Empilements de sphères et bêta-entiers

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THESE DE DOCTORAT DE MATHEMATIQUES
DE L'UNIVERSITE JOSEPH FOURIER (GRENOBLE I)
préparée à l'Institut Fourier
Laboratoire de Mathématiques
UMR 5582 CNRS - UJF

EMPILEMENTS DE SPHERES ET BETA-ENTIERS

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A ma famille

Remerciements

Je souhaite remercier Roland Gillard pour son écoute constante, sa gentillesse, sa disponibilité et les chemins mathématiques qu'il m'a dévoilés, il y a quelques années et encore récemment. Dans le contexte de l'Institut Fourier qui pour moi représente un espace tout-à-fait privilégié de foisonnement de compétences et de concentration de pensée et de liberté, son aide m'a été très précieuse et un encouragement sans faille.

Je suis très reconnaissant à chaque membre du jury d'avoir accepté de participer à mon jury. Si les quelques questions de départ de mes travaux sont d'énoncé simple, les idées qui ont apparu ensuite dans les développements m'ont fait voyager dans des paysages *mathématiques* insoupçonnés qui m'ont montré une toute petite partie de l'étendue de la substance mathématique et la richesse des interactions sur le plan des idées comme sur le plan des personnes. La difficulté de mettre mes travaux récents dans une présentation unifiée, comme j'ai essayé de le faire dans ce mémoire, a alors, du côté du jury, dû avoir comme contrepartie de faire apparaître le texte aride à ses membres, chacun n'étant spécialiste que d'une partie de mes travaux, et je les remercie d'autant.

Je remercie mon rapporteur étranger Karoly Bezdek d'avoir en peu de temps pu apprécier les travaux de Géométrie Discrète que j'ai entrepris ; et à cette occasion je tiens à lui manifester ma reconnaissance pour m'avoir invité l'année dernière à un Colloque sur les Empilements de Sphères à Banff (Canada) qu'il avait organisé, et qui m'a permis de rencontrer les grands spécialistes de ce domaine.

J'ai appris avec plaisir l'acceptation de Michel Waldschmidt de faire partie du jury et d'être rapporteur. Je le remercie bien sincèrement, d'autant que les questions d'Approximation Diophantienne ne me sont apparues clairement que depuis un an et relativement qu'à un des travaux présentés dans ce mémoire. Ce travail s'est révélé être précurseur et se situe maintenant à l'orée de nombreux résultats encore à dégager à la lumière de grands théorèmes d'Approximation Diophantienne. Ce domaine de l'Approximation Diophantienne, qui me semblait bien étranger il y a encore peu de temps, s'est soudainement rapproché.

Je remercie très sincèrement Christine Bachoc de faire partie du jury, de m'avoir permis de m'ouvrir davantage sur la théorie des réseaux, de m'avoir fait découvrir des résultats récents et de rencontrer, en m'invitant au Laboratoire A2X, les spécialistes de cette théorie et des problèmes qui lui sont reliés.

Je suis heureux que Valérie Berthé ait accepté de faire partie du jury. Son écoute, sa disponibilité m'ont permis de m'ouvrir sur les spécialités dynamiques marseillaises des systèmes de substitutions et sa collectivité très vivante, et d'établir des liens inattendus avec mes travaux antérieurs.

Je remercie également Roland Bacher de faire partie du jury, et pour les nombreuses discussions chaleureuses, originales et stimulantes, que j'ai pu avoir autour des empilements de sphères.

Je tiens également à remercier mes coauteurs : Gilbert Muraz, Jean-Pierre Gazeau, Christiane Frougny, dans les collaborations fructueuses que j'ai eues et que je souhaite continuer d'avoir avec eux. Les échanges d'idées, parfois dans des langages différents, finissent par dessiner des résultats nouveaux.

Introduction

Les objets considérés dans cette thèse sont les empilements de sphères égales, principalement de $\mathbb{R}^n$, et les beta-entiers. Un empilement de sphères de $\mathbb{R}^n$ est d'abord un ensemble discret Λ de points de $\mathbb{R}^n$ pour lequel $\|x-y\| \geq r$ dès que $x, y \in \Lambda$ avec $x \neq y$, pour une certaine constante $r > 0$. Le système de sphères est alors donné par $\mathcal{B}(\Lambda) := \{x + B(0, r/2) \mid x \in \Lambda\}$ où $B(c, \epsilon)$ est la boule fermée de centre c et de rayon ϵ. On dit que Λ est un r-code ou un ensemble uniformément discret, de constante r. On utilisera indifféremment dans cette thèse le langage des empilements de sphères ou celui des ensembles uniformément discrets (ou des *quasicristaux mathématiques*, terminologie proposée par Lagarias [La3] [La4]), qui reportent chacun à des littératures distinctes, qui s'enrichissent mutuellement.

Introduite par Minkowski [Min] la Géométrie des Nombres est aujourd'hui considérée comme faisant partie de la Théorie des Nombres. Depuis Minkowski elle s'est arithmétisée et a été développée par de nombreux auteurs ; beaucoup de questions ouvertes existent sur les empilements de sphères égales, ce qui en fait un sujet majeur en Geométrie des Nombres [Bdk] [By2] [Bk] [Ca1] [CS] [GL] [Min] [Ro1] [Z].

Nous nous sommes concentrés sur les problèmes suivants :

(i) aspects métriques et topologiques de l'espace des empilements de sphères, (ii) les trous profonds, la densité et la structure interne asymptotique des empilements les plus denses, (iii) les empilements autosimilaires de type fini, (iv) les empilements de sphères sur beta-entiers et sur beta-réseaux.

L'étude du dernier point (iv) s'est surtout ramenée à l'étude des beta-entiers pour lesquels de nombreuses questions se posent. En effet, Rényi [Re] [Fy1] [Fy2] avait montré que la numération en base x est possible pour tout $x > 1$ dès 1957. Les entiers en base x ou x-entiers, notés $\mathbb{Z}_x$ (notation introduite par Gazeau [G1] [G2]), jouent alors un rôle équivalent à ceux que l'on trouve en base 10, c'est-à-dire $\mathbb{Z}$. Comme l'usage de la base 10 est arbitraire, mais historiquement lié à notre écriture et à son usage [If], comme il n'y a aucune raison mathématique de privilégier la base 10 par rapport à une autre base, il nous a semblé en soi intéressant sinon universel de continuer de dégager les propriétés des beta-entiers, pour tout nombre algébrique $\beta > 1$, à défaut de la faire pour tout $x > 1$ le programme étant alors trop ambitieux.

Il se fait que l'usage de numérations non-classiques procède par le système dynamique sur $[0, 1]$ de multiplication par $\beta > 1$ modulo 1. Nous avons rappelé celui-ci en Section 4 et reconsidéré les liens existant entre Systèmes Dynamiques Symboliques et Théorie des Nombres, étude initiée par Bertrand-Mathis [Be1] [Be2] [Be3] [Be4] [Be5]. Cela nous a amené à proposer en Sous-section 4.2 une classification des nombres algébriques différente de celle de Bertrand-Mathis, reportée dans un article de Blanchard [Bl].

Les travaux faisant l'objet de cette thèse sont les suivants :

[MVG1] (Section 1) "On a generalization of the Selection Theorem of Mahler", J. Théorie Nombres Bordeaux 17 (2005), 237–269,

[MVG2] (Sections 2.1 et 2.2) "On Densest Packings of Equal Spheres of $\mathbb{R}^n$ and Marcinkiewicz Spaces", dans "Interface between Harmonic Analyis and Number Theory", Eds. B. Saffari et al, Birkhaüser (2007), soumis,

[MVG3] (Section 2.3) "On Lower Bounds of the Density of Delone Sets and Holes in Sequences of Sphere Packings", Exp. Math. 14 :1 (2005), 47–57,

[VG1] (Section 2.4) "Covering a Ball with Smaller Equal Balls in $\mathbb{R}^n$", Discrete Comput. Geom. 33 :1 (2005), 143–155,

[VG2] (Section 3) "On Self-Similar Finitely Generated Uniformly Discrete (SFU-) Sets and Sphere Packings", dans "Number Theory and Physics", IRMA Lectures in Mathematics and Theoretical Physics, Eds. L. Nyssen et V. Turaev, E.M.S. Publishing House (2006), accepté,

[GVG1] (Section 4.1) "Geometric Study of the Beta-Integers for a Perron Number and Mathematical Quasicrystals", J. Théorie Nombres Bordeaux 16 (2004), 125–149,

[VG3] (Section 4.2) "On Gaps in Rényi β-expansions of unity for $\beta > 1$ an algebraic number", Annales Institut Fourier (2006), accepté,

[E-VG] (Section 4.3) "Symmetry Groups for Beta-Lattices", Theor. Comput. Sci. 319 (2004), 281–305.

Dans la **Première Partie**, chaque Section présente de manière synthétique les problèmes rencontrés et les résultats apportés en indiquant le chemin général qui unifie l'ensemble. La **Deuxième Partie** de la thèse contient les différents articles, c'est-à-dire toute la mathématique nécessaire à la compréhension détaillée des affirmations de la Première partie.

Definitions.— Un nombre de Pisot β (ou de Pisot - Vijayaraghavan) est un entier algébrique réel > 1 dont tous les conjugués $\beta^{(i)}$ sont dans le disque unité ouvert du plan complexe.

Un nombre de Salem β est un entier algébrique réel > 1 dont tous les conjugués $\beta^{(i)}$ sont dans le disque unité fermé du plan complexe, l'un deux au moins étant sur le cercle unité.

Un nombre de Perron β est un entier algébrique réel > 1 dont tous les conjugués $\beta^{(i)}$ sont de module strictement plus petit que β.

Un nombre de Lind β est un entier algébrique réel > 1 dont tous les conjugués $\beta^{(i)}$ sont de module plus petit que ou égal à β, l'un d'entre eux au moins étant de module égal à β.

Première Partie

1 Espace métrique compact des empilements de sphères égales de $\mathbb{R}^n$

1.1 Contexte et Théorème Principal

En 1946 Mahler [Ma1] a obtenu des résultats importants sur les corps étoilés dans $\mathbb{R}^n$ et leurs réseaux critiques en utilisant le théorème de compacité suivant, appelé maintenant Théorème de Sélection de Mahler ou Théorème de compacité de Mahler [GL].

Théorème 1.1. *Soit (L_r) une suite de réseaux de $\mathbb{R}^n$ telle que, pour tout r :*

(i) $\|x\| \geq c$ pour tout $x \in L_r, x \neq 0$, où c est une constante strictement positive indépendante de r,

(ii) la mesure de Lebesgue $|L_r|$ du domaine fondamental de L_r satisfait $|L_r| \leq M$ avec M une constante $< +\infty$ indépendante de r.

Alors on peut extraire de la suite (L_r) une sous-suite (L'_r) convergente ; si L est la limite de cette suite, on a :

$$|L| = \lim_{r' \to +\infty} |L'_r|.$$

Ce théorème très efficace en Géométrie des Nombres est aussi important que le théorème d'Ascoli-Arzela en Analyse [Ca1] [GL]. Lors d'un séminaire à Princeton [RSD] Mahler a souligné qu'il serait souhaitable de généraliser les grands théorèmes de Géométrie des Nombres, dont ce théorème, à d'autres espaces ambiants que $\mathbb{R}^n$, par exemple à $\mathbb{K}^n$ où $\mathbb{K}$ est un corps de nombres. Plusieurs auteurs ont établi un théorème analogue dans cette optique de généralisation : Chabauty [Cy] en 1950 avec des sous-groupes dans des groupes abéliens localement compacts, Mumford [Mu] dans des groupes de Lie semi-simples sans facteur compact et des espaces de modules de surfaces de Riemann compactes de genre donnée, Macbeath et Swierczkowski [MS] dans des groupes localement compacts et σ-compacts (abéliens ou non) qui sont engendrés par des compacts, McFeat [Mf] dans des espaces d'adèles de corps de nombres, Rogers et Swinnerton-Dyer [RSD] dans des corps de nombres algébriques. La preuve élégante donnée par Groemer [Groe2] de ce théorème est une conséquence du Théorème de Sélection de Blaschke [Ca1], et fait intervenir la correspondance biunivoque qui existe entre un réseau et sa cellule de Voronoi.

La manière dont Chabauty [Cy] démontre le Théorème 1.1[1] est extrêmement riche d'enseignements. En effet, si l'on prend le temps d'observer sa preuve "élémentaire" on voit que la structure de $\mathbb{Z}$-module des réseaux L_r n'est pas nécessaire pour obtenir la convergence de la sous-suite $(L_{r'})$. Puis il prouve

[1] La topologie qu'il utilise est la suivante : soit (E_i) une suite d'ensembles de $\mathbb{R}^n$. Nous disons qu'elle converge vers l'ensemble E de $\mathbb{R}^n$: $\lim_{i \to +\infty} E_i = E$ si, $\forall t > 0, \forall \epsilon > 0$ et tout entier $i \geq i(t, \epsilon)$, chaque point $x \in E$ tel que $\|x\| \leq t$ est à distance $\leq \epsilon$ d'au moins un point de E_i, chaque point $y \in E_i$ tel que $\|y\| \leq t$ est à distance $\leq \epsilon$ d'au moins un point de E.

que la limite L est bien un réseau mais a posteriori une fois la convergence démontrée. Cette remarque permet à Chabauty [Cy] une extension du Théorème de Sélection de Mahler à des espaces ambiants comme des groupes abéliens localement compacts dans une version non métrique du théorème mais topologique. Mumford [Mu] a ensuite amélioré cette approche. Cette remarque de Chabauty, essentielle mais qui semble insignifiante à première vue, ouvre le chemin vers des espaces d'ensembles de points a priori "sans structure" qui sont non-périodiques, au lieu d'être des espaces de réseaux ou de sous-groupes, sous-ensembles de l'espace ambiant munis de structures algébriques additionnelles. Cela suggère que des analogues du Théorème de Sélection de Mahler doivent exister dans des situations bien plus générales.

Dans le travail intitulé "On a generalization of the Selection Theorem of Mahler" [MVG1] nous développons une version du Théorème 1.1 adaptée aux "ensembles de points" (pas seulement de réseau ou de sous-groupe) d'un espace ambiant. Ceci peut être formulé de la façon suivante. Nous nous intéressons aux ensembles d'ensembles de points, disons $\mathcal{UD}(H,\delta)_r$, d'un espace métrique (H,δ), qui est "l'espace ambiant", où δ est une métrique sur H, qui présentent la propriété que leur distance interpoint minimale est plus grande que, ou égale à, une constante strictement positive donnée, disons $r > 0$.

Définition 1.2. On appelle ensemble uniformément discret de (H,δ), de constante r, les ensembles de points $\Lambda \subset H$ qui possèdent la propriété

$$x, y \in \Lambda, x \neq y \implies \delta(x,y) \geq r.$$

Il s'agit par exemple de l'ensemble vide, des sous-ensembles à un point $\{x\}$, avec $x \in H$, ou bien d'un ensemble Λ, admettant au moins deux points, pour lequel cette valeur minimale interpoint r est atteinte au moins pour une paire de points distincts. Il se peut que Λ admette au moins deux points et que cette valeur minimale interpoint r ne soit atteinte pour aucune paire de points distincts, par exemple si $\Lambda = \{x_n = n + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \mid n > 0\} \subset \mathbb{R}$.

Appelons $\mathcal{UD}(H,\delta)_{r,f}$ le sous- ensemble de $\mathcal{UD}(H,\delta)_r$ constitué des ensembles de points finis. La première question qui se pose à partir du (i) du Théorème 1.1 est alors la suivante :

Question 1.1. *Pour quels espaces métriques (H,δ) est-ce que $\mathcal{UD}(H,\delta)_r$ peut être muni d'une topologie telle qu'il soit compact et que la métrique de Hausdorff Δ sur $\mathcal{UD}(H,\delta)_{r,f}$ soit compatible avec la restriction de cette topologie à $\mathcal{UD}(H,\delta)_{r,f}$, et pour quelles valeurs de r ?*

Dans l'objectif de généraliser (ii) dans le Théorème 1.1 rappelons le concept de Besicovitch de relative densité [MVG3].

Définition 1.3. Un sous-ensemble Λ de (H,δ) est dit relativement dense dans H s'il existe $R > 0$ tel que pour tout $z \in H$ il existe $\lambda \in \Lambda$ tel que $\delta(z,\lambda) \leq R$.

L'ensemble Λ est dit relativement dense de constante R si R est choisi minimal pour cette propriété.

En supposant que l'espace ambiant H soit tel que la réponse à la Question 1.1 soit vraie, pour un certain r, on peut formuler la seconde question comme suit.

Question 1.2. *Pour quels espaces métriques (H, δ) est-ce que le sous-ensemble $X(H, \delta)_{r,R}$ de $\mathcal{UD}(H, \delta)_r$ constitués des sous-ensembles relativement denses de constante donnée $R > 0$ est compact, et pour quelles valeurs de R ?*

Définition 1.4. Un sous-ensemble Λ de (H, δ) est un ensemble de Delone s'il existe $r > 0$ et $R > 0$ tels que Λ soit un ensemble uniformément discret de constante r et relativement dense de constante R. Dans ce cas, on dit que Λ est un ensemble de Delone de constantes (r, R).

Les valeurs possibles du rapport R/r lorsque $H = \mathbb{R}^n$ sont envisagées dans "On lower Bounds of the Density of Delone Sets and Holes in Sequences of Sphere Packings" [MVG3]. Par exemple un réseau de $\mathbb{R}^n$ est un ensemble de Delone. La Question 1.2 prend un sens dès que les ensembles de Delone de l'espace H sont infinis. En effet si H est tel que ceux-ci sont tous finis, alors on peut répondre à la Question 1.2 par les propriétés de la métrique de Hausdorff sur l'espace des sous-espaces compacts de H.

Le Théorème principal 1.5 répond aux Questions 1.1 et 1.2 dans le cas où $H = \mathbb{R}^n$ et δ est la métrique Euclidienne standard. On notera $\mathcal{UD}_r$ au lieu de $\mathcal{UD}(\mathbb{R}^n, \delta)_r$, $\mathcal{UD}_{r,f}$ au lieu de $\mathcal{UD}(\mathbb{R}^n, \delta)_{r,f}$, $X_{r,R}$ au lieu de $X(\mathbb{R}^n, \delta)_{r,R}$.

Théorème 1.5. *Pour tout $r > 0$, l'ensemble $\mathcal{UD}_r$ peut être muni d'une métrique d telle que l'espace topologique $(\mathcal{UD}_r, d)$ soit compact et tel que la métrique de Hausdorff sur $\mathcal{UD}_{r,f}$ soit compatible avec la restriction de la topologie de $(\mathcal{UD}_r, d)$ à $\mathcal{UD}_{r,f}$. Pour tout $R > 0$, le sous-espace $X_{r,R}$ de $(\mathcal{UD}_r, d)$ des ensembles de Delone de constantes (r, R) est fermé.*

Cette topologie sur $\mathcal{UD}_r$ n'est pas classique [Ke] [Ml].

1.2 Construction de métriques sur $\mathcal{UD}$

Il est clair qu'il suffit de prouver le Théorème 1.5 pour $r = 1$ pour obtenir le résultat. Pour simplifier, nous notons alors par $\mathcal{UD}$ l'espace $\mathcal{UD}_1$, resp. par X_R l'espace $X_{1,R}$. La construction de trois classes de métriques sur $\mathcal{UD}$ est faite dans [MVG1].

La *première métrique* est associée à une suite exhaustive de compacts de $\mathbb{R}^n$ et à la métrique de Hausdorff sur chaque compact.

Les deux autres métriques sont très différentes : la *deuxième métrique* est inspirée d'une métrique sur l'espace des ensembles de Delone, et est utilisée en Théorie des Systèmes Dynamiques de Pavages (Radin et Wolff [RW], Robinson [Rn], Solomyak [So2], Gouéré [Go], Baake et Lenz [BLz]). Cette métrique est ici adaptée à l'espace des ensembles uniformément discrets.

La *troisième métrique* est obtenue par des systèmes de comptage finis normalisés par des fonctions distance convenables : cette idée a été formulée par Dworkin [Dw] en 1993 pour la première fois, à notre connaissance (pour des ensembles de Delone), bien que l'auteur n'en ait pas apporté de preuves dans sa contribution originale. Ce dernier cas de métrique admet une construction très éloignée des constructions des deux autres et est "moins classique". Nous étudions un peu plus en détail ses propriétés. Elle est plus adaptée à l'étude de problèmes locaux de cristallographie n-dimensionnelle, comme l'étude des clusters locaux de sphères dans des empilements (apériodiques) très denses de $\mathbb{R}^n$. On sait en effet que la preuve de la Conjecture de Kepler par Hales [Ha1] [Ha2] [Ha3] [VG01] (voir aussi la preuve incomplète de Hsiang [Hg]) repose sur l'étude des majorations des densités locales à moyenne portée, et que Lagarias a donné des majorations utilisables en dimension n quelconque en généralisant les critères de Hales [La5].

Chaque métrique est représentative d'une classe de métrique : la première en faisant varier la famille exhaustive de compacts, la deuxième en faisant varier le paramètre qui la décrit, la troisième en faisant varier la normalisation par les fonctions distance.

Nous montrons que ces trois classes de métriques sont topologiquement équivalentes. Leur point commun, est qu'elles privilégient un *point base* de l'espace ambiant.

La précompacité de l'espace $\mathcal{UD}$ est alors une conséquence de la *convergence uniforme* du processus d'enlèvement de points d'un ensemble uniformément discret (non vide), quelle que soit la manière dont on lui enlève ses points. Ce processus de passage à la limite vers l'élément "ensemble vide" $\emptyset$ de $\mathcal{UD}$ est en effet uniformément convergent et l'on s'est appliqué à le décrire explicitement. Le rôle joué par l'ensemble vide dans $\mathcal{UD}$ et la description explicite de ses voisinages dans $\mathcal{UD}$ par la troisième métrique nous a apparu essentiel et fort peu décrit, sinon inexistant, dans la littérature. L'Appendice de [MVG1] contient une démonstration de la continuité uniforme du processus d'enlèvement de points (à l'infini) pour la troisième métrique ; les calculs s'appuient sur la description et les propriétés des représentations des entiers en sommes de carrés d'entiers (pour un panorama sur ces questions, voir [Gr]) et sur une inégalité due à Stolarsky [St]. On procède sur $\mathbb{Z}^n, n \geq 5$, (étape 1) puis sur tout ensemble uniformément discret de manière universelle avec $n \geq 5$ (étape 2), puis par descente pour $n < 5$ (étape 3).

Un phénomène intéressant donné par la troisième métrique (moins visible pour les autres métriques) est le phénomène d'*appariement local de points* :

si Λ et Λ' sont deux ensembles de $\mathcal{UD}$ tels que la distance entre eux soit très petite ($< \epsilon$), alors leurs points respectifs autour du point base, et sur une distance d'environ $1/(2\epsilon)$, sont *tous groupés par paires uniques.* Cela est donc adapté à l'étude de toute fonction sur $\mathcal{UD}$ qui utilise le comptage local de points, notamment la fonction densité (Sous-sections 2.1 et 2.2).

1.3 Généralisation du Théorème de Sélection de Mahler

L'avantage d'avoir des métriques différentes sur $\mathcal{UD}$ est de pouvoir choisir, à partir de celles-ci, la plus adaptée au problème considéré.

En utilisant la troisième métrique, nous montrons que le Théorème 1.5 implique le Théorème 1.1 de Sélection de Mahler. L'idée pour démontrer cette implication est très simple : un réseau est complètement caractérisé par une base constituée de petits vecteurs, situés autour de l'origine choisie comme point base ; d'autre part le phénomène d'appariement de points autour du point base implique que, pour une famille de réseaux tous proches, tous de Delone, appartenant tous à un $X_{r,R}$ avec r et R fixés, les petits vecteurs de leurs bases respectives convergent vers un ensemble de vecteurs libres, donc qui engendre un réseau à la limite. On retrouve alors le cheminement de Chabauty [Cy].

2 Structure des plus denses empilements de sphères

2.1 Théorèmes d'existence

La notion de complète saturation a été introduite par Fejes-Toth, Kuperberg and Kuperberg [FT-K].

Nous dirons que $\Lambda \in \mathcal{UD}$ est *saturé*, ou *maximal*, s'il est impossible d'ajouter une nouvelle réplique de la boule B (boule de rayon $1/2$, puisque $\mathcal{UD}$ est l'espace des ensembles uniforméments discrets de constante 1) à $\mathcal{B}(\Lambda)$ sans détruire le fait que c'est un empilement de sphères i.e. sans créer un recouvrement quelque part d'au moins deux boules du système de boules. L'espace SS des systèmes de boules de rayon $1/2$, est partiellement ordonné par la relation $\prec$ definie par $\Lambda_1, \Lambda_2 \in \mathcal{UD}$, $\mathcal{B}(\Lambda_1) \prec \mathcal{B}(\Lambda_2) \iff \Lambda_1 \subset \Lambda_2$. Par le lemme de Zorn, des empilements de boules maximaux existent. L'opération de saturation d'un empilement de boules consiste à ajouter des boules pour obtenir un empilement maximal de boules. Elle est assez arbitraire et peut être finie ou infinie.

Plus généralement [FT-K], $\mathcal{B}(\Lambda)$ est dit m*-saturé* si aucun sous-sytème fini de $m-1$ boules ne peut être remplacé par m répliques de la boule B. Evidemment, la 1-saturation signifie saturation, et la m-saturation implique la $(m-1)$-saturation. Ce n'est pas parce qu'un empilement de boules est saturé, ou m-saturé, que sa densité est égale à δ_n, la constante d'empilement. L'empilement $\mathcal{B}(\Lambda)$ est *complètement saturé* s'il est m-saturé pour tout $m \geq 1$. La complète saturation est une version plus fine que le fait d'être de densité maximale [Ku].

Nous donnons dans [MVG2] des démonstrations directes et nouvelles de l'existence d'empilements de sphères de densité δ_n (Théorème 2.1, théorème qui par ailleurs est très classique), et de l'existence d'empilements de sphères complètement saturés de $\mathbb{R}^n$ de densité δ_n (Théorème 2.2).

Rappelons quelques définitions. Soit $\Lambda \in \mathcal{UD}$. La densité du système de boules $\mathcal{B}(\Lambda)$ est donnée par

$$\delta(\mathcal{B}(\Lambda)) := \limsup_{T \to +\infty} \left[\mathrm{vol}\Big(\Big(\bigcup_{\lambda \in \Lambda} (\lambda + B(0,1/2))\Big) \bigcap B(0,T)\Big) / \mathrm{vol}(B(0,T)) \right].$$

On pose $\delta_n = \sup_{\Lambda \in \mathcal{UD}} \delta(\mathcal{B}(\Lambda))$, $\delta_{L,n} = \sup_{\Lambda \in \mathcal{L}_n \cap \mathcal{UD}} \delta(\mathcal{B}(\Lambda))$, où $\mathcal{L}_n$ est l'espace des réseaux de $\mathbb{R}^n$. Le nombre δ_n s'appelle la constante d'empilement.

Fejes-Toth, Kuperberg and Kuperberg ont également donné une preuve de l'existence d'empilements complètement saturés (Theorem 1.1 in [FT-K]), ainsi que Bowen [Bn] en 2003 avec $\mathbb{R}^n$, puis avec l'espace hyperbolique $\mathbb{H}^n$, comme espace ambiant, mais par d'autres techniques.

Les démonstrations données dans [MVG2] des Théorèmes 2.1 et 2.2 utilisent différentes métriques sur $\mathcal{UD}$ (Proposition 2.3 et Théorème 2.4), déduites de celles construites en Section 1 ainsi que la continuité de la fonction densité

(Théorème 2.6) sur $\mathcal{UD}$ pour une topologie donnée par une métrique D invariante par les isométries (affines) de $\mathbb{R}^n$.

Théorème 2.1. *Il existe un élément $\Lambda \in \mathcal{UD}$ tel que l'égalité suivante ait lieu :*

$$\delta(\mathcal{B}(\Lambda)) \;=\; \delta_n. \tag{2.1}$$

Théorème 2.2. *Il existe un empilement complètement saturé de boules de $\mathbb{R}^n$, toutes étant des répliques de $B = B(0, 1/2)$, dont la densité est égale à la constante d'empilement δ_n.*

Pour construire une métrique sur $\mathcal{UD}$ invariante par les isométries (affines) de $\mathbb{R}^n$ on a procédé comme suit (voir [MVG2]).

Appelons $O(n, \mathbb{R})$ le groupe orthogonal (n-dimensionel). Une *isométrie* (ou *déplacement Euclidien*) est une paire ordonnée (ρ, t) avec $\rho \in O(n, \mathbb{R})$ et $t \in \mathbb{R}^n$ [Cp]. La composition de deux isométries est donnée par

$$(\rho, t)(\rho', t') = (\rho\rho', \rho(t') + t)$$

et le groupe des isométries est l'extension scindée de $O(n, \mathbb{R})$ par $\mathbb{R}^n$ (comme produit semi-direct). Il est muni de la topologie habituelle.

Dans le Théorème 1.5 la métrique d construite (la troisième par exemple, mais il en est de même pour les autres) n'est pas invariante par translation. Cela vient du fait que les constructions nécessitent un point base de l'espace ambiant. A partir de d, en ajoutant à la construction de d quelques contraintes en plus de manière à ce que d gagne en propriétés d'invariance (voir le point (iii) dans la Proposition 2.3), on peut construire une nouvelle métrique D invariante par translation et par le groupe des déplacements Euclidiens de $\mathbb{R}^n$ (Théorème 2.4). Elle fournit à l'espace $\mathcal{UD}$ une nouvelle topologie, qui convient à l'étude de la continuité de la fonction densité (Théorème 2.6).

Proposition 2.3. *Il existe une métrique d sur $\mathcal{UD}$ telle que :*

(i) l'espace $(\mathcal{UD}, d)$ est compact,

(ii) la métrique de Hausdorff sur $\mathcal{UD}_f$ est compatible avec la restriction de la topologie de $(\mathcal{UD}, d)$ à $\mathcal{UD}_f$,

(iii) $d(\Lambda, \Lambda') = d(\rho(\Lambda), \rho(\Lambda'))$ pour tout $\rho \in O(n, \mathbb{R})$ et tous $\Lambda, \Lambda' \in \mathcal{UD}$.

Puisque la densité d'un empilement de sphères est invariante par toute affinité (non-singulière) de $\mathbb{R}^n$ (Théorème 1.7 dans Rogers [Ro1]), il est naturel de construire des métriques sur $\mathcal{UD}$ qui sont au moins invariantes par les translations et par le groupe orthogonal de $\mathbb{R}^n$. Une telle métrique est donnée par le Théorème suivant.

Théorème 2.4. *Il existe une métrique D sur $\mathcal{UD}$ telle que :*

(i) $D(\Lambda_1, \Lambda_2) = D(\rho(\Lambda_1) + t, \rho(\Lambda_2) + t)$ *pour tout* $t \in \mathbb{R}^n, \rho \in O(n, \mathbb{R}^n)$ *et tout* $\Lambda_1, \Lambda_2 \in \mathcal{UD}$,

(ii) l'espace $(\mathcal{UD}, D)$ *est complet et localement compact,*

(iii) (propriété d'appariement de points) pour tous $\Lambda, \Lambda' \in \mathcal{UD}$ *non vides tels que* $D(\Lambda, \Lambda') < \epsilon$, *chaque point* $\lambda \in \Lambda$ *est associé à un unique point* $\lambda' \in \Lambda'$ *tel que* $\|\lambda - \lambda'\| < \epsilon/2$,

(iv) l'action du groupe d'isométries $O(n, \mathbb{R}) \ltimes \mathbb{R}^n$ *sur* $(\mathcal{UD}, D)$:

$$((\rho, t), \Lambda) \to (\rho, t) \cdot \Lambda = \rho(\Lambda) + t$$

est telle que son sous-groupe de translations $\mathbb{R}^n$ *agit continûment sur* $\mathcal{UD}$.

2.2 Espaces de Marcinkiewicz et découpages asymptotiques

L'existence de plus denses empilements de sphères de $\mathbb{R}^n, n \geq 2$, pose la question de savoir comment ils peuvent être construits. Le problème de donner des constructions d'empilements de sphères très denses entre les bornes de Kabatjanskii-Levenstein et de type Minkowski-Hlawka (voir Figure 1 dans [MVG3]) reste ouvert [Bdk] [Ca1] [CS] [GL] [GO] [Ro1] [Z].

Il y a deux problèmes : le premier est la détermination de la constante d'empilement δ_n comme fonction de n seulement (pour $n = 2$ et $n = 3$, il s'agit de $\pi/\sqrt{12}$ et de $\pi/\sqrt{18}$ respectivement ; pour $n = 3$, voir Hales [Ha1] [Ha2] [Ha3]) ; le second consiste à caractériser les configurations locales et la configuration globale dans un empilement de sphères très dense, en particulier lorsque la densité est δ_n. En fait il n'y a pas une configuration globale mais une infinité qui donnent la même densité, et la relation d'équivalence de Marcinkiewicz partitionne l'espace des empilements de sphères.

Soit $\Lambda \in \mathcal{UD}$. Soient $B = B(0, 1/2)$ et $p > 0$. Désignons par $\mathcal{L}^p_{loc}$ l'espace des fonctions à valeurs complexes f définies sur $\mathbb{R}^n$ dont la puissance p-ième de la valeur absolue $|f|^p$ est intégrable sur tout ensemble mesurable borné de $\mathbb{R}^n$ pour la mesure de Lebesgue. Le fait que la densité

$$\delta(\mathcal{B}(\Lambda)) := \limsup_{T \to +\infty} \left[\mathrm{vol}\big(\big(\bigcup_{\lambda \in \Lambda} (\lambda + B(0, 1/2)) \big) \bigcap B(0, T) \big) / \mathrm{vol}(B(0, T)) \right]$$

de $\mathcal{B}(\Lambda)$ soit égale à la norme ("norme 1") de Marcinkiewicz de la fonction caractéristique $\chi_{\mathcal{B}(\Lambda)}$ of $\mathcal{B}(\Lambda)$ [Bs+] [PH] [Mz], c'est-à-dire

$$\delta(\mathcal{B}(\Lambda)) = \|\chi_{\mathcal{B}(\Lambda)}\|_1, \tag{2.2}$$

où, pour tout $p \in \mathbb{R}^{+*}$ et toute $f \in \mathcal{L}^p_{loc}$,

$$\|f\|_p := \limsup_{t \to +\infty} |f|_{p,t}, \tag{2.3}$$

avec

$$|f|_{p,t} := \left(\frac{1}{\mathrm{vol}(tB)} \int_{tB} |f(x)|^p dx \right)^{1/p}, \qquad f \in \mathcal{L}^p_{loc}, \tag{2.4}$$

pose la question suivante : que peut dire la Théorie des Espaces de Marcinkiewicz au problème de la construction des très denses empilements de sphères ?

Il est clair que le problème de la détermination de la constante d'empilement ou plus généralement de la densité est associé à l'espace quotient $\mathcal{L}^p_{loc}/\mathcal{R}$ où $\mathcal{R}$ est la relation d'équivalence de Marcinkiewicz : on dit que $f \in \mathcal{L}^p_{loc}$ et $g \in \mathcal{L}^p_{loc}$ sont $\mathcal{M}^p$ - équivalentes si

$$\|f - g\|_p = 0.$$

On note $\mathcal{R}$ cette relation d'équivalence. La fonction densité est une fonction de classe, c'est-à-dire est bien définie sur l'espace de Marcinkiewicz $\mathcal{M}^p$ avec $p = 1$. Par exemple tout cluster fini de sphères a la même densité, égale à zéro, que l'empilement vide (aucune sphère) ; la classe de Marcinkiewicz de l'empilement vide étant beaucoup plus gros que l'ensemble des clusters finis de sphères. Il suffit donc de comprendre la construction d'un empilement de sphères particulier dans chaque classe de Marcinkiewicz. L'object de la note [MVG2] est de préciser les contraintes géométriques données par une telle construction.

Appelons $\overline{f}$ la classe dans $\mathcal{M}^p = \mathcal{L}^p_{loc}/\mathcal{R}$ d'une fonction $f \in \mathcal{L}^p_{loc}$, où $\mathcal{L}^p_{loc}$ est muni de la $\mathcal{M}^p$-topologie et par

$$\begin{array}{rrcl} \nu : & \mathcal{UD} & \rightarrow & \mathcal{L}^1_{loc}, \\ & \Lambda & \rightarrow & \chi_{\mathcal{B}(\Lambda)} \end{array} \qquad \text{resp.} \quad \begin{array}{rrcl} \overline{\nu} : & \mathcal{UD} & \rightarrow & \mathcal{M}^1 \\ & \Lambda & \rightarrow & \overline{\chi_{\mathcal{B}(\Lambda)}} \end{array}$$

le plongement (ensembliste) de $\mathcal{UD}$ dans $\mathcal{L}^1_{loc}$, resp. dans $\mathcal{M}^1$.

Théorème 2.5. *L'image $\nu(\mathcal{UD})$ dans $\mathcal{L}^1_{loc} \cap \mathcal{L}^\infty$, resp. $\overline{\nu}(\mathcal{UD})$ dans $\mathcal{M}^1$, est fermée.*

Les deux Théorèmes 2.1 et 2.2 reposent sur la continuité de la fonction densité $\| \cdot \|_1 \circ \nu$ sur l'espace $(\mathcal{UD}, D)$ (voir Section 2.1) comme suit.

Théorème 2.6. *La fonction densité $\Lambda \rightarrow \delta(\mathcal{B}(\Lambda)) = \|\chi(\mathcal{B}(\Lambda))\|_1$ est continue sur $(\mathcal{UD}, D)$ et y est localement constante.*

Le Théorème 2.5 est une reformulation du Théorème 2.7, puisque $\mathcal{M}^p$ est complet [Bs1] [Bs+]. Pour $0 \leq \lambda \leq \mu$ appelons

$$\mathcal{C}(\lambda, \mu) := \{x \in \mathbb{R}^n \mid \lambda \leq \|x\| \leq \mu\}$$

la portion d'espace annulaire fermée entre les sphères centrées à l'origine de rayons respectifs λ et μ.

Théorème 2.7. *Soit* $(\Lambda_m)_{m\geq 1}$ *une suite d'ensembles uniformément discrets de constante 1 telle que la suite* $(\chi_{\mathcal{B}(\Lambda_m)})_{m\geq 1}$ *soit une suite de Cauchy pour la pseudo-métrique* $\|\cdot\|_1$ *sur* $\mathcal{L}^1_{loc} \cap \mathcal{L}^\infty$*. Alors, il existe*

(i) une suite strictement croissante d'entiers positifs $(m_i)_{i\geq 1}$*,*

(ii) une suite strictement croissante de nombres réels $(\lambda_i)_{i\geq 1}$ *avec* $\lambda_i \geq 1$ *et* $\lambda_{i+1} > 2\lambda_i$*,*

telle que, avec

$$\Lambda = \bigcup_{i\geq 1} \Lambda_{m_i} \cap \mathcal{C}(\lambda_i + 1/2, \lambda_{i+1} - 1/2), \tag{2.5}$$

les deux fonctions

$$\chi_{\mathcal{B}(\Lambda)} \text{ and } \lim_{i\to+\infty} \chi_{\mathcal{B}(\Lambda_{m_i})}$$

sont $\mathcal{M}^1$*-équivalentes. Comme conséquence*

$$\delta(\mathcal{B}(\Lambda)) = \lim_{i\to+\infty} \delta(\mathcal{B}(\Lambda_{m_i})). \tag{2.6}$$

La situation est la suivante pour un empilement de sphères $\mathcal{B}(\Lambda)$ de $\mathbb{R}^n$ pour lequel $\delta(\mathcal{B}(\Lambda)) = \delta_n$:

* soit il n'existe pas d'empilements de sphères comme dans le Théorème 2.7 telle que la suite de Cauchy de leurs fonctions caractéristiques respectives soit non stationnaire, auquel cas il y a un *phénomène d'isolement*,
* ou il existe au moins une suite d'empilements de sphères comme au Théorème 2.7 telle que la suite de Cauchy correspondante de leurs fonctions caractéristiques soit non-stationnaire ; il est alors Marcinkiewicz - équivalent à un empilement de sphères ayant la structure asymptotique annulaire donnée par le Théorème 2.7, où la suite des épaisseurs des anneaux présente une croissance exponentielle.
  Cette opération fait alors apparaître la densité non plus comme une lim-sup mais comme une vraie limite (2.6).

Ce partionnement de l'espace en anneaux d'épaisseurs croissantes permet des constructions couche par couche dans chaque portion annulaire, de manière indépendante, puisque les interstices annulaires résiduels intermédiaires $\mathcal{C}(\lambda_i - 1/2, \lambda_i + 1/2)$ sont tous d'épaisseur égale à 1 qui est le double du rayon commun 1/2 des répliques de B. Ces régions intermédiaires ne contribuent pas à la densité si bien qu'elles peuvent être remplies par des sphères ou non. Cependant, l'existence de telles portions sphériques intermédiaires qui seraient laissées lacunaires est incompatible avec la construction d'empilements complètement saturés, au moins pour $n = 2$ [K-K].

Remarquons que la valeur 2 qui contrôle la suite exponentielle des rayons $(\lambda_i)_i$ par la relation $\lambda_{i+1} > 2\lambda_i$ dans le Théorème 2.7 (ii) peut être remplacée par n'importe quelle valeur $a > 1$ Ceci est important pour itérer une construction depuis un germe en dimension n : en effet, en choisissant $a > 1$

suffisamment petit le problème se ramène à remplir une couche de manière la plus dense possible, de manière équivalente ceci revient à travailler en dimension $n-1$, puis à propager dans la direction orthogonale radialement.

2.3 Trous, rayon de couverture et comportement asymptotique

Le Théorème 2.8 relie la densité et le rayon de couverture (constante de Delone) d'un empilement de sphères. Son contenu est très simple et, curieusement, ne semble pas avoir été vu auparavant puisqu'il ne semble pas figurer, à notre connaissance, dans aucun livre de Géométrie des Nombres. Il s'agirait d'un résultat de base, élémentaire.

La contribution [MVG3] reporte le Théorème 2.8 ainsi que ses conséquences à l'asymptotique de divers empilements de construction connue.

Théorème 2.8. *Soit* $n \geq 2$. *Si* $\Lambda \in \mathcal{UD}$ *est un ensemble de Delone de* $\mathbb{R}^n$ *de constante* R, *alors*

$$(2R)^{-n} \quad \leq \quad \delta(\mathcal{B}(\Lambda)) \quad \leq \quad \delta \qquad \text{pour tout } R \geq R_c = R_c(n), \tag{2.7}$$

où R_c *est défini par* $X_{R_c} \neq \emptyset$ *et* $X_s = \emptyset$ *dès que* $s < R_c$.

Appelons $\mu_n(R) := (2R)^{-n}$. La dépendance en R^{-n} de l'expression $\mu_n(R)$ avec n est très importante. Cela permet d'étudier les valeurs asymptotiques de la constante de Delone $R_c(n)$ lorsque n tend vers l'infini.

Théorème 2.9. *Pour tout* $\epsilon > 0$ *il existe* n_ϵ *tel que pour tout* $n > n_\epsilon$,

$$R_c(n) \geq 2^{-0.401} - \epsilon.$$

Le Théorème 2.9 affirme l'existence d'une collection infinie de cellules de Voronoi de taille moyenne dans tout plus dense empilement de sphères égales de $\mathbb{R}^n$ de rayon $1/2$ de rayon de sphère circonscrite plus grande que $2^{-0.401} + o(1) = 0.757333... + o(1)$.

Les applications de la formule (2.7) concernent les rayons de couverture des suites d'empilements de sphères données par des constructions connues. On obtient en effet des bornes inférieures explicites, en fonction de n, pour le rayon de couverture des réseaux de Barnes-Wall BW_n, Craig $\mathbb{A}_n^{(r)}$, Mordell-Weil MW_n, et des empilements de type BCH. On montre que les trous profonds ont des diamètres qui tendent vers l'infini pour plusieurs de ces suites lorsque n tend vers l'infini, ce qui les empêche d'être très denses.

Il est intéressant de comparer la borne inférieure $\mu_n(R)$, et "sa continuité avec R", avec des bornes asymptotiques connues, notamment dans la zone entre les bornes supérieures de Rogers ou Kabatjanskiĭ-Levenštein et les bornes inférieures de type Minkowski-Hlawka pour la densité d'empilements de réseau. Les petites valeurs de R entre les bornes $\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}$ (Blichfeldt [Bt]) et 1 (Butler [Bu]) correspondent à cette zone, où les sphères se touchent presque toutes sans que l'on ait contrôle sur le nombre de contacts réels de chaque sphère dans l'empilement. L'intervalle de valeurs possibles pour R est même inclus dans $[2^{0.401} - \epsilon, 1]$ par le Théorème 2.9.

La construction d'ensembles de Delone de très petite constante de Delone est un problème qui n'est pas considéré dans [MVG3].

Le comportement exponentiel dominant du minorant, en $(2R)^{-n}$, donné par le Théorème 2.8, s'obtient également simplement à partir de la relation entre la taille des trous profonds ($R > 1/2$), d'un empilement de sphères et le problème du recouvrement d'une boule de rayon R par des petites boules égales de rayon 1/2 (voir Section 2.4) comme suit.

Soit $\Lambda \in \mathcal{UD}$. La densité de l'empilement $\mathcal{B}(\Lambda)$ se calcule à partir d'un point de $\mathbb{R}^n$ choisi, mettons α, comme origine et est indépendante de ce point base [Ro1]. Supposons que Λ soit un ensemble de Delone de constante $R = R(\Lambda) > 0$. On a :

$$R(\Lambda) := \sup_{z\in\mathbb{R}^n} \inf_{\lambda\in\Lambda} \|z - \lambda\|.$$

Cela signifie qu'il existe au moins un trou profond (sphérique) dans Λ qui est un sommet d'une cellule de Voronoi de boule circonscrite de rayon $R(\Lambda)$. Supposons que $B(0, R(\Lambda))$ soit un tel trou profond, par une translation adéquate. Soient $x_1, x_2, \ldots, x_\nu$ des points distincts de $\mathbb{R}^n$ tels que

$$\bigcup_{j=1}^{\nu} \big(x_j + B(0, 1/2)\big) \quad \supset \quad B(0, R(\Lambda)).$$

Alors

$$\mathbb{R}^n \quad \subset \quad \bigcup_{i=1}^{\nu} \big(x_i + B(0, 1/2) + \Lambda\big).$$

En effet, tout $z \in \mathbb{R}^n$ est tel qu'il existe $\lambda \in \Lambda$ tel que $z - \lambda \in B(0, R(\Lambda))$. Par suite, il existe $i \in \{1, 2, \ldots, \nu\}$ tel que

$$z \in x_i + B(0, 1/2) + \Lambda.$$

Ce recouvrement de $\mathbb{R}^n$ conduit à

$$1 \leq \sum_{i=1}^{\nu} \chi_{\mathcal{B}(x_i+\Lambda)}(x) \qquad x \in \mathbb{R}^n$$

où $\chi_{\mathcal{B}(x_i+\Lambda)}(x)$ est la fonction caractéristique de l'ensemble $\mathcal{B}(x_i+\Lambda)$. On en déduit, pour tout $t>0$,

$$1 \leq \sum_{i=1}^{\nu} \frac{1}{\mathrm{Vol}(tB(0,1/2))} \int_{tB(0,1/2)} \chi_{\mathcal{B}(x_i+\Lambda)}(x)dx.$$

Par conséquent

$$1 \leq \sum_{i=1}^{\nu} \limsup_{t\to+\infty} \left(\frac{1}{\mathrm{Vol}(tB(0,1/2))} \int_{tB(0,1/2)} \chi_{\mathcal{B}(x_i+\Lambda)}(x)dx \right).$$

Il en résulte : $1 \leq \nu\delta(\mathcal{B}(\Lambda))$. De manière équivalente cela revient à mettre l'origine (variable) α successivement aux points $x_1, x_2, \ldots, x_\nu$ dans le trou. En particulier, avec les notations de la Section 2.4 :

$$\nu_{R(\Lambda),n}^{-1} \quad \leq \quad \delta(\mathcal{B}(\Lambda)),$$

ce qui donne le comportement dominant du minorant annoncé, en $(2R)^{-n}$, grâce au Théorème 2.11.

2.4 Recouvrement d'une boule par des petites boules égales

Soit $T > 1/2$. Posons

$\nu_{T,n} :=$ le nombre minimal de boules (fermées) de rayon 1/2 qui forment un recouvrement de la boule fermée de rayon T dans $\mathbb{R}^n, n \geq 2$.

Le problème suivant semble fondamental :

que sont les entiers $\nu_{T,n}$ *lorsque* $T > 1/2$, $n \geq 2$, *et quelles sont les configurations correspondantes de boules de rayon* $1/2$ *lorsqu'elles forment le recouvrement le plus économique de la boule fermée* $B(0,T)$ *(à rotation et symétrie près) ?*

Nous ne résolvons pas ce problème dans [VG1] mais nous donnons une borne supérieure explicite à $\nu_{T,n}$ dans le Théorème 2.11, qui s'avère meilleure que celle qui était connue jusqu'à présent : et qui est celle donnée par Rogers dans [Ro2] il y a plus de 40 ans, reproduite dans le Théorème 2.10 en corrigeant les quelques coquilles d'impression de l'article original. Dans la deuxième partie de [VG1] nous donnons l'asymptotique des estimations, *non-effective*, pour les bornes inférieures et supérieures de $\nu_{T,n}$, lorsque T et n tendent vers l'infini, grâce à des résultats récents de Böröczky Jr. et Wintsche [BW] sur l'asymptotique du nombre minimal de sphères égales de $\mathbb{R}^n$ qui recouvrent la sphère $\mathbb{S}^{n-1}$. L'optimalité de ces estimations asymptotiques est commentée et donne lieu à conjectures.

Rogers (pp 163-164 et Théorème 2 dans [Ro2]) a obtenu le résultat suivant :

Théorème 2.10. *(i) Si* $n \geq 3$, *avec* $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$,

$$1 < \nu_{T,n} \leq \begin{cases} e\vartheta_n (2T)^n & \text{if} \quad T \geq n/2, \\ n\vartheta_n (2T)^n & \text{if} \quad \frac{n}{2 \ln n} \leq T < \frac{n}{2}. \end{cases} \tag{2.8}$$

(ii) Si $n \geq 9$,

$$1 < \nu_{T,n} \leq \frac{4e(2T)^n n\sqrt{n}}{\ln n - 2} (n \ln n + n \ln(\ln n) + n \ln(2T) + \frac{1}{2} \ln(144n)) \tag{2.9}$$

pour tout $1/2 < T < \frac{n}{2 \ln n}$.

L'assertion (i) du Théorème 2.10 peut être facilement étendue au cas $n = 2$ en invoquant Rogers [Ro1], p 47, de sorte que la borne supérieure stricte $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$ de la densité de recouvrement par des boules égales dans $\mathbb{R}^n$ est encore valide dans ce cas. Ainsi les inégalités (2.8) sont encore vraies pour $n = 2$. Dans le cas $n = 2$, voir aussi Kershner [K]. Par ailleurs, l'assertion (ii) du Théorème 2.10 ne semble par avoir été améliorée depuis ; voir par exemple [GO], Fejes-Toth [FT], Schramm [Sm], Raigorodski [Ri] ou Bourgain et Lindenstrauss [BLs].

Le problème de la détermination explicite de $\nu_{T,n}$ est relié à l'existence de bornes explicites de la densité d'empilements de sphères égales de $\mathbb{R}^n$ par la taille de leurs trous profonds [MVG3], Section 2.3, ainsi qu'à des problèmes divers [MR] [IM] [FF] [Mw].

La raison pour laquelle nous avons repris les calculs (justes) de Rogers est que la formule proposée dans l'article original par Rogers n'est pas symétrique et fait intervenir des grandeurs non naturelles : un facteur "144", un dénominateur "$\ln n - 2$", pour des valeurs de n supérieures à "9", ... donc pour laquelle nous avons supposé d'office que le majorant devrait être reconsidéré autrement.

Théorème 2.11. *Soit* $n \geq 2$. *Les inégalités suivantes ont lieu :*

(i) $n < \nu_{T,n} \leq$

$$\frac{7^{4(\ln 7)/7}}{4} \sqrt{\frac{\pi}{2}} \frac{n\sqrt{n} \left[(n-1)\ln(2Tn) + (n-1)\ln(\ln n) + \frac{1}{2}\ln n + \ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n} - 2}\right)\right]}{T\,(1 - \frac{2}{\ln n})(1 - \frac{2}{\sqrt{\pi\, n}})(\ln n)^2} (2T)^n$$

$$\text{si} \quad 1 < T < \frac{n}{2 \ln n}, \tag{2.10}$$

(ii) $n < \nu_{T,n} \leq$

$$\sqrt{\frac{\pi}{2}} \frac{\sqrt{n} \left[(n-1)\ln(2Tn) + (n-1)\ln(\ln n) + \frac{1}{2}\ln n + \ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n} - 2}\right)\right]}{T\,(1 - \frac{2}{\ln n})(1 - \frac{2}{\sqrt{\pi\, n}})} (2T)^n$$

$$\text{si} \quad 1/2 < T \leq 1. \tag{2.11}$$

La deuxième amélioration du Théorème 2.10 de Rogers est donnée au Théorème 2.12. Elle n'est pas effective et s'appuie sur des résultats récents de Böröczky Jr. et Wintsche [BW]. Pour synthétiser les résultats (dans une écriture non-effective), prenons les notations suivantes. Etant données deux fonctions f et g réelles positives sur l'ensemble des entiers positifs, on écrira $f(n) \ll g(n)$ s'il existe deux constantes positives n_0 et c telles que, pour tout $n \geq n_0$, $f(n) \leq c \cdot g(n)$. Le point de départ est alors la liste suivante des estimations de Rogers :

$$\nu_{T,n} \ll \; n \ln n \cdot (2T)^n \qquad \text{si} \;\; T \geq \frac{n}{2}, \tag{2.12}$$

$$\nu_{T,n} \ll \; n^2 \ln n \cdot (2T)^n \qquad \text{si} \;\; \frac{n}{2\ln n} \leq T < \frac{n}{2}, \tag{2.13}$$

$$\nu_{T,n} \ll \; n^2 \sqrt{n} \cdot (2T)^n \qquad \text{si} \;\; \frac{1}{2} < T < \frac{n}{2\ln n}. \tag{2.14}$$

Probablement (2.12) ne peut pas être amélioré par les méthodes développées ici, et cette estimation est optimale à un facteur $\ln n$ près puisque [BW]

$$n \cdot (2T)^n \;\ll\; \nu_{T,n} \qquad \text{si } T \geq \frac{n}{2} \text{ ou si } \; T = \frac{\sqrt{n}}{2}. \tag{2.15}$$

Le Théorème 2.11 améliore le Théorème 2.10 par les estimations suivantes :

$$\nu_{T,n} \ll \; \frac{n^2\sqrt{n}}{T \ln n} \cdot (2T)^n \qquad \text{si} \;\; 1 < T \leq \frac{n}{2\ln n} \;\; \text{où} \;\; \frac{n^2\sqrt{n}}{T\ln n} > n\sqrt{n} \tag{2.16}$$

$$\nu_{T,n} \ll \; n\sqrt{n} \ln n \cdot (2T)^{n-1} \qquad \text{si} \;\; \frac{1}{2} < T \leq 1. \tag{2.17}$$

Les estimations asymptotiques sont les suivantes.

Théorème 2.12.

$$\nu_{T,n} \ll \; n \ln n \cdot (2T)^n \qquad \textit{si} \;\; T \geq \frac{\sqrt{n}}{2}, \tag{2.18}$$

$$\nu_{T,n} \ll \; \frac{n\sqrt{n}\ln n}{T} \cdot (2T)^n \qquad \textit{si} \;\; 1 \leq T \leq \frac{\sqrt{n}}{2}, \tag{2.19}$$

$$\frac{\nu_{T,n}}{(2T)^{n-1}} \ll \; n\sqrt{n}\sqrt{T - \frac{1}{2}} \cdot \ln 8(T - \frac{1}{2})n \qquad \textit{si} \;\; \frac{1}{2} + \frac{1}{4n} \leq T \leq 1, \tag{2.20}$$

$$\nu_{T,n} \ll \; 2n \qquad \textit{si} \;\; \frac{1}{2} < T \leq \frac{1}{2} + \frac{1}{4n}. \tag{2.21}$$

3 Empilements de sphères autosimilaires de type fini

L'objet de [VG2] est d'une part de donner un survol des liens existant entre Géométrie des Nombres et Cristaux Apériodiques en Physique de la Matière Condensée, d'autre part de prouver le Théorème 3.1 qui fournit l'existence de schémas de coupe-et-projection canoniques au-dessus d'ensembles uniformément discrets autosimilaires et de type fini, et d'en déduire quelques conséquences pour les empilements de sphères que l'on peut construire à l'aide de ceux-ci (Théorème 3.3).

La mathématique des ensembles de points uniformément discrets et des ensembles de Delone développée récemment a au moins quatre origines différentes : (i) l'évidence expérimentale d'états de la matière non périodiques, comme les quasicristaux [AG] [GM] [HG] [J] [S-C] ou les phases cristallines modulées incommensurables [JJ] [Ja] et leur modélisation géométrique (Appendice de [VG2]), (ii) certains travaux de Delone [De1] [De2] [F-S] [Ry] sur la cristallographie géométrique (comparativement, voir [H] [MM] [Op] [Sg] pour une approche mathématique classique des cristaux périodiques), (iii) certains travaux de Meyer sur les ensembles de coupe-et-projection et les ensembles de Meyer [Me1] [Me2] [Me3] [Pa] (pour un langage plus moderne sur les ensembles de Meyer dans les groupes abéliens localement compacts : [Mo1]), (iv) la théorie des pavages autosimilaires [B] [L-J] [So2] et l'utilisation de la théorie ergodique pour étudier la diffraction [BLz] [Sc] [So2] [So3]. En particulier, l'impact sur les mathématiques de la découverte des quasicristaux en 1984 [S-C], comme phases ordonnées à longue-portée, a été soulignée par Lagarias [La3]. Le terme *quasicristaux mathématiques* [BM] [La4] a été proposé pour appeler les ensembles de Delone qui sont utilisés comme modèles géométriques discrets de ces nouveaux états de la matière ayant des propriétés spectrales ou de diffraction particulières; en particulier les *cristaux* sont ceux pour lesquels le spectre est essentiellement purement ponctuel (voir [IUCr] [Se] et l'Appendice pour une nouvelle définition du mot *cristal*, et [Cw] [Gu] [Ho] pour la théorie de la diffraction). Les ensembles de Delone sont conçus comme généralisation naturelle des réseaux en cristallographie des cristaux apériodiques.

Une classification des ensembles uniformément discrets, qui étend de manière immédiate celle des ensembles de Delone [La1] [La2], est proposée dans [VG2]. Les ensembles uniformément discrets de type fini de $\mathbb{R}^n$ constituent la classe la plus grande sur laquelle une application adresse peut être définie [La1].

La théorie des ensembles uniformément discrets de type fini autosimilaires (UTA) généralise celle des empilements de sphères (égales) de réseaux de $\mathbb{R}^n$ [By2] [Ca1] [CS] [GL] [Mt] [Z] puisqu'un réseau est déjà un ensemble UTA lui-même (les entiers sont des autosimilarités). Les ensembles de Meyer autosimilaires admettent comme autosimilarités des nombres de Pisot ou de Salem [Me2] [Me3], alors que les ensembles de Delone de type fini et autosimilaires ont pour autosimilarités des nombres de Perron ou de Lind [La1]. C'est un problème ouvert de trouver un critère qui assure qu'un ensemble uniformément discret, de type fini ou non, admet au moins une autosimilarité. Pour un ensemble de Delone général on s'attend à ce que son ensemble d'opérations de symétrie et en particulier ses symétries d'inflation soient rares, particulièrement lorsque la dimension de l'espace ambiant est grande, a fortiori pour les ensembles UTA ; à titre comparatif Bannai [Ba] et Collinet [Cl] par exemple ont fait l'étude des symétries des réseaux.

L'existence de schémas de coupe-et-projection au-dessus d'un ensemble de Delone est utile pour caractériser l'ensemble de ses autosimilarités, centres d'inflation, amas de sphères locaux, etc [Cs1] [Cs2] [GVG] [M-P]. Etant donné un ensemble uniformément discret arbitraire, c'est un problème ouvert de savoir si un schéma de coupe-et-projection existe au-dessus de lui. Le Théorème 3.1 répond à ce problème en toute généralité pour un ensemble UTA $\Lambda \subset \mathbb{R}^n$. Appelons $K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$ la $\mathbb{R}$-algèbre étale, $\Sigma : K \to K_{\mathbb{R}}$ l'application canonique.

Théorème 3.1. *Soit $\Lambda \subset \mathbb{R}^n, n \geq 1$, un ensemble uniformément discret tel que $m :=$ rang $\mathbb{Z}[\Lambda - \Lambda] < +\infty$ avec $m \geq 1$. Soit $\lambda > 1$ une autosimilarité (affine) de Λ, i.e. un nombre réel > 1 tel que $\lambda(\Lambda - c) \subset \Lambda - c$ pour un certain $c \in \mathbb{R}^n$. Alors*

(i) λ est un entier algébrique réel de degré $d \geq 1$ et d divise m,

(ii) il existe $r = m/d$ vecteurs $w_1, w_2, \ldots, w_r$ $\mathbb{Q}$-linéairement indépendants dans le $\mathbb{Q}(\lambda)$-espace vectoriel $\mathbb{Q}[\Lambda - \Lambda]$ tel que $\mathbb{Z}[\Lambda - \Lambda]$ soit un $\mathbb{Z}$-sous-module de rang m du $\mathbb{Z}$-module :
$\mathbb{Z}[w_1, \lambda w_1, \ldots, \lambda^{d-1} w_1, w_2, \lambda w_2, \ldots, \lambda^{d-1} w_2, \ldots, w_r, \lambda w_r, \ldots, \lambda^{d-1} w_r]$,

(iii) pour toute $\mathbb{Z}$-base $\{v_1, v_2, \ldots, v_m\}$ de $\mathbb{Z}[\Lambda - \Lambda]$, une relation matricielle : $\lambda V = MV$ a lieu, dans laquelle $V = {}^t[v_1, \ldots, v_m]$ et M est une matrice inversible entière $m \times m$ dont le polynôme caractéristique satisfait $\det(X\,I - M) = (\varphi(X))^{m/d}$ où $\varphi(X)$ désigne le polynôme minimal de λ ; en particulier, $\det M = N_{K/\mathbb{Q}}(\lambda)^{m/d}$, où $N_{K/\mathbb{Q}}(\lambda)$ désigne la norme algébrique de λ,

(iv) il existe un schéma de coupe-et-projection au-dessus de Λ :

$$\Big(\coprod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|} \;\simeq\; H \times \mathbb{R}[\Lambda],\; L,\; \pi,\; \mathrm{pr}_1 \Big)$$

où le réseau $L = \coprod_{i=1}^{r} \Sigma\big(\mathbb{Z}[\lambda]\big) \frac{w_i}{\|w_i\|}$ est tel que $pr_1(L) \supset \mathbb{Z}[\Lambda - \Lambda]$, dont l'espace interne H est le produit de deux espaces :

$$H = \big(\; R_K \setminus \mathbb{R}[\Lambda] \; \big) \times \overline{G}$$

où R_K est l'image de $\mathbb{R}[\Lambda]$ dans $\coprod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|}$ par les plongements réels et complexes de K, et $\overline{G}$ la clôture dans $\coprod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|}$ de l'image par Σ

de l'espace des relations sur K entre les générateurs $w_1, \ldots, w_r$. L'espace $R_K \setminus \mathbb{R}[\Lambda]$ est appelé l'espace ombre de Λ. Ce schéma de coupe-et-projection est muni d'une structure Euclidienne donnée par une forme bilinéaire symétrique de type Trace pour laquelle R_K et $\overline{G}$ sont orthogonaux.

Le *cluster central* de la base $(\lambda^j w_i)_{i=1,\ldots,r, j=0,\ldots,d-1}$ est par définition l'ensemble $\{w_1, w_2, \ldots, w_r\}$. Certains vecteurs dans un cluster central peuvent être $\mathbb{R}$-linéairement dépendants. Il est facile de vérifier que

$$r = 1 \text{ dans le Théorème 3.1} \implies \begin{array}{l}\text{le } \mathbb{R}\text{-espace vectoriel engendré par } \Lambda \\ \text{est de dimension } 1.\end{array}$$

La réciproque est généralement fausse ; les ensembles de beta-entiers $\mathbb{Z}_\beta$ [B-L] (Section 4) sur la droite sont par exemple localement finis et autosimilaires mais on ne sait pas s'ils sont uniformément discrets ou non, de type fini ou non en général. Lorsque β est un nombre de Pisot, $\mathbb{Z}_\beta$ est un ensemble UTA puisqu'il est un ensemble de Meyer.

Corollaire 3.2. *Si $\mathcal{B}(\Lambda)$ est un empilement de sphères de $\mathbb{R}^n$ tel que Λ soit de type fini, admette λ comme autosimilarité et soit tel que $r = 1$, i.e. pour lequel le degré d de λ est égal au rang m de $\mathbb{Z}[\Lambda - \Lambda]$, alors $\mathbb{Z}[\Lambda - \Lambda]$ est la projection d'un sous-réseau d'indice fini d'un réseau idéal de $K = \mathbb{Q}(\lambda)$ dans le schéma de coupe-et-projection au-dessus de Λ. Cet indice est un entier multiple de $(\mathcal{O}_K : \mathbb{Z}[\lambda])$.*

Le Théorème 3.1 donne un cadre pour construire des empilements de sphères (égales) $\mathcal{B}(\Lambda)$ pour lesquels les arrangements locaux peuvent se décrire par des jeux de coordonnées finies, comme pour les réseaux. Cela permet de transporter à ces ensembles de sphères les techniques acquises pour les réseaux : t-designs [BV], etc. Dans le Corollaire 3.2 la terminologie "diviseur d'Arakelov", signifiant que le plongement de $\mathbb{Z}[\Lambda - \Lambda]$ dans le schéma de coupe-et-projection est donné par une structure Euclidienne, peut remplacer "réseau idéal" par la correspondence bijective entre ces deux notions [Sf] (voir aussi Neukirch [Ne]).

Le Théorème 2.8 montre qu'il est important d'obtenir des bornes inférieures intéressantes de la constante de Delone pour un ensemble de Delone général. Pour un ensemble de Delone qui est en outre un ensemble UTA, le fait qu'il existe un corps de nombres engendré par l'autosimilarité fait qu'il est possible de donner canoniquement une borne inférieure à la constante de Delone lorsque l'ensemble UTA provient d'un ensemble modèle de fenêtre Ω positionnée particulièrement par rapport au réseau du schéma de coupe-et-projection canonique associé. Cela donne la notion de *k-minceur* pour un ensemble UTA.

Dans [VG2] on montre la double origine de la constante de Delone d'un ensemble UTA : la première origine vient d'abord des propriétés géométriques du cluster central par le Théorème 3.1, la deuxième origine est purement arithmétique, étant la conséquence de résultats récents de Cerri [C1] [C2] sur les minima Euclidien et inhomogène du corps de nombres K et de ceux de Henk [Hk2] sur l'existence de *plans libres* dans les empilements de sphères de réseaux. Ces deux influences procèdent par un sous-réseau d'un produit de *réseaux idéaux* [By2] en bijection avec $\mathbb{Z}[\Lambda - \Lambda]$ dans le schéma de coupe-et-projection donné par le Théorème 3.1 (iv). On ne traite dans [VG2] que le cas $r = 1$, c'est-à-dire le cas d'empilements de sphères alignés (sur une droite affine dans $\mathbb{R}^n$), en ayant en vue le cas général r quelconque.

Théorème 3.3. *Soit $\Lambda \subset \mathbb{R}^n, n \geq 1$, un ensemble UTA qui est soit un ensemble modèle soit un ensemble de Meyer, de fenêtre Ω, dans le schéma de coupe-et-projection défini par le Théorème 3.1 (iv) où $r = 1$, et L' est un réseau tel que $pr_1(L') = \mathbb{Z}[\Lambda - \Lambda]$.*

Supposons que l'autosimilarité λ soit de degré $d \geq 3$, que le rang du groupe des unités de $K = \mathbb{Q}(\lambda)$ soit > 1 et que ce ne soit pas un corps CM. Alors, si Λ est k-mince, $k \geq 2$, sa constante de Delone $R(\Lambda)$ satisfait :

$$R(\Lambda) \;\geq\; \sqrt{d}\,\left(M(K)^{2/d} - M_k(K)^{2/d}\right)^{\frac{1}{2}}, \tag{3.1}$$

où $M(K)$, resp. $M_k(K)$, est le minimum Euclidien, resp. le k-ième minimum Euclidien, de K.

La *condition de remplissage* de l'espace : $m/d = r \geq n$ pour les couples de valeurs $\{(d, m)\}$ (avec les notations du Théorème 3.1) est nécessaire pour construire des empilements de sphères denses dans $\mathbb{R}^n$.

4 Beta-entiers et beta-réseaux

Gazeau [G1] a introduit dès 1997 la notion d'ensemble de beta-entiers, ou β-entiers, que l'on note $\mathbb{Z}_\beta$. Ses objectifs étaient liés à l'observation de beta-réseaux comme supports uniformément discrets pour l'ensemble des positions atomiques de modèles de quasicristaux en physique de la matière condensée, et visaient à établir des systèmes d'ondelettes adéquats sur ces objets. L'état quasicristallin a été découvert en 1984 [S-C] et a donné lieu à de nombreuses théories pour l'expliquer, dont les ensembles modèles provenant de schémas de coupe-et-projection, invariants par l'action d'un groupe ponctuel interdit en cristallographie "classique", dès 1985 [DK], puis les beta-réseaux [G1] [G2].

Les schémas de coupe-et-projection avaient été introduits auparavant (sous une terminologie différente) par Meyer [Me1] [Me2] [Me3] dans les années 70, mais pour des problèmes de synthèse spectrale. Il s'agissait d'introduire en Analyse Harmonique les ensembles dits *harmonieux* dans des groupes abéliens localement compacts. Les constructions de Meyer ont été redécouvertes par la suite par Moody et Patera [Mo1].

Les valeurs naturelles de β envisagées par Gazeau [G1] [G2] pour les beta-réseaux viennent de celles qui sont observables dans la modélisation des quasicristaux en physique, c'est-à-dire souvent des nombres de Pisot quadratiques. Le développement de l'étude des beta-entiers, pour $\beta > 1$ un nombre de Pisot général, s'est alors renforcé sous l'impulsion de C. Frougny [B-K1] [B-K2] [Fy1] [Fy2] par l'usage de techniques de numération en base non-entière.

Les questions posées par les beta-entiers dépassent en fait de beaucoup les considérations initiales provenant de la physique et sont pleinement d'ordre mathématique.

Les beta-entiers sont complètement contrôlés par le développement de Rényi $d_\beta(1)$ de 1, sur lequel de nombreux problèmes sont ouverts. Nous les présentons dans la suite.

4.1 Etude géométrique des beta-entiers avec β un nombre de Perron

Soit $\beta > 1$. Un beta-entier (ou β-entier) positif est par définition un nombre réel positif dont le développement de Rényi en base β n'admet aucune partie fractionnaire [Re] [Py]. Il s'agit d'un polynôme en β dont les coefficients sont dans un alphabet donné, donc bornés, et constituent une suite de coefficients strictement inférieure lexicographiquement, ainsi que tous ses décalés (Conditions de Parry), à une borne supérieure donnée par le développement de Rényi $d_\beta(1)$ de 1. L'ensemble des beta-entiers se note $\mathbb{Z}_\beta$ et a pour propriétés :

$$\mathbb{Z}_\beta = -\mathbb{Z}_\beta, \qquad \beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta.$$

L'ensemble $\mathbb{Z}_\beta$ est autosimilaire par construction, discret, localement fini. L'ensemble

$$B = \{\beta > 1 \mid \mathbb{Z}_\beta \text{ est uniformément discret}\}$$

contient les nombres de Pisot (Thurston [T]). L'ensemble B n'est pas caractérisé pour le moment.

Nous rappelons dans [GVG1] la numération en base β et les beta-entiers d'après Rényi, Parry et Frougny [Fy1] [Fy2]. Ensuite, nous montrons qu'il existe un schéma de coupe-et-projection canonique au-dessus des beta-entiers pour tout nombre de Perron (degré $m \geq 2$). Ce schéma de coupe-et-projection est plongé naturellement dans la décomposition de Jordan de $\mathbb{R}^m$ sur lequel agit la matrice compagnon de β. Il s'agit d'un cadre géométrique où une direction dilatante et un sous-espace de codimension 1 (dit *espace interne*) permettent de réaliser par projection sur une certaine droite l'ensemble $\mathbb{Z}_\beta$. Ces sous-espaces s'obtiennent en inversant la matrice de Vandermonde associée à β (qui provient de la matrice compagnon de β) par les polynômes d'interpolation de Lagrange. On n'a pas besoin de savoir que $\mathbb{Z}_\beta$ est uniformément discret pour cette construction ; qui donc est indépendante du fait que β appartient à B ou non.

Cette décomposition géométrique de $\mathbb{R}^m$ s'effectue sans invoquer aucun système de substitution sur un alphabet fini [A-I] [PF] ou la théorie de Perron-Frobenius [Mc]. Lorsqu'en particulier β est un nombre de Pisot, cela définit une fenêtre d'acceptation minimale dans l'espace interne qui est le fractal de Rauzy [Ra] [AI]. Nous le montrons sur un exemple (Messaoudi [Mi1] [Mi2]). Nous en déduisons, en utilisant ces constructions, une preuve géométrique du fait que $\mathbb{Z}_\beta$ est un ensemble de Meyer lorsque β est un nombre de Pisot.

A ce point, nous indiquons la différence principale avec l'approche substitutive qui est que les matrices en jeu peuvent avoir des coefficients négatifs (voir Akiyama [Ak1] [Ak2]).

Les propriétés additives de $\mathbb{Z}_\beta$ sont étudiées dans la deuxième partie de [GVG1] au moyen de ces schémas de coupe-et-projection canoniques lorsque β est un nombre de Pisot : en A), nous montrons que les éléments de $\mathbb{Z}_\beta^+ = \mathbb{Z}_\beta \cap \mathbb{R}^+$ peuvent être représentés par des combinaisons à coefficients dans $\mathbb{N}$ d'éléments de $\mathbb{Z}_\beta$ de petite norme, en nombre fini, en utilisant des cônes tronqués dont l'axe de révolution est la direction dominante de la matrice compagnon de β et un Lemme de Lind sur les semi-groupes ; en B) nous donnons une interprétation géométrique de la prépériode du β-développement du nombre réel qui est la somme de deux beta-entiers, des ensembles finis T et T' dans les relations [B-K1] [B-K2]

$$\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + T \quad , \quad \mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta + T'$$

et une borne supérieure de l'entier q qui apparaît dans la relation

$$x, y \in \mathbb{Z}_\beta^+ \Longrightarrow x \pm y \in \frac{1}{\beta^q}\mathbb{Z}_\beta$$

lorsque $x + y$ et $x - y$ ont des β-développements finis.

4.2 Lacunes dans $d_\beta(1)$ et classification des nombres algébriques

L'exploration des liens entre Dynamique Symbolique et Théorie des Nombres, relatifs aux β-entiers lorsque $\beta > 1$ est un nombre algébrique ou plus généralement un nombre réel, a commencé avec Bertrand-Mathis [Be1] [Be2]. Blanchard [Bl] a proposé une classification possible des nombres réels selon leur β-shift, par les propriétés du développement de Rényi $d_\beta(1)$ de 1 (en fait il s'agit d'une classification proposée par Bertrand-Mathis et reportée dans [Bl]). De nombreuses questions restent ouvertes concernant la distribution des nombres algébriques $\beta > 1$ dans cette classification (reportée ci-dessous).

Le développement de Rényi de 1 est important puisqu'il contrôle le β-shift [Py] et l'ensemble discret localement fini $\mathbb{Z}_\beta \subset \mathbb{R}$ des β-entiers.

L'article [VG3] a pour objectif de donner un nouveau théorème (Théorème 4.1) sur les lacunes (plages de zéros) dans $d_\beta(1)$ pour $\beta > 1$ un nombre algébrique, et d'apporter des réponses partielles à certaines questions [Bl], en particulier pour les nombres de Salem (Corollaire 4.2).

Le Théorème 4.1 fournit une borne supérieure asymptotique du quotient de lacune de $d_\beta(1)$. Elle est obtenue par des méthodes classiques “à la Liouville” utilisées par Mahler et Güting, puis améliorée par une version récente de l'inégalité de Liouville (Proposition 3.14 dans Waldschmidt [Wa]). Le cheminement de la démonstration du Théorème 4.1 se révèle très fructueux puisqu'il conduit à une nouvelle classification des nombres algébriques $\beta > 1$ fondée sur l'asymptotique des lacunes dans $d_\beta(1)$ et de propriétés caractéristiques, notamment la mesure de Mahler $\mathrm{M}(\beta)$, de β (la définition de $\mathrm{M}(\beta)$ est rappelée dans [VG3] Section 3). Cette double paramétrisation, symbolique et algébrique, a été devinée dans [Bl] p 137. Cette nouvelle classification est complémentaire de celle de Bertrand-Mathis [Bl] pp 137–139, les deux étant rappelées ci-dessous à titre de comparaison. La question de savoir si un nombre algébrique $\beta > 1$ est contenu dans une classe ou une autre a déjà été étudiée et discutée par de nombreux auteurs [Be1] [Be2] [Be3] [Bl] [Bo1] [Bo2] [D-S] [FS] [Li1] [Li2] [Py] [PF] [Sch] [Sk] et dépend au moins de la distribution des conjugués de β dans le plan complexe. Seuls les conjugués de β de module strictement plus grand que 1 interviennent dans le Théorème 4.1 par la mesure de Mahler de β. On déduit immédiatement de cette remarque le Corollaire 4.2. Par suite tous les nombres de Salem appartiennent à $\mathrm{C}_1 \cup \mathrm{C}_2 \cup \mathrm{Q}_0$, alors que les nombres de Pisot sont dans $\mathrm{C}_1 \cup \mathrm{C}_2$ [T].

Une autre démonstration du Corollaire 4.2 consiste à contrôler les plages de zéros de $d_\beta(1)$ par des Théorèmes de Géométrie Diophantienne permettant de prendre en compte en même temps des familles convenables de places du corps de nombres $K = \mathbb{Q}(\beta)$ associées aux conjugués de β et les propriétés de lacunarité de $d_\beta(1)$. Cette autre preuve du Corollaire 4.2 est juste esquissée dans la Section 4 de [VG3], et est obtenue en utilisant le Théorème de Thue-Siegel-Roth cité par Corvaja [Cj] [Ad].

Théorème 4.1. *Soient* $\beta > 1$ *un nombre algébrique et* $\mathrm{M}(\beta)$ *sa mesure de Mahler. Soit* $d_\beta(1) := 0.t_1t_2t_3\ldots$, *avec* $t_i \in A_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$, *le développement de Rényi de* 1. *Supposons que* $d_\beta(1)$ *soit infini et lacunaire dans le sens suivant : il existe deux suites* $\{m_n\}_{n\geq 1}$ *et* $\{s_n\}_{n\geq 0}$ *telles que*

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \ldots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \ldots$$

avec $(s_n - m_n) \geq 2$, $t_{m_n} \neq 0, t_{s_n} \neq 0$ *et* $t_i = 0$ *si* $m_n < i < s_n$ *pour tout* $n \geq 1$. *Alors*

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} \leq \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}. \tag{4.1}$$

En outre, si $\liminf_{n\to+\infty}(m_{n+1} - m_n) = +\infty$, *alors*

$$\limsup_{n\to+\infty} \frac{s_{n+1} - s_n}{m_{n+1} - m_n} \leq \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}. \tag{4.2}$$

Suivant Ostrowski [Os] le quotient $s_n/m_n \geq 1$ est appelé *quotient de lacune*, relativement à la n-ième lacune (en supposant $t_j \neq 0$ pour tout $s_n \leq j \leq m_{n+1}$ de manière à avoir une description unique des lacunes de zéros). Remarquons que le terme "*lacunaire*" a souvent d'autres significations dans la littérature. Appelons $\mathcal{L}(S_\beta)$ le langage du β-shift [Bl] [Fy1] [Fy2] [Lo]. La classification de Bertrand-Mathis/Blanchard ([Bl] pp 137–139) est la suivante :

C_1 : $d_\beta(1)$ est fini.
C_2 : $d_\beta(1)$ est ultimement périodique mais non fini.
C_3 : $d_\beta(1)$ contient des plages bornées de lacunes de zéros, mais n'est pas ultimement périodique.
C_4 : $d_\beta(1)$ ne contient pas certains mots de $\mathcal{L}(S_\beta)$, mais contient des plages de zéros de longueurs arbitrairement grandes.
C_5 : $d_\beta(1)$ contient tous les mots de $\mathcal{L}(S_\beta)$.

Classes de nombres algébriques proposée dans le présent travail, avec les notations du Théorème 4.1 :

$Q_0^{(1)}$: $\quad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n}$ avec $(m_{n+1} - m_n)$ borné.

$$\mathrm{Q}_0^{(2)} \; : \qquad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n} \qquad \text{avec } (s_n - m_n) \text{ borné et } \lim_{n\to+\infty} (m_{n+1} - m_n) = +\infty.$$

$$\mathrm{Q}_0^{(3)} \; : \qquad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n} \qquad \text{avec } \limsup_{n\to+\infty} (s_n - m_n) = +\infty.$$

$$\mathrm{Q}_1 \; : \qquad 1 < \limsup_{n\to+\infty} \frac{s_n}{m_n} < \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}.$$

$$\mathrm{Q}_2 \; : \qquad \limsup_{n\to+\infty} \frac{s_n}{m_n} = \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}.$$

Quelles sont les proportions relatives de chaque classe dans l'ensemble $\overline{\mathbb{Q}}_{>1}$ des nombres algébriques $\beta > 1$? En comparant C_2, C_3 et Q_0^1, quelles sont les proportions relatives dans Q_0^1 entre le sous-ensemble des β pour lesquels $d_\beta(1)$ est ultimement périodique et celui pour lesquels $d_\beta(1)$ n'est pas ultimement périodique ? Schmeling ([Sch] Theorem A) a montré que la classe C_3 (des nombres réels $\beta > 1$) admet une dimension de Hausdorff égale à 1. Nous avons :

- $\overline{\mathbb{Q}}_{>1} \cap \mathrm{C}_2 \quad \subset \quad \mathrm{Q}_0^{(1)}$,
- $\overline{\mathbb{Q}}_{>1} \cap \mathrm{C}_3 \quad \subset \quad \mathrm{Q}_0^{(1)} \cup \mathrm{Q}_0^{(2)}$, avec $\mathrm{C}_3 \cap \mathrm{Q}_0^{(3)} = \emptyset$,
- $\overline{\mathbb{Q}}_{>1} \cap \mathrm{C}_4 \quad \subset \quad \mathrm{Q}_0^{(3)} \cup \mathrm{Q}_1 \cup \mathrm{Q}_2$.

Les nombres de Pisot β sont contenus dans $\mathrm{C}_1 \cup \mathrm{Q}_0^{(1)}$ puisqu'ils sont tels que $d_\beta(1)$ est fini ou ultimement périodique (Parry [Py], Bertrand-Mathis [Be3]). Rappelons qu'un nombre de Perron est un entier algébrique $\beta > 1$ tel que tous les conjugués $\beta^{(i)}$ de β satisfont à $|\beta^{(i)}| < \beta$. Réciproquement, d'après Lind [Li1], Denker, Grillenberger, Sigmund [D-S] et Bertrand-Mathis [Be2], si $\beta > 1$ est tel que $d_\beta(1)$ est ultimement périodique (fini ou non), alors β est un nombre de Perron. Tous les nombres de Perron ne sont pas atteints de cette manière : un nombre de Perron qui possède un conjugué réel plus grand que 1 ne peut être tel que $d_\beta(1)$ soit ultimement périodique ([Bl] p 138). Quant aux nombres de Parry, ils appartiennent à $\mathrm{C}_1 \cup \mathrm{C}_2$. Soit $\mathrm{Q}_0 = \mathrm{Q}_0^{(1)} \cup \mathrm{Q}_0^{(2)} \cup \mathrm{Q}_0^{(3)}$.

Corollaire 4.2. *Soit $\beta > 1$ un nombre de Salem qui n'appartient pas à C_1. Alors β appartient à la classe Q_0.*

La répartition des nombres de Salem dans les classes C_1, $\mathrm{Q}_0^{(1)}$, $\mathrm{Q}_0^{(2)}$ et $\mathrm{Q}_0^{(3)}$ est un problème ouvert, sauf en petit degré. Boyd [Bo] [Bo3] a montré que les nombres de Salem de degré 4 appartiennent à C_2, par conséquent à $\mathrm{Q}_0^{(1)}$. C'est aussi le cas de certains nombres de Salem de degré 6 et 8 dans le cadre d'un modèle probabiliste [Bo2] [Bo3]. Dans [VG3] nous posons la question de savoir si les nombres de Perron pourraient vérifier le Corollaire 4.2.

Comme la définition de la classe $\mathrm{Q}_0 = \mathrm{Q}_0^{(1)} \cup \mathrm{Q}_0^{(2)} \cup \mathrm{Q}_0^{(3)}$ ne fait pas allusion à β, c'est-à-dire à $\mathrm{M}(\beta)$, aux conjugués de β, au polynôme minimal de β, à sa longueur, etc, mais prend en compte uniquement les quotients de lacunes dans $d_\beta(1)$, la classe Q_0, telle qu'elle est définie, peut s'adresser à tout nombre réel $\beta > 1$, en toute généralité, au lieu de se restreindre aux seuls nombres algébriques > 1. La question de savoir s'il existe des nombres transcendants $\beta > 1$ qui appartiennent à la classe Q_0 a été posée dans [Bl] ; dans quelle proportion dans chaque sous-classe ? On donne dans [VG3] quelques exemples de nombres transcendants (constante de Komornik-Loreti [AC] [KoL], nombres Sturmiens [CK]) dans Q_0.

4.3 Lois de groupes sur les β-entiers avec β un nombre de Pisot quadratique unitaire et groupes de symétrie des beta-réseaux

La contribution [E-VG] est un travail en collaboration et porte sur une construction de *groupe du plan* pour les ensembles de points quasipériodiques de $\mathbb{R}^2$ que sont les beta-réseaux. Cela est rendu possible par les structures algébriques que l'on peut mettre sur les beta-entiers $\mathbb{Z}_\beta$ et leur asymptotique. Les beta-réseaux sont une combinaison vectorielle des beta-entiers selon des directions dans le plan complexe données par des racines de l'unité adéquates. Les nombres de Pisot envisagés sont ici :

$$\frac{1+\sqrt{5}}{2}, \quad 1+\sqrt{2}, \quad 2+\sqrt{3}.$$

Lorsque $\beta > 1$ est un nombre de Pisot unitaire quadratique, $\mathbb{Z}_\beta$ peut en effet être muni canoniquement d'une structure de groupe abélien additif et d'une loi de multiplication interne : la *beta-addition* et la *beta-multiplication*. Les groupes de symétrie du plan qui laissent invariant un beta-réseau sont engendrés par l'équivalent des rotations et des translations, que l'on appelle ici des *beta-rotations* et des *beta-translations*. Avec cette nouvelle arithmétique liée à ces nouvelles lois internes, un beta-réseau peut être considéré comme un réseau. La fonction de comptage $\rho_S(n)$ quasipériodique, définie sur l'ensemble des beta-entiers comme celle qui dénombre les petites tuiles (S) entre l'origine et le n-ième beta-entier, joue un rôle central dans ces nouvelles structures de groupe. En particulier, cette fonction se comporte asymptotiquement comme une fonction linéaire. Il s'ensuit que l'on peut considérer les beta-réseaux et leurs symétries comme des réseaux munis de leurs opérations de symétrie habituelles.

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Deuxième Partie

“On a generalization of the Selection Theorem of Mahler”,

G. Muraz et J.-L. Verger-Gaugry,

J. de Théorie des Nombres de Bordeaux 17 (2005), 237–269.

Journal de Théorie des Nombres
de Bordeaux **17** (2005), 237–269

On a generalization of the Selection Theorem of Mahler

par Gilbert MURAZ et Jean-Louis VERGER-GAUGRY

Résumé. On montre que l'ensemble $\mathcal{UD}_r$ des ensembles de points de $\mathbb{R}^n, n \geq 1$, qui ont la propriété que leur distance interpoint minimale est plus grande qu'une constante strictement positive $r > 0$ donnée est muni d'une métrique pour lequel il est compact et tel que la métrique de Hausdorff sur le sous-ensemble $\mathcal{UD}_{r,f} \subset \mathcal{UD}_r$ des ensembles de points finis est compatible avec la restriction de cette topologie à $\mathcal{UD}_{r,f}$. Nous montrons que ses ensembles de Delaunay (Delone) de constantes données dans $\mathbb{R}^n, n \geq 1$, sont compacts. Trois (classes de) métriques, dont l'une de nature cristallographique, nécessitant un point base dans l'espace ambiant, sont données avec leurs propriétés, pour lesquelles nous montrons qu'elles sont topologiquement équivalentes. On prouve que le processus d'enlèvement de points est uniformément continu à l'infini. Nous montrons que ce Théorème de compacité implique le Théorème classique de Sélection de Mahler. Nous discutons la généralisation de ce résultat à des espaces ambiants autres que $\mathbb{R}^n$. L'espace $\mathcal{UD}_r$ est l'espace des empilements de sphères égales de rayon $r/2$.

Abstract. The set $\mathcal{UD}_r$ of point sets of $\mathbb{R}^n, n \geq 1$, having the property that their minimal interpoint distance is greater than a given strictly positive constant $r > 0$ is shown to be equippable by a metric for which it is a compact topological space and such that the Hausdorff metric on the subset $\mathcal{UD}_{r,f} \subset \mathcal{UD}_r$ of the finite point sets is compatible with the restriction of this topology to $\mathcal{UD}_{r,f}$. We show that its subsets of Delone sets of given constants in $\mathbb{R}^n, n \geq 1$, are compact. Three (classes of) metrics, whose one of crystallographic nature, requiring a base point in the ambient space, are given with their corresponding properties, for which we show topological equivalence. The point-removal process is proved to be uniformly continuous at infinity. We prove that this compactness Theorem implies the classical Selection Theorem of Mahler. We discuss generalizations of this result to ambient spaces other than $\mathbb{R}^n$. The space $\mathcal{UD}_r$ is the space of equal sphere packings of radius $r/2$.

1. Introduction

In 1946 Mahler [Ma] obtained important results on star bodies and their critical lattices in $\mathbb{R}^n$ using the following fundamental result called now Mahler's Selection Theorem or Mahler's compactness Theorem.

Theorem 1.1. *Let (L_r) be a sequence of lattices of $\mathbb{R}^n$ such that, for all r:*

(i) $\|x\| \geq c$ *for all* $x \in L_r, x \neq 0$, *with c a strictly positive constant independent of r,*

(ii) *the Lebesgue measure $|L_r|$ of the fundamental region of L_r satisfies $|L_r| \leq M$ with M a constant $< +\infty$ independent of r.*

Then one can extract from the sequence (L_r) a subsequence $(L_{r'})$ that converges to a lattice L of $\mathbb{R}^n$ such that $|L| = \lim_{r' \to +\infty} |L_{r'}|$.

This Theorem is very efficient in many problems of Geometry of Numbers [Ca], [GL] and is as important as the Ascoli-Arzela Theorem in Analysis. The desirability of extending the main Theorems of Geometry of numbers, whose Mahler's compactness Theorem, to general algebraic number fields and more was emphasized by Mahler in a seminar at Princeton [RSD]. Several authors revisited this Theorem, giving generalizations and analogs for other ambient spaces than $\mathbb{R}^n$: Chabauty [Ch] with subgroups in locally compact abelian groups, Mumford [Mu] in semi-simple Lie groups without compact factors and moduli spaces of compact Riemann surfaces of given genus, Macbeath and Swierczkowski [MS] in locally compact and σ-compact topological groups (abelian or not) which are compactly generated, McFeat [Mf] in adele spaces of number fields, Rogers and Swinnerton-Dyer [RSD] in algebraic number fields. Groemer [Groe] gave an elegant proof of this Theorem by showing that it is a consequence of the Selection Theorem of Blaschke [Ca], by noticing the bicontinuous one-to-one correspondence between lattices and their Voronoi domains.

The way that Chabauty [Ch] proved Theorem 1.1 is extremely instructive. A careful attention to his "elementary" proof reveals the very important following fact that the $\mathbb{Z}$-additive structure of the lattices L_r is not necessary to obtain the convergence of a subsequence. From this essential remark, Chabauty proposed in [Ch] a possible extension of Mahler's compactness Theorem to locally compact abelian groups as ambient spaces with a suitable topology, method which was improved by Mumford [Mu]. Furthermore it opens the way to deal with sequences of non-periodic point sets, that is without any additional algebraic structure, instead of only lattices or subgroups, suggesting that Mahler's Selection Theorem should exist in more general situations.

In the present note we develop a new version of Theorem 1.1 adapted to point sets (i.e. not only lattice or subgroup point sets) in an "ambient space". This can be formulated as follows. We will be interested in sets of point sets, say $\mathcal{UD}(H,\delta)_r$, of a metric space (H,δ), which is the "ambient space", where δ is a metric on H, which have the property that the minimal interpoint distance is greater than or equal to a given strictly positive constant, say r. Point sets of H having this property are said *uniformly discrete sets of constant* r. Denote by $\mathcal{UD}(H,\delta)_{r,f}$ the subset of $\mathcal{UD}(H,\delta)_r$ formed by the finite point sets. Concerning assertion (i) in Theorem 1.1, the fundamental question is now the following:

Question 1.1. For which metric spaces (H,δ) can the set $\mathcal{UD}(H,\delta)_r$ be endowed with a topology such that it is compact and that the Hausdorff metric Δ on $\mathcal{UD}(H,\delta)_{r,f}$ is compatible with the restriction of this topology to $\mathcal{UD}(H,\delta)_{r,f}$ and for which values of r ?

In the objective of generalizing assertion (ii) of Theorem 1.1, let us recall the (Besicovitch) concept of relative denseness [MVG]: a subset Λ of (H,δ) is said relatively dense (for δ) in H if there exists $R>0$ such that for all $z\in H$ there exists $\lambda\in\Lambda$ such that $\delta(z,\lambda)\leq R$. We will say that Λ is relatively dense *of constant* R if R is chosen minimal for that property. Then, assuming H satisfies Question 1.1 for some $r>0$, we can formulate the second question as follows:

Question 1.2. For which metric spaces (H,δ) is the subset $X(H,\delta)_{r,R}$ of $\mathcal{UD}(H,\delta)_r$ of the relatively dense subsets of H of given constant $R>0$ compact, and for which values of R ?

By definition, a subset Λ of (H,δ) is a Delone set if there exist $r>0$ and $R>0$ such that it is uniformly discrete of constant $\geq r$ and relatively dense of constant $R>0$. In this case we say that Λ is a *Delone set of constants* (r,R) (see [MVG] for possible values of R/r when $H=\mathbb{R}^n$). For instance, a lattice in $(\mathbb{R}^n,\|\cdot\|)$ is already a Delone set, where $\|\cdot\|$ is the standard euclidean metric. Note that Question 1.2 makes sense for any ambient space (H,δ) for which Delone sets are infinite, as $(\mathbb{R}^n,\|\cdot\|)$. Indeed, if H is such that its Delone sets are all finite, then Question 1.2 can be answered by the classical properties of the Hausdorff metric on the space of compact subsets of H (see Section 6). The main Theorem of this note is the following (proved in Section 4). It provides answers to Question 1.1 and Question 1.2 when $H=\mathbb{R}^n$ and $\delta=\|\cdot\|$. For short, in this case, let us denote by $\mathcal{UD}_r$, resp. $\mathcal{UD}_{r,f}$, the set $\mathcal{UD}(\mathbb{R}^n,\delta)_r$, resp. $\mathcal{UD}(\mathbb{R}^n,\delta)_{r,f}$, and by $X_{r,R}$ the set $X(\mathbb{R}^n,\delta)_{r,R}$.

Theorem 1.2. *For all $r>0$, the set $\mathcal{UD}_r$ can be endowed with a metric d such that the topological space $(\mathcal{UD}_r,d)$ is compact and such that the Hausdorff metric on $\mathcal{UD}_{r,f}$ is compatible with the restriction of the topology of $(\mathcal{UD}_r,d)$ to $\mathcal{UD}_{r,f}$. For all $R>0$, the subspace $X_{r,R}$ of $(\mathcal{UD}_r,d)$ of the Delone sets of constants (r,R) is closed.*

Note that $\mathcal{UD}_r$ is exactly the space of (equal) sphere packings of radius $r/2$ of $\mathbb{R}^n$ [Ca], [Ro].

In Section 2 a construction of d is given from an averaging sequence of compact sets $(K_k)_{k\geq 1}$ of $\mathbb{R}^n$ and the corresponding Hausdorff metric on $\mathcal{UD}(K_k, \|\cdot\|)_{r,f}, k \geq 0$.

Two other constructions of equivalent metrics are given in Section 3; the first one (Subsection 3.1) is inspired by a metric put on the space of Delone sets, which is used in tiling dynamical systems arising from Delone sets (see Radin and Wolff [RW], Robinson [Ro], Solomyak [So], Gouéré [Go], Baake and Lenz [BL]); this metric is here adapted to uniformly discrete sets. The second one (Subsection 3.2) is obtained by point-counting systems normalized by suitable distances: this idea was first formulated by Dworkin in [Dw] (for Delone sets) though given there without any proof by the author. In this last case, since its construction is far away from the Hausdorff metric, we show in final that it implies compatibility with the Hausdorff metric on $\mathcal{UD}_{r,f}$ (Corollary 3.2). The construction of this last metric may seem overly complicated at first sight, but it is of crytallographic nature, with purposes in Geometry of Numbers, while the two other metrics arise from Analysis. The third metric is adapted to study local clusters of spheres in dense sphere packings, whose geometrical classification reveals to be essential, as in Hales's works on Kepler Conjecture [Ha], [La] (see Remark in §3.3). These three metrics require *a base point* in the ambient space $\mathbb{R}^n$, which will be conveniently taken common and equal to 0. They give a way to create new metrics on $\mathcal{UD}_r$, for instance invariant by translations and crystallographic operations adapted to study local and global properties of aperiodic sphere packings [MVG], [MVG1]. In Subsection 3.3 we show that these metrics are topologically equivalent. This topological equivalence is deeply related to the uniform continuity of the removal process of points of a $\mathcal{UD}$-set at infinity (Proposition 3.10 and Proposition A.1 in the Appendix).

In Section 5 we show that Theorem 1.2 implies Mahler's Selection Theorem 1.1 and comment in Section 6 on the space H to provide positive answers to Question 1.1 and Question 1.2. In particular we extend a theorem of Macbeath and Swierczkowski [MS] to the metric case (see Theorem 6.1).

The Appendix contains a proof of the uniform continuity, for the third metric, of the removal process of points of an arbitrary $\mathcal{U}_{\mathcal{D}}$-set Λ at infinity (Proposition A.1), given in a self-contained and detailed way in Step 2. The computations in Step 1, relative to the case $\Lambda = \mathbb{Z}^n$, useful in Step 2, are treated in the same way as in Step 2, therefore in a detailed way, to help the reader.

In the sequel we assume $r = 1$, the general case being identical, and denote $\mathcal{UD}_1$ by $\mathcal{UD}$, resp. $X_{1,R}$ by X_R, and by $\mathcal{UD}_f$ the space of finite uniformly discrete sets of $\mathbb{R}^n$ of constant 1. Elements of $\mathcal{UD}$ are called $\mathcal{UD}$-sets: they are, either the subset of $\mathbb{R}^n$ which contains no point, i.e. the empty set, denoted by $\emptyset$, or one-point subsets $\{x\}$ of $\mathbb{R}^n$, with $x \in \mathbb{R}^n$, or discrete point subsets Λ of $\mathbb{R}^n$ which contain at least two points such that $\|x - y\| \geq 1$ as soon as $x, y \in \Lambda$, with $x \neq y$. $\mathcal{UD}$-sets of $\mathbb{R}^n$, except the empty set $\emptyset$, may have very different $\mathbb{R}$-spans, with affine dimensions ranging from 0 to n. We denote by $B(c, \epsilon)$ the closed ball of $\mathbb{R}^n$ of center c and radius $\epsilon \geq 0$, by $\overset{\circ}{B}(c, \epsilon)$ its interior, by $\mathrm{diam}(A)$ (resp. $\mathcal{R}_A$) the diameter (resp. the circumscribed radius $:= \sup_{a \in A} \|a\|$) of a nonempty subset A of $\mathbb{R}^n$, and by $\mathrm{dist}(A, B)$ the distance $\inf\{\|a - b\| \mid a \in A, b \in B\}$ between two nonempty subsets A and B of $\mathbb{R}^n$.

2. Construction of a metric from the Hausdorff metric

Denote by Δ the Hausdorff metric on the space of nonempty closed subsets of $\mathbb{R}^n$ and by the same symbol its restriction to the space of nonempty closed subsets of any nonempty compact subset of $\mathbb{R}^n$:

$$\Delta(F, G) := \inf\{\rho \geq 0 \mid F \subset G + B(0, \rho) \text{ and } G \subset F + B(0, \rho)\}. \tag{2.1}$$

Let $(K_k)_{k \geq 1}$ be an averaging sequence of compacts sets of $\mathbb{R}^n$ which contains the base point $p_{\text{base}} := 0$: $K_1 \supset \{p_{\text{base}} = 0\}$ and for all $k \geq 1, K_k \subset K_{k+1}$, with the property $\cup_{k \geq 1} K_k = \mathbb{R}^n$. For all $k \geq 1$ and all $\Lambda, \Lambda' \in \mathcal{UD}$ which are not simultaneously empty, we put

$$d_k(\Lambda, \Lambda') := \Delta(\Lambda \cap K_k, \Lambda' \cap K_k). \tag{2.2}$$

If $\Lambda \cap K_k$ or $\Lambda' \cap K_k$ is empty, then $d_k(\Lambda, \Lambda')$ takes the value $+\infty$. On the contrary, since we use the convention that for all $c \in \mathbb{R}^n$ and all $\epsilon \geq 0$ the $\mathcal{UD}$-set $\emptyset + B(c, \epsilon)$ equals the emptyset $\emptyset$, we have:

$$d_k(\emptyset, \emptyset) = 0 \qquad \text{for all } k \geq 0. \tag{2.3}$$

Then we define the mapping d on $\mathcal{UD} \times \mathcal{UD}$, valued in $[0, 1]$, associated with $(K_k)_{k \geq 1}$, by

$$d(\Lambda, \Lambda') := \sum_{k \geq 1} 2^{-k} \frac{d_k(\Lambda, \Lambda')}{1 + d_k(\Lambda, \Lambda')} \qquad \text{for all } \Lambda, \Lambda' \in \mathcal{UD} \tag{2.4}$$

(with $d_k(\Lambda, \Lambda')/(1 + d_k(\Lambda, \Lambda')) = 1$ when $d_k(\Lambda, \Lambda') = +\infty$).

Proposition 2.1. *The mapping d is a metric on $\mathcal{UD}$. The Hausdorff metric on $\mathcal{UD}_f$ is compatible with the restriction of the topology of $(\mathcal{UD}, d)$ to $\mathcal{UD}_f$.*

Proof. Obvious by (2.3), and by construction for the compatibility with Δ. □

Remark. If $(K'_k)_{k\geq 1}$ is another averaging sequence of compact sets of $\mathbb{R}^n$ such that K'_1 contains the base point 0, the metric d' associated with $(K'_k)_{k\geq 1}$ is topologically equivalent to the above metric d constructed from $(K_k)_{k\geq 1}$: indeed, if $(F_n)_n$ is a sequence of $\mathcal{UD}$-sets which converges to a $\mathcal{UD}$-set F for the metric d', i.e. $d'(F_n, F) \to 0, n \to \infty$, then, for all $k \geq 1$, $(F_n \cap K'_k)_n$ is a Cauchy sequence in $(\mathcal{UD}(K'_k, \|\cdot\|)_1, \Delta)$. If j_k is the greatest integer l such that $K_l \subset K'_k$, then $(F_n \cap K_{j_k})_n$ is a Cauchy sequence in $(\mathcal{UD}(K_{j_k}, \|\cdot\|)_1, \Delta)$ which converges to $F \cap K_{j_k}$. Since $\cup_{k\geq 1} F \cap K'_k = \cup_{k\geq 1} F \cap K_{j_k} = \cup_{k\geq 1} F \cap K_k = F$, for all $k \geq 1$, $d_{j_k}(F_n, F)$ tends to 0 when n tends to infinity. We deduce $\lim_{n\to+\infty} d(F_n, F) = 0$ by (2.4) and Lebesgue dominated convergence theorem. Therefore, to obtain a distance d with properties easy to describe, it suffices to consider an averaging sequence of balls centered at the base point 0 of $\mathbb{R}^n$: for instance, $K_k = B(0, R_k), k \geq 1$, with $(R_k)_{k\geq 1}$ a strictly increasing sequence satisfying $\lim_{k\to+\infty} R_k = +\infty$.

Let us note that if 2^{-k} is replaced by a_k in (2.4) where $0 \leq a_k$ and $\sum_{k\geq 1} a_k < +\infty$, we obtain another metric which is also topologically equivalent to d. All these possibilities constitute the class of metrics of d.

A discrete subset Λ of $\mathbb{R}^n$ is said *locally finite* if $\Lambda \cap B(c, \epsilon)$ is finite for all $c \in \mathbb{R}^n$ and all $\epsilon > 0$. The distance d can be extended to the space of locally finite subsets of $\mathbb{R}^n$. Denote by $\mathcal{D}_{l\,f}(\mathbb{R}^n, \|\cdot\|)$ this space. Note that $\emptyset \in \cup_{r>0} \mathcal{UD}_r$ and that $\cup_{r>0} \mathcal{UD}_r$ is contained in $\mathcal{D}_{l\,f}(\mathbb{R}^n, \|\cdot\|)$.

Proposition 2.2. *The mapping d associated with an averaging sequence of compact sets $(K_k)_{k\geq 1}$ of $\mathbb{R}^n$ is a metric on the space $\mathcal{D}_{l\,f}(\mathbb{R}^n, \|\cdot\|)$ of locally finite discrete subsets of $\mathbb{R}^n$. The Hausdorff metric on $\cup_{r>0} \mathcal{UD}_{r,f}$ is compatible with the restriction of the topology of $(\mathcal{D}_{l\,f}(\mathbb{R}^n, \|\cdot\|), d)$ to $\cup_{r>0} \mathcal{UD}_{r,f}$.*

Proof. Same construction and arguments as in Proposition 2.1. □

3. Equivalent metrics

3.1. From tiling dynamical systems. Let $R_{\min} > 0$ be defined by the following property: $X_R = \emptyset$ if $R < R_{\min}$. It is the smallest possible Delone constant of any Delone set (with minimal interpoint distance ≥ 1) in $\mathbb{R}^n$, and depends only upon n [MVG]. It is linked to packings of equal spheres of radius $1/2$ in $\mathbb{R}^n$ exhibiting spherical holes whose radius is always smaller than or equal to $R_{\min}$ and therefore to densest sphere packings [MVG], [CS]. Let $\lambda \geq 2\,R_{\min}^2$ and $p_{\text{base}} = 0$ the base point of $\mathbb{R}^n$. Then, for all $\Lambda, \Lambda' \in \mathcal{UD}$ denote:

$$\text{(3.1)} \quad \Omega(\Lambda, \Lambda') := \Big\{\alpha > 0 \mid \Lambda \cap B(p_{\text{base}}, \frac{\lambda}{\alpha}) \subset \Lambda' + B(0, \alpha) \\ \text{and } \Lambda' \cap B(p_{\text{base}}, \frac{\lambda}{\alpha}) \subset \Lambda + B(0, \alpha)\Big\}$$

and define

$$\text{(3.2)} \qquad \delta_1(\Lambda, \Lambda') := \min\left\{1, \frac{\inf\ \Omega(\Lambda, \Lambda')}{R_{\min}}\right\}.$$

Observe that, if $\alpha \in \Omega(\Lambda, \Lambda')$, then $[\alpha, +\infty) \subset \Omega(\Lambda, \Lambda')$. We have: $\delta_1(\emptyset, \emptyset) = 0$ and, for all $\Lambda \neq \emptyset$, $\delta_1(\Lambda, \emptyset) = \min\{1, \frac{\lambda}{R_{\min}\, \text{dist}(\{0\}, \Lambda)}\}$.

Proposition 3.1. *The mapping δ_1 is a metric on $\mathcal{UD}$. The Hausdorff metric on $\mathcal{UD}_f$ is compatible with the restriction of the topology of $(\mathcal{UD}, \delta_1)$ to $\mathcal{UD}_f$.*

Proof. It is obviously symmetrical. If $\Lambda = \Lambda'$, then $\delta_1(\Lambda, \Lambda') = 0$. Let us show the converse. Assume $\delta_1(\Lambda, \Lambda') = 0$. If Λ or Λ' is the empty set, then it is easy to show that both are equal to $\emptyset$. Assume now that $\Lambda \neq \emptyset, \Lambda' \neq \emptyset$ and that Λ strictly contains $\Lambda \cap \Lambda'$. Then there exists $x \in \Lambda, x \notin \Lambda'$ such that $\text{dist}(\{x\}, \Lambda') > 0$. Since $\Omega(\Lambda, \Lambda')$ equals $(0, +\infty)$ by assumption, it contains in particular $\frac{1}{2}\,\text{dist}(\{x\}, \Lambda) > 0$ and also $\lambda/(2\|x\|) > 0$ if $x \neq 0$. Take $\beta := \frac{1}{2}\min\{\text{dist}(\{x\}, \Lambda'), \lambda/\|x\|\}$ when $x \neq 0$, and $\beta := \frac{1}{2}\,\text{dist}(\{0\}, \Lambda')$ when $x = 0$. Then we would have: $x \in \Lambda \cap B(0, \frac{\lambda}{\beta})$ but $x \notin \Lambda' + B(0, \beta)$. Contradiction. Therefore, $\Lambda = \Lambda \cap \Lambda'$, equivalently $\Lambda \subset \Lambda'$. In a similar way, by symmetry, we obtain $\Lambda' \subset \Lambda$, hence the equality $\Lambda = \Lambda'$.

Let us show the triangle inequality:

$$\delta_1(\Lambda, \Lambda'') \leq \delta_1(\Lambda, \Lambda') + \delta_1(\Lambda', \Lambda'').$$

If $\delta_1(\Lambda, \Lambda') = 1$ or if $\delta_1(\Lambda', \Lambda'') = 1$, then it is satisfied. Assume now $\delta_1(\Lambda, \Lambda') < 1$ and $\delta_1(\Lambda', \Lambda'') < 1$. Let $a \in \Omega(\Lambda, \Lambda')$ and $b \in \Omega(\Lambda', \Lambda'')$. Then $a < R_{\min}$ and $b < R_{\min}$. Let $e = a + b$. Then

$$\Lambda \cap B(0, \frac{\lambda}{e}) \subset \Lambda \cap B(0, \frac{\lambda}{a}) \subset \Lambda' + B(0, a).$$

This implies:

$$\Lambda \cap B(0, \frac{\lambda}{e}) \subset \Lambda' \cap B(0, \frac{\lambda}{e} + a) + B(0, a).$$

But $a + \frac{\lambda}{e} \leq \frac{\lambda}{b}$: indeed, since $be \leq 2\,R_{\min}^2$, we have: $\frac{\lambda}{b} - \frac{\lambda}{e} - a = \frac{a}{be}(\lambda - be) \geq 0$. Hence,

$$\Lambda \cap B(0, \frac{\lambda}{e}) \subset \Lambda' \cap B(0, \frac{\lambda}{b}) + B(0, a) \subset \Lambda'' + B(0, b) + B(0, a) = \Lambda'' + B(0, e).$$

Therefore $e \in \Omega(\Lambda, \Lambda'')$, that is $\Omega(\Lambda, \Lambda'') \supset \Omega(\Lambda, \Lambda') + \Omega(\Lambda', \Lambda'')$. This implies the triangle inequality.

To prove the compatibility of the Hausdorff metric Δ on $\mathcal{UD}_f$ with the topology arising from δ_1, it suffices to show, given $\Lambda, \Lambda' \in \mathcal{UD}_f$ such that $\delta_1(\Lambda, \Lambda')$ is small enough, that the following equality holds:

$$\delta_1(\Lambda, \Lambda') = \frac{\Delta(\Lambda, \Lambda')}{R_{\min}}. \tag{3.3}$$

Indeed, if $\delta_1(\Lambda, \Lambda')$ is small enough, then there exists $T \in \Omega(\Lambda, \Lambda')$ such that $\Lambda = \Lambda \cap B(0, \frac{\lambda}{T})$ and $\Lambda' = \Lambda' \cap B(0, \frac{\lambda}{T})$. Thus

$$\inf \Omega(\Lambda, \Lambda') = \inf\{\rho \geq 0 \mid \Lambda \subset \Lambda' + B(0, \rho)$$

and

$$\Lambda' \subset \Lambda + B(0, \rho)\} = \Delta(\Lambda, \Lambda').$$

We deduce (3.3). □

Proposition 3.2. *The mapping δ_1 is a metric on the space $\mathcal{D}_{l\,f}(\mathbb{R}^n, \|\cdot\|)$ of locally finite discrete subsets of $\mathbb{R}^n$. The Hausdorff metric on $\cup_{r>0}\mathcal{UD}_{r,f}$ is compatible with the restriction of the topology of $(\mathcal{D}_{l\,f}(\mathbb{R}^n, \|\cdot\|), \delta_1)$ to $\cup_{r>0}\mathcal{UD}_{r,f}$.*

Proof. Same construction as in Proposition 3.1. □

Remark. After Blichfeldt (see [MVG]) we have: $R_{\min} \geq \frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}$, hence $\lambda \geq \frac{n}{n+1}$. Given $\Lambda \in X_R$, with $R \geq R_{\min}$, denote by $\Lambda^{(dh)}$ the uniformly discrete set of $\mathbb{R}^n$ constituted by the deepest holes of Λ [CS]. Consider the class of metrics δ_1 constructed as above when $\lambda \geq 2R_{\min}^2$ varies. The normalization factor $R_{\min}^{-1}$ in the definition of δ_1 comes from the fact that, for all $\Lambda \in X_R$ with $R \geq R_{\min}$, we have: $\delta_1(\Lambda, \Lambda^{(dh)}) = 1$ for all λ large enough.

3.2. From point-counting systems with equal spheres. Contrarily to d and δ_1 the metric δ_2 constructed here on $\mathcal{UD}$ has no natural extension to $\mathcal{D}_{l\,f}(\mathbb{R}^n, \|\cdot\|)$. But it possesses nice properties (Subsection 3.2.2) like the point pairing property (Proposition 3.6).

3.2.1. *Construction.* Let $\mathcal{E} = \{(D, E) \mid D$ countable point set in $\mathbb{R}^n, E$ countable point set in $(0, \frac{1}{2})\}$ and $f : \mathbb{R}^n \to [0, 1]$ a continuous function with compact support in $B(0, 1)$ which satisfies $f(0) = 1$ and $f(t) \leq \frac{1/2+\|\lambda - t/2\|}{1/2+\|\lambda\|}$ for all $t \in B(0, 1)$ and $\lambda \in \mathbb{R}^n$ (for technical reasons which will appear below; it is a weak hypothesis; take for instance $f(t) = 1 - \|t\|$ on $B(0, 1)$ and $f(t) = 0$ elsewhere). Recall that a pseudo-metric δ on a space satisfies all the axioms of a distance except that $\delta(u,v)=0$ does not necessarily imply $u=v$.

With each element $(D,E)\in\mathcal{E}$ and (variable) origin α of the affine euclidean space $\mathbb{R}^n$ we associate a real-valued function $d_{\alpha,(D,E)}$ on $\mathcal{UD}\times\mathcal{UD}$ in the following way. Let $\mathcal{B}_{(D,E)}=\{\mathcal{B}_m\}$ denote the countable set of all possible finite collections $\mathcal{B}_m=\{\overset{\circ}{B}(c_1^{(m)},\epsilon_1^{(m)}),\overset{\circ}{B}(c_2^{(m)},\epsilon_2^{(m)}),\ldots,\overset{\circ}{B}(c_{i_m}^{(m)},\epsilon_{i_m}^{(m)})\}$ (with i_m the number of elements $\#\mathcal{B}_m$ of $\mathcal{B}_m$) of open balls such that $c_q^{(m)}\in D$ and $\epsilon_q^{(m)}\in E$ for all $q\in\{1,2,\ldots,i_m\}$, and such that for all m and any pair of balls in $\mathcal{B}_m$ of respective centers $c_q^{(m)}$ and $c_k^{(m)}$, we have $\|c_q^{(m)}-c_k^{(m)}\|\geq 1$. Since any $\mathcal{UD}$-set Λ is countable, we denote by Λ_i its i-th element. Then we define the following function, with $\Lambda,\Lambda'\in\mathcal{UD}$, and base point $p_{\text{base}}=0$:

(3.4)

$$d_{\alpha,(D,E)}(\Lambda,\Lambda')$$
$$:=\sup_{\mathcal{B}_m\in\mathcal{B}_{(D,E)}}\frac{|\phi_{\mathcal{B}_m}(\Lambda)-\phi_{\mathcal{B}_m}(\Lambda')|}{(1/2+\|\alpha-p_{\text{base}}\|+\|\alpha-c_1^{(m)}\|+\|\alpha-c_2^{(m)}\|+\cdots+\|\alpha-c_{i_m}^{(m)}\|)}$$

where the real-valued function $\phi_{\mathcal{B}_m}$ is given by

$$\phi_{\mathcal{B}_m}(\Lambda):=\sum_{\overset{\circ}{B}(c,\epsilon)\in\mathcal{B}_m}\sum_i\epsilon f\Big(\frac{\Lambda_i-c}{\epsilon}\Big).$$

By convention we put $\phi_{\mathcal{B}_m}(\emptyset)=0$ for all $\mathcal{B}_m\in\mathcal{B}_{(D,E)}$ and all $(D,E)\in\mathcal{E}$. It is clear that, for all m and $\Lambda\in\mathcal{UD}$, inside each ball $\overset{\circ}{B}(c,\epsilon)\in\mathcal{B}_m$, there is at most one point of Λ and therefore the summation $\sum_i\epsilon f\big(\frac{\Lambda_i-c}{\epsilon}\big)$ is reduced to at most one non-zero term. Therefore the sum $\phi_{\mathcal{B}_m}(\Lambda)$ is finite.

Lemma 3.1. *For all $(\alpha,(D,E))$ in $\mathbb{R}^n\times\mathcal{E}$, $d_{\alpha,(D,E)}$ is a pseudo-metric valued in $[0,1]$.*

Proof. Let $\alpha\in\mathbb{R}^n$ and $(D,E)\in\mathcal{E}$. It is easy to check that $d_{\alpha,(D,E)}$ is a pseudo-metric on $\mathcal{UD}$. Let us show it is valued in $[0,1]$. Let us consider $\mathcal{B}_m\in\mathcal{B}_{(D,E)}$ for which the centers of its constitutive balls are denoted by $c_1,c_2,\ldots,c_{i_m}$. Then we have

$$\frac{i_m}{2}\leq 1/2+\|\alpha\|+\|\alpha-c_1\|+\|\alpha-c_2\|+\cdots+\|\alpha-c_{i_m}\|.$$

Indeed, if there exists $j\in\{1,2,\ldots,i_m\}$ such that $\|c_j-\alpha\|\leq\frac{1}{2}$, then for all $k\neq j$, $\|c_k-\alpha\|\geq\frac{1}{2}$. Hence

$$\frac{1}{2}+\|\alpha\|+\|\alpha-c_1\|+\|\alpha-c_2\|+\cdots+\|\alpha-c_{i_m}\|\geq\frac{1}{2}+\|\alpha\|+\frac{i_m-1}{2}\geq\frac{i_m}{2}.$$

If $\|c_k - \alpha\| \geq \frac{1}{2}$ for all $k \in \{1, 2, \ldots, i_m\}$, then

$$\frac{1}{2} + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\| \geq \frac{1}{2} + \|\alpha\| + \frac{i_m}{2} \geq \frac{i_m}{2}.$$

On the other hand, since the radii of the balls $\overset{\circ}{B}(c_j, \epsilon_j) \in \mathcal{B}_m$ are less than $\frac{1}{2}$ by construction, we have $0 \leq \phi_{\mathcal{B}_m}(\Lambda) \leq \frac{i_m}{2}$ for all $\mathcal{UD}$-set Λ. Therefore $|\phi_{\mathcal{B}_m}(\Lambda) - \phi_{\mathcal{B}_m}(\Lambda')| \leq \frac{1}{2} + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\|$, for all $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$ and all $\mathcal{UD}$-sets Λ, Λ'. We deduce the claim. □

The uniform topology on $\mathcal{UD}$ given by the pseudo-metrics $d_{\alpha,(D,E)}$ is generated by the open sets $\{\Lambda \in \mathcal{UD} \mid d_{\alpha,(D,E)}(u, \Lambda) < \epsilon\}$, $u \in \mathcal{UD}$ (Weil [We]). In order to get rid of a peculiar choice of the (variable) origin α and of the element (D, E) of $\mathcal{E}$, the supremum over all choices $(\alpha, (D, E))$ in $\mathbb{R}^n \times \mathcal{E}$ is taken.

Proposition 3.3. *The supremum* $\delta_2 := \sup_{\alpha \in \mathbb{R}^n, (D,E) \in \mathcal{E}} d_{\alpha,(D,E)}$ *is a metric on* $\mathcal{UD}$*, valued in* $[0, 1]$.

Proof. The supremum of the family of pseudo-metrics $d_{\alpha,(D,E)}$ is a pseudo-metric which takes its values in $[0, 1]$. We have to show that δ_2 is a metric. Assume Λ, Λ' are $\mathcal{UD}$-sets which are not empty such that $\delta_2(\Lambda, \Lambda') = 0$ and let us show that $\Lambda = \Lambda'$. We will show that $\Lambda \not\subset \Lambda'$ and $\Lambda' \not\subset \Lambda$ are impossible. Assume that $\Lambda \neq \Lambda'$ and that $\Lambda \not\subset \Lambda'$. Then there exists $\lambda \in \Lambda$ such that $\lambda \notin \Lambda'$. Denote $\epsilon := \frac{1}{2} \min\{\frac{1}{2}, \min\{\|\lambda - u\| \mid u \in \Lambda'\}\}$. We have $\epsilon > 0$ since Λ' is a $\mathcal{UD}$-set. The ball $\overset{\circ}{B}(\lambda, \epsilon)$ contains no point of Λ' and only the point λ of Λ. Take $\alpha = \lambda$, $D = \{\lambda\}, E = \{\epsilon\}$. We have $d_{\lambda,(D,E)}(\Lambda, \Lambda') = \frac{\epsilon}{1/2 + \|\lambda\|} > 0$. Hence $\delta_2(\Lambda, \Lambda')$ would be strictly positive. Contradiction. Therefore $\Lambda \subset \Lambda'$. Then, exchanging Λ and Λ', we have $\Lambda' \subset \Lambda$. We deduce the equality $\Lambda = \Lambda'$. If we assume that one of the $\mathcal{UD}$-sets Λ or Λ' is the empty set, we see that the above proof is still valid. □

3.2.2. *Properties.*

Proposition 3.4. *For all* $A, B, C \in \mathcal{UD}$ *such that* $A \cup B \in \mathcal{UD}$ *and all* $(D, E) \in \mathcal{E}$ *and* $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$*, the following assertions hold:*

(i) $\phi_{\mathcal{B}_m}(A \cup B) + \phi_{\mathcal{B}_m}(A \cap B) = \phi_{\mathcal{B}_m}(A) + \phi_{\mathcal{B}_m}(B)$;

(ii) $\delta_2(A \cup B, C) \leq \delta_2(A, C) + \delta_2(B, A \cap B)$;

(iii) $\delta_2(A \cap B, C) \leq \delta_2(A, C) + \delta_2(B, A \cup B)$.

In particular:

(iv) $\delta_2(A \cup B, \emptyset) \leq \delta_2(A, \emptyset) + \delta_2(B, \emptyset)$ *as soon as* $A \cap B = \emptyset$;

(v) $\delta_2(A \cup B, A \cap B) \le \min\{\delta_2(A, A \cap B) + \delta_2(B, A \cap B), \delta_2(A, A \cup B) + \delta_2(B, A \cup B)\}$;

(vi) *if* B *is reduced to one point, say* $\{\lambda\}$*, such that* $\lambda \notin A$*, we have:* $\delta_2(A \cup \{\lambda\}, C) \le \min\{\delta_2(A, C) + \delta_2(\{\lambda\}, \emptyset), \delta_2(\{\lambda\}, C) + \delta_2(A, \emptyset)\}$.

Proof. Assertion (i) can easily be checked from the definition of $\phi_{\mathcal{B}_m}$. Assertion (ii) is a consequence of (i) and of the inequality

$$\begin{aligned}|\phi_{\mathcal{B}_m}(A \cup B) - \phi_{\mathcal{B}_m}(C)| &= |\phi_{\mathcal{B}_m}(A) + \phi_{\mathcal{B}_m}(B) - \phi_{\mathcal{B}_m}(A \cap B) - \phi_{\mathcal{B}_m}(C)| \\ &\le |\phi_{\mathcal{B}_m}(A) - \phi_{\mathcal{B}_m}(C)| + |\phi_{\mathcal{B}_m}(B) - \phi_{\mathcal{B}_m}(A \cap B)|.\end{aligned}$$

Assertion (iii) follows from (ii) by exchanging "∪" and "∩". Assertions (iv) to (vi) can be deduced from (i), (ii) and (iii). □

Assertions (iv) and (vi) in Proposition 3.4 show the special role played by the "empty set" element $\emptyset$ in the set-theoretic processes of "point addition" and "point removal". The uniform continuity of the "point removal process" at infinity of the points of a $\mathcal{UD}$-set is proved in Section 3.3 and in the Appendix.

Proposition 3.5. *The following equalities hold:*

(i) $\delta_2(\{\lambda\}, \emptyset) = \frac{1}{1+2\|\lambda\|}$, *for all* $\lambda \in \mathbb{R}^n$ *(remarkably this value does not depend upon* $f(x)$*),*

(ii) $\delta_2(\Lambda - \{\lambda\}, \Lambda) = \frac{1}{1+2\|\lambda\|}$ *for all non-empty* $\mathcal{UD}$*-set* Λ *and all* $\lambda \in \Lambda$.

Proof. (i) First, let us show that $\delta_2(\{\lambda\}, \emptyset) \le \frac{1}{1+2\|\lambda\|}$. By definition we have

$$\delta_2(\{\lambda\}, \emptyset) = \sup_{\substack{\alpha \in \mathbb{R}^n \\ (D,E) \in \mathcal{E}}} \sup_{\mathcal{B}_m \in \mathcal{B}_{(D,E)}} \frac{\phi_{\mathcal{B}_m}(\Lambda)}{\left(\frac{1}{2} + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \|\alpha - c_2^{(m)}\| + \cdots + \|\alpha - c_{j_m}^{(m)}\|\right)}.$$

Whatever $(D, E) \in \mathcal{E}$, $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$, a maximum of one ball of $\mathcal{B}_m$ may contain λ. Denote by $\overset{\circ}{B}(c, \epsilon)$ this variable generic ball and put $c = c_1^{(m)}$. The other balls of $\mathcal{B}_m$ have a zero contribution to the numerator $\phi_{\mathcal{B}_m}(\Lambda)$ in the expression of $\delta_2(\{\lambda\}, \emptyset)$. The denominator is such that: $\frac{1}{2} + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \|\alpha - c_2^{(m)}\| + \cdots + \|\alpha - c_{j_m}^{(m)}\| \ge \frac{1}{2} + \|\alpha\| + \|\alpha - c\|$. But $\frac{1}{2} + \|\alpha\| + \|\alpha - c\| \ge \frac{1}{2} + \|c\|$, this minimum being reached on the segment $[0, c]$. Therefore, by definition of the function f, we have

$$d_{\alpha,(D,E)}(\{\lambda\}, \emptyset) \le \frac{\epsilon f(\frac{\lambda - c}{\epsilon})}{\frac{1}{2} + \|c\|} \le \frac{\epsilon}{\frac{1}{2} + \|\lambda\|} \le \frac{1/2}{1/2 + \|\lambda\|} = \frac{1}{1 + 2\|\lambda\|}.$$

Conversely, if we take $\alpha = \lambda$, $D = \{\lambda\}$ and E a dense subset in $(0, \frac{1}{2})$, we see that $\delta_2(\{\lambda\}, \emptyset) \ge d_{\alpha=\lambda,(D=\{\lambda\},E)}(\{\lambda\}, \emptyset) = \frac{1/2}{1/2+\|\lambda\|}$. We deduce the equality and assertion (i);

(ii) The proof is similar as in (i) since Λ and $\Lambda - \{\lambda\}$ differ by only one element λ which belongs to at most one ball in a collection $\mathcal{B}_m$ for any $(D, E) \in \mathcal{E}$ and any $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$. The details are left to the reader. □

Corollary 3.1. *For all $\mathcal{UD}$-set $\Lambda \neq \emptyset$ and all $\lambda \in \Lambda$, the inequality holds:*

$$|\delta_2(\Lambda, \emptyset) - \delta_2(\Lambda - \{\lambda\}, \emptyset)| \leq \frac{1}{1 + 2\|\lambda\|}.$$

Proof. From Proposition 3.4 (ii), we deduce $\delta_2(\Lambda, \emptyset) \leq \delta_2(\Lambda - \{\lambda\}, \emptyset) + \delta_2(\{\lambda\}, \emptyset)$. From (iii) in Proposition 3.4, we obtain

$$\delta_2(\Lambda - \{\lambda\}, \emptyset) \leq \delta_2(\Lambda, \emptyset) + \delta_2(\Lambda - \{\lambda\}, \Lambda)$$

but $\delta_2(\{\lambda\}, \emptyset) = \delta_2(\Lambda - \{\lambda\}, \Lambda) = \frac{1}{1+2\|\lambda\|}$ by Proposition 3.5. We deduce the claim. □

Proposition 3.6 (Point pairing property)**.** *Let Λ, Λ' be two $\mathcal{UD}$-sets assumed to be nonempty. Let $l = \mathrm{dist}(\{0\}, \Lambda) < +\infty$ and $\epsilon \in (0, \frac{1}{1+2l})$. Assume that $\delta_2(\Lambda, \Lambda') < \epsilon$. Then, for all $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$,*

(i) *there exists a unique $\lambda' \in \Lambda'$ such that $\|\lambda' - \lambda\| < \frac{1}{2}$,*

(ii) *this pairing (λ, λ') satisfies the inequality: $\|\lambda' - \lambda\| \leq (\frac{1}{2} + \|\lambda\|)\epsilon$.*

Proof. (i) Let us assume that for all $\lambda' \in \Lambda'$, $\lambda \in \Lambda$, such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$ the inequality $\|\lambda' - \lambda\| \geq \frac{1}{2}$ holds. This will lead to a contradiction. Assume the existence of an element $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$ and take $D = \{\lambda\}$ and let E be a countable dense subset in $(0, \frac{1}{2})$. Each $\mathcal{B}_m$ in $\mathcal{B}_{(D,E)}$ is a set constituted by only one element: the ball (say) $\overset{\circ}{B}(\lambda, e_m)$ with $e_m \in E$. We deduce that $\phi_{\mathcal{B}_m}(\Lambda) = e_m$ and $\phi_{\mathcal{B}_m}(\Lambda') = 0$. Hence

$$d_{\lambda,(D,E)}(\Lambda, \Lambda') = \sup_m \frac{e_m}{1/2 + \|\lambda\|} = \frac{1/2}{1/2 + \|\lambda\|} \leq \delta_2(\Lambda, \Lambda').$$

But $\epsilon < \frac{1}{1+2\|\lambda\|}$ is equivalent to $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$. Since we have assumed $\delta_2(\Lambda, \Lambda') < \epsilon$, we should obtain $\epsilon < d_{\lambda,(D,E)}(\Lambda, \Lambda') \leq \delta_2(\Lambda, \Lambda') < \epsilon$. Contradiction. The uniqueness of λ' comes from the fact that Λ' is a $\mathcal{UD}$-set allowing only one element λ' close to λ. (ii) Let us assume that $\lambda \neq \lambda'$ for all $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$, with $\lambda' \in \Lambda'$ that satisfies $\|\lambda' - \lambda\| < \frac{1}{2}$ (if the equality $\lambda = \lambda'$ holds, there is nothing to prove). Then, for all $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$, let us take $\alpha = \lambda$ as base point, $D = \{\lambda\}$ and E a dense subset in $(0, \|\lambda - \lambda'\|] \subset (0, \frac{1}{2})$. Then $\phi_{\mathcal{B}_m}(\Lambda) - \phi_{\mathcal{B}_m}(\Lambda') = e_m \left(1 - f\left(\frac{\lambda' - \lambda}{e_m}\right)\right)$. The restriction of the function $z \to z(1 - f(\frac{\lambda' - \lambda}{z}))$ to $(0, \|\lambda - \lambda'\|]$ is the identity function and is bounded above by $\|\lambda' - \lambda\|$. Therefore,

$$d_{\lambda,(D,E)}(\Lambda, \Lambda') = \sup_{\mathcal{B}_m} \frac{|\phi_{\mathcal{B}_m}(\Lambda) - \phi_{\mathcal{B}_m}(\Lambda')|}{1/2 + \|\lambda\|} = \frac{\|\lambda' - \lambda\|}{1/2 + \|\lambda\|}.$$

Since $d_{\lambda,(D,E)}(\Lambda, \Lambda') \leq \delta_2(\Lambda, \Lambda') < \epsilon$, we obtain $\|\lambda' - \lambda\| \leq (\frac{1}{2} + \|\lambda\|)\epsilon$. $\square$

In other terms, each time two $\mathcal{UD}$-sets Λ, Λ' are sufficiently close to each other for the metric δ_2, every element of Λ lying in a large ball centered at the origin (base point) in $\mathbb{R}^n$, is automatically associated with a unique element of Λ' which is close to it within distance less than $\frac{1}{2}$. Such pairings of elements occur over larger and larger distances from the origin when Λ' tends to Λ. From (ii) the proximity in the pairings (λ, λ') is much better for the elements $\lambda \in \Lambda$ which are the closest to the base point.

Proposition 3.7. *Let* $\epsilon \in (0, 1)$ *and* $\Lambda \in \mathcal{UD}, \Lambda \neq \emptyset$. *Then*

$$\delta_2(\Lambda, \emptyset) < \epsilon \Rightarrow \Lambda \subset \mathbb{R}^n \setminus B\Big(0, \frac{1-\epsilon}{2\epsilon}\Big).$$

Proof. Let us assume the existence of $\lambda \in \Lambda$ such that $\|\lambda\| \leq \frac{1-\epsilon}{2\epsilon}$ and let us show that this hypothesis implies that the assertion $\delta_2(\Lambda, \emptyset) < \epsilon$ is wrong. Take $D = \{\lambda\}$ and E a dense subset in $(0, \frac{1}{2})$. Each $\mathcal{B}_m$ in $\mathcal{B}_{(D,E)}$ is a set constituted by only one ball: say the ball $\overset{\circ}{B}(\lambda, e_m)$ with $e_m \in E$. We deduce that $\phi_{\mathcal{B}_m}(\Lambda) = e_m$. Since $\phi_{\mathcal{B}_m}(\emptyset) = 0$, the following inequality holds:

$$d_{\lambda,(D,E)}(\Lambda, \emptyset) = \sup_m \frac{e_m}{1/2 + \|\lambda\|} = \frac{1/2}{1/2 + \|\lambda\|} \leq \delta_2(\Lambda, \emptyset).$$

But $\epsilon \leq \frac{1}{1+2\|\lambda\|}$ is equivalent to $\|\lambda\| \leq \frac{1-\epsilon}{2\epsilon}$. Hence, $\epsilon \leq d_{\lambda,(D,E)}(\Lambda, \emptyset)$. We deduce $\delta_2(\Lambda, \emptyset) \geq \epsilon$ as claimed. $\square$

From Proposition 3.6 and Proposition 3.7 we deduce

Corollary 3.2. *For all* $t > 0$ *the Hausdorff metric* Δ *on* $\mathcal{UD}\,(B(0, t), \|\cdot\|)$ *is compatible with the restriction of the topology of* $(\mathcal{UD}, \delta_2)$ *to the space* $\mathcal{UD}\big(B(0, t), \|\cdot\|\big)_f = \mathcal{UD}\,(B(0, t), \|\cdot\|)$.

The converse of Proposition 3.7 is much harder (see Appendix).

3.3. Topological equivalence and point-removal. The "point-removal process" of a subcollection of points of a $\mathcal{UD}$-set is particularly easy to describe with d and δ_2. For all $\Lambda \in \mathcal{UD}$ and $R > 0$, denote by Λ_R the new $\mathcal{UD}$-set $\Lambda \cap \overset{\circ}{B}(0, R)$.

Proposition 3.8. *Let* $\delta = d$ *or* δ_2. *Let* $\Lambda, \Lambda' \in \mathcal{UD}$ *and* C *be an arbitrary subset of* $\Lambda \cap \Lambda'$. *Then* $\delta(\Lambda, \Lambda') = \delta(\Lambda \setminus C, \Lambda' \setminus C)$. *In particular, for all* $R > 0$,

$$\delta(\Lambda, \Lambda') = \delta(\Lambda \setminus (\Lambda \cap \Lambda'), \Lambda' \setminus (\Lambda \cap \Lambda')) \quad \textit{and} \quad \delta(\Lambda \setminus \Lambda_R, \emptyset) = \delta(\Lambda_R, \Lambda).$$

Proof. These results follow readily from the definitions of d and δ_2. $\square$

Remark. The distance (for δ_2) between two dense (equal) sphere packings (radius $\frac{1}{2}$) differing only by a small cluster of spheres lying about the point z is exactly the distance between these two clusters, say $\mathcal{C}_1$ and $\mathcal{C}_2$. Thus it is very easy to see that it is a ratio (from (3.4) in Section 3.2.1) roughly given by $\frac{m}{1/2+\|z\|} f(\mathcal{C}_1, \mathcal{C}_2)$, when $\|z\|$ is large enough, where m is the (common) number of spheres of $\mathcal{C}_1$ (or $\mathcal{C}_2$) and $f(\mathcal{C}_1, \mathcal{C}_2)$ a function which depends upon the relative positions of the spheres inside $\mathcal{C}_1$ and $\mathcal{C}_2$. Such simple expressions are easy to handle, can be made more precise and can be differentiated to study optimal positioning of clusters in dense sphere packings [MVG1].

Let us show that the metrics d, δ_1, δ_2 are topologically equivalent on $\mathcal{UD}$.

Proposition 3.9. *For any averaging sequence $(K_i)_{i\geq 1}$ of compact sets of $\mathbb{R}^n$ which contains the base point* 0*, the metric d associated with it is such that:*

(i) *d and δ_1 are topologically equivalent,*
(ii) *d and δ_2 are topologically equivalent.*

Proof. It suffices to show that the identity map is bicontinuous in each case.

(i) $(\mathcal{UD}, \delta_1) \stackrel{\text{id}}{\to} (\mathcal{UD}, d)$ is continuous: let $\epsilon > 0$ be small enough and assume $\Lambda, \Lambda' \in \mathcal{UD}$ with $\Lambda \neq \emptyset$. Let $\eta \in (0,1)$. Let k be the greatest integer such that $K_k \subset B(0, \frac{\lambda}{R_{\min}\eta})$. The map $\eta \to 2^{-k}$ takes the value 0 at zero and is continuous at zero. Then there exists η_0 such that $\eta \leq \eta_0 \Rightarrow 2^{-k} \leq \epsilon/2$. Now, if $\delta_1(\Lambda, \Lambda') < \eta$, then $R_{\min}\eta \in \Omega(\Lambda, \Lambda')$ and $\Delta(\Lambda \cap B(0, \frac{\lambda}{R_{\min}\eta}), \Lambda' \cap B(0, \frac{\lambda}{R_{\min}\eta})) \leq R_{\min}\eta$. We deduce:

$$d(\Lambda, \Lambda') \;\leq\; \sum_{i=1}^{k} 2^{-i} R_{\min}\eta + \sum_{i\geq k+1} 2^{-i} \;\leq\; R_{\min}\eta + 2^{-k}.$$

Hence, for all $\eta < \min\{\frac{\epsilon}{2R_{\min}}, \eta_0\}$ the inequality $\delta_1(\Lambda, \Lambda') < \eta$ implies $d(\Lambda, \Lambda') < \epsilon/2 + \epsilon/2 = \epsilon$, hence the claim. Assume now $\Lambda = \emptyset$. Given $\epsilon > 0$ small enough, by the definition of d, there exists $R > 0$ such that $d(\emptyset, \Lambda') < \epsilon$ for all $\Lambda' \subset \mathbb{R}^n \setminus B(0, R)$. Take $\eta = \frac{\lambda}{RR_{\min}}$. Then the inequality $\delta_1(\emptyset, \Lambda') < \eta$ implies $R_{\min}\eta \in \Omega(\emptyset, \Lambda')$, hence $\Lambda' \cap B(0, \frac{\lambda}{R_{\min}\eta}) = \Lambda' \cap B(0, R) = \emptyset$. We deduce: $\delta_1(\emptyset, \Lambda') < \eta \Rightarrow d(\emptyset, \Lambda') < \epsilon$, hence the continuity at $\emptyset$.

$(\mathcal{UD}, d) \stackrel{\text{id}}{\to} (\mathcal{UD}, \delta_1)$ is continuous: let $\epsilon > 0$ be small enough and assume $\Lambda \neq \emptyset$. By Proposition 2.2 there exists η such that:

$$d(\Lambda \cap B(0, \tfrac{\lambda}{R_{\min}\epsilon}), \Lambda' \cap B(0, \tfrac{\lambda}{R_{\min}\epsilon})) < \eta$$
$$\Rightarrow \Delta(\Lambda \cap B(0, \tfrac{\lambda}{R_{\min}\epsilon}), \Lambda' \cap B(0, \tfrac{\lambda}{R_{\min}\epsilon})) < R_{\min}\epsilon.$$

Since

$$d(\Lambda \cap B(0, \frac{\lambda}{R_{\min}\epsilon}), \Lambda' \cap B(0, \frac{\lambda}{R_{\min}\epsilon})) \leq d(\Lambda, \Lambda')$$

and that

$$\Delta(\Lambda \cap B(0, \frac{\lambda}{R_{\min}\epsilon}), \Lambda' \cap B(0, \frac{\lambda}{R_{\min}\epsilon})) \leq R_{\min}\epsilon$$

is equivalent to $\delta_1(\Lambda, \Lambda') < \epsilon$, we have: $d(\Lambda, \Lambda') < \eta \Rightarrow \delta_1(\Lambda, \Lambda') < \epsilon$, hence the claim. Assume now $\Lambda = \emptyset$. Let $\epsilon > 0$. Let j the smallest integer such that $B(0, \frac{\lambda}{R_{\min}\epsilon}) \subset K_j$. We have $d(\emptyset, \Lambda') = 2^{-u+1}$ where u is the smallest integer such that $\Lambda' \cap K_u \neq \emptyset$, so that $\Lambda' \cap K_i = \emptyset$ for $i = 1, 2, \ldots, u-1$. Then we choose $\eta \in (0,1)$ small enough such that $d(\emptyset, \Lambda') = 2^{-u+1} < \eta < 2^{-u+2}$ with $u > j$. Hence $\Lambda' \cap K_j = \emptyset$. Then $\Lambda' \subset \mathbb{R}^n \setminus B(0, \frac{\lambda}{R_{\min}\epsilon})$. We deduce $\delta_1(\emptyset, \Lambda') < \epsilon$ and the continuity at $\emptyset$.

(ii) $(\mathcal{UD}, \delta_2) \xrightarrow{\text{id}} (\mathcal{UD}, d)$ is continuous: let us first assume that the condition $\sum_{i \geq 1} 2^{-i} \mathcal{R}_{\mathcal{K}_i} < +\infty$ holds. Let $\epsilon > 0$ be small enough. Let $\eta \in (0,1)$ and assume $\delta_2(\Lambda, \Lambda') < \eta$ where $\Lambda \neq \emptyset$. Define $k = k(\eta)$ by the conditions: $K_k \subset B\left(0, \frac{1-\eta}{2\eta}\right)$ and $K_{k+1} \not\subset B\left(0, \frac{1-\eta}{2\eta}\right)$. Since the map $\eta \to 2^{-k}$ takes the value 0 at zero and is continuous at zero, there exists η_0 such that $\eta \leq \eta_0 \Rightarrow 2^{-k} \leq \epsilon/2$. On the other hand, for all i such that $K_i \subset B\left(0, \frac{1-\eta}{2\eta}\right)$, $\Delta(\Lambda \cap K_i, \Lambda' \cap K_i) = d_i(\Lambda, \Lambda') \leq \left(\frac{1}{2} + \mathcal{R}_{K_i}\right)\eta$ by Proposition 3.6. Then

$$d(\Lambda, \Lambda') \leq \sum_{i=1}^{k} 2^{-i}\left(\frac{1}{2} + \mathcal{R}_{K_i}\right)\eta + \sum_{i \geq k+1} 2^{-i} \leq \left(\frac{1}{2} + \sum_{i \geq 1} 2^{-i} \mathcal{R}_{K_i}\right)\eta + 2^{-k}.$$

There exists η_1 such that $\eta \leq \eta_1 \Rightarrow \left(\frac{1}{2} + \sum_{i \geq 1} 2^{-i} \mathcal{R}_{(K_i)}\right)\eta \leq \epsilon/2$. Then, for $\eta \leq \min\{\eta_0, \eta_1\}$, we have $d(\Lambda, \Lambda') \leq \epsilon/2 + \epsilon/2 = \epsilon$. This proves the continuity at all $\Lambda \neq \emptyset$. Continuity at $\emptyset$ arises readily from the definition of d and Proposition 3.7. Using the Remark in Section 2, we wee that the condition $\sum_{i \geq 1} 2^{-i} \mathcal{R}_{\mathcal{K}_i} < +\infty$ can be removed. Thus we obtain the claim in full generality.

$(\mathcal{UD}, d) \xrightarrow{\text{id}} (\mathcal{UD}, \delta_2)$ is continuous: let $\epsilon > 0$ be small enough and assume $\Lambda \neq \emptyset$. By Proposition A.1 there exists R such that

$$\delta_2(\Lambda, \Lambda_R) = \delta_2(\Lambda \setminus \Lambda_R, \emptyset) \leq \epsilon/3 \quad \text{and} \quad \delta_2(\Lambda', \Lambda'_R) = \delta_2(\Lambda' \setminus \Lambda'_R, \emptyset) \leq \epsilon/3.$$

By Corollary 3.2 there exists η_0 such that

$$\Delta(\Lambda_R, \Lambda'_R) < \eta_0 \quad \Rightarrow \quad \delta_2(\Lambda_R, \Lambda'_R) < \epsilon/3.$$

Let $j \geq 1$ be the smallest integer such that $B(0, R) \subset K_j$. Let us take $\eta \in (0, 2^{-j})$ such that $\frac{\eta}{2^{-j} - \eta} < \eta_0 \Longleftrightarrow \eta < \frac{\eta_0 2^{-j}}{1 + \eta_0}$. Then,

$$d(\Lambda, \Lambda') = \sum_{i \geq 1} \frac{2^{-i} d_i(\Lambda, \Lambda')}{1 + d_i(\Lambda, \Lambda')} < \eta \Rightarrow \frac{2^{-i} d_i(\Lambda, \Lambda')}{1 + d_i(\Lambda, \Lambda')} < \eta \text{ for all } i = 1, 2, \ldots, j.$$

We deduce $\Delta(\Lambda_R, \Lambda'_R) \leq d_j(\Lambda, \Lambda') = \Delta(\Lambda \cap K_j, \Lambda' \cap K_j) < \frac{\eta}{2^{-i}-\eta} < \eta_0$. Thus

$$\delta_2(\Lambda, \Lambda') \leq \delta_2(\Lambda, \Lambda_R) + \delta_2(\Lambda_R, \Lambda'_R) + \delta_2(\Lambda'_R, \Lambda') \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

We deduce the claim for $\Lambda \neq \emptyset$. Assume now $\Lambda = \emptyset$. Let $\epsilon > 0$ be small enough. By Proposition A.1 there exists R such that $\Lambda' \subset \mathbb{R}^n \setminus B(0, R)$ implies $\delta_2(\Lambda', \emptyset) \leq \epsilon$. Let j be the smallest integer such that $\Lambda' \cap K_j \neq \emptyset$. Then $\Lambda' \cap K_{j-1} = \emptyset$ and $B(0, R) \subset K_{j-1}$. Since $d(\emptyset, \Lambda') = \sum_{i \geq j} 2^{-i} = 2^{-j+1}$ (with $\Lambda' \neq \emptyset$) we take η such that $2^{-j+1} < \eta < 2^{-j+2}$, for instance $\eta = 3.2^{-j}$. Then $d(\emptyset, \Lambda') < \eta \Rightarrow \delta_2(\Lambda', \emptyset) \leq \epsilon$, hence the continuity at $\emptyset$. □

The following proposition is fundamental. It shows the uniform continuity of the point-removal process at infinity.

Proposition 3.10. *Let $\Lambda \in \mathcal{UD}$. Denote by δ either d or δ_1. Then*

$$\lim_{R\to\infty} \delta(\Lambda, \Lambda_R) = \lim_{R\to\infty} \delta(\Lambda \setminus \Lambda_R, \emptyset) = 0.$$

Moreover the convergence is uniform in the following sense:

$$\forall \epsilon \in (0,1), \exists R > 0 \text{ such that: } \Lambda \subset \mathbb{R}^n \setminus B(0, R) \Rightarrow \delta(\Lambda, \emptyset) < \epsilon.$$

Proof. If Λ is finite, the limit is obviously zero. Assume Λ infinite. The claim is obvious when $\delta = d$ or $\delta = \delta_1$ by definition of d and δ_1. □

Remark. A direct (and self-contained) proof of Proposition 3.10 with $\delta = \delta_2$ can be found in the Appendix (Proposition A.1).

Corollary 3.3. *The subset $\mathcal{UD}_f$ is dense in $\mathcal{UD}$.*

4. Proof of Theorem 1.2

Let $(K_k)_{k\geq 1}$ be an averaging sequence of compact sets which contains the base point 0 and d the metric associated with it. Let us embedd $\mathcal{UD}$ in the product space

$$\mathcal{UD} \subset \prod_{k\geq 1} \mathcal{UD}\big(K_k, \|\cdot\|\big)_1 = W,$$

each $\mathcal{UD}(K_k, \|\cdot\|)_1$ being a compact metric space with the Hausdorff metric Δ, equivalently with d, δ_1 or δ_2 by Proposition 2.1, resp. Proposition 3.1, or resp. Corollary 3.2. Thus the space W is naturally a compact space by Tychonov's Theorem, and it is clear that $\mathcal{UD}$ is closed inside. Indeed, the image can be identified with the families $(V_k) \in \mathcal{UD}(K_k, \|\cdot\|)_1$ such that $V_k \cap K_{k-1} = V_{k-1}$. This is a special case of a projective limit. Therefore it is compact.

For all $R > 0$ the subspace X_R of the Delone sets of constant R is closed in $(\mathcal{UD}, d)$, since the relative denseness conditions are closed. Nevertheless, let us prove directly this result using δ_2. Let $\Lambda \in \mathcal{UD} \setminus X_R$. We will show that it is contained in an open subset disjoint from X_R that will prove that X_R is closed. Since $\Lambda \notin X_R$, there exists $z \in \mathbb{R}^n$ such that $\|z-\lambda\| > R$ for all $\lambda \in \Lambda$. Let $l = \text{dist}(\{z\}, \Lambda) > R$. Denote $\Lambda - z := \{\lambda - z \mid \lambda \in \Lambda\}$ the translated set. For $\epsilon > 0$ small enough and all Γ in the open δ_2-ball $\{\Omega \in \mathcal{UD} \mid \delta_2(\Omega, \Lambda - z) < \epsilon\}$, all the elements γ of Γ satisfy the inequality: $\|\gamma\| \geq R + \frac{l-R}{2} > R$ by the point pairing property (Proposition 3.6); all these point sets Γ are outside X_R. Since the translation by z is bicontinuous the $\mathcal{UD}$-set Λ is contained in the open subset $z + \{\Omega \in \mathcal{UD} \mid \delta_2(\Omega, \Lambda - z) < \epsilon\}$ which is disjoint of X_R.

5. Theorem 1.2 implies Theorem 1.1

Let $\mathcal{L}_n$ be the space of lattices in $\mathbb{R}^n$, identified with the locally compact homogeneous space $GL(n,\mathbb{R})/GL(n,\mathbb{Z})$ [GL], [Ca] (Recall that a lattice in $\mathbb{R}^n$ is a discrete $\mathbb{Z}$-module of maximal rank of $\mathbb{R}^n$, equivalently a discrete subgroup of the group of translations of $\mathbb{R}^n$ with compact fundamental region). The following proposition is a key result for proving Theorem 1.1 from Theorem 1.2, using δ_2 for the proof and invoking Proposition 3.9 for the other metrics.

Proposition 5.1. *The restriction of the metric δ_2, resp. d or δ_1, to $\mathcal{L}_n \cap \mathcal{UD} \subset \mathcal{UD}$ is compatible with the topology on $\mathcal{L}_n \cap \mathcal{UD}$ induced by the quotient topology of $\mathcal{L}_n = GL(n,\mathbb{R})/GL(n,\mathbb{Z})$.*

Proof. This proposition is a reformulation of the following proposition. □

Proposition 5.2. *Let $L \in \mathcal{L}_n \cap \mathcal{UD}$. Denote by $\{e_1, e_2, \ldots, e_n\}$ a basis of L. Then*

(i) *for all $\epsilon > 0$ small enough there exists $\eta > 0$ such that any $\mathbb{Z}$-module $L' \in \mathcal{UD}$ contained in the open ball $\{\Lambda \in \mathcal{UD} \mid \delta_2(L, \Lambda) < \eta\}$ is of rank n and admits a basis $\{e'_1, e'_2, \ldots, e'_n\}$ which satisfies the property: $\max_{i=1,2,\ldots,n} \|e_i - e'_i\| < \epsilon$;*

(ii) *$\forall \eta \in (0,1), \exists\, \epsilon > 0$ such that any lattice $L' \in \mathcal{UD}$ of $\mathbb{R}^n$ admitting a basis $\{e'_1, e'_2, \ldots, e'_n\}$ which satisfies $\max_{i=1,2,\ldots,n} \|e_i - e'_i\| < \epsilon$ is such that $\delta_2(L, L') < \eta$.*

Proof. (i) First let us chose $\epsilon_0 > 0$ small enough such that any n-tuple $\{a_1, a_2, \ldots, a_n\}$ of points of $\mathbb{R}^n$ with $a_i \in B(e_i, \epsilon_0)$, $i = 1, 2, \ldots, n$, is such that the vectors $\{Oa_1, Oa_2, \ldots, Oa_n\}$ are $\mathbb{Z}$-linearly independant (as usual we identify the point a_i with the vector Oa_i, $i = 1, 2, \ldots, n$). For instance, let us take $\epsilon_0 = \frac{1}{3} \min_{i=1,2,\ldots,n} \{\text{dist}(\{e_i\}, \text{Vect}_i)\}$, if Vect_i, $i = 1, 2, \ldots, n$, is the $\mathbb{R}$-span generated by the vectors $Oe_1, Oe_2, \ldots, Oe_{i-1}, Oe_{i+1}, \ldots, Oe_n$. Let $\epsilon \in (0, \epsilon_0)$.

Assume that Λ is a $\mathcal{UD}$-set such that $\delta_2(L, \Lambda) < \eta$ with η small enough. By Proposition 3.6 a pairing between the points of L and Λ occurs over a certain distance, which is $\frac{1-\eta}{2\eta}$, from the origin. Let us take η_1 small enough in order to have $\frac{1-\eta_1}{2\eta_1} \geq \max_{i=1,2,\dots,n} \|e_i\|$. From Proposition 3.6 the condition $0 < \eta < \eta_1$ implies the existence of n points $e'_1, e'_2, \dots, e'_n$ in Λ, the respective close-neighbours of the points $e_1, e_2, \dots, e_n$ of L, which satisfy $\|e'_i - e_i\| \leq (\frac{1}{2} + \|e_i\|)\eta$ for $i = 1, 2, \dots, n$. Take $\eta < \eta_1$ such that $(\frac{1}{2} + \max_{i=1,2,\dots,n} \|e_i\|)\eta < \epsilon$. Since $\epsilon < \epsilon_0$, the vectors $Oe'_1, Oe'_2, \dots, Oe'_n$ are $\mathbb{Z}$-linearly independant. This means that if $\Lambda \in \mathcal{UD}$ is a $\mathbb{Z}$-module of $\mathbb{R}^n$ (necessarily discrete) which satisfies $\delta_2(L, \Lambda) < \eta$, Λ is necessarily of rank n and contains the lattice $\sum_{i=1}^n \mathbb{Z}e'_i$. Let us show that there is equality. Denote by $\mathcal{V}' = \left\{\sum_{i=1}^n \theta_i e'_i \mid 0 \leq \theta_i < 1 \quad \text{for all } i = 1, 2, \dots, n\right\}$. The adherence $\overline{\mathcal{V}'}$ of $\mathcal{V}'$ contains only the points $\sum_{i=1}^n j_i e'_i$ of Λ , with $j_i = 0$ or 1, by the property of the pairing (Proposition 3.6). Therefore the free system $\{Oe'_1, Oe'_2, \dots, Oe'_n\}$ is a basis of Λ.

(ii) Conversely, let $0 < \eta < 1$ and $L' \in \mathcal{UD} \cap \mathcal{L}_n$. For all $R > 0$ the inequality $\delta_2(L, L') \leq \delta_2(L, L_R) + \delta_2(L_R, L'_R) + \delta_2(L'_R, L')$ holds. By Proposition A.1 let us take R large enough such that $\delta_2(L, L_R) < \eta/3$ and $\delta_2(L', L'_R) < \eta/3$. Let us now show that, if L' admits a basis $\{e'_1, e'_2, \dots, e'_n\}$ which satisfies $\max_{i=1,2,\dots,n} \|e_i - e'_i\| < \epsilon$, then ϵ can be taken small enough to have $\delta_2(L_R, L'_R) < \eta/3$ (R kept fixed). Indeed, L_R and L'_R are finite $\mathcal{UD}$-sets. Denote $N := \#L_R$. For all $\alpha \in \mathbb{R}^n$, all $(D, E) \in \mathcal{E}$ and all $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$, by continuity of the function f, the mapping

$$(x_1, x_2, \dots, x_N) \to \phi_{\mathcal{B}_m}(\{x_1, x_2, \dots, x_N\}) := \sum_{\overset{\circ}{B}(c,\omega) \in \mathcal{B}_m} \sum_{i=1}^{N} \omega f\Big(\frac{x_i - c}{\omega}\Big)$$

is continuous on $B(0, R)^N$ for the standard product topology. Therefore all the mappings $d_{\alpha,(D,E)}(L_R, \cdot)$:

$$(x_1, x_2, \dots, x_N) \to \sup_{\mathcal{B}_m} \frac{|\phi_{\mathcal{B}_m}(L_R) - \phi_{\mathcal{B}_m}(\{x_1, x_2, \dots, x_N\})|}{(\frac{1}{2} + \|\alpha\| + \|\alpha - c_{j_1}\| + \|\alpha - c_{j_2}\| + \dots + \|\alpha - c_{j_N}\|)}$$

are continuous on $B(0, R)^N$. The map

$$(x_1, x_2, \dots, x_N) \to d(L_R, \{x_1, x_2, \dots, x_N\})$$

is then continuous on $B(0, R)^N$. Take for $\{x_1, x_2, \dots, x_N\}$ the point set L'_R. Consequently the quantity $\delta_2(L_R, L'_R)$ is strictly less than $\eta/3$ as soon as ϵ is small enough. Finally $\delta_2(L, L') < 3\eta/3 = \eta$ and we deduce the claim. □

Recall that if L is a lattice in $\mathbb{R}^n$ and A a basis of L, then $|\det(A)|$ is called the determinant of L; we will denote it by $|L|$. It is the volume of its fundamental region.

Proposition 5.3. *The subspace* $\{L \in \mathcal{UD} \cap \mathcal{L}_n \mid 0 < |L| \leq M\} \subset \mathcal{L}_n \cap \mathcal{UD}$ *is compact for all* $M > 0$.

Proof. By Proposition 5.1 and since $(\mathcal{UD}, d)$ is a compact topological space, we have just to show that $\{L \in \mathcal{UD} \cap \mathcal{L}_n \mid 0 < |L| \leq M\}$ is closed. Since the operations $x + y$ and xy are continuous, the determinant function $|\cdot|$ is continuous on $\mathcal{L}_n$. Hence $\{L \in \mathcal{UD} \cap \mathcal{L}_n \mid |L| > M\} = |\cdot|^{-1}((M, +\infty))$ is an open set as reciprocal image of the open interval $(M, +\infty)$ by the continuous application $|\cdot|$. By taking its complementary subspace in $\mathcal{UD} \cap \mathcal{L}_n$ we deduce the claim. □

Let us now prove Theorem 1.1. Let us consider a sequence of lattices (L_r) of $\mathbb{R}^n$ such that: (i) $\|x\| \geq 1$ for all $x \in L_r, x \neq 0$, (ii) the determinant $|L_r|$ of L_r satisfies $|L_r| \leq M$ with M a constant $< +\infty$ independent of r. Then $L_r \in \{L \in \mathcal{UD} \cap \mathcal{L}_n \mid 0 < |L| \leq M\}$, for all r, which is compact by Proposition 5.3. Then, by the Bolzano-Weierstrass property, one can extract from the sequence (L_r) a subsequence $(L_{r'})$ that converges to a lattice L of $\mathbb{R}^n$. By continuity of the determinant function $|\cdot|$ and Proposition 5.1, we obtain: $|L| = \lim_{r' \to +\infty} |L_{r'}|$. This concludes the proof.

6. Arbitrary metric spaces

The topological space $(\mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_1, d)$ is a Polish space [B], but its topology is not classical. It is routine to compare it with the topologies reviewed by Kelley [Ke] and Michael [Mi] on spaces of nonempty closed subsets of $\mathbb{R}^n$ and to conclude that it is none of them (see also §4 in [BL], and [Bo]). The metric space $\cup_{r>0} \mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_{r,f}$ is dense in $\mathcal{D}_{l\,f}(\mathbb{R}^n, \|\cdot\|)$. The metric space $\cup_{r>0} \mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_r$, endowed with d or δ_1, is not compact and is much bigger than the space of lattices of $\mathbb{R}^n$. For instance, if $n = 1$, it contains all the Meyer sets $\mathbb{Z}_\beta$ of β-integers (integers in base β) where β is a Pisot number or a Parry number [GVG] (see [Mo], [MVG] for a modern language on Meyer sets and Delone sets).

Let (H, δ) be a metric space.

Proposition 6.1. *If* $\operatorname{diam}(H) = +\infty$*, then* H *only contains infinite Delone sets.*

Proof. Assume that $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_I\}, 1 \leq I < +\infty$, is a finite Delone set in H of constants (r, R) and let us show the contradiction. Since, for all $z \in H$, there exists $\lambda_i \in \Lambda$ such that $\lambda_i \in z + B(0, R)$, then $z \in B(\lambda_i, R)$ and we would have $H \subset \cup_{i=1}^{I} B(\lambda_i, R)$ which is of diameter less than $2RI$. Thus we would have $\operatorname{diam}(H) \leq 2RI < +\infty$. Contradiction. □

If H is compact, then $\operatorname{diam}(H) < +\infty$ and all the Delone sets of H are finite. For all $r > \operatorname{diam}(H)$, the uniformly discrete subsets of H of constant r are empty. When $r \leq \operatorname{diam}(H)$ a uniformly discrete subset of H of constant r is either the empty set $\emptyset$ or is finite. Thus the set-theoretic equality holds:

$$\bigcup_{0<r\leq \operatorname{diam}(H)} \mathcal{UD}(H,\delta)_r \setminus \{\emptyset\} = \bigcup_{\substack{0<R,\\ 0<r\leq \operatorname{diam}(H)}} X(H,\delta)_{r,R}.$$

This space endowed with the Hausdorff metric Δ is a compact space and $\emptyset$ is an isolated point. This provides positive answers to Question 1.1 and Question 1.2 with $d = \Delta$ since, for all $0 < r \leq \operatorname{diam}(H)$ and $R > 0$, $\mathcal{UD}(H,\delta)_r$ is closed in $\cup_{0<r\leq \operatorname{diam}(H)} \mathcal{UD}(H,\delta)_r$, and $X(H,\delta)_{r,R}$ is closed in $\cup_{0<R,0<r\leq \operatorname{diam}(H)} X(H,\delta)_{r,R}$.

The following Theorem is an improvment of Macbeath and Swierczkowski's Theorem [MS] in the context of ("ambient") metric spaces, providing positive answers to Question 1.1 and Question 1.2.

Theorem 6.1. *Let (H,δ) be a σ-compact and locally compact metric space for which* $\operatorname{diam}(H)$ *is infinite. Then, for all $r > 0$, $\mathcal{UD}(H,\delta)_r$ can be endowed with a metric d such that the topological space $(\mathcal{UD}(H,\delta)_r, d)$ is compact and such that the Hausdorff metric on $\mathcal{UD}(H,\delta)_{r,f}$ is compatible with the restriction of the topology of $(\mathcal{UD}(H,\delta)_r, d)$ to $\mathcal{UD}(H,\delta)_{r,f}$. For all $R > 0$, its subspace of the Delone sets of constants (r,R) is closed.*

Proof. The metric d is the one constructed in Section 2 but now on H instead of $\mathbb{R}^n$. From Section 4 we deduce the compactness of $\mathcal{UD}(H,\delta)_r$ for all $r > 0$. Indeed, the proof in Section 4 is valid for all "ambient" metric spaces which are σ-compact and locally compact. □

Appendix A

This Appendix gives a proof of Proposition A.1. Proposition A.1 is related to the rest of the paper by the fact that it implies the topological equivalence between d, δ_1 and δ_2 on $\mathcal{UD}$ (see proof of Proposition 3.9) and is used in the proof of Proposition 5.2. Though fairly long, the present computations are not necessary for many applications concerning the topological space $\mathcal{UD}$.

Proposition A.1. *Let $\Lambda \in \mathcal{UD}$. Then*

$$\lim_{R\to\infty} \delta_2(\Lambda, \Lambda_R) = \lim_{R\to\infty} \delta_2(\Lambda \setminus \Lambda_R, \emptyset) = 0.$$

Moreover the convergence is uniform in the following sense:

$$\forall \epsilon \in (0,1),\ \exists\ R > 0 \ \text{ such that: } \ \Lambda \subset \mathbb{R}^n \setminus B(0,R) \ \Rightarrow\ \delta_2(\Lambda,\emptyset) < \epsilon.$$

Proof. We assume that Λ is infinite in the sequel since, when Λ is finite, the limit is obviously zero. To prove this result we use Stolarsky's inequality [St] (recalled in Proposition A.2 without proof) which provides an (uniform) upper bound of $\delta_2(\Lambda, \Lambda_R)$. Then we explicitely compute this (uniform) upper bound by means of representations of integers as sums of squares (of integers) (see Grosswald [Gr] for a survey) (Steps 1 and 2). This type of computation provides uniform convergence.

Proposition A.2 (Stolarsky [St]). *Let u, v rational integers such that $u \geq v \geq 1$. Let $\{x_1, x_2, \ldots, x_u\}$ be a finite set of u points of $\mathbb{R}^n$ and $\{y_1, y_2, \ldots, y_v\}$ be another finite set of v points of $\mathbb{R}^n, n \geq 2$. Let us define $h(u,v) = 1$ if $u = v$, $h(u,v) = \frac{u-1}{v}$ if $u > v$. Then*

$$\text{(A.1)} \qquad \sum_{1 \leq i < j \leq u} \|x_i - x_j\| + \sum_{1 \leq i < j \leq v} \|y_i - y_j\| \leq h(u,v) \sum_{i=1}^{u} \sum_{j=1}^{v} \|x_i - y_j\|$$

where the constant $h(u,v)$ is best possible.

Let us apply Proposition A.2. Take $v = 1$ and $u = i_m + 1 \geq 2$ with $x_1 = 0$ and $\|x_i\| \geq R$ for all $i = 2, 3, \ldots, u$; then put $y_1 = \alpha \in \mathbb{R}^n$ arbitrary. The inequality (A.1) gives

$$\sum_{j=2}^{i_m+1} \|x_j\| + \sum_{2 \leq i < j \leq i_m+1} \|x_i - x_j\| \leq h(i_m + 1, 1)\Big(\|\alpha\| + \sum_{i=2}^{i_m+1} \|\alpha - x_i\|\Big).$$

Consequently, setting $c_{i-1} = x_i$ for all $i = 2, 3, \ldots, i_m + 1$ for keeping the notations as close as possible to the definition of $d_{\alpha,(D,E)}$ (see (3.4)), the following inequality holds:

(A.2)

$$\frac{i_m}{1/2 + \|\alpha\| + \sum_{i=1}^{i_m} \|\alpha - c_i\|} \leq \frac{i_m}{\frac{1}{2} + \frac{1}{i_m}\Big(\sum_{j=1}^{i_m} \|c_j\| + \sum_{1 \leq i < j \leq i_m} \|c_i - c_j\|\Big)}.$$

The supremum of the right-hand side expression, over all possible configurations of balls in $\mathcal{B}_{(D,E)}$ and $(D,E) \in \mathcal{E}$ such that their centres c_i satisfy $\|c_i\| \geq R$, is greater than $2\delta_2(\Lambda \setminus \Lambda_R, \emptyset)$ (see Proposition 3.3 for the definition of δ_2). We will show that it goes to zero when R tends to infinity. For this, we will compute explicitely a lower bound of

$$\eta(R, i_m) := \frac{1}{2i_m} + \frac{1}{i_m^2} \sum_{j=1}^{i_m} \|c_j\| + \frac{1}{i_m^2} \sum_{1 \leq i < j \leq i_m} \|c_i - c_j\|$$

as a function of R and i_m, where $\eta(R, i_m)$ is the inverse of the right-hand side term in the inequality (A.2). In order to simplify the notations, we will study the quantity $\eta(R, m)$, what amounts merely to replace m by i_m in the rest of the proof for coming back to the inequality (A.2).

We will proceed as follows, in three steps. The first step (Step 1) will consist in making this computation explicit when the points c_i are on the lattice $\mathbb{Z}^n$ with $n \geq 5$. In other terms, we will prove:

$$\lim_{R\to+\infty} \delta_2(\mathbb{Z}^n, {\mathbb{Z}^n}_R) = 0 \qquad \text{for all } n \geq 5.$$

The second step (Step 2) will describe how to provide a lower bound of $\eta_\Lambda(R,m)$ (see its definition in Step 2) from $\eta(R,m)$ when the points c_i are in a $\mathcal{UD}$-set $\Lambda \subset \mathbb{R}^n$ which is not $\mathbb{Z}^n$ with still $n \geq 5$ for which the dimension of the $\mathbb{R}$-span of Λ is n or less than n. In other terms, we will prove:

$$\lim_{R\to+\infty} \delta_2(\Lambda, \Lambda_R) = 0 \qquad \text{for all } \Lambda \in \mathcal{UD} \text{ and } n \geq 5.$$

The final Step 3 will conclude when $n \in \{1,2,3,4\}$ making use of descent arguments to lower dimensions. In other terms, we will prove:

$$\lim_{R\to+\infty} \delta_2(\Lambda, \Lambda_R) = 0 \qquad \text{for all } \Lambda \in \mathcal{UD} \text{ and } n \leq 4.$$

Step 1.– Let us recall the assumptions: $R > \sqrt{2}$ (for technical reasons) and $c_i \in \mathbb{Z}^n$, $\|c_i\| \geq R$, for all $i = 1,2,\dots,m$ with $i \neq j \Rightarrow c_i \neq c_j$. In order to find a lower bound of $\eta(R,m)$, we will compute a lower bound of $m^{-2}\sum_{1\leq i<j\leq m}\|c_i - c_j\|$ (A.5) and a lower bound of $m^{-2}\sum_{j=1}^{m}\|c_j\|$ (A.6) as a function of R and m. These two bounds will be shown to be dependent (by (A.7) and (A.8)). The sum of these two lower bounds will present a minimum and the main difficulty will consist in showing that this minimum tends to infinity when R tends to infinity.

Let us compute a lower bound of $m^{-2}\sum_{1\leq i<j\leq m}\|c_i - c_j\|$. Let s be a positive integer and consider the equation $s = \sum_{i=1}^{n} c_{q,i}^2$ with $c_{q,i} \in \mathbb{Z}$ for all $i = 1,2,\dots,n$. Any n-tuple $(c_{q,1}, c_{q,2}, \dots, c_{q,n})$ which satisfies this equation is called a solution of this equation. This solution represents the vector $c_q = {}^t(c_{q,1}, c_{q,2}, \dots, c_{q,n})$ in $\mathbb{Z}^n$ of norm $s^{1/2}$. Given s, denote by $r_n(s)$ the number of solutions of the above equation; it is the number of elements of $\mathbb{Z}^n$ which lie on the sphere $S(0,\sqrt{s})$ of centre the origin and radius $\sqrt{s}$. Obviously $r_n(0) = 1, r_n(1) = 2^n$. Now, for any integer $m > 1$, there exists a unique integer k such that

$$\text{(A.3)} \quad r_n(0)+r_n(1)+\cdots+r_n(k) < m \leq r_n(0)+r_n(1)+\cdots+r_n(k)+r_n(k+1)$$

with $r_n(k)r_n(k+1) \neq 0$. We know (Grosswald [Gr], Chapters 9, 12 and 13) the behaviour of $r_n(s)$ when $n \geq 5$: there exists two strictly positive constants $\widehat{K_1}(n)$ and $\widehat{K_2}(n)$ such that $r_n(s) = \rho_n(s) + O\left(s^{n/4}\right)$ with $\widehat{K_1}(n)s^{n/2-1} \leq \rho_n(s) \leq \widehat{K_2}(n)s^{n/2-1}$ for any integer $s > 0$. Therefore, there exists two strictly positive constants K_1, K_2, which depend upon n, such that $K_2 \geq 1$ and $K_1 s^{n/2-1} \leq r_n(s) \leq K_2 s^{n/2-1}$ for any integer $s > 0$. By saturating all the spheres $S(c_1, \sqrt{l}) \cap \mathbb{Z}^n$ for $l = 0, 1, 2, \ldots, k$ we deduce

$$\sum_{j=2}^{m} \|c_j - c_1\| \geq \sum_{j=2}^{r_n(0)+r_n(1)+\cdots+r_n(k)+1} \|c_j - c_1\| \geq \sum_{l=0}^{k} r_n(l)\sqrt{l}.$$

Let us consider that m is equal to $r_n(0) + r_n(1) + \cdots + r_n(k) + r_n(k+1)$. We now proceed with the other sums $\sum_{j=i+1}^{m} \|c_j - c_i\|, i \geq 2$. For all $i = 1, 2, \ldots, r_n(k+1)$, the difference $m - i$ is greater than $r_n(0) + r_n(1) + \cdots + r_n(k)$ and this implies

$$\sum_{j=i+1}^{m} \|c_j - c_i\| \geq \sum_{l=0}^{k} r_n(l)\sqrt{l}.$$

Hence

$$\sum_{i=1}^{r_n(k+1)} \sum_{j=i+1}^{m} \|c_j - c_i\| \geq r_n(k+1)\Big(\sum_{l=0}^{k} r_n(l)\sqrt{l}\Big).$$

Since

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \|c_j - c_i\| = \sum_{i=1}^{r_n(k+1)} \sum_{j=i+1}^{m} \|c_j - c_i\| + \sum_{i=r_n(k+1)+1}^{r_n(k+1)+r_n(k)} \sum_{j=i+1}^{m} \|c_j - c_i\| + \cdots + \sum_{i=r_n(k+1)+r_n(k)+\ldots+r_n(2)+1}^{r_n(k+1)+r_n(k)+\ldots+r_n(1)} \sum_{j=i+1}^{m} \|c_j - c_i\|,$$

by reproducing the same computation term by term, we deduce

$$\begin{aligned} \text{(A.4)} \quad \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \|c_j - c_i\| &\geq r_n(k+1)\Big(\sum_{l=0}^{k} r_n(l)\sqrt{l}\Big) \\ &\quad + r_n(k)\Big(\sum_{l=0}^{k-1} r_n(l)\sqrt{l}\Big) + \cdots + r_n(2)r_n(1) + 2^n \\ &\geq \sum_{p=1}^{k+1} r_n(p)\Big(\sum_{l=0}^{p-1} r_n(l)\sqrt{l}\Big) \\ &\geq K_1^2 \sum_{p=1}^{k+1} p^{\frac{n}{2}-1}\Big(\sum_{l=0}^{p-1} l^{\frac{n-1}{2}}\Big). \end{aligned}$$

Now make use of the following classical inequalities: for all $\beta > 0$ and integer $r \geq 1$, $0 + 1^\beta + 2^\beta + \cdots + (r-1)^\beta \leq \int_0^r x^\beta dx = \frac{r^{\beta+1}}{\beta+1} \leq 1^\beta + 2^\beta + \cdots + (r-1)^\beta + r^\beta$. We deduce the following inequalities

$$\begin{aligned}\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \|c_j - c_i\| &\geq \frac{2K_1^2}{n+1} \sum_{p=1}^{k+1} p^{\frac{n}{2}-1}(p-1)^{\frac{n+1}{2}} \\ &\geq \frac{2K_1^2}{n+1} \sum_{p=1}^{k+1} (p-1)^{\frac{n}{2}-1}(p-1)^{\frac{n+1}{2}} \\ &\geq \frac{2K_1^2}{(n+1)} \sum_{p=1}^{k+1} (p-1)^{n-\frac{1}{2}} \\ &\geq \frac{4K_1^2}{(n+1)(2n+1)} \quad k^{n+1/2}\end{aligned}$$

and

$$\begin{aligned}m = r_n(0) + r_n(1) + \cdots + r_n(k) + r_n(k+1) &\leq K_2\Big(1 + \sum_{l=1}^{k+1} l^{\frac{n}{2}-1}\Big) \\ &\leq \frac{2K_2}{n}\left[\frac{n}{2} + (k+2)^{\frac{n}{2}}\right].\end{aligned}$$

Hence

$$m^{-2} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \|c_j - c_i\| \geq \frac{K_1^2 n^2 k^{n+1/2}}{K_2^2 (n+1)(2n+1)(k+2)^n} \left(1 + \frac{n}{2(k+2)^{\frac{n}{2}}}\right)^{-2}.$$

Putting $K_3 := \frac{K_1^2 n^2 2^{n+2}}{K_2^2 (n+1)(2n+1) 3^n (n+2^{\frac{n}{2}+1})^2}$, we deduce

$$m^{-2} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \|c_j - c_i\| \geq K_3 \sqrt{k}. \tag{A.5}$$

It is easy to check that the above computation is still valid when m lies strictly between $r_n(0)+r_n(1)+\cdots+r_n(k)$ and $r_n(0)+r_n(1)+\cdots+r_n(k+1)$. Therefore $\lim_{m\to+\infty} \frac{1}{m^2} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \|c_j - c_i\| = +\infty$. Let us observe that this minimal averaged growth to infinity is in "$\sqrt{k}$", which is extremely slow as compared to the growth of m to infinity.

Let us now compute a lower bound of the sum $m^{-2} \sum_{j=1}^{m} \|c_j\|$. Take for R the square root of an integer, say $R = \sqrt{t}, t \geq 2$. Let us consider that m is equal to $m = r_n(0) + r_n(1) + \cdots + r_n(k+1)$ and let us write it as: $m = r_n(t) + r_n(t+1) + \cdots + r_n(t+u) + w$ for a certain $u \geq 0$ and

$0 \leq w < r_n(t+u+1)$. Then

$$\sum_{j=1}^{m} \|c_j\| \geq \sum_{t}^{t+u} r_n(l)\sqrt{l} \geq K_1 \sum_{t}^{t+u} l^{\frac{n-1}{2}}.$$

As above we will make use of the following classical inequalities: for all positive integers s and $r \geq s+1$ and for any real number $\beta > 0$, $s^\beta + (s+1)^\beta + \cdots + (r-1)^\beta \leq \int_s^r x^\beta dx = \frac{r^{\beta+1} - s^{\beta+1}}{\beta+1} \leq (s+1)^\beta + (s+2)^\beta + \cdots + (r-1)^\beta + r^\beta$. We obtain the following inequalities:

$$\sum_{j=1}^{m} \|c_j\| \geq \frac{2K_1}{n+1}\left[(t+u)^{\frac{n+1}{2}} - (t-1)^{\frac{n+1}{2}}\right]$$

and

$$\begin{aligned}\frac{2K_1}{n}&\left[(t+u)^{n/2} - (t-1)^{n/2}\right] \\ &\leq m \\ &\leq r_n(t) + r_n(t+1) + \cdots + r_n(t+u) + r_n(t+u+1) \\ &\leq \frac{2K_2}{n}\left[(t+u+2)^{n/2} - t^{n/2}\right].\end{aligned}$$

From them we deduce

$$\frac{1}{m}\sum_{j=1}^{m} \|c_j\| \geq \frac{K_1 n\sqrt{u}}{K_2(n+1)}\left((1+\frac{t}{u})^{\frac{n+1}{2}} - \left(\frac{t-1}{u}\right)^{\frac{n+1}{2}}\right) \cdot \left(\left(1+\frac{2+t}{u}\right)^{n/2} - \left(\frac{t}{u}\right)^{n/2}\right)^{-1}.$$

Dividing the above inequality by m once again and changing t into $t-1$ and t into $t+2$ in the corresponding factors gives

$$\frac{1}{m^2}\sum_{j=1}^{m} \|c_j\| \geq \frac{K_1 n^2 u^{\frac{1-n}{2}}}{2K_2^2(n+1)}\left(\left(1+\frac{(t-1)}{u}\right)^{\frac{n+1}{2}} - \left(\frac{t-1}{u}\right)^{\frac{n+1}{2}}\right) \cdot \left(\left(1+\frac{2+t}{u}\right)^{n/2} - \left(\frac{2+t}{u}\right)^{n/2}\right)^{-2}$$

so that, using first-order developments in $(t-1)u^{-1}$, resp. in $(2+t)u^{-1}$, for u^{-1} close to zero, we obtain

$$\frac{1}{m^2}\sum_{j=1}^{m} \|c_j\| \geq \frac{K_1 (t-1)^{\frac{n-1}{2}}}{K_2^2}\frac{1}{u(u+2+t)^{n-2}}. \tag{A.6}$$

This lower bound, as a function of u on $[1, +\infty)$, goes to zero at infinity.

Let us now compute a lower bound of the sum $m^{-2}\sum_{j=1}^{m}\|c_j\|+ m^{-2}\sum_{1\le i<j\le m}\|c_i-c_j\|$. The lower bound given by (A.5) is a function of k and that given by (A.6) a function of u. In order to study their sum, we will deduce from the above a relation between u and $\sqrt{k}$ and replace $\sqrt{k}$ by a lower bound of $\sqrt{k}$ in (A.5) which will only depend upon u. From the above, with $m = r_n(0)+r_n(1)+\cdots+r_n(k+1)$, the following inequalities hold

$$\text{(A.7)} \qquad \frac{2K_1}{n}\left[(t+u)^{n/2}-(t-1)^{n/2}\right] \le m \le \frac{2K_2}{n}\left[\frac{n}{2}+(k+2)^{\frac{n}{2}}\right].$$

Let $h(x) = (t+x)^{n/2}$. Then $h(u)-h(-1) = (u+1)h'(\xi)$ for a certain $\xi\in[-1,u]$. We deduce $h(u)-h(-1)\ge \frac{n}{2}u(t-1)^{\frac{n}{2}-1}$ since the derivative $h'(x)$ is increasing on the interval $[-1,u]$. This last inequality and (A.7) imply

$$\text{(A.8)} \quad u^{1/n}\left[\left(\frac{n}{2}\Big(\frac{K_1}{K_2}(t-1)^{\frac{n}{2}-1}-1\Big)\right)^{2/n} -2\right]^{1/2} \le \sqrt{k}$$

for all $k\ge 1, u\ge 1, t\ge 2$.

Define

$$\text{(A.9)} \qquad g(t,u) := C_1(t)\frac{1}{u(u+2+t)^{n-2}} + C_2(t)u^{1/n}$$

where

$$C_1(t) = K_1K_2^{-2}(t-1)^{\frac{n-1}{2}}$$

and

$$C_2(t) = K_3\left[\left(\frac{n}{2}\Big(\frac{K_1}{K_2}(t-1)^{\frac{n}{2}-1}-1\Big)\right)^{2/n} -2\right]^{1/2}.$$

From (A.5) in which $\sqrt{k}$ is replaced by the above lower bound and from (A.6), we deduce

$$\text{(A.10)} \qquad \eta(\sqrt{t},m) \quad\ge\quad g(t,u).$$

It is routine to compute the value $u_{\min}(t)$ at which the function $u\to g(t,u)$ is minimal and the value $g(t,u_{\min}(t))$ of its minimum. The equation satisfied by $u_{\min}(t)$ is $nC_1(t)(u+2+t)^{1-n}\left[(n-1)u+2+t\right] = C_2(t)u^{1+1/n}$ and

$$\text{(A.11)} \quad g(t,u_{\min}(t)) = C_2(t)\left[\frac{1}{n}\frac{u_{\min}(t)+2+t}{(n-1)u_{\min}(t)+2+t}+1\right](u_{\min}(t))^{1/n}.$$

Since obviously $u_{\min}(t)\ge 1$, $\frac{1}{n}\frac{u+2+t}{(n-1)u+2+t}+1 \ge \frac{1}{n(n-1)}+1$ for $t\ge 2, u\ge 1$ and $\lim_{t\to+\infty} C_2(t) = +\infty$, we obtain: $\lim_{t\to+\infty} g(t,u_{\min}(t)) = +\infty$. We deduce that for any integer m of the form $r_n(0)+r_n(1)+\cdots+r_n(k+1)$ the limit $\lim_{R\to+\infty}\eta(R,m) = +\infty$ holds. It is easy to check that it is so even when m is an arbitrary integer which is not of this form. This implies, after (A.2), that $\lim_{R\to+\infty}\delta_2(\mathbb{Z}^n, {\mathbb{Z}^n}_R) = 0$, for all $n \geq 5$.

Step 2.– We will make use of the results of Step 1 and of the following three Lemmas. The assumption $n \geq 5$ holds. Let us fix the notations: if Γ is a $\mathcal{UD}$-set which contains the origin, then, for all $k \in \mathbb{N}$, denote $\Gamma^{(k)} := \{x \in \Gamma \mid \sqrt{k} \leq \|x\| < \sqrt{k+1}\}$, $r_\Gamma(\sqrt{k})$ the number of elements of $\Gamma^{(k)}$ and $s(\sqrt{k}) := \max_{\Gamma\in\mathcal{UD}}\{r_\Gamma(\sqrt{k})\} < \infty$. Since all the functions $\Gamma \to r_\Gamma(\sqrt{k})$, $k \in \mathbb{N}$, on $\mathcal{UD}$ are valued in $\mathbb{N}$, the maximum $s(\sqrt{k})$ is reached. Since, in particular, $r_{\mathbb{Z}^n}(\sqrt{k}) = r_n(k)$, for any positive integer k, the following Lemma is obvious.

Lemma A.1. *For any positive integer k the inequality $s(\sqrt{k}) \geq r_n(k)$ holds.*

In the following, we will enumerate the elements x_i of a $\mathcal{UD}$-set Λ in such a way that $\|x_j\| \geq \|x_i\|$ as soon as $j \geq i \geq 1$ (with $x_1 = 0$ if Λ contains the origin). The following Lemmas show that the sequence $\{s(\sqrt{k}) \mid k \in \mathbb{N}\}$ is universal for splitting up any $\mathcal{UD}$-set into layers of points with the objective of making use of Stolarsky's inequality (Proposition A.2) in a suitable way.

Lemma A.2. *Let Λ be an infinite $\mathcal{UD}$-set which contains the origin. For all positive integers $M, m \in \mathbb{N}$ such that $\sum_{k=0}^{M} s(\sqrt{k}) < m \leq \sum_{k=0}^{M+1} s(\sqrt{k})$, any point $x_m \in \Lambda$ indexed by such an integer m satisfies $\|x_m\| \geq \sqrt{M+1}$.*

Proof. This fact comes from the way we have enumerated the elements of Λ. Obviously, any point $x_m \in \Lambda$ indexed by such an integer m is such that $\sum_{k=0}^{M} r_\Lambda(\sqrt{k}) \leq \sum_{k=0}^{M} s(\sqrt{k}) < m$. By definition of the function r_Λ we obtain the inequality. □

Lemma A.3. *Let Λ be an infinite $\mathcal{UD}$-set which contains the origin. There exists a subset Λ^* of Λ, with $0 \in \Lambda^*$, and a surjective mapping $\psi_\Lambda : \Lambda \to \mathbb{Z}^n$ such that:*

(i) *$\psi_\Lambda(0) = 0$, $\|\psi_\Lambda(x)\| \leq \|x\|$ for all $x \in \Lambda$;*

(ii) *for all integers $M, m \in \mathbb{N}$ such that $\sum_{k=0}^{M} s(\sqrt{k}) < m \leq \sum_{k=0}^{M+1} s(\sqrt{k})$ the following equalities hold: $\|\psi_\Lambda(x_m)\| = \sqrt{M+1}$ for $x_m \in \Lambda \setminus \Lambda^*$, $\|\psi_\Lambda(x_m)\| = 0$ for $x_m \in \Lambda^*$;*

(iii) *the restriction of ψ_Λ to $\{0\} \cup \Lambda \setminus \Lambda^*$ is a bijection from $\{0\} \cup \Lambda \setminus \Lambda^*$ to $\mathbb{Z}^n$;*

(iv) *when $\Lambda = \mathbb{Z}^n$, then $\Lambda^* = \{0\}$ and ψ_Λ is the identity map up to a re-enumeration of the elements of the layer $(\mathbb{Z}^n)^{(k)}$ of $\mathbb{Z}^n$ for all $k \in \mathbb{N}$.*

Proof. Let us construct the function ψ_Λ. Denote $s^{(M)} := \sum_{k=0}^{M} s(\sqrt{k})$ for all $M \in \mathbb{N}$. The following $s(\sqrt{M+1})$-tuple of points:

$$(x_{s^{(M)}+1}, x_{s^{(M)}+2}, \ldots, x_{s^{(M)}+r_n(M+1)}, x_{s^{(M)}+r_n(M+1)+1},\\ x_{s^{(M)}+r_n(M+1)+2}, \ldots, x_{s^{(M+1)}})$$

of Λ will be splitted up into two parts. Let

$$\Lambda^{*(M)} = \{x_{s^{(M)}+r_n(M+1)+1}, x_{s^{(M)}+r_n(M+1)+2}, \ldots, x_{s^{(M+1)}}\}$$

and $\Lambda^* = \cup_{M\in\mathbb{N}}\Lambda^{*(M)}$. Let us put $\psi_\Lambda(z) = 0$ for all $z \in \Lambda^*$, and, for all $M \in \mathbb{N}$ and for all $i = s^{(M)} + 1, s^{(M)} + 2, \ldots, s^{(M)} + r_n(M+1)$, let us put $\psi_\Lambda(x_i) \in S(0, \sqrt{M+1}) \cap \mathbb{Z}^n$ such that the restriction of ψ_Λ to $\Lambda \setminus \Lambda^*$ is injective. In other terms, the first $r_n(M+1)$ points of the above $s(\sqrt{M+1})$-tuple of points are sent injectively by ψ_Λ to the $r_n(M+1)$ elements of $\mathbb{Z}^n$ of norm $\sqrt{M+1}$ which lie on the sphere $S(0, \sqrt{M+1})$, the remaining points $x_{s^{(M)}+r_n(M+1)+1}, x_{s^{(M)}+r_n(M+1)+2}, \ldots,\ x_{s^{(M+1)}}$ going to the origin of $\mathbb{Z}^n$. There is no uniqueness of such a mapping ψ_Λ: given Λ^*, any re-enumeration e of the elements of $\mathbb{Z}^n$ conserving the norm provides another suitable mapping $e \circ \psi_\Lambda : \Lambda \to \mathbb{Z}^n$. Properties (i) to (iv) of ψ_Λ are easy consequences of its definition. □

Let us now consider an infinite $\mathcal{UD}$-set Λ which contains the origin and let us continue the proof of Proposition A.1 (if Λ does not contain the origin we modify slightly a few points close to the origin for having this property). In a similar way as in Step 1 with (A.2), we are looking for a lower bound of the quantity (with $c_i, c_j \in \Lambda$ and $\|c_i\| \geq R, \|c_j\| \geq R$)

$$\eta_\Lambda(R, m) := \frac{1}{2m} + \frac{1}{m^2}\sum_{j=1}^{m}\|c_j\| + \frac{1}{m^2}\sum_{1\leq i<j\leq m}\|c_j - c_i\|$$

as a function of R and m. Let us observe that the differences $c_j - c_i$ belong to the translated $\mathcal{UD}$-sets $\Lambda - c_i = \{\lambda - c_i \mid \lambda \in \Lambda\}$ of Λ which all contain the origin. Let us now compute a lower bound of $m^{-2}\sum_{1\leq i<j\leq m}\|c_j - c_i\|$. For integers $M, m \in \mathbb{N}$ that satisfy

$$\sum_{k=0}^{M} s(\sqrt{k}) < m \leq \sum_{k=0}^{M+1} s(\sqrt{k}),$$

we deduce the following inequality:

$$\sum_{j=2}^{m}\|c_j - c_1\| \geq \sum_{j=2}^{m}\|\psi_{\Lambda - c_1}(c_j)\| \geq \sum_{l=0}^{M} r_n(l)\sqrt{l}$$

from Lemmas A.1, A.2 and A.3. We now proceed with the other sums $\sum_{j=i+1}^{m}\|c_j - c_i\|$, $i \geq 2$. Let us assume that $m = \sum_{q=0}^{M+1} s(\sqrt{q})$. For all

$i = 1, 2, \dots, s(\sqrt{M})$, the difference $m - i$ is greater than $\sum_{q=0}^{M} s(\sqrt{q})$ and this implies

$$\sum_{j=i+1}^{m} \|c_j - c_i\| \geq \sum_{j=i+1}^{m} \|\psi_{\Lambda - c_i}(c_j)\| \geq \sum_{l=0}^{M} r_n(l)\sqrt{l}.$$

We deduce the inequality

$$\sum_{i=1}^{s(\sqrt{M+1})} \sum_{j=i+1}^{m} \|c_j - c_i\| \geq s(\sqrt{M+1})\Big(\sum_{l=0}^{M} r_n(l)\sqrt{l}\Big)$$
$$\geq r_n(M+1)\Big(\sum_{l=0}^{M} r_n(l)\sqrt{l}\Big).$$

Since for all i, j the inequality $\|c_j - c_i\| \geq \|\psi_{\Lambda - c_i}(c_j)\|$ holds and that

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \|c_j - c_i\| = \sum_{i=1}^{s(\sqrt{M+1})} \sum_{j=i+1}^{m} \|c_j - c_i\|$$
$$+ \sum_{i=s(\sqrt{M+1})+1}^{s(\sqrt{M+1})+s(\sqrt{M})} \sum_{j=i+1}^{m} \|c_j - c_i\| + \dots$$
$$+ \sum_{i=s(\sqrt{M+1})+s(\sqrt{M})+\dots+s(\sqrt{2})+1}^{s(\sqrt{M+1})+s(\sqrt{M})+\dots+s(\sqrt{2})+s(\sqrt{1})} \sum_{j=i+1}^{m} \|c_j - c_i\|,$$

by reproducing the same computation term by term, we deduce

$$\sum_{1 \leq i < j \leq m} \|c_j - c_i\| \geq r_n(M+1)\Big(\sum_{l=0}^{M} r_n(l)\sqrt{l}\Big)$$
$$+ r_n(M)\Big(\sum_{l=0}^{M-1} r_n(l)\sqrt{l}\Big) + \dots + r_n(2)r_n(1) + 2^n.$$

This leads to the same inequality as in (A.5), with $m = \sum_{q=0}^{M+1} s(\sqrt{q})$, except that "$k$" has to be replaced by "M". Therefore, we obtain

$$m^{-2} \sum_{1 \leq i < j \leq m} \|c_j - c_i\| \geq K_3\sqrt{M}. \tag{A.12}$$

Let us now compute a lower bound of $m^{-2}\sum_{j=1}^{m} \|c_j\|$. Take $R = \sqrt{t}$ with $t \geq 2$ an integer and consider $m = \sum_{q=0}^{M+1} s(\sqrt{q})$. This lower bound corresponds to a distribution by layers of the points $c_1, c_2, \dots, c_m$ on Λ so that they are located as close as possible to the sphere $S(0, R)$. Let us write

m as the following sum: $m = s(\sqrt{t}) + s(\sqrt{t+1}) + \cdots + s(\sqrt{t+U}) + W$ for certain integers $U \geq 0$ and $0 < W \leq s(\sqrt{Tt+U+1})$. Then, by Lemma A.3,

$$\sum_{j=1}^{m} \|c_j\| \geq \sum_{j=1}^{m} \|\psi_\Lambda(c_j)\| \geq \sum_{l=t}^{t+U} r_n(l)\sqrt{l}.$$

Hence, by the same type of computation as in Step 1, and by replacing only "u" by "U", we deduce

$$\text{(A.13)} \qquad \frac{1}{m^2}\sum_{j=1}^{m} \|c_j\| \geq \frac{K_1(t-1)^{\frac{n-1}{2}}}{K_2^2}\frac{1}{U(U+2+t)^{n-2}}.$$

In order to compute a lower bound of the sum $m^{-2}\sum_{j=1}^{m}\|c_j\| + m^{-2}\sum_{1\leq i<j\leq m}\|c_j - c_i\|$ as a function of U only from (A.12) and (A.13), it remains to give explicitely a relation between M and U. This relation comes from the computation of a lower bound of m which will be a function of M only and an upper bound of m which will be a function of U only. Let us compute these bounds. First, since $\sum_{k=0}^{M+1} r_n(k) \leq \sum_{k=0}^{M+1} s(\sqrt{k}) = m$ we deduce, by the same type of computation as in Step 1 (with "U" instead of "u"),

$$\text{(A.14)} \qquad \frac{2K_1}{n}\left[(t+U)^{n/2} - (t-1)^{n/2}\right] \leq \sum_{q=0}^{M+1} s(\sqrt{q}) = m.$$

Second, if $\mathrm{vol}(B(0,x))$ denotes the volume of the ball $B(0,x)$, by counting the maximal possible number of points in $\{x \mid \sqrt{k} \leq \|x\| < \sqrt{k+1}\}$ (in this annulus any point should be at a distance from another one greater than unity), we deduce that the term $s(\sqrt{k}), k \geq 1$, is smaller than

$$\Big(\mathrm{vol}\big(B(0,\sqrt{k+1}+\frac{1}{2})\big) - \mathrm{vol}\big(B(0,\sqrt{k}-\frac{1}{2})\big)\Big)\Big(\mathrm{vol}\big(B(0,\frac{1}{2})\big)\Big)^{-1}.$$

Therefore

$$m = \sum_{k=0}^{M+1} s(\sqrt{k}) \leq 1 + 2^n \sum_{k=1}^{M+1}\left[\left(\sqrt{k+1}+\frac{1}{2}\right)^n - \left(\sqrt{k}-\frac{1}{2}\right)^n\right].$$

By a first-order development of each term, we deduce

$$m \leq 1 + n2^n \sum_{k=1}^{M+1}\left[\sqrt{k+1} - \sqrt{k} + 1\right]\left(\sqrt{k+1}+\frac{1}{2}\right)^{n-1}.$$

Since $\sqrt{k+1} - \sqrt{k} + 1 \leq 2$ we obtain that m is certainly exceeded by $n2^{n+1}\sum_{k=1}^{M+1}\left(\sqrt{k+1}+\frac{1}{2}\right)^{n-1}$. Now, for all $1 \leq k \leq M+1$, we have

$\sqrt{k+1}+\frac{1}{2} \leq \sqrt{k+3\sqrt{M+1}}$. We deduce

$$m \leq n2^{n+1} \sum_{k=1}^{M+1} \left(k+3\sqrt{M+1}\right)^{\frac{n-1}{2}}$$
$$\leq \frac{n2^{n+2}}{n+1}\left[(M+2+3\sqrt{M+1})^{\frac{n+1}{2}} - (1+3\sqrt{M+1})^{\frac{n+1}{2}}\right].$$

Denote $l(x) = \left(x+\frac{1+3\sqrt{M+1}}{M+1}\right)^{\frac{n+1}{2}}$ and $\omega = \sup_{M\geq 1}\left(\sup_{x\in[0,1]} l'(x)\right)$. Then it is easy to check, by factorizing $(M+1)^{(n+1)/2}$ and applying a first-order development to the factors in the right-hand side term of the last inequality that this term is smaller than $n2^{n+2}\omega(n+1)^{-1}(M+1)^{\frac{n+1}{2}}$. Hence

$$m \leq n2^{n+2}\omega(M+1)^{\frac{n+1}{2}}. \tag{A.15}$$

From (A.14) and (A.15) (as for (A.7) and (A.8)) we deduce the following inequality

$$U^{\frac{1}{n+1}}\left[\frac{1}{4}\Big(\frac{K_1}{2\omega}\Big)^{2/(n+1)}(t-1)^{\frac{n-2}{n+1}} - 1\right]^{1/2} \leq \sqrt{M}. \tag{A.16}$$

Define

$$g_\Lambda(t,U) := \frac{C_1(t)}{U(U+2+t)^{n-2}} + C_3(t)U^{\frac{1}{n+1}}, \tag{A.17}$$

where $C_3(t) := K_3\left[\frac{1}{4}\left(\frac{K_1}{2\omega}\right)^{\frac{2}{n+1}}(t-1)^{\frac{n-2}{n+1}} - 1\right]^{1/2}$. Then (as in Step 1)

$$\eta_\Lambda(\sqrt{t}, m) \geq g_\Lambda(t, U_{\min}(t)), \tag{A.18}$$

for all $m = \sum_{k=0}^{M+1} s(\sqrt{k})$, where $U_{\min}(t)$ is the value at which the function $U \to g_\Lambda(t,U)$ is minimal. The proof of $\lim_{t\to+\infty} g_\Lambda(t, U_{\min}(t)) = +\infty$ is similar as in Step 1, for any integer m. This implies, after (A.2), that $\lim_{R\to+\infty} \delta_2(\Lambda, \Lambda_R) = 0$ for all $\mathcal{UD}$-set Λ and all $n \geq 5$. This convergence is obviously uniform in the sense stated in Proposition A.1 since the sequence $(s(\sqrt{k}))_k$ is universal and optimal for splitting up any $\mathcal{UD}$-set Λ.

Step 3.– If Λ is a $\mathcal{UD}$-set in $\mathbb{R}^n$ with $n \leq 4$, it can be viewed as a $\mathcal{UD}$-set in $\mathbb{R}^5$. Since Proposition A.1 is true for $n = 5$ by Step 2, it is also true in lower dimensions by descent. □

Acknowledgements. The authors thank the anonymous referee for very helpful comments and suggestions.

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“On densest packings of equal spheres of $\mathbb{R}^n$ and Marcinkiewicz spaces”,

G. Muraz et J.-L. Verger-Gaugry,
dans “Interface between Harmonic Analysis and Number Theory”, Eds. B. Saffari et al, Birkhaüser (2007), soumis.

On densest packings of equal balls of $\mathbb{R}^n$ and Marcinkiewicz spaces

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Abstract. We investigate, by " à la Marcinkiewicz" techniques applied to the (asymptotic) density function, how dense systems of equal spheres of $\mathbb{R}^n, n \geq 1$, can be partitioned at infinity in order to allow the computation of their density as a true limit and not a limsup. The density of a packing of equal balls is the norm 1 of the characteristic function of the systems of balls in the sense of Marcinkiewicz. Existence Theorems for densest sphere packings and completely saturated sphere packings of maximal density are given new direct proofs.

2000 Mathematics Subject Classification: 52C17, 52C23.

Keywords: Delone set, Marcinkiewicz norm, density, sphere packing.

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1 Introduction

The existence of densest sphere packings in $\mathbb{R}^n, n \geq 2$, asked the question to know how they could be constructed. The problem of constructing very dense sphere packings between the bounds of Kabatjanskii-Levenstein and Minkowski-Hlawka type bounds (see Fig. 1 in [MVG1]) remains open [Bz] [Ca] [CS] [GL] [GOR] [R] [Z]. There are two problems: the first one is the determination of the supremum δ_n over all possible densities, δ_n being called the packing constant, as a function of n only (for $n = 3$ see Hales [H]); the second one consists in characterizing the (local, global) configuration of balls in a densest sphere packing, namely for which the density is δ_n.

The notion of complete saturation was introduced by Fejes-Toth, Kuperberg and Kuperberg [FTKK]. Section 2 gives new direct proofs of the existence Theorems for completely saturated sphere packings (see Bowen [Bo] for a proof with $\mathbb{R}^n$ and $\mathbb{H}^n$ as ambient spaces) of maximal density and densest sphere packings in $\mathbb{R}^n$. For this purpose new metrics are introduced (Subsection 2.1) on the space of uniformly discrete sets (space of equal sphere packings), and this leads to a continuity Theorem for the density function (Theorem 7.2).

Let Λ be a uniformly discrete set of $\mathbb{R}^n$ of constant $r > 0$, that is a discrete point set for which $\|x - y\| \geq r$ for all $x, y \in \Lambda$, with equality at least for one couple of elements of Λ, and consider the system of spheres (in fact *balls*) $\mathcal{B}(\Lambda) = \{\lambda + B(0, \frac{r}{2}) \mid \lambda \in \Lambda\}$, where $B(c, t)$ denotes the closed ball of center c and radius t. Let $B = B(0, 1/2)$. The fact that the density

$$\delta(\mathcal{B}(\Lambda)) := \limsup_{T \to +\infty} \left[\mathrm{vol}\big(\big(\bigcup_{\lambda \in \Lambda} (\lambda + B(0, r/2)) \big) \bigcap B(0, T) \big) / \mathrm{vol}(B(0, T)) \right]$$

of $\mathcal{B}(\Lambda)$ is equal to the norm ("norm 1") of Marcinkiewicz of the characteristic function $\chi_{\mathcal{B}(\Lambda)}$ of $\mathcal{B}(\Lambda)$ [B+] [PH] [M], namely

$$\delta(\mathcal{B}(\Lambda)) = \|\chi_{\mathcal{B}(\Lambda)}\|_1, \tag{1.1}$$

where, for all $p \in \mathbb{R}^{+*}$ and all $f \in \mathcal{L}^p_{loc}$ with $\mathcal{L}^p_{loc}$ the space of complex-valued functions f defined on $\mathbb{R}^n$ whose p-th power of the absolute value $|f|^p$ is integrable over any bounded measurable subset of $\mathbb{R}^n$ for the Lebesgue measure,

$$\|f\|_p := \limsup_{t \to +\infty} |f|_{p,t}, \tag{1.2}$$

with

$$|f|_{p,t} := \left(\frac{1}{\mathrm{vol}(tB)} \int_{tB} |f(x)|^p dx\right)^{1/p}, \qquad f \in \mathcal{L}^p_{loc}, \tag{1.3}$$

asks the following question: what can tell the theory of Marcinkiewicz spaces to the problem of constructing very dense sphere packings ? Obviously the problem of the determination of the packing constant or more generally of the density is associated with the quotient space $\mathcal{L}^p_{loc}/\mathcal{R}$ where $\mathcal{R}$ is the Marcinkiewicz equivalence relation (Section 3): the density function is a class function, that is is well defined on the Marcinkiewicz space $\mathcal{M}^p$ with $p = 1$. For instance any finite cluster of spheres has the same density, equal to zero, as the empty packing (no sphere); the Marcinkiewicz class of the empty sphere packing being much larger than the set of finite clusters of spheres. Then it suffices to understand the construction of one peculiar sphere packing per Marcinkiewicz class. It is the object of this note to precise the geometrical constraints given by such a construction.

Since any non-singular affine transformation T on a system of balls $\mathcal{B}(\Lambda)$ leaves its density invariant (Theorem 1.7 in [R]), namely

$$\delta(\mathcal{B}(\Lambda)) = \delta(T(\mathcal{B}(\Lambda))), \tag{1.4}$$

we will only consider packings of spheres of common radius 1/2 in the sequel. It amounts to consider the space $\mathcal{UD}$ of uniformly discrete subsets of $\mathbb{R}^n$ of constant 1. Its elements will be called $\mathcal{UD}$-sets. Denote by $\overline{f}$ the class in $\mathcal{M}^p = \mathcal{L}^p_{loc}/\mathcal{R}$ of $f \in \mathcal{L}^p_{loc}$, where $\mathcal{L}^p_{loc}$ is endowed with the $\mathcal{M}^p$-topology (Section 3), and by

$$\nu: \begin{array}{ccc} \mathcal{UD} & \to & \mathcal{L}^1_{loc}, \\ \Lambda & \to & \chi_{\mathcal{B}(\Lambda)} \end{array} \qquad \text{resp.} \quad \overline{\nu}: \begin{array}{ccc} \mathcal{UD} & \to & \mathcal{M}^1 \\ \Lambda & \to & \overline{\chi_{\mathcal{B}(\Lambda)}} \end{array}$$

the (set-) embedding of $\mathcal{UD}$ in $\mathcal{L}^1_{loc}$, resp. in $\mathcal{M}^1$.

Theorem 1.1. *The image $\nu(\mathcal{UD})$ in $\mathcal{L}^1_{loc} \cap \mathcal{L}^\infty$, resp. $\overline{\nu}(\mathcal{UD})$ in $\mathcal{M}^1$, is closed.*

Theorem 1.1 is a reformulation of the following more accurate theorem, since $\mathcal{M}^p$ is complete [B] [B+]. For $0 \leq \lambda \leq \mu$ denote

$$\mathcal{C}(\lambda, \mu) := \{x \in \mathbb{R}^n \mid \lambda \leq \|x\| \leq \mu\}$$

the closed annular region of space between the spheres centered at the origin of respective radii λ and μ.

Theorem 1.2. *Let $(\Lambda_m)_{m\geq 1}$ be a sequence of $\mathcal{UD}$-sets such that the sequence $(\chi_{\mathcal{B}(\Lambda_m)})_{m\geq 1}$ is a Cauchy sequence for the pseudo-metric $\|\cdot\|_1$ on $\mathcal{L}^1_{loc} \cap \mathcal{L}^\infty$. Then, there exist*

(i) *a strictly increasing sequence of positive integers $(m_i)_{i\geq 1}$,*

(ii) *a strictly increasing sequence of real numbers $(\lambda_i)_{i\geq 1}$ with $\lambda_i \geq 1$ and $\lambda_{i+1} > 2\lambda_i$,*

such that, with

$$\Lambda = \bigcup_{i\geq 1} \Lambda_{m_i} \cap \mathcal{C}(\lambda_i + 1/2, \lambda_{i+1} - 1/2), \tag{1.5}$$

the two functions

$$\chi_{\mathcal{B}(\Lambda)} \text{ and } \lim_{i\to+\infty} \chi_{\mathcal{B}(\Lambda_{m_i})}$$

are $\mathcal{M}^1$-equivalent. As a consequence

$$\delta(\mathcal{B}(\Lambda)) = \lim_{i\to+\infty} \delta(\mathcal{B}(\Lambda_{m_i})). \tag{1.6}$$

The situation is the following for a (densest) sphere packing $\mathcal{B}(\Lambda)$ of $\mathbb{R}^n$ for which $\delta(\mathcal{B}(\Lambda)) = \delta_n$:

* either it cannot be reached by a sequence of sphere packings such as in Theorem 1.2, in which case there is an *isolation phenomenon*,
* or there exists at least one sequence of sphere packings such as in Theorem 1.2, and it is Marcinkiewicz - equivalent to a sphere packing having the asymptotic annular structure given by Theorem 1.2, where the sequence of thicknesses of the annular portions exhibit an exponential growth.

The sharing of space in annular portions as given by Theorem 1.2 may allow constructions of very dense packings of spheres layer-by-layer in each portion independently, since the intermediate regions $\mathcal{C}(\lambda_i - 1/2, \lambda_i + 1/2)$ are all of constant thickness 1 which is twice the common ball radius 1/2. These intermediate regions do not contribute to the density so that they can be filled up or not by spheres. However the existence of such unfilled spherical gaps are not likely to provide completely saturated packings, at least for $n = 2$ [KKK].

Note that the value 2 which controls the exponential sequence of radii $(\lambda_i)_i$ by $\lambda_{i+1} > 2\lambda_i$ in Theorem 1.2 (ii) can be replaced by any value $a > 1$. This is important for understanding constructions of sphere packings iteratively on the dimension n: indeed, chosing $a > 1$ sufficiently small brings the problem back to fill up first one layer in a as dense as possible way, therefore in dimension $n-1$, then propagating towards the orthogonal direction exponentially.

The terminology *density* is usual in the field of lattice sphere packings, while the terminology *asymptotic measure*, therefore *asymptotic density*, is usual in Harmonic Analysis, both meaning the same in the present context.

2 Densest sphere packings and complete saturation

The set SS of systems of equal spheres of radius $1/2$ and the set $\mathcal{UD}$ are in one-to-one correspondence: $\Lambda = (a_i)_{i\in\mathbb{N}} \in \mathcal{UD}$ is the set of sphere centres of $\mathcal{B}(\Lambda) = \{a_i + B \mid i \in \mathbb{N}\} \in SS$. More conveniently we will use the set $\mathcal{UD}$ of point sets of $\mathbb{R}^n$ instead of SS. The subset of $\mathcal{UD}$ of finite uniformly discrete sets of constant 1 of $\mathbb{R}^n$ is denoted by $\mathcal{UD}_f$.

2.1 A metric on $\mathcal{UD}$ invariant by the rigid motions of $\mathbb{R}^n$

Denote by $O(n,\mathbb{R})$ the n-dimensional orthogonal group of $n \times n$ matrices M, i.e. such that $M^{-1} = {}^tM$. A *rigid motion* (or an *Euclidean displacement*) is an ordered pair (ρ, t) with $\rho \in O(n,\mathbb{R})$ and $t \in \mathbb{R}^n$ [Cp]. The composition of two rigid motions is given by $(\rho,t)(\rho',t') = (\rho\rho', \rho(t') + t)$ and the group of rigid motions is the split extension of $O(n,\mathbb{R})$ by $\mathbb{R}^n$ (as a semi-direct product). It is endowed with the usual topology. Theorem 2.1, obtained as a generalization of the Selection Theorem of Mahler [Cy] [GL] [Ma] [Mt], gives the existence of a metric d on $\mathcal{UD}$ [MVG2] which extends the Hausdorff metric on the subspace $\mathcal{UD}_f$. The metric d is not invariant by translation. From it, adding to the construction of d some additional constraints so that it gains in invariant properties (Proposition 2.2 iii) proved in Section 5), a new metric D, invariant by translation and by the group of rigid motions of $\mathbb{R}^n$ (Theorem 2.3 proved in Section 6), can be constructed, giving a new topology to $\mathcal{UD}$, suitable for studying the continuity of the density function (Theorem 7.2).

Theorem 2.1. *The set $\mathcal{UD}$ can be endowed with a metric d such that the topological space $(\mathcal{UD}, d)$ is compact and such that the Hausdorff metric Δ on $\mathcal{UD}_f$ is compatible with the restriction of the topology of $(\mathcal{UD}, d)$ to $\mathcal{UD}_f$.*

Proof. Theorem 1.2 in [MVG2]. □

Proposition 2.2. *There exists a metric d on $\mathcal{UD}$ such that:*

i) *the space* $(\mathcal{UD}, d)$ *is compact,*

ii) *the Hausdorff metric on* $\mathcal{UD}_f$ *is compatible with the restriction of the topology of* $(\mathcal{UD}, d)$ *to* $\mathcal{UD}_f$*,*

iii) $d(\Lambda, \Lambda') = d(\rho(\Lambda), \rho(\Lambda'))$ *for all* $\rho \in O(n, \mathbb{R})$ *and* $\Lambda, \Lambda' \in \mathcal{UD}$*.*

Since the density of a sphere packing is left invariant by any non-singular affine transformation ((1.4); Theorem 1.7 in Rogers [R]), it is natural to construct metrics on $\mathcal{UD}$ which are at least invariant by the translations and by the orthogonal group of $\mathbb{R}^n$. Such a metric is given by the following theorem.

Theorem 2.3. *There exists a metric* D *on* $\mathcal{UD}$ *such that:*

i) $D(\Lambda_1, \Lambda_2) = D(\rho(\Lambda_1) + t, \rho(\Lambda_2) + t)$ *for all* $t \in \mathbb{R}^n, \rho \in O(n, \mathbb{R}^n)$ *and all* $\Lambda_1, \Lambda_2 \in \mathcal{UD}$*,*

ii) *the space* $(\mathcal{UD}, D)$ *is complete and locally compact,*

iii) *(pointwise pairing property) for all non-empty* $\Lambda, \Lambda' \in \mathcal{UD}$ *such that* $D(\Lambda, \Lambda') < \epsilon$*, each point* $\lambda \in \Lambda$ *is associated with a unique point* $\lambda' \in \Lambda'$ *such that* $\|\lambda - \lambda'\| < \epsilon/2$*,*

iv) *the action of the group of rigid motions* $O(n, \mathbb{R}) \ltimes \mathbb{R}^n$ *on* $(\mathcal{UD}, D)$: $((\rho, t), \Lambda) \to (\rho, t) \cdot \Lambda = \rho(\Lambda) + t$ *is such that its subgroup of translations* $\mathbb{R}^n$ *acts continuously on* $\mathcal{UD}$*.*

2.2 Existence Theorems

The two following Theorems rely upon the continuity of the density function $\| \cdot \|_1 \circ \nu$ on the space $(\mathcal{UD}, D)$ (Theorem 2.3 and Theorem 7.2).

Theorem 2.4. *There exists an element* $\Lambda \in \mathcal{UD}$ *such that the following equality holds:*

$$\delta(\mathcal{B}(\Lambda)) = \delta_n. \tag{2.1}$$

Proof. See Groemer [Gr] and Section 7. □

We will say that $\Lambda \in \mathcal{UD}$ is *saturated*, or *maximal*, if it is impossible to add a replica of the ball B (a ball of radius 1/2) to $\mathcal{B}(\Lambda)$ without destroying the fact that it is a packing of balls, i.e. without creating an overlap of balls. The set SS of systems of balls of radius 1/2, is partially ordered by the relation $\prec$ defined by $\Lambda_1, \Lambda_2 \in \mathcal{UD}$, $\mathcal{B}(\Lambda_1) \prec \mathcal{B}(\Lambda_2) \iff \Lambda_1 \subset \Lambda_2$. By Zorn's lemma, maximal packings of balls exist. The saturation operation of a packing of balls consists in adding balls to obtain a maximal packing of balls. It is fairly arbitrary and may be finite or infinite. More generally [FTKK], $\mathcal{B}(\Lambda)$ is said to be *m-saturated* if no finite subsystem of $m-1$ balls of it can be replaced with m replicas of the ball $B(0, r/2)$. The notion of m-saturation was introduced by Fejes-Toth, Kuperberg and Kuperberg [FTKK]. Obviously, 1-saturation means saturation, and m-saturation implies $(m-1)$-saturation. It is not because a packing of balls is saturated, or m-saturated, that its density is equal to δ_n. The packing $\mathcal{B}(\Lambda)$ is *completely saturated* if it is m-saturated for every $m \geq 1$. Complete saturation is a sharper version of maximum density [Ku].

Theorem 2.5. *Every ball in $\mathbb{R}^n$ admits a completely saturated packing with replicas of the ball, whose density is equal to the packing constant δ_n.*

Proof. Theorem 1.1 in [FTKK]. See also Bowen [Bo]. A direct proof is given in Section 7, where we prove that there always exists a completely saturated sphere packing in the Marcinkiewicz class of a densest sphere packing. □

3 Marcinkiewicz spaces and norms

Let $p \in \mathbb{R}^{+*}$. The Marcinkiewicz p-th space $\mathcal{M}^p$ is the quotient space of the subspace $\{f \in \mathcal{L}^p_{loc} \mid \|f\|_p < +\infty\}$ of $\mathcal{L}^p_{loc}$ by the equivalence relation $\mathcal{R}$ which identifies f and g as soon as $\|f-g\|_p = 0$ (Marcinkiewicz [M], Bertrandias [B], Vo Khac [VK]):

$$\mathcal{M}^p := \{\overline{f} \mid f \in \mathcal{L}^p_{loc}, \|f\|_p < +\infty\}. \tag{3.1}$$

This equivalence relation is called Marcinkiewicz equivalence relation. It is usual to introduce, with $|f|_{p,t}$ given by (1.3), the two semi-norms

$$\|f\|_p := \limsup_{t \to +\infty} |f|_{p,t}$$

and

$$|||f|||_p := \sup_{t>0} |f|_{p,t}$$

on $\mathcal{L}^p_{loc}$. The vector space $\mathcal{M}^p$ is then normed with $\|\overline{f}\|_p = \|f\|_p$.

Theorem 3.1. *The space $\mathcal{M}^p$ is complete.*

Proof. Marcinkiewicz [M], [B], [VK]. □

We call $\mathcal{M}^p$-topology the topology induced by this norm on $\mathcal{M}^p$ or on $\mathcal{L}^p_{loc}$ itself. Both spaces will be endowed with this topology.

Following Bertrandias [B] we say that a function $f \in \mathcal{L}^p_{loc}$ is $\mathcal{M}^p$-regular if

$$\lim_{t\to+\infty} \frac{1}{\text{vol(t B)}} \int_{tB\setminus(t-l)B} |f(x)|^p dx = 0 \qquad \text{for all real number } l.$$

Since all functions $f \in \mathcal{L}^p_{loc}$ such that $\|f\|_p = 0$ are $\mathcal{M}^p$-regular, we consider classes of $\mathcal{M}^p$-regular functions of $\mathcal{L}^p_{loc}$ modulo the Marcinkiewicz equivalence relation. We call $\mathcal{M}^p_r$ the set of classes of Marcinkiewicz equivalent $\mathcal{M}^p$-regular functions.

Proposition 3.2. *The set $\mathcal{M}^p_r$ is a complete vector subspace of $\mathcal{M}^p$.*

Proof. [B]. □

4 Proof of Theorem 1.2

Theorem 1.2 is the n-dimensional version of the remark of Marcinkiewicz [M] in the case $p = 1$. We prove a theorem slightly stronger than Theorem 1.2 (Theorem 4.2), by making the assumption in Lemma 4.1 and in Theorem 4.2 that p is ≥ 1 in full generality.

Lemma 4.1. *Let $p \geq 1$. Let $(\lambda_i)_{i\geq 1}$ be a sequence of real numbers such that*

$$\lambda_i \geq 1, \quad \lambda_{i+1} > 2\lambda_i, \qquad\qquad i \geq 1.$$

Let $\mathcal{C}_i := \mathcal{C}\big(\lambda_i + 1/2, \lambda_{i+1} - 1/2\big)$. Then for all bounded function $f \in \mathcal{L}^p_{loc}$ such that $f_{|\mathcal{C}_i} \equiv 0$ for all $i \geq 1$, we have

$$\|f\|_p = 0$$

Proof. Immediate. □

Lemma 4.1 is the special case of $\mathcal{M}^p$-regularity applied to the characteristic functions of systems of spheres which eventually lie within the spherical intermediate regions $\mathcal{C}_i$. It proves that such spheres do not contribute to the density anyway.

Theorem 4.2. *Let $p \geq 1$. Let $(\Lambda_m)_{m\geq 1}$ be a sequence of $\mathcal{UD}$-sets such that the sequence $(\chi_{\mathcal{B}(\Lambda_m)})_{m\geq 1}$ is a Cauchy sequence for the pseudo-metric $\|\cdot\|_p$ on $\mathcal{L}^p_{loc} \cap \mathcal{L}^\infty$. Then, there exist*

(i) *a strictly increasing sequence of positive integers* $(m_i)_{i\geq 1}$,

(ii) *a strictly increasing sequence of real numbers* $(\lambda_i)_{i\geq 1}$ *with* $\lambda_i \geq 1$ *and* $\lambda_{i+1} > 2\lambda_i$,

such that, with

$$\Lambda = \bigcup_{i\geq 1} \Lambda_{m_i} \cap \mathcal{C}(\lambda_i + 1/2, \lambda_{i+1} - 1/2), \tag{4.1}$$

the two functions

$$\chi_{\mathcal{B}(\Lambda)} \qquad \textit{and} \qquad \lim_{i\to+\infty} \chi_{\mathcal{B}(\Lambda_{m_i})} \tag{4.2}$$

are $\mathcal{M}^p$*-equivalent.*

Proof. Since the sequence $(\chi_{\mathcal{B}(\Lambda_m)})$ is a Cauchy sequence, let us chose a subsequence of $\mathcal{UD}$-sets $(\Lambda_{m_i})_{i\geq 1}$ which satisfies

$$\|\chi_{\mathcal{B}(\Lambda_{m_i})} - \chi_{\mathcal{B}(\Lambda_{m_{i+1}})}\|_p \leq 2^{-(i+1)}\,.$$

Then, denoting

$$R_\lambda(f) := \sup_{\lambda+1/2\leq t<+\infty} \left(\frac{1}{\mathrm{vol}(tB)}\int_{tB} |f(x)|^p dx\right)^{1/p}, \qquad f \in \mathcal{L}^p_{loc},$$

let us chose a sequence of real numbers $(\lambda_i)_{i\geq 1}$ for which $\lambda_i \geq 1, \lambda_{i+1} > 2\lambda_i$, and such that

$$R_{\lambda_i}(\chi_{\mathcal{B}(\Lambda_{m_i})} - \chi_{\mathcal{B}(\Lambda_{m_{i+1}})}) \leq 2^{-i}.$$

Let us define the function

$$H(x) := \begin{cases} \chi_{\mathcal{B}(\Lambda_{m_i})}(x) & \text{if} \quad \lambda_i + 1/2 \leq \|x\| \leq \lambda_{i+1} - 1/2 \quad (i = 1, 2, \dots), \\ 0 & \text{if} \quad \lambda_i - 1/2 < \|x\| < \lambda_i + 1/2 \quad (i = 1, 2, \dots), \\ 0 & \text{if} \quad \|x\| \leq \lambda_1 - 1/2. \end{cases}$$

The function $H(x)$ is exactly the characteristic function of $\mathcal{B}(\Lambda)$ on $J := \bigcup_{j=1}^{+\infty} \mathcal{C}_j$ the portion of space occupied by the closed annuli $\mathcal{C}_j$. Let us prove that the function $H(x)$ satisfies:

$$\lim_{i\to+\infty} \|H - \chi_{\mathcal{B}(\Lambda_{m_i})}\|_p = 0. \tag{4.3}$$

Let us fix i and take t and k such that

$$\lambda_k + 1/2 \leq 2t \leq \lambda_{k+1} - 1/2 \tag{4.4}$$

holds with $k \geq i + 1$. Then

$$\int_{tB} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx \;=\; \int_{tB\cap J} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx$$

$$+ \int_{tB\cap(\mathbb{R}^n\setminus J)} \chi_{\mathcal{B}(\Lambda_{m_i})}(x)^p dx.$$

By Lemma 4.1,

$$\|\chi_{\mathcal{B}(\Lambda_{m_i})} \cap \chi_{\mathbb{R}^n\setminus J}\|_p = 0.$$

Hence, we have just to consider the portion of space occupied by the spheres $\mathcal{B}(\Lambda_{m_i})$ in $tB \cap J$. We have

$$\int_{tB\cap J} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx = \sum_{\nu=1}^{i} \int_{tB\cap\mathcal{C}_\nu} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx$$

$$+ \sum_{\nu=i+1}^{k-1} \int_{tB\cap\mathcal{C}_\nu} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx + \int_{tB\cap\mathcal{C}_k} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx$$

$$= A + E + C.$$

Let us now transform the sum A:

$$\sum_{\nu=1}^{i} \int_{tB\cap\mathcal{C}_\nu} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx = \sum_{\nu=1}^{i} \int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\nu})}(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx.$$

But, for all $\nu \in \{1, 2, \ldots, i-1\}$,

$$\left(\int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\nu})}(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx\right)^{1/p}$$

$$\leq \sum_{\omega=\nu}^{i-1} \left(\int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\omega})}(x) - \chi_{\mathcal{B}(\Lambda_{m_{\omega+1}})}(x)|^p dx\right)^{1/p}$$

$$\leq \sum_{\omega=\nu}^{i-1} (\mathrm{vol}((\lambda_\omega + 1/2)B))^{1/p}\ R_{\lambda_\omega}(\chi_{\mathcal{B}(\Lambda_{m_\omega})} - \chi_{\mathcal{B}(\Lambda_{m_{\omega+1}})})$$

$$\leq \sum_{\omega=\nu}^{i-1} (\mathrm{vol}((\lambda_i + 1/2)B))^{1/p}\ 2^{-\omega} \leq (\mathrm{vol}((\lambda_i + 1/2)B))^{1/p}.$$

Hence

$$A \leq i\,\mathrm{vol}((\lambda_i + 1/2)B). \tag{4.5}$$

Let us transform the sum E:

$$\sum_{\nu=i+1}^{k-1} \int_{tB\cap\mathcal{C}_\nu} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx = \sum_{\nu=i+1}^{k-1} \int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\nu})}(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx.$$

But, for all $\nu \in \{i+1, i+2, \dots, k-1\}$,

$$\left(\int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\nu})}(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx\right)^{1/p}$$

$$\leq \sum_{\omega=i}^{\nu-1} \left(\int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\omega})}(x) - \chi_{\mathcal{B}(\Lambda_{m_{\omega+1}})}(x)|^p dx\right)^{1/p}$$

$$\leq \sum_{\omega=i}^{\nu-1} (\mathrm{vol}((\lambda_\omega + 1/2)B))^{1/p}\ R_{\lambda_\omega}(\chi_{\mathcal{B}(\Lambda_{m_\omega})} - \chi_{\mathcal{B}(\Lambda_{m_{\omega+1}})})$$

$$\leq \sum_{\omega=i}^{\nu-1} (\mathrm{vol}((\lambda_\omega + 1/2)B))^{1/p}\ 2^{-\omega} \leq \mathrm{vol}((\lambda_\nu + 1/2)B)^{1/p}\, 2^{-i+1}.$$

Hence,

$$E \leq 2^{-(i-1)p}\,(\mathrm{vol}((\lambda_{i+1}+1/2)B) + \mathrm{vol}((\lambda_{i+2}+1/2)B) + \dots$$

$$+ \mathrm{vol}((\lambda_{k-1}+1/2)B))$$

$$\leq 2^{-(i-1)p}\,\mathrm{vol}((\lambda_k + 1/2)B)\left(\frac{1}{2^n} + \frac{1}{2^{2n}} + \dots\right)$$

$$\leq 2^{-(i-1)p}\,\mathrm{vol}(2tB) = 2^{-(i-1)p+n}\,\mathrm{vol}(tB). \tag{4.6}$$

Let us transform the sum C:

$$C \leq 2^{-(i-1)p+n}\,\mathrm{vol}(tB). \tag{4.7}$$

From (4.5), (4.6) and (4.7) we deduce

$$\left(\frac{1}{\mathrm{vol}(tB)} \int_{tB\cap J} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx\right)^{1/p}$$

$$\leq \left[\frac{i\,\mathrm{vol}((\lambda_i + 1/2)B)}{\mathrm{vol}(tB)} + 2^{-(i-1)p+n+1}\right]^{1/p}.$$

Using (4.4) we deduce, for a certain constant $c > 0$,

$$\|H - \chi_{\mathcal{B}(\Lambda_{m_i})}\|_p \leq c\,2^{-(i-1)}.$$

This implies (4.3). Now, if $m_i \leq q < m_{i+1}$, $i \geq 1$, we have

$$\|H - \chi_{\mathcal{B}(\Lambda_q)}\|_p \leq \|\chi_{\mathcal{B}(\Lambda_{m_i})} - \chi_{\mathcal{B}(\Lambda_q)}\|_p + \|H - \chi_{\mathcal{B}(\Lambda_{m_i})}\|_p = o(1) + o(1) = o(1)$$

when i tends to $+\infty$. The proof of the $\mathcal{M}^p$-equivalence (4.2) between H and $\lim_{i\to+\infty} \chi_{\mathcal{B}(\Lambda_i)}$ is now complete.

The thickness of the empty annular intermediate regions $\mathcal{C}_i$ is equal to 1: it ensures that the limit point set Λ is uniformly discrete of constant 1. □

5 Proof of Proposition 2.2

The metric d on $\mathcal{UD}$ was constructed in [MVG2], §3.2.1, as a kind of counting system normalized by a suitable distance function. In order to make explicit the statement iii) of Proposition 2.2, we recall the construction of d, adding the ingredient (5.2) in order to obtain the claim. The metric d is given in Lemma 5.1.

For all $\Lambda \in \mathcal{UD}$, we denote by Λ_i its i-th element. Let

$$\mathcal{E} = \{(D,E) \mid D \text{ countable point set in } \mathbb{R}^n, E \text{ countable point set in } (0,1/2)\}$$

and $f : \mathbb{R}^n \to [0,1]$ a continuous function with compact support in $B(0,1)$ which satisfies:

$$f(0) = 1, \tag{5.1}$$

$$f(\rho(t)) = f(t) \qquad \text{for all } t \in \mathbb{R}^n \text{ and all } \rho \in O(n,\mathbb{R}), \tag{5.2}$$

$$f(t) \leq \frac{1/2 + \|\lambda - t/2\|}{1/2 + \|\lambda\|} \qquad \text{for all } t \in B(0,1) \text{ and } \lambda \in \mathbb{R}^n. \tag{5.3}$$

It is remarkable that the topology of $(\mathcal{UD}, d)$ does not depend upon f once (5.1) and (5.3) are simultaneously satisfied ([MVG2] Proposition 3.5 and §3.3). Therefore adding (5.2) does not change the topology of $(\mathcal{UD}, d)$ but only the invariance properties of the metric d.

For f for instance, let us take $f(t) = 1 - 2\|t\|$ for $t \in B(0,1/2)$ and $f(t) = 0$ elsewhere.

With each element $(D,E) \in \mathcal{E}$ and origin α of $\mathbb{R}^n$ we associate a real-valued function $d_{\alpha,(D,E)}$ on $\mathcal{UD} \times \mathcal{UD}$ in the following way (denoting by $\overset{\circ}{B}(c,v)$ the interior of the closed ball $B(c,v)$ of centre c and radius $v > 0$). Let $\mathcal{B}_{(D,E)} = \{\mathcal{B}_m\}$ denote the countable set of all possible finite collections

$$\mathcal{B}_m = \{\overset{\circ}{B}(c_1^{(m)}, \epsilon_1^{(m)}), \overset{\circ}{B}(c_2^{(m)}, \epsilon_2^{(m)}), \ldots, \overset{\circ}{B}(c_{i_m}^{(m)}, \epsilon_{i_m}^{(m)})\}$$

of open balls such that $c_q^{(m)} \in D$ and $\epsilon_q^{(m)} \in E$ for all $q \in \{1, 2, \ldots, i_m\}$, and such that for all m and any two distinct balls in $\mathcal{B}_m^{(r)}$ of respective centers

$c_q^{(m)}$ and $c_k^{(m)}$, we have

$$\|c_q^{(m)} - c_k^{(m)}\| \geq 1.$$

Then we define the following function, with $\Lambda, \Lambda' \in \mathcal{UD}$,

$$d_{\alpha,(D,E)}(\Lambda, \Lambda') :=$$

$$\sup_{\mathcal{B}_m \in \mathcal{B}_{(D,E)}} \frac{|\phi_{\mathcal{B}_m}(\Lambda) - \phi_{\mathcal{B}_m}(\Lambda')|}{(1/2 + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \|\alpha - c_2^{(m)}\| + \cdots + \|\alpha - c_{i_m}^{(m)}\|)} \tag{5.4}$$

where the function $\phi_{\mathcal{B}_m}$ is given by

$$\phi_{\mathcal{B}_m}(\Lambda) := \sum_{\overset{\circ}{B}(c,\epsilon) \in \mathcal{B}_m} \sum_i \epsilon f\left(\frac{\Lambda_i - c}{\epsilon}\right),$$

putting $\phi_{\mathcal{B}_m}(\emptyset) = 0$ for all $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$ and all $(D,E) \in \mathcal{E}$ by convention.

Lemma 5.1. *For all $(\alpha, (D,E))$ in $\mathbb{R}^n \times \mathcal{E}$, $d_{\alpha,(D,E)}$ is a pseudo-metric on $\mathcal{UD}$. The supremum $d := \sup_{\substack{\alpha \in \mathbb{R}^n \\ (D,E) \in \mathcal{E}}} d_{\alpha,(D,E)}$ is a metric on $\mathcal{UD}$, valued in $[0,1]$.*

Proof. See Muraz and Verger-Gaugry [MVG2]. □

Let us show that d is invariant by the action of the orthogonal group $O(n, \mathbb{R})$.

Lemma 5.2. *For all $(D,E) \in \mathcal{E}, \alpha \in \mathbb{R}^n, \rho \in O(n, \mathbb{R})$ and $\Lambda, \Lambda' \in \mathcal{UD}$, the following equality holds:*

$$d_{\alpha,(D,E)}(\Lambda, \Lambda') = d_{\rho(\alpha),(\rho(D),E)}(\rho(\Lambda), \rho(\Lambda')).$$

Proof. Let $(D,E) \in \mathcal{E}$ and $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$ with

$$\mathcal{B}_m = \{\overset{\circ}{B}(c_1^{(m)}, \epsilon_1^{(m)}), \overset{\circ}{B}(c_2^{(m)}, \epsilon_2^{(m)}), \ldots, \overset{\circ}{B}(c_{i_m}^{(m)}, \epsilon_{i_m}^{(m)})\}.$$

The following inequalities hold:

$$\|c_q^{(m)} - c_k^{(m)}\| \geq 1 \quad \text{ for all } 1 \leq q, k \leq i_m \text{ with} q \neq k.$$

Let $\rho \in O(n, \mathbb{R})$. The collection $\mathcal{B}_m$ is in one-to-one correspondence with the collection of open balls

$$\mathcal{B}_m^{(\rho)} := \{\overset{\circ}{B}(\rho(c_1^{(m)}), \epsilon_1^{(m)}), \overset{\circ}{B}(\rho(c_2^{(m)}), \epsilon_2^{(m)}), \ldots, \overset{\circ}{B}(\rho(c_{i_m}^{(m)}), \epsilon_{i_m}^{(m)})\} \in \mathcal{B}_{(\rho(D),E)},$$

where the following inequalities

$$\|\rho(c_q^{(m)}) - \rho(c_k^{(m)})\| \geq 1$$

are still true for all $1 \leq q, k \leq i_m$ with $q \neq k$. By (5.2) the following equalities hold:

$$\phi_{\mathcal{B}_m}(\Lambda) = \phi_{\mathcal{B}_m^{(\rho)}}(\rho(\Lambda)).$$

Hence, for a given $\alpha \in \mathbb{R}^n$, by taking the supremum over all the collections $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$ of the following identity:

$$\frac{|\phi_{\mathcal{B}_m}(\Lambda) - \phi_{\mathcal{B}_m}(\Lambda')|}{\frac{1}{2} + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \ldots + \|\alpha - c_{i_m}^{(m)}\|} = \frac{\left|\phi_{\mathcal{B}_m^{(\rho)}}(\rho(\Lambda)) - \phi_{\mathcal{B}_m^{(\rho)}}(\rho(\Lambda')\right|}{\frac{1}{2} + \|\rho(\alpha)\| + \|\rho(\alpha) - \rho(c_1^{(m)})\| + \ldots + \|\rho(\alpha) - \rho(c_{i_m}^{(m)})\|}$$

we deduce the claim. □

By taking now the supremum of $d_{\alpha,(D,E)}(\Lambda, \Lambda')$ over all $\alpha \in \mathbb{R}^n$ and $(D, E) \in \mathcal{E}$ we deduce from Lemma 5.2 that

$$d(\Lambda, \Lambda') = d(\rho(\Lambda), \rho(\Lambda'))$$

for all $\Lambda, \Lambda' \in \mathcal{UD}$ and $\rho \in O(n, \mathbb{R})$ as claimed.

6 Proof of Theorem 2.3

The metric d on $\mathcal{UD}$ (Theorem 2.1) has the advantage to make compact the metric space $(\mathcal{UD}, d)$ but, by the way it is constructed, the disadvantage to use a base point (the origin) in the ambient space $\mathbb{R}^n$. We now remove this disadvantage but the counterpart is that the precompactness of the metric space $\mathcal{UD}$ will be lost. In order to do this, let us first define the new collection of metrics (d_x) on $\mathcal{UD}$ indexed by $x \in \mathbb{R}^n$ by

$$d_x(\Lambda, \Lambda') = d(\Lambda - x, \Lambda' - x), \qquad \Lambda, \Lambda' \in \mathcal{UD}.$$

Let us remark that the metric spaces $(\mathcal{UD}, d_x), x \in \mathbb{R}^n$, are all compact (by Theorem 2.1).

Definition 6.1. Let D be the metric on $\mathcal{UD}$, valued in $[0, 1]$, defined by

$$D(\Lambda, \Lambda') := \sup_{x \in \mathbb{R}^n} d_x(\Lambda, \Lambda'), \qquad \text{for } \Lambda, \Lambda' \in \mathcal{UD}.$$

The metric D is called *the metric of the proximity of points*, or *pp-metric*.

Proof of i): By construction, D is invariant by the translations of $\mathbb{R}^n$. Let us prove its invariance by the orthogonal group $O(n, \mathbb{R})$. Let $\Lambda, \Lambda' \in \mathcal{UD}$ and

$x \in \mathbb{R}^n, \rho \in O(n, \mathbb{R})$. Since

$$d(\Lambda, \Lambda') = d(\rho(\Lambda), \rho(\Lambda'))$$

by Lemma 5.2, we deduce

$$d_x(\Lambda, \Lambda') = d(\Lambda - x, \Lambda' - x) = d(\rho(\Lambda) - \rho(x), \rho(\Lambda') - \rho(x)) = d_{\rho(x)}(\rho(\Lambda), \rho(\Lambda')).$$

Hence,

$$\sup_{x \in \mathbb{R}^n} d_x(\Lambda, \Lambda') = \sup_{x \in \mathbb{R}^n} d_{\rho(x)}(\rho(\Lambda), \rho(\Lambda')).$$

This implies

$$D(\Lambda, \Lambda') = D(\rho(\Lambda), \rho(\Lambda')).$$

Proof of ii): any Cauchy sequence for the pp-metric D is in particular a Cauchy sequence for the metric d_x for all $x \in \mathbb{Q}^n$. But $\mathbb{Q}^n$ is countable. Therefore, from any Cauchy sequence for D, a subsequence which converges for all the metrics $d_x, x \in \mathbb{Q}^n$, can be extracted by a diagonalisation process over all $x \in \mathbb{Q}^n$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, that

$$\sup_{x \in \mathbb{Q}^n} d_x(\Lambda, \Lambda') = \sup_{x \in \mathbb{R}^n} d_x(\Lambda, \Lambda') \qquad \text{for all } \Lambda, \Lambda' \in \mathcal{UD}_r$$

this subsequence, extracted by diagonalization, also converges for the metric D. This prove the completeness of the metric space $(\mathcal{UD}, D)$.

Proof of iii): we will use the pointwise pairing property of the metrics d_x recalled in the following Lemma.

Lemma 6.2. *Let $x \in \mathbb{R}^n$. Let $\Lambda, \Lambda' \in \mathcal{UD}$ assumed non-empty and define $l_x := \inf_{\lambda \in \Lambda} \|\lambda - x\| < +\infty$. Let $\epsilon \in (0, \frac{1}{1+2l_x})$ and let us assume that $d_x(\Lambda, \Lambda') < \epsilon$. Then, for all $\lambda \in \Lambda$ such that $\|\lambda - x\| < \frac{1-\epsilon}{2\epsilon}$,*

(i) *there exists a unique $\lambda' \in \Lambda'$ such that $\|\lambda' - \lambda\| < 1/2$,*

(ii) *this pairing satisfies the inequality $\|\lambda' - \lambda\| \leq (1/2 + \|\lambda - x\|)\epsilon$.*

Proof. See Proposition 3.6 in [MVG2]. □

Let $0 < \epsilon < 1$ and suppose that $\Lambda, \Lambda' \in \mathcal{UD}$ are non-empty and satisfy $D(\Lambda, \Lambda') < \epsilon$. This implies

$$d_\lambda(\Lambda, \Lambda') < \epsilon \qquad \text{for all } \lambda \in \Lambda.$$

From Lemma 6.2, restricting x to all the elements λ of Λ, we deduce

$$\forall \lambda \in \Lambda, \exists \lambda' \in \Lambda' \text{ (unique) such that } \|\lambda - \lambda'\| < \epsilon/2.$$

This proves the existence of unique pointwise pairings of points and the pointwise pairing property for D.

Proof of iv): let us show that

$$\mathcal{UD} \times \mathbb{R}^n \quad \to \quad \mathcal{UD}$$

$$(\Lambda, t) \quad \to \quad \Lambda + t$$

is continuous. Let $\Lambda_0 \in \mathcal{UD}$ and $t_0 \in \mathbb{R}^n$. First, by the pointwise pairing property given by iii), we deduce

$$\lim_{t \to 0} D(\Lambda_0 + t, \Lambda_0) = 0.$$

Let $0 < \epsilon < 1$. Then, there exists $\eta > 0$ such that

$$|t - t_0| < \eta \implies D(\Lambda_0 + (t - t_0), \Lambda_0) < \epsilon/2.$$

Hence, for all $\Lambda \in \mathcal{UD}$ such that $D(\Lambda, \Lambda_0) < \epsilon/2$ and $t \in \mathbb{R}^n$ such that $|t - t_0| < \eta$, we have:

$$D(\Lambda + t, \Lambda_0 + t_0) = D(\Lambda + (t - t_0), \Lambda_0)$$

$$\leq D(\Lambda + (t - t_0), \Lambda_0 + (t - t_0)) + D(\Lambda_0 + (t - t_0), \Lambda_0)$$

$$= D(\Lambda, \Lambda_0) + D(\Lambda_0 + (t - t_0), \Lambda_0) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

We deduce the claim.

Proof of (ii) (continuation): let us prove that $(\mathcal{UD}, D)$ is locally compact. The Hausdorff metric Δ is defined on the set $\mathcal{F}(\mathbb{R}^n)$ of the non-empty closed subsets of $\mathbb{R}^n$ as follows:

$$\Delta(\Lambda, \Lambda') := \max\{\inf\{\epsilon \mid \Lambda' \subset \Lambda + B(0, \epsilon)\}, \inf\{\epsilon \mid \Lambda \subset \Lambda' + B(0, \epsilon)\}\}$$

in particular for $\Lambda, \Lambda' \in \mathcal{UD} \setminus \{\emptyset\}$. $\mathcal{UD} \setminus \{\emptyset\}$ is closed in the complete space $(\mathcal{F}(\mathbb{R}^n), \Delta)$. Then $\mathcal{UD} \setminus \{\emptyset\}$ is complete for Δ. On the space $\mathcal{UD} \setminus \{\emptyset\}$, the two metrics D and Δ are equivalent. The element $\emptyset$ (system of spheres with no sphere) is isolated in $\mathcal{UD}$ for D. Hence, it possesses a neighbourhood (reduced to itself) whose closure is compact. Now, if $\Lambda \in \mathcal{UD} \setminus \{\emptyset\}$ and $0 < \epsilon < 1$, the open neighbourhood $\{\Lambda' \in \mathcal{UD} \mid \Lambda' \subset \Lambda + \overset{\circ}{B}(0, \epsilon)\}$ of Λ admits $\{\Lambda' \in \mathcal{UD} \mid \Lambda' \subset \Lambda + B(0, \epsilon)\}$ as closure which is obviously precompact, hence compact, for D or Δ. We deduce the claim.

7 Proofs of Theorem 2.4 and Theorem 2.5

Assume that there does not exist $\Lambda \in \mathcal{UD}$ such that (2.1) holds. Then, by definition, there exists a sequence $(\Lambda_i)_{i\geq 1}$ such that $\Lambda_i \in \mathcal{UD}$ and

$$\lim_{i\to+\infty} \delta(\mathcal{B}(\Lambda_i)) = \delta_n$$

(as a sequence of real numbers).

Lemma 7.1. *There exists a subsequence* $(\Lambda_{i_j})_{j\geq 1}$ *of the sequence* $(\Lambda_i)_{i\geq 1}$ *which converges for* D.

Proof. Indeed, the sequence $(\Lambda_i)_{i\geq 1}$ may be viewed as a sequence in the compact space $(\mathcal{UD}, d_x)$ for any $x \in \mathbb{Q}^n$. Therefore, for all $x \in \mathbb{Q}^n$, we can extract a subsequence from it which converges for the metric d_x. Iterating this extraction by a diagonalization process over all $x \in \mathbb{Q}^n$, since $\mathbb{Q}^n$ is countable, shows that we obtain a subsequence which converges for all the metrics d_x. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we obtain a convergent sequence $(\Lambda_{i_j})_{j\geq 1}$ for D since

$$\sup_{x\in\mathbb{R}^n} d_x = \sup_{x\in\mathbb{Q}^n} d_x.$$

□

Theorem 7.2. *The density function* $\Lambda \to \delta(\mathcal{B}(\Lambda)) = \|\chi(\mathcal{B}(\Lambda))\|_1$ *is continuous on* $(\mathcal{UD}, D)$ *and locally constant.*

Proof. Let $\Lambda_0 \in \mathcal{UD}$, $T > 0$ large enough and $0 < \epsilon < 1$. By Lemma 6.2 and the pointwise pairing property Theorem 2.3 iii), any $\Lambda \in \mathcal{UD}$ such that $D(\Lambda, \Lambda_0) < \epsilon$ is such that the number of elements $\#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\}$ of Λ within $B(0,T)$ satisfies the following inequalities:

$$\#\{\lambda \in \Lambda_0 \mid \lambda \in B(0, T - \epsilon/2)\} \leq \#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\}$$

$$\leq \#\{\lambda \in \Lambda_0 \mid \lambda \in B(0, T + \epsilon/2)\}.$$

The density of the system of balls $\mathcal{B}(\Lambda)$ is equal to

$$\delta(\mathcal{B}(\Lambda)) = \limsup_{T\to+\infty} \#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\} \left(\frac{1}{2T}\right)^n.$$

Since the contribution - to the calculation of the density - of the points of Λ_0 which lie in the annulus $B(0, T + \epsilon/2) \setminus B(0, T - \epsilon/2)$ tends to zero when T tends to infinity by Theorem 1.8 in Rogers [R], we deduce that

$$\delta(\mathcal{B}(\Lambda)) = \delta(\mathcal{B}(\Lambda_0)),$$

hence the claim. □

Let us now finish the proof of Theorem 2.4. Since the metric space $(\mathcal{UD}, D)$ is complete by Theorem 2.3, the subsequence $(\Lambda_{i_j})_{j\geq 1}$ given by Lemma 7.1 is such that there exists a limit point set

$$\Lambda = \lim_{j\to+\infty} \Lambda_{i_j} \in \mathcal{UD}$$

which satisfies, by Theorem 7.2,

$$\delta_n = \lim_{j\to+\infty} \delta(\mathcal{B}(\Lambda_{i_j})) = \delta(\mathcal{B}(\Lambda)).$$

Contradiction.

Let us remark that, in this proof, we did not need assume that the elements Λ_{i_j} are saturated (the same Remark holds for m-saturation).

Let us prove Theorem 2.5. From Theorem 2.4 there exists at least one element of $\mathcal{UD}$, say Λ, of density the packing constant δ_n. Let us assume that there is no completely saturated packing of equal balls of density δ_n and let us show the contradiction. In particular we assume that Λ is not completetly saturated.

Then there would exist an application $i \to m_i$ from $\mathbb{N} \setminus \{0\}$ to $\mathbb{N} \setminus \{0\}$ and a non-stationary sequence $(\Lambda_i)_{i\geq 1}$ such that

(i) $\Lambda_i \in \mathcal{UD}$ with $\Lambda_1 = \Lambda$,

(ii) Λ_{i+1} is obtained from Λ_i by removing m_i balls and placing $m_i + 1$ balls in the holes formed by this removal process,

(iii) $\delta(\mathcal{B}(\Lambda_i)) = \delta_n$ for all $i \geq 1$.

This corresponds to a constant adding of new balls by (ii), but since the density of $\mathcal{B}(\Lambda_1)$ is already maximal, equal to δ_n, this process ocurs at constant density (iii).

As in the proof of Theorem 2.4, we can extract from the sequence $(\Lambda_i)_{i\geq 1}$ a subsequence $(\Lambda_{i_j})_{j\geq 1}$ which is a Cauchy sequence for D. Since $(\mathcal{UD}, D)$ is complete, there exists $\Lambda \in \mathcal{UD}$ such that

$$\Lambda = \lim_{j\to+\infty} \Lambda_{i_j}.$$

The contradiction comes from the pointwise pairing property (iii) in Theorem 2.3 and the continuity of the density function (Theorem 7.2). Indeed, for all j large enough, $D(\Lambda_{i_j}, \Lambda)$ is sufficiently small so that the pointwise pairing property for D prevents the adding of new balls to Λ_{i_j} whatever their number by the process (ii). Therefore, the subsequence $(\Lambda_{i_j})_{j\geq 1}$ would be stationary, which is excluded by assumption. This gives the claim.

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"On Lower Bounds of the Density of Delone Sets and Holes in Sequences of Sphere Packings",

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Exp. Math. 14 :1 (2005), 47–57.

On Lower Bounds of the Density of Delone Sets and Holes in Sequences of Sphere Packings

G. Muraz and J. -L. Verger-Gaugry

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2000 AMS Subject Classification: Primary 52C17, 52C23

Keywords: Delone set, sphere packing, density, hole

We study lower bounds of the packing density of a system of nonoverlapping equal spheres in $\mathbb{R}^n, n \geq 2$, as a function of the maximal circumradius of its Voronoi cells. Our viewpoint, using Delone sets, allows us to investigate the gap between the upper bounds of Rogers or Kabatjanskii-Levenstein and the Minkowski-Hlawka type lower bounds for the density of lattice-packings, without entering the fundamental problem of constructing Delone sets with Delone constants between $2^{-0.401}$ and 1. As a consequence we provide explicit asymptotic lower bounds of the covering radii (holes) of the Barnes-Wall, Craig, and Mordell-Weil lattices, respectively BW_n, $\mathbb{A}_n^{(r)}$, and MW_n, and of the Delone constants of the BCH packings, when n goes to infinity.

1. INTRODUCTION

The maximal packing density of equal spheres in $\mathbb{R}^n$ has received a lot of attention [Rogers 64, Goodman and O'Rourke 97, Cassels 59, Martinet 96, Conway and Sloane 88, Oesterlé 90, Gruber and Lekkerkerker 87, Zong 99]. Similar problems are encountered in coding theory, data transmission, combinatorial geometry, and cryptology [Hoffstein et al. 01]. We will consider the problem through the context of Delone sets. We will give explicit lower bounds of the density of a Delone set as a function of n and its so-called Delone constant R expressing the maximal size of its holes.

Blichfeldt, Rogers, Levenštein, Sidel'nikov, Kabatjanskiĭ, and Levenštein [Goodman and O'Rourke 97, Gruber and Lekkerkerker 87, Conway and Sloane 88] have given upper bounds of the packing density, while lower bounds of the latice-packing density were given by Minkowski, Davenport-Rogers, Ball [Ball 92], etc. (see Section 2). In between the situation is considered fairly vague. The present paper contributes to our knowledge of the range between both types of bounds although the fundamental problem, far from obvious, of constructing Delone sets of

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very small Delone constant, namely less than 1, is not considered here.

For this we will recall the language of uniformly discrete sets and Delone sets instead of that of systems of spheres. A discrete subset Λ of $\mathbb{R}^n$ is said to be uniformly discrete if there exists a constant $r > 0$ such that $x, y \in \Lambda, x \neq y$ implies $\|x - y\| \geq r$. Thus a uniformly discrete set is either the empty set, a subset $\{x\}$ reduced to one element, or, if it contains at least two points, they satisfy such an inequality. If r is equal to the minimal interpoint distance $\inf\{\|x-y\| \mid x, y \in \Lambda, x \neq y\}$, Λ is said to be a uniformly discrete set of constant r of $\mathbb{R}^n$. Uniformly discrete sets of constant 1 will be called $\mathcal{UD}$-sets and the set of $\mathcal{UD}$-sets will be denoted by $\mathcal{UD}$ (without mentioning the dimension n of the ambient space). There is a one-to-one correspondence between the set SS, of systems of equal spheres of radius 1/2, and the set $\mathcal{UD}$: $\Lambda = (a_i)_{i\in\mathbb{N}} \in \mathcal{UD}$ is the set of sphere centres of $\mathcal{B}(\Lambda) = \{a_i + B \mid i \in \mathbb{N}\} \in SS$ where $B(z,t)$ generically denotes the closed ball centred at $z \in \mathbb{R}^n$ of radius $t > 0$, and $B := B(0, 1/2)$. We will take 1/2 in the sequel for the common radius of spheres to be packed and will consider $\mathcal{UD}$-sets instead of systems of equal spheres of radius 1/2.

Let $\Lambda \in \mathcal{UD}$. The density of the system of spheres $\mathcal{B}(\Lambda)$ is defined by

$$\delta(\mathcal{B}(\Lambda)) := \limsup_{R\to+\infty}\left[\frac{\mathrm{vol}\left(\left(\bigcup_{i\in\mathbb{N}}(a_i+B)\right)\bigcap B(0,R)\right)}{\mathrm{vol}(B(0,R))}\right].$$

Let us denote by $\mathcal{L}$ the space of (n-dimensional) lattices of $\mathbb{R}^n$. We will denote:

$$\delta := \sup_{\Lambda\in\mathcal{UD}} \delta(\mathcal{B}(\Lambda)), \quad \delta_L := \sup_{\Lambda\in\mathcal{L}\cap\mathcal{UD}} \delta(\mathcal{B}(\Lambda))$$

and will call them respectively the *packing density* and the *lattice-packing density*.

A $\mathcal{UD}$-set Λ is said to be a Delone set if there exists a constant $R > 0$ such that, for all $z \in \mathbb{R}^n$, there exists an element $\lambda \in \Lambda$ such that $\|z - \lambda\| \leq R$ (property of *relative denseness* of Besicovitch). If Λ is a Delone set, then $R(\Lambda) := \sup_{z\in\mathbb{R}^n} \inf_{\lambda\in\Lambda} \|z - \lambda\|$ is called the *Delone constant* of Λ. Let $R_c = R_c(n) := \inf\{R(\Lambda) | \Lambda \in \mathcal{UD}\}$. This lower bound is an invariant of the ambient space which is only a function of n and the Euclidean metric on $\mathbb{R}^n$. We will call it the *Delone covering constant.*

In Section 2, we will recall the asymptotic expressions of the classical upper bounds of the packing density and the lower bounds of the lattice-packing density, when n goes to infinity.

In Section 3, we will recall known lower bounds of the minimal hole constant, in the case of lattice packings, and state some results concerning lower bounds of R_c in the general case of arbitrary packings.

The Delone constant of a Delone set $\Lambda \in \mathcal{UD}$ is the maximal circumradius of the Voronoi cells in the Voronoi decomposition of space by Λ (Section 3); if Λ is a lattice, it is the covering radius of the lattice, if Λ is a nonperiodic $\mathcal{UD}$-set, it is the "maximal size of the holes in Λ." In Section 4, we will prove Theorem 1.1.

Theorem 1.1. *Let $n \geq 2$. If Λ is a Delone set of $\mathbb{R}^n$ of Delone constant R, then*

$$(2R)^{-n} \leq \delta(\mathcal{B}(\Lambda)) \leq \delta \text{ for all } R_c \leq R. \qquad (1\text{–}1)$$

Let us denote $\mu_n(R) := (2R)^{-n}$. The $(2R)^{-n}$ dependence of the expression of $\mu_n(R)$ with n is very important and constitutes a key result. It allows us to study the minimal asymptotic values of the Delone covering constant $R_c(n)$ when n tends to infinity. Namely, we will prove Theorem 1.2.

Theorem 1.2. *For all $\epsilon > 0$ there exists $n(\epsilon)$ such that for $n > n(\epsilon), R_c(n) \geq 2^{-0.401} - \epsilon$.*

Remark 1.3. Theorem 1.2 asserts the existence of an infinite collection of *middle-sized* Voronoi cells in any densest or saturated packing of equal spheres of $\mathbb{R}^n$ of radius 1/2 of circumradii greater than

$$2^{-0.401} + o(1) = 0.757333... + o(1).$$

The small values of R between the bound

$$\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}$$

and 1 are discussed in Section 3

In Section 5, as an application of Theorem 1.1, we will obtain explicit lower bounds as a function of n of the covering radii (holes) of known lattices, namely Barnes-Wall BW_n, Craig $\mathbb{A}_n^{(r)}$, Mordell-Weil MW_n, and of the Delone constants of BCH packings.

In Section 6, we will show the pertinency of the lower bound $\mu_n(R)$, and "its continuity with R" by comparing it to known classical asymptotic bounds. The construction of Delone sets with very small Delone constants is a difficult problem which is not considered here. Concerning lattice packings, our results give credit to the conjecture stating that (recall that the space $\mathcal{UD}$ depends

$cn\,2^{-n/2}$ (c a const.)	Blichfeldt [Blichfeldt 29]
$\frac{n}{e}\,2^{-n/2}$	Rogers [Rogers 58]
$2^{(-0.5096+o(1))n}$	Sidel'nikov [Sidel'nikov 73]
$2^{(-0.5237+o(1))n}$	Levenštein [Levenštein 79]
$2^{(-0.5990+o(1))n}$	Kabatjanskiĭ and Levenštein [Kabatjanskiĭ and Levenštein 78]

TABLE 1. Upper bounds of δ as a function of n.

upon n): for all $\epsilon > 0$, there exists $n_L(\epsilon)$ such that for $n > n_L(\epsilon)$ and for all (n-dimensional) lattices $L \in \mathcal{UD}$, $R(L) \geq 1 - \epsilon$.

2. ASYMPTOTIC BEHAVIOUR OF THE UPPER BOUNDS OF δ AND OF THE LOWER BOUNDS OF δ_L

The upper bounds of δ, as a function of n, are recalled in Table 1, the best one being the one of Kabatjanskiĭ and Levenštein ([Rogers 64], [Gruber and Lekkerkerker 87, Section 19 and Section 38, pages 390–391], [Conway and Sloane 88, Chapters 1 and 9], [Zong 99, Chapter 3]).

Their asymptotic expressions, when n goes to infinity, all exhibit a dominant exponential term of the type $2^{-\alpha n}$ where α is close to $1/2$. As for lower bounds, non-trivial lower bounds of the packing constant δ do not seem to exist yet (see Section 6); [Elkies 00a]). The basic result is concerned with lattice packings: the Conjecture of Minkowski (1905) proved by Hlawka [Cassels 59, Gruber and Lekkerkerker 87] states

$$\frac{\zeta(n)}{2^{n-1}} \leq \delta_L \qquad (2\text{–}1)$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ denotes the Riemann ζ-function. Proofs of this lower bound do not provide explicit constructions of very dense lattices. This lower bound was improved by Davenport and Rogers [Davenport and Rogers 47] who gave: $(\ln\sqrt{2} + o(1))\,n\,2^{-n}$, for n sufficiently large, and by Ball [Ball 92] who recently obtained better: $2(n-1)\zeta(n)2^{-n}$. For details, see [Goodman and O'Rourke 97], Chapter VI in [Cassels 59], Chapter 9 in [Conway and Sloane 88], [Gruber and Lekkerkerker 87], or [Zong 99]. One can remark that these asymptotic expressions all exhibit a dominant exponential term in $2^{-\alpha' n}$ with $\alpha' = 1$, and that there exists a close similarity between the asymptotic expressions of the lower and upper bounds and Theorem 1.1. Theorem 1.1 will allow to "go continuously" in some sense from the first type ("$\alpha \simeq 1/2$" case) to the second type ("$\alpha' = 1$" case) of bounds; see Section 6

3. LOWER BOUNDS OF THE MINIMAL HOLE CONSTANT $R_L(n)$ AND OF THE DELONE COVERING CONSTANT $R_C(n)$

Bounds for the (lattice-)packing density are obviously linked to holes. Let us recall some definitions. If a lattice $\Lambda \in \mathcal{UD}$ of $\mathbb{R}^n$ is a Delone set of Delone constant R, then classically the quantity R is called the *covering radius* of Λ. Given a $\mathcal{UD}$-set $\Lambda := \{\lambda_i\}$, to each element $\lambda_i \in \Lambda$ is associated its local cell $C(\lambda_i, \Lambda)$, also denoted by $C(\lambda_i, \mathcal{B}(\Lambda))$, defined by the closed subset (not necessarily bounded), called Voronoi cell at λ_i,

$$C(\lambda_i, \Lambda) := \left\{ \begin{array}{l} x \in \mathbb{R}^n \mid \|x - \lambda_i\| \leq \|x - \lambda_j\| \\ \text{for all } j \neq i \end{array} \right\}.$$

As soon as Λ is a Delone set of Delone constant $R > 0 (R < +\infty)$, all the Voronoi cells at its points are bounded closed convex polyhedra. In this case, for all $\lambda_i \in \Lambda$, we have

$$C(\lambda_i, \Lambda) := \left\{ \begin{array}{l} x \in \mathbb{R}^n \mid \|x - \lambda_i\| \leq \|x - \lambda_j\| \\ \text{for all } j \neq i \text{ with } \|\lambda_j - \lambda_i\| < 2R \end{array} \right\}.$$

By definition the circumradius of the Voronoi cell at λ_i is $\rho_i := \max_v \|\lambda_i - v\|$ where the supremum (reached) is taken over all the vertices v of the Voronoi cell $C(\lambda_i, \Lambda)$ at λ_i and the Delone constant R of Λ is equal to $\max_i \rho_i$. The elements $z \in \mathbb{R}^n$ lying at a distance $R(\Lambda)$ of Λ will be called (spherical) deep holes (or deepest holes) of Λ. The other vertices of Voronoi cells will be called holes.

In the particular case of a lattice L the covering radius $R(L)$ is the circumradius of the Voronoi cell of the lattice L at the origin. Any vertex of this Voronoi cell at a distance of L less than $R(L)$ from L is called shallow hole [Conway and Sloane 88]. All the vertices of the Voronoi cell of a lattice at the origin may be simultaneously deepest holes when this Voronoi cell is highly symmetrical [Verger-Gaugry 97].

Let us define *the minimal hole constant* by

$$R_L = R_L(n) := \min_{L \in \mathcal{UD} \cap \mathcal{L}} R(L)$$

over all lattices L of $\mathbb{R}^n$ which are $\mathcal{UD}$-sets. Its determination is an important problem, already mentioned by

$n = 3$	Böröczky [Böröczky 86]	$= \sqrt{5}/(2\sqrt{3}) \simeq 0.645497...$
$n = 4$	Horvath [Horvath 82]	$= (\sqrt{3}-1)3^{1/4}/\sqrt{2} \simeq 0.68125...$
$n = 5$	Horvath [Horvath 82]	$= \sqrt{9+\sqrt{13}}/(2\sqrt{6}) \simeq 0.72473...$
$n \geq 2$	Rogers [Rogers 50]	< 1.5
$n \geq 2$	Henk [Henk 95]	$\leq \sqrt{21}/4 \simeq 1.1456...$
$n >> 1$	Butler [Butler 72]	$\leq n^{(\log_2 \ln n + c)/n} = 1 + o(1)$ (c is a constant)

TABLE 2. Minimal hole constant $R_L(n)$ for lattice-packings of spheres of radius 1/2 in $\mathbb{R}^n$.

Fejes-Toth [Fejes-Toth 79]. It corresponds to the smallest possible holes in lattice packings $L + B$. Our knowledge about it is comparatively limited and the lattices for which the covering radius is equal to the minimal hole constant are unknown as soon as n is large enough. In Table 2 we summarize some values and known upper bounds of $R_L(n)$.

The following theorem is fundamental but nonconstructive.

Theorem 3.1. (Butler.) [Butler 72]

$R_L(n) \leq 1 + o(1)$ *when n is sufficiently large.*

This leads to the following question:

Question 3.2. For all $\epsilon > 0$, does there exist $n_0(\epsilon)$ such that the inequality $R_L(n) \geq 1-\epsilon$ holds for all $n \geq n_0(\epsilon)$?

If the answer to this fundamental question is yes, then Butler's Theorem [Butler 72] would imply that $R_L(n) = 1+o(1)$. Then this result would be a very important step towards a proof of the conjecture stating that the strict inequality "$\delta > \delta_L$" holds for n large enough. The affirmative answer to Question 3.2 is a conjecture [Conway and Sloane 88]. Consequently, the search for lower bounds of $R_L(n)$ is crucial.

The lower bound $\sqrt{2}/2 + o(1)$ for $R_L(n)$ when n is large enough was given by Blichfeldt (see [Butler 72, page 722]). Let us note that the normalized (see Section 5) Leech lattice $\Lambda_{24}/\sqrt{N(\Lambda_{24})}$ [Elkies 00b] has a small value of its covering radius by the theorem of Conway, Parker, and Sloane (in [Conway and Sloane 88, Chapter 23]): $R(\Lambda_{24}/\sqrt{N(\Lambda_{24})}) = \sqrt{2}/2$. In low dimension, this value is rarely reached [Conway and Sloane 88]. In general, for lattices, the information about its holes is limited (see Chapter 22 by Norton in [Conway and Sloane 88]) because of the difficulty of computing explicitely the Voronoi cells of a lattice from the lattice itself when n is large.

Let us now turn to the notion of *saturation*, linked to the possible filling of holes. We will say that a $\mathcal{UD}$-set Λ is *saturated*, or *maximal*, if it is impossible to add a sphere to $\mathcal{B}(\Lambda)$ without destroying the fact that it is a packing of spheres, i.e., without creating an overlap of spheres. The set SS of systems of spheres of radius 1/2, is partially ordered by the relation $\prec$, defined by

$$\Lambda_1, \Lambda_2 \in \mathcal{UD}, \quad \mathcal{B}(\Lambda_1) \prec \mathcal{B}(\Lambda_2) \iff \Lambda_1 \subset \Lambda_2.$$

By Zorn's Lemma, maximal sphere packings exist. The saturation operation of a sphere packing consists of adding spheres to obtain a maximal sphere packing. It is fairly arbitrary and may be finite or infinite. Note that it is not because a sphere packing is maximal (saturated) that its density is equal to δ.

Let $X_R \subset \mathcal{UD}$ be the subset of Delone sets of Delone constant $R > 0$ of $\mathbb{R}^n$. By saturating a Delone set of Delone constant $R > 0$ we will always obtain a Delone set of constant less than 1, but not a Delone set of Delone constant $= R_c$ in general. Let $R^{(s)} := \sup\{R(\Lambda) \mid \Lambda \text{ saturated}\}$. It is obvious that $1/2 \leq R_c < R^{(s)} \leq 1$, $R_c(n) \leq R_L(n)$ and that the subset of saturated Delone sets of $\mathbb{R}^n$ is included in $\bigcup_{R_c \leq R \leq R^{(s)}} X_R$. More precisely we have the following facts.

Lemma 3.3.

(i) $R^{(s)} = 1$;

(ii) $R_c(n) \geq \frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}} = \frac{\sqrt{2}}{2}(1 + O(1/n))$ *for n large.*

Proof:

(i) Let us assume $R^{(s)} < 1$ and that $R^{(s)}$ is the Delone constant of a saturated Delone set Λ. We will obtain a contradiction. Then there exists $z \in \mathbb{R}^n$ such that $\inf_{\lambda \in \Lambda} \|z - \lambda\| = R^{(s)}$. Up to a translation, we may assume $z = 0$. Let $\epsilon > 0$ be small enough such that $(1+\epsilon)R^{(s)} < 1$. Let $\eta \in (3, 4)$ such that the system of spheres

$$\mathcal{B}\left(\Lambda \cap B(0,\eta R^{(s)})\right) := \{B(c_1,1/2),\ldots,B(c_m,1/2)\}$$

(with $m \geq 1$) is such that $\|hc_j\| < \eta R^{(s)}$ for all $j = 1,2,\ldots,m$ and all $h \in [1, 1+\epsilon)$.

Now let $h \in (1, 1+\epsilon)$ and let us create the new Delone set Λ_h from Λ as follows: first, $\Lambda_h \cap B(0,\eta R^{(s)})$ is exactly equal to the set $\{hc_1, hc_2, \ldots, hc_m\}$ so that, "inside" the ball $B(0,\eta R^{(s)})$, $\mathcal{B}\left(\Lambda_h \cap B(0,\eta R^{(s)})\right) = \{B(hc_1,1/2),\ldots,B(hc_m,1/2)\}$. For constructing $\mathcal{B}\left(\Lambda_h \cap (\mathbb{R}^n \setminus B(0,\eta R^{(s)}))\right)$, we take any infinite packing $\mathcal{B}_1$ of balls of radius $1/2$ centred at points which lie in $\mathbb{R}^n \setminus B(0,\eta R^{(s)})$ so that: (1)$\mathcal{B}_1 \cup \mathcal{B}\left(\Lambda_h \cap B(0,\eta R^{(s)})\right)$ is a packing of balls of $\mathbb{R}^n$, and (2) $\mathcal{B}_1$ is saturated. We obtain the Delone set $\Lambda_h \in \mathcal{UD}$ defined by $\mathcal{B}(\Lambda_h) := \mathcal{B}_1 \cup \mathcal{B}\left(\Lambda_h \cap B(0,\eta R^{(s)})\right)$.

We will take ϵ small enough such that all the Voronoi cells at the points hc_j, with $j = 1,2,\ldots,m$ and $h \in [1,1+\epsilon)$, have a circumradius always strictly less than 1 (this is always possible because of the continuity of the maps defining the vertices of Voronoi cells as functions of the centres of balls). Since the restriction of the system of balls $\mathcal{B}(\Lambda_h)$ to the portion of space outside the cluster $\{B(hc_1,1/2),\ldots,B(hc_m,1/2)\} \cup B(0,\eta R^{(s)})$, is saturated, all the Voronoi cells at the centres of the balls of $\mathcal{B}_1$ have a circumradius $\leq R^{(s)} < 1$.

Then, on one hand, since the distance between 0 and Λ_h is $hR^{(s)} > R^{(s)}$, for $h > 1$, the Delone set Λ_h has a Delone constant strictly greater than $R^{(s)}$. Hence it is not saturated, by definition of $R^{(s)}$. On the other hand, since all the Voronoi cells of Λ_h, at the centres of balls located "outside" and "inside" $B(0,\eta R^{(s)})$, have a circumradius strictly less than 1, it is impossible to add a sphere at any of their vertices to saturate Λ_h, and therefore there is no place in $\mathbb{R}^n$ to add a ball of radius $1/2$ to saturate Λ_h. Contradiction.

In the case where the supremum $R^{(s)} = \sup\{R(\Lambda) \mid \Lambda \text{ saturated}\}$ is not reached, let us still assume that $R^{(s)} < 1$ and let us show the contradiction. Then, necessarily [Verger-Gaugry 01], there exist a sequence of points $(z_i)_{i\geq 1}$ and a sequence of Delone sets $(\Lambda_i)_{i\geq 1}$ such that: $\|z_i\|$ tends to $+\infty$ when i goes to infinity with the property that, for all $\epsilon > 0$ there exists $i_0(\epsilon)$ such that $i \geq i_0(\epsilon)$ implies $R^{(s)} - \epsilon \leq \inf_{\lambda\in\Lambda_i} \|z_i - \lambda\| \leq R^{(s)}$. Let R_i be the Delone constant of Λ_i. We now take ϵ small enough in order to have $1/R^{(s)} > 1/(1-\epsilon/R^{(s)})$. It corresponds to values of i large enough. Then, as above, we will consider a new Delone set $\Lambda_{h,i}$ created from Λ_i by a local dilation of scalar factor h about the point z_i. When $1/R^{(s)} > h > 1/(1-\epsilon/R^{(s)})$ then $hR_i \leq hR^{(s)} < 1$ and $hR_i \geq h(R^{(s)} - \epsilon) > (R^{(s)} - \epsilon)/(1-\epsilon/R^{(s)}) = R^{(s)}$. As above, we obtain a Delone set $\Lambda_{h,i}$ which is such that its Delone constant is strictly greater than $R^{(s)}$ and strictly smaller than 1, thus not saturated and impossible to saturate. Contradiction.

(ii) Let us show that, if Λ is a Delone set of $\mathbb{R}^n$ of constant R, $n \geq 1$, then $\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}} \leq R$. This inequality comes from an inequality of Blichfeldt (Lemma 1 in [Rogers 64, page 79]; or [Blichfeldt 29]) since the distance from the centre of a Voronoi cell to any point of its $(n-i)$-dimensional plane, in the Voronoi decomposition of space by Λ, is at least $\frac{1}{2}\sqrt{\frac{2i}{i+1}}$ for all $1 \leq i \leq n$. Taking $i = n$ in the above inequality gives the result. Note that in the constructions of Rogers, packings of equal ball of radius 1, and not $1/2$, are considered; this justifies the factor $1/2$ in front of the expression.

□

We will call $\sqrt{2}/2$ the Blichfeldt bound.

If $n = 1$, $X_{R_c} = X_{1/2}$ is not empty since it contains $\mathbb{Z}$. If $n = 2$, the set $X_{R_c} = X_{\frac{1}{\sqrt{3}}}$ is not empty since it contains the lattice generated by the points with coordinates $(1,0)$ and $(1/2, \sqrt{3}/2)$ in the plane (extreme lattice) in an orthonormal basis [Kerschner 39]. What happens for $n \geq 3$? The set $X_{\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}}$ is certainly empty since, as soon as $n \geq 3$, the minimal Voronoi cell is not tiling the ambient space periodically [Rogers 64]. (McLaughlin's Theorem is cited in [Hales 00, Oesterlé 99, Verger-Gaugry 01, Hales 97a], and for $n = 3$ [Hales 97b].)

Question 3.4. For which values of n and R is X_R not empty?

This fairly old question (see [Ryshkov 75]) is partially answered by Theorem 1.2.

4. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1: Let $R_c \leq R$ and $T > R$ be a real number. If Λ is a Delone set of constant R of $\mathbb{R}^n$, then $(B(0,R) + \Lambda) \cap B(0,T)$ covers the ball $B(0, T-R)$.

Hence, the number of elements of $\Lambda \cap B(0,T)$ is at least $((T-R)/R)^n$. On the other hand, since all the balls of radius $1/2$ centred at the elements of $\Lambda \cap B(0,T)$ lie within $B(0,T+1/2)$, the proportion of space they occupy in $B(0,T+1/2)$ is at least

$$\left(\frac{T-R}{R}\right)^n \frac{\mathrm{vol}(B(0,1/2))}{\mathrm{vol}(B(0,T+1/2))} = \left(\frac{T-R}{2R(T+1/2)}\right)^n.$$

When T tends to infinity the above quantity tends to $(2R)^{-n}$ which is a lower bound of the density $\delta(\mathcal{B}(\Lambda))$. □

Proof of Theorem 1.2: Let $\sigma_{KL}(n) = 2^{-0.599n}$ be the upper bound of Kabatjanskiĭ-Levenštein of the packing density δ. By Theorem 1.1 we deduce that, with $R_c \leq R \leq 1$,

$$\mu_n(R) \leq \delta \leq 2^{-0.599n}.$$

Raising this equation to the power $1/n$ gives readily $2R \geq 2^{0.599} + o(1)$ that is $R \geq 2^{-0.401} + o(1)$. □

5. ASYMPTOTIC BEHAVIOUR OF HOLES IN SEQUENCES OF LATTICES AND PACKINGS

The expression of the bound $\mu_n(R)$ will be used to compute a lower bound of the Delone constant of a Delone set, or a lower bound of the covering radius of a given lattice $L \in \mathcal{UD} \cap \mathcal{L}$, when its density and its minimal interpoint distance are known.

In the case of a lattice L, the minimal interpoint distance of L is the square root of the norm $N(L)$ of the lattice [Martinet 96]. We will consider the normalized lattice

$$\frac{1}{\sqrt{N(L)}} L$$

instead of the lattice L to apply the preceding considerations with packings of spheres of common radius $1/2$. The situation is similar for a Delone set which will be normalized by its minimal interpoint distance. We will denote by $\mathrm{dens}(L) := \delta(\mathcal{B}(L/\sqrt{N(L)}))$ (Theorem 1.7 in [Rogers 64]) the density of the system of spheres $L + B(0, \sqrt{N(L)}/2)$ if L is a lattice and by $\mathrm{dens}(\Lambda) := \delta(\mathcal{B}(\Lambda/n(\Lambda)))$ (Theorem 1.7 in [Rogers 64]) the density of the system of spheres $\Lambda + B(0, n(\Lambda)/2)$ if Λ is a Delone set of minimal interpoint distance $n(\Lambda)$.

Let us observe that, for all Delone sets Λ and all nonnegative scalar factors λ such that $\Lambda \in \mathcal{UD}$ and $\lambda\Lambda \in \mathcal{UD}$, the equality $R(\lambda\Lambda) = \lambda R(\Lambda)$ holds. Then, from Theorem 1.1, we readily obtain the following inequalities:

(i) $\dfrac{n(\Lambda)}{2}\,\mathrm{dens}(\Lambda)^{-1/n} \leq R(\Lambda)$, for all Delone sets $\Lambda \in \mathcal{UD}$ of minimal interpoint distance $n(\Lambda)$, and

(ii) $\dfrac{\sqrt{N(L)}}{2}\,\mathrm{dens}(L)^{-1/n} \leq R(L)$, for all lattice $L \in \mathcal{UD} \cap \mathcal{L}$ of norm $N(L)$.

In the sequel the following notations will be used: $t_L := \sqrt{N(L)}\,\tilde{t_L}$ with $\tilde{t_L} := \frac{1}{2}\,\mathrm{dens}(L)^{-1/n}$; and $t_\Lambda := n(\Lambda)\,\tilde{t_\Lambda}$ with $\tilde{t_\Lambda} := \frac{1}{2}\,\mathrm{dens}(\Lambda)^{-1/n}$ for L and Λ as above.

Let us now apply these inequalities to some known sequences of lattices and packings, as given by [Conway and Sloane 88, Chapters 5 and 8] and [Martinet 96, Chapter V], to obtain an estimation of the size of the deep holes.

5.1 Leech Lattice

For the Leech lattice Λ_{24} in $\mathbb{R}^{24}$ the density $\delta(\Lambda_{24}) = \pi^{12}/479001600 = 0.001930...$ and the covering radius $R(\Lambda_{24}/\sqrt{N(\Lambda_{24})}) = \sqrt{2}/2$ are both known [Conway and Sloane 88, Elkies 00a]. We obtain $t_{\Lambda_{24}} = 0.6487....$ This numerical value is within 10% of the true value 0.707.... This estimation of the size of the deep hole in Λ_{24} is fairly realistic.

5.2 Barnes-Wall Lattices

The density of the Barnes-Wall lattice BW_n ([Leech 64], [Conway and Sloane 88, page 234 or page 151]), in $\mathbb{R}^n$, $n = 2^m$, $m \geq 2$, is equal to $2^{-5n/4}n^{n/4}\pi^{n/2}/\Gamma(1+n/2)$. The norm $N(BW_n)$ is equal to n ([Leech 64, page 678]).

Proposition 5.1. *Let $n = 2^m$ with $m \geq 2$. The covering radius $R(BW_n) \geq t_{BW_n}$ of the Barnes-Wall lattice BW_n is such that the size of its (deepest) hole tends to infinity as (and better than)*

$$t_{BW_n} := \frac{2^{-1/4}}{\sqrt{\pi e}} n^{3/4}\,(1 + o(1))$$

when n goes to infinity.

Proof: Raising the equation

$$2^{-5n/4}n^{n/4}\pi^{n/2}/\Gamma(1+n/2) = \delta(\mathcal{B}(BW_n/\sqrt{n})) = \mu_n(t)$$

to the power $1/n$ and allowing n to tend to infinity leads easily to the claimed asymptotic expression of $t_{BW_n/\sqrt{n}}$ as a function of n. The multiplication of $t_{BW_n/\sqrt{n}} = \tilde{t_{BW_n}}$ by the minimal interpoint distance $\sqrt{n}$ gives the claimed lower bound t_{BW_n} of the covering radius $R(BW_n)$ of BW_n. □

5.3 BCH Packings

In this section, the reference will be [Conway and Sloane 88, page 155]. Let $n = 2^m, m \geq 4$. The packings of equal spheres considered below are obtained using extended BCH codes in construction C of length n. They are not lattices. There are two packings (a and b) which use two different codes of the Hamming distances. Let us denote the second one by P_{nb}. Its density $\mathrm{dens}(P_{nb})$ satisfies

$$\log_2 \mathrm{dens}(P_{nb}) \simeq -\frac{1}{2} n \log_2 \log_2 n, \quad \text{as } n \to +\infty$$

and its minimal interpoint distance is [Conway and Sloane 88, page 150] $n(P_{nb}) = \sqrt{\gamma}\, 2^a$ with $\gamma = 2$ and $a = [(m-1)/2]$. We deduce the following proposition.

Proposition 5.2. *Let $n = 2^m$ with $m \geq 4$. The Delone constant $R(P_{nb}) \geq t_{P_{nb}}$ of the BCH packing P_{nb} tends to infinity as (and better than)*

$$t_{P_{nb}} = 2^{-\frac{1}{2}+[(-1+\log_2 n)/2]} \sqrt{\log_2 n}\,(1+o(1))$$
$$\simeq \frac{1}{\sqrt{2}} \log_2 n\,(1+o(1))$$

when n goes to infinity.

The proof can be made with the same arguments as in the proof of Proposition 5.1.

5.4 Craig Lattices

These lattices are known to be among the densest ones (see [Martinet 96, pages 163–171], [Conway and Sloane 88, pages 222–224]). The density $\mathrm{dens}(\mathbb{A}_n^{(r)})$ of the Craig lattice $\mathbb{A}_n^{(r)}, n \geq 1, r \geq 1$, in $\mathbb{R}^n$ is at least

$$\frac{(r/2)^{n/2}}{(n+1)^{r-1/2}} \frac{\pi^{n/2}}{\Gamma(1+n/2)},$$

with equality if the norm of the lattice is $2r$. The norm of Craig lattices is not known in general and lower bounds of $N(\mathbb{A}_n^{(r)})$ were obtained by Craig (see [Martinet 96, Bachoc and Batut 92, Craig 78]). The determination of $N(\mathbb{A}_n^{(r)})$ is equivalent to the so-called Tarry-Escott problem in combinatorics and does not seem to be solved yet. However, for some values of n and r this norm is known.

Theorem 5.3. *Let $n \geq 2$.*

(i) [Craig 78] *If $n+1$ is a prime number p and $r < n/2$, then $N(\mathbb{A}_n^{(r)}) \geq 2r$.*

(ii) [Bachoc and Batut 92] *If $n+1$ is a prime number p with r a strict divisor of $n = p-1$, then $N(\mathbb{A}_n^{(r)}) = 2r$.*

Bachoc and Batut [Bachoc and Batut 92] made an exhaustive investigation of Craig lattices for the prime numbers $p \leq 23$. The equality $N(\mathbb{A}_{p-1}^{(r)}) = 2r$ holds for $r = 1$, $r = 2$, $r = 3$ and also for $r = (p+1)/4$ with $p \equiv 3 \bmod 4$. This last case was proved by Elkies (cited in [Gross 90]), from the general theory of Mordell-Weil lattices developed by Elkies and Shioda concerning the groups of rational points of elliptic curves over function fields [Shioda 92]. The equality $N(\mathbb{A}_{p-1}^{(r)}) = 2r$ was also proved to be true for $p \leq 37$ and $r \in [1, \frac{p+1}{4}]$ [Martinet 96, page 169], but wrong for higher values of p.

Using the assertion (ii) in Theorem 5.3 we obtain the following proposition.

Proposition 5.4. *Let $n \geq 2$ such that $n+1$ is a prime number and r a strict divisor of n. Then, the covering radius $R(\mathbb{A}_n^{(r)}) \geq t_{\mathbb{A}_n^{(r)}}$ of the Craig lattice $\mathbb{A}_n^{(r)}$ is such that the size of its (deepest) hole tends to infinity as (and better than)*

$$t_{\mathbb{A}_n^{(r)}} := \frac{1}{\sqrt{2\pi e}} \sqrt{n}(1+o(1))$$

when n goes to infinity.

Let us remark that $t_{\mathbb{A}_n^{(r)}}$ is independent of r when n is large enough.

As shown by Propositions 5.1 and 5.4 the deep holes of the Barnes-Wall and Craig lattices, BW_n and $\mathbb{A}_n^{(r)}$, have sizes which goes to infinity with n (r fixed). In order to allow comparison between them and with Butler's Theorem (Theorem 3.1), we have to consider the normalized lattices

$$\frac{1}{\sqrt{n}} BW_n \text{ and } \frac{1}{\sqrt{2r}} \mathbb{A}_n^{(r)},$$

assuming that n is such that $n+1$ is a prime number. In the first case, the covering radius tends to infinity with n leaving no hope to obtain very dense packings of spheres from the lattices BW_n when n is large enough. In the second case, since

$$t_{\mathbb{A}_n^{(r)}/\sqrt{2r}} = \frac{1}{2\sqrt{\pi e}} \sqrt{\frac{n}{r}}$$

we see that $t_{\mathbb{A}_n^{(r)}/\sqrt{2r}} > 1$ if $r < \frac{1}{4\pi e} n$. Let us recall, from Theorem 3.1, that the existence of very dense lattices (of minimal interpoint distance one) of covering radius as close as 1 is expected. Therefore we can expect to find very dense Craig lattices satisfying this condition when $r = r(n)$ is a suitable function of n and large enough, namely: $r(n) > \frac{1}{4\pi e} n$ for which the lower bound $t_{\mathbb{A}_n^{(r)}/\sqrt{2r}}$ of $R(\mathbb{A}_n^{(r)}/\sqrt{2r})$ is then less than unity. On the other hand, the density $\text{dens}(\mathbb{A}_n^{(r)})$ reaches its maximum when r is the integer the closest to $\frac{n}{2\ln(n+1)}$ (obtained by cancelling the derivative of $\text{dens}(\mathbb{A}_n^{(r)})$ with respect to r, with n fixed, assuming that the norm of the lattice $\mathbb{A}_n^{(r)}$ is exactly $2r$).

Since $\frac{n}{2\ln(n+1)} \leq \frac{1}{4\pi e}\, n$, as soon as n is large enough (for $n \geq e^{2\pi e} - 1$), a good compromise for the value of r, assuming that the norm of the lattice $\mathbb{A}_n^{(r)}$ is exactly $2r$, would be $r :=$ the smallest integer $> \frac{1}{4\pi e}\, n$.

Question 5.5. Do there exist normalized Craig lattices

$$\mathbb{A}_n^{(r)}/\sqrt{N(\mathbb{A}_n^{(r)})}$$

(for general n and r) which exhibit a Delone constant (covering radius) smaller than 1?

5.5 Mordell-Weil Lattices

We will refer here to the class of Mordell-Weil lattices given by the following theorem of Shioda [Shioda 91, Theorem 1.1].

Theorem 5.6. [Shioda 91] *Let p be a prime number such that $p + 1 \equiv 0 (\text{mod } 6)$ and k any field containing $\mathbb{F}_{p^2}$. The Mordell-Weil lattice $E(K)$ of the elliptic curve E*

$$y^2 = x^3 + 1 + u^{p+1} \qquad (5\text{–}1)$$

defined over the rational function field K, where $K = k(u)$, is a positive-definite even integral lattice with the following invariants:

$$\begin{aligned} \text{rank} &= 2p-2 \\ \det &= p^{\frac{p-5}{3}} \\ N(E(K)) &= \tfrac{p+1}{3} \\ \text{centre density } \Delta &= \frac{\left(\frac{p+1}{12}\right)^{p-1}}{p^{(p-5)/6}} \\ \text{kissing number} &\geq 6p(p-1). \end{aligned}$$

Recall that the centre density Δ is the quotient of the density of the lattice divided by the volume $\pi^{n/2}/\Gamma(1 + n/2)$ of the unit ball of $\mathbb{R}^n$. Such a lattice in $\mathbb{R}^{2p-2}$, denoted by MW_n with $n = 2p - 2$, has a minimal interpoint distance equal to $\sqrt{(p+1)/3}$ and a density equal to $\text{dens}(MW_n) = \Delta\frac{\pi^{p-1}}{\Gamma(p)}$. We deduce that

$$\tilde{t}_{MW_n} \simeq \frac{1}{2}\frac{\sqrt{\pi}}{(\Gamma(p))^{1/(2p-2)}}\frac{\left(\frac{p+1}{12}\right)^{1/2}}{p^{(p-5)/(12(p-1))}}$$

$$\simeq \frac{\sqrt{\pi\, e}}{4\sqrt{3}}\,p^{-1/12} \simeq 2^{-2+1/12}\frac{\sqrt{\pi\, e}}{\sqrt{3}}\,n^{-1/12}$$

This value goes to zero while

$$t_{MW_n} \simeq 2^{1/12}\frac{\sqrt{\pi\, e}}{12\sqrt{2}}\,n^{5/12}$$

goes to infinity when p (or n) tends to infinity. This result indicates that the deep holes of the normalized Mordell-Weil lattice $MW_n/\sqrt{N(MW_n)}$ are in fact very shallow, and probably may be bounded above independently of n. This leads to the following question.

Question 5.7. Do there exist normalized Mordell-Weil lattices $MW_n/\sqrt{N(MW_n)}$ which exhibit a Delone constant (covering radius) smaller than 1?

6. COMMENTS AND CONJECTURE

The lower bound $\mu_n(R)$ of δ is particularly interesting for saturated Delone sets of Delone constant R of $\mathbb{R}^n$, that is for $R \leq R^{(s)}$. Since $R^{(s)} = 1$ by Lemma 3.3, we readily obtain a lower bound for δ which is 2^{-n} [Elkies 00a, Elkies 00b]. More generally, the lower bound $\mu_n(R)$ exhibits a dependence with n which is in

$$(2R)^{-n} = 2^{-n(1+\log_2 R)}.$$

Taking $R = R^{(s)} = 1$, gives a 2^{-n} dependence typical of the Minkowski-Hlawka type lower bounds of δ_L, while taking $R = \sqrt{2}/2$ (the Blichfeldt bound, Lemma 3.3) provides a $2^{-n/2}$ dependence typical of the Rogers bound σ_n. In between, all values of R are formally possible but the range is limited (Theorem 1.2).

Here the viewpoint does not include explicit constructions. Working with packings of spheres arising from Delone sets for which we only control the constant R would seem, a priori, to give more freedom to the constructions. Very dense packings are likely to occur with 'almost-touching' spheres everywhere, that is from Delone sets of Delone constants R, as small as possible, close to $R_c(n)$. The corresponding kissing numbers deduced from all the local clusters of spheres would lie between the Coxeter-Böröczky/ Kabatjanskiĭ-Levenštein upper bounds [Böröczky 78, Kabatjanskiĭ and Levenštein 78] and the lower bound of Wyner [Wyner 65], probably closer to the upper bounds. Local arrangements of spheres in a densest sphere packing can be extremely diversified (see [Hales 00, Hales 97a, Hales 97b] on Hales-Ferguson Theorem, for $n = 3$).

In this sense, Theorem 1.1 gives a partial answer to old expectations when R lies between $R_c(n)$ and 1. Indeed, recall [Gruber and Lekkerkerker 87, page 391]: "the best known upper and lower bounds for δ differ by

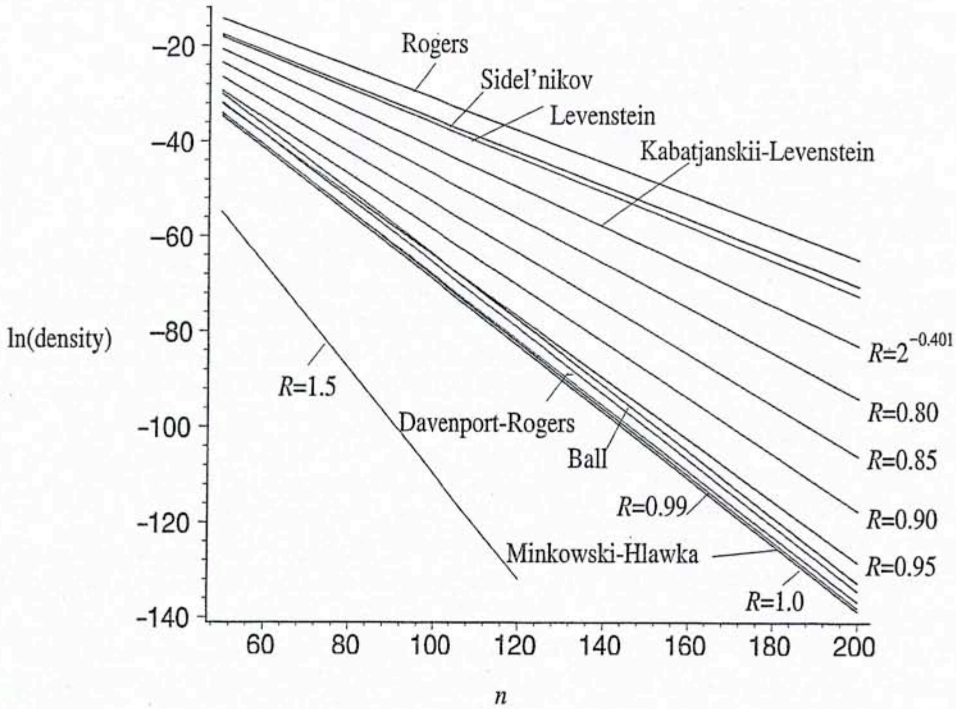

FIGURE 1. Upper bounds of the packing density δ and lower bounds of the lattice-packing density δ_L. The R-dependent lower bounds $\mu_n(R)$ are plotted for $R = 2^{-0.401}, 0.8, 0.85, 0.90, 0.95, 0.99, 1.5$ as a function of the dimension n.

Type	Name	$\log_2 \Delta$
constructions	Barnes-Wall BW_{65536}	180224
	B_{65536}	290998
	$\eta(\overline{\Lambda_{32}})$	295120
	Craig $\mathbb{A}_{65536}^{(2954)}$	297740
(existence) lower bounds	Minkowski-Hlawka	324603
of δ_L	Davenport-Rogers	324616
	Ball	324620
$\mu_{65536}(R)$	$R = 1.5$	286266
	$R = 1.0$	324602
lower bounds	$R = 0.99$	325553
from Theorem 1.1	$R = 0.95$	329452
	$R = 0.90$	334564
	$R = 0.85$	339968
	$R = 0.80$	345700
	$R = 2^{-0.401}$	350882
upper bounds	Kabatjanskiĭ-Levenštein	350882
	Levenštein	355818
of δ	Sidel'nikov	356742
	Rogers	357385

TABLE 3. Table 1.4 of [Conway and Sloane 88, Chapter 1] to which we have added the lower bounds $\mu_{65536}(R)$ for different values of R (the values of the centre density $\log_2 \Delta$ are recomputed from the original references).

a factor which is approximately $2^{n/2}$. This means that the problem of closest packing of spheres is still far from its solution (except for low values of n)." Also recall [Rogers 64, page 9]: we were still, up till now, in the situation where "There remains a wide gap between the results of the Minkowski-Hlawka type, ..., and the results of Blichfeldt type, ..."

In Figure 1 we plot the R-dependent bound $\mu_n(R)$ for several values of R, the upper bounds of Rogers, Sidel'nikov, Levenštein, Kabatjanskiĭ-Levenštein; the lower bounds of Davenport-Rogers, Ball, and of Minkowski-Hlawka, as a function of the dimension n. All values between these two types of bounds can be reached by $\mu_n(R)$ when R is suitably chosen below 1.

The curve $n \to \mu_n(R)$ for $R = 1$ is slightly below the Minkowski-Hlawka bound. When R is greater than 1, the curves $n \to \mu_n(R)$ are entirely below the Minkowski-Hlawka bound. On the contrary, when $R < 1$ is close to unity, the curve $\mu_n(R)$ lies below the Minkowski-Hlawka bound up till a certain value of n and then, as expected, dominates it asymptotically. When $2^{-0.401} < R < 1$ lies far enough from 1 the entire curve $n \to \mu_n(R)$ lies strictly between the two types of bounds (Kabatjanskiĭ-Levenštein and Minkowki-Hlawka).

Theorem 1.2 does not say anything about the frequency and the density of such middle-sized Voronoi cells of circumradius R approximately equal to $2^{-0.401}$ in a general saturated Delone set of $\mathbb{R}^n$ of constant R when n is sufficiently large, in particular in the densest ones.

To allow comparison with known results in literature and to follow Conway and Sloane [Conway and Sloane 88] we have taken n fairly large, namely $n = 65536$. To appreciate the pertinency of the formula given by Theorem 1.1 we have reproduced in Table 3 the Table 1.4 of [Conway and Sloane 88, Chapter 1] and added therein the values of the centre density Δ deduced from $\mu_{65536}(R)$ for $R = 2^{-0.401}, 0.8, 0.85, 0.90, 0.95, 0.99, 1.0, 1.5$. The value of (the logarithm in base 2 of) the centre density Δ computed from $\mu_{65536}(R)$ now sticks to the Kabatjanskiĭ-Levenštein's bound when R is at its asymptotic maximum $R = 2^{-0.401}$. Is this value reached by the Delone constant of a Delone set?

When n is large enough, the sensitivity of $\mu_n(R)$ to the Delone constant R can be perceived by the following comparison (see Table 3): the centre density 324602 relative to the bound $\mu_{65536}(1)$ is slightly below the lower bound 324603 of Minkowski-Hlawka, as expected, whereas the centre density 325553 relative to $\mu_{65536}(0.99)$ is slightly above the best lower bound 324620 of Ball. This gives credit to the conjecture (see Question 3.2 for a precise formulation) that lattices do not exhibit a covering radius less than 1 when n is sufficiently large.

ACKNOWLEDGMENTS

The authors would like to thank Roland Bacher for very helpful comments and suggestions.

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Received December 23, 2003; accepted September 16, 2004.

"Covering a Ball with Smaller Equal Balls in $\mathbb{R}^n$",

J.-L. Verger-Gaugry,

Discrete Comput. Geom. 33 :1 (2005), 143–155.

Discrete Comput Geom 33:143–155 (2005)
DOI: 10.1007/s00454-004-2916-2

Discrete & Computational
Geometry

Covering a Ball with Smaller Equal Balls in $\mathbb{R}^n$

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Abstract. We give an explicit upper bound of the minimal number $\nu_{T,n}$ of balls of radius $\frac{1}{2}$ which form a covering of a ball of radius $T > \frac{1}{2}$ in $\mathbb{R}^n$, $n \geq 2$. The asymptotic estimates of $\nu_{T,n}$ we deduce when n is large are improved further by recent results of Böröczky, Jr. and Wintsche on the asymptotic estimates of the minimal number of equal balls of $\mathbb{R}^n$ covering the sphere $\mathbb{S}^{n-1}$. The optimality of the asymptotic estimates is discussed.

1. Introduction

Let $T > \frac{1}{2}$ and let $\nu_{T,n}$ be the minimal number of (closed) balls of radius $\frac{1}{2}$ which can cover a (closed) ball of radius T in $\mathbb{R}^n$, $n \geq 2$. In [R2, pp. 163–164 and Theorem 2] Rogers has obtained the following result:

Theorem 1.1.

(i) *If* $n \geq 3$, *with* $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$, *we have*

$$1 < \nu_{T,n} \leq \begin{cases} e\vartheta_n (2T)^n & \text{if } \ T \geq \dfrac{n}{2}, \\ n\vartheta_n (2T)^n & \text{if } \ \dfrac{n}{2\ln n} \leq T < \dfrac{n}{2}. \end{cases} \tag{1.1}$$

(ii) *If* $n \geq 9$ *we have*

$$1 < \nu_{T,n} \leq \frac{4e(2T)^n n\sqrt{n}}{\ln n - 2}\left(n \ln n + n \ln(\ln n) + n \ln(2T) + \tfrac{1}{2}\ln(144n)\right) \tag{1.2}$$

for all $\frac{1}{2} < T < n/(2\ln n)$.

Assertion (i) can easily be extended to the case $n = 2$ by invoking [R3, p. 47] so that the strict upper bound $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$ of the covering density of equal balls in $\mathbb{R}^n$ is still a valid one in this case. Thus the inequalities (1.1) are still true for $n = 2$. In the case $n = 2$ (see also [K]), on the other hand, the result (ii) does not seem to have been improved since then, see for instance [GO], [F], [S], [R1] or [BL]. This problem is linked to the existence of explicit lower bounds of the packing constant of equal spheres in $\mathbb{R}^n$ [MVG] and to various problems [MR], [IM], [FF], [M].

In this contribution we give an improvement of the upper bound of $\nu_{T,n}$ given by assertion (ii), i.e. when the radius T is less than $n/(2\ln n)$. Namely, we will prove

Theorem 1.2. *Let $n \geq 2$. The following inequalities hold:*

(i)

$$\begin{aligned} n < \nu_{T,n} &\leq \frac{7^{4(\ln 7)/7}}{4}\sqrt{\frac{\pi}{2}} \\ &\times \frac{n\sqrt{n}[(n-1)\ln(2Tn) + (n-1)\ln(\ln n) + \frac{1}{2}\ln n + \ln(\frac{\pi\sqrt{2n}}{\sqrt{\pi n}-2})]}{T\,(1-2/\ln n)(1-2/\sqrt{\pi\,n})(\ln n)^2} \\ &\times (2T)^n \\ \textit{if}\quad & 1 < T < \frac{n}{2\ln n}, \end{aligned} \tag{1.3}$$

(ii)

$$\begin{aligned} n < \nu_{T,n} &\leq \sqrt{\frac{\pi}{2}} \\ &\times \frac{\sqrt{n}[(n-1)\ln(2Tn) + (n-1)\ln(\ln n) + \frac{1}{2}\ln n + \ln(\frac{\pi\sqrt{2n}}{\sqrt{\pi n}-2})]}{T\,(1-2/\ln n)(1-2/\sqrt{\pi\,n})} \\ &\times (2T)^n \\ \textit{if}\quad & \tfrac{1}{2} < T \leq 1. \end{aligned} \tag{1.4}$$

The following question seems fundamental: What are the integers $\nu_{T,n}$ when $\frac{1}{2} < T,\ 2 \leq n$ and the corresponding configurations of balls of radius $\frac{1}{2}$ when they form the most economical covering of the closed ball $B(0, T)$ of radius T centred at the origin ?

In Section 3 we recall the recent results of Böröczky, Jr. and Wintsche [BW] on the asymptotic estimates in the sphere covering problem by smaller equal balls when n is large. These estimates allow us to make further improvements on the upper bounds of $\nu_{T,n}$ (Theorem 3.1), to appreciate the optimality of these upper bounds with respect to known lower bounds and to state some conjectures.

2. Proof of Theorem 1.2

The idea of the proof is simple: (i) when T is small enough, it amounts to showing that the sphere $S(0, T)$ can be covered by a collection of N balls of radius $\frac{1}{2}$ suitably placed equidistant from the origin, and that this covering to which we add the central ball $B(0, \frac{1}{2})$ actually covers the ball $B(0, T)$ itself; in subsection 2 an upper bound of the minimal value of N is calculated from the results given by the lemmas of subsection 1; (ii) when T is larger, we proceed recursively using (i) to give an upper bound of N. The configuration of balls of radius $\frac{1}{2}$ covering $B(0, T)$ is then ordered by layers, the last layer of balls of radius $\frac{1}{2}$ being at an optimal distance from the origin so as to cover the sphere $S(0, T)$.

1. *Caps and Sectors.* Let $T > \frac{1}{2}$ and $n \geq 2$ in the following. If the closed ball $B(0, T)$ is covered by N smaller balls of radius $\frac{1}{2}$, the smaller balls will intersect the sphere $S(0, T)$, for a certain proportion of them. The intersection of a closed ball of radius $\frac{1}{2}$ and the sphere $S(0, T)$, if it is not empty, is called a (spherical) cap. To fix the notations we define properly what a cap is and the sector it generates in $B(0, T)$.

Let $h \geq 0$ and let u be a unit vector of $\mathbb{R}^n$. We denote by $H_{h,u}$ the affine hyperplane $\{z + hu \mid z \in \mathbb{R}^n, z \cdot u = 0\}$ of $\mathbb{R}^n$. Assume that $H_{h,u}$ intersects the ball $B(0, T)$, i.e. $h \leq T$. We denote

$$C_{T,h,u} := \left\{ z \in S(0, T) \mid \frac{z \cdot u}{\|z\|} \geq \frac{h}{T} \right\}.$$

The $(n-2)$-dimensional sphere $H_{h,u} \cap C_{T,h,u}$ admits $x = \sqrt{T^2 - h^2}$ for the radius. The correspondence between $x \in [0, T]$ and $h \in [0, T]$ is one-to-one. We say that $C_{T,h,u}$ is the cap of chord $2x$ and of centre Tu. If a subset Y of $S(0, T)$ is such that there exist $h \geq 0$ and a unit vector u of $\mathbb{R}^n$ such that $Y = C_{T,h,u}$, then we say that Y is a cap of chord $2x$ of $S(0, T)$.

Every cap $C_{T,h,u}$ of chord $2x$ of $S(0, T)$ generates a sector in $B(0, T)$. We denote it by

$$\mathscr{S}(T, h, u) := \left\{ z \in B(0, T) \mid \frac{z \cdot u}{\|z\|} \geq \frac{h}{T} \right\}.$$

We denote by $V_{(T,x)}$ (indexing with x instead of h) the volume of a sector generated by a cap of chord $2x$ in $S(0, T)$ with $x \leq T$. Let $\omega_n := \pi^{n/2}/\Gamma(1 + n/2)$ so that the (n-dimensional) volume of a ball of radius T in $\mathbb{R}^n$ is $\omega_n T^n$.

Lemma 2.1. *We have*

$$\frac{\omega_{n-1}}{\omega_n} \geq \frac{1}{\sqrt{2\pi}} \sqrt{n} \left(1 - \frac{2}{\sqrt{\pi n}}\right). \tag{2.1}$$

Proof. The following inequalities are classical [V, p. 171]:

$$\frac{\omega_{n-1}}{\omega_n} \geq \begin{cases} \dfrac{1}{\sqrt{2\pi}} \sqrt{n} \left(1 + \left(\dfrac{n}{2e}\right)^{n/2} \dfrac{1}{(n/2)!}\right)^{-1} & \text{if } n \text{ is even,} \\ \dfrac{1}{\sqrt{2\pi}} \sqrt{n+1} \left(1 - \left(\dfrac{n+1}{2e}\right)^{(n+1)/2} \dfrac{1}{((n+1)/2)!}\right) & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

By Stirling's formula we deduce the result. □

Lemma 2.2. *Let $0 < x < T$. Let n be odd and put $\gamma = (n-1)/2$. The volume $V_{(T,x)}$ of a sector in $B(0,T) \subset \mathbb{R}^n$ generated by a cap of chord $2x$ in $S(0,T)$ is equal to*

$$\omega_{n-1}x^{n-1}\left[\frac{\sqrt{T^2-x^2}}{n} + \frac{2(T-\sqrt{T^2-x^2})}{n+1}\sum_{j=0}^{\gamma}\frac{\gamma!\,(\gamma+1)!}{(\gamma+1+j)!\,(\gamma-j)!}\right.$$
$$\left.\times\left(\frac{T-\sqrt{T^2-x^2}}{T+\sqrt{T^2-x^2}}\right)^j\right]. \tag{2.3}$$

It satisfies the relations

(i)

$$V_{(T,x)} = x^n V_{(T/x,1)}, \tag{2.4}$$

(ii)

$$\frac{T}{n\,x} \le \frac{2n(T/x) + (1-n)\sqrt{(T/x)^2-1}}{n(n+1)} \le \frac{1}{\omega_{n-1}}\, V_{(T/x,1)}. \tag{2.5}$$

Proof. Let us show (2.3). The first term $\omega_{n-1}x^{n-1}(\sqrt{T^2-x^2}/n)$ is the volume of the truncated cone $\{z \in \mathscr{S}(T,h,u) \mid z\cdot u \le h\}$ with $h = \sqrt{T^2-x^2}$. The second term in (2.3) is the volume of $\{z \in \mathscr{S}(T,h,u) \mid z\cdot u \ge h\}$: any point of $C_{T,\sqrt{T^2-x^2},u}$ which is at distance t from $H_{\sqrt{T^2-x^2},u}$ is at distance $(x^2 - t^2 - 2t\sqrt{T^2-x^2})^{1/2}$ from the line $\mathbb{R}u$. Hence, this volume equals

$$\int_0^{T-\sqrt{T^2-x^2}} \omega_{n-1}[x^2 - t^2 - 2t\sqrt{T^2-x^2}]^{(n-1)/2}dt.$$

It is obtained by integration by parts,γ times, of the integral

$$\omega_{n-1}\int_0^{\alpha}(\alpha - t)^{\gamma}(t-\beta)^{\gamma}dt \tag{2.6}$$

with $\alpha = T - \sqrt{T^2-x^2}$ and $\beta = -T - \sqrt{T^2-x^2}$.

Relation (2.4) is obvious. Let us show (2.5). We deduce it from the fact that the summation in (2.3) has positive terms and is greater than its first term which is 1. □

Lemma 2.3. *Assume $n \ge 2$ even and $0 < x < 1$. The volume $V_{(T,x)}$ of a sector in $B(0,T) \subset \mathbb{R}^n$ generated by a cap of chord $2x$ in $S(0,T)$ satisfies the relations:*

(i)

$$V_{(T,x)} = x^n V_{(T/x,1)}, \tag{2.7}$$

(ii)

$$\frac{T}{n\,x} \le \frac{2n(T/x) + (2-n)\sqrt{(T/x)^2-1}}{n(n+2)} \le \frac{1}{\omega_{n-1}}\, V_{(T/x,1)}. \tag{2.8}$$

Proof. Equality (2.7) is obvious. In order to prove (2.8), observe that the function $t \to (\alpha - t)(t - \beta)$ defined on the interval $[0, \alpha]$ is valued in the interval $[0, 1]$ since it lies below the horizontal line of y-coordinate $-\alpha\beta = x^2 < 1$. We deduce the following inequalities:

$$(\alpha - t)^{(n+1)/2}(t - \beta)^{(n+1)/2} \leq (\alpha - t)^{n/2}(t - \beta)^{n/2} \leq (\alpha - t)^{(n-1)/2}(t - \beta)^{(n-1)/2}$$

for all $t \in [0, \alpha]$. From (2.6) in the proof of Lemma 2.2 we deduce a lower bound of the volume of the convex hull of $C_{T,\sqrt{T^2-x^2},u}$ for n even using the preceding n odd case of Lemma 2.2: changing n to $n + 1$ now odd in the computation of the lower bound of the summation in (2.3). Note that the computation of the volume of $\{z \in \mathcal{S}(T, h, u) \mid z \cdot u \leq h\}$ with $h = \sqrt{T^2 - x^2}$ still gives $\omega_{n-1}x^{n-1}(\sqrt{T^2 - x^2}/n)$ for n even so that the first term of $V_{(T,x)}$ remains the same as in the n odd case. We deduce inequality (2.8). □

Lemma 2.4. *Let $0 < x \leq \frac{1}{2}$. Let D be a point of the cap $C_{T,\sqrt{T^2-1/4},u} \subset S(0, T) \subset \mathbb{R}^n$ at a distance x from the line $\mathbb{R}u$. Let B denote the unique point which lies in the intersection of $C_{T,\sqrt{T^2-1/4},u} \cap H_{\sqrt{T^2-1/4},u}$ with the plane $(0, D, Tu)$ with the property that it is the closest to D. If η denotes the distance between D and the line OB, we have the following relation between x, T and η:*

$$x = \frac{1}{2}\sqrt{1 - \left(\frac{\eta}{T}\right)^2} - \frac{\eta}{2}\sqrt{4 - \frac{1}{T^2}}, \quad \textit{equivalently}$$

$$\eta = \frac{1}{2}\sqrt{1 - \left(\frac{x}{T}\right)^2} - \frac{x}{2}\sqrt{4 - \frac{1}{T^2}}. \tag{2.9}$$

Proof. Let ψ be the angle between lines OB and OD, and let ψ' be the angle between lines OD and $\mathbb{R}u$, so that $\sin(\psi) = \eta/T$ and $\sin(\psi') = x/T$. Since $\sin(\psi + \psi') = 1/2T$ we obtain

$$1 = 2x\sqrt{1 - (\eta/T)^2} + 2\eta\sqrt{1 - (x/T)^2}.$$

This expression is symmetrical in x and η. It is now easy to deduce, from it, the expression of x as a function of η, as stated by (2.9). □

Lemma 2.5. *Let us assume that a collection of N balls $(B(c_j, \frac{1}{2}))_{j=1,2,\ldots,N}$ of $\mathbb{R}^n$ is such that* (i) *for all $j = 1, 2, \ldots, N$, $B(c_j, \frac{1}{2}) \cap S(0, T)$ is a cap of chord 1 in $S(0, T)$ and* (ii) *these N caps form a covering of $S(0, T)$. Then* (i) *if $T > \sqrt{2}/2$, the union*

$$\bigcup_{j=1}^{N} B(c_j, \tfrac{1}{2}) \quad \textit{covers the annulus} \quad \left\{z \in \mathbb{R}^n \mid T - \frac{1}{2T} \leq \|z\| \leq T\right\}$$

of the ball $B(0, T)$; (ii) *if $\frac{1}{2} < T \leq \sqrt{2}/2$ this union covers $B(0, T)$.*

Proof. Any such ball $B(c_j, \frac{1}{2})$ covers the part of the sector

$$\{z \in \mathcal{S}(T, \sqrt{T^2 - \tfrac{1}{4}}, Oc_j/\|Oc_j\|) \mid \alpha T \leq \|z\|\}$$

with α to be determined. To compute α, we consider two adjacent balls, say $B(c_1, \frac{1}{2})$ and $B(c_2, \frac{1}{2})$, such that the intersection of the respective caps $B(c_1, \frac{1}{2}) \cap S(0, T)$ and $B(c_2, \frac{1}{2}) \cap S(0, T)$ is reduced to one point. Then, on the line $O((c_1 + c_2)/2)$, it is easy to check that all points z such that $T - 1/2T \leq \|z\| \leq T$ are covered. This gives $\alpha = 1 - 1/2T^2$. Now, since the caps $B(c_j, \frac{1}{2}) \cap S(0, T)$ form a covering of $S(0, T)$, the balls $B(c_j, \frac{1}{2})$ form a covering of the annulus $\{z \in \mathbb{R}^n \mid \alpha T \leq \|z\| \leq T\}$. The last assertion is obvious. □

Let us consider $N(\geq 1)$ distinct points $M_1, M_2, \dots, M_N$ of $S(0, T) \subset \mathbb{R}^n$. We consider that they are the respective centres of caps of chord $2x$ of $S(0, T)$. We denote by $\theta_{(T,x)}(M_1, M_2, \dots, M_N)$ the proportion of $S(0, T)$ occupied by these caps. In other terms, with $u_i := OM_i/\|OM_i\|$ for all $i = 1, 2, \dots, N$, we have

$$\theta_{(T,x)}(M_1, M_2, \dots, M_N) := \frac{\mathrm{Vol}_{n-1}(\bigcup_{i=1}^N C_{T,\sqrt{T^2-x^2},u_i})}{\mathrm{Vol}_{n-1}(S(0,T))}.$$

Lemma 2.6. *Let $N \geq 1$ and $x \in (0, \frac{1}{2}]$. The mean $E\theta(N, T, x)$ of $\theta_{(T,x)}(M_1, M_2, \dots, M_N)$ over all possibilites of collections of N distinct points $(M_1, M_2, \dots, M_N)$ of $S(0, T)$ is equal to*

$$E\theta(N, T, x) = 1 - \left(1 - \frac{V_{(T,x)}}{\omega_n T^n}\right)^N.$$

Proof. Let $M_1, M_2, \dots, M_N$ be N points of $S(0, T)$. We define

$$p_i = \frac{\mathrm{Vol}_{n-1}(C_{T,\sqrt{T^2-x^2},u_i})}{\mathrm{Vol}_{n-1}(S(0,T))}, \qquad i = 1, 2, \dots, N,$$

the probability that a point $M \in S(0, T)$ belongs to the cap of chord $2x$ of centre M_i. It is the probability, hence independent of i, that M_i belongs to the cap of chord $2x$ of centre M. We have $p_i = V_{(T,x)}/\omega_n T^n$. Therefore, the probability that M belongs to none of the caps of chord $2x$ of centre M_i for all $i = 1, 2, \dots, N$ is, by the independence of the points, the product of the probabilities that none of the M_i's belongs to the cap of chord $2x$ of centre M, that is the product

$$\left(1 - \frac{V_{(T,x)}}{\omega_n T^n}\right)^N.$$

This value is independent of the collection of points $\{M_i\}$. We deduce the mean $E\theta(N, T, x)$ by complementarity. □

2. *Proof of Theorem* 1.2

Proposition 2.7. *Let $0 < x < \frac{1}{2}$. With $\eta(x) = \frac{1}{2}\sqrt{1 - (x/T)^2} - (x/2)\sqrt{4 - 1/T^2}$, if*

$$N \geq \frac{\omega_n T^n}{V_{(T,x)}} \ln\left(\frac{\omega_n T^n}{V_{(T,\eta(x))}}\right), \tag{2.10}$$

then there exists a collection of N distinct caps of centres $M_1, M_2, \dots, M_N$ of chord 1 *of $S(0, T) \subset \mathbb{R}^n$ satisfying*

$$\ln\left(\frac{1}{1-\theta_{(T,x)}(M_1, M_2, \dots, M_N)}\right) > N\frac{V_{(T,x)}}{\omega_n T^n}, \tag{2.11}$$

which covers $S(0, T)$.

Proof. Given $x \in (0, \frac{1}{2}]$ there exists at least one collection of caps $\{C_{T,\sqrt{T^2-x^2},u_i} \mid i = 1, 2, \dots, N\}$ of centres $M_1, M_2, \dots, M_N$, where the unit vectors $u_i := OM_i/\|OM_i\|$ are all distinct, such that relation (2.11) is true since, after Lemma 2.6, the mean $E\theta(N, T, x)$ is equal to $1 - (1 - V_{(T,x)}/\omega_n T^n)^N$ and that

$$\ln\left(\frac{1}{1-E\theta(N, T, x)}\right) = -N\ln\left(1-\frac{V_{(T,x)}}{\omega_n T^n}\right) > N\frac{V_{(T,x)}}{\omega_n T^n}. \tag{2.12}$$

Note that the points $M_1, M_2, \dots, M_n$ depend upon x. Keeping the centres $M_1, M_2, \dots, M_N$ fixed and putting caps of chord 1 instead of $2x$ around them, we obtain a new collection of caps. Let us show that this new collection of caps of chords 1 of $S(0, T)$ forms a covering. We assume that it does not and will show the contradiction.

Then there exists a point $M \in S(0, T)$ such that

$$M \notin \bigcup_{i=1}^{N} C_{T,\sqrt{T^2-1/4},u_i}.$$

We write $u := OM/\|OM\|$ for the unit vector on the line OM. At worse, M lies close to the boundary of the domain $\bigcup_{i=1}^{N} C_{T,\sqrt{T^2-1/4},u_i}$, hence close to the boundary of one of the caps $C_{T,\sqrt{T^2-1/4},u_i}$ of chord 1. We can now apply Lemma 2.4 as if M were on this boundary: $\eta = \eta(x)$ is strictly positive since $x < \frac{1}{2}$ by (2.9). Therefore the cap $C_{T,\sqrt{T^2-\eta(x)^2},u}$ is not trivial and is disjoint from the union

$$\bigcup_{i=1}^{N} C_{T,\sqrt{T^2-x^2},u_i}.$$

This means that

$$1 - \theta_{(T,x)}(M_1, M_2, \dots, M_N) > \theta_{(T,\eta(x))}(M) > 0.$$

Therefore

$$\ln\left(\frac{1}{1-\theta_{(T,x)}(M_1, M_2, \dots, M_N)}\right) < \ln\left(\frac{1}{\theta_{(T,\eta(x))}(M)}\right).$$

From (2.12) we deduce the relation

$$N\frac{V_{(T,x)}}{\omega_n T^n} < \ln\left(\frac{\omega_n T^n}{V_{(T,\eta(x))}}\right).$$

Hence the contradiction. □

By Lemma 2.1 and (2.4), (2.5), (2.7), and (2.8), we deduce

$$\frac{\omega_n T^n}{V_{(T,x)}} \ln\left(\frac{\omega_n T^n}{V_{(T,\eta(x))}}\right) = \frac{\omega_n}{\omega_{n-1}} \frac{n\,T^n}{x^n} \frac{\omega_{n-1}}{n\,V_{(T/x,1)}} \ln\left(\frac{\omega_n}{\omega_{n-1}} \frac{n\,T^n}{(\eta(x))^n} \frac{\omega_{n-1}}{n\,V_{(T/(\eta(x)),1)}}\right)$$
$$\leq \sqrt{\frac{\pi}{2}} \frac{\sqrt{n}\,(2T)^n\,(1-4\eta(x))^{-n/2}}{T\,(1-2/\sqrt{\pi n})}$$
$$\times \left[-(n-1)\ln(\eta(x)) + (n-1)\ln T + \ln\left(\frac{\sqrt{2}\pi\, n}{\sqrt{\pi n}-2}\right)\right]. \tag{2.13}$$

In Proposition 2.7 we can take any x, hence any η, in the open interval $(0, \frac{1}{2})$ such that condition (2.11) is satisfied. We chose η and $x = x(\eta)$ as functions of n only with η tending monotonically to zero when n goes to infinity, hence x tending to $\frac{1}{2}$. This gives a minimal integer

$$\left\lfloor \frac{\omega_n T^n}{V_{(T,x)}} \ln\left(\frac{\omega_n T^n}{V_{(T,\eta(x))}}\right)\right\rfloor + 1$$

for obtaining the covering property of $S(0, T)$ as a function of n and T only.

We now state the central problem (P).

(P) The problem consists now in finding, in the set of strictly positive monotone decreasing functions $f(x)$ defined on $(\frac{1}{4}, +\infty)$ such that $\lim_{x\to+\infty} f(x) = 0$, one function for which $-(1-4f(x))^{-x/2}\ln(f(x))$ goes the slowest to $+\infty$ when x tends to $+\infty$.

We do not solve this problem here. We simply take $f(x) = 1/(2xu(x))$ with $u(x)$ an increasing monotone continuous function such that $\lim_{x\to+\infty} u(x) = +\infty$, in particular, $u(x) = \ln x$. By reporting this function in (2.13) we take $\eta = 1/(2n\ln n)$, $n \geq 3$. This gives an expression of x as a function of n from (2.9). This function represents a fairly good compromise.

The second member of inequality (2.10) appears as a configurational entropy which has to be exceeded for the existence of a certain configuration (at least one) of equal caps of chord 1 for covering $S(0, T)$. However, condition (2.11) is non-constructive.

We now make explicit the second member of inequality (2.13) with $\eta = 1/(2n\ln n)$. Thus, for all $n \geq 2$, since $(1-2/(n\ln n))^{-n/2} < (1-2/(\ln n))^{-1}$, we obtain

$$\sqrt{\frac{\pi}{2}} \frac{\sqrt{n}}{T} \left(\frac{(2T)^n}{(1-2/\ln n)(1-2/\sqrt{\pi n})}\right)$$
$$\times \left[(n-1)\ln(2Tn\ln n) + \tfrac{1}{2}\ln n + \ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n}-2}\right)\right] \tag{2.14}$$

for $\frac{1}{2} < T \leq 1$.

By Lemma 2.5, if $\frac{1}{2} < T \leq 1$, then, in order to cover the ball $B(0, T)$ by balls of radius $\frac{1}{2}$, it suffices to put a ball of radius $\frac{1}{2}$ centred at the origin (not necessary if $\frac{1}{2} < T \leq \sqrt{2}/2$) and to put a collection of N balls (with N chosen minimal) given by Proposition 2.7 around such that their intersections with $S(0, T)$ are caps of chord 1 which cover $S(0, T)$. This total number of balls, $N + 1$, is certainly exceeded by (2.14). This proves assertion (i) in Theorem 1.2.

Let us prove assertion (ii) in Theorem 1.2. If $T > 1$, we proceed inductively using Lemma 2.5. We cover $B(0, T)$ as follows. We put a ball of radius $\frac{1}{2}$ centred at the origin. Then we put balls of radius $\frac{1}{2}$ in such a way that their intersections with the spheres $S(0, T_m)$ are caps of chord 1 which cover $S(0, T_m)$, where the decreasing sequence $\{T_m\}$ is defined by $T_0 = T, T_1 = T_0 - 1/2T_0, \ldots, T_m = T_{m-1} - 1/2T_{m-1}, \ldots$ with $m \in \{0, 1, \ldots, m_0\}$ and m_0 defined by the condition that $T_{m_0} \leq 1$ and $T_{m_0-1} > 1$. Since, for all integers $m \in \{0, 1, \ldots, m_0\}$, we have

$$T - \frac{m}{2T} \geq T_m$$

the total number of balls of radius $\frac{1}{2}$ disposed in such a configuration required for covering $B(0, T)$ is certainly less than

$$\begin{aligned}
&\sum_{m=0}^{m_0} \left(2\left(T - \frac{m}{2T}\right)\right)^n \frac{\sqrt{\pi n}\,(1 - 2/\ln n)^{-1}}{T\sqrt{2}\,(1 - 2/\sqrt{\pi n})} \\
&\quad \times \left[(n-1)\ln\left(2\left(T - \frac{m}{2T}\right) n \ln n\right) + \frac{\ln n}{2} + \ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n} - 2}\right)\right] \\
&\leq \frac{\sqrt{\pi n}\,(1 - 2/\ln n)^{-1}}{T\sqrt{2}\,(1 - 2/\sqrt{\pi n})}\left[(n-1)\ln(2Tn\ln n) + \frac{\ln n}{2} + \ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n} - 2}\right)\right] \\
&\quad \times \sum_{m=0}^{m_0} \left(2\left(T - \frac{m}{2T}\right)\right)^n.
\end{aligned}$$

However,

$$\sum_{m=0}^{m_0} \left(2\left(T - \frac{m}{2T}\right)\right)^n \leq (2T)^n \sum_{m=0}^{m_0} e^{-n\,m/T^2} \leq (2T)^n \sum_{m=0}^{+\infty} e^{-n\,m/T^2} = \frac{(2T)^n}{1 - e^{-n/T^2}}.$$

Since $T < n/(2\ln n)$, we have

$$\frac{e^{n/T^2}}{e^{n/T^2} - 1} < \frac{e^{4(\ln n)^2/n}}{e^{4(\ln n)^2/n} - 1} < \frac{n}{4(\ln n)^2} e^{4(\ln n)^2/n}.$$

The function $t \to (\ln t)^2/t$ reaches its maximum on $[2, +\infty)$ at $t = e^2$. Hence, for all integers $n \geq 2$, we have $(\ln n)^2/n \leq (\ln 7)^2/7$. We deduce that

$$\sum_{m=0}^{m_0} \left(2\left(T - \frac{m}{2T}\right)\right)^n \leq \frac{e^{4(\ln 7)^2/7}}{4} \frac{n\,(2T)^n}{(\ln n)^2}$$

with a constant $e^{4(\ln 7)^2/7}/4 = 2.176\ldots$. This gives assertion (ii).

As for the strict lower bound n in (2.3) and (2.4), it obviously comes from the dimension of the ambiant space: n balls being placed along the n coordinates axis of any basis of $\mathbb{R}^n$ never cover $B(0, T)$ when $T > \frac{1}{2}$.

3. Asymptotic Estimates: Results and Conjectures

Rogers [R2] constructed certain economic coverings of a larger Euclidean ball by equal smaller balls. When T is large ($T > n/2$) he has computed an upper bound on the quantity $\nu_{T,n}$ that is close to being optimal up to a $\ln n$ factor. On the other hand, his upper bounds are of higher order when $T < n/2$, and it is the object of Theorem 1.2 to improve them in the case $T < n/2ln(n)$.

In this section we reformulate Theorem 1.2 in terms of asymptotic estimates. Then we state further improved upper bounds on the quantity $\nu_{T,n}$ for $T < n/2$ that are most probably close to being optimal up to a $\ln n$ factor. In addition lower bounds on the quantity $\nu_{T,n}$ are discussed. The arguments for further improvements use recent results of [BW].

What follows only discusses the order of the bounds, hence we introduce corresponding notation: given non-negative functions f and g, if $f(n) < c \cdot g(n)$ for a positive absolute constant c, then we write $f(n) \ll g(n)$, or $g(n) \gg f(n)$, or $f(n) = O(g(n))$.

The starting point is the following list of estimates by Rogers [R2]:

$$\nu_{T,n} \ll n \ln n \cdot (2T)^n \qquad \text{if} \quad T \geq \frac{n}{2}; \tag{3.1}$$

$$\nu_{T,n} \ll n^2 \ln n \cdot (2T)^n \qquad \text{if} \quad \frac{n}{2\ln n} \leq T < \frac{n}{2}; \tag{3.2}$$

$$\nu_{T,n} \ll n^2 \sqrt{n} \cdot (2T)^n \qquad \text{if} \quad \frac{1}{2} < T < \frac{n}{2\ln n}. \tag{3.3}$$

Most probably (3.1) cannot be improved with the present methods, and it is actually optimal up to a $\ln n$ factor (see (3.10)). Theorem 1.2 improves (3.3) into the following estimates:

$$\nu_{T,n} \ll \frac{n^2\sqrt{n}}{T \ln n} \cdot (2T)^n \qquad \text{if } 1 < T \leq \frac{n}{2\ln n} \quad \text{where} \quad \frac{n^2\sqrt{n}}{T\ln n} > n\sqrt{n}; \tag{3.4}$$

$$\nu_{T,n} \ll n\sqrt{n} \ln n \cdot (2T)^{n-1} \qquad \text{if} \quad \frac{1}{2} < T \leq 1. \tag{3.5}$$

Using some bounds of [BW, subsection 3.2] estimates (3.2) and (3.3) of Rogers and the present estimates (3.4) and (3.5) can be further improved as follows.

Theorem 3.1. *The following asymptotic estimates hold*:

$$\nu_{T,n} \ll n \ln n \cdot (2T)^n \qquad \textit{if} \quad T \geq \frac{\sqrt{n}}{2}; \tag{3.6}$$

$$\nu_{T,n} \ll \frac{n\sqrt{n}\ln n}{T} \cdot (2T)^n \qquad \textit{if} \quad 1 \leq T \leq \frac{\sqrt{n}}{2}; \tag{3.7}$$

$$\frac{\nu_{T,n}}{(2T)^{n-1}} \ll n\sqrt{n} \cdot \sqrt{T - \tfrac{1}{2}} \cdot \ln 8(T - \tfrac{1}{2})n \qquad \textit{if} \quad \frac{1}{2} + \frac{1}{4n} \leq T \leq 1; \tag{3.8}$$

$$\nu_{T,n} < 2n \qquad \textit{if} \quad \frac{1}{2} < T \leq \frac{1}{2} + \frac{1}{4n}. \tag{3.9}$$

Observe that the estimates in the list change continuously as T increases (up to absolute constant factors).

In order to provide a feeling about the optimality of the estimates above, we list the corresponding known and conjectured lower bounds:

$$\nu_{T,n} \gg n \cdot (2T)^n \qquad \text{if} \quad T \ge \frac{n}{2} \quad \text{or} \quad T = \frac{\sqrt{n}}{2}; \tag{3.10}$$

$$\nu_{T,n} \gg \frac{n\sqrt{n}}{T} \cdot (2T)^n \qquad \text{if} \quad 1 \le T \le \frac{\sqrt{n}}{2}, \quad \textbf{conjectured;} \tag{3.11}$$

$$\frac{\nu_{T,n}}{(2T)^n} \gg n\sqrt{n} \cdot \sqrt{T - \tfrac{1}{2}} \qquad \text{if} \quad \frac{1}{2} + \frac{1}{4n} \le T \le 1, \quad \textbf{conjectured;} \tag{3.12}$$

$$\nu_{T,n} > n \qquad \text{if} \quad \frac{1}{2} < T \le \frac{1}{2} + \frac{1}{4n}. \tag{3.13}$$

It is conjectured [BW] that (3.10) holds for any $T \ge \sqrt{n}/2$ but Böröczkr, Jr. and Wintsche only verified that

$$\nu_{T,n} \gg \frac{n\sqrt{n}}{T} \cdot (2T)^n \qquad \text{if} \quad \frac{\sqrt{n}}{2} \le T \le \frac{n}{2}. \tag{3.14}$$

The quantity $\nu_{T,n}/(2T)^n$ is the minimal density of a covering of a ball of radius T by balls of radius $\frac{1}{2}$. Since $\nu_{T,n}$ balls of radius $\frac{1}{2}$ cover the ball of radius T, readily

$$\nu_{T,n} > (2T)^n.$$

3.1. *Covering a Sphere*

The arguments for (3.6)–(3.14) depend on estimates on the minimal number of equal balls covering a sphere. Let $\tilde{\nu}_n(T,\varrho)$ denote the minimal number of balls of radius ϱ in $\mathbb{R}^n$ that cover the sphere $S(0,T)$ of radius T. The number $\tilde{\nu}_n(T,\varrho)$ corresponds to an optimal function in problem (P) (in Section 2). A better upper estimate of $\tilde{\nu}_n(T,\varrho)$ is given by Corollary 1.2 of [BW]:

$$\tilde{\nu}_n(T,\varrho) \ll n\sqrt{n}\ln n \cdot \left(\frac{T}{\varrho}\right)^{n-1} \qquad \text{if} \quad T \ge 2\varrho; \tag{3.15}$$

$$\frac{\tilde{\nu}_n(T,\frac{1}{2})}{(2T)^{n-1}} \ll n\sqrt{n} \cdot \sqrt{T - \tfrac{1}{2}} \cdot \ln 8(T - \tfrac{1}{2})n \qquad \text{if} \quad \frac{1}{2} + \frac{1}{4n} \le T \le 1. \tag{3.16}$$

Concerning lower bounds, Example 6.3 of [BW] says

$$\tilde{\nu}_n(T,\tfrac{1}{2}) \gg n\sqrt{n} \cdot (2T)^{n-1} \qquad \text{if} \quad T \ge \frac{\sqrt{n}}{2}. \tag{3.17}$$

It is conjectured [BW] that

$$\tilde{\nu}_n(T,\tfrac{1}{2}) \gg n\sqrt{n} \cdot (2T)^{n-1} \qquad \text{if} \quad 1 \le T < \frac{\sqrt{n}}{2}; \tag{3.18}$$

$$\frac{\tilde{\nu}_n(T,\frac{1}{2})}{(2T)^{n-1}} \gg n\sqrt{n} \cdot \sqrt{T - \tfrac{1}{2}} \qquad \text{if} \quad \frac{1}{2} + \frac{1}{4n} \le T \le 1. \tag{3.19}$$

3.2. *Proofs of the Improved Upper Bounds in Theorem* 3.1

If $T \geq \sqrt{n}/2$, then let $\varrho = (1 - 1/n)\frac{1}{2}$. Given any $R \geq \frac{1}{2}$, we cover $S(0, R)$ with $\tilde{\nu}_n(T, \varrho)$ balls of radius ϱ in a way that each ball intersects $S(0, R)$ in an $(n-2)$-sphere of radius ϱ. Since $\sqrt{\frac{1}{4} - \varrho^2} > 1/2\sqrt{n}$, balls of the same centre and of radius $\frac{1}{2}$ cover the annulus between $S(0, R)$ and $S(0, R - 1/2\sqrt{n})$. Writing m to denote the maximal integer such that $T - m/2\sqrt{n} \geq \frac{1}{2}$, it follows by (3.15) that

$$\begin{aligned}
\nu_{T,n} &\leq 1 + \sum_{i=0}^{m} \tilde{\nu}_n\left(T - \frac{i}{2\sqrt{n}}, \varrho\right) \\
&\ll \varrho^{-(n-1)} n\sqrt{n}\ln n \sum_{i=0}^{m}\left(T - \frac{i}{2\sqrt{n}}\right)^{n-1} \\
&\ll 2^{n-1} n\sqrt{n}\ln n \cdot 2\sqrt{n}\int_0^{T+1/2\sqrt{n}} x^{n-1}dx \\
&\ll n\ln n \cdot (2T)^n.
\end{aligned}$$

If $1 \leq T \leq \sqrt{n}/2$, then the argument is based on Lemma 2.5, which actually holds for any $R \geq 1$: we cover $S(0, R)$ with $\tilde{\nu}_n(T, \frac{1}{2})$ balls of radius $\frac{1}{2}$ in a way that each ball intersects $S(0, R)$ in an $(n-2)$-sphere of radius $\frac{1}{2}$. Then the balls cover the annulus between $S(0, R)$ and $S(0, R - 1/2R)$. Writing m to denote the maximal integer such that $T - m/2T \geq \frac{1}{2}$, it follows by (3.15) that

$$\begin{aligned}
\nu_{T,n} &\leq 1 + \sum_{i=0}^{m} \tilde{\nu}_n\left(T - \frac{i}{2T}, \frac{1}{2}\right) \\
&\ll 2^{n-1} n\sqrt{n}\ln n \sum_{i=0}^{m}\left(T - \frac{i}{2T}\right)^{n-1} \\
&< 2^{n-1} n\sqrt{n}\ln n \cdot \frac{T^{n-1}}{1 - e^{-(n-1)/2T^2}} \\
&\ll \frac{n\sqrt{n}\ln n}{T} \cdot (2T)^n,
\end{aligned}$$

using $(n-1)/2T^2 > 1$ in the last step.

If $\frac{1}{2} + 1/4n \leq T \leq 1$, then it is essentially sufficient to cover $S(0, T)$, hence (3.16) yields (3.8).

Finally, if $\frac{1}{2} < T \leq \frac{1}{2} + 1/4n$, then the balls centred at the vertices of the inscribed regular crosspolytope show (3.9) [BW].

3.3. *About the Lower Bounds*

The lower bound (3.10) for $T \geq n/2$ follows essentially directly from the celebrated lower bound of order n on the covering density of a ball, which bound is due to Coxeter et al. [CFR]. In addition, (3.14) is a consequence of (3.17) because the balls covering $B(0, T)$ cover $S(0, T)$ as well. Now (3.10) for $T = \sqrt{n}/2$ is a consequence of (3.14).

If conjectures (3.18) and (3.19) hold, then they yield (3.11) and (3.12).

Acknowledgments

The author thanks Roland Bacher and Gilbert Muraz for very helpful suggestions and the anonymous referee for very useful comments concerning the asymptotic estimates.

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Received December 13, 2002, *and in revised form July* 1, 2003. *Online publication June* 4, 2004.

"On Self-Similar Finitely Generated Uniformly Discrete (SFU-) Sets and Sphere Packings",

J.-L. Verger-Gaugry,
dans "Number Theory and Physics", IRMA Lectures in Mathematics and Theoretical Physics, Eds. L. Nyssen et V. Turaev, E.M.S. Publishing House (2006), accepté.

On Self-Similar Finitely Generated Uniformly Discrete (SFU-) Sets and Sphere Packings

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Abstract. The first part of this paper is a survey on links between Geometry of Numbers and aperiodic crystals in Physics, viewed from the mathematical side. In a second part, we prove the existence of a canonical cut-and-project scheme above a (SFU - set) self-similar finitely generated packing of (equal) spheres Λ in $\mathbb{R}^n$ and investigate its consequences, in particular the role played by the Euclidean and inhomogeneous minima of the algebraic number field generated by the self-similarity on the Delone constant of the sphere packing. We discuss the isolation phenomenon. The degree d of this field divides the $\mathbb{Z}$-rank of $\mathbb{Z}[\Lambda - \Lambda]$. We give a lower bound of the Delone constant of a k-thin SFU - set (sphere packing) which arises from a model set or a Meyer set when d is large enough.

Keywords: Delone set, Meyer set, mathematical quasicrystal, self-similarity, algebraic integer, Euclidean minimum, inhomogeneous minimum, hole, density, sphere packing.

2000 Mathematics Subject Classification: 52C17, 52C23.

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1 Introduction

The mathematics of uniformly discrete point sets and Delone sets developped recently has at least four different origins: (i) the experimental evidence of nonperiodic states of matter in condensed matter physics, so-called aperiodic crystals, like quasicrystals [4] [55] [62] [68] [103] incommensurate modulated crystals phases [67] [69] and their geometric modelization (cf Appendix), (ii) works of Delone (Delone) [36] [37] [42] [97] on geometric crystallography (comparatively, see [58] [83] [90] [101] for a classical mathematical approach of periodic crystals), (iii) works of Meyer on now called cut-and-project sets and Meyer sets [80] [81] [82] [92] (for a modern language of Meyer sets in locally compact Abelian groups: [84]), (iv) the theory of self-similar tilings [10] [75] [109] and the use of ergodic theory to understand diffractivity [5] [98] [109]. In particular, the impact on mathematics of the discovery of quasicrystals in 1984 [103], as long-range ordered phases, was outlined by Lagarias [71]. The term *mathematical quasicrystals* [6] [72] was proposed to name these Delone sets which are used as discrete geometrical models of these new states of matter which have particular spectral or diffraction properties; in particular *crystals* those for which the spectrum is essentially pure point (see [66] [102] and the Appendix for the new definition of what is a *crystal*, and [34] [57] [63] [64] for spectral/diffraction theory). Delone sets are conceived as natural generalizations of lattices in modern crystallography.

In this note we will briefly review these notions (Section 2) and will consider more generally uniformly discrete sets of $\mathbb{R}^n$, in particular (SFU-) self-similar finitely generated uniformly discrete sets (Definition 2.19). A uniformly discrete set of $\mathbb{R}^n$ of constant $r > 0$ is a packing of (equal) spheres of $\mathbb{R}^n$ of (common) radius $r/2$. There are several advantages to consider uniformly discrete sets instead of Delone sets only: their $\mathbb{R}$-spans may take arbitrary dimensions between 0 and n, while that of a Delone set is only n, they can be finite sets which is forbidden for Delone sets, they may exhibit (spherical) holes of arbitrary size at infinity whereas the size of holes in Delone sets is limited by the Delone constant. For instance, see [7] for a nonclassical example. A classification of uniformly discrete sets, hence of Delone sets, which extends that given in [70], is proposed in Subsection 2.3. Finitely generated uniformly discrete sets of $\mathbb{R}^n$ constitute the largest class on which an address map (Subsection 2.3) can be defined.

The theory of SFU - sets generalizes that of lattice packings of (equal) spheres of $\mathbb{R}^n$ [22] [25] [29] [56] [77] [113] since a lattice is already a SFU - set itself (integers are self-similarities: if $m \in \mathbb{Z}$ and L is a lattice, $mL \subset L$), where lattices are or not $\mathcal{O}_F$-lattices for F an algebraic number field with involution [15] [33], and makes use of algebraic integers of certain types (Subsection 2.4). Self-similar Meyer sets only admit self-similarities which are Pisot or Salem numbers [80], while self-similar finitely generated Delone sets only provide Perron or Lind numbers as self-similarities [70]. It is an open problem to find a criterium which ensures that a given uniformly discrete set admits at least one self-similarity. For a general Delone set symmetries and in particular inflation symmetries are expected to be rare, especially when the dimension of the ambient space is large. For lattices, Bannai [11] has shown the existence of many unimodular $\mathbb{Z}$-lattices with trivial (point) automorphism group in a given genus of positive definite unimodular $\mathbb{Z}$-lattices of sufficiently large rank (see also [28]).

The existence of cut-and-project schemes above Delone sets is useful to characterize the set of its self-similarities, inflation centers, local clustering, etc [30] [31] [52] [78]. Given an arbitrary uniformly discrete set it is an open problem whether a cut-and-project scheme lies above it (Subsection 2.1.2). Theorem 1.1 answers in full generality this problem for a given SFU - set $\Lambda \subset \mathbb{R}^n$ (with $\Lambda \in \mathcal{UD}_{fg}$, see Subsection 2.3) with self-similarity λ. The constructions use the Archimedean embeddings of the number field $K := \mathbb{Q}(\lambda)$ generated by the self-similarity λ (Section 3) in a vectorial way, as a product of copies of the étale $\mathbb{R}$-algebra $K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$. Denote by $\Sigma : K \to K_{\mathbb{R}}$

the canonical map. The structure of the lattice in the cut-and-project scheme above Λ arises as a consequence of the Jordan invariants of $\mathbb{R}^{\mathrm{rk}\,\Lambda}$ as a $K[X]$-module (from (ii) in Theorem 1.1).

Theorem 1.1. *Let $\Lambda \subset \mathbb{R}^n, n \geq 1$, be a uniformly discrete set such that $m := rk\ \mathbb{Z}[\Lambda - \Lambda] < +\infty$ with $m \geq 1$. Let $\lambda > 1$ be a (affine) self-similarity of Λ, i.e. a real number > 1 such that $\lambda(\Lambda - c) \subset \Lambda - c$ for a certain $c \in \mathbb{R}^n$. Then*

(i) *λ is a real algebraic integer of degree $d \geq 1$ and d divides m,*

(ii) *there exist $r = m/d$ $\mathbb{Q}$-linearly independent vectors $w_1, w_2, \ldots, w_r$ in the $\mathbb{Q}(\lambda)$-vector space $\mathbb{Q}[\Lambda - \Lambda]$ such that $\mathbb{Z}[\Lambda - \Lambda]$ is a rank m $\mathbb{Z}$-submodule of the $\mathbb{Z}$-module:* $\mathbb{Z}[w_1, \lambda w_1, \ldots, \lambda^{d-1} w_1, w_2, \lambda w_2, \ldots, \lambda^{d-1} w_2, \ldots, w_r, \lambda w_r, \ldots, \lambda^{d-1} w_r]$,

(iii) *for every $\mathbb{Z}$-basis $\{v_1, v_2, \ldots, v_m\}$ of $\mathbb{Z}[\Lambda - \Lambda]$, a matrix relation: $\lambda V = MV$ holds, where $V = {}^t[v_1, \ldots, v_m]$ and M is an invertible integral $m \times m$ matrix with characteristic polynomial $det(X\,I - M) = (\varphi(X))^{m/d}$ in which $\varphi(X)$ is the minimal polynomial of λ; in particular, $det\,M = N_{K/\mathbb{Q}}(\lambda)^{m/d}$, where $N_{K/\mathbb{Q}}(\lambda)$ is the algebraic norm of λ,*

(iv) *there exists a cut-and-project scheme above Λ:*

$$\Big(\coprod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|} \ \simeq\ H \times \mathbb{R}[\Lambda],\ L,\ \pi,\ \mathrm{pr}_1 \Big)$$

where the lattice $L = \coprod_{i=1}^{r} \Sigma\big(\mathbb{Z}[\lambda]\big) \frac{w_i}{\|w_i\|}$ is such that $pr_1(L) \supset \mathbb{Z}[\Lambda - \Lambda]$, whose internal space H is the product of two spaces:

$$H = \big(\ R_K \setminus \mathbb{R}[\Lambda]\ \big) \times \overline{G}$$

where R_K is the image of $\mathbb{R}[\Lambda]$ in $\coprod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|}$ by the real and imaginary embeddings of K, and $\overline{G}$ the closure in $\coprod_{i=1}^{r} K_{\mathbb{R}} \frac{w_i}{\|w_i\|}$ of the image by Σ of the space of relations over K between the generators $w_1, \ldots, w_r$. The space $R_K \setminus \mathbb{R}[\Lambda]$ is called the shadow space of Λ. This cut-and-project scheme is endowed with an Euclidean structure given by a real Trace-like symmetric bilinear form for which R_K and $\overline{G}$ are orthogonal.

The *central cluster* of the basis $(\lambda^j w_i)_{i=1,\ldots,r, j=0,\ldots,d-1}$ is by definition the set $\{w_1, w_2, \ldots, w_r\}$. Note that some vectors in a central cluster may be $\mathbb{R}$-linearly dependent. When $w_1, w_2, \ldots, w_r$ have identical norms and constitute orbits (i.e. F-clusters) under the action of a finite group, say F, constructions in (iv) in Theorem 1.1 can be deduced from [32]. It is easy to check that

$$r = 1 \text{ in Theorem 1.1} \implies \text{the } \mathbb{R}\text{-span of } \Lambda \text{ is one-dimensional .}$$

The converse to this is not always true: in Subsection 2.5 we give the example of the sets $\mathbb{Z}_\beta$ of beta-integers [12] on the line for which open problems exist.

Conversely, what is the set of all self-similarities obtained by Theorem 1.1 (i) ? The answer is simple. If $\beta > 1$ is an algebraic integer there exists at least one SFU - set S admitting β as self-similarity:

$$S = \{\ldots, -\beta^2, -\beta, -1, 0, +1, \beta, \beta^2, \ldots\},$$

on the line, is of finite type and satisfies $\beta S \subset S$ by construction.

Corollary 1.2. *If Λ is a self-similar finitely generated sphere packing in $\mathbb{R}^n$, with self-similarity λ, such that $r = 1$, i.e. for which the degree d of λ equals the rank m of $\mathbb{Z}[\Lambda - \Lambda]$, then $\mathbb{Z}[\Lambda - \Lambda]$ is the projection of a sublattice of finite index of an ideal lattice of $K = \mathbb{Q}(\lambda)$ in the cut-and-project scheme above Λ, of index an integer multiple of $(\mathcal{O}_K : \mathbb{Z}[\lambda])$.*

Theorem 1.1 gives a framework for constructing aperiodic (equal) sphere packings $\mathcal{B}(\Lambda)$ for which local arrangements, for instance like t-designs [9], can be computed from a lattice in higher dimension above Λ. In Corollary 1.2 the terminology "Arakelov divisor", meaning that the embedding of $\mathbb{Z}[\Lambda - \Lambda]$ into the cut-and-project scheme is given with an Euclidean structure, could be substituted to "ideal lattice" by the one-to-one correspondance given in [99] (see also Neukirch [88]).

Dense sphere packings of $\mathbb{R}^n$ are of general interest [20] [25] [29] [54] [56] [86] [113]. For a sphere packing whose set of centers is a Delone set Λ which is a uniformly discrete of constant $r > 0$, of Delone constant $R(\Lambda) := \sup_{z \in \mathbb{R}^n} \inf_{\mu \in \Lambda} \|z - \mu\|$, the density $\delta(\mathcal{B}(\Lambda))$ of $\mathcal{B}(\Lambda)$ satisfies [86]: $\delta(\mathcal{B}(\Lambda)) \geq (\,2R(\Lambda)/r\,)^{-n}$. If Λ is only a uniformly discrete set, no equivalent formula for bounding from below the density $\delta(\mathcal{B}(\Lambda))$ exists in general. However, it is not the case for uniformly discrete sets of $\mathbb{R}^n$, called *pseudo-Delone sets*, which behave in some sense like Delone sets (Subsection 5.1).

Theorem 1.3. *Let Λ be a uniformly discrete set of $\mathbb{R}^n$, of constant $r > 0$, which is pseudo-Delone of pseudo-Delone constant $R(\Lambda)$. Then the density of the sphere packing $\mathcal{B}(\Lambda)$ (of common radius $r/2$) whose set of centers is Λ satisfies:*

$$\delta(\mathcal{B}(\Lambda)) \geq \left(\frac{2R(\Lambda)}{r}\right)^{-n}. \tag{1.1}$$

Theorem 1.3 shows that it is important to obtain interesting lower bounds of the Delone constant (or pseudo-Delone constant) $R(\Lambda)$ to control dense sphere packings, in particular, sphere packings whose set of centers is a SFU - set.

In Section 6 we comment on the two origins of the (pseudo-) Delone constant of a sphere packing whose set of centers is a SFU - set: the first one lies in the geometrical properties of the central cluster $\{w_1, w_2, \ldots, w_r\}$ as given by Theorem 1.1 (ii), the second one is of purely arithmetical nature; it comes from the Euclidean and inhomogeneous minima associated with a sublattice of a product of ideal lattices [14] [26] [27] in bijection with $\mathbb{Z}[\Lambda - \Lambda]$ in the cut-and-project scheme given by Theorem 1.1 (iv). Only the case $r = 1$ is reported in Section 6.

Theorem 1.4. *Let $\Lambda \subset \mathbb{R}^n, n \geq 1$, be a SFU - set which is either a model set or a Meyer set in the cut-and-project scheme defined by Theorem 1.1 (iv) with $r = 1$, Ω as window and lattice L' such that $pr_1(L') = \mathbb{Z}[\Lambda - \Lambda]$.*

Assume that the self-similarity λ is of degree $d \geq 3$, that $K = \mathbb{Q}(\lambda)$ has a unit rank > 1 and is not a CM-field. Then, if Λ is k-thin, $k \geq 2$, its Delone constant $R(\Lambda)$ satisfies:

$$R(\Lambda) \;\geq\; \sqrt{d}\left(M(K)^{2/d} - M_k(K)^{2/d}\right)^{\frac{1}{2}} \;> 0, \tag{1.2}$$

where $M(K)$, resp. $M_k(K)$, is the Euclidean minimum, resp. the k-th Euclidean minimum, of K.

The space-filling condition $m/d = r \geq n$ for couples of values $\{(d, m)\}$ (with the notations of Theorem 1.1) is necessary to construct dense sphere packings of $\mathbb{R}^n$.

The $75^{i\grave{e}mes}$ *Rencontres between Mathematicians and Physicists* held at IRMA - Strasbourg on the Thema “Number Theory and Physics” have offered to the author the opportunity of writing this brief note, initially conceived as a short survey, on the relationships between sphere packings, the mathematics of aperiodic crystals, algebraic number theory and numeration in base an algebraic integer > 1.

2 Uniformly discrete sets and Delone sets

2.1 Definitions and Topology

Let us define *uniformly discrete sets* and *Delone sets* in two different contexts: in the metric case when the ambient space is a metric space which is σ-compact and locally compact, like $\mathbb{R}^n$, and when the ambient space is $\mathbb{R}^n$ with a cut-and-project scheme that lies above it with a locally compact abelian group as internal space.

2.1.1 Metric Case Let (H, δ) be a σ-compact and locally compact metric space with infinite diameter (for δ). A discrete subset Λ of H is said to be *uniformly discrete* if there exists a real number $r > 0$ such that

$$x, y \in \Lambda, x \neq y \text{ implies } \delta(x, y) \geq r.$$

A uniformly discrete set is either the empty set, or a subset $\{x\}$ of H reduced to one element, or, if it contains at least two points, they satisfy such an inequality. If r is equal to the minimal interpoint distance

$$\inf\{\delta(x, y) \mid x, y \in H, x \neq y\}$$

(when $\mathrm{Card}(\Lambda) \geq 2$) Λ is said to be a uniformly discrete set *of constant* r. The space of uniformly discrete sets of constant $r > 0$ of (H, δ) is denoted by $\mathcal{UD}(H, \delta)_r$. It is the space $SS(H, \delta)_r$ of systems of equal spheres (or space of sphere packings) of radius $r/2$ of (H, δ): $\Lambda = (a_i)_{i \in \mathbb{N}} \in \mathcal{UD}(H, \delta)_r$ is the set of sphere centers of

$$\mathcal{B}(\Lambda) = \{B(a_i, r/2) \mid i \in \mathbb{N}\} \in SS(H, \delta)_r$$

where $B(z, t)$ denotes generically the closed ball centered at $z \in H$ of radius $t > 0$.

An element $\Lambda \in \cup_{r>0} \mathcal{UD}(H, \delta)_r$ is said to be a *Delone set* if there exists $R > 0$ such that, for all $z \in H$, there exists an element $\lambda \in \Lambda$ such that $\delta(z, \lambda) \leq R$ (relative denseness property). Then a Delone set is never empty. If Λ is a Delone set, then

$$R(\Lambda) := \sup_{z \in H} \inf_{\lambda \in \Lambda} \delta(z, \lambda) \tag{2.1}$$

is called the *Delone constant* of Λ. In [86] the range of values of the ratio $R(\Lambda)/r$ in the case $H = \mathbb{R}^n, n \geq 1$, is shown to be the continuum

$$\Big[\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}, +\infty\Big). \tag{2.2}$$

In the context of lattices which are $\mathcal{O}_K$-modules, with K a number field, the "ambient space" is obtained in a canonical way via the real and complex embeddings of K and the Delone constant is reminicent of the Euclidean minimum or inhomogeneous minimum, with possible isolated values instead of the continuum (2.2) (Section 6).

The Delone constant of Λ is the maximal circumradius of all its Voronoi cells. If $\Lambda \in \mathcal{UD}(H, \delta)_r$ is a Delone set of Delone constant R, the discrete set Λ is also called a (r, R)-system [97]. Let $X(H, \delta)_{r,R} \subset \mathcal{UD}(H, \delta)_r$ be the subset of uniformly discrete sets of constant r which are Delone sets of H of Delone constant $\leq R$.

Theorem 2.1. *Let (H, δ) be a σ-compact and locally compact metric space for which* $\mathrm{diam}(H)$ *is infinite. Then, for all $r > 0$, $\mathcal{UD}(H, \delta)_r$ can be endowed with a metric d such that the topological space $(\mathcal{UD}(H, \delta)_r, d)$ is compact and such that the Hausdorff metric on $\mathcal{UD}(H, \delta)_{r,f}$ is compatible with the restriction* *of the topology of* $(\mathcal{UD}(H,\delta)_r, d)$ *to* $\mathcal{UD}(H,\delta)_{r,f}$. *For all* $R > 0$ *the subspace* $X(H,\delta)_{r,R}$ *is closed.*

In [87] several (classes of equivalent) metrics on H are constructed. In such constructions a base point, say $\alpha \in \mathbb{R}^n$, is required. When $H = \mathbb{R}^n$, endowed with the Euclidean norm $\|\cdot\|$, the topology on $\mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_r, r > 0$, is expressed by "unique local pairings of points in big balls centered at the base point α", as follows (Proposition 3.6 in [87]). Let $r = 1$, the general case being the same.

Proposition 2.2. *Let* $\Lambda, \Lambda' \in \mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_1$ *with* Λ *and* Λ' *nonempty. Let* $l = \inf\{\|t-\alpha\| \mid t \in \Lambda\} < +\infty$ *and* $\epsilon \in (0, (1+2l)^{-1})$. *Assume* $d(\Lambda, \Lambda') < \epsilon$. *Then, for all* $\lambda \in \Lambda$ *such that* $\|\lambda - \alpha\| < \frac{1-\epsilon}{2\epsilon}$,

(i) *there exists a unique* $\lambda' \in \Lambda'$ *such that* $\|\lambda - \lambda'\| < \frac{1}{2}$,

(ii) *this pairing* (λ, λ') *satisfies the inequality:* $\|\lambda - \lambda'\| \leq (\frac{1}{2} + \|\lambda - \alpha\|)\epsilon$.

2.1.2 Cut-and-Project Schemes Above Uniformly Discrete Sets A locally compact abelian (lca) group is an abelian group G endowed with a topology for which G is a Hausdorff space, each point admits a compact neighbourhood, and such that the mapping $G \times G \to G, (x,y) \to x - y$ is continuous. In the sequel we will denote additively the additive law of G so that 0 is the neutral element of G.

Definition 2.3. Let G be a lca group.

(i) A subset Λ of G is *uniformly discrete* if there exists an open neighbourhood W of 0 so that $(\Lambda - \Lambda) \cap W = \{0\}$,

(ii) a subset Λ of G is *relatively dense* if there exists a compact subset K of G such that $G = \Lambda + K$,

(iii) a Delone set of G is a subset Λ of G which is relatively dense and uniformly discrete.

Definition 2.4. A lattice of $\mathbb{R}^n, n \geq 1$, is a discrete $\mathbb{Z}$-module of rank n. A lattice in a lca group G is a subgroup L of G such that:

(i) L is discrete, i.e. the topology on L induced by that of G is the discrete topology,

(ii) L is cocompact, i.e. G/L is compact.

In the sequel we will only define cut-and-project schemes over uniformly discrete sets Λ which lie in finitely dimensional Euclidean spaces $\mathbb{R}^n$, leaving aside the general case where the ambient space of Λ is a lca group. Such more general constructions can be found in [80], Chap. II, and in [100]. Denote by $\mathcal{L}_n$ the space of (affine) lattices of $\mathbb{R}^n, n \geq 1$.

Definition 2.5. A cut-and-project scheme (over $\mathbb{R}^n$) is given by a 4-tuple $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ where:

(i) $G \times \mathbb{R}^n$ is the direct product of a lca group G and the n-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$,

(ii) L is a lattice in $G \times \mathbb{R}^n$,

such that the natural projections $\pi_1 : G \times \mathbb{R}^n \to G$ and $\pi_2 : G \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy

(1) the restriction $\pi_2|_L$ of π_2 to L is a bijection from L to $\pi_2(L)$,

(2) the image $\pi_1(L)$ is dense in G.

G is called the *internal space*.

Definition 2.6. Let Λ be a uniformly discrete set in the n-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$. A cut-and-project scheme given by the 4-tuple $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ is said to lie *above* Λ if there exists $t \in \mathbb{R}^n$ such that

$$\Lambda - t \ \subset \ \pi_2(L).$$

Remark 2.7. The need to introduce the translation t in the last definition comes from the fact that the uniformly discrete set Λ does not necessarily contain the base point of the cut-and-project scheme, which is the origin of $\mathbb{R}^n$ and at the same time the origin of G. Being a uniformly discrete set, or a Delone set, is an affine notion in the ambient space $\mathbb{R}^n$, while cut-and-project schemes priviledge a base point. For instance the lattice $u + \mathbb{Z}, u = 1/2$, of $\mathbb{R}$ admits $(\{0\} \times \mathbb{R}, \mathbb{Z}, 0, Id)$ as cut-and-project scheme above it; the translation t being $1/2$ in this case. If Λ is a Delone set, the translation t can be chosen such that: $\|t\| \ \leq \ R(\Lambda)$ the Delone constant of Λ.

In the last definition, the image $\pi_2(L)$ of the discrete subgroup $L \subset G \times \mathbb{R}^n$ is a $\mathbb{Z}$-module in $\mathbb{R}^n$ (the classical structure of $\mathbb{Z}$-modules in $\mathbb{R}^n$ is given for instance in [38], Theorem 2.3.7).

Cut-and-project sets, also called model sets, of $\mathbb{R}^n$ form a particular class of Delone sets.

Definition 2.8. A discrete subset Λ of $\mathbb{R}^n, n \geq 1$, is a cut-and-project set, or model set, if there exists a cut-and-project scheme $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ over Λ, with G a lca group, and a relatively compact subset Ω of the internal space G, with nonempty interior, such that:

$$\Lambda - t = \{\pi_2(\omega) \mid \pi_1(\omega) \in \Omega\},$$

for a certain $t \in \mathbb{R}^n$. The set Ω is called the *window* of the cut-and-project set $\Lambda = \Lambda(\Omega)$.

Model sets which arise from cut-and-schemes $(G\times\mathbb{R}^n, L, \pi_1, \pi_2)$ with a lca group G as internal space do not differ too much from model sets that come from cut-and-project sets where the internal space is $\mathbb{R}^m$, for a certain m, by the following proposition.

Proposition 2.9. *Let $\Lambda(\Omega)$ be a cut-and-project set in the cut-and-project scheme $(G\times\mathbb{R}^n, L, \pi_1, \pi_2)$ where G is a lca group. Then there exists a subgroup of G isomorphic to $\mathbb{R}^m$, for a certain $m \geq 0$, and a model set $\Lambda' \subset \mathbb{R}^n$ having $(\mathbb{R}^m \times \mathbb{R}^n, L', \pi_1, \pi_2)$ as cut-and-project scheme above it such that $\Lambda(\Omega)$ is contained in a finite number of translates of Λ'.*

Proof. Proposition 2.7 in [84]. □

Proposition 2.10. *Let $\Lambda = \Lambda(\Omega)$ be a model set in a cut-and-project scheme $(G \times \mathbb{R}^n, L, \pi_1, \pi_2)$ where G is a lca group. Then*

(i) *Λ is Delone set of $\mathbb{R}^n$,*

(ii) *if $\Omega \subset \overline{\mathrm{int}(\Omega)}$ (adherence of its interior) and Ω generates G as a group, the following equality holds:*

$$\mathbb{Z}[\Lambda - \Lambda] \ = \ \pi_2(L).$$

Proof. Proposition 2.6 in [84]. □

Remark 2.11. In [84] and [70], the origin implicitely belongs to the Delone set Λ, whereas in the present note we do not assume this minor fact. That is why we refer to $\mathbb{Z}[\Lambda - \Lambda]$ everywhere instead of $\mathbb{Z}[\Lambda]$, as in Theorem 1.1 or in Proposition 2.10 for instance.

2.2 Continuity of sphere packings arising from model sets

Let Λ be a model set in $\mathbb{R}^n$, viewed as set of centers of a sphere packing, and consider the cut-and-project scheme $(\mathbb{R}^k \times \mathbb{R}^n, L, \pi_1, \pi_2)$ above Λ which allows the construction of Λ by means of a window $\Omega \subset \mathbb{R}^k$. Let us fix the direct product $\mathbb{R}^k \times \mathbb{R}^n$. Let us write $\Lambda = \Lambda_L(\Omega)$ and consider how the model set $\Lambda_L(\Omega)$ varies when Ω and L vary continuously.

Let $\mathcal{W}(\mathbb{R}^k)$ be the uniform space of nonempty open relatively compact subsets of $\mathbb{R}^k$ (set of acceptance windows in $\mathbb{R}^k$) whose affine hull is $\mathbb{R}^k$, endowed with the pseudo-metric

$$\Delta_{\mathcal{W}}(\Omega_1, \Omega_2) := \Delta(\overline{\Omega_1}, \overline{\Omega_2})$$

where Δ is the Hausdorff metric on the space of nonempty closed subsets of $\mathbb{R}^k$. The space of lattices $\mathcal{L}_{n+k}$ in $\mathbb{R}^{n+k}$ is equipped with the quotient topology of $GL(n + k, \mathbb{R})/GL(n + k, \mathbb{Z})$. The metric d built on $\bigcup_{r>0} \mathcal{UD}(\mathbb{R}^{n+k}, \|\cdot\|)_r$

with 0 as base point [87] is compatible with the quotient topology of $GL(n+k,\mathbb{R})/GL(n+k,\mathbb{Z})$. Let $\mathcal{UD} = \bigcup_{r>0} \mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_r$ be the space of uniformly discrete subsets of $\mathbb{R}^n$, endowed with the metric d where here an arbitrary base point $\alpha \in \mathbb{R}^n$ is taken (see [87], Theorem 2.1 and Proposition 2.2). Denote by d_α the metric d in this paragraph only. The two origins, of the cut-and-project scheme and of $\mathbb{R}^n$ for the construction of the metric d_α on $\mathcal{UD}$, are taken a priori different.

Theorem 2.12. *For any base point $\alpha \in \mathbb{R}^n$, the mapping*

$$\mathcal{W}(\mathbb{R}^k) \times \mathcal{L}_{n+k} \to (\mathcal{UD}, d_\alpha) \quad : \quad (\Omega, L) \to \Lambda_L(\Omega) \tag{2.3}$$

is continuous.

Proof. Let $\epsilon > 0$. Let $L_0 \in \mathcal{L}_{n+k}$ and $\Omega_0 \in \mathcal{W}(\mathbb{R}^k)$. Let us show the continuity at (Ω_0, L_0). Let $t = \|\alpha\| + \frac{1-\epsilon/2}{\epsilon}$. Since Ω_0 is open, there exists $\eta_1 > 0$ such that all the sets $\{x \in L \mid \pi_1(x) \in \Omega_0, \|\pi_2(x)\| \le t\}$ have the same cardinality if L belongs to the open set $\{L \mid d(L, L_0) < \eta_1\}$. Since π_2 is continuous and ${\pi_2}_{|_{L_0}}$ is assumed to be a bijection from L_0 onto $\pi_2(L_0)$, then ${\pi_2}_{|_L}$ is also a bijection from L onto $\pi_2(L)$ as soon as $d(L, L_0)$ is small enough. Then, using Proposition 2.2 and invoking the continuity of π_2, there exists $\eta' \le \eta_1$ such that $d(L, L_0) < \eta'$ implies $d_\alpha(\Lambda_L(\Omega_0), \Lambda_{L_0}(\Omega_0)) < \epsilon/2$.

The subset $\{x \in L \mid \pi_1(x) \in \Omega, \|\pi_2(x)\| \le t\}$ of L is such that its projection by π_1 is made of a finite collection of points which lie inside Ω (which is open), and its projection by π_2 is a finite subset of $\pi_2(L)$ which contains $\Lambda_L(\Omega_0) \cap B(\alpha, \frac{1-\epsilon/2}{\epsilon})$ (see Proposition 2.2). Since the projection mappings π_1 and π_2 are continuous and that ${\pi_2}_{|_L}$ is a bijection from L onto $\pi_2(L)$, the mapping $L \to \pi_1 \circ ({\pi_2}_{|_L})^{-1}$ is continuous on the open set $\{L \mid d(L, L_0) < \eta'\}$. Then there exists $\eta'' > 0$ such that $\Delta_{\mathcal{W}}(\Omega, \Omega_0) < \eta''$ implies $d_\alpha(\Lambda_L(\Omega), \Lambda_L(\Omega_0)) < \epsilon/2$ (the value of t is chosen according to this last inequality and Proposition 2.2).

Then, as soon as $\Delta_{\mathcal{W}}(\Omega, \Omega_0) < \eta''$ and $d(L, L_0) < \eta'$ hold, we have:

$$d_\alpha(\Lambda_L(\Omega), \Lambda_{L_0}(\Omega_0)) \le d_\alpha(\Lambda_L(\Omega), \Lambda_L(\Omega_0)) + d_\alpha(\Lambda_L(\Omega_0), \Lambda_{L_0}(\Omega_0)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We deduce the continuity of (2.3). □

Note that the assumption "open" for windows in $\mathcal{W}(\mathbb{R}^k)$ is essential to obtain the continuity in Theorem 2.12. If we consider a collection of model sets parametrized by a sequence of windows which are not necessarily open, but with nonempty interiors, then Theorem 2.12 should be applied with the collections of the interiors of the windows.

For deformations of model sets, see [8] [19].

2.3 A classification of uniformly discrete sets

Classes of Uniformly Discrete Sets and Delone sets in $\mathbb{R}^n$, and their relative inclusions, are given in Theorem 2.16, following Lagarias [70] for Delone sets. We first define some classes of uniformly discrete sets intrinsically, i.e. without any cut-and-project scheme formalism above them. Then we indicate the definitions of point sets which invoke cut-and-project schemes.

Definition 2.13. Let Λ be a nonempty uniformly discrete set of $\mathbb{R}^n$.

(i) Λ is *finitely generated* if the $\mathbb{Z}$-module

$$\mathbb{Z}[\Lambda - \Lambda] := \big\{ \sum_{\text{finite}} \alpha_i (x_i - y_i) \mid \alpha_i \in \mathbb{Z}, x_i, y_i \in \Lambda \big\}$$

is finitely generated, i.e. $\dim_{\mathbb{Q}} \mathbb{Q} \otimes \mathbb{Z}[\Lambda - \Lambda] < +\infty$,

(ii) Λ is *of finite type* if, for all $t > 0$, the intersection

$$(\Lambda - \Lambda) \cap B(0,t)$$

is a finite set.

If Λ is a nonempty finitely generated uniformly discrete set, the *rank of* Λ, denoted by rk Λ, is by definition the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes \mathbb{Z}[\Lambda - \Lambda] = \mathbb{Q}[\Lambda - \Lambda]$. The rank rk Λ is an invariant of Λ. Let $c \in \mathbb{R}^n$. The rank of $\mathbb{Z}[\Lambda - c]$ varies with c and may be different of that of $\mathbb{Z}[\Lambda - \Lambda]$. For instance, with $c = 0$ and $\Lambda = \sqrt{2} + \mathbb{Z}$ in $\mathbb{R}$ we have: rk $\Lambda = 1$ while the rank of $\mathbb{Z}[\Lambda] = \mathbb{Z}[\Lambda - c]$ equals 2 (the notations rk Λ and rk $\mathbb{Z}[\Lambda]$ should not be confused); moreover $\Lambda = \Lambda - c$ and $\mathbb{Z}[\Lambda - \Lambda] = \mathbb{Z}$ are disjoint.

Theorem 2.14 (Lagarias). *Let Λ be a Delone set of finite type of $\mathbb{R}^n, n \geq 1$. Then*

$$\textit{rk}\ \Lambda \leq \textit{Card}\,((\Lambda - \Lambda) \cap B(0, 2R(\Lambda))) \ < \ +\infty \tag{2.4}$$

where $R(\Lambda)$ is the Delone constant of Λ.

Proof. Theorem 2.1 in [70]. □

Definition 2.15. Let Λ be a relatively dense discrete subset of $\mathbb{R}^n$. Λ is a Meyer set if one of the following equivalent assertions is satisfied:

(i) $\Lambda - \Lambda$ is uniformly discrete,

(ii) Λ is a Delone set and there exists a finite set $F \subset \mathbb{R}^n$ such that

$$\Lambda - \Lambda \subset \Lambda + F, \tag{2.5}$$

(iii) Λ is a subset of a model set.

Proof. Theorem 9.1 and Proposition 9.2 in [84]. □

Conditions (i) and (ii) in the definition of Meyer sets are given independently of any "cut-and-project scheme above Λ" consideration while condition (iii) asserts the existence of such a cut-and-project scheme above it. In a similar way a (affine) lattice $L \in \mathcal{L}_n$ in $\mathbb{R}^n$ is intrinsically defined in $\mathbb{R}^n$, without any help of cut-and-project schemes, admits also $(\{0\} \times \mathbb{R}^n, L, 0, \pi_2)$ as cut-and-project scheme above it and is a model set in this cut-and-project scheme. The objectives of Theorem 1.1 consist in showing the existence of general constructions of cut-and-project schemes above SFU - sets in $\mathbb{R}^n$.

Let $n \geq 1$. Denote:

$\mathrm{M}^{(\mathbb{R})}$:= { Model sets in $\mathbb{R}^n$ arising from cut-and-project schemes having a m-dimensional Euclidean space $\mathbb{R}^m$ as internal space }

$\mathrm{M}^{(\mathrm{lcag})}$:= { Model sets in $\mathbb{R}^n$ arising from cut-and-project schemes having a lca group G as internal space }

$\mathcal{M}^{(\mathbb{R})}$:= { Meyer sets in $\mathbb{R}^n$ arising from cut-and-project schemes having a m-dimensional Euclidean space $\mathbb{R}^m$ as internal space }

$\mathcal{M}^{(\mathrm{lcag})}$:= { Meyer sets in $\mathbb{R}^n$ arising from cut-and-project schemes having a lca group G as internal space }

$\mathcal{UD}$:= { Uniformly discrete sets in $\mathbb{R}^n$ }

$\mathcal{UD}_{\mathrm{fg}}$:= { Finitely generated uniformly discrete sets in $\mathbb{R}^n$ } $\subset \cup_{r>0} \mathcal{UD}(\mathbb{R}^n, \|\cdot\|)_r$

$\mathcal{UD}_{\mathrm{ft}}$:= { Uniformly discrete sets of finite type in $\mathbb{R}^n$ }

X_{fg} := { Finitely generated Delone sets in $\mathbb{R}^n$ }

X_{ft} := { Delone sets of finite type in $\mathbb{R}^n$ }

Theorem 2.16. *The following inclusions hold:*

$$\begin{array}{ccccccccccccc} & & & & & & & & & & \mathcal{UD}_{\mathrm{ft}} & \subset & \mathcal{UD}_{\mathrm{fg}} \\ & & & & & & & & & & \cup & & \cup \\ \mathcal{M}^{(\mathrm{lcag})} & \supset & \mathrm{M}^{(\mathrm{lcag})} & \supset & \mathcal{L}_n & \subset & \mathrm{M}^{(\mathbb{R})} & \subset & \mathcal{M}^{(\mathbb{R})} & \subset & X_{\mathrm{ft}} & \subset & X_{\mathrm{fg}} \end{array} \tag{2.6}$$

Proof. Theorem 9.1 in [84], Theorem 2.1 and Theorem 3.1 in [70]. □

Definition 2.17. Let $\Lambda \in \mathcal{UD}_{fg}$. If $\{e_1, e_2, \ldots, e_v\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\Lambda]$, i.e. $\mathbb{Z}[\Lambda] = \mathbb{Z}[e_1, e_2, \ldots, e_v]$, then the address map $\varphi : \mathbb{Z}[\Lambda] \to \mathbb{Z}^v$ of Λ associated to this basis is by definition

$$\varphi\Big(\sum_{i=1}^{v} m_i e_i \Big) = (m_1, m_2, \ldots, m_v).$$

In Section 3 we will mainly use address maps of difference sets $\Lambda - \Lambda$ for the elements Λ of $\mathcal{UD}_{fg}$.

2.4 Algebraic integers, inflation centers and self-similarities

Given Λ a uniformly discrete set of $\mathbb{R}^n$, an (affine) *self-similarity* of Λ is by definition a real number $\lambda > 1$ such that

$$\lambda(\Lambda - c) \quad \subset \quad \Lambda - c \tag{2.7}$$

for a certain point c in $\mathbb{R}^n$ (note that c need not belong to Λ). A point $c \in \mathbb{R}^n$ for which (2.7) occurs for a certain $\lambda > 1$ is called an *inflation center* of Λ. The concept of self-similarity is an affine notion and λ depends upon c. Denote by

$$\mathcal{C}(\Lambda) := \{c \mid \exists \lambda > 1 \text{ such that } \lambda(\Lambda - c) \quad \subset \quad \Lambda - c\} \tag{2.8}$$

the set of inflation centers of Λ and by

$$\mathcal{S}(c) := \{\lambda > 1 \mid \lambda(\Lambda - c) \quad \subset \quad \Lambda - c\}, \qquad \text{for } c \in \mathcal{C}(\Lambda), \tag{2.9}$$

the set of self-similarities associated with the point c.

Proposition 2.18. *Let $\Lambda = t + \mathbb{Z}$ be a (affine) lattice of $\mathbb{R}$ of period 1 with $t \in [0, 1)$. Then*

(i) $\mathcal{C}(\Lambda) = t + \mathbb{Q}$,

(ii) $$\mathcal{S}(c) = \begin{cases} \mathbb{N} \setminus \{0\} & \text{if } c \in t + \mathbb{Z}, \\ (1 - q\mathbb{Z}) \cap \mathbb{N} \setminus \{0\} & \text{if } c \in \mathcal{C}(\Lambda),\ c = t + \frac{p}{q}, \text{ with } p, q \\ & \text{relatively prime } (p \in \mathbb{Z}, q \geq 2). \end{cases}$$

The set of inflation centers of Λ of given (point) density $1/(2q)$ of self-similarities, with q an integer ≥ 2, is exactly the uniformly discrete set

$$t + \{\frac{p}{q} \mid p \in \mathbb{Z}, gcd(p, q) = \pm 1\}.$$

Proof. Routine, with the following definition of the *(point) density of self-similarities* of an inflation center $c \in \mathcal{C}(\Lambda)$:

$$\text{dens}(\mathcal{S}(c)) = \limsup_{t \to \infty} \frac{1}{2t} \# \left(\mathcal{S}(c) \cap (1, t]\right). \tag{2.10}$$

□

In the general case, given a uniformly discrete set, the characterization of $\mathcal{C}(\Lambda)$ and $\mathcal{S}(c)$ with $c \in \mathcal{C}(\Lambda)$ remains an open problem, even for Delone sets; see [30], [31] and [78] for Penrose tilings and sets $\mathbb{Z}_\beta$ of β-integers [52], with

β a quadratic Pisot number. At least, $\mathcal{C}(\Lambda)$ is expected to be far from being everywhere dense as in Proposition 2.18.

Definition 2.19. A uniformly discrete of $\mathbb{R}^n, n \geq 1$, is a (SFU - set) self-similar finitely generated uniformly discrete set if it is finitely generated and admits at least one (affine) self-similarity.

Although a uniformly discrete set of $\mathbb{R}^n, n \geq 1$, may be finite, let us observe that a nonempty (SFU -) self-similar finitely generated uniformly discrete set in $\mathbb{R}^n, n \geq 1$, is always infinite.

Definition 2.20. Let $\lambda > 1$ be a real algebraic integer. Denote by $\lambda^{(i)}$ its conjugates. We say that λ is

(i) a Pisot number if all its conjugates $\lambda^{(i)}$ satisfy $|\lambda^{(i)}| < 1$,

(ii) a Salem number if all its conjugates $\lambda^{(i)}$ satisfy $|\lambda^{(i)}| \leq 1$, with at least one on the unit circle,

(iii) a Perron number if all its conjugates $\lambda^{(i)}$ satisfy $|\lambda^{(i)}| < \lambda$,

(iv) a Lind number if all its conjugates $\lambda^{(i)}$ satisfy $|\lambda^{(i)}| \leq \lambda$, with at least one on the circle $\{|z| = \lambda\}$.

Theorem 2.21 (Meyer). *Let $\Lambda \subset \mathbb{R}^n, n \geq 1$, be a Meyer set. If Λ is a SFU - set, then all its self-similarities are Pisot or Salem numbers.*

Proof. See Theorem 6 in Meyer [82]. □

If $\beta > 1$ is a Pisot number or a Parry number, then the sets $\mathbb{Z}_\beta$ are Meyer sets that admit by construction the self-similarity β [52]. However, there exist many Meyer sets which have no self-similarities at all.

Theorem 2.22 (Lagarias). *Let $\Lambda \subset \mathbb{R}^n, n \geq 1$, be a Delone set. If Λ is a SFU - set, then all its (affine) self-similarities λ are algebraic integers such that* degree(λ) *divides* rk Λ. *Moreover, if Λ is of finite type, then all the self-similarities are Perron or Lind numbers.*

Proof. See Theorem 4.1 in Lagarias [70]. □

The concept of (affine) self-similarity is extended as follows in a natural way [84].

Definition 2.23. Let Λ be a nonempty uniformly discrete set of $\mathbb{R}^n$. A self-similarity of Λ is given by a triple (c, λ, Q) where λ is a real number > 1, Q an element of the orthogonal group $O(n, \mathbb{R})$ such that

$$\lambda Q(\Lambda - c) \quad \subset \quad \Lambda - c \tag{2.11}$$

for a certain $c \in \mathbb{R}^n$ (note that c belongs or not to Λ).

A point c for which (2.11) occurs for certain couple (λ, Q) is called an *inflation center* of Λ, as in the affine case. Problems on self-similar sets are reported in [93].

2.5 Sets $\mathbb{Z}_{\beta}$ of beta-integers and Rauzy fractals

Meyer sets $\mathbb{Z}_\beta \subset \mathbb{R}$ of β-integers with β a Pisot number, and their vectorial extension to $\mathbb{R}^n$ - so-called β-grids -, are useful tools for modeling quasicrystals in physics [39] [40] [50] [51]. Indeed, Penrose tilings in the plane and in space play a fundamental role in this modelling process, with suitable positioning of atoms in the tiles. Gazeau [51] has observed that Penrose tilings can easily be deduced from τ-grids, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean, quadratic Pisot number. Therefore it is natural to extend the constructions of Penrose tilings and β-grids with Pisot numbers β (or more generally with algebraic integers) of higher degree which could be used in the objective of providing possibly new models of aperiodic crystals in crystallography to physicists.

Let us recall the mathematical construction of $\mathbb{Z}_\beta$ on the line and its properties when $\beta > 1$ is a real number, in a general way, and some open questions related to them when β is in particular an algebraic integer. We refer to [47] [48] [91] [95] and [12] for an overview on recent studies on Numeration and its applications.

2.5.1 Construction and Properties For a real number $x \in \mathbb{R}$, the integer part of x will be denoted by $\lfloor x \rfloor$ and its fractional part by $\{x\} = x - \lfloor x \rfloor$. The smallest integer larger than or equal to x will be denoted by $\lceil x \rceil$. For $\beta > 1$ a real number and $z \in [0,1]$ we denote by $T_\beta(z) = \beta z \pmod 1$ the β-transform on $[0,1]$ associated with β, and iteratively, for all integers $j \geq 0$, $T_\beta^{j+1}(z) := T_\beta(T_\beta^j(z))$, where by convention $T_\beta^0 = Id$.

Let $\beta > 1$ be a real number. A beta-representation (or β-representation, or representation in base β) of a real number $x \geq 0$ is given by an infinite sequence $(x_i)_{i \geq 0}$ and an integer $k \in \mathbb{Z}$ such that $x = \sum_{i=0}^{+\infty} x_i \beta^{-i+k}$, where the digits x_i belong to a given alphabet ($\subset \mathbb{N}$). Among all the beta-representations of a real number $x \geq 0, x \neq 1$, there exists a particular one called Rényi β-expansion, which is obtained through the greedy algorithm [47] [48]: in this case, k satisfies $\beta^k \leq x < \beta^{k+1}$ and the digits

$$x_i := \lfloor \beta T_\beta^i(\frac{x}{\beta^{k+1}}) \rfloor \qquad i = 0, 1, 2, \ldots, \tag{2.12}$$

belong to the finite canonical alphabet $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$. If β is an integer, then $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \beta - 1\}$; if β is not an integer, then $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}$. We denote by

$$\langle x \rangle_\beta := x_0 x_1 x_2 \ldots x_k \,.\, x_{k+1} x_{k+2} \ldots \tag{2.13}$$

the couple formed by the string of digits $x_0x_1x_2\dots x_kx_{k+1}x_{k+2}\dots$ and the position of the dot, which is at the kth position (between x_k and x_{k+1}). By definition the integer part (in base β) of x is $\sum_{i=0}^{k} x_i\beta^{-i+k}$ and its fractional part (in base β) is $\sum_{i=k+1}^{+\infty} x_i\beta^{-i+k}$. If a Rényi β-expansion ends in infinitely many zeros, it is said to be finite and the ending zeros are omitted. If it is periodic after a certain rank, it is said to be eventually periodic (the period is the smallest finite string of digits possible, assumed not to be a string of zeros).

There is a particular Rényi β-expansion which plays an important role in the theory, which is the Rényi β-expansion of 1, denoted by $d_\beta(1)$ and defined as follows: since $\beta^0 \leq 1 < \beta$, the value $T_\beta(1/\beta)$ is here set (by convention) to 1. Then using (2.12) for all $i \geq 1$, we obtain: $t_1 = \lfloor\beta\rfloor, t_2 = \lfloor\beta\{\beta\}\rfloor, t_3 = \lfloor\beta\{\beta\{\beta\}\}\rfloor$, etc. The equality $d_\beta(1) = 0.t_1t_2t_3\dots$ corresponds to $1 = \sum_{i=1}^{+\infty} t_i\beta^{-i}$.

Definition 2.24. A real number $\beta > 1$ such that $d_\beta(1)$ is finite or eventually periodic is called a *beta-number* or more recently *a Parry number* (this new name appears in [40]). In particular, it is called a *simple beta-number* or *a simple Parry number* (after [40]) when $d_\beta(1)$ is finite.

Beta-numbers (Parry numbers) are algebraic integers [91] and all their conjugates lie within a compact subset which looks like a fractal in the complex plane [44] [108]. The conjugates of Parry numbers are all bounded above in modulus by the golden mean $\frac{1}{2}(1+\sqrt{5})$ [44] [108].

Definition 2.25. The set

$$\mathbb{Z}_\beta := \{x \in \mathbb{R} \mid |x| \text{ is equal to its integer part in base } \beta\}$$

is called set of beta-integers, or set of β-integers, or set of integers in base β.

By construction, the set $\mathbb{Z}_\beta$ is discrete, relatively dense and locally finite (its intersection with any interval of the line is finite), self-similar, with β as self-similarity (with inflation center the origin), and symmetrical with respect to the origin: $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$, $\mathbb{Z}_\beta = -\mathbb{Z}_\beta$. Its complete set of self-similarities is unknown. Thurston [110] has shown that it is uniformly discrete, hence a Delone set, when β is a Pisot number. From [52], for β a Pisot number, the relatively dense set $\mathbb{Z}_\beta \cap \mathbb{R}^+$ is finitely generated over $\mathbb{N}$.

Theorem 2.26. *If β is a Pisot number, the Delone set $\mathbb{Z}_\beta$ is a Meyer set which is a SFU - set in $\mathbb{R}$.*

Proof. [24], [52]. For the definition of a SFU - set, see Definition 2.19. It is a SFU - set since $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$. ☐

Open problem (P$_1$).— What is the class of real numbers $\beta > 1$ for which $\mathbb{Z}_\beta$ is uniformly discrete, equivalently a Delone set ?

We know that this class contains Pisot numbers [21] [52] and beta-numbers. It also contains some Salem numbers [23]. It is unknown whether it contains all Salem numbers and all Perron [111]. Problem (P$_1$) is linked to the specification of the β-shift [21] [52] [111].

The set $\mathbb{Z}_\beta$ contains $\{0, \pm 1\}$ and all the polynomials in β for which the coefficients are given by the equations (2.12). Parry [91] has shown that the knowledge of $d_\beta(1)$ suffices to exhaust all the possibilities of such polynomials by the so-called "Conditions of Parry (CP$_\beta$)". Let us recall them. Let $(c_i)_{i \geq 1} \in \mathbb{A}_\beta^{\mathbb{N}}$ be the following sequence:

$$c_1 c_2 c_3 \cdots \quad = \begin{cases} t_1 t_2 t_3 \cdots & \text{if } d_\beta(1) = 0.t_1 t_2 \cdots \text{ is infinite,} \\ (t_1 t_2 \cdots t_{m-1}(t_m - 1))^\omega & \text{if } d_\beta(1) \text{ is finite and} \\ & \qquad \text{equal to } 0.t_1 t_2 \cdots t_m, \end{cases} \tag{2.14}$$

where $(\ \ )^\omega$ means that the word within ( ) is indefinitely repeated. When the degree of β is ≥ 2 , we have $c_1 = t_1 = \lfloor \beta \rfloor$. Then the polynomial $\sum_{i=0}^{v} y_i \beta^{-i+v} \geq 0$, with $v \geq 0, y_i \in \mathbb{Z}$ arbitrary, belongs to $\mathbb{Z}_\beta^+ := \mathbb{Z}_\beta \cap \mathbb{R}^+$ if and only if $y_i \in \mathbb{A}_\beta$ and the following $v+1$ inequalities are satisfied:

$$(\mathrm{CP}_\beta): \quad (y_j, y_{j+1}, y_{j+2}, \ldots, y_{v-1}, y_v, 0, 0, 0, \ldots) \prec (c_1, c_2, c_3, \ldots), \quad \text{for all} j = 0, 1, 2, \ldots, v, \tag{2.15}$$

where "$\prec$" means lexicographical smaller. For a negative polynomial, we consider the above criterium applied to its opposite. Conditions of Parry (CP$_\beta$) sieve the elements of the ring $\mathbb{Z}[\beta]$ in the number field $\mathbb{Q}(\beta)$.

The set $\mathbb{Z}_\beta$ can be viewed as the set of vertices of the tiling $\mathcal{T}_\beta$ of the real line for which the tiles are the closed intervals whose extremities are two successive β-integers. When β is a Pisot number, the number of (non-congruent) tiles in $\mathcal{T}_\beta$ is finite [110]. If V is a tile of $\mathcal{T}_\beta$ we denote by $l(V)$ its length. If β is a Pisot number and $d_\beta(1)$ is finite, say $d_\beta(1) = 0.t_1 t_2 \ldots t_m$, then the set of the lengths of the tiles of $\mathcal{T}_\beta$ is exactly $\{T_\beta^i(1) \mid 0 \leq i \leq m-1\} = \{1, \beta - t_1, \beta^2 - t_1\beta - t_2, \ldots, \beta^{m-1} - t_1\beta^{m-2} - t_2\beta^{m-3} - \ldots - t_{m-1}\}$. If β is a Pisot number with $d_\beta(1)$ eventually periodic, say $d_\beta(1) = 0.t_1 t_2 \ldots t_m (t_{m+1} t_{m+2} \ldots t_{m+p})^\omega$, then the set of the lengths of the tiles of $\mathcal{T}_\beta$ is exactly $\{T_\beta^i(1) \mid 0 \leq i \leq m+p-1\} = \{1, \beta - t_1, \beta^2 - t_1\beta - t_2, \ldots, \beta^{m-1} - t_1\beta^{m-2} - t_2\beta^{m-3} - \ldots - t_{m-1}, \beta^m - t_1\beta^{m-1} - t_2\beta^{m-2} - \ldots - t_m, \ldots, \beta^{m+p-1} - t_1\beta^{m+p-2} - t_2\beta^{m+p-3} - \ldots - t_{m+p-1}\}$. Hence, when β is a Pisot number, the set $\mathbb{Z}_\beta$ is a Delone set of (sharp) constants (r, R) with $r = \min\{l(V) \mid V \in \mathcal{T}_\beta\} > 0$ and $R = \frac{1}{2}\max\{l(V) \mid V \in \mathcal{T}_\beta\} = \frac{1}{2}$. The tiling $\mathcal{T}_\beta$ can be obtained directly from a substitution system on a finite alphabet which is associated to β in a canonical way [41] [45] [48].

2.5.2 Rauzy fractals and Meyer sets of beta-integers for β a Pisot number Rauzy fractals were introduced by Rauzy [1] [2], [45] (Chapter 7), [79] [94] [104] [105] to provide geometric interpretations and geometric representations of symbolic dynamical systems, in the general objective of understanding whether substitutive dynamical systems are isomorphic to already known dynamical systems or if they are new. Rauzy [94] generalized the dynamical properties of the Fibonacci substitution [45] to a three-letter alphabet substitution, called Tribonacci substitution or Rauzy substitution, defined by: $1 \to 12, 2 \to 13, 3 \to 1$. The incidence matrix of this substitution is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

its characteristic polynomial is $X^3 - X^2 - X - 1$ with $\beta > 1$ a Pisot number as dominant root, and two complex conjugates roots α and $\overline{\alpha}$ in the unit disc. The incidence matrix admits as eigenspaces in $\mathbb{R}^3$ an expanding one-dimensional direction and a contracting plane [52].

Theorem 2.27 (Rauzy). *The Rauzy fractal generates a self-similar periodic tiling of the plane. The symbolic dynamical system generated by the Tribonacci substitution is measure-theoretically isomorphic to a toral substitution. The Tribonacci substitutive dynamical system has a purely discrete spectrum.*

Proof. See Rauzy [94]. □

Properties of the Rauzy fractal (connectedness, interiors, boundary, etc) were obtained in [2] [35] [65] [79] [104] [105] [106] [107]. Gazeau and Verger-Gaugry [52] proved that the set $\mathbb{Z}_\beta$ of integers in base β are in close relation with the Rauzy fractal, within the framework of a canonical cut-and-project scheme $(\mathbb{R}^3 = G \times E, L = \mathbb{Z}^3, \pi_1, \pi_2)$ above $\mathbb{Z}_\beta$, where E is a line in $\mathbb{R}^3$ and G the corresponding internal space (hyperplane): the Rauzy fractal is the adherence of the image of $\mathbb{Z}_\beta$ by the map $\pi_1 \circ ({\pi_2}_{|_L})^{-1}$ in the internal space G. This situation is quite general for Pisot numbers β and the Rauzy fractal appears as a compact canonical window [52]. However, all the points of L are not selected by this window and only some of them which satisfy the conditions of Parry are projected on E, $\mathbb{Z}_\beta$ being a Meyer set [24] [52].

For all Perron numbers β, the construction of the cut-and-project scheme $(\mathbb{R}^d = G \times E, L = \mathbb{Z}^d, \pi_1, \pi_2)$ over $\mathbb{Z}_\beta$, where d is the degree of β, E a line in $\mathbb{R}^d$ and G the corresponding internal space (hyperplane), is canonical, and does not use the fact that $\mathbb{Z}_\beta$ should be uniformly discrete [52]. If β is a Pisot number then the image of L by $\pi_1 \circ ({\pi_2}_{|_L})^{-1}$ is relatively compact and its adherence is the (geometric) Rauzy fractal. Whether this image is relatively compact for β a general Salem number is not known.

The $\mathbb{Z}$-module $\mathbb{Z}[\mathbb{Z}_\beta - \mathbb{Z}_\beta]$ is finitely generated for β a Pisot number, but it is not known whether it is the case for Perron numbers in general (which are not Pisot numbers).

Open problem (P$_2$).— What is the class of real numbers $\beta > 1$ for which $\mathbb{Z}_\beta$ is uniformly discrete and is not finitely generated, i.e. for which

$$\operatorname{rank} \mathbb{Z}[\mathbb{Z}_\beta - \mathbb{Z}_\beta] = +\infty?$$

3 Proof of Theorem 1.1 - Characterization of SFU - sets

(i) (same proof as [70] Theorem 4.1 (i)) Let $s = \dim_{\mathbb{R}} \mathbb{R}[\Lambda]$ be the dimension of the $\mathbb{R}$-span of Λ (by $\mathbb{R}$-span of Λ, we mean the intersection of all the real affine subspaces of $\mathbb{R}^n$ which contain Λ). Then $1 \leq s \leq n$ and $m := \operatorname{rk} \Lambda \geq s$. By definition the $\mathbb{Z}$-module $\mathbb{Z}[\Lambda - \Lambda] := \{\sum_{\text{finite}} a_i(x_i - x_j) \mid a_i \in \mathbb{Z}, x_i, x_j \in \Lambda\}$ admits a set of m generators, say $\{v_1, v_2, \ldots, v_m\}$, which are $\mathbb{Q}$-linearly independent (nonzero) vectors of $\mathbb{R}^n$. Then

$$\mathbb{Z}[\Lambda - \Lambda] = \mathbb{Z}[v_1, v_2, \ldots, v_m].$$

If $\lambda > 1$ is a self-similarity of Λ then there exists $c \in \mathbb{R}^n$ such that $\lambda(\Lambda - c) \subset \Lambda - c$. Since $\Lambda - \Lambda = \Lambda - c - (\Lambda - c)$, this implies

$$\lambda \mathbb{Z}[\Lambda - \Lambda] \subset \mathbb{Z}[\Lambda - \Lambda]. \tag{3.1}$$

We deduce that there exist integers $a_{i,j} \in \mathbb{Z}$ such that

$$\lambda v_i = \alpha_{i,1} v_1 + \alpha_{i,2} v_2 + \ldots + \alpha_{i,m} v_m \qquad i = 1, 2, \ldots, m$$

with $M = \left(\alpha_{i,j}\right)_{i,j} \in \operatorname{Mat}_m(\mathbb{Z})$ the space of $m \times m$ integral matrices. Hence

$$\lambda V = MV \ , \qquad \text{with} \quad V = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}. \tag{3.2}$$

Equivalently the transposed matrix ${}^tM$ is the matrix associated with the $\mathbb{Q}$-linear map which sends $\{v_1, v_2, \ldots, v_m\}$ to the system $\{\lambda v_1, \lambda v_2, \ldots, \lambda v_m\}$ with respect to the $\mathbb{Q}$-free system $\{v_1, v_2, \ldots, v_m\}$. Since the polynomial $h(X) := \det(X\,\mathrm{I} - M) \in \mathbb{Z}[X]$ is monic and cancels at λ, the real number λ is an algebraic integer of degree less than m.

Let d be the degree of λ and

$$\varphi(X) = X^d + a_1 X^{d-1} + a_2 X^{d-2} + \ldots + a_d, \qquad \text{with } a_i \in \mathbb{Z}, a_d \neq 0,$$

be the minimal polynomial of λ. From (3.2) we deduce $\lambda^j V = M^j V$ for all $j \in \mathbb{N}$. Hence, since $\varphi(\lambda) = 0$,

$$\varphi(M)V = (M^d + a_1 M^{d-1} + a_2 M^{d-2} + \ldots + a_d)V = 0. \tag{3.3}$$

Since $\varphi(M) \in \mathrm{Mat}_m(\mathbb{Z})$ and that the vectors $v_1, v_2, \ldots, v_m$ are $\mathbb{Q}$-linearly independent, we deduce $\varphi(M) = \varphi({}^tM) = 0$. Hence the minimal polynomial $\psi(X) \in \mathbb{Z}[X]$ of the matrix ${}^tM$ divides $\varphi(X)$ in $\mathbb{Z}[X]$. Since $\varphi(X)$ is irreducible over $\mathbb{Q}$, there is equality: $\psi(X) = \varphi(X)$.

Denote by K the number field $\mathbb{Q}(\lambda)$. Equation (3.1) implies that $\mathbb{Z}[\Lambda - \Lambda]$ is a module over the ring $\mathbb{Z}[\lambda]$ and that $\mathbb{Q}[\Lambda - \Lambda]$ is a K-vector space. The ring $\mathbb{Z}[\lambda]$ is a subring of finite index of the ring of integers $\mathcal{O}_K$ of K. The $m \times m$ integral matrix ${}^tM$ corresponds to an endomorphism of $\mathbb{R}^m$, say u, expressed in the canonical basis $\{e_1, e_2, \ldots, e_m\}$. Since $v_1, v_2, \ldots, v_m$ are $\mathbb{Q}$-linearly independent, the base $\{e_1, e_2, \ldots, e_m\}$ of $\mathbb{R}^m$ and the system $\{v_1, v_2, \ldots, v_m\}$ of $\mathbb{R}^n$ can be identified as well as the two $\mathbb{Q}$- vector spaces $\oplus_{i=1}^m \mathbb{Q}e_i$ and $\oplus_{i=1}^m \mathbb{Q}v_i$. There are two cases: $m = 1$ and $m > 1$. When $m = 1$, then necessarily λ is an integer > 1 and $d = 1$. When $m > 1$ and $d = 1$, then the matrix M is the diagonal matrix $\lambda\mathrm{I}$ and λ is an integer > 1. This case occurs for instance for (affine) lattices Λ of $\mathbb{R}^n$. Now, if $m > 1$ and $d \geq 2$, then the endomorphism u induces a Jordan decomposition of $\mathbb{R}^m$ as $K[X]$-module as follows (for instance [46] pp 295-301). We assume $d \geq 2$ in the sequel. Let

$$A(\psi) = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_d \\ 1 & \ddots & 0 & \vdots & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & -a_2 \\ 0 & \ldots & & 1 & -a_1 \end{pmatrix}, \text{ resp. } \quad U_d := \begin{pmatrix} 0 & \ldots & 1 \\ \vdots & 0 & \vdots \\ 0 & \ldots & 0 \end{pmatrix},$$

be the $d \times d$ integral matrix whose terms are 0 except the last column which is composed of the coefficients of $\psi(X)$ (up to sign) and the diagonal under the main diagonal, which is composed with 1, respectively the $d \times d$ matrix whose terms are 0 except the term of the first row and the last column, which is 1 (this makes sense since $d > 1$). Then there exists a basis of $\mathbb{R}^m$, say $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$, in which the matrix of the endomorphism u takes the diagonal form

$$\mathcal{J} := \begin{pmatrix} J_{i_1} & 0 & \ldots & 0 \\ 0 & J_{i_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & J_{i_r} \end{pmatrix}$$

where $i_1 + i_2 + \ldots + i_r = m$, $i_1 \geq i_2 \geq \ldots \geq i_r \geq d$, and $J_{i_q}, 1 \leq q \leq r$, is the $i_q \times i_q$ integral matrix given by

$$J_{i_q} \; := \begin{pmatrix} A(\psi) & 0 & \ldots & 0 \\ U_d & A(\psi) & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & U_d & A(\psi) \end{pmatrix}$$

with 0 everywhere except $A(\psi)$ on the main diagonal and U_d on the diagonal under the main diagonal. Since all the diagonal terms of the matrix J_{i_q} are $A(\psi)$, they are identical, and therefore d divides i_q. Consequently d divides $\sum_{q=1}^{r} i_q = m$.

(ii) Let us pull back this Jordan decomposition to the ambient space $\mathbb{R}^n$ of the uniformly discrete set Λ, block by block. Let us consider the first block J_{i_1}, the situation being the same for the others. The system $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{i_1}\}$ satisfies the following relations:

- for $1 \leq \beta < d$, $0 \leq \alpha \leq \frac{i_1}{d} - 1$,

$$u(\epsilon_{\alpha d+\beta}) = \epsilon_{\alpha d+\beta+1}, \tag{3.4}$$

- for $\beta = d$, $0 \leq \alpha < \frac{i_1}{d} - 1$,

$$u(\epsilon_{(\alpha+1)d}) = \epsilon_{(\alpha+1)d+1} - a_1\epsilon_{(\alpha+1)d} - a_2\epsilon_{(\alpha+1)d-1} - \ldots - a_d\epsilon_{\alpha d+1}, \tag{3.5}$$

- for $\beta = d$, $\alpha = \frac{i_1}{d} - 1$,

$$u(\epsilon_{i_1}) = -a_1\epsilon_{i_1} - a_2\epsilon_{i_1-1} - \ldots - a_d\epsilon_{i_1-d+1}. \tag{3.6}$$

Now the matrices ${}^tM$ and $\mathcal{J}$ have coefficients in $\mathbb{Q}$ and are such that there exist a $m \times m$ invertible matrix $C \in GL(m, \mathbb{R})$ such that ${}^tM = CJC^{-1}$. Then (Corollary 2 in [73], Chap. XV, §3) there exists a $m \times m$ invertible matrix C' in $\mathbb{Q}$ such that:

$${}^tM = C'JC'^{-1}.$$

The matrix C' is the matrix associated with the linear map which sends $\{e_1, e_2, \ldots, e_m\}$ to $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ with respect to the basis $\{e_1, e_2, \ldots, e_m\}$. Let

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} = {}^tC'V \; , \qquad \text{with} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}. \tag{3.7}$$

The $\mathbb{Q}$-free system $\{f_1, f_2, \ldots, f_m\}$ of nonzero vectors of $\mathbb{R}^n$, identified with the basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ of $\mathbb{R}^m$, admits the following structure: from (3.2)

$$\lambda V = MV = {}^t(C'JC'^{-1}) \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \Leftrightarrow {}^tC'^{-1}(\lambda\,\mathrm{I} - {}^tJ){}^tC' \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = 0; \quad (3.8)$$

hence

$$\lambda \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = {}^tJ \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}. \quad (3.9)$$

From (3.9), considering the first block J_{i_1}, the situation being the same with the other blocks $J_{i_q}, 1 \leq q \leq r$, we deduce (see (3.4), (3.5)):

- for $1 \leq \beta < d$, $0 \leq \alpha \leq \frac{i_1}{d} - 1$,

$$f_{\alpha d+\beta+1} = \lambda f_{\alpha d+\beta} = \lambda^{\beta} f_{\alpha d+1}, \quad (3.10)$$

- for $\beta = d$, $0 \leq \alpha < \frac{i_1}{d} - 1$,

$$f_{(\alpha+1)d+1} = \lambda f_{(\alpha+1)d} + a_1 f_{(\alpha+1)d} + a_2 f_{(\alpha+1)d-1} + \ldots + a_d f_{\alpha d+1}. \quad (3.11)$$

Let us show that the assumption $d < i_1$ leads to a contradiction. Assume $d < i_1$. Then we would have $f_{d+1} \neq 0$ from (3.7) since C' is invertible. But, from (3.10) and (3.11), with $\alpha = 0$,

$$\begin{aligned} f_{d+1} &= (\lambda + a_1)\, f_d + a_2\, f_{d-1} + \ldots + a_d\, f_1 \\ &= (\lambda + a_1)\, \lambda^{d-1}\, f_1 + a_2\, \lambda^{d-2}\, f_1 + \ldots + a_d\, f_1 \quad = \quad \varphi(\lambda)\, f_1 = \quad 0. \end{aligned}$$

Contradiction. Hence $d = i_1$. Proceeding now with the other blocks in the same way leads to the equalities $d = i_1 = i_2 = \ldots = i_r$.

The matrix C' belongs to $GL(m, \mathbb{Q})$. If $C' \in GL(m, \mathbb{Z})$ we take

$$w_1 = f_1, w_2 = f_{d+1}, w_3 = f_{2d+1}, \ldots, w_r = f_{(r-1)d+1}.$$

Then we deduce from (3.7) and (3.10) (and its analogs for the other blocks) that $\oplus_{i=1}^{m} \mathbb{Z}\, v_i$ and $\oplus_{q=1}^{r} \oplus_{i=1}^{m} \mathbb{Z}\lambda^i\, w_q$ are isomorphic as $\mathbb{Z}$-modules. We deduce the result in this case. If $C' \in GL(m, \mathbb{Q}) \setminus GL(m, \mathbb{Z})$, let us denote by μ the lcm of the m^2 denominators of the coefficients of C'^{-1} and take

$$w_1 = f_1/\mu, w_2 = f_{d+1}/\mu, w_3 = f_{2d+1}/\mu, \ldots, w_r = f_{(r-1)d+1}/\mu.$$

The coefficients of $D := \mu\, {}^t C'^{-1}$ are in $\mathbb{Z}$ and relatively prime so that

$$D \begin{bmatrix} f_1/\mu \\ \lambda f_1/\mu \\ \vdots \\ \lambda^{d-1} f_1/\mu \\ \vdots \\ f_r/\mu \\ \lambda f_r/\mu \\ \vdots \\ \lambda^{d-1} f_r/\mu \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}. \tag{3.12}$$

From (3.12), the $\mathbb{Z}$-module $\oplus_{i=1}^{m} \mathbb{Z}\, v_i$ is a $\mathbb{Z}$-submodule of $\oplus_{q=1}^{r} \oplus_{i=0}^{d-1} \mathbb{Z}\lambda^i\, w_q$. Hence the result.

(iii) Since $d = i_1 = i_2 = \ldots = i_r$ and $r = \frac{m}{d}$ by (ii), we deduce that $\mathcal{J}$ is the $m \times m$ diagonal matrix for which the diagonal terms are all identical and equal to $A(\psi)$:

$$\mathcal{J} = \begin{pmatrix} A(\psi) & 0 & \ldots & 0 \\ 0 & A(\psi) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & A(\psi) \end{pmatrix}.$$

Since $\det M = \det \mathcal{J}$, we obtain the characteristic polynomial of M:

$$\det(X\, \mathrm{I}_m - M) = \det(X\, \mathrm{I}_d - A(\psi))^{m/d} = (\varphi(X))^{m/d};$$

in particular, $\det M = (\det A(\psi))^{m/d} = N(\lambda)^{m/d}$, where $N(\lambda) = (-1)^d a_d$, product of the conjugates of λ, is the algebraic norm of λ. This formula is reminiscent of the algebraic norm of an element in a ring extension [73].

(iv) There are two cases: either (iv-1) $w_1, w_2, \ldots, w_r$ are K-linearly independent, or (iv-2) they are K-linearly dependent. In the first case the cut-and-project scheme above Λ will admit an internal space reduced to its *shadow space* (see below and [70]), while, in the second case, the internal space will come from the shadow space and the space of relations over K between the vectors w_i.

First let us fix some notations. Denote by $\mathcal{C}$ the finite set $\{w_1, w_2, \ldots, w_r\}$ and call it the *central cluster* of the basis $(\ \lambda^j\, w_i\ )_{i=1,\ldots,r, j=0,1,\ldots,d-1}$. For $i = 1, 2, \ldots, r$, let $\widetilde{w_i} := \|w_i\|^{-1} w_i$. Let $\widetilde{\mathcal{C}} := \{\widetilde{w_1}, \ldots, \widetilde{w_r}\}$ the image of $\mathcal{C}$ on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. We have $\mathrm{Card}(\widetilde{\mathcal{C}}) \leq r$. We assume that the signature of the field $K = \mathbb{Q}(\lambda)$, of degree d, is (r_1, r_2). Then $d = r_1 + 2r_2$. Denote by $\sigma_j, 1 \leq j \leq r_1$, the real embeddings of K in $\mathbb{R}$, and by $\sigma_j, \sigma_{r_2+j} = \overline{\sigma_j}$, where $r_1 + 1 \leq j \leq r_1 + r_2$, the imaginary embeddings of K in $\mathbb{C}$. Assume

$\sigma_1(\lambda) = \lambda$. Let Σ be the embedding of K in $\mathbb{R}^{r_1} \times \mathbb{C}^{2r_2}$ defined by

$$\forall \xi \in K, \quad \Sigma(\xi) = (\sigma_1(\xi), \sigma_2(\xi), \ldots, \sigma_{r_1+2r_2}(\xi))\,.$$

Let $(g_j)_{1\leq j\leq d}$ be a $\mathbb{Z}$-basis of the ring of integers $\mathcal{O}_K$ of K. We identify the field K to $\mathbb{Q}^d$ via the mapping Ψ defined by $\Psi(z) = \sum_{i=1}^{d} z_i g_i$ if $z \in \mathbb{Q}^d$, resp. $\mathcal{O}_K$ to $\mathbb{Z}^d$ if $z \in \mathbb{Z}^d$. The composed mapping $\Phi := \Sigma \circ \Psi$ is then extended in a continuous way from $\mathbb{Q}^d$ to $\mathbb{R}^d$, and denoted in the same way:

$$\forall z \in \mathbb{R}^d, \quad \Phi(z) := \left(\sum_{i=1}^{d} z_i \sigma_1(g_i), \sum_{i=1}^{d} z_i \sigma_2(g_i), \ldots, \sum_{i=1}^{d} z_i \sigma_{r_1+2r_2}(g_i)\right).$$

Σ is an injective homorphism for the ring structures while Ψ is $\mathbb{Q}$-vector space isomorphism. Thus Φ is a $\mathbb{R}$-vector space isomorphism from $\mathbb{R}^d$ onto the étale $\mathbb{R}$-vector space

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \{z \in \mathbb{C}^{2r_2} \mid z_{r_2+j} = \overline{z_j} \text{ for all } j = 1, 2, \ldots, r_2\}.$$

The $\mathbb{R}$-subspace $\Sigma(\mathcal{O}_K)$ of $K_{\mathbb{R}}$ is a lattice. Let us extend Φ to $\mathbb{C}^d$ as a $\mathbb{C}$-endomorphism, keeping the same notation, by (with $\mathrm{I} = \sqrt{-1}$):

$$\Phi(x + \mathrm{I}\ y) = \Phi(x) + \mathrm{I}\ \Phi(y), \quad \text{for all } \ x, y \in \mathbb{R}^d.$$

Let us construct the cut-and-project scheme above Λ. Let $x \in \mathbb{R}[\Lambda] \subset \mathbb{R}^n$ be in the $\mathbb{R}$-span of Λ. For $i = 1, 2, \ldots, r$, denote by $p_i(x)$ the orthogonal projection of x onto the line $\mathbb{R}w_i$ and $\widetilde{w_i} := \|w_i\|^{-1} w_i$. Then $p_i(x)$ can be written

$$p_i(x) = \frac{\langle x, w_i\rangle}{\langle w_i, w_i\rangle} w_i = \langle x, \widetilde{w_i}\rangle \widetilde{w_i},$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. The point $x \in \mathbb{R}^n$ will be said *K-rational* if the r coefficients $\langle x, \widetilde{w_i}\rangle$ belongs to K. In this case, for all $i = 1, \ldots, r$, $\langle x, \widetilde{w_i}\rangle = \sum_{j=1}^{d} \alpha_{i,j}(x) g_j$ with all coefficients $\alpha_{i,j}(x) \in \mathbb{Q}$. Let us define $\Sigma_i : K \to K_{\mathbb{R}}\widetilde{w_i}$ by $\Sigma_i(\xi) = \Sigma(\xi)\widetilde{w_i}$, for all $i = 1, \ldots, r$, and $\Phi_i = \Sigma_i \circ \Psi$ the $\mathbb{R}$-vector space isomorphism from $\mathbb{R}^d$ onto the $\mathbb{R}$-vector space

$$K_{\mathbb{R}}\widetilde{w_i} := (\mathbb{R}\widetilde{w_i})^{r_1} \times \{\{z_j \widetilde{w_i}\} \mid z = (z_j) \in \mathbb{C}^{2r_2}, z_{r_2+j} = \overline{z_j} \text{ for all } j = 1, 2, \ldots, r_2\}.$$

The $\mathbb{R}$-subspace $\Sigma_i(\mathcal{O}_K)$ of $K_{\mathbb{R}}\widetilde{w_i}$ is a lattice. For any set A, denote by $\mathrm{pr}_k : A^d \to A$ the k-th projection, so that $\mathrm{pr}_k(K_{\mathbb{R}}\widetilde{w_i}) = \sigma_k(K)\widetilde{w_i}$ for all $i = 1, \ldots, r$ and $k = 1, \ldots, d$. Since the mapping

$$\mathbb{R}[\Lambda] \to \coprod_{i=1}^{r} \mathrm{pr}_1\left(K_{\mathbb{R}}\widetilde{w_i}\right), \qquad x \to (\ \langle x, \widetilde{w_i}\rangle\ )_i \tag{3.13}$$

is injective, as $\mathbb{R}$-morphism of vector spaces, the $\mathbb{R}$-span $\mathbb{R}[\Lambda]$ of Λ is identified by (3.13) with a s-dimensional $\mathbb{R}$-subspace of the first component $\coprod_{i=1}^{r} K\widetilde{w_i} = \coprod_{i=1}^{r} \mathrm{pr}_1(K_{\mathbb{R}}\widetilde{w_i})$. Denote by R_K the subspace of $\coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i}$ which is the closure of

$$\{(\Sigma_i(\langle x, \widetilde{w_i}\rangle))_i \mid \; x \in \mathbb{R}[\Lambda] \text{ is } K\text{- rational}\}.$$

It is a product of r_1 copies of $\mathbb{R}[\Lambda]$ and r_2 copies of $\mathbb{C}[\Lambda]$, with $\mathrm{pr}_1(R_K) = \mathbb{R}[\Lambda]$.

The space $\coprod_{i=1}^r K_{\mathbb{R}}\widetilde{w_i}$ is a K-vector space, the external law being given by

$$\begin{array}{ccc} K \times \coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i} & \to & \coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i} \\ (\mu, u) & & \Sigma(\mu)\cdot u \end{array} \tag{3.14}$$

with componentwise multiplication, where $u = (u_k)_{k=1,\ldots,d}$, so that the external law, on the k-th component, is given by:

$$\begin{array}{ccc} K \times \coprod_{i=1}^{r} \mathrm{pr}_k\,(K_{\mathbb{R}}\widetilde{w_i}) & \to & \coprod_{i=1}^{r} \mathrm{pr}_k\,(K_{\mathbb{R}}\widetilde{w_i})\,. \\ (\mu, u_k) & & \sigma_k(\mu)\cdot u_k \end{array} \tag{3.15}$$

The actions (3.14) and (3.15) are extended from K to $\mathbb{R}$ by continuity. Thus, by (3.15) and since the conjugate fields $\sigma_j(K), \sigma_{r_2+j}(K)$ are not subfields of $\mathbb{R}$ for $j = 1, 2, \ldots, r_2$ (if $r_2 \neq 0$), the usual scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ should be considered as the restriction to $\mathbb{R}^n$ of the standard hermitian form on $\mathbb{C}^n$; in particular, it is anti-linear for the second variable.

We now construct a real positive definite symmetric bilinear form on $\coprod_{i=1}^r K_{\mathbb{R}}\widetilde{w_i}$. Since

$$\coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i} = \prod_{k=1}^{d} \left(\coprod_{i=1}^{r} \mathrm{pr}_k\,(K_{\mathbb{R}})\,\widetilde{w_i} \right)$$

it suffices to construct it on the k-th component $\coprod_{i=1}^r \sigma_k\,(K)\,\widetilde{w_i}$. Let us define $q_k : \coprod_{i=1}^r \sigma_k(K)\widetilde{w_i} \times \coprod_{i=1}^r \sigma_k(K)\widetilde{w_i} \to \mathbb{R}$ by

$$q_k(U,V) := \left\{ \begin{array}{ll} \sum_{i=1}^r u_i v_i & k = 1, 2, \ldots, r_1, \\ \sum_{i=1}^r (\overline{u_i} v_i + u_i \overline{v_i}) & k = r_1+1, \ldots, r_1+r_2. \end{array} \right. \tag{3.16}$$

where $U = (u_i\widetilde{w_i})_i$ and $V = (v_i\widetilde{w_i})_i$. Then we define

$$q : \coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i} \times \coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i} \to \mathbb{R}, \quad q(U,V) := \sum_{k=1}^{r_1+r_2} q_k(\mathrm{pr}_k(U), \mathrm{pr}_k(V)). \tag{3.17}$$

Let

$$\mathcal{G} := \{\varphi \in \mathbb{R}^{\mathcal{C}} \mid \; \frac{\varphi(w)}{\|w\|} \in K \text{ for all } w \in \mathcal{C}, \; \sum_{i=1}^{r} \|w_i\|^{-1}\varphi(w_i)\,\widetilde{w_i} = 0 \;\}.$$

We call $\mathcal{G}$ the *space of relations over K between the generators $w_1, \ldots, w_r$*. The space $\mathcal{G}$ can be identified with a subspace of $K^{\mathcal{C}} \simeq K^r$, therefore with a subspace of $\coprod_{i=1}^{r} K\widetilde{w_i}$. Denote by G its image by $\coprod_{i=1}^{r} \Sigma_i$ in $\coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i}$ and by $\overline{G}$, resp. $\overline{\mathcal{G}}$, the closure of G, resp. of $\mathcal{G}$. For all $x \in \mathbb{R}[\Lambda]$ and all $\varphi \in \mathcal{G}$,

$$\langle x, \sum_{i=1}^{r} \|w_i\|^{-1}\varphi(w_i)\,\widetilde{w_i}\rangle = \sum_{i=1}^{r} \|w_i\|^{-1}\varphi(w_i)\,\langle x, \widetilde{w_i}\rangle = 0. \tag{3.18}$$

(3.18) implies that $q_1(\ (\|w_i\|^{-1}\varphi(w_i)\widetilde{w_i}\ )_i,\ (\langle x, \widetilde{w_i}\rangle\widetilde{w_i}\ )_i) = 0$. Then the two subspaces $\mathbb{R}[\lambda]$ and $\overline{\mathcal{G}}$ are orthogonal and complementary in $\coprod_{i=1}^{r} \mathbb{R}\,\widetilde{w_i} \simeq \mathbb{R}^r$, of respective dimensions s and $r-s$.

Let us prove that R_K and $\overline{G}$ are orthogonal and complementary, of respective dimension sd and $(r-s)d$, in $\coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i}$. It suffices to prove $q_k(U,V) = 0$ with $U = (\sigma_k(\|w_i\|^{-1}\varphi(w_i))\widetilde{w_i}\ )_i$ and $V = (\sigma_k(\langle x, \widetilde{w_i}\rangle)\,\widetilde{w_i}\ )_i$, for all $k = 2, 3, \ldots, d$, $x \in \mathbb{R}[\Lambda]$ K-rational and $\varphi \in \mathcal{G}$. We have

$$\frac{1}{2}\, q_k(U,V) = Re\left[\sum_{i=1}^{r} \overline{\sigma_k\left(\|w_i\|^{-1}\varphi(w_i)\right)}\,\sigma_k\left(\langle x, \widetilde{w_i}\rangle\right)\right] =$$

$$Re\left[\sigma_k\left(\sum_{i=1}^{r} \overline{\|w_i\|^{-1}\varphi(w_i)}\,\langle x, \widetilde{w_i}\rangle\right)\right] = Re\left[\sigma_k\left(\langle x, \sum_{i=1}^{r} \|w_i\|^{-1}\varphi(w_i)\,\widetilde{w_i}\rangle\right)\right] = 0.$$

We deduce the claim.

The cut-and-project scheme above Λ we have constructed is the following:

$$\left(\coprod_{i=1}^{r} K_{\mathbb{R}}\widetilde{w_i}\ \simeq\ \overline{G} \times R_K\ \simeq\ H \times \mathbb{R}[\Lambda], L, \pi, \mathrm{pr}_1\right)$$

where $L = \coprod_{i=1}^{r} \Sigma_i(\mathcal{O}_K)$ is a lattice in $\overline{G} \times R_K$ and pr_1 such that $\mathrm{pr}_1(R_K) = \mathbb{R}[\Lambda]$, $\mathrm{pr}_1(\overline{G}) = 0$. Because of the structure of the $\mathbb{Z}$-module $\mathbb{Z}[\Lambda - \Lambda]$ given by (ii), it suffices to take $L = \coprod_{i=1}^{r} \Sigma_i(\mathbb{Z}[\lambda])$. The projection mapping π is $\mathrm{Id}-\mathrm{pr}_1$. The internal space, say H, is $\overline{G} \times (R_K \setminus \mathbb{R}[\Lambda])$. By construction, $\pi(L)$ is dense in H and pr_1 is one-to-one on L, onto $\mathrm{pr}_1(L) = \mathbb{Z}[\lambda][w_1, w_2, \ldots, w_r] \supset \mathbb{Z}[\Lambda - \Lambda]$. Note that $\mathbb{Z}[\Lambda - \Lambda]$ is not necessarily a free $\mathbb{Z}[\lambda]$-module, but it is of finite index in $\mathrm{pr}_1(L) = \mathbb{Z}[\lambda][w_1, w_2, \ldots, w_r]$. The Euclidean structure on the cut-and-project scheme given by q is such that $\mathbb{R}[\Lambda]$ and H are orthogonal. The component $R_K \setminus \mathbb{R}[\Lambda]$ of the internal space H is called the *shadow space* in [70].

If the vectors $w_1, w_2, \ldots, w_r$ are K-linearly independent (case (iv-1)) then $\overline{G}$ is trivial and the internal space H is $R_K \setminus \mathbb{R}[\Lambda]$.

This cut-and-project scheme lies above Λ since, for all $\nu \in \Lambda$, $\Lambda - \nu \subset \Lambda - \Lambda \subset \mathbb{Z}[\Lambda - \Lambda] \subset \mathrm{pr}_1(L)$. We deduce the claim.

4 Proof of Corollary 1.2 - Ideal lattices

The objectives of this section are the following: (i) to recall some definitions concerning ideal lattices, referring to [13] [14] [15] [16] [17], (ii) to show that the sublattice (L', q) of (L, q) such that $\mathrm{pr}_1(L') = \mathbb{Z}[\Lambda - \Lambda]$ in the cut-and-project scheme above the SFU - set Λ given by Theorem 1.1 (iv) is a sublattice of an ideal lattice.

It will suffice to show that the canonical bilinear form q defined by (3.17) has suitable properties.

The *canonical involution* (or complex conjugation) of the algebraic number field K generated by the self-similarity λ is the involution $^{-} : K_{\mathbb{R}} \to K_{\mathbb{R}}$ that is the identity on $\mathbb{R}^{r_1}$ and complex conjugation on $\mathbb{C}^{r_2}$. Let $\mathcal{P} := \{\alpha \in K_{\mathbb{R}} \mid \overline{\alpha} = \alpha$ and all components of α are $> 0\}$. Let us denote by $\mathrm{Tr} : K_{\mathbb{R}} \to \mathbb{R}$ the trace map, i.e. $Tr(x_1, x_2, \ldots, x_d) = x_1 + \ldots + x_d$.

A *generalized ideal* will be by definition a sub $\mathcal{O}_K$-module of K-rank one of $K_{\mathbb{R}}$. As examples, fractional ideals of K are generalized ideals; ideals of the type uI where I is an $\mathcal{O}_K$-ideal and $u \in K_{\mathbb{R}}$ are also generalized ideals.

Proposition 4.1. *Let $b : K_{\mathbb{R}} \times K_{\mathbb{R}} \to \mathbb{R}$ be a symmetric bilinear form. The following statements are equivalent:*

(i) *there exists $\alpha \in K_{\mathbb{R}}$ with $\alpha = \overline{\alpha}$ such that*

$$b(x, y) \;=\; Tr(\alpha\, x\overline{y})$$

for all $x, y \in K_{\mathbb{R}}$,

(ii) *the identity*

$$b(\mu x, y) \;=\; b(x, \overline{\mu} y)$$

holds for all $x, y, \mu \in K_{\mathbb{R}}$.

Proof. [13] Proposition 2.1, [14] Proposition 1. □

An *ideal lattice* is a lattice (I, b) where I is a generalized ideal, and $b : K_{\mathbb{R}} \times K_{\mathbb{R}} \to \mathbb{R}$ which satisfies the equivalent conditions of Proposition 4.1 with $\alpha \in \mathcal{P}$. Ideal lattices with respect to the canonical involution correspond bijectively to Arakelov divisors of the number field K [13] [14].

It is easy to check that Proposition 4.1 is satisfied by the real symmetric bilinear form q defined by (3.16) and (3.17) in Section 3, where $\alpha = (\alpha_i)_{1 \leq i \leq d}$ with $\alpha_i = 1$ for all i. We assume $r = 1$, i.e. that the degree of the self-similarity λ is equal to the rank $\mathrm{rk}\,\Lambda$. The lattice L' given by Theorem 1.1 (ii) such that $\mathrm{pr}_1(L') = \mathbb{Z}[\Lambda - \Lambda]$ is of finite index in the $\mathcal{O}_K$-module $\Sigma(\mathcal{O}_K)\widetilde{w_1}$, of K-rank one in $K_{\mathbb{R}}\widetilde{w_1}$. We have:

$$q(\mu\, U, V) = q(U, \overline{\mu} V) \qquad \text{forall } U, V \in K_{\mathbb{R}}\widetilde{w_1} \quad \text{and all } \mu \in K_{\mathbb{R}}.$$

Then, by Proposition 4.1, the bilinear form q has the following expression:

$$q(U,V) = \mathrm{Tr}(\alpha\, U\overline{V}) \qquad \text{forall } U,V \in K_{\mathbb{R}}\widetilde{w_1}. \tag{4.1}$$

Therefore (L', q) is a sublattice of finite index of an Arakelov divisor of K in bijection with $\mathbb{Z}[\Lambda - \Lambda]$ by the projection mapping $\mathrm{pr}_{1_{|_L}}$. We deduce Corollary 1.2.

The construction of the real bilinear form q in Theorem 1.1 (iv) which provides the Euclidean structure to the cut-and-project scheme is obtained with $\alpha = (1)_{1\leq i\leq d}$ in Proposition 4.1. Other choices of α are possible and the parametrization of the set of possible constants $\alpha = \alpha(\widetilde{w_1})$ in q in (4.1) is studied for instance in Schoof ([99] and related works); see also Neukirch [88].

5 Lower bounds of densities and pseudo-Delone constants

5.1 Pseudo-Delone sphere packings

The following definition is inspired by the "empty sphere" method of Delone [37]. If $A \subset \mathbb{R}^n$ is any nonempty subset of $\mathbb{R}^n$ and Λ is a uniformly discrete set of constant $r > 0$, we define the density of $\mathcal{B}(\Lambda)$ in A by

$$\delta_A(\mathcal{B}(\Lambda)) := \limsup_{t\to+\infty} \frac{\mathrm{vol}(\bigcup_{z_i\in\Lambda, \|z_i\|\leq t} B(z_i, r/2) \cap A)}{\mathrm{vol}(B(0,t)\cap A)}. \tag{5.1}$$

We omit the subscript "$\mathbb{R}^n$" when $A = \mathbb{R}^n$.

Definition 5.1. A uniformly discrete set Λ of $\mathbb{R}^n, n \geq 1$, of constant $r > 0$ is *pseudo-Delone* of constant $R_\xi > 0$ when there exists a sequence $\xi := (x_i, T_i)_i$ where $(x_i)_i$ is a sequence of points of $\mathbb{R}^n$ and $(T_i)_i$ a sequence of real numbers such that, with the notation $A_\xi := \mathbb{R}^n \setminus \bigcup_i \overset{\circ}{B}(x_i, T_i)$:

(i) $\forall i,\ T_{i+1} \geq T_i$, with $\ T_i \geq r/2$,

(ii) $\forall i,\ \overset{\circ}{B}(x_i, T_i) \cap \mathcal{B}(\Lambda) = \emptyset$,

(iii) $\forall x \in A_\xi$, $\exists \lambda \in \Lambda$ such that $\|x - \lambda\| \leq R_\xi$,

(iv) $\lim_{T\to+\infty} \frac{\mathrm{vol}(A_\xi \cap B(0,T))}{\mathrm{vol}(B(0,T))} = 1$.

If it is finite, the infimum $\inf\{R_\xi\}$ over all possibilites of point sets $(x_i)_i$ in $\mathbb{R}^n$ and collections of radii $(T_i)_i$, such that (i) to (iv) are satisfied, is called the *pseudo-Delone constant* of Λ and denoted by $R(\Lambda)$. Let us call *optimal* a collection ξ such that R_ξ is equal to $R(\Lambda)$.

By Zorn's Lemma, optimal collections exist. The portion of space $\cup_i \overset{\circ}{B}(x_i, T_i)$ defined by an optimal collection is an invariant, independent of the optimal collection used for defining it. Definition 5.1 means that we can remove the portion of ambient space which does not intervene at infinity for the computation of the density of Λ (in $\mathbb{R}^n$). Note that, for a SFU - set of $\mathbb{R}^n$, this portion of space does not contribute to the determination of the generators w_i in Theorem 1.1 since $\bigcup_i \overset{\circ}{B}(x_i, T_i)$ contains no point of Λ, and it is legitimate to remove it.

5.2 Proof of Theorem 1.3

Let $R_c := \inf\{R(\Lambda) \mid \Lambda \text{ is uniformly discrete of } \mathbb{R}^n \text{ of constant } 1\}$ be the infimum of possible Delone constants over sphere packings of common radius $1/2$. R_c is only a function of n. Then, for all $r > 0$, rR_c is the infimum of Delone constants of uniformly discrete sets of constant r.

Let $r > 0$ and ω_n be the volume of the unit ball of $\mathbb{R}^n$. Let $R \geq rR_c$ and $T > R$ be a real number. Let Λ be a uniformly discrete set of $\mathbb{R}^n$ of constant $r > 0$ which is pseudo-Delone of pseudo-Delone constant R. Let $\xi := (x_i, T_i)_i$ be an optimal sequence and $A_\xi := \mathbb{R}^n \setminus \cup_i \overset{\circ}{B}(x_i, T_i)_i$. For all $\epsilon > 0$, the pseudo-Delone constant of Λ in A_ξ is smaller than $R + \epsilon$. Then $(B(0, R+\epsilon) + \Lambda) \cap B(0, T)$ covers the set $B(0, T - R - \epsilon) \cap A_\xi$. The number of elements of $\Lambda \cap B(0, T)$ is equal to the number of elements of $\Lambda \cap B(0, T) \cap A_\xi$. This number is at least

$$\frac{\omega_n (T - R - \epsilon)^n - \mathrm{vol}((\mathbb{R}^n \setminus A_\xi) \cap B(0, T - R - \epsilon))}{\omega_n (R + \epsilon)^n}$$

$$= \frac{(T - R - \epsilon)^n}{(R + \epsilon)^n} \left(1 - \frac{\mathrm{vol}((\mathbb{R}^n \setminus A_\xi) \cap B(0, T - R - \epsilon))}{\omega_n (T - R - \epsilon)^n} \right).$$

On the other hand, since all the balls of radius $r/2$ centered at the elements of $\Lambda \cap B(0, T)$ lie within $B(0, T + r/2)$ and also within A_ξ, the proportion of space they occupy in $B(0, T + r/2) \cap A_\xi$ is at least

$$\left(\frac{T - R - \epsilon}{R + \epsilon} \right)^n \left(1 - \frac{\mathrm{vol}((\mathbb{R}^n \setminus A_\xi) \cap B(0, T - R - \epsilon))}{\omega_n (T - R - \epsilon)^n} \right) \frac{\mathrm{vol}(B(0, r/2))}{\mathrm{vol}(B(0, T + r/2) \cap A_\xi)} \tag{5.2}$$

But, for all $\epsilon > 0$,

$$\lim_{T \to +\infty} \frac{\mathrm{vol}((\mathbb{R}^n \setminus A_\xi) \cap B(0, T - R - \epsilon))}{\omega_n (T - R - \epsilon)^n} = 0$$

$$\text{and} \qquad \lim_{T \to +\infty} \frac{\mathrm{vol}(B(0, T + r/2))}{\mathrm{vol}(B(0, T + r/2) \cap A_\xi)} = 1.$$

Hence, if T is large enough, the quantity (5.2) is greater than

$$\left(\frac{r(T-R-\epsilon)}{2(R+\epsilon)(T+r/2)}\right)^n.$$

When T tends to infinity, this quantity tends to $(2(R+\epsilon)/r)^{-n}$, for all $\epsilon > 0$, which is a lower bound of $\delta(\mathcal{B}(\Lambda))$. We deduce the claim.

6 Lower bounds of the Delone constant of a SFU - set

The field $K = \mathbb{Q}(\lambda)$ generated by the self-similarity λ of the SFU - set Λ in Theorem 1.1 has its own Euclidean spectrum [26] [27] which leads to specific geometric properties of the Voronoi cell of the lattice L' [33] of the cut-and-project scheme above Λ, where L' such that $\mathrm{pr}_1(L') = \mathbb{Z}[\Lambda - \Lambda]$. By projection by pr_1, in this cut-and-project scheme above Λ, the Delone (or pseudo-Delone) constant of Λ, whatever the occupation of the elements of $\Lambda - \Lambda$ in $\mathbb{Z}[\Lambda - \Lambda]$, reflects the arithmetical features of K (Euclidean minimum, Euclidean spectrum ... [26] [27]) as well as the geometrical characteristics of the central cluster $\{w_1, \ldots, w_r\}$; in particular if Λ is a model set (see Proposition 2.10) or a Meyer set (see Definition 2.15 (iii)), since, in both cases, a window in the internal space controls the thickness of the band around the $\mathbb{R}$-span of Λ which is used for selecting the points of the lattice L'. Recall that the Delone (or pseudo-Delone) constant $R(\Lambda)$ of the SFU - set Λ, if finite, "measures" the maximal size of (spherical) holes in Λ with respect to the portion of space where the density is computed (see §2.1.1, §5 and [86]) In the sequel, we recall these notions and refer to [13] [16] [17] [26] [27] [33] [74].

In the following we will only consider the case $r = 1$, i.e. the case where the degree of the self-similarity λ is equal to the rank of $\mathbb{Z}[\Lambda - \Lambda]$, leaving aside the case $r > 1$. Let us give the name of "spanning self-similarity" to a self-similarity for which $r = 1$. Let us observe that Proposition 2.18 gives answers in the case of a lattice and a spanning self-similarity.

Theorem 6.1, resp. Theorem 6.2, Corollary 6.3 and Theorem 6.4, is a reformulation in the present context of Theorem 3, resp. Theorem 5, Corollary 6 and Theorem 4 (Remark 2), obtained by Cerri [26]. Recall that, for all $\xi \in K$, $\Sigma_1(\xi) = \Sigma(\xi)\widetilde{w_1}$, with $\widetilde{w_1} = \|w_1\|^{-1} w_1$ the unit vector.

6.1 Euclidean and inhomogeneous spectra of the number field generated by the self-similarity

Let $N_{K/\mathbb{Q}}$ be the norm defined on K by

$$\forall \xi \in K, \quad N_{K/\mathbb{Q}}(\xi) = \prod_{i=1}^{d} \sigma_i(\xi) = \prod_{i=1}^{r_1} \sigma_i(\xi) \prod_{i=r_1+1}^{r_1+r_2} |\sigma_i(\xi)|^2 . \tag{6.1}$$

The field K is said to be *norm-Euclidean* if:

$$\forall \xi \in K, \ \exists y \in \mathcal{O}_K \text{ such that } \left|N_{K/\mathbb{Q}}(\xi - y)\right| < 1.$$

Following the notations of Section 3 and [26] [27], we extend $N_{K/\mathbb{Q}} \circ \Psi$ from $\mathbb{Q}^d$ to $\mathbb{R}^d$ by the map denoted by $\mathcal{N}$ as follows:

$$\forall x \in \mathbb{R}^d, \quad \mathcal{N}(x) = \prod_{i=1}^{d} \left(\sum_{j=1}^{d} x_j \sigma_i(g_j) \right). \tag{6.2}$$

Let $\xi \in K$. The *Euclidean minimum of* ξ (relatively to the norm $N_{K/\mathbb{Q}}$) is the real number $m_K(\xi) := \inf\{ \, |N_{K/\mathbb{Q}}(\xi - y)| \ \mid y \in \mathcal{O}_K\}$. The *Euclidean minimum of* K (for the norm $N_{K/\mathbb{Q}}$) is denoted by $M(K)$ and is by definition:

$$M(K) := \sup_{\xi \in K} m_K(\xi). \tag{6.3}$$

The mapping $m_K \circ \Psi$ defined on $\mathbb{Q}^d$ is extended to $\mathbb{R}^d$ and is denoted by m:

$$m(z) := \inf\{ \, |\mathcal{N}(z - l)| \ \mid l \in \mathbb{Z}^d\} \qquad \text{for } z \in \mathbb{R}^d.$$

The *inhomogeneous minimum of* K is denoted by $M(\overline{K})$ and is defined by

$$M(\overline{K}) := \sup_{x \in \mathbb{R}^d} m(x). \tag{6.4}$$

The mapping $m_K \circ \Sigma_1^{-1}$ is extended to $K_{\mathbb{R}}\widetilde{w_1}$ and is denoted by $m_{\overline{K}}$:

$$m_{\overline{K}}(U) := \inf \left\{ \, \left|\prod_{i=1}^{d} (U_i - Z_i)\right| \ \mid Z = (Z_i)_i \widetilde{w_1} \in \Sigma(\mathcal{O}_k)\widetilde{w_1} \right\}$$

for $U = (U_i)_i \widetilde{w_1} \in K_{\mathbb{R}}\widetilde{w_1}$. The set of values of m_K, resp. of m, is called the *Euclidean spectrum*, resp. the *inhomogeneous spectrum of* K. Successive minima are enumerated: the *second inhomogeneous minimum of* K is defined by

$$M_2(\overline{K}) := \sup_{\substack{x \in \mathbb{R}^d \\ m(x) < M(\overline{K})}} m(x),$$

and the *second Euclidean minimum of* K by

$$M_2(K) := \sup_{\substack{\xi \in K \\ m_K(\xi) < M(K)}} m_K(\xi).$$

Iteratively we define ($p \geq 2$):

$$M_{p+1}(\overline{K}) := \sup_{\substack{x \in \mathbb{R}^d \\ m(x) < M_p(\overline{K})}} m(x),$$

and

$$M_{p+1}(K) := \sup_{\substack{\xi \in K \\ m_K(\xi) < M_p(K)}} m_K(\xi).$$

The inhomogeneous minimum $M(\overline{K})$ of K is said to be *isolated* if

$$M_2(\overline{K}) < M(\overline{K}).$$

This isolation phenomenon has been conjectured for $d = 2$ and K totally real by Barnes and Swinnerton-Dyer. Corollary 6.3 below shows that it occurs frequently.

If the inhomogeneous minimum $M(\overline{K})$ of K satisfies the following property:

$$\forall x \in \mathbb{R}^d,\ \exists l \in \mathbb{Z}^d \text{ such that } |\mathcal{N}(x - l)| \leq M(\overline{K}), \tag{6.5}$$

we will say that $M(\overline{K})$ is *attained*. Note that (6.5) is not verified for the quadratic field $K = \mathbb{Q}(\sqrt{13})$ [74].

Theorem 6.1. *Assume that the degree d of the field K generated by the self-similarity λ of the SFU - sphere packing Λ is ≥ 3 and is equal to the rank of $\mathbb{Z}[\Lambda - \Lambda]$. If the unit rank $r_1 + r_2 - 1$ of K is > 1, in particular if K is totally real, then*

(i) *there exists $\xi \in K$ such that*

$$M(\overline{K}) = m_{\overline{K}}(\Sigma_1(\xi)),$$

(ii)

$$M(K) = M(\overline{K}) \in \mathbb{Q}.$$

Proof. Theorem 3 in [26]. □

The question whether ξ is unique under some assumptions is not clear [26] [27].

Theorem 6.2. *Assume that the degree d of the field K generated by the self-similarity λ of the SFU - sphere packing Λ is ≥ 3 and is equal to the rank of $\mathbb{Z}[\Lambda - \Lambda]$. If the unit rank $r_1 + r_2 - 1$ of K is > 1 and if K is not a CM-field, in* *particular if K is totally real, there exists a strictly decreasing sequence $(y_p)_{p\geq 1}$ of positive rational integers, which satisfies:*

(i) $\lim_{p\to+\infty} y_p = 0$,

(ii) $m(\mathbb{R}^d) = \bigcup_{p\geq 1} \{y_p\}$,

(iii) *for each $p \geq 1$, the set $\{x+\mathbb{Z}^d \mid m(x) = y_p\}$ of classes modulo the lattice $\mathbb{Z}^d$ is finite and lifts up to points of $\mathbb{Q}^d$, i.e. $m(x) = 0$ for all $x \notin \mathbb{Q}^d$.*

Proof. Theorem 5 in [26]. □

From the definitions, the inequality $M(K) \leq M(\overline{K})$ holds for an arbitrary number field, with equality if $d = 2$ (Barnes and Swinnerton-Dyer [74]). Recently Cerri [26] (Corollary 3 of Theorem 3) proved that the equality $M(K) = M(\overline{K})$ does hold true for every number field.

Corollary 6.3. *Under the same hypotheses $M(\overline{K})$ is attained and*

(i) $M_p(K) = M_p(\overline{K})$ *for all $p > 1$,*

(ii) $M_2(\overline{K}) < M(\overline{K})$ ($M(\overline{K})$ *is isolated*),

(iii) $\forall p > 1,\ M_{p+1}(\overline{K}) < M_p(\overline{K})$ *and* $\lim_{p\to+\infty} M_p(\overline{K}) = 0$.

What are the possible fundamental regions of the sublattices L' of $L = \Sigma_1(\mathbb{Z}[\lambda])$ in $K_{\mathbb{R}}\widetilde{w_1}$?

Theorem 6.4. *Denote, for all $t > 0$,*

$$\mathcal{A}_t := \{U = (U_i)\widetilde{w_1} \in K_{\mathbb{R}}\widetilde{w_1} \mid \ \big|\prod_{i=1}^{d} U_i\big| \leq t\}.$$

If (unit rank) $r_1 + r_2 - 1 > 1$, then

$$K \text{ is norm-Euclidean} \iff \exists t \in (0,1) \text{ such that } \Sigma_1(\mathcal{O}_K) + \mathcal{A}_t = K_{\mathbb{R}}\widetilde{w_1}.$$

Proof. Remark 2 after Theorem 4 in [26]. □

Then, if the unit rank $r_1 + r_2 - 1$ of K is strictly greater than 1 and K is norm-Euclidean, there exists $t \in (0,1)$ such that the number of copies of $\mathcal{A}_t$ to be considered for obtaining the fundamental region of L' is equal to the index of L' in $\Sigma_1(\mathcal{O}_K)\widetilde{w_1}$, an integer multiple of $(\mathcal{O}_K : \mathbb{Z}[\lambda])$. Recall that ([26] Proposition 4):

(i) $M(K) < 1 \implies K$ is norm-Euclidean,

(ii) $M(K) > 1 \implies K$ is not norm-Euclidean,

If $M(K) = 1$, it is not possible to conclude except if there exists $\xi \in K$ such that $M(K) = m_K(\xi)$; in this case K is not norm-Euclidean. See [27] for computations of $M_p(K), p \geq 1$, with the conventions: $M(K) = M_1(K)$, $M(\overline{K}) = M_1(\overline{K})$.

6.2 Proof of Theorem 1.4

Let us now deduce lower bounds of the Delone constant of the SFU - set $\Lambda \subset \mathbb{R}^n, n \geq 1$. We will assume that Λ is a Meyer set (i.e. a Delone set that is a subset of a model set), defined by a window Ω (chosen minimal) in the internal space $\mathbb{R}_K \setminus \mathbb{R}[\Lambda]$ of the cut-and-project scheme

$$\big(K_{\mathbb{R}}\widetilde{w_1} = (\mathbb{R}_K \setminus \mathbb{R}[\Lambda]) \times \mathbb{R}[\Lambda], L', \pi, \mathrm{pr}_1 \big) \tag{6.6}$$

given by Theorem 1.1 (iv) with L' such that $\mathrm{pr}_1(L') = \mathbb{Z}[\Lambda - \Lambda]$ and the window Ω nonempty, open and relatively compact such that

$$\Lambda \subset \nu + \{\mathrm{pr}_1(U) \mid U \in L', \pi(U) \in \Omega\} \subset \nu + \mathbb{R}\,\widetilde{w_1} \qquad \text{with } \nu \in \Lambda. \tag{6.7}$$

Λ is a Delone set. The central cluster is $\{w_1\}$. Denote by $\mathbb{R}[\Lambda] + \Omega$ the band $\{P = (U, V) \in K_{\mathbb{R}}\widetilde{w_1} \mid U = \mathrm{pr}_1(P) \in \mathbb{R}[\Lambda], V = \pi(P) \in \Omega\}$ parallel to the one-dimensional $\mathbb{R}$-span $\mathbb{R}[\Lambda]$ of Λ. Note that $\mathbb{R}[\Lambda] = \nu + \mathbb{R}\,\widetilde{w_1}$ in $\mathbb{R}^n$.

In the sequel we will use the notations of Subsection 6.1 and the assumptions of Theorem 6.2.

Definition 6.5. Let $k \geq 2$ be an integer. The self-similar finitely generated Delone set Λ defined by (6.6) and (6.7) is called

(i) *thin* if the following condition on Ω and L' holds:

$$0 < \|\pi\big(\Sigma_1(x - t)\big)\| < d\, M(K)^{2/d}, \tag{6.8}$$

for all $t \in \mathcal{O}_K$ such that $\Sigma_1(t) \in L' \cap \big(\mathbb{R}[\Lambda] + \Omega\big)$, and all $x \in m_K^{-1}(M(K))$,

(ii) *k-thin* if

$$d\,(M_{k+1}(K))^{2/d} \leq \|\pi\big(\Sigma_1(x - t)\big)\| < d\,(M_k(K))^{2/d} \tag{6.9}$$

for all $t \in \mathcal{O}_K$ such that $\Sigma_1(t) \in L' \cap \big(\mathbb{R}[\Lambda] + \Omega\big)$ and $x \in \bigcup_{p=1}^{k-1} m_K^{-1}(M_p(K))$.

This definition is consistent with the following facts:

(a) the values of m constitute a strictly decreasing sequence of positive rational integers (Theorem 6.2 (ii)) which are reached a finite number of times modulo L' (Theorem 6.2 (iii)),

(b) the infinite (band) cylinder $\mathbb{R}[\Lambda] + \Omega$, parallel to the one-dimensional space $\mathbb{R}[\Lambda]$, can be made sufficiently narrow in order to avoid the set of points z of $\mathbb{Q}^d$ such that $m_K \circ \Psi(z) \in \bigcup_{p=1}^{k-1} M_p(K)$, for all $k \geq 2$, which is a finite union of translates of L',

(c) Assertion (b) is possible since *free planes*, a fortiori free lines, do exist in any lattice (equal) sphere packings in $\mathbb{R}^d$ once d is large enough: Henk [61] has proved the existence of an $\frac{d}{\log_2(d)}$-dimensional affine plane (called free plane) which does not meet any of the spheres in their interiors. Hence, provided d is large enough, free lines exist in the lattice sphere packing $\mathcal{B}(L')$ corresponding to L' in $K_{\mathbb{R}}\widetilde{w_1}$. Equivalently narrow bands, with section a nonempty open set, about free lines, exist in $K_{\mathbb{R}}\widetilde{w_1}$ which do not intersect L'. By continuity, there exist narrow bands, with nonempty open cross-sections, about free lines, which do not intersect any finite union of translates of L'.

For all $x \in K, t \in \mathcal{O}_K$, by the geometric mean inequality, we have:

$$|N_{K/\mathbb{Q}}(x)|^2 = |N_{K/\mathbb{Q}}(x-t)|^2 = \prod_{i=1}^{r_1} |\sigma_i(x-t)|^2 \prod_{i=r_1+1}^{r_1+r_2} |\sigma_i(x-t)|^4$$

$$\leq \left(\frac{1}{d}\sum_{i=1}^{r_1} |\sigma_i(x-t)|^2 + \frac{2}{d}\sum_{i=r_1+1}^{r_1+r_2} |\sigma_i(x-t)|^2\right)^d = \left(\frac{1}{d}\, q\big(\Sigma_1(x-t), \Sigma_1(x-t)\big)\right)^d \tag{6.10}$$

where $q\big(\Sigma_1(x-t), \Sigma_1(x-t)\big) = \|\mathrm{pr}_1(\Sigma_1(x-t))\|^2 + \|\pi(\Sigma_1(x-t))\|^2$ with the notations of Section 3. We have:

$$\|\mathrm{pr}_1(\Sigma_1(x-t))\|^2 = \|(x-t)\widetilde{w_1}\|^2 = \|(\nu + x\widetilde{w_1}) - (\nu + t\widetilde{w_1})\|^2.$$

The set $m_K^{-1}(M(K)) = \{x \in K \mid m_K(x) = M(K) = M(\overline{K})\}$ is such that $\Sigma_1\big(m_K^{-1}(M(K))\big) = \{\Sigma_1(x) \mid m_K(x) = M(K) = M(\overline{K})\}$ is finite modulo L' by Theorem 6.1 and Theorem 6.2.

Let us take x in $m_K^{-1}(M(K))$. Then, from (6.10), for all $t \in \mathcal{O}_K$,

$$M(K)^2 \leq \left(\frac{1}{d}\left(\|(\nu + x\widetilde{w_1}) - (\nu + t\widetilde{w_1})\|^2 + \|\pi(\Sigma_1(x-t))\|^2\right)\right)^d.$$

Hence, for all $t \in \mathcal{O}_K$ such that $\Sigma_1(t) \in L' \cap \big(\mathbb{R}[\Lambda] + \Omega\big)$,

$$\|(\nu + x\widetilde{w_1}) - (\nu + t\widetilde{w_1})\|^2 \geq d\, M(K)^{2/d} - \|\pi(\Sigma_1(x-t))\|^2.$$

Since by hypothesis $\nu + x\widetilde{w_1}$ does not belong to Λ and that we consider the elements $t \in \mathcal{O}_k$ for which $\nu + t\widetilde{w_1}$ belongs to Λ (with $\Sigma_1(t) \in L' \cap (\mathbb{R}[\Lambda] + \Omega)$), we have:

$$R(\Lambda) \;\geq\; \|(\nu + x\widetilde{w_1}) - (\nu + t\widetilde{w_1})\|.$$

Hence

$$R(\Lambda)^2 \;\geq\; d\, M(K)^{2/d} - \sup \|\pi(\Sigma_1(x-t))\|^2 > 0 \tag{6.11}$$

where the supremum is taken over all $x \in m_K^{-1}(M(K))$ and all $t \in \mathcal{O}_k$ for which $\nu + t\widetilde{w_1}$ belongs to Λ, with $\Sigma_1(t) \in L' \cap (\mathbb{R}[\Lambda] + \Omega)$.

Let us assume that Λ is k-thin and take x in $\bigcup_{p=1}^{k-1} m_K^{-1}(M_p(K))$. Then the supremum in (6.11) is bounded from above by $d\, M_k(K)^{\frac{2}{d}}$ which allows to deduce the claim.

Using Theorem 1.3 or [86] we deduce a lower bound of the density of the SFU - set Λ.

The assumption "d large enough", from fact (c) and [61], can be replaced by "$d \geq 3$" (after Theorem 6.2) since λ is a spanning self-similarity and that it is easy to check that $\mathbb{Z}^3 + B(0, 1/2)$, a fortiori $\mathbb{Z}^n + B(0, 1/2)$, already contains a free line.

The isolation phenomenon which frequently occurs (Corollary 6.3 (ii) of Theorem 6.2) in higher dimension is likely to occur in $\mathbb{R}^n$ as well by projection for Λ.

Appendix.— Crystallography of Aperiodic Crystals and Delone sets

New aperiodic states of matter call for mathematical idealizations of packings of atoms. A deep understanding of the mathematics which lies behind the usual experimental techniques such as: diffraction (X-rays, electrons, neutrons, synchrotron radiation, etc) and inverse problems (crystal reconstruction with satisfying local atom clustering, long-range order and self-similarities, etc), adapted to this new context, is required. Indeed, the situation is well-known for (periodic) crystals [34] [57] [58] [68] [102] but fairly unknown, or at least badly understood for nonperiodic crystals. Quasicrystals and modulated crystals constitute exceptions since the use of cut-and-project sets allows periodization in higher dimension [3] [4] [55] [67] [69] [102]. The parts of mathematics concerned with the crystallography of aperiodic crystals are mainly Geometry of Numbers and Discrete Geometry [25] [56] [113], N-dimensional crystallography when periodization in higher dimension is concerned [83] [90] [101], Spectral Theory, Ergodic Theory and Fourier Transform of Delone sets as far as diffraction is concerned [5] [53] [63] [64] [109], Harmonic Analysis as far as density is concerned (as an asymptotic measure). Atoms are viewed as *hard spheres* and aperiodic crystals as Delone sets (sphere centers). Implicitely this means that atoms behave like spheres, that is do have a spherical potential. This is far from covering the large variety of possibilities of chemical boundings between atomic species (see [112] for quasicrystalline models of pure Boron for instance). This provides first-order crystalline models from which the computation of the electron density is made possible. Then comparison with experimental data (densities, physical properties, ...) leads to refinement of the models.

Looking for a detailed hierarchy of Delone sets, from the mathematical side, either from arithmetics [40] [50] [70] or from tiling theory [6] [10] [85] [108], leads to interesting and new questions concerning crystals, without knowing whether these crystals will exist or not. To finish up let us recall the new definition of a crystal (in $\mathbb{R}^3$) which was recently chosen by the International Union of Crystallography [66] and the former one [101].

Definition 6.6 (former definition). Any solid for which the set of atom positions is a finite union of orbits under the action of a crystallographic group.

Definition 6.7 (new definition). Any solid having an essentially discrete diffraction diagram.

Definition 6.7 covers all cases of solids defined by Definition 6.6 by Poisson formula (see [72] for a proof).

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“Geometric Study of the Beta-integers for a Perron Number and Mathematical Quasicrystals”,

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J. de Théorie des Nombres de Bordeaux 16 (2004), 125–149.

Journal de Théorie des Nombres
de Bordeaux **16** (2004), 1–25

Geometric study of the beta-integers for a Perron number and mathematical quasicrystals

par Jean-Pierre GAZEAU et Jean-Louis VERGER-GAUGRY

Résumé. Nous étudions géométriquement les ensembles de points de $\mathbb{R}$ obtenus par la β-numération que sont les β-entiers $\mathbb{Z}_\beta \subset \mathbb{Z}[\beta]$ où β est un nombre de Perron. Nous montrons qu'il existe deux schémas de coupe-et-projection canoniques associés à la β-numération, où les β-entiers se relèvent en certains points du réseau $\mathbb{Z}^m$ (m = degré de β), situés autour du sous-espace propre dominant de la matrice compagnon de β. Lorsque β est en particulier un nombre de Pisot, nous redonnons une preuve du fait que $\mathbb{Z}_\beta$ est un ensemble de Meyer. Dans les espaces internes les fenêtres d'acceptation canoniques sont des fractals dont l'une est le fractal de Rauzy (à quasi-homothétie près). Nous le montrons sur un exemple. Nous montrons que $\mathbb{Z}_\beta \cap \mathbb{R}^+$ est de type fini sur $\mathbb{N}$, faisons le lien avec la classification de Lagarias des ensembles de Delaunay et donnons une borne supérieure effective de l'entier q dans la relation : $x, y \in \mathbb{Z}_\beta \implies x + y$ (respectivement $x - y$) $\in \beta^{-q}\mathbb{Z}_\beta$ lorsque $x + y$ (respectivement $x - y$) a un β-développement de Rényi fini.

Abstract. We investigate in a geometrical way the point sets of $\mathbb{R}$ obtained by the β-numeration that are the β-integers $\mathbb{Z}_\beta \subset \mathbb{Z}[\beta]$ where β is a Perron number. We show that there exist two canonical cut-and-project schemes associated with the β-numeration, allowing to lift up the β-integers to some points of the lattice $\mathbb{Z}^m$ ($m =$ degree of β) lying about the dominant eigenspace of the companion matrix of β. When β is in particular a Pisot number, this framework gives another proof of the fact that $\mathbb{Z}_\beta$ is a Meyer set. In the internal spaces, the canonical acceptance windows are fractals and one of them is the Rauzy fractal (up to quasi-dilation). We show it on an example. We show that $\mathbb{Z}_\beta \cap \mathbb{R}^+$ is finitely generated over $\mathbb{N}$ and make a link with the classification of Delone sets proposed by Lagarias. Finally we give an effective upper bound for the integer q taking place in the relation: $x, y \in \mathbb{Z}_\beta \implies x + y$ (respectively $x - y$) $\in \beta^{-q}\mathbb{Z}_\beta$ if $x + y$ (respectively $x - y$) has a finite Rényi β- expansion.

Manuscrit reçu le 7 mai 2002.

1. Introduction

Gazeau [Gaz], Burdik et al [Bu] have shown how to construct a discrete set $\mathbb{Z}_\beta \subset \mathbb{Z}[\beta] \subset \mathbb{R}$ which is a Delone set [Mo], called set of β-integers (or beta-integers), when $\beta > 1$ is a Pisot number of degree greater than 2. A beta-integer has by definition no fractional part in its Rényi β-expansion [Re] [Pa]. As basic feature, this Delone set is self-similar, namely $\beta\,\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$.

Since the general notion of β-expansion of real numbers (see section 2 for definitions) was created by Rényi for any real number $\beta > 1$, the set of beta-integers $\mathbb{Z}_\beta$, defined as the set of real numbers equal to the integer part of their β-development, is defined without ambiguity in full generality and is self-similar by construction: $\beta\,\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$. The main questions we may address are the following: (Q1) For which $\beta > 1$ is $\mathbb{Z}_\beta$ a Delone set ? or equivalently (Q1') for which $\beta > 1$ is $\mathbb{Z}_\beta$ a uniformly discrete set ? since the sets $\mathbb{Z}_\beta$ of beta-integers are always relatively dense by construction. Now Delone sets are classified into several types (see the definitions in the Appendix) so that the following question is also fundamental: (Q2) For which class of $\beta > 1$ is $\mathbb{Z}_\beta$ a Delone set of a given type ?

The uniform discretness property of $\mathbb{Z}_\beta$ is a crucial property which is not obtained for all real number β, but very few general results are known nowadays. Thurston has shown that it is the case when β is a Pisot number [Th]. It is conjectured that it is also the case when β is a Perron number. Apart from the Pisot case, many open questions remain (Bertrand-Matthis [Be4], Blanchard [Bl]) and are expressed in terms of the β - shift. Schmeling [Sc] has proved that the class C_3 of real numbers $\beta > 1$ such that the Renyi-expansion $d_\beta(1)$ of 1 in base β contains bounded strings of zeros, but is not eventually periodic, has Hausdorff dimension 1. For all β in this class C_3, the β-shift is specified [Bl]. It is obvious that the specification of the β-shift is equivalent to the fact that $\mathbb{Z}_\beta$ is uniformly discrete. So that the class C_3 would contain all Perron numbers. The idea of exploring relationships between the β-shift and the algebraic properties of β in number theory is due to A. Bertrand-Matthis [Be3]. In this direction, some results are known (Akiyama [Ak] [Ak1]). Parry [Pa] has proved that the β-shift is sofic when β is a Pisot number. Lind [Li] conversely has shown that β is a Perron number if the β-shift is sofic.

In section 2 we will recall some basic facts about the β-numeration and the beta-integers. In section 3, we will establish the geometrical framework which is attached to the algebraic construction of the set of the beta-integers when β is a Perron number in general (of degree $m \geq 2$). Namely, by geometric framework, we mean that we will show the existence of two cut-and-project schemes (see the definitions in the Appendix) embedded in a canonical way in the Jordan real decomposition of $\mathbb{R}^m$ where this decomposition is obtained by the action of the companion matrix of β, respectively of its adjoint, the second cut-and-project scheme being the dual of the first one. This will be done without invoking any substitution system on a finite alphabet [AI] or the theory of Perron-Frobenius [Mi]. These cut-and-project schemes will consist of an internal space which will be an hyperplane of $\mathbb{R}^m$ complementary to a one-dimensional line on which the set of β-integers will be set up in a natural way, together with the usual lattice $\mathbb{Z}^m$ in $\mathbb{R}^m$. The constituting irreducible subspaces of the internal spaces will appear by construction as asymptotic linear invariants. This will allow us to deduce several results when β is a Pisot number: a minimal acceptance window in the internal space closely related to the Rauzy fractal, a geometrical proof that $\mathbb{Z}_\beta$ is a Meyer set, the fact that $\mathbb{Z}_\beta$ is finitely generated over $\mathbb{N}$. We will make a link on an example with the Rauzy fractal when the beta-integers arise from substitution systems of Pisot type (for instance Rauzy [Ra], Arnoux and Ito [AI], Messaoudi [Me] [Me1], Ito and Sano [IS], Chap. 7 in Pytheas Fogg [PF]). At this point, we should outline that the main difference with the substitutive approach is that the matrices involved may have negative coefficients (compare with the general approach of Akiyama [Ak] [Ak1]).

The additive properties of $\mathbb{Z}_\beta$ will be studied in section 4 by means of the canonical cut-and-project schemes when β is a Pisot number: in A), we shall show that the elements of $\mathbb{Z}_\beta \cap \mathbb{R}^+$ can be generated over $\mathbb{N}$ by elements of $\mathbb{Z}_\beta$ of small norm, in finite number, using truncated cones whose axis of revolution is the dominant eigenspace of the companion matrix of β and a Lemma of Lind [Li] on semigroups; in B), we will provide a geometrical interpretation of the maximal preperiod of the β-expansion of some real numbers coming from the addition of two beta-integers, of the finite sets T and T' in the relations [Bu] $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + T$, $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta + T'$ and an upper bound of the integer q taking place in the relation $x, y \in \mathbb{Z}_\beta^+ \implies x \pm y \in \beta^{-q}\mathbb{Z}_\beta$ when $x + y$ and $x - y$ have finite β-expansions.

2. Beta-numeration and beta-integers

Let $\beta \in (1, +\infty) \setminus \mathbb{N}$. We will refer in the following to Rényi [Re], Parry [Pa] and Frougny [Fro] [Fro1] [Bu]. For all $x \in \mathbb{R}$ we will denote by $\lfloor x \rfloor$, resp. $\{x\} = x - \lfloor x \rfloor$, the usual integer part of x, resp. its fractional part. Let us denote by $T(x) = \{\beta x\}$ the ergodic transformation sending $[0, 1]$ into itself. For all $x \in [0, 1]$, the iterates $T^n(x) := T(T^{n-1}(x)), n \geq 1$, with $T^0 := Id$ by convention, provide the sequence $(x_{-i})_{i \geq 1}$ of digits, with $x_{-i} := \lfloor \beta T^{i-1}(x) \rfloor$, in the finite alphabet $A = \{0, 1, \cdots, \lfloor \beta \rfloor\}$. The element x is then equal to its Rényi β-expansion $\sum_{j=1}^{+\infty} x_{-j}\beta^{-j}$ also denoted by $0.x_{-1}x_{-2}x_{-3}\ldots$. The Rényi β-expansion of 1 will be denoted by $d_\beta(1)$. The operator T on $[0,1]$ induces the shift $\sigma : (x_{-1}, x_{-2}, \ldots) \to (x_{-2}, x_{-3}, \ldots)$ on the compact set $A^{\mathbb{N}}$ (with the usual product topology). The closure of the subset of $A^{\mathbb{N}}$ invariant under σ takes the name of β-shift. The knowledge of $d_\beta(1)$ suffices to exhaust all the elements in the β-shift (Parry [Pa]). For this let us define the following sequence $(c_i)_{i\geq 1}$ in $A^{\mathbb{N}}$:

$$c_1 c_2 c_3 \cdots \quad = \begin{cases} t_1 t_2 t_3 \cdots & \text{if the Rényi } \beta- \text{ expansion } d_\beta(1) = 0.t_1 t_2 \cdots \text{ is infinite,} \\ (t_1 t_2 \cdots t_{r-1}(t_r - 1))^\omega & \text{if } d_\beta(1) \text{ is finite and equal to } 0.t_1 t_2 \cdots t_r, \end{cases}$$

where $(\ )^\omega$ means that the word within ( ) is indefinitely repeated. Then the sequence $(y_{-i})_{i\geq 1}$ in $A^{\mathbb{N}}$ is exactly the sequence of digits provided by the iterates of $y = \sum_{i=1}^{+\infty} y_{-i}\beta^{-i}$ by T^n if and only if the following inequalities are satisfied: $(y_{-n}, y_{-(n+1)}, \ldots) < (c_1, c_2, c_3, \ldots)$ for all $n \geq 1$ where " $<$ " means lexicographical smaller. These inequalities will be called conditions of Parry. We will now use finite subsets of the β-shift.

Definition 2.1. Let $\mathbb{Z}_\beta^+ = \{x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 \mid x_i \in A,\ k \geq 0, \text{ and } (x_j, x_{j-1}, \ldots, x_1, x_0, 0, 0, \cdots) < (c_1, c_2, \cdots) \text{ for all } j, 0 \leq j \leq k\ \}$ be the discrete subset of $\mathbb{R}^+$ of the real numbers equal to the integer part of their Rényi β-expansion. The set $\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup \left(-\mathbb{Z}_\beta^+\right)$ is called the set of β- integers.

For all $x \in \mathbb{R}^+$, if $x = \sum_{i=-\infty}^{p} x_i\beta^i$ with $p \geq 0$, is obtained by the greedy algorithm, then $(x_i)_{i\leq p}$ will satisfy the conditions of Parry. We will denote by $\mathrm{int}(x) = \sum_{i=0}^{p} x_i\beta^i$ the integer part of its Rényi β-expansion, respectively by $\mathrm{frac}(x) = \sum_{i=-\infty}^{-1} x_i\beta^i$ its fractional part. The element $1 = \beta^0$ belongs to $\mathbb{Z}_\beta^+$.

Let us now turn to the case where β is a positive real algebraic integer. Then there exists an irreducible polynomial $P(X) = X^m - \sum_{i=0}^{m-1} a_i X^i, a_i \in \mathbb{Z}$ with $m = \mathrm{degree}(\beta)$ such that $P(\beta) = 0$. Then $\beta = \sum_{i=0}^{m-1} a_{m-1-i}\beta^{-i}$. If $a_j \geq 0$ for all j and $(a_n, a_{n+1}, \ldots) < (a_{m-1}, a_{m-2}, \ldots, a_0, 0, 0, \ldots)$ for all $n \leq m-2$, then the Rényi β-expansion of β would be $\sum_{i=0}^{m-1} a_{m-1-i}\beta^{-i}$ from which we would deduce $d_\beta(1) = \sum_{i=0}^{m-1} a_{m-1-i}\beta^{-i-1}$ as well. But the coefficients a_i do not obey the conditions of Parry in general. More considerations on the relations between β-expansions and algebraicity can be found in [Be] [Be1] [Be2] [Be3] [Fro1] [Ak] [Ak1] [Sch]. Bertrand-Matthis [Be] and Schmidt [Sch] have proved that, when β is a Pisot number, $x \in \mathbb{Q}(\beta)$ if and only if the Rényi β-expansion of x is eventually periodic; in particular the Rényi β-expansion of any Pisot number is eventually periodic.

Let us recall that a Perron number β, resp. a Lind number, resp. a Salem number, will be a real algebraic integer $\beta > 1$ whose conjugates $\beta^{(i)}$ are of modulus strictly less than β, resp. of modulus less than β with at least one conjugate of modulus β [La], resp. of modulus less than 1 with at least one conjugate of modulus one. A Pisot number β will be a real algebraic integer $\beta > 1$ for which all the conjugates are in the open unit disc in the complex plane.

3. Canonical cut-and-project schemes over $\mathbb{Z}_\beta$

Assume that $\beta > 1$ is a Perron number of degree $m \geq 2$, dominant root of the irreducible polynomial $P(X) = X^m - a_{m-1}X^{m-1} - a_{m-2}X^{m-2} - \cdots - a_1X - a_0$, $a_i \in \mathbb{Z}, a_0 \neq 0$. All the elements $r\beta^k$ with $k \geq 1, r \in \{1, 2, \ldots, \lfloor\beta\rfloor\}$ are obviously in $\mathbb{Z}_\beta$. We are looking for asymptotic linear invariants associated with them, hence, by linearity, associated with the powers $\beta^k, k \geq 1$, of β, when k tends to infinity. By linearity, they will be also associated to the beta-integers. Let us set up the general situation. For all $k \geq 0$, write $\beta^k = z_{m-1,k}\beta^{m-1} + z_{m-2,k}\beta^{m-2} + \cdots + z_{1,k}\beta + z_{0,k}$, where all the integers $z_{0,k}, z_{1,k}, \cdots, z_{m-1,k}$ belong to $\mathbb{Z}$. Denote $Z_k = {}^t\left(z_{0,k}\ z_{1,k}\ z_{2,k}\ \ldots z_{m-1,k}\right)$, $B = B^{(0)} = {}^t\left(1\ \beta\ \beta^2\ \ldots \beta^{m-1}\right)$, $B^{(j)} = {}^t\left(1\ \beta^{(j)}\ {\beta^{(j)}}^2\ \ldots {\beta^{(j)}}^{m-1}\right)$, where ^{t} means transposition and the elements $\beta^{(j)}, j \in \{1, 2, \cdots, m-1\}$, are the conjugate roots of $\beta = \beta^{(0)}$ in the minimal polynomial of β . Set

$$\mathcal{B}_k = \begin{pmatrix} \beta^k \\ {\beta^{(1)}}^k \\ {\beta^{(2)}}^k \\ \vdots \\ {\beta^{(m-1)}}^k \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & & \cdots & 1 \\ a_0 & a_1 & & \cdots & a_{m-1} \end{pmatrix}$$

the $m \times m$ matrix with coefficients in $\mathbb{Z}$. The transposed matrix of Q is denoted by ${}^tQ$. It is the companion matrix of $P(X)$ (and of β). For all $p, k \in \{0, 1, \cdots, m-1\}$, we have: $z_{p,k} = \delta_{p,k}$ the Kronecker symbol. It is obvious that, for all $k \geq 0$, we have $Z_{k+1} = {}^tQ\ Z_k$. Denote

$$C = \begin{pmatrix} 1 & \beta & \beta^2 & \cdots & \beta^{m-1} \\ 1 & \beta^{(1)} & {\beta^{(1)}}^2 & \cdots & {\beta^{(1)}}^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \beta^{(m-1)} & {\beta^{(m-1)}}^2 & \cdots & {\beta^{(m-1)}}^{m-1} \end{pmatrix}$$

the Vandermonde matrix of order m. We obtain $C\ Z_k = \mathcal{B}_k$ by the real and complex embeddings of $\mathbb{Q}[\beta]$ since all the coefficients $z_{j,k}, j \in \{0, 1, \cdots, m-1\}$, are integers and remain invariant under the conjugation operation.

Theorem 3.1. *If V_1 denotes the vector defined by the first column of C^{-1}, then the limit $\lim_{k\to+\infty} \|Z_k\|^{-1} Z_k$ exists and is equal to the unit vector $u := \|V_1\|^{-1}V_1$. Moreover, all the components of V_1 are real and belong to the $\mathbb{Z}$ - module $\dfrac{\mathbb{Z}[\beta]}{\beta^{m-1}P'(\beta)}$.*

Proof. Since $P(X)$ is minimal, all the roots of $P(X)$ are distinct. Hence, the determinant of C is $\prod_{i<j}(\beta^{(i)} - \beta^{(j)})$ and is not zero. Let $C^{-1} = (\xi_{ij})$. Then $C \cdot C^{-1} = I$, that is
(1)

$$\xi_{1i} + \xi_{2i}\beta^{(j)} + \xi_{3i}\beta^{(j)^2} + \cdots + \xi_{mi}\beta^{(j)^{m-1}} = \delta_{i,j+1}, \begin{cases} i = 1, 2, \ldots, m, \\ j = 0, 1, \ldots, m-1 \end{cases}$$

On the other hand, the Lagrange interpolating polynomials associated with $\{\beta, \beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(m-1)}\}$ are given by

$$L_s(X) = \prod_{\substack{j=0 \\ j\neq s}}^{m-1} \frac{X - \beta^{(j)}}{\beta^{(s)} - \beta^{(j)}} \qquad s = 0, 1, \ldots, m-1.$$

For m arbitrary complex numbers $y_1, y_2, \cdots, y_m$, let us denote by $\sigma_r = \sigma_r(y_1, y_2, \cdots, y_m) = \sum_{1\leq i_1\leq i_2\leq\cdots\leq i_r\leq m} \prod_{j=1}^{r} y_{i_j}$ the r-th elementary symmetric function of the m numbers y_1, y_2, $\cdots$, y_m. The degree of $L_s(X)$ is $m-1$ and $L_s(X)$ can be expressed as

$$L_s(X) = \sum_{r=0}^{m-1} (-1)^r \sigma_r^{(s)} X^{m-r-1} / \prod_{\substack{r=0 \\ r\neq s}}^{m-1} (\beta^{(s)} - \beta^{(r)})$$

where $\sigma_r^{(s)} = \sigma_r(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, \beta^{(s+1)}, \cdots, \beta^{(m-1)})$ denotes the r-th elementary symmetric function of the $m-1$ numbers $\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}$, $\beta^{(s+1)}, \cdots, \beta^{(m-1)}$ where $\beta^{(s)}$ is missing. Since these polynomials satisfy $L_s(\beta^{(k)}) = \delta_{s,k}$ for all $s, k = 0, 1, \cdots, m-1$, comparing with (1), we obtain, by identification of the coefficients

$$\xi_{ji} = \frac{(-1)^{m-j}\sigma_{m-j}^{(i-1)}}{\prod_{\substack{r=0 \\ r\neq i-1}}^{m-1} (\beta^{(i-1)} - \beta^{(r)})} = \frac{(-1)^{m-j}\sigma_{m-j}^{(i-1)}}{P'(\beta^{(i-1)})}$$

for all $i, j = 1, 2, \cdots, m$. We have: $L_s(X) = \sum_{j=1}^{m} \xi_{j,s+1} X^{j-1}$, $s = 0, 1, \cdots, m-1$. Now $C \cdot Z_k = \mathcal{B}_k$ for all $k \geq 0$, hence $Z_k = C^{-1} \cdot \mathcal{B}_k$. Each component $z_{i,k}$, $0 \leq i \leq m-1, k \geq 0$ of Z_k can be expressed as $z_{i,k} = \sum_{j=1}^{m} \xi_{i+1,j} \beta^{(j-1)^k}$. Since β is a Perron number, we have $|\beta^{(j)}| < \beta$ for all

$j,\ 1 \le j \le m-1$. Hence, for all $j,\ 1 \le j \le m-1,\ \lim_{k\to+\infty} \left(\frac{\beta^{(j)}}{\beta}\right)^k = 0$, and therefore $\lim_{k\to+\infty} \frac{z_{i,k}}{\beta^k} = \xi_{i+1,1}$, $i = 0, 1, \cdots, m-1$. Moreover,

$$\lim_{k\to+\infty} \frac{\left(\sum_{i=0}^{m-1} |z_{i,k}|^2\right)^{1/2}}{\beta^k} = \lim_{k\to+\infty} \frac{\|Z_k\|}{\beta^k} = \sqrt{\sum_{i=0}^{m-1} |\xi_{i+1,1}|^2} = \|V_1\|$$

hence the result. The fact that all the components of V_1 are real and belong to the $\mathbb{Z}$ - module $\mathbb{Z}[\beta]/(\beta^{m-1}P'(\beta))$ comes from the following more precise Proposition. □

Proposition 3.1. *The components* $(\xi_{j,1})_{j=1,\ldots,m}$ *of* V_1 *are given by the following explicit functions of the coefficients* a_i *of* $P(X)$*:* $\xi_{j,1} = \frac{a_{j-1}\beta^{j-1}+a_{j-2}\beta^{j-2}+\cdots+a_1\beta+a_0}{\beta^j P'(\beta)}$. *In particular,* $\xi_{m,1} = \frac{1}{P'(\beta)}$.

Proof. We have $L_0(X) = \sum_{j=1}^{m} \xi_{j,1} X^{j-1}$ and $P(X) = \prod_{j=0}^{m-1}(X - \beta^{(j)}) = L_0(X)(X-\beta)P'(\beta)$. All the coefficients of $L_0(X)$ satisfy the following relations: $-\beta P'(\beta)\xi_{1,1} = -a_0$, $-\beta P'(\beta)\xi_{2,1} + \xi_{1,1}P'(\beta) = -a_1$, $-\beta P'(\beta)\xi_{3,1} + \xi_{2,1}P'(\beta) = -a_2$, $\cdots$, $-\beta P'(\beta)\xi_{m,1} + \xi_{m-1,1}P'(\beta) = -a_{m-1}$, $\xi_{m,1}P'(\beta) = 1$. Hence the result recursively from $\xi_{1,1}$ noting that $P'(\beta) \in \mathbb{R} - \{0\}$. □

Theorem 3.2. *Let* $u_B := B/\|B\|$. *Then: (i)* $u \cdot u_B = \|B\|^{-1}\|V_1\|^{-1} > 0$, *(ii) the limit* $\lim_{k\to+\infty} \frac{\|Z_{k+1}\|}{\|Z_k\|}$ *exists and is equal to* β , *(iii)* u *is an eigenvector of* ${}^tQ$ *of eigenvalue* β *and the eigenspace of* $\mathbb{R}^m$ *associated with the eigenvalue* β *of* ${}^tQ$ *is* $\mathbb{R}u$, *(iv)* u_B *is an eigenvector of the adjoint matrix* $({}^tQ)^* = Q$ *associated with the eigenvalue* β *and for all* $x \in \mathbb{C}^m$*:* $\lim_{k\to+\infty} \beta^{-k}\,({}^tQ)^k(x) = (x \cdot B)\ V_1$.

Proof. (i) and (ii): From the relation $C \cdot C^{-1} = Id$ we deduce the equality $V_1 \cdot B = 1$. Hence $u \cdot B = \|V_1\|^{-1} > 0$. Then, for all $k \ge 0$, ${}^tZ_k \cdot B = \beta^k = \|Z_k\|^t(\frac{Z_k}{\|Z_k\|} - u + u) \cdot B > 0$ which tends to infinity when k tends to $+\infty$. Since $u - Z_k/\|Z_k\|$ tends to zero when k goes to infinity, $\|Z_k\|$ behaves at infinity like $\beta^k/\ (u \cdot B)$, hence the limit; (iii): for all $k \ge 0$, ${}^tQ(u) = {}^tQ(u - \frac{Z_k}{\|Z_k\|} + \frac{Z_k}{\|Z_k\|}) = {}^tQ(u - \frac{Z_k}{\|Z_k\|}) + \frac{\|Z_{k+1}\|}{\|Z_k\|}\frac{Z_{k+1}}{\|Z_{k+1}\|}$. The first term is converging to zero and the second one to βu when k goes to infinity, from Theorem 3.1. Hence, the result since all the roots of $P(X)$ are distinct and the (real) eigenspace associated with β is 1 - dimensional; (iv): it is clear that B is an eigenvector of the adjoint matrix Q . If $h_0, h_1, \cdots, h_{m-1} \in \mathbb{C}$, $x = \sum_{j=0}^{m-1} h_j Z_j$, where $Z_0, Z_1, \cdots, Z_{m-1}$ is the canonical basis of $\mathbb{C}^m$, we have: $\beta^{-k}\,({}^tQ)^k(x) =$

$\sum_{j=0}^{m-1} h_j \beta^{-k} Z_{k+j} = \sum_{j=0}^{m-1} h_j \beta^j \left(\frac{Z_{k+j}}{\beta^{k+j}}\right)$, but, from the proof of Theorem 3.1, $\lim_{k\to+\infty} \frac{Z_{k+j}}{\beta^{k+j}} = V_1$ and $\sum_{j=0}^{m-1} h_j \beta^j = x \cdot B$. We deduce the claim. □

Let us denote by ${}^tQ_{\mathbb{C}}$ the automorphism of $\mathbb{C}^m$ which is the complexification operator of ${}^tQ$. Its adjoint $Q_{\mathbb{C}}$ obviously admits $\{B, B^{(1)}, B^{(2)}, \cdots B^{(m-1)}\}$ as a basis of eigenvectors of respective eigenvalues $\beta, \beta^{(1)}, \beta^{(2)}, \cdots \beta^{(m-1)}$. Let us specify their respective actions on $\mathbb{R}^m$. Let $s \geq 1$, resp. t , be the number of real, resp. complex (up to conjugation), embeddings of the number field $\mathbb{Q}(\beta)$. We have $m = s + 2t$. Assume that the conjugates of β are $\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, \beta^{(s)}, \beta^{(s+1)}, \cdots, \beta^{(m-2)} = \beta^{(s+2t-2)}, \beta^{(m-1)} = \beta^{(s+2t-1)}$ where $\beta^{(q)}$ is real if $q \leq s-1$ and $\beta^{(s+2j)} = \overline{\beta^{(s+2j+1)}} = |\beta^{(s+2j)}| e^{i\theta_j}, j = 0, 1, \cdots, t-1$, is complex with non-zero imaginary part. Let us recall that V_1 denotes the vector defined by the first column of C^{-1} (Theorem 3.1).

Corollary 3.3. *(i) A basis of eigenvectors of* ${}^tQ_{\mathbb{C}}$ *is given by the* m *column vectors* $\{W_k\}_{k=1,2,\cdots,m}$ *of respective components* $\xi_{j,k} = \frac{a_{j-1}\beta^{(k-1)^{j-1}} + a_{j-2}\beta^{(k-1)^{j-2}} + \cdots + a_1\beta^{(k-1)} + a_0}{\beta^{(k-1)^j} P'(\beta^{(k-1)})}$ *with* $j = 1, 2, \cdots, m$*; in particular,* $\xi_{m,k} = \frac{1}{P'(\beta^{(k-1)})}$*; (ii) a real Jordan form for* ${}^tQ$ *is given by the diagonal matrix* $\mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, D_0, D_1, \cdots, D_{t-1})$ *in the basis of eigenvectors* $\{V_j\}_{j=1,\cdots,m}$ *with* $V_2 = W_2, \cdots, V_s = W_s$, $V_{s+2j+1} = Im(W_{s+2j+1}), V_{s+2j+2} = Re(W_{s+2j+1})$, $j = 0, 1, \cdots, t-1$ *and where the* 2×2 *real Jordan blocks* D_j *are*

$$\begin{pmatrix} |\beta^{(s+2j)}| \cos\theta_j & -|\beta^{(s+2j)}| \sin\theta_j \\ |\beta^{(s+2j)}| \sin\theta_j & |\beta^{(s+2j)}| \cos\theta_j \end{pmatrix};$$

(iii) a real Jordan form of the adjoint operator $({}^tQ)^* = Q$ *is given by the same diagonal matrix* $\mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, D_0, D_1, \cdots, D_{t-1})$ *in the basis of eigenvectors* $\{X_j\}_{j=1,\cdots,m}$ *with* $X_1 = B, X_2 = B^{(1)}, X_3 = B^{(2)}, \cdots, X_s = B^{(s-1)}$, $X_{s+2j+1} = Im(B^{(s+2j)}), X_{s+2j+2} = Re(B^{(s+2j)})$, $j = 0, 1, \cdots, t-1$*. The* t *planes* $\mathbb{R} X_{s+2j+1} + \mathbb{R} X_{s+2j+2}, j = 0, 1, \ldots, t-1$ *are all orthogonal to* V_1*, and thus also to* u*.*

Proof. (i): We apply, componentwise in the equation $({}^tQ)V_1 = \beta V_1$, the $\mathbb{Q}$ - automorphisms of $\mathbb{C}$ which are the real and complex embeddings of the number field $\mathbb{Q}(\beta)$. Since ${}^tQ$ has rational entries and V_1 has its components in the $\mathbb{Z}$-module $\beta^{1-m}(P'(\beta))^{-1}\,\mathbb{Z}[\beta]$, we deduce the claim: $({}^tQ)W_j = \beta^{(j-1)} W_j$ with $j = 1, 2, \cdots, m$ and where $W_1 = V_1$; (ii): the restrictions of ${}^tQ_{\mathbb{C}}$ to the (real) ${}^tQ$ - invariant subspaces of $\mathbb{R}^m$ have no nilpotent parts since all the roots of $P(X)$ are distinct. Hence, a real Jordan form of ${}^tQ$ is the one proposed with Jordan blocks which are 2×2

on the diagonal [HS]. (iii): in a similar way the equation $QB = \beta B$ implies $QB^{(j)} = \beta^{(j)} B^{(j)}$ with $j = 0, 1, \cdots, m-1$. Obviously $Q_{\mathbb{C}}$ and ${}^tQ_{\mathbb{C}}$ have the same eigenvalues and Q and ${}^tQ$ the same real 2×2 Jordan blocks on the diagonal. The corresponding basis of eigenvectors is given by the vectors X_i [HS]. The orthogonality between V_1 and the vector X_{s+2j+1}, resp. $X_{s+2j+2}, j = 0, 1, \ldots, t-1$, arises from the relation $C \cdot C^{-1} = Id$. We deduce the claim for the planes. □

The linear invariants associated with the powers of β are the invariant subspaces given by Corollary 3.3. Let us turn to the beta-integers. Beta-integers are particular $\mathbb{Z}$-linear combinations of powers of β. We will show how to construct the set $\mathbb{Z}_\beta$ using the above linear invariants, namely, the set $\mathbb{Z}_\beta$ will appear in a natural way on the line $\mathbb{R}\, u_B$ as image of a point set close to the expanding line $\mathbb{R}\, u$.

Remark . — The conditions of Parry, used here in the context of matrices ${}^tQ$ without any condition on the signs of the entries, give the same results as those obtained with the Perron-Frobenius theory (Minc [Mi]), when this one is applicable, that is when ${}^tQ$ has non-negative entries: first, the dimensionality one for the dominant eigenspace of ${}^tQ$; second, the equality $\lim_{k \to +\infty} \beta^{-k} ({}^tQ)^k(x) = (x \cdot B)\, V_1$, for $x \in \mathbb{C}^m$, in Theorem 3.2 (compare with Ruelle [Ru] p136 when ${}^tQ$ has non-negative entries), and its consequences.

Theorem 3.4. *Let π_B be the orthogonal projection mapping of $\mathbb{R}^m$ onto $\mathbb{R}B$ and define $\mathcal{L} = \{x_k Z_k + x_{k-1} Z_{k-1} + \cdots + x_1 Z_1 + x_0 Z_0 \mid x_i \in A,\ k \geq 0$, and $(x_j, x_{j-1}, \cdots, x_1, x_0, 0, 0, \cdots) < (c_1, c_2, \cdots)$ for all $j,\ 0 \leq j \leq k\ \}$ the ${}^tQ$-invariant subset of $\mathbb{Z}^m$. Then: (i) the mapping $\sum_{j=0}^k x_j \beta^j \to \sum_{j=0}^k x_j Z_j\ :\ \mathbb{Z}_\beta^+ \to \mathcal{L}$ (with the same coefficients x_j) is a bijection, (ii) the mapping $\pi_{B_{|\mathbb{Z}^m}}$ is one-to-one onto its image $\mathbb{Z}[\beta] \|B\|^{-1} u_B$: for any $k \geq 0$, $a_0, \cdots, a_k \in \mathbb{Z}$, we have $\pi_B \left(\sum_{i=0}^k a_i Z_i \right) = \left(\sum_{i=0}^k a_i \beta^i \right) \|B\|^{-1} u_B$ and conversely, any polynomial in β on the line generated by $\|B\|^{-1} u_B$ can be uniquely lifted up to a $\mathbb{Z}$ -linear combination of the vectors Z_i with the same coefficients; in particular, $\pi_B(\mathcal{L}) = \mathbb{Z}_\beta^+ \|B\|^{-1} u_B$.*

Proof. (i): this mapping $\mathbb{Z}_\beta^+ \to \mathcal{L}$ is obviously surjective. Let us show that it is injective. Assume there exists a non-zero element $\sum_{j=0}^k x_j \beta^j$ in $\mathbb{Z}_\beta^+$ such that $\sum_{j=0}^k x_j Z_j = 0$. Since ${}^t\left(\sum_{j=0}^k x_j Z_j \right) B = 0 = \sum_{j=0}^k x_j \beta^j$, this would mean that zero could be represented by a non-zero element. This is impossible by construction; (ii): for all $k \geq 0$, we have $\pi_B(Z_k) = \beta^k \|B\|^{-1} u_B$, hence the result by linearity. The injectivity of $\pi_{B_{|\mathbb{Z}^m}}$ comes from the assertion (i). □

Proposition 3.2. *Let* $u_{B,i} = \|X_i\|^{-1}X_i$ *if* $i = 1, 2, \cdots, s$, $u_{B,i} = (\|X_i\|^2 + \|X_{i+1}\|^2)^{1/2}(Re(\|B^{(i-1)}\|^{-2})X_i + Im(\|B^{(i-1)}\|^{-2})X_{i+1})$ *if* $i = s+1, \cdots, m$ *with* $i-(s+1)$ *even, and* $u_{B,i} = (\|X_{i-1}\|^2 + \|X_i\|^2)^{1/2}$ $(-Im(\|B^{(i-1)}\|^{-2})X_i + Re(\|B^{(i-1)}\|^{-2})X_{i+1})$ *if* $i = s+1, \cdots, m$ *with* $i-(s+1)$ *odd. Denote by* $\pi_{B,i} : \mathbb{R}^m \to \mathbb{R}\, u_{B,i}, i = 1, 2, \cdots, s$ *the orthogonal projection mappings to the 1-dimensional eigenspaces of* Q , *resp.* $\pi_{B,i} : \mathbb{R}^m \to \mathbb{R}\, u_{B,i} + \mathbb{R}\, u_{B,i+1}, i = s+1, \cdots, m$ *with* $i-(s+1)$ *even, the orthogonal projection mappings to the irreducible 2-dimensional eigenspaces of* Q . *Then, for all* $k \geq 0$, $a_0, \cdots, a_k \in \mathbb{Z}$, *we have* $\pi_{B,i}(\sum_{j=0}^k a_j Z_j) = \left(\sum_{j=0}^k a_j \beta^{(i-1)^j}\right) \|X_i\|^{-1} u_{B,i}$, $i = 1, 2, \cdots, s$ *and, for all* $i = s+1, \cdots, m$ *with* $i-(s+1)$ *even,* $\pi_{B,i}(\sum_{j=0}^k a_j Z_j) =$

$$\frac{\begin{pmatrix} Re(\sum_{j=0}^k a_j \beta^{(i-1)^j}) & Im(\sum_{j=0}^k a_j \beta^{(i-1)^j}) \\ -Im(\sum_{j=0}^k a_j \beta^{(i-1)^j}) & Re(\sum_{j=0}^k a_j \beta^{(i-1)^j}) \end{pmatrix} \begin{pmatrix} u_{B,i} \\ u_{B,i+1} \end{pmatrix}}{\left(\sum_{k=0}^{m-1} |\beta^{(i-1)}|^{2k}\right)^{1/2}} .$$

Proof. It suffices to apply the real and complex embeddings of $\mathbb{Q}(\beta)$ to the relation

$$\pi_B(\sum_{j=0}^k a_j Z_j) = ((\sum_{j=0}^k a_j Z_j) \cdot B)\|B\|^{-2}B = \left(\sum_{j=0}^k a_j \beta^j\right) \|B\|^{-2}B \ :$$

for complex embeddings, $\|X_i\|^2 + \|X_{i+1}\|^2 = \sum_{k=0}^{m-1} |\beta^{(i-1)}|^{2k}$ and $\|B^{(i-1)}\|^{-2}B^{(i-1)}$ means:

$$\begin{pmatrix} Re(\|B^{(i-1)}\|^{-2}) & Im(\|B^{(i-1)}\|^{-2}) \\ -Im(\|B^{(i-1)}\|^{-2}) & Re(\|B^{(i-1)}\|^{-2}) \end{pmatrix} \begin{pmatrix} X_i \\ X_{i+1} \end{pmatrix} = \frac{1}{(\|X_i\|^2 + \|X_{i+1}\|^2)^{1/2}} \begin{pmatrix} u_{B,i} \\ u_{B,i+1} \end{pmatrix}.$$

□

The explicit expressions given above will allow us below to compare the "geometric" Rauzy fractals deduced from the present study and the "algebraic" Rauzy fractal. Before stating the main theorem about the existence of canonical cut-and-projection schemes associated with the beta-integers when β is a general (non-integer) Perron number, let us first consider the case of equality $u = u_B$ and show that it is rarely occuring.

Proposition 3.3. *The equality* $u = u_B$ *holds if and only if* β *is a Pisot number, root* > 1 *of the polynomial* $X^2 - aX - 1$, *with* $a \geq 1$.

Proof. The condition $u = u_B$ is equivalent to V_1 colinear to B, that is $\xi_{j,1}\beta^{-j+1} =$ a non-zero constant, for all $j = 1, 2, \cdots, m$. The condition is sufficient: if β is such a Pisot number, such equalities hold. Conversely, if such equalities hold, this implies in particular that $\xi_{1,1}\beta^{-1+1} = \xi_{m,1}\beta^{-m+1}$. Thus we obtain $a_0\beta^{m-2} = 1$, that is necessarily $m = 2$ and $a_0 = 1$. The Perron number β is then a Pisot number of negative conjugate $-\beta^{-1}$ which satisfies $\beta^2 - a_1\beta - 1 = 0$, where $a_1 = \beta - \beta^{-1}$ is an integer greater than or equal to 1. This is the only possibility of quadratic Pisot number of norm -1 ([Fro1], Lemma 3). □

Theorem 3.5. *Denote by E the line $\mathbb{R}u_B$ in $\mathbb{R}^m$. There exist two canonical cut-and-project schemes $E \xleftarrow{p_1} (E \times D \simeq \mathbb{R}^m, \mathbb{Z}^m) \xrightarrow{p_2} D$ associated with $\mathbb{Z}_\beta \subset E$ (see the definitions in the Appendix). They are given by, in case (i): the orthogonal projection mapping π_B as p_1, $\oplus_F F$ as internal space D, $p_2 = \oplus_F \pi_F$, where the sums are over all irreducible ${}^tQ$-invariant subspaces F of $\mathbb{R}^m$ except $\mathbb{R}u$ and where π_F is the projection mapping to F along its ${}^tQ$-invariant complementary space, in case (ii): as p_1 the orthogonal projection mapping π_B, $\oplus_F F$ as internal space D where the sum is over all irreducible Q-invariant subspaces F of $\mathbb{R}^m$ except E, as p_2 the sum $\oplus_{i\neq 1}\pi_{B,i}$ of all the orthogonal projection mappings except $\pi_{B,1} = \pi_B$; in the case (ii), the internal space D is orthogonal to the line $\mathbb{R}u$.*

Proof. In both cases, the fact that $p_2(\mathbb{Z}^m)$ is dense in D arises from Kronecker's theorem (Appendix B in [Mey]): since β is an algebraic integer of degree m, the m real numbers $1 = \beta^0, \beta^1, \cdots, \beta^{m-1}$ are linearly independent over $\mathbb{Q}$. Hence, for all $\epsilon > 0$ and all m-tuple of real numbers $x_0, x_1, \cdots, x_{m-1}$ such that the vector (say) $x = {}^t(x_0\, x_1\, \ldots\, x_{m-1})$ belongs to D, there exist a real number w and m rational integers $u_0, u_1, \cdots, u_{m-1}$ such that $\mid x_j - \beta^j w - u_j \mid \leq \epsilon/\sqrt{m}$ for all $j = 0, 1, \ldots, m-1$. In other terms, there exists a point $u = {}^t(u_0, u_1, \cdots, u_{m-1}) \in \mathbb{Z}^m$ such that its image $p_1(u)$ is $wB \in \mathbb{R}u_B$ and its image $p_2(u)$ is close to x up to ϵ. Hence the result. As for the restriction of the projection mapping $p_1 = \pi_B = \pi_{B,1} : \mathbb{R}^m \to E$ to the lattice $\mathbb{Z}^m$, it is injective after Theorem 3.4. The orthogonality between D and u comes directly from Corollary 3.3. □

The mapping $p_1(\mathbb{Z}^m) \to D :\ x \to x^* = p_2 \circ (p_{1_{|\mathbb{Z}^m}})^{-1}(x)$ will be denoted by the same symbol $(.)^*$ in the cases (i) and (ii), the context making the difference.

Proposition 3.4. *Let β be a Pisot number, root > 1 of the polynomial $X^2 - aX - 1$, with $a \geq 1$. Put $c_a = \frac{(1+a\beta)\lfloor\beta\rfloor}{\sqrt{2+a\beta(\beta-1)}}$. Then the two canonical cut-and-project schemes given by (i) and (ii) in Theorem 3.5 are* *identical and the inclusion of* $\mathbb{Z}_\beta \, \|B\|^{-1} u_B$ *in the following model set holds:* $\mathbb{Z}_\beta \, \|B\|^{-1} u_B = \pi_B(\mathcal{L} \cup (-\mathcal{L})) \subset \{v \in \pi_B(\mathbb{Z}^2) \mid v^* \in [-c_a u_{B,2}, +c_a u_{B,2}]\}$ *where* $u_{B,2} = {}^t(-\beta\, 1)\|B\|^{-1}$.

Proof. The two cut-and-project schemes are identical: by Proposition 3.3 the equality $u = u_B$ holds and the line $\mathbb{R}\, u_{B,2}$, which is obviously orthogonal to the line $\mathbb{R}\, u_B$, is ${}^tQ$-invariant. Now, if g denotes an arbitrary element of $\mathcal{L}$, it can be written $g = x_k({}^tQ)^k Z_0 + x_{k-1}({}^tQ)^{k-1} Z_0 + \cdots + x_1({}^tQ) Z_0 + x_0 Z_0$ for a certain integer $k \geq 0$ with $x_i \in A$ and $(x_j, x_{j-1}, \cdots, x_1, x_0, 0, 0, \cdots) < (c_1, c_2, \cdots)$ for all j, $0 \leq j \leq k$. We have $Z_0 = su + s^\perp u_{B,2}$ with $s = \|B\|^{-1}$ and $s^\perp = -\beta \|B\|^{-1}$. Then $g = \sum_{j=0}^k x_j ({}^tQ)^j Z_0 = \sum_{j=0}^k x_j \left(s\beta^j u + s^\perp (-1)^j \beta^{-j} u_{B,2}\right)$. Thus $p_2(g) = s^\perp \sum_{j=0}^k x_j (-1)^j \beta^{-j} u_{B,2}$ and $\|p_1(g)^*\| = \|p_2(g)\| \leq |s^\perp| \lfloor \beta \rfloor \sum_{j=0}^{+\infty} \beta^{-j} = |s^\perp| \lfloor \beta \rfloor \frac{1}{1-\beta^{-1}}$ which is equal to c_a since $\|B\| = \sqrt{2 + a\beta}$. This constant is independent of k. Hence we have $p_1(g) \subset \{v \in \pi_B(\mathbb{Z}^2) \mid v^* \in [-c_a u_{B,2}, +c_a u_{B,2}]\}$ and the claim. □

Let $\mathcal{C} = \{ \sum_{j=0}^{m-1} \alpha_j Z_j \mid \alpha_j \in [0;1] \text{ for all } j = 0, 1, \cdots, m-1 \}$ be the m-cube at the origin. For all irreducible ${}^tQ$- invariant subspace F of $\mathbb{R}^m$, put $\delta_F = \max_{x \in \mathcal{C}} \|\pi_F(x)\|$, λ_F the absolute value of the eigenvalue of ${}^tQ$ on F and $c_F = \lfloor \beta \rfloor \frac{\delta_F}{1 - \lambda_F^m}$. Denote by Ω_F the closed interval centred at 0 in F of length $2c_F$ if $\dim F = 1$, resp. the closed disc centred at 0 in F of radius c_F if $\dim F = 2$.

Theorem 3.6. *Let* β *be a Pisot number of degree* $m \geq 2$ *and* $\Omega = \oplus_F \Omega_F$ *where the sum is over all irreducible* ${}^tQ$*- invariant subspace* F *of* $\mathbb{R}^m$ *except* $\mathbb{R}u$*. Then the inclusion of* $\mathbb{Z}_\beta \, \|B\|^{-1} u_B$ *in the following model set defined by* Ω *holds:* $\mathbb{Z}_\beta \, \|B\|^{-1} u_B = p_1(\mathcal{L} \cup (-\mathcal{L})) \subset \{v \in p_1(\mathbb{Z}^m) \mid v^* \in \Omega \}$ *in the cut-and-project scheme given by the case (i) in Theorem 3.5.*

Proof. If $g = \sum_{j=0}^k x_j Z_j \in \mathcal{L}$ with $k = dm - 1$, and $d \geq 1$ an integer, then

$$g = \sum_{q=0}^{d-1} \sum_{l=0}^{m-1} x_{qm+l} ({}^tQ)^{qm} Z_l = \sum_{q=0}^{d-1} ({}^tQ)^{qm} \left(\sum_{l=0}^{m-1} x_{qm+l} Z_l \right).$$

Hence $p_1(g)^* = p_2(g) = \oplus_F \pi_F(g) = \sum_F \sum_{q=0}^{d-1} \left[({}^tQ_{|F})^{qm} \pi_F \left(\sum_{l=0}^{m-1} x_{qm+l} Z_l \right) \right]$

with:

$$\|\pi_F(g)\| \leq \sum_{q=0}^{d-1} \lfloor \beta \rfloor \lambda_F^{qm} \|\pi_F \left(\sum_{l=0}^{m-1} Z_l \right) \| \leq \lfloor \beta \rfloor \, \delta_F \sum_{q=0}^{+\infty} \lambda_F^{qm} = \lfloor \beta \rfloor \frac{\delta_F}{1 - \lambda_F^m} = c_F.$$

This constant is independent of d, hence of $k = dm - 1$. It is easy to check that it is an upper bound for $\|p_2(g)\|$ if $k \not\equiv -1 (\bmod m)$ and also for all $g \in -\mathcal{L}$. We deduce the claim. □

Corollary 3.7. *If* β *is a Pisot number of degree* $m \geq 2$ *, then* $\mathbb{Z}_\beta$ *is a Meyer set.*

Proof. If β is a Pisot number, the set $\mathbb{Z}_\beta$, viewed as the set of vertices of an aperiodic tiling, is obtained by concatenation of prototiles on the line, which are in finite number by Thurston [Th]. And it is relatively dense by construction. Now, by Theorem 3.6 it is included in a model set. This proves the claim (see the Appendix). □

In both cases of cut-and-project scheme, as given by Theorem 3.5 where the duality between the matrices Q and ${}^tQ$ clearly appears, the internal space represents the contracting hyperplane, whereas the line $\mathbb{R}\,u$ is the expanding direction, when β is a Pisot number. The duality between both cut-and-project schemes is connected to the substitutive approach by the following (Arnoux and Ito [AI], Chap. 7 in Pytheas Fogg [PF]): the abelianized Z'_k of the iterates of the substitution satisfy $Z'_{k+1} = QZ'_k$, and gather now about the line $\mathbb{R}\,B$. If one takes the projection on $\mathbb{R}\,B$ of the new set $\mathcal{L}'$ (defined similarly as $\mathcal{L}$) along the other eigenspaces, one recovers $\mathbb{Z}_\beta$ (up to a scalar factor). A striking feature of the internal spaces is that the numeration in base $\beta^{(j)}$ (conjugates of β) appears as canonical ingredient to control the distance between a point of $\mathcal{L}$ and its orthogonal projection to the expanding line $\mathbb{R}\,u$, in particular at infinity.

Definition 3.1. Let β be a Pisot number of degree $m \geq 2$. The closure $\overline{\left(\mathbb{Z}_\beta^+ \, \|B\|^{-1} u_B\right)^*}$ of the set $p_2(\mathcal{L})$ is called the canonical acceptance window associated with the set of beta-integers $\mathbb{Z}_\beta^+$ in both cases (case (i) or (ii) in Theorem 3.5) of cut-and-project scheme: in the case (i) it will be denoted by $\mathcal{R}_i$ and in the case (ii) by $\mathcal{R}$.

The notations $\mathcal{R}$ and $\mathcal{R}_i$ ($\mathcal{R}_i \subset \Omega$ by Theorem 3.6) with an " $\mathcal{R}$ " like Rauzy are used to recall the close similarity between these sets and the Rauzy fractal (Rauzy [Ra], Arnoux and Ito [AI], Messaoudi [Me], Ito and Sano [IS], Chap. 7 in Pytheas Fogg [PF]). The fact is that the set $\mathcal{R}$ is exactly the Rauzy fractal up to the multiplication by a non-zero scalar factor on each irreducible Q-invariant subspace (by definition we will speak of quasi-dilation). Let us show it on an example.

"Tribonacci" case [Me]: let us consider the irreducible polynomial $P(X) = X^3 - X^2 - X - 1$. Its dominant root is denoted by β, and α and $\overline{\alpha}$ are the two other complex conjugates roots of $P(X)$. In this case, the Rauzy fractal is "algebraically" defined by $\mathcal{E} := \{\sum_{i=3}^{\infty} \epsilon_i \alpha^i \mid \epsilon_i \in$

$\{0,1\}$ and $\epsilon_i\epsilon_{i+1}\epsilon_{i+2} = 0$ for all integer $i \geq 3\}$. The condition imposed on the sequence $(\epsilon_i)_{i\geq 3}$ is exactly that given by the conditions of Parry. Indeed ([Fro1] and section 2), $d_\beta(1) = 0.111$ and the lexicographical maximal sequence is $c_1c_2c_3\cdots = (110)^\omega$. Now (Proposition 3.2) $B^{(1)} = {}^t(1\,\alpha\,\alpha^2)$ and $\|X_2\|^2 + \|X_3\|^2 = 1 + \alpha\overline{\alpha} + \alpha^2\overline{\alpha}^2 = \beta$. We deduce that $\mathcal{R} = \beta^{-1/2}\,\mathcal{E} = \pi_{B,2}(\mathcal{L})$ with the following (metric) identification of $\mathbb{C}$: $\phi(\mathbb{C}) = \mathbb{R}\,u_{B,2} + \mathbb{R}\,u_{B,3}$ where ϕ is the isometry which sends the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, resp. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, to $\begin{pmatrix} u_{B,2} \\ u_{B,3} \end{pmatrix}$, resp. to $\begin{pmatrix} u_{B,3} \\ -u_{B,2} \end{pmatrix}$.

Proposition 3.5. *The canonical acceptance window* $\mathcal{R}$ *(relative to the case (ii) of cut-and-project scheme in Theorem 3.5) is compact and connected. Its interior* $int(\mathcal{R})$ *is simply connected, contains the origin. The set* $\mathcal{R}$ *is such that: (i)* $\overline{int(\mathcal{R})} = \mathcal{R}$*; (ii) it induces a tiling of the internal space* D *modulo the lattice* $\phi(\mathbb{Z}+\mathbb{Z}\,\alpha)$*:* $D = \bigcup_{z\in\phi(\mathbb{Z}+\mathbb{Z}\,\alpha)}(\mathcal{R}+z)$*; (iii)* $(\mathcal{R}+z)\cap int(\mathcal{R}+z') = \emptyset$ *for all* $z, z' \in \phi(\mathbb{Z}+\mathbb{Z}\,\alpha)$, $z \neq z'$.

Proof. Since $\mathcal{R} = \beta^{-1/2}\,\mathcal{E}$, we deduce the properties of $\mathcal{R}$ from those of $\mathcal{E}$ already established in Rauzy [Ra], Messaoudi [Me] and [Me1]. □

Proposition 3.6. *The boundary of* $\mathcal{R}$ *is a fractal Jordan curve. A point* z *belongs to the boundary of* $\mathcal{R}$ *if and only if it admits at least 2 distinct Rényi* α*-expansions. A point belonging to the boundary of* $\mathcal{R}$ *admits 2 or 3 distinct Rényi* α*-expansions, never more.*

Proof. The properties of the boundary of $\mathcal{E}$ are given in Ito and Kimura [IK] and Messaoudi [Me1]). Hence the claim. □

The properties of $\mathcal{R}_i$ follow from the equality: $p_2(\mathcal{R}) = \mathcal{R}_i$, where p_2 refers to the case (i) of cut-and-project scheme in Theorem 3.5, and from Proposition 3.5 and 3.6: in particular, it has also a fractal boundary. We will speak of "geometrical" Rauzy fractals for $\mathcal{R}$ and $\mathcal{R}_i$ and of "algebraic" Rauzy fractal for $\mathcal{E}$. They are similar objects as far as they concentrate all the information about the beta-integers and the completions of their real and complex embeddings (Rauzy [Ra]). The respective canonical acceptance windows associated with $\mathbb{Z}_\beta$ are $\mathcal{R}\cup(-\mathcal{R})$ and $\mathcal{R}_i\cup(-\mathcal{R}_i)$ in the two cut-and-project schemes.

4. Additive properties of $\mathbb{Z}_\beta$

In this section, β will be a Pisot number of degree $m \geq 2$.

A) Cones, generators and semi-groups. – We will show that any element of $\mathcal{L}$ is generated by a finite number of elements of $\mathcal{L}$ of small norm, over $\mathbb{N}$. By projection to E by π_B, the ambiant 1-dimensional space of the beta-integers (Theorem 3.5), this will imply the same property for $\mathbb{Z}_\beta$. This finiteness property, stated in Corollary 4.5, constitutes a refinement of Theorem 4.12 (i) (Lagarias) for the Meyer sets $\mathbb{Z}_\beta$.

First let us fix the notations and simplify them somehow. Let $\pi : \mathbb{R}^m \to \mathbb{R}\,u$ be the projection mapping along its ${}^tQ$-invariant complementary space (instead of denoting it by $\pi_{\mathbb{R}\,u}$), and p_2 the projection mapping of the cut-and-project scheme (i) in Theorem 3.5. Let $\pi^{||} : \mathbb{R}^m \to \mathbb{R}\,u$ be the orthogonal projection mapping and $\pi^{\perp} = Id - \pi^{||}$ ($\pi^{\perp}$ is the mapping $\oplus_{i\neq 1}\pi_{b,i}$ in the case (ii) of cut-and-project scheme in Theorem 3.5). The basic ingredient will be the construction of semi-groups of finite type associated with cones whose axis of revolution is the expanding line $\mathbb{R}\,u$, following an idea of Lind [Li] in another context. Truncating them in a suitable way at a certain distance of the origin will be the key for finding generators of $\mathcal{L}$ over $\mathbb{N}$. In the first Lemma we will consider the possible angular openings of these cones around the expanding line $\mathbb{R}\,u$ for catching the points of $\mathcal{L}$. For $\theta > 0$, define the cone $K_\theta := \{x \in \mathbb{R}^m \mid \theta\|p_2(x)\| \leq \|\pi(x)\|, 0 \leq \pi(x)\cdot u\}$. For $r, w > 0$, define $K_\theta(r) := \{x \in K_\theta \mid \|\pi(x)\| \leq r\}$, $K_\theta(r,w) := \{x \in K_\theta \mid r \leq \|\pi(x)\| \leq w\}$. If $\mathcal{A}$ is an arbitrary subset of $\mathbb{R}^m$, denote by $sg(\mathcal{A}) := \{\sum_{\text{finite}} m_i x_i \mid m_i \in \mathbb{N}, x_i \in \mathcal{A}\}$ the semigroup generated by $\mathcal{A}$. Let ρ be the *covering radius* of the subset $\mathcal{L}\cup(-\mathcal{L})$ with respect to the band $\mathcal{R}_i \times \mathbb{R}\,u$: ρ is the smallest positive real number such that for any $z \in \mathbb{R}^m$ such that $p_2(z) \in \mathcal{R}_i$ the closed ball $B(z,\rho)$ contains at least one element of $\mathcal{L}\cup(-\mathcal{L})$. A lower bound of ρ is given by the covering radius $\sqrt{m}/2$ of the lattice $\mathbb{Z}^m$. Referring $\mathbb{R}^m$ to the basis $\{B, V_2, V_3, \cdots, V_{s+2t}\}$ and using Corollary 3.3 and Theorem 3.6 we easily deduce the following upper bound of ρ: $\frac{1}{2}\|B\|^{-1} + \sum_F c_F$, where the sum $\sum_F$ means there and in the following everywhere it will be used "the sum over all irreducible ${}^tQ$-invariant subspaces F of $\mathbb{R}^m$ except $\mathbb{R}\,u$". The notation diam$(\cdot)$ will be put for the diameter of the set $(\cdot)$ in the following.

Proposition 4.1. *(i) For all $\theta > 0$, there exists an integer $j_0 = j_0(\theta) \geq 0$ such that $Z_j \in K_\theta$ for all $j \geq j_0$; (ii) if ${}^tQ$ is nonnegative, and $\min\{\xi_{j,1} \mid j = 1,2,\cdots,m\} > 2\|V_1\|(diam(\mathcal{R}_i))$, then the following equality $j_0(\theta) = 0$ holds for all $0 < \theta < \theta_{min}$, where $\theta_{min} := -2 + (diam(\mathcal{R}_i))^{-1}\|V_1\|^{-1}\min\{\xi_{j,1} \mid j = 1,2,\cdots,m\}$.*

Proof. (i) Let $\theta > 0$. We have just to prove that $\pi(Z_j)\cdot u$ tends to $+\infty$ and not to $-\infty$ when j goes to $+\infty$. Let $j \geq 0$. Write $Z_j = \pi(Z_j) + p_2(Z_j) = \pi^{||}(Z_j) + \pi^{\perp}(Z_j)$; hence $\| \pi(Z_j) - \pi^{||}(Z_j) \| = \| \pi^{\perp}(Z_j) - p_2(Z_j) \| \leq \| \pi^{\perp}(Z_j) \| + \| p_2(Z_j) \| = \| \pi^{\perp}(Z_j - \pi(Z_j)) \| + \| p_2(Z_j) \| \leq 2\| p_2(Z_j) \| \leq 2\text{diam}(\mathcal{R}_i)$. On the other hand $\| \pi_B(\pi^{||}(Z_j)) - \pi_B(Z_j) \| = \| (Z_j\cdot u)(u_B\cdot u)u_B - \|B\|^{-1}\beta^j u_B \| = | (Z_j\cdot u)(u_B\cdot u) - \|B\|^{-1}\beta^j | = \| \pi_B(\pi^{\perp}(Z_j)) \| \leq \| \pi^{\perp}(Z_j) \| \leq \text{diam}(\mathcal{R})$. Hence, since

$u_B \cdot u > 0$ (Theorem 3.2), $| Z_j \cdot u - (u_B \cdot u)^{-1} \|B\|^{-1} \beta^j | \leq (u_B \cdot u)^{-1} \mathrm{diam}(\mathcal{R})$. Consequently $\| \pi(Z_j) - (u_B \cdot u)^{-1} \|B\|^{-1} \beta^j u \| = \| \pi(Z_j) - \pi^{||}(Z_j) + \pi^{||}(Z_j) - (u_B \cdot u)^{-1} \|B\|^{-1} \beta^j u \| \leq \| \pi(Z_j) - \pi^{||}(Z_j) \| + \| \pi^{||}(Z_j) - (u_B \cdot u)^{-1} \|B\|^{-1} \beta^j u \| \leq 2\mathrm{diam}(\mathcal{R}_i) + (u_B \cdot u)^{-1})\mathrm{diam}(\mathcal{R})$. The quantity $\pi(Z_j) \cdot u$ tends to $+\infty$ as $(u_B \cdot u)^{-1} \|B\|^{-1} \beta^j$ when $j \to +\infty$. Then there exists j_0 such that $Z_j \cdot u \geq 2\mathrm{diam}(\mathcal{R}_i) + ((u_B \cdot u)^{-1} + \theta)\mathrm{diam}(\mathcal{R}_i)$, for all $j \geq j_0$. As a consequence $\pi(Z_j) \cdot u \geq Z_j \cdot u - 2\mathrm{diam}(\mathcal{R}_i) \geq ((u_B \cdot u)^{-1} + \theta)\mathrm{diam}(\mathcal{R}_i) > 0$ for all $j \geq j_0$. We claim that $Z_j \in K_\theta$ for all $j \geq j_0$. Indeed, since $\|p_2(Z_j)\| \leq \mathrm{diam}(\mathcal{R}_i)$, the inequalities hold: $\theta \|p_2(Z_j)\| \leq (\theta + (u_B \cdot u)^{-1})\|p_2(Z_j)\| \leq (\theta + (u_B \cdot u)^{-1})\mathrm{diam}(\mathcal{R}_i) \leq \pi(Z_j) \cdot u = \|\pi(Z_j)\|$ for all $j \geq j_0$.

(ii) If ${}^tQ$ is nonnegative the coefficients a_i in $P(X)$ are nonnegative with $a_0 \neq 0$ and at least one of the coefficients a_k, $k \geq 1$, is non-zero since β is assumed to be a Pisot number and not a Salem number. Hence (Proposition 3.1), since $P'(\beta) > 0$, we have $\|V_1\|^{-1} \xi_{1,1} = \|\pi^{||}(Z_0)\| = Z_0 \cdot u = \frac{a_0}{\beta P'(\beta)} > 0$ and $\|V_1\|^{-1} \xi_{j+1,1} = \|\pi^{||}(Z_j)\| = Z_j \cdot u$, for all $j = 1, 2, \cdots m-1$ with $\|V_1\|^{-1} \min\{\xi_{j,1} \mid j = 1, 2, \cdots m-1\} \geq \frac{a_0}{\beta^m P'(\beta)} > 0$. Because $\{Z_0, Z_1, \cdots, Z_{m-1}\}$ is the canonical basis of $\mathbb{R}^m$, any $Z_j, j \geq m$, can be written as a combination of the elements of this basis with positive coefficients. Hence, $Z_j \cdot u \geq \|V_1\|^{-1} \min\{ \xi_{l,1} \mid l = 1, 2, \cdots, m-1 \}$ for all $j \geq 0$. But the relation $Z_j = \pi(Z_j) + p_2(Z_j) = \pi^{||}(Z_j) + \pi^{\perp}(Z_j)$ implies that $\pi(Z_j) - \pi^{||}(Z_j) = \pi^{\perp}(Z_j) - p_2(Z_j)$. Hence, $|\pi(Z_j) \cdot u - Z_j \cdot u| \leq \|\pi^{\perp}(Z_j)\| + \|p_2(Z_j)\| \leq 2\|p_2(Z_j)\| \leq 2(\mathrm{diam}(\mathcal{R}_i))$ for all $j \geq 0$. Therefore $\pi(Z_j) \cdot u \geq \|V_1\|^{-1} \min\{ \xi_{l,1} \mid l = 1, 2, \cdots, m-1 \} - 2\mathrm{diam}(\mathcal{R}_i)$ which is > 0 by assumption for all $j \geq 0$. Hence, by definition of θ_{min} , $\pi(Z_j) \cdot u = \|\pi(Z_j)\| \geq \theta_{min}(\mathrm{diam}(\mathcal{R}_i)) \geq \theta_{min} \|p_2(Z_j)\| \geq \theta \|p_2(Z_j)\|$ for all $j \geq 0$ and $0 < \theta \leq \theta_{min}$. We deduce that $Z_j \in K_\theta$ for all $j \geq 0$ and $0 < \theta \leq \theta_{min}$. Let us observe that the conditions of the present assertion are generally not fulfilled. □

We now turn to the question of generating the elements of $\mathcal{L}$ by a finite number of them over $\mathbb{N}$. The idea we will follow is simple: let us consider the set of the semi-groups generated by a finite number of (arbitrary) elements of $\mathcal{L} \cap K_\theta$ for all $\theta > 0$; in this set, we will show the existence of semi-groups ($\theta > 0$ fixed) containing $K_{2\theta} \cap \mathcal{L}$, that is containing $\mathcal{L}$ except a finite number of elements of $\mathcal{L}$ close to the origin. Then we will minimize this finite number of excluded elements. For this we will consider the maximal possible values of θ. In final this will provide a suitable value of θ and a control of the norms of the generating elements of the semi-group which will contain $\mathcal{L}$.

Lemma 4.1. (Lind [Li]) *Let* $\theta > 0$. *If* $\delta = (2\theta+2)^{-1}$ *and* $x \in K_{2\theta}$ *with* $\|\pi(x)\| = \pi(x)\cdot u > 4$, *then* $[x - K_\theta(1,3)] \cap K_{2\theta}$ *contains a ball of radius* δ.

Proof. [Li] Take $y = 2u + 3(\pi(x)\cdot u)^{-1}p_2(x)$. We will show that the ball centred at $x-y$ and of radius δ satisfies our claim. Suppose $\|z\| < \delta$. Then $x - y + z \in K_{2\theta}$. Indeed,

$$2\theta\ \|p_2(x-y+z)\| \leq 2\theta\ \left[\left(1 - 3(\pi(x)\cdot u)^{-1}\right)\|p_2(x)\| + \delta\right]$$

$\leq \left[1 - 3(\pi(x)\cdot u)^{-1}\right](\pi(x)\cdot u) + 2\theta(2\theta+2)^{-1} = (\pi(x)\cdot u) - 2 - 2(2\theta+2)^{-1}$ but $p_2(y) = 2$. We deduce $2\theta\ \|p_2(x-y+z)\| \leq \pi(x-y+z)\cdot u$. Let us show that $y - z \in K_\theta$. We have $2\theta\|p_2(y)\| = 6\theta(\pi(x)\cdot u)^{-1}\|p_2(x)\| \leq 3(\pi(x)\cdot u)^{-1}(\pi(x)\cdot u) = 3$. Therefore $\theta\|p_2(y-z)\| \leq \theta(\|p_2(y)\| + \delta) \leq \frac{3}{2} + \theta(2\theta+2)^{-1} = 2 - (2\theta+2)^{-1} \leq \pi(y-z)\cdot u$. Now, since $\delta < 1$, we have the inequalities $1 \leq \pi(y-z)\cdot u \leq 3$, establishing the result. □

Theorem 4.2. *Let* $\theta > 0$. *If* r *is such that* $r > \rho(2\theta+2)$, *then* $K_{2\theta} \cap \mathcal{L} \subset sg(K_\theta(r) \cap \mathcal{L})$.

Proof. Lemma 4.1 implies the following assertion: if $x \in K_{2\theta}$ is such that $\pi(x)\cdot u > 4r$ with $r > \rho(2\theta+2)$, then $[x - K_\theta(r,3r)] \cap K_{2\theta}$ contains a ball of radius $r\delta > \rho$. But ρ is by definition the covering radius of $\mathcal{L} \cup (-\mathcal{L})$, hence this ball intersects $\mathcal{L}$. Now, let $\mathcal{A} = K_\theta(4r) \cap \mathcal{L}$ be the finite point set of $\mathcal{L}$ and let us show that $K_{2\theta} \cap \mathcal{L} \subset sg(\mathcal{A})$. First the inclusion $K_{2\theta}(4r) \cap \mathcal{L} \subset sg(\mathcal{A})$ holds. We now proceed inductively. Suppose $K_{2\theta}(r') \cap \mathcal{L} \subset sg(\mathcal{A})$ for some $r' \geq 4r$. We will show that this implies $K_{2\theta}(r'+r) \cap \mathcal{L} \subset sg(\mathcal{A})$, which will suffice by induction. For this, let us take $g \in \mathcal{L} \cap [K_{2\theta}(r'+r)\ K_{2\theta}(r)]$. From Lemma 4.1 and the above, there exists an element, say y , in $\mathcal{L}$, contained in $[g - K_\theta(r,3r)] \cap K_{2\theta}(r')$. By assumption, $y \in sg(\mathcal{A})$ and $y = g - x$ for some $x \in K_\theta(r,3r) \cap \mathcal{L}_{\mathcal{L}} \subset sg(\mathcal{A})$. Therefore $g = x + y \in sg(\mathcal{A}) + sg(\mathcal{A}) \subset sg(\mathcal{A})$. This concludes the induction. □

Lemma 4.3. *For all* $\theta > 0$, *the following set:* $\mathcal{L}(\theta) := \{\ x \in \mathcal{L}\ \mid p_2(x) \in \mathcal{R}_i, x \notin K_\theta(\rho(2\theta+2)), x \notin K_{2\theta}\ \}$ *is finite.*

Proof. The proof is clear since all $g \in \mathcal{L}$ such that $\pi(g)\cdot u > 2\rho(2\theta+2)$ belongs to $K_{2\theta}$. □

Define $\theta_f := \max\{\theta > 0 \mid \#(\mathcal{L}(\theta)) \text{ is minimal}\}$ (where $\#(\cdot)$ denotes the number of elements of the set $(\cdot)$). If ${}^tQ$ is nonnegative and the condition (ii) in Proposition 4.1 satisfied, then the equality $\#(\mathcal{L}(\theta)) = 0$ holds for $\theta < \theta_{min}$ and therefore $\theta_f \geq \theta_{min}/2$.

Theorem 4.4. (Minimal decomposition). — *Any element* $g \in \mathcal{L} \setminus \mathcal{L}(\theta_f)$ *can be expressed as a finite combination over* $\mathbb{N}$ *of elements of the finite point set* $K_{\theta_f}(\rho(2\theta_f+2)) \cap \mathcal{L}$.

Proof. It is a consequence of Theorem 4.2 with $\theta = \theta_f$ and $r = \rho(2\theta_f + 2)$. □

Corollary 4.5. *There exist two disjoint finite subsets* $\mathcal{F} = \{\|B\|\pi_B(g) \cdot u_B \mid g \in \mathcal{L}(\theta_f))\}$ *and* $\mathcal{F}' = \{g_1, g_2, \cdots, g_\eta\} \subset \{\|B\|\pi_B(g) \cdot u_B \mid g \in K_{\theta_f}(\rho(2\theta_f + 2)) \cap \mathcal{L}\}$ *of* $\mathbb{Z}_\beta^+$ *such that*

$$\mathbb{Z}_\beta^+ \subset \mathcal{F} \ \cup\ \mathbb{N}[g_1, g_2, \cdots, g_\eta]. \tag{2}$$

The generating elements $g_i \in \mathcal{F}'$ *satisfy:* $\|g_i\| \leq \rho(2\theta_f+2)\|B\|^{-1}\|V_1\|^{-1} + diam(\mathcal{R}_i)$. *If the couple* $(\mathcal{F}, \mathcal{F}')$ *is such that* $\eta = \#\mathcal{F}'$ *is minimal for the inclusion relation (2) and* $\mathcal{F}$ *is empty, then the degree* m *of* β *divides* η.

Proof. To obtain the inclusion (2) it suffices to project $\mathcal{L}$ by π_B and to apply Theorem 4.2 and 4.4 and Lemma 4.3. Let us show the upper bound on the norms of the elements of $\mathcal{F}'$. If $g \in K_{\theta_f}(\rho(2\theta_f + 2)) \cap \mathcal{L}$ is decomposed as $g = \pi(g) + t$, where $t \in \mathcal{R}_i$, then, by Theorem 3.2 (i), we have: $\|\pi_B(g)\| \leq \|\pi(g)\|\|B\|^{-1}\|V_1\|^{-1} + \mathrm{diam}(\mathcal{R}_i)$. But $\|\pi(g)\| \leq \rho(2\theta_f + 2)$. We deduce the claim. Now if $\mathbb{Z}_\beta^+ \subset \mathbb{N}[g_1, g_2, \cdots, g_\eta]$ the group $\mathbb{Z}[g_1, g_2, \cdots, g_\eta]$ contains $\mathbb{Z}_\beta$ and the equality $\mathbb{Z}[\mathbb{Z}_\beta] = \mathbb{Z}[g_1, g_2, \cdots, g_\eta]$ necessarily holds. By Theorem 4.12 we deduce that m divides η since the rank of $\mathbb{Z}[\mathbb{Z}_\beta] = \mathbb{Z}[\mathbb{N}[\mathbb{Z}_\beta]]$ is η when η is the smallest integer such that the set inclusion (2) holds and that $\mathcal{F}$ is empty. □

B) Preperiods in the addition of beta-integers. – The Delone set $\mathbb{Z}_\beta$ endowed with the usual addition and multiplication cannot have a ring structure otherwise it would contain $\mathbb{Z}$ but it is obvious that $\mathbb{Z}_\beta$ contains no subset of the type $\lambda\,\mathbb{Z}, \lambda > 0$. This absence of ring structure on $\mathbb{Z}_\beta$ for the usual laws can be partially overcome by controlling the fractional parts of the Rényi β-expansions of $x + y$ and $x - y$ when $x, y \in \mathbb{Z}_\beta$. This is the aim of this paragraph to focus on the geometrical meaning of the sets T and T' as stated in Theorem 4.7 and of the exponent q in its Corollary 4.8.

The projection mappings will be the ones redefined (in a simpler way) at the beginning of the subsection A). Let $R > 0$ and I be an interval of $\mathbb{R}$ having compact closure. Let us extend the m-cube $\mathcal{C}$ for reasons which will appear below. Let $\mathcal{C}' = \{\ \sum_{j=0}^{m-1} \alpha_j Z_j \mid \alpha_j \in [-1; 1] \text{ for all } j = 0, 1, \cdots, m-1\ \}$. For all irreducible ${}^tQ$- invariant subspace F of $\mathbb{R}^m$, put $\delta'_F = \max_{x \in \mathcal{C}'} \|\pi_F(x)\|$, λ_F the absolute value of the eigenvalue of ${}^tQ$ on F and $c'_F = \lfloor\beta\rfloor \frac{\delta'_F}{1-\lambda_F^m}$. Denote by Ω'_F the closed interval centred at 0 in F of length $2c'_F$ if dim $F = 1$, resp. the closed disc centred at 0 in F of radius c'_F if dim $F = 2$. Let $\Omega' = \oplus_F \Omega'_F$. We will denote by $\mathcal{T}_{I,R} := \{x \in \mathbb{R}^m \mid p_2(x) \in \lfloor\beta\rfloor^{-1} R\,\Omega', \pi_B(x) \cdot u_B \in \|B\|^{-1} I\}$ the slice of the band defined by $\lfloor\beta\rfloor^{-1}R\,\Omega'$ in the internal space, extended by symmetrization with respect to $\lfloor\beta\rfloor^{-1}R\,\Omega$ (compare the definitions of Ω' and Ω in Theorem 3.6), of axis the expanding line $\mathbb{R}\,u$. Let $F_R := \{\text{frac}(z) \mid z = a_k\beta^k + a_{k-1}\beta^{k-1} + \cdots + a_1\beta + a_0, a_i \in \mathbb{Z}, |a_i| \le R\} \subset [0,1)$.

Lemma 4.6. *The set* $\{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathcal{T}_{[0,1),R+\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$ *is a finite subset of* $\mathbb{Z}[\beta]\cap[0,1)$ *and the following inclusion holds:* $F_R \subset \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathcal{T}_{[0,1),R+\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$.

Proof. The finiteness of the set is obvious: it is a discrete set in a subset of $\mathbb{R}^m$ having compact closure. The inclusion relation is a reformulation of Lemma 2.1 in [Bu]. Let us briefly recall the proof. Let $z = \sum_{j=0}^{k} a_j\beta^j$ with $a_i \in \mathbb{Z}, |a_i| \le R$. We have also $z = \sum_{j=-\infty}^{k} x_j\beta^j$ as β - expansion of z . Therefore $z-\text{int}(z) = \sum_{i=0}^{k} a_j\beta^k - \sum_{j=0}^{k} x_j\beta^j$. Since $0 \le x_j \le \lfloor\beta\rfloor$ and $|a_i| \le R$, $\text{frac}(z) \in [0,1)$ is a polynomial in β , the coefficients of which have their absolute values bounded by $R + \lfloor\beta\rfloor$. Here the coefficients may be negative or positive. This is why we have introduced $\mathcal{C}'$ instead of $\mathcal{C}$. We deduce the result in a similar way as in the proof of Theorem 3.6 for the computation of the upper bound c_F, except that now it is with Ω', c'_F and the fact that the absolute value of the digits is less than $R+\lfloor\beta\rfloor$; this obliges to multiply Ω' by the factor $(R+\lfloor\beta\rfloor)/\lfloor\beta\rfloor$. The set F_R is finite (Lemma 6.6 in [So]), and (Proposition 3.4) is in one-to-one correspondence with a subset of the finite point set $\mathcal{T}_{[0,1),R+\lfloor\beta\rfloor}$. We deduce the claim. □

Let

$$L_{I,R} := \lfloor \min \left\{ [\ln(\beta^{(i-1)^{-1}})]^{-1} \ln\left(\left(\sum_{k=0}^{m-1} (\beta^{(i-1)})^{2k}\right)^{1/2} \psi_{I,R+\lfloor\beta\rfloor} \right) \right\} \rfloor$$

where the minimum is taken over the real positive embeddings of $\mathbb{Q}(\beta)$ ($i = 1,2,\ldots,s$ and $\beta^{(i-1)} > 0$) and where $\psi_{I,R} := \max\{\|y\| \mid y \in \mathcal{T}_{I,R}\}$. Let us consider an element $z \in F_R$. Its β - expansion: $\sum_{j=1}^{+\infty} z_{-j}\beta^{-j}$ is eventually periodic [Be] [Sch] and therefore can be written $\sum_{j=1}^{k_0(z)} z_{-j}\beta^{-j} + \sum_{k=0}^{+\infty} \sum_{j=k_0(z)+kr(z)+1}^{k_0(z)+(k+1)r(z)} z_{-j}\beta^{-j}$ where the integers $k_0(z), r(z) \ge 1$ are minimal. We will denote by $J_R = \max\{k_0(z) \mid z \in F_R\}$ the maximal preperiod of the β - expansions of the elements of F_R. An upper bound of J_R will be computed below.

Theorem 4.7. *(i) For all* $x, y \in \mathbb{Z}_\beta^+$ *such that* $x+y$ *has a finite* β*-expansion the following relation holds:* $x + y \in \beta^{-L}\,\mathbb{Z}_\beta^+$ *where* $L := \min\{L_{[0,1),2\lfloor\beta\rfloor}, J_{2\lfloor\beta\rfloor}\}$*; (ii) the following inclusions hold:* $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + T$, $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta + T'$, *where* $T = \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathcal{T}_{[0,+1),3\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$ *and* $T' = \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathcal{T}_{(-1,+1),2\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$.

Proof. (i) Let $x = x_k\beta^k + \cdots + x_0$ and $y = y_l\beta^l + \cdots + y_0$ denote two elements of $\mathbb{Z}_\beta^+$. Then $z = x + y$ is of the form $z = a_j\beta^j + \cdots + a_0$ with $0 \leq a_j \leq 2\lfloor\beta\rfloor$. Write now the β - expansion of z as $z = \sum_{j=1}^{+\infty} z_{-j}\beta^{-j} + \sum_{j=0}^{e} z_j\beta^j$ and assume it is finite. Then it admits only a β-expansion up till the term indexed by its preperiod $k_0(z)$ and the period has necessarily the form given above with $r(z) = 1$ and $z_{-j} = 0$ as soon as $j > k_0(z)$. Then $\sum_{j=1}^{k_0(z)} z_{-j}\beta^{-j} = (a_j\beta^j + \cdots + a_0) - (\sum_{i=0}^{e} z_i\beta^i)$. This means that the fractional part $\sum_{j=1}^{k_0(z)} z_{-j}\beta^{-j}$ is a polynomial of the type $\sum_{i=0}^{f} b_i\beta^i$ with $-\lfloor\beta\rfloor \leq b_i \leq 2\lfloor\beta\rfloor$ hence with $|b_i| \leq 2\lfloor\beta\rfloor$. The set $F_{2\lfloor\beta\rfloor}$ is finite (Lemma 4.6) and the set of all possible fractional parts of elements of $\mathbb{Z}_\beta^+$ is exactly in one-to-one correspondence with a subset of the finite point set $\mathcal{T}_{[0,1),3\lfloor\beta\rfloor} \cap \mathbb{Z}^m$ of $\mathbb{Z}^m$. Therefore, there exists a unique $g_z = \sum_{i=0}^{f} b_i Z_i \in \mathcal{T}_{[0,1),3\lfloor\beta\rfloor} \cap \mathbb{Z}^m$ such that $\|B\|\pi_B(g_z) \cdot u_B = \sum_{i=0}^{f} b_i\beta^i = \sum_{j=1}^{k_0(z)} z_{-j}\beta^{-j} = \mathrm{frac}(z)$. Let us apply the real and complex embeddings of the number field $\mathbb{Q}(\beta)$. It gives: $\sum_{j=1}^{k_0(z)} z_{-j}(\beta^{(i-1)})^{-j} = \sum_{j=0}^{f} b_j(\beta^{(i-1)})^j$ for all $i = 2, 3, \ldots, m$. For the real embeddings in particular this implies (Proposition 3.2):

$$\pi_{B,i}(g_z) = \pi_{B,i}(\sum_{j=0}^{f} b_j Z_j) = \frac{\sum_{j=0}^{f} b_j(\beta^{(i-1)})^j}{\|X_i\|} \; u_{B,i} = \frac{\sum_{j=1}^{k_0(z)} z_{-j}(\beta^{(i-1)})^{-j}}{\|X_i\|} \; u_{B,i}$$

for all $i = 1, 2, \cdots, s$ with all $z_{-j} \geq 0$. The case of real embeddings will provide a direct computation of the first upper bound $L_{[0;1),2\lfloor\beta\rfloor}$ of the preperiod and merits to be isolated. Indeed, since in this case $0 < \beta^{(i-1)} < 1$ for all $i \in \{2, 3, \cdots, s\}$, with s assumed ≥ 2, and that all the digits z_{-j} are positive, we necessarily have: $\|X_i\|^{-1}\left(\beta^{(i-1)}\right)^{-j} \geq \psi_{[0,1),3\lfloor\beta\rfloor}$ as soon as j is large enough. Recall that $\|X_i\| = \left(\sum_{k=0}^{m-1}(\beta^{(i-1)})^{2k}\right)^{1/2}$. With the definition of $L_{[0;1),2\lfloor\beta\rfloor}$, this implies that the sum of the positive terms $\sum_{j=1}^{k_0(z)} z_{-j}(\beta^{(i-1)})^{-j}$ cannot contain any term indexed by $-j$ with $j > L_{[0;1),2\lfloor\beta\rfloor}$. Hence, $k_0(z) \leq L_{[0;1),2\lfloor\beta\rfloor}$. As for the negative real embeddings and the complex embeddings they will provide the second upper bound of the preperiod by the computation of $J_{2\lfloor\beta\rfloor}$: indeed, its calculation gives an upper bound of the number of terms $k_0(z)$ in the fractional part of z, hence, after reducing $\mathrm{frac}(z)$ to the same denominator, which will be $\beta^{k_0(z)}$, we immediately get the result; (ii) (This is reformulation of Theorem 2.4 in [Bu]) First, we have $F_{\lfloor\beta\rfloor} \subset F_{2\lfloor\beta\rfloor}$, second $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + F_{2\lfloor\beta\rfloor}$, $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta + (F_{\lfloor\beta\rfloor} \cup -F_{\lfloor\beta\rfloor})$. Since Ω' is invariant by inversion and that

$F_{\lfloor\beta\rfloor} \cup -F_{\lfloor\beta\rfloor} \subset \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathcal{T}_{[0,+1),2\lfloor\beta\rfloor} \cap \mathbb{Z}^m\} \cup \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathcal{T}_{(-1,0],2\lfloor\beta\rfloor} \cap \mathbb{Z}^m\} = \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathcal{T}_{(-1,+1),2\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$ (Lemma 4.6), we deduce the claim. □

Corollary 4.8. *Let* $q = \min\{L_{(-1,+1),2\lfloor\beta\rfloor}, J_{2\lfloor\beta\rfloor}\}$. *Then, for all* $x, y \in \mathbb{Z}_\beta$ *such that* $x+y$ *and* $x-y$ *have finite* β*-expansions, the following relations hold:* $x+y$ (resp. $x-y$) $\in \beta^{-q}\,\mathbb{Z}_\beta$.

Proof. Indeed, $T' \subset T \cup (-T)$. Hence $\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + (T \cup (-T))$. Since $T \cup (-T) = \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathcal{T}_{(-1,+1),3\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$, we deduce the exponent q from the definition of $L_{I,R}$ and from Theorem 4.7. □

Computation of an upper bound of the maximal preperiod J_R. — We will use the case (ii) of cut-and-project scheme in Theorem 3.5. Let $\{Z_{-j}\}_{j\geq 0}$ be the sequence of vectors defined by $Z_0 = ({}^tQ)^j Z_{-j}$. We denote as usual the algebraic norm of β by $N(\beta) = N_{\mathbb{Q}(\beta)/\mathbb{Q}}(\beta) = \prod_{i=0}^{m-1}\beta^{(i)}$. Recall that $a_0 = (-1)^{m-1}N(\beta)$.

Lemma 4.9. *(i) The following limit holds:* $\lim_{j\to+\infty}\|Z_{-j}\| = +\infty$ *; (ii) for all* $j \in \mathbb{N}$, $Z_{-j} \in N(\beta)^{-j}\,\mathbb{Z}^m$. *In particular, if* β *is a unit of the number field* $\mathbb{Q}(\beta)$ *, then all the elements* Z_{-j} *belong to* $\mathbb{Z}^m$.

Proof. (i) Since $|\beta^{(i)}|^{-1} > 1$ for all $i = 1, 2, \ldots, m-1$, the inverse operator $({}^tQ)^{-1}$ acts as a dilation by a factor of modulus strictly greater than one on each ${}^tQ$-invariant subspace F in $\mathbb{R}^m$ except $\mathbb{R}\,u$: all the non-zero components of the vector Z_{-j} (which never belongs to $\mathbb{R}\,u$) in the system $\{V_i\}_{i=2,3,\cdots,m}$ diverge when j tends to infinity, hence the claim. (ii) Solving the equation $Z_0 = ({}^tQ)^1 Z_{-1}$ shows that Z_{-1} can be written $Z_{-1} = -a_0^{-1}(a_1 Z_0 + a_2 Z_1 + \cdots + a_{m-1}Z_{m-2} - Z_{m-1}) \in N(\beta)^{-1}\,\mathbb{Z}^m$. Since by construction we have $Z_j = ({}^tQ)^{-1}(Z_{j+1})$ for all $j \in \mathbb{Z}$, applying $({}^tQ)^{-1}$ to the last equality clearly gives $Z_{-2} \in N(\beta)^{-2}\,\mathbb{Z}^m$ and, by induction $Z_{-h} \in N(\beta)^{-h}\,\mathbb{Z}^m$ for all $h \geq 0$. Now it is classical that β is a unit of $\mathbb{Q}(\beta)$ if and only if $N(\beta) = \pm 1$ establishing the result. □

Theorem 4.10. *Denote by*

$$\mathcal{B}_R = \left\{ x \in \mathbb{R}^m \mid \|\pi_{B,i}(x)\| \leq \frac{\psi_{[0;1),R+\lfloor\beta\rfloor}\left(1 - |\beta^{(i-1)}|^m\right) + \lfloor\beta\rfloor}{\left(\sum_{k=0}^{m-1}|\beta^{(i-1)}|^{2k}\right)^{1/2}\left(1 - |\beta^{(i-1)}|\right)}, \quad i = 2, 3, \cdots, m \right\}$$

the cylinder (band) of axis the expanding line $\mathbb{R}\,u$ *and* $\mathcal{V}_R = \{x \in \mathcal{B}_R \mid \|B\|\pi_B(x)\cdot u_B \in [0,1)\}$ *the slice of the band* $\mathcal{B}_R$ *. Then this slice is such that* $J_R \leq \#(\mathcal{V}_R \cap N(\beta)^{-m}\,\mathbb{Z}^m)$.

Proof. Each element $\alpha \in F_R$ can be written $\alpha = \sum_{i=0}^{m-1} p_i \beta^i$ with $p_i \in \mathbb{Z}$ and $\sum_{i=0}^{m-1} p_i Z_i \in \mathcal{T}_{[0,1),R+\lfloor\beta\rfloor} \cap \mathbb{Z}^m$ (Lemma 4.6). Thus, $|p_i| \leq \psi_{[0,1),R+\lfloor\beta\rfloor}$ for all $i = 0, 1, \cdots, m-1$. Now ([Sch] and section 2), the following equality holds for all $n \geq 0$:

$$T^n(\alpha) = \beta^n \cdot \left(\alpha - \sum_{k=0}^{n} \epsilon_k(\alpha)\beta^{-k}\right) = \sum_{k=1}^{m} r_k^{(n)} \beta^{-k}$$

where $(\epsilon_k(\alpha))_{k\geq 0}$ is the sequence of digits of the Rényi β-expansion of α and $(r_1^{(n)}, r_2^{(n)}, \cdots, r_m^{(n)}) \in \mathbb{Z}^m$. Recall that $\epsilon_0(\alpha) = \lfloor\alpha\rfloor = 0$. The real and complex embeddings of the number field $\mathbb{Q}(\beta)$ applied to $T^n(\alpha)$ provide the m equalities, with $j = 1, 2, \cdots, m$:

$$\left(\beta^{(j-1)}\right)^n \cdot \left(\sum_{i=0}^{m-1} p_i \left(\beta^{(j-1)}\right)^i - \sum_{k=1}^{n} \epsilon_k(\alpha) \left(\beta^{(j-1)}\right)^{-k}\right) = \sum_{k=1}^{m} r_k^{(n)} \left(\beta^{(j-1)}\right)^{-k}.$$

We deduce that

$$\begin{aligned}\left(\sum_{k=0}^{m-1} |\beta^{(i-1)}|^{2k}\right)^{1/2} & \|\pi_{B,i}(\sum_{k=1}^{m} r_k^{(n)} (\beta^{(j-1)})^{-k})\| \\ &= \left|\sum_{k=1}^{m} r_k^{(n)} (\beta^{(j-1)})^{-k}\right| \\ &\leq \sum_{i=0}^{m-1} |p_i||\beta^{(j-1)}|^{n+i} + \lfloor\beta\rfloor \sum_{k=0}^{n} |\beta^{(j-1)}|^k \\ &\leq \frac{1}{1-|\beta^{(j-1)}|} \left[\psi_{[0,1),R+\lfloor\beta\rfloor}(1 - |\beta^{(j-1)}|^m) + \lfloor\beta\rfloor\right]\end{aligned}$$

for all $n \geq 0, j = 2, 3, \cdots, m$ with $0 \leq \sum_{k=1}^{m} r_k^{(n)} \beta^{-k} < 1$. From Proposition 3.2 and Lemma 4.9 the element $\sum_{k=1}^{m} r_k^{(n)} \beta^{-k}$ can be uniquely lifted up to the element $\sum_{k=1}^{m} r_k^{(n)} Z_{-k} \in N(\beta)^{-m}\, \mathbb{Z}^m$. Its projections by the projection mappings $\pi_{B,i}$, $i = 2, 3, \cdots, m$ to the Q-invariant subspaces of $\mathbb{R}^m$ are bounded by constants which are independant of n . The restriction of the lifting of the operator T to $\mathcal{V}_R \cap N(\beta)^{-m}\, \mathbb{Z}^m$ has self-avoiding orbits (to have a preperiod) whose length is necessarily smaller than the number of available points in the volume $\mathcal{V}_R$. We deduce the upper bound $\#(\mathcal{V}_R \cap N(\beta)^{-m}\, \mathbb{Z}^m)$ of J_R. □

Appendix.– Classification of Delone sets. We will say that a subset Λ of $\mathbb{R}^n$ is (i) uniformly discrete if there exists $r > 0$ such that $\|x-y\| \geq r$ for all $x, y \in \Lambda, x \neq y$, (ii) relatively dense if there exists $R > 0$ such that, for all $z \in \mathbb{R}^n$, there exists $\lambda \in \Lambda$ such that the ball $B(z, R)$ contains λ, (iii) a Delone set if it is relatively dense and uniformly discrete. Delone sets are basic objects for mathematical quasicrystals [La2] [MVG].

Definition 4.1. A cut-and-project scheme consists of a direct product $E \times D$, where E and D are Euclidean spaces of finite dimension, and a lattice L in $E \times D$ so that, with respect to the natural projections $p_1 : E \times D \to E$, $p_2 : E \times D \to D$: (i) p_1 restricted to L is one-to-one onto its image $p_1(L)$, (ii) $p_2(L)$ is dense in D. We will denote by $*$ the following operation: $* := p_2 \circ (p_{1_{|_L}})^{-1} : p_1(L) \to D$.

Definition 4.2. A subset Λ of a finite dimensional Euclidean space E is a model set (also called a cut-and-project set) if there exist a cut-and-project scheme $(E \times D, L)$ and a subset Ω of D with nonempty interior and compact closure such that $\Lambda = \Lambda(\Omega) = \{p_1(l) \mid l \in L, p_2(l) \in \Omega\}$, equivalently $= \{v \in p_1(L) \mid v^* \in \Omega\}$. The set Ω is called acceptance window.

Meyer sets were introduced in [Mey]. By definition, we will say that Λ, assumed to be a relatively dense subset of $\mathbb{R}^n$, is a Meyer set of $\mathbb{R}^n$ if it is a subset of a model set. Other equivalent definitions can be found in [Mo] or [Mey]. For instance, Λ is a Meyer set if and only if it is a Delone set and there exists a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$; or if and only if it is relatively dense and $\Lambda - \Lambda$ is uniformly discrete. The above definition shows that the class of Meyer sets of $\mathbb{R}^n$ contains the class of model sets of $\mathbb{R}^n$.

Theorem 4.11. *(Meyer* [Mey]*) Let Λ be a Delone set in $\mathbb{R}^n$ such that $\eta\Lambda \subset \Lambda$ for a real number $\eta > 1$. If Λ is a Meyer set, then η is a Pisot or a Salem number.*

Definition 4.3. A Delone set Λ is said to be finitely generated if $\mathbb{Z}[\Lambda - \Lambda]$ is finitely generated. A Delone set Λ is said to be of finite type if $\Lambda - \Lambda$ is such that its intersection with any closed ball of $\mathbb{R}^n$ is a finite set.

The class of finitely generated Delone sets of $\mathbb{R}^n$ is strictly larger than the class of Delone set of finite type of $\mathbb{R}^n$, which is itself larger than the class of Meyer sets of $\mathbb{R}^n$ [La] [La1].

Theorem 4.12. *(Lagarias* [La]*) Let Λ be a Delone set in $\mathbb{R}^n$ such that $\eta\Lambda \subset \Lambda$ for a real number $\eta > 1$. The following assertions hold: (i) If Λ is finitely generated, then η is an algebraic integer. If the rank of $\mathbb{Z}[\Lambda]$ is s, then the degree of η divides s, (ii) If Λ is a Delone set of finite type, then η is a Perron number or is a Lind number.*

Although $\mathbb{Z}_\beta$ is associated with two canonical cut-and-project schemes when β is a non-integer Perron number, the converse of the assertion (ii) of Theorem 4.12 seems to be an open problem. It is at least already related to the question Q1' of the introduction and to various arithmetical and dynamical problems [ABEI].

Acknowledgements.— We are indebted to Christiane Frougny, Valérie Berthé, Pierre Arnoux and a referee for very useful and valuable comments and discussions.

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“On Gaps in Rényi β-expansions of unity for $\beta > 1$ an algebraic number”,

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Annales Institut Fourier (2006), accepté.

On gaps in Rényi β-expansions of unity for $\beta > 1$ an algebraic number

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Abstract. Let $\beta > 1$ be an algebraic number. We study the strings of zeros ("gaps") in the Rényi β-expansion $d_\beta(1)$ of unity which controls the set $\mathbb{Z}_\beta$ of β-integers. Using a version of Liouville's inequality which extends Mahler's and Güting's approximation theorems, the strings of zeros in $d_\beta(1)$ are shown to exhibit a lacunarity bounded above by $\log(\mathrm{M}(\beta))/\log(\beta)$, where $\mathrm{M}(\beta)$ is the Mahler measure of β. The proof of this result provides in a natural way a new classification of algebraic numbers > 1 where classes are called $\mathrm{Q}_i^{(j)}$ that we compare to Blanchard's one with classes C_1 to C_5. This new classification relies upon the maximal asymptotic "quotient of the gap" value of the lacunary power series associated with $d_\beta(1)$. As a corollary, all Salem numbers are in the class $\mathrm{C}_1 \cup \mathrm{Q}_0^{(1)} \cup \mathrm{Q}_0^{(2)} \cup \mathrm{Q}_0^{(3)}$; this result is also directly proved by a recent generalization of the Thue-Siegel-Roth Theorem given by Corvaja.

2000 Mathematics Subject Classification: 11B05, 11Jxx, 11J68, 11R06, 52C23.

Keywords: Beta-integer, beta-numeration, PV number, Salem number, Perron number, Mahler measure, Diophantine approximation, Mahler's series, mathematical quasicrystal.

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1 Introduction

The exploration of the links between symbolic dynamics and number theory concerned with β-expansions, when $\beta > 1$ is an algebraic number or more generally a real number, has started with Bertrand-Mathis [Be1] [Be2]. Blanchard [Bl] reported a possible classification of real numbers according to their β-shift, by the properties of the Rényi β-expansion $d_\beta(1)$ of 1. A lot of questions remain open concerning the distribution of the algebraic numbers $\beta > 1$ in this classification. The Rényi β-expansion of 1 is important since it controls the β-shift [Pa] and the discrete and locally finite set $\mathbb{Z}_\beta \subset \mathbb{R}$ of β-integers [B-K] [E-VG] [Ga] [G1]. The objective of this note is to give a new Theorem (Theorem 1.1) on the gaps (strings of $0'$s) in $d_\beta(1)$ for algebraic numbers $\beta > 1$, and investigate how it brings (partial) answers to some questions of [Bl], in particular for Salem numbers (Corollary 1.2).

Theorem 1.1 provides an upper bound of the asymptotic quotient of the gap of $d_\beta(1)$ and is obtained by a version of Liouville's inequality extending Mahler's and Güting's approximation theorems. The course of the proof of Theorem 1.1 reveals to be extremely instructive since it leads to a new possible classification of the algebraic numbers β as a function of the asymptotics of gaps in $d_\beta(1)$ and "intrinsic features", namely the Mahler measure $\mathrm{M}(\beta)$, of β (the definition of $\mathrm{M}(\beta)$ is recalled in Section 3). This double parametrization, symbolic and algebraic, was guessed in [Bl] p 137. This new classification complements Blanchard's one [Bl] pp 137–139 and both are recalled below for comparison. The question whether an algebraic number $\beta > 1$ is contained in one class or another one was already discussed by many authors [Be1] [Be2] [Be3] [Bl] [Bo] [Bo1] [Bo2] [Bo3] [D-S] [FS] [Li1] [Li2] [Pa] [PF] [Sc] [Sk] and depends at least upon the distribution of the conjugates of β in the complex plane. Only the conjugates of β of modulus strictly greater than unity intervene in Theorem 1.1 via the Mahler measure of β. Corollary 1.2 is readily deduced from this remark. We deduce that Salem numbers belong to $\mathrm{C}_1 \cup \mathrm{C}_2 \cup \mathrm{Q}_0$, while Pisot numbers are in $\mathrm{C}_1 \cup \mathrm{C}_2$ [Th].

Another proof of Corollary 1.2 consists in controlling the gaps of $d_\beta(1)$ by stronger Theorems of Diophantine Geometry which allow suitable collections of places of the number field $\mathbb{Q}(\beta)$ associated with the conjugates of β and the properties of $d_\beta(1)$ to be taken into account simultaneously. This other proof of Corollary 1.2, just sketched in Section 4, is obtained by using the Theorem of Thue-Siegel-Roth given by Corvaja [A] [C].

Theorem 1.1. *Let* $\beta > 1$ *be an algebraic number and* $\mathrm{M}(\beta)$ *be its Mahler measure. Denote by* $d_\beta(1) := 0.t_1t_2t_3\ldots$, *with* $t_i \in A_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$, *the Rényi* β*-expansion of* 1. *Assume that* $d_\beta(1)$ *is infinite and lacunary in the following sense: there exist two sequences* $\{m_n\}_{n\geq 1}$, $\{s_n\}_{n\geq 0}$ *such that*

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \ldots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \ldots$$

with $(s_n - m_n) \geq 2$, $t_{m_n} \neq 0, t_{s_n} \neq 0$ *and* $t_i = 0$ *if* $m_n < i < s_n$ *for all* $n \geq 1$. *Then*

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} \leq \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}. \tag{1.1}$$

Moreover, if $\liminf_{n\to+\infty}(m_{n+1} - m_n) = +\infty$, *then*

$$\limsup_{n\to+\infty} \frac{s_{n+1} - s_n}{m_{n+1} - m_n} \leq \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}. \tag{1.2}$$

Following Ostrowski [Os] the quotient $s_n/m_n \geq 1$ is called *quotient of the gap*, relatively to the nth-gap (assuming $t_j \neq 0$ for all $s_n \leq j \leq m_{n+1}$ to describe uniquely the gaps). Note that the term "*lacunary*" has often other meanings in literature. Denote by $\mathcal{L}(S_\beta)$ the language of the β-shift [Bl] [Fr1] [Fr2] [Lo]. Blanchard's classification ([Bl] pp 137–139) is as follows:

C_1 : $d_\beta(1)$ is finite.
C_2 : $d_\beta(1)$ is ultimately periodic but not finite.
C_3 : $d_\beta(1)$ contains bounded strings of $0'$s, but is not ultimately periodic.
C_4 : $d_\beta(1)$ does not contain some words of $\mathcal{L}(S_\beta)$, but contains strings of $0'$s with unbounded length.
C_5 : $d_\beta(1)$ contains all words of $\mathcal{L}(S_\beta)$.

Present classes of algebraic numbers, with the notations of Theorem 1.1:

$\mathrm{Q}_0^{(1)}$: $1 = \lim_{n\to+\infty} \frac{s_n}{m_n}$ with $(m_{n+1} - m_n)$ bounded.
$\mathrm{Q}_0^{(2)}$: $1 = \lim_{n\to+\infty} \frac{s_n}{m_n}$ with $(s_n - m_n)$ bounded and $\lim_{n\to+\infty}(m_{n+1} - m_n) = +\infty$.
$\mathrm{Q}_0^{(3)}$: $1 = \lim_{n\to+\infty} \frac{s_n}{m_n}$ with $\limsup_{n\to+\infty}(s_n - m_n) = +\infty$.

$$\mathrm{Q}_1 \ : \qquad 1 \ < \ \limsup_{n\to+\infty} \frac{s_n}{m_n} \ < \ \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}.$$
$$\mathrm{Q}_2 \ : \qquad \limsup_{n\to+\infty} \frac{s_n}{m_n} \ = \ \frac{\log(\mathrm{M}(\beta))}{\log(\beta)}.$$

What are the relative proportions of each class in the whole set $\overline{\mathbb{Q}}_{>1}$ of algebraic numbers $\beta > 1$? Comparing C_2, C_3 and $\mathrm{Q}_0^{(1)}$, what are the relative proportions in $\mathrm{Q}_0^{(1)}$ of those β which give ultimate periodicity in $d_\beta(1)$ and those for which $d_\beta(1)$ is not ultimately periodic ? Schmeling ([Sc] Theorem A) has shown that the class C_3 (of real numbers $\beta > 1$) has Hausdorff dimension one. We have:

- $\overline{\mathbb{Q}}_{>1} \cap C_2 \quad \subset \quad \mathrm{Q}_0^{(1)}$,
- $\overline{\mathbb{Q}}_{>1} \cap C_3 \quad \subset \quad \mathrm{Q}_0^{(1)} \cup \mathrm{Q}_0^{(2)}$, with $C_3 \cap \mathrm{Q}_0^{(3)} = \emptyset$,
- $\overline{\mathbb{Q}}_{>1} \cap C_4 \quad \subset \quad \mathrm{Q}_0^{(3)} \cup \mathrm{Q}_1 \cup \mathrm{Q}_2$.

Pisot numbers β are contained in $\mathrm{C}_1 \cup \mathrm{Q}_0^{(1)}$ since they are such that $d_\beta(1)$ is finite or ultimately periodic (Parry [Pa], Bertrand-Mathis [Be3]). Recall that a Perron number is an algebraic integer $\beta > 1$ such that all the conjugates $\beta^{(i)}$ of β satisfy $|\beta^{(i)}| < \beta$. Conversely, after Lind [Li1], Denker, Grillenberger, Sigmund [D-S] and Bertrand-Mathis [Be2], if $\beta > 1$ is such that $d_\beta(1)$ is ultimately periodic (finite or not), then β is a Perron number. Not all Perron numbers are reached in this way: a Perron number which possesses a real conjugate greater than 1 cannot be such that $d_\beta(1)$ is ultimately periodic ([Bl] p 138). Parry numbers belong to $\mathrm{C}_1 \cup \mathrm{C}_2$. Let $\mathrm{Q}_0 = \mathrm{Q}_0^{(1)} \cup \mathrm{Q}_0^{(2)} \cup \mathrm{Q}_0^{(3)}$.

Corollary 1.2. *Let $\beta > 1$ be a Salem number which does not belong to* C_1. *Then β belongs to the class* Q_0.

The dispatching of Salem numbers in C_1, $\mathrm{Q}_0^{(1)}$, $\mathrm{Q}_0^{(2)}$ and $\mathrm{Q}_0^{(3)}$ is an open problem in general, except in low degree. Boyd [Bo] [Bo3] has shown that Salem numbers of degree 4 belong to C_2, hence to $\mathrm{Q}_0^{(1)}$. It is also the case of some Salem numbers of degree 6 and ≥ 8 in the framework of a probabilistic model [Bo2] [Bo3]. In Section 5 we ask the question whether Corollary 1.2 could still be true for Perron numbers.

The definition of the class Q_0 does not make any allusion to β, i.e. to $\mathrm{M}(\beta)$, to the conjugates of β, to the minimal polynomial of β or to its length, etc, but takes only into account the quotients of the gaps in $d_\beta(1)$. Hence this class Q_0 can be addressed to real numbers $\beta > 1$ in full generality instead of only to algebraic numbers > 1. The question whether there exist transcendental numbers $\beta > 1$ which belong to the class Q_0 was asked in [Bl]; in which proportion in each subclass ? Examples of transcendental numbers (Komornik-Loreti constant [AC] [KL], Sturmian numbers [CK]) in Q_0 are given in Section 5.

In the present note, we are dealing with the algebraicity of values of lacunary series, deduced from $d_\beta(1)$, at the algebraic point β^{-1}. In a related context, concerning more transcendency, Nishioka [N] and Corvaja Zannier [CZ] have followed different patterns and applied the Subspace Theorem [Sw] to deduce different results.

2 Definitions

For $x \in \mathbb{R}$ the integer part of x is $\lfloor x \rfloor$ and its fractional part $\{x\} = x - \lfloor x \rfloor$. The smallest integer larger than or equal to x is denoted by $\lceil x \rceil$. For $\beta > 1$ a real number and $z \in [0,1]$ we denote by $T_\beta(z) = \beta z \pmod 1$ the β-transform on $[0,1]$ associated with β [Pa] [Re], and iteratively, for all integers $j \geq 0$, $T_\beta^{j+1}(z) := T_\beta(T_\beta^j(z))$, where by convention $T_\beta^0 = Id$.

Let $\beta > 1$ be a real number. A beta-representation (or β-representation, or representation in base β) of a real number $x \geq 0$ is given by an infinite sequence $(x_i)_{i \geq 0}$ and an integer $k \in \mathbb{Z}$ such that $x = \sum_{i=0}^{+\infty} x_i \beta^{-i+k}$, where the digits x_i belong to a given alphabet ($\subset \mathbb{N}$) [Fr1] [Fr2] [Lo]. Among all the beta-representations of a real number $x \geq 0, x \neq 1$, there exists a particular one called Rényi β-expansion, which is obtained through the greedy algorithm: in this case, k satisfies $\beta^k \leq x < \beta^{k+1}$ and the digits

$$x_i := \lfloor \beta T_\beta^i\left(\frac{x}{\beta^{k+1}}\right) \rfloor \qquad i = 0, 1, 2, \ldots, \tag{2.1}$$

belong to the finite canonical alphabet $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$. If β is an integer, then $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \beta - 1\}$; if β is not an integer, then $\mathbb{A}_\beta := \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}$. We denote by

$$\langle x \rangle_\beta := x_0 x_1 x_2 \ldots x_k \,.\, x_{k+1} x_{k+2} \ldots \tag{2.2}$$

the couple formed by the string of digits $x_0 x_1 x_2 \ldots x_k x_{k+1} x_{k+2} \ldots$ and the position of the dot, which is at the k-th position (between x_k and x_{k+1}). By definition the integer part (in base β) of x is $\sum_{i=0}^{k} x_i \beta^{-i+k}$ and its fractional part (in base β) is $\sum_{i=k+1}^{+\infty} x_i \beta^{-i+k}$. If a Rényi β-expansion ends in infinitely many zeros, it is said to be finite and the ending zeros are omitted. If it is periodic after a certain rank, it is said to be eventually periodic (the period is the smallest finite string of digits possible, assumed not to be a string of zeros); for the substitutive approach see [F] [PF].

The Rényi β-expansion which plays an important role in the theory is the Rényi β-expansion of 1, denoted by $d_\beta(1)$ and defined as follows: since $\beta^0 \leq 1 < \beta$, the value $T_\beta(1/\beta)$ is set to 1 by convention. Then using the formulae (2.1)

$$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\} \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\} \rfloor, \ldots \tag{2.3}$$

The writing

$$d_\beta(1) = 0.t_1 t_2 t_3 \dots$$

corresponds to

$$1 = \sum_{i=1}^{+\infty} t_i \beta^{-i}.$$

Links between the set $\mathbb{Z}_\beta$ of beta-integers and $d_\beta(1)$ are evoked in [E-VG] [F-K] [G1] [G2] [V1] [V2]. A real number $\beta > 1$ such that $d_\beta(1)$ is finite or eventually periodic is called a *beta-number* or more recently *a Parry number* (this recent terminology appears in [E-VG]). In particular, it is called a *simple beta-number* or *a simple Parry number* when $d_\beta(1)$ is finite. Beta-numbers (Parry numbers) are algebraic integers [Pa] and all their conjugates lie within a compact subset which looks like a fractal in the complex plane [So]. The conjugates of beta-numbers are all bounded above in modulus by the golden mean $\frac{1}{2}(1+\sqrt{5})$ [So] [F-P].

3 Proof of Theorem 1.1

Since algebraic numbers $\beta > 1$ for which the Rényi β-expansion $d_\beta(1)$ of 1 is finite are excluded, we may consider that β does not belong to $\mathbb{N}$. Indeed, if $\beta = h \in \mathbb{N}$, then $d_h(1) = 0.h$ is finite (Lothaire [Lo], Chap. 7). If $\beta \notin \mathbb{N}$, then $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$ and the alphabet A_β equals $\{0, 1, 2, \dots, \lfloor \beta \rfloor\}$, where $\lfloor \beta \rfloor$ denotes the greatest integer smaller than or equal to β.

Let $f(z) := \sum_{i=1}^{+\infty} t_i z^i$ be the "lacunary" power series deduced from the representation $d_\beta(1) = 0.t_1 t_2 t_3 \dots$ associated with the β-shift (*lacunary* in the sense of Theorem 1.1). Since $d_\beta(1)$ is assumed infinite, its radius of convergence is 1. By definition, it satisfies

$$f(\beta^{-1}) = 1, \tag{3.1}$$

which means that the function value $f(\beta^{-1})$ is algebraic, equal to 1, at the real algebraic number β^{-1} in the open disk of convergence $D(0,1)$ of $f(z)$ in the complex plane. This fact is a general intrinsic feature of the Rényi expansion process which leads to the following important consequence by the theory of admissible power series of Mahler [Ma].

Proposition 3.1.

$$\limsup_{n \to +\infty} \frac{s_n}{m_n} < +\infty. \tag{3.2}$$

Proof. This is a consequence of Theorem 1 in [Ma]. Indeed, if we assume that there exists a sequence of integers (n_i) which tends to infinity such that $\lim_{i\to+\infty} s_{n_i}/m_{n_i} = +\infty$, then $f(z)$ would be *admissible* in the sense of [Ma]. Since $f(z)$ is a power series with nonnegative coefficients, which is not a polynomial, the function value $f(\beta^{-1})$ should not be algebraic. But it equals 1. Contradiction. □

Let us improve Proposition 3.1. Assume that

$$\limsup \frac{s_n}{m_n} > \frac{\log(\mathrm{M}(\beta))}{\log(\beta)} \tag{3.3}$$

and show the contradiction with (1.1) and (1.2). Recall that, if

$$P_\beta(X) = \sum_{i=0}^{d} \alpha_i X^i \ = \ \alpha_d \prod_{i=0}^{d-1} (X - \beta^{(i)})$$

with $d \geq 1$, $\alpha_0\alpha_d \neq 0$, denotes the minimal polynomial of $\beta = \beta^{(0)} > 1$, having $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d-1)}$ as conjugates, the Mahler measure of β is by definition

$$\mathrm{M}(\beta) := |\alpha_d| \prod_{i=0}^{d-1} \max\{1, |\beta^{(i)}|\}.$$

Güting [Gü] has shown that the approximation of algebraic numbers by algebraic numbers is fairly difficult to be realized by polynomials. In the present proof, we use a version of Liouville's inequality which generalizes approximation theorems obtained by Güting [Gü], and apply it to the values of the "polynomial tails" of the power series $f(z)$ at the algebraic number β^{-1}, to obtain the contradiction. Let us write

$$f(z) = \sum_{n=0}^{+\infty} Q_n(z) \tag{3.4}$$

with

$$Q_n(z) := \sum_{i=s_n}^{m_{n+1}} t_i z^i, \qquad n = 0, 1, 2, \ldots. \tag{3.5}$$

By construction the polynomials $Q_n(z)$, of degree m_{n+1}, are not identically zero and $Q_n(1) > 0$ is an integer for all $n \geq 0$.

Denote by $S_n(z) = -1 + \sum_{i=1}^{m_n} t_i z^i$ the m_nth-section polynomial of the power series $f(z)-1$ for all $n \geq 1$. Recall that, for $R(X) = \sum_{i=0}^{v} \alpha_i X^i \in \mathbb{Z}[X]$, $L(R) := \sum_{i=0}^{v} |\alpha_i|$ denotes the length of the polynomial $R(X)$. We have: $L(S_n) = 1 + \sum_{i=1}^{m_n} t_i = 1 + \sum_{j=0}^{n-1} Q_j(1)$. From Theorem 5 in [Gü] we deduce that only one of the following cases (G-i) or (G-ii) holds, for all $n \geq 1$:

$$(G-i) \qquad S_n(\beta^{-1}) = 0, \tag{3.6}$$

$$(G-ii) \qquad \left|S_n(\beta^{-1})\right| \geq \frac{1}{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \left(L(P_\beta^*)\right)^{m_n}}, \tag{3.7}$$

where $P_\beta^*(X) = X^d P_\beta(1/X)$ is the reciprocal polynomial of the minimal polynomial of β, for which $L(P_\beta) = L(P_\beta^*) \in \mathbb{N} \setminus \{0,1\}$.

Case (G-i) is impossible for any n. Indeed, if there exists an integer $n_0 \geq 1$ such that (G-i) holds, then, since all the digits t_i are positive and that $\beta^{-1} > 0$, we would have $t_i = 0$ for all $i \geq s_{n_0}$. This would mean that the Rényi expansion of 1 in base β is finite, which is excluded by assumption. Contradiction. Therefore, the only possibility is (G-ii), which holds for all integers $n \geq 1$. From Lemma 3.10 and Liouville's inequality (Proposition 3.14) in Waldschmidt [W] the inequality (G-ii) can be improved to

$$(L-ii) \qquad \left|S_n(\beta^{-1})\right| \geq \frac{1}{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} (\mathrm{M}(\beta))^{m_n}}. \tag{3.8}$$

This improvement may be important; recall the well-known inequalities:

$$\mathrm{M}(\beta) \leq L(P_\beta) \leq 2^{\deg(\beta)}\mathrm{M}(\beta)$$

and see [W] p113 for comparison with different heights. On the other hand, since $\left|S_n(\beta^{-1})\right| = \sum_{i=s_n}^{+\infty} t_i \beta^{-i}$ for all integers $n \geq 1$, we deduce

$$\left|S_n(\beta^{-1})\right| \leq \frac{\lfloor\beta\rfloor}{1-\beta^{-1}}\beta^{-s_n} \qquad n = 1, 2, \ldots. \tag{3.9}$$

Putting together (3.8) and (3.9), we deduce that

$$\frac{\beta^{s_n}}{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \mathrm{M}(\beta)^{m_n}} \leq \frac{\lfloor\beta\rfloor}{1-\beta^{-1}} \tag{3.10}$$

should be satisfied for $n = 1, 2, 3, \ldots$. Denote

$$u_n := \frac{\beta^{s_n}}{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \mathrm{M}(\beta)^{m_n}} \qquad \text{for all } n \geq 1.$$

Proof of (1.1)*:* from (3.3) assumed to be true there exists a sequence of integers (n_i) which tends to infinity and an integer i_0 such that

$$\frac{s_{n_i}}{m_{n_i}} > \frac{\log(\mathrm{M}(\beta))}{\log(\beta)} \qquad \text{for all } i \geq i_0.$$

Now,

$$\left(\frac{1}{1+\lfloor\beta\rfloor m_{n_i}}\right)^{d-1}\left(\frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{\mathrm{M}(\beta)}\right)^{m_{n_i}} \leq \frac{1}{\left(1+\sum_{j=0}^{n_i-1} Q_j(1)\right)^{d-1}}\left(\frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{\mathrm{M}(\beta)}\right)^{m_{n_i}} \leq u_{n_i}. \tag{3.11}$$

For $i \geq i_0$ the inequality

$$1 = \frac{\beta^{\frac{\log(\mathrm{M}(\beta))}{\log(\beta)}}}{\mathrm{M}(\beta)} < \frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{\mathrm{M}(\beta)} \tag{3.12}$$

holds. This implies that the left-hand side member of (3.11) tends exponentially to infinity when i tends to infinity. By (3.11) this forces u_{n_i} to tend to infinity. The contradiction now comes from (3.10) since the sequence (u_n) should be uniformly bounded.

Proof of (1.2)*:* for $n = 1, 2, \ldots$, let us rewrite the n-th quotient

$$\frac{u_{n+1}}{u_n} = \frac{\beta^{s_{n+1}-s_n}}{\mathrm{M}(\beta)^{m_{n+1}-m_n}} \frac{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1}}{\left(1+\sum_{j=0}^{n} Q_j(1)\right)^{d-1}} \tag{3.13}$$

as

$$\frac{u_{n+1}}{u_n} = \frac{\left(\frac{\beta^{\frac{s_{n+1}-s_n}{m_{n+1}-m_n}}}{\mathrm{M}(\beta)}\right)^{m_{n+1}-m_n}}{(m_{n+1}-m_n+1)^{(d-1)}}\left[(m_{n+1}-m_n+1)^{(d-1)} \frac{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1}}{\left(1+\sum_{j=0}^{n} Q_j(1)\right)^{d-1}}\right] \tag{3.14}$$

and denote

$$U_n := \frac{1}{(m_{n+1}-m_n+1)^{(d-1)}}\left(\frac{\beta^{\frac{s_{n+1}-s_n}{m_{n+1}-m_n}}}{\mathrm{M}(\beta)}\right)^{m_{n+1}-m_n} \tag{3.15}$$

and

$$W_n := (m_{n+1}-m_n+1)^{(d-1)}\left(\frac{1+\sum_{j=0}^{n-1} Q_j(1)}{1+\sum_{j=0}^{n} Q_j(1)}\right)^{d-1} \tag{3.16}$$

so that $u_{n+1}/u_n = U_n W_n$.

Lemma 3.2.

$$0 < \liminf_{n\to+\infty} \quad W_n \tag{3.17}$$

Proof. Assume the contrary. Then there exists a subsequence (n_i) of integers which tends to infinity such that $\lim_{i\to+\infty} W_{n_i} = 0$. In other terms, for all

$\epsilon > 0$, there exists i_1 such that $i \geq i_1$ implies $W_{n_i} \leq \epsilon$, equivalently

$$(m_{n_i+1} - m_{n_i} + 1)\big(1 + \sum_{j=0}^{n_i-1} Q_j(1)\big) \leq \epsilon^{\frac{1}{d-1}} \times \big(1 + \sum_{j=0}^{n_i} Q_j(1)\big). \tag{3.18}$$

Since, by hypothesis, $t_{s_n} \geq 1$ and $t_{m_n+1} \geq 1$ for all $n \geq 1$, we have: $n_i \leq 1 + \sum_{j=0}^{n_i-1} Q_j(1)$. On the other hand, $Q_{n_i}(1) \leq \lfloor\beta\rfloor(m_{n_i+1} - m_{n_i} + 1)$. Then, from (3.18) with ϵ taken equal to 1, we would have

$$n_i \leq 1+\sum_{j=0}^{n_i-1} Q_j(1) \leq \frac{Q_{n_i}(1)}{(m_{n_i+1} - m_{n_i} + 1) - 1} \leq \lfloor\beta\rfloor\times\frac{m_{n_i+1} - m_{n_i} + 1}{m_{n_i+1} - m_{n_i}} \leq \frac{3}{2}\lfloor\beta\rfloor. \tag{3.19}$$

But the left-hand side member of (3.19) tends to infinity which is impossible. Contradiction. □

Let us assume that (1.2) does not hold and show the contradiction ; that is, assume that $\liminf_{n\to+\infty}(m_{n+1} - m_n) = +\infty$ and $\limsup_{n\to+\infty}(s_{n+1} - s_n)/(m_{n+1} - m_n) > \log(\mathrm{M}(\beta))/\log(\beta)$ hold. Then

$$1 = \frac{\beta^{\frac{\log(\mathrm{M}(\beta))}{\log(\beta)}}}{\mathrm{M}(\beta)} < \frac{\beta^{\frac{s_{n_i+1}-s_{n_i}}{m_{n_i+1}-m_{n_i}}}}{\mathrm{M}(\beta)} \tag{3.20}$$

for some sequence of integers (n_i) which tends to infinity. This proves that $\limsup_{n\to+\infty} U_n = +\infty$ since $\lim_{i\to+\infty} U_{n_i} = +\infty$ exponentially, by (3.15) and (3.20).

From Lemma 3.2 there exists $r > 0$ such that $W_n \geq r$ for all n large enough. Therefore, $u_{n+1}/u_n = U_n W_n \geq rU_n$ for all n large enough. Since $\limsup_{n\to+\infty} U_n = +\infty$ we conclude that $\limsup u_{n+1}/u_n = +\infty$, hence that $\limsup u_n = +\infty$. This contradicts (3.10) and proves (1.2).

4 A direct proof of Corollary 1.2

Let $\beta > 1$ be a Salem number such that $\beta \notin C_1$. Using the notations of Theorem 1.1 we show that the assumption

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} > 1 \tag{4.1}$$

leads to a contradiction.

Denote by $\mathbb{K}$ the algebraic number field $\mathbb{Q}(\beta)$, considered as a multivalued field with the product formula [C] [Sw] (see also [Lg]).

The present proof is merely an adaptation of that of Theorem 1 in [A], though the objectives are different, and therefore does not merit to be published. We just indicate a few hints for the interested reader.

The main result which is used is Corollary 1 of the Main Theorem in [C], as in [A]. This is a version of the Thue-Siegel-Roth Theorem given by Corvaja which is stronger than Roth Theorem for number fields [Le] [Sw] in the extent it allows to introduce a *missing proportion of places* of $\mathbb{K}$ by considering the projective approximation of the point at infinity in $\mathbb{P}^1(\mathbb{K})$. Since β is a Salem number, it is a unit [B-S]. Hence, this missing proportion has just to be chosen among the pairwise distinct Archimedean places of $\mathbb{K}$.

5 On the class Q_0

5.1 Perron numbers

Let us give, after Solomyak ([So], p 483), the example of a Perron number which is not a beta-number therefore which is not in the class C_2, without knowing whether it is in the class Q_0. This example allows to estimate the sharpness of the upper bound $\log(\mathrm{M}(\beta))/\log(\beta)$ in (1.1). Recall that a real number $\beta > 1$ is a beta-number if the orbit of $x = 1$ under the transformation $T_\beta : x \to \beta x \pmod 1$ is finite [Lo] [PF]. The set of all conjugates of all beta-numbers is the union of the closed unit disc in the complex plane and the set of reciprocals of zeros of the function class $\{f(z) = 1 + \sum a_j z^j \mid 0 \leq a_j \leq 1\}$. The closure of this domain, say Φ, is compact and was studied by Flatto, Lagarias and Poonen [F-P] and Solomyak [So]. After [So], the Perron number $\beta = \frac{1}{2}(1+\sqrt{13})$, dominant root of $P_\beta(X) = X^2 - X - 3$, is not a beta-number, though its only conjugate $\beta' = \frac{1}{2}(1 - \sqrt{13})$ lies in the interior $\mathrm{int}(\Phi)$. We have $\mathrm{M}(\beta) = 3$. By Theorem 1.1 the "quotients of the gaps" are asymptotically bounded above by $\log(3)/\log(\beta) = 1.3171\ldots$, a much better bound than the degree $d = 2$ of β (see Lemma 5.1). This does not suffice to conclude that $\frac{1}{2}(1 + \sqrt{13})$ belongs to Q_0.

Do all Perron numbers belong to Q_0 ? Let $\beta > 1$ be a Perron number of degree $d \geq 2$ and denote by $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d-1)}$ the conjugates of $\beta = \beta^{(0)}$, roots of the minimal polynomial $P_\beta(X)$ of β. Let $K_\beta := \max\{|\beta^{(i)}| \mid i = 1, 2, \ldots, d-1\}$.

Lemma 5.1. *Let $n = n_\beta$ (with $2 \leq n_\beta \leq d$) be the number of conjugates of β of modulus strictly greater than unity (including β). Then*

$$\frac{\log(\mathrm{M}(\beta))}{\log(\beta)} \leq n - \frac{n-1}{(d\beta)^{6d^3} \log \beta}. \tag{5.1}$$

Proof. Obvious since (Lemma 2 in [Li2]): $K_\beta < \beta\big(1 - \frac{1}{(d\beta)^{6d^3}}\big)$. □

The upper bound (5.1) does not allow to give a positive answer to the question and has probably to be improved.

5.2 Transcendental numbers

Let us show that the Komornik-Loreti constant [KL] [AC] belongs to $\mathrm{Q}_0^{(1)}$.

Theorem 5.2. *There exists a smallest* $q \in (1,2)$ *for which there exists a unique expansion of* 1 *as* $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$ *, with* $\delta_n \in \{0,1\}$*. Furthermore, for this smallest* q*, the coefficient* δ_n *is equal to* 0 *(respectively,* 1*) if the sum of the binary digits of* n *is even (respectively, odd). This number* q *can then be obtained as the unique positive solution of* $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$*. It is equal to* $1.787231650\ldots$*.*

This constant q is named Komornik-Loreti constant. Allouche and Cosnard [AC] have shown the following result.

Theorem 5.3. *The constant* q *is a transcendental number, where the sequence of coefficients* $(\delta_n)_{n\geq 1}$ *is the Prouhet-Thue-Morse sequence on the alphabet* $\{0,1\}$*.*

The uniqueness of the development of 1 in base q given by Theorem 5.2 allows to write

$$d_q(1) = 0.\delta_1\delta_2\delta_3\ldots,$$

the coefficients δ_n being the digits of the Rényi q-expansion of 1. Since the strings of zeros and 1's in the Prouhet-Thue-Morse sequence are known (Thue, 1906/1912; [AS]) and uniformly bounded, the constant q belongs to the class $\mathrm{Q}_0^{(1)}$.

As second example, let us show that Sturmian numbers in the interval $(1,2)$ (in the sense of [CK]) belong to $\mathrm{Q}_0^{(1)}$.

A real number $\beta > 1$ is called a Sturmian number if $d_\beta(1)$ is a Sturmian word over a binary alphabet $\{a,b\}$, with $0 \leq a < b = \lfloor \beta \rfloor$. Chi and Kwon [CK] have shown the following theorem.

Theorem 5.4. *Every Sturmian number is transcendental.*

Let us consider all the Sturmian numbers $\beta \in (1,2)$ for which the two-letter alphabet is $\{0,1\}$. For such numbers lacunarity appears in $d_\beta(1)$ (in the sense of Theorem 1.1). By Theorem 3.3 in [CK] strings of zeros, resp. of 1's, cannot be arbitrarily long. This gives the claim.

Acknowledgments

The author thanks J.-P. Allouche, B. Adamczewski, J. Bernat, V. Berthé, C. Frougny, J.-P. Gazeau and the anonymous referee for valuable comments, discussions and remarks.

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"Symmetry Groups for Beta-lattices",
A. Elkharrat, C. Frougny, J.-P. Gazeau et J.-L. Verger-Gaugry, Theor. Comp. Sci. 319 (2004), 281–305.

Available online at www.sciencedirect.com
SCIENCE DIRECT
Theoretical Computer Science 319 (2004) 281–305
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Symmetry groups for beta-lattices

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Abstract

We present a construction of symmetry plane-groups for quasiperiodic point-sets named beta-lattices. The framework is issued from beta-integers counting systems. Beta-lattices are vector superpositions of beta-integers. When $\beta > 1$ is a quadratic Pisot–Vijayaraghavan algebraic unit, the set of beta-integers can be equipped with an abelian group structure and an internal multiplicative law. When $\beta = \frac{(1+\sqrt{5})}{2}$, $1+\sqrt{2}$ and $2+\sqrt{3}$, we show that these arithmetic and algebraic structures lead to freely generated symmetry plane-groups for beta-lattices. These plane-groups are based on repetitions of discrete adapted rotations and translations we shall refer to as "beta-rotations" and "beta-translations". Hence beta-lattices, endowed with beta-rotations and beta-translations, can be viewed like lattices. The quasiperiodic function $\rho_S(n)$, defined on the set of beta-integers as counting the number of small tiles between the origin and the nth beta-integer, plays a central part in these new group structures. In particular, this function behaves asymptotically like a linear function. As an interesting consequence, beta-lattices and their symmetries behave asymptotically like lattices and lattice symmetries, respectively.

MSC: 52C22; 52C23

Keywords: Beta-lattices; Pisot numbers; Quasicrystals; Tilings; Plane groups

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0304-3975/$ - see front matter
doi:10.1016/j.tcs.2004.02.013

1. Introduction

Underlying the notion of a *tiling* there is the notion of a *point-set*. In this paper, we assume point-sets to be *Delaunay sets* [16,17]. There exist infinitely many possibilities to build a tiling from a Delaunay set, and conversely, there are infinitely many ways to build a Delaunay set from its associated tiling. A possible method is to consider the set of vertices of a tiling as an associated Delaunay set [12], which is the correspondence we will assume in the following. We will indifferently mention a tiling or its associated Delaunay set, displaying or not the edges in the figures.

In general, there does not exist a symmetry group for a tiling nor for its associated Delaunay set, except for periodic tilings and *lattices*. Historically, the latter merge from crystallography, and are associated with crystals. Note that in 1991, after the discovery of modulated phases and of quasicrystals, crystallography have been divided in two categories: periodic crystallography, and aperiodic crystallography [10]. Let us sketch the general algebraic frame of periodic crystallography.

Definition 1. A crystallographic group in $\mathbb{R}^d$, or a space-group in $\mathbb{R}^d$, is a discrete group of isometries whose maximal translation subgroup is of rank d, hence isomorphic to $\mathbb{Z}^d$.

Definition 2. A periodic crystal is the orbit under the action of a crystallographic group of a finite number of points of $\mathbb{R}^d$.

We can illustrate these definitions with the square lattice $\Lambda = \mathbb{Z} + \mathbb{Z}e^{i\pi/2}$, which is a classical lattice case. This set presents a 4-fold rotational symmetry. The symmetry space-group G associated with Λ is the semi-direct product of the translation-group of Λ by its rotation-group

$$G = \Lambda \rtimes \{1, -1, e^{i\pi/2}, e^{-i\pi/2}\}$$

its internal law being

$$(\lambda, R)(\lambda', R') = (\lambda + R\lambda', RR')$$

with $\lambda, \lambda' \in \Lambda$ and $R, R' \in \{1, -1, e^{i\pi/2}, e^{-i\pi/2}\}$.

In the context of the 18th problem of Hilbert, Bieberbach has shown that the number of isomorphism classes (equivalently of conjugation classes) of crystallographic groups is finite for all d [25]. Therefore, the number of crystallographic groups leaving invariant a fixed crystal of $\mathbb{R}^d$ is finite.

For quasicrystals, as a consequence of aperiodicity, we do not have such a convenient algebraic structure of symmetry space-groups, as in the periodic case. For quasicrystals determined by some quadratic Pisot–Vijayaraghavan (PV) unit, generically denoted by $\beta > 1$, we can introduce an underlying structure, the so-called *beta-lattice* [1]. Experimentally observed quasicrystals are related to well-known PV numbers [11], namely for $\beta = \tau = \frac{1+\sqrt{5}}{2}$, $\beta = \delta = 1 + \sqrt{2}$, and $\beta = \theta = 2 + \sqrt{3}$.

Beta-lattices are based on beta-integers. When β is a PV number, the set of beta-integers, denoted by $\mathbb{Z}_\beta$, is a self-similar Meyer set, with self-similarity factor β. Recall that a Delaunay set is a Meyer set $\Lambda \in \mathbb{R}^d$ if $\Lambda - \Lambda \subset \Lambda + F$, where F is a finite set. We generically define a beta-lattice $\Gamma = \Gamma(\beta) \in \mathbb{R}^d$ by

$$\Gamma = \sum_{i=1}^{d} \mathbb{Z}_\beta \mathrm{e}_i$$

with (e_i) a base of $\mathbb{R}^d$. Therefore, Γ is a self-similar Meyer set with self-similarity factor β. With this respect, beta-lattices are eligible frames in which one could think of the properties of quasiperiodic point-sets and tilings, thus generalizing the notion of lattice in periodic cases.

The aim of the present work is to extend the algebraic frame of periodic crystals to beta-lattices: we construct a space-group matching Definition 1 such that the beta-lattice is the orbit under the action of this space-group of a finite set of points of $\mathbb{R}^2$, as in Definition 2. In other words, we show that a beta-lattice is at least a "crystal" for a "space-group" that we determine explicitly.

We consider the cases in which β is one of the "quasicrystallographic" numbers mentioned above. Since we restrict ourselves to the case $d = 2$, we rather talk of "plane-groups". We proceed by first recalling the internal additive and multiplicative laws on the set of beta-integers $\mathbb{Z}_\beta \subset \mathbb{R}$, which "almost" endow this set with a structure of ordered ring (order induced by that of $\mathbb{R}$) [6], then by establishing a set of algebraic operations, acting on the given beta-lattice by leaving it invariant. We report on the algebraic constructions of such extended plane-groups, leaving aside the delicate questions of compatible metrics and of the number (finite or infinite) of possible "space-groups" leaving invariant a given beta-lattice. However we show that the internal transformations defined on beta-lattices are *compatible* with Euclidean transformations. Compatibility property is given by the following definition.

Definition 3. Let $\top$ be an internal law defined on $\mathbb{R}^d$, and let $\Lambda \subset \mathbb{R}^d$ be a set. We say that an internal operation $\hat{\top}$ defined on Λ is $\top$-*compatible* with the operation $\top$ if for all λ, $\lambda' \in \Lambda$, $\lambda \top \lambda' \in \Lambda$ implies $\lambda \top \lambda' = \lambda \hat{\top} \lambda'$.

The article is organized as follows. In Section 2, we recall some definitions on Delaunay sets, Meyer sets, and on cyclotomic PV numbers. In Section 3, we recall results on the arithmetics and the internal laws on $\mathbb{Z}_\beta$. Most of this material can be found in [6], and is essential for the understanding of the present article. In Section 4, we give the definition of beta-lattices in the plane, together with their rotational and translational properties. A general form for beta-lattices is $\Gamma_1(\beta) = \mathbb{Z}_\beta + \mathbb{Z}_\beta \mathrm{e}^{\mathrm{i}2\pi/N}$ for β a cyclotomic PV unit of symmetry N. Fig. 1 is a possible tiling of such a beta-lattice with $\beta = \tau$, the golden mean, namely a τ-lattice. Section 5 is the central part of the article, with its main result: the construction of the plane-groups associated with the beta-lattices. We use the internal additive and multiplicative laws on $\mathbb{Z}_\beta$ to define a symmetry point-group for $\Gamma_1(\beta)$ in Theorem 1, and the free symmetry plane-group of $\Gamma_1(\beta)$ in Theorem 2. Then we illustrate the action of the symmetry plane-group of

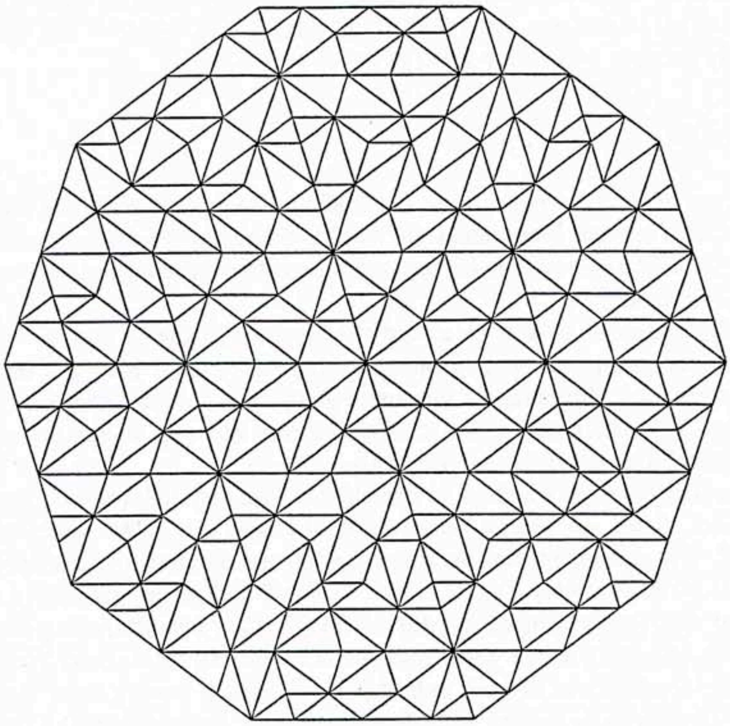

Fig. 1. A tiling of the τ-lattice $\Gamma_1(\tau)$.

$\Gamma_1(\tau)$ on the tiles of a τ-lattice. Section 6 is dedicated to the asymptotic properties of beta-lattices. The striking feature which is shown there is that asymptotically the set of beta-integers behaves like a ring, but with a contraction factor. We touch here the fundamental question of whether a beta-lattice can be considered as a module over an ordered ring. If it were the case, the present construction would enter into the realm of the Artin–Schreier theory ([13, Chapter 6]). Eventually, we make explicit the rotation actions for the quasicrystallographic numbers τ, δ and θ in the appendix.

2. Preliminaries

2.1. Delaunay sets and Meyer sets

Delaunay sets were introduced as a mathematical idealization of a solid-state structure, see [12]. A set $\Lambda \subset \mathbb{R}^d$ is said to be *uniformly discrete* if there exists $r>0$ such that $\|x-y\| \geqslant r$, for all $x, y \in \Lambda$. We can equivalently say that every closed ball of radius r contains at most a point of Λ. A set Λ is said to be *relatively dense* if there exists $R>0$ such that for all $y \in \mathbb{R}^d$, there exists $x \in \Lambda$ such that $\|x-y\|<R$. We can equivalently say that every open ball of radius R contains at least a point of Λ. If both

conditions are satisfied, Λ is said to be a *Delaunay set*. The possible range of ratios R/r is studied in [21] as a function of d. The action of the group of rigid motions (or Euclidean displacements) of $\mathbb{R}^d$ on the set of uniformly discrete sets and Delaunay sets can be found in [22].

The first models of quasicrystal were introduced by Meyer [16–18], and they are now known as *Meyer sets*. A set $\Lambda \subset \mathbb{R}^d$ is said to be a *Meyer set* if it is a Delaunay set and if there exists a finite set F such that

$$\Lambda - \Lambda \subset \Lambda + F.$$

This is equivalent to $\Lambda - \Lambda$ being a Delaunay set. A review on Meyer sets can be found in [19,20].

2.2. Crystals and Bravais lattices

Bravais lattices are used as mathematical models for crystals. A Bravais lattice is an infinite discrete point-set such that the neighborhoods of a point are the same whichever point of the set is considered. Geometrically, a Bravais lattice is characterized by all Euclidean transformations (translations and possibly rotations) that transform the lattice into itself. The condition $2\cos 2\pi/N \in \mathbb{Z}$ characterizes Bravais lattices which are left invariant under rotation of $2\pi/N$, N-fold Bravais lattices, in $\mathbb{R}^2$ (and in $\mathbb{R}^3$). Let us put $\zeta = e^{i2\pi/N}$, $\zeta^N = 1$. If we consider the $\mathbb{Z}$-module in the plane

$$\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}\zeta + \mathbb{Z}\zeta^2 + \cdots + \mathbb{Z}\zeta^{N-1} = \mathbb{Z}\left[2\cos\frac{2\pi}{N}\right] + \mathbb{Z}\left[2\cos\frac{2\pi}{N}\right]\zeta,$$

we get the cyclotomic ring of order N. This N-fold structure is generically dense in $\mathbb{C}$, except precisely for the crystallographic cases. We indeed check that $\mathbb{Z}[\zeta] = \mathbb{Z}$ for $N = 1$ or 2, $\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}i$ for $N = 4$ (square lattice), and $\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}e^{i\pi/3}$ for the triangular and hexagonal cases $N = 3$ and 6. Note that a Bravais lattice is a Meyer set such that $F = \{0\}$.

2.3. Non-crystallographic cases

For a general N, the number $2\cos 2\pi/N$ is an algebraic integer of degree $m = \varphi(N)/2 \leqslant \lfloor (N-1)/2 \rfloor$ where φ is the Euler function and $\lfloor y \rfloor$ denotes the integer part of a real number y. We shall now recall some definitions on numbers.

A *Pisot–Vijayaraghavan number*, or PV number in short, is an algebraic integer $\beta > 1$ such that all its Galois conjugates (i.e. other roots of the involved algebraic equation) have their moduli strictly smaller than 1. A *cyclotomic PV number* with symmetry of order N is a PV number β such that

$$\mathbb{Z}\left[2\cos\frac{2\pi}{N}\right] = \mathbb{Z}[\beta]. \tag{1}$$

Then $\mathbb{Z}[\zeta] = \mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$, with $\zeta = e^{i2\pi/N}$, is a ring invariant under rotation of order N (see [1]). This ring is the natural framework for two-dimensional structures having

β as scaling factor, and $2\pi/N$ rotational symmetry. In this paper, we will focus on quadratic PV units. They are of two kinds. The first kind is such that β is solution of

$$X^2 = aX + 1, \quad a \geqslant 1$$

and its conjugate is $\beta' = -1/\beta$. The second kind is such that β is solution of

$$X^2 = aX - 1, \quad a \geqslant 3$$

and its conjugate is $\beta' = 1/\beta$. Let us give some examples of those numbers, together with their respective Galois conjugates, related to non-crystallographic cyclotomic structures in the plane, and minimal polynomials, the following notations being used throughout the article:

$$N = 5, \quad \beta = \tau = \frac{1+\sqrt{5}}{2} = 1 + 2\cos\frac{2\pi}{5},$$
$$\tau' = -\frac{1}{\tau} = 1 - \tau, \quad X^2 - X - 1 \quad \text{(pentagonal case)},$$
$$N = 10, \quad \beta = \tau = \frac{1+\sqrt{5}}{2} = 2\cos\frac{2\pi}{10},$$
$$\tau' = -\frac{1}{\tau} = 1 - \tau, \quad X^2 - X - 1 \quad \text{(decagonal case)},$$
$$N = 8, \quad \beta = \delta = 1 + \sqrt{2} = 1 + 2\cos\frac{2\pi}{8},$$
$$\delta' = -\frac{1}{\delta} = 2 - \delta, \quad X^2 - 2X - 1 \quad \text{(octogonal case)},$$
$$N = 12, \quad \beta = \theta = 2 + \sqrt{3} = 2 + 2\cos\frac{2\pi}{12},$$
$$\theta' = \frac{1}{\theta} = 4 - \theta, \quad X^2 - 4X + 1 \quad \text{(dodecagonal case)}.$$

Note that in the case $N = 7$, we have $\beta = 1 + 2\cos 2\pi/7$ which is solution of the cubic equation $X^3 - 2X^2 - X + 1 = 0$. At this point, we should be aware that finding a PV number such that the cyclotomic condition (1) is fulfilled for $N \geqslant 16$ is an open problem!

3. Additive and multiplicative properties of beta-integers

3.1. Beta-expansions

When a number $\beta > 1$ appears as a kind of fundamental invariant in a given structure, it is tempting to introduce into the procedure of understanding the latter a *counting system* based precisely on this β. Let us explain here what we mean by counting system.

Among all *beta-representations* of a real number $x \geqslant 0$, i.e. infinite sequences $(x_i)_{i \leqslant k}$, such that $x = \sum_{i \leqslant k} x_i \beta^i$ for a certain integer k, there exists a particular one, called the *beta-expansion*, which is obtained through the "greedy algorithm" (see [23,24]). Recall that $\lfloor y \rfloor$ is the integer part of the real number y, and denote by $\{y\}$ the fractional part of y. There exists $k \in \mathbb{Z}$ such that $\beta^k \leqslant x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. For $i<k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$. Then we get the expansion $x = x_k \beta^k + x_{k-1}\beta^{k-1} + \cdots$. If $x<1$ then $k<0$, and we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. The beta-expansion of x is denoted by

$$\langle x \rangle_\beta = x_k x_{k-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots .$$

The digits x_i obtained by this algorithm are integers from the set $A = \{0, \ldots, \lceil \beta \rceil - 1\}$, called the *canonical alphabet*, where $\lceil \beta \rceil$ denotes the smallest integer larger than β. If an expansion ends in infinitely many zeros, it is said to be *finite*, and the ending zeros are omitted. For instance, if $\beta = \tau \approx 1.618\cdots$, then $x_i \in \{0,1\}$. The τ-expansion of, say, $4 = \tau^2 + 1 + 1/\tau^2$ is $\langle 4 \rangle_\tau = 101.01$. There is a representation which plays an important role in the theory. The *beta-expansion* of 1, denoted by $d_\beta(1)$, is computed by the following process [24]. Let the *beta-transformation* be defined on $[0,1]$ by $T_\beta(x) = \beta x \bmod 1$. Then $d_\beta(1) = (t_i)_{i \geqslant 1}$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$. Bertrand has proved that if β is a PV number, then $d_\beta(1)$ is eventually periodic [2]. For instance, $d_\tau(1) = 11$, $d_\delta(1) = 21$, and $d_\theta(1) = 322\cdots = 3(2)^\omega$, where $(\cdot)^\omega$ means that the digit between parenthesis is repeated an infinite number of times. A number β such that $d_\beta(1)$ is eventually periodic is traditionally called a *beta-number*. Since these numbers were introduced by Parry [23], we propose to call them *Parry numbers*. When $d_\beta(1)$ is finite, β is said to be a *simple* Parry number.

3.2. The set of beta-integers

We now come to the notion of *beta-integer*. The set of beta-integers is the set of real numbers whose beta-expansions are polynomial,

$$\begin{aligned} \mathbb{Z}_\beta &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\} \\ &= \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+), \end{aligned}$$

where $\mathbb{Z}_\beta^+$ is the set of non-negative beta-integers. The set $\mathbb{Z}_\beta$ is self-similar and symmetrical with respect to the origin

$$\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta, \quad \mathbb{Z}_\beta = -\mathbb{Z}_\beta.$$

It has been shown in [3] that if β is a PV number then $\mathbb{Z}_\beta$ is a Meyer set. This means that there exists a finite set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. This beta-dependent set F has to be characterized in order to see to what extent beta-integers differ from ordinary integers with respect to additive and multiplicative structures. This problem is solved in [3,4,6] for all quadratic PV units and for a few higher-degree cases (see also [27]). We now restrict the presentation to quadratic PV units. There are two cases to consider.

Case 1 (*β is solution of $X^2 = aX + 1$, $a \geqslant 1$*): The Galois conjugate is $\beta' = -1/\beta$. The canonical alphabet is equal to $A = \{0, \ldots, a\}$, the beta-expansion of 1 is finite, equal to $d_\beta(1) = a1$, and every positive number of $\mathbb{Z}[\beta]$ has a finite beta-expansion [7]. Denote $\mathbb{A} = \{L, S\}$. Define the substitution σ_β by

$$\sigma_\beta : \begin{cases} L \mapsto L^a S, \\ S \mapsto L. \end{cases}$$

The fixed point of the substitution, denoted by $\sigma_\beta^\infty(L)$, is associated with a tiling of the positive real line, made with the two tiles L and S, where the lengths of the tiles are $\ell(L) = 1$, $\ell(S) = T_\beta(1) = \beta - a = 1/\beta$, see [26,5]. The nodes of this tiling are the positive beta-integers.

Case 2 (*β is solution of $X^2 = aX - 1$, $a \geqslant 3$*): The Galois conjugate is $\beta' = 1/\beta$. The canonical alphabet is equal to $A = \{0, \ldots, a-1\}$, the beta-expansion of 1 is eventually periodic, equal to $d_\beta(1) = (a-1)(a-2)^\omega$, and every positive number of $\mathbb{Z}[\beta]$ has an eventually periodic beta-expansion, which is finite for numbers from $\mathbb{N}[\beta]$, [7]. The substitution σ_β is defined on $\mathbb{A} = \{L, S\}$ by

$$\sigma_\beta : \begin{cases} L \mapsto L^{a-1} S, \\ S \mapsto L^{a-2} S. \end{cases}$$

As in Case 1, the fixed point of the substitution is denoted by $\sigma_\beta^\infty(L)$, and is associated with a tiling of the positive real line, made with the two tiles L and S. The lengths of the tiles are $\ell(L) = 1$, $\ell(S) = T_\beta(1) = \beta - (a-1) = 1 - 1/\beta$ [26,5]. The nodes of this tiling are the positive beta-integers.

In both cases, we shall denote by $|\sigma_\beta^q(L)|$ the number of letters in the word generated by $\sigma_\beta^q(L)$, and by $|\sigma_\beta^q(L)|_L$, respectively $|\sigma_\beta^q(L)|_S$, the number of letters L, respectively S, in the later word.

3.3. Beta-integers arithmetics

Since $\mathbb{Z}_\beta$ is a Meyer set symmetrical with respect to the origin, we have $\mathbb{Z}_\beta - \mathbb{Z}_\beta = \mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Hence, the set $\mathbb{Z}_\beta$ can be qualified as "quasi-additive". It can also be qualified as "quasi-multiplicative". Accordingly, addition and multiplication of beta-integers are characterized below.

- In Case 1, we have

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \left\{0, \pm\left(1 - \frac{1}{\beta}\right)\right\} \subset \mathbb{Z}_\beta/\beta^2, \tag{2}$$

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \left\{0, \pm\frac{1}{\beta}, \ldots, \pm\frac{a}{\beta}\right\} \subset \mathbb{Z}_\beta/\beta^2. \tag{3}$$

For instance, for $\beta = \tau$, $1 + 1 = 2 = \tau + (1 - 1/\tau)$, and $(\tau^2 + 1)(\tau^2 + 1) = \tau^5 + \tau^2 - 1/\tau$.

- In Case 2, we have

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \left\{0, \pm\frac{1}{\beta}\right\} \equiv \tilde{\mathbb{Z}}_\beta, \tag{4}$$

$$\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+/\beta,$$

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \left\{0, \pm\frac{1}{\beta}, \ldots, \pm\frac{a-1}{\beta}\right\} \subset \mathbb{Z}_\beta/\beta. \tag{5}$$

For instance, for $\beta = \theta$, $2 + 2 = \theta + 1/\theta = 2 \times 2$.

The set $\tilde{\mathbb{Z}}_\beta$, introduced in Eq. (4), is called the set of decorated beta-integers. This set plays an important role in the theory of algebraic model sets, and is to be mentioned in the two-dimensional case (Fig. 4).

3.4. Beta-integers as an additive group

Let b_m and b_n be the mth and nth beta-integer.

Definition 4. We define the *beta-addition* as the following internal additive law on the set of beta-integers

$$b_m \oplus b_n = b_{m+n}.$$

The beta-substraction is defined by

$$b_m \ominus b_n = b_{m-n} = b_m \oplus (-b_n).$$

The set of beta-integers endowed with the beta-addition has an abelian group structure [4,6]. Actually, we can endow any countable strictly increasing sequence $\mathscr{S} = (s_n)_{n\in\mathbb{Z}}$ of real numbers, $s_0 = 0$, with such an internal additive law by simple isomorphic transport of the additive group structure of the integers, the additive law of $\mathscr{S}$ being defined by

$$s_m \oplus s_n \stackrel{\text{def}}{=} s_{m+n}.$$

Recall that the internal additive law $\oplus$ defined on $\mathscr{S}$, is said to be *compatible* with addition of real numbers if for all $(m,n) \in \mathbb{Z}^2$, $s_m + s_n \in \mathscr{S}$ implies $s_m + s_n = s_m \oplus s_n$, and obviously, for an arbitrary sequence $\mathscr{S}$, the law $\oplus$ *is not* compatible with the addition of real numbers. Yet this property holds true for $\mathbb{Z}_\beta$!

Lemma 1. *Beta-addition is compatible with addition if β is a quadratic PV unit.*

Proof. It has been proven in [4,6] that beta-addition has the following *minimal distortion property* with respect to addition: for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$ with β a quadratic PV unit,

$$b_m + b_n - (b_m \oplus b_n) \in \begin{cases} \{0, \pm(1-\frac{1}{\beta})\} & \text{in Case 1,} \\ \{0, \pm 1/\beta\} & \text{in Case 2.} \end{cases} \tag{6}$$

Suppose that for a given couple of integers (m,n), there exist q such that $b_m + b_n = b_q$. Then $b_q - (b_m \oplus b_n)$ verifies (6), and this implies $b_q - (b_m \oplus b_n) = 0$. Indeed the distances between two non-equal beta-integers are larger than or equal to $\ell(S) = 1/\beta$ in Case 1, and $\ell(S) = 1 - 1/\beta$ in Case 2. So we have $b_q = b_m \oplus b_n = b_{m+n}$, which gives $q = m+n$. □

For instance, if $\beta = \tau$, then $1 \oplus 1 = \tau$ and $2 - \tau = 1 - 1/\tau$, and if $\beta = \theta$, then $2 \oplus 2 = \theta$ and $4 - \theta = 1/\theta$.

3.5. Internal multiplicative law for beta-integers

We could attempt to play the same game with multiplication by defining

$$b_m \text{``} \times \text{''} b_n \stackrel{\text{def}}{=} b_{mn}$$

for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$. However, we reject this definition of an internal multiplicative law since *it is not* compatible with multiplication in $\mathbb{R}$. For instance, for $\beta = \tau$, $b_2 \times b_2 = \tau \times \tau = \tau^2 = b_3 \neq b_4$.

Definition 5. We define the *quasi-multiplication* as the following internal multiplicative law on the set of beta-integers:

$$b_m \otimes b_n = \begin{cases} b_{(mn - a\rho_S(m)\rho_S(n))} & \text{in Case 1,} \\ b_{(mn - \rho_S(m)\rho_S(n))} & \text{in Case 2,} \end{cases} \tag{7}$$

where, for $n \geqslant 0$, $\rho_S(n)$ denotes the number of tiles S between $b_0 = 0$ and b_n [6]. For instance, for τ, $\rho_S(5) = 2$ while for θ, $\rho_S(5) = 1$. Geometrically, for $n \geqslant 0$, the nth beta-integer is the right vertex of the nth tile of the tiling associated with $\mathbb{Z}_\beta$, which can be expressed as $b_n = n + (-1 + \ell(S))\rho_s(S)$ and from which we derive the following:

$$\rho_S(n) = \frac{1}{1 - 1/\beta}(n - b_n) \quad \text{Case 1,}$$
$$\rho_S(n) = \beta(n - b_n) \quad \text{Case 2.}$$

For $n < 0$, $\rho_S(n) = -\rho_S(-n)$.

Lemma 2. *Quasi-multiplication is compatible with multiplication of real numbers if β is a quadratic PV unit.*

Proof. Quasi-multiplication has minimal distortion property with respect to multiplication [4,6]: for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$ with β quadratic PV unit,

$$b_m b_n - (b_m \otimes b_n) \in \begin{cases} \{(0, \pm 1, \dots, \pm a)(1 - \frac{1}{\beta})\} & \text{Case 1,} \\ \{(0, 1, \dots, a-1)\frac{\operatorname{sgn}(b_m b_n)}{\beta}\} & \text{Case 2.} \end{cases} \tag{8}$$

Suppose that for a given couple of integers (m,n), there exist a q such that $b_m b_n = b_q$. Then $b_q - (b_m \otimes b_n)$ verifies (8), and this implies $b_q - (b_m \otimes b_n) = 0$. Indeed the distances between two non-equal beta-integers are larger than or equal to $\ell(S)=1/\beta$ in Case 1, and $\ell(S)=1-1/\beta$ in Case 2. We then have $b_q=b_m\otimes b_n=b_{mn-a\rho_S(m)\rho_S(n)}$ in Case 1 and $b_q=b_m\otimes b_n=b_{mn-\rho_S(m)\rho_S(n)}$ in Case 2, which gives $q=mn-a\rho_S(m)\rho_S(n)$ in Case 1, and $q=mn-\rho_S(m)\rho_S(n)$ in Case 2. □

An interesting outcome of this multiplicative structure is the following explicit result concerning self-similarity properties of the set of beta-integers.

Let $U=(u_q)_{q\in\mathbb{N}}$ be the linear recurrent sequence of integers associated with β. In Case 1, the u_q are defined by $u_{q+2}=au_{q+1}+u_q$ with $u_0=1$, $u_1=a+1$. In Case 2, the u_q are defined by $u_{q+2}=au_{q+1}-u_n$ with $u_0=1$, $u_1=a$. The recurrence is possibly extended to negative indices.

Proposition 1. *Let β be a quadratic PV unit, and $\mathbb{Z}_\beta$ the corresponding set of beta-integers. Then for $q\in\mathbb{N}$ and $b_n\in\mathbb{Z}_\beta$ we have the self-similarity formulas*:

$$\beta^q\, b_n = b_{u_q}b_n = b_{u_q}\otimes b_n = b_{u_q n-a\rho_S(u_q)\rho_S(n)} = b_{u_q n-(u_q-u_{q-1})\,\rho_S(n)} \quad (\textit{in Case } 1),$$
$$\beta^q\, b_n = b_{u_q}b_n = b_{u_q}\otimes b_n = b_{u_q n-\rho_S(u_q)\rho_S(n)} = b_{u_q n-(u_{q-1})\,\rho_S(n)} \quad (\textit{in Case } 2).$$

The proof is a direct consequence of the definition of the quasi-multiplication and of the following lemma giving some of the properties of the counting function ρ_S.

Lemma 3. *The values assumed by the counting function $\rho_S(n)$ when $n=u_q\in U$ are*

$$\rho_S(u_q)=\frac{u_q-u_{q-1}}{a} \quad (\textit{in Case } 1),$$
$$\rho_S(u_q)=u_{q-1} \quad (\textit{in Case } 2).$$

Proof. *Case* 1: Let $w_q=\rho_S(u_q)$. By construction, $u_q=|\sigma_\beta^q(L)|$ and $w_q=|\sigma_\beta^q(L)|_S$. Therefore, the sequence (w_q) satisfies the same linear recurrence as (u_q), that is $w_q=aw_{q-1}+w_{q-2}$, with $w_0=0$, $w_1=1$. Thus $w_2=aw_1+w_0=(u_2-u_1)/a=a$ and $w_3=aw_2+w_1=(u_3-u_2)/a=a^2+1$. The recurrence is proved through $w_{q+1}=aw_q+w_{q-1}=a(u_q-u_{q-1})/a+(u_{q-1}-u_{q-2})/a=(u_{q+1}-u_q)/a$.

Case 2: Let $w_q=\rho_S(u_q)$. We have $w_q=aw_{q-1}-w_{q-2}$, with $w_0=0$ and $w_1=1$. Then $w_2=aw_1-w_0=u_1=a$ and $w_3=aw_2-w_1=u_2=a^2-1$. The recurrence is proved through $w_{q+1}=aw_q-w_{q-1}=au_{q-1}-u_{q-2}=u_q$. □

It should be noticed that quasi-multiplication does not define a group for not being associative and is not distributive with respect to beta-addition. So it seems hopeless to obtain a ring structure, like we have with integers, with such an internal multiplicative law. Note that beta-addition and quasi-multiplication are related to some operations in numeration systems studied in [9,14,15]. Nevertheless, the set of beta-integers recovers a ring structure asymptotically, as shall be explained in Section 6.

4. Beta-lattices in the plane

4.1. General considerations

We have seen that the condition $2\cos(2\pi/N)\in\mathbb{Z}$, i.e. $N=1,2,3,4$ and 6, characterizes N-fold Bravais lattices in $\mathbb{R}^2$ (and in $\mathbb{R}^3$). We would like to generalize this notion when N is quasicrystallographic i.e. $N=5,10,8$ and 12, respectively, associated with one of the cyclotomic Pisot units $\tau=2\cos(2\pi/10)$, $\delta=1+2\cos(2\pi/8)$ and $\theta=2+2\cos(2\pi/12)$. As a consequence of the results presented above, if (e_i) is a basis of $\mathbb{R}^d$, then the point set

$$\Gamma=\sum_{i=1}^{d}\mathbb{Z}_\beta \mathrm{e}_i$$

is a Meyer set and a lattice for the law $\oplus$. Moreover $\mathbb{Z}_\beta\otimes\Gamma\subset\Gamma$. We shall adopt the generic name of *beta-lattice* for such a Γ. Examples of beta-lattices in the plane are point-sets of the form

$$\Gamma_q(\beta)=\mathbb{Z}_\beta+\mathbb{Z}_\beta\zeta^q,$$

with $\zeta=\mathrm{e}^{\mathrm{i}2\pi/N}$, for $1\leqslant q\leqslant N-1$. Note that the latter are not rotationally invariant. Examples of rotationally invariant point-sets based on beta-integers are

$$\Lambda_q\overset{\mathrm{def}}{=}\bigcup_{j=0}^{N-1}\Gamma_q\zeta^j,\quad 1\leqslant q\leqslant N-1$$

and

$$\mathbb{Z}_\beta[\zeta]\overset{\mathrm{def}}{=}\sum_{j=0}^{N-1}\mathbb{Z}_\beta\zeta^j.$$

These sets Λ_q and $\mathbb{Z}_\beta[\zeta]$ are Meyer sets.

Let us now focus on the simplest case, namely $N=5$ or 10. It is more convenient to introduce the root of unity $\zeta=\mathrm{e}^{\mathrm{i}\pi/5}$, since $\tau=2\cos\pi/5=\zeta+\zeta^c$, where ζ^c is the complex conjugate of ζ. We obtain the set

$$\mathbb{Z}_\tau[\zeta]\;\equiv\;\mathbb{Z}_\tau+\mathbb{Z}_\tau\zeta+\mathbb{Z}_\tau\zeta^2+\mathbb{Z}_\tau\zeta^3+\mathbb{Z}_\tau\zeta^4.$$

Consider now the following τ-lattices in the plane:

$$\Gamma_q=\mathbb{Z}_\tau+\mathbb{Z}_\tau\zeta^q,\quad q=1,\ 2,\ 3,\ \text{or } 4.$$

The following inclusions were proven in [3]:

$$\Gamma_q\subset\mathbb{Z}_\tau[\zeta]\subset\frac{\Gamma_q}{\tau^4}.$$

It has been shown that a large class of aperiodic sets can be embedded in beta-lattices such as $\Gamma_q(\beta)$ (see [3]).

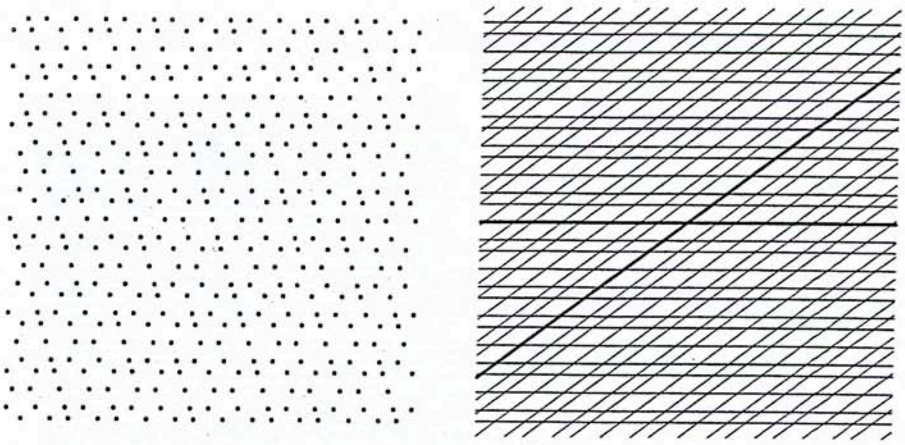

Fig. 2. The τ-lattice $\Gamma_1(\tau)$ with points (left), and its trivial tiling made by joining points along the horizontal axis, and along the direction defined by ζ.

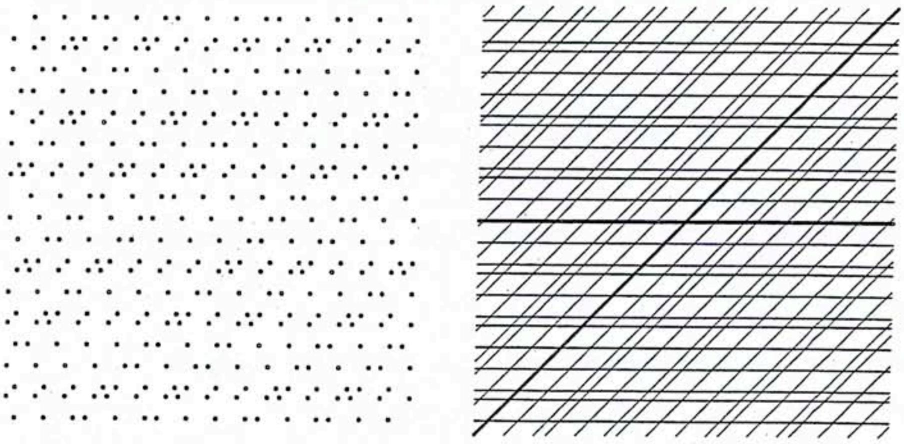

Fig. 3. The δ-lattice $\Gamma_1(\delta)$ with points (left), and its trivial tiling obtained by joining points along the horizontal axis, and along the direction defined by ζ.

On Figs. 2–4, we displayed the τ-lattice $\Gamma_1(\tau)$, the δ-lattice $\Gamma_1(\delta)$ and the decorated θ-lattice, $\tilde{\Gamma}_1(\theta) = \tilde{\mathbb{Z}}_\beta + \tilde{\mathbb{Z}}_\beta\zeta$, respectively, both as point-sets, and as tilings.

4.2. Rotational properties of the beta-lattices $\Gamma_1(\beta)$

Although beta-lattices are not rotationally invariant, we can nevertheless study the action of rotations on them. In this section, and throughout the rest of the article, we focus on $\Gamma_1(\beta)$. For $\beta = \tau$ and δ, any beta-lattice $\Gamma_q(\beta)$ is a subset of the properly scaled beta-lattice $\Gamma_1(\beta)$. Therefore, the rotational properties of $\Gamma_q(\beta)$ can always be

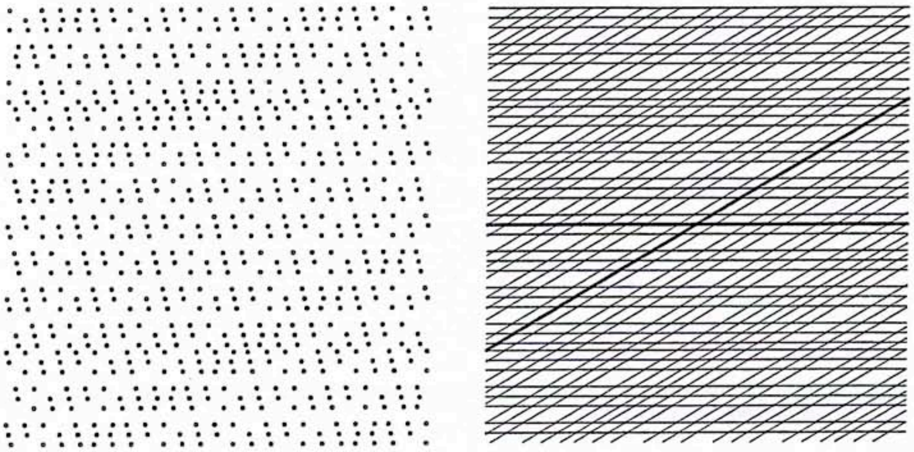

Fig. 4. The decorated θ-lattice $\tilde{\Gamma}_1(\theta)$ with points (left), and its trivial tiling obtained by joining points along the horizontal axis, and along the direction defined by ζ.

reexpressed in terms of the rotational properties of $\Gamma_1(\beta)$. Note that since θ is a quadratic PV unit of the second kind, the statement is slightly different, since the θ-lattices $\Gamma_q(\theta)$ are not subsets of the properly scaled $\Gamma_1(\theta)$, for $q \neq 1$, but of its decorated version $\tilde{\Gamma}_1(\theta)$.

We introduce the algebraic integer associated with ζ, $\chi = \zeta + \bar{\zeta} = 2\cos(2\pi/N)$, which entails $\zeta^2 = -1 + \chi\zeta$, and

$$\zeta^q = \eta_q + \nu_q\zeta, \quad q \in \{0, 1, \dots, N-1\}. \tag{9}$$

A rotation by $q2\pi/N$ on an arbitrary element $b_m + b_n\zeta$ of $\Gamma_1(\beta)$ then gives

$$\zeta^q(b_m + b_n\zeta) = (\eta_q b_m - \nu_q b_n) + (\nu_q b_m + (\eta_q + \nu_q\chi)b_n)\zeta. \tag{10}$$

This is not an element of $\Gamma_1(\beta)$ in general, but belongs to a deflated version of $\Gamma_1(\beta)$ by a certain factor. If we consider the values of the pairs (η_q, ν_q) and of $\eta_q + \nu_q\chi$, when β assumes the specific values τ and δ, we can determine this deflation factor. When $\beta = \theta$, $\zeta^q(b_m + b_n\zeta)$ belongs to the twice decorated θ-lattice $\tilde{\tilde{\Gamma}}_1(\theta)$, as will be shown explicitly.

- When $\beta = \tau$, the results are given for $\zeta = e^{i2\pi/10}$, $\chi = \tau$.

$$\begin{array}{llllll} q & = 0 & 1 & 2 & 3 & 4 \\ (\eta_q, \nu_q) & = (1,0) & (0,1) & (-1,\tau) & (-\tau,\tau) & (-\tau,1) \\ \eta_q + \nu_q\chi & = \ 1 & \tau & \tau & 1 & 0, \end{array}$$

together with $(\eta_{q+5}, \nu_{q+5}) = (-\eta_q, -\nu_q)$. Hence

$$\begin{aligned} \zeta^q\Gamma_1(\tau) &\subset \Gamma_1(\tau) + \left(\left\{0, \pm\left(1 - \frac{1}{\tau}\right)\right\} + \left\{0, \pm\left(1 - \frac{1}{\tau}\right)\right\}\zeta\right) \\ &\subset \frac{\Gamma_1(\tau)}{\tau^2}. \end{aligned}$$

Note that since $\chi=\tau$, $\Gamma_1(\tau)$ is endowed with specific properties which are not encountered in other cases, namely when $\beta=\delta$, and $\beta=\theta$. These properties are given by the following lemma.

Lemma 4. *For $\zeta=e^{i\pi/5}$, all elements of the cyclic group $\{\zeta^q, q\in\{0,1,2,\dots,9\}\}$ are elements of the τ-lattice $\Gamma_1(\tau)$.*

Proof. The demonstration is trivial from the values assumed by η_q and v_q in the case of τ,

$$\zeta^q=\eta_q+v_q\zeta, \quad \text{with } \eta_q, v_q\in\{0,\pm 1,\pm\tau\}. \qquad \square$$

Also note that from the self-similarity property of $\mathbb{Z}_\tau$ we have $\eta_q b_n\in\mathbb{Z}_\tau$, $v_q b_n\in\mathbb{Z}_\tau$ and $(\eta_q+\tau v_q)b_n\in\mathbb{Z}_\tau$, for all q and n.

- When $\beta=\delta$, $\zeta=e^{i2\pi/8}$ and $\chi=\delta-1$.

$$\begin{array}{rcllll} q & = & 0 & 1 & 2 & 3 \\ (\eta_q, v_q) & = & (1,0) & (0,1) & (-1,\delta-1) & (-\delta+1,1) \\ \eta_q+v_q\chi & = & 1 & \delta-1 & 1 & 0, \end{array}$$

together with $(\eta_{q+4}, v_{q+4})=(-\eta_q,-v_q)$. Hence

$$\begin{aligned} \zeta^q\Gamma_1(\delta)\subset\Gamma_1(\delta)+&\left(\left\{0,\pm\left(1-\frac{1}{\delta}\right),\pm 2\left(1-\frac{1}{\delta}\right)\right\}\right.\\ &\left.+\left\{0,\pm\left(1-\frac{1}{\delta}\right),\pm 2\left(1-\frac{1}{\delta}\right)\right\}\zeta\right)\\ \subset&\frac{\Gamma_1(\delta)}{\delta^3}. \end{aligned}$$

Note that $\delta-1=\sqrt{2}$ is not a δ-integer. Its δ-expansion is $\langle\delta-1\rangle_\delta=1\cdot 1$. It turns out that only ζ, ζ^5, ζ^4 and 1 are in $\Gamma_1(\delta)$.

- When $\beta=\theta$, $\zeta=e^{i2\pi/12}$ and $\chi=\theta-2$.

$$\begin{array}{rcllllll} q & = & 0 & 1 & 2 & 3 & 4 & 5 \\ (\eta_q, v_q) & = & (1,0) & (0,1) & (-1,\theta-2) & (-\theta+2,2) & (-2,\theta-2) & (-\theta+2,1) \\ \eta_q+v_q\chi & = & 1 & \theta-2 & 2 & \theta-2 & 1 & 0, \end{array}$$

together with $(\eta_{q+6}, v_{q+6})=(-\eta_q,-v_q)$. Note that $\theta-2=\sqrt{3}$ is not a θ-integer. Moreover, the θ-expansion of $\theta-2$ is infinite: $\langle\theta-2\rangle_\theta=1\cdot(2)^\omega$. Then, only ζ, ζ^6, ζ^7 and 1 are in $\Gamma_1(\theta)$. Let us introduce the decorated θ-lattice $\tilde{\Gamma}_1(\theta)$, as we have done in the one-dimensional case (Eq. (4)),

$$\Gamma_1(\theta)\subset\tilde{\Gamma}_1(\theta)=\tilde{\mathbb{Z}}_\theta+\tilde{\mathbb{Z}}_\theta\zeta.$$

Since $\theta - 2 = 2 - 1/\theta$, then all ζ^q are in $\tilde{\Gamma}_1(\theta)$, and

$$\zeta^q \Gamma_1(\theta) \subset \Gamma_1(\theta) + \left(\left\{ 0, \pm\frac{1}{\theta}, \pm\frac{2}{\theta} \right\} + \left\{ 0, \pm\frac{1}{\theta}, \pm\frac{2}{\theta} \right\} \zeta \right)$$
$$\subset \tilde{\tilde{\Gamma}}_1(\theta) \equiv \tilde{\tilde{\mathbb{Z}}}_\theta + \tilde{\tilde{\mathbb{Z}}}_\theta \zeta, \tag{11}$$

where $\tilde{\tilde{\mathbb{Z}}}_\theta = \mathbb{Z}_\theta + \{0, \pm 1/\theta, \pm 2/\theta\}$.

4.3. Translational properties

They are deduced from Eqs. (2) and (4). In Case 1,

$$\Gamma_q(\beta) + \Gamma_q(\beta) \subset \Gamma_q(\beta)/\beta^2$$

and in Case 2,

$$\Gamma_q(\beta) + \Gamma_q(\beta) \subset \tilde{\Gamma}_q(\beta).$$

5. A plane-group for beta-lattices

Since beta-lattices of the type $\Gamma_q(\beta)$ are not rotationally and translationally invariant, we shall enforce invariance by replacing the usual additive and multiplicative laws by the beta-addition and the quasi-multiplication.

5.1. A point-group for beta-lattices in the plane

Explicit calculations of internal rotation actions on $\Gamma_1(\beta)$, referred to as beta-rotations, are given in the appendix. Note that since the quasi-multiplication is not distributive with respect to beta-addition, we find several candidates for internal rotational operators on $\Gamma_1(\beta)$. The choice for the beta-rotations presented in the following proposition is driven by compatibility property. Other internal rotational operator *are not* compatible with Euclidean rotations!

We formally imitate the expressions of successive rotations given by Eq. (10), by replacing in the equations, $+$ and $-$ by $\oplus$ and $\ominus$, and $\times$ by $\otimes$, when necessary. Proposition 2 below defines the beta-rotations on $\Gamma_1(\beta)$.

Proposition 2.

- *When* $\beta = \tau$, *with the notations of* (9), *the following* 10 *operators* r_q, $q = 0, 1, \ldots, 9$, *leave* $\Gamma_1(\tau)$ *invariant*:

$$r_q \odot (b_m + b_n\zeta) = \eta_q b_m \ominus \nu_q b_n + (\nu_q b_m \oplus (\eta_q + \tau\nu_q) b_n)\zeta.$$

- *When* $\beta = \delta$, *the following operators leave* $\Gamma_1(\delta)$ *invariant*:

$$r_1 \odot (b_m + b_n\zeta) = -b_n + (b_m \oplus \delta b_n \ominus b_n)\zeta = -b_n + b_{m+2n-2\rho_S(n)}\zeta,$$

$$r_2 \odot (b_m + b_n\zeta) = -(b_m \oplus \delta b_n \ominus b_n) + (\delta b_m \ominus b_m \oplus b_n)\zeta$$
$$= -b_{m+2n-2\rho_S(n)} + b_{2m+n-2\rho_S(m)}\zeta,$$
$$r_3 \odot (b_m + b_n\zeta) = -(\delta b_m \ominus b_m \oplus b_n) + b_m\zeta = -b_{2m+n-2\rho_S(m)} + b_m\zeta.$$

- *When $\beta=\theta$, the following operators leave $\Gamma_1(\theta)$ invariant:*

$$r_1 \odot (b_m + b_n\zeta) = -b_n + (b_m \oplus \theta b_n \ominus 2b_n)\zeta = -b_n + b_{m+2n-\rho_S(n)}\zeta,$$
$$r_2 \odot (b_m + b_n\zeta) = -(b_m \oplus \theta b_n \ominus 2b_n) + (\theta b_m \ominus 2b_m \oplus 2b_n)\zeta$$
$$= -b_{m+2n-\rho_S(n)} + b_{2m+2n-\rho_S(m)}\zeta,$$
$$r_3 \odot (b_m + b_n\zeta) = -(\theta b_m \ominus 2b_m \oplus 2b_n) + (2b_m \oplus \theta b_n \ominus 2b_n)\zeta$$
$$= -b_{2m+2n-\rho_S(m)} + b_{2m+2n-\rho_S(n)}\zeta,$$
$$r_4 \odot (b_m + b_n\zeta) = -(2b_m \oplus \theta b_n \ominus 2b_n) + (\theta b_m \ominus 2b_m \oplus b_n)\zeta$$
$$= -b_{2m+2n-\rho_S(m)} + b_{2m+n-\rho_S(n)}\zeta,$$
$$r_5 \odot (b_m + b_n\zeta) = -(\theta b_m \ominus 2b_m \oplus b_n) + b_m\zeta = -b_{n+2m-\rho_S(m)} + b_m\zeta.$$

For $\beta=\tau$, δ or θ, let the composition rule of these operators on $\Gamma_1(\beta)$ be defined by

$$(rr') \odot z = r \odot (r' \odot z)$$

and denote by Id the identity and by $\imath$ the space inversion

$$\imath \odot z = -z.$$

Then, the composition rule $(r,r') \to rr'$ is associative and the following identities hold: $r_0 = Id$ and $r_{q+N/2} = \imath r_q = r_q\imath$ for $q = 0,1,\dots,(N/2)-1$, where N is the symmetry order of β.

Lemma 5. *Beta-rotations defined in Proposition 2 on $\Gamma_1(\beta)$ are compatible with rotations when β assumes one of the specified values τ, δ and θ.*

Proof. We deduce from Eqs. (6) and (8) that beta-rotations have minimal distortion property with respect to ordinary rotations: let $z_{m,n} = b_m + b_n\zeta \in \Gamma_1(\beta)$, then

- $\beta=\tau$, $\zeta^q z_{m,n} - r_q \odot z_{m,n} \in \{0, \pm(1-\frac{1}{\tau})\} + \{0, \pm(1-\frac{1}{\tau})\}\zeta$,
- $\beta=\delta$, $\zeta^q z_{m,n} - r_q \odot z_{m,n} \in \{0, \pm(1-\frac{1}{\delta}), \pm2(1-\frac{1}{\delta})\} + \{0, \pm(1-\frac{1}{\delta}), \pm2(1-\frac{1}{\delta})\}\zeta$,
- $\beta=\theta$, $\zeta^q z_{m,n} - r_q \odot z_{m,n} \in \{0, \pm\frac{1}{\theta}, \pm\frac{2}{\theta}\} + \{0, \pm\frac{1}{\theta}, \pm\frac{2}{\theta}\}\zeta$.

Proposition 2 shows that beta-rotations can be decomposed in terms of beta-additions and quasi-multiplications. Compatibility of beta-rotation with euclidian rotation is thus a consequence of +-compatibility of beta-addition and ×-compatibility of quasi-multiplication. □

Computing the composition of any two of such beta-rotations r_q yields the following important result.

Proposition 3. *For $\beta=\tau$, δ and θ and for $N = 10$, 8 and 12, respectively, let $\mathfrak{R}_N = \mathfrak{R}_N(\beta)$ denote the semi-group freely generated by all r_q, $q \in \{0,1,\dots,N-1\}$. Among all beta-rotations, only $r_0, r_1, r_{N/2-1}, r_{N/2+1}, r_{N-1}$, and $\imath$ have their inverse in $\mathfrak{R}_N$.*

Proof. The following identities are straightforwardly checked:

$$r_1 r_{N/2-1} = r_{N/2-1} r_1 = r_{N/2+1} r_{N-1} = r_{N-1} r_{N/2-1} = \imath,$$
$$r_1 r_{N-1} = r_{N-1} r_1 = r_{N/2-1} r_{N/2+1} = r_{N/2+1} r_{N/2-1} = r_0.$$

A case study of all possible combinations of r_q shows that no other such operators are invertible. □

An immediate consequence is the existence of a symmetry group for $\Gamma_1(\beta)$, i.e. a group of planar transformations leaving $\Gamma_1(\beta)$ invariant.

Theorem 1. *For $\beta = \tau$, δ and θ, the group $\mathscr{R}_N = \mathscr{R}_N(\beta)$, freely generated by the four element set $\{r_0, \imath, r_1, r_{N/2-1}\}$, is a symmetry group for the beta-lattice $\Gamma_1(\beta)$. It is called the* symmetry point-group *of $\Gamma_1(\beta)$.*

Proof. An easy computation shows that the elements of $\mathscr{R}_N$ are invertible. Associativity of the law of internal composition of elements of $\mathscr{R}_N$ is a consequence of Proposition 2. □

5.2. A plane-group for beta-lattices $\Gamma_1(\beta)$

We now introduce into the present formalism the beta-translations acting on $\Gamma_1(\beta)$.

Proposition 4. *Let $z_0 = b_{m_0} + b_{n_0}\zeta$ be an element of the beta-lattice $\Gamma_1(\beta)$. There corresponds to it the internal action $t_{z_0} : \Gamma_1(\beta) \mapsto \Gamma_1(\beta)$*

$$t_{z_0}(z) = z \oplus z_0 \stackrel{\text{def}}{=} b_m \oplus b_{m_0} + (b_n \oplus b_{n_0})\zeta = b_{m+m_0} + b_{n+n_0}\zeta.$$

The set of beta-translations forms an abelian group isomorphic to the beta-lattice $\Gamma_1(\beta)$ considered itself as a group for the law $\oplus$. For this reason it will be also denoted by $\Gamma_1(\beta)$.

Proof. The beta-translation is a simple two-dimensional generalization of the one-dimensional beta-addition. □

As a direct generalization of one-dimensional beta-addition, it is obvious that beta-translation has minimal distortion property with respect to translation, and is compatible with it. Using Proposition 4, we come to the main result of this article.

Theorem 2. *For $\beta = \tau$, δ and θ, and for $N = 10$, 8 and 12, respectively, the group $\mathscr{S}_N = \mathscr{S}_N(\beta)$ freely generated by the five-element set $\{r_0, \imath, r_1, r_{(N/2)-1}, t_1\}$ is a symmetry group for the beta-lattice $\Gamma_1(\beta)$. This group is the semi-direct product of $\Gamma_1(\beta)$ and $\mathscr{R}_N$*

$$\mathscr{S}_n = \Gamma_1(\beta) \rtimes \mathscr{R}_N$$

with the composition rule

$$(b,R)(b',R') = (b \oplus R \odot b', RR').$$

In the present context, $\mathscr{S}_N$ is called the symmetry plane-group *of $\Gamma_1(\beta)$.*

The action of an element of $\mathscr{S}_N$ on $\Gamma_1(\beta)$ is thus defined as

$$(b,R) \cdot z = b \oplus R \odot z = t_b(R \odot z) \in \Gamma_1(\beta).$$

Proof. An easy computation shows that the elements of $\mathscr{S}_N$ are invertible. Associativity of the law of internal composition of elements of $\mathscr{R}_N$ is a consequence of Proposition 2 and of Theorem 1. □

5.3. *Tile transformations using internal operations on $\Gamma_1(\tau)$*

We would like to illustrate the action of $\mathscr{S}_n$ on $\Gamma_1(\beta)$, in the case of τ, by showing how a tile of $\Gamma_1(\tau)$ is transformed under the action of an element of $\mathscr{S}_{10}$.

Let $z = b_m + b_n\zeta \in \Gamma_1(\tau)$. An *elementary quadrilateral tile on z* is the following:

$$\mathbf{T}(z) = \{z, z \oplus 1, z \oplus \zeta, z \oplus (1+\zeta)\}.$$

From the definition of $\Gamma_1(\tau)$, we trivially see that their exist four kinds of elementary tiles, which we shall denote by *LL*, *LS*, *SL* and *SS*, as a reference to the length of their edges (see Fig. 5).

In case of a translation operation by z_0, t_{z_0}, the elementary quadrilateral tile $\mathbf{T}(z)$ is transformed into another elementary quadrilateral tile, whether of the same kind or of another kind, according to

$$t_{z_0}(\mathbf{T}(z)) = \mathbf{T}(z \oplus z_0) = \{z_0 \oplus z, z_0 \oplus z \oplus 1, z_0 \oplus z \oplus \zeta, z_0 \oplus z \oplus (1+\zeta)\}.$$

Another interesting transformation arises when one applies the beta-rotation operator r_1 on $\mathbf{T}(z)$ and around one of the vertex of $\mathbf{T}(z)$. For instance, the beta-rotation around z is given by

$$t_z(r_1 \odot t_{-z}(\mathbf{T}(z))) = \{z, z \oplus \zeta, z \oplus (-1+\tau\zeta), z \oplus (-1+\tau^2\zeta)\}.$$

Examples of such rotation operations are displayed on Fig. 6. This operation not only rotates, but distorts the tiles, in general. Therefore, the beta-rotated tile is not elementary anymore.

Fig. 5. Elementary quadrilateral tiles for the τ-lattice $\Gamma_1(\tau)$. From left to right: *LL*, *LS*, *SL*, *SS*. See also Fig. 2.

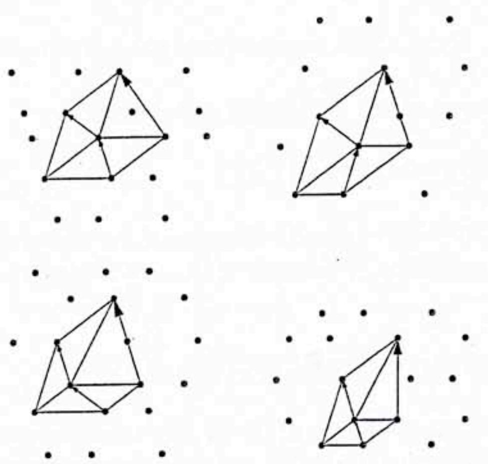

Fig. 6. Rotation operator r_1 applied to elementary tiles of the τ-lattice $\Gamma_1(\tau)$, T(0), T(1) (up), T(ζ) and T($1+\zeta$) (down). Note how the tiles are deformed, by this operation, in order for the vertices to remain in $\Gamma_1(\tau)$. The arrows indicate the vertices of the new tile in which are mapped the vertices of the original tile.

6. Asymptotic properties

An interesting feature of beta-lattices is that they behave like lattices asymptotically.

Lemma 6. *The asymptotic behavior of the counting function* ρ_S *is given by*

$$\rho_S(n) \underset{|n|\to\infty}{\approx} \left(1-\frac{1}{\beta}\right)\frac{n}{a} \quad (\textit{Case } 1),$$
$$\rho_S(n) \underset{|n|\to\infty}{\approx} \frac{n}{\beta} \quad (\textit{Case } 2).$$

Proof. *Case* 1: The proof is based on the development of integers in the linear system $U=(u_q)_{q\in\mathbb{N}}$. We have $n=\sum_{i=0}^{k} u_i d_i$. Then $\rho_S(n)=\sum_{i=0}^{k}\rho_S(u_i)d_i=\sum_{i=0}^{k}(u_i/a)(1-u_{i-1}/u_i)d_i$. When $n\to\infty$ we know that $u_{i-1}/u_i\to 1/\beta$ and $\rho_S(n)\approx(1/a)(1-1/\beta)\sum_{i=0}^{k}u_i d_i=(n/a)(1-1/\beta)$, as $n\to\infty$.

Case 2: As in the first case, the proof is based on the development of integers in (u_q): $n=\sum_{i=0}^{k}u_i d_i$, $\rho_S(n)=\sum_{i=0}^{k}\rho_S(u_i)d_i=\sum_{i=0}^{k}u_{i-1}d_i=\sum_{i=0}^{k}(u_i/a)(1+u_{i-2}/u_i)d_i$. When $n\to\infty$ we know that $u_{i-2}/u_i=(u_{i-2}/u_{i-1})(u_{i-1}/u_i)=1/\beta^2=(a/\beta)-1$. Therefore $\rho_S(n)\approx(1/\beta)\sum_{i=0}^{k}u_i d_i=n/\beta$, as $n\to\infty$. □

Lemma 6 tells us what is the asymptotic behavior of beta-integers for large n, and of the multiplication $\otimes$ for large m and n. From Eq. (7) and Lemma 6 is deduced the following result.

Proposition 5. *Let β be a quadratic PV unit number. Then the following asymptotic behaviour of beta-integers holds true*

$$b_n \underset{|n|\to\infty}{\approx} \gamma n,$$
$$b_m \otimes b_n \underset{|m|,|n|\to\infty}{\approx} \gamma^2 mn,$$

where

$$\gamma = \begin{cases} 1 - \frac{1}{a}(1 - \frac{1}{\beta})^2 = \frac{(a+2)\beta - a^2 - a - 2}{a} & (\textit{Case } 1), \\ 1 - \frac{1}{\beta^2} = a(\beta - a) + 2 & (\textit{Case } 2). \end{cases}$$

Proof. *Case* 1: Any beta-integer b_n can be written $b_n = n - \rho_S(n)(1 - 1/\beta)$. When n becomes large, we can replace $\rho_S(n)$ by its asymptotic value. We then have $b_n \approx n(1 - (1/a)(1 - 1/\beta)^2) = \gamma n$.

Case 2: In the same fashion, we have $b_n = n - \rho_S(n)1/\beta$, and by replacing $\rho_S(n)$ by its asymptotic value for large n we obtain $b_n = n(1 - 1/\beta^2) = \gamma n$.

The second part of the proposition is a direct consequence of the first part. □

We then *almost* recover the definition of multiplication we were thinking about at the beginning of Section 3.5, left alone that in both cases we have a contraction of the resulting index by a factor $\gamma < 1$. We should notice that the multiplication $\otimes$ is *asymptotically* associative and distributive with respect to the addition $\oplus$. In this sense, we can say that $\mathbb{Z}_\beta$ is asymptotically a ring

$$b_m \otimes (b_n \oplus b_p) - (b_m \otimes b_n) \oplus (b_m \otimes b_p) \underset{|m|,|n|,|p|\to\infty}{\approx} 0,$$
$$b_m \otimes (b_n \otimes b_p) - (b_m \otimes b_n) \otimes b_p \underset{|m|,|n|,|p|\to\infty}{\approx} 0.$$

Note that m, n and p must be such that $m \pm n$ and $m \pm p$ are large numbers, otherwise the above equations are not true.

Consequently, we compute the asymptotic behavior of rotational internal laws of beta-lattices, as defined in Section 5.1 in the studied cases.

- When $\beta = \tau$, we have for invertible operators

$$r_1 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-n + (m + \tau n)\zeta),$$
$$r_4 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-\tau m - n - m\zeta).$$

- When $\beta = \delta$, we have for invertible operators

$$r_1 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-n + (m + (\delta - 1)n)\zeta),$$
$$r_3 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-(\delta - 1)m - n + m\zeta).$$

- When $\beta = \theta$, we have for invertible operators

$$r_1 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-n + (m + (\theta - 2)n)\zeta),$$
$$r_5 \odot (b_m + b_n\zeta) \underset{|m|,|n|\to\infty}{\approx} \gamma(-(\theta - 2)m - n + m\zeta).$$

At this point one should be aware that these asymptotic beta-rotations are equivalent to rotations for large $|m|$ and $|n|$, and an easy computation shows that for $z_{m,n} \in \Gamma_1(\beta)$

$$\zeta z_{m,n} - r_1 \odot z_{m,n} \underset{|m|,|n|\to\infty}{\approx} 0,$$
$$\zeta^{N/2-1} z_{m,n} - r_{N/2-1} \odot z_{m,n} \underset{|m|,|n|\to\infty}{\approx} 0$$

with $N = 10, 8$ and 12.

7. Conclusion

The main result of this article is the construction of a symmetry plane-group for beta-lattices for three quadratic PV units. Though preliminary, this study shows the richness of the beta-lattices as far as all the operations of the plane-group can be made arithmetically explicit. Many questions seem to be open, such as the number of possible plane-groups leaving a beta-lattice invariant. Another important issue is to determine whether there is or not a metric left invariant under the action of such groups. It has been shown that a large class of point sets, such as model sets, can be embedded in beta-lattices [8]. A question related to distortion of distances is the action of beta-rotations and beta-translations over a point set embedded in a beta-lattice and over the tiling associated to this point set. The point group $\mathscr{R}_N(\beta)$ also deserves to be carefully studied. The link between beta-lattices and the class of finitely generated modules over ordered rings would deserves to be handled nicely in the framework of the Artin–Schreier theory. The case of PV of higher degree remains open. The present contribution shows the potentiality offered by a class of beta-lattices to provide structure models of more general quasiperiodic crystals, and possibly to predict new crystals.

Appendix A. Explicit internal beta-rotations actions on beta-lattices

In this section, we make the beta-rotation explicit for the quasicrystallographic numbers τ, δ, and θ, and for all the corresponding q, the remaining beta-rotation being deduced from them by combining with space inversion. We give the resulting integer indexes in terms of m, n, and the counting function ρ_S as all involved relations have been introduced in Eqs. (6) and (7).

A.1. Case of the τ-lattice $\Gamma_1(\tau)$

$$r_1 \odot (b_m + b_n\zeta) = b_{-n} + b_{m+2n-\rho_S(n)}\zeta,$$
$$r_2 \odot (b_m + b_n\zeta) = b_{-m-2n+\rho_S(n)} + b_{2(m+n)-\rho_S(m)-\rho_S(n)}\zeta,$$
$$r_3 \odot (b_m + b_n\zeta) = b_{-2(m+n)+\rho_S(m)+\rho_S(n)} + b_{2m+n-\rho_S(m)}\zeta,$$
$$r_4 \odot (b_m + b_n\zeta) = b_{-2m-n+\rho_S(m)} + b_m\zeta.$$

A.2. Case of the δ-lattice $\Gamma_1(\delta)$

For the δ-rotations we would like to proceed to the formal imitation of Eq. (10) as in the case of τ. The case of δ however is slightly more complicated since $\eta_q b_n$ and $v_q b_n$ are not in $\mathbb{Z}_\delta$. When we compute the rotation of an arbitrary element of $\Gamma_1(\delta)$, we need to determine the value of $(\delta-1)b_n$, which is of course not a δ-integer in the general case. Recall that $\otimes$ is not distributive with respect to $\oplus$. Therefore, we have to replace $(\delta-1)b_n$ either by $(\delta\ominus 1)\otimes b_n = b_{2n}$ or by $\delta b_n \ominus b_n = b_{2n-\rho_S(n)}$ (recall that from self-similarity of δ-integers we have $\delta \otimes b_n = \delta b_n$). We then have to make a choice about which operation to choose to build the point-group of $\Gamma_1(\delta)$. We chose to replace $(\delta-1)b_n$ by $\delta b_n \ominus b_n$ in Section 5, since this case satisfies the compatibility property. Other operations may be interesting. For example, the other internal rotation laws do not satisfy the compatibility property and do not have the same asymptotic behavior.

$$r_1 \odot (b_m + b_n\zeta) = \begin{cases} -b_n + b_{m+2n-\rho_S(n)}\zeta, \\ -b_n + b_{m+2n}\zeta, \end{cases}$$

$$r_2 \odot (b_m + b_n\zeta) = \begin{cases} -b_{m+2n-\rho_S(n)} + b_{2m+n-\rho_S(m)}\zeta, \\ -b_{m+2n} + b_{2m+n-\rho_S(m)}\zeta, \\ -b_{m+2n-\rho_S(n)} + b_{2m+n}\zeta, \\ -b_{m+2n} + b_{2m+n}\zeta, \end{cases}$$

$$r_3 \odot (b_m + b_n\zeta) = \begin{cases} -b_{2m+n-\rho_S(m)} + b_m\zeta, \\ -b_{2m+n} + b_m\zeta. \end{cases}$$

A.3. Case of the θ-lattice $\Gamma_1(\theta)$

As in the case of the δ-lattice, we have to decide which operation to use to build the point-group of $\Gamma_1(\theta)$ because of the factor $(\theta-2)b_n$, introduced in the computation of rotations of $\Gamma_1(\theta)$. Once again, we have replaced $(\theta-2)b_n$ by $\theta b_n \ominus 2\otimes b_n = b_{2n-\rho_S(n)}$ in Section 5.2. We give now all possibilities.

$$r_1 \odot (b_m + b_n\zeta) = \begin{cases} -b_n + b_{m+2n-\rho_S(n)}\zeta, \\ -b_n + b_{m+2n}\zeta, \end{cases}$$

$$r_2 \odot (b_m + b_n\zeta) = \begin{cases} -b_{m+2n-\rho_S(n)} + b_{2m+2n-\rho_S(m)}\zeta, \\ -b_{m+2n} + b_{2m+2n-\rho_S(m)}\zeta, \\ -b_{m+2n-\rho_S(n)} + b_{2m+2n}\zeta, \\ -b_{m+2n} + b_{2m+2n}\zeta, \end{cases}$$

$$r_3 \odot (b_m + b_n\zeta) = \begin{cases} -b_{2m+2n-\rho_S(m)} + b_{2m+2n-\rho_S(n)}\zeta, \\ -b_{2m+n} + b_{2m+2n-\rho_S(n)}\zeta, \\ -b_{2m+2n-\rho_S(m)} + b_{2m+2n}\zeta, \\ -b_{2m+2n} + b_{2m+2n}\zeta, \end{cases}$$

$$r_4 \odot (b_m + b_n\zeta) = \begin{cases} b_{2m+2n-\rho_S(n)} + b_{n+2m-\rho_S(m)}\zeta, \\ b_{2m+2n} + b_{n+2m-\rho_S(m)}\zeta, \\ b_{2m+2n-\rho_S(n)} + b_{n+2m}\zeta, \\ b_{2m+2n} + b_{n+2m}\zeta, \end{cases}$$

$$r_5 \odot (b_m + b_n\zeta) = \begin{cases} -b_{2m+n-\rho_S(m)} + b_m\zeta, \\ -b_{2m+n} + b_m\zeta. \end{cases}$$

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RÉSUMÉ

Les objets considérés dans cette thèse sont les empilements de sphères égales, principalement de $\mathbb{R}^n$ et les beta-entiers, pour lesquels on utilise indifféremment le langage des empilements de sphères ou celui des ensembles uniformément discrets pour les décrire. Nous nous sommes concentrés sur les problèmes suivants : (i) aspects métriques et topologiques de l'espace des empilements de sphères pour lequels nous prouvons un théorème de compacité qui généralise le Théorème de Sélection de Mahler relatif aux réseaux, (ii) les relations entre trous profonds et la densité par la constante de Delone ainsi que la structure interne asymptotique, en couches, des empilements les plus denses, (iii) les empilements autosimilaires de type fini pour lesquels nous montrons, pour chacun, l'existence d'un schéma de coupe-et-projection associé à un entier algébrique (l'autosimilarité) dont le degré divise le rang de l'empilement, (iv) les empilements de sphères sur beta-réseaux, dont l'étude a surtout consisté à comprendre l'ensemble discret localement fini $\mathbb{Z}_\beta$ des β-entiers et à proposer une classification des nombres algébriques qui complémente celle de Bertrand-Mathis reportée dans un article de Blanchard, et où la mesure de Mahler de β intervient naturellement.

MOTS-CLES

Beta-entier, beta-réseau, empilement de sphères, recouvrement de sphères, beta-numération, nombre de Pisot, nombre de Salem, nombre de Perron, quasicristal mathématique, ensemble de Meyer, ensemble de Delone, autosimilarité, pavage, groupe cristallographique, norme de Marcinkiewicz, densité, approximation Diophantienne, mesure de Mahler.

CLASSIFICATION MATHEMATIQUE

11B05, 11Jxx, 11J68, 11R06, 30C15, 52C17, 52C22, 52C23.