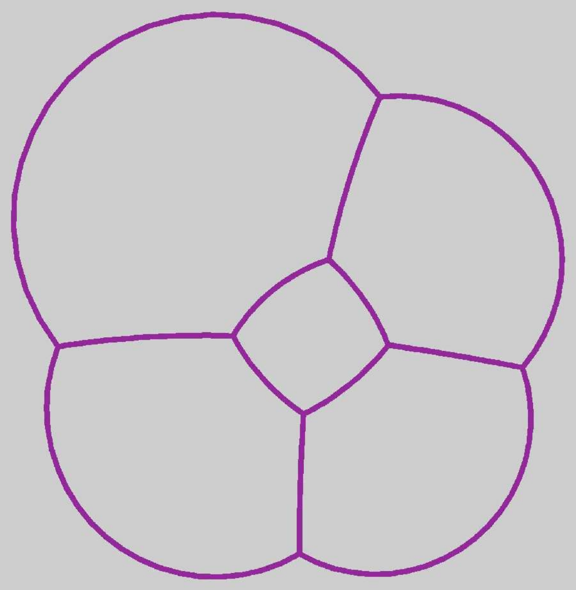
Université de Cergy-Pontoise
Laboratoire de Physique Théorique et Modélisation

Thesis

Structure and Evolution of Soap-Like Foams

by



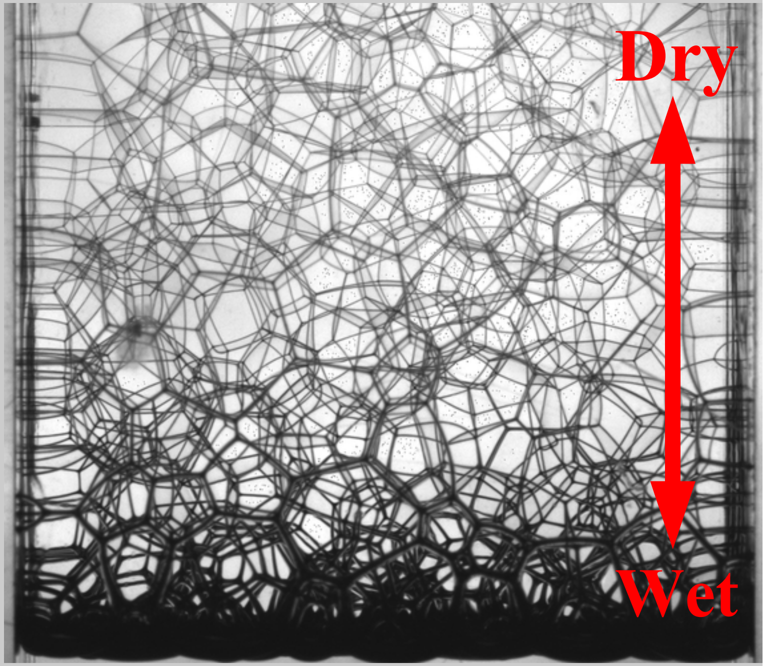
Marco Mancini
13 July 2005

Supervisors:
Christophe Oguey
François Dunlop

Structure and Evolution of Soap-Like Foams

- 2D and 3D Foams at Equilibrium
- Contact with a solid Boundary
- Coarsening of 2D Foams
- Star-Triangle equivalence and T2(3) continuity
- Non-Standard Foams and Decoration
- Star-Triangle equivalence for 3D spherical Foams
- Conclusions

Equilibrated Dry Foams



- Films are two-dimensional surfaces of constant mean curvature H_F:

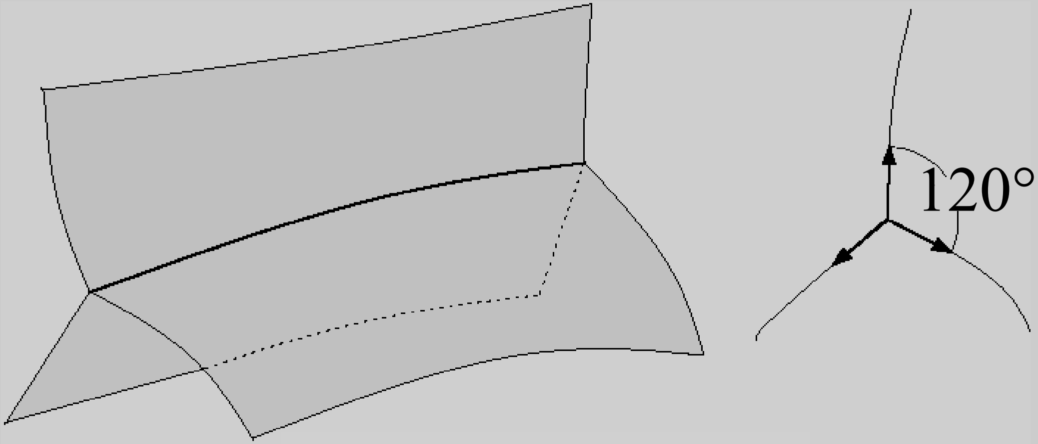
$$\Delta P = 2\gamma H_F$$

Laplace's law

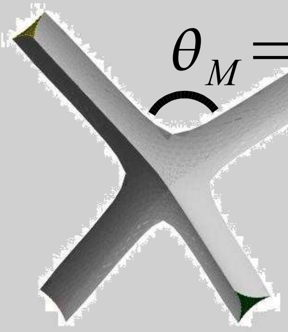
Pressure drop

Constant surface tension

- The films meet 3 by 3 at the borders. The angles between the films are 120° (Plateau's law)



- 4 borders meet in a symmetrical tetrahedral vertex



$\theta_M = \cos^{-1}(-1/3)$

Maraldi angle

Equilibrated Dry Foams

- Different films can have different values of surface tension

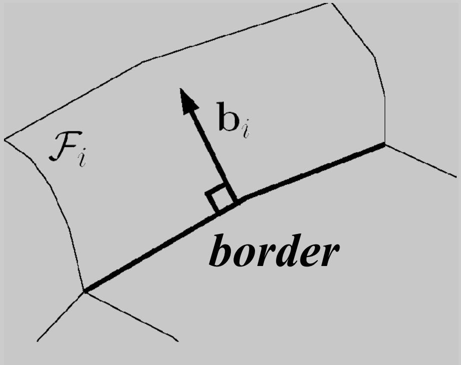
[Adler 1995,2000]

- Equilibrium equations can be rewritten as:

$$\boxed{\begin{aligned} \sum_{i=1}^{3} \gamma_i \boldsymbol{b_i} &= 0 \\ \sum_{i=1}^{3} \gamma_i H_i &= 0 \end{aligned}}$$

$\boldsymbol{b_i}$ is the co-normal

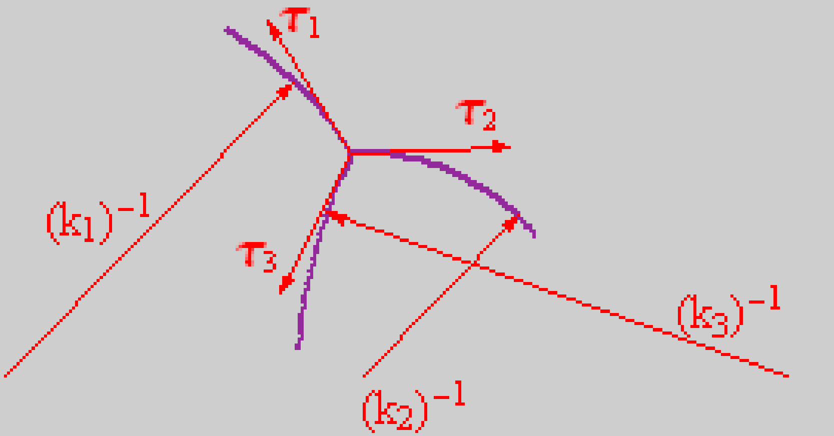
γ_i is the Surface tension



In two Dimensions

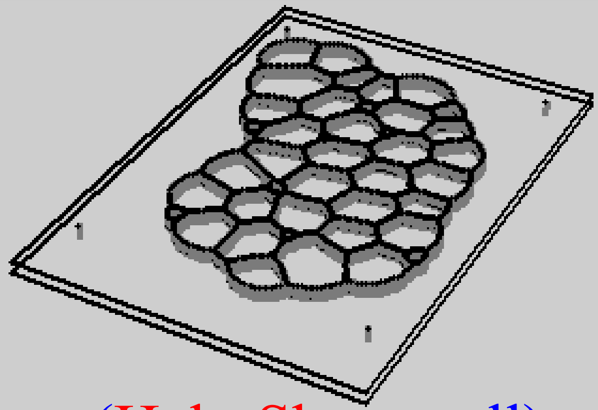
- Equilibrium equations are:

$$\boxed{\begin{aligned} \sum_{i=1}^{3} \gamma_i \boldsymbol{\tau_i} &= 0 \\ \sum_{i=1}^{3} \gamma_i k_i &= 0 \end{aligned}}$$



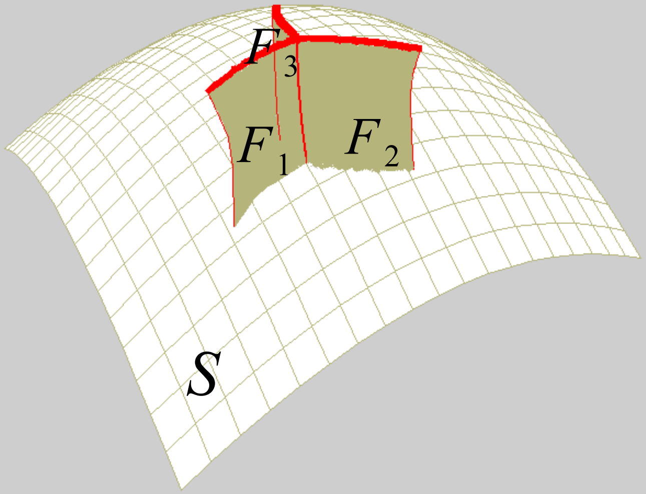
The films are circular arcs

A standard experiment



(Hele-Shaw cell)

Foams in Contact with a Solid Boundary



Dry Plateau border in contact with a rigid (curved) wall

- Equilibrium
- => Normal incidence of the films
- Clean surface

Foams in Contact with a Solid Boundary

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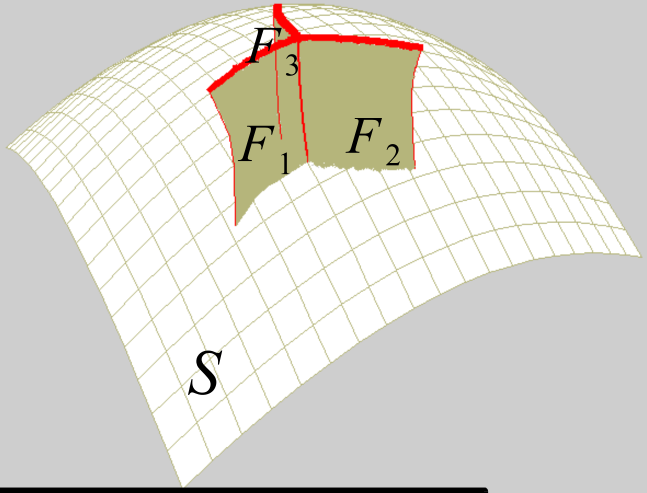
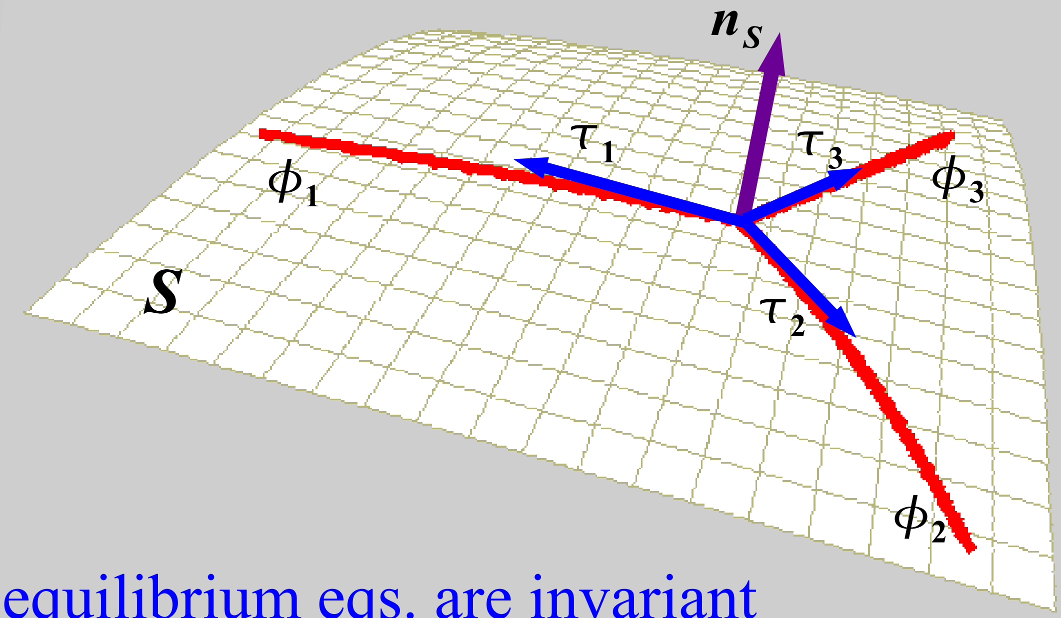


The 3D equilibrium implies that on the surface the equilibrium equations at a vertex are:

$$\boxed{\begin{aligned}&\sum_{i=1}^{3}\gamma_i \boldsymbol{\tau_i}=0\\&\sum_{i=1}^{3}\gamma_i k_g(\boldsymbol{\phi_i},S)=0\end{aligned}}$$



γ_i is surface tension

$k_g(\boldsymbol{\phi_i},S)$ is the geodesic curvature of $\boldsymbol{\phi_i}$ in S

The equilibrium eqs. are invariant under conformal transformations

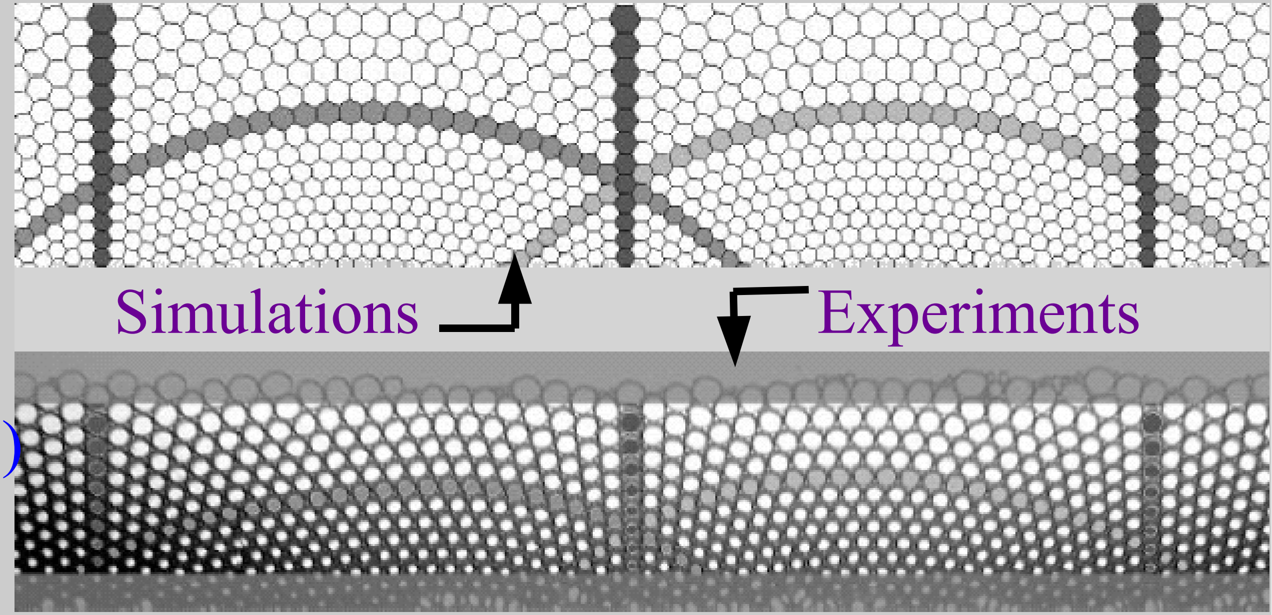
Deformed Hele-Shaw Cell and Conformal Foams

Exp.:

A monodisperse foam in special deformed chambers

Obs.:

The 2D pattern can be related to the hexagonal foam (honeycomb) by a conformal transformation



[Drenckhan *et al*. (2004)]

Observed pattern $X \xrightarrow{f} \tilde{X}$ Hexagonal Reference

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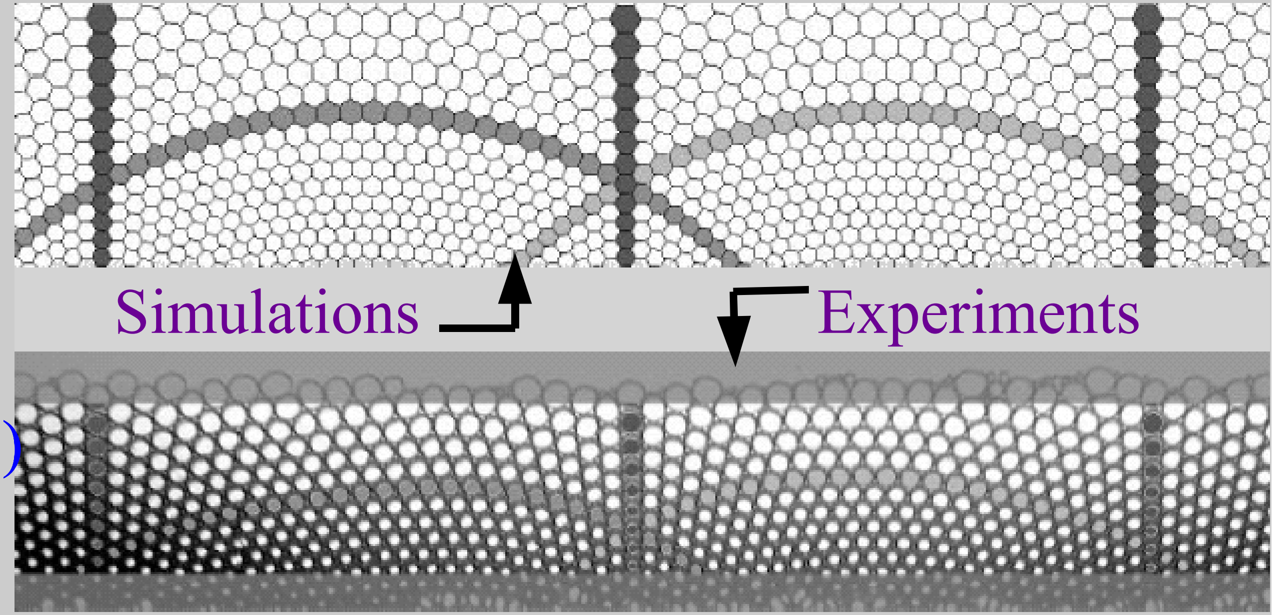
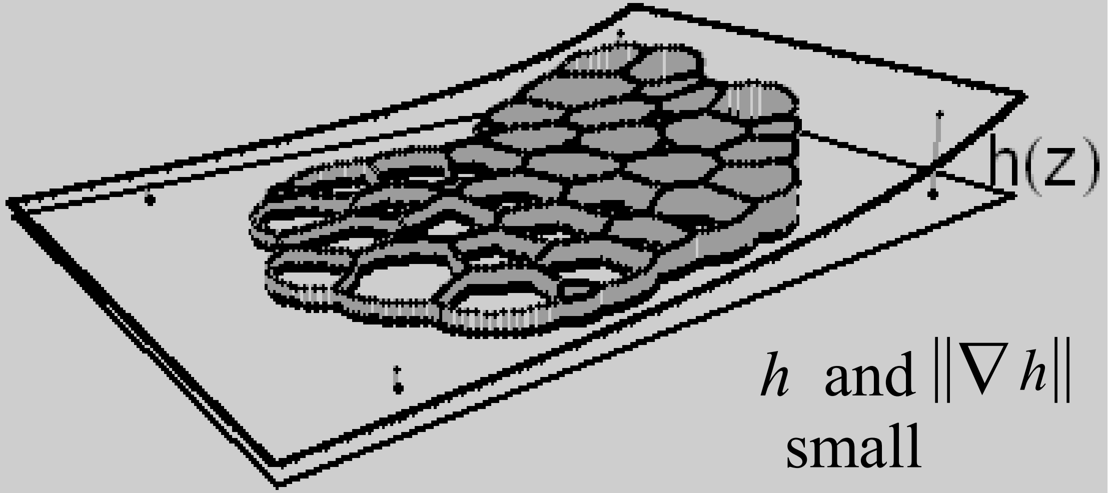


[Drenckhan *et al*. (2004)]

Observed pattern $X \xrightarrow{f} \tilde{X}$ Hexagonal Reference

The bubbles' volume is conserved

$$h \simeq h_0 \| f\,' \|^2$$



h and $\|\nabla h\|$ small

Deformed Hele-Shaw Cell and Conformal Foams

Can the shape of the film be related to the conformal map?

Yes. Imposing the 3D Laplace's law and considering
the film curvature in the third vertical direction k_v

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Laplace's law $\Longrightarrow$ $\Delta P = 2\gamma H_F = \gamma(\underbrace{k}_{\text{Geodesic curvature (horizontal)}} + \overbrace{k_v}^{\text{Vertical curvature}})$

Deformed Hele-Shaw Cell and Conformal Foams

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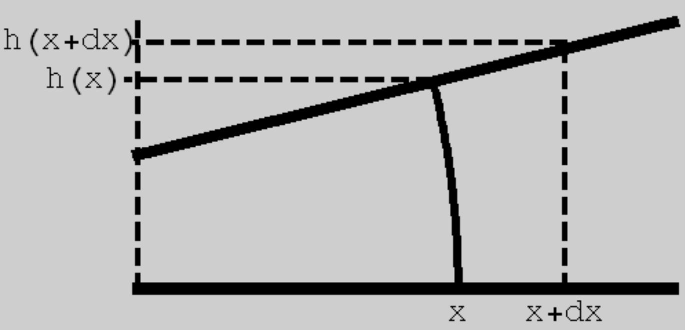
Yes. Imposing the 3D Laplace's law and considering the film curvature in the third vertical direction k_v

Vertical curvature

Laplace's law → $\Delta P = 2\gamma H_F = \gamma(k + k_v)$

Geodesic curvature (horizontal)

Normal to the film



h and $\|\nabla h\|$ small → $k_v \simeq \frac{-n\cdot\nabla h}{h}$

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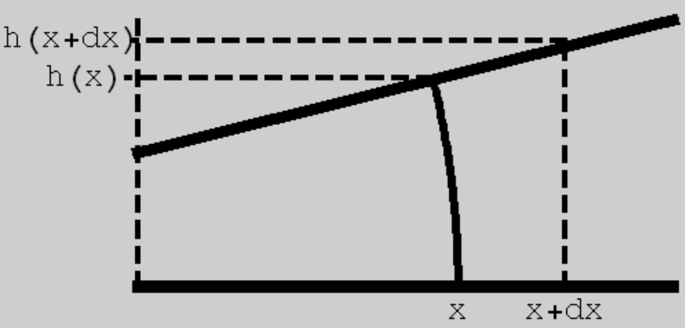


h and $\|\nabla h\|$ small →

$$k_v \simeq \frac{-n \cdot \nabla h}{h}$$

$\tilde{X} = f(X)$ →

The geodesic curvature transforms as:

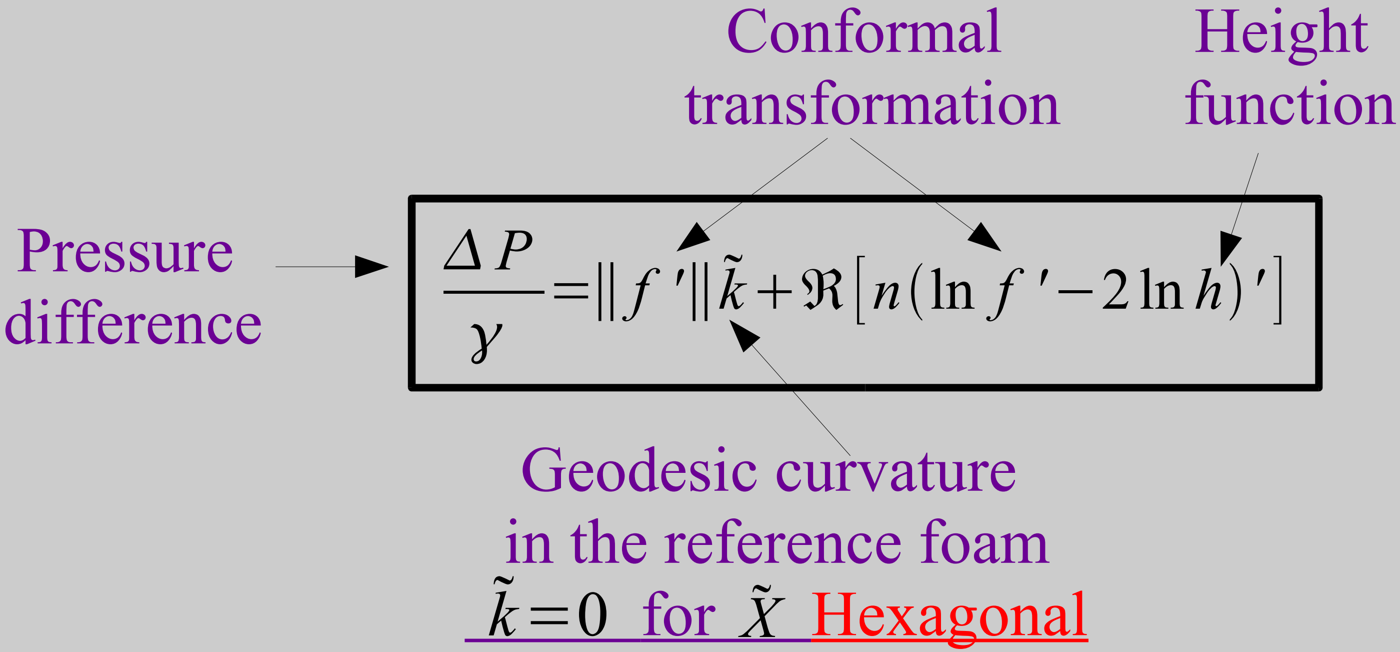
$$k = \|f'\|\tilde{k} + \Re\{n(\ln f')'\}$$

Combining these 3 equations:

Deformed Hele-Shaw Cell and Conformal Foams

$$\boxed{\frac{\Delta P}{\gamma} = \| f\,' \| \tilde{k} + \Re\left[n\left(\ln f\,' - 2 \ln h \right)' \right]}$$

Deformed Hele-Shaw Cell and Conformal Foams



Conformal transformation

Height function

Pressure difference

$$\boxed{\frac{\Delta P}{\gamma}=\|f'\|\tilde{k}+\Re\left[n(\ln f'-2\ln h)'\right]}$$

Geodesic curvature in the reference foam

$\tilde{k}=0$ for $\tilde{X}$ Hexagonal

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Different set-ups:

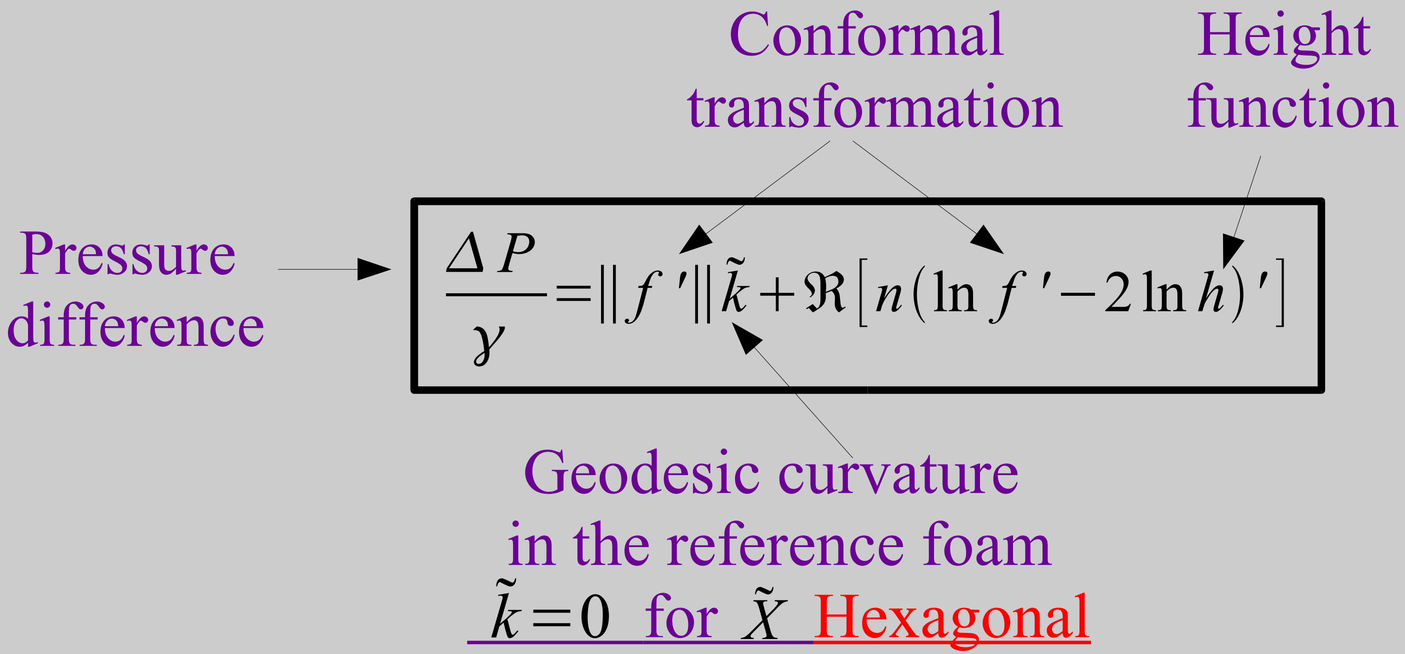
Constant Pressure

$$h \simeq h_0 \|f'\|$$

Constant Volume

$$h \simeq h_0 \|f'\|^2$$

Deformed Hele-Shaw Cell and Conformal Foams



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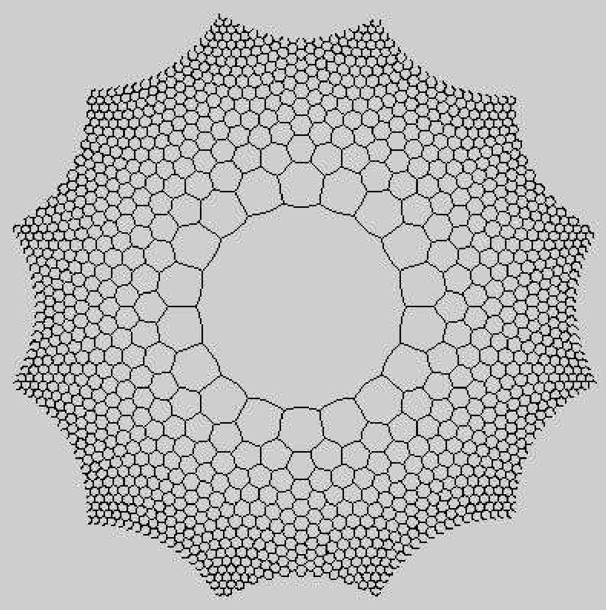
Different set-ups:

	Constant Pressure $h \simeq h_0 \|f'\|$	Constant Volume $h \simeq h_0 \|f'\|^2$
$f(z) = \frac{1}{a^*}\exp(a^* z)$	$h \propto \exp(a \cdot z)$	$h \propto \exp(2a \cdot z)$
$f(z) \propto \frac{z^\alpha}{\alpha}, \quad (\alpha = m/6, \quad m \in \mathbb{N})$	$h \propto \|z\|^{\alpha - 1}$	$h \propto \|z\|^{2\alpha - 2}$

Deformed Hele-Shaw Cell and Conformal Foams

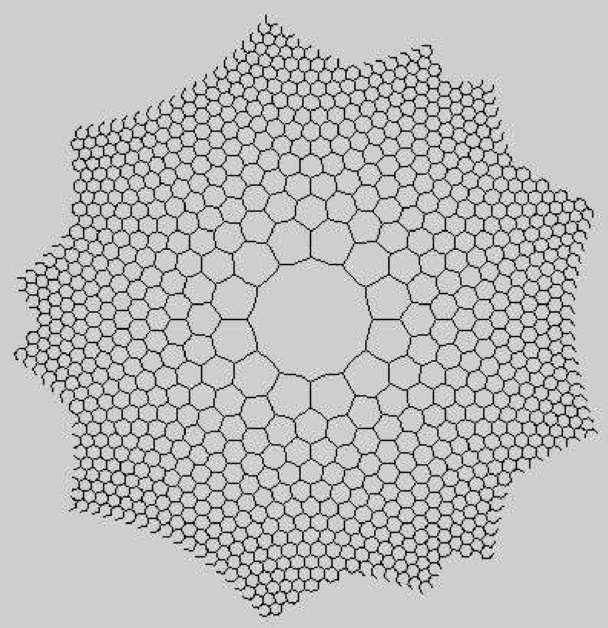
Example: Spherical vessel $h \propto \|z\|^2$

Constant Pressure
$m=18$



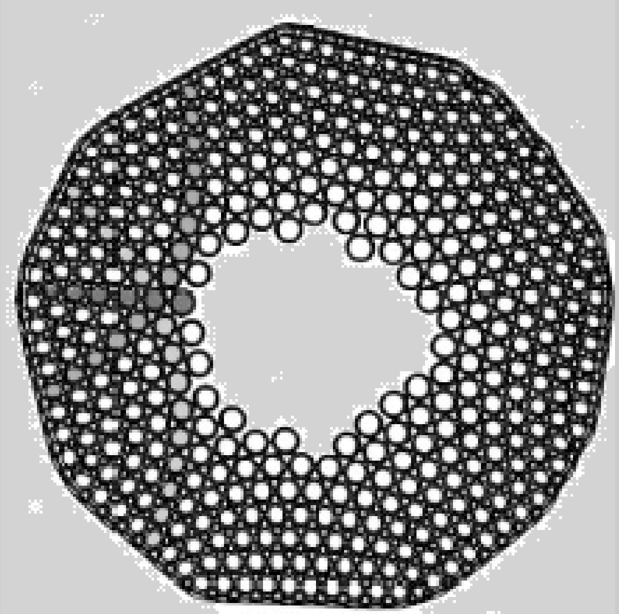
$f \propto z^3$

Constant Volume
$m=12$



$f \propto z^2$

Experimental
$m=9$



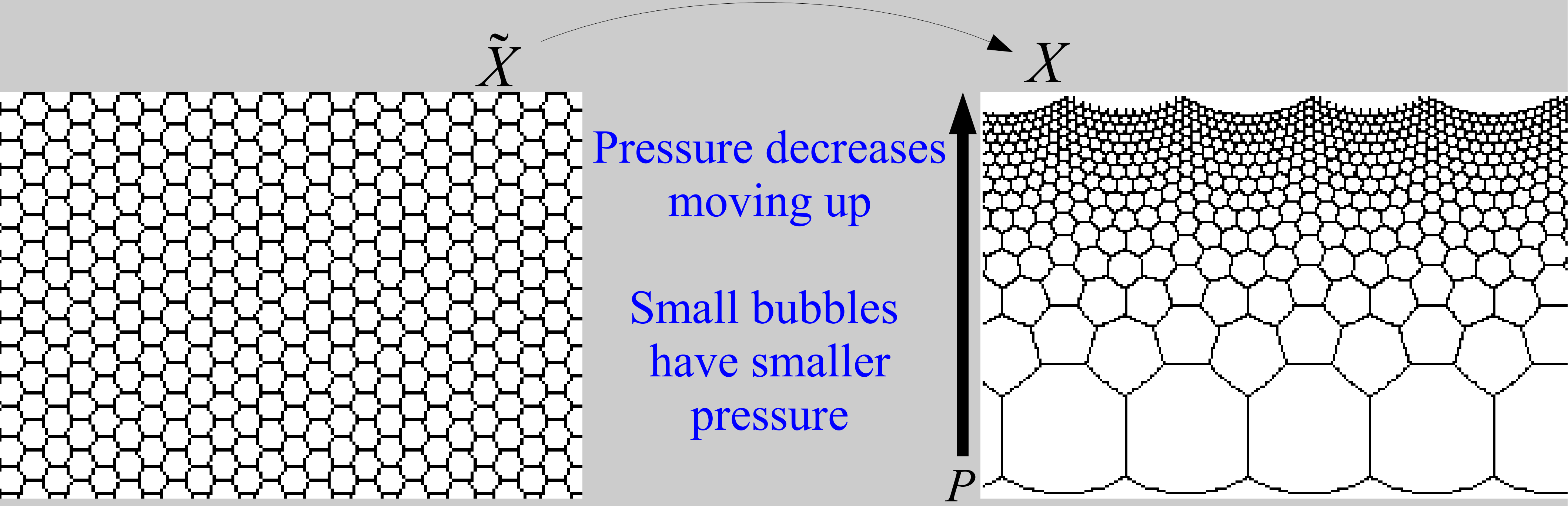
$f \propto z^{2/3}$ (?)

Deformed Hele-Shaw Cell and Conformal Foams

The volume constraint $h \simeq h_0 \| f\,' \|^2$ implies that $k_v = -2\,k$

Laplace's equation $\longrightarrow$ $$\frac{\Delta P}{\gamma} = k + k_v \simeq -k$$

Example: $F(\tilde{z}) = f^{-1}(\tilde{z}) = \frac{1}{a^*} \ln(a^* \tilde{z})$

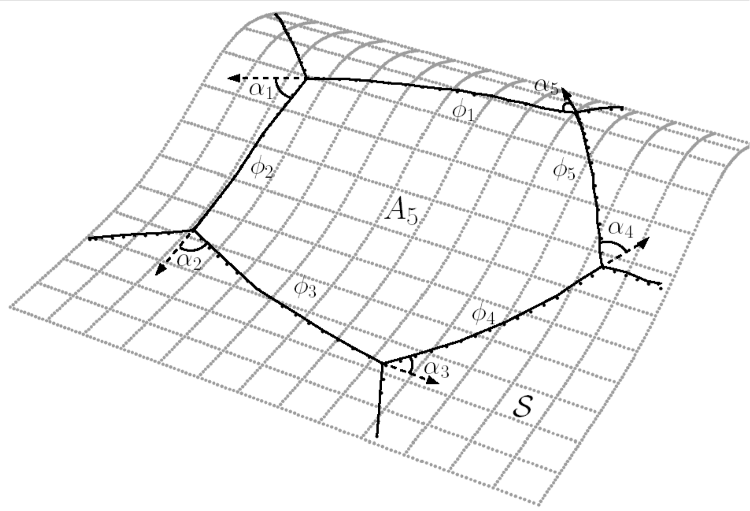


Pressure decreases moving up

Small bubbles have smaller pressure

Slow Evolution by Gas Diffusion of 2D Fomas

A 2D dry foam embedded in a 2-dim manifold S



- Equilibration time << time diffusion
- Incompressibility of the gas
- Validity of the Fick's law $$\frac{dA}{dt}=-\sigma\sum l_i \Delta P_i$$
- No liquid/solid friction force

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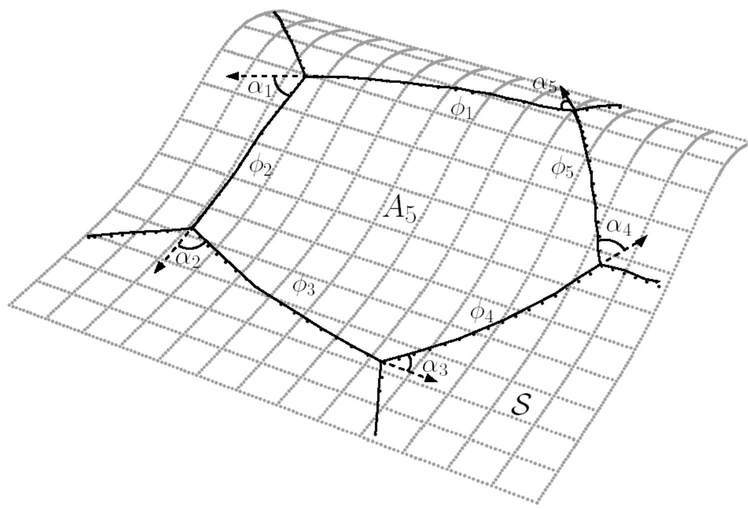

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[Avron, Levine (1992)]

$$\frac{\pi \sigma \gamma}{3}$$

Then the area of a n-sided bubble →

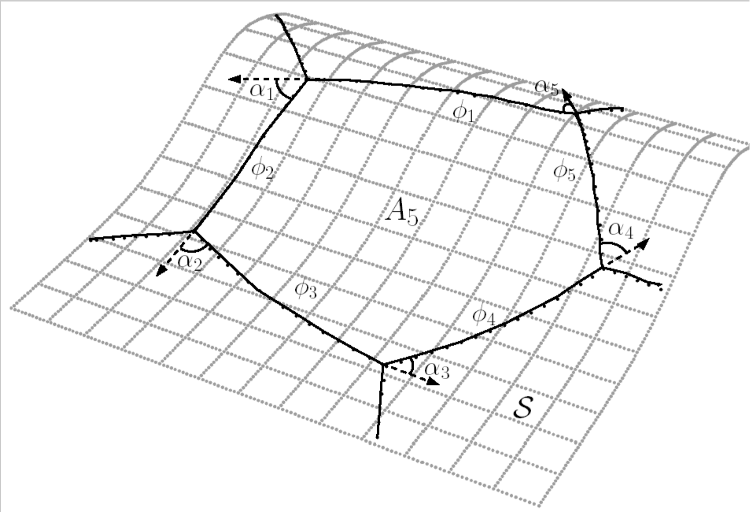
$$\frac{dA_n}{dt} = +\kappa(n-6) + \frac{3\kappa}{\pi} \int_{A_n} G\, dA$$

Topological charge q_n

Gaussian curvature of S

Slow Evolution by Gas Diffusion of 2D Fomas

A 2D dry foam embedded in a 2-dim manifold *S*



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Topological charge q_n

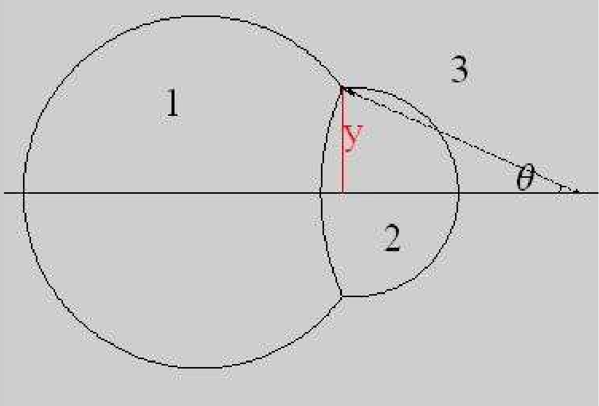
Gaussian curvature of *S*

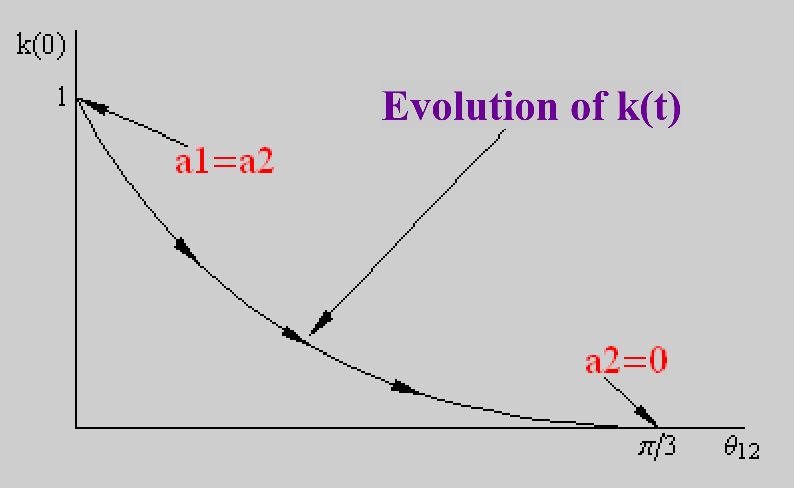
If *S* is flat (G=0) then → $$\frac{dA_n}{dt} = \kappa(n-6)$$ (von Neumann)

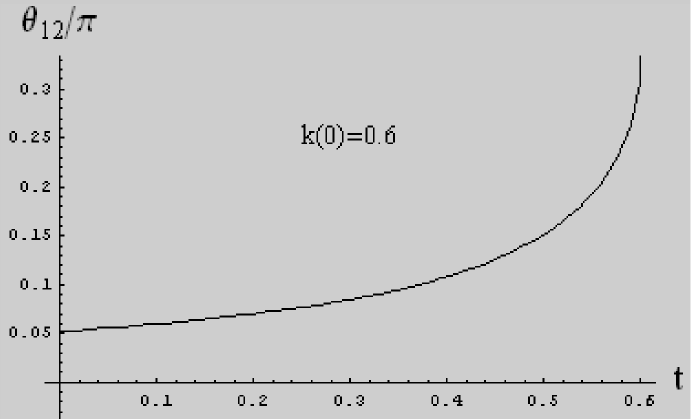
(depends only on the topology of the bubble)

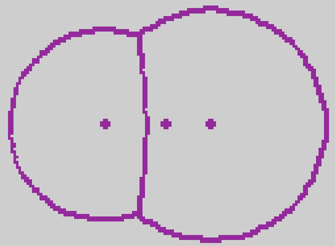
Example:2-Bubble cluster

von Neumann $\longrightarrow$ $\boldsymbol{a}(t)=\{t_1-t,t_2-t\}$ $\longrightarrow$ $k(t)\equiv\frac{a_2(t)}{a_1(t)}=\frac{g(\theta)}{g(-\theta)}$



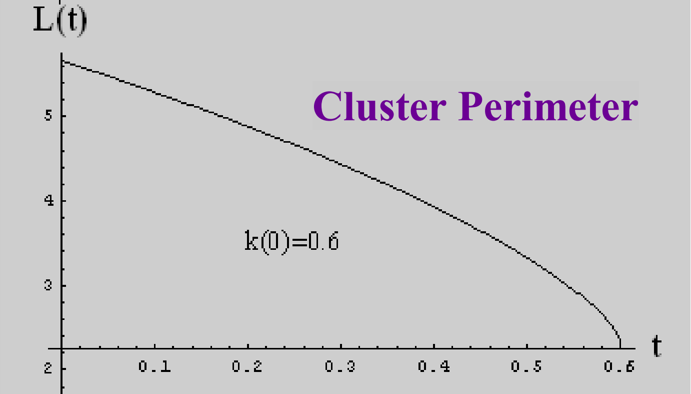






$$L(t)=\frac{2}{\gamma}\lambda\cdot\boldsymbol{a}(t)$$

$$\dot{L}(t)=\frac{1}{\gamma}\lambda\cdot\dot{\boldsymbol{a}}(t)$$



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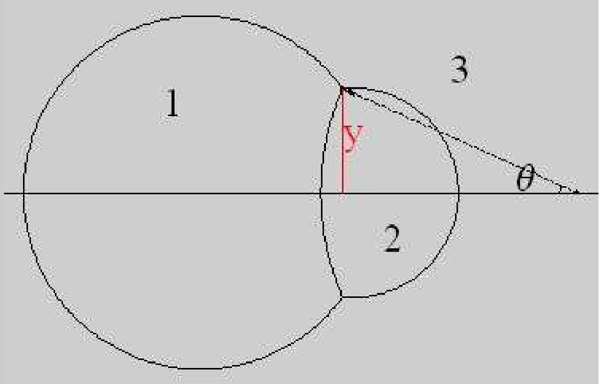


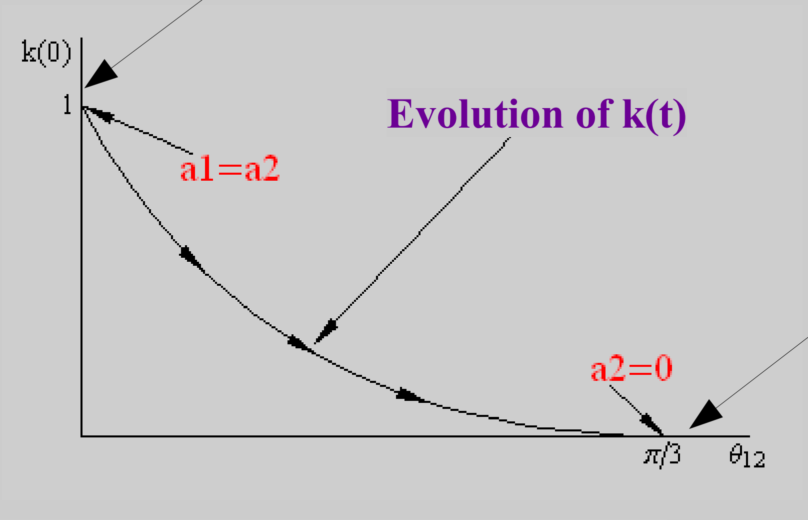
Limit case $a_1=a_2=\tilde{a}$

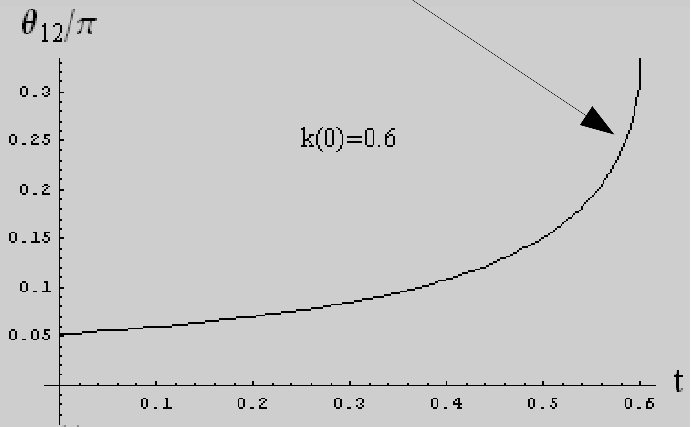
$$L(t)\ \propto\ \sqrt{\tilde{a}(t)}=\sqrt{\tilde{t}-t}$$

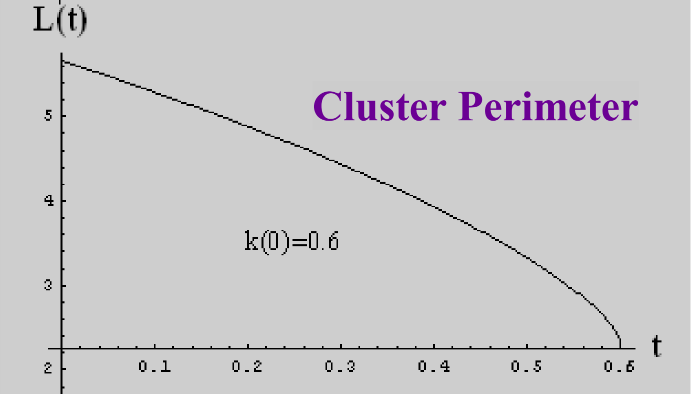
Close to extinction (T2, $t\rightarrow t_2$)

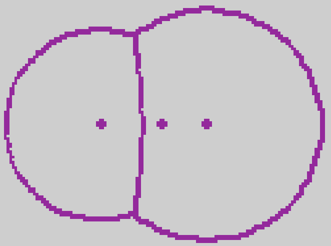
$$L(t)-2\sqrt{\pi\Delta a}\ \sim\ \sqrt{t_2-t}$$

$$\pi/3-\theta\ \sim\ \sqrt{t_2-t}\qquad \text{for}\quad k\neq 1$$









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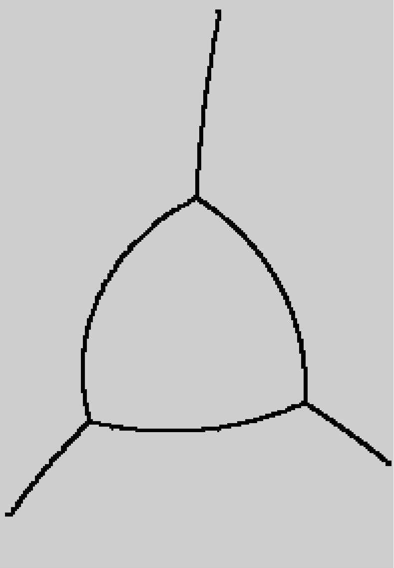
Star-Triangle Equivalence in 2D Foams $(\gamma = cst)$

For any three-sided bubble (triangle) in an equilibrated foam:



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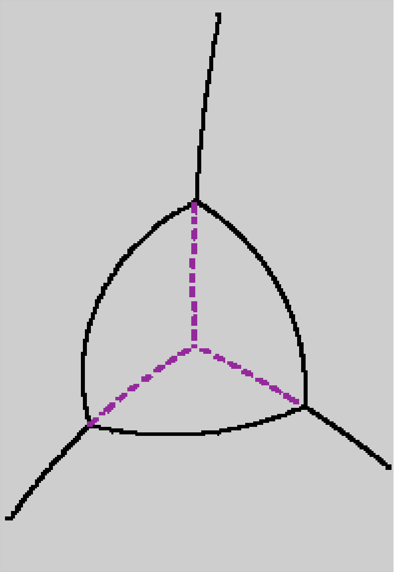
For any three-sided bubble (triangle) in an equilibrated foam:

Exists a virtual equilibrated vertex inside the bubble given by the prologation of the external films



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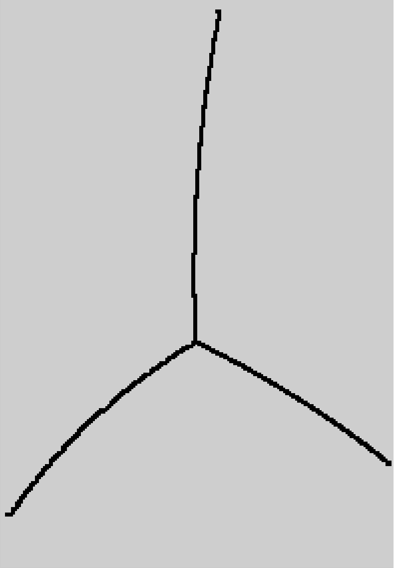
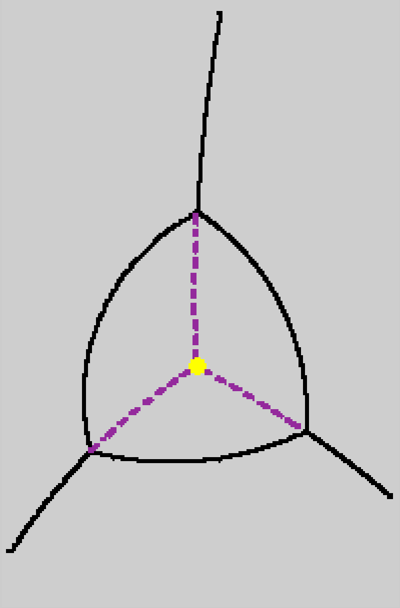
Replacing any triangle by the virtual vertex and its legs (and "viceversa")



The new foam is still equilibrated



Consequences:

- T2 continuity
- Possible Simplification ("Reduction") in computation of equilibrium patterns

Star-Triangle Equivalence in 2D Foams

Continuity at T2

During the gas diffusion
When a 3-sided bubble shrinks
T2(3)

→

The star vertex becomes
a real vertex and the foam
doesn't go out of equilibrium

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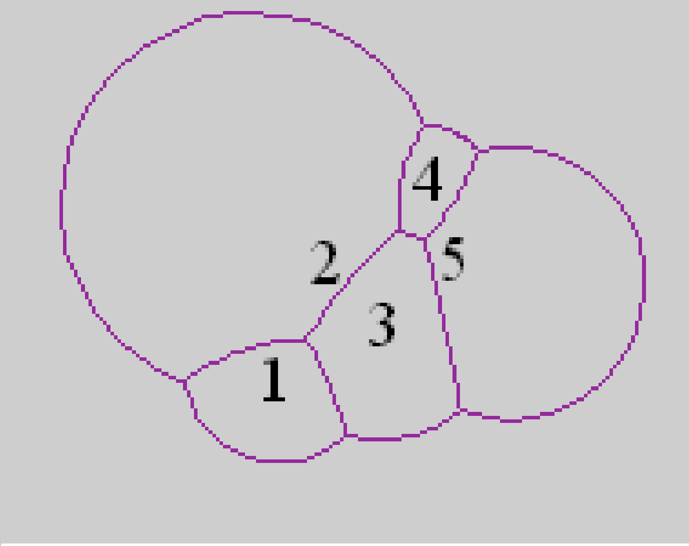

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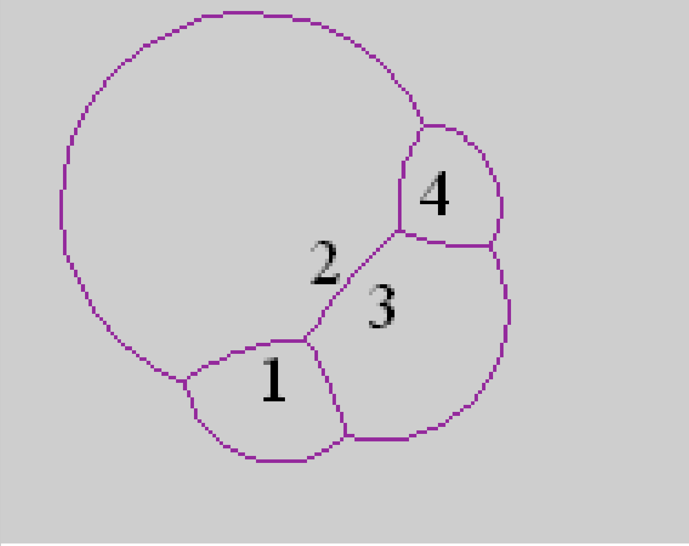

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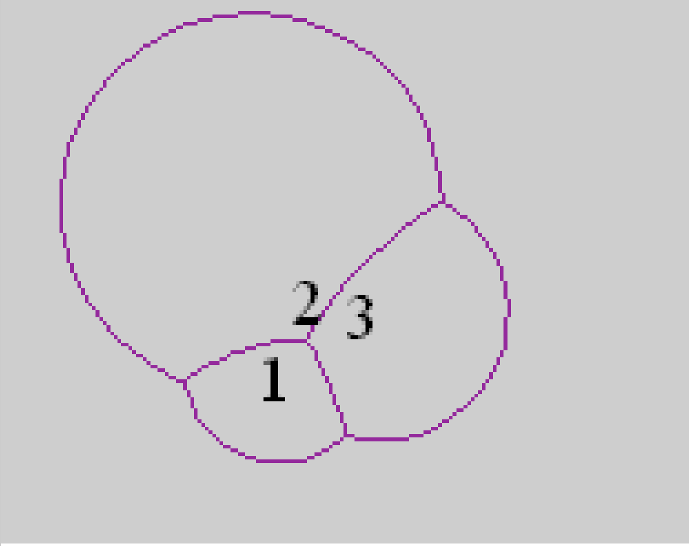

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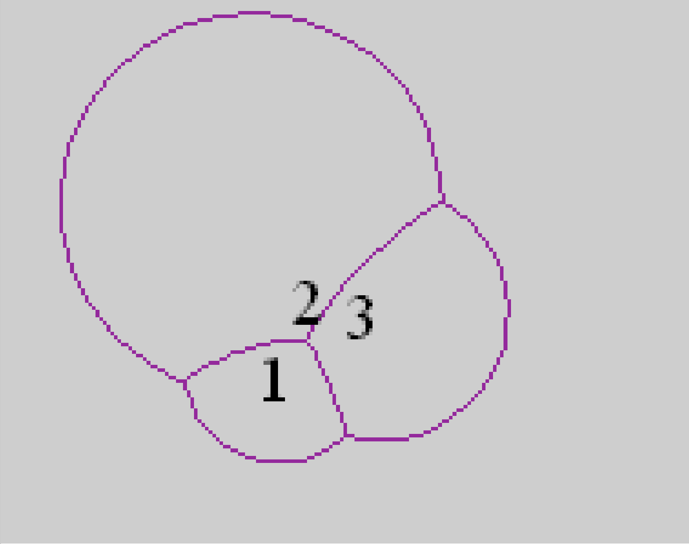

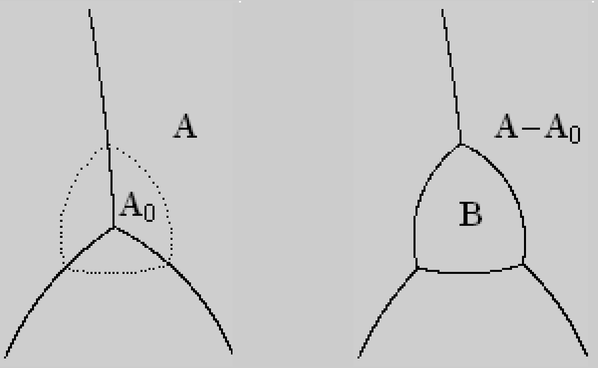

$$t_0 = B/(3\kappa)$$

$$\Delta\ Area = B/3 - A_0$$

Star-Triangle Equivalence in Non-Standard 2D Foams

SMVP [Moukarzel, 1997]

Sources: $\left(P_i, z_i, a_i\right)$

cells →
$$\Omega_i = \left\{ x \in \prod_z : \frac{d(x, P_i)^2 + z_i^2}{a_i} < \frac{d(x, P_j)^2 + z_j^2}{a_j}, \quad \forall\, j \neq i \right\}$$

edges →
$$\Gamma_{ij} = \left\{ x \in \prod_z : \frac{d(x, P_i)^2 + z_i^2}{a_i} = \frac{d(x, P_j)^2 + z_j^2}{a_j} < \frac{d(x, P_k)^2 + z_k^2}{a_k}, \quad \forall\, k \neq i, j \right\}$$

Source projected on the plane $z = 0$

Intensity

Star-Triangle Equivalence in Non-Standard 2D Foams

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Circular partition $\mathcal{F}$
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⟷

- It is aligned
- It admits an oriented reciprocal figure $\mathcal{F}^*$

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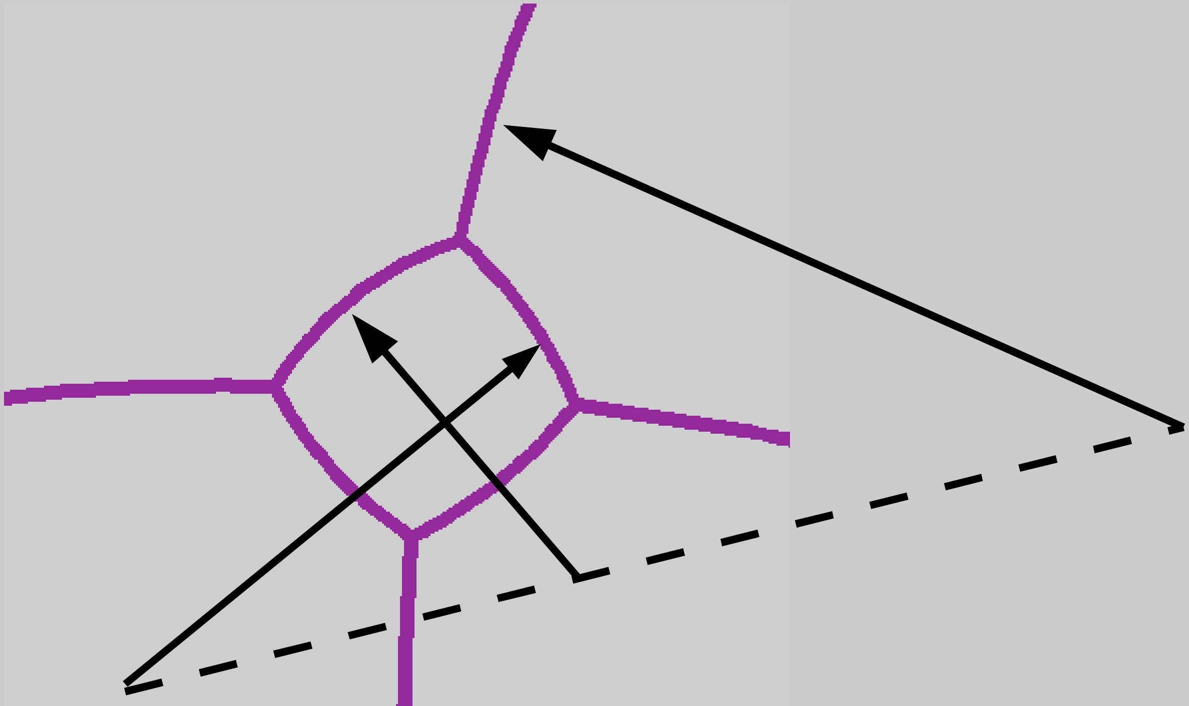

Aligned

The centres of the edges meeting at a vertex are on a line

Star-Triangle Equivalence in Non-Standard 2D Foams

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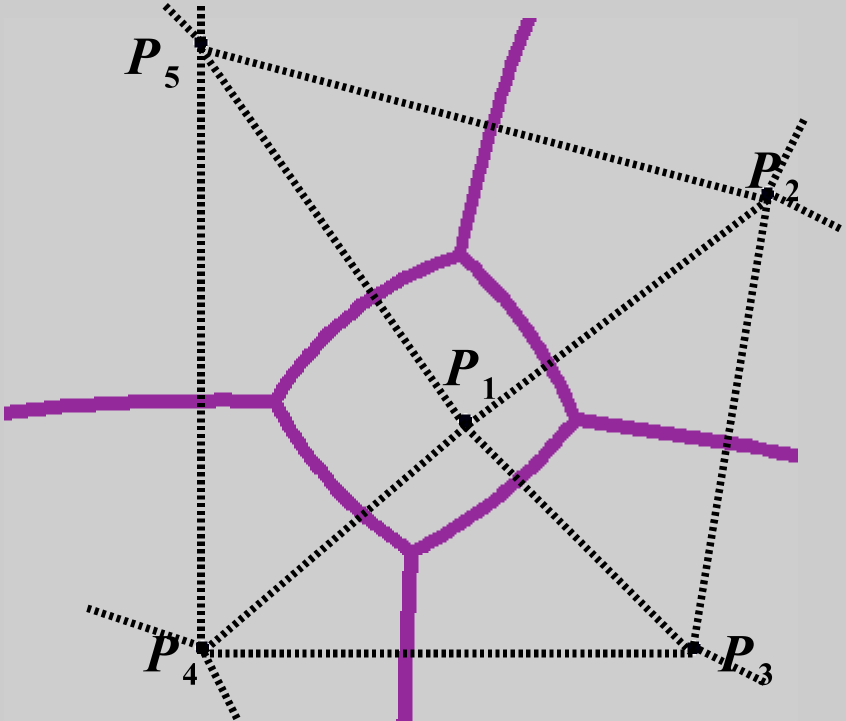

Oriented reciprocal (or dual) figure

- $\mathcal{F}^*$ is a triangulation
- A Source P_i for any cell
- $\overline{P_i P_j} \perp$ edge
- $\left(P_i, P_j, C_{ij}\right)$ ordered

Star-Triangle Equivalence in Non-Standard 2D Foams

Moukarzel's Theorem

A circular partition of the plane, with 3-connectivity,
represents an equilibrated 2D foam $\mathcal{F}$ (non-standard)

If and only if there is an oriented dual figure $\mathcal{F}^*$

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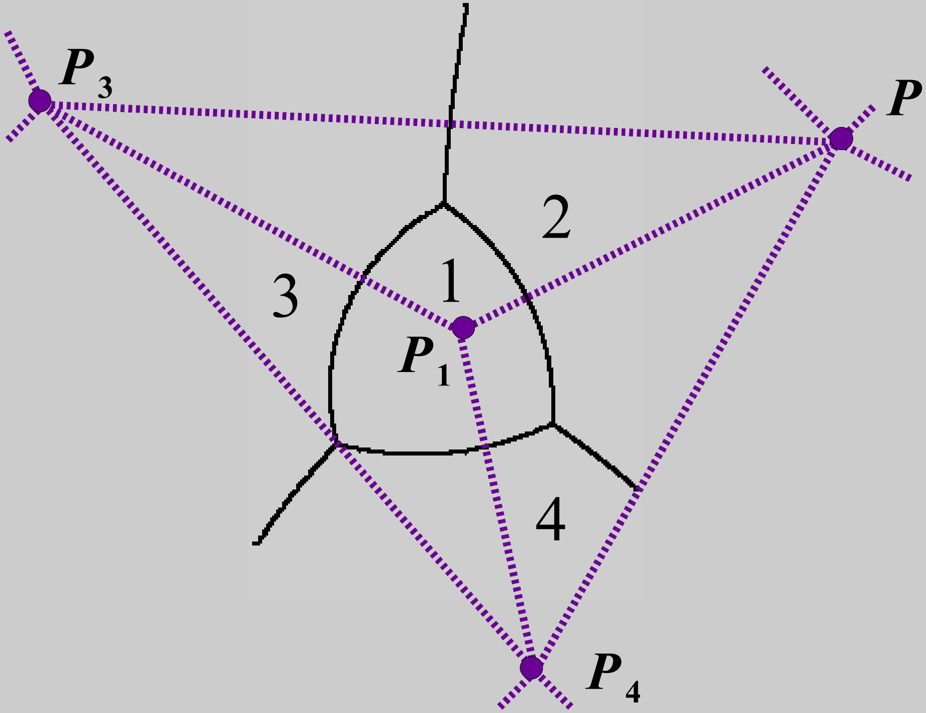
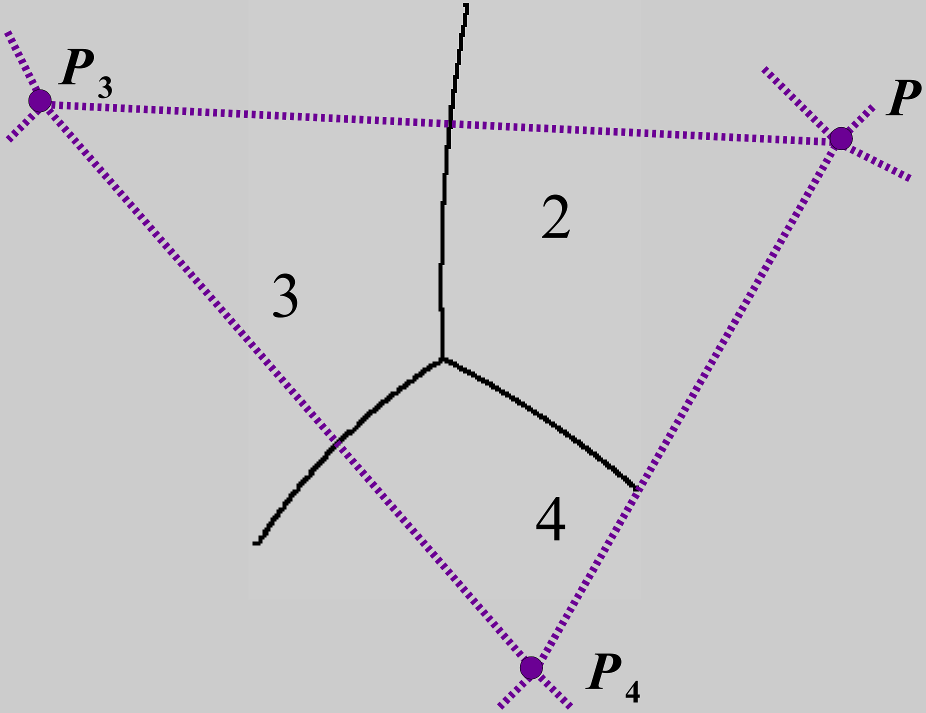
Then, Star-Triangle Equivalence is simply proved:



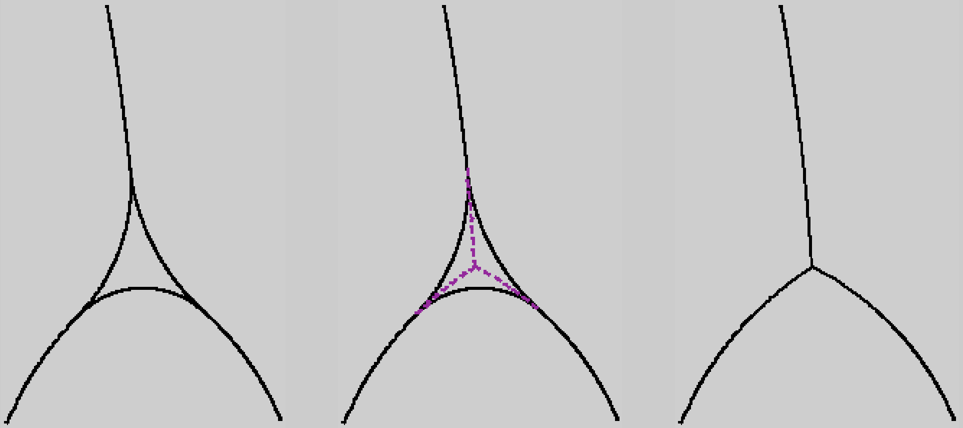


Star-Triangle Equivalence in Non-Standard 2D Foams

Consequences

Consequences 1:

The Decoration Theorem
[Weaire, 1992]



Little liquid at the vertices ↔ Ideal dry

Star-Triangle Equivalence in Non-Standard 2D Foams

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Consequences 1:

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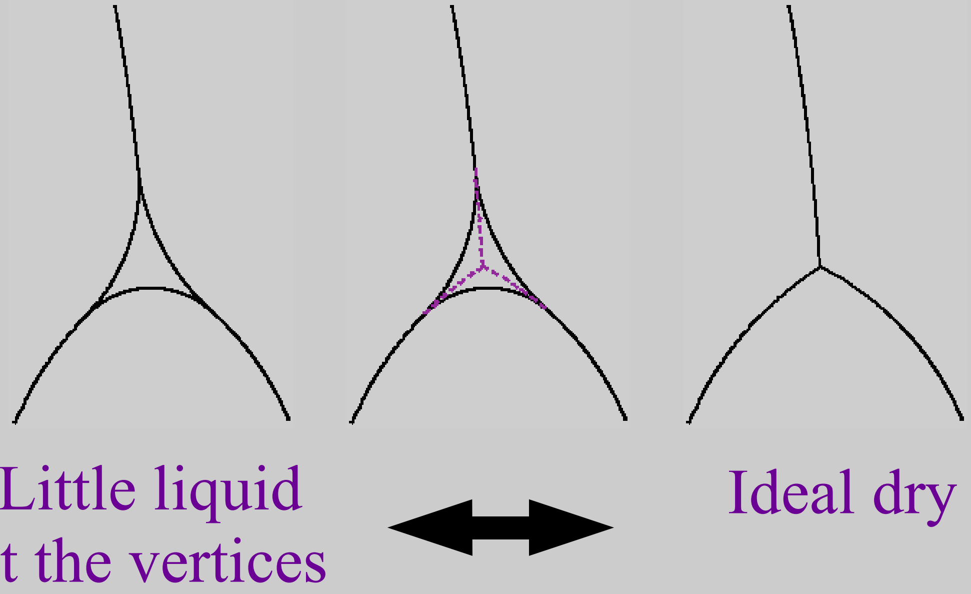

Little liquid
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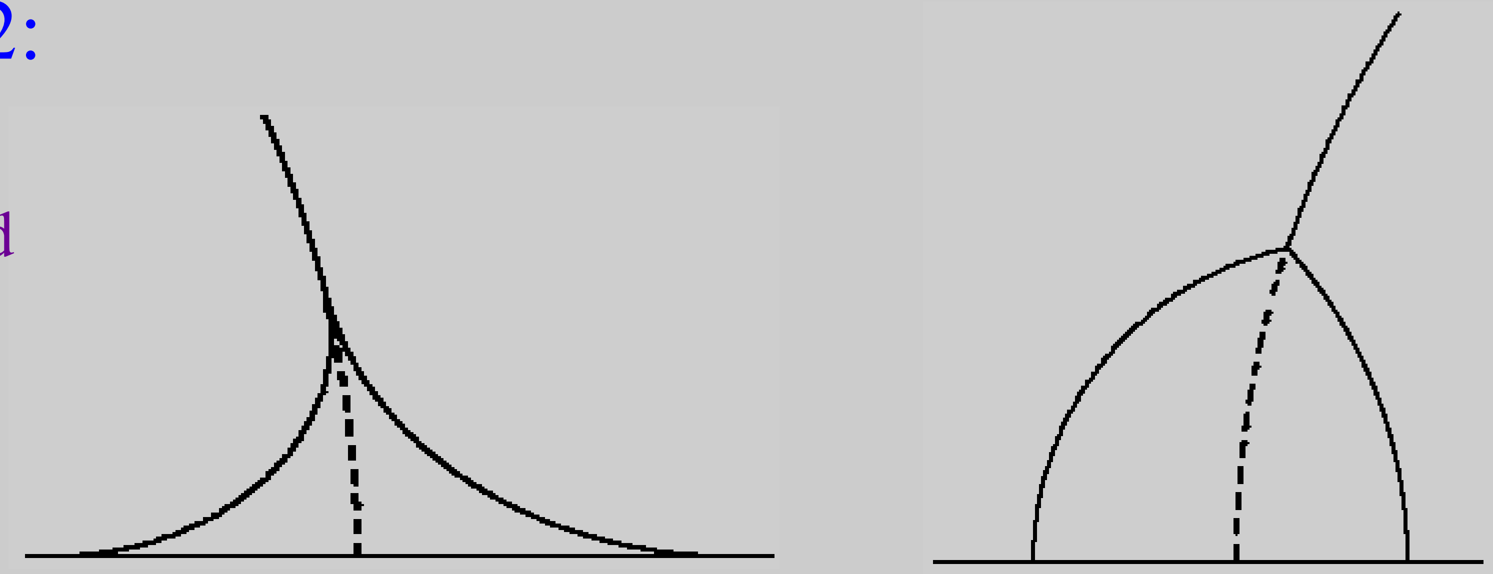
Consequences 2:

Star-Triangle and
decoration at
a flat boundary



Star-Triangle Equivalence for Spherical Foams

Spherical Foams are a subcase of 3D (dry) Foams

- Equilibrium Laws (Plateau+Laplace)
- The Films are spherical caps

Star-Triangle Equivalence → The films of vanishing tetrahedral bubbles are approximately spherical cup [Doornum, Hilgenfeldt , 2003] then T2 is a continuous process

Proof: (Moebius Invariance of spherical foams)

- Existence of a conjugate vertex
- Inversion map toward a symmetrical figure
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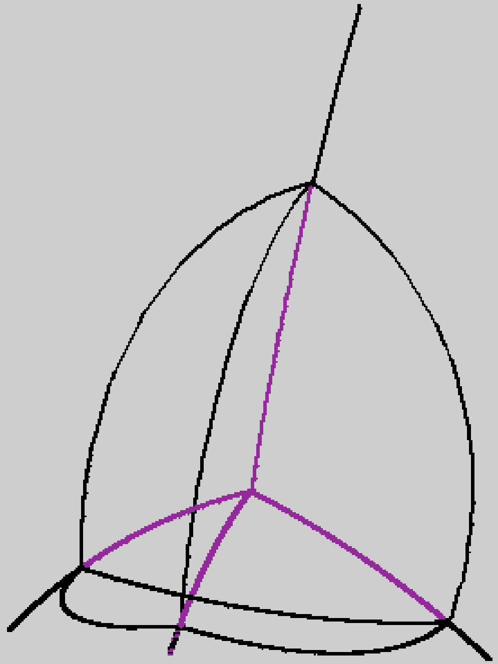


Conclusions

- We have derived the equilibrium equation for 2D Foam at the contact with solid boundaries.
- Invariance under conformal transformations.
- Found an equation linking the conformal transformation to the profile of a deformed Hele-Shaw cell.
- Pressure predicted under constant volume constraints differs from “pure 2D” physics.

- Star-Triangle equivalence in standard and non-standard 2D foams
- New proof and extension of the Decoration theorem
- Similar results along flat boundaries (in 2D)
- T2(3) continuity
- Exact solution in the case of 2-bubbles (2D and 3D)
- Star-Triangle equivalence for 3D spherical bubbles, continuity in tetrahedral bubbles disappearing

Perspective and Projects

- Is the star-triangle equivalence verified by 2D foams embedded in two-dimensional manifold?

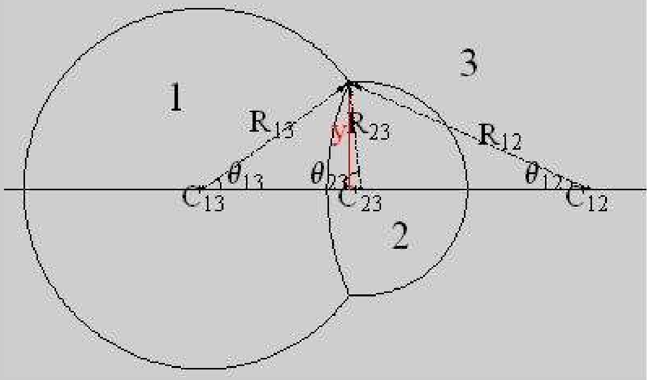
- Is there a generalization of the star-triangle equivalence for 3D foams?

- Develop a program which, given the pressures and the topology of a bubble cluster, would construct the exact equilibrated cluster (example: Flower problem).

- Given a random cluster of N bubbles, how many different star-triangle reductions can one do?

Example:2-Bubble cluster. Minimization and Diffusion

- Hp: circular films, 3-connectivity, mono-contact
- Indipendent Variables: **y**= $\{R_{13}, R_{23}, R_{21}, y\}$
- Enthalpy $G[C[\boldsymbol{y}],\lambda;\boldsymbol{a}]=\gamma L[\boldsymbol{y}]+\lambda\cdot(\boldsymbol{a}-s[\boldsymbol{y}])$



Surface tension

Cell areas vector

Lagrange's multipliers

Constant areas vector

Perimeter

Equilibrium

Minimization:

Minimal parimeter → $\partial_{y_i} G=0$

Fix the areas → $\partial_{\lambda_i} G=0$

- $\theta_{13}=\frac{\pi}{3}-\theta_{12};\ \ \theta_{23}=\frac{\pi}{3}+\theta_{12}$ ← Plateau
- $\frac{1}{R_{21}}+\frac{1}{R_{13}}-\frac{1}{R_{23}}=0$ ← Laplace
- $E=\gamma L=2\lambda\cdot\boldsymbol{a}$ ← Energy

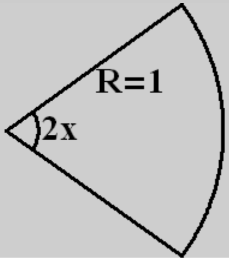
$$\boldsymbol{a}=y^2\{g(-\theta_{12}),g(\theta_{12})\}$$

$$g(\theta)=\frac{f\left(\frac{2\pi}{3}-\theta\right)}{\sin^2\left(\frac{2\pi}{3}-\theta\right)}+\frac{f(\theta)}{\sin^2(\theta)}$$

$\frac{a_1}{a_2}$ Is scale invariant: it doesn't depend on y

$$f(x)=\frac{1}{2}(2x-\sin(2x))$$

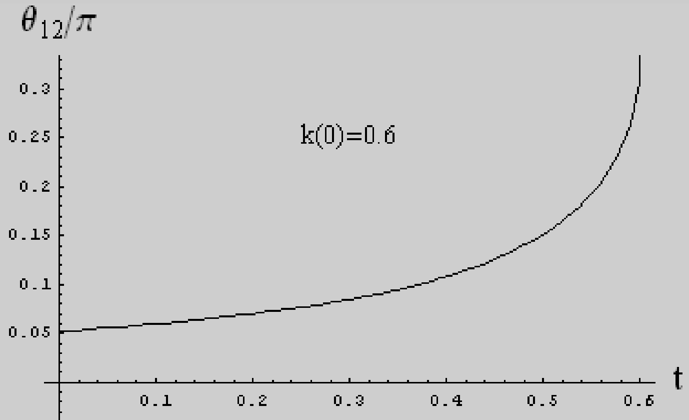
Area of a circular sector of radius = 1

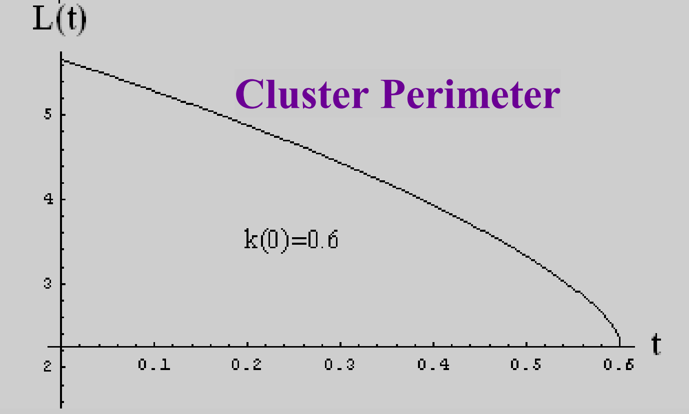


Example:2-Bubble cluster

Rescaled
von Neumann

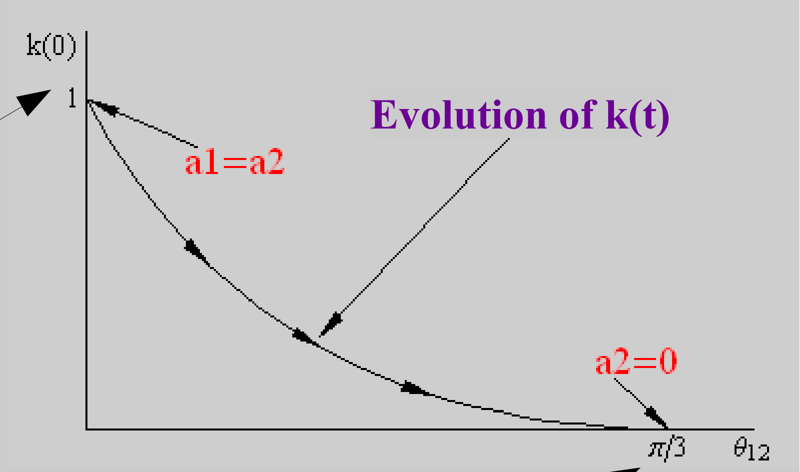
$$\xrightarrow{\begin{pmatrix} a\to a/4\kappa \\ t_i=a_i(0)\end{pmatrix}} \boldsymbol{a}(t)=\{a_1(t),a_2(t)\}=\{t_1-t,t_2-t\} \longrightarrow k(t)\equiv\frac{a_2(t)}{a_1(t)}=\frac{t_2-t}{t_1-t}=\frac{g(\theta_{12})}{g(-\theta_{12})}$$





Limit case $a_1=a_2=\tilde{a}$

$$L(t)\propto\sqrt{\tilde{a}(t)}=\sqrt{\tilde{t}-t}$$
$$R(t)\propto\sqrt{\tilde{a}(t)}=\sqrt{\tilde{t}-t}$$



Limit case $t\to t_2$ (T2)

$$L(t)-2\sqrt{\pi\Delta a}\sim\sqrt{t_2-t}$$
$$\pi/3-\theta_{12}\sim\sqrt{t_2-t}$$

$$L(t)=\frac{2}{\gamma}\lambda\cdot\boldsymbol{a}(t)$$
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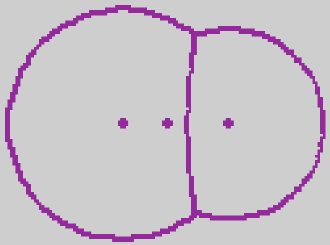

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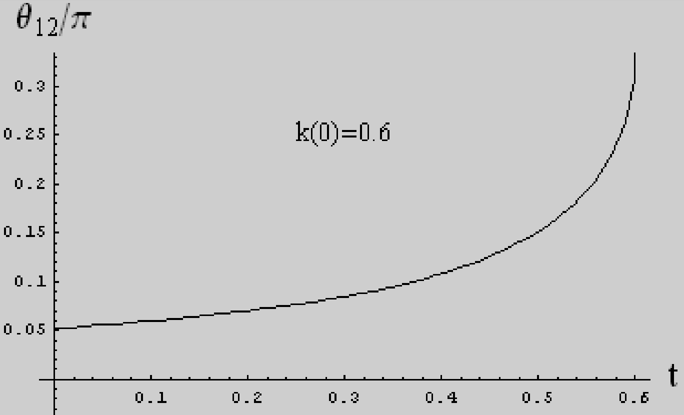

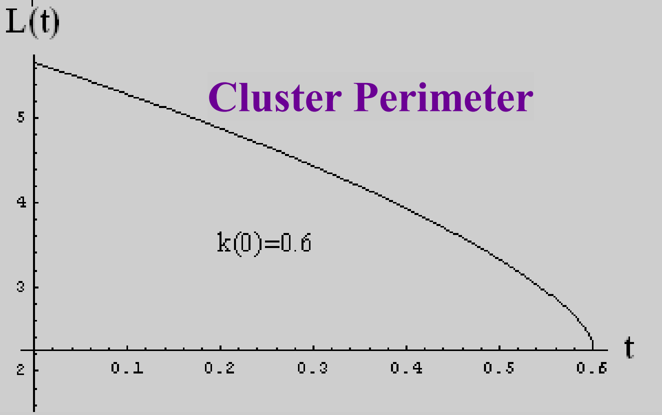

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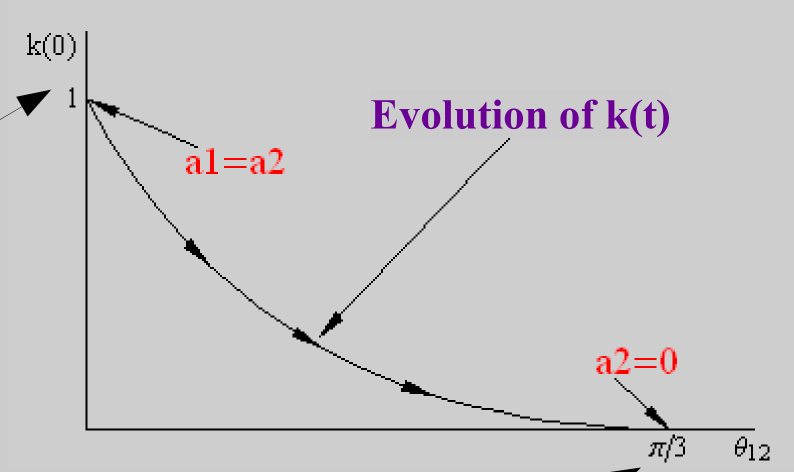
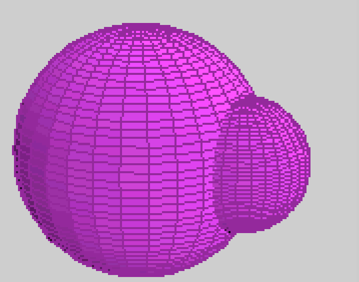


3D 2-Bubble Cluster

$$E(t)=\gamma A(t)=\frac{3}{2}\lambda\cdot\boldsymbol{V}$$
$$\dot{E}(t)=\lambda\cdot\dot{\boldsymbol{V}}$$



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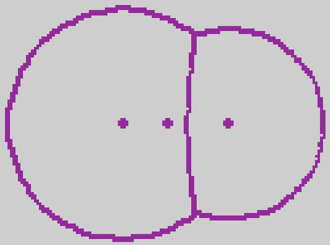

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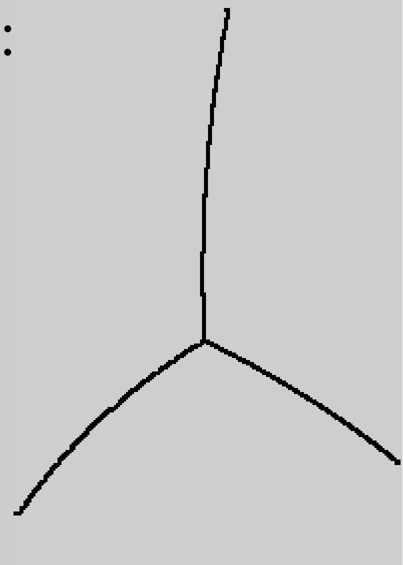
Star-Triangle Equivalence in 2D Foams $(\gamma = cst)$

For any three-sided bubble (triangle) inside an equilibrated foam:



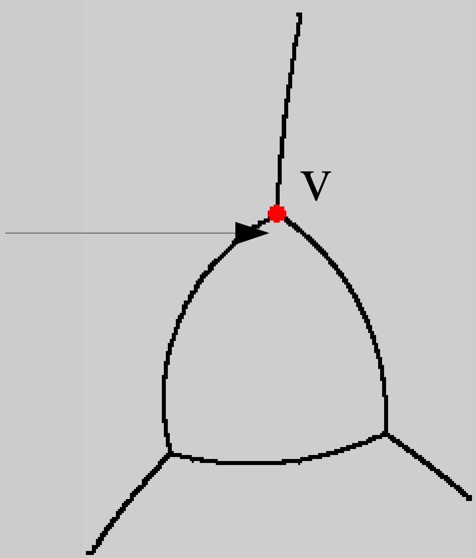
Exists a virtual equilibrated vertex inside the bubble gived by the prologations of the external films joining the bubble

Replacing any triangle by the virtual vertex and its legs
(and "viceversa")

The new foam is still equilibrated

Proof:

Taking
a vertex v of the bubble



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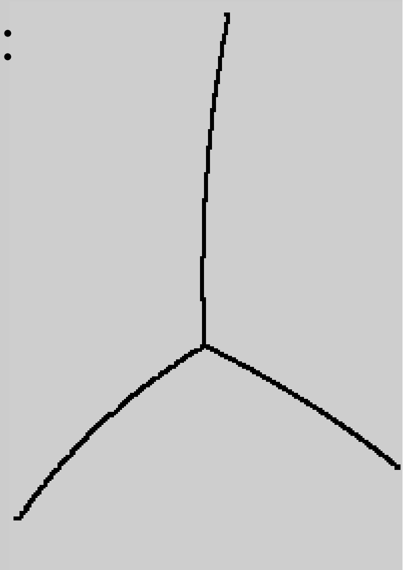

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⬇

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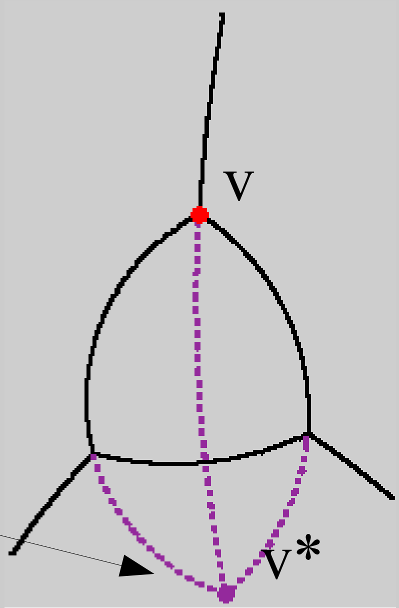
Proof:

Taking
a vertex v of the bubble

It exists an associate
conjugate v*
(equilibrated) vertex



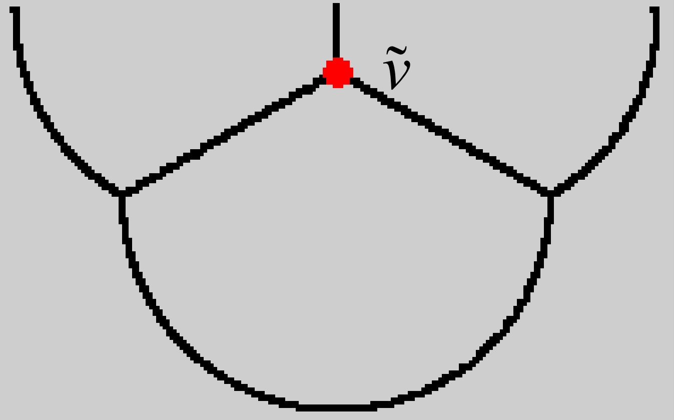
Applying an Inversion
Transformation
to the foam centred in v*

$$z \to \tilde{z} = 1/\left(z - v^*\right)$$

We obtain:

Star-Triangle Equivalence in 2D Foams

..Proof



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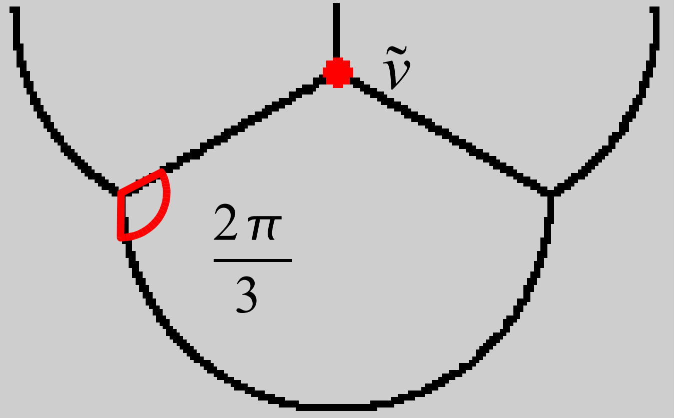

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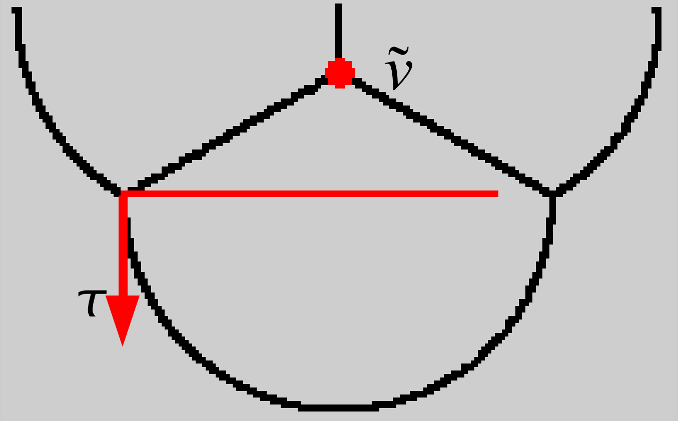

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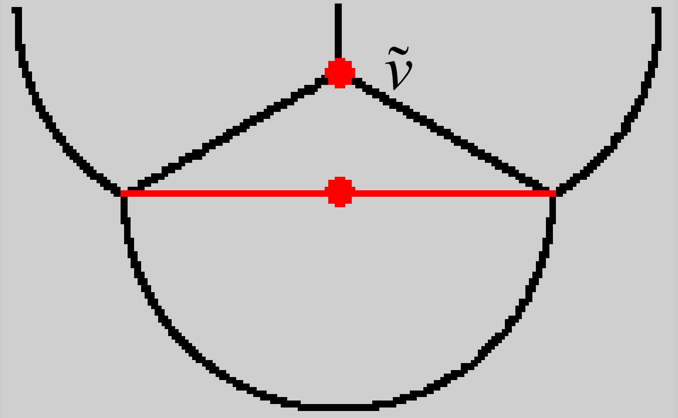

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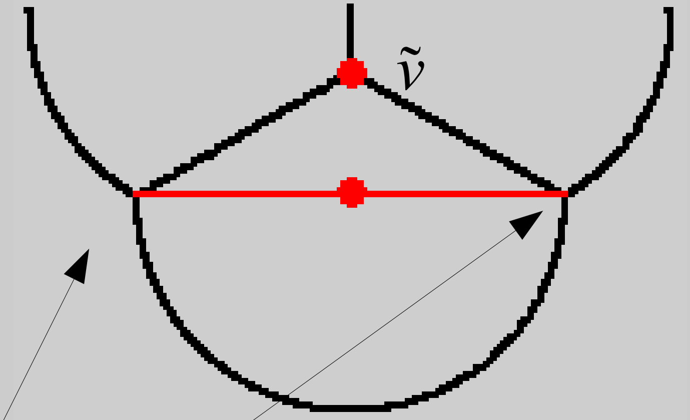
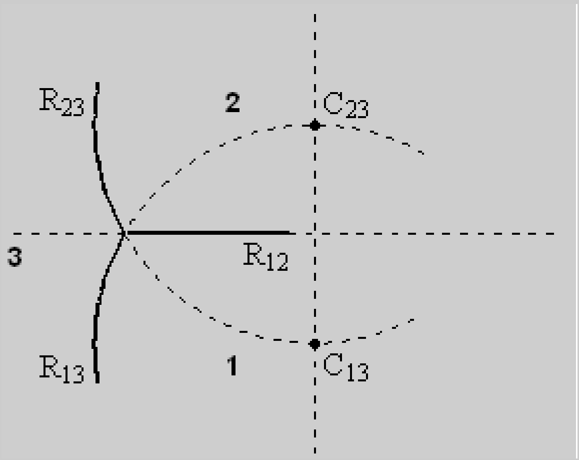


Using the reflexion symmetry
on the 2 flat edges



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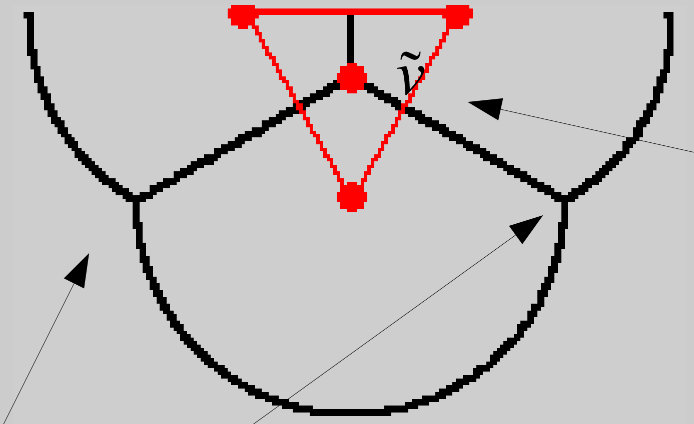

The centres are an equilateral triangle

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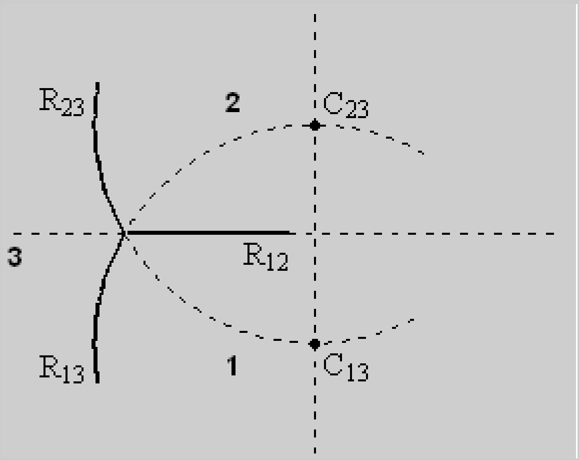

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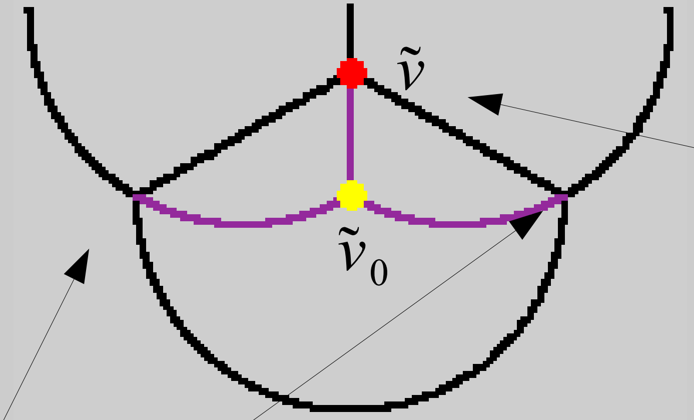

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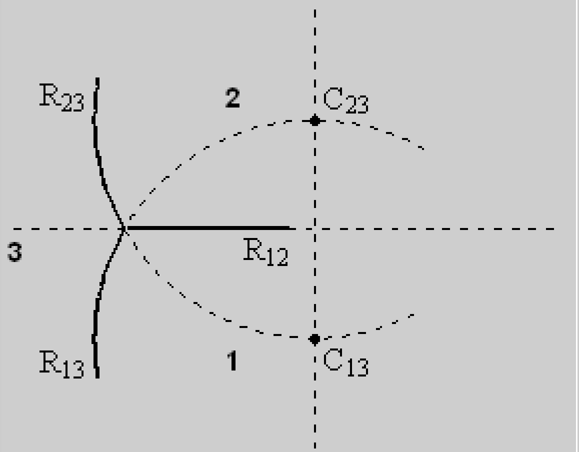
Then the prolongation of the external edges meet at an equilibrated vertex $\tilde{v}_0$



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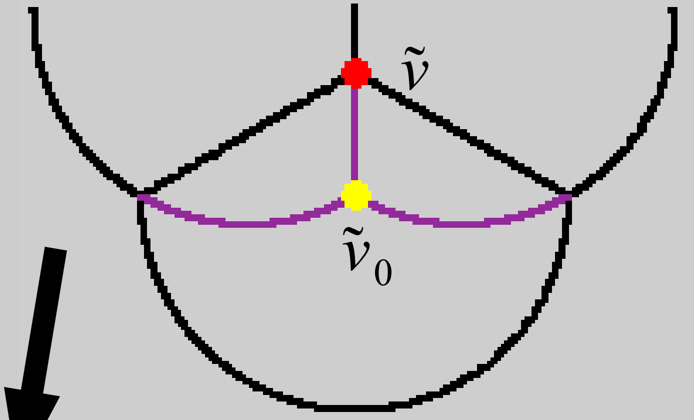

Applying the inverse transformation

$$\tilde{z} \rightarrow z = (1 + \tilde{z}\, v^*)/\tilde{z}$$

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Star-Triangle Equivalence in 2D Foams

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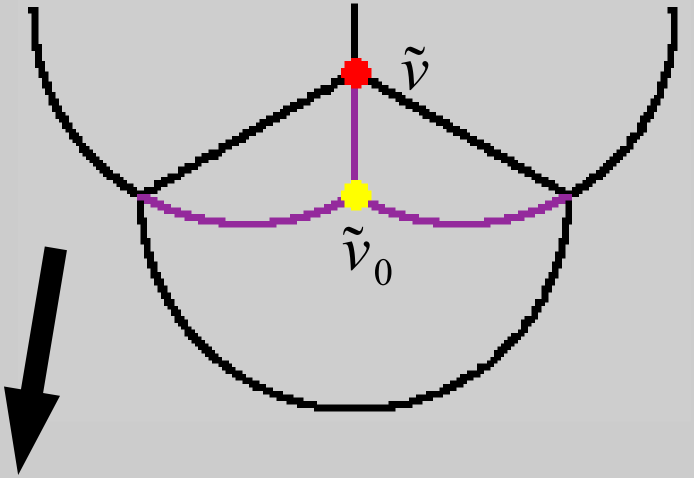

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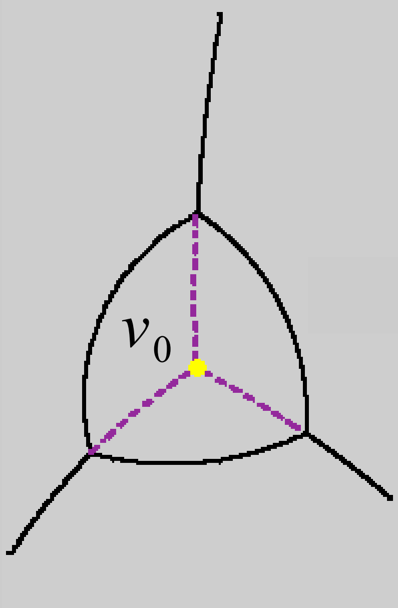

□

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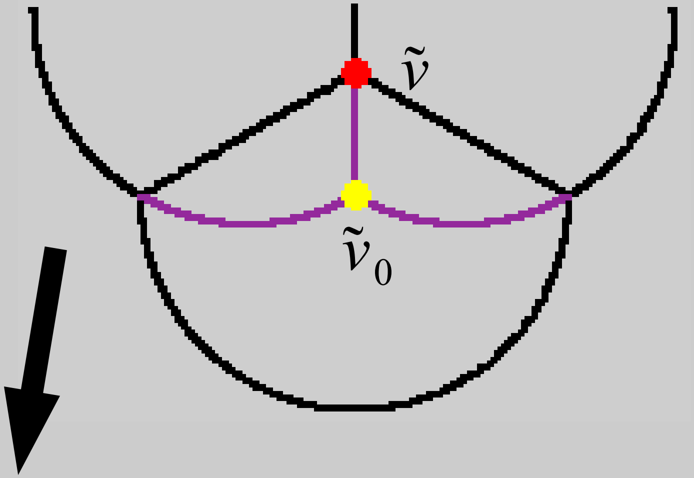

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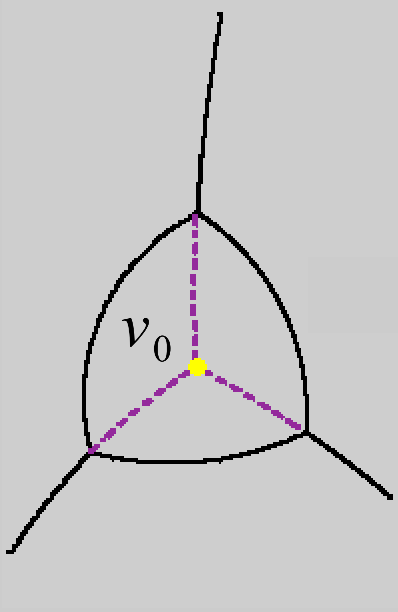

□

Consequences:

- T2 continuity
- Equilibration Reduction

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Spherical Foams are a subcase of 3D (dry) Foams
- Equilibrium Laws (Plateau+Laplace)
- The Films are spherical caps

Star-Triangle Equivalence → The vanishing bubbles are approximately tetrahedral and then T2 is a continuous process

Proof: (Moebius Invariance of spherical foams)

- Existence of a conjugate vertex ← The centres of the 6 films meeting at a vertex are on the same plane
- Inversion map toward a symmetrical figure
- Proof of the existence of a virtual equilibrated vertex
- Inverse transformation

This is a symmetry plane
for the vertex

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- Proof of the existence of a virtual equilibrated vertex
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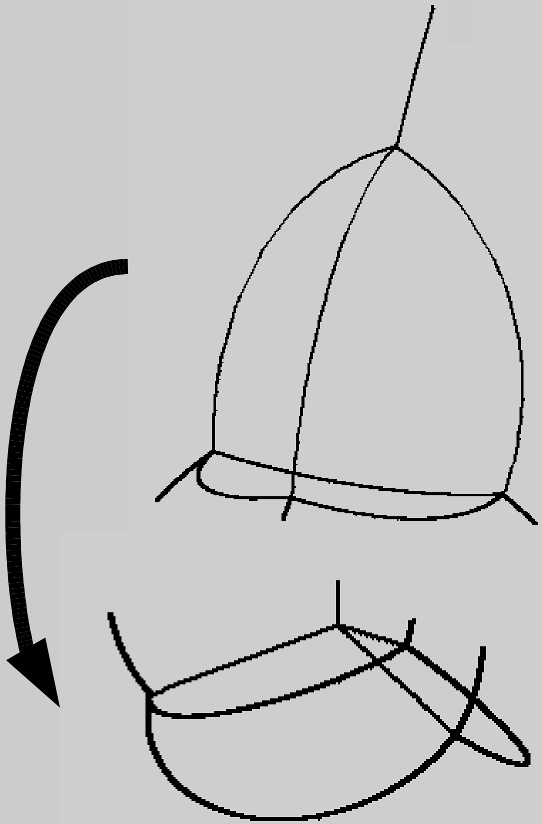


Star-Triangle Equivalence for Spherical Foams

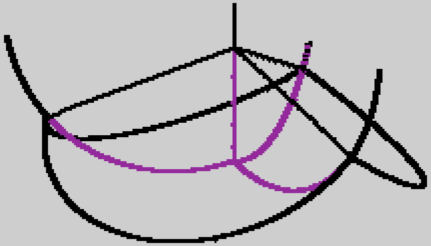
Spherical Foams are a subcase of 3D (dry) Foams

- Equilibrium Laws (Plateau+Laplace)
- The Films are spherical caps

Star-Triangle Equivalence → The vanishing bubbles are approximately tetrahedral and then T2 is a continuous process

Proof: (Moebius Invariance of spherical foams)

- Existence of a conjugate vertex
- Inversion map toward a symmetrical figure
- Proof of the existence of a virtual equilibrated vertex ←
- Inverse transformation



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