Analyse haute fréquence de l'équation de Helmholtz avec terme source

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Introduction

Cette thèse porte sur l'analyse haute fréquence de l'*équation de Helmholtz* avec terme source. L'équation étudiée modélise la propagation, en régime harmonique en temps, d'une onde haute fréquence dans un milieu d'indice variable. Elle s'écrit :

$$-i\frac{\alpha_\varepsilon}{\varepsilon}u^\varepsilon(x) + \Delta u^\varepsilon(x) + \frac{n^2(x)}{\varepsilon^2}u^\varepsilon(x) = \frac{1}{\varepsilon^2}f^\varepsilon(x), \quad x \in \mathbb{R}^d. \tag{1.0.1}$$

Dans toute la suite, la variable x parcourt tout l'espace $\mathbb{R}^d$ en dimension d supérieure ou égale à trois. Dans l'écriture précédente, n désigne l'indice de réfraction du milieu dans lequel les ondes se propagent, et f^ε est la source émettrice. Ce terme source f^ε modélise un signal présentant des phénomènes de concentration et/ou oscillation à l'échelle ε, $\varepsilon > 0$. Le même petit paramètre $\varepsilon > 0$ mesure la fréquence typique $1/\varepsilon$ des modes propres de l'opérateur de Helmholtz $\Delta + n^2(x)/\varepsilon^2$.

Pour cette raison, des *interactions résonantes* peuvent se produire entre les oscillations dues à la source f^ε et celles, à la même fréquence, dictées par l'opérateur de Helmholtz. C'est l'un des phénomènes que nous étudions quantitativement dans cette thèse.

Enfin, le terme $-i\alpha_\varepsilon u^\varepsilon(x)/\varepsilon$, où α_ε est un paramètre strictement positif qui tend vers 0 quand $\varepsilon \to 0$, permet d'assurer l'existence et l'unicité d'une solution u^ε à l'équation (1.0.1) dans $L^2(\mathbb{R}^d)$ pour tout $\varepsilon > 0$. Son signe sélectionne, en un certain sens, u^ε comme la solution *sortante* de (1.0.1). Il s'agit en d'autres termes d'une condition de radiation à l'infini, qui prescrit la nature des oscillations de u^ε à l'infini. En effet, l'une des difficultés de l'étude de l'équation de Helmholtz provient de ce que l'équation

$$\Delta u^\varepsilon(x) + \frac{n^2(x)}{\varepsilon^2}u^\varepsilon(x) = \frac{1}{\varepsilon^2}f^\varepsilon(x), \tag{1.0.2}$$

seule (sans terme d'absorption) ne suffit pas à déterminer la solution u^ε de manière unique. Pour assurer l'unicité de la solution, il est nécessaire de préciser la condition de radiation à l'infini satisfaite par u^ε. Ceci peut se faire de deux manières : soit en ajoutant un terme régularisant (d'absorption) dans le membre de gauche de l'équation (1.0.2), approche que nous avons adoptée ici, soit en précisant une condition de radiation de type Sommerfeld à l'infini.

L'une des questions qui se pose naturellement est celle du lien entre les deux approches. C'est d'ailleurs l'un des points centraux de notre étude. Nous renvoyons à la fin de cette introduction pour des détails sur ce point (ou au Théorème 4.3.2).

Nous étudions la limite haute fréquence $\varepsilon \to 0$ dans l'équation (1.0.1) en terme de mesures de Wigner ou mesures semi-classiques. Ces mesures sont un moyen de décrire la propagation asymptotique de quantités quadratiques, telle la densité locale d'énergie $|u^\varepsilon(x)|^2$, quand $\varepsilon \to 0$. La mesure de Wigner μ est une mesure sur l'espace des phases (position $\times$ fréquence) : $\mu(x,\xi)$ peut-être vue comme l'énergie portée par les rayons au point x à la fréquence ξ. Ces mesures ont été introduites par E. Wigner [Wig], et développées par P.-L. Lions, T. Paul [LP], P. Gérard [Gér1], ce dernier insistant plus particulièrement sur le calcul pseudo-différentiel semi-classique (voir aussi C. Gérard, A. Martinez [GM], et l'article de revue P. Gérard, P. Markowich, N. Mauser, F. Poupaud [GMMP]). Tous ces outils, y compris les opérateurs pseudo-différentiels semi-classiques, sont adaptés à l'étude des phénomènes de concentration et d'oscillation *à une échelle donnée*, ici ε (les mesures semi-classiques sont à rapprocher des mesures de défaut sans échelle introduites par P. Gérard [Gér2] et L. Tartar [Tar]).

Précisément, une mesure de Wigner μ associée à la suite (u^ε) bornée dans $L^2(\mathbb{R}^d)$ (ou dans un espace de type L^2 à poids, comme nous le verrons plus loin) peut être obtenue (à extraction près) comme limite faible de la suite des transformées de Wigner associées à u^ε

$$W^\varepsilon(u^\varepsilon)(x,\xi) = \mathcal{F}_{y\to\xi}\left(u^\varepsilon\left(x+\varepsilon\frac{y}{2}\right)\overline{u^\varepsilon\left(x-\varepsilon\frac{y}{2}\right)}\right), \tag{1.0.3}$$

$$\lim_{\varepsilon\to 0} W^\varepsilon(u^\varepsilon) = \mu \qquad \text{faiblement},$$

où $\mathcal{F}_{y\to\xi}$ désigne la transformée de Fourier en la variable y seulement. De manière équivalente, on obtient μ en testant u^ε contre des opérateurs pseudo-différentiels semi-classiques : pour un symbole $a \in \mathcal{C}_c^\infty(T^*\mathbb{R}^d)$, on a

$$\lim_{\varepsilon\to 0}\langle Op_\varepsilon^w(a)u^\varepsilon,u^\varepsilon\rangle = \int a(x,\xi)d\mu. \tag{1.0.4}$$

De nombreux travaux ont déjà montré l'efficacité de cet outil dans la description de la limite haute fréquence (ou limite semi-classique). Pour des articles dans des contextes proches de ceux étudiés ici, mentionnons [BCKP], [CPR] pour l'étude haute fréquence de l'équation de Helmholtz avec terme source, ainsi que [GL], [Mil2] pour des cas avec interface. Pour des contextes un peu différents, citons par exemple [EY], [PR], [BKR], [Col].

Nos résultats concernent deux cadres d'étude. D'abord (Chapitre 2), nous étudions le cas de *deux sources* quasi-ponctuelles (*i.e.* localisées autour de deux points distincts de $\mathbb{R}^d$) qui envoient des rayons dans toutes les directions. Pour ce cas, nous nous limitons à un indice de réfraction constant dans tout l'espace. Puis (Chapitres 3 et 4), nous considérons le cas d'un indice de réfraction *discontinu* le long d'une interface séparant deux milieux inhomogènes non bornés (ce qui correspond à un problème de transmission), avec une source localisée près de l'origine. Ce dernier point constitue la plus importante partie de notre travail.

Dans les deux cas, nous obtenons le résultat suivant :
sous des hypothèses géométriques appropriées (détaillées plus loin), la mesure de Wigner $\mu(x,\xi)$ associée à u^ε est l'intégrale, le long des rayons de l'optique géométrique et jusqu'en temps infini, d'une source d'énergie, notée $Q(x,\xi)$ dans la suite, qui mesure les interactions résonantes entre la source f^ε et la solution u^ε.

Dans le premier cas, nous montrons de plus que les interactions résonantes entre les deux sources ponctuelles sont négligeables dans l'asymptotique $\varepsilon \to 0$: seule compte l'interférence constructive entre le milieu d'indice n et chaque source prise séparément.

Dans le second cas, nous montrons enfin que l'interface induit un phénomène de réfraction de l'énergie : certains rayons de l'optique géométrique donnent naissance à une partie transmise et une partie réfléchie à la traversée de la discontinuité.

La structure de la preuve, identique dans les deux cas, est la suivante.

Dans un premier temps, nous établissons des *estimations uniformes* sur la suite (u^ε) dans des espaces appropriés (la suite (u^ε) n'est pas uniformément bornée dans L^2). Ceci nous permet de définir une mesure de Wigner associée.

Dans un deuxième temps, nous étudions la *propagation* de cette mesure. Brièvement, nous montrons que la mesure de Wigner μ associée à u^ε est toujours solution d'une équation de transport cinétique (de type Liouville) dans l'espace des phases (ou dans un ouvert de $T^*\mathbb{R}^d$), de la forme

$$\begin{cases} \xi \cdot \nabla_x \mu + \frac{1}{2} \nabla_x n^2 \cdot \nabla_\xi \mu = \text{source}, \\ + \text{ condition de radiation.} \end{cases} \tag{1.0.5}$$

Techniquement, l'obtention de l'équation (1.0.5) nécessite deux étapes principales.

D'abord, on détermine complètement le terme source d'énergie, noté $Q(x,\xi)$. Typiquement, on a besoin pour cela d'identifier la condition de radiation à l'infini satisfaite par la limite de la suite de solutions remises à l'échelle $w^\varepsilon(x) = \varepsilon^{\frac{d-1}{2}} u^\varepsilon(\varepsilon x)$. Comme indiqué plus loin, w^ε satisfait une équation de Helmholtz dont il est délicat de suivre la condition de radiation *uniformément* lorsque $\varepsilon \to 0$.

Ensuite, l'équation de transport sur μ ne suffit pas à déterminer μ de manière unique : il faut ajouter une condition de radiation à l'infini. Celle que l'on obtient est la trace de la condition prescrite sur u^ε à travers le signe de α_ε. On montre que $\mu(x,\xi) \to 0$ lorsque $|x| \to +\infty$ avec $x \cdot \xi < 0$. En d'autres termes, μ est nulle à l'infini dans la direction entrante. Cette information permet alors d'obtenir μ en utilisant l'équation de transport. Si $\big(X(t),\Xi(t)\big)$ est la bicaractéristique passant par (x,ξ) au temps $t = 0$, on peut calculer la valeur de μ en (x,ξ) à partir de $\mu(X(t),\Xi(t))$, $t < 0$, en remontant cette courbe, *tant qu'elle est définie* :

$$\mu(x,\xi) = \mu(X(t),\Xi(t)) + \int_t^0 Q(X(s),\Xi(s))ds.$$

Comme les hypothèses géométriques faites sur l'indice impliquent que $|X(t)| \to +\infty$ et $X(t) \cdot \Xi(t) < 0$ lorsque $t \to -\infty$, on obtient par passage à la limite

$$\mu(x,\xi) = \int_{-\infty}^0 Q(X(s),\Xi(s))ds.$$

Dans la suite de cette introduction, nous détaillons les résultats obtenus dans cette thèse, d'abord dans le cas de deux sources puis dans le cas d'un indice discontinu.

1.1 Cas de deux sources (Chapitre 2, [Fou1])

Dans le chapitre 2, nous nous intéressons à l'analyse haute fréquence de l'équation de Helmholtz (1.0.1) dans le cas de deux sources quasi-ponctuelles, localisées près de deux points distincts de $\mathbb{R}^d$. Pour simplifier les calculs, les résultats de ce chapitre sont présentés en dimension trois (le résultat serait le même en dimension $d \geq 3$). Précisément, on considère l'équation

$$-i\frac{\alpha_\varepsilon}{\varepsilon}u^\varepsilon(x) + \Delta u^\varepsilon(x) + \frac{n(x)^2}{\varepsilon^2}u^\varepsilon(x) = \frac{1}{\varepsilon^2}S^\varepsilon(x), \quad x \in \mathbb{R}^3 \qquad (1.1.1)$$

où

$$S^\varepsilon(x) = S_0^\varepsilon(x) + S_1^\varepsilon(x) = \frac{1}{\varepsilon^3} S_0\Big(\frac{x}{\varepsilon}\Big) + \frac{1}{\varepsilon^3} S_1\Big(\frac{x-q_1}{\varepsilon}\Big)$$

et q_1 est un point de $\mathbb{R}^3$ distinct de l'origine, les profils de concentration S_0 et S_1 étant des fonctions données.

Nous nous restreignons au cas d'un indice de réfraction *constant* dans $\mathbb{R}^3$: $n(x) \equiv 1$. Notre analyse, basée sur une étude en Fourier, repose fortement sur cette hypothèse.

Dans ce problème, deux types d'interactions sont à quantifier : d'une part l'interaction entre les oscillations dues aux sources et celles, à la même fréquence, dictées par l'opérateur de Helmholtz $\Delta + \frac{1}{\varepsilon^2}$; d'autre part, l'interaction entre les deux sources S_0^ε et S_1^ε.

De tels problèmes d'analyse haute fréquence des équations de Helmholtz *avec terme source* ont été étudiés par J.-D. Benamou, F. Castella, B. Perthame, T. Katsaounis, et O. Runborg dans [BCKP, CPR], articles auxquels notre travail fait suite. Dans [BCKP], les auteurs considèrent le cas d'une source ponctuelle et d'un indice de réfraction régulier général alors que dans [CPR], ils traitent le cas d'une source concentrée autour d'une sous-variété générale avec un indice de réfraction constant (le cas d'un indice régulier général pour une telle source a été traité plus récemment par une autre approche par X. P. Wang, P. Zhang [WZ]). Ici, nous empruntons les méthodes utilisées dans les deux articles [BCKP, CPR].

Dans le cas d'une source ponctuelle, *i.e.* quand S^ε se réduit à S_0^ε, et avec un indice de réfraction constant, il est montré dans [BCKP] que la mesure de Wigner correspondante μ_0 est la solution de l'équation de Liouville

$$0^+\mu_0(x,\xi) + \xi \cdot \nabla_x \mu_0(x,\xi) = Q_0(x,\xi) = \frac{1}{(4\pi)^2}\delta(x)\delta(|\xi|^2-1)|\widehat{S_0}(\xi)|^2. \quad (1.1.2)$$

Ici, le terme 0^+ signifie que μ est la solution sortante donnée par

$$\mu_0(x,\xi) = \int_{-\infty}^0 Q_0(x+t\xi,\xi)dt.$$

En particulier, la source d'énergie Q_0 créée par S_0^ε a son support en $x=0$. De même, si la source S^ε se limitait à S_1^ε, la mesure de Wigner correspondante μ_1 serait solution de l'équation de tranport (1.1.2) avec pour terme source

$$Q_1(x,\xi) = \frac{1}{(4\pi)^2}\delta(x-q_1)\delta(|\xi|^2-1)|\widehat{S_1}(\xi)|^2,$$

qui a son support en $x = q_1$. La propriété d'orthogonalité sur les mesures de Wigner nous laisse alors penser que la mesure μ associée à u^ε est solution de l'équation de Liouville avec $Q_0 + Q_1$ pour terme source.

Dans ce sens, nous prouvons le théorème suivant. Notons toutefois que notre preuve ne repose pas sur la propriété d'orthogonalité[1].

Théorème 1 *Si $\langle x\rangle^N S_0$, $\langle x\rangle^N S_1 \in L^2(\mathbb{R}^3)$ pour un certain $N > 1/2$, la mesure de Wigner μ asssociée à (u^ε) vérifie l'équation de transport suivante*

$$\xi \cdot \nabla_x \mu = \frac{1}{(4\pi)^2}\Big(\delta(x)|\widehat{S_0}(\xi)|^2 + \delta(x-q_1)|\widehat{S_1}(\xi)|^2\Big)\delta(|\xi|^2-1) := Q(x). \quad (1.1.3)$$

De plus, μ est la solution sortante de l'équation (1.1.3) dans le sens faible suivant : pour toute fonction test $R \in \mathcal{C}_c^\infty(\mathbb{R}^6)$, si l'on note $g(x,\xi) = \int_0^\infty R(x+\xi t,\xi)dt$, on a

$$\int_{\mathbb{R}^6} R(x,\xi)d\mu(x,\xi) = -\int_{\mathbb{R}^6} Q(x,\xi)g(x,\xi)dxd\xi. \quad (1.1.4)$$

Comme annoncé dans l'introduction, la preuve de ce résultat se fait en deux étapes, la première étant l'obtention de bornes uniformes sur la suite (u^ε), la seconde l'étude de la mesure de Wigner associée. Tous calculs faits, les points délicats de l'analyse se réduisent à l'étude de la suite (a^ε) solution de l'équation

$$-i\alpha_\varepsilon \varepsilon a^\varepsilon + \Delta a^\varepsilon + a^\varepsilon = S_1\Big(x - \frac{q_1}{\varepsilon}\Big).$$

D'une part, nous montrons que (a^ε) est uniformément bornée, ce qui induit des estimations uniformes sur (u^ε). D'autre part, nous montrons que (a^ε) converge faiblement vers 0 quand $\varepsilon \to 0$, ce qui permet de déterminer entièrement le terme source de l'équation (1.1.3). Ce dernier point ne découle pas du fait que $S_1\Big(x - \frac{q_1}{\varepsilon}\Big)$ converge faiblement vers 0 quand $\varepsilon \to 0$ (son support est translaté autour de q_1/ε). En effet, la solution de l'équation de Helmholtz dépend *globalement* de l'information contenue dans la source, y compris de celle qui se trouve à l'infini (en q_1/ε).

Pour montrer les deux points annoncés, nous utilisons la formule explicite pour la transformée de Fourier de a^ε. Cette analyse fait apparaître le rôle particulier des rayons émis par la source localisée en 0 qui pointent dans la direction de la source localisée en q_1 (et réciproquement) : ces rayons sont les

1. Patrick Gérard nous a fait remarqué que les mesures μ_0 et μ_1 sont mutuellement singulières, ceci en utilisant un argument de dimension analogue à celui de notre preuve. Le résultat du Théorème 1 découle alors directement de la propriété d'orthogonalité.

seuls à induire une interférence entre les deux sources. Cette dernière s'avère négligeable à la limite $\varepsilon \to 0$. Pour montrer ce point, on utilise de manière cruciale le fait que les rayons considérés forment un ensemble de dimension un seulement.

Ces résultats étant établis, la preuve du Théorème 1 découle des propriétés prouvées dans [BCKP]. Nous écrivons l'équation satisfaite par la transformée de Wigner associée à (u^ε), et nous passons à la limite $\varepsilon \to 0$ dans les différents termes qui apparaissent dans cette équation. Le seul terme difficile à traiter (et nouveau) est le terme source.

Notons que la condition de radiation (1.1.4) est une version améliorée de la condition de radiation prouvée dans [BCKP]. Notre argument repose sur l'observation que μ est localisée dans l'ensemble d'énergie $\{|\xi|^2 = 1\}$, une propriété qui n'était pas exploitée dans [BCKP].

1.2 Cas d'un indice discontinu le long d'une interface

Les troisième et quatrième chapitres concernent l'analyse de l'équation de Helmholtz (1.0.1)

$$-i\frac{\alpha_\varepsilon}{\varepsilon}u^\varepsilon(x) + \Delta u^\varepsilon(x) + \frac{n^2(x)}{\varepsilon^2}u^\varepsilon(x) = \frac{1}{\varepsilon^2}f^\varepsilon(x), \quad x \in \mathbb{R}^d$$

pour un indice de réfraction *discontinu* le long d'une interface séparant deux milieux inhomogènes non bornés. La source est ici localisée près de l'origine. Plus précisément, l'indice de réfraction est donné par

$$n^2(x) = \begin{cases} n_+^2(x) & \text{si } x \in \Omega_+ \\ n_-^2(x) & \text{si } x \in \Omega_- \end{cases} \tag{1.2.1}$$

où Ω_+, Ω_- sont deux ouverts non bornés de $\mathbb{R}^d$ qui vérifient $\Omega_+ \cup \overline{\Omega_-} = \overline{\Omega_+} \cup \Omega_- = \mathbb{R}^d$, et la source est choisie comme

$$f^\varepsilon(x) = \frac{1}{\varepsilon^{\frac{d-1}{2}}} f\left(\frac{x}{\varepsilon}\right).$$

La première étape de l'analyse (Chapitre 3) est l'obtention de bornes uniformes sur la suite (u^ε). Ensuite, nous étudions la propagation de la mesure de Wigner associée à (u^ε) dans le Chapitre 4. Chacun de ces points séparément est délicat, comme nous le détaillons maintenant.

1.2.1 Estimations uniformes (Chapitre 3, [Fou2])

Dans le chapitre 3, nous prouvons des estimations uniformes sur la solution de l'équation de Helmholtz (sans hautes fréquences)

$$-i\alpha w + \Delta w + n^2(x) w = f(x),\ x \in \mathbb{R}^d, \tag{1.2.2}$$

où l'indice de réfraction n^2 est donné par (1.2.1).

Le lien entre cette équation et l'équation haute fréquence (1.0.1) se fait en introduisant la solution remise à l'échelle

$$w^\varepsilon(x) = \varepsilon^{\frac{d-1}{2}} u^\varepsilon(\varepsilon x), \tag{1.2.3}$$

qui vérifie de manière évidente l'équation

$$-i\alpha_\varepsilon \varepsilon w^\varepsilon + \Delta w^\varepsilon + n^2(\varepsilon x) w^\varepsilon = f(x). \tag{1.2.4}$$

Afin de les appliquer à (1.2.4) ultérieurement, nous souhaitons obtenir des estimations sur la solution w de (1.2.2) qui soient compatibles avec les hautes fréquences, *i.e.* avec le changement d'échelle (1.2.3). Cela nous impose d'une part de faire des hypothèses sur l'indice qui soient transparentes au changement de variable $x \to \varepsilon x$, et d'autre part d'utiliser des normes *homogènes* en espace.

Dans le cas d'un indice discontinu, de telles estimations ne sont pas disponibles dans la littérature. En effet, les résultats antérieurs ne satisfont pas l'invariance d'échelle : il s'agit pour l'essentiel de bornes inhomogènes, dans des espaces L^2 à poids (premier résultat dû à Eïdus [Eïd1], voir aussi [Zha] et [DP]). Classiquement, si l'on note L^2_s l'espace L^2 à poids $\langle x\rangle^s = (1+|x|^2)^{s/2}$, alors pour $s > 1/2$, si $f \in L^2_s$, la solution de l'équation (1.2.2) (avec un indice régulier ou non) est uniformément bornée dans L^2_{-s} :

$$\|w\|_{L^2_{-s}} \leq C\|f\|_{L^2_s}. \tag{1.2.5}$$

De nombreux travaux concernent également l'obtention de telles bornes pour la solution de l'équation haute fréquence (semi-classique) $-i\alpha_\varepsilon \varepsilon u^\varepsilon + \varepsilon^2 \Delta u^\varepsilon + n(x)^2 u^\varepsilon = f(x)$. Mentionnons par exemple [Wan], [Bur].

Nous utilisons la norme de Morrey-Campanato, définie pour $u \in L^2_{loc}$,

$$\|u\|^2_{\dot{B}^*} = \sup_{R>0} \frac{1}{R} \int_{B(R)} |u(x)|^2 dx, \tag{1.2.6}$$

où $B(R)$ est la boule de rayon R, ainsi que la norme duale

$$\|f\|_{\dot{B}} = \sum_{j\in\mathbb{Z}} \left(2^{j+1}\int_{C(j)} |f(x)|^2 dx\right)^{1/2}, \tag{1.2.7}$$

où $C(j) = \{x \in \mathbb{R}^d / 2^j \leq |x| \leq 2^{j+1}\}$. Ces normes, qui sont bien homogènes en la variable d'espace, sont utilisées par B. Perthame, L. Vega [PV1] dans le cas d'un indice de réfraction régulier, et ont été introduites, dans leur version inhomogène, par S. Agmon, L. Hörmander [AH] pour l'étude de tels problèmes dans le cas de coefficients constants.

Nous énonçons maintenant nos hypothèses et résultats.

Nous supposons que l'interface entre les deux milieux $\Gamma = \partial\Omega_+ = \partial\Omega_-$ est une surface régulière (Lipschitz suffit). On note $\nu(x)$ le vecteur normal unitaire en $x \in \Gamma$ orienté de Ω_- vers Ω_+. Nous supposons également que l'interface Γ et l'indice de réfraction satisfont :

(H1) Il existe $\alpha > 0$ tel que la d-ième composante de ν vérifie

$$\nu_d(x) \geq \alpha \quad \text{pour tout} \quad x \in \Gamma. \tag{1.2.8}$$

(H2) le saut de l'indice $[n^2](x) = n_-^2(x) - n_+^2(x)$ est de signe constant pour tout $x \in \Gamma$; on note σ son signe.

(H3) $n^2 \in L^\infty$.

(H4)

$$2\sum_{j\in\mathbb{Z}} \sup_{C(j)} \frac{(x\cdot\nabla n^2(x))_-}{n^2(x)} := \beta_1 < \infty. \tag{1.2.9}$$

(H5)

$$\frac{1}{\alpha}\sum_{j\in\mathbb{Z}} \sup_{C(j)} 2^{j+1}\frac{(\partial_d n^2(x))_\sigma}{n^2(x)} := \beta_2 < \infty. \tag{1.2.10}$$

où ∇n^2 désigne $\nabla n_+^2 \mathbf{1}_{\Omega_+} + \nabla n_-^2 \mathbf{1}_{\Omega_-}$, *i.e.* la dérivée de n^2 en dehors de l'interface, et $\partial_d n^2$ est la dérivée partielle de n^2 par rapport à la variable x_d en dehors de l'interface.

(H6) $\beta_1 + \beta_2 < 1$.

Nous souhaitons commenter ces hypothèses, inspirées de [PV1]. Notons tout d'abord que ces hypothèses autorisent un indice n à longue portée, et même une certaine croissance : n peut *ne pas tendre* vers une constante à l'infini. Par ailleurs, les hypothèses (H2), (H4) et (H5) imposent des restrictions sur le comportement des rayons de l'optique géométrique. La condition (H4) induit la dispersion à l'infini de ces rayons. En effet, on impose que la partie négative de $x \cdot \nabla n^2$, qui contribue à faire "revenir" les rayons, soit petite. Les conditions (H2) et (H5) assurent que les rayons traversent l'interface dans un sens déterminé (de la gauche vers la droite si le saut de l'indice est positif).

Le théorème principal de ce chapitre est le suivant.

Théorème 2 *En dimension $d \geq 3$, supposons (H1)-(H6). Alors, la solution de l'équation de Helmholtz (1.2.2) vérifie les estimations suivantes :*

$$\|\nabla w\|^2_{\dot{B}^*} + \|nw\|^2_{\dot{B}^*} + \int_\Gamma \big|[n^2]\big| \; |w|^2 d\gamma + \int_{\mathbb{R}^d} |\partial_d n^2| \; |w|^2 dx \;\leq\; C\|f\|^2_{\dot{B}}, \tag{1.2.11}$$

où C est une constante qui ne dépend que de α, β_1 et β_2, et $d\gamma$ est la mesure de surface euclidienne sur Γ.

Nous voudrions insister sur différents points de notre analyse.

D'abord, on obtient les bornes uniformes souhaitées dans l'espace homogène $\dot{B}^*$ pour w et pour son gradient (ce sont les deux premiers termes du membre de gauche de (1.2.11)). Ces bornes traduisent de manière *optimale* la décroissance de la solution w à l'infini ($|w(x)| \sim 1/|x|^{\frac{d-1}{2}}$). Elles impliquent les bornes usuelles (1.2.5) dans les espaces L^2 à poids.

D'autre part, les deux derniers termes du membre de gauche de (1.2.11), qui donnent des estimations en norme L^2 *sans poids*, signifient que, dans un certain sens, l'énergie n'est pas captée par l'interface, et qu'elle irradie principalement dans les directions où $\partial_d n^2$ s'annule. Ces estimations d'énergie sont à rapprocher de celle obtenue dans le cas d'un indice régulier par Perthame, Vega [PV2], d'après laquelle l'énergie $|w|^2$ irradie principalement dans les directions des points critiques de n_∞ (où n_∞ est donné par $n(x) \to n_\infty(\frac{x}{|x|})$, quand $|x| \to \infty$).

Dans un cadre plus restrictif, celui d'une interface *plane*, nous complétons le Théorème 2 en montrant aussi une borne uniforme dans L^2 sur la trace du gradient de w sur l'interface. Cette dernière borne s'avére nécessaire dans l'étude haute fréquence faite au Chapitre 4, pour laquelle nous nous limitons donc au cas d'une interface plane.

Théorème 3 *Sous les hypothèses du théorème 2, si l'on suppose de plus que* $\Gamma = \{x_d = 0\}$ *et qu'il existe* $\beta > 0$ *tel que* $\langle x\rangle^{1+\beta}|\nabla_x n^2| \in L^\infty$*, alors*

$$\int_\Gamma \big|[n^2]\big|\, |\nabla w|^2 dx' \leq C\Big(\|f\|_{\dot{B}}^2 + \|\nabla_{x'} f\|_{\dot{B}}^2\Big) \tag{1.2.12}$$

(où l'on note $x = (x', x_d)$*).*
Ici, C *est une constante qui ne dépend que de* α*,* β_1*,* β_2 *et* $\|\langle x\rangle^{1+\beta}\nabla_x n\|_{L^\infty}$*.*

Notre preuve est basée sur une méthode de multiplicateurs empruntée à Perthame et Vega. Comme dans [PV1], nous utilisons d'abord un multiplicateur de type Morawetz, $\nabla\psi\nabla\bar{w} + \frac{1}{2}\Delta\psi\bar{w}$, qui permet de faire apparaître les propriétés dispersives de l'équation. Nous le combinons ensuite avec un multiplicateur elliptique, $\bar{w}$. Suivant Eidus [Eïd1], nous utilisons également un multiplicateur spécifique au cas avec une interface pour laquelle une direction joue un rôle particulier (ici la direction x_d) : il s'agit de $\partial_d\bar{w}$. Les deux premiers multiplicateurs nous permettent de contrôler à la fois ∇w et w localement dans L^2 par $\|f\|_{\dot{B}}$. On borne ensuite l'intégrale de $|w|^2$ sur l'interface grâce au troisième multiplicateur. Pour obtenir l'estimation sur la trace du gradient de w au Théorème 3, nous utilisons de plus le multiplicateur $n\partial_d\bar{w}$.

Notons que le multiplicateur utilisé dans [PV1], ainsi que les bornes qui y sont obtenues, sont à rapprocher, pour l'équation des ondes, du travail de C. Morawetz [Mor], et pour les équations cinétiques de P.-L. Lions, B. Perthame [LPe]. Nous renvoyons aussi à [Per] pour un lien entre les propriétés dispersives de l'équation de Schrödinger et celles de l'équation de Vlasov, ainsi que, dans le même esprit, à [Col] et [CP].

1.2.2 Propagation de la mesure de Wigner (Chapitre 4, [Fou3])

Dans le chapitre 4, nous étudions la propagation de la mesure de Wigner associée à u^ε solution de (1.0.1), quand l'indice est discontinu le long de l'interface *plane* $\Gamma = \{x_d = 1\}$:

$$n^2(x) = \begin{cases} n_+^2(x) & \text{if } x_d \geq 1 \\ n_-^2(x) & \text{if } x_d < 1. \end{cases}$$

On suppose qu'il existe $n_0 > 0$ tel que $n^2(x) \geq n_0^2$ pour tout $x \in \mathbb{R}^d$, de sorte que l'équation (1.0.1) est uniformément de "type Helmholtz". On suppose enfin que le saut de l'indice à l'interface Γ est de signe constant, positif : $[n^2](x) = n_-^2(x) - n_+^2(x) > 0$ pour tout $x \in \Gamma$.

Avant de détailler nos deux principaux résultats, nous introduisons plusieurs mesures : μ et $\mu_\pm$ désignent les mesures de Wigner associées respectivement à u^ε et aux restrictions de u^ε à chacun des deux milieux $u^\varepsilon_\pm = \mathbf{1}_{\{x_d \gtrless 1\}} u^\varepsilon$. Ces trois mesures sont définies sur $T^*\mathbb{R}^d$:

$$\mu = \lim_{\varepsilon\to 0} W^\varepsilon(u^\varepsilon), \quad \mu_\pm = \lim_{\varepsilon\to 0} W^\varepsilon(u^\varepsilon_\pm).$$

Les mesures $\mu_\pm$ sont somme de deux termes : une mesure à support à l'intérieur du milieu $\{x_d \gtrless 1\}$, *i.e.* $\mathbf{1}_{\{x_d\gtrless 1\}}\mu_\pm$, et un terme porté par le bord $\{x_d = 1\}$. La décomposition suivante, déjà observée par L. Miller dans le cas de l'équation de Schrödinger, est encore vraie ici car la trace de la source f^ε à l'interface est nulle à la limite $\varepsilon \to 0$: il existe des mesures $\mu^{\partial_\pm}$ définies sur $T^*\Gamma$ telles que

$$\mu_\pm = \mathbf{1}_{\{x_d\gtrless 1\}}\mu_\pm + \delta(x_d - 1) \otimes \delta(\xi_d) \otimes \mu^{\partial_\pm}.$$

En d'autres termes, la contribution de $\mu_\pm$ à l'interface est portée par les fréquences telles que $\xi_d = 0$, qui correspondent aux rayons dits "glancing".

Notre premier résultat, valide pour un indice de réfraction général, décrit le phénomène de réfraction induit par l'interface. Selon la direction de propagation, la densité d'énergie est soit totalement réfléchie soit partiellement réfléchie et partiellement transmise selon les lois de Snell-Descartes. Plus précisément, nous montrons le théorème suivant.

Théorème 4 *(Cas général)*
Supposons qu'il y a dispersion à l'infini des rayons de l'optique géométrique (voir les hypothèses du Chapitre 3).
Supposons de plus :
(a) hypothèse de non-interférence (il n'y a pas de densité qui arrive en un même point de l'interface à partir des deux milieux),
(b) il n'y a pas d'énergie captée par l'interface ($\mu^{\partial_\pm} = 0$).
Alors, la mesure de Wigner associée à (u^ε) est donnée par

$$\mu(x,\xi) = \int_{-\infty}^{0} (S_t^* Q)(x,\xi)dt, \tag{1.2.13}$$

où S_t^* est le semi-groupe de Snell-Descartes associé à l'indice de réfraction n et Q est donné par

$$Q(x,\xi) = \frac{1}{2^{d+1}\pi^{d-1}}\delta(x)\delta\big(|\xi|^2 - n^2(0)\big)\big(|\hat f(\xi)|^2 + \hat f(\xi)\bar q(\xi)\big). \tag{1.2.14}$$

Ici, q est une densité L^2 sur la sphère $|\xi|^2 = n^2(0)$, dont la valeur est inconnue en général.

Ce résultat appelle plusieurs commentaires.

1- La source d'énergie Q provient de l'interaction résonante entre la source f^ε et la solution u^ε. En particulier, Q est concentrée à l'origine via la masse de Dirac $\delta(x)$ et sur les fréquences résonantes $|\xi|^2 = n^2(0)$ via la masse de Dirac $\delta\big(|\xi|^2 - n^2(0)\big)$.

2- Comme nous l'avons expliqué dans l'introduction, la valeur de la fonction auxiliaire q qui apparaît dans (1.2.14) est liée à la condition de radiation à l'infini satisfaite par la limite faible w de la suite de solutions remises à l'échelle $w^\varepsilon(x) = \varepsilon^{\frac{d-1}{2}} u^\varepsilon(\varepsilon x)$. La valeur $q = 0$ caractérise le fait que w est la solution sortante de l'équation $\Delta w + n(0)^2 w = f$. Dans le cas général, on ne connait pas la valeur de q, car on ne connaît pas la condition de radiation sur w.

3- L'intégrale en temps infini dans l'expression (1.2.13) traduit la condition de radiation à l'infini vérifiée par la mesure μ. Le suivi de cette condition dans le processus de passage à la limite est l'une des difficultés centrales de notre étude.

4- L'hypothèse (b) d'absence d'énergie captée par l'interface est liée à la fois à la condition de radiation à l'infini satisfaite par la trace de la mesure de Wigner μ sur l'interface, et à l'(absence d')énergie portée par les rayons glissants à l'interface.

Dans le cas particulier où les indices n_+ et n_- sont *constants*, que nous appellerons *cas homogène* dans la suite, nous montrons que les hypothèses $(a)-(b)$ du Théorème 4 sont vérifiées. Notons que ces hypothèses, qui sont de nature géométrique, peuvent être ou non satisfaites pour un indice quelconque (penser par exemple au cas d'un indice pour lequel un même rayon revient plusieurs fois à l'interface). La preuve des hypothèses (a) et (b) ainsi que l'identification de q dans le cas homogène constituent ainsi le second résultat important de ce chapitre.

Théorème 5 *(Cas homogène)*
Quand les indices n_+ et n_- sont constants, on a:
(i) l'hypothèse de non-interférence du Théorème 4 est vérifiée,
(ii) $\mu^{\partial_\pm} = 0$, i.e il n'y a pas d'énergie captée par l'interface,
(iii) w est la solution sortante de l'équation de Helmholtz $\Delta w + n_-^2 w = f$, i.e. $q = 0$.

La combinaison des théorèmes 4 et 5 nous permet d'obtenir une expression totalement explicite pour la mesure de Wigner μ dans le cas homogène.

Nous donnons maintenant les principaux ingrédients des preuves de ces deux résultats.

Preuve du Théorème 4.

Notre preuve du théorème 4 est une combinaison de deux méthodes : celle introduite par L. Miller [Mil1] pour l'étude de la limite semi-classique de problèmes de transmission pour les équations de Schrödinger, et celle introduite par Benamou, Castella, Katsaounis, Perthame [BCKP] pour étudier la limite haute fréquence des équations de Helmholtz avec terme source dans le cas d'un indice de réfraction régulier. Nous donnons maintenant quelques détails de la preuve.

Dans une première étape, nous établissons des bornes sur la suite des transformées de Wigner associées à (u^ε). Comme dans [BCKP], nous déduisons ces estimations à partir des bornes uniformes sur la suite (u^ε) que nous avons établies au Chapitre 3. Avant d'aller plus loin, nous voudrions insister sur un point technique important. Pour traiter le phénomène de réfraction de l'énergie, qui met en jeu le transfert de l'énergie à travers l'interface, nous avons besoin de définir les mesures de Wigner associées aux traces de u^ε et $\varepsilon\partial_d u^\varepsilon$ sur l'interface. Pour ce faire, il nous faut des bornes uniformes sur ces traces, bornes dont nous ne disposons pas dans le cas d'une interface générale (*i.e.* lorsque Γ n'est pas un hyperplan).

Dans une deuxième étape, nous étudions la mesure de Wigner μ *hors de l'interface*. Pour cela, nous suivons [BCKP]. Comme l'indice de réfraction est régulier à l'intérieur de chaque milieu, on peut utiliser leurs résultats pour obtenir l'équation de transport satisfaite par la mesure de Wigner μ hors de l'interface. Leur preuve étant basée sur des estimations du type de celles prouvées au Chapitre 3, on obtient

$$0^+\mu + \xi\cdot\nabla_x\mu + \frac{1}{2}\nabla_x n^2(x)\cdot\nabla_\xi\mu = Q, \tag{1.2.15}$$

à l'intérieur de chaque milieu, où Q est donné par (1.2.14). Comme dans le cas d'un indice régulier, le terme $0^+\mu$ est la trace de la condition de radiation sortante sur u^ε. Il a ici un sens particulier : il détermine μ comme la solution sortante de (1.2.15) donnée par

$$\mu(x,\xi) = \int_{-\infty}^{0} Q\big(X(t),\Xi(t)\big)dt,$$

pour les points (x,ξ) tels que la courbe bicaractéristique $(X(t),\Xi(t))$ définie par

$$\left\{\begin{array}{rclrcl} \dot X(t) &=& \Xi(t), & X(0) &=& x \\ \dot\Xi(t) &=& \frac{1}{2}\nabla_x n^2\left(X(t)\right), & \Xi(0) &=& \xi, \end{array}\right.$$

n'atteint pas l'interface pour $t\in(-\infty,0)$.

Dans une troisième étape, nous étudions le comportement de la mesure de

Wigner à la traversée de l'interface en utilisant la méthode de L. Miller [Mil1], [Mil2]. L'approche est la suivante. Nous écrivons d'abord les équations de transport *jusqu'au bord* satisfaites par $\mu_\pm$. Ces équations sont obtenues sous une forme faible, en utilisant seulement des opérateurs test tangentiels, *i.e.* des opérateurs pseudo-différentiels dont le symbole a est polynomial en ξ_d (voir formule (1.0.4)). Nous obtenons ainsi une description du transport de l'énergie jusqu'au bord. La "trace" de ces équations à l'interface nous donne un système d'équations au bord. La résolution de ce système permet de décrire le transfert de l'énergie à l'interface. On obtient en particulier les coefficients de réflection et de transmission partielles.

Enfin, pour obtenir (1.2.13), on utilise l'équation de transport (1.2.15) ainsi que la condition de radiation à l'infini et les relations de propagation au bord obtenues à l'étape précédente. La méthode est la suivante : pour obtenir la valeur de μ au point (x,ξ), on remonte la bicaractéristique passant par ce point en $t = 0$. Plusieurs cas se présentent : si cette courbe n'atteint pas l'interface pour $t \in (-\infty,0)$, on utilise la condition de radiation à l'infini hors de l'interface pour obtenir μ; sinon, on a une relation entre $\mu(x,\xi)$ et la valeur de μ au point où la bicaractéristique atteint l'interface $(\bar{x},\bar{\xi})$. On utilise alors les relations de propagation au bord pour obtenir une relation entre $\mu(\bar{x},\bar{\xi})$ et la valeur de μ le long du rayon transmis ou réfléchi. On itère alors le procédé. Les hypothèses que nous faisons assurent que les rayons partent à l'infini en dehors de l'interface, ce qui nous permet de conclure.

Preuve du Théorème 5.

Pour prouver le point (i), on utilise l'équation de transport vérifiée par μ en dehors de l'interface, ainsi que la condition de radiation à l'infini hors de l'interface. La source f^ε étant concentrée d'un seul côté de l'interface, la source d'énergie Q est nulle à droite de l'interface. Ainsi μ est constante le long des bicaractéristiques du côté droit de l'interface. La condition de radiation à l'infini nous permet alors de conclure que μ est nulle le long des rayons tels que $\xi_d < 0$. Cela signifie qu'il n'y a pas d'énergie qui arrive à l'interface à partir de la droite.

Pour prouver (ii) et (iii), on utilise la formule explicite de la résolvante de l'opérateur de Helmholtz qui est disponible dans ce cas particulier.

Intuitivement, comme la source f^ε se trouve en dehors de l'interface, et comme les rayons glissants ($\xi_d = 0$) ne peuvent provenir de l'un des deux milieux dans le cas homogène (ξ est constant le long des bicaractéristiques), la source ne peut apporter d'énergie à l'intérieur de l'interface (rappelons que $\mu^{\partial_\pm}$ est supportée par $\xi_d = 0$). Cependant, nous ne savons pas montrer que $\mu^{\partial_\pm} = 0$ par des méthodes locales, car l'équation de Helmholtz tient compte des rayons *en temps infini*. Nous avons donc besoin d'informations globales, et c'est la raison pour laquelle nous faisons appel à la formule explicite de la résolvante.

En utilisant cette expression, la preuve du point (ii) repose sur une étude de phase (non-)stationnaire avec singularité. En effet, on sait déjà que μ est à support dans l'ensemble $\{\xi^2 = n(x)^2\}$. Pour cette raison, si l'on note $\xi = (\xi', \xi_d) \in \mathbb{R}^d$, les racines

$$\omega^\varepsilon_\pm(\xi') = \sqrt{\xi'^2 - n^2_\pm \pm i\alpha_\varepsilon \varepsilon}$$

des équations $\xi_d^2 = n^2_\pm - \xi'^2(-i\alpha_\varepsilon\varepsilon)$ apparaissent naturellement à la fois dans la phase et comme fonctions tests. Typiquement, nous devons estimer des termes de la forme

$$\frac{1}{\varepsilon^{\frac{3d+1}{2}}} \int \frac{1}{\omega^\varepsilon_-(\xi')} e^{i\frac{(x'-y')\cdot\zeta'}{\varepsilon} + i\frac{y'\cdot\xi'}{\varepsilon} - \frac{\omega^\varepsilon_-(\xi')}{\varepsilon}} A(x', y', \zeta', \xi') dy' d\xi' d\zeta' \tag{1.2.16}$$

où x' est borné, et l'amplitude A est à support près de $\zeta'^2 = n^2_-$. Tout d'abord, pour $|\xi'|$ loin de n_-, la phase est non-stationnaire par rapport à la variable y'. Il reste donc à traiter le cas où $|\xi'|$ est proche de n_-, *i.e.* près de la singularité de ω^ε_-. Après changement de variable, le terme à estimer est de la forme

$$\frac{1}{\varepsilon^{\frac{3d+1}{2}}} \int \frac{1}{\sqrt{t + i\alpha_\varepsilon\varepsilon}} e^{i\frac{\sqrt{t+n^2_-}}{\varepsilon} - \frac{\sqrt{t+i\alpha_\varepsilon\varepsilon}}{\varepsilon}} B(t) dt, \tag{1.2.17}$$

où B est à support près de $t = 0$.

Maintenant, pour traiter la singularité de $\sqrt{t + i\alpha_\varepsilon\varepsilon}$ quand $\varepsilon \to 0$ dans (1.2.17), l'ingrédient clef est une déformation de contour dans le plan complexe, jointe avec l'utilisation d'extensions presque analytiques : il existe une extension $\tilde{B}$ de B dans le plan complexe (à support compact dans $\mathbb{C}$ si B est à support compact dans $\mathbb{R}$) telle que, pour tout N,

$$\left| \frac{\partial}{\partial \bar{z}} \tilde{B}(z) \right| \leq C_N |\mathcal{I}m\ z|^N. \tag{1.2.18}$$

En utilisant cette extension et la formule de Green-Riemann, on décompose l'intégrale précédente en la somme d'une intégrale de $\tilde{B}$ sur $\{\mathcal{I}m\ z = \beta\}$ ($\beta > 0$ fixé) et d'une intégrale de $\partial\tilde{B}/\partial\bar{z}$ sur $\{\alpha_\varepsilon\varepsilon \leq \mathcal{I}m\ z \leq \beta\}$. La première se majore en utilisant un théorème de phase non-stationnaire usuel, la racine $\sqrt{t + i\beta}$ n'étant plus singulière. Pour estimer la seconde, on sépare les domaines $|\mathcal{I}m\ z| \leq \varepsilon^\delta$ et $|\mathcal{I}m\ z| \geq \varepsilon^\delta$. Pour $|\mathcal{I}m\ z| \leq \varepsilon^\delta$, on utilise la propriété (1.2.18) des extensions presque analytiques. Pour $|\mathcal{I}m\ z| \geq \varepsilon^\delta$, on utilise que $\sqrt{z}$ est minoré par $\varepsilon^{\delta/2}$, de sorte que chaque intégration par parties donne un gain de taille $\varepsilon^{1-\delta/2}$. Toutes ces estimations nous permettent *in fine* de montrer que l'intégrale (1.2.17) est $O(\varepsilon^\infty)$.

Pour finir, nous voudrions détailler la manière dont la valeur de q est liée à la condition de radiation à l'infini satisfaite par la limite w de w^ε. Comme w^ε est solution de

$$-i\alpha_\varepsilon \varepsilon w^\varepsilon + \Delta w^\varepsilon + n^2(\varepsilon x) w^\varepsilon = f(x),$$

il est clair que sa limite faible w est solution de l'équation à coefficients constants

$$\Delta w + n^2(0) w = f. \tag{1.2.19}$$

Cependant, cette équation seule ne suffit pas à déterminer w de manière unique. Dans le cas général, on sait (voir [AH]) qu'il existe une densité $q \in L^2(|\xi|^2 = n^2(0))$ telle que toute solution de (1.2.20) soit donnée en Fourier par

$$\hat{w}(\xi) = \hat{w}_0(\xi) + i\frac{\pi}{2} q(\xi) \delta(|\xi|^2 - n^2(0)),$$

q mesurant le défaut d'unicité. C'est cette densité qui apparaît dans le terme source d'énergie Q du Théorème 4.

Pour assurer l'unicité de la solution à (1.2.19), il faut préciser la condition de radiation à l'infini satisfaite par w. Les conditions à l'infini considérées dans la littérature pour assurer l'unicité sont diverses généralisations des conditions de radiation de Sommerfeld, qui s'écrivent dans le cas d'un indice constant $n(x) \equiv n(0)$:

$$\begin{cases} \frac{\partial w}{\partial r} + in(0) w(x) = o\left(|x|^{-(d-1)/2}\right), \\ u(x) = O\left(|x|^{-(d-1)/2}\right) \end{cases}$$

quand $|x| \to \infty$.
En particulier, il y a existence et unicité d'une solution vérifiant

$$\Delta w + n^2(0) w = f, \tag{1.2.20}$$

$$\lim_{r\to\infty} \frac{1}{r} \int_{S_r} \left| \frac{\partial w}{\partial r} + in(0) w \right|^2 d\sigma = 0. \tag{1.2.21}$$

Cette solution, notée w_0, est appelée solution *sortante* de l'équation (1.2.20). On a

$$w_0(x) = \frac{e^{-in(0)|x|}}{|x|^{\frac{d-1}{2}}} * f,$$

ou, en Fourier,

$$\begin{aligned}\hat{w}_0(\xi) &= (|\xi|^2 - n^2(0) + i0)^{-1}\hat{f}(\xi)\\ &= \Big(p.v.\big(\frac{1}{|\xi|^2 - n^2(0)}\big) + i\frac{\pi}{2}\delta(|\xi|^2 - n^2(0))\Big)\hat{f}(\xi).\end{aligned}$$

Pour cette question de l'unicité d'une solution à l'équation de Helmholtz (1.2.20) avec condition de radiation à l'infini (et le lien avec le principe d'absorption limite), y compris dans le cas d'un indice de réfraction non constant, nous renvoyons par exemple à D. M. Eïdus [Eïd2], Y. Saito [Sai], B. Perthame, L. Vega [PV2].

Le problème de la détermination de la condition de radiation satisfaite par la limite faible w se pose déjà quand l'indice de réfraction est régulier. Dans ce cas, F. Castella [Cas] et X.-P. Wang, P. Zhang [WZ] ont montré récemment par deux approches différentes que la limite faible de la suite de solutions de (4.1.5) est effectivement la solution sortante de (1.2.20).

Chapitre 2

High frequency analysis of Helmholtz equations: case of two point sources

Article [Fou1], soumis pour publication

Contents

2.1 Introduction

In this article, we are interested in the analysis of the high frequency limit of the following Helmholtz equation

$$-i\frac{\alpha_\varepsilon}{\varepsilon}u^\varepsilon + \Delta u^\varepsilon + \frac{n(x)^2}{\varepsilon^2}u^\varepsilon = S^\varepsilon(x), \qquad x \in \mathbb{R}^3 \tag{2.1.1}$$

with

$$S^\varepsilon(x) = S_0^\varepsilon(x) + S_1^\varepsilon(x) = \frac{1}{\varepsilon^3}S_0\Big(\frac{x}{\varepsilon}\Big) + \frac{1}{\varepsilon^3}S_1\Big(\frac{x-q_1}{\varepsilon}\Big)$$

where q_1 is a point in $\mathbb{R}^3$ different from the origin.
In the sequel, we assume that the refraction index n is constant, $n(x) \equiv 1$.

The equation (2.1.1) models the propagation of a source wave in a medium with refraction index $n(x)$. There, the small positive parameter ε is related to the frequency $\omega = \frac{1}{2\pi\varepsilon}$ of u^ε. In this paper, we study the high frequency limit, i.e. the asymptotics $\varepsilon \to 0$. We assume that the regularizing parameter α_ε is positive, with $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. The positivity of α_ε ensures the existence and uniqueness of a solution u^ε to the Helmholtz equation (2.1.1) in $L^2(\mathbb{R}^3)$ for any $\varepsilon > 0$.

The source term S^ε models a source signal that is the sum of two source signals concentrating respectively close to the origin and close to the point q_1 at the scale ε. The concentration profiles S_0 and S_1 are given functions. Since ε is also the scale of the oscillations dictated by the Helmholtz operator $\Delta + \frac{1}{\varepsilon^2}$, resonant interactions can occur between these oscillations and the oscillations due to the sources S_0^ε and S_1^ε. On the other hand, since the two sources are concentrating close to two different points in $\mathbb{R}^3$, one can guess that they do not interact when $\varepsilon \to 0$. These are the phenomena that the present paper aims at studying quantitatively. We refer to Section 3 for the precise assumptions we need on the sources.

In some sense, the sign of the term $-i\alpha_\varepsilon u^\varepsilon/\varepsilon$ prescribes a radiation condition at infinity for u^ε. One of the key difficulty in our problem is to follow this condition in the limiting process $\varepsilon \to 0$.

We study the high frequency limit in terms of Wigner measures (or semi-classical measures). This is a mean to describe the propagation of quadratic quantities, like the local energy density $|u^\varepsilon(x)|^2$, as $\varepsilon \to 0$. The Wigner measure $\mu(x,\xi)$ is the energy carried by rays at the point x with frequency ξ. These measures were introduced by Wigner [Wig] and mathematically developed by P. Gérard [Gér1] and P.-L. Lions and T. Paul [LP] (see also the surveys [Bur] and [GMMP]). They are relevant when a typical length ε is prescribed. They have already proven to be an efficient tool in the study of high frequencies, see for instance [BCKP], [CPR] for Helmholtz equations,

P. Gérard, P.A. Markowich, N.J. Mauser, F. Poupaud [GMMP] for periodic media, G. Papanicolaou, L. Ryzhik [PR] for a formal analysis of general wave equations, L. Erdös, H.T. Yau [EY] for an approach linked to statistical physics, and L. Miller [Mil2] for a study in the case with sharp interface.

The high frequency limit of Helmholtz equations has been studied in Benamou, Castella, Katsaounis, Perthame [BCKP] and Castella, Perthame, Runborg [CPR]. In [BCKP], the authors considered the case of one point source and a general index of refraction whereas in [CPR], they treated the case of a source concentrating close to a general manifold with a constant refraction index. In the present paper, we borrow the methods used in both articles.

In the case of one point source, for instance S_0^ε only, with a constant index of refraction, it is proved in [BCKP] that the corresponding Wigner measure μ_0 is the solution to the Liouville equation

$$0^+\mu_0(x,\xi) + \xi\cdot\nabla_x\mu_0(x,\xi) = Q_0(x,\xi) = \frac{1}{(4\pi)^2}\delta(x)\delta(|\xi|^2-1)|\widehat{S}_0(\xi)|^2,$$

the term 0^+ meaning that μ is the outgoing solution given by

$$\mu_0(x,\xi) = \int_{-\infty}^0 Q_0(x+t\xi,\xi)dt.$$

In particular, the energy source created by S_0^ε is supported at $x=0$. Similarly, the energy source created by the source S_1^ε is supported at $x=q_1$. Thinking of the orthogonality property on Wigner measures, one can guess that the energy source generated by the sum $S_0^\varepsilon + S_1^\varepsilon$ is the sum of the two energy sources created asymptotically by S_0^ε and S_1^ε.

Indeed, we prove in this paper that the Wigner measure μ associated with the sequence (u^ε) satisfies

$$0^+\mu(x,\xi) + \xi\cdot\nabla_x\mu(x,\xi) = Q_0 + Q_1, \tag{2.1.2}$$

where Q_0 and Q_1 are the source terms obtained in [BCKP] in the case of one point source. However, our proof does not rest on the mere orthogonality property[1].

Let us now give some details about our proof. Our strategy is borrowed from [BCKP]. First, we prove uniform estimates on the sequence of solutions

[1]Patrick Gérard pointed out to us that the measures μ_0 and μ_1 are mutually singular, using a dimension argument which is analogous to that of our proof. Hence, the result directly follows from the orthogonality property.

(u^ε). It turns out that we also need to study the limiting behaviour of (and to estimate) the rescaled solutions $\varepsilon u^\varepsilon(\varepsilon x)$ and $\varepsilon u^\varepsilon(q_1+\varepsilon x)$. The latter point is the key difficulty in our paper. It relies on the study of the sequence (a^ε) such that

$$-i\alpha_\varepsilon \varepsilon a^\varepsilon + \Delta a^\varepsilon + a^\varepsilon = S_1\Big(x - \frac{q_1}{\varepsilon}\Big).$$

Using the explicit formula for the Fourier tranform of a^ε, we prove that a^ε is uniformly bounded in a suitable space and that $a^\varepsilon \to 0$ as $\varepsilon \to 0$ weakly. We would like to point out that our analysis, based on a study in Fourier space, strongly rests on the assumption of a constant index of refraction.
Secondly, our results on the Wigner measure then follow from the properties proved in [BCKP]. They are essentially consequences of the uniform bounds on (u^ε): we write the equation satisfied by the Wigner transform associated with (u^ε), and pass to the limit $\varepsilon \to 0$ in the various terms that appear in this equation. The only difficult (and new) term to handle is the source term.
Third, we prove an improved version of the radiation condition of [BCKP]. Our argument relies on the observation that μ is localized on the energy set $\{|\xi|^2 = 1\}$, a property that was not exploited in [BCKP].

The paper is organized as follows. In Section 2, we recall some definitions and state our assumptions. Section 3 is devoted to the proof of uniform bounds on the sequence of solutions (u^ε) and of the convergence of the rescaled solutions. Then, in Section 4, we establish the transport equation satisfied by the Wigner measure μ together with the radiation condition at infinity. In the appendix, we recall the proof of some results established in [BCKP] that we use in our paper.

2.2 Notations and assumptions

In this section, we recall the definitions of Wigner transforms and of the B, B^* norms introduced by Agmon and Hörmander [AH] for the study of Helmholtz equations. Then, we give our assumptions.

2.2.1 Wigner transform and Wigner measures

We use the following definition for the Fourier transform:

$$\hat{u}(\xi) = (\mathcal{F}_{x\to\xi} u)(\xi) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)dx.$$

For $u, v \in \mathcal{S}(\mathbb{R}^3)$ and $\varepsilon > 0$, we define the Wigner transform

$$\begin{aligned} W^\varepsilon(u,v)(x,\xi) &= (\mathcal{F}_{y\to\xi})(u\big(x+\frac{\varepsilon}{2}y\big)\bar{v}\big(x-\frac{\varepsilon}{2}y\big)), \\ W^\varepsilon(u) &= W^\varepsilon(u,u). \end{aligned}$$

In the sequel, we denote $W^\varepsilon = W^\varepsilon(u^\varepsilon)$.

If (u^ε) is a bounded sequence in $L^2(\mathbb{R}^d)$ (or in some weighted L^2 space as we will see later on), it turns out that (see [Gér1], [LP]), up to extracting a subsequence, the sequence $(W^\varepsilon(u^\varepsilon))$ converges weakly to a positive Radon measure μ on the phase space $T^*\mathbb{R}^3 = \mathbb{R}^3_x \times \mathbb{R}^3_\xi$ called Wigner measure (or semiclassical measure) associated with (u^ε):

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^6),\ \lim_{\varepsilon\to 0}\langle W^\varepsilon(u^\varepsilon), \varphi\rangle = \int \varphi(x,\xi)d\mu \tag{2.2.1}$$

We recall that these measures can be obtained using pseudodifferential operators. The Weyl semiclassical operator $a^W(x,\varepsilon D_x)$ (or $Op_\varepsilon^W(a)$) is the continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ associated with the symbol $a \in \mathcal{S}'(T^*\mathbb{R}^d)$ by Weyl quantization rule

$$(a^W(x,\varepsilon D_x)u)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_\xi}\int_{\mathbb{R}^d_y} a\left(\frac{x+y}{2},\varepsilon\xi\right) f(y)e^{i(x-y)\cdot\xi}d\xi dy. \tag{2.2.2}$$

We have the following formula: for $u, v \in \mathcal{S}'(\mathbb{R}^d)$ and $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\langle W^\varepsilon(u,v), a\rangle_{\mathcal{S}',\mathcal{S}} = \langle \bar{v}, a^W(x,\varepsilon D_x)\bar{u}\rangle_{\mathcal{S}',\mathcal{S}}, \tag{2.2.3}$$

where the duality brackets $\langle .,.\rangle$ are semi-linear with respect to the second argument. This formula is also valid for u, v lying in other spaces as we will see in Section 3.

2.2.2 Besov-like norms

In order to get uniform (in ε) bounds on the sequence (u^ε), we shall use the following Besov-like norms, introduced by Agmon and Hörmander [AH]: for $u, f \in L^2_{loc}(\mathbb{R}^3)$, we denote

$$\begin{aligned} \|u\|_{B^*} &= \sup_{j\geq -1}\left(2^{-j}\int_{C(j)}|u|^2dx\right)^{1/2}, \\ \|f\|_B &= \sum_{j\geq -1}\left(2^{j+1}\int_{C(j)}|f|^2dx\right)^{1/2}, \end{aligned}$$

where $C(j)$ denotes the ring $\{x \in \mathbb{R}^3 / 2^j \leq |x| < 2^{j+1}\}$ for $j \geq 0$ and $C(-1)$ is the unit ball.

These norms are adapted to the study of Helmholtz operators. Indeed, Agmon and Hörmander [AH] proved that if v is the solution to

$$-i\alpha v + \Delta v + v = f$$

where $\alpha > 0$, then there exists a constant C independent of α such that

$$\|v\|_{B^*} \leq C\|f\|_B.$$

Perthame and Vega [PV1] generalised this result to Helmholtz equations with general indices of refraction.

We denote for $x \in \mathbb{R}^3$, $|x| = \sqrt{\sum_{j=1}^3 x_j^2}$ and $\langle x\rangle = (1 + |x|^2)^{1/2}$.
For all $\delta > \frac{1}{2}$, we have

$$\|u\|_{L^2_{-\delta}} := \|\langle x\rangle^{-\delta} u\|_{L^2} \leq C(\delta)\|u\|_{B^*}, \tag{2.2.4}$$

and

$$\|f\|_B \leq C(\delta)\|f\|_{L^2_\delta}. \tag{2.2.5}$$

We end this section by stating two properties of these spaces that will be useful for our purpose (the reader can find the proofs in [AH]). The first proposition states that, in some sense, we can define the trace of a function in B on a linear manifold of codimension 1.

Proposition 2.2.1 *There exists a constant C such that for all $f \in B$, we have*

$$\int_{\mathbb{R}} \|f(x_1, .)\|_{L^2(\mathbb{R}^2)} dx_1 \leq C\|f\|_B.$$

The second property gives the stability of the space B by change of variables in Fourier space.

Proposition 2.2.2 *Let $\Omega_1,\ \Omega_2$ be two open sets in $\mathbb{R}^3$, $\psi : \Omega_1 \to \Omega_2$ a $\mathcal{C}^2$ diffeomorphism, $\chi \in \mathcal{C}_c^1(\mathbb{R}^3)$. For all $u \in B$, we denote*

$$Tu = \mathcal{F}^{-1}\big(\chi(\hat{u} \circ \psi)\big).$$

Then

$$\|Tu\|_B \leq C\|\chi\|_{\mathcal{C}_b^1}\|\psi\|_{\mathcal{C}_b^2}\|u\|_B.$$

2.2.3 Assumptions

We are now ready to state our assumptions. Our first assumption, borrowed from [BCKP], concerns the regularizing parameter $\alpha_\varepsilon > 0$.

(H1) $\alpha_\varepsilon \geq \varepsilon^\gamma$ for some $\gamma > 0$.

This assumption is technical and is used to get a radiation condition at infinity in the limit $\varepsilon \to 0$. Next, in order to compute the limit of the energy source, we shall need the assumption

(H2) $\langle x \rangle^N S_0 \in L^2(\mathbb{R}^3)$ and $\langle x \rangle^N S_1 \in L^2(\mathbb{R}^3)$ for some $N > \frac{1}{2} + \frac{3\gamma}{\gamma+1}$.

Note that (H2) implies that the source terms S_0 and S_1 belong to the natural Besov space that is needed to actually solve the Helmholtz equation (4.1.1):

$$\|S_0\|_B, \|S_1\|_B < \infty.$$

2.3 Bounds on solutions to Helmholtz equations

In this section, we first establish uniform bounds on the sequence (u^ε) that will imply estimates on the sequence of Wigner transforms (W^ε). It turns out that we shall also need to compute the limit of the rescaled solutions w_0^ε and w_1^ε defined below in order to obtain the energy source in the equation satisfied by the Wigner measure μ.

Before stating our two results, let us define these rescaled solutions. Following [BCKP] and [CPR], we denote

$$\left\{ \begin{array}{rcl} w_0^\varepsilon(x) & = & \varepsilon u^\varepsilon(\varepsilon x), \\ w_1^\varepsilon(x) & = & \varepsilon u^\varepsilon(q_1 + \varepsilon x). \end{array} \right. \tag{2.3.1}$$

They respectively satisfy

$$\left\{ \begin{array}{rcl} -i\alpha_\varepsilon \varepsilon w_0^\varepsilon + \Delta w_0^\varepsilon + w_0^\varepsilon & = & S_0(x) + S_1\big(x - \frac{q_1}{\varepsilon}\big), \\ -i\alpha_\varepsilon \varepsilon w_1^\varepsilon + \Delta w_1^\varepsilon + w_1^\varepsilon & = & S_0\big(x + \frac{q_1}{\varepsilon}\big) + S_1(x). \end{array} \right.$$

We are ready to state our results on u^ε, w_0^ε and w_1^ε.

Proposition 2.3.1 *Assume S_0, $S_1 \in B$. Then, the solution u^ε to the Helmholtz equation (2.1.1) satisfies the following bound*

$$\|u^\varepsilon\|_{B^*} \leq C(\|S_0\|_B + \|S_1\|_B),$$

where C is a constant independent of ε.

Proposition 2.3.2 *Let w_0^ε and w_1^ε be the rescaled solutions defined by (2.3.1). Then, the sequences (w_0^ε) and (w_1^ε) are uniformly bounded in B^* and they converge weakly-$*$ in B^* to the outgoing solutions w_0 and w_1 to the following Helmholtz equations*

$$\begin{cases} \Delta w_0 + w_0 = S_0 \\ \Delta w_1 + w_1 = S_1, \end{cases}$$

i.e. w_0 and w_1 are given in Fourier space by

$$\widehat{w_j}(\xi) = \frac{-\widehat{S}_j(\xi)}{|\xi|^2 - 1 + i0} = -\Big(p.v.\big(\frac{1}{|\xi|^2-1}\big) + i\pi\delta(|\xi|^2-1)\Big)\widehat{S}_j(\xi), \quad j = 0, 1.$$

Remark: The Helmholtz equation $\Delta w + w = S$ does not uniquely specify the solution w. An extra condition is necessary, for instance the Sommerfeld radiation condition. When the refraction index is constant equal to 1, this condition writes

$$\lim_{r\to\infty} \frac{1}{r}\int_{S_r} \Big|\frac{\partial w}{\partial r} + iw\Big|^2 d\sigma = 0. \tag{2.3.2}$$

Such a solution is called an outgoing solution.
Alternatively, still assuming that the refraction index is constant, the outgoing solution to the Helmholtz equation may be defined as the weak limit w of the sequence (w^δ) such that

$$-i\delta w^\delta + \Delta w^\delta + w^\delta = S(x).$$

We point out that the two points of views are equivalent in the case of a constant index of refraction (which is not true for a general index of refraction).

We prove the two propositions in the following two sections. As we will see in the proofs, our main difficulties are linked to the rays that are emitted by the source at 0 towards the point q_1 (and conversely). Hopefully, the interaction between those rays is "destructive" and not constructive.

2.3.1 Proof of Proposition 2.3.1

In the sequel, C will denote any constant independent of ε.
The scaling invariance

$$\|u^\varepsilon\|_{B^*} \le \|w_0^\varepsilon\|_{B^*},$$

makes it sufficient to prove bounds on w_0^ε. Since w_0^ε is a solution to

$$-i\alpha_\varepsilon \varepsilon w_0^\varepsilon + \Delta w_0^\varepsilon + w_0^\varepsilon = S_0(x) + S_1\big(x - \frac{q_1}{\varepsilon}\big)$$

we may decompose $w_0^\varepsilon = \widetilde{w_0^\varepsilon} + a^\varepsilon$, where $\widetilde{w_0^\varepsilon}$ and a^ε satisfy

$$\begin{cases} -i\alpha_\varepsilon \varepsilon \widetilde{w_0^\varepsilon} + \Delta \widetilde{w_0^\varepsilon} + \widetilde{w_0^\varepsilon} & = & S_0(x), \\ -i\alpha_\varepsilon \varepsilon a^\varepsilon + \Delta a^\varepsilon + a^\varepsilon & = & S_1\big(x - \frac{q_1}{\varepsilon}\big). \end{cases}$$

First, we note that the bound $\|\widetilde{w_0^\varepsilon}\|_{B^*} \leq C\|S_0\|_B$ is established in Agmon-Hörmander [AH] (see also Perthame-Vega [PV1]). Hence, the proof of Proposition 2.3.1 reduces to the proof of the following lemma.

Lemma 2.3.3 *If a^ε is the solution to*

$$-i\alpha_\varepsilon \varepsilon a^\varepsilon + \Delta a^\varepsilon + a^\varepsilon = S_1\big(x - \frac{q_1}{\varepsilon}\big)$$

then a^ε is uniformly (in ε) bounded in $B^\star$:

$$\|a^\varepsilon\|_{B^*} \leq C\|S_1\|_B$$

Proof. We want to prove that

$$\forall v \in B, \quad |\langle a^\varepsilon, v\rangle| \leq C\|S_1\|_B\|v\|_B.$$

Using Parseval's equality, we write

$$\langle a^\varepsilon, v\rangle = \int_{\mathbb{R}^3} \frac{e^{-i\frac{q_1\cdot\xi}{\varepsilon}}\widehat{S_1}(\xi)\bar{\hat{v}}(\xi)}{-|\xi|^2 + 1 - i\varepsilon\alpha_\varepsilon} d\xi. \tag{2.3.3}$$

To estimate this integral, we shall distinguish the values of ξ close to or far from two critical sets: the sphere $\{|\xi|^2 = 1\}$ (the set where the denominator in (2.3.3) vanishes when $\varepsilon \to 0$) and the line $\{\xi$ collinear to $q_1\}$ (the set where we cannot apply directly the stationary phase theorem to (2.3.3)).

More precisely, we first take a small parameter $\delta \in]0,1[$, and we distinguish in the integral (2.3.3), the contributions due to the values of ξ such that $|\xi^2 - 1| \geq \delta$ or $|\xi^2 - 1| \leq \delta$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be a truncation function such that $\chi(\lambda) = 0$ for $|\lambda| \geq 1$. We denote $\chi_\delta(\xi) = \chi\big(\frac{|\xi|^2-1}{\delta}\big)$. We accordingly decompose

$$\begin{aligned} \langle a^\varepsilon, v\rangle &= \int_{\mathbb{R}^3} \frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}\widehat{S_1}(\xi)\bar{\hat{v}}(\xi)\chi_\delta(\xi)}{-|\xi|^2 + 1 - i\varepsilon\alpha_\varepsilon} d\xi + \int_{\mathbb{R}^3} \frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}\widehat{S_1}(\xi)\bar{\hat{v}}(\xi)(1 - \chi_\delta(\xi))}{-|\xi|^2 + 1 - i\varepsilon\alpha_\varepsilon} d\xi \\ &= I^\varepsilon + II^\varepsilon. \end{aligned}$$

First, since the denominator is not singular on the support of χ_δ, we easily bound the first part with the L^2 norms

$$|I^\varepsilon| \leq \frac{\|\chi\|_{L^\infty}}{\delta}\|\widehat{S_1}\|_{L^2}\|\hat{v}\|_{L^2},$$

and using $B \hookrightarrow L^2$, we obtain the desired bound

$$|I^\varepsilon| \leq C\|S_1\|_B\|v\|_B. \tag{2.3.4}$$

Let us now study the second part II^ε where the denominator is singular. Up to a rotation, we may assume $q_1 = |q_1|e_1$, where e_1 is the first vector of the canonical base. We make the polar change of variables

$$\xi = \begin{cases} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{cases}.$$

Remark: In order to make the calculations easier, we write this paper in dimension equal to 3, but the proof would be similar in any dimension $d \geq 3$.

Hence, $q_1 \cdot \xi = |q_1|r\sin\theta\cos\varphi$, and we get

$$II^\varepsilon = \int \frac{e^{-i\frac{|q_1|}{\varepsilon}r\sin\theta\cos\varphi}}{-r^2+1-i\varepsilon\alpha_\varepsilon}\big(\widehat{S}_1\bar{\hat{v}}(1-\chi_\delta)\big)(\xi(r,\theta,\varphi))r^2\sin\theta dr d\theta d\varphi$$

Now, we distinguish the contributions to the integral $d\theta d\phi$ linked to the values close to, or far from, the critical direction $\{\theta = \frac{\pi}{2},\ \varphi = 0 \text{ or } \varphi = \pi\}$ (which corresponds to the case $\{\xi$ collinear to $q_1\}$). To that purpose, let $\eta > 0$ be a small parameter and denote

$$\begin{aligned}
\Omega_0 &= \left\{(r,\theta,\varphi)\;\middle|\;(1-\chi\Big(\frac{r^2-1}{\delta}\Big) \neq 0, \chi\left(\frac{\theta-\frac{\pi}{2}}{\eta}\right) \neq 0, \chi\Big(\frac{\varphi}{\eta}\Big) \neq 0\right\} \\
\Omega_\pi &= \left\{(r,\theta,\varphi)\;\middle|\;(1-\chi\Big(\frac{r^2-1}{\delta}\Big) \neq 0, \chi\left(\frac{\theta-\frac{\pi}{2}}{\eta}\right) \neq 0, \chi\Big(\frac{\varphi-\pi}{\eta}\Big) \neq 0\right\}.
\end{aligned}$$

Let $k_0,\ k_\pi \in \mathcal{C}_c^\infty$ be such that $(1-\chi_\delta)k_0(\theta,\varphi)$ is a localization function on Ω_0 and $(1-\chi_\delta)k_\pi(\theta,\varphi)$ is a localization function on Ω_π. We denote $k = k_0 + k_\pi$. We write

$$\begin{aligned}
II^\varepsilon &= \int \frac{e^{-i\frac{|q_1|}{\varepsilon}r\sin\theta\cos\varphi}}{-r^2+1-i\varepsilon\alpha_\varepsilon}(\widehat{S}_1\bar{\hat{v}})(\xi(r,\theta,\varphi))(1-\chi_\delta(r))k_0(\theta,\varphi)r^2\sin\theta dr d\theta d\varphi \\
&+ \int \frac{e^{-i\frac{|q_1|}{\varepsilon}r\sin\theta\cos\varphi}}{-r^2+1-i\varepsilon\alpha_\varepsilon}(\widehat{S}_1\bar{\hat{v}})(\xi(r,\theta,\varphi))(1-\chi_\delta(r))k_\pi(\theta,\varphi)r^2\sin\theta dr d\theta d\varphi \\
&+ \int \frac{e^{-i\frac{|q_1|}{\varepsilon}r\sin\theta\cos\varphi}}{-r^2+1-i\varepsilon\alpha_\varepsilon}(\widehat{S}_1\bar{\hat{v}})(\xi(r,\theta,\varphi))(1-\chi_\delta(r))(1-k(\theta,\varphi))r^2\sin\theta dr d\theta d\varphi \\
II^\varepsilon &= III_0^\varepsilon + III_\pi^\varepsilon + IV^\varepsilon.
\end{aligned}$$

The two parts III_0^ε and III_π^ε being similar, we only write how to estimate III_0^ε. In order to translate the stationary point in $(0,0)$, we consider the new variable $\alpha = \theta - \frac{\pi}{2}$. The phase function is $rg(\alpha,\varphi) = r\cos\alpha\cos\varphi$ so

$$\begin{aligned}
\frac{\partial g}{\partial \alpha} &= -\sin\alpha\cos\varphi = 0 \quad \text{at } (\alpha,\varphi)=(0,0),\\
\frac{\partial g}{\partial \varphi} &= -\cos\alpha\sin\varphi = 0 \quad \text{at } (\alpha,\varphi)=(0,0).
\end{aligned}$$

and the Hessian at the point $(0,0)$ is

$$D^2 g(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We apply the Morse lemma: upon choosing $\eta > 0$ small enough, there exists a $\mathcal{C}^\infty$ change of variables on Ω_0, $(\alpha,\varphi) \mapsto (\alpha',\varphi')$, such that

$$g(\alpha,\varphi) = 1 - \frac{\alpha'^2}{2} - \frac{\varphi'^2}{2}.$$

Then, we make the change of variables $\alpha'' = \sqrt{\frac{r}{2}}\alpha'$, $\varphi'' = \sqrt{\frac{r}{2}}\varphi'$. Finally, we decompose $(1-\chi_\delta)k_0 = \chi^1\chi^2$, with $\chi^1,\ \chi^2 \in \mathcal{C}_c^\infty$. Thus, we obtain, for the contribution III_0^ε, the formula

$$III_0^\varepsilon = \int \frac{e^{i\frac{q_1}{\varepsilon}(-r+\alpha''^2+\varphi''^2)}}{-r+1+i\varepsilon\alpha_\varepsilon}\widehat{T^1S_1}(r,\alpha'',\varphi'')\overline{\widehat{T^2v}}(r,\alpha'',\varphi'')\,dr\,d\alpha''d\varphi'', \tag{2.3.5}$$

where

$$\begin{aligned}
T^1S_1 &:= \mathcal{F}^{-1}\Big((\chi^1\widehat{S_1})\circ\xi(r,\alpha(\alpha'',\varphi''),\varphi(\alpha'',\varphi''))\Big),\\
T^2v &:= \mathcal{F}^{-1}\Bigg(\frac{-r+1+i\varepsilon\alpha_\varepsilon}{-r^2+1-i\varepsilon\alpha_\varepsilon}(\chi^2\hat{v})\circ\xi(r,\alpha(\alpha'',\varphi''),\varphi(\alpha'',\varphi''))\\
&\qquad \times\frac{2}{r}\Big|\frac{\mathrm{d}\xi}{\mathrm{d}(r,\alpha,\varphi)}\Big|\Big|\frac{\mathrm{d}(\alpha,\varphi)}{\mathrm{d}(\alpha',\varphi')}\Big|\Bigg).
\end{aligned}$$

As a first step, using Proposition 2.2.2, we directly get $T^1S_1 \in B$ with

$$\|T^1S_1\|_B \le C\|S_1\|_B.$$

As a second step, we study T^2v. Since for r close to 1,

$$\Big|\frac{-r+1+i\varepsilon\alpha_\varepsilon}{-r^2+1-i\varepsilon\alpha_\varepsilon}\Big| \le 1,$$

we recover, from Proposition 2.2.2,

$$T^2v \in B \qquad \text{and} \qquad \|T^2v\|_B \le C\|v\|_B.$$

Now, we apply Parseval's equality with respect to the r variable in the formula (2.3.5)

$$\begin{aligned} III_0^\varepsilon &= \int \frac{e^{i\frac{|q_1|}{\varepsilon}(\alpha''^2+\varphi''^2)}}{-r+1+i\varepsilon\alpha_\varepsilon} \widehat{T^1 S_1(. - \frac{|q_1|}{\varepsilon},.,.)} \overline{\widehat{T^2 v}} dr d\alpha'' d\varphi'' \\ &= \int e^{i\frac{|q_1|}{\varepsilon}(\alpha''^2+\varphi''^2)} \mathbf{1}_{\{t>0\}} e^{-(\varepsilon\alpha_\varepsilon - i)t} \mathcal{F}_{r\to\rho}(\widehat{T^1 S_1(. - \frac{|q_1|}{\varepsilon},.,.)})(\rho - t, \alpha'', \varphi'') \\ &\qquad \times \mathcal{F}_{r\to\rho}(\overline{\widehat{T^2 v}})(\rho, \alpha'', \varphi'') dt d\rho d\alpha'' d\varphi''. \end{aligned}$$

where $\mathbf{1}_{\{t>0\}}$ denotes the characteristic function of the set $\{t > 0\}$.
Hence, we obtain

$$\begin{aligned} |III_0^\varepsilon| &\le \Big(\int \| \mathcal{F}_{r\to\rho}(\widehat{T^1 S_1(. - \frac{|q_1|}{\varepsilon},.,.)})(\rho)) \|_{L^2} d\rho \Big) \\ &\qquad \times \Big(\int \| \mathcal{F}_{r\to\rho}(\widehat{T^2 v})(\rho) \|_{L^2} d\rho \Big) \\ |III^\varepsilon| &\le \Big(\int \| T^1 S_1(\rho - \frac{|q_1|}{\varepsilon}) \|_{L^2} d\rho \Big) \Big(\int \| T^2 v(\rho) \|_{L^2} d\rho \Big) \\ |III^\varepsilon| &\le \sum_{j=1}^n \Big(\int \| T^1 S_1(\rho) \|_{L^2} d\rho \Big) \Big(\int \| T^2 v(\rho) \|_{L^2} d\rho \Big). \end{aligned}$$

Now, using Proposition 2.2.1, we get

$$\begin{aligned} |III_0^\varepsilon| &\le C \| T^1 S_1 \|_B \| T^2 v \|_B \\ |III_0^\varepsilon| &\le C \| S_1 \|_B \| v \|_B, \end{aligned}$$

which is the desired estimate.

We are left with the part IV^ε, which corresponds to the directions ξ that are not collinear to q_1. We denote K' the support of $(1 - \chi_\delta)(1 - k)$ which is a compact set. If we denote

$$\eta_1 = |\xi|^2 - 1, \quad \eta_2 = -q_1 \cdot \xi, \tag{2.3.6}$$

then,

$$\frac{\mathrm{d}(\eta_1, \eta_2)}{\mathrm{d}\xi} = \begin{pmatrix} 2\xi \\ -q_1 \end{pmatrix}$$

is of maximal rank 2 for $\xi \in K'$. Hence, there exists a finite covering $(\Omega_j)_{j=1,m}$ ($m \in \mathbb{N}$) of K' such that in Ω_j, we can make the change of variables $\xi \mapsto \eta$, where η_1, η_2 are given by (2.3.6) and η_3 is one of the components of ξ

(depending on Ω_j). We denote $\chi_j = \chi_j^3 \chi_j^4$ some localization functions on Ω_j such that $(1-\chi_\delta)(1-k) = \sum_{j=1}^m \chi_j$. Thus, for $j = 1, \ldots, m$,

$$\int \frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}}{-|\xi|^2+1+i\varepsilon\alpha_\varepsilon} \widehat{S_1}\overline{\hat{v}}\chi_j d\xi = \int \frac{e^{i\frac{\eta_2}{\varepsilon}}}{-\eta_1+i\varepsilon\alpha_\varepsilon} (\widehat{S_1}\overline{\hat{v}}\chi_j)(\xi(\eta)) \Big|\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\Big| d\eta.$$

If we denote

$$\begin{aligned} T_j^3 S_1 &:= \mathcal{F}^{-1}((\chi_j^3 \widehat{S_1}) \circ \xi), \\ T_j^4 v &:= \mathcal{F}^{-1}\big((\chi_j^4 \hat{v}) \circ \xi)\big|\frac{\mathrm{d}\xi}{\mathrm{d}\eta}|\big), \end{aligned}$$

and if $\mathcal{F}_1$ denotes the Fourier transform with respect to the η_1 variable, Parseval's equality with respect to η_1 gives

$$\begin{aligned} \Big|\int \frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}}{-|\xi|^2+1+i\varepsilon\alpha_\varepsilon} \widehat{S_1}\overline{\hat{v}}\chi_j d\xi\Big| &= (2\pi)^d \Big|\int \chi_{\{t>0\}} e^{-\varepsilon\alpha_\varepsilon t} (\mathcal{F}_1^{-1}(\widehat{T_j^3 S_1}))(x_1 - t) \\ &\quad \times (\mathcal{F}_1^{-1}(\overline{\widehat{T_j^4 v}}))(x_1) e^{i\eta_2/\varepsilon} dt dx_1 d\eta_2 d\eta_3\Big| \\ &\leq C\|S_1\|_B \|v\|_B, \end{aligned}$$

using Proposition 2.2.1 again. Summing over j, we obtain

$$|IV^\varepsilon| \leq C\|S_1\|_B \|v\|_B,$$

which ends the proof of the bound

$$|\langle a^\varepsilon, v\rangle| \leq C\|S_1\|_B \|v\|_B.$$

□

2.3.2 Proof of Proposition 2.3.2

As before, we prove the result for the sequence (w_0^ε) only. As we did in the proof of Proposition 2.3.1, we write $w_0^\varepsilon = \widetilde{w_0^\varepsilon} + a^\varepsilon$. Since $\widetilde{w_0^\varepsilon}$ is the solution to a Helmholtz equation with constant index of refraction and fixed source, it converges weakly-$*$ to the outgoing solution w_0 to $\Delta w + w = S_0$. Hence, it suffices to show the following result.

Lemma 2.3.4 *If $a^\varepsilon \in B^\star$ is the solution to*

$$-i\alpha_\varepsilon \varepsilon a^\varepsilon + \Delta a^\varepsilon + a^\varepsilon = S_1\big(x - \frac{q_1}{\varepsilon}\big)$$

then $a^\varepsilon \to 0$ in $B^\star$.

Proof. The proof of this result requires two steps (using a density argument):

1. for $v \in B$, we have the bound $\left|\langle a^\varepsilon, v\rangle\right| \le C\|S_1\|_B\|v\|_B$
2. if S_1 and v are smooth, then $\langle a^\varepsilon, v\rangle \to 0$.

The first point is exactly the result in Lemma 2.3.3. It remains to prove the convergence in the smooth case (the second point above).
We write

$$\langle a^\varepsilon, v\rangle = \int_{\mathbb{R}^3} \frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}\widehat{S}_1(\xi)\bar{\hat{v}}(\xi)}{-|\xi|^2+1-i\varepsilon\alpha_\varepsilon}d\xi.$$

We are thus left with the study of

$$R_\varepsilon(\psi) = \int_{\mathbb{R}^3} \frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}\psi(\xi)}{-|\xi|^2+1-i\varepsilon\alpha_\varepsilon}d\xi \tag{2.3.7}$$

where $\psi = \widehat{S}_1\overline{\hat{v}}$ belongs to $\mathcal{S}(\mathbb{R}^3)$.

As in the proof of Lemma 2.3.3, we distinguish the contributions of various values of ξ. We shall use exactly the same partition, according to the values of ξ close to, or far from, the sphere $|\xi| = 1$ and collinear or not to q_1. We shall use the same notations for the various truncation functions.

We first separate the contributions of ξ such that $|\xi^2-1| \le \delta$ and $|\xi^2-1| \ge \delta$ using the truncation function χ_δ:

$$\begin{aligned} R_\varepsilon(\psi) &= \int_{\mathbb{R}^3} \frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}\psi(\xi)\chi_\delta(\xi)}{-|\xi|^2+1-i\varepsilon\alpha_\varepsilon}d\xi + \int_{\mathbb{R}^3} \frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}\psi(\xi)(1-\chi_\delta(\xi))}{-|\xi|^2+1-i\varepsilon\alpha_\varepsilon}d\xi \\ &= I^\varepsilon + II^\varepsilon. \end{aligned}$$

In the support of χ_δ, since the denominator is not singular, we can apply the non stationary phase method. Since $q_1 \ne 0$, we may assume $q_1^1 \ne 0$ and we have

$$\begin{aligned} I^\varepsilon &= \frac{\varepsilon}{iq_1^1}\int_{\mathbb{R}^3} e^{-i\frac{q_1}{\varepsilon}\cdot\xi}\partial_{\xi_1}\left(\frac{\psi(\xi)\chi_\delta(\xi)}{-|\xi|^2+1-i\varepsilon\alpha_\varepsilon}\right)d\xi \\ &= \frac{\varepsilon}{iq_1^1}\int_{\mathbb{R}^3} e^{-i\frac{q_1}{\varepsilon}\cdot\xi}\left(\frac{\partial_{\xi_1}(\psi(\xi)\chi_\delta(\xi))}{-|\xi|^2+1-i\varepsilon\alpha_\varepsilon} - \frac{2\psi(\xi)\chi_\delta(\xi)\xi_1}{(-|\xi|^2+1-i\varepsilon\alpha_\varepsilon)^2}\right). \end{aligned}$$

Hence, we obtain the bound

$$|I^\varepsilon| \le \frac{\varepsilon}{|q_1^1|}\int_{\mathbb{R}^3}\left(\frac{1}{\delta}|\partial_{\xi_1}(\chi\psi)| + \frac{2}{\delta^2}|\xi_1\chi\psi|\right)d\xi.$$

Since $\partial_{\xi_1}(\chi\psi)$ and $\xi_1\chi\psi$ belongs to $\mathcal{S}$, we have, as $\varepsilon \to 0$,

$$I^\varepsilon \to 0.$$

Let us now study the second term II^ε. As, in Section 2.3.1, we first decompose II^ε into the sum $III_0^\varepsilon + III_\pi^\varepsilon + IV^\varepsilon$. We then use the same changes of variables. It leads to the following formula for III_0^ε

$$III_0^\varepsilon = \int \frac{e^{-i\frac{q_1}{\varepsilon}(r-\alpha''^2-\varphi''^2)}}{-r+1+i\varepsilon\alpha_\varepsilon}\widetilde{\chi}(r,\alpha'',\varphi'')\widetilde{\psi}(r,\alpha'',\varphi'')drd\alpha''d\varphi'',$$

where

$$\begin{aligned}\widetilde{\chi}(r,\alpha'',\varphi'') &= ((1-\chi_\delta)k_0)\circ\xi(r,\alpha(\alpha'',\varphi''),\varphi(\alpha'',\varphi''))\\ &\qquad\times\frac{2(-r+1+i\varepsilon\alpha_\varepsilon)}{r(-r^2+1-i\varepsilon\alpha_\varepsilon)}\left|\frac{\mathrm{d}(\alpha,\varphi)}{\mathrm{d}(\alpha',\varphi')}\right|,\\ \widetilde{\psi}(r,\alpha'',\varphi'') &= \psi\circ\xi(r,\alpha(\alpha'',\varphi''),\varphi(\alpha'',\varphi'')),\end{aligned}$$

are still smooth functions that are bounded independently from ε.
Using Parseval's inequality with respect to the variables (α'',φ''), we obtain the bound

$$|III_0^\varepsilon| \le C\varepsilon\left|\int \frac{e^{-i\frac{|q_1|}{\varepsilon}r}e^{-i\varepsilon(\lambda^2+\mu^2)}}{-r+1+i\varepsilon\alpha_\varepsilon}\mathcal{F}_{\lambda,\mu}(\widetilde{\chi}\widetilde{\psi})drd\lambda d\mu\right|.$$

To obtain the convergence of III_0^ε, it remains to study an integral of the following type

$$\int_{|r-1|\le\delta}\frac{e^{-i\frac{|q_1|}{\varepsilon}r}w(r)}{-r+1+i\varepsilon\alpha_\varepsilon}dr, \text{ where } w\in\mathcal{S}.$$

This is done in the following lemma.

Lemma 2.3.5 $\forall w\in\mathcal{S},\ \forall\theta\in(0,1)$, *we have*

$$\int_{|r|\le\delta}\frac{e^{-i\frac{|q_1|}{\varepsilon}r}w(r)}{-r+i\varepsilon\alpha_\varepsilon}dr = -i\pi w(0) + O_{\varepsilon\to0}(\varepsilon^{-\theta}).$$

Using this lemma, we readily get the estimate

$$|III^\varepsilon| \le C\varepsilon^{1-\theta} \quad \forall\theta\in(0,1), \tag{2.3.8}$$

which proves that $III^\varepsilon \to 0$ as $\varepsilon\to0$.

There remains to give the
Proof of Lemma 2.3.5. We write

$$\begin{aligned}\int_{-\delta}^{\delta}\frac{e^{-i\frac{|q_1|}{\varepsilon}r}w(r)}{-r+i\varepsilon\alpha_\varepsilon}dr &= \int_{-\delta}^{\delta}\frac{e^{-i\frac{|q_1|}{\varepsilon}r}w(r)}{r^2+(\varepsilon\alpha_\varepsilon)^2}(r-i\varepsilon\alpha_\varepsilon)dr\\ &= -i\varepsilon\alpha_\varepsilon\int_{-\delta}^{\delta}\frac{e^{-i\frac{|q_1|}{\varepsilon}r}w(r)}{r^2+(\varepsilon\alpha_\varepsilon)^2}dr + \int_{-\delta}^{\delta}e^{-i\frac{|q_1|}{\varepsilon}r}\frac{rw(r)}{r^2+(\varepsilon\alpha_\varepsilon)^2}dr\\ &= I+II.\end{aligned}$$

We have

$$I = -i\int_{-\frac{\delta}{\varepsilon\alpha_\varepsilon}}^{\frac{\delta}{\varepsilon\alpha_\varepsilon}} \frac{e^{-i|q_1|\alpha_\varepsilon y} w(\varepsilon\alpha_\varepsilon y)}{y^2+1}dy \to -i\pi w(0),$$

and

$$II = \int_{-\delta}^{\delta}\big(e^{-i\frac{|q_1|}{\varepsilon}r}w(r)-w(0)\big)\frac{r}{r^2+(\varepsilon\alpha_\varepsilon)^2}dr + \int_{-\delta}^{\delta} w(0)\frac{r}{r^2+(\varepsilon\alpha_\varepsilon)^2}dr\ .$$

The last term vanishes because the integrand is odd. Moreover, using the smoothness of w, we easily obtain that for all $\theta\in(0,1)$,

$$\big|e^{-i\frac{|q_1|}{\varepsilon}r}w(r)-w(0)\big| \le C_\theta\Big(\frac{r}{\varepsilon}\Big)^\theta$$

Thus,

$$\Big|\int_{-\delta}^{\delta}\big(e^{-i\frac{|q_1|}{\varepsilon}r}w(r)-w(0)\big)\frac{r}{r^2+(\varepsilon\alpha_\varepsilon)^2}dr\Big| \le \frac{C}{\varepsilon^\theta}\int_{-\delta}^{\delta}|r|^{\theta-1}dr$$

and the result is proved. □

We are left with the study of IV^ε. We use the same change of variables as in Section 2.3.1.

$$\begin{aligned} IV^\varepsilon &= \sum_{j=1}^m\int\frac{e^{-i\frac{q_1}{\varepsilon}\cdot\xi}}{-|\xi|^2+1+i\varepsilon\alpha_\varepsilon}\psi(\xi)\chi_j(\xi)d\xi \\ &= \sum_{j=1}^m\int\frac{e^{i\frac{\eta_2}{\varepsilon}}}{-\eta_1+i\varepsilon\alpha_\varepsilon}(\psi\chi_j)(\xi(\eta))\Big|\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\Big|d\eta \\ &= i\varepsilon\sum_{j=1}^m\int\frac{e^{i\frac{\eta_2}{\varepsilon}}}{-\eta_1+i\varepsilon\alpha_\varepsilon}\partial_{\eta_2}\Big((\psi\chi_j)(\xi(\eta))\Big|\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\Big|\Big)d\eta. \end{aligned}$$

The integral obviously converges with respect to all the variables except η_1. It remains to prove the convergence with respect to the η_1 variable, i.e. the convergence of

$$\int\frac{\phi(\eta)}{-\eta_1+i\varepsilon\alpha_\varepsilon}d\eta_1,$$

where

$$\phi = \partial_{\eta_2}\Big((\psi\chi_j)(\xi(\eta))\Big|\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\Big|\Big)$$

is smooth and compactly supported with respect to η. It is a consequence of the fact that the distribution $(x+i0)^{-1}$ is well-defined on $\mathbb{R}$ by

$$\frac{1}{x+i0} = p.v.(\frac{1}{x}) - i\pi\delta(x).$$

We conclude that $IV^\varepsilon\to 0$ and $\langle a^\varepsilon, v\rangle\to 0$ as $\varepsilon\to 0$. □

2.4 Transport equation and radiation condition on μ

In this section, we state and prove our results on the Wigner measure associated with (u^ε). Since we established the uniform bounds on (u^ε) and the convergence of (w_0^ε), (w_1^ε), these results now essentially follows from the results proved in [BCKP]. We first prove bounds on the sequence of Wigner transforms (W^ε) that allow us to define a Wigner measure μ associated to (u^ε). Then, we get the transport equation satisfied by μ together with the radiation condition at infinity, which uniquely determines μ.

2.4.1 Results

Theorem 2.4.1 *Let $S_0,\ S_1 \in B$ and $\lambda > 0$. The sequence (W^ε) is bounded in the Banach space $X_\lambda^\star$ and up to extracting a subsequence, it converges weak-$\star$ to a positive and locally bounded measure μ such that*

$$\sup_{R>0} \frac{1}{R} \int_{|x|<R} \int_{\xi \in \mathbb{R}^3} \mu(x,\xi)\, dx d\xi \leq C(\|S_0\|_B + \|S_1\|_B)^2. \tag{2.4.1}$$

The Banach space X_λ^* is defined as the dual space of the set X_λ of functions $\hat{\varphi}(x,\xi)$ such that $\varphi(x,y) := \mathcal{F}_{\xi\to y}(\hat{\varphi}(x,\xi))$ satisfies

$$\int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} (1+|x|+|y|)^{1+\lambda} |\varphi(x,y)| dy < \infty. \tag{2.4.2}$$

Theorem 2.4.2 *Assume (H1), (H2), (H3). Then the Wigner measure μ asssociated with (u^ε) satisfies the following transport equation*

$$\xi\cdot\nabla_x \mu = \frac{1}{(4\pi)^2}\Big(\delta(x)|\widehat{S_0}(\xi)|^2 + \delta(x-q_1)|\widehat{S_1}(\xi)|^2\Big)\delta(|\xi|^2-1) := Q(x). \tag{2.4.3}$$

Moreover, μ is the outgoing solution to the equation (2.4.3) in the following sense: for all test function $R \in \mathcal{C}_c^\infty(\mathbb{R}^6)$, if we denote $g(x,\xi) = \int_0^\infty R(x-\xi t,\xi)dt$, then

$$\int_{\mathbb{R}^6} R(x,\xi) d\mu(x,\xi) = -\int_{\mathbb{R}^6} Q(x,\xi) g(x,\xi) dx d\xi. \tag{2.4.4}$$

Remark: Here the support of the test function R contains 0, contrary to [BCKP].

2.4.2 Proof of Theorem 2.4.1

This theorem, that is proved in [BCKP], is a consequence of the uniform estimate on the sequence (u^ε) in the space B^* obtained in Proposition 2.3.1. We observe that for any $\lambda > 0$,

$$\|\langle x\rangle^{-\frac{1}{2}-\lambda}u^\varepsilon(x)\|_{L^2} \le C\|u^\varepsilon\|_{B^*} \le C\left(\|S_0\|_B + (\|S_1\|_B\right), \tag{2.4.5}$$

hence, for any function φ satisfying (2.4.2), we have

$$\begin{aligned}&|\langle W^\varepsilon(u^\varepsilon), \hat{\varphi}\rangle|\\ &\le \int_{\mathbb{R}^6} \frac{|u^\varepsilon|(x+\frac{\varepsilon}{2}y)|\overline{u^\varepsilon}|(x-\frac{\varepsilon}{2}y)}{\langle x+\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}\langle x-\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}}\langle x+\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}\langle x-\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}|\varphi|(x,y)dxdy\\ &\le C\|f\|_B^2\int_{\mathbb{R}^3}\sup_{x\in\mathbb{R}^3}\langle |x|+|y|\rangle^{1+0}|\varphi(x,y)|dy.\end{aligned}$$

So $(W^\varepsilon(u^\varepsilon))$ is bounded in X_λ^*, $\lambda > 0$. We deduce that, up to extracting a subsequence, $(W^\varepsilon(u^\varepsilon))$ converges weak-$*$ to a nonnegative measure μ satisfying

$$|\langle \mu, \hat{\varphi}\rangle| \le C\|f\|_B^2\int_{\mathbb{R}^3}\sup_{x\in\mathbb{R}^3}\langle |x|+|y|\rangle^{1+0}|\varphi(x,y)|dy. \tag{2.4.6}$$

We refer for instance to Lions, Paul [LP] for the proof of the nonnegativity of μ.
The bound (2.4.1) is obtained using the following family of functions

$$\varphi_\delta^R(x,y) = \frac{1}{\delta^{3/2}}e^{-|y|^2/\delta}\frac{1}{R}\chi(\langle x\rangle \le R)$$

and letting $\delta \to 0$, $R \to \infty$. □

2.4.3 Proof of the transport equation 2.4.3

This section is devoted to the proof of the transport equation satisfied by μ. We first write the transport equation satisfied by W^ε in a dual form. Then, we study the convergence of the source term (the convergence of the other terms is obvious). Finally, choosing an appropriate test function in the limiting process, we get the radiation condition at infinity satisfied by μ. Proving first a localization property, we improve the radiation condition proved in [BCKP].

Transport equation satisfied by W^ε

W^ε satisfies the following equation

$$\alpha_\varepsilon W^\varepsilon + \xi \cdot \nabla_x W^\varepsilon = Q^\varepsilon, \tag{2.4.7}$$

where, for $\psi \in \mathcal{S}(T^*\mathbb{R}^d)$, if $\varphi(x,y) = \mathcal{F}^{-1}_{y\to\xi}(\psi(x,\xi))$,

$$\begin{aligned}
\langle Q^\varepsilon, \psi\rangle &= \frac{i\varepsilon}{2}\mathcal{I}m\ \langle W^\varepsilon(S^\varepsilon, u^\varepsilon), \psi\rangle \\
&= \frac{i}{2}\mathcal{I}m\left(\int_{\mathbb{R}^6} \overline{w_0^\varepsilon}(x+y)S_0^\varepsilon(y)\varphi\left(\varepsilon\left(x+\frac{y}{2}\right), y\right)dxdy\right) \\
&+ \frac{i}{2}\mathcal{I}m\left(\int_{\mathbb{R}^6} \overline{w_1^\varepsilon}(x+y)S_1^\varepsilon(y)\varphi\left(q_1+\varepsilon\left(x+\frac{y}{2}\right), y\right)dxdy\right)
\end{aligned}$$

This equation can be obtained writing first the equation satisfied by

$$v^\varepsilon(x,y) = u^\varepsilon\big(x+\frac{\varepsilon}{2}y\big)\overline{u^\varepsilon}\big(x-\frac{\varepsilon}{2}y\big).$$

From the equality

$$\nabla_y \cdot \nabla_x v^\varepsilon = \frac{\varepsilon}{2}\big[\Delta u^\varepsilon(x+\frac{\varepsilon}{2}y)\overline{u^\varepsilon}(x-\frac{\varepsilon}{2}y) - \Delta\overline{u^\varepsilon}(x-\frac{\varepsilon}{2}y)u^\varepsilon(x+\frac{\varepsilon}{2}y)\big],$$

we deduce

$$\alpha_\varepsilon v^\varepsilon + i\nabla_y \cdot \nabla_x v^\varepsilon + \frac{i}{2\varepsilon}\big[n^2(x+\frac{\varepsilon}{2}y) - n^2(x-\frac{\varepsilon}{2}y)\big]v^\varepsilon = \sigma_\varepsilon(x,y),$$

where

$$\sigma_\varepsilon(x,y) := \frac{i\varepsilon}{2}\big[S^\varepsilon(x+\frac{\varepsilon}{2}y)\overline{u^\varepsilon}(x-\frac{\varepsilon}{2}y) - \overline{S^\varepsilon}(x-\frac{\varepsilon}{2}y)u^\varepsilon(x+\frac{\varepsilon}{2}y)\big].$$

After a Fourier transform, we obtain the equation (2.4.7).
Then we write the dual form of this equation. Let $\psi \in \mathcal{S}(\mathbb{R}^6)$, we have

$$\alpha_\varepsilon\langle W^\varepsilon, \psi\rangle - \langle W^\varepsilon, \xi\cdot\nabla_x\psi\rangle = \langle Q^\varepsilon, \psi\rangle. \tag{2.4.8}$$

By the definition of the Wigner measure μ, we get

$$\alpha_\varepsilon\langle W^\varepsilon, \psi\rangle \to 0 \qquad \text{and} \qquad \langle W^\varepsilon, \xi\cdot\nabla_x\psi\rangle \to \langle \mu, \xi\cdot\nabla_x\psi\rangle.$$

Hence we are left with the study of the source term $\langle Q^\varepsilon, \psi\rangle$.

Convergence of the source term

In order to compute the limit of the source term in (2.4.7), we develop

$$\langle Q^\varepsilon, \psi\rangle = \frac{i\varepsilon}{2}\mathcal{I}m\Big(\langle W^\varepsilon(S_0^\varepsilon, u^\varepsilon), \psi\rangle + \langle W^\varepsilon(S_1^\varepsilon, u^\varepsilon), \psi\rangle\Big).$$

Thus, the result is contained in the following proposition.

Proposition 2.4.3 *The sequences* $\big(\varepsilon W^\varepsilon(S_0^\varepsilon, u^\varepsilon)\big)$ *and* $\big(\varepsilon W^\varepsilon(S_1^\varepsilon, u^\varepsilon)\big)$ *are bounded in* $\mathcal{S}'(\mathbb{R}^6)$ *and for all real-valued* $\psi \in \mathcal{S}(\mathbb{R}^6)$*, we have*

$$\lim_{\varepsilon\to 0} \varepsilon\langle W^\varepsilon(S_0^\varepsilon, u^\varepsilon), \psi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \overline{\widehat{w_0}}(\xi)\widehat{S}_0(\xi)\psi(0,\xi)d\xi, \quad (2.4.9)$$

$$\lim_{\varepsilon\to 0} \varepsilon\langle W^\varepsilon(S_1^\varepsilon, u^\varepsilon), \psi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \overline{\widehat{w_1}}(\xi)\widehat{S}_1(\xi)\psi(q_1,\xi)d\xi, \quad (2.4.10)$$

where w_0 *and* w_1 *are defined in Proposition 2.3.2.*

Using Proposition 2.4.3, we readily get for any real-valued test function ψ,

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle Q^\varepsilon, \psi\rangle &= \frac{i}{2(2\pi)^3}\mathcal{I}m\left(\int_{\mathbb{R}^3} \overline{\widehat{w_0}}(\xi)\widehat{S}_0(\xi)\psi(0,\xi)d\xi + \int_{\mathbb{R}^3} \overline{\widehat{w_1}}(\xi)\widehat{S}_1(\xi)\psi(q_1,\xi)d\xi\right)\\ &= \frac{1}{(4\pi)^2}\Bigg(\int_{\mathbb{R}^3} |\widehat{S}_0(\xi)|^2\delta(\xi^2-1)\psi(0,\xi)d\xi\\ &\qquad + \int_{\mathbb{R}^3} |\widehat{S}_1(\xi)|^2\delta(\xi^2-1)\psi(q_1,\xi)d\xi\Bigg),\end{aligned}$$

which is the result in Theorem 2.4.2. □

Let us now prove Proposition 2.4.3.
Proof of Proposition 2.4.3. The two terms to study being of the same type, we only consider the first one in our proof. Let $\psi \in \mathcal{S}(T^*\mathbb{R}^d)$ and $\varphi(x,y) = \mathcal{F}^{-1}_{y\to\xi}(\psi(x,\xi))$, then we have

$$\begin{aligned}\varepsilon\langle W^\varepsilon(S_0^\varepsilon, u^\varepsilon), \psi\rangle_{\mathcal{S}',\mathcal{S}} &= \varepsilon\int S_0^\varepsilon\big(x+\frac{\varepsilon}{2}y\big)\overline{u^\varepsilon}\big(x-\frac{\varepsilon}{2}y\big)\varphi(x,y)dxdy\\ &= \int S_0(x)\overline{w_0^\varepsilon}(x+y)\varphi(\varepsilon(x+\frac{y}{2}),y)dxdy.\end{aligned}$$

As a first step, let us prove that $\varepsilon\langle W^\varepsilon(S_0^\varepsilon, u^\varepsilon), \psi\rangle_{\mathcal{S}',\mathcal{S}}$ is bounded. Using that $\psi \in \mathcal{S}(\mathbb{R}^{2d})$, we get

$$\begin{aligned}\big|\varepsilon\langle W^\varepsilon(S_0^\varepsilon, u^\varepsilon), \psi\rangle_{\mathcal{S}',\mathcal{S}}\big| &\le C\int \langle x\rangle^N |S_0(x)|\frac{|w_0^\varepsilon(x+y)|}{\langle x+y\rangle^\beta}\frac{\langle x+y\rangle^\beta}{\langle x\rangle^N\langle y\rangle^k}dxdy\\ &\le C\|\langle x\rangle^N S_0\|_{L^2}\|w_0^\varepsilon\|_{B^*}\int_{\mathbb{R}^3_y}\sup_{x\in\mathbb{R}^3}\frac{\langle x+y\rangle^\beta}{\langle x\rangle^N\langle y\rangle^k}dy\end{aligned}$$

for any $k \geq 0$ and $1/2 < \beta < N$, upon using the Cauchy-Schwarz inequality in x.
Then, we distinguish the cases $|x| \leq |y|$ and $|x| \geq |y|$: the term stemming from the first case gives a contribution which is bounded by $C \int \frac{dy}{\langle y \rangle^{k-\beta}}$ and the second contribution is bounded by $C \int \frac{dy}{\langle y \rangle^{k}}$. Hence, upon choosing k large enough, we obtain

$$|\varepsilon \langle W^\varepsilon(S_0^\varepsilon, u^\varepsilon), \psi \rangle_{\mathcal{S}',\mathcal{S}}| \leq C \| \langle x \rangle^N S_0 \|_{L^2} \| w^\varepsilon \|_{B^*}.$$

As a second step, we compute the limit (2.4.9). We write

$$\begin{aligned} \varepsilon \langle W^\varepsilon(S_0^\varepsilon, u^\varepsilon), \psi \rangle &= \int S_0(x) \overline{w_0^\varepsilon}(x+y) \Big(\varphi\Big(\varepsilon \Big(x + \frac{y}{2} \Big), y \Big) - \varphi(0,y) \Big) dxdy \\ &\quad + \int \overline{w_0^\varepsilon}(x) S_0(x-y) \varphi(0,y) dxdy \\ &= I_\varepsilon + II_\varepsilon. \end{aligned}$$

Reasonning as above, we readily get that $\lim_{\varepsilon \to 0} I_\varepsilon = 0$. Indeed, since $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$, we have, for all $k \in \mathbb{N}$, for all $x,\ y \in \mathbb{R}^d$,

$$\left| \varphi\Big(\varepsilon \Big(x + \frac{y}{2} \Big), y \Big) - \varphi(0,y) \right| \leq C\varepsilon \frac{|x| + |y|}{\langle y \rangle^k} leq C\varepsilon \frac{\langle |x| + |y| \rangle}{\langle y \rangle^k}.$$

Hence,

$$\begin{aligned} |I^\varepsilon| &\leq C\varepsilon \int \langle x \rangle^N |S_0(x)| \frac{|w_0^\varepsilon(x+y)|}{\langle x+y \rangle^\beta} \frac{\langle |x| + |y| \rangle^{\beta+1}}{\langle x \rangle^N \langle y \rangle^k} dxdy \\ &\leq C\varepsilon \| \langle x \rangle^N S_0 \|_{L^2} \| w_0^\varepsilon \|_{B^*} \int_{\mathbb{R}^3_y} \sup_{x \in \mathbb{R}^3} \frac{\langle |x| + |y| \rangle^{\beta+1}}{\langle x \rangle^N \langle y \rangle^k} dy \end{aligned}$$

for any $k \geq 0$ and $1/2 < \beta < N-1$. As above, the previous integral converges for k large enough. Therefore, $I^\varepsilon \to 0$.

We end the proof by proving that the second term II^ε converges to $\int S_0(x) \overline{w_0}(x+y) \widehat{\psi}(0,y) dxdy$. We have

$$II_\varepsilon = \int \overline{w_0^\varepsilon}(x) \big(S_0 * \varphi(0,.) \big)(x) dx.$$

Hence, since w_0^ε converges weakly-$*$ in B^*, it suffices to prove that $S_0 * \varphi(0,.)$ belongs to B. We denote $\phi = \varphi(0,.)$. Then, $\phi \in \mathcal{S}(\mathbb{R}^3)$. Let $1/2 < \beta < N$. We have, using (2.2.5),

$$\| S_0 * \phi \|_B^2 \leq C \| S_0 * \phi \|_{L^2_\beta}^2 = C \int \langle x \rangle^{2\beta} |S_0 * \phi(x)|^2 dx.$$

Moreover, upon using the Cauchy-Schwarz inequality, we get, for all $x \in \mathbb{R}^3$,

$$\begin{aligned} |S_0 * \phi(x)|^2 &\leq \left(\int_{\mathbb{R}^d_y} |S_0(x-y)||\phi(y)|dy \right)^2 \\ &\leq \left(\int_{\mathbb{R}^d_y} |S_0(x-y)|^2|\phi(y)|dy \right) \left(\int_{\mathbb{R}^d_y} |\phi(y)|dy \right) \\ &\leq \|\phi\|_{L^1} |S_0|^2 * |\phi|(x). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \|S_0 * \phi\|_B^2 &\leq C\|\phi\|_{L^1} \int \langle xs\rangle^{2\beta} |S_0(x-y)|^2 |\phi(y)|dy \\ &\leq C\|\langle x\rangle^N S_0\|_{L^2} \int_{\mathbb{R}^3_y} \sup_{x\in\mathbb{R}^3} \frac{\langle x+y\rangle^{2\beta}}{\langle x\rangle^{2N}\langle y\rangle^k} dy, \end{aligned}$$

for any k. As before, this integral converges. Thus, we have established that $S_0 * \widehat{\psi}(0,.)$ belongs to B, which implies that

$$II_\varepsilon \to \int S_0(x)\overline{w_0}(x+y)\widehat{\psi}(0,y)dxdy.$$

□

2.4.4 Proof of the radiation condition (2.4.4)

It remains to prove that μ satisfies the weak radiation condition (2.4.4).

Support of μ

In order to prove the radiation condition without restriction on the test function R (as assumed in [BCKP]), we first prove a localization property on the Wigner measure μ. This property is well-known when u^ε satisfies a Helmholtz equation without source term. It is still valid here thanks to the scaling of S^ε.

Proposition 2.4.4 *Under the hypotheses (H1), (H2), (H3), the Wigner measure μ satisfies*

$$\text{supp}(\mu) \subset \{(x,\xi) \in \mathbb{R}^6 / \ |\xi|^2 = 1\}.$$

Proof. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^6)$ and $\phi^\varepsilon = \phi^W(x, \varepsilon D_x)$. Let us denote $H^\varepsilon = -\varepsilon^2\Delta - 1$. Since u^ε satisfies the Helmholtz equation (2.1.1), we have

$$i\alpha_\varepsilon \varepsilon u^\varepsilon + H^\varepsilon u^\varepsilon = \varepsilon^2 S^\varepsilon. \tag{2.4.11}$$

Moreover, H^ε is a pseudodifferential operator with symbol $|\xi|^2 - 1$. By pseudodifferential calculus, $\phi^\varepsilon H^\varepsilon = Op_\varepsilon^W(\phi(x,\xi)(|\xi|^2-1)) + O(\varepsilon)$ so, using the definition of the measure μ, we get that

$$\begin{aligned}\lim_{\varepsilon\to 0}(\phi^\varepsilon H^\varepsilon u^\varepsilon, u^\varepsilon) &= \lim_{\varepsilon\to 0}(Op_\varepsilon^W(\phi(x,\xi)(|\xi|^2-1))u^\varepsilon, u^\varepsilon)\\ &= \int \phi(x,\xi)(|\xi|^2-1))d\mu.\end{aligned}$$

Using the equation (2.4.11), we write

$$(\phi^\varepsilon H^\varepsilon u^\varepsilon, u^\varepsilon) = \varepsilon^2(\phi^\varepsilon S^\varepsilon, u^\varepsilon) - i\alpha_\varepsilon\varepsilon(\phi^\varepsilon u^\varepsilon, u^\varepsilon) = \varepsilon^2(W^\varepsilon(S^\varepsilon, u^\varepsilon), \phi) - i\alpha_\varepsilon\varepsilon(\phi^\varepsilon u^\varepsilon, u^\varepsilon).$$

On the one hand, Proposition 2.4.3 gives that $\lim_{\varepsilon\to 0}\varepsilon^2(W^\varepsilon(S^\varepsilon, u^\varepsilon), \phi) = 0$. On the other hand, $(\phi^\varepsilon u^\varepsilon, u^\varepsilon)$ is bounded so $\lim_{\varepsilon\to 0}\alpha_\varepsilon\varepsilon(\phi^\varepsilon u^\varepsilon, u^\varepsilon) = 0$. Therefore, for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^6)$, we have $\int \phi(|\xi|^2-1)d\mu = 0$, so $\mathrm{supp}(\mu) \subset \{|\xi|^2 = 1\}$. □

Proof of the condition (2.4.4)

Using the previous localization property, in order to prove the radiation condition (2.4.4), one may only use test functions $R \in \mathcal{C}_c^\infty(\mathbb{R}^6)$ such that $\mathrm{supp}(R) \subset \mathbb{R}^6 \backslash \{\xi = 0\}$.
Let R be such a test function. We associate with R the solution g^ε to

$$-\alpha_\varepsilon g^\varepsilon + \xi \cdot \nabla_x g^\varepsilon = R(x,\xi).$$

By duality, we have

$$\langle Q^\varepsilon, g^\varepsilon\rangle = \langle W^\varepsilon, R\rangle,$$

so that it sufffices to establish the following two convergences:

$$\lim_{\varepsilon\to 0}\langle Q^\varepsilon, g^\varepsilon\rangle = \langle Q, g\rangle, \tag{2.4.12}$$

$$\lim_{\varepsilon\to 0}\langle W^\varepsilon, R\rangle = \langle f, R\rangle, \tag{2.4.13}$$

where Q et g are defined in Theorem 2.4.2.
As before, since $R \in X_\lambda$ for any $\lambda > 0$, the limit (2.4.13) follows from the weak-$*$ convergence of W^ε in $X_\lambda^\star$.
On the other hand,

$$\begin{aligned}\langle Q^\varepsilon, g^\varepsilon\rangle &= \mathcal{I}m\int_{\mathbb{R}^6}\overline{S_0}(x)w_0^\varepsilon(x+y)\widehat{g^\varepsilon}(\varepsilon[x+\frac{y}{2}], y)dxdy \\ &\quad + \mathcal{I}m\int_{\mathbb{R}^6}\overline{S_1}(x)w_1^\varepsilon(x+y)\widehat{g^\varepsilon}(q_1+\varepsilon[x+\frac{y}{2}], y)dxdy,\end{aligned} \tag{2.4.14}$$

so $\langle Q^\varepsilon, g^\varepsilon\rangle$ is the sum of two terms of the same type. Such a term has been studied in [BCKP], where the following result is proved.

Proposition 2.4.5 *Assume (w^ε) is bounded in B^* and that (w^ε) converges weakly-$*$ in B^* to w_0. Assume S_0 satisfy (H3). Let $R \in \mathcal{C}_c^\infty(\mathbb{R}^6)$ be such that $supp(R) \subset \mathbb{R}^6 \backslash \{\xi = 0\}$. Let g^ε be the solution to*

$$-\alpha_\varepsilon g^\varepsilon + \xi \cdot \nabla_x g^\varepsilon = R(x,\xi)$$

and $g(x,\xi) = \int_0^\infty R(x + t\xi, \xi)dt$. Then, we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^6} \overline{S_0}(x) w_0^\varepsilon(x+y)\widehat{g^\varepsilon}(\varepsilon[x+\frac{y}{2}], y)dxdy = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \overline{\widehat{S_0}}(\xi)\widehat{w_0}(\xi) g(0,\xi)d\xi.$$

Proof. The proof of this result is written in the appendix. □

Using the proposition above together with Proposition 2.3.2, we get that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\langle Q^\varepsilon, g^\varepsilon\rangle \\
&= \mathcal{I}m\bigg(\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \overline{\widehat{S_0}}(\xi)\widehat{w_0}(\xi)g(0,\xi)d\xi + \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \overline{\widehat{S_1}}(\xi)\widehat{w_1}(\xi)g(q_1,\xi)d\xi\bigg) \\
&= \frac{1}{(4\pi)^2}\bigg(\int_{\mathbb{R}^3} |\widehat{S_0}(\xi)|^2\delta(\xi^2-1)g(0,\xi)d\xi + \int_{\mathbb{R}^3} |\widehat{S_1}(\xi)|^2\delta(\xi^2-1)g(q_1,\xi)d\xi\bigg).
\end{aligned}$$

Thus, the radiation condition (2.4.4) is proved.

A Proof of Proposition 2.4.5

In the sequel, we denote

$$G^\varepsilon = \int_{\mathbb{R}^6} \overline{S_0}(x) w_0^\varepsilon(x+y)\widehat{g^\varepsilon}(\varepsilon[x+\frac{y}{2}], y)dxdy.$$

A.1 Bounds on G^ε

In order to study G^ε, we need a preliminary result on the test function g^ε.

Lemma A.1 *Let $R \in \mathcal{C}_c^\infty(\mathbb{R}^6 \setminus \{\xi = 0\})$. We denote g^ε the solution to*

$$-\alpha_\varepsilon g^\varepsilon + \xi.\nabla_x g^\varepsilon = R(x,\xi). \tag{A.1}$$

It is given by the explicit formula

$$g^\varepsilon(x,\xi) = -\int_0^\infty \exp(-\alpha_\varepsilon|\xi|^{-1}s)\frac{1}{|\xi|}R(x - \frac{\xi}{|\xi|}s, \xi)ds. \tag{A.2}$$

Then we have the estimate

$$\forall M \geq 0, \quad |\widehat{g^\varepsilon}(x,y)| \leq C\frac{\langle x\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M}, \tag{A.3}$$

where .∧. denotes the infimum of two numbers, and C is a constant depending on M and R.

Proof. Let a be a multiindex such that $|a| \leq M$. We denote $\omega = \frac{\xi}{|\xi|}$. We write

$$\begin{aligned} y^a\widehat{g^\varepsilon}(x,y) &= \mathcal{F}_{\xi\to y}\Big(\int_0^\infty (i\partial_\xi)^a\Big[e^{-\alpha_\varepsilon|\xi|^{-1}s}\frac{1}{|\xi|}R(x-\omega s,\xi)\Big]ds\Big) \\ &= \int_{\mathbb{R}^3} d\xi e^{-i\xi . y}\int_{s=0}^{+\infty} ds \sum_{b,c,d,e,f} C_{(a,b,c,d,e,f)}e^{-\alpha_\varepsilon|\xi|^{-1}s}(i\partial_\xi)^b(-\alpha_\varepsilon|\xi|^{-1}s) \\ &\quad \times(-\alpha_\varepsilon|\xi|^{-1}s)^f(i\partial_\xi)^c(-s\omega)(i\partial_x)^d(i\partial_\xi)^e\left(\frac{1}{|\xi|}R\right)(x-\omega s,\xi) \end{aligned} \tag{A.4}$$

Using that
- $R \in \mathcal{C}_0^\infty(\mathbb{R}^6\backslash\{|\xi|=0\})$ so
  - there exists $r_0,\ A,\ B>0$ such that $supp(R) \subset \{|x| \leq r_0\}\times\{A \leq |\xi| \leq B\}$,
  - R and its derivatives belong to $L^1(\mathbb{R}^6)$.
- $|(i\partial_\xi)^b(-\alpha_\varepsilon|\xi|^{-1}s)| \leq Cs, \quad \forall\ |\xi| \geq A.$
- $|(i\partial_\xi)^c(-s\omega)| \leq Cs, \quad \forall\ |\xi| \geq A.$
- in the integral above, $s \in [|x|-r_0, |x|+r_0]$, so for $|x|$ large enough, we can use the equivalence $s \sim |x| \sim \langle x\rangle$ (where we dente for $a,\ b>0$, $a \sim b$ if $\exists\ c_1,\ c_2>0/\ c_1a<b<c_2a$), we get in (A.4),

$$|y^a\widehat{g^\varepsilon}(x,y)| \leq C\langle x\rangle^M \exp(-\frac{\alpha_\varepsilon}{B}\langle x\rangle).$$

The desired estimate follows. □

Using this lemma, we estimate

$$\begin{aligned} |G^\varepsilon| &\leq \Big|\int_{\mathbb{R}^6} w_0^\varepsilon(x+y)S_0(x)\widehat{g^\varepsilon}(\varepsilon(x+\frac{y}{2},y)dxdy\Big| \\ &\leq C\int_{\mathbb{R}^6}\frac{|w_0^\varepsilon(x+y)|}{\langle x+y\rangle^{\frac{1}{2}+0}}\langle x+y\rangle^{\frac{1}{2}+0}\langle x\rangle^{N_1}|S_0(x)|\langle x\rangle^{-N_1}\frac{\langle\varepsilon(x+\frac{y}{2})\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M} \\ &\leq C\|w_0^\varepsilon\|_{B^\star}\|\langle x\rangle^{N_1}S_0(x)\|_{L^2} \\ &\qquad \times\int_{\mathbb{R}^3}\sup_{x\in\mathbb{R}^3}\langle x+y\rangle^{\frac{1}{2}+0}\langle x\rangle^{-N_1}\frac{\langle\varepsilon(|x|+|y|)\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M}dy. \end{aligned}$$

Now, we prove that the integral

$$\mathcal{I}^\varepsilon = \int_{\mathbb{R}^3} \sup_{x\in\mathbb{R}^3} \langle x+y\rangle^{\frac{1}{2}+0}\langle x\rangle^{-N_1}\frac{\langle \varepsilon(|x|+|y|)\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M}dy$$

is bounded uniformly with respect to ε.
Let us define the following three subsets in $\mathbb{R}^6$

$$\begin{aligned} &A_\varepsilon = \{(x,y)\in\mathbb{R}^6/|x|\geq|y|\}, \quad B_\varepsilon = \{|x|\leq|y|, |\varepsilon^{1-0}y|\leq 1\},\\ &C_\varepsilon = \{|x|\leq|y|, |\varepsilon^{1-0}y|\geq 1\}, \end{aligned} \tag{A.5}$$

where ε^{1-0} means $\varepsilon^{1-\delta}$ with $\delta>0$ sufficiently small.
<u>If $(x,y)\in A_\varepsilon$</u>, then

$$\langle x+y\rangle^{\frac{1}{2}+0}\langle x\rangle^{-N_1}\frac{\langle \varepsilon(|x|+|y|)\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M} \leq C\langle x\rangle^{-N_1+\frac{1}{2}+0}\Big(\langle \varepsilon x\rangle^M \wedge \alpha_\varepsilon^{-M}\Big)\langle y\rangle^{-M}.$$

Now, we distinguish the relative size of $\langle \varepsilon x\rangle$ and ε^γ:

- if $\langle \varepsilon x\rangle \geq \varepsilon^{-\gamma}$, we have

$$\langle x\rangle^{\frac{1}{2}+0-N_1}\Big(\langle \varepsilon x\rangle^M \wedge \alpha_\varepsilon^{-M}\Big) \leq \langle x\rangle^{\frac{1}{2}+0-N_1}\varepsilon^{-\gamma M} \leq \varepsilon^{-\gamma(\frac{1}{2}+0-N_1+M)}.$$

- if $\langle \varepsilon x\rangle \leq \varepsilon^{-\gamma}$, we have $\langle x\rangle \leq \frac{1}{\varepsilon}\langle \varepsilon x\rangle$, hence we get

$$\langle x\rangle^{\frac{1}{2}+0-N_1}\Big(\langle \varepsilon x\rangle^M \wedge \alpha_\varepsilon^{-M}\Big) \leq \langle x\rangle^{\frac{1}{2}+0-N_1}\varepsilon^{-\gamma M} \leq \varepsilon^{-\gamma M}\varepsilon^{-(\gamma+1)(\frac{1}{2}+0-N_1)}.$$

Now, we choose $1/2+0$ such that $N_1<\frac{1}{2}+0$. Then, we get the following bound for the contribution of the set A_ε to $\mathcal{I}_\varepsilon$

$$C\varepsilon^{-\gamma M-(\gamma+1)(1/2+0-N_1)}\int_{\mathbb{R}^3}\langle y\rangle^{-M}dy.$$

Since $N_1 > \frac{3\gamma}{\gamma+1}+\frac{1}{2}$, we can choose $M>3$ such that this contribution is uniformly bounded with respect to ε.
<u>If $(x,y)\in B_\varepsilon$</u>, then $|\varepsilon y|\leq 1$ so we obtain

$$\langle x+y\rangle^{\frac{1}{2}+0}\langle x\rangle^{-N_1}\frac{\langle \varepsilon(|x|+|y|)\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M} \leq C\langle y\rangle^{-M+\frac{1}{2}+0}.$$

Thus, the corresponding contribution to $\mathcal{I}^\varepsilon$ is bounded by $C\int_{\mathbb{R}^3}\langle y\rangle^{-M+\frac{1}{2}+0}dy$ which is convergent.
<u>If $(x,y)\in C_\varepsilon$</u>, then

$$\langle x+y\rangle^{\frac{1}{2}+0}\langle x\rangle^{-N_1}\frac{\langle \varepsilon(|x|+|y|)\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M} \leq C\langle y\rangle^{\frac{1}{2}+0}\frac{\langle \varepsilon y\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M}.$$

Since $|y| \geq \varepsilon^{-1+0}$, if we denote $z = \varepsilon y$, we have $|z| \geq \varepsilon^{\delta}$, for some $\delta > 0$, which implies that

$$\varepsilon\langle\frac{z}{\varepsilon}\rangle \geq \frac{\varepsilon^{\delta}}{\sqrt{2}}\langle z\rangle.$$

Thus, we have

$$\begin{aligned}\mathcal{I}_\varepsilon &\leq C\varepsilon^{-3}\int_{\mathbb{R}^3}\langle\frac{z}{\varepsilon}\rangle^{-M+\frac{1}{2}+0}(\langle z\rangle^M \wedge \varepsilon^{-\gamma M})dz\\ &\leq C\varepsilon^{M(1-\delta)-4}\int_{\mathbb{R}^3}\langle z\rangle^{-M+\frac{1}{2}+0}(\langle z\rangle^M \wedge \varepsilon^{-\gamma M})dz\end{aligned}$$

where we used the hypothesis (H1) $\alpha_\varepsilon \geq \varepsilon^{\gamma}$.
We distinguish two cases, according to the relative size of $\langle z\rangle$ and $\varepsilon^{-\gamma}$. We have

$$\begin{aligned}\int_{\langle z\rangle\geq\varepsilon^{-\gamma}}\langle z\rangle^{\frac{1}{2}+0}\frac{\langle z\rangle^M \wedge \varepsilon^{-\gamma M}}{\langle z\rangle^M}dz &\leq \varepsilon^{-M\gamma}\int_{\langle z\rangle\geq\varepsilon^{\gamma}}\langle z\rangle^{\frac{1}{2}+0-M}dz\\ &\leq C\varepsilon^{-\gamma(\frac{3}{2}+0)},\end{aligned}$$

and

$$\begin{aligned}\int_{\langle z\rangle\leq\varepsilon^{\gamma}}\langle z\rangle^{\frac{1}{2}+0}\frac{\langle z\rangle^M \wedge \varepsilon^{-\gamma M}}{\langle z\rangle^M}dz &\leq \int_{\langle z\rangle\leq\varepsilon^{\gamma}}\langle z\rangle^{\frac{1}{2}+0}dz\\ &\leq C\varepsilon^{-\gamma(\frac{3}{2}+0)}.\end{aligned}$$

Hence, we get

$$\mathcal{I}_\varepsilon \leq C\varepsilon^{M(1-\delta)-4-\gamma(\frac{3}{2}+0)}.$$

To conclude, we choose $M > \frac{4+\frac{3}{2}\gamma}{1-\delta}$, which gives that $\mathcal{I}_\varepsilon$ is uniformly bounded with respect to ε.

A.2 Convergence of G^ε

We decompose G^ε in the following way

$$\begin{aligned}G^\varepsilon &= \int_{\mathbb{R}^6} w_0^\varepsilon(x+y)S_0(x)\big(\widehat{g^\varepsilon}(\varepsilon(x+\frac{y}{2}),y) - \widehat{g^\varepsilon}(0,y)\big)dxdy\\ &\quad + \int_{\mathbb{R}^6} w_0^\varepsilon(x+y)S_0(x)\big(\widehat{g^\varepsilon}(0,y) - \widehat{g}(0,y)\big)dxdy\\ &\quad + \int_{\mathbb{R}^6} w_0^\varepsilon(x+y)S_0(x)\widehat{g}(0,y)dxdy\\ &= I_\varepsilon + II_\varepsilon + III_\varepsilon\end{aligned}$$

Using the same method as in Section A.1, we prove that I_ε, $II_\varepsilon \to 0$. Then, we may write

$$III_\varepsilon = \int_{\mathbb{R}^6} w_0^\varepsilon(x)\big(S_0 * \widehat{g}(0,.)\big)(x)dx.$$

Moreover, we established in the proof of Proposition 2.4.3 that if ϕ is rapidly decreasing at infinity, then $S_0 * \phi \in B$. Hence, since (w_0^ε) converges weakly-$*$ in B^* to w_0, we get

$$\lim_{\varepsilon\to 0} III_\varepsilon = \int_{\mathbb{R}^6} w_0(x)\big(S_0 * \widehat{g}(0,.)\big)(x)dx,$$

which ends the proof.

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Chapitre 3

Morrey-Campanato estimates for Helmholtz equations with two unbounded media

Article [Fou2], Proceedings of the Royal Society of Edinburg, Section A

Contents

3.1 Introduction

We consider the following Helmholtz equation

$$i\varepsilon u + \Delta u + n(x)u = f(x),\ x \in \mathbb{R}^d. \tag{3.1.1}$$

We assume that the refraction index n is nonnegative and discontinuous at the interface between two unbounded inhomogeneous media Ω_+, Ω_- such that $\Omega_+ \cup \overline{\Omega_-} = \overline{\Omega_+} \cup \Omega_- = \mathbb{R}^d$. We write

$$n(x) = \begin{cases} n_+(x) & \text{if } x \in \Omega_+ \\ n_-(x) & \text{if } x \in \Omega_-. \end{cases}$$

We study here the limiting absorption principle, i.e. the limit when $\varepsilon \to 0^+$ of equation (3.1.1): our goal is to prove bounds on u that are uniform in ε. A related question is the statement of the Sommerfeld radiation condition for the following equation

$$\Delta u + n(x)u = f(x),\ x \in \mathbb{R}^d. \tag{3.1.2}$$

This is an open problem in the full generality of our assumptions, stated below.

In this article, we prove uniform Morrey-Campanato type estimates for this equation, using a multiplier method borrowed from [PV1]. These bounds encode in the optimal way the decay: $|u(x)| \sim 1/|x|^{\frac{d-1}{2}}$ at infinity of the solution u. They imply weighted L^2-estimates for the solution u. We also prove a uniform L^2-estimate *without weight* for the trace of the solution on the interface, which states that u carries essentially no energy on this set.

In order to state precisely our assumptions and results, we need the following notations.
First, we use the Morrey-Campanato norm, defined for $u \in L^2_{loc}$,

$$\|u\|^2_{\dot{B}^*} = \sup_{R>0} \frac{1}{R} \int_{B(R)} |u(x)|^2 dx, \tag{3.1.3}$$

where $B(R)$ denotes the ball of radius R. We also use the following dual norm

$$\|f\|_{\dot{B}} = \sum_{j\in\mathbb{Z}} \left(2^{j+1} \int_{C(j)} |f(x)|^2 dx\right)^{1/2}, \tag{3.1.4}$$

where $C(j) = \{x \in \mathbb{R}^d / 2^j \le |x| \le 2^{j+1}\}$.
The duality is given by the easy estimate

$$\left|\int fu dx\right| \le \|f\|_{\dot{B}} \|u\|_{\dot{B}^*}. \tag{3.1.5}$$

Note that both norms $\dot{B}$ and $\dot{B}^*$ are homogeneous in space.

We also denote the radial and tangential derivatives by

$$\frac{\partial}{\partial r} = \frac{x}{|x|} \cdot \nabla, \qquad \frac{\partial}{\partial \tau} = \nabla - \frac{x}{|x|}\frac{\partial}{\partial r}.$$

On the other hand, we assume that the interface between the two media $\Gamma = \partial\Omega_+ = \partial\Omega_-$ is a smooth surface (Lipschitz is enough). Let $d\gamma$ be the euclidian surface measure on Γ and $\nu(x)$ be the unit normal vector at $x \in \Gamma$ directed from Ω_- to Ω_+. We denote, for $x \in \Gamma$, the jump

$$[n](x) = n_+(x) - n_-(x).$$

Throughout this paper we will write ∇n instead of $\nabla n_+ \mathbf{1}_{\Omega_+} + \nabla n_- \mathbf{1}_{\Omega_-}$, the derivative of n outside the interface. Similarly, $\partial_d n$ will denote the partial derivative of n with respect to the x_d variable outside the interface.

The key assumptions we make on the interface Γ and the refraction index n are the following. We comment on these assumptions later.
(H1) There is $\alpha > 0$ such that the d-th component of ν satisfies

$$\nu_d(x) \geq \alpha \quad \text{for all} \quad x \in \Gamma. \tag{3.1.6}$$

(H2) $[n](x)$ has the same sign for all $x \in \Gamma$; the following notation will be convenient: let $\sigma = -$ if $[n]$ is non-negative and $\sigma = +$ if $[n]$ is non-positive.
(H3) $n \in L^\infty$, $n \geq 0$.
(H4)

$$2 \sum_{j\in\mathbb{Z}} \sup_{C(j)} \frac{(x \cdot \nabla n(x))_-}{n(x)} := \beta_1 < \infty. \tag{3.1.7}$$

(H5)

$$\frac{1}{\alpha} \sum_{j\in\mathbb{Z}} \sup_{C(j)} 2^{j+1} \frac{(\partial_d n(x))_\sigma}{n(x)} := \beta_2 < \infty. \tag{3.1.8}$$

(H6) $\beta_1 + \beta_2 < 1$.

We are now ready to state our main theorem.
Theorem 3.1.1 *For dimensions $d \geq 3$, assume (H1)-(H6). Then, the solution to the Helmholtz equation (3.1.1) satisfies the following estimates :*

$$\|\nabla u\|_{\dot{B}^*}^2 + \|n^{1/2}u\|_{\dot{B}^*}^2 + \|(x\cdot\nabla n)_+^{1/2}u\|_{\dot{B}^*}^2 + \int_{\mathbb{R}^d} \frac{|\nabla_\tau u|^2}{|x|}dx + \sup_{R>0}\frac{1}{R^2}\int_{S(R)}|u|^2 d\sigma_R$$
$$+ \int_\Gamma \big|[n]\big|\,|u|^2 d\gamma + \int_{\mathbb{R}^d} |\partial_d n|\,|u|^2 dx \leq C\|f\|_{\dot{B}}^2, \tag{3.1.9}$$

where C is a constant depending only on α, β_1 and β_2.

We would like to stress several aspects of our analysis. First, the homogeneity of the estimates and assumptions makes this theorem compatible with the high frequencies. The scaling invariance plays a fundamental role in the high frequency limit of Helmholtz equations as we will see in the next chapter (see Benamou *et al* [BCKP], Castella *et al* [CPR] for the case of a regular index of refraction).

We would like to point out several terms in the left-hand side. The last two say that, in principle, the energy is not trapped at the interface and it mainly radiates in the directions where $\partial_d n$ vanishes. On the other hand, the first two terms essentially assert that u and ∇u belong to the optimal space $\dot{B}^*$, provided $f \in \dot{B}$. This is the same Morrey-Campanato estimates as in the regular case.

Let us now comment our assumptions on n. First, they allow some growth at infinity : n *does not* go to a constant at infinity. The hypotheses (H2), (H4) and (H5) can be understood as conditions on the trajectories of the geometrical optics. The condition (H4) implies the dispersion of these trajectories. The conditions (H2) and (H5) ensure that the energy goes from one side of the interface to the other. The condition (H5) involves both the interface and the index: it becomes a weaker assumption on the index when the interface is close to a hyperplane ($\alpha \sim 1$). This type of assumptions is natural in the study of the high frequency limit where the link with Liouville's equations can be understood through Wigner transform (see L. Miller [Mil2] for a refraction result in the case of a sharp interface for Schrodinger equation, E. Fouassier [Fou3] for high frequency limit of Helmholtz equations with interface; for an account on high frequency limit for wave equations, see P.-L. Lions, T.Paul [LP], and [BCKP], [CPR] for Helmholtz equations without interface).

Taking $[n] = 0$ in Theorem 3.1.1, i.e. the case without interface, our results boil down to the uniform estimate proved by B. Perthame, L. Vega [PV1] when the refraction index satisfies assumptions (H3)-(H4) with $\beta_1 < 1$. Under these assumptions, they also proved in [PV2] an energy estimate saying that the energy $|u|^2$ mainly radiates in the directions of the critical points of n_∞ (where n_∞ is given by $n(x) \to n_\infty(\frac{x}{|x|})$, $|x| \to \infty$). In our case, the corresponding energy estimate corresponds to the last two terms.

In the case of two unbounded media, similar results, but not scaling invariant, were obtained in previous papers. Eidus [Eïd1] first proved weighted-L^2-estimates, and the L^2-estimate on the trace of the solution on the interface, for a piecewise constant index of refraction. To do so, he assumed that the interface satisfied an extra "cone-like" shape condition, $|x \cdot \nu| \leq C$ for $x \in \Gamma$. Under the same assumption on the interface, Bo Zhang [Zha] also proved inhomogeneous B^*-B estimates when n is a long-range perturbation of a piecewise constant function. S. DeBievre, D.W. Pravica [DP] proved weighted-L^2-estimates in a very general context using Mourre's commutator method. In particular they considered the case of an index that is smooth outside of a compact set in the x_d-direction and only bounded in this compact set.

Our proof is based on a multiplier method. Following B. Perthame, L. Vega [PV1], we use a combination of a Morawetz-type multiplier and of an elliptic multiplier. Following Eidus [Eïd1], we combine it with a multiplier specific to the case with an interface for which one direction plays a particular role (the x_d-direction here). The first two multipliers allow us to control both ∇u and u locally in L^2 by $\|f\|_{\dot{B}}$ and we estimate the integral over the interface of $|u|^2$ using the third multiplier.

The article is organized as follows. In Sections 2 and 3, we present the basic multipliers and the particular choice we make here to prove Theorem 3.1.1. Then, in Section 4, we give another estimate containing the trace of ∇u on the interface.

3.2 Basic identities

Lemma 3.2.1 *The solution to the Helmholtz equation (3.1.1) satisfies the following four identities, for smooth real valued test functions φ, ψ,*

$$-\int_{\mathbb{R}^d} \varphi(x)|\nabla u(x)|^2 + \frac{1}{2}\int_{\mathbb{R}^d} \Delta\varphi(x)|u(x)|^2 + \int_{\mathbb{R}^d} \varphi(x)n(x)|u(x)|^2 = \mathcal{R}e\int_{\mathbb{R}^d} f(x)\varphi(x)\bar{u}(x), \quad (3.2.1)$$

$$\varepsilon\int_{\mathbb{R}^d} \varphi(x)|u(x)|^2 - \mathcal{I}m\int_{\mathbb{R}^d} \nabla\varphi(x)\cdot\nabla u(x)\bar{u}(x) = \mathcal{I}m\int_{\mathbb{R}^d} f(x)\varphi(x)\bar{u}(x), \quad (3.2.2)$$

$$\int_{\mathbb{R}^d} \nabla\bar{u}(x)\cdot D^2\psi(x)\cdot\nabla u(x) - \frac{1}{4}\int_{\mathbb{R}^d} \Delta^2\psi(x)|u(x)|^2 + \frac{1}{2}\int_{\mathbb{R}^d} \nabla n(x)\cdot\nabla\psi(x)|u(x)|^2 - \varepsilon\mathcal{I}m\int_{\mathbb{R}^d} \nabla\psi(x)\cdot\nabla u(x)\bar{u}(x) + \frac{1}{2}\int_{\Gamma}[n]\nu(x)\cdot\nabla\psi(x)|u(x)|^2 d\gamma(x) = -\mathcal{R}e\int_{\mathbb{R}^d} f(x)(\nabla\psi(x)\cdot\nabla\bar{u}(x) + \frac{1}{2}\Delta\psi(x)\bar{u}(x)), \quad (3.2.3)$$

$$\frac{1}{2}\int_{\Gamma}[n]\nu_d(x)|u(x)|^2 d\gamma(x) + \frac{1}{2}\int_{\mathbb{R}^d}\partial_d n(x)|u(x)|^2$$
$$= -\mathcal{R}e\int_{\mathbb{R}^d} f(x)\partial_d\bar{u}(x) - \varepsilon\mathcal{I}m\int_{\mathbb{R}^d} u(x)\partial_d\bar{u}(x). \quad (3.2.4)$$

Proof. The identities (3.2.1) and (3.2.2) are obtained by multiplying the Helmholtz equation (4.1.1) by $\varphi\bar{u}$ and then taking the real and imaginary parts. The identity (3.2.3) is obtained using the Morawetz-type multiplier $\nabla\psi(x)\cdot\nabla\bar{u}(x)+\frac{1}{2}\Delta\psi(x)\bar{u}(x)$ and taking the real part. To get the last identity (3.2.4), we use the multiplier $\partial_d\bar{u}$ and take the real part. □

Lemma 3.2.2 *The solution to the Helmholtz equation (3.1.1) satisfies the following estimate*

$$\int_{\Gamma}|[n]||u(x)|^2 d\gamma(x) + \frac{1}{\alpha}\int_{\mathbb{R}^d}(\partial_d n)_{-\sigma}|u|^2$$
$$\leq 2\int_{\mathbb{R}^d}|f\partial_d\bar{u}| + 2\varepsilon\Big|\mathcal{I}m\int_{\mathbb{R}^d} u\partial_d\bar{u}\Big| + \frac{1}{\alpha}\int_{\mathbb{R}^d}(\partial_d n)_{\sigma}|u|^2. \quad (3.2.5)$$

Proof. This lemma follows directly from the identity (3.2.4) using the hypothesis (H1). □

3.3 Proof of theorem 3.1.1

The following proof in which all the details are included for the convenience of the reader is essentially adapted from [PV1], apart from the treatment of the terms involving the interface and the partial derivative with respect to the x_d-direction, which is the key difficulty of this paper.

We derive the proof of theorem 3.1.1 from the above identities. We make the following choice of test functions ψ and φ, for $R > 0$,

$$\nabla\psi(x) = \begin{cases} x/R & \text{for } |x| \leq R \\ x/|x| & \text{for } |x| > R, \end{cases}$$

$$\varphi(x) = \begin{cases} 1/2R & \text{for } |x| \leq R \\ 0 & \text{for } |x| > R. \end{cases}$$

We also need the following calculations (in the distributional sense)

$$D^2_{ij}\psi(x) = \begin{cases} \delta_{ij}/R & \text{for } |x| \leq R \\ (\delta_{ij}|x|^2 - x_i x_j)/|x|^3 & \text{for } |x| > R, \end{cases}$$

$$\Delta\psi(x) = \begin{cases} d/R & \text{for } |x| \le R \\ (d-1)/|x| & \text{for } |x| > R, \end{cases}$$

and the inequality

$$\frac{1}{4}\int_{\mathbb{R}^d} v\Delta(2\varphi - \Delta\psi) \ge \frac{d-1}{4R^2}\int_{S(R)} v d\sigma_R \quad \text{for } v \ge 0. \tag{3.3.1}$$

We add the identity (3.2.1) to (3.2.3), which gives, using the inequality (3.3.1), for the previous choice of ψ, φ,

$$\begin{aligned}\frac{1}{2R}\int_{B(R)} |\nabla u(x)|^2 &+ \frac{1}{2R}\int_{B(R)} n|u(x)|^2 + \int_{|x|>R} \frac{1}{|x|}\left(|\nabla u(x)|^2 - \left|\frac{x}{|x|}\cdot\nabla u\right|^2\right) \\ &+ \frac{d-1}{4R^2}\int_{S(R)} |u|^2 d\sigma_R + \frac{1}{2}\int_{\mathbb{R}^d} (\nabla\psi(x)\cdot\nabla n(x))_+ |u(x)|^2 \\ \le C\int_{\mathbb{R}^d} |f(x)|\frac{|u(x)|}{|x|} &+ C\int_{\mathbb{R}^d} |f(x)||\nabla u(x)| + \frac{1}{2}\int_{\mathbb{R}^d} (\nabla\psi(x)\cdot\nabla n(x))_- |u(x)|^2 \\ &+ C\varepsilon\int_{\mathbb{R}^d} |u(x)||\nabla u(x)| + \frac{1}{2}\int_\Gamma \big|[n]\big|\; |\nu\cdot\nabla\psi||u(x)|^2 d\gamma.\end{aligned}$$

Then, we use the inequality (3.2.5), together with the bound $|\nabla\psi| \le 1$ to estimate the trace term

$$\begin{aligned}\frac{1}{R}\int_{B(R)} |\nabla u(x)|^2 &+ \frac{1}{R}\int_{B(R)} n|u(x)|^2 + \frac{1}{R}\int_{B(R)} (x\cdot\nabla n(x))_+ |u(x)|^2 \\ &+ \int_{|x|>R} \frac{|\nabla_\tau u(x)|^2}{|x|} + \frac{d-1}{2R^2}\int_{S(R)} |u|^2 d\sigma_R \\ \le C\int_{\mathbb{R}^d} |f(x)|\frac{|u(x)|}{|x|} &+ C\int_{\mathbb{R}^d} |f(x)||\nabla u(x)| + C\varepsilon\int_{\mathbb{R}^d} |u(x)||\nabla u(x)| \\ &+ \int_{\mathbb{R}^d} (\nabla\psi(x)\cdot\nabla n(x))_- |u(x)|^2 + \frac{1}{\alpha}\int_{\mathbb{R}^d} (\partial_d n(x))_\sigma |u(x)|^2.\end{aligned}$$

Our task is to estimate the terms in the right-hand side of the last inequality. We separate them in three types : those containing the source f, those containing ε and those containing the index n.
We begin by the two terms containing f. Using the duality estimate (3.1.5), we get for all $\delta > 0$

$$\begin{aligned}\int_{\mathbb{R}^d} |f||\nabla u| &\le \|\nabla u\|_{\dot{B}^*}\|f\|_{\dot{B}} \\ &\le \delta\|\nabla u\|^2_{\dot{B}^*} + C_\delta\|f\|^2_{\dot{B}}.\end{aligned}$$

For the second term, we use a Cauchy-Schwarz inequality to obtain, for all $\delta > 0$

$$\begin{aligned}
\int_{\mathbb{R}^d} |f(x)|\frac{|u(x)|}{|x|} &\leq \sum_{j\in\mathbb{Z}} \left(2^{-j}\int_{C(j)} \frac{|u|^2}{|x|^2}\right)^{1/2}\left(2^j\int_{C(j)} |f|^2\right)^{1/2} \\
&\leq \left(\sup_{R>0}\frac{1}{R^2}\int_{S(R)} |u|^2 d\sigma_R\right)^{1/2}\sum_{j\in\mathbb{Z}}\left(2^j\int_{C(j)} |f|^2\right)^{1/2} \\
&\leq \delta\sup_{R>0}\frac{1}{R^2}\int_{S(R)} |u|^2 d\sigma_R + C_\delta\|f\|^2_{\dot{B}}.
\end{aligned}$$

Next, we turn to the term containing ε. First, using the identities (3.2.2) and (3.2.1) with $\varphi = 1$, one can notice

$$\begin{aligned}
\varepsilon\int_{\mathbb{R}^d} |u|^2 &\leq \int_{\mathbb{R}^d} |f\bar{u}|, \\
\varepsilon\int_{\mathbb{R}^d} |\nabla u|^2 &\leq \|n\|_{L^\infty}\int_{\mathbb{R}^d} |f\bar{u}|.
\end{aligned}$$

Hence, using again the duality estimate (3.1.5), we obtain for all $\delta > 0$

$$\begin{aligned}
\varepsilon\int_{\mathbb{R}^d} |u||\nabla u| &\leq C\|u\|_{\dot{B}^*}\|f\|_{\dot{B}} \\
&\leq \delta\|u\|^2_{\dot{B}^*} + C_\delta\|f\|^2_{\dot{B}}.
\end{aligned}$$

The terms with n remain. We write

$$\begin{aligned}
\int_{\mathbb{R}^d}(\nabla\psi\cdot\nabla n)_-|u|^2 &\leq \sum_{j\in\mathbb{Z}}\int_{C(j)} n|u|^2\frac{(x\cdot\nabla n(x))_-}{n|x|} \\
&\leq \left(\sup_{R>0}\frac{1}{R}\int_{B(R)} |u|^2\right)\sum_{j\in\mathbb{Z}}\sup_{C(j)} 2\frac{(x\cdot\nabla n)_-}{n|x|}.
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^d}(\nabla\psi(x)\cdot\nabla n(x))_-|u(x)|^2 \leq \beta_1\|u\|^2_{\dot{B}^*}.$$

Similarly, we get

$$\frac{1}{\alpha}\int_{\mathbb{R}^d}(\partial_d n(x))_\sigma|u(x)|^2 \leq \beta_2\|u\|^2_{\dot{B}^*}.$$

Putting all these estimates together gives for all $\delta > 0$

$$\begin{aligned}
&\frac{1}{R}\int_{B(R)} |\nabla u(x)|^2 + \frac{1}{R}\int_{B(R)} |u(x)|^2 + \frac{1}{R}\int_{B(R)} (x\cdot\nabla n(x))_+|u(x)|^2 \\
&\qquad + \int_{|x|>R}\frac{|\nabla_\tau u(x)|^2}{|x|} + \frac{d-1}{2R^2}\int_{S(R)} |u|^2 d\sigma_R \\
&\qquad\qquad \leq (\beta_1+\beta_2+\delta)\|u\|^2_{\dot{B}^*} + \delta\|\nabla u\|^2_{\dot{B}^*} + C_\delta\|f\|^2_{\dot{B}}.
\end{aligned}$$

Hence, choosing δ small enough (depending on $\beta_1+\beta_2$) and taking the supremum with respect to R, we obtain

$$\|\nabla u\|^2_{\dot B^*}+\|n^{1/2}u\|^2_{\dot B^*}+\int_{\mathbb{R}^d}\frac{|\nabla_\tau u|^2}{|x|}+\|(x\cdot\nabla n)_+^{1/2}u\|^2_{\dot B^*} + \sup_{R>0}\frac{1}{R^2}\int_{S(R)}|u|^2 d\sigma_R \le C\|f\|^2_{\dot B}. \tag{3.3.2}$$

To end up the proof, it remains to estimate the two last terms in the left-hand side of (3.1.9). We use the inequality (3.2.5) and the previous bounds

$$\begin{aligned}&\int_\Gamma \big|[n]\big|\ |u(x)|^2 d\gamma+\int_{\mathbb{R}^d}(\partial_d n)_{-\sigma}|u|^2\\ &\le C\int_{\mathbb{R}^d}|f\partial_d\bar u|+C\varepsilon\int_{\mathbb{R}^d}|u|\nabla u|+C\frac{1}{\alpha}\int_{\mathbb{R}^d}(\partial_d n)_\sigma|u|^2\\ &\le C(\|f\|^2_{\dot B}+\|u\|^2_{\dot B^*}).\end{aligned}$$

Using (3.3.2) we obtain

$$\int_\Gamma \big|[n]\big|\ |u(x)|^2 d\gamma+\int_{\mathbb{R}^d}(\partial_d n)_{-\sigma}|u|^2\le C\|f\|^2_{\dot B}.$$

This ends the proof.

3.4 Another trace estimate

In this section, we assume that the interface Γ is a hyperplane $\Gamma=\{x_d=0\}$. Then we prove the following extra uniform estimate

Theorem 3.4.1 *Under the assumptions of Theorem 3.1.1, if we assume moreover that $\Gamma=\{x_d=0\}$ and there exists $\beta>0$ such that $\langle x\rangle^{1+\beta}|\nabla_x n|\in L^\infty$, then*

$$\int_\Gamma \big|[n]\big|\ |\nabla u(x)|^2 dx'\le C\Big(\|f\|^2_{\dot B}+\|\nabla_{x'}f\|^2_{\dot B}\Big) \tag{3.4.1}$$

where C is a constant depending only on α, β_1, β_2 and $\|\langle x\rangle^{1+\beta}\nabla_x n\|_{L^\infty}$.

To prove this theorem, we will need the following identity

Lemma 3.4.2 *The solution to the Helmholtz equation (3.1.1) satisfies*

$$\frac{1}{2}\int_\Gamma [n]\,|\nabla_{x'}u(x)|^2dx' - \frac{1}{2}\int_\Gamma [n]\,|\partial_d u(x)|^2dx' - \frac{1}{2}\int_\Gamma [n^2]\,|u(x)|^2dx'$$
$$-\frac{1}{2}\int_{\mathbb{R}^d}\partial_d n\,|\nabla_{x'}u(x)|^2dx' + \frac{1}{2}\int_{\mathbb{R}^d}\partial_d n\,|\partial_d u(x)|^2dx' + \mathcal{R}e\int \nabla_{x'}n\cdot\nabla_{x'}u\,\partial_d\bar{u}dx$$
$$-\int_{\mathbb{R}^d}\partial_d(n^2(x))|u|^2 - \varepsilon\mathcal{I}m\int_{\mathbb{R}^d} n(x)u(x)\partial_d\bar{u}(x)$$
$$= \mathcal{R}e\int_{\mathbb{R}^d} n(x)f(x)\partial_d\bar{u}(x). \quad (3.4.2)$$

Proof. This identity is obtained by multiplying the equation (3.1.1) by $n\partial_d\bar{u}$ and taking the real part. □

Proof of Theorem 3.4.1:

Firstly, note that $\nabla_{x'}n\;u \in \dot{B}$:

$$\begin{aligned}\|\nabla_{x'}n\;u\|_{\dot{B}} &\leq \sum_{j\in\mathbb{Z}}\left(2^{j+1}\int_{C(j)}\langle x\rangle^{-2-2\beta}|u|^2\right)^{1/2}\\ &\leq \sum_{j<0}\left(2^{j+1}\int_{C(j)}\langle x\rangle^{-2-2\beta}|u|^2\right)^{1/2}\\ &\quad+\sum_{j\geq 0}\left(2^{j+1}\int_{C(j)}\langle x\rangle^{-2-2\beta}|u|^2\right)^{1/2}\end{aligned}$$

but, using (3.1.9), we have, for $\delta > 0$,

$$\|\langle x\rangle^{-\frac{1}{2}-\delta}u\|_{L^2} \leq C\|u\|_{\dot{B}^*} \leq C\|f\|_{\dot{B}},$$

hence

$$\begin{aligned}\|\nabla_{x'}n\;u\|_{\dot{B}} &\leq C\|f\|_{\dot{B}}\sum_{j<0}2^{\frac{j+1}{2}} + C\sum_{j\geq 0}2^{-j\beta/2}\|\langle x\rangle^{-\frac{1}{2}-\frac{\beta}{2}}u\|_{L^2}\\ &\leq C\|f\|_{\dot{B}}.\end{aligned}$$

So, we can apply the estimate (3.1.9) to $\nabla_{x'}u$. Indeed, $\nabla_{x'}u$ satisfies the following Helmholtz equation

$$i\varepsilon\nabla_{x'}u + \Delta\nabla_{x'}u + n(x)\nabla_{x'}u = \nabla_{x'}f(x) + \nabla_{x'}n\;u, \quad (3.4.3)$$

so $\nabla_{x'}u$ satisfies

$$\int_\Gamma \big|[n]\big|\,|\nabla_{x'}u|^2dx' \leq C\Big(\|(\nabla_{x'}f\|^2_{\dot{B}} + \|f\|^2_{\dot{B}}\Big).$$

Then, we use (3.4.2) to get

$$\int_\Gamma \big|[n]\big|\,|\partial_d u|^2 dx' \le \int_\Gamma \big|[n]\big|\,|\nabla_{x'} u|^2 dx' + \int_\Gamma \big|[n^2]\big|\,|u|^2 dx' + \int_{\mathbb{R}^d} |\nabla_x n||\nabla_x u|^2 + \int_{\mathbb{R}^d} |\partial_d(n^2)||u|^2 + \varepsilon \int_{\mathbb{R}^d} n|u\partial_d \bar{u}| + \int_{\mathbb{R}^d} n|f\partial_d \bar{u}|. \quad (3.4.4)$$

Since $n \in L^\infty$, we have

$$\int_\Gamma \big|[n^2]\big|\,|u|^2 dx' \le 2\|n\|_\infty \int_\Gamma |[n]||u|^2 dx' \le C\|f\|^2_{\dot{B}}$$

and

$$\int_{\mathbb{R}^d} |\partial_d(n^2)||u|^2 \le 2\|n\|_\infty \int_{\mathbb{R}^d} |\partial_d n||u|^2 \le C\|f\|^2_{\dot{B}}.$$

Moreover, using the hypothesis $\langle x\rangle^{1+\beta}|\nabla_x n| \in L^\infty$, and the estimate (3.1.9), we get

$$\int_{\mathbb{R}^d} |\nabla_x n||\nabla_x u|^2 \le C\|\nabla u\|^2_{\dot{B}^*} \le C\|f\|^2_{\dot{B}}.$$

Hence, estimating the two last terms in the rigth-hand side of (3.4.4) as before and using (3.1.9), we get

$$\int_\Gamma \big|[n]\big|\,|\partial_d u|^2 dx' \le C\Big(\|f\|^2_{\dot{B}} + \|\nabla_{x'} f\|^2_{\dot{B}}\Big),$$

and thus the theorem is proved.

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Chapitre 4

High frequency limit of Helmholtz equations: refraction by sharp interfaces

Article [Fou3], soumis pour publication.

Contents

4.1 Introduction

In this article, we are interested in the analysis of the high frequency limit of the following Helmholtz equation

$$-i\alpha_\varepsilon \varepsilon u^\varepsilon + \varepsilon^2 \Delta u^\varepsilon + n^2(x) u^\varepsilon = -f^\varepsilon(x) = \frac{-1}{\varepsilon^{\frac{d-1}{2}}} f\left(\frac{x}{\varepsilon}\right), \tag{4.1.1}$$

where the variable x belongs to $\mathbb{R}^d$ for some $d \geq 3$.

We assume that the refraction index is given by

$$n^2(x) = \begin{cases} n_+^2(x) & \text{if } x_d \geq 1 \\ n_-^2(x) & \text{if } x_d < 1. \end{cases} \tag{4.1.2}$$

We also assume that there exists $n_0 > 0$ such that $n^2(x) \geq n_0^2$ for all $x \in \mathbb{R}^d$, which means that equation (4.1.1) is uniformly of "Helmholtz type". Problem (4.1.1), (4.1.2) corresponds to a transmission problem across the flat interface $\Gamma = \{x_d = 1\}$. We assume that the jump at the interface Γ satisfies $[n^2](x) = n_-^2(x) - n_+^2(x) > 0$ for all $x \in \Gamma$. This is the only interesting situation, as we explain below.

Equation (4.1.1) modelizes the propagation of a source wave in a medium with scaled refraction index $n^2(x)/\varepsilon^2$. There, the small positive parameter ε is related to the frequency $\omega = \frac{1}{2\pi\varepsilon}$ of u^ε. In this paper, we study the high frequency limit, *i.e.* the asymptotics $\varepsilon \to 0$.

The source term f^ε models a source signal concentrating close to the origin at the scale ε, the concentration profile f being a given function. Since ε is also the scale of the oscillations dictated by the Helmholtz operator $\Delta + \frac{n^2(x)}{\varepsilon^2}$, resonant interactions can occur between these oscillations and the oscillations due to the source f^ε.

Moreover, the interface induces a refraction phenomenon of the energy. As we will see later on, the energy concentrates along the rays of geometrical optics. We choose here the jump of the index at the interface to be positive, which is the interesting case since those rays are attracted by the regions of high index.

These are the two phenomena that the present paper aims at studying quantitatively in the asymptotics $\varepsilon \to 0$. We refer to Section 2 for the precise assumptions we need on the source f, together with the refraction index n^2.

We assume that the regularizing parameter α_ε is positive, with $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. The positivity of α_ε ensures the existence and uniqueness of a solution u^ε to the Helmholtz equation (4.1.1) in $L^2(\mathbb{R}^d)$ for any $\varepsilon > 0$. In some sense, the sign of the term $-i\alpha_\varepsilon \varepsilon u^\varepsilon$ prescribes a radiation condition at infinity for u^ε. One of the key difficulty in our problem is to follow this condition in the limiting process $\varepsilon \to 0$. We will discuss that point later on.

We study the high frequency limit in terms of Wigner measures (or semiclassical measures). This is a mean to describe the propagation of quadratic quantities, like the local energy density $|u^\varepsilon(x)|^2$, as $\varepsilon \to 0$. The Wigner measure $\mu(x, \xi)$ is the energy carried by rays at the point x with frequency ξ. These measures were introduced by E. Wigner [Wig] and then developed by P. Gérard [Gér1], P.-L. Lions, T. Paul [LP] (see also C. Gérard, A. Martinez [GM] and the survey [GMMP]). They are relevant when a typical length ε is prescribed. They have already proven to be an efficient tool in such problems ([BCKP], [CPR], [GL], [Mil2]).

Let us now give a rough idea of our main results. First, we introduce various measures: μ, $\mu_\pm$ denote the Wigner measures associated respectively with u^ε and with the restrictions $u^\varepsilon_\pm$ of u^ε to each medium. These three measures are defined on $T^*\mathbb{R}^d$. Last, we prove that there exist two measures $\mu^{\partial\pm}$ defined on $T^*\Gamma$ that are, in some sense, the traces of $\mu_\pm$ at the interface:

$$\mu_\pm = \mathbf{1}_{\{x_d \gtrless 1\}} \mu_\pm + \delta(x_d - 1) \otimes \delta(\xi_d) \otimes \mu^{\partial\pm}.$$

Our first result, that is valid for a general index of refraction, describes how the sharp interface induces a refraction phenomenon. Depending on the propagation direction, the energy density is either totally reflected, or partially reflected and partially transmitted according to Snell-Descartes's law. More precisely, we prove the following theorem.

Theorem 1 *(General case)*
Assume there is dispersion at infinity of the rays of geometrical optics (which corresponds to geometrical hypotheses on the refraction index n, see (H2)-(H6) page 74).
Assume also:

(a) non-interference (no density comes from both sides at a same point of the interface, see (H13) page 106),

(b) no energy is trapped in the interface ($\mu^{\partial\pm} = 0$, see (H14) page 106).

Then, the Wigner measure associated with (u^ε) is given by

$$\mu(x, \xi) = \int_{-\infty}^{0} (S_t^* Q)(x, \xi) dt, \tag{4.1.3}$$

where S_t^* is the Snell-Descartes semi-group associated with the refraction in-

dex n (see Section 6 for a precise definition) and Q is given by

$$Q(x,\xi) = \frac{1}{2^{d+1}\pi^{d-1}}\delta(x)\delta\big(|\xi|^2 - n^2(0)\big)\big(|\hat{f}(\xi)|^2 + \hat{f}(\xi)\bar{q}(\xi)\big), \qquad (4.1.4)$$

where q is an L^2 density on the sphere $\{|\xi|^2 = n^2(0)\}$.

In this theorem, the energy source Q comes from the resonant interaction between the source f^ε and the solution u^ε. In particular, Q is concentrated at the origin via the Dirac mass $\delta(x)$ and on the resonant frequencies $|\xi|^2 = n^2(0)$. The value of the auxiliary function q is related to the radiation condition at infinity satisfied by the weak limit w of the rescaled sequence of solutions $w^\varepsilon(x) = \varepsilon^{\frac{d-1}{2}} u^\varepsilon(\varepsilon x)$. In the general case, we cannot compute the actual value of q. Also, in the expression (4.1.3), the integral up to infinite time translates the radiation condition at infinity satisfied by the measure μ. The follow-up of this condition in the limiting process is one the key difficulties in our study. Last, the assumption that no energy is trapped in the interface is linked both with the radiation condition at infinity satisfied by the trace of the Wigner measure μ on the interface, and with the (absence of) energy carried by gliding rays at the interface.

In the particular case when the indices n_+ and n_- are *constant*, a situation that we call the *homogeneous case* in the sequel, we prove that the previous assumptions are satisfied. The dispersion at infinity is obvious in that case since the rays are pieces of lines. The proofs of hypotheses *(a)-(b)* together with the identification of q in that case constitute our second main result.

Theorem 2 *(Homogeneous case)*
When the two indices n_+ and n_- are constant, we have:
(i) the non-interference hypothesis is satisfied,
(ii) $\mu^{\partial_\pm} = 0$,
(iii) $q = 0$ (i.e. w is the outgoing solution to the Helmholtz equation $\Delta w + n_-^2 w = f$).

The combination of Theorem 1 and Theorem 2 gives a completely explicit expression for the Wigner measure μ in the homogeneous case.

To prove point (i), we proceed as follows: we first use the fact that the energy source in the transport equation satisfied by μ away from the interface is concentrated on one side of the interface (at $x = 0$), which implies that μ is constant along the rays on the right side of the interface. Next, we use the radiation condition at infinity outside the interface, which gives that μ vanishes at infinity along the incoming rays. From these two facts, we deduce that no energy is carried by incoming rays at the interface from $\{x_d > 1\}$.

To prove points (*ii*) and (*iii*), we exploit the explicit formula for the resolvent of the Helmholtz operator that is available in the particular case of two homogeneous media, which reduces to a study of (non-)stationary phase with singularity. Indeed, if we denote $\xi = (\xi', \xi_d) \in \mathbb{R}^d$, since the measure μ is supported in the set $\xi^2 = n^2(x)$, the roots $\sqrt{\xi'^2 - n_\pm^2 + i\alpha_\varepsilon\varepsilon}$ to the equations $\xi_d^2 = n_\pm^2 - \xi'^2(-i\alpha_\varepsilon\varepsilon)$ naturally appear in the expressions that we consider. In order to treat the singularity of these roots near $\xi'^2 = n_\pm^2$ when $\varepsilon \to 0$, the key ingredients are a contour deformation in the complex plane and the use of almost-analytic extensions.

The method we use to prove Theorem 1 is a combination of two methods: the one introduced by L. Miller [Mil1] for the study of the semiclassical limit of transmission problems for Schrödinger equations, and the one introduced by Benamou, Castella, Katsaounis, Perthame [BCKP] to study the high frequency limit of Helmholtz equations with source term and smooth index of refraction. Let us give some details.

As a first step, we establish bounds on the sequence of Wigner transforms associated with (u^ε), which will ensure the existence of a Wigner measure μ. These bounds are deduced, as in [BCKP], from uniform (in ε) bounds on the sequence (u^ε). To establish the latter, we rather study the rescaled sequence

$$w^\varepsilon(x) = \varepsilon^{\frac{d-1}{2}} u^\varepsilon(\varepsilon x),$$

which obviously satisfies

$$-i\alpha_\varepsilon\varepsilon w^\varepsilon + \Delta w^\varepsilon + n^2(\varepsilon x) w^\varepsilon = -f(x). \tag{4.1.5}$$

We use the results independently proved by the author in [Fou2] using a multiplicator method borrowed from [PV1]. Under some homogeneous dispersive conditions on the refraction index, these provide uniform homogeneous Besov-like estimates, together with uniform $L^2(\Gamma)$ estimates on the traces of w^ε and $\partial_{x_d} w^\varepsilon$ on the interface. Once these bounds are established, we readily obtain bounds on u^ε.

However, as we have already mentionned, it turns out that our method also requires to identify the limit $w = \lim w^\varepsilon$ (it exists up to extraction) in order to determine the source term Q. This limit w clearly satisfies the following Helmholtz equation with constant index

$$\Delta w + n^2(0) w = -f. \tag{4.1.6}$$

Unfortunately, the equation (4.1.6) does not identify w in a unique way. In the general case, we cannot identify w as *the* outgoing solution to this equation. Two difficulties arise: the treatment of the interface and the variability

of the indices $n_\pm(x)$. We only identify w as the outgoing solution to (4.1.6) when the two media are homogeneous (Theorem 2). This problem already appears when the refraction index is smooth. In the latter case, Castella [Cas] and Wang, Zhang [WZ] recently proved by two different approaches that the weak limit of the solution to (4.1.5) is the outgoing solution to (4.1.6).
Before going further, we would like to emphasize here that we cannot obtain the estimates on u^ε for a general interface (*i.e.* if the interface is not a hyperplane), which prevent us from studying the high frequency limit in this more general context. More precisely, we still get the homogeneous bounds on u^ε and $\varepsilon\nabla u^\varepsilon$ in $\dot{B}^*$, together with the uniform bound in $L^2(\Gamma)$ on the trace $u^\varepsilon_{\rceil_\Gamma}$, but we cannot obtain anymore the uniform bound in $L^2(\Gamma)$ on the trace $\varepsilon\partial_d u^\varepsilon_{\rceil_\Gamma}$ that is also necessary in our study.

As a second step, we study the Wigner measure μ outside the interface. This is done following Benamou *et al* [BCKP]. Since the refraction index is smooth in the interior of each medium, we can use their results to get the transport equation satisfied by the Wigner measure μ outside the interface. Their proof is based on estimates of the type we proved in [Fou2], thus we obtain that

$$0^+\mu + \xi \cdot \nabla_x \mu + \frac{1}{2}\nabla_x n^2(x) \cdot \nabla_\xi \mu = Q, \tag{4.1.7}$$

in the interior of each medium, where Q is given by (4.1.4). The term $0^+\mu$ is the track of the outgoing radiation condition on u^ε. It determines μ as the outgoing solution to (4.1.7) in the following particular sense:

$$\mu(x,\xi) = \int_{-\infty}^{0} Q\big(X(t),\Xi(t)\big)dt,$$

for (x,ξ) such that the bicaracteristics $(X(t),\Xi(t))$ defined by

$$\begin{cases} \dot{X}(t) &= \Xi(t), & X(0) &= x \\ \dot{\Xi}(t) &= \frac{1}{2}\nabla_x n^2\left(X(t)\right), & \Xi(0) &= \xi, \end{cases}$$

does not reach the interface for $t \in (-\infty, 0)$. As in [BCKP], we have to handle two specific difficulties: the treatment of the source term (that can be done thanks to the appropriate scaling chosen for f^ε), and the proof of the radiation condition on μ. By proving first a localization property on μ, we improve the radiation condition at infinity proved in [BCKP].

As a third step, we study the behaviour of μ at the interface. For this, we use the method of Miller [Mil1], [Mil2]. We first write the transport equations *up to the boundary* satisfied by μ in a weak form, using only tangential test operators. Next, using these transport equations, we obtain the local propagation relations at the interface (in particular the refraction).

Finally, to obtain (4.1.3), we last use the transport equation (4.1.7) together with the radiation condition at infinity and the propagation relations at the boundary obtained in the previous step.

Our paper is organized as follows. In Section 2, we first recall the two points of view while studying Wigner measures, pseudodifferential operators or Wigner transforms. Then we give our main assumptions on the refraction index and the source profile f. In Section 3, we establish uniform bounds on the sequence (u^ε) and the sequence of Wigner transforms $(W^\varepsilon(u^\varepsilon))$. In Section 4, we obtain the transport equations satisfied by Wigner measures outside the interface and up to the boundary. In Section 5, we prove our refraction result in the case of two homogeneous media, which illustrates our procedure in this easier case (the geometry of rays is explicitely known in this case). Then, we extend the result to the general case in Section 6, *i.e.* for non constant indices. Section 7 is devoted to the proof of the radiation conditions at infinity in the homogeneous case (as we have already seen, the Helmholtz equation and the kinetic transport equation (4.1.7) must be both complemented by such a condition to determine a unique solution). These conditions concern the limit w of the rescaled solution to the Helmholtz equation on the one hand, and the Wigner measure "inside" the interface on the other hand. In Appendix A, we detail the derivation of the explicit formula for the solution u^ε in the homogeneous case. We recall in Appendix B some results about sharp truncation and pseudodifferential operators we use in our study. Finally, in Appendix C, we give the proofs of the properties on tangential test operators.

4.2 Notations and assumptions on the source and the refraction index

4.2.1 Semiclassical measures and Wigner transform

In this section, we recall some usual definitions and notations we will use in the sequel together with the link between the two different points of view in the study of semiclassical measures (using peudodifferential operators or Wigner transforms).

We use the following definition for the Fourier transform:

$$\hat{u}(\xi) = (\mathcal{F}_{x\to\xi}u)(\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}u(x)dx.$$

The Weyl semiclassical operator $a^w(x, \varepsilon D_x)$ (or $Op^w_\varepsilon(a)$) is the continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ associated with the symbol $a \in \mathcal{S}'(T^*\mathbb{R}^d)$ by

Weyl quantization rule

$$(a^w(x,\varepsilon D_x)u)(x) = \frac{1}{(2\pi\varepsilon)^d}\int_{\mathbb{R}^d_\xi}\int_{\mathbb{R}^d_y} e^{i\frac{(x-y)\cdot\xi}{\varepsilon}} a\big(\frac{x+y}{2},\xi\big) f(y)d\xi dy.$$

If $a \in \mathcal{S}(T^*\mathbb{R}^d)$ then $a^w(x,\varepsilon D_x)$ is continuous from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ and it is continuous from $H^s(\mathbb{R}^d)$ to $H^{s'}(\mathbb{R}^d)$ for any real s, s' with as $\varepsilon \to 0$,

$$\begin{aligned} \|a^w(x,\varepsilon D_x)\|_{\mathcal{L}(H^s,H^{s'})} &= O(1) &&\text{if } s \geq s' \\ \|a^w(x,\varepsilon D_x)\|_{\mathcal{L}(H^s,H^{s'})} &= O(\varepsilon^{s-s'}) &&\text{if } s \leq s'. \end{aligned}$$

For $u, v \in \mathcal{S}(\mathbb{R}^d)$ and $\varepsilon > 0$, we define the Wigner transform

$$\begin{aligned} W^\varepsilon(u,v)(x,\xi) &= (\mathcal{F}_{y\to\xi})(u\big(x+\frac{\varepsilon}{2}y\big)\bar{v}\big(x-\frac{\varepsilon}{2}y\big)), \\ W^\varepsilon(u) &= W^\varepsilon(u,u). \end{aligned}$$

We have the following formula: for $u, v \in \mathcal{S}'(\mathbb{R}^d)$ and $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\langle W^\varepsilon(u,v), a\rangle_{\mathcal{S}',\mathcal{S}} = \langle u, a^w(x,\varepsilon D_x)v\rangle_{\mathcal{S}',\mathcal{S}}, \tag{4.2.1}$$

where the duality brackets $\langle .,.\rangle$ are semi-linear with respect to the first argument. This formula is also valid for u, v lying in other spaces as we will see in Section 3.

If (u^ε) is a bounded sequence in $L^2(\mathbb{R}^d)$ (or in some weighted L^2 space as it is the case in our problem), it turns out that, up to extracting a subsequence, there exists a *Wigner measure* (or *semiclassical measure*) μ associated with (u^ε), *i.e.* a positive Radon measure on the phase space $T^*\mathbb{R}^d = \mathbb{R}^d_x \times \mathbb{R}^d_\xi$ satisfying:

$$\forall a \in \mathcal{C}^\infty_c(\mathbb{R}^{2d}),\ \lim_{\varepsilon\to 0}\langle u^\varepsilon, a^w(x,\varepsilon D_x)u^\varepsilon\rangle_{L^2} = \lim_{\varepsilon\to 0}\langle W^\varepsilon(u^\varepsilon), a\rangle = \int a(x,\xi)d\mu$$

4.2.2 Assumptions on the refraction index and the source

In the sequel, we denote $x = (x', x_d)$ a point in $\mathbb{R}^d$.
In order to get uniform (in ε) bounds on the sequence (u^ε), we use the following homogeneous Besov-like norms: for $u, f \in L^2_{loc}$,

$$\begin{aligned} \|u\|^2_{\dot{B}^*} &= \sup_{R>0}\frac{1}{R}\int_{B(R)}|u|^2 dx, \\ \|f\|_{\dot{B}} &= \sum_{j\in\mathbb{Z}}\Big(2^{j+1}\int_{C(j)}|f|^2 dx\Big)^{1/2}, \end{aligned}$$

where $B(R)$ denotes the ball of radius R, and $C(j)$ the ring $\{x \in \mathbb{R}^d / 2^j \leq |x| < 2^{j+1}\}$.
These norms were introduced (in their inhomogeneous version) by Agmon and Hörmander [AH], and they have been used recently by Perthame and Vega [PV1].
They satisfy the following duality relation

$$\left| \int u(x) f(x) dx \right| \leq \|u\|_{\dot{B}^*} \|f\|_{\dot{B}}.$$

We denote for $x \in \mathbb{R}^d$, $|x| = \sqrt{\sum_{j=1}^d x_j^2}$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$.
For all $\delta > \frac{1}{2}$, the space $\dot{B}^*$ is contained in the weighted L^2 space $L^2_{-\delta}$ with inhomogeneous weight $\langle x \rangle^{-\delta}$:

$$\|u\|_{L^2_{-\delta}} := \|\langle x \rangle^{-\delta} u\|_{L^2} \leq C(\delta) \|u\|_{\dot{B}^*}. \tag{4.2.2}$$

Similarly, we have for all $\delta > 1/2$,

$$\|f\|_{\dot{B}} \leq C(\delta) \|f\|_{L^2_\delta}.$$

We are now ready to state our assumptions. Our first (technical) assumption, borrowed from [BCKP], concerns the regularizing parameter:

(H1) $\alpha_\varepsilon \geq \varepsilon^\gamma$ for some $\gamma > 0$.

Next, we need assumptions on the refraction index that are mainly related to the dispersion at infinity of the rays of geometrical optics. The following five are those made in [Fou2] to obtain the estimates on u^ε.

(H2) there exists $c > 0$ such that $[n^2](x) \geq c$ for all $x \in \Gamma$,

(H3) there exists $n_0 > 0$ such that $n \in L^\infty$, $n \geq n_0$.

(H4)

$$2 \sum_{j \in \mathbb{Z}} \sup_{C(j)} \frac{(x \cdot \nabla n^2(x))_-}{n^2(x)} := \beta_1 < \infty,$$

(H5)

$$\sum_{j \in \mathbb{Z}} \sup_{C(j)} 2^{j+1} \frac{(\partial_d n^2(x))_+}{n^2(x)} := \beta_2 < \infty,$$

(H6) $\beta_1 + \beta_2 < 1$,

Next, following Benamou *et al*[BCKP], in order to follow the radiation condition in the limiting process, we assume a stronger decay at infinity on the index:

(H7) $\langle x\rangle^{N_0}\nabla_x n_\pm^2 \in L^\infty$ for some $N_0 > 2$,

(H8) $\nabla_x n_\pm^2$ locally Lipschitz on $\{x \in \mathbb{R}^d / x_d \gtrless 1\}$.

As we will see in Section 3, to get uniform bounds on u^ε, we assume that the source term satisfies

(H9) $\|f\|_{\dot{B}}, \|\nabla f\|_{\dot{B}} < \infty$.

In order to compute the limit of the energy source, we make, as in [BCKP], the stronger assumption

(H10) $\langle x\rangle^N f \in L^2(\mathbb{R}^d)$ for some $N > \frac{1}{2} + \frac{3\gamma}{\gamma+1}$, and $\langle x\rangle^{N_1}\partial_{x_d} f \in L^2(\mathbb{R}^d)$ for some $N_1 > 1/2$.

Finally, we assume

(H11) $f \in H^{\frac{1}{2}+s}(\mathbb{R}^d)$ for some $s > 0$,

so that the traces of f on hyperplanes are well-defined in $L^2(\mathbb{R}^{d-1}_{x'})$, and

(H12) $\lim_{\varepsilon\to 0} \|f\left(\cdot, \frac{1}{\varepsilon}\right)\|_{L^2(\mathbb{R}^{d-1}_{x'})} = 0$.

The last assumption can be rewritten as $\|f^\varepsilon(\cdot, 1)\|_{L^2(\mathbb{R}^{d-1}_{x'})} \to 0$ as $\varepsilon \to 0$, so it means that no source density remains at the interface as $\varepsilon \to 0$.

Let us comment the assumptions we make on the index n. The conditions (H2) and (H5) are specific to the case with interface: they mainly ensure that the energy goes from one side of the interface to the other. The hypothesis (H4) together with (H3), ensures the dispersion at infinity of the rays of geometrical optics outside the interface. (H4) is a kind of *virial assumption*. We would like to point out that we do not require that the index n goes to a constant at infinity.

We recall here how such hypotheses (H3), (H4) induce the dispersion at infinity of the bicharacteristic curves at zero energy, *i.e.* the zero energy is non-trapping (at least without interface). Indeed, when the bicharacteristics does not intersect the interface, $(X(t), \Xi(t))$ is defined by the Hamiltonian system (for instance),

$$\begin{cases} \dot{X}(t) &= \Xi(t), & X(0) &= x \\ \dot{\Xi}(t) &= \frac{1}{2}\nabla_x n_-^2\left(X(t)\right), & \Xi(0) &= \xi, \end{cases} \tag{4.2.3}$$

where the index of refraction n_- is smooth.
Let (x, ξ) be such that $\xi^2 = n^2(x)$. Then, $\frac{\mathrm{d}}{\mathrm{dt}} X(t)^2 = 2X(t) \cdot \Xi(t)$ and $\frac{\mathrm{d}}{\mathrm{dt}} X(t) \cdot \Xi(t) = \Xi(t)^2 + \frac{1}{2} X(t) \cdot \nabla_x n_-^2\left(X(t)\right)$. Since (H4) implies that for all $x \in \mathbb{R}^d$,

$$\frac{(x \cdot \nabla n^2(x))_-}{n^2(x)} \leq \frac{\beta_1}{2} < \frac{1}{2},$$

we get

$$\frac{\mathrm{d}}{\mathrm{dt}} X(t) \cdot \Xi(t) = n_-(X(t))^2 + \frac{1}{2} X(t) \cdot \nabla_x n_-^2\left(X(t)\right) \geq \frac{n_-(X(t))^2}{2} \geq \frac{n_0^2}{2}.$$

Hence, for t sufficiently negative, we have $X(t) \cdot \Xi(t) \leq \frac{n_0^2}{4} t$ and $X(t)^2 \geq \frac{n_0^2}{4} t^2$. Thus we proved that

$$|X(t)| \to \infty \quad \text{with} \quad X(t) \cdot \Xi(t) < 0 \quad \text{as} \quad t \to -\infty.$$

4.3 Bounds on u^ε, $W^\varepsilon(u^\varepsilon)$, $W^\varepsilon(f^\varepsilon, u^\varepsilon)$

The first step in our study is to prove uniform bounds on the sequence of Wigner transforms $(W^\varepsilon(u^\varepsilon))$, which will ensure the existence of a Wigner measure associated with the sequence of solutions (u^ε) (up to extracting a subsequence). As in [BCKP], we deduce these bounds from uniform homogeneous bounds on (u^ε).

4.3.1 Bounds on the solution to the Helmholtz equation

In this part, we give uniform bounds on the sequences (u^ε) and $(\varepsilon \nabla u^\varepsilon)$ and their traces on the interface. This will allow us to define the various Wigner measures that appear in our problem. The following theorem is proved in [Fou2] (using the multiplier method introduced by Perthame and Vega [PV1]):

Theorem 4.3.1 ***(borrowed from [Fou2])*** *Under the hypotheses (H2)-(H7), the solution to the Helmholtz equation (4.1.1) satisfies*

$$\|\varepsilon \nabla u^\varepsilon\|_{\dot{B}^*}^2 + \|u^\varepsilon\|_{\dot{B}^*}^2 + \int_\Gamma \left|[n^2]\right| |u^\varepsilon|^2 dx' + \int_\Gamma \left|[n^2]\right| |\varepsilon \nabla u^\varepsilon|^2 dx' \leq C(\|f\|_{\dot{B}}^2 + \|\nabla f\|_{\dot{B}}^2) \tag{4.3.1}$$

where C does not depend on ε.

Remarks: *Actually, in [Fou2], we proved the result for $\varepsilon = 1$, but thanks to the homogeneity of the norms and assumptions, it also holds for $\varepsilon \in (0,1)$.*

Let us say again that, for more general interfaces (not hyperplanes), we cannot get the uniform bound in $L^2(\Gamma)$ on the trace $\varepsilon\partial_d u^\varepsilon$, which is necessary in our study.

We draw two consequences of these bounds that will be useful for our purpose. First, we study the limit of the rescaled sequence defined by

$$w^\varepsilon(x) = \varepsilon^{\frac{d-1}{2}} u^\varepsilon(\varepsilon x)$$

that appears while computing the limit of the source term in the transport equation satisfied by the Wigner measure μ. One can notice that, thanks to the homogeneity of the norm $\dot{B}^*$, we have the following scaling invariance

$$\begin{aligned} \|w^\varepsilon\|_{\dot{B}^*} &= \|u^\varepsilon\|_{\dot{B}^*} \\ \|\nabla w^\varepsilon\|_{\dot{B}^*} &= \|\varepsilon \nabla u^\varepsilon\|_{\dot{B}^*}. \end{aligned}$$

Theorem 4.3.2 *(i) We may extract from (w^ε) a subsequence which converges weak-$*$ in $\dot{B}^*$ and strongly in $L^2_{loc}(\mathbb{R}^d)$ to a solution w of*

$$\Delta w + n(0)^2 w = -f. \tag{4.3.2}$$

As a consequence, there exists a density $q \in L^2(|\xi|^2 = n^2(0))$ such that

$$\hat{w}(\xi) = \hat{w}_0(\xi) + i\frac{\pi}{2} q(\xi)\delta(|\xi|^2 - n^2(0)), \tag{4.3.3}$$

where w_0 is the outgoing solution to (4.3.2), given by

$$\hat{w}_0(\xi) = (|\xi|^2 - n^2(0) + i0)^{-1}\hat{f}(\xi) = \Big(p.v.\big(\frac{1}{|\xi|^2 - n^2(0)}\big) + i\frac{\pi}{2}\delta(|\xi|^2 - n^2(0))\Big)\hat{f}(\xi).$$

(ii) If n_+ and n_- are constant then w is the *outgoing solution w_0 to (4.3.2), i.e. $q = 0$.*

Remark: *In general, we cannot identify w as the outgoing solution to (4.3.2). This problem already appears in the case of a smooth index of refraction (*i.e. *without interface). It has been solved in that case only recently by two different approaches by Castella [Cas], and Wang, Zhang [WZ].*

Proof. The first part of point (i) can be easily deduced from Theorem 4.3.1 using Rellich's theorem. The formula (4.3.3) can be found in [AH]. Point

(ii) is proved in Section 7. □

The second property we will need in our study is the ε-oscillation of the sequence of solutions (u^ε).

Proposition 4.3.3 *The sequence (u^ε) is strongly ε-oscillating of order $2+s$: for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $|\varepsilon D_x|^s \varphi |\varepsilon D_x|^2 u^\varepsilon$ is bounded in $L^2(\mathbb{R}^d)$.*

Proof. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. From property (4.2.2) and the estimate (4.3.1), we deduce that the sequences $(u^\varepsilon), (\varepsilon \nabla u^\varepsilon)$ are bounded in $L^2_{-\beta}(\mathbb{R}^d)$ for any $\beta > 1/2$. Since u^ε is the solution to the Helmholtz equation (4.1.1), with a source term f^ε bounded in $L^2_{-\beta}(\mathbb{R}^d)$, $(|\varepsilon D_x|^2 u^\varepsilon)$ is also bounded in $L^2_{-\beta}(\mathbb{R}^d)$. Hence, $|\varepsilon D_x|^s \varphi u^\varepsilon$ is bounded in $L^2(\mathbb{R}^d)$ and so is $|\varepsilon D_{x_d}|^s \varphi n^2 u^\varepsilon$ by Lemma B.4. Moreover, $(|\varepsilon D_x|^s f^\varepsilon)$ is bounded in $L^2(\mathbb{R}^d)$. Indeed,

$$\big\| \, |\varepsilon D_x|^s f^\varepsilon \big\|_{L^2} = \big\| \, |\varepsilon \xi|^s \widehat{f^\varepsilon} \big\|_{L^2} = \sqrt{\varepsilon} \big\| \, |\xi|^s \hat{f} \big\|_{L^2},$$

which is bounded since $f \in H^{\frac{1}{2}+s}(\mathbb{R}^d)$. Hence, using the equation (4.1.1), we deduce that $|\varepsilon D_{x_d}|^s \varphi |\varepsilon D_{x_d}|^2 u^\varepsilon$ is bounded in $L^2(\mathbb{R}^d)$. □

4.3.2 Bounds on the Wigner transforms $W^\varepsilon(u^\varepsilon)$ and $W^\varepsilon(f^\varepsilon, u^\varepsilon)$

From Theorem 4.3.1, we now deduce bounds on the sequences of Wigner transforms $(W^\varepsilon(u^\varepsilon))$ and $(W^\varepsilon(f^\varepsilon, u^\varepsilon))$. We obviously need uniform bounds on $(W^\varepsilon(u^\varepsilon))$. The study of the sequence $(W^\varepsilon(f^\varepsilon, u^\varepsilon))$ is also necessary to handle the source term in the high frequency limit. Indeed, $W^\varepsilon(u^\varepsilon)$ satisfies the following equation, where W^ε stands for $W^\varepsilon(u^\varepsilon)$:

$$\alpha_\varepsilon W^\varepsilon + \xi \cdot \nabla_x W^\varepsilon + Z^\varepsilon \star_\xi W^\varepsilon = \frac{i}{2\varepsilon} \mathcal{I}m \, W^\varepsilon(f^\varepsilon, u^\varepsilon) := Q^\varepsilon \tag{4.3.4}$$

with $Z^\varepsilon(x,\xi) = \frac{i}{2\varepsilon} \mathcal{F}_{y\to\xi}\Big(n^2\big(x + \frac{\varepsilon}{2}y\big) - n^2\big(x - \frac{\varepsilon}{2}y\big)\Big)$.
This equation can be obtained writing first the equation satisfied by $v^\varepsilon(x,y) = u^\varepsilon\big(x + \frac{\varepsilon}{2}y\big)\bar{u^\varepsilon}\big(x - \frac{\varepsilon}{2}y\big)$. From the equality

$$\nabla_y \cdot \nabla_x v^\varepsilon = \frac{\varepsilon}{2}\big[\Delta u^\varepsilon(x + \frac{\varepsilon}{2}y)\bar{u^\varepsilon}(x - \frac{\varepsilon}{2}y) - \Delta \bar{u^\varepsilon}(x - \frac{\varepsilon}{2}y) u^\varepsilon(x + \frac{\varepsilon}{2}y)\big],$$

we deduce

$$\alpha_\varepsilon v^\varepsilon + i \nabla_y \cdot \nabla_x v^\varepsilon + \frac{i}{2\varepsilon}\big[n^2(x + \frac{\varepsilon}{2}y) - n^2(x - \frac{\varepsilon}{2}y)\big] v^\varepsilon = \sigma_\varepsilon(x,y),$$

where

$$\sigma_\varepsilon(x,y) := \frac{i}{2\varepsilon}\big[f^\varepsilon(x+\frac{\varepsilon}{2}y)\bar{u^\varepsilon}(x-\frac{\varepsilon}{2}y) - \bar{f}^\varepsilon(x-\frac{\varepsilon}{2}y)u^\varepsilon(x+\frac{\varepsilon}{2}y)\big].$$

After a Fourier transform, we obtain the equation (4.3.4).

The following two results are proved in [BCKP] (we write the proofs below for the convenience of the reader).

Proposition 4.3.4 ***(borrowed from [BCKP])*** *Assume that the sequence (u^ε) is bounded in $\dot{B}^*$. Then, for any $\lambda > 0$, the sequence of Wigner transforms $(W^\varepsilon(u^\varepsilon))$ is bounded in the Banach space X^*_λ below and , extracting a subsequence, converges weak-$*$ to a nonnegative, locally bounded measure μ such that*

$$\sup_{R>0}\frac{1}{R}\int_{|x|<R}\int_{\xi\in\mathbb{R}^d} d\mu(x,\xi) \le C\|f\|^2_{\dot{B}}. \tag{4.3.5}$$

The Banach space X^*_λ is defined as the dual space of the set X_λ of functions $\hat{\varphi}(x,\xi)$ such that $\varphi(x,y) := \mathcal{F}_{\xi\to y}(\hat{\varphi}(x,\xi))$ satisfies

$$\int_{\mathbb{R}^d_y}\sup_{x\in\mathbb{R}^d}(1+|x|+|y|)^{1+\lambda}|\varphi(x,y)|dy < \infty. \tag{4.3.6}$$

The second result is due to the particular choice of the scaling of the source term in the Helmholtz equation (4.1.1).

Proposition 4.3.5 ***(borrowed from [BCKP])*** *Let (u^ε) be a sequence of functions bounded in $\dot{B}^*$, and $f \in L^2_N(\mathbb{R}^d)$ with $N > \frac{1}{2}$. We denote $f^\varepsilon(x) = \frac{1}{\varepsilon^{\frac{d-1}{2}}}f\big(\frac{x}{\varepsilon}\big)$. Then, the sequence $(W^\varepsilon(f^\varepsilon,u^\varepsilon))$ is bounded in $\mathcal{S}'(T^*\mathbb{R}^d)$ and for all $\psi \in \mathcal{S}(T^*\mathbb{R}^d)$, we have*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\langle W^\varepsilon(f^\varepsilon,u^\varepsilon),\psi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\bar{\hat{w}}(\xi)\hat{f}(\xi)\psi(0,\xi)d\xi, \tag{4.3.7}$$

where w is defined in Proposition 4.3.2.

Proof of Proposition 4.3.4.
We observe that

$$\|\langle x\rangle^{-\frac{1}{2}-\lambda}u^\varepsilon(x)\|_{L^2} \le C\|u^\varepsilon\|_{\dot{B}^*} \le C\|f\|_{\dot{B}},$$

hence, for any function φ satisfying (4.3.6), we have

$$\begin{aligned}
&|\langle W^\varepsilon(u^\varepsilon),\hat{\varphi}\rangle| \\
&\le \int_{\mathbb{R}^{2d}}\frac{|u^\varepsilon|(x+\frac{\varepsilon}{2}y)|\overline{u^\varepsilon}|(x-\frac{\varepsilon}{2}y)}{\langle x+\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}\langle x-\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}}\langle x+\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}\langle x-\frac{\varepsilon}{2}y\rangle^{\frac{1}{2}+0}|\varphi|(x,y)dxdy \\
&\le C\|f\|^2_{\dot{B}}\int_{\mathbb{R}^d}\sup_{x\in\mathbb{R}^d}\langle|x|+|y|\rangle^{1+0}|\varphi(x,y)|dy.
\end{aligned}$$

Therefore, $(W^\varepsilon(u^\varepsilon))$ is bounded in X_λ^*, $\lambda > 0$. We deduce that, up to extracting a subsequence, $(W^\varepsilon(u^\varepsilon))$ converges weak-$*$ to a nonnegative measure μ that satisfies

$$|\langle \mu, \hat{\varphi} \rangle| \leq C\|f\|_{\dot{B}}^2 \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} \langle |x| + |y| \rangle^{1+0} |\varphi(x,y)| dy. \tag{4.3.8}$$

We refer to [LP] for the proof of the nonnegativity of μ.
The bound (4.3.5) is obtained using the following family of functions

$$\varphi_\mu^R(x,y) = \frac{1}{\mu^{3/2}} e^{-|y|^2/\mu} \frac{1}{R} \chi(\langle x \rangle \leq R)$$

and letting $\mu \to 0$, $R \to \infty$. □

Proof of Proposition 4.3.5.
Let $\psi \in \mathcal{S}(T^*\mathbb{R}^d)$ and $\varphi(x,y) = \mathcal{F}^{-1}_{y\to\xi}(\psi(x,\xi))$, then we have

$$\begin{aligned}
\frac{1}{\varepsilon}\langle W^\varepsilon(f^\varepsilon, u^\varepsilon), \psi \rangle_{\mathcal{S}',\mathcal{S}} &= \frac{1}{\varepsilon}\int f^\varepsilon\big(x + \frac{\varepsilon}{2}y\big)\overline{u^\varepsilon}\big(x - \frac{\varepsilon}{2}y\big)\varphi(x,y)dxdy \\
&= \int f(x)\overline{w^\varepsilon}(x+y)\varphi(\varepsilon(x+\frac{y}{2}), y)dxdy.
\end{aligned}$$

Hence, using that $\psi \in \mathcal{S}(\mathbb{R}^{2d})$, we get

$$\begin{aligned}
\left|\frac{1}{\varepsilon}\langle W^\varepsilon(f^\varepsilon, u^\varepsilon), \psi \rangle_{\mathcal{S}',\mathcal{S}}\right| &\leq C\int \langle x \rangle^N |f(x)| \frac{|w^\varepsilon(x+y)|}{\langle x+y \rangle^\beta} \frac{\langle x+y \rangle^\beta}{\langle x \rangle^N \langle y \rangle^k} dxdy \\
&\leq C\|\langle x \rangle^N f\|_{L^2} \|w^\varepsilon\|_{\dot{B}^*} \int_{\mathbb{R}^d_y} \sup_{x\in\mathbb{R}^d} \frac{\langle x+y \rangle^\beta}{\langle x \rangle^N \langle y \rangle^k} dy
\end{aligned}$$

for any $k \geq 0$ and $\beta > 1/2$, upon using the Cauchy-Schwarz inequality in x. Then, we distinguish the cases $|x| \leq |y|$ and $|x| \geq |y|$: the term stemming from the first case gives a contribution which is bounded by $C\varepsilon \int \frac{dy}{\langle y \rangle^{k-\beta}}$ and the second contribution is bounded by $C\varepsilon \int \frac{dy}{\langle y \rangle^k}$. Hence, upon choosing k large enough, we obtain that

$$\left|\frac{1}{\varepsilon}\langle W^\varepsilon(f^\varepsilon, u^\varepsilon), \psi \rangle_{\mathcal{S}',\mathcal{S}}\right| \leq C\|\langle x \rangle^N f\|_{L^2} \|w^\varepsilon\|_{\dot{B}^*}.$$

Now, in order to compute the limit (4.3.7), we write

$$\begin{aligned}
\frac{1}{\varepsilon}\langle W^\varepsilon(f^\varepsilon, u^\varepsilon), \psi \rangle &= \int f(x)\overline{w^\varepsilon}(x+y)\left(\hat{\psi}\left(\varepsilon\left(x+\frac{y}{2}\right), y\right) - \hat{\psi}(0,y)\right)dxdy \\
&\quad + \int f(x)\overline{w^\varepsilon}(x+y)\hat{\psi}(0,y)dxdy \\
&= I_\varepsilon + II_\varepsilon.
\end{aligned}$$

Reasonning as above, we readily get that $\lim_{\varepsilon\to 0} I_\varepsilon = 0$. For the second term, we use the strong convergence of (w^ε) in L^2_{loc}, which implies

$$II_\varepsilon \to \int f(x)\bar{w}(x+y)\hat{\psi}(0,y)dxdy.$$

□

4.4 Transport equations on the Wigner measures

The next step in our study is the derivation of the transport equations satisfied by the various Wigner measures that appear in our problem. These equations are of two different types. The first one is the transport equation satisfied by the Wigner measure μ in the *interior* of each medium, it is deduced from the case with a smooth index of refraction studied in [BCKP]. The other two equations concern the Wigner measures associated with the restrictions of (u^ε) to each side of the interface *up to the boundary*: the presence of the interface induces some extra source terms in these equations that involve the Wigner measures associated with the traces of u^ε and $\varepsilon\partial_d u^\varepsilon$ on the interface.
As we have already noted for the Helmholtz equation, the kinetic transport equation (of Liouville type) satisfied by the Wigner measure μ must be complemented by a radiation condition at infinity to determine a unique solution.

4.4.1 Notations

Throughout our study, we shall use the following notations.

For a function φ defined on $\mathbb{R}^d \times \mathbb{R}^k$ for some $k \geq 0$, we denote $\varphi_{\rceil\Gamma}$ the trace of φ on $\Gamma \times \mathbb{R}^k$.

(u, v) denote the scalar product of u and v in $L^2(\mathbb{R}^d)$, $(u, v)_\pm$ the scalar product of their restrictions to $L^2(x_d \gtrless 1)$, and $(u, v)_\Gamma$ the scalar product of their traces on Γ if they are defined in $L^2(\Gamma)$.

In the sequel, we denote $H^\varepsilon = -\varepsilon^2\Delta - n^2(x)$. H^ε is a selfadjoint semiclassical operator with symbol $|\xi|^2 - n^2(x)$. The equation (4.1.1) can be rewritten

$$i\alpha_\varepsilon \varepsilon u^\varepsilon + H^\varepsilon u^\varepsilon = f^\varepsilon(x).$$

Similarly, we denote $H^\varepsilon_\pm = -\varepsilon^2\Delta - n^2_\pm(x)$.

For all $x \in \mathbb{R}^d$, $\xi' \in \mathbb{R}^{d-1}$, we denote $\omega_\pm(x, \xi') = n^2_\pm(x) - |\xi'|^2$.

We denote $u^\varepsilon_\pm = \mathbf{1}_{\{x_d \gtrless 1\}} u^\varepsilon$ the restrictions of u^ε in each medium, defined on $\mathbb{R}^d$. Next, the sequences $(u^\varepsilon_\pm)$ are bounded in $\dot{B}^*(\mathbb{R}^d)$. Thus, we can associate with them two Wigner measures μ_- and μ_+ on $T^*\mathbb{R}^d$ as defined in Theorem 4.3.4.

Since the sequences of traces $(u^\varepsilon_{\rceil\Gamma})$ and $((\varepsilon\partial_{x_d}u^\varepsilon)_{\rceil\Gamma})$ are bounded in $L^2(\Gamma)$, we can also associate with the sequence $(u^\varepsilon_{\rceil\Gamma}, (\varepsilon\partial_{x_d}u^\varepsilon)_{\rceil\Gamma})$ a matrix valued Wigner measure $\begin{pmatrix} \nu & \bar{\nu}^J \\ \nu^J & \dot{\nu} \end{pmatrix}$.

Crucial property

As pointed out by Luc Miller [Mil2], the hermitian positivity of this matrix measure will be crucial in our proof. We have the following property

$$|\nu^J| \leq (\nu)^{1/2}(\dot{\nu})^{1/2}. \tag{4.4.1}$$

4.4.2 Behavior of the Wigner measure in the interior of each medium

In the interior of each medium, the refraction index is smooth. The behavior of the Wigner measure in that case is studied in Benamou *et al* [BCKP]. We recall their result in Theorem 4.4.2. Actually, they proved the analogous of Theorem 4.4.2 with a weaker radiation condition at infinity. The condition we state here can be easily deduced from the one they proved together with the localization property stated in Propositon 4.4.1.

Support of μ

The following localization property is well-known without source term. It is still valid here thanks to the particular scaling of f^ε.

Proposition 4.4.1

$$supp(\mathbf{1}_{T^*(\mathbb{R}^d\setminus\Gamma)}\mu) \subset \{(x,\xi) \in T^*\mathbb{R}^d/\ |\xi|^2 = n(x)^2\}.$$

Proof. Let $\phi \in \mathcal{C}^\infty_c(T^*(\mathbb{R}^d \setminus \Gamma))$ and $\phi^\varepsilon = \phi^w(x, \varepsilon D_x)$. By pseudodifferential calculus, $\phi^\varepsilon H^\varepsilon = Op^w_\varepsilon(\phi(x,\xi)(|\xi|^2 - n(x)^2)) + O(\varepsilon)$ hence, using the definition of the measure μ, we get

$$\begin{aligned} \lim_{\varepsilon\to 0}(\phi^\varepsilon H^\varepsilon u^\varepsilon, u^\varepsilon) &= \lim_{\varepsilon\to 0}(Op^w_\varepsilon(\phi(x,\xi)(|\xi|^2 - n(x)^2))u^\varepsilon, u^\varepsilon) \\ &= \int \phi(x,\xi)(|\xi|^2 - n^2(x))d\mu. \end{aligned}$$

Using the equation (4.1.1), we write

$$(\phi^\varepsilon H^\varepsilon u^\varepsilon, u^\varepsilon) = (\phi^\varepsilon f^\varepsilon, u^\varepsilon) - i\alpha_\varepsilon\varepsilon(\phi^\varepsilon u^\varepsilon, u^\varepsilon) = (W^\varepsilon(f^\varepsilon, u^\varepsilon), \phi) - i\alpha_\varepsilon\varepsilon(\phi^\varepsilon u^\varepsilon, u^\varepsilon).$$

On the first hand, Proposition 4.3.5 gives that $\lim_{\varepsilon\to 0}(W^\varepsilon(f^\varepsilon, u^\varepsilon), \phi) = 0$. On the other hand, $(\phi^\varepsilon u^\varepsilon, u^\varepsilon)$ is bounded, hence $\lim_{\varepsilon\to 0} \alpha_\varepsilon \varepsilon(\phi^\varepsilon u^\varepsilon, u^\varepsilon) = 0$. Therefore, for any $\phi \in \mathcal{C}_c^\infty(T^*(\mathbb{R}^d \setminus \Gamma))$, we have $\int \phi(|\xi|^2 - n^2(x))d\mu = 0$, thus $\text{supp}(\mathbf{1}_{T^*(\mathbb{R}^d\setminus\Gamma)}\mu) \subset \{|\xi|^2 = n(x)^2\}$. $\square$

Transport equation on μ away from the interface

Theorem 4.4.2 *Under the assumptions (H1)-(H10), the measure μ satisfies the following transport equation as a distribution in* $T^*(\mathbb{R}^d \setminus \Gamma)$

$$\xi \cdot \nabla_x \mu + \frac{1}{2}\nabla_x n^2(x) \cdot \nabla_\xi \mu = Q(x, \xi) \quad \text{in } T^*(\mathbb{R}^d \setminus \Gamma), \tag{4.4.2}$$

where $Q(x,\xi) = \frac{1}{2^{d+1}\pi^{d-1}}\delta(x)\delta(|\xi|^2 - n^2(0))\hat{f}(\xi)(\bar{\hat{f}}(\xi) + \bar{q}(\xi))$, *and* $q \in L^2(\xi^2 = n^2(0))$ *is given in Proposition 4.3.2.*
Moreover, μ satisfies the following outgoing condition at infinity : for all functions $R \in \mathcal{C}_c^\infty(T^*(\mathbb{R}^d \setminus \Gamma))$ such that $g(x,\xi) = \int_0^{+\infty} R(x + t\xi, \xi)dt$ is supported in one side of the interface, we have

$$\langle \mu, R\rangle = \int_{\{\xi^2 = n(0)^2\}} |\hat{f}(\xi)|^2 g(0,\xi)d\xi + \int_{\mathbb{R}^{2d}} \frac{1}{2}\nabla_x n^2 \cdot \nabla_\xi g \, d\mu. \tag{4.4.3}$$

Proof. The proof of (4.4.2) is a straighforward adaptation of [BCKP]. In [BCKP], the radiation condition (4.4.3) is stated in a weaker form, using only test functions R such that $\text{supp}(R) \subset \mathbb{R}^{2d}\setminus\{\xi = 0\}$. Actually, as we see from the previous localization property, this is not a restriction. $\square$

From the previous radiation condition (4.4.3), we deduce the following.
Corollary 4.4.3

$$\mu(x,\xi) \to 0 \quad \textit{when } |x| \to \infty \quad \textit{with } x \cdot \xi < 0 \textit{ and } \quad x_d \neq 1.$$

Proof. Let δ be positive. We look for M such that for all test function $R \in \mathcal{C}_c^\infty(T^*(\mathbb{R}^d \setminus \Gamma))$ with support in $\{|x| \geq M,\ x \cdot \xi < 0\}$, we have: $|\langle \mu, R\rangle| \leq \delta$. Let $R \in \mathcal{C}_c^\infty(T^*(\mathbb{R}^d \setminus \Gamma))$ be as in (4.4.3). We may assume that R has support in $\{|\xi| \geq n(0)/2\}$. Then, the radiation condition (4.4.3) gives

$$\langle \mu, R\rangle = \int_{\{\xi^2 = n(0)^2\}} |\hat{f}(\xi)|^2 g(0,\xi)d\xi + \int_{\mathbb{R}^{2d}} \frac{1}{2}\nabla_x n^2 \cdot \nabla_\xi g \, d\mu.$$

As a first step, we prove that the first term vanishes if R is supported in $\{x \cdot \xi < 0\}$. Indeed, $g(0,\xi) = \int_0^\infty R(t\xi, \xi)dt$ and $t\xi \cdot \xi \geq 0$ for all $t \geq 0$. Thus, if $\text{supp}(R) \subset \{x \cdot \xi < 0\}$, then $g(0, \xi) = 0$.

As a second step, let us study the term $\int_{\mathbb{R}^{2d}} \nabla_x n^2 \cdot \nabla_\xi g \, d\mu$.

First of all, let us show that $\nabla_x n^2 \cdot \nabla_\xi g$ is integrable for the measure μ. In order to prove that point, we will use that μ satisfies (4.3.5), so that, for all $N > 1$, $1/\langle x \rangle^N$ is integrable for the measure μ. Let us bound $\nabla_\xi g$. Since there exists M_1 such that R has support in $\{M \le |x| \le M_1\}$, the only non vanishing contribution in the integral defining $g(x, \xi)$ comes from t such that

$$2\frac{M - |x|}{n(0)} \le t \le 2\frac{M_1 + |x|}{n(0)}. \tag{4.4.4}$$

Thus, we can compute

$$\nabla_\xi g(x, \xi) = \int_{2\frac{M-|x|}{n(0)}}^{2\frac{M_1+|x|}{n(0)}} (t\nabla_x + \nabla_\xi) R(x + t\xi) dt.$$

Now, the derivatives of R are uniformly bounded. Hence, we get

$$\begin{aligned} |\nabla_\xi g(x, \xi)| &\le C \int_{2\frac{M-|x|}{n(0)}}^{\frac{M_1+|x|}{n(0)}} t dt \\ &\le C(1 + |x|) \end{aligned}$$

where C denotes any constant independent of x and ξ.
Therefore, since we assumed that there exists $N_0 > 2$ such that $\langle x \rangle^{N_0} \nabla_x n^2 \in L^\infty$, we get

$$\left| \nabla_x n^2 \cdot \nabla_\xi g \right| \le C / \langle x \rangle^{N_0 - 1}.$$

Since $1/\langle x \rangle^{N_0-1}$ is integrable for μ, there exists M_δ such that

$$\left| \int_{|x| \ge M_\delta} \nabla_x n^2 \cdot \nabla_\xi g d\mu \right| \le \delta. \tag{4.4.5}$$

There now remains to estimate the part coresponding to $|x| \le M_\delta$. When $|x| \le M_\delta$, then for M large enough, we have $|x + t\xi| \ge c|t|$ for t satisfying (4.4.4). Hence, we get, for all $l \in \mathbb{N}$,

$$|(t\nabla_x + \nabla_\xi) R(x + t\xi)| \le \frac{C_l}{\langle t \rangle^l}.$$

Thus, there exists $M'_\delta \ge M_\delta$ such that

$$\int_{2\frac{M'_\delta - M_\delta}{n(0)}}^{\infty} |(t\nabla_x + \nabla_\xi) R(x + t\xi) dt| \le \delta.$$

Using that $\nabla_x n^2$ is bounded, we get, for R with support in $\{|x| \geq M'_\delta\}$,

$$\left| \int_{|x| \leq M_\delta} \nabla_x n^2 \cdot \nabla_\xi g d\mu \right| \leq C\delta. \qquad (4.4.6)$$

This last estimate together with (4.4.5) gives

$$|\langle \mu, R \rangle| \leq C\delta,$$

which ends the proof. □

4.4.3 Study up to the boundary

The above result does not say anything about the Wigner measure μ close to the boundary Γ, where refraction occurs. In order to write the transport equations up to the boundary, we first define tangential test operators. These operators, that act as differential operators in the d-th variable, will be adapted to the treatment of the interface (by integration by parts). Using these test operators, we then study the propagation of the Wigner measure up to the boundary. More precisely, since the behaviour at the boundary depends on the side from which the rays come, we study separately the measures associated with the restrictions of (u^ε) to each medium, $\mu_\pm$. In the second paragraph, we first prove a localization property on $\mu_\pm$ similar to that of Proposition 4.4.1. Then, in the third paragraph, we write the transport equation up to the boundary in a weak form.

Tangential test operators

Following L. Miller [Mil2], we introduce the class $\mathcal{T}^n(\mathbb{R}^d)$ of tangential test operators of order n. We denote $\hat{\mathcal{C}}_c^\infty(\mathbb{R}^{2d-1}) := \{\omega : \mathbb{R}_x^d \times \mathbb{R}_{\xi'}^{d-1} \to \mathbb{R} \ / \ \mathcal{F}_{\xi' \to y'}\omega \in \mathcal{C}_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_{y'}^{d-1})\}$.

Definition 4.4.4 *The semiclassical operator $\phi^\varepsilon = \varphi^W(x, \varepsilon D_x)$ is said to be in $\mathcal{T}^n(\mathbb{R}^d)$ if $\varphi(x,\xi) = \sum_{k=1}^n \varphi_k(x,\xi')\xi_d^k$ with $\varphi_k \in \hat{\mathcal{C}}_c^\infty(\mathbb{R}^{2d-1})$.*

In other words, the tangential test operators have symbols that are polynomial in the ξ_d variable. We will denote $\phi_k^\varepsilon = (\varphi_k)^W(x, \varepsilon D_x)$.

Actually, we will only use tangential test operators of order 1. Indeed, as usual, in order to obtain transport equations on Wigner measures, we test u^ε against commutators involving H^ε. Thinking of the euclidian division of a tangential symbol (considered as a polynomial in ξ_d) by the symbols $|\xi|^2 - n_\pm^2$, that are of degree 2 in ξ_d, one can understand that no information is lost using only tangential test operators of order 1.

Moreover, since they are differential operators in the x_d-variable, these tangential operators have "good" properties concerning the sharp truncations on $\{x_d \gtrless 1\}$, translated in Lemma 4.4.5.

Remark: *The transmission problem we consider here can be rewritten as two boundary value problems. The first propagation result concerning Wigner measures for these problems was obtained by P. Gérard and E. Leichtnam [GL] who were concerned with the Helmholtz equation with constant index of refraction and Dirichlet boundary condition on a convex domain. As pointed out by L. Miller, the method we use here avoids one of their delicate tool: a euclidian division of symbols.*

We give here the properties of these operators we shall need in the sequel. The reader can find the proofs of these results (borrowed from [Mil1]) in Appendix B.

Lemma 4.4.5 ***(borrowed from [Mil1])*** *For any $\phi^\varepsilon \in \mathcal{T}^2(\mathbb{R}^d)$, any truncating function $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ which equals 1 on $[-1,1]$, and any $\chi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$,*
(i) $\lim_{\varepsilon\to 0} \left(\chi_0(\varepsilon D_{x_d})\phi^\varepsilon u_\pm^\varepsilon, u_\pm^\varepsilon\right) - \left(\chi_0(\varepsilon D_{x_d})\phi^\varepsilon u^\varepsilon, u^\varepsilon\right)_\pm = 0$
(ii) $\limsup_{\varepsilon\to 0} \left(\chi\left(\frac{\varepsilon}{\rho}D_{x_d}\right)\phi^\varepsilon u_\pm^\varepsilon, u_\pm^\varepsilon\right) - \left(\phi^\varepsilon u^\varepsilon, u^\varepsilon\right)_\pm \to 0$ *as* $\rho \to +\infty$.

This lemma is a consequence of the ε-oscillation of (u^ε) and the L^2-estimates on the traces of u^ε and $\varepsilon\nabla u^\varepsilon$. Its proof does not use the fact that u^ε is a solution to the Helmholtz equation (4.1.1). The second lemma corresponds to an integration by parts with respect to the x_d variable.

Lemma 4.4.6 ***(borrowed from [Mil1])*** *For all $\phi^\varepsilon \in \mathcal{T}^2(\mathbb{R}^d)$, for all $u, v \in \mathcal{C}^\infty(\mathbb{R}_{x_d}, \mathcal{D}'(\mathbb{R}^{d-1}_{x'}))$,*

$$-\frac{i}{\varepsilon}\Big((\phi^\varepsilon v, u)_\pm - (v, (\phi^\varepsilon)^* u)_\pm\Big) = \pm(\phi_1^\varepsilon v, u)_\Gamma \pm (\phi_2^\varepsilon \varepsilon D_{x_d} v, u)_\Gamma \pm (\phi_2^\varepsilon v, \varepsilon D_{x_d} u)_\Gamma.$$

Support of $\mu_\pm$

As for the Wigner measure μ, we have the following localization property for the measures associated with the restrictions $(u_\pm^\varepsilon)$.

Proposition 4.4.7 *(i) $supp(\mu_\pm) \subset \{|\xi|^2 = n_\pm^2(x)\}$.*
(ii) $\mu = \mu_+ + \mu_-$.

Proof. Point (ii) is consequence of point (i) together with the orthogonality property on Wigner measures. Indeed, since $[n^2] \neq 0$ on the boundary, the measures μ_- and μ_+ are mutually singular. Hence, the Wigner measure associated with $u^\varepsilon = u_-^\varepsilon + u_+^\varepsilon$, *i.e.* μ, is the sum of the measures associated with u_-^ε and u_+^ε.

Now, let us prove point (i). Let $\chi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$, $\omega \in \hat{\mathcal{C}}_c^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$ and $\Omega^\varepsilon =$

$\omega^W(x, \varepsilon D_{x'})$. From the definition of the measures $\mu_\pm$ and pseudodifferential calculus, we have

$$(\chi_0(\varepsilon D_{x_d})\Omega^\varepsilon H^\varepsilon_\pm u^\varepsilon_\pm, u^\varepsilon_\pm) \to \int \chi_0 \omega(\xi^2 - n^2_\pm(x)) d\mu_\pm. \tag{4.4.7}$$

On the one hand, by Lemma 4.4.5, the left-hand side of (4.4.7) has the same limit, as $\varepsilon \to 0$, as $(\chi_0(\varepsilon D_{x_d})\Omega^\varepsilon H^\varepsilon u^\varepsilon, u^\varepsilon)_\pm$. Using the Helmholtz equation (4.1.1), we may then write

$$\begin{aligned}(\chi_0(\varepsilon D_{x_d})\Omega^\varepsilon H^\varepsilon u^\varepsilon, u^\varepsilon)_\pm &= (\chi_0(\varepsilon D_{x_d})\Omega^\varepsilon f^\varepsilon, u^\varepsilon)_\pm - i\alpha_\varepsilon \varepsilon (\chi_0(\varepsilon D_{x_d})\Omega^\varepsilon u^\varepsilon, u^\varepsilon)_\pm \\ &= (W^\varepsilon(f^\varepsilon, u^\varepsilon), \chi_0\omega)_\pm - i\alpha_\varepsilon \varepsilon (\chi_0(\varepsilon D_{x_d})\Omega^\varepsilon u^\varepsilon, u^\varepsilon)_\pm\end{aligned} \tag{4.4.8}$$

Let us now study the two terms in the right hand side of (4.4.8). Reasonning as in the proof of proposition 4.3.5, we have

$$\lim_{\varepsilon \to 0} (W^\varepsilon(f^\varepsilon, u^\varepsilon), \chi_0\omega)_\pm = 0.$$

On the other hand, $\lim_{\varepsilon\to 0} \alpha_\varepsilon \varepsilon (\chi_0(\varepsilon D_{x_d})\Omega^\varepsilon u^\varepsilon, u^\varepsilon)_\pm = 0$. Therefore, for any χ_0, ω, we have $\int \chi_0\omega(\xi^2 - n^2_\pm(x))d\mu_\pm = 0$. Hence, $(\xi^2 - n^2_\pm(x))\mu_\pm = 0$ and $\mathrm{supp}(\mu_\pm) \subset \{|\xi|^2 = n^2_\pm(x)\}$. □

Transport equation on $\mu_\pm$ up to the boundary

The following property specifies what happens at the boundary.

Proposition 4.4.8 *For all φ_0, φ_1 in $\hat{\mathcal{C}}^\infty_c(\mathbb{R}^d_x \times \mathbb{R}^{d-1}_{\xi'})$, we have*

$$\begin{aligned}-\Big\langle \mu_\pm, \Big(\xi\cdot\nabla_x + \frac{1}{2}\nabla_x n^2_\pm \cdot \nabla_\xi\Big)(\varphi_0 + \varphi_1\xi_d)\Big\rangle &= \langle Q_\pm, \varphi_0 + \varphi_1\xi_d\rangle \\ \pm \frac{1}{2}\langle \nu, (|\xi'|^2 - n^2_\pm)\varphi_{1\rceil_\Gamma}\rangle_{T^*\Gamma} &\pm \langle \mathcal{R}e\,\nu^J, \varphi_{0\rceil_\Gamma}\rangle_{T^*\Gamma} \pm \frac{1}{2}\langle \dot\nu, \varphi_{1\rceil_\Gamma}\rangle_{T^*\Gamma}\end{aligned} \tag{4.4.9}$$

where $Q_+ = 0$, and $Q_- = \frac{1}{2^{d+1}\pi^{d-1}}\delta(x)\delta(|\xi|^2 - n^2(0))\Big(|\hat f(\xi)|^2 + \hat f(\xi)\bar q(\xi)\Big)$, q being given in Proposition 4.3.2.

Proof. As usual, in order to get the transport equations satisfied by $\mu_\pm$, we test $u^\varepsilon_\pm$ against commutators that involve H^ε. Specifically, we take $\phi^\varepsilon \in \mathcal{T}^1(\mathbb{R}^d)$, $\varphi(x,\xi) = \varphi_0(x,\xi') + \varphi_1(x,\xi')\xi_d$ denoting its symbol. We apply Lemma 4.4.5 with $[H^\varepsilon, \phi^\varepsilon] \in \mathcal{T}^2(\mathbb{R}^d)$. This gives

$$\lim_{\rho\to+\infty} \limsup_{\varepsilon\to 0} \frac{i}{2\varepsilon}\Big(\chi\Big(\frac{\varepsilon}{\rho}D_{x_d}\Big)[H^\varepsilon, \phi^\varepsilon]u^\varepsilon_\pm, u^\varepsilon_\pm\Big)$$
$$= \limsup_{\varepsilon\to 0}\frac{i}{2\varepsilon}\Big([H^\varepsilon, \phi^\varepsilon]u^\varepsilon, u^\varepsilon\Big)_\pm, \tag{4.4.10}$$

whenever χ is in $\mathcal{C}_c^\infty(\mathbb{R})$ and equals 1 on $[-1,1]$. The limit of the left-hand side in (4.4.10) equals

$$\lim_{\rho\to+\infty}\int \chi\Big(\frac{\xi_d}{\rho}\Big)\left(\xi\cdot\nabla_x+\frac{1}{2}\nabla_x n_\pm^2\cdot\nabla_\xi\right)\varphi\, d\mu_\pm = \int\left(\xi\cdot\nabla_x+\frac{1}{2}\nabla_x n_\pm^2\cdot\nabla_\xi\right)\varphi\, d\mu_\pm.$$

The last inequality uses that $\xi\cdot\nabla_x\varphi+\frac{1}{2}\nabla_x n_\pm^2\cdot\nabla_\xi\varphi$ is bounded on the support of $\mu_\pm$, together with the dominated convergence theorem. Now, let us study the right-hand-side of (4.4.10). It reads, expanding the commutator and using that $H^\varepsilon u^\varepsilon = f^\varepsilon - i\alpha_\varepsilon\varepsilon u^\varepsilon$,

$$\begin{aligned}\frac{i}{2\varepsilon}\left([H^\varepsilon,\phi^\varepsilon]u^\varepsilon,u^\varepsilon\right)_\pm &= \frac{i}{2\varepsilon}\left(H^\varepsilon\phi^\varepsilon u^\varepsilon,u^\varepsilon\right)_\pm-\frac{i}{2\varepsilon}\left(\phi^\varepsilon H^\varepsilon u^\varepsilon,u^\varepsilon\right)_\pm\\ &= \frac{i}{2\varepsilon}\left(H^\varepsilon\phi^\varepsilon u^\varepsilon,u^\varepsilon\right)_\pm-\frac{i}{2\varepsilon}\left(\phi^\varepsilon f^\varepsilon,u^\varepsilon\right)_\pm-\frac{\alpha_\varepsilon}{2}\left(\phi^\varepsilon u^\varepsilon,u^\varepsilon\right)_\pm.\end{aligned}$$

In order to make the "adjoint" terms appear, we use that $(H^\varepsilon)^*u^\varepsilon = H^\varepsilon u^\varepsilon = f^\varepsilon - i\alpha_\varepsilon\varepsilon u^\varepsilon$ to write the following equality

$$-\frac{\alpha_\varepsilon}{2}\left(\phi^\varepsilon u^\varepsilon,u^\varepsilon\right)_\pm = -\frac{i}{2\varepsilon}\left(\phi^\varepsilon u^\varepsilon,(H^\varepsilon)^*u^\varepsilon\right)_\pm+\frac{i}{2\varepsilon}\left(\phi^\varepsilon u^\varepsilon,f^\varepsilon\right)_\pm.$$

We thus get

$$\begin{aligned}\frac{i}{2\varepsilon}\left([H^\varepsilon,\phi^\varepsilon]u^\varepsilon,u^\varepsilon\right)_\pm &= \frac{i}{2\varepsilon}\left(H^\varepsilon\phi^\varepsilon u^\varepsilon,u^\varepsilon\right)_\pm-\frac{i}{2\varepsilon}\left(\phi^\varepsilon u^\varepsilon,(H^\varepsilon)^*u^\varepsilon\right)_\pm\\ &\quad+\frac{i}{2\varepsilon}\left(\phi^\varepsilon u^\varepsilon,f^\varepsilon\right)_\pm-\frac{i}{2\varepsilon}\left(\phi^\varepsilon f^\varepsilon,u^\varepsilon\right)_\pm.\end{aligned}\tag{4.4.11}$$

Next, we study separately the terms involving the source f^ε and the terms involving H^ε in the right-hand side of (4.4.11).

First step: study of $\frac{i}{2\varepsilon}\left(\phi^\varepsilon u^\varepsilon,f^\varepsilon\right)_\pm-\frac{i}{2\varepsilon}\left(\phi^\varepsilon f^\varepsilon,u^\varepsilon\right)_\pm$

We have

$$\begin{aligned}\frac{i}{2\varepsilon}\left(\phi^\varepsilon u^\varepsilon,f^\varepsilon\right)_\pm-\frac{i}{2\varepsilon}\left(\phi^\varepsilon f^\varepsilon,u^\varepsilon\right)_\pm &= \frac{i}{2\varepsilon}\Big[\big(W^\varepsilon(f^\varepsilon,u^\varepsilon),\varphi\big)_\pm-\big(W^\varepsilon(u^\varepsilon,f^\varepsilon),\varphi\big)_\pm\Big]\\ &= \frac{i}{2\varepsilon}\left(\mathcal{I}m\, W^\varepsilon(f^\varepsilon,u^\varepsilon),\varphi\right)_\pm.\end{aligned}$$

The limit of this last term is given by the following lemma.

Lemma 4.4.9

$$\lim_{\varepsilon\to0}\frac{i}{2\varepsilon}\left(\mathcal{I}m\, W^\varepsilon(f^\varepsilon,u^\varepsilon),\varphi\right)_\pm = \langle Q_\pm,\varphi\rangle,$$

where $Q_\pm$ are defined above.

We postpone the proof of this lemma. From this lemma, we obtain the source term coming from f in (4.4.9), *i.e.* $\langle Q_\pm, \varphi\rangle$.

<u>**Second step:**</u> **study of** $\frac{i}{2\varepsilon}\left(H^\varepsilon \phi^\varepsilon u^\varepsilon, u^\varepsilon\right)_\pm - \frac{i}{2\varepsilon}\left(\phi^\varepsilon u^\varepsilon, (H^\varepsilon)^* u^\varepsilon\right)_\pm$

This is done in the lemma below.

Lemma 4.4.10 *For all $\phi^\varepsilon \in \mathcal{T}^1(\mathbb{R}^d)$, we have*

$$
\begin{aligned}
&-\frac{i}{\varepsilon}\Big[\big(H^\varepsilon \phi^\varepsilon u^\varepsilon, u^\varepsilon\big)_\pm - \big(\phi^\varepsilon u^\varepsilon, (H^\varepsilon)^* u^\varepsilon\big)_\pm\Big] \\
&= \pm\big(\phi_0^\varepsilon (\varepsilon D_{x_d} u^\varepsilon)_{\rceil\Gamma}, u^\varepsilon_{\rceil\Gamma}\big)_\Gamma \pm \big(\phi_0^\varepsilon u^\varepsilon_{\rceil\Gamma}, (\varepsilon D_{x_d} u^\varepsilon)_{\rceil\Gamma}\big)_\Gamma \\
&\quad \pm\big(\phi_1^\varepsilon (\varepsilon D_{x_d} u^\varepsilon)_{\rceil\Gamma}, (\varepsilon D_{x_d} u^\varepsilon)_{\rceil\Gamma}\big)_\Gamma \pm \big((n_\pm - \xi'^2)^W \phi_1^\varepsilon u^\varepsilon_{\rceil\Gamma}, u^\varepsilon_{\rceil\Gamma}\big)_\Gamma + O(\varepsilon).
\end{aligned}
$$

Let us again postpone the proof of this lemma and first end the proof of Proposition 4.4.8.

From Lemma 4.4.10 and the definition of $\nu, \dot\nu, \nu^J$, we directly deduce that

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} -\frac{i}{\varepsilon}\Big[\big(H^\varepsilon \phi^\varepsilon u^\varepsilon, u^\varepsilon\big)_\pm - \big(\phi^\varepsilon u^\varepsilon, (H^\varepsilon)^* u^\varepsilon\big)_\pm\Big] \\
&\qquad = \pm\int 2\varphi_{0\rceil\Gamma}\mathcal{R}e(d\nu^J) \pm \int (n_\pm^2 - \xi'^2)\varphi_{1\rceil\Gamma} d\nu \pm \int \varphi_{1\rceil\Gamma} d\dot\nu.
\end{aligned}
$$

Putting together the results of the first and second step, we obtain the limit of the right-hand side of (4.4.10):

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} -\frac{i}{\varepsilon}\left([H^\varepsilon, \phi^\varepsilon] u^\varepsilon, u^\varepsilon\right)_\pm \\
&\qquad = \langle Q_\pm, \varphi\rangle \pm \int 2\varphi_{0\rceil\Gamma}\mathcal{R}e(d\nu^J) \pm \int (n_\pm^2 - \xi'^2)\varphi_{1\rceil\Gamma} d\nu \pm \int \varphi_{1\rceil\Gamma} d\dot\nu.
\end{aligned}
$$

Finally, using the equality (4.4.10), the proposition is proved.

There remains to prove the two lemmas 4.4.9 and 4.4.10.

Proof of Lemma 4.4.9.

As in the proof of Proposition 4.3.5, we can write

$$
\begin{aligned}
\frac{1}{\varepsilon}\left(W^\varepsilon(f^\varepsilon, u^\varepsilon), \varphi\right)_\pm &= \int f(x)\overline{w^\varepsilon}(x+y')\varphi_0\left(\varepsilon\left(x' + \frac{y'}{2}\right), \varepsilon x_d, y'\right) dx dy' \\
&\quad - \int \partial_d f(x)\overline{w^\varepsilon}(x+y')\varphi_1\left(\varepsilon\left(x' + \frac{y'}{2}\right), \varepsilon x_d, y'\right) dx dy' \\
&\quad - \int f(x)\overline{\partial_d w^\varepsilon}(x+y')\varphi_1\left(\varepsilon\left(x' + \frac{y'}{2}\right), \varepsilon x_d, y'\right) dx dy'.
\end{aligned}
$$

Since w^ε, $\partial_d w^\varepsilon$ are bounded in $\dot{B}^*$, and $f \in L^2_N$, $\partial_d f \in L^2_{N_1}$ for some $N,\ N_1 > 1/2$, we get that $\frac{1}{\varepsilon}\left(W^\varepsilon(f^\varepsilon, u^\varepsilon), \varphi\right)_\pm$ is uniformly bounded with respect to ε. Then, we argue as in the end of the proof of Proposition 4.3.5. We need the strong convergence of w^ε and $\partial_d w^\varepsilon$ in $L^2_{loc}(\mathbb{R}^d)$ to w and $\partial_d w$ respectively. These two convergences are consequences of the uniform estimate for w^ε in $H^2_{loc}(\mathbb{R}^d)$ together with the Rellich's theorem. We deduce

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(W^\varepsilon(f^\varepsilon, u^\varepsilon), \varphi\right)_\pm &= \int_{x_d \gtrless 1} f(x)\overline{w}(x+y')\varphi_0\left(0,y'\right)dxdy' \\ &\quad - \int_{x_d\gtrless 1} \partial_d f(x)\overline{w}(x+y')\varphi_1\left(0,y'\right)dxdy' \\ &\quad - \int_{x_d\gtrless 1} f(x)\overline{\partial_d w}(x+y')\varphi_1\left(0,y'\right)dxdy'.\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(W^\varepsilon(f^\varepsilon, u^\varepsilon), \varphi\right)_- = \int \hat{f}(\xi)\overline{\hat{w}}(\xi)\varphi(0,\xi)d\xi,$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(W^\varepsilon(f^\varepsilon, u^\varepsilon), \varphi\right)_+ = 0.$$

We conclude using $\mathcal{I}m\,\hat{w}(\xi) = \frac{\pi}{2}\left(\hat{f}(\xi) + q(\xi)\right)\delta(\xi^2 - n^2(0))$, where q is an L^2 density on the sphere $\xi^2 = n^2(0)$. □

Proof of Lemma 4.4.10.

We use Lemma 4.4.6 with $v = \phi^\varepsilon u^\varepsilon$ and $u = u^\varepsilon$ to get

$$\begin{aligned}&-\frac{i}{\varepsilon}\Big[\big(H^\varepsilon\phi^\varepsilon u^\varepsilon, u^\varepsilon\big)_\pm - \big(\phi^\varepsilon u^\varepsilon, (H^\varepsilon)^* u^\varepsilon\big)_\pm\Big] \\ &= \pm\big(\phi^\varepsilon u^\varepsilon, \varepsilon D_{x_d} u^\varepsilon\big)_\Gamma \pm \left(\varepsilon D_{x_d}\phi^\varepsilon u^\varepsilon, u^\varepsilon\right)_\Gamma \\ &= \pm\big(\phi^\varepsilon u^\varepsilon, \varepsilon D_{x_d} u^\varepsilon\big)_\Gamma \pm \left(\phi^\varepsilon \varepsilon D_{x_d} u^\varepsilon, u^\varepsilon\right)_\Gamma + \big(\varepsilon (D_{x_d}\varphi)^w u^\varepsilon, u^\varepsilon\big)_\Gamma,\end{aligned}$$

where we use that $\varepsilon D_d \phi^\varepsilon = \phi^\varepsilon(\varepsilon D_{x_d}) + \varepsilon(D_{x_d}\varphi)^w$ (see relation (C.1) in Appendix C for $k=0$ and $k=1$).

This calculation readily gives the result for tangential test operators of order 0. Indeed, since the trace of u^ε is bounded in $L^2(\Gamma)$, we have

$$\big(\varepsilon(D_{x_d}\varphi_0)^w u^\varepsilon, u^\varepsilon\big)_\Gamma = O(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Thus, the result is proved for $\phi^\varepsilon = \phi_0^\varepsilon$. Let us now study the case when $\phi^\varepsilon = (\varphi_1\xi_d)^w(x, \varepsilon D_x)$. We are left with the following two terms:

$\big((\varphi_1\xi_d)^w u^\varepsilon, \varepsilon D_{x_d}u^\varepsilon\big)_\Gamma$ and $((\varphi_1\xi_d)^w(\varepsilon D_{x_d})u^\varepsilon, u^\varepsilon)_\Gamma$. Using again the relation (C.1), we get

$$\begin{aligned}\big((\varphi_1\xi_d)^w u^\varepsilon, \varepsilon D_{x_d}u^\varepsilon\big)_\Gamma &= \big(\phi_1^\varepsilon(\varepsilon D_{x_d}u^\varepsilon), \varepsilon D_{x_d}u^\varepsilon\big)_\Gamma + \frac{\varepsilon}{2}\big((D_{x_d}\varphi_1)^w u^\varepsilon, \varepsilon D_{x_d}u^\varepsilon\big)_\Gamma\\ \big((\varphi_1\xi_d)^w(\varepsilon D_{x_d})u^\varepsilon, u^\varepsilon\big)_\Gamma &= \big(\phi_1^\varepsilon(\varepsilon D_{x_d})^2u^\varepsilon, u^\varepsilon\big)_\Gamma + \frac{\varepsilon}{2}\big((D_{x_d}\varphi_1)^w(\varepsilon D_{x_d}u^\varepsilon), u^\varepsilon\big)_\Gamma.\end{aligned}$$

As before, $\varepsilon\big((D_{x_d}\varphi_1)^w u^\varepsilon, \varepsilon D_{x_d}u^\varepsilon\big)_\Gamma = O(\varepsilon)$ and $\varepsilon\big((D_{x_d}\varphi_1)^w(\varepsilon D_{x_d}u^\varepsilon), u^\varepsilon\big)_\Gamma = O(\varepsilon)$. Hence,

$$\begin{aligned}\big((\varphi_1\xi_d)^w u^\varepsilon, \varepsilon D_{x_d}u^\varepsilon\big)_\Gamma &= \big(\phi_1^\varepsilon(\varepsilon D_{x_d}u^\varepsilon), \varepsilon D_{x_d}u^\varepsilon\big)_\Gamma + O(\varepsilon)\\ \big((\varphi_1\xi_d)^w(\varepsilon D_{x_d})u^\varepsilon, u^\varepsilon\big)_\Gamma &= \big(\phi_1^\varepsilon(\varepsilon D_{x_d})^2u^\varepsilon, u^\varepsilon\big)_\Gamma + O(\varepsilon).\end{aligned}$$

Moreover, $(\varepsilon D_{x_d})^2 = H^\varepsilon + (n_\pm^2 - \xi'^2)^w$, from which we deduce

$$\begin{aligned}(\varphi_1^w(\varepsilon D_{x_d})^2u^\varepsilon, u^\varepsilon)_\Gamma &= (\varphi_1^w H^\varepsilon u^\varepsilon, u^\varepsilon)_\Gamma + (\varphi_1^w(n_\pm^2 - \xi'^2)^w u^\varepsilon, u^\varepsilon)_\Gamma\\ &= (\varphi_1^w f^\varepsilon, u^\varepsilon)_\Gamma + ((n_\pm^2 - \xi'^2)^w\varphi_1^w u^\varepsilon, u^\varepsilon)_\Gamma + O(\varepsilon).\end{aligned}$$

Now, since

$$\begin{aligned}(\varphi_1^w(x,\varepsilon D_x)f^\varepsilon, u^\varepsilon)_\Gamma &= ((\varphi_{1\rceil\Gamma})^w(x,\varepsilon D_{x'})f^\varepsilon_{\rceil\Gamma}, u^\varepsilon_{\rceil\Gamma})_\Gamma\\ &= (W^\varepsilon(f^\varepsilon_{\rceil\Gamma}, u^\varepsilon_{\rceil\Gamma}), \varphi_{1_{\rceil\Gamma}}),\end{aligned}$$

we obtain

$$|(\varphi_1^w(x,\varepsilon D_x)f^\varepsilon, u^\varepsilon)_\Gamma| \le C\|f^\varepsilon_{\rceil\Gamma}\|_{L^2(\Gamma)}\|u^\varepsilon_{\rceil\Gamma}\|_{L^2(\Gamma)}.$$

Hence, the assumption (H11) and the boundedness of $(u^\varepsilon_{\rceil\Gamma})$ in $L^2(\Gamma)$ imply

$$(\varphi_1^w(x,\varepsilon D_x)f^\varepsilon, u^\varepsilon)_\Gamma \to 0 \quad \text{as } \varepsilon \to 0.$$

This ends the proof of the lemma. □

4.5 Refraction result in the case of two homogeneous media

In this section, we assume that n_+ and n_- are two constants with $n_- > n_+ > 0$. We choose to first detail our method in the easier case of two homogeneous media. Indeed, the strategy of proof is exactly the same as in the general case but the geometry of the rays is easy to treat in that particular case (the rays are pieces of lines). Moreover, in this special case, we get a completely explicit formula for the Wigner measure associated with

(u^ε), in particular because we can identify the various radiation conditions at infinity that are necessary to entirely determine the Wigner measure μ.

Now we state our main result in the case of two homogeneous media.

Theorem 4.5.1 *Assume (H1) and (H9)-(H12). Let u^ε be the solution to the Helmholtz equation (4.1.1). Assume that the refraction indices n_+ and n_- are constant, with $n_- > n_+$. Then, the Wigner measure associated with (u^ε) is given by*

$$\begin{aligned}
&\mu(x,\xi) = \\
&\quad \mathbf{1}_{\{x_d<1,\xi_d\geq 0\}} \int_{-\infty}^{0} Q(x+t\xi,\xi)dt \\
&+ \mathbf{1}_{\{x_d\leq 1, -\sqrt{[n^2]}\leq \xi_d<0\}} \left(\int_{\frac{1-x_d}{\xi_d}}^{0} Q(x+t\xi,\xi)dt + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} Q(\check{x}+t\check{\xi},\check{\xi})dt \right) \\
&+ \mathbf{1}_{\{x_d\leq 1, \xi_d< -\sqrt{[n^2]}\}} \left(\int_{\frac{1-x_d}{\xi_d}}^{0} Q(x+t\xi,\xi)dt + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} \alpha^R(\xi')Q(\check{x}+t\check{\xi},\check{\xi})dt \right) \\
&+ \mathbf{1}_{\{x_d\geq 1,\xi_d>0\}} \left(\int_{\frac{1-x_d}{\xi_d}}^{0} Q(x+t\xi,\xi)dt + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} \alpha^T(\xi')Q(\tilde{x}+t\tilde{\xi},\tilde{\xi})dt \right),
\end{aligned}$$

where

$$\begin{aligned}
&Q(x,\xi) = \frac{1}{2^{d+1}\pi^{d-1}}\delta(x)\delta(\xi^2-n_-^2)|\hat{f}(\xi)|^2, \\
&\check{\xi} = (\xi', -\xi_d), \;\; \check{x} = (x', 2-x_d), \\
&\tilde{\xi} = \left(\xi', sgn(\xi_d)\sqrt{\xi_d^2+[n^2]}\right), \;\; \tilde{x} = \left(x', 1+(x_d-1)\frac{\tilde{\xi}_d}{|\xi_d|}\right),
\end{aligned}$$

and the coefficients of partial reflection and partial transmission are

$$\alpha^R(\xi') = \left|\frac{2\sqrt{\omega_-(\xi')}}{\sqrt{\omega_+(\xi')}+\sqrt{\omega_-(\xi')}}\right|^2, \quad \alpha^T(\xi') = \left|\frac{\sqrt{\omega_+(\xi')}-\sqrt{\omega_-(\xi')}}{\sqrt{\omega_+(\xi')}+\sqrt{\omega_-(\xi')}}\right|^2.$$

Before going further, let us comment Theorem 4.5.1 with the help of Figure 1 (where the regions V_j, $j = 1,\dots 4$ are defined in Section 5.2). In order to compute the value of μ at the point (x,ξ), we first use the transport equation (4.4.2) to obtain the relation between $\mu(x,\xi)$ and the value of μ along the bicaracteristics $(x+t\xi,\xi)$ until the time when this curve reaches the interface:

$$\mu(x,\xi) = \mu(x+t\xi,\xi) + \int_t^0 Q(x+s\xi,\xi)ds.$$

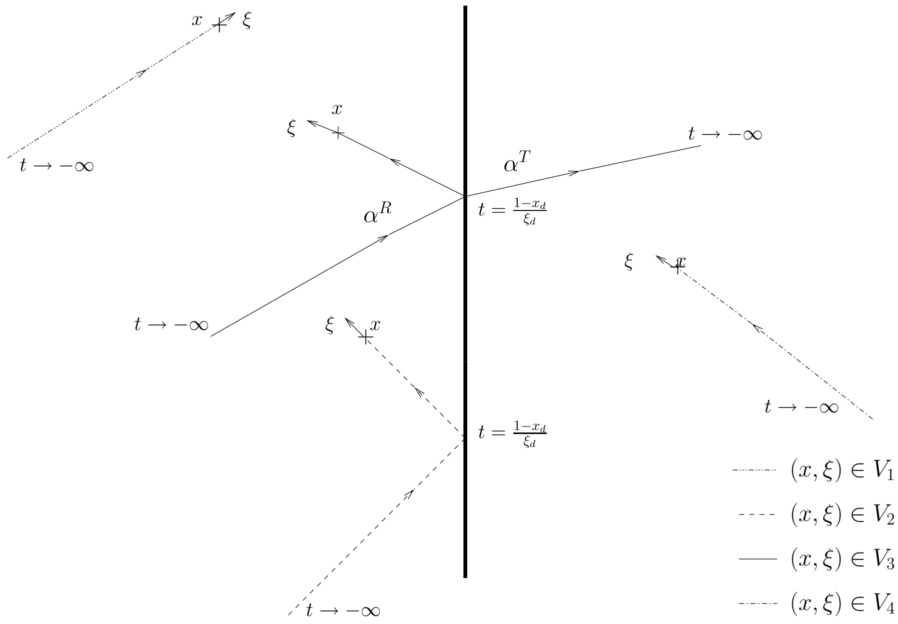

Figure 4.1: *Rays of geometrical optics in the homogeneous case.*

The first part of μ in Theorem 4.5.1, i.e. when $(x,\xi) \in V_1$ in Figure 1, corresponds to points (x,ξ) on the left side of the interface such that the bicaracteristics passing through (x,ξ) at $t = 0$ does not reach the interface for $t \in (-\infty, 0)$. The value of μ at such points is obtained using the radiation condition at infinity stated in Lemma 4.4.3. The second part of μ, i.e. when $(x,\xi) \in V_2$ in Figure 1, corresponds to points (x,ξ) on the left side of the interface such that the bicaracteristics passing through (x,ξ) at $t = 0$ reaches the interface at a point where the ray is totally reflected (at time $(1 - x_d)/\xi_d$). Finally, the third and fourth parts of μ correspond to the two parts of the ray drawn for $(x,\xi) \in V_3$ in Figure 1. For such points, the energy is partially reflected and partially transmitted at the interface.

Theorem 4.5.1 is proved in the subsequent sections 4.5.1 and 4.5.2. We first define the boundary measures (related to the traces of the measures $\mu_\pm$ at the boundary). Then, we obtain the propagation relations at the boundary (total reflexion and refraction) using the transport equations up to the boundary obtained in the previous section. Finally, we get the Wigner measure μ by solving the transport equation satisfied by μ and using both the radiation condition at infinity and the propagation relations at the boundary.

4.5.1 Boundary measures

In this section, we introduce the boundary measures related to the trace of μ on the interface and we give relations between these measures and the semiclassical measures ν, $\dot\nu$ and ν^J associated with the traces of u^ε and its derivative $\varepsilon\partial_d u^\varepsilon$ on the interface. This task is performed using the transport equations on $\mu_\pm$ up to the boundary (4.4.9).

Existence and notations

Outside the interface, μ is a solution to the transport equation

$$\xi \cdot \nabla_x \mu = Q(x,\xi) = \frac{1}{2^{d+1}\pi^{d-1}}\delta(x)\delta(|\xi|^2 - n_-^2)|\hat f(\xi)|^2,$$

which we can rewrite, when $\xi_d \neq 0$,

$$\partial_{x_d}\mu + \frac{1}{\xi_d}\xi' \cdot \nabla_{x'}\mu = \frac{1}{\xi_d}Q(x,\xi).$$

In the last equation, the coefficients are smooth in $\mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus \{\xi_d = 0\})$. Moreover, since $f \in \dot B$, its Fourier transform $\hat f$ belongs to $L^2(|\xi|^2 = n_-^2)$, hence $Q \in \mathcal{C}\left(\mathbb{R}_{x_d}, \mathcal{D}'(\mathbb{R}^{d-1}_{x'} \times \mathbb{R}^d_\xi)\right)$ and $\frac{1}{\xi_d}Q(x,\xi) \in \mathcal{C}(\mathbb{R}_{x_d}, \mathcal{D}'\big(\mathbb{R}^{d-1}_{x'} \times (\mathbb{R}^d_\xi \setminus \{\xi_d = 0\})\big)$. Therefore, using Theorem 4.4.8' in Hörmander [Hör1] we deduce

$$\mu \in \mathcal{C}\left(\mathbb{R}_{x_d}, \mathcal{D}'\big(\mathbb{R}^{d-1}_{x'} \times (\mathbb{R}^d_\xi \setminus \{\xi_d = 0\})\big)\right).$$

For this reason, we can define, in $\{\xi_d \neq 0\}$, the traces

$$\mu^0_\pm = \mu_{\rceil x_d = 1^\pm}. \tag{4.5.1}$$

These measures inherit the positivity of μ and they satisfy the jump formula:

$$\partial_{x_d}(\mathbf{1}_{x_d \gtrless 1}\mu) = \mathbf{1}_{x_d \gtrless 1}\partial_{x_d}\mu \pm \delta(x_d - 1) \otimes \mu^0_\pm.$$

Since we have the localization property $\mathrm{supp}(\mu_\pm) \subset \{|\xi|^2 = n_\pm^2\}$, there exist four nonnegative measures $\mu_\pm^{out}, \mu_\pm^{in}$ (see Figure 2) such that

$$\mu^0_+ = \delta(\xi_d + \sqrt{\omega_+}) \otimes \mu_+^{in} + \delta(\xi_d - \sqrt{\omega_+}) \otimes \mu_+^{out} \tag{4.5.2}$$
$$\mu^0_- = \delta(\xi_d - \sqrt{\omega_-}) \otimes \mu_-^{in} + \delta(\xi_d + \sqrt{\omega_-}) \otimes \mu_-^{out} \tag{4.5.3}$$

where $\omega_\pm$ has been defined in Section 4.1, $\omega_\pm = n_\pm^2 - \xi'^2$.

Our goal is now to find relations between μ_-^{in}, μ_+^{in}, μ_-^{out} and μ_+^{out} that translate the transmission/reflection phenomena at the interface.

First, let us introduce the following last measures.

Lemma 4.5.2 *There exist two nonnegative measures $\mu^{\partial\pm}$ on $T^*\Gamma$ with support in the set $\{|\xi'|^2 = n_\pm^2\}$ such that*

$$\mu_\pm = \mathbf{1}_{\{x_d \gtrless 1\}}\mu_\pm + \delta(x_d - 1) \otimes \delta(\xi_d) \otimes \mu^{\partial\pm}.$$

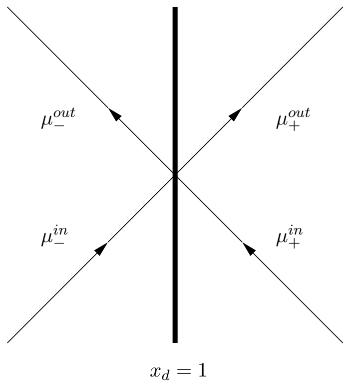

Figure 4.2: *Boundary measures*

Remark: *This means that the density at the interface ($x_d = 1$) can be only carried by the gliding rays $\xi_d = 0$. In the particular case we are studying, these rays don't "come from" one medium since ξ is constant along a ray. Hence, we will have to study separately the density inside the interface.*

Proof. Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathcal{S}(\mathbb{R}^{d-1}))$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\chi(0) = \chi'(0) = 1$, $\chi(\lambda) = 0$ if $\lambda \leq -1$ and $\chi(\lambda) = 2$ if $\lambda \geq 1$. Let $\eta > 0$. We use the transport equation given by the proposition 4.4.8 with $\varphi_0 = 0$ and $\varphi_1(x, \xi') = \eta\, \chi\big(\frac{x_d-1}{\eta}\big)\theta(x, \xi')$. It gives

$$- \Big\langle \mu_\pm, \xi \cdot \nabla_x \Big(\eta\, \chi\Big(\frac{x_d - 1}{\eta}\Big)\theta\xi_d\Big)\Big\rangle = \Big\langle Q_\pm, \eta\, \chi\Big(\frac{x_d - 1}{\eta}\Big)\theta\xi_d\Big\rangle + \eta\, \langle \dot\nu + (n_\pm^2 - \xi'^2)\nu, \theta_0\rangle_{T^*\Gamma},$$

where θ_0 is the trace of θ at the interface Γ.
Since $Q_+ = 0$, $Q_- = \frac{1}{2^{d+1}\pi^{d-1}}\delta(x)\delta(|\xi|^2 - n_-^2)|\hat f(\xi)|^2$ and $\chi\big(\frac{x_d-1}{\eta}\big)$ has support near $x_d = 1$, we deduce that $\langle Q_\pm, \eta\, \chi\big(\frac{x_d-1}{\eta}\big)\theta\xi_d\rangle = 0$ for η small enough. Moreover, $\xi \cdot \nabla_x\Big(\eta\, \chi\Big(\frac{x_d-1}{\eta}\Big)\theta\xi_d\Big) = \chi'\Big(\frac{x_d-1}{\eta}\Big)\xi_d^2\,\theta + \xi' \cdot \nabla_{x'}\theta\;\eta\, \chi\Big(\frac{x_d-1}{\eta}\Big)\xi_d$ converges pointwise to $\mathbf{1}_{\{x_d=1\}}\xi_d^2\,\theta$ as $\eta \to 0$ and it is uniformly bounded with respect to η on the support of $\mu_\pm$. Thus, using the dominated convergence theorem, we deduce $\langle \mathbf{1}_{\{x_d=1\}}\xi_d^2\mu_\pm, \theta\rangle = 0$. Since the test function θ is arbitrary, we get $\mathrm{supp}(\mathbf{1}_{\{x_d=1\}}\mu_\pm) \subset \{x_d = 1, \xi_d = 0\} \cap \{|\xi|^2 = n_\pm^2\}$. □

Next, we obtain the relations that we are looking for, depending on the regions of $T^*\Gamma$.

Lemma 4.5.3 *For $\xi_d \neq 0$, in the set $\{\omega_\pm > 0\}$, we have*
(i) $\pm\mathcal{R}e\, \nu^J = \sqrt{\omega_\pm}(\mu_\pm^{out} - \mu_\pm^{in})$,

(ii) $\frac{1}{2}\dot{\nu} = \omega_\pm(\mu_\pm^{out} + \mu_\pm^{in} - \frac{1}{2}\nu)$.

Proof. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be a nonnegative function such that $\chi(\lambda) = 0$ if $\lambda \leq 1$ and $\chi(\lambda) = 1$ if $\lambda \geq 2$. We denote $\chi_\eta(\xi') = \chi\big(\frac{n_\pm^2 - \xi'^2}{\eta}\big)$. Let $\chi_0 \in \mathcal{C}_c^\infty$ be a nonnegative function such that $\chi_0(1) = 1$ and $\text{supp}(\chi_0) \subset [1/2, 3/2]$. Let $\theta \in \hat{\mathcal{C}}_c^\infty(\mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{\xi'}^{d-1})$.
We successively use the transport equation given by Proposition 4.4.8 with the choice $\varphi_0 = \chi_\eta(\xi')\theta(x',\xi')\chi_0(x_d)$, $\varphi_1 = 0$, next with $\varphi_0 = 0$, $\varphi_1 = \chi_\eta(\xi')\theta(x',\xi')\chi_0(x_d)$. This gives the two relations

$$\begin{aligned} -\langle \mu_\pm, \xi \cdot \nabla_x \varphi_0 \rangle &= \langle Q_\pm, \varphi_0 \rangle \pm \langle \mathcal{R}e\, \nu^J, \chi_\eta \theta \rangle_{T^*\Gamma}, \\ -\langle \mu_\pm, \xi \cdot \nabla_x \varphi_1 \xi_d \rangle &= \langle Q_\pm, \varphi_1 \xi_d \rangle \pm \frac{1}{2} \langle \dot{\nu} + \omega_\pm \nu, \chi_\eta \theta \rangle_{T^*\Gamma}. \end{aligned}$$

Since $Q_+ = 0$, and $Q_- = \frac{1}{(2\pi)^d}\delta(x)\delta(|\xi|^2 - n_-^2)|\hat{f}(\xi)|^2$, we have

$$\langle Q_\pm, \varphi_0 \rangle = \langle Q_\pm, \varphi_1 \xi_d \rangle = 0.$$

Moreover, using Lemma 4.5.2, we get

$$\begin{aligned} -\langle \mu_\pm, \xi \cdot \nabla_x \varphi_0 \rangle &= -\langle \mathbf{1}_{\{x_d \gtrless 1\}} \mu, \xi \cdot \nabla_x \varphi_0 \rangle - \langle \delta(x_d - 1) \otimes \delta(\xi_d) \otimes \mu^{\partial_\pm}, \xi \cdot \nabla_x \varphi_0 \rangle \\ &= \langle \xi \cdot \nabla_x (\mathbf{1}_{\{x_d \gtrless 1\}} \mu), \varphi_0 \rangle - \langle \mu^{\partial_\pm}, \chi_\eta\ \xi' \cdot \nabla_{x'} \theta \rangle. \end{aligned}$$

Since the support of $\mu^{\partial_\pm}$ lies in $\{\xi'^2 = n_\pm^2\}$, and since χ_η vanishes on this set, we deduce

$$\lim_{\eta \to 0} \langle \mu^{\partial_\pm}, \chi_\eta\ \xi' \cdot \nabla_{x'} \theta \rangle = 0.$$

In the same way, we may prove $\langle \delta(x_d - 1) \otimes \delta(\xi_d) \otimes \mu^{\partial_\pm}, \xi \cdot \nabla_x(\varphi_1 \xi_d) \rangle \to 0$ as $\eta \to 0$.
Finally, since for all $(x', \xi') \in T^*\Gamma$, $\chi_\eta(\xi')\theta(x',\xi') \to \mathbf{1}_{\{n_\pm^2 - \xi'^2 = 0\}}\theta(x',\xi')$ as $\eta \to 0$, we obtain

$$\begin{aligned} \langle (\mu_\pm^{out} - \mu_\pm^{in})\sqrt{\omega_\pm}, \theta \rangle_{T^*\Gamma} &= \pm \langle \mathcal{R}e\, \nu^J, \theta \rangle_{T^*\Gamma} \\ \langle (\mu_\pm^{out} + \mu_\pm^{in})\omega_\pm, \theta \rangle_{T^*\Gamma}, &= \frac{1}{2} \langle \dot{\nu} + \omega_\pm \nu, \theta \rangle_{T^*\Gamma}. \end{aligned}$$

Last, θ being arbitrary, the lemma is proved. □

Lemma 4.5.4 *(i)* $\dot{\nu} = 0$ *on* $\{\omega_+ = 0\}$.
(ii) $\mathcal{R}e(\nu^J) = 0$ *on* $\{\omega_+ \leq 0\}$.

Proof. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be a nonnegative function such that $\chi(\lambda) = 0$ if $\lambda \leq 1$ and $\chi(\lambda) = 1$ if $\lambda \geq 2$. Let $\chi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ be a nonnegative function such that $\chi_0(1) = 1$ and $\text{supp}(\chi_0) \subset [1/2, 3/2]$. Let $\theta \in \hat{\mathcal{C}}_c^\infty(\mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{\xi'}^{d-1})$.

We use the transport equation 4.4.9 with $\varphi_0 = \chi\big(\frac{|\xi'|^2-n_+^2}{\eta}\big)\theta(x',\xi')\chi_0(x_d)$, $\varphi_1 = 0$. Since supp $\mu_+ \subset \{|\xi|^2 = n_+^2\}$, we have

$$\langle \mu_\pm, \xi\cdot\nabla_x\varphi_0\rangle = 0,$$

hence

$$\Big\langle \mathcal{R}e(\nu^J), \chi\Big(\frac{|\xi'|^2-n_+^2}{\eta}\Big)\theta(x',\xi')\Big\rangle_{T^*\Gamma} = 0.$$

Then taking the limit $\eta\to 0$, we get

$$\langle \mathcal{R}e(\nu^J), \mathbf{1}_{\{\omega_+<0\}}\theta(x',\xi')\rangle_{T^*\Gamma} = 0.$$

Thus, $\mathcal{R}e(\nu^J) = 0$ on $\{\omega_+ < 0\}$.

To obtain the result on $\{\omega_+ = 0\}$, we again use the transport equation 4.4.9. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be nonnegative, $\chi(0) = 1$, $\chi'(0) = 0$, and $\chi_\eta(\xi') = \chi\big(\frac{|\xi'|^2-n_+^2}{\eta}\big)$; θ and χ_0 are as before. We write the transport equation (4.4.9) with $\varphi_0 = 0$ and $\varphi_1(x,\xi') = \theta(x',\xi')\chi_0(x_d)\chi_\eta(\xi')$:

$$\langle \mu_+, \xi\cdot\nabla_x(\theta\,\chi_0\chi_\eta\xi_d)\rangle = \langle Q, \theta\,\chi_0\chi_\eta\xi_d\rangle + \langle \dot\nu + \omega_+\nu, \theta\,\chi_\eta\xi_d\rangle_{T^*\Gamma}.$$

This gives

$$\langle \dot\nu, \theta\,\chi_\eta\xi_d\rangle_{T^*\Gamma} = -\langle \omega_+\nu, \theta\,\chi_\eta\xi_d\rangle_{T^*\Gamma} + \langle \mu_+, \xi'\cdot\nabla_{x'}\theta\,\chi_0\chi_\eta\xi_d\rangle + \langle \mu_+, \xi_d^2\theta\,\chi_0\chi_\eta\rangle \tag{4.5.4}$$

where we have used $\langle Q, \theta\,\chi_0\chi_\eta\xi_d\rangle = 0$. This last relation comes from the fact that χ_0 vanishes on the support of Q (i.e. when $x = 0$).

Now, we study the three terms in the right hand side of (4.5.4). The first term is easily bounded by

$$\Big|\int \omega_+\chi_\eta\theta d\nu\Big| \le \eta\|\lambda\chi(\lambda)\|_{L^\infty}\Big|\int \theta d\nu\Big|.$$

To bound the other two terms, we use that the support of μ_+ lies in $\{|\xi|^2 = n_+^2\} = \{\xi_d^2 = \omega_+\}$. We obtain

$$\begin{aligned}\Big|\int \xi'\cdot\nabla_{x'}\theta\,\chi_0\chi_\eta\xi_d\,d\mu_+\Big| &\le \sqrt{\eta}\|\lambda\chi(\lambda^2)\|_{L^\infty}\Big|\int \xi'\cdot\nabla_{x'}\theta\,d\mu_+\Big|,\\ \Big|\int \xi_d^2\chi_\eta\chi_0'\theta\,d\mu_+\Big| &\le \eta\|\lambda\chi(\lambda)\|_{L^\infty}\int |\theta| d\mu_+.\end{aligned}$$

Hence, the right hand side of (4.5.4) tends to 0 as $\eta\to 0$. Since χ_η converges pointwise to $\mathbf{1}_{\{\omega_+=0\}}$, taking the limit $\eta\to 0$ in the equation (4.5.4), we get $\dot\nu = 0$ on the set $\{\omega_+ = 0\}$.

Moreover, by the hermitian positivity of the semiclassical matrix valued measure associated to the traces of u^ε and $\varepsilon\partial_{x_d}u^\varepsilon$, we have $|\nu^J| \leq \nu^{1/2}\dot{\nu}^{1/2}$. Therefore, ν^J vanishes on the set $\{\omega_+ = 0\}$ as well. □

Let us end this section with the following result on the measures $\mu^{\partial_\pm}$. Unfortunately, the equation (4.5.5) does not suffice to determine $\mu^{\partial_\pm} = 0$. This last point is linked with the radiation condition at infinity satisfied by $\mu^{\partial_\pm}$ (see Theorem 4.5.6 and Section 7.2).

Lemma 4.5.5 *The measures $\mu^{\partial_\pm}$ satisfy the following transport equation in $T^*\Gamma$ ("inside the boundary"):*

$$\xi' \cdot \nabla_{x'}\mu^{\partial_\pm} = 0. \tag{4.5.5}$$

Proof. The proof is similar to that of Lemma 4.5.4. To obtain the result, we use the transport equation 4.4.9. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be nonnegative, $\chi(0) = 1$, $\chi'(0) = 0$, and $\chi_\eta(\xi') = \chi\big(\frac{|\xi'|^2 - n_+^2}{\eta}\big)$. Let $\theta \in \hat{\mathcal{C}}_c^\infty(\mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{\xi'}^{d-1})$. Let $\eta > 0$ be a small parameter.

We write the transport equation (4.4.9) with $\varphi_0(x,\xi') = \theta(x',\xi')\chi\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta(\xi')$ and $\varphi_1 = 0$:

$$\langle\mu_\pm, \xi\cdot\nabla_x(\theta\,\chi\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta\xi_d)\rangle = \langle Q_\pm, \theta\,\chi\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta\rangle + \langle \mathcal{R}e\,\nu^J, \theta\,\chi_\eta\rangle_{T^*\Gamma}. \tag{4.5.6}$$

Let us first prove that the right-hand side of (4.5.6) vanishes. Indeed, since $Q_+ = 0$ and Q_- has support at $x = 0$, we get $\langle Q_\pm, \theta\,\chi\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta\rangle = 0$. Moreover, $\mathcal{R}e\,\nu^J$ vanishes on $\{\omega_+ \leq 0\} = \{\xi'^2 \geq n_+^2\}$, hence it vanishes on the support of χ_η. Thus, we obtain $\langle\mathcal{R}e\,\nu^J, \theta\,\chi_\eta\rangle_{T^*\Gamma} = 0$.

Now, we prove that the left hand-side of (4.5.6) is equal to $\langle\mu^{\partial_\pm}, \xi'\cdot\nabla_{x'}\theta\rangle$. We have

$$\begin{aligned}\langle\mu_\pm, \xi\cdot\nabla_x(\theta\,\chi\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta(\xi'))\rangle &= \langle\mu_\pm, \theta\,\frac{\xi_d}{\eta^{1/3}}\chi'\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta(\xi'))\rangle \\ &\quad + \langle\mathbf{1}_{\{x_d \gtrless 1\}}\mu_\pm, \xi'\cdot\nabla_{x'}\theta\,\chi\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta(\xi'))\rangle \\ &\quad + \langle\mu^{\partial_\pm}, \xi'\cdot\nabla_{x'}\theta\chi_\eta(\xi')\rangle\end{aligned}$$

First, as in the proof of Lemma 4.5.4, we write

$$\left|\int \frac{\xi_d}{\eta^{1/3}}\chi_\eta(\xi')\chi'\left(\frac{x_d}{\eta^{1/3}}\right)\theta\,d\mu_\pm\right| \leq \eta^{\frac{1}{2}-\frac{1}{3}}\|\lambda\chi(\lambda^2)\|_{L^\infty}\left|\int \xi'\cdot\nabla_{x'}\theta\,d\mu_\pm\right|.$$

Hence, we get, as $\eta \to 0$,

$$\langle\mu_\pm, \theta\,\frac{\xi_d}{\eta^{1/3}}\chi'\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta(\xi'))\rangle \to 0.$$

Second, since $\xi' \cdot \nabla_{x'}\theta\, \chi\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta(\xi')\rangle)$ weakly converges to 0 when $\eta \to 0$ almost everywhere for the measures $\mathbf{1}_{\{x_d \gtrless 1\}}\mu_\pm$, we have

$$\langle \mathbf{1}_{\{x_d \gtrless 1\}}\mu_\pm, \xi' \cdot \nabla_{x'}\theta\, \chi\left(\frac{x_d}{\eta^{1/3}}\right)\chi_\eta(\xi')\rangle) \to 0 \quad \text{as } \eta \to 0.$$

Third, since $\mu^{\partial_\pm}$ has support in $\{\xi'^2 = n_\pm^2\}$, we get

$$\langle \mu^{\partial_\pm}, \xi' \cdot \nabla_{x'}\theta \chi_\eta(\xi')\rangle \to \langle \mu^{\partial_\pm}, \xi' \cdot \nabla_{x'}\theta\rangle \qquad \text{as } \eta \to 0.$$

In conclusion, we obtain $\langle \mu^{\partial_\pm}, \xi' \cdot \nabla_{x'}\theta\rangle = 0$. □

4.5.2 Reflexion/transmission at the interface

In this section, we end the proof of our main theorem in the case of two homogeneous media. We prove it by solving Cauchy problems with respect to the x_d variable. These problems are of two types: in the regions where the rays of geometrical optics do not reach the interface when $t \to -\infty$, we solve Cauchy problems with boundary conditions at infinity in space; in the other regions, we solve Cauchy problems with initial data at $x_d = 1$.

We use the following partition of phase space

$$\begin{aligned} T^*\mathbb{R}^d &= \{x_d < 1, \xi_d \geq 0\} \cup \{x_d \leq 1, -\sqrt{[n^2]} \leq \xi_d < 0\} \\ &\quad \cup \{x_d \leq 1, \xi_d < -\sqrt{[n^2]}\} \cup \{x_d > 1, \xi_d \leq 0\} \\ &\quad \cup \{x_d \geq 1, \xi_d > 0\} \cup \{x_d = 1, \xi_d = 0\} \\ &= V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6. \end{aligned}$$

The value of μ in the first five regions will be obtained by solving the transport equation (4.4.2) on each region V_j ($j = 1, \ldots, 5$), using the radiation condition at infinity and Lemmas 4.5.3 and 4.5.4 to get the values at the boundary. At variance, the value of μ in V_6 cannot be obtained using a transport equation since no ray coming from one media reaches the interface with $\xi_d = 0$ (the rays are given by $(x+t\xi, \xi)$ in the homogeneous case). Thus, we have to study directly $\mu^{\partial_\pm}$. The following proposition implies that $\mu = 0$ in the region V_6.

Theorem 4.5.6 *When n_- and n_+ are constant, we have*

$$\mu^{\partial_\pm} = 0.$$

Proof. The reader can find the proof of this theorem in Section 4.7.2. Using the explicit formula known for the resolvent of the Helmholtz operator in that particular case, the study reduces to a (non-)stationary phase method with singularities. These singularities come from the roots $\sqrt{\xi'^2 - n_\pm^2 + i\alpha_\varepsilon\varepsilon}$ that appear both in the phase function and as test functions and that are singular near $\xi'^2 = n_\pm^2$ when $\varepsilon \to 0$. In order to treat this problem, the key ingredients are a contour deformation in the complex plane and the use of almost-analytic extensions. □

In the first region V_1, μ is the solution to $\xi \cdot \nabla_x \mu = Q$ with the outgoing condition at infinity $\mu(x,\xi) \to 0$ as $|x| \to \infty$ with $x \cdot \xi < 0$ (it is a consequence of the radiation condition 4.4.3). On the other hand, if $(x,\xi) \in V_1$, then for all $t < 0$, $(x + t\xi, \xi) \in V_1$. We deduce

$$\mu(x,\xi) = \mu(x + t\xi, \xi) + \int_t^0 Q(x + s\xi, \xi) ds.$$

Taking the limit $t \to -\infty$, we obtain the value of μ in V_1:

$$\mu(x,\xi) = \int_{-\infty}^0 Q(x + s\xi, \xi) ds.$$

As a consequence,

$$\delta(\xi_d - \sqrt{\omega_-}) \otimes \mu_-^{in}(x',\xi') = \int_{-\infty}^0 Q(x' + s\xi', 1 + s\xi_d, \xi) dt.$$

In particular, the measure μ_-^{in} is known.

Now, we compute μ in the region V_2. We consider the following part of the interface: $\{\omega_- > 0\} \cap \{\omega_+ \leq 0\}$, which corresponds to $0 < \omega_- \leq [n^2]$. On this set, since $\mathcal{R}e\, \nu^J = 0$, from Propositions 4.5.3 and 4.5.4, we get $\mu_-^{out} = \mu_-^{in}$. Hence, for $-\sqrt{[n^2]} \leq \xi_d < 0$, we recover

$$\begin{aligned} \delta(\xi_d + \sqrt{\omega_-}) \otimes \mu_-^{out} &= \delta(\check{\xi}_d - \sqrt{\omega_-}) \otimes \mu_-^{in}(x', \check{\xi}') \\ &= \int_{-\infty}^0 Q(x' + s\check{\xi}', 1 + s\check{\xi}_d, \check{\xi}) ds \end{aligned} \tag{4.5.7}$$

where $\check{\xi} = (\xi', -\xi_d)$. Hence, we are left with the following Cauchy problem in the x_d variable with initial data (4.5.7) at the interface $x_d = 1$: for $(x,\xi) \in V_2$,

$$\begin{cases} \partial_{x_d}\mu + \xi_d^{-1}\xi' \cdot \nabla_{x'}\mu = \xi_d^{-1} Q, & x_d < 1 \\ \mu|_{\{x_d = 1\}}(x',\xi) = \int_{-\infty}^0 Q(x' + s\check{\xi}', 1 + s\check{\xi}_d, \check{\xi}) ds. \end{cases}$$

This problem is explicitely solvable. For $(x,\xi) \in V_2$, we obtain

$$\begin{aligned}
\mu(x,\xi) &= \mu|_{\{x_d=1\}}(x' + \frac{1-x_d}{\xi_d}\xi', \xi) - \int_{x_d}^{1} Q(x' + \frac{s-x_d}{\xi_d}\xi', s, \xi)\frac{ds}{\xi_d} \\
&= \int_{-\infty}^{0} Q\big(x' + \big(\frac{1-x_d}{\xi_d} + s\big)\check{\xi}', 1 + s\check{\xi}_d, \check{\xi}\big)ds \\
&\qquad - \int_{x_d}^{1} Q(x' + \frac{s-x_d}{\xi_d}\xi', s, \xi)\frac{ds}{\xi_d} \\
&= \int_{-\infty}^{\frac{1-x_d}{\xi_d}} Q(x' + t\check{\xi}', 2 - x_d + t\check{\xi}_d, \check{\xi})ds \\
&\qquad + \int_{\frac{1-x_d}{\xi_d}}^{0} Q(x' + t\xi', x_d + t\xi_d, \xi)ds
\end{aligned}$$

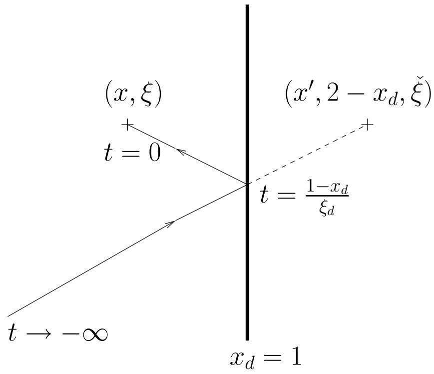

Figure 4.3: *Total reflection*

Remark: *One can notice that* $\frac{1-x_d}{\xi_d}$ *is the time at which the bicharacteristics reaches the interface. The point* $(x', 2-x_d)$ *is the symmetric of* x *with respect to the interface (see Figure 3).*

Next, we consider the part $\{\omega_- > 0\} \cap \{\omega_+ > 0\}$ of the interface. In the region V_4, μ satisfies the equation $\xi \cdot \nabla_x \mu = 0$ with the outgoing radiation condition at infinity $\mu(x,\xi) \to 0$ as $|x| \to \infty$, $x \cdot \xi < 0$. Hence, $\mu = 0$ in this region and

$$\mu_+^{in} = 0.$$

In the next lemma, we write the relations between the other three measures μ_-^{in}, μ_-^{out}, μ_+^{out}. These relations translate the refraction phenomenon. The proof of this result is borrowed from [Mil1].

Lemma 4.5.7 *Let $\mathcal{B} \subset \{\omega_- > 0\} \cap \{\omega_+ > 0\}$ be a Borel set. If $\mu_+^{in} = 0$ in $\mathcal{B}$, then $\mu_+^{out} = \alpha^T \mu_-^{in}$ and $\mu_-^{out} = \alpha^R \mu_-^{in}$.*

Proof. Using lemma 4.5.3, we get

$$\dot{\nu} + \omega_+ \nu = 2\omega_+ \mu_+^{out} = 2\sqrt{\omega_+}\, \mathcal{R}e\, \nu^J.$$

But the matrix measure $\begin{pmatrix} \nu & \bar{\nu}^J \\ \nu^J & \dot{\nu} \end{pmatrix}$ is hermitian so

$$|\nu^J| \leq (\nu)^{\frac{1}{2}} (\dot{\nu})^{\frac{1}{2}}.$$

Hence, we recover

$$2(\omega_+ \nu)^{\frac{1}{2}} (\dot{\nu})^{\frac{1}{2}} \leq \dot{\nu} + \omega_+ \nu = 2\sqrt{\omega_+}\, \mathcal{R}e\, \nu^J \leq 2(\omega_+ \nu)^{\frac{1}{2}} (\dot{\nu})^{\frac{1}{2}}$$

and

$$\dot{\nu} = \omega_+ \nu \qquad \text{in } \{\omega_- > 0\} \cap \{\omega_+ > 0\}.$$

Thus, we now have five equations (the equation above and the four equations in Lemma 4.5.3) involving the six unknown measures ν, $\dot{\nu}$, ν^J, μ_+^{out}, μ_+^{in} and μ_-^{in}. After some calculations, we deduce

$$\begin{cases} \mu_+^{out} &= \alpha^T \mu_-^{in}, \\ \mu_+^{in} &= \alpha^R \mu_-^{in}, \end{cases}$$

where the coefficients α^R and α^T are defined in Theorem 4.5.1. □

Using this lemma, we can now determine μ in the remaining regions V_3 and V_5 by solving Cauchy problems with initial data at $x_d = 1$.

In the region V_3, making the same calculations as in the second region, we obtain the reflected part (with a partial reflexion coefficient):

$$\mu(x, \xi) = \int_{\frac{1-x_d}{\xi_d}}^{0} Q(x' + t\xi', x_d + t\xi_d, \xi) ds + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} \alpha^R(\xi)\, Q(x' + t\check{\xi}', 2 - x_d + t\check{\xi}_d, \check{\xi}) ds.$$

In the region V_5, we have

$$\mu(x, \xi) = \int_{\frac{1-x_d}{\xi_d}}^{0} Q(x + t\xi, \xi) dt + \int_{-\infty}^{0} \alpha^T(\xi) Q\big(x' + \frac{1 - x_d}{\xi_d}\xi' + t\xi', 1 + t\tilde{\xi}_d, \tilde{\xi}\big) dt,$$

so

$$\mu(x,\xi) = \int_0^{\frac{1-x_d}{\xi_d}} Q(x+s\xi,\xi)ds + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} \alpha^T(\xi) Q\big(x'+s\xi', 1+(x_d-1)\frac{\tilde{\xi}_d}{\xi_d}+s\tilde{\xi}_d,\tilde{\xi}\big)ds.$$

This ends the proof of Theorem 4.5.1. □

4.6 Refraction result in the general case

This section is devoted to the proof of our main result in the case of a general index of refraction. We use the same method as in the homogeneous case studied in Section 5. The main extra difficulty is of course the geometry of the rays. As it is usual in such problems, we first define the induced geometry of the boundary, i.e. the elliptic, hyperbolic and glancing regions. One of the main differences is that, in the general case, there exist glancing rays that come from the media at the interface and may carry some energy.

4.6.1 Geometry of the boundary

In order to state our assumptions (in particular what we mean by "non gliding condition"), we need to define a geometry of the interface. From each side of the interface, we can define an induced partition of the boundary (that is usual in the study of boundary problems).

We recall that $\omega_\pm(x',\xi') = n_\pm^2(x',1) - \xi'^2$, where $n_\pm^2(x',1) = \lim_{x_d\to 1^\pm} n_\pm^2(x)$.

Let $\pi_\pm$ be the restriction of the projection map $T^*\mathbb{R}^d\rceil_\Gamma \to T^*\Gamma$ to the characteristic set $\Sigma_\pm = \{\xi^2 = n^2(x), x_d \gtrless 1\}$. Then, from each side, $T^*\Gamma$ can be decomposed as the union of the following regions:

- the elliptic region $\mathcal{E}_\pm = \{(x',\xi') \in T^*\Gamma \,|\, \omega_\pm(x',\xi') < 0\}$ is such that $\pi_\pm^{-1}(\mathcal{E}_\pm) = \emptyset$,
- the hyperbolic region $\mathcal{H}_\pm = \{(x',\xi') \in T^*\Gamma \,|\, \omega_\pm(x',\xi') > 0\}$ is the set of points which possess two distinct inverse images by $\pi_\pm$: $(\pi_\pm^{in})^{-1}(x',\xi') = \mp\sqrt{\omega_\pm(x',\xi')}$ and $(\pi_\pm^{out})^{-1}(x',\xi') = \pm\sqrt{\omega_\pm(x',\xi')}$,
- the glancing region $\mathcal{G}_\pm = \{(x',\xi') \in T^*\Gamma \,|\, \omega_\pm(x',\xi') = 0\}$ is the set of points which possess only one inverse image. There, the hamiltonian vector field is tangent to $T^*\Gamma$.

The last region can be decomposed into the following three subregions:

- the diffractive region $\mathcal{G}_\pm^d = \mathcal{G}_\pm \cap \{\pm\partial_d n_\pm^2 > 0\}$ of points at which the hamiltonian vector field and its opposite are pointing into the considered side,

– the gliding region $\mathcal{G}^g_\pm = \mathcal{G}_\pm \cap \{\pm\partial_d n^2_\pm < 0\}$

– the gliding region of higher order $\mathcal{G}^0_\pm = \mathcal{G}_\pm \cap \{\pm\partial_d n^2_\pm = 0\}$

Remark: *In the constant coeficient case, the glancing region $\mathcal{G}$ is reduced to the gliding region of higher order $\mathcal{G}^0$.*

We define $\mathcal{H}^{out}_\pm = \pi^{-1}_\pm(\mathcal{H}_\pm) \cap \{\pm\xi_d > 0\}$ and $\mathcal{H}^{in}_\pm = \pi^{-1}_\pm(\mathcal{H}_\pm) \cap \{\pm\xi_d < 0\}$.
Let $R : \mathcal{H}^{out}_\pm \rightleftharpoons \mathcal{H}^{in}_\pm$ be the reflection map such that $R = (\pi^{in})^{-1} \circ \pi$ on $\mathcal{H}^{out}$ and $R = (\pi^{out})^{-1} \circ \pi$ on $\mathcal{H}^{in}$.
Finally, we define the transmission map $T_+ : \pi^{-1}_+(\mathcal{H}_+) \to \pi^{-1}_-(\mathcal{H}_-)$: for all $X \in T^*\mathbb{R}^d$,

$$\begin{cases} \pi_+(X) = \pi_-(T_+(X)), \\ (X, T_+(X)) \in \mathcal{H}^{in}_+ \times \mathcal{H}^{out}_- \cup \mathcal{H}^{out}_+ \times \mathcal{H}^{in}_-, \end{cases}$$

and we denote $T_- = T^{-1}_+$.

4.6.2 Boundary measures

As in the case of two homogeneous media studied in Section 5.1, we can define boundary measures $\mu^{in}_\pm$, $\mu^{out}_\pm$, as in formulas (4.5.2), (4.5.3), together with the measures $\mu^{\partial_\pm}$ as in Lemma 4.5.2. The measures $\mu^{in}_\pm$, $\mu^{out}_\pm$ are nonnegative measures defined on the hyperbolic regions $\mathcal{H}_\pm$ and $\mu^{\partial_\pm}$ are nonnegative measures on $T^*\Gamma$ with support in $\mathcal{G}_\pm$. Using the same method as in the homogeneous case, we can prove the following two lemmas similar to Lemma 4.5.3 and Lemma 4.5.7.

Lemma 4.6.1 *In the set $\mathcal{H}_+ \cap \mathcal{H}_-$, we have*
(i) $\pm\mathcal{R}e\,\nu^J = \sqrt{\omega_\pm}(\mu^{out}_\pm - \mu^{in}_\pm)$,
(ii) $\frac{1}{2}\dot{\nu} = \omega_\pm(\mu^{out}_\pm + \mu^{in}_\pm - \frac{1}{2}\nu)$.

Lemma 4.6.2 *Let $\mathcal{B}$ be a Borel set included in $\mathcal{H}_+ \cap \mathcal{H}_-$.*
(i) If $\mu^{in}_- = 0$ on $\mathcal{B}$, then $\mu^{out}_+ = \alpha^R \mu^{in}_+$ and $\mu^{out}_- = \alpha^T \mu^{in}_+$ on $\mathcal{B}$,
(ii) If $\mu^{in}_+ = 0$ on $\mathcal{B}$, then $\mu^{out}_- = \alpha^R \mu^{in}_-$ and $\mu^{out}_+ = \alpha^T \mu^{in}_-$ on $\mathcal{B}$,
where

$$\alpha^R = 1 - \alpha^T = \left| \frac{2\sqrt{\omega_-}}{\sqrt{\omega_+} - \sqrt{\omega_-}} \right|^2$$

Proof. The proof is the same as Lemma 4.5.7. □

The third result is the analogous to Lemma 4.5.4 in the inhomogeneous case. Indeed, in the homogeneous case, the glancing region coincides with the gliding region of higher order $\mathcal{G}^0$, therefore the following lemma reduces to $\dot\nu = \nu^J = 0$ on $\mathcal{G}$.

Lemma 4.6.3 *On the glancing region $\mathcal{G}_\sigma$, $\sigma \in \{+,-\}$, we have:*
(i) $-\sigma(\partial_d n_\sigma^2)\,\mu^{\partial_\sigma} = \frac{1}{2}\dot\nu$,
(ii) $\mu^{\partial_\sigma} = 0$ on $\mathcal{G}_\sigma^d$ and $\dot\nu = \nu^J = 0$ on $\mathcal{G}_\sigma^d \cup \mathcal{G}_\sigma^0$.

Proof. The second point is is an easy consequence of point (i), since $\dot\nu$ and μ^{∂_σ} are nonnegative measures, $\sigma\partial_d n_\sigma^2 > 0$ on $\mathcal{G}_\sigma^d$ and $\partial_d n_\sigma^2 = 0$ on $\mathcal{G}_\sigma^0$. Let us prove point (i). We use the same multipliers as in the proof of Lemma 4.5.4. Let $\chi_0 \in \mathcal{C}_c^\infty$ be a nonnegative function such that $\chi_0(0) = 1$ and $\mathrm{supp}(\chi_0) \subset [-1/2, 1/2]$, let $\theta \in \hat{\mathcal{C}}_c^\infty(\mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{\xi'}^{d-1})$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be nonnegative, $\chi(0) = 1$, $\chi'(0) = 0$, and $\chi_\eta(x,\xi') = \chi\big(\frac{\omega_\sigma}{\eta}\big) = \chi\big(\frac{n_\sigma^2(x)-\xi'^2}{\eta}\big)$. We write the transport equation (4.4.9) with $\varphi_0 = 0$ and $\varphi_1 = \theta\,\chi_0(x_d - 1)\chi_\eta(x,\xi')$. It gives:

$$-\langle\mu_\sigma, \left(\xi\cdot\nabla_x + \frac{1}{2}\nabla_x n_\sigma^2\cdot\nabla_\xi\right)(\theta\,\chi_0\chi_\eta\xi_d)\rangle$$
$$= \langle Q, \theta\,\chi_0\chi_\eta\xi_d\rangle + \sigma\frac{1}{2}\langle\dot\nu + \omega_\sigma\nu, \theta\,\chi_\eta\xi_d\rangle_{T^*\Gamma}.$$

Since $\big(\xi'\cdot\nabla_{x'} + \frac{1}{2}\nabla_{x'}n_\sigma^2\cdot\nabla_{\xi'}\big)\,\omega_\sigma = 0$, there are only two extra terms to handle in comparison with the proof of Lemma 4.5.4. These are $\langle\mu_\sigma, \frac{\xi_d^2}{\eta}\chi'\big(\frac{\omega_\sigma}{\eta}\big)\theta\chi_0\rangle$ and $\langle\mu_\sigma, \frac{1}{2}\partial_d n_\sigma^2\theta\chi_0\chi_\eta\rangle$.
Using the Lebesgue's dominated convergence theorem, the last one is easily seen to tend to $\frac{1}{2}\langle\mu_\sigma\mathbf{1}_{\{\omega_\sigma=0\}}, \partial_d n_\sigma^2\theta\chi_0\rangle$ as $\eta \to 0$.
We turn to the study of the first term. The support of μ_σ lies in $\{\xi_d^2 = \omega_\sigma\}$, then $\big|\frac{\xi_d^2}{\eta}\chi'\big(\frac{\omega_\sigma}{\eta}\big)\theta\,\chi_0\big|$ is bounded by $\|\lambda\chi'(\lambda)\|_{L^\infty}|\theta\,\chi_0|$ on the support of μ_σ. Now, using the dominated convergence theorem, together with the fact that $\frac{\omega_\sigma}{\eta}\chi'\big(\frac{\omega_\sigma}{\eta}\big)$ converges pointwise to 0, we obtain the devined

$$\langle\mu_\sigma, \frac{\xi_d^2}{\eta}\chi'\big(\frac{\omega_\sigma}{\eta}\big)\theta\chi_0\rangle \to 0 \quad \text{as } \eta \to 0.$$

Eventually, we have obtained

$$-\langle\mu_\sigma\mathbf{1}_{\{\omega_\sigma=0\}}, \sigma\partial_d n_\sigma^2\theta\chi_0\rangle = \langle\dot\nu_{|\mathcal{G}}, \theta\rangle_{T^*\Gamma}.$$

Now, we replace $\chi_0(x_d - 1)$ by $\chi_0\big(\frac{x_d-1}{\eta}\big)$ in the previous equation. Since there are no derivatives with respect to x_d, we can apply again the dominated convergence theorem as η tends to 0. This yields: $-\langle\mu^{\partial_\sigma}, \sigma\partial_d n_\sigma^2\theta\rangle = \langle\dot\nu_{|\mathcal{G}}, \theta\rangle_{T^*\Gamma}$. The lemma is proved. □

4.6.3 Snell-Descartes semi-group and last assumptions

As in the constant coefficient case, our proof of the refraction result will use the transport equation (4.4.2) in the interior of each medium, the propagation properties at the interface given in Lemma 4.6.2 together with the radiation condition at infinity (4.4.3). For this purpose, we need to define the "past" of any point $(x,\xi) \in T^*\mathbb{R}^d$, i.e. a "trajectory" from $-\infty$ to 0 that passes through (x,ξ) at $t=0$ (because of the *outgoing* radiation condition at infinity, we need the "past" of (x,ξ) and not its "future").

The lemmas 4.6.2 and 4.6.3 allow us to study the propagation of the measure at the interface except when

- density comes upon $\mathcal{G}_+^0 \cup \mathcal{G}_+^g$ from $\{x_d > 1\}$ or upon $\mathcal{G}_-^0 \cup \mathcal{G}_-^g$ from $\{x_d < 1\}$,
- density comes upon $\mathcal{H}_+ \cap \mathcal{H}_-$ from both sides at the same point.

For this reason, we need to assume the *non gliding condition*:

(H13) $\mu^{\partial_+}(\mathcal{G}_+^0 \cup \mathcal{G}_+^g) = \mu^{\partial_-}(\mathcal{G}_-^0 \cup \mathcal{G}_-^g) = 0$,

and the *non interference condition*:

(H14) μ_+^{in} and μ_-^{in} are mutually singular.

Note that hypothesis (H13) ensures that no density can be trapped in the interface. Indeed, (H13) together with Lemma 4.6.3 imply that $\mu^{\partial_\pm} = 0$.

Now, we define the Snell-Descartes semi-group (see for instance L. Miller [Mil2]).
Let $(x,\xi) \in T^*\mathbb{R}^d$ and $(b_n) \in \{R,T\}^{\mathbb{N}^*}$ be given. We construct a map $\gamma :]-\infty,0] \to T^*\mathbb{R}^d$ by a recursive process. For all $t \leq 0$, we denote $\gamma(t) = (x^\gamma(t), \xi^\gamma(t))$.
Initially, we set $n=0$. If $(x,\xi) \notin \big(\Sigma_+ \setminus (\mathcal{G}_+ \cup \mathcal{H}_+^{out})\big) \cup \big(\Sigma_- \setminus (\mathcal{G}_- \cup \mathcal{H}_-^{out})\big)$ then γ is stationary: $\gamma \equiv (x,\xi)$. If not, then $(x,\xi) \in \Sigma_\sigma \setminus (\mathcal{G}_\sigma^0 \cup \mathcal{G}_\sigma^d \cup \mathcal{H}_\sigma^{out})$ for some $\sigma \in \{+,-\}$ and γ is identified with the bicharacteristical flow $(X(t),\Xi(t))$ from (x,ξ) on a maximal interval $(\bar{t},0]$. When the interval is finite, γ has a limit from the right $(\bar{x},\bar{\xi}) \in \mathcal{G}_\sigma^d \cup \mathcal{G}_\sigma^0 \cup \mathcal{H}_\sigma^{out}$ at $\bar{t}$. Then, we iterate the previous step from $\gamma(\bar{t})$ defined as $\gamma(\bar{t}) = (\bar{x},\bar{\xi})$ when $(\bar{x},\bar{\xi}) \in \mathcal{G}_\sigma^d \cup \mathcal{G}_\sigma^0$, and otherwise as: if $\pi_\sigma(\bar{x},\bar{\xi}) \notin \mathcal{H}_+ \cap \mathcal{H}_-$ then $\gamma(\bar{t}) = R(\bar{x},\bar{\xi})$; if $\pi_\sigma(\bar{x},\bar{\xi}) \in \mathcal{H}_+ \cap \mathcal{H}_-$ then n is replaced by $n+1$, and$\gamma(\bar{t}) = R(\bar{x},\bar{\xi})$ if $b_n = R$, $\gamma(\bar{t}) = T_\sigma(\bar{x},\bar{\xi})$ if $b_n = T$. The trajectory thus defined is continuous from the left with limit from the right.

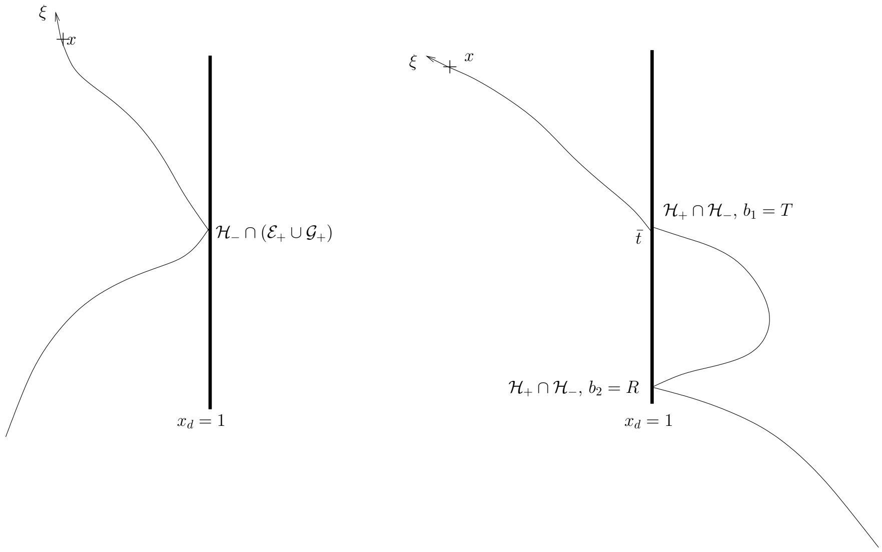

Figure 4.4: *Rays of geometrical optics in the general case.*

The past of $(x, \xi) \in T^*\mathbb{R}^d$ in the term $t \in \mathbb{R}_-$ is the set $\mathcal{A}^t_{(x,\xi)}$ of the various restrictions to $(t, 0]$ of the trajectories γ obtained by the above process starting from (x, ξ) and a choice of a sequence $(b_n) \in \{R, T\}^{\mathbb{N}^*}$. Given the positive bounded continuous reflection and transmission coefficients α^R and α^T on $\mathcal{H}_+ \cap \mathcal{H}_-$ (see Lemma 4.6.2), we assign to each trajectory $\gamma \in \mathcal{A}^t_{(x,\xi)}$ the weight $\alpha(\gamma)$ obtained by mutiplying, for each choice $b \in \{R, T\}$ made during its construction, the value taken by α^b at the corresponding point of $\mathcal{H}_+ \cap \mathcal{H}_-$. Moreover, if γ is stationary for $t' \leq t$, we let $\alpha(\gamma) = 0$ for $t' \leq t$. By construction, the pasts satisfy the following semigroup property: the trajectories γ of $\mathcal{A}^t_{(x,\xi)}$ which coincide on $(t_0, 0]$ (where $t_0 > t_1$) with $\gamma_0 \in \mathcal{A}^{t_0}_{(x,\xi)}$ are obtained by gluing to it the trajectories $\gamma' \in \mathcal{A}^{t_1 - t_0}_{\gamma_0(t_0)}$, i.e. setting $\gamma(t) = \gamma_0(t)$ for all $t \in (t_0, 0]$ and $\gamma(t) = \gamma'(t - t_0)$ for all $t \in (t_1, t_0]$. Moreover, $\alpha(\gamma) = \alpha(\gamma_0)\alpha(\gamma')$, and $\sum_{\gamma \in \mathcal{A}^t_{(x,\xi)}} \alpha(\gamma) = 1$. Therefore, we may define a positive contraction semigroup $(S_t)_{t \leq 0}$ on bounded Borel functions f on $T^*\mathbb{R}^d$ by:

$$S_t f(x, \xi) = \sum_{\gamma \in \mathcal{A}^t_{(x,\xi)}} \alpha(\gamma) f\left(x^\gamma(t^+), \xi^\gamma(t^+)\right).$$

We call Snell-Descartes semigroup the dual semigroup $(S_t^*)_{t \leq 0}$ acting on the set $\mathcal{M}(T^*\mathbb{R}^d)$ of positive Radon measures on phase space.

We can now state our last assumption: in order to use the radiation condition at infinity, we need that the rays go at infinity away from the interface, so we assume that

(H15) for all $(x,\xi) \in T^*\mathbb{R}^d$, for all choice of $(b_n) \in \{R,T\}^{\mathbb{N}^*}$, the map γ constructed by the above process satisfies:

$$\exists\, T(\gamma) < 0 \text{ such that } \quad \forall t \leq T(\gamma),\ \gamma(t) \text{ is stationary or } x_d^\gamma(t) \neq 1,$$

i.e. $\gamma(t)$ coincides with the bicharacteristic curve for $t \leq T(\gamma)$ if γ is not stationnary.

4.6.4 Refraction result

Let us first state precisely the main result we are going to prove in the general case.

Theorem 4.6.4 *Assume (H1)-(H15). Then, the Wigner measure associated with the sequence* (u^ε) *is given by*

$$\mu = \int_{-\infty}^{0} (S_t^* Q)\, dt,$$

where $Q(x,\xi) = \frac{1}{2^{d+1}\pi^{d-1}}\delta(x)\delta(\xi^2 - n^2(0))\hat{f}(\xi)\left(\bar{\hat{f}}(\xi) + q(\xi)\right)$, *$q$ being as in Theorem 4.3.2, and S_t^* is the Snell-Descartes semi-group defined in Section 4.6.3.*

Now, we can end the proof of this result. First of all, as in Section 5.2, we have two results concerning the propagation at the boundary. The first one, that will imply the refraction result, is stated in Lemma 4.6.2. The second one, that will give the total reflexion result, is contained in the following lemma.

Lemma 4.6.5 *(total reflection)*
On the set $\mathcal{H}_- \cap \left(\mathcal{E}_+ \cup \mathcal{G}_+\right)$, *we have* $\mu_-^{in} = \mu_-^{out}$.

Proof. From Lemma 4.6.3, we have $\dot{\nu} = \nu^J = 0$ on $\mathcal{G}_+^d \cup \mathcal{G}_+^0$, and $-(\partial_d n_+^2)\,\mu^{\partial_+} = \frac{1}{2}\dot{\nu}$. Hence, using the hypothesis (H14) ($\mu^{\partial_+}(\mathcal{G}_+^g) = 0$), we get $\dot{\nu} = 0$ on $\mathcal{G}_+^g$. Hence, $\nu^J = 0$ on $\mathcal{G}_+^g$. Thus, the first identity in Lemma 4.6.1 gives that $\mu_-^{in} = \mu_-^{out}$ on $\mathcal{H}_- \cap \left(\mathcal{E}_+ \cup \mathcal{G}_+\right)$. □

Proof of Theorem 4.6.4.
Let $(x,\xi) \in T^*\mathbb{R}^d$. If $(x,\xi) \in T^*\mathbb{R}^d \setminus (\Sigma_+ \cup \Sigma_-)$, $(x,\xi) \notin \mathrm{supp}(\mu)$, so $\mu(x,\xi) = 0$ and looking at the construction of the semi-group S_t^*, we have also $S_t^* = 0$ for all $t \leq 0$. Hence, $\mu(x,\xi) = \int_{-\infty}^0 S_t^* Q\, dt$.

If $(x, \xi) \in (\mathcal{G}_+ \cup \mathcal{G}_- \cup \mathcal{H}_+^{out} \cup \mathcal{H}_-^{out})$, then we have $S_t^* = 0$ for all $t \leq 0$ and $\mu(x, \xi) = 0$ (since $\mu^\partial = 0$).

Last assume $(x, \xi) \in \Sigma_- \cap \{x_d \leq 1\}$ (the case $(x, \xi) \in \Sigma_+ \cap \{x_d \geq 1\}$ can be treated similarly). One can define the bicharacteristic curve. Let $(\bar{t}, 0]$ be the maximal interval on which the bicharacteristic curve passing through (x, ξ) at $t = 0$ is defined.

If $\bar{t} = -\infty$, using the transport equation in the interior of the medium, we obtain for all $t \leq \bar{t}$,

$$\mu(x, \xi) = \mu(X(t), \Xi(t)) + \int_t^0 Q(X(s), \Xi(s))ds.$$

Hence, using the outgoing radiation condition stated in Corollary 4.4.3 and the fact that $|X(t)| \to \infty$ with $X(t) \cdot \Xi(t) < 0$, we get that

$$\mu(x, \xi) = \int_{-\infty}^0 Q(X(s), \Xi(s))ds = \int_{-\infty}^0 S_t^* Q dt.$$

If $\bar{t}$ is finite, let $(\bar{x}, \bar{\xi}) = \lim_{t \to \bar{t}^+} (X(t), \Xi(t))$. Then, $(\bar{x}, \bar{\xi}) \in \mathcal{H}_-^{out} \cup \mathcal{G}_-^0 \cup \mathcal{G}_-^d$. As before, we have

$$\mu(x, \xi) = \mu(\bar{x}, \bar{\xi}) + \int_{\bar{t}}^0 Q(X(s), \Xi(s))ds.$$

If $(\bar{x}, \bar{\xi}) \in \mathcal{H}_+ \cap \mathcal{H}_-$, then using the hypothesis (H14), we can assume that $\mu_+^{in} = 0$ and from Lemma 4.6.2, we obtain, letting $t \to -\infty$, $\mu_-^{out} = \alpha^R \mu_-^{in}$. Thus,

$$\begin{aligned} \mu(x, \xi) &= \alpha^R R^* \left(\delta(\xi_d - \sqrt{\omega_-}) \otimes \mu_-^{in}(x', \xi') \right) + \int_{\bar{t}}^0 Q(X(s), \Xi(s))ds \\ &= \alpha(\gamma) \mu(x^\gamma(\bar{t}), \xi^\gamma(\bar{t})) + \int_{\bar{t}}^0 Q(x^\gamma(s), \xi^\gamma(s))ds, \end{aligned}$$

where γ is defined as in the previous section with $b_1 = R$.

If $(\bar{x}, \bar{\xi}) \in (\mathcal{G}_+ \cup \mathcal{E}_+) \cap \mathcal{H}_-$, from Lemma 4.6.5 (total reflection), we get $\mu_-^{out} = \mu_-^{in}$. Thus,

$$\begin{aligned} \mu(x, \xi) &= R^* \left(\delta(\xi_d - \sqrt{\omega_-}) \otimes \mu_-^{in}(x', \xi') \right) + \int_{\bar{t}}^0 Q(X(s), \Xi(s))ds \\ &= \mu(x^\gamma(\bar{t}), \xi^\gamma(\bar{t})) + \int_{\bar{t}}^0 Q(x^\gamma(s), \xi^\gamma(s))ds. \end{aligned}$$

If $(\bar{x}, \bar{\xi}) \in \mathcal{G}_-^d \cup \mathcal{G}_-^0$, since we assume that no density is trapped in the interface (hypothesis (H13) and Lemma 4.6.3), we have that $\mu_-(\bar{x}, \bar{\xi}) = 0$.

Moreover, in the construction of the Snell-Descartes semi-group we let $S_t^* = 0$ for $t \leq \bar{t}$ if γ is stationary for $t \leq \bar{t}$, which is the case here. Hence, we directly get

$$\mu(x,\xi) = \int_{-\infty}^{0} Q(x^\gamma(s), \xi^\gamma(s))ds.$$

From now on, we may iterate the process from the point $(\bar{x}, \bar{\xi})$. In this way, we obtain, for all $t \leq 0$,

$$\mu(x,\xi) = \alpha(\gamma)(t)\mu(x^\gamma(t), \xi^\gamma(t)) + \int_t^0 \alpha(\gamma)(s)Q(x^\gamma(s), \xi^\gamma(s))ds,$$

with $\gamma \in \mathcal{A}^t_{(x,\xi)}$.

Now, we use the hypothesis (H15): there exists $T(\gamma) < 0$ such that $\forall t \leq T(\gamma)$, $\gamma(t)$ is stationary or $x_d^\gamma(t) \neq 1$. We have already proved that, if γ is stationary from some time $T(\gamma)$, then

$$\mu(x,\xi) = +\int_{-\infty}^{0} \alpha(\gamma)(s)Q(x^\gamma(s), \xi^\gamma(s))ds.$$

If there exists $T(\gamma)$ such that $\forall t \leq T(\gamma)$, $x_d^\gamma(t) \neq 1$ then γ coincides with the bicaracteristic curve for $t \leq T(\gamma)$ so that $|x^\gamma(t)| \to \infty$ with $x^\gamma(t) \cdot \xi^\gamma(t) < 0$. Hence, the radiation condition 4.4.3 implies that $\mu(x^\gamma(t), \xi^\gamma(t))$ tends to 0 as $t \to -\infty$. In that case, we conclude

$$\mu(x,\xi) = \int_{-\infty}^{0} \alpha(\gamma)(s)Q(x^\gamma(s), \xi^\gamma(s))ds.$$

This ends the proof of our main theorem in the general case. □

4.7 Proofs of the radiation conditions in the case of two homogeneous media

In this section, we assume that $n^2(x) = \begin{cases} n_+^2 & \text{if } x_d > 1 \\ n_-^2 & \text{if } x_d < 1, \end{cases}$, n_+, n_- being two constants such that $n_- > n_+ > 0$.

This section is devoted to the proofs of Proposition 4.3.2 (last statement) and Proposition 4.5.6. Our proofs use the explicit formula available in the homogeneous case for the resolvent of the Helmholtz operator. They rest on a precise study of oscillatory integrals with singularities, which is performed using (non)-stationary phase methods. We need the following two theorems that are proved in [Hör1]. The first one is a (complex) stationary phase theorem and the second one is a non stationary phase theorem. We would like to point out that, in both statements, the phase function may depend on a parameter lying in a compact set.

Theorem 4.7.1 *Let $K \subset \mathbb{R}^N$ be a compact set, X an open neighbourhood of K. If $u \in \mathcal{C}_c^\infty(K)$, $\varphi \in \mathcal{C}_c^\infty(X)$, $Im(\varphi) \geq 0$ in X, and $Det(\varphi'') \neq 0$ at the critical points of φ, then*

$$\Big|\int e^{i\lambda\varphi(x)} u(x)dx\Big| \leq \frac{C}{\lambda^{N/2}} \sup_{\substack{x\in K,\\ |\alpha|\leq 2N}} |\partial^\alpha u(x)|.$$

Moreover, this bound is uniform if φ depends smoothly on a parameter in a compact set.

Theorem 4.7.2 *Let $u \in \mathcal{C}_c^l(\mathbb{R}^N)$ with support in the conpact set K, and $\varphi \in \mathcal{C}^{l+1}$ such that $Im(\varphi) \geq 0$ and $\varphi' \neq 0$ on K. Then, we have the following estimate*

$$\Big|\int e^{i\lambda\varphi(x)} u(x)dx\Big|$$
$$\leq C meas(K)\lambda^{-l} \sum_{j=0}^{l} \sup_{x\in K} |\varphi'(x)|^{-l-j} \left(\sup_{2\leq|\alpha|\leq l+1} |\partial^\alpha \varphi(x)| \right)^j \sup_{|\alpha|\leq l-j} |\partial^\alpha u(x)|,$$

where C is bounded when φ stays in a bounded set in $\mathcal{C}^{l+1}$.

4.7.1 Proof of the radiation condition on w (Proposition 4.3.2)

Let w^ε be the solution to

$$-i\alpha_\varepsilon \varepsilon w^\varepsilon + \Delta w^\varepsilon + n_\varepsilon(x)^2 w^\varepsilon = -f(x), \tag{4.7.1}$$

where $n_\varepsilon(x) = n(\varepsilon x)$.
We aim at proving that the weak limit w of the sequence (w^ε) is *the* outgoing solution to the equation

$$\Delta w + n_-^2 w = f, \tag{4.7.2}$$

given in Fourier space by

$$\hat{w}(\xi) = \frac{\hat{f}(\xi)}{\xi^2 - n_-^2 + i0}.$$

Thus, we want to prove that for all $f \in \dot{B}$, $\phi \in \dot{B}$,

$$\lim_{\varepsilon\to 0}\langle w^\varepsilon, \phi\rangle = \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\hat{\phi}(\xi)}{\xi^2 - n_-^2 + i0} d\xi. \tag{4.7.3}$$

Since we already proved in [Fou2] that for $f \in \dot{B}$ and $\phi \in \dot{B}$, the following bound holds:

$$|\langle w^\varepsilon, \phi\rangle| \le C\|f\|_{\dot{B}}\|\phi\|_{\dot{B}},$$

where C is a constant independent of f, we only have to prove relation (4.7.3) when f and ϕ are *smooth*.

Let us denote by $\mathcal{F}'$ the Fourier transform with respect to x' only, namely: $\mathcal{F}'f(\xi', x_d) = (\mathcal{F}_{x'\to\xi'}f)(\xi', x_d)$. With this notation, $\mathcal{F}'w^\varepsilon(\xi', x_d)$ satisfies

$$\partial^2_{x_d}(\mathcal{F}'w^\varepsilon)(\xi', x_d) + (n_\varepsilon^2(x_d) - \xi'^2 + i\alpha_\varepsilon\varepsilon)(\mathcal{F}'w^\varepsilon)(\xi', x_d) = (\mathcal{F}'f)(\xi', x_d).$$

Let $\omega_\pm^\varepsilon(\xi') = \sqrt{\xi'^2 - n_\pm^2 + i\alpha_\varepsilon\varepsilon}$, where we choose the square root with a nonnegative real part. In the sequel, we will often write $\omega_\pm^\varepsilon$ instead of $\omega_\pm^\varepsilon(\xi')$.

The calculation that is detailed in Appendix A leads to the following formula for the kernel of the resolvent:

$$\mathcal{F}'w^\varepsilon(\xi', x_d) = \int R^\varepsilon(\xi', x_d, y_d)\mathcal{F}'f(\xi', y_d)dy_d$$

where

$$\begin{aligned}
R^\varepsilon(\xi', s, t) &= \mathbf{1}_{\{s>\frac{1}{\varepsilon}, t>\frac{1}{\varepsilon}\}} \frac{1}{2\omega_+^\varepsilon}\Big(e^{-\omega_+^\varepsilon|s-t|} + \frac{(\omega_-^\varepsilon - \omega_+^\varepsilon)^2}{[n^2]} e^{-\omega_+^\varepsilon(|s-\frac{1}{\varepsilon}|+|t-\frac{1}{\varepsilon}|)}\Big) \\
&+ \mathbf{1}_{\{s>\frac{1}{\varepsilon}, t<\frac{1}{\varepsilon}\}} \frac{\omega_+^\varepsilon - \omega_-^\varepsilon}{[n^2]} e^{-\omega_+^\varepsilon|s-\frac{1}{\varepsilon}|+\omega_-^\varepsilon|t-\frac{1}{\varepsilon}|} \\
&+ \mathbf{1}_{\{s<\frac{1}{\varepsilon}, t>\frac{1}{\varepsilon}\}} \frac{\omega_+^\varepsilon - \omega_-^\varepsilon}{[n^2]} e^{-\omega_-^\varepsilon|s-\frac{1}{\varepsilon}|+\omega_+^\varepsilon|t-\frac{1}{\varepsilon}|} \\
&+ \mathbf{1}_{\{s<\frac{1}{\varepsilon}, t<\frac{1}{\varepsilon}\}} \frac{1}{2\omega_-^\varepsilon}\Big(\frac{(\omega_-^\varepsilon - \omega_+^\varepsilon)^2}{[n^2]} e^{-\omega_-^\varepsilon(|s-\frac{1}{\varepsilon}|+|t-\frac{1}{\varepsilon}|)} + e^{-\omega_-^\varepsilon|s-t|}\Big) \\
&= (R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon + R_4^\varepsilon + R_5^\varepsilon + R_6^\varepsilon)(\xi', s, t). \qquad (4.7.4)
\end{aligned}$$

With the help of this formula, we obtain

$$w^\varepsilon = \sum_{k=1}^{6} w_k^\varepsilon$$

where

$$\mathcal{F}'w_k^\varepsilon(\xi', x_d) = \int R_k^\varepsilon(\xi', x_d, y_d)\mathcal{F}'f(\xi', y_d)dy_d.$$

The last part of the kernel R_6^ε will give the outgoing solution in the limit $\varepsilon \to 0$. We begin by proving that the contributions of the first five terms vanish when $\varepsilon \to 0$, the only difficult term to handle being the fifth term R_5^ε.

Let ϕ, $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then, $\mathcal{F}'\phi, \mathcal{F}'f \in \mathcal{C}_c^\infty(\mathbb{R}_{x_d}, \mathcal{S}(\mathbb{R}_{\xi'}^{d-1}))$. Let $M > 0$ be such that the supports of ϕ and f with respect to the x_d variable are contained in the interval $[-M, M]$. Then, if ε is small enough ($\varepsilon < \frac{1}{M}$), the first four terms vanish because of the truncation $x_d > 1/\varepsilon$. For $1 \leq k \leq 4$, we indeed have

$$\langle w_k^\varepsilon, \phi\rangle = \int R_k^\varepsilon(\xi', x_d, y_d)\mathcal{F}'f(\xi', y_d)\mathcal{F}'\phi(\xi', x_d)d\xi' dx_d dy_d = 0.$$

Let us now study the fifth term

$$\langle w_5^\varepsilon, \phi\rangle = \int \frac{1}{\omega_-^\varepsilon}\frac{(\omega_-^\varepsilon(\xi') - \omega_+^\varepsilon(\xi'))^2}{2[n^2]} e^{-\omega_-^\varepsilon(\xi')(\frac{2}{\varepsilon} - x_d - y_d)}\mathcal{F}'f(\xi', y_d)\mathcal{F}'\phi(\xi', x_d)d\xi' dx_d dy_d.$$

We have several difficulties to handle in the treatment of this term:

- first, the phase function is stationary at $\xi' = 0$,
- second, the phase function $\omega_-^\varepsilon(\xi') = \sqrt{\xi'^2 - n_-^2 + i\alpha_\varepsilon \varepsilon}$ is singular near $\xi'^2 = n_-^2$ when $\varepsilon \to 0$,
- third, the test function $(\omega_-^\varepsilon(\xi') - \omega_+^\varepsilon(\xi'))^2/\omega_-^\varepsilon(\xi')$ is both singular near $\xi'^2 = n_+^2$ and $\xi'^2 = n_-^2$ when $\varepsilon \to 0$.

Hence, we first decompose the previous integral with respect to size of ξ', in order to separate the stationary point and the singularities of ω_-^ε and ω_+^ε. Near the stationary point, since there is no singularity anymore, we apply a usual stationary phase theorem. When ξ' is far from 0, we have to treat the singularities of ω_-^ε and ω_+^ε. In that case, we write the test function as the sum of a function that is singular near $\xi'^2 = n_+^2$ and a function that is smooth near $\xi'^2 = n_+^2$. To estimate the latter, we only make integrations by parts, the phase function being non-stationary here. To estimate the part with ω_+^ε as test function, we decompose it into two parts: an integral over $|\xi'^2 - n_+^2| \leq \varepsilon^\delta$ and an integral over $|\xi'^2 - n_+^2| \geq \varepsilon^\delta$. The first integral is directly bounded by $C\varepsilon^{\delta/2}$ and we estimate the second one by making integrations by parts.

Let $\chi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ be a truncation function such that $\chi_0(r) = 0$ for $|r| \geq 1$ and $\chi_0(r) = 1$ for $|r| \leq 1/2$. Let $\delta \in (0, 1)$. We define

$$\chi(\xi') = \chi_0\left(\frac{2|\xi'|^2}{n_+^2}\right).$$

Using the truncation function χ , we decompose the term w_5^ε into

$$\begin{aligned}
&\langle w_5^\varepsilon, \phi\rangle \\
&= \frac{1}{2[n^2]}\int \chi(\xi')\frac{(\omega_-^\varepsilon-\omega_+^\varepsilon)^2}{\omega_-^\varepsilon}e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}\mathcal{F}'f(\xi',y_d)\mathcal{F}'\phi(\xi',x_d) \\
&+ \frac{1}{2[n^2]}\int (1-\chi(\xi'))\frac{(\omega_-^\varepsilon-\omega_+^\varepsilon)^2}{\omega_-^\varepsilon}e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}\mathcal{F}'f(\xi',y_d)\mathcal{F}'\phi(\xi',x_d) \\
&= I_\varepsilon + II_\varepsilon.
\end{aligned}$$

First case: ξ' close to 0

We first bound the part I_ε. In this term, ω_-^ε and ω_+^ε are both smooth, hence we may directly use the complex stationary phase theorem 4.7.1. In our case, we apply this theorem with large parameter $\lambda = \frac{2}{\varepsilon} - x_d - y_d$ (for ε small enough, $\frac{2}{\varepsilon} - x_d - y_d \geq \frac{1}{\varepsilon}$) and with phase function $i\omega_-^\varepsilon(\xi')$. We denote $\varphi_\eta(\xi') = i\sqrt{\xi'^2 - n_-^2 + i\eta}$ (the phase function is $\varphi_{\alpha_\varepsilon\varepsilon}$). The function φ_η then depends smoothly on $\eta \in [0,1]$ when $\xi' \in \mathrm{supp}(\chi)$. Moreover, the only critical point of φ_η is $\xi' = 0$ and it satisfies

$$\mathrm{Det}(\varphi_\eta''(0)) = \frac{i^{d-1}}{\left(\sqrt{-n_-^2+i\eta}\right)^{d-1}}.$$

Hence, we can apply Theorem 4.7.1 to get the uniform bound (for ε small enough)

$$\begin{aligned}
|I_\varepsilon| &\leq C\int \frac{dx_d dy_d}{\left(\frac{2}{\varepsilon}-x_d-y_d\right)^{\frac{d-1}{2}}} \\
&\quad \times \sup_{\xi',|\alpha|\leq 2(d-1)}\left|\partial_{\xi'}^\alpha\Big(\frac{(\omega_-^\varepsilon-\omega_+^\varepsilon)^2}{\omega_-^\varepsilon}\chi(\xi')\mathcal{F}'f(\xi',y_d)\mathcal{F}'\phi(\xi',x_d)\Big)\right| \\
&\leq C\varepsilon^{\frac{d-1}{2}}.
\end{aligned}$$

Second case: ξ' far from 0

In this set, expanding the square in $\frac{(\omega_-^\varepsilon-\omega_+^\varepsilon)^2}{\omega_-^\varepsilon}$, we decompose the test function into two parts: a part that is smooth near $\xi'^2 = n_+^2$, corresponding to $\omega_-^\varepsilon + \frac{(\omega_+^\varepsilon)^2}{\omega_-^\varepsilon}$ and a part that is singular near $\xi'^2 = n_+^2$, corresponding to $-2\omega_+^\varepsilon$. We obtain

$$\begin{aligned} II_\varepsilon &= \frac{1}{2[n^2]}\int(1-\chi(\xi'))\big(\omega_-^\varepsilon + \frac{(\omega_+^\varepsilon)^2}{\omega_-^\varepsilon}\big)e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}\mathcal{F}'f(\xi',y_d)\mathcal{F}'\phi(\xi',x_d) \\ &\quad -\frac{1}{[n^2]}\int(1-\chi(\xi'))\omega_+^\varepsilon e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}\mathcal{F}'f(\xi',y_d)\mathcal{F}'\phi(\xi',x_d) \\ &= III_\varepsilon + IV_\varepsilon. \end{aligned}$$

1- Study of III_ε:

We first consider the part III_ε, where the test function is smooth near $\xi'^2 = n_+^2$. In order to treat the singularity of ω_-^ε, we make integrations by parts with respect to the x_d variable to make the term $\frac{e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}}{\omega_-^\varepsilon}$ appear:

$$III_\varepsilon = \int \frac{e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}}{2[n^2]\ \omega_-^\varepsilon}(1-\chi)\mathcal{F}'f(\xi',y_d)(\partial^2_{x_dx_d}(\mathcal{F}'\phi) + (\omega_+^\varepsilon)^2\mathcal{F}'\phi)$$

Next, we use the following formula

$$\frac{e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}}{\omega_-^\varepsilon} = \frac{1}{(\frac{2}{\varepsilon}-x_d-y_d)}\frac{\xi'}{|\xi'|^2}\cdot\nabla_{\xi'}e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}$$

to write

$$III_\varepsilon = \int \frac{e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}}{2[n^2](\frac{2}{\varepsilon}-x_d-y_d)}\nabla_{\xi'}\cdot\big(\frac{\xi'}{|\xi'|^2}(1-\chi)\mathcal{F}'f(\partial^2_{x_dx_d}\mathcal{F}'\phi + (\omega_+^\varepsilon)^2\mathcal{F}'\phi)\big)$$

Thus, using that the test function is bounded and the fact that $\left|\frac{2}{\varepsilon}-x_d-y_d\right| \geq \frac{1}{\varepsilon}$ for ε small enough, we directly obtain

$$|III_\varepsilon| \leq C\varepsilon.$$

2- Study of IV_ε:

In this term, the test function ω_+^ε is singular near $\xi'^2 = n_+^2$. We use the following truncation function: $\chi^\varepsilon(\xi') = \chi_0\left(\frac{|\xi'|^2-n_+^2}{\varepsilon^\delta}\right)$ to decompose IV_ε.

$$\begin{aligned} IV_\varepsilon &= -\frac{1}{[n^2]}\int(1-\chi(\xi'))(1-\chi^\varepsilon(\xi'))\omega_+^\varepsilon e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}\mathcal{F}'f(\xi',y_d)\mathcal{F}'\phi(\xi',x_d) \\ &\quad -\frac{1}{[n^2]}\int(1-\chi(\xi'))\chi^\varepsilon(\xi')\omega_+^\varepsilon e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}\mathcal{F}'f(\xi',y_d)\mathcal{F}'\phi(\xi',x_d) \\ &= V_\varepsilon + VI_\varepsilon. \end{aligned}$$

(a) ξ'^2 **far from** n_+^2 (at scale ε^δ)

Let us first study the term V_ε. We proceed as for the estimate of III_ε. We write

$$\begin{aligned} V_\varepsilon &= -\int \frac{e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}}{[n^2]\ \omega_-^\varepsilon}(1-\chi)(1-\chi^\varepsilon)\omega_+^\varepsilon(\xi')\mathcal{F}'f(\xi',y_d)\partial_{x_d}(\mathcal{F}'\phi) \\ &= -\int \frac{e^{-\omega_-^\varepsilon(\frac{2}{\varepsilon}-x_d-y_d)}}{[n^2](\frac{2}{\varepsilon}-x_d-y_d)}\nabla_{\xi'}\cdot\big(\frac{\xi'}{|\xi'|^2}(1-\chi)(1-\chi^\varepsilon)\omega_+^\varepsilon(\xi')\mathcal{F}'f\partial_{x_d}\mathcal{F}'\phi\big). \end{aligned}$$

We must handle two singular terms: the term involving $\nabla\omega_+^\varepsilon$ and the term involving $\nabla\chi^\varepsilon$. The second one is bounded by $C\varepsilon^{1-\delta}$. For the first one, we use the fact that, on the support of $1-\chi^\varepsilon$,

$$|\omega_+^\varepsilon(\xi')| = ((\xi'^2-n_+^2)^2+(\alpha_\varepsilon\varepsilon)^2)^{1/4} \geq \varepsilon^{\delta/2}.$$

Thus, $|\nabla\omega_+^\varepsilon| = \left|\frac{\xi'}{\omega_+^\varepsilon}\right| \leq C\varepsilon^{-\delta/2}$ on the support of $\mathcal{F}'f$, and we obtain

$$|V_\varepsilon| \leq C\varepsilon^{(1-\frac{\delta}{2})}.$$

(b) ξ'^2 close to n_+^2 (at scale ε^δ)

Finally, the estimate of the term VI_ε directly follows from the bound

$$|\omega_+^\varepsilon(\xi')| \leq (\varepsilon^{2\delta}+(\alpha_\varepsilon\varepsilon)^2)^{1/4} \quad \text{for } \xi' \in \text{supp}(\chi^\varepsilon).$$

We get

$$|VI_\varepsilon| \leq C(\varepsilon^{2\delta}+(\alpha_\varepsilon\varepsilon)^2)^{1/4}.$$

Thus, we obtain $\langle w_5^\varepsilon, \phi\rangle \to 0$ as $\varepsilon \to 0$.

There remains to compute the limit of the sixth contribution

$$\begin{aligned} \langle w_6^\varepsilon, \phi\rangle &= \int \frac{1}{2\omega_-^\varepsilon}e^{-\omega_-^\varepsilon|x_d-y_d|}\mathcal{F}'f(\xi',y_d)\mathcal{F}'\phi(\xi',x_d)d\xi'dx_ddy_d \\ &= \int \hat{\phi}(\xi)\hat{f}(\xi)\Big(\int e^{-ix_d\xi_d}\frac{e^{-\omega_-^\varepsilon|x_d|}}{2\omega_-^\varepsilon}dx_d\Big)d\xi \end{aligned}$$

using Parseval's formula. But, a direct computation gives

$$\int e^{-ix_d\xi_d}\frac{e^{-\omega_-^\varepsilon|x_d|}}{2\omega_-^\varepsilon}dx_d = \frac{1}{\xi^2-n_-^2+i\alpha_\varepsilon\varepsilon},$$

so

$$\langle w_6^\varepsilon, \phi\rangle = \int \frac{\hat{f}(\xi)\hat{\phi}(\xi)}{\xi^2-n_-^2+i\alpha_\varepsilon\varepsilon}d\xi \underset{\varepsilon\to 0}{\longrightarrow} \int \frac{\hat{f}(\xi)\hat{\phi}(\xi)}{\xi^2-n_-^2+i0}d\xi$$

i.e. w_6^ε converges to the outgoing solution to the Helmholtz equation with constant coefficient (4.3.2).

In conclusion, we have obtained that w^ε converges weakly to the outgoing solution to (4.7.2).

4.7.2 Proof of the radiation condition for $\mu^{\partial_\pm}$ in the case of two homogeneous media

This section is devoted to the proof of Proposition 4.5.6, *i.e.* $\mu^{\partial_\pm}=0$, in the case of two homogeneous media.

Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^{d-1}_{x'}\times\mathbb{R}^{d-1}_{\xi'})$. We want to prove that

$$\int \theta(x',\xi')d\mu^{\partial_\pm}=0.$$

In order to do this, let $\chi_0\in\mathcal{C}_c^\infty(\mathbb{R})$ be again a truncation function such that $\chi_0(r)=0$ for $|r|\geq 1$, $\chi_0(r)=1$ for $|r|\leq 1/2$, and let $\eta>0$ be a small parameter. We denote $\Psi_\eta(x,\xi)=\theta(x',\xi')\widehat{\chi_0}(\xi_d)\chi_0\Big(\frac{x_d-1}{\eta}\Big)$. Then, by definition of the Wigner measure μ, we have for any fixed $\eta>0$

$$\lim_{\varepsilon\to 0}\langle Op_\varepsilon^w(\Psi_\eta)u^\varepsilon,u^\varepsilon\rangle=\int \Psi_\eta(x,\xi)d\mu.$$

Moreover, since $\mu=\mathbf{1}_{\{x_d>1\}}\mu_+ +\mathbf{1}_{\{x_d<1\}}\mu_- +\delta(x_d-1)\otimes\delta(\xi_d)\otimes(\mu^{\partial_+}+\mu^{\partial_-})$, we get, passing to the limit $\eta\to 0$,

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\langle Op_\varepsilon^w(\Psi_\eta)u^\varepsilon,u^\varepsilon\rangle=\|\chi_0\|_{L^1}\int\theta(x',\xi')d(\mu^{\partial_+}+\mu^{\partial_-}),\tag{4.7.5}$$

where $\|\chi_0\|_{L^1}\in[1,2]$. Indeed,

$$\begin{aligned}\int\Psi_\eta(x,\xi)d\mu &= \int\theta(x',\xi')\widehat{\chi_0}(\xi_d)\chi_0\Big(\frac{x_d-1}{\eta}\Big)d\left(\mathbf{1}_{\{x_d>1\}}\mu_+ +\mathbf{1}_{\{x_d<1\}}\mu_-\right)\\ &\quad +\widehat{\chi_0}(0)\int\theta(x',\xi')d(\mu^{\partial_+}+\mu^{\partial_-})\end{aligned}$$

and $\chi_0\Big(\frac{x_d-1}{\eta}\Big)\to 0$ as $\eta\to 0$ almost everywhere on the support of $\mathbf{1}_{\{x_d>1\}}\mu_+ + \mathbf{1}_{\{x_d<1\}}\mu_-$, from which it follows that the first term in the right hand side converges to 0 when $\eta\to 0$. Using the equality (4.7.5), our problem first reduces to proving that

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\langle Op_\varepsilon^w(\Psi_\eta)u^\varepsilon,u^\varepsilon\rangle=0.\tag{4.7.6}$$

Indeed, relation (4.7.6) readily implies $\mu^{\partial +} + \mu^{\partial -} = 0$, from which we deduce $\mu^{\partial +} = \mu^{\partial -} = 0$, using the fact that $\mu^{\partial +}$ and $\mu^{\partial -}$ are nonnegative.
Alternatively, the fact that $\mu^{\partial +} + \mu^{\partial -} = 0$ implies $\mu^{\partial +} = \mu^{\partial -} = 0$ can be seen using the localisation property satisfied by $\mu^{\partial \pm}$. Indeed, we have $\mathrm{supp}(\mu^{\partial \pm}) \subset \{\xi'^2 = n_\pm^2\}$, hence $\mu^{\partial -}$ and $\mu^{\partial +}$ have disjoint supports. Note that, using this last property, at some point of our proof, we will study separately the two measures $\mu^{\partial -}$ and $\mu^{\partial +}$, choosing first the test function $\theta(x', \xi')$ with support close to $\xi'^2 = n_-^2$ and then close to $\xi'^2 = n_+^2$.

Let us make two other reductions. First, (u^ε) being uniformly bounded in $\dot{B}^*$, it suffices to prove

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0} \| Op_\varepsilon^w(\Psi_\eta) u^\varepsilon \|_{\dot{B}} = 0. \tag{4.7.7}$$

Last, we may assume, using a density argument, that the source f is smooth: $\mathcal{F}' f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Indeed, we have, for $\lambda > 0$ (the space X_λ is defined in Proposition 4.3.4),

$$|\langle Op_\varepsilon^w(\Psi_\eta) u^\varepsilon, u^\varepsilon \rangle| \le C \|f\|_{\dot{B}}^2 \|\Psi_\eta\|_{X_\lambda} \le C \|f\|_{\dot{B}}^2,$$

(where we have used that $\|\Psi_\eta\|_{X_\lambda} \le C$ uniformly with respect to η). This is our last reduction.

In the sequel, we will actually prove the following stronger result.

Proposition 4.7.3 *There exists $\eta_0 > 0$ such that for all $\eta < \eta_0$,*

$$\| Op_\varepsilon^w(\Psi_\eta) u^\varepsilon \|_{\dot{B}} = O(\varepsilon^\infty).$$

The end of this section is devoted to the proof of Proposition 4.7.3.

We use the following explicit formula for u^ε, where the kernel of the resolvent R^ε is given by (4.7.4):

$$\begin{aligned} u^\varepsilon(y) &= \frac{1}{\varepsilon^{\frac{d-1}{2}}} \int e^{i \frac{y' \cdot \zeta'}{\varepsilon}} R^\varepsilon\Big(\zeta', \frac{y_d}{\varepsilon}, z_d\Big) \mathcal{F}' f(\zeta', z_d) dz_d d\zeta' \\ &= \sum_{k=1}^{6} u_k^\varepsilon(y), \end{aligned}$$

with, for $k = 1, \ldots, 6$,

$$u_k^\varepsilon(y) = \frac{1}{\varepsilon^{\frac{d-1}{2}}} \int e^{i \frac{y' \cdot \zeta'}{\varepsilon}} R_k^\varepsilon\Big(\zeta', \frac{y_d}{\varepsilon}, z_d\Big) \mathcal{F}' f(\zeta', z_d) dz_d d\zeta'.$$

We recall that

$$\| Op_\varepsilon^w(\Psi_\eta) u^\varepsilon \|_{\dot{B}} = \sum_{j \in \mathbb{Z}} 2^{j/2} \left(\int_{C(j)} |Op_\varepsilon^w(\Psi_\eta) u^\varepsilon(x)|^2 \, dx \right)^{1/2}.$$

In order to bound $\|Op_\varepsilon^w(\Psi_\eta)u^\varepsilon\|_{\dot{B}}$, our strategy is the following: we look for an estimate of $Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x)$ in L^∞ depending on the annulus $C(j)$ where x lies.

Lemma 4.7.4 *Let $M > 0$ be such that supp$(\mathcal{F}'f) \subset B(0,M)$ and supp$(\theta) \subset B(0,M)$, $B(0,M)$ being the ball of radius M centered at the origin.*
Let us denote $J = \lceil \log_2 M + 3\rceil$. We have the following properties:

(i) For all $j < 0$, for all $x \in C(j)$, $Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x) = 0$.

(ii) For all $j \geq J$, for all integer $p \geq 0$, for all $x \in C(j)$, there exists an integer $k > \frac{d+1}{2}$ such that

$$|Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x)| \leq C\varepsilon^p 2^{-jk},$$

where C is a constant independent of j, x.

(iii) For all $0 \leq j < J$, for all integer $p \geq 0$, for all $x \in C(j)$,

$$|Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x)| \leq C\varepsilon^p,$$

where C is a constant independent of j, x.

Remark: *Note that if ε is small enough ($\frac{1}{\varepsilon} > M$), the first, second and fourth terms u_1^ε, u_2^ε, u_4^ε vanish (because of the truncation $\mathbf{1}_{\{z_d > 1/\varepsilon\}}$ in R_1^ε, R_2^ε, R_4^ε).*

Before proving this lemma, let us show how it implies Proposition 4.7.3. Using Lemma 4.7.4, we obtain the following bound: for all integer p, there exists an integer $k > \frac{d+1}{2}$ such that

$$\begin{aligned} \|Op_\varepsilon^w(\Psi_\eta)u^\varepsilon\|_{\dot{B}} &\leq C\left(\sum_{j=0}^{J-1} \varepsilon^p + \sum_{j\geq J} \varepsilon^p 2^{j\left(\frac{d+1}{2}-k\right)}\right) \\ &\leq C\varepsilon^p \end{aligned}$$

where we have used that the series $\sum_{j\geq J} 2^{j\left(\frac{d+1}{2}-k\right)}$ converges for $k > \frac{d+1}{2}$.

There now remains to prove Lemma 4.7.4.

Proof of Lemma 4.7.4.
In the sequel, we omit the coefficient $\frac{1}{(2\pi)^d}$ in front of the integral defining $Op_\varepsilon^w(\Psi_\eta)u^\varepsilon$. We have, performing the integration with respect to the ξ_d

variable,

$$\begin{aligned}
&Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x)\\
&\quad= \frac{1}{\varepsilon^{d+\frac{d-1}{2}}}\int e^{i\frac{(x-y)\cdot\xi}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}} R^\varepsilon\Big(\zeta', \frac{y_d}{\varepsilon}, z_d\Big)\mathcal{F}'f(\zeta',z_d)\theta\Big(\frac{x'+y'}{2},\xi'\Big)\\
&\qquad\qquad \times\widehat{\chi_0}(\xi_d)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi d\zeta' dz_d\\
&\quad= \frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}} R^\varepsilon\Big(\zeta', \frac{y_d}{\varepsilon}, z_d\Big)\mathcal{F}'f(\zeta',z_d)\theta\Big(\frac{x'+y'}{2},\xi'\Big)\\
&\qquad\qquad \times\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi' d\zeta' dz_d. \qquad (4.7.8)
\end{aligned}$$

Proof of point (i).

Let $j<0$ and $x\in C(j)$. Then $|x_d|\le 2^j\le\frac12$. For such a value of x_d, $y_d\mapsto\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)$ has support in $\{|y_d|\le\frac12+\varepsilon\}$ and $y_d\mapsto\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)$ has support in $\{y_d\ge\frac32-2\eta\}$. Hence, if η and ε are small enough, we have

$$\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)=0,$$

which gives, using the formula (4.7.8),

$$Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x)=0.$$

Proof of point (ii).

Let $j\ge J$, and $x\in C(j)$. First of all, if $|x_d|\ge 2$, then one can easily check that for η and ε small enough, we have

$$\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)=0,$$

which gives

$$Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x)=0.$$

Now, assume $|x_d|\le 2$. In the previous integral, we have $\Big|\frac{x'+y'}{2}\Big|\le M$. Hence,

$$\begin{aligned}
&|x'|\ge|x|-|x_d|\ge 2^{j-1}-2\ge 2^{j-2},\\
&|x'-y'|=|2x'-(x'+y')|\ge 2^{j-1}-|x'+y'|\ge 2^{j-1}-2M.
\end{aligned}$$

Thus, since $M\le 2^{j-3}$ for $j\ge J$, we get $|x'-y'|\ge 2^{j-2}$. This allows us to use a non-stationary phase method to bound $Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x)$ for $x\in C(j)$. Indeed, if we denote

$$L=\frac{x'-y'}{|x'-y'|^2}\cdot\nabla_{\xi'},$$

we get

$$e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}} = \varepsilon L e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}},$$

which implies, for any $k \in \mathbb{N}$,

$$Op^w_\varepsilon(\Psi_\eta)u^\varepsilon(x) = \varepsilon^{k-\frac{3d-1}{2}} \int e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}} R^\varepsilon\Big(\zeta', \frac{y_d}{\varepsilon}, z_d\Big)\mathcal{F}'f(\zeta', z_d)$$
$$\times ({}^tL)^k\left(\theta\Big(\frac{x'+y'}{2},\xi'\Big)\right)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d.$$

As a consequence, using that $R_1^\varepsilon = R_2^\varepsilon = R_4^\varepsilon = 0$ for ε small enough, that R_3^ε is uniformly bounded with respect to ε, and that R_5^ε, R_6^ε are bounded by $C/\sqrt{\alpha_\varepsilon\varepsilon}$, we get

$$\begin{aligned} |Op^w_\varepsilon(\Psi_\eta)u^\varepsilon(x)| &\leq C\frac{\varepsilon^{k-\frac{3d-1}{2}}}{2^{(j-2)k}\sqrt{\alpha_\varepsilon\varepsilon}} \\ &\leq C\frac{\varepsilon^{k-\frac{3d+\gamma}{2}}}{2^{(j-2)k}}. \end{aligned}$$

Now, let $p \in \mathbb{N}$. We choose $k > p + \frac{3d+\gamma}{2}$ (which implies that $k > \frac{d+1}{2}$) and we obtain,

$$|Op^w_\varepsilon(\Psi_\eta)u^\varepsilon(x)| \leq C_p\varepsilon^p 2^{-jk}.$$

Proof of point (iii).

This is the most difficult case. We are left with the terms with $0 \leq j \leq J$. Here, we treat separately the measures μ^{∂_-} and μ^{∂_+}. Indeed, in order to prove that $\mu^{\partial_-} = 0$ (respectively $\mu^{\partial_+} = 0$), using that μ^{∂_-} (resp. μ^{∂_+}) is supported in $\{\xi'^2 = n_-^2\}$ (resp. $\{\xi'^2 = n_+^2\}$), it suffices to choose a test function θ supported near $\{\xi'^2 = n_-^2\}$ (resp. $\{\xi'^2 = n_+^2\}$).

Let a be a small positive parameter such that $a \leq \min\left(\frac{n_+^2}{2}, \frac{[n^2]}{2}\right)$.

We begin with the study of μ^{∂_-}. Let us explain our strategy (it is the same for the study of μ^{∂_+}). We have to treat the singularity of the root $\omega_-^\varepsilon = \sqrt{\xi'^2 - n_-^2 + i\alpha_\varepsilon\varepsilon}$. In order to do this, we make a contour deformation in the complex plane. If we denote $z = \xi'^2 - n_-^2 + i\alpha_\varepsilon\varepsilon$, the expression that we have to study corresponds to an integral over $\mathcal{I}m\, z = \alpha_\varepsilon\varepsilon$ ($\mathcal{R}e\, z$ being bounded). Using almost-analytic extensions and the Green-Riemann formula, we write it as the sum of an integral over $\mathcal{I}m\, z = \beta$, where β is a positive constant, and an integral over $\alpha_\varepsilon\varepsilon \leq \mathcal{I}mz \leq \beta$. In the integral over $\mathcal{I}m\, z = \beta$, the root $\sqrt{\xi'^2 - n_-^2 + i\beta}$ is no longer singular, thus we can apply a usual non-stationary phase theorem. In order to treat the integral over $\alpha_\varepsilon\varepsilon \leq \mathcal{I}mz \leq \beta$, we separate the sets $|\mathcal{I}m\ z| \leq \varepsilon^\delta$ and $|\mathcal{I}m\ z| \geq \varepsilon^\delta$. For

$|\mathcal{I}m\ z| \leq \varepsilon^{\delta}$, we use the property of almost-analytic extensions (see (4.7.9)). For $|\mathcal{I}m\ z| \geq \varepsilon^{\delta}$, we use that $\sqrt{z}$ is bounded from below by $\varepsilon^{\delta/2}$, so that each integration by part gives a power of $\varepsilon^{1-\delta/2}$.

First case: study of μ^{∂_-} (θ is supported close to $\xi'^2 = n_-^2$)

We assume that $\text{supp}(\theta) \subset \{|\xi'^2 - n_-^2| \leq a/2\}$. We consider separately the contributions due to $|\zeta'|$ close to, or far from, n_-. For that purpose, we define

$$\chi_-(\zeta') = \chi_0(\frac{\zeta'^2 - n_-^2}{a}).$$

We write

$$\begin{aligned} Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x) &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}} R^\varepsilon\Big(\zeta', \frac{y_d}{\varepsilon}, z_d\Big) \\ &\qquad \times \mathcal{F}' f(\zeta', z_d)\,(1-\chi_-(\zeta'))\,\theta\Big(\frac{x'+y'}{2}, \xi'\Big) \\ &\qquad \times \chi_0\Big(\frac{x_d - y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d \\ &+ \frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}} R^\varepsilon\Big(\zeta', \frac{y_d}{\varepsilon}, z_d\Big) \\ &\qquad \times \mathcal{F}' f(\zeta', z_d)\chi_-(\zeta')\theta\Big(\frac{x'+y'}{2}, \xi'\Big) \\ &\qquad \times \chi_0\Big(\frac{x_d - y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d \\ &= I^{\varepsilon,-}(x) + II^{\varepsilon,-}(x). \end{aligned}$$

<u>1- Contribution of $|\zeta'|$ far from n_-</u>

Let us first study the part I^ε where the root ω_-^ε is not singular. The gradient of the phase function $(x'-y')\cdot\xi' + y'\cdot\zeta' + i\omega_-^\varepsilon(\zeta')|y_d - \varepsilon z_d|$ with respect to the y'-variable is $\zeta' - \xi'$. Since we have

$$|\zeta' - \xi'| \geq \min\left(\sqrt{n_-^2 + a} - \sqrt{n_-^2 + \frac{a}{2}}, \sqrt{n_-^2 - \frac{a}{2}} - \sqrt{n_-^2 - a}\right),$$

the operator

$$L = \frac{\zeta' - \xi'}{|\zeta' - \xi'|^2}\cdot\nabla_{y'}$$

is well-defined. Moreover it satisfies

$$\varepsilon L\left(e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}}\right) = e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}}.$$

Hence, we get, for all $p \in \mathbb{N}$,

$$\begin{aligned} I^{\varepsilon,-}(x) &= \varepsilon^{p-\frac{3d-1}{2}} \int_{y_d<1} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}-\frac{\omega_-^\varepsilon(\zeta')}{\varepsilon}|y_d-\varepsilon z_d|} R^\varepsilon\left(\zeta',\frac{y_d}{\varepsilon},z_d\right) \\ &\quad \times \mathcal{F}'f(\zeta',z_d)2\omega_-^\varepsilon(\zeta')\big(1-\chi_-(\zeta')\big)({}^tL)^p\left(\theta\Big(\frac{x'+y'}{2},\xi'\Big)\right) \\ &\quad \times \chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d, \end{aligned}$$

and

$$|I^{\varepsilon,-}| \le C\varepsilon^{p-\frac{3d+\gamma}{2}}.$$

Thus, for such θ,

$$I^{\varepsilon,-}(x) = O(\varepsilon^\infty).$$

2- Contribution of $|\zeta'|$ close to n_-

Now, we are left with the part $II^{\varepsilon,-}$, for which we have to treat the singularity of ω_-^ε, that appears both in the phase and in the test functions, near $|\zeta'| = n_-$. In that case, we study separately the contributions due to R_6^ε, R_5^ε, and R_3^ε. We denote them $II_6^{\varepsilon,-}$, $II_5^{\varepsilon,-}$ and $II_3^{\varepsilon,-}$ respectively. We begin with the term $II_6^{\varepsilon,-}$.

a- Estimate of $II_6^{\varepsilon,-}$

We have

$$\begin{aligned} &II_6^{\varepsilon,-}(x) \\ &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}} \int_{y_d<1} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}-\frac{\omega_-^\varepsilon(\zeta')}{\varepsilon}|y_d-\varepsilon z_d|} \frac{\mathcal{F}'f(\zeta',z_d)}{2\omega_-^\varepsilon(\zeta')}\chi_-(\zeta') \\ &\qquad \times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d. \end{aligned}$$

We first make the polar change of variables $\zeta' = \rho\omega'$ and then $t = \rho^2 - n_-^2$. We get

$$\begin{aligned} &II_6^{\varepsilon,-}(x) \\ &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}} \int_{y_d<1}\int_{\mathbb{S}^{d-2}_{\omega'}} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{t+n_-^2}-\frac{\sqrt{t+i\alpha_\varepsilon\varepsilon}}{\varepsilon}|y_d-\varepsilon z_d|} \frac{g(t,\omega',z_d)}{2\sqrt{t+i\alpha_\varepsilon\varepsilon}} \\ &\qquad \times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dtd\omega'dyd\xi'dz_d, \end{aligned}$$

where $g(t,\omega',z_d) = \mathcal{F}'f\left(\sqrt{t+n_-^2}\omega',z_d\right)\chi_0\left(\frac{t}{a}\right)(t+n_-^2)^{(d-3)/2}$ is smooth and compactly supported with respect to t.

Next, in order to make a contour deformation to avoid the singularity of ω_-^ε, we use the almost-analytic extension of g, $\tilde{g}(t,\omega',z_d)$, defined as in the following proposition. This object was first introduced by Hörmander [Hör2] (also see for instance [DG], [DS]).

Proposition 4.7.5 *Let $g \in \mathcal{C}_c^\infty(\mathbb{R})$. Then, there exists a function $\tilde{g} \in \mathcal{C}_c^\infty(\mathbb{C})$, such that*

$$\tilde{g}_{|\mathbb{R}} = g, \qquad \left|\frac{\partial \tilde{g}}{\partial \bar{z}}(z)\right| \le C_N |\mathcal{I}m\, z|^N,\ N \in \mathbb{N}. \tag{4.7.9}$$

Since $\tilde{g}$ may be constructed as follows

$$\tilde{g}(t+is) = \sum_{n=0}^{\infty} i^n \partial_t^n g(t) \frac{s^n}{n!} \chi(\lambda_n s),$$

for an appropriate sequence (λ_n), the behavior (smoothness and support) of $\tilde{g}$ with respect to the variables ω' and z_d is not modified. We get

$$\begin{aligned} & II_6^{\varepsilon,-}(x) \\ &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}} \int_{y_d<1} \int_{\mathbb{S}_{\omega'}^{d-2}} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon} + i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{t+n_-^2} - \frac{\sqrt{t+i\alpha_\varepsilon\varepsilon}}{\varepsilon}|y_d-\varepsilon z_d|} \frac{\tilde{g}(t,\omega',z_d)}{2\sqrt{t+i\alpha_\varepsilon\varepsilon}} \\ &\qquad \times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta_s}\Big) dt d\omega' dy d\xi' dz_d, \end{aligned}$$

Now, we apply the Green-Riemann theorem on the following set of the complex plane: $\Omega = \{z = t+is \ /\ t \in [-a,a],\ s \in [\alpha_\varepsilon \varepsilon, \beta]\}$. We also denote

$$\begin{aligned} \Gamma_{\alpha_\varepsilon\varepsilon} &= \{t + i\alpha_\varepsilon\varepsilon \ /\ t \in [-a,a]\}, \\ \Gamma_\beta &= \{t + i\beta \ /\ t \in [-a,a]\}. \end{aligned}$$

The complex version of the Green-Riemann formula is: for all $G \in \mathcal{C}^1$,

$$\int_{\partial\Omega} G(z)dz = 2i \int_\Omega \frac{\partial}{\partial \bar{z}} G(z) d\bar{z} \wedge dz$$

(where $d\bar{z} \wedge dz = 2idtds$ if $z = t+is$).
This yields, using that a is outside the support of $\tilde{g}$ with respect to the variable t,

$$
\begin{aligned}
II_6^{\varepsilon,-}(x) &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int_{y_d<1}\int_{\Gamma_\beta} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{z-i\alpha_\varepsilon\varepsilon+n_-^2}-\frac{\sqrt{z}}{\varepsilon}|y_d-\varepsilon z_d|} \\
&\qquad\times\frac{\tilde{g}(z-i\alpha_\varepsilon\varepsilon,\omega',z_d)}{2\sqrt{z}}\theta\Big(\frac{x'+y'}{2},\xi'\Big) \\
&\qquad\times\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dzd\omega'd\xi'dz_d \\
&\quad+\frac{2i}{\varepsilon^{\frac{3d-1}{2}}}\int_{y_d<1}\int_{\Omega}\frac{e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{z-i\alpha_\varepsilon\varepsilon+n_-^2}-\frac{\sqrt{z}}{\varepsilon}|y_d-\varepsilon z_d|}}{2\sqrt{z}} \\
&\qquad\times\frac{\partial}{\partial\bar{z}}\tilde{g}(z-i\alpha_\varepsilon\varepsilon,\omega',z_d)\theta\Big(\frac{x'+y'}{2},\xi'\Big) \\
&\qquad\times\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)d\mathcal{A}d\omega'd\xi'dz_d \\
&= III_6^{\varepsilon,-}(x)+IV_6^{\varepsilon,-}(x)
\end{aligned}
$$

where we used that $\frac{e^{i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{z-i\alpha_\varepsilon\varepsilon+n_-^2}-\frac{\sqrt{z}}{\varepsilon}|y_d-\varepsilon z_d|}}{2\sqrt{z}}$ is holomorphic in an open set containing Ω when $\varepsilon>0$ is fixed.

We first study the term $III_6^{\varepsilon,-}$. We write

$$
\begin{aligned}
III_6^{\varepsilon,-}(x) &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int_{y_d<1} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{t+i\beta-i\alpha_\varepsilon\varepsilon+n_-^2}-\frac{\sqrt{t+i\beta}}{\varepsilon}|y_d-\varepsilon z_d|} \\
&\qquad\times\frac{\tilde{g}(t+i\beta-i\alpha_\varepsilon\varepsilon,\omega',z_d)}{2\sqrt{t+i\beta}}\theta\Big(\frac{x'+y'}{2},\xi'\Big) \\
&\qquad\times\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta_s}\Big)dtd\omega'd\xi'dz_d
\end{aligned}
$$

In this integral, the phase is non-stationary with respect to t. Indeed, if we denote $\varphi(t)=iy'\cdot\omega'\sqrt{t+i\beta-i\alpha_\varepsilon\varepsilon+n_-^2}-\sqrt{t+i\beta}|y_d-\varepsilon z_d|$ (we consider $\alpha_\varepsilon\varepsilon$, β, $y'\cdot\omega'$ and $|y_d-\varepsilon z_d|$ as parameters, which lie in a compact set), we have

$$
\partial_t\varphi(t)=\frac{iy'\cdot\omega'}{\sqrt{t+i\beta-i\alpha_\varepsilon\varepsilon+n_-^2}}-\frac{|y_d-\varepsilon z_d|}{\sqrt{t+i\beta}}.
$$

On the one hand, if we denote $M_1=2^J+2M$, we have

$$
\left|\frac{iy'\cdot\omega'}{\sqrt{t+i\beta-i\alpha_\varepsilon\varepsilon+n_-^2}}\right|\le\frac{M_1}{(n_-^2-a)^{1/2}},
$$

on the other hand,

$$
\left|\frac{|y_d-\varepsilon z_d|}{\sqrt{t+i\beta}}\right|\ge\frac{1}{2\sqrt{\beta}}.
$$

We choose β small enough such that $\frac{M_1}{(n_-^2-a)^{1/2}} \leq \frac{1}{4\sqrt{\beta}}$. With this choice, we get

$$|\partial_t \varphi(t)| \geq \frac{1}{2\sqrt{\beta}}.$$

Moreover, the derivatives of φ are uniformly bounded. To conclude, we use the non-stationary phase theorem 4.7.2, thus obtaining

$$III_6^{\varepsilon,-} = O(\varepsilon^\infty).$$

We end the proof by studying the term $IV_6^{\varepsilon,-}$. We decompose this term into two parts, according to the relative size of $|\mathcal{I}m(z)|$ with respect to ε^δ, where $\delta > 0$ is a small parameter to be chosen later. We still use χ_0 as truncation function. We write

$$\begin{aligned} IV_6^{\varepsilon,-}(x) &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}} \int_{y_d<1} \int_\Omega \frac{e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon} + i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{z-i\alpha_\varepsilon\varepsilon+n_-^2} - \frac{\sqrt{z}}{\varepsilon}|y_d-\varepsilon z_d|}}{2\sqrt{z}} \\ &\quad \times \frac{\partial}{\partial \bar{z}} \tilde{g}(z-i\alpha_\varepsilon\varepsilon, \omega', z_d)\chi_0\left(\frac{\mathcal{I}m(z)}{\varepsilon^\delta}\right) \\ &\quad \times \theta\Big(\frac{x'+y'}{2}, \xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big) dz\wedge d\bar{z}d\omega' d\xi' dz_d \\ &+ \frac{1}{\varepsilon^{\frac{3d-1}{2}}} \int_{y_d<1} \int_\Omega \frac{e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon} + i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{z-i\alpha_\varepsilon\varepsilon+n_-^2} - \frac{\sqrt{z}}{\varepsilon}|y_d-\varepsilon z_d|}}{2\sqrt{z}} \\ &\quad \times \frac{\partial}{\partial \bar{z}} \tilde{g}(z-i\alpha_\varepsilon\varepsilon, \omega', z_d)(1-\chi_0)\left(\frac{\mathcal{I}m(z)}{\varepsilon^\delta}\right) \\ &\quad \times \theta\Big(\frac{x'+y'}{2}, \xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big) dz\wedge d\bar{z}d\omega' d\xi' dz_d \\ &= V_6^{\varepsilon,-}(x) + VI_6^{\varepsilon,-}(x). \end{aligned}$$

Now, to estimate $V_6^{\varepsilon,-}$, we use the property (4.7.9) of the almost-analytic extension $\tilde{g}$. We readily get

$$|V_6^{\varepsilon,-}(x)| \leq C\varepsilon^{\delta N - \frac{3d+\gamma}{2}}. \tag{4.7.10}$$

For the term $VI_6^{\varepsilon,-}$, we write

$$\begin{aligned} VI_6^{\varepsilon,-}(x) &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}} \int_{y_d<1} \int_{\mathbb{S}^{d-2}_{\omega'}} \frac{e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon} + i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{t+is-i\alpha_\varepsilon\varepsilon+n_-^2} - \frac{\sqrt{t+is}}{\varepsilon}|y_d-\varepsilon z_d|}}{2\sqrt{t+is}} \\ &\quad \times \frac{\partial \tilde{g}}{\partial \bar{z}}(t+is-i\alpha_\varepsilon\varepsilon, \omega', z_d)(1-\chi_0)\left(\frac{s}{\varepsilon^\delta}\right) \\ &\quad \times \theta\Big(\frac{x'+y'}{2}, \xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big) dt ds d\omega' dy d\xi' dz_d \end{aligned}$$

and make integrations by parts with respect to the variable t. Since $\tilde{g}$ is compactly supported and a is outside the support of $\tilde{g}$, no boundary term appears. Let us denote

$$L = \frac{\frac{iy'\cdot\omega'}{\sqrt{t+is-i\alpha_\varepsilon\varepsilon+n_-^2}} - \frac{|y_d-\varepsilon z_d|}{\sqrt{t+is}}}{\left|\frac{iy'\cdot\omega'}{\sqrt{t+is-i\alpha_\varepsilon\varepsilon+n_-^2}} - \frac{|y_d-\varepsilon z_d|}{\sqrt{t+is}}\right|^2}\partial_t.$$

We have, using that $|t| \leq a$ and $s \geq \varepsilon^\delta$,

$$\left|\frac{iy'\cdot\omega'}{\sqrt{t+is-i\alpha_\varepsilon\varepsilon+n_-^2}}\right| \leq \frac{M_1}{\sqrt{n_-^2-a}} \quad \text{and} \quad \left|\frac{|y_d-\varepsilon z_d|}{\sqrt{t+is}}\right| \geq \frac{1}{2\sqrt{\beta}},$$

hence for β small enough,

$$\left|\frac{iy'\cdot\omega'}{\sqrt{t+is-i\alpha_\varepsilon\varepsilon+n_-^2}} - \frac{|y_d-\varepsilon z_d|}{\sqrt{t+is}}\right| \geq \frac{1}{4\sqrt{\beta}}. \tag{4.7.11}$$

Thus, L is well defined and we have

$$\varepsilon L\left(e^{i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{t+is-i\alpha_\varepsilon\varepsilon+n_-^2}-\frac{\sqrt{t+is}}{\varepsilon}|y_d-\varepsilon z_d|}\right) = e^{i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{t+is-i\alpha_\varepsilon\varepsilon+n_-^2}-\frac{\sqrt{t+is}}{\varepsilon}|y_d-\varepsilon z_d|}.$$

Hence, we get, for all $P \in \mathbb{N}$,

$$\begin{aligned} VI_6^{\varepsilon,-}(x) = \varepsilon^{P-\frac{3d-1}{2}} &\int_{y_d<1}\int_\Omega e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\omega'}{\varepsilon}\sqrt{t+is-i\alpha_\varepsilon\varepsilon+n_-^2}-\frac{\sqrt{t+is}}{\varepsilon}|y_d-\varepsilon z_d|} \\ &\times({}^tL)^P\left(\frac{1}{2\sqrt{t+is}}\frac{\partial}{\partial\bar{z}}\tilde{g}(t+is-i\alpha_\varepsilon\varepsilon,\omega',z_d)\right)(1-\chi_0)\left(\frac{s}{\varepsilon^\delta}\right) \\ &\times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta_s}\Big)dtdsd\omega'dyd\xi'dz_d \end{aligned} \tag{4.7.12}$$

Now, we use the fact that, in this integral, $s \geq \varepsilon^\delta$.

Lemma 4.7.6 *For all $P \in \mathbb{N}^*$, there exist $k_P \in \mathbb{N}^*$, $C_P > 0$ and $g_P \in \mathcal{C}_c^\infty$ such that*

$$\begin{aligned} &\left|({}^tL)^P\left(\frac{1}{2\sqrt{t+is}}\frac{\partial}{\partial\bar{z}}\tilde{g}(t+is-i\alpha_\varepsilon\varepsilon,\omega',z_d)\right)(1-\chi_0)\left(\frac{s}{\varepsilon^\delta}\right)\right| \\ &\qquad\qquad \leq C_P\varepsilon^{-k_P\delta}g_P(t,s,\omega',z_d). \end{aligned}$$

Proof. The term $\varepsilon^{-k_P\delta/2}$ comes from the fact that, on the support of $(1-\chi_0)\left(\frac{s}{\varepsilon^\delta}\right)$, we have $\left|\sqrt{t+is}\right| \geq \varepsilon^{\delta/2}$. □

From this lemma together with the estimates (4.7.10) and (4.7.12), we deduce, for all N and P,

$$|IV_6^{\varepsilon,-}(x)| \leq |V_6^\varepsilon(x)| + |VI_6^\varepsilon(x)| \leq C\varepsilon^{\delta N - \frac{3d+\gamma}{2}} + C\varepsilon^{P - \frac{3d-1}{2} - \frac{k_P\delta}{2}},$$

which implies that

$$IV_6^{\varepsilon,-}(x) = O(\varepsilon^\infty).$$

(Indeed, let $K \in \mathbb{N}^*$. We first choose $P > \frac{3d-1}{2} + 2K$, then δ sufficiently small such that $k_P\delta \leq 2K$, hence $P - \frac{3d-1}{2} - \frac{k_P\delta}{2} \geq K$ and finally, N large enough such that $\delta N - \frac{3d+\gamma}{2} \geq K$. Then $|IV_6^{\varepsilon,-}(x)| \leq C\varepsilon^K$).

Thus, we have obtained that for θ supported close to $\xi'^2 = n_-^2$, for $x \in C(j)$, $0 \leq j \leq J$,

$$Op_\varepsilon^w(\Psi_\eta)u_6^\varepsilon(x) = O(\varepsilon^\infty).$$

b- Estimate of $II_5^{\varepsilon,-}(x)$

Now, let us study the term corresponding to u_5^ε.

$$\begin{aligned} II_5^{\varepsilon,-}(x) &= \frac{1}{2[n^2]\varepsilon^{\frac{3d-1}{2}}} \int_{y_d<1} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon} + i\frac{y'\cdot\zeta'}{\varepsilon} - \frac{\omega_-^\varepsilon(\zeta')}{\varepsilon}(|y_d-1|+|\varepsilon z_d-1|)} \\ &\quad \times \frac{(\omega_-^\varepsilon(\zeta') - \omega_+^\varepsilon(\zeta'))^2}{\omega_-^\varepsilon(\zeta')} \mathcal{F}' f(\zeta', z_d) \\ &\quad \times \theta\Big(\frac{x'+y'}{2}, \xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big) dy d\xi' d\zeta' dz_d \end{aligned}$$

Since we are considering $|\zeta'|$ close to n_-, the root $\omega_+^\varepsilon(\zeta')$, that appears as a test function, is not singular. Moreover, the coefficient in front of ω_-^ε in the phase function, $|y_d-1| + |\varepsilon z_d - 1|$ is still bounded from below by $1/2$ for ε and η small enough. Thus, we may use exactly the same method as for the estimate of the term $II_6^{\varepsilon,-}$ (using almost-analytic extensions). We get, as before,

$$II_5^{\varepsilon,-} = O(\varepsilon^\infty).$$

c- Estimate of $II_3^{\varepsilon,-}(x)$

We are left with the term corresponding to u_3^ε. We have

$$\begin{aligned} II_3^{\varepsilon,-}(x) &= \frac{1}{[n^2]\varepsilon^{\frac{3d-1}{2}}}\int_{y_d>1} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}-\frac{\omega_-^\varepsilon(\zeta')}{\varepsilon}|y_d-1|-\frac{\omega_+^\varepsilon(\zeta')}{\varepsilon}|\varepsilon z_d-1|} \\ &\quad\times\left(\omega_-^\varepsilon(\zeta')-\omega_+^\varepsilon(\zeta')\right)\mathcal{F}'f(\zeta',z_d)\chi_-(\zeta') \\ &\quad\times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\eta'dz_d \end{aligned}$$

The different point here is that the two roots ω_-^ε and ω_+^ε appear in the phase function. Thanks to the localization of θ near $\xi'^2=n_-^2$ and to the fact that the parameter $|\varepsilon z_d-1|$ is bounded from below, this term is exponentially small with respect to ε. Indeed, in this integral, we have $n_-^2-a\leq|\zeta'|^2\leq n_-^2+a$, so that

$$\mathcal{R}e\,\omega_+^\varepsilon(\zeta')=\mathcal{R}e\sqrt{\zeta'^2-n_+^2+i\alpha_\varepsilon\varepsilon}\geq\sqrt{[n^2]-a}.$$

Hence, since $|\varepsilon z_d-1|\geq 1/2$, we get

$$|II_3^{\varepsilon,-}(x)|\leq C\frac{e^{-\frac{\sqrt{[n^2]-a}}{2\varepsilon}}}{\varepsilon^{\frac{3d-1}{2}}},$$

so that

$$II_3^{\varepsilon,-}(x)=O(\varepsilon^\infty).$$

We have obtained, for θ with support close to $\xi'^2=n_-^2$,

$$\lim_{\varepsilon\to0}\|Op_\varepsilon^w(\Psi_\eta)u^\varepsilon\|_{\dot B}=0.$$

In conclusion of our study, $\mu^{\partial_-}=0$.

<u>Second case</u>: study of μ^{∂_+} (θ is supported close to $\xi'^2=n_+^2$)

We assume that $\mathrm{supp}(\theta)\subset\{|\xi'^2-n_+^2|\leq a/2\}$. Our strategy is similar to the one we used in the first case. We consider separately the contributions due to $|\zeta'|$ close to, or far from, n_+. For that purpose, we define

$$\chi_+(\zeta')=\chi_0(\frac{\zeta'^2-n_+^2}{a}).$$

As before, we write

$$
\begin{aligned}
&Op_\varepsilon^w(\Psi_\eta)u^\varepsilon(x)\\
&= \frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}} R^\varepsilon\Big(\zeta',\frac{y_d}{\varepsilon},z_d\Big)\mathcal{F}'f(\zeta',z_d)\,(1-\chi_+(\zeta'))\\
&\quad\times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d\\
&+\frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}} R^\varepsilon\Big(\zeta',\frac{y_d}{\varepsilon},z_d\Big)\mathcal{F}'f(\zeta',z_d)\chi_+(\zeta')\\
&\quad\times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{x_d-y_d}{\varepsilon}\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d\\
&= I^{\varepsilon,+}(x)+II^{\varepsilon,+}(x).
\end{aligned}
$$

1- Contribution of $|\zeta'|$ far from n_+

Let us first study the part $I^{\varepsilon,+}$. As for the term $I^{\varepsilon,-}$, the gradient of the phase function with respect to the y'-variable, $\zeta'-\xi'$, is bounded from below. Hence, making integrations by part with respect to this variable, we get

$$I^{\varepsilon,+}(x) = O(\varepsilon^\infty).$$

2- Contribution of $|\zeta'|$ close to n_+

Now, we are left with the part $II^{\varepsilon,+}$. In that case, we again study separately the contributions due to R_6^ε, R_5^ε, and R_3^ε. We denote them $II_6^{\varepsilon,+}$, $II_5^{\varepsilon,+}$ and $II_3^{\varepsilon,+}$ respectively. The structure of the proof of the estimate for each term is the same as for the term $II_6^{\varepsilon,-}$:

• we first make the polar change of variable $\zeta' = \rho\omega'$ and then $t = \rho^2 - n_+^2$,

• we then use the almost-analytic extension of the test function and the Green-Riemann formula to decompose the integral into the sum of three terms:

• an integral over $\mathcal{I}m\, z = \beta$ that we estimate using the non-stationary phase theorem 4.7.2. Here, neither the phase function nor the test function are singular anymore, hence we only have to prove that the gradient of the phase function satisfies

$$|\partial_t\varphi| \geq \frac{C}{\sqrt{\beta}}. \tag{4.7.13}$$

• an integral over $\alpha_\varepsilon\varepsilon \leq |\mathcal{I}m\, z| \leq \varepsilon^\delta$ that we estimate using the property (4.7.9) of almost-analytic extensions.

• and an integral over $\beta \geq |\mathcal{I}m\, z| \geq \varepsilon^\delta$ that we estimate using the non-stationary phase theorem 4.7.2. Here, the phase function function still satisfies

$$|\partial_t\varphi| \geq \frac{C}{\sqrt{\beta}}. \tag{4.7.14}$$

On this set, we have to treat the singularity of the test function, which is done as in Lemma 4.7.6.

Thus, to estimate each term $II_6^{\varepsilon,+}$, $II_5^{\varepsilon,+}$ and $II_3^{\varepsilon,+}$, it remains to prove the estimates (4.7.13) and (4.7.14).

a- Estimate of $II_6^{\varepsilon,+}(x)$
We have

$$\begin{aligned} II_6^{\varepsilon,+}(x) &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int_{y_d<1} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}-\frac{\omega_-^\varepsilon(\zeta')}{\varepsilon}|y_d-\varepsilon z_d|}\frac{\mathcal{F}'f(\zeta',z_d)}{2\omega_-^\varepsilon(\zeta')}\chi_+(\zeta') \\ &\quad \times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{1}{\varepsilon}(x_d-y_d)\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d. \end{aligned}$$

With the announced changes of variables, the phase function rewrites

$$\varphi(t,s) = iy'\cdot\omega'\sqrt{t+n_+^2+is-i\alpha_\varepsilon\varepsilon} - \sqrt{t-[n^2]+is}|y_d-\varepsilon z_d|$$

where $s = \mathcal{I}m\, z \in [\alpha_\varepsilon\varepsilon,\beta]$ ($\alpha_\varepsilon\varepsilon$, β, $y'\cdot\omega'$ and $|y_d-\varepsilon z_d|$ are considered as parameters lying in a compact set).
Since

$$\left|\frac{iy'\cdot\omega'}{\sqrt{t+n_+^2+is-i\alpha_\varepsilon\varepsilon}}\right| \leq \frac{M_1}{\sqrt{n_+^2-a}} \text{ and } \left|\frac{|y_d-\varepsilon z_d|}{\sqrt{t-[n^2]+is}}\right| \geq \frac{1}{2\sqrt{s}},$$

we get, for β small enough, $s\leq\beta$,

$$|\partial_t\varphi(t)| \geq \frac{1}{4\sqrt{\beta}},$$

which implies (4.7.13) and (4.7.14).

b- Estimate of $II_5^{\varepsilon,+}(x)$
We have

$$\begin{aligned} II_5^{\varepsilon,+}(x) &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}}\int_{y_d<1} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon}+i\frac{y'\cdot\zeta'}{\varepsilon}-\frac{\omega_-^\varepsilon(\zeta')}{\varepsilon}(|y_d-1|+|\varepsilon z_d-1|)} \\ &\quad \times\frac{\big(\omega_-^\varepsilon(\zeta')-\omega_+^\varepsilon(\zeta')\big)^2}{2\omega_-^\varepsilon(\zeta')}\mathcal{F}'f(\zeta',z_d)\chi_+(\zeta') \\ &\quad \times\theta\Big(\frac{x'+y'}{2},\xi'\Big)\chi_0\Big(\frac{1}{\varepsilon}(x_d-y_d)\Big)\chi_0\Big(\frac{\frac{x_d+y_d}{2}-1}{\eta}\Big)dyd\xi'd\zeta'dz_d. \end{aligned}$$

Here,

$$\varphi(t,s) = iy'\cdot\omega'\sqrt{t+n_+^2+is-i\alpha_\varepsilon\varepsilon} - \sqrt{t-[n^2]+is}\,(|y_d-1|+|\varepsilon z_d-1|)\,.$$

Similarly, we get, for β small enough, and $s \leq \beta$,

$$|\partial_t \varphi(t)| \geq \frac{1}{4\sqrt{\beta}},$$

which implies (4.7.13) and (4.7.14).
Here, ω_+^ε, that appears as a test function, is singular. We treat this problem as in the proof of the estimate of $VI_6^{\varepsilon,-}$ (see Lemma 4.7.6).

c- Estimate of $II_3^{\varepsilon,+}$
We have

$$\begin{aligned} II_3^{\varepsilon,+} &= \frac{1}{\varepsilon^{\frac{3d-1}{2}}} \int_{y_d<1} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon} + i\frac{y'\cdot\zeta'}{\varepsilon} - \frac{\omega_-^\varepsilon(\zeta')}{\varepsilon}|y_d-1| - \frac{\omega_+^\varepsilon(\zeta')}{\varepsilon}|\varepsilon z_d - 1|} \\ &\quad \times \left(\omega_-^\varepsilon(\zeta') - \omega_+^\varepsilon(\zeta')\right) \mathcal{F}' f(\zeta', z_d) \chi_+(\zeta') \\ &\quad \times \theta\Big(\frac{x'+y'}{2}, \xi'\Big) \chi_0\Big(\frac{1}{\varepsilon}(x_d - y_d)\Big) \chi_0\Big(\frac{\frac{x_d+y_d}{2} - 1}{\eta}\Big) dy d\xi' d\zeta' dz_d. \end{aligned}$$

Hence, the phase function, after the changes of variables, writes

$$\varphi(t,s) = iy'\cdot\omega'\sqrt{t + n_+^2 + is - i\alpha_\varepsilon\varepsilon} - \sqrt{t - [n^2] + is}|y_d - 1| - \sqrt{t + is}|\varepsilon z_d - 1|.$$

Then,

$$\partial_t \varphi = \frac{iy'\cdot\omega'}{\sqrt{t + n_+^2 + is - i\alpha_\varepsilon\varepsilon}} - \frac{|y_d - 1|}{\sqrt{t - [n^2] + is}} - \frac{|\varepsilon z_d - 1|}{\sqrt{t + is}}.$$

On the one hand,

$$\left|\frac{|\varepsilon z_d - 1|}{\sqrt{t + is}}\right| \geq \frac{1}{2\sqrt{s}}$$

and on the other hand,

$$\left|\frac{iy'\cdot\omega'}{\sqrt{t + n_+^2 + is - i\alpha_\varepsilon\varepsilon}} - \frac{|y_d - 1|}{\sqrt{t - [n^2] + is}}\right| \leq \frac{M_1}{\sqrt{n_+^2 - a}} + \frac{2}{\sqrt{[n^2] - a}}.$$

Thus, for β small enough and $s \leq \beta$, we have

$$\partial_t \varphi \geq \frac{1}{4\sqrt{\beta}}.$$

In conclusion, we have obtained that $\mu^{\partial_+} = 0$.

A The resolvent in the homogeneous case

We give here some details about the derivation of the explicit formula for the solution to the Helmholtz equation (4.7.1). Since we can apply the Fourier transform with respect to the x' variable, it is sufficient to make the calculations when the dimension d equals to 1. We may also assume that $\varepsilon = 1$. Hence, we are left with the following equation

$$\begin{aligned} -w'' + \omega_+^2 w &= f \quad \text{for } x > 1, &\text{(A.1)}\\ -w'' + \omega_-^2 w &= f \quad \text{for } x \leq 1, &\text{(A.2)} \end{aligned}$$

where ω_+ and ω_- are chosen with a positive real part.

Let use first calculate w when $x > 1$. We can write

$$w(x) = a(x)e^{\omega_+ x} + b(x)e^{-\omega_+ x},$$

where a and b satisfy

$$a'(x)e^{\omega_+ x} + b'(x)e^{-\omega_+ x} = 0. \tag{A.3}$$

Then, $w''(x) = \omega_+^2 w + \omega_+(a'(x)e^{\omega_+ x} - b'(x)e^{-\omega_+ x})$ so, using (A.1) and (A.3) we obtain the following system satisfied by a' and b':

$$\left\{ \begin{array}{rcrcl} a'(x)e^{\omega_+ x} & + & b'(x)e^{-\omega_+ x} & = & 0 \\ a'(x)\omega_+ e^{\omega_+ x} & - & b'(x)\omega_+ e^{-\omega_+ x} & = & -f \end{array} \right.$$

Thus, we get

$$a'(x) = \frac{-1}{2\omega_+} e^{-\omega_+ x} f(x) \qquad b'(x) = \frac{1}{2\omega_+} e^{\omega_+ x} f(x).$$

Integrating these equalities, we obtain

$$\begin{aligned} a(x) &= \frac{1}{2\omega_+} \left(\int_x^{+\infty} e^{-\omega_+ y} f(y)dy + C_+^+ \right), \\ b(x) &= \frac{1}{2\omega_+} \left(\int_{-\infty}^{x} e^{\omega_+ y} f(y)dy + C_+^- \right), \end{aligned}$$

where C_+^- and C_+^+ are two contants.
Thus, we have for $x > 1$,

$$w(x) = \frac{1}{2\omega_+} \Bigg(\int_x^{+\infty} e^{-\omega_+(y-x)} f(y)dy + C_+^+ e^{\omega_+ x} + \int_{-\infty}^{x} e^{-\omega_+(x-y)} f(y)dy + C_+^- e^{-\omega_+ x} \Bigg).$$

Similarly, we get for $x \leq 1$,

$$w(x) = \frac{1}{2\omega_+}\bigg(\int_x^{+\infty} e^{-\omega_-(y-x)} f(y)dy + C_-^+ e^{\omega_- x} + \int_{-\infty}^{x} e^{-\omega_-(x-y)} f(y)dy + C_-^- e^{-\omega_- x} \bigg).$$

Now, we use that $w \in H^2(\mathbb{R})$ to determine the constants. First, it implies that $C_+^- = C_-^+ = 0$. On the other hand, since $H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$, we write the continuity of w and w' at the point $x = 1$. This gives the following system

$$\begin{aligned}
\frac{e^{-\omega_+}}{2\omega_+} C_+^- - \frac{e^{\omega_-}}{2\omega_-} C_-^+ &= \frac{1}{2\omega_-}\int_{-\infty}^{+\infty} e^{-\omega_-|y-1|} f(y)dy \\
&\quad - \frac{1}{2\omega_+}\int_{-\infty}^{+\infty} e^{-\omega_+|y-1|} f(y)dy \\
e^{-\omega_+} C_+^- + e^{\omega_-} C_-^+ &= -\int_1^{+\infty} e^{-\omega_-|y-1|} f + \int_{-\infty}^{1} e^{-\omega_-|y-1|} f \\
&\quad + \int_1^{+\infty} e^{-\omega_+|y-1|} f - \int_{-\infty}^{1} e^{-\omega_+|y-1|} f
\end{aligned}$$

Hence,

$$\begin{aligned}
C_+^- &= \frac{\omega_+ - \omega_-}{\omega_+ + \omega_-} e^{\omega_+} \int_1^{+\infty} e^{-\omega_+|y-1|} f(y)dy - \int_{-\infty}^{1} e^{\omega_+ y} f(y)dy \\
&\quad + \frac{2\omega_+}{\omega_+ + \omega_-} e^{\omega_+} \int_{-\infty}^{1} e^{-\omega_-|y-1|} f(y)dy \\
C_-^+ &= \frac{\omega_- - \omega_+}{\omega_+ + \omega_-} e^{-\omega_-} \int_{-\infty}^{1} e^{-\omega_-|y-1|} f(y)dy - \int_1^{+\infty} e^{-\omega_- y} f(y)dy \\
&\quad + \frac{2\omega_-}{\omega_+ + \omega_-} e^{-\omega_-} \int_1^{+\infty} e^{-\omega_+|y-1|} f(y)dy
\end{aligned}$$

Finally, we obtain for $x > 1$,

$$\begin{aligned}
w(x) &= \frac{1}{2\omega_+}\bigg(\int_1^{+\infty} e^{-\omega_+|y-x|} f(y)dy \\
&\qquad - \int_1^{+\infty} \frac{(\omega_+ - \omega_-)^2}{[n^2]} e^{-\omega_+|x-1| - \omega_+|y-1|} f(y)dy \bigg) \\
&\quad + \int_{-\infty}^{1} \frac{\omega_+ - \omega_-}{[n^2]} e^{-\omega_+|x-1| - \omega_-|y-1|} f(y)dy
\end{aligned}$$

and for $x \leq 1$,

$$
\begin{aligned}
w(x) \;=\; & \frac{1}{2\omega_-}\Big(\int_{-\infty}^{1} e^{-\omega_-|y-x|} f(y)dy \\
& \qquad + \int_{-\infty}^{1} \frac{(\omega_+ - \omega_-)^2}{[n^2]} e^{-\omega_-|x-1|-\omega_-|y-1|} f(y)dy \Big) \\
& + \int_{1}^{+\infty} \frac{\omega_- - \omega_+}{[n^2]} e^{-\omega_-|x-1|-\omega_+|y-1|} f(y)dy.
\end{aligned}
$$

B Sharp truncation and ε-oscillation

Definition B.1 *A sequence of functions (u^ε) is ε-oscillating if it is bounded in $L^2_{loc}(\mathbb{R}^d)$ and for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, $\limsup_\varepsilon \int_{|\varepsilon\xi|>\rho} |\widehat{\varphi u^\varepsilon}|^2 d\xi \to 0$ as $\rho \to \infty$.*

Definition B.2 *A sequence of functions (u^ε) is strongly ε-oscillating if it is bounded in $L^2_{loc}(\mathbb{R}^d)$ and for some order $s > 0$: for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, $(|\varepsilon D_x|^s \varphi u^\varepsilon)$ is bounded in $L^2(\mathbb{R}^d)$.*

Definition B.3 *A sequence of functions (u^ε) is ε-oscillating in x_d if it is bounded in $L^2_{loc}(\mathbb{R}^d)$ and for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, $\limsup_\varepsilon \int_{|\varepsilon\xi_d|>\rho} |\widehat{\varphi u^\varepsilon}|^2 d\xi \to 0$ as $\rho \to \infty$. A sequence of functions (u^ε) is strongly ε-oscillating in x_d if it is bounded in $L^2_{loc}(\mathbb{R}^d)$ and for some order $s > 0$: for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, $(|\varepsilon D_{x_d}|^s \varphi u^\varepsilon)$ is bounded in $L^2(\mathbb{R}^d)$.*

Lemma B.4 *(i) For all $s \in [0, 1/2[$, for all $f \in H^s(\mathbb{R})$,*

$$\| |D_x|^s \mathbf{1}_{x>0} f\|_{L^2(\mathbb{R})} \leq C_s \| |D_x|^s f\|_{L^2(\mathbb{R})}.$$

(ii) If $\chi \in \mathcal{C}^\infty(\mathbb{R})$ satisfies $\chi(\xi) \leq C_M \langle\xi\rangle^{-M}$ then

$$\forall k \in \mathbb{N}^*,\; k \leq M + s,\; \|\chi(\varepsilon D_x)(\varepsilon D_x)^k \mathbf{1}_{x>0}\|_{L^2(\mathbb{R})} = O(\varepsilon^s).$$

Lemma B.5 *(i) If the sequence (u^ε) is ε-oscillating (respectively strongly ε-oscillating, ε-oscillating in x_d), then its truncation $(\mathbf{1}_{x_d>1} u^\varepsilon)$ is ε-oscillating (respectively strongly ε-oscillating, ε-oscillating in x_d).*
(ii) If (u^ε) and $(\varepsilon D_{x_d})^K u^\varepsilon)$ are bounded in $L^2_{loc}(\mathbb{R}^d_+)$ for some $K \in \mathbb{N}^*$, and if for all $k \in \mathbb{N}$ such that $k < K$, the sequences of traces $(u^\varepsilon_k|_{x_d=1})$ of $u^\varepsilon_k = (\varepsilon D_{x_d})^k u^\varepsilon$ are bounded in $L^2_{loc}(\mathbb{R}^{d-1})$, then for all $k < K$, the sequence of extensions $(\underline{u^\varepsilon}_k)$ of u^ε_k by zero is ε-oscillating in x_d.

The reader can find the proofs of these results for instance in L. Miller [Mil2].

C Tangential test operators

We give here the proofs of the results on tangential test operators.

Proof of Lemma 4.4.5:
Let $\omega \in \hat{\mathcal{C}}_c^\infty(\mathbb{R}^{2d-1}_{x,\xi'})$. Then, for all $k \in \mathbb{N}$,

$$(\omega\xi_d^{k+1})^w(x,\varepsilon D_x) = (\omega\xi_d^k)^w(x,\varepsilon D_x)\varepsilon D_{x_d} + \frac{\varepsilon}{2}(D_{x_d}\omega\xi_d^k)^w(x,\varepsilon D_x) \quad \text{(C.1)}$$

$$= \varepsilon D_{x_d}(\omega\xi_d^k)^w(x,\varepsilon D_x) - \frac{\varepsilon}{2}(D_{x_d}\omega\xi_d^k)^w(x,\varepsilon D_x). \quad \text{(C.2)}$$

Hence, by induction, we get that for all $\phi^\varepsilon \in \mathcal{T}^n(\mathbb{R}^d)$, there exist $\omega_k \in \hat{\mathcal{C}}_c^\infty(\mathbb{R}^{2d-1})$ such that

$$\phi^\varepsilon = \sum_{k=0}^n \Omega_k^\varepsilon(\varepsilon D_{x_d})^k + \varepsilon\Theta^\varepsilon, \quad \text{(C.3)}$$

where $\Omega_k^\varepsilon = \omega_k^w(x,\varepsilon D_{x'})$ and $\Theta^\varepsilon = \sum_{k=0}^{n-1}\varepsilon^{n-1-k}\theta_k^\varepsilon$, with $\theta_k^\varepsilon \in \mathcal{T}^k(\mathbb{R}^d)$.
Thanks to (C.3), it suffices to prove the lemma for operators of the form $\phi^\varepsilon = \sum_{k=0}^2 \Omega_k^\varepsilon(\varepsilon D_{x_d})^k$. Using lemma B.5 i) and ii), we obtain that for $0 \le k \le 2$, $u_{\pm,k}^\varepsilon := \mathbf{1}_{x_d \gtrless 1}(\varepsilon D_{x_d})^k u^\varepsilon$ is ε-oscillating. Since Ω_k^ε contains no derivative with respect to the x_d variable, we get that $\Omega_k^\varepsilon u_{\pm,k}^\varepsilon$ is ε-oscillating, and it is compactly supported uniformly with respect to ε. Thus, using Lemma B.4(i), we get

$$\left\|\chi_0(\varepsilon D_{x_d})\Omega_k^\varepsilon\big((\varepsilon D_{x_d})^k u_\pm^\varepsilon - u_{\pm,k}^\varepsilon\big)\right\|_{L^2} = O(\varepsilon^s)$$

for some $s \in (0,\frac{1}{2})$. Thus, point (i) is proved.
Since $\Omega_k^\varepsilon u_{\pm,k}^\varepsilon$ is ε-oscillating, we have $\limsup_{\varepsilon\to 0}\left\|\Big(\chi\big(\frac{\varepsilon}{\rho}D_{x_d}\big)-1\Big)\Omega_k^\varepsilon u_{\pm,k}^\varepsilon\right\|_{L^2} \to 0$ as $\rho \to +\infty$. Hence, $\limsup_{\varepsilon\to 0}\left\|\Big(\chi\big(\frac{\varepsilon}{\rho}D_{x_d}\big) - 1\Big)\phi^\varepsilon u_\pm^\varepsilon\right\|_{L^2} \to 0$ as $\rho \to +\infty$ and point (ii) follows from point (i) with $\chi_0(\xi_d) = \chi\big(\frac{\xi_d}{\rho}\big)$. □

Proof of Lemma 4.4.6:
Let $\omega \in \mathcal{C}_c^\infty(\mathbb{R}^{2d-1})$. We denote $\Omega^\varepsilon = \omega^w(x,\varepsilon D_{x'})$.
At the first order, using the relation (C.2) and integrating by parts, we obtain

$$\begin{aligned}
\big((\omega\xi_d)^w(x,\varepsilon D_x)v,u\big)_\pm &= \big(\varepsilon D_{x_d}\Omega^\varepsilon v,u\big)_\pm - \frac{\varepsilon}{2}\big((D_{x_d}\omega)^w v,u\big)_\pm \\
&= \pm i\varepsilon\big(\Omega^\varepsilon v,u\big)_\Gamma + \big(\Omega^\varepsilon v,\varepsilon D_{x_d}u\big)_\pm - \frac{\varepsilon}{2}\big((D_{x_d}\omega)^w v,u\big)_\pm \\
&= \pm i\varepsilon\big(\Omega^\varepsilon v,u\big)_\Gamma + \Big(v,\big[(\Omega^\varepsilon)^*\varepsilon D_{x_d} - \frac{\varepsilon}{2}\big((D_{x_d}\omega)^w(x,\varepsilon D_{x'})\big)^*\big]u\Big)_\pm \\
&= \pm i\varepsilon\big(\Omega^\varepsilon v,u\big)_\pm + \Big(v,\big((\omega\xi_d)^w(x,\varepsilon D_x)\big)^* u\Big)_\Gamma.
\end{aligned}$$

At order 2, we first use the relations (C.1) and (C.2) to compute

$$(\omega\xi_d^2)^w = (\varepsilon D_{x_d})\Omega^\varepsilon(\varepsilon D_{x_d}) + \frac{\varepsilon^2}{4}(D_{x_d}^2\omega)^w.$$

Hence,

$$\begin{aligned}
\big((\omega\xi_d^2)^w(x,\varepsilon D_x)v,u)_\pm &= \big(\varepsilon D_{x_d}\Omega^\varepsilon\varepsilon D_{x_d}v,u\big)_\pm + \frac{\varepsilon^2}{4}\big((D_{x_d}^2\omega)^w v,u\big)_\pm \\
&= \pm i\varepsilon\big(\Omega^\varepsilon\varepsilon D_{x_d}v,u)_\Gamma + \big(\varepsilon D_{x_d}v,(\Omega^\varepsilon)^*\varepsilon D_{x_d}u\big)_\pm + \frac{\varepsilon^2}{4}\big((D_{x_d}^2\omega)^w v,u\big)_\pm \\
&= \pm i\varepsilon\big(\Omega^\varepsilon\varepsilon D_{x_d}v,u)_\Gamma \pm i\varepsilon\big(v,(\Omega^\varepsilon)^*\varepsilon D_{x_d}u)_\Gamma + \big(v,\varepsilon D_{x_d}(\Omega^\varepsilon)^*\varepsilon D_{x_d}u)_\pm \\
&\qquad + \frac{\varepsilon^2}{4}\big((D_{x_d}^2\omega)^w v,u\big)_\pm \\
&= \pm i\varepsilon\big(\Omega^\varepsilon\varepsilon D_{x_d}v,u)_\Gamma \pm i\varepsilon\big(\Omega^\varepsilon v,\varepsilon D_{x_d}u)_\Gamma + \big(v,\big((\omega\xi_d^2)^w\big)^*u)_\pm,
\end{aligned}$$

which ends the proof. □

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