

Université d'Aix-Marseille I — U.F.R. de Mathématiques, Informatique et Mécanique
École Doctorale de Mathématiques et d'Informatique de Marseille — E.D. 184
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On the Characterization of Common Sense Reasoning in the Presence of Uncertain Information

Thesis

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Jonathan Ben-Naim

Rapporteurs:

Mr. Dov Gabbay Professor, King's College
Mr. Andreas Herzig CNRS researcher, Université Paul Sabatier

Jury:

Mr. Serge Grigorieff Professor, Université Paris 7 (president)
Mr. Andreas Herzig CNRS researcher, Université Paul Sabatier
Mr. David Makinson Professor, King's College
Mr. Pierre Marquis Professor, Université d'Artois
Mr. Karl Schlechta Professor, Université d'Aix-Marseille I (director)

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Abstract

The essence of the present thesis is to provide characterizations (or show the nonexistence of characterizations) for families of consequence relations and revision operators.

First, we will investigate consequence relations that are both paraconsistent and plausible (sometimes monotonic, sometimes non-monotonic) in a general framework that covers e.g. the ones of the well-known paraconsistent logics J_3 and $FOUR$. We will lay the focus on *preferential* and *pivotal consequence relations*. The former are defined by binary preference relations on states labelled by valuations (in the style of e.g. Kraus, Lehmann, and Magidor). The latter are defined by pivots (in the style of e.g. Makinson). A pivot is a fixed subset of valuations which are considered to be the important ones in the absolute sense. We will provide characterizations for these consequence relations. We will also provide characterizations for *preferential-discriminative* and *pivotal-discriminative consequence relations*. They are defined exactly as the plain versions, except that among the conclusions, a formula is rejected if its negation is also present. On the other hand, we will show that the family of all pivotal consequence relations does not admit a characterization containing only conditions universally quantified and of limited size. Finally, we will put in evidence a connexion between pivotal relations and X -logics (from Forget, Risch, and Siegel).

Second, we will study in a general framework an approach to belief revision based on distances between any two valuations (introduced by Lehmann, Magidor, and Schlechta). Suppose we are given such a distance d . This defines an operator $|_d$, called a *distance operator*, which transforms any two sets of valuations V and W into the set $V|_dW$ of all those elements of W that are closest to V . This operator $|_d$ defines naturally the revision of K by α as the set of all formulas satisfied in $M_K|_dM_\alpha$ (i.e. the set of all those models of α that are closest to the models of K). This constitutes a *distance-based revision operator*. Lehmann *et al.* characterized families of such operators using a “loop” condition of arbitrarily big size. An interesting question is to know whether this loop condition can be replaced by a finite one. Extending the results of Schlechta, we will provide elements of negative answer. In fact, we will show that for families of distance operators, there is no characterization containing only finite and universally quantified conditions. We are quite confident that this can be used to show similar impossibility results for families of distance-based revision operators. For instance, the families of Lehmann *et al.* might well be concerned with this, which suggests that their big loop condition cannot be replaced by a finite and universally quantified condition.

Keywords: common sense reasoning, non-monotonic logics, preferential logics, pivotal logics, paraconsistent logics, many-valued logics, iterated belief revision, distance-based revision.

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Introduction

Plausible and paraconsistent consequence relations

In many situations, an agent is confronted with incomplete and/or inconsistent information and then the classical consequence relation proves to be insufficient. Indeed, in case of inconsistent information, it leads to accept every formula as a conclusion, which amounts to loose the whole information. Therefore, the agent needs another relation leading to non-trivial conclusions in spite of the presence of contradictions. So, several paraconsistent consequence relations have been developed, e.g. [Bel77b, Bel77a, Gin88, Sub90a, Sub90b, Pri91, KL92, Loz94, Sub94, BDP95, PH98]. In the present thesis, we will pay attention in particular to certain *many-valued* ones [Bel77b, Bel77a, DdC70, AA94, AA96, AA98, CMdA00, dACM02]. They are defined in frameworks where valuations can assign more than two different truth values to formulas. In fact, they tolerate contradictions within the conclusions, but reject the principle of explosion according to which a single contradiction entails the deduction of every formula.

In case of incomplete information, the classical consequence relation also shows its limits. Indeed, no risk is taken, the conclusions are sure, but too few. The agent often needs to draw conclusions which are more daring, not necessarily sure, but still plausible. Eventually, some “hasty” conclusions will be rejected later, in the presence of additional information. To meet this need, several non-monotonic systems have been developed, e.g. the *default logic* of Reiter [Rei80], the *autoepistemic logic* of Moore [Moo85], McCarthy’s *circumscription* [McC80], and Reiter’s *closed world assumption* [Rei78]. Then, Gabbay, Makinson, Kraus, Lehmann, and Magidor investigated extensively properties for plausible non-monotonic consequence relations [Gab85, Mak89, Mak94, KLM90, LM92]. Central tools to define such relations are *choice functions* [Che54, Arr59, Sen70, AM81, Leh02, Leh01, Sch92b, Sch04]. Indeed, suppose we have at our disposal a function μ , called a choice function, which chooses in any set of valuations V , those elements that are preferred, not necessarily in the absolute sense, but when the valuations in V are the only ones under consideration. Then, it is natural to conclude α (a formula) from Γ (a set of formulas) iff every model for Γ chosen by μ is a model for α . This constitutes a plausible (generally, non-monotonic) consequence relation.

In the present thesis, we will lay the focus on two particular families of choice functions. Let’s present the first one. Suppose we are given a *binary preference relation* \prec on states labelled by valuations (in the style of e.g. [KLM90, Sch04]). This defines naturally a choice function. Indeed, choose in any set of valuations V , each element which labels a state which is \prec -preferred among all the states labelled by an element of V . Those choice functions which can be defined in this manner constitute the first family. The consequence relations defined by this family are called *preferential consequence relations*.

We turn to the second family. Suppose some valuations are considered to be the important ones

in the absolute sense and collect them in a set \mathcal{I} , called a *pivot*. Again, this defines naturally a choice function. Indeed, simply choose in any set of valuations, those elements which belong to \mathcal{I} . Those choice functions that can be defined in this way constitute the second family and the consequence relations defined by it are called *pivotal consequence relations*. Their importance has been put in evidence by D. Makinson in [Mak03, Mak05] where it is shown that they constitute an easy conceptual passage from classical to plausible non-monotonic relations. Indeed, they are perfectly monotonic but already display some of the distinctive features (i.e. the choice functions) of plausible non-monotonic relations.

The reader may have noticed that preferential and pivotal consequence relations could have been introduced directly by binary preference relations and pivots, without speaking of choice functions (as in fact was done historically). An advantage of the latter is that they provide a unified way to present those consequence relations, though they are based on very different objects. In addition, this will enable us to use similar techniques of proof for both kinds of relations (i.e. we will work with those properties which characterize the first family and then work in a similar manner with those properties that characterize the second family).

For a long time, research efforts on paraconsistent relations and plausible relations were separated. However, in many applications, the information is both incomplete and inconsistent. For instance, the semantic web or big databases inevitably contain inconsistencies. This can be due to human or material imperfections as well as contradictory sources of information. On the other hand, neither the web nor big databases can contain “all” information. Indeed, there are rules of which the exceptions cannot be enumerated. Also, some information might be left voluntarily vague or in concise form. Consequently, consequence relations that are both paraconsistent and plausible are useful to reason in such applications.

Such relations first appear in e.g. [Pri91, Bat98, KL92, AA00, KM02]. The idea begins by taking a many-valued framework to get paraconsistency. Then, only those models that are most preferred according to some particular *binary preference relation on valuations* (in the style of [Sho88, Sho87]) are relevant for making inference, which provides plausibility (and in fact also non-monotonicity). In [AL01b, AL01a], A. Avron and I. Lev generalized the study to families of binary preference relations which compare two valuations using, for each of them, this part of a certain set of formulas it satisfies. The present thesis follows this line of research by combining many-valued frameworks and choice functions.

More explicitly, we will investigate preferential and pivotal consequence relations in a general framework. According to the different assumptions which will be made about the latter, it will cover various kinds of frameworks, including e.g. the classical propositional one as well as some many-valued ones. Moreover, in the many-valued frameworks, preferential and pivotal relations lead to rational and non-trivial conclusions in spite of the presence of contradictions and are thus useful to deal with both incomplete and inconsistent information. However, they will not satisfy the *Disjunctive Syllogism* (from α and $\neg\alpha \vee \beta$ we can conclude β), whilst they satisfy it in classical frameworks.

In addition, it is in the many-valued frameworks that new relations, which we will investigate in detail, are really interesting: *preferential-discriminative* and *pivotal-discriminative consequence relations*. They are defined exactly as the plain versions, except that among the conclusions, a formula is rejected if its negation is also present. This kind of approach has been investigated for instance in [BDP95, KM02] (under the name of argumentative approach). In classical frameworks, these discriminative relations do not bring something really new. Indeed, instead of concluding

everything in the face of inconsistent information, we will simply conclude nothing. On the other hand, in many-valued frameworks, where the conclusions are reasonable even from inconsistent information, the discriminative versions will reject the contradictions among them, rendering them all the more reasonable.

The main contribution of the present thesis can now be summarized in one sentence: we characterized, in a general framework, several (sub)families of preferential(-discriminative) and pivotal(-discriminative) consequence relations. When the choice functions under consideration will satisfy a certain property of *definability preservation*, our characterizations will involve only purely syntactic conditions. This has a lot of advantages, let's quote some important ones. Take some syntactic conditions that characterize a family of those consequence relations. This gives a syntactic point of view on this family defined semantically, which enables us to compare it to conditions on the “market”, and thus to other consequence relations. This can also give rise to questions like: if we modified the conditions in such and such a natural-looking way, what would happen on the semantic side? More generally, this can open the door to questions that would not easily come to mind otherwise or to techniques of proof that could not have been employed in the semantic approach. Finally, this can help to find or improve proof systems based on the family, like a Gentzen proof system for instance.

Several characterizations can be found in the literature for preferential relations (e.g. [KLM90, LM92, Leh02, Leh01, Sch92b, Sch96, Sch00, Sch04]) and pivotal relations (e.g. [Rot01, Mak03, Mak05]). We will provide some new ones, though, to do so, we have been inspired by techniques of K. Schlechta [Sch04]. In fact, our innovation is rather related to the discriminative version. To the author knowledge, we accomplished the first systematic work of characterization for preferential-discriminative and pivotal-discriminative relations.

In addition, we will answer negatively a representation problem that was left open by Makinson. More precisely, suppose \mathcal{F} is set of formulas and \mathcal{R} a set of relations on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then, approximately, a characterization of \mathcal{R} will be called *normal* iff it contains only conditions universally quantified and of size smaller than $|\mathcal{F}|$ (note that $|\mathcal{F}|$ may be infinite). We will show, in an infinite classical propositional framework, that there is no normal characterization for the family of all pivotal consequence relations. For that, we will adapt some techniques which have been developed by Schlechta in [Sch04] to show similar impossibility results for preferential relations. Finally, we will investigate the *X-logics* from Forget, Risch, and Siegel [FRS01]. In fact, we will show that those pivotal consequence relations which satisfy a certain property of *universe-coddefinability* correspond precisely to those *X-logics* for which *X* is closed under the basic inference.

Distance-based revision

Another type of common sense reasoning in the presence of uncertain information is belief revision. It is the study of how an intelligent agent may replace its current epistemic state by another one which is non-trivial and incorporates new information (eventually contradicting the previous beliefs of the agent). This area of research finds applications for instance in multi-agent systems. In short, belief revision can be used to model the epistemic states of the agents (taken individually).

In [AGM85], Alchourrón, Gärdenfors, and Makinson proposed an approach now well-known as the AGM approach. An epistemic state is modelled there by a deductively closed set of formulas K and a new information by a formula α . A revision operator is then a function that transforms K and α into a new deductively closed set of formulas (intuitively, the revised epistemic state).

One of the contributions of the AGM approach is that it provides well-known postulates that any reasonable revision operator should satisfy. These postulates have been defended by their authors. But, doubts have been expressed as to their “soundness” (e.g. [KM92]) and especially “completeness” (e.g. [FL94, DP94, Leh95, DP97]). In particular, to be accepted, an operator never needs to put some coherence between the revisions of two different sets K and K' . As a consequence, some operators are accepted though they are not well-behaved when iterated. For example, take three sequences of revisions that differ only at some step in which the new information is α in the first sequence, β in the second, and $\alpha \vee \beta$ in the third. Now, suppose γ is concluded after both the first and the second sequences. Then, it should intuitively be the case that γ is concluded after third sequence too. But, one can find in [LMS01] an AGM operator which does not satisfy this intuitive property.

In addition, modelling an epistemic state by just a deductively closed set of formulas has been rejected by many researchers e.g. [BG93, Bou93, DP97, Wil94, NFPS96]. It is argued in [Leh95, FH96] that this modelling is not sufficient in many AI applications.

This provides motivations for another approach, based on distances between any two valuations, introduced by Lehmann, Magidor, and Schlechta in [SLM96, LMS01]. Their approach is in line with the AGM modelling of an epistemic state, but it defines well-behaved iterated revisions. More precisely, suppose we have at our disposal a distance d between any two valuations. This defines an operator $|_d$, called a *distance operator*, which transforms any ordered pair (V, W) of sets of valuations into the set $V|_dW$ of all those elements of W that are closest to V according to d .

This operator $|_d$ defines naturally the revision of K by α as the set of all formulas satisfied in $M_K|_dM_\alpha$ (i.e. the set of all those models of α that are closest to the models of K). This constitutes a *distance-based revision operator*, which is interesting for its natural aspect and for it is well-behaved when iterated. This is due to the fact that the revisions of the different K 's are all defined by the same distance, which ensures a strong coherence between them. Note that this is not the case with other definitions. For instance, with sphere systems [Gro88] and epistemic entrenchment relations [GM88], the revision of each K is defined by a different structure, without any “glue” relating them.

In [LMS01], Lehmann *et al.* characterized families of distance-based revision operators by the AGM postulates together with new ones that deal with iterated revisions. However, the latter postulates include a “loop” condition of arbitrarily big size. An interesting question is to know whether it can be replaced by a finite condition. In [Sch04], Schlechta provided elements of negative answer. More precisely, suppose \mathcal{V} is a set of valuations and \mathcal{O} a set of binary operators on $\mathcal{P}(\mathcal{V})$. Then, approximatively, a characterization of \mathcal{O} is *S-normal* (i.e. called normal by Schlechta) iff it contains only conditions which are finite, universally quantified (like e.g. the AGM postulates), and simple (i.e. using only elementary operations like e.g. \cup , \cap , \setminus). Schlechta showed that for families of distance operators, there is no S-normal characterization.

Now, there is a strong connexion between the distance operators (which apply to valuations) and the distance-based revision operators (which apply to formulas). It is quite reasonable to think that the work of Schlechta can be continued to show similar impossibility results for families of distance-based revision operators. For instance, the families investigated in [LMS01] might well be concerned with this, which suggests that the arbitrarily big loop condition cannot be replaced by a finite, universally quantified, and simple condition.

We will extend the work of Schlechta in two directions. First, we will use a more general definition of normality. Approximatively, a characterization of \mathcal{O} will be called *normal* iff it contains only conditions which are finite and universally quantified, but not necessarily simple (i.e. the conditions can involve complex structures or functions, etc., we are not limited to elementary operations). We

will show that the families which Schlechta investigated do not admit a normal characterization (in our larger sense). This is therefore a generalization of his negative results. Second, we will extend the negative results (always in our larger sense) to new families of distance operators, in particular to some that respect of the Hamming distance. Note that we will show these results in a very general framework which covers for instance the classical and the many-valued ones.

Though they are negative, these results help to understand more clearly the limits of what is possible in this area. They have therefore an interest of their own. Now, we are quite confident that the present work can be continued, like the work of Schlechta, to show similar impossibility results for families of distance-based revision operators. But, we will cover more families and with a more general definition of normality. This is the main motivation. Moreover, as we will work in a general framework, this direction for future work is still valid if we define revision in a non-classical framework, for instance, a many-valued one. Revision will then benefit from several advantages. For example, it will be possible to represent and revise inconsistent beliefs. We will discuss it in detail in the conclusion.

We finish the introduction with links between the two parts of the present thesis. First, both paraconsistent logics and belief revision are useful to deal with inconsistent information. However, each have adopted a different approach. In particular, belief revision separates new information from old information, whilst paraconsistent logics consider that all information is current. There are also close connexions between revision and plausible non-monotonic logics. Indeed, Gärdenfors and Makinson showed how to define a family of non-monotonic consequence relations from a revision operator and vice versa [Mak89, GM88, G88]. The idea is essentially to consider that β is a consequence of α iff the revision of a given K by α contains β . Postulates for revision on the one hand and that of non-monotonic reasoning on the other hand have been put in correspondence via this equivalence. Part I and Part II are therefore connected at the level of outlines. But, there are also links at the level of the formal machinery. Indeed, both choice functions and distance operators entail a selection of models and in both parts we will work in a general framework that covers at least the classical and the many-valued ones.

Structure of the thesis

In Chapter 1 (beginning of Part I), we will recall some early non-monotonic systems conceived to draw plausible conclusions from incomplete information: the default logic of Reiter, the autoepistemic logic of Moore, and McCarthy's circumscription.

In Chapter 2, we will introduce the fundamental definitions of Part I. More precisely, in Section 2.1, we will introduce our general framework. We will see that it covers in particular the many-valued frameworks of the well-known paraconsistent logics J_3 and $FOUR$. In Section 2.2, we will present the choice functions. We will see which properties characterize those choice functions that can be defined by a binary preference relation on states labelled by valuations. We will do the same work with pivots. In Section 2.3, we will define the preferential(-discriminative) consequence relations and give examples in classical and many-valued frameworks. We will also recall a characterization which involves the well-known system \mathbf{P} of Kraus, Lehmann, and Magidor. In Section 2.4, we will define the pivotal(-discriminative) consequence relations and give again examples in different frameworks.

In Chapters 3 and 4, we will provide, in our general framework, several characterizations of

preferential(-discriminative) and pivotal(-discriminative) consequence relations. These results will be placed into four different categories, depending on the version considered (plain or discriminative) and the property of definability preservation (satisfied or not).

In Chapter 5, we will define the normal characterizations for a set of consequence relations, i.e. those that involve only conditions universally quantified and of limited size. Then, we will show in an infinite classical framework, that there is no normal characterization for the family of all pivotal consequence relations.

In Chapter 6, we will present the X -logics from Forget, Risch, and Siegel. Then, we will show that those pivotal consequence relations which satisfy the property of *universe-coddefinability* are precisely those X -logics for which X is closed under the basic inference.

In Chapter 7 (beginning of Part II), we will recall the AGM approach to belief revision. We will present the expansion, contraction, and revision postulates as well as the Levi and Harper identities. We will also recall two important representation theorems, one with epistemic entrenchment relations from Gärdenfors and Makinson, another with sphere systems from Grove.

In Chapter 8, we will present the pseudo-distances from Lehmann, Magidor, and Schlechta. We will see how they can be used to define naturally the distance-based revision. And we will recall the characterizations of Lehmann *et al.* which contain an arbitrarily big loop condition.

In Chapter 9, we will define the normal characterizations for a set of binary operators, i.e. those that involve only finite and universally quantified conditions. Then, we will show that for several families of distance operators (in particular some related to the Hamming distance), there is no normal characterization.

To finish the introduction, I would like to say that I am the author of all the results shown in the present thesis. Here is an exhaustive list: Propositions 32, 34, 35, 37, 41, 43, 44, 45, and 47 have been published in [BN05b]; Propositions 22, 50, 52, 53, 55, 57, 59, 62, and 67 in [BN05a]; Propositions 84 and 85 in [BN06].

Part I

On the Characterization of Plausible and Paraconsistent Consequence Relations

Chapter 1

Early non-monotonic systems

Among the first non-monotonic systems conceived to draw plausible conclusions from incomplete information, we can find in particular the default logic of Reiter, the autoepistemic logic of Moore, and McCarthy's circumscription. Let's present them briefly.

1.1 Default logic

Let's start with the default logic of Reiter [Rei80]. Essentially, it is classical logic augmented by *default rules* of the form $\frac{\alpha:\beta_1,\dots,\beta_n}{\gamma}$. Intuitively, such a rule means "if α is true and β_1, \dots, β_n are consistent with what is known, then γ can be concluded". For the needs of this short presentation, it will be sufficient to deal just with *singular* rules, i.e. those for which $n = 1$. Now, suppose Γ is a set of formulas and D a set of singular default rules. Then, $\langle \Gamma, D \rangle$ is called a *default theory*. We turn to the *extensions* of $\langle \Gamma, D \rangle$. Suppose Δ is a set of formulas. Then, $\vdash(\Delta)$ denotes the classical closure of Δ and $G(\Delta)$ denotes the smallest set of formulas such that:

- $\Gamma \subseteq G(\Delta)$,
- $G(\Delta) = \vdash(G(\Delta))$, and
- if $\frac{\alpha:\beta}{\gamma} \in D$, $\alpha \in G(\Delta)$, and $\neg\beta \notin G(\Delta)$, then $\gamma \in G(\Delta)$.

Δ is called an extension of $\langle \Gamma, D \rangle$ iff $\Delta = G(\Delta)$. Any such extension represents a rational way to draw plausible conclusions from Γ , according to the default rules of D .

Default logic suffers from some drawbacks. One of them is that the property of *cumulativity* [Mak89] is not necessarily satisfied. Another one is that a default theory may have no extension. For instance, this is the case if $\Gamma = \emptyset$ and $D = \{\frac{:\neg\beta}{\beta}\}$. Reiter put in evidence a particular class of default theories for which there is always at least one extension. More precisely, a singular rule $\frac{\alpha:\beta}{\gamma}$ is *normal* iff β is equivalent to γ . A default theory is normal iff it contains only normal rules, in which case it has at least one extension.

In addition, the role of the default rules can be given different interpretations, which led to different opinions about properties like e.g. *semi-monotonicity* [Rei80] or *commitment to assumptions* [Poo89]. This remark and the one about drawbacks led to many variants of default logic. Among them, we can find: *constrained default logic* [Sch92a, DSJ95], *cumulative default logic* [Bre91], *justified default logic* [Łuk88], and *rational default logic* [MT95]. However, Delgrande and Schaub

demonstrated that all of these variants can already been expressed within the original framework of default logic [DS03, DS05].

Reiter and Criscuolo investigated *semi-normal default theories* [RC81]. A singular rule $\frac{\alpha:\beta}{\gamma}$ is *semi-normal* iff β implies γ . A default theory is semi-normal iff it contains only semi-normal rules. Such theories are interesting for they avoid some counter-intuitive aspects of *transitivity*. Here is an example: first (old) pentium computers are rare (at least nowadays) and rare things are usually expensive. Therefore, by transitivity, first pentiums are expensive, which is of course false.

Touretzky, Poole, Froidevaux, and Kayser developed further default logic in order to take into account priorities between default rules [Tou87, Poo85, FK88]. We finish with Lin, Shoham, Siegel, and Schwind who developed modal semantics for default logic [LS90, Sie90, SS91].

1.2 Autoepistemic logic

The autoepistemic logic of Moore [Moo85] aims to formalize reasoning on knowledge about knowledge. While propositional logic can only express facts, autoepistemic logic can express knowledge and lack of knowledge about facts. Moore developed this logic as a reconstruction of McDermott and Doyle's non-monotonic logic [MD80, McD82] to avoid some peculiarities of the latter.

The syntax of autoepistemic logic extends that of propositional logic by a modal (unary) connective \Box . Intuitively, $\Box\alpha$ means: “ α is believed to be true”. A typical formula is e.g. $(\neg\Box\neg\alpha) \rightarrow \alpha$, which means: “if α is not believed to be false, then α is assumed to be true”. It express therefore a kind of a closed world assumption. Another example is $(\Box\alpha \wedge \neg\Box\neg\beta) \rightarrow \beta$, which means: “if α is believed to be true and β is not believed to be false, then β is assumed to be true”.

Suppose Γ is a set of formulas like the two above. Moore introduced the *stable expansions* of Γ . A set of formulas Δ is a stable expansion of Γ iff

$$\Delta = \{\alpha : \Gamma \cup \{\Box\beta : \beta \in \Delta\} \cup \{\neg\Box\beta : \beta \notin \Delta\} \vdash \alpha\}.$$

Then, intuitively, Δ is a good candidate for the belief set of an ideal introspective agent with premises Γ . This is all the more true as Δ satisfies the three following *stability conditions*:

- (S1) if $\Delta \vdash \alpha$, then $\alpha \in \Delta$;
- (S2) if $\alpha \in \Delta$, then $\Box\alpha \in \Delta$;
- (S3) if $\alpha \notin \Delta$, then $\neg\Box\alpha \in \Delta$.

Konolige and Truszczyński showed that stable expansions in autoepistemic logic are closely related to extensions in default logic [Kon88, Tru91].

We finish with an interesting remark about *stable theories*, i.e. those sets of formulas which satisfy (S1), (S2), and (S3). Any stable theory T contains the following modal axiom schemata:

$$K \quad \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta);$$

$$T \quad \Box\alpha \rightarrow \alpha;$$

$$4 \quad \Box\alpha \rightarrow \Box\Box\alpha;$$

$$5 \quad \neg\Box\alpha \rightarrow \Box\neg\Box\alpha.$$

In addition, T contains all tautologies and is closed under the following rules:

Necessity $\frac{\alpha}{\Box\alpha}$;

Modus Ponens $\frac{\alpha \rightarrow \beta \quad \alpha}{\beta}$.

As a consequence, every theorem of the modal logic $S5$ belongs to T . And actually, Moore showed that the intersection of all stable theories is exactly the set of all theorems of $S5$ [Moo84].

1.3 Circumscription

Circumscription was first proposed by John McCarthy in [McC80, McC86]. It formalizes the common sense assumption that things are as expected unless otherwise specified. For example, if Tweety is a bird and nothing is said about the abnormality of Tweety, then it is natural to assume that Tweety is a normal bird and therefore conclude that it flies (as normal birds fly).

Circumscription formalizes this example as follows. First, the situation is represented by the following set Φ of first-order formulas:

$$\forall x, (bird(x) \wedge \neg abnormal(x)) \rightarrow flies(x);$$

$$bird(Tweety).$$

Then, the idea is to restrict the consideration to those models of Φ where the extension of *abnormal* is minimal in some sense. In other words, *abnormal* is “circumscribed”. But, in these selected models, *abnormal(Tweety)* does not hold (which would not have been the case if *abnormal(Tweety)* was specified in Φ , of course). This allows us to draw the defeasible conclusion *flies(Tweety)*, as desired.

More generally, suppose Φ is a set of first-order formulas (representing what is known about the domain of interest, i.e. facts, rules, etc.), P is a set of predicates (representing the exceptions of some rules), and Z a set of predicates (representing the conclusions of these rules). To fix ideas, in the Tweety example, $P = \{abnormal\}$ and $Z = \{flies\}$. It is customary to call the elements of P the predicates to be minimized and to call the elements of Z the predicates to be varied. Then, consider the binary preference relation \prec on the first-order models defined as follows. Suppose σ and σ' are two first-order models. Then, $\sigma \prec \sigma'$ iff

- σ and σ' have the same domain;
- σ and σ' do not differ in the interpretation of those constants, function symbols, and predicate symbols which belong neither to P nor Z ;
- $\forall p \in P$, the extension of p in σ is a subset of the extension of p in σ' ;
- $\exists p \in P$, the extension of p in σ is a proper subset of the extension of p in σ' .

As usual, a model σ of Φ is \prec -minimal iff there is no model σ' of Φ such that $\sigma' \prec \sigma$. Approximately, in the \prec -minimal models of Φ , the predicates of P will not hold (unless the contrary is specified in Φ , of course), which entails that the predicates of Z will be more likely to hold. Now,

only the \prec -minimal models of Φ will be taken into account for drawing conclusions from Φ . Intuitively, this amounts to apply rules while ignoring their exceptions (except those that are explicitly specified) and therefore draw plausible (and defeasible) conclusions.

This form of minimal inference is equivalent to second-order circumscription [Lif85]. The original definition of McCarthy was syntactical rather than semantical, i.e. the restriction of models is enforced by adding a certain second-order formula. For instance, suppose Φ is finite, $P = \{p\}$, $Z = \emptyset$, and the arity of p is 1. Then, the second-order formula is:

$$(\bigwedge \Phi) \wedge \forall p', \neg[(\bigwedge \Phi)[p/p'] \wedge (\forall x, p'(x) \rightarrow p(x)) \wedge \neg(\forall x, p(x) \rightarrow p'(x))]$$

where p' is a predicate of arity 1, $\bigwedge \Phi$ is the conjunction of the formulas in Φ , and $(\bigwedge \Phi)[p/p']$ is the formula obtained by replacing, in this conjunction, p by p' .

In the original version of circumscription, first-order circumscription, the restriction of models is enforced by adding first-order formulas generated by a certain “circumscription schema”. For instance, suppose Φ is finite, $P = \{p\}$, $Z = \{q\}$, and p, q have arity 1. Then, the circumscription schema is:

$$\neg[(\bigwedge \Phi)[p/p', q/q'] \wedge (\forall x, p'(x) \rightarrow p(x)) \wedge \neg(\forall x, p(x) \rightarrow p'(x))]$$

where p' and q' are new predicates with arity 1. Actually, the \prec -minimal models of Φ are not the only ones remaining after adding such first-order formulas. In fact, the remaining models are exactly those that are \prec' -minimal, where \prec' is the binary preference relation such that $\sigma \prec' \sigma'$ iff $\sigma \prec \sigma'$ and (approximatively) for each predicate r in $P \cup Z$, the extension of r in σ can be syntactically described in σ' (see e.g. [Bes88, Bra88] for details).

Chapter 2

Fundamental definitions

The present chapter introduces fundamental definitions for Part I.

2.1 Semantic structures

2.1.1 Definitions and assumptions

We will work with general formulas, valuations, and satisfaction. A similar approach has been taken in two well-known papers [Mak05, Leh01].

Definition 1 We say that \mathcal{S} is a *semantic structure* iff $\mathcal{S} = \langle \mathcal{F}, \mathcal{V}, \models \rangle$ where \mathcal{F} is a set, \mathcal{V} is a set, and \models is a relation on $\mathcal{V} \times \mathcal{F}$.

Intuitively, \mathcal{F} is a set of formulas, \mathcal{V} a set of valuations for these formulas, and \models a satisfaction relation for these objects (i.e. $v \models \alpha$ means the formula α is satisfied in the valuation v , i.e. v is a model for α).

Notation 2 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure, $\Gamma \subseteq \mathcal{F}$, and $V \subseteq \mathcal{V}$. Then,

$$M_\Gamma := \{v \in \mathcal{V} : \forall \alpha \in \Gamma, v \models \alpha\},$$

$$T(V) := \{\alpha \in \mathcal{F} : V \subseteq M_\alpha\},$$

$$\mathbf{D} := \{V \subseteq \mathcal{V} : \exists \Gamma \subseteq \mathcal{F}, M_\Gamma = V\}.$$

Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , and \mathcal{F} the set of all well-formed formulas (wffs) of \mathcal{L} . Then,

$$T_d(V) := \{\alpha \in \mathcal{F} : V \subseteq M_\alpha \text{ and } V \not\subseteq M_{\neg\alpha}\},$$

$$T_c(V) := \{\alpha \in \mathcal{F} : V \subseteq M_\alpha \text{ and } V \subseteq M_{\neg\alpha}\},$$

$$\mathbf{C} := \{V \subseteq \mathcal{V} : \forall \alpha \in \mathcal{F}, V \not\subseteq M_\alpha \text{ or } V \not\subseteq M_{\neg\alpha}\}.$$

Intuitively, M_Γ is the set of all models for Γ and $T(V)$ the set of all formulas satisfied in V . Every element of $T(V)$ belongs either to $T_d(V)$ or $T_c(V)$, according to whether its negation is also in $T(V)$. \mathbf{D} is the set of all those sets of valuations that are definable by a set of formulas and \mathbf{C} the set of all those sets of valuations that do not satisfy both a formula and its negation. As usual, $M_{\Gamma, \alpha}$, $T(V, v)$, etc. stand for respectively $M_{\Gamma \cup \{\alpha\}}$, $T(V \cup \{v\})$, etc.

Remark 3 The notations M_Γ , $T(V)$, etc. should contain the semantic structure on which they are based. To increase readability, we will omit it. There will never be any ambiguity. We will omit similar things with other notations in the sequel, for the same reason.

A semantic structure defines a basic consequence relation:

Notation 4 We denote by \mathcal{P} the power set operator.

Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure.

We denote by \vdash the relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$ such that $\forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}$,

$$\Gamma \vdash \alpha \text{ iff } M_\Gamma \subseteq M_\alpha.$$

Let \sim be a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then,

$$\sim(\Gamma) := \{\alpha \in \mathcal{F} : \Gamma \sim \alpha\}.$$

Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , and $\Gamma \subseteq \mathcal{F}$.

Then, we say that Γ is *consistent* iff $\forall \alpha \in \mathcal{F}, \Gamma \not\vdash \alpha$ or $\Gamma \not\vdash \neg\alpha$.

We will sometimes need to speak about the basic consequence relation in the sense of Scott:

\Vdash denotes the relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{P}(\mathcal{F})$ such that $\forall \Gamma, \Delta \subseteq \mathcal{F}$,

$$\Gamma \Vdash \Delta \text{ iff } \forall v \in M_\Gamma, \exists \alpha \in \Delta, v \in M_\alpha.$$

The following trivial facts hold, we will use them implicitly in the sequel:

Remark 5 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and $\Gamma, \Delta \subseteq \mathcal{F}$. Then:

$$M_{\Gamma, \Delta} = M_\Gamma \cap M_\Delta;$$

$$\vdash(\Gamma) = T(M_\Gamma);$$

$$M_\Gamma = M_{\vdash(\Gamma)};$$

$$\Gamma \subseteq \vdash(\Delta) \text{ iff } \vdash(\Gamma) \subseteq \vdash(\Delta) \text{ iff } M_\Delta \subseteq M_\Gamma.$$

Sometimes, we will need to make some of the following assumptions about a semantic structure:

Definition 6 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure.

Then, define the following assumptions:

$$(A0) \quad M_{\mathcal{F}} = \emptyset;$$

$$(A1) \quad \mathcal{V} \text{ is finite.}$$

Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , and \mathcal{F} the set of all wffs of \mathcal{L} . Then, define:

$$(A2) \quad \forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}, \text{ if } \alpha \notin T(M_\Gamma) \text{ and } \neg\alpha \notin T(M_\Gamma), \text{ then } M_\Gamma \cap M_\alpha \not\subseteq M_{\neg\alpha}.$$

Suppose \vee and \wedge are binary connectives of \mathcal{L} . Then, define:

$$(A3) \quad \forall \alpha, \beta \in \mathcal{F}, \text{ we have:}$$

$$M_{\alpha \vee \beta} = M_\alpha \cup M_\beta;$$

$$M_{\alpha \wedge \beta} = M_\alpha \cap M_\beta;$$

$$M_{\neg\neg\alpha} = M_\alpha;$$

$$M_{\neg(\alpha \vee \beta)} = M_{\neg\alpha \wedge \neg\beta};$$

$$M_{\neg(\alpha \wedge \beta)} = M_{\neg\alpha \vee \neg\beta}.$$

Clearly, those assumptions are satisfied by classical semantic structures, i.e. structures where \mathcal{F}, \mathcal{V} , and \models are classical. In addition, we will see, in Sections 2.1.2 and 2.1.3, that they are satisfied also by certain many-valued semantic structures.

2.1.2 The semantic structure defined by *FOUR*

The logic *FOUR* was introduced by N. Belnap in [Bel77a, Bel77b]. This logic is useful to deal with inconsistent information. Several presentations are possible, depending on the language under consideration. For the needs of the present thesis, a classical propositional language will be sufficient. The logic has been investigated intensively in e.g. [AA94, AA96, AA98], where richer languages, containing an implication connective \supset (first introduced by A. Avron [Avr91]), were considered.

Notation 7 We denote by \mathcal{A} a set of propositional symbols (or atoms).

We denote by \mathcal{L}_c the classical propositional language containing \mathcal{A} , the usual constants *false* and *true*, and the usual connectives \neg , \vee , and \wedge .

We denote by \mathcal{F}_c the set of all wffs of \mathcal{L}_c .

We recall one possible intuitive interpretation of the logic *FOUR* (more details can be found in [CLM99, Bel77a, Bel77b]). Consider a system in which there are, on the one hand, sources of information and, on the other hand, a processor that listens to them. The sources provide information about the atoms only, not about the compound formulas. For each atom p , there are exactly four possibilities: either the processor is informed (by the sources, taken as a whole) that p is true; or he is informed that p is false; or he is informed of both; or he has no information about p .

Notation 8 Denote by 0 and 1 the classical truth values and define:

$$\mathbf{f} := \{0\}; \quad \mathbf{t} := \{1\}; \quad \top := \{0, 1\}; \quad \perp := \emptyset.$$

The global information given by the sources to the processor can be modelled by a function s from \mathcal{A} to $\{\mathbf{f}, \mathbf{t}, \top, \perp\}$. Intuitively, $1 \in s(p)$ means the processor is informed that p is true, whilst $0 \in s(p)$ means he is informed that p is false.

Then, the processor naturally builds information about the compound formulas from s . Before he starts to do so, the situation can be modelled by a function v from \mathcal{F}_c to $\{\mathbf{f}, \mathbf{t}, \top, \perp\}$ which agrees with s about the atoms and which assigns \perp to all compound formulas. Now, take p and q in \mathcal{A} and suppose $1 \in v(p)$ or $1 \in v(q)$. Then, the processor naturally adds 1 to $v(p \vee q)$. Similarly, if $0 \in v(p)$ and $0 \in v(q)$, then he adds 0 in $v(p \vee q)$. Of course, such rules hold for \neg and \wedge too.

Suppose all those rules are applied recursively to all compound formulas. Then, v represents the “full” (or developed) information given by the sources to the processor. Now, the valuations of the logic *FOUR* can be defined as exactly those functions that can be built in this manner (i.e. like v) from some of these sources-processor systems. More formally,

Definition 9 We say that v is a *four-valued valuation* iff v is a function from \mathcal{F}_c to $\{\mathbf{f}, \mathbf{t}, \top, \perp\}$ such that $v(\text{true}) = \mathbf{t}$, $v(\text{false}) = \mathbf{f}$, and $\forall \alpha, \beta \in \mathcal{F}_c$,

$$\begin{aligned} 1 \in v(\neg\alpha) &\text{ iff } 0 \in v(\alpha); \\ 0 \in v(\neg\alpha) &\text{ iff } 1 \in v(\alpha); \\ 1 \in v(\alpha \vee \beta) &\text{ iff } 1 \in v(\alpha) \text{ or } 1 \in v(\beta); \\ 0 \in v(\alpha \vee \beta) &\text{ iff } 0 \in v(\alpha) \text{ and } 0 \in v(\beta); \\ 1 \in v(\alpha \wedge \beta) &\text{ iff } 1 \in v(\alpha) \text{ and } 1 \in v(\beta); \\ 0 \in v(\alpha \wedge \beta) &\text{ iff } 0 \in v(\alpha) \text{ or } 0 \in v(\beta). \end{aligned}$$

We denote by \mathcal{V}_4 the set of all four-valued valuations.

The definition may become more accessible if we see the four-valued valuations as those functions that satisfy Tables 1, 2, and 3 below:

$v(\alpha)$	$v(\neg\alpha)$
f	t
t	f
\top	\top
\perp	\perp

Table 1.

		$v(\beta)$			
		f	t	\top	\perp
$v(\alpha)$	f	f	t	\top	\perp
	t	t	t	t	t
	\top	\top	\top	\top	t
	\perp	\perp	t	t	\perp

Table 2.

		$v(\beta)$			
		f	t	\top	\perp
$v(\alpha)$	f	f	f	f	f
	t	f	t	\top	\perp
	\top	f	\top	\top	f
	\perp	f	\perp	f	\perp

Table 3.

In the logic *FOUR*, a formula α is considered to be satisfied iff the processor is informed that it is true (it does not matter whether he is also informed that α is false).

Notation 10 We denote by \models_4 the relation on $\mathcal{V}_4 \times \mathcal{F}_c$ such that $\forall v \in \mathcal{V}_4, \forall \alpha \in \mathcal{F}_c$, we have $v \models_4 \alpha$ iff $1 \in v(\alpha)$.

Proof systems for the consequence relations \vdash and \Vdash based on the semantic structure $\langle \mathcal{F}_c, \mathcal{V}_4, \models_4 \rangle$ (i.e. the semantic structure defined by *FOUR*) can be found in e.g. [AA94, AA96, AA98]. Here is one of them:

Axioms:

$$\begin{aligned} \Gamma, \alpha &\Rightarrow \Delta, \alpha \\ \Gamma, \neg true &\Rightarrow \Delta & \Gamma &\Rightarrow \Delta, true \\ \Gamma, false &\Rightarrow \Delta & \Gamma &\Rightarrow \Delta, \neg false \end{aligned}$$

Rules: Exchange, Contraction, and the following rules:

$$\begin{array}{l} \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \neg\neg\alpha \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg\neg\alpha} \\ \frac{\Gamma, \alpha, \beta \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \\ \frac{\Gamma, \neg\alpha \Rightarrow \Delta \quad \Gamma, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \wedge \beta) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\alpha, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \wedge \beta)} \\ \frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \\ \frac{\Gamma, \neg\alpha, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\alpha \quad \Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta)} \end{array}$$

Note that the *FOUR* semantic structure satisfies (A0) and (A3). In addition, if \mathcal{A} is finite, then (A1) is also satisfied. However, (A2) is not satisfied by this structure. In Section 2.1.3, we turn to a many-valued semantic structure which satisfies (A2).

2.1.3 The semantic structure defined by J_3

The logic J_3 was introduced in [DdC70] to answer a question posed in 1948 by S. Jaśkowski, who was interested in systematizing theories capable of containing contradictions, especially if they occur in dialectical reasoning. The step from informal reasoning under contradictions and formal reasoning with databases and information was done in [CMdA00] (also specialized for real database models in [dACM02]), where another formulation of J_3 called **LFII** was introduced, and its first-order version, semantics and proof theory were studied in detail. Investigations of J_3 have also been made in e.g. [Avr91], where richer languages than our \mathcal{L}_c were considered.

The valuations of the logic J_3 can be given the same meaning as those of the logic *FOUR*, except that the consideration is restricted to those systems where the sources, taken as a whole, always give some information about an atom. More formally,

Definition 11 We say that v is a *three-valued valuation* iff v is a function from \mathcal{F}_c to $\{\mathbf{f}, \mathbf{t}, \top\}$ such that $v(\text{true}) = \mathbf{t}$, $v(\text{false}) = \mathbf{f}$, and $\forall \alpha, \beta \in \mathcal{F}_c$,

- $1 \in v(\neg\alpha)$ iff $0 \in v(\alpha)$;
- $0 \in v(\neg\alpha)$ iff $1 \in v(\alpha)$;
- $1 \in v(\alpha \vee \beta)$ iff $1 \in v(\alpha)$ or $1 \in v(\beta)$;
- $0 \in v(\alpha \vee \beta)$ iff $0 \in v(\alpha)$ and $0 \in v(\beta)$;
- $1 \in v(\alpha \wedge \beta)$ iff $1 \in v(\alpha)$ and $1 \in v(\beta)$;
- $0 \in v(\alpha \wedge \beta)$ iff $0 \in v(\alpha)$ or $0 \in v(\beta)$.

We denote by \mathcal{V}_3 the set of all three-valued valuations.

As previously, the definition may become more accessible if we see the three-valued valuations as those functions that satisfy Tables 4, 5, and 6 below:

$v(\alpha)$	$v(\neg\alpha)$
\mathbf{f}	\mathbf{t}
\mathbf{t}	\mathbf{f}
\top	\top

Table 4.

	$v(\beta)$		
	\mathbf{f}	\mathbf{t}	\top
$v(\alpha)$	\mathbf{f}	\mathbf{t}	\top
	\mathbf{t}	\mathbf{t}	\mathbf{t}
	\top	\mathbf{t}	\top

Table 5.

	$v(\beta)$		
	\mathbf{f}	\mathbf{t}	\top
$v(\alpha)$	\mathbf{f}	\mathbf{f}	\mathbf{f}
	\mathbf{t}	\mathbf{f}	\top
	\top	\mathbf{f}	\top

Table 6.

We turn to the satisfaction relation.

Notation 12 We denote by \models_3 the relation on $\mathcal{V}_3 \times \mathcal{F}_c$ such that $\forall v \in \mathcal{V}_3, \forall \alpha \in \mathcal{F}_c$, we have $v \models_3 \alpha$ iff $1 \in v(\alpha)$.

Proof systems for the consequence relations \vdash and \Vdash based on the semantic structure $\langle \mathcal{F}_c, \mathcal{V}_3, \models_3 \rangle$ (i.e. the semantic structure defined by J_3) have been provided in e.g. [Avr91, DdC70] and chapter IX of [Eps90]. We recall one of them:

Axioms:

$$\begin{array}{ll} \Gamma, \alpha \Rightarrow \Delta, \alpha & \Gamma \Rightarrow \Delta, \alpha, \neg\alpha \\ \Gamma, \neg\text{true} \Rightarrow \Delta & \Gamma \Rightarrow \Delta, \text{true} \\ \Gamma, \text{false} \Rightarrow \Delta & \Gamma \Rightarrow \Delta, \neg\text{false} \end{array}$$

Rules: Exchange, Contraction, and the following rules:

$$\begin{array}{c}
\frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \neg\neg\alpha \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg\neg\alpha} \\
\frac{\Gamma, \alpha, \beta \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \\
\frac{\Gamma, \neg\alpha \Rightarrow \Delta \quad \Gamma, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \wedge \beta) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\alpha, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \wedge \beta)} \\
\frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \\
\frac{\Gamma, \neg\alpha, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\alpha \quad \Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta)}
\end{array}$$

The J_3 structure satisfies (A0), (A3) and (A2). In addition, if \mathcal{A} is finite, then it satisfies (A1) too.

2.2 Choice functions

2.2.1 Definitions and properties

In many situations, an agent has some way to choose in any set of valuations V , those elements that are preferred (the bests, the more normal, etc.), not necessarily in the absolute sense, but when the valuations in V are the only ones under consideration. In Social Choice, this is modelled by choice functions [Che54, Arr59, Sen70, AM81, Leh02, Leh01].

Definition 13 Let \mathcal{V} be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ a function from \mathbf{V} to \mathbf{W} . We say that μ is a *choice function* iff $\forall V \in \mathbf{V}, \mu(V) \subseteq V$.

Several properties for choice functions have been put in evidence by researchers in Social Choice. Let's present two important ones. Suppose W is a set of valuations, V is a subset of W , and $v \in V$ is a preferred valuation of W . Then, a natural requirement is that v is a preferred valuation of V . Indeed, in many situations, the larger a set is, the harder it is to be a preferred element of it, and he who can do the most can do the least. This property appears in [Che54] and has been given the name Coherence in [Mou85].

We turn to the second property. Suppose W is a set of valuations, V is a subset of W , and suppose all the preferred valuations of W belong to V . Then, they are expected to include all the preferred valuations of V . The importance of this property has been put in evidence by [Aiz85, AM81] and has been given the name Local Monotonicity in e.g. [Leh01].

Definition 14 Let \mathcal{V} be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ a choice function from \mathbf{V} to \mathbf{W} . We say that μ is *coherent* iff $\forall V, W \in \mathbf{V}$,

$$\text{if } V \subseteq W, \text{ then } \mu(W) \cap V \subseteq \mu(V).$$

We say that μ is *locally monotonic* (LM) iff $\forall V, W \in \mathbf{V}$,

$$\text{if } \mu(W) \subseteq V \subseteq W, \text{ then } \mu(V) \subseteq \mu(W).$$

In addition to their intuitive meanings, these properties are important because, as was shown by K. Schlechta in [Sch00], they characterize those choice functions that can be defined by a binary preference relation on states labelled by valuations (in the style of e.g. [KLM90, Sch04]). We will take a closer look at this in Section 2.2.2.

When we have a semantic structure and a choice function on the valuations, two new properties can be defined. Each of them conveys a simple and natural meaning.

Definition 15 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ a choice function from \mathbf{V} to \mathbf{W} .

We say that μ is *definability preserving* (DP) iff

$$\forall V \in \mathbf{V} \cap \mathbf{D}, \mu(V) \in \mathbf{D}.$$

Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , and \mathcal{F} the set of all wffs of \mathcal{L} .

We say that μ is *coherency preserving* (CP) iff

$$\forall V \in \mathbf{V} \cap \mathbf{C}, \mu(V) \in \mathbf{C}.$$

Definability Preservation has been put in evidence first in [Sch92b]. One of its advantages is that when the choice functions under consideration satisfy it, we will provide characterizations with purely syntactic conditions.

An advantage of Coherency Preservation is that when the choice functions under consideration satisfy it, we will not need to assume (A2) to show our characterizations (in the discriminative case).

Now, we provide properties which characterize those choice functions that can be defined by a pivot (in the style of e.g. D. Makinson [Mak03, Mak05]). A pivot is a fixed subset of valuations which are considered to be the important ones in the absolute sense. Details will be given in Section 2.2.3.

Definition 16 Let \mathcal{V} be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ a choice function from \mathbf{V} to \mathbf{W} .

We say that μ is *strongly coherent* (SC) iff $\forall V, W \in \mathbf{V}$,

$$\mu(W) \cap V \subseteq \mu(V).$$

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ is a semantic structure and $\mathcal{V} \in \mathbf{V}$.

We say that μ is *universe-coddefinable* (UC) iff

$$\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D}.$$

Note that a second advantage of Universe-coddefinability (in addition to the connexion with pivots) is that it provides a link with X -logics [FRS01]. We will see it in Chapter 6.

2.2.2 Preference structures

Binary preference relations on valuations have been investigated by e.g. B. Hansson to give semantics for deontic logics [Han69]. Y. Shoham rediscovered them to give semantics for plausible non-monotonic logics [Sho88, Sho87]. Then, it seems that Imielinski is one of the first persons to introduce binary preference relations on states labelled by valuations [Imi87]. They have been used to give more general semantics for plausible non-monotonic logics, see e.g. [KLM90, LM92, Sch92b, Sch96, Sch00, Sch04]. Let's present them.

Definition 17 We say that \mathcal{R} is a preference structure on a set \mathcal{V} iff $\mathcal{R} = \langle \mathcal{S}, l, \prec \rangle$ where \mathcal{S} is a set, l is a function from \mathcal{S} to \mathcal{V} , and \prec is a relation on $\mathcal{S} \times \mathcal{S}$.

In fact, preference structures are essentially Kripke structures. The difference lies in the interpretation of \prec . In a Kripke structure, it is seen as an accessibility relation, whilst, in a preference structure, it is seen as a preference relation.

We recall a possible meaning for preference structures. Intuitively, \mathcal{V} is a set of valuations for some language \mathcal{L} and \mathcal{S} a set of valuations for some language \mathcal{L}' richer than \mathcal{L} . The elements of \mathcal{S} are called states. $l(s)$ corresponds precisely to this part of s that is about the formulas of \mathcal{L} only. We call l a labelling function. Finally, \prec is a preference relation, i.e. $s \prec s'$ means s is preferred to s' .

We turn to well-known properties for preference structures.

Definition 18 Suppose \mathcal{V} is a set, $\mathcal{R} = \langle \mathcal{S}, l, \prec \rangle$ is a preference structure on \mathcal{V} , $T \subseteq \mathcal{S}$, $s \in T$, $V \subseteq \mathcal{V}$, and $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$.

We say that \mathcal{R} is *transitive* (resp. *irreflexive*) iff \prec is transitive (resp. irreflexive).

We say that s is *preferred* in T iff $\forall s' \in T$, $s' \not\prec s$.

$L(V) := \{s \in \mathcal{S} : l(s) \in V\}$.

We say that \mathcal{R} is **V-smooth** (alias **V-stoppered**) iff $\forall V \in \mathbf{V}$, $\forall s \in L(V)$, either s is preferred in $L(V)$ or there exists s' preferred in $L(V)$ such that $s' \prec s$.

Intuitively, $L(V)$ is the set of all those states that are labelled by a valuation of V . A preference structure defines naturally a choice function. The idea is to choose in any set of valuations V , each element which labels a state which is preferred among all the states labelled by an element of V . More formally:

Definition 19 Suppose $\mathcal{R} = \langle \mathcal{S}, l, \prec \rangle$ is a preference structure on a set \mathcal{V} .

We denote by $\mu_{\mathcal{R}}$ the function from $\mathcal{P}(\mathcal{V})$ to $\mathcal{P}(\mathcal{V})$ such that $\forall V \subseteq \mathcal{V}$,

$$\mu_{\mathcal{R}}(V) = \{v \in V : \exists s \in L(v), s \text{ is preferred in } L(V)\}.$$

In [Sch00], Schlechta showed that Coherence and Local Monotonicity characterize those choice functions that can be defined by a preference structure. Details are given in the proposition just below. It is an immediate corollary of Proposition 2.4, Proposition 2.15, and Fact 1.3 of [Sch00].

Proposition 20 [Sch00] Let \mathcal{V} be a set, \mathbf{V} and \mathbf{W} subsets of $\mathcal{P}(\mathcal{V})$, and μ a choice function from \mathbf{V} to \mathbf{W} . Then,

(0) μ is coherent iff there exists a transitive and irreflexive preference structure \mathcal{R} on \mathcal{V} such that $\forall V \in \mathbf{V}$, we have $\mu(V) = \mu_{\mathcal{R}}(V)$.

Suppose $\forall V, W \in \mathbf{V}$, we have $V \cup W \in \mathbf{V}$ and $V \cap W \in \mathbf{V}$. Then,

(1) μ is coherent and LM iff there exists a **V-smooth**, transitive, and irreflexive preference structure \mathcal{R} on \mathcal{V} such that $\forall V \in \mathbf{V}$, we have $\mu(V) = \mu_{\mathcal{R}}(V)$.

In fact, in [Sch00], the codomain of μ is required to be its domain: \mathbf{V} . However, this plays no role in the proofs. Therefore, verbatim the same proofs are valid when the codomain of μ is an arbitrary subset \mathbf{W} of $\mathcal{P}(\mathcal{V})$. Both myself and Schlechta checked it.

2.2.3 Pivots

Suppose some valuations are considered to be the important ones in the absolute sense. Then, collect them in a set \mathcal{I} , called a pivot. \mathcal{I} defines naturally a choice function $\mu_{\mathcal{I}}$ which chooses in any set of valuations, simply those elements which belong to \mathcal{I} . More formally,

Definition 21 Let \mathcal{V} be a set.

We say that \mathcal{I} is a pivot on \mathcal{V} iff $\mathcal{I} \subseteq \mathcal{V}$.

Let \mathcal{I} be a pivot on \mathcal{V} .

We denote by $\mu_{\mathcal{I}}$ the function from $\mathcal{P}(\mathcal{V})$ to $\mathcal{P}(\mathcal{V})$ such that $\forall V \subseteq \mathcal{V}$,

$$\mu_{\mathcal{I}}(V) = V \cap \mathcal{I}.$$

Pivots have been investigated extensively by D. Makinson in [Mak03, Mak05]. In the present section, we show that the properties of Strong Coherence, Definability Preservation, and Universe-coddefinability characterize those choice functions that can be defined by a pivot. More precisely:

Proposition 22 Let \mathcal{V} be a set, $\mathbf{V}, \mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ a choice function from \mathbf{V} to \mathbf{W} . Then:

(0) μ is SC iff there exists a pivot \mathcal{I} on \mathcal{V} such that $\forall V \in \mathbf{V}$, $\mu(V) = \mu_{\mathcal{I}}(V)$.

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ is a semantic structure and $\mathcal{V} \in \mathbf{V}$. Then:

(1) μ is SC and DP iff there exists a pivot \mathcal{I} on \mathcal{V} such that $\mathcal{I} \in \mathbf{D}$ and $\forall V \in \mathbf{V}$, $\mu(V) = \mu_{\mathcal{I}}(V)$;

(2) μ is SC and UC iff there exists a pivot \mathcal{I} on \mathcal{V} such that $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$ and $\forall V \in \mathbf{V}$, $\mu(V) = \mu_{\mathcal{I}}(V)$.

Proof *Proof of (0).* Direction: “ \rightarrow ”.

Let $\mathcal{I} = \{v \in \mathcal{V} : \exists V \in \mathbf{V}, v \in \mu(V)\}$ and suppose $V \in \mathbf{V}$.

If $v \in \mu(V)$, then $v \in V$ and, by definition of \mathcal{I} , $v \in \mathcal{I}$. Consequently, $\mu(V) \subseteq V \cap \mathcal{I}$.

If $v \in V \cap \mathcal{I}$, then $\exists W \in \mathbf{V}$, $v \in \mu(W)$, thus, by SC, $v \in \mu(W) \cap V \subseteq \mu(V)$.

Consequently, $V \cap \mathcal{I} \subseteq \mu(V)$.

Direction: “ \leftarrow ”.

There exists $\mathcal{I} \subseteq \mathcal{V}$ such that $\forall V \in \mathbf{V}$, $\mu(V) = V \cap \mathcal{I}$.

We show that μ satisfies SC.

Let $V, W \in \mathbf{V}$. Then, $\mu(W) \cap V = W \cap \mathcal{I} \cap V \subseteq \mathcal{I} \cap V = \mu(V)$.

Proof of (1). Direction: “ \rightarrow ”.

Take the same \mathcal{I} as for (0). Then, by verbatim the same proof, $\forall V \in \mathbf{V}$, $\mu(V) = V \cap \mathcal{I}$.

It remains to show that $\mathcal{I} \in \mathbf{D}$.

As $M_{\emptyset} = \mathcal{V}$, $\mathcal{V} \in \mathbf{D}$. Thus, as μ is DP, $\mu(\mathcal{V}) \in \mathbf{D}$. But, $\mu(\mathcal{V}) = \mathcal{V} \cap \mathcal{I} = \mathcal{I}$.

Direction: “ \leftarrow ”.

Verbatim the proof of (0), except that in addition we have $\mathcal{I} \in \mathbf{D}$.

We show that μ is DP. Let $V \in \mathbf{V} \cap \mathbf{D}$.

Then, $\exists \Gamma \subseteq \mathcal{F}$, $M_{\Gamma} = V$. Similarly, as $\mathcal{I} \in \mathbf{D}$, $\exists \Delta \subseteq \mathcal{F}$, $M_{\Delta} = \mathcal{I}$.

Therefore, $\mu(V) = V \cap \mathcal{I} = M_{\Gamma} \cap M_{\Delta} = M_{\Gamma \cup \Delta} \in \mathbf{D}$.

Proof of (2). Direction: “ \rightarrow ”.

Take the same \mathcal{I} as for (0). Then, by verbatim the same proof, $\forall V \in \mathbf{V}, \mu(V) = V \cap \mathcal{I}$. It remains to show $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$. As μ is UC, $\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D}$. But, $\mathcal{V} \setminus \mu(\mathcal{V}) = \mathcal{V} \setminus (\mathcal{V} \cap \mathcal{I}) = \mathcal{V} \setminus \mathcal{I}$.

Direction: “ \leftarrow ”.

Verbatim the proof of (0), except that in addition we have $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$.

We show that μ is UC: $\mathcal{V} \setminus \mu(\mathcal{V}) = \mathcal{V} \setminus (\mathcal{V} \cap \mathcal{I}) = \mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$. ■

2.3 Preferential(-discriminative) consequence relations

2.3.1 Definitions

Suppose we are given a semantic structure and a choice function μ on the valuations. Then, it is natural to conclude a formula α from a set of formulas Γ iff every model for Γ chosen by μ is a model for α . More formally:

Definition 23 Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ is a semantic structure and \vdash a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

We say that \vdash is a *preferential consequence relation* iff there exists a coherent choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}$,

$$\Gamma \vdash \alpha \text{ iff } \mu(M_\Gamma) \subseteq M_\alpha.$$

In addition, if μ is LM, DP, etc., then so is \vdash .

These consequence relations are called “preferential” because, in the light of Proposition 20, they can be defined equivalently with preference structures, instead of coherent choice functions. We opted for choice functions for two reasons.

First, they give a clearer meaning. Indeed, properties like Coherence have simple intuitive justifications, whilst preference structures contain “states”. But, it is not perfectly clear what a state is in daily life. By the way, in [KLM90], Kraus, Lehmann, and Magidor did not consider preference structures to be ontological justifications for their interest in the formal systems investigated, but to be technical tools to study those systems and in particular settle questions of interderivability and find efficient decision procedures (see the end of Section 1.2 of [KLM90]).

Second, thanks to choice functions and their properties, we can present in a unified way preferential (here) and pivotal (in Section 2.4) consequence relations, though they are initially based on different tools (i.e. preference structures or pivots). This will enable us to use similar techniques of proof for both kinds of relations.

Preferential relations lead to plausible conclusions which will eventually be withdrawn later, in the presence of additional information. Therefore, they are useful to deal with incomplete information. We will give an example with a classical semantic structure in Section 2.3.3.

In addition, if a many-valued semantic structure is considered, they lead to rational and non-trivial conclusions in spite of the presence of contradictions and are thus useful to treat both incomplete and inconsistent information. However, they will not satisfy the Disjunctive Syllogism (from α and $\neg\alpha \vee \beta$ we can conclude β). We will give an example with the *FOUR* semantic structure in Section 2.3.4.

Now, we turn to a qualified version of preferential consequence. It captures the idea that the contradictions in the conclusions should be rejected.

Definition 24 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and \vdash a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

We say that \vdash is a *preferential-discriminative consequence relation* iff there is a coherent choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}$,

$$\Gamma \vdash \alpha \text{ iff } \mu(M_\Gamma) \subseteq M_\alpha \text{ and } \mu(M_\Gamma) \not\subseteq M_{\neg\alpha}.$$

In addition, if μ is LM, DP, etc., then so is \vdash .

If a classical semantic structure is considered, the discriminative version does not bring something really new. Indeed, the only difference will be to conclude nothing instead of everything in the face of inconsistent information. On the other hand, with a many-valued structure, the conclusions are rational even from inconsistent information. The discriminative version will then reject the contradictions in the conclusions, rendering the latter all the more rational.

In Definitions 23 and 24, the domain of the choice function is \mathbf{D} . This is natural as only the elements of \mathbf{D} play a role in the definition of a preferential(-discriminative) consequence relation. This point of view has been adopted in e.g. [Leh01] (see Section 6). Now, one might want a definition with choice functions of which the domain is $\mathcal{P}(\mathcal{V})$. In fact, some families of relations can be defined equivalently with \mathbf{D} or $\mathcal{P}(\mathcal{V})$. For instance, as is noted in [Leh01], if μ is a coherent choice function from \mathbf{D} to $\mathcal{P}(\mathcal{V})$, then the function μ' from $\mathcal{P}(\mathcal{V})$ to $\mathcal{P}(\mathcal{V})$ defined by $\mu'(V) = V \cap \mu(M_T(V))$ is a coherent choice function which agrees with μ on \mathbf{D} .

Several characterizations for preferential consequence relations can be found in the literature (e.g. [KLM90, LM92, Leh02, Leh01, Sch92b, Sch96, Sch00, Sch04]). In particular, we will recall (in Section 2.3.2) a characterization that involves the well-known system \mathbf{P} of [KLM90].

2.3.2 The system \mathbf{P}

Gabbay, Makinson, Kraus, Lehmann, and Magidor investigated extensively properties for plausible non-monotonic consequence relations [Gab85, Mak89, Mak94, KLM90, LM92]. A certain set of properties, called the system \mathbf{P} , plays a central role in this area. It is essentially due to Kraus, Lehmann, and Magidor [KLM90] and has been investigated further in [LM92]. Let's present it.

Definition 25 Suppose \mathcal{L} is a language containing the usual connectives \neg and \vee , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and \vdash a relation on $\mathcal{F} \times \mathcal{F}$.

Then, the system \mathbf{P} is the set of the six following conditions: $\forall \alpha, \beta, \gamma \in \mathcal{F}$,

Reflexivity $\alpha \vdash \alpha$

Left Logical Equivalence $\frac{\vdash \alpha \leftrightarrow \beta \quad \alpha \vdash \gamma}{\beta \vdash \gamma}$

Right Weakening $\frac{\vdash \alpha \rightarrow \beta \quad \gamma \vdash \alpha}{\gamma \vdash \beta}$

Cut $\frac{\alpha \wedge \beta \vdash \gamma \quad \alpha \vdash \beta}{\alpha \vdash \gamma}$

Cautious Monotonicity $\frac{\alpha \vdash \beta \quad \alpha \vdash \gamma}{\alpha \wedge \beta \vdash \gamma}$

$$\text{Or } \frac{\alpha \sim \gamma \quad \beta \sim \gamma}{\alpha \vee \beta \sim \gamma}$$

Note that $\alpha \wedge \beta$ is a shorthand for $\neg(\neg\alpha \vee \neg\beta)$. Similarly, $\alpha \rightarrow \beta$ and $\alpha \leftrightarrow \beta$ are shorthands. Note again that \mathbf{P} without **Or** is called \mathbf{C} . The system \mathbf{C} is closely related to the cumulative inference which was investigated by Makinson in [Mak89]. In addition, it seems to correspond to what Gabbay proposed in [Gab85]. Concerning the rule **Or**, it corresponds to the axiom CA of conditional logic.

All the properties in \mathbf{P} are sound if we read $\alpha \sim \beta$ as “ β is a plausible consequence of α ”. In addition, \mathbf{P} is complete in the sense that it characterizes those consequence relations that can be defined by a smooth transitive irreflexive preference structure. This is what makes \mathbf{P} central. More formally:

Definition 26 Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ is a semantic structure.

Then, $\mathbf{D}_f := \{V \subseteq \mathcal{V} : \exists \alpha \in \mathcal{F}, V = M_\alpha\}$.

Suppose \mathcal{L} is a language containing the usual connectives \neg and \vee , and \mathcal{F} the set of all wffs of \mathcal{L} .

Then define the following condition: $\forall v \in \mathcal{V}, \forall \alpha, \beta \in \mathcal{F}, \forall \Gamma \subseteq \mathcal{F}$,

(KLM0) $v \models \neg\alpha$ iff $v \not\models \alpha$;

(KLM1) $v \models \alpha \vee \beta$ iff $v \models \alpha$ or $v \models \beta$.

(KLM2) if for every finite subset Δ of Γ , $M_\Delta \neq \emptyset$, then $M_\Gamma \neq \emptyset$.

Note that (KLM2) is called “assumption of compactness” in [KLM90].

Proposition 27 [KLM90] Suppose \mathcal{L} is a language containing the usual connectives \neg and \vee , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (KLM0)–(KLM2), and \sim a relation on $\mathcal{F} \times \mathcal{F}$.

Then, \sim satisfies all the properties of \mathbf{P} iff there exists a \mathbf{D}_f -smooth transitive irreflexive preference structure \mathcal{R} on \mathcal{V} such that $\forall \alpha, \beta \in \mathcal{F}, \alpha \sim \beta$ iff $\mu_{\mathcal{R}}(M_\alpha) \subseteq M_\beta$.

Note that \sim is a relation on $\mathcal{F} \times \mathcal{F}$, not $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. This difference is crucial. Indeed, if we adapt the conditions of \mathbf{P} in the obvious way to relations on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$ and if we replace \mathbf{D}_f by \mathbf{D} in Proposition 27, then the latter does no longer hold. This negative result was shown by Schlechta in [Sch92b].

Now, by Propositions 20 and 27, we immediately get the following representation theorem:

Proposition 28 Suppose Definition 23 (of preferential consequence relations) is adapted in the obvious way to relations on $\mathcal{F} \times \mathcal{F}$ (essentially, replace \mathbf{D} by \mathbf{D}_f). Suppose also \mathcal{L} is a language containing the usual connectives \neg and \vee , \mathcal{F} the set of all wffs of \mathcal{L} , \sim a relation on $\mathcal{F} \times \mathcal{F}$, and $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure such that (KLM0)–(KLM2) hold and $\forall V, W \in \mathbf{D}_f, V \cup W \in \mathbf{D}_f$ and $V \cap W \in \mathbf{D}_f$.

Then, LM preferential consequence relations are precisely those relations that satisfy the system \mathbf{P} .

Recall that LM choice functions have been defined in Definition 14.

2.3.3 Example in a classical framework

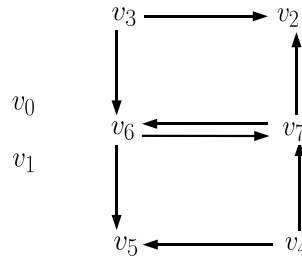
Let \mathcal{L} be a classical propositional language of which the atoms are r , q , and p . They are given the following meanings: r stands for Nixon is a republican, q stands for Nixon is a quaker, and p stands for Nixon is a pacifist. Let \mathcal{F} be the set of all wffs of \mathcal{L} , \mathcal{V} the set of all classical two-valued valuations of \mathcal{L} , and \models the classical satisfaction relation for these objects. Then, \mathcal{V} is the set of the 8 following valuations: $v_0, v_1, v_2, v_3, v_4, v_5, v_6$, and v_7 , which are defined in the obvious way by the following table:

	r	q	p
v_0	0	0	0
v_1	0	0	1
v_2	0	1	0
v_3	0	1	1
v_4	1	0	0
v_5	1	0	1
v_6	1	1	0
v_7	1	1	1

Now, consider the class of all republicans and the class of all quakers. Consider that a republican is normal iff he is not a pacifist and that a quaker is normal iff he is a pacifist. And, consider that a valuation v is more normal than a valuation w from the point of view of a class C iff

- Nixon is an individual of C in both v and w ;
- Nixon is normal in v ;
- Nixon is not normal in w .

In the following graph, there is an arrow from a valuation v to a valuation w iff v is more normal than w from the point of view of some class:



Given those considerations a natural preference structure on \mathcal{V} is $\mathcal{R} = \langle \mathcal{V}, l, \prec \rangle$, where l is identity and \prec is the relation such that $\forall v, w \in \mathcal{V}$, we have $v \prec w$ iff (1) or (2) below holds (i.e. there is an arrow from v to w):

- (1) $v \models r$ and $v \models \neg p$ and $w \models r$ and $w \not\models \neg p$;
- (2) $v \models q$ and $v \models p$ and $w \models q$ and $w \not\models p$.

Finally, let \sim be the preferential consequence relation defined by the coherent choice function $\mu_{\mathcal{R}}$.

Then, \vdash leads us to “jump” to plausible conclusions from incomplete information and to revise previous “hasty” conclusions in the face of new and fuller information. For instance, $r \vdash \neg p$ and $\{r, p\} \not\vdash \neg p$ and $q \vdash p$ and $\{q, \neg p\} \not\vdash p$.

However, \vdash is not paraconsistent. In addition, some sets of formulas are rendered useless, because there is no preferred model for them, though there are models for them. For instance, $\{q, r\} \vdash \alpha, \forall \alpha \in \mathcal{F}$.

2.3.4 Example in the *FOUR* framework

Consider the *FOUR* semantic structure $\langle \mathcal{F}_c, \mathcal{V}_4, \models_4 \rangle$ and suppose $\mathcal{A} = \{r, q, p\}$ (these objects have been defined in Section 2.1.2). In addition, make the same considerations about Nixon, the classes, normality, etc., as in Section 2.3.3, except that this time a valuation v is considered to be more normal than a valuation w from the point of view of a class C iff

- in both v and w , the processor is informed that Nixon is an individual of C ;
- in v , he is informed that Nixon is normal and not informed of the contrary;
- in w , he is not informed that Nixon is normal.

See Section 2.1.2 for recalls about the sources-processor systems. Given those considerations a natural preference structure on \mathcal{V}_4 is $\mathcal{R} = \langle \mathcal{V}_4, l, \prec \rangle$, where l is identity and \prec is the relation such that $\forall v, w \in \mathcal{V}_4$, we have $v \prec w$ iff (1) or (2) below holds (i.e. v is more normal than w from the point of view of some class):

- (1) $v \models_4 r$ and $v \models_4 \neg p$ and $v \not\models_4 p$ and $w \models_4 r$ and $w \not\models_4 \neg p$;
- (2) $v \models_4 q$ and $v \models_4 p$ and $v \not\models_4 \neg p$ and $w \models_4 q$ and $w \not\models_4 p$.

Let \vdash be the preferential consequence relation defined by the coherent choice function $\mu_{\mathcal{R}}$.

Then, again we “jump” to plausible conclusions and revise previous “hasty” conclusions. For instance, $r \vdash \neg p$ and $\{r, p\} \not\vdash \neg p$ and $q \vdash p$ and $\{q, \neg p\} \not\vdash p$.

In addition, \vdash is paraconsistent. For instance, $\{p, \neg p, q\} \vdash p$ and $\{p, \neg p, q\} \vdash \neg p$ and $\{p, \neg p, q\} \vdash q$ and $\{p, \neg p, q\} \not\vdash \neg q$. And, it happens less often that a set of formulas is rendered useless because there is no preferred model for it, though there are models for it. For instance, this time, $\{q, r\} \vdash p$ and $\{q, r\} \vdash \neg p$ and $\{q, r\} \vdash q$ and $\{q, r\} \not\vdash \neg q$ and $\{q, r\} \vdash r$ and $\{q, r\} \not\vdash \neg r$.

However, \vdash does not satisfy the Disjunctive Syllogism. Indeed, for instance, $\{\neg r, r \vee q\} \not\vdash q$.

2.4 Pivotal(-discriminative) consequence relations

2.4.1 Definitions

We turn to pivotal consequence relations. They model plausible ways to draw conclusions from incomplete information. Unlike preferential consequence relations, they are monotonic.

Definition 29 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

We say that \sim is a *pivotal consequence relation* iff there exists a SC choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}$,

$$\Gamma \sim \alpha \text{ iff } \mu(M_\Gamma) \subseteq M_\alpha$$

In addition, if μ is UC, CP, etc., then so is \sim .

We called these relations ‘‘pivotal’’ because, in the light of Proposition 22, they can be defined equivalently with pivots, instead of SC choice functions. Their importance has been put in evidence by D. Makinson in e.g. [Mak03, Mak05], where he showed that they constitute easy conceptual passage from basic to plausible non-monotonic consequence relations. Indeed, they are perfectly monotonic but already display some of the distinctive features (i.e. the choice functions) of plausible non-monotonic relations. Note that pivotal (resp. DP pivotal) consequence relations correspond to Makinson’s pivotal-valuation (resp. pivotal-assumption) relations. We will give an example of how they can be used to draw plausible conclusions from incomplete information in Section 2.4.2.

Moreover, if a many-valued semantic structure is considered, they lead to rational and non-trivial conclusions in spite of the presence of contradictions and are thus useful to treat both incomplete and inconsistent information. However, they will not satisfy the Disjunctive Syllogism. We will give an example with the *FOUR* semantic structure in Section 2.4.3.

Characterizations of pivotal consequence relations, valid in classical frameworks, can be found in the literature. For instance, the following result appears to be part of folklore for decades: the DP pivotal consequence relations correspond precisely to those supraclassical closure operations that are compact and satisfy Disjunction in the premisses. For more details see e.g. [Rot01, Mak03, Mak05].

Now, we turn to a qualified version of pivotal consequence. It captures the idea that the contradictions in the conclusions should be rejected.

Definition 30 Let \mathcal{L} be a language, \neg a unary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

We say that \sim is a *pivotal-discriminative consequence relation* iff there exists a SC choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}$,

$$\Gamma \sim \alpha \text{ iff } \mu(M_\Gamma) \subseteq M_\alpha \text{ and } \mu(M_\Gamma) \not\subseteq M_{\neg\alpha}$$

In addition, if μ is UC, CP, etc., then so is \sim .

As in the preferential case, this discriminative version brings something new in many-valued semantic structures for it removes, in non-trivial cases, the contradictions in the conclusions.

2.4.2 Example in a classical framework

Let \mathcal{L} be a classical propositional language of which the atoms are r , q , and p . Let \mathcal{F} be the set of all wffs of \mathcal{L} , \mathcal{V} the set of all classical two-valued valuations of \mathcal{L} , and \models the classical satisfaction relation for these objects. Now, make the same considerations about Nixon, the classes, normality, etc., as previously. In addition, consider that a valuation is negligible iff (in it) Nixon is a non-normal individual of some class. Then, collect the non-negligible valuations in a pivot \mathcal{I} . More formally:

$$\mathcal{I} = \{v \in \mathcal{V} : \text{if } v \models r, \text{ then } v \models \neg p; \text{ and if } v \models q, \text{ then } v \models p\}.$$

Finally, let \vdash be the pivotal consequence relation defined by the SC choice function $\mu_{\mathcal{I}}$.

Then, \vdash leads to “jump” to plausible conclusions from incomplete information. For instance, $r \vdash \neg p$ and $q \vdash p$. But, we fall into triviality if we face new information that contradict previous “hasty” conclusions. For instance, $\{r, p\} \vdash \alpha, \forall \alpha \in \mathcal{L}$, and $\{q, \neg p\} \vdash \alpha, \forall \alpha \in \mathcal{L}$. This is, in some sense, the price to pay for being monotonic, whereas plausible conclusions are accepted.

In addition, \vdash is not paraconsistent and some sets of formulas are rendered useless because there is no model in the pivot for them, though there are models for them. For instance, $\{q, r\} \vdash \alpha, \forall \alpha \in \mathcal{L}$.

2.4.3 Example in the *FOUR* framework

Consider the *FOUR* semantic structure $\langle \mathcal{F}_c, \mathcal{V}_4, \models_4 \rangle$ and suppose $\mathcal{A} = \{r, q, p\}$. In addition, make the same considerations about Nixon, the classes, normality, etc. as previously, except that this time a valuation is considered to be negligible iff (in it) the processor is informed that Nixon is an individual of some class, but he is not informed that Nixon is a normal individual of that class. Again, collect the non-negligible valuations in a pivot \mathcal{I} . More formally:

$$\mathcal{I} = \{v \in \mathcal{V}_4 : \text{if } v \models_4 r, \text{ then } v \models_4 \neg p; \text{ and if } v \models_4 q, \text{ then } v \models_4 p\}.$$

Let \vdash be the pivotal consequence relation defined by the SC choice function $\mu_{\mathcal{I}}$.

Then, again \vdash leads to “jump” to plausible conclusions from incomplete information. For instance, $r \vdash \neg p$ and $q \vdash p$. Moreover, though “hasty” conclusions are never withdrawn, we do not fall into triviality when we face new information that contradict them. For instance, $\{r, p\} \vdash p$ and $\{r, p\} \vdash \neg p$ and $\{r, p\} \vdash r$ and $\{r, p\} \not\vdash \neg r$.

In addition, \vdash is paraconsistent. For instance, $\{p, \neg p, q\} \vdash p$ and $\{p, \neg p, q\} \vdash \neg p$ and $\{p, \neg p, q\} \vdash q$ and $\{p, \neg p, q\} \not\vdash \neg q$. And, less sets of formulas are rendered useless because there is no model in the pivot for them, though there are models for them. For instance, this time, $\{q, r\} \vdash p$ and $\{q, r\} \vdash \neg p$ and $\{q, r\} \vdash q$ and $\{q, r\} \not\vdash \neg q$ and $\{q, r\} \vdash r$ and $\{q, r\} \not\vdash \neg r$.

However, \vdash does not satisfy the Disjunctive Syllogism. Indeed, for instance, $\{\neg r, r \vee q\} \not\vdash q$.

Chapter 3

Characterizations of preferential consequence relations

In the present chapter, we will characterize families of preferential and preferential-discriminative consequence relations. These results have been published in [BN05b]. Sometimes, we will need to make some assumptions (defined in Section 2.1.1) about the semantic structure under consideration. However, no assumption will be needed for the three following families:

- the preferential consequence relations (Section 3.2);
- the DP preferential consequence relations (Section 3.1);
- the DP LM preferential consequence relations (Section 3.1).

We will assume (A1) and (A3) for:

- the CP preferential-discriminative consequence relations (Section 3.4);
- the CP DP preferential-discriminative consequence relations (Section 3.3);
- the CP DP LM preferential-discriminative consequence relations (Section 3.3).

And, we will need (A1), (A2), and (A3) for:

- the preferential-discriminative consequence relations (Section 3.4);
- the DP preferential-discriminative consequence relations (Section 3.3);
- the DP LM preferential-discriminative consequence relations (Section 3.3).

3.1 The non-discriminative and DP case

The characterizations in this section have already been given in Proposition 3.1 of [Sch00], under the assumption that a classical propositional semantic structure is considered. Using the same techniques as those of Schlechta, we show that his characterizations hold with any semantic structure.

Notation 31 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then, consider the following conditions: $\forall \Gamma, \Delta \subseteq \mathcal{F}$,

- (\sim 0) if $\vdash(\Gamma) = \vdash(\Delta)$, then $\sim(\Gamma) = \sim(\Delta)$;
- (\sim 1) $\vdash(\sim(\Gamma)) = \sim(\Gamma)$;
- (\sim 2) $\Gamma \subseteq \sim(\Gamma)$;
- (\sim 3) $\sim(\Gamma, \Delta) \subseteq \vdash(\sim(\Gamma), \Delta)$;
- (\sim 4) if $\Gamma \subseteq \vdash(\Delta) \subseteq \sim(\Gamma)$, then $\sim(\Gamma) \subseteq \sim(\Delta)$.

Note that those conditions are purely syntactic when there is a proof system available for \vdash (which is the case with e.g. the classical, *FOUR*, and J_3 semantic structures).

Proposition 32 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then,

- (0) \sim is a DP preferential consequence relation iff (\sim 0), (\sim 1), (\sim 2), and (\sim 3) hold;
- (1) \sim is a DP LM preferential consequence relation iff (\sim 0), (\sim 1), (\sim 2), (\sim 3), and (\sim 4) hold.

Proof *Proof of (0).* Direction: “ \rightarrow ”.

By hypothesis, there exists a DP coherent choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\sim(\Gamma) = T(\mu(M_\Gamma))$. We will show:

- (0.0) \sim satisfies (\sim 0);
- (0.1) \sim satisfies (\sim 1);
- (0.2) \sim satisfies (\sim 2);
- (0.3) $\forall \Gamma \subseteq \mathcal{F}$, we have $\mu(M_\Gamma) = M_{\sim(\Gamma)}$;
- (0.4) \sim satisfies (\sim 3).

Direction: “ \leftarrow ”.

Suppose \sim satisfies (\sim 0), (\sim 1), (\sim 2), and (\sim 3).

Let μ be the function from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_{\sim(\Gamma)}$.

Then, μ is well-defined.

Indeed, if $\Gamma, \Delta \subseteq \mathcal{F}$ and $M_\Gamma = M_\Delta$, then $\vdash(\Gamma) = \vdash(\Delta)$, thus, by (\sim 0), $\sim(\Gamma) = \sim(\Delta)$.

In addition, μ is obviously DP. We show the following which ends the proof:

- (0.5) μ is a choice function;
- (0.6) μ is coherent;
- (0.7) $\forall \Gamma \subseteq \mathcal{F}$, we have $\sim(\Gamma) = T(\mu(M_\Gamma))$.

Proof of (0.0). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $\vdash(\Gamma) = \vdash(\Delta)$.

Then, $M_\Gamma = M_\Delta$. Thus, $\sim(\Gamma) = T(\mu(M_\Gamma)) = T(\mu(M_\Delta)) = \sim(\Delta)$.

Proof of (0.1). Let $\Gamma \subseteq \mathcal{F}$. Then, $\vdash(\sim(\Gamma)) = \vdash(T(\mu(M_\Gamma))) = T(M_{T(\mu(M_\Gamma))}) = \sim(\Gamma)$.

Proof of (0.2). Let $\Gamma \subseteq \mathcal{F}$. Then, $\Gamma \subseteq T(M_\Gamma) \subseteq T(\mu(M_\Gamma)) = \sim(\Gamma)$.

Proof of (0.3). Let $\Gamma \subseteq \mathcal{F}$. As μ is DP, $\mu(M_\Gamma) \in \mathbf{D}$.

Thus, $\exists \Gamma' \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_{\Gamma'}$. Therefore, $\mu(M_\Gamma) = M_{\Gamma'} = M_{T(M_{\Gamma'})} = M_{T(\mu(M_\Gamma))} = M_{\sim(\Gamma)}$.

Proof of (0.4). Let $\Gamma, \Delta \subseteq \mathcal{F}$. As $M_{\Gamma, \Delta} \subseteq M_\Gamma$ and μ is coherent, $\mu(M_\Gamma) \cap M_{\Gamma, \Delta} \subseteq \mu(M_{\Gamma, \Delta})$. Therefore, $\sim(\Gamma, \Delta) = T(\mu(M_{\Gamma, \Delta})) \subseteq T(\mu(M_\Gamma) \cap M_{\Gamma, \Delta}) = T(\mu(M_\Gamma) \cap M_\Delta)$. Thus, by (0.0), $\sim(\Gamma, \Delta) \subseteq T(M_{\sim(\Gamma)} \cap M_\Delta) = T(M_{\sim(\Gamma), \Delta}) = \vdash(\sim(\Gamma), \Delta)$.

Proof of (0.5). Let $\Gamma \subseteq \mathcal{F}$. Then, $\mu(M_\Gamma) = M_{\sim(\Gamma)}$, which is a subset of M_Γ , by (\sim 2).

Proof of (0.6). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $M_\Gamma \subseteq M_\Delta$. Then, $\mu(M_\Delta) \cap M_\Gamma = M_{\sim(\Delta)} \cap M_\Gamma = M_{\sim(\Delta), \Gamma}$. But, by (\sim 3), $M_{\sim(\Delta), \Gamma} \subseteq M_{\sim(\Delta, \Gamma)} = \mu(M_{\Delta, \Gamma}) = \mu(M_\Gamma)$.

Proof of (0.7). Let $\Gamma \subseteq \mathcal{F}$. Then, by (\sim 1), $\sim(\Gamma) = \vdash(\sim(\Gamma)) = T(M_{\sim(\Gamma)}) = T(\mu(M_\Gamma))$.

Proof of (1). Direction: “ \rightarrow ”.

Verbatim the same proof as for (0), except that in addition we have μ is LM.

We use it to show that \sim satisfies (\sim 4).

Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $\Gamma \subseteq \vdash(\Delta) \subseteq \sim(\Gamma)$.

Then, by (0.3), $\mu(M_\Gamma) = M_{\sim(\Gamma)} \subseteq M_{\vdash(\Delta)} = M_\Delta \subseteq M_\Gamma$.

Therefore, as μ is locally monotonic, $\mu(M_\Delta) \subseteq \mu(M_\Gamma)$.

Thus, $\sim(\Gamma) = T(\mu(M_\Gamma)) \subseteq T(\mu(M_\Delta)) = \sim(\Delta)$.

Direction: “ \leftarrow ”.

Verbatim the same proof as for (0), except that in addition we have (\sim 4) is satisfied.

We use it to show that μ is locally monotonic.

Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $\mu(M_\Gamma) \subseteq M_\Delta \subseteq M_\Gamma$.

Then, $M_{\sim(\Gamma)} \subseteq M_\Delta \subseteq M_\Gamma$. Therefore, $\Gamma \subseteq T(M_\Gamma) \subseteq T(M_\Delta) = \vdash(\Delta)$.

On the other hand, $\vdash(\Delta) = T(M_\Delta) \subseteq T(M_{\sim(\Gamma)}) = \vdash(\sim(\Gamma))$ which is equal to $\sim(\Gamma)$, by (\sim 1).

Thus, by (\sim 4), we have $\sim(\Gamma) \subseteq \sim(\Delta)$. Therefore, $\mu(M_\Delta) = M_{\sim(\Delta)} \subseteq M_{\sim(\Gamma)} = \mu(M_\Gamma)$. ■

3.2 The non-discriminative and not necessarily DP case

In this section, we will characterize the family of all preferential consequence relations. Unlike in Section 3.1, our conditions will not be purely syntactic (i.e. using only \vdash , \sim , etc.). In fact, we did not succeed in translating properties like Coherence in syntactic terms because the choice functions under consideration are not necessarily definability preserving. Indeed, we do no longer have at our disposal the important equality: $\mu(M_\Gamma) = M_{\sim(\Gamma)}$, which is of great help to perform the translation and which holds precisely because of Definability Preservation.

In Proposition 5.2.11 of [Sch04], K. Schlechta provided a characterization of the family mentioned above, under the assumption that a classical propositional semantic structure is considered. Note that most of his work is done in a very general, in fact algebraic, framework. Only at the end, he applied his general lemmas in a classical framework to get the characterization. The conditions he gave, as ours, are not purely syntactic (e.g. they involve the notion of model, etc.).

However, some limits of what can be done in this area have been put in evidence by Schlechta. Approximatively, he showed in Proposition 5.2.15 of the same book that, in an infinite classical framework, there does not exist a characterization containing only conditions which are universally

quantified, of limited size, and using only simple operations (like e.g. \cup, \cap, \setminus).

The purpose of the present section is to provide a new characterization, more elegant than the one of Schlechta and that hold with any semantic structure. To do so, we have been inspired by the algebraic part of the work of Schlechta (see Proposition 5.2.5 of [Sch04]). Technically, the idea begins by building from any function f , a coherent choice function μ_f such that whenever f “covers” some coherent choice function, it necessarily covers μ_f .

Definition 33 Let \mathcal{V} be a set, \mathbf{V} and \mathbf{W} subsets of $\mathcal{P}(\mathcal{V})$, and f a function from \mathbf{V} to \mathbf{W} . We denote by μ_f the function from \mathbf{V} to $\mathcal{P}(\mathcal{V})$ such that $\forall V \in \mathbf{V}$,

$$\mu_f(V) = \{v \in V : \forall W \in \mathbf{V}, \text{ if } v \in W \subseteq V, \text{ then } v \in f(W)\}.$$

Lemma 34 Let \mathcal{V} be a set, \mathbf{V} and \mathbf{W} subsets of $\mathcal{P}(\mathcal{V})$, and f a function from \mathbf{V} to \mathbf{W} . Then, μ_f is a coherent choice function.

Proof μ_f is obviously a choice function. It remains to show that it is coherent.

Suppose $V, W \in \mathbf{V}$, $V \subseteq W$, and $v \in \mu_f(W) \cap V$. We show $v \in \mu_f(V)$.

To do so, suppose the contrary, i.e. $v \notin \mu_f(V)$.

Then, as $v \in V$, we have $\exists Z \in \mathbf{V}$, $Z \subseteq V$, $v \in Z$, and $v \notin f(Z)$.

But, $V \subseteq W$, thus $Z \subseteq W$. Therefore, by definition of μ_f , $v \notin \mu_f(W)$, which is impossible. ■

Lemma 35 Let \mathcal{V} be a set, \mathbf{V} , \mathbf{W} , and \mathbf{X} subsets of $\mathcal{P}(\mathcal{V})$, f a function from \mathbf{V} to \mathbf{W} , and μ a coherent choice function from \mathbf{V} to \mathbf{X} such that $\forall V \in \mathbf{V}$, $f(V) = M_{T(\mu(V))}$. Then, $\forall V \in \mathbf{V}$, $f(V) = M_{T(\mu_f(V))}$.

Proof Let $V \in \mathbf{V}$. We show $f(V) = M_{T(\mu_f(V))}$.

Case 1: $\exists v \in \mu(V)$, $v \notin \mu_f(V)$.

As $\mu(V) \subseteq V$, we have $v \in V$.

Thus, by definition of μ_f , $\exists W \in \mathbf{V}$, $W \subseteq V$, $v \in W$, and $v \notin f(W) = M_{T(\mu(W))} \supseteq \mu(W)$.

On the other hand, as μ is coherent, $\mu(V) \cap W \subseteq \mu(W)$. Thus, $v \in \mu(W)$, which is impossible.

Case 2: $\mu(V) \subseteq \mu_f(V)$.

Case 2.1: $\exists v \in \mu_f(V)$, $v \notin f(V)$.

Then, $\exists W \in \mathbf{V}$, $W \subseteq V$, $v \in W$, and $v \notin f(W)$. Indeed, just take V itself for the choice of W .

Therefore, $v \notin \mu_f(V)$, which is impossible.

Case 2.2: $\mu_f(V) \subseteq f(V)$.

Then, $f(V) = M_{T(\mu(V))} \subseteq M_{T(\mu_f(V))} \subseteq M_{T(f(V))} = M_{T(M_{T(\mu(V))})} = M_{T(\mu(V))} = f(V)$. ■

Now, everything is ready to show the representation result.

Notation 36 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

Then, consider the following condition: $\forall \Gamma \subseteq \mathcal{F}$,

$$(\sim 5) \quad \sim(\Gamma) = T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta \subseteq M_\Gamma, \text{ then } v \in M_{\sim(\Delta)}\}).$$

Proposition 37 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

Then, \sim is a preferential consequence relation iff (~ 5) holds.

Proof *Direction: “ \rightarrow ”.*

There exists a coherent choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\vdash(\Gamma) = T(\mu(M_\Gamma))$.

Let f be the function from \mathbf{D} to \mathbf{D} such that $\forall V \in \mathbf{D}$, we have $f(V) = M_{T(\mu(V))}$.

By Lemma 35, $\forall V \in \mathbf{D}$, we have $f(V) = M_{T(\mu_f(V))}$.

Note that $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{T(\mu(M_\Gamma))} = M_{\vdash(\Gamma)}$.

We show that $(\vdash 5)$ holds. Let $\Gamma \subseteq \mathcal{F}$.

$$\begin{aligned} \text{Then, } \vdash(\Gamma) &= T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(f(M_\Gamma)) = T(M_{T(\mu_f(M_\Gamma))}) = T(\mu_f(M_\Gamma)) = \\ &= T(\{v \in M_\Gamma : \forall W \in \mathbf{D}, \text{ if } v \in W \subseteq M_\Gamma, \text{ then } v \in f(W)\}) = \\ &= T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta \subseteq M_\Gamma, \text{ then } v \in f(M_\Delta)\}) = \\ &= T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta \subseteq M_\Gamma, \text{ then } v \in M_{\vdash(\Delta)}\}). \end{aligned}$$

Direction: “ \leftarrow ”.

Suppose \vdash satisfies $(\vdash 5)$.

Let f be the function from \mathbf{D} to \mathbf{D} such that $\forall \Gamma \subseteq \mathcal{F}$, we have $f(M_\Gamma) = M_{\vdash(\Gamma)}$.

Note that f is well-defined. Indeed, if $\Gamma, \Delta \subseteq \mathcal{F}$ and $M_\Gamma = M_\Delta$, then, by $(\vdash 5)$, $\vdash(\Gamma) = \vdash(\Delta)$.

In addition, by $(\vdash 5)$, we clearly have $\forall \Gamma \subseteq \mathcal{F}$, $\vdash(\Gamma) = T(\mu_f(M_\Gamma))$.

Finally, by Lemma 34, μ_f is a coherent choice function. \blacksquare

3.3 The discriminative and DP case

In this section, we will characterize certain families of DP preferential-discriminative consequence relations. For that, we need first some standard notations and an inductive construction presented in Definition 39:

Notation 38 \mathbb{N} denotes the natural numbers including 0: $\{0, 1, 2, \dots\}$.

\mathbb{N}^+ denotes the strictly positive natural numbers: $\{1, 2, \dots\}$.

\mathbb{Z} denotes the integers.

Let $i, j \in \mathbb{Z}$. Then, $[i, j]$ denotes the set of all $k \in \mathbb{Z}$ (not in \mathbb{R}) such that $i \leq k \leq j$.

Let \mathcal{L} be a language, \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , and $\beta_1, \beta_2, \dots, \beta_r \in \mathcal{F}$.

Whenever we write $\beta_1 \vee \beta_2 \vee \dots \vee \beta_r$, we mean $(\dots((\beta_1 \vee \beta_2) \vee \beta_3) \vee \dots \vee \beta_{r-1}) \vee \beta_r$.

Definition 39 Let \mathcal{L} be a language, \neg a unary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, \vdash a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$, and $\Gamma \subseteq \mathcal{F}$. Then,

$$H_1(\Gamma) := \{\neg\beta \in \mathcal{F} : \beta \in \vdash(\Gamma, \vdash(\Gamma)) \setminus \vdash(\Gamma) \text{ and } \neg\beta \notin \vdash(\Gamma, \vdash(\Gamma))\}.$$

Let $i \in \mathbb{N}$ with $i \geq 2$. Then,

$$H_i(\Gamma) := \{\neg\beta \in \mathcal{F} : \left\{ \begin{array}{l} \beta \in \vdash(\Gamma, \vdash(\Gamma), H_1(\Gamma), \dots, H_{i-1}(\Gamma)) \setminus \vdash(\Gamma) \text{ and} \\ \neg\beta \notin \vdash(\Gamma, \vdash(\Gamma), H_1(\Gamma), \dots, H_{i-1}(\Gamma)) \end{array} \right\}.$$

$$H(\Gamma) := \bigcup_{i \in \mathbb{N}^+} H_i(\Gamma).$$

Definition 40 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and \vdash a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

Then, consider the following conditions: $\forall \Gamma, \Delta \subseteq \mathcal{F}, \forall \alpha, \beta \in \mathcal{F}$,

- (\sim 6) if $\beta \in \vdash(\Gamma, \sim(\Gamma)) \setminus \sim(\Gamma)$ and $\neg\alpha \in \vdash(\Gamma, \sim(\Gamma), \neg\beta)$, then $\alpha \notin \sim(\Gamma)$;
- (\sim 7) if $\alpha \in \vdash(\Gamma, \sim(\Gamma)) \setminus \sim(\Gamma)$ and $\beta \in \vdash(\Gamma, \sim(\Gamma), \neg\alpha) \setminus \sim(\Gamma)$, then $\alpha \vee \beta \notin \sim(\Gamma)$;
- (\sim 8) if $\alpha \in \sim(\Gamma)$, then $\neg\alpha \notin \vdash(\Gamma, \sim(\Gamma))$;
- (\sim 9) if $\Delta \subseteq \vdash(\Gamma)$, then $\sim(\Gamma) \cup H(\Gamma) \subseteq \vdash(\Delta, \sim(\Delta), H(\Delta), \Gamma)$;
- (\sim 10) if $\Gamma \subseteq \vdash(\Delta) \subseteq \vdash(\Gamma, \sim(\Gamma), H(\Gamma))$, then $\sim(\Gamma) \cup H(\Gamma) \subseteq \vdash(\Delta, \sim(\Delta), H(\Delta))$;
- (\sim 11) if Γ is consistent, then $\sim(\Gamma)$ is consistent, $\Gamma \subseteq \sim(\Gamma)$, and $\vdash(\sim(\Gamma)) = \sim(\Gamma)$.

Note that those conditions are purely syntactic when there is a proof system available for \vdash .

Proposition 41 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee and \wedge binary connectives of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A1) and (A3), and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then,

- (0) \sim is a CP DP preferential-discriminative consequence relation iff (\sim 0), (\sim 6), (\sim 7), (\sim 8), (\sim 9), and (\sim 11) hold;
- (1) \sim is a CP DP LM preferential-discriminative consequence relation iff (\sim 0), (\sim 6), (\sim 7), (\sim 8), (\sim 9), (\sim 10), and (\sim 11) hold.

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies (A2) too. Then,

- (2) \sim is a DP preferential-discriminative consequence relation iff (\sim 0), (\sim 6), (\sim 7), (\sim 8), and (\sim 9) hold;
- (3) \sim is a DP LM preferential-discriminative consequence relation iff (\sim 0), (\sim 6), (\sim 7), (\sim 8), (\sim 9), and (\sim 10) hold.

The proof of Proposition 41 has been relegated at the end of Section 3.3. We need first Definition 42 and Lemmas 43, 44, and 45 below. Here are some purely technical tools:

Definition 42 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A1), \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$, and $\Gamma \subseteq \mathcal{F}$. Then,

$$M_{\Gamma}^1 := \{v \in M_{\Gamma, \sim(\Gamma)} : \exists \beta \in T(M_{\Gamma, \sim(\Gamma)}) \setminus \sim(\Gamma), v \notin M_{\neg\beta}\}.$$

Let $i \in \mathbb{N}$ with $i \geq 2$. Then,

$$M_{\Gamma}^i := \{v \in M_{\Gamma, \sim(\Gamma)} \setminus M_{\Gamma}^1 \cup \dots \cup M_{\Gamma}^{i-1} : \exists \beta \in T(M_{\Gamma, \sim(\Gamma)} \setminus M_{\Gamma}^1 \cup \dots \cup M_{\Gamma}^{i-1}) \setminus \sim(\Gamma), v \notin M_{\neg\beta}\}.$$

$$M'_{\Gamma} := \bigcup_{i \in \mathbb{N}^+} M_{\Gamma}^i$$

$$n(\Gamma) := |\{i \in \mathbb{N}^+ : M_{\Gamma}^i \neq \emptyset\}|$$

Suppose $M_\Gamma^1 \neq \emptyset$. Then, we denote by β_Γ^1 an element of \mathcal{F} (chosen arbitrarily) such that $\exists r \in \mathbb{N}^+, \exists v_1, v_2, \dots, v_r \in \mathcal{V}$, and $\exists \beta_1, \beta_2, \dots, \beta_r \in \mathcal{F}$ with $M_\Gamma^1 = \{v_1, v_2, \dots, v_r\}$,

$$\beta_\Gamma^1 = \beta_1 \vee \beta_2 \vee \dots \vee \beta_r,$$

and $\forall j \in [1, r], \beta_j \notin \vdash(\Gamma), M_{\Gamma, \vdash(\Gamma)} \subseteq M_{\beta_j}$, and $v_j \notin M_{\neg\beta_j}$.
As $M_\Gamma^1 \neq \emptyset$ and M_Γ^1 is finite (thanks to (A1)), such an element exists.

Suppose $i \in \mathbb{N}, i \geq 2$, and $M_\Gamma^i \neq \emptyset$.

Then, we denote by β_Γ^i an element of \mathcal{F} (chosen arbitrarily) such that $\exists r \in \mathbb{N}^+, \exists v_1, v_2, \dots, v_r \in \mathcal{V}$, and $\exists \beta_1, \beta_2, \dots, \beta_r \in \mathcal{F}$ with $M_\Gamma^i = \{v_1, v_2, \dots, v_r\}$,

$$\beta_\Gamma^i = \beta_1 \vee \beta_2 \vee \dots \vee \beta_r,$$

and $\forall j \in [1, r], \beta_j \notin \vdash(\Gamma), M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{i-1} \subseteq M_{\beta_j}$, and $v_j \notin M_{\neg\beta_j}$.
As $M_\Gamma^i \neq \emptyset$ and M_Γ^i is finite, such an element exists.

Suppose $M_\Gamma^l \neq \emptyset$. Then,

$$\beta_\Gamma := \beta_\Gamma^1 \vee \beta_\Gamma^2 \vee \dots \vee \beta_\Gamma^{n(\Gamma)}$$

As $M_\Gamma^l \neq \emptyset, n(\Gamma) \geq 1$. In addition, we will show in Lemma 43 below that $n(\Gamma)$ is finite and $\forall i \in \mathbb{N}^+$ with $i \leq n(\Gamma), M_\Gamma^i \neq \emptyset$. Thus, β_Γ is well-defined.

$$F(\Gamma) := \begin{cases} \{\neg\beta_\Gamma\} & \text{if } M_\Gamma^l \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

$$G(\Gamma) := \{\alpha \in \mathcal{F} : \alpha \notin \vdash(\Gamma), \neg\alpha \notin \vdash(\Gamma), \text{ and } T_d(M_{\Gamma, \vdash(\Gamma)}, \alpha) \subseteq \vdash(\Gamma)\}$$

Here are some quick results on the purely technical tools defined just above:

Lemma 43 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A1), \vdash a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$, $\Gamma \subseteq \mathcal{F}$, and $i, j \in \mathbb{N}^+$. Then,

- (0) if $i \neq j$, then $M_\Gamma^i \cap M_\Gamma^j = \emptyset$;
- (1) if $M_\Gamma^i = \emptyset$, then $M_\Gamma^{i+1} = \emptyset$;
- (2) $T_d(M_{\Gamma, \vdash(\Gamma)}) \subseteq \vdash(\Gamma)$ iff $M_\Gamma^1 = \emptyset$;
- (3) if $i \geq 2$, then $T_d(M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{i-1}) \subseteq \vdash(\Gamma)$ iff $M_\Gamma^i = \emptyset$;
- (4) $n(\Gamma)$ is finite;
- (5) if $i \leq n(\Gamma)$, then $M_\Gamma^i \neq \emptyset$;
- (6) if $i > n(\Gamma)$, then $M_\Gamma^i = \emptyset$;
- (7) if $M_\Gamma^l \neq \emptyset$, then $M_\Gamma^l = M_\Gamma^1 \cup \dots \cup M_\Gamma^{n(\Gamma)}$;
- (8) $T_d(M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^l) \subseteq \vdash(\Gamma)$.

Proof *Proofs of (0), (1), (2), and (3).* Trivial.

Proof of (4). Obvious by (0) and (A1).

Proof of (5). Suppose $\exists i \in \mathbb{N}^+, M_\Gamma^i = \emptyset$ and $i \leq n(\Gamma)$.

Then, by (1), $\forall j \in \mathbb{N}^+, j \geq i, M_\Gamma^j = \emptyset$.

Thus, $|\{j \in \mathbb{N}^+ : M_\Gamma^j \neq \emptyset\}| \leq i - 1 < n(\Gamma)$, which is impossible.

Proof of (6). Suppose $\exists i \in \mathbb{N}^+, M_\Gamma^i \neq \emptyset$ and $i > n(\Gamma)$.

Then, by (1), $\forall j \in \mathbb{N}^+, j \leq i, M_\Gamma^j \neq \emptyset$.

Thus, $|\{j \in \mathbb{N}^+ : M_\Gamma^j \neq \emptyset\}| \geq i > n(\Gamma)$, which is impossible.

Proof of (7). Obvious by (6).

Proof of (8). Case 1: $M'_\Gamma = \emptyset$.

Then, $T_d(M_{\Gamma, \sim(\Gamma)} \setminus M'_\Gamma) = T_d(M_{\Gamma, \sim(\Gamma)})$. In addition, $M_\Gamma^1 = \emptyset$. Thus, by (2), we are done.

Case 2: $M'_\Gamma \neq \emptyset$.

Then, by (7), $T_d(M_{\Gamma, \sim(\Gamma)} \setminus M'_\Gamma) = T_d(M_{\Gamma, \sim(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n(\Gamma)})$.

In addition, $n(\Gamma) + 1 \geq 2$ and, by (6), $M_\Gamma^{n(\Gamma)+1} = \emptyset$. Thus, by (3), we are done. ■

We turn to an important lemma. Its main goal is to show that the conditions (\sim 6), (\sim 7), and (\sim 8) are sufficient to establish the following important equality: $\sim(\Gamma) = T_d(M_{\Gamma, \sim(\Gamma), H(\Gamma)})$, which provides a semantic definition of \sim (in the discriminative manner).

Lemma 44 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee and \wedge binary connectives of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A1) and (A3), \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$ satisfying (\sim 6), (\sim 7), and (\sim 8), and $\Gamma \subseteq \mathcal{F}$. Then,

- (0) if $M'_\Gamma \neq \emptyset$, then $\beta_\Gamma \notin \sim(\Gamma)$;
- (1) if $M'_\Gamma \neq \emptyset$, then $M_{\Gamma, \sim(\Gamma)} \subseteq M_{\beta_\Gamma}$;
- (2) if $M'_\Gamma \neq \emptyset$, then $M'_\Gamma \cap M_{\neg\beta_\Gamma} = \emptyset$;
- (3) if $M'_\Gamma \neq \emptyset$, then $M_{\Gamma, \sim(\Gamma)} \setminus M'_\Gamma \subseteq M_{\neg\beta_\Gamma}$;
- (4) $M_{\Gamma, \sim(\Gamma)} \setminus M'_\Gamma = M_{\Gamma, \sim(\Gamma), F(\Gamma)}$;
- (5) $\sim(\Gamma) = T_d(M_{\Gamma, \sim(\Gamma), F(\Gamma)})$;
- (6) $M_{\Gamma, \sim(\Gamma), H(\Gamma)} = M_{\Gamma, \sim(\Gamma), F(\Gamma)}$;
- (7) $\sim(\Gamma) = T_d(M_{\Gamma, \sim(\Gamma), H(\Gamma)})$.

Proof *Proofs of (0), (1), and (2).* Suppose $M'_\Gamma \neq \emptyset$.

Then, it suffices to show by induction: $\forall i \in [1, n(\Gamma)]$,

$$p_3(i) \quad (M_\Gamma^1 \cup \dots \cup M_\Gamma^i) \cap M_{\neg(\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i)} = \emptyset;$$

$$p_2(i) \quad M_{\Gamma, \sim(\Gamma)} \subseteq M_{\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i};$$

$p_1(i) \quad \beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i \notin \vdash(\Gamma).$

As $M_\Gamma^1 \neq \emptyset, \exists r \in \mathbb{N}^+, \exists v_1, v_2, \dots, v_r \in \mathcal{V},$ and $\exists \beta_1, \beta_2, \dots, \beta_r \in \mathcal{F}, M_\Gamma^1 = \{v_1, \dots, v_r\},$
 $\beta_\Gamma^1 = \beta_1 \vee \dots \vee \beta_r,$ and $\forall j \in [1, r], \beta_j \notin \vdash(\Gamma), M_{\Gamma, \vdash(\Gamma)} \subseteq M_{\beta_j},$ and $v_j \notin M_{\neg\beta_j}.$

Then, it can be shown that:

(0.0) $p_3(1)$ holds;

(0.1) $p_2(1)$ holds;

(0.2) $p_1(1)$ holds.

Now, let $i \in [1, n(\Gamma) - 1]$ and suppose $p_1(i), p_2(i),$ and $p_3(i)$ hold.

As $M_\Gamma^{i+1} \neq \emptyset, \exists r \in \mathbb{N}^+, \exists v_1, v_2, \dots, v_r \in \mathcal{V},$ and $\exists \beta_1, \beta_2, \dots, \beta_r \in \mathcal{F},$

$M_\Gamma^{i+1} = \{v_1, \dots, v_r\}, \beta_\Gamma^{i+1} = \beta_1 \vee \dots \vee \beta_r,$ and

$\forall j \in [1, r], \beta_j \notin \vdash(\Gamma), M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^i \subseteq M_{\beta_j},$ and $v_j \notin M_{\neg\beta_j}.$

Then, it can be shown that:

(0.3) $p_3(i+1)$ holds;

(0.4) $p_2(i+1)$ holds;

(0.5) $\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i \vee \beta_1 \vee \beta_2 \vee \dots \vee \beta_r \notin \vdash(\Gamma);$

(0.6) $p_1(i+1)$ holds.

Proof of (0.0). If $v_j \in M_\Gamma^1,$ then $v_j \notin M_{\neg\beta_j}.$ But, by (A3), $M_{\neg\beta_\Gamma^1} \subseteq M_{\neg\beta_j}.$

Proof of (0.1). We have $M_{\Gamma, \vdash(\Gamma)} \subseteq M_{\beta_1}$ which is a subset of $M_{\beta_\Gamma^1},$ by (A3).

Proof of (0.2). It suffices to show by induction: $\forall j \in [1, r],$

$q(j) \quad \beta_1 \vee \dots \vee \beta_j \notin \vdash(\Gamma).$

Obviously, $q(1)$ holds.

Let $j \in [1, r - 1].$ Suppose $q(j).$ We show $q(j+1).$

By (A3), we have $M_{\Gamma, \vdash(\Gamma)} \subseteq M_{\beta_1 \vee \dots \vee \beta_j}.$

On the other hand, $M_{\Gamma, \vdash(\Gamma), \neg(\beta_1 \vee \dots \vee \beta_j)} \subseteq M_{\Gamma, \vdash(\Gamma)} \subseteq M_{\beta_{j+1}}.$

Thus, by $q(j)$ and (~ 7) (take $\beta_1 \vee \dots \vee \beta_j$ for α and β_{j+1} for β), we get $\beta_1 \vee \dots \vee \beta_{j+1} \notin \vdash(\Gamma).$

Proof of (0.3). Let $v \in M_\Gamma^1 \cup \dots \cup M_\Gamma^{i+1}.$ We show $v \notin M_{\neg(\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^{i+1})}.$

Case 1: $v \in M_\Gamma^1 \cup \dots \cup M_\Gamma^i.$

Then, by $p_3(i),$ we have $v \notin M_{\neg(\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i)}.$ But, by (A3), $M_{\neg(\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^{i+1})} \subseteq M_{\neg(\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i)}.$

Case 2: $v \in M_\Gamma^{i+1}.$

Then, $\exists j \in [1, r], v = v_j.$ Thus, $v \notin M_{\neg\beta_j}.$ But, by (A3), $M_{\neg(\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^{i+1})} \subseteq M_{\neg\beta_\Gamma^{i+1}} \subseteq M_{\neg\beta_j}.$

Proof of (0.4). By $p_2(i), M_{\Gamma, \vdash(\Gamma)} \subseteq M_{\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i}$ which is a subset of $M_{\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^{i+1}},$ by (A3).

Proof of (0.5). It suffices to show by induction $\forall j \in [1, r]:$

$q(j) \quad \beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i \vee \beta_1 \vee \dots \vee \beta_j \notin \vdash(\Gamma).$

We will show:

(0.5.0) $M_{\Gamma, \vdash(\Gamma), \neg(\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i)} \subseteq M_{\beta_1}.$

Then, by $p_1(i), p_2(i),$ (0.5.0), and (~ 7) (take $\beta_\Gamma^1 \vee \dots \vee \beta_\Gamma^i$ for α and β_1 for β), $q(1)$ holds.

Now, let $j \in [1, r - 1]$ and suppose $q(j).$

Then, we will show:

$$(0.5.1) \quad M_{\Gamma, \sim(\Gamma), \neg(\beta_1^1 \vee \dots \vee \beta_1^i \vee \beta_1 \vee \dots \vee \beta_j)} \subseteq M_{\beta_{j+1}}.$$

In addition, by $p_2(i)$ and (A3), we get:

$$(0.5.2) \quad M_{\Gamma, \sim(\Gamma)} \subseteq M_{\beta_1^1 \vee \dots \vee \beta_1^i \vee \beta_1 \vee \dots \vee \beta_j}.$$

By (0.5.1), (0.5.2), $q(j)$, and (\sim 7) (take $\beta_1^1 \vee \dots \vee \beta_1^i \vee \beta_1 \vee \dots \vee \beta_j$ for α and β_{j+1} for β), we get that $q(j+1)$ holds.

Proof of (0.5.0). Let $v \in M_{\Gamma, \sim(\Gamma), \neg(\beta_1^1 \vee \dots \vee \beta_1^i)}$. Then, $v \in M_{\neg(\beta_1^1 \vee \dots \vee \beta_1^i)}$.

Thus, by $p_3(i)$, $v \notin M_{\Gamma}^1 \cup \dots \cup M_{\Gamma}^i$. Therefore, $v \in M_{\Gamma, \sim(\Gamma)} \setminus M_{\Gamma}^1 \cup \dots \cup M_{\Gamma}^i \subseteq M_{\beta_1}$.

Proof of (0.5.1). Let $v \in M_{\Gamma, \sim(\Gamma), \neg(\beta_1^1 \vee \dots \vee \beta_1^i \vee \beta_1 \vee \dots \vee \beta_j)}$. Then, by (A3), $v \in M_{\neg(\beta_1^1 \vee \dots \vee \beta_1^i)}$.

Therefore, by $p_3(i)$, $v \notin M_{\Gamma}^1 \cup \dots \cup M_{\Gamma}^i$. Therefore, $v \in M_{\Gamma, \sim(\Gamma)} \setminus M_{\Gamma}^1 \cup \dots \cup M_{\Gamma}^i \subseteq M_{\beta_{j+1}}$.

Proof of (0.6). By $p_2(i)$ and (A3), we get $M_{\Gamma, \sim(\Gamma)} \subseteq M_{\beta_1^1 \vee \dots \vee \beta_1^i} \subseteq M_{\beta_1^1 \vee \dots \vee \beta_1^i \vee \beta_1 \vee \dots \vee \beta_r}$.

In addition, by (A3), we get $M_{\neg(\beta_1^1 \vee \dots \vee \beta_1^i \vee \beta_1 \vee \dots \vee \beta_r)} = M_{\neg(\beta_1^1 \vee \dots \vee \beta_1^{i+1})}$.

Therefore, by (0.5) and (\sim 6) (take $\beta_1^1 \vee \dots \vee \beta_1^{i+1}$ for α and $\beta_1^1 \vee \dots \vee \beta_1^i \vee \beta_1 \vee \dots \vee \beta_r$ for β), we get that $p_1(i+1)$ holds.

Proof of (3). Suppose $M_{\Gamma}^i \neq \emptyset$, $v \in M_{\Gamma, \sim(\Gamma)} \setminus M_{\Gamma}^i$, and $v \notin M_{\neg\beta_{\Gamma}}$.

Then, by (0), (1), and definition of M_{Γ}^i , we get $v \in M_{\Gamma}^{n(\Gamma)+1}$, which is impossible by Lemma 43 (6).

Proof of (4). Case 1: $M_{\Gamma}^i \neq \emptyset$.

By (3), we get one direction: $M_{\Gamma, \sim(\Gamma)} \setminus M_{\Gamma}^i \subseteq M_{\Gamma, \sim(\Gamma), \neg\beta_{\Gamma}}$.

By (2), we get the other direction: $M_{\Gamma, \sim(\Gamma), \neg\beta_{\Gamma}} \subseteq M_{\Gamma, \sim(\Gamma)} \setminus M_{\Gamma}^i$.

Case 2: $M_{\Gamma}^i = \emptyset$.

Then, obviously, $M_{\Gamma, \sim(\Gamma)} \setminus M_{\Gamma}^i = M_{\Gamma, \sim(\Gamma)} = M_{\Gamma, \sim(\Gamma), F(\Gamma)}$.

Proof of (5). Direction: “ \subseteq ”.

Case 1: $M_{\Gamma}^i \neq \emptyset$.

Suppose the contrary of what we want to show, i.e. $\exists \alpha \in \sim(\Gamma)$, $\alpha \notin T_d(M_{\Gamma, \sim(\Gamma), \neg\beta_{\Gamma}})$.

Then, $M_{\Gamma, \sim(\Gamma), \neg\beta_{\Gamma}} \subseteq M_{\sim(\Gamma)} \subseteq M_{\alpha}$. Thus, $M_{\Gamma, \sim(\Gamma), \neg\beta_{\Gamma}} \subseteq M_{\neg\alpha}$.

Consequently, by (0), (1), and (\sim 6), we get $\alpha \notin \sim(\Gamma)$, which is impossible.

Case 2: $M_{\Gamma}^i = \emptyset$.

Let $\alpha \in \sim(\Gamma)$. Then, $M_{\Gamma, \sim(\Gamma)} \subseteq M_{\sim(\Gamma)} \subseteq M_{\alpha}$. In addition, by (\sim 8), $M_{\Gamma, \sim(\Gamma)} \not\subseteq M_{\neg\alpha}$.

Consequently, $\alpha \in T_d(M_{\Gamma, \sim(\Gamma)}) = T_d(M_{\Gamma, \sim(\Gamma), F(\Gamma)})$.

Direction: “ \supseteq ”. Obvious by (4) and Lemma 43 (8).

Proof of (6). Direction: “ \subseteq ”.

Case 1: $M_{\Gamma}^i = \emptyset$.

Case 1.1: $H_1(\Gamma) \neq \emptyset$.

Then, $\exists \alpha \in \mathcal{F}$, $\alpha \notin \sim(\Gamma)$, $M_{\Gamma, \sim(\Gamma)} \subseteq M_{\alpha}$, and $M_{\Gamma, \sim(\Gamma)} \not\subseteq M_{\neg\alpha}$. Thus, $\alpha \in T_d(M_{\Gamma, \sim(\Gamma)})$.

Therefore, by (5), $\alpha \in \sim(\Gamma)$, which is impossible.

Case 1.2: $H_1(\Gamma) = \emptyset$.

Clearly, $\forall i \in \mathbb{N}^+$, if $H_i(\Gamma) = \emptyset$, then $H_{i+1}(\Gamma) = \emptyset$. Therefore, $H(\Gamma) = \emptyset = F(\Gamma)$.

Case 2: $M_{\Gamma}^i \neq \emptyset$.

As $M'_\Gamma \subseteq M_{\Gamma, \sim(\Gamma)}$, we get, by (2), $M_{\Gamma, \sim(\Gamma)} \not\subseteq M_{\neg\beta_\Gamma}$.
Thus, by (0) and (1), we get $\neg\beta_\Gamma \in H_1(\Gamma) \subseteq H(\Gamma)$. Therefore, $M_{H(\Gamma)} \subseteq M_{F(\Gamma)}$.

Direction: “ \supseteq ”.

Case 1: $M'_\Gamma = \emptyset$.

Verbatim the proof of Case 1 of direction “ \subseteq ”.

Case 2: $M'_\Gamma \neq \emptyset$.

Then, the following holds:

$$(6.0) \quad \forall i \in \mathbb{N}^+, M_{\Gamma, \sim(\Gamma), \neg\beta_\Gamma} \subseteq M_{\Gamma, \sim(\Gamma), H_1(\Gamma), \dots, H_i(\Gamma)}.$$

Now, suppose the contrary of what we want to show, i.e.

$\exists v \in M_{\Gamma, \sim(\Gamma), \neg\beta_\Gamma}$, $v \notin M_{\Gamma, \sim(\Gamma), H(\Gamma)}$. Then, $v \notin M_{H(\Gamma)}$. But, clearly, $M_{H(\Gamma)} = \bigcap_{i \in \mathbb{N}^+} M_{H_i(\Gamma)}$.
Therefore, $\exists i \in \mathbb{N}^+$, $v \notin M_{H_i(\Gamma)}$, which is impossible by (6.0).

Proof of (6.0). We show by induction: $\forall i \in \mathbb{N}^+$,

$$p(i) \quad M_{\Gamma, \sim(\Gamma), \neg\beta_\Gamma} \subseteq M_{\Gamma, \sim(\Gamma), H_1(\Gamma), \dots, H_i(\Gamma)}.$$

We will show

$$(6.0.0) \quad p(1) \text{ holds.}$$

Let $i \in \mathbb{N}^+$, suppose $p(i)$ holds, and suppose $p(i+1)$ does not hold.

Then, $\exists v \in M_{\Gamma, \sim(\Gamma), \neg\beta_\Gamma}$, $v \notin M_{\Gamma, \sim(\Gamma), H_1(\Gamma), \dots, H_{i+1}(\Gamma)}$.

Thus, $\exists j \in [1, i+1]$, $v \notin M_{H_j(\Gamma)}$.

Case 1: $j = 1$.

Then, $\exists \beta \in \mathcal{F}$, $M_{\Gamma, \sim(\Gamma)} \subseteq M_\beta$, $\beta \notin \sim(\Gamma)$, and $v \notin M_{\neg\beta}$.

Thus, $v \in M_\Gamma^1 \cap M_{\neg\beta_\Gamma}$, which is impossible by (2).

Case 2: $j \geq 2$.

Then, $\exists \beta \in \mathcal{F}$, $M_{\Gamma, \sim(\Gamma), H_1(\Gamma), \dots, H_{j-1}(\Gamma)} \subseteq M_\beta$, $\beta \notin \sim(\Gamma)$, and $v \notin M_{\neg\beta}$.

But, by Lemma 43 (7), by (4), and $p(i)$, we get

$$M_{\Gamma, \sim(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n(\Gamma)} = M_{\Gamma, \sim(\Gamma)} \setminus M'_\Gamma = M_{\Gamma, \sim(\Gamma), \neg\beta_\Gamma} \subseteq M_{\Gamma, \sim(\Gamma), H_1(\Gamma), \dots, H_i(\Gamma)} \subseteq M_{\Gamma, \sim(\Gamma), H_1(\Gamma), \dots, H_{j-1}(\Gamma)} \subseteq M_\beta.$$

Therefore, $v \in M_\Gamma^{n(\Gamma)+1}$, which is impossible by Lemma 43 (6).

Proof of (6.0.0). Suppose the contrary of what we want to show, i.e.

suppose $\exists v \in M_{\Gamma, \sim(\Gamma), \neg\beta_\Gamma}$, $v \notin M_{\Gamma, \sim(\Gamma), H_1(\Gamma)}$.

Then, $v \notin M_{H_1(\Gamma)}$. Thus, $\exists \beta \in \mathcal{F}$, $M_{\Gamma, \sim(\Gamma)} \subseteq M_\beta$, $\beta \notin \sim(\Gamma)$, and $v \notin M_{\neg\beta}$.

Thus, $v \in M_\Gamma^1$. Therefore, $v \in M'_\Gamma \cap M_{\neg\beta_\Gamma}$, which is impossible by (2).

Proof of (7). Obvious by (5) and (6). ■

We turn to a second important lemma. Its main purpose is to show that any DP choice function μ representing (in the discriminative manner) a relation \sim satisfies the following equality: $\mu(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$, which enables us to define μ from \sim .

Lemma 45 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee and \wedge binary connectives of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A1) and (A3), $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, μ a DP choice function from \mathbf{D} to \mathbf{V} , \sim the relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\sim(\Gamma) = T_d(\mu(M_\Gamma))$, and $\Gamma \subseteq \mathcal{F}$. Then:

$$(0) \quad \mu(M_\Gamma) \subseteq M_{\Gamma, \sim(\Gamma)};$$

- (1) \vdash satisfies ($\vdash 6$);
- (2) \vdash satisfies ($\vdash 7$);
- (3) \vdash satisfies ($\vdash 8$);
- (4) $M'_\Gamma \cap \mu(M_\Gamma) = \emptyset$;
- (5) $M_{\Gamma, \vdash(\Gamma), T_c(\mu(M_\Gamma))} = \mu(M_\Gamma)$;
- (6) if $M'_\Gamma \neq \emptyset$, then $M_{\Gamma, \vdash(\Gamma), H(\Gamma)} = \mu(M_\Gamma)$.

If $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies (A2) too, then:

- (7) if $M'_\Gamma = \emptyset$, then $M_{G(\Gamma)} = M_{T_c(\mu(M_\Gamma))}$;
- (8) if $M'_\Gamma = \emptyset$, then $M_{\Gamma, \vdash(\Gamma)} \subseteq M_{G(\Gamma)}$;
- (9) $M_{\Gamma, \vdash(\Gamma), H(\Gamma)} = \mu(M_\Gamma)$.

If μ is coherency preserving, then again:

- (10) $M_{\Gamma, \vdash(\Gamma), H(\Gamma)} = \mu(M_\Gamma)$.

Proof Proof of (0). We show $\mu(M_\Gamma) \subseteq M_{\vdash(\Gamma)}$. Let $v \in \mu(M_\Gamma)$ and $\alpha \in \vdash(\Gamma)$.

Then, $\alpha \in T_d(\mu(M_\Gamma))$. Thus, $\mu(M_\Gamma) \subseteq M_\alpha$. Thus, $v \in M_\alpha$ and we are done.

In addition, obviously, $\mu(M_\Gamma) \subseteq M_\Gamma$. Therefore, $\mu(M_\Gamma) \subseteq M_\Gamma \cap M_{\vdash(\Gamma)} = M_{\Gamma, \vdash(\Gamma)}$.

Proof of (1). Let $\alpha, \beta \in \mathcal{F}$ and suppose $\beta \in \vdash(\Gamma, \vdash(\Gamma)) \setminus \vdash(\Gamma)$ and $\neg\alpha \in \vdash(\Gamma, \vdash(\Gamma), \neg\beta)$. Then, by (0), $\mu(M_\Gamma) \subseteq M_{\Gamma, \vdash(\Gamma)} \subseteq M_\beta$. But, $\beta \notin \vdash(\Gamma) = T_d(\mu(M_\Gamma))$. Thus, $\mu(M_\Gamma) \subseteq M_{\neg\beta}$. Consequently, $\mu(M_\Gamma) \subseteq M_{\Gamma, \vdash(\Gamma), \neg\beta} \subseteq M_{\neg\alpha}$. Therefore, $\alpha \notin T_d(\mu(M_\Gamma)) = \vdash(\Gamma)$.

Proof of (2). Let $\alpha, \beta \in \mathcal{F}$ and suppose $\alpha \in \vdash(\Gamma, \vdash(\Gamma)) \setminus \vdash(\Gamma)$ and $\beta \in \vdash(\Gamma, \vdash(\Gamma), \neg\alpha) \setminus \vdash(\Gamma)$. Then, by (0), $\mu(M_\Gamma) \subseteq M_{\Gamma, \vdash(\Gamma)} \subseteq M_\alpha$. But, $\alpha \notin T_d(\mu(M_\Gamma))$. Thus, $\mu(M_\Gamma) \subseteq M_{\neg\alpha}$. Thus, $\mu(M_\Gamma) \subseteq M_{\Gamma, \vdash(\Gamma), \neg\alpha} \subseteq M_\beta$. But, $\beta \notin T_d(\mu(M_\Gamma))$. Therefore $\mu(M_\Gamma) \subseteq M_{\neg\beta}$. Thus, by (A3), $\mu(M_\Gamma) \subseteq M_{\neg\alpha} \cap M_{\neg\beta} = M_{\neg(\alpha \vee \beta)}$. Consequently, $\alpha \vee \beta \notin T_d(\mu(M_\Gamma)) = \vdash(\Gamma)$.

Proof of (3). Let $\alpha \in \vdash(\Gamma)$. Then, $\alpha \in T_d(\mu(M_\Gamma))$. Thus, $\mu(M_\Gamma) \not\subseteq M_{\neg\alpha}$.

Thus, by (0), $M_{\Gamma, \vdash(\Gamma)} \not\subseteq M_{\neg\alpha}$.

Proof of (4). Case 1: $M'_\Gamma = \emptyset$. Obvious.

Case 2: $M'_\Gamma \neq \emptyset$.

It is sufficient to show by induction: $\forall i \in [1, n(\Gamma)]$,

$$p(i) \quad (M_\Gamma^1 \cup \dots \cup M_\Gamma^i) \cap \mu(M_\Gamma) = \emptyset.$$

We will show:

$$(4.0) \quad p(1) \text{ holds.}$$

Let $i \in [1, n(\Gamma) - 1]$. Suppose $p(i)$. We show $p(i + 1)$.

$$\text{Case 1: } M_\Gamma^{i+1} \cap \mu(M_\Gamma) = \emptyset.$$

Then, by $p(i)$, we obviously get $p(i + 1)$.

Case 2: $\exists v \in M_\Gamma^{i+1} \cap \mu(M_\Gamma)$.

Then, $\exists \beta \in \mathcal{F}$, $\beta \notin \vdash(\Gamma)$, $M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^i \subseteq M_\beta$, and $v \notin M_{-\beta}$.

Therefore, by (0) and $p(i)$, $\mu(M_\Gamma) \subseteq M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^i \subseteq M_\beta$. But, $\mu(M_\Gamma) \not\subseteq M_{-\beta}$.

Consequently, $\beta \in T_d(\mu(M_\Gamma)) = \vdash(\Gamma)$, which is impossible.

Proof of (4.0). Suppose the contrary of $p(1)$, i.e. $\exists v \in M_\Gamma^1 \cap \mu(M_\Gamma)$.

Then, $\exists \beta \in \mathcal{F}$, $\beta \notin \vdash(\Gamma)$, $M_{\Gamma, \vdash(\Gamma)} \subseteq M_\beta$ and $v \notin M_{-\beta}$.

Therefore, by (0), $\mu(M_\Gamma) \subseteq M_\beta$. On the other hand, $\mu(M_\Gamma) \not\subseteq M_{-\beta}$.

Therefore, $\beta \in T_d(\mu(M_\Gamma)) = \vdash(\Gamma)$, which is impossible.

Proof of (5). As $\mu(M_\Gamma) \in \mathbf{D}$, $\exists \Gamma' \subseteq \mathcal{F}$, $M_{\Gamma'} = \mu(M_\Gamma)$.

Therefore, $M_{T(\mu(M_\Gamma))} = M_{T(M_{\Gamma'})} = M_{\Gamma'} = \mu(M_\Gamma)$.

Thus, $M_{\Gamma, \vdash(\Gamma), T_c(\mu(M_\Gamma))} = M_{\Gamma, T_d(\mu(M_\Gamma)), T_c(\mu(M_\Gamma))} = M_{\Gamma, T(\mu(M_\Gamma))}$. But, $\Gamma \subseteq T(\mu(M_\Gamma))$.

Therefore, $M_{\Gamma, T(\mu(M_\Gamma))} = M_{T(\mu(M_\Gamma))} = \mu(M_\Gamma)$.

Proof of (6). Suppose $M'_\Gamma \neq \emptyset$. Direction: “ \subseteq ”.

Case 1: $\exists v \in M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n(\Gamma)}$, $v \notin M_{T_c(\mu(M_\Gamma))}$.

Then, $\exists \alpha \in T_c(\mu(M_\Gamma))$, $v \notin M_\alpha$.

By Lemma 44 (3), Lemma 43 (7), and (A3), $M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n(\Gamma)} \subseteq M_{-\beta_\Gamma} \subseteq M_{-(\beta_\Gamma \wedge \alpha)}$.

By (0) and Lemma 44 (1), $\mu(M_\Gamma) \subseteq M_{\beta_\Gamma} \cap M_\alpha = M_{\neg(\beta_\Gamma \wedge \alpha)}$.

Therefore, $\neg(\beta_\Gamma \wedge \alpha) \notin T_d(\mu(M_\Gamma)) = \vdash(\Gamma)$.

In addition, $v \notin M_\alpha \supseteq M_{\neg(\beta_\Gamma \wedge \alpha)}$.

Consequently, $v \in M_\Gamma^{n(\Gamma)+1}$ (take $\neg(\beta_\Gamma \wedge \alpha)$ for the β of the definition of M_Γ^i).

Therefore, by Lemma 43 (6), we get a contradiction.

Case 2: $M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n(\Gamma)} \subseteq M_{T_c(\mu(M_\Gamma))}$.

Then, by Lemma 44 (6), Lemma 44 (4), Lemma 43 (7), and by (5), we get

$M_{\Gamma, \vdash(\Gamma), H(\Gamma)} = M_{\Gamma, \vdash(\Gamma)} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n(\Gamma)} \subseteq M_{\Gamma, \vdash(\Gamma), T_c(\mu(M_\Gamma))} = \mu(M_\Gamma)$.

Direction: “ \supseteq ”.

By (0), (4), Lemma 44 (4), and Lemma 44 (6), we get

$\mu(M_\Gamma) \subseteq M_{\Gamma, \vdash(\Gamma)} \setminus M'_\Gamma = M_{\Gamma, \vdash(\Gamma), F(\Gamma)} = M_{\Gamma, \vdash(\Gamma), H(\Gamma)}$.

Proof of (7). Suppose $M'_\Gamma = \emptyset$. Direction: “ \supseteq ”.

Suppose the contrary of what we want to show, i.e. $\exists v \in M_{T_c(\mu(M_\Gamma))}$, $v \notin M_{G(\Gamma)}$.

Then, $\exists \alpha \in G(\Gamma)$, $v \notin M_\alpha$.

Case 1: $\alpha \in T(M_{\Gamma, \vdash(\Gamma)})$.

As $\alpha \in G(\Gamma)$, $\alpha \notin \vdash(\Gamma)$. Thus, by Lemma 44 (5), $\alpha \notin T_d(M_{\Gamma, \vdash(\Gamma)})$.

Therefore, $\alpha \in T_c(M_{\Gamma, \vdash(\Gamma)})$. Consequently, by (0), $\alpha \in T_c(\mu(M_\Gamma))$.

Thus, $v \in M_\alpha$, which is impossible.

Case 2: $\neg\alpha \in T(M_{\Gamma, \vdash(\Gamma)})$.

As $\alpha \in G(\Gamma)$, $\neg\alpha \notin \vdash(\Gamma)$. Thus, by Lemma 44 (5), $\neg\alpha \notin T_d(M_{\Gamma, \vdash(\Gamma)})$.

Therefore, $\neg\alpha \in T_c(M_{\Gamma, \vdash(\Gamma)})$. Consequently, by (A3), $\alpha \in T_c(M_{\Gamma, \vdash(\Gamma)})$.

Therefore, by (0), $\alpha \in T_c(\mu(M_\Gamma))$. Thus, $v \in M_\alpha$, which is impossible.

Case 3: $\alpha \notin T(M_{\Gamma, \vdash(\Gamma)})$ and $\neg\alpha \notin T(M_{\Gamma, \vdash(\Gamma)})$.

Then, by (A2), $M_{\Gamma, \vdash(\Gamma), \alpha} \not\subseteq M_{-\alpha}$. Therefore, $\alpha \in T_d(M_{\Gamma, \vdash(\Gamma), \alpha})$.

But, $\alpha \in G(\Gamma)$. Thus, $T_d(M_{\Gamma, \sim(\Gamma), \alpha}) \subseteq \sim(\Gamma)$. Thus, $\alpha \in \sim(\Gamma)$. Thus, $\alpha \notin G(\Gamma)$, impossible.

Direction: “ \subseteq ”.

Suppose the contrary of what we want to show, i.e. $\exists v \in M_{G(\Gamma)}, v \notin M_{T_c(\mu(M_\Gamma))}$.

Then, we will show:

$$(7.0) \quad \exists \alpha \in T_c(\mu(M_\Gamma)), |M_{\Gamma, \sim(\Gamma), \alpha}| < |\mu(M_\Gamma)|$$

But, $\mu(M_\Gamma) \subseteq M_\alpha$ and, by (0), $\mu(M_\Gamma) \subseteq M_{\Gamma, \sim(\Gamma)}$. Therefore, $\mu(M_\Gamma) \subseteq M_{\Gamma, \sim(\Gamma), \alpha}$.

Thus, $|\mu(M_\Gamma)| \leq |M_{\Gamma, \sim(\Gamma), \alpha}|$, which is impossible by (7.0).

Proof of (7.0). We have $\exists \delta \in T_c(\mu(M_\Gamma)), v \notin M_\delta$.

By (A1), $|M_{\Gamma, \sim(\Gamma), \delta}|$ is finite. To show (7.0), it suffices to show by induction (in the decreasing direction): $\forall i \in \mathbb{Z}$ with $i \leq |M_{\Gamma, \sim(\Gamma), \delta}|$,

$$p(i) \quad \exists \alpha \in T_c(\mu(M_\Gamma)), v \notin M_\alpha \text{ and } |M_{\Gamma, \sim(\Gamma), \alpha}| - |\mu(M_\Gamma)| \leq i.$$

Obviously, $p(|M_{\Gamma, \sim(\Gamma), \delta}|)$ holds (take δ).

Let $i \in \mathbb{Z}$ with $i \leq |M_{\Gamma, \sim(\Gamma), \delta}|$ and suppose $p(i)$ holds. We show $p(i-1)$.

We have $\exists \alpha \in T_c(\mu(M_\Gamma)), v \notin M_\alpha$ and $|M_{\Gamma, \sim(\Gamma), \alpha}| - |\mu(M_\Gamma)| \leq i$.

Case 1: $T_d(M_{\Gamma, \sim(\Gamma), \alpha}) \subseteq \sim(\Gamma)$.

As $\alpha \in T_c(\mu(M_\Gamma))$ and (A3) holds, we get $\neg\alpha \in T_c(\mu(M_\Gamma))$.

But, $T_c(\mu(M_\Gamma)) \cap T_d(\mu(M_\Gamma)) = \emptyset$. Thus, neither α nor $\neg\alpha$ belongs to $T_d(\mu(M_\Gamma)) = \sim(\Gamma)$.

Consequently, $\alpha \in G(\Gamma)$. Thus, $v \in M_\alpha$, which is impossible.

Case 2: $\exists \beta \in T_d(M_{\Gamma, \sim(\Gamma), \alpha}), \beta \notin \sim(\Gamma)$.

By (0), $\mu(M_\Gamma) \subseteq M_{\Gamma, \sim(\Gamma)}$. On the other hand, $\mu(M_\Gamma) \subseteq M_\alpha$. Thus, $\mu(M_\Gamma) \subseteq M_{\Gamma, \sim(\Gamma), \alpha} \subseteq M_\beta$.

But, $\beta \notin \sim(\Gamma) = T_d(\mu(M_\Gamma))$. Therefore, $\mu(M_\Gamma) \subseteq M_{-\beta}$.

Consequently, $\mu(M_\Gamma) \subseteq M_\alpha \cap M_{-\beta} = M_{\alpha \wedge \neg\beta}$ and $\mu(M_\Gamma) \subseteq M_{-\alpha} \subseteq M_{\neg(\alpha \wedge \neg\beta)}$.

Therefore, $\alpha \wedge \neg\beta \in T_c(\mu(M_\Gamma))$.

Moreover, $v \notin M_\alpha \supseteq M_{\alpha \wedge \neg\beta}$.

In addition, $M_{\Gamma, \sim(\Gamma), \alpha \wedge \neg\beta} \subseteq M_{\Gamma, \sim(\Gamma), \alpha}$, whilst $M_{\Gamma, \sim(\Gamma), \alpha} \not\subseteq M_{-\beta} \supseteq M_{\Gamma, \sim(\Gamma), \alpha \wedge \neg\beta}$.

Thus, $|M_{\Gamma, \sim(\Gamma), \alpha \wedge \neg\beta}| \leq |M_{\Gamma, \sim(\Gamma), \alpha}| - 1$. Thus, $|M_{\Gamma, \sim(\Gamma), \alpha \wedge \neg\beta}| - |\mu(M_\Gamma)| \leq i - 1$.

Therefore, $p(i-1)$ holds (take $\alpha \wedge \neg\beta$).

Proof of (8). Suppose $M'_\Gamma = \emptyset$.

Now, suppose the contrary of what we want to show, i.e. $\exists v \in M_{\Gamma, \sim(\Gamma)}, v \notin M_{G(\Gamma)}$.

Then, $\exists \alpha \in G(\Gamma), v \notin M_\alpha$.

Case 1: $\alpha \in T(M_{\Gamma, \sim(\Gamma)})$.

As $\alpha \in G(\Gamma), \alpha \notin \sim(\Gamma)$. Therefore, by Lemma 44 (5), $\alpha \notin T_d(M_{\Gamma, \sim(\Gamma)})$.

Thus, $\alpha \in T_c(M_{\Gamma, \sim(\Gamma)})$. Therefore, $M_{\Gamma, \sim(\Gamma)} \subseteq M_\alpha$. Consequently, $v \in M_\alpha$, which is impossible.

Case 2: $\neg\alpha \in T(M_{\Gamma, \sim(\Gamma)})$.

As $\alpha \in G(\Gamma), \neg\alpha \notin \sim(\Gamma)$. Therefore, by Lemma 44 (5), $\neg\alpha \notin T_d(M_{\Gamma, \sim(\Gamma)})$.

Thus, $\neg\alpha \in T_c(M_{\Gamma, \sim(\Gamma)})$. Therefore, by (A3), $M_{\Gamma, \sim(\Gamma)} \subseteq M_{\neg\alpha} = M_\alpha$.

Consequently, $v \in M_\alpha$, which is impossible.

Case 3: $\alpha \notin T(M_{\Gamma, \sim(\Gamma)})$ and $\neg\alpha \notin T(M_{\Gamma, \sim(\Gamma)})$.

Then, by (A2), $M_{\Gamma, \sim(\Gamma), \alpha} \not\subseteq M_{-\alpha}$. Thus, $\alpha \in T_d(M_{\Gamma, \sim(\Gamma), \alpha})$. But, $\alpha \in G(\Gamma)$. Thus, $\alpha \notin \sim(\Gamma)$.

Therefore, $T_d(M_{\Gamma, \sim(\Gamma), \alpha}) \not\subseteq \sim(\Gamma)$. Consequently, $\alpha \notin G(\Gamma)$, which is impossible.

Proof of (9). Case 1: $M'_\Gamma = \emptyset$.

By Lemma 44 (6), $M_{\Gamma, \sim(\Gamma), H(\Gamma)} = M_{\Gamma, \sim(\Gamma), F(\Gamma)} = M_{\Gamma, \sim(\Gamma)}$.

But, by (8), (7), and (5), $M_{\Gamma, \sim(\Gamma)} = M_{\Gamma, \sim(\Gamma), G(\Gamma)} = M_{\Gamma, \sim(\Gamma), T_c(\mu(M_\Gamma))} = \mu(M_\Gamma)$.
Case 2: $M'_\Gamma \neq \emptyset$. Obvious by (6).

Proof of (10).

Case 1: $M'_\Gamma = \emptyset$.

Case 1.1: $\exists v \in M_{\Gamma, \sim(\Gamma)}, v \notin M_{T_c(\mu(M_\Gamma))}$.

Case 1.1.1: Γ is not consistent.

Then, $\exists \alpha \in T_c(\mu(M_\Gamma)), v \notin M_\alpha$ and, as Γ is not consistent, $\exists \beta \in \mathcal{F}, M_\Gamma \subseteq M_\beta$ and $M_\Gamma \subseteq M_{\neg\beta}$.

We have $M_{\Gamma, \sim(\Gamma)} \subseteq M_\Gamma \subseteq M_\beta \subseteq M_{\beta \vee \neg\alpha}$.

Moreover, $\mu(M_\Gamma) \subseteq M_\Gamma \subseteq M_{\neg\beta}$. Thus, $\mu(M_\Gamma) \subseteq M_{\neg\beta} \cap M_\alpha = M_{\neg(\beta \vee \neg\alpha)}$.

Therefore, $\beta \vee \neg\alpha \notin T_d(\mu(M_\Gamma)) = \sim(\Gamma)$.

In addition, $v \notin M_\alpha \supseteq M_{\neg(\beta \vee \neg\alpha)}$.

Consequently, $v \in M_\Gamma^1$ (take $\beta \vee \neg\alpha$ for the β of the definition of M_Γ^1).

Thus, $v \in M'_\Gamma$, which is impossible.

Case 1.1.2: Γ is consistent.

Thus, $M_\Gamma \in \mathbf{C}$. Therefore, as μ is coherency preserving, $\mu(M_\Gamma) \in \mathbf{C}$. Thus, $T_c(\mu(M_\Gamma)) = \emptyset$.

Therefore, $M_{T_c(\mu(M_\Gamma))} = \mathcal{V}$. Thus, $v \in M_{T_c(\mu(M_\Gamma))}$, which is impossible.

Case 1.2: $M_{\Gamma, \sim(\Gamma)} \subseteq M_{T_c(\mu(M_\Gamma))}$.

Then, by Lemma 44 (6), $M_{\Gamma, \sim(\Gamma), H(\Gamma)} = M_{\Gamma, \sim(\Gamma), F(\Gamma)} = M_{\Gamma, \sim(\Gamma)} = M_{\Gamma, \sim(\Gamma), T_c(\mu(M_\Gamma))}$.

Therefore, by (5), $M_{\Gamma, \sim(\Gamma), H(\Gamma)} = \mu(M_\Gamma)$.

Case 2: $M'_\Gamma \neq \emptyset$. Obvious by (6). ■

Now comes the proof of **Proposition 41** (which is stated at the beginning of Section 3.3).

Proof *Proof of (0).* Direction: “ \rightarrow ”.

There exists a CP DP coherent choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that

$\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T_d(\mu(M_\Gamma))$.

We will show:

(0.0) \sim satisfies (~ 0).

By Lemma 45 (1), (2), and (3), \sim satisfies (~ 6), (~ 7), and (~ 8).

By Lemma 45 (10) and Coherence of μ , \sim satisfies (~ 9).

We will show:

(0.1) \sim satisfies (~ 11).

Direction: “ \leftarrow ”.

Suppose \sim satisfies (~ 0), (~ 6), (~ 7), (~ 8), (~ 9), and (~ 11).

Then, let μ be the function from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}, \mu(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$.

We will show:

(0.2) μ is well-defined.

Clearly, μ is a DP choice function.

In addition, as \sim satisfies (~ 9), μ is coherent.

We will show:

(0.3) μ is CP.

Finally, by Lemma 44 (7), $\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T_d(\mu(M_\Gamma))$.

Proof of (0.0). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $\vdash(\Gamma) = \vdash(\Delta)$. Then, $M_\Gamma = M_\Delta$.

Therefore, $\sim(\Gamma) = T_d(\mu(M_\Gamma)) = T_d(\mu(M_\Delta)) = \sim(\Delta)$.

Proof of (0.1). Let $\Gamma \subseteq \mathcal{F}$ and suppose Γ is consistent.

Then, $M_\Gamma \in \mathbf{D} \cap \mathbf{C}$. Thus, as μ is CP, $\mu(M_\Gamma) \in \mathbf{C}$. Therefore, $T_d(\mu(M_\Gamma)) = T(\mu(M_\Gamma))$.

Consequently, $\Gamma \subseteq T(M_\Gamma) \subseteq T(\mu(M_\Gamma)) = T_d(\mu(M_\Gamma)) = \vdash(\Gamma)$.

In addition, $M_{\vdash(\Gamma)} = M_{T_d(\mu(M_\Gamma))} = M_{T(\mu(M_\Gamma))}$. But, $\mu(M_\Gamma) \in \mathbf{C}$. Thus, $M_{T(\mu(M_\Gamma))} \in \mathbf{C}$.

Consequently, $\vdash(\Gamma)$ is consistent.

Finally, $\vdash(\Gamma) = T_d(\mu(M_\Gamma)) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(M_{\vdash(\Gamma)}) = \vdash(\vdash(\Gamma))$.

Proof of (0.2). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $M_\Gamma = M_\Delta$.

Then, $\vdash(\Gamma) = \vdash(\Delta)$. Thus, by (~ 0), $\vdash(\Gamma) = \vdash(\Delta)$.

Consequently, $H(\Gamma) = H(\Delta)$. Therefore, $M_{\Gamma, \vdash(\Gamma), H(\Gamma)} = M_{\Delta, \vdash(\Delta), H(\Delta)}$.

Proof of (0.3). Suppose $V \in \mathbf{D} \cap \mathbf{C}$. Then, $\exists \Gamma \subseteq \mathcal{F}, V = M_\Gamma$.

Case 1: $H_1(\Gamma) \neq \emptyset$.

Thus, $\exists \beta \in \mathcal{F}, \beta \notin \vdash(\Gamma)$ and $M_{\Gamma, \vdash(\Gamma)} \subseteq M_\beta$.

By (~ 11), $\Gamma \subseteq \vdash(\Gamma)$ and $\vdash(\vdash(\Gamma)) = \vdash(\Gamma)$. Thus, $M_{\Gamma, \vdash(\Gamma)} = M_{\vdash(\Gamma)}$. Thus, $M_{\vdash(\Gamma)} \subseteq M_\beta$.

Therefore, $\beta \in T(M_{\vdash(\Gamma)}) = \vdash(\vdash(\Gamma)) = \vdash(\Gamma)$, which is impossible.

Case 2: $H_1(\Gamma) = \emptyset$.

Then, $H(\Gamma) = \emptyset$. Thus, $\mu(V) = \mu(M_\Gamma) = M_{\Gamma, \vdash(\Gamma), H(\Gamma)} = M_{\vdash(\Gamma)}$.

But, by (~ 11), $\vdash(\Gamma)$ is consistent. Therefore, $M_{\vdash(\Gamma)} \in \mathbf{C}$.

Proof of (1). Direction: “ \rightarrow ”.

Verbatim the proof of (0), except that in addition we have μ is LM.

Then, by Lemma 45 (10) and LM, \vdash satisfies (~ 10).

Direction: “ \leftarrow ”.

Verbatim the proof of (0), except that in addition we have \vdash satisfies (~ 10).

Then, by definition of μ and (~ 10), μ is LM.

Proof of (2). Direction: “ \rightarrow ”.

Verbatim the proof of (0), except that μ is no longer CP, whilst (A2) now holds.

Note that, in (0), CP was used only to show (~ 11) and (~ 9).

But, (~ 11) is no longer required to hold.

In addition, by Lemma 45 (9) and Coherence of μ , (~ 9) holds.

Direction: “ \leftarrow ”.

Verbatim the proof of (0), except that (~ 11) does no longer hold, whilst (A2) now holds.

However, in (0), (~ 11) was used only to show that μ is CP, which is no longer required.

Note that we do not need to use (A2) in this direction.

Proof of (3). Direction “ \rightarrow ”.

Verbatim the proof of (0), except that μ is no longer CP, whilst μ is now LM and (A2) now holds.

Note that, in (0), CP was used only to show (~ 11) and (~ 9).

But, (~ 11) is no longer required.

In addition, by Lemma 45 (9) and Coherence of μ , (~ 9) holds.

Similarly, by Lemma 45 (9) and Local Monotonicity of μ , (~ 10) holds.

Direction: “ \leftarrow ”.

Verbatim the proof of (0), except that (~ 11) does no longer hold, whilst (~ 10) and (A2) now holds. Note that, in (0), (~ 11) was used only to show that μ is CP, which is no longer required. Now, by definition of μ and by (~ 10), μ is LM. Note that we do not need to use (A2) in this direction. ■

3.4 The discriminative and not necessarily DP case

Unlike in Section 3.3, the conditions of this section will not be purely syntactic. The translation of properties like Coherence in syntactic terms is blocked because we do no longer have the following useful equality: $\mu(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$, which holds when the choice functions under consideration are definability preserving (but this is not the case here). Thanks to Lemmas 34 and 35 (stated in Section 3.2), we will provide a solution with “semi-syntactic” conditions.

Notation 46 Let \mathcal{L} be a language, \neg a unary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

Then, consider the following condition: $\forall \Gamma \subseteq \mathcal{F}$,

$$(\sim 12) \quad \vdash(\Gamma, \sim(\Gamma), H(\Gamma)) = T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta \subseteq M_\Gamma, \text{ then } v \in M_{\sim(\Delta), H(\Delta)}\}).$$

Proposition 47 Let \mathcal{L} be a language, \neg a unary connective of \mathcal{L} , \vee and \wedge binary connectives of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A1) and (A3), and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then,

(0) \sim is a CP preferential-discriminative consequence relation iff (~ 0), (~ 6), (~ 7), (~ 8), (~ 11) and (~ 12) hold.

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies (A2) too. Then,

(1) \sim is a preferential-discriminative consequence relation iff (~ 0), (~ 6), (~ 7), (~ 8), and (~ 12) hold.

Proof *Proof of (1).* Direction: “ \rightarrow ”.

There exists a coherent choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\sim(\Gamma) = T_d(\mu(M_\Gamma))$. Then, \sim satisfies obviously (~ 0).

Let f be the function from \mathbf{D} to \mathbf{D} such that $\forall V \in \mathbf{D}$, $f(V) = M_{T(\mu(V))}$.

Then, by Lemma 35, $\forall V \in \mathbf{D}$, $f(V) = M_{T(\mu_f(V))}$.

Moreover, $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{T(\mu(M_\Gamma))} \subseteq M_{T(M_\Gamma)} = M_\Gamma$.

Therefore, f is a choice function.

Obviously, f is DP.

In addition, $\forall \Gamma \subseteq \mathcal{F}$, $\sim(\Gamma) = T_d(\mu(M_\Gamma)) = T_d(M_{T(\mu(M_\Gamma))}) = T_d(f(M_\Gamma))$.

Consequently, by Lemma 45 (1), (2), and (3), \sim satisfies (~ 6), (~ 7), and (~ 8).

In addition, by Lemma 45 (9), $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$.

We show that \sim satisfies (~ 12). Let $\Gamma \subseteq \mathcal{F}$.

$$\begin{aligned} \text{Then, } \vdash(\Gamma, \sim(\Gamma), H(\Gamma)) &= T(M_{\Gamma, \sim(\Gamma), H(\Gamma)}) = T(f(M_\Gamma)) = T(M_{T(\mu_f(M_\Gamma))}) = T(\mu_f(M_\Gamma)) = \\ &= T(\{v \in M_\Gamma : \forall W \in \mathbf{D}, \text{ if } v \in W \subseteq M_\Gamma, \text{ then } v \in f(W)\}) = \\ &= T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta \subseteq M_\Gamma, \text{ then } v \in f(M_\Delta)\}) = \end{aligned}$$

$T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta \subseteq M_\Gamma, \text{ then } v \in M_{\Delta, \sim(\Delta), H(\Delta)}\}) =$
 $T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta \subseteq M_\Gamma, \text{ then } v \in M_{\sim(\Delta), H(\Delta)}\}).$

Direction: “ \leftarrow ”.

Suppose (~ 0) , (~ 6) , (~ 7) , (~ 8) , and (~ 12) hold.

Let f be the function from \mathbf{D} to \mathbf{D} such that $\forall \Gamma \subseteq \mathcal{F}, f(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$.

By (~ 0) , f is well-defined.

By Lemma 44 (7), $\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T_d(M_{\Gamma, \sim(\Gamma), H(\Gamma)})$.

Therefore, $\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T_d(f(M_\Gamma))$.

By (~ 12) , $\forall \Gamma \subseteq \mathcal{F}, f(M_\Gamma) = M_{T(\mu_f(M_\Gamma))}$.

Therefore, $\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T_d(f(M_\Gamma)) = T_d(M_{T(\mu_f(M_\Gamma))}) = T_d(\mu_f(M_\Gamma))$.

But, by Lemma 34, μ_f is a coherent choice function.

Proof of (0). Direction: “ \rightarrow ”.

Verbatim the proof of (1), except that (A2) does no longer hold, whilst μ is now CP.

Note that (A2) was used only to apply Lemma 45 (9) to get $\forall \Gamma \subseteq \mathcal{F}, f(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$.

But, we will get this equality by another mean.

Indeed, if $V \in \mathbf{D} \cap \mathbf{C}$, then, as μ is CP, $\mu(V) \in \mathbf{C}$, thus $M_{T(\mu(V))} \in \mathbf{C}$, thus $f(V) \in \mathbf{C}$.

Therefore, f is CP.

Consequently, by Lemma 45 (10), we get $\forall \Gamma \subseteq \mathcal{F}, f(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$.

In addition, by verbatim the proof of (0.1) of Proposition 41, \sim satisfies (~ 11) .

Direction: “ \leftarrow ”.

Verbatim the proof of (1), except that (A2) does no longer hold, whilst \sim now satisfies (~ 11) .

But, in this direction, (A2) was not used in (0).

It remains to show that μ_f is CP.

By verbatim the proof of (0.3) of Proposition 41, we get that f is CP.

Let $V \in \mathbf{D} \cap \mathbf{C}$. Then, $f(V) \in \mathbf{C}$. Thus, $M_{T(\mu_f(V))} \in \mathbf{C}$. Thus, $\mu_f(V) \in \mathbf{C}$ and we are done. ■

Chapter 4

Characterizations of pivotal consequence relations

We will provide in this chapter characterizations for several families of pivotal and pivotal-discriminative consequence relations. These results have been published in [BN05a].

Remark 48 The work of characterization in the present chapter is very similar to the one of Chapter 3. Sometimes, the proofs will be almost verbatim the same. Note that Chapter 3 is about coherent choice functions, whilst the present chapter is about strongly coherent choice functions. Beyond the characterizations, a contribution of the present chapter is to give an example of how the techniques developed in Chapter 3 (in particular in the discriminative case) can be adapted to new properties (here Strong Coherence in the place of Coherence).

Sometimes, we will need to make some assumptions (defined in Section 2.1.1) about the semantic structure under consideration. However, no assumption will be needed for the two following families:

- the pivotal consequence relations (Section 4.2);
- the DP pivotal consequence relations (Section 4.1).

We will assume (A0) for:

- the UC pivotal consequence relations (Section 4.2).

We will need (A3) and (A1) for:

- the CP pivotal-discriminative consequence relations (Section 4.4);
- the CP DP pivotal-discriminative consequence relations (Section 4.3).

We will assume (A3), (A1), and (A2) for:

- the pivotal-discriminative consequence relations (Section 4.4);
- the DP pivotal-discriminative consequence relations (Section 4.3).

We will assume (A0), (A3), and (A1) for:

- the CP UC pivotal-discriminative consequence relations (Section 4.4).

We will need (A0), (A3), (A1), and (A2) for:

- the UC pivotal-discriminative consequence relations (Section 4.4).

4.1 The non-discriminative and DP case

In the present section, we provide a characterization for the family of all DP pivotal consequence relations. We will use similar techniques as those in Section 3.1. First, we will establish the equality: $\mu(M_\Gamma) = M_{\sim(\Gamma)}$. Then, thanks to it, properties like Strong Coherence will be translated in syntactic terms.

Definition 49 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim be a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then, consider the following condition: $\forall \Gamma, \Delta \subseteq \mathcal{F}$,

$$(\sim 13) \quad \sim(\Gamma) \subseteq \vdash(\sim(\Delta), \Gamma).$$

Note that this condition is purely syntactic when there is a proof system available for \vdash .

Proposition 50 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim be a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then, \sim is an DP pivotal consequence relation iff \sim satisfies (~ 0) , (~ 1) , (~ 2) , and (~ 13) .

Proof Direction: “ \rightarrow ”.

There exists an DP SC choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\sim(\Gamma) = T(\mu(M_\Gamma))$.

We will show:

- (0) $\forall \Gamma \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_{\sim(\Gamma)}$;
- (1) \sim satisfies (~ 0) ;
- (2) \sim satisfies (~ 1) ;
- (3) \sim satisfies (~ 2) ;
- (4) \sim satisfies (~ 13) .

Direction: “ \leftarrow ”.

Suppose \sim satisfies (~ 0) , (~ 1) , (~ 2) , and (~ 13) .

Let μ be the function from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_{\sim(\Gamma)}$.

We will show:

- (5) μ is well-defined;
- (6) μ is a DP choice function;
- (7) μ is SC;
- (8) $\forall \Gamma \subseteq \mathcal{F}$, $\sim(\Gamma) = T(\mu(M_\Gamma))$.

Proof of (0). Let $\Gamma \subseteq \mathcal{F}$. As μ is DP, $\mu(M_\Gamma) \in \mathbf{D}$. Thus, $\exists \Delta \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_\Delta$.

Therefore, $\mu(M_\Gamma) = M_\Delta = M_{T(M_\Delta)} = M_{T(\mu(M_\Gamma))} = M_{\sim(\Gamma)}$.

Proof of (1). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $\vdash(\Gamma) = \vdash(\Delta)$.

Then, $M_\Gamma = M_\Delta$. Thus, $\sim(\Gamma) = T(\mu(M_\Gamma)) = T(\mu(M_\Delta)) = \sim(\Delta)$.

Proof of (2). Let $\Gamma \subseteq \mathcal{F}$. Then, $\sim(\Gamma) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(M_{\sim(\Gamma)}) = \vdash(\sim(\Gamma))$.

Proof of (3). Let $\Gamma \subseteq \mathcal{F}$. Then, $\Gamma \subseteq T(M_\Gamma) \subseteq T(\mu(M_\Gamma)) = \vdash(\Gamma)$.

Proof of (4). Let $\Gamma, \Delta \subseteq \mathcal{F}$. Then, by (0) and SC,

$$M_{\vdash(\Delta), \Gamma} = M_{\vdash(\Delta)} \cap M_\Gamma = \mu(M_\Delta) \cap M_\Gamma \subseteq \mu(M_\Gamma) = M_{\vdash(\Gamma)}.$$

Therefore, by ($\vdash 1$), we get $\vdash(\Gamma) = \vdash(\vdash(\Gamma)) = T(M_{\vdash(\Gamma)}) \subseteq T(M_{\vdash(\Delta), \Gamma}) = \vdash(\vdash(\Delta), \Gamma)$.

Proof of (5). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $M_\Gamma = M_\Delta$.

Then, $\vdash(\Gamma) = \vdash(\Delta)$. Thus, by ($\vdash 0$), $M_{\vdash(\Gamma)} = M_{\vdash(\Delta)}$.

Proof of (6). Let $\Gamma \subseteq \mathcal{F}$. Then, by ($\vdash 2$), $\mu(M_\Gamma) = M_{\vdash(\Gamma)} \subseteq M_\Gamma$.

Consequently, μ is a choice function. In addition, μ is obviously DP.

Proof of (7). Let $\Gamma, \Delta \subseteq \mathcal{F}$.

Then, by ($\vdash 13$), we get $\mu(M_\Delta) \cap M_\Gamma = M_{\vdash(\Delta)} \cap M_\Gamma = M_{\vdash(\Delta), \Gamma} \subseteq M_{\vdash(\Gamma)} = \mu(M_\Gamma)$.

Proof of (8). Let $\Gamma \subseteq \mathcal{F}$. Then, by ($\vdash 1$), $\vdash(\Gamma) = \vdash(\vdash(\Gamma)) = T(M_{\vdash(\Gamma)}) = T(\mu(M_\Gamma))$. ■

4.2 The non-discriminative and not necessarily DP case

In the present section, we will investigate in particular the family of all pivotal consequence relations. Unlike in Section 4.1, the choice functions considered here are not necessarily definability preserving. As a consequence, again, we will no longer have the equality: $\mu(M_\Gamma) = M_{\vdash(\Gamma)}$. Therefore, again, we cannot translate properties like Strong Coherence in syntactic terms. Moreover, we will put in evidence in Chapter 5 some limits of what can be done in this area. Approximatively, we will show, in an infinite classical framework, that there does not exist a characterization (of the family mentioned) containing only conditions universally quantified and of limited size.

We provide a solution with semi-syntactic conditions. We will develop similar techniques as those of Section 3.2. Technically, the idea begins by building from any function f , a SC choice function ν_f such that whenever f “covers” some SC choice function, it necessarily covers ν_f .

Definition 51 Let \mathcal{V} be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$ and f a function from \mathbf{V} to \mathbf{W} .

We denote by ν_f the function from \mathbf{V} to $\mathcal{P}(\mathcal{V})$ such that $\forall V \in \mathbf{V}$,

$$\nu_f(V) = \{v \in V : \forall W \in \mathbf{V}, \text{ if } v \in W, \text{ then } v \in f(W)\}.$$

Lemma 52 Let \mathcal{V} be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$ and f a function from \mathbf{V} to \mathbf{W} .

Then, ν_f is a SC choice function.

Proof ν_f is obviously a choice function. We show that it satisfies Strong Coherence.

Suppose the contrary, i.e. $\exists V, W \in \mathbf{V}$ and $\exists v \in \nu_f(W) \cap V$ such that $v \notin \nu_f(V)$.

Then, as $v \in V$ and $v \notin \nu_f(V)$, we have $\exists Z \in \mathbf{V}$, $v \in Z$, and $v \notin f(Z)$.

Therefore, simply by definition of ν_f , $v \notin \nu_f(W)$, which is impossible. ■

Lemma 53 Let \mathcal{V} be a set, \mathbf{V} , \mathbf{W} , and \mathbf{X} subsets of $\mathcal{P}(\mathcal{V})$, f a function from \mathbf{V} to \mathbf{W} , and μ a SC choice function from \mathbf{V} to \mathbf{X} such that $\forall V \in \mathbf{V}$, $f(V) = M_{T(\mu(V))}$. Then:

(0) $\forall V \in \mathbf{V}, f(V) = M_{T(\nu_f(V))}$.

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ is a semantic structure satisfying (A0), $\mathbf{D} \subseteq \mathbf{V}$, and μ is UC. Then:

(1) $\nu_f(\mathcal{V}) = \mu(\mathcal{V})$.

Proof *Proof of (0).* Suppose $V \in \mathbf{V}$. We show $f(V) = M_{T(\nu_f(V))}$.

Case 1: $\exists v \in \mu(V), v \notin \nu_f(V)$.

As $\mu(V) \subseteq V$, we have $v \in V$.

Thus, by definition of $\nu_f, \exists W \in \mathbf{V}, v \in W$ and $v \notin f(W) = M_{T(\mu(W))} \supseteq \mu(W)$.

On the other hand, as μ is SC, $\mu(V) \cap W \subseteq \mu(W)$. Thus, $v \in \mu(W)$, which is impossible.

Case 2: $\mu(V) \subseteq \nu_f(V)$.

Case 2.1: $\exists v \in \nu_f(V), v \notin f(V)$.

Then, $\exists W \in \mathbf{V}, v \in W$, and $v \notin f(W)$. Indeed, just take V itself for the choice of W .

Therefore, by definition of $\nu_f, v \notin \nu_f(V)$, which is impossible.

Case 2.2: $\nu_f(V) \subseteq f(V)$.

Then, $f(V) = M_{T(\mu(V))} \subseteq M_{T(\nu_f(V))} \subseteq M_{T(f(V))} = M_{T(M_{T(\mu(V))})} = M_{T(\mu(V))} = f(V)$.

Proof of (1). Direction: “ \subseteq ”.

Suppose the contrary, i.e. $\exists v \in \nu_f(\mathcal{V}), v \notin \mu(\mathcal{V})$.

Then, $v \in \mathcal{V} \setminus \mu(\mathcal{V})$. But, as μ is UC, $\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D} \subseteq \mathbf{V}$.

On the other hand, as $v \in \nu_f(\mathcal{V})$, we get $\forall W \in \mathbf{V}$, if $v \in W$, then $v \in f(W)$.

Therefore, $v \in f(\mathcal{V} \setminus \mu(\mathcal{V})) = M_{T(\mu(\mathcal{V} \setminus \mu(\mathcal{V})))}$.

But, we will show:

(1.0) $\mu(\mathcal{V} \setminus \mu(\mathcal{V})) = \emptyset$.

Therefore, $M_{T(\mu(\mathcal{V} \setminus \mu(\mathcal{V})))} = M_{T(\emptyset)} = M_{\mathcal{F}}$.

But, by (A0), $M_{\mathcal{F}} = \emptyset$. Therefore, $v \in \emptyset$, which is impossible.

Direction: “ \supseteq ”.

Suppose the contrary, i.e. $\exists v \in \mu(\mathcal{V}), v \notin \nu_f(\mathcal{V})$.

As $v \in \mathcal{V}$ and $v \notin \nu_f(\mathcal{V})$, we get $\exists W \in \mathbf{V}, v \in W$ and $v \notin f(W) = M_{T(\mu(W))} \supseteq \mu(W)$.

But, as μ is SC, $\mu(\mathcal{V}) \cap W \subseteq \mu(W)$. Therefore, $v \in \mu(W)$, which is impossible.

Proof of (1.0). Suppose the contrary, i.e. $\exists v \in \mu(\mathcal{V} \setminus \mu(\mathcal{V}))$.

As μ is SC, $\mu(\mathcal{V} \setminus \mu(\mathcal{V})) \cap \mathcal{V} \subseteq \mu(\mathcal{V})$. Thus, $v \in \mu(\mathcal{V})$. Therefore, $v \notin \mathcal{V} \setminus \mu(\mathcal{V})$.

But, $\mu(\mathcal{V} \setminus \mu(\mathcal{V})) \subseteq \mathcal{V} \setminus \mu(\mathcal{V})$. Thus, $v \notin \mu(\mathcal{V} \setminus \mu(\mathcal{V}))$, which is impossible. ■

Definition 54 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim be a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

Then, consider the following conditions: $\forall \Gamma \subseteq \mathcal{F}$,

(\sim 14) $\sim(\Gamma) = T(\{v \in M_{\Gamma} : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_{\Delta}, \text{ then } v \in M_{\sim(\Delta)}\})$;

(\sim 15) $\mathcal{V} \setminus \{v \in \mathcal{V} : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_{\Delta}, \text{ then } v \in M_{\sim(\Delta)}\} \in \mathbf{D}$.

Proposition 55 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then:

(0) \sim is a pivotal consequence relation iff \sim satisfies (\sim 14).

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies (A0). Then:

(1) \sim is a UC pivotal consequence relation iff \sim satisfies (\sim 14) and (\sim 15).

Proof *Proof of (0).* Direction: “ \rightarrow ”

There exists a SC choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\sim(\Gamma) = T(\mu(M_\Gamma))$.

Let f be the function from \mathbf{D} to \mathbf{D} such that $\forall V \in \mathbf{D}$, we have $f(V) = M_{T(\mu(V))}$.

By Lemma 53, $\forall V \in \mathbf{D}$, we have $f(V) = M_{T(\nu_f(V))}$.

Note that $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{T(\mu(M_\Gamma))} = M_{\sim(\Gamma)}$.

We show that (\sim 14) holds. Let $\Gamma \subseteq \mathcal{F}$.

Then, $\sim(\Gamma) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(f(M_\Gamma)) = T(M_{T(\nu_f(M_\Gamma))}) = T(\nu_f(M_\Gamma)) =$

$T(\{v \in M_\Gamma : \forall W \in \mathbf{D}, \text{ if } v \in W, \text{ then } v \in f(W)\}) =$

$T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in f(M_\Delta)\}) =$

$T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in M_{\sim(\Delta)}\})$.

Direction: “ \leftarrow ”.

Suppose \sim satisfies (\sim 14).

Let f be the function from \mathbf{D} to \mathbf{D} such that $\forall \Gamma \subseteq \mathcal{F}$, we have $f(M_\Gamma) = M_{\sim(\Gamma)}$.

Note that f is well-defined. Indeed, if $\Gamma, \Delta \subseteq \mathcal{F}$ and $M_\Gamma = M_\Delta$, then, by (\sim 14), $\sim(\Gamma) = \sim(\Delta)$.

In addition, by (\sim 14), we clearly have $\forall \Gamma \subseteq \mathcal{F}$, $\sim(\Gamma) = T(\nu_f(M_\Gamma))$.

Finally, by Lemma 52, ν_f is a SC choice function.

Proof of (1). Direction: “ \rightarrow ”.

Verbatim the proof of (0), except that in addition we have (A0) holds and μ is UC.

We show that \sim satisfies (\sim 15). As μ is UC, $\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D}$. But, by Lemma 53, $\mu(\mathcal{V}) = \nu_f(\mathcal{V}) =$

$\{v \in \mathcal{V} : \forall W \in \mathbf{D}, \text{ if } v \in W, \text{ then } v \in f(W)\} =$

$\{v \in \mathcal{V} : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in f(M_\Delta)\} =$

$\{v \in \mathcal{V} : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in M_{\sim(\Delta)}\}$.

Direction: “ \leftarrow ”.

Verbatim the proof of (0), except that in addition we have (A0) holds and \sim satisfies (\sim 15).

But, because of (\sim 15), $\mathcal{V} \setminus \nu_f(\mathcal{V}) \in \mathbf{D}$. Therefore, ν_f is UC.

Note that in this direction (A0) is not used. \blacksquare

4.3 The discriminative and DP case

In the present section, we will characterize certain families of DP pivotal-discriminative consequence relations. We will apply Lemmas 44 and 45 (stated in Section 3.3).

Definition 56 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

Then, consider the following condition: $\forall \Gamma, \Delta \subseteq \mathcal{F}, \forall \alpha, \beta \in \mathcal{F}$,

(\sim 16) $\sim(\Gamma) \cup H(\Gamma) \subseteq \vdash(\Delta, \sim(\Delta), H(\Delta), \Gamma)$.

Note that this condition is purely syntactic when there is a proof system available for \vdash .

Proposition 57 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee and \wedge binary connectives of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A3) and (A1), and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then:

(0) \vdash is a CP DP pivotal-discriminative consequence relation iff \vdash satisfies (\vdash 0), (\vdash 6), (\vdash 7), (\vdash 8), (\vdash 16), and (\vdash 11).

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies (A2). Then:

(1) \vdash is a DP pivotal-discriminative consequence relation iff \vdash satisfies (\vdash 0), (\vdash 6), (\vdash 7), (\vdash 8) and (\vdash 16).

Proof *Proof of (0).* Direction: “ \rightarrow ”.

There exists a CP DP SC choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that

$\forall \Gamma \subseteq \mathcal{F}, \vdash(\Gamma) = T_d(\mu(M_\Gamma))$.

We will show:

(0.0) \vdash satisfies (\vdash 0).

By Lemma 45 (1), (2), and (3), \vdash satisfies (\vdash 6), (\vdash 7), and (\vdash 8).

By Lemma 45 (10) and Strong Coherence of μ , \vdash satisfies (\vdash 16).

We will show:

(0.1) \vdash satisfies (\vdash 11).

Direction: “ \leftarrow ”.

Suppose \vdash satisfies (\vdash 0), (\vdash 6), (\vdash 7), (\vdash 8), (\vdash 16), and (\vdash 11).

Then, let μ be the function from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}, \mu(M_\Gamma) = M_{\Gamma, \vdash(\Gamma), H(\Gamma)}$.

We will show:

(0.2) μ is well-defined.

Clearly, μ is a DP choice function.

In addition, as \vdash satisfies (\vdash 16), μ is strongly coherent.

We will show:

(0.3) μ is CP.

Finally, by Lemma 44 (7), $\forall \Gamma \subseteq \mathcal{F}, \vdash(\Gamma) = T_d(\mu(M_\Gamma))$.

Proof of (0.0). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $\vdash(\Gamma) = \vdash(\Delta)$. Then, $M_\Gamma = M_\Delta$.

Therefore, $\vdash(\Gamma) = T_d(\mu(M_\Gamma)) = T_d(\mu(M_\Delta)) = \vdash(\Delta)$.

Proof of (0.1). Let $\Gamma \subseteq \mathcal{F}$ and suppose Γ is consistent.

Then, $M_\Gamma \in \mathbf{D} \cap \mathbf{C}$. Thus, as μ is CP, $\mu(M_\Gamma) \in \mathbf{C}$. Therefore, $T_d(\mu(M_\Gamma)) = T(\mu(M_\Gamma))$.

Consequently, $\Gamma \subseteq T(M_\Gamma) \subseteq T(\mu(M_\Gamma)) = T_d(\mu(M_\Gamma)) = \vdash(\Gamma)$.

In addition, $M_{\vdash(\Gamma)} = M_{T_d(\mu(M_\Gamma))} = M_{T(\mu(M_\Gamma))}$. But, $\mu(M_\Gamma) \in \mathbf{C}$. Thus, $M_{T(\mu(M_\Gamma))} \in \mathbf{C}$.

Consequently, $\vdash(\Gamma)$ is consistent.

Finally, $\vdash(\Gamma) = T_d(\mu(M_\Gamma)) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(M_{\vdash(\Gamma)}) = \vdash(\vdash(\Gamma))$.

Proof of (0.2). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $M_\Gamma = M_\Delta$.

Then, $\vdash(\Gamma) = \vdash(\Delta)$. Thus, by (\vdash 0), $\vdash(\Gamma) = \vdash(\Delta)$.

Consequently, $H(\Gamma) = H(\Delta)$. Therefore, $M_{\Gamma, \vdash(\Gamma), H(\Gamma)} = M_{\Delta, \vdash(\Delta), H(\Delta)}$.

Proof of (0.3). Suppose $V \in \mathbf{D} \cap \mathbf{C}$. Then, $\exists \Gamma \subseteq \mathcal{F}, V = M_\Gamma$.

Case 1: $H_1(\Gamma) \neq \emptyset$.

Thus, $\exists \beta \in \mathcal{F}, \beta \notin \vdash(\Gamma)$ and $M_{\Gamma, \vdash(\Gamma)} \subseteq M_\beta$.

By (\vdash 11), $\Gamma \subseteq \vdash(\Gamma)$ and $\vdash(\vdash(\Gamma)) = \vdash(\Gamma)$. Thus, $M_{\Gamma, \vdash(\Gamma)} = M_{\vdash(\Gamma)}$. Thus, $M_{\vdash(\Gamma)} \subseteq M_\beta$.

Therefore, $\beta \in T(M_{\sim(\Gamma)}) = \vdash(\sim(\Gamma)) = \sim(\Gamma)$, which is impossible.

Case 2: $H_1(\Gamma) = \emptyset$.

Then, $H(\Gamma) = \emptyset$. Thus, $\mu(V) = \mu(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)} = M_{\sim(\Gamma)}$.

But, by (~ 11), $\sim(\Gamma)$ is consistent. Therefore, $M_{\sim(\Gamma)} \in \mathbf{C}$.

Proof of (1). Direction: “ \rightarrow ”.

Verbatim the proof of (0), except that μ is no longer CP, whilst (A2) now holds.

Note that, in (0), CP was used only to show (~ 16) and (~ 11).

But, (~ 11) is no longer required to hold and we are going to get (~ 16) by another mean.

Indeed, by Lemma 45 (9) and Strong Coherence of μ , (~ 16) holds.

Direction: “ \leftarrow ”.

Verbatim the proof of (0), except that (~ 11) does no longer hold, whilst (A2) now holds.

However, in (0), (~ 11) was used only to show that μ is CP, which is no longer required.

Note that we do not need to use (A2) in this direction. ■

4.4 The discriminative and not necessarily DP case

Again, as we do not have Definability Preserving, we do not have the useful equality: $\mu(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$. As a consequence, we will provide again characterizations with semi-syntactic conditions, thanks to Lemmas 52 and 53 (stated in Section 4.2).

Definition 58 Let \mathcal{L} be a language, \neg a unary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

Then, consider the following conditions: $\forall \Gamma \subseteq \mathcal{F}$,

(~ 17) $\vdash(\Gamma, \sim(\Gamma), H(\Gamma)) = T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in M_{\sim(\Delta), H(\Delta)}\})$;

(~ 18) $\mathcal{V} \setminus \{v \in \mathcal{V} : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in M_{\sim(\Delta), H(\Delta)}\} \in \mathbf{D}$.

Proposition 59 Suppose \mathcal{L} is a language, \neg a unary connective of \mathcal{L} , \vee and \wedge binary connectives of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A3) and (A1), and \sim a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then:

(0) \sim is a CP pivotal-discriminative consequence relation iff \sim satisfies (~ 0), (~ 6), (~ 7), (~ 8), (~ 11), and (~ 17).

If $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies (A0) too, then:

(1) \sim is a CP UC pivotal-discriminative consequence relation iff \sim satisfies (~ 0), (~ 6), (~ 7), (~ 8), (~ 11), (~ 17), and (~ 18).

If $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies (A2) too, then:

(2) \sim is a pivotal-discriminative consequence relation iff \sim satisfies (~ 0), (~ 6), (~ 7), (~ 8), and (~ 17).

If $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies (A0) and (A2) too, then:

(3) \vdash is a UC pivotal-discriminative consequence relation iff \vdash satisfies ($\vdash 0$), ($\vdash 6$), ($\vdash 7$), ($\vdash 8$), ($\vdash 17$), and ($\vdash 18$).

Proof *Proof of (2).* Direction: “ \rightarrow ”.

There exists a SC choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}$, $\vdash(\Gamma) = T_d(\mu(M_\Gamma))$.

Then, \vdash satisfies obviously ($\vdash 0$).

Let f be the function from \mathbf{D} to \mathbf{D} such that $\forall V \in \mathbf{D}$, $f(V) = M_{T(\mu(V))}$.

Then, by Lemma 53, $\forall V \in \mathbf{D}$, $f(V) = M_{T(\nu_f(V))}$.

Moreover, $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{T(\mu(M_\Gamma))} \subseteq M_{T(M_\Gamma)} = M_\Gamma$.

Therefore, f is a choice function.

Obviously, f is DP.

In addition, $\forall \Gamma \subseteq \mathcal{F}$, $\vdash(\Gamma) = T_d(\mu(M_\Gamma)) = T_d(M_{T(\mu(M_\Gamma))}) = T_d(f(M_\Gamma))$.

Consequently, by Lemma 45 (1), (2), and (3), \vdash satisfies ($\vdash 6$), ($\vdash 7$), and ($\vdash 8$).

In addition, by Lemma 45 (9), $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \vdash(\Gamma), H(\Gamma)}$.

We show that \vdash satisfies ($\vdash 17$). Let $\Gamma \subseteq \mathcal{F}$.

Then, $\vdash(\Gamma, \vdash(\Gamma), H(\Gamma)) = T(M_{\Gamma, \vdash(\Gamma), H(\Gamma)}) = T(f(M_\Gamma)) = T(M_{T(\nu_f(M_\Gamma))}) = T(\nu_f(M_\Gamma)) = T(\{v \in M_\Gamma : \forall W \in \mathbf{D}, \text{ if } v \in W, \text{ then } v \in f(W)\}) = T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in f(M_\Delta)\}) = T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in M_{\Delta, \vdash(\Delta), H(\Delta)}\}) = T(\{v \in M_\Gamma : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in M_{\vdash(\Delta), H(\Delta)}\})$.

Direction: “ \leftarrow ”.

Suppose ($\vdash 0$), ($\vdash 6$), ($\vdash 7$), ($\vdash 8$), and ($\vdash 17$) hold.

Let f be the function from \mathbf{D} to \mathbf{D} such that $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \vdash(\Gamma), H(\Gamma)}$.

By ($\vdash 0$), f is well-defined.

By Lemma 44 (7), $\forall \Gamma \subseteq \mathcal{F}$, $\vdash(\Gamma) = T_d(M_{\Gamma, \vdash(\Gamma), H(\Gamma)}) = T_d(f(M_\Gamma))$.

By ($\vdash 17$), $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{T(\nu_f(M_\Gamma))}$.

Therefore, $\forall \Gamma \subseteq \mathcal{F}$, $\vdash(\Gamma) = T_d(f(M_\Gamma)) = T_d(M_{T(\nu_f(M_\Gamma))}) = T_d(\nu_f(M_\Gamma))$.

But, by Lemma 52, ν_f is a SC choice function.

Proof of (3). Direction: “ \rightarrow ”.

Verbatim the proof of (2), except that in addition we have (A0) holds and μ is UC.

We show that ($\vdash 18$) holds. As μ is UC, $\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D}$. But, by Lemma 53 (1), $\mu(\mathcal{V}) = \nu_f(\mathcal{V}) =$

$\{v \in \mathcal{V} : \forall W \in \mathbf{D}, \text{ if } v \in W, \text{ then } v \in f(W)\} =$

$\{v \in \mathcal{V} : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in f(M_\Delta)\} =$

$\{v \in \mathcal{V} : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in M_{\Delta, \vdash(\Delta), H(\Delta)}\} =$

$\{v \in \mathcal{V} : \forall \Delta \subseteq \mathcal{F}, \text{ if } v \in M_\Delta, \text{ then } v \in M_{\vdash(\Delta), H(\Delta)}\}$.

Direction: “ \leftarrow ”.

Verbatim the proof of (2), except that in addition we have (A0) holds and \vdash satisfies ($\vdash 18$).

But, because of ($\vdash 18$), $\mathcal{V} \setminus \nu_f(\mathcal{V}) \in \mathbf{D}$. Therefore, ν_f is UC.

Note that (A0) is not used in this direction.

Proof of (0). Direction: “ \rightarrow ”.

Verbatim the proof of (2), except that (A2) does no longer hold, whilst μ is now CP.

Note that (A2) was used, in (2), only to apply Lemma 45 (9) to get $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \vdash(\Gamma), H(\Gamma)}$.

But, we will get this equality by another mean.

Indeed, if $V \in \mathbf{D} \cap \mathbf{C}$, then, as μ is CP, $\mu(V) \in \mathbf{C}$, thus $M_{T(\mu(V))} \in \mathbf{C}$, thus $f(V) \in \mathbf{C}$.

Therefore, f is CP.

Consequently, by Lemma 45 (10), we get $\forall \Gamma \subseteq \mathcal{F}, f(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$.

In addition, by verbatim the proof of (0.1) of Proposition 57, \sim satisfies (\sim 11).

Direction: " \leftarrow ".

Verbatim the proof of (2), except that (A2) does no longer hold, whilst \sim satisfies now (\sim 11).

But, in this direction, (A2) was not used in (2).

It remains to show that ν_f is CP.

By verbatim the proof of (0.3) of Proposition 57, we get that f is CP.

Let $V \in \mathbf{D} \cap \mathbf{C}$. Then, $f(V) \in \mathbf{C}$. Thus, $M_{T(\nu_f(V))} \in \mathbf{C}$. Thus, $\nu_f(V) \in \mathbf{C}$ and we are done.

Proof of (1). Direction: " \rightarrow ".

Verbatim the proof of (2), except that (A2) does no longer hold, whilst (A0) now holds and μ is now UC and CP.

Note that (A2) was used, in (2), only to apply Lemma 45 (9) to get $\forall \Gamma \subseteq \mathcal{F}, f(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$.

But, by verbatim the proof of (0), we get anyway $\forall \Gamma \subseteq \mathcal{F}, f(M_\Gamma) = M_{\Gamma, \sim(\Gamma), H(\Gamma)}$.

In addition, by verbatim the proof of (0.1) of Proposition 57, \sim satisfies (\sim 11).

And, by verbatim the proof of (3), \sim satisfies (\sim 18).

Direction: " \leftarrow ".

Verbatim the proof of (2), except that (A2) does no longer hold, whilst (A0) now holds and \sim satisfies now (\sim 11) and (\sim 18).

But, in this direction, (A2) was not used in (2).

In addition, by verbatim the proof of (0), ν_f is CP.

And, because of (\sim 18), $\mathcal{V} \setminus \nu_f(\mathcal{V}) \in \mathbf{D}$. Therefore, ν_f is UC.

Note that (A0) is not used in this direction. ■

Chapter 5

Nonexistence of normal characterizations

5.1 Definition

Let \mathcal{F} be a set, \mathcal{R} a set of relations on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$, and \vdash a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
Approximatively, a characterization of \mathcal{R} will be called “normal” iff it contains only conditions which are universally quantified and “apply” \vdash at most $|\mathcal{F}|$ times. More formally,

Definition 60 Let \mathcal{F} be a set and \mathcal{R} a set of relations on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
We say that \mathcal{C} is a *normal characterization* of \mathcal{R} iff $\mathcal{C} = \langle \lambda, \Phi \rangle$, where $\lambda \leq |\mathcal{F}|$ is a (finite or infinite) cardinal and Φ is a relation on $\mathcal{P}(\mathcal{F})^{2\lambda}$ such that for every relation \vdash on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$,

$$\vdash \in \mathcal{R} \text{ iff } \forall \Gamma_1, \dots, \Gamma_\lambda \subseteq \mathcal{F}, (\Gamma_1, \dots, \Gamma_\lambda, \vdash(\Gamma_1), \dots, \vdash(\Gamma_\lambda)) \in \Phi.$$

Now, suppose there is no normal characterization of \mathcal{R} . Here are examples (i.e. (C1), (C2), and (C3) below) that will give the reader (we hope) a good idea which conditions cannot characterize \mathcal{R} . This will thus make clearer the range of our impossibility result (Proposition 62 below). To begin, consider the following condition:

$$(C1) \quad \forall \Gamma, \Delta \in \mathbf{F} \subseteq \mathcal{P}(\mathcal{F}), \vdash(\Gamma \cup \vdash(\Delta)) = \emptyset.$$

Then, (C1) cannot characterize \mathcal{R} . Indeed, suppose the contrary, i.e.

suppose $\vdash \in \mathcal{R}$ iff $\forall \Gamma, \Delta \in \mathbf{F}, \vdash(\Gamma \cup \vdash(\Delta)) = \emptyset$.

Then, take $\lambda = 3$ and Φ such that $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6) \in \Phi$ iff

$(\Gamma_1, \Gamma_2 \in \mathbf{F} \text{ and } \Gamma_3 = \Gamma_1 \cup \Gamma_5)$ entails $\Gamma_6 = \emptyset$.

Then, $\langle 3, \Phi \rangle$ is a normal characterization of \mathcal{R} . We give the easy proof of this, so that the reader can check that a convenient relation Φ can be found quickly for all simple conditions like (C1).

Proof Direction: “ \rightarrow ”.

Suppose $\vdash \in \mathcal{R}$.

Then, $\forall \Gamma, \Delta \in \mathbf{F}, \vdash(\Gamma \cup \vdash(\Delta)) = \emptyset$.

Let $\Gamma_1, \Gamma_2, \Gamma_3 \subseteq \mathcal{F}$.

We show $(\Gamma_1, \Gamma_2, \Gamma_3, \vdash(\Gamma_1), \vdash(\Gamma_2), \vdash(\Gamma_3)) \in \Phi$.

Suppose $\Gamma_1, \Gamma_2 \in \mathbf{F}$ and $\Gamma_3 = \Gamma_1 \cup \vdash(\Gamma_2)$.

Then, as $\Gamma_1, \Gamma_2 \in \mathbf{F}$, we get $\vdash(\Gamma_1 \cup \vdash(\Gamma_2)) = \emptyset$.

But, $\vdash(\Gamma_1 \cup \vdash(\Gamma_2)) = \vdash(\Gamma_3)$. Therefore, $\vdash(\Gamma_3) = \emptyset$.

Direction: “ \leftarrow ”.

Suppose $\forall \Gamma_1, \Gamma_2, \Gamma_3 \subseteq \mathcal{F}$, $(\Gamma_1, \Gamma_2, \Gamma_3, \vdash(\Gamma_1), \vdash(\Gamma_2), \vdash(\Gamma_3)) \in \Phi$.

We show $\vdash \in \mathcal{R}$. Let $\Gamma, \Delta \in \mathbf{F}$.

Take $\Gamma_1 = \Gamma, \Gamma_2 = \Delta, \Gamma_3 = \Gamma_1 \cup \vdash(\Gamma_2)$.

Then, $\Gamma_1 \in \mathbf{F}, \Gamma_2 \in \mathbf{F}$, and $\Gamma_3 = \Gamma_1 \cup \vdash(\Gamma_2)$.

But, we have $(\Gamma_1, \Gamma_2, \Gamma_3, \vdash(\Gamma_1), \vdash(\Gamma_2), \vdash(\Gamma_3)) \in \Phi$.

Therefore, by definition of Φ , $\vdash(\Gamma_3) = \emptyset$.

But, $\vdash(\Gamma_3) = \vdash(\Gamma_1 \cup \vdash(\Gamma_2)) = \vdash(\Gamma \cup \vdash(\Delta))$. ■

But actually, we are not limited to simple operations (like e.g. \cup, \cap, \setminus). More complex conditions than (C1) are also excluded. For instance, let f be any function from $\mathcal{P}(\mathcal{F})$ to $\mathcal{P}(\mathcal{F})$ and consider the following condition:

$$(C2) \quad \forall \Gamma, \Delta \in \mathbf{F}, \vdash(f(\Gamma) \cup \vdash(\Delta)) = \emptyset.$$

Then, (C2) cannot characterize \mathcal{R} . Indeed, suppose it characterizes \mathcal{R} .

Then, take $\lambda = 3$ and Φ such that $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6) \in \Phi$ iff

$(\Gamma_1, \Gamma_2 \in \mathbf{F}$ and $\Gamma_3 = f(\Gamma_1) \cup \Gamma_5)$ entails $\Gamma_6 = \emptyset$.

It can be checked that $\langle 3, \Phi \rangle$ is a normal characterization of \mathcal{R} . We leave the easy proof to the reader.

We can even go further combining universal (not existential) quantifiers and functions like f . For instance, let \mathcal{G} be a set of functions from $\mathcal{P}(\mathcal{F})$ to $\mathcal{P}(\mathcal{F})$ and consider the following condition:

$$(C3) \quad \forall \Gamma, \Delta \in \mathbf{F}, \forall f \in \mathcal{G}, \vdash(f(\Gamma) \cup \vdash(\Delta)) = \emptyset.$$

Then, (C3) cannot characterize \mathcal{R} . Indeed, suppose it characterizes \mathcal{R} .

Then, take $\lambda = 3$ and Φ such that $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6) \in \Phi$ iff

$\forall f \in \mathcal{G}$, if $(\Gamma_1, \Gamma_2 \in \mathbf{F}$ and $\Gamma_3 = f(\Gamma_1) \cup \Gamma_5)$, then $\Gamma_6 = \emptyset$.

It can be checked that $\langle 3, \Phi \rangle$ is a normal characterization of \mathcal{R} . The easy proof is left to the reader.

Finally, a good example of a condition which is not excluded is (~ 14). We have seen in Proposition 55 that it characterizes the family of all pivotal consequence relations.

5.2 Impossibility results

We will show, in an infinite classical framework, that there is no normal characterization for the family of all pivotal consequence relations (in other words, (~ 14) cannot be replaced by a simpler condition in Proposition 55). This result has been published in [BN05a]. In the same vein, K. Schlechta showed that there does not exist a normal characterization for the family of all preferential consequence relations (Proposition 5.2.15 of [Sch04]).

Note that he used the word “normal” in a more restrictive sense (see Section 1.6.2.1 of [Sch04]). Approximatively, a characterization of \mathcal{R} is called normal by Schlechta iff it contains only conditions like (C1), i.e. conditions which are universally quantified, “apply” \vdash at most $|\mathcal{F}|$ times, and use only elementary operations like e.g. \cup, \cap, \setminus (complex structures or functions, etc. are not allowed). We have been inspired by the techniques of Schlechta. We will need Lemma 5.2.14 of [Sch04]:

Lemma 61 [Sch04] Suppose \mathcal{A} is infinite, $\langle \mathcal{F}_c, \mathcal{V}, \models \rangle$ is a classical propositional semantic structure, and $\mathbf{V} \subseteq \{V \subseteq \mathcal{V} : |V| \leq |\mathcal{A}|\}$ is such that:

if $V \in \mathbf{V}$ and $W \subseteq V$, then $W \in \mathbf{V}$;

$\forall V, W \in \mathbf{V}$, if $|V \cup W| \leq |\mathcal{A}|$, then $V \cup W \in \mathbf{V}$.

Then, $\forall \Gamma \subseteq \mathcal{F}_c, \exists V_\Gamma \in \mathbf{V}$,

$$(0) \quad T(\bigcap_{V \in \mathbf{V}} M_{T(M_\Gamma \setminus V)}) = T(M_\Gamma \setminus V_\Gamma);$$

$$(1) \quad \forall V \in \mathbf{V}, T(M_\Gamma \setminus V) \subseteq T(M_\Gamma \setminus V_\Gamma).$$

Recall that \mathcal{A} and \mathcal{F}_c have been introduced in Section 2.1.2. Note that the subscript in V_Γ is written just to keep in mind that V_Γ depends on Γ .

Proposition 62 Suppose \mathcal{A} is infinite and $\langle \mathcal{F}_c, \mathcal{V}, \models \rangle$ is a classical propositional semantic structure. Then, there doesn't exist a normal characterization for the family of all pivotal consequence relations.

Proof Suppose the contrary, i.e. there exist a cardinal $\lambda \leq |\mathcal{F}_c|$ and a relation Φ on $\mathcal{P}(\mathcal{F}_c)^{2\lambda}$ such that for every relation \sim on $\mathcal{P}(\mathcal{F}_c) \times \mathcal{F}_c$, \sim is a pivotal consequence relation iff $\forall \Gamma_1, \dots, \Gamma_\lambda \subseteq \mathcal{F}_c$, $(\Gamma_1, \dots, \Gamma_\lambda, \sim(\Gamma_1), \dots, \sim(\Gamma_\lambda)) \in \Phi$. Then, define:

$$\mathbf{V} := \{V \subseteq \mathcal{V} : |V| \leq |\mathcal{A}|\}.$$

In addition, let \sim be the relation on $\mathcal{P}(\mathcal{F}_c) \times \mathcal{F}_c$ such that $\forall \Gamma \subseteq \mathcal{F}_c$,

$$\sim(\Gamma) = T(\bigcap_{V \in \mathbf{V}} M_{T(M_\Gamma \setminus V)}).$$

We will show:

$$(0) \quad \forall V \subseteq \mathcal{V}, \text{ if } |V| \leq |\mathcal{A}|, \text{ then } T(\mathcal{V}) = T(\mathcal{V} \setminus V);$$

$$(1) \quad \exists \Gamma_1, \dots, \Gamma_\lambda \subseteq \mathcal{F}_c \text{ such that } (\Gamma_1, \dots, \Gamma_\lambda, \sim(\Gamma_1), \dots, \sim(\Gamma_\lambda)) \notin \Phi.$$

Now, by lemma 61, we get:

$$(2) \quad \forall \Gamma \subseteq \mathcal{F}_c, \exists V_\Gamma \in \mathbf{V}, \sim(\Gamma) = T(M_\Gamma \setminus V_\Gamma) \text{ and } \forall V \in \mathbf{V}, T(M_\Gamma \setminus V) \subseteq T(M_\Gamma \setminus V_\Gamma).$$

Then, define:

$$\mathcal{X} := \bigcup_{\Gamma \in \{\Gamma_1, \dots, \Gamma_\lambda\}} V_\Gamma.$$

Then, we will show:

$$(3) \quad \forall \Gamma \in \{\Gamma_1, \dots, \Gamma_\lambda\}, \sim(\Gamma) = T(M_\Gamma \setminus \mathcal{X}).$$

Let μ be the function from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall V \in \mathbf{D}, \mu(V) = V \setminus \mathcal{X}$.

We will show:

$$(4) \quad \mu \text{ is a SC choice function.}$$

Let \sim' be the pivotal consequence relation defined by μ .

We will show the following, which entails a contradiction:

$$(5) \quad \sim' \text{ is not a pivotal consequence relation.}$$

Proof of (0). Let $V \subseteq \mathcal{V}$ and suppose $|V| \leq |\mathcal{A}|$.

Obviously, $T(\mathcal{V}) \subseteq T(\mathcal{V} \setminus V)$.

We show $T(\mathcal{V} \setminus V) \subseteq T(\mathcal{V})$.

Suppose the contrary, i.e. $\exists \alpha \in T(\mathcal{V} \setminus V), \alpha \notin T(\mathcal{V})$.

Then, $\exists v \in \mathcal{V}, v \notin M_\alpha$.

Now, define:

$$W := \{w \in \mathcal{V} : \text{for all atom } q \text{ occurring in } \alpha, w(q) = v(q)\}.$$

Then, $\forall w \in W$, we have $w(\alpha) = v(\alpha)$ and thus $w \notin M_\alpha$.

As the number of atoms occurring in α is finite and \mathcal{A} is infinite, we get $|W| = 2^{|\mathcal{A}|}$.

Therefore, $|\mathcal{V}| \leq |\mathcal{A}| < |\mathcal{W}|$. Thus, $\exists w \in \mathcal{W} \setminus \mathcal{V} \subseteq \mathcal{V} \setminus \mathcal{V}$.
Thus, $\mathcal{V} \setminus \mathcal{V} \not\subseteq M_\alpha$. Therefore, $\alpha \notin T(\mathcal{V} \setminus \mathcal{V})$, which is impossible.

Proof of (1). It suffices to show that \sim is not a pivotal consequence relation.

Suppose the contrary, i.e. there exists a SC choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}_c, \sim(\Gamma) = T(\mu(M_\Gamma))$.

As \mathcal{A} is infinite, $\exists p \in \mathcal{A}$. We show that all cases are impossible.

Case 1: $\exists v \in \mu(\mathcal{V}), v \notin M_p$.

Let $\Gamma = T(v)$. Then, $M_\Gamma = \{v\}$.

By SC of μ , we have $\mu(M_\Gamma) = \mu(M_\Gamma) \cap \mathcal{V} \subseteq \mu(\mathcal{V})$. Thus, $\mu(M_\Gamma) \subseteq \mu(\mathcal{V}) \cap M_\Gamma$.

On the other hand, again by SC, $\mu(\mathcal{V}) \cap M_\Gamma \subseteq \mu(M_\Gamma)$. Consequently, $\mu(\mathcal{V}) \cap M_\Gamma = \mu(M_\Gamma)$.

Therefore, $\sim(\Gamma) = T(\mu(M_\Gamma)) = T(\mu(\mathcal{V}) \cap M_\Gamma) = T(\mu(\mathcal{V}) \cap \{v\}) = T(v)$.

But, $p \notin T(v)$. Thus, $p \notin \sim(\Gamma)$.

However, $M_\Gamma \in \mathbf{V}$. Therefore, $\bigcap_{V \in \mathbf{V}} M_{T(M_\Gamma \setminus V)} \subseteq M_{T(M_\Gamma \setminus M_\Gamma)} = M_{T(\emptyset)} = M_{\mathcal{F}_c} = \emptyset$.

Therefore, by definition of \sim , we have $\sim(\Gamma) = T(\emptyset) = \mathcal{F}_c$.

Thus, $p \in \sim(\Gamma)$, which is impossible.

Case 2: $\mu(\mathcal{V}) \subseteq M_p$.

Then, by (0), $\sim(\emptyset) = T(\bigcap_{V \in \mathbf{V}} M_{T(\mathcal{V} \setminus V)}) = T(\bigcap_{V \in \mathbf{V}} M_{T(\mathcal{V})}) = T(M_{T(\mathcal{V})}) = T(\mathcal{V})$.

But, $\mathcal{V} \not\subseteq M_p$. Thus, $p \notin T(\mathcal{V}) = \sim(\emptyset)$.

On the other hand, $\sim(\emptyset) = T(\mu(M_\emptyset)) = T(\mu(\mathcal{V}))$.

But, $\mu(\mathcal{V}) \subseteq M_p$. Thus, $p \in T(\mu(\mathcal{V})) = \sim(\emptyset)$, which is impossible.

Proof of (3). Let $\Gamma \in \{\Gamma_1, \dots, \Gamma_\lambda\}$. Direction: “ \subseteq ”.

We have $V_\Gamma \subseteq \mathcal{X}$. Thus, $M_\Gamma \setminus \mathcal{X} \subseteq M_\Gamma \setminus V_\Gamma$.

Therefore, by (2), $\sim(\Gamma) = T(M_\Gamma \setminus V_\Gamma) \subseteq T(M_\Gamma \setminus \mathcal{X})$.

Direction: “ \supseteq ”.

As \mathcal{A} is infinite, $|\mathcal{A}| = |\mathcal{F}_c|$. Therefore, $\lambda \leq |\mathcal{A}|$. Thus, $|\mathcal{X}| \leq |\mathcal{A}|^2 = |\mathcal{A}|$.

Thus, $\mathcal{X} \in \mathbf{V}$. Thus, by (2), $T(M_\Gamma \setminus \mathcal{X}) \subseteq T(M_\Gamma \setminus V_\Gamma) = \sim(\Gamma)$.

Proof of (4). μ is clearly a choice function. We show that μ satisfies SC. Let $V, W \subseteq \mathcal{V}$.

Then, $\mu(W) \cap V = (W \setminus \mathcal{X}) \cap V = (W \cap V) \setminus \mathcal{X} \subseteq V \setminus \mathcal{X} = \mu(V)$.

Proof of (5). By (3), $\forall \Gamma \in \{\Gamma_1, \dots, \Gamma_\lambda\}, \sim'(\Gamma) = T(\mu(M_\Gamma)) = T(M_\Gamma \setminus \mathcal{X}) = \sim(\Gamma)$.

But, $(\Gamma_1, \dots, \Gamma_\lambda, \sim(\Gamma_1), \dots, \sim(\Gamma_\lambda)) \notin \Phi$. Therefore, $(\Gamma_1, \dots, \Gamma_\lambda, \sim'(\Gamma_1), \dots, \sim'(\Gamma_\lambda)) \notin \Phi$.

Consequently, as $\langle \lambda, \Phi \rangle$ is a normal characterization, \sim' is not a pivotal consequence relation. ■

Chapter 6

A link with X -logics

In this chapter, we investigate a link (that we published in [BN05a]) between pivotal consequence relations and pertinence-based consequence relations (alias X -logics) which were first introduced by Forget, Risch, and Siegel [FRS01]. Suppose some formulas are considered to be the pertinent ones and collect them in a set \mathcal{E} . Then, it is natural to conclude a formula α from a set of formulas Γ iff every pertinent basic consequence of $\Gamma \cup \{\alpha\}$ is a basic consequence of Γ (i.e. the addition of α to Γ does not yield more pertinent formulas than with Γ alone). This constitutes a pertinence-based consequence relation. More formally,

Definition 63 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and \vdash a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.

We say that \vdash is a *pertinence-based consequence relation* (alias X -logic) iff there exists $\mathcal{E} \subseteq \mathcal{F}$ such that $\forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}$,

$$\Gamma \vdash \alpha \text{ iff } \vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma).$$

In addition, if $\vdash(\mathcal{E}) = \mathcal{E}$, we say that \vdash is *closed*.

We introduce a new assumption about semantic structures (in fact, simply a weak version of (A3)):

Definition 64 Suppose \mathcal{L} is a language, \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , and $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure. Then, define the following condition:

$$(A4) \quad \forall \alpha, \beta \in \mathcal{F}, M_{\alpha \vee \beta} = M_{\alpha} \cup M_{\beta}.$$

We will show that when (A4) is assumed, UC pivotal consequence relations are precisely closed pertinence-based consequence relations. For that, we introduce Notation 65 and the very easy Proposition 66 (which we will use implicitly in the sequel).

Notation 65 Suppose \mathcal{L} is a language, \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\Gamma \subseteq \mathcal{F}$ and $\Delta \subseteq \mathcal{F}$. Then:

$$\Gamma \vee \Delta := \{\alpha \vee \beta : \alpha \in \Gamma \text{ and } \beta \in \Delta\}.$$

Proposition 66 Suppose \mathcal{L} is a language, \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A4), $\Gamma \subseteq \mathcal{F}$, and $\Delta \subseteq \mathcal{F}$.

Then, $M_{\Gamma} \cup M_{\Delta} = M_{\Gamma \vee \Delta}$.

Proof Direction: “ \subseteq ”.

Suppose the contrary, i.e. $\exists v \in M_\Gamma \cup M_\Delta, v \notin M_{\Gamma \vee \Delta}$.

Then, $\exists \alpha \in \Gamma, \exists \beta \in \Delta, v \notin M_{\alpha \vee \beta}$.

But, by (A4), $v \in M_\Gamma \cup M_\Delta \subseteq M_\alpha \cup M_\beta = M_{\alpha \vee \beta}$, which is impossible.

Direction: “ \supseteq ”.

Suppose the contrary, i.e. $\exists v \in M_{\Gamma \vee \Delta}, v \notin M_\Gamma \cup M_\Delta$.

Then, $\exists \alpha \in \Gamma, v \notin M_\alpha$ and $\exists \beta \in \Delta, v \notin M_\beta$.

Therefore, by (A4), $v \notin M_\alpha \cup M_\beta = M_{\alpha \vee \beta}$.

However, $\alpha \vee \beta \in \Gamma \vee \Delta$. Thus, $v \notin M_{\Gamma \vee \Delta}$, which is impossible. ■

Proposition 67 Suppose \mathcal{L} is a language, \vee a binary connective of \mathcal{L} , \mathcal{F} the set of all wffs of \mathcal{L} , and $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying (A4).

Then, UC pivotal consequence relations are precisely closed pertinence-based consequence relations.

Proof Direction: “ \subseteq ”.

Let \sim be an UC pivotal consequence relation.

Then, there is an UC SC choice function μ from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T(\mu(M_\Gamma))$.

Thus, by Proposition 22, there exists $\mathcal{I} \subseteq \mathcal{V}$ such that $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$ and $\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T(M_\Gamma \cap \mathcal{I})$.

Define: $\mathcal{E} := T(\mathcal{V} \setminus \mathcal{I})$.

Then, $\vdash(\mathcal{E}) = T(M_\mathcal{E}) = T(M_{T(\mathcal{V} \setminus \mathcal{I})}) = T(\mathcal{V} \setminus \mathcal{I}) = \mathcal{E}$.

In addition, as $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$, we have $M_\mathcal{E} = M_{T(\mathcal{V} \setminus \mathcal{I})} = \mathcal{V} \setminus \mathcal{I}$.

We show:

(0) $\forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}, \Gamma \sim \alpha$ iff $\vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma)$.

Consequently, \sim is a closed pertinence-based consequence relation.

Direction: “ \supseteq ”.

Let \sim be a closed pertinence-based consequence relation.

Then, there is $\mathcal{E} \subseteq \mathcal{F}$ such that $\mathcal{E} = \vdash(\mathcal{E})$ and $\forall \Gamma \subseteq \mathcal{F}, \forall \alpha \in \mathcal{F}, \Gamma \sim \alpha$ iff $\vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma)$.

Define: $\mathcal{I} := \mathcal{V} \setminus M_\mathcal{E}$.

Then, $\mathcal{V} \setminus \mathcal{I} = M_\mathcal{E} \in \mathbf{D}$.

We will show:

(1) $\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T(M_\Gamma \cap \mathcal{I})$.

Let μ be the choice function from \mathbf{D} to $\mathcal{P}(\mathcal{V})$ such that $\forall V \in \mathbf{D}, \mu(V) = V \cap \mathcal{I}$.

Then, $\forall \Gamma \subseteq \mathcal{F}, \sim(\Gamma) = T(\mu(M_\Gamma))$.

In addition, by Proposition 22, μ is a UC SC choice function.

Consequently, \sim is an UC pivotal consequence relation.

Proof of (0). Let $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$. Then:

$\Gamma \sim \alpha$ iff

$M_\Gamma \cap \mathcal{I} \subseteq M_\alpha$ iff

$M_\Gamma \subseteq M_\alpha \cup (\mathcal{V} \setminus \mathcal{I})$ iff

$M_\Gamma \subseteq M_\alpha \cup M_\mathcal{E}$ iff

$M_\Gamma \subseteq M_{\Gamma \cup \{\alpha\}} \cup M_\mathcal{E}$ iff

$M_\Gamma \subseteq M_{(\Gamma \cup \{\alpha\}) \vee \mathcal{E}}$ iff

$T(M_{(\Gamma \cup \{\alpha\}) \vee \mathcal{E}}) \subseteq T(M_\Gamma)$ iff

$T(M_{\Gamma \cup \{\alpha\}} \cup M_\mathcal{E}) \subseteq T(M_\Gamma)$ iff

$T(M_{\Gamma \cup \{\alpha\}}) \cap T(M_{\mathcal{E}}) \subseteq T(M_{\Gamma})$ iff
 $\vdash(\Gamma, \alpha) \cap \vdash(\mathcal{E}) \subseteq \vdash(\Gamma)$ iff
 $\vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma)$.

Proof of (1). Let $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$. Then:

$\Gamma \vdash \alpha$ iff
 $\vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma)$ iff
 $\vdash(\Gamma, \alpha) \cap \vdash(\mathcal{E}) \subseteq \vdash(\Gamma)$ iff
 $T(M_{\Gamma \cup \{\alpha\}}) \cap T(M_{\mathcal{E}}) \subseteq T(M_{\Gamma})$ iff
 $T(M_{\Gamma \cup \{\alpha\}} \cup M_{\mathcal{E}}) \subseteq T(M_{\Gamma})$ iff
 $T(M_{(\Gamma \cup \{\alpha\}) \vee \mathcal{E}}) \subseteq T(M_{\Gamma})$ iff
 $M_{\Gamma} \subseteq M_{(\Gamma \cup \{\alpha\}) \vee \mathcal{E}}$ iff
 $M_{\Gamma} \subseteq M_{\Gamma \cup \{\alpha\}} \cup M_{\mathcal{E}}$ iff
 $M_{\Gamma} \subseteq M_{\alpha} \cup M_{\mathcal{E}}$ iff
 $M_{\Gamma} \cap (\mathcal{V} \setminus M_{\mathcal{E}}) \subseteq M_{\alpha}$ iff
 $M_{\Gamma} \cap \mathcal{I} \subseteq M_{\alpha}$. ■

Part II

Lack of Finite Characterizations for the Distance-based Revision

Chapter 7

The AGM approach

7.1 Basic operators

Belief revision is the study of how an intelligent agent may replace its current epistemic state by another one which is non-trivial and incorporates new information. In [AGM85], C. Alchourrón, P. Gärdenfors, and D. Makinson proposed an approach now well-known as the AGM approach. We will present it in a very simple classical framework.

Suppose \mathcal{L} is a classical propositional language containing the usual connectives $\neg, \wedge, \vee, \rightarrow,$ and \leftrightarrow , \mathcal{F} the set of all wffs of \mathcal{L} , \vdash the classical consequence relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$, and \mathbf{T} the set of all $K \subseteq \mathcal{F}$ such that K is \vdash -closed (i.e. $K = \{\alpha \in \mathcal{F} : K \vdash \alpha\}$). Intuitively, each element of \mathbf{T} represents an epistemic state and each element of \mathcal{F} new information. Three fundamental operations were introduced: *expansion*, *contraction*, and *revision*.

An expansion operator $+$ is a function from $\mathbf{T} \times \mathcal{F}$ to $\mathcal{P}(\mathcal{F})$. Intuitively, $K + \alpha$ is the result of adding α to K without checking the non-triviality of $K + \alpha$. Alchourrón *et al.* proposed the following rationality postulates: $\forall K, K' \in \mathbf{T}, \forall \alpha \in \mathcal{F}$,

$$(K+1) \quad K + \alpha \in \mathbf{T};$$

$$(K+2) \quad \alpha \in K + \alpha;$$

$$(K+3) \quad K \subseteq K + \alpha;$$

$$(K+4) \quad \text{if } \alpha \in K, \text{ then } K + \alpha = K;$$

$$(K+5) \quad \text{if } K \subseteq K', \text{ then } K + \alpha \subseteq K' + \alpha;$$

$$(K+6) \quad \text{if } +' \text{ is an expansion operator satisfying } (K+1)\text{--}(K+5), \text{ then } K + \alpha \subseteq K +' \alpha.$$

The intuitive justifications of $(K+1)$ – $(K+4)$ are obvious. $(K+5)$ ensures monotonicity. $(K+6)$ ensures that the addition of information is minimal. In fact, these postulates are very constraining. Indeed, the only operator which satisfies them is the one such that $K + \alpha$ is the \vdash -closure of $K \cup \alpha$. So, from now on, $+$ will denote this operator.

A contraction operator \div is a function from $\mathbf{T} \times \mathcal{F}$ to $\mathcal{P}(\mathcal{F})$. Intuitively, $K \div \alpha$ is the result of removing α from K . The following postulates have been proposed: $\forall K \in \mathbf{T}, \forall \alpha, \beta \in \mathcal{F}$,

$$(K\div 1) \quad K \div \alpha \in \mathbf{T};$$

- ($K \div 2$) $K \div \alpha \subseteq K$;
- ($K \div 3$) if $\alpha \notin K$, then $K \div \alpha = K$;
- ($K \div 4$) if $\not\vdash \alpha$, then $\alpha \notin K \div \alpha$;
- ($K \div 5$) if $\alpha \in K$, then $K \subseteq (K \div \alpha) + \alpha$;
- ($K \div 6$) if $\vdash \alpha \leftrightarrow \beta$, then $K \div \alpha = K \div \beta$;
- ($K \div 7$) $(K \div \alpha) \cap (K \div \beta) \subseteq K \div (\alpha \wedge \beta)$;
- ($K \div 8$) if $\alpha \notin K \div (\alpha \wedge \beta)$, then $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$.

The intuitive justifications of ($K \div 1$)–($K \div 4$) are obvious. ($K \div 5$) has been strongly criticized in e.g. [Han91, Fuh91, Nie91, LR91] and is also known as the ‘‘Principle of recovery’’. ($K \div 6$) makes the contraction independent of the syntactic form of the formulas. ($K \div 7$) and ($K \div 8$) constrain in a natural way the contraction by a conjunction.

Finally, a revision operator \star is a function from $\mathbf{T} \times \mathcal{F}$ to $\mathcal{P}(\mathcal{F})$. Intuitively, $K \star \alpha$ is the result of adding α to K while maintaining the non-triviality of $K \star \alpha$. The well-known AGM postulates are the following: $\forall K \in \mathbf{T}, \forall \alpha, \beta \in \mathcal{F}$

- ($K \star 1$) $K \star \alpha \in \mathbf{T}$;
- ($K \star 2$) $\alpha \in K \star \alpha$;
- ($K \star 3$) $K \star \alpha \subseteq K + \alpha$;
- ($K \star 4$) if $\neg\alpha \notin K$, then $K + \alpha \subseteq K \star \alpha$;
- ($K \star 5$) $K \star \alpha = \mathcal{F}$ iff $\vdash \neg\alpha$;
- ($K \star 6$) if $\vdash \alpha \leftrightarrow \beta$, then $K \star \alpha = K \star \beta$;
- ($K \star 7$) $K \star (\alpha \wedge \beta) \subseteq (K \star \alpha) + \beta$;
- ($K \star 8$) if $\neg\beta \notin K \star \alpha$, then $(K \star \alpha) + \beta \subseteq K \star (\alpha \wedge \beta)$.

There are obvious motivations for ($K \star 1$)–($K \star 4$). Concerning ($K \star 5$), it ensures that the revised set of formulas is non-trivial. ($K \star 6$) makes the revision independent of the syntactic form of the formulas. ($K \star 7$) and ($K \star 8$) ensure that the loss of information is minimal. Katsuno and Mendelzon reformulated these postulates in a framework where \mathcal{L} is a finite propositional language (see [KM91] for details).

We can define naturally revision from contraction and vice versa (see e.g. [G88]). Indeed, suppose \div is a contraction operator satisfying ($K \div 1$)–($K \div 4$) and ($K \div 6$) (not necessarily the controversial ($K \div 5$)). Then, the revision operator \star defined by the *Levi Identity*:

$$K \star \alpha = (K \div \neg\alpha) + \alpha$$

satisfies ($K \star 1$)–($K \star 6$). In addition, if \div satisfies ($K \div 7$) (resp. ($K \div 8$)), then \star satisfies ($K \star 7$) (resp. ($K \star 8$)).

Conversely, suppose \star is a revision operator that satisfies $(K\star 1)$ – $(K\star 6)$, then the contraction operator \div defined by the *Harper Identity*:

$$K \div \alpha = K \cap (K \star \neg\alpha)$$

satisfies $(K\div 1)$ – $(K\div 6)$. In addition, if \star satisfies $(K\star 7)$ (resp. $(K\star 8)$), then \div satisfies $(K\div 7)$ (resp. $(K\div 8)$).

7.2 Epistemic entrenchment

Several representation theorems related to the AGM approach have been given. Let's present a first one. Suppose we have some way to decide for all α and β , whether β is at least as “epistemically entrenched” as α . In [GM88], this is modelled by an *epistemic entrenchment relation*, i.e. a relation \preceq on $\mathcal{F} \times \mathcal{F}$. The following postulates were proposed for a given $K: \forall \alpha, \beta, \gamma \in \mathcal{F}$,

- (EE1) if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$;
- (EE2) if $\vdash \alpha \rightarrow \beta$, then $\alpha \preceq \beta$;
- (EE3) $\alpha \preceq \alpha \wedge \beta$ or $\beta \preceq \alpha \wedge \beta$;
- (EE4) if $K \neq \mathcal{F}$, then $\alpha \notin K$ iff $\forall \beta \in \mathcal{F}, \alpha \preceq \beta$;
- (EE5) if $\forall \beta \in \mathcal{F}, \beta \preceq \alpha$, then $\vdash \alpha$.

Details about the justifications of these postulates can be found in [GM88] as well as the following representation theorem: a contraction operator \div satisfies $(K\div 1)$ – $(K\div 8)$ iff $\forall K \in \mathbf{T}$, there exists an epistemic entrenchment relation \preceq satisfying (EE1)–(EE5) for K and such that $\forall \alpha \in \mathcal{F}, K \div \alpha = \{\beta \in K : \text{either } \alpha \prec \alpha \vee \beta \text{ or } \vdash \alpha\}$.

Results of this kind put in evidence the importance of the AGM approach. However, a weakness is that there is no necessary connexion between the different epistemic entrenchment relations. It would have been better if the contractions of the different K 's were all defined by the same object.

7.3 Sphere systems

In [Gro88], Grove provided an important characterization using *sphere systems*. Then, Boutilier, Katsuno, and Mendelzon modified it slightly [Bou94, KM91] using *rankings* instead of sphere systems. Let's present the modified version. Suppose \mathcal{V} is the set of all classical valuations of \mathcal{L} , $\alpha \in \mathcal{F}$, and $V \subseteq \mathcal{V}$. Then, M_α denotes the set of all models for α and $T(V)$ the set of all formulas satisfied in V . Now, suppose \preceq is a ranking on \mathcal{V} , i.e. a total, reflexive, and transitive relation on $\mathcal{V} \times \mathcal{V}$. Intuitively, $v \preceq w$ means: “ v is at least as important as w ”. Then, \preceq defines the most important elements of V as follows: $\min_{\preceq}(V) := \{v \in V : \forall w \in V, w \not\prec v\}$.

The following elegant representation theorem holds: a revision operator \star satisfies $(K\star 1)$ – $(K\star 8)$ iff $\forall K \in \mathbf{T} \setminus \{\mathcal{F}\}$, there exists a ranking \preceq on \mathcal{V} such that $K = T(\min_{\preceq}(\mathcal{V}))$ and $\forall \alpha \in \mathcal{F}, K \star \alpha = T(\min_{\preceq}(M_\alpha))$.¹

¹We omitted the case where $K = \mathcal{F}$, which gets special treatments (see [Bou94, KM91] for details).

This theorem constitutes another evidence of the importance of the AGM approach. However, there is again no necessary “glue” between the different rankings. This is not surprising because the AGM postulates do not require an operator to put some coherence between the revisions of two different sets K and K' . As a consequence, some operators are accepted though they are not well-behaved when iterated. To remedy this problem, there are at least two (compatible) kinds of solutions.

First, one could model an epistemic state by something richer than just a set of formulas, for instance, a ranking or an *ordinal conditional function* [Spo88]. Approximatively, the idea is to model not only the current beliefs, but also the strategy of revision, with an aim of constraining them both to get properties of iterated revisions. Some well-known proposals of this kind are: *Boutilier's natural revision* [Bou93, Bou96]; *Freund and Lehmann's approach* [FL94]; *Darwiche and Pearl's approach* [DP94, DP97]; *Lehmann's revised approach* [Leh95]. We will not go into detail. In fact, Part II is essentially related to the other kind of solutions.

Second, one could imagine that a unique object defines all the revisions of the different K 's. This will entail a strong coherence between them. Therefore, properties of iterated revisions will naturally emerge (without necessarily considering rich epistemic states). Schlechta proposed a definition based on *measures* [Sch91, Sch04]. Approximatively, the idea is to associate (in an independent way) to each propositional symbol a measurable subset of the real interval $[0, 1]$, and thus a probability. This probability measure is then extended to arbitrary formulas and gives a “weigh” to each formula. This ordering results in a “pre-EE relation” (see Definition 7.4.1 of [Sch91]) which does not mention any K , and generates epistemic entrenchment relations for arbitrary K 's (see Definition 7.4.2 and Proposition 7.4.2 of [Sch91]). This defines completely a revision operator well-behaved in case of iteration.

An alternative approach of this second kind is the distance-based revision which is presented in detail in Chapter 8.

Chapter 8

Distance-based Revision

The present chapter introduces fundamental definitions for Part II.

8.1 Pseudo-distances

In many circumstances, it is reasonable to assume that an agent can evaluate for any two valuations v and w , how far is the situation described by w from the situation described by v , or how difficult or unexpected the transition from v to w is, etc. In [LMS01], this is modelled by pseudo-distances:

Definition 68 Let \mathcal{V} be a set.

We say that \mathcal{D} is a *pseudo-distance* on \mathcal{V} iff $\mathcal{D} = \langle C, \prec, d \rangle$, where C is a non-empty set, \prec is a strict total order on C , and d is a function from $\mathcal{V} \times \mathcal{V}$ to C .

Intuitively, \mathcal{V} is a set of valuations. Each element of C represents a “cost”. $c \prec c'$ means the cost c is strictly smaller than the cost c' . And, $d(v, w)$ is the cost of the move from v to w . Natural properties that come to mind are those of usual distances. Before introducing them, we need standard notations:

Notation 69 Let $r \in \mathbb{R}$. Then, $abs(r)$ denotes the absolute value of r .

Let $n, m \in \mathbb{N}$. Then, $[n, m]$ denotes the set of every k in \mathbb{N} (not in \mathbb{R}) such that $n \leq k \leq m$.

Definition 70 Let $\mathcal{D} = \langle C, \prec, d \rangle$ be a pseudo-distance on a set \mathcal{V} .

\mathcal{D} is *symmetric* iff $\forall v, w \in \mathcal{V}, d(v, w) = d(w, v)$.

\mathcal{D} is *identity respecting* (IR) iff

- (1) $C = \mathbb{R}$;
- (2) \prec is the usual strict total order on \mathbb{R} ;
- (3) $\forall v, w \in \mathcal{V}, d(v, w) = 0$ iff $v = w$.

\mathcal{D} is *positive* iff (1), (2), and

- (4) $\forall v, w \in \mathcal{V}, 0 \preceq d(v, w)$.

\mathcal{D} is *triangle-inequality respecting* (TIR) iff (1), (2), and

- (5) $\forall v, w, x \in \mathcal{V}, d(v, x) \preceq d(v, w) + d(w, x)$.

These properties have not been imposed from start because natural circumstances could then no longer be modelled. For instance, non-symmetric pseudo-distances are useful when moving from v to w may be “cheaper” than moving from w to v . There are also circumstances where staying

the same requires effort and then non-IR pseudo-distances will be helpful. We can also imagine scenarios where some costs can be seen as “benefits”, we will then turn to non-positive pseudo-distances.

In addition, the costs are not required to be necessarily the real numbers. Indeed, for instance, we could need $|\mathbb{N}|$ to model an “infinite cost” useful when a move is impossible or extremely difficult. If we add $|\mathbb{N}|$ to the reals, then we can define naturally “liberal” versions of identity respect, positivity, and triangle-inequality respect:

Definition 71 Let $\mathcal{D} = \langle C, \prec, d \rangle$ be a pseudo-distance on a set \mathcal{V} .

\mathcal{D} is *liberally IR* iff

- (1) $C = \mathbb{R} \cup \{|\mathbb{N}|\}$;
- (2) $\forall c, c' \in C, c \prec c'$ iff $(c, c' \in \mathbb{R} \text{ and } c < c')$ or $(c \in \mathbb{R} \text{ and } c' = |\mathbb{N}|)$;
- (3) $\forall v, w \in \mathcal{V}, d(v, w) = 0$ iff $v = w$.

\mathcal{D} is *liberally positive* iff (1), (2), and

- (4) $\forall v, w \in \mathcal{V}, 0 \preceq d(v, w)$.

\mathcal{D} is *liberally TIR* iff (1), (2), and

- (5) $\forall v, w, x \in \mathcal{V}$: if $d(v, x), d(v, w), d(w, x) \in \mathbb{R}$, then $d(v, x) \preceq d(v, w) + d(w, x)$;
if $d(v, x) = |\mathbb{N}|$, then $d(v, w) = |\mathbb{N}|$ or $d(w, x) = |\mathbb{N}|$.

The Hamming distance between propositional valuations has been considered by M. Dalal [Dal88] and investigated further by many authors. Respecting this distance is an important property. We need before to present the matrices for a propositional language [Urq01]:

Definition 72 Let $\mathcal{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ be a propositional language (\mathcal{A} denotes the atoms and \mathcal{C} the connectives), let \mathcal{F} be the set of all well-formed formulas of \mathcal{L} , and $\forall \diamond \in \mathcal{C}$, let $n(\diamond)$ be the arity of \diamond .

We say that \mathcal{M} is a *matrix* on \mathcal{L} iff $\mathcal{M} = \langle T, D, f \rangle$, where T is a set, D is a non-empty proper subset of T , and f is a function (whose domain is \mathcal{C}) such that $\forall \diamond \in \mathcal{C}$, f_\diamond (i.e. $f(\diamond)$) is a function from $T^{n(\diamond)}$ to T .

We say that v is a *\mathcal{M} -valuation* iff v is a function from \mathcal{F} to T such that

$$\forall \diamond \in \mathcal{C}, \forall \alpha_1, \dots, \alpha_{n(\diamond)} \in \mathcal{F}, \text{ we have } v(\diamond(\alpha_1, \dots, \alpha_{n(\diamond)})) = f_\diamond(v(\alpha_1), \dots, v(\alpha_{n(\diamond)})).$$

Intuitively, T is a set of truth values and D contains all the designated truth values.

Definition 73 Let $\mathcal{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ be a propositional language, \mathcal{M} a matrix on \mathcal{L} , \mathcal{V} the set of all \mathcal{M} -valuations, and $\mathcal{D} = \langle C, \prec, d \rangle$ a pseudo-distance on \mathcal{V} .

We use the following notation: $\forall v, w \in \mathcal{V}, h(v, w) := \{p \in \mathcal{A} : v(p) \neq w(p)\}$.

We say that \mathcal{D} is *Hamming-inequality respecting* (HIR) iff $\forall v, w, x \in \mathcal{V}$,

$$\text{if } |h(v, w)| < |h(v, x)|, \text{ then } d(v, w) \prec d(v, x).$$

Recall that $h(v, w)$ may be infinite and thus $<$ has to be understood as the usual order on the cardinal numbers. We turn to crucial operators introduced in [LMS01]. They are central in the definition of the distance-based revision. They transform any two sets of valuations V and W into the set of every element w of W such that a global move from V to w is of minimal cost. Note that concerning this point, [LMS01] has its roots in [KM92] and especially in [Lew73].

Definition 74 Let $\mathcal{D} = \langle C, \prec, d \rangle$ be a pseudo-distance on a set \mathcal{V} .

We denote by $|_{\mathcal{D}}$ the binary operator on $\mathcal{P}(\mathcal{V})$ such that $\forall V, W \subseteq \mathcal{V}$,

$$V|_{\mathcal{D}}W = \{w \in W : \exists v \in V, \forall v' \in V, \forall w' \in W, d(v, w) \preceq d(v', w')\}.$$

8.2 Distance-based revision operators

The ontological commitments endorsed in [LMS01] are close to the AGM ones: a classical propositional language is considered and both epistemic states and new information are modelled by consistent sets of formulas (not necessarily deductively closed).

Notation 75 We denote by \mathcal{L}_c some classical propositional language and by \vdash_c , \mathcal{V}_c , \models_c , and \mathcal{F}_c respectively the classical consequence relation, valuations, satisfaction relation, and wffs of \mathcal{L}_c .

Let $\Gamma, \Delta \subseteq \mathcal{F}_c$ and $V \subseteq \mathcal{V}_c$, then standardly:

$$\begin{aligned} \Gamma \vee \Delta &:= \{\alpha \vee \beta : \alpha \in \Gamma, \beta \in \Delta\}; \\ \vdash_c(\Gamma) &:= \{\alpha \in \mathcal{F}_c : \Gamma \vdash_c \alpha\}; \\ M_\Gamma &:= \{v \in \mathcal{V}_c : \forall \alpha \in \Gamma, v \models_c \alpha\}; \\ T(V) &:= \{\alpha \in \mathcal{F}_c : V \subseteq M_\alpha\}; \\ \mathbf{C} &:= \{\Gamma \subseteq \mathcal{F}_c : \vdash_c(\Gamma) \neq \mathcal{F}_c\}; \\ \mathbf{D} &:= \{V \subseteq \mathcal{V}_c : \exists \Gamma \subseteq \mathcal{F}_c, V = M_\Gamma\}. \end{aligned}$$

Remark 76 Some notations in Part II override those of Part I (new notations would have been too cumbersome).

In this classical framework, two new properties for the pseudo-distances can be defined. They convey natural meanings. Their importance has been put in evidence in [LMS01].

Definition 77 Let $\mathcal{D} = \langle C, \prec, d \rangle$ be a pseudo-distance on \mathcal{V}_c .

\mathcal{D} is *definability preserving* (DP) iff $\forall V, W \in \mathbf{D}, V|_{\mathcal{D}}W \in \mathbf{D}$.

\mathcal{D} is *consistency preserving* (CP) iff $\forall V, W \in \mathcal{P}(\mathcal{V}_c) \setminus \{\emptyset\}, V|_{\mathcal{D}}W \neq \emptyset$.

Now, suppose we are given a pseudo-distance \mathcal{D} on \mathcal{V}_c . Then, the revision of a consistent set of formulas Γ by a second one Δ can be defined naturally as the set of all formulas satisfied in $M_\Gamma|_{\mathcal{D}}M_\Delta$ (i.e. the set of all those models of Δ that are “closest” to the models of Γ).

Definition 78 Let \star be an operator from $\mathbf{C} \times \mathbf{C}$ to $\mathcal{P}(\mathcal{F}_c)$.

We say that \star is a *distance-based revision operator* iff there exists a pseudo-distance \mathcal{D} on \mathcal{V}_c such that $\forall \Gamma, \Delta \in \mathbf{C}$,

$$\Gamma \star \Delta = T(M_\Gamma|_{\mathcal{D}}M_\Delta).$$

In addition, if \mathcal{D} is symmetric, HIR, DP etc., then so is \star .

The authors of [LMS01] rewrote the AGM postulates in their framework as follows.

Let \star be an operator from $\mathbf{C} \times \mathbf{C}$ to $\mathcal{P}(\mathcal{F}_c)$ and $\Gamma, \Gamma', \Delta, \Delta' \in \mathbf{C}$. Then, define:

- ($\star 0$) if $\vdash_c(\Gamma) = \vdash_c(\Gamma')$ and $\vdash_c(\Delta) = \vdash_c(\Delta')$, then $\Gamma \star \Delta = \Gamma' \star \Delta'$;
- ($\star 1$) $\Gamma \star \Delta \in \mathbf{C}$ and $\Gamma \star \Delta = \vdash_c(\Gamma \star \Delta)$;
- ($\star 2$) $\Delta \subseteq \Gamma \star \Delta$;
- ($\star 3$) if $\Gamma \cup \Delta \in \mathbf{C}$, then $\Gamma \star \Delta = \vdash_c(\Gamma \cup \Delta)$;
- ($\star 4$) if $(\Gamma \star \Delta) \cup \Delta' \in \mathbf{C}$, then $\Gamma \star (\Delta \cup \Delta') = \vdash_c((\Gamma \star \Delta) \cup \Delta')$.

Then, it can be checked that every positive, IR, CP and DP distance-based revision operator \star satisfies ($\star 0$)-($\star 4$), i.e. the AGM postulates. More importantly, \star satisfies also certain properties that deal with iterated revisions. This is not surprising as the revisions of the different Γ 's are all defined by a unique pseudo-distance, which ensures a strong coherence between them. For example, \star satisfies two following properties: $\forall \Gamma, \Delta, \{\alpha\}, \{\beta\} \in \mathbf{C}$,

- if $\gamma \in (\Gamma \star \{\alpha\}) \star \Delta$ and $\gamma \in (\Gamma \star \{\beta\}) \star \Delta$, then $\gamma \in (\Gamma \star \{\alpha \vee \beta\}) \star \Delta$;
- if $\gamma \in (\Gamma \star \{\alpha \vee \beta\}) \star \Delta$, then $\gamma \in (\Gamma \star \{\alpha\}) \star \Delta$ or $\gamma \in (\Gamma \star \{\beta\}) \star \Delta$.

These properties are not entailed by the AGM postulates, a counter-example can be found in [LMS01]. But, they seem intuitively justified. Indeed, take three sequences of revisions that differ only at some step in which the new information is α in the first sequence, β in the second, and $\alpha \vee \beta$ in the third. Now, suppose γ is concluded after both the first and the second sequences. Then, it should intuitively be the case that γ is concluded after third sequence too.

Similar arguments can be given for the second property. Now, to characterize the full distance-based revision more is needed. This is discussed in Section 8.3.

8.3 Characterizations with conditions of arbitrarily big size

The authors of [LMS01] provided characterizations for families of distance-based revision operators. They proceed in two steps. First, they defined the distance operators, in a very general framework.

Definition 79 Let \mathcal{V} be a set, $\mathbf{V}, \mathbf{W}, \mathbf{X} \subseteq \mathcal{P}(\mathcal{V})$, and $|$ an operator from $\mathbf{V} \times \mathbf{W}$ to \mathbf{X} . We say that $|$ is a *distance operator* iff there exists a pseudo-distance \mathcal{D} on \mathcal{V} such that $\forall V \in \mathbf{V}, \forall W \in \mathbf{W}, V|W = V|_{\mathcal{D}}W$.

In addition, if \mathcal{D} is symmetric, HIR, DP, etc., then so is $|$.

Then, they characterized families of such distance operators (with the least possible assumptions about \mathbf{V} , \mathbf{W} , and \mathbf{X}). This is the essence of their work. Here is an example:

Proposition 80 [LMS01] Let \mathcal{V} be a non-empty set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, and $|$ an operator from $\mathbf{V} \times \mathbf{V}$ to \mathbf{V} . Assume $\emptyset \notin \mathbf{V}$ and $\forall V, W \in \mathbf{V}, V \cup W \in \mathbf{V}$ and if $V \cap W \neq \emptyset$, then $V \cap W \in \mathbf{V}$ too.

Then, $|$ is a symmetric distance operator iff $\forall k \in \mathbb{N}^+, \forall V_0, V_1, \dots, V_k \in \mathbf{V}$, we have $V_0|V_1 \subseteq V_1$ and

$$(\text{loop}) \text{ if } \begin{cases} (V_1|(V_0 \cup V_2)) \cap V_0 \neq \emptyset, \\ (V_2|(V_1 \cup V_3)) \cap V_1 \neq \emptyset, \\ \dots, \\ (V_k|(V_{k-1} \cup V_0)) \cap V_{k-1} \neq \emptyset, \end{cases} \text{ then } (V_0|(V_k \cup V_1)) \cap V_1 \neq \emptyset.$$

In a second step only, they applied these results to characterize families of distance-based revision operators. For instance, they applied Proposition 80 to get Proposition 81 below. We should say immediately that they chose a classical framework to define the distance-based revision. But, if we choose now another framework, there are quite good chances that Proposition 80 can be still applied, thanks to its algebraic nature.

Proposition 81 [LMS01] Let \star be an operator from $\mathbf{C} \times \mathbf{C}$ to $\mathcal{P}(\mathcal{F}_c)$.

Then, \star is a symmetric CP DP distance-based revision operator iff \star satisfies $(\star 0)$, $(\star 1)$, $(\star 2)$, and $\forall k \in \mathbb{N}^+, \forall \Gamma_0, \Gamma_1, \dots, \Gamma_k \in \mathbf{C}$,

$$(\star loop) \text{ if } \begin{cases} \Gamma_0 \cup (\Gamma_1 \star (\Gamma_0 \vee \Gamma_2)) \in \mathbf{C}, \\ \Gamma_1 \cup (\Gamma_2 \star (\Gamma_1 \vee \Gamma_3)) \in \mathbf{C}, \\ \dots, \\ \Gamma_{k-1} \cup (\Gamma_k \star (\Gamma_{k-1} \vee \Gamma_0)) \in \mathbf{C}, \end{cases} \text{ then } \Gamma_1 \cup (\Gamma_0 \star (\Gamma_k \vee \Gamma_1)) \in \mathbf{C}.$$

Chapter 9

Nonexistence of normal characterizations

9.1 Definition

Let \mathcal{V} be a set, \mathcal{O} a set of binary operators on $\mathcal{P}(\mathcal{V})$, and $|$ a binary operators on $\mathcal{P}(\mathcal{V})$.

Approximatively, a characterization of \mathcal{O} is S-normal (i.e. called normal in [Sch04]) iff it contains only conditions which are universally quantified, apply $|$ only a finite number of times, and use only elementary operations (like e.g. \cup, \cap, \setminus), see Section 1.6.2.1 of [Sch04] for details. Here is an example of such a condition:

$$(C1) \quad \forall V, W \in \mathbf{U} \subseteq \mathcal{P}(\mathcal{V}), V|((V \cup W)|W) = \emptyset.$$

Now, we introduce a new, more general, definition of normality with an aim of providing more general impossibility results. Approximatively, in the present thesis, a characterization of \mathcal{O} will be called normal iff it contains only conditions which are universally quantified and apply $|$ only a finite number of times. Then, the conditions can involve complex structures or functions, etc., we are not limited to elementary operations. More formally:

Definition 82 Let \mathcal{V} be a set and \mathcal{O} a set of binary operators on $\mathcal{P}(\mathcal{V})$.

We say that \mathcal{C} is a *normal characterization* of \mathcal{O} iff $\mathcal{C} = \langle n, \Phi \rangle$ where $n \in \mathbb{N}^+$ and Φ is a relation on $\mathcal{P}(\mathcal{V})^{3n}$ such that for every binary operator $|$ on $\mathcal{P}(\mathcal{V})$,

$$| \in \mathcal{O} \text{ iff } \forall V_1, \dots, V_n, W_1, \dots, W_n \subseteq \mathcal{V}, (V_1, \dots, V_n, W_1, \dots, W_n, V_1|W_1, \dots, V_n|W_n) \in \Phi.$$

Remark 83 A notion of normal characterization has already been given in Definition 60. Though Definition 60 and Definition 82 have the same spirit, they are slightly different. In particular, the former covers both finite and infinite characterizations, whilst the latter covers only finite characterizations. The reader should not confuse them as Definition 60 is about consequence relations, whilst Definition 82 is about binary operators.

Now, suppose there is no normal characterization of \mathcal{O} . Here are examples (i.e. (C1), (C2), and (C3) below) that will give the reader (we hope) a good idea which conditions cannot characterize \mathcal{O} . This will therefore make clearer the range of our impossibility results (Propositions 84 and 85 below).

To begin, (C1) cannot characterize \mathcal{O} . Indeed, suppose the contrary, i.e. suppose $| \in \mathcal{O}$ iff $\forall V, W \in \mathbf{U}, V|((V \cup W)|W) = \emptyset$. Then, take $n = 4$ and Φ such that $(V_1, \dots, V_4, W_1, \dots, W_4, X_1, \dots, X_4) \in \Phi$ iff

$$\left\{ \begin{array}{l} V_1, V_2 \in \mathbf{U}, \\ V_3 = V_1 \cup V_2, \\ W_3 = V_2, \\ V_4 = V_1, \\ W_4 = X_3, \end{array} \right. \text{ entail } X_4 = \emptyset.$$

Then, $\langle 4, \Phi \rangle$ is a normal characterization of \mathcal{O} . We give the easy proof of this, so that the reader can check that a convenient relation Φ can be found immediately for all simple conditions like (C1).

Proof Direction: “ \rightarrow ”.

Suppose $| \in \mathcal{O}$.

Then, $\forall V, W \in \mathbf{U}, V|((V \cup W)|W) = \emptyset$.

Let $V_1, \dots, V_4, W_1, \dots, W_4 \subseteq \mathcal{V}$.

We show $(V_1, \dots, V_4, W_1, \dots, W_4, V_1|W_1, \dots, V_4|W_4) \in \Phi$.

Suppose $V_1, V_2 \in \mathbf{U}, V_3 = V_1 \cup V_2, W_3 = V_2, V_4 = V_1$, and $W_4 = V_3|W_3$.

Then, as $V_1, V_2 \in \mathbf{U}$, we get $V_1|((V_1 \cup V_2)|V_2) = \emptyset$.

But, $V_1|((V_1 \cup V_2)|V_2) = V_1|(V_3|W_3) = V_4|W_4$. Therefore, $V_4|W_4 = \emptyset$.

Direction: “ \leftarrow ”.

Suppose $\forall V_1, \dots, V_4, W_1, \dots, W_4 \subseteq \mathcal{V}, (V_1, \dots, V_4, W_1, \dots, W_4, V_1|W_1, \dots, V_4|W_4) \in \Phi$.

We show $| \in \mathcal{O}$. Let $V, W \in \mathbf{U}$.

Then, take $V_1 = V, V_2 = W, V_3 = V_1 \cup V_2, W_3 = V_2, V_4 = V_1, W_4 = V_3|W_3$.

Take any values for W_1 and W_2 .

Then, $V_1 \in \mathbf{U}, V_2 \in \mathbf{U}, V_3 = V_1 \cup V_2, W_3 = V_2, V_4 = V_1$, and $W_4 = V_3|W_3$.

But, we have $(V_1, \dots, V_4, W_1, \dots, W_4, V_1|W_1, \dots, V_4|W_4) \in \Phi$.

Therefore, by definition of Φ , $V_4|W_4 = \emptyset$.

But, $V_4|W_4 = V_1|(V_3|W_3) = V_1|((V_1 \cup V_2)|V_2) = V|((V \cup W)|W)$. ■

At this point, we excluded all those conditions which are excluded by the nonexistence of a S-normal characterization of \mathcal{O} , i.e. all conditions like (C1). But actually, more complex conditions are also excluded. For instance, let f be any function from $\mathcal{P}(\mathcal{V})$ to $\mathcal{P}(\mathcal{V})$. Then, the following condition:

$$(C2) \quad \forall V, W \in \mathbf{U}, f(V)|((V \cup W)|W) = \emptyset.$$

cannot characterize \mathcal{O} . Indeed, suppose it characterizes \mathcal{O} .

Then, take $n = 4$ and Φ such that $(V_1, \dots, V_4, W_1, \dots, W_4, X_1, \dots, X_4) \in \Phi$ iff

$$\left\{ \begin{array}{l} V_1, V_2 \in \mathbf{U}, \\ V_3 = V_1 \cup V_2, \\ W_3 = V_2, \\ V_4 = f(V_1), \\ W_4 = X_3, \end{array} \right. \text{ entail } X_4 = \emptyset.$$

Then, $\langle 4, \Phi \rangle$ is a normal characterization of \mathcal{O} . We leave the easy proof of this to the reader. On the other hand, (C2) is not excluded by Schlechta, if f cannot be constructed from elementary operations. But, even if there exists such a construction, showing that it is indeed the case might well be a difficult problem.

We can even go further combining universal (not existential) quantifiers and functions like f . For instance, let \mathcal{G} be a set of functions from $\mathcal{P}(\mathcal{V})$ to $\mathcal{P}(\mathcal{V})$ and consider the following condition:

$$(C3) \quad \forall f \in \mathcal{G}, \forall V, W \in \mathbf{U}, f(V) | ((V \cup W) | W) = \emptyset.$$

Then, (C3) cannot characterize \mathcal{O} . Indeed, suppose (C3) characterizes \mathcal{O} .

Then, take $n = 4$ and Φ such that $(V_1, \dots, V_4, W_1, \dots, W_4, X_1, \dots, X_4) \in \Phi$ iff

$$\forall f \in \mathcal{G}, \text{ if } \begin{cases} V_1, V_2 \in \mathbf{U}, \\ V_3 = V_1 \cup V_2, \\ W_3 = V_2, \\ V_4 = f(V_1), \\ W_4 = X_3, \end{cases} \text{ then } X_4 = \emptyset.$$

It can be checked that $\langle 4, \Phi \rangle$ is a normal characterization of \mathcal{O} . The easy proof is left to the reader. On the other hand, (C3) is not excluded by Schlechta.

Finally, a good example of a condition which is not excluded (neither by myself nor Schlechta) is of course the arbitrarily big loop condition ($|loop$) of Section 8.3.

9.2 Impossibility results

Given the characterizations of Section 8.3, an interesting question arises: is it possible to replace ($\star loop$) by a finite condition? Obviously, the presence of ($\star loop$) is due to the presence of ($|loop$). So, to solve the problem one might attack its source, i.e. try to replace ($|loop$) by a finite condition. But, in the present chapter, we will show that for families of distance operators, there is no normal characterization (these results have been published in [BN06]). The symmetric family will be concerned with this and therefore ($|loop$) cannot be replaced by a finite and universally quantified condition.

Now, we can go further. Indeed, there is a strong connexion between the distance operators and the distance-based revision operators. Lehmann *et al.* used this connexion to get their results on the latter from their results on the former. It is reasonable to think that the same thing can be done with our negative results, i.e. this thesis can certainly be continued in future work to show that for families of distance-based revision operators, there is no either normal characterization. For instance, the family which is symmetric, CP, and DP might well be concerned with this, which suggests that ($\star loop$) cannot be replaced by a finite and universally quantified condition.

We provide our first impossibility result. It generalizes Proposition 4.2.11 of [Sch04]. Our proof will be based on a slight adaptation of a particular pseudo-distance invented by Schlechta, called “Hamster Wheel”.

Proposition 84 Let \mathcal{V} be an infinite set, \mathcal{N} the set of all symmetric IR positive TIR distance operators from $\mathcal{P}(\mathcal{V})^2$ to $\mathcal{P}(\mathcal{V})$, and \mathcal{O} a set of distance operators from $\mathcal{P}(\mathcal{V})^2$ to $\mathcal{P}(\mathcal{V})$ such that $\mathcal{N} \subseteq \mathcal{O}$. Then, there does not exist a normal characterization of \mathcal{O} .

Proof Suppose the contrary, i.e. there exist $n \in \mathbb{N}^+$ and a relation Φ on $\mathcal{P}(\mathcal{V})^{3n}$ such that

- (0) for every binary operator $|$ on $\mathcal{P}(\mathcal{V})$, we have $| \in \mathcal{O}$ iff
 $\forall V_1, \dots, V_n, W_1, \dots, W_n \subseteq \mathcal{V}, (V_1, \dots, V_n, W_1, \dots, W_n, V_1 | W_1, \dots, V_n | W_n) \in \Phi.$

As \mathcal{V} is infinite, there exist distinct $v_1, \dots, v_m, w_1, \dots, w_m \in \mathcal{V}$, with $m = n + 3$.

Let $X = \{v_1, \dots, v_m, w_1, \dots, w_m\}$.

Let \mathcal{D} be the pseudo-distance on \mathcal{V} such that $\mathcal{D} = \langle \mathbb{R}, <, d \rangle$, where $<$ is the usual order on \mathbb{R} and d is the function defined as follows. Let $v, w \in \mathcal{V}$. Consider the cases that follow:

Case 1: $v = w$.

Case 2: $v \neq w$.

Case 2.1: $\{v, w\} \not\subseteq X$.

Case 2.2: $\{v, w\} \subseteq X$.

Case 2.2.1: $\{v, w\} \subseteq \{v_1, \dots, v_m\}$.

Case 2.2.2: $\{v, w\} \subseteq \{w_1, \dots, w_m\}$.

Case 2.2.3: $\exists i, j \in [1, m], \{v, w\} = \{v_i, w_j\}$.

Case 2.2.3.1: $i = j$.

Case 2.2.3.2: $\text{abs}(i - j) \in \{1, m - 1\}$.

Case 2.2.3.3: $1 < \text{abs}(i - j) < m - 1$.

Then,

$$d(v, w) = \begin{cases} 0 & \text{if Case 1 holds;} \\ 1 & \text{if Case 2.1 holds;} \\ 1.1 & \text{if Case 2.2.1 holds;} \\ 1.1 & \text{if Case 2.2.2 holds;} \\ 1.4 & \text{if Case 2.2.3.1 holds;} \\ 2 & \text{if Case 2.2.3.2 holds;} \\ 1.2 & \text{if Case 2.2.3.3 holds.} \end{cases}$$

Note that \mathcal{D} is essentially, but not exactly, the Hamster Wheel of [Sch04]. The main difference is Case 2.1, which was not treated by Schlechta. The reader can find a picture of \mathcal{D} in Figure 1.

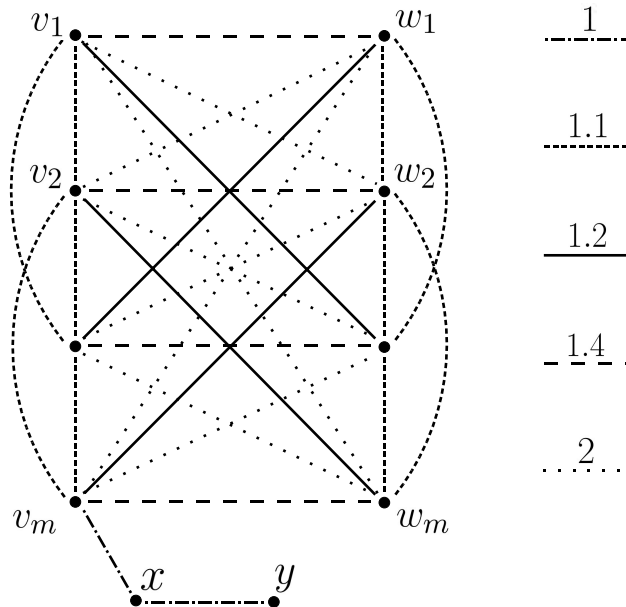


Figure 1: a slight adaptation of Hamster Wheel.

Let $|$ be the binary operator on $\mathcal{P}(\mathcal{V})$ such that $\forall V, W \subseteq \mathcal{V}$,

$$V|W = \begin{cases} \{w_m\} & \text{if } V = \{v_m, v_1\} \text{ and } W = \{w_m, w_1\}; \\ \{v_m\} & \text{if } V = \{w_m, w_1\} \text{ and } W = \{v_m, v_1\}; \\ V|_{\mathcal{D}}W & \text{otherwise.} \end{cases}$$

The difference between $|$ and $|_{\mathcal{D}}$ is sufficiently big so that

(1) $|$ is not a distance operator.

The proof of this will be given later. Thus, $| \notin \mathcal{O}$. Thus, by (0), we get that

(2) $\exists V_1, \dots, V_n, W_1, \dots, W_n \subseteq \mathcal{V}, (V_1, \dots, V_n, W_1, \dots, W_n, V_1|W_1, \dots, V_n|W_n) \notin \Phi$.

In addition, we took m sufficiently big so that

(3) $\exists r \in [1, m-1], \forall i \in [1, n], \{V_i, W_i\} \neq \{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}$.

Again, we will give the proof of this later, for a better readability.

Let $|'$ be the binary operator on $\mathcal{P}(\mathcal{V})$ such that $\forall V, W \subseteq \mathcal{V}$,

$$V|'W = \begin{cases} \{w_{r+1}\} & \text{if } V = \{v_r, v_{r+1}\} \text{ and } W = \{w_r, w_{r+1}\}; \\ \{v_{r+1}\} & \text{if } V = \{w_r, w_{r+1}\} \text{ and } W = \{v_r, v_{r+1}\}; \\ V|W & \text{otherwise.} \end{cases}$$

The difference between $|'$ and $|$ is “invisible” for Φ .

More formally, by (3), we have that: $\forall i \in [1, n], V_i|'W_i = V_i|W_i$.

Therefore, by (2), we get that $(V_1, \dots, V_n, W_1, \dots, W_n, V_1|'W_1, \dots, V_n|'W_n) \notin \Phi$.

Thus, by (0), we obtain that

(4) $|' \notin \mathcal{O}$.

But, at the same time, there is a convenient pseudo-distance that represents $|'$.

Indeed, let \mathcal{D}' be the pseudo-distance on \mathcal{V} such that $\mathcal{D}' = \langle \mathbb{R}, <, d' \rangle$, where d' is the function such that $\forall v, w \in \mathcal{V}$,

$$d'(v, w) = \begin{cases} 1.3 & \text{if } \exists i \in [r+1, m], \{v, w\} = \{v_i, w_i\}; \\ d(v, w) & \text{otherwise.} \end{cases}$$

Then, it can be shown that \mathcal{D}' represents $|'$. More formally,

(5) $|' = |_{\mathcal{D}'}$.

The proof will be given later. Now, \mathcal{D}' is obviously symmetric, IR, and positive.

In addition, \mathcal{D}' is TIR, because \mathcal{D}' is IR and $\forall v, w \in \mathcal{V}, d'(v, w) = 0$ or $1 \leq d'(v, w) \leq 2$.

Therefore, $|'$ is a symmetric IR positive TIR distance operator.

Consequently, $|' \in \mathcal{N}$ and thus

(6) $|' \in \mathcal{O}$.

So, we get the final contradiction by (4) and (6).

Proof of (1). Suppose the contrary, i.e.

suppose there exists a pseudo-distance $\mathcal{S} = \langle C, \prec, g \rangle$ on \mathcal{V} such that $|\mathcal{S}| = |\mathcal{S}|$.

Then, the costs of the moves from the v_i 's to the w_i 's are all equals:

$$(1.1) \quad \forall i \in [1, m-1], g(v_i, w_i) = g(v_{i+1}, w_{i+1}).$$

On the other hand, there is an inequality among them:

$$(1.2) \quad g(v_m, w_m) \prec g(v_1, w_1).$$

But, by (1.1) and (1.2), we get an obvious contradiction.

Proof of (1.1). Let $i \in [1, m-1]$.

Then, $\{v_i, v_{i+1}\} |_{\mathcal{S}} \{w_i, w_{i+1}\} = \{v_i, v_{i+1}\} |_{\mathcal{D}} \{w_i, w_{i+1}\} = \{w_i, w_{i+1}\}$.

Case 1: $g(v_i, w_i) \prec g(v_{i+1}, w_{i+1})$.

As $\{v_i\} |_{\mathcal{S}} \{w_i, w_{i+1}\} = \{v_i\} |_{\mathcal{D}} \{w_i, w_{i+1}\} = \{w_i\} \not\equiv w_{i+1}$, we get $g(v_i, w_i) \prec g(v_i, w_{i+1})$.

Thus, $w_{i+1} \notin \{v_i, v_{i+1}\} |_{\mathcal{S}} \{w_i, w_{i+1}\}$, which is impossible.

Case 2: $g(v_{i+1}, w_{i+1}) \prec g(v_i, w_i)$.

As $\{v_{i+1}\} |_{\mathcal{S}} \{w_i, w_{i+1}\} = \{v_{i+1}\} |_{\mathcal{D}} \{w_i, w_{i+1}\} = \{w_{i+1}\} \not\equiv w_i$,

we get $g(v_{i+1}, w_{i+1}) \prec g(v_{i+1}, w_i)$.

Therefore, $w_i \notin \{v_i, v_{i+1}\} |_{\mathcal{S}} \{w_i, w_{i+1}\}$, which is impossible.

Case 3: $g(v_i, w_i) \not\prec g(v_{i+1}, w_{i+1})$ and $g(v_{i+1}, w_{i+1}) \not\prec g(v_i, w_i)$.

Then, as \prec is total, $g(v_i, w_i) = g(v_{i+1}, w_{i+1})$.

Proof of (1.2). As $\{v_m, v_1\} |_{\mathcal{S}} \{w_m, w_1\} = \{v_m, v_1\} |_{\mathcal{D}} \{w_m, w_1\} = \{w_m\} \not\equiv w_1$,

we have $\exists v \in \{v_m, v_1\}, \exists w \in \{w_m, w_1\}, g(v, w) \prec g(v_1, w_1)$.

Case 1: $g(v_m, w_m) \prec g(v_1, w_1)$. We are done.

Case 2: $g(v_m, w_1) \prec g(v_1, w_1)$.

As $\{v_m\} |_{\mathcal{S}} \{w_m, w_1\} = \{v_m\} |_{\mathcal{D}} \{w_m, w_1\} = \{w_m\} \not\equiv w_1$, we get $g(v_m, w_m) \prec g(v_m, w_1)$.

Thus, by transitivity of \prec , $g(v_m, w_m) \prec g(v_1, w_1)$.

Case 3: $g(v_1, w_m) \prec g(v_1, w_1)$.

Then, $\{v_1\} |_{\mathcal{S}} \{w_m, w_1\} = \{w_m\}$.

However, $\{v_1\} |_{\mathcal{S}} \{w_m, w_1\} = \{v_1\} |_{\mathcal{D}} \{w_m, w_1\} = \{w_1\}$, which is impossible.

Case 4: $g(v_1, w_1) \prec g(v_1, w_1)$. Impossible by irreflexivity of \prec .

Proof of (3). For all $s \in [1, m-1]$, let $I_s := \{i \in [1, n] : \{V_i, W_i\} = \{\{v_s, v_{s+1}\}, \{w_s, w_{s+1}\}\}\}$.

Suppose the opposite of what we want to show, i.e. $\forall s \in [1, m-1], I_s \neq \emptyset$.

As $v_1, \dots, v_m, w_1, \dots, w_m$ are distinct, $\forall s, t \in [1, m-1]$, if $s \neq t$, then $I_s \cap I_t = \emptyset$.

Therefore, $m-1 \leq |I_1 \cup \dots \cup I_{m-1}|$.

On the other hand, $\forall s \in [1, m-1], I_s \subseteq [1, n]$. Thus, $|I_1 \cup \dots \cup I_{m-1}| \leq n$.

Thus, $m-1 \leq n$, which is impossible as $m = n+3$.

Proof of (5). Let $V, W \subseteq \mathcal{V}$.

Case 1: $V = \{v_r, v_{r+1}\}$ and $W = \{w_r, w_{r+1}\}$.

Then, $V |_{\mathcal{D}'} W = \{w_{r+1}\} = V |_{\mathcal{D}'} W$.

Case 2: $V = \{w_r, w_{r+1}\}$ and $W = \{v_r, v_{r+1}\}$.

Then, $V |_{\mathcal{D}'} W = \{v_{r+1}\} = V |_{\mathcal{D}'} W$.

Case 3: $V = \{v_m, v_1\}$ and $W = \{w_m, w_1\}$.
Then, $V|_D W = V|W = \{w_m\} = V|_{\mathcal{D}'} W$.

Case 4: $V = \{w_m, w_1\}$ and $W = \{v_m, v_1\}$.
Then, $V|_D W = V|W = \{v_m\} = V|_{\mathcal{D}'} W$.

Case 5: $\{V, W\} \notin \{\{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}, \{\{v_m, v_1\}, \{w_m, w_1\}\}\}$.
Then, $V|_D W = V|W = V|_{\mathcal{D}'} W$.

Case 5.1: $V = \emptyset$ or $W = \emptyset$.
Then, $V|_{\mathcal{D}'} W = \emptyset = V|_{\mathcal{D}'} W$.

Case 5.2: $V \cap W \neq \emptyset$.
Then, $V|_{\mathcal{D}'} W = V \cap W = V|_{\mathcal{D}'} W$.

Case 5.3: $V \neq \emptyset, W \neq \emptyset$, and $V \cap W = \emptyset$.

Case 5.3.1: $V \not\subseteq X$.
Then, $V|_{\mathcal{D}'} W = W = V|_{\mathcal{D}'} W$.

Case 5.3.2: $V \subseteq X$.
Case 5.3.2.1: $W \not\subseteq X$.
Then, $V|_{\mathcal{D}'} W = W \setminus X = V|_{\mathcal{D}'} W$.

Case 5.3.2.2: $W \subseteq X$.
Case 5.3.2.2.1: $V \not\subseteq \{v_1, \dots, v_m\}$ and $V \not\subseteq \{w_1, \dots, w_m\}$.
Then, $V|_{\mathcal{D}'} W = W = V|_{\mathcal{D}'} W$.

Case 5.3.2.2.2: $V \subseteq \{v_1, \dots, v_m\}$ and $W \not\subseteq \{w_1, \dots, w_m\}$.
Then, $V|_{\mathcal{D}'} W = W \cap \{v_1, \dots, v_m\} = V|_{\mathcal{D}'} W$.

Case 5.3.2.2.3: $V \subseteq \{v_1, \dots, v_m\}$ and $W \subseteq \{w_1, \dots, w_m\}$.
Case 5.3.2.2.3.1: $\exists v_i \in V, \exists w_j \in W, 1 < \text{abs}(i - j) < m - 1$.
Then, $V|_{\mathcal{D}'} W = \{w_j \in W : \exists v_i \in V, 1 < \text{abs}(i - j) < m - 1\} = V|_{\mathcal{D}'} W$.

Case 5.3.2.2.3.2: $\forall v_i \in V, \forall w_j \in W, \text{abs}(i - j) \in \{0, 1, m - 1\}$.
Case 5.3.2.2.3.2.1: $|V \cup W| \geq 5$.
As $m \geq 4$, we get $\exists v_i \in V, \exists w_j \in W, 1 < \text{abs}(i - j) < m - 1$, which is impossible.

Case 5.3.2.2.3.2.2: $|V \cup W| \in \{2, 3, 4\}$.
Case 5.3.2.2.3.2.2.1: $\{k \in [1, m] : v_k \in V \text{ and } w_k \in W\} = \emptyset$.
Then, $V|_{\mathcal{D}'} W = W = V|_{\mathcal{D}'} W$.

Case 5.3.2.2.3.2.2.2: $\exists i \in [1, m], \{k \in [1, m] : v_k \in V \text{ and } w_k \in W\} = \{i\}$.
Then, $V|_{\mathcal{D}'} W = \{w_i\} = V|_{\mathcal{D}'} W$.

Case 5.3.2.2.3.2.2.3: $\exists i, j \in [1, m], i < j, \{k \in [1, m] : v_k \in V \text{ and } w_k \in W\} = \{i, j\}$.
Then, $V = \{v_i, v_j\}$ and $W = \{w_i, w_j\}$.

Case 5.3.2.2.3.2.2.3.1: $r < i$ or $j \leq r$.
Then, $V|_{\mathcal{D}'} W = \{w_i, w_j\} = V|_{\mathcal{D}'} W$.

Case 5.3.2.2.3.2.2.3.2: $i \leq r < j$.
As $\text{abs}(i - j) \in \{1, m - 1\}, \langle V, W \rangle \in \{\{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}, \{\{v_1, v_m\}, \{w_1, w_m\}\}\}$,
which is impossible.

Case 5.3.2.2.3.2.2.4: $|\{k \in [1, m] : v_k \in V \text{ and } w_k \in W\}| \geq 3$.
Then, $|V \cup W| \geq 6$, impossible.

Case 5.3.2.2.4: $V \subseteq \{w_1, \dots, w_m\}$ and $W \not\subseteq \{v_1, \dots, v_m\}$.
Then, $V|_{\mathcal{D}'} W = W \cap \{w_1, \dots, w_m\} = V|_{\mathcal{D}'} W$.

Case 5.3.2.2.5: $V \subseteq \{w_1, \dots, w_m\}$ and $W \subseteq \{v_1, \dots, v_m\}$.
Similar to Case 5.3.2.2.3. ■

We extend the negative results to the “liberal” and Hamming properties. The proof will be based on an adaptation of the Hamster Wheel. Note that the Hamming distance is a realistic distance which has been investigated by many researchers. This strengthens the importance of Proposition 85 in the sense that not only abstract but also concrete cases do not admit a normal characterization.

Proposition 85 Let $\mathcal{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ be a propositional language with \mathcal{A} infinite and countable, \mathcal{M} a matrix on \mathcal{L} , \mathcal{V} the set of all \mathcal{M} -valuations, \mathcal{N} the set of all symmetric, HIR, liberally IR, liberally positive, and liberally TIR distance operators from $\mathcal{P}(\mathcal{V})^2$ to $\mathcal{P}(\mathcal{V})$, and \mathcal{O} a set of distance operators from $\mathcal{P}(\mathcal{V})^2$ to $\mathcal{P}(\mathcal{V})$ such that $\mathcal{N} \subseteq \mathcal{O}$.

Then, there does not exist a normal characterization of \mathcal{O} .

Proof Suppose the contrary, i.e. there are $n \in \mathbb{N}^+$ and a relation Φ on $\mathcal{P}(\mathcal{V})^{3n}$ such that

- (0) for every binary operator $|$ on $\mathcal{P}(\mathcal{V})$, we have $| \in \mathcal{O}$ iff
 $\forall V_1, \dots, V_n, W_1, \dots, W_n \subseteq \mathcal{V}, (V_1, \dots, V_n, W_1, \dots, W_n, V_1|W_1, \dots, V_n|W_n) \in \Phi$.

As \mathcal{A} is infinite, there are distinct $p_1, \dots, p_m, q_1, \dots, q_m \in \mathcal{A}$, with $m = n + 3$.

Let's pose $\mathcal{M} = \langle T, D, f \rangle$.

As $D \neq \emptyset$ and $T \setminus D \neq \emptyset$, there are distinct $0, 1 \in T$.

Now, $\forall i \in [1, m]$, let v_i be the \mathcal{M} -valuation that assigns 1 to p_i and 0 to each other atom of \mathcal{A} .

Similarly, $\forall i \in [1, m]$, let w_i be the \mathcal{M} -valuation that assigns 1 to q_i and 0 to each other atom of \mathcal{A} .

In addition, let $X = \{v_1, \dots, v_m, w_1, \dots, w_m\}$.

Note that $\forall v, w \in X$, with $v \neq w$, we have $|h(v, w)| = 2$.

Finally, let \mathcal{D} be the pseudo-distance on \mathcal{V} such that $\mathcal{D} = \langle \mathbb{R} \cup \{|\mathbb{N}|\}, \prec, d \rangle$,

where \prec and d are defined as follows.

Let $c, c' \in \mathbb{R} \cup \{|\mathbb{N}|\}$.

Then, $c \prec c'$ iff $(c, c' \in \mathbb{R} \text{ and } c < c')$ or $(c \in \mathbb{R} \text{ and } c' = |\mathbb{N}|)$.

Let $v, w \in \mathcal{V}$ and consider the cases which follow:

Case 1: $v = w$.

Case 2: $v \neq w$.

Case 2.1: $\{v, w\} \not\subseteq X$.

Case 2.1.1: $|h(v, w)| = 1$.

Case 2.1.2: $|h(v, w)| \geq 2$.

Case 2.2: $\{v, w\} \subseteq X$.

Case 2.2.1: $\{v, w\} \subseteq \{v_1, \dots, v_m\}$.

Case 2.2.2: $\{v, w\} \subseteq \{w_1, \dots, w_m\}$.

Case 2.2.3: $\exists i, j \in [1, m], \{v, w\} = \{v_i, w_j\}$.

Case 2.2.3.1: $i = j$.

Case 2.2.3.2: $abs(i - j) \in \{1, m - 1\}$.

Case 2.2.3.3: $1 < abs(i - j) < m - 1$.

The proof is verbatim the same as for (1) in the proof of Proposition 84.

Consequently, $| \notin \mathcal{O}$, thus, by (0), we get that

$$(1) \exists V_1, \dots, V_n, W_1, \dots, W_n \subseteq \mathcal{V}, (V_1, \dots, V_n, W_1, \dots, W_n, V_1|W_1, \dots, V_n|W_n) \notin \Phi.$$

Moreover, we chose m big enough so that

$$(2) \exists r \in [1, m-1], \forall i \in [1, n], \{V_i \cap X, W_i \cap X\} \neq \{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}.$$

The proof is verbatim the same as for (3) in the proof of Proposition 84, except that V_i is replaced by $V_i \cap X$ and W_i is replaced by $W_i \cap X$.

Let now $|'$ be the binary operator on $\mathcal{P}(\mathcal{V})$ defined as follows.

Let $V, W \subseteq \mathcal{V}$ and consider the cases that follow:

Case 1: $\forall v \in V, \forall w \in W, \{v, w\} \subseteq X$ or $3 \leq |h(v, w)|$.

Case 1.1: $V \cap X = \{v_r, v_{r+1}\}$ and $W \cap X = \{w_r, w_{r+1}\}$.

Case 1.2: $V \cap X = \{w_r, w_{r+1}\}$ and $W \cap X = \{v_r, v_{r+1}\}$.

Case 1.3: $\{V \cap X, W \cap X\} \neq \{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}$.

Case 2: $\exists v \in V, \exists w \in W, \{v, w\} \not\subseteq X$ and $|h(v, w)| < 3$.

Then,

$$V|'W = \begin{cases} \{w_{r+1}\} & \text{if Case 1.1 holds;} \\ \{v_{r+1}\} & \text{if Case 1.2 holds;} \\ V|W & \text{if Case 1.3 or Case 2 holds.} \end{cases}$$

The difference between $|'$ and $|$ is “invisible” for Φ .

More formally, $\forall i \in [1, n], V_i|'W_i = V_i|W_i$.

The proof is obvious by (2).

Therefore, by (1), we get $(V_1, \dots, V_n, W_1, \dots, W_n, V_1|'W_1, \dots, V_n|'W_n) \notin \Phi$, thus, by (0):

$$(3) |' \notin \mathcal{O}.$$

But, at the same time, there is convenient pseudo-distance that represents $|'$.

Indeed, let \mathcal{D}' be the pseudo-distance on \mathcal{V} such that $\mathcal{D}' = \langle \mathbb{R} \cup \{|\mathbb{N}|\}, \prec, d' \rangle$, where d' is the function such that $\forall v, w \in \mathcal{V}$,

$$d'(v, w) = \begin{cases} 2.3 & \text{if } \exists i \in [r+1, m], \{v, w\} = \{v_i, w_i\}; \\ d(v, w) & \text{otherwise.} \end{cases}$$

Note that $\forall v, w \in \mathcal{V}, |h(v, w)| \in \mathbb{N}$ iff $d(v, w) \in \mathbb{R}$ iff $d'(v, w) \in \mathbb{R}$.

Therefore, $|h(v, w)| = |\mathbb{N}|$ iff $d(v, w) = |\mathbb{N}|$ iff $d'(v, w) = |\mathbb{N}|$.

Note again that $\forall v, w \in \mathcal{V}$, with $|h(v, w)| \in \mathbb{N}$, $|h(v, w)| \leq d'(v, w) \leq d(v, w) \leq |h(v, w)| + 0.5$.

Now, it can be shown that \mathcal{D}' represents $|'$. More formally,

$$(4) |' = |_{\mathcal{D}'}$$

Now, clearly, \mathcal{D}' is symmetric, liberally IR, and liberally positive.

In addition, we will show:

$$(5) \mathcal{D}' \text{ is HIR};$$

$$(6) \mathcal{D}' \text{ is liberally TIR.}$$

So, $|'$ is a symmetric, liberally IR, liberally positive, liberally TIR, and HIR distance operator. Therefore, $|' \in \mathcal{N}$ and thus:

$$(7) \quad |' \in \mathcal{O}.$$

Finally, we get a contradiction by (3) and (7), which ends the proof.

Proof of (4). Let $V, W \subseteq \mathcal{V}$.

Case 1: $\forall v \in V, \forall w \in W, \{v, w\} \subseteq X$ or $3 \leq |h(v, w)|$.

Case 1.1: $V \cap X = \{v_r, v_{r+1}\}$ and $W \cap X = \{w_r, w_{r+1}\}$.

Then, $V|'W = \{w_{r+1}\} = V|_{\mathcal{D}'}W$.

Case 1.2: $V \cap X = \{w_r, w_{r+1}\}$ and $W \cap X = \{v_r, v_{r+1}\}$.

Then, $V|'W = \{v_{r+1}\} = V|_{\mathcal{D}'}W$.

Case 1.3: $V \cap X = \{v_m, v_1\}$ and $W \cap X = \{w_m, w_1\}$.

Then, $V|'W = \{w_m\} = V|_{\mathcal{D}'}W$.

Case 1.4: $V \cap X = \{w_m, w_1\}$ and $W \cap X = \{v_m, v_1\}$.

Then, $V|'W = \{v_m\} = V|_{\mathcal{D}'}W$.

Case 1.5: $\{V \cap X, W \cap X\} \notin \{\{\{v_m, v_1\}, \{w_m, w_1\}\}, \{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}\}$.

Then, $V|'W = V|W = V|_{\mathcal{D}'}W$.

Case 1.5.1: $V \cap W \neq \emptyset$.

Then, $V|_{\mathcal{D}'}W = V \cap W = V|_{\mathcal{D}'}W$.

Case 1.5.2: $V \cap W = \emptyset$.

Case 1.5.2.1: $V \cap X = \emptyset$ or $W \cap X = \emptyset$.

Then, $\forall v \in V, \forall w \in W, d'(v, w) = d(v, w)$. Thus, $V|_{\mathcal{D}'}W = V|_{\mathcal{D}'}W$.

Case 1.5.2.2: $V \cap X \neq \emptyset$ and $W \cap X \neq \emptyset$.

Then, we will show:

$$(4.1) \quad V|_{\mathcal{D}'}W = V \cap X|_{\mathcal{D}'}W \cap X;$$

$$(4.2) \quad V|_{\mathcal{D}'}W = V \cap X|_{\mathcal{D}'}W \cap X.$$

But, we have $V \cap X|_{\mathcal{D}'}W \cap X = V \cap X|_{\mathcal{D}'}W \cap X$.

The proof of this is verbatim the same as for Case 5.3.2.2, in the proof of (5), in the proof of Proposition 84, except that V is replaced by $V \cap X$ and W is replaced by $W \cap X$.

Case 2: $\exists v \in V, \exists w \in W, (\{v, w\} \not\subseteq X$ and $|h(v, w)| \leq 2)$.

Then, $V|'W = V|W = V|_{\mathcal{D}'}W$.

Case 2.1. $V \cap W \neq \emptyset$.

Then, $V|_{\mathcal{D}'}W = V \cap W = V|_{\mathcal{D}'}W$.

Case 2.2. $V \cap W = \emptyset$.

Case 2.2.1. $\exists v' \in V, \exists w' \in W, |h(v, w)| = 1$.

Then, $V|_{\mathcal{D}'}W = \{w \in W : \exists v \in V, |h(v, w)| = 1\} = V|_{\mathcal{D}'}W$.

Case 2.2.2. $\forall v' \in V, \forall w' \in W, |h(v, w)| \geq 2$.

Then, $V|_{\mathcal{D}'}W = \{w \in W : \exists v \in V, \{v, w\} \not\subseteq X$ and $|h(v, w)| = 2\} = V|_{\mathcal{D}'}W$.

Proof of (4.1). Direction: " \subseteq ".

Let $w \in V|_{\mathcal{D}'}W$. Then, $\exists v \in V, \forall v' \in V, \forall w' \in W, d(v, w) \preceq d(v', w')$.

Case 1: $\{v, w\} \subseteq X$.

Then, $w \in V \cap X |_{\mathcal{D}} W \cap X$.

Case 2: $\{v, w\} \not\subseteq X$.

We have $\exists v' \in V \cap X$ and $\exists w' \in W \cap X$. In addition, $d(v', w') \in \mathbb{R}$ and $d(v', w') \leq 2.5$.

Case 2.1: $|h(v, w)| = |\mathbb{N}|$.

Then, $d(v, w) = |\mathbb{N}|$. Therefore, $d(v', w') \prec d(v, w)$, which is impossible.

Case 2.2: $|h(v, w)| \in \mathbb{N}$.

Then, $d(v, w) \in \mathbb{R}$ and $3 \leq |h(v, w)| \leq d(v, w)$. Thus, $d(v', w') < d(v, w)$.

Thus, $d(v', w') \prec d(v, w)$, which is impossible.

Direction: “ \supseteq ”.

Let $w \in V \cap X |_{\mathcal{D}} W \cap X$.

Then, $\exists v \in V \cap X, \forall v' \in V \cap X, \forall w' \in W \cap X, d(v, w) \preceq d(v', w')$.

Let $v' \in V, w' \in W$.

Case 1: $\{v', w'\} \subseteq X$.

Then, $d(v, w) \preceq d(v', w')$.

Case 2: $\{v', w'\} \not\subseteq X$.

As $v, w \in X$, we have $d(v, w) \in \mathbb{R}$ and $d(v, w) \leq 2.5$.

Case 2.1: $|h(v', w')| = |\mathbb{N}|$.

Then, $d(v', w') = |\mathbb{N}|$, thus $d(v, w) \prec d(v', w')$.

Case 2.2: $|h(v', w')| \in \mathbb{N}$.

Then, $d(v', w') \in \mathbb{R}$ and $3 \leq |h(v', w')| \leq d(v', w')$. Thus, $d(v, w) < d(v', w')$.

Thus, $d(v, w) \prec d(v', w')$.

Consequently, in any case $d(v, w) \preceq d(v', w')$. Thus, $w \in V |_{\mathcal{D}} W$.

Proof of (4.2). Verbatim the proof of (4.1), except that $|_{\mathcal{D}}$ and d are replaced by $|_{\mathcal{D}'}$ and d' .

Proof of (5). Let $v, w, x \in \mathcal{V}$ with $|h(v, w)| < |h(v, x)|$.

Case 1: $|h(v, x)| = |\mathbb{N}|$.

Then, $|h(v, w)| \in \mathbb{N}$. Thus, $d'(v, w) \in \mathbb{R}$ and $d'(v, x) = |\mathbb{N}|$. Therefore, $d'(v, w) \prec d'(v, x)$.

Case 2: $|h(v, x)| \in \mathbb{N}$.

Then, $|h(v, w)| \in \mathbb{N}$. Therefore, $d'(v, x) \in \mathbb{R}, d'(v, w) \in \mathbb{R}$, and

$d'(v, w) \leq |h(v, w)| + 0.5 < |h(v, w)| + 1 \leq |h(v, x)| \leq d'(v, x)$. Thus, $d'(v, w) \prec d'(v, x)$.

Proof of (6). Let $v, w, x \in \mathcal{V}$.

First, note that $h(v, x) \subseteq h(v, w) \cup h(w, x)$, thus $|h(v, x)| \leq |h(v, w) \cup h(w, x)|$.

Case 1: $d'(v, x) = |\mathbb{N}|$.

Then, $|h(v, x)| = |\mathbb{N}|$. Now, suppose $d'(v, w) \in \mathbb{R}$ and $d'(w, x) \in \mathbb{R}$.

Then, $|h(v, w)|, |h(w, x)| \in \mathbb{N}$. Thus, $|h(v, w) \cup h(w, x)| \in \mathbb{N}$.

Therefore, $|h(v, x)| \in \mathbb{N}$, which is impossible.

Thus, $d'(v, w) = |\mathbb{N}|$ or $d'(w, x) = |\mathbb{N}|$.

Case 2: $d'(v, x), d'(v, w), d'(w, x) \in \mathbb{R}$.

Case 2.1: $|h(v, w)| = 0$ or $|h(w, x)| = 0$. Trivial.

Case 2.2: $|h(v, w)| \geq 1$ and $|h(w, x)| \geq 1$.

Case 2.2.1: $|h(v, w)| \geq 2$ or $|h(w, x)| \geq 2$.

Case 2.2.1.1: $|h(v, x)| \in \{0, 1, 2\}$.

Then, $d'(v, x) \leq |h(v, x)| + 0.5 \leq 2.5 < 3 \leq |h(v, w)| + |h(w, x)| \leq d'(v, w) + d'(w, x)$.

Case 2.2.1.2: $|h(v, x)| \geq 3$.

Then, $d'(v, x) = |h(v, x)| \leq |h(v, w)| + |h(w, x)| \leq d'(v, w) + d'(w, x)$.

Case 2.2.2: $|h(v, w)| = 1$ and $|h(w, x)| = 1$.

Case 2.2.2.1: $|h(v, x)| \in \{0, 1, 2\}$.

Then, $d'(v, x) \leq |h(v, x)| + 0.5 \leq 2.5 < 1.4 + 1.4 = d'(v, w) + d'(w, x)$.

Case 2.2.2.2: $|h(v, x)| \geq 3$.

Then, $|h(v, x)| > |h(v, w)| + |h(w, x)|$, which is impossible. ■

Conclusion

First, let's summarize what has been achieved in Part I. We provided, in a general framework, characterizations for preferential(-discriminative) and pivotal(-discriminative) consequence relations. We showed, in an infinite classical framework, that the family of all pivotal consequence relations does not admit a characterization containing only conditions universally quantified and of limited size. Finally, we showed that universe-codifiability pivotal consequence relations are precisely those X -logics for which X is deductively closed. Concerning Part II, our contributions are the following: extending the work of Schlechta, we showed in a general framework that for several families of distance operators, there is no characterization containing only finite and universally quantified conditions.

Now, we turn to the perspectives. The general idea is to rebuild different kinds of common sense reasoning in paraconsistent frameworks (just as we investigated preferential and pivotal consequence relations in many-valued-frameworks). A first thing that comes to mind is belief revision.

Indeed, most of the approaches to belief revision treat in a trivial way inconsistent sets of beliefs (if they are treated at all). However, people may be rational despite inconsistent beliefs. Indeed, there may be overwhelming evidence for both something and its contrary. There are also inconsistencies in principle impossible to eliminate like the "Paradox of the Preface" [Mak65]. The latter says that a conscientious author has reasons to believe that everything written in his book is true. But, because of human imperfection, he believes also that his book contains errors, and thus that something must be false. Consequently, he has (in the absolute sense) both reasons to believe that everything is true and that something is false. So, principles of rational belief revision must work on inconsistent sets of beliefs. Standard approaches to belief revision (e.g. AGM) all fail to do this as they are based on classical logic. Paraconsistent logics (e.g. *FOUR*) could be the bases of more adequate approaches.

Another advantage of these paraconsistent approaches is that they will not be forced to eliminate a contradiction even when there is no good way to do it. Contradictions could be tolerated until new information eventually comes to justify one or another way of elimination. Finally, these approaches will benefit from an extended field of application which includes multi-agent systems where the agents can have individually inconsistent beliefs.

More widely, we can investigate in paraconsistent frameworks other kinds of reasoning, for instance : update, merging, counterfactual conditionals or simply new plausible relations.

Several sub-results of Part I can be useful to provide further characterizations (in particular, for new plausible relations). More precisely, in the most general cases, i.e. when the choice functions are not necessarily definability preserving, we obtained results thanks to Lemmas 34 and 35 (for preferential relations) and Lemmas 52 and 53 (for pivotal relations). An interesting point is that we used them both in the plain and the discriminative versions. This suggests that they can be used in yet other versions.

In addition, when the choice functions under consideration were definability preserving, we applied Lemmas 44 and 45 both in the preferential-discriminative and the pivotal-discriminative cases to get characterizations. They can probably be applied again to characterize new families of consequence relations defined in the discriminative manner by definability preserving choice functions (not necessarily coherent or strongly coherent, unlike all the families investigated in this thesis).

Part II offers also possibilities. First, recall that the negative results on distance operators can probably be used to show similar impossibility results for distance-based revision operators.

Moreover, this direction of future work can still be followed if we redefine the distance-based revision in a non-classical framework (as we suggested it in the general idea for perspectives). Indeed, as Lehmann *et al.* did, we worked in a general framework. For example, if we redefine the revision in the *FOUR* framework, then we can probably use the results of [LMS01] and our results respectively to demonstrate characterizations of revision operators and show that they cannot be really improved.

Finally, certain update operators, like that of Winslett [Win88] (or that of Forbus), are based on the Hamming distance. The question to know whether the negative results of Part II have repercussions on these operators constitutes an interesting direction of future research.

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