Existence of solutions and asymptotic limits of the Euler-Poisson and the quantum drift-diffusion systems. Applications to semiconductors and plasmas.

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ÉCOLE DOCTORALE DES SCIENCES FONDAMENTALES
N° : 503

THÈSE EN COTUTELLE

présentée pour obtenir le grade de

DOCTEUR D'UNIVERSITÉ
(Spécialité : MATHÉMATIQUES APPLIQUÉES)

par

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Diplomée d'Etudes Approfondies

EXISTENCE OF SOLUTIONS AND ASYMPTOTIC LIMITS OF THE EULER-POISSON AND THE QUANTUM DRIFT-DIFFUSION SYSTEMS. APPLICATIONS TO SEMICONDUCTORS AND PLASMAS.

Soutenue publiquement le 21 novembre 2006, devant la commission d'examen

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Remerciements

Tout d'abord, je tiens à remercier mes directeurs de thèse, Yue-Jun Peng et Ansgar Jüngel. Leur soutien, leurs connaissances, leur inspiration et leur enthousiasme m'ont été d'une aide très précieuse dans l'accomplissement de ce travail.

Je remercie Thierry Goudon et Ingenuin Gasser d'avoir bien voulu être les rapporteurs de ma thèse. Je remercie également Florian Méhats et Gerhard Huber d'avoir accepté de faire partie du jury.

Je remercie tous les membres pédagogiques et administratifs du Laboratoire de mathématiques de l'Université Blaise Pascal (Clermont-Ferrand) et de l'Université Johannes Gutenberg (Mainz). Il m'est malheureusement impossible de citer tous ceux, de ces deux Universités, qui des secrétaires aux enseignants-chercheurs, en passant par les ingénieurs systèmes m'ont facilité la tâche durant toutes ces années. Merci spécialement à Claire Chainais-Hillairet et Olivier Bodart, de l'Université Blaise Pascal, pour toutes les conversations très utiles que nous avons eues ainsi que pour avoir bien voulu relire mon manuscrit.

Merci à tous les doctorants en mathématiques de l'Université Blaise Pascal pour leur amitié : Vivien Clauzon, Malcolm Djenno, Mathieu Gourcy, Jérémy Ruiz, Rémy Sart,..., sans oublier ceux qui sont partis : Fabrice Blache, Liang Zhen Lei et Benoît Testud. Merci également à tous mes collègues et amis de l'Université Johannes Gutenberg : Bertram Düring, Stefan Holst, Daniel Matthes et Josipa Pina Milišić pour leur formidable accueil et toute l'aide qu'ils m'ont apportée durant mes séjours en Allemagne.

Je n'oublie pas de dire un grand merci à tous mes amis : Mickaël Ansart, Nicolas Beck, Carole Blanchon, Jérôme Coudert, François Grand, Gilla Julien, Nathalie Mathevon, Nicolas Reymond, Dominique Sahuc et tous les autres... pour tous les bons moments de détente passés à leurs cotés. Leur amitié m'a apporté ces dernières années une aide indirecte mais précieuse.

Enfin je ne peux finir sans adresser un merci très spécial à mon fiancé et à toute ma famille (et plus particulièrement mes parents, mon frère et mes grands-parents) pour leur amour, leur compréhension et leurs encouragements durant toutes ces années. Leur soutien inconditionnel m'a permis de toujours donner le meilleur de moi-même.

Abstract

My work concerns two different systems of equations used in the mathematical modeling of semiconductors and plasmas: the Euler-Poisson system and the quantum drift-diffusion system. The first is given by the Euler equations for the conservation of mass and momentum, with a Poisson equation for the electrostatic potential. The second one takes into account the physical effects due to the smallness of the devices (quantum effects). It is a simple extension of the classical drift-diffusion model which consists of two continuity equations for the charge densities, with a Poisson equation for the electrostatic potential.

Using an asymptotic expansion method, we study (in the steady-state case for a potential flow) the limit to zero of the three physical parameters which arise in the Euler-Poisson system: the electron mass, the relaxation time and the Debye length. For each limit, we prove the existence and uniqueness of profiles to the asymptotic expansion and some error estimates. For a vanishing electron mass or a vanishing relaxation time, this method gives us a new approach in the convergence of the Euler-Poisson system to the incompressible Euler equations. For a vanishing Debye length (also called quasineutral limit), we obtain a new approach in the existence of solutions when boundary layers can appear (i.e. when no compatibility condition is assumed). Moreover, using an iterative method, and a finite volume scheme or a penalized mixed finite volume scheme, we numerically show the smallness condition on the electron mass needed in the existence of solutions to the system, condition which has already been shown in the literature.

In the quantum drift-diffusion model for the transient bipolar case in one-space dimension, we show, by using a time discretization and energy estimates, the existence of solutions (for a general doping profile). We also prove rigorously the quasineutral limit (for a vanishing doping profile). Finally, using a new time discretization and an algorithmic construction of entropies, we prove some regularity properties for the solutions of the equation obtained in the quasineutral limit (for a vanishing pressure). This new regularity permits us to prove the positivity of solutions to this equation for at least times large enough.

Résumé

Mes travaux concernent deux systèmes d'équations différents utilisés dans la modélisation mathématique des semi-conducteurs et des plasmas : le système d'Euler-Poisson, et, le système de dérive-diffusion quantique. Le premier est constitué des équations d'Euler pour la conservation de la masse et de la quantité de mouvement et de l'équation de Poisson pour le potentiel électrostatique. Le second prend en compte les effets physiques dus à la petitesse des appareils (effets quantiques). Il s'agit d'une extension simple du modèle de dérive-diffusion classique qui est constitué de deux équations de continuité pour les densités de charge et de l'équation de Poisson pour le potentiel électrostatique.

En utilisant une technique de développement asymptotique, nous étudions les limites en zéro, dans le cas stationnaire pour un flot potentiel, des trois paramètres physiques intervenants dans le système d'Euler-Poisson : la masse d'électron, le temps de relaxation et la longueur de Debye. Pour chacune de ces limites, nous démontrons l'existence et l'unicité des profils ainsi que des estimations d'erreur. Pour les limites de masse d'électron et du temps de relaxation, cette méthode nous donne une nouvelle approche pour la convergence du système d'Euler-Poisson vers les équations d'Euler incompressibles. Pour la limite de la longueur de Debye, aussi appelée limite de quasi-neutralité, nous obtenons ainsi une nouvelle approche dans l'existence de solution du système lorsque des couches limites peuvent apparaître. De plus, en utilisant une méthode itérative, et des schémas volumes finis classiques et volumes finis mixtes pénalisés, nous pouvons faire apparaître numériquement la condition de petitesse sur la masse d'électron nécessaire à l'existence de solution de ce sytème, et qui a déjà été démontrée dans la littérature.

En utilisant une semi-discrétisation en temps et des estimations d'énergie, nous démontrons l'existence de solution (pour un profil de dopage général), ainsi que la limite de quasi-neutralité (pour un profil de dopage nul), dans le modèle évolutif de dérive-diffusion quantique pour le cas bipolaire uni-dimensionnel. Enfin, en utilisant une semi-discrétisation différente et une méthode algorithmique de construction d'entropie, nous montrons des propriétés de régularité des solutions de l'équation obtenue par la limite de quasi-neutralité (dans le cas de pressions nulles). Cette nouvelle régularité nous permet, de plus, de démontrer la stricte positivité des solutions de cette équation au moins pour des temps assez grands.

Zusammenfassung

Meine Arbeit behandelt zwei unterschiedliche Systeme von Gleichungen, die in der mathematischen Modellierung von Halbleitern und Plasmen verwendet werden. Dies sind das Euler-Poisson-Modell und das Quantum-Drift-Diffusionsmodell. Das erste ist durch die Euler-Gleichungen für die Erhaltung der Masse und des Impulses, sowie die Poisson-Gleichung für das elektrostatische Potential gegeben. Das zweite Modell besteht ebenso aus der Poisson-Gleichung und eine Erweiterung der Driftdiffusionsgleichung zur Berücksichtigung von in diesen Bauteildimensionen auftretenden Quanteneffekten.

Für den stationären Fall eines Potentialflusses werden verschiedene Grenzwerte des Euler-Poisson-Modells mittels asymptotischer Entwicklungen untersucht. Speziell werden die Grenzwerte verschwindender Elektronenmasse, Relaxationszeit und verschindender Debyelänge analysiert. Für jeden dieser Grenzwerte werden Existenz und Eindeutigkeit der Lösung der asymptotischen Entwicklungen sowie entsprechende A-Priori-Abschätzungen bewiesen. Die Grenzwerte verschwindender Elektronenmasse sowie verschwindender Relaxationszeit stellen neue Ergebnisse da, die den Übergung vom Euler-Poisson-System zu den inkompressiblen Eulergleichungen beschreiben. Im Falle der asymptotischen Entwicklung für kleine Debye-Längen ermöglicht der hier verwendete Ansatz einen Beweis der Existenz, selbst wenn Randgrenzschichten auftreten. Das Euler-Poisson-System wird mit einem iterativen Verfahren mit Finiten-Volumen- und gemischten Finiten-Volumen-Methoden diskretisiert. Damit konnte die in der Analysis notwendige Bedingung für die Existenz der Lösung - eine Beschränkung auf kleine Elektronenmassen - numerisch verifiziert werden.

Für das bipolare Quantum-Drift-Diffusionsmodell in einer Raumdimension haben wir mit einer Zeitdiskretisierung und Energieabschätzungen die Existenz der Lösung für ein allgemeines Dotierungsprofil gezeigt sowie den quasineutralen Grenzwert für ein verschwindendes Dotierungsprofil analysiert. Abschließend wird eine Verbesserung des Regularitätsresultats für den quasineutralen Grenzwert (für verschwindenden Druck) bewiesen, welches durch eine veränderte Zeitdiskretisierung und neue algorithmisch konstruierte Entropien gewonnen wird. Das neue Regularitätresultat erlaubt den Beweis der Positivität der Lösungen dieser Gleichung zumindest für hinreichend "grosse" Zeiten.

Contents

Part I

Introduction

In this work we are interested in mathematical analysis and numerical simulation of semiconductors and plasmas. The modern computer and telecommunication industry relies strongly on the use of semiconductors. Depending on the devices structure, the transport of particles can be very different, due to several physical phenomena, like drift, diffusion, scattering or quantum effects. Therefore, there are several mathematical models, which can be used in the modeling of such devices. The different models vary for complexity and mathematical properties and build a hierarchy, in which we can distinguish three classes: the kinetic models, the fluid-dynamical models and the quantum models.

In the first part of this work, we study a fluid-dynamical model called the Euler-Poisson system in which the unknowns are the charge densities, the charge velocities and the electrostatic potential. More specifically, this is a hydrodynamic model, since it consists of Euler equations of conservation laws (conservation of mass and momentum), plus a Poisson equation for the electrostatic potential. It can be considered in the bipolar as well as in the unipolar case. This means that we can take into account, for the charge transport, either the two species of charges (electrons and holes for a semiconductor, electrons and ions for a plasma), or only the electrons. For a plasma, the unipolar case means that the ion background density is fixed. Let us mention that the ballistic diodes are an example of semiconductors in which the charge transport is only due to the electrons. Such semiconductors consist of one weakly doped region n between two highly doped regions n^+ (see figure 1)

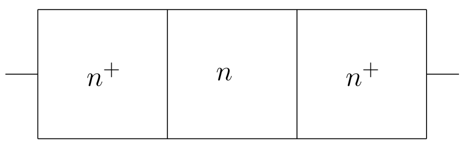

Figure 1: Ballistic diode

In the unipolar Euler-Poisson model three physical parameters appear: the electron mass, the relaxation time and the Debye length. Since they are small compared to the characteristic length of physical interest, it is important to study their limits to zero. It is the goal of Part II for a potential flow in the steady state case.

We refer to Chapter 1 for more details on the Euler-Poisson model and the references therein, and, to Part II (Chapters 3, 4, 5 and 6) for the results obtained on this model.

In the second part of this work, we study the quantum drift-diffusion model. Due to the ongoing miniaturization of semiconductor devices in the microelectronics industry, quantum effects play a more and more dominant role. An example of these very small devices are the tunneling diodes, which have a structure of one length of only few nanometers. It is then important to study models which also take into account these quantum phenomena.

The quantum drift-diffusion model is a simple extension of the classical drift-diffusion model, with only a quantum term of fourth-order in more, called the quantum Bohm potential. Recall that the drift-diffusion system consists of two continuity equations for the charge densities with a Poisson equation for the electrostatic potential.

We are particularly interested in the existence of solutions and the quasineutral limit (i.e. vanishing Debye length) in the bipolar quantum drift-diffusion equations in one space dimension.

Note that the equation obtained in the quasineutral limit corresponds, for vanishing pressure, to the so-called Derrida-Lebowitz-Speer-Spohn equation. This equation has recently attracted a lot of attention due to its remarkable mathematical properties. In particular, the existence of nonnegative solutions has already been shown. However, to our knowledge the positivity of solutions has not been proved yet. There exists only a partial result presented in [10] for small times.

Here we study the regularity of solutions to this limit equation and we obtain better results than in the literature. Moreover, we use these new regularity properties to prove the positivity of solutions at least for times large enough.

We refer to Chapter 2 for more details and the references therein on the quantum drift-diffusion model and to Part III (Chapters 7 and 8) for the results and their proofs.

Chapter 1

Euler-Poisson model

In this chapter, we present the Euler-Poisson system which is studied in Part II. Here we only consider the unipolar model for which we will study the asymptotic limits (see Chapters 4 and 5). These results could be extended to the bipolar case. But here the bipolar model is only treated in Chapter 6 from a numerical point of view and with the same kind of assumptions as for the unipolar model. Note that treating the bipolar system is not very different from treating the unipolar system.

1.1 General Presentation

Let Ω be an open and bounded domain of $\mathbb{R}^d$ ($d = 2$ or $d = 3$ in practice) which stands for the geometry of the semiconductor device or the domain occupied by the plasma. In the unipolar case, the Euler-Poisson system consists of Euler equations of hyperbolic conservation laws:

- mass conservation:

$$\partial_t n + \text{div}(nu) = 0, \tag{1.1}$$

- momentum conservation:

$$\varepsilon\partial_t(nu) + \varepsilon\text{div}(nu \otimes u) + \nabla p(n) = n\nabla\phi - \frac{\varepsilon nu}{\tau}, \tag{1.2}$$

and a linear Poisson equation:

$$-\lambda^2\Delta\phi = C - n. \tag{1.3}$$

Here, the unknowns of the system are the electron density $n = n(t, x)$, the electron velocity $u = u(t, x)$ and the electrostatic potential $\phi = \phi(t, x)$. The function

$C = C(x)$, which only depends on the space variable, represents the doping profile for a semiconductor and the fixed background ion density for a plasma. The quantity

$$J = nu,$$

is the electron current density. The function $p = p(s)$ stands for the pressure function. Indeed, since we only consider isentropic flow, the energy equation of the hydrodynamic model is replaced by a pressure-density relation $p = p(n)$. In all the following, the pressure function is supposed to be sufficiently smooth and strictly increasing for $n > 0$. In practice, the pressure function is typically governed by the $\gamma-$law $p(s) = cs^\gamma$, where $c > 0$ and $\gamma \geq 1$ are constants. The case $\gamma = 1$ corresponds to the isothermal flow, since in this case the temperature is constant. The physical parameters are the scaled electron mass ε, relaxation time τ and Debye length λ. They are small compared to the characteristic length of physical interest (e.g. the length of the device). Therefore it is important to study the asymptotic limits of the system when ε or τ or λ tends to zero independently.

For (n, u, ϕ) smooth enough and $n > 0$, equation (1.2) can be rewritten as

$$\varepsilon\partial_t u + \varepsilon(u \cdot \nabla)u + \nabla h(n) = \nabla\phi - \frac{\varepsilon u}{\tau}, \tag{1.4}$$

where $h = h(s)$ is the enthalpy function of the system and is defined by

$$h'(s) = \frac{p'(s)}{s}, \quad h(1) = 0.$$

Indeed, for $n > 0$, let us divide equation (1.2) by n. We obtain

$$\frac{\varepsilon}{n}\partial_t(nu) + \frac{\varepsilon}{n}\mathrm{div}(nu \otimes u) + \frac{1}{n}\nabla p(n) = \nabla\phi - \frac{\varepsilon u}{\tau}. \tag{1.5}$$

Then, using (1.1),

$$\frac{\varepsilon}{n}\partial_t(nu) = -\frac{\varepsilon u}{n}\mathrm{div}(nu) + \varepsilon u_t. \tag{1.6}$$

Moreover, a straightforward calculation shows that

$$\frac{\varepsilon}{n}\mathrm{div}(nu \otimes u) = \frac{\varepsilon u}{n}\mathrm{div}(nu) + \varepsilon(u \cdot \nabla)u. \tag{1.7}$$

Combining (1.5)-(1.7) we obtain (1.4).

Here we are only interested in the steady-state Euler-Poisson system. In this case the system becomes

$$\mathrm{div}(nu) = 0, \tag{1.8}$$

$$\varepsilon(u \cdot \nabla)u + \nabla h(n) = \nabla\phi - \frac{\varepsilon u}{\tau}, \tag{1.9}$$

$$-\lambda^2 \Delta\phi = C - n, \quad \text{in } \Omega. \tag{1.10}$$

There exist, to my knowledge, no results on system (1.8)-(1.10) without the supplemented assumption of a potential flow. Hence, we assume that the flow in the device is irrotationnal (which corresponds to the case of a potential flow). Then rot $u = 0$ and there exists a velocity potential ψ such that $u = -\nabla\psi$, so that we can rewrite equation (1.9) in the following way

$$\frac{\varepsilon}{2}|\nabla\psi|^2 + h(n) = \phi + \varepsilon\psi. \tag{1.11}$$

Indeed, by using the property rot $u = 0$, we can show

$$(u \cdot \nabla)u = \frac{1}{2}\nabla(|u|^2) = \frac{1}{2}\nabla(|\nabla\psi|^2). \tag{1.12}$$

Then (1.9) gives by using (1.12)

$$\nabla\Big(\frac{\varepsilon}{2}|\nabla\psi|^2 + h(n)\Big) = \nabla\Big(\phi + \frac{\varepsilon\psi}{\tau}\Big). \tag{1.13}$$

Then, by integrating (1.13), we obtain (1.11).

Finally, the steady-state Euler-Poisson system for a potential flow reads:

$$-\text{div}(n\nabla\psi) = 0, \tag{1.14}$$

$$\frac{\varepsilon}{2}|\nabla\psi|^2 + h(n) = \phi + \frac{\varepsilon\psi}{\tau}, \tag{1.15}$$

$$-\lambda^2 \Delta\phi = C - n, \quad \text{in } \Omega. \tag{1.16}$$

The electron current density is now given by

$$J = -n\nabla\psi. \tag{1.17}$$

The existence of irrotational subsonic steady-state flows in the gas-dynamics case (i.e. (1.15) with $\tau = \infty$, $\phi \equiv 0$ and (1.14)) is well known (see e.g. [57]). In [36], the authors employ a different analytical approach, which makes an explicit use of the Poisson equation (1.16) and allows to incorporate boundary conditions appropriate for semiconductors.

1.2 Boundary conditions

Let us now discuss the boundary conditions associated to (1.14)-(1.16). The boundary $\Gamma = \partial\Omega$ is assumed to be split into N disjoint, closed and connected "contact" segments $\Gamma_1, ..., \Gamma_N$ and "insulating" segments, whose the union is denote by Γ_{ins}.

We set Neumann boundary conditions on the insulating segments and Dirichlet boundary conditions on the contact segments. More precisely, let us define the following boundary conditions:

$$n\Big|_{\Gamma_i} = n_D\Big|_{\Gamma_i}, \quad i = 1, ..., N, \tag{1.18}$$

$$(\nabla\psi \times \nu_{norm})\Big|_{\Gamma_i} = 0, \quad i = 1, ..., N, \tag{1.19}$$

$$\nabla n \cdot \nu_{norm}\Big|_{\Gamma_{ins}} = 0, \tag{1.20}$$

$$\nabla\psi \cdot \nu_{norm}\Big|_{\Gamma_{ins}} = 0, \tag{1.21}$$

where ν_{norm} stands for the outward unit normal to Γ. Note that (1.21) means that the normal component of the velocity vanishes along Γ_{ins}, which implies no current flow through Γ_{ins}, and (1.19) implies that ψ is constant on each segment Γ_i. Indeed,

$$\nabla\psi = (\nabla\psi \cdot \nu_{tan})\nu_{tan} + (\nabla\psi \cdot \nu_{norm})\nu_{norm},$$

where ν_{tan} stands for the unit tangential vector to Γ. Then,

$$\nabla\psi \times \nu_{norm} = (\nabla\psi \cdot \nu_{tan})\nu_{tan} \times \nu_{norm}.$$

Using (1.19), since $\nu_{tan} \times \nu_{norm} \neq 0$, we obtain

$$\nabla\psi \cdot \nu_{tan}\Big|_{\Gamma_i} = 0 \implies \frac{\partial\psi}{\partial\nu_{tan}}\Big|_{\Gamma_i} = 0 \implies \psi\Big|_{\Gamma_i} = \text{const.}$$

For a current driven device, the values of these constants are given by imposing the currents $I_i, \ i = 1, ..., N$ flowing out of the contacts ([36]):

$$I_i = -\int_{\Gamma_i} n\nabla\psi \cdot \nu_{norm} ds, \ i = 1, ..., N. \tag{1.22}$$

Due to the conservation equation (1.14), we have:

$$0 = \int_\Omega \text{div}(n\nabla\psi)dx = \int_\Gamma n\nabla\psi \cdot \nu_{norm} ds. \tag{1.23}$$

Since

$$\Gamma = \left(\bigcup_{i=1}^N \Gamma_i\right) \bigcup \Gamma_{ins},$$

using (1.20),

$$\int_\Gamma n\nabla\psi \cdot \nu_{norm} ds = \sum_{i=1}^N \int_{\Gamma_i} n\nabla\psi \cdot \nu_{norm} ds.$$

Then, the data $I_i, \ i = 1, ..., N$ must satisfy

$$\sum_{i=1}^{N} I_i = 0. \tag{1.24}$$

Let us now show that the boundary conditions (1.19) and (1.22) lead to Dirichlet data for the velocity potential ψ ([36]). To this end, we introduce the functions χ_i solutions to

$$\operatorname{div}(n\nabla\chi_i) = 0, \quad \text{in } \Omega, \tag{1.25}$$

$$\chi_i\big|_{\Gamma_j} = \delta_{ij}, \ j = 1, ..., N, \tag{1.26}$$

$$\nabla\chi_i \cdot \nu_{norm}\big|_{\Gamma_{ins}} = 0, \tag{1.27}$$

for $i = 1, ..., N$. Here, δ_{ij} is the Kronecker symbol. Let us define the influence matrix $D = (D_{ij})_{i,j=1,...,N}$ by

$$D_{ij} = \int_{\Gamma_j} n\nabla\chi_i \cdot \nu_{norm} ds.$$

Since (with as yet unknown constants ψ_i)

$$\psi = \sum_{j=1}^{N} \psi_j \chi_j, \tag{1.28}$$

then from (1.22)

$$-\sum_{i=1}^{N} D_{ji}\psi_j = I_i. \tag{1.29}$$

Let us show that D is a symmetric nonnegative matrix. Indeed, multiplying (1.25) by χ_j and integrating over Ω, we obtain

$$\begin{aligned} 0 = \int_\Omega \operatorname{div}(n\nabla\chi_i)\chi_j dx &= \int_\Gamma n\chi_j \nabla\chi_i \cdot \nu_{norm} ds - \int_\Omega n\nabla\chi_i \cdot \nabla\chi_j dx \\ &= \sum_{i=1}^{N} \int_{\Gamma_k} n\chi_j \nabla\chi_i \cdot \nu_{norm} ds - \int_\Omega n\nabla\chi_i \cdot \nabla\chi_j dx \\ &= \int_{\Gamma_j} n\nabla\chi_i \cdot \nu_{norm} ds - \int_\Omega n\nabla\chi_i \cdot \nabla\chi_j dx. \end{aligned}$$

Hence,

$$D_{ij} = \int_\Omega n\nabla\chi_i \cdot \nabla\chi_j dx. \tag{1.30}$$

Then the matrix D is symmetric. Moreover, let $(\xi_1, ..., \xi_N)^t \in \mathbb{R}^N$. Using (1.30):

$$\sum_{i,j=1}^{N} D_{ij}\xi_i\xi_j = \int_\Omega n\nabla\Big(\sum_{i=1}^{N}\xi_i\chi_i\Big)\cdot\nabla\Big(\sum_{j=1}^{N}\xi_j\chi_j\Big)dx = \int_\Omega n|\nabla\chi|^2 dx \geq 0,$$

where

$$\chi = \sum_{i=1}^{N}\xi_i\chi_i$$

satisfy, by using (1.25), (1.26) and (1.27):

$$\mathrm{div}(n\nabla\chi) = 0, \quad \text{in } \Omega, \tag{1.31}$$

$$\nabla\chi\cdot\nu_{norm}\Big|_{\Gamma_{ins}} = 0, \tag{1.32}$$

$$\chi\Big|_{\Gamma_j} = \xi_j. \tag{1.33}$$

Hence, D is a nonnegative matrix.

Furthermore, we can also show that D is a matrix of rank $N-1$. Indeed, if

$$D\xi = 0,$$

we have

$$0 = \xi^t D\xi = \sum_{i,j=1}^{N} D_{ij}\xi_i\xi_j = \int_\Omega n|\nabla\chi|^2 dx,$$

and then

$$\chi = \sum_{i=1}^{N}\xi_i\chi_i = \text{const}.$$

Since for all $1 \leq j \leq N$

$$\chi\Big|_{\Gamma_j} = \sum_{i=1}^{N}\xi_i\chi_i\Big|_{\Gamma_j} = \sum_{i=1}^{N}\xi_i\delta_{ij} = \xi_j,$$

we deduce that $\xi = (1, ..., 1)^t$. Then:

$$\mathrm{Ker}(D) \subset \mathrm{Vect}\{(1, ..., 1)^t\}.$$

By uniqueness of solutions to the problem (1.25)-(1.27), it is easy to see that:

$$\sum_{j=1}^{N}\chi_j = 1.$$

Then,

$$(D\xi)_i = \sum_{j=1}^{N} D_{ij} = \int_\Omega n\nabla\chi_i \cdot \nabla\Big(\sum_{j=1}^{N}\chi_j\Big)dx = 0, \quad \forall\, 1 \le i \le N.$$

This implies that $\xi = (1, ..., 1)^t \in \mathrm{Ker}(D)$, and then

$$\mathrm{Ker}(D) = \mathrm{Vect}\{(1, ..., 1)^t\},$$

and consequently the matrix D is of rank $N - 1$.

Thus, D is a symmetric nonnegative matrix of rank $N-1$. Using condition (1.24), system (1.29) can be solved if we give one of the values ψ_i (e.g. $\psi_1 = 0$). We then obtain:

$$\psi\Big|_{\Gamma_i} = \psi_i, \tag{1.34}$$

which is a Dirichlet boundary condition and in which ψ_i depends on I_j for $j = 1, ..., N$, and on χ_i and n by definition

Remark 1.1. *More information on the influence matrix and their use in vectorial decomposition can be found in [39].*

Remark 1.2. *By analogy to the drift-diffusion model for semiconductors (see [77, 91]), the velocity potential ψ can be seen as a quasi-Fermi level for electrons (the quasi-Fermi potential being defined as: $F = h(n) - \phi$ for electrons and the quasi-Fermi level as $F \equiv 0$). Indeed, if all the ψ_i are vanishing, then there is no current flow and the semiconductor is on thermal equilibrium. We obtain in this case the following equation for the electrostatic potential ϕ, using* (1.15)-(1.16)*:*

$$\phi = h(n), \quad \Delta\phi = n - C \tag{1.35}$$

which is in agreement with the equilibrium of the drift-diffusion model if $h(n) = K\ln n,\ K > 0$, i.e. if $p(n) = Kn$ (case $\gamma = 1$ for the $\gamma-$law).

For a voltage driven device, the values $\psi_i,\ i = 1, ..., N$, (applied potentials) are prescribed (with say $\psi_1 = 0$) and the outflow currents I_i can be computed a posteriorly from (1.24), (1.29).

1.3 Existence and uniqueness of solutions

Let us first give some definitions.

Definition 1.1. • *The electron flow in the device is called* **subsonic** *if*

$$|u| = |\nabla\psi| < \sqrt{p'(n)/\varepsilon}, \quad \text{in } \Omega, \tag{1.36}$$

or, equivalently,

$$\varepsilon|J|^2 < n^2 p'(n), \quad \text{in } \Omega. \tag{1.37}$$

- *The electron flow in the device is called* **supersonic** *if*

$$|u| = |\nabla\psi| > \sqrt{p'(n)/\varepsilon}, \quad in\ \Omega, \tag{1.38}$$

or, equivalently,

$$\varepsilon|J|^2 > n^2 p'(n), \quad in\ \Omega. \tag{1.39}$$

- *The electron flow in the device is called* **transonic** *if it is alternatively supersonic and subsonic.*

- *The quantity* $\sqrt{p'(n)/\varepsilon}$ *is called the* **electron sound speed** *[30].*

Obviously, shocks may occur if the flow is transonic or supersonic. To our knowledge the existence of a purely supersonic solution has not been performed yet.

In Chapter 3, we give existence and uniqueness results of a supersonic solution to the one-dimensional unipolar steady state Euler-Poisson system for a potential flow (1.14)-(1.16) supplemented with Dirichlet boundary conditions.

In [47, 48], the stationary transonic solutions have been studied by using an artificial viscosity method. The existence of such solutions is proved by passing to the limit in the approximate Euler-Poisson system as the viscosity coefficient tends to zero. In [2], the subsonic solutions to a one-dimensional non isentropic model have been studied. In the steady state isentropic case, with Dirichlet boundary conditions, the existence and uniqueness of smooth solutions, in the subsonic region, are obtained for a one-dimensional flow in [35] or a potential flow in [36]. The main assumption of these results is a restriction on the magnitude of the boundary data for ψ, which implies a fully subsonic flow. In one-space dimension, this condition can be verified (see [35]), but for the multi-dimensional case, the smallness condition can not be explicitly verified anymore.

P.Degond and P.Markowich have shown, in [36], the existence and uniqueness of solutions to the system (1.14)-(1.16) (with $\Gamma_{ins} = \emptyset$ and for $\varepsilon = \tau = \lambda = 1$) in the space

$$\mathcal{B} = \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega}) \times \mathcal{C}^{1,\delta}(\bar{\Omega}).$$

Let us now describe the method to prove this result. First of all, using (1.15), we have:

$$\frac{\varepsilon}{2}\Delta(|\nabla\psi|^2) + \Delta h(n) = \Delta\phi + \frac{\varepsilon}{\tau}\Delta\psi. \tag{1.40}$$

The first term on the left-hand side is of third order. However it can be rewritten as a sum of at most second order terms. Indeed, from (1.14) we have:

$$\Delta\psi = -\frac{1}{n}\nabla n \cdot \nabla\psi. \tag{1.41}$$

Then a straightforward calculation and the use of (1.41) yield to

$$\begin{aligned}\Delta(|\nabla\psi|^2) \quad &= \frac{2}{n^2}(\nabla n\cdot\nabla\psi)^2 - \frac{2}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\frac{\partial^2 n}{\partial x_i\partial x_j}\\ &\quad -\frac{2}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial n}{\partial x_j} + 2\sum_{i,j=1}^{d}\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)^2. \qquad (1.42)\end{aligned}$$

Using (1.16) and (1.40), this gives

$$\begin{aligned}-\Delta h(n) + \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\frac{\partial^2 n}{\partial x_i\partial x_j} - \frac{\varepsilon}{\tau n}\nabla\psi\cdot\nabla n - \frac{\varepsilon}{n^2}(\nabla\psi\cdot\nabla n)^2\\ +\frac{\varepsilon}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial n}{\partial x_j} + \frac{1}{\lambda^2}(n-C) = \varepsilon Q(\psi), \qquad (1.43)\end{aligned}$$

where,

$$Q(\psi) = \sum_{i,j=1}^{d}\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)^2.$$

Then, consider the system (1.14), (1.43) (with $\varepsilon = \tau = \lambda = 1$) supplemented with the following Dirichlet boundary conditions (recall that $\Gamma_{ins} = \emptyset$)

$$n = n_D,\ \psi = \psi_D, \quad \text{on } \Gamma. \qquad (1.44)$$

For given n and ψ, the electrostatic potential is given by the relation (1.15). It is easy to see that, for $n > 0$, (n, ψ, ϕ) is a smooth solution to the system (1.14)-(1.16) if and only if (n, ψ) is a smooth solution to the system (1.14), (1.43).

Furthermore, we can show that the smallness condition on the data, which guarantees a subsonic flow in the device, is equivalent to the ellipticity condition for the equation (1.43). Then, under the assumption of a subsonic flow, system (1.14), (1.43) is a nonlinear elliptic problem. The proof given by P.Degond and P.Markowich in [36] makes an explicit use of the regularity properties for second order elliptic problems (see [56]). More precisely, by linearizing the problem and classical results on linear elliptic problems, they prove that the Schauder fixed point Theorem can be applied to obtain the existence of a smooth solution (n, ψ) to the system (1.14), (1.43) subject to the Dirichlet boundary conditions (1.44) (for recall $\varepsilon = \tau = \lambda = 1$). The uniqueness of solutions is also obtained.

The proof of existence relies strongly on regularity results for elliptic equations, which in full generality can only be obtained for Dirichlet problems. That is why the assumption $\Gamma_{ins} = \emptyset$ is made. Nevertheless it is possible to generalize the existence and uniqueness results to mixed Dirichlet-Neumann boundary conditions under additional stringent regularity and geometry assumptions on the boundary segments.

In [80], Y.J.Peng has shown, in a same way, that the smallness condition on the data can be replaced by a smallness condition on the electron mass ε. Then the existence and uniqueness of solutions are valid provided that the electron mass is small enough (which is physically the case). As in [36] the result is shown for Dirichlet boundary conditions (i.e. for $\Gamma_{ins} = \emptyset$).

Remark 1.3. *In [80] the existence and uniqueness of solutions is shown for all ε, τ provided that ε is small enough. It is also proved that the result holds for all λ provided that ε is small enough and that the compatibility condition $n_D = C$ on Γ is satisfied. We will give more details on it in the following section.*

In Chapter 6, we are interested in this smallness condition on the electron mass for the existence and uniqueness of solutions from a numerical point of view. However, the equation (1.43) is fully nonlinear and coupled to ψ till its second derivatives, so that its numerical discretization is not an easy task. Note that (1.14), (1.16) are linear for (ψ, ϕ) and (1.15) is nonlinear only algebraically for n. This motivates us to make the following iterative scheme: for a given $n^m > 0$ ($m \geq 0$) we first solve (ψ^m, ϕ^m) by

$$-\mathrm{div}(n^m \nabla \psi^m) = 0, \tag{1.45}$$

$$-\Delta \phi^m = C - n^m, \quad \text{in } \Omega, \tag{1.46}$$

subject to mixed Dirichlet-Neumann boundary conditions. Then we compute n^{m+1} by the algebraic equation

$$h(n^{m+1}) = \phi^m + \varepsilon \psi^m - \frac{\varepsilon}{2} |\nabla \psi^m|^2. \tag{1.47}$$

Equations (1.45), (1.46) are of elliptic type (provided n^m remains positive). There are several numerical methods to solve this kind of equations (e.g. finite element method, mixed finite element method, finite volume methods...). In Chapter 6, some finite volume schemes are used. The first scheme is "classical" with a two point discretization of the fluxes through the edges, see [44]. It leads to piecewise constant approximate solutions and needs to be completed by a reconstruction of the gradients $\nabla \psi^m$, necessary for the computation of n^{m+1} in (1.47). The second scheme is of mixed type as introduced by J.Droniou and R.Eymard in [42], in which the construction of the gradients is intrinsic. Then we can numerically obtain the solution to the steady-state Euler-Poisson system for a potential flow. The smallness condition on ε then clearly appears. Indeed, if ε is not small enough, after some iterations, the program is stopped since the involved matrix in the resolution of (1.45) becomes singular (the system is no more elliptic).

The bipolar steady-state Euler-Poisson system for a potential flow is also treated in Chapter 6. Let us mention that in [1] the existence of smooth global solutions to the multi-dimensional bipolar transient Euler-Poisson model has been shown under the assumption that the initial densities are close to a constant.

1.4 Asymptotic limits

Let us now discuss the asymptotic limits $\varepsilon \to 0$, $\tau \to 0$ and $\lambda \to 0$ in the system (1.14)-(1.16).

1.4.1 Zero-electron-mass limit

First of all, we consider the zero electron mass limit ($\varepsilon \to 0$). To this end, let $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)_{\varepsilon>0}$ be a sequence of solutions to the system (1.14)-(1.16), (1.44). We set $\lambda = \tau = 1$. Then,

$$-\mathrm{div}(n_\varepsilon \nabla \psi_\varepsilon) = 0, \tag{1.48}$$

$$\frac{\varepsilon}{2}|\nabla \psi_\varepsilon|^2 + h(n_\varepsilon) = \phi_\varepsilon + \varepsilon \psi_\varepsilon, \tag{1.49}$$

$$-\Delta \phi_\varepsilon = C - n_\varepsilon, \text{ in } \Omega, \tag{1.50}$$

$$n_\varepsilon = n_D, \ \psi_\varepsilon = \psi_D, \text{ on } \Gamma. \tag{1.51}$$

By performing the formal limit $\varepsilon \to 0$ in (1.48)-(1.51), we obtain

$$-\mathrm{div}(n \nabla \psi) = 0, \tag{1.52}$$

$$h(n) = \phi, \tag{1.53}$$

$$-\Delta \phi = C - n, \text{ in } \Omega, \tag{1.54}$$

$$n = n_D, \ \psi = \psi_D, \text{ on } \Gamma. \tag{1.55}$$

In [80], Y.J.Peng has shown that the sequence $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)_{\varepsilon>0}$ tends to (n, ψ, ϕ), solution of (1.52)-(1.55) in $\mathcal{B}$ when ε tends to zero. More precisely, he has proved that

$$\|n_\varepsilon - n\|_{\mathcal{W}^{2,q}(\Omega)} \le A_0\varepsilon, \ \|\psi_\varepsilon - \psi\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} \le A_0\varepsilon, \ \|\phi_\varepsilon - \phi\|_{\mathcal{C}^{1,\delta}(\bar\Omega)} \le A_0\varepsilon, \tag{1.56}$$

where $A_0 > 0$ is a constant independent of ε.

In Chapter 4, we consider an asymptotic expansion up to order m, for $m \ge 0$, of the solution. More precisely, let

$$n_{a,\varepsilon}^m = \sum_{k=0}^m \varepsilon^k n_k, \ \psi_{a,\varepsilon}^m = \sum_{k=0}^m \varepsilon^k \psi_k, \ \phi_{a,\varepsilon}^m = \sum_{k=0}^m \varepsilon^k \phi_k. \tag{1.57}$$

We are able to prove the existence and uniqueness of each profile (n_k, ψ_k, ϕ_k), $0 \le k \le m$ and the following convergence result.

Theorem 1.1. *Let* $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)$ *be the solution of the problem* (1.48)-(1.50) *supplemented with appropriate Dirichlet boundary conditions and* $(n^m_{a,\varepsilon}, \psi^m_{a,\varepsilon}, \phi^m_{a,\varepsilon})$ *be the approximate solution given by the asymptotic expansion* (1.57). *Under regularity assumptions, there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$, *we have the following estimates:*

$$\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} \le B_0\varepsilon^{m+1},\ \|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \le B_0\varepsilon^{m+1},\ \|\phi_\varepsilon - \phi^m_{a,\varepsilon}\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \le B_0\varepsilon^{m+1}, \tag{1.58}$$

where $B_0 > 0$ *is a constant independent of* ε.

This is a generalization of the result obtained by Y.J.Peng in [80]. Indeed, (1.56) corresponds to the case of an asymptotic expansion up to order 0, i.e., to the case $m = 0$ in (1.58).

The proof of (1.58) uses strongly the existence and uniqueness of a sequence of solutions and the uniform estimates in ε of the sequence (boundedness also shown in [80]). The idea is to subtract the system (1.48)-(1.51) and the one satisfied by $(n^m_{a,\varepsilon},\ \psi^m_{a,\varepsilon},\ \phi^m_{a,\varepsilon})$. Then as in the previous section, by using the Poisson equations to eliminate $\phi_\varepsilon - \phi^m_{a,\varepsilon}$, we can work on the system verified by $(n_\varepsilon - n^m_{a,\varepsilon},\ \psi_\varepsilon - \psi^m_{a,\varepsilon})$. In these two equations, it appears the following term:

$$I_\varepsilon = \frac{1}{2}\Delta(|\nabla\psi_\varepsilon|^2 - |\nabla\psi^m_{a,\varepsilon}|^2),$$

which is of third order. The main point of the proof is to give an estimate of I_ε by showing that it can be written as a function of at most second order derivatives (see Chapter 4 for more details).

As an application of this result, using asymptotic expansion up to second order, we also establish, in Chapter 4, the convergence of the Euler-Poisson system to the incompressible Euler equations. More precisely, we show the following result.

Corollary 1.1. *Let* $b(x) \equiv 1$ *and* $m = 1$. *Under regularity assumptions, the sequence of solutions* $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)_{\varepsilon>0}$ *to the problem* (1.48)-(1.50) *supplemented with appropriate boundary conditions converges, as* ε *tends to* 0, *to* (n_0, ψ_0, ϕ_0) *in* $\mathcal{B}$, *where* (n_0, ψ_0, ϕ_0) *is the first term in the asymptotic expansion. Moreover,* $u_0 = -\nabla\psi_0$ *satisfies the incompressible Euler equation*

$$(u_0 \cdot \nabla)u_0 + u_0 + \nabla P = 0, \quad div\ u_0 = 0,$$

where the pressure P *is defined by:*

$$P = h'(1)n_1 - \phi_1,$$

with (n_1, ψ_1, ϕ_1) *the second term in the asymptotic expansion.*

1.4.2 Zero-relaxation-time limit

Let us now consider the zero relaxation time limit. Let $(\tilde{n}_\tau, \tilde{\psi}_\tau, \tilde{\phi}_\tau)_{\tau>0}$ be a sequence of solutions. We set $\lambda = 1$ and the following change of variables:

$$n_\tau = \tilde{n}_\tau, \ \psi_\tau = \frac{\tilde{\psi}_\tau}{\tau}, \ \phi_\tau = \tilde{\phi}_\tau.$$

We obtain

$$-\mathrm{div}(n_\tau \nabla \psi_\tau) = 0, \tag{1.59}$$

$$\frac{\varepsilon\tau^2}{2}|\nabla\psi_\tau|^2 + h(n_\tau) = \phi_\tau + \varepsilon\psi_\tau, \tag{1.60}$$

$$-\Delta\phi_\tau = C - n_\tau, \ \text{in } \Omega, \tag{1.61}$$

$$n_\tau = n_D, \ \psi_\tau = \psi_D, \ \text{on } \Gamma. \tag{1.62}$$

By performing the formal limit $\tau \to 0$ in (1.59)-(1.62), we obtain the drift-diffusion system

$$-\mathrm{div}(n\nabla\psi) = 0, \tag{1.63}$$

$$h(n) = \phi + \varepsilon\psi, \tag{1.64}$$

$$-\Delta\phi = C - n, \ \text{in } \Omega, \tag{1.65}$$

$$n = n_D, \ \psi = \psi_D, \ \text{on } \Gamma. \tag{1.66}$$

In [80], Y.J.Peng has shown that the sequence $(n_\tau, \psi_\tau, \phi_\tau)_{\tau>0}$ tends to (n, ψ, ϕ), solution of (1.63)-(1.66) in $\mathcal{B}$ when τ tends to zero. More precisely he has obtained the following estimates:

$$\|n_\tau - n\|_{\mathcal{W}^{2,q}(\Omega)} \le A_1\tau^2, \ \|\psi_\tau - \psi\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \le A_1\tau^2, \ \|\phi_\tau - \phi\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \le A_1\tau^2, \tag{1.67}$$

where $A_1 > 0$ is a constant independent of τ.

In Chapter 4, using the asymptotic expansion method, after proving the existence and uniqueness of each profile, we show the following Theorem.

Theorem 1.2. *Under regularity assumptions, for an asymptotic expansion up to order m such that*

$$n_{a,\tau}^m = \sum_{k=0}^m \tau^{2k} n_k, \ \psi_{a,\tau}^m = \sum_{k=0}^m \tau^{2k}\psi_k, \ \phi_{a,\tau}^m = \sum_{k=0}^m \tau^{2k}\phi_k,$$

we have the following estimates

$$\|n_\tau - n_{a,\tau}^m\|_{\mathcal{W}^{2,q}(\Omega)} \le B_1\tau^{2(m+1)}, \ \|\psi_\tau - \psi_{a,\tau}^m\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \le B_1\tau^{2(m+1)},$$
$$\|\phi_\tau - \phi_{a,\tau}^m\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \le B_1\tau^{2(m+1)}, \tag{1.68}$$

where $B_1 > 0$ is a constant independent of τ.

Again, (1.67) corresponds to the case $m = 0$ in (1.68). The proof of (1.68) is similar to the one for the zero-electron-mass limit. As for the zero-electron-mass limit, we use the asymptotic expansion up to second order to show the convergence of the Euler-Poisson system to the incompressible Euler equations when the relaxation time tends to zero and we obtain an analogous result as Corollary 1.1 (see Chapter 4).

Note that the zero-relaxation-time limit has been investigated in the transient one-dimensional model in [76] and in [65, 66] by using compensated compactness arguments for global weak solutions. The limit system, obtained in this case, is the so-called and well-known drift-diffusion model. The multi-dimensional case for local smooth solutions has also been studied (see [1, 74]).

1.4.3 Quasineutral limit

Finally let us consider the quasineutral limit ($\lambda \to 0$). Let $(n_\lambda, \psi_\lambda, \phi_\lambda)_{\lambda>0}$ be a sequence of solutions to the system. We set $\tau = 1$. Then,

$$-\operatorname{div}(n_\lambda \nabla \psi_\lambda) = 0, \tag{1.69}$$

$$\frac{\varepsilon}{2}|\nabla \psi_\lambda|^2 + h(n_\lambda) = \phi_\lambda + \varepsilon \psi_\lambda, \tag{1.70}$$

$$-\Delta \phi_\lambda = C - n_\lambda, \text{ in } \Omega, \tag{1.71}$$

$$n_\lambda = n_D, \ \psi_\lambda = \psi_D, \text{ on } \Gamma. \tag{1.72}$$

The formal limit in the Poisson equation (1.71) gives:

$$n = C, \text{ in } \Omega,$$

and with (1.72) we have

$$n = n_D, \text{ on } \Gamma.$$

Then if $n_D \neq C$ on Γ, a boundary layer Ω_θ appears (see figure 1.1).

Let us first assume the following **compatibility condition**

$$n_D = C, \text{ on } \Gamma. \tag{1.73}$$

It is clear that this condition avoids any boundary layer and, then, the formal limit $\lambda \to 0$ in (1.69)-(1.72) gives

$$-\operatorname{div}(n \nabla \psi) = 0, \tag{1.74}$$

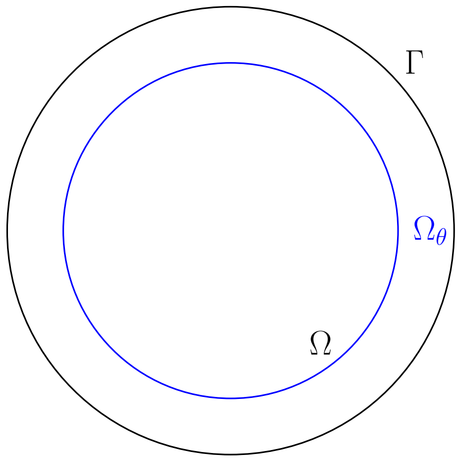

Figure 1.1: Boundary layer representation.

$$\frac{\varepsilon}{2}|\nabla\psi|^2 + h(n) = \phi + \varepsilon\psi, \tag{1.75}$$

$$n = C, \text{ in } \Omega, \tag{1.76}$$

$$n = n_D = C, \ \psi = \psi_D, \text{ on } \Gamma. \tag{1.77}$$

In [80], under the compatibility condition (1.73), Y.J.Peng shows that the sequence $(n_\lambda, \psi_\lambda, \phi_\lambda)_{\lambda>0}$ tends, when λ tends to zero, to (n, ψ, ϕ) in $L^q(\Omega) \times \mathcal{W}^{1,q}(\Omega) \times L^q(\Omega)$, and in $L^\infty(\Omega) \times \mathcal{W}^{1,q}(\Omega) \times L^q(\Omega)$, for all $q \in [1, +\infty[$, if we assume more regularity property on the doping profile. More precisely, by assuming (1.73), it is shown:

$$\|n_\lambda - n\|_{L^q(\Omega)} \leq A_2\lambda^2, \ \|\psi_\lambda - \psi\|_{\mathcal{W}^{1,q}(\Omega)} \leq A_2\lambda^2, \ \|\phi_\lambda - \phi\|_{L^q(\Omega)} \leq A_2\lambda^2, \tag{1.78}$$

or (with the more stringent assumption on the doping profile),

$$\|n_\lambda - n\|_{L^\infty(\Omega)} \leq A_2'\lambda^2, \ \|\psi_\lambda - \psi\|_{\mathcal{W}^{1,q}(\Omega)} \leq A_2'\lambda^2, \ \|\phi_\lambda - \phi\|_{L^q(\Omega)} \leq A_2'\lambda^2, \tag{1.79}$$

where $A_2, \ A_2' > 0$ are constants independent of λ. Then in this case the convergence space obtained is larger than the existence space. Recall that both **existence and convergence results are available only if the compatibility condition** (1.73) **holds**.

In Chapter 5, we are interested in the use of the asymptotic expansion method when the compatibility condition is not satisfied. Here the difficulty is that the existence result does not hold anymore and then we cannot use the same way as for the zero-electron-mass limit or the zero-relaxation-time limit. Indeed we need to prove, on one hand, the existence of each profile for an asymptotic expansion well defined and, on the other hand, in the same time the existence of a sequence $(n_\lambda, \psi_\lambda, \phi_\lambda)_{\lambda>0}$ and its convergence to the asymptotic expansion. Using the same asymptotic expansion as defined in [83] by Y.J.Peng and Y.G.Wang, we are able to prove:

Theorem 1.3. *Under regularity assumptions, for λ small enough, there is an $\varepsilon_0 > 0$ independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$, the problem* (1.69)-(1.71) *subject to appropriate boundary conditions has a solution $(n_\lambda, \psi_\lambda, \phi_\lambda)$ in $\mathcal{B}$ which satisfies:*

$$\|n_\lambda - n_\lambda^a\|_{\mathcal{W}^{2,q}(\Omega)} \leq B_2\lambda^{m-1}, \; \|\psi_\lambda - \psi_\lambda^a\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \leq B_2\lambda^{m-1}, \\ \|\phi_\lambda - \phi_\lambda^a\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \leq B_2\lambda^{m-1}, \tag{1.80}$$

where $B_2 > 0$ stands for a constant independent of λ, and $(n_\lambda^a, \psi_\lambda^a, \phi_\lambda^a)$ for an asymptotic expansion, up to order m with $m \geq 2$, well defined and in which the terms of order 0 and 1 satisfy compatibility conditions.

The last conditions in Theorem 1.3 mean that there are no boundary layers of order 0 and 1. Moreover, with more stringent assumption on the regularity of the boundary data, we show that for all $q \in [1, +\infty[$,

$$\|n_\lambda - n_\lambda^a\|_{L^\infty(\Omega)} \leq B_2'\lambda^{m+1}, \; \|\psi_\lambda - \psi_\lambda^a\|_{\mathcal{W}^{1,q}(\Omega)} \leq B_2'\lambda^{m+1}, \\ \|\phi_\lambda - \phi_\lambda^a\|_{L^q(\Omega)} \leq B_2'\lambda^{m+1}, \tag{1.81}$$

where $B_2' > 0$ is a constant independent of λ.

If we take $m = 1$ in (1.81), we have (1.79) since the term of order 1 in our asymptotic expansion is vanishing. Note that in Chapter 5 we obtain the existence of solutions and the convergence in the same space: $\mathcal{B}$. The proof of this result is really technical. Again the idea is to work on the system obtained after eliminating $\phi_\lambda - \phi_\lambda^a$. Then we define r_λ and p_λ by

$$n_\lambda = n_\lambda^a + \lambda^{m-1} r_\lambda, \; \psi_\lambda = \psi_\lambda^a + \lambda^{m-1} p_\lambda,$$

and we work with the problem satisfied by (r_λ, p_λ). We linearize this system to apply the Schauder fixed point Theorem to show the existence and the boundedness of (r_λ, p_λ) in good spaces. The main point of this proof is to show the existence and boundedness of solutions to the linearized equation on r_λ. To this end, we use the Leray-Schauder fixed point Theorem since the classical results are not available at this step. Indeed, the dependence of the functions in the equation doesn't allow to use them (see Chapter 5 for more details).

Let us mention that there are a lot of study on the quasineutral limit. In [14], it has been performed in the semilinear Poisson equation under the Maxwell-Boltzmann relation. In the steady-state Euler-Poisson system, it has been performed in [95] in the one-dimensional case for well prepared boundary conditions, and, in [79] for general boundary conditions. In [80, 83] the quasineutral limit is used to show the convergence of the Euler-Poisson system to incompressible Euler equations, for well-prepared boundary conditions.

Using pseudo-differential technics, the quasineutral limit has been shown in [29] for local smooth solutions in a one-dimensional model for an isothermal plasma in which the electron density is described by the Maxwell-Boltzmann relation. In [85] the convergence of the transient compressible Euler-Poisson equations (hyperbolic system) to the incompressible Euler equations has also been proved by using the quasineutral limit and the asymptotic expansion method. Numerically, the limit $\lambda \to 0$ has also been studied in the Euler-Poisson system in, for example, [33].

Note that in [53] and the references therein, the combined zero-relaxation-time and quasineutral limit is studied. See also [51, 52] for the study of the quasineutral limit in the bipolar drift-diffusion model.

Finally, let us point out that these three asymptotic limits yield a hierarchy of hydrodynamic models for plasmas given in [65, 66, 67], in which the zero-relaxation-time limit and the quasineutral limit are also shown for the drift-diffusion model.

Chapter 2

Quantum drift-diffusion model

In this Chapter, we present the quantum drift-diffusion model which is studied in Part II. For this model, we consider the bipolar case. This means that the charge transport in the device is due to the two species electrons and positive ions or holes. We are particularly interested in the quasineutral limit in the transient quantum drift-diffusion model and the regularity and positivity of solutions to the obtained limit equation.

2.1 General Presentation

The quantum drift-diffusion model can be derived by the entropy minimization principle from the Wigner-BGK equation in the diffusion limit [37], or, from the so-called quantum hydrodynamic model in the zero-relaxation-time limit [62]. Let us present here the second derivation.

Recall that the quantum hydrodynamic model consists of the isentropic Euler equations of conservation laws for the particle densities and current densities including the quantum Bohm potential and a momentum relaxation term, plus a Poisson equation for the electrostatic potential. It contains highly nonlinear and dispersive terms with third-order derivatives and therefore, its analytical and numerical treatment is quite involved. However, in some physical regimes, this model can be formally reduced to simpler models. More precisely, for example, when performing a diffusive scaling, the convective term can be formally neglected and the model is reduced then to the so-called quantum drift-diffusion model whose analysis and numerical solution are much simpler than that of the original model since it is parabolic and of fourth order.

More specifically, let $Q_T := [0,T] \times \Omega$ for $T > 0$ and $\Omega \subset \mathbb{R}^d$ ($d = 1, 2$ or 3) the domain occupied by the device. The (scaled) **transient isentropic quantum**

hydrodynamic model (in the bipolar case) reads:

$$\partial_t n - \nabla \cdot J_n = 0, \tag{2.1}$$

$$\partial_t J_n - \mathrm{div}\Big(\frac{J_n \otimes J_n}{n}\Big) - \nabla P_n(n) = -\frac{\varepsilon^2}{2} n \nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big) - n\nabla V - \frac{J_n}{\tau}, \tag{2.2}$$

$$\partial_t p + \nabla \cdot J_p = 0, \tag{2.3}$$

$$\partial_t J_p + \mathrm{div}\Big(\frac{J_p \otimes J_p}{p}\Big) - \nabla P_p(p) = \frac{\varepsilon^2}{2} p \nabla\Big(\frac{\Delta\sqrt{p}}{\sqrt{p}}\Big) - p\nabla V - \frac{J_p}{\tau}, \tag{2.4}$$

$$\lambda^2 \Delta V = n - p - C, \quad \text{in } Q_T. \tag{2.5}$$

The unknowns of this system are the electron density $n = n(t,x)$, the positively charged ion (or hole) density $p = p(t,x)$, and the (negative) electrostatic potential $V = V(t,x)$. Here, J_n and J_p stand respectively for the electron current density and the hole current density. The function $C = C(x)$ represents the fixed charged background ions, usually called the doping concentration or doping profile. The functions P_n and P_p are the pressure functions. They are typically of the form $P_\alpha(s) = \theta_\alpha s^{q_\alpha}$, $(\alpha = n,\ p)$ for some $\theta_\alpha > 0$ and $q_\alpha \geq 1$ ($\gamma-$law which we have already mentioned in Chapter 1). Recall that the case $q_\alpha = 1$ corresponds to the isothermal case. Finally, the physical parameters are the scaled Planck constant ε, the ratio of the Debye length and the characteristic length (e.g. the device diameter) $\lambda > 0$, and the relaxation time of the system τ.

Note that the quantum terms $(\varepsilon^2/2)n((\Delta\sqrt{n})/\sqrt{n})$ and $(\varepsilon^2/2)p((\Delta\sqrt{p})/\sqrt{p})$ can be interpreted as quantum self-potential terms with the Bohm potentials $(\Delta\sqrt{n})/\sqrt{n}$ and $(\Delta\sqrt{p})/\sqrt{p}$. The term J_n/τ models interactions of the electrons with the phonons of the semiconductor crystal lattice.

Let us now perform in (2.1)-(2.5) the following diffusion scaling: we substitute t by t/τ and $J_n,\ J_p$ by $\tau J_n,\ \tau J_p$ respectively. After rescaling we obtain the equations:

$$\tau\partial_t n - \tau\nabla \cdot J_n = 0, \tag{2.6}$$

$$\tau^2\partial_t J_n - \tau^2\mathrm{div}\Big(\frac{J_n \otimes J_n}{n}\Big) - \nabla P_n(n) = -\frac{\varepsilon^2}{2} n \nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big) - n\nabla V - J_n, \tag{2.7}$$

$$\tau\partial_t p + \tau\nabla \cdot J_p = 0, \tag{2.8}$$

$$\tau^2\partial_t J_p + \tau^2\mathrm{div}\Big(\frac{J_p \otimes J_p}{p}\Big) + \nabla P_p(p) = \frac{\varepsilon^2}{2} p \nabla\Big(\frac{\Delta\sqrt{p}}{\sqrt{p}}\Big) - p\nabla V - J_p, \tag{2.9}$$

$$\lambda^2 \Delta V = n - p - C, \quad \text{in } Q_T. \tag{2.10}$$

Remark 2.1. *If the constant τ is small, then the system* (2.6)-(2.10) *describes a situation for large time-scale and small current densities.*

Computing the formal limit $\tau \to 0$ in (2.6)-(2.10), the so-called **quantum drift-diffusion model** is derived and reads

$$\partial_t n - \nabla \cdot J_n = 0,\ J_n = -\frac{\varepsilon^2}{2} n \nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big) + \nabla P_n(n) - n\nabla V, \tag{2.11}$$

$$\partial_t p + \nabla \cdot J_p = 0,\ J_p = \frac{\varepsilon^2}{2} p \nabla\Big(\frac{\Delta\sqrt{p}}{\sqrt{p}}\Big) - \nabla P_p(p) - p\nabla V, \tag{2.12}$$

$$\lambda^2 \Delta V = n - p - C, \quad \text{in } Q_T. \tag{2.13}$$

Remark 2.2. *This model is a simple extension of the well-known classical drift-diffusion model. Indeed, if we neglect the recombination-generation rate and set all the physical parameters equal to 1, except for the Debye length, the drift-diffusion system can be written as:*

$$\partial_t n - \nabla \cdot J_n = 0,\ J_n = \nabla P_n(n) - n\nabla V,$$

$$\partial_t p + \nabla \cdot J_p = 0,\ J_p = -\nabla P_p(p) - p\nabla V,$$

$$\lambda^2 \Delta V = n - p - C, \quad \textit{in } Q_T.$$

In this Chapter and in Part III, we are only interested in the **one-dimensional transient quantum drift-diffusion model** which can then be written as:

$$\partial_t n - J_{n,x} = 0,\ J_n = -\frac{\varepsilon^2}{2} n \Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)_x + (P_n(n))_x - nV_x, \tag{2.14}$$

$$\partial_t p + J_{p,x} = 0,\ J_p = \frac{\varepsilon^2}{2} p \Big(\frac{(\sqrt{p})_{xx}}{\sqrt{p}}\Big)_x - (P_p(p))_x - pV_x, \tag{2.15}$$

$$\lambda^2 V_{xx} = n - p - C, \quad \text{in } Q_T. \tag{2.16}$$

It is easy to see that

$$n\Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)_x = \frac{1}{2}(n(\log n)_{xx})_x, \quad \text{and,} \quad p\Big(\frac{(\sqrt{p})_{xx}}{\sqrt{p}}\Big)_x = \frac{1}{2}(p(\log p)_{xx})_x.$$

Then, using these equalities and setting $\varepsilon = \varepsilon/2$, the equations (2.14)-(2.16) can equivalently be rewritten

$$\partial_t n - J_{n,x} = 0, \quad J_n = -\frac{\varepsilon^2}{2}(n(\log n)_{xx})_x + (P_n(n))_x - nV_x, \tag{2.17}$$

$$\partial_t p + J_{p,x} = 0, \quad J_p = \frac{\varepsilon^2}{2}(p(\log p)_{xx})_x - (P_p(p))_x - pV_x, \tag{2.18}$$

$$\lambda^2 V_{xx} = n - p - C, \quad \text{in } Q_T. \tag{2.19}$$

Since we only focus on the one-dimensional case, we can consider the problem in the space interval $\Omega = (0,1)$, without loss of generality. This system can be supplemented with different boundary conditions. In Chapter 7 we consider the following initial and Dirichlet-Neumann boundary conditions given by:

$$n(t,x) = p(t,x) = 1, \;\; n_x(t,x) = p_x(t,x) = 0, \;\; V(t,x) = V_D(x) \;\text{ for } x \in \{0,1\}, \; t > 0, \tag{2.20}$$

$$n(\cdot,0) = n_I, \quad p(\cdot,0) = p_I \;\text{ in } \Omega, \tag{2.21}$$

where $V_D(x) = xU$ and $U \in \mathbb{R}$ is the applied potential. In the case of a vanishing doping profile at the boundary, the Dirichlet boundary conditions for n and p express charge neutrality, whereas Neumann boundary conditions have been employed in numerical simulations of quantum semiconductor devices [49].

2.2 Existence of solutions

In this section, we are interested in the existence of solutions to the system (2.17)-(2.21). Moreover, since n and p are respectively the electron density and the hole density, we also want to show that they are nonnegative. Note that due to the quantum terms, the equations (2.17) and (2.18) are parabolic and of fourth order. In such a situation, no maximum principle is available to prove the nonnegativity of solutions which complicates the analysis (see [69, 70]). Then, we have to find another way to prove this result. To this end, we use a method presented in the literature on a simpler model.

Let us consider the following problem with homogeneous Dirichlet-Neumann boundary conditions

$$\partial_t u + (u(\log u)_{xx})_{xx} = 0, \quad \text{in } Q_T, \tag{2.22}$$

$$u(t,0) = u(t,1) = 1, \;\; u_x(t,0) = u_x(t,1) = 0, \quad t > 0, \tag{2.23}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{2.24}$$

under the assumption that the initial datum u_0 is measurable and satisfies the condition

$$\int_\Omega (u_0 - \log u_0)dx < \infty. \tag{2.25}$$

Let us mention that the problem (2.22)-(2.24) corresponds in fact to the unipolar case of problem (2.17)-(2.21) for a vanishing temperature and a vanishing electric field. Moreover the equation (2.22) is the so-called Derrida-Lebowitz-Speer-Spohn equation derived in the context of fluctuations of a stationary non-equilibrium interface (see [40]).

The first analytical result has been presented in [10] in which the existence of local-in-time classical solutions has been proved. In [69], the existence and the nonnegativity of global-in-time solutions have been shown.

Let us now shortly describe the method used in [69] to obtain this result since we use a same way to prove the existence of nonnegative solutions to the system (2.17)-(2.21).

First, in [69], A.Jüngel and R.Pinnau introduce an exponential change of variables. Setting $u = e^{2y}$, we get from (2.22)

$$\partial_t(e^{2y})_= - 2(e^{2y}y_{xx})_{xx}. \tag{2.26}$$

Hence, the existence of a global-in-time weak solution y of (2.26) implies the existence of a global-in-time nonnegative solution u to (2.22). Note that exponential transformations had already been successfully employed in the study of the stationary quantum hydrodynamic equations [60, 15].

The second step is the semi-discretization in time of equation (2.26), which leads to a sequence of elliptic problems. Using the Leray-Schauder fixed point Theorem, they show the existence of solutions $y(t_k, .)$ in $H^2(\Omega)$ to these resulting elliptic problems. Then, the approximate solutions $y(t_k, .)$ are in $L^\infty(\Omega)$ and the expressions like $e^{y(t_k,.)}$ are well defined.

The third step is the proof of the unique solvability for the semi-discretized problems by using a monotonicity property.

Then, using the fact that (2.22) possesses several Lyapunov functionals and that there are connections to logarithmic Sobolev inequalities (see [41] and references therein), they are able to obtain several a priori estimates.

Finally, in the last step they perform the limit when the time step tends to zero in the weak formulation of the sequence of the resulting elliptic problems. The proof uses, in particular, the a priori estimates obtained in the third step and the Aubin's Lemma, which allow to obtain some necessary strong convergences.

Remark 2.3. *Note that the boundary conditions* (2.23) *simplify considerably the analysis. Indeed, writing* (2.23) *in the new variable y we have*

$$e^{2y(t,0)} = e^{2y(t,1)} = 1, \ y_x(t,0)e^{2y(t,0)} = y_x(t,1)e^{2y(t,1)} = 0, \quad t > 0,$$

which gives

$$y(t,0) = y(t,1) = 0, \ y_x(t,0) = y_x(t,1) = 0, \quad t > 0. \tag{2.27}$$

Hence we obtain **homogeneous** *Dirichlet-Neumann boundary conditions. Thus, using for example the test function y in the weak formulation of* (2.22)*, no integrals with boundary data will appear.*

Let us mention that almost all results for (2.22)-(2.24) *(and for related fourth-order equations like the thin-film model [9]) have been shown only for periodic or* *no-flux boundary conditions or for whole-space problems, in order to avoid integrals boundary data.*

In [59], M.Gualdani, A.Jüngel and G.Toscani have also studied the problem (2.22)-(2.24) but subject to the following more general non-homogeneous Dirichlet-Neumann boundary conditions:

$$u(t,0) = u_0,\ u(t,1) = u_1,\ u_x(t,0) = w_0,\ u_x(t,1) = w_1, \quad t > 0, \tag{2.28}$$

where $u_0,\ u_1 > 0$ and $w_0,\ w_1 \in \mathbb{R}$. In this paper, they first show the existence of the stationary problem by using the change of variables $u = e^y$ and the Leray-Schauder fixed point Theorem. Then to prove the existence of solutions, to the transient problem, under assumption (2.25) for the initial data, they use the existence result for the stationary model and a method similar to the one used in [69] and previously described.

In Chapter 7, we are able to prove the existence of global-in-time solutions to the problem (2.17)-(2.21) and the nonnegativity of n and p. More precisely,

Theorem 2.1. *Let $T > 0$, $U \in \mathbb{R}$, $C \in L^\infty(\Omega)$, and $0 \le n_I, p_I \in L^1(\Omega)$ satisfying*

$$\int_\Omega \big((n_I - \log n_I) + (p_I - \log p_I)\big)dx + \int_\Omega \big(n_I(\log n_I - 1) + p_I(\log p_I - 1)\big)dx < \infty.$$

Furthermore, let $P_n, P_p \in C^1([0,\infty))$ be nondecreasing and assume that there exist $0 < q < 7/2$ and $C_P > 0$ such that

$$|P_\alpha(x)| \le C_P(1 + |x|^q) \quad \textit{for all } x \ge 0,\ \alpha = n, p. \tag{2.29}$$

Then there exists a weak solution $n,\ p \in L^{7/2}(Q_T)$, $V \in L^\infty(0,T;H^2(\Omega))$ to (2.17)-(2.21) such that

$$n, p \ge 0 \ \textit{in } Q_T, \quad \log n, \log p \in L^2(0,T;H_0^2(\Omega)), \quad n_t, p_t \in L^1(0,T;H^{-3}(\Omega)).$$

To prove this result, as seen previously, we first set the following change of variables:

$$n = e^y \ \text{ and } \ p = e^z,$$

which leads to the problem

$$\partial_t(e^y) + \frac{\varepsilon^2}{2}\Big(e^y y_{xx}\Big)_{xx} = \Big((P_n(e^y))_x - e^y V_x\Big)_x, \tag{2.30}$$

$$\partial_t(e^z) + \frac{\varepsilon^2}{2}\Big(e^z z_{xx}\Big)_{xx} = \Big((P_p(e^z))_x - e^z V_x\Big)_x, \tag{2.31}$$

$$\lambda^2 V_{xx} = e^y - e^z - C, \quad \text{in } Q_T, \tag{2.32}$$

subject to the initial condition

$$y(0,.) = y_I = \log n_I, \; z(0,.) = z_I = \log p_I, \quad \text{in } \Omega, \tag{2.33}$$

and homogeneous Dirichlet-Neumann boundary conditions on y and z.

Then, the idea is the same as in [69] or [59]. We first semi-discretize in time the problem, and show existence and uniqueness of solutions to the discrete problems (using the Leray-Schauder fixed point Theorem). Then we obtain some a priori estimates (using energy estimates), and, finally, we perform the limit for vanishing time step (using Aubin's Lemma and our a priori estimates). We refer to Chapter 7 for more details on the proof.

Remark 2.4. *Although the proof of existence of solutions is just an adaptation to the bipolar case of the one presented in [69], we write it completely in Chapter 7 since it makes clear which quantities are uniformly bounded in λ (in appropriate norms). Indeed, the main goal of Chapter 7 is to perform the quasineutral limit in* (2.17)-(2.21) *(see next section).*

Remark 2.5. *Note that in our proof the assumption of one space dimension is strongly used and then adapting our proof to the multi-dimensional case looks very complicated. We refer to [55] for an existence proof in the multi-dimensional equations for vanishing electric field and vanishing pressure with only periodic boundary conditions. To our knowledge, the treatment of the quantum drift-diffusion model in several space dimensions with physically motivated boundary conditions is currently not well known.*

2.3 Quasineutral limit

In charged particle transport, quasineutrality is a commonly used assumption in order to simplify the model equations. Let us first give the definition of the quasineutrality.

Definition 2.1. Quasineutrality *means that the difference between the concentrations of positive ions and electrons is negligible compared to a reference density.*

Formally, quasineutral models are obtained in the limit as the ratio of the Debye length to the characteristic length tends to zero (i.e. $\lambda \to 0$ in the system). Quasineutral models are used, for instance, in simulation of semiconductors and plasmas (see e.g. [31, 32, 34]). The justification of the quasineutral limit in fluid-dynamical models has attracted a lot of attention in the last years and the literature on this problem is huge (see Chapter 1 section 1.4 for references). Recently, the quasineutrality has also been studied in quantum models (see [12]). However, to our knowledge, no analytical results on the quasineutral limit in fluid-type quantum models are available up to now.

In Chapter 7, we give a first result: we rigorously prove the quasineutral limit for a vanishing doping profile.

Performing the formal quasineutral limit, $\lambda \to 0$, in (2.19) we obtain $n = p$ and from (2.17)-(2.18)

$$\partial_t n + \frac{\varepsilon^2}{2}(n(\log n)_{xx})_{xx} = \frac{1}{2}(P_n(n) + P_p(n))_{xx}, \quad \text{in } Q_T, \tag{2.34}$$

with the following initial and boundary conditions

$$n(t,x) = 1,\ n_x(t,x) = 0, \quad \text{for } x \in \{0,1\},\ t > 0, \quad n(0,.) = n_I \quad \text{in } \Omega. \tag{2.35}$$

We show in Chapter 7:

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold and let, in addition,* $C(x) \equiv 0$*,* $q \leq 7/3$ *and* $n_I = p_I$ *in* Ω*. Let* $(n^{(\lambda)}, p^{(\lambda)}, V^{(\lambda)})$ *be a weak solution (in the sense of Theorem 2.1) to* (2.17)-(2.21)*. Then there exists a subsequence of* $(n^{(\lambda)}, p^{(\lambda)}, V^{(\lambda)})$*, which is not relabeled, such that, as* $\lambda \to 0$*,*

$$\begin{aligned} n^{(\lambda)} \to n, \quad p^{(\lambda)} \to n \qquad & \textit{strongly in } L^3(Q_T), \\ n_t^{(\lambda)} \rightharpoonup n_t, \quad p_t^{(\lambda)} \rightharpoonup n_t \qquad & \textit{weakly in } L^{42/41}(0,T;H^{-3}(\Omega)), \\ \log n^{(\lambda)} \rightharpoonup \log n, \quad \log p^{(\lambda)} \rightharpoonup \log n \qquad & \textit{weakly in } L^2(0,T;H^2(\Omega)), \end{aligned}$$

satisfying (2.34)-(2.35).

Let us now shortly explain how to obtain the limit result. The sum of (2.17)-(2.18) leads to the following weak formulation (for ϕ a smooth test function):

$$\int_0^T < \partial_t n + \partial_t p, \phi >_{H^{-2},H^2} dt + \frac{\varepsilon^2}{2}\int_{Q_T}[n(\log n)_{xx} + p(\log p)_{xx}]\phi_{xx} dxdt$$
$$= \int_{Q_T}(P_n(n) + P_p(p))\phi_{xx} dxdt + \int_{Q_T}(n-p)V_x\phi_x dxdt. \tag{2.36}$$

Then to show the convergence of (2.17)-(2.21) to (2.34)-(2.35) when λ tends to zero, we have in particular to prove that the drift term

$$\int_{Q_T}(n-p)V_x\phi_x dxdt,$$

in (2.36) tends to zero when λ tends to zero.

To this end, we use some a priori estimates already obtained for the existence result, and which are uniform in λ, plus other estimates which are only needed for showing this limit and whose the proof is more technical. More precisely, we can show (already for the existence proof) that the "entropy" $\int (n - \log n) dx$ is nonincreasing with respect to time and that the corresponding entropy product terms provide uniform bounds for $\log n$ and $\log p$ with respect to λ in $L^2(0,T,H^2(\Omega))$, and, for n and p in $L^{7/2}(Q_T)$. Also, the entropy $\int n(\log n - 1)dx$ being nonincreasing in time, it provides the following uniform bounds:

$$\|n-p\|_{L^2(Q_T)} \le c\lambda, \quad \|V_x\|_{L^2(Q_T)} \le c\lambda^{-1}, \tag{2.37}$$

where $c > 0$ is a constant independent of λ. These estimates are sufficient for the existence result **but not** for passing to the quasineutral limit. Indeed, using the Hölder inequality, these estimates lead, for the drift term, to

$$\int_{Q_T} (n-p) V_x \phi_x dxdt \le \|n-p\|_{L^2(Q_T)} \|V_x\|_{L^2(Q_T)} \|\phi_x\|_{L^\infty(Q_T)} \le c,$$

where $c > 0$ is still a constant independent of λ. Thus, estimates (2.37) only show that the drift term is uniformly bounded and it is not sufficient to show that it tends to zero! The main problem of the proof of the quasineutral limit is then to obtain that the (negative) electric field V_x is of order $O(\lambda^{-\alpha})$ with $0 < \alpha < 1$. We are able to prove in Chapter 7 the following two estimates

$$\|\sqrt{n} - \sqrt{p}\|_{L^2(Q_T)} \le c\lambda, \quad \|(\sqrt{n} + \sqrt{p}) V_x\|_{L^2(Q_T)} \le c\lambda^{-8/9}, \tag{2.38}$$

which give for the drift term, using Hölder inequality

$$\int_{Q_T} (n-p) V_x \phi_x dxdt = \int_{Q_T} (\sqrt{n} - \sqrt{p})(\sqrt{n} + \sqrt{p}) V_x \phi_x dxdt \le \lambda^{1/9},$$

and hence, this shows that it tends to zero in the quasineutral limit. See Chapter 7 section 7.3 for more details on the proof of estimates (2.38) and the quasineutral limit.

Note that taking the difference of equations (2.17) and (2.18) provides in the formal limit $\lambda \to 0$ an equation for the electrostatic potential,

$$-((n+p)V_x)_x = (P_n(n) - P_p(n))_{xx} \ \text{ in } \Omega, \quad V(t,0) = 0, \ V(t,1) = U.$$

However, since V_x is of the order $O(\lambda^{-1})$ we cannot justify rigorously this limit equation. In the drift-diffusion equations, this is possible under some assumptions (see [70]).

Remark 2.6. *Note that to show the quasineutral limit, we have to assume also that* $n_I = p_I$ *in* Ω, *which, with the boundary conditions chosen, avoids any initial or boundary layer. See for example [67] for the treatment of boundary layers and [52] for the analysis of initial layers in the drift-diffusion model.*

Remark 2.7. *To our knowledge, there is no general uniqueness result on the limit problem.*

2.4 Regularity and positivity of solutions to the limit equation

Let us now consider the equation (2.34) obtained in the quasineutral limit. We assume here vanishing pressures and $\varepsilon = 1$. Then, we obtain, as mentioned in section 2.2, the so-called and well-known Derrida-Lebowitz-Speer-Spohn equation (2.22). Recall that this equation reads

$$\partial_t u + (u(\log u)_{xx})_{xx} = 0. \tag{2.39}$$

In the last few years, (2.39) has attracted a lot of attention in mathematical literature. Indeed, it possesses, for example, several Lyapunov functionals, which allow to obtain a lot of a priori estimates (see [10, 17, 41]), and then it is possible to obtain some regularity properties for the solutions. Moreover, there are also connections to logarithmic Sobolev inequalities (see [41] and references therein). Let us also mention that equation (2.39) can be seen as a one-homogeneous equation which could be a simple example of a generalization to the heat equation to higher order operators since the heat equation can be written

$$\partial_t u - (u(\log u)_x)_x = 0.$$

We have already mentioned, in section 2.2 of this Chapter, the two papers [69] and [59] in which existence of nonnegative global-in-time solutions to (2.39) with homogeneous and non-homogeneous Dirichlet-Neumann boundary conditions has been shown. A global-in-time existence result which is available for periodic boundary conditions has been shown in [41], by using an adaptation of the method used in [69] and shortly presented in section 2.2 of this Chapter. Concerning the multi-dimensional case, there exists, to our knowledge, only one result of existence of global-in-time weak solutions presented in [55].

The long-time behavior of solutions to equation (2.39) has also been studied. In [17], the case of periodic boundary conditions under restrictive regularity conditions on the initial data is presented. In [71] this study is concerned with homogeneous Dirichlet-Neumann boundary conditions and finally in [59] with non-homogeneous

Dirichlet-Neumann boundary conditions. More precisely the exponential decay of the solutions to their steady state has been shown in various norms and in terms of entropy. Numerically, this decay rate has been computed in [18].

Since u in (2.39) stands for a density of charge, it should be possible to show that it is positive and not only nonnegative as already obtained in the literature. This question of nonnegative or positive solutions to fourth-order parabolic equations has already been investigated in the context of lubrication-type equations, like the thin film equation

$$\partial_t u + (f(u)u_{xxx})_x = 0,$$

(see e.g. [7, 8]), where typically, $f(u) = u^\alpha$ for some $\alpha > 0$. However this equation is of degenerate type which makes the analysis easier than for (2.39), at least for the positivity property. Note that (2.39) is **not** of degenerate type, and that to our knowledge, the positivity property problem is still open! There exists only a partial result which establishes the positivity for small times (see [10]).

We are interested in this positivity problem for at least times large enough. To prove it, it seems to be necessary to first show additional regularity results to the ones already proved in the mathematical literature as for example in [41].

Let us consider the equation (2.39) supplemented with periodic boundary conditions and the following initial condition:

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{2.40}$$

where the initial datum u_0 satisfies (2.25).

Using a classical way (already presented in section 2.2) and the algorithmic construction of entropy presented by A.Jüngel and D.Matthes in [63], J.Dolbeault, I.Gentil and A.Jüngel have first shown in [41] the existence of global-in-time nonnegative weak solutions to this problem. Furthermore, they prove that a global-in-time solution u to (2.39)-(2.40) satisfies the regularity properties

$$\sqrt{u} \in L^2(0,T;H^3(\Omega)) \cap L^\infty(0,T;H^1(S^1)).$$

In Chapter 8, we rewrite equation (2.39) under the form

$$\frac{1}{\alpha}u^{1-\alpha}\partial_t(u^\alpha) + (u(\log u)_{xx})_{xx} = 0, \tag{2.41}$$

by using that $\frac{1}{\alpha}u^{1-\alpha}\partial_t(u^\alpha) = \partial_t u$. Then the usual change of variables $u = e^y$ in (2.41) leads to a different time discretization. Therefore, we need to prove again an existence result of nonnegative global-in-time weak solutions. More precisely, we show:

Theorem 2.3. *Let $u_0 : \Omega \to \mathbb{R}$ be a nonnegative measurable function such that $\int_\Omega (u_0 - \log u_0)dx < \infty$ and $u_0 \in H^1(\Omega)$. Let $T > 0$. We assume that $\alpha \in [2/53(25 - 6\sqrt{10}), 1]$. Then there exists a global weak solution u of (2.41)-(2.40) satisfying*

$$u \in L^{5/2}(0,T; W^{1,1}(\Omega)),\ \log u \in L^2(0,T; H^2(\Omega)),$$

$$u \geq 0 \ in\ \Omega \times (0, \infty),$$

and for all $T > 0$ and all smooth functions ϕ,

$$\int_0^T \frac{1}{\alpha} < (u^\alpha)_t, \phi >_{H^{-2},H^2} dt + \int_0^T \int_\Omega u(\log u)_{xx}(u^{\alpha-1}\phi)_{xx} dxdt = 0.$$

The initial datum is satisfied in the sense of $H^{-2}(\Omega) := (H_0^2(\Omega))^*$.

The form of (2.41), its time discretization and the algorithmic construction of entropy, presented by A.Jüngel and D.Matthes in [63], allow us to obtain new a priori estimates. They lead, by performing the vanishing time step limit, to the proof of Theorem 2.3 and to the regularity properties:

Theorem 2.4. *Under the assumptions of Theorem 2.3, the solution u verifies*

$$\begin{aligned} u^\alpha \in L^2(0,T; H^3(\Omega)),\ u^{\alpha/2} \in L^\infty(0,T; H^1(\Omega)), \\ and\ \ u \in L^\infty(0,T; L^\infty(\Omega)), \end{aligned} \tag{2.42}$$

where α is a real number in the interval $I := [2/53(25 - 6\sqrt{10}), 1]$.

We refer to the appendix in Chapter 8 for the justification of the lowest bound of I.

Let us mention that due to the form of (2.41), performing the vanishing time step limit is more complicated than in [41]. Indeed, let us multiply (2.41) by $u^{\alpha-1}$. Then we obtain the following weak formulation:

$$\frac{1}{\alpha}\int_0^T < \partial_t(u^\alpha), \phi >_{H^{-2},H^2} dt + \int_0^T \int_\Omega u(\log u)_{xx}(u^{\alpha-1}\phi)_{xx} dxdt = 0,$$

for a smooth test function ϕ. Then in the discrete problem, the second integral gives a product of three terms and not only two. In this case, the a priori estimates obtained and the Aubin's Lemma are not sufficient to have all the strong necessary convergences (in the good spaces) to pass to the limit on the time step. Therefore we also give in Chapter 8 some preliminary results concerned with the strong convergence of a function sequence in Sobolev spaces (available in one-space dimension) under particular assumptions satisfied by our sequence of solutions to the discrete problem. We refer to Chapter 8 for more details.

Remark 2.8. *The proof given in Chapter 8 uses strongly the one-space dimension property. Then it seems difficult to extend the method to the multi-dimensional case.*

Using the regularities presented in Theorem 2.4, we can prove the positivity of solutions to the problem for times large enough. Indeed, one estimate obtained for the proofs of Theorems 2.3 and 2.4 leads to

$$\|(u^{\alpha/2})_x\|_{L^\infty(0,T;L^2(\Omega))} \leq ce^{-\lambda T},$$

for $c,\ \lambda > 0$ some constants and for all $T > 0$. Using this inequality and convergence properties, in the particular case $\alpha = 1$, we can show that there exist $\gamma > 0$ and $t_0 > 0$ such that for all $t > t_0$ and all $x \in \Omega$,

$$\sqrt{u(t,x)} \geq \gamma > 0.$$

This means that for times large enough, there exists a positive solution to the equation (2.39) (we refer to Chapter 8 for more details).

Part II

Euler-Poisson model

Chapter 3

Example of supersonic solutions to a steady state Euler-Poisson system

This Chapter is an article in collaboration with Yue-Jun Peng published in Applied Mathematical Letters ([82]).

3.1 Introduction

The Euler-Poisson system plays an important role in the mathematical modeling and numerical simulation for plasmas and semiconductors [23, 50, 77]. In the steady state isentropic case the existence and uniqueness of smooth solutions are obtained in the subsonic region for a one-dimensional flow [35] or potential flows [36]. See also [2] for the subsonic solutions to a one-dimensional non-isentropic model. In [47, 48], the stationary transonic solutions are studied by an artificial viscosity approximation. The existence of the transonic solutions is proved by passing to the limit in the approximate Euler-Poisson system as the viscosity coefficient goes to zero. However, the existence of the purely supersonic solutions has not been discussed yet.

In this paper, we give an example of the supersonic solutions in a one-dimensional steady state Euler-Poisson system :

$$\partial_x j = 0, \tag{3.1}$$

$$\partial_x \left(\frac{j^2}{n} + p(n) \right) = n \partial_x \phi - j/\tau, \tag{3.2}$$

$$-\partial_{xx} \phi = b - n. \tag{3.3}$$

Equation (3.1) implies that j is a constant. Here, n, j and ϕ are the electron density, the current density and the electric potential, respectively. The parameter $\tau > 0$

stands for the momentum relaxation time depending on n and j in general. For simplicity, we assume that τ is a constant. The given function $b = b(x)$ is the doping profile for the semiconductors. The pressure function $p = p(n)$ is assumed to be smooth and strictly increasing for $n > 0$. As in [35], we consider equations (3.1)-(3.3) in the interval $(0,1)$ subject to the following Dirichlet boundary conditions:

$$n(0) = n_0, \;\; n(1) = n_1, \;\; \phi(0) = \phi_0, \;\; \phi(1) = \phi_1, \tag{3.4}$$

where $n_0 > 0$, $n_1 > 0$ and ϕ_0, $\phi_1 \in I\!R$ are given data. If $n > 0$ is a smooth function, after eliminating ϕ in (3.2)-(3.3), we obtain a Dirichlet problem for n :

$$-\partial_{xx} F_j(n) - \frac{1}{j}\partial_x\left(\frac{1}{\tau n}\right) + \frac{1}{j^2}(n-b) = 0 \quad \text{in } (0,1), \tag{3.5}$$

$$n(0) = n_0, \;\; n(1) = n_1, \tag{3.6}$$

where

$$F_j(n) = \frac{1}{2n^2} + \frac{h(n)}{j^2} \;\text{ with } h(n) = \int_1^n \frac{p'(y)}{y}\,dy.$$

Once n is solved, from (3.2) ϕ is given explicitly by :

$$\phi(x) = \phi_0 + j^2(F_j(n(x)) - F_j(n_0)) + \int_0^x \frac{j}{\tau n(y)}dy. \tag{3.7}$$

Then ϕ_1 is linked with j by the following relation

$$\phi_1 = \phi_0 + j^2(F_j(n_1) - F_j(n_0)) + \int_0^1 \frac{j}{\tau n(y)}dy. \tag{3.8}$$

It is easy to see that (n,ϕ) with $n > 0$ is a smooth solution of (3.2)-(3.4) if and only if (n,ϕ) is a smooth solution of (3.5)-(3.7). Therefore, we may first solve n to the Dirichlet problem (3.5)-(3.6) and then determine ϕ by (3.7).

Now the equation (3.5) is elliptic if and only if $F_j'(n) \neq 0$. Since p is strictly increasing, there is a unique $n_c(j)$ such that $F_j'(n_c(j)) = 0$, or equivalently

$$\sqrt{p'(n_c(j))} = \frac{|j|}{n_c(j)}.$$

Here the quantities $c = \sqrt{p'(n)}$ and j/n stand for the speed of sound and the electron velocity, respectively. If $n \longrightarrow n^2 p'(n)$ is strictly increasing, we obtain the following alternative :

$$\text{subsonic flow} \Longleftrightarrow F_j'(n) > 0 \Longleftrightarrow n > n_c(j) \Longrightarrow (3.5) \text{ is elliptic,} \tag{3.9}$$

$$\text{supersonic flow} \Longleftrightarrow F_j'(n) < 0 \Longleftrightarrow n < n_c(j) \Longrightarrow (3.5) \text{ is elliptic.} \tag{3.10}$$

Note that the linear term n/j^2 in (3.5) has not a good sign. Nevertheless, it is small as j is large and then can be controlled by the $L^2(0,1)$ norm of $\partial_x n$ by Poincaré's inequality. Similar argument holds for the term $\partial_x(1/j\tau n)$. This is the main feature of the problem to yield the existence and uniqueness of solutions.

3.2 Existence of solutions

Assume $b \in L^\infty(0,1)$. In view of (3.9), the subsonic solutions to (3.2)-(3.4) correspond to the small value of j. They have been considered in [35]. We study here the supersonic solutions which correspond to the case (3.10). To this end, let M_1 and M_2 be any two constants satisfying

$$0 < M_1 < \min(n_0, n_1), \quad \max(n_0, n_1) < M_2. \tag{3.11}$$

Choosing j such that $n_c(j) > M_2$, then (3.10) and (3.11) imply that the boundary data n_0 and n_1 are in the supersonic region. Since the maximum principle can not be applied to (3.5) in the supersonic region, the solutions of (3.5)-(3.6) may not be supersonic flow. To seek for a supersonic solution, we define a smooth and strictly decreasing function $\tilde{F}_j$ on $I\!R^+$ such that

$$\tilde{F}_j(+\infty) = 0, \quad \tilde{F}_j(n) = F_j(n) \text{ for all } n \le M_2.$$

Then we study the following problem instead of (3.5)-(3.6) :

$$-\partial_{xx}\tilde{F}_j(n) - \frac{1}{j}\partial_x\left(\frac{1}{\tau n}\right) + \frac{1}{j^2}(n-b) = 0 \quad \text{in } (0,1), \tag{3.12}$$

$$n(0) = n_0, \ \ n(1) = n_1. \tag{3.13}$$

Our strategy is to prove the existence of a smooth solution n to (3.12)-(3.13) such that $0 < n \le M_2$. Then n is a supersonic solution of (3.5)-(3.6) by the definition of $\tilde{F}_j$.

Since $\tilde{F}_j$ is smooth and strictly decreasing from $I\!R^+$ to $I\!R^+$, we may make a change of variable $v = \tilde{F}_j(n)$ for $n > 0$. Let G_j be the inverse of $\tilde{F}_j$, which is also smooth and strictly decreasing from $I\!R^+$ to $I\!R^+$. Then the problem (3.12)-(3.13) is equivalent to

$$-\partial_{xx}v - \frac{1}{j}\partial_x\left(\frac{1}{\tau G_j(v)}\right) + \frac{1}{j^2}(G_j(v)-b) = 0 \quad \text{in } (0,1), \tag{3.14}$$

$$v(0) = v_{0j} = F_j(n_0), \ \ v(1) = v_{1j} = F_j(n_1). \tag{3.15}$$

To study the problem (3.14)-(3.15), we will apply Schauder's fixed point theorem. For this purpose, let's define a closed convex set

$$S = \{v \in C([0,1]);\ F_j(M_2) \le v \le F_j(M_1)\},$$

and a map T by $v = T(\sigma)$ for $\sigma \in S$, where v solves the linear problem :

$$-\partial_{xx}v + \frac{1}{j\tau}\alpha_j(\sigma)\partial_x v + \frac{1}{j^2}\beta_j(x,\sigma) = 0 \quad \text{in } (0,1), \tag{3.16}$$

$$v(0) = v_{0j}, \quad v(1) = v_{1j}, \tag{3.17}$$

with

$$\alpha_j(\sigma) = \frac{G_j'(\sigma)}{G_j^2(\sigma)} = \frac{1}{G_j^2(\sigma)\tilde{F}_j'(G_j(\sigma))}, \qquad \beta_j(x,\sigma) = G_j(\sigma) - b(x).$$

We observe that $\sigma \in S$ implies that

$$F_j(M_2) \leq \sigma \leq F_j(M_1).$$

From $\tilde{F}_j(\sigma) = F_j(\sigma)$ for $\sigma \leq M_2$, we have

$$M_1 \leq G_j(\sigma) \leq M_2.$$

Therefore, from the definition of F_j, there is a $j_1 > 0$ depending only on M_1 and M_2 such that α_j and β_j are two bounded functions with bounds depending on M_1 and M_2 but independent of j and σ for any $j \in I\!R$ satisfying $|j| \geq j_1$.

For $v \in H^1(0,1)$ and $z \in H_0^1(0,1)$, let

$$a(v,z) = \int_0^1 \left(\partial_x v \partial_x z + \frac{1}{j\tau}\alpha_j(\sigma) z \partial_x v \right) dx, \quad l(z) = -\frac{1}{j^2}\int_0^1 \beta_j(x,\sigma) z dx.$$

It is clear that $l(\cdot)$ is linear and continuous on $H_0^1(0,1)$, and $a(\cdot,\cdot)$ is bilinear and continuous on $H_0^1(0,1) \times H_0^1(0,1)$. Moreover, by Poincaré's inequality,

$$\begin{aligned} a(z,z) &= \int_0^1 \left((\partial_x z)^2 + \frac{1}{j\tau}\alpha_j(\sigma) z \partial_x z \right) dx \\ &\geq \|\partial_x z\|_{L^2(0,1)}^2 - \frac{1}{|j|\tau}\|\alpha_j\|_{L^\infty(0,1)}\|z\|_{L^2(0,1)}\|\partial_x z\|_{L^2(0,1)} \\ &\geq \left(1 - \frac{C_1}{|j|\tau}\|\alpha_j\|_{L^\infty(0,1)}\right)\|\partial_x z\|_{L^2(0,1)}^2, \quad \forall\, z \in H_0^1(0,1), \end{aligned}$$

where $C_1 > 0$ is the constant in Poincaré's inequality. Then there exists a $j_2 \geq \frac{2C_1}{\tau}\|\alpha_j\|_{L^\infty(0,1)}$ depending only on M_1 and M_2 such that

$$a(z,z) \geq \frac{1}{2}\|\partial_x z\|_{L^2(0,1)}^2, \quad \forall\, |j| \geq j_2, \ \forall\, z \in H_0^1(0,1). \tag{3.18}$$

Therefore, $a(\cdot,\cdot)$ is coercive. By Lax-Milgram's theorem, there exists a unique solution $v \in H^1(0,1)$ to the variational problem $a(v,z) = l(z)$, $\forall\, z \in H_0^1(0,1)$ and (3.17). This shows that the map T is well defined.

We prove now that $T(S)$ is a compact set of $C([0,1])$. Indeed, let $\overline{v}_j = (1-x)v_{0j} + xv_{1j}$. Then $v - \overline{v}_j \in H_0^1(0,1)$. From the continuity of $l(\cdot)$ and $a(\cdot,\cdot)$, the coercivity estimate (3.18) and

$$a(v - \overline{v}_j, v - \overline{v}_j) = l(v - \overline{v}_j) - a(\overline{v}_j, v - \overline{v}_j),$$

it is easy to obtain

$$\|\partial_x(v-\overline{v}_j)\|_{L^2(0,1)} \le \frac{2C_1}{j^2}\|\beta_j\|_{L^\infty(0,1)} + \frac{2C_1}{|j|\tau}\|\alpha_j\|_{L^\infty(0,1)}\|\partial_x\overline{v}_j\|_{L^2(0,1)}. \tag{3.19}$$

Recall that α_j and β_j are bounded independent of σ. We conclude from Poincaré's inequality and the compact imbedding from $H^1(0,1)$ into $C([0,1])$ that $T(S)$ is a compact set of $C([0,1])$. Moreover, there are constants $C_2 > 0$ and $j_3 \ge j_2$ which depend only on M_1 and M_2 such that

$$|v(x) - \overline{v}_j(x)| \le \frac{C_2}{|j|}, \quad \forall\, |j| \ge j_3, \quad \forall\, x \in [0,1].$$

Since

$$F_j(\max(n_0,n_1)) \le \overline{v}_j(x) \le F_j(\min(n_0,n_1)), \quad \forall\, x \in [0,1],$$

it follows that

$$F_j(\max(n_0,n_1)) - \frac{C_2}{|j|} \le v(x) \le F_j(\min(n_0,n_1)) + \frac{C_2}{|j|}, \quad \forall\, |j| \ge j_3, \quad \forall\, x \in [0,1].$$

The function $n \longrightarrow F_j(n)$ being strictly decreasing for $n \le M_2$, from (3.11) there is a $j_4 \ge j_3$ depending only on M_1 and M_2 such that

$$F_j(M_2) \le v(x) \le F_j(M_1), \quad \forall\, |j| \ge j_4, \quad \forall\, x \in [0,1]. \tag{3.20}$$

Hence, $v \in S$ and then T is a self map from S to S. Finally, the continuity of T follows from a standard argument. More precisely, for $\sigma_1, \sigma_2 \in S$, we can prove that there is a constant $C_3 > 0$ depending only on M_1 and M_2 such that

$$\left(1 - \frac{C_3}{|j|\tau}\right)\|T(\sigma_1) - T(\sigma_2)\|_{C([0,1])} \le \frac{C_3}{|j|\tau}\|\sigma_1 - \sigma_2\|_{C([0,1])}.$$

Thus, T is continuous for $|j| > j_5 = \max(j_4, C_3/\tau)$. We conclude from Schauder's fixed point theorem the existence of a solution $v \in H^1(0,1) \cap S$ of $v = T(v)$.

This shows the existence of a solution $v \in H^1(0,1) \cap S$ to the problem (3.14)-(3.15), and then the existence of a solution $n = G_j(v) \in H^1(0,1)$ to the problem (3.12)-(3.13). Since $v = \tilde{F}_j(n) = F_j(n)$ for $n \le M_2$, from (3.20) we obtain

$$M_1 \le n(x) \le M_2, \quad \forall\, |j| \ge j_5, \quad \forall\, x \in [0,1]. \tag{3.21}$$

Therefore, $n \in H^1(0,1)$ is a supersonic solution to the problem (3.5)-(3.6). Thus, we have proved

Theorem 3.1. *Let $n_0 > 0$ and $n_1 > 0$. Let M_1, M_2 be two constants satisfying (3.11) and $b \in L^\infty(0,1)$. Then there exists a $j_e > 0$ depending only on M_1 and M_2 such that for any current density j satisfying $|j| \ge j_e$, the problem (3.2)-(3.4) admits a solution $(n,\phi) \in H^1(0,1) \times H^1(0,1)$. This solution is located in the supersonic region and satisfies (3.21).*

3.3 Uniqueness of solutions

There doesn't exist a general result on the uniqueness of solutions when the boundary data are located in the supersonic region. Indeed, for large j the formation of shocks cannot be avoided and the transonic solutions should be investigated. We refer to [47, 48] for the analysis of the transonic solutions. Here we give a uniqueness result in the supersonic region for large j. This result can be stated as follows.

Theorem 3.2. *Let M_1 and M_2 be two constants with $0 < M_1 < M_2$. Let $(n^{(1)}, \phi^{(1)})$ and $(n^{(2)}, \phi^{(2)})$ be two supersonic solutions of (3.2)-(3.3) in $H^1(0,1) \times H^1(0,1)$ with $M_1 \le n^{(1)}, n^{(2)} \le M_2$. Then there exists a $j_u > 0$ depending only on M_1 and M_2 such that for any current density $j \in I\!R$ satisfying $|j| \ge j_u$, we have $(n^{(1)}, \phi^{(1)}) = (n^{(2)}, \phi^{(2)})$.*

Proof. In view of (3.7), it suffices to show that $n^{(1)} = n^{(2)}$. Let $w = n^{(2)} - n^{(1)}$. By subtracting the equation (3.5) satisfied by $n^{(1)}$ and $n^{(2)}$ we obtain :

$$\partial_{xx}(A_j(x)w) + \frac{1}{j\tau}\partial_x(B(x)w) + \frac{1}{j^2}w = 0 \quad \text{in } (0,1), \tag{3.22}$$

where

$$A_j(x) = -\int_0^1 \frac{\partial F_j}{\partial n}\left(n^{(1)}(x) + s\left(n^{(2)}(x) - n^{(1)}(x)\right)\right) ds,$$

$$\frac{1}{M_2^2} \le B(x) = \frac{1}{n^{(1)}n^{(2)}} \le \frac{1}{M_1^2} \quad \text{in } (0,1).$$

From

$$F_j'(n) = -\frac{1}{n^3} + \frac{h'(n)}{j^2},$$

it is easy to check that there are constants $C_4 > 0$ and $j_6 > 0$ which depend only on M_1 and M_2 such that

$$A_j(x) \ge C_4, \quad \forall\, |j| \ge j_6,\ \forall\, x \in [0,1].$$

Multiplying (3.22) by $A_j w \in H_0^1(0,1)$ and integrating over $(0,1)$ give :

$$\int_0^1 \left[\partial_x(A_j(x)w)\right]^2 dx = \int_0^1 \left(-\frac{1}{j\tau}B(x)w\partial_x(A_j(x)w) + \frac{1}{j^2}A_j(x)w^2\right) dx.$$

It follows from Poincaré's inequality that :

$$\|\partial_x(A_j w)\|^2_{L^2(0,1)} \le \frac{1}{C_4}\left(\frac{C_0}{M_1^2|j|\tau} + \frac{C_0^2}{j^2}\right)\|\partial_x(A_j w)\|^2_{L^2(0,1)}.$$

This shows that $A_j w = 0$ and then $w = 0$ provided that $|j| \ge j_7$ for some large $j_7 > 0$ depending only on M_1 and M_2. □

Chapter 4

Asymptotic expansions in a steady state Euler-Poisson system and convergence to incompressible Euler equations

This Chapter is an article in collaboration with Yue-Jun Peng published in the review M3AS ([81]).

4.1 Introduction

The Euler-Poisson system is a hydrodynamic model widely used in the mathematical modeling and numerical simulation for plasmas [23] and semiconductors [77]. It consists of two nonlinear equations given by the conservation of density and momentum, called Euler equations, plus a Poisson equation for the electric potential. Due to the nonlinear hyperbolicity, the weak solution of the transient Euler equations is only studied in one space dimension. In such a situation, the existence of global weak solutions can be shown in the set of bounded functions.

In this paper, we study a steady state unipolar model for electrons. Then the corresponding Euler-Poisson system reads as follows :

$$-\mathrm{div}(nu) = 0, \tag{4.1}$$

$$\varepsilon \mathrm{div}(nu \otimes u) + \nabla p(n) = n\nabla\phi - \frac{\varepsilon nu}{\tau}, \tag{4.2}$$

$$-\lambda^2 \Delta\phi = b(x) - n. \tag{4.3}$$

This system will be studied in an open and bounded domain Ω of $\mathbb{R}^d$ ($d = 2$ or $d = 3$ in practice). Here $n = n(x)$, $u = u(x)$ and $\phi = \phi(x)$ stand for the electron density, electron velocity and electric potential, respectively. The given functions $b = b(x)$ is the doping profile for semiconductors and ion density for the plasmas, and $p = p(n)$ is the pressure function. We assume that b and p are smooth functions, p is strictly increasing for $n > 0$, and there is a constant $\underline{b} > 0$ such that : $b(x) \geq \underline{b}$ for all $x \in \bar{\Omega}$. The physical parameters λ, ε and τ stand for the Debye length, electron mass and relaxation time of the system, respectively. They are small compared with the characteristic length of physical interest. Therefore, it is important to give the mathematical justification of these limits. Notice that for the plasmas, the ions are usually much heavier than the electrons. Therefore the electron mass ε here should be understood as the ratio of the electron mass and ion mass.

In the sequel, we study independently the zero-electron-mass limit $\varepsilon \to 0$ and zero-relaxation-time limit $\tau \to 0$ when $\lambda > 0$ is fixed. For simplicity, we assume that $\lambda = 1$.

We consider the case of a potential flow, $\mathrm{curl} u = 0$. Then by introducing the velocity potential ψ defined by $u = -\nabla\psi$ and using

$$\mathrm{div}(nu \otimes u) = \frac{n}{2}\nabla|\nabla\psi|^2 + u\mathrm{div}(nu),$$

we obtain from (4.1)-(4.2) :

$$\nabla\Big(\frac{1}{2}|\nabla\psi|^2 + h(n) - \phi\Big) = \nabla\psi,$$

where h is the enthalpy of the system defined by :

$$h(n) = \int_1^n \frac{p'(y)}{y}\,dy.$$

Hence, for smooth solutions, the system (4.1)-(4.3) can be written in the form :

$$-\mathrm{div}(n\nabla\psi) = 0, \tag{4.4}$$

$$\frac{\varepsilon}{2}|\nabla\psi|^2 + h(n) = \phi + \frac{\varepsilon\psi}{\tau}, \tag{4.5}$$

$$-\Delta\phi = b(x) - n. \tag{4.6}$$

Equation (4.5) is the Bernoulli law. Eliminating ϕ from (4.4) and (4.6) and using (4.5), we have

$$\begin{aligned} -\Delta h(n) &+ \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\frac{\partial^2 n}{\partial x_i\partial x_j} - \frac{\varepsilon}{\tau n}\nabla\psi.\nabla n - \frac{\varepsilon}{n^2}(\nabla\psi.\nabla n)^2 \\ &+ \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial n}{\partial x_j} + n - b(x) = Q(\psi), \end{aligned} \tag{4.7}$$

where Q is given by

$$Q(\psi) = \sum_{i,j=1}^{d} \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j} \right)^2 . \tag{4.8}$$

For $n > 0$ it is easy to see that (n, ψ, ϕ) is a smooth solution to the system (4.4)-(4.6) if and only if (n, ψ) is a smooth solution to the system (4.4) and (4.7). Moreover, for given ψ, the equation (4.7) is elliptic if and only if the flow is subsonic, i.e., the condition $|\nabla\psi| < \sqrt{p'(n)/\varepsilon}$ holds.

For each limit of the system (4.4)-(4.6), we supplement the Dirichlet boundary conditions. In the case of the zero electron mass limit, we associate to the system (4.4)-(4.6) (where $\tau = 1$) the following boundary conditions :

$$n = \sum_{k=0}^{m} \varepsilon^k \bar{n}_k + n_{D,\varepsilon}^{m+1}, \quad \psi = \sum_{k=0}^{m} \varepsilon^k \bar{\psi}_k + \psi_{D,\varepsilon}^{m+1} \quad \text{on } \Gamma \stackrel{\text{def}}{=} \partial\Omega, \tag{4.9}$$

where $n_{D,\varepsilon}^{m+1}$ and $\psi_{D,\varepsilon}^{m+1}$ are smooth enough and defined in $\bar{\Omega}$ such that $n_{D,\varepsilon}^{m+1} = O(\varepsilon^{m+1})$ and $\psi_{D,\varepsilon}^{m+1} = O(\varepsilon^{m+1})$ uniformly in ε. In the case of the zero relaxation time limit, we first make a change of variable and then associate to the new system the following boundary conditions :

$$n = \sum_{k=0}^{m} \tau^{2k} \bar{n}_k + n_{D,\tau}^{m+1}, \quad \psi = \sum_{k=0}^{m+1} \tau^{2k} \bar{\psi}_k + \psi_{D,\tau}^{m+1} \quad \text{on } \Gamma, \tag{4.10}$$

with similar assumptions on the data to those used in the zero electron mass limit.

For fixed ε and τ, the existence and uniqueness of solutions to the system (4.4)-(4.6) have been already shown in the space

$$B \stackrel{\text{def}}{=} \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega}) \times \mathcal{C}^{1,\delta}(\bar{\Omega})$$

for small Dirichlet data on the velocity potential (see [36]) by using the equivalent system (4.4) and (4.7). The smallness condition on the data guarantees that the problem is located in the subsonic region. In [80] it is shown that this smallness condition corresponds to the smallness of ε. Then the existence and uniqueness of solutions hold for large data provided that ε is small enough. Furthermore, there is given a result of convergence and an error estimate for an asymptotic expansion on ε up to first order of the solution in the same space.

The quasineutral limit $\lambda \to 0$ has been studied by a lot of authors. In one-dimensional steady state Euler-Poisson system it was performed in [95] for well-prepared boundary data and in [79] for general boundary data. The steady problem in several space variables for a potential flow without the formation of boundary layers was investigated in [80]. In [29], by using pseudo-differential techniques, the quasineutral limit was studied for local smooth solutions of an one-dimensional and isothermal model for plasmas in which the electron density is described by the Maxwell-Boltzmann relation. This relation can be obtained in the zero electron mass limit of the Euler-Poisson equations which we will discuss below (Remark 4.1). See also [14] for the study of the quasi-neutral limit in a semi-linear Poisson equation in which the Maxwell-Boltzmann relation is also used.

The zero relaxation time limit in one dimensional transient Euler-Poisson system has been investigated in [76] and [65, 66] by the compensated compactness arguments for global weak solutions. The limit system is governed by the classical drift-diffusion model. In multi-dimensional case and for local smooth solutions this limit has been studied in [74] and [1]. See also [53] for a combined zero relaxation time and vanishing Debye length limit and the references therein.

In this paper, we study the zero-electron-mass limit and zero-relaxation-time limit in the subsonic region by the method of asymptotic expansions. For each limit, we justify the asymptotic expansions in the space B up to any order by using the elliptic properties. As applications of the asymptotic expansions up to second order, we establish in both limits the convergence of the Euler-Poisson system (4.4)-(4.6) to the incompressible Euler equations with explicit pressure expressed by the profiles. Notice that the convergence of the Euler-Poisson system to the incompressible Euler equations has been already shown via the quasi-neutral limit when the Dirichlet boundary data are well prepared (see [80] and [83]).

Finally, let us mention the main differences between this paper and [85] where the convergence of the compressible Euler-Poisson equations to the incompressible Euler equations was also proved. Paper [85] dealt with the quasi-neutral limit in the transient Euler-Poisson system which is hyperbolic. Although the constructions of the profiles in the asymptotic expansions are similar, the limit equations and the justifications of the expansions are quite different in the two papers. In [85], the symmetric hyperbolic property and high order energy estimates are used for local smooth solutions to the Cauchy problem in the space $C(0,T;H^s(\mathbb{R}))$, whereas in this paper, only the elliptic properties are employed for the Dirichlet boundary problem.

The remainder of the paper is arranged as follows. In Sections 4.2, 4.3, we consider the zero electron mass limit. We begin, in Section 4.2, with the asymptotic expansions of solutions to the problem by determining its all order profile. In Section 4.3, we justify the asymptotic expansions up to order m and establish error estimates of order ε^{m+1} for each variable. Section 4.4 is devoted to the zero relaxation time limit. We obtain in a same way the error estimates of order $\tau^{2(m+1)}$ for an asymptotic expansion up to order m. Finally, in the last section, we give applications of these results by showing the convergence of the system (4.4)-(4.6) to the incompressible Euler equations for each limit $\varepsilon \to 0$ and $\tau \to 0$.

4.2 Asymptotic expansion

4.2.1 Derivation of the profile equations

Let $\tau = 1$. We consider the limit $\varepsilon \to 0$ in the steady state Euler-Poisson system for the potential flow, i.e., the system (4.4)-(4.6) associated to the Dirichlet boundary conditions (4.9).

We assume that :

(A1) Ω is a bounded and convex domain of $\mathbb{R}^d$ with $\Gamma = \partial\Omega \in \mathcal{C}^{2,\delta}$, $\delta \in]0,1[$,

(A2) $p \in \mathcal{C}^{m+4}(\mathbb{R}^+)$, $m \in \mathbb{N}$, $p'(n) > 0 \ \forall\, n > 0$,

(A3) $b \in L^\infty(\Omega), 0 < \underline{b} \leq b(x)$,

(A4) $\bar{n}_k \in \mathcal{W}^{2,q}(\Omega)$ for $q > \frac{d}{1-\delta}$ and $\forall\ 0 \leq k \leq m,\ 0 < \underline{n} \leq \bar{n}_0(x) \ \ \forall x \in \Gamma$,

(A5) $\bar{\psi}_k \in \mathcal{C}^{2,\delta}(\bar{\Omega}), \ \ \forall\ 0 \leq k \leq m$,

(A6) the sequence $(\varepsilon^{-(m+1)} n_{D,\varepsilon}^{m+1})_{\varepsilon > 0}$ is bounded in $\mathcal{W}^{2,q}(\Omega)$,

(A7) the sequence $(\varepsilon^{-(m+1)} \psi_{D,\varepsilon}^{m+1})_{\varepsilon > 0}$ is bounded in $\mathcal{C}^{2,\delta}(\bar{\Omega})$.

Let $(n_{a,\varepsilon}, \psi_{a,\varepsilon}, \phi_{a,\varepsilon})$ be defined by the following ansatz :

$$n_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k n_k,\ \ \psi_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k \psi_k,\ \ \phi_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k \phi_k \ \text{ in } \Omega, \tag{4.11}$$

with the boundary conditions :

$$n_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k \bar{n}_k,\ \ \psi_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k \bar{\psi}_k \quad \text{on } \Gamma. \tag{4.12}$$

Plugging the expression (4.11) into the system (4.4)-(4.6), we obtain formally

$$-\mathrm{div}\bigg(\Big(\sum_{k\geq 0} \varepsilon^k n_k\Big) \nabla \Big(\sum_{k\geq 0} \varepsilon^k \psi_k\Big)\bigg) = 0, \tag{4.13}$$

$$\frac{\varepsilon}{2} \left| \nabla \Big(\sum_{k\geq 0} \varepsilon^k \psi_k\Big) \right|^2 + h\Big(\sum_{k\geq 0} \varepsilon^k n_k\Big) = \sum_{k\geq 0} \varepsilon^k \phi_k + \varepsilon \sum_{k\geq 0} \varepsilon^k \psi_k, \tag{4.14}$$

$$-\Delta \left(\sum_{k\geq 0} \varepsilon^k \phi_k \right) = b(x) - \sum_{k\geq 0} \varepsilon^k n_k. \tag{4.15}$$

Now, we seek for the system and boundary conditions for each profile (n_k, ψ_k, ϕ_k). Obviously,

$$\operatorname{div}\Big(\Big(\sum_{k\geq 0}\varepsilon^k n_k\Big)\Big(\sum_{k\geq 0}\varepsilon^k \nabla\psi_k\Big)\Big) = \sum_{k\geq 0}\varepsilon^k \sum_{i=0}^{k} \operatorname{div}(n_i \nabla\psi_{k-i}),$$

$$\Big|\nabla\Big(\sum_{k\geq 0}\varepsilon^k \psi_k\Big)\Big|^2 = \sum_{k\geq 0}\varepsilon^k \Big(\sum_{i=0}^{k} \nabla\psi_i.\nabla\psi_{k-i}\Big)$$

and by the Taylor's formula,

$$h\left(\sum_{k\geq 0}\varepsilon^k n_k\right) = \sum_{k\geq 0}\varepsilon^k h_k(n),$$

where $n = (n_i)_{i\geq 0}$ and

$$h_k(n) = \frac{1}{k!}\frac{d^k h(\sum_{k\geq 0}\varepsilon^k n_k)}{d\varepsilon^k}\Big|_{\varepsilon=0},\ k\geq 0.$$

It is immediate that

$$h_k(n) = h'(n_0)n_k + \bar{h}_k\Big((n_i)_{0\leq i\leq k-1}\Big),\ k\geq 1,$$

where h_k is of class $\mathcal{C}^{m-k+3}$ with $\bar{h}_1 \equiv 0$. It follows that :

$$h\Big(\sum_{k\geq 0}\varepsilon^k n_k\Big) = h(n_0) + \sum_{k\geq 1}\varepsilon^k h'(n_0)n_k + \sum_{k\geq 2}\varepsilon^k \bar{h}_k\Big((n_i)_{0\leq i\leq k-1}\Big).$$

Then by identification of the order in ε in the problem (4.13)-(4.15) and (4.12), we obtain the system for each (n_k, ψ_k, ϕ_k), $k\geq 0$. More precisely, the first order (n_0, ψ_0, ϕ_0) satisfies the nonlinear problem in Ω :

$$-\operatorname{div}(n_0\nabla\psi_0) = 0, \tag{4.16}$$

$$h(n_0) = \phi_0, \tag{4.17}$$

$$-\Delta\phi_0 = b(x) - n_0, \tag{4.18}$$

with the following boundary conditions :

$$n_0 = \bar{n}_0, \quad \psi_0 = \bar{\psi}_0 \quad \text{on } \Gamma. \tag{4.19}$$

For all $k\geq 1$, (n_k, ψ_k, ϕ_k) is obtained by induction on k in the following linear problem in Ω :

$$-\operatorname{div}(n_0\nabla\psi_k) = \sum_{i=1}^{k}\operatorname{div}(n_i\nabla\psi_{k-i}), \tag{4.20}$$

$$h'(n_0)n_k - \phi_k = f_k, \tag{4.21}$$

$$-\Delta\phi_k = -n_k, \tag{4.22}$$

with the boundary conditions :

$$n_k = \bar{n}_k, \quad \psi_k = \bar{\psi}_k \quad \text{on } \Gamma, \tag{4.23}$$

where

$$f_k = \psi_{k-1} - \frac{1}{2}\sum_{i=0}^{k-1} \nabla\psi_{k-1-i}.\nabla\psi_i - \bar{h}_k((n_i)_{0\le i\le k-1}). \tag{4.24}$$

Remark 4.1. *Equation (4.17) expresses a Maxwell-Boltzmann type relation. Indeed, for the isothermal plasma, the pressure is a linear function. Then* $p(n) = a^2 n$ *with* $a > 0$. *This implies from the definition of* h *that* $h(n) = a^2 \log n$ *and hence, from (4.17)* $n_0 = \exp(\phi_0/a^2)$. *This is the classical Maxwell-Boltzmann relation which has been used in [14, 29] and [87] for the study of the quasi-neutral limit.*

4.2.2 Existence and uniqueness of the profiles

Now we show that each problem (4.16)-(4.19) and (4.20)-(4.23) has a unique solution. We start by the problem (4.16)-(4.19). Eliminating ϕ_0 in (4.18) by (4.17), we obtain the nonlinear problem on n_0 :

$$\Delta h(n_0) - n_0 = -b(x) \ \text{ in } \Omega, \tag{4.25}$$

$$n_0 = \bar{n}_0 \quad \text{on } \Gamma. \tag{4.26}$$

Since p is smooth and strictly increasing, so is h. By the assumptions (A2)-(A4) and Lemmas 9.15 and 9.17 in [56] or Lemma 2.2 in [36], this problem admits a unique solution $n_0 \in \mathcal{W}^{2,q}(\Omega)$. Furthermore, the maximum principle gives :

$$n_0(x) \ge \min\left(\min_{x\in\Omega} b(x), \min_{x\in\Gamma} \bar{n}_0(x)\right) \ge n_{min} \overset{\text{def}}{=} \min(\underline{b}, \underline{n}) > 0, \quad \forall\, x \in \overline{\Omega}. \tag{4.27}$$

Then, (4.17) gives a unique $\phi_0 \in \mathcal{W}^{2,q}(\Omega)$. Finally, since the injection $\mathcal{W}^{2,q}(\Omega) \hookrightarrow \mathcal{C}^{1,\delta}(\bar{\Omega})$ is continuous for $q > \frac{d}{1-\delta}$ (see assumption (A4)), the equation (4.16) and the boundary condition in (4.19) provide a unique solution $\psi_0 \in \mathcal{C}^{2,\delta}(\bar{\Omega})$ (see Theorem 6.6, [56]). Hence, we have determined a unique solution (n_0, ψ_0, ϕ_0) to the problem (4.16)-(4.19) in $\mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega}) \times \mathcal{W}^{2,q}(\Omega)$.

Now we consider the problem (4.20)-(4.23). Assume that for some $k \ge 1$ we know all $(n_j, \psi_j, \phi_j) \in B$ for $0 \le j \le k-1$, solutions of the problem (4.16)-(4.19) if $j = 0$ or (4.20)-(4.23) in which k is replaced by $j \ge 1$. Eliminating ϕ_k in (4.21)-(4.22) we obtain the following linear problem for n_k :

$$\Delta(h'(n_0)n_k) - n_k = \Delta f_k \quad \text{in } \Omega, \tag{4.28}$$

$$n_k = \bar{n}_k \quad \text{on } \Gamma. \tag{4.29}$$

Notice that Δf_k contains the third order derivatives of $(\psi_i)_{0\le i\le k-1}$. Since $n \to h(n)$ is strictly increasing for $n > 0$, from (4.27), in order to show the existence of a unique solution $n \in \mathcal{W}^{2,q}(\Omega)$ of the linear problem (4.28)-(4.29), we have to establish the following result.

Lemma 4.1. *Let $m \ge 1$ and $1 \le k \le m$. Assume that $(n_j, \psi_j) \in \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega})$ for all $0 \le j \le k-1$. Then, $f_k \in \mathcal{C}^{1,\delta}(\bar{\Omega})$ and $\Delta f_k \in L^q(\Omega)$.*

Proof. By (4.24) and the continuous injection $\mathcal{W}^{2,q}(\Omega) \hookrightarrow \mathcal{C}^{1,\delta}(\bar{\Omega})$ it is clear that $f_k \in \mathcal{C}^{1,\delta}(\bar{\Omega})$. To prove $\Delta f_k \in L^q(\Omega)$ it suffices to show that Δf_k can be expressed as a function of at most second order derivatives of $(n_i)_{0\le i\le k-1}$ and $(\psi_i)_{0\le i\le k-1}$, and the second order derivative of $(n_i)_{0\le i\le k-1}$ in Δf_k is linear. Since

$$\Delta f_k = \Delta\psi_{k-1} - \frac{1}{2}\Delta\left(\sum_{i=0}^{k-1} \nabla\psi_{k-1-i}.\nabla\psi_i\right) - \Delta\bar{h}_k((n_i)_{0\le i\le k-1})$$

and

$$\begin{aligned}\Delta\left(\nabla\psi_{k-1-i}.\nabla\psi_i\right) &= \nabla(\Delta\psi_{k-1-i}).\nabla\psi_i + \nabla(\Delta\psi_i).\nabla\psi_{k-1-i} \\ &\quad +2\sum_{l,j=1}^{d} \frac{\partial^2\psi_{k-1-i}}{\partial x_l \partial x_j}\frac{\partial^2\psi_i}{\partial x_l\partial x_j},\end{aligned}$$

by the assumptions and the regularity of $\bar{h}_k$, the problem is reduced to show that for all $k \ge 1$, $\Delta\psi_k$ can be expressed as a function of at most first order derivatives of $(n_i)_{0\le i\le k-1}$ and $(\psi_i)_{0\le i\le k-1}$, and the first order derivative of $(n_i)_{0\le i\le k-1}$ in $\Delta\psi_k$ is linear. Indeed, from (4.16) we have :

$$\Delta\psi_0 = \frac{1}{n_0}\nabla n_0.\nabla\psi_0.$$

Then the assertion is true for $k = 0$. Assume it is true for all $0 \le i \le k-1$. From (4.20) we obtain :

$$-\Delta\psi_k = \frac{1}{n_0}\sum_{i=0}^{k}(\nabla n_{k-i}.\nabla\psi_i) + \frac{1}{n_0}\sum_{i=0}^{k-1}(n_{k-i}\Delta\psi_i).$$

This shows the assertion for k and the result follows. □

Consequently, for all $0 \le k \le m$, the linear problem (4.28)-(4.29) has a unique solution $n_k \in \mathcal{W}^{2,q}(\Omega)$. Hence, (4.21) gives a unique $\phi_k \in \mathcal{C}^{1,\delta}(\bar{\Omega})$, and finally the linear equation (4.20) with the boundary condition for ψ_k in (4.23) defines a unique solution $\psi_k \in \mathcal{C}^{2,\delta}(\bar{\Omega})$. In summary, we have :

Theorem 4.1. *Let $m \in \mathbb{N}$ and the assumptions (A1)-(A5) hold. Then there exists a unique asymptotic expansion (4.11) up to order m, i.e., for all $0 \le k \le m$, there exists a unique profile $(n_k, \psi_k, \phi_k) \in B$, solution to the problem (4.16)-(4.19) if $k = 0$ or (4.20)-(4.23) if $1 \le k \le m$. Moreover, $n_0(x) \ge n_{min}$ for all $x \in \overline{\Omega}$.*

4.3 Justification of the asymptotic expansion

4.3.1 The main result

Let $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)$ be a smooth solution of (4.4)-(4.6), (4.9) and $(n^m_{a,\varepsilon}, \psi^m_{a,\varepsilon}, \phi^m_{a,\varepsilon})$ be approximate solution of order m defined by :

$$n^m_{a,\varepsilon} = \sum_{k=0}^{m} \varepsilon^k n_k, \quad \psi^m_{a,\varepsilon} = \sum_{k=0}^{m} \varepsilon^k \psi_k, \quad \phi^m_{a,\varepsilon} = \sum_{k=0}^{m} \varepsilon^k \phi_k, \tag{4.30}$$

where $(n_k, \psi_k, \phi_k)_{0 \le k \le m}$ is the unique solution of (4.16)-(4.19) for $k = 0$ and (4.20)-(4.23) for $1 \le k \le m$.

One of the goals of this paper is to show the following result.

Theorem 4.2. *Let $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)$ be the solution of the problem (4.4)-(4.6), (4.9) and $(n^m_{a,\varepsilon}, \psi^m_{a,\varepsilon}, \phi^m_{a,\varepsilon})$ be the approximate solution given by the asymptotic expansion (4.30). Let the assumptions (A1)-(A7) hold. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, we have the following estimates :*

$$\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} \le A_1 \varepsilon^{m+1}, \|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \le A_1 \varepsilon^{m+1}$$

and

$$\|\phi_\varepsilon - \phi^m_{a,\varepsilon}\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \le A_1 \varepsilon^{m+1},$$

where $A_1 > 0$ is a constant independent of ε.

4.3.2 Derivation of the problem on $(n_\varepsilon - n^m_{a,\varepsilon}, \psi_\varepsilon - \psi^m_{a,\varepsilon}, \phi_\varepsilon - \phi^m_{a,\varepsilon})$

First, since $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)$ is the solution of (4.4)-(4.6) and (4.9), we have in Ω :

$$-\mathrm{div}(n_\varepsilon \nabla \psi_\varepsilon) = 0, \tag{4.31}$$

$$\frac{\varepsilon}{2} |\nabla \psi_\varepsilon|^2 + h(n_\varepsilon) = \phi_\varepsilon + \varepsilon \psi_\varepsilon, \tag{4.32}$$

$$-\Delta \phi_\varepsilon = b(x) - n_\varepsilon \tag{4.33}$$

and

$$n_\varepsilon = \sum_{k=0}^{m} \varepsilon^k \bar{n}_k + n^{m+1}_{D,\varepsilon}, \quad \psi_\varepsilon = \sum_{k=0}^{m} \varepsilon^k \bar{\psi}_k + \psi^{m+1}_{D,\varepsilon} \quad \text{on } \Gamma. \tag{4.34}$$

Now we determine the system verified by the approximate solution $(n^m_{a,\varepsilon}, \psi^m_{a,\varepsilon}, \phi^m_{a,\varepsilon})$. Since

$$\mathrm{div}(n^m_{a,\varepsilon}\nabla\psi^m_{a,\varepsilon}) = \sum_{k=0}^{m}\varepsilon^k\Big(\sum_{i=0}^{k}\mathrm{div}(n_i\nabla\psi_{k-i})\Big) + \varepsilon^{m+1}D_1^\varepsilon,$$

where

$$D_1^\varepsilon = \sum_{k=m+1}^{2m}\Big(\varepsilon^{k-m-1}\sum_{i=k-m}^{m}\mathrm{div}(n_i\nabla\psi_{k-i})\Big), \tag{4.35}$$

from (4.16) and (4.20), we have :

$$-\mathrm{div}(n^m_{a,\varepsilon}\nabla\psi^m_{a,\varepsilon}) = -\varepsilon^{m+1}D_1^\varepsilon. \tag{4.36}$$

Similarly,

$$\frac{\varepsilon}{2}|\nabla\psi^m_{a,\varepsilon}|^2 + h(n^m_{a,\varepsilon}) = \phi^m_{a,\varepsilon} + \varepsilon\psi^m_{a,\varepsilon} - \varepsilon^{m+1}D_2^\varepsilon, \tag{4.37}$$

$$-\Delta\phi^m_{a,\varepsilon} = b(x) - n^m_{a,\varepsilon}, \tag{4.38}$$

$$n^m_{a,\varepsilon} = \sum_{k=0}^{m}\varepsilon^k\bar{n}_k, \quad \psi^m_{a,\varepsilon} = \sum_{k=0}^{m}\varepsilon^k\bar{\psi}_k \quad \text{on } \Gamma, \tag{4.39}$$

where

$$D_2^\varepsilon = -\frac{1}{2}\sum_{k=m}^{2m}\Big(\varepsilon^{k-m}\sum_{i=k-m}^{m}\nabla\psi_i.\nabla\psi_{k-i}\Big) - r_\varepsilon(n) + \psi_m \tag{4.40}$$

and

$$r_\varepsilon(n) = \frac{1}{(m+1)!}\frac{d^{m+1}h(n^m_{a,\xi})}{d\varepsilon^{m+1}} \quad \text{with } \xi \in [0,\varepsilon]. \tag{4.41}$$

By subtraction of the systems (4.31)-(4.34) and (4.36)-(4.39), we obtain in Ω :

$$-\mathrm{div}(n_\varepsilon\nabla\psi_\varepsilon) + \mathrm{div}(n^m_{a,\varepsilon}\nabla\psi^m_{a,\varepsilon}) = \varepsilon^{m+1}D_1^\varepsilon, \tag{4.42}$$

$$\begin{aligned}\frac{\varepsilon}{2}(|\nabla\psi_\varepsilon|^2 - |\nabla\psi^m_{a,\varepsilon}|^2) + h(n_\varepsilon) - h(n^m_{a,\varepsilon}) &= \phi_\varepsilon - \phi^m_{a,\varepsilon} + \varepsilon(\psi_\varepsilon - \psi^m_{a,\varepsilon}) \\ &\quad +\varepsilon^{m+1}D_2^\varepsilon, \end{aligned}\tag{4.43}$$

$$-\Delta(\phi_\varepsilon - \phi^m_{a,\varepsilon}) = -(n_\varepsilon - n^m_{a,\varepsilon}) \tag{4.44}$$

and

$$n_\varepsilon - n^m_{a,\varepsilon} = n^{m+1}_{D,\varepsilon}, \quad \psi_\varepsilon - \psi^m_{a,\varepsilon} = \psi^{m+1}_{D,\varepsilon} \quad \text{on } \Gamma. \tag{4.45}$$

Eliminating $\phi_\varepsilon - \phi^m_{a,\varepsilon}$ in (4.43)-(4.44), we obtain finally the following system in Ω :

$$\begin{aligned}\frac{\varepsilon}{2}\Delta(|\nabla\psi_\varepsilon|^2 - |\nabla\psi^m_{a,\varepsilon}|^2) + \Delta(h(n_\varepsilon) - h(n^m_{a,\varepsilon})) &= n_\varepsilon - n^m_{a,\varepsilon} + \varepsilon\Delta(\psi_\varepsilon - \psi^m_{a,\varepsilon}) \\ &\quad +\varepsilon^{m+1}\Delta D_2^\varepsilon, \end{aligned}\tag{4.46}$$

$$-\mathrm{div}((n_\varepsilon - n^m_{a,\varepsilon})\nabla\psi_\varepsilon + n^m_{a,\varepsilon}\nabla(\psi_\varepsilon - \psi^m_{a,\varepsilon})) = \varepsilon^{m+1}D_1^\varepsilon. \tag{4.47}$$

4.3.3 Some preliminary results

In what follows, $C_i > 0$ $(i \geq 1)$ denote different constants independent of ε. We give some preliminary results for the proof of Theorem 4.2. Lemma 4.2 is a direct consequence of the existence and uniqueness of solutions to the system (4.31)-(4.33) with Dirichlet data bounded in $\mathcal{W}^{2,q}(\Omega)\times\mathcal{C}^{2,\delta}(\bar{\Omega})$. Its proof can be found in [80]. The system (4.46)-(4.47) contains also the third order derivatives which can be treated in a similar way as that for Δf_k (Lemma 4.3). The key estimate is given in Lemma 4.4 These results allow us to justify rigorously the asymptotic expansion (4.11) in B.

Lemma 4.2. *Under the assumptions (A1)-(A7), there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in [0, \varepsilon_1]$ the problem (4.31)-(4.34) has a unique solution $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon) \in B$ with $n_\varepsilon \geq n_{min} > 0$. Furthermore, the sequence of solutions $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)_{\varepsilon>0}$ is bounded in B.*

Lemma 4.3. *Assume that $(n_i, \psi_i) \in \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega})$ for all $0 \leq i \leq m$. Then the sequences $(\nabla D_1^\varepsilon)_{\varepsilon>0}$ and $(\Delta D_2^\varepsilon)_{\varepsilon>0}$ are bounded in $L^q(\Omega)$.*

Proof. We only prove Lemma 4.3 for the sequence $(\nabla D_1^\varepsilon)_{\varepsilon>0}$ since the proof is similar for the sequence $(\Delta D_2^\varepsilon)_{\varepsilon>0}$. By the definition of D_1^ε in (4.35), we have :

$$\nabla D_1^\varepsilon = \sum_{k=m+1}^{2m} \varepsilon^{k-m-1} \sum_{i=k-m}^{m} (\nabla n_i \Delta\psi_{k-i} + \nabla(\nabla n_i.\nabla\psi_{k-i}) + n_i \nabla(\Delta\psi_{k-i})),$$

in which the third order derivatives appear only in the terms $\nabla(\Delta\psi_{k-i})$ for $m+1 \leq k \leq 2m$ and $k-m \leq i \leq m$, i.e., in the terms of the form $\nabla(\Delta\psi_k)$ for $1 \leq k \leq m$. Furthermore, it is easy to check that the second order derivative of $(n_i)_{0\leq i\leq m}$ in ∇D_1^ε is linear. Since $(n_i, \psi_i) \in \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega})$ for all $0 \leq i \leq m$, from the proof of Lemma 4.1 we know that $\Delta\psi_k$ can be expressed as a function of at most first order derivatives of $(n_i)_{0\leq i\leq k-1}$ and $(\psi_i)_{0\leq i\leq k-1}$, and the first order derivative of $(n_i)_{0\leq i\leq k-1}$ in $\Delta\psi_k$ is linear. This shows that the sequence $(\nabla D_1^\varepsilon)_{\varepsilon>0}$ is bounded in $L^q(\Omega)$. □

Lemma 4.4. *Let $I_\varepsilon = \frac{1}{2}\Delta(|\nabla\psi_\varepsilon|^2 - |\nabla\psi_{a,\varepsilon}^m|^2)$. Under the assumption $(n_i, \psi_i) \in \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega})$ for all $0 \leq i \leq m$, there exists a constant $C > 0$ independent of ε such that :*

$$\|I_\varepsilon\|_{L^q(\Omega)} \leq C\Big(\|\psi_\varepsilon - \psi_{a,\varepsilon}^m\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} + \|n_\varepsilon - n_{a,\varepsilon}^m\|_{\mathcal{W}^{2,q}(\Omega)} + \varepsilon^{m+1}\Big). \tag{4.48}$$

Proof. From the relation :

$$\frac{1}{2}\Delta(|\nabla\psi|^2) = \nabla\psi.\nabla(\Delta\psi) + Q(\psi),$$

where $Q(\psi)$ is defined in (4.8), we obtain :

$$\begin{aligned} I_\varepsilon &= \nabla\psi_\varepsilon.\nabla(\Delta\psi_\varepsilon) - \nabla\psi^m_{a,\varepsilon}.\nabla(\Delta\psi^m_{a,\varepsilon}) + Q(\psi_\varepsilon) - Q(\psi^m_{a,\varepsilon}) \\ &= \nabla\psi^m_{a,\varepsilon}.\nabla(\Delta\psi_\varepsilon - \Delta\psi^m_{a,\varepsilon}) + \nabla(\psi_\varepsilon - \psi^m_{a,\varepsilon}).\nabla(\Delta\psi_\varepsilon) + Q(\psi_\varepsilon) - Q(\psi^m_{a,\varepsilon}). \end{aligned}$$

Noting (4.27) and $n_i \in \mathcal{W}^{2,q}(\Omega)$ $(0 \le i \le m)$, for $\varepsilon > 0$ small enough we have $n^m_{a,\varepsilon} > n_*$, where $n_* > 0$ depends only on n_{min} and m. It follows from (4.31) and (4.36) that :

$$\begin{aligned} \Delta\psi_\varepsilon &= -\frac{\nabla n_\varepsilon}{n_\varepsilon}\nabla\psi_\varepsilon, \\ \Delta\psi^m_{a,\varepsilon} &= -\frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}}\nabla\psi^m_{a,\varepsilon} + \varepsilon^{m+1}D^\varepsilon_1. \end{aligned}$$

Then,

$$\begin{aligned} \Delta\psi_\varepsilon - \Delta\psi^m_{a,\varepsilon} &= \frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}}.\nabla\psi^m_{a,\varepsilon} - \frac{\nabla n_\varepsilon}{n_\varepsilon}.\nabla\psi_\varepsilon - \varepsilon^{m+1}D^\varepsilon_1 \\ &= \frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}}.\nabla(\psi^m_{a,\varepsilon} - \psi_\varepsilon) + \Big(\frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}} - \frac{\nabla n_\varepsilon}{n_\varepsilon}\Big).\nabla\psi_\varepsilon - \varepsilon^{m+1}D^\varepsilon_1 \\ &= \frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}}.\nabla(\psi^m_{a,\varepsilon} - \psi_\varepsilon) + \nabla(\ln n^m_{a,\varepsilon} - \ln n_\varepsilon).\nabla\psi_\varepsilon - \varepsilon^{m+1}D^\varepsilon_1. \end{aligned}$$

This gives :

$$\begin{aligned} I_\varepsilon &= \nabla\psi^m_{a,\varepsilon}.\nabla\left[\frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}}.\nabla(\psi^m_{a,\varepsilon} - \psi_\varepsilon) + \nabla(\ln n^m_{a,\varepsilon} - \ln n_\varepsilon).\nabla\psi_\varepsilon - \varepsilon^{m+1}D^\varepsilon_1\right] \\ &\quad +\nabla(\psi^m_{a,\varepsilon} - \psi_\varepsilon).\nabla\Big(\frac{\nabla n_\varepsilon}{n_\varepsilon}.\nabla\psi_\varepsilon\Big) + Q(\psi_\varepsilon) - Q(\psi^m_{a,\varepsilon}). \end{aligned}$$

By Lemma 4.2 and Theorem 4.1, since $(n_\varepsilon)_{\varepsilon>0}$ and $(n^m_{a,\varepsilon})_{\varepsilon>0}$ are bounded in $\mathcal{W}^{2,q}(\Omega)$ and $(\psi_\varepsilon)_{\varepsilon>0}$ and $(\psi^m_{a,\varepsilon})_{\varepsilon>0}$ are bounded in $\mathcal{C}^{2,\delta}(\bar\Omega)$, we deduce from the continuous injection from $\mathcal{W}^{2,q}(\Omega)$ to $\mathcal{C}^{1,\delta}(\bar\Omega)$ that :

$$\|Q(\psi_\varepsilon) - Q(\psi^m_{a,\varepsilon})\|_{L^q(\Omega)} \le C_1\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)}$$

and

$$\|\ln n_\varepsilon - \ln n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} \le C_2\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)}.$$

Hence,

$$\|I_\varepsilon\|_{L^q(\Omega)} \le C_3\Big(\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} + \|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} + \varepsilon^{m+1}\|\nabla D^\varepsilon_1\|_{L^q(\Omega)}\Big).$$

This ends the proof of Lemma 4.4, by using Lemma 4.3 □

4.3.4 Proof of Theorem 4.2

Let

$$F_\varepsilon = I_\varepsilon - \Delta(\psi_\varepsilon - \psi^m_{a,\varepsilon}) - \varepsilon^m \Delta D_2^\varepsilon$$

Then the equation (4.46) can be written as :

$$-\Delta(h(n_\varepsilon) - h(n^m_{a,\varepsilon})) + (n_\varepsilon - n^m_{a,\varepsilon}) = \varepsilon F_\varepsilon. \tag{4.49}$$

By Lemmas 4.3 and 4.4, we have :

$$\|F_\varepsilon\|_{L^q(\Omega)} \leq C_4\Big(\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} + \|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} + \varepsilon^m\Big).$$

Moreover, under the assumption (A6), the Lemmas 9.15 and 9.17 in [56] applied to the equation (4.49) and the boundary condition (4.45) give :

$$\|h(n_\varepsilon) - h(n^m_{a,\varepsilon})\|_{\mathcal{W}^{2,q}(\Omega)} \leq C_5\Big(\varepsilon\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} + \varepsilon\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} + \varepsilon^{m+1}\Big).$$

Noting that h is smooth and strictly increasing, we have :

$$C_6\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} \leq \|h(n_\varepsilon) - h(n^m_{a,\varepsilon})\|_{\mathcal{W}^{2,q}(\Omega)}.$$

We conclude that for ε small enough :

$$\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} \leq C_7\Big(\varepsilon\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} + \varepsilon^{m+1}\Big). \tag{4.50}$$

Next, we rewrite the equation (4.46) under the form :

$$-\Delta(\psi_\varepsilon - \psi^m_{a,\varepsilon}) - \frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}}.\nabla(\psi_\varepsilon - \psi^m_{a,\varepsilon}) = g_\varepsilon, \tag{4.51}$$

where

$$g_\varepsilon = \frac{1}{n^m_{a,\varepsilon}}\mathrm{div}\Big((n_\varepsilon - n^m_{a,\varepsilon})\nabla\psi_\varepsilon\Big) + \frac{\varepsilon^{m+1}}{n^m_{a,\varepsilon}}D_1^\varepsilon.$$

From (4.50) and the continuous injection from $\mathcal{W}^{2,q}(\Omega)$ to $\mathcal{C}^{1,\delta}(\bar\Omega)$, we have :

$$\|g_\varepsilon\|_{C^{0,\delta}(\bar\Omega)} \leq C_8\Big(\varepsilon\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} + \varepsilon^{m+1}\Big).$$

Hence, from the assumption (A7), Theorem 6.6 in [56] shows that the solution ψ_ε of the equation (4.51) associated to the boundary condition given in (4.45) satisfies :

$$\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} \leq C_9\Big(\varepsilon\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} + \varepsilon^{m+1}\Big).$$

We deduce that, for all ε small enough, (for example $\varepsilon \leq 1/2C_9$) :

$$\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \leq C_{10}\varepsilon^{m+1},$$

which yields from (4.50) :

$$\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} \leq C_{11}\varepsilon^{m+1}.$$

Finally, (4.43) gives :

$$\|\phi_\varepsilon - \phi^m_{a,\varepsilon}\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \leq C_{12}\varepsilon^{m+1}.$$

This completes the proof of Theorem 4.2. □

4.4 Zero relaxation time limit

In this section we deal with the zero relaxation time limit $\tau \to 0$ in the system (4.4)-(4.6). We present the results and omit the proofs since they are similar to those of Sections 4.2, 4.3. To simplify the presentation, we make the following change of variable :

$$n_\tau = n, \ \psi_\tau = \frac{\psi}{\tau}, \ \phi_\tau = \phi.$$

Then from (4.4)-(4.6), $(n_\tau, \psi_\tau, \phi_\tau)$ satisfies :

$$-\mathrm{div}(n_\tau \nabla \psi_\tau) = 0, \tag{4.52}$$

$$\frac{\varepsilon}{2}\tau^2|\nabla\psi_\tau|^2 + h(n_\tau) = \phi_\tau + \varepsilon\psi_\tau, \tag{4.53}$$

$$-\Delta\phi_\tau = b(x) - n_\tau. \tag{4.54}$$

We associate to this system the following Dirichlet boundary conditions :

$$n_\tau = \sum_{k=0}^{m} \tau^{2k}\bar{n}_k + n^{m+1}_{D,\tau}, \quad \psi_\tau = \sum_{k=0}^{m} \tau^{2k}\bar{\psi}_k + \psi^{m+1}_{D,\tau} \quad \text{on } \Gamma, \tag{4.55}$$

where $n^{m+1}_{D,\tau}$ and $\psi^{m+1}_{D,\tau}$ satisfy the following conditions :

(A6)' the sequence $(\tau^{-2(m+1)} n^{m+1}_{D,\tau})_{\tau>0}$ is bounded in $\mathcal{W}^{2,q}(\Omega)$.

(A7)' the sequence $(\tau^{-2(m+1)} \psi^{m+1}_{D,\tau})_{\tau>0}$ is bounded in $\mathcal{C}^{2,\delta}(\bar{\Omega})$.

Similar to Lemma 4.2, if the assumptions (A1)-(A5) and (A6)'-(A7)' hold, we can prove that there is a $\varepsilon_2 > 0$ independent of τ such that for all $\varepsilon \in (0, \varepsilon_2]$, the problem (4.52)-(4.55) has a unique solution $(n_\tau, \psi_\tau, \phi_\tau)$ in B with $n_\tau \geq n_{min} > 0$. Moreover, the sequences $(n_\tau)_{\tau>0}$, $(\psi_\tau)_{\tau>0}$ and $(\phi_\tau)_{\tau>0}$ are bounded in $\mathcal{W}^{2,q}(\Omega)$, $\mathcal{C}^{2,\delta}(\bar{\Omega})$ and $\mathcal{C}^{1,\delta}(\bar{\Omega})$, respectively. We refer to [80] for details.

4.4.1 Asymptotic expansion

In view of the system (4.52)-(4.54), it is natural to consider the ansatz defined by :

$$n_{a,\tau} = \sum_{k\geq 0} \tau^{2k} n_k, \quad \psi_{a,\tau} = \sum_{k\geq 0} \tau^{2k} \psi_k, \quad \phi_{a,\tau} = \sum_{k\geq 0} \tau^{2k} \phi_k. \tag{4.56}$$

Plugging (4.56) into the system (4.52)-(4.54) and comparing the orders in τ^2, we have :

The first order profile (n_0, ψ_0, ϕ_0) satisfies the nonlinear drift-diffusion system in Ω :

$$-\text{div}(n_0 \nabla \psi_0) = 0, \tag{4.57}$$

$$h(n_0) = \phi_0 + \varepsilon \psi_0, \tag{4.58}$$

$$-\Delta \phi_0 = b(x) - n_0, \tag{4.59}$$

with the following boundary conditions :

$$n_0 = \bar{n}_0, \quad \psi_0 = \bar{\psi}_0 \quad \text{on } \Gamma. \tag{4.60}$$

For all $k \geq 1$, (n_k, ψ_k, ϕ_k) satisfies the linearized drift-diffusion system in Ω :

$$-\text{div}(n_0 \nabla \psi_k) = \sum_{i=1}^{k} \text{div}(n_i \nabla \psi_{k-i}), \tag{4.61}$$

$$h'(n_0) n_k - \phi_k - \varepsilon \psi_k = -\frac{\varepsilon}{2} \sum_{i=0}^{k-1} \nabla \psi_{k-1-i} . \nabla \psi_i - \bar{h}_k((n_i)_{0\leq i\leq k-1}), \tag{4.62}$$

$$-\Delta \phi_k = -n_k, \tag{4.63}$$

with the boundary conditions :

$$n_k = \bar{n}_k, \quad \psi_k = \bar{\psi}_k \quad \text{on } \Gamma. \tag{4.64}$$

Now we show that each problem (4.57)-(4.60) and (4.61)-(4.64) for all $k \geq 1$ has a unique solution in B when the parameter $\varepsilon > 0$ is small enough. First, eliminating ϕ_0 in (4.58)-(4.59), we have :

$$\Delta h(n_0) - n_0 - \varepsilon \Delta \psi_0 = -b(x).$$

It follows from (4.57) that :

$$\Delta h(n_0) - n_0 - \frac{\varepsilon \nabla n_0}{n_0} . \nabla \psi_0 = -b(x). \tag{4.65}$$

The system (4.57), (4.65) is a simplified version of (4.4) and (4.7). Then applying the Schauder's fixed point Theorem, this system associated to the boundary conditions given in (4.60) admits a unique solution $(n_0, \psi_0) \in \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega})$ provided that $\varepsilon > 0$ is small enough (see [80]). Furthermore, the maximum principle implies :

$$n_0(x) \geq n_{min} > 0, \quad \forall\, x \in \overline{\Omega}.$$

Next, we determine a unique $\phi_0 \in \mathcal{W}^{2,q}(\Omega)$ from (4.58). This shows the existence and uniqueness of (n_0, ψ_0, ϕ_0). By induction and an analogous method used above, we show that the linear problem (4.61)-(4.64), for all $k \geq 1$, has a unique solution $(n_k, \psi_k, \phi_k) \in B$.

Hence, we have proved the following theorem.

Theorem 4.3. *Let $m \in \mathbb{N}$. Assume that the assumptions (A1)-(A5) hold. Then, there exists $\varepsilon_3 > 0$ and a unique asymptotic expansion (4.56) up to order m for all $\varepsilon \in (0, \varepsilon_3]$, i.e., for all $0 \leq k \leq m$, there exists a unique profile $(n_k, \psi_k, \phi_k) \in B$, solution of the problem (4.57)-(4.60) if $k = 0$ or (4.61)-(4.64) if $1 \leq k \leq m$. Moreover, $n_0(x) \geq n_{min}$ for all $x \in \overline{\Omega}$.*

4.4.2 Justification of the asymptotic expansion

Let us define $(n^m_{a,\tau}, \psi^m_{a,\tau}, \phi^m_{a,\tau})$ by :

$$n^m_{a,\tau} = \sum_{k=0}^{m} \tau^{2k} n_k, \quad \psi^m_{a,\tau} = \sum_{k=0}^{m} \tau^{2k} \psi_k, \quad \phi^m_{a,\tau} = \sum_{k=0}^{m} \tau^{2k} \phi_k, \tag{4.66}$$

where (n_0, ψ_0, ϕ_0) is the unique solution of the problem (4.57)-(4.60) and (n_k, ψ_k, ϕ_k) is the unique solution of (4.61)-(4.64) for all $1 \leq k \leq m$. Similar to the arguments used in Section 4.3, we obtain :

$$-\mathrm{div}(n^m_{a,\tau} \nabla \psi^m_{a,\tau}) = -\tau^{2(m+1)} E^\tau_1, \tag{4.67}$$

$$\frac{\varepsilon}{2}\tau^2 |\nabla \psi^m_{a,\tau}|^2 + h(n^m_{a,\tau}) = \phi^m_{a,\tau} + \varepsilon \psi^m_{a,\tau} - \tau^{2(m+1)} E^\tau_2, \tag{4.68}$$

$$-\Delta \phi^m_{a,\tau} = b(x) - n^m_{a,\tau}, \tag{4.69}$$

where

$$E^\tau_1 = \sum_{k=m+1}^{2m} \Big(\tau^{2(k-m-1)} \sum_{i=k-m}^{m} \mathrm{div}(n_i \nabla \psi_{k-i}) \Big),$$

$$E^\tau_2 = -\frac{\varepsilon}{2} \sum_{k=m}^{2m} \Big(\tau^{2(k-m)} \sum_{i=k-m}^{m} \nabla \psi_i . \nabla \psi_{k-i} \Big) - s_\tau(n),$$

$$s_\tau(n) = \frac{1}{(m+1)!}\frac{d^{m+1}h(n^m_{a,\xi})}{d(\tau^2)^{m+1}} \quad \text{with } \xi \in [0,\tau].$$

It follows that

$$-\text{div}(n_\tau \nabla\psi_\tau) + \text{div}(n^m_{a,\tau}\nabla\psi^m_{a,\tau}) = \tau^{m+1}E^\tau_1,$$

$$\frac{\varepsilon}{2}\tau^2(|\nabla\psi_\tau|^2 - |\nabla\psi^m_{a,\tau}|^2) + h(n_\tau) - h(n^m_{a,\tau}) = \phi_\tau - \phi^m_{a,\tau} + \varepsilon(\psi_\tau - \psi^m_{a,\tau}) + \tau^{m+1}E^\tau_2,$$

$$-\Delta(\phi_\tau - \phi^m_{a,\tau}) = -(n_\tau - n^m_{a,\tau})$$

in Ω with the boundary conditions

$$n_\tau - n^m_{a,\tau} = n^{m+1}_{D,\tau}, \quad \psi_\tau - \psi^m_{a,\varepsilon} = \psi^{m+1}_{D,\tau} \quad \text{on } \Gamma.$$

These equations are similar to (4.42)-(4.45). Like Lemma 4.3, we can show that the sequences $(\nabla E^\tau_1)_{\tau>0}$ and $(\Delta E^\tau_2)_{\tau>0}$ are bounded in $L^q(\Omega)$. Then from the assumptions (A6)' and (A7)', we obtain the following convergence result similar to Theorem 4.2.

Theorem 4.4. *Let $(n_\tau, \psi_\tau, \phi_\tau)$ be the solution of the problem (4.52)-(4.55) and $(n^m_{a,\tau}, \psi^m_{a,\tau}, \phi^m_{a,\tau})$ be the approximate solution given by (4.66). Under the assumptions (A1)-(A5) and (A6)'-(A7)', there exists $\varepsilon_4 > 0$ independent of $\tau \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_4]$ we have the following error estimates :*

$$\|n_\tau - n^m_{a,\tau}\|_{\mathcal{W}^{2,q}(\Omega)} \le A_2\tau^{2(m+1)}, \|\psi_\tau - \psi^m_{a,\tau}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} \le A_2\tau^{2(m+1)}$$

and

$$\|\phi_\tau - \phi^m_{a,\tau}\|_{\mathcal{C}^{1,\delta}(\bar\Omega)} \le A_2\tau^{2(m+1)},$$

where $A_2 > 0$ is a constant independent of ε and τ.

4.5 Convergence to the incompressible Euler equations

As applications of Theorems 4.2 and 4.4, we show in this section that when the boundary data are compatible with the function b, the velocity $u = -\nabla\psi$ in each limit satisfies the incompressible Euler equations, in which the pressures are determined as functions of the profiles (n_1, ψ_1, ϕ_1) in both cases. To see this property, we assume in what follows $b(x) \equiv 1$. From the discussion below it is easy to see that for general function b, the velocity u in the zero electron mass limit satisfies formally some compressible type Euler equations with given density determined by a nonlinear Poisson equation. However, the justification of this limit remains open in the time-dependent problem.

4.5.1 Via the zero electron mass limit

Let $(n_\varepsilon, u_\varepsilon, \phi_\varepsilon)$ be a smooth solution of the steady state Euler-Poisson system (4.1)-(4.3) with $\tau = \lambda = 1$. Then :

$$-\text{div}(n_\varepsilon u_\varepsilon) = 0, \tag{4.70}$$

$$\varepsilon\text{div}(n_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p(n_\varepsilon) = n_\varepsilon \nabla\phi_\varepsilon - \varepsilon n_\varepsilon u_\varepsilon, \tag{4.71}$$

$$-\Delta\phi_\varepsilon = 1 - n_\varepsilon. \tag{4.72}$$

For $n_\varepsilon > 0$, the equation (4.71) is equivalent to :

$$(u_\varepsilon.\nabla)u_\varepsilon + \frac{1}{\varepsilon}\nabla(h(n_\varepsilon) - \phi_\varepsilon) + u_\varepsilon = 0.$$

If we take the following ansatz :

$$\begin{aligned} n_\varepsilon &= 1 + \varepsilon n_1 + O(\varepsilon^2), \\ u_\varepsilon &= u_0 + \varepsilon u_1 + O(\varepsilon^2), \\ \phi_\varepsilon &= \phi_0 + \varepsilon\phi_1 + O(\varepsilon^2), \end{aligned}$$

then it is easy to see that $\phi_0 = h(1)$ and u_0 satisfies the incompressible Euler equations :

$$(u_0.\nabla)u_0 + u_0 + \nabla P = 0, \quad \text{div}u_0 = 0, \tag{4.73}$$

where the pressure P is defined by :

$$P = h'(1)n_1 - \phi_1. \tag{4.74}$$

This formal analysis can be easily extended to the transient Euler-Poisson system.

For the potential flow, if we take $\bar{n}_0 = 1$, then by Theorem 4.1 the problem (4.16)-(4.19) has a unique solution (n_0, ψ_0, ϕ_0) given by :

$$n_0 = 1, \ \phi_0 = h(1) \tag{4.75}$$

and

$$-\text{div}(\nabla\psi_0) = 0 \ \text{ in } \Omega, \quad \psi_0 = \bar{\psi}_0 \text{ on } \Gamma. \tag{4.76}$$

Hence, $u_0 = -\nabla\psi_0$ satisfies the incompressible Euler equations (4.73) with $P = h'(1)n_1 - \phi_1$, where (n_1, ψ_1, ϕ_1) is the unique solution of the problem (4.20)-(4.23) for $k = 1$. By Theorem 4.2, we have :

$$\|\psi_\varepsilon - \psi_0\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \leq A_1\varepsilon.$$

Then, the velocity $u_\varepsilon = -\nabla\psi_\varepsilon$ satisfies :

$$\|u_\varepsilon - u_0\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \leq A_1\varepsilon. \tag{4.77}$$

In summary, we have obtained :

Corollary 4.1. *Let $b(x) \equiv 1$, $\bar{n}_0(x) \equiv 1$ and the assumptions (A1)-(A7) hold for $m = 1$. Then the sequence of solutions $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)_{\varepsilon>0}$ of the problem (4.4)-(4.6) and (4.9) converges, as ε tends to 0, to (n_0, ψ_0, ϕ_0) in B, where (n_0, ψ_0, ϕ_0) is the unique solution of the problem (4.75)-(4.76). Moreover, $u_0 = -\nabla\psi_0$ satisfies the incompressible Euler equations (4.73) in which P is defined by (4.74) and (n_1, ψ_1, ϕ_1) is the unique solution of the problem (4.20)-(4.23) for $k = 1$. Furthermore, the estimate (4.77) holds.*

4.5.2 Via the zero relaxation time limit

Let (n, u, ϕ) be a smooth solution of the steady state Euler-Poisson system (4.1)-(4.3) with $\lambda = 1$. As above, we make the following change of variable :

$$n_\tau = n, \ u_\tau = \frac{u}{\tau}, \ \phi_\tau = \phi.$$

If $n_\tau > 0$, then $(n_\tau, u_\tau, \phi_\tau)$ satisfies :

$$-\mathrm{div}(n_\tau u_\tau) = 0, \tag{4.78}$$

$$\varepsilon\tau^2(u_\tau.\nabla)u_\tau + \nabla(h(n_\tau) - \phi_\tau) + \varepsilon u_\tau = 0, \tag{4.79}$$

$$-\Delta\phi_\tau = 1 - n_\tau. \tag{4.80}$$

If we take the following ansatz :

$$\begin{aligned} n_\tau &= 1 + \tau^2 n_1 + O(\tau^4), \\ u_\tau &= u_0 + \tau^2 u_1 + O(\tau^4), \\ \phi_\tau &= \phi_0 + \tau^2 \phi_1 + O(\tau^4), \end{aligned}$$

then it is easy to see that $\Delta\phi_0 = 0$ and u_0 satisfies the incompressible Euler equations :

$$(u_0.\nabla)u_0 + \nabla P = 0, \quad \mathrm{div} u_0 = 0, \tag{4.81}$$

where

$$\nabla P = \frac{1}{\varepsilon}\nabla(h'(1)n_1 - \phi_1) + u_1.$$

This formal analysis can also be extended to the transient Euler-Poisson equations.

For the potential flow, if we take $\bar{n}_0 = 1$, then the problem (4.57)-(4.60) has a unique solution (n_0, ψ_0, ϕ_0) given by :

$$n_0 = 1, \ \phi_0 = h(1) - \varepsilon\psi_0 \tag{4.82}$$

and

$$-\mathrm{div}(\nabla\psi_0) = 0 \ \text{ in } \Omega, \ \psi_0 = \bar{\psi}_0 \ \text{ on } \Gamma. \tag{4.83}$$

Then, $u_0 = -\nabla\psi_0$ satisfies the incompressible Euler equations (4.81) with

$$P = \frac{1}{\varepsilon}(h'(1)n_1 - \phi_1) - \psi_1, \tag{4.84}$$

where (n_1, ψ_1, ϕ_1) is the unique solution of the problem (4.61)-(4.64) for $k = 1$. By Theorem 4.4, we have :

$$\|\psi_\tau - \psi_0\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \leq A_2\tau^2.$$

Therefore, the velocity $u_\tau = -\nabla\psi_\tau$ satisfies :

$$\|u_\tau - u_0\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \leq A_2\tau^2. \tag{4.85}$$

In summary, we have obtained :

Corollary 4.2. *Let $b(x) \equiv 1$, $\bar{n}_0(x) \equiv 1$ and the assumptions (A1)-(A5) and (A6)'-(A7)' hold for $m = 1$. Then for all $\varepsilon \in (0, \varepsilon_4]$, the sequence of solutions $(n_\tau, \psi_\tau, \phi_\tau)_{\tau>0}$ of the problem (4.52)-(4.55) converges, as τ tends to 0, to (n_0, ψ_0, ϕ_0) in B, where (n_0, ψ_0, ϕ_0) is the unique solution of the problem (4.82)-(4.83). Moreover, $u_0 = -\nabla\psi_0$ satisfies the incompressible Euler equations (4.81) with P being defined by (4.84) and (n_1, ψ_1, ϕ_1) being the unique solution of the problem (4.61)-(4.64) for $k = 1$. Furthermore, the estimate (4.85) holds.*

Chapter 5

High order expansions in quasineutral limit of the Euler-Poisson system for a potential flow

This Chapter is an article accepted for publication in the Royal Society of Edinburgh Proceedings A (Mathematics) ([96]).

5.1 Introduction

The hydrodynamic model is widely used in mathematical modeling and numerical simulation for plasmas [23] and semiconductors [77]. It consists in two non linear equations given by the conservation laws of momentum and density, called Euler equations, plus a Poisson equation for the electrostatic potential. Due to the hyperbolicity of the transient non linear Euler equations, the weak solution is only studied in one space dimension. In such a situation, the existence of global weak solution is shown in the set of bounded functions [76].

In this paper we only consider the unipolar steady-state case for a potential flow. Then the Euler-Poisson system reads as follows (see [36, 80, 81] (Chapter 4 here)) :

$$\frac{\varepsilon}{2}|\nabla\psi|^2 + h(n) = \phi + \frac{\varepsilon\psi}{\tau}, \tag{5.1}$$

$$-\mathrm{div}(n\nabla\psi) = 0, \tag{5.2}$$

$$-\lambda^2\Delta\phi = b(x) - n. \tag{5.3}$$

This system will be studied in an open and bounded domain Ω in $\mathbb{R}^d$ (d=2 or d=3 in practice) with sufficiently smooth boundary Γ. Here $n = n(x), \psi = \psi(x), \phi = \phi(x)$

represent respectively the electron density, the velocity potential and the electrostatic potential. The function $h = h(n)$ is the enthalpy for the system and is defined by :

$$h'(n) = \frac{p'(n)}{n}, \ n > 0, \ \text{and} \ h(1) = 0,$$

where $p = p(n)$ is the pressure function, supposed to be sufficiently smooth and strictly increasing for $n > 0$. The function $b = b(x)$ represents the doping profile for a semiconductor and the ion density for a plasma. The parameters λ, ϵ, τ represent respectively the scaled Debye length, electron mass and relaxation time of the system. They are dimensionless and small compared to the characteristic length of physical interest. Then it is important to study the limits $\lambda \to 0, \epsilon \to 0, \tau \to 0$. In [81] (Chapter 4) we used asymptotic expansions to study the zero-electron-mass limit and the zero-relaxation-time limit and to show the convergence of the Euler-Poisson system to the incompressible Euler equations. Here we study the quasineutral limit $\lambda \to 0$. In all the following we take $\tau \equiv 1$ and we keep $\varepsilon > 0$ as a small parameter independent of λ in the equations.

By eliminating ϕ of (5.1) and (5.3) and using (5.2) we have :

$$\begin{aligned} \Delta h(n) &- \frac{\varepsilon}{n}\sum_{i,j=1}^{d} \psi_{x_i}\psi_{x_j}n_{x_ix_j} + \frac{\varepsilon}{n^2}(\nabla\psi.\nabla n)^2 + \frac{\varepsilon}{n}(\nabla\psi.\nabla n) \\ &- \frac{\varepsilon}{n}\sum_{i,j=1}^{d} \psi_{x_i}\psi_{x_ix_j}n_{x_j} - \frac{1}{\lambda^2}(n-b) + \varepsilon Q(\psi) = 0. \end{aligned} \tag{5.4}$$

where Q is given by :

$$Q(\psi) = \sum_{i,j=1}^{d} \psi^2_{x_ix_j}. \tag{5.5}$$

For $n > 0$, it is easy to see that (n, ψ, ϕ) is a smooth solution to the system (5.1)-(5.3) if and only if (n, ψ) is a smooth solution to (5.2) and (5.4). Moreover, for ψ given, the equation (5.4) is elliptic if and only if the flow is subsonic, i.e., the condition $|\nabla\psi| < \sqrt{p'(n)/\varepsilon}$ holds.

We supplement the system (5.1)-(5.3) by Dirichlet boundary conditions :

$$n = \sum_{j=0}^{m} \lambda^j n_D^j + n_{D,\lambda}^m, \ \psi = \sum_{j=0}^{m} \lambda^j \psi_D^j + \psi_{D,\lambda}^m, \ \text{on} \ \Gamma = \partial\Omega, \tag{5.6}$$

where $n_{D,\lambda}^m$ and $\psi_{D,\lambda}^m$ are smooth enough and defined on $\bar{\Omega}$.

The Euler-Poisson system and its asymptotic limits have been studied by a lot of authors. In [36] it is shown the existence and uniqueness of solutions for a potential flow under an assumption on the smallness of data, which implies that the problem is in the subsonic region. In [80] the author shows that the smallness condition corresponds to the smallness of ε. Then the existence and uniqueness hold for large data provided that ε is small enough.

The quasineutral limit has been studied in several special cases. In one-dimensional steady-state Euler-Poisson system it was performed in [95] for well-prepared boundary data. The steady problem in several space variables for a potential flow without the formation of boundary layers was investigated in [80]. In [29] the authors use pseudo-differential techniques to study this limit in transient Euler-Poisson model. The quasineutral limit has also been studied in the bipolar case in the drift-diffusion equations (see [51, 52, 67]). See also [14] for the study of this limit in a semi-linear Poisson equation in which the electron density is described by the Maxwell-Boltzmann relation. This relation is also used in [29].

The zero-electron-mass limit ($\varepsilon \to 0$) and the zero-relaxation-time limit ($\tau \to 0$) have also been studied a lot. See [1, 65, 81] for different results on these two limits. In [53], the authors study the combined relaxation-time limit and the vanishing Debye length.

This article is based on the method of asymptotic expansions presented in [83]. In [83] the justification of these asymptotic expansions is only given up to first order in the one-dimensional case. Here we will give their justification up to any order in the multidimensional case by using the Schauder fixed point Theorem. The main difficulty is to verify the assumptions in this theorem which are achieved by the Leray Schauder fixed point Theorem.

The paper is organized as follow. In Section 2, we give the formal asymptotic expansions and the systems verified by each boundary layer profiles under our assumptions. Section 3 is devoted to the justification up to any order of the asymptotic expansions.

5.2 Formal asymptotic expansions

In this section we study the formal asymptotic expansions of a solution to (5.1)-(5.3). We use for this the method of asymptotic expansions presented in [83]. We assume :

(H1) $b \in C^\infty(\bar\Omega),\ 0 < \underline{n} \leq b(x) \leq \bar n,\ x \in \bar\Omega,\ \underline{n},\ \bar n \in \mathbb{R}$,

(H2) $n_D^j \in C^\infty(\bar\Omega)$ for $0 \leq j \leq m$,

(H3) $\psi_D^j \in C^{2,\delta}(\bar\Omega)$ for $0 \leq j \leq m$,

(H4) $n_D^0(x) = b(x), n_D^1(x) = 0,\ x \in \bar\Omega$,

(H5) $(\lambda^{-m-1} n^m_{D,\lambda})_{\lambda>0}$ is bounded in $W^{2,q}(\Omega)$, $q > \frac{d}{1-\delta}$, $\delta \in (0,1)$,
(H6) $(\lambda^{-m+1} \psi^m_{D,\lambda})_{\lambda>0}$ is bounded in $C^{2,\delta}(\bar{\Omega})$.

The assumption (H4) is a compatibility condition for the first and second order terms. It assures that there will not appear any boundary layers in these two terms. The case without any compatibility conditions presents some difficulties which we didn't succeed in this study (see Remark 5.1).

5.2.1 Internal expansion

Let :

$$n(x) = \sum_{k\geq 0} \lambda^k n_k(x);\ \ \psi(x) = \sum_{k\geq 0} \lambda^k \psi_k(x);\ \ \phi(x) = \sum_{k\geq 0} \lambda^k \phi_k(x).$$

We inject this into (5.1)-(5.3) and obtain :

$$\frac{\varepsilon}{2}|\nabla\Big(\sum_{k\geq 0} \lambda^k \psi_k(x)\Big)|^2 + h\Big(\sum_{k\geq 0} \lambda^k n_k(x)\Big) = \sum_{k\geq 0} \lambda^k \phi_k(x) + \varepsilon \sum_{k\geq 0} \lambda^k \psi_k(x), \tag{5.7}$$

$$-\mathrm{div}\Big(\sum_{k\geq 0} \lambda^k n_k(x) \nabla\Big(\sum_{k\geq 0} \lambda^k \psi_k(x)\Big)\Big) = 0, \tag{5.8}$$

$$-\lambda^2 \Delta\Big(\sum_{k\geq 0} \lambda^k \phi_k(x)\Big) = b(x) - \sum_{k\geq 0} \lambda^k n_k(x),\ \text{ in } \Omega. \tag{5.9}$$

Formally,

$$\mathrm{div}\Big(\sum_{k\geq 0} \lambda^k n_k(x) \nabla\Big(\sum_{k\geq 0} \lambda^k \psi_k(x)\Big)\Big) = \sum_{k\geq 0} \lambda^k \Big(\sum_{i=0}^{k} \mathrm{div}(n_i \nabla \psi_{k-i})\Big),$$

$$|\nabla\Big(\sum_{k\geq 0} \lambda^k \psi_k(x)\Big)|^2 = \sum_{k\geq 0} \lambda^k \Big(\sum_{i=0}^{k} \nabla\psi_i . \nabla\psi_{k-i}\Big),$$

$$h\Big(\sum_{k\geq 0} \lambda^k n_k(x)\Big) = \sum_{k\geq 0} \lambda^k h_k(n),$$

where $n = (n_i)_{i\geq 0}$ and :

$$h_k(n) = \frac{1}{k!} \frac{d^k h\big(\sum_{k\geq 0} \lambda^k n_k\big)}{d\lambda^k}\Big|_{\lambda=0},\ \ k \geq 0.$$

As shown in [81] (Chapter 4): $h_k(n) = h'(n_0) n_k + \bar{h}_k((n_i)_{0\leq i\leq k-1})$, $k \geq 1$ where $\bar{h}_k$ is smooth and $\bar{h}_1 \equiv 0$. Then :

$$h(n_\lambda) = h(n_0) + \sum_{k\geq 1} \lambda^k h'(n_0) n_k + \sum_{k\geq 2} \lambda^k \bar{h}_k((n_i)_{0\leq i\leq k-1}).$$

Hence the system (5.7)-(5.9) becomes :

$$\frac{\varepsilon}{2}\sum_{k\geq 0}\lambda^k\left(\sum_{i=0}^{k}\nabla\psi_i.\nabla\psi_{k-i}\right)+h(n_0) \quad + \quad \sum_{k\geq 1}\lambda^k h'(n_0)n_k+\sum_{k\geq 2}\lambda^k\bar{h}_k((n_i)_{0\leq i\leq k-1})$$
$$= \quad \sum_{k\geq 0}\lambda^k\phi_k(x)+\varepsilon\sum_{k\geq 0}\lambda^k\psi_k(x), \tag{5.10}$$

$$-\sum_{k\geq 0}\lambda^k\left(\sum_{i=0}^{k}\mathrm{div}(n_i\nabla\psi_{k-i})\right)=0, \tag{5.11}$$

$$-\sum_{k\geq 2}\lambda^k\Delta\phi_{k-2}=b(x)-\sum_{k\geq 0}\lambda^k n_k. \tag{5.12}$$

We identify the order in λ and obtain the system verified by (n_k,ψ_k,ϕ_k) for all k. For $k=0$ we have :

$$\phi_0=-\frac{\varepsilon}{2}|\nabla\psi_0|^2-h(n_0)+\varepsilon\psi_0, \tag{5.13}$$

$$\mathrm{div}(n_0\nabla\psi_0)=0, \tag{5.14}$$

$$n_0=b(x). \tag{5.15}$$

For $k=1$ we have :

$$\phi_1=-\varepsilon\nabla\psi_0.\nabla\psi_1-h'(n_0)n_1+\varepsilon\psi_1, \tag{5.16}$$

$$-\mathrm{div}(n_0\nabla\psi_1)=\mathrm{div}(n_1\nabla\psi_0), \tag{5.17}$$

$$n_1=0. \tag{5.18}$$

And for $k\geq 2$:

$$\phi_k=-\frac{\varepsilon}{2}\sum_{i=0}^{k}\nabla\psi_i.\nabla\psi_{k-i}-h'(n_0)n_k-\bar{h}_k((n_i)_{0\leq i\leq k-1})+\varepsilon\psi_k, \tag{5.19}$$

$$-\mathrm{div}(n_0\nabla\psi_k)=\sum_{i=1}^{k}\mathrm{div}(n_i\nabla\psi_{k-i}), \tag{5.20}$$

$$n_k=\Delta\phi_{k-2}. \tag{5.21}$$

All the profiles (n_k,ψ_k,ϕ_k) can be determined, uniquely and sufficiently smooth, by induction on k with boundary conditions given later. First, we obtain n_0 by (5.15), then we have ψ_0 by (5.14) and ϕ_0 by (5.13). We use the same way for determining the solutions of (5.16)-(5.18) and (5.19)-(5.21). Then the internal solution is constructed. For $m\geq 2$ let us denote :

$$n^{\lambda}_{I,m}=\sum_{k=0}^{m}\lambda^k n_k;\ \psi^{\lambda}_{I,m}=\sum_{k=0}^{m}\lambda^k\psi_k;\ \phi^{\lambda}_{I,m}=\sum_{k=0}^{m}\lambda^k\phi_k.$$

By construction, if (n_k, ψ_k, ϕ_k) are smooth enough, then the error equations are of order $O(\lambda^{m+1})$. Since $n_k = \Delta\phi_{k-2}$, for $k \geq 2$, and is not necessarily equal to n_D^k on Γ, then a boundary layer can appear.

5.2.2 External expansion

We follow the notations in [46]. For $x \in \Omega$, we note $t(x)$ the distance from Γ to x and $s(x)$ the point of Γ nearest from x. For $\theta > 0$, let Ω_θ be the boundary layer of size θ :

$$\Omega_\theta = \{x \in \Omega; |x - y| < \theta,\ y \in \Gamma\}.$$

If θ is small enough, $s(x)$ is defined uniquely for all $x \in \Omega_\theta$. In Ω_θ, we define the fast variable by $\xi(x, \lambda) = t(x)/\lambda$. For $x \in \Omega_\theta$, let $\nu(x) = (\nu_1, ..., \nu_d)$ the unit interior-directional normal vector of Γ passing from x. Then from :

$$t(x) = \|x - s(x)\|,\ x - s(x) = t(x)\nu(x),$$

and due to the fact that for all $i = 1, ..., d,\ \partial s(x)/\partial x_i$ is orthogonal to $\nu(x)$, it is easy to see that $\nabla_x t = \nu(x)$. Hence the partial derivative of a function $w(s(x), \xi(x, \lambda))$ may be decomposed as :

$$\frac{\partial w(s(x), \xi(x, \lambda))}{\partial x_i} = \lambda^{-1}\nu_i \frac{\partial w}{\partial \xi} + D_i w, \tag{5.22}$$

where D_i is a first order differential operator in s defined by : $D_i w = \nabla_s w.\frac{\partial s}{\partial x_i}$. Similarly :

$$\frac{\partial^2 w(s(x), \xi(x, \lambda))}{\partial x_i \partial x_j} = \lambda^{-2}\nu_i \frac{\partial^2 w}{\partial \xi^2} + \lambda^{-1} D_{ji} \frac{\partial w}{\partial \xi} + D_j D_i w + \nabla_s w.\frac{\partial^2 s}{\partial x_i \partial x_j}, \tag{5.23}$$

where $D_{ji} = \nu_i D_j + \nu_j D_i + \partial\nu_i/\partial x_j$. Note that for all $i,\ j$ we have : $D_{ji} = D_{ij}$.

For every function $w(x)$ defined in Ω_θ the equivalent function of (s, t) is designated by $\tilde{w}$ i.e. $w(x) = \tilde{w}(s(x), t(x)) = \tilde{w}(s(x), \lambda\xi(x, \lambda))$. We develop $\tilde{w}(s(x), \lambda\xi(x, \lambda))$ formally to obtain :

$$\tilde{w}(s(x), \lambda\xi(x, \lambda)) = \tilde{w}(s(x), 0) + O(\lambda).$$

Let $\bar{w}(s) = \tilde{w}(s, 0)$. Then the ansatz of an approximate solution up to order m of (5.1)-(5.3) in Ω_θ are given by :

$$\tilde{n}^\lambda_{a,m}(x) = n^\lambda_{I,m}(x) + \tilde{n}^\lambda_{B,m}(s(x), \xi(x, \lambda)),$$
$$\tilde{\psi}^\lambda_{a,m}(x) = \psi^\lambda_{I,m}(x) + \tilde{\psi}^\lambda_{B,m}(s(x), \xi(x, \lambda)),$$
$$\tilde{\phi}^\lambda_{a,m}(x) = \phi^\lambda_{I,m}(x) + \tilde{\phi}^\lambda_{B,m}(s(x), \xi(x, \lambda));$$

where the boundary layers $(\tilde{n}^{\lambda}_{B,m}, \tilde{\psi}^{\lambda}_{B,m}, \tilde{\phi}^{\lambda}_{B,m})$ have the expansions :

$$\tilde{n}^{\lambda}_{B,m} = \sum_{k=0}^{m} \lambda^k n^b_k,\ \tilde{\psi}^{\lambda}_{B,m} = \sum_{k=0}^{m+1} \lambda^k \psi^b_k,\ \tilde{\phi}^{\lambda}_{B,m} = \sum_{k=0}^{m} \lambda^k \phi^b_k,$$

in which each term $(n^b_k(s,\xi), \psi^b_k(s,\xi), \phi^b_k(s,\xi))$ will be chosen to decay exponentially when ξ tends to $+\infty$. They are determined by setting $(\tilde{n}^{\lambda}_{a,m}, \tilde{\psi}^{\lambda}_{a,m}, \tilde{\phi}^{\lambda}_{a,m})$ in (5.1)-(5.3) and by identification of the order in λ. Let $\partial/\partial\nu = \sum_{i=1}^{d} \nu_i \partial/\partial x_i$. After computation we obtain :

$$\psi^b_0 \equiv 0,$$

the system for $(n^b_0, \psi^b_1, \phi^b_0)$:

$$(S_0) \qquad \begin{cases} (\bar{n}_0 + n^b_0)\dfrac{\partial^2 \psi^b_1}{\partial \xi^2} + \left(\dfrac{\partial \bar{\psi}_0}{\partial \nu} + \dfrac{\partial \psi^b_1}{\partial \xi}\right)\dfrac{\partial n^b_0}{\partial \xi} = 0, \\ \dfrac{\varepsilon}{2}\left(\dfrac{\partial \psi^b_1}{\partial \xi}\right)^2 + \varepsilon \dfrac{\partial \psi^b_1}{\partial \xi}\dfrac{\partial \bar{\psi}_0}{\partial \nu} + h(\bar{n}_0 + n^b_0) - \phi^b_0 = \bar{\phi}_0 + \varepsilon \bar{\psi}_0, \\ \dfrac{\partial^2 \phi^b_0}{\partial \xi^2} = n^b_0, \end{cases}$$

and the system for $(n^b_k, \psi^b_{k+1}, \phi^b_k)$ with $k \geq 1$:

$$(S_k) \qquad \begin{cases} \varepsilon \dfrac{\partial \psi^b_{k+1}}{\partial \xi}\left(\dfrac{\partial \psi^b_1}{\partial \xi} + \dfrac{\partial \bar{\psi}_0}{\partial \nu}\right) + h'(\bar{n}_0 + n^b_0) n^b_k - \phi^b_k = F_{1,k}(n^b_l, \psi^b_{l+1}, 0 \leq l \leq k-1), \\ (\bar{n}_0 + n^b_0)\dfrac{\partial^2 \psi^b_{k+1}}{\partial \xi^2} + \left(\dfrac{\partial \bar{\psi}_0}{\partial \nu} + \dfrac{\partial \psi^b_1}{\partial \xi}\right)\dfrac{\partial n^b_k}{\partial \xi} + n^b_k \dfrac{\partial^2 \psi^b_1}{\partial \xi^2} + \dfrac{\partial n^b_0}{\partial \xi}\dfrac{\partial \psi^b_{k+1}}{\partial \xi} = \\ \qquad\qquad F_{2,k}(n^b_l, \psi^b_{l+1}, 0 \leq l \leq k-1), \\ -\dfrac{\partial^2 \phi^b_k}{\partial \xi^2} + n^b_k = F_{3,k}(\phi^b_l, k-1 \leq l \leq k-2), \end{cases}$$

where $F_{i,k},\ i = 1,2,3$, are given functions of $(n^b_l, \psi^b_{l+1})_{0 \leq l \leq k-1}$ for $F_{1,k},\ F_{2,k}$, and of $(\phi^b_l)_{k-1 \leq l \leq k-2}$ for $F_{3,k}$.

Hence the approximate solution is constructed in Ω_θ. To complete the definition of the approximate solution in $\bar{\Omega}$, let $\sigma \in C^\infty(0,\infty)$ be a smooth function such that $\sigma(t) = 1$ for $0 \leq t \leq \theta/2$ and $\sigma \equiv 0$ for $t \geq \theta$ and we define :

$$(n^{\lambda}_{B,m}(x), \psi^{\lambda}_{B,m}(x), \phi^{\lambda}_{B,m}(x)) =$$

$$\begin{cases} (\tilde{n}^{\lambda}_{B,m}(s(x), t(x)/\lambda), \tilde{\psi}^{\lambda}_{B,m}(s(x)t(x)/\lambda), \tilde{\phi}^{\lambda}_{B,m}(s(x)t(x)/\lambda))\sigma(t(x)), \quad \text{for } x \in \Omega_\theta, \\ 0, \quad \text{for } x \in \Omega - \Omega_\theta. \end{cases}$$

Then, $(n^{\lambda}_{B,m}, \psi^{\lambda}_{B,m}, \phi^{\lambda}_{B,m})$ has the same regularity as $(\tilde{n}^{\lambda}_{B,m}, \tilde{\psi}^{\lambda}_{B,m}, \tilde{\phi}^{\lambda}_{B,m})$. If each $(n^b_k(s,\xi), \psi^b_k(s,\xi), \phi^b_k(s,\xi))$ decays exponentially when ξ tends to $+\infty$ it is easy to see that the difference between $(n^\lambda_{B,m}, \psi^\lambda_{B,m}, \phi^\lambda_{B,m})$ and $(\tilde{n}^\lambda_{B,m}, \tilde{\psi}^\lambda_{B,m}, \tilde{\phi}^\lambda_{B,m})$ is uniform of order of $e^{-\mu/\lambda}$ for a constant $\mu > 0$.

Finally, the boundary conditions (5.6) give for $s \in \Gamma$:

$$n_0 = n_D^0, n_1 = n_D^1, n_0^b(s,0) = n_1^b(s,0) = 0, \bar{n}_k(s) + n_k^b(s,0) = n_D^k, k \geq 2, \tag{5.24}$$

$$\psi_0 = \psi_D^0, \psi_1 = \psi_D^1, \psi_2 = \psi_D^2, \psi_1^b(s,0) = \psi_2^b(s,0) = 0, \bar{\psi}_k(s) + \psi_k^b(s,0) = \psi_D^k, k \geq 3. \tag{5.25}$$

We refer to [83] for the scheme of determination of $(n_k, \psi_k, \phi_k, n_k^b, \psi_{k+1}^b, \phi_k^b)$. We can show under the assumption (H4) :

$$n_0^b = n_1^b = \psi_1^b = \psi_2^b = 0.$$

The approximate solution up to order m is constructed in the form :

$$(n^a_\lambda, \psi^a_\lambda, \phi^a_\lambda) = (n^\lambda_{I,m} + n^\lambda_{B,m}, \psi^\lambda_{I,m} + \psi^\lambda_{B,m}, \phi^\lambda_{I,m} + \phi^\lambda_{B,m}), \text{ in } \bar{\Omega}. \tag{5.26}$$

By construction : $n^a_\lambda = \sum_{k=0}^m \lambda^k n_D^k$, $\psi^a_\lambda = \sum_{k=0}^m \lambda^k \psi_D^k$ on Γ and :

$$n^a_\lambda = n_0 + \sum_{j=2}^m \lambda^j (n_j + n_j^b), \tag{5.27}$$

$$\psi^a_\lambda = \psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \sum_{j=3}^m \lambda^j(\psi_j + \psi_j^b) + \lambda^{m+1}\psi^b_{m+1}. \tag{5.28}$$

The existence and uniqueness of boundary layers (n_k^b, ψ_{k+1}^b) with exponential decay have been shown in [83] for each $k \geq 0$. Thus we obtain,

Theorem 5.1. *Under the assumptions (H1)-(H6), there exists a unique asymptotic expansion (5.26) up to order m, sufficiently smooth, satisfying (5.27)-(5.28).*

5.3 Justification of the quasineutral limit

We have seen in the introduction that (5.1)-(5.3), (5.6) is equivalent to (5.1)-(5.2), (5.4), (5.6). The main result of this paper is :

Theorem 5.2. *Under the assumption (H1)-(H6), for λ small enough, there is an $\varepsilon_0 > 0$ independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$, the problem (5.2), (5.4), (5.6) has a solution $(n_\lambda, \psi_\lambda) \in W^{2,q}(\Omega) \times C^{2,\delta}(\bar{\Omega})$ which satisfies :*

$$\|n_\lambda - n^a_\lambda\|_{W^{2,q}(\Omega)} \leq A\lambda^{m-1}, \ \|\psi_\lambda - \psi^a_\lambda\|_{C^{2,\delta}(\bar{\Omega})} \leq A\lambda^{m-1}, \tag{5.29}$$

where A is a constant independent of λ.

Remark 5.1. *Using equation (5.1), the continuity of h and the estimates from Theorem 5.2, we can easily show, for λ small enough :*

$$\|\phi_\lambda - \phi_\lambda^a\|_{C^{1,\delta}(\bar\Omega)} \le A\lambda^{m-1},$$

where A is a constant independent of λ.

Remark 5.2. *Without the assumption (H4), some boundary layers of order 0 and 1 appear, and we are not able to estimate $n_\lambda - n_\lambda^a$ and $\psi_\lambda - \psi_\lambda^a$ in the same spaces as in (5.29).*

Proof of Theorem 5.2 : We search a solution of the problem (5.2), (5.4), (5.6) of the form :

$$n_\lambda = n_\lambda^a + \lambda^{m-1} r_\lambda,\ \ \psi_\lambda = \psi_\lambda^a + \lambda^{m-1} p_\lambda,$$

where n_λ^a and ψ_λ^a are the expansions defined in (5.27)-(5.28). To this end we define the two following operators :

$$\begin{aligned} N(n,\psi) &:= L(n,\psi) - \frac{1}{\lambda^2}(n-b) + \varepsilon Q(\psi), \\ M(n,\psi) &:= -\mathrm{div}(n\nabla\psi), \end{aligned}$$

where :

$$\begin{aligned} L(n,\psi) := \Delta h(n) &- \frac{\varepsilon}{n}\sum_{i,j=1}^{3} \psi_{x_i}\psi_{x_j} n_{x_i x_j} + \frac{\varepsilon}{n}\nabla\psi.\nabla n + \frac{\varepsilon}{n^2}(\nabla\psi.\nabla n)^2 \\ &- \frac{\varepsilon}{n}\sum_{i,j=1}^{3} \psi_{x_i}\psi_{x_i x_j} n_{x_j}. \end{aligned}$$

Then the system (5.2),(5.4),(5.6) can be written as :

$$N(n_\lambda,\psi_\lambda)(x,\lambda) = 0, \tag{5.30}$$

$$M(n_\lambda,\psi_\lambda)(x,\lambda) = 0,\ \text{in } \Omega, \tag{5.31}$$

$$n_\lambda = \sum_{k=0}^{m} \lambda^k n_D^k + n_{D,\lambda}^m,\ \ \psi_\lambda = \sum_{k=0}^{m} \lambda^k \psi_D^k + \psi_{D,\lambda}^m,\ \text{on } \Gamma. \tag{5.32}$$

By construction it is easy to check that (see [83]) :

$$N(n_\lambda^a,\psi_\lambda^a) = O(\lambda^{m-1}),\quad M(n_\lambda^a,\psi_\lambda^a) = O(\lambda^m),\ \text{in } L^\infty(\Omega), \tag{5.33}$$

$$N(n_\lambda^a,\psi_\lambda^a) = O(\lambda^{m-2}),\quad M(n_\lambda^a,\psi_\lambda^a) = O(\lambda^{m-1}),\ \text{in } C^1(\bar\Omega), \tag{5.34}$$

uniformly with respect to λ.

We search now the system verified by (r_λ, p_λ) by replacing formally n_λ by $n^a_\lambda + \lambda^{m-1} r_\lambda$ and ψ_λ by $\psi^a_\lambda + \lambda^{m-1} p_\lambda$ in (5.30)-(5.32). From (5.31) we have :

$$-\text{div}[(n^a_\lambda + \lambda^{m-1} r_\lambda)\nabla p_\lambda] = \lambda^{-m+1}[-M(n^a_\lambda, \psi^a_\lambda) + \lambda^{m-1}\text{div}(r_\lambda \nabla \psi^a_\lambda)]. \tag{5.35}$$

Similarly, (5.30) gives :

$$\begin{aligned}
&\Delta H(r_\lambda, x, \lambda) - \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda} \sum_{i,j=1}^{3} \left(\frac{\partial \psi^a_\lambda}{\partial x_i} + \lambda^{m-1}\frac{\partial p_\lambda}{\partial x_i}\right)\left(\frac{\partial \psi^a_\lambda}{\partial x_j} + \lambda^{m-1}\frac{\partial p_\lambda}{\partial x_j}\right)\frac{\partial^2 r_\lambda}{\partial x_i \partial x_j} \\
&+ \frac{\varepsilon}{(n^a_\lambda + \lambda^{m-1} r_\lambda)^2}\Big[\lambda^{m-1}(\nabla(\psi^a_\lambda + \lambda^{m-1} p_\lambda).\nabla r_\lambda)^2 + 2\nabla(\psi^a_\lambda + \lambda^{m-1} p_\lambda).\nabla r_\lambda \times \\
&(\nabla \psi^a_\lambda.\nabla n^a_\lambda + \lambda^{m-1}\nabla p_\lambda.\nabla n^a_\lambda)\Big] + \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda}\nabla(\psi^a_\lambda + \lambda^{m-1} p_\lambda).\nabla r_\lambda \\
&- \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda}\sum_{i,j=1}^{3}\left(\frac{\partial \psi^a_\lambda}{\partial x_i} + \lambda^{m-1}\frac{\partial p_\lambda}{\partial x_i}\right)\left(\frac{\partial^2 \psi^a_\lambda}{\partial x_i \partial x_j} + \lambda^{m-1}\frac{\partial^2 p_\lambda}{\partial x_i \partial x_j}\right)\frac{\partial r_\lambda}{\partial x_j} \qquad (5.36) \\
&- \frac{1}{\lambda^2} r_\lambda (1 - \lambda^2 \beta(r_\lambda, x)) = \frac{-\lambda^{-m+1} n^a_\lambda}{n^a_\lambda + \lambda^{m-1} r_\lambda} N(n^a_\lambda, \psi^a_\lambda) - g_1(n^a_\lambda, \psi^a_\lambda, r_\lambda, p_\lambda) - g_2(n^a_\lambda, \psi^a_\lambda, r_\lambda)
\end{aligned}$$

with :

$$H(r_\lambda, x, \lambda) := r_\lambda \int_0^1 h'(n^a_\lambda(x) + \lambda^{m-1} r_\lambda t)dt,$$

$$\begin{aligned}
g_1(n^a_\lambda, \psi^a_\lambda, r_\lambda, p_\lambda) \;:=\; & -\frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda}\sum_{i,j=1}^{3}\left[\frac{\partial \psi^a_\lambda}{\partial x_i}\frac{\partial p_\lambda}{\partial x_j} + \frac{\partial p_\lambda}{\partial x_i}\left(\frac{\partial \psi^a_\lambda}{\partial x_j} + \lambda^{m-1}\frac{\partial p_\lambda}{\partial x_j}\right)\right]\frac{\partial^2 n^a_\lambda}{\partial x_i \partial x_j} \\
+ \; & \frac{\varepsilon}{(n^a_\lambda + \lambda^{m-1} r_\lambda)^2}\Big[\lambda^{m-1}(\nabla p_\lambda.\nabla n^a_\lambda)^2 + 2(\nabla \psi^a_\lambda.\nabla n^a_\lambda)(\nabla p_\lambda.\nabla n^a_\lambda)\Big] \\
+ \; & \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda}\nabla p_\lambda.\nabla n^a_\lambda - \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda}\left[\sum_{i,j=1}^{3}\frac{\partial \psi^a_\lambda}{\partial x_i}\frac{\partial n^a_\lambda}{\partial x_j}\frac{\partial^2 p_\lambda}{\partial x_i \partial x_j}\right. \\
+ \; & \left.\sum_{i,j=1}^{3}\frac{\partial p_\lambda}{\partial x_i}\frac{\partial n^a_\lambda}{\partial x_j}\left(\frac{\partial^2 \psi^a_\lambda}{\partial x_i \partial x_j} + \lambda^{m-1}\frac{\partial^2 p_\lambda}{\partial x_i \partial x_j}\right)\right] \\
+ \; & \varepsilon\left[\lambda^{m-1} Q(p_\lambda) + 2\sum_{i,j=1}^{3}\frac{\partial^2 \psi^a_\lambda}{\partial x_i \partial x_j}\frac{\partial^2 p_\lambda}{\partial x_i \partial x_j}\right],
\end{aligned}$$

$$g_2(n^a_\lambda, \psi^a_\lambda, r_\lambda) := -\frac{\varepsilon r_\lambda}{n^a_\lambda (n^a_\lambda + \lambda^{m-1} r_\lambda)^2}(\nabla \psi^a_\lambda.\nabla n^a_\lambda)^2 + \frac{\varepsilon r_\lambda}{n^a_\lambda + \lambda^{m-1} r_\lambda} Q(\psi^a_\lambda),$$

and,

$$\beta(r_\lambda, x) = \frac{1}{n^a_\lambda(x) + \lambda^{m-1} r_\lambda} \Delta h(n^a_\lambda(x)) - \frac{1}{\lambda^2 (n^a_\lambda(x) + \lambda^{m-1} r_\lambda)} (n^a_\lambda(x) - b(x)).$$

From (5.6) the boundary conditions associated to (5.35)-(5.36) are :

$$p_\lambda = \lambda^{-m+1} \psi^m_{D,\lambda}, \; r_\lambda = \lambda^{-m+1} n^m_{D,\lambda}, \text{ on } \Gamma. \tag{5.37}$$

Note that by definition, using the assumptions, the application $H(., x, \lambda) : \; r_\lambda \mapsto H(r_\lambda, x, \lambda)$ is continuous and strictly increasing and then invertible.

To prove Theorem 5.2, it suffices to prove the following Lemma.

Lemma 5.1. *Under the assumptions (H1)-(H6), for λ small enough, there is an $\varepsilon_0 > 0$ independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$, the problem (5.35)-(5.37) has a solution $(r_\lambda, p_\lambda) \in W^{2,q}(\Omega) \times C^{2,\delta}(\bar{\Omega})$ which satisfies :*

$$\|r_\lambda\|_{W^{2,q}(\Omega)} \leq A_1, \; \|p_\lambda\|_{C^{2,\delta}(\bar{\Omega})} \leq A_1,$$

where A_1 is a constant independent of λ.

Proof : Let $\sigma_\lambda \in S$ with :

$$S = \{\rho \in C^{1,\delta}(\bar{\Omega}); \; \|\rho\|_{C^{1,\delta}(\bar{\Omega})} \leq K\},$$

where K is a constant independent of λ which will be fixed later. Here S is a closed and convex set. We define the mapping $T : \sigma_\lambda \mapsto p_\lambda \mapsto r_\lambda$ by :
(A) the solution of :

$$\begin{aligned} &-\text{div}[(n^a_\lambda + \lambda^{m-1}\sigma_\lambda)\nabla p_\lambda] = f_1(\sigma_\lambda, \psi^a_\lambda, n^a_\lambda), \text{ in } \Omega, \\ &p_\lambda = \lambda^{-m+1} \psi^m_{D,\lambda}, \text{ on } \Gamma \end{aligned}$$

with

$$f_1(\sigma_\lambda, \psi^a_\lambda, n^a_\lambda) := \lambda^{-m+1} \Big[-M(n^a_\lambda, \psi^a_\lambda) + \lambda^{m-1} \text{div}(\sigma_\lambda \nabla \psi^a_\lambda) \Big];$$

(B) let $r_\lambda = G(v_\lambda, x, \lambda)$ where $G(., x, \lambda) = H^{-1}(., x, \lambda)$ with v_λ being solution of :

$$\Delta v_\lambda - \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\frac{\partial^2 v_\lambda}{\partial x_i \partial x_j}$$
$$-\frac{\varepsilon G''(H(\sigma_\lambda, x, \lambda), x, \lambda)}{(n^a_\lambda + \lambda^{m-1}\sigma_\lambda) G'(H(\sigma_\lambda, x, \lambda), x, \lambda)} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\frac{\partial \sigma_\lambda}{\partial x_i}\frac{\partial v_\lambda}{\partial x_j}$$
$$-\frac{\varepsilon}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\left(\frac{\partial G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{\partial x_j}\frac{\partial v_\lambda}{\partial x_i} + \frac{\partial G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{\partial x_i}\frac{\partial v_\lambda}{\partial x_j}\right)$$
$$-\frac{\varepsilon G''(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\frac{\partial H(\sigma_\lambda, x, \lambda)}{\partial x_i}\frac{\partial v_\lambda}{\partial x_j}$$
$$+\frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{(n^a_\lambda + \lambda^{m-1}\sigma_\lambda)^2}\left[\lambda^{m-1}(\nabla\psi_\lambda.\nabla\sigma_\lambda)\nabla\psi_\lambda + 2(\nabla\psi^a_\lambda.\nabla n^a_\lambda + \lambda^{m-1}\nabla p_\lambda.\nabla n^a_\lambda)\nabla\psi_\lambda\right].\nabla v_\lambda$$
$$+\frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda}\nabla\psi_\lambda.\nabla v_\lambda - \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial^2 \psi_\lambda}{\partial x_i \partial x_j}\frac{\partial v_\lambda}{\partial x_j}$$
$$-\frac{1}{\lambda^2}G(v_\lambda, x, \lambda)(1 - \lambda^2\beta(\sigma_\lambda, x)) = f_2(\sigma_\lambda, x, \lambda),\ x \in \Omega,$$
$$v_\lambda = H(\lambda^{-m+1} n^m_{D,\lambda}, x, \lambda),\ \text{on } \Gamma,$$

where we note :

$$G'(v_\lambda, x, \lambda) = \frac{\partial G}{\partial v_\lambda}(v_\lambda, x, \lambda),\ G''(v_\lambda, x, \lambda) = \frac{\partial^2 G}{\partial v_\lambda^2}(v_\lambda, x, \lambda),$$

$$\begin{aligned}
f_2(\sigma_\lambda, x, \lambda) &= -\lambda^{-m+1}\frac{n^a_\lambda}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda}N(n^a_\lambda, \psi^a_\lambda) - g_1(n^a_\lambda, \psi^a_\lambda, \sigma_\lambda, p_\lambda) \\
&- g_2(n^a_\lambda, \psi^a_\lambda, \sigma_\lambda) + \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\frac{\partial^2 G}{\partial x_i \partial x_j}(H(\sigma_\lambda, x, \lambda), x, \lambda) \\
&- \frac{\varepsilon}{(n^a_\lambda + \lambda^{m-1}\sigma_\lambda)^2}\left[\lambda^{m-1}(\nabla\psi_\lambda.\nabla\sigma_\lambda)\nabla\psi_\lambda + 2(\nabla\psi^a_\lambda.\nabla n^a_\lambda + \lambda^{m-1}\nabla p_\lambda.\nabla n^a_\lambda)\nabla\psi_\lambda\right] \\
&\quad .\nabla_x G(H(\sigma_\lambda, x, \lambda), x, \lambda) \\
&- \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda}\nabla\psi_\lambda.\nabla_x G(H(\sigma_\lambda, x, \lambda), x, \lambda) \\
&+ \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial^2 \psi_\lambda}{\partial x_i \partial x_j}\frac{\partial G}{\partial x_j}(H(\sigma_\lambda, x, \lambda), x, \lambda).
\end{aligned}$$

By construction n_λ^a and ψ_λ^a being sufficiently smooth, $f_1 \in C^{0,\delta}(\bar\Omega)$ and is bounded in $C^{0,\delta}(\bar\Omega)$ uniformly in λ. Moreover by assumption $n_0 \geq \underline{n} > 0$, hence for λ small enough we have : $n_\lambda^a + \lambda^{m-1}\sigma_\lambda \geq \frac{1}{2}\underline{n} > 0$. Finally, using the Theorem 6.6 in [56] we obtain that the problem (A) has a unique solution $p_\lambda \in C^{2,\delta}(\bar\Omega)$ and :

$$\|p_\lambda\|_{C^{2,\delta}(\bar\Omega)} \leq C(K),$$

with $C(K)$ a constant independent of λ. Therefore, there exists $\varepsilon_1 > 0$, independent of λ, such that for all $\varepsilon \in [0, \varepsilon_1]$, (B) is an elliptic problem.

Here, we cannot use the classical result used in [36] due to the fact that the function G depends not only on v_λ, but also on the variable x. The idea here is to use the Leray-Schauder fixed point Theorem to show first the existence and boundedness of solutions v_λ to (B). This implies the existence and boundedness of $r_\lambda = G(v_\lambda, x, \lambda)$ which are needed to prove that T is an application from S to S, and then to apply the Schauder fixed point Theorem. More precisely, we use the Leray-Schauder fixed point Theorem to prove the following Lemma.

Lemma 5.2. *Under the assumptions (H1)-(H6), for λ small enough, there is an $\varepsilon_0 > 0$, independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$, (B) has a unique solution $v_\lambda \in W^{2,q}(\Omega)$ which satisfies :*

$$\|v_\lambda\|_{W^{2,q}(\Omega)} \leq A_2, \ \text{with } A_2 \text{ independent of } \lambda, K.$$

Proof : As mention previously, to prove this lemma, we will use the Leray-Schauder fixed point theorem. Let $\tau \in [0,1]$ and $w \in W^{2,q}(\Omega)$. We define the application $\tilde T : W^{2,q}(\Omega) \times [0,1] \to C^{1,\delta}(\bar\Omega)$ by $(w, \tau) \mapsto v$ where v solves the problem :

$$\begin{aligned}
&\Delta v - \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j} \\
&- \frac{\tau\varepsilon G''(H(\sigma_\lambda, x, \lambda), x, \lambda)}{(n_\lambda^a + \lambda^{m-1}\sigma_\lambda) G'(H(\sigma_\lambda, x, \lambda), x, \lambda)} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\frac{\partial \sigma_\lambda}{\partial x_i}\frac{\partial w}{\partial x_j} \\
&- \frac{\tau\varepsilon}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\left(\frac{\partial G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{\partial x_j}\frac{\partial w}{\partial x_i} + \frac{\partial G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{\partial x_i}\frac{\partial w}{\partial x_j} \right) \\
&- \frac{\tau\varepsilon G''(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial \psi_\lambda}{\partial x_j}\frac{\partial H(\sigma_\lambda, x, \lambda)}{\partial x_i}\frac{\partial w}{\partial x_j} \\
&+ \frac{\tau\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{(n_\lambda^a + \lambda^{m-1}\sigma_\lambda)^2} \Big[\lambda^{m-1}(\nabla\psi_\lambda.\nabla\sigma_\lambda)\nabla\psi_\lambda + 2(\nabla\psi_\lambda^a.\nabla n_\lambda^a + \lambda^{m-1}\nabla p_\lambda.\nabla n_\lambda^a)\nabla\psi_\lambda \Big].\nabla w \\
&+ \frac{\tau\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda}\nabla\psi_\lambda.\nabla w - \frac{\tau\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i}\frac{\partial^2 \psi_\lambda}{\partial x_i \partial x_j}\frac{\partial w}{\partial x_j} \\
&- \frac{\tau}{\lambda^2} G(w, x, \lambda)(1 - \lambda^2\beta(\sigma_\lambda, x)) = \tau f_2(\sigma_\lambda, x, \lambda), \ \ x \in \Omega,
\end{aligned}$$

$$v = \tau H(\lambda^{-m+1} n_{D,\lambda}^m, x, \lambda), \text{ on } \Gamma.$$

This problem is elliptic and admits a unique solution for ε small enough, hence $\tilde{T}$ is well-defined. We have : $\tilde{T}(w, 0) = v_1$ where v_1 is a solution of the linear problem :

$$\Delta v_1 - \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial^2 v_1}{\partial x_i \partial x_j} = 0$$
$$v_1 = 0, \text{ on } \Gamma.$$

Then $v_1 = 0$ and hence $\tilde{T}(w, 0) = 0$. Furthermore, it is not difficult to check that $\tilde{T}$ is continuous and also compact (due to the compact injection from $W^{2,q}(\Omega)$ to $C^{1,\delta}(\bar{\Omega})$).

To achieve the proof of Lemma 5.2, we need to prove that if v_2 is a fixed point of $\tilde{T}$ then we have

$$\|v_2\|_{W^{2,q}(\Omega)} \leq A_3,$$

with A_3 being independent of λ, K. This is the statement of Lemma 5.3. Since its proof being technical, it is moved to Appendix.

Lemma 5.3. *Assuming (H1)-(H6), let v_2 be a fixed point of $\tilde{T}$. Then for λ small enough there is an $\varepsilon_0 > 0$, independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$:*

$$\|v_2\|_{W^{2,q}(\Omega)} \leq A_3, \ \forall \tau \in [0, 1],$$

where A_3 is a constant independent of λ and K.

Then by the Leray-Schauder fixed point theorem, for λ small enough, and $\varepsilon \leq \varepsilon_0$, ε_0 independent of λ, $\tilde{T}_1 = \tilde{T}(w, 1)$ has a fixed point $v_\lambda \in W^{2,q}(\Omega)$ such that :

$$\|v_\lambda\|_{W^{2,q}(\Omega)} \leq A_2, \text{ with } A_2 \text{ independent of } \lambda, K.$$

By definition of $\tilde{T}, v_\lambda$ is a solution to the problem (B). For the uniqueness of solution, we assume that there exist two solutions $v_{\lambda,1}, v_{\lambda,2}$, we subtract the two systems. Using Theorem 9.15 in [56], we show that the obtained system has the unique solution zero. This gives $v_{\lambda,1} = v_{\lambda,2}$, and then we obtain the uniqueness of solutions for (B). This completes the proof of Lemma 5.2.

Now using Lemma 5.2 and the injection $W^{2,q}(\Omega) \hookrightarrow C^{1,\delta}(\bar{\Omega})$, with $K = cA_2$, where c is the injection constant, we obtain $v_\lambda \in S$ which implies $r_\lambda \in S$ and then $T : S \to S$. To complete the verification of the assumptions in the Schauder fixed point Theorem for T we need to show that T is continuous and compact. This is the statement of the following Lemma.

Lemma 5.4. *The application T, as an application from $C^{1,\delta}(\bar{\Omega})$ to $C^{1,\delta}(\bar{\Omega})$, is continuous and compact.*

Proof :

We define $T = I \circ G(.,x,\lambda) \circ \chi \circ \gamma$ by :

$$I : W^{2,q}(\Omega) \hookrightarrow C^{1,\delta}(\bar{\Omega}), \text{ which is continuous and compact,}$$

$$\begin{aligned} G(.,x,\lambda) &: v_\lambda \mapsto r_\lambda, \text{ which is continuous by assumption,} \\ \gamma : C^{1,\delta}(\bar{\Omega}) &\longrightarrow C^{2,\delta}(\bar{\Omega}), \ \sigma_\lambda \mapsto p_\lambda, \text{ where } p_\lambda \text{ is solution of (A),} \\ \chi : C^{2,\delta}(\bar{\Omega}) &\longrightarrow W^{2,q}(\Omega), \ p_\lambda \mapsto v_\lambda, \text{ where } v_\lambda \text{ is solution of (B).} \end{aligned}$$

We have to show that χ and γ are continuous.
Let $p_{\lambda,1}$ and $p_{\lambda,2}$ be two solutions of (A). Then by subtraction of the two systems we obtain the following one :

$$\begin{aligned} &-\text{div}[(n_\lambda^a + \lambda^{m-1}\sigma_{\lambda,1})\nabla(p_{\lambda,1} - p_{\lambda,2})] = f, \text{ in } \Omega, \\ &p_{\lambda,1} - p_{\lambda,2} = 0, \text{ on } \Gamma, \end{aligned}$$

with :

$$f = f_1(\sigma_{\lambda,1}, \psi_\lambda^a, n_\lambda^a) - f_1(\sigma_{\lambda,2}, \psi_\lambda^a, n_\lambda^a) - \lambda^{m-1}\text{div}[(\sigma_{\lambda,1} - \sigma_{\lambda,2})\nabla p_{\lambda,2}].$$

It is clear that :

$$\|\lambda^{m-1}\text{div}[(\sigma_{\lambda,1} - \sigma_{\lambda,2})\nabla p_{\lambda,2}]\|_{C^{0,\delta}(\bar{\Omega})} \leq \tilde{C}\|\sigma_{\lambda,1} - \sigma_{\lambda,2}\|_{C^{1,\delta}(\bar{\Omega})},$$

since $\|p_{\lambda,2}\|_{C^{2,\delta}(\bar{\Omega})} \leq \tilde{C}$.
Moreover, by definition we have :

$$f_1(\sigma_{\lambda,1}, \psi_\lambda^a, n_\lambda^a) - f_1(\sigma_{\lambda,2}, \psi_\lambda^a, n_\lambda^a) = \text{div}[(\sigma_{\lambda,1} - \sigma_{\lambda,2})\nabla\psi_\lambda^a],$$

and by construction : $\|\psi_\lambda^a\|_{C^{2,\delta}(\bar{\Omega})} \leq \tilde{C}$. Then

$$\|f_1(\sigma_{\lambda,1}, \psi_\lambda^a, n_\lambda^a) - f_1(\sigma_{\lambda,2}, \psi_\lambda^a, n_\lambda^a)\|_{C^{0,\delta}(\bar{\Omega})} \leq \tilde{C}\|\sigma_{\lambda,1} - \sigma_{\lambda,2}\|_{C^{1,\delta}(\bar{\Omega})}.$$

Hence we obtain that :

$$\|f\|_{C^{0,\delta}(\bar{\Omega})} \leq \tilde{C}\|\sigma_{\lambda,1} - \sigma_{\lambda,2}\|_{C^{1,\delta}(\bar{\Omega})}.$$

Using Theorem 6.6 in [56]

$$\|p_{\lambda,1} - p_{\lambda,2}\|_{C^{2,\delta}(\bar{\Omega})} \leq \|f\|_{C^{0,\delta}(\bar{\Omega})}.$$

Hence :

$$\|p_{\lambda,1} - p_{\lambda,2}\|_{C^{2,\delta}(\bar{\Omega})} \leq \tilde{C}\|\sigma_{\lambda,1} - \sigma_{\lambda,2}\|_{C^{1,\delta}(\bar{\Omega})},$$

and the application γ is continuous from $C^{1,\delta}(\bar{\Omega})$ to $C^{2,\delta}(\bar{\Omega})$.

In a same way, with the pressure function p_λ smooth enough we obtain that χ is a continuous application from $C^{2,\delta}(\bar{\Omega})$ to $W^{2,q}(\Omega)$. Therefore, T is continuous. The application I being compact, we have that T is also compact. This completes the proof of Lemma 5.4.

Finally all the assumptions in the Schauder fixed point Theorem are satisfied. As a consequence, for λ small enough and $\varepsilon \leq \varepsilon_0$, ε_0 being independent of λ, T has a fixed point. This completes the proof of Lemma 5.1.

Theorem 5.3. *Assume (H1)-(H6). If in addition, $(\lambda^{-m-1}n^m_{D,\lambda})_{\lambda>0}$ is bounded in $W^{2,\infty}(\Omega)$, and $(\lambda^{-m-1}\psi^m_{D,\lambda})_{\lambda>0}$ is bounded in $W^{1,q}(\Omega)$. Then,*

$$\|n_\lambda - n^a_\lambda\|_{L^\infty(\Omega)} \leq A_4\lambda^{m+1}, \ \|\psi_\lambda - \psi^a_\lambda\|_{W^{1,q}(\Omega)} \leq A_4\lambda^{m+1},$$

where A_4 is a constant independent of λ.

Since the proof uses notations defined in the proof of Lemma 5.3, it is also moved in Appendix.

Appendix :

Proof of Lemma 5.3.

Let v_2 be a fixed point of $\tilde{T}$. Let :

$$
\begin{aligned}
\tilde{L}(\tau,\sigma_\lambda,n^a_\lambda,\psi^a_\lambda,p_\lambda)v_2 &:= -\Delta v_2 + \frac{\varepsilon G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{n^a_\lambda+\lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial\psi_\lambda}{\partial x_j}\frac{\partial^2 v_2}{\partial x_i\partial x_j}\\
&+\frac{\tau\varepsilon G''(H(\sigma_\lambda,x,\lambda),x,\lambda)}{(n^a_\lambda+\lambda^{m-1}\sigma_\lambda)G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial\psi_\lambda}{\partial x_j}\frac{\partial\sigma_\lambda}{\partial x_i}\frac{\partial v_2}{\partial x_j}\\
&+\frac{\tau\varepsilon}{n^a_\lambda+\lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial\psi_\lambda}{\partial x_j}\Big(\frac{\partial G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{\partial x_j}\frac{\partial v_2}{\partial x_i}\\
&\qquad+\frac{\partial G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{\partial x_i}\frac{\partial v_2}{\partial x_j}\Big)\\
&+\frac{\tau\varepsilon G''(H(\sigma_\lambda,x,\lambda),x,\lambda)}{n^a_\lambda+\lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial\psi_\lambda}{\partial x_j}\frac{\partial H(\sigma_\lambda,x,\lambda)}{\partial x_i}\frac{\partial v_2}{\partial x_j}\\
&-\frac{\tau\varepsilon G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{(n^a_\lambda+\lambda^{m-1}\sigma_\lambda)^2}\Big[\lambda^{m-1}(\nabla\psi_\lambda.\nabla\sigma_\lambda)\nabla\psi_\lambda+2(\nabla\psi^a_\lambda.\nabla n^a_\lambda\\
&\qquad+\lambda^{m-1}\nabla p_\lambda.\nabla n^a_\lambda)\nabla\psi_\lambda\Big].\nabla v_2\\
&-\frac{\tau\varepsilon G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{n^a_\lambda+\lambda^{m-1}\sigma_\lambda}\nabla\psi_\lambda.\nabla v_2+\frac{\tau\varepsilon G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{n^a_\lambda+\lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial^2\psi_\lambda}{\partial x_i\partial x_j}\frac{\partial v_2}{\partial x_j}.
\end{aligned}
$$

For $\varepsilon\in[0,\varepsilon_1]$, the differential operator $\tilde{L}$ is elliptic. By definition v_2 is solution of :

$$
\begin{aligned}
&-\tilde{L}(\tau,\sigma_\lambda,n^a_\lambda,\psi^a_\lambda,p_\lambda)v_2-\frac{\tau}{\lambda^2}G(v_2,x,\lambda)(1-\lambda^2\beta(\sigma_\lambda,x))=\tau f_2(\sigma_\lambda,x,\lambda),\ x\in\Omega,\\
&v_2=\tau H(\lambda^{-m+1}n^m_{D,\lambda},x,\lambda),\ \text{on }\Gamma. \qquad (5.38)
\end{aligned}
$$

Let

$$
\begin{aligned}
&f_3(\tau,\sigma_\lambda,x,\lambda):=f_2(\sigma_\lambda,x,\lambda)+\frac{1}{\lambda^2}G(B_\tau,x,\lambda)(1-\lambda^2\beta(\sigma_\lambda,x)),\\
&B_\tau:=\tau H(\lambda^{-m+1}n^m_{D,\lambda},x,\lambda).
\end{aligned}
$$

Then (5.38) can be rewritten as:

$$
\begin{aligned}
&\tilde{L}(\tau,\sigma_\lambda,n^a_\lambda,\psi^a_\lambda,p_\lambda)v_2+\frac{\tau}{\lambda^2}\big(G(v_2,x,\lambda)-G(B_\tau,x,\lambda)\big)(1-\lambda^2\beta(\sigma_\lambda,x))\\
&\qquad\qquad=-\tau f_3(\tau,\sigma_\lambda,x,\lambda),\ \text{in }\Omega,\\
&v_2=B_\tau,\ \text{on }\Gamma.
\end{aligned}
$$

We can show that for λ small enough,

$$
\frac{1}{2}\le 1-\lambda^2\beta(\sigma_\lambda,x)\le C_1,\ \forall x\in\Omega, \qquad (5.39)
$$

with C_1 a constant independent of λ and K. Using the result of [80] we show that, there exists $\varepsilon_2 > 0$, independent of λ, such that for all $\varepsilon \in [0, \varepsilon_2]$:

$$\int_\Omega z|z|^{q-2}\tilde{L}(\tau, \sigma_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda) z dx \geq 0, \quad \forall z \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega).$$

Let $u = v_2 - B_\tau$. We have $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and u is a solution of :

$$\begin{aligned} \tilde{L}(\tau, \sigma_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)u &+ \frac{\tau}{\lambda^2}\big(G(u + B_\tau, x, \lambda) - G(B_\tau, x, \lambda)\big)(1 - \lambda^2\beta(\sigma_\lambda, x)) \\ &= -\tau f_3(\tau, \sigma_\lambda, x, \lambda) - \tilde{L}(\tau, \sigma_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)B_\tau, \text{ in } \Omega \quad (5.40) \\ u &= 0, \text{ on } \Gamma. \end{aligned}$$

For λ small enough, the function $H(., x, \lambda)$ is strictly increasing and so is $G(., x, \lambda)$ by definition. Then there exists two constants C_2 and C_3 independent of λ, τ and K such that :

$$C_2 u^2 \leq \big(G(u+B_\tau, x, \lambda) - G(B_\tau, x, \lambda)\big)u = u^2 \int_0^1 G'(u+tB_\tau, x, \lambda)dt \leq C_3 u^2. \quad (5.41)$$

We multiply (5.40) by $u|u|^{q-2}$ and we integrate on Ω. Then using (5.39), (5.41), for λ small enough, by Hölder inequality,

$$\begin{aligned} \frac{C_2\tau}{2\lambda^2}\int_\Omega |u|^q dx &\leq -\int_\Omega \big(\tau f_3(\tau, \sigma_\lambda, x, \lambda) + \tilde{L}(\tau, \sigma_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)B_\tau\big)u|u|^{q-2}dx \\ &\leq \big(\tau\|f_3(\tau, \sigma_\lambda, x, \lambda)\|_{L^q(\Omega)} + \|\tilde{L}(\tau, \sigma_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)B_\tau\|_{L^q(\Omega)}\big)\|u\|_{L^q(\Omega)}^{q-1}. \end{aligned}$$

Then :

$$\|u\|_{L^q(\Omega)} \leq \frac{2\lambda^2}{C_2}\left(\|f_3(\tau, \sigma_\lambda, x, \lambda)\|_{L^q(\Omega)} + \frac{1}{\tau}\|\tilde{L}(\tau, \sigma_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)B_\tau\|_{L^q(\Omega)}\right).$$

For λ small enough, there exists $\varepsilon_3 > 0$ independent of λ such that for all $\varepsilon \in [0, \varepsilon_3]$:

$$\|f_3(\tau, \sigma_\lambda, x, \lambda)\|_{L^q(\Omega)} \leq C_4, \ \frac{1}{\tau}\|\tilde{L}(\tau, \sigma_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)B_\tau\|_{L^q(\Omega)} \leq C_5,$$

with C_4 and C_5 constants independent of λ, τ, K. Hence :

$$\|u\|_{L^q(\Omega)} \leq C_6\lambda^2,$$

with C_6 independent of λ, τ, K. Like in [80], we obtain, for λ small enough,

$$\|u\|_{W^{2,q}(\Omega)} \leq C_7,$$

with C_7 independent of λ, τ, K. Finally :

$$\|v_2\|_{W^{2,q}(\Omega)} \leq \|u\|_{W^{2,q}(\Omega)} + \|B\|_{W^{2,q}(\Omega)}.$$

This completes the proof of Lemma 5.3 with $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Proof of Theorem 5.3. In all the following, the constants C_i are independent of λ. It is clear that $u_\lambda = v_\lambda - B_1$, where $B_1 = B_\tau$ for $\tau = 1$, is solution of :

$$\begin{aligned} \tilde{L}(1, r_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)u_\lambda \quad &+ \quad \frac{1}{\lambda^2}\big(G(u_\lambda + B_1, x, \lambda) - G(B_1, x, \lambda)\big)(1 - \lambda^2\beta(r_\lambda, x)) \\ &= \quad -f_3(1, r_\lambda, x, \lambda) - \tilde{L}(1, r_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)B_1, \text{ in } \Omega \qquad (5.42) \\ u_\lambda \quad &= \quad 0, \text{ on } \Gamma. \end{aligned}$$

Since by assumption $(\lambda^{-m-1} n_{D,\lambda}^m)_{\lambda > 0}$ is bounded in $W^{2,\infty}(\Omega)$, we have :

$$\|f_3(1, r_\lambda, x, \lambda) + \tilde{L}(1, r_\lambda, n_\lambda^a, \psi_\lambda^a, p_\lambda)B_1\|_{L^\infty(\Omega)} \leq C_8.$$

Let :

$$\bar{u} = \frac{2\lambda^2 C_8}{C_2}, \text{ and } \underline{u} = -\frac{2\lambda^2 C_8}{C_3}.$$

We can show that $\bar{u}$ (resp. $\underline{u}$) is an upper-solution (resp. lower-solution) of (5.42). Hence,

$$\underline{u} \leq u_\lambda \leq \bar{u}, \text{ and, } \|u_\lambda\|_{L^\infty(\Omega)} \leq C_9\lambda^2.$$

Using the assumption on $n_{D,\lambda}^m$,

$$\|B_1\|_{L^\infty(\Omega)} \leq C_{10}\lambda^2.$$

Hence :

$$\|v_\lambda\|_{L^\infty(\Omega)} \leq \|u_\lambda\|_{L^\infty(\Omega)} + \|B_1\|_{L^\infty(\Omega)} \leq C_{11}\lambda^2.$$

Then, using the continuity of $H(., x, \lambda)$, we obtain

$$\|r_\lambda\|_{L^\infty(\Omega)} \leq C_{12}\lambda^2,$$

which gives the first estimate in Theorem 5.3. Using the equations satisfied by the boundary layers profiles, we can show that

$$M(n_\lambda^a, \psi_\lambda^a) = O(\lambda^{m+1}) \text{ in } W^{-1,q}(\Omega).$$

Then in a same way than in [80], using the assumption on $\psi_{D,\lambda}^m$, the boundedness of r_λ in $L^\infty(\Omega)$, $\psi_\lambda^a \in C^{2,\delta}(\bar{\Omega})$, (5.33), and the strict positivity of n_λ, for λ small enough, we have :

$$\|p_\lambda\|_{W^{1,q}(\Omega)} \leq C_{13}\lambda^2.$$

This completes the proof of Theorem 5.3.

Chapter 6

Numerical solutions of Euler-Poisson systems for the potential flows

This Chapter is an article, in collaboration with Claire Chainais-Hillairet and Yue-Jun Peng, submitted for publication.

6.1 Introduction

The Euler-Poisson system is widely used in mathematical modeling and numerical simulation for semiconductor devices [77, 78] and plasmas [23]. Here we are interested in the semiconductor model, which consists of two nonlinear equations given by the conservation laws of momentum and density for each species, called Euler equations, coupled to a Poisson equation for the electrostatic potential.

In the bipolar case the steady-state Euler-Poisson system reads :

$$\mathrm{div}(pu_p) = 0, \tag{6.1}$$

$$\mathrm{div}(pu_p \otimes u_p) + \nabla P_p(p) = -p\nabla\phi - \frac{pu_p}{\tau}, \tag{6.2}$$

$$\mathrm{div}(nu_n) = 0, \tag{6.3}$$

$$\varepsilon\mathrm{div}(nu_n \otimes u_n) + \nabla P_n(n) = n\nabla\phi - \varepsilon\frac{nu_n}{\tau}, \tag{6.4}$$

$$\lambda^2\Delta\phi = n - p - C, \qquad \text{in } \Omega, \tag{6.5}$$

where Ω is an open and bounded domain of $\mathbb{R}^d$ ($d = 2$ or $d = 3$ in practice) representing the geometry of the semiconductor device. The unknowns of the system are the electron and hole densities $n = n(x)$, $p = p(x)$, the electron and hole velocities $u_n = u_n(x)$, $u_p = u_p(x)$, and the electrostatic potential $\phi = \phi(x)$. The function $C = C(x)$ stands for the doping profile of the device. The pressure functions

$P_\alpha = P_\alpha(s),\ \alpha = n, p$ are supposed to be sufficiently smooth and strictly increasing for $s > 0$. In applications and throughout this paper, $P_n(s) = P_p(s) = P(s)$. Typically they are governed by the adiabatic law $P(s) = s^\gamma$ where $\gamma \geq 1$ is a constant. The physical parameters λ, ε and τ are respectively the Debye length, the ratio of the electron mass and hole mass, and the relaxation time. They are small compared to the characteristic lengths of physical interest. In the sequel, λ and τ are supposed to be equal to 1.

In this paper we consider the case of potential flows. Then, $\mathrm{rot}(u_\alpha) = 0,\ \alpha = n, p$, and there exist ψ_n and ψ_p such that $u_n = -\nabla\psi_n,\ u_p = -\nabla\psi_p$, where ψ_n and ψ_p are the electron and hole velocity potentials. Under these conditions and with $\lambda = \tau = 1$, the system (6.1)-(6.5) can be rewritten as (see [36]) :

$$-\mathrm{div}(p\nabla\psi_p) = 0, \tag{6.6}$$
$$H(p) + \frac{1}{2}|\nabla\psi_p|^2 = -\phi + \psi_p, \tag{6.7}$$
$$-\mathrm{div}(n\nabla\psi_n) = 0, \tag{6.8}$$
$$H(n) + \frac{\varepsilon}{2}|\nabla\psi_n|^2 = \phi + \varepsilon\psi_n, \tag{6.9}$$
$$\Delta\phi = n - p - C, \qquad \text{in } \Omega, \tag{6.10}$$

where $H = H(s)$ is the enthalpy function of the system defined by:

$$H'(s) = \frac{P'(s)}{s} \quad \text{and} \quad H(1) = 0.$$

In the unipolar case, the system is reduced to three equations (6.8)-(6.10) in which $p = p(x)$ can be neglected. The system (6.6)-(6.10) or (6.8)-(6.10) is in general supplemented with Dirichlet-Neumann boundary conditions on the densities and velocity potentials.

There are many works for the study of the Euler-Poisson system and its asymptotic limits. In the transient system, we refer to [76, 1] for the existence of solutions and to [65, 1] for the asymptotic limits. The numerical study of the Euler-Poisson system in the quasineutral limit case can be found in [31, 33, 32]. Finally, we mention that the zero relaxation time limit was justified in [65]. The limit equations are classical drift-diffusion models for which the numerical results were given in [19, 21].

In the case of steady-state unipolar model for potential flows the existence and local uniqueness of solutions were proved under a smallness condition on the boundary data, which implies that the problem is in the subsonic region (see [36]). It has been proved in [80] that this smallness condition can be replaced by a smallness condition on ε. The results can be extended to the bipolar case provided that ε and the boundary data for ψ_p are small. In [81, 96], (here [81] is Chapter 4 here), the asymptotic limits are performed by using a method of asymptotic expansions.

The goal of this paper is to develop numerical schemes to compute the solutions of the steady-state Euler-Poisson system for potential flows. In particular, we want to illustrate the smallness condition on ε for the existence of solutions to the problem (6.1)-(6.5). The main idea is to use iterative schemes to solve a system of linear partial differential equations for (ϕ, ψ_n, ψ_p) and nonlinear algebraic equations for (n, p) instead of solving a fully nonlinear system of partial differential equations. We present two numerical schemes of finite volume type with reconstruction of the gradients appearing in (6.7) or (6.9). They are based on the work in [42, 44, 45]. For the numerical analysis and simulation in the drift-diffusion equations we refer to [4, 16, 19, 21, 25, 28, 61, 68, 89] and the references therein.

In Section 2, we present our numerical schemes in the unipolar case. The numerical results in two space dimensions in the unipolar and bipolar cases are given in Section 3.

6.2 Presentation of the numerical schemes

In this section we construct numerical schemes to the system in the unipolar case. The corresponding schemes in the bipolar case are similar. Omitting the subscript n for simplicity, the set of equations (6.8)-(6.10) for the unipolar model can be rewritten as :

$$-\text{div}(n\nabla\psi) = 0, \tag{6.11}$$

$$H(n) + \frac{\varepsilon}{2}|\nabla\psi|^2 = \phi + \varepsilon\psi, \tag{6.12}$$

$$\Delta\phi = n - C, \qquad \text{in } \Omega. \tag{6.13}$$

From a theoretical point of view, to study this system one uses equation (6.11) and (6.13) to eliminate ϕ in (6.12) to obtain a system of two equations of unknowns (n, ψ), supplemented with Dirichlet boundary conditions. The resulting equation for n is

$$\begin{aligned} -\Delta H(n) &+ \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\frac{\partial^2 n}{\partial x_i\partial x_j} - \frac{\varepsilon}{n}\nabla\psi.\nabla n - \frac{\varepsilon}{n^2}(\nabla\psi.\nabla n)^2 \\ &+ \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial n}{\partial x_j} + n - b(x) = Q(\psi), \end{aligned} \tag{6.14}$$

where Q is given by

$$Q(\psi) = \sum_{i,j=1}^{d}\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)^2.$$

The existence and uniqueness of solutions to the system (6.11)-(6.14) can be proved under a smallness condition on the boundary data [36] or on ε [80] which ensures the strict ellipticity of the equation for n. When (n,ψ) are solved one obtains easily ϕ from (6.12). However, the equation (6.14) is fully nonlinear and coupled to ψ till its second derivatives, so that its numerical discretization is not an easy task. Note that the first and last equations in the system (6.11)-(6.13) are linear for (ψ,ϕ) and the second one is nonlinear only algebraically for n. This motivates us to make the following iterative scheme : for a given n^m $(m \geq 0)$, we first solve (ψ^m,ϕ^m) by :

$$-\mathrm{div}(n^m \nabla \psi^m) = 0, \tag{6.15}$$
$$-\Delta \phi^m = C - n^m, \qquad \text{in } \Omega, \tag{6.16}$$

subject to mixed Dirichlet-Neumann boundary conditions :

$$\phi^m = \bar{\phi}, \ \psi^m = \bar{\psi} \overset{def}{=} H(n^0) - \bar{\phi}, \quad \text{on } \Gamma_D, \tag{6.17}$$
$$\nabla \phi^m.\nu = \nabla \psi^m.\nu = 0, \quad \text{on } \Gamma_N, \tag{6.18}$$

where ν is the unit outward normal to $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$, with Γ_D being the ohmic contacts and Γ_N the insulating boundary segments. These boundary conditions are physically motivated in the case of a semiconductor (see [36]). In particular, condition (6.17) means that the system is in equilibrium for the first iteration. Then we compute n^{m+1} by the algebraic equation

$$H(n^{m+1}) + \frac{\varepsilon}{2}|\nabla \psi^m|^2 = \phi^m + \varepsilon \psi^m. \tag{6.19}$$

Equations (6.15) and (6.16) are of elliptic type (provided n^m remains positive). There are several numerical methods to solve this kind of equations (e.g. finite element method, mixed finite element method, finite volume methods...). Here, some finite volume schemes are used. The first scheme is "classical" with a two point discretization of the fluxes through the edges, see [44]. It leads to piecewise constant approximate solutions and needs to be completed by a reconstruction of the gradients $\nabla\psi^m$, necessary for the computation of n^{m+1} in (6.19). The second scheme is of mixed finite volume type as introduced by J. Droniou and R. Eymard in [42], in which the construction of the gradients is intrinsic.

6.2.1 Mesh and notations

First, we introduce some notations that are useful for both schemes. It concerns the mesh, the initial and boundary data.

A mesh of Ω is given by a family $\mathcal{T}$ of control volumes (open polygonal convex disjoint subsets of Ω), a family $\mathcal{E}$ of edges in 2-d (faces in 3-d) and a set $\mathcal{P}$ of points of Ω indexed by $\mathcal{T}$: $\mathcal{P} = (\mathbf{x}_K)_{K\in\mathcal{T}}$. For a control volume $K \in \mathcal{T}$ we denote by $m(K)$

the measure of K and $\mathcal{E}_K$ the set of edges of K. The (d-1)-dimensional measure of an edge σ is denoted $m(\sigma)$. In the case where $\sigma \in \mathcal{E}$ such that $\bar{\sigma} = \bar{K} \cap \bar{L}$ with K and L being two neighboring cells, we note $\sigma = K|L$.

The set of interior (resp. boundary) edges is denoted by $\mathcal{E}^{int}$ (resp. $\mathcal{E}^{ext}$), that is $\mathcal{E}^{int} = \{\sigma \in \mathcal{E}; \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}^{ext} = \{\sigma \in \mathcal{E}; \sigma \subset \partial\Omega\}$). We note $\mathcal{E}_D^{ext}$ (resp. $\mathcal{E}_N^{ext}$) the set of $\sigma \subset \Gamma_D$ (resp. $\sigma \subset \Gamma_N$). For all $K \in \mathcal{T}$, we note $\mathcal{E}_K^{ext} = \mathcal{E}_K \cap \mathcal{E}^{ext}$, $\mathcal{E}_{D,K}^{ext}$ (resp. $\mathcal{E}_{N,K}^{ext}$) the edges of K included in Γ_D (resp. Γ_N), and $\mathcal{E}_K^{int} = \mathcal{E}_K \cap \mathcal{E}^{int}$. Finally, for $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{x}_\sigma$ its barycenter and by $\nu_{K,\sigma}$ the exterior unit normal vector to σ.

Given an initial datum n^0 and boundary data $\bar{\phi}$, $\bar{\psi}$, their approximations on each control volume or on each boundary edge are denoted by

$$n_K^0 = \frac{1}{m(K)} \int_K n^0,$$
$$\bar{\phi}_\sigma = \frac{1}{m(\sigma)} \int_\sigma \bar{\phi},$$
$$\bar{\psi}_\sigma = \frac{1}{m(\sigma)} \int_\sigma \bar{\psi},$$

We also set

$$f_K^m = C_K - n_K^m, \quad \text{with} \quad C_K = \frac{1}{m(K)} \int_K C.$$

6.2.2 Classical finite volume scheme

Let us consider an admissible mesh of Ω given by $\mathcal{T}$, $\mathcal{E}$ and $\mathcal{P}$ which satisfy Definition 3.8 in [44]. We recall that the admissibility of $\mathcal{T}$ implies that the straight line between two neighboring centers of cells $(\mathbf{x}_K, \mathbf{x}_L)$ is orthogonal to the edge $\sigma = K|L$. Finally, let us define the transmissibility coefficients :

$$\tau_\sigma = \frac{m(\sigma)}{d(\mathbf{x}_K, \mathbf{x}_L)} \quad \text{if } \sigma = K|L \in \mathcal{E}_K^{int} \quad \text{and} \quad \tau_\sigma = \frac{m(\sigma)}{d(\mathbf{x}_K, \Gamma)} \quad \text{if } \sigma \in \mathcal{E}_K^{ext}, \tag{6.20}$$

and the size of the mesh :

$$h = \max_{K \in \mathcal{T}} \mathrm{diam}(K). \tag{6.21}$$

In all the sequel, we assume that the points $\mathbf{x}_K$ are located inside each control volume. Let $(\phi_K^m)_{K\in\mathcal{T}}$ and $(\psi_K^m)_{K\in\mathcal{T}}$ be the discrete unknowns. A finite volume scheme to the mixed Dirichlet-Neumann problem (6.15)-(6.18) is defined by the following set of equations (see [44]) :

$$-\sum_{\sigma\in\mathcal{E}_K} d\phi_{K,\sigma}^m = m(K) f_K^m, \tag{6.22}$$

$$-\sum_{\sigma\in\mathcal{E}_K} n_\sigma^m d\psi_{K,\sigma}^m = 0, \tag{6.23}$$

where

$$d\phi_{K,\sigma}^m = \begin{cases} \tau_\sigma(\phi_L^m - \phi_K^m),\ \sigma = K|L, \\ \tau_\sigma(\bar{\phi}_\sigma - \phi_K^m),\ \sigma \in \mathcal{E}_{D,K}^{ext}, \\ 0,\ \sigma \in \mathcal{E}_{N,K}^{ext}, \end{cases}$$

$$d\psi_{K,\sigma}^m = \begin{cases} \tau_\sigma(\psi_L^m - \psi_K^m),\ \sigma = K|L, \\ \tau_\sigma(\bar{\psi}_\sigma - \psi_K^m),\ \sigma \in \mathcal{E}_{D,K}^{ext}, \\ 0,\ \sigma \in \mathcal{E}_{N,K}^{ext}, \end{cases}$$

$$n_\sigma^m = \begin{cases} \dfrac{n_K^m + n_L^m}{2},\ \sigma = K|L, \\ n_K^m,\ \sigma \in \mathcal{E}_K^{ext}. \end{cases}$$

The quantities $d\phi_{K,\sigma}^m$ and $d\psi_{K,\sigma}^m$ are the approximations of the fluxes through each edge for each function i.e.

$$d\phi_{K,\sigma}^m \approx \int_\sigma \nabla\phi^m \cdot \nu_{K,\sigma} \quad \text{and} \quad d\psi_{K,\sigma}^m \approx \int_\sigma \nabla\psi^m \cdot \nu_{K,\sigma}.$$

For given n^m, since the equations (6.15)-(6.16) are linear, we obtain the piecewise constant functions ψ^m and ϕ^m, unique solution of (6.22)-(6.23). Then we need to define the gradient of ψ^m. Therefore, we use the reconstruction proposed in [45] ; the approximate gradient is a piecewise constant function, defined on each control volume by

$$\mathbf{w}_K^m = \frac{1}{m(K)} \sum_{\sigma\in\mathcal{E}_K} d\psi_{K,\sigma}^m(\mathbf{x}_\sigma - \mathbf{x}_K), \quad \forall K \in \mathcal{T}.$$

Finally, from (6.12) we obtain the piecewise constant function n^{m+1} by:

$$n_K^{m+1} = H^{-1}\Big(\phi_K^m + \varepsilon\psi_K^m - \frac{\varepsilon}{2}|\mathbf{w}_K^m|^2\Big), \tag{6.24}$$

with H^{-1} being the inverse function of H (the inversibility of H will be discussed later).

6.2.3 Mixed finite volume scheme

As seen previously, the definition of n^{m+1} through the algebraic equation (6.19) needs construction of an approximate gradient, which is not usual in classical finite volume schemes. However, the penalized mixed finite volume scheme introduced by J.Droniou and R.Eymard in [42] for an elliptic equation is a scheme whose unknowns are the function, its gradient on each control volume and the fluxes through each edge. Then the definition of the piecewise constant gradient of the solution is intrinsic. Furthermore this scheme can be used on very general meshes.

Let us consider a mesh of Ω given by $\mathcal{T}$, $\mathcal{E}$ and $\mathcal{P}$ which satisfy definition 2.1 in [42]. We denote by $(\phi_K^m)_{K\in\mathcal{T}}$ and $(\psi_K^m)_{K\in\mathcal{T}}$ the approximate values of ϕ and ψ, by $(\mathbf{v}_K^m)_{K\in\mathcal{T}}$ and $(\mathbf{w}_K^m)_{K\in\mathcal{T}}$ the approximate gradients of ϕ^m and ψ^m, respectively. Let $\xi = (\xi_K)_{K\in\mathcal{T}}$ be a family of small positive numbers and

$$F_{K,\sigma}^m \approx \frac{1}{m(\sigma)}\int_\sigma \nabla\phi^m\cdot\nu_{K,\sigma} \quad \text{and} \quad G_{K,\sigma}^m \approx \frac{1}{m(\sigma)}\int_\sigma \nabla\psi^m\cdot\nu_{K,\sigma}.$$

The penalized mixed finite volume scheme to the problems (6.15)-(6.16) and (6.17)-(6.18) can be written as (see [42]) :

$$\begin{aligned}
\mathbf{v}_K^m\cdot(\mathbf{x}_\sigma-\mathbf{x}_K)+\mathbf{v}_L^m\cdot(\mathbf{x}_\sigma-\mathbf{x}_L)+\xi_K m(K)F_{K,\sigma}-\xi_L m(L)F_{L,\sigma} &= \phi_L^m-\phi_K^m,\\
&\forall K\in\mathcal{M},\ \forall L\in\mathcal{N}_K \text{ with } \sigma=K|L,\\
\mathbf{v}_K^m\cdot(\mathbf{x}_\sigma-\mathbf{x}_K)+\xi_K m(K)F_{K,\sigma} = \bar{\phi}_\sigma-\phi_K^m,&\quad \forall K\in\mathcal{M},\ \forall\sigma\in\mathcal{E}_{D,K}^{ext},\\
F_{K,\sigma}+F_{L,\sigma}=0,\quad \forall\sigma=K|L\in\mathcal{E}^{int},&\quad F_{K,\sigma}=0\quad\forall\sigma\in\mathcal{E}_{N,K}^{ext},\\
m(K)\mathbf{v}_K^m = \sum_{\sigma\in\mathcal{E}_K}F_{K,\sigma}(\mathbf{x}_\sigma-\mathbf{x}_K),&\quad\forall K\in\mathcal{M},\\
-\sum_{\sigma\in\mathcal{E}_K}F_{K,\sigma} = m(K)f_K^m,&\quad\forall K\in\mathcal{M},
\end{aligned}$$

$$\begin{aligned}
\mathbf{w}_K^m\cdot(\mathbf{x}_\sigma-\mathbf{x}_K)+w_L^m\cdot(\mathbf{x}_\sigma-\mathbf{x}_L)+\xi_K m(K)G_{K,\sigma}-\xi_L m(L)G_{L,\sigma} &= \psi_L^m-\psi_K^m,\\
&\forall K\in\mathcal{M},\ \forall L\in\mathcal{N}_K \text{ with } \sigma=K|L,\\
\mathbf{w}_K^m\cdot(\mathbf{x}_\sigma-\mathbf{x}_K)+\xi_K m(K)G_{K,\sigma} = \bar{\psi}_\sigma-\psi_K^m,&\quad \forall K\in\mathcal{M},\ \forall\sigma\in\mathcal{E}_{D,K}^{ext},\\
G_{K,\sigma}+G_{L,\sigma}=0,\quad \forall\sigma=K|L\in\mathcal{E}^{int},&\quad G_{K,\sigma}=0\quad\forall\sigma\in\mathcal{E}_{N,K}^{ext},\\
m(K)n_K^m\mathbf{w}_K^m = \sum_{\sigma\in\mathcal{E}_K}G_{K,\sigma}(\mathbf{x}_\sigma-\mathbf{x}_K),&\quad\forall K\in\mathcal{M},\\
-\sum_{\sigma\in\mathcal{E}_K}G_{K,\sigma} = 0,&\quad\forall K\in\mathcal{M}.
\end{aligned}$$

Then n_K^{m+1} is given by (6.24).

In both schemes, the iterations on m are stopped when the difference between n^m and n^{m+1} is sufficiently small in $L^2(\Omega)$ or $L^\infty(\Omega)$ norm. Notice that for the penalized mixed finite volume scheme we have to solve two systems whose size is $3\mathrm{Card}(\mathcal{M}) + \mathrm{Card}(\mathcal{E})$. However it is possible to proceed to an algebraic elimination which leads for each one to a system of size $\mathrm{Card}(\mathcal{E}^{int})$, following the same principles as in the hybrid resolution presented in [42].

In all the sequel, the classical finite volume scheme is referred to the VF4-scheme and the penalized mixed finite volume scheme to the DE-scheme. The schemes for the bipolar case are almost the same. The only difference is that there are three linear equations (instead of two) and two nonlinear algebraic equations (instead of one), which can be solved exactly with the same methods.

6.3 Numerical results

We perform the numerical simulations in two space dimensions by taking the domain $\Omega = [0,1] \times [0,1]$. A point $\mathbf{x}$ of Ω is denoted by its coordinates $\mathbf{x} = (x_1, x_2)$. Let us define $\Gamma_N = \{(x_1,x_2), x_1 \in [0,1], x_2 \in \{0,1\}\}$ and $\Gamma_D = \Gamma_{D,l} \cup \Gamma_{D,r}$ with

$$\begin{aligned} \Gamma_{D,l} &= \{(x_1,x_2), x_1 = 0, x_2 \in [0,1]\}, \\ \Gamma_{D,r} &= \{(x_1,x_2), x_1 = 1, x_2 \in [0,1]\}. \end{aligned}$$

The pressure function is taken to be $P(s) = s^\gamma$ with $\gamma = 1$ or $5/3$, which implies for the enthalpy :

$$H(s) = \begin{cases} \ln(s), & \text{if} \quad \gamma = 1, \\ \dfrac{5}{2}(s^{2/3} - 1), & \text{if} \quad \gamma = 5/3. \end{cases}$$

In the case $\gamma = 1$, H is defined from $]0,+\infty[$ to $\mathbb{R}$ and admits an inverse function on all $\mathbb{R}$ defined by $H^{-1}(t) = \exp(t)$. In the case $\gamma = 5/3$, H is only defined from $]0,+\infty[$ to $]-5/2,+\infty[$. In order to define its inverse function on all $\mathbb{R}$ we extend the function by continuity by setting :

$$H^{-1}(t) = \begin{cases} \left(\dfrac{2}{5}t + 1\right)^{3/2}, & \text{if} \quad t > -5/2, \\ 0, & \text{else.} \end{cases}$$

6.3.1 Validity of the schemes

To our knowledge, in the literature there do not exist numerical results to the steady-state Euler-Poisson system. Therefore it is impossible to compare our results with those obtained by other numerical methods. To ensure that our schemes provide a good approximation of the solution, we compare the numerical solutions obtained with each scheme to exact solutions. To this end we consider one test case for the unipolar system (6.11)-(6.13).

The electron density n, velocity potential ψ and electrostatic potential ϕ will be indexed with 1 for the test case.

In this test case, the doping profile is defined as follows :

$$C(x_1, x_2) = 1 - 2\varepsilon A^2 \pi^4 \exp(2\pi x_1), \quad (x_1, x_2) \text{ in } \Omega, \tag{6.25}$$

where A is a given constant. This doping profile is strictly positive on $\bar{\Omega}$ provided that $A^2\varepsilon$ is small (for instance, $\varepsilon = 9,5 \times 10^{-6}$ and $A = 1$). We supplement the system (6.11)-(6.13) with the following mixed Dirichlet-Neumann boundary conditions

$$\begin{gathered}
\nabla\psi_1 \cdot \nu = \nabla\phi_1 \cdot \nu = 0, \quad \text{on } \Gamma_N, \\
\psi_1 = A\cos(\pi x_2),\ \phi_1 = \frac{1}{2}A^2\varepsilon\pi^2 - A\varepsilon\cos(\pi x_2), \quad \text{on } \Gamma_{D,l}, \qquad (6.26) \\
\psi_1 = A\exp(\pi)\cos(\pi x_2),\ \phi_1 = \frac{1}{2}A^2\varepsilon\pi^2\exp(2\pi) - A\varepsilon\exp(\pi)\cos(\pi x_2), \quad \text{on } \Gamma_{D,r}.
\end{gathered}$$

Then it is easy to check that the exact solution of (6.11)-(6.13) and (6.25)-(6.26) is given by

$$\begin{gathered}
n_1(x_1, x_2) = 1, \\
\psi_1(x_1, x_2) = A\exp(\pi x_1)\cos(\pi x_2), \qquad (6.27) \\
\phi_1(x_1, x_2) = \frac{1}{2}A^2\varepsilon\pi^2\exp(2\pi x_1) - A\varepsilon\exp(\pi x_1)\cos(\pi x_2), \qquad (x_1, x_2) \in \bar{\Omega}.
\end{gathered}$$

We choose $\varepsilon = 9,5\times10^{-6}$ and $A = 1$. We start the computation with $n_1^0(x_1, x_2) = 1/2$ on Ω. The computations are stopped when the relative L^2-error between two following iterations is smaller than 10^{-3}.

We compute the errors between the numerical and exact solutions for different mesh size for each scheme and for each value of γ. The results are shown in Figure 6.1. We can see that for the two schemes and for each unknown, the errors are decreasing with the mesh size h. The DE-scheme seems to be more efficient in particular for the velocity and electrostatic potentials. Moreover, the errors are smaller in the case $\gamma = 5/3$ than in the case $\gamma = 1$. They are of order of h for each quantity.

We now consider the initial density $n_1^0 = 1$ and the same values for ε and A as above. The required accuracy is of order $O(h)$ in $L^2(\Omega)$ norm for stopping the iterations. The results are shown in Figure 6.2, from which we see that again the errors are increasing functions of h. They are of the same order as previously for each unknown. Again, they are smaller in the case $\gamma = 5/3$ than in the case $\gamma = 1$.

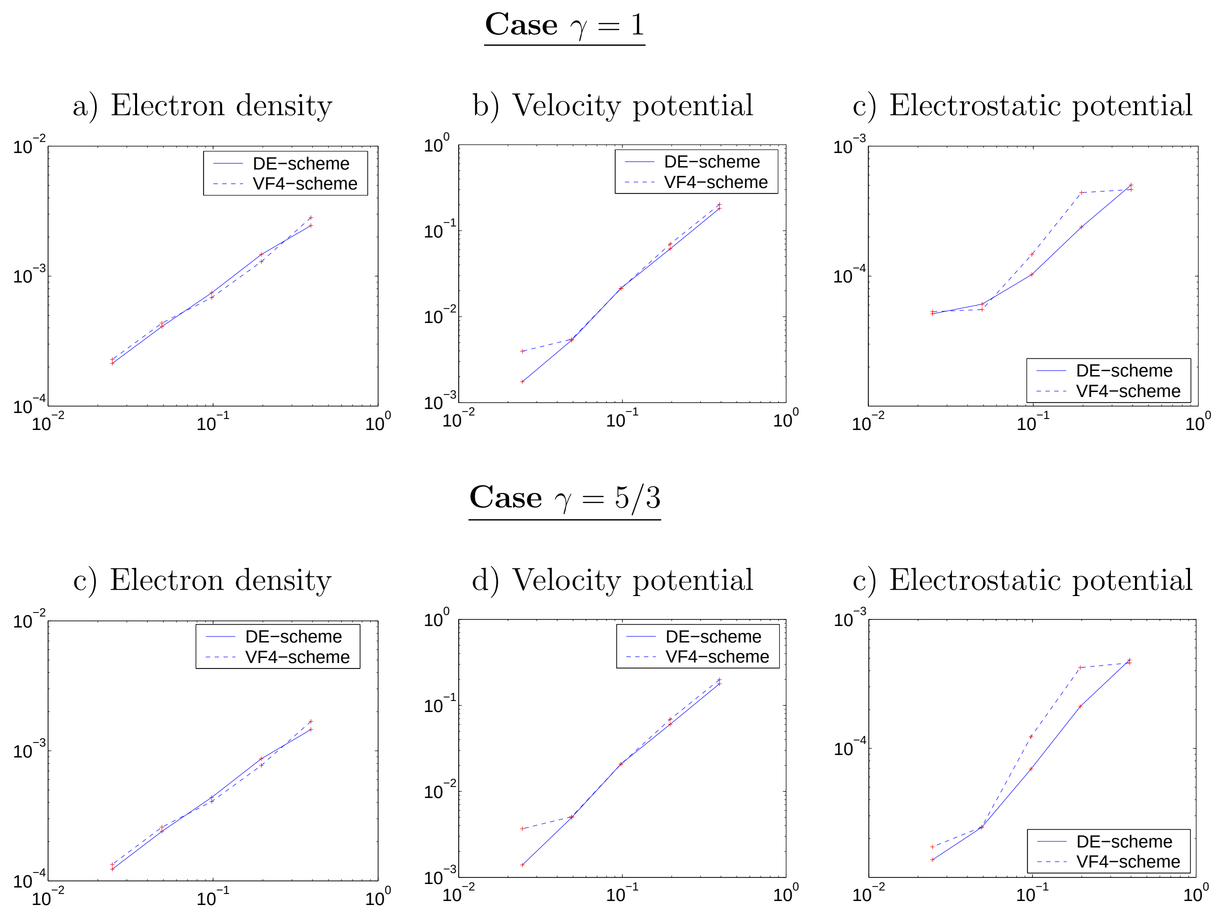

Figure 6.1: Errors of the electron density, the velocity potential and the electrostatic potential as functions of the mesh size for $n_1^0 \equiv 1/2$.

6.3.2 Case of a ballistic diode

A ballistic diode is a semiconductor which consists of a weakly doped n-region S between two highly doped n^+-regions $\Omega \setminus S$. It corresponds to the unipolar case since in such devices the charge transport is only due to electrons. Here we want to compute the numerical solution of the system (6.11)-(6.13) with the doping profile

$$C(\mathbf{x}) = \begin{cases} 1/2, & \text{if} \quad (x_1, x_2) \in S = [1/3, 2/3] \times [0, 1], \\ 1, & \text{else.} \end{cases}$$

Let us take $n^0 = C$ and

$$\phi^m = 0, \quad \text{on } \Gamma_{D,l} \quad \text{and} \quad \phi^m = U, \quad \text{on } \Gamma_{D,r}, \quad m \geq 0. \tag{6.28}$$

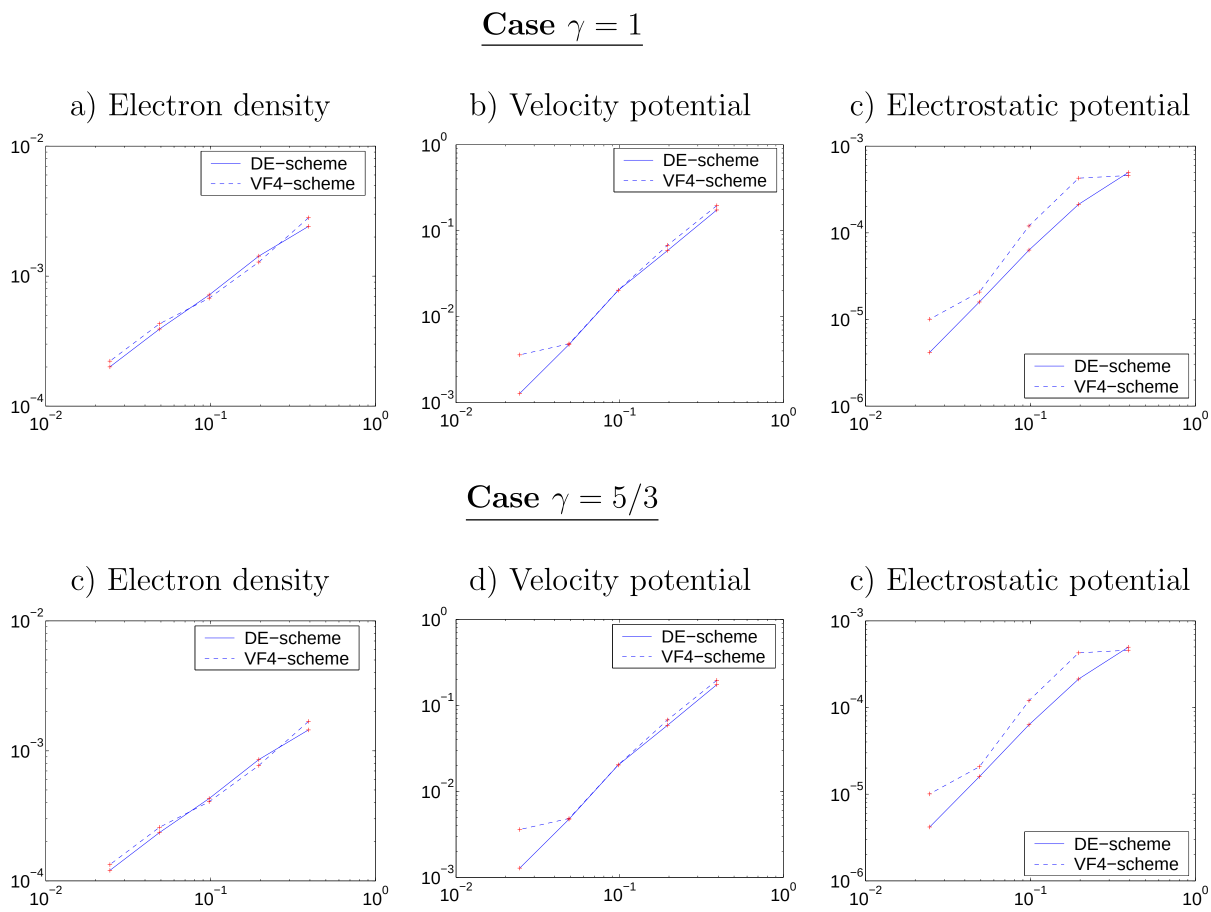

Figure 6.2: Error of the electron density, the velocity potential and the electrostatic potential as a functions of the mesh size for $n_1^0 \equiv 1$.

Here U corresponds to a given applied voltage. Since $H(1) = 0$, from (6.17) we have

$$\psi^m = 0, \quad \text{on } \Gamma_{D,l} \quad \text{and} \quad \psi^m = -U, \quad \text{on } \Gamma_{D,r}, \quad m \geq 0. \tag{6.29}$$

For different values of γ, U and ε, the numerical solutions of the electron density, velocity potential and electrostatic potential are given in Fig. 6.3, 6.4, 6.5, in which the sub-figures a), b) and c) are obtained with the VF4-scheme and d), e) and f) with the DE-scheme. In Fig. 6.3 and Fig. 6.5, we require an accuracy of order 10^{-8} in $L^\infty(\Omega)$ norm for stopping the iteration (the iteration numbers are respectively 11 and 8).

Note that the smallness condition on ε (see [80]), which ensures the strict ellipticity of the system, appears clearly in the numerical simulation. When ε is not small enough, the gradient of the velocity potential becomes more and more large in the iteration. Moreover, due to the negative sign before $|\nabla\psi_K^m|^2$ in the formula (6.24), the condition $n > 0$ is not numerically satisfied and the matrix involved in the computation of ψ^m becomes singular. A numerical example in this case is given in Fig. 6.4 (the computation is stopped after 5 iterations).

In the case $\gamma = 5/3$, due to the definition of the inverse function of H, U should satisfy $U > -5/2$. Indeed, for ε small enough, n_K^{m+1} is nearly given by $H^{-1}(\phi_K^m)$ according to (6.24). If $\phi_K^m \leq -5/2$, then $H^{-1}(\phi_K^m) = 0$ and $n_K^{m+1} \approx 0$, so that the matrix involved in the computation of ψ^{m+1} becomes singular. That is why we choose $U = 1$ in this case (see Fig. 6.5).

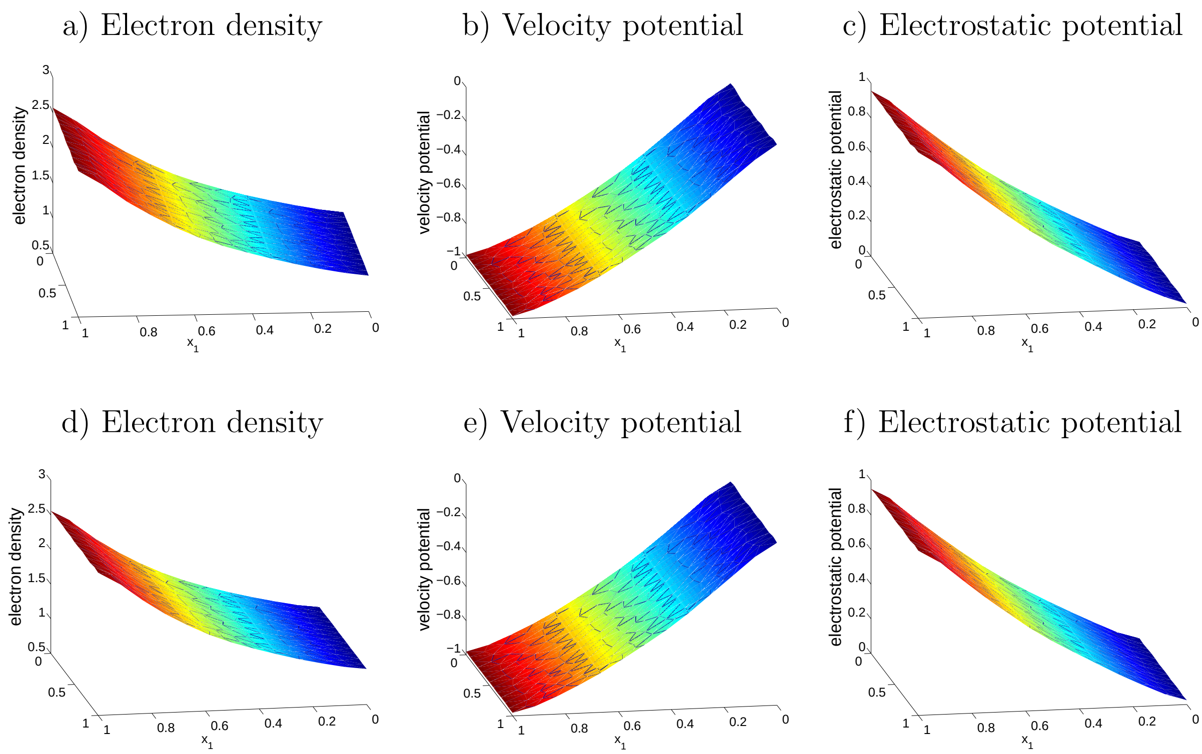

Figure 6.3: Case $\gamma = 1,\ U = 1,\ \varepsilon = 10^{-2}$.

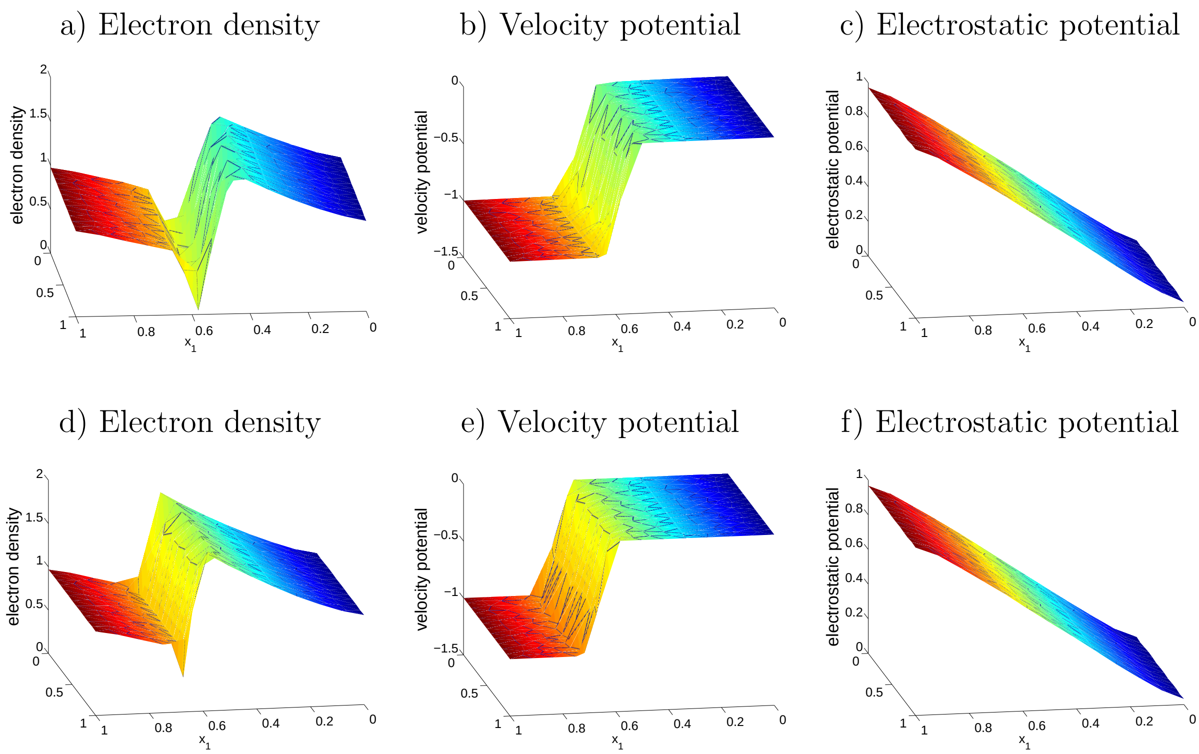

Figure 6.4: Case $\gamma = 1,\ U = 1,\ \varepsilon = 1$.

6.3.3 Bipolar case

Recall that in this case the system reads as follows

$$-\text{div}(p\nabla\psi_p) = 0, \tag{6.30}$$

$$H(p) + \frac{1}{2}|\nabla\psi_p|^2 = -\phi + \psi_p, \tag{6.31}$$

$$-\text{div}(n\nabla\psi_n) = 0, \tag{6.32}$$

$$H(n) + \frac{\varepsilon}{2}|\nabla\psi_n|^2 = \phi + \varepsilon\psi_n, \tag{6.33}$$

$$\Delta\phi = n - p - C, \quad \text{in } \Omega. \tag{6.34}$$

Let us define the initial hole density $p^0 = p^0(x)$ and the initial electron density $n^0 = n^0(x)$ by :

$$p^0(x) = \begin{cases} 1/2, & \text{if} \quad (x_1, x_2) \in [1/3, 2/3] \times [0, 1], \\ 1, & \text{else,} \end{cases}$$

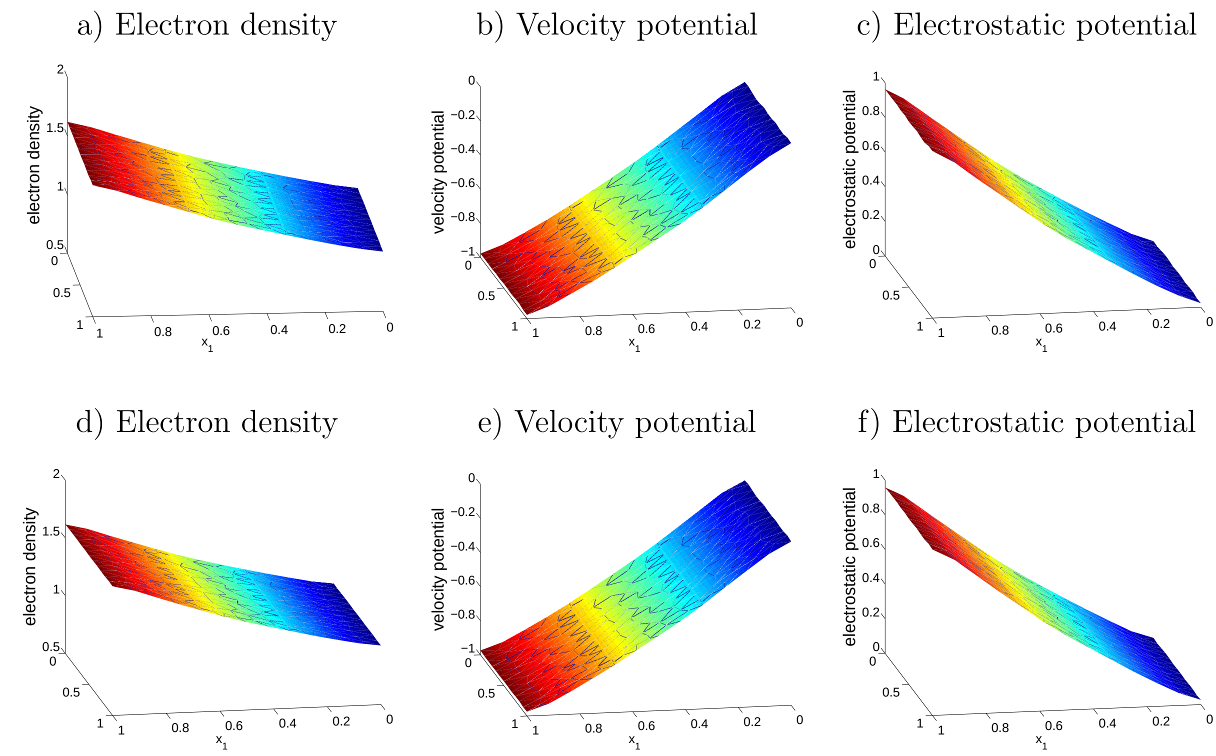

Figure 6.5: Case $\gamma = 5/3,\ U = 1,\ \varepsilon = 10^{-2}$.

$$n^0(x) = \begin{cases} 1/4, & \text{if} \quad (x_1, x_2) \in [1/3, 2/3] \times [0, 1], \\ 1, & \text{else.} \end{cases}$$

For simplicity we take a vanishing doping profile $C \equiv 0$ and supplement the system (6.30)-(6.34) with the boundary conditions (6.28) for $\phi^m,\ m \geq 0$ and the following Dirichlet boundary conditions for ψ_p^m and $\psi_n^m,\ m \geq 0$:

$$\psi_p^m = \psi_n^m = 0, \quad \text{on } \Gamma_{D,l} \quad \text{and} \quad \psi_p^m = U,\ \psi_n^m = -U, \quad \text{on } \Gamma_{D,r}, \tag{6.35}$$

Remark that the formula for computing the hole density p^{m+1} is

$$p_K^{m+1} = H^{-1}\Big(-\phi_K^m + \psi_{p,K}^m - \frac{1}{2}|\nabla\psi_{p,K}^m|^2\Big),\ K \in \mathcal{T} \text{ or } \mathcal{M},\ m \geq 0.$$

In this subsection the smallness condition on ε is still needed because of the same reason as in the unipolar case. The common value of ε is $\varepsilon = 10^{-2}$ and the iterative number is still denoted by N. In Fig 6.6 and Fig. 6.8 the iteration is stopped when the maximum of $p^{m+1} - p^m$ and $n^{m+1} - n^m$ is of order 10^{-8} in $L^\infty(\Omega)$ norm (respectively after 5 and 9 iterations). For $\gamma = 1$, the numerical solutions of the electron density, hole density and hole velocity potential are given in Fig. 6.6 and

Fig. 6.7, in which the sub-figures a), b) and c) are obtained with the VF4-scheme and d), e) and f) with the DE-scheme. In the computation, we need also that the value of $|U|$ to be small. An example for large $|U|$ ($U = 1$) is given in Fig. 6.7 (for 5 iterations), from which we see that the hole density is near zero, so that the matrix involved in the computation of ψ_p is almost singular. However, the computation of the electron density and electron velocity potential can still be carried out, due to the smallness value of ε. Finally, for $\gamma = 5/3$, the numerical solutions of the hole density and electron density are shown in Fig. 6.8, in which the sub-figures a) and b) are obtained with the VF4-scheme and c) and d) with the DE-scheme.

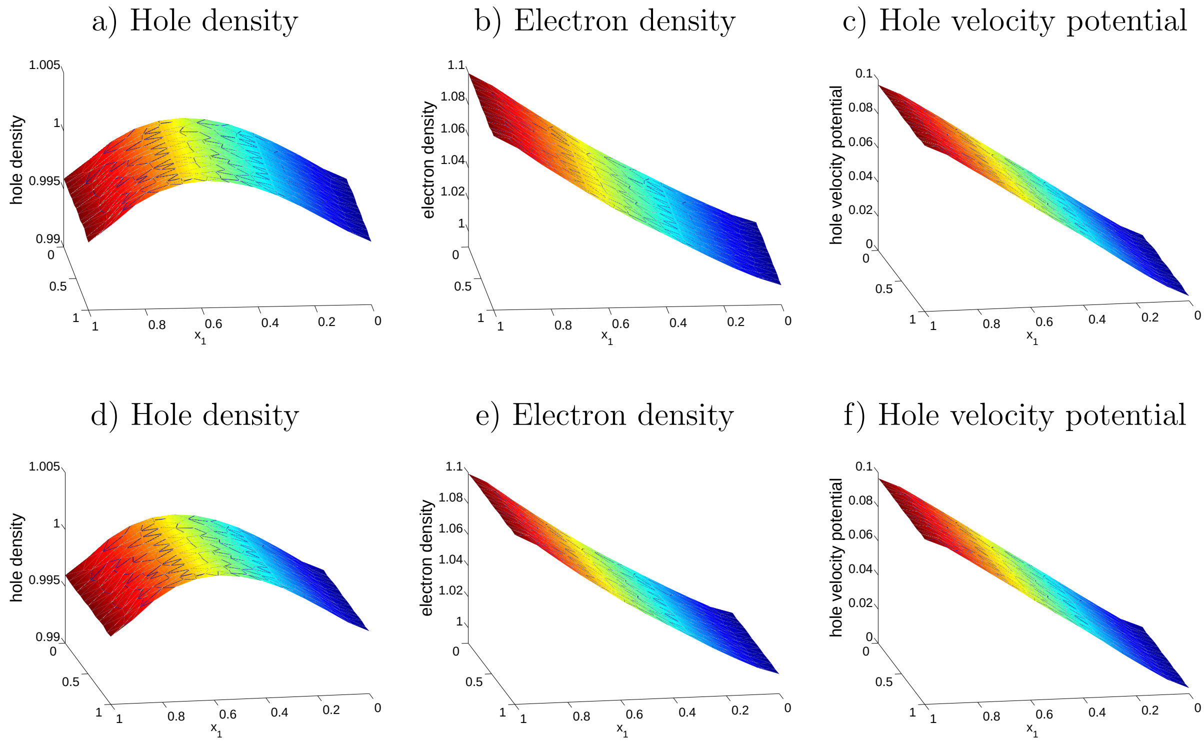

Figure 6.6: Case $\gamma = 1,\ U = 10^{-1},\ \varepsilon = 10^{-2}$.

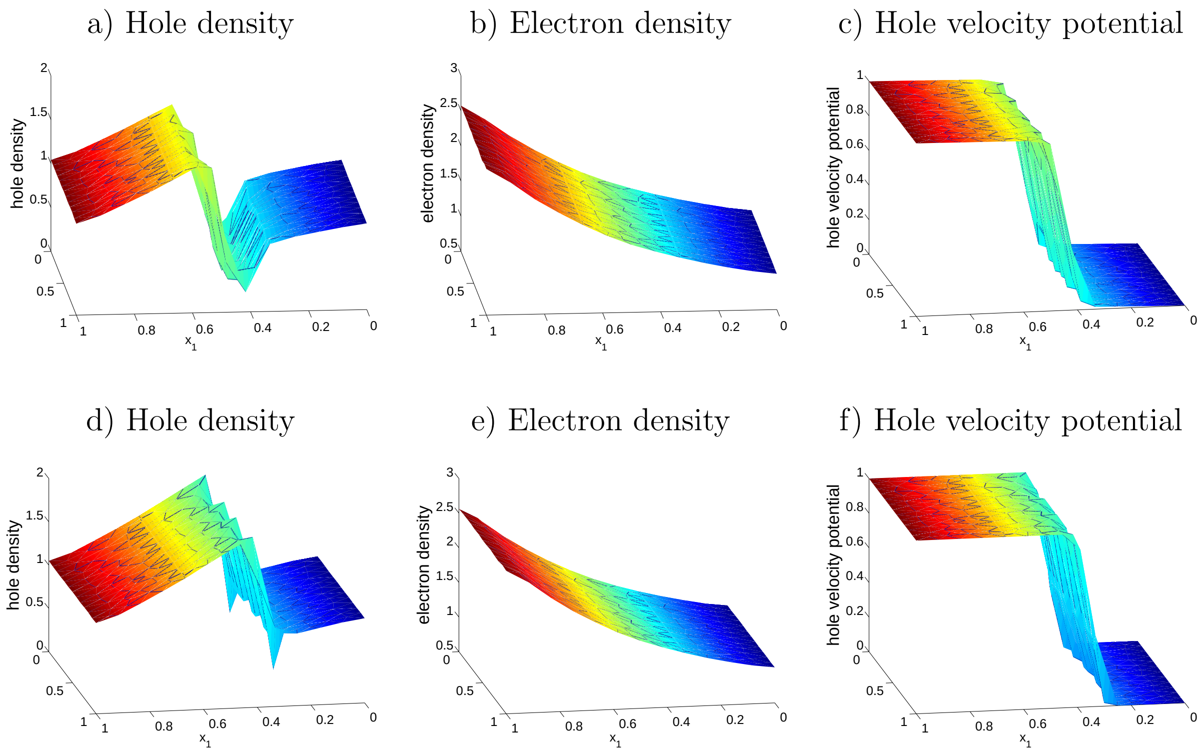

Figure 6.7: Case $\gamma = 1,\ U = 1,\ \varepsilon = 10^{-2}$.

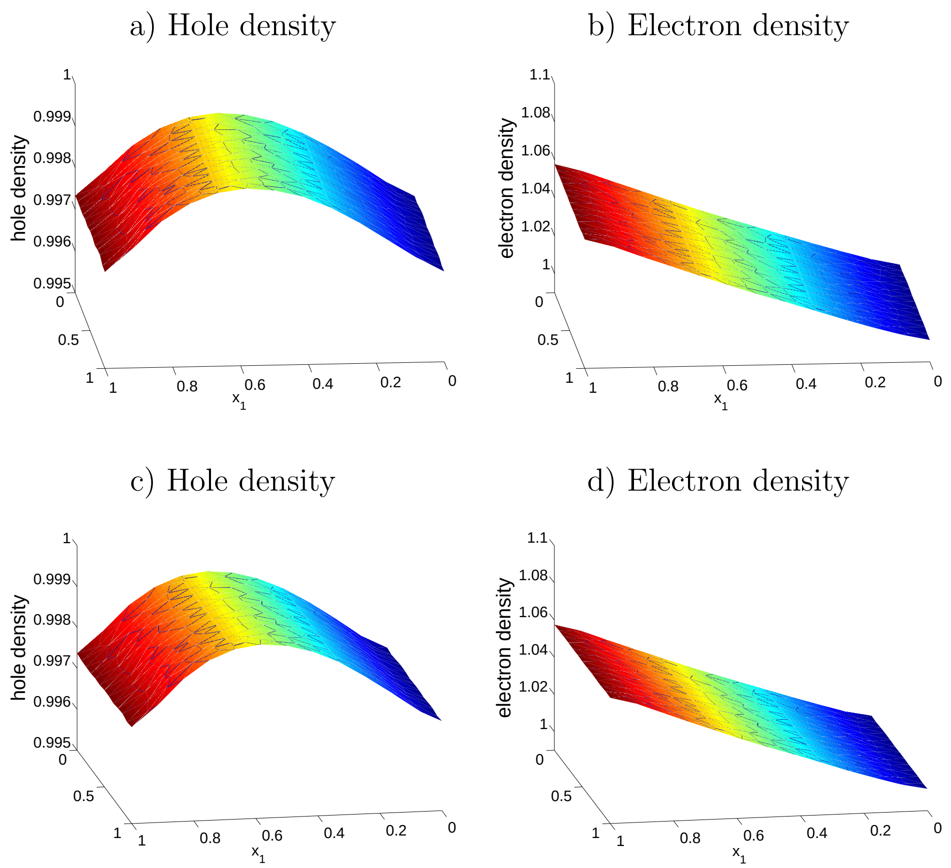

Figure 6.8: Case $\gamma = 5/3,\ U = 10^{-1},\ \varepsilon = 10^{-2}$.

Part III

Quantum drift-diffusion model

Chapter 7

The quasineutral limit in the quantum drift-diffusion equations

This Chapter is an article in collaboration with Ansgar Jüngel submitted for publication.

7.1 Introduction

In charged particle transport, quasineutrality is a commonly used assumption in order to simplify the model equations. Quasineutrality means that the difference between the concentrations of positive ions and electrons is negligible compared to a reference density. Formally, quasineutral models are obtained in the limit as the ratio of the Debye length to a characteristic length tends to zero. Quasineutral models are used, for instance, in semiconductor theory [92] and plasma physics [94]. Recently, quasineutrality has been studied also in quantum models [12]. An important quantum model are the quantum drift-diffusion equations which are a simple quantum extension of the drift-diffusion model used in both semiconductor and plasma theory (see [37] for a derivation and [38, 64] for reviews on macroscopic quantum models).

In this paper we analyze rigorously the quasineutral limit in the (scaled) quantum drift-diffusion equations in one space dimension for the electron density $n(x,t)$, the positively charged ion (or hole) density $p(x,t)$, and the electrostatic potential $V(x,t)$,

$$n_t - J_{n,x} = 0, \quad J_n = -\frac{\varepsilon^2}{2}(n(\log n)_{xx})_x + (P_n(n))_x - nV_x, \tag{7.1}$$

$$p_t + J_{p,x} = 0, \quad J_p = \frac{\varepsilon^2}{2}(p(\log p)_{xx})_x - (P_p(p))_x - pV_x, \tag{7.2}$$

$$\lambda^2 V_{xx} = n - p - C(x), \quad (x,t) \in Q_T = \Omega \times (0,T). \tag{7.3}$$

Here, J_n and J_p are the current densities and $C(x)$ models fixed charged background ions, usually called the doping concentration. The pressure functions P_n and P_p are typically of the form $P_\alpha(x) = \theta_\alpha x^{q_\alpha}$ ($\alpha = n, p$) for some $\theta_\alpha > 0$ and $q_\alpha \geq 1$. The parameter ε is the scaled Planck constant and $\lambda > 0$ is the ratio of the Debye length to the characteristic length (e.g., the device diameter). The equations are supplemented with the initial and boundary conditions

$$n = p = 1, \quad n_x = p_x = 0, \quad V = V_D \qquad \text{for } x \in \{0,1\},\ t > 0, \tag{7.4}$$

$$n(\cdot, 0) = n_I, \quad p(\cdot, 0) = p_I \qquad \text{in } \Omega, \tag{7.5}$$

where $V_D(x) = xU$ and $U \in \mathbb{R}$ is the applied potential. In the case that the doping vanishes at the boundary, the Dirichlet boundary conditions for n and p express charge neutrality, whereas the Neumann boundary conditions have been employed in numerical simulations of quantum semiconductor devices [49].

The quantum drift-diffusion model can be derived by the entropy minimization principle from the Wigner-BGK equation in the diffusion limit [37] or from the so-called quantum hydrodynamic equations in the zero-relaxation-time limit [62]. The existence of weak solutions to the stationary equations have been proved in [6]; the transient equations in one space dimension are analyzed in [70] but only for electrons and isothermal pressure $P_n(n) = \theta_n n$. Numerical simulations can be found in [70, 86].

Mathematically, the parabolic equations (7.1)-(7.2) are of fourth order. In particular, no maximum principles are available which complicates the analysis [69, 70]. In this context, we mention the so-called Derrida-Lebowitz-Speer-Spohn equation [40], obtained from (7.1) for zero pressure and zero electric field. This equation has recently attracted a lot of attention in the mathematical literature since it possesses several Lyapunov functionals and there are connections to logarithmic Sobolev inequalities (see [41] and references therein).

The justification of the quasineutral limit in macroscopic models has been first studied in [14] for a nonlinear Poisson equation (the ion density being fixed). The limit in the drift-diffusion equations (i.e. (7.1)-(7.3) with $\varepsilon = 0$) has been proved in [52, 67] assuming vanishing or at least not sign-changing doping concentrations. Sign-changing doping profiles have been considered in [98]. The quasineutral limit in the steady state Euler-Poisson equations has been investigated in [80, 83, 84, 95], whereas in [29, 53, 54, 97] the time-dependent case has been analyzed. In [13, 58] the limit in the Vlasov-Poisson system has been shown. To our knowledge, no analytical results on the quasineutral limit in fluid-type *quantum* models are available up to now.

In the quasineutral limit $\lambda \to 0$ we obtain formally from (7.3) $n = p$ and from (7.1)-(7.2)

$$n_t + \frac{\varepsilon^2}{2}(n(\log n)_{xx})_{xx} = \frac{1}{2}(P_n(n) + P_p(n))_{xx}, \quad x \in \Omega,\ t > 0, \tag{7.6}$$

with initial and boundary conditions

$$n = 1,\ n_x = 0 \quad \text{for } x \in \{0,1\}, \quad n(\cdot,0) = n_I \quad \text{in } \Omega,\ t > 0. \tag{7.7}$$

In this paper we make the limit rigorous for vanishing doping profile. First we show the existence of weak solutions to (7.1)-(7.4) (for general doping concentrations). In the literature, only results for the unipolar model are available with different boundary conditions [70] or with zero temperature and zero electric field [69]. Therefore, we include a proof for completeness. Moreover, our proof makes clear which quantities are bounded uniformly in the parameter λ (in appropriate norms).

More specifically, we show that the "entropy" $\int (n - \log n)dx$ is nonincreasing with respect to time and that the corresponding entropy production terms provide λ-uniform bounds for $\log n$ and $\log p$ in $L^2(0,T;H^2(\Omega))$ and for n and p in $L^{7/2}(Q_T)$. Also the entropy $\int n(\log n - 1)dx$ is nonincreasing in time, providing the uniform bounds

$$\|n - p\|_{L^2(Q_T)} \le c\lambda, \quad \|V_x\|_{L^2(Q_T)} \le c\lambda^{-1}. \tag{7.8}$$

These estimates are not sufficient to pass to the limit $\lambda \to 0$ in (7.1)-(7.3). Indeed, the sum of (7.1) and (7.2) leads to the drift term in weak formulation

$$\int_{Q_T} (n-p)V_x\phi_x dxdt \le \|n-p\|_{L^2(Q_T)}\|V_x\|_{L^2(Q_T)}\|\phi_x\|_{L^\infty(Q_T)} \le c,$$

where ϕ is some (smooth) test function and $c > 0$ a constant independent of λ. Thus, the estimates (7.8) only show that the above drift term is uniformly bounded; however, we need to prove that it converges to zero as $\lambda \to 0$. The main problem in this limit is that the (negative) electric field V_x is of the order $O(\lambda^{-1})$.

Our idea is to derive (instead of (7.8)) the estimates

$$\|\sqrt{n} - \sqrt{p}\|_{L^2(Q_T)} \le c\lambda, \quad \|(\sqrt{n} + \sqrt{p})V_x\|_{L^2(Q_T)} \le c\lambda^{-8/9}. \tag{7.9}$$

This gives

$$\int_{Q_T} (n-p)V_x\phi_x dxdt \le \|\sqrt{n} - \sqrt{p}\|_{L^2(Q_T)}\|(\sqrt{n} + \sqrt{p})V_x\|_{L^2(Q_T)}\|\phi_x\|_{L^\infty(Q_T)} \le \lambda^{1/9},$$

and hence, the drift term converges to zero as $\lambda \to 0$. The exponent 8/9 in (7.9) is connected with the exponents of some Gagliardo-Nirenberg inequalities (see Lemma 7.9). The first bound in (7.9) is a consequence of the estimate using the "entropy" $\int (n - \log n)dx$. The proof of the second bound in (7.9) is more delicate. It follows from an estimate of the electric energy $\lambda^2 \int (V - W)_x^2 dx$ if W satisfies the boundary data of V *up to first order*, i.e. $W = V$ and $W_x = V_x$ at $x \in \{0,1\}$. Since $V_x(0,t)$ and $V_x(1,t)$ are only of the order $O(\lambda^{-1})$, W is of the same order and prevents an appropriate estimate. To solve this problem, we approximate W by a function W_δ

in such a way that W_δ is of the order $O(1)+O(\delta\lambda^{-1})$ (in the $H^1(\Omega)$ norm). Passing to the limit $\delta \to 0$ then provides the needed estimate in (7.9).

Our main results are the following theorems.

Theorem 7.1. *Let $T > 0$, $U \in \mathbb{R}$, $C \in L^\infty(\Omega)$, and $0 \le n_I, p_I \in L^1(\Omega)$ satisfying*

$$\int_\Omega \big((n_I - \log n_I) + (p_I - \log p_I)\big)dx + \int_\Omega \big(n_I(\log n_I - 1) + p_I(\log p_I - 1)\big)dx < \infty.$$

Furthermore, let $P_n, P_p \in C^1([0,\infty))$ be nondecreasing and assume that there exist $0 < q < 7/2$ and $C_P > 0$ such that

$$|P_\alpha(x)| \le C_P(1+|x|^q) \quad \text{for all } x \ge 0,\ \alpha = n,p. \tag{7.10}$$

Then there exists a weak solution n, $p \in L^{7/2}(Q_T)$, $V \in L^\infty(0,T;H^2(\Omega))$ to (7.1)-(7.5) *such that*

$$n,p \ge 0 \text{ in } Q_T, \quad \log n, \log p \in L^2(0,T;H^2_0(\Omega)), \quad n_t, p_t \in L^1(0,T;H^{-3}(\Omega)).$$

The idea of the proof is to use the exponential transformation $n = e^y$ and $p = e^z$ as in [69] since this automatically gives nonnegative particle densities. First we show the existence of weak solutions to a semi-discrete (elliptic) problem. Appropriate a priori estimates, which are also useful for the quasineutral limit, allow to pass to the limit of vanishing approximation parameter. We stress the fact that, although we employ ideas of [69], the existence theorem is needed since first, there is no existence result for the bipolar quantum drift-diffusion model in the literature; and secondly, the approximation argument is needed in the proof of the quasineutral limit due to the lack of regularity of solutions to (7.1)-(7.5).

It is possible to obtain an existence result for more general (non-homogeneous) boundary data but the proof is very technical; we refer to [59] for a related problem providing the needed mathematical tools.

Theorem 7.2. *Let the assumptions of Theorem 7.1 hold and let, in addition, $C(x) \equiv 0$, $q \le 7/3$ and $n_I = p_I$ in Ω. Let $(n^{(\lambda)}, p^{(\lambda)}, V^{(\lambda)})$ be a weak solution (in the sense of Theorem 7.1) to* (7.1)-(7.5). *Then there exists a subsequence of $(n^{(\lambda)}, p^{(\lambda)}, V^{(\lambda)})$, which is not relabeled, such that, as $\lambda \to 0$,*

$$\begin{aligned} n^{(\lambda)} \to n, \quad p^{(\lambda)} \to n \qquad & \text{strongly in } L^3(Q_T),\\ n_t^{(\lambda)} \rightharpoonup n_t, \quad n_t^{(\lambda)} \rightharpoonup n_t \qquad & \text{weakly in } L^{42/41}(0,T;H^{-3}(\Omega)),\\ \log n^{(\lambda)} \rightharpoonup \log n, \quad \log p^{(\lambda)} \rightharpoonup \log n \qquad & \text{weakly in } L^2(0,T;H^2(\Omega)), \end{aligned}$$

satisfying (7.6)-(7.7).

Our assumptions avoid boundary and initial layers. We refer to [67] for the treatment of boundary layers and to [52] for the analysis of initial layers in the drift-diffusion model (cf. [83] and Remark 7.2).

Taking the difference of equations (7.1) and (7.2) provides in the limit $\lambda \to 0$ formally an equation for the electrostatic potential,

$$-((n+p)V_x)_x = (P_n(n) - P_p(n))_{xx} \quad \text{in } \Omega, \quad V(0,t) = 0, \quad V(1,t) = U.$$

However, since V_x is of the order $O(\lambda^{-1})$ we cannot justify this limit equation rigorously. In the drift-diffusion equations, this is possible under certain assumptions (see [67]).

If uniqueness of solutions holds for the problem (7.6)-(7.7), the whole sequence $(n^{(\lambda)}, p^{(\lambda)}, V^{(\lambda)})$ converges. However, there is no general uniqueness result for the limit problem. For a uniqueness theorem in the case of vanishing pressure under additional assumptions, we refer to [41].

The proof of Theorems 7.1 and 7.2 uses in several places the fact that we consider the one-dimensional equations. An existence proof for the multi-dimensional equations for vanishing pressure and vanishing electric field has been shown in [55] but only using periodic boundary conditions. The treatment of the quantum drift-diffusion model in several dimensions with physically motivated boundary conditions is currently not known.

Another interesting limit is the semiclassical limit $\varepsilon \to 0$. For a result in the stationary equations we refer to [6]. In [26] the limit has been shown in the transient case with homogeneous Neumann boundary conditions. Our a priori estimates seem to be not sufficient to perform the limit for the boundary conditions (7.4) (see Remark 7.1).

The paper is organized as follows. In section 7.2 we derive some a priori estimates needed for the existence result and we prove Theorem 7.1. The estimates are also useful for the quasineutral limit. Section 7.3 is devoted to the derivation of additional estimates independent of λ and the proof of Theorem 7.2.

7.2 Existence of solutions

7.2.1 A priori estimates

We divide the time interval $(0, T]$ for some $T > 0$ in N subintervals $(t_{k-1}, t_k]$ with $t_k = \tau k$, $k = 0, \ldots, N$, and $\tau = T/N$ is the time step. For given $k \in \{1, \ldots, N\}$ and

y_{k-1}, $z_{k-1} \in H_0^2(\Omega)$ we solve the semi-discrete system

$$\begin{aligned}
\frac{1}{\tau}(e^{y_k} - e^{y_{k-1}}) + \frac{\varepsilon^2}{2}(e^{y_k} y_{k,xx})_{xx} &= \big((P_n(e^{y_k}))_x - e^{y_k} V_{k,x}\big)_x, && (7.11)\\
\frac{1}{\tau}(e^{z_k} - e^{z_{k-1}}) + \frac{\varepsilon^2}{2}(e^{z_k} z_{k,xx})_{xx} &= \big((P_p(e^{z_k}))_x + e^{z_k} V_{k,x}\big)_x, && (7.12)\\
\lambda^2 V_{k,xx} &= e^{y_k} - e^{z_k} - C(x) \quad \text{in } \Omega, && (7.13)
\end{aligned}$$

for y_k, $z_k \in H_0^2(\Omega)$, $V_k - V_D \in H^1(\Omega)$, where $V_D(x) = xU$, $x \in \Omega$. We introduce the piecewise constant functions

$$y^{(N)}(x,t) = y_k(x), \quad z^{(N)}(x,t) = z_k(x), \quad V^{(N)}(x,t) = V_k(x) \quad \text{for } x \in \Omega,\ t \in (t_{k-1}, t_k], \tag{7.14}$$

where $k = 1, \dots, N$. First we show that the entropy

$$E_k^{(1)} = \int_\Omega ((e^{y_k} - y_k) + (e^{z_k} - z_k))\, dx$$

is non-increasing. Let y_k, $z_k \in H_0^2(\Omega)$, $V_k - V_D \in H_0^1(\Omega)$ be a solution to (7.11)-(7.13).

Lemma 7.1. *There exists a constant $c(\lambda) > 0$ which is independent of λ if $C(x) \equiv 0$ such that*

$$E_k^{(1)} + \frac{\varepsilon^2}{2}\sum_{j=1}^k \tau \int_\Omega (y_{j,xx}^2 + z_{j,xx}^2)dx + \frac{1}{\lambda^2}\sum_{j=1}^k \tau \int_\Omega (e^{y_j} - e^{z_j})(y_j - z_j)dx \le c(\lambda) E_0^{(1)}. \tag{7.15}$$

Proof. We employ $1 - e^{-y_k} \in H_0^2(\Omega)$ as a test function in the weak formulation of (7.11) to obtain

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega (e^{y_k} - e^{y_{k-1}})(1 - e^{-y_k})dx + \frac{\varepsilon^2}{2}\int_\Omega (y_{k,xx}^2 - y_{k,x}^2 y_{k,xx})dx \\
&\qquad = -\int_\Omega (P_n'(e^{y_k}) y_{k,x}^2 - V_{k,x} y_{k,x})dx. && (7.16)
\end{aligned}$$

With the elementary inequality $e^x \ge 1 + x$ for $x \in \mathbb{R}$ we can write

$$\begin{aligned}
(e^{y_k} - e^{y_{k-1}})(1 - e^{-y_k}) &= e^{y_k} - e^{y_{k-1}} + e^{y_{k-1} - y_k} - 1 \\
&\ge (e^{y_k} - y_k) - (e^{y_{k-1}} - y_{k-1}).
\end{aligned}$$

Since $y_{k,x} = 0$ on the boundary, the second integral on the left-hand side of (7.16) becomes

$$\frac{\varepsilon^2}{2}\int_\Omega \Big(y_{k,xx}^2 - \frac{1}{3}(y_{k,x}^3)_x\Big)dx = \frac{\varepsilon^2}{2}\int_\Omega y_{k,xx}^2 dx.$$

Thus, it follows from (7.16), taking into account that $P_n'(x) \geq 0$ by assumption,

$$\frac{1}{\tau}\int_\Omega (e^{y_k} - y_k)dx + \frac{\varepsilon^2}{2}\int_\Omega y_{k,xx}^2 dx \leq \frac{1}{\tau}\int_\Omega (e^{y_{k-1}} - y_{k-1})dx + \int_\Omega V_{k,x} y_{k,x} dx.$$

We obtain a similar equation for z_k. Then, adding both inequalities and using the Poisson equation (7.13), we arrive at

$$\begin{aligned}
\frac{1}{\tau}E_k^{(1)} + \frac{\varepsilon^2}{2}\int_\Omega (y_{k,xx}^2 + z_{k,xx}^2)dx &\leq \frac{1}{\tau}E_{k-1}^{(1)} + \int_\Omega V_{k,x}(y_{k,x} - z_{k,x})dx \\
&= \frac{1}{\tau}E_{k-1}^{(1)} - \frac{1}{\lambda^2}\int_\Omega (e^{y_k} - e^{z_k} - C(x))(y_k - z_k)dx \\
&\leq \frac{1}{\tau}E_{k-1}^{(1)} - \frac{1}{\lambda^2}\int_\Omega (e^{y_k} - e^{z_k})(y_k - z_k)dx + \frac{1}{\lambda^2}\|C\|_{L^\infty(\Omega)}\int_\Omega (|y_k| + |z_k|)dx.
\end{aligned}$$

Since $|x| \leq e^x - x$ for all $x \in \mathbb{R}$, this yields

$$E_k^{(1)} + \tau\frac{\varepsilon^2}{2}\int_\Omega (y_{k,xx}^2 + z_{k,xx}^2)dx + \frac{\tau}{\lambda^2}\int_\Omega (e^{y_k} - e^{z_k})(y_k - z_k)dx \leq E_{k-1}^{(1)} + \frac{\tau}{\lambda^2}\|C\|_{L^\infty(\Omega)}E_k^{(1)}.$$

Hence, choosing $\tau > 0$ small enough, we obtain (7.15). □

An immediate consequence of the entropy estimate (7.15) (and the Poincaré inequality) are the following uniform bounds for the functions $y^{(N)}$ and $z^{(N)}$ (see (7.14)):

$$\begin{aligned}
\|y^{(N)}\|_{L^\infty(0,T;L^1(\Omega))} + \|z^{(N)}\|_{L^\infty(0,T;L^1(\Omega))} &\leq c(\lambda), && (7.17)\\
\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))} + \|e^{z^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))} &\leq c(\lambda), && (7.18)\\
\|y^{(N)}\|_{L^2(0,T;H^2(\Omega))} + \|z^{(N)}\|_{L^2(0,T;H^2(\Omega))} &\leq c(\lambda). && (7.19)
\end{aligned}$$

Again, if $C(x) \equiv 0$, the constant $c(\lambda)$ does not depend on λ. From these estimates we are able to deduce more uniform bounds.

Lemma 7.2. *There exists a constant $c(\lambda) > 0$ which does not depend on λ if $C(x) \equiv 0$ such that*

$$\begin{aligned}
\|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} + \|e^{z^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} &\leq c(\lambda), && (7.20)\\
\|e^{y^{(N)}}\|_{L^{7/2}(Q_T)} + \|e^{z^{(N)}}\|_{L^{7/2}(Q_T)} &\leq c(\lambda), && (7.21)
\end{aligned}$$

where we recall that $Q_T = \Omega \times (0,T)$.

Proof. We employ the Gagliardo-Nirenberg inequality and the estimates (7.17), (7.19) to find

$$\begin{aligned}\|y^{(N)}\|_{L^{5/2}(0,T;W^{1,\infty}(\Omega))} &\leq \|y_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(\Omega))} \\ &\leq \|y^{(N)}\|_{L^\infty(0,T;L^1(\Omega))}^{1/5}\|y^{(N)}\|_{L^2(0,T;H^2(\Omega))}^{4/5} \leq c(\lambda).\end{aligned}$$

Therefore, with (7.17) and (7.18),

$$\begin{aligned}\|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} &\leq c\left(\|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^1(\Omega))} + \|(e^{y^{(N)}})_x\|_{L^{5/2}(0,T;L^1(\Omega))}\right) \\ &\leq c\|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^1(\Omega))} + c\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))}\|y_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(\Omega))} \\ &\leq c(\lambda).\end{aligned}$$

This shows (7.20). In order to prove (7.21) we use again the Gagliardo-Nirenberg inequality:

$$\begin{aligned}\|e^{y^{(N)}}\|_{L^{7/2}(Q_T)}^{7/2} &\leq c\int_0^T \|e^{y^{(N)}}\|_{L^1(\Omega)}\|e^{y^{(N)}}\|_{W^{1,1}(\Omega)}^{5/2}dt \\ &\leq c\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))}\|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))}^{5/2} \leq c(\lambda).\end{aligned}$$

The bounds for $e^{z^{(N)}}$ are derived in a similar way. □

Remark 7.1. The constants in (7.20)-(7.21) depend on ε since the estimates for $y^{(N)}$ and $z^{(N)}$ in $L^2(0,T;H^2(\Omega))$ do so. Hence, most of the subsequent bounds also depend on ε.

Lemma 7.3. *There exists a constant $c(\lambda) > 0$ depending on λ such that*

$$\|V^{(N)}\|_{L^2(0,T;H^1(\Omega))} \leq c(\lambda). \tag{7.22}$$

Proof. By elliptic estimates,

$$\begin{aligned}\lambda^2\|V_x^{(N)}\|_{L^2(Q_T)} &\leq c\left(\|e^{y^{(N)}} - e^{z^{(N)}} - C(x)\|_{L^2(0,T;H^{-1}(\Omega))} + 1\right) \\ &\leq c\left(\|e^{y^{(N)}} - e^{z^{(N)}} - C(x)\|_{L^2(0,T;L^1(\Omega))} + 1\right) \leq c(\lambda),\end{aligned}$$

since $L^1(\Omega)$ injects continuously into $H^{-1}(\Omega)$ in one space dimension. □

Finally, we need an estimate for the discrete time derivative. For this, we introduce the shift operator

$$(\sigma_N e^{y^{(N)}})(x,t) = e^{y_{k-1}(x)}, \quad (\sigma_N e^{z^{(N)}})(x,t) = e^{z_{k-1}(x)} \quad \text{for } x \in \Omega,\ t \in (t_{k-1}, t_k]. \tag{7.23}$$

Lemma 7.4. *There exists a constant $c(\lambda) > 0$ depending on λ such that for $s = \min\{7/2q,\ 14/11\} > 1$,*

$$\|e^{y^{(N)}} - \sigma_N e^{y^{(N)}}\|_{L^s(0,T;H^{-3}(\Omega))} + \|e^{z^{(N)}} - \sigma_N e^{z^{(N)}}\|_{L^s(0,T;H^{-3}(\Omega))} \le \tau c(\lambda). \qquad (7.24)$$

Proof. We estimate the semi-discrete equation (7.11) in the norm of $L^s(0,T;H^{-3}(\Omega))$. This gives

$$\tau^{-1}\|e^{y^{(N)}} - \sigma_N e^{y^{(N)}}\|_{L^s(0,T;H^{-3}(\Omega))} \le \varepsilon^2 \|e^{y^{(N)}} y_{xx}^{(N)}\|_{L^s(0,T;H^{-1}(\Omega))} + \|P_n(e^{y^{(N)}})\|_{L^s(0,T;H^{-1}(\Omega))} + \|e^{y^{(N)}} V_x^{(N)}\|_{L^s(0,T;H^{-1}(\Omega))}.$$

The first term on the right-hand side is bounded by Hölder's inequality and (7.19), (7.21):

$$\begin{aligned}\|e^{y^{(N)}} y_{xx}^{(N)}\|_{L^s(0,T;H^{-1}(\Omega))} &\le c\|e^{y^{(N)}} y_{xx}^{(N)}\|_{L^s(0,T;L^s(\Omega))} \le c\|e^{y^{(N)}}\|_{L^{2s/(2-s)}(Q_T)} \|y_{xx}^{(N)}\|_{L^2(Q_T)} \\ &\le c\|e^{y^{(N)}}\|_{L^{7/2}(Q_T)} \|y_{xx}^{(N)}\|_{L^2(Q_T)} \le c(\lambda),\end{aligned}$$

since $2s/(2-s) \le 7/2$ is equivalent to $s \le 14/11$. For the second term on the above right-hand side we employ the growth condition on the pressure functions and (7.21):

$$\|P_n(e^{y^{(N)}})\|_{L^s(0,T;H^{-1}(\Omega))} \le c\|P_n(e^{y^{(N)}})\|_{L^s(0,T;L^s(\Omega))} \le c\left(1 + \|e^{y^{(N)}}\|^q_{L^{7/2}(Q_T)}\right) \le c(\lambda).$$

Finally, the last term on the right-hand side can be estimated by using (7.21) and (7.22):

$$\begin{aligned}\|e^{y^{(N)}} V_x^{(N)}\|_{L^s(0,T;H^{-1}(\Omega))} &\le c\|e^{y^{(N)}} V_x^{(N)}\|_{L^s(0,T;L^s(\Omega))} \\ &\le c\|e^{y^{(N)}}\|_{L^{2s/(2-s)}(Q_T)} \|V_x^{(N)}\|_{L^2(Q_T)} \le c(\lambda).\end{aligned}$$

Putting together the three inequalities gives (7.24). The proof for $z^{(N)}$ is analogous. □

7.2.2 Proof of Theorem 7.1

First we show that the semi-discrete problem (7.11)-(7.13) admits a solution.

Lemma 7.5. *Under the hypotheses of Theorem 7.1 there exists a sequence $(y_k, z_k, V_k) \in (H_0^2(\Omega))^2 \times H^2(\Omega)$ with $V_k(0) = 0$ and $V_k(1) = U$ satisfying (7.11)-(7.13).*

Proof. Let y_{k-1}, $z_{k-1} \in H_0^2(\Omega)$ be given. Let v, $w \in H^1(\Omega)$ and solve first

$$\lambda^2 V_{k,xx} = e^v - e^w - C(x) \quad \text{in } \Omega, \quad V_k(0) = 0,\ V_k(1) = U.$$

This problem admits a unique solution $V_k \in H^2(\Omega)$. Then we solve in $H_0^2(\Omega)$ the linear problems

$$\begin{aligned}\frac{\sigma}{\tau}(e^v - e^{y_{k-1}}) + \frac{\varepsilon^2}{2}(e^v y_{k,xx})_{xx} &= \sigma\big((P_n(e^v))_x - e^v V_{k,x}\big)_x,\\ \frac{\sigma}{\tau}(e^w - e^{z_{k-1}}) + \frac{\varepsilon^2}{2}(e^w z_{k,xx})_{xx} &= \sigma\big((P_p(e^w))_x + e^w V_{k,x}\big)_x,\end{aligned}$$

where $\sigma \in [0,1]$. There exists a unique solution $(y_k, z_k) \in (H_0^2(\Omega))^2$. This defines the fixed-point operator $S : (H^1(\Omega))^2 \times [0,1] \to (H^1(\Omega))^2$, $(v, w, \sigma) \mapsto (y_k, z_k)$. Then S is well defined and satisfies $S(v, w, 0) = (0, 0)$. Furthermore, it is not difficult to check that S is continuous and, in view of the compact embedding $H_0^2(\Omega) \hookrightarrow H^1(\Omega)$, also compact. It remains to show that there is a uniform bound for all fixed points of $S(\cdot,\cdot,\sigma)$. The estimates of section 7.2.1 establish the case $\sigma = 1$. The estimates for $\sigma < 1$ are similar (and, in fact, independent of σ). This provides the wanted bound in $H^1(\Omega)$ and the Leray-Schauder fixed-point theorem can be applied to yield the existence of a solution to (7.11)-(7.13). □

Now we are able to prove Theorem 7.1. For this, we have to perform the limit $\tau \to 0$ in (7.11)-(7.13). Actually, the uniform bounds (7.20) and (7.24) and the compact embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ allow to apply Theorem 5 of [93] (Aubin's lemma) yielding the existence of a subsequence of $e^{y^{(N)}}$ and $e^{z^{(N)}}$ (not relabeled) such that $e^{y^{(N)}} \to v$, $e^{z^{(N)}} \to w$ strongly in $L^1(Q_T)$ as $N \to \infty$ or, equivalently, $\tau \to 0$. Moreover, again for a subsequence which is not relabeled,

$$y^{(N)} \rightharpoonup y,\ z^{(N)} \rightharpoonup z \quad \text{weakly in } L^2(0,T;H^2(\Omega)) \tag{7.25}$$

as $\tau \to 0$. The bounds (7.17) and (7.18) allow to use the same arguments as in the proof of Theorem 1.2 in [59] showing that $v = e^y$ and $w = e^z$. Since, by (7.21), $(e^{y^{(N)}})$ is bounded in $L^{7/2}(Q_T)$ and $e^{y^{(N)}} \to e^y$ a.e., the result in [75, Ch. 1.3 and p. 144] yields

$$e^{y^{(N)}} \to e^y \quad \text{strongly in } L^2(Q_T). \tag{7.26}$$

Moreover, the same bound and hypothesis (7.10) imply that $(P_n(e^{y^{(N)}}))$ is bounded in $L^s(0,T;L^s(\Omega))$ for $s = 7/2q > 1$ and hence, by the same argument as before,

$$P_n(e^{y^{(N)}}) \to P_n(e^y) \quad \text{strongly in } L^1(Q_T). \tag{7.27}$$

Finally, the bound (7.24) gives, up to a subsequence,

$$\frac{1}{\tau}(e^{y^{(N)}} - \sigma_N e^{y^{(N)}}) \rightharpoonup (e^y)_t \quad \text{weakly in } L^s(0,T;H^{-3}(\Omega)). \tag{7.28}$$

The same limits hold for $z^{(N)}$. Moreover, by (7.22),

$$V^{(N)} \rightharpoonup V \quad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{7.29}$$

The limits (7.25)-(7.29) allow to pass to the limit $\tau \to 0$ in the weak formulation of (7.11),

$$\int_0^T \int_\Omega \frac{1}{\tau}(e^{y^{(N)}} - \sigma_N e^{y^{(N)}})\phi dxdt + \frac{\varepsilon^2}{2} \int_0^T \int_\Omega e^{y^{(N)}} y_{xx}^{(N)} \phi_{xx} dxdt$$
$$= \int_0^T \int_\Omega \left(P_n(e^{y^{(N)}})\phi_{xx} + e^{y^{(N)}} V_x^{(N)} \phi_x \right) dxdt$$

for all $\phi \in L^\infty(0,T; H^3(\Omega) \cap H_0^2(\Omega))$. The limit functions satisfy y, $z \in L^2(0,T; H_0^2(\Omega))$, $V - V_D \in L^\infty(0,T; H_0^1(\Omega))$, which shows that the boundary conditions are satisfied. Furthermore, the initial conditions hold in the sense of $H^{-3}(\Omega)$. This finishes the proof of Theorem 7.1.

7.3 The quasi-neutral limit

7.3.1 A priori estimates

For the quasi-neutral limit $\lambda \to 0$ we need additional estimates. We recall that the condition $C(x) \equiv 0$ implies that the uniform bounds (7.15)-(7.21) are independent of λ.

Lemma 7.6. *There exists a constant $c > 0$ independent of λ such that*

$$\|e^{y^{(N)}/2} - e^{z^{(N)}/2}\|_{L^2(Q_T)} \le c\lambda. \tag{7.30}$$

Proof. The entropy estimate (7.15) gives

$$\int_{Q_T} (e^{y^{(N)}} - e^{z^{(N)}})(y^{(N)} - z^{(N)}) dxdt \le c\lambda^2.$$

Then the assertion follows if we can show that

$$2(\sqrt{x} - \sqrt{y})^2 \le (x - y)(\log x - \log y) \quad \text{for all } x, y \ge 0. \tag{7.31}$$

This inequality can be seen as follows. It is sufficient to consider $x \ge y > 0$. Then (7.31) is equivalent to

$$2(\sqrt{x} - \sqrt{y}) \le (\sqrt{x} + \sqrt{y}) \log \frac{x}{y}$$

and

$$2\frac{\sqrt{x/y} - 1}{\sqrt{x/y} + 1} = 2\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \le 2 \log \sqrt{\frac{x}{y}}.$$

Thus we only need to prove that

$$\frac{z-1}{z+1} \le \log z \quad \text{for all } z \ge 1.$$

But this is a consequence of $\log z \ge z-1 \ge (z-1)/(z+1)$ for $z \ge 1$, thus proving the lemma. □

The following estimates are derived from the boundedness of the entropy

$$E_k^{(2)} = \int_\Omega (e^{y_k}(y_k - 1) + e^{z_k}(z_k - 1) + 2)\, dx > 0.$$

Lemma 7.7. *The following estimate holds:*

$$E_k^{(2)} + \frac{\varepsilon^2}{2}\sum_{j=1}^k \tau \int_\Omega (e^{y_j} y_{j,xx}^2 + e^{z_j} z_{j,xx}^2) dx + \frac{1}{\lambda^2}\sum_{j=1}^k \tau \int_\Omega (e^{y_j} - e^{z_j})^2 dx \le E_0^{(2)}. \quad (7.32)$$

Proof. We employ the test function $y_k \in H_0^2(\Omega)$ in the weak formulation of (7.11) to obtain

$$\frac{1}{\tau}\int_\Omega (e^{y_k} - e^{y_{k-1}}) y_k dx + \frac{\varepsilon^2}{2}\int_\Omega e^{y_k} y_{k,xx}^2 dx = -\int_\Omega (P_n'(e^{y_k}) e^{y_k} y_{k,x}^2 - V_{k,x} e^{y_k} y_{k,x}) dx. \quad (7.33)$$

The convexity of $x \mapsto e^x$ implies that $e^x - e^y - e^y(x-y) \ge 0$ and hence,

$$\begin{aligned} (e^{y_k} - e^{y_{k-1}}) y_k &\ge (e^{y_k} - e^{y_{k-1}}) y_k + e^{y_{k-1}}(y_k - y_{k-1}) - e^{y_k} + e^{y_{k-1}} \\ &= e^{y_k}(y_k - 1) - e^{y_{k-1}}(y_{k-1} - 1). \end{aligned}$$

Thus it follows

$$\frac{1}{\tau}\int_\Omega e^{y_k}(y_k - 1) dx + \frac{\varepsilon^2}{2}\int_\Omega e^{y_k} y_{k,xx}^2 dx \le \frac{1}{\tau}\int_\Omega e^{y_{k-1}}(y_{k-1} - 1) dx + \int_\Omega V_{k,x}(e^{y_k})_x dx.$$

A similar inequality holds for z_k. Adding both inequalities and then employing the Poisson equation (7.13) gives

$$\frac{1}{\tau} E_k^{(2)} + \frac{\varepsilon^2}{2}\int_\Omega (e^{y_k} y_{k,xx}^2 + e^{z_k} z_{k,xx}^2) dx \le \frac{1}{\tau} E_{k-1}^{(2)} - \frac{1}{\lambda^2}\int_\Omega (e^{y_k} - e^{z_k})^2 dx.$$

This gives the assertion. □

From Lemma 7.7 immediately follows that

$$\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^2(Q_T)} \le c\lambda$$

and therefore, for sufficiently small $\lambda > 0$,

$$\|V_x^{(N)}\|_{L^2(Q_T)} \le c(1 + \lambda^{-2}\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^2(Q_T)}) \le c\lambda^{-1}.$$

In the L^3 norm the exponent in λ is smaller as shown in the following lemma.

Lemma 7.8. *There exists a constant $c > 0$ independent of λ such that*

$$\begin{aligned} \|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^3(Q_T)} &\leq c\lambda^{2/9}, && (7.34) \\ \|V^{(N)}\|_{L^3(0,T;W^{2,3}(\Omega))} &\leq c\lambda^{-16/9}. && (7.35) \end{aligned}$$

Proof. By Hölder's inequality,

$$\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^3(Q_T)} \leq \|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^2(Q_T)}^{2/9} \|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^{7/2}(Q_T)}^{7/9} \leq c\lambda^{2/9},$$

employing (7.21) and (7.32), which shows (7.34). The estimate (7.35) is a consequence from (7.34):

$$\|V_{xx}^{(N)}\|_{L^3(Q_T)} = \lambda^{-2}\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^3(Q_T)} \leq c\lambda^{-16/9}.$$

This finishes the proof. □

The following lemma is our key result.

Lemma 7.9. *There exists a constant $c(\varepsilon) > 0$ independent of λ such that, for sufficiently small $\lambda > 0$,*

$$\|(e^{y^{(N)}/2} + e^{z^{(N)}/2})V_x^{(N)}\|_{L^2(Q_T)} \leq c\lambda^{-8/9}. \qquad (7.36)$$

Proof. The key idea is to define a special extension $W_k(x)$ of the boundary data such that $W_k - V_k \in H_0^2(\Omega)$ becomes an admissible test function in the weak formulation of (7.11)-(7.12). The problem is that $V_{k,x}(0)$ and $V_{k,x}(1)$ are unbounded as $\lambda \to 0$. Therefore, we need to take special care in the definition of W_k. We define

$$W_k(x) = \begin{cases} \delta(V_{k,x}(0) - U)\left(\frac{x}{\delta}\right)^3 + 2\delta(U - V_{k,x}(0))\left(\frac{x}{\delta}\right)^2 + \delta V_{k,x}(0)\frac{x}{\delta} & : \ x \in [0, \delta] \\ xU & : \ x \in [\delta, 1-\delta] \\ \delta(U - V_{k,x}(1))\left(\frac{1-x}{\delta}\right)^3 & \\ \quad + 2\delta(V_{k,x}(1) - U)\left(\frac{1-x}{\delta}\right)^2 - \delta V_{k,x}(1)\frac{1-x}{\delta} + U & : \ x \in [1-\delta, 1]. \end{cases}$$

This function is continuously differentiable, is an element of $H^2(\Omega)$ and satisfies

$$W_k(0) = 0, \quad W_k(1) = U, \quad W_{k,x}(0) = V_{k,x}(0), \quad W_{k,x}(1) = V_{k,x}(1).$$

Let $W^{(N)}(\cdot, t) = W_k$ if $t \in (t_{k-1}, t_k]$. We claim that, for sufficiently small $\lambda > 0$,

$$\begin{aligned} \|W_x^{(N)}\|_{L^3(Q_T)} &\leq c\delta\lambda^{-16/9}, && (7.37) \\ \|W_{xx}^{(N)}\|_{L^3(Q_T)} &\leq c\lambda^{-16/9}. && (7.38) \end{aligned}$$

Indeed, by elliptic estimates and (7.34), we have

$$\begin{aligned}\|W_x^{(N)}\|_{L^3(Q_T)} &\leq c(1+\delta\|V_x^{(N)}(0,\cdot)\|_{L^3(0,T)}+\delta\|V_x^{(N)}(1,\cdot)\|_{L^3(0,T)})\\&\leq c(1+\delta\|V^{(N)}\|_{L^3(0,T;W^{1,\infty}(\Omega))})\leq c(1+\delta\|V^{(N)}\|_{L^3(0,T;W^{2,1}(\Omega))})\\&\leq c(1+\delta\lambda^{-2}\|e^{y^{(N)}}-e^{z^{(N)}}\|_{L^3(0,T;L^1(\Omega))})\leq c\delta\lambda^{-16/9}.\end{aligned}$$

This shows (7.37). In order to prove (7.38), we use (7.35):

$$\begin{aligned}\|W_{xx}^{(N)}\|_{L^3(Q_T)} &\leq c(1+\|V_x^{(N)}(0,\cdot)\|_{L^3(0,T)}+\|V_x^{(N)}(1,\cdot)\|_{L^3(0,T)})\\&\leq c(1+\|V^{(N)}\|_{L^3(0,T;W^{1,\infty}(\Omega))})\leq c\lambda^{-16/9}.\end{aligned}$$

Now we employ $W_k - V_k \in H_0^2(\Omega)$ as a test function in (7.11)-(7.12) and take the difference of the resulting equations to obtain

$$\begin{aligned}&\frac{1}{\tau}\int_\Omega\left((e^{y_k}-e^{z_k})-(e^{y_{k-1}}-e^{z_{k-1}})\right)(W_k-V_k)dx\\&\qquad+\frac{\varepsilon^2}{2}\int_\Omega(e^{y_k}y_{k,xx}-e^{z_k}z_{k,xx})(W_k-V_k)_{xx}dx \qquad (7.39)\\&=\int_\Omega(P_n(e^{y_k})-P_p(e^{z_k}))(W_k-V_k)_{xx}dx+\int_\Omega(e^{y_k}+e^{z_k})V_{k,x}(W_k-V_k)_xdx.\end{aligned}$$

The first integral on the left-hand side can be estimated by means of the Poisson equation (7.13) and Young's inequality:

$$\begin{aligned}&\frac{1}{\tau}\int_\Omega\left((e^{y_k}-e^{z_k})-(e^{y_{k-1}}-e^{z_{k-1}})\right)(W_k-V_k)dx=\frac{\lambda^2}{\tau}\int_\Omega(V_k-V_{k-1})_x(V_k-W_k)_xdx\\&=\frac{\lambda^2}{\tau}\int_\Omega(V_k-W_k)_x^2dx-\frac{\lambda^2}{\tau}\int_\Omega(V_{k-1}-W_{k-1})_x(V_k-W_k)_xdx\\&\quad+\frac{\lambda^2}{\tau}\int_\Omega(W_k-W_{k-1})_x(V_k-W_k)_xdx\\&\geq\frac{\lambda^2}{2\tau}\int_\Omega(V_k-W_k)_x^2dx-\frac{\lambda^2}{2\tau}\int_\Omega(V_{k-1}-W_{k-1})_x^2dx+\frac{\lambda^2}{\tau}\int_\Omega(W_k-W_{k-1})_x(V_k-W_k)_xdx.\end{aligned}$$

Applying Young's inequality to the last integral in (7.39) gives

$$\int_\Omega(e^{y_k}+e^{z_k})V_{k,x}(W_k-V_k)_xdx\leq-\frac{1}{2}\int_\Omega(e^{y_k}+e^{z_k})V_{k,x}^2dx+\frac{1}{2}\int_\Omega(e^{y_k}+e^{z_k})W_{k,x}^2dx.$$

Thus, summation over k in (7.39) yields

$$\begin{aligned}
&\frac{\lambda^2}{2}\int_\Omega (V_N - W_N)_x^2 dx + \frac{1}{2}\sum_{k=1}^N \tau \int_\Omega (e^{y_k} + e^{z_k}) V_{k,x}^2 dx \\
&\quad\leq \frac{\lambda^2}{2}\int_\Omega (V_0 - W_0)_x^2 dx - \lambda^2 \sum_{k=1}^N \int_\Omega (W_k - W_{k-1})_x (V_k - W_k)_x dx \\
&\qquad + \frac{1}{2}\sum_{k=1}^N \tau \int_\Omega (e^{y_k} + e^{z_k}) W_{k,x}^2 dx + \frac{\varepsilon^2}{2}\sum_{k=1}^N \tau \int_\Omega (e^{y_k} y_{k,xx} + e^{z_k} z_{k,xx})(V_k - W_k)_{xx} dx \\
&\qquad - \sum_{k=1}^N \tau \int_\Omega (P_n(e^{y_k}) - P_p(e^{z_k}))(V_k - W_k)_{xx} dx \\
&\quad= I_1 + \cdots + I_5.
\end{aligned}$$

In the following, we write the integrals $I_1, \ldots, I_5$ in terms of $y^{(N)}$, $z^{(N)}$, $V^{(N)}$, and $W^{(N)}$.

For the first integral I_1 we notice that our assumption on the initial data gives $\lambda^2 V_{0,xx} = e^{y_0} - e^{z_0} = 0$ in Ω which, together with the boundary conditions $V_0(0) = 0$, $V_0(1) = U$, shows that V_0 is a linear function and in particular independent of λ. Thus, also $W_{0,x}$ does not depend on λ and

$$I_1 \leq \lambda^2 \|V_x^{(N)}(\cdot, 0)\|_{L^2(\Omega)}^2 + \lambda^2 \|W_x^{(N)}(\cdot, 0)\|_{L^2(\Omega)}^2 \leq c.$$

For I_2 we use (7.37):

$$I_2 \leq \frac{2\lambda^2}{\tau}\|W_x^{(N)}\|_{L^2(Q_T)}\left(\|V_x^{(N)}\|_{L^2(Q_T)} + \|W_x^{(N)}\|_{L^2(Q_T)}\right) \leq \frac{c\delta\lambda^{2/9}}{\tau}(\lambda^{-1} + \delta\lambda^{-16/9}) \leq \frac{c\delta}{\tau\lambda^{7/9}},$$

choosing $\delta \leq \lambda^{7/9}$. Taking into account (7.21) and (7.37) gives

$$I_3 \leq \frac{1}{2}\left(\|e^{y^{(N)}}\|_{L^3(Q_T)} + \|e^{z^{(N)}}\|_{L^3(Q_T)}\right)\|W_x^{(N)}\|_{L^3(Q_T)}^2 \leq c\delta^2\lambda^{-32/9},$$

and an application of Hölder's inequality and (7.21), (7.32), (7.35), and (7.38) yield

$$\begin{aligned}
I_4 &\leq \frac{\varepsilon^2}{2}\left(\|e^{y^{(N)}/2} y_{xx}^{(N)}\|_{L^2(Q_T)}\|e^{y^{(N)}/2}\|_{L^6(Q_T)} + \|e^{z^{(N)}/2} z_{xx}^{(N)}\|_{L^2(Q_T)}\|e^{y^{(N)}/2}\|_{L^6(Q_T)}\right) \\
&\quad \times \left(\|V_{xx}^{(N)}\|_{L^3(Q_T)} + \|W_{xx}^{(N)}\|_{L^3(Q_T)}\right) \leq c\lambda^{-16/9}.
\end{aligned}$$

We proceed with the integral I_5 which we estimate using the growth condition on P_n and P_p and (7.35), (7.38):

$$\begin{aligned}
I_5 &\leq \left(\|P_n(e^{y^{(N)}})\|_{L^{3/2}(Q_T)} + \|P_n(e^{y^{(N)}})\|_{L^{3/2}(Q_T)}\right)\left(\|V_{xx}^{(N)}\|_{L^3(Q_T)} + \|W_{xx}^{(N)}\|_{L^3(Q_T)}\right) \\
&\leq c\left(1 + \|e^{y^{(N)}}\|_{L^{3q/2}(Q_T)}^q + \|e^{z^{(N)}}\|_{L^{3q/2}(Q_T)}^q\right)\lambda^{-16/9} \leq c\lambda^{-16/9},
\end{aligned}$$

since $3q/2 \le 7/2$ is equivalent to our assumption $q \le 7/3$.

The above estimates yield, for sufficiently small $\lambda > 0$,

$$\int_{Q_T} (e^{y^{(N)}} + e^{z^{(N)}})(V^{(N)})_x^2 dxdt \le c\left(1 + \lambda^{-16/9} + \delta^2\lambda^{-32/9} + \delta\tau^{-1}\lambda^{-7/9}\right)$$

Letting $\delta \to 0$ then gives the assertion. □

Remark 7.2. In order to avoid an initial time layer we have assumed that $n_I = p_I$. The above proof shows that it is enough to require that $\|n_I - p_I\|_{H^{-1}(\Omega)}$ is of the order $O(\lambda^{1/9})$. Indeed, the estimate

$$\lambda^2\|V_{0,x}\|^2_{L^2(\Omega)} \le c(1 + \lambda^{-2}\|e^{y_0} - e^{z_0}\|^2_{H^{-1}(\Omega)}) \le c\lambda^{-16/9}$$

shows that $I_1 \le c\lambda^{-16/9}$ holds.

Remark 7.3. The assumption $q \le 7/3$ can be improved to $q < 5/2$ by more technical effort. Indeed, this condition is only needed in the computation of the integral I_5. In order to show how I_5 can be estimated assuming only $q < 5/2$, we proceed as follows.

By the same arguments as in the proof of Lemma 7.8, we can derive

$$\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^r(Q_T)} \le c\lambda^\theta, \quad \|V^{(N)}\|_{L^r(0,T;W^{2,r}(\Omega))} \le c\lambda^{\theta-2}$$

for $2 < r < 7/2$ and $\theta = 2(7-2r)/3r \in (0,1)$. Then

$$I_5 \le c\left(1 + \|e^{y^{(N)}}\|^q_{L^{qr/(r-1)}(Q_T)} + \|e^{z^{(N)}}\|^q_{L^{qr/(r-1)}(Q_T)}\right)\lambda^{\theta-2} \le c\lambda^{\theta-2},$$

since $qr/(r-1) \le 7/2$ is equivalent to $q < 5/2$. This yields

$$\|(e^{y^{(N)}} + e^{z^{(N)}})V_x^{(N)}\|_{L^2(Q_T)} \le c\lambda^{\theta/2-1},$$

which is sufficient for the proof of Theorem 7.2. However, the proof of Lemma 7.10 below becomes more involved. Therefore, and since the improvement is only marginal, we have assumed the stronger condition $q \le 7/3$.

Lemma 7.10. *There exists a constant $c > 0$ independent of λ such that for $s = 42/41$,*

$$\|e^{y^{(N)}} + e^{z^{(N)}} - \sigma_N(e^{y^{(N)}} + e^{z^{(N)}})\|_{L^s(0,T;H^{-3}(\Omega))} \le c\tau.$$

Recall that σ_N is the shift operator defined in (7.23).

Proof. We estimate the sum of equations (7.11) and (7.12):

$$\begin{aligned}
&\frac{1}{\tau}\|e^{y^{(N)}}+e^{z^{(N)}}-\sigma_N(e^{y^{(N)}}+e^{z^{(N)}})\|_{L^s(0,T;H^{-3}(\Omega))}\\
\le\;& \frac{\varepsilon^2}{2}\|e^{y^{(N)}}y_{xx}^{(N)}+e^{z^{(N)}}z_{xx}^{(N)}\|_{L^s(Q_T)}+\|P_n(e^{y^{(N)}})+P_p(e^{z^{(N)}})\|_{L^s(Q_T)}\\
&+\|(e^{y^{(N)}}-e^z)V_x^{(N)}\|_{L^s(Q_T)}.
\end{aligned}$$

The first term on the right-hand side is bounded by (7.19) and (7.21):

$$\|e^{y^{(N)}}y_{xx}^{(N)}\|_{L^s(Q_T)}\le\|e^{y^{(N)}}\|_{L^{2s/(2-s)}(Q_T)}\|y_{xx}^{(N)}\|_{L^2(Q_T)}\le\|e^{y^{(N)}}\|_{L^{21/10}(Q_T)}\|y_{xx}^{(N)}\|_{L^2(Q_T)}\le c,$$

and similarly for the expression for $z^{(N)}$. Taking into account the growth assumption on P_n and (7.21) we find

$$\|P_n(e^{y^{(N)}})\|_{L^s(Q_T)}\le c(1+\|e^{y^{(N)}}\|_{L^q(Q_T)}^q)\le c,$$

and analogously for $z^{(N)}$. For the drift term we need Lemma 7.6 and (7.21):

$$\|e^{y^{(N)}/2}-e^{z^{(N)}/2}\|_{L^{21/10}(Q_T)}\le\|e^{y^{(N)}/2}-e^{z^{(N)}/2}\|_{L^2(Q_T)}^{8/9}\|e^{y^{(N)}/2}-e^{z^{(N)}/2}\|_{L^{7/2}(Q_T)}^{1/9}\le c\lambda^{8/9}.$$

This yields, together with Lemma 7.9,

$$\begin{aligned}
\|(e^{y^{(N)}}-e^{z^{(N)}})V_x^{(N)}\|_{L^s(Q_T)}&\le\|e^{y^{(N)}/2}-e^{z^{(N)}/2}\|_{L^{21/10}(Q_T)}\|(e^{y^{(N)}/2}+e^{z^{(N)}/2})V_x^{(N)}\|_{L^2(Q_T)}\\
&\le c\lambda^{8/9}\lambda^{-8/9}=c.
\end{aligned}$$

Putting together the above bounds gives the assertion. □

7.3.2 Proof of Theorem 7.2

The results of section 7.2.2 allow to pass to the limit $\tau\to 0$ in the uniform estimates of the previous section. This yields weak solutions $y^{(\lambda)}$, $z^{(\lambda)}$, and $V^{(\lambda)}$ satisfying the equations

$$(e^{y^{(\lambda)}})_t+\frac{\varepsilon^2}{2}(e^{y^{(\lambda)}}y_{xx}^{(\lambda)})_{xx}=\big((P_n(e^{y^{(\lambda)}}))_x-e^{y^{(\lambda)}}V^{(\lambda)}\big)_x,\quad(7.40)$$

$$(e^{z^{(\lambda)}})_t+\frac{\varepsilon^2}{2}(e^{z^{(\lambda)}}z_{xx}^{(\lambda)})_{xx}=\big((P_p(e^{z^{(\lambda)}}))_x+e^{z^{(\lambda)}}V^{(\lambda)}\big)_x,\quad(7.41)$$

the boundary and initial conditions (7.4)-(7.5) and the following uniform bounds:

$$\begin{aligned}
\|e^{y^{(\lambda)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))}+\|e^{z^{(\lambda)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))}&\le c,\\
\|(e^{y^{(\lambda)}}+e^{z^{(\lambda)}})_t\|_{L^{42/41}(0,T;H^{-3}(\Omega))}&\le c,\\
\|y^{(\lambda)}\|_{L^2(0,T;H^2(\Omega))}+\|z^{(\lambda)}\|_{L^2(0,T;H^2(\Omega))}&\le c,\\
\|e^{y^{(\lambda)}}\|_{L^{7/2}(Q_T)}+\|e^{z^{(\lambda)}}\|_{L^{7/2}(Q_T)}&\le c
\end{aligned}$$

as well as, by Lemmas 7.6 and 7.9,

$$\|e^{y^{(\lambda)}/2} - e^{z^{(\lambda)}/2}\|_{L^2(Q_T)} \leq c\lambda, \qquad \|(e^{y^{(\lambda)}/2} + e^{z^{(\lambda)}/2})V_x^{(\lambda)}\|_{L^2(Q_T)} \leq c\lambda^{-8/9}. \quad (7.42)$$

Thus, Aubin's lemma and the arguments of section 7.2.2 show the existence of a subsequence (not relabeled) such that, as $\lambda \to 0$,

$$\begin{aligned}
e^{y^{(\lambda)}} \to e^y, \quad e^{z^{(\lambda)}} \to e^y \quad & \text{strongly in } L^3(Q_T) \text{ and weakly in } L^{7/2}(Q_T), \\
y^{(\lambda)} \rightharpoonup y, \quad z^{(\lambda)} \rightharpoonup y \quad & \text{weakly in } L^2(0,T;H^2(\Omega)), \\
(e^{y^{(\lambda)}} + e^{z^{(\lambda)}})_t \rightharpoonup 2(e^y)_t \quad & \text{weakly in } L^{42/41}(0,T;H^{-3}(\Omega)).
\end{aligned}$$

These convergence results imply for all sufficiently smooth ϕ, as $\lambda \to 0$,

$$\begin{aligned}
\int_0^T \langle (e^{y^{(\lambda)}} + e^{z^{(\lambda)}})_t, \phi \rangle_{H^{-3},H^3} dt \quad &\to \quad 2\int_0^T \langle (e^y)_t, \phi \rangle_{H^{-3},H^3} dt, \\
\int_{Q_T} (e^{y^{(\lambda)}} y_{xx}^{(\lambda)} + e^{z^{(\lambda)}} z_{xx}^{(\lambda)})\phi_{xx} dxdt \quad &\to \quad 2\int_{Q_T} e^y y_{xx} \phi_{xx} dxdt, \\
\int_{Q_T} (P_n(e^{y^{(\lambda)}}) + P_p(e^{z^{(\lambda)}}))\phi_{xx} dxdt \quad &\to \quad \int_{Q_T} (P_n(e^y) + P_p(e^y))\phi_{xx} dxdt.
\end{aligned}$$

The delicate integral is the expression containg the drift term. Here we need (7.42):

$$\begin{aligned}
&\int_{Q_T} (e^{y^{(\lambda)}} - e^{z^{(\lambda)}}) V_x^{(\lambda)} \phi_x dxdt \\
&\leq \|e^{y^{(\lambda)}/2} - e^{z^{(\lambda)}/2}\|_{L^2(Q_T)} \|(e^{y^{(\lambda)}/2} + e^{z^{(\lambda)}/2})V_x^{(\lambda)}\|_{L^2(Q_T)} \|\phi_x\|_{L^\infty(Q_T)} \\
&\leq c\lambda \cdot \lambda^{-8/9} \leq c\lambda^{1/9} \to 0.
\end{aligned}$$

These results allow to pass to the limit in the sum of the equations (7.40) and (7.41),

$$\begin{aligned}
&\int_0^T \langle (e^{y^{(\lambda)}} + e^{z^{(\lambda)}})_t, \phi \rangle_{H^{-3},H^3} dt + \frac{\varepsilon^2}{2} \int_{Q_T} (e^{y^{(\lambda)}} y_{xx}^{(\lambda)} + e^{z^{(\lambda)}} z_{xx}^{(\lambda)})\phi_{xx} dxdt \\
&= \int_{Q_T} (P_n(e^{y^{(\lambda)}}) + P_p(e^{z^{(\lambda)}}))\phi_{xx} dxdt + \int_{Q_T} (e^{y^{(\lambda)}} - e^{z^{(\lambda)}}) V_x^{(\lambda)} \phi_x dxdt,
\end{aligned}$$

which proves Theorem 7.2.

Chapter 8

Regularity and positivity of solutions for a logarithmic fourth-order parabolic equation

This Chapter is a work in collaboration with Ansgar Jüngel.

8.1 Introduction

The goal of this paper is to study the regularity properties of weak solutions to a nonlinear fourth-order equation with periodic boundary conditions. More precisely, we consider the following problem

$$u_t + (u(\log u)_{xx})_{xx} = 0, \quad x \in \ \Omega := (0,1), \ t > 0, \tag{8.1}$$

with periodic boundary conditions and the following initial condition

$$u(x,0) = u_0(x), \quad x \in \ \Omega. \tag{8.2}$$

Such an equation arises in the modeling of semiconductors which take into account the quantum effects due to the small size of the devices. Indeed, this kind of equation appears, for instance, for zero pressure, in the quasineutrality limit of the quantum drift-diffusion model (i.e. the limit as the ratio of Debye length to a characteristic length tends to zero, see [72] (Chapter 7 here)). Moreover, it is a one-homogeneous equation which is a simple example of a generalization of the heat equation to higher-order operators. It is then important to study its solutions and their regularity.

Mathematically, the parabolic equation (8.1) is of fourth-order and, in particular, no maximum principle is available, which complicates the analysis [69, 70]. We mention that (8.1) is the so-called Derrida-Lebowitz-Speer-Spohn equation derived in the context of fluctuations of a stationary non-equilibrium interface [40]. It has recently been widely studied because of its remarkable properties. For instance, the solutions are nonnegative and there are several Lyapunov functionals (see [41, 10, 17]). We refer to [63] for a systematic study. Moreover, there are connections to logarithmic Sobolev inequalities (see [41] and references therein).

The first analytical result has been presented in [10]. In this work, the existence of local-in-time classical solutions with periodic boundary conditions has been proved. A global-in-time existence result has been obtained for homogeneous Dirichlet-Neumann boundary conditions in [69] and for periodic boundary conditions in [41] with the same kind of method. Here, we also show a global-in-time existence result using a time discretization (as done in [41]) **but** the approximation scheme is different. This new scheme allows us to obtain additional regularity properties for the solutions.

The long-time behavior of solutions has been studied in [17] in the case of periodic boundary conditions under restrictive regularity conditions on the initial data, in [71] with homogeneous Dirichlet-Neumann boundary conditions and finally, in [59] with non-homogeneous Dirichlet-Neumann boundary conditions. In particular, it has been shown that the solutions converge exponentially fast to their steady-state in various norms and in terms of entropy (see [69]). The decay rate has been numerically computed in [18]. We also mention [70] in which a positivity-preserving numerical scheme for the quantum drift-diffusion model has been proposed.

Concerning the multi-dimensional case, the only result, to our knowledge, is the existence of global-in-time weak solutions (see [55]).

In the last years, the question of nonnegative or positive solutions of fourth-order parabolic equations has also been investigated in the context of lubrication-type equations, e.g. the thin film equation

$$u_t + (f(u)u_{xxx})_x = 0$$

(see, e.g., [7, 8]), where typically, $f(n) = n^\alpha$ for some $\alpha > 0$. This equation is of degenerate type which makes the analysis easier than it is for (8.1), at least concerning the positivity property. Notice that (8.1) is **not** of degenerate type. Note that, up to now, the positivity property for solutions to equation (8.1) has only been shown for times small enough (see [10]). Here using the regularity properties, we are also able to prove the positivity for times large enough.

As said above, the main goal of this paper is to improve the known regularity results for this equation. To this end, we first rewrite, for $\alpha \in [2/53(25 - 6\sqrt{10}), 1]$, (see appendix for the justification of the lowest bound) the equation (8.1) into

$$\frac{1}{\alpha} u^{1-\alpha} \partial_t (u^\alpha) + (u(\log u)_{xx})_{xx} = 0. \tag{8.3}$$

This new form of the equation (8.1) allows us to extend the results given in [41]. Clearly a solution of (8.3) is also a solution of (8.1). Here, we need to show the existence of solutions to equation (8.3). Indeed, it leads to a time discretization which is different from the one used in [41] and we can not use the results shown in this paper. In particular this new discretization allows us to prove that a solution u of (8.1) satisfies $u^\alpha \in L^2(0,T;H^3(\Omega))$ and $u^{\alpha/2} \in L^\infty(0,T,H^1(\Omega))$. Moreover we can show that $u \in L^\infty(0,T;L^\infty(\Omega))$. Our main results are the following Theorems 8.1 and 8.2.

Theorem 8.1. *Let $u_0 : \Omega \to \mathbb{R}$ be a nonnegative measurable function such that $\int_\Omega (u_0 - \log u_0)dx < \infty$ and $u_0 \in H^1(\Omega)$. Let $T > 0$. We assume that $\alpha \in [2/53(25 - 6\sqrt{10}), 1]$. Then there exists a global weak solution u of (8.3)-(8.2) satisfying*

$$u \in L^{5/2}(0,T;W^{1,1}(\Omega)),\ \log u \in L^2(0,T;H^2(\Omega)),$$

$$u \geq 0 \text{ in } \Omega \times (0,\infty),$$

and for all $T > 0$ and all smooth functions ϕ,

$$\int_0^T \frac{1}{\alpha} < (u^\alpha)_t, \phi >_{H^{-2},H^2} dt + \int_0^T \int_\Omega u(\log u)_{xx}(u^{\alpha-1}\phi)_{xx} dxdt = 0.$$

The initial datum is satisfied in the sense of $H^{-2}(\Omega) := (H_0^2(\Omega))^*$.

To show this result, we use a change of variable ($u = e^y$), and a time discretization of (8.3). Then, we prove a priori estimates on the constructed sequence of solutions, which allow us to pass to the limit to zero on the time step. Some of these a priori estimates are obtained by using the method presented in [63]. However, due to the form of (8.3), performing the vanishing time step limit is more complicated than in [41]. Indeed, multiplying (8.3) by $u^{\alpha-1}$, we obtain the weak formulation

$$\frac{1}{\alpha}\int_0^T < \partial_t(u^\alpha), \phi >_{H^{-2},H^2} dt + \int_0^T \int_\Omega u(\log u)_{xx}(u^{\alpha-1}\phi)_{xx} dxdt = 0,$$

for any smooth test function ϕ. Then in the discrete problem, the second integral yields a product of three terms (and not only two as in [41]). In such a situation, a priori estimates and Aubin's Lemma are not sufficient to obtain the strong convergences needed to perform the limit. Therefore, we also give some preliminary results which give strong convergence in one-space dimension with particular assumptions.

Theorem 8.2. *Under the assumptions of Theorem 8.1, the solution u of* (8.3)-(8.2) *verifies*

$$u^\alpha \in L^2(0,T;H^3(\Omega)),\ u^{\alpha/2} \in L^\infty(0,T;H^1(\Omega)),$$

$$\text{and } \ u \in L^\infty(0,T;L^\infty(\Omega)).$$

The proof of this Theorem uses some a priori estimates obtained for the proof of Theorem 8.1 and, in this sense, is a direct consequence of Theorem 8.1.

The paper is organized as follow. In section 8.2, we give some preliminary results on the semi-discretized in time problem: existence of solution to the discrete problem and a priori estimates needed for the proofs of Theorems 8.1 and 8.2. These proofs are given in section 8.3. Section 8.4 is devoted to the positivity property for times large enough. Finally, technical Lemmas are proved in the appendix.

8.2 Preliminary results

The goal of this section is to obtain some preliminary results, and in particular some a priori estimates, needed in the next section for the proofs of Theorems 8.1 and 8.2. First of all, let us set $u = e^y$ in problem (8.3)-(8.2). This gives

$$\frac{1}{\alpha\tau} e^{(1-\alpha)y} \partial_t (e^{\alpha y}) + (e^y y_{xx})_{xx} = 0, \quad x \in \Omega, \ t > 0, \tag{8.4}$$

with periodic boundary conditions and the following initial condition

$$y(0,x) = \log u_0(x), \quad \forall\, x \in \Omega. \tag{8.5}$$

Now we semi-discretize in time the equation (8.4). To this end, let $T > 0$. We divide the time interval $[0,T]$ in N subintervals $(t_{k-1}, t_k]$ with $t_k = \tau k, \ k = 0, ..., N$ where $\tau = T/N$ is the time step. For $k \in \{1, ..., N\}$ and y_{k-1} given, we solve the following equation

$$\frac{1}{\alpha\tau} e^{(1-\alpha)y_k} (e^{\alpha y_k} - e^{\alpha y_{k-1}}) + (e^{y_k} y_{k,xx})_{xx} = 0. \tag{8.6}$$

Let $k = 1, ..., N$, we introduce the piecewise constant function

$$y^{(N)}(t,x) = y_k(x), \text{ for } x \in \Omega, \ t \in (t_{k-1}, t_k],$$

and the shift operator

$$(\sigma_N(y^{(N)}))(t,x) = y_{k-1}(x), \text{ for } x \in \Omega, \ t \in (t_{k-1}, t_k].$$

Using (8.6), $y^{(N)}$ is solution of the following equation

$$\frac{1}{\alpha\tau} (e^{y^{(N)}})^{1-\alpha} \Big((e^{y^{(N)}})^{\alpha} - (\sigma_N(e^{y^{(N)}}))^{\alpha} \Big) + (e^{y^{(N)}} y^{(N)}_{xx})_{xx} = 0. \tag{8.7}$$

To prove the existence of solutions to problem (8.3)-(8.2), and then Theorem 8.1, we need to let τ tends to 0, or equivalently $N \to \infty$, in (8.7). To this end we need first to prove the existence of solutions to (8.6), and then to obtain a priori estimates which allow to pass to the limit.

8.2.1 Existence of solution to (8.6)

Lemma 8.1. *There exists a solution $y_k \in C^\infty(\bar{\Omega})$ of (8.6).*

Proof. The proof of this lemma is similar to the one given for Lemma 2 in [41]. Setting $z = y_{k-1}$ and $y = y_k$, we first consider, for a given $\varepsilon > 0$, the equation

$$(e^y y_{xx})_{xx} - \varepsilon y_{xx} + \varepsilon y = \frac{1}{\alpha\tau} e^{(1-\alpha)y}(e^{\alpha z} - e^{\alpha y}), \quad x \in \Omega. \tag{8.8}$$

In order to prove the existence of a solution to this approximate problem we employ the Leray-Schauder fixed point Theorem. To this end, let $w \in H^1(\Omega)$ and $\sigma \in [0,1]$ be given, we consider

$$a(y,\phi) = F(\phi) \text{ for all } \phi \in H^2(\Omega), \tag{8.9}$$

where,

$$a(y,\phi) = \int_\Omega (e^w y_{xx}\phi_{xx} + \varepsilon y_x \phi_x + \varepsilon y\phi)dx, \;\; F(\phi) = \frac{\sigma}{\alpha\tau}\int_\Omega e^{(1-\alpha)w}(e^{\alpha z} - e^{\alpha w})\phi dx$$

The application $a(.,.)$ is bilinear, continuous and coercive on $H^2(\Omega)$ (see [41]), and F is linear and continuous on $H^2(\Omega)$. Therefore the Lax-Milgram Lemma provides the existence of a solution $y \in H^2(\Omega)$ of (8.9). This defines a fixed-point operator $S : H^1(\Omega) \times [0,1] \to H^1(\Omega)$, $(w,\sigma) \mapsto y$. It holds $S(w,0) = 0$ for all $w \in H^1(\Omega)$. Moreover, the functional S is continuous and compact (since the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ is compact). We need to prove a uniform bound for all fixed points of $S(.,\sigma)$.

Let y be a fixed point of $S(.,\sigma)$, i.e., $y \in H^2(\Omega)$ solvers for all $\phi \in H^2(\Omega)$

$$\int_\Omega \Big(e^y y_{xx}\phi_{xx} + \varepsilon y_x\phi_x + \varepsilon y\phi\Big)dx = \frac{\sigma}{\alpha\tau}\int_\Omega e^{(1-\alpha)y}(e^{\alpha z} - e^{\alpha y})\phi dx. \tag{8.10}$$

Using the test function $\phi = 1 - e^{-y}$ yields

$$\begin{aligned}\int_\Omega y_{xx}^2 dx - \int_\Omega y_{xx}y_x^2 dx + \varepsilon\int_\Omega e^{-y}y_x^2 dx \;\; &+ \;\; \varepsilon\int_\Omega y(1-e^{-y})dx \\ &= \;\; \frac{\sigma}{\alpha\tau}\int_\Omega e^{(1-\alpha)y}(e^{\alpha z} - e^{\alpha y})(1-e^{-y})dx\end{aligned}$$

The second term on the left-hand side vanishes due to the periodicity condition since $y_{xx}y_x^2 = (y_x^3)_x/3$. The third and fourth terms on the left-hand side are nonnegative. Furthermore, we have

$$e^{(1-\alpha)y}(e^{\alpha z} - e^{\alpha y})(1-e^{-y}) = e^y(e^{\alpha z}e^{-\alpha y} - 1) + (1 - e^{\alpha z}e^{-\alpha y}),$$

and using $e^x \geq x+1,\ \forall x>0$ and $1-x^\beta \geq \beta(1-x),\ \forall x \geq 0,\ \beta \in (0,1]$, we obtain

$$1-e^{\alpha z}e^{-\alpha y} \leq -\alpha(z-y),\ e^y(e^{\alpha z}e^{-\alpha y}-1) \leq \alpha(e^z-e^y).$$

Then,

$$\frac{\sigma}{\alpha\tau}\int_\Omega e^{(1-\alpha)y}(e^{\alpha z}-e^{\alpha y})(1-e^{-y})dx \leq \frac{\sigma}{\tau}\int_\Omega [(e^z-z)-(e^y-y)]dx,$$

and we obtain

$$\frac{\sigma}{\tau}\int_\Omega (e^y-y)dx + \int_\Omega y_{xx}^2 dx \leq \frac{\sigma}{\tau}\int_\Omega (e^z-z)dx. \tag{8.11}$$

As in [41], this shows that all fixed points of the operator $S(.,\sigma)$ are uniformly bounded in $H^1(\Omega)$. Notice that a uniform bound for y in $H^2(\Omega)$ is also obtained. The Leray-Schauder fixed point Theorem finally ensures the existence of a fixed point of $S(.,1)$, i.e., the existence of a solution $y \in H^2(\Omega)$ to (8.8).

The limit $\varepsilon \to 0$ can be performed in (8.8) and we obtain the existence of a solution $y_k \in H^2(\Omega)$ to (8.6).

It remains now to prove that the solution y_k to (8.6) lies in $C^\infty(\bar{\Omega})$. To this end, let us set $u_k = e^{y_k}$. We can rewrite (8.6) into

$$u_{k,xxxx} = \left(\frac{2u_{k,xx}u_{k,x}}{u_k} - \frac{u_{k,x}^3}{u_k^2}\right)_x - \frac{1}{\alpha\tau}u_k^{1-\alpha}(u_k^\alpha - u_{k-1}^\alpha). \tag{8.12}$$

Since $y_k \in H^2(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, then $u_k \in H^2(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and $u_k^\beta \in L^\infty(\Omega)$ for all $\beta > 0$. Moreover, u_k is strictly positive in Ω and in particular $1/u_k \in L^\infty(\Omega)$. Thus $u_{k,xx}u_{k,x}/u_k \in L^2(\Omega)$ and $u_{k,x}^3/u_k^2 \in L^\infty(\Omega)$. This implies, in view of (8.12), that $u_{k,xxxx} \in H^{-1}(\Omega)$ and then $u_k \in H^3(\Omega)$. In a same way, using (8.12), we can now show that $u_{k,xxxx} \in L^2(\Omega)$ and $u_k \in H^4(\Omega)$. By bootstrapping, we finally obtain $u_k \in H^n(\Omega)$ for all $n \in \mathbb{N}$ and hence $u_k \in C^\infty(\bar{\Omega})$. Since u_k is strictly positive, this shows that $y_k = \log u_k \in C^\infty(\bar{\Omega})$. This completes the proof of Lemma 8.1. □

8.2.2 A priori estimates

In this part we will give some estimates on the sequence $y^{(N)}$. They will be needed to pass to the limit $N \to \infty$ (or equivalently $\tau \to 0$). First of all, as a direct consequence of the proof of lemma 8.1 we have the following inequalities

$$\|y^{(N)}\|_{L^\infty(0,T;L^1(\Omega))} + \|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \tag{8.13}$$

$$\|y^{(N)}\|_{L^2(0,T;H^2(\Omega))} \leq c. \tag{8.14}$$

Then we have:

Lemma 8.2. *There exists a constant $c(\alpha) > 0$ such that the following inequalities hold*

$$\|(e^{y^{(N)}})^{\alpha/2}\|_{L^\infty(0,T;H^1(\Omega))} + \|(e^{y^{(N)}})^{\alpha/2}\|_{L^2(0,T;H^3(\Omega))} \leq c(\alpha), \tag{8.15}$$

$$\|[(e^{y^{(N)}})^{\alpha/6}]_x\|_{L^6(0,T;L^6(\Omega))} \leq c(\alpha). \tag{8.16}$$

Lemma 8.3. *The following inequality holds*

$$\|(e^{y^{(N)}})^{\alpha}\|_{L^2(0,T;H^3(\Omega))} \leq c(\alpha), \tag{8.17}$$

with $c(\alpha) > 0$ a constant.

For the sake of clarity, these technical Lemmas will be proved in the appendix.

Lemma 8.4. *There exists a constant $c(\alpha) > 0$ such that*

$$\|[(e^{y^{(N)}})^{\alpha/4}]_x^4\|_{L^{3/2}(0,T;L^{3/2}(\Omega))} + \|[(e^{y^{(N)}})^{\alpha/3}]_x^3\|_{L^2(0,T;L^2(\Omega))} \leq c(\alpha), \tag{8.18}$$

$$\|[(e^{y^{(N)}})^{\alpha/2}]_x\|_{L^6(0,T;L^6(\Omega))} + \|[(e^{y^{(N)}})^{\alpha/2}]_{xx}\|_{L^3(0,T;L^3(\Omega))} \leq c(\alpha). \tag{8.19}$$

Proof. We have

$$\begin{aligned}
\|[(e^{y^{(N)}})^{\alpha/4}]_x^4\|_{L^{3/2}(0,T;L^{3/2}(\Omega))}^{3/2} &= \int_0^T \left(\int_\Omega \left(\frac{3}{2}\right)^6 [(e^{y^{(N)}})^{\alpha/6}]_x^6 (e^{y^{(N)}})^{\alpha/2} dx\right) dt, \\
&\leq \left(\frac{3}{2}\right)^6 \|(e^{y^{(N)}})^{\alpha/2}\|_{L^\infty(0,T;L^\infty(\Omega))} \|[(e^{y^{(N)}})^{\alpha/6}]_x\|_{L^6(0,T;L^6(\Omega))}^6, \\
&\leq \left(\frac{3}{2}\right)^6 \|(e^{y^{(N)}})^{\alpha/2}\|_{L^\infty(0,T;H^1(\Omega))} \|[(e^{y^{(N)}})^{\alpha/6}]_x\|_{L^6(0,T;L^6(\Omega))}^6, \\
&\leq c(\alpha),
\end{aligned}$$

from the Sobolev injection theorem, (8.15) and (8.16). In the same way, we have

$$\begin{aligned}
\|[(e^{y^{(N)}})^{\alpha/3}]_x^3\|_{L^2(0,T;L^2(\Omega))}^2 &= \int_0^T \left(\int_\Omega 2^6 [(e^{y^{(N)}})^{\alpha/6}]_x^6 [(e^{y^{(N)}})^{\alpha/2}]^2 dx\right) dt, \\
&\leq 2^6 \|(e^{y^{(N)}})^{\alpha/2}\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \|[(e^{y^{(N)}})^{\alpha/6}]_x\|_{L^6(0,T;L^6(\Omega))}^6, \\
&\leq 2^6 \|(e^{y^{(N)}})^{\alpha/2}\|_{L^\infty(0,T;H^1(\Omega))}^2 \|[(e^{y^{(N)}})^{\alpha/6}]_x\|_{L^6(0,T;L^6(\Omega))}^6, \\
&\leq c(\alpha),
\end{aligned}$$

which gives (8.18). Finally

$$\begin{aligned}
\|[(e^{y^{(N)}})^{\alpha/2}]_x\|_{L^6(0,T;L^6(\Omega))}^6 &= \int_0^T \left(\int_\Omega 3^6 [(e^{y^{(N)}})^{\alpha/6}]_x^6 [(e^{y^{(N)}})^{\alpha/2}]^4 dx\right) dt, \\
&\leq 3^6 \|(e^{y^{(N)}})^{\alpha/2}\|_{L^\infty(0,T;L^\infty(\Omega))}^4 \|[(e^{y^{(N)}})^{\alpha/6}]_x\|_{L^6(0,T;L^6(\Omega))}^6, \\
&\leq 3^6 \|(e^{y^{(N)}})^{\alpha/2}\|_{L^\infty(0,T;H^1(\Omega))}^4 \|[(e^{y^{(N)}})^{\alpha/6}]_x\|_{L^6(0,T;L^6(\Omega))}^6, \\
&\leq c(\alpha).
\end{aligned}$$

Then, using the Gagliardo-Nirenberg inequality for the function $f = [(e^{y^{(N)}})^{\alpha/2}]_x$ we obtain

$$\begin{aligned}\|[(e^{y^{(N)}})^{\alpha/2}]_{xx}\|_{L^3(\Omega)} &\leq \|[(e^{y^{(N)}})^{\alpha/2}]_x\|_{H^2(\Omega)}^{1/2}\|[(e^{y^{(N)}})^{\alpha/2}]_x\|_{L^6(\Omega)}^{1/2}, \\ &\leq \|(e^{y^{(N)}})^{\alpha/2}\|_{H^3(\Omega)}^{1/2}\|[(e^{y^{(N)}})^{\alpha/2}]_x\|_{L^6(\Omega)}^{1/2},\end{aligned}$$

that is

$$\|[(e^{y^{(N)}})^{\alpha/2}]_{xx}\|_{L^3(0,T;L^3(\Omega))}^3 \leq c\int_0^T \|(e^{y^{(N)}})^{\alpha/2}\|_{H^3(\Omega)}^{3/2}\|[(e^{y^{(N)}})^{\alpha/2}]_x\|_{L^6(\Omega)}^{3/2}dt,$$

In view of this, (8.15) and using the Hölder inequality, we have

$$\begin{aligned}\|[(e^{y^{(N)}})^{\alpha/2}]_{xx}\|_{L^3(0,T;L^3(\Omega))}^3 &\leq c\|[(e^{y^{(N)}})^{\alpha/2}]_x\|_{L^2(0,T;H^3(\Omega))}^{3/2}\|[(e^{y^{(N)}})^{\alpha/2}]_x\|_{L^6(0,T;L^6(\Omega))}^{3/2}, \\ &\leq c(\alpha).\end{aligned}$$

This gives (8.19) and completes the proof of Lemma 8.4. □

Lemma 8.5. *Let $s = 14/11$. Then there exists a constant $c(\alpha) > 0$ such that*

$$\|(e^{y^{(N)}})^\alpha - (\sigma_N e^{y^{(N)}})^\alpha\|_{L^s(0,T;H^{-3}(\Omega))} \leq c(\alpha)\tau. \tag{8.20}$$

Proof. By (8.6) we have

$$\frac{1}{\tau}(e^{\alpha y_k} - e^{\alpha y_{k-1}}) = -\alpha e^{(\alpha-1)y_k}(e^{y_k}y_{k,xx})_{xx}. \tag{8.21}$$

For clarity of presentation we note $v = e^{y_k}$ and $w = e^{y_{k-1}}$. We can write

$$\begin{aligned}\alpha v^{\alpha-1}\Big(v(\log v)_{xx}\Big)_{xx} &= 2(v^\alpha(\log v^\alpha)_{xx})_{xx} - (v^\alpha)_{xxxx} + \frac{4}{\alpha}((v^{\alpha/2})_x^2)_{xx} \\ &\quad -c_1(\alpha)(v^{\alpha/2})_{xx}^2 - c_2(\alpha)(v_x^3 v^{\alpha-3})_x + c_3(\alpha)(v^{\alpha/4})_x^4,\end{aligned}$$

where,

$$c_1(\alpha) = \frac{4}{\alpha}(1-\alpha),\ c_2(\alpha) = \frac{\alpha}{3}(1-\alpha)(3\alpha-1),\ c_3(\alpha) = \alpha^2\Big(-\frac{\alpha^2}{4} + \frac{7\alpha}{12} - \frac{1}{3}\Big)\Big(\frac{4}{\alpha}\Big)^4.$$

Then using (8.21) we obtain:

$$\begin{aligned}\frac{1}{\tau}(v^\alpha - w^\alpha) = (v^\alpha)_{xxxx} - 2(v^\alpha(\log v^\alpha)_{xx})_{xx} - \frac{4}{\alpha}((v^{\alpha/2})_x^2)_{xx} + c_1(\alpha)(v^{\alpha/2})_{xx}^2 \\ +c_2(\alpha)((v^{\alpha/3})_x^3)_x - c_3(\alpha)(v^{\alpha/4})_x^4,\end{aligned}$$

which gives,

$$\begin{aligned}
\tau^{-1}\|(e^{y^{(N)}})^{\alpha} - (\sigma_N e^{y^{(N)}})^{\alpha}\|_{L^s(0,T;H^{-3}(\Omega))} \\
\leq\ & \|[(e^{y^{(N)}})^{\alpha}]_{xxxx}\|_{L^s(0,T;H^{-3}(\Omega))} + 2\alpha\|[(e^{y^{(N)}})^{\alpha} y^{(N)}_{xx}]_{xx}\|_{L^s(0,T;H^{-3}(\Omega))} \\
+\ & \frac{4}{\alpha}\left\|\left[((e^{y^{(N)}})^{\alpha/2})_x^2\right]_{xx}\right\|_{L^s(0,T;H^{-3}(\Omega))} + |c_1(\alpha)|\ \|[(e^{y^{(N)}})^{\alpha/2}]^2_{xx}\|_{L^s(0,T;H^{-3}(\Omega))} \\
+\ & |c_2(\alpha)|\ \left\|\left[(e^{y^{(N)}})^{\alpha/3})_x^3\right]_x\right\|_{L^s(0,T;H^{-3}(\Omega))} + |c_3(\alpha)|\ \|[(e^{y^{(N)}})^{\alpha/4}]^4_x\|_{L^s(0,T;H^{-3}(\Omega))}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\tau^{-1}\ \|(e^{y^{(N)}})^{\alpha} - (\sigma_N e^{y^{(N)}})^{\alpha}\|_{L^s(0,T;H^{-3}(\Omega))} \\
&\leq \|[(e^{y^{(N)}})^{\alpha}]_{xx}\|_{L^s(0,T;H^{-1}(\Omega))} + 2\alpha\|(e^{y^{(N)}})^{\alpha} y^{(N)}_{xx}\|_{L^s(0,T;H^{-1}(\Omega))} \\
&+\frac{4}{\alpha}\|[(e^{y^{(N)}})^{\alpha/2}]^2_x\|_{L^s(0,T;H^{-1}(\Omega))} + |c_1(\alpha)|\ \|[(e^{y^{(N)}})^{\alpha/2}]^2_{xx}\|_{L^s(0,T;H^{-3}(\Omega))} \\
&+|c_2(\alpha)|\ \|[(e^{y^{(N)}})^{\alpha/3}]^3_x\|_{L^s(0,T;H^{-2}(\Omega))} + |c_3(\alpha)|\ \|[(e^{y^{(N)}})^{\alpha/4}]^4_x\|_{L^s(0,T;H^{-3}(\Omega))}. \quad (8.22)
\end{aligned}$$

As in [72] (Chapter 7) for the proof of Lemma 7.4, we can show that

$$\|(e^{y^{(N)}})^{\alpha} y^{(N)}_{xx}\|_{L^s(0,T;H^{-1}(\Omega))} \leq c(\alpha), \quad (8.23)$$

for $s = 14/11$. (Notice that the assumption $s = 14/11$ was not needed before). The domain Ω being bounded in one space dimension, by the Sobolev injection Theorem, since $L^1(\Omega)$ is included in $(L^{\infty}(\Omega))'$,

$$\|[(e^{y^{(N)}})^{\alpha/2}]^2_x\|_{L^s(0,T;H^{-1}(\Omega))} \leq c\|[(e^{y^{(N)}})^{\alpha/2}]^2_x\|_{L^s(0,T;L^1(\Omega))}. \quad (8.24)$$

Moreover,

$$\|[(e^{y^{(N)}})^{\alpha/2}]^2_x\|_{L^1(\Omega)} = \int_{\Omega}[(e^{y^{(N)}})^{\alpha/2}]^2_x dx = \|[(e^{y^{(N)}})^{\alpha/2}]_x\|^2_{L^2(\Omega)} \leq \|(e^{y^{(N)}})^{\alpha/2}\|^2_{H^1(\Omega)}.$$

Thus,

$$\begin{aligned}
\|[(e^{y^{(N)}})^{\alpha/2}]^2_x\|^s_{L^s(0,T;L^1(\Omega))} = \int_0^T \|[(e^{y^{(N)}})^{\alpha/2}]^2_x\|^s_{L^1(\Omega)} dt &\leq \int_0^T \|(e^{y^{(N)}})^{\alpha/2}\|^{2s}_{H^1(\Omega)} dt, \\
&\leq T\|(e^{y^{(N)}})^{\alpha/2}\|^{2s}_{L^{\infty}(0,T;H^1(\Omega))},
\end{aligned}$$

which gives, using (8.24) and (8.15),

$$\|[(e^{y^{(N)}})^{\alpha/2}]^2_x\|_{L^s(0,T;H^{-1}(\Omega))} \leq c(\alpha). \quad (8.25)$$

Now, proceeding in a same way than in [72] (Chapter 7) for the proof of Lemma 7.4, since $s < 2$, we have

$$\|[(e^{y^{(N)}})^\alpha]_{xx}\|_{L^s(0,T;H^{-1}(\Omega))} \leq c\|[(e^{y^{(N)}})^\alpha]_{xx}\|_{L^2(0,T;L^2(\Omega))} \leq c\|(e^{y^{(N)}})^\alpha\|_{L^2(0,T;H^2(\Omega))},$$

which gives, using (8.17),

$$\|[(e^{y^{(N)}})^\alpha]_{xx}\|_{L^s(0,T;H^{-1}(\Omega))} \leq c(\alpha). \tag{8.26}$$

We have, as above,

$$\|[(e^{y^{(N)}})^{\alpha/2}]^2_{xx}\|_{L^s(0,T;H^{-3}(\Omega))} \leq c\|[(e^{y^{(N)}})^{\alpha/2}]^2_{xx}\|_{L^s(0,T;L^1(\Omega))},$$

and,

$$\begin{aligned}\|[(e^{y^{(N)}})^{\alpha/2}]^2_{xx}\|^s_{L^s(0,T;L^1(\Omega))} &= \int_0^T \|[(e^{y^{(N)}})^{\alpha/2}]^2_{xx}\|^s_{L^1(\Omega)}dt = \int_0^T \|[(e^{y^{(N)}})^{\alpha/2}]_{xx}\|^{2s}_{L^2(\Omega)}dt,\\ &= \|[(e^{y^{(N)}})^{\alpha/2}]_{xx}\|^{2s}_{L^{2s}(0,T;L^2(\Omega))},\end{aligned}$$

which gives

$$\|[(e^{y^{(N)}})^{\alpha/2}]^2_{xx}\|_{L^s(0,T;H^{-3}(\Omega))} \leq c\|[(e^{y^{(N)}})^{\alpha/2}]_{xx}\|^2_{L^{2s}(0,T;L^2(\Omega))}.$$

Since $s < 3/2$, we get

$$\|[(e^{y^{(N)}})^{\alpha/2}]^2_{xx}\|_{L^s(0,T;H^{-3}(\Omega))} \leq c\|[(e^{y^{(N)}})^{\alpha/2}]_{xx}\|^2_{L^3(0,T;L^3(\Omega))} \leq c(\alpha), \tag{8.27}$$

using inequality (8.19). In a same way, using (8.18), we can show

$$\begin{aligned}\|[(e^{y^{(N)}})^{\alpha/3}]^3_x\|_{L^s(0,T;H^{-2}(\Omega))} &\leq c\|[(e^{y^{(N)}})^{\alpha/3}]^3_x\|_{L^s(0,T;L^2(\Omega))},\\ &\leq \|[(e^{y^{(N)}})^{\alpha/3}]^3_x\|_{L^2(0,T;L^2(\Omega))},\\ &\leq c(\alpha),\end{aligned} \tag{8.28}$$

and,

$$\begin{aligned}\|[(e^{y^{(N)}})^{\alpha/4}]^4_x\|_{L^s(0,T;H^{-3}(\Omega))} &\leq c\|[(e^{y^{(N)}})^{\alpha/4}]^4_x\|_{L^s(0,T;L^1(\Omega))},\\ &\leq \|[(e^{y^{(N)}})^{\alpha/4}]^4_x\|_{L^{3/2}(0,T;L^{3/2}(\Omega))},\\ &\leq c(\alpha).\end{aligned} \tag{8.29}$$

Now using (8.22), (8.23), (8.25), (8.26), (8.27), (8.28) and (8.29) we finally obtain the inequality (8.20). □

Lemma 8.6. *The following inequality holds*

$$\|(e^{y^{(N)}})^{\alpha/4}\|_{L^2(0,T;H^2(\Omega))} \leq c(\alpha),$$

with $c(\alpha) > 0$ *a constant.*

Proof. By definition

$$\|(e^{y^{(N)}})^{\alpha/4}\|_{L^2(0,T;H^2(\Omega))} = \int_0^T \Big[\|(e^{y^{(N)}})^{\alpha/4}\|^2_{L^2(\Omega)} + \|((e^{y^{(N)}})^{\alpha/4})_x\|^2_{L^2(\Omega)} + \|((e^{y^{(N)}})^{\alpha/4})_{xx}\|^2_{L^2(\Omega)}\Big] dt.$$

By using (8.15), we can easily show that

$$\int_0^T \|(e^{y^{(N)}})^{\alpha/4}\|^2_{L^2(\Omega)} dt \leq c(\alpha),$$

and by using (8.18)

$$\int_0^T \|((e^{y^{(N)}})^{\alpha/4})_x\|^2_{L^2(\Omega)} dt \leq c(\alpha).$$

Moreover,

$$\|((e^{y^{(N)}})^{\alpha/4})_{xx}\|^2_{L^2(Q_T)} = \int_0^T \int_\Omega \left(\frac{\alpha}{4} y^{(N)}_{xx} (e^{y^{(N)}})^{\alpha/4} + \left(\frac{\alpha}{4}\right)^2 (y^{(N)}_x)^2 (e^{y^{(N)}})^{\alpha/4}\right)^2 dxdt,$$

$$\leq 2\left(\frac{\alpha}{4}\right)^2 \int_0^T \int_\Omega \left(y^{(N)}_{xx}\right)^2 (e^{y^{(N)}})^{\alpha/2} dxdt + 2\left(\frac{\alpha}{4}\right)^4 \int_0^T \int_\Omega (y^{(N)}_x)^4 (e^{y^{(N)}})^{\alpha/2} dxdt. \quad (8.30)$$

We have by using (8.14) and (8.15)

$$2\left(\frac{\alpha}{4}\right)^2 \int_0^T \int_\Omega \left(y^{(N)}_{xx}\right)^2 (e^{y^{(N)}})^{\alpha/2} dxdt \leq 2\left(\frac{\alpha}{4}\right)^2 \|(e^{y^{(N)}})^{\alpha/2}\|_{L^\infty(0,T;L^\infty(\Omega))} \|y^{(N)}_{xx}\|^2_{L^2(0,T;L^2(\Omega))} \leq c(\alpha). \quad (8.31)$$

Furthermore, using the Hölder inequality in view of (8.14), (8.15) and (8.16), we have

$$2\left(\frac{\alpha}{4}\right)^4 \int_0^T \int_\Omega (y^{(N)}_x)^4 (e^{y^{(N)}})^{\alpha/2} dxdt = 2\left(\frac{\alpha}{4}\right)^4 \left(\frac{6}{\alpha}\right)^3 \int_0^T \int_\Omega y^{(N)}_x ((e^{y^{(N)}})^{\alpha/6})^3_x dxdt,$$

$$\leq 2\left(\frac{\alpha}{4}\right)^4 \left(\frac{6}{\alpha}\right)^3 \int_0^T \|y^{(N)}_x\|_{L^2(\Omega)} \|((e^{y^{(N)}})^{\alpha/6})_x\|^3_{L^6(\Omega)} dt,$$

$$\leq 2\left(\frac{\alpha}{4}\right)^4 \left(\frac{6}{\alpha}\right)^3 \|y^{(N)}_x\|_{L^2(0,T;L^2(\Omega))} \|((e^{y^{(N)}})^{\alpha/6})_x\|^3_{L^6(0,T;L^6(\Omega))} \leq c(\alpha). \quad (8.32)$$

Finally, combining (8.31), (8.32) and (8.30) ends the proof. □

8.3 Existence and regularity of solutions

This section is devoted to the proofs of Theorems 8.1 and 8.2. First of all let us give three results which establish strong convergence properties for well-prepared sequences.

8.3.1 Convergence results

Lemma 8.7. *Let Ω be bounded in $\mathbb{R}$. Let $(g_n)_n$ be a sequence in $Q_T = [0,T] \times \Omega$ such that*

(A1) $g_n \to g$ *a.e. in* Q_T,

(A2) $\|g_n\|_{L^\infty(Q_T)} \leq c$.

Then, $g_n \to g$ *strongly in* $L^p(0,T;L^p(\Omega))$ *for all* $1 \leq p < \infty$.

Proof. Let $1 \leq p < \infty$. Setting $f_n = g_n - g$, by assumption, $f_n \to 0$ a.e. in Q_T. Moreover, since Ω is bounded, $g_n \to g$ a.e. in Q_T and $\|g_n\|_{L^\infty(Q_T)} \leq c$ imply $\|g\|_{L^\infty(Q_T)} \leq c$. Hence, $|f_n(t,\dot{})|^p \leq c$ for all $t \in [0,T]$. We apply now the Lebesgue Theorem to the sequence $(|f_n(t,.)|^p)_n$ in Q_T and we obtain

$$\int_{Q_T} |f_n(t,x)|^p dx \longrightarrow 0.$$

Then,

$$\int_{Q_T} |g_n(t,x) - g(t,x)|^p dx \longrightarrow 0. \tag{8.33}$$

Hence,

$$\|g_n - g\|_{L^p(0,T;L^p(\Omega))} \longrightarrow 0.$$

Then, $g_n \to g$ strongly in $L^p(0,T;L^p(\Omega))$. Finally, since we fixed any p in $[1,\infty[$, the result holds for all $1 \leq p < \infty$. □

Theorem 8.3. *Let $(g_n)_n$ be a sequence of functions in Q_T such that*

(A1) $g_n \rightharpoonup g$ *weakly in* $L^2(0,T;H^2(\Omega))$,

(A2) $g_n \to g$ *strongly in* $L^2(0,T;L^p(\Omega))$, $\forall\ 1 \leq p < \infty$.

Then $g_n \to g$ *strongly in* $L^2(0,T;W^{1,q}(\Omega))$ *for all* $\frac{4}{3} \leq q < \infty$.

Proof. Let $\frac{4}{3} \leq q < \infty$. We have to show that

$$\|g_n - g\|_{L^2(0,T;W^{1,q}(\Omega))} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

We have,

$$\begin{aligned}
&\|g_n - g\|_{L^2(0,T;W^{1,q}(\Omega))} \\
&\quad = \int_0^T \left(\|g_n(t,.) - g(t,.)\|^q_{L^q(\Omega)} + \|(g_n(t,.) - g(t,.))_x\|^q_{L^q(\Omega)} \right)^{2/q} dt, \\
&\quad \leq 2 \int_0^T \left(\|g_n(t,.) - g(t,.)\|^2_{L^q(\Omega)} + \|(g_n(t,.) - g(t,.))_x\|^2_{L^q(\Omega)} \right) dt, \\
&\quad \leq 2\|g_n - g\|^2_{L^2(0,T;L^q(\Omega))} + 2\int_0^T \|(g_n(t,.) - g(t,.))_x\|^2_{L^q(\Omega)} dt. \qquad (8.34)
\end{aligned}$$

Let $\theta = (4q-2)/(5q) \in [0,1]$. Using the Gagliardo-Nirenberg inequality, we obtain

$$\|(g_n(t,.) - g(t,.))_x\|^2_{L^q(\Omega)} \leq c\|g_n(t,.) - g(t,.)\|^{2\theta}_{H^2(\Omega)} \|g_n(t,.) - g(t,.)\|^{2(1-\theta)}_{L^1(\Omega)}.$$

Hence, using the Hölder inequality and assumption (A1),

$$\begin{aligned}
&\int_0^T \|(g_n(t,.) - g(t,.))_x\|^2_{L^q(\Omega)} dt \\
&\quad \leq c \int_0^T \|g_n(t,.) - g(t,.)\|^{2\theta}_{H^2(\Omega)} \|g_n(t,.) - g(t,.)\|^{2(1-\theta)}_{L^1(\Omega)} dt, \\
&\quad \leq \left(\int_0^T \|g_n(t,.) - g(t,.)\|^2_{H^2(\Omega)} dt \right)^{\theta} \left(\int_0^T \|g_n(t,.) - g(t,.)\|^2_{L^1(\Omega)} dt \right)^{1-\theta}, \\
&\quad \leq \|g_n - g\|^{2\theta}_{L^2(0,T;H^2(\Omega))} \|g_n - g\|^{2(1-\theta)}_{L^2(0,T;L^1(\Omega))} \leq c\|g_n - g\|^{2(1-\theta)}_{L^2(0,T;L^1(\Omega))}.
\end{aligned}$$

In view of assumption (A2), we then have

$$\int_0^T \|(g_n(t,.) - g(t,.))_x\|^2_{L^q(\Omega)} dt \longrightarrow 0,$$

and using (8.34), we deduce that $g_n \to g$ strongly in $L^2(0,T;W^{1,q}(\Omega))$ for all $q \in [4/3, \infty[$.

□

We also prove the following result

Theorem 8.4. *Let $(g_n)_n$ a sequence of functions in Q_T such that*

(A1) $g_n \rightharpoonup g$ *weakly in* $L^2(0,T;H^3(\Omega))$,

(A2) $g_n \to g$ *strongly in* $L^2(0,T;L^p(\Omega))$, $\forall\ 1 \le p < \infty$.

Then $g_n \to g$ *strongly in* $L^2(0,T;W^{2,q}(\Omega))$ *for all* $\frac{3}{2} \le q < \infty$.

The proof of this Theorem only differs from the one of Theorem 8.3 by the choice of θ (here we set $\theta = (6q-2)/(7q)$).

8.3.2 Proofs of Theorems 8.1 and 8.2

It now remains to pass to the limit $\tau \to 0$ (or equivalently $N \to +\infty$) to prove Theorems 8.1 and 8.2. First of all let us rewrite equation (8.7) into

$$\frac{1}{\alpha\tau}\Big(e^{\alpha y^{(N)}} - \sigma_N\big(e^{\alpha y^{(N)}}\big)\Big) + e^{(\alpha-1)y^{(N)}}\Big(e^{y^{(N)}} y^{(N)}_{xx}\Big)_{xx} = 0.$$

This last equation has the following weak formulation

$$\frac{1}{\alpha}\int_0^T\int_\Omega \frac{1}{\tau}\Big(e^{\alpha y^{(N)}} - \sigma_N\big(e^{\alpha y^{(N)}}\big)\Big)\phi dxdt + \int_0^T\int_\Omega e^{y^{(N)}} y^{(N)}_{xx}\Big(e^{(\alpha-1)y^{(N)}}\phi\Big)_{xx} dxdt = 0 \tag{8.35}$$

We set $w^{(N)} = e^{\alpha y^{(N)}/4}$. Then

$$e^{y^{(N)}} = (w^{(N)})^{4/\alpha},\ e^{(\alpha-1)y^{(N)}} = (w^{(N)})^{4-4/\alpha},\ \text{and } y^{(N)} = \frac{4}{\alpha}\ln w^{(N)}.$$

The goal is now to rewrite the second integral in (8.35) as integrals of a function of $w^{(N)}$, $w^{(N)}_x$ and $w^{(N)}_{xx}$. First, we have

$$\begin{aligned}
& e^{y^{(N)}} y^{(N)}_{xx}\Big(e^{(\alpha-1)y^{(N)}}\phi\Big)_{xx} = (w^{(N)})^{4/\alpha}\Big(\frac{4}{\alpha}\ln w^{(N)}\Big)_{xx}\Big((w^{(N)})^{4-4/\alpha}\phi\Big)_{xx} \\
= & \frac{4}{\alpha}\Big(4-\frac{4}{\alpha}\Big)(w^{(N)}_{xx})^2(w^{(N)})^2\phi - \frac{4}{\alpha}\Big(4-\frac{4}{\alpha}\Big)w^{(N)}_{xx}(w^{(N)}_x)^2 w^{(N)}\phi \\
+ & \frac{4}{\alpha}\Big(4-\frac{4}{\alpha}\Big)\Big(3-\frac{4}{\alpha}\Big)w^{(N)}_{xx}(w^{(N)}_x)^2 w^{(N)}\phi - \frac{4}{\alpha}\Big(4-\frac{4}{\alpha}\Big)\Big(3-\frac{4}{\alpha}\Big)(w^{(N)}_x)^4\phi \\
+ & \frac{8}{\alpha}\Big(4-\frac{4}{\alpha}\Big)w^{(N)}_{xx}w^{(N)}_x(w^{(N)})^2\phi_x - \frac{8}{\alpha}\Big(4-\frac{4}{\alpha}\Big)(w^{(N)}_x)^3 w^{(N)}\phi_x \\
+ & \frac{4}{\alpha}w^{(N)}_{xx}(w^{(N)})^3\phi_{xx} - \frac{4}{\alpha}(w^{(N)}_x)^2(w^{(N)})^2\phi_{xx}
\end{aligned} \tag{8.36}$$

Moreover, we can write

$$\begin{aligned}
\frac{4}{\alpha}\Big(4-\frac{4}{\alpha}\Big)(w^{(N)}_{xx})^2(w^{(N)})^2\phi \ = \ & \frac{2}{\alpha}\Big(4-\frac{4}{\alpha}\Big)w^{(N)}_{xx}(e^{\alpha y^{(N)}/2})_{xx}w^{(N)}\phi \\
& -\frac{4}{\alpha}\Big(4-\frac{4}{\alpha}\Big)w^{(N)}_{xx}(w^{(N)}_x)^2 w^{(N)}\phi
\end{aligned}$$

Then (8.35) can be rewritten as

$$\begin{aligned}
&\frac{1}{\alpha}\int_0^T\int_\Omega \frac{1}{\tau}\Big(e^{\alpha y^{(N)}} - \sigma_N e^{\alpha y^{(N)}}\Big)\phi dxdt \\
= &-\frac{2}{\alpha}\Big(4-\frac{4}{\alpha}\Big)\Bigg[\int_0^T\int_\Omega w_{xx}^{(N)}(e^{\alpha y^{(N)}/2})_{xx} w^{(N)}\phi dxdt - 4\int_0^T\int_\Omega w_{xx}^{(N)}(w_x^{(N)})^2 w^{(N)}\phi dxdt \\
&+4\int_0^T\int_\Omega w_{xx}^{(N)} w_x^{(N)}(w^{(N)})^2\phi_x dxdt - 4\int_0^T\int_\Omega (w_x^{(N)})^3 w^{(N)}\phi_x dxdt\Bigg] \\
&-\frac{4}{\alpha}\Big(4-\frac{4}{\alpha}\Big)\Big(3-\frac{4}{\alpha}\Big)\Bigg[\int_0^T\int_\Omega w_{xx}^{(N)}(w_x^{(N)})^2 w^{(N)}\phi dxdt - \int_0^T\int_\Omega (w_x^{(N)})^4\phi dxdt\Bigg] \\
&-\frac{4}{\alpha}\Bigg[\int_0^T\int_\Omega w_{xx}^{(N)}(w^{(N)})^3\phi_{xx} dxdt - \int_0^T\int_\Omega (w_x^{(N)})^2(w^{(N)})^2\phi_{xx} dxdt\Bigg] \qquad (8.37)
\end{aligned}$$

Then to complete the proof of Theorem 8.1, we now have to perform the limit $\tau \to 0$ or equivalently $N \to \infty$ in (8.37). Actually, the uniform bounds (8.17) and (8.20) and the compact embedding $W^{3,2}(\Omega) \hookrightarrow W^{2,q}(\Omega)$, for all $1 \leq q < \infty$, allow to apply Theorem 5 of [93] (Aubin's lemma) yielding the existence of a subsequence of $(e^{y^{(N)}})^\alpha$ (not relabeled) such that $(e^{y^{(N)}})^\alpha \to v$ strongly in $L^2(0,T;W^{2,q}(\Omega))$, for all $q \in [1,\infty[$, as $N \to \infty$. Hence, $(e^{y^{(N)}})^\alpha \to v$ strongly in $L^1(Q_T)$. Moreover, from (8.14), we have (up to a subsequence)

$$y^{(N)} \rightharpoonup y \ \text{ weakly in } L^2(0,T,H^2(\Omega)), \quad N \to \infty. \qquad (8.38)$$

Inequality (8.13) allows to use the same argument as in the proof of Theorem 1.2 in [59], to prove that $v = (e^y)^\alpha$. Hence,

$$(e^{y^{(N)}})^\alpha \longrightarrow (e^y)^\alpha \text{ strongly in } L^2(0,T;W^{2,q}(\Omega)) \ \forall\, q \in [1,\infty[,$$

and,

$$(e^{y^{(N)}})^\alpha \longrightarrow (e^y)^\alpha \text{ a.e..}$$

Then

$$(e^{y^{(N)}})^{\alpha/4} \longrightarrow (e^y)^{\alpha/4} \text{ a.e..}$$

Using (8.15), we have

$$\|(e^{y^{(N)}})^{\alpha/4}\|_{L^\infty(Q_T)} \leq c(\alpha).$$

The assumptions of Lemma 8.7 then hold and we obtain the following convergence

$$(e^{y^{(N)}})^{\alpha/4} \longrightarrow (e^y)^{\alpha/4} \text{ strongly in } L^2(0,T;L^p(\Omega)) \ \forall\, p \in [1,\infty[.$$

Moreover, Lemma 8.6 implies

$$(e^{y^{(N)}})^{\alpha/4} \rightharpoonup (e^y)^{\alpha/4} \text{ weakly in } L^2(0,T;H^2(\Omega)). \tag{8.39}$$

Then the assumptions of Theorem 8.3 hold and we have

$$(e^{y^{(N)}})^{\alpha/4} \longrightarrow (e^y)^{\alpha/4} \text{ strongly in } L^2(0,T;W^{1,q}(\Omega)) \ \forall\ q \in [4/3,\infty[,$$

or equivalently, noting $w = (e^y)^{\alpha/4}$,

$$w^{(N)} \longrightarrow w \text{ strongly in } L^2(0,T;W^{1,q}(\Omega)) \ \forall\ q \in [4/3,\infty[. \tag{8.40}$$

Furthermore, using (8.15) and Lemma 8.7 applied to the sequence $(e^{\alpha y^{(N)}/2})_N$, we have

$$(e^{y^{(N)}})^{\alpha/2} \rightharpoonup (e^y)^{\alpha/2} \text{ weakly in } L^2(0,T;H^3(\Omega)),$$

and,

$$(e^{y^{(N)}})^{\alpha/2} \longrightarrow (e^y)^{\alpha/2} \text{ strongly in } L^2(0,T;L^p(\Omega)) \ \forall p \in [1,\infty[.$$

Then, Theorem 8.4 yields

$$(e^{y^{(N)}})^{\alpha/2} \longrightarrow (e^y)^{\alpha/2} \text{ strongly in } L^2(0,T;W^{2,q}(\Omega)) \ \forall\ q \in [3/2,\infty[, \tag{8.41}$$

and, using lemma 8.5, we have

$$\frac{1}{\tau}\Big(e^{\alpha y^{(N)}} - \sigma_N e^{\alpha y^{(N)}}\Big) \rightharpoonup (e^{\alpha y})_t \ \text{ weakly in } L^s(0,T;H^{-3}(\Omega)). \tag{8.42}$$

Finally, (8.40), (8.41) and (8.42) allow us to pass to the limit in (8.37). As in (8.36) the obtained right-hand side can be rewritten as

$$-\int_0^T \int_\Omega e^y y_{xx}\Big(e^{(\alpha-1)y}\phi\Big)_{xx} dxdt.$$

This completes the proof of Theorem 8.1.

Theorem 8.2 is a direct consequence from the a priori estimates obtained in section 8.2.2. Indeed, the convergence shown in the proof of Theorem 8.1 allows us to pass to the limit in the estimates (8.17) and (8.15), and we then obtain that

$$u^\alpha \in L^2(0,T;H^3(\Omega)) \ \text{ and } \ u^{\alpha/2} \in L^\infty(0,T;H^1(\Omega)).$$

Finally, using the Sobolev injection Theorem, we have $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ for Ω bounded in one dimension. Hence

$$u \in L^\infty(0,T;L^\infty(\Omega)).$$

8.4 Positivity of solutions

This section is devoted to the proof, for times large enough, of the positivity of a solution to problem (8.1)-(8.2) or (8.3)-(8.2) in the case when $\alpha = 1$. More precisely, we prove the following result:

Theorem 8.5. *Let $\alpha = 1$ and u be a solution of problem* (8.3)-(8.2). *We assume that $\int_\Omega u_0(x)dx > 0$. Then there exists $t_1 > 0$ such that:*

$$\forall\ t > t_1,\ \forall\ x \in \Omega,\ u(t,x) > 0.$$

To prove this Theorem, we use the two following Lemmas.

Lemma 8.8. *Let $\alpha = 1$ and u be a solution of problem* (8.3)-(8.2). *Then, for all $t \in [0,T]$,*

$$\int_\Omega u(t,x)dx = \int_\Omega u_0(x)dx = A, \tag{8.43}$$

where $A > 0$ is a constant independent of x and t.

Proof. Setting $u_k = e^{y_k}$ in (8.6) and integrating over Ω, we obtain

$$\frac{1}{\tau}\int_\Omega (u_k(x) - u_{k-1}(x))dx = -\int_\Omega \big(u_k(x)(\log u_k(x))_{xx}\big)_{xx} dx = 0,$$

since for all $k = 1, ..., N,\ u_k$ is assumed to have periodic boundary conditions. This gives that for all $k = 1, ..., N$,

$$\int_\Omega u_k(x)dx = \int_\Omega u_0(x)dx = A,$$

and then for all $t \in [0,T]$,

$$\int_\Omega u^{(N)}(t,x)dx = A. \tag{8.44}$$

We pass to the limit $N \to \infty$ in (8.44) and we obtain:

$$\int_\Omega u(t,x)dx = A,$$

for all $t \in [0,T]$. This completes the proof of Lemma 8.8. □

Lemma 8.9. *Let $\alpha = 1$ and u be a solution of* (8.3)-(8.2). *Then:*

$$\forall\ T > 0,\ \|(\sqrt{u})_x\|_{L^\infty(0,T;L^2(\Omega))} \le ae^{-\beta T}, \tag{8.45}$$

where $a = \|(\sqrt{u_0})_x\|_{L^2(\Omega)}$ and $\beta > 0$ is a constant independent of x and t.

Proof. Using inequality (8.51) given in Lemma 8.10 (in Appendix), we have for all $\alpha \in [2/53(25-6\sqrt{10}),1]$:

$$\int_\Omega (u_k^{\alpha/2})_x^2 dx + \mu\alpha \sum_{j=1}^{k} \tau \int_\Omega (u_j^{\alpha/2})_{xxx} dx \leq \int_\Omega (u_0^{\alpha/2})_x^2 dx.$$

Using Poincaré inequality, we obtain

$$\int_\Omega (u_k^{\alpha/2})_x^2 dx \leq \int_\Omega (u_0^{\alpha/2})_x^2 dx - c\alpha\ \mu\ \tau\ \sum_{j=1}^{k} \int_\Omega (u_j^{\alpha/2})_x dx,$$

where c is the Poincaré constant. Now, using the Gronwall Lemma, this gives

$$\max_{0\leq k\leq N} \|(u_k^{\alpha/2})_x\|_{L^2(\Omega)} \leq a_1(\alpha) e^{-\beta(\alpha)T},$$

where $a_1(\alpha) = \|(u_0^{\alpha/2})_x\|_{L^2(\Omega)}$ and $\beta(\alpha) = c\mu\ \alpha$. This gives:

$$\forall\ T > 0,\ \|((u^{(N)})^{\alpha/2})_x\|_{L^\infty(0,T;L^2(\Omega))} \leq a_1(\alpha) e^{-\beta(\alpha)T}.$$

Then, letting $N \to \infty$, we have

$$\forall\ T > 0,\ \|(u^{\alpha/2})_x\|_{L^\infty(0,T;L^2(\Omega))} \leq a_1(\alpha) e^{-\beta(\alpha)T},$$

for all $\alpha \in [2/53(25-6\sqrt{10}),1]$. In particular, for $\alpha = 1$, we obtain (8.45) with $\beta = \beta(1)$ and $a = a_1(1)$. This completes the proof of Lemma 8.9. □

We are now able to prove Theorem 8.5. Using (8.45), for all $T > 0$,

$$\left\| \left(\sqrt{u(t,x)} \right)_x \right\|_{L^2(\Omega)} \leq a e^{-\beta T},\quad \forall\ t \in [0,T].$$

Then, it is clear that for all $\varepsilon > 0$, there exists $T_0 > 0$ such that for $T > T_0$,

$$\left\| \left(\sqrt{u(t,x)} \right)_x \right\|_{L^2(\Omega)} \leq \varepsilon,\quad \forall t \in [0,T].$$

Hence

$$\left(\sqrt{u(t,.)} \right)_x \longrightarrow 0,\ \text{when}\ t \to \infty,\ \text{strongly in}\ \ L^2(\Omega),$$

and we can then deduce that (maybe for a subsequence)

$$\left(\sqrt{u(t,.)} \right)_x \longrightarrow 0\ \text{when}\ t \to \infty,\ \text{a.e. in}\ \Omega. \tag{8.46}$$

Hence, there exists $B > 0$ a constant independent of x such that:

$$\sqrt{u(t,x)} \longrightarrow B\ \text{when}\ t \to \infty,\ \text{a.e. in}\ \Omega. \tag{8.47}$$

In particular,

$$\int_\Omega \sqrt{u(t,x)}dx \longrightarrow B \text{ when } t \to \infty, \tag{8.48}$$

since $mes(\Omega) = 1$ by definition. Moreover, by (8.47) we also have

$$u(t,x) \longrightarrow B^2 \text{ a.e. in } \Omega, \text{ and, } \int_\Omega u(t,x)dx \longrightarrow B^2, \text{ when } t \to \infty. \tag{8.49}$$

Using (8.43), we obtain from (8.49) that $A = B^2$ and then $B = \sqrt{A}$. Hence, using (8.48):

$$\int_\Omega \sqrt{u(t,x)}dx \longrightarrow \sqrt{A} \text{ when } t \to \infty. \tag{8.50}$$

Now, using (8.45), we have for all $T > 0$:

$$\left\| \sqrt{u} - \int_\Omega \sqrt{u(\cdot,x)}dx \right\|_{L^\infty(0,T;L^\infty(\Omega))} \le ae^{-\beta T},$$

and then for any $T > 0$,

$$\left| \sqrt{u(t,x)} - \int_\Omega \sqrt{u(t,x)}dx \right| \le ae^{-\beta T}, \quad \forall t \in [0,T], \ \forall x \in \Omega.$$

As previously, this implies that

$$\lim_{t\to\infty} [\sqrt{u(t,x)} - \int_\Omega \sqrt{u(t,x)}dx] = 0,$$

which gives:

$$\lim_{t\to\infty} \sqrt{u(t,x)} = \lim_{t\to\infty} \int_\Omega \sqrt{u(t,x)}dx = \sqrt{A} > 0.$$

Then, finally, in the case $\alpha = 1$, for times large enough, the solution u to problem (8.3)-(8.2) is strictly positive. This ends the proof of Theorem 8.5.

Appendix :

Proof of Lemma 8.2.

First of all, let $u_k = e^{y_k}$ for all $k \in \{0,...,N\}$. Then, Lemma 8.2 is a direct consequence from the following result:

Lemma 8.10. *Let* $k = 0,...,N$. *Then*

$$\int_\Omega (u_k^{\alpha/2})_x^2 dx + \alpha\mu \sum_{j=1}^k \tau \int_\Omega (u_j^{\alpha/2})_{xxx}^2 dx + \alpha\mu \sum_{j=1}^k \tau \int_\Omega (u_j^{\alpha/6})_x^6 dx \le \int_\Omega (u_0^{\alpha/2})_x^2 dx \tag{8.51}$$

Proof. The proof is based on the method presented in [63]. First, multiply (8.6) by the test function $u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}$ and integrate on Ω. Then,

$$0 = \frac{1}{\alpha\tau}\int_\Omega u_k^{1-\alpha}(u_k^\alpha - u_{k-1}^\alpha)u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx + \int_\Omega (u_k(\log u_k)_{xx})_{xx}u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx$$
$$:= I_1 + I_2. \tag{8.52}$$

Using integration by parts one has

$$\begin{aligned} I_1 &= \frac{1}{\alpha\tau}\int_\Omega u_k^{1-\alpha}(u_k^\alpha - u_{k-1}^\alpha)u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx \\ &= -\frac{1}{\alpha\tau}\int_\Omega [(u_k^{\alpha/2})_x^2 - (u_{k-1}^{\alpha/2})_x^2]dx - \frac{1}{\alpha\tau}\int_\Omega [(u_{k-1}^{\alpha/2})_x^2 - (u_{k-1}^\alpha u_k^{\alpha/2})_x(u_k^{\alpha/2})_x]dx \\ &= -\frac{1}{\alpha\tau}\int_\Omega [(u_k^{\alpha/2})_x^2 - (u_{k-1}^{\alpha/2})_x^2]dx - \frac{\alpha}{4\tau}\int_\Omega u_{k-1}^\alpha\left(\frac{u_{k-1,x}}{u_{k-1}} - \frac{u_{k,x}}{u_k}\right)^2 dx, \end{aligned} \tag{8.53}$$

and

$$\begin{aligned} I_2 &= \int_\Omega (u_k(\log u_k)_{xx})_{xx}u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx \\ &= -\int_\Omega (u_k(\log u_k)_{xx})_x(u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx})_x dx. \end{aligned} \tag{8.54}$$

Proceeding as in [63], we denote $\xi_i = \partial_x^i u_k/u_k$. We have

$$\begin{aligned} &\int_\Omega (u_k(\log u_k)_{xx})_x(u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx})_x dx \\ &\quad= \int_\Omega u_k^\alpha\left(\frac{u_{k,xxx}}{u_k} - 2\frac{u_{k,x}u_{k,xx}}{u_k^2} + \frac{u_{k,x}^3}{u_k^3}\right)\times \\ &\qquad\quad \left(\frac{\alpha}{2}\frac{u_{k,xxx}}{u_k} + 4\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\frac{u_{k,xx}u_{k,x}}{u_k^2} + \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)(\alpha-3)\frac{u_{k,x}^3}{u_k^3}\right)dx, \end{aligned}$$

$$\begin{aligned} &\int_\Omega (u_k^{\alpha/2})_{xxx}^2 dx \\ &\quad= \int_\Omega u_k^\alpha\left(\frac{\alpha}{2}\frac{u_{k,xxx}}{u_k} + 3\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\frac{u_{k,xx}u_{k,x}}{u_k^2} + \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\left(\frac{\alpha}{2}-2\right)\frac{u_{k,x}^3}{u_k^3}\right)^2 dx, \end{aligned}$$

$$\int_\Omega (u_k^{\alpha/6})_x^6 dx = \int_\Omega u_k^\alpha\left(\left(\frac{\alpha}{6}\right)^6\frac{u_{k,x}^6}{u_k^6}\right)dx,$$

$$\int_\Omega \left(\frac{u_{k,x}^5}{u_k^{5-\alpha}}\right)_x dx = \int_\Omega u_k^\alpha\left(5\frac{u_{k,xx}u_{k,x}^4}{u_k^5} - (5-\alpha)\frac{u_{k,x}^6}{u_k^6}\right)dx = 0,$$

$$\int_\Omega \left(\frac{u_{k,x}^3 u_{k,xx}}{u_k^{4-\alpha}}\right)_x dx = \int_\Omega u_k^\alpha\left(3\frac{u_{k,xx}^2u_{k,x}^2}{u_k^4} + \frac{u_{k,xxx}u_{k,x}^3}{u_k^4} - (4-\alpha)\frac{u_{k,x}^4u_{k,xx}}{u_k^5}\right)dx = 0.$$

Now, set

$$\begin{aligned}
P(\xi) &= (\xi_3 - 2\xi_1\xi_2 + \xi_1^3)\left(\frac{\alpha}{2}\xi_3 + 2\alpha\left(\frac{\alpha}{2}-1\right)\xi_2\xi_1 + \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)(\alpha-3)\xi_1^3\right)\\
&= \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)(\alpha-3)\xi_1^6 - \frac{\alpha}{2}(\alpha-2)(\alpha-5)\xi_1^4\xi_2 + \frac{\alpha}{4}(\alpha^2-5\alpha+8)\xi_1^3\xi_3\\
&\quad -4\alpha\left(\frac{\alpha}{2}-1\right)\xi_1^2\xi_2^2 + \alpha(\alpha-3)\xi_1\xi_2\xi_3 + \frac{\alpha}{2}\xi_3^2,\\
Q_1(\xi) &= \left(\frac{\alpha}{2}\xi_3 + 3\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\xi_1\xi_2 + \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\left(\frac{\alpha}{2}-2\right)\xi_1^3\right)^2\\
&= \frac{\alpha^2}{4}\left(\frac{\alpha}{2}-1\right)^2\left(\frac{\alpha}{2}-2\right)^2\xi_1^6 + 3\frac{\alpha^2}{2}\left(\frac{\alpha}{2}-1\right)^2\left(\frac{\alpha}{2}-2\right)\xi_1^4\xi_2\\
&\quad +\frac{\alpha^2}{2}\left(\frac{\alpha}{2}-1\right)\left(\frac{\alpha}{2}-2\right)\xi_1^3\xi_3 + \frac{9\alpha^2}{4}\left(\frac{\alpha}{2}-1\right)^2\xi_1^2\xi_2^2\\
&\quad +3\frac{\alpha^2}{2}\left(\frac{\alpha}{2}-1\right)\xi_1\xi_2\xi_3 + \frac{\alpha^2}{4}\xi_3^2,\\
Q_2(\xi) &= \left(\frac{\alpha}{6}\right)^6\xi_1^6,\\
T_1(\xi) &= -(5-\alpha)\xi_1^6 + 5\xi_1^4\xi_2,\\
T_2(\xi) &= -(4-\alpha)\xi_1^4\xi_2 + 3\xi_1^2\xi_2^2 + \xi_1^3\xi_3.
\end{aligned}$$

With the notations used in [63], we can write

$$\begin{aligned}
\int_\Omega (u_k(\log u_k)_{xx})_x(u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx})_x dx &= \int_\Omega u_k^\alpha D_P(u_k)dx,\\
\int_\Omega (u_k^{\alpha/2})_{xxx}^2 dx &= \int_\Omega u_k^\alpha D_{Q_1}(u_k)dx,\\
\int_\Omega (u_k^{\alpha/6})_x^6 dx &= \int_\Omega u_k^\alpha D_{Q_2}(u_k)dx,\\
\int_\Omega u_k^\alpha D_{T_1}(u_k)dx &= 0,\\
\int_\Omega u_k^\alpha D_{T_2}(u_k)dx &= 0.
\end{aligned}$$

According to [63], it remains now to prove that there exists $\mu > 0$ such that there exist $\theta_1,\ \theta_2 \in \mathbb{R}$ such that for all $\xi \in \mathbb{R}$

$$P(\xi) - \mu Q_1(\xi) - \mu Q_2(\xi) + \theta_1 T_1(\xi) + \theta_2 T_2(\xi) \geq 0. \tag{8.55}$$

Indeed, in this case there exists $\mu > 0$ such that there exist $\theta_1,\ \theta_2 \in \mathbb{R}$ such that

$$\int_\Omega u_k^\alpha D_{P+\theta_1 T_1+\theta_2 T_2}(u_k)dx \geq \mu \int_\Omega u_k^\alpha D_{Q_1}(u_k)dx + \mu \int_\Omega u_k^\alpha D_{Q_2}(u_k)dx,$$

and then in view of (8.52)-(8.54)

$$\frac{1}{\alpha\tau}\int_\Omega ((u_k^{\alpha/2})_x^2 - (u_{k-1}^{\alpha/2})_x^2)dx + \mu\int_\Omega (u_k^{\alpha/2})_{xxx}^2 dx + \mu\int_\Omega (u_k^{\alpha/6})_x^6 dx \leq 0,$$

we get (8.51). In our case, we have

$$\begin{aligned} P(\xi) - \mu Q_1(\xi) - \mu Q_2(\xi) &+ \theta_1 T_1(\xi) + \theta_2 T_2(\xi) \\ &= a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + a_6\xi_3^2, \end{aligned}$$

with

$$\begin{aligned} a_1 &= \frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)(\alpha-3) - \mu\frac{\alpha^2}{4}\Big(\frac{\alpha}{2}-1\Big)^2\Big(\frac{\alpha}{2}-2\Big)^2 - \mu\Big(\frac{\alpha}{6}\Big)^6 - \theta_1(5-\alpha) \\ a_2 &= -\frac{\alpha}{2}(\alpha-2)(\alpha-5) - \mu\frac{3\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big)^2\Big(\frac{\alpha}{2}-2\Big) + 5\theta_1 - \theta_2(4-\alpha) \\ a_3 &= \frac{\alpha}{4}(\alpha^2 - 5\alpha + 8) - \mu\frac{\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big)\Big(\frac{\alpha}{2}-2\Big) + \theta_2 \\ a_4 &= -4\alpha\Big(\frac{\alpha}{2}-1\Big) - \mu\frac{9\alpha^2}{4}\Big(\frac{\alpha}{2}-1\Big)^2 + 3\theta_2 \\ a_5 &= \alpha(\alpha-3) - \mu\frac{3\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big) \\ a_6 &= \frac{\alpha}{2} - \mu\frac{\alpha^2}{4} \end{aligned}$$

Using Mathematica (see Figure 8.1 at the end of Appendix), we obtain that for $2/53(25-6\sqrt{10}) < \alpha \leq 1 < 2/53(25+6\sqrt{10})$, there exists $\mu > 0$ such that there exist $\theta_1,\ \theta_2 \in \mathbb{R}$ satisfying the condition (ii) of lemma 7 in [41]. In particular, for $2/53(25-6\sqrt{10}) < \alpha \leq 1$, there exists $\mu > 0$ such that there exist $\theta_1,\ \theta_2 \in \mathbb{R}$ such that (8.55) is verified for all $\xi \in \mathbb{R}$. This completes the proof of lemma 8.10. □

Proof of Lemma 8.3.

Let us denote $u_k = e^{y_k}$, for all $k \in \{0, ..., N\}$, and $U = e^{y^{(N)}}$, for simplicity of the presentation. To prove Lemma 8.3, we need the following result

Lemma 8.11. *There exists a constant $c(\alpha) > 0$ such that*

$$\|[U^{\alpha/2}]_{xx}^2 [U^{\alpha/2}]_x^2\|_{L^1(0,T;L^1(\Omega))} \leq c(\alpha), \tag{8.56}$$

with $c(\alpha) > 0$ a constant.

Proof. We have,

$$\|(u_k^{\alpha/2})_{xx}^2\|_{H^1(\Omega)}^2 = \|(u_k^{\alpha/2})_{xx}^2\|_{L^2(\Omega)}^2 + \|[(u_k^{\alpha/2})_{xx}^2]_x\|_{L^2(\Omega)}^2. \tag{8.57}$$

First,

$$\|[(u_k^{\alpha/2})^2_{xx}]_x\|^2_{L^2(\Omega)} = 4\int_\Omega (u_k^{\alpha/2})^2_{xxx}(u_k^{\alpha/2})^2_{xx}dx$$
$$\leq 4\|(u_k^{\alpha/2})_{xx}\|^2_{L^\infty(\Omega)}\|(u_k^{\alpha/2})_{xxx}\|^2_{L^2(\Omega)}. \tag{8.58}$$

Furthermore,

$$\|(u_k^{\alpha/2})^2_{xx}\|^2_{L^2(\Omega)} = \|(u_k^{\alpha/2})_{xx}\|^4_{L^4(\Omega)},$$

and using the Gagliardo-Nirenberg inequality

$$\|(u_k^{\alpha/2})_{xx}\|^4_{L^4(\Omega)} \leq c\|u_k^{\alpha/2}\|^3_{H^3(\Omega)}\|u_k^{\alpha/2}\|_{L^2(\Omega)}. \tag{8.59}$$

Then using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a,\ b > 0$ and (8.57)-(8.59):

$$\|(u_k^{\alpha/2})^2_{xx}\|_{H^1(\Omega)} \leq c\left[\|u_k^{\alpha/2}\|^{3/2}_{H^3(\Omega)}\|u_k^{\alpha/2}\|^{1/2}_{L^2(\Omega)} + 2\|(u_k^{\alpha/2})_{xx}\|_{L^\infty(\Omega)}\|(u_k^{\alpha/2})_{xxx}\|^2_{L^2(\Omega)}\right],$$

and:

$$\sum_{k=1}^N \tau\|(u_k^{\alpha/2})^2_{xx}\|_{H^1(\Omega)} \leq c\left[\sum_{k=1}^N \tau\|u_k^{\alpha/2}\|^{3/2}_{H^3(\Omega)}\|u_k^{\alpha/2}\|^{1/2}_{L^2(\Omega)}\right.$$
$$\left. +2\sum_{k=1}^N \tau\|(u_k^{\alpha/2})_{xx}\|_{L^\infty(\Omega)}\|(u_k^{\alpha/2})_{xxx}\|_{L^2(\Omega)}\right].$$

Furthermore,

$$\sum_{k=1}^N \tau\|u_k^{\alpha/2}\|^{3/2}_{H^3(\Omega)}\|u_k^{\alpha/2}\|^{1/2}_{L^2(\Omega)} = \int_0^T \|U^{\alpha/2}\|^{3/2}_{H^3(\Omega)}\|U^{\alpha/2}\|^{1/2}_{L^2(\Omega)}dt$$
$$\leq \|U^{\alpha/2}\|^{1/2}_{L^\infty(0,T;L^2(\Omega))}\|U^{\alpha/2}\|^{3/2}_{L^{3/2}(0,T;H^3(\Omega))}$$
$$\leq \|U^{\alpha/2}\|^{1/2}_{L^\infty(0,T;H^1(\Omega))}\|U^{\alpha/2}\|^{3/2}_{L^2(0,T;H^3(\Omega))}$$
$$\leq c(\alpha) \tag{8.60}$$

using (8.15). Now, using the Cauchy-Schwartz inequality

$$\sum_{k=1}^N \tau\|(u_k^{\alpha/2})_{xx}\|_{L^\infty(\Omega)}\|(u_k^{\alpha/2})_{xxx}\|_{L^2(\Omega)}$$
$$\leq \left(\sum_{k=1}^N \tau\|(u_k^{\alpha/2})_{xx}\|^2_{L^\infty(\Omega)}\right)^{1/2}\left(\sum_{k=1}^N \tau\|(u_k^{\alpha/2})_{xxx}\|^2_{L^2(\Omega)}\right)^{1/2},$$
$$\leq \|[U^{\alpha/2}]_{xx}\|_{L^2(0,T;L^\infty(\Omega))}\|[U^{\alpha/2}]_{xxx}\|_{L^2(0,T;L^2(\Omega))}.$$

Then, using the Sobolev injection Theorem, since Ω is bounded, $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$, which gives in the preceding inequality

$$\begin{aligned}\sum_{k=1}^{N} \tau \|(u_k^{\alpha/2})_{xx}\|_{L^\infty(\Omega)}\|(u_k^{\alpha/2})_{xxx}\|_{L^2(\Omega)} \\ \leq \; & c\|[U^{\alpha/2}]_{xx}\|_{L^2(0,T;H^1(\Omega))}\|[U^{\alpha/2}]_{xxx}\|_{L^2(0,T;L^2(\Omega))}, \\ \leq \; & c(\alpha), \end{aligned} \tag{8.61}$$

with (8.15). Finally, using (8.60) and (8.61) we obtain

$$\|[U^{\alpha/2}]^2_{xx}\|_{L^1(0,T;H^1(\Omega))} \leq c(\alpha). \tag{8.62}$$

Using (8.15), we have $[U^{\alpha/2}]^2_x \in L^\infty(0,T;L^1(\Omega))$. Since by (8.62), $[U^{\alpha/2}]^2_{xx} \in L^1(0,T;H^1(\Omega))$, using again the injection from $H^1(\Omega)$ to $L^\infty(\Omega)$, the product $[U^{\alpha/2}]^2_{xx}[U^{\alpha/2}]^2_x$ is in $L^1(0,T;L^1(\Omega))$, which gives (8.56). This completes the proof of Lemma 8.11. □

We are now able to prove Lemma 8.3. We have

$$\|u_k^\alpha\|^2_{H^3(\Omega)} = \|u_k^\alpha\|^2_{L^2(\Omega)} + \|(u_k^\alpha)_x\|^2_{L^2(\Omega)} + \|(u_k^\alpha)_{xx}\|^2_{L^2(\Omega)} + \|(u_k^\alpha)_{xxx}\|^2_{L^2(\Omega)}.$$

First,

$$\|u_k^\alpha\|^2_{L^2(\Omega)} = \int_\Omega (u_k^\alpha)^2 dx = \int_\Omega (u_k^{\alpha/2})^4 dx \leq \|u_k^{\alpha/2}\|^4_{L^\infty(\Omega)}.$$

Moreover, since $(u_k^\alpha)_x = 2(u_k^{\alpha/2})_x u_k^{\alpha/2}$,

$$\|(u_k^\alpha)_x\|^2_{L^2(\Omega)} \leq 4\|u_k^{\alpha/2}\|^2_{L^\infty(\Omega)}\|(u_k^{\alpha/2})_x\|^2_{L^2(\Omega)},$$

and using the Gagliardo-Nirenberg inequality

$$\|(u_k^\alpha)_x\|^2_{L^2(\Omega)} \leq c\|u_k^{\alpha/2}\|_{L^\infty(\Omega)}\|u_k^{\alpha/2}\|^{2/3}_{H^3(\Omega)}\|u_k^{\alpha/2}\|^{4/3}_{L^2(\Omega}.$$

In a same way, since $(u_k^\alpha)_{xx} = 2(u_k^{\alpha/2})_{xx}u_k^{\alpha/2} + 2(u_k^{\alpha/2})^2_x$, we can show:

$$\|(u_k^{\alpha/2})_{xx}\|^2_{L^2(\Omega)} \leq c\left[\|u_k^{\alpha/2}\|^2_{L^\infty(\Omega)}\|(u_k^{\alpha/2})_{xx}\|^2_{L^2(\Omega)} + \|u_k^{\alpha/2}\|^{5/3}_{H^3(\Omega)}\|u_k^{\alpha/2}\|^{7/3}_{L^2(\Omega)}\right].$$

Furthermore:

$$(u_k^\alpha)_{xxx} = \alpha u_{k,xxx}u_k^{\alpha-1} + 3\alpha(\alpha-1)u_{k,xx}u_{k,x}u_k^{\alpha-2} + \alpha(\alpha-1)(\alpha-2)u^3_{k,x}u_k^{\alpha-3} \tag{8.63}$$

$$(u_k^{\alpha/2})_{xxx} = \frac{\alpha}{2}u_{k,xxx}u_k^{\alpha/2-1} + 3\frac{\alpha}{2}(\frac{\alpha}{2}-1)u_{k,xx}u_{k,x}u_k^{\alpha/2-2} + \frac{\alpha}{2}(\frac{\alpha}{2}-1)(\frac{\alpha}{2}-2)u^3_{k,x}u_k^{\alpha/2-3} \tag{8.64}$$

Using (8.63) and (8.64) we can write:

$$(u_k^\alpha)_{xxx} = 2u_k^{\alpha/2}\bigg((u_k^{\alpha/2})_{xxx} + \frac{3\alpha^2}{4}u_{k,xx}u_{k,x}u_k^{\alpha/2-2} + \frac{3\alpha^2}{4}(\frac{\alpha}{2}-1)u_{k,x}^3u_k^{\alpha/2-3}\bigg), \quad (8.65)$$

and

$$(u_k^\alpha)_{xxx}^2 = 4(u_k^{\alpha/2})^2\Big[(u_k^{\alpha/2})_{xxx} + \frac{3\alpha^2}{4}u_{k,xx}u_{k,x}u_k^{\alpha/2-2} + \frac{3\alpha^2}{4}(\frac{\alpha}{2}-1)u_{k,x}^3u_k^{\alpha/2-3}\Big]^2,$$

which gives the following inequalities:

$$\begin{aligned}(u_k^\alpha)_{xxx}^2 &\le 4(u_k^{\alpha/2})^2\bigg[2(u_k^{\alpha/2})_{xxx}^2 + 2\bigg(\frac{3\alpha^2}{4}u_{k,xx}u_{k,x}u_k^{\alpha/2-2} + \frac{3\alpha^2}{4}(\frac{\alpha}{2}-1)u_{k,x}^3u_k^{\alpha/2-3}\bigg)^2\bigg]\\ &\le 8(u_k^{\alpha/2})^2(u_k^{\alpha/2})_{xxx}^2\\ &\quad +8(u_k^{\alpha/2})^2\bigg(\frac{3\alpha^2}{4}u_{k,xx}u_{k,x}u_k^{\alpha/2-2} + \frac{3\alpha^2}{4}(\frac{\alpha}{2}-1)u_{k,x}^3u_k^{\alpha/2-3}\bigg)^2. \quad (8.66)\end{aligned}$$

We can show

$$\bigg(\frac{3\alpha^2}{4}u_{k,xx}u_{k,x}u_k^{\alpha/2-2} + \frac{3\alpha^2}{4}(\frac{\alpha}{2}-1)u_{k,x}^3u_k^{\alpha/2-3}\bigg)^2 = \frac{9\alpha^4}{64}(u_k^{-\alpha/2})^2\bigg(\frac{u_{k,x}^2}{u_k^{2-\alpha}}\bigg)_x^2,$$

which gives in (8.66):

$$(u_k^\alpha)_{xxx}^2 \le 8(u_k^{\alpha/2})^2(u_k^{\alpha/2})_{xxx}^2 + \frac{9\alpha^4}{8}\bigg(\frac{u_{k,x}^2}{u_k^{2-\alpha}}\bigg)_x^2.$$

Then,

$$\begin{aligned}\|(u_k^\alpha)_{xxx}\|_{L^2(\Omega)}^2 &\le 8\int_\Omega (u_k^{\alpha/2})^2(u_k^{\alpha/2})_{xxx}^2dx + \frac{9\alpha^4}{8}\int_\Omega\bigg(\frac{u_{k,x}^2}{u_k^{2-\alpha}}\bigg)_x^2dx\\ &\le 8I_3 + I_4.\end{aligned}$$

Moreover, we have:

$$I_4 = \frac{9\alpha^4}{8}\int_\Omega\bigg(\frac{4}{\alpha^2}(u_k^{\alpha/2})_x^2\bigg)_x^2dx = 18\int_\Omega\Big((u_k^{\alpha/2})_x^2\Big)_x^2dx = 64\int_\Omega(u_k^{\alpha/2})_{xx}^2(u_k^{\alpha/2})_x^2dx,$$

and,

$$\begin{aligned}I_3 = \int_\Omega(u_k^{\alpha/2})^2(u_k^{\alpha/2})_{xxx}^2dx &\le \|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^2\|(u_k^{\alpha/2})_{xxx}\|_{L^2(\Omega)}^2\\ &\le c\|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^2\|u_k^{\alpha/2}\|_{H^3(\Omega)}^2,\end{aligned}$$

Then using all the above inequalities,

$$\|u_k^\alpha\|_{H^3(\Omega)}^2 \le c\Bigg[\|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^4 + \|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^2\|u_k^{\alpha/2}\|_{H^3(\Omega)}^{2/3}\|u_k^{\alpha/2}\|_{L^2(\Omega)}^{4/3} + \|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^2\|(u_k^{\alpha/2})_{xx}\|_{L^2(\Omega)}^2 + \|u_k^{\alpha/2}\|_{H^3(\Omega)}^{5/3}\|u_k^{\alpha/2}\|_{L^2(\Omega)}^{7/3} + \|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^2\|u_k^{\alpha/2}\|_{H^3(\Omega)}^2 + \|(u_k^{\alpha/2})_{xx}^2(u_k^{\alpha/2})_x^2\|_{L^1(\Omega)}\Bigg],$$

which gives

$$\|U^\alpha\|_{L^2(0,T;H^3(\Omega))} \le c\Bigg[\sum_{k=1}^N \tau\|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^4 + \sum_{k=1}^N \tau\|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^2\|u_k^{\alpha/2}\|_{H^3(\Omega)}^{2/3}\|u_k^{\alpha/2}\|_{L^2(\Omega)}^{4/3} + \sum_{k=1}^N \tau\|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^2\|(u_k^{\alpha/2})_{xx}\|_{L^2(\Omega)}^2 + \sum_{k=1}^N \tau\|u_k^{\alpha/2}\|_{H^3(\Omega)}^{5/3}\|u_k^{\alpha/2}\|_{L^2(\Omega)}^{7/3} + \sum_{k=1}^N \tau\|u_k^{\alpha/2}\|_{L^\infty(\Omega)}^2\|u_k^{\alpha/2}\|_{H^3(\Omega)}^2 + \sum_{k=1}^N \tau\|(u_k^{\alpha/2})_{xx}^2(u_k^{\alpha/2})_x^2\|_{L^1(\Omega)}.\Bigg]$$

Thus, using the Sobolev injection Theorem ($H^1(\Omega) \hookrightarrow L^\infty(\Omega)$) and the Hölder inequality:

$$\|U^\alpha\|_{L^2(0,T;H^3(\Omega))} \le c\Bigg[\|U^{\alpha/2}\|_{L^\infty(0,T;H^1(\Omega))}^2\|U^{\alpha/2}\|_{L^2(0,T;H^1(\Omega))}^2 + \|U^{\alpha/2}\|_{L^\infty(0,T;H^1(\Omega))}^2\|U^{\alpha/2}\|_{L^2(0,T;L^2(\Omega))}^{4/3}\|U^{\alpha/2}\|_{L^2(0,T;H^3(\Omega))}^{2/3} + \|U^{\alpha/2}\|_{L^\infty(0,T;H^1(\Omega))}^2\|U^{\alpha/2}\|_{L^2(0,T;H^2(\Omega))}^2 + \|U^{\alpha/2}\|_{L^\infty(0,T;L^2(\Omega))}^2\|U^{\alpha/2}\|_{L^2(0,T;L^2(\Omega))}^{1/3}\|U^{\alpha/2}\|_{L^2(0,T;H^3(\Omega))}^{5/3} + \|U^{\alpha/2}\|_{L^\infty(0,T;H^1(\Omega))}^2\|U^{\alpha/2}\|_{L^2(0,T;H^3(\Omega))}^2 + \|(U^{\alpha/2})_{xx}^2(U^{\alpha/2})_x^2\|_{L^1(0,T;L^1(\Omega))}\Bigg].$$

Using (8.15) and (8.56) this finally gives (8.17). This completes the proof of Lemma 8.3.

Untitled−4 1

```
In[126]:=
        a1 = (a - 1) (2 a - 3) - m a (a - 1) ^2 (a - 2) ^2 - m / 3 (a / 3) ^5 - c1 (5 - 2 a);
        a2 = -2 (a - 1) (2 a - 5) - 6 m a (a - 1) ^2 (a - 2) + 5 c1 - 2 c2 (2 - a);
        a3 = (2 a^2 - 5 a + 4) - 2 m a (a - 1) (a - 2) + c2;
        a4 = -8 (a - 1) - 9 m a (a - 1) ^2 + 3 c2;
        a5 = 2 (2 a - 3) - 6 m a (a - 1);
        a6 = 1 - m a;

        (* Check for which alpha there exists SOME m>0 *)

In[142]:=
        Reduce[Exists[{m, c1, c2}, a > 0 && m > 0 &&
           (a6 > 0 && 4 a4 a6 > a5^2 && 4 a1 a4 a6 + a2 a3 a5 ≥ a1 a5^2 + a2^2 a6 + a3^2 a4)], Reals]

Out[142]=
```

$$\frac{1}{53}\left(25-6\sqrt{10}\right) < a < \frac{1}{53}\left(25+6\sqrt{10}\right)$$

```
        (* Calculate the optimal m in terms of alpha *)

In[146]:=
        FullSimplify[N[Reduce[Exists[{c1, c2}, a > 0 && m > 0 &&
             (a6 > 0 && 4 a4 a6 > a5^2 && 4 a1 a4 a6 + a2 a3 a5 ≥ a1 a5^2 + a2^2 a6 + a3^2 a4)], Reals]]]

Out[146]=
        0 < m &&
         ((m < Root[-405 + 5670 a - 22113 a^2 + 33372 a^3 - 17172 a^4 + 810 a #1 - 11340 a^2 #1 + 40500 a^3 #1 -
                 58158 a^4 #1 + 29140 a^5 #1 - 405 a^2 #1^2 + 5670 a^3 #1^2 - 15471 a^4 #1^2 + 21870 a^5 #1^2 -
                 12333 a^6 #1^2 - 2916 a^5 #1^3 + 2916 a^6 #1^3 - 60 a^7 #1^3 + 20 a^8 #1^4 &, 1] &&
             (0.113704 < a < 0.318835 || 0.611681 < a < 0.829692)) ||
           (m < Root[-405 + 5670 a - 22113 a^2 + 33372 a^3 - 17172 a^4 + 810 a #1 - 11340 a^2 #1 +
                 40500 a^3 #1 - 58158 a^4 #1 + 29140 a^5 #1 - 405 a^2 #1^2 + 5670 a^3 #1^2 -
                 15471 a^4 #1^2 + 21870 a^5 #1^2 - 12333 a^6 #1^2 - 2916 a^5 #1^3 + 2916 a^6 #1^3 -
                 60 a^7 #1^3 + 20 a^8 #1^4 &, 3] && 0.318835 ≤ a && a ≤ 0.611681))
```

Figure 8.1: Mathematica's program for the calculation of λ in the proof of Lemma 8.10. Here a stands for $\alpha/2$ and m for μ.

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RESUME

Cette thèse concerne deux systèmes d'équations différents utilisés dans la modélisation mathématique des semi-conducteurs et des plasmas.

Dans une première partie, nous considérons un modèle hydrodynamique appelé système d'Euler-Poisson. En utilisant une technique de développement asymptotique, nous étudions les limites en zéro, dans le cas stationnaire pour un flot potentiel, des trois paramètres physiques de ce système : la masse d'électrons, le temps de relaxation et la longueur de Debye. Pour chacune de ces limites, nous démontrons l'existence et l'unicité des profils ainsi que des estimations d'erreur.

Dans une seconde partie, nous considérons le système de dérive-diffusion quantique. Nous démontrons dans un premier temps l'existence de solutions (pour un profil de dopage général) ainsi que la limite de quasi-neutralité (pour un profil de dopage nul), dans le modèle évolutif bipolair uni-dimensionnel. Dans un second temps, nous montrons de nouvelles propriétés de régularité des solutions de l'équation obtenue dans la limite de quasi-neutralité. Ces nouvelles propriétés nous permettent de démontrer, de plus, la stricte positivité des solutions de cette équation pour des temps suffisamment grands.

ABSTRACT

This thesis is devoted to two different systems of equations used in the mathematical modeling of semiconductors and plasmas.

In a first part, we consider a fluid-dynamical model called the Euler-Poisson system. Using an asymptotic expansion method, we study the limit to zero of the three physical parameters which arise in this system: the electron mass, the relaxation time and the Debye length. For each limit, we prove the existence and uniqueness of profiles to the asymptotic expansion and some error estimates.

In a second part, we consider the quantum drift-diffusion model. First, we show the existence of solutions (for a general doping profile) and the quasineutral limit (for a vanishing doping profile), for the transient bipolar model in one-space dimension. Then we prove some new regularity properties for the solutions of the equation obtained in the quasineutral limit. These new properties allow us to also show the positivity of solutions to this equation for times large enough.