove the existence of orbitally stable standing waves, see e.g. Cazenave and Lions [CL82] or more recently Cid and Felmer [CF01].

In his review on nonlinear Dirac models, Ranada [Ran] writes that physicists first claimed that PLS (particle-like Solutions) of the nonlinear Dirac equation couldn't be stable since the second derivative of the energy functional is not positive-definite. Actually, in a very general setting (not related to the Dirac case), Weinstein [Wei85, Wei86], Shatah and Straus [SS85] and Grillakis, Shatah and Straus [GSS87] proved a general orbital stability condition even if the hessian of the energy functional is not positive-definite. Their conditions allow only one simple negative eigenvalue (and a kernel of dimension one also) for the second variation. It therefore cannot be directly applied to the Dirac case. However, it gave rise to an interesting discussion about the application of this method to the Dirac equation in some physical papers [SV86, AS86, BSV87]. Rañada also refers to numerical experiments which seem to confirm that some PLS are asymptotically stable in the Dirac case.

In the Schrödinger case, the asymptotic stability has been extensively studied during the last decade. A fundamental work is the one of Soffer and Weinstein [SW90, SW92], which is devoted to the study of a small nonlinear perturbation of a Schrödinger operator having one simple eigenvalue. They proved that the perturbed small PLS relaxes to a PLS. Later, Pillet and Wayne [PW97] proposed a different proof in the spirit of the central manifold theorem. In all these works, asymptotic stability is a direct consequence of propagation or dispersive estimates on the Schrödinger operator. In order to be able to use these estimates, one has to to consider the initial state (at time $t = 0$) of the perturbation as localized *i.e.* in L^1 or in L^2 weighted spaces with growing weight. To avoid such an assumption, Gustafson, Nakanishi and Tsai [GNT04] proposed to use Strichartz estimates.

Generalizations have been considered for instance by Tsai and Yau [TY02a, TY02c, TY02d, TY02b, Tsa03], who treated the case of a Schrödinger operator having two simple eigenvalues. An interesting phenomenon appeared: if the two eigenvalues are sufficiently distant one from the other, then after linearization around the excited state, one obtains a resonance. Tsai and Yau showed that if there is no resonance, the manifold of ground state has stable directions. In the resonant case, the manifold of ground states is asymptotically stable, whereas the manifold of excited states has stable and unstable directions (in case of instability, under some conditions, one has relaxation to the ground state). For a similar result, see also [SW04, SW05]. Notice that earlier Soffer and Weinstein [SW99] studied a similar resonance phenomenon in the case of the Klein-Gordon equation with a simple eigenvalue; they showed that it induced "metastability". Another problem has been studied by Cuccagna [Cuc01, Cuc03, Cuc05]. He considered the case of big PLS, when the linearized operator has only one eigenvalue and obtained the asymptotical stability of the manifold of ground states. Tsai, Yau and Cuccagna also need propagation or dispersive estimates. The latter is proved by generalizing the work of Yajima [Yaj95] on wave operator.

Interesting development are also given by Rodnianski, Schlag and Soffer [RSS05a] who proved asymptotic stability of an arbitrary number of weakly interacting big PLS. Schlag [Sch04] and Krieger and Schlag [KS05] proved the existence of stable direction for unstable big PLS. We point out that some of the works of Schlag [ES04, GS04, RSS05b] or Soffer [HSS99, JSS91, RSS05b] are dedicated to prove dispersive estimates.

We also would like to mention the works of Buslaev and Perel'mann [BP95, BP92b, BP92c, BP92a], Buslaev and Sulem [BS03, BS02] or Weder [Wed00], in the one dimensional Schrödinger case.

Here, we study a nonlinear Dirac equation as a perturbation of a linear Dirac equation with a Dirac operator possessing only two simple eigenvalues sufficiently close to each other. Hence, we avoid problems of resonance after linearization around a PLS. The paper is organized as follows.

In section 1, we define the important objects and state our main results. We start with the propagation and dispersive linear estimates which will be crucial tools for this study. Then, we consider the nonlinear equation (NLDE) and state the existence of the PLS manifold. Eventually, we present our main theorem in which the stable manifold is constructed.

The section 2 is devoted to the proof of the propagation estimate, which uses spectral techniques. This is a time decay estimate in weighted L^2 spaces, expressing the fact that states associated with the continuous spectrum are not stationary. We use Mourre estimate similarly to Hunziker, Sigal and Soffer [HSS99] (for a generalization of the method, see e.g. [BdMGS96]). This method cannot be used in the neighborhood of the thresholds which needs a specific treatment. In particular, problems can occur in the presence of eigenvalues at thresholds or resonances at thresholds, and we shall assume in the whole paper that we are not in this situation. For the Schrödinger case, a similar problem has been studied by Jensen and Kato [JK79], Jensen and Nenciu [JN01, JN04]. Our arguments near the thresholds are inspired of these works. For a related study, see the article of Fournais and Skibsted [FS04] dealing with long range perturbations of Schrödinger operators.

In Section 3, we then prove the dispersive estimate, using the propagation estimate established in Section 2. For an interesting survey on dispersive estimates for Schrödinger operators, see Schlag [Sch05]. We have not been able to generalize the methods used in the Schrödinger case, in fact it seems that the Dirac equation with a potential behaves like a Klein-Gordon equation with a magnetic potential. This fact has already been noticed by D'Anconna and Fanelli in [DF], where they proved simultaneously dispersive estimates for a massless Dirac equation with a potential and for a wave equation with a magnetic potential. Our method is here inspired of the work by Cuccagna and Schirmer [CS01].

Finally, the last sections are devoted to the proof of our main result concerning the stability of the stationary solutions of (NLDE). We assume that the Dirac operator $D_m + V$ have only two simple eigenvalues and that it has no eigenvalues at thresholds nor resonances at thresholds. Note that our assumptions exclude electric potentials, for which the theorem of Kramers states that the eigenvalues are always degenerate, see [Par90, BH92]. In Section 4, this permits us to construct a manifold of PLS and then to study the spectrum of the linearized operator. This in turn, in Section 5, will allow us to decompose a solution of (NLDE) in three parts: the PLS part, the dispersive part associated with the continuous spectrum and a part corresponding to "excited states". This last part needs a particular treatment since it is not dispersive and hence disturbs the relaxation towards the PLS manifold.

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1 Main results

This section is devoted to the presentation of the model and the statement of our main results.

1.1 Decay estimates for a Dirac operator with potential

Let us first state our results concerning the time decay of $e^{-\mathrm{i}t(D_m+V)}$ in weighted L^2 spaces and Besov spaces. This kind of estimates are called respectively propagation and dispersive estimates. As mentioned in the introduction, these results will be very important tools for the study of our nonlinear time-dependent Dirac equation.

The following spaces will be needed to state the main result of this subsection.

Definition B.1 (Weighted Sobolev space). *The weighted Sobolev space is defined by*

$$H^t_\sigma(\mathbb{R}^3, \mathbb{C}^4) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3),\ \|\langle Q\rangle^\sigma \langle P\rangle^t f\|_2 < \infty \right\}$$

for $\sigma, t \in \mathbb{R}$. *We endow it with the norm*

$$\|f\|_{H^t_\sigma} = \|\langle Q\rangle^\sigma \langle P\rangle^t f\|_2.$$

If $t = 0$, *we write* L^2_σ *instead of* H^0_σ.

We have used the usual notations $\langle u\rangle = \sqrt{1+u^2}$, $P = -i\nabla$, and Q is the operator of multiplication by x in $\mathbb{R}^3$. For the sake of clarity, let us also recall the

Definition B.2 (Besov space). *For* $s \in \mathbb{R}$ *and* $1 \leq p, q \leq \infty$, *the Besov space* $B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4)$ *is the space of all* $f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ *(dual of the Schwartz space) such that*

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j\in\mathbb{N}} 2^{jsq} \|\varphi_j * f\|_p^q \right)^{\frac{1}{q}} < +\infty$$

with $\widehat{\varphi} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ *such that* $\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) = 1$ *for all* $\xi \in \mathbb{R}^3 \setminus \{0\}$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$ *for all* $j \in \mathbb{N}^*$ *and for all* $\xi \in \mathbb{R}^3$, *and* $\widehat{\varphi_0} = 1 - \sum_{j\in\mathbb{N}^*} \widehat{\varphi}_j$. *We endow it with the norm* $f \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4) \mapsto \|f\|_{B^s_{p,q}}$.

In the whole chapter, we shall work within the following

Assumption 1.1. *The potential* $V : \mathbb{R}^3 \mapsto S_4(\mathbb{C})$ *(self-adjoint* 4×4 *matrices) is a* $\mathcal{C}^\infty$ *function such that there exists* $\rho > 5$ *with*

$$\forall \alpha \in \mathbb{N}^3,\ \exists C > 0,\ \forall x \in \mathbb{R}^3,\ |\partial^\alpha V|(x) \leq \frac{C}{\langle x\rangle^{\rho+|\alpha|}}.$$

Notice that by the Kato-Rellich Theorem , the operator

$$H := D_m + V$$

is essentially self-adjoint on $\mathcal{C}^\infty_0(\mathbb{R}^3, \mathbb{C}^4)$ and self-adjoint on $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

We notice that the Weyl's criterion (see [RS78, Theorem XIII.14, Corollary 1]) give that the essential spectrum of H is given by $\mathbb{R} \setminus (-m; m)$. Then the results of Berthier and Georgescu [BG87][Theorem 6,Theorem A] give that there is no eigenvalue embedded in the essential spectrum.

We also work with the

Assumption 1.2. *The operator* H *presents no resonance at thresholds and no eigenvalue at thresholds.*

A resonance is an eigenvector in $H^{1/2}_{-\sigma}(\mathbb{R}^3, \mathbb{C}^4) \setminus H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ for any $\sigma \in (1/2, \rho - 1/2)$ here. Let

$$\mathbf{P}_c(H) = \mathbb{1}_{(-\infty,-m]\cup[+m,+\infty)}(H) \tag{1}$$

be the projector associated with the continuous spectrum of H and

$$\mathcal{H}_c = \mathbf{P}_c(H) L^2(\mathbb{R}^3, \mathbb{C}^4). \tag{2}$$

We are now able to state our

Theorem B.3 (Propagation for perturbed Dirac dynamics)**.** *Assume that Assumptions 1.1 and 1.2 hold and let be* $\sigma > 5/2$*. Then one has*

$$\|e^{-\mathrm{i}tH}\mathbf{P}_c(H)\|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C\langle t\rangle^{-3/2}.$$

The proof of this result will be given in Section 2. We notice that it is still true if we assume $\rho > 3$ in Assumption 1.1.

Our next result is the following theorem, proved in Section 3.

Theorem B.4 (Dispersion for perturbed Dirac dynamics)**.** *Assume that Assumptions 1.1 and 1.2 hold. Then for* $p \in [1,2]$*,* $\theta \in [0,1]$*,* $s - s' \geq (2+\theta)(\frac{2}{p} - 1)$ *and* $q \in [1,\infty]$ *there exists a constant* $C > 0$ *such that we have*

$$\|e^{-\mathrm{i}tH}\mathbf{P}_c(H)\|_{B^s_{p,q}, B^{s'}_{p',q}} \leq C\,(K(t))^{\frac{2}{p}-1}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$*, and*

$$K(t) = \begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \in (0,1], \\ |t|^{-1-\theta/2} & \text{if } |t| \in [1,\infty). \end{cases}$$

1.2 The stable manifold around the PLS for the nonlinear Dirac equation

We now want to study the following nonlinear Dirac equation

$$\begin{cases} \mathrm{i}\partial_t\psi = H\psi + \nabla F(\psi) \\ \psi(0,\cdot) = \psi_0. \end{cases} \tag{3}$$

with $\psi \in \mathcal{C}^1(I, H^1(\mathbb{R}^3, \mathbb{C}^4))$ for some open interval I which contains 0 and where we recall that $H = D_m + V$. The nonlinearity $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a differentiable map for the real structure of $\mathbb{C}^4$ and hence the ∇ symbol has to be understood for the real structure of $\mathbb{C}^4$. For the usual hermitian product of $\mathbb{C}^4$, one has

$$DF(v)h = \Re\langle\nabla F(v), h\rangle.$$

We work within the following

Assumption 1.3. *The operator* H *has only two simple eigenvalues* $\lambda_0 < \lambda_1$*, with* ϕ_0 *and* ϕ_1 *as associated normalized eigenvectors. Moreover, the non resonant condition*

$$|\lambda_1 - \lambda_0| < \min\{|\lambda_0 + m|, |\lambda_0 - m|\}$$

holds.

This assumption is useful in our study of the spectrum of the linearized operator around a stationary state (see Section 4.2), it gives us that this operator has two simple non zero eigenvalue (see Lemma B.38).

Assumption 1.4. *The function* $F : \mathbb{C}^4 \mapsto \mathbb{R}$ *is in* $\mathcal{C}^\infty(\mathbb{R}^8, \mathbb{R})$ *satisfies* $F(z) = O(|z|^4)$ *as* $z \to 0$*. Moreover, it has the gauge invariance property:*

$$F(e^{i\theta}z) = F(z), \forall z \in \mathbb{C}^4,\ \forall\theta \in \mathbb{R}.$$

We will prove in Theorem B.6 that some solutions of the equation (3) are global and can be decomposed as the sum of a PLS plus a remainder part which is vanishing. Since the PLS part may change during the evolution, we need to track it. So we prove that around the origin, PLS form a manifold. We have the

Proposition B.5 (PLS manifold). *Suppose that Assumptions 1.1–1.4 hold. Then for any $\sigma \in \mathbb{R}^+$, there exist Ω a neighborhood of $0 \in \mathbb{C}$, a $\mathcal{C}^\infty$ map*

$$h : \Omega \mapsto \{\phi_0\}^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4)$$

and a $\mathcal{C}^\infty$ map $E : \Omega \mapsto \mathbb{R}$ such that $S(u) = u\phi_0 + h(u)$ satisfy, for all $u \in \Omega$, the identity

$$HS(u) + \nabla F(S(u)) = E(u)S(u), \tag{4}$$

with the following properties

$$\begin{cases} h(e^{i\theta}u) = e^{i\theta}h(u), \quad \forall \theta \in \mathbb{R}, \\ h(u) = O(|u|^2), \\ E(u) = E(|u|), \\ E(u) = \lambda_0 + O(|u|^2). \end{cases}$$

Proof – This kind of results is now classical and left to the reader. For more details, see Subsection 4.1. □

Let us introduce the linearized operator $JH(u)$ around a stationary state $S(u)$

$$H(u) = H + d^2F(S(u)) - E(u)$$

where d^2F is the differential of ∇F. We notice that the operator $H(u)$ is only $\mathbb{R}$-linear but not $\mathbb{C}$-linear. Hence we work with the space $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ instead of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by writing

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}$$

instead of ϕ. Hence the multiplication by $-$i becomes the operator

$$J = \begin{pmatrix} 0 & -I_{\mathbb{R}^4} \\ I_{\mathbb{R}^4} & 0 \end{pmatrix}.$$

The operator $JH(u)$ has a two dimensional geometric kernel and two simple eigenvalues $E_1(u)$ and $-E_1(u)$ which are purely imaginary. The associated eigenspaces are conjugated one to each other. Working on the real part of their direct sum, we introduce a family of basis of this last real space : $\left(\phi_1^1(u), \phi_1^2(u)\right)$.

The rest of the spectrum is the essential spectrum. We write $\mathcal{H}_c(u)$ for the space associated with the continuous spectrum. This space $\mathcal{H}_c(u)$ is the orthogonal of the previous eigenspaces with respect to the product $(f, g) \mapsto \Re\langle f, Jg\rangle$ where $\langle \cdot, \cdot\rangle$ is the standard product of $L^2(\mathbb{R}^3, \mathbb{C}^4)$. The space $\mathcal{H}_c(u)$ is isomorphic to $\mathcal{H}_c$ and the isomorphism is P_c, defined in 2, for any reals s and σ. Moreover P_c is a bounded operator from $H^s_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ to itself for any reals s and σ. These facts will be proven in subsection 4.2.

We are now able to write the main theorem of this paper. Its proof is given in Section 6.

Theorem B.6 (Stable manifold)**.** *Suppose that Assumptions 1.1–1.4 hold. Let* $s, s', \beta \in \mathbb{R}_+^*$ *be such that* $s' \geq s+3 \geq \beta+6$ *and* $\sigma > 5/2$*. There exist* $\varepsilon_0 > 0$*,* $R > 0$*,* $K > 0$*,* $T_0 > 0$ *and Lipschitz map*

$$\Psi : \mathcal{S} \mapsto \mathbb{R}^2$$

where $\mathcal{S} = \left\{(V,\xi)\,;\ V \in B_{\mathbb{C}^2}(0,\varepsilon),\ \xi \in \mathcal{H}_c(u) \cap B_{H^{s'}_\sigma}(0,R)\right\}$ *endowed with the metric of* $\mathbb{C}^2 \times H^{s'}_\sigma$ *with* $\Psi(u,0) = 0$ *for all* $u \in B_{\mathbb{C}}(0,\varepsilon)$*,*

$$|\Psi(u,z)| \leq K\left(|u| + \|z\|_{H^{s'}_\sigma}\right)^2,$$

such that the following holds. For any initial condition of the form

$$\psi_0 = S(u_0) + z_0 + \Psi(u_0,z_0)\cdot\phi_1(u_0)$$

with $(u_0,z_0) \in \mathcal{S}$*, one has*

(i) there exists a unique solution ψ *of* (3) *with initial condition* ψ_0*, and this solution is in*

$$\mathcal{C}\left((-T_0;+\infty), H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\right) \cap \mathcal{C}^1\left(]-T_0;+\infty[, H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right);$$

(ii) there exist $(u_\infty, z_\infty) \in \mathcal{S}$ *and* $E_\infty \in \mathbb{R}$ *with*

$$|u_\infty - u_0| \leq K\|z_0\|^2_{H^{s'}_\sigma}, \quad |E_\infty| \leq K\|z_0\|^2_{H^{s'}_\sigma}, \quad \|z_\infty - z_0\|_{H^{s'}} \leq K\|z_0\|^2_{H^{s'}_\sigma}$$

such that

$$\psi(t) = e^{-\mathrm{i}(tE(u_\infty)+E_\infty+r(t))}\left(S(u_\infty) + e^{JtH(u_\infty)}z_\infty + \varepsilon(t)\right),$$

where

$$\begin{cases} \|\varepsilon(t)\|_{H^{s'}} \leq \dfrac{K}{\langle t\rangle}\|z_0\|^2_{H^{s'}_\sigma} \\ \|\varepsilon(t)\|_{H^{s}_{-\sigma}} \leq \dfrac{K}{\langle t\rangle^2}\|z_0\|^2_{H^{s'}_\sigma} \\ \|\varepsilon(t)\|_{B^{\beta}_{\infty,2}} \leq \dfrac{K}{\langle t\rangle^2}\|z_0\|^2_{H^{s'}_\sigma} \end{cases}$$

and $|r(t)| \leq \frac{K}{\langle t\rangle}\|z_0\|^2_{H^{s'}_\sigma}$ *as* $t \to +\infty$*.*

Remark B.6.1. *The proof of this theorem works also if we want to obtain an expansion of the form*

$$\psi(t) = e^{-\mathrm{i}(tE(u_\infty)+E_\infty)}S(u_\infty) + e^{-\mathrm{i}tD_m}z_\infty + \varepsilon(t)$$

or

$$\psi(t) = e^{-\mathrm{i}(tE(u_\infty)+E_\infty)}S(u_\infty) + e^{-\mathrm{i}tH}z_\infty + \varepsilon(t)$$

with $z_\infty \in \mathcal{H}_c$*. But in this cases, we only have the estimates*

$$\|z_\infty - z_0\|_{H^{s'}} \leq K\|z_0\|_{H^{s'}_\sigma}$$

and

$$\begin{cases} \|\varepsilon(t)\|_{H^{s'}} \leq K\|z_0\|_{H^{s'}_\sigma} \\ \|\varepsilon(t)\|_{H^{s}_{-\sigma}} \leq \dfrac{K}{\langle t\rangle^{3/2}}\|z_0\|_{H^{s'}_\sigma} \\ \|\varepsilon(t)\|_{B^{\beta}_{\infty,2}} \leq \dfrac{K}{\langle t\rangle^{3/2}}\|z_0\|_{H^{s'}_\sigma}, \end{cases}$$

as $t \to +\infty$*, see the remark following Lemma B.50.*

We notice that the stabilization is “faster” than the propagation and the dispersion: it is of order $\langle t\rangle^{-2}$ whereas $e^{JtH(u)}z_\infty$ is of order $\langle t\rangle^{-3/2}$ (due to the estimates of Theorems B.3 and B.4, see Section 5). Hence the theorem states the existence of a family of initial states which form a manifold tangent at the origin to the sum of the eigenspace of H associated with λ_0 and the subspace associated with the continuous spectrum of H: $\mathcal{H}_c$. This family of initial states gives rise to solutions of (3) which asymptotically split in two parts. The first one is a PLS: $e^{-\mathrm{i}(tE(u_\infty)+E_\infty)}S(u_\infty)$ the other is a dispersive perturbation: $e^{JtH(u)}z_\infty$. Hence if one perturbs a PLS in the direction of the continuous spectrum then this PLS relaxes to another PLS by emitting a dispersive wave.

This phenomenon is due to the propagation and the dispersion properties of the subspace associated with the continuous spectrum of H. We don't think that such a phenomenon could take place for perturbations in the direction of the excited states ϕ_1. Indeed, on this subspace, the dynamic seems to be conservative. The fact that we use propagation and dispersive estimates restricts the family of perturbations to regular and localized ones.

We now turn to the proof of our results.

2 Proof of Theorem B.3: propagation estimates

Here we prove the propagation estimates of Theorem B.3. The method used by Jensen and Kato [JK79] to prove this kind of estimates for Schrödinger operator works only for initial states which are spectrally localized near the thresholds $\pm m$. They used the spectral density as the Fourier transform of the propagator. But the Dirac resolvent

$$R_V(\lambda \pm \mathrm{i}\varepsilon) = (H - \lambda \mp \mathrm{i}\varepsilon)^{-1}$$

does not decay in $B(L^2_\sigma, L^2_{-\sigma})$ as $|\lambda| \to +\infty$ for any $\sigma > 0$, see [Yam93]. So we cannot use its Fourier transform. To our knowledge, this method is the only one that permits to treat the problem of propagation for energies near thresholds. Hence with this method, we only prove (in the section 2.1) the

Proposition B.7 (Propagation near thresholds)**.** *Suppose that Assumptions 1.1 and 1.2 hold and let $\chi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ be such that its support is in a sufficiently small neighborhood of $[-m;\ m]$. Then, for $\sigma > 5/2$, one has*

$$\|e^{-\mathrm{i}tH}\mathbf{P}_c(H)\,\chi(H)\,\|_{B(L^2_\sigma, L^2_{-\sigma})} \le C\,\langle t\rangle^{-3/2}\,.$$

We recall that $\mathbf{P}_c(H)$ is defined by (1).

We also need to treat the propagation estimates for initial state whose spectrum does not contain any threshold. We cannot use the spectral density. So we work directly with the propagator. This is exactly the method used by Hunziker, Sigal and Soffer in [HSS99]. But in our case, their result needs some adaptation. Hence we need to generalize [HSS99, Theorem 1.1] to the case of unbounded energy. In Section 2.2, we prove the

Proposition B.8 (Propagation far from thresholds)**.** *Suppose that Assumption 1.1 holds. Then for any $\chi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ bounded with support in $\mathbb{R} \setminus (-m; m)$ and for any $\sigma \ge 0$, there exists $C > 0$ such that*

$$\|e^{-\mathrm{i}tH}\chi(H)\,\|_{B(L^2_\sigma, L^2_{-\sigma})} \le C\,\langle t\rangle^{-\sigma}\,.$$

The proof of Theorem B.3 is then a consequence of the above propositions
Proof – [Proof of Theorem B.3]

We choose $\chi_0 \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ satisfying the assumptions of Proposition B.7, $\chi_\infty \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ satisfying assumptions of Proposition B.8 such that $\chi_0 + \chi_\infty = 1$. Hence the continuous spectrum of H is divided in two parts. We obtain the inequality

$$\|e^{-\mathrm{i}tH}\mathbf{P}_c(H)\|_{B(L^2_\sigma, L^2_{-\sigma})} \leq \|e^{-\mathrm{i}tH}\chi_0(H)\mathbf{P}_c(H)\|_{B(L^2_\sigma, L^2_{-\sigma})} + \|e^{-\mathrm{i}tH}\chi_\infty(H)\|_{B(L^2_\sigma, L^2_{-\sigma})}.$$

Hence from Proposition B.7, and B.8, we deduce Theorem B.3. □

It therefore remains to prove Propositions B.7, and B.8.

2.1 Step 1 : Propagation near thresholds

2.1.1 Proof of Proposition B.7

We now prove Proposition B.7. Let χ be in $\mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$, then the operator $e^{-\mathrm{i}tH}\mathbf{P}_c(H)\chi(H)$ as a function of t is the Fourier transform with respect to λ of

$$\lambda \mapsto \Im R_V^+(\lambda)\mathbb{1}_{(-\infty,-m]\cup[m,\infty)}(\lambda)\chi(\lambda),$$

where

$$R_V^\pm(\lambda) = \lim_{\varepsilon\to 0^+} R_V(\lambda \pm \mathrm{i}\varepsilon), \tag{5}$$

we will prove in Section 2.2 that the limit exists in $\mathcal{B}(L^2_\sigma, L^2_{-\sigma})$. So Proposition B.7 is a consequence of the

Proposition B.9. *Suppose that Assumptions 1.1 and 1.2 hold. Then for $\lambda > m$ close enough to m, one has that*

$$R_V^\pm(\lambda) = \lim_{\varepsilon\to 0^+} R_V(\lambda \pm \mathrm{i}\varepsilon)$$

exists in $\mathcal{B}(\mathcal{H}_\sigma^{-1/2}, \mathcal{H}_{-\sigma}^{1/2})$ for $\sigma > 3/2$. It is $\mathcal{C}^l$ if $\sigma > 1/2 + l$ and $0 < l \leq 2$ with

$$\frac{d^l}{d\lambda^l}\Im R_V^\pm(\lambda) = O(\sqrt{\lambda - m}^{1/2-l}), \tag{6}$$

as $\lambda \to m^+$.

The same holds for $\lambda < -m$ if m is replaced by $-m$.

We prove it in Section 2.1.2. The idea is then to apply to

$$\lambda \mapsto \Im R_V^+(\lambda)\mathbb{1}_{(-\infty,-m]\cup[m,\infty)}(\lambda)\chi(\lambda). \tag{7}$$

with $k = 1$ and $\theta = 1/2$, the following

Lemma B.10 (Lemma 10.2 of [JK79])**.** *Suppose $F(\lambda) = 0$ for $\lambda > a > 0$, $F^{(k+1)} \in L^1([\delta, +\infty[)$ for any $\delta > 0$ and an integer $k \geq 0$ and that $F^{(k+1)}(\lambda) = O(\lambda^{\theta-2})$ near 0 for some $\theta \in (0,1)$. Assume further that $F^{(j)}(0) = 0$ for $j \leq k-1$, then one has*

$$\widehat{F}(t) = O(t^{-k-\theta}).$$

The symbol O may be replaced by o throughout.

We refer to [JK79] for the proof of Lemma B.10. In fact to apply this lemma to (7), one should split this function in two parts, one supported in $\mathbb{R}^+$ and the other in $\mathbb{R}^-$. Then one translates the first one by $-m$ and applies the lemma. To deal with the other part, one works exactly in the same way after a symmetry with respect to the origin. To end the proof of Proposition B.7, it remains to prove Proposition B.9. This the goal of the next section.

2.1.2 Behavior near thresholds of the Dirac resolvent: proof of Proposition B.9

In this section, our aim is to prove Proposition B.9. First of all, we notice that if the limits (5) exist then we have

$$R_V^-(\lambda)^* = R_V^+(\lambda),$$

and since

$$\alpha_5(D_m + V - z)^{-1}\alpha_5 = -(D_m + \alpha_5 V \alpha_5 + z)^{-1},$$

for $\alpha_5 = \prod_{i=1}^{3} \alpha_i \beta$, one obtains

$$\alpha_5 R_V^{\pm}(\lambda)^{-1}\alpha_5 = -R^{\mp}_{\alpha_5 V \alpha_5}(-\lambda).$$

So we only need to study the behavior of $R_V^+(\lambda)$ near $+m$. Let us introduce

$$\mathbb{C}_{++} = \{z \in \mathbb{C},\ \Im z > 0,\ \Re z > 0\}$$

then the behavior for the free case ($V = 0$) is given by the

Proposition B.11 (Dirac's resolvent expansion)**.** *Let $s, s' > 1/2$ with $s+s' > 2$ and $t \in \mathbb{R}$. Then $R_0(z) \in \mathcal{B}(H_s^{t-1}, H_{-s'}^t)$ is uniformly continuous in $\mathbb{C}_{++}$ and so it can be continuously extended to $\overline{\mathbb{C}_{++}}$.*

Moreover, if we introduce for each $j \in \mathbb{N}$, G_j the operator with kernel $\frac{|x-y|^{j-1}}{4\pi j!}$, the formal series $z \in \mathbb{C}_{++}$,

$$R_0(z) = \sum_{j=0}^{\infty} (\mathrm{i}\sqrt{z^2 - m^2})^j D_m G_j + \sum_{j=0}^{\infty} z(\mathrm{i}\sqrt{z^2 - m^2})^j G_j$$

with $\Im(\sqrt{z^2 - m^2}) > 0$ is an asymptotic expansion for $z \to m$ in the following sense:

Let $k \in \mathbb{N}$, if $R_0(z)$ is approximated by the corespondent finite series up to $j = k$, the remainder is $o(|z-m|^{k/2})$, as $z \to m$, in the norm of $\mathcal{B}\left(H_s^{t-1}, H_{-s'}^t\right)$ with s, $s' > k + 1/2$ (and $s + s' > 2$ if $k = 0$) and $t \in \mathbb{R}$.

In the same sense, this identity can be differentiated in z any number of times. More precisely, for $l \in \mathbb{N}^*$ the l^{th} derivative in z of the said finite series is equal to $\frac{d^l}{dz^l} R(z)$ up to an error $o(|z-m|^{k/2-l})$, as $z \to m$, in the norm of $\mathcal{B}\left(H_s^{t-1}, H_{-s'}^t\right)$ with s, $s' > k + l + 1/2$ and $t \in \mathbb{R}$.

Proof – It is an adaptation of lemmas of [JK79]. We rewrite [JK79, Lemma 2.1], [JK79, Lemma 2.2] and [JK79, Lemma 2.3] in the Dirac case with help of the identity

$$(D_m - z)^{-1}(D_m + z)^{-1} = (-\Delta + m^2 - z^2)^{-1},$$

or in $\left(\mathbb{C}^2\right)^2$

$$(D_m - z)^{-1} = \begin{pmatrix} \dfrac{z+m}{-\Delta - z^2 + m^2} & \dfrac{\sigma \cdot \nabla}{-\Delta - z^2 + m^2} \\ \dfrac{\sigma \cdot \nabla}{-\Delta - z^2 + m^2} & \dfrac{z-m}{-\Delta - z^2 + m^2} \end{pmatrix}$$

where σ are the two dimensional Pauli matrices. □

To obtain the behavior of the Dirac resolvent in the general case, we would like to use the formula

$$R_V(z) = M(z)^{-1} R_0(z) \tag{8}$$

with

$$M(z) = (1 + R_0(z)V) .$$

To give a meaning to Identity (8), we have to prove that $M(z)$ is invertible in $\mathcal{B}(H^{1/2}_{-\sigma})$ for $\sigma > 1/2$ with $\sigma + 1/2 < \rho$, where ρ is introduced in assumption 1.1. We will also give the asymptotic behavior of $R_V^+(z)$ and some of its derivatives as $\lambda \to m^+$. By means of Proposition B.11, one has

$$z \in \overline{\mathbb{C}_{++}} \mapsto M(z) \in \mathcal{B}\left(H^{1/2}_{-\sigma}\right)$$

is uniformly continuous for $1/2 < \sigma$ and $2 < \sigma + \sigma' \leq \rho$ and some $\sigma' > 1/2$. We also have

$$M(z) = M(m) + A(z),$$

with $A(z)$ uniformly continuous in $\mathcal{B}\left(H^{1/2}_{-\sigma'}, H^{1/2}_{-\sigma}\right)$ near m in $\mathbb{C}_{++}$ and tending to 0 as $\lambda \to m$ for $1/2 < \sigma$ and $2 < \sigma + \sigma' \leq \rho$ and some $\sigma' > 1/2$. We now prove the

Lemma B.12 (Threshold's eigenvector and resonance)**.** *Suppose that Assumption 1.1 holds. Let $\mathcal{M}(s)$ be the kernel of $M(m)$ in $H^{1/2}_{-s}$ and $\mathcal{K}(s)$ the kernel of $(H - m)$ in $H^{1/2}_{-s}$. Then $\mathcal{M}(s)$ and $\mathcal{K}(s)$ are finite dimensional and do not depend on $s \in (1/2, \rho - 1/2)$. So we write $\mathcal{M}$ and $\mathcal{K}$ and we have*

$$\mathcal{M} = \mathcal{K}.$$

Proof – See also [JK79, Lemma 3.1].

Let $u \in \mathcal{K}(s)$, then $(D_m + V - m)u = 0$ and $u \in H^{1/2}_{-s}$, so $Vu \in H^{-1/2}_{\rho-s}$ and since $\rho - s > 1/2$, $s > 1/2$, and $s + \rho - s > 2$, we obtain, by Proposition B.11, $(D_m - m)^{-1}(D_m - m)u = (D_m - m)^{-1} Vu \in H^{1/2}_{-s}$. For any $\phi \in \mathcal{C}_0^\infty$,

$$\langle \phi, (D_m - m)^{-1}(D_m - m)u\rangle = \langle (D_m - m)(D_m - m)^{-1}\phi, u\rangle = \langle \phi, u\rangle,$$

we obtain $(D_m - m)(D_m - m)^{-1}Vu = Vu$ and $(D_m - m)(u + (D_m - m)^{-1}Vu) = 0$. Since $D_m - m$ has no kernel in $H^{1/2}_{-s}$, because there's no harmonic function in L^2_{-s}, we obtain $u + (D_m - m)^{-1}Vu = 0$. Hence, we have $\mathcal{K}(s) \subset \mathcal{M}(s)$.

Conversely, $I + (D_m - m)^{-1}V$ defines a Fredholm operator of $\mathcal{B}(H^{1/2}_{-s})$. If $u \in \mathcal{M}(s)$ then $u \in H^{1/2}_{-s}$ and $(D_m - m)^{-1}Vu \in H^{1/2}_{-s}$. So we write $0 = (D_m - m)(u + (D_m - m)^{-1}Vu) = (D_m - m + V)u$ and we obtain $\mathcal{M}(s) \subset \mathcal{M}(s)$.

Now we introduce $I+V\,(D_m-m)^{-1}\in\mathcal{B}(H_s^{-1/2})$, and its kernel $\mathcal{N}(s)$ which is finite dimensional is a Fredholm operator. We have that $\mathcal{N}(s)$ is decreasing with s and $\mathcal{M}(s)$ is increasing. Since, by duality, $\dim\mathcal{M}(s)=\dim\mathcal{N}(s)$, we deduce that $\mathcal{N}(s)$ and $\mathcal{K}(s)=\mathcal{M}(s)$ do not depend on s. □

We are now able to conclude the proof of Proposition B.9.

Proof – [Proof of Proposition B.9] Assumption 1.2 gives $\mathcal{K}=0$ and so with Lemma B.12, one obtains $\mathcal{M}=0$. Hence $M(m)$ is invertible since it is a Fredholm operator. We use Von Neumann series to obtain that $M(z)$ is invertible in $\mathcal{B}\left(H^{1/2}_{-\sigma}\right)$ for $\sigma>1/2$, $2<\sigma+\sigma'\leq\rho$ and some $\sigma'>1/2$ and

$$M(z)^{-1}=M(m)^{-1}\sum_{j\geq 0}(A(z)M(m)^{-1})^j.$$

So for $\lambda\geq m$ close enough to m,

$$M^+(\lambda)^{-1}=\lim_{\varepsilon\to 0^+}M(\lambda+\mathrm{i}\varepsilon)^{-1}$$

exists in $\mathcal{B}\left(H^{1/2}_{-\sigma}\right)$ for $\sigma>1/2$ with $2<\sigma+\sigma'\leq\rho$ and some $\sigma'>1/2$. We obtain that $\lim_{\varepsilon\to 0^+}R_V\,(\lambda+i\varepsilon)$ exists in $\mathcal{B}\left(H^{-1/2}_{\sigma''},H^{1/2}_{-\sigma}\right)$ for $\sigma>1/2$ and $\sigma\geq\sigma''>1/2$ with $\sigma+\sigma''>2$, $2<\sigma+\sigma'\leq\rho$ and some $\sigma'>1/2$.

Using Proposition B.11, we prove that if $1/2+k<\sigma$, and $\sigma'+1/2+k<\rho$ then

$$\frac{d^k}{d\lambda^k}M^+(\lambda)=O(\sqrt{\lambda-m}^{1/2-k})$$

in $\mathcal{B}\left(H^{-1/2}_{-\sigma'},H^{1/2}_{-\sigma}\right)$ for $k\in\mathbb{N}^*$ as $\lambda\to m^+$. Since we have

$$\frac{d}{d\lambda}F(\lambda)^{-1}=-F(\lambda)^{-1}\left(\frac{d}{d\lambda}F(\lambda)\right)F(\lambda)^{-1},$$

for matrix valued differentiable function F with invertible values, we obtain for $k\in\mathbb{N}^*$ the estimate

$$\frac{d^k}{d\lambda^k}M^+(\lambda)^{-1}=O(\sqrt{\lambda-m}^{1/2-k}),$$

in $\mathcal{B}\left(H^{1/2}_{-\sigma}\right)$ with $1/2+k<\sigma$ and $\sigma+1/2+k<\rho$ as $\lambda\to m^+$. So by Leibniz formula, we also have for $k\in\mathbb{N}^*$

$$\frac{d^k}{d\lambda^k}R_V^+(\lambda)=O(\sqrt{\lambda-m}^{1/2-k}),$$

in $\mathcal{B}\left(H^{-1/2}_{\sigma'},H^{1/2}_{-\sigma}\right)$ with $1/2+j<\sigma$, $1/2+k-j<\sigma'$ and $1/2+k-j+\sigma<\rho$ for all $j\in\{0,\dots,k\}$ as $\lambda\to m^+$. For the case $k=0$, we have the formula

$$R_V(z)=R_0(z)\left(1+VR_0(z)\right)^{-1}.$$

Since $R^+(m)=R^-(m)$, this leads to

$$\Im R_V^+(m)=0,$$

and so

$$\Im R_V^{\pm}(\lambda) = O(\sqrt{\lambda - m}^{1/2}),$$

as $\lambda \to m^+$ in $\mathcal{B}\left(H_\sigma^{-1/2}, H_{-\sigma}^{1/2}\right)$ with $3/2 < \sigma$ and $\sigma + 3/2 < \rho$. Hence (6) is proved. □

2.2 Step 2: Propagation far from thresholds

In this section, we prove Proposition B.8. We prove the propositions for $t \geq 0$. Then using $\left(e^{-\mathrm{it}H}\right)^* = e^{\mathrm{it}H}$, the result easily follows for $t \leq 0$.

2.2.1 Proof of Proposition B.8

Let us introduce

$$A = \frac{1}{2}\left\{D_m^{-1} P \cdot Q + Q \cdot P D_m^{-1}\right\}.$$

[IM99, Lemma 3.1] gives that A is an essentially self-adjoint operator and the domain of its closure contains the domain of $\langle Q \rangle$. Proposition B.8 is then a consequence of the

Theorem B.13 (Minimal escape velocity)**.** *Suppose that Assumption 1.1 holds. Then for any $\chi \in \mathcal{C}_0^\infty$ bounded with support in $(-\infty, -m) \cup (m, +\infty)$, there exists $\theta > 0$ such that for any $l \in \mathbb{R}$, for any $v \in (0, \theta)$, and any $a \in \mathbb{R}$, one has*

$$\forall t > 0, \ \left\| \mathbb{1}_{A-a-vt \leq 0} e^{-itH} \chi(H) \mathbb{1}_{A-a \geq 0} \right\| \leq C t^{-l},$$

where C do not depend a and t.

The proof will be given in Section 2.2.2. Let us now show that Theorem B.13 implies Proposition B.8.

Proof – [Proof of Proposition B.8] We notice that for $c \geq 0$

$$\begin{aligned} \langle A \rangle^{-\alpha} &= \langle A \rangle^{-\alpha} \mathbb{1}_{A \leq ct} + O(t^{-\alpha}), \\ \langle A \rangle^{-\alpha} &= \langle A \rangle^{-\alpha} \mathbb{1}_{A \geq -ct} + O(t^{-\alpha}), \end{aligned}$$

when $t > 0$, this leads to

$$\langle A \rangle^{-\alpha} e^{-itH} \chi(H) \langle A \rangle^{-\alpha} = \langle A \rangle^{-\alpha} \mathbb{1}_{A \leq \frac{(\theta-\varepsilon)t}{2}} e^{-itH} \chi(H) \mathbb{1}_{A \geq -\frac{\theta t}{2}} \langle A \rangle^{-\alpha} + O(t^{-\alpha}),$$

So if we choose $a = -\frac{\theta t}{2}$ and $v = \theta - \frac{\varepsilon}{2}$ in Theorem B.13, we obtain

$$\|\langle A \rangle^{-\alpha} e^{-itH} \chi(H) \langle A \rangle^{-\alpha}\| \leq C t^{-\min(\alpha,\, l)}.$$

Then we prove that $\langle A \rangle^{\alpha} \langle Q \rangle^{-\alpha}$ is bounded for any positive α. It is quite immediate for integer α using multi-commutator expansion [HS00, Identity (B.24)]. To prove it for any positive real, we use [SS98, Identity (1.2)]. This identity states that for a self adjoint with $B \geq 1$ and a positive real β, we have on domain of $B^{[\alpha]+1}$

$$B^{\beta} = \frac{\sin(\pi\{\beta\})}{\pi} \int_0^{+\infty} \frac{w^{\{\beta\}-1}}{B + w} dw B^{[\beta]+1},$$

where $\{\beta\} = \beta - [\beta]$ and $[\beta]$ is the integer part. With this formula for $B = \langle A \rangle^{2k}$ for any $k \in \mathbb{N}$, we prove for any $\beta \in]0,\, 1[$ that

$$\langle A \rangle^{2k\beta} \leq C \langle Q \rangle^{2k\beta}.$$

This ends the proof of Proposition B.8. □

2.2.2 Proof of Theorem B.13

Our proof of Theorem B.13 is an adaptation of the one of [HSS99], we make some modifications.

For any self-adjoint operator B with domain $D(B)$, we write $Ad_A(B)$ for the operator $[A, B]$ with domain $D(A) \cap D(B)$ dense in $D(B)$, defined by

$$\forall u, v \in D(A) \cap D(B),\ \langle \mathrm{i}[A, B]u, v\rangle = \mathrm{i}(\langle Bu, Av\rangle - \langle Au, Bv\rangle).$$

First of all, we have

Lemma B.14. *Suppose that Assumption 1.1 holds. Then $Ad_A^k(H)$ is bounded and can be written as a finite sum of terms of the form*

$$f(P)g(Q)h(P)$$

where f and h are rational fractions with coefficients in $\mathcal{M}_4(\mathbb{C})$ of degree at most 0 with no poles, and g is a function that satisfies, like V, the assumption 1.1.

Proof – The proof is a simple calculation based on the fact that $Ad_{P_j}(f(Q)) = -\mathrm{i}(\nabla_j f)(Q)$. □

We can state the

Lemma B.15 (Mourre estimate)**.** *If V satisfies Assumption 1.1. Then for any $\theta \in (0, 1)$ there exists $\nu \geq 0$ such that one has*

$$\mathbb{1}_{|H|\geq m+\nu}\mathrm{i}[H, A]\mathbb{1}_{|H|\geq m+\nu} \geq \theta\mathbb{1}_{|H|\geq m+\nu}.$$

For any $\lambda \in (-\infty, -m) \cup (+m, +\infty)$, for any $\delta > 0$ there exists $\varepsilon > 0$ such that one has

$$\mathbb{1}_{|H-\lambda|\leq\varepsilon}\mathrm{i}[H, A]\mathbb{1}_{|H-\lambda|\leq\varepsilon} \geq \left(\frac{\lambda^2}{\lambda^2 + m^2} - \delta\right)\mathbb{1}_{|H-\lambda|\leq\varepsilon}.$$

Proof – This is a consequence of $\mathrm{i}[D_m, A] = \frac{-\Delta}{-\Delta+m^2}$ and $[V, A]$ is a compact operator in $B(L^2(\mathbb{R}^3, \mathbb{C}^4))$.

For the last statement, one should ensure that λ is not an eigenvalue. For λ in the continuous spectrum, it follows from [BG87][Theorem 6,Theorem A]. □

We now adapt [HSS99, Theorem 1.1] to the case of unbounded energy since here the multi-commutators $Ad_A^k(H)$ are bounded operators. We need the We introduce the

Definition B.16. *We call generalized indicator function of $\mathbb{R}^-$ a function of the form*

$$x \mapsto e^{\frac{u(x)}{x}}\mathbb{1}_{\mathbb{R}^-}(x)$$

with $u \in \mathcal{C}_0^\infty(\mathbb{R})$ supported in $[-\eta, \eta]$ (for some $\eta > 0$), nonnegative, and such that $u(0) = 1$.

Note that our generalized indicator function of $\mathbb{R}^-$ are of infinite order in the sense of [HSS99, Section 2]. Using the commutators expansion presented in [HS00, Section B] and the Mourre estimate of Lemma B.15, we have the

Lemma B.17. *Suppose V satisfies Assumption 1.1. Let f be a generalized indicator function of $\mathbb{R}^-$ and $g \in \mathcal{C}^\infty(\mathbb{R})$ with support sufficiently far from thresholds $\pm m$ or support sufficiently small in $(-\infty, -m) \cup (+m, +\infty)$. Let $A_s = s^{-1}\{A - a\}$ and $0 < \varepsilon \leq 1$. Then for any $n \in \mathbb{N}$ and $\delta > 0$ there exists a $C > 0$, independent of $a \in \mathbb{R}$, such that, for $s \geq 1$, one has*

$$g(H)i[H, f(A_s)]g(H) \leq s^{-1}\theta g(H)f'(A_s)g(H) + Cs^{-1-\varepsilon}g(H)f^{1-\delta}(A_s)g(H) + Cs^{-(2n-1-\varepsilon)}g^2(H).$$

Proof – See [HSS99, Lemma 2.1], in our case we don't need to replace H by $b(H)H$ with $b \in \mathcal{C}_0^\infty$. Indeed, our commutators $Ad_A^k(H)$ are bounded by means of Lemma B.14. Then we replace the notion of function of order p by the one of generalized indicator function. Finally, we use the fact that a generalized indicator function f satisfies

$$\forall k \in \mathbb{N}, \forall \delta \in (0,1), \exists C > 0, \quad \left| f^{(k)} \right| \leq C \left| f \right|^{1-\delta}.$$

□

We are now able to give the
Proof – [Proof of Theorem B.13] We write χ as a finite sum of function $g_j \in \mathcal{C}^\infty(\mathbb{R})$ with support sufficiently far from thresholds $\pm m$ or support sufficiently small in $(-\infty, -m) \cup (+m, +\infty)$. If we prove the theorem for g_j instead of χ the theorem follows by summing each estimates for g_j since the sum is finite. In the rest of the proof, we will not write the index j of g.

We notice that if $0 < v < \theta - \eta$ and if F is a positive non increasing $\mathcal{C}^\infty$-function which equals 0 on $\mathbb{R}^+$ and 1 on $(-\infty, -\eta)$, we have

$$\mathbb{1}_{(A-a-vs)<0} \leq F\left(\frac{A-a}{s} - \theta\right).$$

Now suppose F is a generalized indicator function of R^-. We consider

$$F(s^{-1}\{A - a - \theta t\})$$

and study the time evolution of the observable $g(H)f(A_{ts})g(H)$, where $f = F^2$, with respect to $e^{-itH}\mathbb{1}_{A-a>0}$. That is to say we study

$$\langle e^{-\text{it}H}\mathbb{1}_{A-a>0}\psi, g(H)f(A_{ts})g(H)e^{-\text{it}H}\mathbb{1}_{A-a>0}\psi\rangle.$$

We work exactly as in the proof of [HSS99, Theorem 1.1]. Hence using Lemma B.17 we obtain for $0 \leq t \leq s$ and $s > 1$

$$\langle e^{-\text{it}H}\mathbb{1}_{A-a>0}\psi, g(H)f(A_{ts})g(H)e^{-\text{it}H}\mathbb{1}_{A-a>0}\psi\rangle \leq Cs^{-(2n-2-\varepsilon)}\|\psi\|^2 + Cs^{-1-\varepsilon}\int_0^t \langle e^{-\text{it}H}\mathbb{1}_{A-a>0}\psi, g(H)f(A_{ts})g(H)e^{-\text{it}H}\mathbb{1}_{A-a>0}\psi\rangle^{1-\delta}||\psi||^{2\delta}.$$

Then using the Gronwall's lemma (see [ABdMG96, Lemma 7.A.1]), we obtain

$$\langle e^{-\text{it}H}\mathbb{1}_{A-a>0}\psi, g(H)f(A_{ts})g(H)e^{-\text{it}D_m}\mathbb{1}_{A-a>0}\psi\rangle \leq \left\{Cs^{-\delta(2n-2-\varepsilon)}\|\psi\|^{2\delta} + \delta Cs^{-\varepsilon}\|\psi\|^{2\delta}\right\}^{1/\delta},$$

so if we choose a small δ and a big n, the proof is done if we choose $s = \max\{1, t\}$. □

3 Proof of Theorem B.4: dispersive estimates

Dispersive estimates for Schrödinger operators with electric potentials take place in Lebesgue spaces. This fact permits to use simple perturbation methods (like Duhamel's formula) to prove the decay estimates for perturbed Schrödinger equations. Unfortunately, we have only been able to prove dispersive estimates for Dirac operators in Besov spaces, so it was not possible for us to use Duhamel's formula or other perturbation method used for Schrödinger operators.

We notice that in the case of a Dirac operator with scalar potentials (matrix valued functions colinear with β), the square of the Dirac equation gives four coupled Klein-Gordon equations with an electrostatic potential. This permits to use results on the Klein-Gordon equation. For example, Yajima [Yaj95] proved dispersive estimates for the Klein-Gordon equation by using wave operators associated with Schrödinger operators including an electrostatic potential. But in the general case, by taking the square of a Dirac operator with a potential, we obtain also a magnetic potential. Hence the method used by Yajima does not work in our case.

To our knowledge the only one study of the dispersive estimates associated with the Dirac equation, is the the work of D'Anconna and Fanelli [DF] for the massless case. For non zero mass we have not been able to found any reference. Even for the free case for which dispersive estimates can be deduced from those of Klein-Gordon equation. Here, to give a sketch of the proof for the general case, we first prove the free case estimates (see Section 3.2), using estimates on oscillatory integrals of Section 3.1. In Section 3.3, following Cuccagna and Schirmer [CS01], we introduce the distorted plane waves. This permits us to tackle the proof of the general case in Section 3.3.2.

3.1 Estimates on some oscillatory integrals

Here, we state some stationary phase type results which will be useful for the rest of the proof. We denote by S^2 the unit sphere of $\mathbb{R}^3$.

Lemma B.18. *Let be $f \in \mathcal{C}^1(S^2)$ and for any $v \in S^2$ and any $k \in \mathbb{R}$ define*

$$J_v(k) = \int_{S^2} e^{\mathrm{i}k\{1-v\cdot\omega\}} f(\omega)\, d\omega.$$

Then we have

$$|J_v(k)| \leq \frac{C}{\langle k\rangle}\left\{\sum_{|\alpha|\leq 1}\int_{S^2}\frac{|\nabla^\alpha f(\omega)|}{|\omega-v|^{|\alpha|}}\,d\omega + \sum_{|\alpha|\leq 1}\int_{S^2}\frac{|\nabla^\alpha f(\omega)|}{|\omega+v|^{|\alpha|}}\,d\omega\right\}, \tag{9}$$

where C does not depend on f, k or v.

If f is in $\mathcal{C}^2(S^2)$ with $f(v) = f(-v) = 0$, we have

$$|J_v(k)| \leq \frac{C}{\langle k\rangle^{3/2}}\left\{\sum_{|\alpha|\leq 1}\int_{S^2}\frac{\left|\nabla^{2\alpha} f(\omega)\right|}{|\omega-v|^{|\alpha|}}\,d\omega + \sum_{|\alpha|\leq 1}\int_{S^2}\frac{\left|\nabla^{2\alpha} f(\omega)\right|}{|\omega+v|^{|\alpha|}}\,d\omega\right\}, \tag{10}$$

where C does not depend on f, k or v.

If f is in $\mathcal{C}^2(S^2)$ and vanishes in a neighborhood of v and $-v$, we have

$$|J_v(k)| \leq \frac{C}{\langle k\rangle^{2}}\left\{\frac{\sum_{|\alpha|\leq 1}\int_{S^2}\frac{\left|\nabla^{2\alpha} f(\omega)\right|}{|\omega-v|^{|\alpha|}}\,d\omega}{\operatorname{dist}(\operatorname{supp}(f), v)} + \frac{\sum_{|\alpha|\leq 1}\int_{S^2}\frac{\left|\nabla^{2\alpha} f(\omega)\right|}{|\omega+v|^{|\alpha|}}\,d\omega}{\operatorname{dist}(\operatorname{supp}(f), -v)}\right\}, \tag{11}$$

where C does not depend on f, k or v.

Proof – We can suppose $v = (0, 0, 1)$ since estimate (9), (10) and (11) are invariant under the action of rotations. We have

$$J_v(k) = \int_0^{2\pi} \int_0^{\pi} e^{\mathrm{i}k\{1-\cos(\phi)\}} f(\theta, \phi) \sin(\phi)\, d\phi d\theta,$$

then we make an integration by parts in ϕ

$$J_v(k) = -\frac{\mathrm{i}}{k} \int_0^{2\pi} \left[e^{\mathrm{i}k\{1-\cos(\phi)\}} f(\theta, \phi) \right]_0^{\pi} d\theta \\ + \frac{\mathrm{i}}{k} \int_0^{2\pi} \int_0^{\pi} e^{\mathrm{i}k\{1-\cos(\phi)\}} \partial_\phi f(\theta, \phi)\, d\phi d\theta,$$

If we suppose that f vanishes in a neighborhood of v or $-v$, then we use that for any ϕ'

$$|f(\theta, \phi')| \leq \int_0^{\pi} |\partial_\phi f(\theta, \phi)|\, d\phi$$

to obtain (9) in this case. Otherwise with help of a smooth cut-off, we split the integral in two parts, each one has a support far from v or $-v$. Repeating the previous proof for each part, we prove the estimate (9) in the general case.

If moreover we have $f(v) = f(-v) = 0$ then we have for any $\alpha > 0$

$$J_v(k) = \frac{\mathrm{i}}{k} \int_0^{2\pi} \int_0^{\alpha} e^{\mathrm{i}k\{1-\cos(\phi)\}} \partial_\phi f(\theta, \phi)\, d\phi d\theta \\ + \frac{\mathrm{i}}{k} \int_0^{2\pi} \int_{\pi-\alpha}^{\pi} e^{\mathrm{i}k\{1-\cos(\phi)\}} \partial_\phi f(\theta, \phi)\, d\phi d\theta \\ + \frac{\mathrm{i}}{k} \int_0^{2\pi} \int_{\alpha}^{\pi-\alpha} e^{\mathrm{i}k\{1-\cos(\phi)\}} \partial_\phi f(\theta, \phi)\, d\phi d\theta.$$

We use an integration by parts to obtain for the second term of the right hand side

$$\int_{\alpha}^{\pi-\alpha} e^{\mathrm{i}k\{1-\cos(\phi)\}} \partial_\phi f(\theta, \phi)\, d\phi = \frac{i}{k} \left[e^{\mathrm{i}k\{1-\cos(\phi)\}} \frac{\partial_\phi f(\theta, \phi)}{\sin(\phi)} \right]_{\alpha}^{\pi-\alpha} \\ - \frac{i}{k} \int_{\alpha}^{\pi-\alpha} e^{\mathrm{i}k\{1-\cos(\phi)\}} \left\{ \frac{\partial_\phi^2 f(\theta, \phi)}{\sin(\phi)} - \frac{\cos(\phi) \partial_\phi f(\theta, \phi)}{\sin(\phi)^2} \right\} d\phi,$$

for the other terms of the right hand side direct estimations give us

$$|J_v(k)| \leq \frac{C\alpha}{|k|} \int_0^{2\pi} \sup_\phi |\partial_\phi f(\theta, \phi)|\, d\theta \\ + \frac{C}{\alpha |k|^2} \int_0^{2\pi} \left\{ \sup_\phi |\partial_\phi f(\theta, \phi)|\, d\theta + \int_0^{2\pi} \int_0^{\pi} \left| \partial_\phi^2 f(\theta, \phi) \right|\, d\phi d\theta \right\},$$

choosing $\alpha = \sqrt{|k|}^{-1}$ and working like in the proof of the estimate (9), we obtain estimate (10). The reader recognized the proof of the well known Van der Corput Lemma with modification in order to give precise estimates.

For the estimate (11), we first split the integral $J_v(k)$ in two hemispheres with respect to the pole v and we choose $\alpha = \mathrm{dist}(\mathrm{supp}(f), v)$ or $\alpha = \mathrm{dist}(\mathrm{supp}(f), -v)$. □

We obtain first the

Proposition B.19. *Let $h \in \mathcal{C}(\mathbb{R})$ and $g \in \mathcal{C}^2(\mathbb{R}^3)$ be such that the integrals appearing in the following estimate are finite. Then defining*

$$I(k,u) = \int_{\mathbb{R}^3} e^{\mathrm{i}k\{h(|\xi|)-\xi\cdot u\}} g(\xi)\, d\xi,$$

for any $u \in \mathbb{R}^3$ and any $k \in \mathbb{R}$, we have

$$|I(k,u)| \le \frac{C}{|ku|} \max_{|\alpha|\le 1} \left\{ \int_{\mathbb{R}^3} |\xi|^{|\alpha|-1} \frac{|\nabla^\alpha g(\xi)|}{\left|\frac{u}{|u|} - \frac{\xi}{|\xi|}\right|^{|\alpha|}}\, d\xi, \int_{\mathbb{R}^3} |\xi|^{|\alpha|-1} \frac{|\nabla^\alpha g(\xi)|}{\left|\frac{u}{|u|} + \frac{\xi}{|\xi|}\right|^{|\alpha|}}\, d\xi \right\} \tag{12}$$

where C does not depend on h, g, k or u.

If moreover g vanishes in a cone of axis $D = \mathrm{Span}(u)$, we have

$$|I(k,u)| \le \frac{C}{|ku|^2 \,\mathrm{dist}(\mathrm{supp}(g)\cap S^2, D \cap S^2)} \times$$
$$\times \max_{|\alpha|\le 1} \left\{ \int_{\mathbb{R}^3} |\xi|^{2|\alpha|-2} \frac{|\nabla^{2\alpha} g(\xi)|}{\left|\frac{u}{|u|} - \frac{\xi}{|\xi|}\right|^{|\alpha|}}\, d\xi, \int_{\mathbb{R}^3} |\xi|^{2|\alpha|-2} \frac{|\nabla^{2\alpha} g(\xi)|}{\left|\frac{u}{|u|} + \frac{\xi}{|\xi|}\right|^{|\alpha|}}\, d\xi \right\}$$

where C does not depend on h, g, k or u.

Proof – We write

$$I(k,u) = \int_{\mathbb{R}^3} e^{\mathrm{i}k\{h(|\xi|)-\xi\cdot u\}} g(\xi)\, d\xi = \int_{\mathbb{R}^+} e^{\mathrm{i}k\{h(\rho)-\rho|u|\}} J_{\frac{u}{|u|},\rho}(\rho k|u|)\rho^2\, d\rho,$$

where $J_{v,\rho}(k) = \int_{S^2} e^{\mathrm{i}k\{1-v\cdot\omega\}} g(\rho\omega)\, d\omega$ and we apply Lemma B.18. □

We introduce a first useful variant with the

Proposition B.20. *Let $g \in \mathcal{C}^{1+k}(\mathbb{R}^3)$ be such that the integrals appearing in the following estimate are finite. We introduce*

$$F(x) = \int_{\mathbb{R}^3} e^{\mathrm{i}\{|\xi||x|-\xi\cdot x\}} g(\xi)\, d\xi$$

for any $x \in \mathbb{R}^3$. Then for all $\alpha \in \mathbb{N}^3$ such that $|\alpha| \le k$, we have

$$|\nabla^\alpha F(x)| \le \frac{C}{|x|^{|\alpha|+1}} \max_{|\beta|\le 1+|\alpha|} \left\{ \int_{\mathbb{R}^3} |\xi|^{|\beta|-1} \frac{|\nabla^\beta g(\xi)|}{\left|\frac{x}{|x|} - \frac{\xi}{|\xi|}\right|}\, d\xi \right\}. \tag{13}$$

If moreover g vanishes in a half cone of axis $D^+ = \left\{\rho\frac{x}{|x|},\ \rho \in \mathbb{R}^+\right\}$. Then for all $\alpha \in \mathbb{N}^3$ such that $|\alpha| \leq k$, we have

$$|\nabla^\alpha F(x)| \leq \frac{C}{|x|^{|\alpha|+2}\mathrm{dist}(\mathrm{supp}(g)\cap S^2, D^+\cap S^2)}\times$$
$$\times \max_{|\beta|\leq 2+|\alpha|}\left\{\int_{\mathbb{R}^3}|\xi|^{|\beta|-2}\frac{|\nabla^\beta g(\xi)|}{\left|\frac{x}{|x|}-\frac{\xi}{|\xi|}\right|}\,d\xi\right\}. \quad (14)$$

Proof – The critical points correspond to the the semi axis spanned by x. We treat the part of the integrals which is far from critical points by using an integration by parts with help of the operator $L = \frac{\frac{\xi}{|\xi|}-\frac{x}{|x|}}{|x|\left|\frac{\xi}{|\xi|}-\frac{x}{|x|}\right|^2}\cdot\nabla_\xi$. Let be $\phi(x,\xi) = \{|\xi||x| - \xi\cdot x\}$, we have

$$F(x) = \langle Le^{\mathrm{i}\phi(x,\cdot)}, g\rangle = \langle e^{\mathrm{i}\phi(\xi,\cdot)}, L^* g\rangle,$$

with

$$L^* = -L - \frac{2}{|x||\xi|\left|\frac{\xi}{|\xi|}-\frac{x}{|x|}\right|^2}.$$

This gives the bound

$$\frac{C}{|x|}\max_{|\beta|\leq 1}\left\{\int_{\mathbb{R}^3}|\xi|^{|\beta|-1}|\nabla^\beta g(\xi)|\,d\xi\right\},$$

or after an iteration

$$\frac{C}{|x|^2}\max_{|\beta|\leq 2}\left\{\int_{\mathbb{R}^3}|\xi|^{|\beta|-1}|\nabla^\beta g(\xi)|\,d\xi\right\},$$

we obtain Estimate (13) for $\alpha = 0$. The method to treat the other part of the integral is exactly the one we used in the proof of Proposition B.19.

For higher order derivatives, we have

$$\nabla_x e^{\mathrm{i}k\phi(x,\xi)} = \frac{|\xi|}{|x|}\nabla_\xi e^{\mathrm{i}k\phi(x,\xi)}$$

and so

$$\langle\nabla_x e^{\mathrm{i}\phi(x,\cdot)}, g\rangle = -\frac{1}{|x|}\langle e^{\mathrm{i}\phi(x,\cdot)}, \nabla|Q|g\rangle.$$

the result is then obtained by applying this trick $|\alpha|$ times and then repeating our proof for the case $\alpha = 0$, we obtain Estimates (13) and (14) for $\nabla^\alpha F(x)$. □

And finally, we need the

Proposition B.21. *Let be $g \in \mathcal{C}^2(\mathbb{R}^3)$ with compact support. Then for any $u \in \mathbb{R}^3$, $k \in \mathbb{R}$ and*

$$I(k,u) = \int_{\mathbb{R}^3} e^{\mathrm{i}k\{\sqrt{\xi^2+m^2}-\xi\cdot u\}}g(\xi)\,d\xi,$$

we have

$$|I(k,u)| \leq \frac{C}{|k|^{3/2}} \max\left[\max_{|\alpha|\leq 2} \left\{ \int_{\mathbb{R}^3} \left| \langle \xi \rangle^{|\alpha|-1} \nabla^\alpha g(\xi) \right| \, d\xi \right\} ; \right.$$
$$\left. \frac{1}{|u| \sqrt{\inf_{x\in \mathrm{supp}(g)} \left\{ \frac{m^2}{\sqrt{x^2+m^2}^3} \right\}}} \max_{\substack{l\leq 1\\ n\leq 1}} \left\{ \int_{\mathbb{R}^3} |\xi|^{l-n-1} \frac{\left|\partial^l_{|\xi|} \partial^n_\omega g(\xi)\right|}{\left|\frac{u}{|u|} - \frac{\xi}{|\xi|}\right|} d\xi \right\} \right]. \quad (15)$$

Proof – We can suppose $u = (0, 0, |u|)$ since estimate (15) is invariant under the action of rotations.

The oscillatory integral $I(k, u)$ is bounded and critical points of the phase of $I(k, u)$ are supported by the semi axis spanned by u. With help of a smooth cut-off function χ , we split the integral in two parts $I(k, u) = I_1(k, u) + I_2(k, u)$, where $I_1(k, u)$ is supported in a half cone around u. We then use multiple integrations by parts with help of the operator

$$L = \frac{\frac{\xi}{\sqrt{\xi^2+m^2}} - u}{|k| \left| \frac{\xi}{\sqrt{\xi^2+m^2}} - u \right|^2} \cdot \nabla_\xi.$$

Since $(1-\chi)\, g \in \mathcal{C}^2(\mathbb{R}^3)$ has support far from critical points and since for $\lambda(\xi) = \sqrt{\xi^2+m^2}$ we have $\left\| \nabla^\alpha_\xi \lambda(\xi) \right\| \leq C_\alpha \lambda(\xi)^{1-|\alpha|}$, we obtain

$$|I_1(k,u)| \leq \frac{C}{|k|} \sum_{|\alpha|\leq 1} \|\lambda(Q)^{|\alpha|-1} \nabla^\alpha g\|_{L^1}$$

and

$$|I_1(k,u)| \leq \frac{C}{k^2} \sum_{|\alpha|\leq 2} \|\lambda(Q)^{|\alpha|-1} \nabla^\alpha g\|_{L^1}.$$

Otherwise $I_2(k, u)$ has support in a small cone around u, and we have

$$\begin{aligned} I_2(k,u) &= \int_{\mathbb{R}^3} e^{\mathrm{i}k\{\sqrt{\xi^2+m^2} - \xi\cdot u\}} \widetilde{g}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^+} e^{\mathrm{i}k\{\sqrt{\xi^2+m^2} - |\xi||u|\}} J_{\frac{u}{|u|},\rho}(\rho k|u|) \rho^2 \, d\rho, \end{aligned}$$

with

$$J_{v,\rho}(k) = \int_{S^2} e^{\mathrm{i}k\{1-v\cdot\omega\}} \widetilde{g}(\rho\omega)\, d\omega,$$

where $\widetilde{g} = \chi g$. We obtain after an integration by parts

$$J_{v,\rho}(k) = -\frac{\mathrm{i}}{k} \int_0^{2\pi} \left[e^{\mathrm{i}k\{1-\cos(\phi)\}} \widetilde{g}(\rho\omega(\theta,\phi)) \right]_0^\pi d\theta$$
$$+ \frac{\mathrm{i}}{k} \int_0^{2\pi} \int_0^\pi e^{\mathrm{i}k\{1-\cos(\phi)\}} \partial_\phi \widetilde{g}(\rho\omega(\theta,\phi))\, d\phi d\theta,$$

Since we assumed $\widetilde{g}$ is supported in half cone around u, we have $\widetilde{g}(\rho\omega(\theta,\pi)) = 0$. Hence we obtain

$$J_{v,\rho}(k) \quad = \quad \frac{\mathrm{i}}{k}\int_0^{2\pi}\int_0^{\pi} \partial_\phi \widetilde{g}(\rho\omega(\theta,\phi))\, d\theta \;+\; \frac{\mathrm{i}}{k}\int_0^{2\pi}\int_0^{\pi} e^{\mathrm{i}k\{1-\cos(\phi)\}}\partial_\phi \widetilde{g}(\rho\omega(\theta,\phi))\, d\phi d\theta,$$

and so

$$\begin{aligned} I_2(k,u) = {} & \frac{\mathrm{i}}{|k|\,|u|}\int_{\mathbb{R}^+}\int_0^{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}k\{\sqrt{\rho^2+m^2}-\rho|u|\}}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\, d\phi d\theta \rho\, d\rho \\ & + \frac{\mathrm{i}}{|k|\,|u|}\int_{\mathbb{R}^+}\int_0^{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\{\sqrt{\rho^2+m^2}-\rho|u|\cos(\phi)\}}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\, d\phi d\theta\rho\, d\rho. \end{aligned} \tag{16}$$

Let us now study the decay resulting from the dispersive behavior of the radial part. To this end, we follow the proof of the well-known Van Der Corput lemma. We study

$$L(k,u,\phi,\phi',\theta) \quad = \quad \int_{\mathbb{R}^+} e^{\mathrm{i}|k|\{\sqrt{\rho^2+m^2}-\rho|u|\cos(\phi')\}}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\rho\, d\rho.$$

Notice that, in view of (16), we are only interested by $L(k,u,\phi,\phi,\theta)$ and $L(k,u,\phi,0,\theta)$. First, for any differentiable function on $\mathbb{R}$ such that $|f'| \geq 1$, we have for any $\alpha \in \mathbb{R}^+$

$$\lambda\left(\{t\in\mathbb{R};\ |f(t)|\leq\alpha\}\right) \leq \alpha, \tag{17}$$

for λ the Lebesgue measure. We introduce

$$h(\rho) = \sqrt{\rho^2+m^2} - \rho|u|\cos(\phi'),$$

and we apply (17) to

$$f(\rho) = \frac{\partial_\rho h(\rho)}{\inf\limits_{x\in\mathrm{supp}(g)}\left\{|\partial^2_{|x|}h(|x|)|\right\}}.$$

We notice that $\partial^2_\rho h(\rho)$ does not depend on u or ϕ'. With help of a smooth cut-off function, we split the integral L in two parts, one has support

$$\left\{\rho\in\mathbb{R}^+;\ |f(\rho)|<\alpha\right\}$$

and the other is its complementary. In fact, we obtain exactly three interval corresponding to

$$\left\{\rho\in\mathbb{R}^+;\ f(\rho)<-\alpha\right\},\ \left\{\rho\in\mathbb{R}^+;\ -\alpha\leq f(\rho)\leq\alpha\right\},\ \left\{\rho\in\mathbb{R}^+;\ \alpha<f(\rho)\right\}.$$

In the first and third interval, we make an integration by parts and in the second interval, we use Estimate (17). Hence we obtain the bound

$$\begin{aligned} & \left|L(k,u,\phi,\phi',\theta)\right| \leq \\ & \quad \max\left\{\alpha,\ \frac{1}{\alpha|k|\inf\limits_{x\in\mathrm{supp}(\widetilde{g})}\left\{|\partial^2_{|x|}h(|x|)|\right\}}\right\}\sup_{\rho\in\mathbb{R}^+}\left\{\rho\,|\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))|\right\} \\ & \qquad + \frac{1}{\alpha|k|\inf\limits_{x\in\mathrm{supp}(\widetilde{g})}\left\{|\partial^2_{|x|}h(|x|)|\right\}}\times \\ & \qquad \times\max\left\{\int_{\mathbb{R}^+}|\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))|\ d\rho,\ \int_{\mathbb{R}^+}\rho\,|\partial_\rho\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))|\ d\rho\right\}. \end{aligned}$$

We use

$$\rho\,|\partial_\phi \widetilde{g}(\rho\omega(\theta,\phi))| \leq 2\max\left\{\int_{\mathbb{R}^+} |\partial_\phi \widetilde{g}(r\omega(\theta,\phi))|\;dr,\ \int_{\mathbb{R}^+} r\,|\partial_\rho\partial_\phi \widetilde{g}(r\omega(\theta,\phi))|\;dr\right\}$$

and then choose

$$\alpha = \frac{1}{\sqrt{|k| \inf\limits_{x\in \mathrm{supp}(\widetilde{g})} \left\{|\partial^2_{|x|} h(|x|)|\right\}}},$$

and plugging the resulting estimates for $\phi' = 0$ and $\phi' = \phi$ in (16), we obtain estimate (15). □

3.2 Dispersive estimates for the free case equation

Thanks to the tools introduced in Section 3.1, we are able to state the

Theorem B.22 (Dispersive estimates for free Dirac operator)**.** *For any $p \in [1,2]$, for all $\theta \in [0;1]$, for all $s, s' \in \mathbb{R}$, such that $s - s' \geq (\frac{2}{p} - 1)(2+\theta)$ and any $q \in [1,\infty]$, we have*

$$\|e^{-\mathrm{i}tD_m}\|_{B^s_{p,q}, B^{s'}_{p',q}} \leq (K(t))^{\frac{2}{p}-1},$$

with

$$K(t) = \begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \in [0,1], \\ |t|^{-1-\theta/2} & \text{if } |t| \in [1,\infty), \end{cases}$$

and $p' = \frac{p}{p-1}$.

Proof – The proof is a straightforward adaptation of the one of Brenner in Appendix 2 of [Bre85] for the Klein-Gordon equation. We give a sketch of the proof for the reader's convenience. Note that the proof of Theorem B.4, for non zero potential, is based on the same ideas.

We only need to prove the case $p = 1$, since the general case follows by interpolation of the case $p = 1$ and the charge conservation which corresponds to the case $p = 2$. Then using $D_m = \sqrt{-\Delta + m^2}(\pi_+ - \pi_-)$ with $\pi_\pm = \mathbb{1}_{\mathbb{R}^\pm}(D_m) = \frac{1}{2}\left\{1 \pm |D_m|^{-1}D_m\right\}$, we obtain the estimates from those relative to the relativistic Schrödinger operator $\sqrt{-\Delta + m^2}$:

$$\|e^{-\mathrm{i}t\sqrt{-\Delta+m^2}}\|_{B^s_{1,q}, B^{s'}_{\infty,q}} \leq K(t)$$

which in turn follow from

Proposition B.23. *For any $\chi \in \mathcal{D}(\mathbb{R}^3, \mathbb{C}^4)$, we define $\chi_j(x) = \chi(2^{-j}|x|)$. Then for $\theta' \in [0,1]$, we have:*

1. *if $0 \notin \mathrm{supp}(\chi)$,*

$$\|e^{-\mathrm{i}t\sqrt{-\Delta+m^2}}\chi_j\left(\sqrt{-\Delta+m^2}\right)\|_{L^1, L^\infty} \leq C2^{(2+\theta')j}|t|^{-(1+\theta'/2)} \tag{18}$$

where C is independent of t and j;

2. *if* $0 \in \operatorname{supp}(\chi)$,

$$\|e^{-\mathrm{i}t\sqrt{-\Delta+m^2}}\chi(-\Delta+m^2)\|_{L^1,L^\infty} \leq C\langle t\rangle^{-3/2} \tag{19}$$

where C *is independent* t.

We postpone the proof of Proposition B.23 until the end of the proof of Theorem B.22.

We have

$$\left\|e^{-\mathrm{i}t\sqrt{-\Delta+m^2}}\chi_j(\sqrt{-\Delta+m^2})f\right\|_\infty \leq C2^{3j}\|f\|_1$$

interpolating with Estimate (18) of Proposition B.23 for $\theta' = 0$ when $t \leq 1$ and using Estimate (18) with $\theta' = \theta$ for $t \geq 1$, one obtains

$$2^{js'}\left\|e^{-\mathrm{i}t\sqrt{-\Delta+m^2}}\chi_j(\sqrt{-\Delta+m^2})f\right\|_\infty \leq C\kappa_j(t)2^{js}\|f\|_1$$

with

$$\kappa_j(t) = 2^{j(2+\theta+s'-s)}\begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \leq 1, \\ |t|^{-1-\theta/2} & \text{if } |t| \geq 1. \end{cases}$$

We use

$$\sup_{j\in\mathbb{N}} \kappa_j \leq CK(t)$$

if $2+\theta \leq s-s'$ and Estimate (19) to prove Theorem B.22. Hence to conclude the proof, we need to give the

Proof – [Proof of Proposition B.23] Estimates of the same type, but for $\mathcal{B}\left(L^p, L^{p'}\right)$ spaces with $p \in [4/3, 2]$ can be found in [MSW80, MSW79]. In the present case $p = 1$ a proof can be found in [Bre77]. This proof, which covers a much more general situation, is quite complicated. We propose here a simpler proof inspired by [CS01]. The kernel of $e^{-\mathrm{i}t\sqrt{-\Delta+m^2}}\chi_j(\sqrt{-\Delta+m^2})$ is given by $(2\pi)^{-3/2}K_j(x-y)$ where

$$K_j(x,t) = \int_{\mathbb{R}^3} e^{-\mathrm{i}t\sqrt{\xi^2+m^2}+x\cdot\xi}\chi_j(\sqrt{\xi^2+m^2})\,d\xi$$

Hence, we estimate the L^∞ norm of K_j.

If $|x|/|t| \ll 2^{j-1}/\sqrt{2^{2j-2}+m^2}$ or $|x|/|t| \gg 1$, we use non stationary phase lemma in $\mathbb{R}^3$ with help of the operator $L = \dfrac{\frac{\xi}{\sqrt{\xi^2+m^2}}-x}{t\left|\frac{\xi}{\sqrt{\xi^2+m^2}}-x\right|^2}\cdot\nabla$. Hence, in this case, we obtain the estimate

$$\left|\int_{\mathbb{R}^3} e^{-\mathrm{i}t\sqrt{\xi^2+m^2}+x\cdot\xi}\chi_j(\sqrt{\xi^2+m^2})\,d\xi\right| \leq C_n 2^{-(n-3)j}|t|^{-n},$$

for any $n \in \mathbb{N}$. Otherwise, we apply Proposition B.19 with $h(r) = \sqrt{r^2+m^2}$, $k = t$, $u = x/t$ and $g(x) = \chi_j(|x|)$. So if $0 \notin \operatorname{supp}(\chi)$, to obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} e^{-\mathrm{i}t\sqrt{\xi^2+m^2}+x\cdot\xi}\chi_j(\sqrt{\xi^2+m^2})\,d\xi\right| \\ &\leq \frac{C}{|t|}\max_{|\beta|\leq 1}\int_{\mathbb{R}^3}|\xi|^{|\beta|-1}\frac{\left|\nabla^\beta\chi_j(\sqrt{\xi^2+m^2})\right|}{\left|\frac{x}{|x|}\pm\frac{\xi}{|\xi|}\right|}d\xi \\ &\leq \frac{C2^{2j}}{|t|}.\end{aligned}$$

Notice that in this case, $|x|/|t| \geq c' > 0$. If instead of Proposition B.19, we use Proposition B.21 with $g = \chi_j$, $k = t$ and $u = x/t$, we prove the estimate

$$|K_j(x,t)| \leq \frac{C2^{3j}}{|t|^{3/2}}.$$

The estimate (18) is then obtained by interpolation. For (19), we use the classical stationary (Morse lemma) and non-stationary phase methods (integration by parts) in $\mathbb{R}^3$. For more details about the method one can look at the end of the proof of Proposition B.30. This ends the proof of B.23. □

This ends the proof of Theorem B.22. □

3.3 Distorted Plane Waves

Our aim is now to generalize the previous method to the perturbed case. Let us introduce the wave operators

$$W^{\pm} = \lim_{t\to\pm\infty} e^{\mathrm{i}t(D_m+V)}e^{-\mathrm{i}tD_m} \tag{20}$$

(for the existence and the completeness : $\mathrm{Ran}(W^{\pm}) = \mathrm{Ran}(\mathbf{P}_c(H))$ of these operator see [GM01, Theorem 1.5]). With the intertwining property

$$f(H)W^{\pm} = W^{\pm}f(D_m), \tag{21}$$

for any bounded borelian function f, and Fourier transform $\mathcal{F}$, we shall obtain for $h(\xi) = \alpha\cdot\xi + m\beta$

$$e^{-\mathrm{i}tH}\mathbf{P}_c(H) = W^{\pm}e^{-\mathrm{i}tD_m}W^{\pm *} = W^{\pm}\mathcal{F}e^{-\mathrm{i}th(Q)}\left(W^{\pm}\mathcal{F}\right)^*.$$

So we can adapt the previous method if we are able to prove some estimates about the kernel ψ_V of $W^{\pm}\mathcal{F}$. The kernel ψ_V is called *distorted plane wave*. We notice that ψ_V is a 4×4 matrix valued function.

We will show that the previous method works with $\psi_V\psi_V^*\chi_j$ in place of χ_j with small modifications. So we need estimates on ψ_V. Generally, distorted plane waves are studied like perturbations of free plane waves. So we will prove estimates on the perturbative part, written w in the sequel.

3.3.1 Definition and properties

We need to introduce the free plane wave. Let

$$M(k) = \sqrt{k^2+m^2}\beta$$

for any $k \in \mathbb{R}^3$.We introduce the vectors

$$\psi_0^j(k,x) = e^{\mathrm{i}k\cdot x}u(k)e_j$$

where

$$u(k) = \frac{(m+\lambda(k))Id - \beta\alpha\cdot k}{\sqrt{2\lambda(k)(m+\lambda(k))}} \tag{22}$$

with $\lambda(k) = \sqrt{k^2 + m^2}$ and $(e_j)_j$ are vectors of the canonical basis of $\mathbb{C}^4$.

By definition, a distorted plane wave is a solution of the PDO equation

$$(D_m + V)\,\psi = \pm\sqrt{k^2 + m^2}\psi \tag{23}$$

with for some j and any $k \in \mathbb{R}^3$, $\psi(k,x) - \psi_0^j(k,x)$ tending to zero as x goes to infinity (in some sense), see [Agm75, section 5].

A solution of (23) is a function $\psi(k,x)$ of two variables here k is a 3-dimensional vector which is called the wave vector. A free plane wave ψ_0^j satisfies the PDO equation (23) in the case $V = 0$, for more details see [Tha92, Section 1.4, Section 1.F]. Following [Agm75], we introduce two families of function

$$\psi_V^j(k,x) = \psi_0^j(k,x) - \left\{R_V^+(\lambda(k))V(\cdot)\psi_0^j(k,\cdot)\right\}(x)$$

for $j \in \{1,2\}$ and

$$\psi_V^j(k,x) = \psi_0^j(k,x) - \left\{R_V^+(-\lambda(k))V(\cdot)\psi_0^j(k,\cdot)\right\}(x)$$

for $j \in \{3,4\}$. The rest of the proof works also for R_V^- instead of R_V^+ (the trace of the resolvent $R_V^\pm$ was introduced in (5)).

In case there is no resonance at thresholds and no eigenvalue at thresholds, Theorem B.3 gives us that $R_V^+(\lambda(p))$ is in $B(L^2_\sigma, L^2_{-\sigma})$ for any $\sigma > 5/2$, this also work if $\sigma \geq 1$ see Proposition B.33 below. So the previous definition make sense if Assumption 1.1 holds and we have the

Proposition B.24. *Suppose that Assumptions 1.1 and 1.2 hold. Then for any $k \in \mathbb{R}^3 \setminus \{0\}$, $\psi_V^j(k,x)$ satisfies equation* (23)*.*

Distorted plane waves define a generalized Fourier transform. We introduce $\psi_V(k,x) \in \mathcal{M}_4(\mathbb{C})$ the matrix with vector column $\psi_V^j(k,x)$ and we define

$$(\mathcal{F}_V f)(k) = \int_{\mathbb{R}^3} \psi_V(k,x) f(x)\,dx,$$

which is *a priori* defined on the Schwartz space $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$ but will be extended to L^2.

Distorted plane waves are also called generalized eigenfunctions, since they correspond to "eigenvalues" associated with the continuous spectrum. Indeed, we can prove the

Theorem B.25 (Eigenfunction Expansion)**.** *Suppose that Assumptions 1.1 and 1.2 hold. Then the operator $\mathcal{F}_V$ defines a bounded linear map from L^2 into itself. Its kernel is given by the the sum of the eigenspaces of H. Moreover it is a unitary map from $\mathbf{P}_c(H)L^2$ onto $L^2(\mathbb{R}^3)$ with*

$$(\mathcal{F}_V^* f)(x) = \lim_{n\to\infty} \int_{K_n} \psi_V(k,x)^* f(k)\,dk,$$

for any $(K_n)_{n\in\mathbb{N}}$ a family of compact sets with $K_n \subset K_{n+1}$ and $\cup_{n\in\mathbb{N}} K_n = \mathbb{R}^3$. Finally, for any interval $I \subset \mathbb{R}$, one has

$$\|\mathbb{1}_I(H)f\|^2 = \int_{\sigma(h(k))\cap I\neq\emptyset} |\mathcal{F}_V f|^2\,dk \tag{24}$$

where $\sigma(h(k))$ is the spectrum of $h(k)$.

Proof – The proof is an easy adaptation of the proof of [Agm75, Theorem 6.2] (see also [RS79, Theorem XI.41]), the main difference is that here we insert the unitary matrix u defined in (22). Formula (24) is nothing more than an adaptation of [Agm75, Formula (6.6)] or [RS79, Formula 82e']. □

We also have the

Lemma B.26 (Intertwining Property)**.** *Suppose that Assumptions 1.1 and1.2 hold. Then let g be a bounded borelian function with support in $\mathbb{R} \setminus (-m, m)$, we have*

$$\mathcal{F}_V g(H) = (g \circ M)\mathcal{F}_V. \tag{25}$$

Proof – Using (24), we obtain that (25) is true for $g = \mathbb{1}_I$ with I an interval of $\mathbb{R}\setminus(-m, m)$. We then obtain it for bounded borelian function with support in $\mathbb{R} \setminus (-m, m)$, usual density arguments and properties of functional calculus. More precisely, we use the fact that a bounded sequence of borelian functions which converges everywhere gives a sequence of bounded operators which converge strongly. □

Hence we deduce that, for any $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, the kernel of $e^{-\mathrm{i}tH}\chi(H)$ is given by

$$\left[e^{-\mathrm{i}tH}\chi(H)\mathbf{P}_c(H)\right](x, y) = \int_{\mathbb{R}^3} \psi_V(k, x)^* e^{-\mathrm{i}tM(k)}\chi(M(k))\psi_V(k, y)\, dk.$$

which exactly means

$$e^{-\mathrm{i}tH}\chi(H)\mathbf{P}_c(H) = (\mathcal{F}_V)^* e^{-\mathrm{i}tM}\chi(M)\mathcal{F}_V \tag{26}$$

We recall that we want to prove the decay of $e^{-\mathrm{i}tH}\chi(H)$ as $t \to +\infty$ in some Besov spaces. We observe that

$$e^{-\mathrm{i}tM(k)}\chi(M(k)) = e^{-\mathrm{i}t\lambda(k)}\chi(\lambda(k))P_+ + e^{\mathrm{i}t\lambda(k)}\chi(-\lambda(k))P_-,$$

where P_+ (resp. P_-) is the projector associated with the positive (resp. negative) part of the spectrum of $M(k)$, *i.e.*

$$P_\pm = \frac{1}{2}(1 \pm \beta).$$

Hence, in the following we study the functions

$$(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \int_{\mathbb{R}^3} e^{\mp \mathrm{i}t\lambda(k)} \left(P_\pm \psi_V(k, x)\right)^* \left(P_\pm \psi_V(k, y)\right) \chi(h(k))\, dk.$$

3.3.2 End of the proof of Theorem B.4

We now prove Theorem B.4 with help of three propositions which will be proven in Section 3.3.3. These propositions give some estimates on the perturbed part of the distorted plane wave. Following Cuccagna and Schirmer in [CS01], we write $\psi_V(k, x) = e^{\mathrm{i}k\cdot x}(u(k) + w(k, x))$ where w is the perturbation part which satisfies

$$w(k, x)_j = \begin{cases} e^{-\mathrm{i}k\cdot x}\left\{R_V^+(+\lambda(k))\left\{V e^{\mathrm{i}k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if} \quad j \in \{1, 2\}, \\ e^{-\mathrm{i}k\cdot x}\left\{R_V^+(-\lambda(k))\left\{V e^{\mathrm{i}k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if} \quad j \in \{3, 4\}, \end{cases} \tag{27}$$

and we now state our propositions.

Proposition B.27. *Suppose that Assumptions 1.1 and 1.2 hold. Then there exists $C > 0$ such that for any k, $x \in \mathbb{R}^3 \setminus \{0\}$, and any $\beta \in \mathbb{N}^3$ with $|\beta| \leq 1$, one has*

$$\left|\nabla_k^\beta w(k,x)\right| \leq \frac{C}{\langle k\rangle^{|\beta|}} \frac{\langle x\rangle^{|\beta|}}{\left\langle |x| \left|\frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle}. \tag{28}$$

Moreover, one has

$$|\nabla_k w(k,x)| \leq C \frac{\langle \min\{|x|,\ |k|\}\rangle}{\langle k\rangle \left\langle \min\{|x|,\ |k|\} \left|\frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle^2}. \tag{29}$$

We use this to prove the time decay in $|t|^{-1}$. Unfortunately this doesn't work for the $|t|^{-3/2}$ decay, hence we then study

$$v(k,x)_j = \begin{cases} e^{\mathrm{i}|k||x|}\left\{R_V^+(+\lambda(k))\left\{Ve^{\mathrm{i}k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if } j \in \{1,\ 2\},\\ e^{-\mathrm{i}|k||x|}\left\{R_V^+(-\lambda(k))\left\{Ve^{\mathrm{i}k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if } j \in \{3,\ 4\},\end{cases} \tag{30}$$

and

$$\widetilde{v}(k,x)_j = \begin{cases} e^{\mathrm{i}|k||x|}\left\{\nabla_k R_V^+(+\lambda(k))\left\{Ve^{\mathrm{i}k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if } j \in \{1,\ 2\},\\ e^{-\mathrm{i}|k||x|}\left\{\nabla_k R_V^+(-\lambda(k))\left\{Ve^{\mathrm{i}k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if } j \in \{3,\ 4\}.\end{cases} \tag{31}$$

One has the

Proposition B.28. *Suppose that Assumptions 1.1 and 1.2 hold. Then if $\rho > 3 + |\beta|$ for some $\beta \in \mathbb{N}^3$, there exists $C > 0$ such that for any k, $x \in \mathbb{R}^3 \setminus \{0\}$, one has*

$$|\nabla_k^\beta v(k,x)| \leq \frac{C}{\langle |x| \left|\frac{x}{|x|} - \frac{k}{|k|}\right|\rangle}.$$

Proposition B.29. *Suppose that Assumptions 1.1 and 1.2 hold. Then if $\rho > 3 + |\beta|$ for some $\beta \in \mathbb{N}^3$, there exists $C > 0$ such that for any k, $x \in \mathbb{R}^3 \setminus \{0\}$, one has*

$$|\nabla_k^\beta \widetilde{v}(k,x)| \leq C \frac{\langle \min\{|x|,\ |k|\}\rangle}{\langle k\rangle \left\langle \min\{|x|,\ |k|\} \left|\frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle^2}.$$

Using Proposition B.27, B.28 and B.29 (which are proved in Section 3.3.3 below), let us prove the following

Proposition B.30. *Suppose that Assumptions 1.1 and 1.2 hold. Then for $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, with support in $\mathbb{R} \setminus [-m; m]$, for any $\theta \in [0,1]$ and $j \in \mathbb{N}$, we have*

$$\left\|e^{-\mathrm{i}tH}\chi(2^{-j}H)\right\|_{L^1 \to L^\infty} \leq \frac{C 2^{(2+\theta)j}}{|t|^{1+\theta/2}}, \tag{32}$$

with C independent of t and j.

For $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, for any $\theta \in [0,1]$, we also have

$$\left\|e^{-\mathrm{i}tH}\chi(H)\mathbf{P}_c(H)\right\|_{L^1 \to L^\infty} \leq \frac{C}{|t|^{1+\theta/2}}, \tag{33}$$

with C independent of t.

Proof – The proof works like the one of Proposition B.23 with some modifications due to the fact that high derivatives in k of $w(k,x)$ grow with respect to x.

We need the L^∞ norm of the kernel of $e^{-\mathrm{i}tH}\chi(2^{-j}H)$. This kernel thanks to (26) is given by

$$\begin{aligned} I_j(t,x,y) = \int_{\mathbb{R}^3} & e^{-\mathrm{i}t\sqrt{\xi^2+m^2}} e^{-i\xi\cdot(x-y)} \{P_+(u^*(\xi)+w^*(x,\xi))\} \times \\ & \times \{P_+(u(\xi)+w(y,\xi)\}\,\chi(2^{-j}\lambda(\xi))\,d\xi \\ + \int_{\mathbb{R}^3} & e^{+\mathrm{i}t\sqrt{\xi^2+m^2}} e^{-i\xi\cdot(x-y)} \{P_-(u^*(\xi)+w^*(x,\xi))\} \times \\ & \times \{P_-(u(\xi)+w(y,\xi)\}\,\chi(-2^{-j}\lambda(\xi))\,d\xi. \end{aligned}$$

We notice that if we expand each integrand in terms of u and w, we obtain the sum of the integrals

$$\begin{aligned} & I_j^{\pm}[z,z'](t,x,y) \\ & \quad = \int_{\mathbb{R}^3} e^{\mp\mathrm{i}t\sqrt{\xi^2+m^2}} e^{-\mathrm{i}\xi\cdot(x-y)} \left\{P_\pm z^*(x,\xi)P_\pm)z'(y,\xi)\right\} \chi(\pm 2^{-j}\lambda(\xi))\,d\xi. \end{aligned}$$

with $z,z' \in \{u,w\}$. We notice that $I_j^+[u,u](t,x,y) + I_j^-[u,u](t,x,y)$ is the kernel of $e^{-itD_m}\chi_j(D_m)$, hence we only treat the other integrals.

For the $|t|^{-1}$ decay, if $|x-y|/|t| \ll 2^{j-1}/\sqrt{2^{2j-2}+m^2}$ or $|x-y|/|t| \gg 1$, the phase has no critical point. We use an integration by parts in $\mathbb{R}^3$ with help of the operator

$$L = \frac{\left(\frac{\xi}{\sqrt{\xi^2+m^2}} - x\right)}{t\left|\frac{\xi}{\sqrt{\xi^2+m^2}} - x\right|^2} \cdot \nabla_\xi.$$

So with the estimate (28) of Proposition B.27 and with

$$\left|\partial_i \frac{\xi}{\sqrt{\xi^2+m^2}}\right| \le \frac{C}{|\xi|},$$

we obtain the estimate

$$\left|I_j^{\pm}[z,z'](t,x,y)\right| \le C2^{2j}|t|^{-1},$$

with C independent of j and t.

Otherwise if $|x-y|/|t| \ge c > 0$, using first (12) of Proposition B.19 and then (29) of Proposition B.27, we infer

$$\left|I_j^{\pm}[z,z'](t,x,y)\right| \le C2^{2j}|t|^{-1},$$

with C independent of j and t.

For the $|t|^{-3/2}$ decay, first if $|x-y|/|t| \ge c > 0$, we write

$$I_j^{\pm}[z,z'](t,x,y) = \int_{\mathbb{R}^+} e^{\mp\mathrm{i}t\sqrt{\rho^2+m^2}-\mathrm{i}\rho|x-y|} J_{\frac{x-y}{|x-y|}}(\rho|x-y|)\,\rho^2 d\rho.$$

where

$$J_v(k) = \int_{S^2} e^{\mathrm{i}k(1-\omega\cdot v)} \left\{P_\pm z^*(x,\rho\omega) P_\pm z'(y,\rho\omega)\right\} \chi(\pm 2^{-j}\lambda(\rho\omega))\, d\omega.$$

We can suppose $v = (0;0;1)$ and so

$$J_v(k) = \int_0^{2\pi}\int_0^{\pi} e^{\mathrm{i}k(1-\cos(\phi))}\Big\{P_\pm z^*(x,\rho\omega(\theta,\phi))\times \\ \times P_\pm z'(y,\rho\omega(\theta,\phi))\Big\}\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi)))\,\sin(\phi) d\phi d\theta.$$

An integration by parts in ϕ gives

$$\begin{aligned} J_v(k) = \frac{1}{\mathrm{i}k}\int_0^{2\pi} &\left[e^{\mathrm{i}k(1-\cos(\phi))}\Big\{P_\pm z^*(x,\rho\omega(\theta,\phi))\times \right.\\ &\left.\times P_\pm z'(y,\rho\omega(\theta,\phi))\Big\}\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi)))\right]_0^\pi d\theta \\ &- \frac{1}{\mathrm{i}k}\int_0^{2\pi}\int_0^{\pi} e^{\mathrm{i}k(1-\cos(\phi))}\partial_\phi\Big\{z^*(x,\rho\omega(\theta,\cdot))\times \\ &\times P_\pm z'(y,\rho\omega(\theta,\cdot))\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\cdot)))\Big\}(\phi)\, d\phi d\theta. \end{aligned}$$

The integrand of the first term can be rewritten in order to obtain a sum of two integral in ϕ over the interval $[0,\pi]$. To this end, we introduce a smooth cut-off function which splits $[0,\pi]$ in two parts one is a neighborhood of 0 and the other a neighborhood of π. Then most of the terms obtained after derivation can be treated by the method used for the $|t|^{-1}$ decay. Only the two terms where derivatives of z and z' appear need a particular treatment. Now we have to distinguish the case $z = z' = w$ from the two others where $z = u$ or $z' = u$. If $z = z' = w$, the terms which need a particular treatment are bounded by $C|t|^{-1}$ times the supremum in ϕ' of the $L^1_{\phi,\,\theta}([0,\pi]\times[0,2\pi])$ of

$$L^\pm_{j,n,m}(t,x,y,\phi,\phi') = \int_{\mathbb{R}^+} e^{\mp\mathrm{i}t\sqrt{\rho^2+m^2}-\mathrm{i}\rho|x-y|\cos(\phi')} \left\{P_\pm \partial_\phi^n z^*(x,\rho\omega)\right\}\times \\ \times \left\{P_\pm \partial_\phi^m z'(y,\rho\omega(\theta,\phi))\right\}\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi)))\,\rho d\rho,$$

with $n,m\in\mathbb{N}$ such that $n+m=1$. It is a sum of terms of the form

$$\int_{\mathbb{R}^+} e^{\mathrm{i}t\left\{\mp\sqrt{\rho^2+m^2}-\rho\frac{|x-y|}{t}(\cos(\phi')-\cos(\phi))-\varepsilon_i\rho\frac{|x|}{t}+\varepsilon_{i'}\rho\frac{|y|}{t}\right\}}\times \\ \times\left\{(P_\pm)_{k,i}\left(e^{-\mathrm{i}\psi(k,x)}\partial_\phi^n z^*(x,\rho\omega)\right)_{i,l}\right\}\times \\ \times\left\{(P_\pm)_{l,k'}\left(\partial_\phi^m z'(y,\rho\omega(\theta,\phi))e^{\mathrm{i}\psi(k,y)}\right)_{k',i'}\right\}\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi)))\,\rho d\rho.$$

where ϕ is the angle between $\frac{x-y}{|x-y|}$ and the $z-$axis and $\psi(x,k)\in\mathcal{M}_4(\mathbb{C})$ is given by

$$\begin{pmatrix} (|x||k|+x\cdot k)\, I_{\mathbb{C}^2} & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & (-|x||k|+x\cdot k)\, I_{\mathbb{C}^2}\end{pmatrix},$$

and $\varepsilon_i, {\varepsilon_i}'\in\{\pm 1\}$. We introduce

$$K(\rho) = \left\{(P)_{k,i}\left(e^{-\mathrm{i}\psi(k,x)}\partial_\phi^n z^*(x,\rho\omega)\right)_{i,l}\right\}$$

and

$$\left\{ (P_\pm)_{l,k'} \left(\partial_\phi^m z'(y, \rho\omega(\theta,\phi)) e^{\mathrm{i}\psi(k,y)} \right)_{k',i'} \right\},$$

$$\phi(\rho) = \mp\sqrt{\rho^2+m^2} - \rho\frac{|x-y|}{t}\left(\cos(\phi') - \cos(\phi)\right) - \varepsilon_i \rho \frac{|x|}{t} + \varepsilon_{i'}\rho\frac{|y|}{t}$$

and

$$f(\rho) = \frac{\partial_\rho \phi(\rho)}{\inf\limits_{x\in \mathrm{supp}(\chi_j)} \left\{ |\partial_\rho^2 \phi(|x|,\lambda)| \right\}}.$$

With help of a smooth cut-off function, we split the integral in two parts. One has support $\{t\in\mathbb{R};\, |f(t)|\le\alpha\}$ on which we use the estimate

$$\lambda\left(\{t\in\mathbb{R};\, |f(t)|\le\alpha\}\right) \le \alpha,$$

for λ the Lebesgue measure, since $|f'|>1$. The other is its complementary, in which we make an integration by parts. We obtain the estimate

$$\begin{aligned}\left| J_j^+[r,r'](t,x,y) \right| \le C\alpha \sup_{\rho\in A_j}\{\rho\, |K(\rho)|\} + \frac{1}{\alpha t \inf\limits_{\rho\in A_j}\left\{|\partial_\rho^2\phi(\rho)|\right\}} \times \\ \times \max\left\{ \int_{A_j} \{\rho\, |(\partial_\rho K)(\rho)|\}\, ;\ \int_{A_j} |K(\rho)|\, d\rho;\ 2^{-j}\int_{A_j}\{\rho\, |K(\rho)|\} \right\},\end{aligned}$$

where $A_j = g^{-1}\{\mathrm{supp}(\chi_j)\}$ with $g(\rho)=\sqrt{\rho^2+m^2}$. Hence with (29) of Proposition B.27 and Proposition B.29, we can choose $\alpha = 2^{2j}\sqrt{t}^{-1}$ and we obtain the bound of (32) in this case.

For the case $(z,z') = (u,w)$ (the case $(z,z')=(w,u)$ is similar), we study by the same way the integral

$$\begin{aligned}\int_{\mathbb{R}^+} e^{\mathrm{i}t\left\{\mp\sqrt{\rho^2+m^2} - \rho\frac{|x-y|}{t}\left(\cos(\phi')-\cos(\widetilde{\phi})\right) - \varepsilon_{i'}\rho\frac{|y|}{t}\right\}} \Bigg\{ \left\{ (P_\pm)_{k,i} \left(\partial_\phi^n z^*(x,\rho\omega)\right)_{i,l} \right\} \times \\ \times \left\{ (P_\pm)_{l,k'} \left(\partial_\phi^m z'(y,\rho\omega(\theta,\phi)) e^{\mathrm{i}\psi(k,y)} \right)_{k',i'} \right\} \times \\ \times \chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi))) \Bigg\} \rho d\rho,\end{aligned}$$

where $\widetilde{\phi}$ is the angle between $\frac{y}{|y|}$ and the $z-$axis.

If $|x-y|/|t| \ll 1$, we can suppose $|x-y|/|t| < |\xi|/(2\sqrt{\xi^2+m^2})$ for any $\xi\in\mathrm{supp}(\chi_j)$ and instead of applying the trick of the proof of Lemma B.18 (integration by parts with respect to an angular variables) to the integral $I_j^\pm[z,z'](t,x,y)$, we make an integration by parts with help of

$$\frac{\partial_{|\xi|}}{\frac{|\xi|}{\sqrt{\xi^2+m^2}} \pm \frac{\xi}{|\xi|}\cdot\frac{x-y}{t}}.$$

The rest of the proof is the same.

We now turn to the proof of estimate (33), the kernel of the operator is given by a sum of terms of the form

$$I_j^{\pm}[z,z'](t,x,y) = \int_{\mathbb{R}^3} e^{\mp \mathrm{i}t\sqrt{\xi^2+m^2}} e^{-\mathrm{i}\xi\cdot(x-y)} \left\{P_{\pm} z^*(x,\xi) P_{\pm} z'(y,\xi)\right\} \chi(\pm\lambda(\xi))\, d\xi.$$

We first notice that Proposition B.27 implies that this integral is bounded. Then we split the integral in two parts. One is supported in a small neighborhood of the critical point of the phase, the other is its complementary. To treat this last integral we work exactly like the case "$\frac{|x-y|}{t} \ll 1$", just mentioned above. For the other one, we apply the Morse lemma to reduced the study to

$$\int_{\mathbb{R}^3} e^{\mp \mathrm{i}t\xi^2} \left\{P_{\pm} z^*(x,f(\xi)) P_{\pm} z'(y,f(\xi))\right\} \widetilde{\chi}(f(\xi))\, d\xi = \int_{S^2}\int_{\mathbb{R}^+} \rho e^{\mp \mathrm{i}t\rho^2} \Big\{ P_{\pm} z^*(x,f(\rho\omega)) \times \times P_{\pm} z'(y,f(\rho\omega)) \Big\} \widetilde{\chi}(f(\rho\omega))\, d\rho d\omega,$$

where $\widetilde{\chi}$ is the product of an indicator of a small neighborhood of the critical point with $\chi(\pm\lambda(\cdot))$. Then an integration by parts in ρ and the Van Der Corput lemma give (33) when $\theta = 1$. Since we have that the integral is bounded the general case easily follows. □

We are now able to write the proof of Theorem B.4, using Proposition B.30.

Proof – [Proof of Theorem B.4] We notice that

$$\begin{aligned}\phi_k(D_m)\phi_j(H) &= D_m^{-1}\phi_k(D_m)H\phi_j(H) - D_m^{-1}\phi_k(D_m)V\phi_j(H) \\ &= 2^{-k}\widetilde{\phi}_k(D_m)2^j\widetilde{\phi}_j(H) - 2^{-k}\widetilde{\phi}_k(D_m)V\phi_j(H)\end{aligned}$$

We can also use H^{-1} since the support of ϕ_j is far from 0

$$\phi_k(D_m)\phi_j(H) = D_m\phi_k(D_m)H^{-1}\phi_j(H) - \phi_k(H)VH^{-1}\phi_j(H)$$

or higher power in D_m^{-1} or H^{-1} to obtain with (33)

$$\|\phi_i(D_m)e^{-\mathrm{i}t(H)}\phi_j(H)\phi_k(D_m)\|_{L^1,\,L^\infty} \le C2^{-r'|j-i|}\frac{C2^{(2+\theta)j}}{t^{1+\theta}}2^{-r|j-k|}$$

for any reals r, r' with C independent of i, j. Hence if r, $r' > 0$, we work like in the proof of Theorem B.22 (*i.e.* like in Appendix 2 of [Bre77]) to conclude the proof. □

It now remains to prove Proposition B.27, B.28 and B.29.

3.3.3 Some estimates

Estimates for w We remind us of the definition of w in (27) and we introduce

$$\widetilde{R}_V^{\pm}(k) = e^{-ik\cdot Q}R_V^{+}(\pm\lambda(k))e^{ik\cdot Q}. \tag{34}$$

We have

Lemma B.31. *Suppose that Assumptions 1.1 and 1.2 hold. For any $\alpha \in \mathbb{N}^3$, let be $\sigma > 4+|\alpha|$. Then there exists $C>0$ such that for any k, $x \in \mathbb{R}^3 \setminus \{0\}$, we have*

$$\left|\left\{\nabla_k^\alpha \widetilde{R}_0^\pm(k)\langle Q\rangle^{-\sigma} q\right\}(x)\right| \leq \frac{C}{\langle k\rangle^{|\alpha|}} \frac{\langle x\rangle^{|\alpha|}}{\left\langle |x| \left| \frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle} \|q\|_{W^{2+|\alpha|,\infty}}. \tag{35}$$

There exists $C>0$ such that for any k, $x \in \mathbb{R}^3 \setminus \{0\}$, we also have

$$\left|\left\{\nabla_k^\alpha \widetilde{R}_0^\pm(k)\langle Q\rangle^{-\sigma} q\right\}(x)\right| \;\leq\; C\frac{\langle x\rangle^{\alpha-1}}{\langle k\rangle^{\alpha-1}} \frac{\langle \min\{|x|,\ |k|\}\rangle}{\langle k\rangle \left\langle \min\{|x|,\ |k|\} \left| \frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle^2} \|q\|_{W^{2+|\alpha|,\infty}}. \tag{36}$$

Proof – We write

$$\left(\widetilde{R}_0^\pm(k)\langle Q\rangle^{-\sigma} q\right)(x) = \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\{\pm|k||y|+k\cdot y\}}}{4\pi|y|} \left\{\frac{\alpha\cdot(x-y)q(x-y)}{\langle x-y\rangle^{\sigma+2}} + \frac{\alpha\cdot\nabla q(x-y)}{\langle x-y\rangle^{\sigma}}\right\} dy$$
$$+ (\alpha\cdot k + m\beta \pm \lambda(k)) \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\{\pm|k||y|+k\cdot y\}}}{4\pi|y|} \frac{q(x-y)}{\langle x-y\rangle^{\sigma}}\, dy.$$

We restrict our study to $\widetilde{R}_0^+(k)$ since the two cases are similar. Hence we only need to estimate integrals of the form

$$R(k)(x) = \int_{\mathbb{R}^3} e^{\mathrm{i}\{|k||y|+k\cdot y\}} \frac{u(x-y)}{|y|}\, dy$$

with $u \in W_\sigma^{1+|\alpha|,\infty}(\mathbb{R}^3, \mathbb{C})$.

In a first step, a straightforward calculation shows that

$$|\nabla_k^\alpha R(k)(x)| \leq C\langle x\rangle^{|\alpha|-1} \|u\|_{L^\infty_\sigma} \tag{37}$$

if $\sigma > 3+\max\{|\alpha|-1;0\}$. Then using the trick we used in the proof of Proposition B.20, we obtain

$$\nabla_k^\alpha R(k)(x) = \frac{\mathrm{i}^{|\alpha|}}{|k|^{|\alpha|}} \int_{\mathbb{R}^3} e^{\mathrm{i}\{|k||y|+k\cdot y\}} \left\{(\nabla|Q|)^\alpha \frac{u(x-\cdot)}{|Q|}\right\}(y)\, dy,$$

and so with (37), we infer

$$|\nabla_k^\alpha R(k)(x)| \leq \frac{C\langle x\rangle^{|\alpha|-1}}{\langle k\rangle^{|\alpha|}} \|u\|_{W_\sigma^{|\alpha|,\infty}}, \tag{38}$$

since $\sigma > 3+\max\{|\alpha|-1,0\}$.

In a second step, we apply Estimate (13) of Proposition B.20 to $R(k)(x)$, this gives

$$|\nabla_k^\alpha R(k)(x)| \quad\leq\quad \frac{C}{|k|^{|\alpha|+1}} \max_{|\beta|\leq 1+|\alpha|} \left\{\int_{\mathbb{R}^3} |y|^{|\beta|-1} \frac{1}{\left|\frac{k}{|k|} - \frac{y}{|y|}\right|} \left|\nabla^\beta \left\{\frac{u(x-\cdot)}{|Q|}\right\}(y)\right| dy\right\}.$$

Hence we need to estimate on integral of the form

$$G(x,\omega) = \int_{\mathbb{R}^3} \frac{1}{|y|^n} \frac{1}{\langle x-y\rangle^s} \frac{1}{\left|\frac{y}{|y|} - \omega\right|}\, dy$$

with $\omega \in S^2$, $-|\alpha|+1 \le n \le 2$ and $s > \sigma$. To obtain appropriate estimates, we use

$$|x-y| \ge \frac{1}{4}\max\{|y|,|x|\}\left|\frac{x}{|x|}-\frac{y}{|y|}\right| + \frac{1}{2}\,||x|-|y||$$

to write for $\theta, \theta' \ge 0$ such that $\theta + \theta' = 1$,

$$\begin{aligned} G(x,\omega) &\le C\int_{\mathbb{R}^3} \frac{1}{|y|^n}\frac{1}{\langle |x|-|y|\rangle^{\theta s}}\frac{1}{\left\langle |x|\left|\frac{x}{|x|}-\frac{y}{|y|}\right|\right\rangle^{\theta' s}}\frac{1}{\left|\frac{y}{|y|}-\omega\right|}\,dy \\ &\le \frac{C}{|x|\left\langle |x|\left|\omega-\frac{x}{|x|}\right|\right\rangle}\int_{\mathbb{R}^+}\frac{1}{r^{n-2}}\frac{1}{\langle |x|-r\rangle^{\theta s}}\,dr \\ &\le \frac{C\langle x\rangle^{|\alpha|+1}}{|x|\left\langle |x|\left|\omega-\frac{x}{|x|}\right|\right\rangle}, \end{aligned}$$

if $\theta' s > 2$ and $\theta s > 1 + \max\{2-n;0\}$. Since $G(0,\omega)$ is bounded, we obtain

$$G(x,\omega) \le \frac{C\langle x\rangle^{|\alpha|}}{\left\langle |x|\left|\omega-\frac{x}{|x|}\right|\right\rangle},$$

Hence, we obtain with estimate (38)

$$|\nabla_k^\alpha R(k)(x)| \le \frac{C}{\langle k\rangle^{|\alpha|+1}}\frac{\langle x\rangle^{|\alpha|}}{\left\langle |x|\left|\frac{k}{|k|}-\frac{x}{|x|}\right|\right\rangle}\|u\|_{W_\sigma^{|\alpha|+1,\infty}}, \tag{39}$$

which gives estimate (35).

In a third step, if $k/|k| \neq x/|x|$, we split the integral for $\nabla_k^\alpha R(k)(x)$ in two parts with help of a smooth cut-off function defined in S^2 the support of which is a half cone determined by the bisector plane of the couple $(k/|k|; x/|x|)$. So we obtain $\nabla_k^\alpha R(k)(x) = R_1(k)(x) + R_2(k)(x)$ with $R_1(k)(x)$ having a support containing $x/|x|$ and $R_2(k)(x)$ having a support containing $k/|k|$. We then apply the estimate (14) of Proposition B.20 to $R_1(k)(x)$ to obtain

$$\begin{aligned} |\nabla_k^\alpha R_1(k)(x)| \le{}& \frac{C}{|k|^{|\alpha|+2}\left|\frac{k}{|k|}-\frac{x}{|x|}\right|}\times \\ &\times \max_{|\beta|\le 2+|\alpha|}\left\{\int_{\mathbb{R}^3}|y|^{|\beta|-2}\frac{1}{\left|\frac{k}{|k|}-\frac{y}{|y|}\right|}\left|\nabla^\beta\left\{\frac{u(x-\cdot)}{|Q|}\right\}(y)\right|\,dy\right\}. \end{aligned}$$

This gives the estimate

$$|\nabla_k^\alpha R_1(k)(x)| \le \frac{C}{|k|^{|\alpha|+2}}\frac{\langle x\rangle^{|\alpha|-1}}{\left|\frac{k}{|k|}-\frac{x}{|x|}\right|^2}\|u\|_{W_\sigma^{|\alpha|+2,\infty}},$$

since $\sigma > 2+|\alpha|$. Using (39), we infer

$$|\nabla_k^\alpha R_1(k)(x)| \le \frac{C}{\langle k\rangle^{|\alpha|+1}}\frac{\langle x\rangle^{|\alpha|}}{\langle\sqrt{|k||x|}\left|\frac{k}{|k|}-\frac{x}{|x|}\right|\rangle^2}\|u\|_{W_\sigma^{|\alpha|+2,\infty}},$$

or, using (38),

$$|\nabla_k^\alpha R_1(k)(x)| \leq \frac{C}{\langle k\rangle^{|\alpha|}} \frac{\langle x\rangle^{|\alpha|-1}}{\langle |k| \left|\frac{k}{|k|} - \frac{x}{|x|}\right|\rangle^2} \|u\|_{W_\sigma^{|\alpha|+2,\infty}}.$$

For $R_2(x)(k)$, we use the inequality $|x-y| \geq \frac{|x|\left|\frac{x}{|x|}-\frac{y}{|y|}\right|}{2}$ to obtain

$$R_2(k)(x) \leq \frac{C\langle x\rangle^{|\alpha|-1}}{\langle k\rangle^{|\alpha|}\langle |x| \left|\frac{k}{|k|} - \frac{x}{|x|}\right|\rangle^s} \|u\|_{W_\sigma^{|\alpha|,\infty}},$$

since $\sigma > 3 + s + \max\{|\alpha| - 1, 0\}$. So now we easily deduce estimate (36). □

For the sequel, we need the following

Lemma B.32. *Let be $s \in \mathbb{R}$ and ϕ a $\mathcal{C}^\infty$ function such that there is $\sigma > 0$ with*

$$\forall \alpha \in \mathbb{N}^3,\ |\nabla^\alpha \phi(x)| \leq \frac{C_\alpha}{\langle x\rangle^\sigma}.$$

We have that $[\langle P\rangle^s, \phi(Q)]$ is bounded from H_q^t into $H_{q'}^{t'}$ with $q' + \sigma \geq q$ and $t' + 1 \geq t + s$.

Proof – We want to prove that

$$\langle Q\rangle^q \langle P\rangle^t [\langle P\rangle^s, \phi(Q)] \langle P\rangle^{-t'} \langle Q\rangle^{-q'} \tag{40}$$

is bounded in $\mathcal{B}(L^2)$. Using the identity

$$[\langle P\rangle^s, \phi(Q)] = [\langle P\rangle^{s/2}, \phi(Q)]\langle P\rangle^{s/2} + \langle P\rangle^{s/2}[\langle P\rangle^{s/2}, \phi(Q)]$$

we reduce the proof to the case $|s| < 1$. And with the identity

$$[\langle P\rangle^s, \phi(Q)] = -\langle P\rangle^s [\langle P\rangle^{-s}, \phi(Q)] \langle P\rangle^s$$

we only need to study the case $-1 < s < 0$. The proof in this case is based on the following identity for $-1 < s < 0$

$$\langle P\rangle^s = (-\Delta + 1)^{s/2} = \frac{-\sin(\pi\left\{\frac{s}{2}\right\})}{\pi} \int_0^{+\infty} \frac{w^{\left\{\frac{s}{2}\right\}}}{-\Delta + 1 + w} dw.$$

So we have

$$[\langle P\rangle^s, \phi(Q)] = \sum_{k=1}^m \frac{\Gamma(s/2+1)}{\Gamma(s/2+1-k)} (-\Delta+1)^{s/2-k} Ad^k_{-\Delta+1}(\phi(Q)) + R_m$$

with

$$R_m = \frac{(-1)^m \sin(\pi\left\{\frac{s}{2}\right\})}{\pi} \int_0^{+\infty} \frac{w^{\left\{\frac{s}{2}\right\}}}{(-\Delta+1+w)^{m+1}} Ad^{m+1}_{-\Delta+1}(\phi(Q)) \frac{dw}{-\Delta+1+w}.$$

Then we use $\frac{-\Delta+1}{-\Delta+1+w} = 1 - \frac{w}{-\Delta+1+w}$, and we commute powers of $\langle P\rangle$ with operators of the form $\nabla^\alpha \phi(Q)$. Hence we can repeat the previous computation until we obtain only non positive powers of $\langle P\rangle$ in (40). So we only need to prove that operators of the form

$$[\langle Q\rangle^q, \phi(P)]\langle Q\rangle^{-q'}$$

with $q \leq q' + 1$ and ϕ satisfying the assumption of the lemma are bounded in $\mathcal{B}(L^2)$, we just repeat the previous calculation but we switch the role of P and Q. This ends the proof. □

We now state a particular version of the Limiting Absorption Principle for H.

Proposition B.33. *We assume that Assumptions 1.1 and 1.2 hold. Then for any $\sigma \geq 1$ there exists $C > 0$ such that for any $k \in \mathbb{R}^3$, we have*

$$\|\widetilde{R}_V^{\pm}(k)\|_{\mathcal{B}(L^2_\sigma, L^2_{-\sigma})} \leq C.$$

Proof – In fact, we just need to prove that for any $\sigma \geq 1$ there exists $C > 0$ such that for any $\lambda \in \mathbb{R} \setminus (-m, m)$

$$\|R_V^{+}(\lambda)\|_{\mathcal{B}(L^2_\sigma, L^2_{-\sigma})} \leq C.$$

Using Theorem B.3, we have that it is true if $\sigma > 5/2$. Then we use Born expansion

$$R_V^+(\lambda) = R_0^+(\lambda) - R_0^+(\lambda) V R_0^+(\lambda) + R_0^+(\lambda) V R_V^+(\lambda) V R_0^+(\lambda)$$

and [IM99, Theorem 2.1(i)] to end the proof. □

We are now able to give the
Proof – [Proof of Proposition B.27] We only give a the general idea of the proof and we leave the details to the reader. We notice that with $\widetilde{R}_V^{\pm}$ defined by (34), we obtain

$$w = \widetilde{R}_V V u$$

with an abuse of notation since we avoid to distinguish the case where we have $\widetilde{R}_V^+$ or $\widetilde{R}_V^-$. We recall the identities

$$\widetilde{R}_V V = \widetilde{R}_0 V - \widetilde{R}_0 V \widetilde{R}_V V = \widetilde{R}_0 V - \widetilde{R}_0 V \widetilde{R}_0 + \widetilde{R}_0 V \widetilde{R}_V V \widetilde{R}_0 V. \tag{41}$$

Since, we have

$$\widetilde{R}_V = (1 + \widetilde{R}_0 V)^{-1} \widetilde{R}_0, \quad (1 + \widetilde{R}_0 V)^{-1} = 1 - \widetilde{R}_V V,$$

for $|\alpha| = 1$, we obtain

$$\nabla_k^\alpha \widetilde{R}_V = (1 - \widetilde{R}_V V) \nabla_k^\alpha \widetilde{R}_0 (1 - V \widetilde{R}_V). \tag{42}$$

Using (42), we obtain a formula where only derivatives of $\widetilde{R}_0$ appear (if there is derivatives). Then between a derivative of $\widetilde{R}_0$ and $\widetilde{R}_V$, we insert a $\widetilde{R_0}$ with the identity (41):

$$\widetilde{R}_V V \nabla_k^\alpha \widetilde{R}_0 V = \widetilde{R}_0 V \nabla_k^\alpha \widetilde{R}_0 V - \widetilde{R}_0 V \widetilde{R}_0 V \nabla_k^\alpha \widetilde{R}_0 V + \widetilde{R}_0 V \widetilde{R}_V V \widetilde{R}_0 V \nabla_k^\alpha \widetilde{R}_0 V.$$

This ensures that if $\rho > 5$, V or its derivatives decays enough to use Estimate (35) and Proposition B.33. Since these estimates need derivatives and Sobolev's injections, we apply Lemma B.32 to conclude the proof. □

Estimates for v We remind us of the definition of v in (30) and we introduce

$$S_V^{\varepsilon_1,\varepsilon_2}(k) = e^{-\varepsilon_1\varepsilon_2 \mathrm{i}|k||Q|} R_V^{\varepsilon_1}(\varepsilon_2\lambda(k)) e^{\mathrm{i}k\cdot Q},$$

where $\varepsilon_i \in \{-1, 1\}$. With an abuse of notation, we will write $v = S_V V u$. We have

Lemma B.34. *There exists $C > 0$, such that for any $k \in \mathbb{R}^3 \setminus \{0\}$ and $\beta \in \mathbb{N}^3$, we have*

$$\left| \left(\nabla_k^\beta S_0^{\varepsilon_1,\varepsilon_2}(k) \langle Q\rangle^{-\sigma} q \right)(x) \right| \leq \frac{C}{\left\langle |x| \left| \frac{x}{|x|} - \frac{k}{|k|} \right| \right\rangle} \|q\|_{W^{2+|\beta|,\infty}}$$

for any $\sigma > 3 + |\beta|$.

Proof –

$$\begin{aligned}
&\left(S_0^{\varepsilon_1,\varepsilon_2}(k) \langle Q\rangle^{-\sigma} q \right)(x) = \\
&\quad \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\varepsilon_1\varepsilon_2\{|k||x-y| - |k||x| + \varepsilon_1\varepsilon_2 k\cdot y\}}}{4\pi|x-y|} \left\{ \frac{\alpha\cdot y q(y)}{\langle y\rangle^{\sigma+2}} + \frac{\alpha\cdot\nabla q(y)}{\langle y\rangle^{\sigma}} \right\} dy \\
&\qquad + (\alpha\cdot k + m\beta \pm \lambda(k)) \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\varepsilon_1\varepsilon_2\{|k||x-y| - |k||x| + \varepsilon_1\varepsilon_2 k\cdot y\}}}{4\pi|x-y|} \frac{q(y)}{\langle y\rangle^{\sigma}}\, dy.
\end{aligned}$$

For the sake of simplicity, we only write the proof when $\beta = 0$. The proof for derivatives works in the same way using $||x-y| - |x|| \leq |y|$ and $\sigma > 3 + |\beta|$. But the proof for the case $\beta = 0$, has been already done since

$$\left| \left(S_0^{\varepsilon_1,\varepsilon_2}(k) \langle Q\rangle^{-\sigma} q \right)(x) \right| = \left| \left(R_0^{\varepsilon_1,\varepsilon_2}(k) \langle Q\rangle^{-\sigma} q \right)(x) \right|.$$

□

Hence using Proposition B.33, we able to write the
Proof – [Proof of Proposition B.28] We write with an abuse of notation

$$v = S_V V u,$$

and we use the Born formula

$$S_V V = S_0 V - S_0 V \widetilde{R}_V V,$$

together with Lemma B.34, Propositions B.32 and B.33. The proof works like the one for w. □

Estimates for $\widetilde{v}$ We remind us of the definition of $\widetilde{v}$ in (31) and we introduce

$$T_V^{\varepsilon_1,\varepsilon_2}(k) = e^{-\varepsilon_1\varepsilon_2 \mathrm{i}|k||Q| + \mathrm{i}k\cdot Q} \nabla_k \widetilde{R}_V^{\varepsilon_1}(\varepsilon_2\lambda(k)) e^{\mathrm{i}k\cdot Q},$$

where $\varepsilon_i \in \{-1, 1\}$. With another abuse of notation, here we will write $\widetilde{v} = T_V V u$. We have

Lemma B.35. *There exists $C > 0$, such that for any $k \in \mathbb{R}^3 \setminus \{0\}$ and $\beta \in \mathbb{N}^3$, we have*

$$\left| \left(\nabla_k^{\beta} T_0^{\varepsilon_1,\varepsilon_2}(k) \langle Q \rangle^{-\sigma} q \right)(x) \right| \leq C \frac{\langle \min\{|x|,\ |k|\} \rangle}{\langle k \rangle \left\langle \min\{|x|,\ |k|\} \left| \frac{x}{|x|} - \frac{k}{|k|} \right| \right\rangle^2} \|q\|_{W^{2+|\beta|,\infty}},$$

for any $\sigma > 4 + |\beta|$.

Proof – This is an obvious adaptation of the proof of Lemma B.34, we just notice that one has

$$\left| \left(T_0^{\varepsilon_1,\varepsilon_2}(k) \langle Q \rangle^{-\sigma} q \right)(x) \right| = \left| \left(\nabla_k \widetilde{R}_0^{\varepsilon_1}(\varepsilon_2 \lambda(k)) \langle Q \rangle^{-\sigma} q \right)(x) \right|.$$

□

Hence, we have
Proof – [Proof of Proposition B.29] One more time, we write with an abuse of notation

$$v = T_V V u + S_V V \nabla_k u,$$

The second term of the right hand side could be studied exactly as we done in proof of Proposition B.28 and for the first one we use the formula

$$T_V V = T_0 V - T_0 V \widetilde{R}_V V + S_0 V \nabla_k \widetilde{R}_V V,$$

together with Lemma B.35, Propositions B.32 and B.33. The proof works like the one for w. □

4 The linearized operator

In this section, we study the spectral properties of the linearized operator, associated with Equation (3), around a stationary state. This will be useful since we compare the dynamics associated with Equation (3) to the dynamic of the linear Dirac equation associated with H. This comparison is possible only because when the PLS is small, the linearized operator is a small perturbation of H.

4.1 The manifold of the particle-like solutions

First we notice that Proposition B.5, which gives the existence of stationary states, is a consequence of

Proposition B.36. *Let H be a self adjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with a simple eigenvalue λ_0 associated with a normalized eigenvector ϕ_0. Assume that there is a neighborhood $\mathcal{O} \subset \mathbb{R}$ of λ_0 such that for all $\lambda \in \mathcal{O}$ the operator $(H - \lambda)^{-1} P_0$ is in $\mathcal{B}(L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4))$ for any $\sigma \in \mathbb{R}^+$, and in $\mathcal{B}(H^l(\mathbb{R}^3, \mathbb{C}^4), H^{l+1}(\mathbb{R}^3, \mathbb{C}^4))$ for any $l \in \mathbb{N}$, where P_0 is the projector into the orthogonal space of ϕ_0. Let $F \in \mathcal{C}^{k+1}(\mathbb{C}^4, \mathbb{C}^4)$ be such that $F(z) = O(|z|^3)$.*

Then for any $\sigma \in \mathbb{R}^+$, there exist Ω a neighborhood of $0 \in \mathbb{C}$, a $\mathcal{C}^k$ map

$$h : \Omega \mapsto \{\phi_0\}^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4)$$

and a $\mathcal{C}^k$ *map* $E : \Omega \mapsto \mathbb{R}$ *such that* $S(u) = u\phi_0 + h(u)$ *satisfy for all* $u \in \Omega$,

$$HS(u) + \nabla F(S(u)) = E(u)S(u),$$

with the following properties

$$\begin{cases} h(e^{i\theta}u) = e^{i\theta}h(u), \quad \forall \theta \in \mathbb{R}, \\ h(u) = O(|u|^2), \\ E(u) = E(|u|), \\ E(u) = \lambda_0 + O(|u|^2). \end{cases}$$

The proof of this proposition is an obvious adaptation of the one of [PW97, Proposition 2.2], and we don't repeat it here. One can also obtain it by means of the Crandall-Rabinowitz theorem but it doesn't give immediately the decomposition associated to the spectrum of $H = D_m + V$.

Suppose that Assumptions 1.1–1.4 hold. To show that $(H-\lambda)^{-1}P_0$ is in $\mathcal{B}(L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4))$ for any $\sigma > 0$, we just need to prove that $\alpha \mapsto e^{\alpha\langle Q\rangle}(H-\lambda)^{-1}P_0 e^{-\alpha\langle Q\rangle}$ is of class $\mathcal{C}^k$ near 0 in $\mathcal{B}(L^2(\mathbb{R}^3, \mathbb{C}^4))$ for any $k \in \mathbb{N}$, this can be proved with help of of [His00, Lemma 5.1]. To prove that $(H-\lambda)^{-1}P_0$ for any $l \in \mathbb{N}$ is in $\mathcal{B}(H^l(\mathbb{R}^3, \mathbb{C}^4), H^{l+1}(\mathbb{R}^3, \mathbb{C}^4))$ for any $l \in \mathbb{N}$, we first notice that $(D_m - \lambda)^{-1}$ is in $\mathcal{B}(H^l(\mathbb{R}^3, \mathbb{C}^4), H^{l+1}(\mathbb{R}^3, \mathbb{C}^4))$ then we use wave operator, see 20 and [GM01, Theorem 1.5], and the intertwining property, see 21, to conclude.

We shall need some properties of stationary solutions of (3). Following [His00], we have the

Lemma B.37 (exponential decay)**.** *Suppose that Assumptions 1.1–1.4 hold. For all* $\beta \in \mathbb{N}^2$, $s \in \mathbb{R}^+$ *and* $p, q \in [1, \infty]$. *There is* $\gamma > 0$, $\varepsilon > 0$ *and* $C > 0$ *such that for all* $u \in B_{\mathbb{C}}(0, \varepsilon)$, *one has*

$$\|e^{\gamma\langle Q\rangle}\partial_u^\beta S(u)\|_{B^s_{p,q}} \le C\|S(u)\|_2,$$

where $\partial_u^\beta = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u \partial^{\beta_2}\Im u}$.

Proof – In fact we prove that for any k in $\mathbb{N}$ there is $\gamma > 0$ and $\varepsilon > 0$ and $C > 0$ such that for all $u \in B_{\mathbb{C}}(0, \varepsilon)$ one has

$$\|e^{\gamma\langle Q\rangle}\partial_u^\beta S(u)\|_{H^k} \le C\|S(u)\|_2.$$

Then interpolation and the following property of Besov spaces over $\mathbb{R}^3$ permit to conclude: $B^s_{2,2} = H^s$, $B^s_{p,r} \subset B^{s'}_{p,q}$ if $s' < s$, $B^u_{r,q} \subset B^s_{p,q}$ if $1 \le r \le p \le \infty$ and $u - n/r = s - n/p$ and $\|uv\|_{B^s_{p,q}} \le C\|u\|_{B^s_{q,t}}\|v\|_{B^s_{r,t}}$ if $\frac{1}{p} + \frac{s}{3} > \frac{1}{q} + \frac{1}{r}$.

We only prove the lemma for $\beta = 0$, the other cases are similar. We have

$$D_m S(u) + VS(u) + \nabla F(S(u)) = E(u)S(u).$$

Let us introduce the $\mathbb{R}-$linear operator W of multiplication by the matrix valued function $x \in \mathbb{R}^3 \mapsto -\mathrm{i}D\nabla F(S(u)(x))\mathrm{i} + V(x)$. We obtain, with the gauge invariance of F, the identity

$$WS(u) = \nabla F(S(u)) + VS(u).$$

The "potential" W tends to zero as x goes to ∞. In fact, as a function of x, W is in $L^1 \cap L^\infty$; we can write $W = W_c + W_\delta$ where W_c is compactly supported and $\|W_\delta\|_{L^1\cap L^\infty} \le \delta$.

We have that $D_m + W_\delta - E(u)$ is invertible for δ sufficiently small and

$$e^{\gamma\langle Q\rangle}S(u) = e^{\gamma\langle Q\rangle}\left(D_m + W_\delta - E(u)\right)^{-1} e^{-\gamma\langle Q\rangle}\{e^{\gamma\langle Q\rangle}W_c S(u)\}.$$

For γ small, $\left(D_m + \gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)$ is invertible in L^2 and

$$e^{\gamma\langle Q\rangle}S(u) \quad = \quad \left(D_m + \gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)^{-1} e^{\gamma\langle Q\rangle}W_c S(u).$$

This proves the lemma for $k=0$ since $e^{\gamma\langle Q\rangle}W_c$ is bounded. Now we notice that

$$|P|\left(D_m + \gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)^{-1}$$
$$= \frac{|P|}{D_m} - \frac{|P|}{D_m}\left(2\gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)\left(D_m + \gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)^{-1}.$$

Hence we obtain

$$\|e^{\gamma\langle Q\rangle}S(u)\|_{H^k} \leq C\|S(u)\|_{H^{k-1}}.$$

This identity proves the lemma by induction. □

4.2 The spectrum of the linearized operator

Here we study the spectrum of the linearized operator associated with Equation (3) around a stationary state $S(u)$. Let us introduce

$$H(u) = H + d^2F(S(u)) - E(u)$$

where d^2F is the differential of ∇F. The operator $H(u)$ is $\mathbb{R}-$linear but not $\mathbb{C}-$linear. Replacing $L^2(\mathbb{R}^3,\mathbb{C}^4)$ by $L^2(\mathbb{R}^3,\mathbb{R}^4\times\mathbb{R}^4)$ with the inner product obtained by taking the real part of the inner product of $L^2(\mathbb{R}^3,\mathbb{C}^4)$, we obtain a symmetric operator. We then complexify this real Hilbert space and obtain $L^2(\mathbb{R}^3,\mathbb{C}^4\times\mathbb{C}^4)$ with its canonical hermitian product. This process transforms the operator $-\mathrm{i}$ into

$$J = \begin{pmatrix} 0 & -Id_{\mathbb{C}^4} \\ Id_{\mathbb{C}^4} & 0 \end{pmatrix}.$$

For $\phi \in L^2(\mathbb{R}^3,\mathbb{R}^4\times\mathbb{R}^4) \subset L^2(\mathbb{R}^3,\mathbb{C}^4\times\mathbb{C}^4)$, we still write ϕ instead of

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}.$$

The extension of $H(u)$ over $L^2(\mathbb{R}^3,\mathbb{C}^4\times\mathbb{C}^4)$ is also written $H(u)$ and is now a real operator.

The linearized operator associated with Equation (3) around the stationary state $S(u)$ is given by $JH(u)$. We shall now study its spectrum. Differentiating (4), we have that

$$\mathcal{H}_0 = \mathrm{Span}\left\{\frac{\partial}{\partial\Re u}S(u),\ \frac{\partial}{\partial\Im u}S(u)\right\}$$

is invariant under the action of $JH(u)$. We notice (see [GNT04]) that

$$\mathcal{H}_0(u) = \mathrm{Span}\left\{JS(u), \partial_{|u|}S(u)\right\}.$$

Using gauge invariance and differentiating, we obtain

$$JH(u)JS(u) = 0 \quad \text{and} \quad JH(u)\partial_{|u|}S(u) = \partial_{|u|}E(u)JS(u).$$

Hence $\mathcal{H}_0(u)$ is contained in the geometric null space of $JH(u)$, in fact it is exactly the geometric null space as proved in the sequel of this subsection. First, with Assumption 1.3, we obtain that $JH(u)$ has two other simple eigenvalues, as stated in the following

Lemma B.38. *Let be*

$$S_1^+(0) = \begin{pmatrix} \phi_1 \\ -\mathrm{i}\phi_1 \end{pmatrix} \text{ and } S_1^-(0) = \begin{pmatrix} \overline{\phi_1} \\ \mathrm{i}\overline{\phi_1} \end{pmatrix}.$$

Suppose that Assumptions 1.1–1.4 hold. Then there are $\varepsilon > 0$ and four $\mathcal{C}^\infty$ maps $E_1^\pm : B_{\mathbb{C}}(0,\varepsilon) \mapsto \mathbb{C}$ and $k_1^\pm : B_{\mathbb{C}}(0,\varepsilon) \mapsto \left\{S_1^\pm(0)\right\}^\perp$ such that

$$JH(u)S_1^\pm(u) = E_1^\pm(u)S_1^\pm(u),$$

with $\|S_1^\pm(u)\| = 1$, $S_1^\pm(u) = S_1^\pm(0) + k_1^\pm(u)$, $E_1^\pm(u) = \pm i(\lambda_1 - \lambda_0) + O(|u|^2)$ and $k_1^\pm(0) = 0$.

Proof – This can be proved in the same fashion as [PW97, Proposition 2.2] using Assumption 1.3. □

We also obtain

Lemma B.39 (exponential decay in Besov spaces)**.** *Suppose that Assumptions 1.1–1.4 hold. Then for any $\beta \in \mathbb{N}^2$, $s \in \mathbb{R}$ and $p, q \in [1,\infty]$ there is $\gamma > 0$, $\varepsilon > 0$ and a positive constant C such that for all $u \in B_{\mathbb{C}}(0,\varepsilon)$, we have*

$$\|e^{\gamma\langle Q\rangle}\partial_u^\beta S_1^\pm(u)\|_{B^s_{p,q}} \leq C\|S_1^\pm(u)\|_2,$$

where $\partial_u^\beta = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u\partial^{\beta_2}\Im u}$.

Proof – The proof is exactly the same as the one of Lemma B.37. □

Let $\mathcal{H}_{\pm 1}(u)$ be the space spanned by $S_1^\pm(u)$. Let us now prove that the orthogonal space with respect to the hermitian product associated to J

$$\mathcal{H}_c(u) = \{\mathcal{H}_0(u) \oplus \mathcal{H}_{+1}(u) \oplus \mathcal{H}_{-1}(u)\}^\perp$$

contains no eigenvector. We notice that $\mathcal{H}_c(u)$ is invariant under the action of $JH(u)$. We have

Lemma B.40 (Continuous subspace property)**.** *If Assumptions 1.1–1.4 hold, let $\mathbf{P}_c(u)$ be the orthogonal projector onto $\mathcal{H}_c(u)$. Then there exists $\varepsilon > 0$ such that for u', $u \in B_{\mathbb{C}}(0,\varepsilon)$*

$$\mathbf{P}_c((u))|_{\mathcal{H}_c(u')} : \mathcal{H}_c(u') \mapsto \mathcal{H}_c(u)$$

is an isomorphism from $B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^8) \cap \mathcal{H}_c(u')$ into $B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^8) \cap \mathcal{H}_c(u)$, for any $s \in \mathbb{R}^+$ and any $p, q \in [1,\infty]$. The inverse $R(u',u)$ is continuous with respect to u and u'.

Proof – This proof is a straightforward adaptation of the one of [GNT04, Lemma 2.2]. □

So we have

Lemma B.41. *Under the assumptions of Proposition B.5, there exists $\varepsilon > 0$ such that for any $u \in B_{\mathbb{C}}(0, \varepsilon)$, we have*

$$\begin{aligned} \|\langle Q\rangle^{-\sigma} e^{sJH(u)} \mathbf{P}_c(u)\psi\| &\leq C\langle s\rangle^{-\min\{\sigma,\, 3/2\}} \|\langle Q\rangle^{\sigma}\psi\|,\ \forall\psi \in L^2_\sigma \\ \int_{\mathbb{R}} \|\langle Q\rangle^{-\sigma} e^{sJH(u)} \mathbf{P}_c(u)\psi\|^2\, ds &\leq C\|\psi\|,\ \forall\psi \in L^2. \end{aligned}$$

As a consequence, $\mathcal{H}_c(u)$ does not contain any eigenvector.

Proof – For the sake of clarity, we introduce

$$\zeta(u) = \left(J\frac{\partial}{\partial \Re u} S(u),\ J\frac{\partial}{\partial \Im u} S(u),\ JS_1^+(u),\ JS_1^-(u) \right).$$

Writing Duhamel's formula for $H(u)$ with respect to $H - E(u)$, we obtain

$$e^{tJH(u)}\mathbf{P}_c(u) = e^{tJ(H-E(u))}\mathbf{P}_c(u) + \int_0^t e^{(t-s)J(D_m+V-E(u))} Jd^2F(S(u)) e^{sJH(u)} \mathbf{P}_c(u)\, ds.$$

We have

$$\mathbf{P}_c(0)|^{-1}_{\mathcal{H}_c(u)} = R(u,0) = Id_{\mathcal{H}_c(0)} + \sum_i |\alpha_i(u,0)\rangle\, \langle\zeta_i(0)|$$

where the coordinates of $\alpha_i(u', u)$ are linear combination of the coordinates of $\zeta(u)$, so it can be extended to $L^2_{-\sigma}$ and we have

$$\begin{aligned} &\|\langle Q\rangle^{-\sigma} e^{-tJH(u)}\mathbf{P}_c(u)\psi\| \\ &\leq \|\langle Q\rangle^{-\sigma}\ \mathbf{P}_c(0)|^{-1}_{\mathcal{H}_c(u)}\, \langle Q\rangle^{\sigma}\| \Big\{ \|\langle Q\rangle^{-\sigma}\mathbf{P}_c(0) e^{-tJ(D_m+V-E(u))}\mathbf{P}_c(u)\psi\| \\ &\quad + \int_0^t \Big\| \langle Q\rangle^{-\sigma}\mathbf{P}_c(0) e^{-J(t-s)(H-E(u))} JD\nabla F(S(u)) e^{-sJH(u)}\mathbf{P}_c(u)\psi \Big\|\, ds \Big\} \\ &\leq C\langle t\rangle^{-\min\{\sigma,\, 3/2\}} \|\langle Q\rangle^{\sigma}\psi\| \\ &\quad + C\int_0^t \langle t-s\rangle^{-\min\{\sigma,\, 3/2\}} \|\langle Q\rangle^{2\sigma} D\nabla F(S(u))\| \|\langle Q\rangle^{-\sigma} e^{-isH(u)}\mathbf{P}_c(u)\psi\|\, ds \end{aligned}$$

We then introduce

$$M(t) = \sup_{s\in[0,t]} \{\langle s\rangle^{-\min\{\sigma,\, 3/2\}} \|\langle Q\rangle^{-\sigma} e^{-sJH(u)}\mathbf{P}_c(u)\psi\|\}$$

and we obtain for $|z| \leq \varepsilon$

$$M(t) \leq C(\|\langle Q\rangle^{\sigma}\psi\| + \varepsilon M(t))$$

which gives for ε sufficiently small

$$M(t) \leq C\|\langle Q\rangle^{\sigma}\psi\|,$$

or

$$\|\langle Q\rangle^{-\sigma}e^{-sJH(u)}\mathbf{P}_c(u)\psi\| \leq C\langle s\rangle^{-\min\{\sigma,\,3/2\}}\|\langle Q\rangle^{\sigma}\psi\|.$$

With the same method, see Lemma B.58, we obtain the second estimate.

Then we obtain with the second estimate that there is no eigenvector in the range of $\mathbf{P}_c(u)$ that is to say $\mathcal{H}_c(u)$. □

This gives

Lemma B.42. *Suppose that Assumptions 1.1–1.4 hold. We have, for sufficiently small* $u \in \mathbb{C}$, $E_1^{\pm}(u) \in \mathrm{i}\mathbb{R}$ *with* $E_1^{\pm}(u) = -E_1^{\mp}(u)$ *and* $S_1^{-}(u) = \overline{S_1^{+}(u)}$ *for the conjugation of* $\mathbb{C}^8$.

Proof – The last statement straightforwardly follows from

$$JH(u)S_1^{\pm}(u) = E_1^{\pm}(u)S_1^{\pm}(u),$$

since there is no more eigenvalues than the 0 and $E_1^{\pm}(u)$, we obtain $\overline{E_1^{\pm}(u)} = E_1^{\mp}(u)$.

Then we specify the essential spectrum of $JH(u)$. A classical study gives that the continuous spectrum of $JH(0)$ is given by

$$\{\mathrm{i}\lambda;\ \lambda \in \mathbb{R},\, |\lambda| \geq \min\{|m-\lambda_0|, |m+\lambda_0|\}\}\,.$$

Using Weyl's criterion (see [RS78, Theorem XIII.14, Corollary 1], the adaptation is quite easy in our case), we obtain that the essential spectrum is

$$\{\mathrm{i}\lambda;\ \lambda \in \mathbb{R},\, |\lambda| \geq \min\{|m-E|, |m+E|\}\}\,.$$

Hence $E_1^{\pm}(u)$ are necessarily purely imaginary. Indeed if $H(u) - E_1^{\pm}(u)J$ is not invertible then $H(u) + \overline{E_1^{\pm}(u)}J$ is not invertible too. Since $-\overline{E_1^{\pm}(u)}$ is not in the essential spectrum, it is necessarily an eigenvalue in the neighborhood of $\pm\mathrm{i}\,(\lambda_1 - \lambda_0)$. Hence this gives $-\overline{E_1^{\pm}(u)} = E_1^{\pm}(u)$. □

4.3 Decomposition of the system

We want to decompose a solution ϕ of the equation (3) with respect to the spectrum of $JH(u)$. And in fact, we only study the resulting equations for these different parts of the decomposition. First we isolate a part which corresponds to a PLS. For any solution of (3) over an interval of time I containing 0, we write for $t \in I$

$$\phi(t) = e^{-\mathrm{i}\int_0^t E(u(s))\,ds}\left(S(u(t)) + \eta(t)\right).$$

In order to give an equation for η, we introduce the following space

$$\mathcal{H}_0^{\perp}(u) = \left\{\eta \in L^2(\mathbb{R}^3, \mathbb{C}^8), \left\langle J\eta, \frac{\partial}{\partial \Re u}S(u)\right\rangle = 0,\ \left\langle J\eta, \frac{\partial}{\partial \Im u}S(u)\right\rangle = 0\right\}.$$

In fact it is the space

$$\mathcal{H}_{+1}(u) \oplus \mathcal{H}_{-1}(u) \oplus \mathcal{H}_c(u)$$

which is invariant under the action of $JH(u)$ and we state the

Lemma B.43 (decomposition lemma)**.** *Suppose that Assumptions 1.1–1.4 hold. Let be $s \geq 0$ and $p \geq 1$ there exist $\delta > 0$ and a $\mathcal{C}^\infty$ map $U : B_{W^{s,p}}(0,\delta) \mapsto B_{\mathbb{C}}(0,\varepsilon)$ which satisfies, for $\psi \in B_{W^{s,p}}(0,\delta)$,*

$$\psi = S(u) + \eta, \text{ with } \eta \in \mathcal{H}_0^\perp(u) \Longleftrightarrow u = U(\psi).$$

Proof – It is [GNT04, Lemma 2.3]. □

This lemma ensures that we can impose the orthogonality condition

$$\eta(t) \in \mathcal{H}_0^\perp(u(t)). \tag{43}$$

So instead of solving the Equation (3) in ϕ, we want to solve the equation

$$\begin{aligned} \mathrm{i}\partial_t\eta &= \{H - E(u)\}\,\eta + \{\nabla F(S(u)+\eta) - \nabla F(S(u))\} - \mathrm{i}dS(u)\dot{u} \\ &= \left\{H + d^2F(S(u)) - E(u)\right\}\eta + N(u,\eta) - \mathrm{i}dS(u)\dot{u} \end{aligned} \tag{44}$$

for $\eta \in \mathcal{H}_0^\perp(u(t))$. Here d^2F is the differential of ∇F and dS the differential of S in $\mathbb{R}^2$. To close the system, we need an equation for u. Let us now derive an equation for the path u, by means of (43):

$$\langle \eta(t), JdS(u(t))\rangle = 0.$$

After a time derivation, we obtain

$$0 = \langle JH(u(t))\eta(t) + JN(u(t)),\eta(t)) + dS(u(t))\dot{u}(t), JdS(u(t))\rangle - \langle \eta, Jd^2S(u(t))\dot{u}(t)\rangle.$$

Since $S(u) \in J\mathcal{H}_0(u)$, we have

$$\langle H(u)\eta, dS(u)\rangle = \langle \eta, H(u)dS(u)\rangle = \langle \eta, dE(u)S(u)\rangle = 0,$$

we obtain

$$[\langle JdS(u(t)), dS(u(t))\rangle - \langle J\eta(t), d^2S(u(t))\rangle]\dot{u}(t) = -\langle N(u(t),\eta(t)), dS(u(t))\rangle.$$

So we notice that

$$[\langle JdS(u(t)), dS(u(t))\rangle - \langle J\eta(t), d^2S(u(t))\rangle] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + O(|u(t)| + \|\eta(t)\|_2),$$

which proves that $[\langle JdS(u(t)), dS(u(t))\rangle - \langle J\eta(t), d^2S(u(t))\rangle]$ is invertible for small $|u(t)|$ and $\|\eta(t)\|_2$, we therefore introduce its inverse

$$A(u,\eta) = [\langle JdS(u), dS(u)\rangle - \langle J\eta, d^2S(u)\rangle]^{-1}$$

and write

$$\partial_t u(t) = -A(u(t),\eta(t))\langle N(u(t),\eta(t)), dS(u(t))\rangle.$$

Plugging in Equation (44), and similarly to the linear case we decompose η with respect to the spectral decomposition of $H(u) = H + D\nabla F(S(u)) - E(u)$

$$\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$$

with $z \in \mathcal{H}_c(u) \cap L^2(\mathbb{R}^3, \mathbb{R}^8)$ and $\alpha^- = \overline{\alpha^+}$. We obtain the system

$$\begin{cases} \dot{u} &= -A(u,\eta)\langle N(u,\eta), dS(u)\rangle \\ \dot{\alpha}^{\pm} &= E^{\pm}(u)\alpha^{\pm} + \langle JN(u,\eta), JS_1^{\pm}(u)\rangle \\ & \quad +\langle dS(u)A(u,\eta)\langle N(u,\eta), dS(u)\rangle\, JS_1^{\pm}(u)\rangle \\ & \quad\quad +\langle (dS_1^{\pm}(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle,\ JS_1^{\pm}(u)\rangle\alpha^{\pm} \\ & \quad\quad +\langle (dS_1^{\mp}(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle,\ JS_1^{\pm}(u)\rangle\alpha^{\mp} \\ \partial_t z &= JH(u)z + \mathbf{P}_c(u)JN(u,\eta) \\ & \quad +\mathbf{P}_c(u)dS(u)A(u,\eta)\langle N(u,\eta), dS(u)\rangle \\ & \quad\quad +(D\mathbf{P}_c(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle\eta \end{cases},$$

which we will now study. We notice that this equation is defined only for z small with real values, $\alpha^- = \overline{\alpha^+}$ small and u small.

5 The stabilization towards the PLS manifold

We now build a solution which stabilizes towards the manifold of the stationary states. To this end, we will use Theorem B.3 and Theorem B.4 to prove that z tends to zero in L^∞ and L^2_{loc}. It is possible here since we build solutions for which we ensure that α^+ and α^- also tend to zero. We do not think that this convergence holds for all initial states but we do not know any counterexample.

We also notice that we look for a real solution $\phi = S(u)\mu + \eta$, hence η should be real and therefore $\alpha^- = \overline{\alpha^+}$.

We impose the following condition

$$|\alpha| \leq \frac{C}{\langle t\rangle^2}.$$

Under the assumptions of Theorem B.6, let us define for any ε, $\delta > 0$

$$\mathcal{U}(\varepsilon,\delta) = \left\{ u \in \mathcal{C}^\infty((-T_0;+\infty), B_{\mathbb{C}}(0,\varepsilon)),\ |\dot{u}(t)| \leq \frac{\delta^2}{\langle t\rangle^3}, \forall t > -T_0 \right\}.$$

For any $u \in \mathcal{U}(\varepsilon)$, let s, s', β be such that $s' \geq s+3 \geq \beta+6 > 6$ and let $\sigma > 5/2$. We define

$$\mathcal{Z}(u,\delta) = \Bigg\{ z \in \mathcal{C}^\infty((-T_0;+\infty), L^2(\mathbb{R}^3,\mathbb{R}^8)),\ z(t) \in \mathcal{H}_c(u(t)),$$
$$\max\Bigg[\sup_{v\in(-T_0;+\infty)} \{\|z(v)\|_{H^{s'}}\}, \sup_{v\in(-T_0;+\infty)} \{\langle v\rangle^{3/2}\|z(v)\|_{B^\beta_{\infty,2}}\},$$
$$\sup_{v\in(-T_0;+\infty)} \{\langle v\rangle^{3/2}\{\|z(v)\|_{H^s_{-\sigma}}\}\Bigg] \leq \delta \Bigg\}.$$

Then we define the set

$$\Omega(\delta) = \Big\{ \alpha = \big(\alpha^+,\alpha^-\big) \in \mathcal{C}^\infty((-T_0;+\infty)),\ \alpha^- = \overline{\alpha^+},$$
$$\sup_{t\in(-T_0;+\infty)} \Big\{\langle t\rangle^{3/2}|\alpha(t)|\Big\} \leq \delta^2 \Big\}.$$

5.1 Step 1: Construction of z

First we solve in $\mathcal{Z}(u,\delta)$ the equation relative to z for $u \in \mathcal{U}(\delta,\varepsilon)$ and $\alpha \in \Omega(\delta)$. We first prove the

Lemma B.44. *Suppose that Assumptions 1.1–1.4 hold. There exist $\delta > 0$ and $\varepsilon > 0$ such that for any $u \in \mathcal{U}(\delta,\varepsilon)$, any $\alpha \in \Omega(\delta)$ and any $z_0 \in B_{H^{s'}_\sigma}(0,\delta)$, there are a $T^+(z_0) > 0$ and a solution $z \in \mathcal{C}^1\left((-T_0, T^+(z_0)); H^{s'}\right)$ of the equation*

$$\begin{cases} \partial_t z &= JH(u)z + \mathbf{P}_c(u)JN(u,\eta) \\ & \qquad +\mathbf{P}_c(u)dS(u)A(u,\eta)\langle N(u,\eta), dS(u)\rangle \\ & \qquad -(D\mathbf{P}_c(u))\dot{u}\eta, \\ z(0) &= z_0, \end{cases} \tag{45}$$

where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$.

We have $T^+(z_0)$ such that $T^+(z_0) = +\infty$ or $\lim_{t\to T^+(z_0)} \|z(t)\|_{H^{s'}} \geq \delta$.

If $z_0 \in \mathcal{H}_c(u(0))$ then $z(t) \in \mathcal{H}_c(u(t))$ for $t \in [0, T^+(z_0))$.

Let be $u \in \mathcal{U}(\varepsilon,\delta)$, $\alpha \in \Omega(\delta)$ and $z_0 \in \mathcal{H}_c(u(0)) \cap H^{s'}_\sigma$. Let us write

$$u_\infty = \lim_{t\to+\infty} u(t),$$

we define $\mathcal{T}_{u,\alpha,z_0}(z)$ by

$$\begin{aligned} \mathcal{T}_{u,\alpha,z_0}(z)(t) &= e^{JtH(u_\infty)}z_0 \\ &- \int_0^t e^{J(t-v)H(u_\infty)}\mathbf{P}_c(u(v))J\left\{E(S(u(v))) - E(S(u_\infty))\right\}z(v)\,dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)}\mathbf{P}_c(u(v))J\left\{d^2F(S(u(v))) - d^2F(S(u_\infty))\right\}z(v)\,dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)}\mathbf{P}_c(u(v))JN(u(v),\eta(v))\,dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)}\mathbf{P}_c(u(v))dS(u(s))A(u(v),\eta)\langle N(u(v),\eta(v)), dS(u(v))\rangle\,dv \\ &- \int_0^t e^{J(t-v)H(u_\infty)}(d\mathbf{P}_c(u(v)))\dot{u}\eta(v)\,dv. \end{aligned}$$

Lemma B.44 is a consequence of the fix point theorem applied to $\mathcal{T}_{u,\alpha,z_0}$. The map $\mathcal{T}_{u,\alpha,z_0}$ leaves a small ball of $\mathcal{C}((-T_0;T), H^{s'})$ invariant for small ε and δ. This fact follows from Lemma B.57 and the

Lemma B.45. *If F satisfies Assumption 1.4. Then for $\sigma \in \mathbb{R}$, $s > 1$, $p, p_1, p_2 \in [1,\infty]$ and $q \in [1,\infty]$ satisfying*

$$\frac{1}{p} + \frac{s}{3} \geq \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p},$$

there exist $\varepsilon > 0$ and $C > 0$ such that for all $u \in B_{\mathbb{C}}(0,\varepsilon)$ and $\eta \in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8) \cap L^\infty(\mathbb{R}^3,\mathbb{R}^8)$, we have

$$\|\langle Q\rangle^\sigma N(u,\eta)\|_{B^s_{p,q}} \leq C\left(|u| + \|\eta\|_{B^s_{p_2,q}}\right)\|\eta\|_{L^\infty}\|\langle Q\rangle^\sigma\eta\|_{B^s_{p_1,q}}. \tag{46}$$

Proof – We recall the definition

$$N(u,\eta) = \nabla F(S(u)+\eta) - \nabla F(S(u)) - d^2F(S(u))\eta.$$

We have

$$N(u,\eta) = \int_0^1 \int_0^1 d^3F(S(u)+\theta'\theta\eta)\cdot\eta\cdot\theta\eta\, d\theta' d\theta.$$

Then we use for $s\in\mathbb{R}_+^*$, p, p_1, p_2, $\in[1,\infty]$ such that $\frac{1}{p}+\frac{s}{3}\geq\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{1}{p}$,

$$\|uv\|_{B^s_{p,q}} \leq C\|u\|_{B^s_{p_1,q}}\|v\|_{B^s_{p_2,q}},$$

and for $s>1$, we use [EV97, Proposition 2.1]

$$\|d^3F(\psi)\|_{B^s_{p_2,q}} \leq C\left(s,F,\|\psi\|_\infty\right)\|\psi\|_{B^s_{p_2,q}},$$

eventually using Lemma B.37, and

$$\left\|\langle Q\rangle^\sigma|\eta|^2\right\|_{B^s_{p_1,q}} \leq C\|\eta\|_{L^\infty}\|\langle Q\rangle^\sigma\eta\|_{B^s_{p_1,q}}$$

we conclude the proof. □

The map $\mathcal{T}_{u,\alpha,z_0}$ is also a contraction for the norm of $\mathcal{C}((-T_0;T),H^{s'})$ for small ε and δ. This follows from Lemma B.57 and the two following lemmas:

Lemma B.46. *If F satisfies Assumption 1.4. Then for any $\sigma\in\mathbb{R}$, $s>1$, $p,q\in[1,\infty]$ with $sp\geq 3$, there exist $\varepsilon>0$ and $C>0$ such that for all u, $u'\in B_{\mathbb{C}}(0,\varepsilon)$ and η, $\eta'\in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)$, we have*

$$\begin{aligned}\left\|\langle Q\rangle^\sigma\left\{N(u,\eta)-N(u',\eta')\right\}\right\|_{B^s_{p,q}} &\leq C\left(s,F,|u|+|u'|+\|\eta\|_{B^s_{p,q}}+\|\eta'\|_{B^s_{p,q}}\right)\times\\ &\times\Bigg\{\left(\|\langle Q\rangle^{\sigma_1}\eta\|_{B^s_{p,q}}+\|\langle Q\rangle^{\sigma_1}\eta'\|_{B^s_{p,q}}\right)^2\left(|u-u'|+\left\|\langle Q\rangle^{\sigma_2}(\eta-\eta')\right\|_{B^s_{p,q}}\right)\\ &\quad+\left(|u|+|u'|+\left\|\langle Q\rangle^{\sigma_1'}\eta\right\|_{B^s_{p,q}}+\left\|\langle Q\rangle^{\sigma_1'}\eta'\right\|_{B^s_{p,q}}\right)\times\\ &\quad\times\left(\|\langle Q\rangle^{\sigma_2'}\eta\|_{B^s_{p,q}}+\|\langle Q\rangle^{\sigma_2'}\eta'\|_{B^s_{p,q}}\right)\|\langle Q\rangle^{\sigma_3'}\left(\eta-\eta'\right)\|_{B^s_{p,q}}\Bigg\},\end{aligned}$$

with $2\sigma_1+\sigma_2=\sigma_1'+\sigma_2'+\sigma_3'=\sigma$.

Proof – Using the identity

$$N(u,\eta) = \int_0^1 \int_0^1 d^3F(S(u)+\theta'\theta\eta)\cdot\eta\cdot\theta\eta\, d\theta' d\theta.$$

we can restrict the study to $d^3F(\phi)-d^3F(\phi')$. If $F=O(|z|^5)$, we have

$$\|\langle Q\rangle^\sigma\left(d^3F(\phi)-d^3F(\phi')\right)\|_{B^s_{p,q}} \leq \int_0^1 \|d^4F(\phi+t(\phi-\phi'))\|_{B^s_{p,q}}\|\langle Q\rangle^\sigma(\phi-\phi')\|_{B^s_{p,q}}\, dt.$$

Then since $s > 0$, we use

$$\|d^4F(\psi)\|_{B^s_{p,q}} \leq C(s, F, \|\psi\|_{B^s_{p,q}}).$$

Using Lemma B.6, we conclude the proof when $F = O(|z|^5)$.

Otherwise, if F is an homogeneous polynomial of order 4, the proof is easily adaptable since d^4F is a constant matrix of $\mathcal{M}_8(\mathbb{R})$.

The case $F = O(|z|^4)$ follows by summing the two previous one since as a function of $u \in \mathbb{R}^8$, $F(u) = Au^{\otimes 4} + O(|u|^5)$. □

We also need the

Lemma B.47. *If F satisfies Assumption 1.4, then for any $\sigma \in \mathbb{R}$, $s \in \mathbb{R}$, $p, q \in [1, \infty]$ and $M > 0$ there exist $\varepsilon > 0$ and $C > 0$ such that for all u, $u' \in B_{\mathbb{C}}(0, \varepsilon)$ and η, $\eta' \in B_{B^s_{p,q}(\mathbb{R}^3, \mathbb{R}^8)}(0, M)$, we have*

$$\left|A(u, \eta) - A(u', \eta')\right| \leq C \left\{ \left|u - u'\right| + \left\| \langle Q\rangle^\sigma \left\{\eta - \eta'\right\}\right\|_{B^s_{p,q}} \right\}$$

Proof – We recall that

$$A(u, \eta) = [\langle JdS(u), dS(u)\rangle - \langle J\eta, d^2S(u)\rangle]^{-1}.$$

We have

$$\begin{aligned} A(u, \eta) - A(u', \eta') = &-[\langle JdS(u), dS(u)\rangle - \langle J\eta, d^2S(u)\rangle]^{-1}\times \\ &\times \left\{\langle JdS(u), dS(u)\rangle - \langle J\eta, d^2S(u)\rangle - \langle JdS(u'), dS(u')\rangle + \langle J\eta', d^2S(u')\rangle\right\} \times \\ &\times [\langle JdS(u'), dS(u')\rangle - \langle J\eta', d^2S(u')\rangle]^{-1}. \end{aligned}$$

The lemma then follows from Lemma B.37. □

Proof – [of Lemma B.44] Hence we can apply the fixed point theorem to $\mathcal{T}_{u,\alpha,z_0}$ in a small ball of $\mathcal{C}((-T_0; T), H^{s'})$ and we obtain a fixed point z which is a solution of (45). Then with classical arguments, we prove the existence of $T^+(z_0)$ such that for $t \in (-T_0, T^+(z_0))$, we have $\|z(t)\|_{H^{s'}} < \delta$ and $\lim_{t\to T^+(z_0)}$. A derivation of $(1 - \mathbf{P}_c(u(t)))z(t)$ shows that $(1 - \mathbf{P}_c(u(t)))z(t) = 0$ for $t \in (-T_0, T^+(z_0))$ if $(1 - \mathbf{P}_c(u(0)))z_0 = 0$. □

Now, we prove that

Lemma B.48. *Suppose that Assumptions 1.1–1.4 hold. There exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in \Omega(\delta)$, $u \in \mathcal{U}(\varepsilon, \delta)$, and $z_0 \in B_{H^{s'}_\sigma}(0, \delta) \cap \mathcal{H}_c(u(0))$, we have $T^+(z_0) = +\infty$ and $z \in \mathcal{Z}(u, \delta)$.*

Proof – Since z is a fixed point of $\mathcal{T}_{u,\alpha,z_0}$, for any $t \in [0, T^+(z_0))$, with Lemma B.57 and

Lemma B.40, we obtain

$$\begin{aligned}
&\|z(t)\|_{H^{s'}}\\
&\quad\le C\|\mathbf{P}_c(u_\infty)z(t)\|_{H^{s'}}\\
&\quad\le C\|z_0\|_{H^{s'}}+C\int_0^t \|\{E(S(u(v)))-E(S(u_\infty))\}\,z(v)\|_{H^{s'}}\,dv\\
&\qquad+C\int_0^t \left\|\left\{d^2F(S(u(v)))-d^2F(S(u_\infty))\right\}z(v)\right\|_{H^{s'}}\,dv\\
&\qquad+C\int_0^t \|N(u(v),\eta(v))\|_{H^{s'}}\,dv\\
&\qquad+C\int_0^t \|dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)),dS(u(v))\rangle\|_{H^{s'}}\,dv\\
&\qquad+C\int_0^t \|(d\mathbf{P}_c(u(v)))\dot u(v)\eta(v)\|_{H^{s'}}\,dv.
\end{aligned}$$

Now, with Lemma B.45, we obtain

$$\begin{aligned}
\|z(t)\|_{H^{s'}} \;\le\;& C\|z_0\|_{H^{s'}}+C\varepsilon\int_0^t |u(v)-u_\infty|\,\|z\|_{H^{s'}}\,dv\\
&+C\int_0^t \left(|u(v)|+\|\eta(v)\|_{H^{s'}}\right)\|\eta(v)\|_{L^\infty}\|\eta(v)\|_{H^{s'}}\,dv\\
&+C\int_0^t |\dot u(v)|\,\|\eta(v)\|_{H^{s'}}\,dv,
\end{aligned}$$

and so

$$\|z(t)\|_{H^{s'}} \;\le\; C\|z_0\|_{H^{s'}}+C\varepsilon\delta^3+C\left(\varepsilon+\delta\right)\delta^2+C\delta^3.$$

Then, we also have

$$\begin{aligned}
z(t)=\;& e^{-\mathrm{i}tH+\mathrm{i}\int_0^t E(u(r))\,dr}z_0\\
&+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\mathbf{P}_c(u(v))Jd^2F(S(u(v)))z(v)\,dv\\
&+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\mathbf{P}_c(u(v))JN(u(v),\eta(v))\,dv\\
&+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\mathbf{P}_c(u(v))\times\\
&\times dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)),dS(u(v))\rangle\,dv\\
&-\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}(d\mathbf{P}_c(u(v)))\dot u(v)\eta(v)\,dv.
\end{aligned}$$

Hence by Lemma B.40 and Theorem B.4, we have

$$\begin{aligned}
&\|z(t)\|_{B^{\beta}_{\infty,2}} \\
&\leq C\langle t\rangle^{-3/2}\|z_0\|_{B^{\beta+3}_{1,2}} + C\int_0^t \langle t-v\rangle^{-3/2}\left\|d^2F(S(u(v)))z(v)\right\|_{B^{\beta+3}_{1,2}}\,dv \\
&\quad +C\int_0^t \langle t-v\rangle^{-3/2}\|N(u(v),\eta(v))\|_{B^{\beta+3}_{1,2}}\,dv \\
&\quad +C\int_0^t \langle t-v\rangle^{-3/2}\|dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)),dS(u(v))\rangle\|_{B^{\beta+3}_{1,2}}\,dv \\
&\quad +C\int_0^t \langle t-v\rangle^{-3/2}\|(d\mathbf{P}_c(u(v)))\dot{u}(v)\eta(v)\|_{B^{\beta+3}_{1,2}}\,dv.
\end{aligned}$$

With Lemma B.45, we infer

$$\begin{aligned}
&\|z(t)\|_{B^{\beta}_{\infty,2}} \leq \\
&C\langle t\rangle^{-3/2}\|z_0\|_{B^{\beta+3}_{1,2}} + C\int_0^t \langle t-v\rangle^{-3/2}|u(v)|^2\|z(v)\|_{H^{\beta+3}_{-\sigma}}\,dv \\
&+C\int_0^t \langle t-v\rangle^{-3/2}\left(|u(v)|+\|\eta(v)\|_{H^{\beta+3}}\right)\|\eta(v)\|_{L^\infty}\|\eta(v)\|_{H^{\beta+3}}\,dv \\
&+C\int_0^t \langle t-v\rangle^{-3/2}\left(|u(v)|+\|\eta(v)\|_{H^{\beta+3}}\right)\|\eta(v)\|_{L^\infty}\|\eta(v)\|_{H^{\beta+3}}\,dv \\
&+C\int_0^t \langle t-v\rangle^{-3/2}|\dot{u}(v)|\,\|\eta(v)\|_{H^{\beta+3}}\,dv.
\end{aligned}$$

With the estimate

$$\int_0^t \langle t-v\rangle^{-3/2}\langle v\rangle^{-3/2}\,dv \leq C\langle t\rangle^{-3/2},$$

we infer

$$\langle t\rangle^{3/2}\|z(t)\|_{B^{\beta}_{\infty,2}} \leq C\|z_0\|_{B^{\beta+3}_{1,2}} + C\varepsilon^2\delta + C(\varepsilon+\delta)\delta^2 + C\delta^3.$$

Then we also have

$$\begin{aligned}
z(t) = e^{-\mathrm{i}tH+\mathrm{i}\int_0^t E(u(r))\,dr}z_0 &+ \int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\times \\
&\times \mathbf{P}_c(u(v))J\nabla F(\eta(v))\,dv \\
&+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\times \\
\times \mathbf{P}_c(u(v))J\{\nabla F(S(u(v))+\eta(v)) &- \nabla F(S(u(v)) - \nabla F(\eta(v))\}\,dv \\
&+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\times \\
\times \mathbf{P}_c(u(v))dS(u(v))A(u(v),\eta(v))&\langle N(u(v),\eta(v)),dS(u(v))\rangle\,dv \\
&-\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}(d\mathbf{P}_c(u(v)))\dot{u}(v)\eta(v)\,dv.
\end{aligned}$$

We now use Lemma B.40 and Theorem B.3, except for the second term of the right hand side for which we used Theorem B.4 since $\sigma > 3/2$. We also use Lemma B.45, except for the third term of the right hand side for which an obvious adaptation of the proof of Lemma B.46 gives

$$\|\nabla F(S(u(v))+\eta(v))-\nabla F(S(u(v))-\nabla F(\eta(v))\|_{H^s_\sigma} \\ \leq C\left(|u(v)|+\|\eta(v)\|_{H^s}\right)|u(v)|\,\|\eta(v)\|_{H^s_{-\sigma}}$$

and so we obtain

$$\langle t\rangle^{3/2}\|z(t)\|_{H^s_{-\sigma}} \leq C\|z_0\|_{H^s_\sigma}+C\delta^3+C\left(\varepsilon+\delta\right)\delta+C\left(\varepsilon+\delta\right)\delta^2+C\delta^3.$$

Therefore we have that $T^+(z_0)=+\infty$ and z belongs to $\mathcal{Z}(u,\delta)$ if $\|z_0\|_{H^{s'}_\sigma}$, δ and ε are small enough. □

The global solution z just found is a function of z_0, u and α and we write $z=z[z_0,u,\alpha]$. As a function of these parameters, it has the useful property given by the

Lemma B.49. *Suppose that Assumptions 1.1–1.4 hold. There exist* $\delta_0>0$, $\varepsilon_0>0$ *and* $\kappa\in(0,1)$ *such that for any* $\delta\in(0,\delta_0)$, $\varepsilon\in(0,\varepsilon_0)$, $u,\ u'\in\mathcal{U}(\varepsilon,\delta)$, $\alpha,\ \alpha'\in\Omega(\delta)$, $z_0\in B_{H^{s'}_\sigma}(0,\delta)\cap\mathcal{H}_c(u(0))$, *and* $z_0'\in B_{H^{s'}_\sigma}(0,\delta)\cap\mathcal{H}_c(u'(0))$, *we have*

$$\left\|e^{-\mathrm{i}\int_0^t E(u(r))\,dr}z[z_0',u',\alpha']-e^{-\mathrm{i}\int_0^t E(u'(r))\,dr}z[z_0,u,\alpha]\right\|_{L^\infty(\mathbb{R}^+,H^s_{-\sigma})} \leq \left\|z_0-z_0'\right\|_{L^\infty(\mathbb{R}^+,H^{s'})} \\ +\kappa\left\{\left\|u-u'\right\|_{L^\infty}+\left\|\dot u-\dot u'\right\|_{L^1}+\left\|\alpha-\alpha'\right\|_{L^\infty}\right\}.$$

Proof – It is an easy consequence of straightforward estimates on the following identity

$$\begin{aligned} z(t) &= e^{-\mathrm{i}tH+\mathrm{i}\int_0^t E(u(r))\,dr}z_0 \\ &\quad+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\mathbf{P}_c(u(v))Jd^2F(S(u(v)))z(v)\,dv \\ &\quad+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\mathbf{P}_c(u(v))JN(u(v),\eta(v))\,dv \\ &\quad+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}\mathbf{P}_c(u(v))\times \\ &\quad\times dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)),dS(u(v))\rangle\,dv \\ &\quad-\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(u(r))\,dr}(d\mathbf{P}_c(u(v)))\dot u(v)\eta(v)\,dv. \end{aligned}$$

based only on Lemma B.46 and B.47. □

We also have the following

Lemma B.50. *Under the assumptions of Lemma B.48, take an arbitrary* $u\in\mathcal{U}(\delta,\varepsilon)$ *and an arbitrary* $\alpha\in\Omega(\delta)$, *and consider the solution* z *of* (45), *with initial condition* $z_0\in B_{H^{s'}_\sigma}(0,\delta)\cap\mathcal{H}_c(u(0))$ *small. Then the following limit*

$$z_\infty=\lim_{t\to\infty}e^{-JtH(u_\infty)}z(t)$$

exists in $H^{s'} \cap B^{\beta}_{\infty,2} \cap H^s_{-\sigma}$*. Moreover, we have* $z_\infty \in \mathcal{H}_c(u_\infty)$ *and*

$$\begin{aligned} \|e^{JtH(u_\infty)} z_\infty - z(t)\|_{H^{s'}} &\leq C\frac{\delta^2}{\langle t\rangle}, \\ \|e^{JtH(u_\infty)} z_\infty - z(t)\|_{B^{\beta}_{\infty,2}} &\leq C\frac{\delta^2}{\langle t\rangle^2}, \\ \|e^{JtH(u_\infty)} z_\infty - z(t)\|_{H^s_{-\sigma}} &\leq C\frac{\delta^2}{\langle t\rangle^2}. \end{aligned}$$

Proof – Using exactly the same method as in the proof of Lemma B.48, applied to

$$\begin{aligned} \mathcal{T}_{u,\alpha,z_0}(z)(t) &= e^{JtH(u_\infty)} z_0 \\ &- \int_0^t e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) J \left\{E(S(u(v))) - E(S(u_\infty))\right\} z(v)\, dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) J \left\{d^2F(S(u(v))) - d^2F(S(u_\infty))\right\} z(v)\, dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) J N(u(v),\eta(v))\, dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) dS(u(s)) A(u(v),\eta) \langle N(u(v),\eta(v)), dS(u(v))\rangle\, dv \\ &- \int_0^t e^{J(t-v)H(u_\infty)} (d\mathbf{P}_c(u(v)))\dot{u}\eta(v)\, dv. \end{aligned}$$

we prove that the limit exists using the estimates on z. Using the technics used to obtain the estimates on z, we obtain the same estimates for $e^{JtH(u_\infty)} z_\infty$. Since $e^{JtH(u_\infty)} z_\infty$ tends to zero, z_∞ necessarily belongs to $\mathcal{H}_c(u_\infty)$.

Using same ideas, we also obtain the convergence rate. The only problem lies the term:

$$\int_t^{+\infty} e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) J \left\{E(S(u(v))) - E(S(u_\infty))\right\} z(v)\, dv.$$

To treat it we write

$$\begin{aligned} &\int_t^{+\infty} e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) J \left\{E(S(u(v))) - E(S(u_\infty))\right\} z(v)\, dv = \\ &\quad \int_t^{+\infty} e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) J \left\{E(S(u(v))) - E(S(u_\infty))\right\} \left(e^{JvH(u_\infty)} z(v) - z_\infty\right) dv \\ &\quad + \int_t^{+\infty} e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) J \left\{E(S(u(v))) - E(S(u_\infty))\right\} e^{JvH(u_\infty)} z_\infty\, dv. \end{aligned}$$

When estimating the decay rate, the first term of the right hand side can be treated by absorbing it on the left side of the inequality for the $H^{s'}$ estimate or using the $H^{s'}$ estimate for the other estimates. To treat the second term, we write

$$\begin{aligned} &\int_t^{+\infty} e^{J(t-v)H(u_\infty)} \mathbf{P}_c(u(v)) J \left\{E(S(u(v))) - E(S(u_\infty))\right\} e^{JvH(u_\infty)} z_\infty\, dv = \\ &\quad \int_t^{+\infty} \mathbf{P}_c(u(v)) J \left\{E(S(u(v))) - E(S(u_\infty))\right\} dv e^{JtH(u_\infty)} z_\infty \\ &\quad + \int_t^{+\infty} \mathbf{P}_c(u(v)) [e^{J(t-v)H(u_\infty)}, J] \left\{E(S(u(v))) - E(S(u_\infty))\right\} e^{JvH(u_\infty)} z_\infty\, dv. \end{aligned}$$

When estimating the decay rate, we treat the first term of the right hand side we use estimates on $e^{JtH(u_\infty)}$, the Duhamel's formula and estimates on $e^{-\mathrm{i}tH}$. To deal with the other term, we use

$$[e^{JsH(u_\infty)}, J] = \int_0^s e^{JrH(u_\infty)}[J, d^2F(S(u_\infty))]e^{J(s-r)H(u_\infty)}\,dr,$$

this gives us for the second term of the right hand side

$$\int_t^{+\infty} \mathbf{P}_c(u(v)) \int_0^{t-v} e^{JrH(u_\infty)}[J, d^2F(S(u_\infty))]e^{J(t-r)H(u_\infty)} z_\infty\,dr(E(S(u(v)))-E(S(u_\infty)))\,dv.$$

To estimate the decay rate, we use estimates on $e^{JtH(u_\infty)}$, the Duhamel's formula and estimates on $e^{-\mathrm{i}tH}$. □

Remark B.50.1. *The preceding proof also works with the formula*

$$\begin{aligned}
e^{\mathrm{i}tD_m-\mathrm{i}\int_0^t E(u(r))\,dr}z(t) = z_0 &+ \int_0^t e^{\mathrm{i}vD_m-\mathrm{i}\int_0^v E(u(r))\,dr}\mathbf{P}_c(u(v))Vz(v)\,dv \\
&+ \int_0^t e^{\mathrm{i}vD_m-\mathrm{i}\int_v^t E(u(r))\,dr}\mathbf{P}_c(u(v))Jd^2F(S(u(v)))z(v)\,dv \\
&+ \int_0^t e^{\mathrm{i}vD_m-\mathrm{i}\int_0^v E(u(r))\,dr}\mathbf{P}_c(u(v))JN(u(v),\eta(v))\,dv \\
&+ \int_0^t e^{\mathrm{i}vD_m-\mathrm{i}\int_0^v E(u(r))\,dr}\times \\
&\times \mathbf{P}_c(u(v))dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\,dv \\
&- \int_0^t e^{\mathrm{i}vD_m-\mathrm{i}\int_0^v E(u(r))\,dr}(d\mathbf{P}_c(u(v)))\dot{u}(v)\eta(v)\,dv.
\end{aligned}$$

Hence we obtain the same result with

$$e^{-\mathrm{i}tD_m+\mathrm{i}\int_0^t E(u(r))\,dr}z_\infty$$

instead of

$$e^{JtH(u_\infty)}z_\infty.$$

But we obtain the estimates

$$\begin{aligned}
\|e^{-\mathrm{i}tD_m+\mathrm{i}\int_0^t E(u(r))\,dr}z_\infty - z(t)\|_{H^{s'}} &\leq C\delta, \\
\|e^{-\mathrm{i}tD_m+\mathrm{i}\int_0^t E(u(r))\,dr}z_\infty - z(t)\|_{B^\beta_{\infty,2}} &\leq C\frac{\delta}{\langle t\rangle^{3/2}}, \\
\|e^{-\mathrm{i}tD_m+\mathrm{i}\int_0^t E(u(r))\,dr}z_\infty - z(t)\|_{H^s_{-\sigma}} &\leq C\frac{\delta}{\langle t\rangle^{3/2}}.
\end{aligned}$$

The same holds with

$$e^{-\mathrm{i}tH+\mathrm{i}\int_0^t E(u(r))\,dr}z_\infty$$

instead of

$$e^{JtH(u_\infty)}z_\infty,$$

and we can prove $z_\infty \in \mathcal{H}_c$.

5.2 Step 2: Construction of α

For any $u \in \mathcal{U}(\varepsilon,\delta)$ and $z_0 \in B_{H^{s'}_\sigma}(0,\delta) \cap \mathcal{H}_c(u(0))$, let us define a map $\mathcal{G}_{u,z_0}$ on $\Omega(\delta)$ by

$$\begin{aligned}\mathcal{G}_{u,z_0}(\alpha)^{\pm}(t) = -\int_t^{\infty} e^{\int_s^t E_1^{\pm}(u(w))\,dw} \Big\{ &\langle JN(u(v),\eta(v)), S_1^{\pm}(u(v))\rangle \\ &+ \langle dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\, S_1^{\pm}(u(v))\rangle \\ &- \langle (dS_1^{\pm}(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle,\, S_1^{\pm}(u(v))\rangle \alpha^{\pm}(v) \\ &- \langle (dS_1^{\mp}(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle,\, S_1^{\pm}(u(v))\rangle \alpha^{\mp}(v) \Big\}\, dv,\end{aligned}$$

where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$ and $z = z[z_0,u,\alpha]$ is the solution found in the previous paragraph.

We have that $\mathcal{G}_{u,z_0}$ leaves $\Omega(\delta)$ invariant as stated in the

Lemma B.51. *Suppose that Assumptions 1.1–1.4 hold. There exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\delta \in (0,\delta_0)$, $\varepsilon \in (0,\varepsilon_0)$, $u \in \mathcal{U}(\varepsilon,\delta)$ and $z_0 \in B_{H^{s'}_\sigma}(0,\delta) \cap \mathcal{H}_c(u(0))$, we have that $\mathcal{G}_{u,z_0}(\alpha)$ maps $\Omega(\delta)$ into itself.*

Proof – We have by means of Estimate (46) with e.g. $\sigma < -3$, $s = 0$, $p = q = 2$ and $p_1 = p_2 = 4$, if $u_0 \in \mathbb{C}$ and $z_0 \in \mathcal{H}_c(u_0) \cap H^{s'}_\sigma(\mathbb{R}^3,\mathbb{R}^8)$ are small enough

$$\begin{aligned}&\left|\mathcal{G}_{u,z_0}(\alpha)^{\pm}(t)\right| \\ \le\;& C\int_t^{\infty}\Big\{ \left|\langle JN(u(s),\eta(s)), JS_1^{\pm}(u(s))\rangle\right| \\ +\;& \left|\langle dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\, JS_1^{\pm}(u(v))\rangle\right| \\ +\;& \left|\langle (dS_1^{\pm}(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle,\, JS_1^{\pm}(u(v))\rangle\alpha^{\pm}(v)\right| \\ +\;& \left|\langle (dS_1^{\mp}(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle,\, JS_1^{\pm}(u(v))\rangle\alpha^{\mp}(v)\right|\Big\}\, dv \\ \le\;& C\delta^2\langle t\rangle^{-2}.\end{aligned}$$

Hence for small δ and small ε, we have $\mathcal{G}_{u,z_0}(\Omega(\delta)) \subset \Omega(\delta)$. □

Then $\mathcal{G}_{u,z_0}$ is a contraction for the L^∞ norm, as stated in the

Lemma B.52. *Suppose that Assumptions 1.1–1.4 hold. There exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and $\kappa \in (0,1)$ such that for any $\delta \in (0,\delta_0)$, $\varepsilon \in (0,\varepsilon_0)$, $u, u' \in \mathcal{U}(\varepsilon)$, $z_0 \in B_{H^{s'}_\sigma}(0,\delta) \cap \mathcal{H}_c(u(0))$, $z_0' \in B_{H^{s'}_\sigma}(0,\delta) \cap \mathcal{H}_c(u'(0))$ and $\alpha, \alpha' \in \Omega(\delta)$, we have*

$$\begin{aligned}&\left\|\mathcal{G}_{u',z_0'}(\alpha') - \mathcal{G}_{u,z_0}(\alpha)\right\|_{L^\infty(\mathbb{R}^+)} \\ &\qquad\le \kappa\left(\left|u'-u\right|_{L^\infty(\mathbb{R}^+)} + \left|\alpha'-\alpha\right|_{L^\infty(\mathbb{R}^+)} + \left\|\dot u - \dot u'\right\|_{L^1} + \left\|z_0' - z_0\right\|_{H^{s'}}\right).\end{aligned}$$

Proof – It is a straightforward computation based on Lemma B.49, on Lemma B.46 with e.g. $\sigma < -6$, $\sigma_2, \sigma_3' < -3$ and $s = 0$, $p = q = 2$, on Lemma B.47, on Lemma B.45 with e.g. $\sigma < -3$, $p = q = 2$ and $p_1 = p_2 = 4$ and on Lemma B.39. □

We now state the

Lemma B.53. *Suppose that Assumptions 1.1–1.4 hold. There exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, $u \in \mathcal{U}(\varepsilon)$ *and* $z_0 \in B_{H^{s'}_\sigma}(0, \delta) \cap \mathcal{H}_c(u(0))$, *the equation*

$$\begin{aligned}\dot{\alpha}^{\pm} = E^{\pm}(u)\alpha^{\pm} + \langle JN(u,\eta), JS_1^{\pm}(u)\rangle \\ + \langle dS(u)A(u,\eta)\langle N(u,\eta), dS(u)\rangle\, JS_1^{\pm}(u)\rangle \\ - \langle (dS_1^{\pm}(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle,\ JS_1^{\pm}(u)\rangle\alpha^{\pm} \\ - \langle (dS_1^{\mp}(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle,\ JS_1^{\pm}(u)\rangle\alpha^{\mp},\end{aligned}$$

where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z[z_0, u, \alpha](t)$, *has a unique solution in* $\Omega(\delta)$.

Proof – The proof is classical since we proved that the integral equation

$$\alpha(t) = \mathcal{G}_{u,z_0}(\alpha)(t)$$

can be solved by means of the fixed point theorem. □

5.3 Step 3: Construction of u

Here we want to solve the equation for u. We notice that z and α have been built in the previous section and are functions of u and $z_0 \in \mathcal{H}_c(u(0))$. Let us introduce for any $\alpha \in \Omega(\delta)$ and $u_0 \in B_{\mathbb{C}}(0, \varepsilon)$ the function on $\mathcal{U}(\varepsilon, \delta)$:

$$f_{u_0,z_0}(u)(t) = u_0 - \int_0^t A(u(v), \eta(v))\langle N(u(v), \eta(v)), dS(u(v))\rangle\, dv,$$

where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$. We have the

Lemma B.54. *Suppose that Assumptions 1.1–1.4 hold. There exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $\delta \in (0, \delta_0)$ *and* $\varepsilon \in (0, \varepsilon_0)$, *the function* f_{u_0,z_0} *maps* $\mathcal{U}(\varepsilon, \delta)$ *into itself if* u_0 *and* $z_0 \in H^{s'}_\sigma \cap \mathcal{H}_c(u_0)$ *are small enough.*

Proof – By means of Lemma B.45, we obtain

$$|f_{u_0,z_0}(u)(t)| \leq |u_0| + C\int_0^t \|N(u(v), \eta(v))\|_{H^s_{-\sigma}} \leq |u_0| + C\,(\varepsilon + \delta)\,\delta^2.$$

Hence for u_0 and δ small $f_{u_0,z_0}(u)(t) \in B_{\mathbb{C}}(0, \varepsilon)$. Estimate (46) also gives the existence of $(f_{u_0,z_0}(u))_\infty = \lim_{t\to+\infty} f_{u_0,z_0}(u)(t)$ and then

$$\left|\frac{d}{dt} f_{u_0,z_0}(u)\right| \leq C\,\|N(u,\eta)\|_{H^s_{-\sigma}} \leq \frac{C}{\langle t\rangle^3}\,(\varepsilon + \delta)\,\delta^2.$$

□

The function f_{u_0,z_0} has also a Lipschitz property as stated by the

Lemma B.55. *Suppose that Assumptions 1.1–1.4 hold. There exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *and* $\kappa \in (0,1)$ *such that for any* $\delta \in (0,\delta_0)$, $\varepsilon \in (0,\varepsilon_0)$, $u,\ u' \in \mathcal{U}(\varepsilon,\delta)$, $z_0 \in \mathcal{H}_c(u(0)) \cap H^{s'}_\sigma$, $z_0' \in \mathcal{H}_c(u'(0)) \cap H^{s'}_\sigma$ *small enough, and* u_0, u_0' *small enough, we have*

$$\left| f_{u_0,z_0}(u) - f_{u_0',z_0'}(u') \right|_{L^\infty(\mathbb{R}^+)} + \left| \partial_t f_{u_0,z_0}(u) - \partial_t f_{u_0',z_0'}(u') \right|_{L^1(\mathbb{R}^+)}$$
$$\leq |u_0 - u_0'| + \kappa \left(\|u - u'\|_{L^\infty(\mathbb{R}^+)} + \|\dot{u} - \dot{u}'\|_{L^1} + \|z_0 - z_0'\|_{H^{s'}_\sigma} \right).$$

Proof – This a straightforward consequence of Lemma B.46, B.47, B.49 and B.52. □

We are now able to prove the

Lemma B.56. *Suppose that Assumptions 1.1–1.4 hold. There exists* $\delta_0 > 0$ *such that for any* $\delta \in (0,\delta_0)$ $u_0 \in \mathbb{C}$ *small enough and* $z_0 \in \mathcal{H}_c(u_0) \cap H^{s'}_\sigma$ *small enough, the equation*

$$\begin{cases} \dot{u} & = & -A(u,\eta)\langle N(u,\eta), dS(u)\rangle, \\ u(0) & = & u_0, \end{cases}$$

where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$, *has a unique solution in* $\mathcal{U}(\delta,\varepsilon)$.

Proof – This is also a straightforward consequence of the fixed point theorem for f_{u_0,z_0}. □

5.4 Step 4: End of the proof of Theorem B.6

We now conclude our proof We write $\psi_0 = S(u_0) + \alpha^+(0)S_1^+(u_0) + \alpha^-(0)S_1^-(u_0) + z_0$ where $z_0 \in \mathbf{H}_c(u_0)$ then $\psi(t) = S(u(t)) + \alpha^+(u_0,z_0)(t)S_1^+(u(t)) + \alpha^-(u_0,z_0)(t)S_1^-(u(t)) + z(t)$ where $z(t) \in \mathbf{H}_c(u(t))$. Hence if we choose $\phi_1^1(u) = \frac{S_1^+(u)+S_1^-(u)}{2}$, $\phi_1^2(u) = \frac{S_1^+(u)-S_1^-(u)}{2i}$ we have

$$\Psi_u(z) = \left(\Re\alpha^+(u_0,z_0), \quad -\Im\alpha^+(u_0,z_0)\right).$$

We also have

$$E_\infty = \int_0^\infty \{E(u(s)) - E(u_\infty)\}\ ds.$$

In the proof of Lemma B.48, we see that δ is of the same order as $\|z_0\|_{H^{s'}_\sigma}$. The rest of Theorem B.6 easily follows.

□

APPENDIX

6 The wave operator and similarity for the linearized operator

Inspired by [Kat66], we use an argument of similarity to prove the

Lemma B.57. *Suppose that Assumptions 1.1–1.4 hold. For all $s \in \mathbb{R}^+$, there exists $C_s > 0$, such that*

$$\forall t \in \mathbb{R},\ \|e^{tJH(z)}\|_{\mathcal{L}(H^s)} \leq C_s.$$

We prove this lemma by using the boundedness in H^s of the wave operator:

$$W_\pm = s - \lim_{t\to\pm\infty} e^{-tH(z)^*J} e^{-\mathrm{i}t(H-E(z))}\mathbf{P}_c(H)$$

and the intertwining property:

$$e^{-tH(z)^*J}\mathbf{P}_c(z)^* = W^\pm e^{-\mathrm{i}t(H-E(z))}\mathbf{P}_c(H)(W^\pm)^{-1}.$$

This boundedness follows from the

Lemma B.58 (Smooth and small non-selfadjoint perturbations)**.** *Suppose that Assumptions 1.1–1.4 hold. Let be $\psi \in L^2$ and $\sigma \geq 1$. Then there exist $\varepsilon > 0$ and $C > 0$ such that*

$$\forall z \in B_{\mathbb{C}}(0,\varepsilon),\ \int_0^\infty \| < Q >^{-\sigma} e^{sJH(z)}\mathbf{P}_c(z)\psi\|_2^2\, ds \leq C\|\psi\|_2^2. \tag{47}$$

Proof – By Lemma (B.40), we have $R(z,0)\mathbf{P}_c(0)\mathbf{P}_c(z) = \mathbf{P}_c(z)$.

$$\begin{aligned}
&\|\langle Q\rangle^{-\sigma} e^{tJH(z)}\mathbf{P}_c(z)\|\\
&= C\|\langle Q\rangle^{-\sigma} R(z,0)\mathbf{P}_c(0)e^{tJH(z)}\mathbf{P}_c(z)\|\\
&\leq \|\langle Q\rangle^{-\sigma} R(z,0)\mathbf{P}_c(0)e^{-\mathrm{i}t(H-E(z))}\mathbf{P}_c(z)\|\\
&\quad + \int_0^t \|\langle Q\rangle^{-\sigma} R(z,0)\mathbf{P}_c(0)e^{-\mathrm{i}(t-s)(H-E(z))}D\nabla F(S(z))e^{sJH(z)}\mathbf{P}_c(z)\|\, ds\\
&\leq \|\langle Q\rangle^{-\sigma} R(z,0)\langle Q\rangle^{\sigma}\|\|\langle Q\rangle^{-\sigma}\mathbf{P}_c(0)e^{-\mathrm{i}t(H-E(z))}\|\|\mathbf{P}_c(z)\|\\
&\quad + C|z|^2\|\langle Q\rangle^{-\sigma} R(z,0)\langle Q\rangle^{\sigma}\| \times\\
&\quad \times \int_0^t \|\langle Q\rangle^{-\sigma}\mathbf{P}_c(0)e^{-\mathrm{i}(t-s)(H-E(z))}\langle Q\rangle^{-\sigma}\|\|\langle Q\rangle^{-\sigma}e^{sJH(z)}\mathbf{P}_c(z)\|\, ds.
\end{aligned}$$

Using Proposition B.33, we obtain the claim (47) for z sufficiently small. □

This give us the existence and the boundedness of the wave operator, as stated by the following

Lemma B.59. *Suppose that Assumptions 1.1–1.4 hold. Let be*

$$W_t = e^{-tH(z)^*J} e^{-it(H-E(z))}\mathbf{P}_c(H).$$

Then the limits

$$W^\pm = \lim_{t\to\pm\infty} W_t$$

exist in $B(H^s)$ *and their ranges are equal to* $\mathrm{Ran}\,(\mathbf{P}_c(z))$*. The same is true for* W_t^* *and*

$$\left(W^{\pm}\right)^{-1} = \lim_{t\to\pm\infty} (W_t)^{-1}\,.$$

Proof – Let us define $W_t = e^{-tH(z)^*J}e^{-it(H-E(z))}$. we have for $\phi \in H_c(z)$ and $\psi \in H_c(0)$

$$\langle \phi, W_t\psi\rangle = \langle \phi, \psi\rangle + \int_0^t \left\langle \phi, \frac{d}{ds}W_s\psi\right\rangle ds,$$

Since we have

$$\begin{aligned}\left\langle \phi, \frac{d}{ds}W_s\psi\right\rangle &= \left\langle e^{-tJH(z)}\phi, D\nabla F(S(z))e^{-\mathrm{it}(H-E(z))}\psi\right\rangle \\ &\leq C|z|^2\|\langle Q\rangle^{-\sigma}e^{tJH(z)}\phi\|\|\langle Q\rangle^{-\sigma}e^{-\mathrm{it}(H-E(z))}\psi\|.\end{aligned}$$

which gives $\langle \phi, \frac{d}{ds}W_s\psi\rangle \in L^1(\mathbb{R})$, so $W_\pm$ exists and is bounded in $\mathcal{L}(H_c(0), H_c(z))$ by the previous lemma. Since for any vector ϕ in an eigenspace of $JH(z)$, $W_t^*\phi$ tends weakly to zero, we obtain that the range of $W^\pm$ is a subspace of the range of $\mathbf{P}_c(z)$. Then the same statements about $(W_t)^{-1}$ follows by the same way. The invertibility is then immediate. □

Proof – [of Lemma B.57] The L^2 bound follows from the intertwining property as explained before Lemma B.58.

The proof of the H^k bounds follows from commutation argument, we apply the same scheme to

$$\begin{aligned}\partial_i e^{-tJH(z)}\mathbf{P}_c(z) &= [\partial_i, \mathbf{P}_c(z)]e^{-tJH(z)}\mathbf{P}_c(z) + \mathbf{P}_c(z)e^{-tJH(z)}[\partial_i, \mathbf{P}_c(z)] \\ &\quad +\mathbf{P}_c(z)[\partial_i, e^{-tJH(z)}]\mathbf{P}_c(z) \\ &= [\partial_i, \mathbf{P}_c(z)]e^{-tJH(z)}\partial_i + \mathbf{P}_c(z)e^{-tJH(z)}[\partial_i, \mathbf{P}_c(z)] \\ &\quad + \int_0^t e^{-(t-s)JH(z)}\mathbf{P}_c(z)(\partial_i D\nabla F(S))e^{-sJH(z)}\mathbf{P}_c(z)dz.\end{aligned}$$

□

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B.III

Sur la stabilité asymptotique de petites ondes stationnaires d'une équation de Dirac non linéaire dans un cas résonnant

Dans ce chapitre, on étudie le comportement des perturbations de petites ondes stationnaires d'une équation de Dirac non linéaire dans le cas résonnant. On suppose que l'opérateur de référence $H = D_m + V$ a seulement deux valeurs propres et celles-ci sont doubles, cette dégénérescence est due au théorème de Kramers.

Dans ce cas, nous pouvons construire une petite variété d'états stationnaires tangente au premier espace propre de H. Puis on suppose qu'une condition de résonance est vérifiée pour la première valeur propre. Nous pouvons alors construire autour de chaque état stationnaire une variété centrale. Nous montrons alors que toute perturbation dans $H^{s'}$, avec $s' > 2$, de cette variété se stabilise sur la variété des états stationnaires. Nous construisons aussi une variété centrale stable et une variété centrale instable autour de chaque état stationnaire. Dans chaque variété, on obtient la stabilisation sur la variété centrale dans un sens du temps et l'instabilité dans l'autre sens. Finalement, à l'extérieur de toute ces variétés, nous obtenons l'instabilité dans les deux sens du temps.

On the asymptotic stability of small nonlinear Dirac standing waves in a resonant case

Abstract

In this work, we study the behavior of perturbations of small nonlinear Dirac standing waves. We assume that the linear Dirac operator of reference $H = D_m + V$ has only two double eigenvalues, this degeneracy is due to a theorem of Kramers.

In this case, we can build a small manifold of stationary solutions tangent to the first eigenspace of H. Then we assume that a resonance condition holds for the first eigenvalue. We build a center manifold of real codimension 8 around each stationary solution. Inside this center manifold any $H^{s'}$ perturbation of stationary solutions, with $s' > 2$, stabilizes towards a standing wave. We also build center-stable and center-unstable manifolds each one of real codimension 4. Inside each manifold, we obtain stabilization towards the center manifold in one direction of time, while in the other, we have instability. Eventually, outside all these manifolds, we have instability in the two directions of time.

Introduction

We study the asymptotic stability of stationary solutions of a time-dependent nonlinear Dirac equation.

A localized stationary solution of a given time-dependent equation represents a bound state of a particle. Like Rañada [Ran], we call it a *particle-like solution* (PLS). Many works have been devoted to the proof of the existence of such solutions for a wide variety of equations. Although their stability is a crucial problem (in particular in numerical computation or experiment), a smaller attention has been deserved to this issue.

In this paper, we deal with the problem of stability of small PLS of the following nonlinear Dirac equation:

$$\mathrm{i}\partial_t \psi = (D_m + V)\psi + \nabla F(\psi) \qquad \text{(NLDE)}$$

where ∇F is the gradient of $F : \mathbb{C}^4 \mapsto \mathbb{R}$ for the standard scalar product of $\mathbb{R}^8$. Here, D_m is the usual Dirac operator, see Thaller [Tha92], acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$D_m = \alpha \cdot (-\mathrm{i}\nabla) + m\beta = -\mathrm{i}\sum_{k=1}^{3} \alpha_k \partial_k + m\beta$$

where $m \in \mathbb{R}_+^*$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, β are $\mathbb{C}^4$ hermitian matrices satisfying the following properties:

$$\begin{cases} \alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik} \mathbf{1}_{\mathbb{C}^4}, & i, k \in \{1, 2, 3\}, \\ \alpha_i \beta + \beta \alpha_i = \mathbf{0}_{\mathbb{C}^4}, & i \in \{1, 2, 3\}, \\ \beta^2 = \mathbf{1}_{\mathbb{C}^4}. \end{cases}$$

Here, we choose

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}$$

$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In (NLDE), V is the external potential field and $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a nonlinearity with the following gauge invariance:

$$\forall(\theta, z) \in \mathbb{R} \times \mathbb{C}^4, \quad F(e^{i\theta} z) = F(z). \tag{1}$$

Some additional assumptions on F and V will be made in the sequel. Stationary solutions (PLS) of (NLDE) take the form $\psi(t, x) = e^{-\mathrm{i}Et}\phi(x)$ where ϕ satisfies

$$E\phi = (D_m + V)\phi + \nabla F(\phi). \tag{PLSE}$$

There exists a manifold of small solutions to (PLSE) tangent to the first eigenspace of $D_m + V$ (see Proposition B.5 below).

In [Bou05], we prove that there are stable directions for the PLS manifold under a non resonance assumption on the spectrum of $H := D_m + V$. This gives a stable manifold, containing the PLS manifold. But we were not able to say anything about solutions starting outside the stable manifold.

In the Schrödinger case, orbital stability results (see e.g [CL82], [Wei85, Wei86] or [SS85, GSS87]) give that any solution stays near the PLS manifold. Unfortunately, orbital stability criteria applied to Schrödinger equations use the fact that Schrödinger operators are bounded from below. Hence the question of orbital stability for Dirac standing waves cannot be solved by a straightforward application of the methods used in the Schrödinger case.

Concerning the asymptotic stability, in the Schrödinger equation, the question has been solved in several cases. For small stationary solutions in the simple eigenvalue case it has been studied by Soffer and Weinstein [SW90, SW92], Pillet and Wayne [PW97] or Gustafson, Nakanishi and Tsai [GNT04]. For the two eigenvalue case under a resonance condition for an excited state, the problem has been studied by Tsai and Yau [TY02a, TY02c, TY02d, TY02b, Tsa03] or Soffer and Weinstein [SW04, SW05]. Another problem has been studied by Cuccagna [Cuc01, Cuc03, Cuc05], he considered the case of big PLS, when the linearized operator has only one eigenvalue and obtained the asymptotic stability of the manifold of ground states. Schlag [Sch04] proved that any ground state of the cubic nonlinear Schrödinger equation in dimension 3 is orbitally unstable but posses a stable manifold of codimension 9.

We also would like to mention the works of Buslaev and Perel'mann [BP95, BP92b, BP92c, BP92a], Buslaev and Sulem [BS03, BS02], Weder [Wed00] or Krieger and Schlag [KS05] in the one dimensional Schrödinger case. Krieger and Schlag [KS05] proved a result similar to [Sch04] in the one dimensional case.

The result we present here, states the existence of a stable manifold and describes the behavior of solutions starting outside of it. In fact, we prove the instability of the stable manifold. We also prove stabilization towards stationary solutions inside the stable manifold for $H^{s'}$ perturbation with $s' > 2$. We have been able to obtain it since we impose a resonance condition for the first eigenvalue, while in [Bou05], we assumed there is no resonance phenomena.

This paper is organized as follow.

In Section 1, we present our main results and the assumptions we need. Subsection 1.1, is devoted to the statement of the time decay estimates of the propagator associated with $H = D_m + V$ on the continuous subspace. One is a kind of smoothness result, in the sense of Kato (see e.g. [Kat66]), the other is a Strichartz type result. We prove these estimates with the propagation and dispersive estimates proved in [Bou05]. In subsection 1.2, we state the existence of small stationary states forming a manifold tangent to an eigenspace of H. The study of the dynamics around such states leads us to our main result, see Subsection 1.3, where we split a neighborhood of a stationary state in different parts, each one giving rise to stabilization or instability.

To prove our main theorem, we consider our nonlinear system as a small perturbation of a linear equation. More precisely in Subsection 2.2, we show that the spectral properties of the linearized operator around a stationary state, presented in Section 2, permits to obtain, like in the linear case, some properties of the dynamics around a stationary state. Eventually, in Section 3, we obtain, with our time decay estimates, a stabilization result for $H^{s'}$ perturbation with $s' > 2$. Contrarily to [Bou05], we do not restrict ourself to localized perturbations. Our results are an adaptation to the Dirac case of some results of Tsai and Yau [TY02d] and Gustafson, Nakanishi and Tsai [GNT04].

1 Assumptions and statements

1.1 Time decay estimates

We generalize to small nonlinear perturbations, stability results for linear systems. These results, like in [Bou05], follow from linear decay estimates. Here we use smoothness type and Strichartz type estimates deduced from propagation and dispersive estimates of [Bou05]. Hence, we work within the

Assumption 1.1. *The potential $V : \mathbb{R}^3 \mapsto S_4(\mathbb{C})$ (self-adjoint 4×4 matrices) is a $\mathcal{C}^\infty$ function such that there exists $\rho > 5$ with*

$$\forall \alpha \in \mathbb{N}^3,\ \exists C > 0,\ \forall x \in \mathbb{R}^3,\ |\partial^\alpha V|(x) \leq \frac{C}{\langle x \rangle^{\rho + |\alpha|}}.$$

We notice that by the Kato-Rellich theorem, the operator

$$H := D_m + V$$

is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ and self-adjoint on $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

We also mention that Weyls's theorem gives us that the essential spectrum of H is $(-\infty, -m] \cup [+m, +\infty)$ and the work of Berthier and Ggeorgescu [BG87, Theorem 6, Theorem A], gives us that there is no embedded eigenvalue. Hence the thresholds $\pm m$ are the only points of the continuous spectrum which can be associated with wave of zero velocity. These waves perturb the spectral density and diminish the decay rate in the propagation and the dispersive estimates. Hence, we also work within the

Assumption 1.2. *The operator H presents no resonance at thresholds and no eigenvalue at thresholds.*

A resonance is a stationary solution in $H^{1/2}_{-\sigma} \setminus H^{1/2}$ for any $\sigma \in (1/2, \rho - 2)$, where H^t_σ is given by

Definition B.1 (Weighted Sobolev space). *The weighted Sobolev space is defined by*

$$H^t_\sigma(\mathbb{R}^3, \mathbb{C}^4) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3),\ \|\langle Q \rangle^\sigma \langle P \rangle^t f\|_2 < \infty \right\}$$

for $\sigma, t \in \mathbb{R}$. We endow it with the norm

$$\|f\|_{H^t_\sigma} = \|\langle Q\rangle^\sigma \langle P\rangle^t f\|_2.$$

If $t = 0$, we write L^2_σ instead of H^0_σ.

We have used the usual notations $\langle u\rangle = \sqrt{1+u^2}$, $P = -i\nabla$, and Q is the operator of multiplication by x in $\mathbb{R}^3$.

Now let

$$\mathbf{P}_c(H) = \mathbb{1}_{(-\infty,-m|\cup[+m,+\infty)}(H)$$

be the projector associated with the continuous spectrum of H and $\mathcal{H}_c$ its range. Using [Bou05, Theorem 1.1], we obtain a Limiting Absorption Principle which gives the

Theorem B.2 (H-smoothness of $\langle Q\rangle^{-1}$)**.** *If Assumptions 1.1 and 1.2 hold. Then for any $\sigma \geq 1$ and $s \in \mathbb{R}$, one has:*

$$\int_{\mathbb{R}} \left\|\langle Q\rangle^{-\sigma} e^{-\mathrm{i}tH}\mathbf{P}_c(H)\,\psi\right\|^2_{H^s} dt \leq C\,\|\psi\|^2_{H^s}, \tag{i}$$

$$\left\|\int_{\mathbb{R}} e^{\mathrm{i}tH}\mathbf{P}_c(H)\,\langle Q\rangle^{-\sigma} F(t)\,dt\right\|_{H^s} \leq C\,\|F\|_{L^2(\mathbb{R},H^s)}, \tag{ii}$$

$$\left\|\int_{s<t} \langle Q\rangle^{-\sigma} e^{-\mathrm{i}(t-s)H}\mathbf{P}_c(H)\,\langle Q\rangle^{-\sigma} F(s)\,ds\right\|_{L^2(\mathbb{R},H^s)} \leq C\,\|F\|_{L^2(\mathbb{R},H^s)}, \tag{iii}$$

Proof – We first prove (i). For $s = 0$, it is (see *e.g.* [ABdMG96, Proposition 7.11] or [RS78, Theorem XIII.25]) a consequence of the limiting absorption principle:

$$\sup_{\Im z\in(0,1)} \left\{\left\|\langle Q\rangle^{-\sigma}(H-z)^{-1}P_c(H)\,\langle Q\rangle^{-\sigma}\right\|_2\right\} < \infty \tag{2}$$

this follows from [Bou05, Theorem 1.1] for $\sigma > 5/2$.Then we use Born expansion

$$R^+_V(\lambda) = R^+_0(\lambda) - R^+_0(\lambda)VR^+_0(\lambda) + R^+_0(\lambda)VR^+_V(\lambda)VR^+_0(\lambda)$$

and [IM99, Theorem 2.1(i)] to conclude the proof for $s = 0$ and $\sigma \geq 1$. For $s \in 2\mathbb{Z}$ it follows from the previous cases using boudedness of $< H >^s < D_m >^{-s}$ and $< H >^{-s} < D_m >^s$ which follow from the boundedness of V and its derivatives. The rest follows by interpolation.

Estimates (i) and (ii) are equivalent by duality.

To prove estimate (iii) when $s = 0$, we notice, using the Fourier transform in time, that it is equivalent to

$$\left|\int_{\mathbb{R}} \left\langle \widehat{G}(\lambda), \langle Q\rangle^{-\sigma} R^+_V(\lambda)\mathbf{P}_c(H)\,\langle Q\rangle^{-\sigma}\widehat{F}(\lambda)\right\rangle d\lambda\right| \leq C\left\|\widehat{G}\right\|_{L^2_\lambda(\mathbb{R},L^2(\mathbb{R}^3,\mathbb{C}^4))}\left\|\widehat{F}\right\|_{L^2_\lambda(\mathbb{R},L^2(\mathbb{R}^3,\mathbb{C}^4))}$$

for any F, G in $L^2_t(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))$. This in turn follows from the Limiting Absorption just proved.

□

To state the next result, we need the

Definition B.3 (Besov space). *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the Besov space $B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4)$ is the space of all $f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ (dual of the Schwartz space) such that*

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j \in \mathbb{N}} 2^{jsq} \|\varphi_j * f\|_p^q \right)^{\frac{1}{q}} < +\infty$$

with $\widehat{\varphi} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that $\sum_{j \in \mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$ for all $j \in \mathbb{N}^*$ and for all $\xi \in \mathbb{R}^3$, and $\widehat{\varphi_0} = 1 - \sum_{j \in \mathbb{N}^*} \widehat{\varphi}_j$. It is endowed with the natural norm $f \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4) \mapsto \|f\|_{B^s_{p,q}}$.

Using the Dispersive estimates of [Bou05, Theorem 1.2] and [KT98, Theorem 10.1], we obtain the

Theorem B.4 (Strichartz-type estimates)**.** *If Assumptions 1.1 and 1.2 hold. Then for any $2 \le p, q \le \infty$, $\theta \in [0, 1]$, $\beta \in [-\theta/2, \theta/2]$ with $(1 - \frac{2}{q})(1 + \beta) = \frac{2}{p}$ and $(p, \beta) \neq (2, 0)$, and for any reals s, s' with $s' - s \ge \alpha(q)$ where $\alpha(q) = (1+\beta)(1-\frac{2}{q})$, there exists a positive constant C such that*

$$\|e^{-\mathrm{i}tH} P_c(H)\psi\|_{L^p_t(\mathbb{R}, B^s_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \le C\|\psi\|_{H^{s'}(\mathbb{R}^3,\mathbb{C}^4)}, \tag{i}$$

$$\| \int e^{\mathrm{i}tH} P_c(H) F(t)\, dt \|_{H^s} \le C\|F\|_{L^{p'}_t(\mathbb{R}, B^{s'}_{q',2}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{ii}$$

$$\| \int_{s<t} e^{-\mathrm{i}(t-s)H} P_c(H) F(s)\, ds \|_{L^p_t(\mathbb{R}, B^{-s}_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \le C\|F\|_{L^{\tilde{p}'}_t(\mathbb{R}, B^{\tilde{s}}_{\tilde{q}',2}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{iii}$$

for any $r \in [1, \infty]$, $(\tilde{q}, \tilde{p})$ chosen like (q, p) and $s + \tilde{s} \ge \alpha(q) + \alpha(\tilde{q})$.

Proof – This is a consequence of [KT98, Theorem 10.1] applied to $U(t) = e^{-\mathrm{i}tH} P_c(H)$, using [Bou05, Theorem 1.2] and

$$B^{(1+\frac{\theta}{2})(1-\frac{2}{q})}_{q,2} \hookrightarrow (L^2, B^{1+\theta/2}_{1,2})_{2/((1\pm\theta/2)p),2}$$

continuously for $p \ge 2$ ($p \neq 2$ if $\theta = 0$) and $1/q = 1 - 1/((1 \pm \theta/2)p)$, which follows from the proof of s[BL76, Theorem 6.4.5]. □

1.2 The manifold of PLS

We study the following nonlinear Dirac equation

$$\begin{cases} \mathrm{i}\partial_t \psi = H\psi + \nabla F(\psi) \\ \psi(0, \cdot) = \psi_0. \end{cases} \tag{3}$$

with $\psi \in \mathcal{C}^1(I, H^1(\mathbb{R}^3, \mathbb{C}^4))$ for some open interval I which contains 0 and $H = D_m + V$. The nonlinearity $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a differentiable map for the real structure of $\mathbb{C}^4$ and hence the ∇ symbol has to be understood for the real structure of $\mathbb{C}^4$. For the usual hermitian product of $\mathbb{C}^4$, one has

$$DF(v)h = \Re\langle \nabla F(v), h\rangle.$$

If F has a gauge invariance (see Equation (1) or Assumption 1.4 below), this equation can have stationary solutions *i.e.* solution of the form $e^{-\mathrm{i}Et}\phi_0$ where ϕ_0 satisfies the nonlinear stationary equation

$$E\phi_0 = H\phi_0 + \nabla F(\phi_0).$$

Then we notice the Dirac operator D_m have an interesting invariance property due to its matrix structure. This invariance can be shared by some perturbed Dirac operators and gives a consequence of a theorem of Kramers, see [Par90]. Let us precise it, we introduce K an antilinear operator defined by:

$$K\begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \sigma_2\overline{\psi} \\ \sigma_2\overline{\chi} \end{pmatrix} \text{ with } \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{4}$$

The operator D_m commutes with K. So if V also commutes with K, we obtain that the eigenspaces of H are always of even dimension. Here we work with the

Assumption 1.3. *The potential V commutes to K. The operator $H := D_m + V$ has only two double eigenvalues $\lambda_0 < \lambda_1$, with $\{\phi_0, K\phi_0\}$ and $\{\phi_1, K\phi\}$ as associated orthonormalized basis.*

We also need the

Assumption 1.4. *The function $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is in $\mathcal{C}^\infty(\mathbb{R}^8, \mathbb{R})$ and satisfies $F(z) = O(|z|^4)$ as $z \to 0$. Moreover, it has the following invariance properties:*

$$\forall z \in \mathbb{C}^4,\ \forall \theta \in \mathbb{R},\ F(Kz) = F(z),\ F(e^{i\theta}z) = F(z).$$

We obtain the

Proposition B.5 (PLS manifold)**.** *If Assumptions 1.1–1.4 hold. Then for any $\sigma \in \mathbb{R}^+$, there exists Ω a neighborhood of $0 \in \mathbb{C}^2$, a $\mathcal{C}^\infty$ map*

$$h : \Omega \mapsto \{\phi_0, K\phi_0\}^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4)$$

and a $\mathcal{C}^\infty$ map $E : \Omega \mapsto \mathbb{R}$ such that $S((u_1, u_2)) = u_1\phi_0 + u_2K\phi_0 + h((u_1, u_2))$ satisfy for all $U \in \Omega$,

$$HS(U) + \nabla F(S(U)) = E(U)S(U), \tag{5}$$

with the following properties

$$\begin{cases} h((u_1,u_2)) = \left(\frac{u_1}{|(u_1,u_2)|} Id_{\mathbb{C}^4} + \frac{u_2}{|(u_1,u_2)|}K\right) h\left((|(u_1,u_2)|, 0)\right), \quad \forall U = (u_1,u_2) \in \Omega, \\ h(U) = O(|U|^2), \\ E(U) = E(|U|), \\ E(U) = \lambda_0 + O(|U|^2). \end{cases}$$

Proof – This result is adapted from [PW97, Proposition 2.2] after the reduction due to the invariance of the problem with respect to K. □

Moreover, we have

Lemma B.6 (exponential decay)**.** *For any $\beta \in \mathbb{N}^4$, $s \in \mathbb{R}^+$ and $p, q \in [1, \infty]$. There is $\gamma > 0$, $\varepsilon > 0$ and $C > 0$ such that for all $U \in B_{\mathbb{C}^2}(0, \varepsilon)$ one has*

$$\|e^{\gamma\langle Q\rangle}\partial_U^\beta S(U)\|_{B^s_{p,q}} \leq C\|S(U)\|_2,$$

where $\partial^\beta_{(u_1,u_2)} = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u_1 \partial^{\beta_2}\Im u_1 \partial^{\beta_3}\Re u_2 \partial^{\beta_4}\Im u_2}$.

Proof – This is like in [Bou05, Lemma 4.1], where we used ideas of [His00]. □

1.3 The unstable manifold, the stabilization and the nonlinear scattering

Each stationary solution previously introduced has, like in [Bou05], a stable manifold. Under the following assumptions, we can prove that the stable manifold is unstable, that is to say that a small perturbation of a stationary solution starting outside of this manifold leaves from any small neighborhood of this stationary solution. We work with the

Assumption 1.5. *The resonant condition*

$$|\lambda_1 - \lambda_0| > \min\{|\lambda_0 + m|, |\lambda_0 - m|\}$$

holds. Moreover, we have the Fermi Golden Rule

$$\Gamma(\phi) = \lim_{\substack{\varepsilon\to 0,\\ \varepsilon>0}} \left\langle d^2F(\phi)\phi_1, \Im\left((H-\lambda_0)+(\lambda_1-\lambda_0)-\mathrm{i}\varepsilon\right)^{-1} P_c(H) d^2F(\phi)\phi_1 \right\rangle > 0 \qquad (6)$$

for any non zero eigenvector ϕ associated with λ_0.

In this assumption, the notation d^2F denotes the differential of ∇F.

Let us introduce the linearized operator $JH(U)$ around a stationary state $S(U)$:

$$H(U) = H + d^2F(S(U)).$$

We notice that the operator $H(U)$ is only $\mathbb{R}$-linear but not $\mathbb{C}$-linear. Hence we work with the space $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ instead of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by writing

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}$$

instead of ϕ. The multiplication by $-\mathrm{i}$ gives the operator

$$J = \begin{pmatrix} 0 & -I_{\mathbb{R}^4} \\ I_{\mathbb{R}^4} & 0 \end{pmatrix}.$$

Now we mention some spectral properties of $JH(U)$ in $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ which are needed to state and to understand our main theorem. These properties will be proven in subsection **??** below.

The operator $JH(U)$ has a four dimensional geometric kernel and four double eigenvalues $E_1(U)$, $\overline{E_1}(U)$, $-E_1(U)$ and $-\overline{E_1}(U)$ with $\Re E_1(U) > 0$.

The eigenspaces associated with $E_1(U)$ and $\overline{E_1}(U)$ are conjugated via the complex conjugation, we will work on the real part of their sum: $\mathcal{H}^1_+(U)$, we introduce a basis $(\xi_i(U))_{i=1,\dots,4}$ of $\mathcal{H}^1_+(U)$.

The same holds for $-E_1(U)$ and $-\overline{E_1}(U)$ and working on the real part of the sum of the associated eigenspaces : $\mathcal{H}^1_-(U)$, we introduce a basis $(\xi_i(U))_{i=5,\dots,8}$ of $\mathcal{H}^1_-(U)$.

The rest of the spectrum is the essential or continuous spectrum. We write $\mathcal{H}_c(U)$ for the space associated with the continuous spectrum. The space $\mathcal{H}_c(U)$ is the orthogonal of the previous eigenspaces with respect to the product $(f,g) \mapsto \Re\langle f, Jg\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$. We introduce $\mathcal{H}^r_c(U)$ the intersection of $\mathcal{H}_c = \mathcal{H}_c(0)$ with

$$\left\{ \Im\left(H - E(U) + \lambda_1 - \lambda_0 - \frac{\mathrm{i}}{2}\Gamma(S(U))\right)^{-1} d^2F(S(U))|U| \left(\frac{u_1}{|U|} I_{\mathbb{C}^4} + \frac{u_2}{|U|} K\right)\phi_1, \right.$$
$$\left. \forall U = (u_1, u_2) \in \mathbb{C}^2 \right\}^{\perp}.$$

The space $\mathcal{H}_c(U)$ is isomorphic to $\mathcal{H}_c^r(U)$. The isomorphism is the orthogonal projector onto $\mathcal{H}_c^r(U)$ with respect to the product $(f,g) \mapsto \Re \langle f, Jg \rangle_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$ and this projector is bounded from $H^s_\sigma(\mathbb{R}^3,\mathbb{C}^4)$ into itself for any reals s and σ.

We can state our

Theorem B.7. *If Assumptions 1.1–1.5 hold. Then, for $s' > \beta + 2 > 2\,\sigma > 3/2$, there exist $\varepsilon > 0$ and a continuous map $r : B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \mathbb{R}$ with $r(V) = O(\Gamma(V))$, such that for any initial condition of the form*

$$\psi_0 = S(V_0) + \xi_0 + A \cdot \xi(V_0)$$

with $V_0 \in B_{\mathbb{C}^2}(0,\varepsilon)$, $\xi_0 \in \mathcal{H}_c(V_0) \cap B_{H^{s'}}(0, r(V))$ and $A \in \mathbb{C}^4$, the following holds.

(i) For the set

$$\mathcal{S} = \left\{ (V,\xi)\,;\; V \in B_{\mathbb{C}^2}(0,\varepsilon),\; \xi \in \mathcal{H}_c(V) \cap B_{H^{s'}}(0, r(V)) \right\}$$

endowed with the metric of $\mathbb{C}^2 \times H^{s'}$, there exist $C > 0$, $\mathcal{V}$ a neighborhood of $(0,0)$ in $\mathcal{S}$, and a smooth map $\Psi : \mathcal{V} \mapsto \mathbb{C}^4$ with graph $\mathcal{CM}$ and $\Psi(\cdot,0) = 0$ and $\|\Psi(0,0)\| = \|D\Psi(0,0)\| = 0$ such that if $(V_0,\xi_0) \in \mathcal{V}$ if $A = \Psi(V_0,\xi_0)$, we have:

(a) there exists a unique solution ψ of (3) with initial condition ψ_0 and this solution is in $\mathcal{C}\left(\mathbb{R}, H^{s'}\right) \cap \mathcal{C}^1\left(]-T_0; +\infty[, H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right)$;

(b) there exist $\mathcal{V}^\pm$ open neighborhoods of $(0,0)$ in $\mathcal{S}$ and bijective maps

$$(V_0;\xi_0) \in \mathcal{V} \mapsto (V_{\pm\infty};\xi_{\pm\infty}) \in \mathcal{V}^\pm,$$

such that

$$|V_{\pm\infty} - V_0| \le C\|\xi_0\|^2_{H^{s'}},\; \|\xi_{\pm\infty} - \xi_0\|_{H^{s'}} \le C\,|V_0|\,\|\xi_0\|_{H^{s'}},$$

and for all $t \in \mathbb{R}$

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(V(v))\,dv} S(V(t)) + e^{JtE(V_{\pm\infty})} e^{JtH(V_{\pm\infty})} \xi_{\pm\infty} + \varepsilon_\pm(t)$$

with $\dot V \in L^q(\mathbb{R}^\pm)$ for all $q \in [1,\infty]$, $\lim\limits_{t\to\pm\infty} V(t) = V_{\pm\infty}$,

$$\max\left\{ \|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm,H^{s'})},\; \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,H^s_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,B^\beta_{\infty,2})} \right\} \le C\,|V_0|\,\|\xi_0\|_{H^{s'}},$$

and

$$\lim_{t\to\pm\infty} \|\varepsilon_\pm(t)\|_{H^{s'}} = 0.$$

(ii) For

$$\widetilde{S_+} = \left\{ (U,z,p)\,;\; U \in B_{\mathbb{C}^2}(0,\varepsilon),\; z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0, r(U)),\right.$$
$$\left. p \in B_{\mathbb{R}^8}(0, r(U)),\; \text{with } p_i = 0 \text{ for } i = 1,\dots,4 \right\}$$

and

$$\widetilde{S_-} = \Big\{ (U,z,p)\,;\ U \in B_{\mathbb{C}^2}(0,\varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0,r(U)),$$
$$p \in B_{\mathbb{R}^8}(0,r(U)),\ \text{with } p_i = 0 \text{ for } i = 5,\dots,8 \Big\}$$

endowed with the metric of $\mathbb{C}^2 \times H^{s'} \times \mathbb{R}^8$, *there exist neighborhoods* $\mathcal{W}_\pm$ *of* $(0,0,0)$ *in* $\tilde{S}_\pm$ *and smooth maps* $\Phi_\pm : \mathcal{W}_\pm \mapsto \mathbb{C}^4$ *with* $\|\Phi_\pm(0,0,0)\| = \|\Phi_\pm(0,0,0)\| = 0$, $(\Phi_+(\cdot,\cdot,p))_i = p_i$ *if* $i = 5,\dots,8$, $(\Phi_-(\cdot,\cdot,p))_i = p_i$ *if* $i = 1,\dots,4$ *and*

(a) if $A \in \Phi_\pm(\mathcal{W}_\pm)$ *and* $A \notin \Psi(\mathcal{V})$ *then there exists a unique solution* ψ *of* (3) *with initial condition* ψ_0 *and for any small neighborhood* $\mathcal{O}$ *of* $S(V_0)$ *containing* ψ_0, *there exist* $t_\pm(\psi_0) > 0$ *such that* ϕ *is in*

$$\mathcal{C}\left([-t_+;+\infty), H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\right) \cap \mathcal{C}^1\left((-t_+;+\infty), H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right)$$

(resp.

$$\mathcal{C}\left((-\infty;t_-], H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\right) \cap \mathcal{C}^1\left((-\infty;t_-), H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right)),$$

and

$$\mathrm{dist}_{L^2}(\psi(t),\mathcal{CM}) = O(e^{\mp\gamma t}) \text{ as } t \to \pm\infty \quad \text{and} \quad \psi(\mp t_\pm) \notin \mathcal{O}$$

where γ *is in a ball around* $1/2\Gamma(S(V_0))$, *the radius of which is* $O(|U_0|^6)$*);*

(b) if $A \notin \Phi_+(\mathcal{W}^+) \cup \Phi_-(\mathcal{W}_-)$ *then there exists a unique solution* ψ *of* (3) *with initial condition* ψ_0 *and for any small neighborhood* $\mathcal{O}$ *of* $S(V_0)$ *containing* ψ_0 *there exist* $t_+(\psi_0) > 0$, $t_-(\psi_0) < 0$ *such that* ϕ *is in* $\mathcal{C}\left([t_-;t_+], H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\right) \cap \mathcal{C}^1\left((t_-;t_+), H^{s'-1}(\mathbb{R}^3,\mathbb{C}^4)\right)$, *and we have* $\psi(t_+) \notin \mathcal{O}$ *and* $\psi(t_-) \notin \mathcal{O}$.

The first part of this theorem shows, as in [Bou05], that perturbations in the direction of the continuous subspace, except four directions, relax towards a stationary solution. We have excluded four directions in the continuous subspace, which, due to resonance phenomena, induce orbital instability. The second part of the theorem tells us what happens for perturbations in the directions of an excited state and in the four directions of the continuous spectrum for which we haven't proved stabilization. We thus study eight directions: four of them give a manifold on which there holds exponential stabilization in positive time and orbital instability in negative time, while the four others give a manifold on which there holds exponential stabilization in negative time and orbital instability in positive time. Outside these manifolds, we have orbital instability in both negative and positive time.

Remark B.7.1. *We can also obtain expansions of the form* $\psi(t) = e^{-\mathrm{i}\int_0^t E(V(v))\,dv} S(V_\pm) + e^{-\mathrm{i}tH}\xi_\pm + \varepsilon_\pm(t)$ *or* $\psi(t) = e^{-\mathrm{i}\int_0^t E(V(v))\,dv} S(V_\pm) + e^{-\mathrm{i}tD_m}\xi_\pm + \varepsilon_\pm(t)$ *with the same conclusion but we will only have* $\|\xi_{\pm\infty} - \xi_0\|_{H^{s'}} \le C\|\xi_0\|_{H^{s'}}$,

$$\max\left\{ \|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm,H^{s'})},\ \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,H^s_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,B^\beta_{\infty,2})} \right\} \le C\|\xi_0\|_{H^{s'}}.$$

Notice that when $V_0 = 0$ then $\xi_0 = 0$ and $p = 0$ so the theorem do not say anything for this case. In fact, the charge conservation gives the orbital stability of 0*. But one cannot obtain asymptotic stability since we can build a manifold of stationary states tangent to the eigenspace associated with λ_1 similarly to Proposition B.5.*

2 Linearized operator and exponentially stable and unstable manifolds

We study the dynamics associated with (3) around a stationary state. We will use spectral properties of the linearized operator around a stationary state.

2.1 The spectrum of the linearized operator

Here we study the spectrum of the linearized operator associated with Equation (3) around a stationary state $S(U)$. Let us introduce

$$H(U) = H + d^2F(S(U)) - E(U)$$

where d^2F is the differential of ∇F. The operator $H(U)$ is $\mathbb{R}-$linear but not $\mathbb{C}-$linear. Replacing $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ with the inner product obtained by taking the real part of the inner product of $L^2(\mathbb{R}^3, \mathbb{C}^4)$, we obtain a symmetric operator. We then complexify this real Hilbert space and obtain $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ with its canonical hermitian product. This process transforms the operator $-\mathrm{i}$ into

$$J = \begin{pmatrix} 0 & Id_{\mathbb{C}^4} \\ -Id_{\mathbb{C}^4} & 0 \end{pmatrix}.$$

For $\phi \in L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4) \subset L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$, we still write ϕ instead of

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}.$$

The extension of $H(U)$ over $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ is also written $H(U)$ and is now a real operator. The extension of K is also written K.

The linearized operator associated with Equation (3) around the stationary state $S(U)$ is given by $JH(U)$. We shall now study its spectrum.

Differentiating (5), we have that for $U = (u_1, u_2)$

$$\mathcal{H}_0(u_1, u_2) = \mathrm{Span}\left\{\frac{\partial}{\partial \Re u_1}S(u_1, u_2),\ \frac{\partial}{\partial \Im u_1}S(u_1, u_2), \frac{\partial}{\partial \Re u_2}S(u_1, u_2),\ \frac{\partial}{\partial \Im u_2}S(u_1, u_2)\right\}$$

is invariant under the action of $JH(U)$. Using the gauge invariance, we notice that $S(U) \in \mathcal{H}_0(U)$ and differentiating (5), we obtain for any $\beta \in \mathbb{N}^4$ with $|\beta| = 1$:

$$JH(U)JS(U) = 0 \quad \text{and} \quad JH(U)\partial_U^\beta S(U) = (\partial_U^\beta E)(U)JS(U).$$

The space $\mathcal{H}_0(U)$ is contained in the geometric null space of $JH(U)$, in fact it is exactly the geometric null space as proved in the sequel.

Now we state our results on the spectrum of $JH(U)$. The first deals with the excited states part. We have the

Proposition B.8. *If Assumptions 1.1–1.5 hold. Then there exists a map $E_1 : B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \mathbb{R}$ with*

$$\begin{cases} \Im E_1(U) = (\lambda_1 - E(U)) + O(|U|^4) \\ \Re E_1(U) = 1/2\Gamma(S(U)) + O(|U|^6). \end{cases}$$

such that $E_1(U)$, $\overline{E_1}(U)$, $-E_1(U)$ and $-\overline{E_1}(U)$ are double eigenvalue of $JH(U)$. For any $s \in \mathbb{R}$, there exist smooth maps $k^\pm : B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \left\{ \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} \right\}^\perp \cap H^s$ such that

$$k_\pm(U) \\ - \left((H - E(U)) + \mathrm{i}E_1(U)\right)^{-1} P_c(H) d^2F(S(U))|U| \left(\frac{u_1}{|U|} Id_{\mathbb{C}^4} + \frac{u_2}{|U|} K \right) \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} \tag{7}$$

is $o(|U|^2)$ in $B(L^2_{-\sigma})$ for any $\sigma \in \mathbb{R}$. For

$$\Phi_\pm(U) = |U| \left(\frac{u_1}{|U|} Id_{\mathbb{C}^4} + \frac{u_2}{|U|} K \right) \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} + k_\pm(U),$$

we have

- $\{\Phi_+(U), K\Phi_+(U)\}$ *is a basis of the eigenspace associated with $E_1(U)$,*
- $\left\{\overline{\Phi_+(U)}, K\overline{\Phi_+(U)}\right\}$ *is a basis of the eigenspace associated with $\overline{E_1(U)}$,*
- $\{\Phi_-(U), K\Phi_-(U)\}$ *is a basis of the eigenspace associated with $-E_1(U)$,*
- $\left\{\overline{\Phi_-(U)}, K\overline{\Phi_-(U)}\right\}$ *is a basis of the eigenspace associated with $-\overline{E_1(U)}$.*

Moreover for any $\beta \in \mathbb{N}^4$, $s \in \mathbb{R}^+$ and $p, q \in [1,\infty]$. There is $\gamma > 0$, $\varepsilon > 0$ and $C > 0$ such that for all $U \in B_{\mathbb{C}^2}(0,\varepsilon)$, for

$$\xi(U) \in \left\{ \Phi_+(U), K\Phi_+(U), \overline{\Phi_+(U)}, K\overline{\Phi_+(U)}, \Phi_-(U), K\Phi_-(U), \overline{\Phi_-(U)}, K\overline{\Phi_-(U)} \right\}$$

one has

$$\|e^{\gamma\langle Q\rangle} \partial_U^\beta \xi(U)\|_{B^s_{p,q}} \le C\|S(U)\|_2, \tag{8}$$

where $\partial^\beta_{u_1,u_2} = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u_1 \partial^{\beta_2}\Im u_1 \partial^{\beta_1}\Re u_2 \partial^{\beta_2}\Im u_2}$.

Proof – Using Weyl's sequences, we prove that the essential spectrum of $JH(U)$, for small U, is the essential spectrum of JH. So z with non zero real part is in the spectrum of $JH(U)$ if and only if it is an isolated eigenvalue. The equation to solve for excited states is:

$$(JH(U) - z)\,\phi = 0. \tag{9}$$

Since the proof is similar for all cases, we restrict the study to solutions the form $\phi = S_1 + \eta$ where S_1 is the normalized eigenvector:

$$S_1 = \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix}$$

and $\eta \in \{S_1\}^\perp$, the orthogonal relation is taken in fact with respect to J, but since $JS_1 = \mathrm{i}S_1$, we can take it in the standard way. We obtain the equation

$$\eta = (JH - z)^{-1} P_1^\perp W(U)\{S_1 + \eta\} \tag{10}$$

with $P_1^\perp$ the orthogonal projector, with respect to J, into $\{S_1\}^\perp$ and $W(U) = JH(U) - J(H - E(U))$. We remark that $\{S_1\}^\perp$ is invariant under the action of $J(H - E(U))$. To solve this equation in η for a fixed u and z, we notice that if

$$\Re z > 0 \text{ and } |\Im z| \geq m,$$

the series

$$k(U,z) = (JH - z)^{-1} P_1^\perp \sum_{k\geq 0} \left(-W(U)(JH - z)^{-1} P_1^\perp\right)^k W(U)S_1,$$

is convergent for sufficiently small $|U|$ using the Limiting Absorption Principle (2). Hence, we have a solution of (10).

Then we solve the equation in z, we obtain from Equation (9) the equation

$$\langle (JH(U) - z)\,\phi, S_1\rangle = 0.$$

With $\phi = S_1 + k(U,z)$, we infer

$$\begin{aligned}
z &= \langle JH(U)S_1, S_1\rangle + \langle JH(U)k(U,z), S_1\rangle \\
&= \mathrm{i}(\lambda_1 - \lambda_0) + \langle W(U)S_1, S_1\rangle \\
&\quad + \sum_{k\geq 0} \left\langle JH(U)(JH - z)^{-1} P_1^\perp \left(-W(U)(J(H - E(U)) - z)^{-1} P_1^\perp\right)^k W(U)S_1, S_1 \right\rangle \\
&= \mathrm{i}(\lambda_1 - \lambda_0) \\
&\quad + \langle W(U)S_1, S_1\rangle + \sum_{k\geq 0} \left\langle P_1^\perp \left(-W(U)(J(H - E(U)) - z)^{-1} P_1^\perp\right)^k W(U)S_1, S_1 \right\rangle \\
&\qquad + \sum_{k\geq 0} \Big\langle (W(U) - z)(J(H - E(U)) - z)^{-1} P_1^\perp \times \\
&\qquad\quad \times \left(-W(U)(J(H - E(U)) - z)^{-1} P_1^\perp\right)^k W(U)S_1, S_1 \Big\rangle.
\end{aligned}$$

Since $P_1^\perp S_1 = 0$, we introduce the function

$$\begin{aligned}
&f(z) = \mathrm{i}(\lambda_1 - \lambda_0) + \langle W(U)S_1, S_1\rangle \\
&+ \sum_{k\geq 0} \left\langle W(U)(J(H - E(U)) - z)^{-1} P_1^\perp \left(-W(U)(J(H - E(U)) - z)^{-1} P_1^\perp\right)^k W(U)S_1, S_1 \right\rangle.
\end{aligned}$$

Since $JS_1 = -\mathrm{i}S_1$ we obtain that $\Re\langle W(U)S_1, S_1\rangle = 0$, so with the Limiting Absorption Principle (2), we have

$$\begin{aligned}
\Re f(z) &= \Re \left\langle W(U)(J(H - E(U)) - z)^{-1} P_1^\perp W(U)S_1, S_1 \right\rangle + O(|U|^6) \\
&= \Im \left\langle d^2F(S(U))\left((H - E(U)) + zJ\right)^{-1} P_1^\perp d^2F(S(U))S_1, S_1 \right\rangle + O(|U|^6)
\end{aligned}$$

using (6) and

$$((H - E(U)) + zJ)^{-1} = \frac{1}{2}\Big(((H - E(U)) - \mathrm{i}z)^{-1}(I_{C^2} + \mathrm{i}J) + ((H - E(U)) + \mathrm{i}z)^{-1}(I_{C^2} - \mathrm{i}J)\Big),$$

we obtain

$$\begin{aligned}\Im\left\langle d^2F(S(U))\,((H - E(U)) + zJ)^{-1}\, d^2F(S(U))S_1, S_1\right\rangle &= \frac{1}{2}\Im\left\langle d^2F(S(U))\,((H - E(U)) - \mathrm{i}z)^{-1}\, d^2F(S(U))S_1, S_1\right\rangle \\ &+ \frac{1}{2}\Im\left\langle d^2F(S(U))\,((H - E(U)) + \mathrm{i}z)^{-1}\, d^2F(S(U))S_1, S_1\right\rangle \\ &+ \Im\left\langle d^2F(S(U))\,\left((H - E(U))^2 + z^2\right)^{-1} zJd^2F(S(U))S_1, S_1\right\rangle,\end{aligned}$$

and so

$$\begin{aligned}\Im\left\langle d^2F(S(U))\,((H - E(U)) + (\mathrm{i}(\lambda_1 - \lambda_0) + 0)\,J)^{-1}\, d^2F(S(U))S_1, S_1\right\rangle \\ = \frac{1}{2}\Im\left\langle d^2F(S(U))\,((H - E(U)) + (\lambda_1 - \lambda_0) - \mathrm{i}0)^{-1}\, d^2F(S(U))S_1, S_1\right\rangle\end{aligned}$$

Using Assumption 1.5, the local limiting absorption principle (2) and regularity results of [GM01][Theorem 1.7], we obtain

$$\Re f(z) = 1/2\Gamma(S(U)) + O(|U|^6)$$

for z in a ball of radius of order $|U|^2$ around $\mathrm{i}\,(\lambda_1 - \lambda_0)$ and for small U. We also prove by the same way

$$\Im f(z) = (\lambda_1 - \lambda_0) + O(|U|^4)$$

for z in a ball of radius of order $|U|^2$ around $\mathrm{i}\,(\lambda_1 - \lambda_0)$ and for small U.

So we have proved that for sufficiently small U, f leaves a ball around $\mathrm{i}\,(\lambda_1 - \lambda_0) + 1/2\Gamma(S(U))$ invariant and is a contraction. Therefore, we have a fixed point $E_1(U)$ of each U. Then we choose $k_+(U) = k(U, E_1(U))$. Using the complex conjugation, we obtain the eigenvalue $\overline{E_1(U)}$ and its associated eigenspace.

To obtain $-E_1(U)$ and $-\overline{E_1(U)}$, we notice that $E_1(U)$ and $\overline{E_1(U)}$ are eigenvalues of $(JH(U))^*$. Hence using the symmetry : $J(JH(U)) = -(JH(U))^*J$, we show that any eigenvector ϕ of $(JH(U))^*$ associated with λ, $J\phi$ is an eigenvector of $JH(U)$ associated with $-\lambda$ and this gives k_-.

The exponential decay works like in Lemma B.6 □

We introduce $\mathcal{H}_c^r(U)$ is the intersection of $\mathcal{H}_c$ with

$$\left\{\Im\left(H - E(U) + \lambda_1 - \lambda_0 + \frac{\mathrm{i}}{2}\Gamma(S(U))\right)^{-1} d^2F(S(U))|V|\left(\frac{v_1}{|V|}I_{\mathbb{C}^4} + \frac{v_2}{|V|}K\right)\begin{pmatrix}\phi_1\\ -i\phi_1\end{pmatrix}, \quad \forall V = (v_1, v_2) \in \mathbb{C}^2\right\}^{\perp_J},$$

and $P_c^r(U)$ the orthogonal projector with respect to J onto $\mathcal{H}_c^r(U)$. The following result deals with the essential spectrum of our linearized operator.

Proposition B.9. *If Assumptions 1.1–1.5 hold. For any sufficiently small non zero $U \in \mathbb{C}^2$, let*

$$\mathcal{H}_1(U) = \mathrm{span}\left\{\Phi_+(U), K\Phi_+(U), \overline{\Phi_+(U)}, K\overline{\Phi_+(U)}, \Phi_-(U), K\Phi_-(U), \overline{\Phi_-(U)}, K\overline{\Phi_-(U)}\right\}.$$

The orthogonal space of $\mathcal{H}_0(U) \oplus \mathcal{H}_c(U)$ with respect to the hermitian product associated to J

$$\mathcal{H}_c(U) = \{\mathcal{H}_0(U) \oplus \mathcal{H}_1(U)\}^\perp$$

is invariant under the action of $JH(U)$.

We have for $U' \in B_{\mathbb{C}^2}(U, \varepsilon)$, with sufficiently small $\varepsilon > 0$, that

$$\mathbf{P}_c^r(U)|_{\mathcal{H}_c(U')} : \mathcal{H}_c(U') \mapsto \mathcal{H}_c^r(U)$$

is an isomorphism and is a bounded operator from $H^s_\sigma(\mathbb{R}^3, \mathbb{C}^8)$ to itself for any reals s and σ, the inverse $R^r(U', U)$ is continuous with respect to U and U' for these norm.

We also have for $\mathbf{P}_c(U)$, the orthogonal projector onto $\mathcal{H}_c(U)$ with respect to J, and for $U' \in B_{\mathbb{C}^2}(U, \varepsilon)$, with sufficiently small $\varepsilon > 0$, that

$$\mathbf{P}_c(U)|_{\mathcal{H}_c(U')} : \mathcal{H}_c(U') \mapsto \mathcal{H}_c(U)$$

is an isomorphism and is a bounded operator from $H^s_\sigma(\mathbb{R}^3, \mathbb{C}^8)$ to itself for any reals s and σ, the inverse $R(U', U)$ is continuous with respect to U and U' for these norm.

Moreover, we have

$$\int_{\mathbb{R}} \|\langle Q\rangle^{-\sigma} e^{sJH(U)} \mathbf{P}_c(U)\psi\|^2 \, ds \leq C\|\psi\|_2^2, \ \forall \psi \in L^2. \tag{11}$$

and $\mathcal{H}_c(U)$ contains no eigenvector. We also have

$$\|e^{tJH(U)} \mathbf{P}_c(U)\psi\| \leq C\|\psi\|_2, \ \forall t \in \mathbb{R}, \ \forall \psi \in L^2. \tag{12}$$

Proof – We prove that there is no other eigenvector, by proving that smoothness estimate (11) takes place over

$$\mathcal{H}_c(U) = \{\mathcal{H}_0(U) \oplus \mathcal{H}_1(U)\}^\perp.$$

First we prove that

$$\mathbf{P}_c((U))|_{\mathcal{H}_c(U')} : \mathcal{H}_c(U') \mapsto \mathcal{H}_c(U)$$

is an isomorphism. To prove it, we exhibit an inverse $R(U', U)$ which is of the form

$$R(U', U) = Id + \sum_i \left|\alpha_i(U', U)\right\rangle \langle J\xi_i(U)|$$

where $\xi_i(U)$ is a basis of the eigenspaces of $JH(U)$ and $\alpha_i(U', U) \in \left\{\xi_i(U) + (\mathcal{H}_c(U'))^\perp\right\} \cap \mathcal{H}_c(U)$. Such a α exists because this constraint reduce to invert a Gramm matrix when $U = U'$ and a small perturbation of such matrices if for U and U' close one to each other. One then check that $R(U', U)$ sends $\mathcal{H}_c(U)$ in $\mathcal{H}_c(U')$ and is the inverse of $\mathbf{P}_c((U))|_{\mathcal{H}_c(U')}$. The boundedness of R follows from the exponential decay of eigenvectors and their derivatives, the continuity of R follows from the contiuity of the eigenvector with respect to the parameters U and U' see Proposition B.8, the paragraph above and Proposition B.6.

The statement for $\mathbf{P}_c(U)^r|_{\mathcal{H}_c(U')}$ follows by the same way.

Therefore since $R(U,U)\mathbf{P}_c^r(U)\mathbf{P}_c(U) = \mathbf{P}_c(U)$.

$$
\begin{aligned}
&\|\langle Q\rangle^{-\sigma} e^{tJH(U)}\mathbf{P}_c(U)\psi\| \\
&= \; C\|\langle Q\rangle^{-\sigma} R(U,U)\mathbf{P}_c^r(U) e^{tJH(U)}\mathbf{P}_c(U)\psi\| \\
&\le \; \|\langle Q\rangle^{-\sigma} R(U,U)\mathbf{P}_c^r(U) e^{-\mathrm{i}t(H-E(U))}\mathbf{P}_c(U)\psi\| \\
&\quad + \int_0^t \|\langle Q\rangle^{-\sigma} R(U,U)\mathbf{P}_c^r(U) e^{-\mathrm{i}(t-s)(H-E(U))} D\nabla F(S(U)) e^{sJH(U)}\mathbf{P}_c(U)\psi\|\, ds \\
&\le \; \|\langle Q\rangle^{-\sigma} R(U,U)\langle Q\rangle^{\sigma}\| \|\langle Q\rangle^{-\sigma}\mathbf{P}_c^r(U) e^{-\mathrm{i}t(H-E(U))}\mathbf{P}_c(U)\psi\| \\
&\quad + C|z|^2 \|\langle Q\rangle^{-\sigma} R(U,U)\langle Q\rangle^{\sigma}\| \times \\
&\quad \times \int_0^t \|\langle Q\rangle^{-\sigma}\mathbf{P}_c^r(U) e^{-\mathrm{i}(t-s)(H-E(U))}\langle Q\rangle^{-\sigma}\| \|\langle Q\rangle^{-\sigma} e^{sJH(U)}\mathbf{P}_c(U)\psi\|\, ds. \qquad (13)
\end{aligned}
$$

Using estimate (i), we obtain the estimate (11) for U sufficiently small:

$$\int_{\mathbb{R}} \|\langle Q\rangle^{-\sigma} e^{-sJH(U)}\mathbf{P}_c(U)\psi\|^2\, ds \le C\|\psi\|^2.$$

Hence there is no eigenvector in the range of $P_c(U)$.

Using the inequalities (13) and the conservation law for H, we prove the estimate (12).

□

We obtain immediately the

Proposition B.10. *Suppose that Assumptions 1.1–1.3 and 1.5 hold. Then the space $\mathcal{H}_0(U)$ is the geometric null space of $JH(U)$.*

2.2 Stable, unstable and center manifold

We can now obtain results similar to those of Bates and Jones [BJ89].

We have that $JH(U)$ as an operator in $L^2(\mathbb{R}^3,\mathbb{R}^8)$ is a closed densely defined operator that generates a continuous semigroup on $L^2(\mathbb{R}^3,\mathbb{R}^8)$. The spectrum of $JH(U)$ in $L^2(\mathbb{R}^3,\mathbb{R}^8)$ is the same as $JH(U)$ in $L^2(\mathbb{R}^3,\mathbb{C}^8)$ and so it splits in three parts:

$$
\begin{aligned}
\sigma_s(U) &= \{\lambda \in \sigma(JH(u)),\ \Re\lambda < 0\} = \left\{-E_1(U), -\overline{E_1(U)}\right\} \\
\sigma_c(U) &= \{\lambda \in \sigma(JH(u)),\ \Re\lambda = 0\} = \mathrm{i}\,\{\mathbb{R}\setminus(-E(U),E(U))\} \\
\sigma_u(U) &= \{\lambda \in \sigma(JH(u)),\ \Re\lambda > 0\} = \left\{E_1(U), \overline{E_1(U)}\right\}
\end{aligned}
$$

each one is associated with a spectral real subspace

$$
\begin{aligned}
X_s(U) &= \mathrm{span}_{\mathbb{R}}\{\Re\Phi_-(U), \Im\Phi_-(U), K\Re\Phi_-(U), K\Im\Phi_-(U)\} \\
X_u(U) &= \mathrm{span}_{\mathbb{R}}\{\Re\Phi_+(U), \Im\Phi_+(U), K\Re\Phi_+(U), K\Im\Phi_+(U)\} \\
X_c(U) &= \Re\mathcal{H}_0(U) \oplus \Re\mathcal{H}_c(U)
\end{aligned}
$$

where we used the notation $\Re\Psi = (1/2)\left(\Psi + \overline{\Psi}\right)$ and $\Im\Psi = -(\mathrm{i}/2)\left(\Psi - \overline{\Psi}\right)$ and $\Re X = \{\Re\Psi, \Psi \in X\}$. The spaces $X_s(U)$ and $X_u(U)$ are finite dimensional. Let us write $\pi^c(U)$, $\pi^s(U)$ and $\pi^u(U)$ for the projector associated with the decomposition $X_c(U)\oplus X_s(U)\oplus X_u(U)$. Since the eigenvectors belongs also to H^s_σ for any $s,\sigma\in\mathbb{R}$, the projector $P_c(U)$ and $\pi^c(U)$ can be defined in H^s_σ for any $(s,\sigma\in\mathbb{R}$. We can hence defined an extend the spaces $\mathcal{H}_c(U)$ and $X_c(U)$ to H^s_σ for any $s,\sigma\in\mathbb{R}$. We have the

Lemma B.11. *If Assumptions 1.1–1.5 hold. Then there exist $C_1, C_2 > 0$ such that for all $t \in \mathbb{R}$, we have*

$$C_1 e^{-\gamma(U)t} \leq \left\| e^{tJH(U)} \pi^s(U) \right\|_{\mathcal{B}(H^s_\sigma)} \leq C_2 e^{-\gamma(U)t}, \tag{14}$$

$$C_1 e^{\gamma(U)t} \leq \left\| e^{tJH(U)} \pi^u(U) \right\|_{\mathcal{B}(H^s_\sigma)} \leq C_2 e^{\gamma(U)t}, \tag{15}$$

$$\left\| e^{tJH(U)} \pi^c(U) \right\|_{\mathcal{B}(H^s_\sigma)} \leq C_2 \langle t \rangle^r, \tag{16}$$

for some power r, for any $s, \sigma \in \mathbb{R}$ and where $\gamma(U) = \Re E_1(U)$.

Proof – The statements for the spaces $X^s(U)$ and $X^u(U)$ follows from (8).

The statement about $X^c(U)$ is a few more complicate. We are not looking for an optimal r. Using the fact that for $s \in \mathbb{R}$, $< H >^s < D_m >^{-s}$ and $< H >^{-s} < D_m >^s$ are bounded in L^2, we reduce the study to the case $s = 0$. First, let us prove the result for $e^{-\mathrm{i}t(D_m+V)}$ in L^2_σ with $\sigma \in 2\mathbb{N}$, the general case follows by duality and interpolation. In fact, the result in this case follows from iterating several times the proof of [Tha92, Theorem 1.3], which is based on the charge conservation.

Then for $e^{tJH(U)}\pi^c(U)$ follows by the same way using also the charge conservation of $e^{tJH(U)}P_c(U)$ (see (12)) and the fact that $e^{tJH(U)}S(U) = S(U)$, $e^{tJH(U)}\partial_U^\beta S(U) = \partial_U^\beta S(U) + t\partial_U^\beta E(U) S(U)$ and Lemma B.6. □

By now we do not restrict to the norm of $L^2(\mathbb{R}^3, \mathbb{R}^8)$, we extend our study to $H^s_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ for any $s, \sigma \in \mathbb{R}$, but we still write $\mathcal{H}_c(U)$ and $X_c(U)$ for the extensions of these spaces to $H^s_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ for any $s, \sigma \in \mathbb{R}$.

We now study the behavior of the solutions in H^s_σ of (3) centered around $S(U)$:

$$\partial_t \phi = JH(U)\phi + JN(U, \phi) \tag{17}$$

where $H(U) = H + d^2F(S(U)) - E(U)$ and $N(U, \phi) = \nabla F(S(U) + \phi) - \nabla F(S(U)) - d^2F(S(U))\phi$ and d^2F is the differential of ∇F.

In this subsection, we study a modified equation which coincides with (17) as long as the solutions stays small:

$$\partial_t \phi = JH(U)\phi + JN_\varepsilon(U, \phi) \tag{18}$$

where $N_\varepsilon(U, \eta) = \rho(\varepsilon^{-1}\eta) N(U, \eta)$ and ρ is a smooth function with compact support around 0.

We state the

Proposition B.12 (Center-Stable Manifold)**.** *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique invariant smooth center-stable manifold $W^{cs}(U)$ for* (18) *build as a graph and tangent to $S(U) + X_c(U) \oplus X_s(U)$ at $S(U)$.*

Any solution ϕ of (18) *in the neighborhood of $S(U)$ tends as $t \to -\infty$ to $W^{cs}(U)$ with*

$$dist(\phi(t), W^{cs}(U)) = O(e^{\gamma t}) \text{ as } t \to -\infty$$

for any $\gamma \in (0, \gamma(U))$ and for any sufficiently small neighborhood V of $S(U)$ any solution in V not in $W^{cs}(U)$ leaves V in finite positive time.

Remark B.12.1. *If we only consider small solutions, we obtain a locally invariant manifold for the equation* (17)*, that is to say that for any initial condition in the manifold there exist an associated solution of* (17) *which stays in this manifold in a small interval of time around* 0*.*

Proof – Our proof is an adaptation of the one of Bressan [Bre] and we refer to it for more details. First we prove that there is a global solution of the equation (18) which do not grow much as $t \to +\infty$. We look for solution as a fixed point:

$$y(t) = \mathcal{G}_\varepsilon(y_0, y)(t)$$

for any $y_0 \in X_s(U) \oplus X_c(U)$ where for small positive ε

$$\mathcal{G}_\varepsilon(y_0,\eta)(t) = e^{tJH(U)}y_0 + \int_0^t e^{(t-s)JH(U)}\pi^s(U)JN_\varepsilon(U,\eta(s))\,ds \\ + \int_0^t e^{(t-s)JH(U)}\pi^c(U)JN_\varepsilon(U,\eta(s))\,ds - \int_t^{+\infty} e^{(t-s)JH(U)}\pi^u(U)JN_\varepsilon(U,\eta(s))\,ds$$

when $t \geq 0$ and

$$\mathcal{G}_\varepsilon(y_0,\eta)(t) = e^{tJH(U)}y_0 + \int_t^0 e^{(t-s)JH(U)}\pi^s(U)JN_\varepsilon(U,\eta(s))\,ds \\ + \int_t^0 e^{(t-s)JH(U)}\pi^c(U)JN_\varepsilon(U,\eta(s))\,ds - \int_t^{+\infty} e^{(t-s)JH(U)}\pi^u(U)JN_\varepsilon(U,\eta(s))\,ds$$

when $t \leq 0$, with $\pi^*(U)$ the projector into $X^*(U)$ with respect to the decomposition $\oplus_{*\in\{c,s,u\}}X^*(U)$.

Let us introduce for $\gamma(U) = \Re E_1(U)$ and any Γ smaller than $\gamma(U)$, the space

$$Y_\Gamma = \left\{\eta : \mathbb{R} \mapsto L^2(\mathbb{R}^3,\mathbb{C}^4),\ \exists C>0,\ \|\eta(t)\|_2 \leq e^{\Gamma|t|},\ \forall t \in \mathbb{R}\right\}.$$

For sufficiently small $\varepsilon > 0$, the map $\mathcal{G}_\varepsilon(y_0,\cdot)$ leaves Y_Γ invariant and is continuous for the norm

$$y \mapsto \sup_{t\in\mathbb{R}}\left\{\|\eta(t)\|_2 e^{-\Gamma|t|}\right\}.$$

Moreover, it is a strict contraction for sufficiently small U and $\varepsilon > 0$. In fact, ε is a $o(\gamma(U))$. This proves the existence of the fixed point y.

Then we fix $h_U^{cs}(y_0) = y(0) - y_0$. The invariance of the graph of h_U^{cs} by the flow is immediate.

Now we prove the smoothness property. In fact $N_\varepsilon(U,\eta)$ is l times differentiable in η from $Y_{\Gamma'}$ to Y_Γ if $(l+1)\Gamma' \leq \Gamma$ and $\mathcal{G}_\varepsilon$ is l times differentiable from $X_c(U) \oplus X_s(U) \times Y_{\Gamma''}$ to Y_Γ if $2l\Gamma'' \leq \Gamma$, see [Bre]. We introduce the family y_n satisfying

$$\eta_0 = 0 \text{ and } \eta_{n+1} = \mathcal{G}_\varepsilon(y_0,\eta_n).$$

This sequence converge to y in Y_Γ. Moreover, as functions of y_0, the convergence is uniform in Y_Γ on bounded sets of $X_s(U) \oplus X_c(U)$.

We want to prove that the sequence of their derivatives of order k with respect to η also converges in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. We prove it by induction in k.

So suppose that $(\partial^j \eta_n)_{n\in\mathbb{N}}$ is converging in Y_Γ for all $j < k$ and any $\Gamma < \gamma(U)$. Then we have that (see [Bre])

$$\begin{aligned}\partial\eta_n &= \partial\mathcal{G}_\varepsilon(y_0, \eta_{n-1}) = L + M\left(\partial_\eta N_\varepsilon(U, \eta_{n-1})\partial\eta_{n_1}\right)\\ \partial^k\eta_n &= \partial^k\mathcal{G}_\varepsilon(y_0, \eta_{n-1}) = M\left(\partial_\eta N_\varepsilon(U, \eta_{n-1})\partial^k\eta_{n-1} + \Psi_k(\eta_{n-1}, \dots, \partial^{k-1}\eta_{n-1})\right), \ \forall k \geq 2\end{aligned}$$

with $L = e^{tJH(U)}$ and

$$(M\eta)(t) = -\int_0^t e^{(t-s)JH(U)}\pi^{cs}(U)J\eta(s)\, ds + \int_t^{+\infty} e^{(t-s)JH(U)}\pi^{u}(U)J\eta(s)\, ds$$

and Ψ_k a smooth function of k parameter. Hence since $M \circ \partial_\eta N_\varepsilon(U, y_{n-1})$ is a strict contraction in Y_Γ for sufficiently small ε and U (once more ε is a $o(\gamma(U))$), this proves the convergence of the sequence of k-th derivatives in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. Hence the sequences of derivatives of $(\eta_n)_{n\in\mathbb{N}}$ in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. This gives the differentiability at any order of $y(0) = h(y_0)$. This also gives, since $N(U, \eta) = O(|\eta|^2)$ around zero, that $h(y_0) = O(|y_0|^2)$ around zero.

Now we want prove that $W^{cs}(U)$ is attractive in negative time. In fact $W^{cs}(U)$ is the graph of a smooth function $h : X^{cs} \mapsto X^u(U)$. Let η be such that $S(U) + \eta$ is a solution of (3), we have

$$\partial_t\eta = JH(U)\eta + JN(U, \eta).$$

$$\eta = y + r = y + h(y) + z$$

with $y = \pi^{cs}(U)\eta$ and we have the following equation for z

$$\partial_t z = JH(U)z + M(U, y, z)$$

where

$$\begin{aligned}M(U, y, z) = \pi^u(U)\left\{JN(U, \eta) - JN(U, y + h(y))\right\}\\ - Dh(y)\pi^{cs}(U)\left\{JN(U, \eta) - JN(U, y + h(y))\right\}.\end{aligned}$$

Using Duhamel's formula, we obtain

$$z(t) = e^{tJH(U)}z(0) + \int_0^t e^{(t-s)JH(U)}M(U, y(s), z(s))\, ds$$

We obtain

$$\|z(t)\| \leq e^{\gamma(U)t}\|z(0)\| + C\int_0^t e^{(t-s)\gamma(U)}\|M(U, y(s), z(s))\|\, ds$$

and so for $\gamma \in (0, \gamma(U))$

$$e^{-\gamma t}\|z(t)\| \leq \|z(0)\| + C|U| \sup_{s\in[0,t]}\left\{e^{-\gamma s}\|z(s)\|\right\} + o(\sup_{s\in[0,t]}\left\{e^{-\gamma s}\|z(s)\|\right\})$$

where C do not depend of U and z. Hence if $z(0)$ is small, we have that there exists $c > 0$ such that $\|z(t)\| \leq ce^{\gamma t}$ for all $t \leq 0$.

Now choose V a sufficiently small neighborhood of 0 and ϕ a solution of (18) initially in V but not in $W^{cs}(U)$. Suppose it stays in V in positive time. We obtain that $\phi \in Y_\Gamma$. We have

$$\begin{aligned}\phi(t) = e^{tJH(U)}\left(\pi^s(U)+\pi^c(U)\right)\phi(0) + \int_0^t e^{-sJH(U)}\pi^s(U)JN_\varepsilon(U,\phi(s))\,ds \\ + \int_0^t e^{-sJH(U)}\pi^c(U)JN_\varepsilon(U,\phi(s))\,ds \\ + e^{tJH(U)}\pi^u(U)\left(y(0) + \int_0^\infty e^{-sJH(U)}(U)JN_\varepsilon(U,\phi(s))\,ds\right) \\ - \int_t^\infty e^{-sJH(U)}\pi^u(U)JN_\varepsilon(U,\phi(s))\,ds.\end{aligned}$$

with

$$\phi(0) + \int_0^\infty e^{-sJH(U)}(U)JN_\varepsilon(U,\phi(s))\,ds \neq 0.$$

Hence we obtain with (15), that $\phi(t)$ exponentially tends to infinity in norm. This is a contradiction so ϕ leaves V in finite time. □

Then reversing the time direction that is to say replacing H by $-H$ and F by $-F$, we obtain with this theorem a locally invariant center unstable manifold with the corresponding properties:

Proposition B.13 (Center-Unstable Manifold)**.** *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique smooth invariant center unstable manifold $W^{cu}(U)$ for* (18)*, build as a graph and tangent to $S(U)+X_c(U)\oplus X_u(U)$ at $S(U)$.*

Any solution ϕ of (18) *in the neighborhood of $S(U)$ tends as $t \to +\infty$ to $W^{cu}(U)$ with*

$$dist(\phi(t), W^{cu}(U)) = O(e^{-\gamma t}) \text{ as } t \to +\infty$$

and for $\gamma \in (0,\gamma(U))$, for any V sufficiently small neighborhood of $S(U)$ any solution in V not in $W^{cu}(U)$ leaves V in finite negative time.

We can build by the same way a center manifold which is the intersection of the two previous:

Proposition B.14 (Center Manifold)**.** *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique smooth invariant center manifold $W^c(U)$ for* (18)*, build as graph and tangent to $S(U)+X_c(U)$ at $S(U)$.*

Moreover, we have $W^c(U) = W^{cs}(U) \cap W^{cu}(U)$ and $W^c(U)$ contains the part of the PLS manifold which is in a neighborhood of $S(U)$.

Proof – We build the center manifold with the same method as in the previous cases. We can also build a center-unstable manifold inside center-stable manifold. More precisely, let $h_U^s : X_c(U) \oplus X_s(U) \mapsto X_u(U)$ be the map defining center-stable manifold and $h_U^u : X_c(U) \oplus X_u(U) \mapsto X_s(U)$ be the map defining center-unstable manifold. A solution $y = S(U)+y_c+y_s+y_u$ with $y_* \in X_*(U)$ for $* \in \{c,s,u\}$ is in the center-stable manifold if $y_u = h_U^s(y_c, y_s)$. Hence to obtain a center-unstable manifold inside center-stable manifold one has to solve, for each y_c, the equation

$$y_s = h_U^u(y_c, h_U^s(y_c, y_s))$$

this could be solve inside a small ball for small y_c and small U by means of the fixed point theorem, since $h^*_U(y_c, z)$ is a $O(|y_c|^2 + |z|^2)$ around zero for $* \in \{s, u\}$.

By the same way, we can also build a center-stable manifold inside the center-unstable manifold.

Using the uniqueness of the center manifold, we obtain that this two manifolds are equal to the center manifold and $W^c(U) = W^{cs}(U) \cap W^{cu}(U)$.

Then any stationary states in a small neighborhood of $S(U)$ converges to $W^{cs}(U)$ and $W^{cu}(U)$ using the stabilization results of the previous lemmas. Hence, we have that it belongs to $W^{cs}(U) \cap W^{cu}(U) = W^c(U)$. □

Remark B.14.1. *We notice that in the previous proofs we obtained ε as a function of U and this function is $O(\gamma(U))$.*

In the following section, we precise the dynamic inside the center manifold.

3 The dynamic on the center manifold

In this section, we prove that the dynamic on the center manifold around $S(V_0)$, for small non zero V_0, relaxes towards the PLS manifold. To this end, we use Theorem B.2 and Theorem B.4 about the time decay of the propagator associated with H.

3.1 Decomposition of the system

Like in [Bou05], we decompose a solution $\phi \in W^c(V_0)$ of the equation (3) with respect to the spectrum of $JH(U)$, with U specified in the sequel, and we study the equations for these different parts of the decomposition.

For any solution of (3) over an interval of time I containing 0, we write for $t \in I$

$$\phi(t) = e^{-\mathrm{i}\int_0^t E(U(s))\,ds}\left(S(U(t)) + \eta(t)\right).$$

where $\eta \in \mathcal{H}_0^\perp(U)$ and

$$\mathcal{H}_0^\perp(u_1,u_2) = \Bigg\{\eta \in L^2(\mathbb{R}^3,\mathbb{C}^8), \left\langle J\eta, \frac{\partial}{\partial \Re u_1} S(u_1,u_2)\right\rangle = \left\langle J\eta, \frac{\partial}{\partial \Im u_1} S(u_1,u_2)\right\rangle$$
$$= \left\langle J\eta, \frac{\partial}{\partial \Re u_2} S(u_1,u_2)\right\rangle = \left\langle J\eta, \frac{\partial}{\partial \Im u_2} S(u_1,u_2)\right\rangle = 0\Bigg\},$$

we notice that the orthgonality is taken with repsect to J. In fact, we have

$$\mathcal{H}_0^\perp(U) = \mathcal{H}_1(U) \oplus \mathcal{H}_c(U)$$

which is invariant under the action of $JH(U)$. We recall that $\mathcal{H}_1(U)$ is defined in Proposition B.8 and $\mathcal{H}_c(U)$ in Proposition B.9. We impose the orthogonality condition

$$\eta(t) \in \mathcal{H}_0^\perp(U(t)). \tag{19}$$

So we want to solve the equation

$$\begin{aligned} \mathrm{i}\partial_t \eta &= \{H - E(U)\}\eta + \{\nabla F(S(U)+\eta) - \nabla F(S(U))\} - \mathrm{i}dS(U)\dot{U} \\ &= \left\{H + d^2F(S(U)) - E(U)\right\}\eta + N(U,\eta) - \mathrm{i}dS(U)\dot{U} \end{aligned} \tag{20}$$

for $\eta \in \mathcal{H}_0^{\perp}(U(t))$. Here d^2F is the differential of ∇F and dS the differential of S in $\mathbb{R}^2$. To close the system, we need the equation for U. This follows from the condition

$$\langle \eta(t), JdS(U(t))\rangle = 0.$$

After a time derivation (like in[Bou05]), we obtain the desired equation:

$$\dot{U}(t) = -A(U(t), \eta(t))\langle N(U(t), \eta(t)), dS(U(t))\rangle.$$

where

$$A(U, \eta) = [\langle JdS(U), dS(U)\rangle - \langle J\eta, d^2S(U)\rangle]^{-1}$$

the matrix $[\langle JdS(U(t)), dS(U(t))\rangle - \langle J\eta(t), d^2S(U(t))\rangle]$ is invertible for small $|U(t)|$ and $\|\eta(t)\|_2$ since we have

$$[\langle JdS(U(t)), dS(U(t))\rangle - \langle J\eta(t), d^2S(U(t))\rangle] = \begin{pmatrix} J & 0_2 \\ 0_2 & J \end{pmatrix} + O(|U(t)| + \|\eta(t)\|_2),$$

Lemma B.15. *For any $s, s', \sigma \in \mathbb{R}$ and $p, q \in [1, \infty]$, let*

$$\mathcal{S}(V_0, \varepsilon) = \left\{(U, z);\ U \in B_{\mathbb{C}^2}(V_0, \varepsilon), z \in \mathcal{H}_c(U) \cap B_{H^{s'}_\sigma}(0, \varepsilon')\right\}$$

There exist $\varepsilon, \varepsilon' > 0$ and a unique map $g : S(V_0, \varepsilon) \mapsto B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4)$ which is smooth and such that for all $(U, z) \in S(V_0, \varepsilon)$, $g(U, z) \in \mathcal{H}_1(U)$, $z + g(U, z) \in \mathcal{H}_0(U)^{\perp}$, and $S(U) + z + g(U, z) \in W^c(V_0)$. Moreover, we have $g(U, 0) = 0$.

Proof – if h^c is the function for which $W^c(V_0)$ is the graph. Any $\phi \in L^2(\mathbb{R}^3, \mathbb{R}^8)$ can be written in the form $S(V_0) + \tilde{U} \cdot DS(V_0) + \xi + \rho$ where $\rho \in \mathcal{H}_1(V_0)$ and $\xi \in \mathcal{H}_c(V_0)$. It can be also written in the form $S(U) + z + r$ $r \in \mathcal{H}_1(U)$ and $z \in \mathcal{H}_c(U)$. These two decompositions in fact defines two bijective smooth maps in sufficiently small sets. We write Ψ for the first and Φ for the second. Then $f = \Psi \circ \Phi^{-1}$ has 3 components following the decomposition $\mathcal{H}_0(V_0) \oplus \mathcal{H}_1(V_0) \oplus \mathcal{H}_c(V_0)$, we write them (f_1, f_2, f_3). Then g is the solution of the implicit function in r

$$F(U, z, r) = f_2(U, z, r) - h^c(f_1(U, z, r), f_3(U, z, r)) = 0$$

which can be solved by solved by the implicit function theorem in $H^{s'}_\sigma$ for $s', \sigma \in \mathbb{R}$ since $\partial_r F(V_0, 0, 0)$ is invertible from $\mathcal{H}_1(V_0)$ to itself because $\partial_r f_2(V_0, 0, 0)$ is invertible from $\mathcal{H}_1(V_0)$ to itself and $\partial_r h_c(0, 0) = 0$.

The smoothness of g follows from the fact that $g(U, z) \in \mathcal{H}_1(U)$ and (8).

The last assertion follows by the fact that around $S(V_0)$, the PLS manifold is contained in center manifold associated with $S(V_0)$. □

Hence decomposing η with respect to the spectrum of $JH(U)$, we write

$$\eta(t) = g(U(t), z(t)) + z(t)$$

with $z \in \mathcal{H}_c(U) \cap L^2(\mathbb{R}^3, \mathbb{R}^8)$. We obtain the system

$$\begin{cases} \dot{U} = -A(U, \eta)\langle N(U, \eta), dS(U)\rangle \\ \partial_t z = JH(U)z + \mathbf{P}_c(U)JN(U, \eta) + (dP_c(U))A(U, \eta)\langle N(U, \eta), dS(U)\rangle\eta \end{cases}$$

with

$$\eta(t) = z(t) + g\left(U(t), z(t)\right).$$

We notice that this equation is defined only for z small with real values and U small. We now study this system.

3.2 The stabilization towards the PLS manifold

We now show that any solution of (3) which belongs to the center manifold $W^c(V_0)$ stabilizes as $t \to +\infty$ towards the manifold of the stationary states inside $W^c(V_0)$. The proof for $t \to -\infty$ is similar. To this end, we will use Theorem B.2 and Theorem B.4 to prove that z tends to zero in some sense.

Let us define for any ε, $\delta > 0$

$$\mathcal{U}(\varepsilon, \delta) = \left\{ U \in \mathcal{C}^\infty(\mathbb{R}, B_{\mathbb{C}^2}(V_0, \varepsilon)),\ \|\dot{U}\|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})} \leq \delta^2 \right\}$$

and for any $U \in \mathcal{U}(\varepsilon)$, let s', β be such that $s' > \beta + 2 > 2$ and $\sigma > 3/2$,

$$\mathcal{Z}(U, \delta) = \Bigg\{ z \in \mathcal{C}^\infty(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{R}^8)),\ z(t) \in \mathcal{H}_c(U(t)),$$
$$\max \left[\|z\|_{L^\infty(\mathbb{R}, H^{s'})}, \|z\|_{L^2(\mathbb{R}, H^{s'}_{-\sigma})}, \|z\|_{L^2(\mathbb{R}, B^\beta_{\infty,2})} \right] < \delta \Bigg\},$$

and ε, δ are small enough to ensure that for $U \in \mathcal{U}(\varepsilon, \delta)$ and $z \in \mathcal{Z}(U, \delta)$

$$S(U) + z + g(U, z) \in W^c(V_0).$$

where g is defined by Lemma B.15.

3.2.1 some useful lemma

We will need some technical lemmas, which we collect here.

Lemma B.16. *If Assumptions 1.1–1.5 hold. Let $\sigma, \sigma' \in \mathbb{R}$, $s > 1$ and $p, \tilde{p_1}, p_1, p_2, q \in [1, \infty]$ such that*

$$\frac{1}{p} + \frac{s}{3} \geq \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p}.$$

and

$$\frac{1}{p} + \frac{s}{3} \geq \frac{1}{\tilde{p_1}}.$$

Then there exist $\varepsilon > 0$ and $C > 0$ such that for all $U \in B_{\mathbb{C}}(0, \varepsilon)$ and $\eta \in B^s_{p_2,q}(\mathbb{R}^3, \mathbb{R}^8) \cap L^\infty(\mathbb{R}^3, \mathbb{R}^8)$ with $\langle Q \rangle^\sigma \eta \in B^s_{p_1,q}(\mathbb{R}^3, \mathbb{R}^8)$ and $\langle Q \rangle^{\sigma'} \eta \in B^s_{\tilde{p}_1,q}(\mathbb{R}^3, \mathbb{R}^8)$, we have

$$\|\langle Q \rangle^\sigma N(U, \eta)\|_{B^s_{p,q}} \leq C\left(s, F, |U| + \|\eta\|_{L^\infty}\right) |U|\, \|\eta\|_{L^\infty} \left\| \langle Q \rangle^{\sigma'} \eta \right\|_{B^s_{\tilde{p}_1,q}}$$
$$+ C\left(s, F, |U| + \|\eta\|_{L^\infty \cap B^s_{p_2,q}}\right) \|\eta\|^2_{L^\infty} \|\langle Q \rangle^\sigma \eta\|_{B^s_{p_1,q}}. \quad (21)$$

Proof – We recall the definition

$$N(U, \eta) = \nabla F(S(U) + \eta) - \nabla F(S(U)) - d^2 F(S(U))\eta.$$

We have

$$N(U, \eta) = \int_0^1 \int_0^1 d^3 F(S(U) + \theta' \theta \eta) \cdot \eta \cdot \theta \eta \, d\theta' d\theta,$$

or

$$N(U,\eta) = \int_0^1\int_0^1 d^3F(S(U))\cdot\eta\cdot\theta\eta\,d\theta' d\theta + \int_0^1\int_0^1 d^4F(S(U)+\theta''\theta'\theta\eta)\cdot\theta'\theta\eta\cdot\eta\cdot\theta\eta\,d\theta'' d\theta' d\theta,$$

Then we use for $s\in\mathbb{R}_+^*$, p, p_1, p_2, $\in[1,\infty]$ such that $\frac{1}{p}+\frac{s}{3}\geq\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{1}{p}$,

$$\|uv\|_{B^s_{p,q}}\leq C\|u\|_{B^s_{p_1,q}}\|v\|_{B^s_{p_2,q}},$$

and for $s>1$, we use [EV97, Proposition 2.1]

$$\|d^kF(\psi)\|_{B^s_{p_2,q}}\leq C\left(s,F,\|\psi\|_{L^\infty}\right)\|\psi\|_{B^s_{p_2,q}},$$

for $k=3$ or $k=4$ and $d^4F(z)=O(|z|)$, otherwise we decompose $d^4F(z)=A+O(|z|)$ where A is the constant operator.

Eventually using Lemma B.6 and

$$\left\|\langle Q\rangle^\sigma|\eta|^l\right\|_{B^s_{p_1,q}}\leq C\|\eta\|^{l-1}_{L^\infty}\|\langle Q\rangle^\sigma\eta\|_{B^s_{p_1,q}},$$

for $l\in\mathbb{N}$, we conclude the proof. □

Lemma B.17. *If Assumptions 1.1–1.5 hold. Let be $\sigma\in\mathbb{R}$, $s>1$, $p,p_1,p_2,q\in[1,\infty]$ and $\sigma_1,\sigma_2\in\mathbb{R}$ such that*

$$\frac{1}{p}+\frac{s}{3}\geq\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{1}{p}.$$

Then there exist $\varepsilon>0$ and $C>0$ such that for all $U\in B_{\mathbb{C}}(0,\varepsilon)$ and $\eta\in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)\cap L^\infty(\mathbb{R}^3,\mathbb{R}^8)$ with $\langle Q\rangle^{\sigma_1}\eta\in B^s_{p_1,q}(\mathbb{R}^3,\mathbb{R}^8)$ and $\langle Q\rangle^{\sigma_2}\eta\in B^s_{p_2,q}(\mathbb{R}^3,\mathbb{R}^8)$, we have

$$\begin{aligned}&\|< Q >^\sigma\left(\nabla F(S(U)+\eta)-\nabla F(S(U)-\nabla F(\eta))\right\|_{B^s_{p,q}}\\&\qquad\leq C(s,F,|U|+\|\eta\|_{L^\infty})\left(|U|+\|< Q >^{\sigma_1}\eta\|_{B^s_{p_1,q}}\right)|U|\,\|< Q >^{\sigma_2}\eta\|_{B^s_{p_2,q}}.\end{aligned}$$

Lemma B.18. *If Assumptions 1.1–1.5 hold. Let be $\sigma\in\mathbb{R}$, $s>1$ and $p,q\in[1,\infty]$ such that $sp\geq3$. Then there exist $\varepsilon>0$ and $C>0$ such that for all U, $U'\in B_{\mathbb{C}^2}(0,\varepsilon)$ and η, $\eta'\in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)$, we have*

$$\begin{aligned}&\left\|\langle Q\rangle^\sigma\left\{N(U,\eta)-N(U',\eta')\right\}\right\|_{B^s_{p,q}}\leq C\left(s,F,|U|+|U'|+\|\eta\|_{B^s_{p,q}}+\|\eta'\|_{B^s_{p,q}}\right)\times\\&\times\Bigg\{\left(\|\langle Q\rangle^{\sigma_1}\eta\|_{B^s_{p,q}}+\|\langle Q\rangle^{\sigma_1}\eta'\|_{B^s_{p,q}}\right)^2\left(|U-U'|+\left\|\langle Q\rangle^{\sigma_2}(\eta-\eta')\right\|_{B^s_{p,q}}\right)\\&\quad+\left(|U|+|U'|+\left\|\langle Q\rangle^{\sigma_1'}\eta\right\|_{B^s_{p,q}}+\left\|\langle Q\rangle^{\sigma_1'}\eta'\right\|_{B^s_{p,q}}\right)\times\\&\quad\times\left(\|\langle Q\rangle^{\sigma_2'}\eta\|_{B^s_{p,q}}+\|\langle Q\rangle^{\sigma_2'}\eta'\|_{B^s_{p,q}}\right)\|\langle Q\rangle^{\sigma_3'}(\eta-\eta')\|_{B^s_{p,q}}\Bigg\},\end{aligned}$$

with $2\sigma_1+\sigma_2=\sigma_1'+\sigma_2'+\sigma_3'=\sigma$ if $< Q >^w\eta$, $< Q >^w\eta'\in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)$ for $w\in\{\sigma_1,\sigma_2,\sigma_1',\sigma_2',\sigma_3'\}$.

Proof – Using the identity

$$N(u,\eta)=\int_0^1\int_0^1 d^3F(S(u)+\theta'\theta\eta)\cdot\eta\cdot\theta\eta\,d\theta' d\theta.$$

we can restrict the study to $d^3F(\phi)-d^3F(\phi')$. If $F=O(|z|^5)$, we have

$$\|\langle Q\rangle^\sigma\left(d^3F(\phi)-d^3F(\phi')\right)\|_{B^s_{p,q}}\leq\int_0^1\|d^4F(\phi+t(\phi-\phi'))\|_{B^s_{p,q}}\|\langle Q\rangle^\sigma(\phi-\phi')\|_{B^s_{p,q}}\,dt.$$

Then since $s>1$ and $sp\geq 3$, we use

$$\|d^4F(\psi)\|_{B^s_{p,q}}\leq C(s,F,\|\psi\|_{B^s_{p,q}}).$$

Using Lemma B.6, we conclude the proof when $F=O(|z|^5)$.

Otherwise, if F is an homogeneous polynomial of order 4, the proof is easily adaptable since d^4F is a constant tensor.

The case $F=O(|z|^4)$ follows by summing the two previous one since as a function of $u\in\mathbb{R}^8$, $F(u)=Au^{\otimes 4}+O(|u|^5)$. □

Lemma B.19. *If Assumptions 1.1–1.5 hold. Let be $\sigma\in\mathbb{R}$, $s\in\mathbb{R}$ and $p,q\in[1,\infty]$. Then there exist $\varepsilon>0$, $M>0$ and $C>0$ such that for all U, $U'\in B_{\mathbb{C}^2}(0,\varepsilon)$ and η, $\eta'\in B_{L^2(\mathbb{R}^3,\mathbb{R}^8)}(0,M)$ with $\langle Q\rangle^\sigma\{\eta-\eta'\}\in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)$, one has*

$$\left|A(U,\eta)-A(U',\eta')\right|\leq C\left\{\left|U-U'\right|+\left\|\langle Q\rangle^\sigma\{\eta-\eta'\}\right\|_{B^s_{p,q}}\right\}\tag{22}$$

Proof – We recall that

$$A(U,\eta)=[\langle JdS(U),dS(U)\rangle-\langle J\eta,d^2S(U)\rangle]^{-1}.$$

We have

$$\begin{aligned}A(U,\eta)-A(u',\eta')=-[\langle JdS(U),dS(U)\rangle-\langle J\eta,d^2S(U)\rangle]^{-1}\times\\ \times\left\{\langle JdS(U),dS(U)\rangle-\langle J\eta,d^2S(U)\rangle-\langle JdS(U'),dS(U')\rangle+\langle J\eta',d^2S(U')\rangle\right\}\times\\ \times[\langle JdS(U'),dS(U')\rangle-\langle J\eta',d^2S(U')\rangle]^{-1}.\end{aligned}$$

The lemma then follows from Lemma B.6. □

3.2.2 Global wellposedness for z and stabilization

Let be $U\in\mathcal{U}(\varepsilon,\delta)$ and $z_0\in\mathcal{H}_c(U(0))\cap H^{s'}$. Let us write $U_\infty=\lim_{t\to+\infty}U(t)$, we define $\mathcal{T}_{U,z_0}(z)$ by

$$\begin{aligned}\mathcal{T}_{U,z_0}(z)(t)=e^{-\mathrm{i}tH+\mathrm{i}\int_0^t E(U(r))\,dr}z_0+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\nabla F(\eta(v))\,dv\\ +\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\{\nabla F(S(U(v))+\eta(v))-\nabla F(S(U(v))-\nabla F(\eta(v))\}\,dv\\ +\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))dS(U(v))A(U(v),\eta(v))\langle N(U(v),\eta(v)),dS(U(v))\rangle\,dv\\ -\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}(d\mathbf{P}_c(U(v))\dot U(v)\eta(v)\,dv.\end{aligned}$$

with

$$\eta(t) = z(t) + g\left(U(t), z(t)\right)$$

First, we have a local wellposedness result for z with the

Lemma B.20. *If Assumptions 1.1–1.5 hold. Then there exist $\delta > 0$ and $\varepsilon > 0$ such that for any $U \in \mathcal{U}(\delta, \varepsilon)$ and $z_0 \in B_{H^{s'}(0,\delta)} \cap \mathcal{H}_c(U(0))$ there are $T^{\pm}(z_0, U) > 0$ and a unique solution $z \in \mathcal{C}((-T^-(z_0,U); +T^+(z_0,U)), H^{s'}(0,\delta))$ of the equation*

$$\begin{cases} \partial_t z = JH(U)z + \mathbf{P}_c(U)JN(u,\eta) - (d\mathbf{P}_c(U))\dot{U}\eta, \\ z(0) = z_0, \end{cases} \tag{23}$$

where $\eta(t) = z(t) + g\left(U(t), z(t)\right)$.

Moreover, z is smooth and we have $T^+(z_0, U) = +\infty$ or $\lim_{t\to T^+(z_0,U)} \|z(t)\|_{H^{s'}} \geq \delta$ and $T^-(z_0,U) = +\infty$ or $\lim_{t\to -T^-(z_0,U)} \|z(t)\|_{H^{s'}} \geq \delta$.

Proof – It is a consequence of the fix point theorem applied to $\mathcal{T}_{U,z_0}$.

Using Lemmas B.16, B.18 and B.19 with the Estimate (14)–(16) and the properties of g given by Lemma B.15 we obtain that $\mathcal{T}_{U,z_0}$ leaves a small ball in $H^{s'}$ invariant and is a contraction inside this ball.

Hence there exists a unique solution defined on the interval $[-T, T]$. Classical arguments permit to extend the solution over a maximal interval $(-T^-(z_0,U), T^+(z_0,U))$ such that if $T^+(z_0,U) < \infty$ then necessarily the solution should leave a small ball in $H^{s'}$ at time $T^+(z_0,U)$. □

We have now a global wellposedness result as stated in the

Lemma B.21. *If Assumptions 1.1–1.5 hold. There exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that for any $\delta \in (0,\delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, $U \in \mathcal{U}(\varepsilon,\delta)$ and $z_0 \in B_{H^{s'}}(0,\delta) \cap \mathcal{H}_c(U(0))$ we obtain for the Cauchy problem (23), $T^+(U,z_0) = +\infty$, $T^+(U,z_0) = -\infty$ and $z \in \mathcal{Z}(U,\delta)$.*

Proof – The smoothness of z follows from a bootstrapping argument on the fixed point theorem and the fact that $(1 - P_c(U))z \equiv 0$ follows from a derivation and $(1 - P_c(U(0)))z(0) = 0$.

Let us introduce for any $0 < T < T^+(U, z_0)$, the function

$$m(T) = \sup_{t\in(-T,T)} \left\{ \|z\|_{L^\infty((-T,T),H^{s'})},\ \|z\|_{L^2((-T,T),H^{s'}_{-\sigma})},\ \|z\|_{L^2((-T,T),B^{\beta}_{\infty,2})} \right\}$$

First, we study the estimation of $L^2((-T,T), H^{s'}_{-\sigma})$. We use only the estimates of the Theorem B.1.

$$\begin{aligned} &\|z\|_{L^2((-T,T),H^{s'}_{-\sigma})} \\ &\leq\ C_0 \|z_0\|_{H^{s'}_{-\sigma}} + C \left\| \mathbf{P}_c \int_0^t e^{-\mathrm{i}(t-v)H + \mathrm{i}\int_v^t E(U(r))\,dr} \mathbf{P}_c(U(v)) J\nabla F(\eta(v))\,dv \right\|_{L^2((-T,T),H^{s'}_{-\sigma})} \\ &\quad + C \left\| \nabla F(S(U) + \eta) - \nabla F(S(U) - \nabla F(\eta) \right\|_{L^2((-T,T),H^{s'}_{\sigma})} \\ &\quad + C \left\| dS(U) A(U,\eta) \langle N(U,\eta), dS(U) \rangle \right\|_{L^2((-T,T),H^{s'}_{\sigma})} \\ &\quad + C \left\| (d\mathbf{P}_c(U)\dot{U}\eta \right\|_{L^2((-T,T),H^{s'}_{\sigma})} . \end{aligned}$$

We now study the estimation of the third term of the right hand side

$$\begin{aligned}
&\left\| \int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr} \mathbf{P}_c\mathbf{P}_c(U(v))J\nabla F(\eta(v))\,dv \right\|_{L^2_t((-T,T),H^{s'}_{-\sigma})} \\
&\leq \int_{-T}^T \left\| e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr} \mathbf{P}_c\mathbf{P}_c(U(v))J\nabla F(\eta(v)) \right\|_{L^2_t((-T,T),H^{s'}_{-\sigma})} dv \\
&\leq C(U)\left\|\nabla F(\eta)\right\|_{L^1((-T,T),H^{s'})} \\
&\leq C(U)\left\|\eta\right\|^2_{L^2((-T,T),L^\infty)} \left\|\eta\right\|_{L^\infty((-T,T),H^{s'})},
\end{aligned}$$

where we used Theorem B.1 Estimate (ii).

Hence for the $L^2H^{s'}_{-\sigma}$ estimate, we obtain

$$\begin{aligned}
\|z\|_{L^2((-T,T),H^{s'}_{-\sigma})} \leq\ & C_0\|z_0\|_{H^{s'}_{-\sigma}} + C\|\eta\|^2_{L^2((-T,T),L^\infty)}\|\eta\|_{L^\infty((-T,T),H^{s'})} \\
&+C\left(\|U\|_\infty + \|\eta(v)\|_{L^2((-T,T),H^{s'}_{-\sigma})}\right)\|U\|_\infty\|\eta\|_{L^2((-T,T),H^{s'}_{-\sigma})} \\
&+C\|\eta\|^2_{L^2((-T,T),L^\infty)} + C\left\|\dot{U}\right\|_{L^2}\|\eta\|_{L^\infty((-T,T),H^{s'})},
\end{aligned}$$

using Lemma B.15, we obtain

$$\|z\|_{L^2((-T,T),H^{s'}_{-\sigma})} \leq C_0\|z_0\|_{H^{s'}_{-\sigma}} + Cm(T)^3 + Cm(T)^2 + C\sqrt{m(T)}m(T)^2$$

Then, we estimate the $H^{s'}$ norm

$$\begin{aligned}
\|z(t)\|_{H^{s'}} \leq \|z_0\|_{H^{s'}} &+ \int_{-T}^T \|\nabla F(\eta(v))\|_{H^{s'}}\,dv \\
&+\Big\| \int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))\times \\
&\quad \times J\{\nabla F(S(U(v))+\eta(v)) - \nabla F(S(U(v)) - \nabla F(\eta(v))\}\,dv\Big\|_{H^{s'}} \\
&+\int_{-T}^T \|dS(U(v))A(U(v),\eta(v))\langle N(U(v),\eta(v)), dS(U(v))\rangle\|_{H^{s'}}\,dv \\
&+\int_{-T}^T \left\|(d\mathbf{P}_c(U(v))\dot{U}(v)\eta(v)\right\|_{H^{s'}}\,dv.
\end{aligned}$$

to estimate the third term of the right hand side, we use the H-smoothness estimates, more precisely Theorem B.1 Estimate (ii) and then we use Lemma B.14:

$$\begin{aligned}
&\left\| \int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\{\nabla F(S(U(v))+\eta(v)) - \nabla F(S(U(v))) - \nabla F(\eta(v))\}\,dv\right\|_{H^{s'}} \\
&\qquad\leq C\left\|\{\nabla F(S(U)+\eta) - \nabla F(S(U) - \nabla F(\eta)\right\|_{L^2((-T,T),H^{s'}_\sigma)} \\
&\qquad\leq C\left(\|U\|_\infty + \|\eta(v)\|_{L^\infty((-T,T),H^{s'})}\right)\|U\|_\infty\|\eta\|_{L^2((-T,T),H^{s'}_{-\sigma})}
\end{aligned}$$

Hence for the $L^\infty H^{s'}$ estimate, we obtain

$$\begin{aligned}
\|z(t)\|_{H^{s'}} \leq \|z_0\|_{H^{s'}} &+ C\|\eta\|_{L^\infty((-T,T),H^{s'})}\|\eta\|^2_{L^2((-T,T),L^\infty)} \\
C&\left(\|U\|_{L^\infty((-T,T))} + \|\eta(v)\|_{L^\infty((-T,T),H^{s'})}\right)\|U\|_{L^\infty((-T,T))}\|\eta\|_{L^2((-T,T),H^{s'}_{-\sigma})} \\
&+C\|\eta\|^2_{L^2((-T,T),L^\infty)} + \left\|\dot{U}\right\|_{L^1((-T,T))}\|\eta\|_{L^\infty((-T,T),H^{s'})},
\end{aligned}$$

using Lemma B.15, we obtain

$$\|z(t)\|_{H^{s'}} \leq \|z_0\|_{H^{s'}} + C\varepsilon m(T) + Cm(T)^3 + Cm(T)^2.$$

For the $L^2 B^{\beta}_{\infty,2}$ estimate, by Proposition B.9 and Theorem B.4, we have

$$\begin{aligned}\|z\|_{L^2((-T,T),B^{\beta}_{\infty,2})} \leq\ & C_0\|z_0\|_{H^{\beta+1+\theta/2}} + C\left\|d^2F(S(U))\cdot\eta\right\|_{L^2((-T,T),B^{\beta+2+\theta}_{1,2})} + C\left\|N(U,\eta)\right\|_{L^1((-T,T),H^{\beta+1+\theta/2})} \\ & + C\left\|dS(U)A(U,\eta)\langle N(U,\eta), dS(U)\rangle\right\|_{L^1((-T,T),H^{\beta+1+\theta/2})} \\ & + C\left\|(d\mathbf{P}_c(U))\dot{U}\eta\right\|_{L^1((-T,T),H^{\beta+1+\theta/2})} dv.\end{aligned}$$

With Lemma B.16 and B.17, we infer

$$\begin{aligned}\|z\|_{L^2(\mathbb{R},B^{\beta}_{\infty,2})} \leq\ & C_0\|z_0\|_{H^{\beta+1+\theta/2}} + C|U|_\infty\|\eta\|_{L^2((-T,T),H^{\beta+2+\theta}_{-\sigma})} \\ & + C|U|_\infty\|z\|_{L^2((-T,T),L^\infty)}\|z\|_{L^2((-T,T),H^{\beta+1+\theta/2})} \\ & + C(|U|_\infty + \|\eta\|_{L^\infty((-T,T),H^{\beta+1+\theta/2})})\|\eta\|^2_{L^2((-T,T),L^\infty)}\|z\|_{L^\infty((-T,T),H^{\beta+1+\theta/2})} \\ & + C(|U|_\infty + \|\eta\|_{L^\infty((-T,T),H^{\beta+1+\theta/2})})\|\eta\|_{L^2((-T,T),H^{\beta+1+\theta/2}_{-\sigma})}\|\eta\|_{L^\infty((-T,T),H^{\beta+1+\theta/2})} \\ & + C|\dot{U}|_\infty\|\eta\|_{L^2((-T,T),H^{\beta+1+\theta/2}_{-\sigma})}\end{aligned}$$

we infer since $s' \geq \beta + 2 + \theta$ and using Lemma B.15,

$$\|z\|_{L^2((-T,T),B^{\beta}_{\infty,2})} \leq C_0\|z_0\|_{H^{\beta+1+\theta/2}} + \varepsilon m(T) + Cm(T)^3 + Cm(T)^2.$$

Hence we obtain

$$m(T) \leq C_0\|z_0\|_{H^{\beta+1+\theta/2}} + C\varepsilon m(T) + Cm(T)^3 + Cm(T)^2 + C\sqrt{m(T)}m(T)^2,$$

where C_0 do not depend of m and C is a nondecreasing function of $\|z\|_{L^\infty((-T,T),H^{\beta+1+\theta/2})}$ and hence it can be bounded by a nondecreasing function of m.

If $\|z_0\|_{H^{s'}}$ is small then $m(0)$ is small and $m(T)$ stay small. Therefore we have that $z \in \mathcal{Z}(U,\delta)$ if $\|z_0\|_{H^{s'}}$is small enough for any δ and ε are small enough. □

Assume that Assumption 1.1–1.5 hold. The solution z just found is a function of z_0 and U, writing it $z[z_0,U]$, we have the following important property given by the

Lemma B.22. *There exists $\delta_0 > 0$, $\varepsilon_0 > 0$, $C > 0$, $\kappa \in (0,1)$ such that for any $\delta \in (0,\delta_0)$, $\varepsilon \in (0,\varepsilon_0)$, U, $U' \in \mathcal{U}(\varepsilon,\delta)$, $z_0 \in \mathcal{H}_c(U(0))$, $z'_0 \in \mathcal{H}_c(U'(0))$, $z \in \mathcal{Z}(U,\delta)$ and $z' \in \mathcal{Z}(U',\delta)$, one has*

$$\begin{aligned}&\left\|e^{-\mathrm{i}\int_0^t E(U(r))\,dr} z[z'_0,U'] - e^{-\mathrm{i}\int_0^t E(U'(r))\,dr} z[z_0,U]\right\|_{L^2\infty(\mathbb{R},H^{s'})\cap L^2(\mathbb{R},L^\infty)\cap L^2(\mathbb{R},H^{s'}_{-\sigma})} \\ &\qquad \leq C\left\|z_0 - z'_0\right\|_{H^{s'}} + \kappa\left\{\left\|U-U'\right\|_{L^\infty} + \left\|\dot{U}-\dot{U}'\right\|_{L^\infty}\right\}.\end{aligned}$$

Proof – Using the technics of the previous lemma on the identity

$$e^{-\mathrm{i}\int_0^t E(U(r))\,dr} z - e^{-\mathrm{i}\int_0^t E(U'(r))\,dr} z' = e^{-\mathrm{i}\int_0^t E(U(r))\,dr}\mathcal{T}_{U,z_0}(z) - e^{-\mathrm{i}\int_0^t E(U'(r))\,dr}\mathcal{T}_{U',z'_0}(z').$$

□

3.2.3 Global wellposedness for U and stabilization

Here we want to solve the equation for U. We notice that z and α have been built in the previous section and are functions of U and $z_0 \in \mathcal{H}_c(U(0))$. Let us introduce for any $U_0 \in B_{\mathbb{C}}(0,\varepsilon)$ the function on $\mathcal{U}(\varepsilon,\delta)$:

$$f_{U_0}(U)(t) = U_0 - \int_0^t A(U(v),\eta(v))\langle N(U(v),\eta(v)), dS(U(v))\rangle\, dv,$$

where $\eta = z(t) + g\,[U(t), z(t)]$. We have the

Lemma B.23. *If Assumptions 1.1–1.5 hold. There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$, the function f_{U_0} maps $\mathcal{U}(\varepsilon,\delta)$ into itself if U_0 and $z_0 \in H^{s'} \cap \mathcal{H}_c(U_0)$ are small enough.*

Proof – By means of Lemma B.16, we obtain

$$\|\partial_t f_{U_0}(U)\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})} \leq C\,\|N(U(v),\eta(v))\|_{L^1(\mathbb{R},H^{s'}_{-\sigma})\cap L^\infty(\mathbb{R},H^{s'})} \leq \delta^2.$$

and

$$\|f_{U_0}(U)\|_{L^\infty(\mathbb{R})} \leq |U_0| + C\,\|N(U(v),\eta(v))\|_{L^1(\mathbb{R},H^{s'})} \leq |U_0| + \delta^2,$$

hence for sufficiently small U_0 and δ, we obtain the lemma. □

The function f_{U_0} has also a local Lipshitz property as stated by the

Lemma B.24. *If Assumptions 1.1–1.5 hold. For any $T > 0$, there exists $\delta_0 > 0$, $\varepsilon_0 > 0$ and $\kappa \in (0,1)$ such that for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$, for any u, $u' \in \mathcal{U}(\varepsilon,\delta)$, for any $z_0 \in \mathcal{H}_c(U(0)) \cap H^{s'}$, for any $z_0' \in \mathcal{H}_c(U'(0)) \cap H^{s'}$ small enough, for U_0, U_0' small enough, such that*

$$\left|f_{U_0}(U) - f_{U_0'}(U')\right|_{L^\infty((-T;T))} + \left|\partial_t f_{U_0}(U) - \partial_t f_{U_0'}(U')\right|_{L^1((-T;T))}$$
$$\leq |U_0 - U_0'| + \kappa\left(\|U - U'\|_{L^\infty((-T;T))} + \left\|\dot{U} - \dot{U}'\right\|_{L^1((-T;T))} + \|z_0 - z_0'\|_{H^{s'}}\right).$$

Proof – This a straightforward consequence of Lemma B.18, B.19 and B.22. □

We now obtain the

Lemma B.25. *If Assumptions 1.1–1.5 hold. There exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$ such that for any $U_0 \in \mathbb{C}$ small and $z_0 \in \mathcal{H}_c(U_0) \cap H^{s'}_\sigma$ small, the equation*

$$\begin{cases} \dot{U} &= -A(U,\eta)\langle N(U,\eta), dS(U)\rangle, \\ U(0) &= U_0, \end{cases}$$

where $\eta(t) = z(t) + g\,[U(t), z(t)]$, has a unique solution in $\mathcal{U}(\delta,\varepsilon)$.

Proof – This is also a the fixed point result for f_{U_0}. Let us fix $T > 0$ and consider, for any $V \in \mathcal{U}(\delta,\varepsilon)$ with sufficiently small $\delta > 0$ and $\varepsilon > 0$, the sequence:

$$\begin{cases} V_{n+1} = f_{U_0}(V_n),\ \forall n \in \mathbb{N} \\ V_0 = V; \end{cases}$$

for any $n \in \mathbb{N}$, $V_n \in \mathcal{U}(\delta, \varepsilon)$. With Lemma B.24, the fixed point theorem give us the convergence for the norms of $L^\infty((-T,T))$ and $\dot{W}^{1,1}((-T,T))$ of (V_n).

Then we notice that for any $T' \in \mathbb{R}$, we have

$$V_{n+1}(t) = f_{f_{U_0}(V_n)(T')}(V_n)(t - T').$$

Since for $T' \in (-T;T)$, $(f_{U_0}(V_n)(T'))$ is a Cauchy sequence, the Lemma B.24 give us the convergence of (V_n) for the norms of $L^\infty((T'-T;T'+T))$ and $\dot{W}^{1,1}((T'-T;T'+T))$.

Iterating this process, we prove the Lemma since the other statements are classical.

□

3.3 The nonlinear scattering and the end of the proof of Theorem B.7

We can conclude the proof of Theorem B.7 with our scattering result. First, we notice that we have an asymptotic profile for U, since $U \in \mathcal{U}(\varepsilon, \delta)$. Then we can also obtain an asymptotic profile for $e^{\mathrm{i}tH + \mathrm{i}\int_0^t E(U(v))\,dv} z$. But we prefer to obtain a scattering result with respect to $e^{JtH(U_\infty))\,dv}$. Hence we replace U by

$$t \mapsto V_\pm(t) = e^{-\mathrm{i}\int_0^t (E(U(v)) - E(U_{\pm\infty}))\,dv} U(t)$$

and z by

$$t \mapsto \xi_\pm(t) = e^{J\int_0^t (E(U(v)) - E(U_{\pm\infty}))\,dv} z(t).$$

We notice that $V_\pm(0) = U_0 := V_0$ and $\xi_\pm(0) = z_0 := \xi_0$.

We have the

Lemma B.26. *If Assumptions 1.1–1.5 hold. Then for any $U \in \mathcal{U}(\delta, \varepsilon)$ and solution z of (23), with $z_0 \in H^{s'}$ small, the following limit*

$$\xi_\infty = \lim_{t \to \pm\infty} e^{-JtH(V_{\pm\infty})} \xi_\pm(t)$$

exists in $H^{s'}$. Moreover, we have $z_\infty \in \mathcal{H}_c(0)$ and

$$\| e^{JtH(U_\infty)\,dr} \xi_{\pm\infty} - \xi_\pm(t) \|_{L_t^\infty(\mathbb{R}, H^{s'}) \cap L_t^2(\mathbb{R}, H^{s'}_{-\sigma}) \cap L_t^2(\mathbb{R}, B^\beta_{\infty,2})} \le C\delta^2$$

and

$$\lim_{t \to \pm\infty} \| e^{JtH(U_\infty)\,dr} \xi_{\pm\infty} - \xi_\pm(t) \|_{L_t^\infty(\mathbb{R}, H^{s'})} = 0.$$

Proof – Using exactly the same method as the one of Lemma B.21, applied to

$$\begin{aligned}
e^{-JtH(V_{\pm\infty})\,dr} \xi_\pm(t) = z_0 \\
&+ \int_0^t e^{JvH(V_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) J \left(\nabla F(S(V_\pm(v))) - \nabla F(S(V_{\pm\infty})) \right) \xi_\pm(v)\,dv \\
&\quad + \int_0^t e^{JvH(V_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) J N(V_\pm(v), \tilde{\eta}_\pm(v))\,dv \\
&+ \int_0^t e^{JvH(V_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) dS(V(v)) A(V_\pm(v), \tilde{\eta}_\pm(v)) \langle N(V_\pm(v), \tilde{\eta}_\pm(v)), dS(V_\pm(v)) \rangle\,dv \\
&- \int_0^t e^{JvH(V_\infty)} (d\mathbf{P}_c(V_\pm(v))) A(V_\pm(v), \tilde{\eta}_\pm(v)) \langle N(V_\pm(v), \tilde{\eta}_\pm(v)), dS(V_\pm(v)) \rangle \tilde{\eta}_\pm(v)\,dv,
\end{aligned}$$

with $\tilde{\eta}_\pm(t) = e^{J\int_0^t (E(V_\pm(v))-E(V_{\pm\infty}))\,dv}\,(z(t)+g(U(t),z(t)))$, we prove that the limit exist by the same way we also obtain the estimates and the convergence of

$$e^{JtH(U_\infty)\,dr}\xi_{\pm\infty} - \xi_\pm(t).$$

Since $(1-P_c(U(t)))\,z(t)=0$, we have $(1-P_c(V_\pm\infty))\,\xi_{\pm\infty}=0$ and hence $\xi_{\pm\infty}$ belongs to $\mathcal{H}_c(V_{\pm\infty})$. □

The scattering result follows from a one to one correspondence of the initial profile with the asymptotic profile as stated in the

Proposition B.27. *If Assumptions 1.1–1.5 hold. There exist $\varepsilon > 0$ and a continuous map $r : B^2_{\mathbb{C}}(0,\varepsilon) \mapsto \mathbb{R}^+$ with $r(U) = O(\Gamma(U))$ and $\mathcal{V}_0$, $\mathcal{V}_\pm$ neighborhoods of $(0,0)$ in*

$$S = \left\{(V,z)\,;\; V \in B_{\mathbb{C}^2}(0,\varepsilon),\; z \in \mathcal{H}_c(V) \cap B_{H^{s'}}(0, r(V))\right\}$$

which is endowed with the norm of $\mathbb{C}^2 \times H^{s'}$ such that the maps

$$\mathcal{P}_\pm : \begin{pmatrix} V_0 \\ \xi_0 \end{pmatrix} \in \mathcal{V}_0 \mapsto \begin{pmatrix} V_{\pm\infty} \\ z_{\pm\infty} \end{pmatrix} \in \mathcal{V}_\pm$$

are smooth bijections.

Proof – We choose

$$\mathcal{V}_0 = \left\{(V,z)\,;\; V \in B_{\mathbb{C}^2}(0,r),\; z \in \mathcal{H}_c(V) \cap B_{H^{s'}}(0, r(V))\right\}$$

for some positive r. We write

$$\mathcal{P}_+\left(V_0,\xi_0\right) = \left(V_0,\xi_0\right) + \mathcal{R}\left(V_0,\xi_0\right)$$

with $\mathcal{R}$ a lipshitz function. With the Proposition B.22 and B.24, we obtain that the Lipshitz norm of $\mathcal{R}$ is a $O\left(r\right)$. The proposition follows for sufficiently small positive r.

The statements for r follow from Remark B.14.1. □

Proof – **End of proof of Theorem B.7** Then we notice that the small *locally invariant* center manifold build in Section 2.2 for Equation (17) is now a small *invariant* (globally in time) center manifold. Indeed , we have just proved the stabilization towards the PLS manifold, this ensures that a solution in the center manifold will stay inside this manifold in the two direction of time.

For the same reasons the small locally invariant center-unstable manifold build in Section 2.2 is invariant and attractive in positive time. The corresponding conclusion holds for the center stable manifold.

The statements on the map r follows from Remark B.14.1.

□

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B.IV

Un résultat de stabilité pour des solutions stationnaires d'une classe d'équations de Dirac non linéaires

Résumé

Nous présentons, pour un opérateur de Dirac, des estimations de Strichartz et de régularité au sens de Kato. Puis nous nous intéressons à une classe d'équations de Dirac non linéaires. Celle-ci est associée à un opérateur de Dirac n'ayant qu'une seule valeur propre, dont la multiplicité est deux. Nous construisons une petite variété d'états stationnaires et, appliquant nos estimations, nous énonçons un théorème sur la stabilité asymptotique de ces états.

A stability result for small stationary solutions of a class of nonlinear Dirac equations

Abstract

We present Strichartz estimates for a Dirac operator with a potential and other estimates of the same type in weighted L^2 spaces. Then we study a class of nonlinear Dirac equations associated with a Dirac operator having only one double eigenvalue. We build a small manifold of stationary solutions and, as an application of our decay estimates, we state a theorem on the asymptotic stability of these solutions.

1 Version française abrégée

Nous étudions la stabilité des petites solutions stationnaires de l'équation de Dirac non linéaire (1). Dans cette note, nous reprenons les résultats de [Bou05] sous une hypothèse différente : que notre opérateur linéaire de référence n'a qu'une valeur propre de multiplicité deux. Cette dégénérescence est due à une invariance du système linéaire par un opérateur antilinéaire K défini en (3). L'opérateur K commute aux potentiels électriques et scalaires. Le doublement des valeurs propres apparaît donc dans d'importants modèles physiques. Notons que pour des potentiels matriciels plus généraux, il pourrait arriver que notre valeur propre soit simple. Dans ce cas, les résultats qui suivent restent valables avec des adaptations évidentes.

En supposant que la non linéarité est elle aussi invariante par K, nous obtenons des solutions stationnaires qui forment une petite variété tangente à l'espace propre de notre opérateur linéaire de référence. Nous pouvons alors montrer que toute perturbation d'un élément de cette famille se stabilise asymptotiquement sur la variété. Pour cela nous utilisons des estimations de propagation et de dispersion dont on trouvera une preuve dans [Bou05]. Nous obtenons ainsi un résultat qui est l'analogue de [GNT04][Theorem 1.7, Theorem 1.8].

Pour faire nos estimations, nous faisons l'hypothèse suivante :

Hypothèse 1.1. *Le potentiel* $V : \mathbb{R}^3 \mapsto S_4(\mathbb{C})$ *(matrice symétrique* 4×4 *) est une fonction* $\mathcal{C}^\infty$ *telle qu'il existe* $\rho > 5$ *vérifiant* $\forall \alpha \in \mathbb{N}^3$, $\exists C > 0$, $\forall x \in \mathbb{R}^3$, $|\partial^\alpha V|(x) \leq C\langle x \rangle^{-\rho-|\alpha|}$.

La notation $< x >$ désigne la quantité $\sqrt{1+x^2}$.

L'opérateur $H := D_m + V$ est alors autoadjoint sur $H^1(\mathbb{R}^3, \mathbb{C}^4)$. Dans notre cas, les seuils $\pm m$ sont les seuls points du spectre continu qui peuvent être associés à des ondes de vitesse nulle. La présence de telles ondes peut alors ralentir la propagation et la dispersion des ondes libres. Nous supposerons :

Hypothèse 1.2. *L'opérateur* $H = D_m + V$ *n'a pas de résonance aux seuils ni de vecteur propre aux seuils.*

Une résonance est ici un vecteur propre dans $H^{1/2}_{-\sigma}(\mathbb{R}^3, \mathbb{C}^4) \setminus H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ pour un $\sigma \in (1/2, \rho - 1/2)$. Les espaces L^2_σ et H^s_σ sont définis en B.2.

On introduit $\mathbf{P}_c(H) = \mathbb{1}_{\mathbb{R}\setminus(-m;m)}(H)$ le projecteur associé au spectre continu de H et $\mathcal{H}_c$ son image. En utilisant [Bou05, Theorem 1.1], on a le

Théorème 17 (H-régularité de $\langle Q\rangle^{-1}$)**.** *Supposons que les hypothèses 1.1 et 1.2 soient vérifiées. On a*

$$\int_{\mathbb{R}} \left\| \langle Q\rangle^{-1} e^{-\mathrm{i}tH} \mathbf{P}_c(H)\psi \right\|_{H^s}^2 dt \leq C \|\psi\|_{H^s}^2, \tag{i}$$

$$\left\| \int_{s<t} \langle Q\rangle^{-1} e^{-\mathrm{i}(t-s)H} \mathbf{P}_c(H) \langle Q\rangle^{-1} F(s)\, ds \right\|_{L^2(\mathbb{R},H^s)} \leq C \|F\|_{L^2(\mathbb{R},H^s)}, \tag{iii}$$

Nous obtenons également avec [Bou05, Theorem 1.2] et [KT98, Theorem 10.1] le

Théorème 18 (Estimations de Strichartz)**.** *Supposons que les hypothèses 1.1 et 1.2 soient vérifiées. Alors pour tout* $2 \leq p, q \leq \infty$, $\theta \in [0,1]$, $\beta \in [-\theta/2, \theta/2]$ *tels que* $(1-\frac{2}{q})(1+\beta) = \frac{2}{p}$ *avec* $(p,\beta) \neq (2,0)$, *et* $s' - s \geq \alpha(q)$, *avec* $\alpha(q) = (1+\beta)(1-\frac{2}{q})$, *il existe une constante strictement positive* C *telle que*

$$\|e^{-\mathrm{i}tH} P_c(H)\psi\|_{L_t^p(\mathbb{R},B^s_{q,r}(\mathbb{R}^3,\mathbb{C}^4))} \leq C\|\psi\|_{H^{s'}(\mathbb{R}^3,\mathbb{C}^4)}, \tag{i}$$

$$\left\| \int_{s<t} e^{-\mathrm{i}(t-s)H} P_c(H) F(s)\, ds \right\|_{L^p(\mathbb{R},B^{-s}_{q,r}(\mathbb{R}^3,\mathbb{C}^4))} \leq C\|F\|_{L^{\tilde{p}'}(\mathbb{R},B^{\tilde{s}}_{\tilde{q}',r}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{iii}$$

pour tout $r \in [1,\infty]$ *et* $(\tilde{q},\tilde{p})$ *choisi comme* (q,p) *et* $s + \tilde{s} \geq \alpha(q) + \alpha(\tilde{q})$.

Avant d'énoncer nos principaux résultats, nous rappelons ici les idées du théorème de Kramers, voir [BH92, Par90]. L'opérateur D_m commute avec l'opérateur antilinéaire K défini en (3). Si V commute lui aussi avec K, les espaces propres sont alors nécessairement de dimensions paires. Nous faisons l'hypothèse suivante :

Hypothèse 1.3. *Le potentiel* V *commute à* K *et* H *n'a qu'une seule valeur propre* λ_0 *qui est double.*

Nous introduisons alors une base orthonormée de vecteurs propres donnée par $(\phi_0, K\phi_0)$. Pour construire notre variété, nous supposerons que le problème non linéaire est lui aussi invariant par l'action de K :

Hypothèse 1.4. *La fonction* $F : \mathbb{C}^4 \mapsto \mathbb{R}$ *de classe* $\mathcal{C}^\infty$ *vérifie* $F(z) = O(|z|^5)$ *as* $z \to 0$. *De plus, on a* $F(Kz) = F(z)$ *et* $F(e^{i\theta}z) = F(z), \forall z \in \mathbb{C}^4$, $\forall\theta \in \mathbb{R}$.

On obtient alors

Proposition B.1. *Supposons que les hypothèses 1.1 et 1.4 soient vérifiées. Pour tout* $\sigma \in \mathbb{R}^+$, *il existe* Ω *un voisinage de* $0 \in \mathbb{C}^2$, *une fonction de classe* $\mathcal{C}^\infty$

$$h : \Omega \mapsto \mathcal{H}_c \cap H^2(\mathbb{R}^3,\mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3,\mathbb{C}^4),$$

et une fonction $E \in \mathcal{C}^\infty(\Omega,\mathbb{R})$ *tels que* $S((u_1,u_2)) = u_1\phi_0 + u_2K\phi_0 + h((u_1,u_2))$ *vérifie, pour tout* $U = (u_1,u_2) \in \Omega$,

$$HS(U) + \nabla F(S(U)) = E(U)S(U),$$

avec $h((u_1,u_2)) = \left(\frac{u_1}{|(u_1,u_2)|} Id_{\mathbb{C}^4} + \frac{u_2}{|(u_1,u_2)|} K\right) h\left((|(u_1,u_2)|, 0)\right)$, $h(U) = O(|U|^2)$, $E(U) = E(|U|)$ *et* $E(U) = \lambda_0 + O(|U|^2)$.

De plus, pour tout $\alpha \in \mathbb{N}^4$, $s \in \mathbb{R}^+$ *et* $p,q \in [1,\infty]$, *il existe* $\gamma > 0$, $\varepsilon > 0$ *et* $C > 0$ *tels que pour tout* $U \in B_{\mathbb{C}^2}(0,\varepsilon)$, *on a* $\|e^{\gamma\langle Q\rangle}\partial_U^\alpha S(U)\|_{B^s_{p,q}} \leq C\|S(U)\|_2$.

On introduit alors i$H(U)$ l'opérateur linéarisé au voisinage d'un état stationnaire $S(U)$ avec $H(U) = \left\{H + d^2F(S(U)) - E(U)\right\}$, où d^2F est la différentielle of ∇F. Cet opérateur a un noyau géométrique de dimension réelle 4. Le reste du spectre est le spectre essentiel et on note $\mathcal{H}_c(U)$ l'espace associé qui est aussi l'orthogonal du noyau pour le produit $(f,g) \mapsto \Re\,(f, \mathrm{i}g)_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$.

On introduit également l'ensemble :

$$S = \{(U,z)\,;\; U \in \Omega,\; z \in \mathcal{H}_c(U)\}\,,$$

que l'on muni de la norme de $\mathbb{C}^2 \times H^{s'}$. Nous pouvons donc maintenant présenter notre

Théorème 19 (Stabilisation et diffusion non linéaire)**.** *Supposons que les hypothèses 1.1 et 1.4 soient vérifiées. Soient $s' > \beta + 2 > 2$ et $\sigma > 3/2$, il existe un voisinage $\mathcal{V}_0$ de $(0,0)$ dans S et $C > 0$ tels que pour tout condition initiale $\psi_0 = S(U_0) + \xi_0$ avec $(U_0, \xi_0) \in S$, on a :*

(i) il existe une unique solution globale ψ et cette solution est dans $\mathcal{C}\left(\mathbb{R}, H^{s'}\right) \cap \mathcal{C}^1\left(\mathbb{R}, H^{s'-1}\right)$;

(ii) il existe des bijections de classe $\mathcal{C}^\infty$

$$(U_{\pm\infty}; \xi_\pm) : \mathcal{V}_0 \mapsto \mathcal{V}^\pm,$$

où $\mathcal{V}^\pm$ sont des voisinages ouverts de $(0,0)$ dans S telles que

$$|U_{\pm\infty} - U_0| \le C\|\xi_0\|^2_{H^{s'}},\; \|\xi_\pm - \xi_0\|_{H^{s'}} \le C\,|U_0|\,\|\xi_0\|_{H^{s'}},$$

telles que pour tout $t \in \mathbb{R}$, $\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U(t)) + e^{JtE(U_{\pm\infty})} e^{JtH(U_{\pm\infty})}\xi_\pm + \varepsilon_\pm(t)$ avec $\dot U \in L^p(R^\pm)$ pour tout $p \in [1,\infty]$, $\lim\limits_{t\to\pm\infty} U(t) = U_{\pm\infty}$,

$$\max\left\{\|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^{s'})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^{s'}_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})}\right\} \le C\,|U_0|\,\|\xi_0\|_{H^{s'}}$$

et

$$\lim_{t\to\pm\infty} \|\varepsilon_\pm(t)\|_{H^{s'}} = 0.$$

Nous pouvons aussi obtenir le développement

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U_\pm) + e^{-\mathrm{i}tD_m}\xi_\pm + \varepsilon_\pm(t)$$

avec les mêmes conclusions mais dans ce cas nous avons seulement $\|\xi_\pm - \xi_0\|_{H^{s'}} \le C\|\xi_0\|_{H^{s'}}$ et

$$\max\left\{\|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^{s'})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^{s'}_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})}\right\} \le C\|\xi_0\|_{H^{s'}}$$

ou encore

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U_\pm) + e^{-\mathrm{i}tH}\xi_\pm + \varepsilon_\pm(t)$$

avec $\xi_\pm \in \mathcal{H}_c$ et les mêmes conclusions mais seulement $\|\xi_\pm - \xi_0\|_{H^{s'}} \le C\|\xi_0\|_{H^{s'}}$ et

$$\max\left\{\|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^{s'})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^{s'}_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})}\right\} \le C\|\xi_0\|_{H^{s'}}.$$

2 English version

We study the stability of small stationary solutions of the following nonlinear Dirac equation:

$$\mathrm{i}\partial_t\psi = (D_m + V)\psi + \nabla F(\psi). \tag{1}$$

Here, D_m is the usual Dirac operator acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ defined with help of Pauli matrices, see [Tha92]. In (1), V is an external potential field and F is a nonlinearity with some natural invariance properties which will be specified in the sequel. Stationary solutions of (1) take the form $\psi(t,x) = e^{-\mathrm{i}Et}\phi(x)$ where ϕ satisfies

$$E\phi = (D_m + V)\phi + \nabla F(\phi). \tag{2}$$

We state our results about the existence of a manifold of small stationary solutions of (1). Then, we present our theorem about stabilization towards this manifold. We shall need linear decay estimates associated with the continuous spectral subspace of $H := D_m + V$. Proofs of these results can be adapted from [Bou05] with small modifications. Our result is the analogue in the Dirac case of the one of [GNT04][Theorem 1.7, Theorem 1.8].

We first state our results concerning the time decay of $e^{-\mathrm{i}t(D_m+V)}$ in weighted L^2 spaces and Besov spaces. This decays will be expressed in term of Lebegue spaces with respect to the time. This kind of estimates, respectively the H-smoothness of $\langle Q\rangle^{-1}$ and Strichartz estimates, are useful in the study of our nonlinear time-dependent Dirac equation. To state our results, we need the

Definition B.2 (Weighted Sobolev space). *The weighted Sobolev space is defined by*

$$H^t_\sigma(\mathbb{R}^3, \mathbb{C}^4) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3),\ \|\langle Q\rangle^\sigma \langle P\rangle^t f\|_2 < \infty \right\}$$

for $\sigma, t \in \mathbb{R}$. It is endowed with the natural norm $\|f\|_{H^t_\sigma} = \|\langle Q\rangle^\sigma\langle P\rangle^t f\|_2$. If $t = 0$, we write L^2_σ instead of $\mathcal{H}^0_\sigma$.

We write $\langle u\rangle$ for $\sqrt{1+u^2}$, P for $-i\nabla$ and Q for the operator of multiplication by x in $\mathbb{R}^3$.

We work within the

Assumption 2.1. *The potential $V : \mathbb{R}^3 \mapsto S_4(\mathbb{C})$ (self-adjoint 4×4 matrices) is a $\mathcal{C}^\infty$ function such that there exists $\rho > 5$ with $\forall\alpha \in \mathbb{N}^3$, $\exists C > 0$, $\forall x \in \mathbb{R}^3$, $|\partial^\alpha V|(x) \le C\langle x\rangle^{-\rho-|\alpha|}$.*

By the Kato-Rellich theorem $H := D_m + V$ is essentially self-adjoint on $\mathcal{C}^\infty_0(\mathbb{R}^3, \mathbb{C}^4)$ and self-adjoint on $H^1(\mathbb{R}^3, \mathbb{C}^4)$. In this case, the thresholds $\pm m$ are the only points of the continuous spectrum which can be associated with wave of zero velocity. These phenomenons can perturb the propagation and the dispersion. Hence we also work within the

Assumption 2.2. *The operator H presents no resonance at thresholds and no eigenvalue at thresholds.*

A resonance is an eigenvector in $H^{1/2}_{-\sigma}(\mathbb{R}^3, \mathbb{C}^4)\backslash H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ for some $\sigma \in (1/2, \rho-1/2)$ here. Let $\mathbf{P}_c(H) = \mathbb{1}_{(-\infty,-m|\cup[+m,+\infty)}(H)$ be the projector associated with the continuous spectrum of H and $\mathcal{H}_c$ its range. Using [Bou05, Theorem 1.1], we obtain a Limiting Absorption Principle which gives the following theorem

Theorem B.3 (H-smoothness of $\langle Q\rangle^{-1}$). *Suppose that Assumptions 2.1– 2.2. Then one has*

$$\int_{\mathbb{R}} \| \langle Q\rangle^{-1} e^{-\mathrm{i}tH} \mathbf{P}_c(H)\psi\|^2_{H^s}\, dt \le C\,\|\psi\|^2_{H^s}, \tag{i}$$

$$\left\| \int_{s<t} \langle Q\rangle^{-1} e^{-\mathrm{i}(t-s)H} \mathbf{P}_c(H) \langle Q\rangle^{-1} F(s)\, ds \right\|_{L^2(\mathbb{R},H^s)} \le C\,\|F\|_{L^2(\mathbb{R},H^s)}, \tag{iii}$$

Then using [Bou05, Theorem 1.2], [KT98, Theorem 10.1] and interpolation in Besov spaces

Theorem B.4 (Strichartz-type estimates). *Suppose that assumption 2.1 and 2.2 hold. Then for any* $2 \le p,q \le \infty$, $\theta \in [0,1]$, $\beta \in [-\theta/2, \theta/2]$ *such that* $(1-\frac{2}{q})(1+\beta) = \frac{2}{p}$ *with* $(p,\beta) \neq (2,0)$ *and* $s'-s \ge \alpha(q)$, *where* $\alpha(q) = (1+\beta)(1-\frac{2}{q})$, *there exists a positive constant* C *such that*

$$\|e^{-\mathrm{i}tH} P_c(H)\psi\|_{L^p_t(\mathbb{R},B^s_{q,r}(\mathbb{R}^3,\mathbb{C}^4))} \le C\|\psi\|_{H^{s'}(\mathbb{R}^3,\mathbb{C}^4)}, \tag{i}$$

$$\|\int_{s<t} e^{-\mathrm{i}(t-s)H} P_c(H)F(s)\,ds\|_{L^p(\mathbb{R},B^{-s}_{q,r}(\mathbb{R}^3,\mathbb{C}^4))} \le C\|F\|_{L^{\tilde{p}'}(\mathbb{R},B^{\tilde{s}}_{\tilde{q}',r}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{iii}$$

for any $r \in [1,\infty]$ *and* $(\tilde{q},\tilde{p})$ *chosen like* (q,p) *and* $s+\tilde{s} \ge \alpha(q)+\alpha(\tilde{q})$.

We now state our main results. We first mention the content of the theorem of Kramers, see e.g. [BH92, Theorem 6.3.] or [Par90]. The operator D_m commutes with K an antilinear operator defined by:

$$K\begin{pmatrix}\psi\\ \chi\end{pmatrix} = \begin{pmatrix}\sigma_2\overline{\psi}\\ \sigma_2\overline{\chi}\end{pmatrix} \text{ with } \sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \imath & 0\end{pmatrix}. \tag{3}$$

If V also commutes with K the eigenspaces of H are of even dimension. The class of potentials commuting to K contains the class of electric and scalar potentials. Hence the operator K appears naturally in some important physical systems. We work with the following

Assumption 2.3. *The potential* V *commutes with* K *and* H *has only one double eigenvalue* λ_0.

We hence introduce an orthonormal basis of eigenvectors given by $(\phi_0, K\phi_0)$. Notice that the following results are still true, with straightforward adaptations, if we suppose λ_0 is simple. We also mention that results of [Bou05] are true with obvious modifications if the assumptions of the present notes hold with two double eigenvalues.

To build our manifold, we suppose that the nonlinear problem is also invariant with respect to K:

Assumption 2.4. *The function* $F : \mathbb{C}^4 \mapsto \mathbb{R}$ *is in* $\mathcal{C}^\infty(\mathbb{R}^8,\mathbb{R})$ *satisfies* $F(z) = O(|z|^5)$ *as* $z \to 0$. *Moreover, we have* $F(Kz) = F(z)$ *and* $F(e^{i\theta}z) = F(z), \forall z \in \mathbb{C}^4$, $\forall \theta \in \mathbb{R}$.

We therefore have the

Proposition B.5. *Suppose that Assumptions 2.1– 1.4. For any* $\sigma \in \mathbb{R}^+$, *there exist a neighborhood* Ω *of* $0 \in \mathbb{C}^2$, *a* $\mathcal{C}^\infty$ *map*

$$h : \Omega \mapsto \mathcal{H}_c \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4)$$

and a $\mathcal{C}^\infty$ *map* $E : \Omega \mapsto \mathbb{R}$ *such that* $S((u_1, u_2)) = u_1\phi_0 + u_2 K\phi_0 + h((u_1, u_2))$ *satisfies for all* $U = (u_1, u_2) \in \Omega$,

$$HS(U) + \nabla F(S(U)) = E(U)S(U),$$

$h((u_1, u_2)) = \left(\frac{u_1}{|(u_1,u_2)|} Id_{\mathbb{C}^4} + \frac{u_2}{|(u_1,u_2)|} K\right) h\left((|(u_1, u_2)|, 0)\right)$, $h(U) = O(|U|^2)$ *and* $E(U) = E(|U|)$ *and* $E(U) = \lambda_0 + O(|U|^2)$.

Moreover, for any $\alpha \in \mathbb{N}^4$, $s \in \mathbb{R}^+$ *and* $p, q \in [1, \infty]$ *there exist* $\gamma > 0$, $\varepsilon > 0$ *and* $C > 0$ *such that for all* $U \in B_{\mathbb{C}^2}(0, \varepsilon)$ *one has* $\|e^{\gamma\langle Q\rangle} \partial^\alpha_U S(U)\|_{B^s_{p,q}} \leq C\|S(U)\|_2$.

The proof of this proposition is an adaptation of the one of [PW97, Proposition 2.2].

We introduce $\mathrm{i}H(U)$ the linearized operator around a stationary state $S(U)$ with $H(U) = \left\{H + d^2F(S(U)) - E(U)\right\}$, where d^2F is the differential of ∇F. This operator has a geometric kernel of dimension 4. The rest of the spectrum is the essential spectrum, we write $\mathcal{H}_c(U)$, for the associated space which is the orthogonal of the kernel for the product $(f, g) \mapsto \Re\,(f, \mathrm{i}g)_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$.

We introduce the set:

$$S = \{(U, z)\,;\; U \in \Omega,\; z \in \mathcal{H}_c(U)\}\,,$$

endowed with the norm of $\mathbb{C}^2 \times H^{s'}$. Now we can state our

Theorem B.6 (Stabilization and nonlinear scattering)**.** *Suppose that Assumptions 2.1– 1.4. Let* $s' > \beta + 2 > 2$ *and* $\sigma > 3/2$, *there exist* $\mathcal{V}_0$ *a neighborhood of* $(0, 0)$ *in* S *and* $C > 0$ *such that for any initial condition of the form* $\psi_0 = S(U_0) + \xi_0$ *with* $(U_0, \xi_0) \in S$, *one has*

- *(i) there exists a unique global solution* ψ *and this solution is in* $\mathcal{C}\left(\mathbb{R}, H^{s'}\right) \cap \mathcal{C}^1\left(\mathbb{R}, H^{s'-1}\right)$;
- *(ii) there exist* $\mathcal{C}^\infty$ *bijective maps* $(U_{\pm\infty}; \xi_\pm) : \mathcal{V}_0 \mapsto \mathcal{V}^\pm$, *where* $\mathcal{V}^\pm$ *are open neighborhoods of* $(0, 0)$ *in* S *with*

$$|U_{\pm\infty} - U_0| \leq C\|\xi_0\|^2_{H^{s'}},\ \|\xi_\pm - \xi_0\|_{H^{s'}} \leq C\,|U_0|\,\|\xi_0\|_{H^{s'}},$$

such that for all $t \in \mathbb{R}$ $\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U(t)) + e^{JtE(U_{\pm\infty})} e^{JtH(U_{\pm\infty})} \xi_\pm + \varepsilon_\pm(t)$ *with* $\dot{U} \in L^p(\mathbb{R}^\pm)$ *for all* $p \in [1, \infty]$, $\lim_{t\to\pm\infty} U(t) = U_\pm$ *and*

$$\max\left\{\|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^{s'})}\,,\ \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^{s'}_{-\sigma})}\,,\ \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})}\right\} \leq C\,|U_0|\,\|\xi_0\|_{H^{s'}}$$

and

$$\lim_{t\to\pm\infty} \|\varepsilon_\pm\|_{H^{s'}} = 0.$$

We also obtain the expansion

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U_\pm) + e^{-\mathrm{i}tD_m}\xi_\pm + \varepsilon_\pm(t)$$

with the same conclusion but we will only have $\|\xi_\pm - \xi_0\|_{H^{s'}} \leq C\|\xi_0\|_{H^{s'}}$ and

$$\max\left\{\|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm,H^{s'})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,H^{s'}_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,B^\beta_{\infty,2})}\right\} \leq C\|\xi_0\|_{H^{s'}}$$

in this case or

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U_\pm) + e^{-\mathrm{i}tH}\xi_\pm + \varepsilon_\pm(t)$$

with $\xi_\pm \in \mathcal{H}_c$ and the same conclusion but $\|\xi_\pm - \xi_0\|_{H^{s'}} \leq C\|\xi_0\|_{H^{s'}}$ and

$$\max\left\{\|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm,H^{s'})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,H^{s'}_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm,B^\beta_{\infty,2})}\right\} \leq C\|\xi_0\|_{H^{s'}}.$$

Sketch of the proof of Theorem B.6. We decompose a function ϕ with respect to the spectrum of $H(U) = \{H + d^2F(S(U)) - E(U)\}$, where d^2F is the differential of ∇F with respect to $\mathbb{R}^4$. Then we solve the resulting equations instead of the equation (1). We introduce the following space for $U = (u_1, u_2)$

$$\mathcal{H}_c((u_1,u_2)) = \Bigg\{\eta \in L^2(\mathbb{R}^3,\mathbb{C}^4), \Re\left(\mathrm{i}\eta, \frac{\partial}{\partial \Re u_i} S(u_1,u_2)\right)_{L^2(\mathbb{R}^3,\,\mathbb{C}^4)} = 0,$$
$$\Re\left(\mathrm{i}\eta, \frac{\partial}{\partial \Im u_i} S(u_1,u_2)\right)_{L^2(\mathbb{R}^3,\,\mathbb{C}^4)} 0,\ i \in \{1,2\}\Bigg\}.$$

which is stable under the action of $\mathrm{i}H(u)$. For any function over an interval of time I containing 0, we write for $t \in I$, $\phi(t) = exp\left(-\mathrm{i}\int_0^t E(u(s))\,ds\right)(S(U(t)) + \eta(t))$. If ϕ is small in L^2, as in [GNT04, Lemma 2.3], we can prove that there exists a unique couple (U,η) with $U(t) \in B_{\mathbb{C}^2}(0,\varepsilon)$ such that $\eta(t) \in \mathcal{H}_c(U(t))$. So we solve the equation $\mathrm{i}\partial_t\eta = H(U)\eta + N(U,\eta) - \mathrm{i}dS(U)\dot{U}$ for $\eta \in \mathcal{H}_c(U(t))$ with $N(U,\eta) = \nabla F(S(U)+\eta) - \nabla F(S(U)) - d^2F(S(U))\eta$, where dS is the differential of S in $\mathbb{R}^4$. We have the orthogonality condition $\langle \eta(t), \mathrm{i}dS(U(t))\rangle = 0$ and a time derivation gives

$$\partial_t U(t) = -A(U(t),\eta(t))\langle N(U(t),\eta(t)), dS(U(t))\rangle \tag{4}$$

with

$$A(U,\eta) = [\langle \mathrm{i}dS(U), dS(U)\rangle + \langle \mathrm{i}\eta, d^2S(U)\rangle]^{-1}$$

since $[\langle \mathrm{i}dS(U(t)), dS(U(t))\rangle + \langle \mathrm{i}\eta(t), d^2S(U(t))\rangle]$ is invertible for small $|U(t)|$ and $\|\eta(t)\|_2$. This gives

$$\partial_t\eta = \mathrm{i}H(U)\eta + \mathrm{i}N(U,\eta) - dS(U)A(U,\eta)\langle N(U,\eta)), dS(U)\rangle = \mathrm{i}H(U)\eta + \mathcal{G}(U,\eta). \tag{5}$$

We build solutions of the system (4)–(5) with a fixed point theorem. For any $U \in B_{\mathbb{C}^2}(0,\varepsilon)$, we define $\mathcal{Z}(U,\delta)$ a subset of $\mathcal{C}^\infty(\mathbb{R}, L^2(\mathbb{R}^3,\mathbb{R}^8))$ by

$$\left\{\eta,\ \eta(t) \in \mathcal{H}_c(U(t)),\ \max\left[\{\|\eta\|_{L^\infty(\mathbb{R},H^{s'})}, \|\eta\|_{L^2(\mathbb{R},B^\beta_{\infty,2})}, \|\eta\|_{L^2(\mathbb{R},H^s_{-\sigma})}\right] < \delta\right\}.$$

For any $\eta_0 \in \mathcal{H}_c(U(0))$, let $(s,t) \in \mathbb{R}^2 \mapsto P(t,s)$ be the propagator associated with the time dependent operator $\mathrm{i}\,(H + D\nabla F(S(U)))$, we study

$$\mathcal{T}_{U,\eta_0}(\eta) = P(t,0)\eta_0 + \int_0^t P(t,v)\mathcal{G}(U(v),\eta(v))\,dv.$$

We have that for sufficiently small ε and δ the application $\mathcal{T}_{U,z_0}$ maps $\mathcal{Z}(U,\delta)$ into itself if $\|\eta_0\|_{H^{s'}}$ is small enough. Moreover, there exists $\kappa \in (0,1)$ such that

$$\left\|\mathcal{T}_{U',\eta_0'}(\eta') - \mathcal{T}_{U,\eta_0}(\eta)\right\|_{L^\infty(\mathbb{R}^+,H^{s'})} \leq \|\eta_0 - \eta_0'\|_{H^{s'}} + \kappa \left\{\|U - U'\|_{L^\infty(\mathbb{R})} + \|\eta - \eta'\|_{L^\infty(\mathbb{R},H^{s'})}\right\}.$$

Hence there exists a solution $\eta \in \mathcal{Z}(U,\delta)$ with $\eta(0) = \eta_0$ of the equation (5). We also prove that the following limit $\eta_\infty = \lim_{t\to\infty} e^{\mathrm{i}tH}\eta(t)$ exists in $\mathcal{H}_c \cap H^{s'}$.

We notice that η is a function of U and η_0. Let us introduce for $U_0 \in B_{\mathbb{C}^2}(0,\varepsilon)$ the function on $B_{\mathbb{C}^2}(0,\varepsilon)$:

$$f(U)(t) = U_0 - \int_0^t A(U(v),\eta[U,\eta_0](v))\langle N(U(v),\eta[U,\eta_0](v)), dS(U(v))\rangle\, dv.$$

For sufficiently small ε, δ, U_0 and η_0, f maps $B_{\mathbb{C}^2}(0,\varepsilon)$ into itself and is a contraction. Hence the equation (4) has a unique solution in $B_{\mathbb{C}^2}(0,\varepsilon)$ with $U(0) = U_0$. □

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Étude de la stabilité des petites solutions stationnaires pour une classe d'équations de Dirac non linéaires

Résumé : Cette thèse est consacrée à l'étude de la stabilité de petits états stationnaires d'une équation d'évolution non linéaire issue de la mécanique quantique relativiste : l'équation de Dirac non linéaire.

Tout le long de notre étude, les équations non linéaires sont vues comme des petites perturbations non linéaires de systèmes linéaires. Une partie de cette thèse est donc consacrée à l'étude de problèmes linéaires. Nous montrons que, pour un opérateur de Dirac n'ayant pas de résonance aux seuils ni de valeur propre aux seuils, le propagateur vérifie des estimations de propagation et de dispersion. Nous en déduisons également des estimations de régularité au sens de Kato et des estimations de Strichartz.

En faisant des hypothèses *ad hoc* sur le spectre discret d'un opérateur de Dirac, nous construisons des petites variétés formées d'états stationnaires. Puis en faisant varier ces hypothèses, nous faisons apparaître des phénomènes de stabilisation et d'instabilité orbitale pour certains de ces états.

Mots clés : équations aux dérivées partielles, opérateur de Dirac, estimations de propagation, estimations de dispersion, estimations de régularité, estimations de Strichartz, équation de Dirac non linéaire, états stationnaires, stabilité, stabilité orbitale, stabilité asymptotique, directions stables.

A study of the stability of small stationary solutions for a class of nonlinear Dirac equations

Abstract: This thesis is devoted to the study of the stability of small stationary solutions of a nonlinear time dependent equation coming from relativistic quantum mechanic: the nonlinear Dirac equation.

In this study, non linear equations are viewed as small nonlinear perturbations of linear system. A part of this thesis is hence devoted to the study of linear problems. We prove that for a Dirac operator, with no resonance at thresholds nor eigenvalue at thresholds, the propagator satisfies propagation and dispersive estimates. We also deduce smoothness estimates in sense of Kato and Strichartz estimates.

With some *ad hoc* assumptions on the discret spectrum of a Dirac operator, we build small manifolds of stationary states. Then with small variations on the these assumptions, we can emphase some stabilization process and orbital instability phenomenons for some stationary states.

Keywords: Partial Differential Equations, Dirac Operator, Propagation estimates, Dispersive estimates, Smoothness estimates, Strichartz estimates, Nonlinear Dirac equation, Stationary states, Stabilty, Orbital stabilty, Asymptotic stability, Stable directions.

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