Quelques propriétés des algèbres de von Neumannengendrées par des q-Gaussiens

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Introduction

Dans ce travail nous nous sommes intéressés aux propriétés des algèbres de von Neumann engendrées par des Gaussiens non commutatifs q-déformés. Ces algèbres sont apparues pour la première fois en 1969 dans un article de Frisch et Bourret [FB]. Soit T un ensemble et Γ une fonction symétrique à valeurs réelles et de type positif sur $T \times T$ (fonction de covariance), Frisch et Bourret [FB] introduisent des algèbres d'opérateurs où sont réalisées les relations de q-commutation ($q \in [-1, 1]$) :

$$a(s)a^*(t) - qa^*(t)a(s) = \Gamma(s, t) \tag{1}$$

pour une certaine famille d'opérateurs $(a(t))_{t\in T}$. Rappelons tout d'abord qu'une fonction symétrique Γ sur $T\times T$ est de type positif si et seulement s'il existe un espace de Hilbert réel H et $(\xi_s)_{s\in T}$ une famille de vecteurs de H tels que pour tout $s, t \in T$ on a $\Gamma(s, t) = \langle \xi_s, \xi_t\rangle$. On peut donc réécrire les relations (1) sous la forme :

$$a(\xi_s)a^*(\xi_t) - qa^*(\xi_t)a(\xi_s) = \langle \xi_s, \xi_t\rangle. \tag{2}$$

Pour $q = 1$ (respectivement $q = -1$), les relations (2) sont les célèbres relations de commutation (respectivement d'anti-commutation) canoniques qui sont réalisées par les opérateurs de création et d'annihilation bosoniques (respectivement fermioniques) sur l'espace de Fock symétrique (respectivement anti-symétrique). Pour $q = 0$, les relations (2) tombent dans le domaine des probabilités libres de Voiculescu. Dans ce dernier cas également, les relations (2) sont vérifiées par des opérateurs de création et d'annihilation libres sur l'espace de Fock libre. Le problème général d'existence d'une algèbre d'opérateurs où sont réalisées les relations (2) n'a trouvé sa solution qu'au début des années 90 dans les travaux de Bożejko et Speicher [BS3]. Dans cet article, les auteurs construisent un espace de Fock muni d'un produit scalaire tordu, où les relations de q-commutation sont également réalisées par des opérateurs de création et d'annihilation. Nous commençons par rappeler cette construction.

Dans la suite, q désigne un nombre réel dans l'intervalle $[-1, 1]$. Étant donné un espace de Hilbert réel $H_{\mathbb{R}}$ muni d'un produit scalaire $\langle ., .\rangle$, nous désignerons par $H_{\mathbb{C}}$ son complexifié (dont le produit scalaire est anti-linéaire par rapport à la première variable) et par $H_{\mathbb{C}}^{\otimes n}$ le produit tensoriel canonique de $H_{\mathbb{C}}$ avec lui-même n fois ($n \geqslant 1$). Soit $P_q^{(n)} : H_{\mathbb{C}}^{\otimes n} \longrightarrow H_{\mathbb{C}}^{\otimes n}$ l'opérateur défini par :

$$P_q^{(n)} = \sum_{\sigma\in S_n} q^{i(\sigma)} U_\sigma$$

où S_n est le groupe des permutations de n éléments, $i(\sigma)$ le nombre d'inversions de la permutation σ et $\sigma \mapsto U_\sigma$ est l'action naturelle de S_n sur $H_{\mathbb{C}}^{\otimes n}$. Un des résultats principaux de [BS3] garantit que, pour tout $n \geqslant 1$, l'opérateur $P_q^{(n)}$ est strictement positif sur $H_{\mathbb{C}}^{\otimes n}$ lorsque $q \in (-1,1)$ et positif pour $q \in \{-1,1\}$. Dans le cas où $q \in (-1,1)$, on peut donc définir un nouveau produit scalaire sur $H_{\mathbb{C}}^{\otimes n}$ par :

$$\langle \xi, \eta \rangle_q = \langle P_q^{(n)} \xi, \eta \rangle, \quad \xi, \eta \in H_{\mathbb{C}}^{\otimes n} \tag{3}$$

Lorsque $q \in \{-1,1\}$, $P_q^{(n)}$ a un noyau non trivial sur $H_{\mathbb{C}}^{\otimes n}$ et nous désignerons encore par $H_{\mathbb{C}}^{\otimes n}$ le quotient de $H_{\mathbb{C}}^{\otimes n}$ par ce noyau, que nous munirons du produit scalaire q-déformé défini comme dans (3). L'espace de Fock q-déformé, $\mathcal{F}_q(H_{\mathbb{C}})$, est par définition la somme Hilbertienne :

$$\mathcal{F}_q(H_{\mathbb{C}}) = \mathbb{C}\Omega \bigoplus_{n \geqslant 1} H_{\mathbb{C}}^{\otimes n},$$

où Ω est un vecteur unité appelé vacuum. Étant donné $f \in H_{\mathbb{C}}$, l'opérateur de création $a^*(f) \in B(\mathcal{F}_q(H_{\mathbb{C}}))$ est défini par

$$a^*(f)\Omega = f, \quad a^*(f)(f_1 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n \quad \text{où} \quad f_1, \ldots, f_n \in H_{\mathbb{C}}.$$

L'opérateur d'annihilation associé à f, noté $a(f)$, est par définition l'adjoint de l'opérateur $a^*(f)$. On a plus explicitement :

$$a(f)\Omega = 0, \quad a(f)(f_1 \otimes \cdots \otimes f_n) = \sum_{k=1}^{n} q^{k-1} \langle f, f_k \rangle f_1 \otimes \cdots \otimes \hat{f}_k \otimes \cdots \otimes f_n \ \text{ où } \ f_1, \ldots, f_n \in H_{\mathbb{C}},$$

et où $\hat{f}_k$ signifie que f_k a été omis. Les opérateurs de création et d'annihilation satisfont aux relations de q-commutation suivantes et répondent au problème d'existence de Frisch et Bourret :

$$a(f)a^*(g) - q a^*(g)a(f) = \langle f, g \rangle Id, \quad f, g \in H_{\mathbb{C}}.$$

Les variables non-commutatives qui sont au centre de notre étude, sont les opérateurs Hermitiens q-Gaussiens définis par

$$G(f) = a^*(f) + a(f), \quad f \in H_{\mathbb{R}}.$$

Nous nous intéresserons plus spécifiquement à l'algèbre de von Neumann qu'engendrent les q-Gaussiens :

$$\Gamma_q(H_{\mathbb{R}}) = \{G(f) = a^*(f) + a(f),\ f \in H_{\mathbb{R}}\}'' \subset B\left(\mathcal{F}_q(H_{\mathbb{C}})\right).$$

Pour $q = 1$, l'algèbre $\Gamma_1(H_{\mathbb{R}})$ est une algèbre commutative affiliée à des variables Gaussiennes classiques, et pour $q = -1$, l'algèbre $\Gamma_{-1}(H_{\mathbb{R}})$ est le facteur hyperfini de type II_1. Notons également que pour $q = 0$, l'algèbre $\Gamma_0(H_{\mathbb{R}})$ est l'algèbre engendrée par des opérateurs semi-circulaires libres, introduite par Voiculescu. D'après un résultat célèbre de Voiculescu [VDN], l'algèbre $\Gamma_0(H_{\mathbb{R}})$ est isomorphe à l'algèbre de von Neumann réduite du groupe libre $\mathbb{F}_I$ (où I est la dimension Hilbertienne de $H_{\mathbb{R}}$) qui est au coeur des probabilités libres. La famille d'algèbres $\left(\Gamma_q(H_{\mathbb{R}})\right)_{-1<q<1}$ est, en quelque sorte, une déformation "continue" de l'algèbre libre $\Gamma_0(H_{\mathbb{R}})$. L'étude de ces algèbres s'est considérablement développée ces dernières années. Malgré les nombreux progrès, beaucoup de questions restent non résolues. En particulier, cette déformation du cas libre introduit-elle effectivement des algèbres qui sortent du cadre des algèbres de groupes libres ? Plus précisément, $\Gamma_q(H_{\mathbb{R}})$ est-elle ou non isomorphe à l'algèbre libre $\Gamma_0(H_{\mathbb{R}})$ pour tout $q \in (-1,1)$? Les propriétés des algèbres q-Gaussiennes, que nous établissons dans ce mémoire, sont des extensions des propriétés déjà connues pour les algèbres réduites de groupes libres (i.e pour $q = 0$).

Faisons l'inventaire de quelques unes des propriétés de $\Gamma_q(H_{\mathbb{R}})$ en rapport avec les résultats de ce mémoire. Tout d'abord, ces algèbres sont des algèbres finies, munies d'une trace normale et normalisée τ qui est l'état vectoriel associé à Ω :

$$\tau(x) = \langle \Omega, x\Omega \rangle_q, \quad x \in \Gamma_q(H_{\mathbb{R}}).$$

Bożejko, Kümmerer et Speicher [BKS] ont prouvé que ces algèbres sont des facteurs de type II_1, lorsque $\dim(H_{\mathbb{R}})$ est infinie, en montrant l'unicité d'une telle trace. Ceci a ensuite été amélioré par Śniady [Sn2] qui a étendu ce résultat pour $\dim(H_{\mathbb{R}}) \geqslant f(q)$, où f est une certaine fonction de q. Récemment, Ricard [Ri] a apporté la solution complète au problème, en prouvant la factorialité pour tout $q \in (-1,1)$ et dès que $\dim(H_{\mathbb{R}}) \geqslant 2$.

Rappelons qu'une algèbre de von Neumann $\mathcal{N} \subset B(H)$ est dite injective lorsqu'il existe une espérance conditionnelle de $B(H)$ sur $\mathcal{N}$ (ce qui est indépendant de l'inclusion $\mathcal{N} \subset B(H)$ considérée). La non-injectivité de $\Gamma_q(H_{\mathbb{R}})$ (dès que $\dim(H_{\mathbb{R}}) \geqslant 2$) est un des résultats principaux de ce mémoire (cf. Théorème 1.3.4). Il s'agit d'une amélioration d'un résultat partiel dû à Bożejko et Speicher [BS1]. Dans certains cas particuliers, Śniady [Sn2] a amélioré le résultat du Théorème 1.3.4, en prouvant que les algèbres q-Gaussiennes n'ont pas la propriété Γ lorsque $\dim(H_{\mathbb{R}}) \geqslant f(q)$ (où f est une certaine fonction de q déjà introduite plus haut). Parallèlement, Shlyakhtenko [Sh2] a établi que les algèbres q-Gaussiennes sont solides lorsque $|q| < \sqrt{2}-1$ et $\dim(H_{\mathbb{R}}) < +\infty$. Or, d'après un résultat d'Ozawa [Oz2], un facteur solide et non injectif est premier et n'a pas la propriété Γ. Il s'ensuit, en combinant les faits précédents, que pour $|q| < \sqrt{2}-1$ et $\dim(H_{\mathbb{R}}) < +\infty$, les algèbres q-Gaussiennes sont des facteurs premiers n'ayant pas la propriété Γ.

Dans le cas libre, la distribution semi-circulaire apparaît comme loi limite de matrices aléatoires de Wigner de grande taille. Il était intéressant de se demander si de tels modèles aléatoires asymptotiques matriciels existent également pour les algèbres q-déformées. Un des premiers résultats dans ce sens est le Théorème limite de Speicher [Sp] qui permet de voir la loi q-Gaussienne comme loi limite de sommes normalisées de matrices qui sont aléatoirement commutantes ou anti-commutantes. Différents autres modèles ont été proposés. En particulier, Śniady [Sn] a fourni un modèle asymptotique matriciel Gaussien pour $0 \leqslant q \leqslant 1$. Dans [MN], Mingo et Nica donnent un modèle asymptotique matriciel pour les systèmes q-circulaires en établissant un théorème limite semblable au théorème de Speicher et en l'appliquant à des matrices unitaires.

Outre son intérêt probabiliste, ce type de modèles et théorèmes limites a permis à Biane [Bi] (en utilisant la construction de Speicher) de caractériser l'hypercontractivité des opérateurs de quantification seconde relatifs aux algèbres q-Gaussiennes. Sa démarche passe par un résultat correspondant sur des algèbres engendrées par des matrices qui commutent ou anti-commutent. Ce dernier résultat s'inspire lui-même de ceux déjà connus dans les cas bosonique et fermionique. Pour les algèbres bosoniques, $\Gamma_1(H_{\mathbb{R}})$, le résultat remonte à Nelson [Ne]. Quant au cas des algèbres fermioniques, il a été résolu partiellement par Gross [Gro], puis complètement par Carlen et Lieb [CL].

L'existence de modèles asymptotiques matriciels aléatoires induit également que les algèbres q-Gaussiennes sont approchables, en un sens faible, par des algèbres matricielles :

nous prouvons que les algèbres q-Gaussiennes sont des sous-algèbres d'ultraproduits d'algèbres matricielles. En particulier les algèbres q-Gaussiennes ont la propriété QWEP (cf. Théorème 2.3.3). Nous avons obtenu l'analogue de ce dernier résultat (et des précédents) pour deux autres types de déformations qui généralisent dans différentes directions la q-déformation. Nous les décrivons dans les paragraphes suivants.

Dans [BS1], Bożejko et Speicher ont construit des représentations sur des espaces de Fock d'une large classe de relations de commutation englobant les relations (2). Celles-ci sont de la forme :

$$a(i)a^*(j) - \sum_{r,s \in I} t_{j\,s}^{i\,r} a^*(r)a(s) = \delta_{ij} Id \tag{4}$$

pour certaines familles de coefficients complexes $t_{j\,s}^{i\,r}$. Soit $H_{\mathbb{C}} = l^2(I)$, l'espace de Hilbert complexe des familles indexées par I de carré sommable, et T l'opérateur défini sur $H_{\mathbb{C}} \otimes H_{\mathbb{C}}$ par

$$T(e_i \otimes e_j) = \sum_{r,s \in I} t_{j\,s}^{i\,r} e_r \otimes e_s$$

(où $(e_i)_{i \in I}$ est la base Hilbertienne canonique de $H_{\mathbb{C}}$). Dans la suite, nous supposerons que T est un opérateur de Yang-Baxter borné sur $H_{\mathbb{C}} \otimes H_{\mathbb{C}}$, ce qui signifie que $T \in B(H_{\mathbb{C}} \otimes H_{\mathbb{C}})$ est un opérateur vérifiant la condition nattée :

$$(T \otimes I_{H_{\mathbb{C}}})(I_{H_{\mathbb{C}}} \otimes T)(T \otimes I_{H_{\mathbb{C}}}) = (I_{H_{\mathbb{C}}} \otimes T)(T \otimes I_{H_{\mathbb{C}}})(I_{H_{\mathbb{C}}} \otimes T).$$

Désignons alors par T_k (pour $1 \leqslant k \leqslant n-1$) l'opérateur de $H_{\mathbb{C}}^{\otimes n}$ défini par

$$T_k = I_{H_{\mathbb{C}}^{\otimes k-1}} \otimes T \otimes I_{H_{\mathbb{C}}^{\otimes n-k-1}}.$$

Il est alors possible d'étendre à S_n, par quasi-multiplicativité, l'application φ suivante :

$$\varphi(\pi_k) = T_k$$

où π_k est la transposition échangeant k et $k+1$. Soit $P_T^{(n)}$ l'opérateur défini sur $H_{\mathbb{C}}^{\otimes n}$ par

$$P_T^{(n)} = \sum_{\sigma \in S_n} \varphi(\sigma).$$

Lorsque T est un opérateur de Yang-Baxter hermitien et strictement contractant sur $H \otimes H$, Bożejko et Speicher ont prouvé que $P_T^{(n)}$ est un opérateur strictement positif sur $H_{\mathbb{C}}^{\otimes n}$. On peut donc déformer le produit scalaire canonique sur $H_{\mathbb{C}}^{\otimes n}$ en définissant un nouveau produit scalaire $\langle ., . \rangle_T$ par

$$\langle \xi, \eta \rangle_T = \langle P_T^{(n)} \xi, \eta \rangle, \quad \xi, \eta \in H_{\mathbb{C}}^{\otimes n}.$$

De manière tout à fait analogue à la q-déformation, on peut alors définir un espace de Fock T-déformé (noté $\mathcal{F}_T(H_{\mathbb{C}})$), des opérateurs de création et d'annihilation, des T-Gaussiens ainsi qu'une algèbre de von Neumann T-déformée $\Gamma_T(H_{\mathbb{R}}) = \{G(f),\ f \in H_{\mathbb{R}}\}''$. On vérifie alors que les relations (4) sont satisfaites par les opérateurs de création $(a^*(e_i))_{i \in I}$ et leurs adjoints. Si, de plus, les coefficients $t_{j\,s}^{i\,r}$ satisfont à la condition de cyclicité

$$t_{j\,r}^{i\,s} = t_{i\,j}^{s\,r}, \tag{5}$$

alors l'état vectoriel τ associé au vecteur Ω est une trace normale et normalisée sur $\Gamma_T(H_{\mathbb{R}})$. Lorsque la condition (5) est réalisée nous dirons que l'opérateur T est tracial. Notre résultat de non-injectivité (cf. Théorème 1.3.4) est valable dans ce contexte et vient compléter le résultat partiel de Bożejko et Speicher [BS1]. En parallèle, Kròlak [K] a étendu au cas de la T-déformation les résultats de [BKS] et [B2], en prouvant que les algèbres T-déformées sont des facteurs de type II_1, dès que la dimension de l'espace de Hilbert sous-jacent est infinie. Toujours sous la condition $\dim(H_{\mathbb{R}}) = +\infty$, Kròlak [KPhD] a également montré que l'algèbre $\Gamma_T(H_{\mathbb{R}})$ n'a pas la propriété Γ, ce qui renforce notre résultat de non-injectivité (qui lui est valable dès que $\dim(H_{\mathbb{R}}) \geqslant 2$). L'hypercontractivité du T-semi-groupe d'Ornstein-Uhlenbeck a été résolue par Kròlak dans sa thèse [KPhD] en ramenant le problème au cas q-Gaussien, où $\|T\| < q$. Signalons qu'on ne sait toujours pas si ces algèbres sont des facteurs dans le cas où $\dim(H_{\mathbb{R}}) < +\infty$.

Le dernier type d'algèbres déformées, auquel nous nous sommes intéressés, généralise à la fois les algèbres q-Gaussiennes de Bożejko et Speicher, et les algèbres libres quasi-libres de Shlyakhtenko [Sh] ; ce sont les q-algèbres d'Araki-Woods introduites par Hiaï dans [Hi]. Une telle algèbre peut être vue comme une sous-algèbre de $B(\mathcal{F}_q(H))$ engendrée par des q-Gaussiens $G(f)$ pour $f \in H_{\mathbb{R}}$, où $H_{\mathbb{R}} \subset H$ est ici un espace de Hilbert réel quelconque, inclus dans l'espace de Hilbert complexe H. Une définition équivalente (cf. [Hi],[Sh] et [RvD]) est la suivante. Soit $q \in (-1,1)$, $H_{\mathbb{R}}$ un espace de Hilbert réel et $(U_t)_{t\in\mathbb{R}}$ un groupe fortement continu de transformations orthogonales de $H_{\mathbb{R}}$. Soit $H_{\mathbb{C}} = \mathbb{C}\otimes H_{\mathbb{R}}$ le complexifié de $H_{\mathbb{R}}$. Nous désignerons encore par $(U_t)_{t\in\mathbb{R}}$ l'extension du groupe précédent en un groupe d'unitaires de $H_{\mathbb{C}}$. Soit alors A l'opérateur (non borné) strictement positif sur $H_{\mathbb{C}}$ qui est le générateur infinitésimal du groupe unitaire $(U_t)_{t\in\mathbb{R}}$:

$$U_t = A^{it}, \quad \text{pour tout } t \in \mathbb{R}.$$

On définit un nouveau produit scalaire sur $H_{\mathbb{C}}$ par

$$\langle \xi, \eta \rangle_H = \langle 2A(I+A)^{-1}\xi, \eta \rangle_{H_{\mathbb{C}}}, \quad \xi, \eta \in H_{\mathbb{C}}.$$

Soit H l'espace de Hilbert complexe qui est la complétion de $H_{\mathbb{C}}$ par rapport à ce produit scalaire, et soit $\mathcal{F}_q(H)$ l'espace de q-Fock associé à H. La q-algèbre d'Araki-Woods $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ est par définition l'algèbre de von Neumann engendrée par les q-Gaussiens associés aux vecteurs de $H_{\mathbb{R}}$:

$$\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}}) = \{G(f), f \in H_{\mathbb{R}}\}'' \subset B\left(\mathcal{F}_q(H)\right).$$

L'état vectoriel τ associé à Ω n'est pas une trace sur $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ sauf dans le cas trivial où $U_t \equiv I$ (qui correspond au cas de la q-déformation). Les q-algèbres d'Araki-Woods sont des facteurs lorsque A possède un nombre infini de vecteurs propres deux à deux orthogonaux, ou bien lorsque le spectre ponctuel de A est vide [Hi].

La classification de ces facteurs a été donnée par Hiaï dans [Hi]. Lorsque le spectre ponctuel de A est vide, $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ est un facteur de type III_1. Dans le cas où A possède un nombre infini de vecteurs propres deux à deux orthogonaux, désignons par G le sous-groupe multiplicatif de $\mathbb{R}_+^*$ engendré par le spectre ponctuel de A. On est alors dans l'un des trois cas suivants [Hi] :

- $G = \{1\}$, et alors $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ est de type II_1 (cas trivial).
- $G = \{\lambda^n,\ n \in \mathbb{Z}\}$, où $\lambda > 1$. $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ est alors un facteur de type III_λ.

- $G = \mathbb{R}_+^*$, et dans ce cas $\Gamma_q(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$ est un facteur de type III_1.

La question concernant la non-injectivité de $\Gamma_q(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$ a été partiellement résolue dans [Hi] : $\Gamma_q(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$ est non injective dès qu'il existe $T \geqslant 1$ tel que

$$\frac{\dim\left(E_A([1,T])(H)\right)}{T} > \frac{16}{(1-|q|)^2},$$

où E_A est la résolution de l'identité associée à l'opérateur A. Nous démontrons un critère analogue mais indépendant de q (cf. Corollaire 1.4.2).

Pisier et Shlyakhtenko [PS] ont établi que les algèbres libres quasi-libres $\Gamma_0(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$ ont la propriété QWEP lorsque $(U_t)_{t\in\mathbb{R}}$ est presque périodique. Ce résultat est un point clef dans leur preuve du Théorème de Grothendieck version espaces d'opérateurs. Nous avons étendu ce résultat aux q-algèbres d'Araki-Woods, sans condition supplémentaire sur le groupe $(U_t)_{t\in\mathbb{R}}$. Notre preuve passe par la construction d'un modèle asymptotique matriciel pour les q-algèbres d'Araki-Woods, qui peut être d'un intérêt indépendant.

Dans le premier chapitre, nous prouvons que les algèbres $\Gamma_T(H_\mathbb{R})$ sont non injectives dès que T est un opérateur de Yang-Baxter hermitien strictement contractant et tracial, et dès que la dimension de l'espace de Hilbert réel $H_\mathbb{R}$ est plus grande que 2.

Théorème 1.3.4 $\Gamma_T(H_\mathbb{R})$ *est non injective dès que* $\dim(H_\mathbb{R}) \geqslant 2$.

Nous obtenons ce résultat comme conséquence asymptotique du théorème suivant qui donne une généralisation, à coefficients opérateurs, des inégalités de Bożejko-Haagerup à coefficients scalaires [B2].

Théorème 1.3.3 *Soit K un espace de Hilbert complexe. Pour tout $n \geqslant 0$ et pour tout $\xi \in B(K) \underset{\min}{\otimes} H^{\otimes n}$ on a*

$$\max_{0\leqslant k\leqslant n} \|(id \otimes R_{n,\,k}^*)(\xi)\| \leqslant \|(id \otimes W)(\xi)\|_{\min} \leqslant C_q(n+1) \max_{0\leqslant k\leqslant n} \|(id \otimes R_{n,\,k}^*)(\xi)\|$$

où id est l'identité de $B(K)$, et où les normes $\|(id \otimes R_{n,\,k}^*)(\xi)\|$ sont prises dans $B(K) \underset{\min}{\otimes} H_c^{\otimes n-k} \underset{\min}{\otimes} H_r^{\otimes k}$.

Ici $W(\xi)$ est le produit de Wick associé à ξ, i.e. l'élément de $\Gamma_T(H_\mathbb{R})$ déterminé de manière unique par $W(\xi)\Omega = \xi$. Les ingrédients principaux de la preuve du Théorème 1.3.3 sont la formule de Wick (cf. Lemme 1.2.2)

$$W(\xi) = \sum_{k=0}^{n} U_k R_{n,k}^*(\xi),$$

et l'estimation de la norme complètement bornée de l'application bilinéaire U_k (cf. sections 1.2 et 1.3 pour les définitions précises des opérateurs et des notions qui entrent en jeu). Comme corollaire de cette preuve, nous avons obtenu le résultat suivant :

Corollaire 1.3.5 *$C_T^*(H_\mathbb{R})$ n'a pas la propriété d'espérance conditionnelle faible (WEP) dès que* $\dim H_\mathbb{R} \geqslant 2$.

Ici $C_T^*(H_{\mathbb{R}})$ désigne la C^*-algèbre engendrée par les T-Gaussiens. Dans la section 1.4, nous étendons les résultats précédents au cas des q-algèbres d'Araki-Woods. Nous commençons par énoncer des inégalités de Bożejko-Haagerup à coefficients opérateurs (cf. Théorème 1.4.1), analogues aux inégalités du Théorème 1.3.3. Nous en déduisons le critère de non-injectivité suivant :

Théorème 1.4.2

$$\textit{Si} \quad \dim E_A(\{1\})\, H_{\mathbb{C}} \geqslant 2 \quad \textit{ou, pour un certain } T > 1, \quad \frac{\dim E_A(]1,T])\, H_{\mathbb{C}}}{T^2} > \frac{1}{2},$$

où E_A est la résolution spectrale de l'identité associée à A, alors $\Gamma_q(H_{\mathbb{R}}, U_t)$ est non-injective.

Dans le deuxième chapitre, nous prouvons que les q-algèbres d'Araki-Woods sont QWEP. Nous commençons par prouver ce résultat dans le cas q-Gaussien classique (cf. Théorème 2.3.3 énoncé ci-dessous) où la preuve découle de l'existence d'un modèle asymptotique matriciel. Nous rappelons à cette occasion le Théorème limite de Speicher [Sp] et la construction du modèle asymptotique aléatoire des Bébés-Fock [Bi].

Théorème 2.3.3 *Soit $H_{\mathbb{R}}$ un espace de Hilbert réel et $q \in (-1,1)$. L'algèbre de von Neumann $\Gamma_q(H_{\mathbb{R}})$ est QWEP.*

En utilisant l'ultracontractivité du q-semi-groupe d'Ornstein Uhlenbeck, nous établissons ensuite que, $C_q^*(H_{\mathbb{R}})$, la C^*-algèbre engendrée par les q-Gaussiens est "faiblement ucp complémentée" dans l'algèbre de von Neumann $\Gamma_q(H_{\mathbb{R}})$. Ceci nous conduit au renforcement suivant du Théorème 2.3.3 :

Corollaire 2.3.4 *Soit $H_{\mathbb{R}}$ un espace de Hilbert réel et $q \in (-1,1)$. La C^*-algèbre $C_q^*(H_{\mathbb{R}})$ est QWEP.*

La preuve du Théorème 2.3.3 se ramène, par un argument de limite inductive, au cas où $H_{\mathbb{R}}$ est de dimension finie. Nous utilisons alors le modèle asymptotique matriciel des Bébés-Fock afin de construire un isomorphisme algébrique entre une sous-$*$-algèbre préfaiblement dense de $\Gamma_q(H_{\mathbb{R}})$ et une sous-algèbre d'un ultraproduit d'algèbres matricielles. Dans le cas tracial, cet isomorphisme algébrique (qui préserve les traces) s'étend naturellement en un isomorphisme de $\Gamma_q(H_{\mathbb{R}})$ sur une sous-algèbre de von Neumann (nécessairement complètement complémentée) d'un ultraproduit d'algèbres matricielles, ce qui permet de conclure.

Dans le cas général nous commençons par considérer la situation où l'espace de Hilbert sous-jacent est de dimension finie. Dans la section 2.5, nous construisons un analogue du modèle des Bébés-Fock muni d'un état non-tracial dont nous étudions la théorie modulaire. Nous appliquons le Théorème de Speicher à ce modèle de Bébés-Fock afin d'obtenir un modèle aléatoire asymptotique matriciel qui nous permet de construire un isomorphisme algébrique, qui préserve les états, entre une sous-$*$-algèbre préfaiblement dense de $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ et une sous-algèbre d'un ultraproduit d'algèbres matricielles. Nous utilisons le critère de la section 2.4 qui nous permet d'étendre l'isomorphisme algébrique précédent en un isomorphisme d'algèbres de von Neumann entre $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ et une sous-algèbre complètement complémentée d'un ultraproduit d'algèbres matricielles. L'outil principal qui nous permet de pallier le défaut supplémentaire de commutativité dans le cas non-tracial nous est donné par les différentes théories modulaires qui entrent en jeu dans cette construction. Nous obtenons alors le résultat principal de la section 2.5 :

Théorème 2.5.8 *Si $H_{\mathbb{R}}$ est un espace de Hilbert réel de dimension finie muni d'un groupe de transformations orthogonales $(U_t)_{t\in\mathbb{R}}$, alors l'algèbre de von Neumann $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ est QWEP.*

Par limite inductive, on en déduit immédiatement que le résultat du théorème précédent subsiste lorsque le groupe $(U_t)_{t\in\mathbb{R}}$ est presque périodique. Nous concluons dans le cas général grâce à un procédé de discrétisation du spectre du générateur infinitésimal A. Ce procédé nous permet de réaliser (via le critère de la section 2.4) une algèbre $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ quelconque comme sous-algèbre complètement complémentée d'un ultraproduit d'algèbres discrétisées presque périodiques $\Gamma_q\left(H_{\mathbb{R}}, (U_t^n)_{t\in\mathbb{R}}\right)$. Nous en déduisons alors notre résultat principal :

Théorème 2.6.3 *Soit $H_{\mathbb{R}}$ un espace de Hilbert réel muni d'un groupe de transformations orthogonales $(U_t)_{t\in\mathbb{R}}$. Alors, pour tout $q \in (-1, 1)$, la q-algèbre d'Araki-Woods $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ est QWEP.*

Chapter 1

Haagerup-Bożejko Inequality and non-injectivity of deformed Gaussian algebras

1.1 Introduction

Let $H_{\mathbb{R}}$ be a real Hilbert space and $H_{\mathbb{C}}$ its complexification. Let T be a Yang-Baxter operator on $H_{\mathbb{C}} \otimes H_{\mathbb{C}}$ with $\|T\| < 1$. Let $\mathcal{F}_T(H_{\mathbb{C}})$ be the associated deformed Fock space and $\Gamma_T(H_{\mathbb{R}})$ the von Neumann algebra generated by the corresponding deformed gaussian random variables, introduced by Bożejko and Speicher [BS1] (also see [BKS]). In addition, we will assume that T is tracial, i.e that the vacuum expectation is a trace on $\Gamma_T(H_{\mathbb{R}})$ (see [BS1]). Under these assumptions, it was proved in [BS1] that $\Gamma_T(H_{\mathbb{R}})$ is not injective as soon as dim $H_{\mathbb{R}} > \frac{16}{(1-q)^2}$, where $\|T\| = q$. Since then the problem whether $\Gamma_T(H_{\mathbb{R}})$ is not injective as soon as dim $H_{\mathbb{R}} \geqslant 2$ had been left open. We emphasize that this problem remained open even in the particular case of the q-deformation, that is when $T = q\sigma$, where σ is the reflexion: $\sigma(\xi \otimes \eta) = \eta \otimes \xi$. Recall that the free von Neumann algebra $\Gamma_0(H_{\mathbb{R}})$ (corresponding to $T = 0$) is not injective as soon as $n =$ dim $H_{\mathbb{R}} \geqslant 2$, for $\Gamma_0(H_{\mathbb{R}})$ is isomorphic to the free group von Neumann algebra $VN(\mathbb{F}_n)$ (see [VDN]). The main result of this paper solves the above problem.

To explain the idea of our proof we first recall the main ingredient of the proof of the non injectivity theorem in [BS1]. It is the following vector-valued non-commutative Khintchine inequality. Let $(e_i)_{i\in I}$ be an orthonormal basis of $H_{\mathbb{R}}$. Let K be a complex Hilbert space and $B(K)$ the space of all bounded operators on K. Then for any finitely supported family $(a_i)_{i\in I} \subset B(K)$

$$\max\left\{\|\sum_{i\in I} a_i^* a_i\|^{\frac{1}{2}}_{B(K)}, \|\sum_{i\in I} a_i a_i^*\|^{\frac{1}{2}}_{B(K)}\right\} \leqslant \left\|\sum_{i\in I} a_i \otimes G(e_i)\right\|$$
$$\leqslant \frac{2}{\sqrt{1-q}} \max\left\{\|\sum_{i\in I} a_i^* a_i\|^{\frac{1}{2}}_{B(K)}, \|\sum_{i\in I} a_i a_i^*\|^{\frac{1}{2}}_{B(K)}\right\}$$

where $G(e) = a^*(e) + a(e)$ is the deformed gaussian variable associated with a vector $e \in H_{\mathbb{R}}$. Using this Khintchine inequality and the equivalence between the injectivity and the semi-discreteness, one easily deduces the non-injectivity of $\Gamma_T(H_{\mathbb{R}})$ as soon as $\dim H_{\mathbb{R}} > \frac{16}{(1-q)^2}$.

The proof of our non-injectivity theorem follows the same pattern. We will first need to extend the preceding vector-valued non-commutative Khintchine inequality to Wick products. It is well known that for any ξ, a finite linear combination of elementary tensors, there is a unique operator $W(\xi) \in \Gamma_T(H_{\mathbb{R}})$ such that $W(\xi)\Omega = \xi$. Instead of the previous inequality, the main ingredient of our proof is the following. Let $n \geqslant 1$. Let $(\xi_{\underline{i}})_{|\underline{i}|=n}$ be an orthonormal basis of $H_{\mathbb{C}}^{\otimes n}$ and $(\alpha_{\underline{i}}) \subset B(K)$ a finitely supported family. Then

$$\max_{0\leqslant k\leqslant n}\{\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes R^*_{n,k}\xi_{\underline{i}}\|\} \leqslant \|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes W(\xi_{\underline{i}})\| \leqslant (n+1)C_q \max_{0\leqslant k\leqslant n}\{\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes R^*_{n,k}\xi_{\underline{i}}\|\} \quad (1.1)$$

where the norms in the left and right handside have to be taken in $B(K) \underset{\min}{\otimes} H_c^{\otimes n-k} \otimes_h H_c^{\otimes k}$ (see Theorem 1.3.3 below for the precise statement). Inequality (1) is the vector-valued version of Bożejko's ultracontractivity inequality proved in [B2] and thus it solves a problem posed in [B2]. Using (1) and a careful analysis on the norms of Wick products on a same level, we deduce our non-injectivity result.

The plan of this paper is as follows. The first section is devoted to necessary definitions and preliminaries on the deformation by a Yang-Baxter operator and the associated von Neumann algebra. In this section, we also include a brief discussion on the simplest case, the free case, i.e. when $T = 0$. All our results and arguments become very simple in this case, for instance, inequality (1) above is then easy to state and prove. The proof of the non-injectivity of $\Gamma_0(H_{\mathbb{R}})$ can be done in just a few lines. The reason why we have decided to include such a discussion on the free case is the fact that it already contains the main idea for the general case. In the second section we will establish (1) and prove the non-injectivity of $\Gamma_T(H_{\mathbb{R}})$. The last section aims at proving the non-injectivity of the Araki-Woods factors $\Gamma_q(H, U_t)$ introduced by Hiai in [Hi]. Note that Hiai proved a non-injectivity result with a condition on the dimension of the spectral sets of the positive generator of U_t, which is similar to that of [BS1]. The problem is left open whether the dimension can go down to 2. Although we cannot completely solve this, our method permits to improve in some sense the criterion for non-injectivity given in [Hi].

1.2 Preliminaries

Recall that the free Fock space associated with $H_{\mathbb{R}}$ is given by

$$\mathcal{F}_0(H_{\mathbb{C}}) = \bigoplus_{n\geqslant 0} H_{\mathbb{C}}^{\otimes n}$$

where $H_{\mathbb{C}}^{\otimes 0}$ is by definition $\mathbb{C}\Omega$ with Ω a unit vector called the vacuum.

A Yang-Baxter operator on $H_{\mathbb{C}}\otimes H_{\mathbb{C}}$ is a self-adjoint contraction satisfying the following braid relation:

$$(I\otimes T)(T\otimes I)(I\otimes T) = (T\otimes I)(I\otimes T)(T\otimes I)$$

For $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$ we define T_k on $H_{\mathbb{C}}^{\otimes n}$ by

$$T_k = I_{H_{\mathbb{C}}^{\otimes k-1}} \otimes T \otimes I_{H_{\mathbb{C}}^{\otimes n-k-1}}$$

Let S_n be the group of permutations on a set of n elements. A function φ is defined on S_n by quasi-multiplicative extension of:

$$\varphi(\pi_k) = T_k$$

where $\pi_k = (k, k+1)$ is the transposition exchanging k and $k+1$, $1 \leqslant k \leqslant n-1$. The symmetrizator $P_T^{(n)}$ is the following operator defined on $H_{\mathbb{C}}^{\otimes n}$ by:

$$P_T^{(n)} = \sum_{\sigma \in S_n} \varphi(\sigma)$$

$P_T^{(n)}$ is a positive operator on $H_{\mathbb{C}}^{\otimes n}$ for any Yang-Baxter operator T and is strictly positive if T is strictly contractive (see [BS1]). In the latter case we are allowed to define a new scalar product on $H_{\mathbb{C}}^{\otimes n}$ (for $n \geqslant 2$) by :

$$\langle \xi, \eta \rangle_T = \langle \xi, P_T^{(n)} \eta \rangle$$

The associated norm is denoted by $\|.\|_T$. The deformed Fock space associated with T is then defined by

$$\mathcal{F}_T\left(H_{\mathbb{C}}\right) = \bigoplus_{n \geqslant 0} H_{\mathbb{C}}^{\otimes n}$$

where $H_{\mathbb{C}}^{\otimes n}$ is now equipped with our deformed scalar product for $n \geqslant 2$. From now on we will only consider a strictly contractive Yang-Baxter T and $\|T\| \leqslant q < 1$.

For $f \in H_{\mathbb{R}}$, $a^*(f)$ will denote the creation operator associated with f, and $a(f)$ its adjoint with respect to the T-scalar product:

$$a^*(f)(f_1 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n$$

For $f \in H_{\mathbb{R}}$ the deformed gaussian is the following hermitian operator:

$$G(f) = a^*(f) + a(f)$$

Throughout this paper we are interested in $\Gamma_T(H_{\mathbb{R}})$ which is the von Neumann algebra generated by all gaussians $G(f)$ for $f \in H_{\mathbb{R}}$:

$$\Gamma_T(H_{\mathbb{R}}) = \{G(f) : f \in H_{\mathbb{R}}\}'' \subset B(\mathcal{F}_T\left(H_{\mathbb{C}}\right))$$

Let $(e_i)_{i \in I}$ be an orthonormal basis of $H_{\mathbb{R}}$ and set

$$t_{ij}^{sr} = \langle e_s \otimes e_r, T(e_i \otimes e_j) \rangle$$

Then the following deformed commutation relations hold:

$$a(e_i)a^*(e_j) - \sum_{r,\, s \in I} t_{js}^{ir} a^*(e_r) a(e_s) = \delta_{ij}$$

Moreover if the following condition holds

$$\langle e_s \otimes e_r, T e_i \otimes e_j \rangle = \langle e_r \otimes e_j, T e_s \otimes e_i \rangle$$

which is equivalent to the cyclic condition:

$$t_{ij}^{sr} = t_{si}^{rj}$$

then the vacuum is cyclic and separating for $\Gamma_T(H_\mathbb{R})$ and the vacuum expectation is a faithful trace on $\Gamma_T(H_\mathbb{R})$ that will be denoted by τ. If this cyclic condition holds we say that T is tracial, and from now on we will always assume that T has this property.

We will denote by $\Gamma_T^\infty(H_\mathbb{R})$ the subspace $\Gamma_T(H_\mathbb{R})\Omega$ of $\mathcal{F}_T(H_\mathbb{C})$. Since Ω is separating for $\Gamma_T(H_\mathbb{R})$, for every $\xi \in \Gamma_T^\infty(H_\mathbb{R})$ there exists a unique operator $W(\xi) \in \Gamma_T(H_\mathbb{R})$ such that

$$W(\xi)\Omega = \xi$$

W is called Wick product.

The right creation operator, $a_r^*(f)$, is defined by the following formula:

$$a_r^*(f)(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_n \otimes f$$

We will also denote by $a_r(f)$ the right annihilation operator, which is its adjoint with respect to the T-scalar product, by $G_r(f)$ the right gaussian operator, and by $\Gamma_{T,r}(H_\mathbb{R})$ the von Neumann algebra generated by all right gaussians. It is easy to see that $\Gamma_{T,r}(H_\mathbb{R}) \subset \Gamma_T(H_\mathbb{R})'$. Actually, by Tomita's theory, we have

$$\Gamma_{T,r}(H_\mathbb{R}) = S\Gamma_T(H_\mathbb{R})S = \Gamma_T(H_\mathbb{R})'$$

where S is the anti linear operator on $\mathcal{F}_T(H_\mathbb{C})$ (which is actually an anti unitary) defined by

$$S(f_1 \otimes \cdots \otimes f_n) = f_n \otimes \cdots \otimes f_1$$

for any $f_1, \cdots, f_n \in H_\mathbb{R}$. Since Ω is also separating for $\Gamma_{T,r}(H_\mathbb{R})$ we can define the right Wick product, that will be denoted by $W_r(\xi)$. For any $\xi \in \Gamma_T^\infty(H_\mathbb{R})$ we have

$$(W(\xi))^* = W(S\xi) \qquad \text{and} \qquad SW(\xi)S = W_r(S\xi)$$

Some particular cases of deformation have been studied in the literature. Let $(q_{ij})_{i,\, j\in I}$ be a hermitian matrix such that $\sup_{i,j} |q_{ij}| < 1$. Define

$$Te_i \otimes e_j = q_{ij} e_j \otimes e_i$$

Then T is a strictly contractive Yang-Baxter operator, and it is tracial if and only if the q_{ij} are real. Our deformed Fock space is then a realization of the following q_{ij}-relations:

$$a(e_i)a^*(e_j) - q_{ij}a^*(e_j)a(e_i) = \delta_{ij}$$

In the special case where all q_{ij} are equal, we obtain the well known q-relations.

Let us define the following selfadjoint unitary on the free Fock space:

$$\forall\, f_1, \ldots, f_n \in H_\mathbb{C}, \quad U(f_1 \otimes \cdots \otimes f_n) = f_n \otimes \cdots \otimes f_1$$

Since $UP_T^{(n)} = P_T^{(n)}U$ (see [K]), U is also a selfadjoint unitary on each T-Fock space.

Given vectors $f_1, \dots, f_n$ in $H_{\mathbb{R}}$ we define:

$$a^*(f_1 \otimes \cdots \otimes f_n) = a^*(f_1) \dots a^*(f_n) \quad \text{and} \quad a(f_1 \otimes \cdots \otimes f_n) = a(f_1) \dots a(f_n)$$

For $0 \leqslant k \leqslant n$, let $R_{n,k}$ be the operator on $H_{\mathbb{C}}^{\otimes n}$ given by

$$R_{n,k} = \sum_{\sigma \in S_n / S_{n-k} \times S_k} \varphi(\sigma^{-1})$$

where the sum runs over the representatives of the right cosets of $S_{n-k} \times S_k$ in S_n with minimal number of inversions. Then

$$P_T^{(n)} = R_{n,k} \left(P_T^{(n-k)} \otimes P_T^{(k)} \right) \quad \text{and} \quad \|R_{n,k}\| \leqslant C_q \tag{1.2}$$

where $C_q = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$ (see [B2] and [K]). It follows that

$$P_T^{(n)} \leqslant C_q P_T^{(n-k)} \otimes P_T^{(k)} \tag{1.3}$$

It also follows that a^*, respectively a, extend linearly, respectively antilinearly, and continuously to $H_{\mathbb{C}}^{\otimes n}$ for every $n \geqslant 1$. Then for each vector $\xi \in H_{\mathbb{C}}^{\otimes n}$ we have

$$\|a^*(\xi)\| \leqslant C_q^{\frac{1}{2}} \|\xi\|_T \quad \text{and} \quad (a^*(\xi))^* = a(U\xi). \tag{1.4}$$

Let $n \geqslant 1$ and $1 \leqslant k \leqslant n$, $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$ will be the Hilbert tensor product of the Hilbert spaces $H_{\mathbb{C}}^{\otimes k}$ and $H_{\mathbb{C}}^{\otimes n-k}$ where both $H_{\mathbb{C}}^{\otimes k}$ and $H_{\mathbb{C}}^{\otimes n-k}$ are equipped with the T-scalar product.

Lemma 1.2.1 *There is a positive constant $D_{q,n,k}$ such that*

$$P_T^{(n-k)} \otimes P_T^{(k)} \leqslant D_{q,n,k} P_T^{(n)}$$

Consequently for every $n \geqslant 1$ and $1 \leqslant k \leqslant n$, $H_{\mathbb{C}}^{\otimes n}$ and $H_{\mathbb{C}}^{\otimes k} \otimes H_{\mathbb{C}}^{\otimes n-k}$ are algebraically the same and their norms are equivalent.

Remark. It is still not known whether one can choose $D_{q,n,k}$ independent of n and k.

Proof. It was shown in [B1] that there is a positive constant $\omega(q)$ such that

$$P_T^{(n-1)} \otimes I \leqslant \omega(q)^{-1} P_T^{(n)}$$

Since $U(P_T^{(n-1)} \otimes I)U = I \otimes P_T^{(n-1)}$ we also have

$$I \otimes P_T^{(n-1)} \leqslant \omega(q)^{-1} P_T^{(n)} \tag{1.5}$$

Fix some k, $2 \leqslant k \leqslant n-1$, using (3) and (4) we get:

$$\begin{aligned} P_T^{(n-k+1)} \otimes P_T^{(k-1)} &\leqslant C_q P_T^{(n-k)} \otimes I \otimes P_T^{(k-1)} \\ &\leqslant C_q \omega(q)^{-1} P_T^{(n-k)} \otimes P_T^{(k)} \end{aligned}$$

Thus by iteration it follows that for $0 \leqslant k \leqslant n$:

$$P_T^{(n-k)} \otimes P_T^{(k)} \leqslant \omega(q)^{-1}(C_q\omega(q)^{-1})^{n-k}P_T^{(n)} \tag{1.6}$$

Since $U(P_T^{(n-k)} \otimes P_T^{(k)})U = P_T^{(k)} \otimes P_T^{(n-k)}$ it follows from (6) that

$$P_T^{(k)} \otimes P_T^{(n-k)} \leqslant \omega(q)^{-1}(C_q\omega(q)^{-1})^{n-k}P_T^{(n)}$$

Combining this last inequality and (6) we finally obtain:

$$P_T^{(n-k)} \otimes P_T^{(k)} \leqslant \omega(q)^{-1}(C_q\omega(q)^{-1})^{\min(k,n-k)}P_T^{(n)} \tag{1.7}$$

Then the desired result follows from (3) and (7). □

For $k \geqslant 0$ let us now define on the family of finite linear combinations of elementary tensors of length not less than k the following operator U_k:

$$U_k(f_1 \otimes \cdots \otimes f_n) = a^*(f_1 \otimes \cdots \otimes f_{n-k})a(\overline{f}_{n-k+1} \otimes \cdots \otimes \overline{f}_n)$$

where $\overline{\xi + i\eta} = \xi - i\eta$ for all $\xi,\ \eta \in H_{\mathbb{R}}$.

Fix n and k with $n \geqslant k$. Let $\mathcal{J} :\ H_{\mathbb{C}}^{\otimes k} \to \overline{H_{\mathbb{C}}^{\otimes k}}$ be the conjugation (which is an anti isometry). For any $f_1, \cdots, f_n$, $\mathcal{J}$ is defined by $\mathcal{J}(f_1 \otimes \cdots \otimes f_n) = \overline{f_1} \otimes \cdots \otimes \overline{f_n}$. It is clear that U_k extends boundedly to $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$ by the formula:

$$U_k = M(a^* \otimes a\mathcal{J})$$

where M is the multiplication operator from $B(\mathcal{F}_T(H_{\mathbb{C}})) \underset{\min}{\otimes} B(\mathcal{F}_T(H_{\mathbb{C}}))$ to $B(\mathcal{F}_T(H_{\mathbb{C}}))$ defined by $M(A \otimes B) = AB$. Moreover, by (1.4) we have

$$\|U_k\| \leqslant \|M\|.\|a^* \otimes a\mathcal{J}\| \leqslant C_q$$

where U_k is viewed as an operator from $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$ to $B(\mathcal{F}_T(H_{\mathbb{C}}))$.

In the following lemma we state an extension of the Wick formula (Theorem 3 in [K]). We deduce it as an easy consequence of the original Wick formula and of our previous discussion.

Lemma 1.2.2 *Let $n \geqslant 1$ and $\xi \in H_{\mathbb{C}}^{\otimes n}$, then $H_{\mathbb{C}}^{\otimes n} \subset \Gamma_T^{\infty}(H_{\mathbb{R}})$ and we have the following Wick formula:*

$$W(\xi) = \sum_{k=0}^{n} U_k R_{n,k}^*(\xi) \tag{1.8}$$

Moreover

$$\|\xi\|_q \leqslant \|W(\xi)\| \leqslant C_q^{\frac{3}{2}}(n+1)\|\xi\|_q \tag{1.9}$$

Remark. (1.9) is the well known Bożejko's inequality discussed in [B2] and [K], and which implies the ultracontractivity of the q-Ornstein Uhlenbeck semigroup. We include an elementary and simple proof.

Proof. The usual Wick formula is the following (see [B2] and [K]): $\forall f_1, \ldots, f_n \in H_{\mathbb{C}}$ we have

$$W(f_1 \otimes \cdots \otimes f_n) = \sum_{k=0}^{n} \sum_{\sigma \in S_n/S_{n-k}\times S_k} U_k \varphi(\sigma)(f_1 \otimes \cdots \otimes f_n)$$

Hence (1.8) holds for every $\xi \in \mathcal{A}_n$, the vector space of linear combinations of elementary tensors of length n. By Lemma 1.2.1 and our previous discussion, the right handside of (1.8) is continuous from $H_{\mathbb{C}}^{\otimes n}$ to $B(\mathcal{F}_T(H_{\mathbb{C}}))$. Since Ω is separating, it follows that $H_{\mathbb{C}}^{\otimes n} \subset \Gamma_T^{\infty}(H_{\mathbb{R}})$ and that (1.8) extends by density from $\mathcal{A}_n$ to $H_{\mathbb{C}}^{\otimes n}$. Actually, our argument shows that for any $\xi \in H^{\otimes n}$, $W(\xi)$ belongs to $C_T^*(H_{\mathbb{R}})$ which is the C^*-algebra generated by the T-gaussians.

Since for any $\xi \in H_{\mathbb{C}}^{\otimes n}$, $W(\xi)\Omega = \xi$, the left inequality in (1.9) holds. We have just showed that W is bounded from $H_{\mathbb{C}}^{\otimes n}$ to $B(\mathcal{F}_T(H_{\mathbb{C}}))$. Hence, there is a constant $B_{q,n}$ such that for any $\xi \in H_{\mathbb{C}}^{\otimes n}$ we have $\|W(\xi)\| \leqslant B_{q,n}\|\xi\|_q$. To end the proof of (1.9) we now give a precise estimate of $B_{q,n}$. Let $\xi \in H_{\mathbb{C}}^{\otimes n}$, by (1.8) and (3) we have

$$\|W(\xi)\| \leqslant \sum_{k=0}^{n} \|U_k R_{n,k}^*(\xi)\| \leqslant C_q \sum_{k=0}^{n} \|R_{n,k}^*(\xi)\|_{H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}} \tag{1.10}$$

It remains to compute the norm of $R_{n,k}^*$ as an operator from $H_{\mathbb{C}}^{\otimes n}$ to $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$. Let $\eta \in H_{\mathbb{C}}^{\otimes n}$ we have, by (2) and (3)

$$\begin{aligned} \|R_{n,k}^*\eta\|^2_{H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}} &= \langle P_T^{(n-k)} \otimes P_T^{(k)} R_{n,k}^*\eta, R_{n,k}^*\eta\rangle_0 \\ &= \langle P_T^{(n)}\eta, R_{n,k}^*\eta\rangle_0 \leqslant \|\eta\|_T \|R_{n,k}^*\eta\|_T \end{aligned}$$

On the other hand,

$$\begin{aligned} \|R_{n,k}^*\eta\|_T^2 &= \langle P_T^{(n)} R_{n,k}^*\eta, R_{n,k}^*\eta\rangle_0 \leqslant C_q \langle P_T^{(n-k)} \otimes P_T^{(k)} R_{n,k}^*\eta, R_{n,k}^*\eta\rangle_0 \\ &\leqslant C_q \langle P_T^{(n)}\eta, R_{n,k}^*\eta\rangle_0 \\ &\leqslant C_q \|\eta\|_T \|R_{n,k}^*\eta\|_T \end{aligned}$$

Hence it follows that $\|R_{n,k}^*\eta\|_T \leqslant C_q\|\eta\|_T$ and $\|R_{n,k}^*\eta\|^2_{H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}} \leqslant C_q\|\eta\|_T^2$. Thus $\|R_{n,k}^*\| \leqslant C_q^{\frac{1}{2}}$ as an operator from $H_{\mathbb{C}}^{\otimes n}$ to $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$. From (1.10) and this last estimate, follows the second inequality in (1.9).

□

The remainder of this section is devoted to a simple proof of the non-injectivity of the free von Neumann algebra $\Gamma_0(H_{\mathbb{R}})$ ($\dim H_{\mathbb{R}} \geqslant 2$). The main ingredient is the vector valued Bożejko inequality (Lemma 1.2.3 below), which is the free Fock space analogue of the corresponding inequality for the free groups proved by Haagerup and Pisier in [HP] and extended by Buchholz in [Bu2]. Note also that the inequality (11) below was first proved in [HP] in the case $n = 1$ (i.e. for free gaussians) and that a similar inequality holds for products of free gaussians (see [Bu2]).

We will need the following notations: $(e_i)_{i\in I}$ will denote an orthonormal basis of $H_{\mathbb{R}}$, and for a multi-index $\underline{i}$ of length n, $\underline{i} = (i_1, \ldots, i_n) \in I^n$, $e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_n}$. $(e_{\underline{i}})_{|\underline{i}|=n}$ is a real orthonormal basis of $H_{\mathbb{C}}^{\otimes n}$ equipped with the free scalar product and $(e_{\underline{i}})_{|\underline{i}|\geqslant 0}$ is a real orthonormal basis of the free Fock space.

Lemma 1.2.3 *Let $n \geqslant 1$, K a complex Hilbert space and $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ a finitely supported family of $B(K)$. Then:*

$$\max_{0\leqslant k\leqslant n}\left\{\left\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\|\right\} \leqslant \left\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes W(e_{\underline{i}})\right\| \leqslant (n+1)\max_{0\leqslant k\leqslant n}\left\{\left\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\|\right\} \qquad (1.11)$$

Remark. Since $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ is finitely supported the operator-coefficient matrix $(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}$ is a finite matrix, say a $r\times s$ matrix, and its norm is the operator norm in $B(l_2^s(K), l_2^r(K))$.

Proof. We write

$$\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes W(e_{\underline{i}}) = \sum_{k=0}^{n} F_k$$

where

$$F_k = \sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\alpha_{\underline{j},\,\underline{l}}\otimes a^*(e_{\underline{j}})a(e_{\underline{l}})$$

we have

$$F_k = (\dots I_K\otimes a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}(\alpha_{\underline{j},\,\underline{l}}\otimes I_{\mathcal{F}_0(H_{\mathbb{C}})})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\begin{pmatrix}\vdots\\ I_K\otimes a(e_{\underline{l}})\\ \vdots\end{pmatrix}_{|\underline{l}|=k}$$

that is, F_k is a product of three matrices, the first is a row indexed by $\underline{j}$, the third a column indexed by $\underline{l}$. Note that

$$\|(\dots a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}\|^2 = \|\sum_{|\underline{j}|=n-k}a^*(e_{\underline{j}})(a^*(e_{\underline{j}}))^*\| = \|\sum_{|\underline{j}|=n-k}a^*(e_{\underline{j}})a(Ue_{\underline{j}})\|$$

It is easy to see that $\sum\limits_{|\underline{j}|=n-k}a^*(e_{\underline{j}})a(Ue_{\underline{j}})$ is the orthogonal projection on $\bigoplus\limits_{p\geqslant n-k}H^{\otimes p}$.

Thus

$$\|(\dots a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}\|\leqslant 1$$

Therefore

$$\begin{aligned}\|F_k\| &\leqslant \|(\dots I_K\otimes a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}\|.\|(\alpha_{\underline{j},\,\underline{l}}\otimes I_{\mathcal{F}_0(H_{\mathbb{C}})})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|.\left\|\begin{pmatrix}\vdots\\ I_K\otimes a(e_{\underline{l}})\\ \vdots\end{pmatrix}_{|\underline{l}|=k}\right\|\\ &\leqslant \|(\dots a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}\|.\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|.\|(\dots a^*(Ue_{\underline{l}})\dots)_{|\underline{l}|=k}\|\\ &\leqslant \|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|\end{aligned}$$

It follows that

$$\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes W(e_{\underline{i}})\| \leqslant \sum_{k=0}^{n} \|F_k\| \leqslant (n+1) \max_{0\leqslant k\leqslant n} \|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}}\|$$

To prove the first inequality, fix $0 \leqslant k_0 \leqslant n$ and consider $(v_{\underline{p}})_{|\underline{p}|=k_0}$ such that $\sum_{|\underline{p}|=k_0} \|v_{\underline{p}}\|^2 < +\infty$. Let $\eta = \sum_{|\underline{p}|=k_0} v_{\underline{p}} \otimes U e_{\underline{p}}$. We have :

$$\begin{aligned}
\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes W(e_{\underline{i}})\eta\|^2 &= \sum_{k=0}^{n} \|F_k \eta\|^2 \geqslant \|F_{k_0}\eta\|^2 \\
&= \| \sum_{\substack{|\underline{j}|=n-k_0 \\ |\underline{l}|=k_0}} \alpha_{\underline{j},\,\underline{l}} v_{\underline{l}} \otimes e_{\underline{j}}\|^2 \\
&= \sum_{|\underline{j}|=n-k_0} \| \sum_{|\underline{l}|=k_0} \alpha_{\underline{j},\,\underline{l}} v_{\underline{l}}\|^2 \\
&= \left\| (\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k_0 \\ |\underline{l}|=k_0}} \begin{pmatrix} \vdots \\ v_{\underline{l}} \\ \vdots \end{pmatrix}_{|\underline{l}|=k_0} \right\|^2
\end{aligned}$$

Then the result follows. □

Using Lemma 1.2.3, it is now easy to prove that $\Gamma_0(H_{\mathbb{R}})$ is not injective as soon as $\dim H_{\mathbb{R}} \geqslant 2$. Suppose that $\Gamma_0(H_{\mathbb{R}})$ is injective and $\dim H_{\mathbb{R}} \geqslant 2$. Choose two orthonormal vectors e_1 and e_2 in $H_{\mathbb{R}}$. For $n \geqslant 1$ we have by semi-discreteness (which is equivalent to the injectivity):

$$\tau\left(\sum_{|\underline{i}|=n} W(e_{\underline{i}})^* W(e_{\underline{i}})\right) \leqslant \| \sum_{|\underline{i}|=n} \overline{W(e_{\underline{i}})} \otimes W(e_{\underline{i}})\|$$

where in the above sums, the index $\underline{i} \in \{1,2\}^n$. However,

$$\begin{aligned}
\tau\left(\sum_{|\underline{i}|=n} W(e_{\underline{i}})^* W(e_{\underline{i}})\right) &= \sum_{|\underline{i}|=n} \langle W(e_{\underline{i}})\Omega, W(e_{\underline{i}})\Omega\rangle_0 \\
&= \sum_{|\underline{i}|=n} \|e_{\underline{i}}\|^2 = 2^n
\end{aligned}$$

On the other hand, by Lemma 1.2.3,

$$\begin{aligned}
\| \sum_{|\underline{i}|=n} \overline{W(e_{\underline{i}})} \otimes W(e_{\underline{i}})\| &\leqslant (n+1) \max_{0\leqslant k\leqslant n} \{\|(\overline{W(e_{\underline{j},\,\underline{l}})})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}}\|\} \\
&\leqslant (n+1) \left(\sum_{|\underline{i}|=n} \|W(e_{\underline{i}})\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned} &\leqslant (n+1)(2^n(n+1)^2)^{\frac{1}{2}} \\ &\leqslant (n+1)^2 2^{\frac{n}{2}} \end{aligned}$$

Combining the preceding inequalities, we get $2^n \leqslant (n+1)^2 2^{\frac{n}{2}}$ which yields a contradiction for sufficiently large n. Therefore, $\Gamma_0(H_{\mathbb{R}})$ is not injective if dim $H_{\mathbb{R}} \geqslant 2$.

1.3 Generalized Haagerup-Bożejko inequality and non injectivity of $\Gamma_T(H_{\mathbb{R}})$

In the following we state and prove the generalized inequality (1). It actually solves a question of Marek Bożejko (in [B2] page 210) whether it is possible to find an operator coefficient version of the following inequality (this is inequality (1.9) in Lemma 1.2.2):

$$\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} e_{\underline{i}}\| \leqslant \|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} W(e_{\underline{i}})\| \leqslant C_q^{\frac{3}{2}}(n+1)\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} e_{\underline{i}}\| \tag{1.12}$$

where $(\alpha_{\underline{i}})_{\underline{i}}$ is a finitely supported family of complex numbers. Inequality (12) was proved in [B2] for the q-deformation, and generalized in [K] for the Yang-Baxter deformation.

First, we need to recall some basic notions from operator space theory. We refer to [ER] and [P] for more information.
Given K a complex Hilbert space, we can equip K with the column, respectively the row, operator space structure denoted by K_c, respectively K_r, and defined by

$$K_c = B(\mathbb{C}, K) \qquad \text{and} \qquad K_r = B(K^*, \mathbb{C}).$$

Moreover, we have $K_c^* = \overline{K}_r$ as operator spaces.

Given two operator spaces E and F, let us briefly recall the definition of the Haagerup tensor product of E and F. $E \otimes F$ will denote the algebraic tensor product of E and F. For $n \geqslant 1$ and $x = (x_{i,j})$ belonging to $M_n(E \otimes F)$ we define

$$\|x\|_{(h,n)} = \inf\{\|y\|_{M_{n,r}(E)}\|z\|_{M_{r,n}(F)}\}$$

where the infimum runs over all $r \geqslant 1$ and all decompositions of x of the form

$$x_{i,j} = \sum_{k=1}^{r} y_{i,k} \otimes z_{k,j}.$$

By Ruan's theorem, this sequence of norms define an operator space structure on the completion of $E \otimes F$ equipped with $\| \,.\, \|_h = \| \,.\, \|_{(h,1)}$. The resulting operator space, which is called the Haagerup tensor product of E and F is denoted by $E \otimes_h F$.
In this setting, a bilinear map $u: E \times F \to B(K)$ is said to be completely bounded, in short c.b, if and only if the associated linear map $\hat{u}: E \otimes F \to B(K)$ extends completely boundedly to $E \otimes_h F$. We define $\|u\|_{cb} = \|\hat{u}\|_{cb}$. This notion goes back to Christensen and Sinclair [CS].

We will often use the following classical identities for Hilbertian operator spaces:

$$K_c \underset{\min}{\otimes} H_r = K_c \otimes_h H_r = \mathcal{K}(\overline{H}, K),$$

where $\mathcal{K}$ stands for the compact operators and

$$K_c \underset{\min}{\otimes} H_c = K_c \otimes_h H_c = (K \otimes_2 H)_c$$

and similarly for rows using duality.

There is another notion of complete boundedness for bilinear maps, called jointly complete boundedness. Let E, F be operator spaces, K a complex Hilbert space, and $u: E \times F \to B(K)$ a bilinear map. u is said to be jointly completely bounded (in short j.c.b) if and only if for any C*-algebras B_1 and B_2, u can be boundedly extended to a bilinear map $(u)_{B_1,B_2}: E \underset{\min}{\otimes} B_1 \times F \underset{\min}{\otimes} B_2 \to B(K) \underset{\min}{\otimes} B_1 \underset{\min}{\otimes} B_2$ taking $(e \otimes b_1,\ f \otimes b_2)$ to $u(e,\ f) \otimes b_1 \otimes b_2$. We put $\|u\|_{jcb} = \sup\limits_{B_1,\, B_2} \|(u)_{B_1,B_2}\|$. Observe that in this definition B_1 and B_2 can be replaced by operator spaces.

We will need the fact that every bilinear c.b map is a j.c.b map with $\|u\|_{jcb} \leqslant \|u\|_{cb}$. Let K be a complex Hilbert space and $u: B(K) \times K_c \to K_c$ the bilinear map taking $(\varphi,\ k)$ to $\varphi(k)$. Then it is easy to see that u is a norm one bilinear cb map.
To simplify our notations, $H_{\mathbb{C}}$ will be, most of the time, replaced by H in the rest of this section. For the same reason we will denote by $H_c^{\otimes n}$ (respectively $H_r^{\otimes n}$) the column Hilbert space $(H_{\mathbb{C}}^{\otimes n})_c$ (respectively the row Hilbert space $(H_{\mathbb{C}}^{\otimes n})_r$).

Lemma 1.3.1 *Let $n \geqslant 1$. The mappings $a^*: H_c^{\otimes n} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ and $a: \overline{H}_r^{\otimes n} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ are completely bounded with cb-norms less than $\sqrt{C_q}$.*

Proof. Let us start with the proof of the statement concerning a^*. Let $n \geqslant 1$, K a complex Hilbert space and $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ a finitely supported family of $B(K)$ such that

$$\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes e_{\underline{i}}\|_{B(K) \underset{\min}{\otimes} H_c} < 1.$$

Then, since the maps $a^*(e_{\underline{i}})$ acts diagonally with respect to degrees of tensors in $\mathcal{F}_T(H_{\mathbb{C}})$,

$$\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(e_{\underline{i}})\|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_T(H_{\mathbb{C}}))} = \sup_{k \geqslant 0} \|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(e_{\underline{i}})\|_{B(K) \underset{\min}{\otimes} B(H^{\otimes k}, H^{\otimes n+k})}$$

To compute the right term, fix $k \geqslant 0$ and let $(\xi_{\underline{j}})_{|\underline{j}|=k}$ be a finitely supported family of vectors in K such that

$$\|\sum_{|\underline{j}|=k} \xi_{\underline{j}} \otimes e_{\underline{j}}\|_{K \otimes_2 H^{\otimes k}} < 1.$$

By (3) we have

$$\|\sum_{\underline{i},\, \underline{j}} \alpha_{\underline{i}}(\xi_{\underline{j}}) \otimes e_{\underline{i}} \otimes e_{\underline{j}}\|_{K \otimes_2 H^{\otimes n+k}} \leqslant C_q^{\frac{1}{2}} \|\sum_{\underline{i},\, \underline{j}} \alpha_{\underline{i}}(\xi_{\underline{j}}) \otimes e_{\underline{i}} \otimes e_{\underline{j}}\|_{K \otimes_2 H^{\otimes n} \otimes_2 H^{\otimes k}}.$$

Let $u: B(K) \times K_c \to K_c$ given by $(\varphi, \xi) \mapsto \varphi(\xi)$. Recall that $\|u\|_{cb} = 1$. Consequently, $\|u\|_{jcb} \leqslant 1$. Therefore, we deduce

$$\|\sum_{\underline{i},\,\underline{j}} \alpha_{\underline{i}}(\xi_{\underline{j}}) \otimes e_{\underline{i}} \otimes e_{\underline{j}}\|_{K\otimes_2 H^{\otimes n}\otimes_2 H^{\otimes k}} = \|\sum_{\underline{i},\,\underline{j}} \alpha_{\underline{i}}(\xi_{\underline{j}}) \otimes e_{\underline{i}} \otimes e_{\underline{j}}\|_{K_c \underset{\min}{\otimes} H_c^{\otimes n} \underset{\min}{\otimes} H_c^{\otimes k}}$$

$$= \|(u)_{H_c^{\otimes n},\, H_c^{\otimes k}}(\sum_{\underline{i}} \alpha_{\underline{i}} \otimes e_{\underline{i}},\ \sum_{\underline{j}} \xi_{\underline{j}} \otimes e_{\underline{j}})\|$$

$$\leqslant \|u\|_{jcb} \|\sum_{\underline{i}} \alpha_{\underline{i}} \otimes e_{\underline{i}}\|_{B(K) \underset{\min}{\otimes} H_c^{\otimes n}} \|\sum_{\underline{j}} \xi_{\underline{j}} \otimes e_{\underline{j}}\|_{K_c \underset{\min}{\otimes} H_c^{\otimes k}}$$

$$\leqslant 1$$

By the result just proved, for any complex Hilbert space K and for any finitely supported family $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ of $B(K)$ we have

$$\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(e_{\underline{i}})\|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_T(H_{\mathbb{C}}))} \leqslant \sqrt{C_q} \|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes e_{\underline{i}}\|_{B(K) \underset{\min}{\otimes} H_c^{\otimes n}}$$

Taking adjoints on both sides we get

$$\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}}^* \otimes a(Ue_{\underline{i}})\|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_T(H_{\mathbb{C}}))} \leqslant \sqrt{C_q} \|\sum_{|\underline{i}|=n} \alpha_{\underline{i}}^* \otimes \overline{e_{\underline{i}}}\|_{B(K) \underset{\min}{\otimes} \overline{H}_r^{\otimes n}}$$

Changing $\alpha_{\underline{i}}^*$ to $\alpha_{\underline{i}}$ and using the fact that U (reversing the order of tensor) is a complete isometry on $H_r^{\otimes n}$, we get that for any finitely supported family $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ of $B(K)$ we have

$$\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a(\overline{e_{\underline{i}}})\|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_T(H_{\mathbb{C}}))} \leqslant \sqrt{C_q} \|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes \overline{e_{\underline{i}}}\|_{B(K) \underset{\min}{\otimes} \overline{H}_r^{\otimes n}}.$$

In other words,

$$a : \overline{H}_r^{\otimes n} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$$

is also completely bounded with norm less than $\sqrt{C_q}$. □

Corollary 1.3.2 *For any $n \geqslant 0$, and any $k \in \{0...n\}$,*

$$U_k : \ H_c^{\otimes n-k} \otimes_h H_r^{\otimes k} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$$

is completely bounded with cb-norm less than C_q.

Proof. Let us denote by M the multiplication map from $B(\mathcal{F}_T(H_{\mathbb{C}})) \otimes_h B(\mathcal{F}_T(H_{\mathbb{C}}))$ into $B(\mathcal{F}_T(H_{\mathbb{C}}))$ given by $A \otimes B \mapsto AB$, M is obviously completely contractive. We have the formula

$$U_k = M(a^* \otimes a\mathcal{J})$$

if $\mathcal{J} : H^{\otimes k} \to \overline{H}^{\otimes k}$ is the conjugation (which is a complete isometry). By injectivity of the Haagerup tensor product and by Lemma 1.3.1 we deduce that

$$\|a^* \otimes a\mathcal{J}\|_{cb} \leqslant C_q$$

Then

$$\|U_k\|_{cb} \leqslant \|M\|_{cb}\|a^* \otimes a\mathcal{J}\|_{cb} \leqslant C_q$$

□

Recall that, by definition, $\Gamma_T^\infty(H_\mathbb{R})$ is identified with $\Gamma_T(H_\mathbb{R})$ by the mapping sending ξ to $W(\xi)$. Thus $\Gamma_T^\infty(H_\mathbb{R})$ inherits the operator space structure of $\Gamma_T(H_\mathbb{R})$. In particular for all $n \geqslant 0$, $H^{\otimes n}$ will be equipped with the operator space structure of $E_n = \{W(\xi),\ \xi \in H^{\otimes n}\}$.

Theorem 1.3.3 below was first obtained via elementary, but long, computations. In the version presented here, we have chosen to follow an approach indicated to us by Eric Ricard. This approach is much more transparent but involves some notions of operator space theory (see the Annexe for the original proofs).

Theorem 1.3.3 *Let K be a complex Hilbert space. Then for all $n \geqslant 0$ and for all $\xi \in B(K) \underset{\min}{\otimes} H^{\otimes n}$ we have*

$$\max_{0\leqslant k\leqslant n} \|(Id \otimes R^*_{n,\,k})(\xi)\| \leqslant \|(Id \otimes W)(\xi)\|_{\min} \leqslant C_q(n+1) \max_{0\leqslant k\leqslant n} \|(Id \otimes R^*_{n,\,k})(\xi)\| \quad (1.13)$$

where Id denotes the identity mapping of $B(K)$, and where the norm $\|(Id \otimes R^*_{n,\,k})(\xi)\|$ is that of $B(K) \underset{\min}{\otimes} H_c^{\otimes n-k} \underset{\min}{\otimes} H_r^{\otimes k}$.

Proof. For the second inequality, we use the Wick formula:

$$W\,|_{H^{\otimes n}} = \sum_{k=0}^{n} U_k\, R^*_{n,k}.$$

Let $\xi \in B(K) \underset{\min}{\otimes} H^{\otimes n}$, then by corollary 1.3.2

$$\|(Id \otimes W)(\xi)\|_{\min} \leqslant C_q \sum_{k=0}^{n} \|(Id \otimes R^*_{n,\,k})(\xi)\|$$

which yields the majoration.

For the minoration, for $x \in H_c^{\otimes n-k} \otimes H_r^{\otimes k} \subset B(\overline{H}^{\otimes k}, H^{\otimes n-k})$, we claim that

$$P_{n-k}U_k(x)\,\big|_{H^{\otimes k}} = x(U\mathcal{J}) \quad (1.14)$$

where P_{n-k} is the projection on tensors of rank $n-k$ in $\mathcal{F}_T\,(H_\mathbb{C})$. Assuming this claim and recalling that U and $\mathcal{J}$ are (anti)-isometry, we get that for any $x \in B(K) \underset{\min}{\otimes} H_c^{\otimes n-k} \underset{\min}{\otimes} H_r^{\otimes k}$

$$\|x\|_{B(K) \underset{\min}{\otimes} H_c^{\otimes n-k} \underset{\min}{\otimes} H_r^{\otimes k}} \leqslant \|P_{n-k}\|_{B(\mathcal{F}_T(H_\mathbb{C}))}\|(Id \otimes U_k)(x)\|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_T(H_\mathbb{C}))}$$

The conclusion follows applying this inequality to $x = (Id \otimes R^*_{n,\,k})(\xi)$

To prove (1.14), it suffices to consider an elementary tensor product with entries in any basis of H, say $x = e_{\underline{i}} \otimes e_{\underline{j}}$. Consider $e_{\underline{l}} \in H^{\otimes k}$, a lenght argument gives that $a(\mathcal{J}e_{\underline{j}}).e_{\underline{l}}$ is of the form $\lambda\Omega$, with

$$\lambda = \langle a(\mathcal{J}e_{\underline{j}}).e_{\underline{l}}, \Omega\rangle = \langle e_{\underline{l}}, \mathcal{J}Ue_{\underline{j}}\rangle$$

We deduce that

$$P_{n-k}U_k(e_{\underline{i}}\otimes e_{\underline{j}}).e_{\underline{l}} = \langle e_{\underline{l}}, U\mathcal{J}e_{\underline{j}}\rangle\, e_{\underline{i}}.$$

On the other hand, viewing x as an operator, we compute

$$x(\mathcal{J}U).e_{\underline{l}} = x.(\mathcal{J}Ue_{\underline{l}}) = \langle e_{\underline{j}}, \mathcal{J}Ue_{\underline{l}}\rangle\, e_{\underline{i}}$$

But since U is unitary and $\mathcal{J}$ antiunitary,

$$\langle e_{\underline{j}}, \mathcal{J}Ue_{\underline{l}}\rangle = \langle e_{\underline{l}}, U\mathcal{J}e_{\underline{j}}\rangle$$

This ends the proof. □

The following theorem is our main result.

Theorem 1.3.4 *$\Gamma_T(H_{\mathbb{R}})$ is not injective as soon as $\dim(H_{\mathbb{R}})\geqslant 2$.*

Proof. Let $d\leqslant \dim H_{\mathbb{R}}$. For all $n\geqslant 0$, $(\xi_{\underline{i}})_{|\underline{i}|=n}$ will denote a real orthonormal family of $H^{\otimes n}$ equipped with the T-scalar product of cardinal d^n. For example one can take $\xi_{\underline{i}} = (P_T^{(n)})^{-\frac{1}{2}}e_{\underline{i}}$.

Suppose that $\Gamma_T(H_{\mathbb{R}})$ is injective. Fix $n\geqslant 1$. By injectivity we have,

$$\tau(\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})^*W(\xi_{\underline{i}})) \leqslant \|\sum_{|\underline{i}|=n}\overline{W(\xi_{\underline{i}})}\otimes W(\xi_{\underline{i}})\|$$

It is clear that

$$\tau(\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})^*W(\xi_{\underline{i}})) = d^n$$

On the other hand, applying twice (13) consecutively

$$\|\sum_{|\underline{i}|=n}\overline{W(\xi_{\underline{i}})}\otimes W(\xi_{\underline{i}})\| \leqslant (n+1)^2C_q^2 \max_{0\leqslant k,k'\leqslant n}\{\|\sum_{|\underline{i}|=n}\overline{R^*_{n,k'}(\xi_{\underline{i}})}\otimes R^*_{n,k}(\xi_{\underline{i}})\|\}$$

The norms are computed in $\overline{H}_c^{\otimes n-k'}\underset{\min}{\otimes}\overline{H}_r^{\otimes k'}\underset{\min}{\otimes}H_c^{\otimes n-k}\underset{\min}{\otimes}H_r^{\otimes k}$ for fixed k and k'. We can rearrange this tensor product and use the comparison with the Hilbert Schmidt norm : Let $t=\sum_{|\underline{i}|=n}\overline{R^*_{n,k'}(\xi_{\underline{i}})}\otimes R^*_{n,k}(\xi_{\underline{i}})$,

$$\begin{aligned}
\|t\|_{\overline{H}_c^{\otimes n-k'}\underset{\min}{\otimes}\overline{H}_r^{\otimes k'}\underset{\min}{\otimes}H_c^{\otimes n-k}\underset{\min}{\otimes}H_r^{\otimes k}} &= \|t\|_{(\overline{H}^{\otimes n-k'}\otimes_2 H^{\otimes n-k})_c\underset{\min}{\otimes}(\overline{H}^{\otimes k'}\otimes_2 H^{\otimes k})_r}\\
&\leqslant \|t\|_{(\overline{H}^{\otimes n-k'}\otimes_2 H^{\otimes n-k})\otimes_2(\overline{H}^{\otimes k'}\otimes_2 H^{\otimes k})}\\
&\leqslant \|t\|_{\overline{H}^{\otimes n-k'}\otimes_2\overline{H}^{\otimes k'}\otimes_2 H^{\otimes n-k}\otimes_2 H^{\otimes k}}
\end{aligned}$$

Finally, we use the estimates on $R^*_{n,k}$:

$$\begin{aligned}
\|t\|_{\overline{H}_c^{\otimes n-k'}\underset{\min}{\otimes}\overline{H}_r^{\otimes k'}\underset{\min}{\otimes}H_c^{\otimes n-k}\underset{\min}{\otimes}H_r^{\otimes k}} &\leqslant \|t\|_{\overline{H}^{\otimes n-k'}\otimes_2\overline{H}^{\otimes k'}\otimes_2 H^{\otimes n-k}\otimes_2 H^{\otimes k}}\\
&\leqslant C_q\|\sum_{|\underline{i}|=n}\overline{\xi_{\underline{i}}}\otimes\xi_{\underline{i}}\|_{\overline{H}^{\otimes n}\otimes_2 H^{\otimes n}}
\end{aligned}$$

But by the choice of $\xi_{\underline{i}}$: $\|\sum_{|\underline{i}|=n}\overline{\xi_{\underline{i}}}\otimes\xi_{\underline{i}}\|_{\overline{H}^n\otimes_2 H^n} = d^{n/2}$.

Combining all inequalities above, we deduce

$$d^n \leqslant C_q^3 (n+1)^2 d^{n/2}$$

which yields a contradiction when n tends to infinity as soon as $d \geqslant 2$. □

Let $C_T^*(H_\mathbb{R})$ be the C^*-algebra generated by all gaussians $G(f)$ for $f \in H_\mathbb{R}$. The preceding theorem implies directly that $C_T^*(H_\mathbb{R})$ is not nuclear as soon as $\dim(H_\mathbb{R}) \geqslant 2$ (see [CE] Corollary 6.5). Actually the preceding argument can be modified to prove that $C_T^*(H_\mathbb{R})$ does not have the weak expectation property as soon as $\dim H_\mathbb{R} \geqslant 2$. Recall that a C^*-algebra A has the weak expectation property (WEP in short) if and only if the canonical inclusion $A \to A^{**}$ factorizes completely contractively through $B(K)$ for some complex Hilbert space K. By the results of Haagerup (see [P] Chapter 15) a C^*-algebra A has the WEP if and only if for all finite family $x_1, \ldots, x_n$ in A

$$\| \sum_{i=1}^n x_i \otimes \overline{x_i} \|_{A \otimes_{\max} \overline{A}} = \| \sum_{i=1}^n x_i \otimes \overline{x_i} \|_{A \underset{\min}{\otimes} \overline{A}} \tag{1.15}$$

Corollary 1.3.5 *$C_T^*(H_\mathbb{R})$ does not have the WEP as soon as* $\dim H_\mathbb{R} \geqslant 2$.

Proof. Let us use the same notations as in the preceding proof and suppose that $C_T^*(H_\mathbb{R})$ has the WEP. Fix $n \geqslant 1$, by (1.15) we have

$$\| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) \otimes \overline{W(\xi_{\underline{i}})} \|_{C_T^*(H_\mathbb{R}) \otimes_{\max} \overline{C_T^*(H_\mathbb{R})}} \leqslant \| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) \otimes \overline{W(\xi_{\underline{i}})} \|_{C_T^*(H_\mathbb{R}) \underset{\min}{\otimes} \overline{C_T^*(H_\mathbb{R})}} \tag{1.16}$$

To estimate from below the left handside of (1.16) observe that $\Phi : \overline{C_T^*(H_\mathbb{R})} \to C_T^*(H_\mathbb{R})'$ taking $\overline{W(\xi)}$ to $\mathcal{J}UW(\xi)\mathcal{J}U = W_r(\mathcal{J}U\xi)$ is a $*$- representation. Thus

$$\begin{aligned}
\| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) \otimes \overline{W(\xi_{\underline{i}})} \|_{C_T^*(H_\mathbb{R}) \otimes_{\max} \overline{C_T^*(H_\mathbb{R})}} &= \| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) \otimes W_r(\mathcal{J}U\xi_{\underline{i}}) \|_{C_T^*(H_\mathbb{R}) \otimes_{\max} C_T^*(H_\mathbb{R})'} \\
&\geqslant \| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) W_r(\mathcal{J}U\xi_{\underline{i}}) \|_{B(\mathcal{F}_T(H_\mathbb{C}))} \\
&\geqslant \sum_{|\underline{i}|=n} \langle \mathcal{J}U\xi_{\underline{i}}, W(\xi_{\underline{i}})^* \Omega \rangle_T \\
&\geqslant \sum_{|\underline{i}|=n} \langle \mathcal{J}U\xi_{\underline{i}}, W(\mathcal{J}U\xi_{\underline{i}}) \Omega \rangle_T \\
&\geqslant \sum_{|\underline{i}|=n} \| \mathcal{J}U\xi_{\underline{i}} \|_T^2 = d^n
\end{aligned}$$

Then we can finish the proof as for Theorem 1.3.4. □

Remark. Non nuclearity of $C_T^*(H_\mathbb{R})$ is equivalent to the fact that $C_T^*(H_\mathbb{R})$ does not have the completely positive approximation property as soon as $\dim(H_\mathbb{R}) \geqslant 2$. However it is possible to prove that $C_T^*(H_\mathbb{R})$ has the metric approximation property, by truncation of the Ornstein-Uhlenbeck semigroup. Arguing by duality and interpolation, it is not difficult to show that $L^p(\Gamma_T(H_\mathbb{R}))$ has the metric approximation property for $1 \leqslant p < \infty$. However, at the time of this writing, we are not able to prove that $C_T^*(H_\mathbb{R})$ has the completely bounded approximation property.

1.4 The case of the $q-$Araki-Woods algebras

For this last section we mainly refer to [Hi] where the $q-$Araki-Woods algebras are defined as a generalization of the $q-$ deformed case of Bożejko and Speicher on the one hand, and the quasi-free case of Shlyakhtenko (see [Sh]) on the other. More precisely, let $H_{\mathbb{R}}$ be a real Hilbert space given with U_t, a strongly continuous group of orthogonal transformations on $H_{\mathbb{R}}$. U_t can be extended to a unitary group on the complexification $H_{\mathbb{C}}$. Let A be its positive non-singular generator on $H_{\mathbb{C}}$: $U_t = A^{it}$. A new scalar product $\langle\ .\ ,\ .\ \rangle_U$ is defined on $H_{\mathbb{C}}$ by the following relation:

$$\langle \xi, \eta \rangle_U = \langle 2A(1+A)^{-1}\xi, \eta \rangle$$

We will denote by H the completion of $H_{\mathbb{C}}$ with respect to this new scalar product.

For a fixed $q \in (-1,1)$, we now consider the $q-$deformed Fock space associated with H and we denote it by $\mathcal{F}_q(H)$. Recall that it is the Fock space with the following Yang-Baxter deformation T defined by:

$$\begin{aligned} T : H \otimes H &\longrightarrow H \otimes H \\ \xi \otimes \eta &\longmapsto q\eta \otimes \xi \end{aligned}$$

Or equivalently, for every $n \geqslant 2$ and $\sigma \in S_n$ we have

$$\varphi(\sigma) = q^{i(\sigma)} U_\sigma$$

where $i(\sigma)$ denotes the number of inversions of the permutation σ and U_σ is the unitary on $H^{\otimes n}$ defined by

$$U_\sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)}$$

In this setting, the $q-$Araki-Woods algebra is the following von Neumann algebra

$$\Gamma_q(H_{\mathbb{R}}, U_t) = \{G(h), h \in H_{\mathbb{R}}\}'' \subset B\left(\mathcal{F}_q\left(H_{\mathbb{C}}\right)\right)$$

Let $H'_{\mathbb{R}} = \{g \in H, \langle g, h\rangle_U \in \mathbb{R} \text{ for all } h \in H_{\mathbb{R}}\}$ and

$$\Gamma_{q,r}(H'_{\mathbb{R}}, U_t) = \{G_r(h), h \in H'_{\mathbb{R}}\}''$$

where $G_r(h)$ is the right gaussian corresponding to the right creation operator.
Since $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t) \subset \Gamma_q(H_{\mathbb{R}}, U_t)'$, $\overline{H_{\mathbb{R}} + iH_{\mathbb{R}}} = H$ and $\overline{H'_{\mathbb{R}} + iH'_{\mathbb{R}}} = H$ (see [Sh]), it is easy to deduce that Ω is cyclic and separating for both $\Gamma_q(H_{\mathbb{R}}, U_t)$ and $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$. So Tomita's theory can apply: recall that the anti-linear operator S is the closure of the operator defined by:

$$S(x\Omega) = x^*\Omega \quad \text{for all } x \in \Gamma_q(H_{\mathbb{R}}, U_t)$$

Let $S = J\Delta^{\frac{1}{2}}$ be its polar decomposition. J and Δ are called respectively the modular conjugation and the modular operator. The following explicit formulas hold (see [Hi] and [Sh])

$$S(h_1 \otimes \cdots \otimes h_n) = h_n \otimes \cdots \otimes h_1 \quad \text{for all } h_1, \ldots, h_n \in H_{\mathbb{R}}$$

Δ is the closure of the operator $\bigoplus_{n=0}^{\infty} (A^{-1})^{\otimes n}$ and

$$J(h_1 \otimes \cdots \otimes h_n) = A^{-\frac{1}{2}} h_n \otimes \cdots \otimes A^{-\frac{1}{2}} h_1 \quad \text{for all } h_1, \ldots, h_n \in H_{\mathbb{R}} \cap \operatorname{dom} A^{-\frac{1}{2}}$$

By Tomita's theory, we have

$$\Gamma_q(H_{\mathbb{R}}, U_t)' = J\Gamma_q(H_{\mathbb{R}}, U_t)J$$

Let $h \in H_{\mathbb{R}}$, as in [Sh] we have $Jh \in H_{\mathbb{R}'}$, then, since Ω is separating for $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$, we obtain that $JG(h)J = G_r(Jh) \in \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$, so that

$$\Gamma_q(H_{\mathbb{R}}, U_t)' = \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$$

Moreover, if $\xi \in \Gamma_q(H_{\mathbb{R}}, U_t)\Omega$, then $J\xi \in \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)\Omega$ and since Ω is separating, we get $JW(\xi)J = W_r(J\xi)$.

Recall that if U_t is non trivial, the vacuum expectation φ is no longer tracial and is called the $q-$quasi-free state. In fact in most cases (see [Hi] Theorem 3.3), Araki-Woods factors are type III von Neumann algebras.

When A is bounded, it is clear that our preliminaries are still valid with minor changes. For example we should get an extra $\|A^{-1}\|^{k/2} = \|A\|^{k/2}$ in the estimation of $\|U_k\|$. Note, in particular, that the Wick formula, as stated in Lemma 1.2.2, is still true, and that the following analogue of Bożejko's scalar inequality holds: (proved in [Hi])
If A is bounded, $(\eta_u)_{u\in U}$ is a family of vectors in $H^{\otimes n}$ and $(\alpha_u)_{u\in U}$ a finitely supported family of complex numbers then:

$$\Big\|\sum_{u\in U}\alpha_u\eta_u\Big\|_q \leqslant \Big\|\sum_{u\in U}\alpha_u W(\eta_u)\Big\| \leqslant C_{|q|}^{\frac{3}{2}}\frac{\|A\|^{\frac{n+1}{2}}-1}{\|A\|^{\frac{1}{2}}-1}\Big\|\sum_{u\in U}\alpha_u\eta_u\Big\|_q \tag{1.17}$$

It is also a straightforward verification that Lemma 1.3.1, still holds in this setting. Observe also that U is a unitary on $\mathcal{F}_q(H)$: this follows from the fact that for every $n \geqslant 1$, $P_q^{(n)}$, $A^{\otimes n}$ and U commute on $H^{\otimes n}$. Note that $\mathcal{J}$ is no more an anti unitary from $H^{\otimes k}$ to $\overline{H^{\otimes k}}$, but since $U_k(I\otimes S) = M(a^*\otimes aU)$, we can deduce, as in the proof of Corollary 1.3.2, that $U_k(I\otimes S):\ H_c^{\otimes n-k}\otimes_h \overline{H_r^{\otimes k}} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ is completely bounded with norm less than C_q, where I stands for the identity of $H_c^{\otimes n-k}$. Following the same lines as in the proof of Theorem 1.3.3 we get:

Theorem 1.4.1 *Assume A is bounded. Let K be a complex Hilbert space. Then for all $n \geqslant 0$ and for all $\xi \in B(K)\underset{\min}{\otimes} H^{\otimes n}$ we have*

$$\max_{0\leqslant k\leqslant n}\|(Id\otimes((I\otimes S)R^*_{n,\,k})(\xi)\| \leqslant \|(Id\otimes W)(\xi)\|_{\min} \tag{1.18}$$

$$\leqslant C_q(n+1)\max_{0\leqslant k\leqslant n}\|(Id\otimes((I\otimes S)R^*_{n,\,k})(\xi)\|$$

where Id denotes the identity mapping of $B(K)$, I the identity of $\underline{H_c^{\otimes n-k}}$, and where the norms of the left and right handsides are taken in $B(K)\underset{\min}{\otimes} H_c^{\otimes n-k}\underset{\min}{\otimes}\overline{H_r^{\otimes k}}$.

It is known (see [Hi]) that if U_t has a non trivial continuous part then $\Gamma_q(H_{\mathbb{R}}, U_t)$ is not injective. Using our techniques we are able to state a non-injectivity criterion similar to that of [Hi] but independent of q.

Corollary 1.4.2 *If either*

$$\dim\, E_A\left(\{1\}\right) H_{\mathbb{C}} \geqslant 2$$

or for some $T > 1$

$$\frac{\dim\, E_A\left(]1,T]\right) H_{\mathbb{C}}}{T^2} > \frac{1}{2}$$

where E_A *is the spectral projection of* A*, then* $\Gamma_q(H_{\mathbb{R}}, U_t)$ *is non injective.*

Proof. We can assume that U_t is almost periodic, then we can write

$$(H_{\mathbb{R}},\ U_t) = (\hat{H}_{\mathbb{R}},\ \mathrm{Id}_{\hat{\mathrm{H}}_{\mathbb{R}}}) \bigoplus_{\alpha\in\Lambda} (H_{\mathbb{R}}^{(\alpha)},\ U_t^{(\alpha)})$$

where

$$H_{\mathbb{R}}^{(\alpha)} = \mathbb{R}^2, \quad U_t^{(\alpha)} = \begin{pmatrix} \cos(t\ln\lambda_\alpha) & -\sin(t\ln\lambda_\alpha) \\ \sin(t\ln\lambda_\alpha) & \cos(t\ln\lambda_\alpha) \end{pmatrix}, \quad \lambda_\alpha > 1$$

Thus the eigenvalues of the generator $A^{(\alpha)}$ of $U_t^{(\alpha)}$ are λ_α and λ_α^{-1}.

If $\dim\, E_A\left(\{1\}\right) H_{\mathbb{C}} \geqslant 2$ then $\dim\, \hat{H}_{\mathbb{R}} \geqslant 2$ and since U_t is trivial on $\hat{H}_{\mathbb{R}}$, the non-injectivity follows from Theorem 1.3.4.

For the remaining case we first suppose that $\dim H_{\mathbb{R}} = 2$, U_t is not trivial and that $\Gamma_q(H_{\mathbb{R}}, U_t)$ is injective. For all $n \geqslant 1$, $A^{\otimes n}$ is a positive operator on $H^{\otimes n}$ equipped with the deformed scalar product, we will denote by λ and λ^{-1} the eigenvalues of A with $\lambda > 1$ and by $(\xi_{\underline{i}})_{|\underline{i}|=n}$ an orthonormal basis of eigenvectors of $A^{\otimes n}$ associated to the eigenvalues $(\lambda_{\underline{i}})_{|\underline{i}|=n}$. Since $\Gamma_q(H_{\mathbb{R}}, U_t)$ is semidiscrete we must have for every $n \geqslant 1$

$$\|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\| \leqslant \|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}}) \otimes W(\xi_{\underline{i}})\| = \|\sum_{|\underline{i}|=n} JW(\xi_{\underline{i}})J \otimes W(\xi_{\underline{i}})\|$$

It is easily seen that

$$\begin{aligned} \|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\| &\geqslant \sum_{|\underline{i}|=n} \langle \Omega, W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\Omega\rangle_q \\ &= \sum_{|\underline{i}|=n} \langle JW(\xi_{\underline{i}})^* J\Omega, W(\xi_{\underline{i}})\Omega\rangle_q \\ &= \sum_{|\underline{i}|=n} \langle \Delta^{\frac{1}{2}}\xi_{\underline{i}}, \xi_{\underline{i}}\rangle_q = \mathrm{Trace}\left(\left(A^{-\frac{1}{2}}\right)^{\otimes n}\right) = (\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}})^n \end{aligned}$$

On the other hand, the map from $J\Gamma_q(H_{\mathbb{R}}, U_t)J$ to $\overline{\Gamma_q(H_{\mathbb{R}}, U_t)}$ taking $JW(\xi)J$ to $\overline{W(\xi)}$ is a $*$-isomorphism, hence

$$\|\sum_{|\underline{i}|=n} JW(\xi_{\underline{i}})J \otimes W(\xi_{\underline{i}})\|_{\min} = \|\sum_{|\underline{i}|=n} \overline{W(\xi_{\underline{i}})} \otimes W(\xi_{\underline{i}})\|_{\min}$$

Applying (1.18) twice, and recalling that on $H^{\otimes k}$, $S = J\Delta^{\frac{1}{2}} = J(A^{\otimes k})^{-\frac{1}{2}}$ and that $J:\ \overline{H_r^{\otimes k}} \to H_r^{\otimes k}$ is completely isometric, we get

$$\|\sum_{|\underline{i}|=n} \overline{W(\xi_{\underline{i}})} \otimes W(\xi_{\underline{i}})\|_{\min} \leqslant C_q^2 (n+1)^2 \max_{0\leqslant k,k'\leqslant n} \|\sum_{|\underline{i}|=n} \overline{(I\otimes S)R_{n,\,k'}^*(\xi_{\underline{i}})} \otimes (I\otimes S)R_{n,\,k}^*(\xi_{\underline{i}})\|$$

$$\leqslant C_q^2(n+1)^2 \max_{0\leqslant k,k'\leqslant n} \| \sum_{|\underline{i}|=n} \overline{(I\otimes (A^{\otimes k'})^{-\frac{1}{2}})R^*_{n,\,k'}(\xi_{\underline{i}})} \otimes (I\otimes (A^{\otimes k})^{-\frac{1}{2}})R^*_{n,\,k}(\xi_{\underline{i}})\|$$

Where the norms are computed in $\overline{H_c^{\otimes n-k'}} \underset{\min}{\otimes} \overline{H_r^{\otimes k'}} \underset{\min}{\otimes} H_c^{\otimes n-k} \underset{\min}{\otimes} H_r^{\otimes k}$. For a fixed (k,k'), let us denote by

$$t=\sum_{|\underline{i}|=n} \overline{(I\otimes (A^{\otimes k'})^{-\frac{1}{2}})R^*_{n,\,k'}(\xi_{\underline{i}})} \otimes (I\otimes (A^{\otimes k})^{-\frac{1}{2}})R^*_{n,\,k}(\xi_{\underline{i}})$$

As in the proof of Theorem 1.3.4, we have the following Hilbert-Schmidt estimate:

$$\|t\|_{\overline{H_c^{\otimes n-k'}} \underset{\min}{\otimes} \overline{H_r^{\otimes k'}} \underset{\min}{\otimes} H_c^{\otimes n-k} \underset{\min}{\otimes} H_r^{\otimes k}} \leqslant \|t\|_{\overline{H^{\otimes n-k'}}\otimes_2 \overline{H^{\otimes k'}}\otimes_2 H^{\otimes n-k}\otimes_2 H^{\otimes k}}$$

Recall that $R^*_{n,\,k}:\ H^{\otimes n}\to H^{\otimes n-k}\otimes_2 H^{\otimes k}$ is of norm less than $C_{|q|}^{\frac{1}{2}}$ and that $\|(A^{\otimes k})^{-\frac{1}{2}}\|_{B(H^{\otimes k})}=\lambda^{\frac{k}{2}}$. Hence,

$$\begin{aligned}\|t\|_{\overline{H^{\otimes n-k'}}\otimes_2 \overline{H^{\otimes k'}}\otimes_2 H^{\otimes n-k}\otimes_2 H^{\otimes k}} &\leqslant C_{|q|}\lambda^n \| \sum_{|\underline{i}|=n} \overline{\xi_{\underline{i}}}\otimes \xi_{\underline{i}}\|_{\overline{H^{\otimes n}}\otimes H^{\otimes n}} \\ &\leqslant C_{|q|}(\sqrt{2}\lambda)^n\end{aligned}$$

Combining all inequalities we get

$$(\lambda^{\frac{1}{2}}+\lambda^{-\frac{1}{2}})^n \leqslant C_{|q|}^3(n+1)^2(\sqrt{2}\lambda)^n.$$

We now return to the general case, we fix $T>1$ and we denote by $\lambda_1,\dots,\lambda_p$ the eigenvalues of A in $]1,T]$ counted with multiplicities. Thus we have $p=\dim\, E_A\,(]1,T])\,H_{\mathbb{C}}$. It is easy to deduce from our first step that for any $n\geqslant 1$ we have

$$(\sum_{i=1}^p \lambda_i^{\frac{1}{2}}+\lambda_i^{-\frac{1}{2}})^n \leqslant C_{|q|}^3(n+1)^2(2p)^{\frac{n}{2}}T^n$$

Since for any i we have $\lambda_i^{\frac{1}{2}}+\lambda_i^{-\frac{1}{2}}\geqslant 2$ we deduce

$$(2p)^n \leqslant C_{|q|}^3(n+1)^2(2p)^{\frac{n}{2}}T^n$$

So we necessarily have

$$\frac{2p}{T^2}\leqslant 1$$

that is to say

$$\frac{\dim\, E_A]1,T]H_{\mathbb{C}}}{T^2}\leqslant \frac{1}{2}$$

□

Chapter 2

Asymptotic matricial models and QWEP property for q-Araki-Woods algebras

2.1 Introduction

Recall that a C^*-algebra has the weak expectation property (in short WEP) if the canonical inclusion from A into A^{**} factorizes completely contractively through some $B(H)$ (H Hilbert). A C^*-algebra is QWEP if it is a quotient by a closed ideal of an algebra with the WEP. The notion of QWEP was introduced by Kirchberg in [Kir]. Since then, it became an important notion in the theory of C^*-algebras. Very recently, Pisier and Shlyakhtenko [PS] proved that Shlyakhtenko's free quasi-free factors are QWEP. This result plays an important role in their work on the operator space Grothendieck Theorem, as well as in the subsequent related works [P1] and [Xu]. On the other hand, in his paper [J] on the embedding of Pisier's operator Hilbertian space OH and the projection constant of OH_n, Junge used QWEP in a crucial way.

Hiai [Hi] introduced the so-called q-Araki-Woods algebras. Let $-1 < q < 1$, and let $H_{\mathbb{R}}$ be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ an orthogonal group on $H_{\mathbb{R}}$. Let $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ denote the associated q-Araki-Woods algebra. These algebras are generalizations of both Shlyakhtenko's free quasi-free factors (for $q = 0$), and Bożejko and Speicher's q-Gaussian algebras (for $(U_t)_{t\in\mathbb{R}}$ trivial). In this paper we prove that $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ is QWEP. This is an extension of Pisier-Shlyakthenko's result for the free quasi-free factor (with $(U_t)_{t\in\mathbb{R}}$ almost periodic), already quoted above.

In the first two sections below we recall some general background on q-Araki-Woods algebras and we give a proof of our main result in the particular case of Bożejko and Speicher's q-Gaussian algebras $\Gamma_q(H_{\mathbb{R}})$. The proof relies on an asymptotic random matrix model for standard q-Gaussians. The existence of such a model goes back to Speicher's central limit Theorem for mixed commuting/anti-commuting non-commutative random variables (see [Sp]). Alternatively, one can also use the Gaussian random matrix model given by Śniady in [Sn]. Notice that the matrices arising from Speicher's central limit Theorem may not be uniformly bounded in norm. Therefore, we have to cut them off in order to define a homomorphism from a dense subalgebra of $\Gamma_q(H_{\mathbb{R}})$ into an ultraproduct of matricial algebras. In this tracial framework it can be shown quite easily that this homomorphism extends to an isometric $*$-homomorphism of von Neumann algebras, simply because it is trace preserving. Thus $\Gamma_q(H_{\mathbb{R}})$ can be seen as a (necessarily completely complemented) subalgebra of an ultraproduct of matricial algebras. This solves the problem in the tracial case.

Moreover, in this (relatively) simple situation, we are able to extend the result to the C^*-algebra generated by all q-Gaussians, $C_q^*(H_{\mathbb{R}})$. Indeed, using the ultracontractivity of the q-Ornstein Uhlenbeck semi-group (see [B2]) we establish that $C_q^*(H_{\mathbb{R}})$ is "weakly ucp complemented" in $\Gamma_q(H_{\mathbb{R}})$. This last fact, combined with the QWEP of $\Gamma_q(H_{\mathbb{R}})$, implies that $C_q^*(H_{\mathbb{R}})$ is also QWEP.

In the remaining of the paper we adapt the proof of section 2.3 to the more general type-*III* q-Araki-Woods algebras. In section 2.4 we start by recalling Raynaud's construction of the von Neumann algebra's ultraproduct when algebras are equipped with non-tracial states (see [Ray]). Then, we give some general conditions in order to define an embedding into such an ultraproduct, whose image is the image of a state preserving conditional expectation.

In section 2.5 we define a twisted Baby Fock model, to which we apply Speicher's central limit Theorem. This provides us with an asymptotic random matrix model for (finite dimensional) q-Araki Woods algebras, generalizing the asymptotic model already introduced by Speicher and used by Biane in [Bi]. Using this asymptotic model, we then define an algebraic $*$-homomorphism from a dense subalgebra of $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ into a von Neumann ultraproduct of finite dimensional C^*-algebras. Notice that the cut off argument requires some extra work (compare the proofs of Lemma 2.3.1 and Lemma 2.5.7), for instance we need to use our knowledge of the modular theory at the Baby Fock level to conclude. We then apply the general results of section 2.4 (Theorem 2.4.3) to extend this algebraic $*$-homomorphism into a $*$-isomorphism from $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ to the von Neumann algebra's ultraproduct, whose image is completely complemented. This allows us to show that $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ is QWEP for $H_{\mathbb{R}}$ finite dimensional (see Theorem 2.5.8). It implies, by inductive limit, that $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ is QWEP when $(U_t)_{t\in\mathbb{R}}$ is almost periodic (see Corollary 2.5.9).

In the last section, we consider a general algebra $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$. We use a discretization procedure on the unitary group $(U_t)_{t\in\mathbb{R}}$ in order to approach $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ by almost periodic q-Araki-Woods algebras. We then apply the general results of section 2.4 and, we recover the general algebra as a complemented subalgebra of the ultraproduct of the discretized ones (see Theorem 2.6.3). From this last fact follows the QWEP of $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$. However we were unable to establish the corresponding result for the C^*-algebra $C_q^*(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$. Indeed, if $(U_t)_{t\in\mathbb{R}}$ is not trivial then the ultracontractivity of the q-Ornstein-Uhlenbeck semi-group never holds in any right-neighborhood of zero (see [Hi]).

We highlight that the modular theory on the twisted Baby Fock algebras, on their ultraproduct, and on the q-Araki Woods algebras, are crucial tools in order to overcome the difficulties arising in the non-tracial case.

2.2 Preliminaries

2.2.1 q-Araki-Woods algebras

We mainly follow the notations used in [Sh], [Hi] and [Nou]. Let $H_{\mathbb{R}}$ be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous group of orthogonal transformations on $H_{\mathbb{R}}$. We denote by $H_{\mathbb{C}}$ the complexification of $H_{\mathbb{R}}$ and still by $(U_t)_{t\in\mathbb{R}}$ its extension to a group of unitaries on $H_{\mathbb{C}}$. Let A be the (unbounded) non degenerate positive infinitesimal generator of $(U_t)_{t\in\mathbb{R}}$.

$$U_t = A^{it} \qquad \text{for all } t \in \mathbb{R}$$

A new scalar product $\langle\ .\ ,\ .\ \rangle_U$ is defined on $H_{\mathbb{C}}$ by the following relation:

$$\langle \xi, \eta\rangle_U = \langle 2A(1+A)^{-1}\xi, \eta\rangle$$

We denote by H the completion of $H_{\mathbb{C}}$ with respect to this new scalar product. For $q \in (-1, 1)$ we consider the q-Fock space associated with H and given by:

$$\mathcal{F}_q(H) = \mathbb{C}\Omega \bigoplus_{n\geqslant 1} H^{\otimes n}$$

where $H^{\otimes n}$ is equipped with Bożejko and Speicher's q-scalar product (see [BS1]). The usual creation and annihilation operators on $\mathcal{F}_q(H)$ are denoted respectively by a^* and a (see [BS1]). For $f \in H_{\mathbb{R}}$, $G(f)$, the q-Gaussian operator associated to f, is by definition:

$$G(f) = a^*(f) + a(f) \in B\left(\mathcal{F}_q(H)\right)$$

The von Neumann algebra that they generate in $B\left(\mathcal{F}_q(H)\right)$ is the so-called q-Araki-Woods algebra: $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$. The q-Araki-Woods algebra is equipped with a faithful normal state φ which is the expectation on the vacuum vector Ω. We denote by W the Wick product ; it is the inverse of the mapping:

$$\begin{aligned} \Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}}) &\longrightarrow \Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})\Omega \\ X &\mapsto X\Omega \end{aligned}$$

Recall that $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}}) \subset B\left(\mathcal{F}_q\left(H\right)\right)$ is the GNS representation of (Γ, φ). The modular theory relative to the state φ was computed in the papers [Hi] and [Sh]. We now briefly recall their results. As usual we denote by S the closure of the operator:

$$S(x\Omega) = x^*\Omega \quad \text{for all } x \in \Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$$

Let $S = J\Delta^{\frac{1}{2}}$ be its polar decomposition. J and Δ are respectively the modular conjugation and the modular operator relative to φ. The following explicit formulas hold :

$$S(h_1 \otimes \cdots \otimes h_n) = h_n \otimes \cdots \otimes h_1 \quad \text{for all } h_1, \ldots, h_n \in H_{\mathbb{R}}$$

Δ is the closure of the operator $\overset{\infty}{\underset{n=0}{\oplus}} (A^{-1})^{\otimes n}$ and

$$J(h_1 \otimes \cdots \otimes h_n) = A^{-\frac{1}{2}}h_n \otimes \cdots \otimes A^{-\frac{1}{2}}h_1 \quad \text{for all } h_1, \ldots, h_n \in H_{\mathbb{R}} \cap \mathrm{dom} A^{-\frac{1}{2}}$$

The modular group of automorphisms $(\sigma_t)_{t\in\mathbb{R}}$ on $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ relative to φ is given by:

$$\sigma_t(G(f)) = \Delta^{it}G(f)\Delta^{-it} = G(U_{-t}f) \qquad \text{for all} \quad t\in\mathbb{R} \quad \text{and all} \quad f\in H_\mathbb{R}$$

In the following Lemma we state a well known formula giving, in particular, all moments of the q-Gaussians.

Lemma 2.2.1 *Let $r\in\mathbb{N}_*$ and $(h_l)_{\substack{-r\leqslant l\leqslant r\\ k\neq 0}}$ be a family of vectors in $H_\mathbb{R}$. For all $l\in\{1,\dots,r\}$ consider the operator $d_l = a^*(h_l)+a(h_{-l})$. For all $(k(1),\dots,k(r))\in\{1,*\}^r$ we have:*

$$\varphi(d_1^{k(1)}\dots d_r^{k(r)}) = \begin{cases} 0 & \textit{if } r \textit{ is odd} \\ \displaystyle\sum_{\substack{\mathcal{V} \text{ 2-partition}\\ \mathcal{V}=\{(s_l,t_l)_{l=1}^{l=p}\} \text{ with } s_l<t_l}} q^{i(\mathcal{V})}\prod_{l=1}^{p}\varphi(d_{s_l}^{k(s_l)}d_{t_l}^{k(t_l)}) & \textit{if } r=2p \end{cases}$$

where $i(\mathcal{V}) = \#\{(k,l),\ s_k<s_l<t_k<t_l\}$ is the number of crossings of the 2-partition $\mathcal{V}$.

Remarks.

• When $(U_t)_{t\in\mathbb{R}}$ is trivial, $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ reduces to Bożejko and Speicher's q-Gaussian algebra $\Gamma_q(H_\mathbb{R})$. This is the only case where φ is a trace on $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$. Actually, $\Gamma_q(H_\mathbb{R})$ is known to be a non-hyperfinite II_1 factor (see [BKS], [BS1], [Nou] and [Ri]). In all other cases $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ turns out to be a type *III* von Neumann algebra (see [Sh] and [Hi]).

• Lemma 2.2.1 implies that for all $n\in\mathbb{N}$ and all $f\in H_\mathbb{R}$:

$$\varphi(G(f)^{2n}) = \sum_{\mathcal{V}\text{ 2-partition}} q^{i(\mathcal{V})}\|f\|_{H_\mathbb{R}}^{2n}$$

Therefore, we see that the distribution of a single gaussian does not depend on the group $(U_t)_{t\in\mathbb{R}}$. In the tracial case (thus in all cases), and when $\|f\| = 1$, this distribution is the absolutely continuous probability measure ν_q supported on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ whose orthogonal polynomials are the q-Hermite polynomials (see [BKS]). In particular, we have :

$$\text{For all} \quad f\in H_\mathbb{R}, \quad \|G(f)\| = \frac{2}{\sqrt{1-q}}\|f\|_{H_\mathbb{R}} \tag{2.1}$$

2.2.2 The finite dimensional case

We now briefly recall a description of the von Neumann algebra $\Gamma_q(H_\mathbb{R},U_t)$ where $H_\mathbb{R}$ is an Euclidian space of dimension $2k$ ($k\in\mathbb{N}_*$). There exists $(H_j)_{1\leqslant j\leqslant k}$ a family of two dimensional spaces, invariant under $(U_t)_{t\in\mathbb{R}}$, and $(\lambda_j)_{1\leqslant j\leqslant k}$ some real numbers greater or equal to 1 such that for all $j\in\{1,\dots,k\}$,

$$H_\mathbb{R} = \bigoplus_{1\leqslant j\leqslant k} H_j \quad \text{and} \quad U_{t|H_j} = \begin{pmatrix} \cos(t\ln(\lambda_j)) & -\sin(t\ln(\lambda_j)) \\ \sin(t\ln(\lambda_j)) & \cos(t\ln(\lambda_j)) \end{pmatrix}$$

We put $I = \{-k, \ldots, -1\} \cup \{1, \ldots, k\}$. It is then easily checked that the deformed scalar product $\langle\, .\,,\, .\,\rangle_U$ on the complexification $H_{\mathbb{C}}$ of $H_{\mathbb{R}}$ is characterized by the condition that there exists a basis $(f_j)_{j\in I}$ in $H_{\mathbb{R}}$ such that for all $(j,l) \in \{1, \ldots, k\}^2$

$$\langle f_j, f_{-l}\rangle_U = \delta_{j,l}.i\frac{\lambda_j - 1}{\lambda_j + 1} \quad \text{and} \quad \langle f_{\pm j}, f_{\pm l}\rangle_U = \delta_{j,l} \tag{2.2}$$

For all $j \in \{1, \ldots, k\}$ we put $\mu_j = \lambda_j^{\frac{1}{4}}$. Let $(e_j)_{j\in I}$ be a real orthonormal basis of $\mathbb{C}^{2k}$ equipped with its canonical scalar product. For all $j \in \{1, \ldots, k\}$ we put

$$\hat{f}_j = \frac{1}{\sqrt{\mu_j^2 + \mu_j^{-2}}}\left(\mu_j e_{-j} + \mu_j^{-1} e_j\right) \quad \text{and} \quad \hat{f}_{-j} = \frac{i}{\sqrt{\mu_j^2 + \mu_j^{-2}}}\left(\mu_j e_{-j} - \mu_j^{-1} e_j\right)$$

It is easy to see that the conditions (2.2) are fulfilled for the family $(\hat{f}_j)_{j\in I}$. We will denote by $H_{\mathbb{R}}$ the Euclidian space generated by the family $(\hat{f}_j)_{j\in I}$ in $\mathbb{C}^{2k}$. This provides us with a realization of $\Gamma_q(H_{\mathbb{R}}, U_t)$ as a subalgebra of $B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$. Indeed, $\Gamma_q(H_{\mathbb{R}}, U_t) = \{G(\hat{f}_j),\, j \in I\}'' \subset B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$. For all $j \in \{1, \ldots, k\}$ put

$$f_j = \frac{\sqrt{\mu_j^2 + \mu_j^{-2}}}{2}\hat{f}_j \quad \text{and} \quad f_{-j} = \frac{\sqrt{\mu_j^2 + \mu_j^{-2}}}{2}\hat{f}_{-j}$$

We define the following generalized semi-circular variable by:

$$c_j = G(f_j) + iG(f_{-j}) = W(f_j + if_{-j})$$

It is clear that $\Gamma_q(H_{\mathbb{R}}, U_t) = \{c_j,\ j \in \{1, \ldots, k\}\}'' \subset B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$ and we can check that

$$c_j = \mu_j a(e_{-j}) + \mu_j^{-1} a^*(e_j) \tag{2.3}$$

Moreover, for all $j \in \{1, \ldots, k\}$, c_j is an entire vector for $(\sigma_t)_{t\in\mathbb{R}}$ and we have, for all $z \in \mathbb{C}$:

$$\sigma_z(c_j) = \lambda_j^{iz} c_j.$$

Recall that all odd $*$-moments of the family $(c_j)_{1\leqslant j\leqslant k}$ are zero. Applying Lemma 2.2.1 to the operators c_j we state, for further references, an explicit formula for the $*$-moments of $(c_j)_{1\leqslant j\leqslant k}$. In the following we use the convention $c^{-1} = c^*$ when there is no possible confusion.

Lemma 2.2.2 *Let* $r \in \mathbb{N}_*$, $(j(1), \ldots, j(2r)) \in \{1, \ldots, k\}^{2r}$ *and* $(k(1), \ldots, k(2r)) \in \{\pm 1\}^{2r}$

$$\begin{aligned}
\varphi(c_{j(1)}^{k(1)} \ldots c_{j(2r)}^{k(2r)}) &= \sum_{\substack{\mathcal{V}\ \text{2-partition} \\ \mathcal{V}=\{(s_l,t_l)_{l=1}^{l=r}\} \text{with } s_l<t_l}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \varphi(c_{j(s_l)}^{k(s_l)} c_{j(t_l)}^{k(t_l)}) \\
&= \sum_{\substack{\mathcal{V}\ \text{2-partition} \\ \mathcal{V}=\{(s_l,t_l)_{l=1}^{l=r}\} \text{with } s_l<t_l}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \mu_{j(s_l)}^{2k(s_l)}\, \delta_{k(s_l),\, -k(t_l)}\, \delta_{j(s_l), j(t_l)}
\end{aligned}$$

Proof. As said above this is a consequence of Lemma 2.2.1 and the explicit computation of covariances. Using (2.3) we have:

$$\begin{aligned}\varphi(c_{j(1)}^{k(1)}c_{j(2)}^{k(2)}) &= \langle c_{j(1)}^{-k(1)}\Omega, c_{j(2)}^{k(2)}\Omega\rangle \\ &= \langle \mu_{j(1)}^{k(1)} e_{-k(1)j(1)}, \mu_{j(2)}^{-k(2)} e_{k(2)j(2)}\rangle \\ &= \mu_{j(1)}^{2k(1)}\, \delta_{k(1),\,-k(2)}\, \delta_{j(1),j(2)}\end{aligned}$$

□

2.2.3 Baby Fock

The symmetric Baby Fock (also known as symmetric toy Fock space) is at some point a discrete approximation of the bosonic Fock space (see [PAM]). In [Bi], Biane considered spin systems with mixed commutation and anti-commutation relations (which is a generalization of the symmetric toy Fock), and used it to approximate q-Fock space (via Speicher central limit Theorem). In this section we recall the formal construction of [Bi]. Let I be a finite subset of $\mathbb{Z}$ and ϵ a function from $I \times I$ to $\{-1, 1\}$ satisfying for all $(i, j) \in I^2$, $\epsilon(i, j) = \epsilon(j, i)$ and $\epsilon(i, i) = -1$. Let $\mathcal{A}(I, \epsilon)$ be the free complex unital algebra with generators $(x_i)_{i\in I}$ quotiented by the relations

$$x_i x_j - \epsilon(i, j)x_j x_i = 2\delta_{i,j} \quad \textit{for} \quad (i, j) \in I^2 \tag{2.4}$$

We define an involution on $\mathcal{A}(I, \epsilon)$ by $x_i^* = x_i$. For a subset $A = \{i_1, \dots , i_k\}$ of I with $i_1 < \cdots < i_k$ we put $x_A = x_{i_1} \dots x_{i_k}$, where, by convention, $x_\varnothing = 1$. Then $(x_A)_{A\subset I}$ is a basis of the vector space $\mathcal{A}(I, \epsilon)$. Let φ^ϵ be the tracial functional defined by $\varphi^\epsilon(x_A) = \delta_{A,\varnothing}$ for all $A \subset I$. $\langle x, y\rangle = \varphi^\epsilon(x^*y)$ defines a positive definite hermitian form on $\mathcal{A}(I, \epsilon)$. We will denote by $L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon)$ the Hilbert space $\mathcal{A}(I, \epsilon)$ equipped with $\langle\, .\, , .\,\rangle$. $(x_A)_{A\subset I}$ is an orthonormal basis of $L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon)$. For each $i \in I$, define the following partial isometries β_i^* and α_i^* of $L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon)$ by:

$$\beta_i^*(x_A) = \begin{cases} x_i x_A & \text{if} \quad i \notin A \\ 0 & \text{if} \quad i \in A \end{cases} \quad \text{and} \quad \alpha_i^*(x_A) = \begin{cases} x_A x_i & \text{if} \quad i \notin A \\ 0 & \text{if} \quad i \in A \end{cases}$$

Note that their adjoints are given by:

$$\beta_i(x_A) = \begin{cases} x_i x_A & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases} \quad \text{and} \quad \alpha_i(x_A) = \begin{cases} x_A x_i & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases}$$

β_i^* and β_i (respectively α_i^* and α_i) are called the left (respectively right) creation and annihilation operators at the Baby Fock level. In the next Lemma we recall from [Bi] the fundamental relations 1. and 2., and we leave the proof of 3., 4. and 5. to the reader.

Lemma 2.2.3 *The following relations hold:*

1. *For all* $i \in I$ $(\beta_i^*)^2 = \beta_i^2 = 0$ *and* $\beta_i\beta_i^* + \beta_i^*\beta_i = Id$.
2. *For all* $(i, j) \in I^2$ *with* $i \neq j$ $\beta_i\beta_j - \epsilon(i, j)\beta_j\beta_i = 0$ *and* $\beta_i\beta_j^* - \epsilon(i, j)\beta_j^*\beta_i = 0$.
3. *Same relations as in 1. and 2. with* α *in place of* β.

4. For all $i \in I$ $\beta_i^*\alpha_i^* = \alpha_i^*\beta_i^* = 0$ and for all $(i,j) \in I^2$ with $i \neq j$ $\beta_i^*\alpha_j^* = \alpha_j^*\beta_i^*$.

5. For all $(i,j) \in I^2$ $\beta_i^*\alpha_j = \alpha_j\beta_i^*$.

It is easily seen, by 1. and 2. of Lemma 2.2.3, that the self adjoint operators defined by: $\gamma_i = \beta_i^* + \beta_i$ satisfy the following relation :

$$\text{for all} \quad (i,j) \in I^2, \quad \gamma_i\gamma_j - \epsilon(i,j)\gamma_j\gamma_i = 2\delta_{i,j}\mathrm{Id} \tag{2.5}$$

Let $\Gamma_I \subset B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ be the $*$-algebra generated by all γ_i, $i \in I$. Still denoting by φ^ϵ the vector state associated to the vector 1, it is known that φ^ϵ is a faithful normalized trace on the finite dimensional C^*-algebra Γ_I (see the remarks below). Moreover, $\Gamma_I \subset B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ is the faithful GNS representation of $(\Gamma_I,\varphi^\epsilon)$ with cyclic and separating vector 1.

Remarks.

• It is clear that we can do the previous construction for some finite sets I that are not given explicitly as subsets of $\mathbb{Z}$. Then, to each total order on I we can associate a basis $(x_A)_{A\subset I}$ of $\mathcal{A}(I,\epsilon)$. But, because of the commutation relations (2.4), the state φ^ϵ, the scalar product on $\mathcal{A}(I,\epsilon)$ and the creation operators do not depend on the chosen total order.

• We can also extend the previous construction to not necessarily finite sets I. Only the faithfulness of φ^ϵ on Γ_I requires some comments. It suffices to see that the vector 1 is separating for Γ_I. Indeed, set $\delta_i = \alpha_i^* + \alpha_i$ for $i \in I$, and

$$\Gamma_{r,I} = \{\delta_i, i \in I\}'' \subset B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)).$$

Then, it is clear from 4. and 5. of Lemma 2.2.3, that $\Gamma_{r,I} \subset \Gamma_I'$ (there is actually equality). Since 1 is clearly cyclic for $\Gamma_{r,I}$, then it is also cyclic for Γ_I', thus 1 is separating for Γ_I.

• Let I and J, $I \subset J$, be some sets together with signs ϵ and ϵ' such that $\epsilon'_{|I\times I} = \epsilon$. It is clear that $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$ embeds isometrically in $L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'})$. Set $K = J \setminus I$. Fix some total orders on I and K and consider the total order on J which coincides with the orders of I and K and such that any element of I is smaller than any element of K. The associated orthonormal basis of $L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'})$ is given by the family $(x_A x_B)_{A\in\mathcal{F}(I),B\in\mathcal{F}(K)}$ (where $\mathcal{F}(I)$, respectively $\mathcal{F}(K)$, denotes the set of finite subsets of I, respectively K). In particular with have the following Hilbertian decomposition:

$$L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}) = \bigoplus_{B\in\mathcal{F}(K)} L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)x_B \tag{2.6}$$

For $j \in I$ we (temporarily) denote by $\widetilde{\beta}_j$ the annihilation operator in $B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ and simply by β_j its analogue in $B(L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}))$. Let $\widetilde{C}_I$ (respectively C_J) be the C^*-algebra generated by $\{\widetilde{\beta}_j,\ j \in I\}$ (respectively $\{\beta_j,\ j \in J\}$) in $B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ (respectively $B(L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}))$). Consider also C_I the C^*-algebra generated by $\{\beta_j,\ j \in I\}$ in $B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$. For $B = \{j_1,\dots,j_k\} \subset K$, with $j_1 < \dots < j_k$, let us denote by α_B the operator $\alpha_{j_1}\dots\alpha_{j_k}$. If $\widetilde{T} \in \widetilde{C}_I$ and if T denotes its counterpart in C_I, then it is easily seen that, with respect to the Hilbertian decomposition (2.6), we have

$$T = \bigoplus_{B\in\mathcal{F}(K)} \alpha_B^*\widetilde{T}\alpha_B. \tag{2.7}$$

It follows that $\widetilde{C}_I$ is $*$-isomorphic to $C_I \subset C_J$.

• It is possible to find explicitly selfadjoint matrices satisfying the mixed commutation and anti-commutation relations (2.5) (see [Sp] and [Bi]). We choose to present this approach because it will be easier to handle the objects of modular theory in this abstract situation when we will deal with non-tracial von Neumann algebras (see section 2.5).

2.2.4 Speicher's central limit Theorem

We recall Speicher's central limit theorem which is specially designed to handle either commuting or anti-commuting (depending on a function ϵ) independent variables. Roughly speaking, Speicher's central limit theorem asserts that such a family of centered noncommutative variables which have a fixed covariance, and uniformly bounded $*$-moments, is convergent in $*$-moments, as soon as a combinatorial quantity associated to ϵ is converging. Moreover the limit $*$-distribution is only determined by the common covariance and the limit of the combinatorial quantity.
We start by recalling some basic notions on independence and set partitions.

Definition 2.2.4 *Let $(\mathcal{A}, \varphi)$ be a $*$-algebra equipped with a state φ and $(\mathcal{A}_i)_{i\in I}$ a family of C^*-subalgebras of $\mathcal{A}$. The family $(\mathcal{A}_i)_{i\in I}$ is said to be independent if for all $r \in \mathbb{N}_*$, $(i_1, \ldots, i_r) \in I^r$ with $i_s \neq i_t$ for $s \neq t$, and all $a_{i_s} \in \mathcal{A}_{i_s}$ for $s \in \{1, \ldots, r\}$ we have:*

$$\varphi(a_{i_1} \ldots a_{i_r}) = \varphi(a_{i_1}) \ldots \varphi(a_{i_r})$$

As usual, a family $(a_i)_{i\in I}$ of non-commutative random variables of $\mathcal{A}$ will be called independent if the family of C^*-subalgebras of $\mathcal{A}$ that they generate is independent.

On the set of p-uples of integers belonging to $\{1, \ldots, N\}$ define the equivalence relation $\sim$ by:

$$(i(1), \ldots, i(p)) \sim (j(1), \ldots, j(p)) \text{ if } \big(i(l) = i(m) \Longleftrightarrow j(l) = j(m)\big)\ \forall\ (l, m) \in \{1, \ldots, p\}^2$$

Then the equivalence classes for the relation $\sim$ are given by the partitions of the set $\{1, \ldots, p\}$. We will often use the notation $\underline{j}$ to denote the p-uple $(j(1), \ldots, j(p))$ and $\mathcal{V}_{\underline{j}}$ to denote its equivalence class for the relation $\sim$. We denote by $V_1, \ldots, V_r$ the blocks of the partition $\mathcal{V}$ and we call $\mathcal{V}$ a 2-partition if each of these blocks is of cardinal 2. The set of all 2-partitions of the set $\{1, \ldots, p\}$ (p even) will be denoted by $\mathcal{P}_2(1, \ldots, p)$. For $\mathcal{V} \in \mathcal{P}_2(1, \ldots, 2r)$ let us denote by $V_l = (s_l, t_l)$, $s_l < t_l$, for $l \in \{1, \ldots, r\}$ the blocks of the partition $\mathcal{V}$. The set of crossings of $\mathcal{V}$ is defined by

$$I(\mathcal{V}) = \{(l, m) \in \{1, \ldots, r\}^2,\ s_l < s_m < t_l < t_m\}$$

The 2-partition $\mathcal{V}$ is said to be crossing if $I(\mathcal{V}) \neq \varnothing$ and non-crossing if $I(\mathcal{V}) = \varnothing$.

Theorem 2.2.5 (Speicher) *Consider k sequences $(b_{i,j})_{(i,j)\in\mathbb{N}_*\times\{1,\ldots,k\}}$ in a noncommutative probability space (B, φ) satisfying the following conditions:*

1. *The family $(b_{i,j})_{(i,j)\in\mathbb{N}_*\times\{1,\ldots,k\}}$ is independent.*

2. *For all $(i, j) \in \mathbb{N}_* \times \{1, \ldots, k\}$, $\varphi(b_{i,j}) = 0$*

3. *For all* $(k(1),k(2)) \in \{-1,1\}^2$ *and* $(j(1),j(2)) \in \{1,\dots,k\}^2$*, the covariance* $\varphi(b_{i,j(1)}^{k(1)} b_{i,j(2)}^{k(2)})$ *is independent of* i *and will be denoted by* $\varphi(b_{j(1)}^{k(1)} b_{j(2)}^{k(2)})$*.*

4. *For all* $w \in \mathbb{N}_*$*,* $(k(1),\dots,k(w)) \in \{-1,1\}^w$ *and all* $j \in \{1,\dots,k\}$ *there exists a constant* C *such that for all* $i \in \mathbb{N}_*$*,* $|\varphi(b_{i,j}^{k(1)} \dots b_{i,j}^{k(w)})| \leqslant C$*.*

5. *For all* $(i(1),i(2)) \in \mathbb{N}_*^2$ *there exists a sign* $\epsilon(i(1),i(2)) \in \{-1,1\}$ *such that for all* $(j(1),j(2)) \in \{1,\dots,k\}^2$ *with* $(i(1),j(1)) \neq (i(2),j(2))$ *and all* $(k(1),k(2)) \in \{-1,1\}^2$ *we have*

$$b_{i(1),j(1)}^{k(1)} b_{i(2),j(2)}^{k(2)} - \epsilon(i(1),i(2)) b_{i(2),j(2)}^{k(2)} b_{i(1),j(1)}^{k(1)} = 0.$$

(notice that the function ϵ *is necessarily symmetric in its two arguments).*

6. *For all* $r \in \mathbb{N}_*$ *and all* $\mathcal{V} \in \mathcal{P}_2(1,\dots,2r)$ *the following limit exists*

$$t(\mathcal{V}) = \lim_{N\to+\infty} \frac{1}{N^r} \sum_{\substack{i(s_1),\dots,i(s_r)=1 \\ i(s_l)\neq i(s_m) \text{ for } l\neq m}}^{N} \prod_{(l,m)\in I(\mathcal{V})} \epsilon\left(i(s_l), i(s_m)\right))$$

Let $S_{N,j} = \dfrac{1}{\sqrt{N}} \sum_{i=1}^N b_{i,j}$*. Then we have for all* $p \in \mathbb{N}_*$*,* $(k(1),\dots,k(p)) \in \{-1,1\}^p$ *and all* $(j(1),\dots,j(p)) \in \{1,\dots,k\}^p$*:*

$$\lim_{N\to+\infty} \varphi(S_{N,j(1)}^{k(1)} \dots S_{N,j(p)}^{k(p)}) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \displaystyle\sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\dots,2r) \\ \mathcal{V}=\{(s_l,t_l)_{l=1}^{l=r}\}}} t(\mathcal{V}) \prod_{l=1}^{r} \varphi(b_{j(s_l)}^{k(s_l)} b_{j(t_l)}^{k(t_l)}) & \text{if } p = 2r \end{cases}$$

Remark. Speicher's Theorem is proved in [Sp] for a single limit variable. One could either convince himself that the proof of Theorem 2.2.5 goes along the same lines, or deduce it from Speicher's usual theorem. Indeed, it suffices to apply Speicher's theorem to the family $\left(\sum_{j=1}^{k} z_j b_{i,j}\right)_{i\in\mathbb{N}}$, for all $(z_1,\dots,z_k) \in \mathbb{T}^k$ and to identify the Fourier coefficients of the limit $*$-moments.

The following Lemma, proved in [Sp], guarantees the almost sure convergence of the quantity $t(\mathcal{V})$ provided that the function ϵ has independent entries following the same 2-points Dirac distribution:

Lemma 2.2.6 *Let* $q \in (-1,1)$ *and consider a family of random variables* $\epsilon(i,j)$ *for* $(i,j) \in \mathbb{N}_*$ *with* $i \neq j$*, such that*

1. *For all* $(i,j) \in \mathbb{N}_*$ *with* $i \neq j$*,* $\epsilon(i,j) = \epsilon(j,i)$

2. *The family* $(\epsilon(i,j))_{i>j}$ *is independent*

3. For all $(i,j) \in \mathbb{N}_*$ with $i \neq j$ the probability distribution of $\epsilon(i,j)$ is

$$\frac{1+q}{2}\delta_1 + \frac{1-q}{2}\delta_{-1}$$

Then, almost surely, we have for all $r \in \mathbb{N}_*$ and for all $\mathcal{V} \in \mathcal{P}_2(1,\dots,2r)$

$$\lim_{N\to+\infty} \frac{1}{N^r} \sum_{\substack{i(s_1),\dots,i(s_r)=1 \\ i(s_l)\neq i(s_m) \text{ for } l\neq m}}^{N} \prod_{(l,m)\in I(\mathcal{V})} \epsilon(i(s_l), i(s_m)) = q^{i(\mathcal{V})}$$

Remark. It is now a straightforward verification to see that Theorem 2.2.5 combined to Lemma 2.2.6 can be applied to families of mixed commuting /anti-commuting Gaussian operators (see Lemma 2.5.4 for the independence condition). The limit moments are those given by the classical q-Gaussian operators (by classical we mean that $(U_t)_{t\in\mathbb{R}}$ is trivial). Alternatively, one can apply directly Speicher's theorem to families of mixed commuting /anti-commuting creation operators as it is done in [Sp] and [Bi]. The limit $*$-moments are in this case the $*$-moments of classical q-creation operators.

2.3 The tracial case

Our goal in this section is to show that $\Gamma_q(H_\mathbb{R})$ is QWEP. In fact, by inductive limit, it is sufficient to prove it for $H_\mathbb{R}$ finite dimensional. Let $k \geqslant 1$. We will consider $\mathbb{R}^k$ as the real Hilbert space of dimension k, with the canonical orthonormal basis $(e_1,\dots,e_k)$, and $\mathbb{C}^k$, its complex counterpart. Let us fix $q \in (-1,1)$ and consider $\Gamma_q(\mathbb{R}^k)$ the von Neumann algebra generated by the q-Gaussians $G(e_1),\dots,G(e_k)$. We denote by τ the expectation on the vacuum vector, which is a trace in this particular case.

By the ending remark of section 2.2., there are Hermitian matrices, $g_{n,1}(\omega),\dots,g_{n,k}(\omega)$, depending on a random parameter denoted by ω and lying in a finite dimensional matrix algebra, such that their joint $*$-distribution converges almost surely to the joint $*$-distribution of the q-Gaussians in the following sense: for all polynomial P in k noncommuting variables,

$$\lim_{n\to\infty} \tau_n(P(g_{n,1}(\omega),\dots,g_{n,k}(\omega))) = \tau(P(G(e_1),\dots,G(e_k))) \quad \text{almost surely in } \omega.$$

We will denote by $\mathcal{A}_n$ the finite dimensional C^*-algebra generated by $g_{n,1}(\omega),\dots,g_{n,k}(\omega)$. We recall that these algebras are equipped with the trace τ_n defined by:

$$\tau_n(x) = \langle 1, x.1\rangle$$

Since the set of all monomials in k noncommuting variables is countable, we have for almost all ω,

$$\lim_{n\to\infty} \tau_n(P(g_{n,1}(\omega),\dots,g_{n,k}(\omega)) = \tau(P(G(e_1),\dots,G(e_k))) \quad \text{for all such monomials } P \tag{2.8}$$

A fortiori we can find an ω_0 such that (2.8) holds for ω_0. We will fix such an ω_0 and simply denote by $g_{n,i}$ the matrix $g_{n,i}(\omega_0)$ for all $i \in \{1,\ldots,k\}$. With these notations, it is clear that, by linearity, we have for all polynomials P in k noncommuting variables,

$$\lim_{n\to\infty} \tau_n(P(g_{n,1},\ldots,g_{n,k})) = \tau(P(G(e_1),\ldots,G(e_k))). \tag{2.9}$$

We need to have a uniform control on the norms of the matrices $g_{n,i}$. Let C be such that $\|G(e_1)\| < C$, we will replace the $g_{n,i}$'s by their truncations $\chi_{]-C,C[}(g_{n,i})g_{n,i}$ (where $\chi_{]-C,C[}$ denotes the characteristic function of the interval $]-C,C[$). For simplicity $\chi_{]-C,C[}(g_{n,i})g_{n,i}$ will be denoted by $\tilde{g}_{n,i}$. We now check that (2.9) is still valid for the $\tilde{g}_{n,i}$'s.

Lemma 2.3.1 *With the notations above, for all polynomials P in k noncommuting variables we have*

$$\lim_{n\to\infty} \tau_n(P(\tilde{g}_{n,1},\ldots,\tilde{g}_{n,k})) = \tau(P(G(e_1),\ldots,G(e_k))). \tag{2.10}$$

Proof. We just have to prove that for all monomials P in k non-commuting variables we have

$$\lim_{n\to\infty} \tau_n\left[P(\tilde{g}_{n,1},\ldots,\tilde{g}_{n,k}) - P(g_{n,1},\ldots,g_{n,k})\right] = 0.$$

Writing $g_{n,i} = \tilde{g}_{n,i} + (g_{n,i} - \tilde{g}_{n,i})$ and developing using multilinearity, we are reduced to show that the L^1-norms of any monomial in $\tilde{g}_{n,i}$ and $(g_{n,i} - \tilde{g}_{n,i})$ (with at least one factor $(g_{n,i} - \tilde{g}_{n,i})$) tend to 0. By the Hölder inequality and the uniform boundedness of the $\|\tilde{g}_{n,i}\|$'s, it suffices to show that for all $i \in \{1\ldots k\}$,

$$\lim_{n\to\infty} \tau_n(|\tilde{g}_{n,i} - g_{n,i}|^p) = 0 \quad \text{for all } p \geqslant 1. \tag{2.11}$$

Let us prove (2.11) for $i = 1$. We are now in a commutative setting. Indeed, let us introduce the spectral resolutions of identity, E_t^n (respectively E_t), of $g_{n,1}$ (respectively $G(e_1)$). By (2.9) we have for all polynomials P

$$\lim_{n\to\infty} \tau_n(P(g_{n,1})) = \tau(P(G(e_1))).$$

We can rewrite this as follows: for all polynomials P

$$\lim_{n\to\infty} \int_{\sigma(g_{n,1})} P(t)\mathrm{d}\langle E_t^n.1, 1\rangle = \int_{\sigma(G(e_1))} P(t)\mathrm{d}\langle E_t.\Omega,\Omega\rangle.$$

Let μ_n (respectively μ) denote the compactly supported probability measure $\langle E_t^n.1, 1\rangle$ (respectively $\langle E_t.\Omega,\Omega\rangle$) on $\mathbb{R}$. With these notations our assumption becomes: for all polynomials P

$$\lim_{n\to\infty} \int P\mathrm{d}\mu_n = \int P\mathrm{d}\mu. \tag{2.12}$$

and (2.11) is equivalent to:

$$\lim_{n\to\infty} \int_{|t|\geqslant C} |t|^p \mathrm{d}\mu_n = 0 \quad \text{for all } p \geqslant 1. \tag{2.13}$$

Then the result follows from the following elementary Lemma. We give a proof for sake of completeness.

Lemma 2.3.2 *Let $(\mu_n)_{n\geqslant 1}$ be a sequence of compactly supported probability measures on $\mathbb{R}$ converging in moments to a compactly supported probability measure μ on $\mathbb{R}$. Assume that the support of μ is included in the open interval $]-C,C[$. Then,*

$$\lim_{n\to\infty}\int_{|t|\geqslant C} \mathrm{d}\mu_n = 0.$$

Moreover, let f be a borelian function on $\mathbb{R}$ such that there exist $M>0$ and $r\in\mathbb{N}$ satisfying $|f(t)|\leqslant M(t^{2r}+1)$ for all $t\geqslant C$. Then,

$$\lim_{n\to\infty}\int_{|t|\geqslant C} f\mathrm{d}\mu_n = 0.$$

Proof. For the first assertion, let $C'<C$ such that the support of μ is included in $]-C',C'[$. Let $\epsilon>0$ and an integer k such that $\left(\frac{C'}{C}\right)^{2k}\leqslant\epsilon$. Let $P(t)=\left(\frac{t}{C}\right)^{2k}$. It is clear that $\chi_{\{|t|\geqslant C\}}(t)\leqslant P(t)$ for all $t\in\mathbb{R}$ and that $\sup_{|t|<C'}P(t)\leqslant\epsilon$. Thus,

$$0\leqslant\limsup_{n\to\infty}\int_{|t|\geqslant C}\mathrm{d}\mu_n\leqslant\lim_{n\to\infty}\int P(t)\mathrm{d}\mu_n=\int P(t)\mathrm{d}\mu\leqslant\epsilon.$$

Since ϵ is arbitrary, we get $\lim\limits_{n\to\infty}\int_{|t|\geqslant C}\mathrm{d}\mu_n=0$.
The second assertion is a consequence of the first one. Let f be a borelian function on $\mathbb{R}$ such that there exist $M>0$ and $r\in\mathbb{N}$ satisfying $|f(t)|\leqslant M(t^{2r}+1)$ for all $t\in\mathbb{R}$. Using the Cauchy-Schwarz inequality we get:

$$\begin{aligned}
0\leqslant\limsup_{n\to\infty}\int_{|t|\geqslant C}|f|\mathrm{d}\mu_n &\leqslant \limsup_{n\to\infty}\int_{|t|\geqslant C}M(t^{2r}+1)\mathrm{d}\mu_n\\
&\leqslant M\lim_{n\to\infty}\left(\int(t^{2r}+1)^2\mathrm{d}\mu_n\right)^{\frac12}\lim_{n\to\infty}\left(\int_{|t|\geqslant C}\mathrm{d}\mu_n\right)^{\frac12}\\
&\leqslant M\left(\int(t^{2r}+1)^2\mathrm{d}\mu\right)^{\frac12}\lim_{n\to\infty}\left(\int_{|t|\geqslant C}\mathrm{d}\mu_n\right)^{\frac12}=0
\end{aligned}$$

□

Remark. Let us define $\mathcal{A}$ as the $*$-algebra generated by $G(e_1),\dots,G(e_k)$. Observe that $\mathcal{A}$ is isomorphic to the $*$-algebra of all polynomials in k non-commuting variables (the free complex $*$-algebra with k generators). Indeed, if $P(G(e_1),\dots,G(e_k))=0$ for a polynomial P in k non-commuting variables, then the equation $P(G(e_1),\dots,G(e_k))\Omega=0$ implies that all coefficients of monomials of highest degree are 0, and thus $P=0$ by induction. More generally, this remains true for the q-Araki Woods algebras $\Gamma_q\left(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}}\right)$: if $(e_i)_{i\in I}$ is a free family of vectors in $H_{\mathbb{R}}$ then the $*$-algebra $\mathcal{A}_I$ generated by the family $(G(e_i))_{i\in I}$ is isomorphic to the free complex $*$-algebra with I generators.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}^*$ and consider the ultraproduct von Neumann algebra (see [P] section 9.10) N defined by

$$N=\left(\prod_{n\geqslant 1}\mathcal{A}_n\right)\Big/ I_{\mathcal{U}}$$

where $I_{\mathcal{U}} = \{(x_n)_{n\geqslant 1} \in \prod_{n\geqslant 1} \mathcal{A}_n,\ \lim_{\mathcal{U}} \tau_n(x_n^* x_n) = 0\}$. The von Neumann algebra N is equipped with the faithful normal and normalized trace $\tau((x_n)_{n\geqslant 1}) = \lim_{\mathcal{U}} \tau_n(x_n)$ (which is well defined).

Using the asymptotic matrix model for the q-Gaussians and by the preceding remark, we can define a $*$-homomorphism φ between the $*$-algebras $\mathcal{A}$ and N in the following way:

$$\varphi(P(G(e_1), \ldots, G(e_k))) = (P(\tilde{g}_{n,1}, \ldots, \tilde{g}_{n,k}))_{n\geqslant 1}$$

for every polynomial P in k non-commuting variables. By Lemma 2.3.1, φ is trace preserving on $\mathcal{A}$. Since the $*$-algebra $\mathcal{A}$ is weak-$*$ dense in $\Gamma_q(\mathbb{R}^k)$, φ extends naturally to a trace preserving homomorphism of von Neumann algebras, that is still denoted by φ (see Lemma 2.4.2 below for a more general result). It follows that $\Gamma_q(\mathbb{R}^k)$ is isomorphic to a sub-algebra of N which is the image of a conditional expectation (this is automatic in the tracial case). Since the $\mathcal{A}_n$'s are finite dimensional, they are injective, hence their product is injective and a fortiori has the WEP, and thus N is QWEP. Since $\Gamma_q(\mathbb{R}^k)$ is isomorphic to a sub-algebra of N which is the image of a conditional expectation, $\Gamma_q(\mathbb{R}^k)$ is also QWEP (see [Oz]). We have obtained the following:

Theorem 2.3.3 *Let $H_{\mathbb{R}}$ be a real Hilbert space and $q \in (-1, 1)$. The von Neumann algebra $\Gamma_q(H_{\mathbb{R}})$ is QWEP.*

Proof. Our previous discussion implies the result for every finite dimensional $H_{\mathbb{R}}$. The general result is a consequence of the stability of QWEP by inductive limit (see [Kir] and [Oz] Proposition 4.1 (iii)). ☐

Let $C_q^*(H_{\mathbb{R}})$ be the C^*-algebra generated by all q-Gaussians:

$$C_q^*(H_{\mathbb{R}}) = C^*\big(\{G(f), f \in H_{\mathbb{R}}\}\big) \subset B(\mathcal{F}_q(H_{\mathbb{C}})).$$

We now deduce the following strengthening of Theorem 2.3.3.

Corollary 2.3.4 *Let $H_{\mathbb{R}}$ be a real Hilbert space and $q \in (-1, 1)$. The C^*-algebra $C_q^*(H_{\mathbb{R}})$ is QWEP.*

Let A, B, with $A \subset B$ be C^*-algebras. Recall (from [Oz]) that A is said to be weakly cp complemented in B, if there exists a unital completely positive map $\Phi : B \longrightarrow A^{**}$ such that $\Phi_{|A} = \mathrm{id}_A$. Corollary 2.3.4 is then a consequence of the following Lemma.

Lemma 2.3.5 *The C^*-algebra $C_q^*(H_{\mathbb{R}})$ is weakly cp complemented in the von Neumann algebra $\Gamma_q(H_{\mathbb{R}})$.*

Proof. For any $t \in \mathbb{R}_+$ denote by Φ_t the unital completely positive maps which are the second quantization of $e^{-t}\mathrm{id} : H_{\mathbb{R}} \longrightarrow H_{\mathbb{R}}$ (see [BKS]):

$$\Phi_t = \Gamma_q(e^{-t}\mathrm{id}) : \Gamma_q(H_{\mathbb{R}}) \longrightarrow \Gamma_q(H_{\mathbb{R}}), \quad \text{for all} \quad t \geqslant 0.$$

$(\Phi_t)_{t\in\mathbb{R}_+}$ is a semi-group of unital completely positive maps which is also known as the q-Ornstein-Uhlenbeck semi-group. By the well-known ultracontractivity of the semi-group $(\Phi_t)_{t\in\mathbb{R}_+}$ (see [B2]), for all $t \in \mathbb{R}_+^*$ and all $W(\xi) \in \Gamma_q(H_{\mathbb{R}})$, we have

$$\|\Phi_t(W(\xi))\| \leqslant C_{|q|}^{\frac{3}{2}} \frac{1}{1 - e^{-t}} \|\xi\|. \tag{2.14}$$

On the other hand, as a consequence of the Haagerup-Bożejko's inequality (see [B2]), for every $n \in \mathbb{N}$ and for every $\xi_n \in H_{\mathbb{C}}^{\otimes n}$, we have $W(\xi_n) \in C_q^*(H_{\mathbb{R}})$. Fix $t \in \mathbb{R}_+^*$, $W(\xi) \in \Gamma_q(H_{\mathbb{R}})$, and write $\xi = \sum_{n\in\mathbb{N}} \xi_n$ with $\xi_n \in H_{\mathbb{C}}^{\otimes n}$ for all n. From our last observation, for all $N \in \mathbb{N}$,

$$T_N = \Phi_t\big(W\big(\sum_{n=0}^{N} \xi_n\big)\big) = \sum_{n=0}^{N} e^{-tn} W(\xi_n) \in C_q^*(H_{\mathbb{R}}).$$

By (2.14), $\Phi_t(W(\xi))$ is the norm limit of the sequence $(T_N)_{N\in\mathbb{N}}$, so $\Phi_t(W(\xi))$ belongs to $C_q^*(H_{\mathbb{R}})$. It follows that Φ_t maps $\Gamma_q(H_{\mathbb{R}})$ into $C_q^*(H_{\mathbb{R}})$. Moreover, it is clear that

$$\lim_{t\to 0} \|\Phi_t(W(\xi)) - W(\xi)\| = 0, \quad \text{for all} \quad W(\xi) \in C_q^*(H_{\mathbb{R}}). \tag{2.15}$$

Take $(t_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to 0 and fix $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$. By w^*-compactness of the closed balls in $\big(C_q^*(H_{\mathbb{R}})\big)^{**}$, we can define the following mapping $\Phi : \Gamma_q(H_{\mathbb{R}}) \longrightarrow \big(C_q^*(H_{\mathbb{R}})\big)^{**}$ by

$$\Phi(W(\xi)) = w^*\text{-}\lim_{n,\mathcal{U}} \Phi_{t_n}(W(\xi)), \quad \text{for all} \quad W(\xi) \in \Gamma_q(H_{\mathbb{R}}).$$

Φ is a unital completely positive map satisfying $\Phi_{|C_q^*(H_{\mathbb{R}})} = \mathrm{id}_{C_q^*(H_{\mathbb{R}})}$ by (2.15). □

Proof of Corollary 2.3.4. This is a consequence of Theorem 2.3.3, Lemma 2.3.5 and Proposition 4.1 (ii) in [Oz]. □

2.4 Embedding into an ultraproduct

The general setting is as follows. We start with a family $((\mathcal{A}_n, \varphi_n))_{n\in\mathbb{N}}$ of von Neumann algebras equipped with normal faithful state φ_n. We assume that $\mathcal{A}_n \subset B(H_n)$, where the inclusion is given by the G.N.S. representation of $(\mathcal{A}_n, \varphi_n)$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let

$$\widetilde{\mathcal{A}} = \prod_{n\in\mathbb{N}} \mathcal{A}_n/\mathcal{U}$$

be the C^*-ultraproduct over $\mathcal{U}$ of the algebras $\mathcal{A}_n$. We canonically identify $\widetilde{\mathcal{A}} \subset B(H)$, where $H = \prod_{n\in\mathbb{N}} H_n/\mathcal{U}$ is the ultraproduct over $\mathcal{U}$ of the Hilbert spaces H_n. Following Raynaud (see [Ray]), we define $\mathcal{A}$, the vN-ultraproduct over $\mathcal{U}$ of the von Neumann algebras $\mathcal{A}_n$, as the w^*-closure of $\widetilde{\mathcal{A}}$ in $B(H)$. Then the predual $\mathcal{A}_*$ of $\mathcal{A}$ is isometrically isomorphic to the Banach ultraproduct over $\mathcal{U}$ of the preduals $(\mathcal{A}_n)_*$:

$$\mathcal{A}_* = \prod_{n\in\mathbb{N}} (\mathcal{A}_n)_* /\mathcal{U} \tag{2.16}$$

Let us denote by φ the normal state on $\mathcal{A}$ associated to $(\varphi_n)_{n\in\mathbb{N}}$. Note that φ is not faithful on $\mathcal{A}$, so we introduce $p \in \mathcal{A}$ the support of the state φ. Recall that for all $x \in \mathcal{A}$ we have $\varphi(x) = \varphi(xp) = \varphi(px)$, and that $\varphi(x) = 0$ for a positive x implies that $pxp = 0$. Denote by $(p\mathcal{A}p, \varphi)$ the induced von Neumann algebra $p\mathcal{A}p \subset B(pH)$ equipped with the restriction of the state φ. For each $n \in \mathbb{N}$, let $(\sigma_t^n)_{t\in\mathbb{R}}$ be the modular group of automorphisms of

φ_n with the associated modular operator given by Δ_n. For all $t \in \mathbb{R}$, let $(\Delta_n^{it})^\bullet$ be the associated unitary in $\prod_{n\in\mathbb{N}} B(H_n)/\mathcal{U} \subset B(H)$. Since $(\sigma_t^n)^\bullet_{n\in\mathbb{N}}$ is the conjugation by $(\Delta_n^{it})^\bullet$, it follows that $(\sigma_t^n)^\bullet_{n\in\mathbb{N}}$ extends by w^*-continuity to a group of $*$-automorphisms of $\mathcal{A}$. Let $(\sigma_t)_{t\in\mathbb{R}}$ be the local modular group of automorphisms of $p\mathcal{A}p$. By Raynaud's result (see Theorem 2.1 in [Ray]), $p\mathcal{A}p$ is stable by $(\sigma_t^n)^\bullet_{n\in\mathbb{N}}$ and the restriction of $(\sigma_t^n)^\bullet_{n\in\mathbb{N}}$ to $p\mathcal{A}p$ coincides with σ_t.

In the following, we consider a von Neumann algebra $\mathcal{N} \subset B(K)$ equipped with a normal faithful state ψ. Let $\widetilde{\mathcal{N}}$ be a w^*-dense $*$-subalgebra of $\mathcal{N}$ and Φ a $*$-homomorphism from $\widetilde{\mathcal{N}}$ into $\mathcal{A}$ whose image will be denoted by $\widetilde{\mathcal{B}}$ with w^*-closure denoted by $\mathcal{B}$:

$$\Phi \;:\; \widetilde{\mathcal{N}} \subset \mathcal{N} \subset B(K) \longrightarrow \widetilde{\mathcal{B}} \subset \mathcal{A} \subset B(H) \quad \text{and} \quad \overline{\widetilde{\mathcal{N}}}^{w^*} = \mathcal{N}, \quad \overline{\widetilde{\mathcal{B}}}^{w^*} = \mathcal{B}$$

By a result of Takesaki (see [Tak]) there is a normal conditional expectation from $p\mathcal{A}p$ onto $p\mathcal{B}p$ if and only if $p\mathcal{B}p$ is stable by the modular group of φ (which is here given by Raynaud's results). Under this condition there will be a normal conditional expectation from $\mathcal{A}$ onto $p\mathcal{B}p$ and $p\mathcal{B}p$ will inherit some of the properties of $\mathcal{A}$. We would like to pull back these properties to $\mathcal{N}$ itself. It turns out that, with good assumptions on Φ (see Lemma 2.4.1 below), the compression from $\mathcal{B}$ onto $p\mathcal{B}p$ is a $*$-homomorphism. If in addition, we suppose that Φ is state preserving, then $p\Phi p$ can be extended into a w^*-continuous $*$-isomorphism between $\mathcal{N}$ and $p\mathcal{B}p$.

Lemma 2.4.1 *In the following,* 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. $\Longleftrightarrow$ 4. $\Longleftrightarrow$ 5.*:*

1. *For all $x \in \widetilde{\mathcal{B}}$ there is a representative $(x_n)_{n\in\mathbb{N}}$ of x such that for all $n \in \mathbb{N}$, x_n is entire for $(\sigma_t^n)_{t\in\mathbb{R}}$ and $(\sigma_{-i}^n(x_n))_{n\in\mathbb{N}}$ is uniformly bounded.*

2. *For all $x \in \widetilde{\mathcal{B}}$ there exists $z \in \mathcal{A}$ such that for all $y \in \mathcal{A}$ we have $\varphi(xy) = \varphi(yz)$.*

3. *For all $(x, y) \in \mathcal{B}^2$: $\varphi(xpy) = \varphi(xy)$*

4. *For all $(x, y) \in \mathcal{B}^2$, $pxyp = pxpyp$, i.e the canonical application from $\mathcal{B}$ to $p\mathcal{B}p$ is a $*$-homomorphism.*

5. *$p \in \mathcal{B}'$.*

Proof. 1. $\Longrightarrow$ 2. Consider $x \in \widetilde{\mathcal{B}}$ with a representative $(x_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, x_n is entire for $(\sigma_t^n)_{t\in\mathbb{R}}$ and $(\sigma_{-i}^n(x_n))_{n\in\mathbb{N}}$ is uniformly bounded. Denote by $z \in \mathcal{A}$ the class $(\sigma_{-i}^n(x_n))^\bullet_{n\in\mathbb{N}}$. By w^*-density and continuity if suffices to consider an element y in $\widetilde{\mathcal{A}}$ with representative $(y_n)_{n\in\mathbb{N}}$. Then,

$$\varphi(xy) = \lim_{n,\mathcal{U}} \varphi_n(x_n y_n) = \lim_{n,\mathcal{U}} \varphi_n(y_n \sigma_{-i}^n(x_n)) = \varphi(yz)$$

2. $\Longrightarrow$ 3. Here again it suffices to consider $(x, y) \in \widetilde{\mathcal{B}}^2$. By assumption there exists $z \in \mathcal{A}$ such that for all $t \in \mathcal{A}$, $\varphi(xt) = \varphi(tz)$. Applying our assumption for $t = py$ and $t = y$ successively, we obtain the desired result:

$$\varphi(xpy) = \varphi(pyz) = \varphi(yz) = \varphi(xy)$$

3. $\Longrightarrow$ 4. Let $x \in \mathcal{B}$. We have, by 3.: $\varphi(x(1-p)x^*) = 0$. Since p is the support of φ and $x(1-p)x^* \geqslant 0$, this implies $px(1-p)x^*p = 0$. Thus for all $x \in \mathcal{B}$ we have

$$pxpx^*p = pxx^*p$$

We conclude by polarization.
4. $\Longrightarrow$ 5. Let q be an orthogonal projection in $\mathcal{B}$. By 4., pqp is again an orthogonal projection and we claim that this is equivalent to $pq = qp$. Indeed, let us denote by x the contraction qp. Then $x^*x = pqp$ and since pqp is an orthogonal projection we have $|x| = pqp$. It follows that the polar decomposition of x is of the form $x = upqp$, with u a partial isometry. Computing x^2, we see that x is a projection:

$$x^2 = upqp(qp) = upqp = x.$$

Since x is contractive, we deduce that x is an orthogonal projection and that $x^* = x$. Thus $pq = qp$. Since $\mathcal{B}$ is generated by its projections, we have $p \in \mathcal{B}'$.
5. $\Longrightarrow$ 3. This is clear. □

We assume that one of the technical conditions of the previous Lemma is fulfilled. Let us denote by $\Theta = p\Phi p$. Θ is a $*$-homomorphism from $\widetilde{\mathcal{N}}$, into $p\mathcal{A}p$.

$$\Theta = p\Phi p \; : \; \widetilde{\mathcal{N}} \longrightarrow p\mathcal{A}p \subset B(pH)$$

We assume that Φ, and hence Θ, is state preserving. Then Θ can be extended into a (w^*-continuous) $*$-isomorphism from $\mathcal{N}$ onto $p\mathcal{B}p$. This is indeed a consequence of the following well known fact:

Lemma 2.4.2 *Let $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ be von Neumann algebras equipped with normal faithful states. Let $\widetilde{\mathcal{M}}$, (respectively $\widetilde{\mathcal{N}}$), be a w^* dense $*$-subalgebra of $\mathcal{M}$ (respectively $\mathcal{N}$). Let Ψ be a $*$-homomorphism from $\widetilde{\mathcal{M}}$ onto $\widetilde{\mathcal{N}}$ such that for all $m \in \widetilde{\mathcal{M}}$ we have $\psi(\Psi(m)) = \varphi(m)$ (Ψ is state preserving). Then Ψ extends uniquely into a normal $*$-isomorphism between $\mathcal{M}$ and $\mathcal{N}$.*

Proof. Since φ is faithful, we have for all $m \in \mathcal{M}$, $\|m\| = \lim_{n\to+\infty} \varphi\left((m^*m)^n\right)^{\frac{1}{2n}}$. Thus, since Ψ is state preserving, Ψ is isometric from $\widetilde{\mathcal{M}}$ onto $\widetilde{\mathcal{N}}$. We put

$$\varphi\widetilde{\mathcal{M}} = \{\varphi.m,\; m \in \widetilde{\mathcal{M}}\} \subset \mathcal{M}_* \quad \text{and} \quad \psi\widetilde{\mathcal{N}} = \{\psi.n,\; n \in \widetilde{\mathcal{N}}\} \subset \mathcal{N}_*.$$

$\varphi\widetilde{\mathcal{M}}$ (respectively $\psi\widetilde{\mathcal{N}}$) is dense in $\mathcal{M}_*$ (respectively $\mathcal{N}_*$). Let us define the following linear operator Ξ from $\psi\widetilde{\mathcal{N}}$ onto $\varphi\widetilde{\mathcal{M}}$:

$$\Xi(\psi.\Psi(m)) = \varphi.m \quad \text{for all } m \in \widetilde{\mathcal{M}}$$

Using Kaplansky's density Theorem and the fact that Ψ is isometric, we compute:

$$\begin{aligned}\|\Xi(\psi.\Psi(m)\| &= \sup_{m_0\in\widetilde{\mathcal{M}},\; \|m_0\|\leqslant 1} \|\varphi(mm_0)\| = \sup_{m_0\in\widetilde{\mathcal{M}},\; \|m_0\|\leqslant 1} \|\psi(\Psi(m)\Psi(m_0))\| \\ &= \sup_{n_0\in\widetilde{\mathcal{N}},\; \|n_0\|\leqslant 1} \|\psi(\Psi(m)n_0))\| = \|\psi.\Psi(m)\|\end{aligned}$$

So that Ξ extends into a surjective isometry from $\mathcal{N}_*$ onto $\mathcal{M}_*$. Moreover Ξ is the preadjoint of Ψ. Indeed we have for all $(m, m_0) \in \widetilde{\mathcal{M}}^2$:

$$\langle \psi.\Psi(m), \Psi(m_0)\rangle = \psi(\Psi(m)\Psi(m_0)) = \varphi(mm_0) = \langle \Xi(\psi.\Psi(m)), m_0\rangle$$

Thus Ψ extends to a normal $*$-isomorphism between $\mathcal{N}$ and $\mathcal{M}$. □

In the following Theorem, we sum up what we have proved in the previous discussion:

Theorem 2.4.3 *Let $(\mathcal{N}, \psi)$ and $(\mathcal{A}_n, \varphi_n)$, for $n \in \mathbb{N}$, be von Neumann algebras equipped with normal faithful states. Let $\mathcal{U}$ be a non trivial ultrafilter on $\mathbb{N}$, and $\mathcal{A}$ the von Neumann algebra ultraproduct over $\mathcal{U}$ of the $\mathcal{A}_n$'s. For all $n \in \mathbb{N}$ let us denote by $(\sigma_t^n)_{t\in\mathbb{R}}$ the modular group of φ_n and by φ the normal state on $\mathcal{A}$ which is the ultraproduct of the states φ_n. $p \in \mathcal{A}$ denote the support of φ. Consider $\widetilde{\mathcal{N}}$ a w^*-dense $*$-subalgebra of $\mathcal{N}$ and a $*$-homomorphism Φ*

$$\Phi \; : \; \widetilde{\mathcal{N}} \subset \mathcal{N} \longrightarrow \mathcal{A} = \prod_{n,\mathcal{U}} \mathcal{A}_n$$

Assume Φ satisfies:

1. *Φ is state preserving: for all $x \in \widetilde{\mathcal{N}}$ we have*

$$\varphi(\Phi(x)) = \psi(x)$$

2. *For all $(x, y) \in \Phi(\widetilde{\mathcal{N}})^2$*

$$\varphi(xy) = \varphi(xpy).$$

   (Or one of the technical conditions of Lemma 2.4.1.)

3. *For all $t \in \mathbb{R}$ and for all $y = (y_n)^\bullet_{n\in\mathbb{N}} \in \Phi(\widetilde{\mathcal{N}})$,*

$$p(\sigma_t^n(y_n))^\bullet_{n\in\mathbb{N}} p \; (= \sigma_t(pyp)) \in p\mathcal{B}p$$

   where $\mathcal{B}$ is the w^*-closure of $\Phi(\widetilde{\mathcal{N}})$ in $\mathcal{A}$.

Then $\Theta = p\Phi p \; : \; \widetilde{\mathcal{N}} \longrightarrow p\mathcal{A}p$ is a state preserving $*$-homomorphism which can be extended into a normal isomorphism (still denoted by Θ) between $\mathcal{N}$ and its image $\Theta(\mathcal{N}) = p\mathcal{B}p$. Moreover there exists a (normal) state preserving conditional expectation from $\mathcal{A}$ onto $\Theta(\mathcal{N})$.

Remarks.

• Conditions 2. is in fact necessary for Θ being a $*$-homomorphism (by Lemma 2.4.1), and condition 3. is necessary for the existence of a state preserving conditional expectation onto $\Theta(\mathcal{N})$ (by [Tak]).

• Let us denote by $(\sigma_t^\psi)_{t\in\mathbb{R}}$ the modular group of $*$-automorphisms of ψ. Provided that $(\sigma_t^\psi)_{t\in\mathbb{R}}$ maps $\widetilde{\mathcal{N}}$ into itself, we can replace condition 2. of the previous Theorem by the following intertwining condition: For all $t \in \mathbb{R}$ and for all $x \in \widetilde{\mathcal{N}}$ we have

$$p(\sigma_t^n(y_n))^\bullet_{n\in\mathbb{N}} p \; (= \sigma_t(p\Phi(x)p)) = p\Phi(\sigma_t^\psi(x))p$$

where $\Phi(x) = (y_n)^\bullet_{n\in\mathbb{N}}$. Moreover, notice that if the conclusion of the Theorem is true, then this condition must be fulfilled for all $t \in \mathbb{R}$ and for all $x \in \mathcal{N}$ (see [Tak2] page 95).

Corollary 2.4.4 *Under the assumptions of the previous Theorem, $\mathcal{N}$ is QWEP provided that each of the $\mathcal{A}_n$ is QWEP.*

Proof. This is a consequence of Kirchberg's results (see [Kir] and [Oz]). First, $\prod_{n\in\mathbb{N}} \mathcal{A}_n$ is QWEP as a product of QWEP C^*-algebras ([Oz] Proposition 4.1 (i)). Since $\tilde{\mathcal{A}}$ is a quotient of a QWEP C^*-algebra, it is also QWEP. It follows that $\mathcal{A}$ which is the w^*-closure of $\tilde{\mathcal{A}}$ in $B(H)$ is QWEP (by [Oz] Proposition 4.1 (iii)). Since there is a conditional expectation from $\mathcal{A}$ onto $p\mathcal{A}p$, $p\mathcal{A}p$ is QWEP (see [Kir]). Finally, by Theorem 2.4.3, $\mathcal{N}$ is isomorphic to a subalgebra of $p\mathcal{A}p$ which is the image of a (state preserving) conditional expectation, thus $\mathcal{N}$ inherits the QWEP property. □

2.5 The finite dimensional case

In this section we show that $\Gamma_q(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$ is QWEP when $H_\mathbb{R}$ is finite dimensional. For notational purpose, it will be more convenient to deal with $\dim(H_\mathbb{R})$ even. This is not relevant in our context (see the remark after Theorem 2.5.8). We put $\dim(H_\mathbb{R}) = 2k$. Notice that $\Gamma_q(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$ only depends on the spectrum of the operator A. The spectrum of A is given by the set $\{\lambda_1, \dots, \lambda_k\} \cup \{\lambda_1^{-1}, \dots, \lambda_k^{-1}\}$ where for all $j \in \{1, \dots, k\}$, $\lambda_j \geqslant 1$. As in subsection 2.2.2 , we use the notation $\mu_j = \lambda_j^{\frac{1}{4}}$.

2.5.1 Twisted Baby Fock

We start by adapting Biane's model to our situation. Let us denote by I the set $\{-k, \dots, -1\} \cup \{1, \dots, k\}$. As in subsection 2.2.4, we give us a function ϵ on $I \times I$ into $\{-1, 1\}$ and we consider the associated complex $*$-algebra $\mathcal{A}(I, \epsilon)$. By analogy with (2.3), for all $j \in \{1, \dots, k\}$ we define the following generalized semi-circular variables acting on $L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon)$:

$$\gamma_i = \mu_i^{-1}\beta_i^* + \mu_i\beta_{-i} \quad \textit{and} \quad \delta_i = \mu_i\alpha_i^* + \mu_i^{-1}\alpha_{-i}$$

We denote by Γ (respectively Γ_r) the von Neumann algebra generated in $B(L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon))$ by the γ_i (respectively δ_i). Γ_r is the natural candidate for the commutant of Γ in $B(L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon))$. We need to show that the vector 1 is cyclic and separating for Γ. To do so we must assume that ϵ satisfies the following additional condition:

$$\text{For all} \quad (i, j) \in I^2, \quad \epsilon(i, j) = \epsilon(|i|, |j|) \tag{2.17}$$

This condition is in fact a necessary condition for $\Gamma_r \subset \Gamma'$ and for condition 1.(a) of Lemma 2.5.2 below.

Lemma 2.5.1 *Under condition (2.17) the following relation holds:*

$$\textit{For all} \quad i \in I, \quad \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} = \beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^*$$

Proof. Let $i \in I$ and $A \subset I$. We have

$$(\alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i})(x_A) = \begin{cases} x_{-i}x_Ax_{-i} & \text{if} \quad i \in A \quad \text{and} \quad -i \in A \\ 0 & \text{if} \quad i \in A \quad \text{and} \quad -i \notin A \\ x_ix_Ax_i + x_{-i}x_Ax_{-i} & \text{if} \quad i \notin A \quad \text{and} \quad -i \in A \\ x_ix_Ax_i & \text{if} \quad i \notin A \quad \text{and} \quad -i \notin A \end{cases}$$

and

$$(\beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^*)(x_A) = \begin{cases} x_i x_A x_i & \text{if} \quad i \in A \quad \text{and} \quad -i \in A \\ x_i x_A x_i + x_{-i} x_A x_{-i} & \text{if} \quad i \in A \quad \text{and} \quad -i \notin A \\ 0 & \text{if} \quad i \notin A \quad \text{and} \quad -i \in A \\ x_{-i} x_A x_{-i} & \text{if} \quad i \notin A \quad \text{and} \quad -i \notin A \end{cases}$$

Thus, we need to study the following cases. Assume that $A = \{i_1, \ldots, i_p\}$ where $i_1 < \cdots < i_p$.

1. If i and $-i$ belong to A then there exists $(l, m) \in \{1, \ldots, p\}$, $l < m$, such that $i_l = -i$ and $i_m = i$. Applying successively relations (2.4) and (2.17), we get:

$$\begin{aligned} x_{-i} x_A x_{-i} &= \left(\prod_{q=1}^{l-1} \epsilon(i_q, -i) \right) x_{i_1} \ldots x_{i_{l-1}} x_{i_{l+1}} \ldots x_{i_p} x_{-i} \\ &= \left(\prod_{q=1}^{l-1} \epsilon(i_q, -i) \right) \left(\prod_{q=l+1}^{p} \epsilon(i_q, -i) \right) x_A = - \left(\prod_{q=1}^{p} \epsilon(i_q, -i) \right) x_A \\ &= - \left(\prod_{q=1}^{p} \epsilon(i_q, i) \right) x_A = x_i x_A x_i \end{aligned}$$

2. If i and $-i$ do not belong to A, we can check in a similar way that:

$$\begin{aligned} x_{-i} x_A x_{-i} &= \left(\prod_{q=1}^{p} \epsilon(i_q, -i) \right) x_A = \left(\prod_{q=1}^{p} \epsilon(i_q, i) \right) x_A \\ &= x_i x_A x_i \end{aligned}$$

3. If $i \in A$ and $-i \notin A$, then there exists $l \in \{1, \ldots, p\}$ such that $i_l = i$. We have:

$$\begin{aligned} x_i x_A x_i &= \left(\prod_{q=1}^{l-1} \epsilon(i_q, i) \right) x_{i_1} \ldots x_{i_{l-1}} x_{i_{l+1}} \ldots x_{i_p} x_i \\ &= \left(\prod_{q=1}^{l-1} \epsilon(i_q, i) \right) \left(\prod_{q=l+1}^{p} \epsilon(i_q, i) \right) x_A = - \left(\prod_{q=1}^{p} \epsilon(i_q, i) \right) x_A \\ &= - \left(\prod_{q=1}^{p} \epsilon(i_q, -i) \right) x_A = -x_{-i} x_A x_{-i} \end{aligned}$$

   This finishes the proof.

□

Lemma 2.5.2 *By construction we have:*

1. *For all $(i, j) \in \{1, \ldots, k\}^2$, $i \neq j$, the following mixed commutation and anti-commutation relations hold:*

(a) $\gamma_i\gamma_j - \epsilon(i,j)\gamma_j\gamma_i = 0$

(b) $\gamma_i^*\gamma_j - \epsilon(i,j)\gamma_j\gamma_i^* = 0$

(c) $(\gamma_i^*)^2 = \gamma_i^2 = 0$

(d) $\gamma_i^*\gamma_i + \gamma_i\gamma_i^* = (\mu_i^2 + \mu_i^{-2})Id.$

2. *Same relations as in 1. for the operators* δ_i.

3. $\Gamma_r \subset \Gamma'$.

4. *The vector* 1 *is cyclic and separating for both* Γ *and* Γ_r.

5. $\Gamma \subset B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ *is the (faithful) G.N.S representation of* $(\Gamma,\varphi^\epsilon)$.

Proof. 1.(a) Thanks to 2. of Lemma 2.2.3 and (2.17) we get:

$$\begin{aligned}
\gamma_i\gamma_j &= \mu_i^{-1}\mu_j^{-1}\beta_i^*\beta_j^* + \mu_i\mu_j\beta_{-i}\beta_{-j} + \mu_i^{-1}\mu_j\beta_i^*\beta_{-j} + \mu_i\mu_j^{-1}\beta_{-i}\beta_j^* \\
&= \epsilon(i,j)\mu_i^{-1}\mu_j^{-1}\beta_j^*\beta_i^* + \epsilon(-i,-j)\mu_i\mu_j\beta_{-j}\beta_{-i} + \epsilon(i,-j)\mu_i^{-1}\mu_j\beta_{-j}\beta_i^* \\
&\qquad + \epsilon(-i,j)\mu_i\mu_j^{-1}\beta_j^*\beta_{-i} \\
&= \epsilon(i,j)\left(\mu_i^{-1}\mu_j^{-1}\beta_j^*\beta_i^* + \mu_i\mu_j\beta_{-j}\beta_{-i} + \mu_i^{-1}\mu_j\beta_{-j}\beta_i^* + \mu_i\mu_j^{-1}\beta_j^*\beta_{-i}\right) \\
&= \epsilon(i,j)\gamma_j\gamma_i
\end{aligned}$$

1.(b) Is analogous to (a) and is left to the reader.
1.(c) Using 1. and 2. of Lemma 2.2.3, and $\epsilon(i,-i) = \epsilon(i,i) = -1$ we get:

$$\begin{aligned}
\gamma_i^2 &= \mu_i^{-2}(\beta_i^*)^2 + \mu_i^2\beta_{-i}^2 + \beta_i^*\beta_{-i} + \beta_{-i}\beta_i^* \\
&= \epsilon(i,-i)\beta_{-i}\beta_i^* + \beta_{-i}\beta_i^* = 0
\end{aligned}$$

1.(d) Using similar arguments, we compute:

$$\begin{aligned}
\gamma_i^*\gamma_i + \gamma_i\gamma_i^* &= \mu_i^{-2}(\beta_i\beta_i^* + \beta_i^*\beta_i) + \mu_i^2(\beta_{-i}^*\beta_{-i} + \beta_{-i}\beta_{-i}^*) + \beta_i\beta_{-i} + \beta_{-i}\beta_i \\
&\qquad + \beta_{-i}^*\beta_i^* + \beta_i^*\beta_{-i}^* \\
&= (\mu_i^{-2} + \mu_i^2)Id + (\epsilon(i,-i)+1)(\beta_i\beta_{-i} + \beta_{-i}^*\beta_i^*) = (\mu_i^{-2} + \mu_i^2)Id
\end{aligned}$$

2. Is now clear from the proof of 1. since the relations for the α_i's are the same as the ones for the β_i's.
3. It suffices to show that for all $(i,j) \in \{1,\dots,k\}^2$ we have $\gamma_i\delta_j = \delta_j\gamma_i$ and $\gamma_i\delta_j^* = \delta_j^*\gamma_i$. If $i \neq j$ then from 5. of Lemma 2.2.3 it is clear that $\gamma_i\delta_j = \delta_j\gamma_i$ and $\gamma_i\delta_j^* = \delta_j^*\gamma_i$.
If $i = j$ then using 4. and 5. of Lemma 2.2.3 and Lemma 2.5.1 we obtain the desired result as follows:

$$\begin{aligned}
\gamma_i\delta_i &= \beta_i^*\alpha_i^* + \beta_{-i}\alpha_{-i} + \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^* = \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^* \\
&= \mu_i^{-2}\alpha_{-i}\beta_i^* + \mu_i^2\alpha_i^*\beta_{-i} = \delta_i\gamma_i
\end{aligned}$$

and

$$\begin{aligned}
\gamma_i\delta_i^* &= \beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^* + \mu_i^{-2}\beta_i^*\alpha_{-i}^* + \mu_i^2\beta_{-i}\alpha_i \\
&= \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} + \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^*
\end{aligned}$$

$$= \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} + \mu_i^{-2}\alpha_{-i}\beta_i^* + \mu_i^2\alpha_i^*\beta_{-i} = \delta_i^*\gamma_i$$

4. It suffices to prove that for any $A \subset I$ we have $x_A \in \Gamma 1 \cap \Gamma_r 1$. Let $A \subset I$ and $(\chi_i)_{i\in I} \in \{0,1\}^I$ such that $\chi_i = 1$ if and only if $i \in A$. Then

$$\begin{aligned} x_A &= x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k} \\ &= (\mu_k^{-1}\gamma_k^*)^{\chi_{-k}} \dots (\mu_1^{-1}\gamma_1^*)^{\chi_{-1}} (\mu_1\gamma_1)^{\chi_1} \dots (\mu_k\gamma_k)^{\chi_k} 1 \\ &= \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} \gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}} \gamma_1^{\chi_1} \dots \gamma_k^{\chi_k} 1 \end{aligned}$$

where by convention $\gamma_i^{-1} = \gamma_i^*$.
The same computation is valid for Γ_r and we obtain:

$$x_A = \mu_1^{\chi_{-1}-\chi_1} \dots \mu_k^{\chi_{-k}-\chi_k} \delta_k^{\chi_k} \dots \delta_1^{\chi_1} \delta_1^{-\chi_{-1}} \dots \delta_k^{-\chi_{-k}} 1$$

It follows that the vector 1 is cyclic for both Γ and Γ_r. Since $\Gamma_r \subset \Gamma'$ then 1 is also cyclic for Γ' and thus separating for Γ. The same argument applies to Γ_r and thus 1 is also a cyclic and separating vector for Γ_r.
5. This is clear from the just proved assertion and the fact that the state φ^ϵ is equal to the vector state associated to the vector 1. □

By the Lemma just proved, we are in a situation where we can apply Tomita-Takesaki theory. As usual we denote by S the involution on $L^2(\mathcal{A}(I,\epsilon), \varphi^\epsilon)$ defined by: $S(\gamma 1) = \gamma^* 1$ for all $\gamma \in \Gamma$. Δ will denote the modular operator and J the modular conjugation. Recall that $S = J\Delta^{\frac{1}{2}}$ is the polar decomposition of the antilinear operator S (which is here bounded since we are in a finite dimensional framework). We also denote by $(\sigma_t)_{t\in\mathbb{R}}$ the modular group of automorphisms of Γ associated to φ. Recall that for all $\gamma \in \Gamma$ and all $t \in \mathbb{R}$ we have $\sigma_t(\gamma) = \Delta^{it}\gamma\Delta^{-it}$.

Notation: In the following, for $A \subset I$ we denote by $(\chi_i)_{i\in I}$ the characteristic function of the set A : $\chi_i = 1$ if $i \in A$ and $\chi_i = 0$ if $i \notin A$. (We will not keep track of the dependance in A unless there could be some confusion.)

Proposition 2.5.3 *The modular operators and the modular group of* $(\Gamma, \varphi^\epsilon)$ *are determined by:*

1. *J is the antilinear operator given by: for all $A \subset I$,*

$$J(x_A) = J(x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k}) = x_{-k}^{\chi_k} \dots x_{-1}^{\chi_1} x_1^{\chi_{-1}} \dots x_k^{\chi_{-k}}$$

2. *Δ is the diagonal and positive operator given by: for all $A \subset I$,*

$$\Delta(x_A) = \Delta(x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k}) = \lambda_k^{(\chi_k-\chi_{-k})} \dots \lambda_1^{(\chi_1-\chi_{-1})} x_A$$

3. *For all $j \in \{1\dots,k\}$, γ_j is entire for $(\sigma_t)_t$ and satisfies $\sigma_z(\gamma_j) = \lambda_j^{iz}\gamma_j$ for all $z \in \mathbb{C}$.*

Proof. Let $A \subset I$. We have

$$x_A = x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k} = \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} \gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}} \gamma_1^{\chi_1} \dots \gamma_k^{\chi_k} 1$$

Thus,

$$S(x_A) = \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} (\gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}} \gamma_1^{\chi_1} \dots \gamma_k^{\chi_k})^* 1$$

$$= \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} \gamma_k^{-\chi_k} \dots \gamma_1^{-\chi_1} \gamma_1^{\chi_{-1}} \dots \gamma_k^{\chi_{-k}} 1$$
$$= \mu_1^{2(\chi_1-\chi_{-1})} \dots \mu_k^{2(\chi_k-\chi_{-k})} x_{-k}^{\chi_k} \dots x_{-1}^{\chi_1} x_1^{\chi_{-1}} \dots x_k^{\chi_{-k}}$$

By uniqueness of the polar decomposition, we obtain the stated result. Let $j \in \{1 \dots k\}$ and $t \in \mathbb{R}$ we have:

$$\sigma_t(\gamma_j)1 = \Delta^{it}\gamma_j\Delta^{-it}1 = \Delta^{it}\gamma_j 1 = \mu_j^{-1}\Delta^{it}x_j = \mu_j^{-1}\mu_j^{4it}x_j$$
$$= \mu_j^{4it}\gamma_j 1$$

It follows, since 1 is separating for Γ, that $\sigma_t(\gamma_j) = \mu_j^{4it}\gamma_j$. □

Remarks.

• We have $\Gamma' = \Gamma_r$. Indeed we have already proved the inclusion $\Gamma_r \subset \Gamma'$ in Lemma 2.5.2 . For the reverse inclusion we can use Tomita-Takesaki theory which ensures that $\Gamma' = J\Gamma J$. But for all $j \in I$ it is easy to see that $J\beta_j J = \alpha_{-j}$. It follows that for all $j \in \{1, \dots, k\}$ we have $J\gamma_j J = \delta_j^*$. Thus $\Gamma' \subset \Gamma_r$. The equality $\Gamma' = \Gamma_r$ can also be seen as a consequence of a general fact in Tomita-Takesaki theory: it suffices to remark that Γ_r is the right Hilbertian algebra associated to Γ in its GNS representation.

• The previous construction can be performed for an infinite set of the form $J \times \{-1, 1\}$ given with a family of eigenvalues $(\mu_j)_{j\in J} \in [1, +\infty[^J$ and a sign function ϵ satisfying

$$\epsilon((j,i),(j',i')) = \epsilon((j,1),(j',1)) \quad \text{for all} \quad ((j,i),(j',i')) \in (J \times \{-1,1\})^2.$$

2.5.2 Central limit approximation of q-Gaussians

In this section we use the twisted Baby Fock construction to obtain an asymptotic random matrix model for the q-Gaussian variables, via Speicher's central limit Theorem. Let us first check the independence condition:

Lemma 2.5.4 *For all $j \in \{1, \dots, k\}$ let us denote by $\mathcal{A}_j$ the C^*-subalgebra of $B(L^2(\mathcal{A}(I,\epsilon), \varphi^\epsilon))$ generated by the operators β_j and β_{-j}. Then the family $(\mathcal{A}_j)_{1\leqslant j\leqslant k}$ is independent in $B(L^2(\mathcal{A}(I,\epsilon), \varphi^\epsilon))$. In particular, the family $(\gamma_j)_{1\leqslant j\leqslant k}$ is independent.*

Proof. The proof proceeds by induction. Changing notation, it suffices to show that

$$\varphi^\epsilon(a_1 \dots a_{r+1}) = \varphi^\epsilon(a_1 \dots a_r)\varphi^\epsilon(a_{r+1})$$

where $a_l \in \mathcal{A}_l$ for all $l \in \{1, \dots, r+1\}$. Since a_{r+1} is a certain non-commutative polynomial in the variables β_{r+1}, β_{r+1}^*, $\beta_{-(r+1)}$, and $\beta_{-(r+1)}^*$, it is clear that there exists $\nu \in \mathrm{Span}\{x_{r+1},\ x_{-(r+1)},\ x_{-(r+1)}x_{r+1}\}$ such that

$$a_{r+1}1 = \langle 1, a_{r+1}1\rangle 1 + \nu$$

It is easy to see that $a_r^* \dots a_1^* 1 \in \mathrm{Span}\{x_B,\ B \subset \{-r, \dots, -1\} \cup \{1, \dots, r\}\}$, which is orthogonal to $\mathrm{Span}\{x_{r+1},\ x_{-(r+1)},\ x_{-(r+1)}x_{r+1}\}$. We compute:

$$\varphi^\epsilon(a_1 \dots a_{r+1}) = \langle 1, a_1 \dots a_r a_{r+1} 1\rangle = \langle a_r^* \dots a_1^* 1, a_{r+1}1\rangle$$
$$= \langle a_r^* \dots a_1^* 1, 1\rangle\langle 1, a_{r+1}1\rangle + \langle a_r^* \dots a_1^* 1, \nu\rangle = \langle 1, a_1 \dots a_r 1\rangle\langle 1, a_{r+1}1\rangle$$

$$= \varphi^{\epsilon}(a_1 \dots a_r)\varphi^{\epsilon}(a_{r+1})$$

□

Remark. It is clear that one can prove, in the same way, that the C^*-algebras generated by the β_j are independent (this is Proposition 3 in [Bi]).

Let $q \in (-1,1)$. Let us choose a family of random variables $(\epsilon(i,j))_{(i,j)\in\mathbb{N}_*^2, i\neq j}$ as in Lemma 2.2.6, and set $\epsilon(i,i) = -1$ for all $i \in \mathbb{N}_*$. As in section 2.5.1, for all $n \in \mathbb{N}_*$ we will consider the complex $*$-algebra $\mathcal{A}(I_n, \epsilon_n)$ where

$$I_n = \{1,\dots,n\} \times \big(\{-k,\dots,-1\} \cup \{1,\dots,k\}\big)$$

and

$$\epsilon_n((i,j),(i',j')) = \epsilon(i,i') \quad \text{for all} \quad ((i,j),(i',j')) \in I_n^2.$$

Notice that the analogue of condition (2.17) is automatically satisfied. Indeed, we have:

$$\epsilon_n((i,j),(i',j')) = \epsilon_n((i,|j|),(i',|j'|)) \quad \text{for all} \quad ((i,j),(i',j')) \in I_n^2.$$

Let us remind that $\mathcal{A}(I_n, \epsilon_n)$ is the unital free complex algebra with generators $(x_{i,j})_{(i,j)\in I_n}$ quotiented by the relations,

$$x_{i,j}x_{i',j'} - \epsilon(i,i')x_{i',j'}x_{i,j} = 2\delta_{(i,j),(i',j')}$$

and with involution given by $x_{i,j}^* = x_{i,j}$. For all $(i,j) \in \{1,\dots,n\} \times \{1,\dots,k\}$ let $\gamma_{i,j}$ be the "twisted semi-circular variable" associated to μ_j

$$\gamma_{i,j} = \mu_j^{-1}\beta_{i,j}^* + \mu_j\beta_{i,j}$$

We denote by $\Gamma_n \subset B(L^2(\mathcal{A}(I_n,\epsilon_n),\varphi^{\epsilon_n}))$ the von-Neumann algebra generated by the $\gamma_{i,j}$ for $(i,j) \in \{1,\dots,n\} \times \{1,\dots,k\}$. Observe that all our notations are consistent since $(\Gamma_n, \varphi^{\epsilon_n})$ is naturally embedded in $(\Gamma_{n+1}, \varphi^{\epsilon_{n+1}})$ (see the remarks following Lemma (2.2.3)). In fact all these algebras $(\Gamma_n, \varphi^{\epsilon_n})$ can be embedded in the bigger von Neumann algebra $(\Gamma, \varphi^{\overline{\epsilon}})$ which is the Baby Fock construction associated to the infinite set $\overline{I}$ and the sign function $\overline{\epsilon}$ given by

$$\overline{I} = \mathbb{N}_* \times \big(\{-k,\dots,-1\} \cup \{1,\dots,k\}\big)$$

and

$$\overline{\epsilon}((i,j),(i',j')) = \epsilon(i,i') \quad \text{for all} \quad ((i,j),(i',j')) \in \overline{I}^2.$$

Let us denote by $s_{n,j}$ the following sum:

$$s_{n,j} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\gamma_{i,j}$$

We now check the hypothesis of Theorem 2.2.5 for the family $(\gamma_{i,j})_{(i,j)\in\mathbb{N}_*\times\{1,\dots,k\}} \subset (\Gamma, \varphi^{\overline{\epsilon}})$.

1. The family is independent by Lemma 2.5.4.

2. It is clear that for all (i,j) we have $\varphi^{\overline{\epsilon}}(\gamma_{i,j}) = 0$.

3. Let $(j(1), j(2)) \in \{1, \ldots, k\}$ and $i \in \mathbb{N}_*$. We compute and identify the covariance thanks to Lemma 2.2.2:

$$\begin{aligned}\varphi^{\overline{\epsilon}}(\gamma_{i,j(1)}^{k(1)}\gamma_{i,j(2)}^{k(2)}) &= \langle \gamma_{i,j(1)}^{-k(1)}1, \gamma_{i,j(2)}^{k(2)}1\rangle = \langle \mu_{j(1)}^{k(1)}x_{-k(1)i,-k(1)j(1)}, \mu_{j(2)}^{-k(2)}x_{k(2)i,k(2)j(2)}\rangle \\ &= \mu_{j(1)}^{2k(1)}\delta_{k(2),-k(1)}\delta_{j(1),j(2)} = \varphi(c_{j(1)}^{k(1)}c_{j(2)}^{k(2)})\end{aligned}$$

4. It is easily seen that $\varphi^{\overline{\epsilon}}(\gamma_{i,j}^{k(1)} \ldots \gamma_{i,j}^{k(w)})$ is independent of $i \in \mathbb{N}_*$.

5. This is a consequence of Lemma 2.5.2.

6. This follows from Lemma 2.2.6 almost surely.

Thus, by Theorem 2.2.5, we have, almost surely, for all $p \in \mathbb{N}_*$, $(k(1), \ldots, k(p)) \in \{-1, 1\}^p$ and all $(j(1), \ldots, j(p)) \in \{1, \ldots, k\}^p$:

$$\lim_{n\to+\infty} \varphi^{\overline{\epsilon}}(s_{n,j(1)}^{k(1)} \ldots s_{n,j(p)}^{k(p)}) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \displaystyle\sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\ldots,2r) \\ \mathcal{V}=\{(s_l,t_l)\}_{l=1}^{l=r}}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \varphi(c_{j(s_l)}^{k(s_l)}c_{j(t_l)}^{k(t_l)}) & \text{if } p = 2r \end{cases}$$

By Lemma 2.2.2 we see that all $*$-moments of the family $(s_{n,j})_{j\in\{1,\ldots,k\}}$ converge when n goes to infinity to the corresponding $*$-moments of the family $(c_j)_{j\in\{1,\ldots,k\}}$:

Proposition 2.5.5 *For all $p \in \mathbb{N}_*$, $(j(1), \ldots, j(p)) \in \{1, \ldots, k\}^p$ and for all $(k(1), \ldots, k(p)) \in \{-1, 1\}^p$ we have:*

$$\lim_{n\to+\infty} \varphi^{\overline{\epsilon}}(s_{n,j(1)}^{k(1)} \ldots s_{n,j(p)}^{k(p)}) = \varphi(c_{j(1)}^{k(1)} \ldots c_{j(p)}^{k(p)}) \quad \textit{almost surely} \tag{2.18}$$

Remark. It is possible (and maybe easier) to apply directly Speicher's Theorem to the independent family $(\beta_{i,j})_{(i,j)\in\overline{I}^2}$. Then, it suffices to follow the analogies between the Baby Fock and the q-Fock frameworks to deduce the previous Proposition.

2.5.3 $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ is QWEP

For all $j \in \{1, \ldots, k\}$ let us denote by $g_{n,j} = \mathrm{Re}\,(s_{n,j})$ and $g_{n,-j} = \mathrm{Im}\,(s_{n,j})$. By (2.18) we have that for all monomials P in $2k$ non-commuting variables:

$$\lim_{n\to+\infty} \varphi^{\overline{\epsilon}}(P(g_{n,-k}, \ldots, g_{n,k})) = \varphi(P(G(f_{-k}), \ldots, G(f_k))) \quad \text{almost surely} \tag{2.19}$$

Since the set of all non-commutative monomials is countable, we can find a choice of signs ϵ such that (2.19) is true for all P. In the sequel we fix such an ϵ and forget about the dependance on ϵ.

Lemma 2.5.6 *For all polynomials P in $2k$ non-commuting variables we have:*

$$\lim_{n\to+\infty} \varphi(P(g_{n,-k}, \ldots, g_{n,k})) = \varphi(P(G(f_{-k}), \ldots, G(f_k))) \tag{2.20}$$

We are now ready to construct an embedding of $\Gamma_q(H_{\mathbb{R}}, U_t)$ into an ultraproduct of the finite dimensional von Neumann algebras Γ_n. To do so we need to have a uniform bound on the operators $g_{n,j}$. Let $C > 0$ such that for all $j \in I$, $\|G(f_j)\| < C$, as in the tracial case, we replace the $g_{n,j}$ by the their truncations $\tilde{g}_{n,j} = \chi_{]-C,C[}(g_{n,j})g_{n,j}$. The following is the analogue of Lemma 2.3.1:

Lemma 2.5.7 *For all polynomials P in $2k$ non-commuting variables we have:*

$$\lim_{n\to+\infty} \varphi(P(\tilde{g}_{n,-k}, \ldots, \tilde{g}_{n,k})) = \varphi(P(G(f_{-k}), \ldots, G(f_k))) \tag{2.21}$$

Remark. For all $n \in \mathbb{N}_*$ and all $j \in I$ the element $g_{n,j}$ is entire for the modular group (this is always the case in a finite dimensional framework). By (3) of proposition 2.5.3, we have for all $j \in \{1, \ldots, k\}$

$$\sigma_z(s_{n,j}) = \lambda_j^{iz} s_{n,j} \quad \text{for all } z \in \mathbb{C}$$

Thus for all $z \in \mathbb{C}$,

$$\sigma_z(g_{n,j}) = \begin{cases} \cos(z\ln(\lambda_j))g_{n,j} - \sin(z\ln(\lambda_j))g_{n,-j} & \text{for all } j \in \{1, \ldots, k\} \\ \sin(z\ln(\lambda_{-j}))g_{n,-j} + \cos(z\ln(\lambda_{-j}))g_{n,j} & \text{for all } j \in \{-1, \ldots, -k\} \end{cases} \tag{2.22}$$

Proof of Lemma 2.5.7. It suffices to show that for all $(j(1), \ldots, j(p)) \in I^p$ we have

$$\lim_{n\to+\infty} \varphi(\tilde{g}_{n,j(1)} \ldots \tilde{g}_{n,j(p)}) = \varphi(G(f_{j(1)}) \ldots G(f_{j(p)}))$$

By (2.20) it is sufficient to prove that

$$\lim_{n\to+\infty} |\varphi(g_{n,j(1)} \ldots g_{n,j(p)}) - \varphi(\tilde{g}_{n,j(1)} \ldots \tilde{g}_{n,j(p)})| = 0$$

Using multi-linearity we can write

$$\begin{aligned} &|\varphi(g_{n,j(1)} \ldots g_{n,j(p)}) - \varphi(\tilde{g}_{n,j(1)} \ldots \tilde{g}_{n,j(p)})| \\ &\qquad = |\sum_{l=1}^{p} \varphi[\tilde{g}_{n,j(1)} \ldots \tilde{g}_{n,j(l-1)}(g_{n,j(l)} - \tilde{g}_{n,j(l)})g_{n,j(l+1)} \ldots g_{n,j(p)}]| \\ &\qquad \leqslant \sum_{l=1}^{p} |\varphi[\tilde{g}_{n,j(1)} \ldots \tilde{g}_{n,j(l-1)}(g_{n,j(l)} - \tilde{g}_{n,j(l)})g_{n,j(l+1)} \ldots g_{n,j(p)}]| \end{aligned}$$

Fix $l \in \{1, \ldots, p\}$, using the modular group we have:

$$\begin{aligned} &|\varphi[\tilde{g}_{n,j(1)} \ldots \tilde{g}_{n,j(l-1)}(g_{n,j(l)} - \tilde{g}_{n,j(l)})g_{n,j(l+1)} \ldots g_{n,j(p)}]| \\ &\qquad = |\varphi[\sigma_i(g_{n,j(l+1)} \ldots g_{n,j(p)})\tilde{g}_{n,j(1)} \ldots \tilde{g}_{n,j(l-1)}(g_{n,j(l)} - \tilde{g}_{n,j(l)})]| \end{aligned}$$

Estimating by Cauchy-Schwarz's inequality we obtain:

$$\begin{aligned} &|\varphi[\sigma_i(g_{n,j(l+1)} \ldots g_{n,j(p)})\tilde{g}_{n,j(1)} \ldots \tilde{g}_{n,j(l-1)}(g_{n,j(l)} - \tilde{g}_{n,j(l)})]| \\ &\qquad \leqslant \varphi[\sigma_i(g_{n,j(l+1)} \ldots g_{n,j(p)})\tilde{g}_{n,j(1)} \ldots \tilde{g}^2_{n,j(l-1)} \ldots \tilde{g}_{n,j(1)}\sigma_{-i}(g_{n,j(p)} \ldots g_{n,j(l+1)})]^{\frac{1}{2}} \end{aligned}$$

$$\times \varphi[(g_{n,j(l)} - \tilde{g}_{n,j(l)})^2]^{\frac{1}{2}}$$
$$\leqslant C^{l-1}\varphi[\sigma_i(g_{n,j(l+1)} \dots g_{n,j(p)})\sigma_{-i}(g_{n,j(p)} \dots g_{n,j(l+1)})]^{\frac{1}{2}}\varphi[(g_{n,j(l)} - \tilde{g}_{n,j(l)})^2]^{\frac{1}{2}}$$

The conclusion follows from the convergence of this last term to 0. Indeed, by (2.22) there exists a polynomial in $2k$ non-commutative variables Q, independent on n, such that $Q(g_{n,-k} \dots g_{n,k}) = \sigma_i(g_{n,j(l+1)} \dots g_{n,j(p)})\sigma_{-i}(g_{n,j(p)} \dots g_{n,j(l+1)})$. It follows by (2.20) that

$$\lim_{n\to+\infty} \varphi[\sigma_i(g_{n,j(l+1)} \dots g_{n,j(p)})\sigma_{-i}(g_{n,j(p)} \dots g_{n,j(l+1)})] = \varphi(Q(G(f_{-k}) \dots G(f_k))).$$

And by Lemma 2.3.2, $\varphi[(g_{n,j(l)} - \tilde{g}_{n,j(l)})^2]$ converges to 0 when n goes to infinity. □

Let us denote by $\mathcal{P}$ the w^*-dense $*$-subalgebra of $\Gamma_q(H_\mathbb{R}, U_t)$ generated by the set $\{G(f_j),\ j \in I\}$. We know that $\mathcal{P}$ is isomorphic to the algebra of non-commutative polynomials in $2k$ variables (see the remark after Lemma 2.3.2). Given $\mathcal{U}$ a non trivial ultrafilter on $\mathbb{N}$, it is thus possible to define the following $*$-homomorphism Φ from $\mathcal{P}$ into the von Neumann ultraproduct $\mathcal{A} = \prod_{n,\mathcal{U}} \Gamma_n$ by:

$$\Phi(P(G(f_{-k}), \dots, G(f_k))) = (P(\tilde{g}_{n,-k}, \dots, \tilde{g}_{n,k}))^\bullet_{n\in\mathbb{N}}$$

Indeed the right term is well defined since it is uniformly bounded in norm. Let us check the hypothesis of Theorem 2.4.3.

1. By Lemma 2.5.7, Φ is state preserving.

2. It is sufficient to check that condition 2. of Lemma 2.4.1 is satisfied for every generator $\Phi(G(f_j))$, $j \in I$. Let us fix $j \in I$ and recall that by (2.22) there are complex numbers ν_j and ω_j (independent of n) such that $\sigma^n_{-i}(g_{n,j}) = \nu_j g_{n,j} + \omega_j g_{n,-j}$. We show that condition 2. of Lemma 2.4.1 is satisfied for $x = \Phi(G(f_j))$ and $z = \nu_j\Phi(G(f_j)) + \omega_j\Phi(G(f_{-j}))$. By w^*-density it is sufficient to consider $y = (y_n)^\bullet_{n\in\mathbb{N}} \in \widetilde{\mathcal{A}}$. Using Lemma 2.5.7 we have:

$$\begin{aligned}\varphi(\Phi(G(f_j))y) &= \lim_{n,\mathcal{U}} \varphi_n(\tilde{g}_{n,j}y_n) = \lim_{n,\mathcal{U}} \varphi_n(g_{n,j}y_n) = \lim_{n,\mathcal{U}} \varphi_n(y_n\sigma^n_{-i}(g_{n,j}))\\ &= \lim_{n,\mathcal{U}} \varphi_n(y_n(\nu_j g_{n,j} + \omega_j g_{n,-j})) = \lim_{n,\mathcal{U}} \varphi_n(y_n(\nu_j \tilde{g}_{n,j} + \omega_j \tilde{g}_{n,-j}))\\ &= \varphi(y(\nu_j\Phi(G(f_j)) + \omega_j\Phi(G(f_{-j}))))\end{aligned}$$

3. It suffices to check that the intertwining condition given in the remark of Theorem 2.4.3 is satisfied for the generators $\Phi(G(f_j)) = (\tilde{g}_{n,j})^\bullet_{n\in N}$:

$$\text{for all } j \in I, \quad \sigma_t(p\Phi(G(f_j))p) = p\Phi(\sigma_t(G(f_j)))p$$

To fix ideas we will suppose that $j \geqslant 0$. Recall that in this case for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, we have

$$\sigma^n_t(g_{n,j}) = \cos(t\ln(\lambda_j))g_{n,j} - \sin(t\ln(\lambda_j))g_{n,-j}.$$

Since the functional calculus commutes with automorphisms, for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, we have:

$$\sigma_t^n(\tilde{g}_{n,j}) = h(\sigma_t^n(g_{n,j})),$$

where $h(\lambda) = \chi_{]-C,C[}(\lambda)\lambda$, for all $\lambda \in \mathbb{R}$. But by Lemma 2.5.6,

$$\sigma_t^n(g_{n,j}) = \cos(t\ln(\lambda_j))g_{n,j} - \sin(t\ln(\lambda_j))g_{n,-j}$$

converges in distribution to

$$\cos(t\ln(\lambda_j))G(f_j) - \sin(t\ln(\lambda_j))G(f_{-j}) = \sigma_t(G(f_j))$$

and $\|\sigma_t(G(f_j))\| = \|G(f_j)\| < C$. Thus, by Lemma 2.3.2, we deduce that $\sigma_t^n(\tilde{g}_{n,j})$ converges in distribution to $\sigma_t(G(f_j))$. On the other hand, by Lemma 2.5.7,

$$\cos(t\ln(\lambda_j))\tilde{g}_{n,j} - \sin(t\ln(\lambda_j))\tilde{g}_{n,-j}$$

also converges in distribution to

$$\cos(t\ln(\lambda_j))G(f_j) - \sin(t\ln(\lambda_j))G(f_{-j}) = \sigma_t(G(f_j)).$$

Let $y \in \mathcal{A}$, using Raynaud's results we compute:

$$\begin{aligned}
\varphi(\sigma_t(p\Phi(G(f_j))p)pyp) &= \varphi((\Delta_n^{it})^\bullet p\Phi(G(f_j))p(\Delta_n^{-it})^\bullet pyp) \\
&= \varphi(p(\Delta_n^{it})^\bullet \Phi(G(f_j))(\Delta_n^{-it})^\bullet pyp) \\
&= \varphi((\Delta_n^{it})^\bullet \Phi(G(f_j))(\Delta_n^{-it})^\bullet py)
\end{aligned}$$

Let $z = (z_n)^\bullet_{n\in\mathbb{N}} \in \widetilde{\mathcal{A}}$. By our previous observations, we have:

$$\begin{aligned}
\varphi((\Delta_n^{it})^\bullet \Phi(G(f_j))(\Delta_n^{-it})^\bullet z) &= \lim_{n,\mathcal{U}} \varphi_n(\Delta_n^{it}\tilde{g}_{n,j}\Delta_n^{-it}z_n) \\
&= \lim_{n,\mathcal{U}} \varphi_n(\sigma_t^n(\tilde{g}_{n,j})z_n) \\
&= \varphi(\sigma_t(G(f_j))z) \\
&= \lim_{n,\mathcal{U}} \varphi_n((\cos(t\ln(\lambda_j))\tilde{g}_{n,j} - \sin(t\ln(\lambda_j))\tilde{g}_{n,-j})z_n) \\
&= \varphi((\cos(t\ln(\lambda_j))\Phi(G(f_j)) - \sin(t\ln(\lambda_j))\Phi(G(f_{-j})))z) \\
&= \varphi((p\Phi(\sigma_t(G(f_j)))p)zp)
\end{aligned}$$

By w^*-density and continuity, we can replace z by py in the previous equality, which gives:

$$\varphi(\sigma_t(p\Phi(G(f_j))p)pyp) = \varphi((p\Phi(\sigma_t(G(f_j)))p)pyp).$$

Thus, taking $y = \sigma_t(p\Phi(G(f_j))p) - p\Phi(\sigma_t(G(f_j)))p$, and by the faithfulness of $\varphi(p\,.\,p)$ we deduce that

$$\sigma_t(p\Phi(G(f_j))p) = p\Phi(\sigma_t(G(f_j)))p \in p\mathrm{Im}(\Phi)p$$

By Theorem 2.4.3, $\Theta = p\Phi p$ can be extended into a (necessarily injective because state preserving) w^*-continuous $*$-homomorphism from $\Gamma_q(H_\mathbb{R}, U_t)$ into $p\mathcal{A}p$ with a completely complemented image. By its corollary 2.4.4, since the algebras Γ_n are finite dimensional and a fortiori are QWEP, it follows that $\Gamma_q(H_\mathbb{R}, U_t)$ is QWEP.

Theorem 2.5.8 *If $H_\mathbb{R}$ is a finite dimensional real Hilbert space equipped with a group of orthogonal transformations $(U_t)_{t\in\mathbb{R}}$, then the von Neumann algebra $\Gamma_q(H_\mathbb{R}, U_t)$ is QWEP.*

Remark. We have only proved the Theorem for $H_\mathbb{R}$ of even dimension over $\mathbb{R}$. We did this only for simplicity of notations. Of course this is not relevant since, if the dimension of $H_\mathbb{R}$ is odd, then we just have to consider the real Hilbert space $H_\mathbb{R} \oplus \mathbb{R}$ equipped with $(U_t \oplus \mathrm{Id})_{t\in\mathbb{R}}$. $\Gamma_q(H_\mathbb{R} \oplus \mathbb{R}, U_t \oplus \mathrm{Id})$ is QWEP by our previous discussion. Let us denote by Q the projection from $H_\mathbb{R} \oplus \mathbb{R}$ onto $H_\mathbb{R}$, then Q intertwines $(U_t \oplus \mathrm{Id})_{t\in\mathbb{R}}$ and $(U_t)_{t\in\mathbb{R}}$. In this situation we can consider $\Gamma_q(Q)$, the second quantization of Q (see [Hi]), which is a conditional expectation from $\Gamma_q(H_\mathbb{R} \oplus \mathbb{R}, U_t \oplus \mathrm{Id})$ onto $\Gamma_q(H_\mathbb{R}, U_t)$. Thus $\Gamma_q(H_\mathbb{R}, U_t)$ is completely complemented into a QWEP von Neumann algebra, so $\Gamma_q(H_\mathbb{R}, U_t)$ is QWEP.

Corollary 2.5.9 *If $(U_t)_{t\in\mathbb{R}}$ is almost periodic on $H_\mathbb{R}$, then $\Gamma_q(H_\mathbb{R}, U_t)$ is QWEP.*

Proof. There exist an invariant real Hilbert space H_1, an orthogonal family of invariant 2-dimensional real Hilbert spaces $(H_\alpha)_{\alpha\in A}$ and real eigenvalues $(\lambda_\alpha)_{\alpha\in A}$ greater than 1 such that

$$H_\mathbb{R} = H_1 \bigoplus_{\alpha\in A} H_\alpha \quad \text{and} \quad U_{t|H_1} = \mathrm{Id}_{H_1}, \quad U_{t|H_\alpha} = \begin{pmatrix} \cos(t\ln(\lambda_\alpha)) & -\sin(t\ln(\lambda_\alpha)) \\ \sin(t\ln(\lambda_\alpha)) & \cos(t\ln(\lambda_\alpha)) \end{pmatrix}$$

In particular it is possible to find a net $(I_\beta)_{\beta\in B}$ of isometries from finite dimensional subspaces $H_\beta \subset H_\mathbb{R}$ into $H_\mathbb{R}$, such that for all $\beta \in B$, H_β is stable by $(U_t)_{t\in\mathbb{R}}$ and $\bigcup_{\beta\in B} H_\beta$ is dense in $H_\mathbb{R}$. By second quantization, for all $\beta \in B$, there exists an isometric $*$-homomorphism $\Gamma_q(I_\beta)$ from $\Gamma_q(H_\beta, U_{t|H_\beta})$ into $\Gamma_q(H_\mathbb{R}, U_t)$, and $\Gamma_q(H_\mathbb{R}, U_t)$ is the inductive limit (in the von Neumann algebra's sense) of the algebras $\Gamma_q(H_\beta, U_{t|H_\beta})$. By the previous Theorem, for all $\beta \in B$, $\Gamma_q(H_\beta, U_{t|H_\beta})$ is QWEP, thus $\Gamma_q(H_\mathbb{R}, U_t)$ is QWEP, as an inductive limit of QWEP von Neumann algebras. □

2.6 The general case

We will derive the general case by discretization and an ultraproduct argument similar to that of the previous section.

2.6.1 Discretization argument

Let $H_\mathbb{R}$ be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ a strongly continuous group of orthogonal transformations on $H_\mathbb{R}$. We denote by $H_\mathbb{C}$ the complexification of $H_\mathbb{R}$ and by $(U_t)_{t\in\mathbb{R}}$ its extension to a group of unitaries on $H_\mathbb{C}$. Let A be the (unbounded) non degenerate positive infinitesimal generator of $(U_t)_{t\in\mathbb{R}}$. For every $n \in \mathbb{N}_*$ let g_n be the bounded Borelian function defined by:

$$g_n = \chi_{]1,1+\frac{1}{2^n}[} + \left(\sum_{k=2^n+1}^{n2^n-1} \frac{k}{2^n} \chi_{[\frac{k}{2^n},\frac{k+1}{2^n}[} \right) + n\chi_{[n,+\infty[}$$

and

$$f_n(t) = g_n(t)\chi_{\{t>1\}}(t) + \frac{1}{g_n(1/t)}\chi_{\{t<1\}}(t) + \chi_{\{1\}}(t) \quad \text{for all } t \in \mathbb{R}_+$$

It is clear that

$$f_n(t) \nearrow t \quad \text{for all} \quad t \geqslant 1 \qquad \text{and} \qquad f_n(t) = \frac{1}{f_n(1/t)} \quad \text{for all} \quad t \in \mathbb{R}_+^*. \tag{2.23}$$

For all $n \in \mathbb{N}_*$, let A_n be the invertible positive and bounded operator on $H_{\mathbb{C}}$ defined by $A_n = f_n(A)$. Denoting by $\mathcal{J}$ the conjugation on $H_{\mathbb{C}}$, we know, by [Sh], that $\mathcal{J}A = A^{-1}\mathcal{J}$. By the second part of (2.23), it follows that for all $n \in \mathbb{N}_*$,

$$\mathcal{J}A_n = \mathcal{J}f_n(A) = f_n(A^{-1})\mathcal{J} = f_n(A)^{-1}\mathcal{J} = A_n^{-1}\mathcal{J} \tag{2.24}$$

Consider the strongly continuous unitary group $(U_t^n)_{t\in\mathbb{R}}$ on $H_{\mathbb{C}}$ with positive non degenerate and bounded infinitesimal generator given by A_n. By definition, we have $U_t^n = A_n^{it}$. By (2.24), and since $\mathcal{J}$ is anti-linear, we have for all $n \in \mathbb{N}_*$ and all $t \in \mathbb{R}$:

$$\mathcal{J}U_t^n = \mathcal{J}A_n^{it} = A_n^{it}\mathcal{J} = U_t^n\mathcal{J}$$

It follows that for all $n \in \mathbb{N}_*$ and for all $t \in \mathbb{R}$, $H_{\mathbb{R}}$ is globally invariant by U_t^n, thus we have

$$U_t^n(H_{\mathbb{R}}) = H_{\mathbb{R}}$$

Hence, $(U_t^n)_{t\in\mathbb{R}}$ induces a group of orthogonal transformations on $H_{\mathbb{R}}$ such that its extension on $H_{\mathbb{C}}$ has infinitesimal generator given by the discretized operator A_n. In the following we will index by $n \in \mathbb{N}_*$ the objects relative to the discretized von Neumann algebra $\Gamma_n = \Gamma_q\left(H_{\mathbb{R}}, (U_t^n)_{t\in\mathbb{R}}\right)$. We simply set $\Gamma = \Gamma_q\left(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}}\right)$.

Remark. Notice that $H_{\mathbb{C}}$ is contractively included in H and all H_n, and that the inclusion $H_{\mathbb{R}} \subset H$ (respectively $H_{\mathbb{R}} \subset H_n$) is isometric since $\mathrm{Re}\left(\langle\,.\,,\,.\,\rangle_U\right)_{|H_{\mathbb{R}}\times H_{\mathbb{R}}} = \langle\,.\,,\,.\,\rangle_{H_{\mathbb{R}}}$ (see [Sh]). Moreover for all $n \in \mathbb{N}_*$ the scalar products $\langle\,.\,,\,.\,\rangle_{U^n}$ and $\langle\,.\,,\,.\,\rangle_{H_{\mathbb{C}}}$ are equivalent on $H_{\mathbb{C}}$ since A_n is bounded.

Scholie 2.6.1 *For all ξ and η in $H_{\mathbb{C}}$ we have:*

$$\lim_{n\to+\infty} \langle \xi, \eta\rangle_{H_n} = \langle \xi, \eta\rangle_H$$

Proof. Let E_A be the spectral resolution of A. Take $\xi \in H_{\mathbb{C}}$ and denote by μ_ξ the finite positive measure on $\mathbb{R}_+$ given by $\mu_\xi = \langle E_A(.)\xi, \xi\rangle_{H_{\mathbb{C}}}$. Since for all $\lambda \in \mathbb{R}_+$, $\lim_{n\to+\infty} g\circ f_n(\lambda) = g(\lambda)$, and $g(\lambda) = 2\lambda/(1+\lambda)$ is bounded on $\mathbb{R}_+$, we have by the Lebesgue dominated convergence Theorem:

$$\begin{aligned}\|\xi\|_H^2 = \langle \frac{2A}{1+A}\xi, \xi\rangle_{H_{\mathbb{C}}} &= \int_{\mathbb{R}_+} g(\lambda)\mathrm{d}\mu_\xi(\lambda) \\ &= \lim_{n\to+\infty}\int_{\mathbb{R}_+} g\circ f_n(\lambda)\mathrm{d}\mu_\xi(\lambda) = \lim_{n\to+\infty}\langle \frac{2A_n}{1+A_n}\xi, \xi\rangle_{H_{\mathbb{C}}} = \lim_{n\to+\infty}\|\xi\|_{H_n}^2\end{aligned}$$

And we finish the proof by polarization. □

Let E be the vector space given by

$$E = \cup_{k\in\mathbb{N}_*}\chi_{[\frac{1}{k},k]}(A)(H_\mathbb{R})$$

We have

$$\mathcal{J}\chi_{[\frac{1}{k},k]}(A) = \chi_{[\frac{1}{k},k]}(A^{-1})\mathcal{J} = \chi_{[\frac{1}{k},k]}(A)\mathcal{J}$$

thus $E \subset H_\mathbb{R}$. Since A is non degenerate,

$$\overline{\cup_{k\in\mathbb{N}_*}\chi_{[\frac{1}{k},k]}(A)(H_\mathbb{C})} = \chi_{]0,+\infty[}(A)(H_\mathbb{C}) = H_\mathbb{C}$$

It follows that E is dense in $H_\mathbb{R}$. Let $(e_i)_{i\in I}$ be an algebraic basis of unit vectors of E and denote by $\mathcal{E}$ the algebra generated by the Gaussians $G(e_i)$ for $i \in I$. $\mathcal{E}$ is w^* dense in Γ and every element in $\mathcal{E}$ is entire for $(\sigma_t)_{t\in\mathbb{R}}$ (because for all $k \in \mathbb{N}_*$, A is bounded and has a bounded inverse on $\chi_{[\frac{1}{k},k]}(A)(H_\mathbb{C})$). Denoting by W the Wick product in Γ, we have for all $i \in I$ and all $z \in \mathbb{C}$:

$$\sigma_z(G(e_i)) = W(U_{-z}e_i) = W(A^{-iz}e_i) \tag{2.25}$$

Since $H_\mathbb{R} \subset H$ and for all $n \in \mathbb{N}_*$, $H_\mathbb{R} \subset H_n$(isometrically), we have by (2.1)

$$\text{For all} \quad (i,n) \in I \times \mathbb{N}_*, \qquad \|G_n(e_i)\| = \frac{2}{\sqrt{1-q}} \tag{2.26}$$

Scholie 2.6.2 *For all $r \in \mathbb{R}$ and for all $i \in I$ we have*

$$\sup_{n\in\mathbb{N}_*} \|\sigma_{ir}^n(G_n(e_i))\| < +\infty$$

Proof. Fix $i \in I$. By (2.25):

$$\begin{aligned}\|\sigma_{ir}^n(G_n(e_i))\| &= \|W(A_n^r e_i)\| = \|a_n^*(A_n^r e_i) + a_n(\mathcal{J}A_n^r e_i)\| \\ &\leqslant C_{|q|}^{\frac{1}{2}}(\|A_n^r e_i\|_{H_n} + \|\mathcal{J}A_n^r e_i\|_{H_n}) \\ &\leqslant C_{|q|}^{\frac{1}{2}}(\|A_n^r e_i\|_{H_n} + \|\Delta_n^{\frac{1}{2}} A_n^r e_i\|_{H_n}) \\ &\leqslant C_{|q|}^{\frac{1}{2}}(\|A_n^r e_i\|_{H_n} + \|A_n^{r-\frac{1}{2}} e_i\|_{H_n})\end{aligned}$$

Thus it suffices to prove that for all $r \in \mathbb{R}$ we have

$$\sup_{n\in\mathbb{N}_*} \|A_n^r e_i\|_{H_n} < +\infty$$

Let us denote by $\mu_i = \langle E_A(.)e_i, e_i\rangle_{H_\mathbb{C}}$ and by $g_r(\lambda) = 2\lambda^{2r+1}/(1+\lambda)$. There exists $k \in \mathbb{N}_*$ such that $e_i \in \chi_{[1/k,k]}(A)(H_\mathbb{R})$, thus we have :

$$\|A_n^r e_i\|_{H_n}^2 = \langle g_r \circ f_n(A)e_i, e_i\rangle_{H_\mathbb{C}} = \int_{[1/k,k]} g_r \circ f_n(\lambda)\mathrm{d}\mu_i(\lambda)$$

It is easily seen that $(g_r \circ f_n)_{n\in\mathbb{N}_*}$ converges uniformly to g_r on $[1/k, k]$. The result follows by:

$$\lim_{n\to+\infty} \|A_n^r e_i\|_{H_n}^2 = \lim_{n\to+\infty} \int_{[1/k,k]} g_r \circ f_n(\lambda)\mathrm{d}\mu_i(\lambda) = \int_{[1/k,k]} g_r(\lambda)\mathrm{d}\mu_i(\lambda) = \|A^r e_i\|_H^2.$$

□

2.6.2 Conclusion

Recall that $\mathcal{E}$ is isomorphic to the complex free $*$-algebra with $|I|$ generators. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}_*$, by (2.26) we can define a $*$-homomorphism Φ from $\mathcal{E}$ into the von Neumann algebra ultraproduct over $\mathcal{U}$ of the algebras Γ_n by:

$$\begin{aligned} \Phi \;:\; \mathcal{E} &\longrightarrow \mathcal{A} = \prod_{n,\mathcal{U}} \Gamma_n \\ G(e_i) &\longmapsto (G_n(e_i))^{\bullet}_{n\in\mathbb{N}_*} \end{aligned}$$

We will now check the hypothesis of Theorem 2.4.3.

1. We first check that Φ is state preserving. It suffices to verify it for a product of an even number of Gaussians. Take $(i_1,\dots,i_{2k}) \in I^{2k}$, we have by Scholie 2.6.1:

$$\begin{aligned} \varphi(G(e_{i_1})\dots G(e_{i_{2k}})) &= \sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\dots,k)\\ \mathcal{V}=((s(l),t(l)))_{l=1}^{l=k}}} q^{i(\mathcal{V})} \prod_{l=1}^{l=k} \langle e_{i_{s(l)}}, e_{i_{t(l)}}\rangle_H \\ &= \lim_{n\to+\infty} \sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\dots,k)\\ \mathcal{V}=((s(l),t(l)))_{l=1}^{l=k}}} q^{i(\mathcal{V})} \prod_{l=1}^{l=k} \langle e_{i_{s(l)}}, e_{i_{t(l)}}\rangle_{H_n} \\ &= \lim_{n\to+\infty} \varphi_n(G_n(e_{i_1})\dots G_n(e_{i_{2k}})) \end{aligned}$$

   This implies, in particular that Φ is state preserving.

2. Condition 1. of lemma 2.4.1 is satisfied by Scholie 2.6.2.

3. It suffices to check that for all $i \in I$ and all $t \in \mathbb{R}$, $(\sigma_t^n(G_n(e_i)))^{\bullet}_{n\in\mathbb{N}_*} \in \overline{\mathrm{Im}\Phi}^{w^*}$. Fix $i \in I$ and $t \in \mathbb{R}$. For all $n \in \mathbb{N}_*$ we have

$$\|A_n^{-it}e_i - A^{-it}e_i\|^2_{H_\mathbb{R}} = \int_{\mathbb{R}_+} |f_n^{-it}(\lambda) - \lambda^{-it}|^2 \mathrm{d}\mu_i(\lambda)$$

   By the Lebesgue dominated convergence Theorem, it follows that

$$\lim_{n\to+\infty} \|A_n^{-it}e_i - A^{-it}e_i\|_{H_\mathbb{R}} = 0.$$

   By (2.26) we deduce that

$$\lim_{n\to+\infty} \|G_n(A_n^{-it}e_i) - G_n(A^{-it}e_i)\| = 0$$

   Thus we have

$$(\sigma_t^n(G_n(e_i)))^{\bullet}_{n\in\mathbb{N}_*} = (G_n(A_n^{-it}e_i))^{\bullet}_{n\in\mathbb{N}_*} = (G_n(A^{-it}e_i))^{\bullet}_{n\in\mathbb{N}_*} \in \overline{\mathrm{Im}\Phi}^{\|\cdot\|} \subset \overline{\mathrm{Im}\Phi}^{w^*}.$$

By Theorem 2.4.3, we deduce our main Theorem:

Theorem 2.6.3 *Let $H_{\mathbb{R}}$ be a real Hilbert space given with a group of orthogonal transformations $(U_t)_{t\in\mathbb{R}}$. Then for all $q \in (-1,1)$ the q-Araki-Woods algebra $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ is QWEP.*

Remark. We were unable to prove that the C^*-algebra $C_q^*(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ (for $(U_t)_{t\in\mathbb{R}}$ non trivial) is QWEP, even for a finite dimensional Hilbert space $H_{\mathbb{R}}$. The proof of Lemma 2.3.5 could not be directly adapted to this case. Indeed, in the non-tracial framework, the ultracontractivity of the q-Ornstein-Uhlenbeck semi-group $(\Phi_t)_{t\in\mathbb{R}_+}$ is known when A is bounded and $t > \frac{\ln(\|A\|)}{2}$ but in any cases it fails for $0 < t < \frac{\ln(\|A\|)}{4}$ (see [Hi] Theorem 4.1 and Proposition 4.5).

Annexe

Dans cette dernière partie nous incluons les démonstrations originelles du Lemme 1.3.1 et des Théorèmes 1.3.3, 1.3.4 et 1.4.1 . Les preuves sont plus calculatoires et n'ont pas la clarté de celles de la première partie, néanmoins elles ont l'avantage d'être plus élémentaires en ne faisant (presque) pas appel à la théorie des espaces d'opérateurs.

Pour tout $n \geqslant 0$ nous commençons par fixer une base orthonormée, $(\xi_{\underline{i}})_{|\underline{i}|=n}$, de $H_{\mathbb{C}}^{\otimes n}$ équipé du T-produit scalaire. $(\xi_{\underline{i}})_{|\underline{i}|\geqslant 0}$ est alors une base orthonormée de $\mathcal{F}_T\,(H_{\mathbb{C}})$, et pour tout $n \geqslant 1$ et $1 \leqslant k \leqslant n$, $(\xi_{\underline{j}} \otimes \xi_{\underline{l}})_{|\underline{j}|=n-k,\ |\underline{l}|=k}$ est une base orthonormée de $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$.

Lemme 1 *Soit $n \geqslant 0$, K un espace de Hilbert complexe et $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ une famille à support fini d'opérateurs sur K. On a alors :*

$$\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}}^* \alpha_{\underline{i}} \|_{B(K)}^{\frac{1}{2}} \leqslant \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(\xi_{\underline{i}}) \|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_T(H_{\mathbb{C}}))} \leqslant C_{|q|}^{\frac{1}{2}} \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}}^* \alpha_{\underline{i}} \|_{B(K)}^{\frac{1}{2}}$$

En particulier $\|a^*\|_{CB(H_c^{\otimes n}, B(\mathcal{F}_T(H_{\mathbb{C}})))} \leqslant C_{|q|}^{\frac{1}{2}}$.

Preuve : Pour démontrer la première inégalité, prenons $v \in K$.

$$\begin{aligned} \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(\xi_{\underline{i}})(v \otimes \Omega) \|^2 &= \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}}(v) \otimes \xi_{\underline{i}} \|^2 = \sum_{|\underline{i}|=n} \|\alpha_{\underline{i}}(v)\|^2 \\ &= \| \sum_{|\underline{i}|=n} (\alpha_{\underline{i}} \otimes \xi_{\underline{i}})(v \otimes 1) \|_{K \otimes H_{\mathbb{C}}^{\otimes n}}^2 \end{aligned}$$

En prenant le supremum sur la boule unité de K on en déduit :

$$\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(\xi_{\underline{i}}) \|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_T(H_{\mathbb{C}}))} \geqslant \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes \xi_{\underline{i}} \|_{B(K) \underset{\min}{\otimes} H_c^{\otimes n}} = \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}}^* \alpha_{\underline{i}} \|_{B(K)}^{\frac{1}{2}}.$$

Pour la seconde inégalité, remarquons que pour tout $k \in \mathbb{N}$ et $\underline{i}$ de longueur n, $a^*(\xi_{\underline{i}})$ envoie $H_{\mathbb{C}}^{\otimes k}$ dans $H_{\mathbb{C}}^{\otimes k+n}$. Il nous suffit donc d'évaluer la norme de $\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(\xi_{\underline{i}})$ dans $B(K) \underset{\min}{\otimes} B(H_{\mathbb{C}}^{\otimes k}, H_{\mathbb{C}}^{\otimes k+n})$. Grâce à (1.3) on a :

$$\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(\xi_{\underline{i}}) \|_{B(K) \underset{\min}{\otimes} B(H_{\mathbb{C}}^{\otimes k}, H_{\mathbb{C}}^{\otimes k+n})} \leqslant C_{|q|}^{\frac{1}{2}} \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(\xi_{\underline{i}}) \|_{B(K) \underset{\min}{\otimes} B(H_{\mathbb{C}}^{\otimes k}, H_{\mathbb{C}}^{\otimes n} \otimes H_{\mathbb{C}}^{\otimes k})}$$

Soit alors $\nu = \sum_{|\underline{j}|=k} v_{\underline{j}} \otimes \xi_{\underline{j}} \in K \otimes H_{\mathbb{C}}^{\otimes k}$ tel que $\|\nu\| \leqslant 1$. Par orthogonalité, on a :

$$\|\sum_{|\underline{i}|=n}(\alpha_{\underline{i}}\otimes a^*(\xi_{\underline{i}}))(\nu)\|^2_{K\otimes H_{\mathbb{C}}^{\otimes n}\otimes H_{\mathbb{C}}^{\otimes k}} = \sum_{|\underline{i}|=n}\|\sum_{|\underline{j}|=k}\alpha_{\underline{i}}(v_{\underline{j}})\otimes\xi_{\underline{j}}\|^2_{K\otimes H_{\mathbb{C}}^{\otimes k}} = \sum_{|\underline{i}|=n}\|\alpha_{\underline{i}}\otimes I_{H_{\mathbb{C}}^{\otimes k}}(\nu)\|^2$$
$$\leqslant \|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes I_{H_{\mathbb{C}}^{\otimes k}}\otimes\xi_{\underline{i}}\|^2_{B(K)\underset{\min}{\otimes}B(H_{\mathbb{C}}^{\otimes k})\underset{\min}{\otimes}H_c^{\otimes n}}$$
$$= \|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes\xi_{\underline{i}}\|^2_{B(K)\underset{\min}{\otimes}H_c^{\otimes n}}$$
$$= \|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}^*\alpha_{\underline{i}}\|_{B(K)}$$

D'où le résultat. □

Remarque : Par adjonction on en déduit :

$$\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\alpha_{\underline{i}}^*\|^{\frac{1}{2}}_{B(K)} \leqslant \|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes a(U\xi_{\underline{i}})\|_{B(K)\underset{\min}{\otimes}B(\mathcal{F}_T(H_{\mathbb{C}}))} \leqslant C_{|q|}^{\frac{1}{2}}\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\alpha_{\underline{i}}^*\|^{\frac{1}{2}}_{B(K)}$$

Plus généralement on a le résultat suivant :

Lemme 2 *Soit $n\in\mathbb{N}^*$, $0\leqslant k\leqslant n$, K un espace de Hilbert complexe et $(\alpha_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}$ une famille à support fini de $B(K)$. On a alors :*

$$\|(\alpha_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|_{B(K)\underset{\min}{\otimes}B(H_{\mathbb{C}}^{\otimes k},H_{\mathbb{C}}^{\otimes n-k})} \leqslant \|\sum_{|\underline{j}|,|\underline{l}|}\alpha_{\underline{j},\underline{l}}\otimes a^*(\xi_{\underline{j}})a(\xi_{\underline{l}})\|_{B(K)\underset{\min}{\otimes}B(\mathcal{F}_T(H_{\mathbb{C}}))}$$
$$\leqslant C_{|q|}\|(\alpha_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|$$

Preuve : Nous commençons par démontrer la première inégalité. Prenons $(v_{\underline{m}})_{|\underline{m}|=k}$ une famille de vecteurs de K telle que $\sum_{|\underline{m}|=k}\|v_{\underline{m}}\|^2<+\infty$. Soit $\nu=\sum_{|\underline{m}|=k}v_{\underline{m}}\otimes U\xi_{\underline{m}}$. Rappelons que $a(\xi_{\underline{l}})U\xi_{\underline{m}}=\delta_{\underline{l},\underline{m}}$. On a donc :

$$\|\sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\alpha_{\underline{j},\underline{l}}\otimes a^*(\xi_{\underline{j}})a(\xi_{\underline{l}})(\nu)\|^2 = \|\sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=|\underline{m}|=k}}\alpha_{\underline{j},\underline{l}}(v_{\underline{m}})\otimes a^*(\xi_{\underline{j}})a(\xi_{\underline{l}})U\xi_{\underline{m}}\|^2$$
$$= \|\sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\alpha_{\underline{j},\underline{l}}(v_{\underline{l}})\otimes\xi_{\underline{j}}\|^2$$
$$= \sum_{|\underline{j}|=n-k}\|\sum_{|\underline{l}|=k}\alpha_{\underline{j},\underline{l}}(v_{\underline{l}})\|^2$$
$$= \left\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\begin{pmatrix}\vdots\\ v_{\underline{l}}\\ \vdots\end{pmatrix}_{|\underline{l}|=k}\right\|^2$$

En prenant le supremum sur les familles $(v_{\underline{m}})_{|\underline{m}|=k}$ telles que $\sum_{|\underline{m}|=k}\|v_{\underline{m}}\|^2\leqslant 1$ on obtient :

$$\|(\alpha_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|_{B(K)\underset{\min}{\otimes}B(H_{\mathbb{C}}^{\otimes k},H_{\mathbb{C}}^{\otimes n-k})}\leqslant\|\sum_{|\underline{j}|,|\underline{l}|}\alpha_{\underline{j},\underline{l}}\otimes a^*(\xi_{\underline{j}})a(\xi_{\underline{l}})\|_{B(K)\underset{\min}{\otimes}B(\mathcal{F}_T(H_{\mathbb{C}}))}$$

Pour établir la seconde inégalité, nous procédons comme dans le Lemme précédent. On peut se restreindre à évaluer la norme de $\sum_{|\underline{j}|,|\underline{l}|}\alpha_{\underline{j},\underline{l}}\otimes a^*(\xi_{\underline{j}})a(\xi_{\underline{l}})$ dans $B(K)\underset{\min}{\otimes}B(H_{\mathbb{C}}^{\otimes p},H_{\mathbb{C}}^{\otimes p+n-2k})$ pour $p\geqslant k$. Grâce à (1.3) on a :

$$\begin{aligned}&\|\sum_{|\underline{j}|,|\underline{l}|}\alpha_{\underline{j},\underline{l}}\otimes a^*(\xi_{\underline{j}})a(\xi_{\underline{l}})\|_{B(K)\underset{\min}{\otimes}B(H_{\mathbb{C}}^{\otimes p},H_{\mathbb{C}}^{\otimes p+n-2k})}\\&\qquad\leqslant C_{|q|}^{\frac{1}{2}}\|\sum_{|\underline{j}|,|\underline{l}|}\alpha_{\underline{j},\underline{l}}\otimes a^*(\xi_{\underline{j}})a(\xi_{\underline{l}})\|_{B(K)\underset{\min}{\otimes}B(H_{\mathbb{C}}^{\otimes p},H_{\mathbb{C}}^{\otimes n-k}\otimes H_{\mathbb{C}}^{\otimes p-k})}\end{aligned}$$

Soit $\nu=\sum_{|\underline{m}|=p}v_{\underline{m}}\otimes\xi_{\underline{m}}$ un vecteur de $K\otimes H_{\mathbb{C}}^{\otimes p}$ tel que $\|\nu\|\leqslant 1$. Par orthogonalité et en utilisant la remarque qui suit le Lemme 1 on a :

$$\begin{aligned}&\|\sum_{|\underline{j}|,|\underline{l}|}\alpha_{\underline{j},\underline{l}}\otimes a^*(\xi_{\underline{j}})a(\xi_{\underline{l}})(\nu)\|^2_{K\otimes H_{\mathbb{C}}^{\otimes n-k}\otimes H_{\mathbb{C}}^{\otimes p-k}}\\
&=\sum_{|\underline{j}|=n-k}\|\sum_{\substack{|\underline{l}|=k\\|\underline{m}|=p}}\alpha_{\underline{j},\underline{l}}(v_{\underline{m}})\otimes a(\xi_{\underline{l}})\xi_{\underline{m}}\|^2_{K\otimes H_{\mathbb{C}}^{\otimes p-k}}\\
&=\sum_{|\underline{j}|=n-k}\|\sum_{|\underline{l}|=k}\alpha_{\underline{j},\underline{l}}\otimes a(\xi_{\underline{l}})(\nu)\|^2_{K\otimes H_{\mathbb{C}}^{\otimes p-k}}\\
&\leqslant\|\sum_{|\underline{j}|,|\underline{l}|}\alpha_{\underline{j},\underline{l}}\otimes a(\xi_{\underline{l}})\otimes\xi_{\underline{j}}\|^2_{B(K)\underset{\min}{\otimes}B(H_{\mathbb{C}}^{\otimes p},H_{\mathbb{C}}^{\otimes p-k})\underset{\min}{\otimes}H_c^{\otimes n-k}}\\
&=\|\sum_{|\underline{l}|=k}\Big(\sum_{|\underline{j}|=n-k}\alpha_{\underline{j},\underline{l}}\otimes\xi_{\underline{j}}\Big)\otimes a(\xi_{\underline{l}})\|^2_{B(K)\underset{\min}{\otimes}H_c^{\otimes n-k}\underset{\min}{\otimes}B(H_{\mathbb{C}}^{\otimes p},H_{\mathbb{C}}^{\otimes p-k})}\\
&\leqslant C_{|q|}\|\sum_{|\underline{l}|=k}\Big(\sum_{|\underline{j}|=n-k}\alpha_{\underline{j},\underline{l}}\otimes\xi_{\underline{j}}\Big)\otimes\xi_{\underline{l}}\|^2_{B(K)\underset{\min}{\otimes}H_c^{\otimes n-k}\underset{\min}{\otimes}H_r^{\otimes k}}\\
&=C_{|q|}\|\sum_{|\underline{j}|,|\underline{l}|}\alpha_{\underline{j},\underline{l}}\otimes(\xi_{\underline{j}}\otimes\xi_{\underline{l}})\|^2_{B(K)\underset{\min}{\otimes}\mathcal{K}(\overline{H_{\mathbb{C}}}^{\otimes k},H_{\mathbb{C}}^{\otimes n-k})}\\
&=C_{|q|}\Big\|(\alpha_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\Big\|^2\end{aligned}$$

D'où le résultat en combinant les estimations précédentes. □

Comme corollaire du Lemme 2 et de la formule de Wick, nous obtenons le théorème suivant qui est une version plus "explicite" du Théorème 1.3.3.

Théorème 3 *Soit $n \geqslant 1$, $(\eta_u)_{u\in U}$ une famille de vecteurs de $H_{\mathbb{C}}^{\otimes n}$, K un Hilbert complexe, et $(\alpha_u)_{u\in U}$ une famille à support fini de $B(K)$. Alors :*

$$\max_{0\leqslant k\leqslant n}\left\{\left\|(\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\|\right\} \leqslant \left\|\sum_{u\in U}\alpha_u\otimes W(\eta_u)\right\| \leqslant (n+1)C_q \max_{0\leqslant k\leqslant n}\left\{\left\|(\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\|\right\} \quad (3.27)$$

où

$$\varphi(\sigma)\eta_u = \sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}} \gamma_{\underline{j},\underline{l}}^{\sigma,u}\xi_{\underline{j}}\otimes\xi_{\underline{l}}, \quad \sigma\in S_n$$

et

$$\beta_{\underline{j},\underline{l}} = \sum_{u\in U}\sum_{\sigma\in S_n/S_{n-k}\times S_k}\gamma_{\underline{j},\underline{l}}^{\sigma,u}\alpha_u$$

Remarque : Pour retrouver la formulation du Théorème 1.3.3 fixons $k\in\{0,\dots,n\}$.

$$\begin{aligned}\|(\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\| &= \|\sum_{|\underline{j}|,|\underline{l}|}\beta_{\underline{j},\underline{l}}\otimes(\xi_{\underline{j}}\otimes\xi_{\underline{l}})\|_{B(K)\underset{\min}{\otimes}H_c^{\otimes n-k}\underset{\min}{\otimes}H_r^{\otimes k}}\\ &= \|\sum_{u\in U}\sum_{|\underline{j}|,|\underline{l}|,\sigma}\gamma_{\underline{j},\underline{l}}^{\sigma,u}\alpha_u\otimes(\xi_{\underline{j}}\otimes\xi_{\underline{l}})\|_{B(K)\underset{\min}{\otimes}H_c^{\otimes n-k}\underset{\min}{\otimes}H_r^{\otimes k}}\end{aligned}$$

Si $u\in U$ est fixé, on a

$$\sum_{|\underline{j}|,|\underline{l}|,\sigma}\gamma_{\underline{j},\underline{l}}^{\sigma,u}\xi_{\underline{j}}\otimes\xi_{\underline{l}} = \sum_{\sigma\in S_n/S_{n-k}\times S_k}\varphi(\sigma)\eta_u = R_{n,k}^*\eta_u$$

Donc

$$\|(\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\| = \|\sum_{u\in U}\alpha_u\otimes R_{n,k}^*\eta_u\|_{B(K)\underset{\min}{\otimes}H_c^{\otimes n-k}\underset{\min}{\otimes}H_r^{\otimes k}}.$$

Preuve : Nous commençons par prouver la seconde inégalité dans (3.27). D'après la formule de Wick :

$$\sum_{u\in U}\alpha_u\otimes W(\eta_u) = \sum_{k=0}^{n}F_k$$

où

$$F_k = \sum_{u,\,\sigma}\alpha_u\otimes U_k\varphi(\sigma)(\eta_u)$$

Puisque $(\xi_{\underline{j}}\otimes\xi_{\underline{l}})_{|\underline{j}|=n-k,\,|\underline{l}|=k}$ est une base orthonormée de $H_{\mathbb{C}}^{\otimes n-k}\otimes H_{\mathbb{C}}^{\otimes k}$ il existe des scalaires $\gamma_{\underline{j},\underline{l}}^{\sigma,u}$ tels que

$$\varphi(\sigma)\eta_u = \sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\gamma_{\underline{j},\underline{l}}^{\sigma,u}\xi_{\underline{j}}\otimes\xi_{\underline{l}}$$

Il s'ensuit que

$$U_k\varphi(\sigma)\eta_u = \sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\gamma_{\underline{j},\underline{l}}^{\sigma,u}U_k(\xi_{\underline{j}}\otimes\xi_{\underline{l}})$$

$$= \sum_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \gamma^{\sigma,u}_{\underline{j},\underline{l}} a^*(\xi_{\underline{j}}) a(\overline{\xi_{\underline{l}}})$$

Donc

$$\begin{aligned} F_k &= \sum_{\substack{u,\sigma,|\underline{l}|=k \\ |\underline{j}|=n-k}} \gamma^{\sigma,u}_{\underline{j},\underline{l}} \alpha_{\underline{u}} \otimes a^*(\xi_{\underline{j}}) a(\overline{\xi_{\underline{l}}}) \\ &= \sum_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \beta_{\underline{j},\underline{l}} \otimes a^*(\xi_{\underline{j}}) a(\overline{\xi_{\underline{l}}}) \end{aligned}$$

où

$$\beta_{\underline{j},\underline{l}} = \sum_{u\in U} \sum_{\sigma \in S_n/S_{n-k}\times S_k} \gamma^{\sigma,u}_{\underline{j},\underline{l}} \alpha_u$$

Il suffit ensuite d'utiliser le Lemme 2 pour obtenir la seconde inégalité.

Pour établir la première inégalité dans (3.27), fixons $k_0 \in \{0,\dots,n\}$. Puisque pour tout $k \in \{k_0,\dots,n\}$, F_k envoie $K \otimes H_{\mathbb{C}}^{\otimes k_0}$ dans $K \otimes H_{\mathbb{C}}^{\otimes k_0+n-2k}$, par orthogonalité, on a

$$\| \sum_{u\in U} \alpha_u \otimes W(\eta_u)_{\big| K\otimes H_{\mathbb{C}}^{\otimes k_0}} \| \geqslant \|F_{k_0 \big| K\otimes H_{\mathbb{C}}^{\otimes k_0}}\|$$

Or d'après la preuve du Lemme 2,

$$\|F_{k_0 \big| K\otimes H_{\mathbb{C}}^{\otimes k_0}}\| \geqslant \|(\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}}\|$$

D'où la première inégalité dans (3.27). □

Nous indiquons ensuite comment modifier la preuve du Théorème 1.3.4 avec, comme outil, les inégalités du Théorème 3. L'argument est le même (écrit dans ce qui suit avec $d = 2$) et il nous suffit de détailler l'estimation de la quantité

$$\| \sum_{\underline{i}\in I_n} \overline{W(\xi_{\underline{i}})} \otimes W(\xi_{\underline{i}}) \|.$$

où $I_n = \{1,2\}^n$. En utilisant (3.27), on obtient

$$\| \sum_{\underline{i}\in I_n} \overline{W(\xi_{\underline{i}})} \otimes W(\xi_{\underline{i}}) \| \leqslant (n+1) C_q \max_{0\leqslant k\leqslant n} \{ \|(\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}}\| \}$$

Pour tout k, $0 \leqslant k \leqslant n$,

$$\|(\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}}\|^2 \leqslant \sum_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \|\beta_{\underline{j},\underline{l}}\|^2 = \sum_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \| \sum_{\underline{i}\in I_n,\sigma} \gamma^{\sigma,\underline{i}}_{\underline{j},\underline{l}} \overline{W(\xi_{\underline{i}})} \|^2$$

En utilisant cette fois-ci l'inégalité de Bożejko-Haagerup scalaire (1.9),

$$\begin{aligned}\|(\beta_{\underline{j},\underline{l}})\|^2 &\leqslant C_q^3(n+1)^2 \sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k,\underline{i}\in I_n}} \Big|\sum_{\sigma}\gamma_{\underline{j},\underline{l}}^{\sigma,\underline{i}}\Big|^2\\ &\leqslant C_q^3(n+1)^2\sum_{\underline{i}\in I_n} Q_{\underline{i}}\end{aligned}$$

où

$$Q_{\underline{i}} = \sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}} \Big|\sum_{\sigma}\gamma_{\underline{j},\underline{l}}^{\sigma,\underline{i}}\Big|^2$$

Rappelons que

$$\varphi(\sigma)\xi_{\underline{i}} = \sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}} \gamma_{\underline{j},\underline{l}}^{\sigma,\underline{i}}\xi_{\underline{j}}\otimes\xi_{\underline{l}}$$

Donc

$$\sum_{\sigma}\sum_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}} \gamma_{\underline{j},\underline{l}}^{\sigma,\underline{i}}\xi_{\underline{j}}\otimes\xi_{\underline{l}} = \sum_{\sigma}\varphi(\sigma)\xi_{\underline{i}} = R_{n,k}^*\xi_{\underline{i}}$$

Nous en déduisons

$$\begin{aligned}Q_{\underline{i}} &= \langle P_T^{(n-k)}\otimes P_T^{(k)} R_{n,k}^*\xi_{\underline{i}}, R_{n,k}^*\xi_{\underline{i}}\rangle_0\\ &= \langle P_T^{(n)}\xi_{\underline{i}}, R_{n,k}^*\xi_{\underline{i}}\rangle_0 = \langle \xi_{\underline{i}}, R_{n,k}^*\xi_{\underline{i}}\rangle_T\\ &\leqslant \|\xi_{\underline{i}}\|_T\|R_{n,k}^*\xi_{\underline{i}}\|_T \leqslant C_q\end{aligned}$$

Nous obtenons finalement

$$\|(\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|^2 \leqslant C_q^4(n+1)^2 2^n$$

et nous concluons comme dans la preuve du Théorème 1.3.4. □

Dans la suite nous reprenons la démarche précédente que nous adaptons au cas non-tracial. Les résultats sont énoncés et les preuves sont (au mieux) rapidement ébauchées.

Plaçons nous dans le cas où le groupe $(U_t)_{t\in\mathbb{R}}$ est presque-périodique. Désignons par $(e_i)_{i\in I}$ une base Hilbertienne de vecteurs propres de A pour le produit scalaire déformé de H. Pour tout $i \in I$, nous noterons λ_i la valeur propre associée au vecteur propre e_i. Étant donné $n \geqslant 0$ et $\underline{i} = (i_1,\dots,i_n)$ un multi-indice de longueur n, nous désignerons par $e_{\underline{i}}$ le tenseur élémentaire $e_{i_1}\otimes\cdots\otimes e_{i_n}$. Il est alors clair que $(e_{\underline{i}})_{|\underline{i}|\geqslant 0}$ est une base Hilbertienne de $\mathcal{F}_0(H)$. De plus, pour tout multi-indice $\underline{i} = (i_1,\dots,i_n)$, $e_{\underline{i}}$ est un vecteur propre de $A^{\otimes n}$ associé à la valeur propre $\lambda_{\underline{i}} = \lambda_{i_1}\dots\lambda_{i_n}$. Puisque pour tout $n \geqslant 0$, $A^{\otimes n}$ et $P_q^{(n)}$ commutent, pour tout multi-indice $\underline{i}$ de longueur n, le vecteur $\xi_{\underline{i}} = (P_q^{(n)})^{-\frac{1}{2}}e_{\underline{i}}$ est vecteur propre de $A^{\otimes n}$ associé à la valeur propre $\lambda_{\underline{i}}$. De plus la famille $(\xi_{\underline{i}})_{|\underline{i}|\geqslant 0}$ est une base Hilbertienne de $\mathcal{F}_q(H)$.

Le lemme suivant est l'équivalent du Lemme 1.

Lemme 4 *Soit $n \geqslant 0$, K un espace de Hilbert complexe et $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ une famille à support fini d'opérateurs sur K. On a alors :*

$$\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}}^* \alpha_{\underline{i}} \|_{B(K)}^{\frac{1}{2}} \leqslant \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(\xi_{\underline{i}}) \|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_q(H))} \leqslant C_{|q|}^{\frac{1}{2}} \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}}^* \alpha_{\underline{i}} \|_{B(K)}^{\frac{1}{2}} \tag{3.28}$$

$$\| \sum_{|\underline{i}|=n} \lambda_{\underline{i}}^{-1} \alpha_{\underline{i}}^* \alpha_{\underline{i}} \|^{\frac{1}{2}} \leqslant \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(S\xi_{\underline{i}}) \|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_q(H))} \leqslant C_{|q|}^{\frac{1}{2}} \| \sum_{|\underline{i}|=n} \lambda_{\underline{i}}^{-1} \alpha_{\underline{i}}^* \alpha_{\underline{i}} \|^{\frac{1}{2}} \tag{3.29}$$

$$\| \sum_{|\underline{i}|=n} \lambda_{\underline{i}}^{-1} \alpha_{\underline{i}} \alpha_{\underline{i}}^* \|^{\frac{1}{2}} \leqslant \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a(\overline{\xi_{\underline{i}}}) \|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_q(H))} \leqslant C_{|q|}^{\frac{1}{2}} \| \sum_{|\underline{i}|=n} \lambda_{\underline{i}}^{-1} \alpha_{\underline{i}} \alpha_{\underline{i}}^* \|^{\frac{1}{2}} \tag{3.30}$$

Preuve : La preuve de (3.28) est la même que celle du Lemme 1. Pour prouver (3.29) il suffit de voir que

$$S\xi_{\underline{i}} = J\Delta^{\frac{1}{2}}\xi_{\underline{i}} = J(A^{-\frac{1}{2}})^{\otimes n}\xi_{\underline{i}} = \lambda_{\underline{i}}^{-\frac{1}{2}} J\xi_{\underline{i}},$$

et $(J\xi_{\underline{i}})_{|\underline{i}|=n}$ étant orthonormée, (3.29) est donc un conséquence de (3.28). Pour démontrer (3.30) il suffit de passer à l'adjoint dans (3.29) et de remarquer que $US\xi_{\underline{i}} = U^2\overline{\xi_{\underline{i}}} = \overline{\xi_{\underline{i}}}$. □

Lemme 5 *Soit $n \in \mathbb{N}^*$, $0 \leqslant k \leqslant n$, K un espace de Hilbert complexe et $(\alpha_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}}$ une famille à support fini de $B(K)$. On a alors :*

$$\|(\lambda_{\underline{l}}^{-\frac{1}{2}} \alpha_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}}\|_{B(K) \underset{\min}{\otimes} B(H^{\otimes k}, H^{\otimes n-k})} \leqslant \| \sum_{|\underline{j}|,|\underline{l}|} \alpha_{\underline{j},\underline{l}} \otimes a^*(\xi_{\underline{j}}) a(\overline{\xi_{\underline{l}}}) \|_{B(K) \underset{\min}{\otimes} B(\mathcal{F}_q(H))}$$

$$\leqslant C_{|q|} \|(\lambda_{\underline{l}}^{-\frac{1}{2}} \alpha_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}}\|$$

Preuve : La preuve découle du Lemme 4 de la même manière que celle du Lemme 2 découle du Lemme 1. Nous indiquons simplement que pour l'inégalité inférieure, il convient de tester sur les vecteurs de la forme $\nu = \sum_{|\underline{m}|=k} v_{\underline{m}} \otimes J\xi_{\underline{m}}$ en ayant à l'esprit que

$$a(\overline{\xi_{\underline{l}}}) J\xi_{\underline{m}} = \lambda_{\underline{l}}^{-\frac{1}{2}} \delta_{\underline{l},\underline{m}}.$$

□

Grâce au Lemme 4, il est clair qu'on obtient l'analogue suivant du Théorème 3.

Théorème 6 *Soit $n \geqslant 1$, $(\eta_u)_{u \in U}$ une famille de vecteurs de $H^{\otimes n}$, K un Hilbert complexe, et $(\alpha_u)_{u \in U}$ une famille à support fini de $B(K)$. Alors :*

$$\max_{0 \leqslant k \leqslant n} \left\{ \left\| (\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \right\| \right\} \leqslant \left\| \sum_{u \in U} \alpha_u \otimes W(\eta_u) \right\| \leqslant (n+1) C_q \max_{0 \leqslant k \leqslant n} \left\{ \left\| (\beta_{\underline{j},\underline{l}})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \right\| \right\} \tag{3.31}$$

où

$$\varphi(\sigma)\eta_u = \sum_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \gamma_{\underline{j},\underline{l}}^{\sigma,u} \xi_{\underline{j}} \otimes \xi_{\underline{l}}, \quad \sigma \in S_n$$

et

$$\beta_{\underline{j},\underline{l}} = \sum_{u \in U} \sum_{\sigma \in S_n / S_{n-k} \times S_k} \lambda_{\underline{l}}^{-\frac{1}{2}} \gamma_{\underline{j},\underline{l}}^{\sigma,u} \alpha_u$$

Le lecteur téméraire qui nous aurait suivi jusque là (ou qui plus probablement, par une curiosité insoutenable, jetterait un oeil sur la fin) se convaincra sans peine qu'il est possible, à partir de la formulation explicite du Théorème 6, d'aboutir au critère de non-injectivité du Corollaire 1.4.2.

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