Dynamique des équations des ondes avec amortissement variable

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UNIVERSITÉ PARIS XI
UFR SCIENTIFIQUE D'ORSAY

N^o D'ORDRE :

THÈSE

Présentée pour obtenir

Le GRADE de DOCTEUR EN SCIENCES
DE L'UNIVERSITÉ PARIS XI ORSAY

PAR

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Dynamique des équations des ondes avec amortissement variable

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Soutenue le jeudi 8 décembre 2005 devant la commission d'examen.

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Remerciements :

En tout premier lieu, ma gratitude va à Geneviève Raugel qui m'a encadré durant ces trois années de thèse, dans une atmosphère de travail à la fois sérieuse et décontractée. Elle a toujours su m'écouter et me guider tout en me laissant libre de choisir les sujets de recherche qui m'intéressaient. Je serai toujours impressionné par sa disponibilité pour répondre à mes questions et la patience et le soin avec lesquels elle arrive à relire mes manuscrits.

Je tiens à remercier Thierry Gallay et Enrique Zuazua qui ont accepté le rôle de rapporteurs, ainsi que Pavol Brunovský, Patrick Gérard et Alain Haraux. Tous, par leurs travaux ou leur aide, ont contribué à cette thèse. Je leur suis reconnaissant d'avoir accepté de faire partie de mon jury.

Un grand merci à l'ensemble de l'équipe Analyse Numérique et ÉDP d'Orsay qui m'a accueilli, spécialement à François Alouges, Nicolas Burq, Patrick Gérard, Bernard Helffer, Marius Paicu, Luc Robbiano et Nicolay Tzvetkov qui ont toujours été disponibles pour me conseiller et répondre à mes questions. Merci à Valérie Lavigne, Danielle Lemeur et Catherine Poupon pour m'avoir guidé dans les labyrinthes administratifs.

Je ne peux pas oublier de citer les cours de l'ÉNS qui m'ont donné envie de travailler sur l'étude qualitative des ÉDP, en particulier ceux de Benoît Perthame, Ivar Ekeland, Fabrice Béthuel et Yves Benoist. Merci à Frédéric Paulin qui m'a orienté vers Orsay. Je tiens aussi à remercier mes camarades de l'ÉNS parmi lesquels j'ai mathématiquement grandi et avec lequels une discussion scientifique peut surgir aussi bien autour d'un baby-foot qu'en haut d'une montagne corse.

J'ai récemment lu que des expériences prouveraient que la bonne humeur stimule la créativité. Il me faut donc remercier l'ensemble des doctorants du bâtiment 430, dont la liste trop longue implique une sélection. Merci à ma ¡¡soeur¿¿ Bouthaina qui m'a accompagné dans ces années de thèse, merci aux autres doctorants de l'équipe ÉDP, parmi lesquels Clément, Guillemette, Karine, Laurent et Ramona et merci au bureau 114 : Ismaël, Nicolae, Huong et le cochon qui chante.

Je remercie de tout cœur mes amis qui m'ont encouragé pendant cette thèse et ont su me distraire des mathématiques.

Enfin, je remercie ma famille qui, depuis que je suis tout petit, m'a donné goût aux sciences en général et aux mathématiques en particulier. Il ne leur reste plus qu'à lire cette thèse.

Romain

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Chapitre 1 : Introduction

Cette thèse a pour sujet l'étude qualitative de la dynamique des équations des ondes amorties. Nous considèrerons en particulier l'équation des ondes amorties à l'intérieur d'un ouvert borné $\Omega \subset \mathbb{R}^d$:

$$\left\{ \begin{array}{l} u_{tt} + \gamma(x) u_t = \Delta u + f(x,u) \quad \text{ sur } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{ sur } \partial\Omega \end{array} \right. \tag{0.1}$$

où $\gamma \geq 0$ est une fonction bornée sur Ω et strictement positive sur un ouvert non vide de Ω, et l'équation des ondes amorties sur le bord de Ω :

$$\left\{ \begin{array}{l} u_{tt} = \Delta u + f(x,u) \quad \text{ sur } \Omega \\ \frac{\partial u}{\partial \nu} + g(x) u_t = 0 \quad \text{ sur } \partial\Omega \end{array} \right. \tag{0.2}$$

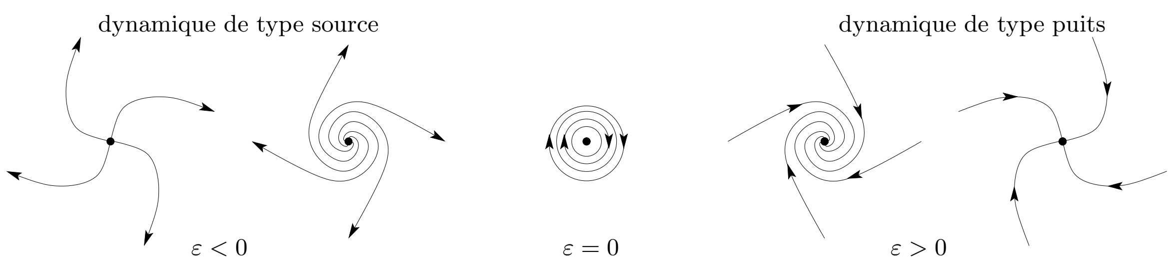
où $g \geq 0$ est une fonction bornée sur $\partial\Omega$ et strictement positive sur un ouvert non vide de $\partial\Omega$.
Ces équations interviennent dans de nombreuses modélisations de phénomènes vibratoires, comme les déplacements d'ondes sonores dans l'air, la propagation de vibrations dans un objet... L'existence locale ou globale des solutions de ces équations est connue depuis longtemps, mais leur étude qualitative ne s'est développée que récemment.
Les résultats principaux de cette thèse concernent la stabilité de la dynamique des équations (0.1) et (0.2) et l'étude de la convergence de la dynamique de l'équation (0.1) vers celle de (0.2) quand $\gamma(x)$ tend vers $g(x) \otimes \delta_{x\in\partial\Omega}$ au sens des distributions. Entre autres, nous démontrons, en dimension $d = 1$, la généricité de la propriété de Morse-Smale par rapport à la non-linéarité $f(x,u)$. Ces résultats permettent en particulier de justifier la modélisation à l'origine de l'équation (0.2).
Cette introduction se veut un aperçu rapide des résultats de cette thèse. Dans un premier temps, nous rappelons les motivations physiques et mathématiques à l'origine de ce travail, puis nous exposons de façon informelle les résultats obtenus, enfin, nous présentons dans un dernier paragraphe le plan général de cette thèse.

1 Problèmes de stabilité de la dynamique

1.1 Motivations physiques

La nature, ou plutôt la modélisation que l'on en fait, regorge de systèmes dynamiques, c'est-à-dire de systèmes qui évoluent dans le temps de façon déterministe. La compréhension de la dynamique de ces systèmes est aujourd'hui nécessaire pour un grand nombre d'applications pratiques. On peut citer par exemple les prévisions météorologiques, l'évolution des écosystèmes, la propagation d'ondes cérébrales, la trajectoire d'une sonde spatiale ou l'effet du vent sur un pont suspendu. Afin de mieux comprendre ces systèmes complexes, il est nécessaire de faire des approximations et des simplifications. On peut par exemple ne prendre en compte que les espèces principales d'un écosystème ou bien considérer un fil, qui est un objet tridimensionnel, comme un objet unidimensionnel en négligeant son épaisseur. Par ailleurs, la discrétisation d'un problème, préalable à tout calcul numérique, est une grande source de simplifications, les ordinateurs n'effectuant pas de calculs sur des objets continus mais sur leur approximation par des points ou des polygônes. On peut aussi citer des approximations d'ordre historique : la mécanique newtonienne n'est que la limite de la mécanique relativiste quand la vitesse de la lumière tend vers l'infini. Enfin, rappelons que toute mesure de paramètres physiques est source d'erreurs.

L'étude du système approché permet-elle de décrire le comportement du système initial ?
La réponse à cette question dépend des applications et de ce que l'on veut savoir sur le système initial. Dans certains cas, on ne s'intéresse qu'à une évolution sur un temps fini donné, par exemple la météo sur plusieurs jours. Il suffit alors que les trajectoires du système approché soient assez proches de celles du système réel dans le laps de temps donné. Dans d'autres cas, on souhaite connaître l'évolution générale du système pour des temps très grands, par exemple savoir si une espèce animale survivra ou finira par s'éteindre dans les décennies à venir, ce qui est a priori différent de savoir si la population grandira ou pas dans les prochaines années. Une telle étude nécessite que la dynamique qualitative globale du système approché soit semblable à celle du système réel, c'est-à-dire que la dynamique du système soit stable sous la perturbation provenant de l'approximation du modèle. Nous insistons sur le fait que la convergence des trajectoires sur des temps finis n'entraîne pas la stabilité de la dynamique, en présentant ci-dessous l'exemple d'une famille de systèmes dynamiques dépendant d'un paramètre ε, qui subit une bifurcation de type Hopf.



Les trajectoires varient de façon continue, mais la dynamique globale change radicalement en $\varepsilon = 0$ et est donc instable en ce point.

1.2 Approche et motivations mathématiques

Il est connu que la question de la stabilité de la dynamique est liée à la propriété de Morse-Smale. Soit X un espace de Banach et soit $S(t)$ un système dynamique sur X qui est de type gradient (comme dans tout le reste de ce paragraphe, nous renvoyons au chapitre 2 pour plus de détails). On dit que le système gradient $S(t)$ vérifie la propriété de Morse-Smale si ses points d'équilibre sont en nombre fini, sont tous hyperboliques et si leurs variétés stables et instables se coupent transversalement. Soient Λ un espace métrique et $(S_\lambda(t))_{\lambda\in\Lambda}$ une famille de systèmes dynamiques gradients sur X vérifiant des propriétés d'unicité rétrograde adéquates. Il a été prouvé que si $S_{\lambda_0}(t)$ vérifie la propriété de Morse-Smale et si $S_\lambda(t)$ converge de façon suffisamment régulière vers $S_{\lambda_0}(t)$ quand λ tend vers λ_0, alors la dynamique est stable par rapport à λ dans un sens qui sera précisé au chapitre 2.
Si les systèmes $S_\lambda(t)$ sont définis sur des variétés compactes de dimension finie, savoir si $S_\lambda(t)$ converge de façon suffisamment régulière vers $S_{\lambda_0}(t)$ est assez simple. De plus, on sait que presque tous les systèmes dynamiques gradients définis sur des variétés compactes de dimension finie vérifient la propriété de Morse-Smale. Il est donc relativement aisé d'obtenir dans ce cas la stabilité de la dynamique sous l'effet d'une perturbation.
Comme les modèles physiques font souvent appel à des équations aux dérivées partielles qui engendrent des systèmes dynamiques définis sur des espaces de dimension infinie, la question de la généralisation des résultats, de la dimension finie à la dimension infinie, se pose tout naturellement. D'une part, on veut savoir quelles sont les équations aux dérivées partielles qui satisfont la propriété de Morse-Smale, d'autre part on doit étudier les perturbations qui nous intéressent pour savoir si la convergence de la famille de systèmes $S_\lambda(t)$ vers $S_{\lambda_0}(t)$ est suffisamment régulière pour obtenir la stabilité de la dynamique. Notons que pour des systèmes de dimension infinie, cette convergence peut être très faible. Ceci nous amènera dans cette thèse à généraliser les résultats de stabilité connus à des perturbations peu régulières.

2 Généricité de la propriété de Morse-Smale

Il est en général très difficile de savoir si une équation aux dérivées partielles donnée engendre un système dynamique $S(t)$ de type Morse-Smale. De plus, on ne connait pas forcément $S(t)$ de manière précise : généralement, on sait simplement que $S(t)$ appartient à une famille de systèmes dynamiques $S_\mu(t)$ où μ est un ensemble de paramètres que l'on ne peut mesurer exactement. C'est pourquoi, on souhaite en général montrer que la propriété de Morse-Smale est vérifiée pour presque tous les éléments d'une famille de systèmes dynamiques $S_\mu(t)$, où μ est un paramètre pertinent du point de vue physique.
Ainsi, on montre dans cette thèse que les systèmes dynamiques engendrés par l'équation

des ondes

$$\begin{cases} u_{tt}(x,t) + \gamma(x)u_t(x,t) = u_{xx}(x,t) + f(x,u(x,t)) & (x,t) \in]0,1[\times\mathbb{R}_+ \\ u(0,t) = 0 \text{ ou } u_x(0,t) = 0 & t \geq 0 \\ u(1,t) = 0 \text{ ou } u_x(1,t) = 0 & t \geq 0 \end{cases} \tag{2.1}$$

où $\gamma \geq 0$ est une fonction bornée sur $]0,1[$ et strictement positive sur un sous-intervalle de $]0,1[$, et par l'équation des ondes amorties sur le bord

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + f(x,u(x,t)) & (x,t) \in]0,1[\times\mathbb{R}_+ \\ -u_x(0,t) + g_0 u_t(0,t) = 0 & t \geq 0 \\ u_x(1,t) + g_1 u_t(1,t) = 0 & t \geq 0 \end{cases} \tag{2.2}$$

où g_0 et g_1 sont deux réels positifs différents de un et pas tous les deux nuls, vérifient la propriété de Morse-Smale génériquement par rapport à la non-linéarité $f(x,u)$ (voir chapitre 4).
Ce résultat est en grande partie lié aux propriétés spectrales des opérateurs linéaires associés à ces deux équations. Par exemple, dans le cas de l'équation (2.1), il est connu que le spectre de l'opérateur

$$A = \begin{pmatrix} 0 & Id \\ \partial_{xx} & -\gamma(x) \end{pmatrix}$$

a une structure très particulière : la suite des parties réelles de ses valeurs propres est bornée et n'a qu'un seul point d'accumulation égal à $-\frac{1}{2}\int_0^1 \gamma(x)dx$, et de plus, les vecteurs propres de A forment une base de Riesz. La preuve de la généricité de la propriété de Morse-Smale passe aussi par l'étude spectrale de l'opérateur

$$A_e = A + \begin{pmatrix} 0 & 0 \\ f'_u(x,e) & 0 \end{pmatrix},$$

qui est la linéarisation de (2.1) autour d'un point d'équilibre $(e,0)$. On montre par exemple que A_e vérifie les mêmes propriétés spectrales que A et que ses valeurs propres sont simples génériquement par rapport à la non-linéarité f.

3 Étude d'une perturbation

Pour montrer la stabilité de la dynamique sous l'effet d'une perturbation, outre le fait que le système limite est de Morse-Smale, il faut montrer que la convergence vers le système limite est suffisamment régulière. Dans le chapitre 5 de cette thèse, nous étudions la convergence de la dynamique de l'équation (0.1) vers celle de (0.2) quand $\gamma(x)$ tend vers $g(x) \otimes \delta_{x\in\partial\Omega}$ au sens des distributions. Cela a pour but de justifier le modèle (0.2) qui découle de la simplification suivante : si la dissipation $\gamma(x)$ de (0.1) est suffisamment localisée sur

le bord du domaine, on peut la remplacer par une dissipation de type fonction de Dirac $g(x)\otimes\delta_{x\in\partial\Omega}$ pour simplifier le problème. L'étude de cette convergence se fait principalement en comparant, quand ils existent, les attracteurs globaux compacts des équations (0.1) et (0.2), les attracteurs étant des objets très caractéristiques de la dynamique globale des systèmes. En dimension $d = 1$ et sous des hypothèses naturelles, nous prouvons la convergence des attracteurs dans $\mathbb{H}^1(\Omega)\times\mathbb{L}^2(\Omega)$ ainsi que la stabilité de la dynamique des flots restreints à ces attracteurs. En dimension supérieure à un, on n'obtient en général qu'une convergence des attracteurs dans $\mathbb{H}^{1-\varepsilon}(\Omega)\times\mathbb{H}^{-\varepsilon}(\Omega)$ pour tout $\varepsilon > 0$.
Le fait que le résultat en dimension supérieure ou égale à deux soit plus faible que celui obtenu en dimension un est lié à un problème de théorie du contrôle. Soit Ω un ouvert borné régulier de $\mathbb{R}^d$, on pose $X = \mathbb{H}^1(\Omega)\times\mathbb{L}^2(\Omega)$. Soit $(\gamma_n(x))_{n\in\mathbb{N}}$ une suite de dissipations convergeant vers $g(x)\otimes\delta_{x\in\partial\Omega}$ et soit A_n l'opérateur

$$A_n = \begin{pmatrix} 0 & Id \\ \Delta & -\gamma_n(x) \end{pmatrix} \quad D(A_n) = \mathbb{H}^2(\Omega)\times\mathbb{H}^1(\Omega)\ .$$

Il est aujourd'hui bien connu que des arguments d'optique géométrique donnent des conditions nécessaires et suffisantes sur le support de $\gamma_n(x)$ pour l'existence de deux familles de constantes strictement positives M_n et λ_n telles que

$$\forall n\in\mathbb{N},\ \forall t\geq 0,\quad \|e^{A_n t}\|_{\mathcal{L}(X)} \leq M_n e^{-\lambda_n t}\ . \tag{3.1}$$

L'estimation (3.1) est suffisante pour comparer les attracteurs de (0.1) et (0.2) dans $\mathbb{H}^{1-\varepsilon}(\Omega)\times\mathbb{H}^{-\varepsilon}(\Omega)$ pour tout $\varepsilon > 0$. Toutefois, la comparaison des attracteurs dans X nécessite une estimation plus précise telle que l'existence de deux constantes strictement positives M et λ, indépendantes de n, telles que

$$\forall n\in\mathbb{N},\ \forall t\geq 0,\quad \|e^{A_n t}\|_{\mathcal{L}(X)} \leq M e^{-\lambda t}\ . \tag{3.2}$$

Dans cette thèse, nous ferons une étude complète de (3.2) dans le cas de la dimension $d = 1$, mais le cas de la dimension supérieure reste essentiellement ouvert.

4 Notes sur les chapitres suivants

Cette thèse a été rédigée en considérant que le lecteur connait principalement la théorie des équations aux dérivées partielles. Le chapitre 2 a pour but d'introduire les notions et définitions classiques de la théorie des systèmes dynamiques qui seront utiles à la compréhension de cette thèse. On y trouvera aussi une revue des résultats antérieurs, ainsi que des références.
Le chapitre 3 consiste en un aperçu rapide des résultats de cette thèse. Y sont résumés les énoncés des principaux théorèmes, les difficultés rencontrées lors de leur démonstration et

les idées nouvelles mises en place pour les surmonter.
Les chapitres 4 et 5 sont le coeur de cette thèse. Ils énoncent les résultats obtenus et donnent leur démonstration. Il s'agit de deux articles destinés à être publiés, aussi sont-ils écrits en anglais. Le premier est déjà publié dans le Journal de Mathématiques Pures et Appliquées n^{o}84 (2005).
Le chapitre 6 est une annexe. Il regroupe des suppléments au chapitre 5 et un nouvel exemple d'équations des ondes de type gradient.

Chapitre 2 : Perturbations et stabilité dans les systèmes dynamiques

Le but de ce chapitre est de présenter les notions classiques de la théorie des systèmes dynamiques qui seront utilisées dans cette thèse. On y passera aussi en revue les problématiques et les résultats connus concernant la stabilité structurelle de la dynamique des équations différentielles ordinaires et des équations aux dérivées partielles.

1 Problèmes de stabilité : le formalisme mathématique

Pour simplifier, nous nous limiterons ici aux systèmes dynamiques appelés semi-groupes.

Définition 1.1. *Soit r un entier positif ou nul. Soit X un espace de Banach et $\mathcal{M}$ une sous-variété de X de classe $\mathcal{C}^r$. La famille $(S(t))_{t\in\mathbb{R}_+}$ d'applications de $\mathcal{M}$ dans $\mathcal{M}$ est appelée* **semi-groupe** *non-linéaire de classe $\mathcal{C}^r$, ou simplement semi-groupe $\mathcal{C}^r$, si :*
i) $S(0) = Id$,
ii) pour tout temps positifs t et s, $S(t)S(s) = S(t+s)$,
iii) l'application $S : (t,U) \longmapsto S(t)U$ est de classe $\mathcal{C}^0(\mathbb{R}_+ \times \mathcal{M}, \mathcal{M})$ et toutes ses dérivées de Fréchet en U jusqu'à l'ordre r sont de classe $\mathcal{C}^0(\mathbb{R}_+ \times X, X)$.
On dit que $(S(t))_{t\in\mathbb{R}}$ est un **groupe** *de classe $\mathcal{C}^r$ si les trois propriétés précédentes sont vraies pour t et s dans $\mathbb{R}$.*

Dorénavant, nous entendrons par système dynamique, ou semi-groupe, un semi-groupe de classe $\mathcal{C}^0$.
Notons que l'on peut aussi définir des systèmes discrets, où la dynamique ne correspond plus à une évolution continue pendant un temps $t \geq 0$ mais un nombre $n \in \mathbb{N}$ d'itérations d'un processus.
Nous rappelons ici quelques définitions classiques.

Définition 1.2. *Une* **trajectoire** *définie sur un intervalle I est une fonction $U \in \mathcal{C}^0(I, X)$ telle que pour tout $t \in I$ et $t' \geq 0$ tel que $t + t' \in I$, $U(t+t') = S(t')U(t)$. Si $I = \mathbb{R}$, on parle de* **trajectoire complète**.
*L'***orbite positive** *d'un borné $\mathcal{B}$ est l'ensemble $\{S(t)x \ / \ t \geq 0, x \in \mathcal{B}\}$.*
On appelle **point d'équilibre** *un élément $e \in X$ tel que pour tout t positif, $S(t)e = e$.*
On appelle **orbite périodique** *de plus petite période T une trajectoire complète $U(t)$*

vérifiant $U(t+T) = U(t)$ pour tout $t \geq 0$, et telle que cette propriété ne soit pas vérifiée pour un temps $T' \in]0, T[$.
*L'***ensemble ω-limite** *d'un point U_0 est l'ensemble des points $x \in X$ tels qu'il existe une suite de temps $(t_n)_{n\in\mathbb{N}}$ croissant vers $+\infty$ telle que $S(t_n)U_0$ converge vers x.*
*L'***ensemble α-limite** *d'un point U_0 est l'ensemble des points $x \in X$ tels qu'il existe une trajectoire $(U(t))_{t\leq 0}$ avec $U(0) = U_0$ et une suite de temps $(t_n)_{n\in\mathbb{N}}$ décroissant vers $-\infty$ telle que $U(t_n)$ converge vers x.*
Une trajectoire **homocline** *est une trajectoire complète $U(t)$ telle que l'ensemble $\omega-$limite de $U(0)$ et l'ensemble $\{x \in X \ / \ \exists (t_n) \to -\infty, \ U(t_n) \to x\}$ sont égaux et sont restreints à un seul point d'équilibre ou à une seule orbite périodique.*
Une trajectoire **hétérocline** *est une trajectoire complète $U(t)$ telle que l'ensemble $\omega-$limite de $U(0)$ et l'ensemble $\{x \in X \ / \ \exists (t_n) \to -\infty, \ U(t_n) \to x\}$ sont distincts et sont chacun restreints à un seul point d'équilibre ou à une seule orbite périodique.*

Soit Λ un espace métrique. Les problèmes de perturbations sont modélisés par une famille $(S_\lambda(t))_{\lambda\in\Lambda}$ de semi-groupes $\mathcal{C}^0$. $S_{\lambda_0}(t)$ correspond au système dynamique initial et pour λ proche de λ_0, $S_\lambda(t)$ correspond au système approché, que l'on espère de plus en plus proche de $S_{\lambda_0}(t)$ quand λ tend vers λ_0. Dire que $S_\lambda(t)$ est une bonne approximation de $S_{\lambda_0}(t)$ sur des temps finis revient par exemple à montrer une estimation du type :

$$\forall T_2 > T_1 \geq 0, \ \forall M > 0, \ \sup_{\|U\|_X \leq M} \ \sup_{t\in[T_1,T_2]} \|S_{\lambda_0}(t)U - S_\lambda(t)U\|_X \longrightarrow 0 \ \text{ quand } \lambda \longrightarrow \lambda_0 \ ,$$

ou une estimation plus faible dans le cas de perturbations singulières.
Si l'on souhaite faire une étude asymptotique, la convergence en temps fini ne suffit pas. Obtenir une convergence de toutes les trajectoires en temps infini est en général illusoire. Il faut alors se restreindre soit à l'étude d'un comportement qualitatif locale, comme l'attractivité d'un point d'équilibre, soit à l'étude de la convergence d'ensembles caractéristiques de la dynamique comme par exemple les attracteurs compacts globaux.

Définition 1.3. *On dit qu'un ensemble $\mathcal{S}_1 \subset X$* **attire** *un ensemble $\mathcal{S}_2 \subset X$ si*

$$\sup_{x\in\mathcal{S}_2} dist_X(S(t)x, \mathcal{S}_1) \longrightarrow 0 \quad \textit{quand } t \to +\infty,$$

où $dist_X$ est la distance d'un point à un ensemble

$$dist_X(x, \mathcal{S}_1) = \inf_{y\in\mathcal{S}_1} \|y - x\|_X \ .$$

Un **attracteur global compact** *$\mathcal{A}$ pour un système $S(t)$ est un ensemble compact, invariant par le flot (pour tout $t \geq 0$, $S(t)\mathcal{A} = \mathcal{A}$) et qui attire tous les bornés de X.*

De nombreux systèmes physiques dissipatifs, autrement dit dont l'énergie décroit avec le temps, possèdent un attracteur compact global. Remarquons que cet attracteur contient toutes les trajectoires complètes.
Soit $S_{\lambda_0}(t)$ un système qui admet un attracteur compact $\mathcal{A}_{\lambda_0}$. Supposons que ses perturbations $S_\lambda(t)$ admettent aussi un attracteur compact $\mathcal{A}_\lambda$. Une première étape dans la comparaison qualitative de la dynamique est l'étude de la convergence de $\mathcal{A}_\lambda$ vers $\mathcal{A}_{\lambda_0}$ en tant qu'ensembles. Ainsi, dans le cas de l'étude d'une population animale, si $\mathcal{A}_{\lambda_0} = \{0\}$ (c'est-à-dire si la population finit toujours par s'éteindre), la continuité des attracteurs indique que $\mathcal{A}_\lambda$ est un ensemble très proche de $\{0\}$ (c'est-à-dire que la population finit toujours par devenir très petite). Enfin, pour affiner l'étude, on peut chercher à comparer la dynamique restreinte à l'attracteur, par exemple en comparant les diagrammes de phase : si l'attracteur limite comprend n points d'équilibres et m orbites périodiques, on souhaite qu'il en soit de même pour l'attracteur perturbé et que les connexions homoclines et hétéroclines soient conservées. Dans le cas de l'exemple simple $\mathcal{A}_{\lambda_0} = \{0\}$, la conservation du diagramme de phase signifie que $\mathcal{A}_\lambda$ se réduit à un point d'équilibre unique.

2 Propriété de Morse-Smale et résultats de stabilité

2.1 Systèmes dynamiques sur des variétés compactes de dimension finie

Soit $\mathcal{M}^d \subset \mathbb{R}^k$ une variété compacte sans bord de dimension finie d et $r \geq 1$. Soit F un champ de vecteurs de classe $\mathcal{C}^r$, c'est-à-dire une fonction de $\mathcal{C}^r(\mathcal{M}^d, \mathbb{R}^k)$ telle que $F(x)$ appartienne à l'espace tangent $T_x\mathcal{M}^d$. En définissant les trajectoires $U(t)$ comme les solutions de l'équation différentielle ordinaire $\frac{d}{dt}U(t) = F(U(t))$, on obtient un système dynamique $S_F(t)$. Dans la suite, nous considérons uniquement les systèmes dynamiques sur $\mathcal{M}^d$ de ce type. Notons que pour $r \geq 1$, les trajectoires de tels systèmes sont toujours prolongeables en des trajectoires complètes et que ces systèmes satisfont la propriété de groupe d'opérateurs. De plus, comme $\mathcal{M}^d$ est compacte, l'ensemble $\omega-$ ou $\alpha-$limite de tout point est non-vide. Pour plus de détails sur les notions de ce paragraphe ou pour les preuves des propositions, nous renvoyons le lecteur au livre de Palis et De Melo [44] et à celui de Shub [61].

Définition 2.1. *Un point d'équilibre e est dit* **hyperbolique** *si la différentielle du système en e $DS_F(e)(1) : T_e\mathcal{M}^d \longrightarrow T_e\mathcal{M}^d$ n'a pas de valeur propre sur le cercle unité. Une orbite périodique $U(t)$ de période T est dite hyperbolique si la différentielle $DS_F(U(0))(T) : T_{U(0)}\mathcal{M}^d \longrightarrow T_{U(0)}\mathcal{M}^d$ de $S_F(T)$ en $U(0)$ n'a pas de valeur propre sur le cercle unité autre que 1 et que 1 est une valeur propre simple.*
Un ensemble $\mathcal{S} \subset \mathcal{M}^d$ est dit **positivement invariant** *si pour tout $t \geq 0$, $S_F(t)\mathcal{S} \subset \mathcal{S}$ et*

invariant *si pour tout* $t \geq 0$, $S_F(t)\mathcal{S} = \mathcal{S}$.
Soit $\mathcal{S} \subset \mathcal{M}^d$ *un ensemble invariant. On appelle respectivement* **ensemble stable** *et* **ensemble instable** *de* $\mathcal{S}$ *les ensembles*

$$W^s(\mathcal{S}) = \{U_0 \in \mathcal{M}^d \ / \ dist_X(S_F(t)U_0, \mathcal{S}) \longrightarrow 0 \ quand \ t \longrightarrow +\infty\}$$

et

$$\begin{aligned} W^u(\mathcal{S}) = \{U_0 \in \mathcal{M}^d \ / \ &il \ existe \ une \ trajectoire \ (U(t))_{t\in\mathbb{R}_-}, \ telle \ que \\ &U(0) = U_0 \ et \ dist_X(U(t), \mathcal{S}) \longrightarrow 0 \ quand \ t \longrightarrow -\infty\} \end{aligned}$$

Soit $U(t)$ *un point d'équilibre hyperbolique ou une orbite périodique hyperbolique et soit* $\mathcal{O} = \cup_{t\in\mathbb{R}} U(t)$. *Soit* $\mathcal{V}$ *un voisinage de* $\mathcal{O}$, *on appelle* **variété stable locale** *l'ensemble*

$$W^s(\mathcal{O}, \mathcal{V}) = \{x_0 \in \mathcal{V} \ / \ \forall t \geq 0, \ S_F(t)x_0 \in \mathcal{V}\} \ ,$$

et on appelle **variété instable locale** *l'ensemble*

$$\begin{aligned} W^u(\mathcal{O}, \mathcal{V}) = \{x_0 \in \mathcal{V} \ / \ &il \ existe \ une \ trajectoire \ (x(t))_{t\in\mathbb{R}_-}, \ telle \ que \\ &x(0) = x_0 \ et \ \ \forall t \leq 0, \ x(t) \in \mathcal{V}\} \ . \end{aligned}$$

Si $W^s(\mathcal{O}, \mathcal{V})$ *est un voisinage de* $\mathcal{O}$, *on dit que* $\mathcal{O}$ *est un* **puits**, *si* $W^u(\mathcal{O}, \mathcal{V})$ *est un voisinage de* $\mathcal{O}$, *on dit que* $\mathcal{O}$ *est une* **source**, *autrement, on dit que* $\mathcal{O}$ *est un* **point selle**.

Remarques : Dans le cas d'un système du type $S_F(t)$ en dimension finie, un point d'équilibre est hyperbolique si et seulement si l'opérateur $DF(e) : T_e\mathcal{M}^d \longrightarrow T_e\mathcal{M}^d$ n'a pas de valeur propre sur l'axe imaginaire.
Toujours dans le cas de la dimension finie, une orbite périodique $U(t)$ de plus petite période T est hyperbolique si et seulement si $U(0)$ est un point d'équilibre hyperbolique de l'application de Poincaré. Rappelons que l'application de Poincaré Φ est l'application de premier retour définie comme suit. Soit θ un disque transverse à la trajectoire périodique et tel que $\theta \cap \{U(t), \ t \in [0, T[\} = U(0)$. Pour $x \in \theta$ suffisamment proche de $U(0)$, il existe un temps $t > 0$ tel que $S(t)x \in \theta$ et on définit $\Phi(x) = S(T_{min}(x))x$ où $T_{min}(x)$ est le plus petit temps $t > 0$ tel que $S(t)x \in \theta$. Cette remarque reste vraie pour certains systèmes dynamiques de dimension infinie comme ceux engendrés par une équation parabolique (voir le livre de Henry [28]).
Dans le cas de systèmes dynamiques définis sur des espaces non compacts, on appelle puits, sources ou points selles les points d'équilibres qui sont des puits, sources ou points selles du système restreint à un compact invariant caractéristique de la dynamique, typiquement un attracteur local ou global.

Les variétés stables et instables sont reliées à leur version locale par la proposition suivante.

Proposition 2.2. *Soit $U(t)$ un point d'équilibre hyperbolique ou une orbite périodique hyperbolique et soit $\mathcal{O} = \{U(t) \ / \ t \in \mathbb{R}\}$. Il existe un voisinage $\mathcal{V}_0$ de $\mathcal{O}$ tel que pour tout voisinage $\mathcal{V} \subset \mathcal{V}_0$ de $\mathcal{O}$, $W^s(\mathcal{O}, \mathcal{V})$ (resp. $W^u(\mathcal{O}, \mathcal{V})$) est une variété et est contenue dans $W^s(\mathcal{O})$ (resp. $W^u(\mathcal{O})$). De plus, on a*

$$W^s(\mathcal{O}) = \bigcup_{t \leq 0} S_F(t) W^s(\mathcal{O}, \mathcal{V}) \quad et \quad W^u(\mathcal{O}) = \bigcup_{t \geq 0} S_F(t) W^u(\mathcal{O}, \mathcal{V}) \ .$$

Par abus de langage, $W^s(\mathcal{O})$ et $W^u(\mathcal{O})$ sont respectivement appelées **variété stable** et **variété instable** de $\mathcal{O}$, même dans les cas où il ne s'agit pas de variétés.
On peut maintenant définir la propriété de Morse-Smale.

Définition 2.3. *On appelle* **ensemble errant** *l'ensemble des points qui possèdent un voisinage $\mathcal{V}$ tel qu'il existe un temps $T > 0$ tel que $S_F(t)\mathcal{V} \cap \mathcal{V} = \emptyset$ pour tout $|t| \geq T$. Son complémentaire est appelé* **ensemble non-errant**.
On dit qu'un système dynamique satisfait la propriété de **Morse-Smale** *si :*
i) son ensemble non-errant est réduit à l'ensemble des points d'équilibre et des orbites périodiques,
ii) les éléments de l'ensemble non-errant sont tous hyperboliques et sont en nombre fini,
iii) les variétés stables et instables des éléments non-errants se coupent transversalement, c'est-à-dire qu'en tout point d'intersection, la somme des espaces tangents est égale à l'espace tangent tout entier.

Soit $S_F(t)$ un système dynamique qui satisfait à la propriété de Morse-Smale. On appelle **diagramme de phase** *de $S_F(t)$ le graphe orienté dont les sommets sont les points d'équilibre et les orbites périodiques et dans lequel deux sommets sont reliés s'il existe une orbite hétérocline entre les deux éléments (le sens de l'arête étant celui du temps sur l'orbite hétérocline).*

Remarque : Dans ii), l'hypothèse que les équilibres et les orbites périodiques sont en nombre fini est redondante avec l'hypothèse d'hyperbolicité dans le cas d'une variété compacte. Nous l'incluons en vue de la généralisation au cas de la dimension infinie.
De plus, un système de Morse-Smale ne peut avoir d'orbite homocline. En effet, si $\mathcal{O}$ est un point d'équilibre ou une orbite périodique, on a $dim(W^s(\mathcal{O})) + dim(W^u(\mathcal{O})) = d$. Il n'est donc pas possible que $W^s(\mathcal{O}) \cap W^u(\mathcal{O})$ contienne une trajectoire et que cette intersection soit transverse.

L'intérêt de la propriété de Morse-Smale est qu'elle est une condition suffisante à la stabilité de la dynamique (voir [43] et [45]).

Théorème 2.4. *Soit $r \geq 1$, soit $F \in \mathcal{C}^r(\mathcal{M}^d)$ et soit $S_F(t)$ le système dynamique associé. Si $S_F(t)$ satisfait la propriété de Morse-Smale, alors la dynamique de $S_F(t)$ est stable dans le sens suivant. Il existe un voisinage $\mathcal{V}$ de F dans $\mathcal{C}^r(\mathcal{M}^d)$ tel que pour tout G dans $\mathcal{V}$, le* *système dynamique associé $S_G(t)$ satisfait la propriété de Morse-Smale et a un diagramme de phase isomorphe à celui de $S_F(t)$. De plus, il existe un homéomorphisme h de $\mathcal{M}^d$ envoyant les trajectoires de $S_G(t)$ sur celles de $S_F(t)$ en conservant le sens du temps.*

La question est de savoir si la propriété de Morse-Smale est vérifiée pour "presque tous" les systèmes, c'est-à-dire si c'est une propriété générique.

Définition 2.5. *Soit X un espace de Banach, un sous-ensemble de X est dit* **générique** *s'il contient une intersection dénombrable d'ouverts denses (en particulier il s'agit d'un ensemble dense d'après le théorème de Baire). Une propriété est dite générique si elle est vérifiée par un ensemble générique d'éléments.*

Sur une variété compacte de dimension $d = 2$, la propriété de Morse-Smale peut être reformulée. Il est alors possible de montrer sa généricité. On trouvera la preuve originale du cas des surfaces orientables dans [47] et celle pour les surfaces quelconques dans [53]. On pourra aussi se référer à [44], qui contient une autre preuve du cas orientable.

Proposition 2.6. *On suppose que $d = 2$. Un système dynamique $S_F(t)$ satisfait la propriété de Morse-Smale si et seulement si :*
i) tous ses points d'équilibre et ses orbites périodiques sont hyperboliques,
ii) tout ensemble $\omega-$ ou $\alpha-$limite d'une trajectoire complète se réduit à un seul point d'équilibre ou une seule orbite périodique,
iii) il n'existe pas d'orbite complète dont les ensembles $\omega-$ et $\alpha-$limite sont tous les deux des points selles.

Si $r \geq 1$ et si la surface $\mathcal{M}^2$ est orientable, alors $S_F(t)$ satisfait la propriété de Morse-Smale pour un ensemble générique de champs de vecteurs F de classe $\mathcal{C}^r$ sur $\mathcal{M}^2$.
Si la surface $\mathcal{M}^2$ est quelconque, alors $S_F(t)$ satisfait la propriété de Morse-Smale pour un ensemble générique de champs de vecteurs F de classe $\mathcal{C}^1$ sur $\mathcal{M}^2$.

Pour obtenir la généricité de la propriété de Morse-Smale en dimension d supérieure ou égale à 3, il faut se restreindre à la classe des champs de vecteurs gradients de classe $\mathcal{C}^r$, c'est-à-dire des champs F tels qu'il existe un champ de vecteurs G de classe $\mathcal{C}^{r+1}$ tel que $F = -\overrightarrow{\nabla} G$. Nous soulignons que la quantité $G(U(t))$ décroît strictement le long des trajectoires $U(t)$ d'un tel système, excepté bien sûr dans le cas où $U(t)$ est un point d'équilibre. En particulier, les systèmes gradients n'ont ni d'orbites périodiques, ni d'orbites homoclines.

Proposition 2.7. *Soit $d \geq 2$ et $r \geq 1$, la propriété de Morse-Smale est générique dans l'ensemble des champs de vecteurs gradients de classe $\mathcal{C}^r$ sur $\mathcal{M}^d$.*

Ce résultat provient de l'équivalence entre la propriété de Morse-Smale et celle de Kupka-Smale pour les systèmes dynamiques gradients sur une variété compacte.

Définition 2.8. *On dit qu'un système dynamique satisfait la propriété de* **Kupka-Smale** *si :*
i) ses équilibres et orbites périodiques sont tous hyperboliques,
ii) les variétés stables et instables des équilibres et orbites périodiques se coupent transversalement.

La proposition 2.7 se déduit donc du résultat de généricité suivant (voir [35], [62] et aussi [44]).

Proposition 2.9. *Soit $d \geq 2$ et $r \geq 1$, la propriété de Kupka-Smale est générique dans l'ensemble des champs de vecteurs de classe $\mathcal{C}^r$ sur $\mathcal{M}^d$.*

2.2 Systèmes dynamiques sur $\mathbb{R}^d$

Soit F une fonction de $\mathcal{C}^r(\mathbb{R}^d)$, on peut lui associer un système dynamique $S_F(t)$ exactement comme dans le paragraphe précédent. La différence avec le cas de la variété compacte est la perte de la compacité. En particulier, les trajectoires peuvent exploser en temps fini. Afin d'éviter cela, il peut être commode de se restreindre aux systèmes dissipatifs.

Définition 2.10. *On dit qu'un système dynamique est* **dissipatif** *sur les points (resp. les bornés), s'il existe un borné $\mathcal{B}$ qui attire tous les points (resp. tous les bornés).*

Il est clair qu'un système dissipatif sur les bornés l'est a fortiori sur les points. La compacité locale de $X = \mathbb{R}^d$ implique que la réciproque est aussi vraie pour un système dynamique sur $\mathbb{R}^d$.

Proposition 2.11. *Soit $S(t)$ un système dynamique continu sur $\mathbb{R}^d$, alors $S(t)$ est dissipatif sur les points si et seulement si il est dissipatif sur les bornés.*

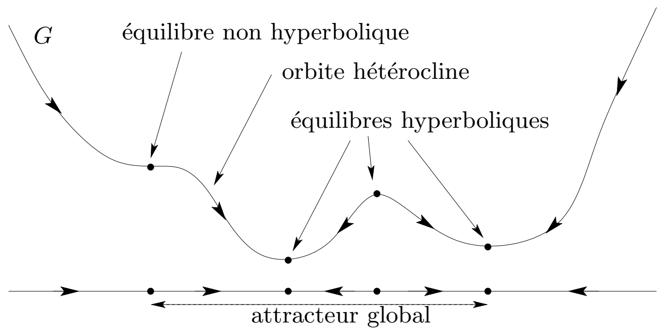
Démonstration : Le seul sens de l'équivalence à montrer est que si $S(t)$ est dissipatif sur les points, il l'est aussi sur les bornés. Supposons donc que $\mathcal{B}$ est un borné qui attire les points sous la dynamique de $S(t)$. Soit $\varepsilon > 0$, on pose $\mathcal{B}_\varepsilon = \{x \in \mathbb{R}^d \ / \ \inf_{b\in\mathcal{B}} \|x - b\| \leq \varepsilon\}$ et $\gamma_\varepsilon = \cup_{t\geq 0} S(t)\mathcal{B}_\varepsilon$. Soit K un borné de $\mathbb{R}^d$. Par dissipation et par continuité, pour tout point $x \in K$, il existe un voisinage $\mathcal{V}_x$ de x et un temps t_x tel que $S(t_x)\mathcal{V}_x \subset \mathcal{B}_\varepsilon$. Comme K est compact, on peut le recouvrir par un nombre fini de voisinages $\mathcal{V}_{x_1}$,..., $\mathcal{V}_{x_p}$ et si on pose $T = \max\{t_{x_i}, i = 1, ..., p\}$, on a

$$\forall t \geq T,\ S(t)K \subset \gamma_\varepsilon \ .$$

Il reste à montrer que γ_ε est borné. En raisonnant comme pour K, on trouve un nombre fini d'ouverts $\mathcal{V}_1$,..., $\mathcal{V}_q$ et de temps t_1,...,t_q tels que $\mathcal{B}_\varepsilon \subset \cup_i \mathcal{V}_i$ et $S(t_i)\mathcal{V}_i \subset \mathcal{B}_\varepsilon$. Soit $T = \max\{t_i\}$, on a donc que pour tout $t \geq 0$, $S(t)\mathcal{B}_\varepsilon \subset \cup_{\tau=0}^{\tau=T} S(\tau)\mathcal{B}_\varepsilon$ et donc que $\gamma_\varepsilon \subset \cup_{\tau=0}^{\tau=T} S(\tau)\mathcal{B}_\varepsilon$. Le système $S(t)$ étant continu, $\cup_{\tau=0}^{\tau=T} S(\tau)\mathcal{B}_\varepsilon$ est borné et γ_ε est donc un ensemble borné qui attire tous les bornés de $\mathbb{R}^d$. $\square$

Dans ce paragraphe, nous parlerons simplement de systèmes dissipatifs. Les orbites d'un système dissipatif étant bornées pour des temps positifs, elles n'explosent pas et possèdent donc un ensemble $\omega-$limite. Par contre, même si l'unicité rétrograde reste vraie pour des systèmes de type $S_F(t)$, une trajectoire n'est pas forcément définie pour tous les temps négatifs.
Dans le cas d'un système gradient, on peut reconnaître le caractère dissipatif par l'allure du potentiel.

Proposition 2.12. *Soit G une fonction de classe $\mathcal{C}^1(\mathbb{R}^d)$. On suppose qu'il existe $M > 0$ telle que pour tout x de norme plus grande que M, $\overrightarrow{\nabla} G(x)$ soit non nul, et on suppose que G est croissante à l'infini dans le sens suivant : il n'existe pas de suite $(x_n) \subset \mathbb{R}^d$ telle que $\|x_n\|$ tende vers $+\infty$ et $G(x_n)$ soit une suite strictement décroisante.*
Alors, si $F = -\overrightarrow{\nabla} G$, le système dynamique $S_F(t)$ associé est dissipatif.



exemple de dynamique dissipative sur la droite

Nous soulignons que pour un système dynamique gradient et dissipatif sur $\mathbb{R}^d$, l'ensemble des points d'équilibre est borné, et les propriétés de Kupka-Smale et de Morse-Smale sont équivalentes.
On obtient, comme dans le cas de la variété compacte un résultat de généricité, voir [48].

Théorème 2.13. *Soit $r \geq 1$. La propriété de Kupka-Smale est générique dans l'ensemble des champs de vecteurs de classe $\mathcal{C}^r$ sur $\mathbb{R}^d$. En particulier, la propriété de Morse-Smale est générique dans l'ensemble des champs de vecteurs gradients dissipatifs de classe $\mathcal{C}^r$ sur $\mathbb{R}^d$*

Toujours par manque de compacité, nous ne pouvons pas espérer que la propriété de Morse-Smale entraîne une stabilité de la dynamique sur tout l'espace. Il faut donc, par exemple, perturber le système de façon compacte et n'observer que la dynamique restreinte à l'attracteur global.

Théorème 2.14. *Soit $F \in \mathcal{C}^1(\mathbb{R}^d)$ et soit $S_F(t)$ le système dynamique associé. Le système $S_F(t)$ est dissipatif si et seulement s'il admet un attracteur global compact $\mathcal{A}_F$. En outre, si $S_F(t)$ est dissipatif et satisfait la propriété de Morse-Smale, alors la dynamique de $S_F(t)$ est stable dans le sens suivant. Pour tout compact $K \subset \mathbb{R}^d$ tel que $\mathcal{A}_F$ est contenu dans l'intérieur de K, il existe un voisinage $\mathcal{V}$ de $F_{|K}$ dans $\mathcal{C}^1(K)$ tel que pour tout $G \in \mathcal{C}^1(\mathbb{R}^d)$ tel que $G_{|K} \in \mathcal{V}$ et $G = F$ en dehors de K, le système dynamique associé $S_G(t)$ est dissipatif, satisfait la propriété de Morse-Smale et a un diagramme de phase isomorphe à celui de $S_F(t)$. De plus, il existe un homéomorphisme h de $\mathcal{A}_G$ dans $\mathcal{A}_F$ envoyant les trajectoires de $S_G(t)_{|\mathcal{A}_G}$ sur celles de $S_F(t)_{|\mathcal{A}_F}$ en conservant le sens du temps.*

2.3 Systèmes dynamiques de dimension infinie

Les systèmes dynamiques qui nous intéressent dans cette thèse sont les systèmes dynamiques de dimension infinie engendrés par des équations aux dérivées partielles. Ces dernières engendrent généralement des systèmes dynamiques $S(t)$ sur un espace de Banach X de dimension infinie si on associe à toute donnée initiale $U_0 \in X$ l'unique solution $S(t)U_0 = U(t)$ de l'équation dans X. Toutes les définitions introduites dans les paragraphes précédents se généralisent au cas de la dimension infinie sans difficultés et il est naturel de se demander si les résultats obtenus dans le cadre de la dimension finie se généralisent de même. Comme auparavant, le premier problème est d'obtenir un système dynamique global, c'est-à-dire que les solutions n'explosent pas en temps fini. De la même façon que pour $\mathbb{R}^d$, cela peut se faire en supposant que le système généré par l'équation est dissipatif dans un certain sens. Soulignons que X n'étant plus localement compact, la dissipation sur les points n'est plus équivalente à la dissipation sur les bornés. Il y a de plus de nouveaux problèmes : on ne peut pas définir en général les trajectoires pour des temps négatifs, les trajectoires bornées ne sont plus forcément précompactes et la notion de système gradient ne peut pas se définir aussi simplement. Cela nous pousse à introduire les notions suivantes.

Définition 2.15. *Soit X un espace de Banach. Un système dynamique $S(t) : X \longrightarrow X$ satisfait la propriété d'***unicité rétrograde** *si pour tout $t \geq 0$, et pour tous U et V dans X, l'égalité $S(t)U = S(t)V$ implique que $U = V$.*
Un système dynamique $S(t) : X \longrightarrow X$ est dit **asymptotiquement compact** *ou* **asymptotiquement régulier** *si pour tout fermé borné $\mathcal{B}$ de X, il existe un compact non vide $K(\mathcal{B})$ tel que $K(\mathcal{B})$ attire l'ensemble $\{x \in \mathcal{B} \ / \ (\cup_{t\geq 0} S(t)x) \subset \mathcal{B}\}$.*
Un système dynamique $S(t) : X \longrightarrow X$ est dit **gradient** *s'il existe une fonctionnelle $\Phi \in \mathcal{C}^0(X, \mathbb{R})$ telle que pour tout $U_0 \in X$, la fonction $t \longmapsto \Phi(S(t)U_0)$ est décroissante au sens large et telle que, si $U_0 \in X$ vérifie l'égalité $\Phi(S(t)U_0) = \Phi(U_0)$ pour tout $t \geq 0$, alors*

U_0 est un point d'équilibre.

La compacité asymptotique peut se reformuler ainsi.

Proposition 2.16. *Un système dynamique $S(t)$ est asymptotiquement compact si et seulement si la propriété suivante est satisfaite. Si $\mathcal{B}$ est un borné de X tel qu'il existe un temps $T \geq 0$ tel que l'orbite $\cup_{t\geq T} S(t)\mathcal{B}$ est bornée, alors pour toutes suites $(b_n) \subset \mathcal{B}$ et $t_n \longrightarrow +\infty$, l'ensemble $\{S(t_n)b_n\}$ est relativement compact.*

On obtient les résultats d'existence d'attracteur suivants.

Théorème 2.17. *Soit X un espace de Banach. Un système dynamique $S(t)$ admet un attracteur global compact $\mathcal{A}$ si et seulement si*
i) $S(t)$ est asymptotiquement compact,
ii) $S(t)$ est dissipatif sur les points,
iii) pour tout borné $\mathcal{B} \subset X$, il existe un temps T tel que l'orbite $\cup_{t\geq T} S(t)\mathcal{B}$ est bornée.
De plus, $\mathcal{A}$ est alors la réunion des ensembles $\omega-$limite des bornés de X et est un ensemble connexe.

De nombreuses démonstrations de théorèmes à peu près équivalents au théorème ci-dessus se trouvent dans les références suivantes : [18], [19], [36] et [5] qui contient aussi une version généralisée aux systèmes multivalués (voir également [66]). Le théorème précédent et le principe de La Salle entraînent le résultat suivant.

Proposition 2.18. *Soit $S(t)$ un système gradient asymptotiquement compact tel que pour tout borné $\mathcal{B} \subset X$, il existe un temps T tel que l'orbite $\cup_{t\geq T} S(t)\mathcal{B}$ soit bornée. Si l'ensemble $\mathcal{E}$ des points d'équilibre est borné, alors $S(t)$ admet un attracteur global compact $\mathcal{A}$ qui est l'ensemble instable de $\mathcal{E}$. En outre, si $\mathcal{E}$ est un ensemble discret, alors $\mathcal{E}$ est un ensemble fini $\{e_1, e_2, ..., e_p\}$ et $\mathcal{A} = \cup_{i=1}^{p} W^u(e_i)$.*

La comparaison de la dynamique d'un système $S_0(t)$ à celle d'un système perturbé $S_\lambda(t)$ est beaucoup plus difficile dans le cadre de la dimension infinie que dans celui de $\mathbb{R}^d$. Tout d'abord, la comparaison des attracteurs en tant qu'ensemble n'est pas aussi simple, d'autant plus que la définition de la proximité des systèmes $S_0(t)$ et $S_\lambda(t)$ est plus complexe car plusieurs topologie sont possibles. Notons de plus que, afin de simplifier les énoncés des théorèmes, nous supposerons toujours que les systèmes $S_\lambda(t)$ sont définis sur un même espace de Banach X. Toutefois, dans les applications pratiques, ils sont souvent définis sur des espaces X_λ variant avec le paramètre.
Voici par exemple deux résultats de convergence d'attracteurs tirés de [57] (voir aussi [4] pour des résultats semblables). En général, la semi-continuité supérieure des attracteurs découle rapidement de la comparaison des orbites sur des temps finis grâce aux propriétés d'attraction et d'invariance des attracteurs.

Théorème 2.19. *Soit $(S_\lambda(t))_{\lambda\in\Lambda}$ une famille de systèmes dynamiques telle que pour tout $\lambda \in \Lambda$, $S_\lambda(t)$ admette un attracteur global compact $\mathcal{A}_\lambda$. Soit $\lambda_0 \in \Lambda$, on suppose que l'une des deux propriétés suivantes est vérifiée :*

i) il existe $\eta > 0$, $T > 0$ et un compact $K \subset X$ tels que

$$\left(\bigcup_{\|\lambda_0-\lambda\|_\Lambda<\eta} \mathcal{A}_\lambda \right) \subset K$$

et tels que pour toutes suites $\lambda_k \longrightarrow \lambda_0$ et $x_k \in \mathcal{A}_{\lambda_k} \longrightarrow x_0$, on ait $S_{\lambda_k}(T)x_k \longrightarrow S_{\lambda_0}(T)x_0$,

ou

ii) il existe $\eta > 0$, $T > 0$ et un borné $\mathcal{B} \subset X$ tels que

$$\left(\bigcup_{\|\lambda_0-\lambda\|_\Lambda<\eta} \mathcal{A}_\lambda \right) \subset \mathcal{B}$$

et pour tout $\varepsilon > 0$ et $t \geq T$, il existe $\theta \in]0, \eta[$ tel que pour tout λ tel que $\|\lambda_0 - \lambda\| < \theta$ et pour tout $x_\lambda \in \mathcal{A}_\lambda$, on ait $\|S_\lambda(t)x_\lambda - S_{\lambda_0}(t)x_\lambda\|_X \leq \varepsilon$.

Alors, les attracteurs globaux sont **semi-continus supérieurement** *en λ_0, c'est-à-dire que*

$$\sup_{x_\lambda\in\mathcal{A}_\lambda} dist_X(x_\lambda, \mathcal{A}_{\lambda_0}) \longrightarrow 0 \quad quand\ \lambda \longrightarrow \lambda_0 \ .$$

Dans le cas d'un système dynamique gradient, l'attracteur n'est rien d'autre que l'ensemble instable de l'ensemble des points d'équilibre. Ainsi, si les points d'équilibres sont isolés, un résultat de convergence des variétés instables locales des points d'équilibres permet d'obtenir un résultat de convergence globale de l'attracteur.

Théorème 2.20. *Soit $(S_\lambda(t))_{\lambda\in\Lambda}$ une famille de systèmes dynamiques gradients telle que pour tout $\lambda \in \Lambda$, $S_\lambda(t)$ admette un attracteur global compact $\mathcal{A}_\lambda$. Soit $\lambda_0 \in \Lambda$, on suppose que l'hypothèse ii) de la proposition précédente est satisfaite, que le système dynamique $S_{\lambda_0}(t)$ n'a qu'un nombre fini de points d'équilibre e_1, e_2...e_p et que $\mathcal{A}_{\lambda_0} = \cup_{i=1}^p W^u_{\lambda_0}(e_i)$. On suppose en outre que pour tout i, il existe un voisinage $\mathcal{V}_i$ de e_i tel que la variété locale instable vérifie*

$$\sup_{x\in W^u_{\lambda_0}(e_i,\mathcal{V}_i)} dist_X(x, \mathcal{A}_\lambda) \longrightarrow 0 \quad quand\ \lambda \longrightarrow \lambda_0 \ .$$

Alors, les attracteurs sont **semi-continus inférieurement** *en λ_0, c'est-à-dire que*

$$\sup_{x_{\lambda_0}\in\mathcal{A}_{\lambda_0}} dist_X(x_{\lambda_0}, \mathcal{A}_\lambda) \longrightarrow 0 \quad quand\ \lambda \longrightarrow \lambda_0 \ .$$

Remarque : On peut obtenir des estimations pour ces convergences mais, même dans le cas de la semi-continuité supérieure, il faut se restreindre aux systèmes gradients dont les équilibres sont hyperboliques. En effet, pour de tels systèmes, on peut montrer que l'attracteur attire les bornés à une vitesse exponentielle (voir [57] et [4]).

Si le système $S_{\lambda_0}(t)$ satisfait la propriété de Morse-Smale, on peut comparer dans certains cas les dynamiques du système initial et du système perturbé. Comme dans le cas des champs de vecteurs sur $\mathbb{R}^d$, il faut se limiter aux restrictions de la dynamique sur l'attracteur.

Théorème 2.21. *Soit $k \geq 1$ et soit $(S_\lambda(t))_{\lambda\in\Lambda}$ une famille de semi-groupes de classe $\mathcal{C}^k$ sur un espace de Banach X. On suppose que les systèmes dépendent de manière $\mathcal{C}^k$ du temps et du paramètre dans le sens où pour tout $t \geq 0$, la fonction*

$$(\lambda, U) \in \Lambda \times X \mapsto S_\lambda(t)U \in X$$

est de classe $\mathcal{C}^k$ et il existe $T > 0$ tel que la fonction

$$(\lambda, t, U) \in \Lambda \times]T, +\infty[\times X \mapsto S_\lambda(t)U \in X$$

est de classe $\mathcal{C}^k$.
Soit $\lambda_0 \in \Lambda$ tel que $S_{\lambda_0}(t)$ satisfait la propriété de Morse-Smale. On suppose que pour tout $\lambda \in \Lambda$, le système $S_\lambda(t)$ admet un attracteur global compact $\mathcal{A}_\lambda$. On suppose que ces attracteurs sont semi-continus supérieurement en $\lambda = \lambda_0$, et que la restriction du flot de $S_\lambda(t)$ à l'attracteur $\mathcal{A}_\lambda$, ainsi que sa différentielle d'ordre 1, vérifient la propriété d'unicité rétrograde.
Alors $S_{\lambda_0}(t)$ est structurellement stable dans le sens suivant. Il existe un voisinage $\mathcal{V}$ de λ_0 dans Λ tel que, pour tout $\lambda \in \mathcal{V}$, $S_\lambda(t)$ satisfait la propriété de Morse-Smale et a un diagramme de phase isomorphe à celui de $S_{\lambda_0}(t)$. De plus, il existe un homéomorphisme $h : \mathcal{A}_\lambda \longrightarrow \mathcal{A}_{\lambda_0}$ qui envoie les trajectoires du système restreint $S_\lambda(t)_{|\mathcal{A}_\lambda}$ sur celles de $S_{\lambda_0}(t)_{|\mathcal{A}_{\lambda_0}}$ en préservant le sens du temps.

La démonstration de ce théorème peut se trouver dans [42] (voir aussi la seconde édition de [21]). Le problème est que l'hypothèse que $S_\lambda(t)$ est de classe $\mathcal{C}^1$ en temps est trop forte pour qu'on puisse appliquer ce théorème à des systèmes engendrés par des équations aux dérivées partielles non régularisantes en temps finis, telle que l'équation des ondes amorties. Cette hypothèse entraîne que l'application de Poincaré d'une orbite périodique de $S_\lambda(t)$ (voir la remarque suivant la définition 2.1) est une fonction lipschitzienne de t. Pour montrer l'existence d'une orbite périodique pour $S_\lambda(t)$ qui soit proche d'une orbite périodique hyperbolique de $S_{\lambda_0}(t)$, la méthode classique utilise la propriété de Lipschitz en temps de l'application de Poincaré. Cela devient un problème difficile quand on ne suppose pas que $S_\lambda(t)$ est de classe $\mathcal{C}^1$ en temps. Heureusement, les systèmes qui nous intéresseront ici sont tous gradients et ne possèdent donc pas d'orbites périodiques. En reprenant alors les arguments de [42] ou [21], on obtient le résultat suivant.

Théorème 2.22. *Soit $k \geq 1$ et soit $(S_\lambda(t))_{\lambda\in\Lambda}$ une famille de semi-groupes gradients de classe $\mathcal{C}^k$ sur un espace de Banach X. On suppose que les systèmes dépendent de manière continue du paramètre dans le sens où la fonction*

$$(\lambda, t, U) \in \Lambda \times \mathbb{R}_+ \times X \mapsto S_\lambda(t)U \in X$$

est continue ainsi que sa dérivée par rapport à U,

$$(\lambda, t, U) \in \Lambda \times \mathbb{R}_+ \times X \mapsto DS_\lambda(t)U \in X \ .$$

Soit $\lambda_0 \in \Lambda$ tel que $S_{\lambda_0}(t)$ satisfait la propriété de Morse-Smale. On suppose que pour tout $\lambda \in \Lambda$, le système $S_\lambda(t)$ admet un attracteur global compact $\mathcal{A}_\lambda$. On suppose que ces attracteurs sont semi-continus supérieurement en $\lambda = \lambda_0$, et que la restriction du flot de $S_\lambda(t)$ à l'attracteur $\mathcal{A}_\lambda$, ainsi que sa différentielle d'ordre 1, vérifient la propriété d'unicité rétrograde.
Alors $S_{\lambda_0}(t)$ est structurellement stable dans le sens suivant. Il existe un voisinage $\mathcal{V}$ de λ_0 dans Λ tel que, pour tout $\lambda \in \mathcal{V}$, $S_\lambda(t)$ satisfait la propriété de Morse-Smale et a un diagramme de phase isomorphe à celui de $S_{\lambda_0}(t)$. De plus, il existe un homéomorphisme $h : \mathcal{A}_\lambda \longrightarrow \mathcal{A}_{\lambda_0}$ qui envoie les trajectoires du système restreint $S_\lambda(t)_{|\mathcal{A}_\lambda}$ sur celles de $S_{\lambda_0}(t)_{|\mathcal{A}_{\lambda_0}}$ en préservant le sens du temps.

3 Exemples de systèmes dynamiques engendrés par des équations aux dérivées partielles

Le but de ce paragraphe est de rappeler très brièvement les propriétés des systèmes dynamiques engendrés par certaines familles d' équations aux dérivées partielles (EDP) dissipatives.
Les EDP considérées ici s'écrivent de façon abstraite comme suit. Soient X un espace de Hilbert, A un opérateur de domaine $D(A) \subset X$ qui engendre un semi-groupe linéaire continu e^{At} et F une fonction de X dans X. On considère l'équation

$$\begin{cases} \frac{\partial}{\partial t}U(t) = AU(t) + F(U(t)) \quad t > 0 \\ U(0) = U_0 \in X \end{cases} \tag{3.1}$$

On note $S(t)$ le semi-groupe engendré par l'équation (3.1). Nous ne nous attarderons pas ici sur les problèmes de semi-groupes linéaires continus ou analytiques et nous renvoyons le lecteur intéressé au livre de Pazy [46].

3.1 Équation parabolique en dimension un d'espace

Soient $L > 0$ et f une fonction localement lipschitzienne de $]0, L[\times\mathbb{R}\times\mathbb{R}$ dans $\mathbb{R}$ et soient g_0 et g_1 deux fonctions de classe $\mathcal{C}^0(\mathbb{R},\mathbb{R})$. On considère l'équation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + f(x,u,u_x) & (x,t) \in]0,L[\times\mathbb{R}_+ \\ u(0,t) = 0 \text{ (ou } u_x(0,t) = g_0(u)) & t \geq 0 \\ u(L,t) = 0 \text{ (ou } u_x(L,t) = g_1(u)) & t \geq 0 \\ u(x,0) = u_0(x) \in \mathbb{H}^1_0(]0,L[) \text{ (ou } \mathbb{H}^1(]0,L[)) & \end{cases} \tag{3.2}$$

Soit $X = \mathbb{H}^1_0(]0,L[)$ (ou $\mathbb{H}^1(]0,L[)$ selon les conditions aux bords choisies). Le laplacien $A = \Delta$ avec condition au bord de type Dirichlet (resp. l'opérateur $A = \Delta - Id$ avec condition au bord de type Neumann) engendre un semi-groupe analytique e^{At}. Si on impose une condition de croissance sur f du type : il existe $\gamma \in [0,2[$, et $k \in \mathcal{C}^0(\mathbb{R}_+,\mathbb{R}_+)$ tels que

$$\forall r > 0,\ \forall (x,u,p) \in [0,L]\times[-r,r]\times\mathbb{R},\quad |f(x,u,p)| \leq k(r)\,(1 + |p|^\gamma)\ , \tag{3.3}$$

alors $F : u \in X \longmapsto f(x,u,u_x)$ est localement lipschitzienne de X dans $\mathbb{L}^2(]0,L[)$ et l'équation (3.2) engendre un semi-groupe $\mathcal{C}^0$ sur X (voir [46]).
Pour obtenir des propriétés de dissipation, il faut rajouter une condition du type

$$\exists K > 0,\ \forall (x,u) \in [0,L]\times\mathbb{R},\quad |u| > K \Longrightarrow uf(x,u,0) \leq 0\ . \tag{3.4}$$

Si f vérifie les hypothèses (3.3) et (3.4), alors l'équation (3.2) engendre un système dynamique global $S(t)$ pour lequel les trajectoires des bornés de X sont bornées. De plus, le système est compact, puisqu'il est bien connu que l'équation parabolique est régularisante : pour tout borné $\mathcal{B} \subset X$ et tout temps $T > 0$, il existe $M = M(\mathcal{B}) > 0$ tel que

$$\sup_{u_0\in\mathcal{B}} \sup_{t\geq T} \|S(t)u_0\|_{\mathbb{H}^2} \leq M\ . \tag{3.5}$$

On peut alors obtenir l'existence d'un attracteur global compact grâce au théorème 2.17 par exemple.
Enfin, Zelenyak a montré dans [69] que $S(t)$ est un système gradient.

3.2 Équation parabolique en dimension supérieure

En dimension $d \geq 2$, le système dynamique engendré par l'équation parabolique n'est plus gradient en général si la non-linéarité f dépend de u_x (voir par exemple [50]). On se limite donc à l'équation parabolique suivante.
Soit Ω un ouvert borné régulier de $\mathbb{R}^d$, soit $p > d$ et soit f une fonction localement lipschitzienne de $\Omega\times\mathbb{R}$ dans $\mathbb{R}$. On considère l'équation

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + f(x,u) & (x,t) \in \Omega\times\mathbb{R}_+ \\ u(x,t) = 0 \text{ (ou } \frac{\partial u}{\partial\nu}(x,t) = 0) & (x,t) \in \partial\Omega\times\mathbb{R}_+ \\ u(x,0) = u_0(x) \in W^{1,p}_0(\Omega) \text{ (ou } W^{1,p}(\Omega)) & \end{cases} \tag{3.6}$$

Comme $p > d$, $W^{1,p}(\Omega)$ s'injecte de manière continue dans $\mathbb{L}^\infty(\Omega)$ et $F : u \in W^{1,p}(\Omega) \longmapsto f(x,u) \in \mathbb{L}^p(\Omega)$ est lipschitzienne sur les bornés de $X = W_0^{1,p}(\Omega)$ (ou $X = W^{1,p}(\Omega)$). L'équation (3.6) engendre alors un système dynamique local $S(t)$ sur X.
On peut aussi considérer l'équation parabolique sur $X = \mathbb{H}_0^1(\Omega)$ (ou $X = \mathbb{H}^1(\Omega)$). Dans ce cas, afin que $F : u \in \mathbb{H}^1(\Omega) \longmapsto f(x,u) \in \mathbb{L}^2(\Omega)$ soit une fonction lipschitzienne sur les bornés de $\mathbb{H}^1(\Omega)$, on suppose qu'il existe $C > 0$ et $\alpha \in [0, \frac{2}{d-2}]$ tels que

$$\forall (u_1, u_2) \in \mathbb{R}^2, \quad \sup_{x\in\Omega} |f(x,u_1) - f(x,u_2)| \leq C\left(1 + |u_1|^\alpha + |u_2|^\alpha\right)|u_1 - u_2| \ . \tag{3.7}$$

Dans tous les cas, le système dynamique $S(t)$ engendré par l'équation parabolique devient global si on ajoute une condition de dissipation du type

$$\limsup_{u\to\pm\infty} \sup_{x\in\Omega} \frac{f(x,u)}{u} < 0 \ . \tag{3.8}$$

La condition précédente implique que les trajectoires des bornés sont bornées. Comme en dimension un, le système $S(t)$ est compact car pour tout borné $\mathcal{B} \subset X$ et tout temps $T > 0$, il existe $M = M(\mathcal{B}) > 0$ tel que

$$\sup_{u_0\in\mathcal{B}} \sup_{t\geq T} \|S(t)u_0\|_{W^{2,p}} \leq M$$

(ou bien tel que la propriété (3.5) soit vérifiée, si $X = \mathbb{H}_0^1(\Omega)$). De plus, $S(t)$ est gradient, sa fonctionnelle de Lyapounov étant donnée par

$$\Phi : \left(\begin{array}{ccc} X & \longrightarrow & \mathbb{R} \\ u(x) & \longmapsto & \frac{1}{2}\|u\|_{\mathbb{H}^1}^2 - \int_\Omega \int_0^{u(x)} f(x,\xi)d\xi dx \end{array} \right) \ . \tag{3.9}$$

En effet, le long d'une trajectoire $u(t) = S(t)u_0$, Φ vérifie

$$\frac{\partial}{\partial t}\Phi(u(t)) = -\int_\Omega |u_t(x,t)|^2 dx \ .$$

Si f vérifie les hypothèses (3.7) et (3.8), le théorème 2.18 implique directement que l'équation (3.6) possède un attracteur global compact.
Enfin, signalons que Bardos et Tartar ont montré que l'équation (3.6) satisfait la propriété d'unicité rétrograde (voir [7]).

3.3 Équation des ondes amorties à l'intérieur du domaine

La seconde famille d'EDP qui nous intéressera ici sont les équations hyperboliques du type équation des ondes amorties. Soit Ω un domaine borné régulier de $\mathbb{R}^d$, soit $\gamma(x) \geq 0$

une fonction bornée sur Ω et strictement positive sur un ouvert non vide de Ω et soit f une fonction localement lipschitzienne de $\Omega \times \mathbb{R}$ dans $\mathbb{R}$. On considère l'équation

$$\left\{ \begin{array}{ll} u_{tt}(x,t) + \gamma(x) u_t(x,t) = \Delta u(x,t) + f(x,u) & (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 \text{ (ou } \frac{\partial u}{\partial \nu}(x,t) = 0) & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \in \mathbb{H}^1_0(\Omega) \times \mathbb{L}^2(\Omega) & (\text{ou } \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)) \end{array} \right. \tag{3.10}$$

Pour mettre l'équation des ondes amorties (3.10) sous la forme classique (3.1), on pose $X = \mathbb{H}^1_0(\Omega) \times \mathbb{L}^2(\Omega)$ (ou $X = \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)$), $U(t) = (u(t), u_t(t))$,

$$A = \begin{pmatrix} 0 & Id \\ \Delta & -\gamma(x) \end{pmatrix} \qquad D(A) = \left(\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_0(\Omega)\right) \times \mathbb{H}^1_0(\Omega) \ \ (\text{ou } D(A) = \mathbb{H}^2(\Omega) \times \mathbb{H}^1(\Omega)) \ ,$$

et $F : U = \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto F(U) = \begin{pmatrix} 0 \\ f(x,u) \end{pmatrix}$.

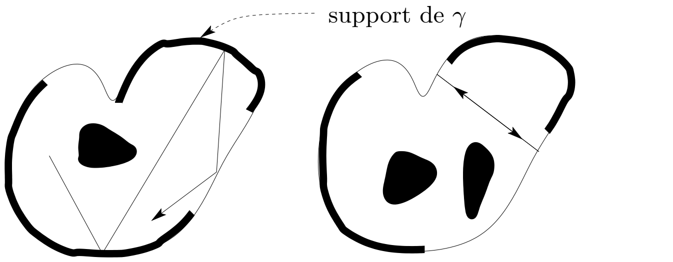
Il est bien connu que A engendre un semi-groupe de contractions e^{At} qui est aussi un groupe d'opérateurs. Afin de s'assurer que F est une fonction lipschitzienne sur les bornés de X, on peut imposer par exemple qu'il existe $C > 0$ et $\alpha \in [0, \frac{2}{d-2}]$ ($\alpha \in \mathbb{R}$ si $d = 1$ ou 2) tels que (3.7) soit vérifié. Dans ce cas, (3.10) engendre un groupe local $S(t)$ de classe $\mathcal{C}^0$ qui devient global si on ajoute une condition de dissipation du type (3.8) qui implique que les trajectoires des bornés sont bornées. Remarquons que comme $S(t)$ est un groupe, la propriété d'unicité rétrograde est vérifiée.

Si la fonction f est telle que F soit définie de $D(A)$ dans $D(A)$ et lipschitzienne sur les bornés de $D(A)$, on peut aussi considérer $S(t)$ comme un système dynamique sur $D(A)$. En dimension $d = 2$ ou $d = 3$, la première composante de $D(A)$ s'injecte dans $\mathbb{L}^\infty(\Omega)$ et il suffit donc que f soit localement lipschitzienne pour que F soit lipschitzienne sur les bornés. Il suffit alors de vérifier que F envoie $D(A)$ sur $D(A)$ pour obtenir que $S(t)$ engendre un semi-groupe local sur $D(A)$.

Le problème de la compacité asymptotique est plus difficile que dans le cas parabolique, puisque l'équation n'est pas régularisante en temps fini. Toutefois, si on impose qu'il existe $C > 0$ et $\alpha \in [0, \frac{2}{d-2}[$ ($\alpha \in \mathbb{R}$ si $d = 1$ ou 2) tels que (3.7) soit vérifié, alors F devient une fonction compacte de X dans X. On peut alors montrer (voir [19]) que $S(t)$ est asymptotiquement compact si e^{At} est exponentiellement décroissant, c'est-à-dire si

$$\exists M > 0,\ \exists \lambda > 0,\ \forall t \geq 0, \quad \|e^{At}\|_{\mathcal{L}(X)} \leq M e^{-\lambda t} \ . \tag{3.11}$$

En dimension $d = 1$, (3.11) équivaut à l'existence d'un ouvert sur lequel γ est strictement positif (c'est-à-dire que le support de $\gamma(x)$ est non vide). En dimension $d \geq 2$, (3.11) est plus restrictive et équivaut à la propriété suivante. Il existe une longueur $L > 0$ telle que toute géodésique de longueur L (i.e. toute ligne droite de longueur L rebondissant sur les bords de Ω selon la loi de Descartes) rencontre le support de $\gamma(x)$ (voir [6]).



On peut noter en particulier que si le support de $\gamma(x)$ contient un voisinage de tout le bord $\partial\Omega$ et s'il existe $C > 0$ et $\alpha \in [0, \frac{2}{d-2}[$ ($\alpha \in \mathbb{R}$ si $d = 1$ ou 2) tels que (3.7) soit vérifié, alors $S(t)$ est asymptotiquement compact.
Finalement, nous allons nous intéresser au caractère gradient de l'équation des ondes (3.10). Tout d'abord, remarquons que la fonctionnelle

$$\Phi : \left(\begin{array}{ccc} X & \longrightarrow & \mathbb{R} \\ (u, v) & \longmapsto & \frac{1}{2}(\|u\|^2_{\mathbb{H}^1} + \|v\|^2_{\mathbb{L}^2}) - \int_\Omega \int_0^{u(x)} f(x, \xi) d\xi dx \end{array} \right) \tag{3.12}$$

décroit le long des trajectoires du système puisque si $U(t) = (u, u_t)$ est une solution de (3.10), alors

$$\Phi(U(t_2)) - \Phi(U(t_1)) = - \int_{t_1}^{t_2} \int_\Omega \gamma(x) |u_t|^2(x, t) dx dt \leq 0 \ . \tag{3.13}$$

On aura donc prouvé que le système (3.10) est gradient si on montre que $\Phi(U(t)) = \Phi(U(0))$ pour tout $t \geq 0$ implique que U est un point d'équilibre. Cela nous ramène à un problème de prolongement unique : il faut montrer que si $U(t) = (u, u_t)(t)$ est une solution telle que $u_t(x, t)$ est nul pour tout t positif et pour tout x dans le support de γ, alors $u_t(x, t)$ est nul pour tout $(x, t) \in \Omega \times \mathbb{R}_+$. Dans le cas de la dimension $d = 1$, ceci est vrai dès que le support de γ est non vide. En dimension supérieure, le problème n'est pas encore totalement résolu bien que plusieurs conditions suffisantes soient connues. Soit $\omega \subset \partial\Omega$ une partie du bord dont un voisinage est contenu dans le support de γ. Si $\omega = \partial\Omega$, (3.10) engendre un système gradient (voir [60]). Dans le cas de conditions au bord de type Dirichlet, s'il existe un point $x_0 \in \mathbb{R}^d$ tel que

$$\{x \in \partial\Omega \ / \ (x - x_0).\nu > 0\} \subset \omega \ ,$$

où ν est le vecteur normal sortant, alors le système (3.10) est gradient (voir [33]). Enfin, dans le cas de conditions au bord de type Neumann, [37] donne diverses conditions suffisantes pour avoir un système gradient. Par exemple si Ω est un disque, il suffit que ω couvre un peu plus qu'un demi-périmètre.

3.4 Équation des ondes amorties sur le bord du domaine

La seconde équation hyperbolique dont il sera question ici est l'équation des ondes amorties sur le bord. Soit Ω un domaine borné régulier de $\mathbb{R}^d$, soit $\gamma(x) \geq 0$ une fonction bornée sur le bord $\partial\Omega$ et strictement positive sur un ouvert non vide du bord, et soit f une fonction localement lipschitzienne de $\Omega \times \mathbb{R}$ dans $\mathbb{R}$. On considère l'équation

$$\left\{ \begin{array}{ll} u_{tt}(x,t) = \Delta u(x,t) + f(x,u) & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x,t) + \gamma(x) u_t(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \in \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega) & \end{array} \right. \tag{3.14}$$

Le modèle physique d'où (3.14) est issu sera discuté au paragraphe 2 du chapitre 3.
Pour écrire l'équation des ondes amorties sur le bord (3.14) sous la forme (3.1), on pose $X = \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)$, $U(t) = (u(t), u_t(t))$,

$$F : U = \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto F(U) = \begin{pmatrix} 0 \\ f(x,u) \end{pmatrix}$$

et $A = \begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix}$ $\quad D(A) = \left\{ (u,v) \in \mathbb{H}^2(\Omega) \times \mathbb{H}^1(\Omega) \;\Big/\; \frac{\partial u}{\partial \nu} + \gamma(x) v = 0 \text{ sur } \partial\Omega \right\}$.

Contrairement à l'équation (3.10), le semi-groupe linéaire e^{At} engendré par A n'est généralement pas prolongeable en un groupe d'opérateurs. A part cela, tout ce qui a été dit dans le paragraphe précédent, concernant la compacité asymptotique et les caractères gradient et dissipatif, reste vrai. En dimension $d = 1$, par exemple pour $\Omega =]0,1[$, il est prouvé dans [14] que e^{At} peut être prolongé en un groupe d'opérateurs si et seulement si $\gamma(0) \neq 1$ et $\gamma(1) \neq 1$. Dans le cas, $\gamma(0) = 1$ ou $\gamma(1) = 1$, l'unicité rétrograde ne peut pas être vraie pour (3.14) en général. Dans [40], il est montré qu'en dimension $d \geq 2$, e^{At} ne peut pas être prolongé en un groupe d'opérateurs et on trouvera dans [39] une discussion sur la propriété d'unicité rétrograde.

4 Généricité de la propriété de Morse-Smale dans les équations aux dérivées partielles

D'après les théorèmes 2.21 et 2.22, la propriété de Morse-Smale est une caractérisation pertinente de la stabilité de la dynamique en dimension infinie. Peut-on prouver que, comme dans le cas des champs de vecteurs gradients en dimension finie, presque toutes les EDP de type gradient satisfont la propriété de Morse-Smale ? En fait, il faut commencer par définir ce que signifie le "presque tout" dans le cadre de la dimension infinie. Pour cela, on se fixe un espace de Banach Λ et une famille d'EDP dépendant d'un paramètre $\lambda \in \Lambda$, et on cherche à savoir si les systèmes dynamiques $S_\lambda(t)$ associés à cette famille satisfont la propriété de

Morse-Smale génériquement par rapport à λ. Pour que le résultat soit pertinent d'un point de vue mathématique, les systèmes $S_\lambda(t)$ doivent provenir d'EDP qui appartiennent à une même famille : équations de réaction-diffusion, équations des ondes... Bien sûr, les système doivent dépendre d'une façon raisonnablement continue du paramètre λ. Enfin, du point de vue physique, il faut que le paramètre λ soit effectivement une variable du problème physique qui peut fluctuer légèrement. Typiquement, il peut s'agir des forces extérieures et internes, de coefficients de diffusions ou bien de la forme ou de la position d'un objet, mais il ne peut pas s'agir d'une constante physique universelle ou d'un paramètre touchant à la structure même de l'équation.

4.1 Équation parabolique

Les premiers résultats de généricité de la propriété de Morse-Smale ont été obtenus pour l'équation parabolique (3.2) en dimension un.
On suppose que g_0 et g_1 sont de classe $\mathcal{C}^4(\mathbb{R})$ et que la fonction $f : (x,u,p) \longmapsto f(x,u,p)$ est de classe $\mathcal{C}^3([0,L] \times (\mathbb{R} \cup \{+\infty\}) \times (\mathbb{R} \cup \{-\infty\}), \mathbb{R})$. D. Henry a montré dans [29] (voir aussi [1]) que la transversalité des variétés stables et instables est automatique pour (3.2).

Théorème 4.1. *Soit $S(t)$ le système dynamique engendré par l'équation parabolique (3.2). Sous les hypothèses de régularité précédentes, si e_- et e_+ sont deux équilibres hyperboliques de $S(t)$, alors la variété instable de e_- intersecte transversalement la variété stable de e_+.*

Ainsi, dès que l'on impose des conditions d'existence globale et de dissipation du type (3.3) et (3.4), la propriété de Morse-Smale pour (3.2) est équivalente à l'hyperbolicité de tous ses points d'équilibre. On obtient donc que (3.2) engendre un système de Morse-Smale génériquement par rapport à f, ou par rapport à L, ou par rapport aux g_i, en utilisant les résultats de généricité de l'hyperbolicité des équilibres (voir par exemple [8], [30] ou [63]). Le caractère automatique de la transversalité est très particulier à l'équation parabolique et à la dimension un. Il provient des théorèmes de type Sturm-Liouville sur le nombre de zéros des fonctions propres d'un opérateur elliptique et du fait que le nombre de zéros des solutions de l'équation linéarisée associée à (3.2) décroit avec le temps.

Le cas de l'équation parabolique (3.6) en dimension supérieure d'espace est plus difficile car la transversalité n'est plus automatique. Dans [9], Brunovský et Poláčik ont montré que la propriété de Kupka-Smale est générique par rapport à une non-linéarité $f(x,u)$ dépendant de x et de u.
Soit Ω un ouvert borné régulier de $\mathbb{R}^d$ et soit $X = W_0^{1,p}(\Omega)$ avec $p > d$ (de telle sorte que $X \hookrightarrow \mathcal{C}^0(\Omega)$). Soit $k \geq 1$, on note $\mathfrak{G}$ l'espace $\mathcal{C}^k(\Omega \times \mathbb{R}, \mathbb{R})$ muni de la topologie de Whitney, c'est-à-dire la topologie engendrée par les ouverts

$$\{g \in \mathfrak{G} \ / \ |D^i f(x,u) - D^i g(x,u)| \leq \delta(u), \ i = 0, ..., k, (x,u) \in \overline{\Omega} \times \mathbb{R}\} \ , \qquad (4.1)$$

où f est une fonction de $\mathfrak{G}$, $D^i f$ sa différentielle d'ordre i et δ est une fonction continue strictement positive sur $\mathbb{R}$.

Théorème 4.2. *L'ensemble des non-linéarités $f \in \mathfrak{G}$, telles que tous les équilibres de (3.6) sont hyperboliques et que leurs variétés stables et instables se coupent transversalement, est générique dans $\mathfrak{G}$.*

Remarquons qu'on obtient comme conséquence directe la généricité de la propriété de Morse-Smale dans l'ensemble des non-linéarités vérifiant une condition de dissipation du type (3.8). Ce théorème est optimal dans le sens où Poláčik a montré dans [51] que l'ensemble des non-linéarités $f(x,u) = f(u)$ ne dépendant pas de x pour lesquelles les variétés stables et instables des équilibres se coupent transversalement n'est pas dense dans l'ensemble des fonctions $f(u)$. Remarquons que le théorème 4.2 reste vrai pour d'autres conditions au bord.

La démonstration du théorème 4.2 repose sur une caractérisation fonctionnelle de la transversalité inspirée de [49]. La preuve de cette caractérisation utilise fortement la théorie des dichotomies exponentielles (voir [28]).
Si e est un équilibre instable de (3.6), on note $m(e)$ l'indice de Morse de e, c'est-à-dire la dimension de la variété instable locale de e.

Proposition 4.3. *Soient e_- et e_+ deux équilibres hyperboliques de (3.6) reliés par une orbite hétérocline u telle que $\lim_{t\to\pm\infty} u(t) = e_\pm$. Soit $\delta \in]0, 1/2[$, alors l'opérateur*

$$L : \left(\begin{array}{ccc} \mathcal{X} = \mathcal{C}^{1,\delta}(\mathbb{R}, \mathbb{L}^p(\Omega)) \cap \mathcal{C}^{0,\delta}(\mathbb{R}, W^{2,p}(\Omega)) & \longrightarrow & \mathcal{Z} = \mathcal{C}^{0,\delta}(\mathbb{R}, \mathbb{L}^p(\Omega)) \\ v(t) & \longmapsto & v_t(t) - \Delta v(t) - f'_u(x, u(t))v(t) \end{array} \right) \tag{4.2}$$

est un opérateur de Fredholm d'indice $m(e_-) - m(e_+)$. De plus, l'orbite hétérocline u est transverse si et seulement si L est surjectif. Enfin, $h \in \mathcal{Z}$ est dans l'image de L si et seulement si $\int_{-\infty}^{+\infty} \int_\Omega h(x,t)\psi(x,t)dxdt = 0$ pour toute solution faible ψ bornée sur $\mathbb{R}$ de l'équation adjointe

$$\psi_t(x,t) + \Delta\psi(x,t) + f'_u(x, u(x,t))\psi(x,t) = 0 \ . \tag{4.3}$$

En général, L n'est pas surjective, mais on veut montrer que cela est le cas pour une non-linéarité f générique. Pour cela, on introduit l'application

$$\Phi : \left(\begin{array}{ccc} \mathfrak{G} \times \mathcal{X} & \longrightarrow & \mathcal{Z} \\ (f,u) & \longmapsto & u_t - \Delta u - f(x,u) \end{array} \right) \ . \tag{4.4}$$

Pour un f fixé, montrer que l'équation parabolique (3.6) vérifie la propriété de Morse-Smale revient grosso-modo à montrer que 0 est une valeur régulière de $u \longmapsto \Phi(f,u)$, c'est-à-dire que, pour toute solution u de $\Phi(f,u) = 0$, l'application $L = D_u\Phi(f,u)$ est surjective. Pour montrer que 0 est une valeur régulière pour un ensemble générique de non-linéarités $f(x,u)$, on applique alors un théorème de Sard-Smale (voir annexe du chapitre 4). Tout revient alors à montrer que pour tout (f,u) tel que $\Phi(f,u) = 0$, la différentielle

$$D\Phi(f,u) \ : \quad (g,v) \longmapsto Lv - g(x,u)$$

est surjective. En fait, l'application du théorème de Sard-Smale doit être précédée d'une longue série de réductions techniques (voir le théorème 4.c.1 de [9], sa généralisation rappelée en annexe au chapitre 4 et son application donnée au paragraphe 5.2 de ce même chapitre). En utilisant entre autres la caractérisation de l'image de L énoncée dans la proposition 4.3, on ramène alors la démonstration du théorème 4.2 au fait de trouver, pour un ensemble dense de fonctions $f(x,u)$ analytiques en u et régulières en x, pour toute solution hétérocline $u(x,t)$ de (3.6) (i.e. solution de $\Phi(f,u) = 0$) et pour toute solution faible $\psi(x,t)$ non nulle et bornée sur $\mathbb{R}$ de l'équation adjointe (4.3), une fonction $g(x,u)$ telle que

$$\int_{-\infty}^{+\infty} \int_\Omega g(x,u(x,t))\psi(x,t)dxdt \neq 0 \ . \tag{4.5}$$

L'avantage de s'être restreint à des fonctions $f(x,u)$ analytiques en u est que toute solution u de (3.6) ou ψ de (4.3) est alors analytique en temps. On cherche ensuite la fonction g parmi les fonctions du type $b(x)g_{\varepsilon,\zeta}(u)$ où b est localisé près d'un point $x_0 \in \Omega$ à choisir et

$$g_{\varepsilon,\zeta}(u) = \frac{1}{\varepsilon}\Theta\left(\frac{u-\zeta}{\varepsilon}\right) \quad \text{où } \Theta(z) = \begin{cases} e^{-1/(1-z^2)} & \text{si } |z| < 1 \\ 0 & \text{sinon.} \end{cases}$$

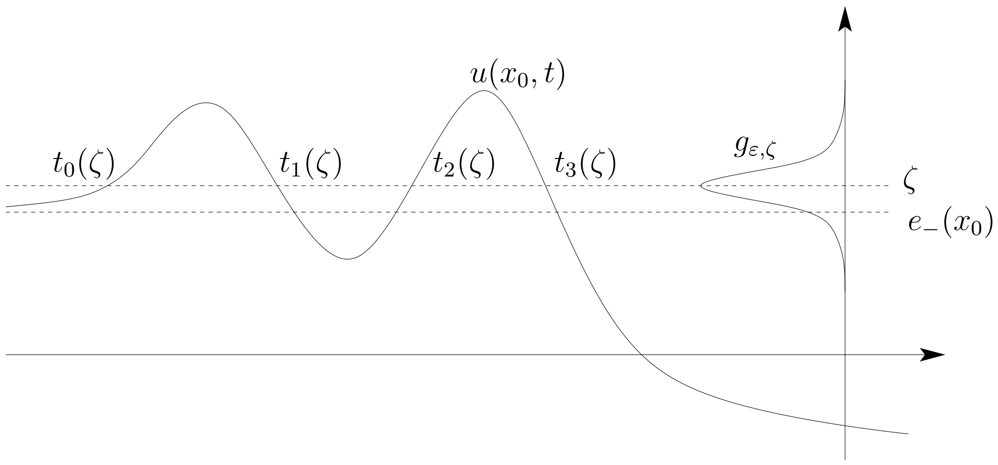
Il faut alors trouver des paramètres ε et ζ tels que

$$I = \int_{-\infty}^{+\infty} g_{\varepsilon,\zeta}(u(x_0,t))\psi(x_0,t)dt \neq 0 \ . \tag{4.6}$$

Soit e_- et e_+ les deux points d'équilibres de (3.6) tels que $u(.,t) \longrightarrow e_\pm$ quand $t \longrightarrow \pm\infty$. On montre que l'ensemble des points $x_0 \in \Omega$ tels que $u_t(x_0,t) \neq 0$ pour $|t|$ grand et que $e_-(x_0) \neq e_+(x_0)$ est générique dans Ω. En outre, une propriété nodale particulière à l'équation parabolique (voir [27]) et l'analyticité de u impliquent qu'il existe un ensemble générique de points $x_0 \in \Omega$ tel que

$$u(x_0,t) = e_-(x_0) \text{ entraîne } u_t(x_0,t) \neq 0 \ . \tag{4.7}$$

On choisit alors x_0 dans l'ensemble générique des points de Ω vérifiant les propriétés ci-dessus et on arrive à la situation suivante.



Pour simplifier, on raisonne maintenant sur la figure ci-dessus (les autres cas, comme par exemple $e_-(x_0) < e_+(x_0)$, sont semblables). Puisque $u(x_0,t)$ est analytique en temps, il existe un nombre fini de temps $t_1,...,t_p$ tels que $u(x_0,t_i) = e_-(x_0)$. Quand ζ est assez proche de $e_-(x_0)$, par le théorème des fonctions implicites, on trouve des temps $t_i(\zeta)$, analytiques en ζ, tels que $u(x_0,t_i(\zeta)) = \zeta$. De plus, quand ζ est assez proche de $e_-(x_0)$ et $\zeta > e_-(x_0)$, il existe un temps $t_0(\zeta)$ proche de $-\infty$ tel que $u(x_0,t_0(\zeta)) = \zeta$. La fin de la démonstration du théorème 4.2 consiste en un raisonnement par l'absurde. Supposons que pour tous paramètres ϵ et ζ, la propriété (4.6) ne soit pas vérifiée. En faisant tendre ϵ vers 0, on trouve que

$$\text{si } \zeta > e_-(x_0),\ \sum_{i=0}^{p} \psi(x_0,t_i(\zeta))t_i'(\zeta) = 0 \quad \text{et si } \zeta < e_-(x_0),\ \sum_{i=1}^{p} \psi(x_0,t_i(\zeta))t_i'(\zeta) = 0\ .$$

Comme les fonctions $t_i(\zeta)$ sont analytiques en ζ et que $\psi(x_0,t)$ est analytique en temps, on obtient que $\psi(x_0,t_0(\zeta))t_0'(\zeta) = 0$ pour ζ proche de $e_-(x_0)$ et $\zeta > e_-(x_0)$, ce qui implique que $\psi(x_0,t)$ est nulle sur un intervalle de temps ouvert et donc est identiquement nulle par analyticité.
Le même raisonnement peut être fait pour tout x_0 dans un sous-ensemble générique, et donc a fortiori dense, de Ω. Donc, si on ne peut pas trouver x_0, ε et ζ tels que (4.6) soit vérifiée, $\psi(x,t)$ doit être identiquement nulle, ce qui est absurde.

4.2 Équation des ondes

L'équation des ondes amorties la plus simple est celle où la dissipation est constante. Brunovský et Raugel ont obtenu dans [10] un résultat semblable au théorème 4.2 dans le cas de la dissipation constante et de conditions au bord de type Dirichlet ou Neumann.
Soit $d = 1, 2$ ou 3 et soit Ω un domaine borné régulier de $\mathbb{R}^d$. Si $d = 1$, on pose $X =$

$\mathbb{H}^1_0(\Omega) \times \mathbb{L}^2(\Omega)$ dans le cas de conditions au bord de type Dirichlet et $X = \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)$ dans le cas de conditions au bord de type Neumann. Si $d = 2$ ou 3, on pose $X = (\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_0(\Omega)) \times \mathbb{H}^1_0(\Omega)$ dans le cas de conditions au bord de type Dirichlet et $X = \{u \in \mathbb{H}^2(\Omega)\ /\ \forall x \in \partial\Omega,\ \frac{\partial u}{\partial \nu}(x) = 0\} \times \mathbb{H}^1(\Omega)$ dans le cas de conditions au bord de type Neumann. Soit $k \geq 3$, si $d = 1$ ou dans le cas de conditions au bord de type Neumann, on pose $\mathfrak{G} = \mathcal{C}^k(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ muni de la topologie de Whitney engendrée par les ouverts du type (4.1). Si $d \geq 2$ et dans le cas de conditions au bord de type Dirichlet, on pose $\mathfrak{G} = \{f \in \mathcal{C}^k(\overline{\Omega} \times \mathbb{R}, \mathbb{R})\ /\ \forall x \in \partial\Omega,\ f(x,0) = 0\}$ muni de la topologie de Whitney.
Soit $f \in \mathfrak{G}$ et soit γ un nombre réel strictement positif, on considère l'équation

$$\begin{cases} u_{tt}(x,t) + \gamma u_t(x,t) = \Delta u(x,t) + f(x,u) & (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 \text{ (ou } \frac{\partial u}{\partial \nu}(x,t) = 0) & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \in X & \end{cases} \tag{4.8}$$

Le résultat de Brunovský et Raugel est le suivant.

Théorème 4.4. *L'ensemble des non-linéarités $f \in \mathfrak{G}$, telles que tous les équilibres de (4.8) sont hyperboliques et que leurs variétés stables et instables se coupent transversalement, est générique dans $\mathfrak{G}$.*

Remarquons qu'on se limite aux petites dimensions mais que la preuve fonctionne en toute dimension, sauf que des lourdeurs techniques apparaissent car il faut que la première composante de l'espace X s'injecte dans $\mathcal{C}^0(\Omega)$. On pourrait aussi choisir d'autres conditions au bord ou un opérateur plus général que le Laplacien Δ, même d'ordre supérieur.

La démonstration du théorème 4.4 suit un schéma similaire à celle du théorème 4.2. Toutefois, l'équation des ondes n'a pas d'effet régularisant en temps fini, contrairement à l'équation parabolique, et cela amène trois difficultés nouvelles. D'abord, on ne peut pas travailler avec des fonctions L et Φ définies dans les espaces $\mathcal{X}$ et $\mathcal{Z}$ de la proposition 4.3; ensuite, même si $f(x,u)$ est analytique, les solutions de (4.8) ne sont pas forcément analytiques et, enfin, la propriété nodale (4.7) n'est pas connue pour l'équations des ondes. On résout la première difficulté en utilisant des versions discrétisées de Φ et L pour l'équation des ondes. Plus précisément, on pose

$$\Phi : \begin{pmatrix} \mathfrak{G} \times l^\infty(\mathbb{Z}, X) & \longrightarrow & l^\infty(\mathbb{Z}, X) \\ (f, U_n) & \longmapsto & U_{n+1} - e^A U_n - \int_0^1 e^{A(1-t)} F(S(t)U_n) dt \end{pmatrix},$$

où $S(t)$ est le système dynamique engendré par (4.8) et A l'opérateur linéaire associé. On obtient alors une variante de la proposition 4.3 grâce à des dichotomies exponentielles discrètes. Pour la deuxième, il est montré dans [26] que si $f(x,u)$ est analytique en u, alors les solutions $u(x,t)$ de (4.8) qui sont définies et bornées sur tout $\mathbb{R}$ sont analytiques en temps. Or, les fonctions u et ψ qui interviennent dans la preuve du théorème 4.4 sont

justement de ce type. A l'inverse des deux premières, la troisième difficulté n'est pas directement surmontable et Brunovský et Raugel ont dû modifier les dernières étapes de la preuve du théorème 4.2 afin d'obtenir leur résultat.
Reprenons les notations du paragraphe précédent, avec $U = (u, u_t)$ trajectoire hétérocline de l'équation (4.8) et $\Psi = (\varphi, \psi)$ solution de l'équation adjointe

$$\frac{\partial}{\partial t}\Psi(t) = -A^*\Psi(t) - (DF(U(t)))^*\Psi(t) \ .$$

Il faut trouver un point $x_0 \in \Omega$ et des paramètres ε et ζ tels que la propriété (4.6) soit vérifiée. Quand ζ est proche de $e_-(x_0)$ et $\zeta > e_-(x_0)$, l'intégrale I de (4.6) peut se décomposer en deux parties : une intégrale I_0 pour des temps t proches de $t_0(\zeta)$ et donc de $-\infty$ et une intégrale I_1 pour les temps t proches des temps $t_1, ..., t_p$. En utilisant l'analyticité de u et ψ, on effectue un développement de Puiseux qui aboutit à l'estimation suivante

$$I_1(\zeta, \varepsilon) = (\zeta - e_-(x_0))^r + o(\zeta - e_-(x_0))^r + \omega(\zeta, \varepsilon) \ , \tag{4.9}$$

où $r \in \mathbb{Q}$ et $\lim_{\epsilon \to 0} \omega(\zeta, \varepsilon) = 0$.
D'autre part, on peut montrer qu'il existe deux valeurs propres λ et μ de l'opérateur linéaire

$$A_{e_-} = \begin{pmatrix} 0 & Id \\ \Delta + f'_u(x, e_-) & -\gamma \end{pmatrix} ,$$

telles que $\lambda > 0$ et $Re(\mu) < 0$ et, quand $t \longrightarrow -\infty$,

$$\|(u, u_t)(t) - (e_-, 0)\|_X = e^{\lambda t} + o(e^{\lambda t}) \ \text{ et } \ \|(\psi, \psi_t)(t)\|_X = e^{-Re(\mu)t} + o(e^{-Re(\mu)t}) \ .$$

On souhaite montrer que, pour presque tout $x_0 \in \Omega$, ces comportements asymptotiques sont les mêmes pour les fonctions réelles $t \mapsto u(x_0, t)$ et $t \mapsto \psi(x_0, t)$. Ce point est assez délicat pour ce qui est de la fonction $t \mapsto \psi(x_0, t)$ et Brunovský et Raugel le surmontent en faisant appel à des propriétés de presque-périodicité de l'équation des ondes libres. Une fois obtenu le comportement asymptotique de $u(x_0, .)$ et $\psi(x_0, .)$, on peut développer l'intégrale I_0 :

$$I_0(\zeta, \varepsilon) = (\zeta - e_-(x_0))^{-\frac{Re(\mu)}{\lambda} - 1} p(t_0(\zeta)) + o(\zeta - e_-(x_0))^{\frac{Re(\mu)}{\lambda} - 1} + \omega(\zeta, \varepsilon) \ , \tag{4.10}$$

où $\lim_{\epsilon \to 0} \omega(\zeta, \varepsilon) = 0$ et p est une fonction réelle telle que $\limsup_{t \to -\infty} |p(t)| > 0$.
Il suffit alors de montrer que pour une fonction f générique, le rapport des parties réelles de deux valeurs propres distinctes de A_{e_-} est irrationnel et donc que $-Re(\mu)/\lambda - 1 \neq r$. En réunissant (4.10) et (4.9), on obtient l'existence de x_0, ε et ζ tels que $I = I_0 + I_1$ est non nulle, c'est-à-dire que (4.6) est vérifié. Ceci conclut la démonstration.

5 Exemples de perturbations

Comme nous l'avons mentionné dans le chapitre précédent, les modèles qui nous permettent d'étudier les phénomènes physiques sont souvent obtenus par des simplifications et des approximations. Ainsi, si une famille de systèmes dynamiques $S_\lambda(t)$ sur X dépend d'un paramètre $\lambda \in \Lambda$ qui est petit dans le phénomène étudié, il est tentant de faire l'approximation $\lambda = 0$ et d'utiliser le système $S_0(t)$ pour comprendre le comportement qualitatif du système réel. Pour justifier cette approximation, il convient de savoir si la dynamique de $S_0(t)$ est qualitativement la même que celle de $S_\lambda(t)$ pour λ petit.
Ainsi que nous l'avons vu dans le paragraphe 1, il y a plusieurs niveaux de réponse à la question de la stabilité de la dynamique. La première est de montrer une convergence des trajectoires en temps finis du type

$$\forall T \geq 0,\ \forall M > 0,\quad \sup_{\|U\|_X \leq M} \sup_{t\in[0,T]} \|S_0(t)U - S_\lambda(t)U\|_X \longrightarrow 0 \ \text{ quand } \lambda \longrightarrow 0\ . \tag{5.1}$$

Ensuite, si les systèmes $S_\lambda(t)$ possèdent un attracteur compact, on peut essayer de montrer la semi-continuité supérieure ou inférieure des attracteurs dans X, en utilisant par exemple les théorèmes 2.19 et 2.20.
Enfin, on cherche à appliquer le théorème 2.21 ou le théorème 2.22 pour obtenir la stabilité de la dynamique. C'est ici que les résultats de généricité de la propriété de Morse-Smale trouvent leur application. En effet, si le paramètre, par rapport auquel la généricité est obtenue, est pertinent du point de vue physique, on peut supposer que le modèle $S_0(t)$ satisfait la propriété de Morse-Smale (quitte à faire une nouvelle approximation) et appliquer les théorèmes 2.21 ou 2.22.
Il faut souligner que tous les problèmes de perturbation ne satisfont pas (5.1). Dans certains cas, on ne peut montrer que la propriété de convergence suivante

$$\forall T \geq 0,\ \forall M > 0,\quad \sup_{\|U\|_Y \leq M} \sup_{t\in[0,T]} \|S_0(t)U - S_\lambda(t)U\|_X \longrightarrow 0 \ \text{ quand } \lambda \longrightarrow 0\ , \tag{5.2}$$

où Y est un sous-espace dense de X s'injectant continuement dans X. Dans les exemples, Y est typiquement un espace de fonctions plus régulières que celles de X. Si l'équation étudiée est régularisante en temps fini, on obtient de (5.2) que, pour tous les temps $T_2 \geq T_1$ strictement positifs, on a

$$\forall M > 0,\quad \sup_{\|U\|_X \leq M} \sup_{t\in[T_1,T_2]} \|S_0(t)U - S_\lambda(t)U\|_X \longrightarrow 0 \ \text{ quand } \lambda \longrightarrow 0\ , \tag{5.3}$$

Les perturbations vérifiant (5.1) ou (5.3) seront dites régulières, celles vérifiant seulement (5.2) seront dites singulières ou irrégulières.
Remarquons que dans le cas de certaines perturbations, le système $S_\lambda(t)$ est défini sur un espace X_λ qui peut varier avec λ. La comparaison des systèmes $S_\lambda(t)$ et $S_0(t)$ s'effectue alors le plus souvent dans un espace dans lequel X_λ et X_0 peuvent s'injecter ou se projeter.

Enfin, il va sans dire que la liste des problèmes de perturbation présentés ici, ainsi que celle des références associées, ne sont pas exhaustives.

5.1 Perturbation de la non-linéarité ou de l'amortissement

Perturber la non-linéarité d'une équation est peut-être la perturbation la plus simple qui soit, car il s'agit le plus souvent d'une perturbation très régulière. Elle se rencontre dans diverses applications pratiques, en particulier dans le cas très fréquent où la non-linéarité dépend de paramètres physiques qui ne peuvent être mesurés précisément.
L'exemple typique est celui de l'équation parabolique (3.6). Soit $\Omega \subset \mathbb{R}^d$ un ouvert borné régulier. On rappelle que $\mathfrak{G}$ est l'espace de fonctions défini au paragraphe 4.1 et on pose

$$\mathfrak{G}_{diss} = \{f \in \mathfrak{G} \ / \ \limsup_{|u|\longrightarrow+\infty} \sup_{x\in\Omega} \frac{f(x,u)}{u} < 0\} \ .$$

En utilisant les théorèmes de continuité des attracteurs 2.19 et 2.20, le théorème 2.21 sur la stabilité de la dynamique et le résultat de généricité de la propriété de Morse-Smale du paragraphe 4.1, on obtient le résultat suivant.

Théorème 5.1. *Pour tout $f \in \mathfrak{G}_{diss}$, le système dynamique $S_f(t)$ engendré par l'équation parabolique (3.6) admet un attracteur global compact $\mathcal{A}_f$. La fonction qui à tout $f \in \mathfrak{G}_{diss}$ associe $\mathcal{A}_f \subset X$ est semi-continue supérieurement en tout point $f \in \mathfrak{G}_{diss}$, et continue en presque tout point $f \in \mathfrak{G}_{diss}$. De plus, autour de presque tout $f_0 \in \mathfrak{G}_{diss}$, il existe un voisinage $\mathcal{V}_{f_0}$ tel que pour tout $f \in \mathcal{V}_{f_0}$, $S_f(t)$ satisfait la propriété de Morse-Smale, $S_f(t)$ a un diagramme de phase isomorphe à celui de $S_{f_0}(t)$ et, il existe un homéomorphisme $h : \mathcal{A}_f \longrightarrow \mathcal{A}_{f_0}$ qui envoie les trajectoires du système restreint $S_f(t)_{|\mathcal{A}_f}$ sur celles de $S_{f_0}(t)_{|\mathcal{A}_{f_0}}$ en préservant le sens du temps.*

On peut aussi étudier la stabilité de la dynamique de l'équation des ondes amorties à l'intérieur d'un ouvert Ω par rapport à une perturbation de l'amortissement $\gamma(x) \in \mathbb{L}^\infty(\Omega)$. Ce problème peut être vu comme une pertubation de la non-linéarité dans la mesure où l'équation (3.10) n'est rien d'autre que l'équation des ondes libres $u_{tt} = \Delta u + g(x, u, u_t)$ avec une non-linéarité $g(x, u, u_t) = f(x, u) - \gamma(x)u_t$. Il s'agit là aussi d'une pertubation régulière et des résultats de continuité des attracteurs ou de la dynamique peuvent être facilement obtenus, en particulier en utilisant la généricité de la propriété de Morse-Smale énoncée dans le théorème 1.1 du chapitre 3.
Par exemple, prenons $\Omega =]0, 1[$ et $(\gamma_\varepsilon(x)) \subset \mathbb{L}^\infty(]0, 1[, \mathbb{R}_+)$ une famille d'amortissements telles que $\gamma_0(x)$ soit strictement positif sur un ouvert non vide de $]0, 1[$.

Théorème 5.2. *On suppose que γ_ε converge vers γ_0 dans $\mathbb{L}^\infty(]0, 1[, \mathbb{R}_+)$ quand ε tend vers 0. Pour tout $f(x, u) \in \mathfrak{G}_{diss}$ et tout $\varepsilon \in \mathbb{R}$ assez petit, le semi-groupe $S_\varepsilon(t)$ engendré* *par l'équation des ondes (3.10) avec un amortissement γ_ε admet un attracteur compact global $\mathcal{A}_\varepsilon$. De plus, les attracteurs $\mathcal{A}_\varepsilon$ sont semi-continus supérieurement en $\varepsilon = 0$ et, pour presque toute non-linéarité $f \in \mathfrak{G}_{diss}$ semi-continus inférieurement en $\varepsilon = 0$. En outre, pour presque toutes les fonctions $f \in \mathfrak{G}_{diss}$, il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$, $S_\varepsilon(t)$ satisfait la propriété de Morse-Smale et a un diagramme de phase isomorphe à celui de $S_0(t)$. De plus, il existe un homéomorphisme $h : \mathcal{A}_\varepsilon \longrightarrow \mathcal{A}_0$ qui envoie les trajectoires du système restreint $S_\varepsilon(t)_{|\mathcal{A}_\varepsilon}$ sur celles de $S_0(t)_{|\mathcal{A}_0}$ en préservant le sens du temps.*

5.2 Discrétisations et calculs numériques

Considérons une équation aux dérivées partielles engendrant un système dynamique sur un espace de Banach X qui est un ensemble de fonctions définies sur un ouvert $\Omega \subset \mathbb{R}^d$. Pour pouvoir simuler par ordinateur le comportement du système $S(t)$, on discrétise généralement le problème en introduisant des paramètres τ et h et en considérant un système discret $(S^n_{\tau,h})_{n\in\mathbb{N}}$ sur un espace de dimension finie X_h. Le domaine Ω devient alors un ensemble fini Ω_h de points de Ω, chacun éloigné d'une distance au plus h de l'ensemble des autres, le temps t est remplacé par des temps discrets $n\tau$ et les fonctions $U \in X$ par l'échantillonage $\tilde{U}_h = U_{|\Omega_h}$. On espère alors obtenir pour τ et h assez petits une bonne idée de la dynamique de $S(t)$ grâce à la simulation numérique $(S^n_{\tau,h})_{n\in\mathbb{N}}$. Bien sûr, les résultats de convergence de la dynamique dépendent de l'équation et du schéma numérique choisis. Pour une revue de résultats et une liste de références, nous renvoyons à [20].
Pour illustrer ce qui précède, nous allons considérer l'exemple suivant.
Soient $\tau > 0$, $N \in \mathbb{N}^*$ et $h = 1/N$, on discrétise l'équation parabolique unidimensionnelle

$$u_t(x,t) = u_{xx}(x,t) + f(u(x,t)) \qquad (x,t) \in]0,1[\times\mathbb{R}_+ \ , \tag{5.4}$$

par un schéma semi-implicite

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + f(u_j^n) \qquad 0 \le n \le N, \ \ j \in \mathbb{N} \ . \tag{5.5}$$

Il s'agit d'une perturbation régulière et il faut signaler que (5.5) engendre un système de type gradient, qui de plus vérifie, comme (5.4), la propriété de transversalité automatique énoncée dans le théorème 4.1 (voir [38]). On peut alors appliquer les théorèmes 2.19, 2.20 et 2.22 et obtenir la convergence de la dynamique dans le cas générique où les équilibres de (5.4) sont tous hyperboliques. Ainsi, le comportement qualitatif de la simulation numérique est représentatif de celui de l'équation parabolique pour des pas de discrétisation assez petits. Pour plus de précisions sur les discrétisations de l'équation parabolique, et les comparaisons des dynamiques associés, voir [15], [64] et [65].

5.3 La limite parabolique de l'équation des ondes

Soient $d \geq 1$ et $\Omega \subset \mathbb{R}^d$ un domaine borné régulier. Soient $\gamma > 0$ et $\varepsilon > 0$. On s'intéresse à la convergence quand ε tend vers 0 de la dynamique de

$$(S_\varepsilon(t)) \quad \left\{ \begin{array}{ll} \varepsilon u^\varepsilon_{tt}(x,t) + \gamma u^\varepsilon_t(x,t) = \Delta u^\varepsilon(x,t) + f(x, u^\varepsilon(x,t)) & (x,t) \in \Omega \times \mathbb{R}_+ \\ u^\varepsilon(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u^\varepsilon(x,0), u^\varepsilon_t(x,0)) = (u_0, u_1)(x) \in \mathbb{H}^1_0(\Omega) \times \mathbb{L}^2(\Omega) & \end{array} \right.$$

vers celle de

$$(S_0(t)) \quad \left\{ \begin{array}{ll} \gamma u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)) & \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ u(x,0) = u_0(x) \in \mathbb{H}^1_0(\Omega) & \end{array} \right.$$

Le problème de la limite parabolique de l'équation des ondes a un intérêt théorique évident, mais il a aussi un intérêt pratique. En effet, l'équation parabolique est utilisée dans de nombreux problèmes de modélisation, par exemple en biologie des populations, en électricité, en physique. Or, elle autorise de la propagation d'information à des vitesses infinies, ce qui n'est pas très réaliste pour une population animale ou la plupart des objets physiques (voir [11]). Dans certains modèles, l'évolution des populations est décrite par une équation des ondes, qui, elle, n'autorise que de la propagation à vitesse bornée (voir par exemple [17]). L'étude de la convergence de $S_\varepsilon(t)$ vers $S_0(t)$ permet donc de retrouver les modèles paraboliques quand le paramètre ϵ du modèle hyperbolique est suffisamment petit pour être choisi nul en première approximation.

Sous les hypothèses (3.7) et (3.8), les semi-groupes $S_\varepsilon(t)$ (resp. $S_0(t)$) sont bien définis sur $X = \mathbb{H}^1_0(\Omega) \times \mathbb{L}^2(\Omega)$ (resp. $\mathbb{H}^1_0(\Omega)$). À première vue, $S_\varepsilon(t)$ apparaît comme une perturbation singulière de $S_0(t)$, d'autant plus que les espaces, sur lesquels $S_\varepsilon(t)$ et $S_0(t)$ sont définis, sont différents. En fait, il n'en est rien et on peut montrer qu'il s'agit d'une perturbation régulière du type (5.3) (cf [55]). Plus précisément, pour tout $(u_0, v_0) \in X$, on peut comparer les trajectoires $u(t) = S_0(t)u_0$ et $(u^\varepsilon, u^\varepsilon_t)(t) = S_\varepsilon(t)(u_0, v_0)$ de la façon suivante. Pour tout $M > 0$ et pour tout $\varepsilon > 0$ assez petit, il existe une constante $C = C(M)$ telle que, si $\|u_0\|^2_{\mathbb{H}^1} + \varepsilon\|v_0\|^2_{\mathbb{L}^2} \leq M$, alors, pour tout $t \geq 0$,

$$\varepsilon\|\frac{\partial}{\partial t}(tu^\varepsilon(t) - tu(t))\|^2_{\mathbb{L}^2} + \|t(u^\varepsilon - u)(t)\|^2_{\mathbb{H}^1} \leq \varepsilon^2 C e^{Ct}(1 + \|(u_0, v_0)\|^2_X) \ . \tag{5.6}$$

Cette estimation amène à introduire le système dynamique $\tilde{S}(t)$, non continu en $t = 0$, défini par

$$\left\{ \begin{array}{l} \tilde{S}(t)(u_0, v_0) = (S_0(t)u_0, \frac{\partial}{\partial t}(S_0(t)u_0)) \ , \quad t > 0 \\ \tilde{S}(0)(u_0, v_0) = (u_0, v_0) \end{array} \right.$$

L'estimation (5.6) montre que la convergence de $S_\epsilon(t)$ vers $\tilde{S}(t)$ est régulière.

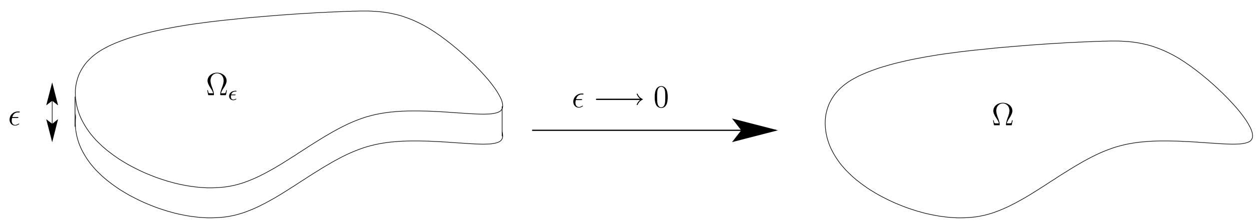
Pour comparer les attracteurs, on note $\mathcal{A}_\varepsilon$ l'attracteur global compact de $S_\varepsilon(t)$ et on pose

$$\tilde{\mathcal{A}} = \{(u,v) \in X \ / \ u \in \mathcal{A}_0, v = \Delta u + f(x,u)\} \ ,$$

où $\mathcal{A}_0 \subset \mathbb{H}_0^1(\Omega)$ est l'attracteur global compact de $S_0(t)$. Les théorèmes 2.19, 2.20 et 2.22 sont applicables et on peut montrer la convergence des attracteurs $\mathcal{A}_\varepsilon$ vers $\tilde{\mathcal{A}}$. Enfin, la dynamique est stable si on suppose que $S_0(t)$ est un système de Morse-Smale, ce qui est une condition générique par rapport à $f(x,u)$ d'après le théorème 4.2.
Les résultats que nous citons sont partiellement prouvés dans [55] (pour les détails, voir [54]). Il faut bien sûr souligner que la limite parabolique de l'équation des ondes a été l'objet de nombreuses études, parmi lesquelles [71], [3], [22], [23], [41], [34], [67], [70] et [16]. On trouvera la mise en perspective historique de ces références dans [55].
Enfin, signalons que la limite parabolique de l'équation des ondes a aussi été étudiée dans le cas d'une dissipation $\gamma(x)$ variable, et même d'une dissipation sur le bord, voir [59] et [58].

5.4 Les domaines minces

Quand on veut modéliser les vibrations de la membrane d'un tambour ou la propagation d'un courant électrique dans un fil, on se place naturellement dans des espaces de dimension deux et un respectivement. Pourtant, la membrane, comme le fil, sont des objets de dimension trois, mais leur épaisseur, très petite, est négligée. Cela est-il justifié ?
Le problème des domaines minces a fait l'objet de nombreuses études. Nous nous limiterons ici à des cas modèles.
Nous utiliserons dans ce paragraphe les notations suivantes. Soit $d \geq 1$ et soit Ω un domaine borné régulier de $\mathbb{R}^d$. Soit $\varepsilon > 0$ et soit $g \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$ une fonction strictement positive. On pose $\Omega_\varepsilon = \{(x,y) \ / \ x \in \Omega, \ y \in]0, \varepsilon g(x)[\}$.



Équation parabolique

On étudie la convergence quand ε tend vers 0 de la dynamique de

$$(S_\varepsilon(t)) \quad \begin{cases} u_t^\varepsilon(x,y,t) = \Delta u^\varepsilon(x,y,t) + f(x, u^\varepsilon(x,y,t)) & (x,y,t) \in \Omega_\varepsilon \times \mathbb{R}_+ \\ \frac{\partial u^\varepsilon}{\partial \nu}(x,y,t) = 0 & (x,y,t) \in \partial\Omega_\varepsilon \times \mathbb{R}_+ \\ u^\varepsilon(x,y,0) = u_0^\varepsilon(x,y) \in \mathbb{H}^1(\Omega_\varepsilon) & \end{cases}$$

vers celle de

$$(S_0(t)) \quad \begin{cases} u_t(x,t) = \frac{1}{g(x)}\mathrm{div}(g(x)\overrightarrow{\nabla} u(x,t)) + f(x,u(x,t)) & (x,t)\in\Omega\times\mathbb{R}_+ \\ \frac{\partial u}{\partial\nu}(x,t) = 0 & (x,t)\in\partial\Omega\times\mathbb{R}_+ \\ u(x,0) = u_0(x)\in\mathbb{H}^1(\Omega) \end{cases}$$

Bien entendu, les systèmes dynamiques $S_\varepsilon(t)$ et $S_0(t)$ sont définis sur des espaces différents. Afin de les comparer, on introduit l'opérateur de moyenne $\mathbb{M}$ défini de $\mathbb{L}^2(\Omega_\varepsilon)$ dans $\mathbb{L}^2(\Omega)$ par $\mathbb{M}u(x) = \frac{1}{\varepsilon g(x)}\int_0^{\varepsilon g(x)} u(x,y)dy$. La perturbation du type domaine mince pour l'équation parabolique est régulière. Plus précisemment, il est montré dans [24] (voir aussi [56]) que pour tout $M>0$, il existe une constante $C=C(M)$ telle que si $\frac{1}{\sqrt{\varepsilon}}\|u_0^\varepsilon\|_{\mathbb{H}^1(\Omega_\varepsilon)}\leq M$ alors

$$\frac{1}{\sqrt{\varepsilon}}\|t(S_\varepsilon(t)u_0^\varepsilon - S_0(t)\mathbb{M}u_0^\varepsilon)\|_{\mathbb{H}^1(\Omega_\varepsilon)} \leq \varepsilon C e^{Ct} \ .$$

Si les attracteurs globaux existent, on obtient alors leur semi-continuité supérieure par une application directe du théorème 2.19. En outre, les théorèmes 2.20, 2.21 et 4.2 impliquent que, pour une fonction f générique, les attracteurs sont continus et les diagrammes de phases de $S_\varepsilon(t)$ et $S_0(t)$ sont isomorphes pour ε assez petit.
On peut étudier cette perturbation dans un cadre plus général, en particulier quand le domaine mince est raccordé à un autre domaine (mince ou pas). On obtient alors un système d'équations couplées. On peut citer [2] pour le raccord d'un domaine mince à un domaine fixe, [25] pour un domaine mince en forme de L, [12] pour des domaines minces d'épaisseurs différentes, [68] pour des domaines minces tubulaires et [52] pour la compression dans une direction de domaines généraux, c'est-à-dire pas forcément de la forme $\{(x,y)\ /\ x\in\Omega,\ y\in]0,\varepsilon g(x)[\}$.

Équation hyperbolique

Soit $\gamma>0$, on étudie la convergence quand ε tend vers 0 de la dynamique de

$$(S_\varepsilon(t)) \quad \begin{cases} u_{tt}^\varepsilon(x,y,t) + \gamma u_t^\varepsilon(x,y,t) = \Delta u^\varepsilon(x,y,t) + f(x,u^\varepsilon(x,y,t)) & (x,y,t)\in\Omega_\varepsilon\times\mathbb{R}_+ \\ \frac{\partial u^\varepsilon}{\partial\nu}(x,y,t) = 0 & (x,y,t)\in\partial\Omega_\varepsilon\times\mathbb{R}_+ \\ (u^\varepsilon(x,y,0),u_t^\varepsilon(x,y,0)) = (u_0^\varepsilon,u_1^\varepsilon)(x)\in\mathbb{H}^1(\Omega_\varepsilon)\times\mathbb{L}^2(\Omega_\varepsilon) \end{cases}$$

vers celle de

$$(S_0(t)) \quad \begin{cases} u_{tt}(x,t) + \gamma u_t(x,t) = \frac{1}{g(x)}\mathrm{div}(g(x)\overrightarrow{\nabla} u(x,t)) + f(x,u(x,t)) & (x,t)\in\Omega\times\mathbb{R}_+ \\ \frac{\partial u}{\partial\nu}(x,t) = 0 & (x,t)\in\partial\Omega\times\mathbb{R}_+ \\ (u(x,0),u_t(x,0)) = (u_0,u_1)(x)\in\mathbb{H}^1(\Omega)\times\mathbb{L}^2(\Omega) \end{cases}$$

La grande différence avec le cas parabolique est que la perturbation du type domaine mince est singulière dans le cas hyperbolique. Pour tout $M>0$, il existe une constante $C=$

$C(M)$ telle que pour tout $U_0^\varepsilon = (u_0^\varepsilon, u_1^\varepsilon) \in \mathbb{H}^2(\Omega_\varepsilon) \times \mathbb{H}^1(\Omega_\varepsilon)$, l'inégalité $\frac{1}{\sqrt{\varepsilon}} \|U_0^\varepsilon\|_{\mathbb{H}^2 \times \mathbb{H}^1} \leq M$ implique

$$\frac{1}{\sqrt{\varepsilon}} \|S_\varepsilon(t) U_0^\varepsilon - S_0(t) \mathbb{M} U_0^\varepsilon)\|_{\mathbb{H}^1 \times \mathbb{L}^2} \leq \sqrt{\varepsilon} C e^{Ct} \ . \tag{5.7}$$

Bien que cette perturbation soit singulière, on peut tout de même appliquer les théorèmes du paragraphe 2.3 et obtenir la continuité des attracteurs. En effet, si les attracteurs des systèmes $S_\varepsilon(t)$ et $S_0(t)$ existent, on peut montrer qu'ils sont bornés dans $\mathbb{H}^2 \times \mathbb{H}^1$ et donc utiliser l'estimation (5.7). Pour ce qui concerne la stabilité de la dynamique, le théorème 2.22 n'est pas applicable tel quel, voir la remarque à la fin du chapitre 3.

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Chapitre 3 : Contributions dans le cadre de cette thèse

Ce chapitre a pour but d'énoncer les résultats principaux de cette thèse, ainsi que plusieurs résultats intermédiaires obtenus, qui ont leur intérêt propre en dehors du problème étudié.

1 Généricité de la propriété de Morse-Smale

Dans le cadre de la dimension $d = 1$, nous avons généralisé le théorème 4.4 du chapitre 2 au cas de la dissipation non-constante, y compris pour une dissipation sur le bord. Ce résultat est l'objet du chapitre 4 de cette thèse.
Soit $k \geq 2$, on note $\mathfrak{G}$ l'espace $\mathcal{C}^k(]0,1[\times\mathbb{R},\mathbb{R})$ muni de la topologie de Whitney engendrée par les ouverts

$$\{g \in \mathfrak{G} \ / \ |D^i f(x,u) - D^i g(x,u)| \leq \delta(u),\ i = 0,...,k, (x,u) \in \overline{\Omega} \times \mathbb{R}\} \ .$$

On considère l'équation des ondes amorties à l'intérieur d'un intervalle

$$\left\{\begin{array}{ll} u_{tt}(x,t) + \gamma(x)u_t(x,t) = u_{xx}(x,t) + f(x,u) & (x,t) \in]0,1[\times\mathbb{R}_+ \\ u(0,t) = 0 \text{ (ou } u_x(0,t) = 0) & t \geq 0 \\ u(1,t) = 0 \text{ (ou } u_x(1,t) = 0) & t \geq 0 \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \in \mathbb{H}^1_0(]0,1[) \times \mathbb{L}^2(\Omega) & \text{(ou } \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(\Omega)) \end{array}\right. \tag{1.1}$$

où $\gamma \in \mathbb{L}^\infty(]0,1[)$ est une fonction positive qui est strictement positive sur un ouvert de $]0,1[$, et l'équation des ondes amorties sur le bord de l'intervalle

$$\left\{\begin{array}{ll} u_{tt}(x,t) = \Delta u(x,t) + f(x,u) & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x,t) + \gamma(x)u_t(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \in \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega) & \end{array}\right. \tag{1.2}$$

où $\gamma(0)$ et $\gamma(1)$ sont deux nombres positifs, dont au moins un est strictement positif et tels que $\gamma(0) \neq 1$ et $\gamma(1) \neq 1$.

Théorème 1.1. *L'ensemble des non-linéarités $f \in \mathfrak{G}$, telles que tous les équilibres de l'équation des ondes avec amortissement interne (1.1) sont hyperboliques et que leurs*

variétés stables et instables se coupent transversalement, est générique dans $\mathfrak{G}$.
L'ensemble des non-linéarités $f \in \mathfrak{G}$, telles que tous les équilibres de l'équation des ondes amorties sur le bord (1.2) sont hyperboliques et que leurs variétés stables et instables se coupent transversalement, est générique dans $\mathfrak{G}$.

Notons que le résultat du chapitre 4 est en fait plus général que le théorème 1.1 et est valable pour une certaine classe d'EDP de dimension un, dont les équations des ondes (1.2) et (1.1) ne sont que les exemples principaux.

La structure de la preuve du théorème 1.1 est semblable à celle du résultat de [1]. La plupart des difficultés nouvelles provient de la structure du spectre de l'opérateur linéaire A associé aux équations (1.1) et (1.2). Dans le cas de la dissipation γ constante, les valeurs propres non réelles sont toutes sur la même droite verticale $\{z \in \mathbb{C} \ / \ Re(z) = -\frac{\gamma}{2}\}$ et les valeurs propres ainsi que les vecteurs propres sont liés de manière explicite à ceux du Laplacien. En dimension 1, dans le cas de la dissipation $\gamma(x)$ non constante, ou d'un amortissement à support dans le bord de l'intervalle, le spectre de A est plus complexe. On sait toutefois que la suite des parties réelles des valeurs propres a un seul point d'accumulation qui est strictement négatif et que les vecteurs propres forment une base de Riesz (voir [2] et [3]). Cette structure spectrale plus complexe amène des problèmes nouveaux par rapport au cas de la dissipation constante. Par exemple, il faut montrer que si $(e, 0)$ est un point d'équilibre, la linéarisation du système dynamique près de $(e, 0)$, donnée par l'opérateur

$$A_e = A + \begin{pmatrix} 0 & 0 \\ f'_u(x,e) & 0 \end{pmatrix} ,$$

possède les même propriétés spectrales que A, ce qui était trivial dans le cas γ constant. D'autre part, la démonstration de [1] utilise le fait que, si γ est constant, pour une fonction $f(x, u)$ générique, le rapport d'une valeur propre positive λ de A_e et de la partie réelle de n'importe quelle autre valeur propre μ distincte de λ est irrationnel. Dans le cas $\gamma(x)$ variable, nous ne pouvons le montrer que si μ est réel. Il faut donc adapter les arguments finaux de la preuve du cas γ constant et montrer que si μ n'est pas réel, même si les exposants r et $-Re(\mu)/\lambda - 1$ des développements (4.9) et (4.10) du chapitre précédent sont égaux, la somme des termes dominants de I_0 et I_1 ne s'annule pas.
L'autre difficulté principale concerne l'étude asymptotique de la fonction $t \longmapsto \psi(x_0, t)$ que l'on utilise dans le développement (4.10) de la preuve du théorème 4.4 du chapitre précédent. En effet, on ne peut plus utiliser les propriétés de presque-périodicité de l'équation des ondes libres comme cela était le cas pour un amortissement constant. Il faut utiliser en remplacement un argument de transformée de Laplace.
En outre, comme dans la démonstration de Brunovský et Raugel, il faut montrer que si $f(x, u)$ est analytique en u, alors les solutions bornées sur tout $\mathbb{R}$ sont analytiques en temps. Si cette propriété était connue pour l'équation des ondes amorties à l'intérieur (1.1), c'est

la première fois qu'elle est signalée dans le cas d'un amortissement sur le bord (1.2), en supposant $\gamma \neq 1$.
Enfin, nous soulignons aussi qu'au cours de la démonstration du théorème 1.1, plusieurs résultats de généricité sont démontrés. Les preuves ont été réalisées dans un cadre un peu plus large que nécessaire, en particulier, elles sont valables en dimension quelconque.

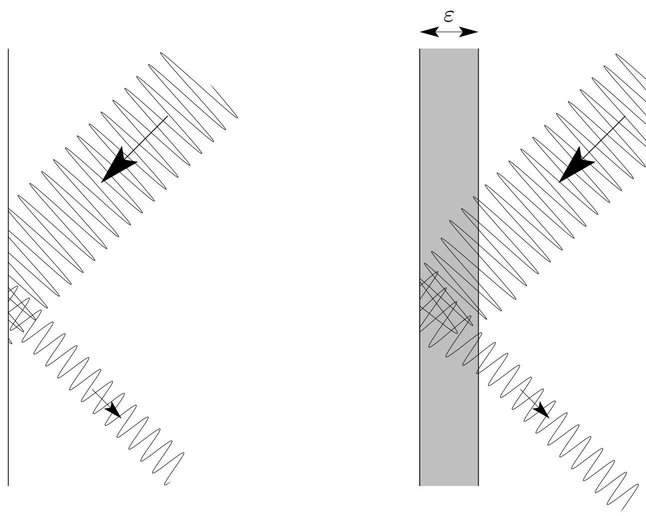
Proposition 1.2. *On considère $S(t)$ le semi-groupe engendré par l'équation des ondes amorties à l'intérieur ou amorties sur le bord en dimension d quelconque et on note A l'opérateur linéaire associé. On suppose qu'il existe deux constante strictement positives M et λ telles que le semi-groupe linéaire e^{At} vérifie*

$$\forall t \geq 0, \ \|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t} \ .$$

Alors, génériquement par rapport à la non-linéarité f, les points d'équilibre $(e, 0)$ de $S(t)$ sont hyperboliques et les valeurs propres des linéarisations A_e sont algébriquement simples.

2 Convergence vers la dissipation sur le bord

L'étude de cette perturbation fait l'objet du chapitre 5 de cette thèse. Cette étude repose sur l'idée suivante : l'équation des ondes amorties sur le bord est un modèle de la propagation des ondes dans un endroit où les ondes sonores ne seraient pas amorties à l'intérieur du domaine, mais à chaque fois qu'elles rebondissent sur la paroi (typiquement, une chambre insonorisée par de la moquette posée sur les murs). Le modèle mathématique consiste à mettre une distribution de Dirac comme dissipation sur le bord, c'est-à-dire une dissipation “infinie” sur une épaisseur nulle. Bien entendu, dans la réalité, la dissipation n'est pas une fonction de Dirac, mais est très forte sur un voisinage mince du bord.



modèle : amortissement sur le bord   réalité : amortissement sur un voisinage du bord

Afin de justifier que l'équation des ondes amorties sur le bord est un bon modèle, il convient d'étudier la perturbation correspondante. Soient $d \geq 1$, $\Omega \subset \mathbb{R}^d$ un domaine borné régulier et $X = \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)$. Soit $\gamma > 0$, on pose $\gamma_\infty = \gamma\delta_{x\in\partial\Omega}$ où $\delta_{x\in\partial\Omega}$ est la fonction de Dirac à support dans le bord de Ω, et pour $n \in \mathbb{N}$, on pose

$$\gamma_n(x) = \left\{ \begin{array}{ll} n\gamma & \text{si } dist(x, \partial\Omega) < \frac{1}{n} \\ 0 & \text{sinon} \end{array} \right.$$

On considère la convergence quand n tend vers $+\infty$ de la dynamique de

$$(S_n(t)) \quad \left\{ \begin{array}{ll} u_{tt}(x,t) + \gamma_n(x)u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)) & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u(x,0), u_t(x,0)) = (u_0, u_1)(x) \in X & \end{array} \right.$$

vers celle de

$$(S_\infty(t)) \quad \left\{ \begin{array}{ll} u_{tt}(x,t) = \Delta u(x,t) + f(x,u) & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x,t) + \gamma u_t(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x))) \in X & \end{array} \right.$$

Nous soulignons que nous ne présentons ici qu'un cadre simplifié pour faciliter l'exposition des résultats. On pourrait bien sûr choisir une dissipation $\gamma(x) \in \mathbb{L}^\infty(\partial\Omega)$ variable, pouvant même s'annuler sur une partie du bord, voir le chapitre 5 pour plus de détails.
La perturbation présentée ici est singulière, c'est-à-dire que l'on ne peut avoir mieux que des estimations de convergence du type : pour tous T et s strictement positifs et pour tout borné $\mathcal{B}$ de $X^s = \mathbb{H}^{1+s}(\Omega) \times \mathbb{H}^s(\Omega)$,

$$\sup_{t\in[0,T],\ U\in\mathcal{B}} \|S_\infty(t)U - S_n(t)U\|_X \longrightarrow 0 \ \text{ quand } n \longrightarrow +\infty \ . \tag{2.1}$$

Contrairement au cas de la perturbation régulière de l'amortissement mentionnée dans le paragraphe 5.1 du chapitre 2, la norme $\|\gamma_n\|_{\mathbb{L}^\infty}$ des amortissements n'est pas bornée. Cette différence est à l'origine de la singularité de ce problème et des diverses difficultés mathématiques rencontrées dans son étude.
Pour tout $n \in \mathbb{N} \cup \{+\infty\}$, on note A_n l'opérateur linéaire associé à $S_n(t)$ et défini dans les paragraphes 3.3 et 3.4 du chapitre 2. Pour montrer une estimation telle que (2.1), on prouve la convergence de A_n^{-1} vers A_∞^{-1}, puis des résultats abstraits sur les semi-groupes de contractions montrant que cette convergence des inverses implique une convergence du type (2.1) pour les semi-groupes linéaires $e^{A_n t}$.
Malgré la singularité de la perturbation, on peut appliquer le théorème 2.20 du chapitre 2 et montrer que la famille $(\mathcal{A}_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ des attracteurs compacts associés aux systèmes dynamiques $S_n(t)$ est semi-continue inférieurement dans X quand n tend vers $+\infty$. Pour cela, on doit comparer les variétés instables locales des systèmes à l'aide d'un théorème de point fixe à paramètre. Nous soulignons que, à cause de la singularité de la perturbation, cette comparaison n'est possible que grâce à la régularité des variétés instables locales. De même, le théorème 2.20 du chapitre 2 n'est applicable que parce que l'on peut montrer que l'attracteur $\mathcal{A}_\infty$ est borné dans un espace plus régulier du type X^s pour un certain $s > 0$.
A l'inverse de la semi-continuité inférieure, la semi-continuité supérieure ne peut être montrée en toute généralité que dans un espace moins régulier comme $X^{-s} = \mathbb{H}^{1-s}(\Omega) \times \mathbb{H}^{-s}(\Omega)$, pour $s > 0$. En effet, on ne sait pas si l'ensemble des attracteurs $\cup_{n\in\mathbb{N}\cup\{+\infty\}}\mathcal{A}_n$ est borné dans un espace régulier du type X^s avec $s > 0$. Cette propriété serait vérifiée si on pouvait prouver l'existence de deux constantes strictement positives M et λ, indépendantes de n, telles que

$$\forall n \in \mathbb{N},\ \forall t \geq 0,\ \|e^{A_n t}\|_{\mathcal{L}(X)} \leq M e^{-\lambda t} . \tag{2.2}$$

Dans le cas de la dimension $d = 1$, on fait dans le chapitre 5 une étude exhaustive de la propriété (2.2) et on obtient des critères concrets nécessaires pour qu'elle soit vérifiée. On montre ainsi que dans le cas simple présenté ici et en dimension $d = 1$, l'estimation (2.2) est satisfaite et, par conséquent, que les attracteurs $\mathcal{A}_n$ sont semi-continus supérieurement dans X. Dans le cas de la dimension supérieure ou égale à deux, la question de la décroissance exponentielle uniforme (2.2) est essentiellement ouverte.
Enfin, nous montrons dans le chapitre 5 que la propriété de Morse-Smale implique la stabilité de la dynamique, même dans le cas de la perturbation irrégulière présentée ici. Autrement dit, nous généralisons le théorème 2.22 du chapitre 2 au cas d'une perturbation singulière. Notre preuve n'est pas une simple généralisation de celle de [4]. En effet, nous utilisons des arguments différents qui reposent très fortement sur la structure gradient.
Ainsi, en dimension $d = 1$, dans le cadre simplifié présenté ici, nous pouvons énoncer un résultat complet de convergence. Soit $\Omega =]0, 1[$ et soit $\mathfrak{G}_{diss}$ l'ensemble des fonctions $f \in \mathfrak{G}$ telles que

$$\limsup_{|u|\to+\infty} \sup_{x\in\Omega} \frac{f(x,u)}{u} < 0 .$$

En utilisant le résultat de généricité de la propriété de Morse-Smale du paragraphe précédent, on obtient le théorème suivant.

Théorème 2.1. *Pour tout* $n \in \mathbb{N} \cup \{+\infty\}$ *et* $f \in \mathfrak{G}_{diss}$*, le système dynamique* $S_n(t)$ *admet un attracteur global compact* $\mathcal{A}_n$*. Pour presque toutes les fonctions* $f \in \mathfrak{G}_{diss}$*, les attracteurs* $\mathcal{A}_n$ *convergent vers* $\mathcal{A}_\infty$ *au sens de la distance de Haussdorf, quand* n *tend vers* $+\infty$*. En outre, si* $\gamma \neq 1$*, pour presque toutes les fonctions* $f \in \mathfrak{G}_{diss}$*, il existe* $n_0 \in \mathbb{N}$ *tel que pour tout* $n \geq n_0$*,* $S_n(t)$ *satisfait la propriété de Morse-Smale et a un diagramme de phase isomorphe à celui de* $S_\infty(t)$*. De plus, il existe un homéomorphisme* $h : \mathcal{A}_n \longrightarrow \mathcal{A}_\infty$ *qui envoie les trajectoires du système restreint* $S_n(t)_{|\mathcal{A}_n}$ *sur celles de* $S_\infty(t)_{|\mathcal{A}_\infty}$ *en préservant le sens du temps.*

Remarque : La méthode que nous avons utilisée pour généraliser le théorème 2.22 est applicable à d'autres perturbations singulières de systèmes gradients. En particulier, elle peut être appliquée à la perturbation de type domaine mince d'une équation hyperbolique, qui a été présentée dans le paragraphe 5.4 du chapitre 2. On obtient alors, pour cette perturbation, un nouveau résultat de stabilité de la dynamique similaire au théorème 2.1.

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Chapitre 4 : Propriété de transversalité générique pour une classe d'équations des ondes avec amortissement variable

1 Introduction

In the qualitative study of partial differential equations, the Morse-Smale property plays an important role as it ensures in some sense the stability of the qualitative behaviour of solutions. It is possible to define the Morse-Smale property for general dynamical systems. However, since few facts are known concerning non-gradient Morse-Smale systems, even in the finite-dimensional case, we will restrict ourselves to gradient systems, that is to dynamical systems, for which there exists a strict Lyapunov function. We recall that the $\omega-$limit set of any compact trajectory of a gradient system consists only on equilibrium points. A gradient system is said to have the Morse-Smale property if it has a finite number of equilibria, which are all hyperbolic, and if all the stable and unstable manifolds of the equilibria intersect transversally (we shall recall these notions before introducing Theorem 1.1 ; for more details, see [18] or [32]). For gradient systems on finite-dimensional compact manifolds, it is well-known that the Morse-Smale property implies the stability of the system, that is that, for small perturbations of the vector-field, the flow remains qualitatively the same. The Kupka-Smale theorem implies that the Morse-Smale property is generic in the class of gradient systems on finite-dimensional compact manifolds (see [32], p.152). We recall that a set is said to be generic if it contains a countable intersection of open dense sets, and that a property is said to be generic if it holds on a generic set. The genericity of a property in Banach spaces implies that this property holds on a dense set ; it corresponds to the "almost everywhere" notion in measurable spaces. So we can say that almost all the gradient systems on finite-dimensional compact manifolds have the Morse-Smale property.

It is therefore natural to wonder what can be generalised to infinite-dimensional gradient systems, in particular to those generated by partial differential equations. This is not only a theoretical question. Indeed, when one works for example on a numerical simulation, or on a physical system with parameters, which were imprecisely measured, one deals with a pertubation of the original system. If the last one is stable, one can consider that the qualitative behaviour, which is observed on the approximative system, holds for the exact one.

In general, infinite-dimensional gradient systems, and also gradient systems defined on finite-dimensional non-compact manifolds, have an infinite number of equilibria. For this reason, we consider here the Kupka-Smale property. A gradient system is said to have the Kupka-Smale property if all its equilibria are hyperbolic and all its stable and unstable manifolds intersect transversally. Under additional dissipative hypotheses, all the equilibria of a Kupka-Smale gradient system S belong to a compact set, which implies that the system is Morse-Smale. In particular, if a Kupka-Smale gradient system S has a compact global attractor $\mathcal{A}$, then it is Morse-Smale. The structural stability result concerning Morse-Smale systems on finite-dimensional compact manifolds has been extended as follows by Oliva (see [18]). Let $S_\epsilon(t)$, $t \in \mathbb{R}_+$, be a parametrized family of gradient systems, having a compact global attractor $\mathcal{A}_\epsilon$ on X. If the attractors $\mathcal{A}_\epsilon$ are upper-semicontinuous in ϵ at $\epsilon = 0$, and if $S_0(t)$, $t \in \mathbb{R}_+$, is Morse-Smale, then, under additional reversibility hypotheses, the restrictions to $\mathcal{A}_\epsilon$ of the discrete dynamical systems $(S_\epsilon(1))^n$ are conjugated to the restriction to $\mathcal{A}_0$ of $(S_0(1))^n$, for $\epsilon > 0$ small enough. More precisely, there exists a diffeomorphism h_ϵ from $\mathcal{A}_0$ onto $\mathcal{A}_\epsilon$ such that $h \circ (S_0(1))^n = (S_\epsilon(1))^n \circ h$. This property implies in particular that $S_\epsilon(t)$, $t \in \mathbb{R}_+$, is still a Morse-Smale system and that its phase-diagram (that is the description of its equilibria and the trajectories connecting them) is equivalent to the one of $S_0(t)$. This shows that the Morse-Smale property is still relevant for infinite-dimensional gradient systems.

Next, one can wonder if the property of genericity of the gradient Kupka-Smale systems on finite-dimensional manifolds extends to the infinite-dimensional case in a meaningful way. The first results on genericity of the Kupka-Smale property for partial differential equations concern the parabolic equation

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)) \ , & (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 \ , & (x,t) \in \partial\Omega \times \mathbb{R}_+ \end{cases} , \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$. Henry proved that in space dimension $N = 1$, a heteroclinic orbit connecting two equilibria is necessarily transversal (see [23] and also [1]). This implies the genericity of the Kupka-Smale property, since the equilibria of (1.1) are all hyperbolic generically with respect to the non-linearity f (see [4]). Unfortunately, this transversality property is not true in higher dimension. Brunovský and Polàčik proved that the equation (1.1) satisfies the Kupka-Smale property generically with respect to the non-linearity f, for any dimension $N \geq 1$ (see [5]).
Later, Brunovský and Raugel considered the wave equation with constant damping

$$\begin{cases} u_{tt}(x,t) + \gamma u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)) \ , & (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 \ , & (x,t) \in \partial\Omega \times \mathbb{R}_+ \end{cases} , \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ and γ is a positive constant.
They proved that this equation satisfies the Kupka-Smale property generically with respect

to the non-linearity f, for any dimension $N \geq 1$ (see [6]). Notice that, as the dynamics of hyperbolic equations are richer than the ones of parabolic equations, the transversality result of Henry is no longer true even in the one-dimensional case. So, the genericity result of [6] is also meaningful in this case. In their proofs, Brunovský and Polàčik used an abstract genericity theorem, that Brunovský and Raugel improved, in order to be able to apply it in the hyperbolic case. We recall this improved version in Appendix A. This theorem is very useful for showing genericity of the Kupka-Smale property for partial differential equations with respect to a class of non-linearities depending on a parameter. The key points of its proof are a version of the Sard-Smale theorem (a slightly stronger version than Theorem A.2 given here), and a functional characterisation of the transversality. Applying this abstract theorem, Brunovský and Polàčik, as well as Brunovský and Raugel, reduced their genericity problem to the construction of a perturbation $h(x, u)$ of the nonlinearity $f(x, u)$, such that a certain integral, depending on h, the heteroclinic orbits and the global bounded solutions of the adjoint linearized equation to (1.1) and (1.2) respectively, does not vanish (see the integrals I and J given by (5.7) and (5.8) in our case). In both papers, the suitable perturbation $h(x, u)$ is localized in the neighborhood of an appropriate point $x = x_0$ of the domain Ω. In [5], the choice of x_0 was the consequence of delicate properties of the singular nodal set of the reaction-diffusion equation. Since corresponding nodal properties are not available in the case of the hyperbolic equation, other techniques had to be used. The proof in [6] uses the development of the respective globally defined and bounded solutions $u(x_0, t)$ and $\psi(x_0, t)$, of the equation (1.2) and its corresponding adjoint linearized equation, into fractional series in the neighborhood of certain points and also their asymptotic development in the neighborhood of $t = -\infty$. The asymptotic development of these bounded solutions strongly depends on spectral properties of the linearized equations around the equilibria.

A natural extension of the above results is the study of wave equations with non-constant damping. As said above, spectral properties of the equation play an important role in the proof of [6]. In particular, the existence of a Riesz basis composed by the eigenvectors of the linearized operator is used there. The existence of such a Riesz basis is known in general only in the one-dimensional case. For this reason, we restrict our study here to this case. The most classical example of wave equation with non-constant damping is the wave equation with internal local damping

$$\left\{ \begin{array}{ll} u_{tt}(x,t) + \gamma(x)u_t(x,t) = u_{xx}(x,t) + f(x, u(x,t)) \ , & (x,t) \in [0,1] \times \mathbb{R}_+ \\ u(x,t) = 0 \ , & (x,t) \in \{0,1\} \times \mathbb{R}_+ \end{array} \right. \tag{1.3}$$

where γ is a nonnegative bounded function which is positive on an open subset of $[0, 1]$. This equation generates a dynamical system $S(t) : U_0 \in X \longmapsto S(t)U_0$ on the Banach space $X = \mathbb{H}^1_0(]0, 1[) \times \mathbb{L}^2(]0, 1[)$. An equilibrium E of (1.3) is said hyperbolic if the spectrum of the linearization $D_U S(1)E$ does not intersect the unit circle in the complex plane. We also recall that two submanifolds of X intersect transversally if, at any point of intersection, one of the two tangent spaces contains a closed complement of the other. For $k \geq 1$, we denote $\mathfrak{G}^k$ the set $\mathcal{C}^k([0,1] \times \mathbb{R}, \mathbb{R})$ endowed with the Whitney topology, that is, the topology generated by the sets

$$\{g \in \mathfrak{G}^k \ / \ |D^i f(x,u) - D^i g(x,u)| < \delta(u), i = 0, ..., k, x \in [0,1], u \in \mathbb{R}\},$$

where δ is any positive function on $\mathbb{R}$ and f is any function of $\mathcal{C}^k([0,1] \times \mathbb{R}, \mathbb{R})$. A sequence of functions f_n converges to f in the Whitney topology if and only if there exists a compact set $K \subset \mathbb{R}$ such that for all $i = 0, \ldots, k$, the derivatives $D^i f_n$ converge uniformly to $D^i f$ on $[0,1] \times K$, and for all n, except maybe a finite number, $f_n = f$ on $[0,1] \times (\mathbb{R} \setminus K)$. Moreover, the space $\mathfrak{G}^k$ is a Baire space, which means that a generic set of $\mathfrak{G}^k$ is also a dense set. For more details concerning this topology, see [15].

One of the main results of this paper is the following theorem.

Theorem 1.1. *Let $k \geq 2$ and let $\mathfrak{G}^{KS}$ be the set of functions $f \in \mathfrak{G}^k$ such that the damped wave equation (1.3) satisfies the Kupka-Smale property, that is, such that all the equilibria of (1.3) are hyperbolic and their stable and unstable manifolds intersect transversally. Then $\mathfrak{G}^{KS}$ is a generic subset of $\mathfrak{G}^k$.*

Restricting the above space $\mathfrak{G}^k$, we show the genericity of the Morse-Smale property. For example, let $\mathfrak{G}^k_{\text{diss}}$ be the open subset of $\mathfrak{G}^k$ defined by

$$\mathfrak{G}^k_{\text{diss}} = \{f \in \mathfrak{G}^k \ / \ \limsup_{u \to \pm\infty} \frac{f(x,u)}{u} < 0\} \ .$$

When f belongs to $\mathfrak{G}^k_{\text{diss}}$, we show in Corollary 2.7 that Equation (1.3) admits a compact global attractor. Thus, the number of equilibria is finite and the Kupka-Smale property is equivalent to the Morse-Smale one.

Corollary 1.2. *If $k \geq 2$, the damped wave equation (1.3) satisfies the Morse-Smale property for a generic dissipative non-linearity $f \in \mathfrak{G}^k_{diss}$.*

The knowledge of the asymptotic behaviour of the spectrum of the linear operator associated to the wave equation with constant damping (1.2) is a key-point in the proof of [6]. In the constant damping case, explicit relations between the eigenvalues and the eigenfunctions of the damped wave operator and those of the Laplacian operator are known. In particular, the eigenvalues are either real or belong to the same vertical line. In the case of the one-dimensional wave equation with non-constant damping (1.3), one only knows that the generalized eigenvectors of the operator form a Riesz basis and that the real part of the eigenvalues have only one point of accumulation. As these spectral properties are weaker, the proof of Theorem 1.1 is more involved. We emphasize that the Riesz basis property is central in our proofs. This property is strongly linked to the dimension one as it is known not to hold in general in higher dimensional cases. The few examples in higher dimensions in which the eigenvectors form a Riesz basis are very particular (e.g. the constant damping case or equation with radial symmetry, see [36]). For these reasons we only consider here the one dimensional case.

Another important example of hyperbolic equation with non-constant damping is the following wave equation damped on the boundary,

$$\left\{ \begin{array}{ll} u_{tt}(x,t) = u_{xx}(x,t) - u(x,t) + f(x,u(x,t)) \ , & (x,t) \in [0,1] \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x,t) + \gamma(x) u_t(x,t) = 0 \ , & (x,t) \in \{0,1\} \times \mathbb{R}_+ \end{array} \right. \tag{1.4}$$

where γ is a nonnegative bounded function which is positive on at least one point of $\{0,1\}$. When $\gamma(0) \neq 1$ and $\gamma(1) \neq 1$, the structure of the spectrum of the linearized operator is similar to the one of Equation (1.3). Thus, we obtain the following result.

Theorem 1.3. *If we assume that $k \geq 2$, $\gamma(0) \neq 1$ and $\gamma(1) \neq 1$, then the set of functions $f \in \mathfrak{G}^k$, such that Equation (1.4) has the Kupka-Smale property, is a generic subset of $\mathfrak{G}^k$. Moreover, the set of functions $f \in \mathfrak{G}^k_{diss}$, such that Equation (1.4) has the Morse-Smale property, is a generic subset of $\mathfrak{G}^k_{diss}$.*

When $\gamma(0) = 1$ or $\gamma(1) = 1$, the spectrum of the linearized operator obtained by choosing $f = 0$ is empty. See Subsection 3.1 for a discussion of this case.

In order to enhance the common structures of Equations (1.3) and (1.4) and to make easier the understanding of the mechanism of the proofs, we have chosen to work with an abstract wave equation

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ u_t \end{pmatrix} = A \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ f(x,u) \end{pmatrix} = \begin{pmatrix} u_t \\ -B(u + \Gamma u_t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(x,u) \end{pmatrix} , \tag{1.5}$$

where the operators B and Γ will be defined in Section 2. We assume that the eigenvectors of A form a Riesz basis. Adding other hypotheses, we prove the genericity of Kupka-Smale property for Equation (1.5) (see Theorem 2.6). Theorems 1.1 and 1.3 will be a direct consequences of this abstract theorem. Moreover, Theorem 2.6 can be applied to other one-dimensional dissipative wave equations.
Notice that, in showing the genericity of the transversality, we need to prove auxiliary properties, which have their own interest and were not known for Equation (1.4) and the abstract version (1.5). For example, to define the global manifold structure of the stable and unstable sets of equilibria (see [22]), we need to prove that (1.5), as well as the linearized equation (and its adjoint) satisfy a backward uniqueness property. We will also prove that Equation (1.5) generates an asymptotically smooth dynamical system. This property

ensures the compactness of any bounded set of trajectories. Besides, we use the asymptotic smoothness to prove that the globally bounded solutions of (1.5) are analytic in time, when the nonlinearity f is analytic in u. This property plays a central role in the last part of the proof of the abstract theorem 2.6. In this paper, we also show how to use the structure of the spectrum of the linear operator A to obtain all these auxiliary results. Indeed, the fact that the set of rootvectors of A is a Riesz basis of the functional space enables simple and elegant proofs of such qualitative properties.
To prove the genericity of Kupka-Smale property, we follow the lines of [6]. The main new difficulties appear at the end of the proof, when we must prove that the integral introduced in [5] and [6] (see (5.8) in our case) does not always vanish. Indeed, this step strongly uses the properties of the spectrum, and so had to be modified. Moreover, the developpment of the bounded solutions ψ of the adjoint equation in the neighborhood of $t = -\infty$ is more involved (see Proposition 5.5 and Lemma 5.8). In order to estimate the decay of the real function $\psi(x_0, t)$ at some point $x_0 \in \Omega$, Brunovský and Raugel used a property of almost-periodicity coming from the fact that the non-real eigenvalues of the operator all lay on a same vertical line. In the non-constant damping case, the spectrum is more complicated and such argument cannot be applied. We replaced it by a Laplace transform argument.

Section 2 of this paper is devoted to the statement of the abstract theorem (Theorem 2.6) and the proof of Theorem 1.1. In Section 3, we deal with the case of Equation (1.4) and introduce another example. We also discuss which properties still hold in cases where the hypotheses of Theorem 2.6 are only partially satisfied. The proof of our main theorem is split into two parts. The first part deals with generic properties of the spectrum, including hyperbolicity or simplicity of the eigenvalues. These properties, given in Section 4, are worth a separated section, as they have their own interest. The second part of the proof of our main theorem is given in Section 5. It shows how to apply the Brunovský-Polàčik-Raugel theorem A.1.

2 Abstract genericity theorem

In this section, we are going to state our abstract theorem. We first introduce the frame in which we will work. In particular, we describe the space X and the operators A, B and Γ that we introduced in the introduction.
We want our theorem to be as simple as possible and, at the same time, to be directly applicable to as many situations as possible. For these reasons, our assumptions on A will be as basic as possible and thus we will have to do some preliminaries to be able to state

our main Theorem 2.6.
In order to make the reading easier, we illustrate each abstract hypothesis with the corresponding property in the case of the internal damped wave equation (1.3). As a result, Theorem 1.1 will be a direct corollary of Theorem 2.6. We briefly recall that the damped wave equation (1.3) and its analogue in higher dimension have been extensively studied for a long time ([11], [35], [13], [38]...). In the one-dimensional case, the exponential decay of the linear semigroup has been proved in [21] (see also [8] and [11]). In higher dimensions, the exponential decay is still true under additional conditions ([2], [35] and [38]). In these cases, the regularity of the complete bounded orbits is proved in [20].

2.1 Introduction of the abstract wave equation

We work in $\mathbb{L}^2(]0,1[)$ with the usual scalar product

$$< u|v >_{\mathbb{L}^2} = \int_0^1 u(x)\overline{v}(x)dx \ .$$

In order to define the operator A, we introduce the operators B and Γ, which satisfy the following hypotheses.

(B) B is a real positive self-adjoint operator from its domain $D(B)$ into $\mathbb{L}^2(]0,1[)$. Moreover, we assume that $B^{-1/2}$ is smoothing in the sense that $B^{-1/2}$ defines a continuous linear mapping from $\mathbb{H}^\alpha(]0,1[)$ into $\mathbb{H}^{\alpha+1}(]0,1[)$ for all $\alpha \geq 0$ (in particular $D(B^{1/2})$ is continuously imbedded in $\mathbb{H}^1(]0,1[)$).

(Gam) Γ is a continuous linear operator from $D(B^{1/2})$ into $D(B^{1/2})$. In addition, Γ is a compact nonnegative self-adjoint operator on $D(B^{1/2})$. In particular, for any φ and ψ in $D(B^{1/2})$, we have

$$< B^{1/2}\Gamma\varphi|B^{1/2}\psi >_{\mathbb{L}^2} = < B^{1/2}\varphi|B^{1/2}\Gamma\psi >_{\mathbb{L}^2} \ .$$

We introduce the Banach space $X = D(B^{1/2}) \times \mathbb{L}^2$ endowed with the natural scalar product :

$$\begin{aligned} \left\langle \begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_X &= < B^{1/2}u|B^{1/2}\varphi >_{\mathbb{L}^2} + < v|\psi >_{\mathbb{L}^2} \\ &:= < u|\varphi >_{D(B^{1/2})} + < v|\psi >_{\mathbb{L}^2} \end{aligned} \tag{2.1}$$

Let A be the operator

$$A : \ D(A) \longrightarrow X \ , \quad A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -B(u+\Gamma v) \end{pmatrix} \ , \tag{2.2}$$

where $D(A)$ is defined as follows

$$D(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X \Big/ \; v \in D(B^{\frac{1}{2}}),\; (u + \Gamma v) \in D(B) \right\}.$$

Example of Equation (1.3) : *We set $B = -\Delta_D$, where Δ_D is the Laplacian with homogeneous Dirichlet boundary condition. We have*

$$D(B) = \mathbb{H}^2(]0,1[) \cap \mathbb{H}^1_0(]0,1[) \text{ and } X = \mathbb{H}^1_0(]0,1[) \times \mathbb{L}^2(]0,1[) \;.$$

The operator Γ is defined as follows.

$$\Gamma : \begin{pmatrix} \mathbb{H}^1_0(]0,1[) & \longrightarrow & \mathbb{H}^1_0(]0,1[) \\ v & \longmapsto & B^{-1}(\gamma(x)v) \end{pmatrix} \;.$$

The operator Γ is compact. Moreover it is nonnegative and self-adjoint on $\mathbb{H}^1_0(]0,1[)$, since for any $(v,v') \in \mathbb{H}^1_0(]0,1[)^2$, we have

$$< \Gamma v | v' >_{D(B^{1/2})} = \int_0^1 \gamma(x) v(x) \overline{v'(x)}\, dx \;.$$

The operator A is simply the classical damped wave operator

$$A = \begin{pmatrix} 0 & Id \\ \Delta_D & -\gamma(x) \end{pmatrix} \;.$$

Remark : Actually, we do not need to assume that B is positive. It suffices to suppose that there exists a positive number λ such that $B' = B + \lambda Id$ satisfies the property (B). An example is $B = -\Delta_N$, where Δ_N is the Laplacian with Neumann boundary conditions. In such cases, we can work with B' by replacing $f(x,u)$ by $f(x,u) - \lambda u$ in Equation (2.4).

The hypotheses (B) and (Gam) imply directly the following lemma.

Lemma 2.1. *For any function $h \in \mathbb{L}^\infty([0,1],\mathbb{C})$ and any complex number λ, the operator L from $D(B^{1/2})$ into $D(B^{1/2})$ defined by*

$$L\varphi = \varphi + \lambda \Gamma \varphi - B^{-1}(h\varphi),$$

is a Fredholm operator of index 0 and $L^* = \overline{L}$ on $D(B^{1/2})$. In particular, g is in the range of L if and only if, for all φ in the kernel of L, $< g | \overline{\varphi} >_{D(B^{1/2})} = 0$.

Proof : The operator $B^{-1}(h.)$ is defined from $D(B^{1/2})$ into $D(B)$, so it is compact from $D(B^{1/2})$ into $D(B^{1/2})$. By assumption, Γ is also compact on $D(B^{1/2})$, so, as L is a compact perturbation of the identity on $D(B^{1/2})$, L is a Fredholm operator of index 0 (see for example [3]). $L^* = \overline{L}$, indeed if v and φ are two functions of $D(B^{1/2})$, we have

$$\begin{aligned}
< L\varphi|v >_{D(B^{1/2})} &= < \varphi|v >_{D(B^{1/2})} +\lambda < \Gamma\varphi|v >_{D(B^{1/2})} - < B^{-1/2}(h\varphi)|B^{1/2}v >_{\mathbb{L}^2} \\
&= < \varphi|v >_{D(B^{1/2})} +\lambda < \varphi|\Gamma v >_{D(B^{1/2})} - < B^{1/2}\varphi|B^{-1/2}(\overline{h}v) >_{\mathbb{L}^2} \\
&= < \varphi|v + \overline{\lambda}\Gamma v - B^{-1}(\overline{h}v) >_{D(B^{1/2})} \\
&= < \varphi|\overline{L}v >_{D(B^{1/2})}
\end{aligned}$$

The last claim of the lemma is just the Fredholm alternative. □

As a consequence of Lemma 2.1, we can prove that A generates a $\mathcal{C}^0-$semigroup.

Proposition 2.2. *A is a nonpositive operator and $Id - A$ is surjective from $D(A)$ onto X. As a consequence, A generates a $\mathcal{C}^0-$semigroup e^{At} of contractions on X. Moreover, A has a compact resolvent.*

Proof : If $(u, v) \in D(A)$, then

$$\left\langle A\begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_X = - < \Gamma v|v >_{D(B^{1/2})} ,$$

so A is nonpositive and thus dissipative. We next prove the maximality, that is that $Id-A$ is surjective. If $(h, g) \in X$, there exists a vector $(u, v) \in D(A)$ such that

$$(Id - A)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h \\ g \end{pmatrix}$$

if and only if

$$\left\{ \begin{array}{l} u - v = h \\ v + B(u + \Gamma v) = g \end{array} \right. \tag{2.3}$$

This is equivalent to find a function $u \in D(B^{1/2})$ such that

$$(u - h) + \Gamma(u - h) + B^{-1}(u - h) = B^{-1}g - h.$$

Lemma 2.1 implies that this is possible if $Ker(Id + \Gamma + B^{-1}) = \{0\}$. But if $\varphi \in D(B^{1/2})$ is such that

$$\varphi + \Gamma\varphi + B^{-1}\varphi = 0,$$

then
$$\|\varphi\|^2_{D(B^{1/2})} + < \Gamma\varphi|\varphi >_{D(B^{1/2})} + \|\varphi\|^2_{\mathbb{L}^2} = 0 \ ,$$
and since $< \Gamma\varphi|\varphi >_{D(B^{1/2})} \geq 0$, it follows that $\varphi = 0$. As a consequence of the Lumer-Phillips Theorem (see [33]), A generates a $\mathcal{C}^0-$semigroup e^{At} on X.
Finally, we can easily prove that $(Id - A)^{-1}$ is compact, using the equalities (2.3), and the fact that B^{-1} and Γ are compact on $D(B^{1/2})$. $\square$

Let f be any function of $\mathcal{C}^k([0,1] \times \mathbb{R})$. Since the map
$$\begin{pmatrix} u \\ v \end{pmatrix} \in X \longmapsto \begin{pmatrix} 0 \\ f(x,u) \end{pmatrix} \in X$$
is of class $\mathcal{C}^1$, the equation
$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ u_t \end{pmatrix} = A\begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ f(x,u) \end{pmatrix} \quad t > 0, \qquad \begin{pmatrix} u \\ u_t \end{pmatrix}_{\big|t=0} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X \tag{2.4}$$
has a unique local solution $(u, u_t) \in \mathcal{C}^0([0,T[, X)$ where T is a positive time, which depends on the initial data (u_0, u_1). We denote by $S(t)(u_0, u_1) = (u, u_t)$ the solution in $\mathcal{C}^0([0,T[, X)$ of (2.4) and remark that $S(t)$ is a local nonlinear semigroup on X. If (u, u_t) is a solution in $\mathcal{C}^0([0,T[, X)$ of Equation (2.4), the linearized equation along the solution (u, u_t) is given by
$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ w_t \end{pmatrix} = A\begin{pmatrix} w \\ w_t \end{pmatrix} + \begin{pmatrix} 0 \\ f'_u(x,u(x,t))w \end{pmatrix}, t > 0, \ \begin{pmatrix} w \\ w_t \end{pmatrix}_{\big|t=0} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \in X. \tag{2.5}$$

Now, we will look at the adjoint operator A^*. First notice that, with our choice of the scalar product, we have $X^* = X$. Due to Hypothesis (Gam), we verify that
$$A^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6}$$
Notice that (2.6) means that the adjoint equation $\frac{\partial}{\partial t}(u, u_t) = -A^*(u, u_t)$ is "equivalent" to the equation $\frac{\partial}{\partial t}(u, u_t) = A(u, u_t)$ where the time is reversed. Finally, the adjoint equation of (2.4) is given by
$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} \theta \\ \psi \end{pmatrix} = -A^*\begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} B^{-1}(f'_u(x,u)\psi) \\ 0 \end{pmatrix} & t < 0 \\ \begin{pmatrix} \theta \\ \psi \end{pmatrix}_{t=0} = \begin{pmatrix} \theta_0 \\ \psi_0 \end{pmatrix} \in X & \end{cases} \tag{2.7}$$
To obtain our main theorem, we need to introduce spectral hypotheses.

2.2 Spectral assumptions

As A has a compact resolvant, its spectrum consists only of isolated eigenvalues λ_n with finite multiplicity. With each λ_n, we can associate an orthonormalized Jordan Chain

$$(V^n_{j,k})_{j\leq m_n-1,k\leq m_{n,j}-1}$$

with

$$\forall\ 0\leq j\leq m_n-1,\quad AV^n_{j,0}=\lambda_n V^n_{j,0}$$
$$\forall\ 0\leq j\leq m_n-1,\ 1\leq k\leq m_{n,j}-1,\quad AV^n_{j,k}=\lambda_n V^n_{j,k}+V^n_{j,k-1}$$

We will assume that the rootvectors form a Riesz basis of X. We say that a set $(\Psi_n)_{n\in\mathbb{N}}$ is a Riesz basis of X if there exist two positive constants a_1 and a_2, such that for any $U\in X$, there is a unique sequence of complex numbers (α_n) such that

$$U=\sum_{n\in\mathbb{N}}\alpha_n\Psi_n,$$

and

$$a_1\|U\|^2_X\leq\sum_{n\in\mathbb{N}}|\alpha_n|^2\leq a_2\|U\|^2_X\ . \tag{2.8}$$

See [14] for details.
We assume the following spectral properties

(Spec) the operator A is such that

a) the rootvectors $(V^n_{j,k})$ of A form a Riesz basis of X.

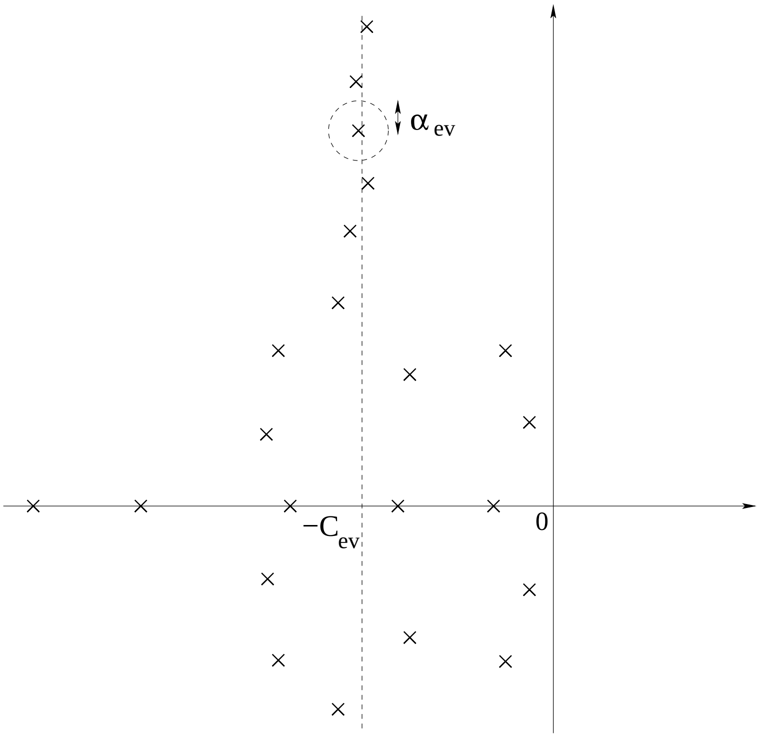
b) there exist two positive constants M_{ev} and C_{ev} such that all eigenvalues λ with $|\lambda|>M_{ev}$ are simple and can be written as

$$\lambda_{\pm n}=a_n\pm ib_n\ \text{ with }\ a_n\longrightarrow -C_{ev}\ . \tag{2.9}$$

c) the eigenvalues are uniformly isolated in the sense that there exists a constant $\alpha_{ev}>0$ such that

$$\inf_{n\neq m}|\lambda_n-\lambda_m|\ >\ \alpha_{ev}\ .$$

In short, the spectrum looks like



Example of Equation (1.3) : *It has been proved by Cox and Zuazua in [11] that the spectrum of Equation (1.3) satisfies Hypothesis (Spec). In particular, they showed the following high-frequency estimate*

$$\lambda_{\pm n} = -\frac{\gamma_0}{2} \pm in\pi + O(\frac{1}{n}) \ ,$$

where γ_0 *is the average of* γ *on* $[0, 1]$*, that is,*

$$\gamma_0 = \int_0^1 \gamma(x)\ dx \ > \ 0 \ .$$

We will use an important property which easily follows from Hypothesis (Spec), but has to be enhanced.

Proposition 2.3. *Under Hypothesis (Spec), the eigenvalues of A satisfy*

$$\sum_{\lambda \in \sigma(A)} \frac{1}{|\lambda|^2} \ < \ \infty.$$

We assume furthermore that

(UCP) B and Γ satisfy the following weak unique continuation property : for any given $\lambda \in \mathbb{C}$, any function $h \in \mathbb{L}^\infty$, and any $\varphi \in D(B^{1/2}) \setminus \{0\}$ with $(\varphi + \lambda\Gamma\varphi) \in D(B)$ which satisfy

$$B(\varphi + \lambda\Gamma\varphi) = h(x)\varphi \ ,$$

a) φ vanishes only at a finite number of points of $[0, 1]$.

b) $< \Gamma\varphi|\varphi >_{D(B^{1/2})} > 0$.

Example of Equation (1.3) : *Notice that Property (UCP)a) for Equation (1.3) corresponds to the following well-known fact. If* $\varphi \in \mathbb{H}^1_0(]0, 1[) \setminus \{0\}$ *satisfies*

$$\varphi_{xx}(x) = (\lambda\gamma(x) - h(x))\varphi(x) \ ,$$

then φ *vanishes only at a finite number of points of* $[0, 1]$. *In particular,*

$$< \Gamma\varphi|\varphi >_{D(B^{1/2})} = \int_0^1 \gamma(x)|\varphi(x)|^2 > 0 \ ,$$

and thus Hypothesis (UCP)b) obviously holds.

From the assumptions (Spec) and (UCP), we deduce the exponential decay property of the linear semigroup e^{At}. We deduce as well that the semigroup e^{At} is in fact a group.

Proposition 2.4. *Under the assumptions (B), (Gam), (Spec) and (UCP), the linear semigroup* e^{At} *is decreasing with an exponential decay rate. More precisely, there exist* $\delta^1_d > \delta^2_d > 0$ *and two positive constants* K^1_d *and* K^2_d, *such that for any* $U_0 \in X$ *and* $t > 0$,

$$K^1_d \ e^{-\delta^1_d t} \ \|U_0\|_X \ \leq \ \|e^{At}U_0\|_X \ \leq \ K^2_d \ e^{-\delta^2_d t} \ \|U_0\|_X \ . \tag{2.10}$$

Besides, the $\mathcal{C}^0-$*semigroup* $(e^{At})_{t\in\mathbb{R}_+}$ *can be extended to a* $\mathcal{C}^0-$*group* $(e^{At})_{t\in\mathbb{R}}$. *In the same way, the wave equation (2.4), the linearized equation (2.5) and the adjoint equation (2.7) generate* $\mathcal{C}^0-$*groups and thus satisfy the backward uniqueness property.*

Proof : Since we assumed that there exists a Riesz basis $(V^n_{j,k})$ composed by rootvectors of A, we can write

$$U_0 = \sum_{n,j,k} \alpha_{n,j,k} V^n_{j,k} \ ,$$

where the series converges normally in X. As

$$e^{tA} V^n_{j,k} = (V^n_{j,k} + t \ V^n_{j,k-1} + ... + t^k \ V^n_{j,0})e^{\lambda_n t} \ ,$$

we have that

$$e^{tA}U_0 = \sum_{n,j,k} \alpha_{n,j,k}(V^n_{j,k} + t\ V^n_{j,k-1} + ... + t^k\ V^n_{j,0})e^{\lambda_n t}\ .$$

By assumption (Spec)b), there is only a finite number of eigenvalues which are not simple, so the multiplicity of an eigenvalue is uniformly bounded and the polynomial terms do not matter. Since we have the equivalence of norms (2.8), to conclude, we have to show that there exist $\delta^1_d > \delta^2_d > 0$ such that all the eigenvalues λ_n of A are in the strip

$$-\delta^1_d < \mathrm{Re}(\lambda_n) < -\delta^2_d < 0.$$

As Property (Spec)b) holds, it is sufficient to prove that all the eigenvalues have negative real part. We proved in Proposition 2.2 that A is nonpositive and so its eigenvalues have nonpositive real part. If $i\lambda \in i\mathbb{R}$ is an eigenvalue of A with eigenvector $(\varphi, i\lambda\varphi)$ $(\varphi \neq 0)$, then

$$\varphi + i\lambda\Gamma\varphi - \lambda^2 B^{-1}\varphi = 0,$$

and

$$\|\varphi\|^2_{D(B^{1/2})} + i\lambda < \Gamma\varphi|\varphi >_{D(B^{1/2})} - \lambda^2\|\varphi\|^2_{\mathbb{L}^2} = 0\ .$$

Due to the assumption (UCP)b), $< \Gamma\varphi|\varphi >_{D(B^{1/2})} \neq 0$, which implies that $\lambda = 0$, and thus $\varphi = 0$. This proves that all the eigenvalues have negative real part.
To show that e^{At} can be extended to a group, we formally define e^{-At}. Let

$$U_0 = \sum_{n,j,k} \alpha_{n,j,k} V^n_{j,k}\ .$$

We denote by $\beta^t_{n,j,0}$,..., $\beta^t_{n,j,m_{n,j}-1}$, the solutions of the system

$$\begin{pmatrix} 1 & t & \dots & t^{m_{n,j}-1} \\ 0 & 1 & \dots & t^{m_{n,j}-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta^t_{n,j,0} \\ \beta^t_{n,j,1} \\ \vdots \\ \beta^t_{n,j,m_{n,j}-1} \end{pmatrix} = \begin{pmatrix} \alpha_{n,j,0} \\ \alpha_{n,j,1} \\ \vdots \\ \alpha_{n,j,m_{n,j}-1} \end{pmatrix}\ .$$

Notice that the above system is well-defined for all time $t \geq 0$. We set

$$e^{-At}U_0 = \sum_n e^{-\lambda_n t} \left(\sum_{j,k} \beta^t_{n,j,k} V^n_{j,k} \right)\ .$$

The property (2.8), together with the fact that the real part and the multiplicity of the eigenvalues are bounded, imply that the operator e^{-At} is linear continuous from X into X, and it continuously depends on t. Besides, $e^{-At} = (e^{At})^{-1}$ by construction. This shows that e^{At} is a group (see [33]). $\square$

The following proposition shows that the exponential decay of e^{At} implies that the dynamical system $S(t)$ generated by Equation (2.4) is asymptotically smooth. We recall that it means that any bounded positively invariant set $\mathcal{B}$ of X is attracted by a compact set $K(\mathcal{B}) \subset \overline{\mathcal{B}}$ (see [17]).

Proposition 2.5. *The dynamical system $S(t)$ is asymptotically smooth.*

Proof : Let f be a function of $\mathfrak{G}^k$ and (u_0, v_0) be initial data in X. We denote $(u(t), u_t(t))$ the trajectory $S(t)(u_0, v_0)$. By definition of a mild solution, we have

$$S(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = e^{tA}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{(t-s)A}\begin{pmatrix} 0 \\ f(x, u(t)) \end{pmatrix} ds.$$

The linear term satisfies $\|e^{tA}\|_{\mathcal{L}(X)} \leq K_d^2 e^{-\delta_d^2 t}$. Moreover, the mapping $F : (u, v) \longmapsto (0, f(x, u))$, defined from X to X, is completely continuous in the sense of [17]. Indeed, if (u, v) belongs to a bounded set $\mathcal{B} \subset X$ then $f(x, u)$ belongs to a bounded set of $\mathbb{H}^1(]0, 1[)$ and thus $F(\mathcal{B})$ is a precompact set of X. Applying Theorem 4.6.1 of [17] yields that $S(t)$ is asymptotically smooth. $\square$

2.3 The main theorem

As the backward uniqueness property is satisfied by the linearized equation (2.5) and the adjoint equation (2.7), the stable and unstable sets of the hyperbolic equilibria of the equation (2.4) are imbedded manifolds in X (see [22]), so wondering if their intersection is transversal or not has a sense. We recall that (2.4) is said to have the Kupka-Smale property, if all its equilibria are hyperbolic and all its stable and unstable manifolds intersect transversally. We also recall that for $k \geq 1$, $\mathfrak{G}^k$ is the set of the functions $f \in \mathcal{C}^k([0, 1] \times \mathbb{R}, \mathbb{R})$ endowed with the Whitney topology, that is, the topology generated by the sets

$$\{g \in \mathfrak{G}^k \ / \ |D^i f(x, u) - D^i g(x, u)| < \delta(u), i = 0, 1, ..., k, x \in [0, 1], u \in \mathbb{R}\},$$

where δ is any positive function on $\mathbb{R}$.
We are now able to state our main theorem.

Theorem 2.6. *Let f be a function of $\mathfrak{G}^k$. We assume that the operator A defined by (2.2) satisfies all the above properties (B), (Gam), (Spec) and (UCP). We assume in addition that the following properties hold :*

(Grad) *The Equation (2.4) is a gradient system.*

(Loc) *For any function $\varphi \in D(B^{1/2})$, the scalar product $< \Gamma\varphi|\varphi >_{D(B^{1/2})}$ depends only on the values of φ^2.*

If $k \geq 2$, then the set $\mathfrak{G}^{KS}$ of all the functions $f \in \mathfrak{G}^k$ such that (2.4) has the Kupka-Smale property is a generic subset of $\mathfrak{G}^k$.

If f satisfies an additional dissipative condition, Equation (2.4) has a compact global attractor. In this case, if (2.4) is Kupka-Smale, then it is Morse-Smale; that is, (2.4) has a finite number of equilibria which are all hyperbolic and all its stable and unstable manifolds intersect transversally. Let $\mathfrak{G}^k_{\text{diss}}$ be the open subset of $\mathfrak{G}^k$ defined by

$$\mathfrak{G}^k_{\text{diss}} = \{f \in \mathfrak{G}^k \ / \ \limsup_{u \to \pm\infty} \frac{f(x,u)}{u} < 0\} \ .$$

When f belongs to $\mathfrak{G}^k_{\text{diss}}$, the above theorem can be improved as follows.

Corollary 2.7. *We assume that all the hypotheses of Theorem 2.6 are satisfied. If f belongs to $\mathfrak{G}^k_{diss}$, then Equation (2.4) admits a compact global attractor. As a consequence, the set of non-linearities $f \in \mathfrak{G}^k_{diss}$ such that (2.4) has the Morse-Smale property is generic in $\mathfrak{G}^k_{diss}$.*

Proof : We only have to check that, if f belongs to $\mathfrak{G}^k_{\text{diss}}$, then Equation (2.4) admits a compact global attractor. We follow the proof given in [19]. We have already shown that (2.4) generates an asymptotically smooth system $S(t)$. Next, we have to prove that $S(t)$ is point-dissipative, that is that there exists a bounded set which attracts each point of X. As f belongs to $\mathfrak{G}^k_{\text{diss}}$, there exist two positive constants α and C such that

$$f(x,u)u \leq C - \alpha u^2 \ .$$

This implies that the equilibrium points $(e, 0)$ of Equation (2.4) are uniformly bounded, since

$$\|(e,0)\|^2_X = \|e\|^2_{D(B^{1/2})} = \int (Be)(x)e(x)dx = \int f(x,e(x))e(x)dx \leq C \ .$$

The asymptotically smooth system $S(t)$ is assumed to be gradient, so all the points of X are attracted by the equilibria, which belong to a bounded set. Thus, $S(t)$ is point-dissipative. Finally, we have to verify that the trajectories of the bounded sets of X are bounded. More precisely, let $\mathcal{B}$ be a bounded set of X, we set $\gamma_+(\mathcal{B}) = \cup_{t\geq 0} S(t)\mathcal{B}$. To obtain the existence of a compact global attractor, it remains to check that $\gamma_+(\mathcal{B})$ is bounded. We define the functional Φ from X into $\mathbb{R}$ as follows.

$$\Phi((u_0,v_0)) = \frac{1}{2}\|(u_0,v_0)\|^2_X - \int_0^1 F(x,(u_0,v_0))dx \ ,$$

where

$$F(x,(u_0,v_0)) = \int_0^{u_0} f(x,\zeta)d\zeta \ .$$

If $(u_0,v_0) \in D(A)$, then $(u,v)(t) = S(t)(u_0,v_0)$ belongs to $\mathcal{C}^0(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, X)$, and Φ is non-increasing along this trajectory since we can write

$$\frac{\partial}{\partial t}\Phi((u,v)(t)) = - < \Gamma v | v >_{D(B^{1/2})} \leq 0 \ .$$

As $S(t)$ and Φ are continuous on X, and that $D(A)$ is dense in X, we deduce that Φ is non-increasing along all the trajectories of $S(t)$. In fact, Φ is a strict Lyapunov function in the concrete examples. The first components u_0 of the elements (u_0,v_0) of $\mathcal{B}$ are uniformly bounded in $\mathbb{H}^1(]0,1[)$, so Φ is also bounded on $\mathcal{B}$ by a constant $C(\mathcal{B})$. As f belongs to $\mathfrak{G}^k_{\text{diss}}$, there exists two positive constants $\tilde{\alpha}$ and $\tilde{C}$ such that

$$F(x,u) \leq \tilde{C} - \tilde{\alpha} u^2 \ .$$

This implies that

$$\frac{1}{2}\|(u,v)\|^2_X - \tilde{C} \leq \Phi((u,v)) \leq \Phi((u_0,v_0)) \leq C(\mathcal{B}) \ .$$

This shows that $\gamma_+(\mathcal{B})$ is bounded in X. □

Proof of Theorem 1.1 and Corollary 1.2 : As already indicated, Theorem 1.1 and Corollary 1.2 are direct consequences of Theorem 2.6 and Corollary 2.7. While introducing the hypotheses (B), (Gam), (Spec) and (UCP), we have checked that Equation (1.3) satisfies all these conditions. Moreover, it is known that (1.3) is a gradient system (see for example [19]). In addition, the property (Loc) is obviously satisfied as

$$\forall \varphi \in D(B^{1/2}), \ < \Gamma\varphi | \varphi >_{D(B^{1/2})} = \int_0^1 \gamma(x) |\varphi(x)|^2 dx \ .$$

So we can apply Theorem 2.6 and Corollary 2.7. □

3 Other examples of applications

In this section, we apply the abstract theorem to the case of the boundary damping and thus prove Theorem 1.3. We give also another example, which illustrates the case of operators of higher order and which is interesting in the sense that we will need to generalize Theorem 2.6. Of course, these examples are not exhaustive. In particular, other boundary conditions can be taken in Equation (1.3). We also notice that Hypothesis (Spec) has been proved for many other one-dimensional equations.
In the last subsection, we enhance which properties are still true for equations which do not satisfy all the hypotheses of Theorem 2.6.

3.1 The wave equation with damping on the boundary

Like the internal damped wave equation (1.3), the equation with boundary damping (1.4) has also attracted much attention (see for example [9], [12], [26], [27], [28], [37] and [39]).
Theorem 1.3 is a direct consequence of Theorem 2.6 and Corollary 2.7.

Proof of Theorem 1.3 : We can assume without loss of generality that $\gamma(0) \neq 0$ (the case $\gamma(0) = 0$ and $\gamma(1) \neq 0$ is similar). Equation (1.4) can be written in the frame of (2.4) with A defined by (2.2). Indeed, we set

$X = \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[)$, $B = -\partial^2_{xx} + Id$, $D(B) = \{u \in \mathbb{H}^2(]0,1[) \ / \ u_x(0) = u_x(1) = 0\}$,

and, for any $v \in \mathbb{H}^1(]0,1[)$, we denote by Γv the solution in $\mathbb{H}^2(]0,1[)$ of

$$\left\{ \begin{array}{ll} (\partial^2_{xx} - Id)(\Gamma v)(x) = 0 & x \in]0,1[\\ \frac{\partial}{\partial \nu}(\Gamma v)(x) = \gamma(x) v(x) & x \in \{0,1\} \end{array} \right. .$$

We recall that we equipp the space $X = D(B^{1/2}) \times \mathbb{L}^2(]0,1[)$ with the inner product defined by (2.1). For any $v \in \mathbb{H}^1(]0,1[)$, we have

$$< \Gamma v | v >_{D(B^{1/2})} = \gamma(0)|v(0)|^2 + \gamma(1)|v(1)|^2 \ . \tag{3.1}$$

Thus, the operators B and Γ clearly satisfy Hypotheses (B), (Gam) and (Loc). Following Cox and Zuazua (see [12]), we prove that Hypothesis (Spec) holds. Let

$$\omega = \left(\frac{\gamma(0) + 1}{\gamma(0) - 1} \right) \left(\frac{\gamma(1) + 1}{\gamma(1) - 1} \right) \ .$$

We have the following high-frequency estimate

$$\lambda_{\pm n} = -\ln|\omega| + \left\{ \begin{array}{ll} \pm i n \pi + O(\frac{1}{n}) & \text{if } \omega > 0 \\ \pm i (n + \frac{1}{2}) \pi + O(\frac{1}{n}) & \text{if } \omega < 0 \end{array} \right. .$$

Hypothesis (UCP)a) is a well-known unique continuation property (see [31]). To show the property (UCP)b), we assume that $\lambda \in \mathbb{C}$, $h \in \mathbb{L}^\infty$, and $\varphi \neq 0$ is such that $(\varphi + \lambda\Gamma\varphi) \in D(B)$ and

$$\left\{ \begin{array}{l} B(\varphi + \lambda\Gamma\varphi) = h(x)\varphi \\ < \Gamma\varphi|\varphi >_{D(B^{1/2})} = 0 \end{array} \right. \quad \text{that is} \quad \left\{ \begin{array}{l} -\varphi_{xx}(x) = h(x)\varphi(x), \ x \in]0,1[\\ \frac{\partial\varphi}{\partial\nu}(x) = -\lambda\gamma(x)\varphi(x), \ x = 0,1 \\ \gamma(0)|\varphi(0)|^2 + \gamma(1)|\varphi(1)|^2 = 0 \end{array} \right. .$$

Thus, φ satisfies both Neumann and Dirichlet conditions at the end point x=0. Simply using Cauchy-Lipschitz Theorem, we find that $\varphi = 0$, that is that Hypothesis (UCP)b) is satisfied.
Finally, it remains to show that (2.4) generates a gradient system. Although the proof is classical (see [27]), we quickly recall it. The Lyapounov function associated to (2.4) is given by

$$\Phi((u_0, v_0)) = \frac{1}{2}\|(u_0, v_0)\|_X^2 - \int_0^1 F(x, (u_0, v_0))dx ,$$

where

$$F(x, (u_0, v_0)) = \int_0^{u_0} f(x, \zeta)d\zeta .$$

Along a trajectory $(u, v) = S(t)(u_0, v_0)$ of Equation (1.4), we have

$$\frac{\partial}{\partial t}\Phi((u, v)(t)) = - < \Gamma v|v >_{D(B^{1/2})} \leq 0.$$

Moreover, if the trajectory is such that $\frac{\partial}{\partial t}\Phi((u, v)(t)) = 0$ for $t \in [0, T]$, then v satisfies $v(0) = \frac{\partial}{\partial x}v(0) = 0$. We set $v_0(x) = v(x, 0)$ and $v_T(x) = v(x, T)$. If we reverse the role of time and space, v is a solution of

$$\left\{ \begin{array}{ll} \frac{\partial^2}{\partial x^2}v(x,t) = (\frac{\partial^2}{\partial t^2} + Id - f'_u(x, u(x,t)))v(x,t) & 0 < x < 1, \ t \in]0, T[\\ v(x, 0) = v_0(x), \ v(x, T) = v_T(x) & 0 < x < 1 \\ v(x = 0, t) = \frac{\partial}{\partial x}v(x = 0, t) = 0 & t \in]0, T[\ . \end{array} \right.$$

The uniqueness of the solutions of the wave equation gives that $v(x, t) = u_t(x, t) = 0$ for $x \in]0, 1[$ and $t \in]0, T[$. This implies that Φ is a strict Lyapounov function, and so that (2.4) generates a gradient system. $\square$

Remarks : • When $\gamma(0) = 1$ or $\gamma(1) = 1$, the spectrum of the linear operator defined in Equation (1.4) by setting $f = 0$ is empty. Moreover, it is well-known that, in this case, all solutions of the linear equation vanish in finite time, that is that Equation (1.4) does not satisfy the backward uniqueness property. Thus, we cannot ensure that the stable and unstable manifolds are immersed manifolds as they may have self-intersections. In this

case, we are not able even to define the notion of transversality of these manifolds.
• In [12], Cox and Zuazua considered Equation (1.4) with $\rho(x)^2 u_{tt}$ instead of u_{tt}, where ρ is a measurable function with bounded variation and satisfies $0 < \alpha \leq \rho \leq \beta < \infty$. We can apply Theorem 2.6 to this case if we notice that our theorem is also valid when $\mathbb{L}^2(]0,1[)$ is replaced by $\mathbb{L}^2_\rho(]0,1[)$, that is the space $\mathbb{L}^2$ endowed with the equivalent norm $\|f\|^2_\rho = \int \rho^2 |f|^2$.

3.2 A beam equation with joint feedback control

In Hypothesis (Spec)b), we assumed that there exists a constant C_{ev} such that the eigenvalues of A satisfy

$$\mathrm{Re}\lambda_n \longrightarrow -C_{ev} \ .$$

Obviously, a careful look at the proof of Theorem 2.6 shows that it is also valid if we assume that the sequence $(\mathrm{Re}(\lambda_n))_{n\in\mathbb{Z}}$ has a finite number of accumulation points. The following example, described in [16], illustrates this slight generalization.
Let f be a function in $\mathfrak{G}^k$, γ and K be two positive constants and let d be a point of $]0,1[$. We study in

$$X = (\mathbb{H}^2(]0,1[) \cap \mathbb{H}^1_0(]0,1[)) \times \mathbb{L}^2(]0,1[)$$

the equation

$$\begin{cases} u_{tt}(x,t) + \gamma u_t(x,t) = -u_{xxxx}(x,t) + f(x,u) & 0 < x < d, \ d < x < 1, \ t > 0 \\ u(x,t) = u_{xx}(x,t) = 0 & x = 0,1 \ , \ t > 0 \\ \partial^k_x u(d_+,t) = \partial^k_x u(d_-,t) & k = 0,1,2 \ , \ t > 0 \\ u_{xxx}(d_-,t) - u_{xxx}(d_+,t) = K u_t(d,t) & t > 0 \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) \in X & x \in [0,1] \end{cases} \tag{3.2}$$

where, if h is a piecewise continuous function, we denote

$$h(d^+) = \lim_{x\to d, \ x>d} h(x) \ \text{ and } \ h(d^-) = \lim_{x\to d, \ x<d} h(x) \ .$$

Theorem 3.1. *If d is a rational number, then the set of functions $f \in \mathfrak{G}^k$, such that Equation (3.2) has the Kupka-Smale property, is a generic subset of $\mathfrak{G}^k$.*

Proof : Equation (3.2) can be written in the frame of (2.4) with A defined by (2.2). Indeed, we set

$$B = \Delta^2, \quad D(B) = \{u \in \mathbb{H}^4(]0,1[) \ / \ u(0) = u(1) = u_{xx}(0) = u_{xx}(1) = 0\} \ ,$$

and

$$\Gamma : \begin{pmatrix} \mathbb{H}^2(]0,1[) \cap \mathbb{H}^1_0(]0,1[) & \longrightarrow & \mathbb{H}^2(]0,1[) \cap \mathbb{H}^1_0(]0,1[) \\ v & \longmapsto & \gamma B^{-1} v + \kappa(v(d)) \end{pmatrix} \ ,$$

where, for any $\beta \in \mathbb{R}$, $\kappa = \kappa(\beta)$ is the solution of

$$\left\{\begin{array}{ll} \kappa_{xxxx}(x) = 0 & 0 < x < d \text{ and } d < x < 1 \\ \kappa(x) = \kappa_{xx}(x) = 0 & x = 0, 1 \\ \partial_x^k \kappa(d_-) = \partial_x^k \kappa(d_+) & k = 0, 1, 2 \\ \kappa_{xxx}(d_-) - \kappa_{xxx}(d_+) = -K\beta & \end{array}\right. .$$

For any $v \in \mathbb{H}^2(]0,1[) \cap \mathbb{H}_0^1(]0,1[)$, an easy computation gives

$$< \Gamma v | v >_{D(B^{1/2})} = K |v(d)|^2 + \gamma \int_0^1 |v(x)|^2 \, dx \; .$$

Thus, the operators B and Γ clearly satisfy Hypotheses (B), (Gam) and (Loc). In [16], the constant γ was taken to be zero and the system (3.2) was not gradient. For this reason, we add a dissipative term γu_t, which gives a gradient structure to (3.2) and implies Property (UCP). Finally, by adapting the proofs of [16], we find that the rootvectors of A form a Riesz basis of X and that the eigenvalues of A satisfy

$$\lambda_{\pm n} = -\frac{\gamma}{2} \pm i(n\pi)^2 - K \sin^2(nd\pi) + O(\frac{1}{n}) \; .$$

If d belongs to $\mathbb{Q}$, the real parts of the eigenvalues have only a finite number of accumulation points, and so we have constructed an example satisfying a generalized condition (Spec). $\square$

3.3 Equations which satisfy only part of the hypotheses

Some examples of damped wave equations do not satisfy all the hypotheses of Theorem 2.6. Although we cannot prove the generic Kupka-Smale property, some of the propositions proved here are still valid. The hypotheses, which generally fail to be satisfied are (Grad), (UCP)b), (Spec)b) and (Spec)c). We point out that the assumptions (Spec)b) and (Spec)c) are crucial in the last steps of the proof of Theorem 2.6. However, such precise asymptotic estimates of the spectrum of A are not needed to prove Proposition 2.4. Actually, in the proof of Proposition 2.4, we only used the hypothesis (Spec)a) and the following property

(Spec') All the eigenvalues of A belong to a strip

$$\{z \in \mathbb{C} \; / \; -\beta < \mathrm{Re}(z) < -\alpha < 0\} \; ,$$

where α and β are two positive constants.

In this proof, the assumptions (Spec)b), (Spec)c) and (UCP)b) have been used only to show that (Spec') holds. Likewise, other interesting properties do not require these assumptions.

Theorem 3.2. *Let A be the operator defined by (2.2). Under the hypotheses (B), (Gam), (Spec)a), (Spec') and (UCP)a), the conclusions of Proposition 2.4, Proposition 2.5, Theorem 4.3 and Proposition 5.2 are still true.*

One application of this theorem is the beam equation with non-constant damping. In [29], Li, Yu, Liang and Zhu proved that the equation

$$\left\{ \begin{array}{l} \left(\frac{\partial^2}{\partial t^2} + \gamma(x)\frac{\partial}{\partial t} + \frac{\partial^4}{\partial x^4}\right) u(x,t) = f(x,u) \ , \quad (x,t) \in]0,1[\times\mathbb{R}_+ \\ u(0,t) = \frac{\partial}{\partial x}u(0,t) = \frac{\partial^2}{\partial x^2}u(1,t) = \frac{\partial^3}{\partial x^3}u(1,t) = 0 \end{array} \right.$$

satisfies Hypotheses (Spec)a) and (Spec'), if an appropriate positivity condition on γ holds. They also enhanced that (Spec)b) is still not known in this case. As a consequence, Theorem 3.2 is valid for the beam equation with non-constant damping.
We end this section by remarking that there are other cases of equations, which do not exactly fit in our frame; however, mimicing our proofs, we can show that the properties given in Theorem 3.2 are still true. This is for example the case for the beam equations with boundary damping, where one needs an additional equation on the boundary in order to describe the system (see [10]).

4 Proof of the main theorem : generic spectral properties

Let $(e,0)$ be an equilibrium of Equation (2.4), that is a solution $e \in D(B)$ of the equation $Be = f(x,e)$. The linearized operator at the equilibrium $(e,0)$ is

$$A_e \begin{pmatrix} u \\ v \end{pmatrix} = \left(A + \begin{pmatrix} 0 & 0 \\ f'_u(x,e) & 0 \end{pmatrix}\right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -B(u+\Gamma v) + f'_u(x,e)u \end{pmatrix} , \qquad (4.1)$$

with $D(A_e) = D(A)$. Notice that $(e,0) \in X$ implies that e belongs to $\mathbb{H}^1(]0,1[)$ and so $f'_u(x,e)$ is in $\mathcal{C}^0(]0,1[)$.

In this section, we study generic properties of the spectrum of the linearized operator A_e. We need generic properties with respect to the function f, such as hyperbolicity of the equilibria $(e,0)$ or simplicity of the eigenvalues of A_e. We point out that these properties

deserve a separate section as they do not only play an important role in the proof of Theorem 2.6, but have also their own interest. Their proofs, which mainly consist in applying Sard-Smale Theorem (Theorem A.2), need almost nothing else as the facts that the eigenfunctions of A_e have some smoothness and satisfy a unique continuation property such as (UCP)a). Thus, we would be able to prove the generic hyperbolicity of the equilibria and the generic simplicity of the eigenvalue for a larger class of operators. But, as this is not the main purpose of this paper, we keep most of the hypotheses of Theorem 2.6.
We only want to enhance that all the properties proved in this section are not specific to the one-dimensional case. For this reason, we do not assume Hypothesis (Spec), which seems to be an one-dimensional property. Instead, we will assume that A satisfies the following exponential decay property, which is often true in higher dimension.

(ED) There exist two positive constants δ_d and K_d such that, for all $U \in X$, and all $t > 0$,

$$\|e^{At}U\|_X \;\leq\; K_d e^{-\delta_d t}\|U\|_X \;.$$

Remarks : • We assume Property (ED) to ensure the hyperbolicity of the equilibria. If Hypothesis (ED) does not hold, we can say that, generically with respect to f, the equilibria and the eigenvalues are simple. But, (ED) is needed to say that the equilibria are not only simple, but also hyperbolic.
• We will only use one property related to the one-dimensional case, which is the fact that $\mathbb{H}^1(]0,1[)$ is continuously imbedded into $\mathcal{C}^0(]0,1[)$. For the higher dimensional case, replacing our space $X = D(B^{1/2}) \times \mathbb{L}^2(]0,1[)$ by an adequate subspace of $D(A^n)$ (n large enough), so that $D(A^n)$ is continuously imbedded in $\mathcal{C}^0(]0,1[)\times\mathbb{L}^2(]0,1[)$, we prove mutatis mutandis the same generic results. The only problem is that $(0, f(x,u))$ has to be in $D(A^n)$, and so we must be more careful and show that the perturbations of $(0, f(x,u))$ are still in $D(A^n)$. We refer to [6] where the cases of dimensions two and three are considered.

First, notice that an eigenvector of A_e corresponding to the eigenvalue λ is of the form $(\varphi, \lambda\varphi) \in D(A)$ and satisfies

$$-B(\varphi + \lambda\Gamma\varphi) + f'_u(x,e)\varphi = \lambda^2\varphi.$$

We will have to study functionals depending on φ and λ, so the last formulation is not very convenient as we must have $(\varphi + \lambda\Gamma\varphi) \in D(B)$, that is that φ belongs to a space which depends on the parameter λ. That is why we will use the equivalent formulation : $\varphi \in D(B^{1/2})$ and

$$\varphi + \lambda\Gamma\varphi = B^{-1}\left[(f'_u(x,e) - \lambda^2)\varphi\right],$$

which is much easier to handle.

Proposition 4.1. *Let A be the operator defined by (2.2), satisfying Properties (B), (Gam) and (UCP) defined in Section 2. If $(e,0)$ is an equilibrium of (2.4), and $f \in \mathfrak{G}^k$, then the eigenvalues of A_e with non-negative real part are all real.*

Proof : As e is in $\mathbb{L}^\infty(]0,1[)$, $f'_u(x,e(x))$ belongs to $\mathbb{L}^\infty(]0,1[)$. Let $(\varphi,\lambda\varphi)$ be an eigenfunction of A_e that is

$$-B(\varphi+\lambda\Gamma\varphi)+f'_u(x,e)\varphi=\lambda^2\varphi.$$

By multiplying this equality by $\overline{\varphi}$ and integrating, we find

$$-\|\varphi\|^2_{D(B^{1/2})}+\int_0^1 f'_u(x,e)|\varphi|^2=\lambda^2\|\varphi\|^2_{\mathbb{L}^2}+\lambda<\Gamma\varphi|\varphi>_{D(B^{1/2})}.$$

If λ is not real, by taking the imaginary part of the above equality, we obtain

$$\mathrm{Re}(\lambda)\|\varphi\|^2_{\mathbb{L}^2}=-\frac{1}{2}<\Gamma\varphi|\varphi>_{D(B^{1/2})}.$$

We assumed in (UCP)b) that Γ is strictly positive on the eigenfunctions, which implies that if λ is not real, $\mathrm{Re}(\lambda)<0$. $\square$

In what follows, we will denote f'_u for $f'_u(x,e)$ when no confusion is possible.
Our last preliminary concerns the algebraic simplicity of the eigenvalues. Assume that there is an element $(u,v)\in D(A)$ such that

$$(A_e-\lambda Id)\begin{pmatrix} u\\ v\end{pmatrix}=\begin{pmatrix}\varphi\\ \lambda\varphi\end{pmatrix},\ \text{that is}\ \begin{cases} v-\lambda u=\varphi\\ -B(u+\Gamma v)+f'_u u-\lambda v=\lambda\varphi\end{cases}$$

So $v=\lambda u+\varphi$ and

$$u+\lambda\Gamma u+B^{-1}(\lambda^2u-f'_u u)=-(\Gamma\varphi+2\lambda B^{-1}\varphi).$$

We deduce that the algebraic multiplicity of λ is higher than one if and only if there exists an eigenvector $(\varphi,\lambda\varphi)$ with

$$\Gamma\varphi+2\lambda B^{-1}\varphi\in R\left(Id+\lambda\Gamma.+B^{-1}\left[(\lambda^2-f'_u).\right]\right) \tag{4.2}$$

4.1 Genericity of the hyperbolicity

We recall that an equilibrium $(e,0)$ of the wave equation (2.4) is hyperbolic if the spectrum of e^{A_e} does not intersect the unit circle of $\mathbb{C}$.
We recall also that $\mathfrak{G}^k$ is the set of the functions $f\in\mathcal{C}^k([0,1]\times\mathbb{R},\mathbb{R})$ (with $k\geq 2$) endowed with the Whitney topology defined in Section 2.

Theorem 4.2. *Let A be the operator defined by (2.2), satisfying Properties (B), (Gam) and (UCP) defined in Section 2, and the exponential decay property (ED).*
Let $\mathfrak{G}^H$ be the set of all functions $f\in\mathfrak{G}^k$, such that all the equilibria of the wave equation (2.4) are hyperbolic. Then $\mathfrak{G}^H$ is a generic subset of $\mathfrak{G}^k$.

Proof : We assumed that A satisfies the exponential decay property (ED), which means that the radius of the spectrum of e^A in strictly less than one. Let U_0 be an element of X, and $U(t) = e^{At}U_0$. We have

$$e^{A_e}U = e^AU + \int_0^1 e^{A(1-t)}(0, f'_u(x,e)u(t))dt \ .$$

As $f'_u(x,e)u(s)$ is bounded in $\mathbb{H}^1(]0,1[)$, which is compactly imbedded in $\mathbb{L}^2(]0,1[)$, the operator e^{A_e} is a compact perturbation of e^A, and so the radius of its essential spectrum is the same as the one of e^A, thus it is strictly less than one.
Hence, concerning the hyperbolicity, we are reduced to consider the point spectrum. To show that no eigenvalue of e^{A_e} belongs to the unit circle, we have to prove that no eigenvalue of A_e belongs to the imaginary axis. By Proposition 4.1, the only possible eigenvalue of A_e on the imaginary axis is 0. In the case where 0 is an eigenvalue, there exists a function $\varphi \in D(B^{1/2})$ such that $\varphi = B^{-1}(f'_u\varphi)$. Notice that, as $Id - B^{-1}(f'_u.)$ is a Fredholm operator of index 0, its injectivity is equivalent to its surjectivity.
The following proof can be found in [5]. We just give it in our frame for sake of completeness. Let $\mathfrak{G}^H_n$ be the set of all functions $f \in \mathfrak{G}^k$ such that all equilibria $(e,0)$ with $\|e\|_{\mathbb{L}^\infty} \leq n$ are hyperbolic. We only have to prove that $\mathfrak{G}^H_n$ is an dense open subset of $\mathfrak{G}^k$, as $\mathfrak{G}^H = \cap_n \mathfrak{G}^H_n$.
First, $\mathfrak{G}^H_n$ is open. Indeed, if (f_k) is a sequence of functions of $\mathfrak{G}^k \setminus \mathfrak{G}^H_n$ which converges to some $f \in \mathfrak{G}^k$, then we have a sequence of equilibria $(e_k, 0)$ with $\|e_k\|_{\mathbb{L}^\infty} \leq n$, and a sequence of functions $\varphi_k \in D(B^{1/2})$ with $\|\varphi_k\|_{\mathbb{L}^2} = 1$, such that

$$e_k = B^{-1}f_k(x, e_k(x))$$

$$\varphi_k = B^{-1}((f_k)'_u(x, e_k(x))\varphi_k).$$

As $(e_k)_k$ is bounded in $\mathbb{L}^\infty$, the sequence $(f_k(x, e_k(x)))_k$ is bounded in $\mathbb{L}^\infty$ too and thus (e_k) is bounded in $D(B^{1/2})$. Since $D(B^{1/2})$ is compactly imbedded in $\mathbb{L}^\infty$, by extracting a subsequence, we can assume that $e_k \longrightarrow e$ in $\mathbb{L}^\infty$. Using the equation once more, we find that $e_k \longrightarrow e$ in $D(B^{1/2})$. The same argument shows that $\varphi_k \longrightarrow \varphi$ in $D(B^{1/2})$. Continuity arguments imply that $(e,0)$ is an equilibrium of Equation (2.4) corresponding to the nonlinearity f, that $\|e\|_{\mathbb{L}^\infty} \leq n$, and that $\varphi = B^{-1}((f'_u)(x, e(x))\varphi)$. This means that $f \notin \mathfrak{G}^H_n$.
We now have to prove the density of $\mathfrak{G}^H_n$. Let f be a function of $\mathfrak{G}^k$, let $\eta : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth compactly supported function which is equal to 1 on $[-n-1; n+1]$. It is sufficient to show that there exists a dense set of functions $b(x) \in \mathcal{C}^k$ such that $f(x,u) + b(x)\eta(u)$ is in $\mathfrak{G}^H_n$. We apply the Sard-Smale Theorem (see Theorem A.2) with

$$U = \{e \in D(B^{1/2}) \ / \ \|e\|_{\mathbb{L}^\infty} < n+1\}, \quad V = \mathcal{C}^k([0,1]), \quad Z = D(B^{1/2}),$$

and

$$\Phi(e,b)(x) = e - B^{-1}(f(x,e) + b(x)\eta(e)) = e - B^{-1}(f(x,e) + b(x)).$$

Let $z = 0$. If $(e, b) \in \Phi^{-1}(z)$, then $(e, 0)$ is an equilibrium, moreover, as we have noticed before, the surjectivity of $D_e\Phi(e, b) = Id - B^{-1}(f'_u(x, e).)$ is equivalent to the hyperbolicity of $(e, 0)$. That is why, if the three hypotheses of Theorem A.2 are satisfied, $\mathfrak{G}^H_n$ will be a generic, and a fortiori a dense subset of $\mathfrak{G}^k$.
We next verify that the hypotheses of Theorem A.2 hold. The space V is obviously separable and $D(B^{1/2})$ is separable since it is the image of the separable space $\mathbb{L}^2(]0, 1[)$ by the bounded operator $B^{-1/2}$. By Lemma 2.1, the operator $D_e\Phi = Id - B^{-1}(f'_u(x, e).)$ is a Fredholm operator of index 0 from $D(B^{1/2})$ into itself. It remains to check that, if $(e, b) \in \Phi^{-1}(0)$, $D\Phi(e, b)$ is a surjective map from $D(B^{1/2}) \times V$ into $D(B^{1/2})$; that is, for any $g \in D(B^{1/2})$, we must find $(\varphi, c) \in D(B^{1/2}) \times V$ such that

$$D\Phi(e, b).(\varphi, c) = \varphi - B^{-1}(f'_u(., e)\varphi + c) = g \ .$$

According to Lemma 2.1, we need to find a function c such that for any ψ in the kernel of $Id - B^{-1}(f'_u(x, e).)$, the function $g + B^{-1}c$ is orthogonal to ψ in $D(B^{1/2})$, that is that

$$\forall \psi \in \mathrm{Ker}(Id - B^{-1}(f'_u(x, e).)), \quad - < c|\psi >_{\mathbb{L}^2} = < g|\psi >_{D(B^{1/2})} .$$

As the kernel of $Id - B^{-1}(f'_u(x, e).)$ is finite-dimensional, this is clearly possible. $\square$

4.2 Genericity of the simplicity of the eigenvalues

Let $(e, 0)$ be an equilibrium of (2.4). The simplicity of the eigenvalues of A_e is not directly required in the Kupka-Smale Property. But we will see that it plays a crucial role in the proof of Theorem 2.6 (see Theorem 4.4 or Proposition 5.3).

Theorem 4.3. *We assume that the operator A, defined by (2.2), satisfies Properties (B), (Gam) and (UCP) defined in Section 2, and the exponential decay property (ED).
Let $\mathfrak{G}^S$ be the set of all functions $f \in \mathfrak{G}^k$, such that, for any equilibrium $(e, 0)$ of (2.4), all the eigenvalues of the linearized operator A_e are simple. Then $\mathfrak{G}^S$ is a generic subset of $\mathfrak{G}^k$.*

Proof : Let $\mathfrak{G}^S_{n,m}$ be the set of all nonlinearities $f \in \mathfrak{G}^k$ such that for any equilibrium $(e, 0)$ of (2.4) with $\|e\|_{\mathbb{L}^\infty} \leq n$, all the eigenvalues λ with $|\lambda| \leq m$ of the linearized operator A_e are simple. As $\mathfrak{G}^S = \cap \mathfrak{G}^S_{n,m}$, we only need to prove that $\mathfrak{G}^S_{n,m}$ is an open dense set.
We can prove that $\mathfrak{G}^S_{n,m}$ is an open set by proving that its complementary is closed by using the same method as in the proof of Theorem 4.2.
Like in the proof of Theorem 4.2, we apply Sard-Smale Theorem in order to prove the density. Let f be a function of $\mathfrak{G}^k$. By perturbing f, we can assume that f is of class $\mathcal{C}^3$ and belongs to $\mathfrak{G}^H_{n+1}$. Let η_1 and η_2 be two regular functions with compact support such that for any $u \in [-n - 1, n + 1]$, $\eta_1(u) = 1$ and $\eta_2(u) = u$. We apply Theorem A.2 in order

to prove that we can find two functions b_1 and b_2 in $\mathcal{C}^k([0,1])$ as small as wanted, such that $f(x,u)+b_1(x)\eta_1(u)+b_2(x)\eta_2(u)$ is in $\mathfrak{G}^S_{n,m}$. This is sufficient to prove that $\mathfrak{G}^S_{n,m}$ is a dense set. We set

$$U = \{e \in D(B^{1/2}) \ / \ \|e\|_{\mathbb{L}^\infty} < n+1\} \times (D(B^{1/2}) \setminus \{0\}) \times \mathbb{C},$$

$$V = \{b = (b_1, b_2) \in (\mathcal{C}^k([0,1])^2 \ / \ f + b_1\eta_1 + b_2\eta_2 \in \mathfrak{G}^H_{n+1}\},$$

$$Z = (D(B^{1/2}))^2, \quad z = (0,0).$$

And we apply Theorem A.2 to the functional

$$\Phi(e,\varphi,\lambda,b)(x) = \begin{pmatrix} e - B^{-1}(f(x,e) + b_1(x) + b_2(x)e) \\ \varphi + \lambda\Gamma\varphi + B^{-1}((\lambda^2 - f'_u(x,e) - b_2(x))\varphi) \end{pmatrix}.$$

First, we notice that U and V are open subsets of separable metric spaces. Moreover, since f is of class $\mathcal{C}^3$, Φ is of class $\mathcal{C}^2$.
We next prove that $D\Phi$ is surjective from $(D(B^{1/2}))^2 \times \mathbb{C} \times (\mathcal{C}^k([0,1])^2$ into $(D(B^{1/2}))^2$ at each point of $\Phi^{-1}(0)$. More precisely, for each $(g,h) \in (D(B^{1/2}))^2$ and $(e,\varphi,\lambda,b) \in \Phi^{-1}(0)$, we must find $(\tilde{e},\tilde{\varphi},\tilde{\lambda},\tilde{b}) \in (D(B^{1/2}))^2 \times \mathbb{C} \times (\mathcal{C}^k([0,1])^2$ such that $D\Phi.(\tilde{e},\tilde{\varphi},\tilde{\lambda},\tilde{b}) = (g,h)$. We choose $\tilde{\lambda} = 0$ and $\tilde{b} = (-a(x)e(x), a(x))$ where $a \in \mathcal{C}^k([0,1])$ has to be determined. Notice that $e = B^{-1}(f(x,e(x)))$, so, as $f \in \mathcal{C}^k$ and $B^{-1/2}$ is smoothing, e belongs to $\mathbb{H}^{k+1}(]0,1[)$ and a fortiori to $\mathcal{C}^k$. So our choice $\tilde{b} = (-a(x)e(x), a(x))$ belongs as claimed to $(\mathcal{C}^k([0,1]))^2$. We introduce the operator

$$L = Id + \lambda\Gamma + B^{-1}((\lambda^2 - f'_u - b_2).).$$

We have to find $\tilde{e}$, $\tilde{\varphi}$ and a such that

$$\tilde{e} - B^{-1}((f'_u(x,e) + b_2)\tilde{e}) = g$$

$$L\tilde{\varphi} = B^{-1}(f''_{uu}(x,e)\varphi\tilde{e}) + B^{-1}(a\varphi) + h \ .$$

Since the equilibrium $(e,0)$ is hyperbolic, there exists $\tilde{e} \in D(B^{1/2})$ such that the first equality holds. By the Fredholm alternative given in Lemma 2.1, the second equality will hold if we find a function $a \in \mathcal{C}^k([0,1])$ such that $B^{-1}(f''_{uu}\varphi\tilde{e}) + B^{-1}(a\varphi) + h$ is orthogonal in $D(B^{1/2})$ to the kernel of $\overline{L}$, which is finite-dimensional. Let $(\varphi_1, ...\varphi_p)$ be a basis of this kernel. We have to find a function a such that

$$\forall i = 1,...,p \ , \quad < a\varphi|\varphi_i >_{\mathbb{L}^2} = - < h + B^{-1}(f''_{uu}\varphi\tilde{e})|\varphi_i >_{D(B^{1/2})} := c_i \ . \qquad (4.3)$$

We easily deduce from the unique continuation hypothesis (UCP)a) that the set $\{a\varphi \ / \ a \in \mathcal{C}^k([0,1])\}$ is dense in $\mathbb{L}^2([0,1])$, so we can find a function $a \in \mathcal{C}^k$ such that (4.3) is satisfied. So Hypothesis (ii) of Theorem A.2 is fulfilled.

It remains to prove that Hypothesis (i) of Theorem A.2 holds. We will show that, for any $(e, \varphi, \lambda, b) \in \Phi^{-1}(0)$, the operator

$$D_{(e,\varphi,\lambda)}\Phi : \ (\tilde{e}, \tilde{\varphi}, \tilde{\lambda}) \mapsto \begin{pmatrix} \tilde{e} - B^{-1}((f'_u(x,e) + b_2)\tilde{e}) \\ L\tilde{\varphi} + \tilde{\lambda}(\Gamma\varphi + 2\lambda B^{-1}\varphi) - B^{-1}(f''_{uu}\tilde{e}) \end{pmatrix}$$

is a Fredholm operator of index 1. Let m_λ be the multiplicity of the eigenvalue λ that is the dimension of the kernel of L. If $(\tilde{e}, \tilde{\varphi}, \tilde{\lambda})$ belongs to the kernel of $D_{(e,\varphi,\lambda)}\Phi$, then $\tilde{e} = 0$ since $(e, 0)$ is a hyperbolic equilibrium point. Hence we have

$$L\tilde{\varphi} = -\tilde{\lambda}(\Gamma\varphi + 2\lambda B^{-1}\varphi) \ ,$$

the dimension of the kernel of $D_{(e,\varphi,\lambda)}\Phi$ is m_λ if $(\Gamma\varphi + 2\lambda B^{-1}\varphi)$ does not belongs to the range of L, and $m_\lambda + 1$ if it does. To determine the codimension of $D_{(e,\varphi,\lambda)}\Phi$, we use the same arguments once more. As $(e, 0)$ is a hyperbolic equilibrium, $(Id - B^{-1}((f'_u(x,e) + b_2).)$ is bijective; hence, the codimension of the range of $D_{(e,\varphi,\lambda)}\Phi$ is equal to the codimension of the range of

$$(\tilde{\varphi}, \tilde{\lambda}) \longmapsto L\tilde{\varphi} + \tilde{\lambda}(\Gamma\varphi + 2\lambda B^{-1}\varphi)$$

which is $m_\lambda - 1$ if $(\Gamma\varphi + 2\lambda B^{-1}\varphi)$ does not belongs to the range of L, and m_λ if it does. In both cases, it follows that $D_{(e,\varphi,\lambda)}\Phi$ is a Fredholm operator of index 1. Thus, all the hypotheses of Theorem A.2 hold.
It follows, that for a generic set of functions $b = (b_1, b_2)$, for any (e, φ, λ) such that $(e, \varphi, \lambda, b) \in \Phi^{-1}(0)$, $D_{(e,\varphi,\lambda)}\Phi$ is surjective, that is, the codimension of its range is 0. This implies that $m_\lambda = 1$ and that $(\Gamma\varphi + 2\lambda B^{-1}\varphi)$ does not belongs to the range of L. In other terms, this means that, for a generic set of functions (b_1, b_2), for any equilibrium $(e, 0)$ of Equation (2.4), all the eigenvalues λ of A_e are geometrically and algebraically simple. $\square$

4.3 Genericity of the irrationality of some ratio

We will now prove that if $(e, 0)$ is a hyperbolic equilibrium of Equation (2.4), then the ratio between two distinct real eigenvalues of A_e is irrational for a generic set of nonlinearities $f \in \mathfrak{G}^k$. Of course, this property is not really intuitive, but we use it in the proof of Theorem 2.6. Notice that, in [6], Brunovský and Raugel proved also that the ratio of a positive eigenvalue and the real part of one with negative real part is generically irrational. But, in our case, we could not prove it, that is why our generic result is a little weaker. However, this is not a problem since we could modify the method of [6].
We want also to point out that the proof of the following theorem is the only proof which presents complications due to the choice of dealing with an abstract frame. In the case of Equations (1.3) and (1.4), the proof is only slightly more involved than in the constant damping case. In order to prove the result in the abstract frame, we had to introduce the

hypothesis (Loc), which is of course satisfied in the case of Equations (1.3) and (1.4). This hypothesis is used only in the proof of the following result.

We recall that C_{ev} is a real number defined in Hypothesis (Spec)b) of Section 2, but can obviously be replaced by any real number.

Theorem 4.4. *We assume that the operator A, defined by (2.2), satisfies Properties (B), (Gam), (UCP) and (Loc) defined in Section 2, and the exponential decay property (ED). Let $\mathfrak{G}^I$ be the set of all functions $f \in \mathfrak{G}^k$ such that, for any equilibrium $(e,0)$ of (2.4) and any two distinct real eigenvalues λ and μ of the linearisation A_e, there is no rational number $r \in \mathbb{Q}$ such that $\lambda = r\mu$ or $\lambda = rC_{ev}$. Then $\mathfrak{G}^I$ is a generic subset of $\mathfrak{G}^k$.*

Proof : Let $\mathfrak{G}^I_{r,n,m}$ be the set of all the functions $f \in \mathfrak{G}^k$ such that, for any equilibrium $(e,0)$ of (2.4) with $\|e\|_{\mathbb{L}^\infty} \leq n$, and any two distinct real eigenvalues λ and μ of the linearized operator A_e with $|\lambda| \leq m$ and $|\mu| \leq m$, we have $\frac{\lambda}{\mu} \neq r$ and $\frac{\lambda}{C_{ev}} \neq r$. As the set of the rational numbers r is countable, we only need to prove that $\mathfrak{G}^I_{r,n,m}$ is a dense open subset of $\mathfrak{G}^k$.
We can show that $\mathfrak{G}^I_{r,n,m}$ is open exactly as we do for $\mathfrak{G}^H_n$ in the proof of Theorem 4.2.
Here, to prove the density of $\mathfrak{G}^I_{r,n,m}$, we will not use the Sard-Smale theorem. We point out that the density can be proved by using a version of the Sard-Smale Theorem as it is done in [6]. But, we chose to use another method, which is possible once the generic simplicity of the eigenvalues is proved.
First notice that, as B and A have compact resolvents, there are only a finite number of equilibria $(e,0)$ with $\|e\|_{\mathbb{L}^\infty} \leq n$ and a finite number of eigenvalues λ with $|\lambda| \leq m$. For this reason, we only have to prove that for any equilibrium $(e,0)$ and any two real eigenvalues λ and μ, we can perturb f in such a way that the perturbed eigenvalues λ and μ satisfy $\lambda \neq r.\mu$ and $\lambda \neq r.C_{ev}$. First, by perturbing f, we can assume that $f \in \mathfrak{G}^S_{n+1,m+1}$, that is that λ and μ are simple. We consider perturbations of f of the form $f(x,u) + \tau.a(x)(u(x) - e(x))$, where $a \in \mathcal{C}^k([0,1])$ will be determined later. Notice that $(e,0)$ is still an equilibrium of (2.4) for these perturbation, and that, as $\mathfrak{G}^S_{n+1,m+1}$ is open, there exists a number $\varepsilon > 0$, such that if $|\tau| < \varepsilon$, λ and μ are simple eigenvalues. By the implicit function theorem, $\lambda(\tau)$ and $\mu(\tau)$ are $\mathcal{C}^1-$functions of τ and the same property holds for the associated real normalized eigenvectors $(\varphi, \lambda\varphi)$ and $(\psi, \mu\psi)$ with $\|\varphi\|_{\mathbb{L}^2} = \|\psi\|_{\mathbb{L}^2} = 1$. We differentiate the equality

$$\varphi(\tau) + \lambda(\tau)\Gamma\varphi(\tau) + B^{-1}(\lambda^2(\tau) - f'_u(x,e) - \tau.a(x))\varphi(\tau) = 0$$

with respect to τ, to obtain that, at $\tau = 0$,

$$D_\tau\lambda(\Gamma\varphi + 2\lambda B^{-1}\varphi) = -(Id + \lambda\Gamma + B^{-1}(\lambda^2 - f'_u(x,e)))D_\tau\varphi + B^{-1}(a(x)\varphi). \qquad (4.4)$$

The algebraic simplicity of λ implies that $(\Gamma\varphi(0)+2\lambda B^{-1}\varphi(0))$ is not orthogonal to $\varphi(0)$ in $D(B^{1/2})$. By taking the scalar product in $D(B^{1/2})$ of the above equality with φ, we obtain :

$$D_\tau\lambda = \frac{\int_0^1 a(x)\varphi^2}{<\Gamma\varphi|\varphi>_{D(B^{1/2})} + 2\lambda}.$$

Thus, we can easily find functions $a(x)$ (for example $a(x)=1$) for which $D_\tau\lambda(0)$ is strictly positive. This means that, if f is such that $\lambda(0) = rC_{ev}$, by perturbing $f(x,u)$ by the function $\tau(u(x)-e(x))$, we have $\lambda(\tau) \neq rC_{ev}$ for τ small enough.
Assume now that $\lambda(0) = r\mu(0)$. We shall prove that there exists a perturbation of f of the form $\tau a(x)(u(x)-e(x))$ such that $\lambda(\tau) \neq r\mu(\tau)$ for τ small enough. We argue by contradiction : assume that, for any function $a \in \mathcal{C}^k([0,1])$ and any $\tau \in]-\varepsilon;\varepsilon[$, we have $\lambda(\tau) = r\mu(\tau)$. This implies that, for any a and any τ, $D_\tau\lambda(\tau) = rD_\tau\mu(\tau)$, that is

$$\frac{\int_0^1 a(x)\varphi(\tau)^2}{<\Gamma\varphi(\tau)|\varphi(\tau)>_{D(B^{1/2})} + 2\lambda(\tau)} = r.\frac{\int_0^1 a(x)\psi(\tau)^2}{<\Gamma\psi(\tau)|\psi(\tau)>_{D(B^{1/2})} + 2\mu(\tau)}\ .$$

This means that $\varphi(\tau)^2$ is proportionnal to $\psi(\tau)^2$, and because both are real normalized functions, we must have, for any $x \in [0,1]$, $\varphi(\tau)^2(x) = \psi(\tau)^2(x)$. Thus

$$<\Gamma\psi(\tau)|\psi(\tau)>_{D(B^{1/2})} + 2\mu(\tau) = r(<\Gamma\varphi(\tau)|\varphi(\tau)>_{D(B^{1/2})} + 2\lambda(\tau)).$$

Since $r \neq 0$, by the hyperbolicity assumption, and since $r \neq 1$, as λ and μ are distinct eigenvalues, Assumption (Loc) of Theorem 2.6 implies that, for any function $a \in \mathcal{C}^k([0,1])$ and any $\tau \in]-\varepsilon;\varepsilon[$,

$$2\lambda(\tau)(1+\frac{1}{r}) + <\Gamma\varphi(\tau)|\varphi(\tau)>_{D(B^{1/2})} = 0\ . \tag{4.5}$$

We will show that this is impossible. Indeed, we will prove that we can find a function $a \in \mathcal{C}^k([0,1])$ such that the derivative of (4.5) satisfies

$$2D_\tau\lambda(0)(1+\frac{1}{r}) + 2 <\Gamma\varphi(0)|D_\tau\varphi(0)>_{D(B^{1/2})} \neq 0\ . \tag{4.6}$$

Thus, this will imply that we can perturb f in such a way that $\lambda(\tau) \neq r\mu(\tau)$ for τ small enough. To find a function a satisfying (4.6) we just have to work at the point $\tau = 0$, that is why, until the end of the proof, we omit the dependance in τ, and write for example λ instead of $\lambda(0)$.
First, we try to find a perturbation $a(x)$ such that $D_\tau\lambda = 0$ and $<\Gamma\varphi|D_\tau\varphi>_{D(B^{1/2})} \neq 0$. Let $a(x)$ be in the orthogonal space of φ^2 in $\mathbb{L}^2(]0,1[)$, which ensures that $D_\tau\lambda = 0$. As we have seen in the proof of Theorem 4.3, Hypothesis (UCP)a) implies that the set $\{a(x)\varphi\ /\ <a(x)\varphi|\varphi>_{\mathbb{L}^2} = 0\}$ is dense in the orthogonal of φ in $\mathbb{L}^2(]0,1[)$. In other words,

the set $\{B^{-1}(a(x)\varphi)\ /\ < a(x)\varphi|\varphi >_{\mathbb{L}^2}= 0\}$ is dense in the orthogonal of φ in $D(B^{1/2})$, which is exactly the range of $L = Id + \lambda\Gamma + B^{-1}(\lambda^2 - f'_u)$. Equality (4.4) together with the fact that $\|\varphi\|_{\mathbb{L}^2} = 1$ and $D_\tau\lambda = 0$, imply that $D_\tau\varphi$ is uniquely determined by

$$\left\{ \begin{array}{l} L(D_\tau\varphi) = B^{-1}(a(x)\varphi) \\ < \varphi|D_\tau\varphi >_{\mathbb{L}^2}=< B^{-1}\varphi|D_\tau\varphi >_{D(B^{1/2})}= 0 \ . \end{array} \right.$$

We have seen that $\{B^{-1}(a(x)\varphi)\ /\ < a(x)\varphi|\varphi >_{\mathbb{L}^2}= 0\}$ is dense in the range of L. So, we can find a perturbation $a(x)$ such that $< \Gamma\varphi|D_\tau\varphi >_{D(B^{1/2})}\neq 0$ unless $\Gamma\varphi$ is proportionnal to $B^{-1}\varphi$. If this is the case, (4.5) implies that

$$\Gamma\varphi = -2\lambda(1 + \frac{1}{r})B^{-1}\varphi \ . \tag{4.7}$$

Finally, we will show that even if (4.7) holds, we can find a function $a(x) \in \mathcal{C}^k([0, 1])$ such that (4.6) is satisfied. Using (4.4) and (4.7), $D_\tau\varphi$ and $D_\tau\lambda$ are uniquely determined by

$$\left\{ \begin{array}{l} L(D_\tau\varphi) - \frac{2\lambda}{r}D_\tau\lambda B^{-1}\varphi = B^{-1}(a(x)\varphi) \\ < \varphi|D_\tau\varphi >_{\mathbb{L}^2}= 0 \end{array} \right. \ .$$

If we choose $a(x) = 1$, we obtain $D_\tau\lambda = -\frac{r}{2\lambda} \neq 0$ and $D_\tau\varphi = 0$, so Property (4.6) holds. $\square$

5 Proof of the main theorem : generic transversality

We begin this section with auxiliary results which will be used in the proof of Theorem 2.6. We would like to point out that these preliminary lemmas have some interest by themselves. Once they are obtained, we are able to apply the Brunovský-Polàčik-Raugel Theorem and thus to prove Theorem 2.6.

5.1 Preliminary lemmas

We recall that, if $(e, 0)$ is an equilibrium of (2.4), we set

$$A_e = \left(A + \begin{pmatrix} 0 & 0 \\ f'_u(x, e) & 0 \end{pmatrix}\right) \ .$$

The following proposition is a consequence of the fact that A_e is a compact perturbation of A.

Proposition 5.1. *If A satisfies Hypothesis (Spec) and if $(e,0)$ is an equilibrium of (2.4) and $f \in \mathfrak{G}^k$, then A_e also satisfies Properties (Spec).*

Proof : Let (λ_n) be the eigenvalues of A and let $(V^n_{j,k})$ be the associated rootvectors defined in Subsection 2.2. We assume that A satisfies Hypothesis (Spec). Let K be the function

$$K : \left(\begin{array}{ccc} X & \longrightarrow & X \\ (u,v) & \longmapsto & (0, f'_u(x,e(x))u) \end{array} \right) .$$

As $D(B^{1/2})$ is compactly imbedded in $\mathbb{L}^2(]0,1[)$, and $f'_u(x,e(x))$ belongs to $\mathbb{L}^\infty(]0,1[)$, K is a compact operator. We want to prove that $A+K$ also satisfies Hypothesis (Spec). Such a property has been extensively studied for operators of the form $A+C$ where C is a small bounded perturbation. We will see that similar results hold for compact perturbations.
Let $R(\lambda,A)$ (resp. $R(\lambda,A+K)$) be the resolvant of A (resp. $A+K$). As $(V^n_{j,k})$ is a Riesz basis of X,

$$V^n_{j,k} \rightharpoonup 0 \text{ weakly in } X \text{ when } n \longrightarrow \infty .$$

Thus, because K is compact, we have

$$KV^n_{j,k} \longrightarrow 0 \quad \text{when } n \longrightarrow \infty . \tag{5.1}$$

For any $U \in X$, and any $\lambda \neq \lambda_n$, we introduce the sequences $(\alpha^n_{j,k})$ and $(\beta^n_{j,k})$ defined by

$$U = \sum_{n=0}^{+\infty} \sum_{j=0}^{m_n} \sum_{k=0}^{m_{n,j}} \alpha^n_{j,k} V^n_{j,k} ,$$

and

$$\begin{pmatrix} (\lambda_n - \lambda) & 1 & 0 & \dots & 0 \\ 0 & (\lambda_n-\lambda) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & (\lambda_n - \lambda) \end{pmatrix} \begin{pmatrix} \beta^n_{j,0} \\ \beta^n_{j,1} \\ \vdots \\ \vdots \\ \beta^n_{j,m_{n,j}-1} \end{pmatrix} = \begin{pmatrix} \alpha^n_{j,0} \\ \alpha^n_{j,1} \\ \vdots \\ \vdots \\ \alpha^n_{j,m_{n,j}-1} \end{pmatrix} . \tag{5.2}$$

We have by construction

$$R(\lambda,A)U = \sum_n \sum_{j,k} \beta^n_{j,k} V^n_{j,k} .$$

Notice that, because of Hypothesis (Spec)b), the multiplicities $m_{n,j}$ are bounded. For $\lambda \neq \lambda_n$, Equality (5.2) implies that there exists a positive constant C, independent of j and n, such that

$$\sum_{k=0}^{m_{n,j}-1} |\beta^n_{j,k}|^2 \leq \frac{C}{|\lambda-\lambda_n|^2} \left(1 + \frac{1}{|\lambda-\lambda_n|^{2m_{n,j}-2}}\right) \sum |\alpha^n_{j,k}|^2 .$$

Using the equivalence of the norms (2.8), we find that there is a constant $C' > 0$ such that

$$\|KR(\lambda, A)U\|_X^2 \leq C' \max_{n,j,k} \left\{ \frac{\|KV_{j,k}^n\|_X^2}{|\lambda - \lambda_n|^2} \left(1 + \frac{1}{|\lambda - \lambda_n|^{2m_{n,j}-2}}\right) \right\} \|U\|_X^2 \qquad (5.3)$$

Let $B(\lambda_n, r)$ be the ball of center λ_n and radius r in $\mathbb{C}$. Let α_{ev} be the constant introduced in Hypothesis (Spec). For $r < \alpha_{ev}$, $R(\lambda, A)$ is a well-defined bounded operator for any $\lambda \in \partial B(\lambda_n, r)$. Properties (5.1) and (5.3) imply that, for $r < \alpha_{ev}$,

$$\sup_{\lambda \in \partial B(\lambda_n, r)} \|KR(\lambda, A)\|_{\mathcal{L}(X)} \longrightarrow 0 \ , \quad \text{when } n \longrightarrow \infty \ .$$

That is why, for n large enough and for any $\lambda \in \partial B(\lambda_n, r)$, the operator $(\lambda + KR(\lambda, A))$ is invertible. Since we have $\lambda - (A+K) = (Id + KR(\lambda, A))(\lambda - A)$, the operator $(\lambda - (A+K))$ is invertible and

$$R(\lambda, A+K) = (\lambda - (A+K))^{-1} = R(\lambda, A)(Id + KR(\lambda, A))^{-1} \ .$$

Using the previous equality and arguing as in Theorems 3.16 and 3.18 of chapter IV of [25], we obtain that, for all $r_0 < \alpha_{ev}$, there exists a constant $C'' > 0$ such that, for n large enough, $r < r_0$ and $\lambda \in \partial B(\lambda_n, r)$, $R(\lambda, A+K)$ is a compact operator bounded by $\frac{C''}{r}$. Moreover, if (μ_n) is the sequence of eigenvalues of $A+K$, then

$$|\mu_n - \lambda_n| \longrightarrow 0, \text{ as } n \longrightarrow 0 \ ,$$

and, since λ_n is simple for n large enough, so is μ_n. In conclusion, $A+K$ also satisfies Hypotheses (Spec)b) and (Spec)c).
To show that Hypothesis (Spec)a) also holds for $A+K$, we mimic the proofs concerning perturbations by small bounded operators. For example, adapting the proof of Theorem 4 of [30] (see also Th 4.15 of [25]), we show that the sequence of rootvectors of $A+K$ is equivalent to the one of A. More precisely, let $(U_i)_{i\geq 0}$, $(V_i)_{i\geq 0}$, $(U_i^*)_{i\geq 0}$ and $(V_i^*)_{i\geq 0}$, be respectively the rootvectors of A and $A+K$ and their biorthonormalized sequences (see [14]). There exists an integer N such that, for $i \geq N$, the eigenvectors U_i and V_i correspond to simple eigenvalues. For $i \geq N$, let P_i and Q_i be the following eigenprojections

$$P_i = < .|U_i^* > U_i \qquad Q_i = < .|V_i^* > V_i \ ,$$

and let P_0 (resp. Q_0) be the eigenprojection onto the space spanned by $U_0, ..., U_{N-1}$ (resp. $V_0, ..., V_{N-1}$). The method given in the proof of Theorem 4 of [30] shows that there exists a bounded invertible operator D in $\mathcal{L}(X)$ such that, for $i = 0$ and $i \geq N$,

$$Q_i = D^{-1} P_i D \ .$$

Scaling in an appropriate way, we obtain for all i

$$V_i = D^{-1} U_i \text{ and } V_i^* = D^* U_i^* \ .$$

According to Theorem 2.1 of chapter VI of [14], these equalities imply that the sequence of rootvectors of $A + K$ forms a Riesz basis of X. $\square$

We recall that $S(t)$ denotes the local $\mathcal{C}^0$-group generated by Equation (2.4). We denote by $S_u^*(t, s)$ the evolution operator defined by Equation (2.7).

Proposition 5.2. *Let f be a function of $\mathcal{C}^k([0, 1] \times \mathbb{R}, \mathbb{R})$ ($k \geq 1$). We assume that the hypotheses of Theorem 2.6 hold. If $\mathcal{I} \subset X$ is a bounded closed invariant set of $S(t)$ (for example a bounded complete solution), then, for any $U_0 \in \mathcal{I}$, the map $t \in \mathbb{R} \longmapsto S(t)U_0 \in X$ is of class $\mathcal{C}^k$ and $S(t)U_0$ is a classical solution. If $f(x, u)$ is analytic in $u \in \mathbb{R}$ uniformly in $x \in [0, 1]$, then the map $t \longmapsto S(t)U_0$ is analytic. The same regularity properties also hold for $S_u^*(t, 0)$.*
Moreover, $\mathcal{I}$ is bounded in $D(A)$ and $S(t)U_0$ belongs to $\mathcal{C}^{k-1}(\mathbb{R}, D(A))$. Furthermore, for any $f_0 \in \mathfrak{G}^k$ and any positive number R, there exists a neighborhood $N(f_0) \subset \mathfrak{G}^k$ and a positive constant $C = C(R, f_0)$, such that, if $f \in N(f_0)$ and $\mathcal{I}$ is bounded in X by R, then for any $U_0 \in \mathcal{I}$,

$$\|S(t)U_0\|_{\mathcal{C}_b^0(\mathbb{R}, D(A)) \cap \mathcal{C}_b^1(\mathbb{R}, X)} \leq C(R, f_0).$$

Proof : This regularity property is a direct consequence of Theorem 1.1 of [20]. This theorem says that the trajectories in a compact invariant set are as regular as f (in the sense of our proposition) once we can find projections $(P_N)_{N \in \mathbb{N}}$, which commute with A and converge to the identity in X, such that AP_N is a bounded operator on X and the semigroup e^{At} is exponentially decreasing on $(Id - P_N)X$ for N large enough, with constants independent of N.
First notice that Proposition 2.5 implies that $\mathcal{I}$ is compact.
Let $(V_{j,k}^n)$ be the Riesz basis composed of rootvectors of A. There exists a biorthonormalised basis $(W_{j,k}^n)_{n \in \mathbb{N}, j < m_n, k < m_{n,j}}$ such that

$$< V_{j,k}^n | W_{j',k'}^{n'} >_X = \delta_{n,n'} \delta_{j,j'} \delta_{k,k'}$$

(see [14] for details). Let

$$P_N = \sum_{n=0}^{N} \sum_{j=0}^{m_n - 1} \sum_{k=0}^{m_{n,j} - 1} \left\langle . \left| W_{j,k}^n \right. \right\rangle V_{j,k}^n$$

be the projection onto the subspace generated by the eigenspaces corresponding to the first $N + 1$ eigenvalues. Obviously, P_N converges strongly to the identity, and AP_N is bounded. Since the exponential decay property has been proved in Proposition 2.4, all the hypotheses of Theorem 1.1 of [20] are satisfied once we have proved that $AP_N = P_N A$ in $D(A)$. Looking at

$$< A^* W_{j,k}^n | V_{j',k'}^{n'} > = < W_{j,k}^n | A V_{j',k'}^{n'} >,$$

we find that

$$A^*W^n_{j,k} = \left\{ \begin{array}{ll} \overline{\lambda}_n W^n_{j,k} & \text{if } k = m_{n,j} - 1 \\ \overline{\lambda}_n W^n_{j,k} + W^n_{j,k+1} & \text{if } 0 \leq k \leq m_{n,j} - 2. \end{array} \right. . \tag{5.4}$$

Then, an easy computation gives the expected commutation between A and P_N.
We prove with the same arguments that the above properties are satisfied by $S^*_u(t, 0)$.
Finally, notice that the second part of our proposition is also a consequence of Theorem 1.1 of [20]. Although these properties are not enhanced in the statement of Theorem 1.1, they can be deduced from its proof. □

Let M be a positive constant. Let $f \in \mathfrak{G}^k$ and let (u, u_t) be a complete bounded solution of (2.4) with $\|u(t)\|_{\mathbb{L}^\infty} \leq M$ for all time $t \in \mathbb{R}$. As the system is gradient and asymptotically smooth, there are two distinct equilibria e_+ and e_- such that $(u, u_t) \longrightarrow (e_\pm, 0)$ when $t \longrightarrow \pm\infty$.

Proposition 5.3. *We assume that the hypotheses of Theorem 2.6 are fulfilled.*
Let f be a function in $\mathfrak{G}^k$ such that $f(x, u)$ is analytic with respect to $u \in [-M, M]$, uniformly in x, and such that, for any equilibrium $(e, 0)$ of (2.4), $(e, 0)$ is hyperbolic and all the eigenvalues of the linearized operator A_e are simple. Let (u, u_t) be the trajectory defined above. Then, there exist a positive eigenvalue λ of A_{e_-} with corresponding eigenfunction $(\varphi, \lambda\varphi)$, a nonzero number b and a positive constant δ such that

$$\left\| \begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} e_- \\ 0 \end{pmatrix} - be^{\lambda t} \begin{pmatrix} \varphi \\ \lambda\varphi \end{pmatrix} \right\|_X = o\left(e^{(\lambda+\delta)t}\right) \qquad \text{when } t \longrightarrow -\infty \ .$$

Moreover,

$$\left\| u_t - b\lambda e^{\lambda t}\varphi \right\|_{D(B^{1/2})} = o\left(e^{(\lambda+\delta)t}\right) \qquad \text{when } t \longrightarrow -\infty \ .$$

Proof : The first property is classical since the spectrum of A_{e_-} contains only a finite number of eigenvalues with positive real part. To obtain the second estimate, we use the regularity proposition 5.2 to differentiate Equation (2.4) with respect to the time variable. Then the second estimate is shown as the first one. □

The next proposition gives a similar result for the adjoint equation (2.7). However, new difficulties come from the fact that the concerned part of the spectrum contains an infinite number of eigenvalues.
Let f be as in Proposition 5.3 (actually, $f \in \mathfrak{G}^k$ with $k \geq 3$ is enough). Let (u, u_t) be a complete bounded solution of Equation (2.4), that is a solution of (2.4) which is uniformly bounded in X for all time $t \in \mathbb{R}$. Let (θ, ψ) be a complete bounded solution of the adjoint equation (2.7), that is

$$\left\{ \begin{array}{l} \theta_t = \psi - B^{-1}(f'_u(x, u)\psi) \\ \psi_t = -B(\theta - \Gamma\psi) \end{array} \right. . \tag{5.5}$$

Proposition 5.2 implies that (θ, ψ) belongs to $\mathcal{C}^2(\mathbb{R}, X) \cap \mathcal{C}^1(\mathbb{R}, D(A^*))$. We deduce from (5.5) that

$$\psi_{tt} = -B(\psi - B^{-1}(f'_u(x,u)\psi) - \Gamma\psi_t) \ .$$

This can be written

$$\frac{\partial}{\partial t}\begin{pmatrix} \psi \\ \psi_t \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left[A + \begin{pmatrix} 0 & 0 \\ f'_u(x,u) & 0 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi \\ \psi_t \end{pmatrix} \ . \qquad (5.6)$$

Proposition 5.2 implies that (ψ, ψ_t) belongs to $\mathcal{C}^1(\mathbb{R}, X) \cap \mathcal{C}^0(\mathbb{R}, D(A^*))$. Thus, we can formulate the following proposition.

Proposition 5.4. *Let f and (u, u_t) be as in Proposition 5.3. If (θ, ψ) is a complete bounded solution of the adjoint equation (2.7), there exists a positive real number μ such that*

$$\lim_{t \to -\infty} \ln \left\| \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} \right\|_X^{1/t} = \mu \ .$$

Moreover, there exist a positive constant δ and a solution

$$(\psi^\infty, \psi_t^\infty) \in \mathcal{C}^1(\mathbb{R}, X) \cap \mathcal{C}^0(\mathbb{R}, D(A^*))$$

of the limit equation

$$\frac{\partial}{\partial t}\begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_{e_-} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} \ ,$$

such that

$$\left\| \begin{pmatrix} \psi \\ \psi_t \end{pmatrix} - \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} \right\|_X = o\left(e^{(\mu+\delta)t}\right) \qquad \textit{when } t \longrightarrow -\infty.$$

In particular

$$\lim_{t \to -\infty} \ln \left\| \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} \right\|_X^{1/t} = \mu \ .$$

Proof : We refer here to the proof of Propositions 5.3 and 5.4 of [6], which are proved by using Theorems B.5, B.6 and B.7 of [6]. The theorems of Appendix B of [6] give sufficient conditions, under which a solution of an equation of type $V_t = C(t)V$, with $C(t) \longrightarrow C(\infty)$ when $t \longrightarrow \infty$, converges to a solution of the limit equation $\Psi_t = C(\infty)\Psi$.
We only want to enhance that Hypothesis (Spec) implies that for any real number l, we can find a gap in the real part of the spectrum as near as needed from l. This property is necessary to define the spectral projections and to apply the theorems of Appendix B of [6]. $\square$

The last proposition will be used to find a point $x \in [0,1]$ such that the asymptotic speed of a trajectory (ψ, ψ_t) of Equation (5.6) in X, and the asymptotic speed of the function $\psi(x,.)$ at the chosen point x are equal.

Proposition 5.5. *Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of nonzero complex numbers in $\ell^1(\mathbb{C})$, and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that* $\sup\{Re(\lambda_n)\} < \infty$*. Let*

$$f(t) = \sum_{n\in\mathbb{N}} c_n e^{\lambda_n t} \ .$$

The function f is well defined since $(c_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{C})$ and since for each t, $|e^{\lambda_n t}|$ is uniformly bounded with respect to n. Moreover, we have

$$\inf\left\{\lambda \in \mathbb{R} \ / \ \lim_{t\to\infty} |f(t)|e^{-\lambda t} = 0\right\} = \sup\{Re(\lambda_n)\}.$$

Proof : It is clear that if $\lambda > \sup\{\mathrm{Re}(\lambda_n)\}$, then $|f(t)|e^{-\lambda t} \longrightarrow 0$. So

$$\inf\left\{\lambda \in \mathbb{R} \ / \ \lim_{t\to\infty} |f(t)|\, e^{-\lambda t} = 0\right\} \leq \sup\{\mathrm{Re}(\lambda_n)\}.$$

Now, assume that there exist a number λ and a positive constant ε such that $\lambda < \lambda + \varepsilon < \sup\{\mathrm{Re}(\lambda_n)\}$ and $|f(t)|e^{-\lambda t} \longrightarrow 0$. As $f(t)e^{-(\lambda+\varepsilon)t} = o(e^{-\varepsilon t})$, the Laplace transform of f,

$$Lf(z) = \int_0^\infty f(t)e^{-zt}dt \ ,$$

is defined on the half-plane $H = \{z \in \mathbb{C} \ / \ \mathrm{Re}(z) \geq \lambda + \varepsilon\}$ and is holomorphic on H. But if z is such that $\mathrm{Re}(z) > \sup\{\mathrm{Re}(\lambda_n)\}$, then we can develop Lf as a sum of meromorphic functions as follows :

$$Lf(z) = \int_0^\infty \sum c_n e^{(\lambda_n - z)t} = \sum c_n \int_0^\infty e^{(\lambda_n - z)t} = -\sum \frac{c_n}{\lambda_n - z}.$$

As Lf is holomorphic on H, this expression must be valid on the whole half-plane H. But, because $\sup\{\mathrm{Re}(\lambda_n)\} > \lambda + \varepsilon$ and $c_n \neq 0$, Lf has poles in H, which contradicts the fact that Lf is holomorphic in H. So,

$$\inf\left\{\lambda \in \mathbb{R} \ / \ \lim_{t\to\infty} |f(t)|\, e^{-\lambda t} = 0\right\} \geq \sup\{\mathrm{Re}(\lambda_n)\}.$$

□

5.2 Proof of Theorem 2.6

In this subsection, we prove Theorem 2.6 by using the Brunovský-Polàčik-Raugel Theorem, recalled in Appendix A. The way to apply it has already been explained in [5] and [6]. Whereas the verification of the hypotheses (h1)-(h6) does not change, significant changes appear in verifying condition (h7).

First step : Application of Theorem A.1
We have to put our problem in the framework of Theorem A.1. Let $Z = \mathbb{L}^\infty(]0,1[) \times \mathbb{L}^2(]0,1[)$, let $\Lambda = \mathfrak{G}^k$ ($k \geq 2$), $U = (u, u_t)$ and $F((u,v), f) = (0, f(x,u))$. Equation (2.4) becomes

$$U_t = AU + F(U, f) \ .$$

We set for any $r \in \mathbb{N}$,

$$\Lambda^r = \mathcal{C}^r([0,1] \times [-n,n]) \ .$$

Let R be the restriction operator

$$Rf = f_{\big|[0,1]\times[-n,n]} \ ,$$

which is continuous, open and surjective. We recall that, in the proof of Theorem 4.2, we have introduced the set $\mathfrak{G}^H_n$ of all the functions f such that all the equilibria $(e,0)$ of (2.4) with $\|e\|_{\mathbb{L}^\infty} \leq n$ are hyperbolic. We proved that $\mathfrak{G}^H_n$ is an open dense subset of $\mathfrak{G}^k$. Let L be the open dense subspace of Λ^k defined by

$$L = R\mathfrak{G}^H_n .$$

We also set $M = n$. Let $\mathfrak{G}^{KS}_n$ be the set of all the functions in $\mathfrak{G}^k$ for which all the heteroclinic orbits (u, u_t) of Equation (2.4) with $\|(u,u_t)\|_Z < n$ are transverse. Assume that Theorem A.1 can be applied. If $\mathcal{L}$ is the generic subset of L given in the conclusion of Theorem A.1, then $R^{-1}\mathcal{L} \subset \mathfrak{G}^{KS}_n$ is a generic subset of $\mathfrak{G}^H_n$ and so a generic subset of $\mathfrak{G}^k$. As

$$\mathfrak{G}^{KS} = \bigcap_{n\in\mathbb{N}} \mathfrak{G}^{KS}_n,$$

Theorem 2.6 will be proved. Thus, it remains to prove that all the hypotheses of Theorem A.1 are satisfied.

Hypotheses (FP), (AP), (BUP1), (BUP2), (h1), (h2), (h3), (h4) of Theorem A.1 are obvious or were proved in Subsections 2.1 and 2.2. Condition (h6) is a consequence of Proposition 5.2 ; (h5) comes from the gradient structure and the asymptotic smoothness of (2.4). Finally, we have only to prove that Hypothesis (h7) is satisfied.

Let $\lambda_0 = f_0$ and let $\mathcal{V}$ be a neighborhood of f_0 in L. Let $(e, 0)$ be an equilibrium of (2.4) with $\|e\|_{\mathbb{L}^\infty} < M$. We recall that the Morse index of $(e, 0)$ is the dimension of its unstable manifold, that is the number of eigenvalues with positive real part of A_e. As f_0 belongs to $\mathcal{C}^0([0, 1] \times [-n, n])$ and $(e, 0)$ satisfies the equality $e = B^{-1} f_0(x, e)$, the equilibrium $(e, 0)$ is bounded in $D(A)$ by a constant which only depends on M and f_0. As the equilibria of (2.4) have the hyperbolicity property, they are isolated in X. So, due to the compact imbedding of $D(A)$ into X, the number of equilibria $(e, 0)$ of (2.4) with $\|e\|_{\mathbb{L}^\infty} < M$ is finite. We deduce that there exists an integer r such that $r - 1$ is strictly larger than all the Morse indices of the equilibria of (2.4). Then, we set

$$\hat{\Lambda} = \Lambda^r.$$

The Morse indices depend continuously on the non-linearity f. As the number of equilibria is finite, we can restrict the neighborhood $\mathcal{V}$ without loss of generality, such that for any $f \in \mathcal{V}$, the Morse indices of the equilibria stay strictly less than $r - 1$. By density, we can find a function $f_1 \in \mathcal{V}$ which belongs to $R\mathfrak{G}_n^H \cap \Lambda^r$. As $R\mathfrak{G}_n^H \cap \Lambda^r \cap \mathcal{V}$ is open in Λ^r, by a simple perturbation, we can find a function $f_2 \in R\mathfrak{G}_n^H \cap \Lambda^r \cap \mathcal{V}$ which is analytic in u, uniformly with respect to x. In the proof of Theorems 4.3 and 4.4, we showed that for any $f \in \mathfrak{G}^k$, we can find a perturbation of the form $\eta(a(x) + b(x)u)$, with a and b as smooth as f, such that $f(x, u) + a(x) + b(x)u$ belongs to $\mathfrak{G}^I \cap \mathfrak{G}^S$. That is why, by perturbing f_2 in this way, we can find $\hat{f} \in R\mathfrak{G}_n^H \cap \Lambda^r \cap \mathcal{V}$, which belongs to $\mathfrak{G}^I \cap \mathfrak{G}^S$ and which is analytic in u, uniformly with respect to x. Indeed, f_2 was assumed to be analytic in u, uniformly with respect to x, and a perturbation of the form $\eta(a(x) + b(x)u)$ does not alter this property. By construction, Hypotheses (h7)(a) and (h7)(b) of Theorem A.1 hold. It remains to prove that Hypothesis (h7)(c) is satisfied for our $\hat{f}$.

Second step : Hypothesis (h7)(c)

Let (u, u_t) be a heteroclinic solution of Equation (2.4) with $\sup_{t\in\mathbb{R}} \|(u, u_t)\|_Z < n$ and (θ, ψ) be a nontrivial complete bounded solution of the adjoint equation (2.7). We must find a function $h \in \mathcal{C}^r([0, 1] \times \mathbb{R}, \mathbb{R})$ such that

$$I = \int_0^\infty \left\langle (\theta, \psi) \middle| D_f F(U, \hat{f})h \right\rangle_X dt \neq 0,$$

that is such that

$$I = \int_0^\infty \int_0^1 \psi(x, t) h(x, u(x, t)) dx dt \neq 0.$$

In the particular case where $h(x, u) = b(x)g(u)$, the above condition becomes

$$I = \int_0^1 b(x) \left(\int_{\mathbb{R}} \psi(x, t) g(u(x, t)) dt \right) dx \neq 0 \ . \tag{5.7}$$

So it is sufficient to find a point x and a function $g \in \mathcal{C}^r(\mathbb{R}, \mathbb{R})$ such that

$$J = \int_{\mathbb{R}} \psi(x,t) g(u(x,t)) dt \neq 0 \ . \tag{5.8}$$

As u is a heteroclinic solution, there are two distinct equilibria e_+ and e_- such that $u \longrightarrow e_\pm$ when $t \longrightarrow \pm\infty$. We choose x such that $(e_+ - e_-)(x) \neq 0$ and all the eigenfunctions $(\varphi, \lambda\varphi)$ of A_{e_-} satisfy $\varphi(x) \neq 0$. This is possible due to Hypothesis (UCP)a) of Theorem 2.6 and the fact that the set of eigenfunctions is a countable set. Such a choice is essential in the remaining part of the proof. Indeed, we will see that it ensures that the asymptotic behaviour in time of the real functions $u(x,t)$ and $\psi(x,t)$ is similar to those of $u(.,t)$ and $\psi(.,t)$ in $D(B^{1/2})$. To the end of the proof, we will only consider the functions at this chosen point x.
As $\hat{f}(\xi, u)$ is analytic in u, uniformly in ξ, we can apply Proposition 5.2, and so $u(x,.)$ and $\psi(x,.)$ are analytic functions of t. Due to Proposition 5.3, there exist a nonzero number b and a positive real eigenvalue λ of A_{e_-} with eigenvector $(\varphi, \lambda\varphi)$ such that

$$\left\| \begin{pmatrix} u \\ u_t \end{pmatrix} - \begin{pmatrix} e_- \\ 0 \end{pmatrix} - be^{\lambda t} \begin{pmatrix} \varphi \\ \lambda\varphi \end{pmatrix} \right\|_X = o(e^{(\lambda+\delta)t}) \ .$$

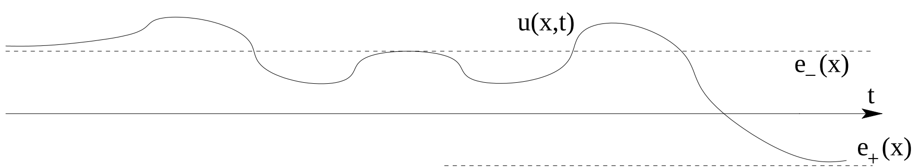
As $D(B^{1/2})$ is imbedded in $\mathcal{C}^0$, we obtain

$$u(x,t) = e_-(x) + be^{\lambda t}\varphi(x) + o(e^{(\lambda+\delta)t}) \ .$$

Using the second estimate of Proposition 5.3, we also have

$$u_t(x,t) = b\lambda\varphi(x)e^{\lambda t} + o(e^{(\lambda+\delta)t}),$$

when t goes to $-\infty$. Because of the choice of x, $\varphi(x) \neq 0$, so we know that $u_t(x,t)$ does not vanish on a neighborhood of $-\infty$. Without loss of generality, we can assume for example that $b\varphi(x) > 0$ and $e_-(x) > e_+(x)$. To summarize, $u(x,t)$ looks like



We choose the function g of the form

$$g(u) = g_{\zeta,\varepsilon}(u) = \frac{1}{\varepsilon}\Theta(\frac{u-\zeta}{\varepsilon}) \ ,$$

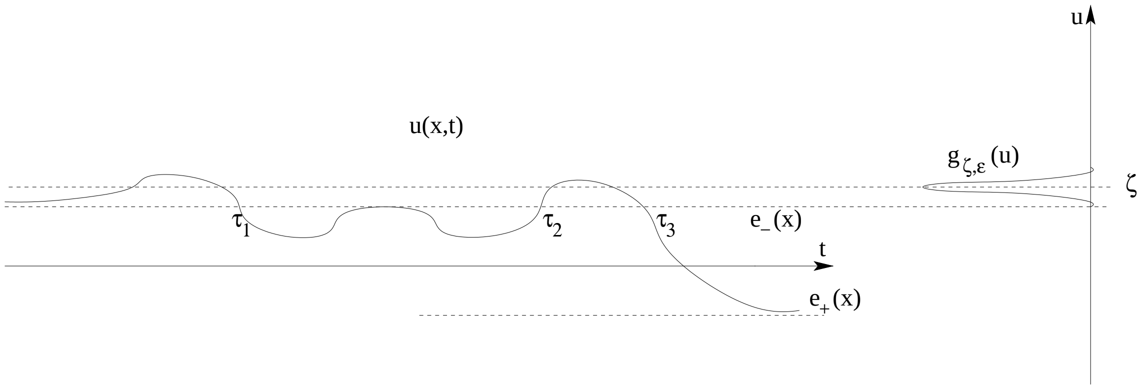
where Θ is a smooth normalized bump function. For example, we take

$$\Theta(s) = \left\{ \begin{array}{ll} 0 & \text{if } |s| > 1 \\ \frac{1}{C}\, e^{-\frac{1}{1-s^2}} & \text{if } |s| \leq 1 \end{array} \right. ,$$

where

$$C = \int_{-1}^{1} e^{-\frac{1}{1-\sigma^2}} d\sigma \ .$$

In what follows, we always assume that $0 < \varepsilon < \zeta - e_-(x)$. As $u(x,.)$ is strictly increasing in a neighborhood of $t = -\infty$, $e_-(x) \neq e_+(x)$, and that $u(x,.)$ is analytic in time, there is only a finite number of solutions of the equation $u(x,\tau) = e_-(x)$. We denote by τ_1,..., τ_m all the solutions τ of $u(x,\tau) = e_-(x)$, for which we do not have $u(x,t) \leq e_-(x)$ in a neighborhood of $t = \tau$.



Let $(N_i)_{i=0,..,m}$ be disjoint neighborhoods of $t = -\infty$ for $i = 0$ and τ_i for $i = 1,..,m$; then for $\zeta - e_-(x)$ and ε small enough we can split the integral J into a finite sum of integrals :

$$J = \int_{\mathbb{R}} \psi(x,t) g(u(x,t)) dt = \sum_{i=0}^{m} \int_{N_i} \psi(x,t) g_{\zeta,\varepsilon}(u(x,t)) dt = \sum_{i=0}^{m} J_i \ .$$

In their paper [5], Brunovský and Polàčik conclude quickly from this splitting, as a property of the parabolic equation ensures that $\frac{d}{dt}u(x,\tau_i) \neq 0$ for a generic x. In our case, we cannot be sure that $\frac{d}{dt}u(x,\tau_i) \neq 0$, so we need to estimate the integrals J_i in order to conclude. This method was introduced by Brunovský and Raugel in [6].

Third step : Estimations of the integrals J_i

Lemma 5.6. *If $0 < \varepsilon < \zeta - e_-(x)$ with $\zeta - e_-(x)$ small enough, there exists a rational number $r \in \mathbb{Q}$ such that*

$$\sum_{i=1}^{m} J_i = S((\zeta - e_-(x))^r) + \omega(\zeta,\varepsilon),$$

where $S(z)$ is a power series of z and

$$\lim_{\varepsilon \to 0} \omega(\zeta,\varepsilon) = 0.$$

Proof : For sake of completeness, we repeat here the proof of Lemma 5.6 of [6].
As $u(x,.)$ and $\psi(x,.)$ are analytic functions of the time, we may write, when t is near τ_i,

$$u(x,t) = e_-(x) + \sum_{l=k}^{+\infty} a_l(t-\tau_i)^l,$$

and

$$\psi(x,t) = \sum_{l=k'}^{+\infty} d_l(t-\tau_i)^l.$$

In what follows, we assume that k is odd. Denote

$$z = (\zeta - e_-(x))^{-1/k}(t-\tau_i) \ , \tag{5.9}$$

then, for $\zeta \neq e_-(x)$, $u(t) = \zeta$ if and only if

$$a_k(\zeta - e_-(x))z^k + \sum_{l=k+1}^{\infty} a_l(\zeta - e_-(x))^{\frac{l}{k}} z^l = \zeta - e_-(x) \ ,$$

or,

$$H(z, (\zeta - e_-(x))^{1/k}) \equiv a_k z^k + \sum_{l=k+1}^{\infty} a_l(\zeta - e_-(x))^{\frac{l}{k}-1} z^l = 1 \ .$$

Since $a_k \neq 0$, we may apply the implicit function theorem to the equation

$$H(z, (\zeta - e_-(x))^{1/k}) - 1 = 0$$

in the neighbourhood of $(a_k^{-1/k}, 0)$. Hence, locally near $z = a_k^{-1/k}$, the above equation has a unique solution

$$z = a_k^{-1/k} + \sum_{l=1}^{+\infty} c_l(\zeta - e_-(x))^{l/k} \ .$$

Let t_ζ be the solution of $u(x,t_\zeta) = \zeta$ near τ_i. Substituting the above expression of z into (5.9), we obtain that, for ζ close to $e_-(x)$,

$$t_\zeta = \tau_i + a_k^{-1/k}(\zeta - e_-(x))^{1/k} + (\zeta - e_-(x))^{1/k} S((\zeta - e_-(x))^{1/k}) \ ,$$

where $S(z)$ is a power series of z. Further, we deduce that

$$\begin{aligned} u_t(t_\zeta) &= k a_k (t_\zeta - \tau_i)^{k-1} + \sum_{l=k+1}^{+\infty} l a_l (t_\zeta - \tau_i)^{l-1} \\ &= k a_k^{1-1/k} (\zeta - e_-(x))^{1-1/k} S((\zeta - e_-(x))^{1/k}) \ . \end{aligned}$$

Due to the change of variables $t = t(u)$ in J_i and the fact that $g_{\zeta,\varepsilon}$ converges to the Dirac function at the point ζ when $\varepsilon \to 0$, we obtain

$$\begin{aligned} J_i &= \int_{|u(t)-\zeta|\le\varepsilon} \psi(x,t)g(u(t))\, dt \\ &= \int_{\zeta-\varepsilon}^{\zeta+\varepsilon} \psi(x,t(u))g_{\zeta,\varepsilon}(u)\, \frac{du}{u_t(x,t(u))} \\ &= \frac{\psi(x,t_\zeta)}{u_t(x,t_\zeta)} + \omega(\zeta,\varepsilon)\ , \end{aligned}$$

with $\omega(\zeta,\varepsilon) \longrightarrow 0$ when $\varepsilon \longrightarrow 0$. Finally, using the analyticity of $u(x,.)$ and $\psi(x,.)$ we obtain that

$$J_i = (\zeta - e_-(x))^{\frac{k'+1}{k}-1} S((\zeta - e_-(x))^{\frac{1}{k}}) + \omega(\zeta,\varepsilon).$$

In the case where k is even, the only difference would be that we must split J_i into two parts, the one when $t < \tau_i$ and the one when $t > \tau_i$. Next, we can deal with each part as we do with the whole integral J_i when k is odd (see [6]). $\square$

We recall that when t goes to $-\infty$ and $\delta > 0$ is small enough

$$u(x,t) = e_-(x) + be^{\lambda t}\varphi(x) + o(e^{(\lambda+\delta)t}),$$

and

$$u_t(x,t) = b\lambda\varphi(x)e^{\lambda t} + o(e^{(\lambda+\delta)t}).$$

Moreover, according to Proposition 5.4, there exist a positive real number μ and a solution $(\psi^\infty, \psi_t^\infty) \in \mathcal{C}^0(\mathbb{R}, D(A^*))$ of the limit equation

$$\frac{\partial}{\partial t}\begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_{e_-} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix}$$

such that

$$\lim_{t\to-\infty} \ln \left\| \begin{pmatrix} \psi^\infty \\ \psi_t^\infty \end{pmatrix} \right\|_X^{1/t} = \mu,$$

and

$$\psi(x,t) = \psi^\infty(x,t) + O\left(e^{(\mu+\delta)t}\right),$$

when t goes to $-\infty$ and $\delta > 0$ is small enough.

Lemma 5.7. *If $0 < \varepsilon < \zeta - e_-(x)$ are small enough, then*

$$\begin{aligned} J_0 &= \frac{\psi^\infty(t_\zeta) + O\left((\zeta - e_-(x))^{\frac{\mu+\delta}{\lambda}}\right)}{u_t(t_\zeta)} + \omega(\zeta, \varepsilon) \\ &= \frac{\psi^\infty(t_\zeta)}{\lambda\phi(x)(\zeta - e_-(x))} + O\left(\frac{\psi^\infty(t_\zeta)}{(\zeta - e_-(x))^{1-\frac{\delta}{\lambda}}}\right) + O\left((\zeta - e_-(x))^{\frac{\mu}{\lambda}-1+\frac{\delta}{\lambda}}\right) + \omega(\zeta, \varepsilon) \end{aligned}$$

where

$$t_\zeta = \frac{1}{\lambda} \ln\left[\frac{\zeta - e_-(x)}{b\varphi(x)} + O(\zeta - e_-(x))^2\right] , \tag{5.10}$$

and

$$\lim_{\varepsilon \to 0} \omega(\zeta, \varepsilon) = 0.$$

Proof : The proof is exactly the same as the one of Lemma 5.6. Here, the implicit function theorem gives us that, if t_ζ is the unique solution in N_0 of $u(x,t) = \zeta$, then

$$t_\zeta = \frac{1}{\lambda} \ln\left[\frac{\zeta - e_-(x)}{b\varphi(x)} + O(\zeta - e_-(x))^2\right] .$$

Finally, we use the same change of variables as in Lemma 5.6 to obtain the result (see Lemma 5.7 of [6]). □

We recall that, by the choice of $\hat{f}$, the spectrum of A_{e_-} consists only of simple eigenvalues λ_n with eigenvectors $(\varphi_n, \lambda_n\varphi_n)$. Moreover, Proposition 5.1 implies that this set of eigenvectors is a Riesz basis of X.

Lemma 5.8. *There exists a sequence of coefficients $(c_n) \in \ell^2(\mathbb{C})$ such that*

$$\psi^\infty(x,t) = \sum_{n\in\mathbb{N}} c_n e^{-\lambda_n t}\varphi_n(x) .$$

Moreover, if μ is the real number defined in Proposition 5.4,

$$\mu = -\sup\{Re(\lambda_n) \ / \ c_n \neq 0\} ,$$

and we have that, for all $\eta > 0$,

$$\psi^\infty(x,t) = o(e^{(\mu-\eta)t}) \quad \text{and} \quad e^{(\mu+\eta)t} = o(\psi^\infty(x,t)) , \tag{5.11}$$

when t goes to $-\infty$.

Proof : The set $(\varphi_n, \lambda_n \varphi_n)_{n\in\mathbb{N}}$ is a Riesz basis of X. As the eigenvalues are simple, from (5.4) we deduce that the associated biorthonormalised basis consists only of eigenvectors of A^*. We can assume that the eigenfunctions $(\varphi_n, \lambda_n \varphi_n)_{n\in\mathbb{N}}$ are conveniently normalized so that $(\overline{\varphi}_n, -\overline{\lambda}_n \overline{\varphi}_n)_{n\in\mathbb{N}}$ is the associated biorthonormalised basis. We recall that this biorthonormalised basis is also a Riesz basis of X (see [14]). So there exists $(d_n) \in \ell^2(\mathbb{C})$ such that

$$\begin{pmatrix} \psi^\infty \\ \psi^\infty_t \end{pmatrix}(0) = \sum_{n\in\mathbb{N}} d_n \begin{pmatrix} \overline{\varphi}_n \\ -\overline{\lambda}_n \overline{\varphi}_n \end{pmatrix} .$$

More precisely

$$d_n = \left\langle \begin{pmatrix} \psi^\infty \\ \psi^\infty_t \end{pmatrix}(0) \middle| \begin{pmatrix} \varphi_n \\ \lambda_n \varphi_n \end{pmatrix} \right\rangle_X .$$

As $(\overline{\varphi}_n, -\overline{\lambda}_n \overline{\varphi}_n)$ is an eigenvector of

$$-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_{e_-} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

for the eigenvalue $-\overline{\lambda}_n$, we have

$$\begin{pmatrix} \psi^\infty \\ \psi^\infty_t \end{pmatrix}(t) = \sum_{n\in\mathbb{N}} d_n e^{-\overline{\lambda}_n t} \begin{pmatrix} \overline{\varphi}_n \\ -\overline{\lambda}_n \overline{\varphi}_n \end{pmatrix} . \tag{5.12}$$

As $D(B^{1/2})$ is continuously imbedded in $\mathcal{C}^0([0,1])$, we can write

$$\psi^\infty(x,t) = \sum_{n\in\mathbb{N}} d_n e^{-\overline{\lambda}_n t}\, \overline{\varphi}_n(x) . \tag{5.13}$$

Since the spectrum of A_{e_-} is symmetric with respect to the real axis, the set of eigenpairs $\{(\lambda_n, \phi_n)\}$ is equal to the set $\{(\overline{\lambda}_n, \overline{\phi}_n)\}$. So, we can reorder the sum (5.13) to obtain a set of coefficients $\{c_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$ such that

$$\psi^\infty(x,t) = \sum_{n\in\mathbb{N}} c_n e^{-\lambda_n t} \varphi_n(x) .$$

As μ is defined by

$$\mu = \lim_{t\to-\infty} \ln \left\| \begin{pmatrix} \psi^\infty \\ \psi^\infty_t \end{pmatrix} \right\|_X^{1/t} ,$$

the decomposition (5.12) implies that

$$\mu = -\sup\{\mathrm{Re}(\overline{\lambda}_n) \ / \ d_n \neq 0\} = -\sup\{\mathrm{Re}(\lambda_n) \ / \ c_n \neq 0\} .$$

Finally, the last claim of the lemma will directly follow from Proposition 5.5, if we prove that $(d_n\overline{\varphi}_n(x))_{n\in\mathbb{N}}$ belongs to $\ell^1(\mathbb{C})$.
Due to Proposition 5.4, $(\psi^\infty,\psi_t^\infty)\in D(A^*)$. So we can write in X

$$\begin{aligned}
A^*\begin{pmatrix}\psi^\infty\\ \psi_t^\infty\end{pmatrix}(0) &= \sum\left\langle A^*\begin{pmatrix}\psi^\infty\\ \psi_t^\infty\end{pmatrix}(0)\middle|\begin{pmatrix}\varphi_n\\ \lambda_n\varphi_n\end{pmatrix}\right\rangle_X\begin{pmatrix}\overline{\varphi}_n\\ -\overline{\lambda}_n\overline{\varphi}_n\end{pmatrix}\\
&= \sum\left\langle\begin{pmatrix}\psi^\infty\\ \psi_t^\infty\end{pmatrix}(0)\middle|A\begin{pmatrix}\varphi_n\\ \lambda_n\varphi_n\end{pmatrix}\right\rangle_X\begin{pmatrix}\overline{\varphi}_n\\ -\overline{\lambda}_n\overline{\varphi}_n\end{pmatrix}\\
&= \sum d_n\overline{\lambda}_n\begin{pmatrix}\overline{\varphi}_n\\ -\overline{\lambda}_n\overline{\varphi}_n\end{pmatrix}.
\end{aligned}$$

which implies that $|d_n\overline{\lambda}_n|^2$ is summable since $((\overline{\varphi}_n,-\overline{\lambda}_n\overline{\varphi}_n))_{n\in\mathbb{N}}$ is a Riesz basis. In addition, applying the equivalence of norms (2.8) to the vector $(\overline{\varphi}_n,-\overline{\lambda}_n\overline{\varphi}_n)$, we have that $\|(\overline{\varphi}_n,-\overline{\lambda}_n\overline{\varphi}_n)\|_X$ is uniformly bounded by $\frac{1}{a_1}$. So, we can write

$$\sum|d_n\overline{\varphi}_n(x)|\le\frac{1}{a_1}\sum|d_n|\le\frac{1}{a_1}\sqrt{\sum\frac{1}{|\lambda_n|^2}}\sqrt{\sum|\lambda_n d_n|^2}.$$

As we know, by Proposition 5.1, that Hypothesis (Spec)b) is also valid for A_{e_-}, we have that $\sum(\frac{1}{|\lambda_n|^2})$ is convergent, and thus $(d_n\overline{\varphi}_n(x))_{n\in\mathbb{N}}$ belongs to $\ell^1(\mathbb{C})$. $\square$

Fourth step : Conclusion
Summarizing the above arguments and computations, we get

$$\begin{aligned}
J &= J_0+\sum_{i=1}^m J_i\\
&= \frac{\psi^\infty(t_\zeta)}{\lambda\phi(x)(\zeta-e_-(x))}+O\left(\frac{\psi^\infty(t_\zeta)}{(\zeta-e_-(x))^{1-\frac{\delta}{\lambda}}}+(\zeta-e_-(x))^{\frac{\mu}{\lambda}-1+\frac{\delta}{\lambda}}\right)\\
&\quad +S((\zeta-e_-(x))^r)+\omega(\zeta,\varepsilon)\\
&= G(\zeta)+\omega(\zeta,\varepsilon)\ ,
\end{aligned}$$

where $\omega(\zeta,\varepsilon)\longrightarrow 0$ when $\varepsilon\longrightarrow 0$, r is a rational number, and

$$t_\zeta=\frac{1}{\lambda}\ln\left[\frac{\zeta-e_-(x)}{b\varphi(x)}+O(\zeta-e_-(x))^2\right]\ . \tag{5.14}$$

To prove that Hypothesis (h7)(c) of Theorem A.1 is satisfied, and so to complete our proof, we must find ζ and ε such that

$$J=G(\zeta)+\omega(\zeta,\varepsilon)\neq 0.$$

As $\omega(\zeta,\varepsilon) \longrightarrow 0$ when ε goes to 0, we only need to find ζ such that $G(\zeta) \neq 0$, and then choose a positive number ε small enough to ensure that $G(\zeta) + \omega(\zeta,\varepsilon) \neq 0$. Assume that the first term of the series $S(z)$ is of order k (where k may be $+\infty$ if $S(z) = 0$). There are three cases.

1) If $kr < \frac{\mu}{\lambda} - 1$, then, using (5.14) and (5.11), we have that, for all $\eta > 0$,

$$\psi^{\infty}(t_\zeta) = o(\zeta - e_-(x))^{\frac{\mu}{\lambda}-\eta} \ ,$$

and in particular,

$$\frac{\psi^{\infty}(t_\zeta)}{\zeta - e_-(x)} = o(\zeta - e_-(x))^{kr}.$$

The dominant terms of J_0 and $\sum_{i=1}^{m} J_i$ are different, and so we can find ζ as small as needed such that $G(\zeta) \neq 0$.

2) Assume now that $kr > \frac{\mu}{\lambda} - 1$, we can conclude just as in the preceding case since we have

$$(\zeta - e_-(x))^{kr} = o\left(\frac{\psi^{\infty}(t_\zeta)}{\zeta - e_-(x)}\right).$$

3) Finally, we assume that $kr = \frac{\mu}{\lambda} - 1$. Notice that, by construction, $\hat{f}$ is assumed to satisfy the irrational ratio property of Theorem 4.4. As $\frac{\mu}{\lambda} = kr + 1$ is a rational number, $-\mu$ cannot be a real negative eigenvalue of A_{e_-} or the number $-C_{ev}$. We know, applying Lemma 5.1, that A_{e_-} satisfies Hypothesis (Spec)b). Using Lemma 5.8, as

$$\mu = -\sup\{\mathrm{Re}(\lambda_n) \ / \ c_n \neq 0\} \ ,$$

the only possibility is that $-\mu$ is the real part of nonreal eigenvalues. Thus, there exists a finite number of nonreal eigenvalues λ_1,..., λ_p, with $\mathrm{Re}(\lambda_l) = -\mu$, such that

$$\psi^{\infty}(t) = \sum_{l=1}^{p} d_l e^{-\lambda_l t} + \sum_{\mathrm{Re}\lambda < -\mu} c_\lambda e^{-\lambda t} \phi_\lambda(x) \ ,$$

where the coefficients d_l are not zero. Since $-C_{ev}$ is the only accumulation point of the real part of the spectrum of A_{e_-} and $\mu \neq -C_{ev}$, we deduce from this equality that

$$\psi^{\infty}(t) = e^{\mu t} P(t) + o(e^{\mu t}) \ ,$$

where

$$P(t) = \sum_{l=1}^{p} d_l e^{-i\mathrm{Im}(\lambda_l)t}.$$

So we have,

$$\frac{\psi^\infty(t_\zeta)}{\lambda\phi(x)(\zeta - e_-(x))} = \frac{(\zeta - e_-(x))^{\frac{\mu}{\lambda}-1}}{\lambda\phi(x)} P(t_\zeta) + o(\zeta - e_-(x))^{\frac{\mu}{\lambda}-1}.$$

Let Cz^k be the dominant term of the series $S(z)$. We must prove, as in the other cases, that we can find ζ as small as needed such that

$$G(\zeta) = (C - \frac{P(t_\zeta)}{\lambda\phi(x)})(\zeta - e_-(x))^{kr} + o(\zeta - e_-(x))^{kr} \neq 0.$$

As $P(t)$ is a non-constant almost-periodic function, we can find a constant $C' \neq C$ and a sequence of times $(t_n) \longrightarrow +\infty$ such that $P(t_n) = C'\lambda\phi(x)$. So, we have a sequence $(\zeta_n) \longrightarrow e_-(x)$ with

$$G(\zeta_n) = (C - C')(\zeta_n - e_-(x))^{kr} + o(\zeta_n - e_-(x))^{kr} \neq 0,$$

and obviously, for n large enough, $G(\zeta_n) \neq 0$.

We have proved that, in all the cases, we can find ζ and ε small enough, such that

$$J = \int_{\mathbb{R}} \psi(x,t) g_{\zeta,\varepsilon}(u(x,t))dt \neq 0.$$

Our proof is now complete.

A Appendix

A.1 Brunovský-Polàčik-Raugel Theorem

In this subsection, we recall Theorem 4.8 of [6]. This abstract theorem is the key point of [6], as it gives the genericity of the transversality of the heteroclinic orbits. Our Theorem 2.6 is a more concrete result but concerns only damped wave equations and so it is less general than the Brunovský-Polàčik-Raugel abstract theorem. Since we refer to it to prove Theorem 2.6, we recall Theorem 4.8 of [6]. Notice that the version of the Brunovský-Polàčik-Raugel theorem, that we recall here, is not exactly the one which can be found in [6], but is a slightly stronger version given in [7].
We recall that $\mathrm{Ind}(E)$ denotes the Morse index of a hyperbolic equilibrium E, that is the dimension of the local unstable manifold of E.

We consider the abstract semilinear equation with a nonlinearity F depending on a parameter $\lambda \in \Lambda$, where Λ is a Banach space,

$$\frac{\partial U}{\partial t}(t) = AU(t) + F(U(t), \lambda) \ , \quad t > 0 \ , \quad U(0) = U_0 \ . \tag{A.1}$$

We also introduce a Banach space Z, with $X \subset Z$. We assume that

(FP) X is a reflexive Banach space and the inclusion of X into Z is continuous.

(AP) A is the generator of a $\mathcal{C}^0$-semigroup on the Banach space X.

For $M > 0$ fixed, we introduce the open set $G = \{v \in X \mid \|v\|_Z < M\}$ in X and a mapping $F \in \mathcal{C}_b^r(G \times L, X)$, $r \geq 1$, where L is open in the Banach space Λ. The mapping $F(.,\lambda) : x \in G \longmapsto F(x,\lambda) \in X$ is of class $\mathcal{C}^{1,1}_{loc}(X,X)$ uniformly in $\lambda \in L$.

(BUP1) If $\lambda \in L$ and $U_1(t)$ and $U_2(t)$ are two solutions in $\mathcal{C}^0([0,T],X)$ of (A.1), and if there exists τ, $0 \leq \tau \leq T$ such that $U_1(\tau) = U_2(\tau)$, then $U_1(t) = U_2(t)$ for all $t \in [0,T]$.

(BUP2) If $\lambda \in L$ and $\tilde{U}(t)$ is a solution of (A.1) on an interval (t_1, t_2), then the evolution operator $T_{\tilde{U}}(t,s) \in L(X,X)$ defined by the linear variational equation

$$\frac{\partial Y}{\partial t}(t) = AY(t) + DF(\tilde{U}(t),\lambda)Y(t) \ , \quad t > s \ , \quad Y(s) = Y_0 \ . \tag{A.2}$$

is injective and its image is dense in X for any $t_1 < s \leq t < t_2$.

Theorem A.1. *Assume that (AP), (FP), (BUP1) and (BUP2), together with the following additional assumptions are satisfied.*

(h1) The Banach space Λ is separable.

(h2) A has a compact resolvent.

(h3) For any bounded set $L_0 \subset L$, F belongs to the space of $\mathcal{C}^2$-functions of $G \times L_0$ into X whose derivatives up to order 2 are bounded on $G \times L_0$.

(h4) For any $\lambda \in L$, all equilibria of (A.1) are hyperbolic.

(h5) For any $\lambda \in L$, all nonconstant bounded (in the norm of X) solutions on $\mathbb{R}$ of (A.1) are heteroclinic orbits.

(h6) For any $\lambda_0 \in L$ and any $R > 0$, there exist a neighborhood $\mathcal{V}_0$ of λ_0 in L and a positive constant $C = C(\lambda_0, R)$ such that, if $U(t)$ is a heteroclinic orbit of (A.1) for $\lambda \in \mathcal{V}_0$ and if $\max_{t\in\mathbb{R}} \|U(t)\|_X \leq R$, then $U(t)$ is a classical solution of (A.1) and

$$\|U(t)\|_{\mathcal{C}^0_b(\mathbb{R},D(A))\cap\mathcal{C}^1_b(\mathbb{R},X)} \leq C(\lambda_0, R) \ .$$

(h7) Given any $\lambda_0 \in L$ and any neighborhood $\mathcal{V}$ of λ_0 in L, there exist $\hat{\lambda} \in \mathcal{V}$ and a Banach space $\hat{\Lambda}$ with the following properties

(a) $\hat{\lambda} \in \hat{\Lambda}$ and $\hat{\Lambda}$ is continuously embedded in Λ.

(b) $\hat{\lambda}$ has an open neighborhood $\hat{\mathcal{V}}$ in $\hat{\Lambda}$ such that $\hat{\mathcal{V}} \subset \mathcal{V}$ and

$$F|_{G\times\hat{\mathcal{V}}} \in \mathcal{C}^r(G \times \hat{\mathcal{V}}, X) \text{ with the derivatives up to order } r \text{ bounded} \ ,$$

where $r > Ind(E) + 1$ for any equilibrium point E of (A.1) with $\lambda = \hat{\lambda}$.

(c) If $\tilde{U}$ is a heteroclinic solution of (A.1) with $\lambda = \hat{\lambda}$ and $\Psi(t)$, $t \in \mathbb{R}$, is a nontrivial bounded mild solution of

$$\frac{\partial \Psi}{\partial t}(s) = -(A^* + DF^*(\tilde{U}(s), \hat{\lambda}))\Psi(s) \ , \quad s < t \ , \quad \Psi(t) = \Psi_0 \ .$$

then there exists $\lambda \in \hat{\Lambda}$ such that

$$\int_{-\infty}^{+\infty} \langle \Psi(t), D_\lambda F(U(t), \hat{\lambda})\lambda \rangle_X \, \mathrm{d}t \neq 0 \ .$$

Under these assumptions, there is a generic (or residual) subset $\mathcal{L} \subset L$ such that for any $\lambda \in \mathcal{L}$, any heteroclinic orbit of (A.1) contained in G is transverse.

A.2 Sard-Smale theorem

The following theorem is a main tool to prove genericity results. We give here the simplest version (see for example [24] for a proof or stronger versions).
We recall that, if f is a differentiable function from X into Z, a value $z \in Z$ is said to be regular for f if for any $x \in f^{-1}(z)$, the differential $Df(x)$ is surjective.

Theorem A.2. *Let X, Y, Z be three Banach spaces, $U \subset X$ and $V \subset Y$ two open sets, and $\Phi : U \times V \longrightarrow Z$ be a mapping of class $\mathcal{C}^r$ ($r \geq 1$). Let z be an element of Z. Assume that the following hypotheses hold :*
(i) for each $(x, y) \in \Phi^{-1}(z)$, $D_x\Phi(x, y)$ is a Fredholm operator of index strictly less than r.
(ii) for each $(x, y) \in \Phi^{-1}(z)$, $D\Phi(x, y)$ is surjective.
(iii) X and Y are separable metric spaces.
Then the set $\{y \in Y/z \text{ is a regular value of } \Phi(., y)\}$ is a generic subset of Y.

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Chapitre 5 : Convergence de l'équation des ondes amorties à l'intérieur vers l'équation des ondes amorties sur le bord

1 Introduction

This article is devoted to the comparison of the dynamics of the wave equation damped in the interior of the domain Ω with the dynamics of the wave equation damped on the boundary of Ω, when the interior damping converges to a Dirac distribution supported by the boundary.

One of the physical motivation is the following. We consider a soundproof room, where carpet covers all the walls. This situation is modeled as follows. Let Ω be a smooth bounded domain of $\mathbb{R}^d$ ($d = 1, 2$ or 3) and let γ be a non-negative function in $\mathbb{L}^\infty(\partial\Omega)$ (the effective dissipation of the carpet at a point of the wall). The propagation of waves in the room is modeled by the wave equation damped in the boundary

$$\begin{cases} u_{tt}(x,t) = (\Delta - Id)u(x,t) + f(x,u(x,t)) \ , & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x,t) + \gamma(x)u_t(x,t) = 0 \ , & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u,u_t)_{|t=0} = (u_0,u_1) \in \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega) \end{cases} \tag{1.1}$$

Notice that, in this model, the waves are not dissipated in the interior of the room but instantaneously damped at each rebound on the walls. This corresponds to a ponctual dissipation of the form $\gamma(x) \otimes \delta_{x\in\partial\Omega}$, where $\delta_{x\in\partial\Omega}$ is the Dirac function supported by the boundary. Of course, this is an approximation of the reality, as the carpet has some thickness. Thus, we can model more precisely the propagation of waves in the soundproof room by the equation

$$\begin{cases} u_{tt}(x,t) + \gamma_n(x)u_t(x,t) = (\Delta - Id)u(x,t) + f(x,u(x,t)) \ , & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial}{\partial \nu}u(x,t) = 0 \ , & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u,u_t)_{|t=0} = (u_0,u_1) \in \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega) \end{cases} \tag{1.2}$$

where γ_n is a bounded function, which is positive on a small neighborhood of $\partial\Omega$ and vanishes elsewhere.

The purpose of this paper is to study the relevance of the model equation (1.1), that is to understand in which sense the dynamics of Equation (1.2) converge to the ones of Equation

(1.1) when γ_n converges to $\gamma_\infty = \gamma(x) \otimes \delta_{x\in\partial\Omega}$ in the sense of distributions. This paper is also an opportunity to present in a different way some classical proofs on stability of gradient Morse-Smale systems.

Both equations have been extensively studied, we refer for example to [8], [10], [14], [23], [40] and [45] for the wave equation with internal damping (1.2) ; and [9], [11], [29], [31], [32], [44] and [46] for the wave equation with boundary damping (1.1). However, the convergence of the dynamics of Equation (1.2) to these of Equation (1.1) has apparently not yet been studied. The only work in this direction is the convergence of the internal control of the wave equation towards boundary control in the one-dimensional case (see [13]). In this paper, we have chosen to focus on the convergence of the compact global attractor of (1.2) to the one of (1.1), when they exist, and on the comparison of the respective dynamics on them. Indeed, the compact global attractor, which consists of all the globally bounded solutions on $\mathbb{R}$, is somehow representative of the dynamics of the equation. We note that the study of convergence of attractors for other less regular perturbations is classical ; the main tools can be found for example in [19], [3], [4] and [41].
We introduce the spaces $X = \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)$ and $X^s = \mathbb{H}^{1+s}(\Omega) \times \mathbb{H}^s(\Omega)$ $(s \in \mathbb{R})$. In the general case, we are able to prove results similar to the following one.

Let Ω be a smooth bounded domain of $\mathbb{R}^2$, let $\gamma_\infty = \delta_{x\in\partial\Omega}$ and $\gamma_n(x) = n$ if $\mathrm{dist}(x, \partial\Omega) < 1/n$ and 0 elsewhere. Let $f \in \mathcal{C}^2(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ be such that $\sup_{x\in\Omega} \limsup_{|u|\to+\infty} \frac{f(x,u)}{u} < 0$ and that there exist two constants $C > 0$ and $p \in \mathbb{R}_+$ so that $|f''_{uu}(x, u)| + |f''_{x,u}(x, u)| < C(1 + |u|^p)$ for $(x, u) \in \Omega \times \mathbb{R}$.

Theorem 1.1. *Let Ω, γ_n, γ_∞ and f be as described above. Then, Equations (1.1) and (1.2) have compact global attractors $\mathcal{A}_\infty$ and $\mathcal{A}_n$ respectively. Moreover, the union of the attractors $(\cup_{n\in\mathbb{N}\cup\{+\infty\}}\mathcal{A}_n)$ is bounded in X and the attractors $(\mathcal{A}_n)$ are upper-semicontinuous at $\mathcal{A}_\infty$ in X^{-s}, for any $s > 0$, that is,*

$$\sup_{U_n\in\mathcal{A}_n} \inf_{U_\infty\in\mathcal{A}_\infty} \|U_n - U_\infty\|_{X^{-s}} \longrightarrow 0 \ .$$

If all the equilibrium points of (1.1) are hyperbolic, the attractors $(\mathcal{A}_n)$ are lower- semicontinuous in X at $\mathcal{A}_\infty$. Moreover, the upper and lower semicontinuity can be estimated in the sense that there exists $\delta > 0$ such that

$$\max\left(\sup_{U_n\in\mathcal{A}_n} \inf_{U_\infty\in\mathcal{A}_\infty} \|U_n - U_\infty\|_{X^{-s}} \ ; \ \sup_{U_\infty\in\mathcal{A}_\infty} \inf_{U_n\in\mathcal{A}_n} \|U_\infty - U_n\|_X \right) \leq \frac{1}{n^\delta} \ .$$

In general, we cannot prove upper-semicontinuity in X because the perturbation is too singular. Let A_n and A_∞ be the linear operators associated respectively with the equations (1.2) and (1.1). The perturbation is not regular in the sense that $e^{A_n t}$ does not converge to $e^{A_\infty t}$ in $\mathcal{L}(X)$. However, we can prove that, in general, A_n^{-1} converges to A_∞^{-1} in $\mathcal{L}(X)$ and

that this convergence of the inverses implies convergence of the trajectories in X^{-s} for any initial data in X, and convergence of the trajectories in X if the initial data (u_0, u_1) are bounded in a more regular space X^s $(s > 0)$.
The proof of the lower-semicontinuity in X uses as main arguments the gradient structure of (1.1) and (1.2), as well as the convergence of the local unstable manifolds of the equilibria. To prove this property, we identify the local unstable manifolds with local strongly unstable manifolds and show the continuity of these manifolds with respect to the parameter n. Although our perturbation is irregular, we can prove lower-semicontinuity in X due to the regularity of the local unstable manifolds of the equilibria of the limit problem.
The upper-semicontinuity instead cannot be shown in X in general. Indeed, we know that the union $\cup_n \mathcal{A}_n$ is bounded in X, but we do not know if it is bounded in a more regular space X^s. Thus, for initial data in $\cup_n \mathcal{A}_n$, we are able to compare the trajectories only in the norm of X^{-s}.
To prove upper-semicontinuity in X, we need to bound $\cup_n \mathcal{A}_n$ in X^s for some $s > 0$. The main way to prove this property is to show a uniform decay rate for the semigroups, that is that there exist constants $M > 0$ and $\lambda > 0$ such that, for all $U \in X$ and $t \geq 0$, we have

$$\forall n \in \mathbb{N},\ \|e^{A_n t} U\|_X \leq M e^{-\lambda t} \|U\|_X \ . \tag{1.3}$$

Such estimate is well-known for fixed n. However, the methods for proving the exponential decay for fixed n often give constants M and λ depending on $\|\gamma_n\|_{\mathbb{L}^\infty}$, or are based on a contradiction argument. Thus, they are not adaptable to the proof of a uniform estimate in the case of our irregular perturbation, where $\|\gamma_n\|_{\mathbb{L}^\infty}$ goes to $+\infty$. In dimension two and higher dimension, the uniform bound (1.3) is not known to hold, except for some very particular examples presented here. In the one-dimensional case, we give necessary and sufficient conditions for (1.3) to hold. The proof uses a multiplier method and is inspired by [13] and [23] (other methods are also possible, see the result of [2] in the appendix). Thus, in dimension one, we can show a more precise result, which is typically the following.

Let $\Omega =]0,1[$, $\gamma_\infty = 2\delta_{x=0}$ and $\gamma_n(x) = 2n$ if $x \in]0, \frac{1}{n}[$ and 0 elsewhere. Let $f \in \mathcal{C}^2([0,1] \times \mathbb{R}, \mathbb{R})$ be such that $\sup_{x\in\Omega} \limsup_{|u|\to+\infty} \frac{f(x,u)}{u} < 0$. Notice that we do not choose $\gamma_\infty = \delta_{x=0}$ because, with this dissipation, Equation (1.1) does not satisfy the backward uniqueness property. Without backward uniqueness result, we cannot properly define the Morse-Smale property (see [11] and the remarks preceding Theorem 2.12).

Theorem 1.2. *Let Ω, γ_n, γ_∞ and f be as described above. Then, Equations (1.1) and (1.2) have compact global attractors $\mathcal{A}_\infty$ and $\mathcal{A}_n$ respectively. Moreover, the union of the attractors $(\cup_{n\in\mathbb{N}\cup\{+\infty\}} \mathcal{A}_n)$ is bounded in X^s for $s \in]0, 1/2[$. As a consequence, the attractors $\mathcal{A}_n$ are upper-semicontinuous at $\mathcal{A}_\infty$ in the space X.*
If all the equilibrium points of (1.1) are hyperbolic, then the sequence of attractors $(\mathcal{A}_n)$ is

continuous in X in the sense that there exists $\delta > 0$ such that

$$\max\left(\sup_{U_n\in\mathcal{A}_n}\inf_{U_\infty\in\mathcal{A}_\infty}\|U_n-U_\infty\|_X \ ; \ \sup_{U_\infty\in\mathcal{A}_\infty}\inf_{U_n\in\mathcal{A}_n}\|U_\infty-U_n\|_X\right)\leq\frac{1}{n^\delta} \ .$$

In dimension one, we can even go further and compare the dynamics on the attractors $\mathcal{A}_n$ and $\mathcal{A}_\infty$. A part of this comparison is described by the notion of equivalence of phase-diagrams. Let $S(t)$ be a gradient dynamical system which admits a compact global attractor with only hyperbolic equilibrium points. If E and E' are two equilibrium points of $S(t)$, we say that $E \leq E'$ if and only if there exists a trajectory $U(t) \in \mathcal{C}^0(\mathbb{R}, X)$ such that

$$\lim_{t\to-\infty} U(t) = E' \text{ and } \lim_{t\to+\infty} U(t) = E \ .$$

The phase-diagram of $S(t)$ is the above oriented graph on the set of equilibria. Two phase-diagrams are equivalent if there exists an isomorphism between the set of equilibria, which preserves the oriented edges.
It is proved in [19], [37] and [38] that the stability of phase-diagrams is related to the Morse-Smale property. We recall that a gradient dynamical system $S(t)$ has the Morse-Smale property if it has a finite number of equilibrium points which are all hyperbolic and if the stable and unstable manifolds of these equilibria intersect transversally. The result of [19] says that if $S_0(t)$ is a dynamical system, which satisfies the Morse-Smale property, and if $S_\varepsilon(t)$ is a "regular" perturbation of $S_0(t)$ such that the compact global attractors of $S_\varepsilon(t)$ are upper-semicontinuous at $\varepsilon = 0$, then $S_\varepsilon(t)$ satisfies the Morse-Smale property for ε small enough and its phase-diagram is equivalent to the one of $S_0(t)$. Unfortunately, our perturbation is not regular enough for a direct application of [19]. However, using the smoothness of the attractors, we can adapt the proof of [19] to show the following result.

Theorem 1.3. *Let Ω, γ_n, γ_∞ and f be as in Theorem 1.2. If the dynamical system generated by (1.1) satisfies the Morse-Smale property, then, for n large enough, the dynamical system generated by (1.2) satisfies the Morse-Smale property and its phase-diagram is equivalent to the one of (1.1). Moreover, there exists a homeomorphism h defined from $\mathcal{A}_n$ into $\mathcal{A}_\infty$ which maps the trajectories of $S_n(t)_{|\mathcal{A}_n}$ onto the trajectories of $S_\infty(t)_{|\mathcal{A}_\infty}$ preserving the sense of time.*

We notice that (1.1) satisfies the Morse-Smale property for a generic non-linearity f (see [26]). We also enhance that we give a proof of Theorem 1.3 presented in a way, which is different from [19], and, which extensively uses the gradient structure of (1.1) and (1.2).

Of course, in this paper, we do not only consider the particular situations of Theorems 1.2 and 1.1, but more general cases. The general frame, the main hypotheses and the main results are stated in Section 2. The abstract result of convergence for semigroups of contractions and the study of the convergence of the trajectories of Equation (1.1) to those of Equation (1.2) are given in Section 3. Continuity of the local unstable manifolds and of part of the local stable manifolds as well as stability of phase-diagrams are studied in Sections 4 and 5 respectively. In Section 6, we give concrete conditions under which the inequality (1.3) holds. In Section 7, we describe examples of applications. Finally, in the Appendix, we state the above-mentionned result of [2] and study another one-dimensional case.

2 Setting of the problem and main results

In this section, we first introduce the notation. We immediately prove a first result of convergence, without which nothing can be done. This leads to a condition, which will be implicitely assumed in all what follows. Finally, in the last part of this section, we put together the main hypotheses, which will be used, and state the most important results.

2.1 The abstract frame

We introduce an abstract frame for Equations (1.1) and (1.2). This has two purposes. The first one is to give results, which concern a larger family of equations than (1.1) and (1.2) (for example, other boundary conditions can be chosen). The second advantage of the abstract setting is to gather Equations (1.1) and (1.2) into a common frame, which makes the comparison easier.

Let Ω be a smooth bounded domain of $\mathbb{R}^d$ ($d = 1, 2$ or 3) and let ω_N be a non-empty smooth open subset of $\partial\Omega$. We denote by ω_D the largest open subset of $\partial\Omega \setminus \omega_N$.
If $\omega_D \neq \emptyset$, we set $B = -\Delta_{BC}$ where Δ_{BC} is the Laplacian with Neumann boundary condition on ω_N and Dirichlet condition on ω_D. If ω_N covers the whole boundary, we set $B = -\Delta_N + Id$ where Δ_N is the Laplacian with Neumann boundary condition. In all the cases, B is a positive self-adjoint operator from $D(B)$ into $\mathbb{L}^2(\Omega)$.
Let (λ_k, φ_k) be the set of eigenvalues of B and corresponding eigenvectors normalized in $\mathbb{L}^2(\Omega)$. We denote $D(B^{s/2})$ the Hilbert space

$$D(B^{s/2}) = \left\{ u = \sum c_k \varphi_k \ / \ \|u\|^2_{D(B^{s/2})} = \sum |c_k|^2 \lambda_k^s < +\infty \right\} \ .$$

We notice that for $s \in [0, 1/2[$, $D(B^{s/2}) = \mathbb{H}^s(\Omega)$ and for $s \in]1/2, 5/4]$, $D(B^{s/2}) = \mathbb{H}^s(\Omega) \cap \{u \in \mathbb{H}^s(\Omega) / u_{|\omega_D} = 0\}$ (see Proposition 2.1). For larger s, the domain of $B^{s/2}$ can be less simple due to the regularity problem induced by mixed boundary conditions. We set

$$X = D(B^{1/2}) \times \mathbb{L}^2(\Omega) \ ,$$

endowed with the product topology. We also set $X^s = D(B^{(1+s)/2}) \times D(B^{s/2})$. Let γ be a non-negative function in $\mathbb{L}^\infty(\omega_N)$, which is positive on an open subset of ω_N. We set $\gamma_\infty(x) = \gamma(x)\delta_{x\in\omega_N}$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of non-negative functions in $\mathbb{L}^\infty(\Omega)$, which are positive on an open subset of Ω and which converge to γ_∞ in the sense of distributions, that is that

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d),\ \int_\Omega \gamma_n\varphi \longrightarrow \int \gamma_\infty\varphi = \int_{\omega_N} \gamma\varphi\ .$$

For each $n \in \mathbb{N}$, we introduce the linear continuous operator Γ_n, defined from $D(B^{1/2})$ into $D(B^{1/2})$ by $\Gamma_n = B^{-1}(\gamma_n .)$. We also introduce the operator Γ_∞ defined from $D(B^{1/2})$ into $D(B^{1/2})$ by

$$\forall u \in D(B^{1/2}),\ \Gamma_\infty u \text{ is the solution of } \left\{\begin{array}{ll} (\Delta - \kappa Id)\Gamma_\infty u = 0 & \text{on } \Omega \\ \frac{\partial}{\partial\nu}\Gamma_\infty u = \gamma(x)u & \text{on } \omega_N \\ \Gamma_\infty u = 0 & \text{on } \omega_D \end{array}\right. , \tag{2.1}$$

where $\kappa = 1$ if $\omega_D = \emptyset$ and $\kappa = 0$ if not. We remark that

$$\forall n \in \mathbb{N},\ \forall \varphi,\psi \in D(B^{1/2}),\ <\Gamma_n\varphi|\psi>_{D(B^{1/2})} = \int_\Omega \gamma_n\varphi\overline{\psi}\ ,$$

and

$$\forall \varphi,\psi \in D(B^{1/2}),\ <\Gamma_\infty\varphi|\psi>_{D(B^{1/2})} = \int_{\partial\Omega} \gamma\varphi\overline{\psi}\ .$$

We set

$$s_0 = \left\{\begin{array}{ll} s_0 = 1/2 & \text{for } d = 1 \text{ or } d = 2 \\ s_0 = 1/4 & \text{for } d = 3 \end{array}\right. \tag{2.2}$$

Proposition 2.1. *For all $\varepsilon > 0$, $s \in [0, s_0[$ and $n \in \mathbb{N}\cup\{+\infty\}$, the operator Γ_n can be extended to a continuous linear operator from $D(B^{\varepsilon+1/4})$ into $D(B^{(1+s)/2})$. In particular, Γ_n is a compact non-negative selfadjoint operator from $D(B^{\varepsilon+1/4})$ into $D(B^{1/2})$.*

Proof : The proposition follows from the regularity properties of the operator B. If $\overline{\omega}_N \cap \overline{\omega}_D = \emptyset$, then the regularity is clear since $D(B^{(1+s)/2}) = \{u \in \mathbb{H}^{1+s}(\Omega)/u_{|\omega_D} = 0\}$ if $s < 1/2$ for any d. If we have mixed boundary conditions with $\overline{\omega}_N \cap \overline{\omega}_D \neq \emptyset$, then the regularity is more difficult to obtain. In dimension $d = 2$ (resp. $d = 3$), we refer to [16] (resp. [12]). $\square$

For all $n \in \mathbb{N}\cup\{+\infty\}$, let A_n be the unbounded operator defined on X by

$$\forall \begin{pmatrix} u \\ v \end{pmatrix} \in X,\ A_n\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -B(u+\Gamma_n v) \end{pmatrix}\ ,$$

$$D(A_n) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X \Big/ v \in D(B^{1/2}) \text{ and } u + \Gamma_n v \in D(B) \right\}\ .$$

We enhance that, if n is finite, A_n is the classical wave operator

$$\forall n \in \mathbb{N},\ A_n = \begin{pmatrix} 0 & Id \\ -B & -\gamma_n \end{pmatrix},\ D(A_n) = D(B) \times D(B^{1/2}) \ .$$

Using the Hille-Yosida theorem, one shows that the operator A_n generates a linear $\mathcal{C}^0-$semigroup $e^{A_n t}$ of contractions (see [29] for $n = +\infty$, see also [26] for a proof in the given abstract frame). In particular, A_n is dissipative since

$$\forall U = (u, v) \in D(A_n),\ < A_n U | U >_X = - < \Gamma_n v | v >_{D(B^{1/2})} \leq 0 \ . \tag{2.3}$$

For $U = (u, v)$, we set

$$F(U) = \begin{pmatrix} 0 \\ f(x, u) \end{pmatrix} \ . \tag{2.4}$$

We are interested in the convergence of the following family of equations, when n goes to $+\infty$

$$\left\{ \begin{array}{l} U_t = A_n U + F(U) \\ U_{|t=0} = U_0 \in X \end{array} \right. \ . \tag{2.5}$$

We first introduce conditions so that the above equations are be well-posed.
In the whole paper, we assume that the non-linearity f satisfies the following hypothesis.

(NL) $f \in \mathcal{C}^2(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and if the dimension is

d=2 there exist $C > 0$ and $\alpha \geq 0$ such that

$$|f''_{uu}(x, u)| + |f''_{ux}(x, u)| \leq C(1 + |u|^\alpha) \ .$$

d=3 there exist $C > 0$ and $\alpha \in [0, 1[$ such that

$$|f''_{uu}(x, u)| \leq C(1 + |u|^\alpha) \text{ and } |f''_{ux}(x, u)| \leq C(1 + |u|^{3+\alpha}) \ .$$

Since the regularity of f is not the main purpose of this paper, we choose to state Hypothesis (NL) in a simple but surely too strong way. For example, the condition $f \in \mathcal{C}^2$ could be relaxed to the condition $f \in \mathcal{C}^1$ with Hölder continuous derivatives. We can also assume an exponential growth rate for the non-linearity if $d = 2$ (see [21] or [5]). We notice that, for most of our results, weaker hypotheses on f are sufficient. For example, the critical case of a cubic non-linearity in dimension $d = 3$ is studied in [27].
To obtain global existence of solutions and existence of a compact global attractor, we also need to assume a dissipative condition for f, for example,

(Diss) $$\sup_{x \in \Omega} \limsup_{|u| \to +\infty} \frac{f(x, u)}{u} < 0 \ .$$

Classical Sobolev imbeddings (see for example [1]) show that Hypothesis (NL) implies the following properties (see Chapters 4.7 and 4.8 of [17] for a proof).

Lemma 2.2. *Assume that Hypothesis (NL) holds. Then, there exists a positive number p such that for any u, v in $\mathbb{H}^1(\Omega)$, we have*

$$\|f(x,u) - f(x,v)\|_{\mathbb{L}^2} \leq C(1 + \|u\|_{\mathbb{H}^1}^p + \|v\|_{\mathbb{H}^1}^p)\|u - v\|_{\mathbb{H}^1} \ .$$

Moreover, if $\mathcal{B}$ is a bounded set of $\mathbb{H}^1(\Omega)$, then $\{f(x,u)|u \in \mathcal{B}\}$ and $\{f'_u(x,u)v|(u,v) \in \mathcal{B}^2\}$ are bounded subsets of $\mathbb{H}^\sigma(\Omega)$, where $\sigma \in]0,1[$ when $d = 1$ or $d = 2$ and $\sigma \in]0, \frac{1-\alpha}{2}[$ when $d = 3$. In addition, we have

$$\forall u \in \mathcal{B}, \ \|f(x,u)\|_{\mathbb{H}^\sigma} \leq C_\sigma \|u\|_{\mathbb{H}^1} \text{ and } \|f'_u(x,u)v\|_{\mathbb{H}^\sigma} \leq C_\sigma \|v\|_{\mathbb{H}^1} \ ,$$

where the constant C_σ depends on σ, except if $d = 1$.
In particular, $F : (u,v) \in X \mapsto (0, f(x,u))$ is of class $\mathcal{C}^{1,1}_{loc}(X,X)$ and is a compact and Lipschitz-continuous function on the bounded sets of X.

Using a classical result of local existence (see [39], Chapter 6, Theorem 1.2), we deduce from Hypothesis (NL) that for each $n \in \mathbb{N} \cup \{+\infty\}$, Equation (2.5) generates a local dynamical system $S_n(t)$ on X.

Proposition 2.3. *If f satisfies (NL), then for all $M > 0$ and $K > 0$, there exists a time $T > 0$ such that, for all $n \in \mathbb{N} \cup \{+\infty\}$ and U_0 with $\|U_0\|_X \leq M$, Equation (2.5) has a unique mild solution $U_n(t) = S_n(t)U_0 \in \mathcal{C}^0([0,T], X)$, which satisfies*

$$\forall t \in [0,T], \ \|U_n(t)\|_X \leq M + K \ .$$

Moreover, there exists a constant $C > 0$ such that for all U_0 and U'_0 with $\|U_0\|_X \leq M$ and $\|U'_0\|_X \leq M$ we have

$$\forall n \in \mathbb{N} \cup \{+\infty\}, \ \forall t \in [0,T], \ \|S_n(t)(U_0 - U'_0)\|_X \leq Ce^{Ct}\|U_0 - U'_0\|_X \ .$$

The hypothesis (Diss) implies global existence of trajectories, that is that $S_n(t) : X \longrightarrow X$ are global dynamical systems.

Proposition 2.4. *Assume that f satisfies (NL) and (Diss). Then, for any bounded set $\mathcal{B}$ of X, for any $n \in \mathbb{N} \cup \{+\infty\}$ and for any $U_0 \in \mathcal{B}$, $S_n(t)U_0$ ($t \geq 0$) is a global mild solution of (2.5) and is uniformly bounded in X with respect to t and U_0.*

Proof : For $U = (u,v) \in X$, we set

$$\Phi(U) = \frac{1}{2}\|U\|_X^2 - \int_\Omega \int_0^u f(x,\zeta)d\zeta \ . \tag{2.6}$$

From (2.3) and the density of $D(A_n)$ in X, we deduce that the functional Φ is non-increasing along the trajectories of the dynamical systems $S_n(t)$ ($n \in \mathbb{N} \cup \{+\infty\}$). Indeed, let $U_0 \in D(A_n)$ and $U(t) = (u(t), v(t)) = S_n(t)U_0$, we have

$$\Phi(U(t_2)) - \Phi(U(t_1)) = \int_{t_1}^{t_2} < A_n U(t) | U(t) >_X \, dt = - \int_{t_1}^{t_2} < \Gamma_n v(t) | v(t) >_{D(B^{1/2})} \leq 0 \ . \tag{2.7}$$

Hypothesis (Diss) implies that there exist two positive constants C and μ such that

$$f(x,u)u \leq C - \mu u^2 \text{ and } \int_0^u f(x,\zeta) d\zeta \leq C - \mu u^2 \ . \tag{2.8}$$

So, for any $U_0 \in \mathcal{B}$ and any positive time t such that $S_n(t)U_0$ exists, we have

$$\frac{1}{2} \| S_n(t) U_0 \|_X^2 - C \leq \Phi(S_n(t)U_0) \leq \Phi(U_0) \ .$$

Sobolev imbeddings show that $\Phi(U_0)$ is bounded uniformly with respect to $U_0 \in \mathcal{B}$. Thus, the trajectories cannot blow up and are defined and bounded for all times. $\square$

For $U(t) \in \mathcal{C}^0([0,T], X)$, we can also consider the trajectory $V_n(t) = DS_n(U)(t)V_0$ of the linearized dynamical system $DS_n(U)$ along U, that is the solution of

$$\left\{ \begin{array}{l} \partial_t V_n(t) = A_n V_n(t) + F'(U(t)) V_n(t) \\ V_n(0) = V_0 \in X \end{array} \right. \tag{2.9}$$

Due to Lemma 2.2, $W \in X \longmapsto F'(U)W$ is locally Lipschitzian and Proposition 2.3 is also valid for $DS_n(U)(t)$. Moreover the trajectories $DS_n(U)(t)V_0$ exist for all $t \in [0,T]$ since $DS_n(U)(t)$ is a linear dynamical system.

2.2 Convergence of the inverses

If the inverses A_n^{-1} do not converge to A_∞^{-1}, then one cannot hope any convergence result, since we cannot even ensure that a part of the spectrum of the operators is continuous when n goes to $+\infty$. That is why, we immediatly show that this convergence holds in natural situations. In the rest of the paper, this convergence of the inverses will be assumed.
A simple calculation shows that A_n is invertible of compact inverse and that A_n^{-1} is given by

$$\forall \begin{pmatrix} u \\ v \end{pmatrix} \in X, \ A_n^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Gamma_n u - B^{-1} v \\ u \end{pmatrix} \ . \tag{2.10}$$

We present here a typical situation.
Let θ be a bounded open subset of $\mathbb{R}^{d-1}$ with a boundary of class $\mathcal{C}^\infty$. We set $\tilde{\Omega} =]0,1[\times \theta$.

Let γ be a nonnegative function in $\mathbb{L}^\infty(\theta)$ and let γ_n be a sequence of nonnegative functions in $\mathbb{L}^\infty(\tilde{\Omega})$, which converges to $\gamma \otimes \delta_{x=0}$ in the sense of distributions, that is that

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d),\ \int_{\tilde{\Omega}} \gamma_n(x,y)\varphi(x,y)dxdy \longrightarrow \int_\theta \gamma(y)\varphi(0,y)dy\ .$$

We assume moreover that

$$\sup_{y\in\theta}\left(\left|\gamma(y) - \int_0^1 \gamma_n(x,y)dx\right| + \int_0^1 \gamma_n(x,y)\sqrt{|x|}dx\right) \longrightarrow 0\ . \tag{2.11}$$

Notice that Hypothesis (2.11) is always fullfilled in the one-dimensional case $d = 1$. We have the following result.

Theorem 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^d$. Assume that there exists a covering Ω_1,...,Ω_p of Ω such that the description of the dissipations γ_n on Ω_i is $\mathcal{C}^1-$diffeomorphic to the typical situation described previously. Then, there exists a sequence of positive numbers (c_n) converging to zero such that*

$$\forall \varphi \in \mathbb{H}^1(\Omega),\ \|(\Gamma_\infty - \Gamma_n)\varphi\|_{D(B^{1/2})} \le c_n \|\varphi\|_{\mathbb{H}^1}\ . \tag{2.12}$$

As a consequence, A_n^{-1} converges to A_∞^{-1} in $\mathcal{L}(X)$.

Proof : We recall that, on $D(B^{1/2})$, the norms $\|.\|_{D(B^{1/2})}$ and $\|.\|_{\mathbb{H}^1}$ are equivalent. We have to show that, for all φ and ψ in $D(B^{1/2})$, there exists a sequence of positive numbers (c_n) converging to zero such that

$$< (\Gamma_\infty - \Gamma_n)\phi|\overline{\psi} >_{D(B^{1/2})} \le c_n \|\varphi\|_{\mathbb{H}^1}\|\psi\|_{\mathbb{H}^1}\ ,$$

that is that

$$\int_{\omega_N} \gamma\varphi\psi - \int_\Omega \gamma_n\varphi\psi \le c_n\|\varphi\|_{\mathbb{H}^1}\|\psi\|_{\mathbb{H}^1}\ . \tag{2.13}$$

Clearly, it is sufficient to prove (2.13) in the typical situation introduced above and with smooth functions. Let φ and ψ be two functions of $\mathcal{C}^\infty(\tilde{\Omega})$, and let

$$I_n = \left|\int_\theta \gamma(y)\varphi(0,y)\psi(0,y)dy\ -\ \int_{\tilde{\Omega}} \gamma_n(x,y)\varphi(x,y)\psi(x,y)dxdy\right|\ .$$

We have $I_n \le J_n + K_n$, where

$$J_n = \left|\int_\theta \varphi(0,y)\psi(0,y)\left(\gamma(y) - \int_0^1 \gamma_n(x,y)dx\right)dy\right|$$

and

$$K_n = \left|\int_{\tilde{\Omega}} \gamma_n(x,y)(\varphi(x,y)\psi(x,y) - \varphi(0,y)\psi(0,y))dxdy\right|\ .$$

Let

$$d_n = \sup_{y \in \theta} \left(\left| \gamma(y) - \int_0^1 \gamma_n(x,y) dx \right| + \int_0^1 \gamma_n(x,y) \sqrt{|x|} dx \right) .$$

Using the control of the norm $\mathbb{L}^2(\theta)$ by the norm $\mathbb{H}^1(\tilde{\Omega})$, we obtain

$$J_n \leq d_n \|\varphi\|_{\mathbb{H}^1} \|\psi\|_{\mathbb{H}^1} .$$

For the second term, we write

$$\begin{aligned} K_n \quad &\leq \left| \int_\Omega \gamma_n(x,y) \varphi(x,y) (\psi(x,y) - \psi(0,y)) dxdy \right| \\ &\quad + \left| \int_\Omega \gamma_n(x,y) \psi(0,y) (\varphi(x,y) - \varphi(0,y)) dxdy \right| . \end{aligned} \tag{2.14}$$

We deal with the first term of (2.14) by using the Cauchy-Schwarz inequality

$$\begin{aligned} K_n^1 \quad &= \left| \int_{\tilde{\Omega}} \gamma_n(x,y) \varphi(x,y) (\psi(x,y) - \psi(0,y)) dxdy \right| \\ &= \left| \int_{\tilde{\Omega}} \gamma_n(x,y) \varphi(x,y) \left(\int_0^x \frac{\partial \psi}{\partial x}(\xi, y) d\xi \right) dxdy \right| \\ &\leq \int_{\tilde{\Omega}} \gamma_n(x,y) |\varphi(x,y)| \sqrt{x} \left(\int_0^x \left| \frac{\partial \psi}{\partial x}(\xi,y) \right|^2 d\xi \right)^{1/2} dxdy \\ &\leq \int_\theta \left(\int_0^1 \left| \frac{\partial \psi}{\partial x}(\xi,y) \right|^2 d\xi \right)^{1/2} \left(\sup_{\xi \in]0,1[} |\varphi(\xi,y)| \right) \left(\int_0^1 \gamma_n(x,y) \sqrt{x} dx \right) dy \\ &\leq d_n \int_\theta \left(\int_0^1 \left| \frac{\partial \psi}{\partial x}(\xi,y) \right|^2 d\xi \right)^{1/2} \left(\sup_{\xi \in]0,1[} |\varphi(\xi,y)| \right) dy . \end{aligned}$$

Using the control of the $\mathbb{L}^\infty$-norm by the $\mathbb{H}^1$-norm in the one-dimensional space, we find

$$\begin{aligned} K_n^1 \quad &\leq d_n \int_\theta \left(\int_0^1 \left| \frac{\partial \psi}{\partial x}(x,y) \right|^2 dx \right)^{1/2} \left(\int_0^1 |\varphi(x,y)|^2 + \left| \frac{\partial \varphi}{\partial x}(x,y) \right|^2 dx \right)^{1/2} dy \\ &\leq d_n \left(\int_{\tilde{\Omega}} \left| \frac{\partial \psi}{\partial x}(x,y) \right|^2 dxdy \right)^{1/2} \left(\int_{\tilde{\Omega}} |\varphi(x,y)|^2 + \left| \frac{\partial \varphi}{\partial x}(x,y) \right|^2 dxdy \right)^{1/2} \\ &\leq d_n \|\varphi\|_{\mathbb{H}^1} \|\psi\|_{\mathbb{H}^1} . \end{aligned}$$

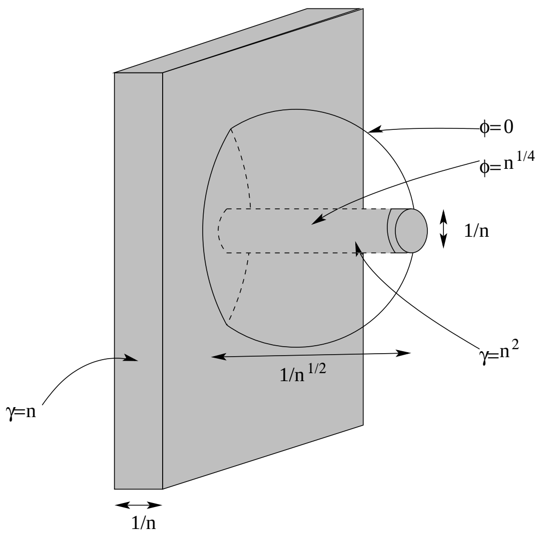
Applying the same argument to the second term of (2.14), we complete the proof of the estimate (2.13).
Thus, we have shown that Γ_n converges to Γ_∞ in $\mathcal{L}(D(B^{1/2}))$. From (2.10) and (2.12), we

deduce that A_n^{-1} converges to A_∞^{-1} in $\mathcal{L}(X)$. $\square$

To show that the natural Hypothesis (2.11) is necessary, we give a counter-example to Theorem 2.5 when (2.11) is omitted.
Let $\Omega =]0,1[\times]-1,1[^2$. Let

$$\gamma_n(x,y) = \begin{cases} n & \text{if } 0 \le x \le \frac{1}{n} \\ n^2 & \text{if } \frac{1}{n} \le x \le \frac{1}{\sqrt{n}}, |y| < \frac{1}{n} \\ 0 & \text{elsewhere.} \end{cases}$$

We notice that γ_n converges to $\gamma = \delta_{x=0}$ in the sense of the distributions. Let $\varphi_n(x,y)$ be the function with support in the ball B of center $(\frac{1}{2\sqrt{n}}, 0, 0)$ and of radius $R = \frac{1}{2\sqrt{n}}$ with $\varphi_n(r,\theta) = \frac{1}{2}n^{1/4} - rn^{3/4}$ in it, where $r = ((x - \frac{1}{2\sqrt{n}})^2 + y^2)^{1/2}$.



In the support of φ_n, the norm of the gradient of φ_n is $n^{3/4}$, so $\|\varphi\|_{\mathbb{H}^1} \sim 1$. We have $\int \gamma_\infty |\varphi_n|^2 = 0$ and

$$\int \gamma_n |\varphi_n|^2 \sim \frac{n^2 (n^{1/4})^2}{n^2 \sqrt{n}} \sim 1 \ .$$

So Γ_n does not converge to Γ_∞ in $\mathcal{L}(D(B^{1/2}))$.

Using the same arguments as in the proof of Theorem 2.5, we obtain the following property.

Proposition 2.6. *We assume that the same hypotheses as in Theorem 2.5 hold. Let* $\frac{1}{2} > s \ge 0$. *There exists* M *independent of* n *such that*

$$\forall n \in \mathbb{N} \cup \{+\infty\},\ \forall U \in D(A_n),\ \|U\|_{X^s} \le M \|U\|_{D(A_n)} \ .$$

Proof : Assume that the proposition is not satisfied. Then, there exists a sequence $U_k = (u_k, v_k)$ such that

$$\|U_k\|_{X^s} = 1 \text{ and } \|U_k\|_{D(A_{n_k})} \longrightarrow 0 \ .$$

This implies that $v_k \longrightarrow 0$ in $D(B^{1/2})$ and $Bu_k + \gamma_{n_k} v_k \longrightarrow 0$ in $\mathbb{L}^2(\Omega)$. If we prove that $\gamma_{n_k} v_k \longrightarrow 0$ in $D(B^{(-1+s)/2})$, then we will have $u_k \longrightarrow 0$ in $D(B^{(1+s)/2})$. But the properties $u_k \longrightarrow 0$ in $D(B^{(1+s)/2})$ and $v_k \longrightarrow 0$ in $D(B^{1/2})$ contradict the fact that $\|U_k\|_{X^s} = 1$.
It remains to show that $\gamma_{n_k} v_k \longrightarrow 0$ in $D(B^{(-1+s)/2})$. Let $\varphi \in D(B^{(1-s)/2})$, we have

$$\left| \int_\Omega \gamma_{n_k} v_k \varphi \right|^2 \leq \int_\Omega \gamma_{n_k} |v_k|^2 \int_\Omega \gamma_{n_k} |\varphi|^2 \ . \tag{2.15}$$

As $v_k \longrightarrow 0$ in $D(B^{1/2})$, we have $\int_\Omega \gamma_{n_k} |v_k|^2 \longrightarrow 0$ by Theorem 2.5. In order to prove that $\int_\Omega \gamma_{n_k} |\varphi|^2$ is bounded, we come back to the typical situation introduced before Theorem 2.5. We have

$$\int_\theta \int_0^1 \gamma_{n_k}(x,y) |\varphi(x,y)|^2 dxdy \leq \left(\sup_{y\in\theta} \int_0^1 \gamma_{n_k}(x,y) dx \right) \left(\int_\theta | \sup_{x\in[0,1]} \varphi(x,y)|^2 dy \right) \ .$$

We know that φ is bounded in $D(B^{(1-s)/2})$ and so in $\mathbb{H}^{1-s}(\Omega)$. In the typical case $\tilde{\Omega} =]0,1[\times\theta$, and thus $\mathbb{H}^{1-s}(\tilde{\Omega}) \hookrightarrow \mathbb{L}^2(\theta, \mathbb{H}^{1-s}(]0,1[))$. Using the fact that $\mathbb{H}^{1-s}(]0,1[)$ is embedded in $\mathcal{C}^0(]0,1[)$, we obtain that $\int_\theta |\sup_{x\in[0,1]} \varphi(x,y)|^2 dy < +\infty$. On the other hand, (2.11) implies that $\sup_{y\in\theta} \int_0^1 \gamma_{n_k}(x,y) dx < +\infty$, which implies the proposition. □

2.3 Main hypotheses and results

In this section, we put together all the main hypotheses and theorems.
We recall that $S_n(t)$ denotes the local dynamical system generated by (2.5). In what follows, we will assume that

$$\varepsilon_n = \|A_\infty^{-1} - A_n^{-1}\|_{\mathcal{L}(X)} \longrightarrow 0 \ . \tag{2.16}$$

Moreover, we also assume in the whole article that f satisfies Hypothesis (NL). In addition, Hypothesis (Diss) will be assumed when we deal with global results.
In Section 3, we show that the convergence of the inverses implies some weak convergence for the trajectories. The convergence is weak in the sense that, in order to compare $S_n(t)U_0$ with $S_\infty(t)U_0$ in the space X^s, U_0 has to belong to a more regular space $X^{s+\varepsilon}$. For example, we will obtain the following results.

Proposition 2.7. *Assume that Hypothesis (Diss) is satisfied. Let $\mathcal{B}$ be a bounded set of X and $s \in [0,1]$, there exists a positive constant C such that*

$$\forall U \in \mathcal{B},\ \forall t \geq 0,\ \|S_\infty(t)U - S_n(t)U\|_{X^{-s}} \leq Ce^{Ct}\varepsilon_n^{s/8} \ . \tag{2.17}$$

If $\mathcal{B}^s$ is a bounded set of X^s ($s \in]0, s_0[$), then there exists a positive constant C such that

$$\forall U \in \mathcal{B}^s,\ \forall t \geq 0,\ \|S_\infty(t)U - S_n(t)U\|_X \leq Ce^{Ct}\varepsilon_n^\beta\ , \tag{2.18}$$

where $\beta = \frac{s^2}{2}$ if d=1 or d=2, and $\beta = \min(\frac{s^2}{2}, \frac{1-\alpha}{4})$ if d=3.

To obtain existence of compact global attractors, we will have to assume that the linear semigroups $e^{A_n t}$ are exponentially decreasing :

(ED) There exists a family of positive constants M_n and λ_n ($n \in \mathbb{N} \cup \{+\infty\}$) such that

$$\|e^{A_n t}\|_{\mathcal{L}(X)} \leq M_n e^{-\lambda_n t}\ .$$

As discussed in the introduction, we will need the uniform version of (ED) in order to obtain uniform regularity of the attractors :

(UED) There exist two positive constants M and λ such that for any $t \geq 0$ and $U \in X$,

$$\forall n \in \mathbb{N},\ \|e^{A_n t}U\|_X \leq Me^{-\lambda t}\|U\|_X\ .$$

Finally, we introduce hypotheses on the dynamical systems.
The dynamical systems $S_n(t)$ are gradient systems if we show that the function Φ introduced in (2.6) is a strict Lyapounov function. We already know that Φ is not increasing along the trajectories because of (2.7). To prove that Φ is a strict Lyapounov function, it remains to show that, if, for some $n \in \mathbb{N} \cup \{+\infty\}$, U_0 satisfies $\Phi(S_n(t)U_0) = \Phi(U_0)$ for all $t \geq 0$, then U_0 is an equilibrium point, that is $S_n(t)U_0 = U_0$ for all $t \geq 0$. We will assume that this property is fulfilled :

(Grad) the dynamical systems $S_n(t)$ ($n \in \mathbb{N} \cup \{+\infty\}$) are all gradient.

Our last assumption is the following :

(Hyp) All the equilibrium points E of $S_\infty(t)$ are hyperbolic, that is, that the spectrum of $DS_\infty(t)E$ does not intersect the unit circle of $\mathbb{C}$.

A discussion about the hypotheses (ED), (UED) and (Grad) is given in Section 6. We also enhance that Hypothesis (Hyp) is not very restrictive since it is satisfied for a generic non-linearity f (see for example [43] and [7]) or a generic domain Ω (see [24]).

We introduce the distance between a point $U \in X$ and a set $\mathcal{S} \subset X$ as

$$\operatorname{dist}_X(U, \mathcal{S}) = \inf_{V \in \mathcal{S}} \|U - V\|_X\ . \tag{2.19}$$

We also define the Hausdorff distance of two sets $\mathcal{S}_1 \subset X$ and $\mathcal{S}_2 \subset X$ as

$$d_X(\mathcal{S}_1, \mathcal{S}_2) = \max\left(\sup_{U_1 \in \mathcal{S}_1} \operatorname{dist}_X(U_1, \mathcal{S}_2)\ ;\ \sup_{U_2 \in \mathcal{S}_2} \operatorname{dist}_X(U_2, \mathcal{S}_1)\right)\ , \tag{2.20}$$

We denote $\operatorname{dist}_{X^{-s}}$ and $d_{X^{-s}}$ the same notions in the norm $\|.\|_{X^{-s}}$. We have the following theorem.

Theorem 2.8. *We assume that (Diss), (Grad) and (ED) hold. Then, the dynamical system $S_n(t)$, for $n \in \mathbb{N} \cup \{+\infty\}$, has a compact global attractor $\mathcal{A}_n$. Moreover, these attractors are composed by the union of the equilibrium points (denoted by $\mathcal{E}$) and the complete bounded trajectories coming from $\mathcal{E}$, that is that*

$$\mathcal{A}_n = \{U_0 \in X \ / \ \exists U(t) \in \mathcal{C}_b^0(\mathbb{R}, X), \text{ solution of (2.5) such that} \\ U(0) = U_0 \text{ and } \lim_{t \to -\infty} dist_X(U(t), \mathcal{E}) = 0 \ \} \ . \tag{2.21}$$

The set $(\bigcup_n \mathcal{A}_n)$ is bounded in X, and, for any $s \in]0, 1/2[$, the attractors are upper-semicontinuous in X^{-s}, that is

$$\sup_{U_n \in \mathcal{A}_n} dist_{X^{-s}}(U_n, \mathcal{A}_\infty) \longrightarrow 0 \text{ when } n \longrightarrow +\infty \ .$$

Proof : The existence and boundedness of attractors for Equation (2.5) is classical, we briefly recall the outline of the proof. According to Theorem 2.4.6 of [17], $S_n(t)$ has a compact global attractor if $S_n(t)$ is asymptotically smooth and point-dissipative and if the orbits of bounded sets are bounded. Proposition 2.4 implies that the orbits of bounded sets are bounded. Since $e^{A_n t}$ is exponentially decreasing and that the map $F : X \to X$ is compact, $S_n(t)$ is asymptotically smooth (see [17]). The property (2.8) implies that the equilibria $E = (e, 0)$ of (2.5) are bounded independently of n. By LaSalle's principle (see Lemma 3.8.2 of [17]), the gradient structure and the asymptotic smoothness imply that any trajectory is attracted by the set of equilibrium points. Because of the boundedness of the set of equilibria, $S_n(t)$ is point dissipative. Thus $S_n(t)$ has a compact global attractor, which is bounded in X uniformly in n and which, due to the gradient structure, is described by (2.21). For proofs or details about these notions, see [17].
Following the arguments of [18] (see also [41] or [3]), we prove the upper-semicontinuity in X^{-s}. Let $\varepsilon > 0$, as $\mathcal{A}_\infty$ is a global attractor for $S_\infty(t)$ and as the union $\bigcup_n \mathcal{A}_n$ is bounded in X, there exists a time $T > 0$ such that

$$\forall U \in \bigcup_n \mathcal{A}_n, \ \forall t \geq T, \ \mathrm{dist}_X(S_\infty(t)U, \mathcal{A}_\infty) \leq \varepsilon/2 \ . \tag{2.22}$$

As $\mathcal{A}_n$ is uniformly bounded in X, using (2.17), we have that, for n large enough,

$$\forall U_n \in \mathcal{A}_n, \ \|(S_n(T) - S_\infty(T))U_n\|_{X^{-s}} \leq \frac{\varepsilon}{2} \ . \tag{2.23}$$

The estimates (2.22) and (2.23) imply, for n large enough, that

$$\sup_{U_n \in \mathcal{A}_n} \mathrm{dist}_{X^{-s}}(S_n(T)U_n, \mathcal{A}_\infty) \leq \varepsilon \ .$$

As $S_n(T)\mathcal{A}_n = \mathcal{A}_n$, this proves the upper-semicontinuity. $\square$

Remark : The existence of attractors for critical non-linearities (that is cubic-like non-linearities) has been studied in dimension $d = 3$, see for example [14] and [9]. We notice that the above proof shows upper-semicontinuity in X^{-s} for these attractors. See [27] for the lower-semicontinuity.

If we assume a uniform exponential decay for the linear semigroups $e^{A_n t}$, we obtain the upper-semicontinuity in X. Indeed, we have the following regularity result.

Proposition 2.9. *Assume that (Diss), (Grad) and (UED) hold. Then, there exists a constant M such that the attractors $\mathcal{A}_n$ of $S_n(t)$, for $n \in \mathbb{N} \cup \{+\infty\}$, satisfy*

$$\sup_{n\in\mathbb{N}\cup\{+\infty\}} \sup_{U_n\in\mathcal{A}_n} \|U_n\|_{D(A_n)} \leq M \ . \tag{2.24}$$

In particular, the union $\cup_n \mathcal{A}_n$ is bounded in X^s ($s \in]0, 1/2[$).

Proof : If (2.24) holds, then $\cup_n \mathcal{A}_n$ is bounded in X^s as a direct consequence of Proposition 2.6. Thus, we only have to show that (2.24) is satisfied.
It is well-known that, for fixed n, $\mathcal{A}_n$ is bounded in $D(A_n)$. We only have to show that (UED) implies that $\mathcal{A}_n$ is bounded in $D(A_n)$, uniformly with respect to $n \in \mathbb{N} \cup \{+\infty\}$.
We already know that the attractors $\mathcal{A}_n$ are bounded in X by a constant K. Moreover, they are a union of complete trajectories. Let $U(t) = (u, u_t) \subset \mathcal{A}_n$ be such a trajectory, we have

$$U(t) = \int_{-\infty}^{t} e^{A_n(t-s)} F(U(s)) ds \ .$$

Notice that this integral has a sense since (UED) holds. Let $\delta > 0$, we write

$$U(t+\delta) - U(t) = \int_{-\infty}^{t} e^{A_n(t-s)} (F(U(s+\delta)) - F(U(s))) ds \ .$$

And so, since (UED) is satisfied, there exist M and λ independent of n such that

$$\|(U(t+\delta) - U(t))\|_X \leq M \int_{-\infty}^{t} e^{-\lambda(t-s)} \|f(x, u(s+\delta)) - f(x, u(s))\|_{\mathbb{L}^2} ds \ . \tag{2.25}$$

Due to the assumption (NL), there exists $\sigma \in]0, 1[$ such that

$$\begin{aligned} \|f(x, u(s+\delta)) - f(x, u(s))\|_{\mathbb{L}^2} &\leq C \, \|u(s+\delta) - u(s)\|_{\mathbb{H}^\sigma} \\ &\leq C \, \|u(s+\delta) - u(s)\|_{\mathbb{H}^1}^{\sigma} \, \|u(s+\delta) - u(s)\|_{\mathbb{L}^2}^{1-\sigma} \end{aligned}$$

The Young inequality implies that, for any $\varepsilon > 0$, there exists a constant C_ε such that

$$\|f(x, u(s+\delta)) - f(x, u(s))\|_{\mathbb{L}^2} \leq \varepsilon \|u(s+\delta) - u(s)\|_{\mathbb{H}^1} + C_\varepsilon \|u(s+\delta) - u(s)\|_{\mathbb{L}^2} \ .$$

As $\|u_t\|_{\mathbb{L}^2}$ is bounded by K, $\|\delta^{-1}(u(s+\delta)-u(s))\|_{\mathbb{L}^2}$ is uniformly bounded. So, combining the above inequality with (2.25), we obtain, for any $t \in \mathbb{R}$,

$$\|\delta^{-1}(U(t+\delta)-U(t))\|_X \leq \varepsilon\frac{M}{\lambda}\sup_{s\in\mathbb{R}}\|\delta^{-1}(U(s+\delta)-U(s))\|_X + \frac{M}{\lambda}C_\varepsilon K \ .$$

Thus, for ε sufficiently small, we get

$$\sup_{s\in\mathbb{R}}\|\delta^{-1}(U(s+\delta)-U(s))\|_X \leq C \ ,$$

where C does not depend on δ or on n. When δ converges to 0, we find that $U(s)$ satisfies $\sup_{s\in\mathbb{R}}\|U_t(s)\|_X \leq C$. Finally, writing $A_nU = U_t - F(U)$, we obtain that U is bounded in $D(A_n)$ by a constant which does not depend on n. □

Thus, if we mimic the proof of Theorem 2.8, using (2.18) instead of (2.17), we show the upper-semicontinuity in X.

Theorem 2.10. *We assume that all the hypotheses of Proposition 2.9 hold. Then, the attractors are upper-semicontinuous in X, that is*

$$\sup_{U_n\in\mathcal{A}_n} dist_X(U_n,\mathcal{A}_\infty) \longrightarrow 0 \ \textit{when} \ n \longrightarrow +\infty \ .$$

If we assume in addition that all the equilibria are hyperbolic, then we can prove the lower-semicontinuity of attractors. In this case, we can give not only an estimate of the rate of the lower-semicontinuity in X, but also of the upper-semicontinuity in X^{-s}. Notice that we do not need Hypothesis (UED) to obtain the lower-semicontinuity in X.

Theorem 2.11. *We assume that (Diss), (Grad), (ED) and (Hyp) are satisfied. Then, the attractors $\mathcal{A}_n$ are lower-semicontinuous in X.*
Moreover, there exist two positive constants C and δ such that

$$\sup_{U_\infty\in\mathcal{A}_\infty} dist_X(U_\infty,\mathcal{A}_n) \leq C\varepsilon_n^\delta \ . \tag{2.26}$$

and

$$\sup_{U_n\in\mathcal{A}_n} dist_{X^{-s}}(U_n,\mathcal{A}_\infty) \leq C\varepsilon_n^\delta \ . \tag{2.27}$$

Furthermore, if we assume in addition that Hypothesis (UED) holds, then the family of attractors is continuous in X and there exist two positive constants C and δ such that, for any n,

$$d_X(\mathcal{A}_\infty,\mathcal{A}_n) \leq C\varepsilon_n^\delta \ . \tag{2.28}$$

Our last theorem concerns the stability of the phase-diagrams. We have briefly recalled the notion of phase diagrams and its link with the Morse-Smale property in the introduction. First, notice that, in dimension higher than one or if $d = 1$ and $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$ with $a = 1$ or $b = 1$, the Morse-Smale property is not relevant. Indeed, in these cases, $e^{A_n t}$ is not a group (see [11] for $d = 1$ and [35] for $d \geq 2$). Thus, we cannot ensure that the backward uniqueness property is satisfied and that the stable sets of equilibria are well-defined manifolds, which is needed to define the transversality (for more details, see [19]). In the cases where we can define the Morse-Smale property, we prove the following theorem in Section 5.

Theorem 2.12. *We assume that $d = 1$, $\Omega =]0, 1[$ and $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$ with $a \neq 1$ and $b \neq 1$. We also assume that (Diss) and (UED) are satisfied and that the dynamical system $S_\infty(t)$ satisfies the Morse-Smale property. Then, for n large enough, the dynamical system $S_n(t)$ satisfies the Morse-Smale property and its phase-diagram is equivalent to the one of $S_\infty(t)$.*

We underline that Theorem 2.12 has applications since it is proved in [26] that, if $\Omega =]0, 1[$, $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$ with $a \neq 1$ and $b \neq 1$, the Morse-Smale property holds for $S_\infty(t)$, generically with respect to the non-linearity f.

Remark : We can readely adapt the proof of Theorem 3.2 of [36] to show the existence of a homeomorphism h defined from $\mathcal{A}_n$ to $\mathcal{A}_\infty$ which maps the trajectories of $S_n(t)_{|\mathcal{A}_n}$ onto the trajectories of $S_\infty(t)_{|\mathcal{A}_\infty}$ preserving the sense of time. The properties needed to adapt the proof of Theorem 3.2 of [36] are shown in Sections 4 and 5. They namely are the isomorphism of phase-diagrams of Theorem 2.12, the comparison of the local stable and unstable manifolds stated in Theorems 4.7 and 4.13 and the results of Section 5.1.

3 Convergence of the trajectories

3.1 Some abstract results of convergence

The difference between two linear semigroups of contractions can be estimated by the difference between the inverses of the infinitesimal generators.

Proposition 3.1. *Let X be a Hilbert space. Let A_1 and A_2 be two maximal dissipative operators of bounded inverse in $\mathcal{L}(X)$. Then, the operator A_i generates a $\mathcal{C}^0$-semigroup in X and we have, for all $U \in D(A_1)$ and $t \in \mathbb{R}_+$,*

$$\|e^{A_1 t}U - e^{A_2 t}U\|_X \leq \sqrt{\alpha}\left(\sqrt{\alpha} + \sqrt{\alpha + 4t}\right)\|U\|_{D(A_1)} ,$$

where $\alpha = ||A_1^{-1} - A_2^{-1}||_{\mathcal{L}(X)}$.

Proposition 3.1 is a direct consequence of the next proposition. The stronger version, where projections are added, is useful to prove convergence of stable and unstable manifolds of a hyperbolic equilibrium of the dynamical systems, or to estimate convergence of semigroups, which are not defined on the same space.

Proposition 3.2. *Let P_1 and P_2 be two continuous projections on a Hilbert space X. For $i = 1, 2$, let A_i be a linear operator with $D(A_i) \subset P_i X$ and $A_i \in \mathcal{L}(D(A_i), P_i X)$, which is dissipative, invertible and of bounded inverse. Then, A_i generates a $\mathcal{C}^0-$semigroup $e^{A_i t}$ on $P_i X$ and for all $U \in D(A_1) \subset P_1 X$ and $t \in \mathbb{R}_+$,*

$$\|e^{A_1 t} P_1 U - e^{A_2 t} P_2 U\|_X \leq \left(C\alpha + \sqrt{\alpha^2 + 4C^2 t\,(\alpha + \beta)} \right) \|U\|_{D(A_1)} ,$$

where $\alpha = ||A_1^{-1} P_1 - A_2^{-1} P_2||_{\mathcal{L}(X)}$, $\beta = \|A_1^{-1}\|_{\mathcal{L}(P_1 X)} \|P_1 - P_2\|_{\mathcal{L}(X)}$ and $C = \max_{i=1,2}\{\|P_i\|_{\mathcal{L}(X)}\}$.

Proof : As the operator A_i is invertible, it is onto $P_i X$ and thus A_i is a maximal operator. Since it is also dissipative on $P_i X$, it generates a $\mathcal{C}^0-$semigroup $e^{A_i t}$ on $P_i X$, which satisfies

$$\forall U \in X, t \in \mathbb{R}_+, \ \|e^{A_i t} P_i U\|_X \leq \|P_i U\|_X \tag{3.1}$$

(see for example [39]). We write that

$$\begin{aligned} \|e^{A_1 t} P_1 U - e^{A_2 t} P_2 U\|_X \leq \|e^{A_1 t} P_1 U - e^{A_2 t}(A_2^{-1} P_2 A_1 U)\|_X \\ + \|e^{A_2 t} P_2 (A_2^{-1} P_2 - A_1^{-1} P_1) A_1 U\|_X \ . \end{aligned} \tag{3.2}$$

Using (3.1), we easily bound the last term of (3.2) by $C\alpha \|U\|_{D(A_1)}$. To estimate the derivative of the first term of (3.2), we set

$$D = \frac{1}{2} \frac{d}{dt} \|e^{A_1 t} P_1 U - e^{A_2 t}(A_2^{-1} P_2 A_1 U)\|_X^2 \ . \tag{3.3}$$

Since $U \in D(A_1)$ and $A_2^{-1} P_2 A_1 U \in D(A_2)$, we have

$$D = < A_1 e^{A_1 t} P_1 U - A_2 e^{A_2 t}(A_2^{-1} P_2 A_1 U) | e^{A_1 t} P_1 U - e^{A_2 t}(A_2^{-1} P_2 A_1 U) >_X ,$$

where $< .|. >_X$ is the scalar product associated with the norm $\|.\|_X$.
We set $V = e^{A_1 t} P_1 A_1 U \in P_1 X$ and $W = e^{A_2 t} P_2 A_1 U \in P_2 X$. We have

$$\begin{aligned} D &= < V - W | A_1^{-1} V - A_2^{-1} W >_X \\ &= < V - W | A_1^{-1} P_1 (V - W) >_X + < V - W | (A_1^{-1} P_1 - A_2^{-1} P_2) W >_X \ . \end{aligned}$$

Since $P_1 V = V$ and $P_2 W = W$, we obtain

$$D = < P_1(V - W) | A_1^{-1} P_1 (V - W) >_X + < (P_1 - P_2) W | A_1^{-1} P_1 (V - W) >_X$$

$$+ < V - W|(A_1^{-1}P_1 - A_2^{-1}P_2)W >_X \ .$$

As A_1 is dissipative on P_1X, the first scalar product is nonpositive. Since $\|V\|_X \leq C\|U\|_{D(A_1)}$ and $\|W\|_X \leq C\|U\|_{D(A_1)}$, we obtain $D \leq 2C^2(\alpha+\beta)\|U\|^2_{D(A_1)}$, where α, β and C are as in the statement of the proposition. The integration of (3.3) gives

$$\|e^{A_1t}P_1U - e^{A_2t}(A_2^{-1}P_2A_1U)\|_X \leq \sqrt{\alpha^2 + 4C^2t\,(\alpha+\beta)}\|U\|_{D(A_1)} \ ,$$

Coming back to the estimate (3.2), we finish the proof. □

Corollary 3.3. *Let P_1 and P_2 be two continuous projections on a Hilbert space X. For $i = 1, 2$, let A_i be a linear operator with $D(A_i) \subset P_iX$ and $A_i \in \mathcal{L}(D(A_i), P_iX)$. We assume that there exist a constant μ such that $A_i - \mu Id$ is dissipative, invertible and of bounded inverse (which implies that A_i generates a $\mathcal{C}^0-$semigroup). Moreover, we assume that there exist two positive constants M and λ such the semigroup generated by A_i satisfies*

$$\forall t \geq 0, \ \|e^{A_it}\|_{\mathcal{L}(P_iX)} \leq Me^{-\lambda t} \ .$$

Then, for all $\eta \in]0, \lambda[$, there exists M_η, independent of the operator A_i, such that for all $U \in D(A_1) \subset P_1X$ and $t \in \mathbb{R}_+$,

$$\|e^{A_1t}P_1U - e^{A_2t}P_2U\|_X \leq C(\alpha + \sqrt{\alpha+\beta})M_\eta e^{-\eta t}\|U\|_{D(A_1)} \ , \tag{3.4}$$

where $\alpha = ||(A_1 - \mu Id)^{-1}P_1 - (A_2 - \mu Id)^{-1}P_2||_{\mathcal{L}(X)}$, $\beta = \|(A_1 - \mu Id)^{-1}\|_{\mathcal{L}(P_1X)}\|P_1 - P_2\|_{\mathcal{L}(X)}$ and $C = \max\limits_{i=1,2}\{\|P_i\|_{\mathcal{L}(X)}\}$.

Proof : Changing A_i into $A_i - \mu Id$ and λ into $\lambda + \mu$, we can assume that $\mu = 0$. Let $p \in \mathbb{N}^*$, $U \in D(A_1)$ and $t \in \mathbb{R}_+$. We have

$$\|e^{A_1t}P_1U - e^{A_2t}P_2U\|_X \leq \|e^{A_2(1-\frac{1}{p})t}P_2(e^{A_1\frac{t}{p}}P_1 - e^{A_2\frac{t}{p}}P_2)U\|_X$$

$$+\|e^{A_2(1-\frac{2}{p})t}P_2(e^{A_1\frac{t}{p}}P_1 - e^{A_2\frac{t}{p}}P_2)e^{A_1\frac{t}{p}}P_1U\|_X + ... + \|(e^{A_1\frac{t}{p}}P_1 - e^{A_2\frac{t}{p}}P_2)e^{A_1(1-\frac{t}{p})}P_1U\|_X \ .$$

Using Proposition 3.2, we obtain

$$\|e^{A_1t}P_1U - e^{A_2t}P_2U\|_X \leq pMe^{-\lambda(1-\frac{1}{p})t}\left(C\alpha + \sqrt{\alpha^2 + 4C^2\frac{t}{p}\,(\alpha+\beta)}\right)\ \|U\|_{D(A_1)} \ .$$

Thus, for all $\eta \in]0, \lambda[$ given, we can choose p and M_η large enough such that (3.4) holds. □

Our fourth result concerns the convergence in a weaker norm than the norm of X.

Proposition 3.4. *Let X be a Hilbert space and let A_1 and A_2 be two maximal dissipative operators of bounded inverse in $\mathcal{L}(X)$. Then, for all $U \in X$ and $t \in \mathbb{R}_+$,*

$$\|A_1^{-1}(e^{A_1 t}U - e^{A_2 t}U)\|_X \leq \sqrt{\alpha}\left(3\sqrt{\alpha} + \sqrt{\alpha + 4t}\right)\|U\|_X \ ,$$

where $\alpha = ||A_1^{-1} - A_2^{-1}||_{\mathcal{L}(X)}$.

Proof : We have

$$\|A_1^{-1}(e^{A_1 t}U - e^{A_2 t}U)\|_X \leq \|(e^{A_1 t} - e^{A_2 t})A_1^{-1}U\|_X + \|(A_1^{-1} - A_2^{-1})e^{A_2 t}U\|_X$$
$$+\|e^{A_2 t}(A_1^{-1} - A_2^{-1})U\|_X \ .$$

We finish the proof by applying Proposition 3.1. $\square$

3.2 Convergence of the trajectories

We recall that $\varepsilon_n = \|A_\infty^{-1} - A_n^{-1}\|_{\mathcal{L}(X)}$ is assumed to converge to zero. In this section, we compare $S_n(t)U_0$ with $S_\infty(t)U_0$ on finite time intervals.
In the previous section, we have seen that the convergence of the linear semigroups $e^{A_n t}$ can be estimated if the initial data are in $D(A_n)$, $n \in \mathbb{N} \cup \{+\infty\}$. Using interpolation arguments, we see that actually less regularity is needed. We recall that s_0 is the positive number defined by (2.2).

Proposition 3.5. *For all $s \in]0, s_0[$, there exists $C > 0$ such that, for all time $T > 0$, for all $t \in [0, T]$ and $U_0 \in X^s$, we have*

$$\forall t \in [0,T], \forall U_0 \in X^s, \ \|(e^{A_\infty t} - e^{A_n t})U_0\|_X \leq C(1 + T^{s^2/2})\varepsilon_n^{s^2/2}\|U_0\|_{X^s} \ . \qquad (3.5)$$

Moreover, if the initial data have zero as first component, we can improve the above estimate as follows : for all $s \in [0, 1/2[$, there exists $C > 0$ such that, for all time $T > 0$, for all $t \in [0, T]$ and $(0, v_0) \in X^s$, we have

$$\|(e^{A_\infty t} - e^{A_n t})(0, v_0)\|_X \leq C(1 + T^{s/2})\varepsilon_n^{s/2}\|(0, v_0)\|_{X^s} \ . \qquad (3.6)$$

Proof : In this proof, C denotes a generic positive constant, which does not depend on n or T.
If $U_0 = (u_0, v_0) \in D(A_\infty)$, then, using Proposition 3.1, we have

$$\|(e^{A_\infty t} - e^{A_n t})U_0\|_X \leq C(1 + T^{1/2})\varepsilon_n^{1/2}\|U_0\|_{D(A_\infty)}$$
$$\leq C(1 + T^{1/2})\varepsilon_n^{1/2}(\|u_0 + \Gamma_\infty v_0\|_{D(B)} + \|v_0\|_{D(B^{1/2})}) \ . \qquad (3.7)$$

On the other hand, we have

$$\|(e^{A_\infty t} - e^{A_n t})U_0\|_X \le C\|U_0\|_X \le C(\|u_0\|_{D(B^{1/2})} + \|v_0\|_{\mathbb{L}^2}) . \tag{3.8}$$

Since Γ_∞ is a bounded operator on $D(B^{1/2})$, we have, if $U_0 \in D(B^{1/2}) \times D(B^{1/2})$,

$$\begin{aligned} \|(e^{A_\infty t} - e^{A_n t})U_0\|_X &\le C(\|u_0\|_{D(B^{1/2})} + \|v_0\|_{D(B^{1/2})}) && (3.9) \\ &\le C(\|u_0 + \Gamma_\infty v_0\|_{D(B^{1/2})} + \|v_0\|_{D(B^{1/2})}) . && (3.10) \end{aligned}$$

Interpolating between (3.7) and (3.10), we obtain

$$\|(e^{A_\infty t} - e^{A_n t})U_0\|_X \le C(1 + T^{s/2})\varepsilon_n^{s/2}(\|u_0 + \Gamma_\infty v_0\|_{D(B^{(1+s)/2})} + \|v_0\|_{D(B^{1/2})}) .$$

Due to Proposition 2.1, if s belongs to $]0, s_0[$, then $\Gamma_\infty v_0$ is in $D(B^{(1+s)/2})$ and we have $\|\Gamma_\infty v_0\|_{D(B^{(1+s)/2})} \le C\|v_0\|_{D(B^{1/2})}$. Thus,

$$\|(e^{A_\infty t} - e^{A_n t})U_0\|_X \le C(1 + T^{s/2})\varepsilon_n^{s/2}(\|u_0\|_{D(B^{(1+s)/2})} + \|v_0\|_{D(B^{1/2})}) .$$

We interpolate again with (3.8) and we find that, for all $U_0 \in X^s$,

$$\begin{aligned} \|(e^{A_\infty t} - e^{A_n t})U_0\|_X &\le C(1 + T^{s^2/2})\varepsilon_n^{s^2/2}(\|u_0\|_{D(B^{(1+s)/2})} + \|v_0\|_{D(B^{s/2})}) \\ &\le C(1 + T^{s^2/2})\varepsilon_n^{s^2/2}\|U_0\|_{X^s} . \end{aligned}$$

The proof of (3.6) is similar. Let $(0, v_0) \in D(A_\infty)$. Since $\Gamma_\infty v_0 \in D(B)$, v_0 vanishes on the part of the boundary $\{x \in \omega_N \ / \ \gamma(x) \neq 0\}$. Therefore, $\Gamma_\infty v_0 = 0$ and (3.7) gives that

$$\|(e^{A_\infty t} - e^{A_n t})(0, v_0)\|_X \le C(1 + T^{1/2})\|v_0\|_{D(B^{1/2})} .$$

Interpolating with (3.8), we obtain that (3.6) holds for all $(0, v_0) \in D(A_\infty)$. If $s < 1/2$, the set $\{(u, v) \in D(A_\infty) \ / \ u = 0\}$ is dense in $\{(u, v) \in X^s \ / \ u = 0\}$. Using this density, we conclude that (3.6) holds for all $(0, v_0) \in X^s$. $\square$

Remarks : As noticed in the previous section, if the semigroups $e^{A_n t}$ have a uniform exponential decay rate, then the constant C does not depend on T.
Of course, one can expect that the decay rate $\varepsilon_n^{s^2/2}$ can be replaced by $\varepsilon_n^{s/2}$, when $s < s_0$. To obtain this better decay rate, one has to show that X^s is the interpolated space between X and $D(A_\infty)$, which is not a so easy result.

Proposition 3.4 implies a result similar to the above one.

Proposition 3.6. *For any $s \in [0,1]$ and any positive time T, there exists a positive constant C such that*

$$\forall t \in [0,T], \forall U_0 \in X,\ \|(e^{A_\infty t} - e^{A_n t})U_0\|_{X^{-s}} \leq C(1+T^{s/8})\varepsilon_n^{s/8}\|U_0\|_X\ . \tag{3.11}$$

Proof : Proposition 3.4 implies that

$$\|A_\infty^{-1}((e^{A_\infty t} - e^{A_n t})U_0)\|_X \leq C(1+T^{1/2})\varepsilon_n^{1/2}\|U_0\|_X\ .$$

We set $(\varphi,\psi) = (e^{A_\infty t} - e^{A_n t})U_0$. We have

$$\|\Gamma_\infty \varphi + B^{-1}\psi\|_{D(B^{1/2})} + \|\varphi\|_{\mathbb{L}^2} \leq C(1+T^{1/2})\varepsilon_n^{1/2}\|U_0\|_X\ . \tag{3.12}$$

On the other hand, the dissipativeness of A_n implies that

$$\|\psi\|_{\mathbb{L}^2} + \|\varphi\|_{D(B^{1/2})} \leq C\|U_0\|_X\ . \tag{3.13}$$

Since $\|\varphi\|_{D(B^{\theta/2})} \leq \|\varphi\|_{\mathbb{L}^2}^{1-\theta}\|\varphi\|_{D(B^{1/2})}^{\theta}$, (3.12) and (3.13) give

$$\forall \eta \in [0,1],\ \|\varphi\|_{D(B^{(1-\eta)/2})} \leq C(1+T^{\eta/2})\varepsilon_n^{\eta/2}\|U_0\|_X\ . \tag{3.14}$$

As Γ_∞ is linear continuous from $D(B^{(1-\eta)/2})$ into $D(B^{1/2})$ for all $\eta \in [0,1/2[$, (3.12) and (3.14) imply that

$$\|\psi\|_{D(B^{-1/2})} \leq C(1+T^{1/2})\varepsilon_n^{1/2}\|U_0\|_X + \|\varphi\|_{D(B^{(1-\eta)/2})} \leq C(1+T^{\eta/2})\varepsilon_n^{\eta/2}\|U_0\|_X\ .$$

As $\|\psi\|_{D(B^{-s/2})} \leq \|\psi\|_{D(B^{-1/2})}^{s}\|\psi\|_{\mathbb{L}^2}^{1-s}$, the above inequality and (3.13) yield that

$$\|\psi\|_{D(B^{-s/2})} \leq C(1+T^{\eta s/2})\varepsilon_n^{\eta s/2}\|U_0\|_X\ .$$

The estimate (3.11) follows from the above result for $\eta = 1/4$ and (3.14) for $\eta = s$. $\square$

The comparison of trajectories is based on the following lemma.

Lemma 3.7. *Let $\mathcal{B}$ be a bounded set of X^s, $s \in]0,s_0[$. Let $T > 0$, $M > 0$ and $n_0 \in \mathbb{N}$ be such that, for all $U \in \mathcal{B}$, $n \geq n_0$ (including $n = +\infty$) and $t \in [0,T]$, the integral solution $S_n(t)U \in \mathcal{C}^0([0,T],X)$ of (2.5) exists and satisfies*

$$\|S_n(t)U\|_X \leq M\ .$$

Then, there exists a constant $C = C(M)$ such that

$$\forall U \in \mathcal{B},\ \forall t \in [0,T],\ \|S_\infty(t)U - S_n(t)U\|_X \leq Ce^{CT}\varepsilon_n^{\beta}\ , \tag{3.15}$$

where $\beta = \frac{s^2}{2}$ if d=1 or d=2, and $\beta = \min(\frac{s^2}{2}, \frac{1-\alpha}{4})$ if d=3.

Proof : In this proof, C denotes a positive constant which does not depend on n or T, but may depend on M. We have

$$\|S_\infty(t)U - S_n(t)U\|_X \leq \|(e^{A_\infty t} - e^{A_n t})U\|_X + \int_0^t \|(e^{A_\infty(t-\tau)} - e^{A_n(t-\tau)})F(S_\infty(\tau)U)\|_X d\tau$$
$$+ \int_0^t \|e^{A_n(t-\tau)}(F(S_\infty(\tau)U) - F(S_n(\tau)U))\|_X d\tau \ . \qquad (3.16)$$

We bound the three terms of the previous inequality as follows.
Using Proposition 3.5, we have

$$\|(e^{A_\infty t} - e^{A_n t})U\|_X \leq C(1 + T^{s^2/2})\varepsilon_n^{s^2/2} \ .$$

As for $\tau \in [0, T]$, $S_\infty(\tau)U$ is bounded in X, Lemma 2.2 and Proposition 3.5 imply that

$$\int_0^t \|(e^{A_\infty(t-\tau)} - e^{A_n(t-\tau)})F(S_\infty(\tau)U)\|_X \leq C(M)(1 + T^\eta)\varepsilon_n^\eta \ ,$$

with $\eta < 1/4$ if $d = 1$ or $d = 2$, and $\eta = \frac{1-\alpha}{4}$ if $d = 3$. As F is locally Lipschitzian, we have

$$\begin{aligned}\int_0^t \|e^{A_n(t-\tau)}F(S_\infty(\tau)U) - F(S_n(\tau)U)\|_X d\tau \quad &\leq \int_0^t \|F(S_\infty(\tau)U) - F(S_n(\tau)U)\|_X d\tau \\ &\leq C(M)\int_0^t \|S_\infty(\tau)U - S_n(\tau)U\|_X d\tau \ .\end{aligned}$$

We finish the proof by applying Gronwall's lemma to (3.16). $\square$

Remark : In fact, we can show that, if U belongs to X^s for some $s > 0$, then $S_\infty(t)U \in \mathbb{L}^\infty([0, T], X^{s^2})$. Thus, we can prove that (3.15) holds for all $\beta \leq s^2/2$, even if $d = 3$ and if f is cubic-like (see [27]).

We deduce from Lemma 3.7 a stronger result.

Theorem 3.8. *Let $\mathcal{B}$ be a bounded set of X^s, $s \in]0, s_0[$, and let T be a positive time. There exists $M > 0$ such that, for all $U \in \mathcal{B}$ and $t \in [0, T]$, $S_\infty(t)U$ exists and satisfies $\|S_\infty(t)U\|_X \leq M$, if and only if there exists $M' > 0$ such that, for n large enough, $U \in \mathcal{B}$ and $t \in [0, T]$, $S_n(t)U$ exists and satisfies $\|S_n(t)U\|_X \leq M'$.*
Moreover, if one of these equivalent properties is satisfied, then there exists a constant $C = C(M)$ such that, for n large enough,

$$\forall U \in \mathcal{B}, \ \forall t \in [0, T], \ \|S_\infty(t)U - S_n(t)U\|_X \leq Ce^{CT}\varepsilon_n^\beta \ ,$$

where $\beta = \frac{s^2}{2}$ if d=1 or d=2, and $\beta = \min(\frac{s^2}{2}, \frac{1-\alpha}{4})$ if d=3.

Proof : Once the equivalence is proved, the estimate is a consequence of Lemma 3.7. Assume that, for all $U \in \mathcal{B}$ and $t \in [0,T]$, $S_\infty(t)U$ exists and satisfies

$$\|S_\infty(t)U\|_X \leq M \ . \tag{3.17}$$

Assume that there exist sequences $U_k \in \mathcal{B}$, $t_k \in [0,T]$ and $n_k \longrightarrow +\infty$ such that

$$\forall t \in [0, t_k[, \ \|S_{n_k}(t)U_k\|_X < 2M \text{ and } \|S_{n_k}(t_k)U_k\|_X = 2M \ .$$

We have

$$\|S_\infty(t_k)U_k\|_X \geq \|S_{n_k}(t_k)U_k\|_X - \|(S_{n_k}(t_k) - S_\infty(t_k))U_k\|_X \ .$$

For k large enough, applying Lemma 3.7 (with M replaced by $2M$), we find that $\|S_\infty(t_k)U_k\|_X \geq \frac{3}{2}M$, which contradicts (3.17). Thus, for n large enough, for any U in $\mathcal{B}$ and any $t \in [0,T]$, $S_n(t)U$ exists and satisfies $\|S_n(t)U\|_X \leq M' = 2M$.
This proves the "only if" part. The "if" part is shown in the same way. □

The previous theorem together with the density of X^s in X imply the convergence of the trajectories in X for any initial data U in X. However, the convergence is not uniform on a bounded set of X.

Corollary 3.9. *Let U be an initial datum in X and let T be a positive time. Then the mild solution $S_\infty(t)U \in \mathcal{C}^0([0,T],X)$ of (2.5) with $n = \infty$ exists if and only if there exists M such that, for n large enough, the mild solution $S_n(t)U \in \mathcal{C}^0([0,T],X)$ of (2.5) exists and $\|S_n(t)U\|_X \leq M$ for $t \in [0,T]$.*
Moreover, if one of the equivalent properties is satisfied, then

$$\sup_{t\in[0,T]} \|(S_\infty(t) - S_n(t))U\|_X \longrightarrow 0 \ \textit{when} \ n \longrightarrow +\infty \ . \tag{3.18}$$

In the following theorem, we obtain the convergence of trajectories in X^{-s} for initial data in X. Notice that, contrary to Theorem 3.8, we cannot prove existence of trajectories in $\mathcal{C}^0([0,T],X)$ for n large enough assuming only the existence of trajectories for the limit case $n = \infty$.

Theorem 3.10. *Let $\mathcal{B}$ be a bounded set of X. We assume that there exist $T > 0$, $M > 0$ and $n_0 \in \mathbb{N}$ such that, for all $U \in \mathcal{B}$, $n \geq n_0$ (and also $n = \infty$) and $t \in [0,T]$, the solution $S_n(t)U$ of (2.5) exists in $\mathcal{C}^0([0,T],X)$ and satisfies $\|S_n(t)U\|_X \leq M$. Then, there exists a constant C such that*

$$\forall U \in \mathcal{B}, \forall t \in [0,T], \ \|A_\infty^{-1}(S_\infty(t) - S_n(t))U\|_X \leq Ce^{CT}\varepsilon_n^{1/2} \ . \tag{3.19}$$

Moreover, for any $s \in [0,1]$, there exists a constant C' such that

$$\forall U \in \mathcal{B}, \forall t \in [0,T], \ \|(S_\infty(t) - S_n(t))U\|_{X^{-s}} \leq C'e^{C'T}\varepsilon_n^{s/8} \ . \tag{3.20}$$

Proof : As usual, C denotes a generic positive constant, which may vary from line to line. We recall that A_∞^{-1} is given by (2.10). We set $S_n(t)U = (u_n(t), v_n(t))$. We write

$$\begin{aligned}\|A_\infty^{-1}(S_\infty(t)-S_n(t))U\|_X \quad &\leq \|A_\infty^{-1}(e^{A_\infty t}-e^{A_n t})U\|_X \\ &\quad + \int_0^t \|A_\infty^{-1}(e^{A_\infty(t-\tau)}-e^{A_n(t-\tau)})F(S_n(\tau)U)\|_X d\tau \\ &\quad + \int_0^t \|e^{A_\infty(t-\tau)}A_\infty^{-1}(F(S_\infty(\tau)U)-F(S_n(\tau)U))\|_X d\tau \ .\end{aligned}$$

Using Proposition 3.4, we find

$$\begin{aligned}\|A_\infty^{-1}(S_\infty(t)-S_n(t))U\|_X \quad &\leq C(1+\sqrt{T})\varepsilon_n^{1/2}\|U\|_X \\ &\quad + \int_0^t C(1+\sqrt{T})\varepsilon_n^{1/2}\|f(x,u_n(x,\tau))\|_{\mathbb{L}^2} d\tau \\ &\quad + \int_0^t \|B^{-1/2}(f(x,u_\infty(x,\tau))-f(x,u_n(x,\tau)))\|_{\mathbb{L}^2} d\tau \ .\end{aligned}$$

Using (NL), we obtain that $\|f(x,u_n)\|_{\mathbb{L}^2}$ is bounded. We next show that

$$I = \|B^{-1/2}(f(x,u_\infty)-f(x,u_n))\|_{\mathbb{L}^2} \leq C\|u_\infty - u_n\|_{\mathbb{L}^2} \ . \tag{3.21}$$

Indeed, if for example the dimension is equal to 3, we have

$$\begin{aligned} I \quad &= \sup_{\|\varphi\|_{D(B^{1/2})}=1} \int_\Omega (f(x,u_\infty)-f(x,u_n))\varphi dx \\ &\leq \sup_{\|\varphi\|_{D(B^{1/2})}=1} C\left|\int_\Omega (1+|u_\infty|^\alpha+|u_n|^\alpha)|u_\infty-u_n|\varphi dx\right| \\ &\leq \sup_{\|\varphi\|_{D(B^{1/2})}=1} C\|u_\infty-u_n\|_{\mathbb{L}^2}\left(\int_\Omega|\varphi|^6\right)^{1/6}\left(\int_\Omega(1+|u_\infty|^\alpha+|u_n|^\alpha)^3\right)^{1/3}\end{aligned}$$

Since $\mathbb{H}^1(\Omega)$ (and thus $D(B^{1/2})$) is continuously imbedded in $\mathbb{L}^6(\Omega)$, we obtain (3.21) and we finish the proof of Inequality (3.19) by using Gronwall's lemma .
We enhance that, to obtain (3.20), we cannot directly use Proposition 3.6. This is linked to the fact that A_∞ does not generate a semigroup on X^{-s}. However, we can deduce (3.20) from (3.19) with the same arguments as in the proof of Proposition 3.6. □

With the same arguments, we obtain similar results for the linearized dynamical system.

Proposition 3.11. *Let $U(t) \in \mathcal{C}^0([0,T],X)$. The conclusions of Theorems 3.8 and 3.10 are also valid if $S_n(t)$ is replaced by $DS_n(U)(t)$, the linearized dynamical system defined by (2.9). In particular, let $\mathcal{B}$ be a bounded set of X^s, $s \in]0,s_0[$, and T be a positive time, there exists a positive constant C such that if $U_0 \in \mathcal{B}$ and $U_n(t) \in \mathcal{C}^0([0,T],X)$ is the solution of (2.5) with initial data U_0, then*

$$\forall t \in [0,T],\ \|DS_\infty(U_\infty)(t) - DS_n(U_n)(t)\|_{\mathcal{L}(X^s,X)} \leq Ce^{CT}\varepsilon_n^\beta\ ,$$

where $\beta = \frac{s^2}{2}$ if d=1 or d=2, and $\beta = \min(\frac{s^2}{2}, \frac{1-\alpha}{4})$ if d=3.

4 Comparison of local stable and unstable manifolds

In the previous section, we have proved the convergence of trajectories for a given initial datum. Theorem 3.8 shows that, if we want to study the convergence of orbits for initial data in a bounded set of X, this set must have compactness properties. Thus, it is natural to wonder, in the case where Equation (2.5) has a compact global attractor $\mathcal{A}_n$, if the attractors $\mathcal{A}_n$ converge to $\mathcal{A}_\infty$. The existence, boundedness, regularity and upper-semicontinuity of the attractors have already been discussed in Theorem 2.8, Proposition 2.9 and Theorem 2.10. In this section, we study the convergence of the local unstable manifolds and the convergence of regular parts of the local stable manifolds. Then, we deduce the lower-semicontinuity of the attractors from the convergence of the local unstable manifolds. Notice that the convergence of regular parts of the local stable manifolds is not needed to show the lower-semicontinuity.

We begin by recalling some classical notions. An equilibrium point $E \in X$ is said to be hyperbolic for the dynamical system $S(t)$ if the spectrum of the linearization $DS(E)(1)$ does not intersect the complex unit circle. Let P^u be the spectral projection onto the part of the spectrum of modulus larger than 1, and $P^s = Id - P^u$ the spectral projection onto the part of the spectrum of modulus smaller than 1. If E is hyperbolic, there exist two positive constants λ_u and λ_s and two positive constants M_u and M_s such that

$$\forall t \geq 0,\ \|DS(E)(t)P^s\|_{\mathcal{L}(X)} \leq M_s e^{-\lambda_s t} \ \text{ and }\ \forall t \leq 0,\ \|DS(E)(t)P^u\|_{\mathcal{L}(X)} \leq M_u e^{\lambda_u t}\ .$$

We set $B^u(r) = P^uX \cap B(E,r)$ and $B^s(r) = P^sX \cap B(E,r)$. The following theorem is classical in the theory of dynamical systems (see for example the Appendix of [17]).

Theorem 4.1. *We assume that $S(t)$ is of class $\mathcal{C}^{1,1}$ from X into X and that E is a hyperbolic equilibrium point of $S(t)$. For $r > 0$ small enough, there exists a unique function h^s from $B^s(r)$ into $B^u(M_s r)$, which is of class $\mathcal{C}^{1,1}$, satisfies $h^s(E) = E$ and $Dh^s(E) = 0$. Moreover, its graph $W^s(E,r)$ (called the local stable manifold) satisfies the following properties.*

i) $W^s(E,r) = \{U \in B(E, 2M_s r) \mid P^s U \in B^s(r)$ *and* $\forall t \geq 0,\ S(t)U \in B(E, 2M_s r)\}$,
ii) if $U \in W^s(E,r)$ *then*

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \|S(t)U - E\|_X \leq -\lambda_s \ .$$

There exists also a unique function h^u *from* $B^u(r)$ *into* $B^s(M_u r)$*, which is of class* $\mathcal{C}^{1,1}$*, satisfies* $h^u(E) = E$ *and* $Dh^u(E) = 0$*. Moreover, its graph* $W^u(E,r)$ *(called the local unstable manifold) satisfies the following properties.*
iii) $W^u(E,r) = \{U \in B(E, 2M_u r) \mid P^u U \in B^u(r)$ *and there exists a negative trajectory* $U(t) \in \mathcal{C}^0(]-\infty, 0], X)$ *such that* $\forall t \leq 0,\ U(t) \in B(E, 2M_u r)\}$,
iv) if $U \in W^u(E,r)$ *then there exists a unique negative trajectory* $U(t) \in \mathcal{C}^0(]-\infty,0],X)$ *such that* $U(t) \in B(2M_u r)$ *for any* $t \leq 0$*, and*

$$\limsup_{t\to-\infty} \frac{1}{|t|} \ln \|U(t) - E\|_X \leq -\lambda_u \ .$$

We also introduce some classical definitions and the corresponding notations.

Definition 4.2. *Let* E *be a hyperbolic equilibrium. The dimension of* $P^u X$*, which is also the one of* $W^u(E,r)$*, is called the Morse index of* E *and is denoted by* $m(E)$*.*
We also define the stable and unstable sets of E*, which are not necessarily well-defined manifolds, by* $W^s(E) = \{U \in X \mid \lim_{t\to+\infty} S(t)U = E\}$ *and* $W^u(E) = \{U \in X \mid \exists$ *a negative trajectory* $U(t) \in \mathcal{C}^0(]-\infty,0],X)$ *such that* $\lim_{t\to-\infty} U(t) = E\}$ *respectively.*

4.1 Preliminary results and spectral study

In what follows, we use the notations of Theorem 4.1 with a subscript n for the dependance with respect to n.
Let $E = (e, 0)$ be an equilibrium point of (2.5). We set

$$\forall n \in \mathbb{N} \cup \{+\infty\},\ \tilde{A}_n = A_n + \begin{pmatrix} 0 & 0 \\ f'_u(x, e(x)) & 0 \end{pmatrix} \ .$$

Notice that the linearization of $S_n(t)$ at the equilibrium point E is $DS_n(E)(t) = e^{\tilde{A}_n t}$. We also set, for any $U = (u,v)$ in X,

$$g(U) = \begin{pmatrix} 0 \\ f(x,u) - f'_u(x, e(x))u \end{pmatrix} \ .$$

Equation (2.5) becomes

$$U_t = \tilde{A}_n + g(U) \ . \tag{4.1}$$

When no confusion is possible, we denote $f'_u(x,e)$ by f'_u. Hypothesis (NL) implies the following properties.

Lemma 4.3. *The function g is a compact Lipschitz-continuous function on the bounded sets of X. More precisely, we have*

$$\forall U, U' \in B_X(E, r),\ \|g(U) - g(U')\|_X \leq l(r)\|U - U'\|_X\ ,$$

where $l(r)$ is a non-negative and non-decreasing function which tends to 0 when r goes to 0. In addition, g is of class $\mathcal{C}^{1,1}$ and if $\mathcal{B}$ is a bounded set of X, then there exists a positive constant $C = C(\mathcal{B})$ such that

$$\forall U \in \mathcal{B}, \forall V \in X,\ \|g(U)\|_{X^\sigma} \leq C\|U\|_X \quad \text{and } \|g'(U)V\|_{X^\sigma} \leq C\|V\|_X, \tag{4.2}$$

where $\sigma \in]0, 1[$ when $d = 1$ or $d = 2$ and $\sigma \in]0, \frac{1-\alpha}{2}[$ when $d = 3$.
Moreover, $\tilde{A}_n$ and $e^{\tilde{A}_n}$ are compact perturbations of A_n and e^{A_n} respectively.

Proof : The first part of the Theorem is a consequence of Lemma 2.2 and of classical Sobolev imbeddings. In particular, Lemma 2.2 shows that if $u \in \mathbb{H}^1(\Omega)$, then $f'_u(x, e)u \in \mathbb{H}^\sigma(\Omega)$. Thus, the map $(u, v) \mapsto (0, f'_u(x, e)u)$ is compact from X into X and $\tilde{A}_n$ is a compact perturbation of A_n. To show that $e^{\tilde{A}_n}$ is a compact perturbation of e^{A_n}, we remark that if $U_0 \in X$ and $(u(t), u_t(t)) = e^{\tilde{A}_n t}U_0$, then

$$e^{\tilde{A}_n}U_0 = e^{A_n}U_0 + \int_0^1 e^{A_n(1-t)} \begin{pmatrix} 0 \\ f'_u(x, e(x))u(t) \end{pmatrix} dt\ .$$

□

The behaviour of the spectrum of $\tilde{A}_n$ is described in the following proposition.

Proposition 4.4. *Assume that Hypothesis (ED) holds. Let $\lambda \in \mathbb{C}$ be such that the operator $(\tilde{A}_\infty - \lambda Id) \in \mathcal{L}(X)$ is invertible. Then, for n large enough, $(\tilde{A}_n - \lambda Id)$ is also invertible and there exists a positive constant C_λ such that*

$$\|(\tilde{A}_\infty - \lambda Id)^{-1} - (\tilde{A}_n - \lambda Id)^{-1}\|_{\mathcal{L}(X)} \leq C_\lambda \varepsilon_n\ .$$

As a consequence, the point spectrum of $\tilde{A}_n$ converges to the one of $\tilde{A}_\infty$ on every bounded set of $\mathbb{C}$. Moreover, if E is a hyperbolic equilibrium point of the dynamical system $S_\infty(t)$, then, for n large enough, it is a hyperbolic equilibrium point of the dynamical system $S_n(t)$ and there exists a positive constant C such that

$$\|P^u_\infty - P^u_n\|_{\mathcal{L}(X)} \leq C\varepsilon_n\ . \tag{4.3}$$

In addition, the part of the spectrum of $\tilde{A}_n$ ($n \in \mathbb{N} \cup \{+\infty\}$) with non-negative real part is composed by a finite number of real positive eigenvalues. Finally, the Morse index of E for

$S_n(t)$, which is the number of positive eigenvalues of $\tilde{A}_n$ is equal for n large enough to the Morse index of E for $S_\infty(t)$.

Proof : We denote by $K_\lambda \in \mathcal{L}(D(B^{1/2}))$ the operator $Id + \lambda^2 B^{-1} - B^{-1} f'_u$. A straightforward computation shows that $(\tilde{A}_n - \lambda Id)$ is invertible if and only if $(K_\lambda + \lambda \Gamma_n)$ is invertible in $\mathcal{L}(D(B^{1/2}))$ and in this case

$$(\tilde{A}_n - \lambda Id)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (K_\lambda + \lambda\Gamma_n)^{-1}(-B^{-1}v - \lambda B^{-1}u + B^{-1}f'_u u - \Gamma_n u) \\ (K_\lambda + \lambda\Gamma_n)^{-1}(-\lambda B^{-1}v + u) \end{pmatrix} .$$

If $(\tilde{A}_\infty - \lambda Id)$ is invertible, then $(K_\lambda + \lambda\Gamma_\infty)$ is invertible in $\mathcal{L}(D(B^{1/2}))$ and we have

$$(K_\lambda + \lambda\Gamma_n) = (K_\lambda + \lambda\Gamma_\infty)(Id - \lambda(K_\lambda + \lambda\Gamma_\infty)^{-1}(\Gamma_\infty - \Gamma_n)) .$$

For n large enough, $\|\lambda(K_\lambda + \lambda\Gamma_\infty)^{-1}(\Gamma_\infty - \Gamma_n)\|_{\mathcal{L}(D(B^{1/2}))} \leq \frac{1}{2}$, and $(K_\lambda + \lambda\Gamma_n)$ is invertible. Moreover,

$$\begin{aligned} (K_\lambda + \lambda\Gamma_n)^{-1} \quad - \quad & (K_\lambda + \lambda\Gamma_\infty)^{-1} = \lambda(K_\lambda + \lambda\Gamma_\infty)^{-1}(\Gamma_\infty - \Gamma_n) \\ & \times \left(\sum_{k\geq 0} \lambda^k((K_\lambda + \lambda\Gamma_\infty)^{-1}(\Gamma_\infty - \Gamma_n))^k\right)(K_\lambda + \lambda\Gamma_\infty)^{-1} , \end{aligned}$$

and so, for n large enough,

$$\|(K_\lambda + \lambda\Gamma_n)^{-1} - (K_\lambda + \lambda\Gamma_\infty)^{-1}\|_{\mathcal{L}(D(B^{1/2}))} \leq 2\varepsilon_n \|(K_\lambda + \lambda\Gamma_\infty)^{-1}\|^2_{\mathcal{L}(D(B^{1/2}))} .$$

This gives the first assertion of the proposition. It is well-known that this implies the convergence of the point spectrum.
Assume that E is a hyperbolic equilibrium point for the dynamical system $S_\infty(t)$, we want to prove that for n large enough, it is also a hyperbolic equilibrium point for the dynamical system $S_n(t)$. As Hypothesis (ED) holds, the radius of the spectrum of e^{A_n} is strictly less than one. Since, by Lemma 4.3, $e^{\tilde{A}_n}$ is a compact perturbation of e^{A_n}, the radius of the essential spectrum of $e^{\tilde{A}_n}$ is strictly less than one. As a consequence, for each n, there exists $\delta_n > 0$ such that the spectrum of $\tilde{A}_n$ with real part greater than $-\delta_n$ is only composed by a finite number of eigenvalues of finite multiplicity. We next prove that an eigenvalue of $\tilde{A}_n$ with non-negative real part must be real. Then, the proof of the hyperbolicity of E for $S_n(t)$ is reduced to the proof that $\lambda = 0$ is not an eigenvalue of $\tilde{A}_n$. The local convergence of the spectrum of $\tilde{A}_n$ to the one of $\tilde{A}_\infty$, together with the hyperbolicity of E for $S_\infty(t)$, ensure that $\lambda = 0$ is not an eigenvalue of $\tilde{A}_n$, for n large enough.
We finish the proof by showing that the eigenvalues of $\tilde{A}_\infty$ with non-negative real part are real. The proof in the case of $n < \infty$ is similar and even easier.

Let λ be a non-real eigenvalue of $\tilde{A}_\infty$ with eigenvector $(\varphi, \lambda\varphi)$ such that $\|\varphi\|_{\mathbb{L}^2} = 1$. We have

$$\left\{ \begin{array}{l} \lambda^2\varphi = \Delta\varphi - \kappa\varphi + f'_u(x,e)\varphi \\ \frac{\partial\varphi}{\partial\nu} + \lambda\gamma\varphi = 0 \quad \text{on } \omega_N \\ \varphi = 0 \quad \text{on } \omega_D \end{array} \right. \tag{4.4}$$

where ω_N (resp. ω_D) is the part of $\partial\Omega$ where B has Neumann (resp. Dirichlet) boundary conditions, and where $\kappa = 1$ if $\omega_D = \emptyset$ and $\kappa = 0$ in the other case.
Multiplying the first equation by $\overline{\varphi}$ and integrating, we obtain

$$-\|\overrightarrow{\nabla}\varphi\|^2_{\mathbb{L}^2} - \kappa\|\varphi\|^2_{\mathbb{L}^2} + \int_\Omega f'_u|\phi|^2 = \lambda^2\|\varphi\|^2_{\mathbb{L}^2} + \lambda\int_{\omega_N}\gamma|\varphi|^2 \ .$$

Taking the imaginary part and using the fact that $Im(\lambda) \neq 0$, we find

$$Re(\lambda) = -\frac{1}{2}\int_{\omega_N}\gamma|\varphi|^2 \ . \tag{4.5}$$

To prove that $Re(\lambda) < 0$, we argue by contradiction. Assume that $\int_{\omega_N}\gamma|\varphi|^2 = 0$. There exists an open subset ω of the boundary such that $\varphi_{|\omega} \equiv 0$ and Equation (4.4) shows that $\frac{\partial\varphi}{\partial\nu}_{|\omega} \equiv \varphi_{|\omega} \equiv 0$. Let θ be an open connected subset of Ω such that $(\overline{\omega}_N \cap \overline{\omega}_D) \cap \overline{\theta} = \emptyset$, and $\overline{\theta} \cap \omega \neq \emptyset$. The set θ is defined such that it is distant from the points of the boundary where the Neumann boundary condition meets the Dirichlet one. Regularity theorems for problems with mixed boundary conditions imply that e belongs to $\mathbb{H}^2(\theta)$ and so to $\mathbb{L}^\infty(\theta)$ (see [16]). Thus, as φ is a solution of (4.4), φ satisfies in θ

$$\left\{ \begin{array}{l} \lambda^2\varphi = \Delta\varphi + h\varphi \\ \frac{\partial\varphi}{\partial\nu} = \varphi = 0 \quad \text{on } \omega \cap \overline{\theta} \end{array} \right. \tag{4.6}$$

with some additional boundary conditions, where $h = -\kappa Id + f'_u(x, e(x))$ belongs to $\mathbb{L}^\infty(\theta)$. The classical unique continuation property implies that φ identically vanishes on θ and thus on Ω, which is absurd. $\square$

Let E be a hyperbolic equilibrium point. Using the above proposition, we know that there exist two constants μ and η with $0 < \eta < \mu$ such that the spectrum of $\tilde{A}_\infty$ has the following decomposition.

$$\sigma(\tilde{A}_\infty) = \Big(\sigma(\tilde{A}_\infty) \cap \{z \in \mathbb{C}/Re(z) < 0\}\Big) \cup \Big(\sigma(\tilde{A}_\infty) \cap \{z \in \mathbb{C}/Re(z) \geq \mu + 2\eta\}\Big) \ .$$

Proposition 4.4 implies that, for n large enough, we have

$$\sigma(\tilde{A}_n) = \Big(\sigma(\tilde{A}_n) \cap \{z \in \mathbb{C}/Re(z) < 0\}\Big) \cup \Big(\sigma(\tilde{A}_n) \cap \{z \in \mathbb{C}/Re(z) \geq \mu + \eta\}\Big) \ . \tag{4.7}$$

For $n \in \mathbb{N} \cup \{+\infty\}$, we denote by P_n^u the spectral projection onto the space generated by the eigenvectors corresponding to the part of the spectrum of $\tilde{A}_n$ with real part larger than μ. We set $P_n^s = Id - P_n^u$.

Proposition 4.5. *There exist two positive constants M_u and M_s such that*

$$\forall n \in \mathbb{N} \cup \{+\infty\}, \begin{cases} \forall t \geq 0, & \|e^{\tilde{A}_n t} P_n^s\|_{\mathcal{L}(X)} \leq M_s e^{(\mu-\eta)t} \\ \forall t \leq 0, & \|e^{\tilde{A}_n t} P_n^u\|_{\mathcal{L}(X)} \leq M_u e^{(\mu+\eta)t} \end{cases} \tag{4.8}$$

The proof of the above result is based on the following equivalence.

Theorem 4.6. *Let H_n be a sequence of Hilbert spaces. Let D_n be the generator of a $\mathcal{C}^0-$semigroup of contractions $e^{D_n t}$ on H_n, and let $\lambda > 0$. There exist two positive constants ε and C such that*

$$\forall t \geq 0, \ \|e^{D_n t}\|_{\mathcal{L}(H_n)} \leq C e^{-(\lambda+\varepsilon)t} \tag{4.9}$$

if and only if there exists $\varepsilon' > 0$ such that for all $n \in \mathbb{N}$, the spectrum of D_n satisfies $\sigma(D_n) \subset \{z \in \mathbb{C} \ / \ Re(z) < -\lambda - \varepsilon'\}$ and such that we have

$$\exists M > 0 \text{ such that } \sup_{n\in\mathbb{N}} \sup_{\nu\in\mathbb{R}} \|(D_n + (\lambda + i\nu)Id)^{-1}\|_{\mathcal{L}(H_n)} \leq M \ . \tag{4.10}$$

This result is proved in [34]. Although the theorems given in [34] are stated less precisely, it can be deduced from their proofs.

Proof of Proposition 4.5 : First, notice that $e^{\tilde{A}_n t}$ is well-defined on $P_n^u X$ even if $t \leq 0$ and that there exists M such that for any $t \leq 0$, $\|e^{\tilde{A}_n t}\|_{\mathcal{L}(P_n^u X)} \leq M e^{(\mu+\eta)t}$, since $P_n^u X$ is a subspace spanned by a finite number of eigenvectors of $\tilde{A}_n$ corresponding to eigenvalues larger than $\mu + \eta$, this number of eigenvectors being independent of n. Thus, the second estimate of (4.8) is a direct consequence of the convergence of P_n^u to P_∞^u. Let $H_n = P_n^s X$ and let $\tilde{D}_n$ be the restriction to H_n of the operator $\tilde{A}_n - \|f_u'\|_\infty Id$. Notice that $\tilde{D}_n$ is a dissipative operator on H_n and thus that $e^{\tilde{D}_n t}$ is a semigroup of contractions. We set $\lambda = \|f_u'\|_\infty - (\mu - \eta)$. If we prove that (4.10) holds for $\tilde{D}_n$, we will obtain that

$$\|e^{\tilde{D}_n t}\|_{\mathcal{L}(X)} \leq M e^{-(\|f_u'\|_\infty - (\mu-\eta))t} \ ,$$

and so that

$$\|e^{\tilde{A}_n t}\|_{\mathcal{L}(X)} \leq M e^{(\mu-\eta)t} \ .$$

Then the first estimate of (4.8) will be a direct consequence of the convergence of P_n^s to P_∞^s.

The spectral condition of Theorem 4.6 is clear due to the definition of H_n and the fact that $\mu - \eta$ is positive. To show that (4.10) holds, we argue by contradiction and assume that there exist sequences (ν_k) and $(n_k) \to +\infty$ such that

$$\|(\tilde{D}_{n_k} + (\lambda + i\nu_k)Id)^{-1}\|_{\mathcal{L}(H_{n_k})} \longrightarrow +\infty \ . \tag{4.11}$$

As E is hyperbolic for $S_\infty(t)$, Proposition 4.4 implies that $|\nu_k| \longrightarrow +\infty$.
Assume that $\nu_k \longrightarrow +\infty$ and that $\nu_k > 0$ (the case $\nu_k \longrightarrow -\infty$ is similar). We set $D_n = A_n - \|f'_u\|_\infty Id$. As $e^{A_n t}$ is a semigroup of contractions, for all $n \in \mathbb{N} \cup \{+\infty\}$, we have that $\|e^{D_n t}\|_{\mathcal{L}(X)} \leq e^{-\|f'_u\|_\infty t}$ and thus Theorem 4.6 show that

$$\exists M > 0,\ \sup_{\nu_k} \|(D_{n_k} + (\lambda + i\nu_k)Id)^{-1}\|_{\mathcal{L}(X)} \leq M \ . \tag{4.12}$$

Let K be the compact operator $(u, v) \in X \mapsto (0, f'_u(x, e(x))u)$. We have

$$(\tilde{D}_{n_k} + (\lambda + i\nu_k)) = (Id + K(D_{n_k} + (\lambda + i\nu_k))^{-1})(D_{n_k} + (\lambda + i\nu_k)) \ . \tag{4.13}$$

A straightforward calculus shows that if $(\varphi_k, \psi_k) = (D_{n_k} + (\lambda + i\nu_k))^{-1}(u, v)$, then

$$-\varphi_k + (\lambda + i\nu_k)\Gamma_{n_k}\varphi_k - (\lambda + i\nu_k)^2 B^{-1}\varphi_k = B^{-1}v - (\lambda + i\nu_k)B^{-1}u + \Gamma_{n_k}u \ .$$

Multiplying by $B\overline{\varphi}_k$ and integrating, we find

$$\begin{aligned} \nu_k^2 \|\varphi_k\|^2_{\mathbb{L}^2} &= \ < \varphi_k - (\lambda + i\nu_k)\Gamma_{n_k}\varphi_k + \Gamma_{n_k}u | \varphi_k >_{D(B^{1/2})} \\ &\quad + < v - (\lambda + i\nu_k)u | \varphi_k >_{\mathbb{L}^2} + (\lambda^2 + 2i\lambda\nu_k)\|\varphi_k\|^2_{\mathbb{L}^2} \ . \end{aligned}$$

So, there exists a positive constant C such that

$$\nu_k{}^2 \|\varphi_k\|^2_{\mathbb{L}^2} \leq C(1 + \nu_k)\left(\|(u, v)\|_X + \|\varphi_k\|_{D(B^{1/2})}\right)\|\varphi_k\|_{D(B^{1/2})} \ .$$

As (4.12) holds, we have $\|\varphi_k\|_{D(B^{1/2})} \leq M\|(u, v)\|_X$ and so $\|\varphi_k\|_{\mathbb{L}^2} \leq \frac{C}{\sqrt{\nu_k}}\|(u, v)\|_X$. Using (NL), we find that there exists $s \in]0, 1/2[$ such that

$$\|K(D_{n_k} + (\lambda + i\nu_k))^{-1}(u, v)\|_X = \|f'_u(x, e)\varphi_k\|_{\mathbb{L}^2} \leq \frac{C}{\nu_k^s}\|(u, v)\|_X \ ,$$

and so $\|K(D_{n_k} + (\lambda + i\nu_k))^{-1}\|_{\mathcal{L}(X)} \longrightarrow 0$ as $k \longrightarrow +\infty$. Thus, (4.13) implies that $\tilde{D}_{n_k} + (\lambda + i\nu_k)$ is invertible for k large enough and satisfies (4.10) with a constant $\tilde{M}$ independent of ν_k. This contradicts the above assumption (4.11) and proves the proposition. □

4.2 Convergence of the local unstable manifolds

As above, we will use the notations of Theorem 4.1 with a subscript n for the dependance with respect to n. In particular, we recall that $B^u_n(r) = P^u_n X \cap B_X(E, r)$ and $B^s_n(r) = P^s_n X \cap B_X(E, r)$.
The whole section is devoted to the proof of the following theorem.

Theorem 4.7. *Let E be a hyperbolic equilibrium point of the dynamical system $S_\infty(t)$. We assume that the exponential decay (ED) holds. Then, E is a hyperbolic equilibrium point of $S_n(t)$ for n large enough and there exists a radius $r > 0$ such that the function h_n^u and its derivative Dh_n^u are defined in $B_n^u(r)$. In other words, the local unstable manifolds $W_n^u(E, r)$ are defined for n large enough in a neighborhood of E independent of n. Moreover, the decay rate λ_u of Property iv) of Theorem 4.1 and the Lipschitz-constants of h_n^u and Dh_n^u are uniform in n. In addition, there exists a positive constant C such that, for all $\xi \in B_\infty^u(r)$,*

$$\|h_\infty^u(\xi) - h_n^u(P_n^u \xi)\|_X \le C\varepsilon_n^\beta \quad \text{and} \quad \|Dh_\infty^u(\xi)P_\infty^u - Dh_n^u(P_n^u \xi)P_n^u\|_{\mathcal{L}(X,X)} \le C\varepsilon_n^\beta \ , \tag{4.14}$$

where β is any number in $]0, 1/8[$ if $d = 1$ or $d = 2$ or any number in $]0, \min(\frac{1}{32}, \frac{1-\alpha}{4})[$ if $d = 3$. In particular, we have that

$$d_X(W_n^u(E, r); W_\infty^u(E, r)) \le C\varepsilon_n^\beta \ .$$

Til the end of this section, we assume that (ED) holds. For sake of simplicity, we may set without loss of generality that $E = 0$ and $f(x, 0) = 0$. We also assume that $E = 0$ is a hyperbolic equilibrium of the dynamical system $S_n(t)$ and that the spectral decomposition (4.7) holds for any $n \in \mathbb{N} \cup \{+\infty\}$.

The outline of the proof of Theorem 4.7 is as follows. We know that, for each n, there exists a local unstable manifold $W_n^u(E, r_n)$. We will construct, for each $n \in \mathbb{N} \cup \{+\infty\}$, the local strongly unstable manifold $W_n^{su}(E, r_n)$ in $B_X(0, r_n)$, corresponding to the spectral decomposition (4.7). This construction is done with a fixed point theorem, using the method of Lyapounov-Perron (see [17]). We will show that this construction can be made in a ball $B_X(0, r)$ independent of n. Next, we will compare $W_n^{su}(E, r)$ and $W_\infty^{su}(E, r)$, using the continuity of the fixed point with respect to the parameter n. Finally, as $E = 0$ is hyperbolic for each n, and as (4.7) holds, we know that the local strongly unstable manifold $W_n^{su}(E, r)$ is in fact the local unstable manifold $W_n^u(E, r)$ defined in Theorem 4.1.

We introduce the space

$$Y_\mu = \{U \in \mathcal{C}^0(] - \infty, 0], \mathbb{C}) \ / \ \sup_{t \le 0} \|U(t)\|_X e^{-\mu t} < +\infty\} \ .$$

We endow Y_μ with the norm $\|.\|_\mu$ defined by

$$\|U\|_\mu = \sup_{t \le 0} \|U(t)\|_X e^{-\mu t} \ .$$

We set $B_\mu(R) = \{U \in Y_\mu \ / \ \|U\|_\mu \le R\}$. We recall that the integral equation associated to $U_t = \tilde{A}_n U + g(U)$ is

$$U(t) = e^{\tilde{A}_n(t-t_0)}U(t_0) + \int_{t_0}^t e^{\tilde{A}_n(t-s)}g(U(s))ds \ . \tag{4.15}$$

We next prove the following result.

Theorem 4.8. *We assume that the hypotheses of Theorem 4.7 hold. For $r > 0$ small enough, there exists a family $(h_n^u)_{n\in\mathbb{N}\cup\{+\infty\}}$ of functions of class $\mathcal{C}^1$, defined from $B_n^u(r)$ into $B_n^s(M_u r)$, such that $h_n^u(0) = 0$. The graph $W_n^{su}(0,r)$ of h_n^u satisfies*

$$W_n^{su}(0,r) = \{U_0 \in B_X(0, 2M_u r) \ / \ P_n^u U_0 \in B_n^u(r) \text{ and there exists } U \in B_\mu(2M_u r) \text{ solution of (4.15) such that } U(0) = U_0\} \ .$$

Moreover, there exists a positive constant $C = C(\beta)$ such that

$$d_X(W_n^{su}(0,r), W_\infty^{su}(0,r)) \le C\varepsilon_n^\beta \ ,$$

where β is any number in $]0, 1/8[$ if $d = 1$ or $d = 2$ or any number in $]0, \min(\frac{1}{32}, \frac{1-\alpha}{4})[$ if $d = 3$.

The proof of this theorem consists of several lemmas.
The solutions of (4.15) are characterized as follows.

Lemma 4.9. *Let $R > 0$ and $U \in B_\mu(R)$. For any $n \in \mathbb{N}\cup\{+\infty\}$, U is a negative trajectory of (4.15) if and only if, for all $t \le 0$,*

$$U(t) = \int_{-\infty}^t e^{\tilde{A}_n(t-s)} P_n^s g(U(s)) ds + e^{\tilde{A}_n t} P_n^u \xi - \int_t^0 e^{\tilde{A}_n(t-s)} P_n^u g(U(s)) ds \tag{4.16}$$

where $\xi = U(0)$.

Proof : Since the proof is classical, we omit it (see [17]). □

Let $\xi \in X$, we introduce the functional T_n^ξ defined from Y_μ into Y_μ by

$$(T_n^\xi U)(t) = \int_{-\infty}^t e^{\tilde{A}_n(t-s)} P_n^s g(U(s)) ds + e^{\tilde{A}_n t} P_n^u \xi - \int_t^0 e^{\tilde{A}_n(t-s)} P_n^u g(U(s)) ds \ . \tag{4.17}$$

Lemma 4.9 shows that $U(0) \in W_n^{su}(E, r)$ if and only if $T_n^\xi U = U$. It remains to prove that T_n^ξ is a contraction.

Lemma 4.10. *There exists a positive constant r_0, independent of n, such that for all $n \in \mathbb{N}\cup\{+\infty\}$, for all $r \in]0, r_0[$ and $\xi \in X$ with $\|P_n^u \xi\|_X \le r$, T_n^ξ is defined from $B_\mu(2M_u r)$ into $B_\mu(2M_u r)$. Moreover,*

$$\forall n \in \mathbb{N}\cup\{+\infty\}, \forall U, U' \in B_\mu(2r), \ \|T_n^\xi U - T_n^\xi U'\|_\mu \le \frac{1}{2}\|U - U'\|_\mu \ .$$

Proof : To see that T_n^ξ maps $B_\mu(2M_ur)$ into $B_\mu(2M_ur)$, we bound the three terms of (4.17). Let $U \in B_\mu(2M_ur)$. We have

$$\begin{aligned}\|e^{-\mu t}\int_{-\infty}^t e^{\tilde{A}_n(t-s)}P_n^s g(U(s))ds\|_X ds \quad &\leq \int_{-\infty}^t e^{-\mu t}M_s e^{(\mu-\eta)(t-s)}l(2M_ur)\|U(s)\|_X ds\\ &\leq M_s l(2M_ur)\int_{-\infty}^t e^{-\eta(t-s)}\|U\|_\mu ds\\ &\leq \frac{M_s}{\eta}l(2M_ur)\|U\|_\mu\end{aligned}$$

Using (4.8), we obtain $\|e^{-\mu t}e^{\tilde{A}_n t}P_n^u\xi\|_X \leq M_u\|\xi\|_X$. To bound the last term, we write

$$\begin{aligned}\|e^{-\mu t}\int_t^0 e^{\tilde{A}_n(t-s)}P_n^u g(U(s))ds\|_X \quad &\leq \int_t^0 e^{-\mu t}e^{(\mu+\eta)(t-s)}M_u l(2M_ur)\|U(s)\|_X ds\\ &\leq M_u l(2M_ur)\int_t^0 e^{\eta(t-s)}\|U\|_\mu ds\\ &\leq \frac{M_u}{\eta}l(2M_ur)\|U\|_\mu\end{aligned}$$

Thus, using the fact that $l(2M_ur) \longrightarrow 0$, we can choose r_1 small enough so that

$$\frac{M_u+M_s}{\eta}l(2M_ur)2M_ur \leq M_ur\ ,$$

and thus T_n^ξ is defined from $B_\mu(2M_ur)$ into $B_\mu(2M_ur)$. The fact that T_n^ξ is a contraction for r small enough is proved by the same way. We will choose $r_0 \in]0, r_1]$ so that T_n^ξ is a contraction with constant of contraction equal to $1/2$. $\square$

The previous lemma implies that, if r is small enough, for any $n \in \mathbb{N}\cup\{+\infty\}$ and any $\xi \in B_n^u(r)$ there exists a unique solution $U_n^\xi(t) \in B_\mu(2M_ur)$ of (4.15) such that $P_n^u\xi = P_n^uU_n^\xi(0)$. We define the function h_n^u by

$$h_n^u\ :\ \left(\begin{array}{ccc} B_n^u(r) & \longrightarrow & P_n^sX\\ \xi & \longmapsto & P_n^sU_n^\xi(0)\end{array}\right)\ .$$

To be more precise, $P_n^sU_n^\xi(0) = \int_{-\infty}^0 e^{-\tilde{A}_n s}P_n^s g(U(s))ds$ and so, the choice of r in the preceding proof implies that $\|P_n^sU_n^\xi(0)\|_X \leq M_ur$. Therefore, h_n^u is defined from $B_n^u(r)$ into $B_n^s(M_ur)$. Moreover, using the same arguments as in the proof of Lemma 4.10, we can show that h_n^u is Lipschitzian. To finish the proof of Theorem 4.8, we show the following two lemmas.

Lemma 4.11. *There exists a positive constant C such that for any $U \in D(A_\infty)$ and $t \le 0$, we have*

$$\left\| \left(e^{(\tilde{A}_n - \mu)t} P_n^u - e^{(\tilde{A}_\infty - \mu)t} P_\infty^u \right) U \right\|_X \le C \varepsilon_n^{1/2} \|U\|_{D(A_\infty)} \ . \tag{4.18}$$

There exists a positive constant C such that, for any $U \in X$ and any $t \le 0$, we have

$$\|(e^{\tilde{A}_n t} P_n^u - e^{\tilde{A}_\infty t} P_\infty^u) g(U(s))\|_X \le C e^{(\mu + \eta/2)t} \varepsilon_n^\beta \|U(s)\|_X \ , \tag{4.19}$$

with β as in Theorem 4.8, and for any $t \ge 0$

$$\|(e^{\tilde{A}_n (t-s)} P_n^s - e^{\tilde{A}_\infty (t-s)} P_\infty^s) g(U(s))\|_X \le C e^{-(\mu - \eta/2)t} \varepsilon_n^\beta \|U(s)\|_X \ . \tag{4.20}$$

Proof : We notice that $-\tilde{A}_n$ is a bounded operator on $P_n^u X$, since $P_n^u X$ is spanned by a finite number of eigenvectors of $-\tilde{A}_n$. This number and the associated eigenvalues being bounded with respect to n, there exists a positive constant C such that for all $n \in \mathbb{N} \cup \{+\infty\}$, $-\tilde{A}_n - C$ is a dissipative operator on $P_n^u X$. We also remark that the operators $(\tilde{A}_n - \|f_u'(x,0)\|_{\mathbb{L}^\infty} Id)$ are dissipative on $P_n^s X$.
Thus, (4.18) is a direct consequence of Corollary 3.3, Propositions 4.4 and 4.5. The estimates (4.19) and (4.20) are proved in the same way, using the regularity property (4.2) of g and interpolation arguments similar to the proof of Proposition 3.5. $\square$

Lemma 4.12. *Let $r \in]0, r_0[$, where r_0 has been defined in Lemma 4.10, and let $\xi \in X$ such that $\|P_\infty^u \xi\|_X \le r$. There exists a positive constant C such that*

$$\|U_\infty^\xi - U_n^\xi\|_\mu \le C \varepsilon_n^\beta \ , \tag{4.21}$$

where β is given in Theorem 4.8. Moreover, if we set, for $n \in \mathbb{N} \cup \{+\infty\}$, $\xi_n = P_n^u \xi$, then

$$\|h_\infty^u(\xi_\infty) - h_n^u(\xi_n)\|_X \le C \varepsilon_n^\beta \ . \tag{4.22}$$

Proof : We have

$$\begin{aligned} \|U_\infty^\xi - U_n^\xi\|_\mu \quad &= \|T_\infty^\xi U_\infty^\xi - T_n^\xi U_n^\xi\|_\mu \\ &\le \|T_n^\xi U_\infty^\xi - T_n^\xi U_n^\xi\|_\mu + \|T_\infty^\xi U_\infty^\xi - T_n^\xi U_\infty^\xi\|_\mu \\ &\le \frac{1}{2} \|U_\infty^\xi - U_n^\xi\|_\mu + \|T_\infty^\xi U_\infty^\xi - T_n^\xi U_\infty^\xi\|_\mu \ , \end{aligned}$$

and thus,

$$\|U_\infty^\xi - U_n^\xi\|_\mu \le 2 \|T_\infty^\xi U_\infty^\xi - T_n^\xi U_\infty^\xi\|_\mu \ . \tag{4.23}$$

To simplify the notations, we set $U = U_\infty^\xi$. We have

$$\begin{aligned} T_n^\xi U - T_\infty^\xi U \quad &= (e^{\tilde{A}_n t} P_n^u - e^{\tilde{A}_\infty t} P_\infty^u)\xi - \int_t^0 (e^{\tilde{A}_n (t-s)} P_n^u - e^{\tilde{A}_\infty (t-s)} P_\infty^u) g(U(s)) ds \\ &\qquad + \int_{-\infty}^t (e^{\tilde{A}_n (t-s)} P_n^s - e^{\tilde{A}_\infty (t-s)} P_\infty^s) g(U(s)) ds \\ &= K_1 - K_2 + K_3 \ . \end{aligned} \tag{4.24}$$

To estimate the term $\|T_n^\xi U - T_\infty^\xi U\|_\mu = \sup_{t\leq 0} e^{-\mu t}\|T_n^\xi U - T_\infty^\xi U\|_X$, we proceed as follows.

$$e^{-\mu t}K_1 = \left(e^{(\tilde{A}_n-\mu)t}P_n^u - e^{(\tilde{A}_\infty-\mu)t}P_\infty^u\right)P_\infty^u\xi + e^{-\mu t}e^{\tilde{A}_n t}P_n^u(P_n^u - P_\infty^u)\xi \ . \tag{4.25}$$

As P_∞^u is a projection on a finite number of eigenvalues, $P_\infty^u\xi$ belongs to $D(\tilde{A}_\infty)$ and $\|P_\infty^u\xi\|_{D(\tilde{A}_\infty)} \leq C\|\xi\|_X$. Thus, Lemma 4.11 implies that there exists a positive constant C such that for any $t \leq 0$,

$$\left\|\left(e^{(\tilde{A}_n-\mu)t}P_n^u - e^{(\tilde{A}_\infty-\mu)t}P_\infty^u\right)P_\infty^u\xi\right\|_X \leq C\varepsilon_n^{1/2}\|\xi\|_X \ .$$

For the second term of (4.25), we use (4.8) and (4.3) to get

$$\|e^{-\mu t}e^{\tilde{A}_n t}P_n^u(P_n^u - P_\infty^u)\xi\|_X \leq C\varepsilon_n\|\xi\|_X \ ,$$

and thus, gathering the terms of (4.25), we obtain

$$\|K_1\|_\mu \leq C\varepsilon_n^{1/2}\|\xi\|_X \ .$$

We bound the second term of (4.24) by using (4.19) as follows

$$\begin{aligned}
\|e^{-\mu t}K_2\|_X \quad &= \|e^{-\mu t}\int_t^0 (e^{\tilde{A}_n(t-s)}P_n^u - e^{\tilde{A}_\infty(t-s)}P_\infty^u)g(U(s))ds\|_X \\
&\leq \int_t^0 Ce^{\frac{\eta}{2}(t-s)}e^{-\mu s}\varepsilon_n^\beta\|U(s)\|_X ds \\
&\leq C\varepsilon_n^\beta\|U\|_\mu\int_t^0 e^{\frac{\eta}{2}(t-s)}ds \ \leq \frac{2C}{\eta}\varepsilon_n^\beta \ .
\end{aligned}$$

To bound the third term of (4.24), we use (4.20) :

$$\begin{aligned}
\|e^{-\mu t}K_3\|_X \quad &= \|e^{-\mu t}\int_{-\infty}^t (e^{\tilde{A}_n(t-s)}P_n^s - e^{\tilde{A}_\infty(t-s)}P_\infty^s)g(U(s))ds\|_X \\
&\leq C\varepsilon_n^\beta\int_{-\infty}^t e^{-\frac{\eta}{2}(t-s)}e^{-\mu t}\|U(s)\|_X ds \\
&\leq C\varepsilon_n^\beta\|U\|_\mu\int_{-\infty}^t e^{-\frac{\eta}{2}(t-s)}ds \ \leq \frac{2C}{\eta}\varepsilon_n^\beta \ .
\end{aligned}$$

Due to the decomposition (4.24), the inequality (4.23) and the above bounds of $\|K_i\|_\mu$ $(i = 1, 2, 3)$ imply the estimate (4.21).
The inequality (4.22) is a direct consequence of (4.21) and of (4.3). □

Proof of Theorem 4.7 : Lemma 4.12 completes the proof of Theorem 4.8. By Proposition 4.4, for n large enough, E is a hyperbolic equilibrium for $S_n(t)$. Proposition 4.4 together with the decay property (4.8) also imply that there exists a local unstable manifold $W^u_n(E,r)$ which is equal to the strong unstable manifold $W^{su}_n(E,r)$ we have constructed. Thus, the estimate (4.22) of Lemma 4.12 implies the first estimate of (4.14).
It is well-known that, if g is of class $\mathcal{C}^p$, then the mapping $(\xi,U)\longmapsto T^\xi_n U$ is of class $\mathcal{C}^p$ and the fixed point U^ξ_n is a $\mathcal{C}^p$-mapping from $P_nB_X(0,r)$ into Y_μ (see [17]). In particular, we notice that, like in (4.16), we have

$$\begin{aligned} DU^\xi_n\zeta &= e^{\tilde{A}_n t}P^u_n\zeta+\int_{-\infty}^{t} e^{\tilde{A}_n(t-s)}P^s_n g'(U^\xi_n(s))DU^\xi_n(s)\zeta ds \\ &\quad -\int_t^0 e^{\tilde{A}_n(t-s)}P^u_n g'(U^\xi_n(s))DU^\xi_n(s)\zeta ds \;. \end{aligned}$$

Thus, arguing as in Lemma 4.10, one shows that DU^ξ_n is defined in a ball $P^u_nB_X(0,r)$, where r does not depend on n. Arguing as in Lemma 4.12 and using property (4.3) several times, one shows the convergence of DU^ξ_n towards DU^ξ_∞ as well as the second estimate in (4.14).
Finally, the proof of the fact that the Lipschitz-constants of Dh^u_n is uniform with respect to n is similar to the proof of Lemma 4.10. □

4.3 Convergence of the regular part of the local stable manifolds

We can also study the convergence of the local stable manifolds. Notice that this theorem is not needed for the convergence of the attractors $\mathcal{A}_n$ but will be required for the proof of stability of phase-diagrams (see Theorem 2.12).

Theorem 4.13. *Assume that the uniform exponential decay property (UED) holds. Let E be a hyperbolic equilibrium point of the dynamical system $S_\infty(t)$. Then E is also a hyperbolic equilibrium point of $S_n(t)$ for n large enough. Moreover, there exists $n_0\in\mathbb{N}$, such that, for $n\geq n_0$, the local stable manifold $W^s_n(E,r)$ satisfies the properties i) and ii) of Theorem 4.1 with positive constants r, M_s and λ_s independent of n and such that, for $n\geq n_0$, $W^s_n(E,r)$ is the graph of a function h^s_n which is of class $\mathcal{C}^{1,1}(B^s_n(r),P^u_nX)$. Furthermore, the Lipschitz-constants of Dh^s_n is bounded uniformly with respect to n.*
In addition, if $\mathcal{B}$ is a bounded set of X^σ ($\sigma\in]0,s_0[$), there exists a positive constant $C=C(\mathcal{B},\beta)$ such that

$$\forall\xi\in B^s_\infty(r)\cap\mathcal{B},\;\|h^s_\infty(\xi)-h^s_n(P^s_n\xi)\|_X\leq C\varepsilon_n^\beta\;, \tag{4.26}$$

and

$$\forall\xi\in B^s_\infty(r)\cap\mathcal{B},\;\|Dh^s_\infty(\xi)P^s_\infty-Dh^s_n(P^s_n\xi)P^s_n\|_{\mathcal{L}(X^\sigma,X)}\leq C\varepsilon_n^\beta\;, \tag{4.27}$$

where β is any number in $]0,\frac{\sigma^2}{2}[$ if $d=1$ or $d=2$ or any number in $]0,\min(\frac{\sigma^2}{2},\frac{1-\alpha}{4})[$ if $d=3$. In particular, the regular part of the local stable manifold converges in the following sense :

$$d_X(W^s_n(E,r)\cap\mathcal{B};W^s_\infty(E,r)\cap\mathcal{B})\leq C\varepsilon_n^\beta\ . \tag{4.28}$$

Proof : We underline that the important point is the independance of r and λ_s with respect to n. This property is closely linked to Hypothesis (UED). Indeed, assuming the uniform exponential decay (UED), we can improve the estimates (4.8) as follows : there exist positive constants M_s, λ_s and η such that

$$\forall n\in\mathbb{N}\cup\{+\infty\},\forall t\geq 0,\ \|e^{\tilde{A}_n t}P^s_n\|_{\mathcal{L}(X)}\leq M_s e^{-(\lambda_s+\eta)t}\ . \tag{4.29}$$

The outline of the proof is exactly the same as Theorem 4.7, but here, instead of Y_μ, we consider the space

$$Z_{\tilde{\mu}}=\{U\in\mathcal{C}^0([0,+\infty[,\mathbb{C})\ /\ \sup_{t\geq 0}\|U(t)\|_X e^{\tilde{\mu}t}<+\infty\}\ ,$$

where $0<\tilde{\mu}<\lambda_s$ and we remplace T^ξ_n by the functional

$$R^\xi_n : U\in Z_{\tilde{\mu}}\longmapsto\int_t^\infty e^{\tilde{A}_n(t-s)}P^u_n g(U(s))ds+e^{\tilde{A}_n t}P^s_n\xi-\int_0^t e^{\tilde{A}_n(t-s)}P^s_n g(U(s))ds\ .$$

We would like to insist on the modifications in the proof of Lemma 4.12. In this proof, we used the fact that, for all $\xi\in X$, $P^u_\infty\xi$ belongs to $D(\tilde{A}_\infty)$, which is not the case of $P^s_\infty\xi$. As a consequence, we cannot prove the convergence of the whole local stable manifold $W^s_n(E,r)$. Fortunately, we only need the convergence of the subset $W^s_n(E,r)\cap\mathcal{B}$. If we choose $\xi\in W^s_n(E,r)\cap\mathcal{B}$, $P^s_\infty\xi=\xi-P^u_\infty\xi$ is bounded in X^σ and the arguments of Lemma 4.12 are valid in our case. In the same way, we can only prove the convergence of the regular part of the tangent spaces and this convergence is shown with the same arguments as the convergence of the tangent spaces of the local unstable manifolds. Finally, notice that the Lipschitz-constants of h^s_n and Dh^s_n are uniform in n because of Estimate (4.29). □

4.4 Lower-semicontinuity and estimates of the convergence.

Proof of Theorem 2.11 : The lower-semicontinuity of the attractors follows from the convergence of the local unstable manifolds proved in the previous section. In fact, we can be more precise and prove Estimate (2.26). Proofs of such an estimate of the lower-semicontinuity can be found in [20] and [3]. Although the presentation of these proofs is different, the ideas are the same, in particular the gradient structure is strongly used. We also underline that the proof of the estimate for the lower-semicontinuity can be made, by

using the notion of chain of equilibria that we introduce in Section 5.2.
Hypothesis (Hyp) allows us to prove estimates for the upper-semicontinuity due to the following result. If all the equilibria of $S_\infty(t)$ are hyperbolic, then any bounded set $\mathcal{B}$ of X is exponentially attracted by $\mathcal{A}_\infty$, that is that there exist positive constants M and λ such that

$$\forall t \geq 0,\ \sup_{U\in\mathcal{B}} \mathrm{dist}_X(S_\infty(t)U, \mathcal{A}_\infty) \leq Me^{-\lambda t}\ . \tag{4.30}$$

The proof of this property, and the fact that it implies an estimate of the upper-semicontinuity can be found in [30], [22] or [41]. Once again, the proof of this exponential attraction strongly uses the gradient structure of the dynamical system.

To obtain an estimate of the upper-semicontinuity from (4.30), we modify the proof of Theorem 2.8 as follows. The attracting property (2.22) is replaced by the stronger property

$$\forall U \in \bigcup_n \mathcal{A}_n,\ \mathrm{dist}_X(S_\infty(t)U, \mathcal{A}_\infty) \leq Me^{-\lambda t}\ . \tag{4.31}$$

On the other hand, Theorem 3.10 and the fact that $\cup\mathcal{A}_n$ is bounded in X imply that

$$\forall U_n \in \mathcal{A}_n,\ \|(S_n(t) - S_\infty(t))U_n\|_{X^{-s}} \leq Ce^{Ct}\varepsilon_n^{s/8}\ . \tag{4.32}$$

Replacing t by $-\frac{s}{16C}\ln\varepsilon_n$ in (4.32), which is positive for n large enough, we deduce from (4.31) and (4.32), that

$$\sup_{U_n\in\mathcal{A}_n} \mathrm{dist}_{X^{-s}}(S_n(t)U_n, \mathcal{A}_\infty) \leq Me^{-\lambda t} + Ce^{Ct}\varepsilon_n^{s/8} = M\varepsilon_n^{\frac{\lambda s}{16C}} + C\varepsilon_n^{\frac{s}{16}}$$

This concludes the proof of the inequality (2.27) since $S_n(t)\mathcal{A}_n = \mathcal{A}_n$. □

5 Stability of phase-diagrams

In this section, we prove Theorem 2.12. We assume in the whole section that $\Omega =]0,1[$ and $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$, with $a \neq 1$ and $b \neq 1$. We recall that these hypotheses imply that $e^{A_n t}$ is a group of operators for all $n \in \mathbb{N}\cup\{+\infty\}$ and that $S_n(t)$ and $DS_n(t)$ are one to one. Thus, if E is a hyperbolic equilibrium of $S_n(t)$, then the stable and unstable sets $W_n^s(E)$ and $W_n^u(E)$, introduced in Definition 4.2, are well-defined global manifolds of X. We also assume that the hypotheses of Theorem 2.12 hold, that is that Hypotheses (Diss) and (UED) and the Morse-Smale property for $S_\infty(t)$ are satisfied.
Let E_- and E_+ be two equilibria of the dynamical systems $S_n(t)$, we say that $S_n(t)$ admits a connecting orbit between E_- and E_+ if there exists a complete trajectory $U_n(t)$ ($t \in \mathbb{R}$),

solution of Equation (2.5) such that $U_n(t)$ converges to E_- (resp. E_+) when t goes to $-\infty$ (resp. $+\infty$). This orbit is said to be transversal if at any point of it, the manifolds $W_n^u(E_-)$ and $W_n^s(E_+)$ intersect transversally, that is that at each point U_n of the trajectory, the tangent space $T_{U_n}W_n^s(E_+)$ has a closed complement and $T_{U_n}W_n^u(E_-) + T_{U_n}W_n^s(E_+) = X$.

The proof of Theorem 2.12 can be split into the following two lemmas.

Lemma 5.1. *We assume that $\Omega =]0,1[$, $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$, with $a \neq 1$ and $b \neq 1$, that $S_\infty(t)$ satisfies the Morse-Smale property and that Hypotheses (Diss) and (UED) hold. Let E_- and E_+ be two hyperbolic equilibria of the dynamical systems $S_n(t)$. If $W_\infty^u(E_-) \cap W_\infty^s(E_+)$ is a manifold of dimension r then, for n large enough, $W_n^u(E_-) \cap W_n^s(E_+)$ is a manifold of dimension r.*

Lemma 5.2. *Assume that the hypotheses of Lemma 5.1 hold. If $\mathcal{O}_n$ is a sequence of connecting orbits for $S_n(t)$ between E_- and E_+, then*

i) $S_\infty(t)$ admits a connecting orbit between E_- and E_+,

ii) there exists a subsequence $\mathcal{O}_{\varphi(n)}$ of $\mathcal{O}_n$ such that, for n large enough, the orbits $\mathcal{O}_{\varphi(n)}$ are transversal.

Remark : We underline that the proof of i) of Lemma 5.2 gives an interesting result even if $S_\infty(t)$ is not a Morse-Smale system. Indeed, the proof shows that there exists a chain of equilibria $E_- = E_0$, $E_1 ... E_p = E_+$ such that $S_\infty(t)$ admits a connecting orbit between E_i and E_{i+1}. The Morse-Smale property is only used to prove that this implies the existence of a connecting orbit between E_- and E_+.

Proof of Theorem 2.12 : Lemmas 5.1 and 5.2 imply Theorem 2.12, that is the stability of phase-diagram and the Morse-Smale property. Indeed, the number of equilibrium points of $S_\infty(t)$ (and thus of $S_n(t)$) is finite since they are bounded in $D(A_\infty)$ and are hyperbolic. Thus, Lemmas 5.1 and 5.2 clearly imply the stability of phase-diagrams. The hyperbolicity of equilibria for $S_n(t)$, for n large enough, has been proved in Proposition 4.4. Finally, assume that $S_n(t)$ is not a Morse-Smale system for n large enough, then we can find a sequence of complete bounded trajectories for $S_n(t)$ which are not transversal. Since the number of equilibria is finite, we can assume that the trajectories connect the same equilibria and this contradicts Lemma 5.2. Thus, $S_n(t)$ has the Morse-Smale property for n large enough. $\square$

5.1 Proof of Lemma 5.1

Let E_- and E_+ be two equilibria of $S_\infty(t)$. In Theorems 4.7 and 4.13, we have shown that there exist two radii r_- and r_+ such that the local manifolds $W_n^u(E_-, r_-)$ and $W_n^s(E_+, r_+)$

are well-defined. We denote P_n^{u+} (resp. P_n^{u-}) the projection onto the unstable part of the spectrum of the linearization $\tilde{A}_n$ at the equilibrium point E^+ (resp.E^-). Similarly, $P_n^{s\pm}$ are the projections onto the stable part. We set $B_n^u(E_\pm, r_\pm) = B(E_\pm, r_\pm) \cap P_n^{u\pm}X$ and $B_n^s(E_\pm, r_\pm) = B(E_\pm, r_\pm) \cap P_n^{s\pm}X$. We denote

$$h_n^u : B_n^u(E_-, r_-) \longrightarrow B_n^s(E_-, M_- r_-) \text{ and } h_n^s : B_n^s(E_+, r_+) \longrightarrow B_n^u(E_+, M_+ r_+)$$

the functions given in Theorem 4.1, whose graphs are $W_n^u(E_-, r_-)$ and $W_n^s(E_+, r_+)$ respectively.
For any time $T \geq 0$, we introduce the map

$$\Psi_T^n : \left(\begin{array}{ccc} B_\infty^u(E_-, r_-) & \longrightarrow & X \\ \xi & \longmapsto & S_n(T) \circ [Id + h_n^u(.)]P_n^{u-}\xi \end{array} \right) .$$

The union of the ranges $\bigcup_{T\geq 0} R(\Psi_T^n)$ is equal to the unstable manifold $W_n^u(E_-)$. Assume that $S_\infty(t)$ admits a connecting orbit between E_- and E_+, and let U_0 be a point of this trajectory such that $P_\infty^{u-}U_0$ belongs to $B_\infty^u(E_-, r_-)$. There exists a neighborhood θ of $P_\infty^{u-}U_0$ in $B_\infty^u(E_-, r_-)$ such that $\Psi_T^\infty(\theta) \subset B(E_+, r_+)$ for some T large enough. For $n = \infty$ and for any n large enough, we set

$$\Phi^n : \left(\begin{array}{ccc} \theta & \longrightarrow & B_\infty^u(E_+, r_+) \\ \xi & \longmapsto & P_\infty^{u+} \circ [P_n^{u+} - h_n^s(P_n^{s+}.)] \circ \Psi_T^n(\xi) \end{array} \right) .$$

Since, for n large enough, P_∞^{u+} is an isomorphism from $P_n^{u+}X$ onto $P_\infty^{u+}X$, it follows that, by construction, the equality $\Phi^n(\xi) = 0$ is equivalent for n large enough to the existence of a trajectory for $S_n(t)$ between E_- and E_+, which intersects the subset $[Id + h_n^u(.)]P_n^{u-}(\theta)$ of the unstable manifold $W_n^u(E_-, r_-)$.
Using Proposition 3.11 and Theorems 4.7 and 4.13, we obtain the following properties.

Lemma 5.3. *The function Φ^n and the derivatives $D\Psi_T^n$ and $D\Phi^n$ are well-defined for n large enough. Moreover, Ψ_T^n, Φ^n, $D\Psi_T^n$ and $D\Phi^n$ are continuous with respect to $\xi \in \theta$, uniformly in $n \in \mathbb{N} \cup \{+\infty\}$ and converge respectively to Ψ_T^∞, Φ^∞, $D\Psi_T^\infty$ and $D\Phi^\infty$, when n goes to $+\infty$, uniformly in $\xi \in \theta$.*

We recall that $m(E_\pm)$ is the Morse index of $E_\pm$, that is the dimension of the linear unstable space $P_\infty^{u+}X$. As $S_\infty(t)$ and $DS_\infty(t)$ are one-to-one, $\Psi_T^\infty(\theta)$ is an open subset of dimension $m(E_-)$ of $W_\infty^u(E_-)$. By assumption, it has a non-empty transversal intersection with $W_\infty^s(E_+, r_+)$. The classical $\lambda-$lemma (see [38] and [19]) implies that for all $\varepsilon > 0$, we can find T large enough and a submanifold $\tilde{\theta}$ of θ, which contains $P_\infty^{u-}U_0$ and which is of dimension $m(E_+)$, such that $\psi_T^\infty(\tilde{\theta})$ is $\varepsilon - \mathcal{C}^1-$close to $B_\infty^u(E_+, r_+)$. Thus, $P_\infty^{u+} \circ D\psi_T^\infty(P_\infty^{u-}U_0)$ is onto $P_\infty^{u+}X$, and by Lemma 5.3, if θ is choosen small enough, then for any $\xi \in \theta$, $P_\infty^{u+} \circ D\psi_T^\infty(\xi)$ is onto $P_\infty^{u+}X$. As $Dh_\infty^s(E_+) = 0$, if r_+ is small enough, Dh_∞^s is small and $D\Phi^\infty(\xi)$ is onto $P_\infty^{u+}X$, that is that Φ^∞ is a submersion. Using Theorem 2.8

of Chapter one of [15], we see that Φ^∞ is an open function, ie $\Phi^\infty(\theta)$ is a neighborhood of 0. Lemma 5.3 implies that $\Phi^n(\theta)$ is also a neighborhood of 0 for n large enough and that Φ^n is also a submersion. Theorem 2.8 of chapter one of [15] implies that $(\Phi^n)^{-1}(0)$ is a submanifold of θ of dimension $m(E_-) - m(E_+)$. Since $S_\infty(t)$ and $DS_\infty(t)$ are one-to-one, the dimension of the intersection $W^u_n(E_-) \cap W^s_n(E_+)$ is $m(E_-) - m(E_+)$.

5.2 The notion of chain of equilibria

We introduce in this section the notion of chain of equilibria. The ideas behind it are not really new since this notion is close to the one of *family of combined limit trajectories* given in [3] and [4], which was used to show lower-semicontinuity of attractors.
This notion enables us to give a proof of Lemma 5.2, which is different from [19]. In particular, we do not need any result of convergence of the local stable manifolds to prove the property i) of Lemma 5.2. On the other hand, we extensively use the gradient structure, that is that the Lyapounov function Φ given by (2.6) is non-increasing along the trajectories of $S_n(t)$ and that,

$$\text{if, for any } t \geq 0,\ \Phi(S_\infty(t)U) = \Phi(U), \text{ then } U \text{ is an equilibrium point.} \tag{5.1}$$

In the proof of Lemma 5.2, we will use several times the following result. We recall that the upper-semicontinuous in X of the attractors has been shown in Theorem 2.10.

Lemma 5.4. *Assume that the attractors $\mathcal{A}_n$ are upper-semicontinuous in X at $n = +\infty$. For any positive time T and any sequence $(U_n)_{n\in\mathbb{N}}$, such that $U_n \in \mathcal{A}_n$, there exists $U_\infty \in \mathcal{A}_\infty$ and a subsequence (U_{n_k}) of (U_n) satisfying*

$$\sup_{t\in[0,T]} \|S_{n_k}(t)U_{n_k} - S_\infty(t)U_\infty\|_X \longrightarrow 0 \textit{ when } n \longrightarrow +\infty\ .$$

Proof : Due to the upper-semicontinuity of the attractors, there exists a sequence of points $V_n \in \mathcal{A}_\infty$ such that $\|U_n - V_n\|_X \to 0$. As $\mathcal{A}_\infty$ is compact, we can extract a subsequence V_{n_k} which converges to $U_\infty \in \mathcal{A}_\infty$. Proposition 2.3 implies that $\sup_{t\in[0,T]} \|S_{n_k}(t)U_{n_k} - S_{n_k}(t)U_\infty\|_X \longrightarrow 0$. On the other hand, Theorem 3.8 and the regularity of $\mathcal{A}_\infty$ imply that $\sup_{t\in[0,T]} \|S_{n_k}(t)U_\infty - S_\infty(t)U_\infty\|_X \longrightarrow 0$ and the proof is complete. □

To avoid heavy notations, we do not reindex subsequences in what follows. We recall that $\mathcal{E}$ denotes the set of all equilibria. We choose a small enough radius r such that the balls $B(E, 2r)$ $(E \in \mathcal{E})$ are disjoint and such that the local stable and unstable manifolds $W^s_\infty(E, 2r)$ and $W^u_\infty(E, 2r)$ are well-defined. Let E_- and E_+ be two equilibrium points. Assume that for n large enough, $S_n(t)$ has a connecting orbit between E_- and E_+. There exist U^0_n in the local unstable manifold $W^u_n(E_-, r_n)$ $(r_n \leq r)$ and t_n such that U^0_n converges to E_-, $S_n(t_n)U_n^0$ belongs to $W_n^s(E_+, r'_n)$ ($r'_n \leq r$) and $S_n(t_n)U_n^0$ converges to E_+. We introduce the following notion.

Definition 5.5. *Let $\mathcal{O}_n$ be an orbit of $S_n(t)$. A sequence of equilibria $E_- = E_0$, E_1,..., $E_p = E_+$ is called a chain of equilibria of length p for the sequence $(\mathcal{O}_n)$ if there exist $U_n^0 \in \mathcal{O}_n$ and $p+1$ sequences of times $0 = t_n^0 < t_n^1 < ... < t_n^p = t_n$ such that, if we set $U_n(t) = S_n(t)U_n^0$, then*

$$U_n(t_n^i) \longrightarrow E_i \ , \ \text{as } n \longrightarrow +\infty$$

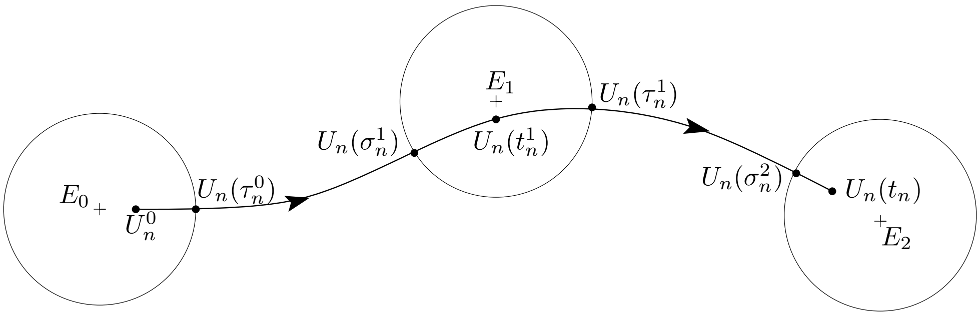
and for all $n \in \mathbb{N}$ and $i < p$, there exists $t \in]t_n^i, t_n^{i+1}[$ such that $U_n(t)$ does not belong to $\cup_{E \in \mathcal{E}} \overline{B(E, r)}$.

If E_i is a chain of equilibria, $U_n(t_n^i) \in B(E_i, r)$ for n large enough and we can assume that this holds for all n. For $i > 0$, we denote the time of entrance in $B(E_i, r)$

$$\sigma_n^i = \sup\{t \leq t_n^i \mid U_n(t) \notin B(E_i, r)\} \ ,$$

and for $i < p$, we denote the time of exit of $B(E_i, r)$

$$\tau_n^i = \inf\{t \geq t_n^i \mid U_n(t) \notin B(E_i, r)\} \ .$$



We obtain the following result.

Lemma 5.6. *There exist $V_i \in \partial B(E_i, r) \cap W_\infty^s(E_i, 2r) \cap \mathcal{A}_\infty$ and $W_i \in \partial B(E_i, r) \cap W_\infty^u(E_i, 2r)$ such that, extracting subsequences, we have*

$$U_n(\sigma_n^i) \longrightarrow V_i \text{ and } U_n(\tau_n^i) \longrightarrow W_i \ .$$

Proof : We use Lemma 5.4 with $T = 0$ to show that there exists a point $W_i \in \mathcal{A}_\infty$ such that $U_n(\tau_n^i) \longrightarrow W_i$. Due to the definition of τ_n^i, it is clear that $U_n(\tau_n^i) \in \partial B(E_i, r)$ and thus $W_i \in \partial B(E_i, r)$. Assume that there exist a time T and $\tilde{W}_i \in X$ such that $S_\infty(T)\tilde{W}_i = W_i$ and $\tilde{W}_i \notin \overline{B}(E_i, r)$. Using Lemma 5.4, we find that $U_n(\tau_n^i - T) \longrightarrow \tilde{W}_i$, otherwise we contradict the backward uniqueness of $S_\infty(t)$, and thus $U_n(\tau_n^i - T) \notin B(E_i, r)$ for n large enough. If $i = 0$, this contradict the fact that $U_n^0 \in W_n^u(E_-, r_n)$. If $i \geq 1$, we must have $\tau_n^i - T < t_n^i < \tau_n^i$, so we can assume that $t_n^i - (\tau_n^i - T) \longrightarrow s$. Lemma 5.4 shows that

$S_\infty(s)\tilde{W}_i = E_i$, which is absurd. We have thus proved that $W_i \in W^u_\infty(E, r)$.
The arguments are similar for σ^i_n. $\square$

The length of a chain of equilibria is bounded, since the number of equilibria is finite and we have the following property.

Lemma 5.7. *If (E_i) is a chain of equilibria, then $i < j$ implies $E_i \neq E_j$.*

Proof : Since the Lyapounov function Φ does not increase along the trajectories of $S_\infty(t)$ and that (5.1) holds, we must have $\Phi(V_j) > \Phi(E_j)$ and $\Phi(W_i) \leq \Phi(E_i)$. Lemma 5.6 and the decay of Φ along the trajectories of $S_n(t)$ imply that $\Phi(E_i) > \Phi(E_j)$. $\square$

Of course, the set of chains of equilibria corresponding to the trajectories $S_n(t)U^0_n$ is not empty as (E_-, E_+) is a trivial chain. So, we can choose a chain of equilibria (E_i) of maximal length since the number of equilibria is finite and since Lemma 5.7 holds.

Lemma 5.8. *If (E_i) is a chain of equilibria of maximal length p, then there exists a finite time T such that*

$$\forall i = 0, .., p-1, \ \sup_{n\in\mathbb{N}} \{\sigma^{i+1}_n - \tau^i_n\} \leq T \ .$$

Proof : Assume that $\sigma^{i+1}_n - \tau^i_n \longrightarrow +\infty$. Let $T_n = \sqrt{\sigma^{i+1}_n - \tau^i_n}$. There exists a sequence of times $s_n \in]\tau^i_n, \sigma^{i+1}_n - T_n[$ such that $\Phi(U_n(s_n)) - \Phi(U_n(s_n + T_n)) \longrightarrow 0$. Indeed, if not, there exists $\varepsilon > 0$ such that for all $s \in]\tau^i_n, \sigma^{i+1}_n - T_n[$ and n large enough, we have $\Phi(U_n(s)) - \Phi(U_n(s + T_n)) > \varepsilon$. If we denote $\lfloor T_n \rfloor$ the largest integer less than T_n, this implies that $\Phi(U_n(\tau^i_n)) - \Phi(U_n(\sigma^{i+1}_n)) > \lfloor T_n \rfloor \varepsilon \longrightarrow +\infty$, which is absurd. Using Lemma 5.4, we find that $U_n(s_n)$ converges to $U \in \mathcal{A}_\infty$ and that for all $t \geq 0$, we have $\Phi(U) - \Phi(S_\infty(t)U) = 0$. This means that U is an equilibrium point which contradicts the fact that the length of the chain of equilibria $E_1, ..., E_p$ is maximal. $\square$

We conclude with the following result.

Lemma 5.9. *If (E_i) is a chain of equilibria of maximal length p between E_- and E_+, then, for all $i < p$, $S_\infty(t)$ admits a connecting orbit between E_i and E_{i+1}.*

Proof : We can assume that $\sigma^{i+1}_n - \tau^i_n \longrightarrow T_i$. Using the notation of Lemma 5.6, we have $W_i \in W^u_\infty(E_i)$ and $V_{i+1} \in W^s_\infty(E_{i+1})$. We obtain

$$\begin{aligned}
\|S_\infty(T_i)W_i - V_{i+1}\|_X \quad & \leq \|S_\infty(T_i)W_i - S_\infty(\sigma^{i+1}_n - \tau^i_n)W_i\|_X \\
& + \|S_\infty(\sigma^{i+1}_n - \tau^i_n)W_i - S_n(\sigma^{i+1}_n - \tau^i_n)W_i\|_X \\
& + \|S_n(\sigma^{i+1}_n - \tau^i_n)W_i - S_n(\sigma^{i+1}_n - \tau^i_n)U_n(\tau^i_n))\|_X \\
& + \|U_n(\sigma^{i+1}_n) - V_{i+1}\|_X \ .
\end{aligned}$$

Taking the limit when n goes to $+\infty$, we find that $S_\infty(T_i)W_i = V_{i+1}$, which yields a connecting orbit for $S_\infty(t)$ between E_i and E_{i+1}. $\square$

5.3 Proof of Lemma 5.2

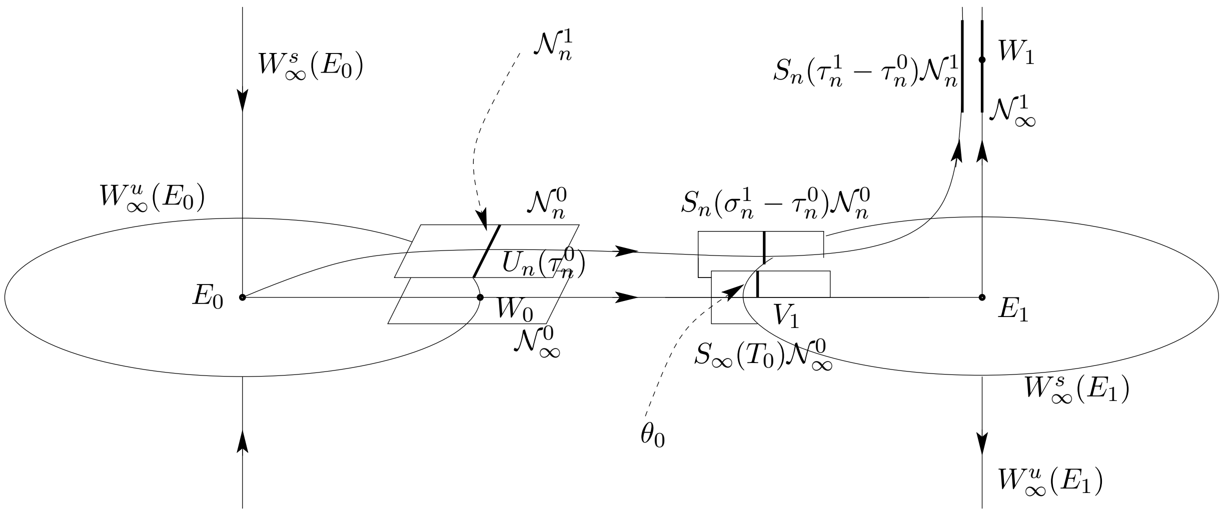
We use the notations of Section 5.2. Assume that there exists a sequence of connecting orbits $\mathcal{O}_n$ for $S_n(t)$ between E_- and E_+. As noticed in the previous section, up to an extraction of a subsequence, there exists a chain of equilibria of maximal length $E_- = E_0, E_1, ..., E_p = E_+$ associated with our sequence $(\mathcal{O}_n)$ of trajectories. Lemma 5.9 shows that $S_\infty(t)$ admits a connecting orbit between E_i and E_{i+1} $(0 \leq i \leq p-1)$. Thus, Property i) of Lemma 5.2 is a direct consequence of the classical cascading property : if $S(t)$ is a Morse-Smale dynamical system which admits a connecting orbit between E_i and E_{i+1} $(0 \leq i \leq p-1)$, then it has a connecting orbit between E_0 and E_p (see for example [38] or [19]).
Next, we prove Property ii). Let θ_1 and θ_2 be two open sets of a Banach space X. We say that two $\mathcal{C}^1-$manifolds $i_1 : \theta_1 \to X$ and $i_2 : \theta_2 \to X$ are $\varepsilon - \mathcal{C}^1-$close if there exists a $\mathcal{C}^1-$diffeomorphism $\varphi : \theta_1 \to \theta_2$, such that $i_1 : \theta_1 \to X$ and $i_2 \circ \varphi : \theta_1 \to X$ are $\varepsilon - \mathcal{C}^1-$maps, that is that $\|i_1 - i_2 \circ \varphi\|_{\mathbb{L}^\infty(\theta_1, X)} < \varepsilon$ and the same for the derivative $\|Di_1 - D(i_2 \circ \varphi)\|_{\mathbb{L}^\infty(\theta_1, X)} < \varepsilon$. We define similarly the $\mathcal{C}^1-$convergence of $\mathcal{C}^1-$manifolds. The classical local $\lambda-$lemma can be extended as follows in our particular frame.

Proposition 5.10. *Let E be a hyperbolic equilibrium point with Morse index $m(E)$. Let $\mathcal{B}$ be a bounded set of X^σ $(\sigma > 0)$. Let q_∞ be a point of $W^s_\infty(E, r) \cap \mathcal{B}$ and let $D_\infty \subset \mathcal{B}$ be a disk of center q_∞, which is transversal to $W^s_\infty(E, r)$ and whose dimension is $m(E)$. Let $(D_n)_{n\in\mathbb{N}}$ be a family of disks with center q_n, bounded in $\mathcal{B}$, and such that D_n $\mathcal{C}^1-$converges to D_∞.*
Then, for all $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $T > 0$ such that for all $n \geq N$ and $t \geq T$, the connected component of $S_n(t)D_n \cap B_X(E, r)$, to which $S_n(t)q_n$ belongs, is $\varepsilon - \mathcal{C}^1-$close to $W^u_\infty(E, r)$.

Proof : The proof of the proposition is a straightforward adaptation of the proof of the classical $\lambda-$lemma (see for example [37] or [38]). Notice that the proof crucially uses Hypothesis (UED), which implies that Property ii) of Theorem 4.1 holds uniformly with respect to n, and the fact that the family of disks belongs to a bounded set $\mathcal{B}$ of X^σ $(\sigma > 0)$. $\square$

We recall that $\lim U_n(\sigma^i_n) = V_i \in \partial B(E_i, r) \cap W^s_\infty(E_i, 2r) \cap \mathcal{A}_\infty$ and $\lim U_n(\tau^i_n) = W_i \in \partial B(E_i, r) \cap W^u_\infty(E_i, 2r)$. Due to the convergence of the local unstable manifolds proved in

Theorem 4.7, there exist a neighborhood $\mathcal{N}_\infty^0$ of W_0 in $W_\infty^u(E_0, 2r)$ and a sequence of neighborhoods $(\mathcal{N}_n^0)$ of $U_n(\tau_n^0)$ in $W_n^u(E_0, 2r)$ such that $\mathcal{N}_n^0$ $\mathcal{C}^1-$converges to $\mathcal{N}_\infty^0$. As, by Lemma 5.8, $\sigma_n^1 - \tau_n^0$ is bounded, we can assume that $\sigma_n^1 - \tau_n^0 \longrightarrow T_0$. Notice that the sequence of manifolds $(\mathcal{N}_n^0)$ is bounded in X^σ for some positive σ and that $\mathcal{N}_n^0$ is finite-dimensional. Thus, Proposition 3.11 implies that the manifold $S_n(\sigma_n^1 - \tau_n^0)\mathcal{N}_n^0$, which contains $U_n(\sigma_n^1)$, $\mathcal{C}^1-$converges in X to the manifold $S_\infty(T_0)\mathcal{N}_\infty^0$, which contains V_1. As $S_\infty(t)$ has the Morse-Smale property, we can find a submanifold θ_0 of $S_\infty(T_0)\mathcal{N}_\infty^0$ of dimension $m(E_1)$ which is tranversal to $W_\infty^s(E_1)$ and which contains V_1. Thus, we can find a submanifold $\mathcal{N}_n^1$ of $\mathcal{N}_n^0$ of dimension $m(E_1)$, which contains $U_n(\tau_n^0)$ and is such that $S_n(\sigma_n^1 - \tau_n^0)\mathcal{N}_n^1$ $\mathcal{C}^1-$converges to θ_0. Using the generalized $\lambda-$lemma of Proposition 5.10, we find that there exists a neighborhood $\mathcal{N}_\infty^1$ of W_1 in $W_\infty^u(E_1, 2r)$ such that $S_n(\tau_n^1 - \tau_n^0)\mathcal{N}_n^1$ $\mathcal{C}^1-$converges to $\mathcal{N}_\infty^1$.



By a finite number of iterations of this process, we obtain that there exists a submanifold $\mathcal{N}_n^{p-1}$ of $\mathcal{N}_n^0$ of dimension $m(E_{p-1})$ such that $S_n(\sigma_n^p - \tau_n^0)\mathcal{N}_n^{p-1}$ $\mathcal{C}^1-$converges to $S_\infty(T_{p-1})\mathcal{N}_\infty^{p-1}$, a neighborhood of V_p in $W_\infty^u(E_{p-1})$. As the union of the attractors $\cup\mathcal{A}_n$ is bounded in X^s for some positive s, there exists a ball $\mathcal{B}$ of X^s such that $S_n(\sigma_n^p - \tau_n^0)\mathcal{N}_n^{p-1} \subset \mathcal{B}$ for all n. The convergence of the regular part of the local stable manifolds (see Theorem 4.13) implies that $W_n^s(E_p, 2r) \cap \mathcal{B}$ $\mathcal{C}^1-$converges to $W_\infty^s(E_p, 2r) \cap \mathcal{B}$. Thus, for n large enough, the dimension of $W_n^s(E_p, 2r) \cap S_n(\sigma_n^p - \tau_n^0)\mathcal{N}_n^{p-1}$ is less than the dimension of $W_\infty^s(E_p, 2r) \cap S_\infty(T_{p-1})\mathcal{N}_\infty^{p-1}$. By assumption, $S_\infty(T_{p-1})\mathcal{N}_\infty^{p-1}$ and $W_\infty^s(E_p, 2r)$ intersect tranversally and so, a dimensional argument implies that $W_n^s(E_p, 2r)$ and $S_n(\sigma_n^p - \tau_n^0)\mathcal{N}_n^{p-1}$ intersect tranversally. As $S_n(\sigma_n^p - \tau_n^0)\mathcal{N}_n^{p-1}$ is a submanifold of $W_n^u(E_0)$, this shows that the orbit $\mathcal{O}_n$ is transversal.

6 Study of the hypotheses

6.1 The one-dimensional case

First, we notice that Hypotheses (ED) and (Grad) are always satisfied in dimension one. Indeed, we have assumed that $\gamma_n \neq 0$ in the space of the measures, which is well-known to imply (ED) and (Grad), even for the case $n = \infty$. Concerning Hypothesis (ED), we refer to [23], [8] and [10] for $n \in \mathbb{N}$; and [11], [29], [32], [44] and [45] for $n = +\infty$. Concerning Hypothesis (Grad), we respectively refer to [21] and [31].
Hypothesis (UED) is the only assumption that we have to verify in dimension one. There exist many methods to prove the exponential decay property for Equation (1.2) when n is fixed. However, the proof of uniform exponential decay for a family of dissipations $(\gamma_n)_{n\in\mathbb{N}}$ is more difficult, especially when the family is not bounded in $\mathbb{L}^\infty$. In the one-dimensional case, we are able to adapt an idea of Haraux (see [23]).

Definition 6.1. *We say that a dissipation γ is effective on the free waves if the following criterium is satisfied.*

(EFW) *There exist a time T and a positive constant C such that, for any $(\varphi_0, \varphi_1) \in X$, the solution of the free wave equation*

$$\begin{cases} \varphi_{tt} + B\varphi = 0 \\ (\varphi, \varphi_t)_{|t=0} = (\varphi_0, \varphi_1) \in X \end{cases} \tag{6.1}$$

satisfies

$$\int_0^T \int_\Omega \gamma(x)|\varphi_t(x,t)|^2 dxdt \geq C\|(\varphi_0, \varphi_1)\|_X^2 \ . \tag{6.2}$$

The following implication is well-known for n fixed (see [23]). We extend it easily to the case of a family of dissipations.

Proposition 6.2. *If (UED) is satisfied, then the family of dissipations $(\gamma_n)_{n\in\mathbb{N}} \subset \mathbb{L}^\infty(\Omega)$ is uniformly effective on the free waves, that is that the property (EFW) is satisfied for each γ_n, with T and C independent of n.*

Proof : Assume that (UED) is satisfied, then there exists a positive time T, independent of n, such that $\|e^{A_n T}\|^2_{\mathcal{L}(X)} \leq \frac{1}{2}$. Thus, for any $U_0 = (\varphi_0, \varphi_1) \in X$, we have,

$$\int_0^T \int_\Omega \gamma_n |u_t|^2 = \frac{1}{2}(\|U_0\|^2 - \|U(T)\|^2) \geq \frac{1}{4}\|U_0\|_X^2 \ , \tag{6.3}$$

where $(u,u_t)(t) = U(t) = e^{A_n t}U_0$, For any $U_0 = (\varphi_0, \varphi_1)$, we denote (φ, φ_t) the solution of the free wave equation (6.1). We set $w = u - \varphi$, which is the solution of the system

$$\begin{cases} w_{tt} + \gamma_n w_t + Bw = -\gamma_n \varphi_t \\ w(0) = 0 \end{cases} .$$

Multipliying by w_t and integrating on $[0,T] \times \Omega$, we obtain

$$\frac{1}{2}\|w(T)\|_X^2 + \int_0^T \int_\Omega \gamma_n |w_t|^2 = -\int_0^T \int_\Omega \gamma_n \varphi_t \overline{w}_t \ ,$$

and thus, using Cauchy-Schwartz inequality, we get

$$\int_0^T \int_\Omega \gamma_n |w_t|^2 \leq \int_0^T \int_\Omega \gamma_n |\varphi_t|^2 \ .$$

Finally, (6.3) implies that

$$\|(\varphi_0,\varphi_1)\|_X^2 \leq 4\int_0^T \int_\Omega \gamma_n |u_t|^2 \leq 4\left(\int_0^T \int_\Omega \gamma_n |w_t|^2 + \int_0^T \int_\Omega \gamma_n |\varphi_t|^2\right) \leq 8\int_0^T \int_\Omega \gamma_n |\varphi_t|^2 \ .$$

$\square$

Of course, the interesting question is to know if the uniform effectiveness on the free waves implies (UED). We give here a way of obtaining this implication in dimension one, by using a multiplier method inspired by [13]. This method is of course not the only one. In the appendix, we recall a theorem of [2], which implies the same result. The crucial point in the following results is to obtain the dependence of the constants on $\|\gamma\|_{\mathbb{L}^1}$ and not on $\|\gamma\|_{\mathbb{L}^\infty}$, since our family of dissipations $(\gamma_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^1(]0,1[)$ but not in $\mathbb{L}^\infty(]0,1[)$.
To simplify, we work here with $B = -\Delta_N + Id$. The same results are true for other boundary conditions with a similar proof.
First, we use the multipliers method to prove the following estimate.

Proposition 6.3. *Let $\gamma \in \mathbb{L}^1(]0,1[)$ and $h \in \mathbb{L}^1_t(\mathbb{R}, \mathbb{L}^2_x(]0,1[))$. Let u be the solution of*

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = h(x,t) \quad (x,t) \in]0,1[\times\mathbb{R}_+ \\ u_x(0,t) = u_x(1,t) = 0 \\ (u,u_t)_{|t=0} = (u_0,u_1) \in \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[) \end{cases}$$

Then, for all $T > 0$, there exists a constant $C = C(T, \|\gamma\|_{\mathbb{L}^1})$ such that

$$\int_0^T \int_0^1 \gamma(x)(|u_x|^2 + |u|^2 + |u_t|^2)dxdt \leq C\left(\int_0^T \int_0^1 |h|(|u_x| + |u_t|)dxdt + \|u_0\|_{\mathbb{H}^1}^2 + \|u_1\|_{\mathbb{L}^2}^2\right) .$$

Proof : We set

$$\rho = \begin{cases} \int_0^x \gamma(\xi)d\xi & 0 \le x \le 1/2 \\ 2(1-x)\int_0^{1/2} \gamma(\xi)d\xi & 1/2 \le x \le 1 \end{cases}$$

Notice that $\|\rho\|_{\mathbb{L}^\infty} \le \|\gamma\|_{\mathbb{L}^1}$ and $\rho(0) = \rho(1) = 0$. We have

$$\int_0^T \int_0^1 (u_{tt} - u_{xx} + u)\rho u_x = \int_0^T \int_0^1 h\rho u_x \ .$$

Using integrations by parts, we find

$$\frac{1}{2}\int_0^T \int_0^1 \rho_x(|u_x|^2 + |u_t|^2) = -\left[\int_0^1 \rho u_t u_x dx\right]_0^T + \int_0^T \int_0^1 h\rho u_x + \frac{1}{2}\int_0^T \int_0^1 \rho_x |u|^2 dxdt \ . \quad (6.4)$$

The classical energy argument gives

$$\forall t \in [0,T],\ \int_0^1 (|u_x|^2 + |u|^2 + |u_t|^2)(t)dx \le \|u_0\|^2_{\mathbb{H}^1} + \|u_1\|^2_{\mathbb{L}^2} + \int_0^T \int_0^1 |h||u_t|dxdt \ . \quad (6.5)$$

As ρ_x is bounded in $\mathbb{L}^1(]0,1[)$ by $\|\gamma\|_{\mathbb{L}^1}$ and $\mathbb{H}^1(]0,1[) \hookrightarrow \mathbb{L}^\infty(]0,1[)$, we have

$$\begin{aligned} \int_0^T \int_0^1 \rho_x |u|^2 \quad & \le CT\|\gamma\|_{\mathbb{L}^1} \sup_{t\in[0,T]} \|u(t)\|^2_{\mathbb{H}^1} \\ & \le CT\|\gamma\|_{\mathbb{L}^1} \left(\int_0^T \int_0^1 |h||u_t|dxdt + \|u_0\|^2_{\mathbb{H}^1} + \|u_1\|^2_{\mathbb{L}^2}\right) \ . \end{aligned}$$

Moreover, ρ is bounded in $\mathbb{L}^\infty$ by $\|\gamma\|_{\mathbb{L}^1}$ and so (6.5) gives

$$\left[\int_0^1 \rho u_t u_x dx\right]_0^T \le C\|\gamma\|_{\mathbb{L}^1} \left(\int_0^T \int_0^1 |h||u_t|dxdt + \|u_0\|^2_{\mathbb{H}^1} + \|u_1\|^2_{\mathbb{L}^2}\right) \ .$$

Using the above estimates in (6.4), we find

$$\int_0^T \int_0^1 \rho_x(|u_x|^2 + |u|^2 + |u_t|^2)dxdt \le C\left(\int_0^T \int_0^1 |h|(|u_x| + |u_t|)dxdt + \|u_0\|^2_{\mathbb{H}^1} + \|u_1\|^2_{\mathbb{L}^2}\right) \ . \quad (6.6)$$

On the other hand, since $\rho_x(x) = \gamma(x)$ for $x \in]0,1/2]$ and since $\rho_x(x)$ is bounded by $\|\gamma\|_{\mathbb{L}^1}$ for $x \in]1/2,1[$, we have

$$\begin{aligned} \int_0^T \int_0^{1/2} \gamma(x)(|u_x|^2 + |u|^2 + |u_t|^2)dxdt \le & \int_0^T \int_0^1 \rho_x(|u_x|^2 + |u_t|^2 + |u_t|^2)dxdt \\ & + \|\gamma\|_{\mathbb{L}^1} \int_0^T \int_{1/2}^1 (|u_x|^2 + |u_t|^2 + |u_t|^2)dxdt \ . \quad (6.7) \end{aligned}$$

The estimates (6.5), (6.6) and (6.7) show that

$$\int_0^T \int_0^{1/2} \gamma(x)(|u_x|^2 + |u|^2 + |u_t|^2)dxdt$$

$$\leq C\left(\|u_{0x}\|^2_{\mathbb{L}^2} + \|u_1\|^2_{\mathbb{L}^2} + \int_0^T \int_0^1 |h|(|u_x| + |u_t|)dxdt\right) ,$$

where C depends on $\|\gamma\|_{\mathbb{L}^1}$ and T only.
In order to estimate the integral $\int_0^T \int_{1/2}^1 \gamma(x)(|u_x|^2 + |u|^2 + |u_t|^2)dxdt$, we argue in the same way with ρ taken as follows :

$$\rho = \begin{cases} 2x \int_{1/2}^1 \gamma(\xi)d\xi & 0 \leq x \leq 1/2 \\ \int_x^1 \gamma(\xi)d\xi & 1/2 \leq x \leq 1 \end{cases}$$

□

We obtain the following criterium for the exponential decay.

Theorem 6.4. *Let γ be a nonnegative function of $\mathbb{L}^\infty(]0,1[)$. Assume that (EFW) is satisfied. Then, there exist two positive constants M and λ depending only on the constants C, T introduced in (6.2) and $\|\gamma\|_{\mathbb{L}^1}$ such that, for each initial data $(u_0, u_1) \in \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[)$, the solution u of*

$$\begin{cases} u_{tt}(x,t) + \gamma(x)u_t(x,t) = u_{xx}(x,t) - u(x,t) \ , \ (x,t) \in]0,1[\times\mathbb{R}_+ \\ u_x(0,t) = u_x(1,t) = 0 \\ (u, u_t)_{|t=0} = (u_0, u_1) \in \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[) \end{cases} \tag{6.8}$$

satisfies

$$\|u(t)\|^2_{\mathbb{H}^1} + \|u_t(t)\|^2_{\mathbb{L}^2} \leq M(\|u_0\|^2_{\mathbb{H}^1} + \|u_1\|^2_{\mathbb{L}^2})e^{-\lambda t} .$$

Proof : We denote the energy $E(t) = \frac{1}{2}(\|u(t)\|^2_{\mathbb{H}^1} + \|u_t(t)\|^2_{\mathbb{L}^2})$. We know that

$$E(0) - E(T) = \int_0^T \int_0^1 \gamma(x)|u_t(x,t)|^2 dxdt . \tag{6.9}$$

Let φ be the solution of the wave equation (6.1) with $\varphi_0 = u_0$ and $\varphi_1 = u_1$. We set $v = u - \varphi$, which is the solution of

$$\begin{cases} v_{tt} - v_{xx} + v = -\gamma u_t \\ v_x(0,t) = v_x(1,t) = 0 \\ v(x,0) = 0 \\ v_t(x,0) = 0 \end{cases}$$

Using Proposition 6.3, we obtain

$$\int_0^T \int_0^1 \gamma(x)(|v_x|^2 + |v|^2 + |v_t|^2)dxdt \leq C \left(\int_0^T \int_0^1 \gamma(x)|u_t|(|v_x| + |v_t|)dxdt \right) ,$$

where C depends on T and $\|\gamma\|_{\mathbb{L}^1}$ only. Thus,

$$\left(\int_0^T \int_0^1 \gamma(x)(|v_x|^2 + |v|^2 + |v_t|^2)dxdt \right)^2$$
$$\leq C \int_0^T \int_0^1 \gamma(x)|u_t|^2 dxdt \;\times\; \int_0^T \int_0^1 \gamma(x)(|v_x| + |v_t|)^2 dxdt ,$$

and also, by the Young inequality,

$$\int_0^T \int_0^1 \gamma(x)(|v_x|^2 + |v|^2 + |v_t|^2)dxdt \leq C \int_0^T \int_0^1 \gamma(x)|u_t|^2 dxdt = C(E(0) - E(T)) . \quad (6.10)$$

Finally, using (6.9), (6.10) and the hypothesis (EFW), we obtain

$$\begin{aligned} E(T) \quad & \leq E(0) \leq C \int_0^T \int_0^1 \gamma(x)|\varphi_t(x,t)|^2 dxdt \\ & \leq C \left(\int_0^T \int_0^1 \gamma(x)|u_t(x,t)|^2 dxdt + \int_0^T \int_0^1 \gamma(x)|v_t(x,t)|^2 dxdt \right) \\ & \leq C(E(0) - E(T)) . \end{aligned}$$

The exponential decay of the energy follows from this inequality (see for example [23]). □

In the last part of this section, we give concrete conditions implying the criterium (EFW) uniformly in n. Thus, we obtain examples of one dimensional equations satisfying the Hypothesis (UED). Notice that our method also gives higher dimensional examples where (EFW) is satisfied uniformly in n, but in these cases, we have no proof that (EFW) implies the uniform exponential decay (UED).
We wonder when Hypothesis (UED) is satisfied for the family of equations

$$\left\{ \begin{array}{l} u_{tt}(x,t) + \gamma_n(x)u_t(x,t) = u_{xx}(x,t) - u(x,t) + f(x,u) \quad (x,t) \in]0,1[\times\mathbb{R}_+ \\ u_x(0,t) = u_x(1,t) = 0 \\ (u,u_t)_{|t=0} = (u_0,u_1) \in \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[) \end{array} \right. \quad (6.11)$$

Remark that Proposition 6.2 and Theorem 6.4 imply that, if the semiflow generated by (6.11) satisfies (UED) for a sequence of dissipations γ_n, then the property (UED) also holds for any sequence of dissipations $\tilde{\gamma}_n \geq \gamma_n$. Thus, we may restrict our study to dissipations of the form $\gamma_n(x) = n\chi_{]a_n;a_n+1/n[}$. Next, we show the following lemma, which replaces

the criterium (EFW), concerning the solutions of the free waves, by a criterium on the eigenfunctions of the free waves operator.
We denote by λ_k^2 ($\lambda_k > 0$, $k \in \mathbb{N}^*$) the eigenvalues of B and φ_k the corresponding eigenfunctions normalized by $\|\varphi_k\|_{\mathbb{L}^2} = 1$.

Lemma 6.5. *We assume that γ_∞ is effective on the free waves, that is that (EFW) holds for γ_∞. We also assume that there exist a family of complex numbers (α_k) and an application h defined from $\mathbb{N}^* \times \mathbb{N}^*$ into $\{0,1\}$ such that*

$$\forall k, k' \in \mathbb{N}^*, \ \int_\Omega \gamma_\infty \varphi_k \overline{\varphi}_{k'} = \alpha_k \overline{\alpha}_{k'} h(k,k') \ .$$

If $h(k,k') = 0$ implies $\int \gamma_n \varphi_k \overline{\varphi}_{k'} = 0$ for all $n \in \mathbb{N}$ and if

$$\inf_{n\in\mathbb{N}} \inf_{k\in\mathbb{N}^*} \frac{1}{|\alpha_k|^2} \int_\Omega \gamma_n |\varphi_k|^2 > 0 \ , \tag{6.12}$$

then the family of dissipation (γ_n) is uniformly effective on the free waves, that is that (EFW) holds uniformly in n.

Proof : For $k \in \mathbb{N}^*$, we set $\lambda_{-k} = -\lambda_k$ and $\varphi_{-k} = \varphi_k$. A solution of (6.1) can be decomposed as follows.

$$\begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix} = \sum_{k\in\mathbb{Z}^*} c_k e^{i\lambda_k t} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{i\lambda_k}\varphi_k \\ \varphi_k \end{pmatrix} \quad \text{where} \quad \|(\varphi,\varphi_t)_{|t=0}\|_X^2 = \sum_{k\in\mathbb{Z}^*} |c_k|^2 \ .$$

As (6.2) holds for γ_∞, we have that $\int_0^T \int_\Omega \gamma_\infty |\varphi_t|^2 \geq C\|(\varphi,\varphi_t)_{|t=0}\|_X^2$, that is that

$$\sum_{k,k'} c_k \overline{c}_{k'} \frac{e^{i(\lambda_k - \lambda_k')T} - 1}{\lambda_k - \lambda_{k'}} \ \alpha_k \overline{\alpha}_{k'} h(|k|,|k'|) \ \geq \ C \sum_{k\in\mathbb{Z}^*} |c_k|^2 \ , \tag{6.13}$$

where by convention $\frac{e^{i(\lambda_k-\lambda_k')T}-1}{\lambda_k-\lambda_{k'}} = T$ when $\lambda_k = \lambda_k'$. Concerning the dissipation γ_n, we have

$$\begin{aligned} \int_0^T \int_\Omega \gamma_n |\varphi_t|^2 \quad &= \sum_{k,k'} c_k \overline{c}_{k'} \frac{e^{i(\lambda_k-\lambda_k')T} - 1}{\lambda_k - \lambda_{k'}} \int_\Omega \gamma_n \varphi_k \overline{\varphi}_{k'} \\ &= \int_\Omega \sum_{k,k'} \left(\frac{c_k}{\alpha_k}\sqrt{\gamma_n}\varphi_k\right) \overline{\left(\frac{c_{k'}}{\alpha_{k'}}\sqrt{\gamma_n}\varphi_{k'}\right)} \frac{e^{i(\lambda_k-\lambda_k')T} - 1}{\lambda_k - \lambda_{k'}} \alpha_k \overline{\alpha}'_k h(|k|,|k'|) \ . \end{aligned}$$

Notice that (6.13) implies that $\inf |\alpha_k| > 0$. Moreover, choosing $(\varphi, \varphi_t)_{|t=0}$ in X^s for s large enough, we have that $\sum(\frac{c_k}{\alpha_k}\|\varphi_k\|_{\mathbb{L}^\infty})^2$ is finite. Thus, due to the inequality (6.13), we obtain

$$\int_0^T \int_\Omega \gamma_n |\varphi_t|^2 \geq \int_\Omega C \sum_{k\in\mathbb{Z}^*} \frac{|c_k|^2}{|\alpha_k|^2}\gamma_n|\varphi_k|^2 \geq C \sum_{k\in\mathbb{Z}^*} \frac{|c_k|^2}{|\alpha_k|^2} \int_\Omega \gamma_n |\varphi_k|^2 \ .$$

Using (6.12) we find that (6.2) holds uniformly in n for any initial data $(\varphi, \varphi_t)_{|t=0}$ in X^s. The density of the space X^s in X then concludes. □

We apply Lemma 6.5 to obtain the following result.

Proposition 6.6. *Let $(a_n) \subset [0, 1[$ be a sequence such that $a_n \longrightarrow 0$ when $n \longrightarrow +\infty$. We set*

$$\gamma_n(x) = \begin{cases} n & \text{if } a_n \leq x \leq a_n + \frac{1}{n} \\ 0 & \text{elsewhere .} \end{cases}$$

Then the family of equations (6.11) satisfies (UED) if and only if $\sup\{na_n\} < +\infty$.

Proof : We have to verify the hypotheses of Lemma 6.5. In our case, we have $\gamma_\infty = \delta_{x=0}$, $\lambda_k = \sqrt{k^2+1}$ and $\varphi_k = \sqrt{2}\cos(k\pi x)$. Thus, $\int_\Omega \gamma_\infty \varphi_k \overline{\varphi}_{k'} = 2$, and we can set $\alpha_k = \sqrt{2}$ and $h \equiv 1$. It is well-known that Equation (6.8) with $\gamma = \gamma_\infty$ generates an exponentially decaying semigroup. So, the criterium (EFW) is satisfied by γ_∞.
To apply Lemma 6.5, it remains to show (6.12). If (6.12) does not hold, then it is clear that (EFW) cannot be satisfied uniformly. Thus, we have to prove that $\sup\{na_n\} < +\infty$ is equivalent to the existence of $\varepsilon > 0$ such that

$$\inf_{n\in\mathbb{N}} \inf_{k\in\mathbb{N}^*} n \int_{a_n}^{a_n+\frac{1}{n}} |\cos(k\pi x)|^2 dx \geq \varepsilon \ . \tag{6.14}$$

We have

$$n \int_{a_n}^{a_n+\frac{1}{n}} |\cos(k\pi x)|^2 dx = \frac{1}{2}\left(1 + \frac{n}{k\pi}\sin(\pi\frac{k}{n})\cos(\pi k(2a_n + \frac{1}{n}))\right) \ .$$

Assume that (6.14) is not true, then there exist two sequences (k_p) and (n_p) such that

$$\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(\pi k_p(2a_{n_p} + \frac{1}{n_p})) \longrightarrow -1 \ .$$

This implies that $\frac{k_p}{n_p} \to 0$ and $2k_pa_{n_p} \longrightarrow 1 \bmod(2)$, and so $|n_pa_{n_p}| \to +\infty$.
Assume now that there exists a subsequence satisfying $|n_pa_{n_p}| \to +\infty$. Let k_p be the smallest integer strictly larger than $\frac{n_p}{2n_pa_{n_p}+1}$. We have

$$\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(\pi k_p(2a_{n_p} + \frac{1}{n_p})) \longrightarrow -1 \ ,$$

and thus (6.14) is not satisfied. □

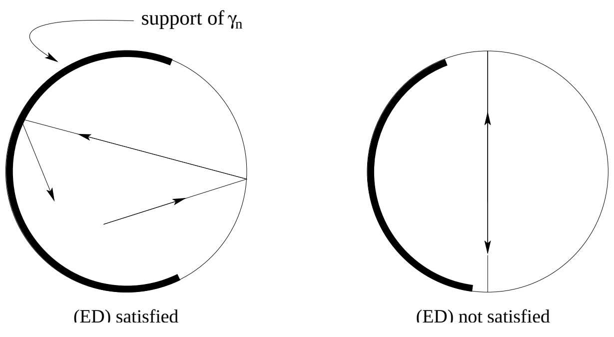
Remark : The same result obviously holds when the Neumann boundary condition at $x = 1$ is replaced by the Dirichlet condition.
We will come back to the case $\sup_n\{na_n\} = +\infty$ in the appendix A.2.

6.2 The two and three-dimensional cases

In dimension higher than one, our hypotheses are less easy to verify. First, Hypotheses (ED) and (Grad) do not always hold. The hypothesis (ED) is equivalent to geometrical conditions on the support of γ_n, which are now well-understood. The case of Hypothesis (UED) is much more difficult and its study in dimension two or higher is still mostly open.

Hypothesis (ED)

It is now well-known that the following geometric condition is equivalent to (ED), see [6]. For each $n \in \mathbb{N} \cup \{+\infty\}$, there is a length L_n such that all geodesics on Ω associated to the operator $\partial^2_{tt} + B$ and of length greater than L_n meet the support of γ_n. In dimension one, the condition is trivially satisfied. In the higher dimensional case, the condition is more restrictive, since, for some examples, there exist geodesics of infinite length, which do not meet the support of γ_n.



Hypothesis (Grad)

Let $n \in \mathbb{N} \cup \{+\infty\}$ be given. Let $U_0 \in X$ be such that for all $t \geq 0$, we have $\Phi(S_n(t)U_0) = \Phi(U_0)$. If $U(t) = S_n(t)U_0 = (u(t), v(t))$, we thus have

$$u_t(t) = v(t) \text{ and } u_{tt} + B(u + \Gamma_n u_t) = f(x, u) .$$

We know that

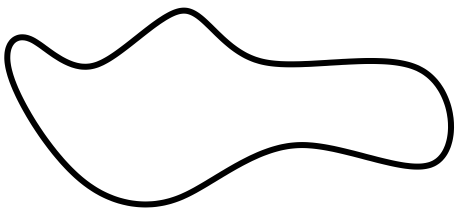
$$\forall t \geq 0,\ \frac{\partial}{\partial t}\Phi(U(t)) =< A_n U|U >= -\int_\Omega \gamma_n |v|^2 = 0\ .$$

Hence, $v = u_t$ satisfies

$$\forall t \geq 0,\ v_{tt} + Bv = f'_u(x,u)v \text{ and } \left\{ \begin{array}{l} v = 0 \text{ on } \mathrm{supp}(\gamma_n) \text{ if } n \in \mathbb{N} \\ \text{or} \\ v = 0 \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \mathrm{supp}(\gamma_\infty) \end{array} \right. . \tag{6.15}$$

To prove that Hypothesis (Grad) holds, we must show that (6.15) implies that $u_t = v = 0$ on $\Omega \times \mathbb{R}_+$. This unique continuation argument holds under geometrical conditions.

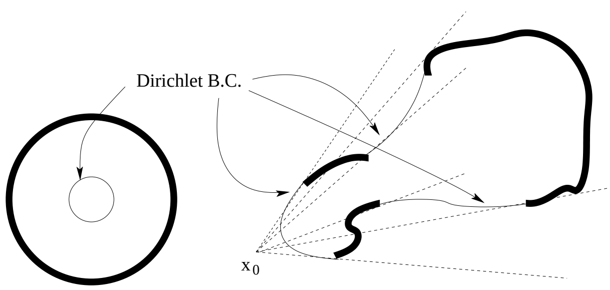
- If the support of γ_n contains a neighborhood of the boundary $\partial\Omega$ for $n \in \mathbb{N}$ and if the support of γ_∞ is equal to $\partial\Omega$, then (Grad) is satisfied (see [42]).



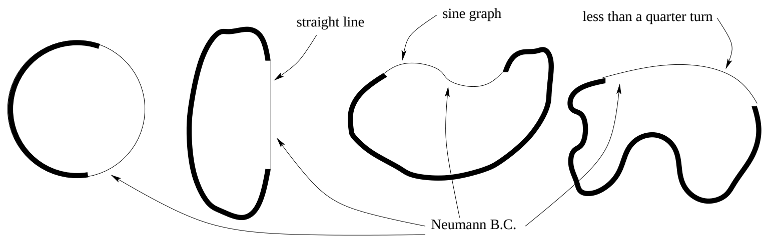
- Assume that $\mathrm{supp}(\gamma_\infty) = \omega_N$, that the support of γ_n contains a neighborhood of ω_N and that there exists a point $x_0 \in \mathbb{R}^d$ such that

$$\{x \in \partial\Omega\ /\ (x - x_0).\nu > 0\} \subset \omega_N\ ,$$

  then (Grad) holds (see [28]).



- Let Ω be a domain with a boundary of class $\mathcal{C}^1$. We assume that the support of γ_n includes a neighborhood of $\mathrm{supp}(\gamma_\infty)$ and that the boundary conditions on the whole boundary $\partial\Omega$ are of Neumann type, that is that $\omega_D = \emptyset$. In this case, [33] gives many sufficient conditions for (Grad) to hold. In particular, if Ω is a disk, the fact that the support of γ_∞ covers slightly more than a half circle is sufficient. Other examples are given.



Remark : For all the examples that we give here, one notices that (ED) is satisfied. However, there is no reason that (Grad) implies (ED) in general.

Hypothesis (UED)

The methods, which were used in the one-dimensional case, cannot be generalized to dimensions two or three. For these dimensions, using an energy method, we obtain here a criterium equivalent to the property (UED). However, except for the particular cases where γ_n is uniformly bounded away from 0, it is very difficult to exhibit examples satisfying this criterium.
The following equivalence is very classical. The property (UED) is satisfied if and only if there exist two positive constants T and C such that, for all $U_0 \in X$ and $n \in \mathbb{N}$, if we set $U(t) = (u, u_t)(t) = e^{A_n t} U_0$, then we have

$$\int_0^T \int_\Omega \gamma_n |u_t|^2 \geq C \|U_0\|_X^2 \ . \tag{6.16}$$

We can weaken this criterium as follows.

Proposition 6.7. *The uniform exponential decay property (UED) is satisfied if and only if there exist two positive constants T and C, independent of n, such that, for all $U_0 \in X$ and $n \in \mathbb{N}$, if we set $U(t) = (u, u_t)(t) = e^{A_n t} U_0$, then we have*

$$\int_0^T \int_\Omega \gamma_n |u_t|^2 \geq C \int_0^T \int_\Omega |u_t|^2 \ . \tag{6.17}$$

Proof : The "only if" part is a direct consequence of the classical criterium (6.16). Indeed, the property (UED) implies that there exist two positive constants T and C such that, for all $U_0 \in X$ and $n \in \mathbb{N}$,

$$\int_0^T \int_\Omega \gamma_n |u_t|^2 \geq C \|U_0\|_X^2 = \frac{C}{T} \int_0^T \|U_0\|_X^2 dt \geq \frac{C}{T} \int_0^T \|U(t)\|_X^2 dt \geq C \int_0^T \int_\Omega |u_t|^2 \ .$$

In order to prove the "if" part of the equivalence, we introduce the following functional. Let α be a positive number to be chosen. For all $U = (u, v) \in X$, we set

$$F(U) = \frac{1}{2} \int_\Omega (|u|^2 + 2uv + \alpha |v|^2 + \alpha |B^{1/2} u|^2) dx \ . \tag{6.18}$$

For α large enough, the functional F is clearly equivalent to the energy in the sense that there exists a positive constant μ such that

$$\forall U \in X,\ \frac{1}{\mu}\|U\|_X^2 \leq F(U) \leq \mu\|U\|_X^2\ . \tag{6.19}$$

Let $U_0 \in D(A_n)$ and $n \in \mathbb{N}$, we set $U(t) = (u, u_t)(t) = e^{A_n t}U_0$. As $U(t) \in \mathcal{C}^1(\mathbb{R}_+, X)$, we can write

$$\begin{aligned}
\partial_t F(U(t)) &= \int_\Omega (uu_t + uu_{tt} + |u_t|^2 + \alpha u_t u_{tt} + \alpha(B^{1/2}u)(B^{1/2}u_t))dx \\
&= \int_\Omega (uu_t - \gamma_n uu_t - (Bu)u + |u_t|^2 - \alpha\gamma_n|u_t|^2 \\
&\qquad -\alpha u_t(Bu) + \alpha(B^{1/2}u)(B^{1/2}u_t))dx\ .
\end{aligned}$$

Thus, for all $\varepsilon > 0$, we have that

$$\partial_t F(U(t)) \leq \int_\Omega \varepsilon(1+\gamma_n)|u|^2 + \frac{1}{\varepsilon}(1+\gamma_n)|u_t|^2 - |B^{1/2}u|^2 + |u_t|^2 - \alpha\gamma_n|u_t|^2\ .$$

As Γ_n converges to Γ_∞ in $\mathcal{L}(D(B^{1/2}))$, we know that there exists a positive constant C, independent of n, such that, for all $u \in D(B^{1/2})$, $\int_\Omega \gamma_n|u|^2 \leq C\|u\|^2_{D(B^{1/2})}$. Therefore, for ε small enough and α large enough, (6.17) implies the existence of a time T and a positive constant C such that

$$\int_0^T \partial_t F(U(t)) \leq -C\int_0^T F(t)\ .$$

Thus, using the density of $D(A_n)$ in X, we obtain that, for all $U_0 \in X$,

$$F(U_0) - F(e^{A_n T}U_0) \geq C\int_0^T F(e^{A_n t}U_0)dt\ .$$

The inequalities (6.19) and the fact that $e^{A_n t}$ is a contraction imply that, for all $U_0 \in X$ and $k \in \mathbb{N}$,

$$\|U_0\|_X^2 \geq \frac{C}{\mu^2}\int_0^{kT}\|e^{A_n t}U_0\|_X^2 dt \geq \frac{C}{\mu^2}kT\|e^{A_n kT}U_0\|_X^2\ .$$

For k large enough, we obtain a time T', independent of n, such that $\|e^{A_n T'}U_0\|_X \leq \frac{1}{2}\|U_0\|_X^2$. This is well-known to imply the uniform exponential decay (UED). □

It is difficult to find examples satisfying the criterium (6.17). Indeed, opposite to the criterium (EFW) obtained in the one-dimensional case, (6.17) involves functions $U(t)$, which are solutions of an equation which depends of n. However, Proposition 6.7 gives some very particular examples where (UED) is satisfied in dimension higher than one. This corollary is stated in a way so that it can be applied to a general sequence of dissipations, which satisfies (6.20) only.

Corollary 6.8. *Let $(\gamma_n^0)_{n\in\mathbb{N}}$ be a sequence of non-negative functions in $\mathbb{L}^\infty(\Omega)$. Assume that there exists a positive constant C independent of n such that*

$$\forall u \in D(B^{1/2}),\ \forall n \in \mathbb{N}, \quad \int_\Omega \gamma_n^0(x)|u(x)|^2 dx \leq C\|u\|^2_{D(B^{1/2})} \ . \tag{6.20}$$

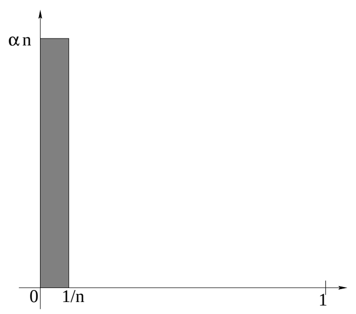
Then, for all $\eta > 0$, the uniform exponential decay property (UED) is satisfied for the sequence of dissipations $\gamma_n = \eta + \gamma_n^0$.

Notice that this result in not a priori trivial, since, as the sequence (γ_n^0) is not necessary bounded in $\mathbb{L}^\infty(\Omega)$, overdamping phenomenas may occur. The fact that $\gamma_n \geq \eta > 0$ seems slightly artificial from a mathematical point of view. However, it is not from the physical point of view since γ_n never really vanishes in the concrete cases. For example, η can be seen as the resistance of air when (2.5) models the propagation of waves in a room.

7 Examples

In this section, we give some examples illustrating our results. For each example, we define Ω, B and γ_n and say if the convergence of the attractors holds in the space X or only in X^{-s} ($s > 0$). Saying that the convergence holds in X^{-s} does not mean that there is no convergence in X. It only means that we are not able to prove it for the moment. Here we do not give explicit non-linearities for which Hypothesis (Hyp) is satisfied.
We recall that we denote by Δ_N the Laplacian with Neumann boundary conditions.

Example 1 :

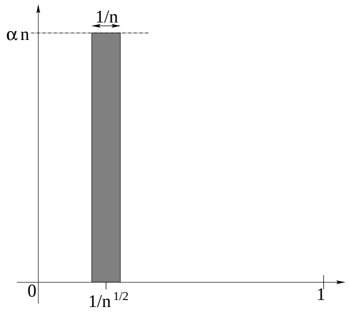


$\Omega =]0,1[$, $B = Id - \Delta_N$, $\alpha > 0$

$$\gamma_n(x) = \begin{cases} \alpha n, & x \in]0, \frac{1}{n}[\\ 0, & \text{elsewhere} \end{cases}, \quad \gamma_\infty = \alpha\delta_{x=0} \ .$$

Convergence in X.

Example 2 :

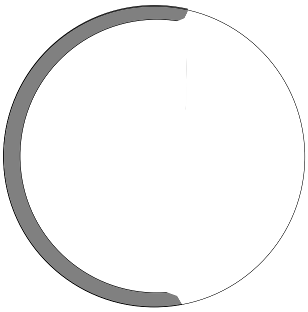


$\Omega =]0,1[$, $B = Id - \Delta_N$, $\alpha > 0$

$$\gamma_n(x) = \begin{cases} \alpha n, & x \in]\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} + \frac{1}{n}[\\ 0, & \text{elsewhere} \end{cases} \quad , \quad \gamma_\infty = \alpha\delta_{x=0} \ .$$

Convergence in X^{-s}.

Example 3 :

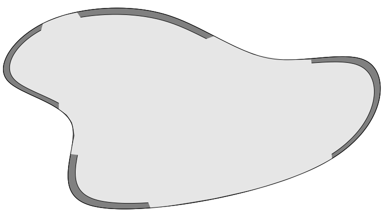


Ω is the disk of $\mathbb{R}^2$, $B = Id - \Delta_N$,
ω is an open subset of $\partial\Omega$ which covers strictly more than half of the circle

$$\gamma_n(x) = \begin{cases} n, & \text{dist}(x,\omega) < \frac{1}{n} \\ 0, & \text{elsewhere} \end{cases} \quad , \quad \gamma_\infty = \delta_{x\in\omega} \ .$$

Convergence in X^{-s}.

Example 4 :

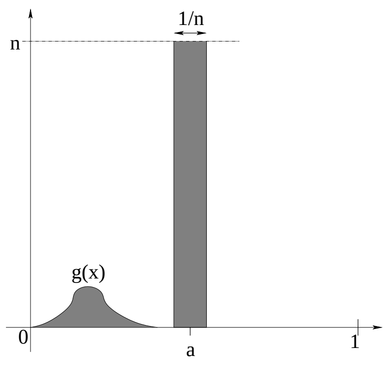


$\Omega \subset \mathbb{R}^2$, $\eta > 0$, ω is any subset of $\partial\Omega$
$B = Id - \Delta_N$

$$\gamma_n(x) = \begin{cases} n, & \text{dist}(x,\omega) < \frac{1}{n} \\ \eta, & \text{elsewhere} \end{cases} \quad , \quad \gamma_\infty = \eta + \delta_{x\in\omega} \ .$$

Convergence in X.

Example 5 : For sake of simplicity, the abstract frame of this paper has not been defined so that this example fits in it. However, all the results given here are valid for this case. Notice that we need the additional dissipation g, since the singular internal dissipation $\delta_{x=a}$ is not sufficient to obtain exponential decay (see [25]), or the gradient structure. We denote by Δ_D the Laplacian with Dirichlet boundary conditions.



$\Omega =]0,1[$, $a \in]0,1[$ and g is a nonnegative function, in $\mathbb{L}^\infty(]0,1[)$ which is positive on an open subset, $B = -\Delta_D$

$$\gamma_n(x) = \begin{cases} g(x) + n, & x \in]a - \frac{1}{2n}, a + \frac{1}{2n}[\\ g(x), & \text{elsewhere} \end{cases}$$

$\gamma_\infty(x) = g(x) + \delta_{x=a}$.

Convergence in X.

A Appendix

A.1 A result of Ammari and Tucsnak

We have proved in Section 6.1 that if $\Omega =]0,1[$ and if the dissipations γ_n satisfy uniformly the property (EFW), then (UED) is satisfied. We show here how to obtain the same implication with a different method, which has been introduced in [2]. To simplify the notation, we state the results of [2] in our frame.
Let γ be a function in $\mathbb{L}^\infty(\Omega)$. We introduce the following hypothesis

(H) If $\beta > 0$ is fixed and $C_\beta = \{\lambda \in \mathbb{C}/Re(\lambda) = \beta\}$, then the function

$$H(\lambda) = \sqrt{\gamma}\,\lambda(\lambda^2 Id + B)^{-1}\sqrt{\gamma}$$

defined from C_β into $\mathcal{L}(\mathbb{L}^2)$ is bounded and we set

$$M_\beta = \sup_{\lambda \in C_\beta} \|H(\lambda)\|_{\mathcal{L}(\mathbb{L}^2)} < \infty \ .$$

In our case, Theorem 2.2 of [2] can be stated as follows.

Theorem A.1. *Assume that the hypothesis (H) holds and that γ is effective on the free waves, i.e. that (EFW) is satisfied. Then, there exist two positive constants M and λ depending only on the constants C, T introduced in (6.2) and on the family of constants M_β introduced in (H) such that, for any initial data $(u_0, u_1) \in X$, the solution u of*

$$\begin{cases} u_{tt} + \gamma(x)u_t + Bu = 0 \\ (u, u_t)_{|t=0} = (u_0, u_1) \in X \end{cases} \tag{A.1}$$

satisfies

$$\|(u, u_t)(t)\|_X \le M\|(u_0, u_1)\|_X e^{-\lambda t} \ .$$

The idea of the proof of Theorem 2.2 of [2] is to replace the multipliers method by a Laplace transform argument to obtain a result similar to Proposition 6.3.
Theorem 6.4 is then a direct consequence of Theorem A.1 and of the following property.

Proposition A.2. *Let $\Omega =]0,1[$ and $B = -\Delta_N + Id$. For $\gamma \in \mathbb{L}^\infty(]0,1[)$, Hypothesis (H) is satisfied and the bound M_β depends on $\|\gamma\|_{\mathbb{L}^1}$ only.*

Proof : We notice that $\sqrt{\gamma}$ is bounded in $\mathbb{L}^2(]0,1[)$ by $\|\gamma\|_{\mathbb{L}^1}^{1/2}$. So, the operator of multiplication by $\sqrt{\gamma}$ is bounded in $\mathcal{L}(\mathbb{L}^2,\mathbb{L}^1)$ and in $\mathcal{L}(\mathbb{L}^\infty,\mathbb{L}^2)$. It remains to show that, on C_β, the operator $\lambda(\lambda^2 Id + B)^{-1}$ is uniformly bounded in $\mathcal{L}(\mathbb{L}^1,\mathbb{L}^\infty)$.
Let $f \in \mathbb{L}^1(]0,1[)$ and u be the solution of

$$-u_{xx} + u + \lambda^2 u = f \ , \quad u_x(0) = u_x(1) = 0 \ . \tag{A.2}$$

We set $\theta = (-\lambda^2 - 1)^{1/2}$. The solution of (A.2) is given by

$$u(x) = C\cos(\theta x) - \sin(\theta x)\int_0^x \frac{\cos(\theta s)}{\theta} f(s)ds + \cos(\theta x)\int_0^x \frac{\sin(\theta s)}{\theta} f(s)ds \ , \tag{A.3}$$

where

$$C = -\int_0^1 \frac{\sin(\theta s)}{\theta} f(s)ds - \frac{\operatorname{cotg}(\theta)}{\theta}\int_0^1 \cos(\theta s) f(s)ds \ .$$

A direct computation shows that, if $\lambda = \beta + i\mu$, then

$$\operatorname{Im}(\theta) = -\left((\mu^2-\beta^2-1)^2 + (2\mu\beta)^2\right)^{1/4} \sin\left(\frac{1}{2}\operatorname{arctg}\left(\frac{2\mu\beta}{\mu^2-\beta^2-1}\right)\right) \ .$$

Thus, $\operatorname{Im}(\theta) \longrightarrow \mp\beta \neq 0$ when $\mu \longrightarrow \pm\infty$. This implies that $\sin(\theta)$, $\cos(\theta)$, $\operatorname{cotg}(\theta)$ and $\frac{1}{\theta}$ are uniformly bounded on C_β. Since $f \in \mathbb{L}^1(]0,1[)$, (A.3) proves that $u \in \mathbb{L}^\infty$ and so $\lambda(\lambda^2 Id + B)^{-1}$ is uniformly bounded in $\mathcal{L}(\mathbb{L}^1,\mathbb{L}^\infty)$. □

Unfortunately, Theorem A.1 is not applicable in dimension higher than one. Indeed, it is shown in [2] that property (H) implies the following fact. For all $T > 0$, there exists $C > 0$ such that all the solutions φ of the free wave equation (6.1) satisfy

$$\int_0^T \int_\Omega \gamma(x)|\varphi_t|^2 dxdt \leq C\|(\varphi_0,\varphi_1)\|_X^2 \ . \tag{A.4}$$

Let $\gamma_\infty = \delta_{x\in\omega}$ be a dissipation on a part of the boundary. In dimension higher than one, we can imagine a wave travelling along the curve ω for which the left-hand side of the inequality (A.4) is infinite. If (A.4) does not hold for the boundary dissipation, we cannot hope that it holds uniformly for the family $(\gamma_n)_{n\in\mathbb{N}}$ when γ_n converges to γ_∞. The following counter-example illustrates this remark.

Proposition A.3. *Let $\Omega =]0,1[^2$, $B = -\Delta_N + Id$. For all time $T > 0$, there exists a sequence of initial data $(\varphi_0^n, \varphi_1^n) \in X$, with $\|(\varphi_0^n, \varphi_1^n)\|_X = 1$, such that the solutions $\varphi_n(x,y,t)$ of the free wave equation (6.1) satisfy*

$$\int_0^T \int_0^1 |\frac{\partial}{\partial t}\varphi_n(0,y,t)|^2 dydt \longrightarrow +\infty \ ,$$

when $n \longrightarrow +\infty$.

Proof : We choose the decomposition of the initial data on the eigenvectors of the free wave operator as follows. Let

$$\begin{pmatrix} \varphi_0^n \\ \varphi_1^n \end{pmatrix} = \sum_{k=0}^{n-1} \sqrt{\frac{2}{n}} \begin{pmatrix} \frac{1}{i\sqrt{n^6+k^2+1}} \cos(n^3\pi y)\cos(k\pi x) \\ \cos(n^3\pi y)\cos(k\pi x) \end{pmatrix} \ .$$

Notice that $\|(\varphi_0^n, \varphi_1^n)\|_X = 1$. A straightforward calculus gives

$$\int_0^T \int_0^1 |\frac{\partial}{\partial t}\varphi_n(0,y,t)|^2 dydt = 2\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \frac{1}{n} \frac{\sin(\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1})T}{\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1}} \ .$$

Since

$$\left|\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1}\right| \leq \frac{|k^2-k'^2|}{2\sqrt{n^6}} \leq \frac{1}{n} \ ,$$

for n large enough, there exists $\varepsilon > 0$ such that

$$\frac{\sin(\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1})T}{\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1}} \geq \varepsilon > 0 \ .$$

And thus,

$$\int_0^T \int_0^1 |\frac{\partial}{\partial t}\varphi_n(0,y,t)|^2 dydt \geq 2n\varepsilon \ .$$

□

A.2 An example of convergence of the attractors in X, when (UED) does not hold

To show the convergence of the attractors $\mathcal{A}_n$ in X, we had to show Proposition 2.9 that is that

$$\exists M \geq 0, \ \forall n \in \mathbb{N}, \ \sup_{U_n \in \mathcal{A}_n} \|U_n\|_{D(A_n)} \leq M \ . \tag{A.5}$$

We have shown that Hypothesis (UED) implies the above bound, but, of course, it is not necessary. The purpose here is to give examples where (UED) is not satisfied but where (A.5) holds.
We set $\Omega =]0,1[$ and $B = -\Delta_N + Id$. Let $\alpha > 0$ and $f \in \mathcal{C}^2([0,1] \times \mathbb{R}, \mathbb{R})$. We study the family of equations

$$\left\{ \begin{array}{ll} u_{tt}(x,t) + \gamma_n u_t(x,t) = u_{xx}(x,t) - u(x,t) + f(x, u(x,t)) & (x,t) \in]0,1[\times\mathbb{R}_+ \\ u_x(0,t) = u_x(1,t) = 0 & t \geq 0 \\ (u, u_t)_{|t=0} = (u_0, u_1) \in \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[) & \end{array} \right. \tag{A.6}$$

where, if $n \in \mathbb{N}$

$$\gamma_n(x) = \left\{ \begin{array}{ll} n & \text{if } \frac{1}{n^\alpha} \leq x \leq \frac{1}{n^\alpha} + \frac{1}{n} \\ 0 & \text{elsewhere .} \end{array} \right.$$

and $\gamma_\infty(x) = \delta_{x=0}$.
In Proposition 6.6, we proved that (UED) holds if and only if $\alpha > 1$. The purpose of this section is the proof of the following result.

Proposition A.4. *We assume that f satisfies Hypothesis (Diss). The dynamical systems generated by (A.6) admit a compact global attractor $\mathcal{A}_n$. Moreover, if $\alpha > \frac{16}{17}$ then (A.5) holds and the conclusions of Theorem 2.10 are valid.*

In what follows, we assume that $\alpha \in]\frac{16}{17}, 1[$, the case $\alpha \geq 1$ has already been considered in Proposition 6.6. The proof of Proposition A.4 is a consequence of the following two lemmas.

Lemma A.5. *There exist a time T and a constant C such that, for any $(\varphi_0, \varphi_1) \in X$, the solution of the free wave equation*

$$\left\{ \begin{array}{l} \varphi_{tt} + B\varphi = 0 \\ (\varphi, \varphi_t)_{|t=0} = (\varphi_0, \varphi_1) \end{array} \right. \tag{A.7}$$

satifies for all $n \in \mathbb{N} \cup \{+\infty\}$

$$\int_0^T \int_\Omega \gamma_n(x) |\varphi_t(x,t)|^2 dx dt \geq C \|(\varphi_0, \varphi_1)\|^2_{X^{1-1/\alpha}} . \tag{A.8}$$

Proof : Using the same arguments as those of Lemma 6.5, we see that (A.8) is satisfied uniformly with respect to ε if there exists a positive constant C such that

$$\forall n \in \mathbb{N},\ \forall k \in \mathbb{N},\ \int_\Omega \gamma_n |\cos(k\pi x)|^2 dx = n \int_{n^{-\alpha}}^{n^{-\alpha}+1/n} |\cos(k\pi x)|^2 \geq \frac{C}{k^{2/\alpha - 2}} , \tag{A.9}$$

that is that

$$\frac{1}{2}\left(1+\frac{n}{k\pi}\sin(\pi\frac{k}{n})\cos(k\pi(\frac{2}{n^\alpha}+\frac{1}{n})\right)\geq\frac{C}{k^{2/\alpha-2}}\ .$$

Assume that the above inequality does not hold. Then, there exist two sequences (k_p) and (n_p) such that

$$\left(1+\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(k_p\pi(\frac{2}{{n_p}^\alpha}+\frac{1}{n_p})\right)k_p^{2-2/\alpha}\longrightarrow 0\ . \tag{A.10}$$

We must have $\frac{k_p}{n_p}\longrightarrow 0$ and

$$\frac{2k_p}{n_p^\alpha}\longrightarrow 1 \bmod 2\ . \tag{A.11}$$

Thus, for p large enough, we have $0\leq\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\leq 1-\frac{1}{6}\left(\frac{k_p}{n_p}\right)^2$. This shows that

$$1+\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(k_p\pi(\frac{2}{{n_p}^\alpha}+\frac{1}{n_p})\geq\frac{1}{6}\left(\frac{k_p}{n_p}\right)^2\ .$$

Using (A.11), we obtain that, for p large enough, $\frac{1}{n_p}>\frac{1}{(4k_p)^{\frac{1}{\alpha}}}$, and thus

$$1+\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(k_p\pi(\frac{2}{{n_p}^\alpha}+\frac{1}{n_p})\geq\frac{1}{6(4)^{\frac{2}{\alpha}}}\left(\frac{1}{(k_p)^{\frac{1}{\alpha}-1}}\right)^2\ .$$

This is a contradiction to the assumption that (A.9) does not hold. □

The second lemma is a direct adaptation of a theorem of [2].

Lemma A.6. *If $\alpha>\frac{16}{17}$, there exist $\lambda>1$, $s\in]0,1/2[$ and $M>0$ such that*

$$\forall U_0\in X,\ \forall n\in\mathbb{N},\ \|e^{A_nt}U_0\|_X\leq\frac{M}{(1+t)^\lambda}\|U_0\|_{X^s}\ .$$

Proof : The outline of the proof is exactly the same as the one of Theorem 2.4 of [2]. First, notice that we have proved in Proposition A.2 that Hypothesis (H) introduced in Section A.1 is satisfied uniformly in n. Arguing as in [2], with some slight modifications, we show that Lemma A.5 and Proposition 2.6 imply that, for all $\sigma\in]0,1/2[$,

$$\forall t\geq 0,\ \|U(t)\|_X\leq\frac{M}{(1+t)^{\frac{1}{2(1/\theta-1)}}}\|U_0\|_{D(A_n)}\ ,$$

where $\theta = \frac{\sigma}{\sigma+1-1/\alpha}$. With the same interpolation methods as in Proposition 3.5, we obtain

$$\forall t \geq 0,\ \|U(t)\|_X \leq \frac{M}{(1+t)^{\frac{s^2}{2(1/\theta-1)}}} \|U_0\|_{X^s}\ .$$

We end the proof by noticing that we can find $s \in]0,1/2[$ and $\sigma \in]0,1/2[$ such that $\lambda = \frac{s^2}{2(1/\theta-1)} > 1$ is equivalent to $\alpha > \frac{16}{17}$. □

We are now able to prove the proposition.
Proof of Proposition A.4 : All we have to prove is that the inequality (A.5) holds. The proof is exactly the same as the one of Proposition 2.9. The only change is the estimate of $e^{A_n(t-\tau)}(F(U(\tau+\delta)) - F(U(\tau)))$ for $\tau \leq t$. Lemma A.6 implies that there exist $s \in]0,1/2[$ and $\lambda > 1$ such that

$$\left\|e^{A_n(t-\tau)}(F(U(\tau+\delta)) - F(U(\tau)))\right\|_X \leq \frac{C}{(1+t-\tau)^\lambda} \|F(U(\tau+\delta)) - F(U(\tau)))\|_{X^s}\ .$$

Hypothesis (NL) implies that there exists $\eta \in]0,1[$ such that

$$\|F(U(\tau+\delta)) - F(U(\tau)))\|_{X^s} \leq \|u(\tau+\delta) - u(\tau)\|_{\mathbb{H}^\eta}\ .$$

Thus,

$$\begin{aligned}&\left\|e^{A_n(t-\tau)}(F(U(\tau+\delta)) - F(U(\tau)))\right\|_X \\ &\qquad\qquad \leq \frac{C}{(1+t-\tau)^\lambda} \|u(\tau+\delta) - u(\tau)\|_{\mathbb{H}^1}^{\eta} \|u(s+\delta) - u(s)\|_{\mathbb{L}^2}^{1-\eta}\ .\end{aligned}$$

Using the fact that $\int_{-\infty}^t \frac{d\tau}{(1+t-\tau)^\lambda}$ is finite, we conclude with the same arguments as in Proposition 2.9. □

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Chapitre 6 : Résultats annexes

1 Convergence d'un amortissement interne vers un amortissement sur le bord : cas d'une non-linéarité critique

Dans le chapitre 5 de cette thèse, nous nous sommes restreints au cas d'une non-linéarité $f(x,u)$ sous-critique, c'est-à-dire que dans le cas de la dimension trois d'espace, nous avons supposé qu'il existe $C>0$ et $\alpha\in[0,1[$ tels que

$$|f''_{uu}(x,u)|\leq C(1+|u|^\alpha) \text{ et } |f''_{ux}(x,u)|\leq C(1+|u|^{3+\alpha})\ .$$

Le but de ce paragraphe est d'étudier le cas critique $\alpha=1$ et d'obtenir des résultats semblables au cas sous-critique : comparaison des trajectoires, existence des attracteurs et semi-continuités supérieure et inférieure des attracteurs. De plus, nous obtiendrons le résultat nouveau suivant : sous des hypothèses génériques, l'attracteur $\mathcal{A}_\infty$ est borné dans $\mathbb{H}^{1+s}\times\mathbb{H}^s$ pour un certain $s>0$.
Pour simplifier, nous nous plaçons dans le cadre suivant, mais les résultats sont généralisables aux cas où une propriété de prolongement unique est connue pour le problème limite. Soient $\Omega\subset\mathbb{R}^3$ un domaine borné régulier et $X=\mathbb{H}^1(\Omega)\times\mathbb{L}^2(\Omega)$. Soit $\gamma>0$, on pose $\gamma_\infty=\gamma\delta_{x\in\partial\Omega}$ où $\delta_{x\in\partial\Omega}$ est la fonction de Dirac à support dans le bord de Ω, et pour $n\in\mathbb{N}$, on pose

$$\gamma_n(x)=\begin{cases} n\gamma & \text{si } dist(x,\partial\Omega)<\frac{1}{n}\\ 0 & \text{sinon}\end{cases}$$

On considère l'équation des ondes amorties à l'intérieur de Ω

$$\begin{cases} u_{tt}(x,t)+\gamma_n(x)u_t(x,t)=\Delta u(x,t)+f(x,u(x,t)) & (x,t)\in\Omega\times\mathbb{R}_+\\ \frac{\partial u}{\partial\nu}(x,t)=0 & (x,t)\in\partial\Omega\times\mathbb{R}_+\\ (u(x,0),u_t(x,0))=(u_0,u_1)(x)\in X \end{cases}\tag{1.1}$$

et l'équation des ondes amorties sur le bord de Ω

$$\begin{cases} u_{tt}(x,t)=\Delta u(x,t)+f(x,u) & (x,t)\in\Omega\times\mathbb{R}_+\\ \frac{\partial u}{\partial\nu}(x,t)+\gamma u_t(x,t)=0 & (x,t)\in\partial\Omega\times\mathbb{R}_+\\ (u(x,0),u_t(x,0))=(u_0(x),u_1(x)))\in X \end{cases}\tag{1.2}$$

On reprend toutes les notations du chapitre 5, en particulier, on rappelle que A_n et $S_n(t)$ ($n \in \mathbb{N} \cup \{+\infty\}$) désignent les opérateurs linéaires et les semi-groupes associés aux équations (1.1) et (1.2).
On suppose que f est une fonction de classe $\mathcal{C}^2(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ et qu'il existe $C > 0$ tel que

$$|f''_{uu}(x,u)| \leq C(1+|u|) \ , \quad |f''_{ux}(x,u)| \leq C(1+|u|^4) \quad \text{et} \quad \limsup_{|u|\to+\infty} \sup_{x\in\Omega} \frac{f(x,u)}{u} < 0 \ . \qquad (1.3)$$

Rappelons que (1.3) implique en particulier que la fonction $F : (u,v) \in X \longmapsto (0, f(x,u)) \in X$ est lipschitzienne sur les bornés de X.
Dans le cas critique, le premier point délicat est l'existence d'un attracteur global compact. En effet, la méthode classique, rappelée dans les chapitres 4 et 5 de cette thèse, est fondée sur le fait que, si f est une non-linéarité sous-critique, F est une application compacte. Dans le cas critique, cela n'est plus vrai. En utilisant un théorème de prolongement unique, Feireisl et Zuazua ont montré que le système dynamique engendré par l'équation (1.1) peut se décomposer en la somme de deux applications, l'une étant une contraction stricte et l'autre étant compacte (voir [4]). Pour le système $S_n(t)$ ($n \in \mathbb{N}$), ils obtiennent alors l'existence d'un attracteur global compact $\mathcal{A}_n$, qui est borné dans X^s pour un certain $s > 0$. Dans le cas d'une dissipation à support sur le bord de Ω, l'existence d'un attracteur $\mathcal{A}_\infty$ est prouvée dans [3] par la même méthode, mais sans obtenir que $\mathcal{A}_\infty$ a une meilleure régularité que X.
Les mêmes arguments que dans le cas sous-critique (voir chapitre 5) et la propriété de prolongement unique impliquent que les systèmes $S_n(t)$ et $S_\infty(t)$ sont gradients et possèdent la même fonctionnelle de Lyapounov. Il s'ensuit que les attracteurs $\mathcal{A}_n$ sont bornés dans X par une constante indépendante de n. Afin de comparer les attracteurs $\mathcal{A}_n$ avec $\mathcal{A}_\infty$, remarquons que la convergence des trajectoires dans X^{-s} énoncée dans le théorème 3.10 du chapitre 5 est encore valable dans le cas critique. En effet, on utilise seulement le fait que F est lipschitzienne sur les bornés de $\mathbb{L}^2(\Omega) \times \mathbb{H}^{-1}(\Omega)$. On obtient donc avec les mêmes arguments que dans le chapitre 5 le théorème suivant.

Théorème 1.1. *Pour tout $s > 0$, les attracteurs $\mathcal{A}_n$ sont semi-continus supérieurement dans X^{-s} quand $n \longrightarrow +\infty$.*

A l'inverse de celle du théorème 3.10, la démonstration du théorème 3.8 donnée dans le chapitre 5 n'est pas directement adaptable au cas d'une non-linéarité critique. Afin de généraliser le théorème 3.8, nous utilisons le théorème 1.5 du chapitre 6 du livre de Pazy (voir [8]) pour obtenir le résultat suivant. Pour tout borné $\mathcal{B}$ de X, il existe une constante C telle que, si $U_0 \in D(A_\infty) \cap \mathcal{B}$, alors $S_\infty(t)U_0 \in \mathcal{C}^0([0,T], D(A_\infty))$ et

$$\|S_\infty(t)U_0\|_{D(A_\infty)} \leq Ce^{CT}\|U_0\|_{D(A_\infty)} \ . \qquad (1.4)$$

Par des arguments d'interpolation semblables à ceux de la proposition 3.5 du chapitre 5, nous déduisons de la propriété ci-dessus le résultat suivant.

Proposition 1.2. *Pour tout $s \in]0, \frac{1}{2}[$ et pour tout borné $\mathcal{B}$ de X, il existe une constante C telle que, si $U_0 \in X^s \cap \mathcal{B}$, alors $S_\infty(t)U_0 \in \mathcal{C}^0([0,T], X^{s^2})$ et*

$$\|S_\infty(t)U_0\|_{X^{s^2}} \leq Ce^{CT}\|U_0\|_{X^s} \ . \tag{1.5}$$

Contrairement à la proposition 3.5 du chapitre 5, nous allons devoir effectuer des interpolations non-linéaires. Pour cela, nous utiliserons le résultat suivant (voir [10]).

Proposition 1.3. *Soient quatre espaces de Banach $E_0 \subset E_1$ et $F_0 \subset F_1$. Soit $T : E_1 \longrightarrow F_1$ un opérateur non-linéaire tel qu'il existe des fonctions continues f_1 et f_2 telles que*

$$\forall (e, e') \in E_1^2,\ \|Te - Te'\|_{F_1} \leq f_1(\|e\|_{E_1}, \|e'\|_{E_1})\|e - e'\|_{E_1} \ ,$$

$$\text{et } \ \forall e \in E_0,\ \|Te\|_{F_0} \leq f_2(\|e\|_{E_1})\|e\|_{E_0} \ .$$

Alors, pour tout $\theta \in [0,1]$, il existe une constante positive C telle que

$$\forall e \in [E_0, E_1]_\theta,\ \|Te\|_{[F_0,F_1]_\theta} \leq Cf_2(2\|e\|_{E_1})^{1-\theta} f_1(\|e\|_{E_1}, 2\|e\|_{E_1})^\theta \|e\|_{[E_0,E_1]_\theta} \ .$$

Démonstration de la proposition 1.2 : Dans cette preuve, nous noterons K les constantes de la forme Ce^{CT}, on supposera $U_0 \in D(A_\infty)$ et on posera $U_0 = (u_0, v_0)$ et $U(t) = (u,v)(t)$.
La propriété (1.4) signifie que

$$\|v(t)\|_{D(B^{1/2})} + \|u(t) + \Gamma_\infty v(t)\|_{D(B)} \leq K(\|v_0\|_{D(B^{1/2})} + \|u_0 + \Gamma_\infty v_0\|_{D(B)}) \ . \tag{1.6}$$

D'autre part, nous savons que

$$\|v(t)\|_{\mathbb{L}^2} + \|u(t)\|_{D(B^{1/2})} \leq K(\|v_0\|_{\mathbb{L}^2} + \|u_0\|_{D(B^{1/2})}) \ . \tag{1.7}$$

Comme Γ_∞ est un opérateur linéaire continu de $D(B^{1/2})$ dans $D(B^{1/2})$, nous déduisons de (1.7) que

$$\|v(t)\|_{\mathbb{L}^2} + \|u(t) + \Gamma_\infty v(t)\|_{D(B^{1/2})} \leq K(\|v_0\|_{D(B^{1/2})} + \|u_0 + \Gamma_\infty v_0\|_{D(B^{1/2})}) \ . \tag{1.8}$$

Soit $s \in [0, \frac{1}{2}[$. En interpolant (1.8) avec (1.6) grâce à la proposition 1.3, on obtient

$$\|v(t)\|_{D(B^{s/2})} + \|u(t) + \Gamma_\infty v(t)\|_{D(B^{(1+s)/2})} \leq K(\|v_0\|_{D(B^{1/2})} + \|u_0 + \Gamma_\infty v_0\|_{D(B^{(1+s)/2})}) \ .$$

Puisque $s < \frac{1}{2}$, on a $\|\Gamma_\infty v\|_{D(B^{1/2+s})} \leq \|v\|_{D(B^{1/2})}$ et donc

$$\|v(t)\|_{D(B^{s/2})} + \|u(t)\|_{D(B^{(1+s)/2})} \leq K(\|v_0\|_{D(B^{1/2})} + \|u_0\|_{D(B^{(1+s)/2})}) \ .$$

En utilisant de nouveau la proposition 1.3, l'interpolation de l'inégalité ci-dessus avec (1.7) donne que

$$\|v(t)\|_{D(B^{s^2/2})} + \|u(t)\|_{D(B^{(1+s^2)/2})} \leq K(\|v_0\|_{D(B^{s/2})} + \|u_0\|_{D(B^{(1+s^2)/2})}) \ .$$

On obtient alors (1.5) par la densité de $D(A_\infty)$ dans X^s. $\square$

La proposition précédente permet de montrer un résultat de convergence des trajectoires semblable à celui du chapitre 5, même dans le cas d'une non-linéarité de type cubique.

Théorème 1.4. *Soit $s \in]0, \frac{1}{2}[$ et soit $\mathcal{B}^s$ un borné de X^s. Il existe une constante $C = C(\mathcal{B})$ telle que*

$$\forall U \in \mathcal{B},\ \forall t \in [0,T], \quad \|S_\infty(t)U - S_n(t)U\|_X \leq Ce^{CT}\varepsilon_n^{s^2/2} \ .$$

Démonstration : La preuve est identique à celle du théorème 3.8 du chapitre 5. La seule différence réside dans la preuve du lemme 3.7, où le caractère sous-critique de la non-linéarité est utilisé dans l'estimation de

$$\int_0^t \|(e^{A_\infty(t-\tau)} - e^{A_n(t-\tau)})F(S_\infty(\tau)U)\|_X d\tau \ .$$

Or, grâce à la proposition 1.2, nous savons que $S_\infty(\tau)U$ est borné dans X^{s^2}. En utilisant les théorèmes d'injections des espaces de Sobolev, on montre alors que, même dans le cas d'une non-linéarité critique, $F(S_\infty(\tau)U)$ est borné dans X^{2s^2}. La proposition 3.5 du chapitre 5 implique alors que

$$\int_0^t \|(e^{A_\infty(t-\tau)} - e^{A_n(t-\tau)})F(S_\infty(\tau)U)\|_X d\tau \leq Ce^{CT}\varepsilon_n^{s^2/2} \ .$$

On obtient donc une estimation indépendante du taux de croissance de f et on finit la démonstration du théorème 1.4 en suivant celle du théorème 3.8 du chapitre 5. $\square$

Afin d'utiliser le théorème 1.4 pour montrer la convergence des variétés instables locales ou la semi-continuité inférieure des attracteurs, il nous faut des résultats de régularité sur $W^u_\infty(E,r)$ ou sur $\mathcal{A}_\infty$.

Théorème 1.5. *Soit $E = (e,0)$ un point d'équilibre hyperbolique de $S_\infty(t)$. Pour r suffisamment petit, la variété instable locale $W^u_\infty(E,r)$ est bornée dans $D(A_\infty)$. De plus, pour tout $\beta \in]0, \frac{1}{8}[$, il existe une constante $C > 0$ telle que, pour n assez grand,*

$$d_X(W^u_n(E,r), W^u_\infty(E,r)) \leq C\varepsilon_n^\beta \ .$$

Démonstration : Pour montrer que $W^u_\infty(E,r)$ est bornée dans $D(A_\infty)$, on considère $\tilde{S}_\infty(t)$ la restriction du système $S_\infty(t)$ à $D(A_\infty)$. Le semi-groupe $\tilde{S}_\infty(t)$ est bien défini d'après (1.4) et on peut alors construire des variétés instables locales $\tilde{W}^u_\infty(E,r)$ pour ce système. Comme les vecteurs propres de la linéarisation dans X de $S_\infty(t)$ près de E sont

dans $D(A_\infty)$, la dimension des variétés instables locales $W^u_\infty(E,r)$ et $\tilde{W}^u_\infty(E,r)$ sont égales. D'où $W^u_\infty(E,r) = \tilde{W}^u_\infty(E,r)$ et donc $W^u_\infty(E,r)$ est borné dans $D(A_\infty)$.
On montre la convergence des variétés instables locales de la même façon que dans le chapitre 5. La seule différence est que les estimations (4.19) et (4.20) du chapitre 5 ne sont pas vraies en toute généralité, mais seulement pour toute trajectoire $U(s)$ bornée dans un espace régulier. Comme ces estimations ne sont utilisées que pour des trajectoires contenues dans $W^u_\infty(E,r)$, cela n'est pas grave. $\square$

En supposant que les points d'équilibre de $S_\infty(t)$ sont tous hyperboliques, on peut utiliser la méthode des chaînes d'équilibres introduites dans le paragraphe 5.2 du chapitre 5 et obtenir le résultat suivant.

Théorème 1.6. *Si tous les points d'équilibre de $S_\infty(t)$ sont hyperboliques, alors $\mathcal{A}_\infty$ est borné dans $D(A_\infty)$.*

Démonstration : La preuve de ce théorème est donnée dans le paragraphe 2.2. $\square$

Une application directe du théorème 2.20 du chapitre 2 donne alors la convergence des attracteurs.

Théorème 1.7. *On suppose que tous les équilibres de $S_\infty(t)$ sont hyperboliques, alors les attracteurs $\mathcal{A}_n$ sont semi-continus inférieurement dans X.*
De plus, pour tout $s > 0$, il existe deux constantes strictement positives C et δ telles que

$$\sup_{U_\infty \in \mathcal{A}_\infty} dist_X(U_\infty, \mathcal{A}_n) \leq C\varepsilon_n^\delta \quad \textit{et} \quad \sup_{U_n \in \mathcal{A}_n} dist_{X^{-s}}(U_n, \mathcal{A}_\infty) \leq C\varepsilon_n^\delta \ .$$

2 Utilisation de la notion de chaîne d'équilibres

Dans le paragraphe 5.2 du chapitre 5, nous avons introduit la notion de chaîne d'équilibres associée à une suite de trajectoires. Cette notion est très proche de celle de *famille de trajectoires limites combinées* introduite par Babin et Vishik pour estimer la vitesse de convergence vers les attracteurs de systèmes gradients (voir [1]). Dans le chapitre 5, pour montrer la stabilité du diagramme de phase, nous avons utilisé des chaînes d'équilibres associées à des suites d'orbites $\mathcal{O}_k$ provenant de systèmes dynamiques $S_{n_k}(t)$ différents. Dans ce paragraphe, nous allons réintroduire la notion de chaîne d'équilibres dans le contexte d'une suite d'orbites $\mathcal{O}_k$ associée à un même système dynamique gradient $S(t)$. Puis nous l'appliquerons à la démonstration du théorème 1.6 du paragraphe précédent.

2.1 Chaînes d'équilibres pour une suite d'orbites d'un même système

Nous allons reprendre ce qui a été fait dans le paragraphe 5.2 du chapitre 5 en modifiant ce qui est nécessaire.
Soient X un espace de Banach et $S(t)$ un système dynamique gradient de classe $\mathcal{C}^1$ sur X qui est asymptotiquement compact. On suppose que $S(t)$ possède un ensemble fini de points d'équilibre $\mathcal{E}$, et que tout équilibre $e \in \mathcal{E}$ est hyperbolique. On choisit un rayon r tel que les boules $B(e, 2r)$ $(e \in \mathcal{E})$ ne se coupent pas et tel que les variétés stables et instables locales $W^s_{loc}(e, r)$ et $W^u_{loc}(e, r)$ soient bien définies pour tout équilibre e.
Soit $(U^0_k)_{k\in\mathbb{N}}$ une suite bornée de points de X. On pose $U_k(t) = S(t)U^0_k$ et on suppose que l'orbite positive de la suite, c'est-à-dire l'ensemble $\{U_k(t) \ / \ k \in \mathbb{N}, \ t \geq 0\}$ est bornée. La compacité asymptotique, le caractère gradient et le fait que l'ensemble $\{U_k(t)/t \geq 0\}$ est borné entraînent que l'ensemble $\omega-$limite de U^0_k est un compact connexe, et donc, puisque le nombre de points d'équilibre est fini, $U_k(t)$ tend vers un unique point d'équilibre quand t tend vers $+\infty$. Vu que $\mathcal{E}$ est fini, quitte à extraire une sous-suite, il existe un équilibre e_+ tel que pour tout k, $\lim_{t\to+\infty} U_k(t) = e_+$. On choisit alors une suite de temps t_k telle que $U_k(t_k) \in W^s_{loc}(e_+, r)$ et $U_k(t_k)$ tende vers e_+ quand k tend vers $+\infty$. On définit les chaînes d'équilibres comme suit.

Définition 2.1. *Une suite $e_0, e_1, \ldots, e_p = e_+$ est appelée une chaîne d'équilibres de longueur p pour (U_k) s'il existe $p+1$ suites de temps $0 \leq t^0_k < t^1_k < \ldots < t^p_k = t_k$ telles que, quitte à extraire des sous-suites,*

$$U_k(t^i_k) \longrightarrow e_i \ \textit{quand} \ k \longrightarrow +\infty \ ,$$

et pour tout $k \in \mathbb{N}$ et $i < p$, il existe $t \in]t^i_n, t^{i+1}_n[$ tel que $U_n(t)$ n'appartienne à aucune boule $B(e, r)$ $(e \in \mathcal{E})$.

L'ensemble des chaînes d'équilibres associées à (U_k) est non vide car (e_+) est une chaîne triviale. Si une chaîne d'équilibres nous est donnée, on note les temps d'entrée et de sortie de $B(e_i, r)$ comme suit

$$\sigma^i_k = \sup\{t \leq t^i_k \mid U_k(t) \notin B(e_i, r)\} \ \text{ et } \ \tau^i_k = \inf\{t \geq t^i_k \mid U_k(t) \notin B(e_i, r)\} \ .$$

On note que le temps d'entrée σ^0_k n'est pas forcément défini et que le temps de sortie τ^p_k ne l'est jamais.
Contrairement au paragraphe 5.2 du chapitre 5, nous ne supposons aucune unicité rétrograde, cette hypothèse est remplacée par le $\lambda-$lemme suivant qui est une conséquence classique de l'hyperbolicité des équilibres, et du fait qu'ils soient en nombre fini (voir par exemple [7] ou [5]).
On rappelle que $\text{dist}_X(U, \mathcal{S})$ désigne la distance d'un point $U \in X$ à un ensemble $\mathcal{S} \subset X$, c'est-à-dire

$$\text{dist}_X(U, \mathcal{S}) = \inf_{x\in\mathcal{S}} \|U - x\|_X \ .$$

Proposition 2.2. *Il existe deux constantes strictement positives M et λ telles que si e est un point d'équilibre hyperbolique de $S(t)$ et U un point de X tel qu'il existe T tel que pour tout $t \in [0,T]$, $S(t)U$ soit dans $B(e,r)$, alors*

$$dist_X(S(T)U, W^u_{loc}(e,2r)) \leq Me^{-\lambda T} \ .$$

On peut alors montrer l'équivalent du lemme 5.6 du chapitre 5.

Lemme 2.3. *Soit $i = 1,...,p$, il existe $V_i \in \partial B(e_i,r) \cap W^s_{loc}(e_i,2r)$ tel que, quitte à extraire des sous-suites, on ait $U_k(\sigma^i_k) \longrightarrow V_i$.*
Soit $i = 0,...,p-1$, il existe $W_i \in \partial B(e_i,r) \cap W^u_{loc}(e_i,2r)$ tel que, quitte à extraire des sous-suites, on ait $U_k(\tau^i_k) \longrightarrow W_i$.

Démonstration : Notons tout d'abord que $\tau^0_k - t^0_k \longrightarrow +\infty$. En effet, si cela n'était pas vrai, on pourrait supposer, quitte à extraire une sous-suite, qu'il existe un temps T fini tel que $\tau^0_k - t^0_k \longrightarrow T$ et on aurait que $\lim_{k\to+\infty} U_k(\tau^0_k) = S(T)e_0 \in \partial B(e_0,r)$, ce qui est absurde. Il s'ensuit donc que τ^0_k tend vers $+\infty$.
En utilisant la compacité asymptotique, on obtient, pour $i = 0,...,p-1$, l'existence d'un point W_i de $\partial B(e_i,r)$ tel que, quitte à extraire des sous-suites, on ait $U_k(\tau^i_k) \longrightarrow W_i$. De même, pour $i = 1,...,p$, il existe $V_i \in \partial B(e_i,r)$ tel que, quitte à extraire, on ait $U_k(\sigma^i_k) \longrightarrow V_i$. On montre que $V_i \in W^s_{loc}(e_i,2r)$ de la même façon que dans le lemme 5.6 du chapitre 5. Pour montrer que $W_i \in W^u_{loc}(e_i,2r)$, la preuve du lemme 5.6 utilisait des propriétés d'unicité rétrograde qui ne sont plus valables ici. Pour contourner ce problème, on montre que, quitte à extraire une sous-suite, $\tau^i_k - t^i_k \longrightarrow +\infty$ de la même façon que l'on a prouvé que $\tau^0_k - t^0_k \longrightarrow +\infty$. On peut alors appliquer la proposition 2.2 et obtenir que $W_i \in W^u_{loc}(e_i,2r)$. □

Remarquons que, même si σ^0_k existe, $U_k(\sigma^0_k)$ ne converge pas en général.
La suite du raisonnement est semblable à celui du paragraphe 5.2 du chapitre 5. On montre ainsi que la longueur d'une chaîne est borné par le nombre de points d'équilibre, et on obtient le même résultat que le lemme 5.8 du chapitre 5.

Lemme 2.4. *Si (e_i) est une chaîne d'équilibres de longueur maximale p, alors il existe un temps $T \in \mathbb{R}_+$ tel que*

$$\forall i = 0,...,p-1, \quad \sup_{k\in\mathbb{N}} \{\sigma^{i+1}_k - \tau^i_k\} \leq T \ ,$$

et tel que, si de plus σ^0_k existe, alors $\sup_{k\in\mathbb{N}} \sigma^0_k \leq T$.

Démonstration : La démonstration est identique à celle du lemme 5.8 du chapitre 5. Un léger changement intervient dans la preuve du fait que $\sup \sigma^0_k \leq T$. Nous allons donc

démontrer ce point.
Soit $\Phi : X \longrightarrow X$ la fonctionnelle de Lyapounov associée au système gradient $S(t)$. Supposons que $\sigma_k^0 \longrightarrow +\infty$ et posons $T_k = \sqrt{\sigma_k^0}$. Quitte à extraire une sous-suite, on montre que la compacité asymptotique implique que les suites $U_k(T_k)$ et $U_k(\sigma_k^0)$ sont convergentes et donc que les suites $\Phi(U_k(T_k))$ et $\Phi(U_k(\sigma_k^0))$ sont bornées. On en déduit l'existence d'une suite de temps $s_k \in [T_k, \sigma_k^0 - T_k]$ tels que $\Phi(U_k(s_k)) - \Phi(U_k(s_k + T_k)) \longrightarrow 0$. Comme $s_k \geq T_k \longrightarrow +\infty$, on peut supposer que $U_k(s_k)$ converge vers un point U de X, qui vérifie alors, par continuité de Φ que pour tout $t \geq 0$, $\Phi(S(t)U) = \Phi(U)$. Donc, U est un point d'équilibre, ce qui contredit la maximalité de la chaîne (e_i). $\square$

Remarque : Tout ceci est encore valable pour des chaîne d'équilibres associées à des suites d'orbites $\mathcal{O}_k$ provenant de systèmes dynamiques $S_{n_k}(t)$ différents dès lors que des estimations semblables à la proposition 2.2 sont vérifiées uniformément en n.

2.2 Application à la régularité de l'attracteur dans le cas d'une non-linéarité critique

Le but de ce paragraphe est la démonstration du théorème 1.6. Il faut souligner que la preuve présentée ici est très générale et peut s'appliquer à de nombreux autres systèmes dynamiques gradients. Nous reprenons toutes les notations du paragraphe 1. En outre, comme nous travaillons sur un système dynamique fixe, nous oublions les indices et écrivons $S(t)$, $\mathcal{A}$, A etc. pour $S_\infty(t)$, $\mathcal{A}_\infty$, A_∞ etc..
On suppose que tous les équilibres de $S(t)$ sont hyperboliques. On sait qu'il existe deux constantes strictement positives M et λ telle que

$$\forall t \geq 0,\ \|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t} \ . \tag{2.1}$$

Si $E = (e, 0)$ est un équilibre de $S(t)$, on pose pour tout $U = (u, v) \in X$

$$G_e(U) = \begin{pmatrix} 0 \\ g_e(x, u) \end{pmatrix} \quad \text{où} \quad g_e(x, u) = f(x, u) - f'_u(x, e)u \ . \tag{2.2}$$

Dans le lemme 4.3 du chapitre 5, on montre que G_e est une fonction lipschitzienne sur les bornés de X. De plus, si on note $l_e(r)$ la constante de Lipschitz de G_e sur la boule $B_X(E, r)$, alors $l_e(r)$ tend vers 0 quand r tend vers 0. Dans toute la suite, nous supposerons que r est assez petit pour que $l_e(r) \leq \frac{\lambda}{2M}$ pour tout équilibre $(e, 0)$ de $S(t)$. Rappelons que cela est possible car les équilibres de $S(t)$ sont en nombre fini.
Pour démontrer le théorème 1.6, nous allons raisonner par l'absurde. Nous savons que l'attracteur $\mathcal{A}$ est borné dans X. Supposons qu'il existe une suite de points $V_k \in \mathcal{A}$ tels que $\|V_k\|_{D(A)} \longrightarrow +\infty$, où par convention $\|V_k\|_{D(A)} = +\infty$ si $V_k \notin D(A)$.

Quitte à extraire une sous-suite, il existe deux points d'équilibre E_- et E_+ de $S(t)$, une suite de points U_k^0 et des temps t_k et T_k tels que $U_k^0 \in W^u_{loc}(E_-, r)$, $U_k^0 \longrightarrow E_-$, $S(t_k)U_k^0 \in W^s_{loc}(E_+, r)$, $S(t_k)U_k^0 \longrightarrow E_+$ et $V_k = S(T_k)U_k^0$. On note $U_k(t) = S(t)U_k^0$ pour $t \geq 0$ et on prolonge $U_k(t)$ en une trajectoire définie sur tout $\mathbb{R}$ telle que pour tout $t \leq 0$, $U_k(t) \in W^u_{loc}(E_-, r)$. Le paragraphe précédent montre qu'il existe une chaîne d'équilibres de longueur maximale $E_- = E_0, ..., E_p = E_+$ associée à la suite de trajectoires U_k et on reprend les notations introduites dans ce paragraphe. Pour simplifier la rédaction, on pose $\sigma_k^0 = -\infty$ et $\tau_k^p = t_k$. Soit i_0 le plus petit des indices i tels que $1 \leq i \leq p$ et

$$\limsup_{k \to +\infty} \sup_{t \in]\sigma_k^i, \tau_k^i]} \|U_k(t)\|_{D(A)} = +\infty \ . \tag{2.3}$$

L'ensemble des indices i vérifiant (2.3) est non vide car, comme la chaîne d'équilibres $E_0, ..., E_p$ est de longueur maximale, les suites $\sigma_k^i - \tau_k^{i-1}$ sont bornés. Donc, si la suite $U_k(\tau_k^i)$ est bornée dans $D(A)$, (1.4) implique que $\sup_k \sup_{t \in [\tau_k^i, \sigma_k^{i+1}]} \|U_k(t)\|_{D(A)} < +\infty$. On en déduit que si aucun indice i ne vérifiait (2.3), alors toute la trajectoire U_k serait bornée dans $D(A)$ ce qui est absurde. D'autre part, i_0 est forcément plus grand que 1 d'après le théorème 1.5.

Comme i_0 est le plus petit indice i tel que la propriété (2.3) soit vérifiée, on sait que la suite $\sup_{t \leq \sigma_k^{i_0}} \|U_k(t)\|_{D(A)}$ est bornée. Comme $\partial_t U_k = AU_k + F(U_k)$, on a donc

$$\sup_{k \in \mathbb{N}} \sup_{t \leq \sigma_k^{i_0}} \|\partial_t U_k(t)\|_X \leq C \ . \tag{2.4}$$

Dans tout le reste de la preuve, C désigne une constante ne dépendant pas de k ou du temps.

Soit $t \in]\sigma_k^i, \tau_k^i]$ et $\delta > 0$. En utilisant la décroissance exponentielle de e^{At}, on peut écrire comme dans la preuve de la proposition 2.9 du chapitre 5 que

$$U_k(t + \delta) - U_k(t) = \int_{-\infty}^{t} e^{A(t-s)}(F(U_k(s + \delta)) - F(U(s)))ds \ ,$$

et donc que

$$\|(U_k(t + \delta) - U_k(t))\|_X \leq M \int_{-\infty}^{t} e^{-\lambda(t-s)} \|F(U_k(s + \delta)) - F(U_k(s))\|_X ds \ . \tag{2.5}$$

Comme F est lipschitzienne sur les bornés de X, l'estimation (2.4) implique que pour tout $s \in] - \infty, \sigma_k^{i_0}]$,

$$\|F(U_k(s + \delta)) - F(U_k(s))\|_X \leq C\|U_k(s + \delta) - U_k(s)\|_X \leq C\delta \ .$$

D'autre part, si $s \in]\sigma_k^{i_0}, \tau_k^{i_0}[$ alors on a $U_k(s) \in B_X(E_{i_0}, r)$ et la décomposition $f(x, u) = f'_u(x, e_{i_0})u + g_{e_{i_0}}(x, u)$ donne que

$$\begin{aligned} \|F(U_k(s + \delta)) - F(U_k(s))\|_X \quad & \leq \|g_{e_{i_0}}(x, u_k(s + \delta)) - g_{e_{i_0}}(x, u_k(s)\|_{\mathbb{L}^2} \\ & + \|f'_u(x, e_{i_0}(x))(u_k(s + \delta) - u_k(s))\|_{\mathbb{L}^2} \ . \end{aligned}$$

Comme $g_{e_{i_0}}$ est lipschitzienne sur la boule $B_X(E_{i_0}, r)$ avec une constante de Lipschitz $l(r)$, on a que

$$\begin{aligned}
\|g_{e_{i_0}}(x, u_k(s+\delta)) - g_{e_{i_0}}(x, u_k(s))\|_{\mathbb{L}^2} &\leq l(r)\|u_k(s+\delta) - u_k(s)\|_{\mathbb{H}^1} \\
&\leq l(r) \sup_{s' \in]\sigma_k^{i_0}, \tau_k^{i_0}]} \|U_k(s'+\delta) - U_k(s')\|_X \\
&\leq \frac{\lambda}{2M} \sup_{s' \in]\sigma_k^{i_0}, \tau_k^{i_0}]} \|U_k(s'+\delta) - U_k(s')\|_X
\end{aligned}$$

Les résultats classiques de régularité des équations elliptiques impliquent que e_{i_0} appartient à $\mathbb{L}^\infty(\Omega)$. On a donc

$$\|f'_u(x, e_{i_0}(x))(u_k(s+\delta) - u_k(s))\|_{\mathbb{L}^2} \leq C\|u_k(s+\delta) - u_k(s)\|_{\mathbb{L}^2} \leq C\delta\|U_k(s)\|_X \ .$$

Comme les trajectoires $U_k(s)$ sont uniformément bornées dans X, en utilisant les estimations ci-dessus dans (2.5), on obtient que, pour tout $t \in]\sigma_k^{i_0}, \tau_k^{i_0}]$

$$\|(U_k(t+\delta) - U_k(t))\|_X \leq \frac{1}{2} \sup_{s \in]\sigma_k^{i_0}, \tau_k^{i_0}]} \|U_k(s+\delta) - U_k(s)\|_X + C\delta \ .$$

On en déduit que

$$\sup_{t \in]\sigma_k^{i_0}, \tau_k^{i_0}]} \|U_k(t+\delta) - U_k(t)\|_X \leq 2C\delta \ . \tag{2.6}$$

D'autre part, on sait que $U_k(\tau_k^{i_0-1})$ est borné dans $D(A)$ et donc que $U_k(t)$ est de classe $\mathcal{C}^1([\sigma_k^{i_0}, \tau_k^{i_0}], X) \cap \mathcal{C}^0([\sigma_k^{i_0}, \tau_k^{i_0}], D(A))$. En faisant tendre δ vers 0 dans (2.6), on trouve que $\partial_t U_k(t)$ est borné dans X pour $t \in]\sigma_k^{i_0}, \tau_k^{i_0}]$. Puis en écrivant $AU_k(t) = \partial_t U_k(t) - F(U_k(t))$, que $U_k(t)$ est borné dans $D(A)$, uniformément en k et $t \in]\sigma_k^{i_0}, \tau_k^{i_0}]$. Ceci contredit (2.3) et finit la démonstration du théorème 1.6.

3 Un nouvel exemple d'équation des ondes amorties de type gradient

Le théorème énoncé dans ce paragraphe n'est que le rapprochement de trois résultats tirés des articles [9], [6] et [2]. Il permet de donner un nouvel exemple de système de type gradient pour la classe des équations des ondes amorties.
Soit Ω un ouvert borné régulier de $\mathbb{R}^d$ ($d = 2$ ou 3). Soit γ une fonction positive et bornée sur Ω et soit $f(x, u)$ une fonction de classe $\mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ que l'on suppose analytique en u

et sous-critique, c'est-à-dire qu'il existe $C > 0$ et $\alpha \geq 0$ si $d = 2$ ou $\alpha \in [0, 1[$ si $d = 3$ tels que

$$|f''_{uu}(x, u)| \leq C(1 + |u|^\alpha) \text{ et } |f''_{ux}(x, u)| \leq C(1 + |u|^{3+\alpha}) \ . \tag{3.1}$$

On pose $X = \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)$ (resp. $X = \mathbb{H}^1_0(\Omega) \times \mathbb{L}^2(\Omega)$) et on considère le système dynamique $S(t)$ engendré par l'équation des ondes amorties

$$\left\{ \begin{array}{ll} u_{tt}(x,t) + \gamma(x)u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)) & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x,t) = 0 \text{ (resp. } u(x,t) = 0) & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u, u_t)(x, 0) = (u_0, u_1) \in X & \end{array} \right. \tag{3.2}$$

Soit A l'opérateur linéaire associé à (3.2) et défini par

$$A = \begin{pmatrix} 0 & Id \\ \Delta & -\gamma \end{pmatrix} \qquad D(A) = \mathbb{H}^2(\Omega) \times \mathbb{H}^1(\Omega) \ \ (\text{resp. } D(A) = (\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_0(\Omega)) \times \mathbb{H}^1_0(\Omega)) \ .$$

Nous supposons que la condition géométrique suivante est vérifiée : il existe une longueur $L > 0$ telle que toute géodésique de longueur L (i.e. toute ligne droite de longueur L rebondissant sur les bords de Ω selon la loi de Descartes) rencontre le support de $\gamma(x)$. D'après [2], ceci est équivalent à l'existence de deux constantes strictement positives M et λ telles que, pour tout $t \geq 0$, $\|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$. Comme f est supposée sous-critique, cela entraine que $S(t)$ est un système asymptotiquement compact.
Dans le paragraphe 3.3 du chapitre 2, nous avons vu que, si l'on impose des conditions géométriques plus fortes, $S(t)$ est un système gradient. Le but de ce paragraphe est de montrer que, dans le cas particulier d'une fonction $f \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ et analytique en u, la condition géométrique de [2] est suffisante.

Théorème 3.1. *On suppose que f est de classe $\mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, est analytique en u et vérifie*

$$\limsup_{u \to \pm\infty} \sup_{x \in \Omega} \frac{f(x,u)}{u} < 0 \ . \tag{3.3}$$

Alors le système $S(t)$ engendré par l'équation (3.2) est de type gradient, c'est-à-dire qu'il vérifie les propriétés suivantes :
i) tout point $U_0 \in X$ a un ensemble $\omega-$limite qui est contenu dans l'ensemble $\mathcal{E}$ des points d'équilibre.
ii) tout point $U_0 \in X$, dont la trajectoire négative est bornée, admet un ensemble $\alpha-$limite qui est contenu dans $\mathcal{E}$.
Si de plus, les points d'équilibre sont isolés, alors ces ensembles $\alpha-$ et $\omega-$limite sont restreints à un seul point d'équilibre.

Démonstration : Nous rappelons que la fonctionnelle

$$\Phi : \left(\begin{array}{ccc} X & \longrightarrow & \mathbb{R} \\ (u, v) & \longmapsto & \frac{1}{2}(\|u\|^2_{\mathbb{H}^1} + \|v\|^2_{\mathbb{L}^2}) - \int_\Omega \int_0^{u(x)} f(x, \xi) d\xi dx \end{array} \right) \tag{3.4}$$

décroit le long des trajectoires. De plus, (3.3) implique qu'il existe une constante C telle que $\frac{1}{2}\|U\|_X^2 - C \leq \Phi(U)$. D'autre part, (3.1) entraine que Φ est bornée sur les bornés de X. En particulier, la trajectoire positive de tout ensemble borné est bornée.
En utilisant le principe de LaSalle et des arguments classiques, on obtient, dans les cas i) et ii) du théorème, l'existence d'ensembles $\alpha-$ et $\omega-$limite bornés, connexes, invariants par le flot et sur lesquels Φ est constante. Tout ce qu'il reste à montrer, est que ces ensembles sont composés uniquement de points d'équilibre.
Soit U_0 un point de l'ensemble $\alpha-$ (resp. $\omega-$limite). Par la propriété d'invariance, on peut prolonger la trajectoire positive de U_0 en une trajectoire complète $(U(t))_{t\in\mathbb{R}}$ qui est contenue dans l'ensemble $\alpha-$ (resp. $\omega-$limite) et qui est donc bornée et sur laquelle Φ est constante. En utilisant le résultat de [6], on montre que, puisque $f(x,u)$ est analytique en u et $U(t)$ est bornée sur $\mathbb{R}$, $U(t)$ est analytique en temps à valeurs dans X et $U(t) \in \mathcal{C}^0(\mathbb{R}, \mathbb{H}^{k+1}(\Omega) \times \mathbb{H}^k(\Omega))$ pour tout $k \geq 0$.
D'autre part, en posant $U(t) = (u, u_t)(t)$, on a

$$\frac{d}{dt}\Phi(U(t)) = -\int_\Omega \gamma(x)|u_t|^2 \ .$$

Donc $v = u_t$ est nulle sur le support de γ et vérifie l'équation

$$v_{tt}(x,t) = \Delta v(x,t) + f'_u(x, u(x,t))v(x,t) \ .$$

D'après le résultat de régularité précédent, $h(x,t) = f'_u(x, u(x,t))$ est de classe $\mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}, \mathbb{R})$ et est analytique en t. Le résultat de [9] montre alors que la nullité de v se propage et donc que $v(x,t)$ est identiquement nulle sur $\Omega \times \mathbb{R}$. Comme $v = u_t$, cela implique que U_0 est un point d'équilibre. $\square$

Remarque : Nous soulignons que $S(t)$ n'est pas un système gradient au sens du chapitre 2. En effet, nous avons montré que, si $U(t)$ est une trajectoire définie et bornée sur tout $\mathbb{R}$, sur laquelle la fonctionnelle de Lyapounov est constante, alors $U(t)$ est un point d'équilibre. Pour obtenir le caractère gradient au sens du chapitre 2, il faudrait généraliser cette implication à toutes les trajectoire définies et bornées sur $\mathbb{R}_+$.

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n° d'impression : 2707
$4^{\text{ème}}$ trimestre 2005

Abstract :
This thesis concerns the qualitative study of the dynamics of the damped wave equations on a bounded domain $\Omega \subset \mathbb{R}^d$. We begin with a chapter describing the various notions of stability and reviewing the known results.
In the next chapter, we prove the genericity, with respect to the non-linearity, of the Morse-Smale property for the one-dimensional wave equation with internal damping $\gamma(x)$ (WEID) and the one with boundary damping $g(x) \otimes \delta_{x \in \partial\Omega}$ (WEBD). The proof mainly uses accurate properties of the asymptotic behaviour of the functions $t \mapsto u(x_0, t)$, where u is a bounded solution of either Equations (WEID) or (WEBD), or their adjoint equations, and where x_0 is a fixed point of Ω. This asymptotic behaviour is deduced from the spectral properties of the linearized operator around an equilibrium point. In particular, the eigenvectors of this operator form a Riesz basis and its eigenvalues are generically simple.
The last part of this thesis concerns the study of the convergence of the dynamics of Equation (WEID) to the ones of Equation (WEBD) when the sequence of internal dampings $\gamma_n(x)$ converges to $g(x) \otimes \delta_{x \in \partial\Omega}$ in the sense of distributions. In dimension $d = 1$, we show that the dynamics of (WEID) converge to the ones of (WEBD). In dimension $d \geq 2$, weaker results of convergence of attractors are obtained. The perturbation studied here is singular and thus, some classical results of stability have to be generalized. To obtain the best results of convergence, one has to show that the linear semigroups associated with (WEID) decay with an exponential rate $\|e^{A_n t}\|_X \leq M e^{-\lambda t}$, uniformly with respect to n. In this thesis, we present a complete study of this uniform exponential decay in dimension $d = 1$.
These results show in particular the relevance of the model of the wave equation with boundary damping.

Key-words : damped wave equation, stability of dynamics, attractors, boundary damping, Morse-Smale property, transversality, singular perturbations.

Résumé :
Cette thèse a pour sujet l'étude qualitative de la dynamique des équations des ondes amorties sur un domaine borné $\Omega \subset \mathbb{R}^d$. Outre un chapitre de présentation des notions de stabilité de la dynamique et des travaux antérieurs, cette thèse s'articule autour de deux parties principales.
Dans la première partie, on démontre, en dimension $d = 1$, que la propriété de Morse-Smale est générique par rapport à la non-linéarité, pour l'équation des ondes avec amortissement interne $\gamma(x)$ (EOAI) et celle avec amortissement sur le bord $g(x) \otimes \delta_{x\in\partial\Omega}$ (EOAB). La démonstration utilise des propriétés fines du comportement asymptotique des fonctions $t \mapsto u(x_0, t)$, où u est une solution bornée des équations (EOAI), (EOAB) ou de leurs équations adjointes et où x_0 est un point fixé de Ω. Ce comportement asymptotique se déduit principalement des propriétés spectrales de l'opérateur linéarisé autour d'un point d'équilibre. En particulier, les vecteurs propres de cet opérateur forment une base de Riesz et ses valeurs propres sont génériquement simples.
La deuxième partie de cette thèse concerne l'étude de la convergence de la dynamique de l'équation (EOAI) vers celle de l'équation (EOAB) quand la suite d'amortissements internes $\gamma_n(x)$ tend vers $g(x) \otimes \delta_{x\in\partial\Omega}$ au sens des distributions. En dimension $d = 1$, on montre que la dynamique de (EOAI) converge vers celle de (EOAB). En dimension $d \geq 2$, des résultats un peu plus faibles de convergence des attracteurs sont obtenus. La perturbation étudiée ici est irrégulière et on doit donc généraliser certains théorèmes classiques de stabilité. Pour obtenir les meilleurs résultats de convergence, il faut montrer que les semi-groupes linéaires associés à (EOAI) satisfont à une décroissance de type exponentiel $\|e^{A_n t}\|_X \leq Me^{-\lambda t}$, uniformément en n. Dans cette thèse, on fait l'étude complète de cette décroissance exponentielle uniforme en dimension $d = 1$.
Ces résultats permettent entre autres de justifier le modèle de l'équation des ondes amorties sur le bord.

Mots-Clés : équation des ondes amorties, stabilité de la dynamique, attracteurs, amortissement sur le bord, propriété de Morse-Smale, transversalité, perturbations singulières.

See the **English abstract** on the previous page.

AMS Classification Codes (2000) : 35B25, 35B30, 35B37, 35B41, 35B65, 35L05, 35L90, 37C20, 37L15, 37L45.