Phénomène de concentration pour desproblèmes non linéaires issus de la géométrie

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À

mon père, ma mère,

mes frères, mes soeurs.

Affectueusement.

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Résumé : L'objet de cette thèse est l'étude d'un phénomène de concentration pour une série de problèmes non linéaires issus de la géométrie : l'existence d'hypersurfaces plongées dans une variété Riemannienne dont la r-courbure moyenne est constante. (La r-courbure moyenne d'une hypersurface est la rième fonction symétrique de la courbure principale de l'hyersurface). Nous donnons dans cette thèse quelques résultats d'existence de telles hypersurfaces. En outre, les exemples que nous construisons mettent en évidence un phénomène de concentration le long de sous variétés, phénomène associé à un phénomène de résonance qui rend l'analyse de ces objets particulièrement délicate et que l'on rencontre dans l'étude de nombreux autres problèmes non-linéaires, équation de Schrödinger non linéaire, problème de perturbations singulière, système de réaction-diffusion, $\cdots$

Dans la première partie on montre qu'étant donné une variété Riemannienne M et p un point critique non-dégénéré de fonction courbure scalaire, il existe une famille d'hypersurfaces à r-courbure moyenne constante qui se concentrent en p lorsque leurs r-courbure moyenne tend vers l'infini. Les éléments de cette famille constituent un feuilletage d'un voisinage de p. Ce résultat tend à toute les r-courbure moyenne un résultat obtenu par Rugang Ye pour le cas de la 1-courbure moyenne (la courbure moyenne).

Dans la deuxième partie, étant donné une variété Riemannienne M de dimension $m+1$ et une sous variété K de dimension k inférieure ou égale a $m+1-r$, on constuit une famille d'hypersurfaces à r-courbure moyenne constante qui se concentrent en p lorsque leurs r-courbure moyenne tend vers l'infini. Cette fois-ci les éléments de cette famille constituent un feuilletage d'un voisinage tubulaire de K. Ce résultat généralise un résultat précédent de Rafe Mazzeo et Frank Pacard lorsque la sous variété K est une géodésique. La démonstration de ce résultat utilise une idée introduite par A. Malchiodi et M. Montenegro.

Dans la troisième partie, étant donné une variété Riemannienne M et une géodésique compacte K, on montre sous des hypothèses géométriques qu'il existe une famille d'hypersurfaces à courbure moyenne constante de "type Delaunay" qui se concentre sur K lorque leur courbure moyenne tend vers l'infini. Les hypersurfaces construite ne sont pas proche d'un voisinage tubulaire de K mais oscillent comme une surface de Delaunay. De plus, contrairement aux précédent résultat de Mazzeo et Pacard aucun phénomène de résonance n'apparaît.

Abstract : In this Thesis, we study concentration phenomena for geometrical nonlinear elliptic equations : the existence of constant r-curvature hypersurfaces in Riemannian manifolds. (The r-mean curvature of a hypersurfave is defined to be the r-th elementary symmetric function of the principal curvature of the hypersurface). We give in this thesis some results of existence of such a submanifolds. Moreover, the examples which we build highlight a concentration phenomena along submanifolds, phenomena associated with a resonance phenomena which returns the analysis of these objects particularly delicate and which one meets in the study of many other nonlinear problems : nonlinear Schrödinger equations, singularly perturbed problems, reaction-diffusion systems, $\cdots$.

In the first part, given a Riemannian manifold M and p a non-degenerate critical point of the scalair curvature function, we prove the existence of a family of constant r-mean curvature hypersurfaces which concentrate at p as their r-mean curvature tends to infinity. The elements of this family constitute the leaves of a foliation of a neigborhood of the point p.

In the second part, given a Riemannian manifold M of dimension $m+1$ and a non-degenerate minimal submanifold K of dimension less than $m + 1 - r$, we construct a family of constant r-mean curvature hypersurfaces which concentrate at K as their r-mean curvature tends to infinity. This time the closure of the elements of this family constitute the leaves of a lamination of tubular neigborhood of K. This result extends a previous result of Rafe Mazzeo and Frank Pacard which holds for geodesics. The proof of this result uses ideas from a paper by A. Malchiodi and M. Montenegro.

In the third part, given a Riemannian manifold M and a compact geodesic K, we construcu under some geometrical assumptions a family of "Dalaunay type" constant mean curvature hypersurfaces which concentrate along the geodesic K as their mean curvature tends to infinity. This time the hypersurfaces constructed are not close to a geodesic tubular neighborhood of K but rather oscillate as Delaunay surfaces do.

0.1 Introduction

L'objet de cette thèse est l'étude d'un phénomène de concentration pour une série de problèmes non linéaires issus de la géométrie : l'existence d'hypersurfaces plongées dans une variété Riemannienne dont la r-courbure moyenne est constante.

Etant donnée une hypersurface Σ^m plongée dans une variété Riemannienne (M^{m+1}, g), la r-courbure moyenne est définie comme étant égale à

$$\sigma_r = \sum_{i_1 < \ldots < i_r} \kappa_{i_1} \ \ldots \ \kappa_{i_r}$$

où les κ_i sont les courbures principales de l'hypersurface Σ^m. Dans le cas particulier où la variété $(M^{m+1}, g) = (\mathbb{R}^{m+1}, g_{eucl})$ est l'espace euclidien, σ_1 correspond (à une constante multiplicative près) à la courbure moyenne de l'hypersurface et nous substituerons fréquemment "courbure moyenne" à "1-courbure moyenne", σ_2 correspond, toujours à une constante multiplicative près, à la courbure scalaire de l'hypersurface et σ_m correspond à la courbure de Gauss-Kronecker.

Les hypersurfaces dont la r-courbure moyenne est constante constituent une classe importante d'objets dans les variétés Riemanniennes même si très peu d'exemples sont connus [36]. Nous donnons dans cette thèse quelques résultats d'existence de telles hypersurfaces. En outre, les exemples que nous construisons mettent en évidence un phénomène de concentration le long de sous variétés, phénomène associé à un phénomène de résonance qui rend l'analyse de ces objets particulièrement délicate et que l'on rencontre dans l'étude de nombreux autres problèmes non-linéaires.

0.1.1 Familles d'hypersurfaces se concentrant en un point

L'existence de familles d'hypersurfaces à courbure moyenne constante se concentrant en un point a été étudiée par R. Ye [42]. Pour tout p point critique non dégénéré de la courbure scalaire $\mathcal{R}$ sur (M^{m+1}, g), R. Ye démontre l'existence d'une famille d'hypersurfaces plongées, à courbure moyenne constante, qui se concentrent en p lorsque la courbure moyenne tend vers l'infini.

Dans le premier chapitre de cette thèse, nous avons généralisé ce résultat à toutes les r-courbures moyennes. Comme dans [42], l'idée est de perturber les sphères géodésiques $\bar{S}_\rho(p)$ de rayon ρ petit, centrées au point p. Un simple calcul montre que $\bar{S}_\rho(p)$ est proche d'une hypersurface à r-courbure moyenne constante lorsque ρ tend vers 0 au sens où

$$\sigma_r(\bar{S}_\rho(p)) = C_m^r \, \rho^{-r} + \mathcal{O}(\rho^{-r+2}).$$

Ainsi, il semble raisonnable de perturber $\bar{S}_\rho(p)$ en une hypersurface dont la r-courbure moyenne est constante égale à $C_m^r\,\rho^{-r}$, pour ρ assez petit. Le résultat ci-dessous montre qu'une telle construction est possible si p est un point critique non dégénéré de la courbure scalaire $\mathcal{R}$ sur (M^{m+1}, g) :

Theorem 0.1.1 *Fixons $r \in \{1, \ldots, m\}$ et p un point critique non-dégénéré de la courbure scalaire $\mathcal{R}$ sur (M^{m+1}, g). Alors, il existe $\rho_0 > 0$ tel que, pour tout $\rho \in (0, \rho_0)$, la sphère géodésique $\bar{S}_\rho(p)$ peut être perturbée en une hypersurface S_ρ dont la r-courbure moyenne est constante égale à $\sigma_r = C_m^r\,\rho^{-r}$. En outre, les hypersurfaces S_ρ forment un feuilletage d'un voisinage de p.*

Ce résultat généralise, pour tout $r = 1, \ldots, m$, le résultat obtenu par R. Ye pour $r = 1$. Précisons que la démonstration de R. Ye utilise de manière essentielle le fait que l'on s'intéresse à la courbure moyenne d'hypersurfaces (i.e. à σ_1) et qu'elle ne semble pas pouvoir s'adapter aux autres courbures. Notre démonstration est basée sur un développement limité de la r-courbure moyenne d'une sphère géodésique perturbée suivant la normale. Plus précisément, étant donnée une fonction w définie sur S^m et un point q proche de p, on considère la sphère géodésique perturbée $S_\rho(q, w)$ qui est paramétrée par

$$x \in S^m \longrightarrow \mathrm{Exp}_q^M\left(\rho\,(1 - w(x))\,\Theta(x)\right)$$

où $\Theta(x) = \sum_j x^j E_j$ et $E_1, \ldots, E_{m+1}$ est une base orthonormée de $T_q\,M$. Nous obtenons un développement de la r-courbure moyenne en puissances de ρ et en puissances de la fonction w et de ses dérivées

$$\begin{aligned} \rho^r\,\sigma_r(S_\rho(q, w)) \;&=\; C_m^r + C_{m-1}^{r-1}\,\Big((\Delta_{S^m} + m)\,w - \tfrac{1}{3}\,\mathrm{Ric}_q(\Theta, \Theta)\,\rho^2 \\ &-\; \tfrac{1}{4}\,\nabla_\Theta \mathrm{Ric}_q(\Theta, \Theta)\,\rho^3\Big) + \mathcal{O}(\rho^4) + \rho^2\,L_q(w) + Q_q(w) \end{aligned}$$

où Ric_q désigne le tenseur de Ricci sur (M^{m+1}, g) calculé en q. L'opérateur L_q est un opérateur différentiel du second ordre dont les coefficients sont bornées indépendamment de ρ et de q. L'opérateur Q_q est un opérateur différentiel du second ordre non linéaire dont le développement de Taylor en puissances de w et de ses dérivées ne contient pas de terme constant ni de terme linéaire et a ses coefficients de Taylor qui sont bornées indépendamment de ρ et de q.

La principale difficulté à surmonter pour résoudre l'équation

$$\rho^r\,\sigma_r(S_\rho(q, w)) = C_m^r \tag{1}$$

est l'existence d'un noyau de dimension $(m+1)$ pour l'opérateur $\Delta_{S^m} + m$. L'idée est de projeter l'équation (1) d'une part sur l'orthogonal de ce noyau et d'autre part sur ce noyau. Nous obtenons ainsi un système couplant une équation aux dérivées partielles elliptique non linéaire avec une équation algébrique. La résolution de l'équations aux dérivées partielles est basée sur l'utilisation d'un théorème de point fixe pour les applications contractantes et ne pose aucune difficulté particulière. Cette première équation étant résolue, il reste à résoudre une équation algébrique de la forme

$$\nabla \mathcal{R}_q = F_\rho(q), \tag{2}$$

où F_ρ est une application non linéaire définie sur M^{m+1} et à valeurs dans TM. De plus, on montre qu'indépendamment du choix de q, F_ρ est majorée par une constante (indépendante de ρ) fois ρ^2. Le gradient de la courbure scalaire de (M^{m+1}, g) apparaît lors de la projection du terme $\nabla_\Theta \operatorname{Ric}(\Theta, \Theta)$ sur le noyau de $\Delta_{S^m} + m$. Une condition suffisante permettant la résolution de l'équation (2) pour toute petite valeur de ρ est que p soit point critique non-dégénéré de la courbure scalaire $\mathcal{R}$.

La méthode de construction permet d'avoir de nombreuses informations quant aux hypersurfaces construites. En particulier, ces hypersurfaces sont des graphes normaux sur $\bar{S}_\rho(q)$ pour une fonction majorée par une constante fois ρ^2. De plus, on obtient le développement limité du volume de S_ρ

$$\operatorname{Vol}_m(S_\rho) = \rho^m \operatorname{Vol}_m(S^m) \left(1 - \frac{1}{2(m+1)} \mathcal{R}_p \, \rho^2 + \mathcal{O}(\rho^4)\right),$$

ainsi que le développement limité du volume du domaine B_ρ de M^{m+1} contenu dans S_ρ et contenant le point p

$$\operatorname{Vol}_{m+1}(B_\rho) = \frac{1}{m+1} \rho^{m+1} \operatorname{Vol}_m(S^m) \left(1 - \frac{m+2}{2m(m+3)} \mathcal{R}_p \, \rho^2 + \mathcal{O}(\rho^4)\right),$$

Remarquons que les premiers termes de ces développements limitées ne dépendent pas de r. Dans le cas où $r = 1$, ces deux développements limités permettent de trouver un développement limité du profil isopérimétrique en puissances de ρ (c.f [35])

La construction d'une famille à un paramètre d'hypersurfaces à r-courbure moyenne constante qui tend vers $+\infty$ est à rapprocher à des nombreux résultats de concentration qui ont été mis en évidence ces dernières années dans l'étude des perturbations singulières d'équations semi-linéaires [3], [23], [25], ...

0.1.2 Familles d'hypersurfaces se concentrant le long d'une sous variété

Dans le deuxième chapitre de cette thèse, nous nous sommes intéressés à l'existence de familles d'hypersurfaces à r-courbure moyenne constante qui se concentrerent sur une sous variété K^k de (M^{m+1}, g), avec $k \in \{1, \dots, m\}$.

Soit K^k une sous variété de dimension k compacte dans M^{m+1}, on note

$$\bar{T}_\rho(K) := \{q \in M^{m+1} : \quad \text{dist}_g(q, K) = \rho\};$$

le tube géodésique de rayon ρ autour de K^k. Pour tout ρ assez petit, $\bar{T}_\rho(K)$ est une hypersurface régulière et un simple calcul montre que

$$\rho^r\, \sigma_r(\bar{T}_\rho(K)) = C^r_{m-k} + \mathcal{O}(1). \tag{3}$$

Comme dans l'analyse précédente, il semble assez naturel d'essayer de perturber cette hypersurface afin de construire une hypersurfaces dont la r-courbure moyenne est constante, du moins lorsque ρ est assez petit. Une simple observation montre que dans le développement limité précédent le terme suivant du développement dépend du vecteur courbure moyenne de la sous variété K^k. En particulier, si K^k est une sous variété minimale, on obtient alors

$$\rho^r\, \sigma_r(\bar{T}_\rho(K)) = C^r_{m-k} + \mathcal{O}(\rho). \tag{4}$$

Dans le cas où $r = 1$ et où K^k est une géodésique non dégénérée ($k = 1$) R. Mazzeo et F. Pacard [28] ont démontré le :

Theorem 0.1.2 *[28] Soit K une géodésique fermée non dégénérée. Il existe $k_0 \in \mathbb{N}$ et deux suites $\rho'_k < \rho''_k$ définies pour $k \geq k_0$ et qui tendent vers 0, telles que, si $\rho \in I_k := (\rho'_k, \rho''_k)$, le tube géodésique $\bar{T}_\rho(K)$ peut être perturbé en une hypersurface dont la courbure moyenne est constante égale à $\sigma_1 = \frac{m-1}{\rho}$.*

De plus, ils obtiennent une estimation des paramètres ρ'_k et ρ''_k

$$\begin{aligned} \rho'_k &= \tfrac{\sqrt{m-1}\,\Lambda}{2\,\pi\,(k+1)} + \mathcal{O}(k^{-9/4}), \\ \rho''_k &= \tfrac{\sqrt{m-1}\,\Lambda}{2\,\pi\,k} + \mathcal{O}(k^{-9/4}), \end{aligned} \tag{5}$$

où Λ est la longueur de la géodésique K.

Remarquons que leur méthode ne permet pas d'obtenir ce résultat de perturbation pour toutes les valeurs (petites) du paramètre ρ, mais simplement pour ρ appartenant à une suite d'intervalles non vides. Ceci est dû à un phénomène de résonance qui apparaît de manière naturelle dans ce problème pour les grandes valeurs de la courbure moyenne (i.e. pour les petites valeurs de ρ). La difficulté principale dans la démonstration vient de l'existence de valeurs de ρ pour lesquelles l'opérateur de Jacobi associé au tube géodésique $\bar{T}_\rho(K)$ n'est pas inversible. La construction s'appuie alors sur une étude précise du spectre de cet opérateur de Jacobi lorsque le paramètre ρ tend vers 0.

Dans la deuxième partie de cette thèse, nous avons généralisé ce résultat à toutes les r-courbures ce qui n'introduit pas vraiment de difficulté supplémentaire et nous avons aussi considéré le cas où l'ensemble de concentration est une sous variété de dimension $k \in \{1, \dots, m\}$ ce qui cette fois-ci introduit des difficultés supplémentaires majeures.

Dans le cas de la courbure moyenne (i.e. de σ_1), notre principal résultat est le :

Theorem 0.1.3 *Étant donnée K^k une sous variété minimale non dégénérée de (M^{m+1}, g) avec $1 \leq k \leq m-1$, il existe $I \subset (0, +\infty)$, réunion dénombrable disjointe d'intervalles ouverts non vides, tels que pour tout $\rho \in I$, le tube géodésique $\bar{T}_\rho(K)$ peut être perturbé en une hypersurface à courbure moyenne constante T_ρ avec $\sigma_1 = \frac{m-k}{\rho}$. De plus, pour tout $q \geq 2$ il existe une constante $c_q > 0$ telle que*

$$|\mathcal{H}^1\left((0, \rho) \cap I\right) - \rho| \leq c_q\, \rho^q,$$

où $\mathcal{H}^1$ dénote la mesure de Lebesgue unidimensionnelle.

Remarquons que cette fois, nous ne disposons que de très peu d'informations sur l'ensemble dans lequel nous pouvons choisir le paramètre ρ, si ce n'est que cet ensemble est en quelque sorte de plus en plus dense dans $(0, \rho)$ quand ρ tend vers 0. La démonstration de ce résultat permet d'améliorer le résultat de R. Mazzeo et F. Pacard au sens où l'on peut montrer que

$$\begin{aligned} \rho'_k &= \tfrac{\sqrt{m-k}\,\Lambda}{2\,\pi\,(k+1)} + \mathcal{O}(k^{-q}), \\ \rho''_k &= \tfrac{\sqrt{m-k}\,\Lambda}{2\,\pi\,k} + \mathcal{O}(k^{-q}), \end{aligned} \tag{6}$$

pour tout $q > 2$.

La condition de non-dégénéresence de K est simplement une condition d'inversibilité de l'opérateur de Jacobi associé à K (i.e. l'opérateur courbure moyenne linéarisé en K). Pour décrire le comportement des hypersurfaces T_ρ obtenues dans ce résultat quand ρ tend vers 0, on peut s'intéresser à la densité de volume associée à T_ρ pour laquelle on a :

$$\rho^{k-m}\, \mathcal{H}^m \llcorner T_\rho \rightharpoonup \omega_{m-k}\, \mathcal{H}^k \llcorner K, \tag{7}$$

lorsque ρ tend vers 0 où $\mathcal{H}^l$ désigne la mesure de Hausdorff l-dimensionnelle .

La démonstration de ce résultat est basée sur le calcul de la courbure moyenne d'un tube géodésique perturbé $T_\rho(K, w, \Phi)$ qui est décrit par

$$(p, x) \in SNK \longrightarrow \mathrm{Exp}_p\, \left(\rho\, (1 - w(p, x))\, \Theta(x) - \Phi(p)\right) \tag{8}$$

où w est une fonction définie sur SNK, le fibré normal en sphères sur K, et Φ est une section de NK, le fibré normal sur K. En fait, Φ permet de décrire localement toutes les sous variétés proches de K (rôle joué par le point q proche du point p dans la démonstration de R. Ye). Quant à la fonction w, elle permet essentiellement de décrire les hypersurfaces proches du tube géodésique de rayon ρ centré sur la sous variété proche de K.

Le développement limité de la courbure moyenne de l'hypersurface paramétrée par (8) en puissances de ρ, w et de Φ est donné par

$$\begin{aligned} \sigma_1(S_\rho(K,w,\Phi)) &= \frac{m-k}{\rho} - B(\Theta,\Theta)\,\rho + \mathcal{O}(\rho^2) \\ &+ \left(\rho\,\Delta_K + \tfrac{1}{\rho}(\Delta_{S^{m-k}} + m - k)\right)\, w + g(\mathfrak{J}_K\,\Phi,\Theta) \\ &+ \rho\,L(w,\Phi) + \tfrac{1}{\rho}\,Q(w,\Phi) \end{aligned}$$

où B est une forme quadratique sur NK et $\mathfrak{J}_K$ est l'opérateur de Jacobi associé à K qui apparaît dans la variation seconde du volume [15]

$$\mathfrak{J}_K := -\,\Delta_K^N - \mathfrak{B}_K + \mathfrak{R}^N$$

où Δ_K^N est le Laplacien sur NK, $\mathfrak{B}_K$ est un potentiel calculé à partir de la deuxième forme fondamentale de K et $\mathfrak{R}^N$ est une contraction du tenseur de courbure associé à la connexion sur NK. Cet opérateur est un opérateur elliptique que nous avons supposé être inversible (hypothèse de non dégénérescence de K).

La résolution de l'équation $\sigma_1 = \frac{m-k}{\rho}$ se traduit par la recherche d'une fonction w et d'un champ de vecteurs Φ solutions de

$$\begin{aligned} &\left(\rho\,\Delta_K + \tfrac{1}{\rho}(\Delta_{S^{m-k}} + m - k)\right)\, w + g(\mathfrak{J}_K\,\Phi,\Theta) \\ &\qquad = B(\Theta,\Theta)\,\rho - \mathcal{O}(\rho^2) - \rho\,L(w,\Phi) - \tfrac{1}{\rho}\,Q(w,\Phi). \end{aligned} \tag{9}$$

On décompose toute fonction v définie sur SNK en

$$v = \rho\,w + g(\Phi,\Theta)$$

où, pour tout $p \in K$

$$w(p,\cdot)\ \perp\ \mathrm{Ker}(\Delta_{S^{m-k}} + m - k)$$

et Φ est une section de NK. Cette décomposition revient à considérer, pour tout $p \in K$, la décomposition de $w(p,\cdot)$ sur les fonctions propres de $\Delta_{S^{m-k}}$ comme cela avait déjà été fait dans le §1. On considère alors l'opérateur

$$\mathbb{L}_\rho\,(\rho\,w + g(\Phi,\Theta)) = \left(\rho\,\Delta_K + \frac{1}{\rho}(\Delta_{S^{m-k}} + m - k)\right)\, w + g(\mathfrak{J}_K\,\Phi,\Theta).$$

Le spectre de cet opérateur est la réunion d'une part de l'ensemble des

$$\Lambda_{ij} := \lambda_i + \frac{1}{\rho^2}\,(\mu_j - m + k),$$

où λ_i sont les valeurs propres de Δ_K et μ_j sont les valeurs propres de $\Delta_{S^{m-k}}$ et d'autre part de l'ensemble des valeurs propres de l'opérateur de Jacobi $\mathfrak{J}_K$ (qui sont non nulles car la sous variété K est supposée non dégénérée).

Remarquons que

$$\Lambda_{i0} := \lambda_i - \frac{1}{\rho^2}\,(m-k) = 0 \qquad \text{si} \qquad \rho = \sqrt{\frac{m-k}{\lambda_i}}. \tag{10}$$

En particulier $\mathbb{L}_\rho$ n'est pas inversible pour ces valeurs de ρ. C'est là l'origine du phénomène de résonance dont nous avons déjà parlé et qui nous empêchera d'entreprendre la résolution de (9) pour toutes les (petites) valeurs de ρ. Dans le cas où

$$\rho \notin \left\{ \sqrt{\frac{m-k}{\lambda_i}} \quad : \quad i \geq 1 \right\}, \tag{11}$$

on peut tout de même évaluer la distance entre 0 et le spectre de $\mathbb{L}_\rho$, ce qui permet d'obtenir une estimation sur la norme de $(\mathbb{L}_\rho)^{-1}$ (disons en tant qu'opérateur de L^2 dans L^2). Malheureusement, dès que $k \geq 2$, des calculs formels montrent que la perturbation $\rho\, L$ qui apparaît dans (9) n'est pas assez petite pour pouvoir considérer $\mathbb{L}_\rho + \rho\, L$ comme une perturbation de $\mathbb{L}_\rho$. De plus, toujours quand $k \geq 2$, le terme d'erreur $B(\Theta,\Theta)\,\rho - \mathcal{O}(\rho^2)$ est bien trop grand et la résolution de (9) semble alors compromise.

Afin de contourner cette difficulté, nous avons utilisé une stratégie mise au point dans un contexte différent par A. Malchiodi and M. Montenegro [26].

Dans un premier temps, nous "améliorons la solution approchée" de notre problème en utilisant un schéma itératif. Supposons que (11) soit vérifiée, nous pouvons alors définir par récurrence (w_i, Φ_i), $i \geq 0$, comme étant la solution de

$$\begin{aligned} &\rho\,\Delta_K w_{i+1} + \tfrac{1}{\rho}(\Delta_{S^{m-k}} + m - k)\, w_{i+1} + g(\mathfrak{J}_K\, \Phi_{i+1}, \Theta) \\ &\qquad = B(\Theta,\Theta)\,\rho - \mathcal{O}(\rho^2) - \rho\, L(w_i, \Phi_i) - \tfrac{1}{\rho}\, Q(w_i, \Phi_i) - \rho\,\Delta_K w_i \end{aligned}$$

avec par exemple $w_0 = 0$, $\Phi_0 = 0$. Malheureusement, comme nous l'avons mentionné ci-dessus, des calculs formels montrent que la norme de l'inverse de l'opérateur qui apparaît à gauche est trop grande pour assurer des estimations raisonnables sur le couple

(w_i, Φ_i). L'idée est alors de remplacer le schéma ci-dessus par

$$\begin{aligned} &\tfrac{1}{\rho}(\Delta_{S^{m-k}} + m - k)\, w_{i+1} + g(\mathfrak{J}_K\, \Phi_{i+1}, \Theta) \\ &\quad = B(\Theta,\Theta)\,\rho - \mathcal{O}(\rho^2) - \rho\, L(w_i, \Phi_i) - \tfrac{1}{\rho}\, Q(w_i,\Phi_i) - \rho\,\Delta_K w_i. \end{aligned} \tag{12}$$

L'avantage de ce schéma itératif réside dans le fait que l'on peut construire un "inverse à droite" borné pour l'opérateur qui apparaît à gauche au sens suivant : Étant donnée une fonction $\rho\, z + g(\Psi, \Theta)$ définie sur SNK où $z(p,\cdot)$ est $L^2(S^{m-k})$-orthogonal à Ker $(\Delta_{S^{m-k}} + m - k)$ et où Ψ est une section de NK, nous commençons par résoudre

$$\mathfrak{J}_K\, \Phi = \Psi$$

en utilisant la non dégénérescence de l'opérateur $\mathfrak{J}_K$. Ensuite, nous résolvons

$$\frac{1}{\rho}\,(\Delta_{S^{m-k}} + m - k)\, w(p,\cdot) = z(p,\cdot)$$

pour chaque $p \in K$ (Nous considérons p comme un paramètre). Nous obtenons ainsi un opérateur

$$\rho\, z + g(\Psi,\Theta) \in L^2(SNK) \longrightarrow \rho\, w + g(\Phi,\Theta) \in L^2(SNK)$$

dont la norme est bornée indépendamment de ρ et qui est un inverse à droite pour l'opérateur

$$\tilde{\mathbb{L}}_\rho(\rho\, w + g(\Phi,\Theta)) := \frac{1}{\rho}(\Delta_{S^{m-k}} + m - k)\, w + g(\mathfrak{J}_K\, \Phi, \Theta)$$

qui apparaît à gauche de (12). Malheureusement, cet "inverse à droite" ne permet pas de "gagner" de régularité suivant K car l'opérateur $\tilde{\mathbb{L}}_\rho$ est clairement non elliptique. En particulier nous ne pouvons pas utiliser cet "inverse à droite" afin de résoudre (9) en utilisant un argument de point fixe.

Toutefois, nous pouvons utiliser cet "inverse à droite" dans le schéma itératif un nombre fini de fois étant donné que l'erreur $B(\Theta,\Theta)\,\rho - \mathcal{O}(\rho^2)$ qui apparaît dans (11), est une fonction $\mathcal{C}^\infty$. On obtient alors les estimations

$$w_i = \mathcal{O}(\rho^2) \qquad \text{et} \qquad \Phi_i = \mathcal{O}(\rho^2)$$

et la courbure moyenne de $T_\rho(K, w_i, \Phi_i)$ est alors donnée par

$$\sigma_1(T_\rho(K, w_i, \Phi_i)) = \frac{m-k}{\rho} + \mathcal{O}(\rho^{2+i})$$

en fonction de $i \geq 1$.

Il reste maintenant à déterminer $w = w_i + \tilde{w}$ et $\Phi = \Phi_i + \tilde{\Phi}$ de telle sorte que $\sigma_1(T_\rho(K, w, \Phi)) = \frac{m-k}{\rho}$. Autrement dit, nous devons résoudre une équation non linéaire de la forme

$$\begin{aligned}\Big(\rho\,\Delta_K + \tfrac{1}{\rho}(\Delta_{S^{m-k}} + m - k)\Big)\ \tilde{w} + g(\mathfrak{J}_K\,\tilde{\Phi}, \Theta) + \rho\,L_i(\tilde{w}, \tilde{\Phi}) \\ = \mathcal{O}(\rho^{2+i}) - \tfrac{1}{\rho}\,Q_i(\tilde{w}, \tilde{\Phi})\end{aligned}$$

La résolution de cette équation par un argument de point fixe pour les applications contractantes ne pose aucun problème si nous pouvons démontrer que, pour certaines valeurs de ρ, l'inverse de l'opérateur qui apparaît dans le membre de gauche a une norme bornée par une puissance (indépendante de i) de $1/\rho$. Pour ce faire nous avons estimé la distance entre 0 et le spectre de l'opérateur auto adjoint

$$\mathbb{L}_{\rho,i}\,\tilde{v} := (\rho\,\Delta_K + \frac{1}{\rho}(\Delta_{S^{m-k}} + m - k))\,\tilde{w} + g(\mathfrak{J}_K\,\tilde{\Phi}, \Theta) + \rho\,L_i(\tilde{w}, \tilde{\Phi}), \qquad \tilde{v} = \rho\,\tilde{w} + g(\tilde{\Phi}, \Theta),$$

en utilisant la formule de Weyl pour estimer le nombre de valeurs propres de $\mathfrak{J}_K$ qui sont plus petites que λ

$$\#\{j \in \mathbb{N}\,:\,\lambda_j \leq \lambda\} \sim \lambda^{-\frac{k}{2}}$$

nous obtenons une estimation de l'indice (nombre de valeurs propres négatives) de l'opérateur $\mathbb{L}_{\rho,i}$

$$\text{Index}\,\mathbb{L}_{\rho,i} \sim c\,\rho^{-k} \tag{13}$$

Ensuite, nous obtenons une minoration de la dérivée par rapport à ρ des petites valeurs propres de $\mathbb{L}_{\rho,i}$

$$\rho\,\partial_\rho \sigma \geq 2\,(m-k) - c\,\rho, \tag{14}$$

Cette estimation est basée sur un résultat de localisation des fonctions propres associées aux petites valeurs propres de $\mathbb{L}_{\rho,i}$ qui stipule que, si $\mathbb{L}_{\rho,i}\,\tilde{v} = -\lambda\,\tilde{v}$ et λ est petite, alors $\tilde{v}$ est essentiellement une fonction définie sur K.

L'estimation (14) nous permet de montrer que pour ρ assez petit, les petites valeurs propres sont des fonctions croissantes de ρ, ce qui montre que l'indice de $\mathbb{L}_{\rho,i}$ est une fonction monotone décroissante de ρ. Nous utilisons alors (13) pour montrer que, étant donné $t \geq 2$, les intervalles $[r_1, r_2] \subset (\rho, 2\rho)$ tels que

$$r_2 - r_1 \geq \rho^{k+t}$$

et pour lesquels $\mathbb{L}_{r,i}$ n'a pas de noyau pour tout $r \in (r_1, r_2)$ recouvre $(\rho, 2\rho)$ à l'exception d'un ensemble de mesure majorée par une constante fois ρ^t.

En utilisant ce dernier fait ainsi que (14), nous obtenons alors une minoration de la distance entre 0 et le spectre de $\mathbb{L}_{\rho,i}$, puis une estimation uniforme pour la norme

$$(\mathbb{L}_{\rho,i})^{-1} : L^2(SNK) \longrightarrow L^2(SNK)$$

par une constante fois ρ^{k+t} lorsque $r \in (r_1 + \rho^{k+t+1}, r_2 - \rho^{k+t+1})$. Cette estimation qui ne dépend pas de i (si ρ est choisi assez petit) est précisement l'estimation recherchée qui permet de résoudre (9) en utilisant un argument de point fixe pour les applications contractantes.

La généralisation de ce résultat aux autres courbures est l'objet de la section §2.8 du chapitre 2 de cette thèse. Le résultat principal de cette partie est le :

Theorem 0.1.4 *On suppose que $1 \leq r < m - k$ et que K^k est une sous variété minimale non dégénérée. Alors, il existe $I \subset (0, +\infty)$, réunion dénombrable d'intervalles non vides disjoints, tel que pour tout $\rho \in I$, le tube géodésique $\bar{T}_\rho(K)$ peut être perturbé en une hypersurface T_ρ dont la r-courbure moyenne est constante égale à $\sigma_r = C^r_{n-k}\, \rho^{-r}$.*

La démonstration est essentiellement la même que dans le cas où $r = 1$, néanmoins un nouveau phénomène de perte d'ellipticité apparaît ici quand $r \geq m - k$. En effet, cette fois ci, l'opérateur $\mathbb{L}_\rho$ est remplacé par

$$\mathbb{L}_\rho(\rho\, w + g(\Phi, \Theta))\, \tilde{v} := C^{r-1}_{m-k} \left(\rho\, \Delta_K + \frac{m-k-r}{m-k-1} \left(\Delta_{S^{m-k}} + m - k \right) \right) w + g(\mathfrak{J}_K\, \Phi, \Theta)$$

La condition $r < m - k$ permet précisément de garantir l'ellipticité de cet opérateur.

0.1.3 Autres problèmes non-linéaires présentant le même phénomène de résonnance

Nous décrivons brièvement des résultats récents pour lesquels un phénomène de résonance associé à un phénomène de concentration a été mis en évidence.

Par exemple, A. Malchiodi [22] a étudié l'existence des solutions périodiques de l'équation

$$\ddot{x} + \frac{1}{\varepsilon^2}\, V(x) = 0, \qquad x \in \mathbb{R}^n \tag{15}$$

pour $\varepsilon > 0$, ici $V : \mathbb{R}^n \to \mathbb{R}$ est une fonction régulière dont l'ensemble des points critiques est une hypersurface $M \subset \mathbb{R}^n$. Il a distingué deux cas suivant la nature du potentiel V : lorsque V est de type répulsif par rapport à M, i.e.

$$V''(x)\, (n_x, n_x) < 0 \qquad \forall x \in\, M, \quad 0 \neq n_x \perp T_x M$$

il a démontré que toute géodésique fermée non dégénérée $x_0 : S^1 \to\, M$ est limite, quand $\varepsilon > 0$ tend vers 0, d'une famille à un paramètre u_ε de solutions de l'équation (15). En particulier, il n'y a aucune restriction sur le paramètre ε. Dans le cas attractif

$$V''(x)\, (n_x, n_x) > 0 \qquad \forall x \in\, M \quad , 0 \neq n_x \perp T_x M$$

la situation est radicalement différente étant donnée l'existence d'un phénomène de résonance et, sous des hypothèses techniques, A. Malchiodi a démontré l'existence d'une suite $(\varepsilon_k)_k$ qui tend vers 0 et une suite de solutions de l'équation (15) qui converge vers x_0 quand k tend vers $+\infty$.

De même, J. Shatah et C. Zeng [38] ont considéré l'équation

$$D_t \dot{p} + \Pi_p \left(w'(p) \right) = 0 \text{ sur } M, \tag{16}$$

où M est une sous variété de dimension k plongée dans $\mathbb{R}^{m+1}$, Π_p est la projection orthogonale de $T_p\mathbb{R}^{m+1}$ sur T_pM et D_t est la dérivée covariante sur M dans la direction de $\dot{p}$. Le problème est de démontrer que les solutions périodiques de l'équation (16) sont des limites d'une suite de solutions périodiques de l'équation pénalisée :

$$\ddot{x} + w'(x) + \frac{1}{\varepsilon^2} G'(x) = 0 \tag{17}$$

où cette fois ci x est une courbe de $\mathbb{R}^{m+1}$ et où le potentiel de pénalisation $G = \text{dist}(\cdot, M)^2$ au voisinage de M. J. Shatah et C. Zeng [38] ont montré qu'étant donnée p_0 une solution périodique non dégénérée de l'équation (16), il existe une suite $(\varepsilon_k)_k$ qui tend vers 0 et une suite de solutions périodiques x_k de l'équation (17) avec $\varepsilon = \varepsilon_k$ qui converge vers p_0. Comme dans [22], l'existence des solutions ne peut pas être prouvée pour tout $\varepsilon > 0$ et ceci est expliqué par la présence d'un phénomène de résonance correspondant à des valeurs de ε pour lesquelles l'opérateur (17) linéarisé en p_0 admet un noyau non trivial. En effet, si on recherche formellement les solutions de (17) comme perturbations des solutions de (16), i.e. des solutions de la forme

$$x = p_0 + y^t + y^n$$

où p_0 est solution de l'équation (16), y^n et y^t désignent des perturbations normales et tangentes à M, l'opérateur linéarisé associé à (17), projeté sur le fibré normal s'écrit

$$L(y^n) = \ddot{y}^n + A(y^n) + \frac{1}{\varepsilon^2} y^n$$

et les modes résonnants correspondent aux valeurs de ε vérifiant

$$\frac{1}{\varepsilon^2} = \lambda_j$$

où les λ_j sont les valeurs propres de l'opérateur $y^n \longmapsto \ddot{y}^n + A(y^n)$.

Dans le domaine des perturbations singulières d'équations aux dérivées partielles semi linéaires, A. Malchiodi et M. Montenegro [25] ont démontré un résultat de concentration pour les équations de la forme :

$$-\varepsilon^2 \Delta u + u = u^p \qquad p \geq 2 \tag{18}$$

où la fonction u est définie dans un domaine borné régulier Ω de $\mathbb{R}^n$ et vérifie $\partial_\nu u = 0$ sur le bord de Ω. Ils démontrent, pour une suite $\varepsilon_n \to 0$, l'existence d'une solution positive qui se concentre le long de $\partial\Omega$ [25], [26] ou bien le long d'une géodésique non dégénérée de $\partial\Omega$ [23]. Comme dans les résultats précédents, la construction n'est possible que pour une suite ε_n qui tend vers 0 et pas pour tout ε assez petit, ceci est une fois de plus dû à la présence d'un phénomène de résonance.

Dans [7], M. Del Pino, M. Kowalczyk et J. Wei ont étudié l'existence de solutions pour l'équation de Schrödinger non linéaire :

$$-i\,\varepsilon\,\frac{\partial\psi}{\partial t} = \varepsilon^2\,\Delta\psi - Q(y)\,\psi + \psi\,|\psi|^p \tag{19}$$

dans $\mathbb{R}^2$, où $p > 1$. Si l'on recherche ψ sous la forme $\psi(t,y) = \exp(-i\,\lambda\,t/\varepsilon)\,u(y)$, alors u est solution de l'équation semi linéaire

$$\varepsilon^2\,\Delta u - (Q(y)+\lambda)\,u + u^p = 0, \quad u > 0, \tag{20}$$

M. Del Pino, M. Kowalczyk et J. Wei ont démontré qu'étant donné une courbe Γ stationnaire, non dégénérée relativement à la fonctionnelle

$$\Gamma \longrightarrow \int_\Gamma V^{\frac{p+1}{p-1}-\frac{1}{2}}\,d\gamma$$

et si $Q+\lambda$ est une fonction uniformément positive, alors pour tout $c > 0$, il existe $\varepsilon_0 > 0$ et $\lambda_* > 0$ telles que, pour tout $\varepsilon < \varepsilon_0$ vérifiant $|\varepsilon^2\,j^2 - \lambda_*| \geq c\varepsilon, \forall\, j \in \mathbb{N}$, l'équation (20) admet une solution positive u_ε qui se concentre le long de Γ. Ce résultat généralise un résultat partiel qui avait été obtenu par A. Ambrosetti, A. Malchiodi et W.M. Ni [2] dans le cas des potentiels V ne dépendant que de la distance à l'origine.

0.1.4 Hypersurfaces de "type Delaunay" dans quelques variétés Riemanniennes

Dans le chapitre 3, on s'intéresse à l'existence d'hypersurfaces de type Delaunay dans quelques variétés Riemanniennes. Commençons par définir, pour tout $m \geq 2$ le paramètre

$$\tau_m := \frac{1}{m}(m-1)^{\frac{m-1}{m}}$$

Il existe, dans $\mathbb{R}^{m+1}$, une famille à un paramètre $S_{\tau,\rho}$, avec $\rho > 0$ et $\tau \in (0,\tau_m]$, d'hypersurfaces à courbure moyenne constante qui sont de révolution autour de l'axe x_{m+1}. Ces hypersurfaces généralisent en toute dimension $m \geq 2$ les surfaces de Delaunay [12]. On peut paramétrer ces hypersurfaces par

$$X(s,\Theta) := \rho\,(\,\tau\,e^{\sigma(s)}\,\Theta,\;\kappa(s)),$$

où $(s,\Theta) \in \mathbb{R} \times S^{m-1}$. Lorsque $\tau \in (0,\tau_m)$, la fonction σ est une solution non constante de

$$(\partial_s \sigma)^2 + \tau^2 (e^\sigma + e^{(1-m)\sigma})^2 = 1,$$

qui vérifie $\sigma(0) < 0$ et $\partial_s \sigma(0) = 0$ alors que la fonction κ est définie par

$$\partial_s \kappa = \tau^2 e^\sigma \, (e^\sigma + e^{(1-m)\sigma}),$$

avec $\kappa(0) = 0$. Lorsque $\tau = \tau_m$ les hypersurfaces correspondantes sont en fait des cylindres droits $S^{m-1}(\frac{m-1}{m}\rho) \times \mathbb{R}$, alors que, quand τ tend vers 0, ces hypersurfaces convergent vers une suite de sphères de rayon ρ qui sont tangentes et centrées sur l'axe x_{m+1}. Ces hypersurfaces sont invariantes par rapport à un groupe discret de translations $t_\tau \, e_{m+1} \, \mathbb{Z}$ le long de l'axe x_{m+1} où $t_\tau > 0$ est la plus petite période verticale de l'hypersurface.

On vérifie qu'il existe une fonction régulière $\tau \longrightarrow c(\tau)$ telle que

$$\rho^{1-m} \, \mathcal{H}^m \llcorner S_{\tau,\rho} \rightharpoonup c(\tau) \, \mathcal{H}^1 \llcorner K, \tag{21}$$

De plus $c(\tau)$ tend vers $\frac{1}{2}|S^m|$ quand τ tend vers 0 et $c(\tau_m) = (\frac{m-1}{m})^{m-1} \, |S^{m-1}|$.

L'objet de ce dernier chapitre est de démontrer l'existence d'hypersurfaces qui, quand leur courbure moyenne tend vers $+\infty$, se concentrent le long d'une géodésique mais pour lesquelles (21) reste valable. Ces hypersurfaces généralisent dans les variétés riemannienne les surfaces de Delaunay. Dans des cas particulier, les sphères, l'espace hyperbolique, ... l'existence de ces hypersurfaces était déjà connue par d'autres méthodes.

On suppose que K est une géodésique compacte de (M^{m+1}, g). On définie un système de coordonnées dans un voisinage tubulaire de K pour lequel on suppose que les coefficients de la métrique ne dépendent pas du point sur K. C'est, par exemple, le cas lorsqu'on considère les métriques de la forme $g = A(x') \, dx_0 + g_{x'}$ sur $M^{m+1} = S^1 \times N^m$. On note ℓ la longueur de K. On suppose que la géodésique K est τ-non dégénérée, c'est à dire que l'on suppose que l'opérateur

$$\mathfrak{J}_\tau \Phi := \nabla^2_{E_0} \Phi + \alpha_\tau \, R(\Phi, X_0) \, X_0$$

défini sur le fibré normal à K est inversible. Ici X_0 est le vecteur unitaire tangent à K et $\alpha_\tau \in \mathbb{R}$ est une constante définie dans le §3 du Chapitre 3. Sous ces hypothèses, nous avons obtenu le :

Theorem 0.1.5 *On suppose que K est τ-non dégénérée. Il existe un ensemble fini $T \subset (0,\tau_m)$ ($T = \emptyset$ lorsque $m = 2$) tel que, pour tout $\tau \in (0,\tau_m) - T$, il existe $i_\tau \in \mathbb{N}$ tel que, pour tout $i \in \mathbb{N}$, $i \geq i_\tau$, si nous définissons $\rho_i > 0$ par*

$$i \, t_\tau \, \rho_i = \ell$$

alors, il existe une hypersurface $\Sigma_{\tau,i}$ dont la courbure moyenne est constante égale à $\sigma = \frac{1}{\rho_i}$ et telle que

$$\rho_i^{1-m}\,\mathcal{H}^m \llcorner \Sigma_{\tau,i} \rightharpoonup c(\tau)\,\mathcal{H}^1 \llcorner K,$$

quand i tend vers $+\infty$.

La constante α_τ est une fonction continue en τ qui converge vers 1 quand τ tend vers τ_m et l'opérateur

$$\mathfrak{J}_{\tau_m}\Phi := \nabla^2_{E_0}\Phi + R(\Phi, X_0)\,X_0$$

est n'est autre que l'opérateur de Jacobi associé à K. En particulier, toute géodésique non dégénérée (au sens usuel) est τ-non dégénérée pour τ proche de τ_m.

Comme dans §3, nous considérons Φ une section de NK et w une fonction définie sur SNK. Pour φ et κ solutions de

$$(\partial_s\varphi)^2 + (\varphi^2 + \tau^m\,\varphi^{2-m})^2 = \varphi^2,$$

et

$$\partial_s\kappa = \varphi^2\,(1 + \tau^m\,\varphi^{-m}),$$

avec $\varphi(0) < \tau$, $\partial_s\varphi(0) = 0$ et $\kappa(0) = 0$, nous définissons (pour tout $\rho > 0$ assez petit) l'hypersurface de Delaunay perturbée :

$$G(s, x') = \mathrm{Exp}^M_{\gamma(\rho\,\kappa(s))}\left(\,\rho\,(\varphi(s) - w(s,x'))\,\Theta(x') - \Phi(\rho\,\kappa(s))\,\right)$$

et nous souhaitons déterminer w et Φ de telle sorte que sa courbure moyenne soit constante. Cette fois ci, aucun phénomène de résonance n'apparaît. Néanmoins, une difficulté supplémentaire surgit. En effet, l'opérateur de Jacobi associé à une hypersurface de Delaunay admet deux champs de Jacobi qui sont invariant par rotation. Un premier champ de Jacobi est associé à la famille à 1-paramètre d'hypersurfaces à courbure moyenne constante obtenue en translatant l'hypersurface de Delaunay le long de son axe. Ce champ de Jacobi est périodique et une fois transporté dans notre contexte Riemannien, il induit une valeur propre proche de 0 pour l'opérateur linéarisé autour de la solution approchée. C'est pour pouvoir s'affranchir de cette difficulté que nous avons supposé que les coefficients de la métrique ne dépendaient pas du point sur K. Un second champ de Jacobi est associé à la famille à 1-paramètre d'hypersurfaces à courbure moyenne constante obtenue en modifiant le paramètre de Delaunay τ, il est obtenu en écrivant pour η assez petit l'hypersurface de Delaunay $S_{\eta+\tau}$ comme un graphe normal sur S_τ pour une fonction w_η et en différentiant w_η par rapport à η en $\eta = 0$ (c.f [12] pour plus de détails). Il est démontré dans [12] que ce champ de Jacobi n'est en général pas périodique (i.e. ne peut être périodique que quand $n \geq 3$ et pour au plus un nombre fini de valeurs de τ).

Ce résultat montre qu'il n'existe pas de résultat de quantification de l'énergie (la densité de volume) pour les hypersurfaces à courbure moyenne constante.

Ce résultat est à comparer aux résultats de quantification qui ont été obtenus pour d'autres problèmes non linéaire comme les applications harmoniques en dimension $n \geq 3$ [17], les équations de Yang-Mills en dimension $n \geq 5$ [34].

Chapitre 1

Constant r-curvature hypersurfaces in riemannian manifolds

Abstract In [42], Rugang Ye proved the existence of family of constant mean curvature hypersurfaces in an $m+1$-dimensional Riemannian manifold (M^{m+1}, g), which concentrate at a point p_0 (which is required to be a nondegenerate critical point of the scalar curvature), moreover he proved that this family constitute a foliation of a neighborhood of p_0. In this chapter we extend this result to the other curvatures (the r-th mean curvature for $1 \leq r \leq m$) and we give the expansion of the m-dimensional volume of the leaves of this foliation as well as the $m+1$-dimensional volume of the sets enclosed by each leaf.

1.1 Introduction

Let S be an oriented embedded (or possibly immersed) hypersurface in a Riemannian manifold (M^{m+1}, g). The shape operator A_S is the symmetric endomorphism of the tangent bundle of S associated with the second fundamental form of S, b_S, by

$$b_S(X,Y) = g_S(A_S\, X, Y), \quad \forall\, X, Y \in TS; \qquad \text{here} \qquad g_S = g|_{TS}\,.$$

The eigenvalues κ_i of the shape operator A_S are the principal curvatures of the hypersurface S. The r-curvature of S is define to be the r-th symmetric function of the principal curvatures of S, i.e.

$$\sigma_r(S) := \sum_{i_1 < \ldots < i_r} \kappa_{i_1} \ldots \kappa_{i_r}.$$

Hence, when $r = 1$, σ_1 is equal to m times the mean curvature of S. When $(M^{m+1}, g) = (\mathbb{R}^{m+1}, g_{eucl})$ is the Euclidean space, σ_2 is equal to $\frac{m(m-1)}{2}$ times the scalar curvature of S and σ_m is equal to the Gauss-Kronecker curvature of S. In this chapter we are interested in the existence of hypersurfaces in M^{m+1} whose r-curvature is constant. Hypersurfaces with constant mean curvature, constant scalar curvature or constant Gauss-Kronecker curvature in Euclidean space or space forms constitute an important class of submanifolds. In Riemannian manifolds very few examples of constant r-curvature hypersurfaces are known, except when $r = 1$.

R. Ye [42], [43] has proved the existence of a local foliation by constant mean curvature hypersurfaces which concentrate at a point (which is required to be a nondegenerate critical point of the scalar curvature function). We extend the result and methods of [42] to handle the case $r = 2, \ldots, m$. No extra curvature hypotheses are required. In particular, we prove the existence of foliations of a neighborhood of any nondegenerate critical point of the scalar curvature of (M^{m+1}, g) by constant Gauss-Kronecker or constant scalar curvature hypersurfaces. As in [42] the idea is to perturb $\bar{S}_\rho(p)$, a geodesic sphere with small radius $\rho > 0$ centered at a point p. A simple computation will show that $\bar{S}_\rho(p)$ is close to being a constant r-curvature hypersurface as ρ tends to 0 and in fact

$$\sigma_r(\bar{S}_\rho(p)) = C_m^r \, \rho^{-r} + \mathcal{O}(\rho^{-r+2}),$$

In this note, we show that it is possible to perturb $\bar{S}_\rho(p)$ for every small radius provided p is close to a nondegenerate critical point of the scalar curvature of M. The analysis follows exactly the analysis performed in [42]. In fact, independently of the value of r, the linearized r-curvature operator about the unit Euclidean sphere is always a multiple of $\Delta_{S^m} + m$, the linearized mean curvature operator about the unit Euclidean sphere. This implies that, as in [42], to perform the perturbation of a small geodesic sphere, one has to overcome the problem of the existence of $(m+1)$-dimensional kernel of $\Delta_{S^m} + m$, kernel which is related to the invariance of r-curvature with respect to the action of isometries (in the case of the unit sphere, this kernel is only generated by translations). This is where, as in [42] we use the fact that we are close to a nondegenerate critical point of the scalar curvature of the ambient manifold.

Even if there is no additional difficulty to treat the general case $r = 2, \ldots, m$, the analysis of [42] is specific to treat the case of mean curvature and, unfortunately, can't be used to treat the general case. The main technical result of this note is a precise expansion of geometric operators (first and second fundamental forms) for perturbed geodesic sphere (see Proposition 2.2.1, Proposition 2.3.1 and Proposition 2.3.3). We believe that these expansions are of independent interest and can be used in many other construction [24]. Our main result is :

Theorem 1.1.1 *Suppose that p_0 is a nondegenerate critical point of the scalar curvature $\mathcal{R}$ of M. Then there exists $\rho_0 > 0$, such that for all $\rho \in (0, \rho_0)$, the geodesic sphere $\bar{S}_\rho(p_0)$ may be perturbed to a constant r-curvature hypersurface S_ρ with $\sigma_r = C_m^r \, \rho^{-r}$. Moreover these r-curvature hypersurfaces constitute a local foliation of a neighborhood of p_0.*

The existence of the hypersurfaces is not so difficult and can be obtained rather easily. The fact that they constitute a local foliation requires more work. The leaves S_ρ are small perturbation of geodesic spheres in the sense that S_ρ is a normal graph over $\bar{S}_\rho(p_0)$ for some function $\bar{w}_\rho$ which is bounded by a constant times ρ^2.

The hypersurface S_ρ is a small perturbation of $\bar{S}_\rho(p_0)$ in the sense that it is the normal graph of some function (with L^∞ norm bounded by a constant times ρ^2) over a geodesic sphere obtained centered at a point at distance bounded by a constant times ρ^2 of p_0.

1.2 Expansion of the metric in geodesic normal coordinates

We now introduce geodesic normal coordinates in a neighborhood of a point $p \in M$. We choose an orthonormal basis E_i, $i = 1, \ldots, m+1$, of T_pM.

Consider, in a neighborhood of p in M, normal geodesic coordinates

$$F(x) := \exp_p^M(x_i \, E_i), \qquad x := (x_1, \ldots, x_{m+1}),$$

where $\exp^M$ is the exponential map on M and summation over repeated indices is understood. This yields the coordinate vector fields $X_i := F_*(\partial_{x_i})$. As usual, the Fermi coordinates above are defined so that the metric coefficients

$$g_{ij} = g(X_i, X_j)$$

equal δ_{ij} at p. We now compute higher terms in the Taylor expansions of the functions g_{ij}. The metric coefficients at $q := F(x)$ are given in terms of geometric data at $p := F(0)$ and $|x| := (x_1^2 + \ldots + x_{m+1}^2)^{1/2}$.

Notation The symbol $\mathcal{O}(|x|^r)$ indicates an analytic function such that it and its partial derivatives of any order, with respect to the vector fields $x^j \, X_i$, are bounded by a constant times $|x|^r$ in some fixed neighborhood of 0.

We now give the well known expansion for the metric in normal coordinates [39], [16], [46], but we briefly recall the proof in the Appendix for completeness.

Proposition 1.2.1 *At the point $q = F(x)$, the following expansions hold*

$$g_{ij} = \delta_{ij} + \frac{1}{3}\, g(R(E_k, E_i)\, E_\ell, E_j)\, x_k\, x_\ell + \frac{1}{6}\, g(\nabla_{E_k} R(E_\ell, E_i) E_s, E_j)\, x_k\, x_\ell\, x_s + \mathcal{O}(|x|^4) \tag{1.1}$$

where all curvature terms are evaluated at p.

1.3 Geometry of spheres

We derive expansions as ρ tends to 0 for the metric, second fundamental form and mean curvature of the sphere $\bar{S}_\rho(p)$ and their perturbations.

Fix $\rho > 0$. We use a local parametrization $z \to \Theta(z)$ of $S^m \subset T_pM$. Now define the map

$$G(z) := F\left(\rho\,(1 - w(z))\,\Theta(z)\right),$$

and denote its image by $S_\rho(p, w)$, so in particular $S_\rho(p, 0) = \bar{S}_\rho(p)$. Because of the definition of these hypersurfaces using the exponential map, various vector fields we shall use may be regarded either as fields along $S_\rho(p, w)$ or as vectors of T_pM. To help allay this confusion, we write

$$\Theta := \Theta^j\, E_j \qquad\qquad \Theta_i := \partial_{z^i}\Theta^j\, E_j.$$

These are all vectors in the tangent space T_pM. On the other hand, the vectors

$$\Upsilon := \Theta^j\, X_j \qquad\qquad \Upsilon_i := \partial_{z^i}\Theta^j\, X_j$$

lie in the tangent space T_qM, where $q = F(z)$. For brevity, we also write

$$w_j := \partial_{z^j} w, \qquad w_{ij} := \partial_{z^i}\, \partial_{z^j} w.$$

In terms of all this notation, the tangent space to $S_\rho(w)$ at any point is spanned by the vectors

$$Z_j = G_*(\partial_{z^j}) = \rho\,((1 - w)\,\Upsilon_j - w_j\,\Upsilon), \qquad j = 1, \ldots, m. \tag{1.2}$$

1.3.1 Notation for error terms

The formulas for the various geometric quantities of $S_\rho(p, w)$ are potentially very complicated, and so it is important to condense notation as much as possible. Fortunately, we do not need to know the full structure of all of these quantities. Because it is so fundamental, we have isolated the notational conventions we shall use in this separate subsection.

Any expression of the form $L^j(w)$ denotes a linear combination of the functions w together with its derivatives with respect to the vector fields Θ_i up to order j. The coefficients are assumed to be smooth functions on S^m which are bounded by a constant independent of $\rho \in (0,1)$ and $p \in M$, in $\mathcal{C}^\infty$ topology.

Similarly, any expression of the form $Q^j(w)$ denotes a nonlinear operator in the functions w together with its derivatives with respect to the vector fields Θ_i up to order j. Again, the coefficients of the Taylor expansion of the corresponding differential operator are smooth functions on S^m which are bounded by a constant independent of $\rho \in (0,1)$ and $p \in M$ in the $\mathcal{C}^\infty$ topology. In addition Q^j vanishes quadratically at $w = 0$.

Finally, any term of the form $L^j \ltimes Q^k$ will denote any finite sum of the product of a linear operators L^j with nonlinear operators Q^k.

We also agree that any term denoted $\mathcal{O}(\rho^d)$ is a smooth function on S^m which is bounded by a constant (independent of p) times ρ^d in the $\mathcal{C}^\infty$ topology.

1.3.2 The first fundamental form

The next step is the computation of the coefficients of the first fundamental form of $S_\rho(p,w)$. We set $q := G(z)$ and $p := G(0)$. We obtain directly from (3.2.4) that

$$\begin{aligned} g(X_i, X_j) &= \delta_{ij} + \tfrac{1}{3}\, g(R(\Theta, E_i)\,\Theta, E_j)\,\rho^2\,(1-w)^2 \\ &+ \tfrac{1}{6}\, g(\nabla_\Theta R(\Theta, E_i)\,\Theta, E_j)\,\rho^3\,(1-w)^3 \\ &+ \mathcal{O}(\rho^4) + \rho^4\, L^0(w) + \rho^4\, Q^0(w). \end{aligned} \tag{1.3}$$

where all the curvature terms are evaluated at p. Observe that we have

$$g(\Upsilon, \Upsilon) \equiv 1 \qquad g(\Upsilon, \Upsilon_j) \equiv 0$$

Using these expansions it is easy to obtain the expansion of the first fundamental form of $S_\rho(p,w)$.

Proposition 1.3.1 *We have*

$$\begin{aligned}\rho^{-2}\,(1-w)^{-2}\,g(Z_i,Z_j) &= g(\Theta_i,\Theta_j)+\tfrac{1}{3}\,g(R(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^2\,(1-w)^2\\ &+ \tfrac{1}{6}\,g(\nabla_\Theta R(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^3\,(1-w)^3+(1-w)^{-2}\,w_i\,w_j\\ &+ \mathcal{O}(\rho^4)+\rho^4 L^0(w)+\rho^4\,Q^0(w).\end{aligned} \tag{1.4}$$

where all curvature terms are evaluated at p.

1.3.3 The normal vector field

Our next task is to understand the dependence on w of the unit normal N to $S_\rho(w)$. Define the vector field

$$\tilde{N} := -\,\Upsilon + A^j\,Z_j,$$

and choose the coefficients A^j so that that $\tilde{N}$ is orthogonal to all of the Z_i. This leads to a linear system for A^j.

$$\sum_j A^j\,g(Z_j,Z_i) = -\rho\,w_i$$

Observe that

$$g(\tilde{N},\tilde{N}) = 1+\rho\,\sum_j A_j\,w_j$$

The unit normal vector field N about $S_\rho(p,w)$ is defined to be

$$N := \frac{\tilde{N}}{g(\tilde{N},\tilde{N})^{1/2}} \tag{1.5}$$

1.3.4 The second fundamental form

We now compute the second fundamental form. To simplify the computations below, we henceforth assume that, at the point $\Theta(z)\in S^m$,

$$g(\Theta_i,\Theta_j)=\delta_{ij} \qquad \text{and} \qquad \overline{\nabla}_{\Theta_i}\Theta_j = 0, \quad i,j=1,\ldots,m \tag{1.6}$$

(where $\overline{\nabla}$ is the connection on TS^{m-1}).

Proposition 1.3.2 *The following expansions hold*

$$\begin{array}{rcl} -g(\nabla_{Z_i} N, Z_j) & = & \rho\,(1-w)\,\delta_{ij} + \rho\, w_{ij} + \frac{2}{3}\, g(R(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^3\,(1-w)^3 \\ & + & \frac{5}{12}\, g(\nabla_\Theta R(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^4\,(1-w)^4 \\ & - & \frac{1}{3}\,\left(g(R(\nabla w,\,\Theta_i)\,\Theta,\Theta_j) + g(R(\Theta,\,\Theta_i)\,\nabla w,\Theta_j)\right)\,\rho^3 \\ & + & \mathcal{O}(\rho^5) + \rho^4\, L^1(w) + \rho\, Q^1(w) + \rho\, L^2(w) \ltimes Q^1(w) \end{array} \tag{1.7}$$

where as usual, all curvature terms are computed at the point p.

Proof : We will first obtain the expansion of $g(\nabla_{Z_i}\tilde{N}, Z_j)$. To this aim, we compute

$$\begin{array}{rcl} -g(\nabla_{Z_i}\tilde{N}, Z_j) & = & g(\nabla_{Z_i}\Upsilon, Z_j) - \sum_k g(\nabla_{Z_i}(A^k\, Z_k), Z_j) \\ & = & \frac{1}{1-w} g(\nabla_{Z_i}((1-w)\,\Upsilon), Z_j) + \frac{1}{1-w}\, w_i\, g(\Upsilon, Z_j) \\ & - & \sum_k g(\nabla_{Z_i}(A^k\, Z_k), Z_j) \\ & = & \frac{1}{1-w} g(\nabla_{Z_i}((1-w)\,\Upsilon), Z_j) - \frac{\rho}{1-w}\, w_i\, w_j \\ & - & \sum_k g(\nabla_{Z_i}(A^k\, Z_k), Z_j) \end{array}$$

Now, recall that

$$\sum_k A^k\, g(Z_k, Z_j) = -\rho\, w_j$$

Hence

$$\sum_k g(\nabla_{Z_i}(A^k\, Z_k), Z_j) = -\rho\, w_{ij} - \sum_k A^k\, g(Z_k, \nabla_{Z_i} Z_j)$$

Using the fact that

$$2\, g(Z_k, \nabla_{Z_i} Z_j) = Z_i\, g(Z_k, Z_j) + Z_j\, g(Z_k, Z_i) - Z_k\, g(Z_i, Z_j)$$

we conclude that

$$\sum_k g(\nabla_{Z_i}(A^k\, Z_k), Z_j) = -\rho\, w_{ij} - \frac{1}{2}\sum_k A^k\, \left(Z_i\, g(Z_k, Z_j) + Z_j g(Z_k, Z_i) - Z_k\, g(Z_i, Z_j)\right)$$

To analyze the term $\nabla_{Z_i}((1-w)\,\Upsilon)$, let us revert for the moment and regard w as functions of the coordinates z and also consider ρ as a variable instead of just a parameter. Thus we consider

$$\tilde{F}(\rho, z) = F\big(\rho(1-w(z))\Theta(z)\big).$$

The coordinate vector fields Z_j are still equal to $\tilde{F}_*(\partial_{z_j})$, but now we also have $Z_0 := (1-w)\,\Upsilon = \tilde{F}_*(\partial_\rho)$, which is the identity we wish to use below. Now, we write

$$\begin{aligned} g(\nabla_{Z_i}((1-w)\,\Upsilon), Z_j) + g(\nabla_{Z_j}((1-w)\,\Upsilon), Z_i) &= g(\nabla_{Z_i} Z_0, Z_j) + g(\nabla_{Z_j} Z_0, Z_i) \\ &= Z_0\, g(Z_i, Z_j) \end{aligned}$$

Collecting the above we have obtained to formula

$$\begin{aligned} -g(\nabla_{Z_i} \tilde{N}, Z_j) &= \tfrac{1}{2(1-w)}\, Z_0\, g(Z_i, Z_j) - \tfrac{1}{1-w}\, \rho\, w_i\, w_j + \rho\, w_{ij} \\ &+ \tfrac{1}{2} \textstyle\sum_k A^k \, \left(Z_i\, g(Z_k, Z_j) + Z_j\, g(Z_k, Z_i) - Z_k\, g(Z_i, Z_j) \right) \end{aligned}$$

We will now expand the first and last term in this expression.

If the coordinates y are chosen so that $g(\Theta_i, \Theta_j) = \delta_{ij}$ at the point where we will compute the shape form, we have, using the result of Proposition 2.3.1,

$$\begin{aligned} \tfrac{1}{2(1-w)}\, Z_0\, g(Z_i, Z_j) &= \rho\,(1-w)\,\delta_{ij} + \tfrac{2}{3}\, g(R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^3\,(1-w)^3 \\ &+ \tfrac{5}{12}\, g(\nabla_\Theta R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^4\,(1-w)^4 + \tfrac{1}{1-w}\,\rho\, w_i\, w_j \\ &+ \mathcal{O}(\rho^5) + \rho^5\, L^0(w) + \rho^5\, Q^0(w). \end{aligned}$$

Using the same Proposition together with the fact that the coordinates y are chosen so that $\bar{\nabla}_{\Theta_i} \Theta_j = 0$ at the point where we will compute the shape form, we also have

$$\begin{aligned} & Z_i\, g(Z_k, Z_j) + Z_j\, g(Z_k, Z_i) - Z_k\, g(Z_i, Z_j) = \\ & \tfrac{2}{3}\, \left(g(R(\Theta_k, \Theta_i)\,\Theta_j, \Theta) + g(R(\Theta_k, \Theta_i)\,\Theta_j, \Theta) \right)\, \rho^4 \\ & + \mathcal{O}(\rho^5) + \rho^2\, L^1(w) + \rho^2\, Q^1(w) + \rho^2\, L^2(w) \ltimes L^1(w) \end{aligned}$$

If the coordinates y are chosen so that $g(\Theta_i, \Theta_j) = \delta_{ij}$ at the point where we will compute the shape form, we have the expansion

$$A^k = -\frac{w_k}{\rho(1-w)^2} + \rho\, L^1(w) + \rho\, Q^1(w)$$

collecting the above estimates, we conclude that

$$\begin{aligned} -g(\nabla_{Z_i} \tilde{N}, Z_j) &= \rho\,(1-w)\,\delta_{ij} + \rho\, w_{ij} + \tfrac{2}{3}\, g(R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^3\,(1-w)^3 \\ &+ \tfrac{5}{12}\, g(\nabla_\Theta R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^4\,(1-w)^4 \\ &- \tfrac{1}{3}\, \left(g(R(\Theta_k,\, \Theta_i)\,\Theta, \Theta_j) + g(R(\Theta,\, \Theta_i)\,\Theta_k, \Theta_j) \right)\, \rho^3\, w_k \\ &+ \mathcal{O}(\rho^5) + \rho^4\, L^1(w) + \rho\, Q^1(w) + \rho\, L^2(w) \ltimes Q^1(w) \end{aligned}$$

It remains to observe that

$$g(\tilde{N}, \tilde{N})^{-1/2} = 1 + Q^1(w)$$

This finishes the proof of the estimate. $\square$

1.3.5 The shape operator of perturbed surfaces

Collecting the estimates of the last subsection we obtain the expansion of the shape operator of the hypersurface $S_\rho(p,w)$. In the coordinate system defined in the previous sections, we get

Proposition 1.3.3 *Under the previous hypothesis, the shape operator of the hypersurface $S_\rho(p,w)$ is given by*

$$\begin{aligned}\rho\, A_{ij}(w) &= (1+w)\,\delta_{ij} + w_{ij} + \tfrac{1}{3}\,g(R(\Theta,\,\Theta_i)\,\Theta,\Theta_j)\,\rho^2 + \tfrac{1}{4}\,g(\nabla_\Theta R(\Theta,\,\Theta_i)\,\Theta,\Theta_j)\,\rho^3 \\ &- \tfrac{1}{3}\,\big[\,g(R(\Theta,\,\Theta_i)\,\Theta,\Theta_j)\,w + \big(g(R(\Theta_k,\,\Theta_i)\,\Theta,\Theta_j) + g(R(\Theta,\,\Theta_i)\,\Theta_k,\Theta_j)\big)\;w_k \\ &\qquad + g(R(\Theta,\Theta_i),\Theta,\Theta_k)\,w_{kj}\big]\;\rho^2 \\ &+ \mathcal{O}(\rho^4) + \rho^3\,L^2(w) + Q^1(w) + L^2(w)\ltimes L^0(w) + L^2(w)\ltimes Q^1(w).\end{aligned} \tag{1.8}$$

where all curvature terms are computed at the point p.

1.4 The r-curvature of the perturbed sphere

Given any symmetric matrix A, and any $r = 0,\ldots,m$, we define

$$\sigma_r(A) := \sum_{i_1<\ldots<i_r} \lambda_{i_1}\ldots\lambda_{i_r}.$$

where $\lambda_1,\ldots,\lambda_m$ are the eigenvalues of A. The r-th Newton transform of A is defined by

$$T_r(A) := \sigma_r(A)\,I - \sigma_{r-1}(A)\,A + \cdots + (-1)^r\,A^r.$$

with $T_m(A) = 0$. Now suppose that $A = A(t)$ depends smoothly on a parameter t, it is proved in [33] that

$$\frac{d}{dt}\sigma_r(A) = \mathrm{Tr}\left(T_{r-1}(A)\,\frac{d}{dt}A\right) \tag{1.9}$$

From this computation, it follows at once that, given any $m\times m$ symmetric matrix H,

$$\sigma_r(I+H) = C^r_m + C^{r-1}_{m-1}\,\mathrm{Tr}(H) + \mathcal{O}(|H|^2)$$

Using this together with the previous expansion of the shape operator, it is not hard to check that the r-curvature of the hypersurface $S_\rho(p,w)$ can be expanded as

$$\begin{aligned}\rho^r\,\sigma_r(S_\rho(p,w)) &= C^r_m + C^{r-1}_{m-1}\,\Big[(\Delta_{S^m}+m)\,w - \tfrac{1}{3}\,\mathrm{Ric}(\Theta,\Theta)\,\rho^2 - \tfrac{1}{4}\,\nabla_\Theta\mathrm{Ric}(\Theta,\Theta)\,\rho^3 \\ &\quad + \tfrac{1}{3}\,\big(\mathrm{Ric}(\Theta,\Theta) + 2\,\mathrm{Ric}(\nabla\cdot,\Theta) - g(R(\Theta,\nabla\cdot)\Theta,\nabla\cdot)\big)\;w\,\rho^2 \\ &\quad + \mathcal{O}(\rho^4) + \rho^3\,L^2(w) + Q^1(w) + L^2(w)\ltimes L^0(w) + L^2(w)\ltimes Q^1(w)\Big]\end{aligned}$$

where as usual, all curvature terms are computed at p. Here we have defined

$$\text{Ric}(\nabla\cdot, \Theta) := \text{Ric}(e_i, \Theta)\, e_i$$

and

$$g(R(\Theta, \nabla\cdot), \Theta, \nabla\cdot) := g(R(\Theta, e_i), \Theta, e_j)\, e_i\, e_j$$

if $e_1, \ldots, e_m$ is an orthonormal frame field of $T_{\bar{q}} S^m$ satisfying $\bar{\nabla}_{e_i} e_j = 0$ at the point $\bar{q} \in S^m$ where these expressions are computed. It will be convenient to set

$$\mathcal{L} := \frac{1}{3}\,\left(\text{Ric}(\Theta, \Theta) + 2\,\text{Ric}(\nabla, \Theta) - g(R(\Theta, \nabla), \Theta, \nabla)\right)$$

Now observe that a similar expansion is valid in Euclidean space and in this case the expansion of $\rho^{-r}\,\sigma_r(p, w)$ does not depend on ρ (nor on p). This means that the nonlinear operator

$$Q^2 := Q^1 + L^2 \ltimes L^0 + L^2 \ltimes Q^1$$

can be decomposed into its value in Euclidean space and a similar operator all of whose coefficients are bounded by ρ. This fact can also be recovered by going through all the above expansions. Therefore, we can write

$$Q^2 = Q^2_e + \rho\, Q^2_r$$

where Q^2_e is the corresponding nonlinear operator when the metric is Euclidean and hence it does not depend on ρ; while $\rho\, Q^2_r$ denotes the discrepancy induced by the curvature of the metric g on M. Both Q^2_e and Q^2_r satisfy the usual properties.

1.5 Existence of foliations by constant r-curvature hypersurfaces

Assume that we are given $p_0 \in M$, a nondegenerate critical point of the scalar curvature $\mathcal{R}$ on M. We would like to find a small function $w \in \mathcal{C}^{2,\alpha}(S^m)$ and a point p close to p_0 such that

$$\sigma_r(S_\rho(p, w)) = C^r_m\, \rho^{-r}$$

In view of the previous expansion, this amount to solve the nonlinear equation

$$\begin{aligned}(\Delta_{S^m} + m)\, w = \frac{1}{3}\,\text{Ric}(\Theta, \Theta)\, \rho^2 + \frac{1}{4}\,\nabla_\Theta \text{Ric}(\Theta, \Theta)\, \rho^3 - \mathcal{O}(\rho^4) \\ - \rho^2\, \mathcal{L}\, w - \rho^3\, L^2(w) - Q^2(w)\end{aligned} \tag{1.10}$$

We denote by Π and $\Pi^\perp$ the L^2-orthogonal projections of $L^2(S^m)$ onto $\mathrm{Ker}(\Delta_{S^m} + m)$ and $\mathrm{Ker}(\Delta_{S^m} + m)^\perp$, respectively. Recall that the kernel of $\Delta_{S^m} + m$ is spanned by φ_i, for $i = 1, \ldots, m+1$, the restriction to the unit sphere of x_i, the coordinates functions in $\mathbb{R}^{m+1}$.

First fixed point argument From now on, we assume that the function $w \in \mathcal{C}^{2,\alpha}(S^m)$ is L^2-orthogonal to $\mathrm{Ker}(\Delta_{S^m} + m)$ and we project the equation (1.10) over $\mathrm{Ker}(\Delta_{S^m} + m)^\perp$. We obtain

$$\begin{aligned}(\Delta_{S^m} + m)\, w = \quad & \Pi^\perp \left[\tfrac{1}{3}\,\mathrm{Ric}(\Theta,\Theta)\,\rho^2 + \tfrac{1}{4}\,\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\,\rho^3 - \mathcal{O}(\rho^4)\right. \\ & \left. -\rho^2\,\mathcal{L}\,w - \rho^3\,L^2(w) - Q^2(w)\right]\end{aligned}$$

We define $w_0 \in \mathrm{Ker}(\Delta_{S^m} + m)^\perp$ to be the unique solution of

$$(\Delta_{S^m} + m)\, w_0 = \frac{1}{3}\,\mathrm{Ric}(\Theta,\Theta) \tag{1.11}$$

since $\Pi^\perp\,(\mathrm{Ric}(\Theta,\Theta)) = \mathrm{Ric}(\Theta,\Theta)$. Similarly, we define $w_1 \in \mathrm{Ker}(\Delta_{S^m} + m)^\perp$ to be the unique solution of

$$(\Delta_{S^m} + m)\, w_1 = \frac{1}{4}\,\Pi^\perp\left[\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\right]$$

It is easy to rephrase the solvability of the nonlinear equation (1.10) as a fixed point problem since the operator $\Delta_{S^m} + m$ is invertible from the space of $\mathcal{C}^{2,\alpha}(S^m)$ functions which are L^2-orthogonal to $\mathrm{Ker}(\Delta_{S^m} + m)$ into the space of $\mathcal{C}^{0,\alpha}(S^m)$ functions which are L^2-orthogonal to $\mathrm{Ker}(\Delta_{S^m} + m)$. We write $w := \rho^2\, w_0 + \rho^3\, w_1 + \rho^4\, v$, so that it remains to solve an equation which can be written for short as

$$(\Delta_{S^m} + m)\, v = -\mathcal{O}(1) - \rho^{-2}\,\mathcal{L}w - \rho^{-1}\,L^2(w) - \rho^{-4}\,Q^2(w)$$

Applying a standard fixed point theorem for contraction mappings, it is easy to check that there exists a constant $\kappa > 0$, which is independent of the choice of the point $p \in M$, such that there exists a unique fixed point in ball of radius κ in $\mathcal{C}^{2,\alpha}(S^m)$, provided ρ is chosen small enough, say $\rho \in (0, \rho_0)$. We denote by v_p this solution and define

$$w_p := \rho^2\, w_0 + \rho^3\, w_1 + \rho^4\, v_p.$$

It is easy to check that, reducing the value of ρ_0 if this is necessary,

$$\|w_p - w_{p'}\|_{\mathcal{C}^{2,\alpha}(S^m)} \le c\,\rho^2\,\mathrm{dist}(p,p'), \tag{1.12}$$

for some constant c which does not depend on $\rho \in (0, \rho_0)$ nor on p or p'. In addition, the mapping

$$(\rho, p) \in (0, \rho_0) \times M \longrightarrow w_p \in \mathcal{C}^{2,\alpha}(S^m)$$

is smooth and

$$\|D_p w_p\|_{\mathcal{C}^{2,\alpha}(S^m)} + \rho\, \|\partial_\rho w_p\|_{\mathcal{C}^{2,\alpha}(S^m)} \leq c\, \rho^2$$

for some constant c which does not depend on $\rho \in (0, \rho_0)$ nor on p.

Second fixed point argument It now remains to project the equation (1.10) where w has been replaced by w_p, over Ker$(\Delta_{S^m} + m)$. To this aim, we recall the nice and key observation from [42].

The problem is to compute the L^2-projection of the quantity $g(\nabla_\Theta R(\Theta,\, \Theta_i)\, \Theta, \Theta_j)$ over the kernel of the operator $\Delta_{S^m} + m$. This amounts to compute, for any $n = 1, \ldots, m+1$, the quantity

$$B_n := \sum_{i,j,k,\ell} g(\nabla_{E_j} R(E_i, E_k) E_i, E_\ell) \int_{S^m} x_j\, x_k\, x_\ell\, x_n$$

Now to evaluate this quantity, simply use the fact that the integral vanishes unless all indices are all equal or constitute two pairs of equal indices. Using this, together with the symmetries of the curvature tensor which imply that $R(E, E) = 0$, we obtain

$$\begin{array}{rcl} B_n & = & g(\nabla_{E_n} R(E_i, E_n) E_i, E_n) \, \left(\int_{S^m} x_1^4 - 3 \, \int_{S^m} x_1^2\, x_2^2 \right) \\ & + & g(\nabla_{E_n} R(E_i, E_j) E_i + 2 \nabla_{E_j} R(E_i, E_n) E_i, E_j) \, \int_{S^m} x_1^2\, x_2^2 \end{array}$$

Now, use second Bianchi identity

$$g(\nabla_{E_n} R(E_i, E_j) E_i, E_j) = 2\, g(\nabla_{E_j} R(E_i, E_n) E_i, E_j)$$

together with the fact that

$$\int_{S^m} x_1^4 = 3 \int_{S^m} x_1^2\, x_2^2 = \frac{3}{(n+3)} \int_{S^m} x_1^2$$

To conclude that

$$\Pi \left(g(\nabla_\Theta R(\Theta,\, \Theta_i)\, \Theta, \Theta_j) \right) = - \frac{1}{m+3}\, g(\nabla \mathcal{R}, x_i\, E_i)$$

where $\mathcal{R}$ denotes the scalar curvature function, computed at p.

Therefore, the projection of the equation (1.10) over Ker $(\Delta_{S^m} + m)$ yields

$$g(\nabla \mathcal{R}, x_i\, E_i) = V_p$$

where we have defined

$$V_p := 4\,(m+3)\, \Pi\, \left[\rho^{-3}\, \mathcal{O}(\rho^4) + \rho^{-1}\, \mathcal{L}\, w_p + L^2(w_p) + \rho^{-3}\, Q^2(w_p) \right]$$

Now, using the fact that p_0 is a nondegenerate critical point of the scalar curvature, we conclude easily (applying for example a topological degree argument) that there exists p close to p_0 satisfying (1.12) provided ρ is close enough to 0. This gives the existence of constant r-curvature leaves for all ρ small enough, unfortunately it follows from (2.10) that the point p is at most at distance a constant times ρ from p_0 and this is not enough to show that the constant r-curvature leaves form a foliation of a neighborhood of p_0.

To improve this estimate, many observations are due. First, observe that we can decompose $\mathcal{O}(\rho^4)$ into the sum of two functions, one of which is homogeneous of degree 4 (in the coordinate functions x_i) and the other one which is bounded by a constant times ρ^5. The L^2-projection of the homogeneous function of degree 4 is equal to 0 since this homogeneous function is invariant under the change of coordinates Θ into $-\Theta$. Hence we conclude that

$$|\Pi\,(\mathcal{O}(\rho^4))| \leq c\,\rho^5$$

Similarly, observe that w_0 and hence $\mathcal{L}\,w_0$ are invariant under the change Θ into $-\Theta$ and hence the L^2 projection of $\mathcal{L}\,w_0$ over Ker$(\Delta_{S^m} + m)$ again identically equal to 0. Therefore, we conclude that

$$|\Pi\,(\mathcal{L}\,w_p)| \leq c\,\rho^3$$

Finally, we use the observation at the end of §4. Since the nonlinear operator Q_e^2 preserves functions which are invariant under the action of $-I$, we conclude that $\Pi(Q_e^2(\rho^2\,w_0)) = 0$ and hence

$$|\Pi(Q^2(w_p))| \leq c\,\rho^5$$

These precise estimates imply that,

$$|V_p| \leq c\,\rho^2$$

for some constant which does not depend on p nor on ρ. With slightly more work, we get using similar arguments that

$$|\Pi\;(V_p - V_{p'})\,| \leq c\,\rho^2\,\mathrm{dist}(p,p') \tag{1.13}$$

Now, for all ρ small enough, we can find a solution of (1.10) using a fixed point argument for contraction mapping, in the geodesic ball of radius $2\,\rho^2$ centered at any nondegenerate critical point of $\mathcal{R}$. Moreover, the solution p_ρ depends smoothly on ρ and

$$|\partial_\rho p_\rho| \leq c\,\rho$$

This later fact, together with (1.13) shows that the solutions constitute a local foliation. This completes the proof of the main result.

Having derived such precise estimates, we can compute the expansion of the m-dimensional volume of the leaves of the foliation as well as the $(m+1)$-dimensional volume enclosed by each leaf.

Proposition 1.5.1 *For all ρ small enough the following expansions hold for the m-dimensional volume of S_ρ*

$$\mathit{Vol}_m(S_\rho) = \rho^m \, \mathit{Vol}_m(S^m) \left(1 - \frac{1}{2(m+1)} \, \mathcal{R} \, \rho^2 + \mathcal{O}(\rho^4)\right)$$

and the $(m+1)$-dimensional volume of the set B_ρ enclosed by S_ρ and containing the point p_0

$$\mathit{Vol}_{m+1}(B_\rho) = \frac{1}{m+1} \, \rho^{m+1} \, \mathit{Vol}_m(S^m) \left(1 - \frac{m+2}{2m(m+3)} \, \mathcal{R} \, \rho^2 + \mathcal{O}(\rho^4)\right)$$

where the scalar curvature is computed at p_0, a nondegenerate critical point of $\mathcal{R}$.

Proof : Integrating (1.11) over S^m we find

$$m \int_{S^m} w_0 = \frac{1}{3} \int_{S^m} \mathrm{Ric}(\Theta, \Theta)$$

Now, plugging the expansion of w_p into the expression of the first fundamental form given in Proposition 2.3.1, we find the expansion of h the induced metric on S_ρ

$$\begin{aligned} \rho^{-2} \, h_{ij} &= (1 - 2\rho^2 w_0 - 2\rho^3 w_1)\delta_{ij} + \frac{1}{3} \, g(R(\Theta, \Theta_i) \, \Theta, \Theta_j) \, \rho^2 \\ &\quad + \frac{1}{6} \, g(\nabla_\Theta R(\Theta, \Theta_i) \, \Theta, \Theta_j) \, \rho^3 + \mathcal{O}(\rho^4) \end{aligned} \tag{1.14}$$

This implies that

$$\rho^{-m} \sqrt{|h|} = 1 - m \, \rho^2 \, w_0 - m \, \rho^3 \, w_1 - \frac{1}{6} \, \mathrm{Ric}(\Theta, \Theta) \, \rho^2 - \frac{1}{12} \, \nabla_\Theta \mathrm{Ric}(\Theta, \Theta) \, \rho^3 + \mathcal{O}(\rho^4)$$

The first estimate follows from integrating this expansion using the fact that the integral of w_1 and the integral $\nabla_\Theta \mathrm{Ric}\,(\Theta, \Theta)$ over S^m vanish together with the fact that

$$\int_{S^m} \mathrm{Ric}(\Theta, \Theta) = \frac{1}{m+1} \, \mathrm{Vol}_m(S^m) \, \mathcal{R}.$$

Next, we consider polar geodesic normal coordinates (r, Θ) centered at p_ρ. In these coordinates the metric g expanded as

$$r^{-2} \, g_{ij} = \delta_{ij} + \frac{1}{3} \, g(R(\Theta, \Theta_i) \, \Theta, \Theta_j) \, r^2 + \frac{1}{6} \, g(\nabla_\Theta R(\Theta, \Theta_i) \, \Theta, \Theta_j) \, r^3 + \mathcal{O}(r^4). \tag{1.15}$$

then, the volume form can be expanded as

$$r^{-m}\,\sqrt{|g|} = 1 - \frac{1}{6}\,\mathrm{Ric}(\Theta,\Theta)\,r^2 - \frac{1}{12}\,\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\,r^3 + \mathcal{O}(r^4).$$

Integration over the set $r \le \rho\,(1-w_p)$ give

$$\begin{aligned}
\mathrm{Vol}_{m+1}(B_\rho) &= \int\int_{r\le\rho\,(1-w_p)} r^m \left(1 - \frac{1}{6}\,\mathrm{Ric}(\Theta,\Theta)\,r^2 - \frac{1}{12}\,\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\,r^3 + \mathcal{O}(r^4)\right) \\
&= \frac{1}{m+1}\,\rho^{m+1} \int_{S^m} (\,1-(m+1)\rho^2\,w_0\,) + \mathcal{O}(\rho^{m+5}) \\
&- \frac{1}{6}\,\frac{1}{m+3}\,\rho^{m+3} \int_{S^m} (\,1-(m+3)\rho^2\,w_0\,)\,\mathrm{Ric}(\Theta,\Theta) \\
&= \frac{1}{m+1}\,\rho^{m+1}\,\mathrm{Vol}_m(S^m) - \rho^{m+3} \int_{S^m} w_0 - \frac{1}{6}\,\frac{\rho^{m+3}}{m+3} \int_{S^m} \mathrm{Ric}(\Theta,\Theta) \\
&+ \mathcal{O}(\rho^{m+5}) \\
&= \frac{1}{m+1}\,\rho^{m+1}\,\mathrm{Vol}_m(S^m) - (\,\frac{1}{3m} + \frac{1}{6}\,\frac{1}{m+3}\,)\rho^{m+3} \int_{S^m} \mathrm{Ric}(\Theta,\Theta) \\
&+ \mathcal{O}(\rho^{m+5}) \\
&= \frac{1}{m+1}\,\rho^{m+1}\,\mathrm{Vol}_m(S^m)\left(1 - \frac{m+2}{2m(m+3)}\,\mathcal{R}\,\rho^2 + \mathcal{O}(\rho^4)\right)
\end{aligned}$$

This gives the second estimate. □

1.6 Appendix : proof of Proposition 2.2.1

The curve $s \to \exp_p^M(sE)$ is a geodesic. Therefore, if X is the unit tangent vector to the curve we have $\nabla_X X = 0$. Hence we also have $(\nabla_X)^n X = 0$ for all $n \ge 1$. In particular, we have, at p,

$$(\nabla_E)^n E = 0$$

for all $E \in T_pM$ and for all $n \ge 1$.

Observe that X_a are coordinate vector fields hence

$$\nabla_{X_a} X_b = \nabla_{X_b} X_a$$

Taking $E = E_a + \varepsilon\, E_b$ and looking for the coefficient of ε in $\nabla_E E = 0$, we get

$$\nabla_{E_a} E_b = 0$$

Looking at the coefficient of ε in $\nabla^2_E E = 0$, we get

$$2\,\nabla^2_{E_a} E_b + \nabla_{E_b} \nabla_{E_a} E_a = 0 \tag{1.16}$$

Finally, looking at the coefficient of ε in $\nabla^3_E E = 0$, we get

$$2\,\nabla^3_{E_a} E_b + (\nabla_{E_b} \nabla_{E_a} + \nabla_{E_a} \nabla_{E_b}) \nabla_{E_a} E_a = 0 \tag{1.17}$$

Recall that, by definition

$$\nabla_X\, \nabla_Y := R(X, Y) + \nabla_Y\, \nabla_X + \nabla_{[X,Y]}$$

Hence, if X and Y are coordinate vector fields we simply have

$$\nabla_X\, \nabla_Y X := R(X, Y) X + \nabla_Y\, \nabla_X\, X, \tag{1.18}$$

We also have

$$\begin{aligned} \nabla_Y\, \nabla_X\, \nabla_Y X \quad &:= \quad \nabla_Y R(X, Y) X + R(\nabla_Y X, Y) X + R(X, \nabla_Y Y) X \\ &+ \quad R(X, Y) \nabla_Y X + \nabla^2_Y \nabla_X X + \nabla_Y \nabla_{[X,Y]} X \end{aligned} \tag{1.19}$$

Now use (1.16) and (1.18) to obtain

$$3\,\nabla^2_{E_a} E_b = R(E_a, E_b)\, E_a, \tag{1.20}$$

Similarly, use (1.17) and (1.19) to obtain

$$2\,\nabla^3_{E_a}\, E_b + R(E_b, E_a)\, \nabla_{E_a} E_a + 2 \nabla_{E_a} \nabla_{E_b} \nabla_{E_a} E_a = 0 \tag{1.21}$$

Since $\nabla_{E_a} E_b = 0$, we get

$$2\,\nabla^3_{E_a}\, E_b + 2 \nabla_{E_a} \nabla_{E_b} \nabla_{E_a} E_a = 0$$

Using this, we conclude that

$$\begin{aligned} 2\,\nabla^3_{E_a}\, E_b = -2\,\nabla_{E_a} \nabla_{E_b} \nabla_{E_a} E_a &= -2\,\nabla_{E_a} (R(E_b, E_a)\, E_a + \nabla_{E_a} \nabla_{E_a} E_b) \\ &= -2\,\nabla_{E_a} (R(E_b, E_a)\, E_a) - 2\,\nabla^3_{E_a}\, E_b \end{aligned}$$

Hence

$$2\,\nabla^3_{E_a}\, E_b = -\nabla_{E_a} R(E_b, E_a)\, E_a \tag{1.22}$$

Now, we have

$$X_c\, g_{ab} = g(\nabla_{X_c} X_a, X_b) + g(X_a, \nabla_{X_c} X_b),$$

and we get $X_c\, g_{ab}\big|_p = 0$. This yields the first order Taylor expansion

$$g_{ab} = \delta_{ab} + \mathcal{O}(|x|^2),$$

To compute the second order terms, it suffices to compute $X_c^2\, g_{ab}$ at p and polarize. We compute

$$X_c^2\, g_{ab} = g(\nabla^2_{X_c} X_a, X_b) + g(X_a, \nabla^2_{X_c} X_b) + 2\, g(\nabla_{X_c} X_a, \nabla_{X_c} X_b)$$

Using (1.21) we get

$$X_c^2\, g_{ab}\big|_p = \frac{2}{3}\, g(R(E_c, E_a)\, E_c, E_b).$$

The formula for the second order Taylor coefficient for g_{ab} now follows at once.

Similarly, we compute

$$X_c^3\, g_{ab}\big|_p = g(\nabla^3_{X_c} X_a, X_b) + 3g(\nabla^2_{X_c} X_a, \nabla_{X_c} X_b) + 3g(\nabla_{X_c} X_a, \nabla^2_{X_c} X_b) + g(X_a, \nabla^3_{X_c} X_b)$$

and using (1.22) this gives

$$X_c^3\, g_{ab}\big|_p = g(\nabla_{E_c} R(E_a, E_c)\, E_b, E_c).$$

the formula for the second order Taylor expansion for g_{ab} holds at once. $\square$

Chapitre 2

Constant r-curvature hypersurfaces condensing along a submanifold

Abstract We are interested in families of constant r-curvature hypersurfaces, with r-curvature varying from one member of the family to another, which 'condense' to a submanifold $K^k \subset M^{m+1}$ of codimension greater than 1. Two cases have been studied previously : R. Ye proved the existence of a local foliation by constant mean curvature hypersurfaces when K is a point (which is required to be a nondegenerate critical point of the scalar curvature function) ; in [28], R. Mazzeo and F. Pacard proved the existence of a lamination when K is a nondegenerate geodesic. In this chapter we extend this last result to handle the general case, when K is an arbitrary nondegenerate minimal submanifold. In particular, this proves the existence of constant r-curvature hypersurfaces with nontrivial topology in any Riemannian manifold. This new approach is inspired by some recent work of A. Malchiodi and M. Montenegro in the contex of semilinear elliptic partial differential equations.

2.1 Introduction

Let S be an oriented embedded (or possibly immersed) hypersurface in a Riemannian manifold (M^{m+1}, g). The shape operator A_S is the symmetric endomorphism of the tangent bundle of S associated with the second fundamental form of S, b_S, by

$$b_S(X,Y) = g_S(A_S\,X,Y), \quad \forall\, X,Y \in TS; \qquad \text{here} \qquad g_S = g|_{TS}\,.$$

The eigenvalues κ_i of the shape form A_S are the principal curvatures of the hypersurface S. We define the r-curvature of S to be the r-th symmetric function of the principal curvatures of S, i.e.

$$\sigma_r(S) := \sum_{i_1<\ldots<i_r} \kappa_{i_1} \ldots \kappa_{i_r}$$

Observe that when $r = 1$, $\sigma_1(S)$ is equal to m times H (the mean curvature of S) and when (M^{m+1}, g) is the Euclidean space $(\mathbb{R}^{m+1}, g_{eucl})$, $\sigma_2(\Sigma)$ is equal to $\frac{m(m-1)}{2}$ times the scalar curvature of S and σ_m is equal to the Gauss-Kronecker curvature of S.

In this chapter we are interested in families of constant r-curvature hypersurfaces, with r-curvature varying from one member of the family to another, which 'condense' to a submanifold $K^k \subset M^{m+1}$ of codimension greater than 1. Under fairly reasonable geometric assumptions [28], the existence of such a family implies that K is minimal. Two cases have been studied previously : Ye [42], [43] proved the existence of a local foliation by constant mean curvature hypersurfaces when K is a point (which is required to be a nondegenerate critical point of the scalar curvature function) ; more recently, the second and third authors [28] proved existence of a closed lamination by constant mean curvature hypersurfaces in a neighborhood of K, when K is a nondegenerate geodesic. In this chapter we extend the result and methods of [28] to handle the general case, when K is an arbitrary nondegenerate minimal submanifold. No extra curvature hypotheses are required. In particular, this proves the existence of constant r-curvature hypersurfaces with nontrivial topology in any Riemannian manifold.

Let us describe our result in more detail. Let K^k be a closed (possibly immersed) submanifold in M^{m+1}, $1 \leq k \leq m-1$, and define the geodesic tube of radius ρ about K by

$$\bar{S}_\rho := \{q \in M^{m+1} : \quad \text{dist}_g(q, K) = \rho\}.$$

This is a smooth (immersed) hypersurface provided ρ is smaller than the radius of curvature of K, and we henceforth always tacitly assume that this is the case. The r-curvature of this tube satisfies

$$\sigma_r(\mathcal{S}_\rho(K)) = C^r_{n-1}\, \rho^{-r} + \mathcal{O}(\rho^{-r}) \qquad \text{as} \qquad \rho \searrow 0, \tag{2.1}$$

where $n = m + 1 - k$ and hence it is plausible that we might be able to perturb this tube to a constant r-curvature hypersurface with $\sigma_r \equiv C^r_{n-1}\, \rho^{-r}$. This is not quite true since the r-curvature of $\bar{S}_\rho$ is not sufficiently close to being constant, but when K is minimal there is a better estimate

$$\sigma_r(\mathcal{S}_\rho(K)) = C^r_{n-1}\, \rho^{-r} + \mathcal{O}(\rho^{1-r}) \qquad \text{as} \qquad \rho \searrow 0,$$

Even in this case, there are other more subtle obstructions to carrying out this procedure at certain radii ρ related to eigenvalues of the linearized mean curvature operator on $\bar{S}_\rho$, which in turn are related to a genuine bifurcation phenomenon, at least when $k = 1$,

[28]. Thus we do not obtain existence of the constant r-curvature perturbation for every small radius.

In the case of the mean curvature (i.e of H), our main result is :

Theorem 2.1.1 *Suppose that K^k is a nondegenerate closed minimal submanifold $1 \leq k \leq m-1$. Then there exists $I \subset (0, +\infty)$, countable union of disjoint nonempty open intervals, such that for all $\rho \in I$, the geodesic tube $\bar{S}_\rho$ may be perturbed to a constant mean curvature hypersurface S_ρ with $H = \frac{n-1}{m}\,\rho^{-1}$. Moreover, for any $q \geq 2$ there exists a $c_q > 0$ such that*

$$|\mathcal{H}^1((0,\rho) \cap I) - \rho| \leq c_q\, \rho^q,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

The nondegeneracy condition on K is simply that the linearized mean curvature operator, also called the Jacobi operator, is invertible ; this restriction is quite mild and holds generically [45]. As noted above, this result was already known when $k = 0, 1$, but the case $k > 1$ requires a more complicated analysis. This new approach is inspired by some recent work of Malchiodi and Montenegro in a somewhat different context [26], [23]. Let is also mention the related work of Shatah and Zeng on the existence of periodic solutions for some penalized Hamiltonian system [38].

The hypersurface S_ρ is a small perturbation of $\bar{S}_\rho$ in the sense that it is the normal graph of some function (with L^∞ norm bounded by a constant times ρ^3) over a submanifold obtained by 'translating' K by a section of its normal bundle (with L^∞ norm bounded by a constant times ρ^2) ; we refer to §3.1 for the precise formulation of the construction of S_ρ. When K is embedded, then so are the hypersurfaces S_ρ for ρ sufficiently small. In addition, the closure of the union of the hypersurfaces in each of the families $\{S_\rho\}_{\rho \in I_i}$ constitutes a local lamination of some annular neighborhood of K.

That the construction fails for certain values of ρ is related to a bifurcation phenomenon. When $k = 1$ the families of surfaces which bifurcate off are (perturbations of) Delaunay unduloids [19] ; however, when $k \geq 2$, this bifurcation is only known to exist in special cases, and the geometry of the surfaces in the putative bifurcating branches is less clear. In any case, such bifurcations are inherent to the problem and occur also in [25] and in many other situations. Furthermore, the index of the hypersurfaces S_ρ, $\rho \in I$, tends to $+\infty$ as $\rho \to 0$.

One way to describe the behavior of S_ρ as ρ tends to 0 is to consider the associated area and curvature densities of S_ρ as ρ tends to 0 ; these quantities, properly rescaled, are extremely close to the corresponding quantities for $\bar{S}_\rho$, which in turn satisfy

$$\rho^{k-m}\, \mathcal{H}^m \llcorner \bar{S}_\rho \rightharpoonup \omega_{m-k}\, \mathcal{H}^k \llcorner K, \tag{2.2}$$

and, for all $q \geq 1$,

$$\rho^{k-m+q}\, |A_{\bar{S}_\rho}|^q\, \mathcal{H}^m \llcorner \bar{S}_\rho \rightharpoonup (m-k)^{q/2}\, \omega_{m-k}\, \mathcal{H}^k \llcorner K, \tag{2.3}$$

as $\rho \searrow 0$. Here $|A_S|^2 := \mathrm{Tr}((A_S)^t\, A_S)$ is the norm squared of the shape operator. From the explicit estimates in the construction of S_ρ one can deduce that (2.2) and (2.3) also hold when $\bar{S}_\rho$ is replaced by S_ρ.

One can ask whether (2.2) and (2.3) hold for any family of constant mean curvature hypersurfaces which condense along K. It turns out that this is not the case : families of constant mean curvature hypersurfaces condensing along a nondegenerate geodesic which do not satisfy (2.2) are constructed in [19]. In another direction, it is plausible that one should be able to construct families of constant mean curvature hypersurfaces which condense along lower dimensional sets which are still minimal in an appropriate sense, but with singularities, for example a Steiner tree with geodesic edges. A simple example of this is when S_ρ is obtained by homothetically rescaling a fixed Delaunay trinoid in $\mathbb{R}^3$. The limit then is a union of three rays meeting at a common vertex, each ray having an associated density coming from the limiting Delaunay necksize on that end; each ray is minimal, of course, and, for the construction to hold the entire configuration has to be 'balanced' in the sense that the weighted sum of the vectors along the rays vanishes.

Keeping these various phenomena in mind, it is an interesting problem to prove whether our main result has a suitable converse, or if it is possible to characterize the possible condensation sets of such families of constant mean curvature hypersurfaces. Let us mention the following results by Rosenberg in this direction :

Theorem 2.1.2 *[36] There exists a constant $H_0 > 0$ only depending on the geometry of (M, g) such that, if S is an embedded constant mean curvature hypersurface with mean curvature greater than H_0 then S is homologically trivial. Moreover, the distance from any point p in the mean convex part of $M - S$ and S is bounded by c/H, where $c > 0$ only depends on the geometry of (M, g).*

In the next section we calculate the asymptotic expansion of the metric on M in Fermi coordinates around K; this is applied in the (quite technical) §3 to derive the expansions of various geometric quantities for the tubes $\bar{S}_\rho$ and their perturbations. This is used in §4 to obtain the expression for the mean curvature of the perturbed tubes, which gives us the equation which must be solved. An iteration scheme is introduced in §5 which allows us to find a preliminary perturbation for which the error term is much better, and estimates for the gaps in the spectrum of the linearization are obtained in §6; finally, the existence of the constant mean curvature hypersurfaces S_ρ is obtained in §7.

2.2 Expansion of the metric in Fermi coordinates near K

2.2.1 Fermi coordinates

We now introduce Fermi coordinates in a neighborhood of K. For a given $p \in K$, there is a natural splitting

$$T_pM = T_pK \oplus N_pK.$$

Choose orthonormal bases E_a, $a = n+1, \dots, m+1$, for T_pK, and E_i, $i = 1, \dots, n$, of N_pK.

Notation : We shall always use the convention that indices $a, b, c, d, \dots \in \{n+1, \dots, m+1\}$, indices $i, j, k, \ell, \dots \in \{1, \dots, n\}$ and indices $\alpha, \beta, \gamma, \dots \in \{1, \dots, m+1\}$.

Consider, in a neighborhood of p in K, normal geodesic coordinates

$$f(y) := \exp_p^K(y^a\, E_a), \qquad y := (y^{n+1}, \dots, y^{m+1}),$$

where $\exp^K$ is the exponential map on K and summation over repeated indices is understood. This yields the coordinate vector fields $X_a := f_*(\partial_{y^a})$. For any $E \in T_pK$, the curve

$$s \longrightarrow \gamma_E(s) := \exp_p^K(sE),$$

is a geodesic in K, so that

$$\nabla_{X_a} X_b|_p \in N_pK.$$

We define the numbers Γ^i_{ab} by

$$\nabla_{X_a} X_b|_p = \Gamma^i_{ab}\, E_i.$$

Now extend the E_i along each $\gamma_E(s)$ so that they are parallel with respect to the induced connection on the normal bundle NK. This yields an orthonormal frame field X_i for NK in a neighborhood of p in K which satisfies

$$\nabla_{X_a} X_i|_p \in T_pK,$$

and hence defines coefficients Γ^b_{ai} by

$$\nabla_{X_a} X_i|_p = \Gamma^b_{ai}\, E_b.$$

A coordinate system in a neighborhood of p in M is now defined by

$$F(x,y) := \exp^M_{f(y)}(x^i\, X_i), \qquad (x,y) := (x^1,\dots,x^n,y^{n+1},\dots,y^{m+1}),$$

with corresponding coordinate vector fields

$$X_i := F_*(\partial_{x^i}) \qquad \text{and} \qquad X_a := F_*(\partial_{y^a}).$$

By construction, $X_\alpha\,|_p = E_\alpha$.

2.2.2 Taylor expansion of the metric

As usual, the Fermi coordinates above are defined so that the metric coefficients

$$g_{\alpha\beta} = g(X_\alpha, X_\beta),$$

equal $\delta_{\alpha\beta}$ at p; furthermore, $g(X_b, X_i) = 0$ in some neighborhood of p in K. This implies that

$$X_a\, g(X_b, X_i) = g(\nabla_{X_a} X_b, X_i) + g(X_b, \nabla_{X_a} X_i) = 0,$$

on K, which yields the identity

$$\Gamma^i_{ab} + \Gamma^b_{ai} = 0. \tag{2.4}$$

at p.

Denote by $\Gamma^b_a : N_pK \longrightarrow \mathbb{R}$ the linear form

$$\Gamma^b_a(\cdot) := g(\nabla_{E_a} E_b, \cdot) = -g(\nabla_{E_a} \cdot, E_b). \tag{2.5}$$

We now compute higher terms in the Taylor expansions of the functions $g_{\alpha\beta}$. The metric coefficients at $q := F(x,0)$ are given in terms of geometric data at $p := F(0,0)$ and $|x| = \mathrm{dist}_g(p,q)$.

Notation The symbol $\mathcal{O}(|x|^r)$ indicates a function such that it and its partial derivatives of any order, with respect to the vector fields X_a and $x^i\, X_j$, are bounded by $c\,|x|^r$ in some fixed neighborhood of 0.

We begin with the expansion of the covariant derivative :

Lemma 2.2.1 *At the point of $q = F(x,0)$, the following expansions hold*

$$\begin{aligned}
\nabla_{X_i} X_j &= \mathcal{O}(|x|) X_\gamma, \\
\nabla_{X_a} X_b &= \Gamma^b_a(E_i)\, X_i + \mathcal{O}(|x|) X_\gamma, \\
\nabla_{X_a} X_i &= \nabla_{X_i} X_a = -\Gamma^b_a(E_i)\, X_b + \mathcal{O}(|x|)^\gamma\, X_\gamma,
\end{aligned} \tag{2.6}$$

Proof : Observe that, because we are using coordinate vector fields, $\nabla_{X_\alpha} X_\beta = \nabla_{X_\beta} X_\alpha$ for any α, β. We also have $\nabla_{X_i} X_j \,|_p = 0$ since any $X \in N_p K$ is tangent to the geodesic $s \longrightarrow \exp_p^M(sX)$, and hence

$$\nabla_{X_i + X_j}(X_i + X_j)\big|_p = 0.$$

Therefore

$$(\nabla_{X_i} X_j + \nabla_{X_j} X_i)\big|_p = 0,$$

and this completes the proof of the first estimate.

We have by construction

$$\nabla_{X_a} X_b = \Gamma^i_{ab}\, X_i + \mathcal{O}(|x|)\, X_\gamma,$$

and

$$\nabla_{X_a} X_i = \nabla_{X_i} X_a = \Gamma^b_{ai}\, X_b + \mathcal{O}(|x|)\, X_\gamma.$$

The next two estimates follow from the definition of Γ^b_a and (2.4). □

We now give the expansion of the metric coefficients. The expansion of the g_{ij}, $i, j = 1, \ldots, n$, agrees with the well known expansion for the metric in normal coordinates [39], [16], [46], but we briefly recall the proof here for completeness.

Proposition 2.2.1 *At the point $q = F(x, 0)$, the following expansions hold*

$$\begin{aligned} g_{ij} &= \delta_{ij} + \tfrac{1}{3}\, g(R(E_k, E_i)\, E_\ell, E_j)\, x^k\, x^\ell + \mathcal{O}(|x|^3), \\ g_{ai} &= \mathcal{O}(|x|^2), \\ g_{ab} &= \delta_{ab} - 2\, \Gamma^b_a(E_i)\, x^i + \left[g(R(E_k, E_a)\, E_\ell, E_b) + \Gamma^c_a(E_k)\, \Gamma^b_c(E_\ell) \right]\, x^k\, x^\ell + \mathcal{O}(|x|^3), \end{aligned} \tag{2.7}$$

where summation over repeated indices is understood.

Proof : By construction, $g_{\alpha\beta} = \delta_{\alpha\beta}$ at p, and so

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(|x|).$$

Now, from

$$X_i\, g_{\alpha\beta} = g(\nabla_{X_i} X_\alpha, X_\beta) + g(X_\alpha, \nabla_{X_i} X_\beta),$$

and Lemma 2.2.1, we get

$$X_i\, g_{aj}\big|_p = 0, \qquad X_i\, g_{jk}\big|_p = 0 \qquad \text{and} \qquad X_i\, g_{ab}\big|_p = \Gamma^b_{ai} + \Gamma^a_{ib} = 2\Gamma^b_{ai}.$$

This yields the first order Taylor expansion

$$g_{aj} = \mathcal{O}(|x|^2), \qquad g_{ij} = \delta_{ij} + \mathcal{O}(|x|^2) \qquad \text{and} \qquad g_{ab} = \delta_{ab} + 2\, \Gamma^b_{ai}\, x^i + \mathcal{O}(|x|^2).$$

To compute the second order terms, it suffices to compute $X_k\, X_k\, g_{\alpha\beta}$ at p and polarize (i.e. replace X_k by $X_i + X_j$, etc.). We compute

$$X_k\, X_k\, g_{\alpha\beta} = g(\nabla^2_{X_k} X_\alpha, X_\beta) + g(X_\alpha, \nabla^2_{X_k} X_\beta) + 2\, g(\nabla_{X_k} X_\alpha, \nabla_{X_k} X_\beta). \qquad (2.8)$$

To proceed, first observe that

$$\nabla_X X\big|_{p'} = \nabla^2_X X\big|_{p'} = 0,$$

at $p' \in K$, for any $X \in N_{p'}K$. Indeed, for all $p' \in K$, $X \in N_{p'}K$ is tangent to the geodesic $s \longrightarrow \exp^M_{p'}(sX)$, and so $\nabla_X X = \nabla^2_X X = 0$ at the point p'.

In particular, taking $X = X_k + \varepsilon\, X_j$, we obtain

$$0 = \nabla_{X_k+\varepsilon X_j} \nabla_{X_k+\varepsilon X_j} (X_k + \varepsilon X_j)\,_{|p},$$

equating the coefficient of ε to 0 gives $\nabla_{X_j} \nabla_{X_k} X_k\,_{|p} = -2 \nabla_{X_k} \nabla_{X_k} X_j\,_{|p}$, and hence

$$3\, \nabla^2_{X_k} X_j\big|_p = R(E_k, E_j)\, E_k,$$

So finally, using (2.8) together with the result of Lemma 2.2.1, we get

$$X_k\, X_k\, g_{ij}\big|_p = \frac{2}{3}\, g(R(E_k, E_i)\, E_k, E_j).$$

The formula for the second order Taylor coefficient for g_{ij} now follows at once.

Recall that, since X_γ are coordinate vector fields, we have from (2.8)

$$\nabla^2_{X_k} X_\gamma = \nabla_{X_k} \nabla_{X_\gamma} X_k = \nabla_{X_\gamma} \nabla_{X_k} X_k + R(X_k, X_\gamma)\, X_k.$$

Using (2.8), this yields

$$\begin{aligned} X_k\, X_k\, g_{ab} &= 2\, g(R(X_k, X_a) X_k, X_b) + 2\, g(\nabla_{X_k} X_a, \nabla_{X_k} X_b) \\ &+ g(\nabla_{X_a} \nabla_{X_k} X_k, X_b) + g(X_a, \nabla_{X_b} \nabla_{X_k} X_k) \end{aligned}$$

Using the result of Lemma 2.2.1 together with the fact that $\nabla_X X = 0$ at $p' \in K$ for any $X \in N_{p'}K$, we conclude that

$$X_k\, X_k\, g_{ab}\big|_p = 2\, g(R(E_k, E_a) E_k, E_b) + 2\, \Gamma^c_{ak}\, \Gamma^c_{bk}$$

and using the definition of Γ^b_a given in (2.5) this gives the formula for the second order Taylor expansion for g_{ab}. □

Later on, we will need an expansion of some covariant derivatives which is more accurate than the one given in Lemma 2.2.1. These are given in the :

Lemma 2.2.2 *At the point $q = F(x, 0)$, the following expansion holds*

$$\begin{aligned} \nabla_{X_a} X_b &= \Gamma_a^b(E_j)\, X_j - g(R(E_i, E_a)\, E_j, E_b)\, x^i\, X_j \\ &+ \tfrac{1}{2}\left(g(R(E_a, E_b)\, E_i, E_j) - \Gamma_a^c(E_i)\, \Gamma_c^b(E_j) - \Gamma_a^c(E_j)\, \Gamma_c^b(E_i)\right)\, x^i\, X_j \\ &+ \mathcal{O}(|x|)^c\, X_c + \mathcal{O}(|x|^2)^j\, X_j, \end{aligned} \tag{2.9}$$

where summation over repeated indices is understood.

Proof : We compute

$$\begin{aligned} X_i\, g(\nabla_{X_a} X_b, X_j) &= g(\nabla_{X_i} \nabla_{X_a} X_b, X_j) + g(\nabla_{X_a} X_b, \nabla_{X_i} X_j) \\ &= g(R(X_i, X_a)\, X_b, X_j) + g(\nabla_{X_a} \nabla_{X_b} X_i, X_j) + g(\nabla_{X_a} X_b, \nabla_{X_i} X_j). \end{aligned}$$

Observe that, by construction, we have arranged in such a way that

$$\nabla_{X_a + \varepsilon X_b} X_i = (\Gamma_{ai}^c + \varepsilon\, \Gamma_{bi}^c)\, X_c,$$

along the geodesic $s \longrightarrow \exp_p^K(s(E_a + \varepsilon E_b))$. Hence, along this geodesic

$$\nabla^2_{X_a + \varepsilon X_b} X_i = ((X_a + \varepsilon\, X_b)(\Gamma_{ai}^c + \varepsilon\, \Gamma_{bi}^c))\; X_c + (\Gamma_{ai}^c + \varepsilon\, \Gamma_{bi}^c)\, \nabla_{X_a + \varepsilon\, X_b}\, X_c. \tag{2.10}$$

Evaluating this at the point p and looking for the coefficient of ε , we obtain

$$(\nabla_{X_a}\, \nabla_{X_b} X_i + \nabla_{X_b}\, \nabla_{X_a} X_i)\big|_p - (\Gamma_{ai}^c\, \nabla_{X_b}\, X_c + \Gamma_{bi}^c\, \nabla_{X_a}\, X_c)\big|_p \in T_p K.$$

Hence we get

$$\begin{aligned} g(\nabla_{X_a}\, \nabla_{X_b} X_i, X_j)\big|_p + g(\nabla_{X_b}\, \nabla_{X_a} X_i, X_j)\big|_p &= \Gamma_{ai}^c\;\, g(\nabla_{X_b}\, X_c, X_j)\big|_p \\ &+ \Gamma_{bi}^c\, g(\nabla_{X_a}\, X_c, X_j)\big|_p \\ &= \Gamma_{ai}^c\, \Gamma_{bc}^j + \Gamma_{bi}^c\, \Gamma_{ac}^j \end{aligned}$$

Finally, we use the fact that

$$g(\nabla_{X_b}\, \nabla_{X_a} X_i, X_j) = g(R(X_b, X_a)\, X_i, X_j) + g(\nabla_{X_a}\, \nabla_{X_b} X_i, X_j)$$

to conclude that, at the point p

$$2\, g(\nabla_{X_a}\, \nabla_{X_b} X_i, X_j)\big|_p = g(R(E_a, E_b)\, E_i, E_j) + \Gamma_{ai}^c\, \Gamma_{bc}^j + \Gamma_{bi}^c\, \Gamma_{ac}^j$$

Collecting these estimates together with the fact that $\nabla_{X_i} X_j\big|_p = 0$ we conclude that

$$2\, X_i\, g(\nabla_{X_a} X_b, X_j)\big|_p = -2g(R(E_i, E_a)\, E_j, E_b) + g(R(E_a, E_b)\, E_i, E_j) + \Gamma_{ai}^c\, \Gamma_{bc}^j + \Gamma_{bi}^c\, \Gamma_{ac}^j.$$

This, together with the fact that $g_{ij} = \delta_{ij} + \mathcal{O}(|x|)^2$, easily implies (2.9). $\square$

2.3 Geometry of tubes

We derive expansions as ρ tends to 0 for the metric, second fundamental form and mean curvature of the tubes $\bar{S}_\rho$ and their perturbations. This is an extension of the computation in [28].

2.3.1 Perturbed tubes

We now describe a suitable class of deformations of the geodesic tubes $\bar{S}_\rho$, depending on a section Φ of NK and a scalar function w on the spherical normal bundle SNK.

Fix $\rho > 0$. It will be convenient to introduce the scaled variable $\bar{y} = y/\rho$; we also use a local parametrization $z \to \Theta(z)$ of S^{n-1}. Now define the map

$$G(z, \bar{y}) := F\left(\rho\,(1 + w(z, \bar{y}))\,\Theta(z) + \Phi(\rho\,\bar{y}), \rho\,\bar{y}\right),$$

and denote its image by $S_\rho(w, \Phi)$, so in particular

$$S_\rho(0, 0) = \bar{S}_\rho.$$

Notation : Because of the definition of these hypersurfaces using the exponential map, various vector fields we shall use may be regarded either as fields along K or along $S_\rho(w, \Phi)$. To help allay this confusion, we write

$$\Phi := \Phi^j\, E_j, \qquad \Phi_a := \partial_{y^a}\,\Phi^j\, E_j, \qquad \Phi_{ab} := \partial_{y^a}\partial_{y^b}\,\Phi^j\, E_j,$$
$$\Theta := \Theta^j\, E_j, \qquad \Theta_i := \partial_{z^i}\Theta^j\, E_j.$$

These are all vectors in the tangent space T_pM at the fixed point $p \in K$. On the other hand, the vectors

$$\Psi := \Phi^j\, X_j, \qquad \Psi_a := \partial_{y^a}\,\Phi^j\, X_j,$$
$$\Upsilon := \Theta^j\, X_j, \qquad \Upsilon_i := \partial_{z^i}\Theta^j\, X_j,$$

lie in the tangent space T_qM, $q = F(z, y)$.

For brevity, we also write

$$w_j := \partial_{z^j} w, \quad w_{\bar{a}} := \partial_{\bar{y}^a} w, \qquad w_{ij} := \partial_{z^i}\,\partial_{z^j} w, \quad w_{\bar{a}\bar{b}} := \partial_{\bar{y}^a}\,\partial_{\bar{y}^b} w, \quad w_{\bar{a}j} := \partial_{\bar{y}^a}\,\partial_{z^j} w.$$

In terms of all this notation, the tangent space to $S_\rho(w, \Phi)$ at any point is spanned by the vectors

$$\begin{aligned} Z_j &= G_*(\partial_{z^j}) = \rho\,((1 + w)\,\Upsilon_j + w_j\,\Upsilon), \qquad j = 1, \dots, n-1, \\ Z_{\bar{a}} &= G_*(\partial_{\bar{y}^a}) = \rho\,(X_a + w_{\bar{a}}\,\Upsilon + \Psi_a), \qquad a = n+1, \dots, m+1. \end{aligned} \tag{2.11}$$

2.3.2 Notation for error terms

The formulas for the various geometric quantities of $S_\rho(w,\Phi)$ are potentially very complicated, and so it is important to condense notation as much as possible. Fortunately, we do not need to know the full structure of all of these quantities. Because it is so fundamental, we have isolated the notational conventions we shall use in this separate subsection.

Any expression of the form $L(w,\Phi)$ denotes a linear combination of the functions w together with its derivatives with respect to the vector fields $\rho\, X_a$ and X_i up to order 2, and Φ^j together with their derivatives with respect to the vector fields X_a up to order 2. The coefficients are assumed to be smooth functions on SNK which are bounded by a constant independent of ρ in the $\mathcal{C}^\infty$ topology (i.e. derivatives taken with respect to X_a and X_i).

Similarly, an expression of the form $Q(w,\Phi)$ denotes a nonlinear operator in the functions w together with its derivatives with respect to the vector fields $\rho\, X_a$ and X_i up to order 2, and Φ^j together with their derivatives with respect to the vector fields X_a up to order 2. Again, the coefficients of the Taylor expansion of the corresponding differential operator are smooth functions on SNK which are bounded by a constant independent of ρ in the $\mathcal{C}^\infty$ topology, and Q which vanishes quadratically at $(w,\Phi)=(0,0)$.

In order to keep notations as simple as possible in the technical proofs, we will use the condensed notations L and Q instead of $L(w,\Phi)$ and $Q(w,\Phi)$.

Finally, any term denoted $\mathcal{O}(\rho^d)$ is a smooth function on SNK which is bounded by a constant by a constant times ρ^d in the $\mathcal{C}^\infty$ topology.

2.3.3 The first fundamental form

The next step is the computation of the coefficients of the first fundamental form of $S_\rho(w,\Phi)$. We set

$$q := G(z,0) = F(\rho(1+w(z,0))\,\Theta(z) + \Phi(0), 0)$$

and $p := G(0,0)$. We obtain directly from (3.2.4) that

$$
\begin{array}{rcl}
g(X_a, X_b) & = & \delta_{ab} - 2\,\rho\,\Gamma_a^b(\Theta) + \mathcal{O}(\rho^2) - 2\,\Gamma_a^b\,(\Phi) + \rho\, L(w,\Phi) + Q(w,\Phi), \\
g(X_i, X_j) & = & \delta_{ij} + \frac{\rho^2}{3}\, g(R(\Theta, E_i)\,\Theta, E_j) + \mathcal{O}(\rho^3), \\
& + & \frac{\rho}{3}\,(g(R(\Theta, E_i)\,\Phi, E_j) + g(R(\Phi, E_i)\,\Theta, E_j)) + \rho^2\, L(w,\Phi) + Q(w,\Phi) \\
g(X_i, X_a) & = & \mathcal{O}(\rho^2) + \rho\, L(w,\Phi) + Q(w,\Phi).
\end{array}
\tag{2.12}
$$

We now explain a simple argument which will be frequently used throughout the chapter. Using the previous expansions, we compute

$$
\begin{array}{rcl}
g(\Upsilon, \Upsilon_j) & = & g(\Theta, \Theta_j) + \frac{\rho^2}{3}\, g(R(\Theta,\Theta)\,\Theta, \Theta_j) + \mathcal{O}(\rho^3) \\
& + & \frac{\rho}{3}\,(g(R(\Theta,\Theta)\,\Phi, \Theta_j) + g(R(\Phi,\Theta)\,\Theta, \Theta_j)) + \rho^2\, L(w,\Phi) + Q(w,\Phi).
\end{array}
$$

However, when $w = 0$ and $\Phi = 0$, $g(\Upsilon, \Upsilon_j) = 0$ since Υ is normal and Υ_j is tangent to $S_\rho(0,0)$ then, so that the sum of the first three terms on the right, which is independent of w and Φ, must also vanish. This, together with the fact that $R(\Theta,\Theta) = 0$ implies that

$$
g(\Upsilon, \Upsilon_j) = \frac{\rho}{3}\, g(R(\Phi,\Theta)\,\Theta, \Theta_j) + \rho^2\, L(w,\Phi) + Q(w,\Phi) \tag{2.13}
$$

Using similar arguments, we have

$$
\begin{array}{rcl}
g(\Upsilon, \Upsilon) & = & g(\Theta, \Theta) + \frac{\rho^2}{3}\, g(R(\Theta,\Theta)\,\Theta, \Theta_j) + \mathcal{O}(\rho^3) \\
& + & \frac{\rho}{3}\,(g(R(\Theta,\Theta)\,\Phi, \Theta) + g(R(\Phi,\Theta)\,\Theta, \Theta)) + \rho^2\, L(w,\Phi) + Q(w,\Phi).
\end{array}
$$

This, together with the fact that $g(\Upsilon,\Upsilon) = 1$ when $w = 0$ and $\Phi = 0$, yields

$$
g(\Upsilon, \Upsilon) = 1 + \rho^2\, L(w,\Phi) + Q(w,\Phi). \tag{2.14}
$$

Using these expansions is is easy to obtain the expansion of the first fundamental form of $S_\rho(w,\Phi)$.

Proposition 2.3.1 *We have*

$$
\begin{array}{rcl}
\rho^{-2}\, g(Z_{\bar a}, Z_{\bar b}) & = & \delta_{ab} - 2\,\rho\,\Gamma_a^b(\Theta) + \mathcal{O}(\rho^2) - 2\,\Gamma_a^b(\Phi) + \rho\, L(w,\Phi) + Q(w,\Phi), \\
\rho^{-2}\, g(Z_{\bar a}, Z_j) & = & \mathcal{O}(\rho^2) + L(w,\Phi) + Q(w,\Phi), \\
\rho^{-2}\, g(Z_i, Z_j) & = & g(\Theta_i, \Theta_j) + \frac{\rho^2}{3}\, g(R(\Theta, \Theta_i)\,\Theta, \Theta_j) + \mathcal{O}(\rho^3) + 2\, g(\Theta_i, \Theta_j)\, w, \\
& + & \frac{\rho}{3}\,(g(R(\Theta,\Theta_i)\Phi, \Theta_j) + g(R(\Theta,\Theta_j)\Phi, \Theta_i)) + \rho^2 L(w,\Phi) + Q(w,\Phi),
\end{array}
\tag{2.15}
$$

where summation over repeated indices is understood.

2.3.4 The normal vector field

Our next task is to understand the dependence on (w, Φ) of the unit normal N to $S_\rho(w, \Phi)$. Define the normal (not unitary) vector field

$$\tilde{N} := -\Upsilon + \frac{1}{\rho}\,(\alpha^j\, Z_j + \beta^a\, Z_{\bar{a}}),$$

where the coefficients α^j and β^a are chosen so that $\tilde{N}$ is orthogonal to all of the $Z_{\bar{b}}$ and Z_i. The unit normal vector field $S_\rho(w, \Phi)$ is defined by

$$N := \frac{\tilde{N}}{|\tilde{N}|}.$$

We have the following :

Proposition 2.3.2 *With the above notations, the coefficients α^j are solutions of the system*

$$g(\Theta_i, \Theta_j)\,\alpha^j = w_i + \frac{\rho}{3}\,g(R(\Phi, \Theta)\,\Theta, \Theta_i) + \rho^2\,L(w, \Phi) + Q(w, \Phi), \qquad i = 1, \ldots, n-1,$$

where summation over j is understood, and the expansion of the coefficients β^a is given by

$$\beta^a = w_{\bar{a}} + g(\Phi_a, \Theta) + \rho\,L(w, \Phi) + Q(w, \Phi).$$

Finally

$$|\tilde{N}|^{-1} = 1 + \rho^2\,L(w, \Phi) + Q(w, \Phi).$$

Proof : We look for coefficients α^j and β^a so that that $\tilde{N}$ is orthogonal to all of the $Z_{\bar{b}}$ and Z_i. This leads to a linear system for α^j and β^a.

We have the following expansions

$$\begin{aligned} g(\Upsilon, Z_{\bar{a}}) &= \rho\,w_{\bar{a}} + \rho\,g(\Phi_a, \Theta) + \rho^2\,L + \rho\,Q, \\ g(\Upsilon, Z_j) &= \rho\,w_j + \tfrac{\rho^2}{3}\,g(R(\Phi, \Theta)\,\Theta, \Theta_j) + \rho^3\,L + \rho\,Q. \end{aligned} \tag{2.16}$$

These follow from (2.12), (2.13) and (2.14), together with the fact that $g(\Upsilon, Z_{\bar{a}}) = 0$ and $g(\Upsilon, Z_j) = 0$ when $w = 0$ and $\Phi = 0$.

Using Proposition 2.3.1, we get with little work the expansions for both β^a and the system α^j satisfy. Collecting these, the estimate for the norm of $\tilde{N}$ follows at once. $\square$

2.3.5 The second fundamental form

We now compute the second fundamental form. To simplify the computations below, we henceforth assume that, at the point $\Theta(z) \in S^{n-1}$,

$$g(\Theta_i, \Theta_j) = \delta_{ij}, \qquad \text{and} \qquad \overline{\nabla}_{\Theta_i} \Theta_j = 0, , \quad i, j = 1, \dots, n-1, \tag{2.17}$$

(where $\overline{\nabla}$ is the connection on TS^{n-1}).

Proposition 2.3.3 *The following expansions hold*

$$\begin{array}{rcl}
\rho^{-2}\, g(N, \nabla_{Z_{\bar a}} Z_{\bar a}) & = & -\Gamma^a_a(\Theta) + \rho\, g(R(\Theta, E_a)\,\Theta, E_a) + \rho\, \Gamma^c_a(\Theta)\, \Gamma^a_c(\Theta) + \mathcal{O}(\rho^2) \\
& - & \frac{1}{\rho}\, w_{\bar a \bar a} - g(\Phi_{aa}, \Theta) + g(R(\Phi, E_a)\,\Theta, E_a) + \Gamma^c_a(\Theta)\, \Gamma^a_c(\Phi) + w_j\, \Gamma^a_a(\Theta_j) \\
& + & \rho\, L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \\
\rho^{-2}\, g(N, \nabla_{Z_j} Z_j) & = & \frac{1}{\rho} + \frac{2}{3}\, \rho\, g(R(\Theta, \Theta_j)\,\Theta, \Theta_j) + \mathcal{O}(\rho^2) \\
& - & \frac{1}{\rho}\, w_{jj} + \frac{1}{\rho}\, w + \frac{2}{3}\, g(R(\Phi, \Theta_j)\,\Theta, \Theta_j) \\
& + & \rho\, L(w, \Phi) + \frac{1}{\rho}\; Q(w, \Phi), \\
\rho^{-2}\, g(N, \nabla_{Z_{\bar a}} Z_{\bar b}) & = & -\Gamma^b_a(\Theta) - \frac{1}{\rho}\, w_{\bar a \bar b} + \mathcal{O}(\rho) + L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \qquad a \neq b, \\
\rho^{-2}\, g(N, \nabla_{Z_{\bar a}} Z_j) & = & \mathcal{O}(\rho) + \frac{1}{\rho}\, L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \\
\rho^{-2}\, g(N, \nabla_{Z_i} Z_j) & = & \mathcal{O}(\rho) + \frac{1}{\rho}\, L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \qquad i \neq j,
\end{array} \tag{2.18}$$

where summation over repeated indices is understood.

Proof : Some preliminary computations are needed. First note that by Lemma 2.2.1, we have

$$\begin{array}{rcl}
\nabla_{X_a} X_b & = & \Gamma^b_a(E_i)\, X_i + (\mathcal{O}(\rho) + L + Q)^\gamma\, X_\gamma, \\
\nabla_{X_i} X_j & = & (\mathcal{O}(\rho) + L + Q)^\gamma\, X_\gamma, \\
\nabla_{X_a} X_i & = & -\Gamma^b_a(E_i)\, X_b + (\mathcal{O}(\rho) + L + Q)^\gamma\, X_\gamma.
\end{array} \tag{2.19}$$

In particular, this, together with the expression of $Z_{\bar a}$, implies that

$$\begin{array}{rcl}
\nabla_{Z_{\bar a}} X_b & = & \rho\, \Gamma^b_a(E_i)\, X_i + (\mathcal{O}(\rho^2) + \rho\, L + \rho\, Q)^\gamma\, X_\gamma, \\
\nabla_{Z_{\bar a}} X_i & = & -\rho\, \Gamma^b_a(E_i)\, X_b + (\mathcal{O}(\rho^2) + \rho\, L + \rho\, Q)^\gamma\, X_\gamma.
\end{array} \tag{2.20}$$

We will also need the following expansion which follows from the result of Lemma 2.2.2

$$\begin{aligned}
\nabla_{X_a} X_b &= +\Gamma_a^b(E_j)\, X_j - g(R(\rho\,\Theta + \Phi, E_a)\, E_j, E_b)\, X_j \\
&+ \tfrac{1}{2}\, \left(g(R(E_a, E_b)\, \rho\,\Theta + \Phi, E_j) - \Gamma_a^c(\rho\,\Theta + \Phi)\, \Gamma_c^b(E_j) - \Gamma_c^b(\rho\,\Theta + \Phi)\, \Gamma_a^c(E_j) \right)\, X_j \\
&+ (\mathcal{O}(\rho) + L + Q)\, X_c + (\mathcal{O}(\rho^2) + \rho\, L + Q)\, X_j.
\end{aligned} \tag{2.21}$$

Finally, we will need the expansions

$$g(\Upsilon, X_a) = \rho\, L + Q, \qquad \text{and} \qquad g(\Upsilon, \Upsilon_j) = \rho\, L + Q, \tag{2.22}$$

whose proof can be obtained as in §3.2, starting from the estimates (2.12) and using the fact that $g(\Upsilon, X_a) = g(\Upsilon, \Upsilon_j) = 0$ when $w = 0$ and $\Phi = 0$.

Observe that it is enough to get these expansions when N is replaced by $\tilde{N}$ and then multiply the expansion by the expansion of $|\tilde{N}|^{-1}$ which is given in Proposition 2.3.2.

First estimate : We estimate $g(\tilde{N}, \nabla_{Z_{\bar{a}}} Z_{\bar{b}})$ when $a = b$ since the corresponding estimate, when $a \neq b$ is not as important and follows from the same proof. We must expand

$$\rho^{-2}\, g(\tilde{N}, \nabla_{Z_{\bar{a}}} Z_{\bar{a}}) = \rho^{-1}\, \left(g(\tilde{N}, \nabla_{Z_{\bar{a}}} X_a) + g(\tilde{N}, \nabla_{Z_{\bar{a}}} (w_{\bar{a}}\, \Upsilon)) + g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Psi_a) \right).$$

The proof is this estimate is broken into three steps :

Step 1 : From Proposition 2.3.2, we get

$$g(\tilde{N}, \Upsilon) = -g(\Upsilon, \Upsilon) + \frac{1}{\rho}\, (\alpha^j\, g(Z_j, \Upsilon) + \beta^a\, g(Z_a, \Upsilon)) = -1 + \rho^2\, L + Q.$$

Substituting $\tilde{N} = -\Upsilon + (\tilde{N} + \Upsilon)$ gives

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Upsilon) = -\frac{1}{2}\, \partial_{\bar{y}^a} g(\Upsilon, \Upsilon) + g(\tilde{N} + \Upsilon, \nabla_{Z_{\bar{a}}} \Upsilon).$$

But it follows from (2.14) that

$$\partial_{\bar{y}^a}\, g(\Upsilon, \Upsilon) = \rho^3\, L + \rho\, Q,$$

and (2.20) together with the expression of $\tilde{N}$ implies that

$$g(\tilde{N} + \Upsilon, \nabla_{Z_{\bar{a}}} \Upsilon) = \rho\, L + \rho\, Q.$$

Collecting these estimates we get

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Upsilon) = \rho\, L + Q.$$

Hence we conclude that

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} (w_{\bar{a}}\, \Upsilon)) = w_{\bar{a}\bar{a}}\, g(\tilde{N}, \Upsilon) + w_{\bar{a}}\, g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Upsilon) = -w_{\bar{a}\bar{a}} + Q.$$

Step 2 : Next,

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Psi_a) = \rho\, g(\tilde{N}, \Psi_{aa}) + \Phi_a^j\, g(\tilde{N}, \nabla_{Z_{\bar{a}}} X_j).$$

From (2.20), we have

$$\Phi_a^j\, g(\tilde{N}, \nabla_{Z_{\bar{a}}} X_j) = \rho^2\, L + \rho\, Q.$$

Also, using the decomposition of $\tilde{N}$ and (2.12), we have

$$g(\tilde{N}, \Psi_{aa}) = -g(\Upsilon, \Psi_{aa}) + g(\tilde{N} + \Upsilon, \Psi_{aa}) = -g(\Theta, \Phi_{aa}) + \rho^2\, L + Q.$$

Collecting these gives

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Psi_a) = -\rho\, g(\Phi_{aa}, \Theta) + \rho^2\, L + \rho\, Q.$$

Step 3 : Expanding $Z_{\bar{a}}$ gives

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} X_a) = \rho\, (g(\tilde{N}, \nabla_{X_a} X_a) + w_{\bar{a}}\, g(\tilde{N}, \nabla_{\Upsilon} X_a) + \Phi_a^j\, g(\tilde{N}, \nabla_{X_j} X_b)). \quad (2.23)$$

With the help of (2.19) and (2.22), we evaluate

$$\begin{aligned} g(\tilde{N}, \nabla_{\Upsilon} X_a) &= \mathcal{O}(\rho) + L + Q, \\ g(\tilde{N}, \nabla_{X_j} X_a) &= \mathcal{O}(\rho) + L + Q, \\ g(\tilde{N} + \Upsilon, \nabla_{X_a} X_a) &= \alpha^j\, \Gamma_a^a\, (\Theta_j) + \rho\, L + Q, \end{aligned}$$

and plugging these into (2.23) already gives

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} X_a) = -\rho\, g(\Upsilon, \nabla_{X_a} X_a) + \rho\, \alpha_j\, \Gamma_a^a(\Theta_j) + \rho^2\, L + \rho\, Q.$$

Using (2.21) we get the expansion

$$\begin{aligned} \nabla_{X_a} X_a &= \Gamma_a^a(E_j)\, X_j - g(R(\rho\Theta + \Phi, E_a)\, E_j, E_a)\, X_j - \Gamma_a^c(\rho\, \Theta + \Phi)\, \Gamma_c^a(E_j)\, X_j \\ &+ (\mathcal{O}(\rho) + L + Q)^c\, X_c + (\mathcal{O}(\rho^2) + \rho\, L + Q)^j\, X_j. \end{aligned}$$

Finally, using (2.12) again together with the fact that $\alpha_j = w_j + \rho\, L$, we conclude that

$$
\begin{aligned}
g(\tilde{N}, \nabla_{Z_{\bar a}} X_a) &= -\rho\, \Gamma^a_a(\Theta) + \rho^2\, g(R(\Theta, E_a)\, \Theta, E_a) + \mathcal{O}(\rho^3) \\
&+ \rho\, g(R(\Phi, E_a)\, \Theta, E_a) + \rho\, \Gamma^c_a(\rho\, \Theta + \Phi)\, \Gamma^a_c(\Theta) + \rho\, w_j\, \Gamma^a_a\, (\Theta_j) \\
&+ \rho^2\, L + \rho\, Q,
\end{aligned}
$$

which, together with the results of Step 1 and Step 2, completes the proof of the first estimate.

Second estimate : We estimate $g(\tilde{N}, \nabla_{Z_i} Z_j)$ when $i = j$ since, just as before, the corresponding estimate, when $i \neq j$ is not as important and follows similarly. This part is taken directly from [28]. Recall that

$$
\tilde{N} = -\Upsilon + \frac{1}{\rho}\, (\alpha^j\, Z_j + \beta^a\, Z_a),
$$

Now write

$$
\begin{aligned}
g(\tilde{N}, \nabla_{Z_j} Z_j) &= -g(\nabla_{Z_j} \tilde{N}, Z_j) \\
&= g(\nabla_{Z_j} \Upsilon, Z_j) - \tfrac{1}{\rho}\, g(\nabla_{Z_j}(\alpha^i\, Z_i), Z_j) \\
&- \tfrac{1}{\rho}\, \beta^a\, g(Z_a, \nabla_{Z_j} Z_j) + \tfrac{1}{\rho}\, \partial_{z^j}\, g(\beta^a\, Z_a, Z_j)
\end{aligned}
$$

Step 1 : We compute

$$
g(Z_a, \nabla_{Z_j} Z_j) = \partial_{y^j}\, g(Z_{\bar a}, Z_j) - g(Z_j, \nabla_{Z_j} Z_{\bar a}) = \partial_{y^j}\, g(Z_{\bar a}, Z_j) - \frac{1}{2}\, \partial_{\bar y^a}\, g(Z_j, Z_j),
$$

and by (2.15), we can estimate

$$
g(Z_a, \nabla_{Z_j} Z_j) = \mathcal{O}(\rho^4) + \rho^2\, L + \rho^2\, Q.
$$

Hence we already obtain

$$
\frac{1}{\rho}\, \beta^a\, g(Z_{\bar a}, \nabla_{Z_j} Z_j) = \rho^3\, L + \rho\, Q.
$$

Step 2 : Next, using the expansion given in Proposition 2.3.2 together with (2.15), we find that

$$
\frac{1}{\rho}\, \partial_{z_j}\, g(\beta^a\, Z_{\bar a}, Z_j) = \rho^3\, L + \rho\, Q.
$$

Step 3 : We now estimate

$$
C := 2\, g(\nabla_{Z_j} \Upsilon, Z_j).
$$

It is convenient to define

$$C' := \frac{2}{1+w}\, g(\nabla_{Z_j}(1+w)\,\Upsilon, Z_j),$$

It follows from (2.16) that

$$C = C' + \rho\, Q,$$

hence it is enough to focuss on the estimate of C'. To analyze this term, let us revert for the moment and regard w and Φ as functions of the coordinates $(z, \bar{y})$ and also consider ρ as a variable instead of just a parameter. Thus we consider

$$\tilde{F}(\rho, z, \bar{y}) = F\big(\rho(1+w(z,\bar{y}))\Upsilon(z) + \Phi(t\,\bar{y}), t\,\bar{y}\big).$$

The coordinate vector fields Z_j are still equal to $\tilde{F}_*(\partial_{z_j})$, but now we also have $(1+w)\Upsilon = \tilde{F}_*(\partial_\rho)$, which is the identity we wish to use below. Now, from (2.15), we write

$$C' = \frac{1}{1+w}\, g(\nabla_{\nabla_{(1+w)\Upsilon}} Z_j, Z_j) = \frac{1}{1+w}\, \partial_\rho g(Z_j, Z_j).$$

Therefore, it follows from (2.15) in Proposition 2.3.1 that

$$\begin{aligned}
C &= \tfrac{1}{1+w}\, \partial_\rho \, [\rho^2\, g(\Theta_j, \Theta_j) + \tfrac{1}{3}\,\rho^4\, g(R(\Theta, \Theta_j)\,\Theta, \Theta_j) + \mathcal{O}(\rho^5) \\
&+ \; 2\,\rho^2\, g(\Theta_j, \Theta_j)\, w + \tfrac{2}{3}\,\rho^3\, g(R(\Theta, \Theta_j)\,\Phi, \Theta_j) + \rho^4\, L + \rho^2\, Q] + \rho\, Q \\
&= \tfrac{1}{1+w}\, [2\,\rho\, g(\Theta_j, \Theta_j) + \tfrac{4}{3}\,\rho^3\, g(R(\Theta, \Theta_j)\,\Theta, \Theta_j) + \mathcal{O}(\rho^4) \\
&+ \; 4\,\rho\, g(\Theta_j, \Theta_j)\, w + 2\,\rho^2\, g(R(\Theta, \Theta_j)\,\Phi, \Theta_j) + \rho^3\, L + \rho\, Q] \\
&= 2\,\rho\, g(\Theta_j, \Theta_j) + \tfrac{4}{3}\,\rho^3\, g(R(\Theta, \Theta_j)\,\Theta, \Theta_j) + \mathcal{O}(\rho^4) \\
&+ \; 2\, w\, g(\Theta_j, \Theta_j)\,\rho + 2\,\rho^2\, g(R(\Theta, \Theta_j)\,\Phi, \Theta_j) + \rho^3\, L + \rho\, Q.
\end{aligned}$$

Step 4 : Finally, we must compute

$$\begin{aligned}
D &:= g(\nabla_{Z_j}(\alpha^i\, Z_i), Z_j) \\
&= g(Z_i, Z_j)\, \partial_{z^j}\alpha^i + \alpha^i\, g(\nabla_{Z_i} Z_j, Z_j) \\
&= g(Z_i, Z_j)\, \partial_{z^j}\alpha^i + \tfrac{1}{2}\,\alpha^i\, \partial_{z^i}\, g(Z_j, Z_j)
\end{aligned}$$

Observe that (2.17) implies

$$\partial_{z^j} g(\Theta_i, \Theta_{j'}) = 0$$

at the point p. Using this together with (2.15) and the expression for the α^i given in Proposition 2.3.2, we get

$$\alpha^i\, \partial_{z^i}\, g(Z_j, Z_j) = \rho^4\, L + \rho^2\, Q.$$

It follows from (2.15) and the definition of α^i again that

$$g(Z_i, Z_j)\,\partial_{z^j}\alpha^i = \rho^2\, g(\Theta_i, \Theta_j)\,\partial_{z^j}\alpha^i + \rho^4\, L + \rho^2\, Q.$$

Therefore, it remains to estimate $g(\Theta_i, \Theta_j)\,\partial_{z^j}\alpha^i$. By definition, we have

$$g(\Theta_i, \Theta_j)\,\alpha^i = w_j + \frac{\rho}{3}\, g(R(\Phi, \Theta)\,\Theta, \Theta_j) + \rho^2\, L + Q.$$

Differentiating with respect to z^j we get

$$\big(g(\Theta_i, \Theta_j)\,\partial_{z^j}\alpha^i + \alpha^i\,\partial_{z^j} g(\Theta_i, \Theta_j)\big) = w_{jj} + \frac{\rho}{3}\,\partial_{z^j} g(R(\Phi, \Theta)\,\Theta, \Theta_j) + \rho^2\, L + Q. \tag{2.24}$$

Again, it follows from (2.17) that $\partial_{z^j} g(\Theta_i, \Theta_j) = 0$. Moreover this also implies that,

$$\nabla_{\Theta_j}\Theta = \Theta_j, \qquad \text{and} \qquad \nabla_{\Theta_j}\Theta_j = a_j\,\Theta,$$

for some $a_j \in \mathbb{R}$. Therefore, we have

$$g(R(\Phi, \Theta)\,\nabla_{\Theta_j}\Theta, \Theta_j) = g(R(\Phi, \Theta)\,\Theta_j, \Theta_j) = 0,$$

and

$$g(R(\Phi, \Theta)\,\Theta, \nabla_{\Theta_j}\Theta_j) = a_j\, g(R(\Phi, \Theta)\,\Theta, \Theta) = 0.$$

Inserting these information into (2.24) yields

$$g(\Theta_i, \Theta_j)\,\partial_{z^j}\alpha^i \;\; = \;\; w_{jj} + \tfrac{\rho}{3}\, g(R(\Phi, \Theta_j)\,\Theta, \Theta_j) + \rho^2\, L + Q.$$

Collecting these estimates, we conclude that

$$D = \rho^2\, w_{jj} \, + \frac{\rho^3}{3}\, g(R(\Phi, \Theta_j)\,\Theta, \Theta_j) + \rho^4\, L + \rho^2\, Q,$$

and with the estimates of the previous steps, this finishes the proof of the estimate.

Third estimate : Decompose

$$\frac{1}{\rho}\, g(\tilde N, \nabla_{Z_{\bar a}} Z_j) = g(\tilde N, \Upsilon_j)\, w_{\bar a} + g(\tilde N, \Upsilon)\, w_{\bar a j} + (1+w)\, g(\tilde N, \nabla_{Z_{\bar a}}\Upsilon_j) + w_j\, g(\tilde N, \nabla_{Z_{\bar a}}\Upsilon).$$

As above we use the expression of $\tilde N$ given in Proposition 2.3.2 to estimate

$$g(\tilde N, \Upsilon_j) = -g(\Upsilon, \Upsilon_j) + g(\tilde N + \Upsilon, \Upsilon_j) = L + Q.$$

Similarly

$$g(\tilde N, \Upsilon) = -1 + L + Q.$$

But now, by (2.20), we have

$$g(\tilde N, \nabla_{Z_{\bar a}}\,\Upsilon_j) = \mathcal{O}(\rho^2) + \rho\, L + \rho\, Q,$$

and, as already shown in the first step of the proof of the first estimate

$$g(\tilde N, \nabla_{Z_{\bar a}}\,\Upsilon) = \rho\, L + Q,$$

and the proof of the estimate follows directly. □

2.4 The mean curvature of perturbed tubes

Collecting the estimates of the last subsection we obtain the expansion of the mean curvature of the hypersurface $S_\rho(w,\Phi)$. In the coordinate system defined in the previous sections, we get

$$
\begin{aligned}
\rho\, m\, H(w,\Phi) &= n-1-\rho\,\Gamma_a^a(\Theta) \\
&+ \Big(g(R(\Theta,\,E_a)\,\Theta,E_a)+\tfrac{1}{3}\,g(R(\Theta,\,E_i)\,\Theta,E_i)-\Gamma_a^c(\Theta)\,\Gamma_c^a(\Theta)\Big)\;\rho^2+\mathcal{O}(\rho^3) \\
&- \left(w_{\bar a\bar a}+\Delta_{S^{n-1}}w+(n-1)\,w\right)-2\,\rho\,\Gamma_a^b(\Theta)\,w_{\bar a\bar b}+\rho\,\Gamma_a^a(\Theta_j)\,w_j \\
&- \rho\, g(\Phi_{aa},\Theta)+\rho\, g(R(E_a,\Phi)\,E_a,\Theta)-\rho\,\Gamma_a^c(\Phi)\,\Gamma_c^a(\Theta)+\rho^2\,L(w,\Phi)+Q(w,\Phi),
\end{aligned}
\tag{2.25}
$$

where summation over repeated indices is understood. We can simplify this rather complicated expression as follows. First, note that

$$
K \text{ minimal} \qquad \Longleftrightarrow \qquad \Gamma_a^a=0,
$$

where summation over a is understood. Next, define

$$
\mathcal{L}_\rho := -\left(\rho^2\,\Delta_K+\Delta_{S^{n-1}}+(n-1)\right), \tag{2.26}
$$

as an operator on the spherical normal bundle SNK with the expression (2.49) in any local coordinates. Also, the Jacobi (linearized mean curvature) operator, for K is defined by

$$
\mathfrak{J} := -\Delta^N+\mathcal{R}^N-\mathcal{B}^N, \tag{2.27}
$$

cf. [15]. To explain the terms here, recall that the Levi-Civita connection for g induces not only the Levi-Civita connection on K, but also a connection ∇^N on the normal bundle NK. The first term here is simply the rough Laplacian for this connection, i.e.

$$
\Delta^N := (\nabla^N)^*\nabla^N=\nabla^N_{E_a}\,\nabla^N_{E_a}-\nabla^N_{(\nabla_{E_a}E_a)^T}.
$$

in the coordinates we have chosen. The second term is the contraction (in normal directions) of the curvature operator for this connection :

$$
\mathcal{R}^N := \left(R(E_a,\cdot)\,E_a\right)^N,
$$

where E_a is (any) orthonormal frame for TK. Finally, the second fundamental form

$$
B: T_pK\times T_pK \longrightarrow N_pK, \qquad B(X,Y):=\left(\nabla_X Y\right)^N, \quad X,Y\in T_pK,
$$

defines a symmetric operator

$$
\mathcal{B}^N := B^t\cdot B;
$$

in terms of the coefficients $\Gamma_a^b := B(E_a, E_b)$,

$$g(\mathcal{B}^N X, Y) = \Gamma_a^b(X)\,\Gamma_b^a(Y).$$

where summation over repeated indices is understood. We also use the Ricci tensor

$$\text{Ric}(X, Y) = g(R(X, E_\gamma)\,E_\gamma, Y), \qquad X, Y \in T_p M.$$

Finally, we introduce the operator

$$g(\cdot, B) \circ \nabla_K^2 = g(\cdot, B(E_a, E_b))\,(\nabla_{E_a}\nabla_{E_b} - \nabla_{(\nabla_{E_a} E_b)^T})$$

in the coordinates we have chosen and the quadratic form

$$\Omega\,(\cdot, \cdot) := -\frac{2}{3}\,g(\mathcal{R}^N \cdot, \cdot) + \frac{1}{3}\,\text{Ric}(\cdot, \cdot) + g(\mathcal{B}^N \cdot, \cdot)$$

acting on $N_p K$. In terms of all of these notations, we have the

Proposition 2.4.1 *Let K be a minimal submanifold. Then the mean curvature of $\mathcal{T}_\rho(w, \Phi)$ can be expanded as*

$$\begin{aligned} \rho\, m\, H(w, \Phi) &= (n-1) - \Omega(\Theta, \Theta)\,\rho^2 + \mathcal{O}(\rho^3) \\ &+ \mathcal{L}_\rho\, w + \rho\, g(\mathfrak{J}\,\Phi, \Theta) - 2\,\rho^3\, g(\Theta, B) \circ \nabla_K^2 w \\ &+ \rho^2\, L(w, \Phi) + Q(w, \Phi). \end{aligned} \tag{2.28}$$

The equation $\rho\, m\, H = n - 1$ can now be written as

$$\mathcal{L}_\rho\, w + \rho\, g(\mathfrak{J}\,\Phi, \Theta) = \Omega(\Theta, \Theta)\,\rho^2 + \mathcal{O}(\rho^3) + 2\,\rho^3\, g(\Theta, B) \circ \nabla_K^2\, w + \rho^2\, L(w, \Phi) + Q(w, \Phi). \tag{2.29}$$

2.4.1 Decomposition of functions on SNK

Before proceeding, we now state more clearly our notation for functions on SNK.

Let (φ_j, λ_j) be the eigendata of $\Delta_{S^{n-1}}$, with eigenfunctions orthonormal and counted with multiplicity. These individual eigenfunctions do not make sense on all of SNK, but their span is a well-defined subspace $\mathcal{S} \subset L^2(SNK)$; thus $v \in \mathcal{S}$ if its restriction to each fibre of SNK lies in the span of $\{\varphi_1, \ldots, \varphi_n\}$. We denote by Π and $\Pi^\perp$ the L^2 orthogonal projections of $L^2(SNK)$ onto $\mathcal{S}$ and $\mathcal{S}^\perp$, respectively.

Now, given any function $v \in L^2(SNK)$, we write

$$\Pi\, v = g(\Phi, \Theta), \qquad \Pi^{\perp} v = \rho\, w,$$

so that

$$v = \rho\, w + g(\Phi, \Theta),$$

here Φ is a section of the normal bundle NK, and the somewhat elaborate notation in the second summand here reflects the fact that any element of $\mathcal{S}$ can be written (locally) as the inner product of a section of NK and the vector Θ, whose components are the linear coordinate functions on each S^{n-1}. We shall often identify this summand with Φ, and thus, in the following, w and Φ will always represent the components of v in $\mathcal{S}^{\perp}$ and $\mathcal{S}$, respectively.

Later on we shall further decompose

$$w = w_0 + w_1, \tag{2.30}$$

where w_0 is a function on K and the integral of w_1 over each fibre of SNK vanishes.

Note that the operator

$$J : v \longrightarrow g(\mathfrak{J}\, \Phi, \Theta),$$

defined for $v = g(\Phi, \Theta)$, preserves $\mathcal{S}$ and is invertible since K is a nondegerate minimal submanifold.

2.5 Improvement of the approximate solution

The first important step in solving (2.29) is to use an iteration scheme to find a sequence of approximate solutions $(w^{(i)}, \Phi^{(i)})$ for which the estimates for the error term are increasingly small. Namely

$$\rho\, m\, H(w^{(i)}, \Phi^{(i)}) = n - 1 + \mathcal{O}(\rho^{i+3}),$$

for all $i \geq 1$.

Letting $(w^{(0)}, \Phi^{(0)}) = (0, 0)$, we define the sequence $(w^{(i+1)}, \Phi^{(i+1)}) \in \mathcal{S}^{\perp} \oplus \mathcal{S}$ inductively as the unique solution to

$$\begin{aligned} \mathcal{L}_0\, w^{(i+1)} + \rho\, g(\mathfrak{J}\, \Phi^{(i+1)}, \Theta) \;&=\; \Omega(\Theta, \Theta)\, \rho^2 + \mathcal{O}(\rho^3) - \rho^2\, \Delta_K w^{(i)} + 2\, \rho^3\, g(\Theta, B) \circ \nabla^2_K w^{(i)} \\ &+\; \rho^2\, L(w^{(i)}, \Phi^{(i)}) + Q(w^{(i)}, \Phi^{(i)}). \end{aligned} \tag{2.31}$$

here

$$\mathcal{L}_0 := -\left(\Delta_{S^{n-1}} + (n-1)\right).$$

Observe, and this is the key point, that the operator Δ_K acting on functions has been moved to the right hand side and hence, the operator on the left hand side is not elliptic anymore. This equation becomes simpler when divided into its $\mathcal{S}^\perp$ and $\mathcal{S}$ components. Thus using that $\mathcal{L}_0$ annihilates $\mathcal{S}$ and

$$\Omega(\Theta, \Theta) \in \mathcal{S}^\perp,$$

since it is quadratic in Θ, (2.31) can be rewritten as the two separate equations :

$$\begin{aligned} \mathcal{L}_0\, w^{(i+1)} &= \Pi^\perp \Big(\Omega(\Theta,\Theta)\rho^2 + \mathcal{O}(\rho^3) - \rho^2\, \Delta_K w^{(i)} + 2\,\rho^3\, g(\Theta, B) \circ \nabla_K^2 w^{(i)} \\ &+ \rho^2\, L(w^{(i)}, \Phi^{(i)}) + Q(w^{(i)}, \Phi^{(i)})\Big)\,, \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} g(\mathfrak{J}\,\Phi^{(i+1)}, \Theta) &= \Pi\,\Big(\mathcal{O}(\rho^2) + 2\rho^2\, g(\Theta, B) \circ \nabla_K^2 w^{(i)} \\ &+ \rho\, L(w^{(i)}, \Phi^{(i)}) + \rho^{-1}\, Q(w^{(i)}, \Phi^{(i)})\Big)\,, \end{aligned}$$

since $\Pi\,(\Delta_K w) = 0$ for all $w \in \mathcal{S}$.

That there is a unique solution now follows directly from the invertibility of the operators J on $\mathcal{S}$ and $\mathcal{L}_0$ on $\mathcal{S}^\perp$, so the only issue is to obtain estimates.

Lemma 2.5.1 *For this sequence $(w^{(i)}, \Phi^{(i)})$, we have the estimates*

$$w^{(i)} = \mathcal{O}(\rho^2), \qquad \Phi^{(i)} = \mathcal{O}(\rho^2),$$

$$w^{(i+1)} - w^{(i)} = \mathcal{O}(\rho^{i+3}), \qquad \Phi^{(i+1)} - \Phi^{(i)} = \mathcal{O}(\rho^{i+2}),$$

for all $i \geq 1$.

Proof : The estimates for $(w^{(1)}, \Phi^{(1)})$ are immediate, and the result for $i > 1$ is proved by a standard induction using the general structure of the operators L and Q. $\Box$

As already mentioned, the operator in the right hand side of (2.32) is not elliptic since $\mathcal{L}_0$ acts on functions defined on SNK and $\mathcal{L}_0$ does not involve any derivatives with respect to y^a. Nevertheless, since we are working with functions in $\mathcal{S}$, the equation

$$\mathcal{L}_0\, w = f,$$

can always be solved for any $f \in \mathcal{S}$ (we have in mind that this equation is solved on each fiber of NK with the base point as a parameter), but without any gain of regularity in the y^a variables and in fact there is a "loss" of two derivatives in the y^a

variables at each iteration. At first glance, it would have been more natural to work with the operator $\mathcal{L}_\rho$, which is elliptic, and solve the equation

$$\mathcal{L}_\rho\, w = f,$$

but the operator $\mathcal{L}_\rho$ has the disadvantage to have a nontrivial kernel in $\mathcal{S}$ each time $\frac{n-1}{\rho^2}$ belongs to the spectrum of $-\Delta_K$. This implies that the corresponding iteration scheme, using the operator $\mathcal{L}_\rho$ instead of $\mathcal{L}_0$ does not work for any value of ρ. In addition, even if $\frac{n-1}{\rho^2}$ is chosen not to belong to the spectrum of $-\Delta_K$, the norm of the inverse of $\mathcal{L}_\rho$ will blow up as ρ tends to 0 and hence the estimates for w_i and Φ_i will not be as good as the one stated in Lemma 2.5.1.

To conclude, the use of the iteration scheme (2.31) allows one to improve the approximate solution to any finite order. Observe that the error $\Omega(\Theta, \Theta)\, \rho^2 + \mathcal{O}(\rho^3)$ in (2.31) is smooth in the y^a variables and hence loosing finitely regularity in these variables is not a real issue.

Finally, replacing (w, Φ) by $(w^{(i)} + w, \Phi^{(i)} + \Phi)$ in (2.29), the equation we must solve becomes

$$\tfrac{1}{\rho}\, \mathcal{L}_\rho\, w + g(\mathfrak{J}\, \Phi, \Theta) - 2\, \rho^2\, g(\Theta, B) \circ \nabla^2_K w + \rho\, L_i(w, \Phi) = \mathcal{O}_i(\rho^{i+2}) + \tfrac{1}{\rho}\, Q_i(w, \Phi). \tag{2.33}$$

This is of course simply the expansion of the equation

$$m\, H(w^{(i)} + w, \Phi^{(i)} + \Phi) = \frac{n-1}{\rho}.$$

The linear and nonlinear operators L_i and Q_i appearing in this equation are different from the ones before, but enjoy similar properties, uniformly in i. The indices i are here to remind the reader that these quantities depend on i.

2.6 Estimating the spectrum of the linearized operators

We now examine the mapping properties of the linear operator

$$(w, \Phi) \longmapsto \frac{1}{\rho}\, \mathcal{L}_\rho\, w + g(\mathfrak{J}\, \Phi, \Theta) - 2\, \rho^2\, g(\Theta, B) \circ \nabla^2_K\, w + \rho\, L_i(w, \Phi) \tag{2.34}$$

which appears in (2.33). This is not precisely the usual Jacobi operator (applied to the function $\rho\, w + g(\Phi, \Theta)$), because we are parametrizing this hypersurface as a graph over $S_\rho(w^{(i)}, \Phi^{(i)})$ using the vector field $-\Upsilon$ rather than the unit normal.

To understand the difference between (2.34) and the Jacobi operator, recall that if N is the unit normal to a hypersurface Σ and $\tilde{N}$ is any other transverse vector field, then hypersurfaces which are $\mathcal{C}^2$ close to Σ can be parameterized as either

$$\Sigma \ni q \mapsto \exp_q^M(vN) \qquad \text{or} \qquad \Sigma \ni q \mapsto \exp_q^M(\tilde{v}\tilde{N}).$$

The corresponding linearized mean curvature operators $\mathbb{L}_{\Sigma,N}$ and $\mathbb{L}_{\Sigma,\tilde{N}}$ are related by

$$\mathbb{L}_{\Sigma,\tilde{N}}\,\tilde{v} = \mathbb{L}_{\Sigma,N}(g(N,\tilde{N})\,\tilde{v}) + m\,(\tilde{N}^T H_\Sigma)\,\tilde{v},$$

here $\tilde{N}^T$ is the orthogonal projection of $\tilde{N}$ onto $T\Sigma$. Since $\mathbb{L}_{\Sigma,N}$ is self-adjoint with respect to the usual inner product, we conclude that $L_{\Sigma,\tilde{N}}$ is self-adjoint with respect to the inner product

$$\langle v, v'\rangle := \int_\Sigma v\,v'\,g(N,\tilde{N})\,dvol_\Sigma.$$

Now suppose that $\Sigma = S_\rho(w^{(i)},\Phi^{(i)})$ and $\tilde{N} = -\Upsilon$. From Lemma 2.5.1 and Proposition 2.3.2 we have

$$g(N,-\Upsilon) = 1 + \mathcal{O}(\rho^4).$$

Furthermore, from Proposition 2.3.1 and Lemma 2.5.1, and the fact that K is minimal, the volume forms of the tubes $S_\rho(w^{(i)},\Phi^{(i)})$ and SNK are related by

$$dvol_{S_\rho(w^{(i)},\Phi^{(i)})} = \rho^{(n-1)/2}\,(1+\mathcal{O}(\rho^2))\,dvol_{SNK}.$$

We define $c_{\rho,i} > 0$ by

$$g(N,-\Upsilon)\,dvol_{S_\rho(w^{(i)},\Phi^{(i)})} = \rho^{(n-1)/2}\,c_{\rho,i}\,dvol_{SNK}. \tag{2.35}$$

and the operator

$$\mathbb{L}_{\rho,i}\,v := c_{\rho,i}\,\left(\frac{1}{\rho}\,\mathcal{L}_\rho\,w + g(\mathfrak{J}\,\Phi,\Theta) - 2\,\rho^2\,g(\Theta,B)\circ\nabla_K^2 w + \rho\,L_i(w,\Phi)\right),$$

where we have decomposed as usual $v = \rho\,w + g(\Phi,\Theta)$. Thanks to (2.35), we can write

$$\mathbb{L}_{\rho,i}\,v = \frac{1}{\rho}\,\mathcal{L}_\rho\,w + g(\mathfrak{J}\,\Phi,\Theta) - 2\,\rho^2\,g(\Theta,B)\circ\nabla_K^2 w + \rho\,\bar{L}_i(w,\Phi), \tag{2.36}$$

where $\bar{L}_i$ enjoys properties similar to the one enjoyed by L_i.

Finally, multiplying (2.33) by $c_{\rho,i}$ gives one further equivalent form of this equation,

$$\mathbb{L}_{\rho,i}\,v = \mathcal{O}_i(\rho^{2+i}) + \frac{1}{\rho}\,\bar{Q}_i\left(\frac{1}{\rho}\Pi^\perp v, \Pi v\right), \tag{2.37}$$

where the nonlinear operator on the right has the same properties as before.

Associated to $\mathbb{L}_{\rho,i}$ is the symmetric bilinear form form

$$\mathcal{C}_{\rho,i}(v,v') := \int_{SNK} v\, \mathbb{L}_{\rho,i}\, v'\, dvol_{SNK},$$

and its associated quadratic form $\mathcal{Q}_{\rho,i}(v) := \mathcal{C}_{\rho,i}(v,v)$.

We shall study these forms as perturbations of the model forms

$$\begin{array}{rcl} \mathcal{C}_0(v,v') & := & -\displaystyle\int_{SNK} w'\,(\rho^2\,\Delta_K w + \Delta_{S^{n-1}} w + (n-1)\,w)\, dvol_{SNK} \\ & + & \dfrac{\omega_{n-1}}{n} \displaystyle\int_K g(\mathfrak{J}\Phi, \Phi')\, dvol_K, \end{array}$$

and associated quadratic form $\mathcal{Q}_0(v) := \mathcal{C}_0(v,v)$, where $\omega_{n-1} = |S^{n-1}|$ is the volume of S^{n-1}. Observe that

$$\int_{SNK} g(\Phi,\Theta)^2 dvol_{SNK} = \frac{\omega_{n-1}}{n} \int_K |\Phi|^2\, dvol_K.$$

To make precise the sense in which $\mathcal{Q}_0$ and $\mathcal{Q}_{\rho,i}$ are close, define the weighted norm

$$\|v\|^2_{H^1_\rho} := \int_{SNK} (\rho^2\, |\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 + |w|^2)\, dvol_{SNK} + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2)\, dvol_K,$$

and also

$$\|v\|^2_{L^2_\rho} := \int_{SNK} |w|^2\, dvol_{SNK} + \int_K |\Phi|^2\, dvol_K.$$

Using (2.35) and the properties of $\bar{L}_i$, we have the important :

Proposition 2.6.1 *There exists a constant $c > 0$ (independent of i) such that*

$$|\mathcal{C}_{\rho,i}(v,v') - \mathcal{C}_0(v,v')| \leq c\,\rho\, \|v\|_{H^1_\rho}\, \|v'\|_{H^1_\rho}. \tag{2.38}$$

Proof : This estimate arises from the fact that, $-2\,\rho\, g(\Theta, B) \circ \nabla^2_K w + \bar{L}_i(w,\Phi)$ certainly involves terms of the form w, $\rho\,\partial_{y^a} w$, $\rho\,\partial_{y^a}\partial_{y^b} w$, $\partial_{z^j} w$, $\partial_{z^j}\partial_{z^{j'}} w$ and also Φ^j, $\partial_{y^a}\Phi^j$ and $\partial_{y^a}\partial_{y^b}\,\Phi^j$. Hence, after integration by parts,

$$\int_{SNK} (-2\,\rho\, g(\Theta,B) \circ \nabla^2_K w + \bar{L}_i(w,\Phi))\,(\rho\, w' + g(\Phi',\Theta))\, dvol_{SNK},$$

can be bounded by a constant times $\|v\|_{H^1_\rho}\, \|v'\|_{H^1_\rho}$. □

2.6.1 Estimates for eigenfunctions with small eigenvalues

We prove that eigenfunctions of $\mathbb{L}_{\rho,i}$ corresponding to small eigenvalues are localized in the sense that they are essentially functions defined on K.

Lemma 2.6.1 *Let σ be an eigenvalue of $\mathbb{L}_{\rho,i}$ and $v = \rho\, w + g(\Phi, \Theta)$ a corresponding eigenfunction. There exist constants $c, c_0 > 0$ such that if $|\sigma| \leq c_0$, then*

$$\|v - \rho\, w_0\|^2_{H^1_\rho} \leq c\, \rho\, \|v\|^2_{H^1_\rho},$$

for all $\rho \in (0,1)$, where $w = w_0 + w_1$ is the decomposition from (2.30).

Proof : For any $v' = \rho\, w' + g(\Phi', \Theta)$, we have

$$\begin{aligned} \mathcal{C}_{\rho,i}(v,v') &= \sigma \int_{SNK} (\rho^2 w\, w' + g(\Phi,\Theta) g(\Phi',\Theta))\, dvol_{SNK} \\ &= \sigma \int_{SNK} \rho^2 w\, w'\, dvol_{SNK} + \sigma\, \frac{\omega_{n-1}}{n} \int_K g(\Phi, \Phi')\, dvol_K. \end{aligned}$$

In addition, (2.38) gives

$$\begin{aligned} \Big| \int_{SNK} (\rho^2\, \nabla_K w\, \nabla_K w' + \nabla_{S^{n-1}} w \nabla_{S^{n-1}} w' - (n-1+\sigma\,\rho^2)\, w\, w')\, dvol_{SNK} \\ - \frac{\omega_{n-1}}{n} \int_K (g(\mathfrak{J}\Phi, \Phi') - \sigma\, g(\Phi,\Phi'))\, dvol_K \Big| \leq c\, \rho\, \|v\|_{H^1_\rho}\, \|v'\|_{H^1_\rho}. \end{aligned} \tag{2.39}$$

Step 1 : Take $w' = 0$ and $\Phi' = \Phi^+$ (resp. $\Phi' = \Phi^-$) in (2.39), where Φ^+ (resp. Φ^-) is the L^2 projection of Φ over the space of eigenfunctions of $\mathfrak{J}$ associated to positive (resp. negative) eigenvalues. This yields

$$\left| \int_K (g(\mathfrak{J}\Phi, \Phi^\pm) - \sigma\, g(\Phi, \Phi^+))\, dvol_K \right| \leq c\, \rho\, \|v\|_{H^1_\rho}\, \|g(\Phi^\pm, \Theta)\|_{H^1_\rho}.$$

Since $\mathfrak{J}$ is invertible, there exists $c_1 > 0$ such that

$$2\, c_1\, \|g(\Phi^\pm, \Theta)\|^2_{H^1_\rho} \leq \left| \int_K g(\mathfrak{J}\Phi, \Phi^\pm)\, dvol_K \right|,$$

hence

$$(2\, c_1 - |\sigma|)\, \|g(\Phi^\pm, \Theta)\|^2_{H^1_\rho} \leq c\, \rho\, \|v\|^2_{H^1_\rho}.$$

Assuming $c_1 \geq |\sigma|$, we conclude that

$$\|g(\Phi^\pm, \Theta)\|^2_{H^1_\rho} \leq c\, \rho\, \|v\|^2_{H^1_\rho}.$$

Step 2 : Now use (2.39) with $\Phi' = 0$ and $w' = w_1$ to get

$$\left| \int_{SNK} (\rho^2 \, |\nabla_K w_1|^2 + |\nabla_{S^{n-1}} w_1|^2 - (n - 1 - \sigma \, \rho^2) \, |w_1|^2) \, dvol_{SNK} \right| \leq c \, \rho \, \|v\|_{H^1_\rho} \, \|\rho \, w_1\|_{H^1_\rho}.$$

However, since

$$\Pi \, w_1 = 0 \qquad \text{and} \qquad \int_{S^{n-1}} w_1 \, dvol_{S^{n-1}} = 0,$$

we have

$$\int_{S^{n-1}} |\nabla_{S^{n-1}} w_1|^2 \, dvol_{S^{n-1}} \geq 2 \, n \int_{S^{n-1}} |w_1|^2 \, dvol_{S^{n-1}},$$

hence

$$\left| \int_{SNK} (\rho^2 \, |\nabla_K w_1|^2 + \frac{1}{2} \, |\nabla_{S^{n-1}} w_1|^2 + (1 - |\sigma| \, \rho^2) \, |w_1|^2) \, dvol_{SNK} \right| \leq c \, \rho \, \|v\|_{H^1_\rho} \, \|\rho \, w_1\|_{H^1_\rho}.$$

This implies that

$$\|\rho \, w_1\|^2_{H^1_\rho} \leq c \, \rho \, \|v\|^2_{H^1_\rho},$$

for all $\rho \in (0, 1)$, provided $|\sigma| \leq 1/2$. This completes the proof if $c_0 = \min(c_1, 1/2)$, since $v - \rho \, w_0 = \rho \, w_1 + g(\Phi, \Theta)$. □

2.6.2 Variation of small eigenvalues with respect to ρ

We shall need to obtain some information about the spectral gaps of $\mathbb{L}_{\rho,i}$ when ρ is small, and to do this, it is necessary to understand the rate of variation of the small eigenvalues of this operator.

Lemma 2.6.2 *There exist constants $c_0, c > 0$ such that, if σ is an eigenvalue of $\mathbb{L}_{\rho,i}$ with $|\sigma| < c_0$, then*

$$\rho \, \partial_\rho \sigma \geq 2 \, (n - 1) - c \, \rho,$$

provided ρ is small enough.

Proof : There is a well-known formula for the variation of a simple eigenvalue

$$\partial_\rho \sigma = \int_{SNK} v \, (\partial_\rho \mathbb{L}_{\rho,i}) \, v \, dvol_{SNK},$$

where $\mathbb{L}_{\rho,i} v = \sigma \, v$, is normalized by $\|v\|_{L^2} = 1$. Here, by definition,

$$\|v\|^2_{L^2} := \int_{SNK} v^2 \, dvol_{SNK}.$$

Complications arise in the presence of multiplicities, but a result of Kato [14] shows that if one considers the derivative of the eigenvalue as a multi-valued function, then an analogue of this same formula holds for self adjoint operators :

$$\partial_\rho \sigma \in \left\{ \int_{SNK} v \, (\partial_\rho \mathbb{L}_{\rho,i}) \, v \, dvol_{SNK} \qquad : \qquad \mathbb{L}_{\rho,i} v = \sigma \, v, \qquad \|v\|_{L^2} = 1 \right\}.$$

Hence we must provide bounds for the set on the right. We do this by comparing to the model case and using the bounds for eigenfunctions obtained in the last subsection.

Assume that $\mathbb{L}_{\rho,i} v = \sigma \, v$, but rather than normalizing the function v by $\|v\|_{L^2} = 1$, assume instead that $\|v\|_{L^2_\rho} = 1$. In order to compute $\partial_\rho \mathbb{L}_{\rho,i}$, recall that

$$w = \rho^{-1} \, \Pi^\perp v \qquad \text{and that} \qquad g(\mathfrak{J}\Phi, \Theta) = \Pi \, v,$$

so we can write

$$\mathbb{L}_{\rho,i} \, v = -\Delta_K \, (\Pi^\perp \, v) + \frac{1}{\rho^2} \, \mathcal{L}_0 \, (\Pi^\perp \, v) + \Pi \, v - 2 \, \rho \, g(\Theta, B) \circ \nabla^2_K \, (\Pi^\perp \, v) + \rho \, \bar{L}_i(\rho^{-1} \, \Pi^\perp v, \Pi \, v).$$

Since Π and $\Pi^\perp$ are independent of ρ, we have

$$\partial_\rho \mathbb{L}_{\rho,i} v = -\frac{2}{\rho^3} \, \mathcal{L}_0 \, (\Pi^\perp v) - 2 \, g(\Theta, B) \circ \nabla^2_K \, (\Pi^\perp \, v) + \, \tilde{L}_i(\rho^{-1} \, \Pi^\perp v, \Pi v),$$

where the operator $\tilde{L}_i$ varies from line to line but satisfies the usual assumptions. This now gives

$$\left| \int_{SNK} v \, (\partial_\rho \mathbb{L}_{\rho,i}) \, v \, dvol_{SNK} - \frac{2}{\rho} \int_{SNK} (|\nabla_{S^{n-1}} w|^2 - (n-1) \, |w|^2) \, dvol_{SNK} \right| \leq c \, \|v\|^2_{H^1_\rho}. \tag{2.40}$$

Now, if v is an eigenfunction of $\mathbb{L}_{\rho,i}$, we have

$$Q_{\rho,i}(v) = \sigma \, \|v\|^2_{L^2} = \sigma \int_{SNK} \rho^2 \, |w|^2 \, dvol_{SNK} + \sigma \, \frac{\omega_{n-1}}{n} \int_K |\Phi|^2 \, dvol_K,$$

and hence by (2.38),

$$\begin{aligned} \Bigg| \int_{SNK} (\rho^2 \, |\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 - (n - 1 + \sigma \, \rho^2) \, |w|^2) \, dvol_{SNK} \\ - \frac{\omega_{n-1}}{n} \int_K (g(\mathfrak{J}\Phi, \Phi) + \sigma \, g(\Phi, \Phi)) \, dvol_K \Bigg| \leq c \, \rho \, \|v\|^2_{H^1_\rho}, \end{aligned} \tag{2.41}$$

By Lemma 2.6.1, if we assume that $|\sigma| \leq c_0$ and if, as usual we decompose $v = \rho \, w + g(\Phi, \Theta)$, we get

$$\int_{SNK} |\nabla_{S^{n-1}} w|^2 \, dvol_{SNK} + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2) \, dvol_K \leq c \, \rho \, \|v\|^2_{H^1_\rho}, \tag{2.42}$$

(observe that $\nabla_{S^{n-1}} w = \nabla_{S^{n-1}} w_1$ if w is decomposed as $w = w_0 + w_1$ as usual) and inserting this in (2.41) gives

$$\left| \int_{SNK} (\rho^2 \, |\nabla_K w|^2 - (n-1+\sigma) \, |w|^2) \, dvol_{SNK} \right| \leq c \, \rho \, \|v\|^2_{H^1_\rho}. \tag{2.43}$$

Adding these last two estimates now implies that

$$\|v\|^2_{H^1_\rho} \leq c \, \rho \, \|v\|^2_{H^1_\rho} + c \int_{SNK} |w|^2 \, dvol_{SNK};$$

Thus, when ρ is small enough,

$$\|v\|^2_{H^1_\rho} \leq c \int_{SNK} w^2 \, dvol_{SNK} \leq c \, \|v\|^2_{L^2_\rho} \leq c,$$

if we normalize v by $\|v\|_{L^2_\rho} = 1$. From (2.42) again

$$\int_{SNK} |\nabla_{S^{n-1}} w|^2 \, dvol_{SNK} + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2) \, dvol_K \leq c \, \rho.$$

Inserting this into (2.40), and using again that $\|v\|_{L^2_\rho} = 1$, we get

$$\left| \int_{SNK} v \, (\partial_\rho \mathbb{L}_{\rho,i}) \, v \, dvol_{SNK} - \frac{2}{\rho} \, (n-1) \right| \leq c \tag{2.44}$$

for all eigenfunction v such that $\mathbb{L}_{\rho,i} v = \sigma \, v$ which is normalized by $\|v\|_{L^2_\rho} = 1$.

This already implies that $\partial_\rho \sigma > 0$ for ρ small enough. But observing that we always have $||v||_{L^2} \leq \|v\|_{L^2_\rho}$, we conclude that

$$\inf_{\substack{\mathbb{L}_\rho v = \sigma \, v \\ \|v\|_{L^2} = 1}} \int_{SNK} v \, (\partial_\rho \mathbb{L}_\rho) \, v \, dvol_{SNK} \quad \geq \inf_{\substack{\mathbb{L}_\rho v = \sigma \, v \\ \|v\|_{L^2_\rho} = 1}} \int_{SNK} v \, (\partial_\rho \mathbb{L}_\rho) \, v \, dvol_{SNK},$$

and (2.44) implies that

$$\partial_\rho \sigma \geq \frac{2}{\rho} \, (n-1) - c.$$

This completes the proof of the result. □

2.6.3 The spectral gap at 0 of $\mathbb{L}_{\rho,i}$

We can now prove a quantitative statement about the clustering of the spectrum at 0 of $\mathbb{L}_{\rho,i}$ as $\rho \searrow 0$. The ultimate goal is to estimate the norm of the inverse of this operator, but by self-adjointness, this is equivalent to an estimate on the size of the spectral gap at 0.

Lemma 2.6.3 *Fix any $q \geq 2$. Then there exists a sequence of disjoint nonempty open intervals $I_\ell = (\rho_\ell^-, \rho_\ell^+)$, $\rho_\ell^\pm \to 0$ and a constant $c_q > 0$ such that when $\rho \in I^q := \cup_\ell I_\ell$, the operator $\mathbb{L}_{\rho,i}$ is invertible and*

$$(\mathbb{L}_{\rho,i})^{-1} : L^2(SNK) \longrightarrow L^2(SNK),$$

has norm bounded by $c_q\, \rho^{-k-q+1}$, uniformly in $\rho \in I$. Furthermore, $I^q := \cup_\ell I_\ell$ satisfies

$$\left|\mathcal{H}^1((0,\rho) \cap I^q) - \rho\right| \leq c\, \rho^q, \qquad \rho \searrow 0.$$

Proof : An estimate for the size of the spectral gap at 0 is related to the spectral flow of $\mathbb{L}_{\rho,i}$, and so it suffices to find an asymptotic estimate for the number of negative eigenvalues of $\mathbb{L}_{\rho,i}$. Define the two quadratic forms

$$\mathcal{Q}^\pm(v) := \mathcal{Q}_0(v) \pm \gamma\, \rho\, \|v\|^2_{H^1_\rho}.$$

From (2.38), if $\gamma > 0$ is sufficiently large, then

$$\mathcal{Q}^- \leq \mathcal{Q}_{\rho,i} \leq \mathcal{Q}^+,$$

and this will give a two-sided bound for the index of $\mathcal{Q}_{\rho,i}$, i.e. the dimension of the largest space where $\mathcal{Q}_{\rho,i}$ is negative.

Given any function w defined on SNK, we write

$$D_0^\pm(w) := (1 \pm \gamma\, \rho) \int_K \rho^2\, |\nabla_K w|^2\, dvol_{SNK} - (n - 1 \mp \gamma\, \rho) \int_K |w|^2\, dvol_K,$$

$$D_1^\pm(w) := (1 \pm \gamma\, \rho) \int_{SNK} (\rho^2\, |\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2)\, dvol_{SNK} - (n - 1 \mp \gamma\, \rho) \int_{SNK} |w|^2\, dvol_K,$$

and finally, we define

$$D^\pm(\Phi) := -(1 \pm \gamma\, \rho) \int_K g(\mathfrak{J}\, \Phi, \Phi)\, dvol_K.$$

With these definitions in mind, we have

$$\mathcal{Q}^\pm(v) = \omega_{n-1}\, D_0^\pm(w_0) + D_1^\pm(w_1) + \frac{\omega_{n-1}}{n}\, D^\pm(\Phi),$$

if we decompose $v = \rho\, w + g(\Phi, \Theta)$ and further decompose $w = w_0 + w_1$ as usual.

If $1-\gamma\,\rho>0$, then the index of $D^\pm$ is equal to the index of the minimal submanifold K, and hence does not depend on ρ. Next, if $2\,n\,(1-\gamma\,\rho)-(n-1+\gamma\,\rho)>0$, then the index of $D_1^\pm$ equals 0 since we have

$$\int_{S^{n-1}} |\nabla_{S^{n-1}} w_1|^2 \, dvol_{S^{n-1}} \geq 2\,n \int_{S^{n-1}} |w_1|^2 \, dvol_{S^{n-1}}.$$

So it remains only to study the index of $D_0^\pm$. We denote by

$$\mu_0 < \mu_1 \leq \ldots \leq \mu_j \leq \ldots$$

the eigenvalues of $-\Delta_K$ which are counted with multiplicity. Weyl's asymptotic formula states that

$$\#\{j \in \mathbb{N} \quad : \quad \mu_j \leq \mu\} \sim c_K\, \mu^{\frac{k}{2}}.$$

where $c_K>0$ only depends on the dimension and the volume of K. Now, the index of $D_0^\pm$ is equal to the largest $j\in\mathbb{N}$ such that

$$(1 \pm \gamma\,\rho)\,\rho^2\,\mu_j < (n-1 \mp c\,\rho).$$

Using Weyl's asymptotic formula, we conclude that

$$\mathrm{Ind}\, D_0^\pm \sim c_K \left(\frac{n-1}{\rho^2}\right)^{\frac{k}{2}},$$

and hence we have proved that the index $\mathcal{Q}_{\rho,i}$ is asymptotic to $c\,\rho^{-k}$, where c only depends on K and m.

Let $\rho_\ell \searrow 0$ be the decreasing sequence corresponding to the values at which the index of $\mathcal{Q}_{\rho,i}$ changes, counted according to the dimension of the nullspace of $\mathbb{L}_{\rho_\ell,i}$, i.e.

$$\rho_{\ell-1} < \rho_\ell = \ldots = \rho_{\ell'} < \rho_{\ell'+1},$$

if $\dim \mathrm{Ker}\, \mathbb{L}_{\rho_\ell,i} = \ell' - \ell + 1$. This is well-defined since, by Lemma 2.6.2 the small eigenvalues of $\mathbb{L}_{\rho,\ell}$ are monotone increasing for ρ small enough and hence, the function

$$\rho \longrightarrow \mathrm{Ind}\, \mathcal{Q}_{\rho,i},$$

is monotone decreasing for ρ small.

The asymptotic estimates for $\mathrm{Ind}\, Q_{2\rho,i}$ and $\mathrm{Ind}\, Q_{\rho,i}$ imply that

$$r_\rho := \#\{\ell \quad : \quad \rho_\ell \in (\rho, 2\rho)\} \sim c\,\rho^{-k}.$$

Letting λ_ρ denote the sum of lengths of intervals $(\rho_{\ell+1}, \rho_\ell)$ for which

$$\rho_{\ell+1} \in (\rho, 2\rho) \qquad \text{and} \qquad (\rho_\ell - \rho_{\ell+1}) \leq \rho^{k+q},$$

then we have $\lambda_\rho \leq c\,\rho^q$. From this we conclude that $\tilde{\lambda}_\rho$, the sum of lengths of all intervals $(\rho_{\ell+1}, \rho_\ell)$ where

$$\rho_{\ell+1} < \rho \qquad \text{and} \qquad (\rho_\ell - \rho_{\ell+1}) \leq \rho^{k+q},$$

is also bounded by $c\,\rho^q$, where the constant $c > 0$ depends on q.

We set

$$\tilde{I}^q := \bigcup_{\ell \in J_q} (\rho_{\ell+1}, \rho_\ell), \qquad \text{where} \qquad \ell \in J_q \qquad \Leftrightarrow \qquad \rho_\ell - \rho_{\ell+1} > \rho_\ell^{k+q}.$$

Then by the above, we have

$$\left| \mathcal{H}^1((0,\rho) \cap \tilde{I}^q) - \rho \right| \leq c_q\, \rho^q.$$

Finally, consider for any $\ell \in J_q$ and $\rho \in (\rho_{\ell+1}, \rho_\ell)$. We denote by

$$\sigma^-(\rho) < 0 < \sigma^+(\rho)$$

the eigenvalues of $\mathbb{L}_{\rho,i}$ which are closest to 0. By construction,

$$\lim_{\rho \searrow \rho_{\ell+1}} \sigma^+(\rho) = \lim_{\rho \nearrow \rho_\ell} \sigma^-(\rho) = 0.$$

By Lemma 2.6.2,

$$\begin{aligned} \sigma^-(\rho) &\leq 2\,(n-1)\, \log(\rho/\rho_\ell) + c\,(\rho_\ell - \rho), \\ \sigma^+(\rho) &\geq 2\,(n-1)\, \log(\rho/\rho_{\ell+1}) - c\,(\rho - \rho_{\ell+1}), \end{aligned} \tag{2.45}$$

for all $\rho \in (\rho_{\ell+1}, \rho_\ell)$. Hence by the monotonicity of small eigenvalues,

$$\sigma^-(\rho) \leq \sigma^-(\rho_\ell - \rho_\ell^{k+q}/4) < 0 < \sigma^+(\rho_{\ell+1} + \rho_\ell^{k+q}/4) \leq \sigma^+(\rho),$$

if

$$\rho \in I^q := \bigcup_\ell (\rho_{\ell+1} + \rho_\ell^{k+q}/4, \rho_\ell - \rho_\ell^{k+q}/4),$$

and, using (2.45) we conclude that the infimum of the absolute value of the eigenvalues of $\mathbb{L}_{\rho,i}$ is bounded from below by a constant (only depending on K and m) times ρ_ℓ^{k+q-1}, provided ρ_ℓ is small enough. Moreover, as above we have

$$\left| \mathcal{H}^1((0,\rho) \cap I^q) - \rho \right| \leq c_q\, \rho^q.$$

The result then follows at once. $\square$

2.7 Existence of constant mean curvature hypersurfaces

We now use the results of the previous sections in order to solve the equation (2.37) which reduces to find a fixed point

$$v = (\mathbb{L}_{\rho,i})^{-1}\left(\mathcal{O}_i(\rho^{2+i}) + \frac{1}{\rho}\,\bar{Q}_i\left(\frac{1}{\rho}\,\Pi^\perp v, \Pi v\right)\right).$$

We start with the following elementary observation

Lemma 2.7.1 *There exists a constant* $c > 0$ *such that*

$$\rho^{2+\alpha}\,\|v\|_{\mathcal{C}^{2,\alpha}(SNK)} \leq c\,\rho^2\,\|\mathbb{L}_{\rho,i}\,v\|_{\mathcal{C}^{0,\alpha}(SNK)} + c\,\rho^{-\frac{k}{2}}\,\|v\|_{L^2(SNK)}$$

Proof : This is a simple application of (rescaled) standard elliptic estimates. We set $f := \mathbb{L}_{\rho,i}\,v$ and, as in §3.1, we use local normal coordinates $\bar{y} = y/\rho$ to parameterize a ball of radius $2\,\rho\,R$ in K, for some fixed small constant $R > 0$, and local coordinates z to parameterize S^{n-1}. Define the functions

$$\bar{v}(z,\bar{y}) := v(z,\rho\,\bar{y}) \qquad \text{and} \qquad \bar{f}(z,\bar{y}) := \rho^2\,f(z,\rho\,\bar{y})$$

It is easy to check that $f := \mathbb{L}_{\rho,i}\,v$ translates into $\bar{\mathbb{L}}_{\rho,i}\bar{v} = \bar{f}$, where $\bar{\mathbb{L}}_{\rho,i}$ is a second order elliptic operator whose coefficients are bounded uniformly as ρ tends to 0. Moreover, the principal part of $\bar{\mathbb{L}}_{\rho,i}$ is the Laplace operator on SNK. Standard elliptic estimates yield

$$\|\bar{v}\|_{\bar{\mathcal{C}}^{2,\alpha}(B_R\times S^{n-1})} \leq c\,\|\bar{f}\|_{\bar{\mathcal{C}}^{0,\alpha}(B_R\times S^{n-1})} + c\,\left(\int_{S^{n-1}}\left(\int_{B_{2R}}|\bar{v}|^2\,d\bar{y}\right)\,dvol_{S^{n-1}}\right)^{1/2}$$

where, to evaluate the Hölder norms in $\bar{\mathcal{C}}^{p,\alpha}$ one takes derivatives with respect to $\bar{y}$ and z. Going back to the functions v and f we have

$$\rho^{2+\alpha}\,\|v\|_{\mathcal{C}^{2,\alpha}(B_{\rho R}\times S^{n-1})} \leq c\,\|\bar{v}\|_{\bar{\mathcal{C}}^{2,\alpha}(B_R\times S^{n-1})}, \qquad \|\bar{f}\|_{\bar{\mathcal{C}}^{0,\alpha}(B_R\times S^{n-1})} \leq c\,\rho^2\,\|f\|_{\mathcal{C}^{0,\alpha}(B_{\rho R}\times S^{n-1})}$$

and

$$\left(\int_{S^{n-1}}\left(\int_{B_{2R}}|\bar{v}|^2\,d\bar{y}\right)\,dvol_{S^{n-1}}\right)^{1/2} \leq c\,\rho^{-\frac{k}{2}}\left(\int_{S^{n-1}}\left(\int_{B_{2\rho R}}|v|^2\,dy\right)\,dvol_{S^{n-1}}\right)^{1/2}$$

the result then follows at once. □

We fix $q \geq 2$ and $\alpha \in (0,1)$ and define

$$D := \frac{3}{2}\,k + q + \alpha + 1, \qquad \text{and} \qquad i = 3\,k + 2q + 4 > 2D + 1.$$

Collecting the result of Lemma 2.6.3 and the result of the previous Lemma, we conclude that, if $\rho \in I^q$, then

$$\|v\|_{\mathcal{C}^{2,\alpha}(SNK)} \leq c\,\rho^{-D}\,\|\mathbb{L}_{\rho,i}\,v\|_{\mathcal{C}^{0,\alpha}(SNK)}, \tag{2.46}$$

where the constant $c > 0$ does not depend on ρ (but depends on i, hence on q).

We define the nonlinear mapping

$$\mathcal{N}_\rho(v) := (\mathbb{L}_{\rho,i})^{-1}\left(\mathcal{O}_i(\rho^{2+i}) + \frac{1}{\rho}\,\bar{Q}_i\left(\frac{1}{\rho}\,\Pi^\perp v, \Pi v\right)\right).$$

It follows from (2.46) that we have

$$\|\mathcal{N}_\rho(0)\|_{\mathcal{C}^{2,\alpha}} \leq \frac{c_q}{2}\,\rho^{2+i-D}$$

for some constant $c_q > 0$ depending on q but independent of $\rho \in I^q$.

Given $\rho > 0$, we set

$$B_\rho := \{v \in \mathcal{C}^{2,\alpha}(SNK) \qquad : \qquad \|v\|_{\mathcal{C}^{2,\alpha}} \leq c_q\,\rho^{2+i-D}\}.$$

Using the properties of the operator $\bar{Q}_i$, it is easy to check that there exists $\rho_q > 0$, only depending on q, such that, for all $\rho \in (0, \rho_q) \cap I^q$,

$$\|\mathcal{N}_\rho(v)\|_{\mathcal{C}^{2,\alpha}(SNK)} \leq c_0\,\rho^{2+i-D}$$

and

$$\|\mathcal{N}_\rho(v) - \mathcal{N}_\rho(v')\|_{\mathcal{C}^{2,\alpha}} \leq c\,\rho^{i-1-2D}\,\|v - v'\|_{\mathcal{C}^{2,\alpha}}$$

for all $v, v' \in B_\rho$. In particular, the mapping $\mathcal{N}_\rho$ admits a (unique) fixed point

$$v_\rho = \rho\,w_\rho + g(\Phi_\rho, \Theta)$$

in B_ρ. This yields the existence of $S_\rho(w^{(i)} + w_\rho, \Phi^{(i)} + \Phi_\rho)$, a constant mean curvature perturbation of the tube $S_\rho(w^{(i)}, \Phi^{(i)})$ for all $\rho \in (0, \rho_q) \cap I^q$. The proof of the Theorem is therefore complete with

$$I := \cup_{q \geq 2}((0, \rho_q) \cap I^q).$$

2.8 Existence of constant r-curvature hypersurfaces

In this section, we extend the result of theorem 2.1.1 to any $1 \leq r < n-1 = m-k$. We prove

Theorem 2.8.1 *Suppose that $1 \leq r < n-1$ and K^k is a nondegenerate closed minimal submanifold $1 \leq k \leq m-1$. Then there exists $I \subset (0, +\infty)$, countable union of disjoint nonempty open intervals, such that for all $\rho \in I$, the geodesic tube $\bar{S}_\rho$ may be perturbed to a constant r-curvature hypersurface S_ρ with $\sigma_r = C^r_{n-1}\, \rho^{-r}$. Moreover, for any $q \geq 2$ there exists a $c_q > 0$ such that*

$$|\mathcal{H}^1((0,\rho) \cap I) - \rho| \leq c_q\, \rho^q,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

As noted above, this result extend the result of theorem 2.1.1 to any r-curvature, for $1 \leq r < n-1$. The prove is the same, but a new phenomena of lost of ellipticity appear which explain the fact that we assume that $1 \leq r < n-1$.

2.8.1 The shape operator of the perturbed tubes

In this subsection, we use estimates given in Proposition 3.2.2 and proposition 2.3.3, we obtain the expansion of the shape operator of the hypersurface $S_\rho(w, \Phi)$. We get

Proposition 2.8.1 *Under the previous hypothesis, the shape operator is given by*

$$\begin{aligned}
\rho\, A_{aa}(w,\Phi) &= \rho^2\, g(R(\Theta, E_a)\Theta, E_a) - \rho^2\, \Gamma^c_a(\Theta)\, \Gamma^a_c(\Theta) + \mathcal{O}(\rho^3) \\
&- w_{aa} - \rho\, g(\Phi_{aa} + R(\Phi, E_a)E_a\,, \Theta) + \rho\, \Gamma^c_a(\Phi)\, \Gamma^a_c(\Theta) \\
&- 2\,\rho\, \Gamma^c_a(\Theta)\, w_{ac} + \rho^2\, L(w,\Phi) + Q(w,\Phi) \\
\rho\, A_{ii}(w,\Phi) &= 1 + \tfrac{1}{3}\, \rho^2\, g(R(\Theta,\Theta_i)\Theta,\Theta_i) - w_{ii} - w + \mathcal{O}(\rho^3) \\
&+ \rho^2\, L(w,\Phi) + Q(w,\Phi) \\
\rho\, A_{aj}(w,\Phi) &= \mathcal{O}(\rho^2) + L(w,\Phi) + Q(w,\Phi) \\
\rho\, A_{ij}(w,\Phi) &= \mathcal{O}(\rho^2) + L(w,\Phi) + Q(w,\Phi) \quad i \neq j \\
\rho\, A_{ab}(w,\Phi) &= -\rho\, \Gamma^b_a(\Theta) - w_{ac} + \mathcal{O}(\rho^2) + L(w,\Phi) + Q(w,\Phi) \quad a \neq b
\end{aligned} \tag{2.47}$$

where all curvature terms are computed at the point p.

2.8.2 The r-curvature of perturbed tubes

Given any symmetric matrix A, and any $r = 0, \ldots, m$, we define

$$\sigma_r(A) := \sum_{i_1 < \ldots < i_r} \lambda_{i_1} \ldots \lambda_{i_r}$$

where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A. The r-th Newton transform of A is defined by

$$T_r(A) := \sigma_r(A)\, I - \sigma_{r-1}(A)\, A + \cdots + (-1)^r\, A^r$$

with $T_m(A) = 0$. Now suppose that $A = A(t)$ depends smoothly on a parameter t, it is proved in [33] that

$$\frac{d}{dt}\sigma_r(A) = \mathrm{Tr}\left(T_{r-1}(A)\, \frac{d}{dt} A \right) \tag{2.48}$$

Assume that

$$\rho\, A = \tilde{I} + H$$

where H is an $m \times m$ symmetric matrix and

$$\tilde{I} = \left(\begin{array}{cc} I_{n-1} & 0 \\ 0 & 0 \end{array} \right)$$

where $n = m + 1 - k$. We define the function

$$f_r(t) := \sigma_r(\tilde{I} + tH)$$

so that

$$\rho^r\, \sigma_r(A) = f_r(0) + f_r'(0) + \mathcal{O}(|\tilde{H}|^2)$$

where $\tilde{H} = \tilde{I}\, H$. Observe that $f_r(0) = C_{n-1}^r \;\; if \;\, r \leq \, n-1$ and $f_r(0) = 0 \;\; if \;\, r \geq \, n$. Using this together with the previous expansions of the shape operator, it is not hard to check that the r-curvature of the hypersurface $S_\rho(w, \Phi)$ can be expanded as

$$\begin{array}{rcl} \rho^r\, \sigma_r(S_\rho(w,\Phi)) & = & C_{n-1}^r + C_{n-1}^{r-1}\, \rho^2 \left(\frac{1}{3}\, \frac{n-r}{n-1}\, g(R(\Theta,\Theta_i)\Theta,\Theta_i) + g(R(\Theta,E_a)\Theta,E_a) \right) \\ & - & C_{n-1}^{r-1}\, \rho^2\, \Gamma_a^c(\Theta)\, \Gamma_c^a(\Theta) + \mathcal{O}(\rho^3) \\ & - & C_{n-1}^{r-1}\, \left(\rho^2\, \Delta_K w + \frac{n-r}{n-1}\, (\Delta_{S^{n-1}} w + (n-1)\, w) \right) - 2\, \rho\, C_{n-1}^{r-1}\, \Gamma_a^c(\Theta)\, w_{ac} \\ & - & \rho\, C_{n-1}^{r-1}\, (\, g(\Delta_K \Phi + R(\Phi, E_a) E_a\, , \Theta) - \Gamma_a^c(\Phi)\, \Gamma_c^a(\Theta)) \\ & + & \rho^2\, L(w,\Phi) + Q(w,\Phi). \end{array}$$

The linear and nonlinear operators appearing on the above expression are different from the ones before, but enjoy similar properties. We define

$$\mathcal{L}_\rho := -C_{n-1}^{r-1} \left(\rho^2\, \Delta_K + \frac{n-r}{n-1}\, (\Delta_{S^{n-1}} + (n-1)) \right), \tag{2.49}$$

as an operator on the spherical normal bundle SNK with the expression (2.49) in any local coordinates. Also, the Jacobi operator, for K is defined by

$$\mathfrak{J} := C_{n-1}^{r-1}\, \left(-\Delta^N - \mathcal{B}^N + \mathcal{R}^N \right), \tag{2.50}$$

where the operators Δ^N, $\mathcal{B}^N$ and $\mathcal{R}^N$ are defined in the section 2.4. Introduce the quadratic form

$$\Omega\,(\cdot,\cdot) := -C_{n-1}^{r-1}\,\left((1-\frac{1}{3}\,\frac{n-r}{n-1})\,g(\mathcal{R}^N\cdot,\cdot) - \frac{1}{3}\,\frac{n-r}{n-1}\,\mathrm{Ric}(\cdot,\cdot) - g(\mathcal{B}^N\cdot,\cdot)\right)$$

acting on N_pK. In terms of all of this notation, we have the

Proposition 2.8.2 *Let K be a minimal submanifold. Then the r-curvature of $\mathcal{S}_\rho(w,\Phi)$ can be expanded as*

$$\begin{aligned}\rho^r\,\sigma_r(S_\rho(w,\Phi)) &= C_{n-1}^r - \Omega(\Theta,\Theta)\,\rho^2 + \mathcal{O}(\rho^3)\\ &+ \mathcal{L}_\rho\,w + \rho\,g(\mathfrak{J}\,\Phi,\Theta) - 2\,C_{n-1}^{r-1}\,\rho^3\,g(\Theta,B)\circ\nabla_K^2 w \qquad (2.51)\\ &+ \rho^2\,L(w,\Phi) + Q(w,\Phi).\end{aligned}$$

The equation $\rho^r\,\sigma_r(S_\rho(w,\Phi)) = C_{n-1}^r$ can now be written as

$$\begin{aligned}\mathcal{L}_\rho\,w + \rho\,g(\mathfrak{J}\,\Phi,\Theta) &= \Omega(\Theta,\Theta)\,\rho^2 + \mathcal{O}(\rho^3) + 2\,C_{n-1}^{r-1}\,\rho^3\,g(\Theta,B)\circ\nabla_K^2 w\\ &+ \rho^2\,L(w,\Phi) + Q(w,\Phi).\end{aligned} \qquad (2.52)$$

The proof of Theorem 2.8.1 follows the corresponding proof when $r = 1$. The main observation is that the operator $\mathcal{L}_\rho$ is elliptic precisely when $r < n-1$. Moreover Lemma 2.6.2 relies on the fact that in (2.52) the linear operator acting on w is self adjoint. This is the case since it is proven in [36] that the linearized r-curvature operator can be written as

$$L_r = div(T_r\,\nabla.)$$

where T_r is the r-th Newton transform associated to the shape form of the hypersurface.

Chapitre 3

Delaunay Type hypersurface in some Riemannian manifold

3.1 Introduction

Let Σ be an oriented embedded (or possibly immersed) hypersurface in a Riemannian manifold (M^{m+1}, g). The shape operator A_Σ is the symmetric endomorphism of the tangent bundle of Σ associated with the second fundamental form of Σ, b_Σ, by

$$b_\Sigma(X, Y) = g_\Sigma(A_\Sigma\, X, Y), \quad \forall\, X, Y \in T\Sigma; \qquad \text{here} \qquad g_\Sigma = g|_{T\Sigma}\,.$$

The eigenvalues κ_i of the shape operator A_Σ are called the principal curvatures of the hypersurface Σ and the mean curvature of Σ is defined to be the average of the principal curvatures of Σ, i.e.

$$H(S) := \frac{1}{m}(\kappa_1 + \ldots + \kappa_m).$$

In this chapter, we are interested in families of constant mean curvature hypersurfaces, with mean curvature varying from one member of the family to another. This study follows the work of R. Ye [42], [43] where the existence of a local foliations by constant mean curvature hypersurfaces which concentrate at a critical point of the scalar curvature function is proven, the work of F. Pacard and R. Mazzeo [28] where the existence of constant mean curvature hypersurfaces which condensate over a closed geodesic K and also the work of R. Mazzeo, F. Pacard and the author where this last result was extended to handle the case where condensation occurs along a minimal submanifold [19]. In particular, these results prove the existence of constant mean curvature hypersurfaces with nontrivial topology in any Riemannian manifold.

Given $m \geq 2$ we set

$$\tau_m := \frac{1}{m}(m-1)^{\frac{m-1}{m}}$$

In $\mathbb{R}^{m+1}$, there exists a family $S_{\tau,\rho}$, for $\rho > 0$ and $\tau \in (0, \tau_m]$, of constant mean curvature hypersurfaces of revolution about the x_{m+1}-axis, which generalize when $m \neq 2$ the well known Delaunay surfaces [12]. These hypersurfaces can be parameterized by

$$X(s, \Theta) := \rho\,(\,\tau\, e^{\sigma(s)}\,\Theta,\ \kappa(s)),$$

for $(s, \Theta) \in \mathbb{R} \times S^{m-1}$. When $\tau \in (0, \tau_m)$ the function σ is a nonconstant solution of

$$(\partial_s \sigma)^2 + \tau^2 (e^{\sigma} + e^{(1-m)\sigma})^2 = 1,$$

with $\sigma(0) < 0$ and $\partial_s \sigma(0) = 0$ and the function κ is the given by

$$\partial_s \kappa = \tau^2 e^{\sigma}\,(e^{\sigma} + e^{(1-m)\sigma}),$$

with $\kappa(0) = 0$. When $\tau = \tau_m$ the corresponding hypersurfaces are cylinders $S^{m-1}(\frac{m-1}{m}\rho) \times \mathbb{R}$, while as τ tends to 0 these converge to spheres of radius ρ which are mutually tangent and arranged along the x_{m+1}-axis. These hypersurfaces are invariant under a discreet one parameter group of translations $t_\tau\, e_{m+1}\,\mathbb{Z}$ along the x_{m+1}-axis, where $t_\tau > 0$ is the least "vertical" period.

It is easy to check that there exists a smooth positive function $\tau \longrightarrow c(\tau)$ such that

$$\rho^{1-m}\,\mathcal{H}^m \llcorner S_{\tau,\rho} \rightharpoonup c(\tau)\,\mathcal{H}^1 \llcorner K, \tag{3.1}$$

Moreover $c(\tau)$ tends to $\frac{1}{2}|S^m|$ as τ tends to 0 and $c(\tau_m) = (\frac{m-1}{m})^{m-1}\,|S^{m-1}|$.

In this chapter, we prove the existence of sequences of constant mean curvature hypersurfaces which condensate, as their mean curvature tends to $+\infty$, to a closed geodesic in some Riemannian manifolds (M^{m+1}, g) and for which (3.1) holds. These hypersurfaces generalize, in Riemannian manifolds, the well known family of Delaunay hypersurfaces in Euclidean space which are rotationally symmetric hypersurfaces with constant mean curvature. They arise in a one parameter family which interpolate between the cylinder and a string of spheres arranged along a common axis. Analogues of Delaunay hypersurfaces are also known to exist in hyperbolic space $\mathbb{H}^{m+1}$ and they basically share similar properties. Passing to appropriate compact quotients of $\mathbb{R}^{m+1}$ or of $\mathbb{H}^{m+1}$, these Delaunay families produce examples of constant mean curvature hypersurfaces which condensate along a closed geodesic in compact hyperblic spaces. Delaunay hypersurfaces are also known to exist in standard spheres S^{n+1} and, in some sense, our result generalizes these explicit examples and in many others where

Delaunay hypersurfaces can be understood as branches of constant mean curvature hypersurfaces which bifurcate from the family of constant mean curvature hypersurfaces $S^{m-1}(r) \times S^1(\sqrt{1-r^2})$ in S^{m+1} as r tends to 0 [13].

Let us describe our result in more detail. Let K be a geodesic in a compact Riemannian manifold (M^{m+1}, g). We can define Fermi coordinates in some tubular neighborhood of K (see §2.1). We assume that, in these coordinates, the coefficients of the metric do not depend on the the point on K (i.e. do not depend on x_0). This is for example the case when one considers a warped product metric $g = A(x')\, dx_0 + g_{x'}$ on $M^{m+1} = S^1 \times N^m$.

The length of K will be denoted by ℓ. We assume that this geodesic is τ-nondegenerate, i.e. that the operator

$$\mathfrak{J}_\tau \Phi := \nabla^2_{E_0} \Phi + \alpha_\tau\, R(\Phi, X_0)\, X_0$$

is invertible where X_0 is the unit tangent vector field to K and $\alpha_\tau \in \mathbb{R}$ is defined in §3. Our main result reads :

Theorem 3.1.1 *Assume that K is a τ-nondegenerate geodesic. We further assume that in Fermi coordinates, the coefficients of the metric do not depend on x_0. There exists a finite set $T \subset (0, \tau_m)$ ($T = \emptyset$ when $m = 2$) such that, for all $\tau \in (0, \tau_m) - T$, there exists $i_\tau \in \mathbb{N}$ such that, for all $i \in \mathbb{N}$, $i \geq i_\tau$, if we define $\rho_i > 0$ by*

$$i\, t_\tau\, \rho_i = \ell$$

then there exists a constant mean curvature hypersurface $\Sigma_{\tau,i}$ with mean curvature $H = \frac{1}{\rho_i}$ and for which

$$\rho_i^{1-m}\, \mathcal{H}^m \llcorner \Sigma_{\tau,i} \rightharpoonup c(\tau)\, \mathcal{H}^1 \llcorner K.$$

The definition of the constant α_τ is quite involved and we have not been able to get a value in terms of τ. However, α_τ is a continuous function of τ which converges to 1 as τ tends to τ_m. In particular the operator

$$\mathfrak{J}_{\tau_m} \Phi := \nabla^2_{E_0} \Phi + R(\Phi, X_0)\, X_0$$

is nothing but the Jacobi operator about K which appears in the second variation of the length functional. In particular, any nondegenerate geodesic (in the usual sense) is τ-nondegenerate for τ close to τ_m.

The case where $\tau = \tau_m$ was already treated in [28] where it was shown that the corresponding result was valid for a carefully chosen sequence $(\rho_i)_i$ tending to 0. In addition it was proven that the corresponding hypersurfaces were leafs of a partial foliations, which is not true anymore when $\tau \neq \tau_m$.

3.2 Expansion of the metric in Fermi coordinates near K

3.2.1 Fermi coordinates

We now introduce Fermi coordinates in a neighborhood of K. For a given $p \in K$, there is a natural splitting

$$T_p M = T_p K \oplus N_p K.$$

In a neighborhood of p in K, we choose a normal frame $E_1 \ \ldots \ E_m$ of NK such that $\nabla_{E_0} E_i = 0$ for all $i = 1, \ldots, m$ (this can be achieved locally by parallel transport of a fixed orthonormal basis of $N_p K$). These determines a coordinate system

$$F \ : \ (x^0, \ldots, x^m) \quad \longmapsto \quad \exp^M_{\gamma(x^0)}(x^i E_i)$$

where $\exp^M$ is the exponential map on M, γ is the arc length parametrization of the geodesic K as in the Introduction and summation over repeated indices is understood.

The corresponding coordinate vector fields are denoted by

$$X_\alpha = F_*(\partial_{x^\alpha}). \qquad \alpha = 0 \ldots, m.$$

Observe that, by construction $X_\alpha = E_\alpha$ along K.

Notation : We shall always use the convention that indices $i, j, k \ \in \{1, \ldots, m\}$ and $\alpha, \beta, \gamma, \ldots, \in \{0, \ldots, m\}$. It will be convenient to adopt the notation

$$x = (x^0, x^1, \ldots, x^m) = (x^0, x')$$

where $x' := (x^1, \ldots, x^m)$.

3.2.2 Taylor expansion of the metric

The material of this section is an extension of the results of chapter 2 since in the present analysis require more precise expansions.

Notation : In the following the symbol $\mathcal{O}(|x'|^n)$ indicates a function f such that it and its partial derivatives of any order, with respect to X_0 and $x^i X_j$ are bounded by $c\,|x'|^n$ in some fixed neighborhood of K.

We denote by $q = F(x^0, x')$. Observe that, by construction, we have

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(|x'|)$$

The next two results are concerned with the expansions of the geometric data at the point $q = F(x^0, x')$ in terms of the geometric data at $p = F(x^0, 0)$. Their proof can be found in [28].

Lemma 3.2.1 *For $\alpha, \beta = 0, \ldots, m$, we have at the point q*

$$\nabla_{X_\alpha} X_\beta = \mathcal{O}(|x'|) X_\gamma \tag{3.2}$$

In addition, we also have

$$\nabla_{X_\alpha} X_0 = g(R(E_k, E_0)E_\alpha, E_\beta)\, x^k\, X_\beta + \mathcal{O}(|x'|^2)\, X_\gamma \tag{3.3}$$

where summation over repeated indices is understood. Here the curvature tensor R is computed at p.

Proof : The first estimate is already proven in chapter 2. The second estimate follows from the following computation

$$X_k\, g(\nabla_{X_\alpha} X_0, X_\beta) = g(\nabla_{X_k} \nabla_{X_0} X_\alpha, X_\beta) + g(\nabla_{X_\alpha} X_0, \nabla_{X_k} X_\beta)$$

Hence

$$X_k\, g(\nabla_{X_\alpha} X_0, X_\beta) = g(R(X_k, X_0)\, X_\alpha, X_\beta) + g(\nabla_{X_0} \nabla_{X_k} X_\alpha, X_\beta) + g(\nabla_{X_\alpha} X_0, \nabla_{X_k} X_\beta)$$

Computed along K we get

$$X_k\, g(\nabla_{X_\alpha} X_0, X_\beta)\, |_K = g(R(E_k, E_0)\, E_\alpha, E_\beta)$$

The result then follows from (3.2). □

We now give the expansion of the metric coefficients in the above defined coordinates

Proposition 3.2.1 *The expansions of the metric coefficients at the point q are given by :*

$$\begin{aligned} g_{ij} &= \delta_{ij} + \tfrac{1}{3} g(R(E_i, E_k)E_j, E_l)\, x^k x^l + \mathcal{O}(|x'|^3) \\ g_{00} &= 1 + g(R(E_k, E_0)E_l, E_0) x^k\, x^l + \mathcal{O}(|x'|^3) \\ g_{0j} &= \tfrac{2}{3}\, g(E_k, E_0)\, E_l\, , E_j)\, x^k\, x^l + \mathcal{O}(|x'|^3) \end{aligned} \tag{3.4}$$

where summation over repeated indices is understood. Here the curvature tensor R is computed at p.

Proof : The proof of the first two estimates is given in chapter 2. We therefore concentrate on the proof of the third estimate. We start with

$$X_k^2\, g(X_0, X_j) = g(\nabla_{X_k} \nabla_{X_0} X_k, X_j) + g(X_0, \nabla_{X_k} \nabla_{X_j} X_k) + 2\, g(\nabla_{X_k} X_0, \nabla_{X_k} X_j)$$

Now it is shown in chapter 2 that, along K

$$\nabla_{X_k} \nabla_{X_j} X_k \,|_K = \frac{1}{3}\, R(E_k, E_j)\, E_k$$

Moreover

$$\nabla_{X_k} \nabla_{X_0} X_k = R(X_k, X_0)\, X_k + \nabla_{X_0} \nabla_{X_k} X_k$$

and hence, along K, we have

$$\nabla_{X_k} \nabla_{X_0} X_k \,|_K = R(E_k, E_0)\, E_k$$

since $\nabla_{X_k} X_k |_K = 0$. We conclude, using the fact that $\nabla_{X_k} X_j \,|_K = 0$ that

$$X_k^2\, g(X_0, X_j) \,|_K = \frac{4}{3}\, g(R(E_k, E_0)\, E_k, E_j)$$

along K. The result then follows at once. □

3.2.3 Geometry of Delaunay type hypersurfaces

We consider, in $\mathbb{R}^{m+1}$ the hypersurface parameterized by

$$D(s, \Theta) = (\varphi(s)\, \Theta,\ \kappa(s)),$$

where $\tau \in (0, \frac{1}{m}(m-1)^{\frac{m-1}{m}}]$ and $(s, \Theta) \in \mathbb{R} \times S^{m-1}$. The functions φ and κ are solutions of

$$(\partial_s \varphi)^2 + (\varphi^2 + \tau^m\, \varphi^{2-m})^2 = \varphi^2,$$

and

$$\partial_s \kappa = \varphi^2\, (1 + \tau^m\, \varphi^{-m}),$$

with initial conditions $\varphi(0) < \tau$, $\partial_s \varphi(0) = 0$ and $\kappa(0) = 0$. It is easy to check that the hypersurface parameterized by D is a constant mean curvature hypersurface equal to 1. We refer to [12] for further details.

Let Φ be a section of NK and w a scalar function on the spherical bundle SNK. We fix $\rho > 0$. We define

$$G(s, x') = exp^M_{\gamma(\rho\, \kappa(s))}\, (\, \rho\, (\varphi(s) - w(s, x'))\, \Theta(x') + \Phi(\rho\, \kappa(s))\,)$$

the image of G, will be called $\Sigma_\rho(w,\Phi)$ and even though it depends on τ we shall not make this dependence explicit in the notation.

Notation : Because of the definition of these hypersurfaces using the exponential map, various vector fields we shall use may be regarded either as fields along K or along $\Sigma_\rho(w,\Phi)$. To distinguish them, we will write

$$\Phi := \Phi^j\, E_j \qquad \Phi' := \partial_{x^0}\, \Phi^j\, E_j \qquad \Phi'' := \partial^2_{x^0}\, \Phi^j\, E_j$$

$$\Theta := x^j\, E_j \qquad \Theta_i := \partial_{y^i} x^j\, E_j.$$

where we use the local parametrization $y \to x'(y)$ of a neighborhood of a point in S^{m-1} These are all vectors in the tangent space T_pM at $p = F(x_0, 0)$ while the vectors

$$\Psi := \Phi^j\, X_j \qquad \Psi' := \partial_{x^0}\, \Phi^j\, X_j,$$

$$\Upsilon := \Theta^j\, X_j \qquad \Upsilon_i := \partial_{y^i}\Theta^j\, X_j$$

lie in the tangent space T_qM, $q = F(x_0, x'(y))$.

For brevity, we will also write $G(s,y)$ instead of $G(s,x'(y))$, $w(s,y)$ instead of $w(s,x'(y))$ and

$$w_s := \partial_s w \qquad w_j := \partial_{y_j} w \qquad w_{sj} := \partial_s \partial_{y_j} w \qquad w_{ss} := \partial^2_s w$$

$$\varphi_s := \partial_s \varphi \qquad \varphi_{ss} := \partial^2_s\, \varphi \qquad \kappa_s := \partial_s \kappa$$

In terms of all this notation, the tangent space of $\Sigma_\rho(w,\Phi)$ is spanned by the vector fields :

$$Z_0 \;=\; \rho\,\left(\kappa_s\, X_0 + (\varphi_s - w_s)\,\Upsilon + \kappa_s\,\Phi'\right) \tag{3.5}$$

$$Z_j \;=\; \rho\,\left((\varphi - w\,)\,\Upsilon_j - w_j\,\Upsilon\right). \tag{3.6}$$

We will use two different types of Hölder spaces for functions on SNK and sections of NK : The ordinary Hölder spaces $\mathcal{C}^{m,\alpha}(SNK)$, $\mathcal{C}^{m,\alpha}(K,NK)$ which are based on differentiations with respect to the vector fields ∂_{x_0} and ∂_{y_j}. The modified Hölder spaces $\mathcal{C}^{m,\alpha}_\rho(SNK)$, $\mathcal{C}^{m,\alpha}_\rho(K,NK)$ which are based on differentiations with respect to the vector fields $\rho\,\partial_{x_0}$ and ∂_{y_j}. We assume that

$$\Phi(x_0) = \sum_{j=1}^n \phi_j(x_0)\, X_j \in \mathcal{C}^{2,\alpha}(K,NK), \qquad \text{and} \qquad w \in \mathcal{C}^{2,\alpha}_\rho(SNK).$$

Definition 3.2.1 *In the following,* $L(w,\Phi)$ *denotes a linear combination of the functions* w *together with its derivatives with respect to* ρX_0 *and* X_i *up to order* 2*, and* Φ^j *together with its derivatives with respect to* X_0 *up to order* 2 *which satisfies*

$$||L(w,\Phi)||_{C^{0,\alpha}_\rho} \leq c\,\left(||w||_{C^{2,\alpha}_\rho(SNK)} + ||\Phi||_{C^{0,\alpha}(K,NK)}\right) \tag{3.7}$$

Similarly, an expression of the form $Q(w,\Phi)$ *denotes a nonlinear operator in the functions* w *together with its derivatives with respect to the vector fields* $\rho\,\partial_{x_0}$ *and* ∂_{y_i} *up to order* 2*, and* Φ^j *together with their derivatives with respect to the vector fields* ∂_{x_0} *up to order* 2*. Again, the coefficients of the Taylor expansion of the corresponding differential operator are smooth on* SNK*, and* Q *which vanishes quadratically at* $(w,\Phi)=(0,0)$ *and which satisfies*

$$\begin{aligned}\|Q(w_2,\Phi_2)-Q(w_1,\Phi_1)\|_{\mathcal{C}^{0,\alpha}_\rho} \quad &\leq \quad c\,\sup_{i=1,2}\left(\|w_i\|_{\mathcal{C}^{2,\alpha}_\rho(SNK)} + \|\Phi_i\|_{\mathcal{C}^{2,\alpha}(K,NK)}\right)\\ \times\left(\|w_2-w_1\|_{\mathcal{C}^{2,\alpha}_\rho(SNK)} \right. &+ \left. \|\Phi_2-\Phi_1\|_{\mathcal{C}^{2,\alpha}(K,NK)}\right)\end{aligned} \tag{3.8}$$

Finally, any term denoted $\mathcal{O}(\rho^d)$ *is a smooth function on* SNK *which is bounded in* $\mathcal{C}^\infty(SNK)$ *by a constant times* ρ^d*.*

Using the expansions of Lemma 3.2.1, we deduce that at the point of coordinate (s,y) and in terms of the operators L and Q defined above, the metric g has the following expansions

Lemma 3.2.2 *At the point* $q=G(s,x'(y))$*, we have*

$$\begin{aligned} g_{00} &= 1 + g(R(\rho\,\varphi\,\Theta+\Phi,E_0)\,\rho\,\varphi\,\Theta+\Phi,E_0) + \mathcal{O}(\rho^3) + \rho^2\,L(w,\Phi) + Q(w,\Phi),\\ g_{0j} &= \frac{2}{3}\,g(R(\rho\,\varphi\,\Theta+\Phi,E_0)\,\rho\,\varphi\,\Theta+\Phi,E_j) + \mathcal{O}(\rho^3) + \rho^2\,L(w,\Phi) + Q(w,\Phi),\\ g_{ij} &= \delta_{ij} + \frac{1}{3}\,g(\,R(\rho\,\varphi\,\Theta+\Phi,E_i)\,\rho\,\varphi\,\Theta+\Phi,E_j) + \mathcal{O}(\rho^3) + \rho^2\,L(w,\Phi) + Q(w,\Phi).\end{aligned}$$

Using these estimates we can provide the expansion of the first fundamental form, the unit normal vectors field and the second fundamental form of $\Sigma_\rho(w,\Phi)$ in both powers of ρ, w and Φ.

We start with the observations that :

$$g(\Upsilon,\Upsilon) = 1+\rho^2\,L(w,\Phi)+Q(w,\Phi) \qquad g(\Upsilon,\Upsilon_j) = \rho\,L(w,\Phi)+Q(w,\Phi)$$

and

$$g(\Upsilon, X_0) = \rho\, L(w,\Phi) + Q(w,\Phi)$$

These follow at once from Lemma 3.2.2 together with the fact that $g(\Upsilon,\Upsilon) = 1$, $g(\Upsilon,\Upsilon_j) = g(\Upsilon, X_0) = 0$ when $w \equiv 0$ and $\Phi \equiv 0$. Using this we get :

Proposition 3.2.2 *The following expansions hold*

$$\begin{aligned}
\rho^{-2}\, g(Z_0,Z_0) &= \varphi^2 + \rho^2\,\varphi^2\,\kappa_s^2\, g(R(\Theta,E_0)\Theta,E_0) - 2\,\varphi_s\, w_s + 2\,\varphi_s\,\kappa_s\, g(\Phi',\Theta) \\
&\quad +2\,\rho\,\varphi\,\kappa_s^2\, g(R(\Theta,E_0)\Phi,E_0) + \frac{4}{3}\,\rho\,\varphi\,\kappa_s\,\varphi_s\, g(R(\Theta,E_0)\Phi,\Theta) \\
&\quad +\mathcal{O}(\rho^3) + \rho^2\, L(w,\Phi) + Q(w,\Phi) \\
\rho^{-2}\, g(Z_0,Z_j) &= O(\rho^2) + L(w,\Phi) + Q(w,\Phi) \\
\rho^{-2}\, g(Z_i,Z_j) &= (\varphi - w)^2\, g(\Theta_i,\Theta_j) + \frac{1}{3}\,\varphi^2\, g(R(\rho\,\varphi\,\Theta + \Phi, \Theta_i)\,\rho\,\varphi\,\Theta + \Phi, \Theta_j) \\
&\quad +\mathcal{O}(\rho^3) + \rho^2 L(w,\Phi) + Q(w,\Phi)
\end{aligned}$$

where summation over repeated indices is understood.

Our next task is to understand the dependence on (w,Φ) of the unit normal N to $\Sigma_\rho(w,\Phi)$. We start by defining the normal vector field (not necessary unitary)

$$\widetilde{N} = \varphi_s X_0 - \kappa_s\,\Upsilon + \frac{1}{\rho}\,(\alpha_0\, Z_0 + \beta_j Z_j)$$

where the coefficient α_0 and β_j are chosen so that $\widetilde{N}$ is orthogonal to Z_0 and Z_j. The unit normal vector field of $\Sigma_\rho(w,\Phi)$ is then defined by

$$N := \frac{\widetilde{N}}{|\widetilde{N}|}.$$

We have the following

Proposition 3.2.3 *Under the above notations, the coefficient α_0 can be expanded as*

$$\begin{aligned}
\alpha_0 &= \tfrac{1}{\varphi^2}\Big(\kappa_s^2\, g(\Phi',\Theta) - \kappa_s\, w_s - \rho^2\,\varphi^2\,\kappa_s\,\varphi_s\, g(R(\Theta,E_0)\Theta,E_0) \\
&\quad -2\,\rho\,\varphi\,\kappa_s\,\varphi_s\, g(R(\Theta,E_0)\Phi,E_0) + \tfrac{2}{3}\,\rho\,\varphi\,(\kappa_s^2 - \varphi_s^2)\, g(R(\Theta,E_0)\Phi,\Theta)\Big) \\
&\quad +\mathcal{O}(\rho^3) + \rho^2\, L(w,\Phi) + Q(w,\Phi)
\end{aligned}$$

and the coefficients β_j are solutions of the system :

$$\begin{aligned}\frac{1}{\rho^2}\,\sum_i \beta_i\, g(Z_i, Z_j) &= -\kappa_s\, w_j + \tfrac{1}{3}\,\rho\,\varphi^2\,\kappa_s\, g(R(\Phi,\Theta)\,\Theta,\Theta_j)\\ &- \tfrac{2}{3}\,\varphi\,\varphi_s\, g(R(\rho\,\varphi\,\Theta+\Phi, E_0)\,\rho\,\varphi\,\Theta+\Phi,\Theta_j)\\ &+ \mathcal{O}(\rho^3) + \rho^2\, L(w,\Phi) + Q(w,\Phi).\end{aligned}$$

where summation over repeated indices is understood. Moreover, we have

$$\begin{aligned}\varphi\,|\tilde{N}|^{-1} &= 1 - \tfrac{1}{2}\,\varphi_s^2\,\rho^2\, g(R(\Theta,E_0),\Theta,E_0) - \varphi_s^2\,\varphi^{-1}\,\rho\, g(R(\Theta,E_0)\Phi,E_0)\\ &+ \tfrac{2}{3}\,\varphi_s\,\kappa_s,\varphi^{-1}\,\rho\, g(R(\Theta,E_0)\Phi\,\Theta) + \mathcal{O}(\rho^3) + \rho^2\, L(w,\Phi) + Q(w,\Phi)\end{aligned}$$

Proof : In the definition of $\tilde{N}$, the coefficients α_0 and β_j have to be chosen so that $\widetilde{N}$ is orthogonal to Z_0 and Z_j. This induces a linear system of equations for α_0 and the β_j.

Using the previous expansions, we easily get

$$\begin{aligned}\frac{1}{\rho}\, g(\kappa_s\Upsilon - \varphi_s\, X_0, Z_0) &= \kappa_s^2\, g(\Phi',\Theta) - \kappa_s\, w_s + \tfrac{2}{3}\,\rho\,\varphi\,(\kappa_s^2 -\ \varphi_s^2)\, g(R(\Theta,E_0)\Phi,\Theta)\\ &-\kappa_s\,\varphi_s\, g(R(\rho\,\varphi\,\Theta+\Phi,E_0)\,\rho\,\varphi\,\Theta+\Phi,E_0)\\ &+\mathcal{O}(\rho^2) + \rho\, L(w,\Phi) + Q(w,\Phi)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\rho}\, g(\kappa_s\Upsilon - \varphi_s\, X_0, Z_j) &= -\kappa_s\, w_j + \tfrac{1}{3}\,\rho\,\varphi^2\,\kappa_s\, g(R(\Phi,\Theta)\,\Theta,\Theta_j)\\ &- \tfrac{2}{3}\,\varphi\,\varphi_s\, g(R(\rho\,\varphi\,\Theta+\Phi,E_0)\,\rho\,\varphi\,\Theta+\Phi,\Theta_j)\\ &+ \mathcal{O}(\rho^3) + \rho^2\, L(w,\Phi) + Q(w,\Phi).\end{aligned}$$

The expansions follows from the above computations and Proposition 3.2.2. □

We are now in a position to compute the coefficients of the second fundamental form of $\Sigma_\rho(w,\Phi)$. To simplify the computations, we assume from now on that, at the point $\Upsilon(y) \in S^{m-1}$ where the computation will be performed, coordinates are chosen so that

$$g(\Theta_i\,,\Theta_j) = \delta_{ij}, \qquad \text{and} \qquad \bar{\nabla}_{\Theta_i}\Theta_j = 0, \qquad i,j = 1,\ldots,m-1. \tag{3.9}$$

where $\bar{\nabla}$ denotes the connection on TS^{m-1}.

Building on the previous expansions, we have the :

Proposition 3.2.4 *The following expansions hold*

$$
\begin{aligned}
\rho^{-2} g(N, \nabla_{Z_0} Z_0) &= \tfrac{1}{\rho}\,\tfrac{1}{\varphi}\,[\varphi_s\,\kappa_{ss} - \kappa_s\,\varphi_{ss}] - \tfrac{1}{\rho}\,\tfrac{\kappa_s\,\varphi_s}{\varphi^2}\,w_s + \tfrac{1}{\rho}\,\tfrac{\kappa_s}{\varphi}\,w_{ss} + \mathcal{O}(\rho^2)\\
&+ \tfrac{1}{\rho}\,\tfrac{\kappa_s}{\varphi^2}\,[\kappa_s\,\varphi_s - \varphi\,\kappa_{ss}]\,g(\Theta, \Phi') - \tfrac{\kappa_s^3}{\varphi}\,g(\Theta, \Phi'')\\
&+ \rho\,\tfrac{1}{\varphi}\,\left[\tfrac{1}{2}\,\varphi_s^3\,\kappa_{ss} + \kappa_s\,\varphi^3 + \varphi\,\kappa_s\,\varphi_s^2 - \tfrac{1}{2}\,\kappa_s\,\varphi_{ss}\,\varphi_s^2\right]\,g(R(\Theta, E_0)\Theta, E_0)\\
&+ \tfrac{1}{\varphi^2}\,[\varphi_s^3\,\kappa_{ss} + \kappa_s\,\varphi^3 + \varphi\,\kappa_s\,\varphi_s^2 - \kappa_s\,\varphi_{ss}\,\varphi_s^2]\,g(R(\Theta, E_0)\Phi, E_0)\\
&+ \tfrac{2}{3}\,\tfrac{1}{\varphi^2}\,[\varphi_s\,\varphi_{ss}\kappa_s^2 - \varphi_s^2\,\kappa_s\,\kappa_{ss} + \varphi\,\varphi_s^3]\,g(R(\Theta, E_0)\Phi, \Theta)\\
&+ \rho\,L(w, \Phi) + \tfrac{1}{\rho}\,Q(w, \Phi).\\
\rho^{-2} g(\tilde{N}, \nabla_{Z_0} Z_i) &= \rho^{-2} g(N, \nabla_{Z_i} Z_0) = \mathcal{O}(\rho) + \tfrac{1}{\rho}\,L(w, \Phi) + \tfrac{1}{\rho}\,Q(w, \Phi).\\
\rho^{-2} g(\tilde{N}, \nabla_{Z_j} Z_j) &= \tfrac{1}{\rho}\,\kappa_s - \tfrac{1}{\rho}\tfrac{\kappa_s}{\varphi}\;w + \tfrac{1}{\rho}\,\tfrac{\kappa_s\,\varphi_s}{\varphi^2}\,w_s + \tfrac{1}{\rho}\,\tfrac{\kappa_s}{\varphi}\,w_{jj} - \tfrac{1}{\rho}\,\tfrac{\varphi_s}{\varphi^2}\,\kappa_s^2\,g(\Theta, \Phi')\\
&+ \tfrac{1}{2}\,\rho\,\varphi_s^2\,\kappa_s\,g(R(\Theta, E_0)\Theta, E_0) + \varphi_s^2\,\tfrac{\kappa_s}{\varphi}\,g(R(\Theta, E_0)\Phi, E_0)\\
&+ \tfrac{2}{3}\,\tfrac{\varphi_s^3}{\varphi}\,g(R(\Theta, E_0)\Phi, \Theta) + \tfrac{2}{3}\,\rho\,\varphi^2\,\kappa_s\,g(R(\Theta, \Theta_j)\Theta, \Theta_j)\\
&+ \tfrac{2}{3}\,\varphi\,\kappa_s\,g(R(\Theta, \Theta_j)\Phi, \Theta_j) + \tfrac{2}{3}\,\rho\,\varphi^2\,\varphi_s\,g(R(\Theta_j, E_0)\Theta, \Theta_j)\\
&+ \tfrac{2}{3}\,\varphi\,\varphi_s\,g(R(\Theta_j, E_0)\Phi, \Theta_j) + \mathcal{O}(\rho^2) + \rho\,L(w, \Phi) + \tfrac{1}{\rho}\,Q(w, \Phi).\\
\rho^{-2}\,g(\tilde{N}, \nabla_{Z_i} Z_j) &= \mathcal{O}(\rho) + \tfrac{1}{\rho}\,L(w, \Phi) + \tfrac{1}{\rho}\,Q(w, \Phi), \qquad i \neq j.
\end{aligned}
$$

where summation over repeated indices is understood.

Proof : Note first that by Lemma 3.2.1, we can write

$$
\nabla_{X_\alpha} X_\beta = \sum_{\gamma=0}^{m-1} \left(\mathcal{O}(\rho) + L(w, \Phi) + Q(w, \Phi)\right) X_\gamma \tag{3.10}
$$

In particular, this together with the expression of Z_0 and Z_j, implies the rough estimate

$$
\nabla_{Z_\alpha} X_\beta = \sum_{\gamma} \left(\mathcal{O}(\rho^2) + \rho\,L(w, \Phi) + \rho\,Q(w, \Phi)\right) X_\gamma \tag{3.11}
$$

We will need a more precise estimate for $\nabla_{Z_0} X_0$.

First estimate : We estimate $g(\widetilde{N}, \nabla_{Z_0} Z_0)$. We must expand

$$
\begin{aligned}
\rho^{-2} g(\widetilde{N}, \nabla_{Z_0} Z_0) &= \rho^{-1}\,\kappa_{ss} g(\widetilde{N}, X_0) + \rho^{-1}\,\kappa_s g(\widetilde{N}, \nabla_{Z_0} X_0)\\
&+ \rho^{-1}\,(\varphi_{ss} - w_{ss})\,g(\widetilde{N}, \Upsilon) + \rho^{-1}\,(\varphi_s - w_s\,)\,g(\widetilde{N}, \nabla_{Z_0} \Upsilon)\\
&+ \rho^{-1}\,\kappa_{ss}\,g(\Phi', \widetilde{N}) + \rho^{-1}\,\kappa_s\,g(\widetilde{N}, \nabla_{Z_0} \Phi')
\end{aligned}
$$

The estimation is broken into 4 steps :

Step 1 From Proposition 3.2.3, we have

$$g(\widetilde{N}, X_0) \;=\; \varphi_s\, g(X_0, X_0) - \kappa_s\, g(\Upsilon, X_0) + \frac{1}{\rho}\,\alpha_0\, g(Z_0, X_0) + \frac{1}{\rho}\,\beta_j\, g(Z_j, X_0).$$

We need the following expansion which follows from the result of Lemma 3.2.2

$$\begin{aligned} g(Z_0, X_0) &= \rho\,\kappa_s\, g(X_0, X_0) + \rho\,(\varphi_s - w_s)\; g(X_0, \Upsilon) + \rho\,\kappa_s\, g(X_0, \Phi') \\ &= \rho\,\kappa_s + \rho^3\,\kappa_s\,\varphi^2\, g(\Theta, E_0)\Theta, E_0) + 2\,\rho^2\,\kappa_s\,\varphi\, g(\Theta, E_0)\Phi, E_0) \\ &+ \frac{2}{3}\,\rho^2\,\varphi_s\,\varphi\, g(\Theta, E_0)\Phi, \Theta) + \mathcal{O}(\rho^4) + \rho^3\, L(w, \Phi) + \rho\, Q(w, \Phi) \end{aligned}$$

we conclude that

$$\begin{aligned} \frac{1}{\rho}\,\alpha_0\, g(Z_0, X_0) = \frac{1}{\varphi^2}\Big[&\kappa_s^3\, g(\Theta, \Phi') - \kappa_s^2\, w_s - \rho^2\,\varphi^2\,\varphi_s\,\kappa_s^2\, g(R(\Theta, E_0)\Theta, E_0) + \mathcal{O}(\rho^3) \\ &- 2\,\rho\,\varphi\,\varphi_s\,\kappa_s^2\, g(R(\Theta, E_0)\,\Phi, E_0) + \frac{2}{3}\,\rho\,\varphi\,\kappa_s\,(\kappa_s^2 - \varphi_s^2)\, g(R(\Theta, E_0)\Phi, \Theta)\Big] \\ &+ \rho^2\, L(w, \Phi) + Q(w, \Phi) \end{aligned} \tag{3.12}$$

A similar computation allows to show that

$$\frac{1}{\rho}\,\beta_j\, g(Z_j, X_0) = \rho^2\, L(w, \Phi) + Q(w, \Phi) \tag{3.13}$$

Collecting the estimates (3.12) and (3.13) together with the result of Lemma 3.2.2, we obtain

$$\begin{aligned} g(\widetilde{N}, X_0) &= \varphi_s - \frac{\kappa_s^2}{\varphi^2}\, w_s + \frac{\kappa_s^3}{\varphi^2}\, g(\Theta, \Phi') + \mathcal{O}(\rho^3) \\ &+ \rho^2\,\varphi_s^3\, g(R(\Theta, E_0)\Theta, E_0) + 2\,\rho\,\frac{\varphi_s^3}{\varphi}\, g(R(\Theta, E_0)\Phi, E_0) \\ &- \frac{4}{3}\,\rho\,\frac{\kappa_s\,\varphi_s^2}{\varphi}\, g(R(\Theta, E_0)\Phi, \Theta) + \rho^2\, L(w, \Phi) + Q(w, \Phi) \end{aligned}$$

Step 2 : We now compute

$$g(\widetilde{N}, \Upsilon) = \varphi_s\, g(X_0, \Upsilon) - \kappa_s\, g(\Upsilon, \Upsilon) + \frac{1}{\rho}\,\alpha_0\, g(Z_0, \Upsilon) + \frac{1}{\rho}\,\beta_j\, g(Z_j, \Upsilon). \tag{3.14}$$

Using the expression of Z_0, we get

$$g(Z_0, \Upsilon) \;=\; \rho\, \kappa_s\, g(X_0, \Upsilon) + \rho\, (\varphi_s - w_s)\, g(\Upsilon, \Upsilon) + \rho\, \kappa_s\, g(\Phi', \Upsilon)$$

Collecting these together with the estimates of Lemma 3.2.2, we obtain

$$\begin{aligned} g(Z_0, \Upsilon) &= \rho\, \varphi_s - \rho\, w_s + \rho\, \kappa_s\, g(\Theta, \Phi') + \mathcal{O}(\rho^4) \\ &+ \frac{2}{3}\, \rho^2\, \varphi\, \kappa_s\, g(R(\Theta, E_0)\Phi, \Theta) + \rho^3\, L(w, \Phi) + \rho\, Q(w, \Phi) \end{aligned}$$

Hence we conclude using the expression of α_0 given in Proposition 3.2.3 that

$$\begin{aligned} \frac{1}{\rho}\, \alpha_0\, g(Z_0, \Upsilon) &= \frac{1}{\varphi^2}\, \kappa_s^2\, \varphi_s\, g(\Theta, \Phi') - \frac{1}{\varphi^2}\, \kappa_s\, \varphi_s\, w_s - \rho^2\, \varphi_s^2\, \kappa_s\, g(R(\Theta, E_0)\Theta, E_0) + \mathcal{O}(\rho^3) \\ &- 2\, \rho\, \varphi_s^2\, \frac{\kappa_s}{\varphi}\, g(R(\Theta, E_0)\Phi, E_0) + \frac{2}{3}\, \rho\, \frac{\kappa_s}{\varphi}\, (\kappa_s^2 - \varphi_s^2)\, g(R(\Theta, E_0)\Phi, \Theta) \\ &+ \rho^2\, L(w, \Phi) + Q(w, \Phi) \end{aligned}$$

Similarly, we have

$$\frac{1}{\rho}\, \beta_j\, g(Z_j, \Upsilon) = \rho^2\, L(w, \Phi) + Q(w, \Phi)$$

Using Lemma 3.2.2 and (3.14), we get

$$\begin{aligned} g(\widetilde{N}, \Upsilon) = -\kappa_s - \frac{\varphi_s\, \kappa_s}{\varphi^2}\, w_s + \frac{\kappa_s^2}{\varphi^2}\, \varphi_s\, g(\Theta, \Phi') - \rho^2\, \varphi_s^2\, \kappa_s\, g(R(\Theta, E_0)\Theta, E_0) \\ - 2\, \rho\, \frac{\kappa_s}{\varphi}\, \varphi_s^2\, g(R(\Theta, E_0)\Phi, E_0) + \frac{4}{3}\, \rho\, \frac{\varphi_s}{\varphi}\, \kappa_s^2\, g(R(\Theta, E_0)\Phi, \Theta) \\ + \mathcal{O}(\rho^3) + \rho^2\, L(w, \Phi) + \, Q(w, \Phi). \end{aligned} \tag{3.15}$$

Step 3 Expanding Z_0 gives

$$\begin{aligned} g(\widetilde{N}, \nabla_{Z_0} X_0) &= \rho\, \kappa_s\, g(\widetilde{N}, \nabla_{X_0} X_0) + \rho\, (\varphi_s - w_s)\, g(\widetilde{N}, \nabla_{\Upsilon} X_0) \\ &+ \rho\, \kappa_s\, g(\widetilde{N}, \nabla_{\Phi'} X_0) \end{aligned}$$

Using Lemma 3.2.1, we get

$$\begin{aligned} g(\widetilde{N}, \nabla_{X_0} X_0) &= \rho\, \varphi\, \kappa_s\, g(R(\Theta, E_0)\Theta, E_0) + \mathcal{O}(\rho^3) \\ &+ \kappa_s\, g(R(\Theta, E_0)\Phi, E_0) + \rho^2\, L(w, \Phi) + \, Q(w, \Phi), \end{aligned}$$

$$
\begin{aligned}
g(\widetilde{N}, \nabla_{\Upsilon} X_0) &= \rho\,\varphi\,\varphi_s\, g(R(\Theta, E_0)\Theta, E_0) + \mathcal{O}(\rho^3) \\
&+ \varphi_s\, g(R(\Theta, E_0)\Phi, E_0) + \rho^2\, L(w, \Phi) + Q(w, \Phi).
\end{aligned}
$$

and

$$
g(\widetilde{N}, \nabla_{\Phi'} X_0) = \rho^2\, L(w, \Phi) + Q(w, \Phi).
$$

This gives

$$
\begin{aligned}
g(\widetilde{N}, \nabla_{Z_0} X_0) &= \rho^2\,\varphi^3\, g(R(\Theta, E_0)\Theta, E_0) + \mathcal{O}(\rho^3) \\
&+ \rho\,\varphi^2\, g(R(\Theta, E_0)\Phi, E_0) + \rho^2\, L(w, \Phi) + Q(w, \Phi).
\end{aligned}
$$

A similar computation and (3.10) gives

$$
\begin{aligned}
g(\widetilde{N}, \nabla_{Z_0} \Upsilon) &= \rho^2\,\varphi\,\varphi_s\,\kappa_s\, g(R(\Theta, E_0)\Theta, E_0) + \mathcal{O}(\rho^3) \\
&+ \rho\,\varphi_s\,\kappa_s\, g(R(\Theta, E_0)\Phi, E_0) + \frac{2}{3}\,\rho\,\varphi_s^2\, g(R(\Theta, E_0)\Phi, \Theta) \\
&+ \rho^2\, L(w, \Phi) + Q(w, \Phi).
\end{aligned}
$$

Step 4 Now, we expand

$$
g(\widetilde{N}, \Phi') = \varphi_s\, g(X_0, \Phi') - \kappa_s\, g(\Upsilon, \Phi') + \frac{1}{\rho}\,\alpha_0\, g(Z_0, \Phi') + \frac{1}{\rho}\,\beta_j\, g(Z_j, \Phi')
$$

using the expansions

$$
g(X_0, \Phi') = \rho^2\, L(w, \Phi) + Q(w, \Phi)
$$

and

$$
g(\Upsilon, \Phi') = g(\Theta, \Phi') + \rho^2\, L(w, \Phi) + Q(w, \Phi)
$$

we conclude using the expansions of α_0 and β_j given in Proposition 3.2.3

$$
\frac{1}{\rho}\,\alpha_0\, g(Z_0, \Phi') = \rho^2\, L(w, \Phi) + Q(w, \Phi)
$$

and

$$
\frac{1}{\rho}\,\beta_j\, g(Z_j, \Phi') = \rho^2\, L(w, \Phi) + Q(w, \Phi)
$$

hence

$$g(\widetilde{N},\Phi') = -\kappa_s\, g(\Theta,\Phi') + \rho^2\, L(w,\Phi) + \, Q(w,\Phi)$$

Step 5 : Finally

$$g(\widetilde{N},\nabla_{Z_0}\Phi') = \rho\,\kappa_s\, g(\widetilde{N},\Phi'') + \partial_{x^0}\Phi_j\, g(\widetilde{N},\nabla_{Z_0}X_j)$$

where summation over repeated indices is understood. From (3.11) we have

$$\partial_{x^0}\Phi_j\, g(\widetilde{N},\nabla_{Z_0}X_j) = \rho^2\, L(w,\Phi) + \rho\, Q(w,\Phi)$$

On the other hand

$$g(\widetilde{N},\Phi'') \;=\; \varphi_s\, g(X_0,\Phi'') - \kappa_s\, g(\Upsilon,\Phi'') + \frac{1}{\rho}\,\alpha_0\, g(Z_0,\Phi'') + \frac{1}{\rho}\,\beta_j\, g(Z_j,\Phi'')$$

Using lemma 3.2.2, we get

$$g(\widetilde{N},\Phi'') = -\kappa_s\, g(\Theta,\Phi'') + \rho^2\, L(w,\Phi) + \, Q(w,\Phi)$$

we hence conclude that

$$g(\widetilde{N},\nabla_{Z_0}\Phi') = -\rho\,\kappa_s^2\, g(\Theta,\Phi'') + \rho^2\, L(w,\Phi) + \rho\, Q(w,\Phi)$$

Collecting the expansions of the steps 1-5, we prove the first estimate.

Second estimate : Decompose

$$\begin{aligned} g(\widetilde{N},\nabla_{Z_0}Z_j) \;&=\; -\rho\, w_s\, g(\widetilde{N},\Upsilon_j) + \rho\, w_{js}\, g(\widetilde{N},\Upsilon) + \rho\,(\varphi - w)\, g(\widetilde{N},\nabla_{Z_0}\Upsilon_j) \\ &\quad -\rho\, w_j\, g(N,\nabla_{Z_0}\Upsilon). \end{aligned}$$

The second estimate follows from the above computations.

Third estimate : Now, we estimate $g(\widetilde{N},\nabla_{Z_i}Z_j)$. Observe that

$$g(\widetilde{N},\nabla_{Z_i}Z_j) = -\frac{1}{2}\,\Big(g(\nabla_{Z_i}\widetilde{N},Z_j) + g(\nabla_{Z_j}\widetilde{N},Z_i)\Big)$$

$$\begin{aligned} g(Z_j,\nabla_{Z_i}\widetilde{N}) \;&=\; \varphi_s\, g(Z_j,\nabla_{Z_i}X_0) - \kappa_s\, g(Z_j,\nabla_{Z_i}\Upsilon) \\ &+\; \frac{1}{\rho}\, g(Z_j,\nabla_{Z_i}(\alpha_0\, Z_0)) + \frac{1}{\rho}\, g(Z_j,\nabla_{Z_i}(\beta_k\, Z_k)) \end{aligned}$$

The estimates is broken into 4 steps.

Step 1 we set

$$A_{ij} := g(\nabla_{Z_j} X_0, Z_i) + g(\nabla_{Z_i} X_0, Z_j)$$

We expand

$$\nabla_{Z_i} X_0 = \rho\,(\varphi - w)\,\nabla_{\Upsilon_i} X_0 - \rho\, w_i\, \nabla_{\Upsilon} X_0$$

We will need the following expansions which follows from the expansions of the lemma 3.2.1

$$\nabla_{X_i} X_0 \;=\; g(R(\rho\,\varphi\,\Theta + \Phi, E_0)E_i, E_k)\, X_k + \big(\mathcal{O}(\rho^2) + \rho\, L(w,\Phi) + Q(w,\Phi)\big)\; X_\gamma$$

In particular, we obtain the following expansions

$$\rho\, w_i\, \nabla_{\Upsilon} X_0 = (\rho^2\, L(w,\Phi) + \rho\, Q(w,\Phi))\, X_k + \rho^2\, Q(w,\Phi)\, X_0.$$

and

$$\nabla_{\Upsilon_i} X_0 \;=\; g(R(\rho\,\varphi\,\Theta + \Phi, E_0)\Theta_i, E_k)\, X_k + \big(\mathcal{O}(\rho^2) + \rho\, L(w,\Phi) + Q(w,\Phi)\big)\; X_\gamma$$

Collecting these together, we deduce

$$\begin{aligned} \nabla_{Z_i} X_0 = \rho\,(\varphi - w)\, g(R(\rho\,\varphi\,\Theta + \Phi, E_0)\Theta_i, E_k)\, X_k \\ + \big(\mathcal{O}(\rho^3) + \rho^2\, L(w,\Phi) + \rho\, Q(w,\Phi)\big)\; X_j. \end{aligned} \tag{3.16}$$

We now compute

$$g(X_k, Z_j) \;=\; \rho\,(\varphi - w)\, g(X_k, \Upsilon_j) - \rho\, w_j\, g(X_k, \Upsilon).$$

Therefore, it follows from lemma 3.2.1 that

$$\begin{aligned} g(X_k, Z_j) \;&=\; \rho\,(\varphi - w)\, g(E_k, \Theta_j) + \frac{1}{3}\,\rho^3\varphi^3\, g(R(\Theta,\Theta_j)\Theta, E_k) \\ &+\; \frac{1}{3}\,\rho^2\,\varphi^2\; (\, g(R(\Theta,\Theta_j)\Phi, E_k) + g(R(\Phi,\Theta_j)\Theta, E_k)) \\ &-\; \rho\, w_j\, g(E_k,\Theta_j) + \mathcal{O}(\rho^4) + \rho^3\, L(w,\Phi) + \rho^2\, Q(w,\Phi). \end{aligned}$$

which together with (3.16) gives

$$\begin{aligned} g(\nabla_{Z_i} X_0, Z_j) \;&=\; \rho^3\varphi^3\, g(R(\Theta, E_0)\Theta_i, \Theta_j) + \rho^2\varphi^2\, g(R(\Phi, E_0)\Theta_i, \Theta_j) \\ &+\; \mathcal{O}(\rho^4) + \rho^3\, L(w,\Phi) + \rho^2\, Q(w,\Phi). \end{aligned}$$

In particular, we deduce using the fact that $g(R(.,E_0)\Theta_i, \Theta_j) = -g(R(.,E_0)\Theta_j, \Theta_i)$

$$A_{ij} = \mathcal{O}(\rho^4) + \rho^3\, L(w,\Phi) + \rho^2\, Q(w,\Phi)$$

Step 2 We now estimate

$$\bar{B}_{ij} := \frac{-\kappa_s}{2}\,(\,g(Z_i, \nabla_{Z_j}\Upsilon) + \,g(\nabla_{Z_i}\Upsilon, Z_j)\,)$$

To this aim, we set

$$B_{ij} = g(Z_i, \nabla_{Z_j}\Upsilon)$$

so that

$$\bar{B}_{ij} = \frac{-\kappa_s}{2}\,(B_{ij} + B_{j\,i})$$

It is convenient to define

$$B'_{ij} = \frac{1}{\varphi - w}\, g\left(Z_i, \nabla_{Z_j}(\varphi - w)\,\Upsilon\right)$$

we have the expansion

$$B'_{ij} = \frac{1}{\varphi}\left(1 + \frac{w}{\varphi} + Q(w, \Phi)\right) g(Z_i, \nabla_{Z_j}(\varphi - w)\Upsilon)$$

Decompose

$$g(Z_i, \nabla_{Z_j}(\varphi - w)\Upsilon) \;=\; (\varphi - w)\; g(Z_i, \nabla_{Z_j}\Upsilon) - w_i\, g(Z_i, \Upsilon)$$

On the other hands, we can easily see that

$$w_i\, g(Z_i, \Upsilon) = \rho\, Q(w, \Phi)$$

We then conclude that

$$B'_{ij} = B_{ij} + \rho\, Q(w, \Phi)$$

hence it is enough to focuss on the estimate of B'_{ij}. To analyze this term, let us revert for the moment and regard w and Φ as functions of the coordinates (s, y) and also consider ρ as a variable instead of just a parameter. Thus we consider

$$\tilde{F}(s, \rho, y) = F\big(s, \rho\,(\varphi - w(s, y))\,\Upsilon(y) + \Phi(\rho\,\kappa(s))\big).$$

The coordinate vector fields Z_j are still equal to $\tilde{F}_*(\partial_{y_j})$, but now we also have

$$\tilde{F}_*(\partial_\rho) = \kappa(s)\,X_0 + (\varphi - w)\,\Upsilon + \kappa(s)\partial_\rho\,\Phi(\rho\,\kappa(s)).$$

which is the identity we wish to use below. Now, we write

$$\begin{aligned}
B'_{ij} &= \frac{1}{\varphi - w}\, g\left(Z_i\,, \nabla_{Z_j}(\tilde{F}_*(\partial_\rho) - \kappa(s)\,X_0 - \kappa(s)\partial_\rho\,\Phi\,)\right) \\
&= \frac{1}{\varphi - w}\, g(Z_i\,, \nabla_{\partial_\rho} Z_j) - \frac{1}{\varphi - w}\, g(Z_i\,, \nabla_{Z_j}(\kappa(s)\,X_0)\,) \\
&- \frac{1}{\varphi - w}\, g(Z_i\,, \nabla_{Z_j}(\kappa(s)\,\partial_\rho\,\Phi))
\end{aligned}$$

Using the following expansion

$$g(\nabla_{Z_j} X_0\,, Z_i) \;\;=\;\; \rho^3\, L(w, \Phi) + \rho^2\, Q(w, \Phi)$$

We deduce that

$$g(Z_i\,, \nabla_{Z_j}(\kappa(s)\, X_0)) = \rho^3\, L(w, \Phi) + \rho\, Q(w, \Phi)$$

On the other hand, it is easy to see that

$$g(Z_i\,, \nabla_{Z_j}(\kappa(s)\, \partial_\rho\, \Phi)) = \rho^3\, L(w, \Phi) + \rho\, Q(w, \Phi)$$

Therefore, we deduce that

$$\begin{aligned} B'_{ij} + B'_{j\,i} \;\;&=\;\; \frac{1}{\varphi - w}\, \big(g(Z_i\,, \nabla_{\partial_\rho} Z_j) + g(Z_j\,, \nabla_{\partial_\rho} Z_i)\,\big) \\ &+\;\; \rho^3\, L(w, \Phi) + \rho^2\, Q(w, \Phi) \\ &=\;\; \frac{1}{\varphi - w}\, \partial_\rho\, g(Z_i\,, Z_j) + \rho^3\, L(w, \Phi) + \rho^2\, Q(w, \Phi) \end{aligned}$$

It follows from the expansions of proposition 3.2.2 and (3.9) that

$$\begin{aligned} B'_{ij} + B'_{j\,i} \;\;=\;\; & \frac{1}{\varphi}\,(\,1 + \frac{w}{\varphi} + Q(w, \Phi)\,)\Big[2\,\rho\,(\varphi - w)^2\,\delta_{ij} + \mathcal{O}(\rho^4) + \frac{4}{3}\,\rho^3\,\varphi^4\, g(R(\Theta, \Theta_i)\Theta, \Theta_j) \\ & + \rho^2\,\varphi^3\,\{\, g(R(\Theta, \Theta_i)\Phi, \Theta_j) + g(R(\Theta, \Theta_j)\Phi, \Theta_i)\,\}\Big] + \rho^3\, L(w, \Phi) + \rho^2\, Q(w, \Phi) \end{aligned}$$

Step 3 : Now, we estimate

$$C_{ij} = g(\,\nabla_{Z_i}(\beta_k\, Z_k),\; Z_j) + g(\nabla_{Z_j}(\beta_k\, Z_k),\; Z_i\,)$$

clearly

$$C_{ij} = \partial_{y_i}\,\beta_k\, g(Z_j, Z_k) + \partial_{y_j}\,\beta_k\, g(Z_i, Z_k) + \beta_k\,\partial_{y_k}\, g(Z_i, Z_j)$$

It follows from the proposition 3.2.3 that

$$
\begin{aligned}
\frac{1}{\rho^2}\,\partial_i(\,\beta_k\, g(Z_j, Z_k)\,) \;\; = \;\; & \kappa_s\, w_{ij} + \frac{1}{3}\,\rho\,\varphi^2\,\kappa_s\,[\, g(R(\Phi,\Theta_i)\Theta,\Theta_j) + g(R(\Phi,\Theta)\Theta_i,\Theta_j)\,] \\
& -\frac{2}{3}\,\rho^2\,\varphi^3\,\varphi_s\,[\, g(R(\Theta_i, E_0)\Theta,\Theta_j) + g(R(\Theta, E_0)\Theta_i,\Theta_j)\,] \\
& -\frac{2}{3}\,\rho\,\varphi^2\,\varphi_s\,[\, g(R(\Theta_i, E_0)\Phi,\Theta_j) + g(R(\Phi, E_0)\Theta_i,\Theta_j)\,] \\
& +\mathcal{O}(\rho^3) + \rho^2\, L(w,\Phi) + \rho^2\, Q(w,\Phi)
\end{aligned}
$$

Now, observe that

$$
(\partial_i\,\beta_k)\, g(Z_j, Z_k) = \partial_i(\,\beta_k\, g(Z_j, Z_k)\,) - \beta_k\,\partial_i\, g(Z_j, Z_k)
$$

On the other hand, using the expression of β_j given in proposition 3.2.3 together with (3.9), we get

$$
\beta_k\,\partial_{y_k}\, g(Z_i, Z_j) = \rho^4\, L(w,\Phi) + \rho^2\, Q(w,\Phi)
$$

Collecting these, we conclude

$$
\begin{aligned}
\frac{1}{\rho^2}\, C_{ij} \;\; = \;\; & -2\,\kappa_s\, w_{ij} + \frac{1}{3}\,\rho\,\varphi^2\,\kappa_s\,[\, g(R(\Phi,\Theta_i)\Theta,\Theta_j) + g(R(\Phi,\Theta)\Theta_i,\Theta_j)\,] \\
& -\frac{2}{3}\,\rho^2\,\varphi^3\,\varphi_s\,[\, g(R(\Theta_i, E_0)\Theta,\Theta_j) + g(R(\Theta_j, E_0)\Theta,\Theta_i)\,] \\
& -\frac{2}{3}\,\rho\,\varphi^2\,\varphi_s\,[\, g(R(\Theta_i, E_0)\Phi,\Theta_j) + g(R(\Theta_j, E_0)\Phi,\Theta_i)\,] \\
& +\mathcal{O}(\rho^3) + \rho^2\, L(w,\Phi) + \rho^2\, Q(w,\Phi)
\end{aligned}
$$

here we used the fact that $R(X,Y) + R(Y,X) = 0$

Step 4 Finally we must compute

$$
D_{ij} = g(\,\nabla_{Z_i}(\alpha_0\, Z_0), Z_j\,).
$$

Clearly, we have

$$
D_{ij} = -g(\,\nabla_{Z_i}\, Z_j, \alpha_0\, Z_0) + \partial_{y_i}\, g(\, Z_j, \alpha_0\, Z_0).
$$

Using lemma 3.2.1 and proposition 3.2.2, we can easily see that

$$
g(\nabla_{Z_i}\, Z_j, \alpha_0\, Z_0) = \rho^4\, L(w,\Phi) + \rho^2\, Q(w,\Phi)
$$

and

$$(\partial_{y_i}\,\alpha_0)\,g(Z_0,Z_j)=\rho^4\,L(w,\Phi)+\rho^2\,Q(w,\Phi)$$

Now, we use the fact that $\bar{\nabla}_{\Upsilon_i}\Upsilon_j=0$ at $p\in K$ and $g(\Theta_i,\Theta_j)=\delta_{ij}$ we deduce

$$\partial_{y_i}\,g(Z_0,Z_j)=\rho^2\,\varphi\,\varphi_s\,\delta_{ij}+\rho^2\,L(w,\Phi)+Q(w,\Phi)$$

Collecting these, we obtain

$$\begin{aligned}
D_{ij} &= -g(\nabla_{Z_i}\,Z_j,\alpha_0\,Z_0)+(\partial_{y_i}\,\alpha_0)\,g(Z_j,Z_0)+\alpha_0\,\partial_{y_i}\,g(Z_0,Z_j)\\
&= -\rho^2\,\frac{\varphi_s}{\varphi}\,\kappa_s\,\delta_{ij}\,w_s+\rho^2\,\frac{\varphi_s}{\varphi}\,\kappa_s^2\,\delta_{ij}\,g(\Theta,\Phi')+\mathcal{O}(\rho^5)\\
&\quad -\rho^4\,\varphi\,\varphi_s^2\,\kappa_s\,\delta_{ij}\,g(R(\Theta,E_0)\Theta,E_0)-2\,\rho^3\,\varphi_s^2\,\kappa_s\,\delta_{ij}\,g(R(\Theta,E_0)\Phi,E_0)\\
&\quad +\frac{2}{3}\,\rho^3\,\varphi_s\,(\kappa_s^2-\varphi_s^2)\,\delta_{ij}\,g(R(\Theta,E_0)\Phi,\Theta)+\rho^3\,L(w,\Phi)+\rho\,Q(w,\Phi)
\end{aligned}$$

This together with the estimates of the previous steps give the desired expansions. □

3.2.4 The mean curvature of the perturbed Delaunay hypersurface

Collecting the estimates of the last subsection together with the fact that

$$\varphi_s\,\kappa_{ss}-\kappa_s\,\varphi_{ss}=m\,\varphi^3-(m-1)\,\varphi\,\kappa_s$$

we obtain the expansion of the mean curvature of the hypersurface $\Sigma_\rho(w,\Phi)$. In the coordinate system defined in the previous sections, we get

$$\begin{aligned}
\rho\,m\,H(w,\Phi) &= m+\tfrac{1}{3}\rho^2\,\kappa_s\,g(R(\Theta,\Theta_i)\Theta,\Theta_i)+\rho^2\,a_\tau\,g(\,R(\Theta,E_0)\Theta,E_0)+\mathcal{O}(\rho^3)\\
&\quad +\kappa_s\,\varphi^{-3}\,\left(\partial_s^2+\Delta_{S^{m-1}}+b_\tau\,\partial_s+m-1\,\right)\,w\\
&\quad -\kappa_s^3\,\varphi^{-3}\,g\left(\rho\,\Phi''+c_\tau\,\Phi'+\rho\,d_\tau\,R(\Phi,E_0)\,E_0,\ \Theta\,\right)\\
&\quad +\tfrac{2}{3}\rho^2\,\varphi_s\,g(R(\Theta,\Theta_i)E_0,\Theta_i)+\tfrac{2}{3}\,\rho\,\tfrac{\varphi_s}{\varphi}\,g(R(\Phi,\Theta_i)\,E_0,\Theta_i)\\
&\quad +\rho\,e_\tau\,g(R(\Phi,\Theta)E_0,\Theta)+\rho^2\,L(w,\Phi)+Q(w,\Phi)
\end{aligned} \tag{3.17}$$

Summation over repeated indices is understood. Here the functions a_τ, b_τ, c_τ, d_τ, and e_τ are respectively given by

$$a_\tau:=\frac{1}{\varphi^2}\,\left(\frac{m}{2}\,\varphi^2\,\varphi_s^2+2\,\kappa_s\,\varphi^2-m\,\varphi^2\,\kappa_s^2+(m-2)\,\kappa_s^3\right),$$

$$b_\tau := \frac{m\,\varphi_s}{\kappa_s}\,\left(\varphi - \tau^m\,\varphi^{1-m}\right),$$

$$c_\tau := \frac{\varphi_s}{\kappa_s^2}\,((m+2)\,\varphi + 2\,(1-m)\,\tau^m\,\varphi^{1-m}),$$

$$d_\tau := \frac{1}{\kappa_s^3}\,\left(\,(2m-1)\,\kappa_s^3 - 2\,m\,\varphi^2\,\kappa_s^2 + 2\,\kappa_s\,\varphi_s^2 + m\,\varphi_s^2\,\varphi^2\right),$$

and

$$e_\tau := \frac{2}{3}\,\frac{\varphi_s}{\varphi^3}\,\left(3\,m\,\varphi^2\,\kappa_s - m\,\varphi_s^2 + 3\,(1-m)\,\kappa_s^2\right)$$

Observe that when $\tau = \tau_m := \frac{1}{m}\,(m-1)^{\frac{m-1}{m}}$ (i.e, the hypersurface parameterized by X is the cylinder $S^{m-1}(\rho\,\frac{m-1}{m}) \times \mathbb{R}$), then

$$\varphi = \varphi_{\tau_m} \equiv \frac{m-1}{m}, \quad \varphi_s = (\varphi_{\tau_m})_s \equiv 0 \quad \kappa_s = (\kappa_{\tau_m})_s \equiv \frac{m-1}{m}$$

and we recover the expression of the mean curvature of the perturbed tubes which was found in [28].

Remark 3.2.1 *As already mentioned the operator $\rho^2\,L + Q$ depends on Φ, Φ' and Φ'' (and w and its derivatives). However, first observe that this expression is affine in Φ'', that is it can be decomposed as*

$$\begin{aligned} \rho^2\,L(w,\ldots,\Phi'') + Q(w,\ldots,\Phi'') \;&=\; \rho^2\,L(w,\ldots,\Phi') + Q(w,\ldots,\Phi') \\ &+\; \textstyle\sum_j(\mathcal{O}(\rho^2) + L(w,\ldots,\Phi') + Q(w,\ldots,\Phi'))\,\Phi''_j \end{aligned}$$

Now, closer inspection of the proof of Proposition 3.2.4 (precisely the analysis done in Step 5 of the first estimate) shows that the exponents in ρ in front of the terms involving Φ'' are slightly better than expected and in fact one has

$$\begin{aligned} \rho^2\,L(w,\ldots,\Phi'') + Q(w,\ldots,\Phi'') \;&=\; \rho^2\,L(w,\ldots,\Phi') + Q(w,\ldots,\Phi') \\ &+\; \textstyle\sum_j(\mathcal{O}(\rho^3) + \rho\,L(w,\ldots,\Phi') + \rho\,Q(w,\ldots,\Phi'))\,\Phi''_j \end{aligned}$$

3.3 Jacobi operators

In this section we study the mapping properties of the linear operators which appear in the expression of the mean curvature of $\Sigma_\rho(w,\Phi)$. The first one is an operator defined on the spherical normal bundle SNK given by

$$\mathcal{L}_\tau w := \partial_s^2\,w + \Delta_{S^{m-1}}\,w + b_\tau\,\partial_s\,w + (m-1)\,w \tag{3.18}$$

The second one is defined by

$$\mathfrak{J}\,\Phi = \Phi'' + \frac{1}{\rho}\,c_\tau\,\Phi' + d_\tau\,R(\Phi, E_0)\,E_0 \tag{3.19}$$

We introduce the quadratic form

$$\Omega(\cdot,\cdot) := -\left(a_\tau - \frac{1}{3}\,\kappa_s\right)\,g(R(\cdot, E_0)\cdot, E_0) + \frac{1}{3}\,\kappa_s\,\mathrm{Ric}(\cdot,\cdot).$$

acting on N_pK. Granted these notations, the equation $\rho\, m\, H(w,\Phi) = m$ can be written in the more compact form :

$$\begin{array}{rcl} \kappa_s\,\mathcal{L}_\tau w - \rho\,\kappa_s^3\,g(\mathfrak{J}\,\Phi,\Theta) & = & \varphi^3\,\Omega(\Theta,\Theta)\,\rho^2 - \frac{2}{3}\,\rho^2\,\varphi_s\,\varphi^3\,g(R(\Theta,\Theta_i)E_0,\Theta_i) + \mathcal{O}(\rho^3) \\ & - & \rho\,e_\tau\,\varphi^3\,g(R(\Phi,\Theta)E_0,\Theta) - \frac{2}{3}\,\rho\,\varphi_s\,\varphi^2\;g(R(\Phi,\Theta_i)\Theta_i, E_0) \\ & + & \rho^2\,L(w,\Phi) + Q(w,\Phi) \end{array} \tag{3.20}$$

In order to analyze the operator $\mathcal{L}_\tau$, we introduce the eigenfunction decomposition

$$w(s,\theta) = \sum_{j\in\mathbb{N}} w_j(s)\,\varphi_j(\theta)$$

where φ_j are the eigenfunctions of $-\Delta_{\mathcal{S}^{m-1}}$, namely

$$-\Delta_{\mathcal{S}^{m-1}}\,\varphi_j = \lambda_j\,\varphi_j$$

and λ_j are given by

$$0 = \lambda_0 < \lambda_1 = \ldots = \lambda_m = m-1 < \lambda_{m+1} = 2m = \ldots$$

w decomposes as $w_0 + \hat{w} + \tilde{w}$, where

$$\hat{w} := \sum_{j=1}^{m} w_j\,\varphi_j \qquad \textit{and} \qquad \tilde{w} = \sum_{j>m} w_j\,\varphi_j$$

We denote by Π_0, $\hat{\Pi}$ and $\tilde{\Pi}$ the projections on these three components respectively. From now on, we assume that we are working with functions w such that $\hat{\Pi}\,w = 0$. We denote by $\mathcal{L}_0$ and $\tilde{\mathcal{L}}$ the operators induced by the first and the third components. In particular

$$\mathcal{L}_0 = \partial_s^2 + b_\tau\,\partial_s + (m-1)$$

3.3.1 Mapping properties

In this subsection, we study the mapping properties of $\mathfrak{J}$ and the components of $\mathcal{L}_\tau$. We first study the operator $\mathfrak{J}$. Recall that

$$t = \rho\, \kappa(s)$$

Using this, observe that, by direct computation

$$(\kappa_s^3\, \varphi^{m-4})^2\, \mathfrak{J}\Phi = \kappa_s^3\, \varphi^{m-4}\, \frac{d}{dt}\left(\kappa_s^3\, \varphi^{m-4}\, \frac{d}{dt}\Phi\right) + (\kappa_s^3\, \varphi^{m-4})^2\, d_\tau\, R(\Phi, E_0)\, E_0$$

We consider the change of variable

$$\zeta_\tau\, d\xi = \frac{1}{\kappa_s^3\, \varphi^{m-4}}\, dt$$

where the constant ζ_τ is defined so that the integral of the right hand side over the length of K is equal to ℓ. Hence

$$\zeta_\tau = \frac{1}{\kappa(\pi_\tau)} \int_0^{\pi_\tau} \frac{1}{\kappa_s^2\, \varphi^{m-4}}\, ds$$

where π_τ is the least period of φ.

Using this change of variable we reduce the study of $\mathfrak{J}$ to the study of the operator

$$\tilde{\mathfrak{J}}\Phi = \frac{d^2\Phi}{d\xi^2} + \tilde{c}_\tau\, R(\Phi, E_0)\, E_0$$

where we have set

$$\tilde{c}_\tau := \zeta_\tau^2\, (\kappa_s^3\, \varphi^{m-4})^2\, d_\tau$$

We define α_τ to be the average of the function $\tilde{c}_\tau$, namely

$$\alpha_\tau = \frac{1}{\kappa(\pi_\tau)} \int_0^{\pi_\tau} \zeta_\tau\, \kappa_s^4\, \varphi^{m-4}\, d_\tau\, ds$$

This is the constant which appears in the introduction. Recall that the geodesic K is said to be τ-nondegenerate when the operator

$$\mathfrak{J}_\tau := \frac{d^2\Phi}{d\xi^2} + \alpha_\tau\, R(\Phi, X_0)\, X_0$$

is invertible. We have the following

Proposition 3.3.1 *Assume that $\mathfrak{J}_\tau$ is τ-nondegenerate. Then, for all ρ small enough, the operator*

$$\mathfrak{J} : \mathcal{C}^{2,\alpha}(K, NK) \longrightarrow \mathcal{C}^{0,\alpha}(K, NK)$$

is an isomorphism. Furthermore, we have

$$\|\Phi\|_{\mathcal{C}^1(K,NK)} + \rho^{1+\alpha} \|\Phi''\|_{\mathcal{C}^{0,\alpha}(K,NK)} \leq c \, \|\mathfrak{J}\, \Phi\|_{\mathcal{C}^{0,\alpha}(K,NK)}$$

Proof : To begin with, we study the operator $\tilde{\mathfrak{J}}$ and more precisely the equation

$$\tilde{\mathfrak{J}}\, \Phi = \Psi$$

Observe that we can write

$$\tilde{c}_\tau - \alpha_\tau = \rho \, \frac{dC_\tau}{d\xi}$$

where C_τ is a bounded function.

We define Φ_0 to be the solution of

$$\mathfrak{J}_\tau \, \Phi_0 = \Psi$$

and next we define F_1 to be the solution of

$$\mathfrak{J}_\tau \, F_1 = -C_\tau R(\Phi_0, E_0)\, E_0$$

It should be clear that

$$\|\Phi_0\|_{\mathcal{C}^2(SNK)} + \|F_1\|_{\mathcal{C}^2(SNK)} \leq c \, \|\Psi\|_{\mathcal{C}^0(SNK)}$$

We define the operator

$$G(\Psi) := \Phi_0 + \rho \, \frac{dF_1}{d\xi}$$

We have

$$\|G(\Psi)\|_{\mathcal{C}^1(SNK)} \leq c \, \|\Psi\|_{\mathcal{C}^0(SNK)}$$

Moreover, direct computation shows that

$$\left\| \tilde{\mathfrak{J}} \circ G(\Psi) - \Psi \right\|_{\mathcal{C}^0(SNK)} \leq c \, \rho \, \|\Psi\|_{\mathcal{C}^0(SNK)}$$

Hence $\tilde{\mathfrak{J}} \circ G = I + R$ where $R : \mathcal{C}^0(SNK) \longrightarrow \mathcal{C}^0(SNK)$ is small. A simple perturbation argument now shows that the operator $\tilde{\mathfrak{J}}$ is invertible and has a right inverse $\tilde{G}$ such that

$$\left\| \tilde{G}(\Psi) \right\|_{\mathcal{C}^1(SNK)} \leq c \, \|\Psi\|_{\mathcal{C}^0(SNK)}$$

Since $\tilde{\mathfrak{J}} \circ \tilde{G}(\Psi) = \Psi$ we conclude that

$$\left\| \tilde{G}(\Psi) \right\|_{\mathcal{C}^{2,\alpha}(SNK)} \leq c \, \|\Psi\|_{\mathcal{C}^{0,\alpha}(SNK)}$$

Now, performing the change of variable $t = t(\xi)$ back we conclude that the operator $\mathfrak{J}$ is invertible. However, observe that one has

$$\xi = t + \rho \, f_\rho(\frac{t}{\rho})$$

where f is a smooth function bounded independently of ρ. And hence, when solving the equation $\mathfrak{J} \, \Phi = \Psi$ one can conclude that

$$\|\Phi\|_{\mathcal{C}^1(SNK)} \leq c \, \|\Psi\|_{\mathcal{C}^0(SNK)}$$

(since $\frac{d\xi}{dt}$ is bounded independently of ρ) but one only has

$$\rho \, \|\Phi''\|_{\mathcal{C}^1(SNK)} \leq c \, \|\Psi\|_{\mathcal{C}^0(SNK)}$$

(since $\rho \, \frac{d^2\xi}{dt^2}$ is bounded independently of ρ) and similar estimate for the Hölder norm of Φ''. This completes the proof of the result. $\square$

Remark 3.3.1 *Let us observe that, in the special case where*

$$\Psi = \varphi_s \, h(\varphi)$$

where h is a smooth function then one get the estimate

$$\|\Phi\|_{\mathcal{C}^1(K,NK)} + \rho^{1+\alpha} \, \|\Phi''\|_{\mathcal{C}^{0,\alpha}(K,NK)} \leq c \, \rho$$

for the solution Φ of $\mathfrak{J} \, \Phi = \Psi$. This follows at once from the proof of the previous Proposition together with the fact that

$$(\kappa_s^3 \, \varphi^{m-4})^2 \, h(\varphi) \, \varphi_s = \rho \, (\kappa_s^3 \, \varphi^{m-4})^3 \, \zeta_\tau \, \kappa_s \, h(\varphi) \, \frac{d\varphi}{d\xi}$$

and hence

$$(\kappa_s^3 \, \varphi^{m-4})^2 \, h(\varphi) \, \varphi_s = \rho \frac{dk(\varphi)}{d\xi}$$

for some smooth bounded function k.

Next, we study the mapping properties of the operator $\tilde{\mathcal{L}}$. We have the :

Proposition 3.3.2 *The operator* $\tilde{\mathcal{L}} \,:\, \tilde{\Pi}\, \mathcal{C}^{2,\alpha}_\rho(SNK) \longrightarrow \tilde{\Pi}\, \mathcal{C}^{0,\alpha}_\rho(SNK)$ *is an isomorphism with inverse uniformly bounded as* $\rho \to 0$.

Proof : We show that the operator is injective. Indeed, if $\tilde{w} = \sum_{j>m} w_j\, \varphi_j$ is a solution of $\tilde{\mathcal{L}}\, w = 0$, then for any $j \geq m+1$, w_j is a solution of the ordinary differential equation

$$\partial_s^2\, w_j + b_\tau\, \partial_s\, w_j + (m-1-\lambda_j)\, w_j = 0 \tag{3.21}$$

We set

$$B_\tau(s) := \int_0^s b_\tau$$

We have explicitly

$$e^{B_\tau} = \kappa_s^2\, \tau^{-m}\, \varphi^{m-4}$$

Observe that B_τ is periodic. Multiplying (3.21) by $e^{B_\tau}\, w_j$ and integrating by parts, we get

$$\begin{aligned} 0 &= \int (\partial_s^2\, w_j)\, e^{B_\tau}\, w_j + \int\, b_\tau\, \partial_s\, w_j\, e^{B_\tau}\, w_j + (m-1-\lambda_j)\, \int e^{B_\tau}\, w_j^2 \\ &= -\int\, \partial_s\, w_j\, \partial_s\, (e^{B_\tau}\, w_j) + \int\, b_\tau\, \partial_s\, w_j\, e^{B_\tau}\, w_j + (m-1-\lambda_j)\, \int e^{B_\tau}\, w_j^2 \\ &= -\int\, e^{B_\tau}\, |\partial_s\, w_j|^2 + (m-1-\lambda_j)\, \int e^{B_\tau}\, w_j^2 \end{aligned}$$

If $j > m$, we have $\lambda_j > m-1$, so all quantities in the right hand side of the last equality are negative. In particular this implies that $\int_{\mathbb{R}} e^{B_\tau}\, w_j^2 = 0$ and hence $w_j \equiv 0\ \forall\, j \geq m+1$ and $\tilde{w} \equiv 0$. This already shows that the operator is an isomorphism.

It remains to show that its inverse is bounded uniformly as ρ tends to 0 (observe that the operator depends on ρ through the change of variable $dt = \rho\, \kappa_s\, ds$). To show this we argue by contradiction and assume that the result is not true. There would exists a sequence ρ_i tending to 0 and a sequence of functions w_i such that

$$\|w_i\|_{L^\infty(SNK)} = 1$$

and

$$\lim_{i\to+\infty} \|\mathcal{L}\, w_i\|_{L^\infty(SNK)} = 0$$

Using elliptic estimates and Ascoli-Arzela Theorem, we can extract some subsequence which converges to w a nontrivial bounded solution of $\mathcal{L}\, w = 0$ in $\mathbb{R} \times S^{m-1}$. In addition, the eigenvalue expansion of w in the spherical variables does not include any low eigenmode, namely

$$w = \sum_{j>m} w_j\, \varphi_j$$

At this point, we can apply the above integration by part to show that $w_j = 0$, hence $w \equiv 0$. A contradiction. This completes the proof of the result. $\square$

Finally, we consider the operator

$$\mathcal{L}_0 \;: \Pi_0 \, \mathcal{C}^{2,\alpha}_\rho(SNK) \longrightarrow \Pi_0 \, \mathcal{C}^{0,\alpha}_\rho(SNK)$$

It will be convenient to define the conjugate operator

$$L_0 := e^{\frac{B_\tau}{2}} \, \mathcal{L}_0 \, e^{-\frac{B_\tau}{2}}$$

which is explicitly given

$$L_0 = \partial_s^2 + \left(m - 1 - \frac{1}{4} \, b_\tau^2 - \frac{1}{2} \, \partial_s \, b_\tau \right)$$

using the expression of b_τ we obtain

$$L_0 = \partial_s^2 - \left(\frac{m-2}{2} \right)^2 + \frac{m(m+2)}{4} \, \varphi^2 + \frac{m(3m-2)}{4} \, \tau^{2m} \, \varphi^{2-2m}$$

There exist two independent solutions of $L_0 \, w = 0$ which are denoted by $w_0^\pm$. One of these solutions is periodic and corresponds to the Jacobi field associated to translation of the Delaunay along its axis

$$w_0^+ = \varphi^{\frac{m-4}{2}} \, \varphi_s$$

The second (independent) solution of the homogeneous problem $L_0 \, w = 0$ corresponds to the Jacobi field associated to the change of Delaunay parameter. We refer to [13] for further details. It is proven in [12] that w_0^- is not periodic for all values of $\tau \in (0, \tau_m)$ except when τ belongs to some finite set $T \subset (0, \tau_m)$. When $m = 2$ it is shown in [13] that w_0^- is not periodic for all values of τ and hence $T = \emptyset$ in this case. We shall from now on assume that w_0^- is not periodic, i.e. $\tau \in (0, \tau_m) - T$.

We define the function space

$$\mathcal{H}^{k,\alpha}_\rho(SNK) := \{ w \in \Pi_0 \, \mathcal{C}^{k,\alpha}_\rho(SNK) \quad : \quad w(s) = w(-s) \qquad \text{and} \qquad w(s + \pi_\tau) = w(s) \}$$

where we recall that π_τ is the least period of φ. This space is endowed with the norm of the space $\mathcal{C}^{k,\alpha}_\rho(SNK)$. Since w_0^+ is an odd function of s, it does not belong to the above space and hence the operator

$$L_0 : \mathcal{H}^{2,\alpha}_\rho(SNK) \longrightarrow \mathcal{H}^{0,\alpha}_\rho(SNK)$$

is an isomorphism. Obviously we have

$$||w||_{\mathcal{C}^{2,\alpha}_\rho(SNK)} \leq \, c \, ||L_0 \, w||_{\mathcal{C}^{0,\alpha}_\rho(SNK)}$$

for a constant $c > 0$ independent of ρ.

3.4 Existence of "Delaunay type" hypersurfaces

We now use the results of the previous sections to perturb $\bar{\Sigma}_\rho(K)$. We will assume from now on that, in Fermi coordinates, the coefficients of the metric do not depend on x_0. We further assume that the geodesic K is τ-nondegenerate and also that $\tau \in (0, \tau_m) - T$.

We must find $(w_0, \Phi, \tilde{w}) \in \mathcal{H}^{2,\alpha}_\rho(SNK) \oplus \mathcal{C}^{2,\alpha}(K, NK) \oplus \tilde{\Pi}\, \mathcal{C}^{2,\alpha}_\rho(SNK)$ such that

$$\rho\, m\, H(w, \Phi) = m \tag{3.22}$$

Let us denote by

$$f \;\; = \;\; \rho^2\, \Omega(\Theta, \Theta) + \frac{2}{3}\rho^2\, \varphi_s\, g(R(\Theta, \Theta_i)E_0, \Theta_i) + \mathcal{O}(\rho^3)$$

the inhomogeneous term appearing in (3.17). We can decompose f into three components, $f_0 + \hat{f} + \tilde{f}$ according to the projections defined in §3. We define a section of the normal bundle Ψ so that

$$\rho\, g(\Psi, \Theta) = \hat{f}$$

The equation (3.22) is then equivalent to the coupled system

$$\left\{ \begin{array}{rcl} L_0\, w_0 & = & f_0 + \rho\, L(w, \Phi) + Q(w, \Phi) \\ \mathfrak{J}\, \Phi & = & \Psi + \rho\, L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi) \\ \tilde{\mathcal{L}}_{SNK} \tilde{w} & = & \tilde{f} + \rho\, L(w, \Phi) + Q(w, \Phi) \end{array} \right. \tag{3.23}$$

Some observations are due. First, concerning the solution of

$$\mathfrak{J}\, \Phi = \Psi$$

one has the estimate

$$\|\Phi_0\|_{\mathcal{C}^1(K,NK)} + \rho^{1+\alpha}\, \|\Phi_0''\|_{\mathcal{C}^{0,\alpha}(K,NK)} \leq c\, \rho^2$$

even though $\|\Psi\|_{\mathcal{C}^{0,\alpha}(K,NK)} \leq c\, \rho$. The rational being that Ψ can be decomposed into $\Psi = \Psi_1 + \Psi_2$ where $\|\Psi_2\|_{\mathcal{C}^{0,\alpha}(K,NK)} \leq c\, \rho^2$ and $\Psi_1 = \rho\, h(\varphi)\, \varphi_s\, \chi$ where χ is some fixed section of NK and $h(\varphi) := \varphi^3\, \kappa_s^{-3}$. The result then follows from Remark 3.3.1.

Granted the properties of the operator involved, this system can be solved easily using a standard fixed point argument for contraction mapping in the space of $\Xi := (w_0, \Phi, \tilde{w})$ such that

$$\|w_0\|_{\mathcal{H}^{2,\alpha}_\rho(SNK)} + \|\Phi|\|_{\mathcal{C}^1(K,NK)} + \rho^{1+\alpha}\, \|\Phi''\|_{\mathcal{C}^{0,\alpha}(K,NK)} + \|\tilde{w}\|_{\tilde{\Pi}\mathcal{C}^{k,\alpha}_\rho(SNK)} \leq c\, \rho^2$$

provided the constant $c > 0$ is fixed large enough. There is no difficulty in applying this fixed point and we shall omit the details. Observe nevertheless that in the right hand side of the second equation, one has to use the special structure of the operator $\rho L + \frac{1}{\rho} Q$ as described in Remark 3.2.1. This completes the proof of the result.

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