Directed polymers in random media and multifractal fields

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Table des matières

Introduction

Ce travail est constitué de deux parties : une première consacrée à l'étude de modèles de polymères dirigés en milieu aléatoire et une seconde consacrée à la modélisation statistique du champ de vitesse d'un écoulement turbulent. Ces deux parties ont pour point commun d'être liées à un modèle de physique statistique introduit par Mandelbrot ([**43**], [**44**]) pour modéliser la dissipation d'énergie cinétique d'un écoulement fluide : le chaos multiplicatif. Plus précisément, Mandelbrot a introduit un modèle continu dans [**43**] (rigoureusement défini par Kahane dans [**32**]) et un modèle discret dans [**44**] (pour le modèle discret, on utilisera le terme de cascades multiplicatives).

Pour définir ces modèles, on se place en premier lieu dans un cadre assez général (cf. [**34**]). On considère un espace métrique séparable, localement compact (T, d) que l'on muni de la tribu borélienne et on considère sur cet espace une mesure de Radon positive σ. Soit $(\Omega, \mathcal{F}, P)$ un espace de probabilité sur lequel est définie une suite de fonctions aléatoires positives $(P_i)_{i \geqslant 1}$ telles que :

(1) pour presque tout ω dans Ω, l'application $t \rightarrow P_i(t, \omega)$ est borélienne et positive.

(2) pour tout t dans T, les $P_i(t, .)$ sont des variables aléatoires positives telles que $E(P_i(t, .)) = 1$.

(3) les P_i, $i \geqslant 1$, forment une séquence indépendante.

On pose :

$$Q_n(t, \omega) = \prod_{i=1}^{n} P_i(t, \omega). \tag{0.1}$$

et on note $\mathcal{F}_n$ la tribu engendrée par les P_i pour $i \leqslant n$:

$$\mathcal{F}_n = \sigma\{P_i(t);\ t \in T,\ i \leqslant n\}$$

A cette suite $(P_i)_{i\geqslant 1}$ et à la mesure σ, on associe la suite de mesures aléatoires $(S_n)_{n\geqslant 1}$ definies pour tout borélien B par :

$$S_n(B) = \int_B Q_n(t)d\sigma(t). \qquad (0.2)$$

Il est facile de voir que, pour toute fonction borélienne positive et bornée f,

$$(\int_T f(t)dS_n(t))_{n\geqslant 1}$$

est une $\mathcal{F}_n$-martingale positive d'espérance $\int_T f(t)d\sigma(t)$. On peut alors montrer (cf. [**34**]) que les mesures $(S_n)_{n\geqslant 1}$ convergent vaguement p.s. vers une mesure aléatoire positive S telle que pour tout borélien B :

$$S_n(B) \underset{n\to\infty}{\longrightarrow} S(B) \qquad p.s. \qquad (0.3)$$

Tout le problème est alors d'étudier la mesure limite S. Les questions que l'on peut se poser entre autres sur S sont dans l'ordre :

– La mesure S est-elle nulle p.s. ?
– La mesure S est-elle p.s. absolument continue par rapport à σ ou p.s. singulière par rapport à σ ? (on est dans l'un des deux scénarios évoqués du fait de la loi du 0-1 pour les évènements de la tribu de queue)
– Si T est un sous-espace de $\mathbb{R}^d$, quelle est la dimension de Hausdorff du support de S en fonction de celle du support de σ ?

On insiste sur le caractère non-trivial de ces questions : par exemple, même si pour tout borélien borné B l'on a la convergence p.s. (0.3), il n'y a aucune raison pour que l'on ait presque sûrement :

$$\text{pour tout borélien borné } B, \quad S_n(B) \underset{n\to\infty}{\longrightarrow} S(B)$$

et donc la mesure S n'est pas nécessairement p.s. absolument continue par rapport à σ.

Comme cas particulier, on peut considérer les cascades multiplicatives, le chaos multiplicatif gaussien et les polymères dirigés en milieu aléatoire.

Les cascades multiplicatives. On expose ci-dessous un modèle très général de cascades (plus général que celles étudiées par Kahane). On prend pour espace (T, d) l'intervalle $[0, 1]$ muni de la valeur absolue et on considère sur cet espace la mesure de Lebesgue. Soit N un entier supérieur ou égal à 2 et q une loi sur $(\mathbb{R}_+^*)^N$. On note U l'ensemble (ou arbre) des suites finies à valeur dans l'intervalle entier $[|\ 0, N-1\ |]$:

$$U = \bigcup_{i\in\mathbb{N}} [|\ 0, N-1\ |]^i.$$

Si $u = u_1 \dots u_i$ et $v = v_1 \dots v_{i'}$ sont deux éléments de U de longueurs respectives i et i', on note uv la suite de longueur $i + i'$ définie par :

$$uv = u_1 \dots u_i v_1 \dots v_{i'}.$$

On considère alors un espace de probabilité $(\Omega, \mathcal{F}, P)$ sur lequel est définie une suite de variables $(A_u)_{u\in U}$ telles que les vecteurs $(A_{u0}, \dots, A_{u(N-1)})_{u\in U}$ soient une suite i.i.d. de loi q. On suppose également que :

$$E(\sum_{i=0}^{N-1} A_i) = 1.$$

Si $u = u_1 \dots u_i$, on note I_u l'intervalle N-adique $[\sum_{j=1}^{i} \frac{u_j}{N^j}, \sum_{j=1}^{i} \frac{u_j}{N^j} + \frac{1}{N^i}[$. Le modèle des cascades multiplicatives est alors la suite de mesures $(S_n)_{n\,\geqslant\,1}$ données par :

$$\text{pour tout borélien B,} \quad S_n(B) = \int_B Q_n(t)dt$$

où la fonction Q_n est définie par la relation (0.1) avec :

$$P_i(t) = \sum_{u=u_1\dots u_i} A_u 1_{I_u}(t).$$

Etudier si la mesure limite S est presque sûrement nulle est équivalent à étudier la fonction de partition du modèle Z_n donnée par $Z_n = S_n([0,1])$: S est presque sûrement nulle si et seulement si Z_n tend p.s. vers 0 lorsque n tend vers l'infini. On verra (voir la partie 2.2 ci-dessous) que l'on peut relier la fonction de partition des modèles de cascades à celle des polymères dirigés.

Le chaos multiplicatif gaussien. On expose ici un cas particulier de la théorie générale introduite dans [**32**]. Soit d un entier supérieur ou égal à 1. On prend pour espace (T, d) l'espace $\mathbb{R}^d$ muni de la norme euclidienne et on considère sur cet espace la mesure de Lebesgue. Soit $(\Omega, \mathcal{F}, P)$ un espace de probabilité sur lequel est définie une suite indépendante de processus gaussiens $(X_i)_{i\,\geqslant\,1}$ centrés de fonction de covariance p_i :

$$p_i(s,t) = E(X_i(s)X_i(t)) \qquad s,t \in \mathbb{R}^d$$

On se propose alors d'étudier la suite de mesures $(S_n)_{n \geqslant 1}$ données par :

$$\text{pour tout borélien B,} \quad S_n(B) = \int_B Q_n(t)dt$$

où la fonction Q_n est définie par la relation (0.1) avec :

$$P_i(t) = e^{X_i(t) - \frac{1}{2}p_i(t,t)}.$$

On peut alors introduire le noyau q, à valeurs dans $[0, \infty]$, défini par :

$$q(s,t) = \sum_{i=1}^{\infty} p_i(s,t), \qquad s,t \in \mathbb{R}^d. \tag{0.4}$$

On dira qu'un noyau q est de type σ-positif s'il admet une décomposition de la forme (0.4). La mesure S limite des mesures S_n est appelée chaos multiplicatif gaussien de noyau q (il est possible de montrer que la loi de S est uniquement déterminé par q ; autrement dit, la loi de S est indépendant de sa décomposition en somme infine de la forme (0.4)).

Dans le cadre de la turbulence homogène et isotrope, Kolmogorov suppose que la dissipation d'énergie cinétique dans une boule de rayon r petit est une variable ϵ_r telle que $\ln \epsilon_r$ est gaussienne avec une variance de l'ordre d'une constante que multiplie $\ln \frac{1}{r}$. Une manière rigoureuse de donner un sens à cette affirmation est de considérer que la dissipation est le chaos multiplicatif gaussien de noyau q donné par :

$$q(s,t) = u \ln^+ \frac{1}{|t-s|} + O(1), \qquad s,t \in \mathbb{R}^d, \tag{0.5}$$

où $\ln^+(.) = \max(\ln(.), 0)$ et u est un paramètre positif. Bien sûr, pour parler d'un tel chaos multiplicatif, il faut montrer que l'expression (0.5) définit un noyau de type σ-positif : c'est bien le cas ! A partir du chaos multiplicatif de noyau donné par (0.5), il est possible d'aborder la construction de champs de vitesses reproduisant les principales propriétés observées expérimentalement dans un écoulement fluide turbulent (voir chapitre 4). En réalité, dans le chapitre 4, on sera amené à introduire une construction plus générale et plus souple du chaos multiplicatif.

Les polymères dirigés en milieu aléatoire. Comme l'étude des polymères sera longuement abordée dans la partie suivante, on se contente ici de montrer que les polymères peuvent s'obtenir à partir de la construction décrite au début de l'introduction. On prend pour espace (T, d) l'espace Ω défini par :

$$\Omega = \{(\omega_i)_{i \geqslant 1} \in (\mathbb{Z}^d)^{\mathbb{N}^*};\ \forall i,\ |\omega_i|_1 \leqslant i\},$$

où $|.|_1$ désigne la norme L^1 standard. On muni Ω de la distance :

$$d(\omega, \widetilde{\omega}) = \sum_{i=1}^{\infty} \frac{|\omega_i - \widetilde{\omega}_i|_1 \wedge 1}{2^i}.$$

Enfin, on choisit pour mesure σ la mesure P de la marche aléatoire simple sur $\mathbb{Z}^d$ et on considère une suite $\eta = (\eta(n, x))_{(n,x) \in \mathbb{N} \times \mathbb{Z}^d}$ de variables aléatoires réelles i.i.d. sur un espace de probabilité $(H, \mathcal{G}, Q)$. On introduit un paramètre réel β et on suppose que :

$$\forall \beta,\ \lambda(\beta) = \ln Q(e^{\beta \eta(n,x)}) < \infty.$$

On peut alors regarder la suite de mesures $(S_n^\beta)_{n \geqslant 1}$ données par :

$$\text{pour tout borélien B,} \quad S_n^\beta(B) = \int_B Q_n^\beta(\omega) dP(\omega)$$

où la fonction Q_n^β est définie par la relation (0.1) avec :

$$P_i(\omega) = e^{\beta \eta(i, \omega_i) - \lambda(\beta)}.$$

Dans le cas des polymères, on travaille avec des probabilités donc on étudie μ_n^β définie par :

$$\mu_n^\beta = S_n^\beta / Z_n^\beta,$$

où $Z_n^\beta = S_n^\beta(\Omega)$ est la fonction de partition.

1. Les Polymères Dirigés en environnement aléatoire

1.1. Les Polymères dirigés et leur lien avec l'équation KPZ. Cette partie est une brève introduction aux polymères du point de vue de la physique (il ne faut donc pas s'attendre à ce que les objets présentés ici soient tous bien définis mathématiquement !).

Le modèle des polymères dirigés en environnement aléatoire a été introduit par C. Henley et D. Huse ([**28**]) pour modéliser dans $\mathbb{R}^d$ la phase de séparation dans le modèle d'Ising perturbé par des impuretés aléatoires entre chaque spin. Les auteurs adoptent une paramétrisation locale de l'interface par $x \in \mathbb{R}^{d-1}$: si l'on suppose que la phase de séparation est décrite par une fonction $z(x)$ avec $x \in \mathbb{R}^{d-1}$, l'energie de la phase de séparation est donnée par :

$$E(z) = \int_{\mathbb{R}^{d-1}} (\frac{1}{2}\sigma|\nabla z|^2 + \eta(x, z(x)))dx, \tag{1.1}$$

où σ est un paramètre positif et η un potentiel aléatoire issu de la présence d'impuretés. Le terme $\int_{\mathbb{R}^{d-1}} \frac{1}{2}\sigma|\nabla z|^2 dx$ est une énergie de Dirichlet pénalisant les oscillations de l'interface et le terme $\int_{\mathbb{R}^{d-1}} \eta(x, z(x))dx$ tient compte des impuretés qui tendent à pousser l'interface là où les coûts sont plus faibles. Vu la taille des systèmes considérés, on adopte une modélisation statistique à travers le formalisme de Boltzmann-Gibbs. Ainsi, si l'on se place dans un système à température T, la phase de séparation est distribuée selon une mesure de Gibbs dont la fonction de partition entre $(0, 0)$ et (x, y) est donnée par :

$$Z(x, y) = \int_{z:(0,0)\to(x,y)} e^{-\frac{E(z)}{k_B T}}, \tag{1.2}$$

où l'intégrale porte sur les chemins dirigés entre $(0, 0)$ et (x, y) et k_B est la constante de Boltzmann. Dans [**29**], les mêmes auteurs (avec D. Fischer) démontrent que sous la mesure de Gibbs associée, en dimension 2, l'interface est superdiffusive :

$$z(x) \approx x^{2/3}. \tag{1.3}$$

L'équation (1.3) reste en toute généralité une conjecture mathématique partiellement confirmée par les travaux de Johansson ([**31**]). Dans le même article, ils démontrent que $Z(x, y)$ vérifie l' équation aux dérivées partielles (EDP) stochastique (équation de la chaleur avec bruit multiplicatif) :

$$\frac{\partial Z(x, y)}{\partial x} = \frac{k_B T}{2\sigma}\Delta_y Z(x, y) + \frac{1}{k_B T}\eta(x, y)Z(x, y). \tag{1.4}$$

On verra un peu plus bas que $Z(x, y)$ est donc trés fortement lié à une équation fondamentale de la physique statistique : l'équation KPZ.

L'équation KPZ (du nom de Kardar-Parisi-Zhang) est une EDP non linéaire et stochastique qui a été introduite dans [**36**] pour décrire la croissance d'une interface dans divers contextes ; si l'on suppose que cette interface est une fonction réelle $h(t, x)$ du temps t et d'un espace de référence dont la variable est $x \in \mathbb{R}^{d-1}$, cette équation s'écrit :

$$\frac{\partial h}{\partial t} = \nu\Delta_x h + \frac{\lambda}{2}|\nabla_x h|^2 + \eta(t, x), \tag{1.5}$$

où η est un bruit blanc gaussien sur $\mathbb{R}^d$.

L'équation KPZ a été utilisée pour modéliser la ligne de propagation d'un feu dans une forêt ou sur du papier, la croissance de colonies de bactéries sur une boite de Petri, etc... (cf. **[27]** pour un résumé sur les applications physiques de KPZ et des polymères dirigés). L'équation KPZ est donc censée décrire la propagation d'une entité dans un milieu inhomogène (dans les exemples décrits, la fôret peut etre plus ou moins dense et, sur la boite de Petri, la concentration en nutriments n'est jamais homogène) : il est donc naturel d'introduire un terme probabiliste (le bruit blanc η) pour rendre compte de ces inhomogénéités rencontrées au cours du temps. Le terme non linéaire déterministe $|\nabla h|^2$ assure une croissance perpendiculaire à la ligne de séparation.

Si l'on définit W par la relation $W(t,x) = e^{(\lambda/2\nu)h(t,x)}$, alors W vérifie l'EDP stochastique :

$$\frac{\partial W(t,x)}{\partial t} = \nu\Delta_x W(t,x) + (\lambda/2\nu)\eta(t,x)W(t,x). \tag{1.6}$$

En comparant (1.4) et (1.6), on voit qu'à des constantes prés, $W(t,x)$ peut être vu comme la fonction de partition point à point d'un polymère dirigé dans un environnement décrit par un bruit blanc gaussien. C'est à cette identification que l'on doit en partie l'intérêt que la communauté physique porte au modèle des polymères dirigés.

1.2. Historique mathématique du domaine. Cette thèse contient l'étude de deux modèles mathématiques de polymères dirigés : un modèle discret et un modèle continu. Sont présentés ici le modèle discret ainsi qu'un historique des principaux résultas obtenus jusqu'en 2004.

Soit d un entier supérieur ou égal à 1. On considère la marche aléatoire simple $((\omega_n)_{n\in\mathbb{N}}, (P^x)_{x\in\mathbb{Z}^d})$ sur $\mathbb{Z}^d$, définie sur un espace mesurable $(\Omega, \mathcal{F})$; plus précisément, si x est dans $\mathbb{Z}^d$, $(\omega_n - \omega_{n-1})_{n \geqslant 1}$ est une suite i.i.d. sous P^x telle que :

$$P^x(\omega_0 = x) = 1, \quad P^x(\omega_n - \omega_{n-1} = \pm\delta_j) = \frac{1}{2d}, \quad j = 1, \dots, d,$$

où $(\delta_j)_{1 \leqslant j \leqslant d}$ est le j-ième vecteur de la base canonique. Dans la suite, P désignera la mesure P^0 et on notera $P^x(X)$ l'espérance d'une variable X par rapport à P^x.

L'environnement avec lequel interagit la marche est défini par une suite $\eta = (\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ de variables aléatoires réelles i.i.d. sur un espace de probabilité $(H, \mathcal{G}, Q)$. On suppose que l'environnement a des moments exponentiels de tout ordre :

$$\forall \beta \in \mathbb{R} \quad \lambda(\beta) \overset{def.}{=} \ln Q(e^{\beta\eta(n,x)}) < \infty. \tag{1.7}$$

On se propose d'étudier pour tout entier $n \geqslant 1$ la mesure (aléatoire en η) μ_n^x donnée par :

$$\mu_n^x(d\omega) = \frac{1}{Z_n^x} \exp(\beta H_n(\omega) - n\lambda(\beta)) P_n^x(d\omega), \tag{1.8}$$

où $\beta \in \mathbb{R}$ est l'inverse de la température,

$$H_n(\omega) \overset{def.}{=} \sum_{j=1}^{n} \eta(j, \omega_j)$$

et

$$Z_n^x = P^x(\exp(\beta H_n(\omega) - n\lambda(\beta)))$$

est la fonction de partition renormalisée de façon à avoir $Q(Z_n^x) = 1$. Dans la suite, μ_n (Z_n) désignera μ_n^0 (Z_n^0).

Il est important de remarquer que, du fait que les variables de désordre soient définies sur $\mathbb{N} \times \mathbb{Z}^d$, on travaille sur le graphe de $(\omega_n)_{n \geqslant 1}$ (modèle dirigé) et donc, **lorsque l'on considère par la suite un polymère en dimension d, on travaille physiquement en dimension $d+1$.**

On considère également la filtration $(\mathcal{G}_n)_{n \geqslant 0}$:

$$\mathcal{G}_n = \sigma\{\eta(j, x); j \leqslant n, x \in \mathbb{Z}^d\}.$$

Il est relativement aisé de vérifier que $(Z_n^x, \mathcal{G}_n)_{n \geqslant 0}$ est une martingale positive, donc converge Q-p.s. vers une variable Z_∞^x. On vérifie également sans peine que l'évènement $\{Z_\infty^x = 0\}$ est dans la tribu de queue $\bigcap_{n \geqslant 1} \sigma\{\eta(j, x); j \geqslant n, x \in \mathbb{Z}^d\}$ et donc, par la loi du 0-1,

$$Q(Z_\infty^x = 0) = 1 \quad \text{ou} \quad Q(Z_\infty^x = 0) = 0. \tag{1.9}$$

Dans le premier cas, on dit qu'on est dans le régime de fort désordre : la marche simple est perturbée par l'environnement et tend à se localiser dans les régions de fort environnement ; dans le deuxième, on dit qu'on est dans le régime de faible désordre : tout se passe comme si $\beta = 0$ et donc que l'environnement n'affecte pratiquement pas la marche.

C'est Bolthausen qui le premier a introduit la martingale $(Z_n^x, \mathcal{G}_n)_{n \geqslant 0}$ et constaté la loi du 0-1 (cf. [**7**]). Il n'est pas évident a priori de justifier cette terminologie. Si l'on regarde l'équation (1.8) qui définit μ_n^x, on constate que chaque trajectoire a une probabilité proportionnelle à l'exponentielle de β que multiplie la somme des environnements rencontrés le long de la trajectoire. De cela, on déduit que la mesure de polymère charge plus fortement les endroits où l'environnement est élevé et ce d'autant plus que l'inverse de la température β est grande. On peut donc s'attendre, si la terminologie (faible désordre)-(fort désordre) est correcte, à observer une transition de phase (faible désordre)-(fort désordre) lorsque l'on fait augmenter β ; c'est ce qu'affirme le théorème suivant que l'on peut trouver dans [**15**] :

THÉORÈME 1.1. *Il existe une valeur $\beta_c = \beta_c(d, \eta)$ telle que :*

$$\begin{cases} \beta_c = 0, \quad d = 1, 2 \\ 0 < \beta_c \leqslant \infty \quad si\ d \geqslant 3 \end{cases}$$

et telle qu'on soit dans le régime de faible désordre si $\beta \in]0, \beta_c[$ et dans le régime de fort désordre si $\beta > \beta_c$.

On va montrer la deuxieme partie de ce théorème ; on suppose que $d \geqslant 3$ et on définit $N_{1,n}$ comme le nombre d'intersections ordonnées de deux marches aléatoires simples ω et $\widetilde{\omega}$ entre 1 et n :

$$N_{1,n} \overset{def.}{=} \sum_{j=1}^{n} 1_{\omega_j = \widetilde{\omega}_j}.$$

On peut montrer en utilisant le théorème de Fubini que :

$$\begin{aligned} Q((Z_n^x)^2) &= P^x \otimes P^x(e^{\lambda_2(\beta) N_{1,n}}) \\ &= P \otimes P(e^{\lambda_2(\beta) N_{1,n}}). \end{aligned}$$

et donc :

$$\sup_{n \geqslant 0} Q((Z_n^x)^2) = P \otimes P(e^{\lambda_2(\beta) N_{1,\infty}}).$$

Si l'on note

$$\pi_d \overset{def.}{=} P(\exists n \geqslant 1,\ \omega_n = 0) < 1,$$

on en déduit l'équivalence suivante :

$$\sup_{n \geqslant 0} Q((Z_n^x)^2) < \infty \iff \lambda_2(\beta) < \ln(\frac{1}{\pi_d}). \tag{L2}$$

Sous la condition (L2) ci-dessus, la martingale Z_n^x est bornée dans L^2 et donc converge dans L^2 vers Z_∞^x. En particulier, $Q(Z_\infty^x) = 1$ et on est dans le régime de faible désordre.

Entre 1988 et 1996, tous les articles mathématiques sur les polymères dirigés (à température positive : $\beta < \infty$) sont consacrés a l'étude de ceux-ci sous la condition

(L2) qui est plus restrictive que la condition de faible désordre (cf. [**6**]). Ces articles ([**1**], [**7**], [**30**], [**53**], [**52**]) ont établis que le polymère avait un comportement diffusif sous la condition (L2) (théorème limite centrale classique et fonctionnel, théorème limite locale). Plus précisément, on peut citer le théorème suivant :

THÉORÈME 1.2 (Théorème limite centrale, [**30**], [**7**], [**53**]). *Supposons que β vérifie la condition (L2). Alors, pour tout $f \in C(\mathbb{R}^d)$ avec une croissance au plus polynomiale,*

$$\mu_n^x\left(f(\frac{\omega_n}{\sqrt{n}})\right) \underset{n\to\infty}{\longrightarrow} \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\frac{x}{\sqrt{d}})e^{-\frac{|x|^2}{2}}dx,\ Q-p.s.$$

Le théorème ci-dessus montre donc que, pour $d \geqslant 3$ et β "petit", la mesure de polymère est diffusive au sens où $\omega_n \approx \sqrt{n}$, un régime à l'encontre de celui conjecturé pour la dimension $d = 1$ (cf. équation (1.3)). Ce comportement diffusif, mis en évidence pour la première fois par J. Imbrie et T. Spencer dans [**30**], a été une grande surprise pour les physiciens théoriciens (les praticiens n'ont guère à s'en soucier puisque la condition (L2) ne peut être valide que pour $d \geqslant 3$ ce qui correspond à une dimension physique supérieure ou égale à 4).

On reviendra dans le paragraphe 2.1 sur le théorème limite locale obtenu sous la condition (L2) par Sinai dans [**52**].

En 2002, à partir d'une étude approfondie de la fonction de partition, P. Carmona et Y. Hu ([**9**]) établissent un certain nombre de résultats de localisation dans le régime de fort désordre dans un environnement gaussien. Ces résultats sont généralisés aux environnements généraux vérifiant (1.7) dans [**11**]. Pour exprimer les résultats obtenus, on se place sur l'espace $(\Omega^2, \mathcal{F}^{\otimes 2})$ et on considère deux marches aléatoires simples indépendantes $(\omega_n, \widetilde{\omega}_n)_{n\in\mathbb{N}}$ qui évoluent dans le même environnement ; on peut donc considérer la mesure $\mu_n^{\otimes 2}$ définie par :

$$\mu_n^{\otimes 2}(d\omega) = \frac{1}{Z_n^2}\exp(\beta H_n(\omega) + \beta H_n(\widetilde{\omega}) - 2n\lambda(\beta))P_n^{\otimes 2}(d\omega, d\widetilde{\omega}). \qquad (1.10)$$

On introduit la quantité I_n définie par :

$$I_n = \mu_{n-1}^{\otimes 2}(\omega_n = \widetilde{\omega}_n). \qquad (1.11)$$

Le fait d'utiliser μ_{n-1} dans l'expression ci-dessus au lieu de μ_n est faite pour des raisons techniques. Pour comprendre la signification de I_n, on peut remarquer en sommant sur les valeurs prises par ω_n que l'on a l'encadrement suivant :

$$\max_{x\in\mathbb{Z}^d}\mu_{n-1}(\omega_n = x)^2 \leqslant I_n \leqslant \max_{x\in\mathbb{Z}^d}\mu_{n-1}(\omega_n = x). \qquad (1.12)$$

Ainsi I_n est une "mesure" de la masse prise par le point préféré de la marche sous la mesure de polymère. On peut relier I_n à l'asymptotique de $\ln Z_n$ (cf. th.2.1 dans [**11**]) :

THÉORÈME 1.3 ([**9**] , [**11**]). *Si $\beta \neq 0$, alors :*

$$\{Z_\infty = 0\} = \{\sum_{n\geqslant 1} I_n = \infty\} \qquad Q-p.s.$$

De plus, si $Q(Z_\infty = 0) = 1$, il existe $c_1, c_2 \in]0,\infty[$ tels que :

$$-c_1 \ln Z_n \leqslant \sum_{1\leqslant j\leqslant n} I_j \leqslant -c_2 \ln Z_n \qquad \text{pour } n \text{ assez grand, } Q-p.s. \tag{1.13}$$

Le comportement de $\ln Z_n$ et en particulier l'energie libre $p(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln Z_n$ (qui existe Q-p.s. et dans $L^1(Q)$, cf. prop. 2.5 dans [**11**]) sont donc fortement liés à la localisation de la marche. La mesure de polymère (1.8) est très complexe à cause de la présence du désordre et il est tout à fait remarquable que l'estimation de Z_n permette d'accéder à des propriétés fines de localisation via (1.13). On cite également le résultat suivant de localisation en dimension 1 et 2 :

THÉORÈME 1.4 ([**9**], [**11**]). *Si $\beta \neq 0$ et $d = 1, 2$, alors il existe $c \in]0,\infty[$ tel que :*

$$\overline{\lim_{n\to\infty}} I_n \geqslant c. \qquad Q-p.s. \tag{1.14}$$

Enfin, on rappelle un résultat récent dû à P. Carmona et Y. Hu qui confirme la terminologie fort désordre dans le cadre d'un modèle voisin : le modèle parabolique d'Anderson. En effet, ils montrent que la propriété de fort désordre exprimée en terme de martingale implique la propriété de localisation (1.14).

Retraduit en terme de point préféré, l'inégalité (1.14) s'écrit :

$$\overline{\lim_{n\to\infty}} \max_{x\in\mathbb{Z}^d} \mu_{n-1}(\omega_n = x) \geqslant c. \qquad Q-p.s.$$

La marche est donc fortement attirée par certaines régions ; à titre de comparaison, on rappelle que la marche aléatoire simple ($\beta = 0$ ici) a tendance à s'étaler uniformémént autour de l'origine dans un rayon de taille $\sqrt{n}$:

$$\max_{x\in\mathbb{Z}^d} P(\omega_n = x) = O(\frac{1}{n^{d/2}}).$$

2. Résultats obtenus sur les polymères dirigés

Cette partie regroupe trois articles sur les polymères dirigés en milieu aléatoire qui constituent les chapitres 1, 2 et 3.

2.1. Un théorème limite locale. Le théorème limite locale constitue le chapitre 1 de cette thèse. Il a fait l'objet d'un article à paraitre dans *Les Annales de l'I.H.P.* On commence par introduire quelques notations ; on note $q^{(n)}(x)$ la probabilité pour la marche aléatoire simple d'être en x au pas n :

$$q^{(n)}(x) \stackrel{def.}{=} P(\omega_n = x).$$

Pour $k \leqslant n$, on note :

$$e_{k,n} \stackrel{def.}{=} \exp((\sum_{j=k}^{n} \beta\eta(j,\omega_j)) - (n-k+1)\lambda(\beta))$$

et l'analogue dans un environnement retourné en temps :

$$\overleftarrow{e}_{k,n} \stackrel{def.}{=} \exp((\sum_{j=0}^{n-k} \beta\eta(n-j,\omega_j)) - (n-k+1)\lambda(\beta)).$$

On peut maintenant énoncer le théorème limite locale :

THÉORÈME 2.1 (Sinai, **[52]**). *Supposons que β vérifie la condition (L2) et soit A un réel strictement positif. Alors, si $(l_n)_{n \geqslant 0}$ est une suite d'entiers qui tend vers l'infini telle que $l_n = o(n^a)$ avec $a < \frac{1}{2}$,*

$$P^x(e_{1,n} \mid \omega_n = y) = P^x(e_{1,l_n}) P^y(\overleftarrow{e}_{n-l_n,n}) + \delta_n^{x,y} \tag{2.1}$$

avec

$$\sup_{|y-x| \leqslant A\sqrt{n}} Q(|\ \delta_n^{x,y}\ |^2) \underset{n\to\infty}{\longrightarrow} 0.$$

On en déduit la formule suivante qui est celle de l'article de Sinai :

$$P^x(e_{1,n} 1_{\omega_n = y}) = q^{(n)}(y-x)(Z_n^x P^y(\overleftarrow{e}_{1,n}) + \bar{\delta}_n^{x,y}) \tag{2.2}$$

avec

$$\sup_{|y-x| \leqslant A\sqrt{n}} Q(|\ \bar{\delta}_n^{x,y}\ |) \underset{n\to\infty}{\longrightarrow} 0.$$

Sinai a démontré ce théorème en utilisant des développements perturbatifs dans $L^2(Q)$ qui peuvent être utilisés seulement dans un cadre discret. J'ai redémontré ce théorème en m'appuyant sur la théorie du potentiel et du calcul dans $L^2(Q)$, ce qui a permis de prouver un résultat analogue pour un modèle continu que l'on va présenter. Le théorème limite locale donne des indications fines sur la mesure de polymère et implique le théorème centrale limite en Q-probabilités. En terme de mesure de polymère, on a :

$$\mu_n^x(\omega_n = y) \approx P^y(\overleftarrow{e}_{n-l_n,n}) q^{(n)}(y-x), \tag{2.3}$$

et donc, dans le régime (L2), sous la mesure de polymère, la marche ne ressent l'environnement que dans une fine couche autour du point d'arrivée.

Un modèle continu. Le modèle continu que l'on considère a été introduit par F. Comets et N. Yoshida dans [**12**]. On renvoit à [**12**] et [**14**] pour une étude de ce modèle : dans ces articles, les auteurs démontrent un théorème limite centrale (voir ci-dessous) et des résultats de localisation semblables au modèle discret. Néanmoins, certains résultats obtenus dans le cas continu sont plus fins que dans le cas discret (avec le mouvement brownien et le processus de Poisson, on dispose d'outils puissants tels que la transformée de Girsanov et le calcul stochastique).

Soit d un entier supérieur ou égal à 1 et $((\omega_t)_{t\in\mathbb{R}_+}, (P^x)_{x\in\mathbb{R}^d})$ le mouvement brownien standard, défini sur un espace mesurable $(\Omega, \mathcal{F})$. Par la suite P désignera la mesure P^0 et on notera $P^x(X)$ l'espérance d'une variable X par rapport à P^x.

L'environnement avec lequel interagit le mouvement brownien est un processus de Poisson sur $\mathbb{R}_+ \times \mathbb{R}^d$, défini sur un espace de probabilité $(M, \mathcal{G}, Q)$ et de mesure caractéristique la mesure de lebesgue. On introduit V_t le "tube" de volume unité autour du graphe $\{(s, \omega_s)\}_{0<s\leqslant t}$ du mouvement brownien :

$$V_t = V_t(\omega) = \{(s,x); s \in]0,t], x \in U(\omega_s)\}$$

où $U(x)$ est la boule fermée de $\mathbb{R}^d$ de volume 1 centrée au point $x \in \mathbb{R}^d$.

L'objet d'étude dans ce cas est la mesure (aléatoire) μ_t^x donnée par :

$$\mu_t^x(d\omega) = \frac{\exp(\beta\eta(V_t) - \lambda(\beta)t)}{Z_t^x} P_t^x(d\omega),$$

où $\beta \in \mathbb{R}$ est l'inverse de la température et

$$Z_t^x = P^x(\exp(\beta\eta(V_t) - \lambda(\beta)t))$$

la fonction de partition renormalisée ($Q(Z_t^x) = 1$). Contrairement au cas discret, on a la valeur explicite de λ (on ne travaille pas avec une mesure aléatoire quelconque mais avec une mesure de Poisson) :

$$\lambda(\beta) = e^\beta - 1.$$

On peut aussi introduire la fonction de partition point à point $Z_t^x(y)$ définie par :

$$Z_t^x(y) = p(t,x,y)P^x(\exp(\beta\eta(V_t) - \lambda(\beta)t) \mid \omega_t = y),$$

où $p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|y-x|^2}{2t}}$ est le semigroupe du mouvement brownien. Il est à signaler que les auteurs de [**12**] ont montré que $Z_t^x(y)$ vérifie l'EDP (1.4) avec bruit poissonien au sens des distributions :

$$dZ_t^x(y) = \frac{1}{2}\Delta_y Z_t^x(y)dt + \lambda(\beta)Z_{t^-}^x(y)\eta(dt \times U(y)).$$

On introduit la filtration $(\mathcal{G}_t)_{t>0}$ definie par :

$$\mathcal{G}_t = \sigma\{\eta(A); A \in \mathcal{B}(]0,t] \times \mathbb{R}^d)\}.$$

Comme dans le cas discret, il est aisé de montrer que $(Z_t^x, \mathcal{G}_t)_{t>0}$ est une martingale positive, donc converge Q-p.s. vers une variable positive ou nulle Z_∞^x qui satisfait la loi suivante du 0-1 :

$$Q(Z_\infty^x = 0) = 1 \quad \text{ou} \quad Q(Z_\infty^x = 0) = 0.$$

Dans le premier cas, on dit qu'on est dans le régime de fort désordre et dans le deuxième, on dit qu'on est dans le régime de faible désordre.

Comme dans le cas discret, on peut considérer un régime (L2). Plus précisément, on peut montrer que, pour $d \geqslant 3$, il existe $\lambda_d > 0$ tel que :

$$\sup_{t \geqslant 0} Q((Z_t^x)^2) < \infty \iff \lambda_2(\beta) < \lambda_d. \tag{L2}$$

Dans le régime (L2), il est possible de montrer un théorème limite centrale :

THÉORÈME 2.2 (Théorème limite centrale, **[14]**). *Supposons que β vérifie la condition (L2). Alors, pour tout $f \in C(\mathbb{R}^d)$ avec une croissance au plus polynomiale,*

$$\mu_n^x\left(f(\frac{\omega_t}{\sqrt{t}})\right) \underset{t\to\infty}{\longrightarrow} \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x)e^{-\frac{|x|^2}{2}}dx, \ Q-p.s.$$

Pour énoncer le théorème limite locale, on introduit, pour $s \leqslant t$:

$$e_{s,t} \overset{def.}{=} e^{\beta\eta(V_{s,t})-\lambda(\beta)(t-s)}$$

où $V_{s,t}$ est le tube de volume unité autour du graphe $\{(u,\omega_u)_{s<u\leqslant t}\}$:

$$V_{s,t} = \{(u,x); u \in]s,t], x \in U(\omega_u)\}.$$

On définit également l'analogue dans un environnement retourné en temps :

$$\overleftarrow{e}_{s,t} \overset{def.}{=} e^{\beta\eta(\overleftarrow{V}_{s,t})-\lambda(\beta)(t-s)}$$

où

$$\overleftarrow{V}_{s,t} = \{(t-u,x); u \in]0,t-s], x \in U(\omega_u)\}.$$

Dans le régime (L2), les méthodes que j'ai employé pour démontrer le théorème limite locale dans le cas discret s'adaptent au cas continu :

THÉORÈME 2.3 (Théorème limite locale, [**55**]). *Supposons que β vérifie la condition (L2) et soit A un réel strictement positif. Alors, si $(l_t)_{t \geqslant 0}$ est une fonction positive qui tend vers l'infini telle que $l_t = o(t^a)$ avec $a < \frac{1}{2}$,*

$$P^x(e_{0,t} \mid \omega_t = y) = P^x(e_{0,l_t}) P^y(\overleftarrow{e}_{t-l_t,t}) + \delta_t^{x,y} \tag{2.4}$$

avec

$$\sup_{|y-x| \leqslant A\sqrt{t}} Q(\mid \delta_t^{x,y} \mid^2) \underset{t\to\infty}{\longrightarrow} 0.$$

On en déduit la formule suivante avec approximation dans L^1 :

$$P^x(e_{0,t} \mid \omega_t = y) = Z_t^x P^y(\overleftarrow{e}_{0,t}) + \bar{\delta}_t^{x,y} \tag{2.5}$$

avec

$$\sup_{|y-x| \leqslant A\sqrt{t}} Q(\mid \bar{\delta}_t^{x,y} \mid) \underset{t\to\infty}{\longrightarrow} 0.$$

2.2. Majoration des polymères dirigés par les cascades multiplicatives. La comparaison du modèle de polymères dirigés avec le modèle des cascades multiplicatives fait l'objet du chapitre 2. Il s'agit d'un travail effectué en collaboration avec Francis Comets, à paraitre dans *ALEA*.

Sous la condition (1.7), la proposition 2.5 dans [**11**] assure l'existence (au sens Q-p.s. et dans $L^p(Q)$ pour tout p) de l'énergie libre du polymère définie par la limite suivante :

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln Z_n(\beta). \tag{2.6}$$

Si l'on applique l'inégalité de Jensen à la fonction concave ln, on obtient $p(\beta) \leqslant 0$. Tout comme pour la transition (faible désordre)-(fort désordre), on constate un changement de phase pour l'énergie libre (cf. théorème 3.2 dans [**15**]) :

THÉORÈME 2.4. *Il existe $\widetilde{\beta}_c \in [0, \infty]$ tel que :*

$$p(\beta) \quad \begin{cases} = 0 & \text{if} \quad \beta \in [0, \widetilde{\beta}_c], \\ < 0 & \text{if} \quad \beta > \widetilde{\beta}_c. \end{cases}$$

L'équivalence (1.13) montre l'importance du point de vue de la localisation de trouver les β tels que $p(\beta) < 0$. En terme d'énergie libre, cette équivalence s'écrit :

$$p(\beta) < 0 \qquad \Longleftrightarrow \qquad \exists c > 0 \quad \underline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I_k \geqslant c \qquad Q-p.s. \tag{2.7}$$

Pour donner des bornes supérieures sur $p(\beta)$, on a cherché à comparer p avec l'énergie libre des cascades multiplicatives. Si m est un entier positif fixé et si l'on note $Z_m(x) = P(e^{\beta H_m(\omega)-m\lambda(\beta)}1_{\omega_m=x})$, on peut définir l'énergie libre de la cascade multiplicative qui a pour loi d'itération la loi de $(Z_m(x))_{|x| \leqslant m}$ (on renvoit à [**42**] pour un article présentant les principales propriétés des cascades multiplicatives). Celle-ci est donnée par la relation (cf. [**23**]) :

$$p_m^{\mathrm{tree}}(\beta) = \inf_{\theta \in]0,1]} \frac{1}{\theta} \ln Q\Big(\sum_{|x| \leqslant m} Z_m(x)^\theta\Big) \tag{2.8}$$

On peut résumer l'article par le théorème suivant :

THÉORÈME 2.5. *Dès que l'environnement vérifie la condition (1.7), on a :*

$$p(\beta) \leqslant \inf_{m \geqslant 1} \frac{1}{m} p_m^{\mathrm{tree}}(\beta).$$

Si de plus l'environnement est gaussien ou borné, l'inégalité ci-dessus est une égalité.

Ce théorème donne donc une formule variationnelle pour l'énergie libre des polymères dirigés. Il est assez remarquable et loin d'être évident a priori que le modèle des polymères dirigés puisse s'obtenir comme limite de modèles arborescents à géométrie plus simple (le même phénomène apparait en percolation, cf. [**21**]).

En corollaire de ce théorème, à partir d'estimations sur la fonction de partition obtenues par les auteurs de [**11**], on a résolu la conjecture suivante, dont les physiciens théoriciens attendaient la preuve depuis deux décennies :

$$\text{En dimension d=1}, \quad \forall \beta \neq 0, \quad p(\beta) < 0.$$

Enfin, on finit cette partie en signalant que, pour les cascades multiplicatives associées à la loi de $(Z_m(x))_{|x| \leqslant m}$, il est possible de définir un seuil β_c^m analogue à β_c (introduit dans le théorème 1.1) et un seuil $\widetilde{\beta}_c^m$ analogue à $\widetilde{\beta}_c$. Dans le cas des cascades, on a $\beta_c^m = \widetilde{\beta}_c^m$ et donc on peut conjecturer la même chose sur les polymères dirigés : c'est une des questions ouvertes sur laquelle s'oriente les recherches actuelles (en tout cas, les miennes !).

2.3. Une nouvelle approche de la localisation : les ϵ-atomes. Dans le chapitre 3 de cette thèse, on utilise une approche simple et générale pour l'étude de la relation entre energie libre et localisation. Ce chapitre a fait l'objet d'un article à paraitre dans *Probability Theory and Related Fields.* On a entrepris dans ce chapitre l'étude des polymères dirigés sous des hypothèses sur l'environnement plus faibles que (1.7). A l'origine, cette étude est justifiée pour deux raisons. D'une part, dans l'équation KPZ (1.6), certains physiciens ont proposé, pour modèliser certains phénomènes physiques, de remplacer le bruit blanc gaussien η par un bruit avec des queues à décroissance polynomiale ([**58**]). D'autre part, sur le plan mathématique, si l'on fait tendre β vers l'infini, on obtient la percolation de dernier passage sur le graphe orienté $\mathbb{N} \times \mathbb{Z}^d$. Pour $d = 1$, Johansson a obtenu dans [**31**] des résultats très précis sur le problème de percolation de dernier passage avec des variables exponentielles. Or les variables exponentielles ne vérifient pas (1.7).

Par souci de simplification, pour exposer les résultats obtenus, on suppose que les variables d'environnement sont positives ou nulles (nous renvoyons au chapitre 3 pour les énoncés précis lorsque les variables ne sont pas supposées positives ou nulles). On suppose également que les variables d'environnement vérifient une condition introduite par J. Martin pour l'étude de la percolation de dernier passage ([**46**], [**47**]) :

$$\int_0^\infty Q(\eta(n,x) > t)^{\frac{1}{d+1}} dt < \infty. \tag{2.9}$$

On n'a pas le droit a priori de considérer la fonction de partition renormalisée puisque on peut avoir $\lambda(\beta) = \infty$. Il est alors impossible d'utiliser les techniques de martingale qui sont employées pour obtenir les théorèmes 1.3, 1.4. On considère donc ici que la fonction de partition n'a pas été renormalisée de façon a être de Q-espérance 1 ; dans cette partie, Z_n est donc définie par la relation :

$$Z_n = P(e^{\beta H_n(\omega)}). \tag{2.10}$$

Sous la condition (2.9), on obtient l'existence de l'énergie libre :

THÉORÈME 2.6. *On suppose que l'environnement vérifie (2.9). Alors il existe un réel $p(\beta)$ tel que l'on ait la convergence suivante :*

$$\frac{\ln Z_n}{n} \underset{n\to\infty}{\longrightarrow} p(\beta) \qquad Q-p.s. \text{ et dans } L^1(Q).$$

Contrairement aux études précédentes, on s'est intéressé à toute la mesure et non plus seulement au point préféré. Plus précisément, on voudrait quantifier la masse de la mesure portée par des "points macroscopiques". On introduit donc les ϵ-atomes comme les points de la mesure qui ont une masse plus grande que ϵ :

$$\mathcal{A}_j^{\epsilon,\beta} = \{x \in \mathbb{Z}^d : \mu_{j-1}(\omega_j = x) > \epsilon\}.$$

On peut maintenant citer le théorème suivant qui montre que, si l'on prend ϵ suffisamment petit mais fixe, les ϵ-atomes portent l'essentiel de la mesure de polymère lorsque $\lambda(\beta) = \infty$:

THÉORÈME 2.7. *On suppose que* $\lambda(\beta) = \infty$. *Alors, quelque soit* $\delta < 1$, *il existe* $\epsilon(\delta) > 0$ *tel que :*

$$\liminf_{n\to\infty} Q(\frac{1}{n}\sum_{j=1}^{n} \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon(\delta),\beta})) \geqslant \delta. \tag{2.11}$$

Nos méthodes permettent aussi de traiter le cas où, si l'on note :

$$R = \sup\{\beta \in \mathbb{R}^+ : \lambda(\beta) < \infty\},$$

R est un réel strictement positif (autrement dit, l'environnement a des moments exponentiels). Dans ce cas, on peut citer une version presque sûre du théorème ci-dessus au voisinage de R :

THÉORÈME 2.8. *On suppose que l'environnement explose au point* R *:*

$$\lambda(R)/R = \infty. \tag{2.12}$$

Alors, quelque soit $\delta < 1$, *il existe* $\epsilon(\delta) > 0$ *et* $\beta(\delta)$ *dans* $]0, R[$ *tels que :*

$$\forall \beta \in [\beta(\delta), R[\qquad \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon(\delta),\beta}) \geqslant \delta \qquad Q-p.s.$$

Enfin, les méthodes utilisées dans le chapitre permettent de redémontrer un sens de (2.7) sous des hypothèses affaiblies ; on peut donc citer le théorème suivant en utilisant la terminologie des ϵ-atomes :

THÉORÈME 2.9. *Quelque soit* β *dans* $]0, R[$, *on a l'implication suivante :*

$$p(\beta) < \lambda(\beta) \qquad \Rightarrow \qquad \exists\, \epsilon > 0, \qquad \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon,\beta}) \geqslant \epsilon. \qquad Q-p.s.$$

3. Vers la construction d'un champ aléatoire de vitesse turbulent

3.1. Pourquoi une théorie probabiliste de la turbulence ? Pour une introduction concise à la turbulence, on renvoit à [**50**].

Le mouvement d'un fluide incompressible, de viscosité $\nu > 0$, confiné dans un domaine $D \subset \mathbb{R}^3$ de l'espace est décrit par les équations de Navier-Stokes (1823) :

$$\text{(N-S)} \begin{cases} \frac{\partial v}{\partial t} + (v.\nabla)v - \nu\Delta v = -\nabla p \\ div(v) = 0 \end{cases}$$

où $v(t, x)$ est le champ de vitesse du fluide et $p(t, x)$ sa pression. Bien sûr, pour être complet, il faut spécifier une condition initiale v_0 et imposer des conditions au bord de D. Cette équation est très généralement considérée par les physiciens comme un modèle pertinent d'écoulement de fluide dans une vaste gamme de régime.

On associe à cette équation le nombre (sans dimension) de Reynolds de l'écoulement :

$$Re = \frac{|(v.\nabla)v|}{\nu\Delta v} \approx \frac{UL}{\nu}, \tag{3.1}$$

où L et U sont repectivement la longueur et la vitesse caractéristiques de l'écoulement. Lorsque le nombre Re est "élevé" (supérieur à 1000 en pratique), le terme convectif non-linéaire $(v.\nabla)v$ devient prépondérant sur le terme linéaire de dissipation $\nu\Delta v$ et on dit que l'écoulement est turbulent. C'est ce régime auquel on s' intéresse. Pour l'instant, on a vu qu'une équation déterministe suffisait selon les physiciens à décrire le champ de vitesse d'un écoulement alors pourquoi introduire des probabilités dans ce domaine ?

Il y a plusieurs raisons d'ordre expérimental à cela. Dans la pratique, le physicien a rarement accès à la condition initiale de l'écoulement et donc poser la question en terme de problème de Cauchy n'est pas nécessairement pertinent. En deuxième lieu, l'objectif du physicien est d'expliquer les faits observés expérimentalement ; or les lois expérimentales de la turbulence sont de nature statistique. Par exemple, on peut considérer les mesures expérimentales faites dans le tunnel S1 de l'ONERA rapportées dans [**24**] : en un point du tunnel a été mesuré à quelques minutes d'intervalle la vitesse v du vent pendant 1 seconde à la fréquence de 5 kHz. En termes plus mathématiques, si l'on mesure le temps t en secondes et si l'on suppose que les séries de mesures ont été faites à 2 minutes d'intervalle, les expérimentateurs ont mesurés $(v_{i/5000})_{1 \leqslant i \leqslant 5000}$ et $(v_{120+i/5000})_{1 \leqslant i \leqslant 5000}$. En approximant un peu, disons que ce sont les courbes $(v_t)_{0 \leqslant t \leqslant 1}$ et $(v_{120+t})_{0 \leqslant t \leqslant 1}$ qui ont été mesurées. Comme le constate l'auteur (p.28 de [**24**]), il semble impossible de prédire la courbe $(v_{120+t})_{0 \leqslant t \leqslant 1}$ à partir de $(v_t)_{0 \leqslant t \leqslant 1}$; en revanche (cf. fig. 3.3 p.30 de [**24**]), on constate que les deux mesures $\int_0^1 \delta_{v_t} dt$ et $\int_{120}^{121} \delta_{v_t} dt$ sont sensiblements identiques !

Une manière naturelle de tenir compte des deux constats ci-dessus est de faire appel à la théorie des probabilités. Puisque l'on ne peut pas connaitre la condition initiale avec précision, on peut supposer que la condition initiale v_0 de l'équation (N-S) est un champ aléatoire et le fait que les deux mesures $\int_0^1 \delta_{v_t} dt$ et $\int_{120}^{121} \delta_{v_t} dt$ sont sensiblements identiques nous conduit à penser que, par ergodicité, pour tout t les mesures aléatoires $\frac{1}{T}\int_t^{T+t} \delta_{v_s} ds$ convergent en loi lorsque $T \to \infty$ vers une unique mesure : on est donc amené a rechercher les mesures invariantes de (N-S).

Cependant, ce programme se heurte à de sérieuses difficultés. En effet, en dimension 3, partant d'une condition initiale v_0 régulière, il n'a pas encore été démontré que l'équation (N-S) admet une solution unique pour tout t ; quant à l'existence de mesures invariantes, il semble que ce soit un probleme encore plus difficile (cf. [**51**]) !

Pour résumer, on a vu qu'il était naturel de considérer le champ de vitesse d'un écoulement turbulent comme un objet aléatoire et que pour obtenir une solution satifaisante au probleme de la turbulence il faudrait construire un champ aléatoire invariant par la dynamique. Malheureusement, cela semble hors de portée pour l'instant et on a donc entrepris une démarche plus modeste, à savoir la construction de champs aléatoires vérifiant les principales propriétés observées expérimentalement.

3.2. Construction de champs multifractaux. Dans cette partie est exposé le chapitre 4 de cette thèse qui résulte d'un travail en collaboration avec Jean Duchon et Raoul Robert. Ce travail précise et prolonge des résultats évoqués dans [**19**], [**20**] ; notamment, on étend la famille de champs symétriques de [**19**] en une famille dissymétrique à 4 paramètres et on aborde le problème de la construction d'un champ vectoriel intermittent incompressible qui vérifie la loi du 4/5 (cf. l'équation (3.2) ci-dessous).

Lorsque le nombre de Reynolds est élevé, si l'on considère un écoulement turbulent dans un domaine D et que l'on se place loin du bord, on observe un certain nombre de symmétries au niveau statistique : le champ de vitesse semble homogène en temps (à condition de laisser suffisamment de temps pour que s'installe un régime stationnaire), homogène en espace et isotrope. Mathématiquement, on cherche à construire un champ (de vitesse) aléatoire $(U(x))_{x\in\mathbb{R}^3}$ incompressible à valeurs dans $\mathbb{R}^3$, défini sur un espace probabilisé $(\Omega, \mathcal{F}, P)$ et qui vérifie :

(1) Pour tout x et ξ dans $\mathbb{R}^3$, $(U(x+\xi))_{x\in\mathbb{R}^3}$ a même loi que $(U(x))_{x\in\mathbb{R}^3}$ (homogénéité en espace).

(2) Pour toute rotation R, $(U(Rx))_{x\in\mathbb{R}^3}$ a même loi que $(RU(x))_{x\in\mathbb{R}^3}$ (isotropie).

A partir de mesures expérimentales, on observe également une forme affaiblie d'invariance d'échelle appelée intermittence qui se caractérise sur les moments du champs par la formule suivante pour tout q positif :

$$E\left(|(U(x+\xi)-U(x)).\frac{\xi}{|\xi|}|^q\right) \underset{|\xi|\to 0}{\sim} C_q|\xi|^{\zeta_q},$$

où C_q est une constante (indépendante de x et ξ par symmétrie du champ) et ζ_q est une fonction strictement concave appelée fonction de structure du champ. Enfin, si l'on note par D la dissipation d'energie cinétique du flot par unité de masse, Kolmogorov a démontré en 1941 ([**35**]) que tout champ aléatoire homogène, isotrope et qui vérifie (N-S) avec $\nu \to 0$ satisfait la loi dite du 4/5 :

$$E\left(((U(x+\xi)-U(x)).\frac{\xi}{|\xi|})^3\right) \underset{\xi\to 0}{\sim} -\frac{4}{5}D|\xi|. \tag{3.2}$$

La démonstration de Kolmogorov n'est pas rigoureuse au sens mathématique mais néanmoins est considérée comme valide par l'ensemble de la communauté physique. Pour une preuve rigoureuse de la loi du 4/5 sous des hypothèses raisonnables sur le champ U, on renvoit à [**18**]. En particulier, la loi du 4/5 implique que la loi des incréments du champ est dissymétrique.

Dans le chapitre 4, on s'est placé en dimension d quelconque. Dans un premier temps, on a construit une famille de champs scalaires $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$ dissymétriques. Le champ $\mathcal{X}$ dépend de 4 paramètres : R, α, γ_1, γ_0^*. R est un paramètre de corrélation, α et γ_1 des paramètres d'intermittence et γ_0^* mesure la dissymétrie. Soit e un vecteur unitaire de $\mathbb{R}^d$ et l un entier positif. Pour le champ $\mathcal{X}$, on a démontré que si $l\gamma_1^2$ est "petit" (cf. les conditions de la proposition 3.5), il existe $C_l \neq 0$ (indépendant de R, e et γ_0^*) tel que :

$$E((\mathcal{X}(x+\lambda e)-\mathcal{X}(x))^{2l}) \underset{\lambda\to 0}{\sim} C_l(\frac{\lambda}{R})^{\zeta_{2l}},$$

où ζ_{2l} est donnée par la relation (ω_d désigne le volume de la sphère unité) :

$$\zeta_{2l} = l(2\alpha-d) - 2\gamma_1^2\omega_d l(l-1).$$

Sous la même condition sur $l\gamma_1^2$, il existe C_l (indépendant de R, e, γ_0^* et tel que $C_3 \neq 0$) tel que :

$$E((\mathcal{X}(x+\lambda e)-\mathcal{X}(x))^{2l+1}) \underset{\lambda\to 0}{\sim} \gamma_0^* R^{d/2} C_l(\frac{\lambda}{R})^{\widetilde{\zeta}_{2l+1}}, \tag{3.3}$$

où $\widetilde{\zeta}_{2l+1}$ est donnée par la relation :

$$\widetilde{\zeta}_{2l+1} = l(2\alpha-d) - 2\gamma_1^2\omega_d l(l-1) + 2. \tag{3.4}$$

Malheureusement, l'expression de $\widetilde{\zeta}_{2l+1}$ est incompatible avec la loi du 4/5 qui impose la condition $\widetilde{\zeta}_3 = 1$. En modifiant la construction de $\mathcal{X}$ de manière à introduire plus de dissymétrie , on a construit un champ vectoriel $\mathcal{X}_0$ (dépendant de α, γ_0 et d'un paramètre de corrélation R) tel que, si e est un vecteur unitaire et q un entier positif, il existe C_q (indépendant de R, e et tel que $C_3 \neq 0$) tel que :

$$E\left(((\mathcal{X}_0(x+\lambda e)-\mathcal{X}_0(x)).e)^q\right) \underset{\lambda\to 0}{\sim} C_q(\frac{\lambda}{R})^{\zeta_q},$$

où ζ_q est donnée par la relation :

$$\zeta_q = q\alpha - \frac{1}{2}q(q-1)\gamma_0^2\omega_d.$$

Du point de vue de la turbulence et notamment de la loi du $4/5$, ce champ est satisfaisant car si on choisit α et γ_0 tels que $\alpha = 1/3 + 4\pi\gamma_0^2$, on a $\zeta_3 = 1$.

Dans un deuxieme temps, on a cherché à modifier la construction de $\mathcal{X}_0$ (dont la divergence n'est pas nulle) pour en faire un champ incompressible. Ainsi, on a obtenu un champ vectoriel U incompressible, homogène, isotrope et intermittent de fonction de structure ζ_q donnée par celle du champ $\mathcal{X}_0$. Cependant, on a trouvé :

$$\lim_{\xi \to 0} \frac{1}{|\xi|} E\left(((U(x+\xi) - U(x)).\frac{\xi}{|\xi|})^3 \right) = 0$$

et donc la construction d'un champ à dissipation positive reste un problème ouvert.

CHAPTER 1

A Local limit theorem in the diffusive case

1. Introduction

Directed polymers in random environment is a model of statistical mechanics in which stochastic processes interact with a random environment, depending on both time and space: one studies the path of the stochastic process under a random Gibbs measure depending on the temperature (as the temperature increases, the influence of the random environment decreases).

In this chapter, we will consider two polymer models: a random walk model of directed polymers and its continuous analogue, a Brownian model of directed polymers. The discrete model first appeared in the physics literature ([**28**]) to modelize the phase boundary of Ising model subject to random impurities and its first mathematical study was undertaken by Imbrie, Spencer in 1988 ([**30**]) and Bolthausen in 1989 ([**7**]). The continuous model we study here was first introduced and studied by Comets and Yoshida in 2004 ([**12**]). These models are related to many models of statistical physics. We refer to the survey paper [**39**] by Krug and Spohn for an account on these models and their relations.

In the sequel, we will suppose that the dimension of the underlying stochastic process is greater than or equal to 3 and that the normalized partition function is bounded in L^2 (see subchapters 1.1-1.2. for the definition of the normalized partition function). Under these assumptions, the polymer is diffusive in the sense that a central limit theorem holds: by scaling by the square root of the length, the discrete and continuous polymer models converge in law to a Gaussian measure (see [**30**], [**7**], [**53**], [**14**]). One can sometimes go a step further than convergence in law by giving an equivalent of the density: this is called a local limit theorem. In [**52**], Sinai obtained a local limit theorem by using a perturbation expansion. Unfortunately, it is not clear how to adapt the strategy to the continuous setting. The object of this work is to give a new method for proving Sinai's theorem; this method is sufficiently general to be easily adapted to prove a similar local limit theorem in the continuous setting. Our approach is simple and relies only on L^2 computations and on properties of the simple random walk bridges (Brownian bridges in the continuous case).

Finally, we recall that some results have been achieved in the case of dimension less than or equal to 2 or when the temperature is low. In these cases, the polymer is non-diffusive (see remark 2.6 below) and many conjectures remain open. For an account on these cases, we refer to **[9]** in a Gaussian environment and to **[13]** in a general environment.

The chapter is organized as follows: each subchapter is divided into two parts, one of them being devoted to the discrete model and the other one being devoted to the continuous model. First, we introduce the two models. In the second subchapter, we will remind the known results at high temperature when the dimension of the underlying process is greater than or equal to 3; we will also formulate an analogue to Sinai's local limit theorem for the Brownian directed polymer. In the third subchapter, we will prove the local limit theorem for both models.

1.1. The random walk model of directed polymers.

- Let $((\omega_n)_{n\in\mathbb{N}}, (P^x)_{x\in\mathbb{Z}^d})$ denote the simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$, defined on a measurable space $(\Omega, \mathcal{F})$; more precisely, for x in $\mathbb{Z}^d$, under the measure P^x, $(\omega_n - \omega_{n-1})_{n\,\geqslant\,1}$ are independent and

$$P^x(\omega_0 = x) = 1, \quad P^x(\omega_n - \omega_{n-1} = \pm\delta_j) = \frac{1}{2d}, \quad j = 1, \dots, d,$$

  where $(\delta_j)_{1\,\leqslant\,j\,\leqslant\,d}$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. In the sequel, P will denote P^0 and P_n^x will denote the simple random walk measure on paths of length n starting from x. For x in $\mathbb{Z}^d$, let $q^{(n)}(x)$ be the probability for the random walk starting from 0 to be in x at time n:

$$q^{(n)}(x) \stackrel{def.}{=} P(\omega_n = x).$$

- The random environment on each lattice site is a sequence $\eta = (\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ of real valued, non-constant and i.i.d. random variables defined on a probability space $(H, \mathcal{G}, Q)$ such that

$$\forall \beta \in \mathbb{R} \ \ \lambda(\beta) \stackrel{def.}{=} \ln Q(e^{\beta\eta(n,x)}) < \infty.$$

- For any $n > 0$, we define the (Q-random) polymer measure μ_n^x on paths of length n starting from x by:

$$\mu_n^x(d\omega) = \frac{1}{Z_n^x} \exp(\beta H_n(\omega) - n\lambda(\beta)) P_n^x(d\omega)$$

where $\beta \in \mathbb{R}$ is the inverse temperature,

$$H_n(\omega) \stackrel{def.}{=} \sum_{j=1}^{n} \eta(j, \omega_j)$$

and

$$Z_n^x = P^x(\exp(\beta H_n(\omega) - n\lambda(\beta)))$$

is the normalized partition function ($Q(Z_n^x) = 1$).

Let $(\mathcal{G}_n)_{n \geqslant 0}$ be the filtration defined by

$$\mathcal{G}_n = \sigma\{\eta(j, x); j \leqslant n, x \in \mathbb{Z}^d\}.$$

For any fixed path ω, $((\sum_{j=1}^{n} \beta\eta(j, \omega_j)) - n\lambda(\beta))_{n \geqslant 1}$ is a random walk with independent increments thus it is not hard to see that $(Z_n^x, \mathcal{G}_n)_{n \geqslant 0}$ is a positive martingale. Therefore, it converges Q-a.s. to a limit Z_∞^x. Since the event $(Z_\infty^x = 0)$ is measurable with respect to the tail σ-field

$$\bigcap_{n \geqslant 1} \sigma\{\eta(j, x); j \geqslant n, x \in \mathbb{Z}^d\},$$

by Kolmogorov's $0 - 1$ law, there are only two possible situations

$$Q(Z_\infty^x = 0) = 1 \quad or \quad Q(Z_\infty^x = 0) = 0.$$

In the former case, we say that strong disorder holds and in the latter case we say that weak disorder holds.

1.2. The Brownian model of directed polymers.

- Let $((\omega_t)_{t \in \mathbb{R}_+}, (P^x)_{x \in \mathbb{R}^d})$ denote a d-dimensional standard Brownian motion, defined on a measurable space $(\Omega, \mathcal{F})$. In the sequel, P will denote P^0 and P_t^x the Brownian measure on paths of length t starting from x. For $t > 0$ and x, y in $\mathbb{R}^d$, let $p(t, x, y)$ be the transition density of the Brownian motion:

$$p(t, x, y) \stackrel{def.}{=} \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|y-x|^2}{2t}}.$$

- The random environment η is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d$ with unit intensity, defined on a probability space $(M, \mathcal{G}, Q)$. We recall that η is an integer valued random measure characterized by the following property: If $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ are disjoint and bounded Borel sets, then

$$Q(\bigcap_{j=1}^{n} (\eta(A_j) = k_j)) = \prod_{j=1}^{n} e^{-|A_j|} \frac{| A_j |^{k_j}}{k_j!}.$$

where $k_1, \dots k_n \in \mathbb{N}$ and $|\ .\ |$ denotes the Lebesgue measure in $\mathbb{R}^{d+1}$. We define V_t to be the unit volume "tube" around the graph $\{(s, \omega_s)\}_{0<s\leqslant t}$ of the Brownian path:

$$V_t = V_t(\omega) = \{(s,x); s \in]0,t], x \in U(\omega_s)\}$$

where $U(x)$ is the closed ball in $\mathbb{R}^d$ with unit volume and centered at $x \in \mathbb{R}^d$.

- For any $t > 0$, we define the (Q-random) polymer measure μ_t^x on paths of length t starting from x by:

$$\mu_t^x(d\omega) = \frac{\exp(\beta\eta(V_t) - \lambda(\beta)t)}{Z_t^x} P_t^x(d\omega),$$

where $\beta \in \mathbb{R}$ is the inverse temperature and

$$Z_t^x = P^x(\exp(\beta\eta(V_t) - \lambda(\beta)t))$$

is the normalized partition function ($Q(Z_t^x) = 1$). In this setting, the random environment is a Poisson point process so we get the explicit value:

$$\lambda(\beta) = e^\beta - 1 \in]-1, \infty[.$$

It is natural to introduce the filtration $(\mathcal{G}_t)_{t>0}$ defined by :

$$\mathcal{G}_t = \sigma\{\eta(A); A \in \mathcal{B}(]0,t] \times \mathbb{R}^d)\}.$$

As in the discrete setting, it is not hard to show that $(Z_t^x, \mathcal{G}_t)_{t>0}$ is a positive martingale wich converges Q-a.s. to a non negative random variable Z_∞^x that has the following property:

$$Q(Z_\infty^x = 0) = 1 \quad or \quad Q(Z_\infty^x = 0) = 0.$$

In the former case, we say that strong disorder holds and in the latter case we say that weak disorder holds.

2. The diffusive case

From now on, in the rest of this chapter, we will only consider the case $d \geqslant 3$ and we will suppose that the normalized partition function is bounded in $L^2(Q)$. In that case, the latter converges Q-a.s. and in $L^2(Q)$ to the random variable Z_∞^x. The L^2-convergence implies that $Q(Z_\infty^x) = 1$ and therefore weak disorder holds. Under these assumptions, the behavior of the typical path under the polymer measure is diffusive (see **[13]** for the discrete case and **[14]** for the continuous case).

2.1. The random walk model. In order to get a nice probabilistic interpretation, we work on the product space $(\Omega^2, \mathcal{F}^{\otimes 2}, (P^x \otimes P^y)_{x,y\in\mathbb{Z}^d})$ and thus consider another simple random walk $(\widetilde{\omega}_n)_{n\in\mathbb{N}}$ independent of the first one $(\omega_n)_{n\in\mathbb{N}}$ under the same environment.

Let $\lambda_2(\beta) \overset{def.}{=} \lambda(2\beta) - 2\lambda(\beta)$ and $N_{k,n} = N_{k,n}(\omega, \widetilde{\omega})$ be the number of ordered intersections of ω and $\widetilde{\omega}$ between k and n:

$$N_{k,n} \overset{def.}{=} \sum_{j=k}^{n} 1_{\omega_j = \widetilde{\omega}_j}.$$

With these notations, the following proposition is straightforward (e.g., [**13**]):

Proposition 2.1. *We have the following identity:*

$$\begin{aligned} Q((Z_n^x)^2) &= P^x \otimes P^x(e^{\lambda_2(\beta)N_{1,n}}) \\ &= P \otimes P(e^{\lambda_2(\beta)N_{1,n}}). \end{aligned}$$

In particular,

$$\sup_{n\geqslant 0} Q((Z_n^x)^2) = P \otimes P(e^{\lambda_2(\beta)N_{1,\infty}}).$$

We have the following equivalence

$$P \otimes P(e^{\lambda_2(\beta)N_{1,\infty}}) < \infty \iff \lambda_2(\beta) < \ln(\frac{1}{\pi_d})$$

where $\pi_d \overset{def.}{=} P(\exists n \geqslant 1,\ \omega_n = 0) < 1$. *Thus, we have the following equivalence:*

$$\sup_{n\geqslant 0} Q((Z_n^x)^2) < \infty \iff \lambda_2(\beta) < \ln(\frac{1}{\pi_d}).$$

A series of articles [**30**], [**7**], [**53**] lead to the following central limit theorem:

Theorem 2.2 (Central limit Theorem). *Suppose that the normalized partition function is bounded in* L^2*:*

$$\lambda_2(\beta) < \ln(\frac{1}{\pi_d}).$$

Then, for all $f \in C(\mathbb{R}^d)$ *with at most polynomial growth at infinity,*

$$\mu_n^x\left(f(\frac{\omega_n}{\sqrt{n}})\right) \underset{n\to\infty}{\longrightarrow} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(\frac{x}{\sqrt{d}}) e^{-\frac{|x|^2}{2}} dx,\ Q-a.s.$$

A step further is to try and prove a local limit theorem: one wants to obtain an expansion of the density $P^x(e^{\beta H_n(\omega)-n\lambda(\beta)}1_{\omega_n=y})$. As mentioned in the introduction, this has been done in **[52]** by Sinai. In this chapter, we will give a different proof of the local limit theorem which can be adapted to prove a continuous analogue in the Brownian setting.

Let us introduce a few notations that we will use in the rest of this chapter. We define for $k \leqslant n$

$$e_{k,n} \stackrel{def.}{=} \exp((\sum_{j=k}^{n} \beta\eta(j,\omega_j)) - (n-k+1)\lambda(\beta))$$

and the time reversed analogue

$$\overleftarrow{e}_{k,n} \stackrel{def.}{=} \exp((\sum_{j=0}^{n-k} \beta\eta(n-j,\omega_j)) - (n-k+1)\lambda(\beta)).$$

We can now recall Sinai's local limit theorem in a suitable form:

THEOREM 2.3 (Sinai, 1995). *Let $d \geqslant 3$, $A > 0$ and β be such that $\lambda_2(\beta) < \ln(\frac{1}{\pi_d})$. Then, if $(l_n)_{n\geqslant 0}$ is a sequence of integers that tend to infinity such that $l_n = o(n^a)$ with $a < \frac{1}{2}$,*

$$P^x(e_{1,n} \mid \omega_n = y) = P^x(e_{1,l_n})P^y(\overleftarrow{e}_{n-l_n,n}) + \delta_n^{x,y} \tag{2.1}$$

with

$$\sup_{|y-x| \leqslant A\sqrt{n}} Q(|\ \delta_n^{x,y}\ |^2) \underset{n\to\infty}{\to} 0.$$

This leads to the following formulation that can be found in Sinai's article:

$$P^x(e_{1,n} \mid \omega_n = y) = Z_\infty^x P^y(\overleftarrow{e}_{1,n}) + \bar{\delta}_n^{x,y} \tag{2.2}$$

with

$$\sup_{|y-x| \leqslant A\sqrt{n}} Q(|\ \bar{\delta}_n^{x,y}\ |) \underset{n\to\infty}{\to} 0.$$

REMARK 2.4. *Intuitively, the local limit theorem asserts that, conditionally to the event $(\omega_n = y)$, the polymer only "feels" the environment at times k small where it stays near x and at times k close to n where it stays near y. In between , the polymer behaves like a conditioned simple random walk.*

REMARK 2.5. *Theorem 2.3 leads to a weak form of theorem 2.2: for all $f \in C(\mathbb{R}^d)$ with compact support,*

$$\mu_n^x\left(f(\frac{\omega_n}{\sqrt{n}})\right) \underset{n\to\infty}{\overset{Q-Proba.}{\longrightarrow}} \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\frac{x}{\sqrt{d}})e^{-\frac{|x|^2}{2}}dx.$$

This derivation can be found in [**52**].

REMARK 2.6. *At a heuristic level, we argue that the local limit theorem is a natural definition for the polymer to be diffusive (more natural than the central limit theorem itself). Roughly, the local limit theorem implies*

$$\begin{aligned} I_n &\stackrel{def.}{=} \sum_{x\in\mathbb{Z}^d} \mu_n(\omega_n = x)^2 \\ &\approx \sum_{x\in\mathbb{Z}^d} (P^x(\overleftarrow{e}_{1,n}))^2 q^{(n)}(x)^2 \\ &\approx Q(Z_n^2) \times \sum_{x\in\mathbb{Z}^d} q^{(n)}(x)^2 \\ &\approx \frac{C}{n^{d/2}}. \end{aligned}$$

With other respects, recall (e.g. [**13**]*) that for $d = 1, 2$ and $\beta \neq 0$ or $d \geqslant 3$ and β large,*

$$\exists \delta > 0, \quad \overline{\lim_{n\to\infty}} I_n \geqslant \delta \;\; Q - a.s.$$

(at least if η is unbounded in the second case). Therefore, it is natural to call these two cases "non-diffusive" as mentioned in the introduction.

2.2. The Brownian model. This subchapter is the continuous analogue of the previous one. We work on the product space $(\Omega^2, \mathcal{F}^{\otimes 2}, (P^x \otimes P^y)_{x,y\in\mathbb{R}^d})$ and thus consider another d-dimensional Brownian motion $(\widetilde{\omega}_t)_{t\in\mathbb{R}_+}$ independent of the first one $(\omega_t)_{t\in\mathbb{R}_+}$ under the same environment.

Let $\lambda_2(\beta) \stackrel{def}{=} \lambda(2\beta) - 2\lambda(\beta)$ where we recall that $\lambda(\beta) = e^\beta - 1$. Let $N_{s,t} = N_{s,t}(\omega, \widetilde{\omega})$ be the volume of the overlap in time $[s, t]$ of unit "tubes" around ω and $\widetilde{\omega}$:

$$N_{s,t} \stackrel{def.}{=} \int_s^t \mid U(\omega_u) \cap U(\widetilde{\omega}_u) \mid du.$$

With these notations, we can find the following proposition in [**12**]:

PROPOSITION 2.7. *We have the following identity:*

$$\begin{aligned} Q((Z_t^x)^2) &= P^x \otimes P^x(e^{\lambda_2(\beta)N_{0,t}}) \\ &= P \otimes P(e^{\lambda_2(\beta)N_{0,t}}). \end{aligned}$$

In particular,

$$\sup_{t\geqslant 0} Q((Z_t^x)^2) = P \otimes P(e^{\lambda_2(\beta)N_{0,\infty}}).$$

There exists $\lambda(d) > 0$ *such that:*

$$\lambda' \in]0, \lambda(d)[\iff P \otimes P(e^{\lambda' N_{0,\infty}}) < \infty.$$

In [**14**], Comets and Yoshida prove the following central limit theorem:

THEOREM 2.8 (Central limit theorem). *Suppose that* β *is such that:*

$$\lambda_2(\beta) < \lambda(d).$$

Then, for all $f \in C(\mathbb{R}^d)$ *with at most polynomial growth at infinity,*

$$\mu_t^x \left(f(\frac{\omega_t}{\sqrt{t}}) \right) \underset{t\to\infty}{\longrightarrow} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-\frac{|x|^2}{2}} dx, \ Q - a.s.$$

As in the discrete setting ,we define for $s \leqslant t$

$$e_{s,t} \overset{def.}{=} e^{\beta\eta(V_{s,t}) - \lambda(\beta)(t-s)}$$

where $V_{s,t}$ is the unit "tube" around the graph $\{(u, \omega_u)_{s<u\leqslant t}\}$:

$$V_{s,t} = \{(u, x); u \in]s, t], x \in U(\omega_u)\}.$$

We also define the time reversed analogue:

$$\overleftarrow{e}_{s,t} \overset{def.}{=} e^{\beta\eta(\overleftarrow{V}_{s,t}) - \lambda(\beta)(t-s)}$$

where

$$\overleftarrow{V}_{s,t} = \{(t-u, x); u \in]0, t-s], x \in U(\omega_u)\}.$$

We can now formulate a new result: the local limit theorem for Brownian polymers.

THEOREM 2.9. *Let* $d \geqslant 3$, $A > 0$ *and* β *be such that* $\lambda_2(\beta) < \lambda(d)$. *Then, if* $(l_t)_{t\geqslant 0}$ *is a positive function that tends to infinity such that* $l_t = o(t^a)$ *with* $a < \frac{1}{2}$,

$$P^x(e_{0,t} \mid \omega_t = y) = P^x(e_{0,l_t}) P^y(\overleftarrow{e}_{t-l_t,t}) + \delta_t^{x,y}$$

with

$$\sup_{|y-x| \leqslant A\sqrt{t}} Q(|\ \delta_t^{x,y}\ |^2) \underset{t\to\infty}{\to} 0.$$

This leads to the following formulation in L^1*:*

$$P^x(e_{0,t} \mid \omega_t = y) = Z_\infty^x P^y(\overleftarrow{e}_{0,t}) + \bar{\delta}_t^{x,y}$$

with

$$\sup_{|y-x| \leqslant A\sqrt{t}} Q(|\ \bar{\delta}_t^{x,y}\ |) \underset{t\to\infty}{\to} 0.$$

REMARK 2.10. *The remarks 2.4 and 2.5 apply here too.*

3. Proofs

Our proof of theorem 2.3 is based on the way bridge measures of the simple random walk relate to the measure of the simple random walk. This proof can be translated in the continuous setting because Brownian bridge measures relate to the Wiener measure in a similar way. The two main relations we use are the absolute continuity result (3.4) (relation (3.10) in the Brownian setting) and the inequality (3.6) (relation (3.13) in the Brownian setting) which can be proved by using potential theory.

3.1. Proof of theorem 2.3. First we state and prove a few results that we will use in the proof of theorem 2.3. We remind the classical local limit theorem for the simple random walk (cf. [**40**]):

THEOREM 3.1 (Local limit theorem). *For $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, we say that n and x have the same parity and write $n \leftrightarrow x$ if $n+\sum_{k=1}^d x_k$ is even and we define $\bar{q}^{(n)}(x)$ to be the Gaussian approximation of $q^{(n)}(x)$:*

$$\bar{q}^{(n)}(x) \overset{def.}{=} 2(\frac{d}{2\pi n})^{\frac{d}{2}} e^{-\frac{d|x|^2}{2n}}.$$

With these notations, we have:

$$\sup_{x:n\leftrightarrow x} \mid q^{(n)}(x) - \bar{q}^{(n)}(x) \mid = O(\frac{1}{n^{\frac{d}{2}+1}}). \tag{3.1}$$

In particular,

$$\sup_{x:n\leftrightarrow x} q^{(n)}(x) = O(\frac{1}{n^{\frac{d}{2}}}). \tag{3.2}$$

and if one fixes $A > 0$, there exists $c > 0$ such that

$$\inf_{\substack{x:n\leftrightarrow x \\ |x| \leqslant A\sqrt{n}}} q^{(n)}(x) \geqslant c\frac{1}{n^{\frac{d}{2}}}. \tag{3.3}$$

We will need the following obvious corollary of theorem 3.1 wich can be understood as an absolute continuity result:

COROLLARY 3.2. *Let $t \in]0,1[$ and $A > 0$. There exists a constant $C(A,d) > 0$ such that:*

$$\forall f \geqslant 0 \ \forall n \sup_{|y-x| \leqslant A\sqrt{n}} P^x \otimes P^x(f((\omega_k, \widetilde{\omega}_k)_{k \leqslant \lfloor nt \rfloor} \mid \omega_n = y, \widetilde{\omega}_n = y)) \\ \leqslant \frac{C(A,d)}{(1-t)^d} P^x \otimes P^x(f((\omega_k, \widetilde{\omega}_k)_{k \leqslant \lfloor nt \rfloor})). \tag{3.4}$$

PROOF. By developping the left hand side of the inequality:

$$
\begin{aligned}
&P^x \otimes P^x(f((\omega_k, \widetilde{\omega}_k)_{k \leqslant \lfloor nt \rfloor}) \mid \omega_n = y, \widetilde{\omega}_n = y) \\
&= \sum_{\substack{z_1,\dots,z_{\lfloor nt \rfloor} \in \mathbb{Z}^d \\ \widetilde{z}_1,\dots,\widetilde{z}_{\lfloor nt \rfloor} \in \mathbb{Z}^d}} q^{(1)}(z_1 - x) \dots q^{(1)}(z_{\lfloor nt \rfloor} - z_{\lfloor nt \rfloor - 1}) \\
&q^{(1)}(\widetilde{z}_1 - x) \dots q^{(1)}(\widetilde{z}_{\lfloor nt \rfloor} - \widetilde{z}_{\lfloor nt \rfloor - 1}) \\
&f(z_1, \dots, z_{\lfloor nt \rfloor}, \widetilde{z}_1, \dots, \widetilde{z}_{\lfloor nt \rfloor}) \frac{q^{(n-\lfloor nt \rfloor)}(y - z_{\lfloor nt \rfloor})}{q^{(n)}(y-x)} \\
&\frac{q^{(n-\lfloor nt \rfloor)}(y - \widetilde{z}_{\lfloor nt \rfloor})}{q^{(n)}(y-x)}.
\end{aligned}
$$

By the local limit theorem 3.1,

$$
\frac{q^{(n-\lfloor nt \rfloor)}(y - z_{\lfloor nt \rfloor})}{q^{(n)}(y-x)} \underset{(3.2,3.3)}{\leqslant} C'\left(\frac{n}{n - \lfloor nt \rfloor}\right)^{\frac{d}{2}} \leqslant \frac{C'}{(1-t)^{\frac{d}{2}}}.
$$

Similarly,

$$
\frac{q^{(n-\lfloor nt \rfloor)}(y - \widetilde{z}_{\lfloor nt \rfloor})}{q^{(n)}(y-x)} \leqslant \frac{C'}{(1-t)^{\frac{d}{2}}}.
$$

□

In order to prove theorem 2.3, we will also need to use a result that comes from discrete potential theory. For a complete overview of potential theory for discrete Markov chains, we refer to [**57**].

LEMMA 3.3. *For $d \geqslant 3$ and $v : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow \mathbb{R}_+$ a bounded and non negative function, define*

$$
\Phi(x,y) = P^x \otimes P^y(e^{\sum_{k=1}^{\infty} v(\omega_k, \widetilde{\omega}_k)}).
$$

Suppose that

$$
\sup_{x,y \in \mathbb{Z}^d} \Phi(x,y) < \infty.
$$

Then there exists a constant $C \in]0, \infty[$ such that

$$
\sup_{x,y \in \mathbb{Z}^d} P^x \otimes P^y(e^{\sum_{k=1}^{n} v(\omega_k, \widetilde{\omega}_k)} \mid f(\omega_n, \widetilde{\omega}_n) \mid) \leqslant \frac{C}{n^d} \sum_{x,y \in \mathbb{Z}^d} \mid f(x,y) \mid \qquad (3.5)
$$

for all f in $L^1(\mathbb{Z}^{2d})$ and $n \geqslant 1$.

PROOF. We will show inequality (3.5) for n even, the case n odd being similar. Let $(\omega_n)_{n\geqslant 0}$ denote the simple random walk on $\mathbb{Z}^d$. By theorem 4.18 in **[57]**, $(\omega_{2n})_{n\geqslant 0}$ satisfies the d-isoperimetric inequality (IS_d therein) on its underlying graph. By remark 4.11 in **[57]**, $(\omega_{2n}, \widetilde{\omega}_{2n})_{n\geqslant 0}$ satisfies the 2d-isoperimetric inequality on its underlying graph. Consider the Markov chain in $\mathbb{Z}^d\times\mathbb{Z}^d$ with kernel:

$$K((x,y),(x',y')) = \sum_{(z,\widetilde{z})\in\mathbb{Z}^{2d}} \frac{1}{\Phi(x,y)} e^{v(z,\widetilde{z})} e^{v(x',y')} p((x,y),(z,\widetilde{z}))p((z,\widetilde{z}),(x',y'))$$

where p is the transition kernel of $(\omega_n,\widetilde{\omega}_n)_{n\geqslant 0}$. The transition kernel K is reversible with invariant measure $m(x,y) = (\Phi(x,y))^2 e^{v(x,y)}$. By assumption, we have

$$0 < \inf_{x,y\in\mathbb{Z}^d} m(x,y) \leqslant \sup_{x,y\in\mathbb{Z}^d} m(x,y) < \infty.$$

By assumption, there exists $c, C > 0$ such that for all $(x,y),(x',y')$ in $\mathbb{Z}^d$

$$cp^{(2)}((x,y),(x',y')) \leqslant K((x,y),(x',y')) \leqslant Cp^{(2)}((x,y),(x',y'))$$

where $p^{(2)}$ is the transition kernel of $(\omega_{2n},\widetilde{\omega}_{2n})_{n\geqslant 0}$. Therefore K satisfies the 2d-isoperimetric inequality on its underlying graph. By corollary 14.5 in **[57]**,

$$\sup_{x,y\in\mathbb{Z}^d} \frac{1}{\Phi(x,y)} P^x\otimes P^y(e^{\sum_{k=1}^{2n} v(\omega_k,\widetilde{\omega}_k)} \mid f(\omega_{2n},\widetilde{\omega}_{2n}) \mid \Phi(\omega_{2n},\widetilde{\omega}_{2n})) \leqslant C\frac{1}{n^d}\sum_{x,y\in\mathbb{Z}^d} \mid f(x,y)\mid$$

for all f in $L^1(\mathbb{Z}^{2d})$ and $n\geqslant 1$. The inequality (3.5) follows by using the boundedness of v and the assumption on Φ. □

We can now state the following useful corollary of lemma 3.3:

COROLLARY 3.4. *Let $A>0$ and $x\in\mathbb{Z}^d$. Under the assumptions of lemma 3.3, there exists $C\in]0,\infty[$ such that:*

$$\forall n \sup_{|y-x|\leqslant A\sqrt{n}} P^x\otimes P^x(e^{\sum_{k=1}^{n} v(\omega_k,\widetilde{\omega}_k)} \mid \omega_n = y, \widetilde{\omega}_n = y) \leqslant C. \tag{3.6}$$

PROOF. Let $y\in\mathbb{Z}^d$ be such that $\mid y-x\mid\ \leqslant A\sqrt{n}$. By applying inequality (3.5) with $f = 1_{\{y,y\}}$, we get:

$$P^x\otimes P^x(e^{\sum_{k=1}^{n} v(\omega_k,\widetilde{\omega}_k)} \mid \omega_n = y,\widetilde{\omega}_n = y) \leqslant \frac{C}{n^d P^x\otimes P^x(\omega_n = y,\widetilde{\omega}_n = y)} \underset{(3.3)}{\leqslant} C'.$$

□

We can now prove theorem 2.3.

Proof of theorem 2.3. Let l_n be a sequence tending to infinity and such that $\forall n \ l_n \leqslant n/2$. First, we compare in L^2 the quantity $P^x(e_{1,n} \mid \omega_n = y)$ with $P^x(e_{1,l_n}e_{n-l_n,n} \mid \omega_n = y)$. Therefore we compute:

$$Q(P^x(e_{1,n} - e_{1,l_n}e_{n-l_n,n} \mid \omega_n = y))^2 = P^x \otimes P^x(e^{\lambda_2(\beta)N_{1,n}} - e^{\lambda_2(\beta)N_{1,l_n}}e^{\lambda_2(\beta)N_{n-l_n,n}} \mid \omega_n = y, \widetilde{\omega}_n = y)$$
$$\leqslant P^x \otimes P^x(e^{\lambda_2(\beta)N_{1,l_n}}e^{\lambda_2(\beta)N_{n-l_n,n}}\Delta_n \mid \omega_n = y, \widetilde{\omega}_n = y)$$

with

$$\Delta_n = e^{\lambda_2(\beta)N_{l_n,n-l_n}} - 1.$$

Let $\delta > 0$ be such that $(1+\delta)\lambda_2(\beta) < \ln(\frac{1}{\pi_d})$. We remind that this implies:

$$\sup_{x,y\in\mathbb{Z}^d} P^x \otimes P^y(e^{(1+\delta)\lambda_2(\beta)\sum_{k=1}^{\infty}1_{\omega_k=\widetilde{\omega}_k}}) = P^x \otimes P^x(e^{(1+\delta)\lambda_2(\beta)\sum_{k=1}^{\infty}1_{\omega_k=\widetilde{\omega}_k}})$$
$$= P \otimes P(e^{(1+\delta)\lambda_2(\beta)N_{1,\infty}}) < \infty.$$

Using inequality (3.6) with $v(x,y) = (1+\delta)\lambda_2(\beta)1_{x=y}$, there exists $C > 0$ such that:

$$\sup_{n,|y-x|\,\leqslant\, A\sqrt{n}} P^x \otimes P^x(e^{(1+\delta)\lambda_2(\beta)N_{1,n}} \mid \omega_n = y, \widetilde{\omega}_n = y) \leqslant C. \tag{3.7}$$

Let ϵ, M be two positive numbers such that $e^{\lambda_2(\beta)\epsilon} - 1 < M$. By writing

$$1 = 1_{\Delta_n < e^{\lambda_2(\beta)\epsilon}-1} + 1_{M<\Delta_n} + 1_{e^{\lambda_2(\beta)\epsilon}-1\,\leqslant\,\Delta_n\,\leqslant\, M},$$

we get

$$P^x \otimes P^x(e^{\lambda_2(\beta)(N_{1,l_n}+N_{n-l_n,n})}\Delta_n \mid \omega_n = y, \widetilde{\omega}_n = y) \leqslant C(e^{\lambda_2(\beta)\epsilon} - 1) + \frac{C}{M^\delta}$$
$$+ MP^x \otimes P^x(1_{\Delta_n\,\geqslant\, e^{\lambda_2(\beta)\epsilon}-1}e^{\lambda_2(\beta)N_{1,n}} \mid \omega_n = y, \widetilde{\omega}_n = y).$$

Let $q > 1$ be such that $\frac{1}{q} + \frac{1}{1+\delta} = 1$. By Holder's inequality and inequality (3.6), we get

$$P^x \otimes P^x(1_{\Delta_n\,\geqslant\, e^{\lambda_2(\beta)\epsilon}-1}e^{\lambda_2(\beta)N_{1,n}} \mid \omega_n = y, \widetilde{\omega}_n = y))$$
$$\leqslant (P^x \otimes P^x(\Delta_n \geqslant e^{\lambda_2(\beta)\epsilon} - 1 \mid \omega_n = y, \widetilde{\omega}_n = y))^{\frac{1}{q}}C^{\frac{1}{1+\delta}}.$$

But, since $N_{l_n,n-l_n}$ is integer valued, we get uniformly on $\mid y-x \mid \leqslant A\sqrt{n}$:

$$\begin{aligned}
&P^x \otimes P^x(\Delta_n \geqslant e^{\lambda_2(\beta)\epsilon} - 1 \mid \omega_n = y, \widetilde{\omega}_n = y) \\
&= P^x \otimes P^x(N_{l_n,n-l_n} \geqslant 1 \mid \omega_n = y, \widetilde{\omega}_n = y) \\
&\leqslant P^x \otimes P^x(N_{l_n,n/2} \geqslant 1 \mid \omega_n = y, \widetilde{\omega}_n = y)) \\
&+ P^x \otimes P^x(N_{n/2,n-l_n} \geqslant 1 \mid \omega_n = y, \widetilde{\omega}_n = y)) \\
&= P^x \otimes P^x(N_{l_n,n/2} \geqslant 1 \mid \omega_n = y, \widetilde{\omega}_n = y)) \\
&+ P^y \otimes P^y(N_{l_n,n/2} \geqslant 1 \mid \omega_n = x, \widetilde{\omega}_n = x)) \ \ (symmetry) \\
&\underset{(3.4)}{\leqslant} C' P^x \otimes P^x(N_{l_n,n/2} \geqslant 1) \\
&+ C' P^y \otimes P^y(N_{l_n,n/2} \geqslant 1) \\
&= 2C' P \otimes P(N_{l_n,n/2} \geqslant 1) \underset{n\to\infty}{\to} 0.
\end{aligned}$$

We have used in the limit above the fact that $N_{0,\infty} < \infty \ \ P \otimes P - a.s.$ and that $l_n \underset{n\to\infty}{\longrightarrow} 0$. Therefore, we get

$$\overline{\lim_{n\to\infty}} \sup_{|y-x| \leqslant A\sqrt{n}} P^x \otimes P^x(e^{\lambda_2(\beta)(N_{1,l_n}+N_{n-l_n,n})}\Delta_n \mid \omega_n = y, \widetilde{\omega}_n = y) \leqslant C(e^{\lambda_2(\beta)\epsilon} - 1) + \frac{C}{M^\delta}.$$

We conclude that the above limit is equal to 0 by letting $\epsilon \downarrow 0$ and $M \uparrow \infty$.

From now on, we suppose that $l_n = o(n^a)$ for some $a < \frac{1}{2}$ and denote by $P^{(z,k)}(.)$ the random walk measure on paths which start at position z at time k. By the Markov property of the simple random walk, we get:

$$P^x(e_{1,l_n} e_{n-l_n,n} \mid \omega_n = y) = \sum_{|z_1-x| \leqslant l_n, |y-z_2| \leqslant l_n} P^x(e_{1,l_n} 1_{\omega_{l_n}=z_1}) \frac{q^{(n-2l_n)}(z_2 - z_1)}{q^{(n)}(y-x)} \times P^{(z_2,n-l_n)}(e_{n-l_n,n} 1_{\omega_n=y}).$$

By symmetry of the simple random walk, we have:

$$\begin{aligned}
\sum_{|y-z_2| \leqslant l_n} P^{(z_2,n-l_n)}(e_{n-l_n,n} 1_{\omega_n=y}) &= \sum_{|y-z_2| \leqslant l_n} P^y(\overleftarrow{e}_{n-l_n,n} 1_{\omega_{l_n}=z_2}) \\
&= P^y(\overleftarrow{e}_{n-l_n,n}).
\end{aligned}$$

Therefore,

$$Q((P^x(e_{1,l_n}e_{n-l_n,n} \mid \omega_n = y) - P^x(e_{1,l_n})P^y(\overleftarrow{e}_{n-l_n,n}))^2)$$
$$= \sum_{\substack{|z_1-x|\leqslant l_n, |y-z_2|\leqslant l_n \\ |z_1'-x|\leqslant l_n, |y-z_2'|\leqslant l_n}} \delta_n^{z_1,z_2,x,y}\delta_n^{z_1',z_2',x,y}\times$$
$$P^x\otimes P^x(e^{\lambda^2(\beta)N_{1,l_n}}1_{\omega_{l_n}=z_1}1_{\widetilde{\omega}_{l_n}=z_1'})P^{(z_2,n-l_n)}\otimes P^{(z_2',n-l_n)}(e^{\lambda^2(\beta)N_{n-l_n,n}}1_{\omega_n=y}1_{\widetilde{\omega}_n=y})$$

where

$$\delta_n^{z,w,x,y} = \frac{q^{(n-2l_n)}(w-z)}{q^{(n)}(y-x)} - 1.$$

The idea is that, by the classical local limit theorem, we get in the previous sum the following estimate:

$$\frac{q^{(n-2l_n)}(z_2-z_1)}{q^{(n)}(y-x)} \approx \frac{\bar{q}^{(n-2l_n)}(z_2-z_1)}{\bar{q}^{(n)}(y-x)} \approx 1.$$

Let us make this statement rigorous and obtain inequality (3.8) below. We use the notations of theorem 3.1 and decompose $\delta_n^{z,w,x,y}$ into three terms:

$$\delta_n^{z,w,x,y} = \delta_{1,n}^{z,w,x,y} + \delta_{2,n}^{z,w,x,y} + \delta_{3,n}^{z,w,x,y}$$

where

$$\delta_{1,n}^{z,w,x,y} = \frac{q^{(n-2l_n)}(w-z)-\bar{q}^{(n-2l_n)}(w-z)}{q^{(n)}(y-x)},\ \delta_{2,n}^{z,w,x,y} = \frac{\bar{q}^{(n-2l_n)}(w-z)-\bar{q}^{(n)}(y-x)}{q^{(n)}(y-x)},$$

$$\delta_{3,n}^{z,w,x,y} = \frac{\bar{q}^{(n)}(y-x)-q^{(n)}(y-x)}{q^{(n)}(y-x)}.$$

An application of (3.1) and (3.3) gives for $j=1,3$:

$$\sup_{\substack{|z-x|\leqslant l_n, |y-w|\leqslant l_n \\ |y-x|\leqslant A\sqrt{n}}} \mid \delta_{j,n}^{z,w,x,y} \mid = O(\frac{1}{n}) \underset{n\to\infty}{\longrightarrow} 0.$$

An application of (3.3) gives:

$$\mid \delta_{2,n}^{z,w,x,y} \mid = \mid \frac{\bar{q}^{(n-2l_n)}(w-z)}{q^{(n)}(y-x)} \mid\mid 1 - \frac{\bar{q}^{(n)}(y-x)}{\bar{q}^{(n-2l_n)}(w-z)} \mid$$
$$\underset{(3.3)}{\leqslant} C \mid 1 - \frac{\bar{q}^{(n)}(y-x)}{\bar{q}^{(n-2l_n)}(w-z)} \mid .$$
$$\leqslant C \mid 1 - (\frac{n-2l_n}{n})^{\frac{d}{2}} e^{\frac{d|w-z|^2}{2(n-2l_n)} - \frac{d|y-x|^2}{2n}} \mid .$$

It is not hard to show that:

$$\sup_{\substack{|z-x|\leqslant l_n,|y-w|\leqslant l_n\\|y-x|\leqslant A\sqrt{n}}} \mid 1-(\frac{n-2l_n}{n})^{\frac{d}{2}} e^{\frac{d|w-z|^2}{2(n-2l_n)}-\frac{d|y-x|^2}{2n}} \mid \underset{n\to\infty}{\longrightarrow} 0$$

so we have

$$\sup_{\substack{|z-x|\leqslant l_n,|y-w|\leqslant l_n\\|y-x|\leqslant A\sqrt{n}}} \mid \delta_{2,n}^{z,w,x,y} \mid \underset{n\to\infty}{\longrightarrow} 0.$$

Finally, we get:

$$\begin{aligned}&\sup_{|y-x|\leqslant A\sqrt{n}} Q((P^x(e_{1,l_n}e_{n-l_n,n} \mid \omega_n=y)-P^x(e_{1,l_n})P^y(\overleftarrow{e}_{n-l_n,n}))^2)\\&\qquad\leqslant \sup_{\substack{|z-x|\leqslant l_n,|y-w|\leqslant l_n\\|y-x|\leqslant A\sqrt{n}}} \mid \delta_n^{z,w,x,y} \mid^2 (P\otimes P(e^{\lambda_2(\beta)(1+N_{1,l_n})}))^2 \underset{n\to\infty}{\longrightarrow} 0. \qquad (3.8)\end{aligned}$$

Therefore, we get the expansion (2.1). To get the expansion (2.2), observe that

$$P^x(e_{1,l_n}) \overset{L^2(Q)}{\underset{n\to\infty}{\longrightarrow}} Z_\infty^x$$

and, by symmetry,

$$\begin{aligned}&\sup_{y\in\mathbb{Z}^d} Q((P^y(\overleftarrow{e}_{n-l_n,n})-P^y(\overleftarrow{e}_{1,n}))^2)\\&\qquad= P\otimes P(e^{\lambda_2(\beta)(1+N_{1,l_n})}-e^{\lambda_2(\beta)(1+N_{1,n-1})}) \underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

□

3.2. Proof of theorem 2.9. In order to prove theorem 2.9, we adapt in detail the previous proof to the Brownian setting. In the discrete setting, there are three key intermediate results: the local limit theorem 3.1, corollary 3.2 and corollary 3.4. In the continuous setting, we do not need any local limit theorem since Brownian motion is already a Gaussian process. Therefore, we only require a Brownian analogue to corollary 3.2 and corollary 3.4. The following construction of the Brownian bridge can be found in the appendix of [**54**]:

PROPOSITION 3.5. *For $x,y\in\mathbb{R}^d$, $t>0$, there exists a unique probability measure $P_t^{x,y}$ on $C([0,1],\mathbb{R}^d)$ such that for $s\in[0,t[, A\in\mathcal{F}_s$:*

$$P_t^{x,y}(A)=\frac{1}{p(t,x,y)}P^x(1_A p(t-s,\omega_s,y)) \qquad (3.9)$$

$y\to P_t^{x,y}$ is a regular conditional probability of P_t^x given $\omega_t=y$.

In the sequel, we will always work with the representation (3.9) of Brownian bridge. With this representation, we can now easily prove the Brownian analogue of corollary 3.2:

COROLLARY 3.6. *Let $s \in]0,1[$ and $A > 0$. There exists a constant $C(A,d) > 0$ such that*

$$\forall f \geqslant 0 \ \ \forall t > 0 \quad \sup_{|y-x| \leqslant A\sqrt{t}} P^x(f((\omega_u)_{u \leqslant st}) \mid \omega_t = y) \leqslant \frac{C(A,d)}{(1-s)^{\frac{d}{2}}} P^x(f((\omega_u)_{u \leqslant st})). \tag{3.10}$$

PROOF. If $\mid y-x \mid \ \leqslant A\sqrt{t}$ then

$$\begin{aligned} P^x(f((\omega_u)_{u \leqslant st}) \mid \omega_t = y) &= \frac{(2\pi t)^{\frac{d}{2}}}{(2\pi t(1-s))^{\frac{d}{2}}} e^{\frac{|y-x|^2}{2t}} P^x(e^{-\frac{|y-\omega_{st}|^2}{2t(1-s)}} f((\omega_u)_{u \leqslant st})) \\ &\leqslant \frac{e^{A^2/2}}{(1-s)^{\frac{d}{2}}} P^x(f((\omega_u)_{u \leqslant st})). \end{aligned}$$

□

The Brownian analogue to lemma 3.3 is a slight variation of Lemma 3.1.3. in [**14**].

LEMMA 3.7. *For $d \geqslant 3$ and $v : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ a bounded, non negative and compactly supported measurable function, define*

$$\Phi(x,y) = P^x \otimes P^y(e^{\int_0^\infty v(\widetilde{\omega}_s - \omega_s)ds}).$$

Suppose that

$$\sup_{x,y \in \mathbb{R}^{2d}} \Phi(x,y) < \infty.$$

Then there exists a constant $C > 0$ such that

$$\sup_{x,y \in \mathbb{R}^{2d}} P^x \otimes P^y(e^{\int_0^t v(\widetilde{\omega}_s - \omega_s)ds}) \mid f(\omega_t, \widetilde{\omega}_t) \mid) \leqslant \frac{C}{t^d} \int_{\mathbb{R}^{2d}} \mid f(x,y) \mid dxdy \tag{3.11}$$

for all f in $L^1(\mathbb{R}^{2d})$ and $t > 0$.

PROOF. By using the same arguments as the ones in the proof of Lemma 3.1.3. in [**14**], all we have to prove is

$$\forall F \in C_c^\infty(\mathbb{R}^{2d}) \quad \int_{\mathbb{R}^{2d}} (\frac{1}{2} \bigtriangledown_{x,y} F. \bigtriangledown_{x,y} \Phi - v(y-x)F(x,y)\Phi(x,y))dxdy = 0.$$

Since $(\widetilde{\omega}_{s/2}-\omega_{s/2})_{s\geqslant 0}$ is a Brownian motion, we have that $\Phi(x,y)=\widetilde{\Phi}(y-x)$ where:

$$\forall z\in\mathbb{R}^d\ \ \widetilde{\Phi}(z)=P^z(e^{\int_0^\infty \frac{1}{2}v(\omega_s)ds}).$$

By equation (3.19) in the proof Lemma 3.1.3. in **[14]**, we have:

$$\forall g\in C_c^\infty(\mathbb{R}^d)\quad \int_{\mathbb{R}^d}(\bigtriangledown_{\widetilde{y}}g(\widetilde{y}).\bigtriangledown_{\widetilde{y}}\widetilde{\Phi}-v(\widetilde{y})g(\widetilde{y})\widetilde{\Phi}(\widetilde{y}))d\widetilde{y}=0. \tag{3.12}$$

By making the change of variable $(\widetilde{x},\widetilde{y})=(y+x,y-x)$, $F(x,y)=f(\widetilde{x},\widetilde{y})$ we get:

$$\bigtriangledown_{x,y}F.\bigtriangledown_{x,y}\Phi=-(\bigtriangledown_{\widetilde{x}}f-\bigtriangledown_{\widetilde{y}}f)\bigtriangledown_{\widetilde{y}}\widetilde{\Phi}+(\bigtriangledown_{\widetilde{x}}f+\bigtriangledown_{\widetilde{y}}f)\bigtriangledown_{\widetilde{y}}\widetilde{\Phi}=2\bigtriangledown_{\widetilde{y}}f\bigtriangledown_{\widetilde{y}}\widetilde{\Phi}$$

Therefore,

$$\begin{aligned}\forall F\in C_c^\infty(\mathbb{R}^{2d})\quad &\int_{\mathbb{R}^{2d}}(\frac{1}{2}\bigtriangledown_{x,y}F.\bigtriangledown_{x,y}\Phi-v(y-x)F(x,y)\Phi(x,y))dxdy\\ &=\frac{1}{2^d}\int_{\mathbb{R}^{2d}}(\bigtriangledown_{\widetilde{y}}f(\widetilde{x},\widetilde{y})\bigtriangledown_{\widetilde{y}}\widetilde{\Phi}(\widetilde{y})-v(\widetilde{y})f(\widetilde{x},\widetilde{y})\widetilde{\Phi}(\widetilde{y}))d\widetilde{x}d\widetilde{y}\\ &=\frac{1}{2^d}\int_{\mathbb{R}^d}(\int_{\mathbb{R}^d}(\bigtriangledown_{\widetilde{y}}f(\widetilde{x},\widetilde{y})\bigtriangledown_{\widetilde{y}}\widetilde{\Phi}(\widetilde{y})-v(\widetilde{y})f(\widetilde{x},\widetilde{y})\widetilde{\Phi}(\widetilde{y}))d\widetilde{y})d\widetilde{x}\\ &\underset{(3.12)}{=}0\end{aligned}$$

□

We can now state the following analogue to corollary 3.4:

COROLLARY 3.8. *Let $A>0$ and $x,y\in\mathbb{R}^d$. Under the above assumptions, there exists $C>0$ such that:*

$$\forall t\quad \sup_{|y-x|\,\leqslant\,A\sqrt{t}}P^x\otimes P^x(e^{\int_0^t v(\widetilde{\omega}_s-\omega_s)ds}\mid \omega_t=y,\widetilde{\omega}_t=y)\leqslant C. \tag{3.13}$$

PROOF. Let $r>0$ and $y\in\mathbb{R}^d$ such that $\mid y-x\mid\ \leqslant A\sqrt{t}$. By applying (3.11) with $f=1_{B((y,y),r)}$, we get:

$$P^x\otimes P^x(e^{\int_0^t v(\widetilde{\omega}_s-\omega_s)ds}1_{B((y,y),r)}(\omega_t,\widetilde{\omega}_t))\leqslant\frac{C}{t^d}\mid B((y,y),r)\mid .$$

Therefore,

$$\begin{aligned}P^x\otimes P^x(e^{\int_0^t v(\widetilde{\omega}_s-\omega_s)ds}1_{B((y,y),r)}(\omega_t,\widetilde{\omega}_t))/P^x\otimes P^x((\omega_t,\widetilde{\omega}_t)\in B((y,y),r))\\ \leqslant\frac{C}{t^d}\frac{\mid B((y,y),r)\mid}{P^x\otimes P^x((\omega_t,\widetilde{\omega}_t)\in B((y,y),r))}.\end{aligned} \tag{3.14}$$

As $r \downarrow 0$, a classical result on Brownian bridges asserts that the left hand side of (3.14) tends to

$$P^x \otimes P^x(e^{\int_0^t v(\widetilde{\omega}_s - \omega_s)ds} \mid \omega_t = y, \widetilde{\omega}_t = y).$$

As $r \downarrow 0$, the right hand side of (3.14) tends to

$$C \frac{e^{\frac{|y-x|^2}{t}} (2\pi t)^d}{t^d} \leqslant (2\pi)^d C e^{A^2}.$$

□

Proof of theorem 2.9. The proof of theorem 2.9 is quite similar but even simpler than the proof of theorem 2.3 since Brownian motion is already Gaussian. We will not repeat the details but we indicate the main steps for convenience. Suppose that β is such that

$$\lambda_2(\beta) < \lambda(d).$$

There exists $\delta > 0$ such that $(1+\delta)\lambda_2(\beta) < \lambda(d)$. Using inequality (3.13) applied to $v(y-x) = (1+\delta)\lambda_2(\beta) \mid U(y-x) \cap U(0) \mid$, we get the following analogue to (3.7): there exists $C > 0$ such that

$$\sup_{t, |y-x| \leqslant A\sqrt{t}} P^x \otimes P^x(e^{(1+\delta)\lambda_2(\beta) N_{0,t}} \mid \omega_t = y, \widetilde{\omega}_t = y) \leqslant C. \tag{3.15}$$

Using inequality (3.15) and inequality (3.10), we get

$$P^x(e_{0,t} \mid \omega_t = y) \approx P^x(e_{0,l_t} e_{t-l_t,t} \mid \omega_t = y).$$

Using the Markov property and the symmetry of Brownian motion, we get

$$P^x(e_{0,l_t} e_{t-l_t,t} \mid \omega_t = y) \approx P^x(e_{0,l_t}) P^y(\overleftarrow{e}_{t-l_t,t}).$$

□

CHAPTER 2

Majorizing directed polymers with multiplicative cascades

1. Introduction

Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be the simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ starting at 0, defined on a probability space $(\Omega, \mathcal{F}, P)$. We also consider a sequence $\eta = (\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ of real valued, non-constant and i.i.d. random variables defined on another probability space $(H, \mathcal{G}, Q)$ with finite exponential moments. The path ω represents the directed polymer and η the random environment.

For any $n > 0$, we define the (random) polymer measure μ_n on the path space $(\Omega, \mathcal{F})$ by:

$$\mu_n(d\omega) = \frac{1}{Z_n}\exp(\beta H_n(\omega))P(d\omega)$$

where $\beta \in \mathbb{R}^+$ is the inverse temperature, where

$$H_n(\omega) \stackrel{\text{def.}}{=} \sum_{j=1}^{n} \eta(j, \omega_j)$$

and where

$$Z_n = P[\exp(\beta H_n(\omega))]$$

is the partition function. We use the notation $P[X]$ for the expectation of a random variable X. By symmetry, we can – and we will – restrict to $\beta \geqslant 0$.

The free energy of the polymer is defined as the limit

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n}\ln(Z_n(\beta)/Q[Z_n(\beta)]) \tag{1.1}$$

where the limit exists Q-a.s. and in L^p for all $p \geqslant 1$ and is constant (cf. [**11**]). An application of Jensen's inequality to the concave function $\ln(\cdot)$ yields $p(\beta) \leqslant 0$. As shown in theorem 3.2 (b) in [**15**], there exists a $\beta_c \in [0, \infty]$ such that

$$p(\beta) \quad \begin{cases} = 0 & \text{if} \quad \beta \in [0, \beta_c], \\ < 0 & \text{if} \quad \beta > \beta_c. \end{cases}$$

An important question in the study of directed polymers is to find the β such that $p(\beta) < 0$. Indeed, one can show that the negativity of $p(\beta)$ is equivalent to a localization property for $(\omega_n)_{n\in\mathbb{N}}$,$(\widetilde{\omega}_n)_{n\in\mathbb{N}}$ two independent random walks under the polymer measure μ_n (cf Corollary 2.2 in [**11**]):

$$p(\beta) < 0 \qquad \Longleftrightarrow \qquad \exists c > 0 \quad \varliminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mu_{k-1}^{\otimes 2}(\omega_k = \widetilde{\omega}_k) \geqslant c \qquad Q.a.s.$$

The statement in the right-hand side means that the polymer localizes in narrow corridors with positive probability. It is not known how to characterize directly these corridors, and therefore this criterion for the transition localization/delocalization is rather efficient since it does not require any knowledge on them. Hence, it is important to get good upper bounds on p in order to spot the transition. Our main result is the following.

THEOREM 1.1. *In dimension $d = 1$, $\beta_c = 0$.*

There is a clear consensus on this fact in the physics literature, but no proof for it, except via the replica method or in the (different) case of a space-periodic environment where much more computations can be performed [**8**].

This result follows from a family of upper bounds, given by the free energies of models on trees depending on an integer parameter m ($m \geqslant 1$). These trees are deterministic and regular, with random weights, they fall in the scope of the generalized multiplicative cascades [**42**] or smoothing transformations [**22**] which are well known generalizations of the random cascades introduced in [**44**] for a statistical description of turbulence. When the environment variables have nice concentration properties – e.g., gaussian or bounded η's –, we prove in theorem 3.6 that the polymer free energy is the infimum over m of the one of the m-tree model. For general environmental distribution we only have an upper bound from theorem 3.3, but it is enough to show the above theorem. This also explains the title of the present paper.

Recall at this point that directed polymers in a Bernoulli random environment are positive temperature versions of oriented percolation. Our bounds here have a flavor similar to the lower bounds for the critical threshold in 2-dimensional oriented percolation (i.e., $d = 1$ in our notations) in section 6 of Durrett [**21**]. In that paper, percolation is compared to Galton-Watson processes obtained in running oriented percolation for m steps ($m \geqslant 1$), and then using the distribution of wet sites as offspring distribution.

Next, we comment on the case of supercritical 1-dimensional oriented percolation. Then, η is Bernoulli distributed with parameter $p > \vec{p}_c(1)$. The infinite cluster is the set of points (t, x) with $t \in \mathbb{N}, x \in \mathbb{Z}, P(\omega_t = x) > 0$, which are connected to ∞ by an open oriented path – i.e., a path ω with $\eta(s, \omega_s) = 1\ \forall s \geqslant t$. It is known that this cluster, at large scale, is approximatively a cone with vertex $(0, 0)$, direction $[0, x)$ and positive angle, and it has a positive density. In words, there is a huge number of oriented paths of length n with energy $H_n = n - \mathcal{O}(1)$. However, according to the theorem, the polymer measure has a strong localization property. This first seems paradoxical, since there are exponentially many suitable paths on the energetic level. Hence, this is essentially an entropic phenomenon, due to large fluctuations in the number of such paths.

For numerics, our upper bounds do not seem very efficient: on the basis of preliminary numerical simulations they converge quite slowly as $m \to \infty$. Finally we mention that lower bounds for the polymer free energy can be obtained from a well-known super-additivity property, see formula (2.3).

2. Notations and preliminaries

We first introduce some further notations.

Let $((\omega_n)_{n\in\mathbb{N}}, (P^x)_{x\in\mathbb{Z}^d})$ denote the simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$, defined on a probability space $(\Omega, \mathcal{F})$: for x in $\mathbb{Z}^d$, under the measure P^x, $(\omega_n - \omega_{n-1})_{n\,\geqslant\,1}$ are independent and

$$P^x(\omega_0 = x) = 1, \quad P^x(\omega_n - \omega_{n-1} = \pm\delta_j) = \frac{1}{2d}, \quad j = 1, \dots, d,$$

where $(\delta_j)_{1\,\leqslant\,j\,\leqslant\,d}$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. Like in the introduction, we will use the notation P for P^0.

For the environment, we assume that for all $\beta \in \mathbb{R}$,

$$\lambda(\beta) \overset{\text{def.}}{=} \ln Q(e^{\beta\eta(n,x)}) < \infty.$$

It is convenient to consider the normalized partition function

$$W_n = Z_n/Q[Z_n] = P[\exp(\beta H_n(\omega) - n\lambda(\beta))].$$

We define for $k < n$, $x, y \in \mathbb{Z}^d$,

$$H_{k,n}(\omega) = \sum_{j=1}^{n-k} \eta(k + j, \omega_j)$$

and

$$W^x_{k,n}(y) = P^x(e^{\beta H_{k,n}(\omega)-(n-k)\lambda(\beta)}1_{\omega_{n-k}=y}). \tag{2.1}$$

In the sequel, $W_n(x)$ will stand for $W^x_{0,n}(0)$. The Markov property of the simple random walk yields

$$W_n = \sum_{x,y\in\mathbb{Z}^d} W_k(x)W^x_{k,n}(y). \tag{2.2}$$

This identity will be extensively used in the sequel.

Finally, we recall ([**11**]) that with p defined by (1.1) it holds

$$p(\beta) = \lim_{n\to\infty}\frac{1}{n}Q(\ln(W_n(\beta))) = \sup_{n\geqslant 1}\frac{1}{n}Q(\ln(W_n(\beta))) \tag{2.3}$$

where the last equality is a consequence of super-additivity arguments.

2.1. Definition and well known facts on generalized multiplicative cascades. In this section, we introduce a model of generalized multiplicative cascades on a tree. For an overview of results, we refer to [**42**]. Let $N \geqslant 2$ be a fixed integer and

$$U = \bigcup_{k\in\mathbb{N}}[|\ 1, N\ |]^k$$

be the set of all finite sequences $u = u_1 \dots u_k$ of elements in $[|\ 1, N\ |]$. With the previous notation, we write $|\ u\ |= k$ for its length. For $u = u_1 \dots u_k, v = v_1 \dots v_k$ two finite sequences, let uv denote the sequence $u_1 \dots u_k v_1 \dots v_k$. Let q be a non degenerate probability distribution on $(\mathbb{R}^*_+)^N$. It is known (cf. [**42**]) that there exist a probability space with probability measure denoted by $\mathtt{P}$ (and expectation $\mathtt{E}$), and random variables $(A_u)_{u\in U}$ defined on this space, such that the random vectors $(A_{u1}, \dots, A_{uN})_{u\in U}$ form an i.i.d. sequence with common distribution q. We assume that the $(A_i)_{1\leqslant i\leqslant N}$ are normalized:

$$\mathtt{E}(\sum_{i=1}^N A_i) = 1$$

and that they have moments of all order: $\mathtt{E}[\sum_{i=1}^N A_i^p] < \infty\ \forall p \in \mathbb{R}$. Consider the process $(W^{\text{casc}}_n)_{n\in\mathbb{N}}$ defined by

$$W^{\text{casc}}_n = \sum_{u_1,\dots,u_n\in[|1,N|]} A_{u_1}A_{u_1u_2}\dots A_{u_1\dots u_n} \tag{2.4}$$

and the filtration

$$\mathcal{G}_n := \sigma\{A_u; |\ u\ | \ \leqslant n\}, \qquad n \geqslant 1.$$

Then $(W_n^{\text{casc}}, \mathcal{G}_n)_{n \geqslant 1}$ is a non negative martingale so the limit $W_\infty^{\text{casc}} = \lim_{n\to\infty} W_n^{\text{casc}}$ exists. We are interested in the behavior of the associated free energy:

$$p_n = \frac{1}{n} \ln W_n^{\text{casc}} \ .$$

In the case where the $(A_i)_{i \leqslant N}$ are i.i.d, the exact limit of p_n as n goes to infinty was derived in [**23**]. In the general case, the proofs in [**23**] can easily be adapted to show the following summary result.

THEOREM 2.1. *The following convergence holds* P*-a.s. and in* L^p *for all* $p \geqslant 1$*:*

$$p_n \underset{n\to\infty}{\longrightarrow} \inf_{\theta\in]0,1]} \frac{1}{\theta} \ln(\mathtt{E} \sum_{i=1}^N A_i^\theta) \leqslant 0,$$

where the inequality is a consequence of the normalization. Finding the limit of p_n as n tends to infinity amounts to studying the function v defined by

$$\forall \theta \in]0,1], \qquad v(\theta) = \frac{1}{\theta} \ln(\mathtt{E} \sum_{i=1}^N A_i^\theta) \ ,$$

which has derivative

$$v'(1) = \mathtt{E} \sum_{i=1}^N A_i \ln(A_i) \ .$$

LEMMA 2.2. *If* $\mathtt{E} \sum_{i=1}^N A_i \ln(A_i) \leqslant 0$*, the function* v *is strictly decreasing on* $]0,1]$ *and thus*

$$\inf_{\theta\in]0,1]} v(\theta) = v(1) = 0.$$

If $\mathtt{E} \sum_{i=1}^N A_i \ln(A_i) > 0$*, there exists a unique* $\theta^* \in]0,1[$ *such that*

$$\inf_{\theta\in]0,1]} v(\theta) = v(\theta^*) < 0.$$

PROOF. For all $\theta \in]0,1]$, we have the following expression for the derivative of v:

$$v'(\theta) = \frac{g(\theta)}{\theta^2}$$

where g is given by

$$g(\theta) = \theta \frac{\mathtt{E} \sum_{i=1}^N A_i^\theta \ln(A_i)}{\mathtt{E} \sum_{i=1}^N A_i^\theta} - \ln(\mathtt{E} \sum_{i=1}^N A_i^\theta).$$

In particular, we obtain the value of $v'(1)$ given above. By direct computation, one can obtain the following expression for g'

$$\forall\theta > 0 \qquad g'(\theta) = \theta\frac{\mathtt{E}(\sum_{i=1}^N A_i^\theta(\ln(A_i) - \mathtt{E}(\ln(A) \mid A^\theta))^2)}{\mathtt{E}(\sum_{i=1}^N A_i^\theta)}$$

where $\mathtt{E}(\ln(A) \mid A^\theta)$ is a notation for

$$\mathtt{E}(\ln(A) \mid A^\theta) = \frac{\mathtt{E}(\sum_{i=1}^N A_i^\theta \ln(A_i))}{\mathtt{E}(\sum_{i=1}^N A_i^\theta)}.$$

In particular, g is strictly increasing and we have

$$g(1) = \mathtt{E}(\sum_{i=1}^N A_i \ln(A_i)).$$

By considering the two cases $g(1) \leqslant 0$ and $g(1) > 0$, we can easily conclude. □

2.2. Concentration of measure in the gaussian and the bounded case. For a complete survey on the concentration of measure phenomenon, we refer to [**41**]. In the gaussian case, we have

THEOREM 2.3. *Let $M \geqslant 1$ be an integer. We consider $\mathbb{R}^M$ equiped with the usual euclidian norm $\|\cdot\|$. If X_M is a standard gaussian vector on some probability space (with a probability measure* $\mathtt{P}$*) and F is a C-lipschitzian function ($|F(x) - F(y)| \leqslant C\|x-y\|$) from $\mathbb{R}^M$ to $\mathbb{R}$ then*

$$\mathtt{E}(e^{\lambda(F(X_M) - \mathtt{E}(F(X_M)))}) \leqslant e^{\frac{C^2\lambda^2}{2}}. \tag{2.5}$$

Therefore, we have the following concentration result

$$\mathtt{P}(\mid F(X_M) - \mathtt{E}(F(X_M)) \mid \; \geqslant r) \leqslant 2e^{-\frac{r^2}{2C^2}} \tag{2.6}$$

In the bounded case, we get a similar concentration result (cf. Corollary 3.3 in [**41**]) .

THEOREM 2.4. *Let $M \geqslant 1$ be an integer and $a < b$ be two real numbers. If X_M is a random vector in $[a,b]^M$ with i.i.d. components on some probability space and F is a convex and C-lipschitzian function from $[a,b]^M$ to $\mathbb{R}$ for the euclidian norm, then*

$$\mathtt{E}(e^{\lambda(F(X_M) - \mathtt{E}(F(X_M)))}) \leqslant e^{C^2(b-a)^2\lambda^2}. \tag{2.7}$$

Therefore, we have the following concentration result

$$\mathtt{P}(F(X_M) - \mathtt{E}(F(X_M)) \geqslant r) \leqslant e^{-\frac{r^2}{4C^2(b-a)^2}} \tag{2.8}$$

We can derive from the above theorems a concentration result for the free energy at time n:

COROLLARY 2.5. *If the environment η is standard gaussian then for all $\lambda \geqslant 0$,*

$$Q(e^{\lambda(\ln(W_n)-Q(\ln(W_n)))}) \leqslant e^{\frac{\beta^2\lambda^2 n}{2}}. \tag{2.9}$$

If the environment η belongs to $[a,b]$ for $a<b$ two real numbers, then for all $\lambda \geqslant 0$,

$$Q(e^{\lambda(\ln(W_n)-Q(\ln(W_n)))}) \leqslant e^{\beta^2(b-a)^2\lambda^2 n}. \tag{2.10}$$

PROOF. As a function of the environment, $\ln(W_n)$ is convex and $\beta\sqrt{n}$-lipschitzian (cf. the proof of proposition 1.4 in [**9**]). Therefore, in the gaussian case, the result is a direct application of (2.5) and, in the bounded case, simply (2.7). □

3. Majorizing polymers with cascades

Let us fix an integer $m \geqslant 1$ and define L_m to be set of points visited by the simple random walk at time m:

$$L_m \stackrel{def}{=} \{x \in \mathbb{Z}^d; P(w_m = x) > 0\}.$$

We introduce $(W^{\text{tree}}_{m,n})_{n\,\geqslant\,1} \equiv (W^{\text{casc}}_n)_{n\,\geqslant\,1}$ the martingale of the multiplicative cascade associated to the random vector $(W_m(x))_{x\in L_m}$, i.e., defined by (2.4) when $N = |L_m|$ and q is the law of $(W_m(x))_{x\in L_m}$ with $W_m(x)$ from (2.1). Let $p^{\text{tree}}_m(\beta)$ denote the associated free energy. In view of theorem 2.1, $p^{\text{tree}}_m(\beta)$ is given by

$$p^{\text{tree}}_m(\beta) = \inf_{\theta\in]0,1]} v_m(\theta) \tag{3.1}$$

where v_m is given by the expression

$$\forall\theta \in]0,1] \qquad v_m(\theta) = \frac{1}{\theta}\ln(Q\sum_{x\in L_m} W_m(x)^\theta). \tag{3.2}$$

We will first need the following monotonicity lemma.

LEMMA 3.1. *Assume that $\phi :]0,\infty[\longrightarrow\ R$ is $\mathcal{C}^1$ and that there are constants $C, p \in [1,\infty[$ such that*

$$\forall u > 0 \qquad |\ \phi'(u)\ | \ \leqslant Cu^p + Cu^{-p}.$$

Then for all $x \in L_m$ $\phi(W_m(x)), \frac{\partial\phi(W_m(x))}{\partial\beta} \in L^1(Q)$, $Q\phi(W_m(x))$ is $\mathcal{C}^1$ in $\beta \in \mathbb{R}$ and

$$\frac{\partial}{\partial\beta}Q\phi(W_m(x)) = Q\frac{\partial}{\partial\beta}\phi(W_m(x)).$$

Suppose in addition that ϕ is concave. Then ,

$$\forall \beta \geqslant 0 \qquad Q\frac{\partial}{\partial \beta}\phi(W_m(x)) \leqslant 0.$$

PROOF. The proof is an immediate adaptation of the proof of lemma 3.3 in **[15]**. □

As a consequence we can define the following

PROPOSITION 3.2. *The function p_m^{tree} is non-increasing in β. There exists a critical value $\beta_c^m \in (0,\infty]$ such that*

$$p_m^{\text{tree}}(\beta) = \begin{cases} 0 & \text{if} \quad \beta \in [0, \beta_c^m], \\ < 0 & \text{if} \quad \beta > \beta_c^m. \end{cases}$$

PROOF. For all $\theta \in]0,1]$, the function $x \to x^\theta$ is concave so by lemma 3.1, we see from expression (3.2) that $v_m(\theta)$ is non-increasing as a function of β. Therefore, we see from (3.1) that p_m^{tree} is itself non-increasing in β and we obtain the existence of β_c^m ($\beta_c^m \in [0,\infty]$). Since

$$v_m'(1) = Q\sum_{x\in L_m} W_m(x)\ln W_m(x) \longrightarrow \sum_{x\in L_m} P(\omega_n = x)\ln P(\omega_n = x) < 0\ ,$$

as $\beta \searrow 0$, we conclude that β_c^m is strictly positive by continuity of $\partial_\theta v_m(\theta,\beta)_{|\theta=1}$ in β and by lemma 3.1.

□

THEOREM 3.3. *We have the following inequality*

$$p(\beta) \leqslant \inf_{m\geqslant 1}\frac{1}{m}p_m^{\text{tree}}(\beta). \tag{3.3}$$

PROOF. Let $\theta \in (0,1)$ and m be a positive integer. By using the subadditive estimate

$$\forall u,v > 0, \qquad (u+v)^\theta < u^\theta + v^\theta, \tag{3.4}$$

we have for all $n \geqslant 1$

$$\begin{aligned}
Q\frac{1}{n}\ln W_{nm} &= Q\frac{1}{\theta n}\ln W_{nm}^{\theta} \\
&\overset{(2.2)}{=} Q\frac{1}{\theta n}\ln\Bigg(\sum_{x_1,\dots,x_n} W_m(x_1)\dots W^{x_{n-1}}_{(n-1)m,nm}(x_n)\Bigg)^{\theta} \\
&\overset{(3.4)}{\leqslant} Q\frac{1}{\theta n}\ln \sum_{x_1,\dots,x_n} W_m(x_1)^{\theta}\dots W^{x_{n-1}}_{(n-1)m,nm}(x_n)^{\theta} \\
&\overset{\text{(Jensen)}}{\leqslant} \frac{1}{\theta n}\ln Q \sum_{x_1,\dots,x_n} W_m(x_1)^{\theta}\dots W^{x_{n-1}}_{(n-1)m,nm}(x_n)^{\theta} \\
&= \frac{1}{\theta n}\ln\Bigg(Q\sum_x W_m(x)^{\theta}\Bigg)^{n} \\
&= \frac{1}{\theta}\ln Q\sum_x W_m(x)^{\theta}
\end{aligned}$$

The proof is complete by taking the limit as $n \to \infty$ and then by taking the infimum over all $\theta \in]0,1]$ and $m \geqslant 1$. □

In particular, to prove $p(\beta) < 0$ it suffices to find $m \geqslant 1$ (in fact, $m \geqslant 2$) and $\theta \in (0,1)$ such that $Q\sum_x W_m(x)^{\theta} < 1$. The theorem is a handy way to obtain upper bounds on the critical β.

Remark 3.4. *Let $\theta \in]0,1[$ and $m \geqslant 1$. Using (3.4), we find by a similar computation that for all $k \geqslant 2$*

$$\begin{aligned}
Q\sum_y W_{km}(y)^{\theta} &= Q\sum_y \big(\sum_{x_1,\dots,x_{k-1}} W_m(x_1)\dots W^{x_{k-1}}_{(k-1)m,km}(y)\big)^{\theta} \\
&< Q\sum_y \sum_{x_1,\dots,x_{k-1}} W_m(x_1)^{\theta}\dots W^{x_{k-1}}_{(k-1)m,km}(y)^{\theta} \\
&= \big(Q\sum_x W_m(x)^{\theta}\big)^{k}. \qquad (3.5)
\end{aligned}$$

In view of (3.1) and of the smoothness of $v_m(\cdot)$, we conclude that

$$\frac{1}{km}p^{\text{tree}}_{km}(\beta) \leqslant \frac{1}{m}p^{\text{tree}}_{m}(\beta).$$

Observe that when $p^{\text{tree}}_m(\beta) < 0$, the infimum in (3.1) is achieved for some $\theta \in (0,1)$, and therefore the above inequality is strict. In particular,

$$\inf_{m\geqslant 1}\frac{1}{m}p^{\text{tree}}_m(\beta) = \underline{\lim}_{m\to\infty}\frac{1}{m}p^{\text{tree}}_m(\beta). \qquad (3.6)$$

The authors do not know if the sequence $(p_m^{\rm tree}(\beta))_{m\geqslant 1}$ *is subadditive. However a simple argument yields the stronger result*

$$\inf_{m\geqslant 1}\frac{1}{m}p_m^{\rm tree}(\beta)=\lim_{m\to\infty}\frac{1}{m}p_m^{\rm tree}(\beta). \tag{3.7}$$

Indeed, by repeating the steps in (3.5), we we see that, for $0\leqslant \ell < m, k\geqslant 1$ *and* $\theta\in(0,1]$,

$$v_{km+\ell}(\theta)\leqslant kv_m(\theta)+v_\ell(\theta)\ ,$$

whereas, by concavity,

$$v_\ell(\theta)\leqslant\frac{1}{\theta}\sum_x\big(QW_\ell(\theta)\big)^\theta=v_\ell(\theta,0)$$

where $v_\ell(\theta,0)=v_\ell(\theta,\beta)_{|\beta=0}\in(0,\infty)$. *Therefore,*

$$\max_{km\leqslant n<(k+1)m}\frac{v_n(\theta)}{n}\leqslant\frac{k}{(k+\varepsilon)m}v_m(\theta)+\frac{1}{km}v_\ell(\theta,0)\ ,$$

where $\varepsilon=0$ *or* 1 *according to the sign of* $v_m(\theta)$. *Now, recalling that* $v_m(\theta)\geqslant p_m^{\rm tree}(\beta)$ *and taking the limit* $k\to\infty$, *leads to*

$$\limsup_n\frac{p_n^{\rm tree}(\beta)}{n}\leqslant\frac{v_m(\theta)}{m},\quad m\geqslant 1,\theta\in(0,1].$$

Combined with (3.6), this implies (3.7).

We add another

REMARK 3.5. *Suppose that there exists* $m\geqslant 1$ *such that*

$$Q\sum_x W_m(x)\ln W_m(x)=0.$$

We have

$$\begin{aligned}
Q\sum_{y} W_{2m}(y)\ln W_{2m}(y) &= Q\sum_{x,y} W_m(x)W^x_{m,2m}(y)\ln W_{2m}(y)\\
&> \sum_{x,y} QW_m(x)W^x_{m,2m}(y)\ln\left(W_m(x)W^x_{m,2m}(y)\right)\\
&= \sum_{x}\left(QW_m(x)\ln W_m(x)\right)\sum_{y} QW^x_{m,2m}(y)\\
&\quad+\sum_{x}\left(QW_m(x)\right)\sum_{y} QW^x_{m,2m}(y)\ln W^x_{m,2m}(y)\\
&= 2\sum_{x} QW_m(x)\ln W_m(x)\\
&= 0
\end{aligned}$$

Hence, by lemma 2.2, $p^{\text{tree}}_{2m}(\beta) < 0$ *and finally* $p(\beta) < 0$.

As a consequence of theorem 3.3, we get our main result

Proof of theorem 1.1: Let $\theta \in]0,1]$ and $\beta > 0$. By using lemma 4.1 in [**11**], there exists a $c(\theta) > 0$ such that

$$\forall m \geqslant 1 \qquad Q(W_m^\theta) \leqslant e^{-c(\theta)m^{\frac{1}{3}}}.$$

Therefore

$$\begin{aligned}
Q(\sum_{x\in L_m}(W_m(x))^\theta) &\leqslant \ \mid L_m \mid Q(W_m^\theta)\\
&\leqslant \ \mid L_m \mid e^{-c(\theta)m^{\frac{1}{3}}} \underset{m\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where we have used the fact that $\mid L_m \mid = O(m)$. In particular, there exists $m \geqslant 1$ such that

$$Q(\sum_{x\in L_m}(W_m(x))^\theta) < 1.$$

We have $p^{\text{tree}}_m(\beta) < 0$ and so by theorem 3.3 $p(\beta) < 0$. □

THEOREM 3.6. *Suppose the environment η is bounded or gaussian. Then the inequality (3.3) is in fact an equality*

$$p(\beta) = \inf_{m\geqslant 1} p^{\text{tree}}_m(\beta).$$

PROOF. The inequality $p(\beta) \leqslant \inf_{m \geqslant 1} p_m^{\text{tree}}(\beta)$ is in fact the conclusion of theorem 3.3 and thus is true for all environments.

We must show that $\inf_{m \geqslant 1} p_m^{\text{tree}}(\beta) \leqslant p(\beta)$. We treat the gaussian case, the bounded case being similar. If $\beta \leqslant \beta_c$, we have by definition $p(\beta) = 0$ and since for all $m \geqslant 1$, $p_m^{\text{tree}}(\beta) \leqslant 0$, the result is obvious. Suppose that β is such that $\beta > \beta_c$. By definition of β_c, $p(\beta) < 0$. Let $\theta \in]0, 1]$. We have by the concentration result (2.9)

$$\begin{aligned} Q(W_m^\theta) &= e^{\theta Q(\ln(W_m))} Q(e^{\theta(\ln W_m - Q(\ln(W_m))}) \\ &\leqslant e^{\theta p(\beta) m + \frac{\beta^2 \theta^2 m}{2}}. \end{aligned}$$

For all $m \geqslant 1$,

$$\begin{aligned} \frac{1}{m} p_m^{\text{tree}}(\beta) &\leqslant \frac{1}{\theta m} \ln(Q(\sum_{x \in L_m} (W_m(x))^\theta)) \\ &\leqslant \frac{1}{\theta m} \ln(\mid L_m \mid) + \frac{1}{\theta m} \ln(Q(W_m^\theta)) \\ &\leqslant \frac{1}{\theta m} \ln(\mid L_m \mid) + p(\beta) + \frac{\beta^2 \theta}{2} \\ &\underset{m \to \infty}{\longrightarrow} p(\beta) + \frac{\beta^2 \theta}{2} \end{aligned}$$

where we have used the fact that $\mid L_m \mid = O(m^d)$. Thus, by remark 3.4

$$\inf_{m \geqslant 1} \frac{1}{m} p_m^{\text{tree}}(\beta) = \lim_{m \to \infty} \frac{1}{m} p_m^{\text{tree}}(\beta) \leqslant p(\beta) + \frac{\beta^2 \theta}{2}.$$

The proof is complete by letting $\theta \downarrow 0$. □

CHAPTER 3

Strong localisation and macroscopic atoms for directed polymers

1. Introduction

In this chapter, we consider a model of directed polymers in random environment introduced by Huse and Henley in 1985 ([**28**]) to modelize impurity-induced domain-wall roughening in the 2D-Ising model. This model relates to many physical models of growing random surfaces including the well known Kardar-Parisi-Zhang equation driven by gaussian noise (we refer to [**39**] for an account on these models and their relations). In [**58**], Zhang proposed to replace the gaussian noise in the KPZ equation by a noise with power-law tail to describe fluid flows. Since then, this model has been used to describe fire fronts, bacterial colonies, etc.... In the field of polymers, the authors of [**26**],[**45**] study the random energy landscape of zero temperature directed polymers in power-law environment distributions.

The first mathematical study of directed polymers at positive temperature was undertaken by Imbrie, Spencer in 1988 ([**30**]) and carried out by numerous authors ([**1**],[**7**],[**9**],[**11**],[**52**],[**53**]); for an overview of the achieved results, we refer to [**13**]. In [**9**] and [**11**], the authors show, using martingale techniques, that the quenched free energy is strictly less than the annealed one if and only if a localization theorem for the polymer's favorite point holds. In all these previous mathematical articles, the authors assume that the environment has exponential moments of all order. When considering a temperature where the moment generating function of the environment is infinite, no martingale technique can be used, making the usual strategy irrelevant. Hence, a natural question in this case is: what is left from the localization picture? In this chapter, our approach is more general than the martingale approach used in the above references and we obtain our localization results under much weaker conditions on the distribution of the environment (including the power tail distributions studied in [**26**],[**45**], exponential distributions...). The case of exponentially distributed environments is of particular interest in view of the exact results derived by Johansson in [**31**] for directed last passage percolation with i.i.d. exponential variables in dimension $d = 1$. Since directed last passage percolation can be recovered from directed polymers by letting the temperature go to 0, one can view the polymer measure as an interpolation between the directed percolation model and the simple random walk.

In this chapter, we go a step further than favorite point localization and derive localization results in terms of ϵ-atoms by using bounds on the free energy. We call ϵ-atoms, atoms of the polymer measure of mass at least ϵ. Roughly, we show, under certain assumptions on the environment, that the whole mass of the polymer measure at "low temperature" is essentially carried by ϵ-atoms (cf. theorems 3.2 and 3.7 below). Our method of proof relies mainly on a simple inequality (cf. lemma 5.1 below) and on an upper bound on greedy lattice animals established in [**46**]. Using lemma 5.3, we also give a different proof for localization in terms of the polymer's favorite point if the quenched free energy is strictly less than the annealed one (cf. theorem 3.6 below).

The chapter is organized as follows: in section 2, we introduce the model and the definition of ϵ-atoms. In section 3, we state an existence theorem for the free energy and our localization theorems. In section 4, we give the proofs.

2. The model and definition of ϵ-atoms

2.1. The model. The model we consider in this paper consists of a simple random walk under a random Gibbs measure depending on the temperature. More precisely,

Let $((\omega_n)_{n\in\mathbb{N}}, P)$ denote the simple random walk starting from 0 on the d-dimensional integer lattice $\mathbb{Z}^d$, defined on a measurable space $(\Omega, \mathcal{F})$; more precisely, under the measure P, $(\omega_n - \omega_{n-1})_{n\,\geqslant\,1}$ are independent and

$$P(\omega_0 = 0) = 1, \quad P(\omega_n - \omega_{n-1} = \pm\delta_j) = \frac{1}{2d}, \quad j = 1, \dots, d,$$

where $(\delta_j)_{1\,\leqslant\,j\,\leqslant\,d}$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$.

The random environment on each lattice site is a sequence $\eta = (\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ of real valued, non-constant and i.i.d. random variables defined on a probability space $(H, \mathcal{G}, Q)$. We denote by F the common distribution function of the sequence $(\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$. In the whole paper, we will suppose the following:

Assumptions:

$$\int_0^\infty (1 - F(x))^{\frac{1}{d+1}} dx < \infty \tag{2.1}$$

and

$$Q(|\eta(n,x)|) < \infty. \tag{2.2}$$

Let λ be the logarithmic moment generating function of $\eta(n,x)$:

$$\forall \beta \in \mathbb{R}_+ \quad \lambda(\beta) \stackrel{def.}{=} \ln Q(e^{\beta\eta(n,x)}) \leqslant \infty.$$

For any $n > 0$, we define the (Q-random) polymer measure μ_n on $(\Omega, \mathcal{F})$ by:

$$\mu_n(d\omega) = \frac{1}{Z_n} \exp(\beta H_n(\omega)) P(d\omega)$$

where $\beta \in \mathbb{R}_+$ is the inverse temperature,

$$H_n(\omega) \stackrel{def.}{=} \sum_{j=1}^{n} \eta(j, \omega_j)$$

is the hamiltonian and

$$Z_n = P(\exp(\beta H_n(\omega)))$$

is the partition function.

The above definition shows that the polymer is attracted to sites where the environment is large and positive and repelled by sites where the environment is large and negative; as the inverse temperature β increases, the influence of the environment increases and tends to push the random walk in a few "corridors" where the environment takes high positive values: we will see in the next sections quantitative statements of these heuristics.

2.2. Definition of ϵ-atoms. The purpose of this chapter is to study where the polymer measure $(\mu_{j-1}(\omega_j = x))_{x \in \mathbb{Z}^d}$ is concentrated for large j; under assumptions on the environment η and on the inverse temperature β (typically β "large"), we show in some sense that the mass carried by a few points is "significant". To give a quantitative statement of this phenomenom, we are naturally lead to introduce the notion of ϵ-atoms. More precisely, let $\epsilon > 0$ be some positive real number; we define $\mathcal{A}_j^{\epsilon,\beta}$ the set of ϵ-atoms to be the points of $\mathbb{Z}^d$ wich carry a mass of at least ϵ:

$$\mathcal{A}_j^{\epsilon,\beta} = \{x \in \mathbb{Z}^d : \mu_{j-1}(\omega_j = x) > \epsilon\}.$$

For $\delta < 1$, we define the event $A_j^{\epsilon,\delta,\beta}$ to be the environments for which $\mathcal{A}_j^{\epsilon,\beta}$ has a mass of at least δ:

$$A_j^{\epsilon,\delta,\beta} = \{\eta : \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon,\beta}) \geqslant \delta\},$$

and $A_j^{\epsilon,\beta}$ to be the environments for which $\mathcal{A}_j^{\epsilon,\beta}$ has at least one element:

$$A_j^{\epsilon,\beta} = \{\eta : \max_{x\in\mathbb{Z}^d} \mu_{j-1}(\omega_j = x) > \epsilon\}.$$

In terms of ϵ-atoms, we state the following localization result derived in [**11**] under the assumption $\lambda(\beta) < \infty$ $(\forall\beta)$ (cf. corollary 2.2 therein):

$$p(\beta) < \lambda(\beta) \;\Leftrightarrow\; \exists\, \epsilon > 0,\ \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon,\beta}) \geqslant \epsilon \qquad Q - a.s.$$

This equivalence asserts that the quenched free energy is strictly less than the annealed one if and only if there exists $\epsilon > 0$ such that the mass carried by the ϵ-atoms is for large n (in the sense of Césaro) bounded below by some positive constant.

We recall that if for all β in $\mathbb{R}$ the moment generating function $\lambda(\beta)$ is finite then for all β different from 0 the strict inequality $p(\beta) < \lambda(\beta)$ holds for dimension $d = 1$ (theorem 1.1 in [**16**]). For dimension $d = 2$, this problem is still open.

Finally, we introduce the following definition for $(\mu_{j-1}(\omega_j \in .))_{j\geqslant 1}$:

Definition 2.1. *The sequence $(\mu_{j-1}(\omega_j \in .))_{j\geqslant 1}$ is asymptotically purely atomic (in Césaro mean) if for all sequence $(\epsilon_j)_{j\geqslant 1}$ tending to 0 as j goes to infinity the following convergence holds:*

$$\frac{1}{n}\sum_{j=1}^{n} \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon_j,\beta}) \underset{n\to\infty}{\longrightarrow} 1 \text{ in } Q\text{-Probab.}$$

Less formally, $(\mu_{j-1}(\omega_j \in .))_{j\geqslant 1}$ is asymptotically purely atomic if, for large j, the polymer measure concentrates on few atoms.

3. Results

3.1. Existence of the free energy. First, we establish the existence of the free energy for all β in $\mathbb{R}_+$. We recall that condition (2.1) implies that $Q[(\eta(n,x)_+)^{d+1}] < \infty$ but is implied by the existence of some $\epsilon > 0$ such that $Q[(\eta(n,x)_+)^{d+1+\epsilon}] < \infty$;

in particular, it is much weaker than the existence of exponential moments. We denote by $\vec{\Pi}_n$ the oriented paths of the simple random walk up to time n:

$$\vec{\Pi}_n = \{(j, \omega_j)_{1 \leqslant j \leqslant n}; \quad \forall j, \quad |\omega_{j+1} - \omega_j| = 1\}$$

We define $\vec{N}(n)$ as the maximum of the Hamiltonian along the paths of $\vec{\Pi}_n$:

$$\vec{N}(n) \stackrel{def.}{=} \max_{(j,\omega_j)_j \in \vec{\Pi}_n} \sum_{j=1}^{n} \eta(j, \omega_j).$$

As a consequence of condition (2.1) and proposition 3.4 in [**47**], we can define α in the following way:

$$\alpha \stackrel{def.}{=} \sup_{n \geqslant 1} Q(\frac{\vec{N}(n)}{n}) < \infty.$$

Since $Q(|\eta(n,x)|) < \infty$, by Kingman's subadditive ergodic theorem, we get that:

$$\frac{\vec{N}(n)}{n} \underset{n\to\infty}{\longrightarrow} \alpha \qquad Q-a.s. \text{ and in } L^1(Q).$$

For all β in $\mathbb{R}_+$, the obvious bound $\ln Z_n \leqslant \beta \vec{N}(n)$ and condition (2.2) ensure the existence in $[Q(\eta(n,x)), \alpha]$ of

$$p(\beta) = \sup_{n \geqslant 1} Q(\frac{\ln Z_n}{n}).$$

To get a strong convergence result, we introduce the following condition:

$$\int_{-\infty}^{0} F(x)^{\frac{1}{d+1}} dx < \infty. \tag{3.1}$$

THEOREM 3.1. *The averaged free energy exists in the following weak sense:*

$$\frac{Q(\ln Z_n)}{n} \underset{n\to\infty}{\longrightarrow} p(\beta).$$

We have the following bound on the free energy:

$$p(\beta) \leqslant \alpha\beta \wedge \lambda(\beta). \tag{3.2}$$

If, in addition, the environment satisfies condition (3.1), one gets the following stronger result:

$$\frac{\ln Z_n}{n} \underset{n\to\infty}{\longrightarrow} p(\beta) \qquad Q-a.s. \textit{ and in } L^1(Q).$$

However, not much is known on the limit p: p is convex and $p(\beta)/\beta \to \alpha$ as $\beta \to \infty$. In subsection 3.3, we will tackle the question of the comparaison of p with its annealed bound λ.

3.2. Strong localization in probability. In this subsection, we fix the inverse temperature $\beta > 0$ and we suppose that:

$$\lambda(\beta) = \infty.$$

Intuitively, when $\lambda(\beta) = \infty$, the environment can take large values and one expects the polymer measure to concentrate in those regions of high environment. A quantitative statement of this is the following theorem:

THEOREM 3.2. *Suppose that $\lambda(\beta) = \infty$. Then, for all $\delta < 1$, there exists $\epsilon(\delta) > 0$ such that:*

$$\liminf_{n\to\infty} Q(\frac{1}{n}\sum_{j=1}^{n} \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon(\delta),\beta})) \geqslant \delta. \tag{3.3}$$

An immediate corollary of the above theorem is the following convergence result:

COROLLARY 3.3. *The sequence $(\mu_{j-1}(\omega_j \in .))_{j\geqslant 1}$ is asymptotically purely atomic.*

3.3. Almost sure strong localization. In order to get almost sure localization results, we will suppose that the environment has non trivial exponential moments. More precisely, let $R = \sup\{\beta \in \mathbb{R}^+ : \lambda(\beta) < \infty\}$. In this subsection, we will suppose that $R > 0$ (possibly $R = \infty$). On the interval $]0, R[$, we want to compare p to its annealed bound λ, a standard procedure in statistical physics. Roughly, we have the following conjectured picture for directed polymers:

(1) when $p(\beta) = \lambda(\beta)$, μ_n spreads out uniformly.
(2) when $p(\beta) < \lambda(\beta)$, μ_n has macroscopic atoms which may concentrate the whole mass.

When $d \geqslant 3$ and β satisfies:

$$\lambda(2\beta) - 2\lambda(\beta) < \ln(1/P(\exists n \geqslant 1,\ \omega_n = 0))$$

(this condition implies $p(\beta) = \lambda(\beta)$), the situation is well understood: the polymer is diffusive in the sense that the measure $\mu_n(\omega_n/\sqrt{n} \in .)$ converges weakly to a gaussian law ([**7**],[**30**],[**53**]) and satisfies a local limit theorem ([**52**],[**55**]). For case (2), we refer to theorem 3.6 below.

First, we give a preliminary lemma wich states that there is a phase transition between case (1) and (2) under some assumptions on the environment.

LEMMA 3.4. *The function $p - \lambda$ is nonincreasing on the interval $[0, R[$. Suppose that one of the two following conditions is satisfied:*

- $R < \infty$ *and* $\alpha < \frac{\lambda(R)}{R}$.
- $R = \infty$ *and defining* $L = essup(\eta(n,x))$*, we have*

$$Q(\eta(n,x) = L) < \vec{p}_c(d),$$

where $\vec{p}_c(d)$ *denotes the site percolation threshold for the oriented graph induced on* $\mathbb{N} \times \mathbb{Z}^d$ *by the simple random walk.*

Then there exists $\beta_c < R$ *such that:*

$$\beta \in [0, \beta_c[\quad \Rightarrow \quad p(\beta) = \lambda(\beta).$$
$$\beta \in]\beta_c, R[\quad \Rightarrow \quad p(\beta) < \lambda(\beta).$$

PROOF. One can adapt the proof of lemma 3.3 in **[15]** to prove that $p - \lambda$ is nonincreasing on the interval $[0, R[$.

If $R < \infty$ and $\alpha < \frac{\lambda(R)}{R}$ then

$$\limsup_{\beta \to R} (p(\beta) - \lambda(\beta)) < 0,$$

and the existence of β_c follows.

If $R = \infty$ and $Q(\eta(n,x) = L) < \vec{p}_c(d)$, then one can show that $\alpha < L$ (cf. Proposition 5.5 in the appendix). Therefore,

$$p(\beta) - \lambda(\beta) \underset{\beta \to \infty}{\sim} \beta(\alpha - L) \underset{\beta \to \infty}{\to} -\infty$$

and the existence of β_c follows. □

REMARK 3.5. *In lemma 3.4, one can have* $\beta_c = 0$*. It is believed that this is the case in dimension* $d = 1$ *and* $d = 2$*.*

In particular, lemma 3.4 gives sufficient conditions for the existence of β in $]0, R[$ such that the strict inequality $p(\beta) < \lambda(\beta)$ holds. Now, we state our first almost sure localization result which generalizes corollary 2.2 in **[11]**:

THEOREM 3.6. *Suppose that the environment satisfies condition (3.1). Then for all* β *in* $]0, R[$*, we have the following implication:*

$$p(\beta) < \lambda(\beta) \qquad \Rightarrow \qquad \exists\, \epsilon > 0, \qquad \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon,\beta}) \geqslant \epsilon. \qquad Q - a.s.$$

In the next theorem, we will make the assumption that η "explodes" at R:

$$\lambda(R)/R = \infty, \tag{3.4}$$

where wet set $\lambda(R)/R = \text{essup}(\eta(n,x))$ if $R = \infty$ and that

$$\exists \theta > 1, \qquad Q(|\eta(n,x)|^\theta) < \infty. \tag{3.5}$$

THEOREM 3.7. *Suppose that the environment satisfies conditions (3.4), (3.5). Then for all $\delta < 1$, there exists $\epsilon(\delta) > 0$ and $\beta(\delta)$ in $]0, R[$ such that:*

$$\forall \beta \in [\beta(\delta), R[\qquad \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon(\delta),\beta}) \geqslant \delta \qquad Q - a.s.$$

The above theorem can be seen as a continuity result in view of theorem 3.2.

4. Proof of theorem 3.1

Proof of theorem 3.1 Let $L \in \mathbb{N}^* \cup \{\infty\}$. We define $Y_{n,L}$ by

$$Y_{n,L} = \frac{1}{n} \ln P(e^{\beta H_n^L(\omega)}).$$

where

$$H_n^L(\omega) = \sum_{j=1}^{n} \eta(j,\omega_j) \wedge L \vee -L.$$

Similarly, we define

$$p_L(\beta) = \sup_{n \geqslant 1} Q(Y_{n,L}).$$

With these notations, $Y_{n,\infty} = \frac{1}{n} \ln Z_n$ and $p_\infty(\beta) = p(\beta)$. It is well known that the sequence $(Q(\ln Z_n))_{n \geqslant 1}$ is superadditive and so we have the following limit:

$$\lim_{n\to\infty} Q(Y_{n,\infty}) = p(\beta).$$

The obvious bound $\ln Z_n \leqslant \beta \overrightarrow{N}(n)$ ensures $p(\beta) \leqslant \alpha\beta$ and an application of Jensen's inequality to ln ensures that $p(\beta) \leqslant \lambda(\beta)$, giving the first two parts of theorem 3.1. From know on, we suppose the environment satisfies condition (3.1).

For all $L \in \mathbb{N}^*$, it is known (cf. proposition 2.5 in [**11**]) that

$$Y_{n,L} \underset{n\to\infty}{\longrightarrow} p_L(\beta) \qquad \text{Q-a.s. and in } L^1(Q).$$

One has the following bounds:

$$\forall n, L \geqslant 1, \qquad |Y_{n,\infty} - Y_{n,L}| = |\frac{1}{n} \ln P(e^{\beta(H_n(\omega) - H_n^L(\omega))})|$$
$$\leqslant \frac{\beta}{n} \max_{\omega \in \vec{\Pi}_n} |H_n(\omega) - H_n^L(\omega)|$$
$$= \frac{\beta}{n} \max_{(j,\omega_j)_j \in \vec{\Pi}_n} \sum_{j=1}^{n} (|\eta|(j,\omega_j) - L)_+.$$

Therefore proposition 3.4 in **[47]** ensures the existence of some constant $c < \infty$ such that the following estimates hold

$$\limsup_{n\to\infty} |Y_{n,\infty} - Y_{n,L}| \leqslant c\beta \int_L^\infty (1 - F(x) + F(-x))^{\frac{1}{d+1}} dx$$
$$\leqslant c\beta \int_L^\infty (1 - F(x))^{\frac{1}{d+1}} dx + c\beta \int_{-\infty}^{-L} F(x)^{\frac{1}{d+1}} dx \qquad Q - a.s.$$

and similarly

$$\limsup_{n\to\infty} Q(|Y_{n,\infty} - Y_{n,L}|) \leqslant c\beta \int_L^\infty (1 - F(x))^{\frac{1}{d+1}} dx + c\beta \int_{-\infty}^{-L} F(x)^{\frac{1}{d+1}} dx.$$

Therefore we have

$$|Y_{n,\infty} - Q(Y_{n,\infty})| \leqslant |Y_{n,\infty} - Y_{n,L}| + |Y_{n,L} - Q(Y_{n,L})| + |Q(Y_{n,L}) - Q(Y_{n,\infty})|$$
$$\leqslant |Y_{n,\infty} - Y_{n,L}| + |Y_{n,L} - Q(Y_{n,L})| + Q(|Y_{n,L} - Y_{n,\infty}|).$$

By letting $n \to \infty$, we get

$$\limsup_{n\to\infty} |Y_{n,\infty} - p(\beta)| \leqslant 2c\beta \int_L^\infty (1 - F(x))^{\frac{1}{d+1}} dx + 2c\beta \int_{-\infty}^{-L} (F(x))^{\frac{1}{d+1}} dx \qquad Q - a.s.$$

By letting $L \to \infty$ above, we conclude

$$Y_{n,\infty} \underset{n\to\infty}{\overset{Q-a.s.}{\longrightarrow}} p(\beta).$$

Similarly, we obtain

$$Y_{n,\infty} \underset{n\to\infty}{\overset{L^1(Q)}{\longrightarrow}} p(\beta).$$

□

5. Proof of theorems 3.2, 3.6, 3.7

5.1. Some preliminary lemmas. We first introduce a few notations we will use in the following two lemmas. For n a postive integer, we define $\mathcal{P}_n$ to be the standard probability simplex in $\mathbb{R}^n$:

$$\mathcal{P}_n = \{(\lambda_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}_+^n; \sum_{i=1}^{n} \lambda_i = 1\}.$$

For $\epsilon, \delta \in]0,1[$, we define

$$\mathcal{P}_n^{\epsilon,\delta} = \{(\lambda_i)_{1 \leqslant i \leqslant n} \in \mathcal{P}_n; \sum_{i=1}^{n} \lambda_i 1_{\lambda_i > \epsilon} \leqslant \delta\}$$

In this section, $(X_i)_{i \geqslant 1}$ will denote an i.i.d. sequence of positive random variables on a probability space $(H, \mathcal{H}, P)$ such that:

$$E[|\ln X_1|] < \infty.$$

LEMMA 5.1. *Let $\delta \in]\frac{1}{2}, 1[$ and $\epsilon \in]0, 1-\delta[$ be such that $\frac{(1-\delta)}{\epsilon}$ is a positive integer. We have for all $n \geqslant \frac{(1-\delta)}{\epsilon} + 1$:*

$$\inf_{(\lambda_i)_{1 \leqslant i \leqslant n} \in \mathcal{P}_n^{\epsilon,\delta}} E[\ln(\sum_{i=1}^{n} \lambda_i X_i)] = E[\ln(\epsilon \sum_{i=1}^{\frac{(1-\delta)}{\epsilon}} X_i + \delta X_{\frac{(1-\delta)}{\epsilon}+1})].$$

PROOF. We can suppose that X_1 is non constant. We first establish an auxiliary result we will use intensively in the rest of the proof. Let k be a integer greater than or equal to 2 and $(\lambda_i)_{1 \leqslant i \leqslant k}$ an element of $\mathcal{P}_k$ such that $0 < \lambda_2 \leqslant \lambda_1 < 1$. One can therefore consider the function $\phi : [0, (1-\lambda_1) \wedge \lambda_2] \longrightarrow \mathbb{R}$ defined by:

$$\forall \rho \in [0, (1-\lambda_1) \wedge \lambda_2] \qquad \phi(\rho) = E[\ln((\lambda_1 + \rho)X_1 + (\lambda_2 - \rho)X_2 + \sum_{i=3}^{k} \lambda_i X_i)].$$

One can compute the derivative of ϕ and we get $\forall \rho \in]0,(1-\lambda_1)\wedge\lambda_2]$:

$$\begin{aligned}
\phi'(\rho) &= E[\frac{X_1 - X_2}{(\lambda_1+\rho)X_1 + (\lambda_2-\rho)X_2 + \sum_{i=3}^k \lambda_i X_i}] \\
&= E[\frac{X_1 - X_2}{(\lambda_1+\rho)X_1 + (\lambda_2-\rho)X_2 + \sum_{i=3}^k \lambda_i X_i} 1_{X_1>X_2}] \\
&\quad + E[\frac{X_1 - X_2}{(\lambda_1+\rho)X_1 + (\lambda_2-\rho)X_2 + \sum_{i=3}^k \lambda_i X_i} 1_{X_1<X_2}] \\
&= E[\frac{X_1 - X_2}{(\lambda_1+\rho)X_1 + (\lambda_2-\rho)X_2 + \sum_{i=3}^k \lambda_i X_i} 1_{X_1>X_2}] \\
&\quad - E[\frac{X_1 - X_2}{(\lambda_1+\rho)X_2 + (\lambda_2-\rho)X_1 + \sum_{i=3}^k \lambda_i X_i} 1_{X_1>X_2}] \\
&< 0
\end{aligned}$$

where the last inequality comes from the fact that $x \longrightarrow \frac{1}{x}$ is decreasing. Therefore, ϕ is a decreasing function and thus one can conclude that $\forall \rho \in]0,(1-\lambda_1)\wedge\lambda_2]$:

$$E[\ln((\lambda_1+\rho)X_1 + (\lambda_2-\rho)X_1 + \sum_{i=3}^k \lambda_i X_i)] < E[\ln(\sum_{i=1}^k \lambda_i X_i)]. \tag{5.1}$$

Let $n \geqslant \frac{(1-\delta)}{\epsilon} + 1$ be a fixed integer and consider the application $f : \mathcal{P}_n^{\epsilon,\delta} \longrightarrow \mathbb{R}$ defined by

$$\forall (\lambda_i) \in \mathcal{P}_n^{\epsilon,\delta} \qquad f((\lambda_i)) = E[\ln(\sum_{i=1}^n \lambda_i X_i)].$$

Since f is continuous on the compact set $\mathcal{P}_n^{\epsilon,\delta}$, there exists $(\lambda_i^*) \in \mathcal{P}_n^{\epsilon,\delta}$ such that:

$$\inf_{(\lambda_i)\in\mathcal{P}_n^{\epsilon,\delta}} f((\lambda_i)) = f((\lambda_i^*)) \tag{5.2}$$

Let $p = \#\{i; \lambda_i^* > 0\}$. Since f is symmetric, we can suppose that $\lambda_1^* \geqslant \lambda_2^* \geqslant \ldots \lambda_p^* > 0$ and that $\lambda_i^* = 0$ for $i > p$. We introduce the following set:

$$F_\epsilon = \{i; \lambda_i^* > \epsilon\}$$

Let $k = \#F_\epsilon$; we have the following identity:

$${}^cF_\epsilon = [|\; k+1, p \;|].$$

If $k \geqslant 2$, for $\rho > 0$ sufficiently small, we have $(\lambda_1^*+\rho, \lambda_2^*-\rho, \lambda_3^*, \ldots, \lambda_p^*, 0, \ldots, 0) \in \mathcal{P}_n^{\epsilon,\delta}$ and by inequality (5.1), we get

$$f(\lambda_1^*+\rho, \lambda_2^*-\rho, \lambda_3^*, \ldots, \lambda_p^*, 0, \ldots, 0) < f((\lambda_i^*)),$$

which contradicts (5.2). Therefore, $k \leqslant 1$. If $\lambda_{p-1}^* < \epsilon$ then for $\rho > 0$ sufficiently small, $(\lambda_1^*, \dots, \lambda_{p-2}^*, \lambda_{p-1}^* + \rho, \lambda_p^* - \rho, 0, \dots, 0) \in \mathcal{P}_n^{\epsilon,\delta}$ and by inequality (5.1) we get

$$f((\lambda_1^*, \dots, \lambda_{p-2}^*, \lambda_{p-1}^* + \rho, \lambda_p^* - \rho, 0, \dots, 0)) < f((\lambda_i^*)),$$

which contradicts (5.2). Therefore, $(\lambda_i^*) = (\lambda_1^*, \epsilon, \dots, \epsilon, \lambda_p^*, 0, \dots, 0)$. If $\lambda_1^* < \delta$, then for $\rho > 0$ sufficiently small,$(\lambda_1^* + \rho, \lambda_2^* - \rho, \lambda_3^*, \dots, \lambda_p^*, 0, \dots, 0) \in \mathcal{P}_n^{\epsilon,\delta}$ and by inequality (5.1) we get

$$f((\lambda_1^* + \rho, \lambda_2^* - \rho, \lambda_3^*, \dots, \lambda_p^*, 0, \dots, 0)) < f((\lambda_i^*)),$$

which contradicts (5.2). Thus $\lambda_1^* = \delta$ and since $\sum_{i=1}^p \lambda_i^* = 1$, we get $p = 1 + \frac{1-\delta}{\epsilon}$ and $\lambda_p^* = \epsilon$. We can conclude

$$\inf_{(\lambda_i) \in \mathcal{P}_n^{\epsilon,\delta}} f((\lambda_i)) = f((\lambda_i^*)) = E[\ln(\epsilon \sum_{i=1}^{\frac{(1-\delta)}{\epsilon}} X_i + \delta X_{\frac{(1-\delta)}{\epsilon}+1})].$$

□

REMARK 5.2. *Under suitable integrability assumptions, the same result holds when one considers a general concave function instead of* ln.

In the same spirit than the above lemma, we state the following lemma without proving it.

LEMMA 5.3. *Let k be some positive integer and $\epsilon = \frac{1}{k}$. Then we have for all $n \geqslant k$:*

$$\inf_{\substack{(\lambda_i)_{1 \leqslant i \leqslant n} \in \mathcal{P}_n \\ \max(\lambda_i) \leqslant \epsilon}} E[\ln(\sum_{i=1}^n \lambda_i X_i)] = E[\ln(\epsilon \sum_{i=1}^{1/\epsilon} X_i)].$$

Finally, we state the following convergence result:

LEMMA 5.4. *Let $a, b > 0$ be two positive numbers such that $a < b$. We have the following convergence:*

$$\inf_{\beta \in [a,b]} E[\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta)] \underset{n \to \infty}{\longrightarrow} \inf_{\beta \in [a,b]} \ln E[X_1^\beta].$$

PROOF. The fact that the left hand side is less than or equal to the right hand side is a consequence of Jensen's inequality.

Let $L > 0$ be such that $-\frac{L}{a} < E[\ln(X_1)]$. Then for all β in $[a,b]$ we have:

$$\begin{aligned}
E[\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta})] &= E[\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta})1_{\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta}) \geqslant -L}] \\
&\quad +E[\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta})1_{\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta})<-L}] \\
&\geqslant E[\ln(\frac{1}{n}\sum_{i=1}^{n}(X_i \wedge L)^{\beta})1_{\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta}) \geqslant -L}] \\
&\quad +\beta E[\frac{1}{n}\sum_{i=1}^{n} \ln X_i 1_{\frac{1}{n}\sum_{i=1}^{n} \ln X_i<-\frac{L}{\beta}}] \\
&\geqslant E[\ln(\frac{1}{n}\sum_{i=1}^{n}(X_i \wedge L)^{\beta})1_{\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta}) \geqslant -L}] \\
&\quad +\beta E[\ln(X_1)1_{\frac{1}{n}\sum_{i=1}^{n} \ln(X_i)<-\frac{L}{\beta}}] \\
&\geqslant E[\ln(\frac{1}{n}\sum_{i=1}^{n}(X_i \wedge L)^{\beta})1_{\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta}) \geqslant -L}] \\
&\quad -bE[|\ln(X_1)|1_{\frac{1}{n}\sum_{i=1}^{n} \ln(X_i)<-\frac{L}{a}}].
\end{aligned}$$

By taking the infimum over all $\beta \in [a,b]$ and using the bounded convergence theorem, we conclude that:

$$\liminf_{n\to\infty} \inf_{\beta\in[a,b]} E[\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta})] \geqslant \inf_{\beta\in[a,b]} \ln E[(X_1 \wedge L)^{\beta}].$$

We obtain the result by letting $L \to \infty$ in the above inequality. □

5.2. Proof of theorem 3.2. Following the notations of lemma 5.1, we consider an i.i.d. sequence $(X_i)_{i\geqslant 1}$ defined on some probability space $(H, \mathcal{H}, P)$ and such that $X_1 \overset{law}{=} e^{\eta(n,x)}$. Let $\delta < 1$ and $c(\delta)$ be some integer we will choose at the end of the proof. Finally, we set $\epsilon = \frac{1-\delta}{c(\delta)}$ (for notational convenience, we write ϵ instead of $\epsilon(\delta)$).

We have the following computation:

$$\begin{aligned}\frac{Q(\ln Z_n)}{n} &= \frac{1}{n}\sum_{j=1}^{n} Q(\ln(\frac{Z_j}{Z_{j-1}}))\\ &= \frac{1}{n}\sum_{j=1}^{n} Q(\ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)})).\\ &\underset{(Jensen)}{\geqslant} \frac{1}{n}\sum_{j=1}^{n} Q(1_{{}^cA_j^{\epsilon,\delta,\beta}} \ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)})) + \beta E[\ln X_1]Q(A_j^{\epsilon,\delta,\beta})\end{aligned}$$

Thus, we get the following inequality:

$$\begin{aligned}\frac{Q(\ln Z_n)}{n} - \beta E[\ln X_1] \quad &\geqslant \quad \sum_{j=1}^{n} Q(1_{{}^cA_j^{\epsilon,\delta,\beta}} \ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}))\\ &\quad -\beta E[\ln X_1]Q(A_j^{\epsilon,\delta,\beta})\end{aligned}$$

By applying lemma 5.1 to the family $(e^{\beta\eta(j,x)})_{x\in\mathbb{Z}^d}$ under the conditional measure $Q(.|\mathcal{G}_{j-1})$, we get:

$$\frac{Q(\ln(Z_n))}{n} \geqslant (\frac{1}{n}\sum_{j=1}^{n} Q({}^cA_j^{\epsilon,\delta,\beta}))(E[\ln((1-\delta)\sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)}X_k^\beta + \delta X_{c(\delta)+1}^\beta)] - \beta E[\ln X_1]).$$

Therefore, using (3.2) and letting n go to infinity, we get

$$\limsup_{n\to\infty}(\frac{1}{n}\sum_{j=1}^{n} Q({}^cA_j^{\epsilon,\delta,\beta})) \leqslant \frac{\alpha\beta - \beta E[\ln X_1]}{E[\ln((1-\delta)\sum_{k=1}^{c(\delta)}\frac{1}{c(\delta)}X_k^\beta + \delta X_{c(\delta)+1}^\beta)] - \beta E[\ln X_1]}.$$

Since $\lambda(\beta) = \infty$, by lemma 5.4, one can choose $c(\delta)$ such that

$$\frac{\alpha\beta - \beta E[\ln X_1]}{E[\ln((1-\delta)\sum_{k=1}^{c(\delta)}\frac{1}{c(\delta)}X_k^\beta + \delta X_{c(\delta)+1}^\beta)] - \beta E[\ln X_1]} \leqslant 1-\delta.$$

Since $\mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon,\beta}) \geqslant \delta 1_{A_j^{\epsilon,\delta,\beta}}$, we get the desired result.

□

5.3. Proof of theorems 3.6,3.7. Both theorems are based on lemma 5.1 or lemma 5.3 and on the law of large numbers for martingales. Following the notations of lemma 5.1, we consider an i.i.d. sequence $(X_i)_{i\geqslant 1}$ defined on some probability space $(H, \mathcal{H}, P)$ and such that $X_1 \overset{law}{=} e^{\eta(n,x)}$. We start by proving theorem 3.7.

Proof of theorem 3.7. Let $\delta < 1$ and $c(\delta)$ be some integer we will choose at the end of the proof. Finally, we set $\epsilon = \frac{1-\delta}{c(\delta)}$ (for notational convenience, we write ϵ instead of $\epsilon(\delta)$). We have the following computation:

$$\begin{aligned}\frac{\ln Z_n}{n} &= \frac{1}{n}\sum_{j=1}^{n} \ln(\frac{Z_j}{Z_{j-1}})) \\ &= \frac{1}{n}\sum_{j=1}^{n} \ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}). \\ &= \frac{1}{n}\sum_{j=1}^{n} 1_{{}^cA_j^{\epsilon,\delta,\beta}} \ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}) \\ &+ \frac{1}{n}\sum_{j=1}^{n} 1_{A_j^{\epsilon,\delta,\beta}} \ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}). \end{aligned} \tag{5.3}$$

(5.4)

Consider the $(\mathcal{G}_n)$-martingale M_n defined by:

$$M_n = \sum_{j=1}^{n} 1_{{}^cA_j^{\epsilon,\delta,\beta}}(\ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}) - Q(\ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)})|\mathcal{G}_{j-1})).$$

By definition of M_n and by applying lemma 5.1 to the family $(e^{\beta\eta(j,x)})_{x\in\mathbb{Z}^d}$ under the conditional measure $Q(.|\mathcal{G}_{j-1})$, we get:

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{n} 1_{{}^cA_j^{\epsilon,\delta,\beta}} \ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}) \\ &= M_n + \sum_{j=1}^{n} 1_{{}^cA_j^{\epsilon,\delta,\beta}} Q(\ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)})|\mathcal{G}_{j-1}) \\ &\geqslant M_n + (\frac{1}{n}\sum_{j=1}^{n} 1_{{}^cA_j^{\epsilon,\delta,\beta}}) E[\ln((1-\delta)\sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta)]\end{aligned}$$

Similarly, consider the $(\mathcal{G}_n)$-martingale N_n defined by:

$$N_n = \sum_{j=1}^{n} 1_{A_j^{\epsilon,\delta,\beta}}(\ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}) - Q(\ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)})|\mathcal{G}_{j-1})).$$

By concavity of ln, we get

$$\begin{aligned}
&\frac{1}{n}\sum_{j=1}^{n} 1_{A_j^{\epsilon,\delta,\beta}} \ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}) \\
&= N_n + \sum_{j=1}^{n} 1_{A_j^{\epsilon,\delta,\beta}} Q(\ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)})|\mathcal{G}_{j-1}) \\
&\geqslant N_n + \beta E[\ln X_1](\frac{1}{n}\sum_{j=1}^{n} 1_{A_j^{\epsilon,\delta,\beta}})
\end{aligned}$$

Plugging the two above inequalities in inequality (5.3), we get:

$$\begin{aligned}
&\frac{\ln Z_n}{n} - \frac{M_n}{n} - \frac{N_n}{n} - \beta E[\ln X_1] \geqslant \\
&(\frac{1}{n}\sum_{j=1}^{n} 1_{{}^cA_j^{\epsilon,\delta,\beta}})(E[\ln((1-\delta)\sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta)] - \beta E[\ln X_1]) \qquad (5.5)
\end{aligned}$$

There exists some constant $C > 0$ such that for all j:

$$\beta \sum_x \mu_{j-1}(\omega_j = x)\eta(j,x) \leqslant \ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}) \leqslant C|\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}|^{\frac{1}{\theta}}$$

Thus there exists some constant $C' > 0$ such that for all j:

$$\begin{aligned}
Q(|\ln(\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)})|^\theta) &\leqslant C'Q(|\sum_x \mu_{j-1}(\omega_j = x)e^{\beta\eta(j,x)}|) \\
&\quad + C'Q(|\sum_x \mu_{j-1}(\omega_j = x)\eta(j,x)|^\theta) \\
&\leqslant C'(e^{\lambda(\beta)} + Q(|\eta(j,x)|^\theta)).
\end{aligned}$$

Therefore, M_n and N_n are of the form $\sum_{j=1}^n (Y_j - Q(Y_j|\mathcal{G}_{j-1}))$ with:

$$\sup_j Q(|Y_j|^\theta) < \infty.$$

By using theorem 2.19 in **[25]**, we conclude that:

$$\lim_{n\to\infty} \frac{M_n}{n} = \lim_{n\to\infty} \frac{N_n}{n} = 0 \qquad Q - a.s.$$

By letting n go to infinity in inequality (5.5) , we get

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} 1_{{}^cA_j^{\epsilon,\delta,\beta}} \leqslant \frac{\alpha\beta - \beta E[\ln X_1]}{E[\ln((1-\delta)\sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta)] - \beta E[\ln X_1]}$$

By using lemma 5.4, one can choose $c(\delta)$ and $\beta(\delta)$ in $]0,R[$ such that:

$$\forall\beta\in[\beta(\delta),R[\qquad \frac{\alpha\beta-\beta E[\ln X_1]}{E[\ln((1-\delta)\sum_{k=1}^{c(\delta)}\frac{1}{c(\delta)}X_k^\beta+\delta X_{c(\delta)+1}^\beta)]-\beta E[\ln X_1]}\leqslant 1-\delta,$$

which implies the result since $\mu_{j-1}(\omega_j\in\mathcal{A}_j^{\epsilon,\beta})\geqslant\delta 1_{A_j^{\epsilon,\delta,\beta}}$.

□

The proof of theorem 3.6 follows a similar strategy to the proof of theorem 3.7. Therefore, we only give a sketch of the proof.

Proof of theorem 3.6.

Suppose that β is such that $p(\beta)<\lambda(\beta)$. Then one can chose a positive integer k sufficiently large for the following inequality to hold with $\epsilon=\frac{1}{k}$:

$$p(\beta)<E[\ln(\epsilon\sum_{i=1}^{1/\epsilon}X_i^\beta)].$$

By the same strategy than for the proof of theorem 3.7 (using lemma 5.3 instead of lemma 5.1), we get

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}1_{{}^cA_j^{\epsilon,\beta}}\leqslant\frac{p(\beta)-\beta E[\ln X_1]}{E[\ln(\epsilon\sum_{i=1}^{\frac{1}{\epsilon}}X_i^\beta)]-\beta E[\ln X_1]}<1. \tag{5.6}$$

This implies easily the desired result.

□

Appendix

In the appendix, we suppose the environment η is bounded and we consider $L=\text{essup}(\eta(n,x))$. Let $\vec{p}_c(d)$ denote the site percolation threshold for the oriented graph $\vec{\Pi}^d$ induced on $\mathbb{N}\times\mathbb{Z}^d$ by the simple random walk. We prove the following result:

PROPOSITION 5.5. *Suppose that the environment satisfies $Q(\eta(n,x)=L)<\vec{p}_c(d)$. Then we have the following strict inequality:*

$$\alpha<L.$$

PROOF. The proof is an adaptation of the proof of proposition 5.8 in [**38**].

We define $\widetilde{\eta}(j,\omega_j)=L-\eta(j,\omega_j)$. The variables $\widetilde{\eta}(j,\omega_j)$ are non negative and such that

$$Q(\widetilde{\eta}(j,\omega_j)=0)<\vec{p}_c(d).$$

We introduce $\widetilde{N}(n)$ by the following formula:

$$\widetilde{N}(n) = \min_{(j,\omega_j)_j \in \vec{\Pi}_n} \sum_{j=1}^{n} \widetilde{\eta}(j,\omega_j).$$

Since $\vec{N} = nL - \widetilde{N}(n)$, we have to prove that there exists some $\epsilon > 0$ such that:

$$\underline{\lim}_n \frac{\widetilde{N}(n)}{n} \geqslant \epsilon, \quad Q - a.s.$$

We consider i.i.d. Bernoulli percolation $(1_{\widetilde{\eta}(j,\omega_j)=0})_{(j,\omega_j)\in \vec{\Pi}^d}$ on $\vec{\Pi}^d$.

For $n \geqslant 1$, we define L_n to be the set of points visited by the simple random walk at time n:

$$L_n = \{x \in \mathbb{Z}^d; P(w_n = x) > 0\}.$$

For $x \in L_n$, we denote by $\{0 \leftrightarrow x\}$ the event that there exists an open path in $\vec{\Pi}^d$ from the origin to (n,x). Since $Q(\widetilde{\eta}(n,x) = 0) < \vec{p}_c(d)$, a standard result in percolation theory tells us that there exists $c > 0$ such that:

$$Q(\cup_{x\in L_n}\{0 \leftrightarrow x\}) \leqslant e^{-cn}.$$

In particular, one can find n such that:

$$Q(\sum_{x\in L_n} 1_{\{0\leftrightarrow x\}}) \leqslant 1/2.$$

In the sequel, we choose a fixed n that satisfies the above condition.

For $l < k$ and $(x,y) \in L_l \times L_k$, we consider:

$$\widetilde{N}_{l,k}^{x,y} = \min_{\substack{(j,\omega_j)_j \in \vec{\Pi}^d \\ \omega_l = x, \omega_k = y}} \sum_{j=l+1}^{k} \widetilde{\eta}(j,\omega_j).$$

Since $Q(\sum_x e^{-\xi \widetilde{N}_{0,n}^{0,x}}) \underset{\xi\to\infty}{\longrightarrow} Q(\sum_{x\in L_n} 1_{\{0\leftrightarrow x\}})$, we can choose a fixed ξ such that:

$$Q(\sum_x e^{-\xi \widetilde{N}_{0,n}^{0,x}}) < 3/4.$$

Finally, we choose $\epsilon > 0$ such that:

$$e^{\xi\epsilon n} Q(\sum_x e^{-\xi \widetilde{N}_{0,n}^{0,x}}) \leqslant 3/4.$$

For all $k \geqslant 1$, we have:

$$\begin{aligned} Q(\widetilde{N}(nk) \leqslant \epsilon nk) &\leqslant Q(\cup_{x_1,\ldots,x_k}(\widetilde{N}_{0,n}^{0,x_1} + \ldots + \widetilde{N}_{n(k-1),nk}^{x_{k-1},x_k} \leqslant \epsilon nk)) \\ &\leqslant \sum_{x_1,\ldots,x_k} e^{\xi\epsilon nk} Q(e^{-\xi(\widetilde{N}_{0,n}^{0,x_1} + \ldots + \widetilde{N}_{n(k-1),nk}^{x_{k-1},x_k})}) \\ &\leqslant (e^{\xi\epsilon n} Q(\sum_x e^{-\xi \widetilde{N}_{0,n}^{0,x}}))^k \\ &\leqslant (\frac{3}{4})^k. \end{aligned}$$

By the lemma of Borel-Cantelli, we conclude that:

$$\underline{\lim}_k \frac{\widetilde{N}(nk)}{nk} \geqslant \epsilon, \quad Q - a.s.$$

This gives the desired result since $\frac{\widetilde{N}(n)}{n}$ converges $Q - a.s.$ as n goes to infinity.

□

CHAPTER 4

Hydrodynamic turbulence and intermittent random fields

1. Introduction

Roughly observed, some random phenomena seem perfectly scale invariant. This is the case for the velocity field of turbulent flows or the (logarithm of) evolution in time of the price of a financial asset. However, a more precise empirical study of these phenomena displays in fact a weakened form of scale invariance commonly called multifractal scale invariance or intermittency (the exponent which governs the power law scaling of the process or field is no longer linear). An important question is therefore to construct intermittent random fields which exhibit the observed characteristics.

Following the work of Kolmogorov and Obukhov ([**37**], [**49**]) on the energy dissipation in turbulent flows, Mandelbrot introduced in [**44**] a "limit-lognormal" model to describe turbulent dissipation or the volatility of a financial asset. This model was rigorously defined and studied in a mathematical framework by Kahane in [**32**]; more precisely, Kahane constructed a random measure called gaussian multiplicative chaos. A natural extension of this work is to use gaussian multiplicative chaos to construct a field (or a process in the financial case) which describes the whole phenomenom: the velocity field in turbulent flows (the price of an asset on a financial market). This extension was first performed by Mandelbrot himself who proposed to modelize the price of a financial asset with a time changed Brownian motion, the time change being random and independant of the Brownian motion. In [**3**], the authors proposed for the time change to take the primitive of multiplicative chaos: this gives the so called multifractal random walk model (MRW) (Bacry and Muzy later generalized the construction of the MRW model in [**4**]). The obtained process accounts for many observed properties of financial assets.

The inconvenient of the above construction and of the MRW model is that the laws of the increments are symmetrical. In the case of finance, this is in contradiction with the skewness property observed for certain asset prices. In the case of turbulence, the laws of the increments must be nonsymmetrical: it is a theoretical necessity and stems from the dissipation of the kinetic energy ([**24**]). In light of these observations, we are naturally led to construct random fields which generalize to any dimension such process and which present multifractal scale invariance as well as nonsymmetrical increments.

We will answer a very natural question: how can one obtain a two parameter family of multifractal fields with nonsymmetrical increments by perturbing a given scale invariant gaussian random field on $\mathbb{R}^d$? Finally, in the last part we will mention the difficulties which arise in trying to construct an incompressible multifractal velocity field that verifies the 4/5-law of Kolmogorov with positive dissipation.

2. Notations and preliminary results

2.1. The underlying gaussian field. Let $dW_0(x)$ denote the gaussian white noise on $\mathbb{R}^d$ and $\varphi : \mathbb{R}^d \to [0,1]$ denote a C^∞, radially symmetric function worth 1 for $|x| \leqslant 1$ and 0 for $|x| > 2$. We also introduce a fixed correlation scale $R > 0$ and α a number which satisfies

$$d/2 < \alpha < d/2 + 1. \tag{2.1}$$

We define the gaussian field $\mathcal{X}^g$ by the following formula:

$$\mathcal{X}^g(x) = \int_{\mathbb{R}^d} \frac{\varphi_R(x-y)}{|x-y|^{d-\alpha}} dW_0(y), \tag{2.2}$$

where we set the following notation:

$$\varphi_R(x) = R^{d/2-\alpha}\varphi(\frac{x}{R}).$$

It is easy to show that (2.2) defines a homogeneous, isotropic gaussian field which is almost surely holderian of order $< \alpha - d/2$. Note that condition (2.1) implies that the integrand in (2.2) is square integrable and the $R^{d/2-\alpha}$ factor ensures that the field is dimensionless.

scaling property. Let e be a unitary vector and $\lambda > 0$. we have the following identity in law:

$$\mathcal{X}^g(x+\lambda e) - \mathcal{X}^g(x) \underset{(law)}{=} \int_{\mathbb{R}^d} (\frac{\varphi_R(y-\lambda e)}{|y-\lambda e|^{d-\alpha}} - \frac{\varphi_R(y)}{|y|^{d-\alpha}}) dW_0(y).$$

From the gaussianity of the above law, we deduce that for all $q > 0$, there exists $c_q > 0$ such that:

$$E(|\mathcal{X}^g(x+\lambda e) - \mathcal{X}^g(x)|^q) = \sigma_{\lambda e}^q c_q,$$

with

$$\sigma^2_{\lambda e} \underset{\lambda\to 0}{\sim} (\frac{\lambda}{R})^{2\alpha-d}\int_{\mathbb{R}^d}(\frac{1}{|y-e|^{d-a}}-\frac{1}{|y|^{d-a}})^2 dy.$$

We thus derive the following scaling

$$E(|\mathcal{X}^g(x+\lambda e)-\mathcal{X}^g(x)|^q) \underset{\lambda\to 0}{\sim} (\frac{\lambda}{R})^{q(\alpha-d/2)}C_q,$$

where the constant C_q is independent of e. One says that $(\mathcal{X}^g(x))_{x\in\mathbb{R}^d}$ is at small scales monofractal with scaling exponent $\alpha-d/2$.

A field $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$ is multifractal if there exists a non linear function ζ_q such that:

$$E(|\mathcal{X}(x+\lambda e)-\mathcal{X}(x)|^q) \underset{\lambda\to 0}{\sim} (\frac{\lambda}{R})^{\zeta_q}C_q.$$

We call ζ_q the structural function of the field $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$.

2.2. Outline of the construction of multifractal fields from the field $\mathcal{X}^g$. Our construction is inspired by the work of Kahane in [**32**]. Let $\epsilon>0$ and $X^\epsilon(y)$ a regular family of gaussian fields (not necessarily independent of dW_0). We consider a family of fields $\mathcal{X}^\epsilon$ defined by:

$$\mathcal{X}^\epsilon(x)=\int_{\mathbb{R}^d}\frac{\varphi_R(x-y)}{|x-y|_\epsilon^{d-\alpha}}e^{X^\epsilon(y)-C_\epsilon}dW_0(y) \tag{2.3}$$

($|x-y|_\epsilon$ is defined in the next subsection and is given by a standard convolution). For an appropriate family X^ϵ, we show that it is possible to find constants C_ϵ such that $\mathcal{X}^\epsilon$ tends to a non trivial field $\mathcal{X}$ as ϵ tends to 0. If one chooses X^ϵ independent of dW_0, we will see that this leads to a field $\mathcal{X}$ that extends the model introduced by Bacry in [**3**] and that has symmetrical increments. Thus, to obtain nonsymmetrical increments, we must introduce correlation between X^ϵ and dW_0.

2.3. Notations and construction of the family X^ϵ. Let k^R be the function

$$k^R(x)=\begin{cases}\frac{1}{|x|^{d/2}} & \text{for } |x|\leqslant R,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\theta(x)$ be a C^∞, non negative and radialy symmetrical function with compact support in $|x|\leqslant 1$ such that

$$\int_{\mathbb{R}^d}\theta(x)dx=1.$$

We define $\theta^\epsilon=\frac{1}{\epsilon^d}\theta(\frac{\cdot}{\epsilon})$ and the corresponding convolutions:

$$k^R_\epsilon=\theta^\epsilon * k^R, \qquad |.|_\epsilon=\theta^\epsilon * |.|.$$

Let γ be a strictly positive parameter and dW be a gaussian white noise on $\mathbb{R}^d$. We consider the following gaussian field:

$$X^\epsilon(y) = \gamma \int_{\mathbb{R}^d} k_\epsilon^R(y-\sigma)dW(\sigma).$$

Its correlation kernel is given by:

$$E(X^\epsilon(x)X^\epsilon(y)) = \gamma^2 \rho_{\epsilon/R}(\frac{x-y}{R}),$$

where $\rho = k^1 * k^1$ and $\rho_\epsilon = \theta^\epsilon * \theta^\epsilon * \rho$. One can prove the following expansion (cf. lemma 5.3 in the appendix):

$$\rho(x) = \omega_d \ln^+ \frac{1}{|x|} + \phi(x),$$

where ω_d denotes the surface of the unit sphere in $\mathbb{R}^d$ and ϕ is a continuous function that vanishes for $|x| \geqslant 2$. We will note $|.|_* = 1 \wedge |.|$ and, with this definition, the previous expansion is equivalent to:

$$e^{\rho(x)} = \frac{e^{\phi(x)}}{|x|_*^{\omega_d}}.$$

One can also prove the following expansions with respect to ϵ (cf. lemma 5.4 in the appendix):

$$k_\epsilon^R(0) = \frac{C_0}{\epsilon^{d/2}} \tag{2.4}$$

with $C_0 = \int_{|u| \leqslant 1} \frac{\theta(u)}{u^{d/2}} du$ and there exists a constant C_1 such that (cf. lemma 5.5 in the appendix):

$$\rho_{\epsilon/R}(0) = \omega_d \ln \frac{R}{\epsilon} + C_1 + o(\epsilon). \tag{2.5}$$

In the sequel, we will consider the case

$$\gamma dW = \gamma_0(\epsilon) dW_0 + \gamma_1 dW_1,$$

where dW_1 is a white noise independant of dW_0 and $\gamma_0(\epsilon)$ is a function of ϵ that will be defined later. Note that the integral in formula (2.3) has a meaning since dW_0 can be viewed as a random distribution.

2.4. Preliminary technical results. We remind the following integration by parts formula for gaussian vectors (cf. lemma 1.2.1 in [**48**]):

LEMMA 2.1. *Let* $(g, g_1, \dots, g_n)$ *be a centered gaussian vector and* $G : \mathbb{R}^n \to \mathbb{R}$ *a* C^1 *function such that its partial derivatives have at most exponential growth. Then we have:*

$$E(gG(g_1, \dots, g_n)) = \sum_{i=1}^{n} E(gg_i)E(\frac{\partial G}{\partial x_i}(g_1, \dots, g_n)). \tag{2.6}$$

From the above formula, one can easily deduce by induction the following lemma which will be frequently used in the sequel:

LEMMA 2.2. *Let* $l \in N^*$ *be some positive integer and* $(g, g_1, \dots, g_{2l})$ *a centered gaussian vector. Then:*

$$E(g_1 \dots g_{2l} e^g) = (\sum_{k=0}^{l} S_{k,l}) e^{\frac{1}{2}E(g^2)},$$

where

$$S_{k,l} = \sum_{\{i_1,\dots,i_{2k}\} \subset \{1,\dots,2l\}} \sum E(gg_{i_1}) \dots E(gg_{i_{2k}}) E(g_{i_{2k+1}} g_{i_{2k+2}}) \dots E(g_{i_{2l-1}} g_{i_{2l}}),$$

where the second sum is taken over all partitions of $\{1, \dots, 2l\}/\{i_1, \dots, i_{2k}\}$ *in subsets of two elements* $\{i_{2k+1}, i_{2k+2}\}$.

Similarly, we get the following formula:

$$E(g_1 \dots g_{2l+1} e^g) = (\sum_{k=0}^{l} \widetilde{S}_{k,l}) e^{\frac{1}{2}E(g^2)},$$

where

$$\widetilde{S}_{k,l} = \sum_{\{i_1,\dots,i_{2k+1}\} \subset \{1,\dots,2l+1\}} \sum E(gg_{i_1}) \dots E(gg_{i_{2k+1}}) E(g_{i_{2k+2}} g_{i_{2k+3}}) \dots E(g_{i_{2l}} g_{i_{2l+1}}),$$

REMARK 2.3. *In* $S_{k,l}$ $(\widetilde{S}_{k,l})$, *the summation is made of* $\frac{2l!}{2k!2^{l-k}(l-k)!}$ $(\frac{(2l+1)!}{(2k+1)!2^{l-k}(l-k)!})$ *terms, number we will denote by* $\alpha_{k,l}$ $(\widetilde{\alpha}_{k,l})$.

We will also use the following lemma essentially due to Kahane ([**32**]).

LEMMA 2.4. *Let* (T, d) *be a metric space and* σ *a finite positive measure on* T *equiped with the borelian* σ*-field induced by* d.

Let $q : T\times T :\to \mathbb{R}_+$ *a symmetric application and* m *a positive integer. Then we have the following inequalities:*

$$\int_{T^{2m}} e^{\sum_{1\leqslant j<k\leqslant 2m} q(t_j,t_k)} d\sigma(t_1)\dots d\sigma(t_{2m}) \leqslant \sigma(T)(\sup_{s\in T}\int_T e^{mq(t,s)}d\sigma(t))^{2m-1}, \quad (2.7)$$

$$\begin{aligned}&\int_{T^{2m+1}} e^{\sum_{1\leqslant j<k\leqslant 2m+1} q(t_j,t_k)} d\sigma(t_1)\dots d\sigma(t_{2m+1})\\ &\leqslant \sigma(T)\sup_{s,\widetilde{s}}(\int_T e^{q(\widetilde{s},t)}d\sigma(t))(\int_T e^{mq(s,t)}e^{q(\widetilde{s},t)}d\sigma(t))^{2m-1}.\end{aligned} \quad (2.8)$$

Proof. The proof of (2.7) can be found in [**32**]. Thus we just prove how to derive inequality (2.8) from (2.7). By integrating with respect to the first $2m$ variables, we get:

$$\begin{aligned}&\int_{T^{2m+1}} e^{\sum_{1\leqslant j<k\leqslant 2m+1} q(t_j,t_k)} d\sigma(t_1)\dots d\sigma(t_{2m+1})\\ &= \int_T d\sigma(t_{2m+1})\int_{T^{2m}} e^{\sum_{1\leqslant j<k\leqslant 2m} q(t_j,t_k)}\prod_{j=1}^{2m} e^{q(t_j,t_{2m+1})}d\sigma(t_1)\dots d\sigma(t_{2m})\\ &\underset{(2.7)}{\leqslant} \int_T d\sigma(t_{2m+1})(\int_T e^{q(t,t_{2m+1})}d\sigma(t))(\sup_s\int_T e^{mq(s,t)}e^{q(t,t_{2m+1})}d\sigma(t))^{2m-1}\\ &\leqslant \sigma(T)\sup_{s,\widetilde{s}}(\int_T e^{q(\widetilde{s},t)}d\sigma(t))(\int_T e^{mq(s,t)}e^{q(\widetilde{s},t)}d\sigma(t))^{2m-1}.\end{aligned}$$

3. The field $\mathcal{X}$

In this section, we will suppose that $d/2<\alpha<(d/2+1)\wedge d$ **and** $\omega_d\gamma_1^2<d$. We consider the field $\mathcal{X}^\epsilon$ defined by formula (2.3) with

$$X^\epsilon(y)=\gamma_0(\epsilon)X_0^\epsilon(y)+\gamma_1X_1^\epsilon(y),$$

where

$$X_i^\epsilon(y)=\int_{\mathbb{R}^d}k_\epsilon^R(y-\sigma)dW_i(\sigma),\qquad i=0,1.$$

We set also

$$C_\epsilon=((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0).$$

and

$$\gamma_0(\epsilon)=\gamma_0^*(\frac{\epsilon}{R})^{\frac{d-\omega_d\gamma_1^2}{2}}.$$

Therefore, we introduce a slight correlation between X^ϵ and dW_0 ($\gamma_0(\epsilon)$ tends to 0 as ϵ goes to 0).

3.1. Multiplicative chaos in dimension d. Multiplicative chaos or the "limit-lognormal" model introduced by Mandelbrot is a generalization of the exponential of a gaussian process. As mentioned in the introduction, it was defined rigorously by Kahane in [**32**]. The construction of Kahane was based on the theory of martingales and thus the generalized correlation kernel (here $\rho(t-s)$) had to verify a condition hard to verify practically (the σ-positivity condition). Our construction is based on L^2-theory and can be carried out without this condition.

Let γ_1 be some real number such that $\gamma_1^2\omega_d < d$ and ϵ a positive number. Let $\mathcal{B}(\mathbb{R}^d)$ denote the standard borelian σ-field; we want to consider the limit as ϵ goes to 0 of the random measures Q^{ϵ,γ_1} defined by:

$$\begin{aligned} Q^{\epsilon,\gamma_1}(dy) &= e^{\gamma_1 X_1^\epsilon(y)-\frac{1}{2}E((X_1^\epsilon(y))^2)}dy \\ &= e^{\gamma_1 X_1^\epsilon(y)-\frac{1}{2}\gamma_1^2\rho_{\epsilon/R}(0)}dy. \end{aligned} \tag{3.1}$$

This leads us to state the following proposition:

PROPOSITION 3.1 (Multiplicative chaos of order γ_1). *There exists a positive random measure $Q^{\gamma_1}(dy)$ independent of the regularizing function θ such that:*

1. *for all A bounded in $\mathcal{B}(\mathbb{R}^d)$, $E(Q^{\gamma_1}(A)) = |A|$.*
2. *Q^{γ_1} has almost surely no atoms.*
3. *Q^{γ_1} is almost surely singular with respect to the Lebesgue measure.*

If q is some positive integer and $f:\mathbb{R}^d\to\mathbb{R}$ a deterministic function that satisfies the following condition:

$$\int_{(\mathbb{R}^d)^{2q}} |f(y_1)|\dots|f(y_{2q})| \prod_{1\leqslant i<j\leqslant 2q} \frac{1}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} dy_1\dots dy_{2q} < \infty, \tag{3.2}$$

then we have the following convergence:

$$\int_{\mathbb{R}^d} f(y)Q^{\epsilon,\gamma_1}(dy) \underset{\epsilon\to 0}{\overset{L^{2q}}{\to}} \int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy).$$

We also have the following expression for the moments of $\int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy)$:

$$\forall k\leqslant 2q,\quad E((\int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy))^k) = \int_{(\mathbb{R}^d)^k} f(y_1)\dots f(y_k) \prod_{1\leqslant i<j\leqslant k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} dy_1\dots dy_k. \tag{3.3}$$

We will call $Q^{\gamma_1}(dy)$ multiplicative chaos of order γ_1.

Proof. We first start by considering a positive integer and a function f that satisfies the corresponding integrability condition (3.2). Let ϵ, ϵ' be two positive numbers. By using Fubini, we get for all $j \leqslant 2q$:

$$E((\int_{\mathbb{R}^d} f(y)Q^{\epsilon,\gamma_1}(dy))^j(\int_{\mathbb{R}^d} f(y)Q^{\epsilon',\gamma_1}(dy))^{2q-j})$$
$$= e^{-\frac{j}{2}\gamma_1^2\rho_{\epsilon/R}(0)-\frac{2q-j}{2}\gamma_1^2\rho_{\epsilon'/R}(0)}\int_{(\mathbb{R}^d)^{2q}} f(y_1)\dots f(y_{2q})\times$$
$$e^{\frac{1}{2}\gamma_1^2 E((\sum_{i=1}^{j}\mathcal{X}_1^{\epsilon}(y_i)+\sum_{i=j+1}^{2q}\mathcal{X}_1^{\epsilon'}(y_i))^2)}dy_1\dots dy_{2q}$$
$$\underset{\epsilon,\epsilon'\to 0}{\to}\int_{(\mathbb{R}^d)^{2q}} f(y_1)\dots f(y_{2q})\prod_{1\leqslant i<j\leqslant 2q}\frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}}dy_1\dots dy_{2q},$$

From this, we deduce that:

$$E((\int_{\mathbb{R}^d} f(y)Q^{\epsilon,\gamma_1}(dy)-\int_{\mathbb{R}^d} f(y)Q^{\epsilon',\gamma_1}(dy))^{2q})\underset{\epsilon,\epsilon'\to 0}{\to} 0$$

and therefore that $\int_{\mathbb{R}^d} f(y)Q^{\epsilon,\gamma_1}(dy)$ is a Cauchy sequence in L^{2q} that converges to some random variable $\widetilde{Q}^{\gamma_1}(f)$. For $k \leqslant 2q$, the moment $E((\widetilde{Q}^{\gamma_1}(f))^k)$ is the limit as ϵ goes to 0 of $E((Q^{\epsilon,\gamma_1}(f))^k)$; from this one can deduce that the moments of $\widetilde{Q}^{\gamma_1}(f)$ are given by formula (3.3).

Since $\gamma_1^2\omega_d < d$, for any bounded set A in $\mathcal{B}(\mathbb{R}^d)$, we deduce from the proof above and lemma 2.4 that $Q^{\epsilon,\gamma_1}(A)$ converges in L^2 to some random variable $\widetilde{Q}^{\gamma_1}(A)$. This defines a family of random variables (indexed by the bounded borelian sets) that satisfies the following properties:

(1) For all disjoint and bounded sets A_1, A_2 in $\mathcal{B}(\mathbb{R}^d)$,

$$\widetilde{Q}^{\gamma_1}(A_1\cup A_2)=\widetilde{Q}^{\gamma_1}(A_1)+\widetilde{Q}^{\gamma_1}(A_2)\qquad a.s.$$

(2) For any bounded sequence $(A_n)_{n\geqslant 1}$ decreasing to $\varnothing$:

$$\widetilde{Q}^{\gamma_1}(A_n)\underset{n\to\infty}{\to} 0\quad a.s.$$

By theorem 6.1.VI. in [**17**], there exists a random measure Q^{γ_1} such that for all bounded A in $\mathcal{B}(\mathbb{R}^d)$ we have:

$$Q^{\gamma_1}(A)=\widetilde{Q}^{\gamma_1}(A)\quad a.s.$$

Property (2) is an immediate consequence of corollary 6.3.VI. in [**17**] and for property (3), we refer to [**32**]. Finally, one can easily show that the limit random variable $\widetilde{Q}^{\gamma_1}(f)$ is almost surely equal to $\int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy)$. □

3.2. Convergence towards a field $\mathcal{X}$. In this subsection, we will prove the following proposition:

Proposition 3.2. *Let α be such that $d/2 < \alpha < (d/2+1) \wedge d$ and γ_1 such that $2\gamma_1^2\omega_d < \alpha - d/2$. There exists a field $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$ such that for all k and $x_1,\dots,x_k \in \mathbb{R}^d$ the following convergence in law holds:*

$$(\mathcal{X}^\epsilon(x_1),\dots,\mathcal{X}^\epsilon(x_k)) \underset{\epsilon\to 0}{\Rightarrow} (\mathcal{X}(x_1),\dots,\mathcal{X}(x_k)). \tag{3.4}$$

Let l be an integer such that one of the following conditions hold:

1. *l is even and $l\gamma_1^2\omega_d < \alpha - d/2$.*
2. *l is odd and $(l+1)\gamma_1^2\omega_d < \alpha - d/2$.*

Then there exists C such that, for all x in $\mathbb{R}^d$, the random variable $\mathcal{X}(x)$ has a moment of order $2l$ given by the following expression:

$$E((\mathcal{X}(x))^{2l}) = \sum_{k=0}^{l} \alpha_{k,l} C^{2k} \int_{(\mathbb{R}^d)^{k+l}} \frac{\varphi_R(y_1)}{|y_1|^{d-\alpha}} \cdots \frac{\varphi_R(y_{2k})}{|y_{2k}|^{d-\alpha}} \frac{\varphi_R(y_{2k+1})^2}{|y_{2k+1}|^{2(d-\alpha)}} \cdots \frac{\varphi_R(y_{k+l})^2}{|y_{k+l}|^{2(d-\alpha)}}$$
$$\prod_{1\leqslant i<j\leqslant 2k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} \prod_{\substack{1\leqslant i\leqslant 2k\\ j>2k}} \frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}} \prod_{2k+1\leqslant i<j\leqslant k+l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_1\dots dy_{k+l}. \tag{3.5}$$

We also have:

$$E((\mathcal{X}(x+h)-\mathcal{X}(x))^{2l}) = \sum_{k=0}^{l} \alpha_{k,l} C^{2k} \int_{(\mathbb{R}^d)^{k+l}} (\frac{\varphi_R(y_1-h)}{|y_1-h|^{d-\alpha}} - \frac{\varphi_R(y_1)}{|y_1|^{d-\alpha}}) \cdots$$
$$(\frac{\varphi_R(y_{2k}-h)}{|y_{2k}-h|^{d-\alpha}} - \frac{\varphi_R(y_{2k})}{|y_{2k}|^{d-\alpha}})(\frac{\varphi_R(y_{2k+1}-h)}{|y_{2k+1}-h|^{d-\alpha}} - \frac{\varphi_R(y_{2k+1})}{|y_{2k+1}|^{d-\alpha}})^2 \cdots (\frac{\varphi_R(y_{k+l}-h)}{|y_{k+l}-h|^{d-\alpha}} - \frac{\varphi_R(y_{k+l})}{|y_{k+l}|^{d-\alpha}})^2$$
$$\prod_{1\leqslant i<j\leqslant 2k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} \prod_{\substack{1\leqslant i\leqslant 2k\\ j>2k}} \frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}} \prod_{2k+1\leqslant i<j\leqslant k+l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_1\dots dy_{k+l}. \tag{3.6}$$

Proof. Let γ_1 be such that $2\gamma_1^2\omega_d < \alpha - d/2$. We set

$$C = \frac{\gamma_0^* C_0 e^{-1/2\gamma_1^2 C_1}}{R^{d/2}}.$$

and define two auxiliary fields $\mathcal{Y}^\epsilon, \mathcal{Z}^\epsilon$ by the following expressions:

$$\mathcal{Y}^\epsilon(x) = \int_{\mathbb{R}^d} \frac{\varphi_R(x-y)}{|x-y|^{d-\alpha}} Q^{\epsilon,\gamma_1}(dy) \tag{3.7}$$

and

$$\mathcal{Z}^\epsilon(x)=\int_{\mathbb{R}^d}\frac{\varphi_R(x-y)}{|x-y|^{d-\alpha}}e^{\gamma_1 X_1^\epsilon(y)-C_\epsilon}dW_0(y). \tag{3.8}$$

Note that $\mathcal{Z}^\epsilon(x)$ exists since X_1^ϵ and dW_0 are independent with:

$$E(\int_{\mathbb{R}^d}\frac{\varphi_R(x-y)^2}{|x-y|^{2(d-\alpha)}}e^{2\gamma_1 X_1^\epsilon(y)-2C_\epsilon}dy)<\infty. \tag{3.9}$$

We can compute, for all x in $\mathbb{R}^d$, $E((\mathcal{X}^\epsilon(x)-C\mathcal{Y}^\epsilon(x)-\mathcal{Z}^\epsilon(x))^2)$ (cf. the more complicated computations in the proof of proposition 3.7) and derive the following limit:

$$\mathcal{X}^\epsilon(x)-(C\mathcal{Y}^\epsilon(x)+\mathcal{Z}^\epsilon(x))\underset{\epsilon\to 0}{\overset{L^2}{\to}}0.$$

Thus, we must show that the finite dimensional distributions of the field $C\mathcal{Y}^\epsilon+\mathcal{Z}^\epsilon$ converge in law. Let k be some positive integer and $x_1,\dots,x_k$ points in $\mathbb{R}^d$. For all $\xi=(\xi_1,\dots,\xi_k)$ in $\mathbb{R}^k$, by conditioning on the field generated by the white noise dW_1 and using proposition 3.1, we get:

$$\begin{aligned}
&E(e^{i\xi.(C\mathcal{Y}^\epsilon(x_1)+\mathcal{Z}^\epsilon(x_1),\dots,C\mathcal{Y}^\epsilon(x_k)+\mathcal{Z}^\epsilon(x_k))})\\
&=E(e^{iC\int_{\mathbb{R}^d}(\sum_{i=1}^k\xi_i\frac{\varphi_R(x_i-y)}{|x_i-y|^{d-\alpha}})Q^{\epsilon,\gamma_1}(dy)-\frac{1}{2}e^{2\gamma_0(\epsilon)^2\rho_\epsilon/R(0)}\int_{\mathbb{R}^d}(\sum_{i=1}^k\xi_i\frac{\varphi_R(x_i-y)}{|x_i-y|^{d-\alpha}})^2Q^{\epsilon,2\gamma_1}(dy)})\\
&\underset{\epsilon\to 0}{\to}E(e^{iC\int_{\mathbb{R}^d}(\sum_{i=1}^k\xi_i\frac{\varphi_R(x_i-y)}{|x_i-y|^{d-\alpha}})Q^{\gamma_1}(dy)-\frac{1}{2}\int_{\mathbb{R}^d}(\sum_{i=1}^k\xi_i\frac{\varphi_R(x_i-y)}{|x_i-y|^{d-\alpha}})^2Q^{2\gamma_1}(dy)}).
\end{aligned}$$

Thus, by applying Levy's theorem, we conclude that the finite dimensional distributions of the field $C\mathcal{Y}^\epsilon+\mathcal{Z}^\epsilon$ converge in law to those of a field $\mathcal{X}$. We also get from the proof above the characteristic function of $(\mathcal{X}(x_1),\dots,\mathcal{X}(x_k))$:

$$E(e^{i\xi.(\mathcal{X}(x_1),\dots,\mathcal{X}(x_k))})=E(e^{iC\int_{\mathbb{R}^d}(\sum_{i=1}^k\xi_i\frac{\varphi_R(x_i-y)}{|x_i-y|^{d-\alpha}})Q^{\gamma_1}(dy)-\frac{1}{2}\int_{\mathbb{R}^d}(\sum_{i=1}^k\xi_i\frac{\varphi_R(x_i-y)}{|x_i-y|^{d-\alpha}})^2Q^{2\gamma_1}(dy)}). \tag{3.10}$$

Suppose l is a positive integer that satisfies the condition of the proposition. By applying (3.10), we get for all ξ in $\mathbb{R}$:

$$E(e^{i\xi.\mathcal{X}(x)})=E(e^{iC\xi\int_{\mathbb{R}^d}\frac{\varphi_R(x-y)}{|x-y|^{d-\alpha}}Q^{\gamma_1}(dy)-\frac{1}{2}\xi^2\int_{\mathbb{R}^d}(\frac{\varphi_R(x-y)}{|x-y|^{d-\alpha}})^2Q^{2\gamma_1}(dy)}).$$

We derive expression (3.5) by computing $\frac{\partial^{2l}E(e^{i\xi.\mathcal{X}(x)})}{\partial\xi^{2l}}|_{\xi=0}$ thanks to the above formula and proposition 3.1. We derive expression (3.6) similarly.

3.3. Scaling of $\mathcal{X}$ and tightness of $\mathcal{X}^\epsilon$. The purpose of this subsection is to show that the field $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$ satisfies the multifractal scaling relation (this is what propositions 3.5 and 3.6 below assert) and to study the tightness of the family $\mathcal{X}^\epsilon$.

We first state two preliminary lemmas we will use in the rest of the paper.

LEMMA 3.3. *Let δ be some real number such that $0 \leqslant \delta < \alpha$ and $\delta \neq \alpha - 1$. There exists $C = C(\delta)$ such that we have the following inequality for $|h| \leqslant R$:*

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left|\frac{\varphi_R(y-h)}{|y-h|^{d-\alpha}} - \frac{\varphi_R(y)}{|y|^{d-\alpha}}\right| \frac{1}{|\frac{x-y}{R}|_*^\delta} dy \leqslant R^{d/2} C \left|\frac{h}{R}\right|^{(\alpha-\delta)\wedge 1}. \tag{3.11}$$

PROOF. By homogenity, we suppose that $R = 1$ and for simplicity, we suppose $d \geqslant 2$. Since $\frac{1}{|x|_*} \leqslant \frac{1}{|x|} + 1$, we have to show that for $\delta \in [0, \alpha[$:

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left|\frac{\varphi(y-h)}{|y-h|^{d-\alpha}} - \frac{\varphi(y)}{|y|^{d-\alpha}}\right| \frac{1}{|x-y|^\delta} dy \leqslant C|h|^{(\alpha-\delta)\wedge 1}.$$

There exists C such that for all y and h, we have:

$$|\varphi(y-h) - \varphi(y)| \leqslant C|h| \text{ and } \varphi(y) \leqslant C 1_{|y|\leqslant 2}. \tag{3.12}$$

We set

$$I(x) = \int_{\mathbb{R}^d} \left|\frac{\varphi(y-h)}{|y-h|^{d-\alpha}} - \frac{\varphi(y)}{|y|^{d-\alpha}}\right| \frac{1}{|x-y|^\delta} dy.$$

Therefore we get

$$\begin{aligned} I(x) &\leqslant C|h| \int_{|y|\leqslant 3} \frac{1}{|y|^{d-\alpha}} \frac{1}{|x-y|^\delta} dy \qquad (3.13)\\ &\quad + C \int_{|y|\leqslant 3} \left|\frac{1}{|y-h|^{d-\alpha}} - \frac{1}{|y|^{d-\alpha}}\right| \frac{1}{|x-y|^\delta} dy \\ &\leqslant C|h| + C \int_{|y|\leqslant 3} \left|\frac{1}{|y-h|^{d-\alpha}} - \frac{1}{|y|^{d-\alpha}}\right| \frac{1}{|x-y|^\delta} dy. \qquad (3.14) \end{aligned}$$

First case: $\delta < \alpha - 1$.

Plugging inequality

$$\left|\frac{1}{|y-h|^{d-\alpha}} - \frac{1}{|y|^{d-\alpha}}\right| \leqslant \frac{(d-\alpha)|h|}{|y-h|^{d-\alpha+1} \wedge |y|^{d-\alpha+1}}$$

in (3.14), we get

$$I(x) \leqslant C|h| \int_{|y|\leqslant 3} \frac{1}{|y-h|^{d-\alpha+1} \wedge |y|^{d-\alpha+1}} \frac{1}{|x-y|^\delta} dy,$$

which concludes the proof.

Second case: $\delta > \alpha - 1$.

By the change of variable $y = |h|u$ and setting $h = |h|e$ with $|e| = 1$, we get:

$$\begin{aligned}
&\int_{|y| \leqslant 3} \Big|\frac{1}{|y-h|^{d-\alpha}} - \frac{1}{|y|^{d-\alpha}}\Big| \frac{1}{|x-y|^{\delta}} dy \\
&= |h|^{\alpha-\delta} \int_{|u| \leqslant \frac{3}{|h|}} \Big|\frac{1}{|u-e|^{d-\alpha}} - \frac{1}{|u|^{d-\alpha}}\Big| \frac{1}{|x/|h| - u|^{\delta}} du \\
&\leqslant |h|^{\alpha-\delta} \sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \Big|\frac{1}{|u-e|^{d-\alpha}} - \frac{1}{|u|^{d-\alpha}}\Big| \frac{1}{|a-u|^{\delta}} du.
\end{aligned}$$

□

LEMMA 3.4. *Let δ be some real number such that $0 \leqslant \delta < 2\alpha - d$ or, if $d = 1$ and $\alpha > 1$, $\delta < 1$. There exists $C = C(\delta)$ such that we have the following inequality for $|h| \leqslant R$:*

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \Big|\frac{\varphi_R(y-h)}{|y-h|^{d-\alpha}} - \frac{\varphi_R(y)}{|y|^{d-\alpha}}\Big|^2 \frac{1}{|\frac{x-y}{R}|_*^{\delta}} dy \leqslant C \Big|\frac{h}{R}\Big|^{2\alpha-d-\delta}. \tag{3.15}$$

PROOF. As in the proof above, we can replace $|.|_*$ by $|.|$ and suppose that $R = 1$; thus we have to show inequality (3.15) with $J(x)$ where we set:

$$J(x) = \int_{\mathbb{R}^d} \Big|\frac{\varphi(y-h)}{|y-h|^{d-\alpha}} - \frac{\varphi(y)}{|y|^{d-\alpha}}\Big|^2 \frac{1}{|x-y|^{\delta}} dy.$$

Using inequality (3.12), we get

$$\begin{aligned}
J(x) &\leqslant C|h|^2 \int_{|y| \leqslant 3} \frac{1}{|y|^{2(d-\alpha)}} \frac{1}{|x-y|^{\delta}} dy \\
&\quad + C \int_{|y| \leqslant 3} \Big|\frac{1}{|y-h|^{d-\alpha}} - \frac{1}{|y|^{d-\alpha}}\Big|^2 \frac{1}{|x-y|^{\delta}} dy \\
&\leqslant C|h|^2 + C \int_{|y| \leqslant 3} \Big|\frac{1}{|y-h|^{d-\alpha}} - \frac{1}{|y|^{d-\alpha}}\Big|^2 \frac{1}{|x-y|^{\delta}} dy.
\end{aligned} \tag{3.16}$$

Since $2 > 2\alpha - d - \delta$, we only have to consider the second term in inequality (3.16). By the change of variable $y = |h|u$ and setting $h = |h|e$ with $|e| = 1$, we get:

$$\begin{aligned}
&\int_{|y| \leqslant 3} \Big|\frac{1}{|y-h|^{d-\alpha}} - \frac{1}{|y|^{d-\alpha}}\Big|^2 \frac{1}{|x-y|^{\delta}} dy \\
&= |h|^{2\alpha-d-\delta} \int_{|u| \leqslant \frac{3}{|h|}} \Big|\frac{1}{|u-e|^{d-\alpha}} - \frac{1}{|u|^{d-\alpha}}\Big|^2 \frac{1}{|x/|h| - u|^{\delta}} du \\
&\leqslant |h|^{2\alpha-d-\delta} \sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \Big|\frac{1}{|u-e|^{d-\alpha}} - \frac{1}{|u|^{d-\alpha}}\Big|^2 \frac{1}{|a-u|^{\delta}} du.
\end{aligned}$$

□

Proposition 3.5. *(Scaling along the even integers)*

Let l be an integer such that one of the following conditions hold:

(1) *l is even and $l\gamma_1^2\omega_d < \alpha - d/2$*
(2) *l is odd and $(l+1)\gamma_1^2\omega_d < \alpha - d/2$.*

Let e be a unit vector ($|e| = 1$). Then there exists $C_l \neq 0$ independent of e such that the following scaling relation holds:

$$E((\mathcal{X}(x+\lambda e) - \mathcal{X}(x))^{2l}) \underset{\lambda \to 0}{\sim} C_l (\frac{\lambda}{R})^{\zeta_{2l}}, \tag{3.17}$$

where we have

$$\zeta_{2l} = l(2\alpha - d) - 2\gamma_1^2\omega_d l(l-1). \tag{3.18}$$

Proof. For simplicity, we will suppose that l is even and that $l\gamma_1^2\omega_d < \alpha - d/2$. We introduce the following notation:

$$f_h(y) = \frac{\varphi_R(y-h)}{|y-h|^{d-\alpha}} - \frac{\varphi_R(y)}{|y|^{d-\alpha}}.$$

We shall see that the scaling at small scale of the sum (3.6) is given by the term $k = 0$. Indeed for all $k \geqslant 1$ let us consider the integral

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^{k+l}} f_h(y_1)\dots f_h(y_{2k})(f_h(y_{2k+1}))^2\dots(f_h(y_{k+l}))^2 \prod_{1 \leqslant i<j \leqslant 2k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} \times \\
&\prod_{\substack{1 \leqslant i \leqslant 2k \\ j>2k}} \frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}} \prod_{2k+1 \leqslant i<j \leqslant k+l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_1\dots dy_{k+l}. \\
&\leqslant I_k J_{k,l},
\end{aligned} \tag{3.19}$$

where we set

$$\begin{aligned}
I_k = \sup_{y_{2k+1},\dots,y_{k+l}} &\int_{(\mathbb{R}^d)^{2k}} f_h(y_1)\dots f_h(y_{2k}) \prod_{\substack{1 \leqslant i \leqslant 2k \\ j>2k}} \frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}} \times \\
&\prod_{1 \leqslant i<j \leqslant 2k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} dy_1\dots dy_{2k}.
\end{aligned}$$

and

$$J_{k,l} = \int_{(\mathbb{R}^d)^{l-k}} (f_h(y_{2k+1}))^2\dots(f_h(y_{k+l}))^2 \prod_{2k+1 \leqslant i<j \leqslant k+l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_{2k+1}\dots dy_{k+l}.$$

By using the estimates (3.11),(3.15) and the inequalities (2.7), (2.8), one can show that for all $k \geqslant 1$, we have

$$I_k J_{k,l} \leqslant C R^{dk} |\frac{h}{R}|^{c_{k,l}},$$

with

$$c_{k,l} = (\alpha - 2(l-k)\gamma_1^2\omega_d)\wedge 1 + ((\alpha-(2l-k)\gamma_1^2\omega_d)\wedge 1)(2k-1) + (2\alpha - d)(l-k) - 2\gamma_1^2\omega_d(l-k)(l-k-1).$$

If $\alpha - 2(l-k)\gamma_1^2\omega_d < 1$, then $c_{k,l} = \zeta_{2l} + k(d - \gamma_1^2\omega_d)$; If $\alpha - 2(l-k)\gamma_1^2\omega_d \geqslant 1$ and $\alpha - (2l-k)\gamma_1^2\omega_d < 1$, then $c_{k,l} = \zeta_{2l} + 1 - \alpha + dk + (2l - 3k)\gamma_1^2\omega_d$; otherwise $c_{k,l} = 2k + (2\alpha - d)(l-k) - 2\gamma_1^2\omega_d(l-k)(l-k-1)$. In all cases, it is easy to show that $c_{k,l} > \zeta_{2l}$ under the conditions of the proposition..

Finally, we study the term where $k = 0$. We get for $h = \lambda e$ with $|e| = 1$:

$$\int_{(\mathbb{R}^d)^l} (f_h(y_1))^2 \dots (f_h(y_l))^2 \prod_{1 \leqslant i<j \leqslant l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_1 \dots dy_l.$$

$$\underset{y_i=\lambda u_i}{=} (\frac{\lambda}{R})^{l(2\alpha-d)} \int_{(\mathbb{R}^d)^l} (\frac{\varphi_R(\lambda(u_1-e))}{|u_1-e|^{d-\alpha}} - \frac{\varphi_R(\lambda u_1)}{|u_1|^{d-\alpha}})^2 \dots (\frac{\varphi_R(\lambda(u_l-e))}{|u_l-e|^{d-\alpha}} - \frac{\varphi_R(\lambda u_l)}{|u_l|^{d-\alpha}})^2 \times$$

$$\prod_{1 \leqslant i<j \leqslant l} \frac{e^{4\gamma_1^2\phi(\frac{\lambda(u_i-u_j)}{R})}}{|\frac{\lambda(u_i-u_j)}{R}|_*^{4\gamma_1^2\omega_d}} du_1 \dots du_l.$$

$$\underset{\lambda\to 0}{\sim} e^{2l(l-1)\gamma_1^2\phi(0)} (\frac{\lambda}{R})^{\zeta_{2l}} \int_{(\mathbb{R}^d)^l} (\frac{1}{|u_1-e|^{d-\alpha}} - \frac{1}{|u_1|^{d-\alpha}})^2 \dots (\frac{1}{|u_l-e|^{d-\alpha}} - \frac{1}{|u_l|^{d-\alpha}})^2 \times$$

$$\prod_{1 \leqslant i<j \leqslant l} \frac{1}{|u_i-u_j|^{4\gamma_1^2\omega_d}} du_1 \dots du_l,$$

and inequality (2.7) shows that this integral is finite when $l\gamma_1^2\omega_d < \alpha - d/2$.

□

In the next proposition, we state the scaling relations of $\mathcal{X}$ along the odd integers.

Proposition 3.6. *(Scaling along the odd integers) We suppose in this proposition that $\alpha > 2$ and thus $d \geqslant 3$. Let l be an integer such that $l+1$ satisfies one of the conditions in proposition 3.5.*

Let e be a unit vector ($|e| = 1$). Then there exists C_l (with $C_3 \neq 0$) independent of e such that the following scaling relation holds:

$$E((\mathcal{X}(x+\lambda e) - \mathcal{X}(x))^{2l+1}) \underset{\lambda\to 0}{\sim} \gamma_0^* R^{d/2} C_l (\frac{\lambda}{R})^{\widetilde{\zeta}_{2l+1}}, \tag{3.20}$$

where we have

$$\widetilde{\zeta}_{2l+1} = l(2\alpha - d) - 2\gamma_1^2\omega_d l(l-1) + 2. \tag{3.21}$$

PROOF. As in proposition 3.2, setting $C=\frac{\gamma_0^* C_0 e^{-1/2\gamma_1^2 C_1}}{R^{d/2}}$, it is possible to show that:

$$
\begin{aligned}
&E((\mathcal{X}(x+h)-\mathcal{X}(x))^{2l+1})\\
&=\sum_{k=0}^{l}\widetilde{\alpha}_{k,l}C^{2k+1}\int_{(\mathbb{R}^d)^{k+l+1}} f_h(y_1)\dots f_h(y_{2k+1})(f_h(y_{2k+2}))^2\dots(f_h(y_{k+l+1}))^2\times\\
&\prod_{1\leqslant i<j\leqslant 2k+1}\frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}}\prod_{\substack{1\leqslant i\leqslant 2k+1\\ j>2k+1}}\frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}}\prod_{2k+2\leqslant i<j\leqslant k+l+1}\frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}}dy_1\dots dy_{k+l+1},
\end{aligned}
$$

where, as usual, we set

$$f_h(y)=\frac{\varphi_R(y-h)}{|y-h|^{d-\alpha}}-\frac{\varphi_R(y)}{|y|^{d-\alpha}}.$$

Similarly to proposition 3.5, to get the main contribution as $|h|$ goes to 0, we examine the term $k=0$. We set:

$$\psi(y)=\frac{\varphi(y)}{|y|^{d-\alpha}}.$$

Note that the condition $2l\gamma_1^2\omega_d<\alpha-2$ ensures that:

$$\int_{\mathbb{R}^d}|\frac{\partial^2\psi}{\partial y^i\partial y^j}|e^{2l\gamma_1^2\rho(y)}dy<\infty.$$

We get for $h=\lambda e$ with $|e|=1$ the following equivalent:

$$
\begin{aligned}
&\int_{(\mathbb{R}^d)^{l+1}} f_h(y_1)(f_h(y_2))^2\dots(f_h(y_{l+1}))^2\prod_{j\geqslant 2}\frac{e^{2\gamma_1^2\phi(\frac{y_1-y_j}{R})}}{|\frac{y_1-y_j}{R}|_*^{2\gamma_1^2\omega_d}}\times\\
&\prod_{2\leqslant i<j\leqslant l+1}\frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}}dy_1\dots dy_{l+1}.\\
&\underset{\substack{y_i=\lambda u_i, i\geqslant 2\\ y_1=Ry}}{=} R^{d/2}(\frac{\lambda}{R})^{l(2\alpha-d)}\int_{(\mathbb{R}^d)^{l+1}}(\psi(y-\frac{\lambda}{R}e)-\psi(y))\prod_{j\geqslant 2}(\frac{\varphi_R(\lambda(u_j-e))}{|u_j-e|^{d-\alpha}}-\frac{\varphi_R(\lambda u_j)}{|u_j|^{d-\alpha}})^2\times\\
&\prod_{j\geqslant 2}\frac{e^{2\gamma_1^2\phi(y-\frac{\lambda u_j}{R})}}{|y-\frac{\lambda u_j}{R}|_*^{2\gamma_1^2\omega_d}}\prod_{2\leqslant i<j\leqslant l+1}\frac{e^{4\gamma_1^2\phi(\frac{\lambda(u_i-u_j)}{R})}}{|\frac{\lambda(u_i-u_j)}{R}|_*^{4\gamma_1^2\omega_d}}dydu_2\dots du_{l+1}.\\
&\underset{\lambda\to 0}{\sim} R^{d/2}\widetilde{C}_l e^{2l(l-1)\gamma_1^2\phi(0)}(\frac{\lambda}{R})^{\widetilde{\zeta}_{2l+1}},
\end{aligned}
$$

with $\widetilde{C}_l = A_l B_l$ where

$$A_l = \sum_{1 \leqslant i,j \leqslant d} e_i e_j \int_{\mathbb{R}^d} \frac{\partial^2 \psi}{\partial y^i \partial y^j} e^{2l\gamma_1^2 \rho(y)} dy$$

and

$$B_l = \int_{(\mathbb{R}^d)^l} \Big(\frac{1}{|u_1 - e|^{d-\alpha}} - \frac{1}{|u_1|^{d-\alpha}}\Big)^2 \dots \Big(\frac{1}{|u_l - e|^{d-\alpha}} - \frac{1}{|u_l|^{d-\alpha}}\Big)^2 \times \prod_{2 \leqslant i<j \leqslant l+1} \frac{1}{|u_i - u_j|^{4\gamma_1^2 \omega_d}} du_2 \dots du_{l+1}.$$

A direct computation shows that $\widetilde{C}_3 \neq 0$ from which we deduce that for $\gamma_0^* \neq 0$ the distribution of $\mathcal{X}(x + \lambda e) - \mathcal{X}(x)$ is nonsymmetrical.

□

PROPOSITION 3.7. *(Tightness) Let l be some positive integer that satisfies the condition of proposition 3.5 and γ a positive parameter such that $\gamma_1^2 < \gamma^2$. Then there exists $\epsilon_0 > 0$ and C independent of ϵ such that for $\epsilon < \epsilon_0$ and $|h| \leqslant R$:*

$$\forall x, \qquad E((\mathcal{X}^\epsilon(x+h) - \mathcal{X}^\epsilon(x))^{2l}) \leqslant C|h|^{l(2\alpha-d)-2\gamma^2\omega_d l(l-1)}, \tag{3.22}$$

and

$$E((\mathcal{X}^\epsilon(0))^{2l}) \leqslant C. \tag{3.23}$$

PROOF. We only prove (3.22) (the proof of 3.23 is similar). We are going to compute the moment

$$E\Big(\Big(\int_{\mathbb{R}^d} f_{\epsilon,h}(y) e^{X^\epsilon(y)-C_\epsilon} dW_0(y)\Big)^{2l}\Big)$$

where we set

$$f_{\epsilon,h}(y) = \frac{\varphi_R(y-h)}{|y-h|_\epsilon^{d-\alpha}} - \frac{\varphi_R(y)}{|y|_\epsilon^{d-\alpha}}.$$

We get

$$E\Big(\Big(\int_{\mathbb{R}^d} f_{\epsilon,h}(y) e^{X^\epsilon(y)-C_\epsilon} dW_0(y)\Big)^{2l}\Big) = e^{-2lC_\epsilon} \int f_{\epsilon,h}(y_1) \dots f_{\epsilon,h}(y_{2l}) E(e^{\hat{X}^\epsilon} dW_0(y_1) \dots dW_0(y_{2l})), \tag{3.24}$$

where

$$\hat{X}^\epsilon = X^\epsilon(y_1) + \dots + X^\epsilon(y_{2l}).$$

The rest of the computation can be performed rigorously by regularizing the white noise dW_0, using lemma 2.2 and going to the limit. It is easy to see that we obtain the same result by introducing the following formal rules:

$$E(dW_0(y)dW_0(y')) = \delta_{y-y'}dy \tag{3.25}$$

and

$$E(dW_0(y)X^\epsilon(y')) = \gamma_0(\epsilon)k_\epsilon^R(y'-y)dy \tag{3.26}$$

As a consequence of lemma 2.2, $E(e^{\hat{X}^\epsilon}dW_0(y_1)\dots dW_0(y_q))$ is the sum of terms of the form

$$E(dW_0(y_1)\hat{X}^\epsilon)\dots E(dW_0(y_k)\hat{X}^\epsilon)E(dW_0(y_{k+1})dW_0(y_{k+2}))\dots E(dW_0(y_{q-1})dW_0(y_q))e^{\frac{1}{2}E((\hat{X}^\epsilon)^2)}. \tag{3.27}$$

We will compute the limit of each one of these terms. By using (3.26), we get

$$\begin{aligned} E(dW_0(y)X^\epsilon(y_l)) &= \gamma_0(\epsilon)(\sum_{i=1}^q k_\epsilon^R(y_i-y_l))dy_l \\ &= \gamma_0(\epsilon)k_\epsilon^R(0)(1+Q_l^\epsilon)dy_l \end{aligned}$$

where

$$Q_l^\epsilon = \frac{1}{k_\epsilon^R(0)}(\sum_{i\neq l} k_\epsilon^R(y_i-y_l)).$$

We also have:

$$e^{\frac{1}{2}E((\hat{X}^\epsilon)^2)} = e^{(\frac{q}{2}\rho_{\epsilon/R}(0)+\sum_{i<j}\rho_{\epsilon/R}(\frac{y_i-y_j}{R}))(\gamma_0(\epsilon)^2+\gamma_1^2)}.$$

By using lemma 2.2, expression (3.24) and the rules above, we get:

$$E((\int_{\mathbb{R}^d} f_{\epsilon,h}(y)e^{X^\epsilon(y)-C_\epsilon}dW_0(y))^{2l}) = \sum_{k=0}^l \alpha_{k,l}(\gamma_0(\epsilon))^{2k}(k_\epsilon^R(0))^{2k}e^{(2l-k)((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)-2lC_\epsilon}$$
$$\int_{(\mathbb{R}^d)^{k+l}} f_{\epsilon,h}(y_1)\dots f_{\epsilon,h}(y_{2k})(f_{\epsilon,h}(y_{2k+1}))^2\dots(f_{\epsilon,h}(y_{k+l}))^2\prod_{i=1}^{2k}(1+Q_{i,k,l}^\epsilon)e^{S_{k,l}^\epsilon}dy_1\dots dy_{k+l}$$

where

$$Q_{i,k,l}^\epsilon = \frac{1}{k_\epsilon^R(0)}(\sum_{\substack{1\leqslant j\leqslant 2k\\ j\neq i}} k_\epsilon^R(y_i-y_j) + 2\sum_{j>2k} k_\epsilon^R(y_i-y_j))$$

and

$$\begin{aligned} S^\epsilon_{k,l} &= ((\gamma_0(\epsilon))^2+\gamma_1^2)(\sum_{1\leqslant i<j\leqslant 2k}\rho_{\epsilon/R}(\frac{y_i-y_j}{R})+2\sum_{1\leqslant i\leqslant 2k}\sum_{j>2k}\rho_{\epsilon/R}(\frac{y_i-y_j}{R}) \\ &+4\sum_{2k+1\leqslant i<j\leqslant k+l}\rho_{\epsilon/R}(\frac{y_i-y_j}{R})). \end{aligned}$$

We first take care of the normalizing constant outside each integral:

$$(\gamma_0(\epsilon)k^R_\epsilon(0)e^{-1/2((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)})^{2k}e^{2l((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)-2lC_\epsilon}.$$

By the choice of C_ϵ, we have $e^{2l((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)-2lC_\epsilon}=1$. Using expansions (2.4) and (2.5), we derive the following equivalent:

$$\gamma_0(\epsilon)k^R_\epsilon(0)e^{-1/2((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)}\underset{\epsilon\to 0}{\sim}\frac{\gamma_0^*C_0e^{-1/2\gamma_1^2C_1}}{R^{d/2}}.$$

In conclusion, the constant outside the integral of term k in the above sum is $\alpha_{k,l}(\frac{\gamma_0^*C_0e^{-1/2\gamma_1^2C_1}}{R^{d/2}})^{2k}$.

Let γ be such that $\gamma_1^2<\gamma^2$. One can choose $\epsilon_0>0$ such that $\gamma_0(\epsilon_0))^2+\gamma_1^2<\gamma^2$. Using the fact that, for all y, $\rho_{\epsilon/R}(y/R)\leqslant\omega_d\ln^+\frac{R}{y}+C$ with C independent of ϵ, we get:

$$\begin{aligned} e^{S^\epsilon_{k,l}}\leqslant C\prod_{1\leqslant i<j\leqslant 2k}\frac{1}{|\frac{y_i-y_j}{R}|_*^{\gamma^2\omega_d}}\prod_{\substack{1\leqslant i<\leqslant 2k\\ j>2k}}\frac{1}{|\frac{y_i-y_j}{R}|_*^{2\gamma^2\omega_d}}\times \\ \prod_{2k+1\leqslant i<j\leqslant k+l}\frac{1}{|\frac{y_i-y_j}{R}|_*^{4\gamma^2\omega_d}}. \end{aligned} \tag{3.28}$$

Finally, we conclude by using inequality (3.11) and (3.15) similarly as in the proof of proposition 3.5.

□

REMARK 3.8. *One can easily deduce from this that for γ_1^2 sufficiently small, by Kolmogorov's compacity theorem, $\mathcal{X}^\epsilon$ tends to $\mathcal{X}$ in the functional sense and that $\mathcal{X}$ is locally hölderian.*

COMMENT 3.9. *Starting with a two parameter (R,α) monofractal gaussian field, we constructed a four parameter $(R,\alpha,\gamma_1,\gamma_0^*)$ multifractal field with nonsymmetrical increments. In dimension $d=1$, this family can be used for financial modeling and, in any case, has it's own interest. Unfortunately, this family is inappropriate to modelize the velocity of turbulent flows where, as we shall see, the 4/5-law of*

Kolmogorov imposes the condition $\widetilde{\zeta}_3 = 1$: indeed, a look at expression (3.21) shows that, for this family, $\widetilde{\zeta}_3 > 2$.

In the case where $\gamma_0^* = 0$, we obtain symmetrical random fields which extend to higher dimensions the model introduced in [**3**].

In the next section, we will study a multifractal field which is not in this family but that can be seen as a limit case where $\gamma_1 = 0$ and γ_0 is constant (independent of ϵ). As we will see, this family will be compatible with the 4/5-law.

4. The field $\mathcal{X}_0$

4.1. Construction of the field $\mathcal{X}_0$. In this section, we only outline the main steps of the construction of $\mathcal{X}_0$. The field $\mathcal{X}_0^\epsilon$ is given by formula (2.3) where X^ϵ is now defined by:

$$X^\epsilon(y) = \gamma_0 \int_{\mathbb{R}^d} k_\epsilon^R(y-\sigma)dW_0(\sigma).$$

We suppose that α is in the interval $]0,1[$. We choose the normalizing constant C_ϵ such that:

$$\gamma_0 k_\epsilon^R(0) e^{-C_\epsilon + \frac{1}{2}\gamma_0^2 \rho_{\epsilon/R}(0)} = 1.$$

We start by stating a lemma we will use in the proof of the proposition below:

LEMMA 4.1. *let δ be some real number different from d. Then there exists $C = C(\delta) > 0$ with:*

$$\int_{|u| \leqslant R} \frac{du}{|u|_\epsilon^\delta} \leqslant C\epsilon^{(d-\delta)\wedge 0}. \tag{4.1}$$

PROOF. We suppose $\delta > d$, the other case being obvious. We have:

$$\begin{aligned}\int_{|u| \leqslant R} \frac{du}{|u|_\epsilon^\delta} &\underset{u=\epsilon\widetilde{u}}{=} \epsilon^{d-\delta} \int_{|\widetilde{u}| \leqslant R/\epsilon} \frac{d\widetilde{u}}{(\int_{|v| \leqslant 1} \theta(v)|v+\widetilde{u}|dv)^\delta} \\ &\leqslant \epsilon^{d-\delta} \int_{\mathbb{R}^d} \frac{d\widetilde{u}}{(\int_{|v| \leqslant 1} \theta(v)|v+\widetilde{u}|dv)^\delta}.\end{aligned}$$

□

We can now state the following proposition:

PROPOSITION 4.2. *Let q be some positive integer satisfying:*

(1) *$q = 1$ or $q = 2$ with $\gamma_0^2 \omega_d < \alpha$.*
(2) *q is even, greater or equal to 4 with $(q - 3/2)\gamma_0^2 \omega_d < \alpha \wedge \frac{d}{2}$.*
(3) *q is odd, greater or equal to 3 with $(q - \frac{1}{2})\gamma_0^2 \omega_d < \alpha \wedge \frac{d}{2}$.*

Under the above condition, for all x, $\mathcal{X}_0^\epsilon(x)$ converges in L^q to a random variable $\mathcal{X}_0(x)$ such that, if e is a unit vector, we get the following scaling:

$$E((\mathcal{X}_0(x+\lambda e)-\mathcal{X}_0(x))^q) \underset{\lambda\to 0}{\sim} C_q(\frac{\lambda}{R})^{\zeta_q}, \tag{4.2}$$

where

$$\zeta_q = q\alpha - \frac{1}{2}q(q-1)\gamma_0^2\omega_d$$

and

$$C_q = e^{\frac{q(q-1)}{2}\phi(0)}\int_{(\mathbb{R}^d)^q}\prod_{1\leqslant i<j\leqslant q}\frac{1}{|u_i-u_j|^{\gamma_0^2\omega_d}}\prod_{1\leqslant i\leqslant q}(\frac{1}{|u_i-e|^{d-\alpha}}-\frac{1}{|u_i|^{d-\alpha}})du_1\dots du_q. \tag{4.3}$$

Proof. In the proof, we suppose that $q=2l$ with $l\geqslant 1$; we will first prove that:

$$E((\mathcal{X}_0^\epsilon(x))^q) \underset{\epsilon\to 0}{\to} \int_{(\mathbb{R}^d)^q}\prod_{1\leqslant i<j\leqslant q}\frac{e^{\gamma_0^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_0^2\omega_d}}\prod_{1\leqslant i\leqslant q}\frac{\varphi_R(y_i)}{|y_i|^{d-\alpha}}dy_1\dots dy_q. \tag{4.4}$$

We remind that the right hand side of the above limit exists by lemma 2.4. In order to prove the above relation, we develop $E((\mathcal{X}_0^\epsilon(x))^q)$ in $l+1$ terms similarly as in the proof of proposition 3.7; then, using formula (2.4) and the fact that, for all y, $\rho_{\epsilon/R}(y)\leqslant \omega_d\ln\frac{1}{\epsilon}+C$, we are led to show that, for all $k\leqslant l-1$, we have the following convergence:

$$\epsilon^{(l-k)(d-\gamma_0^2\omega_d)}\epsilon^{-2(l-k)(l-k-1)\gamma_0^2\omega_d}\epsilon^{-4k(l-k)\gamma_0^2\omega_d}\int_{(\mathbb{R}^d)^{k+l}}\frac{\varphi_R(y_1)}{|y_1|_\epsilon^{d-\alpha}}\dots\frac{\varphi_R(y_{2k})}{|y_{2k}|_\epsilon^{d-\alpha}}\times$$
$$\frac{\varphi_R(y_{2k+1})^2}{|y_{2k+1}|_\epsilon^{2(d-\alpha)}}\dots\frac{\varphi_R(y_{k+l})^2}{|y_{k+l}|_\epsilon^{2(d-\alpha)}}\prod_{1\leqslant i<j\leqslant 2k}\frac{e^{\gamma_0^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_0^2\omega_d}}dy_1\dots dy_{k+l}\underset{\epsilon\to 0}{\to} 0.$$

We apply inequality (4.1) and obtain (if $\alpha=d/2$, one can work with $\alpha-\eta$ for $\eta>0$ sufficiently small) :

$$\int_{\mathbb{R}^d}\frac{\varphi_R(y)^2}{|y|_\epsilon^{2(d-\alpha)}}dy\leqslant \epsilon^{(2\alpha-d)\wedge 0}$$

Therefore the above convergence to 0 amounts to showing that, for all $k\leqslant l-1$, we have the following inequality:

$$d+(2\alpha-d)\wedge 0-\gamma_0^2\omega_d > 2(l-k-1)\gamma_0^2\omega_d+4k\gamma_0^2\omega_d.$$

This is equivalent to $(2l-\frac{3}{2})\gamma_0^2\omega_d<\alpha\wedge\frac{d}{2}$. One can show, for all x, that $(\mathcal{X}_0^\epsilon(x))_{\epsilon>0}$ is a Cauchy sequence in L^q by computing $E((\mathcal{X}_0^\epsilon(x)-\mathcal{X}_0^{\epsilon'}(x))^q)$ and letting ϵ,ϵ' go to 0. Thus, $E((\mathcal{X}_0(x))^q)$ is given by the left hand side of (4.4).

To show the scaling (4.2), observe that we can prove the following analogue to (4.4):

$$E((\mathcal{X}_0(x+\lambda e)-\mathcal{X}_0(x))^q)=\int_{(\mathbb{R}^d)^q}\prod_{1\leqslant i<j\leqslant q}\frac{e^{\gamma_0^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_0^2\omega_d}}\prod_{1\leqslant i\leqslant q}f_{\lambda e}(y_i)dy_1\dots dy_q, \tag{4.5}$$

where

$$f_{\lambda e}(y)=\frac{\varphi_R(y-\lambda e)}{|y-\lambda e|^{d-\alpha}}-\frac{\varphi_R(y)}{|y|^{d-\alpha}}. \tag{4.6}$$

By setting $y_i=\lambda\widetilde{y}_i$ in the integral of (4.5), we deduce easily (4.2). □

REMARK 4.3. *Similarly as in the previous section, for γ_0 sufficiently small, $\mathcal{X}_0^\epsilon$ converges in law to $\mathcal{X}_0$ in the space of continuous fields.*

4.2. Nonsymmetry of the increments of $\mathcal{X}_0$. Let q be some odd integer and consider C_q given by formula (4.3). It is clear that C_q does not depend on the unitary vector e. By making the change of variable $\widetilde{u}_i=u_i-e$ in (4.3), we get:

$$C_q(e)=-C_q(-e)=-C_q(e).$$

Thus, we get $C_q=0$. Therefore, it is not obvious to see if the law of $\mathcal{X}_0(x+h)-\mathcal{X}_0(x)$ is nonsymmetrical or not. Nevertheless, one can repeat the same construction of $\mathcal{X}_0$ as above replacing the kernel $\frac{\varphi_R(x-y)}{|x-y|_\epsilon^{d-\alpha}}$ by the kernel:

$$\varphi_R(x-y)\frac{x_i-y_i}{|x-y|_\epsilon^{d-\alpha+1}} \quad ,i=1\dots d. \tag{4.7}$$

In this case, $C_1=0$ but, by Monte Carlo simulation, one can verify that $C_3\neq 0$. In particular, with the kernel (4.7), the law of $\mathcal{X}_0(x+h)-\mathcal{X}_0(x)$ is nonsymmetrical.

5. A step towards a model of the velocity field of turbulent flows

An acceptable solution to the problem of hydrodynamical turbulence in dimension 3 would be to construct a random velocity field U solution to the dynamics (Euler or Navier Stokes typically) that is stationnary, incompressible, space-homogeneous, isotropic and that satisfies the main statistical properties of the velocity field of turbulent flows. These properties are:

(1) The 4/5-law of Kolmogorov that links the energy dissipation of the turbulent flow to the statistics of the increments of the velocity. This law is widely

accepted since it is the only one that can be proven with the dynamics ([**18**], [**24**], [**50**]). More precisely, this law states:

$$E\left(((U(x+\xi)-U(x)).\frac{\xi}{|\xi|})^3\right) = -\frac{4}{5}D|\xi|. \tag{5.1}$$

In the above formula, D denotes the average dissipation of the kinetic energy per unit mass in the fluid.

REMARK 5.1. *To obtain this law, it is sufficient to suppose that the field U is space homogeneous and isotropic.*

(2) The intermittency of the field U:

$$E\left(|(U(x+\xi)-U(x)).\frac{\xi}{|\xi|}|^q\right) \underset{|\xi|\to 0}{\sim} C_q|\xi|^{\zeta_q}, \tag{5.2}$$

where q is a positive real number and the ζ_q are called the structural exponents.

It is a very challenging task to construct a field with all the aforementioned properties, especially because this field must be invariant by the Euler or Navier-Stokes equation.

Nevertheless, one can in the first place forget the invariance condition and simply try to construct a field that satisfies all the other properties. The 4/5-law shows that the nonsymmetry of the increments is an essential feature: this is one of the main difficulties in trying to extend the previous construction of scalar fields to 3-dimensional incompressible fields.

Quite naturally, we consider the incompressible family U^ϵ defined by:

$$U^\epsilon(x) = \int_{\mathbb{R}^3} \varphi_R(x-y)\frac{x-y}{|x-y|_\epsilon^{d-\alpha+1}} \wedge e^{X^\epsilon(y)-C_\epsilon} dW(y),$$

where $dW(y) = (dW_1(y), dW_2(y), dW_3(y))$ denotes a three dimensional white noise and X^ϵ is defined by the following formula:

$$X^\epsilon(y) = \gamma \int_{\mathbb{R}^3} K_\epsilon^R(y-\sigma).dW(\sigma),$$

with $K^R(x) = \frac{x}{|x|^{1+d/2}}1_{|x|\leqslant R}$. As in the previous sections, we choose the constant C_ϵ such that U^ϵ converges to a non trivial field U as ϵ goes to 0. The vector field U we obtain is incompressible, homogeneous, isotropic and intermittent with structural exponents ζ_q defined by:

$$\zeta_q = q\alpha - 2\pi\gamma^2 q(q-1). \tag{5.3}$$

We can derive the energy dissipation D of the field U by computing the following limit:

$$-\frac{5}{4}\lim_{\lambda\to 0}\frac{1}{\lambda}E\left(((U(x+\lambda e)-U(x)).e)^3\right),$$

where e is some unitary vector. In order to get D finite and $D\neq 0$, we must have $\zeta_3 = 1$ or equivalently:

$$\alpha = 1/3 + 4\pi\gamma^2. \tag{5.4}$$

Unfortunately, since the field $e^{X^\epsilon(y)}dW(y)$ is isotropic with respect to all unitary transformations (and not just the rotations), we get $D = 0$. Thus, the construction of an intermittent incompressible field with positive dissipation remains an open question.

COMMENT 5.2. *If one plugs relation (5.4) into (5.3), we get the following expression for* ζ_q*:*

$$\zeta_q = (1/3 + 6\pi\gamma^2)q - 2\pi\gamma^2 q^2.$$

The small scale behavior of the field depends only on the intermittency parameter γ^2*. One can easily identify it using the experimental curve obtained in* [**2**] *(cf. fig. 8.8 p. 132 in* [**24**]*): with their data, we find* $4\pi\gamma^2 = 0.023$*. Thus, the intermittency parameter is small, a situation that seems to be similar in finance for certain assets; indeed, if one denotes by* γ *the intermittency parameter of the MRW model introduced by Bacry, Delour and Muzy (*[**3**]*), the authors of* [**5**] *estimate* $\gamma^2 \approx 0.03$ *for the S&P500 future index using intraday data over the period 1988-1999.*

Appendix

In the appendix, we prove a few technical lemmas.

LEMMA 5.3. *Let* $x \in \mathbb{R}^d$ *be different from* 0*. Then we get:*

$$\rho(x) = \omega_d \ln^+\frac{1}{|x|} + \phi(x),$$

where ω_d *denotes the surface of the unit sphere in* $\mathbb{R}^d$ *and* ϕ *is a continuous function that vanishes for* $|x| \geqslant 2$*.*

PROOF. Let $e_1 = (1, 0, \ldots, 0)$ denote the first vector of the canonical basis of $\mathbb{R}^d$. By isotropy, we can suppose that $x = \lambda e_1$ with $\lambda > 0$. We also suppose that

$\lambda \leqslant 1$. By changing variables, we get:

$$\rho(x) = \int_{|y| \leqslant 1,\ |y-\lambda e_1| \leqslant 1} \frac{1}{|y|^{d/2}} \frac{1}{|y-\lambda e_1|^{d/2}} dy$$
$$\underset{u=\frac{y}{\lambda}}{=} \int_{|u| \leqslant \frac{1}{\lambda}, |u-e_1| \leqslant \frac{1}{\lambda}} \frac{1}{|u|^{d/2}} \frac{1}{|u-e_1|^{d/2}} du$$

Therefore, we get the following bounds on ρ:

$$\int_{|u| \leqslant \frac{1}{\lambda}-1} \frac{1}{|u|^{d/2}} \frac{1}{|u-e_1|^{d/2}} du \leqslant \rho(x) \leqslant \int_{|u| \leqslant \frac{1}{\lambda}} \frac{1}{|u|^{d/2}} \frac{1}{|u-e_1|^{d/2}} du.$$

The difference of the above bounds goes to 0 as λ goes to 0 so the lemma is a consequence of the following expansion as λ goes to 0:

$$\int_{|u| \leqslant \frac{1}{\lambda}} \frac{1}{|u|^{d/2}} \frac{1}{|u-e_1|^{d/2}} du = \omega_d \ln(\frac{1}{\lambda}) + c + o(1).$$

□

LEMMA 5.4. *We have the following equality for $\epsilon \leqslant R$:*

$$k_\epsilon^R(0) = \frac{C_0}{\epsilon^{d/2}},$$

with $C_0 = \int_{|u| \leqslant 1} \frac{\theta(u)}{u^{d/2}} du$.

PROOF. We have:

$$k_\epsilon^R(0) = \int_{\mathbb{R}^d} \theta^\epsilon(u) k^R(u) du$$
$$\underset{v=\epsilon u}{=} \frac{1}{\epsilon^{d/2}} \int_{|v| \leqslant 1, |v| \leqslant R/\epsilon} \frac{\theta(v) dv}{|v|^{d/2}}.$$

□

LEMMA 5.5. *There exists some constant C_1 such that we have the following expansion:*

$$\rho_{\epsilon/R}(0) = \omega_d \ln \frac{R}{\epsilon} + C_1 + o(\epsilon).$$

PROOF. By changing variables, we can suppose that $R = 1$. By definition, we have:

$$\rho_\epsilon(0) = \int_{\mathbb{R}^d} k_\epsilon^1(x)^2 dx,$$

where k^1_ϵ is given by:

$$\begin{aligned} k^1_\epsilon(x) \quad &= \int_{\mathbb{R}^d} \theta^\epsilon(x-y)k^1(y)dy \\ &\underset{v=(y-x)/\epsilon}{=} \int_{\mathbb{R}^d} \theta(v)k^1(x+\epsilon v)dv. \end{aligned}$$

Therefore, by setting $x = \epsilon u$, we get:

$$\begin{aligned} \rho_\epsilon(0) &= \int_{\mathbb{R}^d} du\Big(\int_{(\mathbb{R}^d)^2} 1_{|u+v| \leqslant \frac{1}{\epsilon}} 1_{|u+\widetilde{v}| \leqslant \frac{1}{\epsilon}} \frac{\theta(v)\theta(\widetilde{v})dvd\widetilde{v}}{|u+v|^{d/2}|u+\widetilde{v}|^{d/2}}\Big) \\ &= \int_{|u| \leqslant 1} (\dots) + \int_{|u|>1} (\dots). \end{aligned}$$

It is easy to see that $\int_{|u| \leqslant 1}(\dots)$ has a limit as ϵ goes to 0 and there exists a constant c such that:

$$\int_{|u|>1} (\dots) = \int_{|u|>1} du\Big(\int_{(\mathbb{R}^d)^2} 1_{|u+v| \leqslant \frac{1}{\epsilon}} 1_{|u+\widetilde{v}| \leqslant \frac{1}{\epsilon}} \frac{\theta(v)\theta(\widetilde{v})dvd\widetilde{v}}{|u|^d}\Big) + c + o(1).$$

Using the fact that $1_{|u| \leqslant \frac{1}{\epsilon}-1} \leqslant 1_{|u+v| \leqslant \frac{1}{\epsilon}} \leqslant 1_{|u| \leqslant \frac{1}{\epsilon}+1}$ and $\int_{\mathbb{R}^d} \theta(v)dv = 1$, we get:

$$\omega_d \ln\Big(\frac{1}{\epsilon} - 1\Big) \leqslant \int_{|u|>1} du\Big(\int_{(\mathbb{R}^d)^2} 1_{|u+v| \leqslant \frac{1}{\epsilon}} 1_{|u+\widetilde{v}| \leqslant \frac{1}{\epsilon}} \frac{\theta(v)\theta(\widetilde{v})dvd\widetilde{v}}{|u|^d}\Big) \leqslant \omega_d \ln\Big(\frac{1}{\epsilon} + 1\Big).$$

This implies the desired result. $\square$

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