Sur quelques questions de géométrie différentielle liées à la théorie des corps et des fils élastiques

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Résumé

Le but de cette thèse est d'étudier certaines questions issues de la théorie de l'élasticité en utilisant des méthodes d'analyse mathématique et de géométrie différentielle. L'idée de notre démarche est de ne plus regarder la déformation du corps solide comme l'inconnue principale du problème, mais d'exprimer ce problème en termes des propriétés géométriques qui caractérisent le corps élastique qui subit la déformation.

Dans un premier temps, nous traitons le cas mono-dimensionnel, qui est lié à l'étude des fils élastiques, et nous prouvons des résultats d'existence, d'unicité et de stabilité d'une courbe dans des espaces de Sobolev. Ce problème nous conduit à étudier le cas plus général d'une immersion de dimension et de co-dimension quelconques d'une sous-variété dans l'espace euclidien. Nous montrons que le résultat classique d'existence et d'unicité d'une telle immersion peut être étendu jusqu'au bord de la sous-variété, sous une certaine hypothèse de régularité peu restrictive sur celui-ci. En outre, nous obtenons que l'application ainsi construite est localement lipschitzienne pour les topologies appropriées. Enfin, nous revenons à l'étude des fils élastiques, pour obtenir des inégalités de Korn linéaires et non linéaires pour les courbes en dimension 3. Ces inégalités jouent un rôle important dans l'analyse mathématique du modèle linéaire et non linéaire de poutres gauches.

Mots-clés : élasticité linéaire et non linéaire, géométrie différentielle, courbes, immersions, sous-variétés, inégalités de Korn, fils élastiques.

Abstract

The aim of this thesis is to study some questions which arise in the theory of elasticity, by using methods of mathematical analysis and differential geometry. The idea of our approach is not to consider the deformation of the solid body as the main unknown of the problem, but to express this problem in terms of the geometrical properties of the elastic body under deformation.

First, we treat the one-dimensional case, related to the study of elastic wires, and we prove some existence, uniqueness and stability results for a curve in Sobolev spaces. This problem leads us to study the more general case of an immersion of arbitrary dimension and co-dimension of a submanifold in an Euclidean space. We show that the classical existence and uniqueness result for such an immersion can be extended up to the boundary of the submanifold, under a specific, but mild, regularity assumption on this set. Moreover, we obtain that the mapping constructed in this fashion is locally Lipschitz-continuous with respect to suitable topologies. Finally, we reconsider the study of elastic wires, to obtain some linear and nonlinear Korn inequalities for curves in dimension 3. These inequalities play an important role in the mathematical analysis of the linear and nonlinear model of curved rods.

Keywords: linear and nonlinear elasticity, differential geometry, curves, immersions, submanifolds, Korn inequalities, elastic wires.

Introduction

Le cadre général de la théorie de l'élasticité.

Le problème statique en élasticité non linéaire consiste à déterminer la déformation d'un corps élastique de configuration de référence $\overline{\Omega}$, où Ω est un domaine (ouvert et connexe) de l'espace euclidien $\mathbb{R}^3$, ou plus généralement une sous-variété de dimension p de $\mathbb{R}^3$. La déformation est une application Φ de Ω à valeurs dans $\mathbb{R}^3$ qui à chaque point x du solide associe sa position $\Phi(x)$ après déformation. Nous allons supposer dans la suite que l'application Φ est suffisamment régulière pour que ce que l'on écrit ait un sens. De plus, pour des raisons physiques, elle doit être globalement injective sur Ω (pour éviter l'interpénétration de la matière) et doit conserver l'orientation de l'espace.

A chaque déformation Φ, on associe l'énergie $J(\Phi)$, définie par la différence entre l'énergie interne $I(\Phi)$ et le travail $l(\Phi)$ des forces extérieures :

$$J(\Phi) = I(\Phi) - l(\Phi). \tag{1.1}$$

Lorsque le solide est constitué d'un matériau hyperélastique homogène, son comportement est régi par une énergie interne exprimée comme l'intégrale sur Ω d'une densité d'énergie $W(\nabla\Phi)$, qui ne dépend que du gradient de la déformation et où l'expression de W dépend du type de matériau constituant le solide. Si le solide est soumis à des forces mortes volumiques de densité f et à des forces mortes surfaciques de densité g, le travail des forces extérieures peut aussi être exprimé sous une forme intégrale. Par conséquent, sous ces hypothèses, l'expression de l'énergie $J(\Phi)$ prend la forme suivante :

$$J(\Phi) = \int_\Omega W(\nabla\Phi)dx - \big(\int_\Omega f \cdot \Phi \; dx + \int_{\partial\Omega} g \cdot \Phi \; d\sigma\big). \tag{1.2}$$

Un état d'équilibre du solide déformé, décrit par une déformation ψ, est ensuite obtenu comme solution du problème de minimisation suivant :

$$J(\psi) = \inf_{\Phi} J(\Phi), \tag{1.3}$$

où $\Phi \colon \Omega \to \mathbb{R}^3$ parcourt l'ensemble des déformations admissibles.

La question de l'existence d'un tel minimiseur dans l'espace des déformations admissibles est restée longtemps ouverte, la difficulté venant du fait que ce problème est non-convexe pour toutes les lois de comportement W usuelles (voir pour plus de détails [**15**, Théorème 4.8.1]).

Cependant, une étape fondamentale dans cette direction a été accomplie avec les résultats d'existence de Ball [**5**], qui établissent l'existence de solutions à ce problème de minimisation lorsque W est polyconvexe et vérifie certaines conditions de croissance et de coercivité. On rappelle qu'une densité d'énergie $W\colon F \in \mathbb{M}^3 \to \mathbb{R}^3$ est polyconvexe si elle est convexe par rapport aux mineurs de F.

Une autre question importante porte sur l'identification et la justification rigoureuse des modèles de plaques, de coques et de fils à partir de l'élasticité tridimensionnelle. Dans cette direction, nous signalons les travaux de Fox, Raoult et Simo [**33**], qui ont justifié de façon formelle les équations des plaques non linéaires élastiques, telles qu'elles sont décrites dans la littérature en mécanique, en utilisant la méthode du développement asymptotique, introduite par Ciarlet et Destuynder [**19**] en élasticité linéaire.

Une autre démarche plus rigoureuse est basée sur l'usage de la Γ-convergence, qui est une notion de convergence sur les fonctionnelles. Cette approche a été utilisée avec succès pour les fils par Acerbi, Butazzo et Percivale [**1**], pour les plaques par Le Dret et Raoult [**41**], Ben Belgacem [**6**], [**7**], Pantz [**50**], Friesecke, James et Müller [**35**], [**36**], [**37**] et pour les coques par Le Dret et Raoult [**42**], Friesecke, James, Mora et Müller [**34**].

Une approche géométrique en théorie de l'élasticité.

Les difficultées rencontrées pour résoudre le problème de minimisation (1.3) ont conduit à une nouvelle direction de recherche, visant à explorer la possibilité de combiner les méthodes de l'analyse mathématique avec des outils géométriques.

Cette nouvelle approche, déjà suggérée par Antman [**4**], a été récemment développée par Ciarlet et Laurent [**21**] et par Ciarlet et C. Mardare [**27**], [**28**] (voir aussi les travaux de Opoka et Pietraszkiewicz [**49**]). L'idée centrale de cette démarche est de ne plus regarder la déformation Φ comme l'inconnue principale du problème, mais d'exprimer ce problème en termes des propriétés géométriques qui caractérisent le corps élastique qui subit la déformation. En effet, de ce point de vue, l'étude des déformations d'un solide en théorie de l'élasticité revient en géométrie différentielle à l'étude des immersions d'une variété différentielle dans l'espace euclidien. Le changement de position d'un point matériel lors d'une déformation est alors modélisé par une métrique sur la variété correspondante et par une notion adaptée de courbure sur cette variété.

Lorsque le corps est constitué d'un matériau hyperélastique, son comportement est régi par une densité d'énergie de la forme

$$\Phi \mapsto W(\cdot, \nabla\Phi^T \nabla\Phi) \tag{1.4}$$

pour toute déformation $\Phi\colon \Omega \to \mathbb{R}^3$, la fonction $W\colon \Omega \times \mathbb{S}^3_> \to \mathbb{R}$ caractérisant le matériau. L'énergie peut alors être calculée avec la formule (1.2) et un état d'équilibre est trouvé comme une solution du problème de minimisation (1.3). Néanmoins, la forme particulière de l'expression (1.4) suggère qu'une autre méthode pourrait aussi être envisagée : au lieu de prendre comme inconnue principale l'application Φ, il est possible de considérer le problème dans l'inconnue $C := \nabla\Phi^T \nabla\Phi$. Le champs de matrices C est connu en élasticité comme étant le tenseur de Cauchy-Green associé à la déformation Φ et représente la métrique de Ω associée à la déformation Φ en géométrie differentielle. Cela conduit donc à un nouveau problème de minimisation, cette fois-ci dans l'inconnue C, qui pourrait être plus facile à résoudre.

Toutefois, cette méthode soulève plusieures questions : Etant donné C, est-il possible de reconstruire Φ ? Est-ce que l'application $C \mapsto \Phi$ a-t-elle de bonnes propriétés de continuité, pour permettre un eventuel passage à la limite dans les termes de force ou bien dans les conditions aux limites ? Dans quels espaces de fonctions faut-il se placer ?

Cette demarche peut aussi s'appliquer dans la théorie des coques, qui sont des solides cylindriques minces ayant la géométrie de leur surface moyenne $S = \theta(\omega)$, où ω est un domaine de $\mathbb{R}^2$ et $\theta\colon \omega \to \mathbb{R}^3$ est une application suffisamment régulière. Dans le cas d'un matériau hyperélastique, la densité d'énergie associée à une déformation $\phi\colon S \to \mathbb{R}^3$ de cette surface moyenne est donnée par

$$\phi \mapsto W(\cdot, a(\phi), b(\phi)), \tag{1.5}$$

où la fonction $W\colon \omega \times \mathbb{S}^2_> \times \mathbb{S}^2 \to \mathbb{R}$ caractérise le matériau et $a(\phi)$ et $b(\phi)$ sont des champs de matrices désignant respectivement la première et la deuxième forme fondamentale de la surface déformée $\phi(\omega)$. Dans ce cas, les nouvelles inconnues sont les deux formes elles-mêmes et les mêmes questions que dans le cas de l'élasticité tridimensionnelle sont à résoudre.

Il est à remarquer que ce nouveau point de vue nécessite l'extension des résultats classiques de géométrie différentielle à un cadre fonctionnel plus adapté aux problèmes issus de la théorie de l'élasticité. Il s'agit d'utiliser la notion de dérivée au sens généralisé et de chercher la solution de ces problèmes dans des espaces de fonctions de faible régularité, en particulier dans des espaces de type Sobolev. Il est aussi important de prendre en compte les conditions aux limites qui apparaissent naturellement dans les équations qui régissent le comportement du solide déformé, d'où le besoin d'étudier les immersions d'une variété à bord.

Cette nouvelle approche a été déjà appliquée avec succès en élasticité linéaire par P.G. Ciarlet et P. Ciarlet Jr. (voir [**18**]). Par contre, dans le cadre de l'élasticité non linéaire on voit apparaître des difficultées supplémentaires. Néanmoins, des resultats partiels dans cette direction ont été obtenus en élasticité tridimensionnelle (voir [**21**], [**25**], [**27**], [**28**], [**43**], [**46**], [**51**]) ainsi qu'en élasticité bidimensionnelle (voir [**17**], [**20**], [**26**], [**29**], [**45**]).

Notre travail s'inscrit dans cette direction. Dans une première partie nous nous sommes intéressés au cas mono-dimensionnel, qui est lié à l'étude de la déformation des fils élastiques. Notre objectif a été de traiter des questions liées à l'existence et l'unicité d'une courbe dans des hypothèses plus faibles que celles que l'on considère usuellement.

Ce problème, lié aux immersions d'une courbe dans un espace euclidien de dimension quelconque, nous a conduit dans le deuxième chapitre à étudier le cas plus général des immersions de dimension et de co-dimension quelconques d'une sous-variété dans l'espace euclidien. Les questions abordées dans cette partie portent sur la reconstruction d'une sous-variété à bord et trouvent leur motivation à la fois en géométrie différentielle et en théorie de l'élasticité.

Dans la troisième partie nous nous sommes placés à nouveau dans le cadre de la théorie des fils élastiques, pour obtenir des inégalités de Korn linéaires et non linéaires pour les courbes, inégalités qui jouent un rôle important dans l'analyse mathématique du problème statique en théorie des fils élastiques.

Présentation générale de la thèse.

Dans cette partie, nous présentons d'une manière générale les principaux objectifs qui ont été poursuivis dans ce travail de thèse, ainsi que les résultats que nous avons obtenus dans chaque chapitre. Le mémoire est divisé en trois chapitres et une annexe.

En général, les fils d'éppaisseur constante sont modélisés par des solides cylindriques minces, occupant au repos un ouvert de la forme $\Omega^\varepsilon = \omega^\varepsilon \times (0, L)$, où $\omega^\varepsilon = \varepsilon\omega$, ω est un domaine de $\mathbb{R}^2$ et ε est un paramètre destiné à tendre vers zéro. Si les fils sont constitués d'un matériau hyperélastique, on peut utiliser ce qui a été présenté précédemment pour décrire leur comportement. Néanmoins, une alternative nettement plus simple consiste à utiliser un modèle mono-dimensionnel, à savoir un modèle qui fait intervenir seulement la déformation de la ligne centrale. Un tel modèle présente de nombreux avantages, car il se prête plus facilement à une analyse physique et mathématique, et nécessite aussi beaucoup moins de calculs lors d'un traitement par simulations numériques.

Exprimé en termes de la géométrie différentielle, ce problème revient à étudier les immersions d'une courbe dans l'espace euclidien. Notre premier objectif dans ce chapitre a été d'établir dans quelles conditions le théorème fondamental de la théorie des courbes reste vrai sous des hypothèses faibles de régularité. Plus précisément, nous avons montré qu'il existe une courbe unique $c \in H^n(I; \mathbb{R}^n)$, $I = (0, T)$, isométriquement immergée dans $\mathbb{R}^n$, dont les fonctions de courbure sont $(n-1)$ fonctions données de $H^{n-2}(I; \mathbb{R}) \times H^{n-3}(I; \mathbb{R}) \times \ldots \times H^1(I; \mathbb{R}) \times L^2(I; \mathbb{R})$. Grâce à ce résultat, l'application qui, à chaque $(n-1)$-uple de fonctions de courbure, associe une courbe immergée dans $\mathbb{R}^n$, est bien définie. Notre deuxième objectif dans ce chapitre a été de prouver que l'application ainsi construite est de classe $\mathcal{C}^\infty$.

Considérons maintenant $\omega \subset \mathbb{R}^p$ un ouvert connexe et simplement connexe, muni d'une métrique riemannienne. Il est connu en géométrie différentielle (voir [**38**], [**39**], [**53**], [**58**]), que ω peut être isométriquement immergé dans l'espace euclidien identifié à $\mathbb{R}^{p+q}$ si et seulement si les tenseurs associés vérifient les relations de Gauss-Ricci-Codazzi et que les immersions ainsi définies sont déterminées aux isométries de $\mathbb{R}^{p+q}$ près. Du point de vue de l'élasticité (voir Betounes [**11**], où l'intérêt de l'étude de la mécanique des solides en dimension quelconque est souligné), l'immersion $\theta\colon \omega \to \mathbb{R}^{p+q}$ ainsi obtenue peut être interprétée comme la déformation d'un solide dont $\overline{\omega}$ est la configuration de référence.

Le premier objectif du Chapitre 2 est d'étendre ce résultat d'existence et d'unicité de l'immersion θ jusqu'au bord de ω. Nous avons montré que cela est possible sous l'hypothèse que ω satisfait la "propriété géodésique", qui est une hypothèse de régularité peu restrictive sur la frontière de ω. Outre sa motivation géométrique, ce résultat permet de prendre en compte les conditions au bord qui apparaissent dans les équations satisfaites par une déformation.

Le théorème précédent de reconstruction jusqu'au bord nous permet de définir une application qui associe aux données géométriques de départ la sous-variété ainsi construite. Le deuxième objectif qui a été poursuivi dans ce chapitre consiste à étudier les propriétés de régularité de cette application. Nous avons montré que, lorsque l'ouvert ω est borné, l'application de reconstruction est localement lipschitzienne pour les topologies usuelles des espaces de Banach $\mathcal{C}^l(\overline{\omega})$, $l \geq 1$.

Les résultats ainsi obtenus constituent une approche unifiée, qui généralise à des dimensions et co-dimensions quelconques des travaux récents qui ont été faits dans le cas d'un ouvert de $\mathbb{R}^n$ (voir [**27**]) et dans le cas d'une surface (voir [**29**]).

Notre objectif dans le troisième chapitre, qui est un travail en commun avec Sorin Mardare, est de poursuivre l'étude des courbes dans l'espace euclidien, pour obtenir des inégalités de Korn linéaires et non linéaires dans ce cas.

Une première motivation de notre travail provient de l'importance des inégalités de Korn linéaires dans la théorie de l'élasticité linéaire, où elles représentent un outil de base dans la démonstration de l'existence d'une solution pour le modèle linéaire de Koiter (voir la Section 1.7, le Chapitre 2 et les références de Ciarlet [**16**] pour l'inégalité de Korn tridimensionnelle et l'inégalité de Korn pour une surface, ainsi que les travaux de Chen et Jost [**13**] pour une version de ce résultat dans le cadre de la géométrie riemannienne). En ce qui concerne les inégalités de Korn non linéaires, notre travail a été motivé par l'approche géométrique de la théorie des fils non linéairement élastiques que nous avons présentée auparavant. De ce point de vue, ces inégalités non linéaires peuvent être interprétées comme des extensions naturelles des inégalités de Korn non linéaires établies par Ciarlet et C. Mardare dans [**28**] pour le cas d'un sous-ensemble ouvert de $\mathbb{R}^n$.

La premiere partie de ce travail consiste à obtenir une inégalité de Korn linéaire pour les courbes de $\mathbb{R}^3$, qui montre que l'on peut contrôler le champ de déplacement de la courbe, mesuré pour une norme de Sobolev appropriée, par les tenseurs linéarisés de changement de métrique, de courbure, et de torsion, de la courbe, mesurés en norme L^2. Dans la deuxième partie, nous avons obtenu des inégalités qui montrent que la norme $W^{m,p}$ du champ de déplacement de la courbe peut être majorée par des expressions qui ne dépendent que des tenseurs de changement de métrique, de courbure, et de torsion de la courbe, mesurés dans des normes appropriées. Les inégalités ainsi obtenues peuvent être vues comme des inégalités de Korn non linéaires pour les courbes.

Dans la dernière partie de la thèse, qui est présentée sous la forme d'un appendice, notre but a été de développer un outil qui permet de reconstruire des courbes immergées dans l'espace euclidien de dimension 2 ou 3 à partir des fonctions de courbure et de torsion dérivables au sens de Sobolev, mais pas nécessairement au sens classique. Nous avons fait des simulations numériques à l'aide du logiciel Scilab et nous avons utilisé les résultats théoriques obtenus dans le premier chapitre.

Nous donnons maintenant une description plus détaillée des résultats obtenus dans chaque chapitre de la thèse.

Présentation du Chapitre 1.

Le but du premier chapitre est l'étude des courbes isométriquement plongées dans un espace euclidien, problème qui est lié en théorie de l'élasticité à l'étude de la déformation des fils élastiques.

Il est connu en géométrie différentielle que l'on peut reconstruire une courbe à partir de ses fonctions de courbure si on peut dériver ces fonctions suffisamment de fois au sens classique. Le cadre géométrique est le suivant : on considère un intervalle $I = (0, T)$ de $\mathbb{R}$ et $c\colon I \to \mathbb{R}^n$ une courbe régulière, i.e., telle que les vecteurs $c^{(1)}(t), c^{(2)}(t), \ldots, c^{(n-1)}(t)$ soient linéairement indépendants pour tout $t \in I$. On montre alors qu'il exite un unique repère de Frenet associé à cette courbe, que l'on note $\{e_1(t), \ldots, e_n(t)\}$, et que les fonctions de courbure sont données par les formules suivantes :

$$k_i(t) = \frac{< e_{i+1}(t), e_i'(t) >}{|c'(t)|}, \ \forall i \in \{1, \ldots, n-1\}. \tag{1.6}$$

Avec ces notations, le théorème fondamental de la théorie des courbes affirme alors qu'on peut reconstruire la courbe c si on prescrit ses $(n-1)$ fonctions de courbure et que cette courbe est unique aux isométries de $\mathbb{R}^n$ près (voir, par exemple Klingenberg [**40**] ou Spivak [**52**], pour une preuve en dimension n, mais sans précisions sur la régularité des données, ou bien Berger et Gostiaux [**8**] pour une démonstration en dimension 3 qui prend en considération l'ordre de dérivabilité au sens classique des fonctions de courbure).

Récemment, divers généralisations de ce problème de reconstruction, motivées par des questions rencontrées en élasticité non linéaire, ont été obtenues pour le cas d'un ouvert de $\mathbb{R}^n$ et celui d'une surface par S. Mardare [**45**], [**46**] pour des données qui se trouvent dans des espaces de Sobolev.

Le premier objectif de ce chapitre est d'établir un résultat analogue pour une courbe, à savoir prouver que le théorème fondamental de la théorie des courbes reste vrai sous des hypothèse de régularité plus faibles que celles que l'on considère en géométrie differentielle classique. L'idée est de considérer que les fonctions de courbure appartiennent à des espaces de Sobolev et d'utiliser la notion de dérivée généralisée au sens des distributions. Le résultat s'énonce alors de la façon suivante :

Théorème 1. *(existence et unicité) Soient* $(F_1, \ldots, F_{n-1}) \in H^{n-2}(I;\mathbb{R})\times\ldots\times H^1(I;\mathbb{R})\times L^2(I;\mathbb{R})$ *des fonctions telles que* $F_1(t) > 0, \ldots, F_{n-2}(t) > 0$ *pout tout* $t \in I$*. Alors :*

(a) Il existe une courbe régulière $c \in H^n(I;\mathbb{R}^n)$ telle que $|c'(t)| = 1$ pour tout $t \in I$ et telle que ses fonctions de courbure soient $F_1, \ldots, F_{n-1}$, i.e., $k_i(t) = F_i(t)$ pour tout $i \in \{1, \ldots, n-1\}$ et pour tout $t \in I$.

(b) Si c et $\tilde{c}$ sont deux courbes qui satisfont les conditions de (a), alors il existe une isométrie propre $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ telle que $\tilde{c} = \varphi \circ c$.

(c) Si $x_0 \in \mathbb{R}^n$ est fixé, alors il existe une unique courbe c telle que $c(0) = x_0$, son repère de Frenet à l'origine soit $e_1(0) = (1, 0, \ldots, 0), \ldots, e_n(0) = (0, 0, \ldots, 1)$ et telle que les conditions de (a) soient satisfaites.

L'idée de la preuve de ce résultat est d'écrire, comme dans le cas classique, les équations de Frénet (mais qui cette fois-ci sont vérifiées seulement au sens des distributions) et de démontrer l'existence et l'unicité de la solution d'un tel système. La forme particulière de ces équations permet ensuite de prouver la régularité de la solution ainsi obtenue, et, par conséquent, de montrer l'existence de la courbe $c \in H^n(I;\mathbb{R}^n)$. L'unicité de la courbe est obtenue grâce à l'unicité de la solution du système précédent.

Notons que l'hypothèse que les premières $(n-2)$ fonctions de courbures sont strictement positives est essentielle, comme le montre le contre-exemple donné dans la dernière section de ce chapitre.

Une autre question concerne la régularité de l'application qui associe aux données géométriques initiales la courbe ainsi reconstruite, qui est bien définie grâce aux résultats d'existence et d'unicité précédents. Dans cette direction, des résultats similaires ont été obtenus dans le cas d'un ouvert de $\mathbb{R}^n$ (voir Ciarlet et Laurent [**21**]) et aussi dans celui d'une surface (voir Ciarlet [**17**]), mais pour des données régulières.

Le deuxième but de cette partie est d'étudier la régularité de l'application qui associe aux $(n-1)$ fonctions de courbure la courbe $c \in H^n(I;\mathbb{R}^n)$ ainsi reconstruite. Le fait que cette application est bien définie est une conséquence du Théorème 1. Pour simplifier la présentation de ce résultat, on introduit les notations suivantes :

$$\mathcal{H}(I;\mathbb{R}) := \prod_{k=0}^{n-2} H^{n-k-2}(I;\mathbb{R}), \tag{1.7}$$

et

$$\mathcal{H}(I;\mathbb{R})_> := \{(F_1, \ldots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R});\ F_i(t) > 0,\ \forall t \in I,\ \forall i \in \{1, \ldots, n-2\}\}. \tag{1.8}$$

Il est à remarquer que l'ensemble $\mathcal{H}(I;\mathbb{R})_>$ est un ouvert de l'espace de Hilbert $\mathcal{H}(I;\mathbb{R})$. Avec ces notations, le Théorème 1(c) montre qu'il existe une application

$$\mathcal{F}\colon (F_1, \ldots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_> \mapsto c \in H^n(I;\mathbb{R}^n), \tag{1.9}$$

où la courbe c est telle que ses fonctions de courbure sont exactement $F_1, \ldots, F_{n-1}$ et satisfait en plus les "conditions initiales" $c(0) = x_0$ et $e_1(0) = (1, 0, \ldots, 0), \ldots, e_n(0) = (0, 0, \ldots, 1)$. Le résultat pricipal de la deuxième partie est alors le suivant :

THÉORÈME 2. *On considère l'application*

$$\mathcal{F}\colon \mathcal{H}(I;\mathbb{R})_{>} \to H^n(I;\mathbb{R}^n),$$

$$(F_1,\dots,F_{n-1}) \mapsto c$$

où la courbe c est définie dans la partie (c) du Théorème 1. L'application $\mathcal{F}$ est alors de classe $\mathcal{C}^\infty$.

La preuve de ce résultat est basée sur le théorème des fonctions implicites, appliqué dans le même cadre fonctionnel que celui utilisé dans la preuve du Théorème 1 pour montrer l'existence de la courbe c.

Le résultat d'existence et unicité contenu dans le Théorème 1 et le résultat de régularité $\mathcal{C}^\infty$ donné dans le Théorème 2 permettent aussi de retrouver comme corollaires les résultats analogues en géométrie différentielle classique, qui utilisaient la notion de dérivée au sens usuel.

Les résultats présentés dans ce chapitre ont fait l'objet des publications [**54**] et [**55**].

Présentation du Chapitre 2.

Le théorème fondamental de la géométrie riemannienne affirme qu'un espace de Riemann connexe et simplement connexe ω de $\mathbb{R}^p$ peut être isométriquement immergé dans l'espace euclidien $\mathbb{R}^{p+q}$ si, et seulement si, il existe des tenseurs satisfaisant les équations de Gauss-Ricci-Codazzi, auquel cas ces immersions sont déterminées aux isométries de $\mathbb{R}^{p+q}$ près. Ce théorème peut être démontré par différentes méthodes : en utilisant les coordonnées locales (voir Eisenhart [**32**]), les connexions du fibré normal (voir Szczarba [**53**]), les formes différentielles (voir Tenenblat [**58**]), ou l'existence d'une métrique plate compatible (voir Jacobowitz [**38**], [**39**], articles dans lesquels on trouve aussi une présentation de l'état de l'art du problème).

Ce deuxième chapitre a deux objectifs : sous une certaine hypothèse de régularité sur la frontière de ω, nous avons d'abord établi un résultat analogue d'existence et unicité d'une sous-variété "jusqu'au bord", puis nous avons montré que l'application qui associe à ces tenseurs la sous-variété ainsi reconstruite est localement lipschitzienne pour les topologies usuelles des espaces de Banach $\mathcal{C}^l(\overline{\omega})$, $l \geq 1$.

Pour présenter ce travail, nous commençons par préciser l'hypothèse de régularité sur la frontière de l'ouvert ω qui sera utilisée dans la suite :

DÉFINITION 1. *On dit qu'un sous-ensemble ouvert ω de $\mathbb{R}^p$ satisfait la propriété géodésique si, et seulement si, il est connexe et, pour tout point $x_0 \in \partial\omega$ et pour tout $\varepsilon > 0$, il existe $\delta = \delta(x_0,\varepsilon) > 0$ tel que*

$$d_\omega(x,y) < \varepsilon \text{ pour tout } x,y \in \omega \cap B(x_0;\delta).$$

Il est à remarquer que cette hypothèse n'est pas très restrictive, car tout ouvert connexe de $\mathbb{R}^p$ à frontière lipschitzienne (cf. Définition 4.5. de [**2**], ou pp. 14-15 de [**48**]), satisfait la propriété géodésique.

Une autre définition dont nous avons besoin pour énoncer nos résultats est la suivante :

DÉFINITION 2. *Soit ω un ouvert de $\mathbb{R}^p$. Pour tout entier $l \geq 1$, on définit l'ensemble $\mathcal{C}^l(\overline{\omega})$ comme étant l'espace de toutes les fonctions $f \in \mathcal{C}^l(\omega)$ qui, en même temps que toutes leurs dérivées partielles $\partial^{\boldsymbol{\alpha}} f$, $1 \leq |\boldsymbol{\alpha}| \leq l$, possèdent des extensions continues sur l'adhérence $\overline{\omega}$ de ω. Des définitions analogues ont lieu pour les espaces $\mathcal{C}^l(\overline{\omega};\mathbb{R}^N)$, $\mathcal{C}^l(\overline{\omega};\mathbb{M}^N)$, $\mathcal{C}^l(\overline{\omega};\mathbb{S}^N)$, etc., pour tout entier $N > 0$. Toutes les extensions continues qui apparaissent dans ces espaces sont identifiées par des lettres surlignées, comme par exemple dans la définition de l'ensemble suivant :*

$$\mathcal{C}^2(\overline{\omega};\mathbb{S}^N_>) := \{A \in \mathcal{C}^l(\overline{\omega};\mathbb{S}^N); \overline{A}(y) \in \mathbb{S}^N_> \text{ pour tout } y \in \overline{\omega}\}.$$

Les espaces $\mathcal{C}^l(\overline{\omega};\mathbb{R}^N)$ et $\mathcal{C}^l(\overline{\omega};\mathbb{M}^N)$, $l \geq 1$, sont munis des normes suivantes :

$$\|F\|_{l,\overline{\omega}} := \sup_{y\in\overline{\omega},\ |\boldsymbol{\alpha}|\leq l} |\overline{\partial^{\boldsymbol{\alpha}} F}(y)| \text{ pour tout } F \in \mathcal{C}^l(\overline{\omega};\mathbb{M}^N), \tag{1.10}$$

et

$$\|\theta\|_{l,\overline{\omega}} := \sup_{y\in\overline{\omega},\ |\boldsymbol{\alpha}|\leq l} |\overline{\partial^{\boldsymbol{\alpha}} \theta}(y)| \text{ pour tout } \theta \in \mathcal{C}^l(\overline{\omega};\mathbb{R}^N). \tag{1.11}$$

Le cadre géométrique, ainsi que toutes les notations, est décrit en détail dans le corps de la thèse. Nous retenons ici juste le fait que les équations Gauss-Ricci-Codazzi ont la forme suivante :

$$(\partial_\alpha \Gamma^\tau_{\beta\delta} - \partial_\beta \Gamma^\tau_{\alpha\delta} + \Gamma^\sigma_{\beta\delta}\Gamma^\tau_{\alpha\sigma} - \Gamma^\sigma_{\alpha\delta}\Gamma^\tau_{\beta\sigma})a_{\tau\gamma} = b^i_{\gamma\alpha}b^i_{\delta\beta} - b^i_{\gamma\beta}b^i_{\delta\alpha}, \tag{1.12}$$

$$\partial_\alpha T^{ij}_\beta - \partial_\beta T^{ij}_\alpha + T^{kj}_\beta T^{ik}_\alpha - T^{kj}_\alpha T^{ik}_\beta + a^{\sigma\tau}(b^j_{\alpha\tau}b^i_{\beta\sigma} - b^j_{\beta\tau}b^i_{\alpha\sigma}) = 0, \tag{1.13}$$

$$\partial_\alpha b^j_{\gamma\beta} - \partial_\beta b^j_{\gamma\alpha} = \Gamma^\tau_{\alpha\gamma}b^j_{\tau\beta} - \Gamma^\tau_{\beta\gamma}b^j_{\tau\alpha} + b^i_{\gamma\beta}T^{ij}_\alpha - b^i_{\gamma\alpha}T^{ij}_\beta, \tag{1.14}$$

où $\alpha,\beta,\gamma,\delta,\sigma,\tau \in \{1,\ldots,p\}$ et $i,j,k \in \{1,\ldots,q\}$. Ce sont les équations classiques de Gauss-Ricci-Codazzi satisfaites pour une sous-variété de l'espace euclidien, données sous cette forme par exemple dans [**32**, page 190].

Avec toutes ces notations, nous pouvons maintenant énoncer le théorème fondamental de la géométrie riemannienne, qui établit l'existence et l'unicité d'une immersion isométrique d'un sous-ensemble ouvert $\omega \subset \mathbb{R}^p$, muni d'une métrique riemannienne, dans l'espace euclidien $\mathbb{R}^{p+q}$:

THÉORÈME 3. *Soit ω un ouvert connexe et simplement connexe de $\mathbb{R}^p$. Soient $A = (a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^p_>)$ un champ de matrices symétriques définies positives, $B^i = (b^i_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^p)$, $i \in \{1,\ldots,q\}$, q champs de matrices symétriques, et $T_\alpha = (T^{ij}_\alpha) \in \mathcal{C}^1(\omega;\mathbb{A}^q)$, $\alpha \in \{1,\ldots,p\}$, p champs de matrices anti-symétriques, qui satisfont les relations de Gauss-Ricci-Codazzi (1.12-1.14). Etant donné $y_0 \in \omega$ fixé, on note par $G_0 \in \mathbb{M}^{p+q}$ la matrice* *suivante :*

$$G_0 := \begin{pmatrix} A(y_0) & 0 \\ 0 & I_q \end{pmatrix}. \tag{1.15}$$

Alors il existe une immersion $\theta \in \mathcal{C}^3(\omega, \mathbb{R}^{p+q})$ unique et une famille unique orthonormée de q champs de vecteurs $N^1, \ldots, N^q \in \mathcal{C}^2(\omega, \mathbb{R}^{p+q})$, normaux à $\theta(\omega)$, satisfaisant les relations

$$(i)\ \langle \partial_\alpha \theta(y), \partial_\beta \theta(y) \rangle = a_{\alpha\beta}(y),\ \forall y \in \omega,\ \forall \alpha, \beta \in \{1, \ldots, p\}, \tag{1.16}$$

$$(ii)\ \big\langle \partial_{\alpha\beta} \theta(y), N^i(y) \big\rangle = -\, b^i_{\alpha\beta}(y)\ \forall y \in \omega,\ \forall \alpha, \beta \in \{1, \ldots, p\},\ \forall i \in \{1, \ldots, q\}, \tag{1.17}$$

$$(iii)\ \big\langle \partial_\alpha N^i(y), N^j(y) \big\rangle = T^{ij}_\alpha(y)\ \forall y \in \omega,\ \forall \alpha \in \{1, \ldots, p\},\ \forall i, j \in \{1, \ldots, q\}, \tag{1.18}$$

tels que $\theta(y_0) = 0$, et tels que la α-ième colonne de la matrice $G_0^{1/2}$ soit $\partial_\alpha \theta(y_0)$, pour tout $\alpha \in \{1, \ldots, p\}$.

Notre premier objectif est de montrer que l'immersion isométrique $\theta \colon \omega \subset \mathbb{R}^p \to \mathbb{R}^n$ donnée par le théorème précédent peut être prolongée, sous certaines hypothèses, jusqu'à la frontière $\partial\omega$ de l'ouvert ω. Le théorème suivant étend ainsi des résultats obtenus précédemment pour un ouvert de $\mathbb{R}^n$ (voir [**27**, Théorème 3.3]) et pour une surface (voir [**29**, Théorème 2]) :

Théorème 4. *Soit ω un ouvert connexe et simplement connexe de $\mathbb{R}^p$ qui satisfait la propriété géodésique (cf. Définition 1). On considère un champs de matrices symétriques définies positives $A = (a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>)$ (cf. Définition 2), q champs de matrices symétriques $B^i = (b^i_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)$, $i \in \{1, \ldots, q\}$, et p champs de matrices anti-symétriques $T_\alpha = (T^{ij}_\alpha) \in \mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)$, $\alpha \in \{1, \ldots, p\}$, tels que les relations* (1.12), (1.13) *et* (1.14) *soient satisfaites. Alors il existe une application $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$ et une famille orthonormée de q champs de vecteurs $N^1, \ldots, N^q \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ normaux à $\theta(\overline{\omega})$ tels que*

$$(i)\ \langle \overline{\partial_\alpha \theta}(y), \overline{\partial_\beta \theta}(y) \rangle = \overline{a_{\alpha\beta}}(y),\ \forall y \in \overline{\omega},\ \forall \alpha, \beta \in \{1, \ldots, p\}, \tag{1.19}$$

$$(ii)\ \Big\langle \overline{\partial_{\alpha\beta} \theta}(y), \overline{N^i}(y) \Big\rangle = -\, \overline{b^i_{\alpha\beta}}(y)\ \forall y \in \overline{\omega},\ \forall \alpha, \beta \in \{1, \ldots, p\},\ \forall i \in \{1, \ldots, q\}, \tag{1.20}$$

$$(iii)\ \Big\langle \overline{\partial_\alpha N^i}(y), \overline{N^j}(y) \Big\rangle = \overline{T^{ij}_\alpha}(y)\ \forall y \in \overline{\omega},\ \forall \alpha \in \{1, \ldots, p\},\ \forall i, j \in \{1, \ldots, q\}, \tag{1.21}$$

tels que $\theta(y_0) = 0$, et tels que la α-ième colonne de la matrice $G_0^{1/2}$ soit $\partial_\alpha \theta(y_0)$ pour tout $\alpha \in \{1, \ldots, p\}$, où la matrice G_0 est donnée par la relation (1.15).

La preuve de ce théorème se fait en montrant que l'application $\theta \colon \omega \subset \mathbb{R}^p \to \mathbb{R}^n$, ainsi que ses dérivées partielles d'ordre ≤ 3 ont des prolongements continus à $\overline{\omega}$. L'idée est d'utiliser le fait que ω satisfait la propriété géodésique pour déduire des estimations permettant d'appliquer le lemme de Gronwall et de montrer l'existence de tels prolongements.

Pour rendre la présentation de la deuxième partie plus claire, nous introduisons la notation suivante :

$$\begin{aligned} X(\overline{\omega}) := \{ \big(A, (B^i), (T_\alpha)\big) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)\big)^p \\ \text{tels que les relations (1.12), (1.13) et (1.14) soient satisfaites}\}, \end{aligned} \tag{1.22}$$

où on rappelle que (1.12), (1.13) et (1.14) sont les relations de Gauss-Ricci-Codazzi écrites en coordonnées locales. Considérons un point $y_0 \in \omega$ fixé et une matrice G_0 donnée comme dans la relation (1.15). Nous pouvons alors construire l'application

$$\mathcal{F}\colon X(\overline{\omega}) \to \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q, \tag{1.23}$$

qui associe à chaque élément $\big(A, (B^i), (T_\alpha)\big) \in X(\overline{\omega})$ l'unique élément $\big(\theta, (N^i)\big) \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q$ qui satisfait les conditions ci-dessous :

$$\Big\langle \overline{\partial_\alpha \theta}(y), \overline{N^i}(y) \Big\rangle = 0 \ \forall y \in \overline{\omega},\ \forall \alpha \in \{1, \dots, p\},\ \forall i \in \{1, \dots, q\}, \tag{1.24}$$

les relations (1.19), (1.20), (1.21), $\theta(y_0) = 0$, et la α-ième colonne de la matrice $G_0^{1/2}$ est $\partial_\alpha \theta(y_0)$, pour tout $\alpha \in \{1, \dots, p\}$. Grâce au Théorème 4, cette application est alors bien définie. Notre second objectif est d'étudier les propriétés de cette application.

Notons que si, de plus, l'ensemble ω est borné, les espaces $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p)$, $\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)$, $\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)$, $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ et $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$, munis de leurs normes usuelles, sont des espaces de Banach. L'espace produit $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)\big)^p$, muni de la norme

$$\|\big(A, (B^i), (T_\alpha)\big)\|_{2,1,1,\overline{\omega}} = \|A\|_{2,\overline{\omega}} + \max_i \|B^i\|_{1,\overline{\omega}} + \max_\alpha \|T_\alpha\|_{1,\overline{\omega}} \tag{1.25}$$

est un espace de Banach et l'ensemble $X(\overline{\omega})$ devient un espace métrique lorsqu'il est équipé de la topologie induite. L'espace produit $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q$, muni de la norme

$$\|\big(\theta, (N^i)\big)\|_{3,2,\overline{\omega}} = \|\theta\|_{3,\overline{\omega}} + \max_i \|N^i\|_{2,\overline{\omega}} \tag{1.26}$$

est aussi un espace de Banach.

Avec toutes ces notations, nous pouvons maintenant énoncer le deuxième résultat principal de ce chapitre, qui généralise le Théorème 5.2. de [**27**] et le Théorème 3 de [**29**].

THÉORÈME 5. *Soit ω un ouvert borné, connexe et simplement connexe de $\mathbb{R}^p$, qui satisfait la propriété géodésique. On munit les espaces $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)\big)^p$ et $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q$ des normes $\|\cdot\|_{2,1,1,\overline{\omega}}$ et $\|\cdot\|_{3,2,\overline{\omega}}$ respectivement, et on munit l'espace $X(\overline{\omega})$ de la métrique induite par la norme $\|\cdot\|_{2,1,1,\overline{\omega}}$. Alors l'application*

$$\mathcal{F}\colon \big(A, (B^i), (T_\alpha)\big) \in X(\overline{\omega}) \to \big(\theta, (N^i)\big) \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q,$$

est continue. Elle est même localement lipschitzienne sur l'ensemble $X(\overline{\omega})$.

Pour établir ce résultat, nous considérons un point arbitraire $\hat{X} := \big(\hat{A}, (\hat{B}^i), (\hat{T}_\alpha)\big)$ de l'espace $X(\overline{\omega})$ et nous démontrons qu'il existe des constantes $c(\hat{X}) > 0$ et $\delta(\hat{X}) > 0$ telles que

$$\|\big(\theta, (N^i)\big) - \big(\tilde{\theta}, (\tilde{N}^i)\big)\|_{3,2,\overline{\omega}} \leq c(\hat{X})\|X - \tilde{X}\|_{2,1,1,\overline{\omega}} \tag{1.27}$$

pour tous $X, \tilde{X} \in B(\hat{X}; \delta(\hat{X}))$, où $\big(\theta, (N^i)\big) = \mathcal{F}(X)$ et $\big(\tilde{\theta}, (\tilde{N}^i)\big) = \mathcal{F}(\tilde{X})$. Ensuite, la stratégie consiste à écrire $\mathcal{F}$ comme une application composée et à montrer que chacune des applications composantes est une application localement lipschitzienne.

Le travail de ce chapitre fait l'objet des publications [**56**] et [**57**].

Présentation du Chapitre 3.

L'inégalité de Korn tridimensionnelle joue un rôle fondamental dans l'analyse mathématique du problème de l'élasticité tridimensionnelle linéarisé. Cette inégalité, qui peut être établie ou bien en coordonnées cartésiennes (voir, par exemple Duvaut et Lions [**31**]), ou bien directement en coordonnées curvilignes (voir Ciarlet [**14**]), affirme que l'on peut contrôler la norme du champs de déplacements d'un ouvert par la norme du tenseur linéarisé de changement de métrique de l'ouvert (pour un choix approprié de ces normes).

De même, dans le cas d'une coque, l'inégalité de Korn sur une surface est l'outil de base pour établir l'existence d'une solution pour le modèle linéaire de Koiter (voir Bernadou, Ciarlet et Miara [**10**]) et intervient aussi dans l'analyse asymptotique des coques élastiques (voir Ciarlet et Lods [**22**], [**23**] et Ciarlet, Lods, Miara [**24**]). Grâce à cette inégalité, on peut majorer la norme du champs de déplacements d'une surface par une expression qui dépend de la norme du tenseur linéarisé de métrique et de la norme du tenseur linéarisé de courbure de la surface. Cette inégalité a été établie pour la première fois par Bernadou et Ciarlet [**9**] ; une autre preuve, sous des hypothèses plus faibles, a été donnée par Blouza et Le Dret dans [**12**]. Elle peut aussi être établie comme une conséquence de l'inégalité de Korn tridimensionnelle en coordonnées curvilignes (voir Ciarlet et S. Mardare [**30**]) ou bien dans un cadre plus général, pour une surface compacte sans bord (voir S. Mardare [**44**]).

Ces travaux posent naturellement la question de savoir s'il y a une inégalité analogue dans le cas d'une courbe. Notre objectif dans le troisième chapitre, qui est un travail fait en collaboration avec Sorin Mardare, est de donner une réponse à cette question, en établissant des inégalités de Korn linéaires et non linéaires pour des courbes de $\mathbb{R}^3$.

Pour obtenir les inégalités linéaires, nous nous sommes placés dans un cadre similaire à celui de Blouza et Le Dret [**12**], dans le sens où les hypothèses de régularité sur la courbe de référence font intervenir les dérivées généralisées au lieu des dérivées classiques. Compte-tenu des analogies avec le cas tridimensionnel et le cas bidimensionnel, on s'attend à voir apparaître dans ces inégalités le tenseur linéarisé de changement de métrique, le tenseur linéarisé de changement de courbure, et le tenseur linéarisé de changement de torsion associés à un champ $\boldsymbol{\eta}$ de déplacement de la courbe. La première etape consiste donc à calculer explicitement l'expression de ces tenseurs, le résultat étant donné dans l'énoncé suivant :

LEMME 1. *Le tenseur linéarisé de changement de métrique associé au champ de vecteurs* $\boldsymbol{\eta}$ *est donné par*

$$a_{lin}(\boldsymbol{\eta}) = \boldsymbol{\eta}' \cdot \boldsymbol{T}, \tag{1.28}$$

le tenseur linéarisé de changement de courbure associé au champ de vecteurs $\boldsymbol{\eta}$ est donné par

$$\begin{aligned} k_{lin}(\boldsymbol{\eta}) &= \boldsymbol{\eta}'' \cdot \boldsymbol{N} - 2k\boldsymbol{\eta}' \cdot \boldsymbol{T} \\ &= (\boldsymbol{\eta}' \cdot \boldsymbol{N})' - k\boldsymbol{\eta}' \cdot \boldsymbol{T} - \tau\boldsymbol{\eta} \cdot \boldsymbol{B}, \end{aligned} \tag{1.29}$$

et le tenseur linéarisé de changement de torsion associé au champ de vecteurs $\boldsymbol{\eta}$ est donné par

$$\begin{aligned} \tau_{lin}(\boldsymbol{\eta}) &= -\tau\boldsymbol{\eta}' \cdot \boldsymbol{T} - \frac{\tau}{k}\boldsymbol{\eta}'' \cdot \boldsymbol{N} + (\frac{1}{k}\boldsymbol{\eta}''' - \frac{k'}{k^2}\boldsymbol{\eta}'' + k\boldsymbol{\eta}') \cdot \boldsymbol{B} \\ &= -\tau\boldsymbol{\eta}' \cdot \boldsymbol{T} + (k\boldsymbol{\eta}' - \frac{k'}{k^2}\boldsymbol{\eta}'') \cdot \boldsymbol{B} + \frac{1}{k}(\boldsymbol{\eta}'' \cdot \boldsymbol{B})'. \end{aligned} \tag{1.30}$$

Les notations utilisées ici sont les suivantes : $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ est le repère de Frenet associé à la courbe régulière (au sens du Chapitre 1) $\boldsymbol{\gamma} \in W^{3,\infty}(I;\mathbb{R}^3)$, k est la courbure de la courbe et τ est sa torsion. Il est à remarquer que, si $\boldsymbol{\eta} \in W^{1,p}(I;\mathbb{R}^3)$, $1 \leq p \leq +\infty$, satisfait $\boldsymbol{\eta}' \cdot \boldsymbol{N},\ \boldsymbol{\eta}'' \cdot \boldsymbol{B} \in W^{1,p}(I)$, alors le tenseur linéarisé de changement de métrique, le tenseur linéarisé de changement de courbure, et le tenseur linéarisé de changement de torsion associés au champs de déplacement $\boldsymbol{\eta}$ appartiennent à $L^p(I)$.

Avec ces notations, le premier résultat établi est une inégalité de Korn sans conditions au bord, qui s'énonce ainsi :

THÉORÈME 6. *Soit $\boldsymbol{\gamma} \in W^{3,\infty}(I;\mathbb{R}^3)$ une courbe régulière et soit $\boldsymbol{V}$ l'espace défini par*

$$\boldsymbol{V} := \{\boldsymbol{\eta} \in W^{1,p}(I;\mathbb{R}^3)\ ;\ \boldsymbol{\eta}' \cdot \boldsymbol{N} \in W^{1,p}(I) \text{ et } \boldsymbol{\eta}'' \cdot \boldsymbol{B} \in W^{1,p}(I)\,\}.$$

Alors il existe une constante $C > 0$ qui dépend seulement de la courbe $\boldsymbol{\gamma}$ telle que

$$\begin{aligned} \|\boldsymbol{\eta}\|_{W^{1,p}} &+ \|\boldsymbol{\eta}' \cdot \boldsymbol{N}\|_{W^{1,p}} + \|\boldsymbol{\eta}'' \cdot \boldsymbol{B}\|_{W^{1,p}} \\ &\leq C\Big(\|\boldsymbol{\eta}\|_{L^p} + \|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|k_{lin}(\boldsymbol{\eta})\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta})\|_{L^p}\Big) \end{aligned} \tag{1.31}$$

pour tout $\boldsymbol{\eta} \in \boldsymbol{V}$.

L'étape suivante consiste à démontrer un lemme des déplacements rigides infinitésimaux, sous des hypothèses faibles de régularité (pour un point de vue similaire dans le cas d'une surface, voir Blouza et Le Dret [**12**] et Anicic, Le Dret et Raoult [**3**]).

LEMME 2. *Soit $\boldsymbol{\eta} \in \boldsymbol{V}$ tel que $a_{lin}(\boldsymbol{\eta}) = k_{lin}(\boldsymbol{\eta}) = \tau_{lin}(\boldsymbol{\eta}) = 0$. Alors il existe deux vecteurs $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ tels que*

$$\boldsymbol{\eta}(s) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{\gamma}(s) \text{ pour tout } s \in I\,.$$

La preuve de ce lemme peut se faire ou bien directement à partir du Lemme 1, ou bien en utilisant une approche analogue à celle de Ciarlet et S. Mardare [**30**], qui consiste à l'obtenir comme une conséquence du lemme des déplacements rigides infinitésimaux tridimensionnel en coordonnées curvilignes.

L'inégalité de Korn linéaire pour les courbes est alors obtenue (par un raisonnement par contradiction) comme une conséquence de l'inégalité de Korn sans conditions au bord et du lemme des déplacements rigides infinitésimaux précédent :

THÉORÈME 7. *Soit* $\boldsymbol{\gamma} \in W^{3,\infty}(I;\mathbb{R}^3)$ *une courbe régulière et soit* $\boldsymbol{V_0}$ *l'espace défini par*

$$\boldsymbol{V_0} := \Big\{ \boldsymbol{\eta} \in W^{1,p}(I;\mathbb{R}^3)\ ;\ \boldsymbol{\eta}' \cdot \boldsymbol{N},\ \boldsymbol{\eta}'' \cdot \boldsymbol{B} \in W^{1,p}(I) \text{ et } \boldsymbol{\eta}(0) = \boldsymbol{0},\ (\boldsymbol{\eta}' \cdot \boldsymbol{N})(0) = (\boldsymbol{\eta}' \cdot \boldsymbol{B})(0) = (\boldsymbol{\eta}'' \cdot \boldsymbol{B})(0) = 0 \Big\}.$$

Alors il existe une constante $C > 0$ *qui dépend seulement de la courbe* $\boldsymbol{\gamma}$ *telle que*

$$\|\boldsymbol{\eta}\|_{W^{1,p}} + \|\boldsymbol{\eta}' \cdot \boldsymbol{N}\|_{W^{1,p}} + \|\boldsymbol{\eta}'' \cdot \boldsymbol{B}\|_{W^{1,p}} \leq C\Big(\|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|k_{lin}(\boldsymbol{\eta})\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta})\|_{L^p} \Big) \tag{1.32}$$

pour tout $\boldsymbol{\eta} \in \boldsymbol{V_0}$.

Dans la deuxième partie de ce chapitre, nous établissons des inégalités de Korn non linéaires pour les courbes. Le cadre est le suivant : $\boldsymbol{\gamma} \in W^{3,p}(I;\mathbb{R}^3)$, $1 \leq p \leq +\infty$, est considérée comme la courbe de référence, donc fixée, et $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I;\mathbb{R}^3)$ est considérée comme la courbe déformée, *i.e.,* une courbe définie par $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} + \boldsymbol{\eta}$, où $\boldsymbol{\eta}$ est un champ de déplacement. Notre but est d'obtenir pour des couples (p, m) appropriés, des estimations de la norme $W^{m,p}$ de $(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})$, qui peuvent être interprétées comme des versions non linéaires de l'inégalité de Korn linéaire obtenue dans la première partie. Une première inégalité de ce type est donnée dans le théorème suivant :

THÉORÈME 8. *Soit* $p \in [1, +\infty]$ *et soit* $\boldsymbol{\gamma} \in W^{3,p}(I;\mathbb{R}^3)$ *une courbe régulière telle que* $\boldsymbol{\gamma}(0) = 0$ *et* $\boldsymbol{X}(0) = I_3$. *Alors il existe une constante* $C_1(\boldsymbol{\gamma}) > 0$ *telle que*

$$\|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{1,p}(I;\mathbb{R}^3)} \leq C_1(\boldsymbol{\gamma}) \big\{ \|\tilde{a} - a\|_{L^p(I)} + \|\tilde{a}\tilde{k} - ak\|_{L^p(I)} + \|\tilde{a}\tilde{\tau} - a\tau\|_{L^p(I)} \big\} \tag{1.33}$$

pour toute courbe régulière $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I;\mathbb{R}^3)$ *qui satisfait les conditions* $\tilde{\boldsymbol{\gamma}}(0) = 0$ *et* $\tilde{\boldsymbol{X}}(0) = I_3$.

Si l'on veut obtenir une inégalité faisant intervenir les dérivées généralisées d'ordre deux de la différence $(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})$, on utilise alors le résultat suivant :

THÉORÈME 9. *Soit* $p \in [1, +\infty]$ *et soit* $\boldsymbol{\gamma} \in W^{3,p}(I;\mathbb{R}^3)$ *une courbe régulière telle que* $\boldsymbol{\gamma}(0) = 0$ *et* $\boldsymbol{X}(0) = I_3$. *Alors il existe une constante* $C_2(\boldsymbol{\gamma}) > 0$ *telle que*

$$\begin{aligned} \|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{2,p}(I;\mathbb{R}^3)} \leq\ & C_2(\boldsymbol{\gamma}) \big\{ \|\tilde{a} - a\|_{W^{1,p}(I)} + \|\tilde{a}^2\tilde{k} - a^2k\|_{L^p(I)} \\ & + \|\tilde{a}\tilde{k} - ak\|_{L^p(I)} + \|\tilde{a}\tilde{\tau} - a\tau\|_{L^p(I)} \big\} \end{aligned} \tag{1.34}$$

pour toute courbe régulière $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I;\mathbb{R}^3)$ *qui satisfait les conditions* $\tilde{\boldsymbol{\gamma}}(0) = 0$ *et* $\tilde{\boldsymbol{X}}(0) = I_3$.

Les inégalités précédentes peuvent être généralisées jusqu'à l'ordre trois pour les dérivées généralisées de $(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})$, comme le montre le théorème ci-dessous :

THÉORÈME 10. *Soit* $p \in [1, +\infty]$ *et soit* $\boldsymbol{\gamma} \in W^{3,p}(I;\mathbb{R}^3)$ *une courbe régulière telle que* $\boldsymbol{\gamma}(0) = 0$ *et* $\boldsymbol{X}(0) = I_3$. *Alors il existe une constante* $C_3(\boldsymbol{\gamma}) > 0$ *telle que*

$$\begin{aligned} \|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{3,p}(I;\mathbb{R}^3)} \leq\ & C_3(\boldsymbol{\gamma}) \big\{ \|\tilde{a} - a\|_{W^{2,p}(I)} + \|\tilde{a}^2\tilde{k} - a^2k\|_{W^{1,p}(I)} \\ & + \|\tilde{a}\tilde{k} - ak\|_{L^p(I)} + \|(\tilde{a}^2)'\tilde{k} - (a^2)'k\|_{L^p(I)} + \|\tilde{a}\tilde{\tau} - a\tau\|_{L^p(I)} \\ & + \|\tilde{a}^3\tilde{k}^2 - a^3k^2\|_{L^p(I)} + \|\tilde{a}^3\tilde{k}\tilde{\tau} - a^3k\tau\|_{L^p(I)} \big\} \end{aligned} \tag{1.35}$$

pour toute courbe régulière $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I;\mathbb{R}^3)$ *qui satisfait les conditions* $\tilde{\boldsymbol{\gamma}}(0) = 0$ *et* $\tilde{\boldsymbol{X}}(0) = I_3$.

Les démonstrations de ces théorèmes reposent sur des inégalités obtenues à partir des équations de Frenet vérifiées par les applications $\boldsymbol{\gamma}$ et $\tilde{\boldsymbol{\gamma}}$, en utilisant le lemme de Gronwall.

Les résultats que nous avons présentés dans ce chapitre font l'objet de l'article [**47**].

Résumé de l'Annexe.

Des résultats classiques de géométrie différentielle permettent de reconstruire des courbes dont on connait les fonctions de courbure et de torsion : les droites, qui ont une courbure nulle, les cercles, qui ont une courbure donnée par l'inverse du rayon (ces courbes ont une torsion nulle, car ce sont des courbes planes), les hélices circulaires, qui ont une courbure et une torsion constantes, etc.

Le résultat de reconstruction d'une courbe qui a été décrit dans le Théorème 1 nous a conduit à essayer de visualiser d'autres courbes, qui ont une courbure et une torsion données par des fonctions qui, cette fois-ci, ne sont plus dérivables au sens classique. Grâce à ce résultat, il est possible de considérer une fonction de courbure positive dans $H^1(I)$, une fonction de torsion dans $L^2(I)$, et d'obtenir une courbe c dans $H^3(I;\mathbb{R}^3)$ dont les courbures sont exactement les fonctions données.

Pour mettre en pratique cette idée, nous avons fait des simulations numériques à l'aide du Scilab, qui est un logiciel libre développé par l'INRIA (www.scilab.org). Nous avons d'abord testé notre méthode sur des exemples classiques, pour valider notre construction. Dans un deuxième temps, grâce à ce moyen, nous avons construit, par exemple, une courbe plane dont la courbure est une fonction du type “dents de scie”, ou bien une courbe gauche (*i.e.* plongée dans $\mathbb{R}^3$) dont la torsion est définie par $\tau(s) = |s|$, etc.

L'intêret de cette Annexe est de montrer que les résultats théoriques exposés dans la première partie de la thèse peuvent être aussi visualisés graphiquement.

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CHAPITRE 1

Sur la reconstruction d'une courbe isométriquement immergée dans un espace Euclidien

On the recovery of a curve isometrically immersed in $\mathbb{E}^n$

ABSTRACT. It is known from classical differential geometry that one can reconstruct a curve with $(n-1)$ prescribed curvature functions, if these functions can be differentiated a certain number of times in the usual sense and if the first $(n-2)$ functions are strictly positive. We establish here that this result still holds under the assumption that the curvature functions belong to some Sobolev spaces, by using the notion of derivative in the distributional sense. We also show that the mapping which associates with such prescribed curvature functions the reconstructed curve is of class $\mathcal{C}^\infty$.

1. Introduction

Physically, we can think of a curve as being obtained from a straight line by bending and twisting. After reflecting on this construction, we are led to conjecture that, roughly speaking, the curvature functions describe completely the behavior of the curve. This statement is true: we can find a proof of this classical result of differential geometry for example in Klingenberg [**4**], where the case of a curve isometrically immersed in the n-dimensional Euclidean space is treated, without specifying the regularity requirements for the initial data.

The same reconstruction problem can be posed for a surface or for an open set of $\mathbb{R}^n$ and there exist different methods to prove this result of differential geometry. Recently, motivated by some problems encountered in nonlinear elasticity, some extensions have been obtained for the case of Sobolev-type functions. More specifically, the classical result for an open set states that if the metric tensor is of class $\mathcal{C}^2$ and satisfies the Riemann compatibility conditions, then it is induced by an immersion (see for instance [**10**] for a "local" version, or [**2**] for the proof of the existence of a global immersion if in addition the open set is simply-connected). Then, it has been proved by C. Mardare in [**7**] that the same result holds under the assumption that the metric is of class $\mathcal{C}^1$; moreover, we can find in S. Mardare [**8**] an even stronger result, for the case when the metric is only of class $W^{1,\infty}_{loc}$.

Another interesting question concerns the regularity of the mapping that can be defined by associating with the prescribed data (metric and curvature) the reconstructed manifold in $\mathbb{R}^n$. In this direction, it has been established that this mapping is continuous for certain natural metrizable topologies, in the case of an open set of $\mathbb{R}^n$ by Ciarlet and Laurent in [**3**] and in the case of a surface (using a different method) by Ciarlet in [**1**].

The purpose of this paper is twofold. First we provide a proof of the existence and uniqueness of a curve immersed in $\mathbb{R}^n$, whose curvature functions are $(n-1)$ prescribed functions in $H^{n-2}(I;\mathbb{R})\times H^{n-3}(I;\mathbb{R})\times\ldots\times H^1(I;\mathbb{R})\times L^2(I;\mathbb{R})$; we emphasize that, instead of the classical framework of differential geometry, where all functions are considered to be indefinitely derivable, our setting will be that of distributions and we will always use the notion of derivative in the general sense. Second, we show that the mapping constructed in this fashion is of class $\mathcal{C}^\infty$. As corollaries, we derive the same results in the classical setting, where derivatives are considered in the usual sense.

The paper is organized as follows. In Section 2, we present some technical results which will be used in the sequel. In Section 3, we prove the existence and uniqueness (or uniqueness up to rigid motions, if the assumptions are weakened) of a curve with prescribed curvatures and in Section 4 we show that the mapping constructed in this manner is of class $\mathcal{C}^\infty$. Finally, in Section 5 we gather some additional commentaries about our problem.

The results of this paper have been announced in [**11**].

2. Preliminaries

To begin with, we introduce some conventions and notations that will be used throughout the article. For any $n \geq 1$, the n-dimensional Euclidean space $\mathbb{E}^n$ will be identified with $\mathbb{R}^n$ and will be endowed with the Euclidean norm defined by $|a| = \sqrt{< a, a >}$, where $< a, b >$ denotes the Euclidean inner product of $a, b \in \mathbb{R}^n$. The notations $\mathbb{M}^{n\times n}$ and $\mathbb{O}^n_+$, respectively designate the set of all real square matrices and of all proper orthogonal matrices of order n (a matrix Q is proper orthogonal if Q is orthogonal and $det\ Q = 1$). A mapping $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\varphi(x) = x_0 + Qx$, where $x_0 \in \mathbb{R}^n$ and $Q \in \mathbb{O}^n_+$ is called a proper isometry or rigid motion. We denote by

$$|A| := \sup_{v\in\mathbb{R}^n\setminus\{0\}} \frac{|Av|}{|v|}$$

the operator norm of a matrix $A \in \mathbb{M}^{n\times n}$ and by I_n the identity matrix of order n.

If X is a Hilbert space, we denote by $|\cdot|_X$ its norm induced by the inner product and by $\mathcal{D}'((0,T);X)$ the space of X-valued distributions. For all integer $m \geq 0$, the m-th derivative of $f \in \mathcal{D}'((0,T);X)$ is denoted $f^{(m)}$ and the first derivative is denoted f' or $f^{(1)}$. The function spaces used in this paper are denoted as follows: $L^p((0,T);X)$ for

$1 \leq p < +\infty$ is the space of all mesurable functions $f\colon (0,T) \to X$ such that

$$|f|_{L^p((0,T);X)} := \Big(\int_0^T |f(t)|_X^p dt \Big)^{\frac{1}{p}} < +\infty$$

and

$$H^m((0,T);X) := \{v \in L^2((0,T);X);\ v^{(k)} \in L^2((0,T);X)\ \forall k \leq m\}.$$

Let $H^0((0,T);X) := L^2((0,T);X)$. For all integer $m \geq 0$, the space $H^m((0,T);X)$ endowed with the inner product

$$< u, v >_{H^m((0,T);X)} := \sum_{k=0}^{m} \int_0^T < u^{(k)}(t), v^{(k)}(t) >_X dt$$

is a Hilbert space.

In the sequel, we present three lemmas that will be key ingredients in the proof of our main results (Theorems 3.1 and 4.1). The first lemma establishes the existence and uniqueness of the solution of a differential system and the second and third lemmas are about mappings between Sobolev spaces. In what follows, the derivatives are to be understood in the distributional sense and classes of functions in $H^1((0,T);\mathbb{M}^{n\times n})$ are identified with their continuous representative, as allowed by the Sobolev imbedding theorem. In particular, it makes sense to consider $Y(0)$ in the system below, since $Y \in \mathcal{C}^0([0,T];\mathbb{M}^{n\times n})$.

LEMMA 2.1. *Consider the system of differential equations:*

$$\begin{aligned} &Y'(t) = A(t)Y(t) + B(t) \text{ for almost all } t \text{ in } (0,T), \\ &Y(0) = Y_0, \end{aligned} \tag{2.1}$$

where A and B belong to the space $L^2((0,T);\mathbb{M}^{n\times n})$ and Y_0 is a matrix of $\mathbb{M}^{n\times n}$.

Then there exists a unique solution $Y \in H^1((0,T);\mathbb{M}^{n\times n})$ to this system.

Proof: It is a direct consequence of Theorem 4.1 and Remark 4.3 of [**5**]. □

LEMMA 2.2. *Let $k \geq 0$ and $m \geq 1$ be two integers such that $k \leq m$. Then the mapping*

$$(f,g) \in H^k((0,T);\mathbb{R}) \times H^m((0,T);\mathbb{R}^n) \to (fg) \in H^k((0,T);\mathbb{R}^n)$$

is of class $\mathcal{C}^\infty$.

Proof: The proof is straightforward: this application being bilinear, we only have to prove its continuity, in order to prove that it is $\mathcal{C}^\infty$. To do this, we distinguish two situations: $1 \leq k \leq m$ and $0 = k < m$ and then use the Sobolev imbeddings and their consequence that $H^k((0,T);\mathbb{R}^n)$ is a Banach algebra for $k \geq 1$. □

LEMMA 2.3. *Let k be a positive integer; then the mapping*

$$f \in H^k((0,T);\mathbb{R}^n) \to g \in H^{k+1}((0,T);\mathbb{R}^n),$$

where $g(t) = \int_0^t f(s)ds$, is of class $\mathcal{C}^\infty$.

Proof: The continuity of this mapping is a classical result (see, for example, [**6**]). Since this mapping is also linear, it is of class $\mathcal{C}^\infty$. □

For the sake of completeness, we state the implicit function theorem in the functional setting of Banach spaces.

THEOREM 2.1. *(implicit function theorem) Let there be given three Banach spaces X_1, X_2 and Y, an open subset Ω of the space $X_1 \times X_2$ containing a point (a_1, a_2), and a mapping $\varphi\colon \Omega \subset X_1 \times X_2 \to Y$ satisfying*

$$\varphi \in \mathcal{C}^1(\Omega, Y),\ \partial_2\varphi(a_1, a_2) \in Isom(X_2, Y).$$

Let $\varphi(a_1, a_2) = b \in Y$. Then there exist open subsets O_1 and O_2 of the spaces X_1 and X_2 respectively, such that $(a_1, a_2) \in O_1 \times O_2 \subset \Omega$, and there exists an implicit function $f\colon O_1 \subset X_1 \to O_2 \subset X_2$ such that

$$\{(x_1, x_2) \in O_1 \times O_2, \varphi(x_1, x_2) = b\} = \{(x_1, x_2) \in O_1 \times O_2, x_2 = f(x_1)\},$$

$f \in \mathcal{C}^1(O_1, X_2)$ and $f'(x_1) = -\{\partial_2\varphi(x_1, f(x_1))\}^{-1}\partial_1\varphi(x_1, f(x_1))$ for all $x_1 \in O_1$; moreover, f is unique, provided that O_1 is taken sufficiently small.

If in addition the mapping $\varphi\colon \Omega \subset X_1 \times X_2 \to Y$ is of class $\mathcal{C}^m$, $m \geq 2$, the implicit function $f\colon O_1 \to X_2$ is also of class $\mathcal{C}^m$.

Proof: This is a well-known result (see for example [**9**, Chap. 3, Sect. 8]). □

3. Existence and uniqueness of the curve c

To begin with, we present the geometrical framework of our problem. The integer $n \geq 3$ is fixed throughout this section. If $v_1, \ldots, v_k$, $k \geq 1$, are vectors in $\mathbb{R}^n$, we denote by $sp\{v_1, \ldots, v_k\}$ the vector space spanned by these vectors.

Let $I = (0, T)$ be a bounded interval of $\mathbb{R}$. Let $c\colon I \to \mathbb{R}^n$ be a *regular curve* of class $\mathcal{C}^{n-1}$ over $(0, T)$, *i.e.*, the vectors $c^{(1)}(t), c^{(2)}(t), \ldots, c^{(n-1)}(t)$ are linearly independent for all $t \in I$. In this setting, one can show (see, for example, [**4**]) that there exists a unique Frenet frame associated with this curve, denoted $\{e_1(t), \ldots, e_n(t)\}$, which is a family of vector fields along the curve c such that

$$< e_i(t), e_j(t) >= \delta_{ij},\ \forall i, j \in \{1, \ldots, n\},\ \forall t \in I,$$

$$sp\{e_1(t), \ldots e_k(t)\} = sp\{c^{(1)}(t), \ldots, c^{(k)}(t)\},\ \forall k \in \{1, \ldots, n-1\},\ \forall t \in I,$$

the two bases having the same orientation (*i.e.,* the matrix of change of basis has positive determinant) and such that $\{e_1(t), \ldots e_n(t)\}$ is positively oriented for all $t \in I$ (*i.e.,* it has the same orientation as the natural ordered basis $\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$).

Then Frenet formulas read:

$$e_i'(t) = \sum_{j=1}^{n} a_{ij}(t) e_j(t),\ \forall i \in \{1, \ldots, n\},$$

where the functions $a_{ij}: I \to \mathbb{R}$ satisfy

$$a_{ij}(t) + a_{ji}(t) = 0, \ \forall i, j \in \{1, \dots, n\}, \ \forall t \in I,$$

$$a_{ij}(t) = 0, \ \forall j \geq i+2, \ \forall t \in I.$$

The curvature functions of c at the point $t \in I$ are then defined by:

$$k_i(t) := \frac{a_{i,i+1}(t)}{|c'(t)|}, \ \forall i \in \{1, \dots, n-1\} \tag{3.1}$$

or equivalently, by means of the Frenet formulas, as

$$k_i(t) = \frac{< e_{i+1}(t), e_i'(t) >}{|c'(t)|}. \tag{3.2}$$

We recall that the curvature functions are invariant under the action of rigid motions, *i.e.*, if $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ is a proper isometry and $\tilde{c} = \varphi \circ c$, then $\tilde{k}_i(t) = k_i(t)$ for all $i \in \{1, \dots, n-1\}$ and for all $t \in I$ (with self-explanatory notations).

The objective of this section is to prove the existence and uniqueness up to rigid motions of a curve isometrically immersed in the n-dimensional space and with prescribed curvature functions. While this result is classical if the curvature functions are regular enough (see the assumptions of Corollary 3.2), it will be assumed here that they only belong to some specific Sobolev spaces.

THEOREM 3.1. *(existence and uniqueness) Let* $(F_1, \dots, F_{n-1}) \in H^{n-2}(I;\mathbb{R}) \times \dots \times H^1(I;\mathbb{R}) \times L^2(I;\mathbb{R})$ *be such that* $F_1(t) > 0, \dots, F_{n-2}(t) > 0$ *for all* $t \in I$. *Then:*

(a) There exists a regular curve $c \in H^n(I;\mathbb{R}^n)$ *such that* $|c'(t)| = 1$ *for all* $t \in I$ *and its curvature functions are* $F_1, \dots, F_{n-1}$, *i.e.,* $k_i(t) = F_i(t)$ *for all* $i \in \{1, \dots, n-1\}$ *and* $t \in I$.

(b) If c *and* $\tilde{c}$ *are two curves satisfying the conditions of part (a), then there exists a rigid motion* $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\tilde{c} = \varphi \circ c$.

(c) If $x_0 \in \mathbb{R}^n$ *is fixed, then there exists a unique curve* c *satisfying the properties of part (a) and such that* $c(0) = x_0$ *and its Frenet frame at the origin is given by* $e_1(0) = (1, 0, \dots, 0), \dots, e_n(0) = (0, 0, \dots, 1)$.

Proof : The proof is broken into six steps, in order to provide a more clear presentation. Throughout the proof, the elements of the space $H^1(I;\mathbb{R}^n)$, which are classes of functions with respect to the equality almost everywhere, are identified with their continuous representative (as allowed by the Sobolev imbedding theorem).

(i) We show that there exists a unique solution $(e_1, \dots, e_n) \in H^1(I;\mathbb{R}^n) \times \dots \times H^1(I;\mathbb{R}^n)$ *to the Cauchy problem:*

$$e_i'(t) = - F_{i-1}(t) e_{i-1}(t) + F_i(t) e_{i+1}(t) \text{ a.e. in } I, \ i \in \{1, \dots, n\}, \tag{3.3}$$

$$e_1(0) = (1, 0, \dots, 0), \dots, e_n(0) = (0, 0, \dots, 1), \tag{3.4}$$

where $F_0 = F_n := 0, e_0 = e_{n+1} := 0$.

We rewrite the system of ordinary differential equations as a matrix equation, *viz.*,

$$\begin{pmatrix} e_1^1 & \dots & e_1^n \\ e_2^1 & \dots & e_2^n \\ \vdots & & \\ e_n^1 & \dots & e_n^n \end{pmatrix}' = \begin{pmatrix} 0 & F_1 & \dots & 0 \\ -F_1 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & -F_{n-1} & 0 \end{pmatrix} \begin{pmatrix} e_1^1 & \dots & e_1^n \\ e_2^1 & \dots & e_2^n \\ \vdots & & \\ e_n^1 & \dots & e_n^n \end{pmatrix}.$$

The Cauchy problem given by the relations (3.3) and (3.4) can be written as a linear system of ordinary differential equations of the form

$$\begin{aligned} Y' &= AY \text{ a.e. in } I, \\ Y(0) &= I_n. \end{aligned}$$

By assumption, the mapping $A\colon I \to \mathbb{M}^{n\times n}$ belongs to the space $L^2(I;\mathbb{M}^{n\times n})$, since the functions $F_1,\dots,F_{n-1}$ are at least in $L^2(I;\mathbb{R})$. Then Lemma 2.1 applied to the above system shows that it possesses a unique solution $Y \in H^1(I;\mathbb{M}^{n\times n})$.

Thus, we have obtained the existence of a unique family $\{e_1,\dots,e_n\}$ that satisfies the Cauchy problem given by (3.3) and (3.4).

(*ii*) *We show that this family is orthonormal at all* $t \in I$.

Let $\alpha_{ij}(t) :=< e_i(t), e_j(t) >$ for all $t \in I$ and for all $i,j \in \{1,\dots,n\}$. Then the system (3.3) shows that:

$$\begin{aligned} \alpha_{ij}'(t) &=< e_i(t)', e_j(t) > + < e_i(t), e_j'(t) > \\ &=< -F_{i-1}(t)e_{i-1}(t) + F_i(t)e_{i+1}(t), e_j(t) > + < e_i(t), -F_{j-1}(t)e_{j-1}(t) + F_j(t)e_{j+1}(t) >, \end{aligned}$$

with the convention made in step (i) that $F_0 = F_n = 0$ and $e_0 = e_{n+1} = 0$. This implies on the one hand that the functions $\alpha_{ij}\colon I \to \mathbb{R}$ satisfy the following system:

$$\alpha_{ij}'(t) = -F_{i-1}(t)\alpha_{i-1,j}(t) + F_i(t)\alpha_{i+1,j}(t) - F_{j-1}(t)\alpha_{i,j-1}(t) + F_j(t)\alpha_{i,j+1}(t),$$

$$\alpha_{ij}(0) = \delta_{ij}.$$

On the other hand, one can see that the functions $\beta\colon I \to \mathbb{R}$, $\forall i,j \in \{1,\dots,n\}$, defined by $\beta_{ij}(t) = \delta_{ij}$ for all $t \in I$ satisfy the same system (to this end, one can distinguish the following three possible situations: $i-1=j$ or $i+1=j$ or $i-1 \neq j$ and $i+1 \neq j$). Consequently, the uniqueness of the solution to the above system implies that $\alpha_{ij}(t) = \delta_{ij}$ for all $t \in I$ and for all $i,j \in \{1,\dots,n\}$. In other words, $< e_i(t), e_j(t) >= \delta_{ij}$ for all $t \in I$ and for all $i,j \in \{1,\dots,n\}$. Hence the family $\{e_1(t),\dots,e_n(t)\}$ is orthonormal for all $t \in I$.

(*iii*) *We show a regularity result for the solution* $\{e_1,\dots,e_n\}$ *to the system* (3.3)-(3.4).

Since for all $m > \frac{1}{2}$, $H^m(I;\mathbb{R})$ is an algebra, it follows that the product (fe) between $f \in H^m(I;\mathbb{R})$ and $e \in H^m(I;\mathbb{R}^n)$ belongs to $H^m(I;\mathbb{R}^n)$. We shall use this fact in the proof below.

We first infer from (3.3) that:

$$e'_m = -F_{m-1}e_{m-1} + F_m e_{m+1} \text{ for } m = 1, 2, \dots, n-2.$$

Since $e_{m-1}, e_{m+1} \in H^1(I;\mathbb{R}^n)$ and $F_{m-1}, F_m \in H^1(I;\mathbb{R})$, we obtain that $e_m \in H^2(I;\mathbb{R}^n)$ for all $m \in \{1, 2, \dots, n-2\}$.

Next, we again infer from (3.3) that:

$$e'_m = -F_{m-1}e_{m-1} + F_m e_{m+1} \text{ for } m = 1, 2, \dots, n-3.$$

Since $e_{m-1}, e_{m+1} \in H^2(I;\mathbb{R}^n)$ and F_{m-1} and F_m are at least in $H^2(I;\mathbb{R})$, we deduce that $e_{n-3} \in H^3(I;\mathbb{R})$ for all $m \in \{1, 2, \dots, n-3\}$.

We continue the same argument, using each time relation (3.3) for $m = 1, 2, \dots, n-k$ and $k = 2, 3, \dots, n-1$. In this fashion, we eventually obtain that

$$e'_1 = F_1 e_2.$$

Noting that $F_1 \in H^{n-2}(I;\mathbb{R})$ and $e_2 \in H^{n-2}(I;\mathbb{R}^n)$, we finally deduce that $e_1 \in H^{n-1}(I;\mathbb{R}^n)$.

In conclusion, we have shown that $e_i \in H^{n-i}(I;\mathbb{R}^n)$ and $e_n \in H^1(I;\mathbb{R}^n)$, where $i = 1, 2, \dots, n-1$.

(iv) We establish the existence of a curve satisfying the part (a) of the theorem.

Define the function $c\colon [0,T] \to \mathbb{R}^n$ by $c(t) := \int_0^t e_1(s)ds + x_0$, $0 \le t \le T$. This integral is well-defined since we know from step *(iii)* that $e_1 \in H^{n-1}(I;\mathbb{R}^n)$, which also implies that $c \in H^n(I;\mathbb{R}^n)$.

This allows to compute the successive derivatives, by proceeding recursively. For $k = 1$ we have:

$$c^{(1)} = e_1$$

Assume now that for an arbitrary $k \le n-1$, the k-th derivative is given by:

$$c^{(k)} = \sum_{i=1}^{k} a_i^k e_i, \tag{3.5}$$

where $a_i^k \in H^{n-k}(I;\mathbb{R}^n)$. We deduce that:

$$\begin{aligned} c^{(k+1)} &= \sum_{i=1}^{k} (a_i^k)' e_i + \sum_{i=1}^{k} a_i^k(-F_{i-1}e_{i-1} + F_i e_{i+1}) \\ &= \sum_{i=1}^{k+1} ((a_i^k)' - a_{i+1}^k F_i + a_{i-1}^k F_{i-1}) e_i, \end{aligned}$$

with the convention that $a_0^k = 0$ and $a_k^i = 0$ for all $i > k$. Hence

$$c^{(k+1)} = \sum_{i=1}^{k+1} a_i^{k+1} e_i,$$

where the functions $a_i^{k+1} := (a_i^k)' - a_{i+1}^k F_i + a_{i-1}^k F_{i-1}$ belong to the space $H^{n-k-1}(I;\mathbb{R}^n)$ for all $i \in \{1,\dots,k+1\}$. Consequently, relations (3.5) hold for all $k \in \{1,\dots,n-1\}$.

From these relations, we first infer that $|c'(t)| = |e_1(t)| = 1$ for all $t \in I$. Then, if $A^k(t)$ denotes the lower triangular matrix in $\mathbb{M}^{n\times n}$ whose entries are a_i^k if $i \le k$, these relations can be written as the following matrix equation

$$\begin{pmatrix} c^{(1)}(t) \\ c^{(2)}(t) \\ \vdots \\ c^{(k)}(t) \end{pmatrix} = A^k(t) \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_k(t) \end{pmatrix}.$$

Note that the matrices $A^k(t)$, $k \le n-1$, have only positive terms on the principal diagonal, *viz.*, 1, $F_1(t) > 0$, $F_1(t)F_2(t) > 0$,.., $F_1(t)F_2(t)\dots F_{k-1}(t) > 0$ for all $t \in I$. This shows that these matrices are invertible and their determinant is strictly positive. Then, since the matrix $A^{n-1}(t)$ is invertible, the vectors $\{c^{(1)}(t),\dots,c^{(n-1)}(t)\}$ are linearly independent for all $t \in I$, so that the curve c is regular.

We now show that the orthonormal family of vectors $\{e_1(t),\dots,e_n(t)\}$ constitutes the Frenet frame of the curve c at $t \in I$. First, since the matrices $A^k(t)$, $k \le n-1$ are invertible and with strictly positive determinant, it follows that

(3.6) $$sp\{e_1(t),\dots,e_k(t)\} = sp\{c^{(1)}(t),\dots,c^{(k)}(t)\},\ \forall k \in \{1,\dots,n-1\},\ \forall t \in I$$

and the two bases have the same orientation.

Let $\Delta(t)$ be the determinant of the $n \times n$ matrix whose k-th row is the vector $e_k(t)$, $1 \le k \le n$. We notice that $\Delta\colon I \to \mathbb{R}$ is a continuous function, since $Y \in H^1(I;\mathbb{M}^{n\times n}) \subset \mathcal{C}^0(\bar{I};\mathbb{M}^{n\times n})$. Then, thanks to step (*ii*), the family $\{e_1(t),\dots,e_n(t)\}$ is orthonormal for all $t \in I$, hence in particular these vectors form a linearly independent system for each t. This implies on the one hand that $\Delta(t) \neq 0$, for all $t \in I$. On the other hand, by using relation (3.4), we obtain that $\Delta(0) = 1$. Consequently, $\Delta(t) > 0$ for all $t \in I$, which means that the basis $\{e_1(t),\dots,e_n(t)\}$ is positively oriented for all $t \in I$.

From all these relations, we conclude that $\{e_1(t),\dots,e_n(t)\}$ is the Frenet frame of the curve c. Consequently, its curvatures are given by (see relation (3.2))

$$\begin{aligned} k_i(t) =& \frac{< e_{i+1}(t), e_i'(t) >}{|c'(t)|} \\ =& < e_{i+1}(t), -F_{i-1}(t)e_{i-1}(t) + F_i(t)e_{i+1}(t) > \\ =& F_i(t), \end{aligned}$$

where we used relation (3.3) and step (*ii*). This establishes part (*a*) of Theorem 3.1.

(*v*) *We prove part (c) of Theorem 3.1.*

Let c and $\tilde{c}$ be two regular curves of class H^n over I, parametrized by their arc length (*i.e.*, $|c'(t)| = |\tilde{c}'(t)| = 1$ for all $t \in I$), such that $k_i = \tilde{k}_i$ for all $i \in \{1,\dots,n-1\}$. The

Frenet equations for the curves c and $\tilde{c}$ are respectively given by

$$e_i' = \sum_{j=1}^{n} a_{ij} e_j, \text{ a.e. in } I, \ \forall i \in \{1, \ldots, n\}$$

and

$$\tilde{e}_i' = \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{e}_j, \text{ a.e. in } I, \ \forall i \in \{1, \ldots, n\}.$$

Since $k_i = \tilde{k}_i$, formula (3.1) shows that $a_{ij} = \tilde{a}_{ij}$ a.e. in I and for all $i, j \in \{1, \ldots, n\}$. Noting that $e_i(0) = \tilde{e}_i(0)$ for all $i \in \{1, \ldots, n\}$ (thanks to relation (3.4)), Lemma 2.1 implies that $e_i = \tilde{e}_i$ for all $i \in \{1, \ldots, n\}$. In particular, $e_1 = \tilde{e}_1$, hence $c' = \tilde{c}'$. Since $c(0) = \tilde{c}(0)$, we finally obtain that $c = \tilde{c}$ in I.

(vi) We establish part (b) of Theorem 3.1.

Let c and $\tilde{c}$ be two regular curves parametrized by their arc length, such that $k_i = \tilde{k}_i$ for all $i \in \{1, \ldots, n-1\}$. Let $\{e_1(0), \ldots, e_n(0)\}$ be the Frenet frame of c at $c(0)$ and let $\{\tilde{e}_1(0), \ldots, \tilde{e}_n(0)\}$ be the Frenet frame of the curve $\tilde{c}$ at $\tilde{c}(0)$. Clearly, there exist a vector $a \in \mathbb{R}^n$ and a matrix $Q \in \mathbb{O}_+^n$, such that

$$\tilde{c}(0) = a + Qc(0) \tag{3.7}$$

and

$$\tilde{e}_i(0) = Qe_i(0), \text{ for all } i \in \{1, \ldots, n\}. \tag{3.8}$$

The Frenet equations for the curves c and $\tilde{c}$ respectively read

$$e_i' = \sum_{j=1}^{n} a_{ij} e_j, \ \forall i \in \{1, \ldots, n\} \tag{3.9}$$

and

$$\tilde{e}_i' = \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{e}_j, \ \forall i \in \{1, \ldots, n\}. \tag{3.10}$$

Since

$$a_{ij} = \begin{cases} -k_i & \text{if } i = j-1, \\ k_i & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{a}_{ij} = \begin{cases} -\tilde{k}_i & \text{if } i = j-1, \\ \tilde{k}_i & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

we deduce that $a_{ij} = \tilde{a}_{ij}$. Consequently,

$$Qe_i' = \sum_{j=1}^{n} a_{ij} Qe_j \Rightarrow (Qe_i)' = \sum_{j=1}^{n} a_{ij}(Qe_j).$$

This last relation, combined with the relations (3.8) and (3.10) show that $(\tilde{e}_i)$ and (Qe_i) satisfy the same Cauchy problem. Then the uniqueness result of Lemma 2.1 implies that

$$Qe_i = \tilde{e}_i, \ \forall i \in \{1, \dots, n\}.$$

In particular, the first relation (*i.e.,* corresponding to $i = 1$ in the above relation) shows that

$$(Qc)' = Qc' = Qe_1 = \tilde{e}_1 = (\tilde{c})'.$$

Therefore, there exists a vector $V \in \mathbb{R}^n$ such that $Qc(t) = \tilde{c}(t) + V$ for all $t \in I$. Then relation (3.7) shows that $V = -a$, so that $\tilde{c}(t) = Qc(t) + a$ for all $t \in I$. This means that $\tilde{c} = \varphi \circ c$, where $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\varphi(x) = a + Qx$ is a rigid motion. □

We now restate the above result in the special, but most commonly encountered in practice, case of dimension 3.

COROLLARY 3.1. *Let* $k \in H^1(I; \mathbb{R}_+^*)$ *and* $\tau \in L^2(I; \mathbb{R})$.

(a) There exists a curve $c \in H^3(I; \mathbb{R}^3)$*, unique up to rigid motions in* $\mathbb{R}^3$*, parametrized by its arc length, such that* k *and* τ *are its curvature and torsion functions.*

(b) If $x_0 \in \mathbb{R}^3$ *is fixed, then there exists a unique curve* c *satisfying the properties of part (a) and such that* $c(0) = x_0$ *and its Frenet frame at the origin is given by* $e_1(0) = (1, 0, 0)$, $e_2(0) = (0, 1, 0)$, $e_3(0) = (0, 0, 1)$. □

We also can use Theorem 3.1 to obtain the analogous statement for curves of class $\mathcal{C}^n$. More specifically, the following result holds:

COROLLARY 3.2. *Let* $(F_1, \dots, F_{n-1}) \in \mathcal{C}^{n-2}(I; \mathbb{R}) \times \dots \times \mathcal{C}^1(I; \mathbb{R}) \times \mathcal{C}^0(I; \mathbb{R})$ *be such that* $F_1(t) > 0, \dots, F_{n-2}(t) > 0$ *for all* $t \in I$*. Then:*

(a) There exists a regular curve $c \in \mathcal{C}^n(I; \mathbb{R}^n)$ *such that* $|c'(t)| = 1$ *for all* $t \in I$ *and its curvature functions are* $F_1, \dots, F_{n-1}$*, i.e.,* $k_i(t) = F_i(t)$ *for all* $i \in \{1, \dots, n-1\}$ *and* $t \in I$.

(b) If c *and* $\tilde{c}$ *are two curves satisfying the conditions of part (a), then there exists a rigid motion* $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\tilde{c} = \varphi \circ c$.

(c) If $x_0 \in \mathbb{R}^n$ *is fixed, then there exists a unique curve* c *satisfying the conditions of part (a) and such that* $c(0) = x_0$ *and its Frenet frame at the origin is given by* $e_1(0) = (1, 0, \dots, 0), \dots, e_n(0) = (0, 0, \dots, 1)$.

Sketch of proof: In order to prove this corollary, we can use two different approaches: we can either carry out the same computations as in the proof of Theorem 3.1 and use a classical result of existence and uniqueness for ordinary differential equations (instead of Lemma 2.1), or we can derive these results from Theorem 3.1, by using in particular the Sobolev imbedding $H^m(I; \mathbb{R}^n) \subset \mathcal{C}^{m-1}(I; \mathbb{R}^n)$. For this second approach, we also need some further analysis which makes the proof rather lengthy. By contrast, the first approach leads to the result in a simpler way. □

4. Regularity of a curve as a mapping of its curvatures

In order to simplify the presentation, we introduce the following notations:

$$\mathcal{H}(I;\mathbb{R}) := \prod_{k=0}^{n-2} H^{n-k-2}(I;\mathbb{R}),$$

$$\mathcal{H}(I;\mathbb{R})_> := \{(F_1,\dots,F_{n-1}) \in \mathcal{H}(I;\mathbb{R});\ F_i(t) > 0,\ \forall t \in I,\ \forall i \in \{1,\dots,n-2\}\},$$

and

$$\mathbf{H}(I;\mathbb{R}^n) := \Big(\prod_{k=1}^{n-1} H^{n-k}(I;\mathbb{R}^n)\Big) \times H^1(I;\mathbb{R}^n).$$

The set $\mathcal{H}(I;\mathbb{R})_>$ is open in the Hilbert space $\mathcal{H}(I;\mathbb{R})$, endowed with the inner product

$$< (F_1,\dots,F_{n-1}),(G_1,\dots,G_{n-1}) >_{\mathcal{H}(I;\mathbb{R})} := \sum_{k=0}^{n-2} < F_k, G_k >_{H^{n-k-2}(I;\mathbb{R})} .$$

The space $\mathbf{H}(I;\mathbb{R}^n)$, endowed with the inner product

$$< (e_1,\dots,e_n),(f_1,\dots,f_n) >_{\mathbf{H}(I;\mathbb{R}^n)} := \sum_{k=1}^{n-1} < e_k, f_k >_{H^{n-k}(I;\mathbb{R}^n)} + < e_n, f_n >_{H^1(I;\mathbb{R}^n)},$$

is a Hilbert space.

In the previous section, under some appropriate assumptions, we have proved the existence and uniqueness of a curve c with prescribed curvature functions. More specifically, Theorem 3.1 asserts that with each $(n-1)$-uple of functions $(F_1,\dots,F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_>$, one can associate a unique curve $c \in H^n(I;\mathbb{R}^n)$ parametrized by its arc length, satisfying some *ad hoc* "initial" conditions, and whose curvatures are the given functions $(F_1,\dots,F_{n-1})$. In this way, we have constructed a mapping

$$\mathcal{F}\colon (F_1,\dots,F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_> \to c \in H^n(I;\mathbb{R}^n).$$

The aim of this section is to study the regularity properties of this mapping. Our main result is the following:

THEOREM 4.1. *Define the mapping*

$$\mathcal{F}\colon (F_1,\dots,F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_> \to c \in H^n(I;\mathbb{R}^n),$$

where the curve c is defined in part (c) of Theorem 3.1. Then the mapping $\mathcal{F}$ is of class $\mathcal{C}^\infty$.

Proof : For clarity, we break the proof in three steps: in step (i) we construct a function f (related to $\mathcal{F}$) and prove that it is of class $\mathcal{C}^\infty$, in step (ii) we apply the implicit function theorem 2.1 to this function, and in step (iii) we conclude the proof.

(i) *Let* $e_1^0 = (1,0,\dots,0)$, $e_2^0 = (0,1,\dots,0),\dots,e_n^0 = (0,0,\dots,1)$. *Define the function*

$$f\colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to \mathbf{H}(I;\mathbb{R}^n)$$

by

$$f\Big((F_1,\dots,F_{n-1}),(e_1,\dots,e_n)\Big) = (w^1,\dots,w^n)$$

where, for all $t \in I$,

$$\begin{aligned}
w^1(t) &:= -e_1(t) + e_1^0 + \int_0^t F_1(s)e_2(s)ds,\\
w^2(t) &:= -e_2(t) + e_2^0 + \int_0^t (F_2(s)e_3(s) - F_1(s)e_1(s))ds,\\
&\vdots\\
w^n(t) &:= -e_n(t) + e_n^0 + \int_0^t (-F_{n-1}(s)e_{n-1}(s))ds.
\end{aligned} \tag{4.1}$$

Then the function f is well-defined and of class $\mathcal{C}^\infty$.

To see this, it suffices to prove that each component of f is well-defined and of class $\mathcal{C}^\infty$. More specifically, we have to show that, for each $k \in \{1,2,\dots,n\}$, the function defined by

$$f^k \colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to H^{n-k}(I;\mathbb{R}^n)$$

$$f^k\Big((F_1,\dots,F_{n-1}),(e_1,\dots,e_n)\Big) = w^k$$

where

$$w^k(t) = -e_k(t) + e_k^0 + \int_0^t (F_k(s)e_{k+1}(s) - F_{k-1}(s)e_{k-1}(s))ds, \text{ for all } t \in I,$$

is well-defined and of class $\mathcal{C}^\infty$. Note that we use the convention that $F_0 = 0$ and $e_0 = e_{n+1} = 0$.
Since $F_k \in H^{n-k-1}(I;\mathbb{R})$ and $e_{k+1} \in H^{n-k-1}(I;\mathbb{R}^n)$, we deduce that

$$F_k e_{k+1} \in H^{n-k-1}(I;\mathbb{R}^n).$$

In the same manner, we infer from the relations $F_{k-1} \in H^{n-k}(I;\mathbb{R})$ and $e_{k-1} \in H^{n-k+1}(I;\mathbb{R}^n)$ that

$$F_{k-1}e_{k-1} \in H^{n-k}(I;\mathbb{R}^n).$$

Consequently, the last two relations imply that

$$F_k e_{k+1} - F_{k-1}e_{k-1} \in H^{n-k-1}(I;\mathbb{R}^n).$$

We infer from Lemma 2.3 and from the relations $e_k \in H^{n-k}(I;\mathbb{R}^n)$ that $w^k \in H^{n-k}(I;\mathbb{R}^n)$. This shows that the functions f^k are well-defined for all $k \in \{1,\dots,n\}$.

We next show that the mapping f^k is of class $\mathcal{C}^\infty$. We start by writing f^k as a sum of four terms, *viz.*, $f^k = f_1^k + f_2^k + f_3^k + f_4^k$, where:

$$\begin{aligned} f_1^k\Big((F_1,\dots,F_{n-1}),(e_1,\dots,e_n)\Big) &= -\, e_k,\\ f_2^k\Big((F_1,\dots,F_{n-1}),(e_1,\dots,e_n)\Big) &= e_k^0,\\ f_3^k\Big((F_1,\dots,F_{n-1}),(e_1,\dots,e_n)\Big) &= \{t \mapsto \int_0^t F_k(s)e_{k+1}(s)ds\},\\ f_4^k\Big((F_1,\dots,F_{n-1}),(e_1,\dots,e_n)\Big) &= \{t \mapsto -\int_0^t F_{k-1}(s)e_{k-1}(s)ds\}. \end{aligned}$$

The mappings f_k^1 and f_k^2 are of class $\mathcal{C}^\infty$ since the first one is a projection and the second one is constant.

In order to prove that f_3^k is of class $\mathcal{C}^\infty$, note that the projection

$$\mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to H^{n-k-1}(I;\mathbb{R}) \times H^{n-k-1}(I;\mathbb{R}^n)$$

$$\Big((F_1,\dots,F_{n-1}),(e_1,\dots,e_n)\Big) \mapsto (F_k, e_{k+1})$$

is of class $\mathcal{C}^\infty$. Then, applying succesively Lemma 2.2 and Lemma 2.3, one can show that the mapping

$$H^{n-k-1}(I;\mathbb{R}) \times H^{n-k-1}(I;\mathbb{R}^n) \to H^{n-k-1}(I;\mathbb{R}^n) \to H^{n-k}(I;\mathbb{R}^n),$$

defined by

$$(F_k, e_{k+1}) \mapsto F_k e_{k+1} \mapsto \{t \mapsto \int_0^t F_k(s)e_{k+1}(s)ds\},$$

is of class $\mathcal{C}^\infty$. The mapping f_3^k, being the composition of two mappings of class $\mathcal{C}^\infty$, is thus also of class $\mathcal{C}^\infty$.

The same argument can be used to show that the function f_4^k is of class $\mathcal{C}^\infty$. We write f_4^k as a composite mapping made of a projection and a mapping that is of class $\mathcal{C}^\infty$, as shown by applying successively Lemmas 2.2 and 2.3. Hence the mapping f_4^k is also of class $\mathcal{C}^\infty$.

The mapping f^k being the sum of four applications of class $\mathcal{C}^\infty$, is of class $\mathcal{C}^\infty$ too. Letting k vary in the set $\{1,\dots,n\}$ shows that the mapping f is of class $\mathcal{C}^\infty$, as claimed.

(*ii*) *The implicit function theorem can be applied to the function f defined in step* (*i*).

The functional framework is that presented in Theorem 2.1. Let $X_1 = \mathcal{H}(I;\mathbb{R})_>$, let $X_2 = \mathbf{H}(I;\mathbb{R}^n)$, and let

$$\tilde{p} := \Big((\tilde{F}_1,\dots,\tilde{F}_{n-1}),(\tilde{e}_1,\dots,\tilde{e}_n)\Big) \in \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n)$$

such that $f(\tilde{p}) = 0$. Note that such an element $\tilde{p}$ always exists, as showed in steps (*i*) and (*iii*) of the proof of Theorem 3.1. More specifically, it was shown there that for any

$(\tilde{F}_1, \dots, \tilde{F}_{n-1}) \in \mathcal{H}(I;\mathbb{R})_>$, there exists a unique n-tuple $(\tilde{e}_1, \dots, \tilde{e}_n) \in \mathbf{H}(I;\mathbb{R}^n)$ such that $f\Big((\tilde{F}_1, \dots, \tilde{F}_{n-1}), (\tilde{e}_1, \dots, \tilde{e}_n)\Big) = 0$.

The partial derivatives of the function f are denoted

$$f_{F_i} := \frac{\partial f}{\partial F_i} \colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to \mathcal{L}\Big(H^{n-i-1}(I;\mathbb{R}), \mathbf{H}(I;\mathbb{R}^n)\Big)$$

for $i = 1, \dots, n-1$, and

$$f_{e_j} := \frac{\partial f}{\partial e_j} \colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to \mathcal{L}\Big(H^{n-j}(I;\mathbb{R}^n), \mathbf{H}(I;\mathbb{R}^n)\Big)$$

for $j = 1, \dots, n$.

The gradient matrix of the function f is denoted:

$$Df = \begin{pmatrix} f^1_{F_1} & \cdots & f^1_{F_{n-1}} & f^1_{e_1} & \cdots & f^1_{e_n} \\ \vdots & & & & & \\ f^n_{F_1} & \cdots & f^n_{F_{n-1}} & f^n_{e_1} & \cdots & f^n_{e_n} \end{pmatrix}_{n\times(2n-1)},$$

and the derivative:

$$D_{(e_1,\dots,e_n)}f \colon \mathcal{H}(I;\mathbb{R})_> \times \mathbf{H}(I;\mathbb{R}^n) \to \mathcal{L}\Big(\mathcal{H}(I;\mathbb{R})_>, \mathbf{H}(I;\mathbb{R}^n)\Big)$$

can be written using matrix notation under the form:

$$D_{(e_1,\dots,e_n)}f = \begin{pmatrix} f^1_{e_1} & \cdots & f^1_{e_n} \\ \vdots & & \\ f^n_{e_1} & \cdots & f^n_{e_n} \end{pmatrix}_{n\times n}.$$

We have already seen that the mapping f is of class $\mathcal{C}^\infty$. In order to apply the implicit function theorem (see Theorem 2.1), we have to prove that $D_{(e_1,\dots,e_n)}f(\tilde{p})$ is an isomorphism between the spaces $\mathbf{H}(I;\mathbb{R}^n)$ and $\mathbf{H}(I;\mathbb{R}^n)$.

First, we claim that this mapping is one-to-one, which means that for any $(w_1, \dots, w_n) \in \mathbf{H}(I;\mathbb{R}^n)$, there exists a unique $(v_1, \dots, v_n) \in \mathbf{H}(I;\mathbb{R}^n)$ such that

$$\begin{pmatrix} f^1_{e_1}(\tilde{p}) & \cdots & f^1_{e_n}(\tilde{p}) \\ \vdots & & \\ f^n_{e_1}(\tilde{p}) & \cdots & f^n_{e_n}(\tilde{p}) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}. \tag{4.2}$$

Equivalently, this can be written as

$$\begin{aligned} & f^1_{e_1}(\tilde{p})v_1 + f^1_{e_2}(\tilde{p})v_2 + \ldots + f^1_{e_n}(\tilde{p})v_n = w_1, \\ & \vdots \\ & f^n_{e_1}(\tilde{p})v_1 + f^n_{e_2}(\tilde{p})v_2 + \ldots + f^n_{e_n}(\tilde{p})v_n = w_n, \end{aligned}$$

or, according to the definition of f given in (4.1), as

$$\begin{aligned}
-v_1(t) + \int_0^t \tilde{F}_1(s)v_2(s)ds =& w_1(t),\\
-\int_0^t \tilde{F}_1(s)v_1(s)ds - v_2(t) + \int_0^t \tilde{F}_2(s)v_3(s)ds =& w_2(t),\\
&\vdots\\
-\int_0^t \tilde{F}_{n-2}(s)v_{n-2}(s)ds - v_{n-1}(t) + \int_0^t \tilde{F}_{n-1}(s)v_n(s)ds =& w_{n-1}(t),\\
-\int_0^t \tilde{F}_{n-1}(s)v_{n-1}(s)ds - v_n(t) =& w_n(t),
\end{aligned}$$

for all $t \in I$.
By derivation, this implies that the n-tuple $(v_1, \dots, v_n)$ satisfies the following system:

$$\begin{aligned}
v_1' =& \tilde{F}_1 v_2 - w_1',\\
v_2' =& \tilde{F}_2 v_3 - \tilde{F}_1 v_1 - w_2',\\
&\vdots\\
v_{n-1}' =& \tilde{F}_{n-1} v_n - \tilde{F}_{n-2} v_{n-2} - w_{n-1}',\\
v_n' =& -\tilde{F}_{n-1} v_{n-1} - w_n',
\end{aligned}$$

and the initial conditions:

$$\begin{aligned}
v_1(0) =& -w_1(0),\\
v_2(0) =& -w_2(0),\\
&\vdots\\
v_{n-1}(0) =& -w_{n-1}(0),\\
v_n(0) =& -w_n(0).
\end{aligned}$$

Note that this is a system of ordinary differential equations, which in matrix form becomes:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}' = \begin{pmatrix} 0 & \tilde{F}_1 & \dots & 0 & 0 \\ -\tilde{F}_1 & 0 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & \tilde{F}_{n-1} \\ 0 & 0 & \dots & -\tilde{F}_{n-1} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} + \begin{pmatrix} -w_1' \\ -w_2' \\ \vdots \\ -w_{n-1}' \\ -w_n' \end{pmatrix}$$

with the initial condition:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}(0) = \begin{pmatrix} -w_1 \\ -w_2 \\ \vdots \\ -w_n \end{pmatrix}(0)$$

Thanks to Lemma 2.1, this system has a unique solution in $H^1(I; \mathbb{M}^{n\times n})$. Applying the same method as that used in step (iii) of the proof of Theorem 3.1, and taking into account the fact that $(w_1, \ldots, w_n) \in \mathbf{H}(I;\mathbb{R}^n)$, one can see that $(v_1, \ldots, v_n) \in \mathbf{H}(I;\mathbb{R}^n)$. Thus, we have proved that $D_{(e_1,\ldots,e_n)}f(\tilde{p})$ is one-to-one, hence an isomorphism, between the spaces $\mathbf{H}(I;\mathbb{R}^n)$ and $\mathbf{H}(I;\mathbb{R}^n)$.

We are now in a position to apply the implicit function theorem 2.1 to the function f, which is of class $\mathcal{C}^\infty$. Accordingly, there exist an open subset U of $\mathcal{H}(I;\mathbb{R})_>$ containing $(\tilde{F}_1, \ldots, \tilde{F}_{n-1})$, an open subset V of $\mathbf{H}(I;\mathbb{R}^n)$ containing $(\tilde{e}_1, \ldots, \tilde{e}_n)$, and an implicit function $g\colon U \to V$ such that

$$f\Big((F_1, \ldots, F_{n-1}), (e_1, \ldots, e_n)\Big) = 0 \text{ and } \Big((F_1, \ldots, F_{n-1}), (e_1, \ldots, e_n)\Big) \in U \times V$$

is equivalent to

$$(e_1, \ldots, e_n) = g(F_1, \ldots, F_{n-1}) \text{ for all } (F_1, \ldots, F_{n-1}) \in U.$$

Moreover, the same theorem shows that the mapping $g\colon U \to V$ is of class $\mathcal{C}^\infty$.

But, in the proof of Theorem 3.1, we have seen that for any $(F_1, \ldots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_>$ the equation

$$f\Big((F_1, \ldots, F_{n-1}), (e_1, \ldots, e_n)\Big) = 0$$

has a unique solution $(e_1, \ldots, e_n) \in \mathbf{H}(I;\mathbb{R}^n)$. This shows that the mapping $\bar{g}\colon \mathcal{H}(I;\mathbb{R})_> \to \mathbf{H}(I;\mathbb{R}^n)$ defined by $\bar{g}(F_1, \ldots, F_{n-1}) = (e_1, \ldots, e_n)$, is well-defined. Therefore, the uniqueness part of the implicit function theorem 2.1, shows that $\bar{g} = g$ on U, hence that $\bar{g}$ is of class $\mathcal{C}^\infty$ over U. Since the $(n-1)$-tuple $(\tilde{F}_1, \ldots, \tilde{F}_{n-1})$ was arbitrarily chosen in $\mathcal{H}(I;\mathbb{R})_>$, we deduce that the mapping $\bar{g}$ is of class $\mathcal{C}^\infty$ over $\mathcal{H}(I;\mathbb{R})_>$.

(iii) We now conclude our proof.

First, the previous step shows that the mapping

$$\bar{g}\colon (F_1, \ldots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R})_> \to (e_1, \ldots, e_n) \in \mathbf{H}(I;\mathbb{R}^n)$$

is of class $\mathcal{C}^\infty$.

Second, the mapping

$$(e_1, \ldots, e_n) \in \mathbf{H}(I;\mathbb{R}^n) \to e_1 \in \mathbf{H}(I;\mathbb{R}^n)$$

is a projection, hence of class $\mathcal{C}^\infty$.

Third, the mapping

$$\phi\colon H^{n-1}(I;\mathbb{R}^n) \to H^n(I;\mathbb{R}^n)$$

defined by

$$e_1 \mapsto \{t \mapsto c(t) = \int_0^t e_1(s)ds + x_0\}$$

is also of class $\mathcal{C}^\infty$ thanks to Lemma 2.3 (a translation by a constant vector x_0 is clearly of class $\mathcal{C}^\infty$).

Since the mapping

$$\mathcal{F}\colon (F_1, \dots, F_{n-1}) \in \mathcal{H}(I; \mathbb{R})_> \to c \in H^n(I; \mathbb{R}^n)$$

is the composition of the three above mappings, it is also of class $\mathcal{C}^\infty$. The proof is now complete. □

In the special case of dimension 3, which is the most encountered in practice, the theorem above leads to the following corollary:

COROLLARY 4.1. *The mapping*

$$\mathcal{F}\colon (k, \tau) \in H^1(I; \mathbb{R}_+^*) \times L^2(I; \mathbb{R}) \to c \in H^3(I; \mathbb{R}^3),$$

where the curve c is defined in part (b) of Corollary 3.1, is of class $\mathcal{C}^\infty$*.*

□

For curves of class $\mathcal{C}^n$, the following result, similar to that of Theorem 4.1, holds:

COROLLARY 4.2. *Let the mapping*

$$\mathcal{F}\colon \mathcal{C}^{n-2}(I; \mathbb{R}_+^*) \times \mathcal{C}^{n-3}(I; \mathbb{R}_+^*) \times \ldots \times \mathcal{C}^1(I; \mathbb{R}_+^*) \times \mathcal{C}^0(I; \mathbb{R}) \to \mathcal{C}^n(I; \mathbb{R}^n)$$

be defined by

$$(F_1, \dots, F_{n-1}) \to c,$$

where the curve c is defined in part (c) of Corollary 3.2. Then $\mathcal{F}$ *is of class* $\mathcal{C}^\infty$*.*

Sketch of proof: The two methods that we have already mentioned for the proof of Corollary 3.2 can be also used to prove this result. Accordingly, we can either use the classical theory for ordinary differential equations, or we can derive the result from Theorem 4.1 and from the fact that the imbedding $H^m(I; \mathbb{R}^n) \subset \mathcal{C}^{m-1}(I; \mathbb{R}^n)$ is continuous for all $m > \frac{1}{2}$ and linear, hence of class C^∞. □

5. Commentaries

(1) In the statements of Theorems 3.1 and 4.1, we have considered the case of a curve parametrized by its arc length. This restriction is not essential however. To see this, let $\alpha\colon I \to \mathbb{R}^n$ be a given curve, not necessarily parametrized by its arc length, such that $\alpha'(t) \neq 0$ for all $t \in I$. Then, it is always possible to obtain another curve $\beta\colon J \to \mathbb{R}^n$ this time parametrized by its arc length, which has the same image and the same curvature functions as the curve α. Indeed, let

$$s(t) := \int_0^t |\alpha'(\tau)| d\tau \text{ for all } t \in I.$$

Since $s'(t) = |\alpha'(t)| \neq 0$, the inverse function theorem shows that there exists an inverse function $s \mapsto t(s)$, defined on $J := s(I)$. It is then easily seen that the curve $\beta := \alpha \circ t \colon J \to \mathbb{R}^n$ satisfies the required properties.

(2) In this paper, we have restricted our attention to curves such that $\{c^{(1)}(t), \ldots, c^{(n-1)}(t)\}$ are linearly independent at each point $t \in I$. In fact, if $c^{(k)}(t)$ is linearly dependent on $\{c'(t), \ldots, c^{(k-1)}(t)\}$ along a whole interval $[a, b] \subset I$, then one can prove that the image of c lies in a $(k-1)$-dimensional subspace of $\mathbb{R}^n$, so that we can establish a result similar to that of Theorems 3.1 and 4.1 on this interval, but in a lower dimension. More difficulties arise in some other cases (for example if the property above holds only at isolated points or at some sequence of points); for details, see [**10**, Chap. 1].

(3) Another natural question arises: What happens (for curves immersed in the three-dimensional space, for simplicity) at the points where the curvature vanishes and consequently the torsion is not defined ? More specifically, assume that $\alpha : [a, b] \to \mathbb{R}^3$ is a curve whose torsion vanishes everywhere, save at one point $t_0 \in]a, b[$, where the torsion is undefined; then it seems reasonable to say that α has zero torsion everywhere, by extension. However, if we accept this convention, we can see that the hypothesis $k > 0$ of Corollary 3.1 is an essential one, as shown by the following example:

Let

$$\alpha_1(t) = \begin{cases} 0; & \text{if } t = 0 \\ (t, 0, 5e^{-\frac{1}{t^2}}); & \text{if } t \neq 0 \end{cases},$$

and

$$\alpha_2(t) = \begin{cases} (t, 5e^{-\frac{1}{t^2}}, 0); & \text{if } t < 0 \\ 0; & \text{if } t = 0 \\ (t, 0, 5e^{-\frac{1}{t^2}}); & \text{if } t > 0 \end{cases}.$$

Then α_1 and α_2 have the same curvature and the same torsion, but there is no rigid transformation mapping α_1 onto α_2. To see this, note that the curvatures of α_1 and α_2 are also vanishing in $t = 0$, since the function $\varphi \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) := \begin{cases} e^{-\frac{1}{t^2}} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0 \end{cases}$$

has the property that all its derivatives vanish at 0.

The curve α_1 is a planar curve, hence its torsion vanishes, *i.e.*, $\tau_{\alpha_1}(t) = 0$ for all $t \neq 0$ and, by the convention above, $\tau_{\alpha_1}(t) = 0$ for all $t \in \mathbb{R}$. In the same way, one can see that $\tau_{\alpha_2}(t) = 0$ for all $t \in \mathbb{R}$, by the same convention. Therefore, the curves α_1 and α_2 have the same curvature and the same torsion. On the other hand, any rigid motion would have to be the identity on one portion of $\mathbb{R}^3$ and a rotation on the other one, which is impossible.

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CHAPITRE 2

Sur la reconstruction et la continuité d'une sous-variété à bord

On the Recovery and Continutity of a Submanifold with Boundary

ABSTRACT. The fundamental theorem of Riemannian geometry asserts that a connected and simply-connected Riemannian space ω of $\mathbb{R}^p$ can be isometrically immersed into the Euclidean space $\mathbb{R}^{p+q}$ if and only if there exist tensors satisfying the Gauss-Ricci-Codazzi equations, in which case these immersions are uniquely determined up to isometries in $\mathbb{R}^{p+q}$. In this fashion, we can define a mapping which associates with these prescribed tensors the reconstructed submanifold. The purpose of this paper is twofold: under a smoothness assumption on the boundary of ω, we first establish an analogous result for the existence and uniqueness of a submanifold "with boundary" and then show that the mapping constructed in this fashion is locally Lipschitz-continuous with respect to the topology of the Banach spaces $\mathcal{C}^l(\overline{\omega})$, $l \geq 1$.

1. Introduction

The fundamental theorem of surface theory asserts that if we consider a symmetric positive definite matrix field of order two and of class $\mathcal{C}^2$ and a symmetric matrix field of order two and of class $\mathcal{C}^1$ that satisfy together the Gauss and Codazzi-Mainardi equations in a connected open subset of $\mathbb{R}^2$, then there exists an immersed surface in the three-dimensional Euclidean space with these fields as its first and second fundamental forms (see, for instance, [**3**] for a *local* version, or [**7**] for a *global* version if in addition the open set is simply-connected).

In *multi-dimensional differential geometry*, a similar result holds for immersions with arbitrary codimension. In this case, besides the induced first and second fundamental forms, another tensor needs to be added, which has to do with the induced connection in the normal bundle. Then, it is possible to state an analogous theorem to the fundamental theorem of surface theory, known as the *fundamental theorem of Riemannian geometry* (recalled here for convenience in Theorem 3.1).

More precisely, this theorem asserts that a connected and simply-connected open subset ω of $\mathbb{R}^p$ endowed with a Riemannian metric of class $\mathcal{C}^2$ can be isometrically immersed in a

Euclidean space $\mathbb{R}^{p+q}$ if and only if there exist tensors satisfying the Gauss-Ricci-Codazzi equations and that these immersions are uniquely determined up to rigid motions. The proof of this result has been considered from several different viewpoints: local coordinates (see, for example, [**12**]), connections in the normal bundle (see [**20**]), differential forms (see [**21**]) or flat compatible metric connections (see [**14**], paper which also includes a general discussion on this subject).

Note that the particular choice of the Euclidean *flat* embedding space may simplify the result, but a *curved* embedding space can always be further embedded in a higher-dimensional flat space, by using a theorem of Whitney (see [**4**, p. 30]).

Our *first objective* is to extend this classical existence and uniqueness up to rigid motions result *"up to the boundary"* of the set ω. More specifically, under a smoothness assumption on the boundary of the set ω (see Definition 2.1 of the so-called *geodesic property*), we prove that if the prescribed geometrical data are such that the corresponding functions and their partial derivatives can be extended by continuity to the closure $\overline{\omega}$, then the reconstructed isometric immersion and the corresponding normal basis share the same property (cf. Theorem 3.2).

The above existence and uniqueness result shows that there exists a mapping $\mathcal{F}$ that associates with matrix fields defined "up to the boundary" a class of isometric immersions that are also defined "up to the boundary".

Our *second objective* is to study the continuity of the mapping $\mathcal{F}$ constructed in this fashion between appropriate Banach spaces. In this direction, we establish in Theorem 4.1 that, if the set ω is *bounded* and satisfies the *geodesic property*, the mapping $\mathcal{F}$ is *locally Lipschitz-continuous*, in the sense given by Definition 2.3.

Note that the issue of continuity for a similar type of mapping, but this time constructed in the case when the boundary of the set is ignored, was recently studied for an open subset of $\mathbb{R}^3$ in [**6**] and for a surface in [**5**].

We emphasize that the study of those properties that hold "up to, or beyond, the boundary", has been recently considered: the case of a manifold with boundary as a function of its metric tensor is found in [**8**] and the case of a surface with boundary as a function of its metric and curvature tensors is treated in [**9**]. So our result generalizes in effect those of [**8**] and [**9**] to "arbitrary dimensions and co-dimensions"; although the methodology of the proofs is similar, its implementation requires considerably more work, however. As noted in these articles, the theoretical aspect of extensions "up to the boundary" does not seem to have been previously considered in the existing literature of differential geometry.

We mention that another field of interest for the Gauss-Ricci-Codazzi equations and their integrability conditions is the theory of *soliton equations* of mathematical physics, where the submanifolds leading in this way to soliton equations are called soliton submanifolds. In this direction, we note that the theory of soliton surfaces was extensively treated in [**19**] and the case of soliton immersions of arbitrary dimension was developed in [**11**].

An "applied" motivation of this work is given by some questions encountered in *nonlinear elasticity*. From this viewpoint, the immersion $\theta\colon \omega \to \mathbb{R}^{p+q}$ can be interpreted as a deformation of a solid occupying the set ω under the action of some applied forces, provided that the mapping θ is injective (since we want to avoid the interpenetrability of matter). The case $p = 3$, $q = 0$ is related to the theory of nonlinear three-dimensional elasticity and the case $p = 2$, $q = 1$ constitutes a model for the nonlinear shell theory.

In this paper, we treat the general case of a p-dimensional submanifold ω of $\mathbb{R}^{p+q}$, in order to provide a unified description of the geometrical framework for continuum mechanics for strings, membranes and solids. For a more detailed treatment of the geometrical aspects of solid mechanics in higher dimensions, see, for example, [**2**].

Note that in this case, the boundary conditions that are found in the traditional boundary-value problems in nonlinear elasticity are aptly expressed in terms of the boundary values of the deformation or of its derivatives; hence the need to study this type of dependence "up to the boundary" of the set to be deformed. In this direction, we remark that it should be interesting to consider also a Sobolev-type framework, as for instance in [**16**], where the case of domains in $\mathbb{R}^n$ is studied, or in [**10**], where a continuity result for deformations in H^1 is established.

The paper is organized as follows. In Section 2, we introduce some notations and results which will be used in the sequel. In Section 3, we first present the classical geometrical framework and then prove our existence and uniqueness theorem "up to the boundary". Finally, in Section 4, we establish that the mapping constructed in the previous section is locally Lipschitz-continuous.

2. Preliminaries

To begin with, we introduce some conventions and notations that will be used throughout the article. For any $N \geq 1$, the N-dimensional Euclidean space $\mathbb{E}^N$ will be identified with $\mathbb{R}^N$ and will be endowed with the Euclidean norm defined by $|a| = \sqrt{\langle a, a\rangle}$, where $\langle a, b\rangle$ denotes the Euclidean inner product of $a, b \in \mathbb{R}^N$. The notations $\mathbb{M}^N$, $\mathbb{O}^N$, $\mathbb{S}^N$, $\mathbb{A}^N$ and $\mathbb{S}^N_>$ respectively designate the set of all real square matrices, of all orthogonal matrices, of all symmetric matrices, of all anti-symmetric matrices and of all positive definite symmetric matrices of order N.

We denote by

$$|A| := \sup_{v \in \mathbb{R}^N \setminus \{0\}} \frac{|Av|}{|v|}$$

the operator norm of a matrix $A \in \mathbb{M}^N$ and by I_N the identity matrix of order N. It is well known that $|A|$ is also given by the square root of the largest eigenvalue of the matrix $A^T A$, where A^T denotes the transpose of the matrix A.

In any metric space, the open ball with center x and radius $\delta > 0$ is denoted $B(x;\delta)$.

Let there be given an integer $p > 0$ and let ω be a connected open subset of $\mathbb{R}^p$. Given two points $x, y \in \omega$, a *path joining x to y in ω* is a mapping $\gamma \in \mathcal{C}^1([0,1];\mathbb{R}^p)$ that satisfies $\gamma(t) \in \omega$ for all $t \in [0,1]$ and such that $\gamma(0) = x$, $\gamma(1) = y$. The *geodesic distance in ω* between two points $x, y \in \omega$ is then defined by

$$d_\omega(x,y) := \inf\{L(\gamma); \gamma \text{ is a path joining } x \text{ to } y \text{ in } \omega\},$$

where $L(\gamma) = \int_0^1 |\gamma'(t)|dt$ is the length of the path γ.

The *geodesic diameter* of ω is defined by

$$D_\omega := \sup_{x,y\in\omega} d_\omega(x,y).$$

The results of this paper will be established under a specific, but mild, regularity assumption on the boundary of the open subset ω, according to the following definition, which was also used in [**8**]:

DEFINITION 2.1. *An open subset ω of $\mathbb{R}^p$ satisfies the geodesic property if it is connected and, given any point $x_0 \in \partial\omega$ and any $\varepsilon > 0$, there exists $\delta = \delta(x_0,\varepsilon) > 0$ such that*

$$d_\omega(x,y) < \varepsilon \text{ for all } x, y \in \omega \cap B(x_0;\delta).$$

Remark. Notice that this assumption is not a very restrictive one, since any connected open subset of $\mathbb{R}^p$ with a Lipschitz-continuous boundary (cf. Definition 4.5. of [**1**], or pp. 14-15 of [**15**]), satisfies the geodesic property. □

Let $y = (y_1, \ldots, y_p)$ denote a generic point of $\mathbb{R}^p$, $p \in \mathbb{N}^*$. Partial derivativ operators of order $l \geq 1$ are denoted $\partial^{\boldsymbol{\alpha}} f$, where $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{N}^p$ is a multi-index satisfying $|\boldsymbol{\alpha}| := \sum_{i=1}^p \alpha_i = l$. Partial derivative operators of the first and second order are also denoted $\partial_\alpha := \frac{\partial}{\partial y_\alpha}$ and $\partial_{\alpha\beta} := \frac{\partial^2}{\partial y_\alpha \partial y_\beta}$.

We introduce in the following definition the notion of spaces of functions, vector fields, or matrix fields "of class $\mathcal{C}^l$ up to the boundary of ω".

DEFINITION 2.2. *Let ω be an open subset of $\mathbb{R}^p$. For any integer $l \geq 1$, we define the space $\mathcal{C}^l(\overline{\omega})$ as the subspace of the space $\mathcal{C}^l(\omega)$ that consists of all functions $f \in \mathcal{C}^l(\omega)$ that, together with all their partial derivatives $\partial^{\boldsymbol{\alpha}} f$, $1 \leq |\boldsymbol{\alpha}| \leq l$, possess continuous extensions to the closure $\overline{\omega}$ of ω. Equivalently, a function $f\colon \omega \to \mathbb{R}$ belongs to $\mathcal{C}^l(\overline{\omega})$ if $f \in \mathcal{C}^l(\omega)$ and, at each point y_0 of the boundary $\partial\omega$ of ω, $\lim_{y\in\omega\to y_0} f(y)$ and $\lim_{y\in\omega\to y_0} \partial^{\boldsymbol{\alpha}} f(y)$ for all $1 \leq |\boldsymbol{\alpha}| \leq l$ exist. Analogous definitions hold for the spaces $\mathcal{C}^l(\overline{\omega};\mathbb{R}^N)$, $\mathcal{C}^l(\overline{\omega};\mathbb{M}^N)$, $\mathcal{C}^l(\overline{\omega};\mathbb{S}^N)$, etc., for all integer $N > 0$.*

All the continuous extensions appearing in such spaces will be identified by a bar. Thus for instance, we shall denote by $\overline{f} \in \mathcal{C}^0(\overline{\omega})$ and $\overline{\partial^{\boldsymbol{\alpha}} f} \in \mathcal{C}^0(\overline{\omega})$, $1 \leq |\boldsymbol{\alpha}| \leq l$, the continuous extensions to $\overline{\omega}$ of the functions f and $\partial^{\boldsymbol{\alpha}} f$ if $f \in \mathcal{C}^l(\overline{\omega})$; etc.

The spaces $\mathcal{C}^l(\overline{\omega};\mathbb{R}^N)$ and $\mathcal{C}^l(\overline{\omega};\mathbb{M}^N)$, $l \geq 1$, are endowed with their natural norms, defined by:

$$\|F\|_{l,\overline{\omega}} := \sup_{y\in\overline{\omega},\ |\boldsymbol{\alpha}|\leq l} |\overline{\partial^{\boldsymbol{\alpha}} F}(y)| \text{ for all } F \in \mathcal{C}^l(\overline{\omega};\mathbb{M}^N),$$

and

$$\|\theta\|_{l,\overline{\omega}} := \sup_{y\in\overline{\omega},\ |\boldsymbol{\alpha}|\leq l} |\overline{\partial^{\boldsymbol{\alpha}} \theta}(y)| \text{ for all } \theta \in \mathcal{C}^l(\overline{\omega};\mathbb{R}^N).$$

Next, we define the set

$$\mathcal{C}^2(\overline{\omega};\mathbb{S}^N_>) := \{A \in \mathcal{C}^l(\overline{\omega};\mathbb{S}^N); \overline{A}(y) \in \mathbb{S}^N_> \text{ for all } y \in \overline{\omega}\}.$$

We also introduce the following class of functions:

Definition 2.3. *Let X and Y be normed vector spaces and let A be a subset of X. A mapping $\chi\colon A \to Y$ is said to be locally Lipschitz-continuous over A if, given any $\hat{u} \in A$, there exists constants $c(\hat{u}) > 0$ and $\delta(\hat{u}) > 0$ such that*

$$\|\chi(u) - \chi(v)\|_Y \leq c(\hat{u})\|u - v\|_X \textit{ for all } u, v \in A \cap B(\hat{u};\delta(\hat{u})).$$

Remark. A direct consequence of the mean value theorem applied to the mapping $\chi \in \mathcal{C}^1(U;Y)$, where U is an open subset of X, is that χ is locally Lipschitz-continuous over U if at least one of the following additional hypotheses is satisfied:

- the mapping χ is the restriction to U of a continuous linear mapping from X into Y;
- the space X is finite-dimensional;
- given any $\hat{u} \in U$, there exists $\delta(\hat{u}) > 0$ such that

$$\sup_{v\in B(\hat{u};\delta(\hat{u}))} \|D\chi(v)\|_{\mathcal{L}(X;Y)} < +\infty,$$

where $D\chi(v) \in \mathcal{L}(X;Y)$ denotes the Fréchet derivative of χ at v. □

Finally, we present three lemmas that will be used in the proof of our main results (Theorems 3.2 and 4.1). In the first lemma we give a characterization of boundedness of the subset ω in terms of its geodesic diameter, in the second lemma we state some properties of the square root of a symmetric positive-definite matrix and in the last lemma, we give an estimate for the norm of the solution to a particular ordinary differential system.

Lemma 2.1. *An open subset ω of $\mathbb{R}^p$ that satisfies the geodesic property is bounded if and only if $D_\omega < +\infty$.*

Proof: See, for example, Lemma 2.3 of [**8**]. □

Lemma 2.2. *Given any matrix $A \in \mathbb{S}^N_>$, there exists a unique matrix $A^{1/2} \in \mathbb{S}^N_>$ such that $(A^{1/2})^2 = A$ and the mapping*

$$\phi\colon A \in \mathbb{S}^N_> \to \phi(A) = A^{1/2} \in \mathbb{S}^N_>$$

defined in this fashion is of class $\mathcal{C}^\infty$.

Proof: The proof of this well-known property can be found in, e.g., [**13**, Sect. 3]. □

LEMMA 2.3. *Let there be given matrix fields* $A, B \in \mathcal{C}^0([0,1];\mathbb{M}^N)$ *and* $Z \in \mathcal{C}^1([0,1];\mathbb{M}^N)$ *that satisfy*

$$Z'(t) = Z(t)A(t) + B(t),\ 0 \le t \le 1.$$

Then

$$|Z(t)| \le |Z(0)| \exp\Big(\int_0^t |A(\tau)| d\tau\Big) + \int_0^t |B(s)| \exp\Big(\int_s^t |A(\tau)| d\tau\Big) ds,\ 0 \le t \le 1.$$

Proof: This is a direct application of Gronwall's Lemma for vector fields (see, e.g., [**17**, Lemma 15.2.6]). □

3. Recovery of a submanifold with boundary

A classical result of differential geometry states that a connected and simply-connected Riemannian space can be isometrically immersed into a given Euclidean space if and only if there exist tensors satisfying the Gauss-Ricci-Codazzi equations, in which case the immersion is uniquely determined up to rigid motions by these tensors. This assertion is known as the *fundamental theorem of Riemannian geometry* and it will be recorded here for convenience (see Theorem 3.1).

The first objective of this paper is to establish, under some specific assumptions, that this result can be extended "up to the boundary" of the submanifold (see Theorem 3.2). To this end, we first introduce in Section 3.1 the geometrical framework of our problem, and then we show how the classical results of differential geometry previously stated can be extended up to the boundary of the Riemann space (see Section 3.2).

3.1. Isometric immersions of a Riemannian space in Euclidean space.

Two integers $p, q \ge 1$ are fixed throughout this article. Let ω be an open subset of $\mathbb{R}^p$. We denote by $y = (y_1, \dots, y_p) \in \omega$ a generic point of ω and by $\{e_1, \dots, e_p\}$ the canonical basis of $\mathbb{R}^p$.

In what follows, Greek indices vary in the set $\{1, \dots, p\}$, Latin indices vary in the set $\{1, \dots, q\}$ and the summation convention with respect to repeated indices is systematically used.

Let $\theta\colon \omega \to \mathbb{R}^{p+q}$ be an immersion of the open set ω in the $(p+q)$-dimensional Euclidean space, identified with $\mathbb{R}^{p+q}$, that is, the mapping θ is differentiable and its Fréchet dérivative $\mathrm{d}\theta_y\colon T_y\omega \to T_{\theta(y)}\mathbb{R}^{p+q}$ is injective for all $y \in \omega$. Here $T_y\omega = \mathbb{R}^p$ denotes the tangent space to ω at y and $T_{\theta(y)}\mathbb{R}^{p+q} = \mathbb{R}^{p+q}$ denotes the tangent space to $\mathbb{R}^{p+q}$ at $\theta(y)$. Then, for each $y \in \omega$, there exists a neighborhood $U \subset \omega$ of y such that $\theta(U) \subset \mathbb{R}^{p+q}$ is a submanifold of $\mathbb{R}^{p+q}$ (see for details, [**4**, Chap. 6, Section 2]). To simplify the notations, we shall identify ω with $\theta(\omega)$ and each vector $V \in T_y\omega,\ y \in \omega$, with $\mathrm{d}\theta_y(V) \in T_{\theta(y)}\mathbb{R}^{p+q}$. A basis in the tangent space $T_{\theta(y)}\theta(\omega)$ is given by

$$\partial_1\theta(y) := \mathrm{d}\theta_y(e_1), \dots, \partial_p\theta(y) := \mathrm{d}\theta_y(e_p),$$

a vector field N is called *normal* if

$$\langle N(y), \partial_\alpha \theta(y) \rangle = 0 \text{ for all } y \in \omega \text{ and for all } \alpha \in \{1, \dots, p\}$$

and its derivatives are given by $\partial_\alpha N(y) := \mathrm{d}N_y(e_\alpha)$.

We denote by $\mathcal{X}(\omega)$ and $\mathcal{X}(\omega)^\perp$ the set of differentiable vector fields that are tangent, respectively normal, to $\theta(\omega)$.

The Euclidean metric of $\mathbb{R}^{p+q}$ induces a Riemannian metric on ω, also called the *first fundamental form* of the immersion, given by its covariant components

$$a_{\alpha\beta}(y) := \langle \partial_\alpha \theta(y), \partial_\beta \theta(y) \rangle, \ \forall y \in \omega, \ \forall \alpha, \beta \in \{1, \dots, p\}.$$

If ω is endowed with this metric, θ becomes an isometric immersion of ω into $\mathbb{R}^{p+q}$.

Let $\overline{\nabla}$ denote the usual Riemann connection on $\mathbb{R}^{p+q}$ (see, for details, [**4**, Chap. 2, Section 3]). If X and Y are vector fields on ω and $\overline{X}$, $\overline{Y}$ are local extensions to $\mathbb{R}^{p+q}$, we define the *induced Riemann connection* on $\theta(\omega)$ by

$$\nabla_X Y := (\overline{\nabla}_{\overline{X}} \overline{Y})^T,$$

where by V^T we denote the tangential part of a vector V. The remaining normal part is denoted $B(X, Y)$, where $B \colon \mathcal{X}(\omega) \times \mathcal{X}(\omega) \to \mathcal{X}^\perp(\omega)$ is a symmetric bilinear mapping. This mapping is called the *second fundamental form* of the immersion $\theta \colon \omega \to \mathbb{R}^{p+q}$. We thus have:

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y) \tag{3.1}$$

for all tangent vector fields X and Y.

To define the induced *normal connection* of the immersion, we take the projection of $\overline{\nabla}_X N$ into the normal space, where X is a tangent vector field and N is a normal vector field. Denoting this induced connection by $\nabla_X^\perp N$, we thus have:

$$\overline{\nabla}_X N = A(X, N) + \nabla_X^\perp N, \tag{3.2}$$

where the mapping $A \colon \mathcal{X}(\omega) \times \mathcal{X}^\perp(\omega) \to \mathcal{X}(\omega)$ is related to the mapping B by the relation

$$\langle A(X, N), Y \rangle = - \langle B(X, Y), N \rangle \ \text{ for all } Y \in \mathcal{X}(\omega). \tag{3.3}$$

We now write these expressions in the local coordinates $(y_1, \dots, y_p)$ on ω. Denote by $X_\alpha = \partial_\alpha \theta$ for all $\alpha \in \{1, \dots, p\}$ the basis in the tangent space; we fix in the normal space an orthogonal basis, denoted $\{N^1, \dots, N^q\}$. Hence:

$$\left\langle N^i, N^j \right\rangle = \delta^{ij} \ \forall i, \ j \in \{1, \dots, q\}$$

and

$$\left\langle N^i, X_\alpha \right\rangle = 0 \ \forall i \in \{1, \dots, q\}, \ \forall \alpha \in \{1, \dots, p\}.$$

The Riemann connection induced on $\theta(\omega)$ and the normal connection of the immersion define the *Christoffel symbols* by the following relations:

$$\nabla_{X_\alpha} X_\beta = \Gamma^\tau_{\alpha\beta} X_\tau \tag{3.4}$$

and

$$\nabla^{\perp}_{X_\beta} N^i = T^{ij}_\beta N^j. \tag{3.5}$$

Remarks. 1. If θ is of class $\mathcal{C}^2$ over ω, since the connection ∇ is compatible with the metric induced by θ, the Christoffel coefficients $\Gamma^\tau_{\alpha\beta}$ satisfy

$$\Gamma^\tau_{\alpha\beta} = \frac{1}{2} a^{\sigma\tau} (\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\alpha\sigma} - \partial_\sigma a_{\alpha\beta}),$$

where $(a^{\sigma\tau}) = (a_{\alpha\beta})^{-1}$.

2. Note that we have $\Gamma^\gamma_{\beta\alpha} = \Gamma^\gamma_{\alpha\beta}$ since the connection ∇ is symmetric.

3. The Christoffel symbols T^{ij}_β associated with the connection $\nabla^\perp$ are anti-symmetric in i, j, i.e., $T^{ij}_\beta = -T^{ji}_\beta$, since the basis $\{N^i\}_i$ is orthonormal. □

We define the coefficients of the second fundamental form by

$$b^i_{\alpha\beta} := -\left\langle B(X_\alpha, X_\beta), N^i \right\rangle \tag{3.6}$$

and we note that $b^i_{\alpha\beta} = b^i_{\beta\alpha}$ since the bilinear mapping B is symmetric. By using the relation (3.3), the mapping A can be expressed in local coordinates as follows:

$$A(X_\alpha, N^i) = b^{i\tau}_\alpha X_\tau, \tag{3.7}$$

where $b^{i\tau}_\alpha := a^{\sigma\tau} b^i_{\alpha\sigma}$ denote the mixed components of the second fundamental form.

Assume now that the immersion θ is of class $\mathcal{C}^3$ over ω. We want to relate the Riemann curvature tensor associated with the connection $\overline{\nabla}$ on $\mathbb{R}^{p+q}$, which is defined by

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} := \overline{\nabla}_{\overline{X}} \overline{\nabla}_{\overline{Y}} \overline{Z} - \overline{\nabla}_{\overline{Y}} \overline{\nabla}_{\overline{X}} \overline{Z} - \overline{\nabla}_{[\overline{X}, \overline{Y}]} \overline{Z} \tag{3.8}$$

where $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are vector fields on $\mathbb{R}^{p+q}$, to the Riemann curvature tensor associated with ∇ on $\theta(\omega)$ and to the normal curvature tensor associated with $\nabla^\perp$, which are respectively defined by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \ \forall X, Y, Z \in \mathcal{X}(\omega), \tag{3.9}$$

and

$$R^\perp(X,Y)N := \nabla^\perp_X \nabla^\perp_Y N - \nabla^\perp_Y \nabla^\perp_X N - \nabla^\perp_{[X,Y]} N, \ \forall X, Y \in \mathcal{X}(\omega), \ \forall N \in \mathcal{X}^\perp(\omega). \tag{3.10}$$

Since the Euclidean space $\mathbb{R}^{p+q}$ is flat, the Riemann curvature tensor $\overline{R}(\overline{X}, \overline{Y})\overline{Z}$ vanishes. Using relations (3.1) and (3.2) and expressing the fact that the tangential part and the normal part of $\overline{R}(X,Y)Z$ and $\overline{R}(X,Y)N$ vanish, we deduce that the following relations hold for all $X, Y, Z, W \in \mathcal{X}(\omega)$ and for all $N, \tilde{N} \in \mathcal{X}^\perp(\omega)$:

$$\langle R(X,Y)Z, W \rangle = \langle B(X,W), B(Y,Z) \rangle - \langle B(X,Z), B(Y,W) \rangle \tag{3.11}$$

$$\left\langle R^\perp(X,Y)N, \tilde{N} \right\rangle = \left\langle A(X,N), A(Y,\tilde{N}) \right\rangle - \left\langle A(X,\tilde{N}), A(Y,N) \right\rangle \tag{3.12}$$

$$(\nabla_X A)(Y, N) = (\nabla_Y A)(X, N), \tag{3.13}$$

$$(\nabla_Y B)(X, Z) = (\nabla_X B)(Y, Z), \tag{3.14}$$

where

$$(\nabla_X A)(Y, N) := \nabla_X(A(Y, N)) - A(\nabla_X Y, N) - A(Y, \nabla^{\perp}_X N),$$

$$(\nabla_X B)(Y, Z) := \nabla^{\perp}_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(X, \nabla_Y Z).$$

Since the mappings A and B are related by relation (3.3), equations (3.13) and (3.14) are equivalent. Relation (3.11) is called *Gauss equation*, relation (3.12) is the *Ricci equation* and relation (3.13) is the *Codazzi equation.* Together, these relations form the set of necessary conditions that have to be satisfied in order that a Riemannian space be isometrically immersed in Euclidean space.

Notice that the Gauss and Ricci equations are algebraic expressions that relate the curvatures of the tangent, respectively, normal bundle, with the second fundamental form of the immersion, but for the Codazzi equation we need to differentiate the second fundamental form considered as a tensor.

In order to obtain the expression of these equations in local coordinates, we set $X = X_\alpha$, $Y = X_\beta$, $Z = X_\delta$, $W = X_\gamma$ and $N = N^i$, $\tilde{N} = N^j$ in (3.11-3.13). Then, using relations (3.4), (3.5), (3.6) and (3.7), the Gauss-Ricci-Codazzi equations become:

$$(\partial_\alpha \Gamma^\tau_{\beta\delta} - \partial_\beta \Gamma^\tau_{\alpha\delta} + \Gamma^\sigma_{\beta\delta}\Gamma^\tau_{\alpha\sigma} - \Gamma^\sigma_{\alpha\delta}\Gamma^\tau_{\beta\sigma}) a_{\tau\gamma} = b^i_{\gamma\alpha} b^i_{\delta\beta} - b^i_{\gamma\beta} b^i_{\delta\alpha}, \tag{3.15}$$

$$\partial_\alpha T^{ij}_\beta - \partial_\beta T^{ij}_\alpha + T^{kj}_\beta T^{ik}_\alpha - T^{kj}_\alpha T^{ik}_\beta + a^{\sigma\tau}(b^j_{\alpha\tau} b^i_{\beta\sigma} - b^j_{\beta\tau} b^i_{\alpha\sigma}) = 0, \tag{3.16}$$

$$\partial_\alpha b^j_{\gamma\beta} - \partial_\beta b^j_{\gamma\alpha} = \Gamma^\tau_{\alpha\gamma} b^j_{\tau\beta} - \Gamma^\tau_{\beta\gamma} b^j_{\tau\alpha} + b^i_{\gamma\beta} T^{ij}_\alpha - b^i_{\gamma\alpha} T^{ij}_\beta, \tag{3.17}$$

where $\alpha, \beta, \gamma, \delta, \sigma, \tau \in \{1, \ldots, p\}$ and $i, j, k \in \{1, \ldots, q\}$. These are the classical Gauss-Ricci-Codazzi equations satisfied for a submanifold of the Euclidean space, as given, for example, in [**12**, page 190].

Note that the left-hand side of relation (3.11), which is usually denoted $R_{\alpha\beta\gamma\delta}$, defines the Riemann curvature tensor of $\theta(\omega)$ in its covariant components.

The following theorem establishes the existence and uniqueness, under some additional assumptions, of an isometric immersion of an open subset $\omega \subset \mathbb{R}^p$ endowed with a Riemannian metric into the Euclidean space $\mathbb{R}^{p+q}$.

THEOREM 3.1. *Let ω be a connected and simply connected open subset of $\mathbb{R}^p$. Let there be given a positive definite symmetric matrix field $A = (a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^p_>)$, q symmetric matrix fields $B^i = (b^i_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^p)$, $i \in \{1, \ldots, q\}$, and p anti-symmetric matrix fields*

$T_\alpha = (T_\alpha^{ij}) \in \mathcal{C}^1(\omega; \mathbb{A}^q)$, $\alpha \in \{1, \dots, p\}$, satisfying the Gauss-Ricci-Codazzi equations (3.15-3.17). Let $y_0 \in \omega$ be fixed and denote by $G_0 \in \mathbb{M}^{p+q}$ the following matrix:

$$G_0 := \begin{pmatrix} a_{11}(y_0) & a_{12}(y_0) & \dots & a_{1p}(y_0) & 0 & 0 & \dots & 0 \\ a_{21}(y_0) & a_{22}(y_0) & \dots & a_{2p}(y_0) & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{p1}(y_0) & a_{p2}(y_0) & \dots & a_{pp}(y_0) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{3.18}$$

Then there exist a unique immersion $\theta \in \mathcal{C}^3(\omega; \mathbb{R}^{p+q})$ and a unique orthonormal family of q vector fields $N^1, \dots, N^q \in \mathcal{C}^2(\omega; \mathbb{R}^{p+q})$, normal to $\theta(\omega)$, satisfying

$$(i)\ \langle \partial_\alpha \theta(y), \partial_\beta \theta(y) \rangle = a_{\alpha\beta}(y),\ \forall y \in \omega,\ \forall \alpha, \beta \in \{1, \dots, p\}, \tag{3.19}$$

$$(ii)\ \big\langle \partial_{\alpha\beta} \theta(y), N^i(y) \big\rangle = -\, b^i_{\alpha\beta}(y)\ \forall y \in \omega,\ \forall \alpha, \beta \in \{1, \dots, p\},\ \forall i \in \{1, \dots, q\}, \tag{3.20}$$

$$(iii)\ \big\langle \partial_\alpha N^i(y), N^j(y) \big\rangle = T^{ij}_\alpha(y)\ \forall y \in \omega,\ \forall \alpha \in \{1, \dots, p\},\ \forall i, j \in \{1, \dots, q\}, \tag{3.21}$$

such that $\theta(y_0) = \mathbf{0}$ and the α-th column of the matrix $G_0^{1/2}$ is $\partial_\alpha \theta(y_0)$ for all $\alpha \in \{1, \dots, p\}$.

Proof: This is a classical result, save for the choice of conditions at y_0: a proof using local coordinates can be found in [**12**] and in [**14**]. We summarize this proof here and, at the same time, we introduce some new matrix notations. Our interest is to provide a framework that will be used in the sequel, as illustrated in the proofs of Theorems 3.2 and 4.1.

Step 1. Let the matrix field $\Gamma_\alpha \colon \omega \to \mathbb{M}^{p+q}$ be defined by

$$\Gamma_\alpha := \begin{pmatrix} \Gamma^1_{1\alpha} & \Gamma^1_{2\alpha} & \dots & \Gamma^1_{p\alpha} & b^{11}_\alpha & b^{21}_\alpha & \dots & b^{q1}_\alpha \\ \Gamma^2_{1\alpha} & \Gamma^2_{2\alpha} & \dots & \Gamma^2_{p\alpha} & b^{12}_\alpha & b^{22}_\alpha & \dots & b^{q2}_\alpha \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Gamma^p_{1\alpha} & \Gamma^p_{2\alpha} & \dots & \Gamma^p_{p\alpha} & b^{1p}_\alpha & b^{2p}_\alpha & \dots & b^{qp}_\alpha \\ -b^1_{1\alpha} & -b^1_{2\alpha} & \dots & -b^1_{p\alpha} & T^{11}_\alpha & T^{12}_\alpha & \dots & T^{1q}_\alpha \\ -b^2_{1\alpha} & -b^2_{2\alpha} & \dots & -b^2_{p\alpha} & T^{21}_\alpha & T^{22}_\alpha & \dots & T^{2q}_\alpha \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -b^q_{1\alpha} & -b^q_{2\alpha} & \dots & -b^q_{p\alpha} & T^{q1}_\alpha & T^{q2}_\alpha & \dots & T^{qq}_\alpha \end{pmatrix}. \tag{3.22}$$

The regularity assumptions made on the initial data show that $\Gamma_\alpha \in \mathcal{C}^1(\omega; \mathbb{M}^{p+q})$ for all $\alpha \in \{1, \dots, p\}$.

Step 2. We define the matrix field $F \in \mathcal{C}^2(\omega; \mathbb{M}^{p+q})$ as being the unique solution to the problem

$$\partial_\alpha F = F\Gamma_\alpha \text{ in } \omega \tag{3.23}$$

$$F(y_0) = G_0^{1/2}. \tag{3.24}$$

We emphasize that this system is obtained from the equations (3.1) and (3.2) written in local coordinates, i.e.,

$$\partial_\alpha X_\beta = \Gamma^\tau_{\alpha\beta} X_\tau - b^i_{\alpha\beta} N^i \tag{3.25}$$

$$\partial_\alpha N^i = T^{ij}_\alpha N^j + b^{i\tau}_\alpha X_\tau. \tag{3.26}$$

We also note that the equations of the matrix equation (3.23) form an overdetermined system of partial differential equations, so we must verify compatibility conditions in order to obtain the existence of a solution. The compatibility conditions can be written as a system of matrix equations of the form

$$\partial_\beta \Gamma_\alpha - \partial_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha + \Gamma_\alpha \Gamma_\beta = \mathbf{0}$$

for all $\alpha, \beta \in \{1, \ldots, p\}$.

The fact that these compatibility conditions are verified is a consequence of the Gauss-Ricci-Codazzi equations. More precisely, if L_{mn} denotes the coefficient of the matrix appearing in the left-hand side of the above relation, where m is the row index, then $L_{mn} = 0$ for $m, n \in \{1, \ldots, p\}$ is a consequence of relation (3.15), the conditions $L_{mn} = 0$ for $m \in \{1, \ldots, p\}$, $n \in \{p+1, \ldots, p+q\}$ and for $m \in \{p+1, \ldots, p+q\}$, $n \in \{1, \ldots, p\}$ are satisfied because of relation (3.17), and the conditions $L_{mn} = 0$ for $m, n \in \{p+1, \ldots, p+q\}$ are verified since the relation (3.16) is true.

Step 3. We define the mapping $\theta \in \mathcal{C}^3(\omega; \mathbb{R}^{p+q})$ as being the unique solution to the problem

$$\partial_\alpha \theta = \mathbf{f}_\alpha \text{ in } \omega \tag{3.27}$$

$$\theta(y_0) = \mathbf{0}, \tag{3.28}$$

where $\mathbf{f}_\alpha(y)$ is the α-th column of the matrix $F(y)$ found in Step 2.

We also define $N^i \in \mathcal{C}^2(\omega; \mathbb{R}^{p+q})$ by the relation

$$N^i(y) := \mathbf{f}_{p+i}(y), \tag{3.29}$$

for all $y \in \omega$ and for all $i \in \{1, \ldots, q\}$, where $\mathbf{f}_{p+i}(y)$ is the $(p+i)$-th column of the matrix $F(y)$.

We remark that the compatibility conditions which have to be verified in order to obtain a solution to the problem (3.27-3.28) are $\partial_\beta \mathbf{f}_\alpha = \partial_\alpha \mathbf{f}_\beta$ for all $\alpha, \beta \in \{1, \ldots, p\}$. These equalities hold as a consequence of the symmetry properties of the Christoffel coefficients $\Gamma^\tau_{\alpha\beta}$ and of the coefficients of the second fundamental form $b^i_{\alpha\beta}$.

Step 4. We show that the fields θ and $N^1, \dots, N^p$ found in the previous step satisfy the conditions (3.19), (3.20) and (3.21) of the theorem.

To this end, we first show that the matrix fields $(\langle \mathbf{f}_m, \mathbf{f}_n \rangle)$ and (f_{mn}), where $f_{\alpha\beta} = a_{\alpha\beta}$, $f_{p+i,p+j} = \delta_{ij}$, $f_{p+i,\alpha} = f_{\alpha,p+i} = 0$, satisfy the same Cauchy problem:

$$\partial_\alpha X = \Gamma_\alpha X + X \Gamma_\alpha$$
$$X(y_0) = G_0^{1/2}.$$

Consequently, the uniqueness of the solution for this system shows that $\langle \mathbf{f}_m, \mathbf{f}_n \rangle = f_{mn}$ for all $m, n \in \{1, \dots, p+q\}$, which implies in particular relation (3.19). The relations (3.20) and (3.21) are then obtained by combining relations (3.25) and (3.26) with the fact that $\langle \mathbf{f}_m, \mathbf{f}_n \rangle = f_{mn}$ for all $m, n \in \{1, \dots, p+q\}$. □

While the immersion θ and the vector fields $N^1, \dots, N^p$ found in Theorem 3.1 are uniquely determined, this is no longer true if they are not fixed at the point y_0, according to the following corollary to Theorem 3.1.

COROLLARY 3.1. *Let the assumptions on the set ω, on the symmetric matrix fields $A = (a_{\alpha\beta})$, $B^i = (b^i_{\alpha\beta})$, $i \in \{1, \dots, q\}$, and on the anti-symmetric matrix fields $T_\alpha = (T^{ij}_\alpha)$, $\alpha \in \{1, \dots, p\}$, be as in Theorem 3.1. Then there exists a mapping $\theta \in \mathcal{C}^3(\omega; \mathbb{R}^{p+q})$ and an orthonormal family of q vector fields $N^1, \dots, N^q \in \mathcal{C}^2(\omega; \mathbb{R}^{p+q})$ normal to $\theta(\omega)$ such that*

(3.30) $\quad (i) \ \langle \partial_\alpha \theta(y), \partial_\beta \theta(y) \rangle = a_{\alpha\beta}(y), \ \forall y \in \omega, \ \forall \alpha, \beta \in \{1, \dots, p\},$

(3.31) $\quad (ii) \ \big\langle \partial_{\alpha\beta} \theta(y), N^i(y) \big\rangle = - b^i_{\alpha\beta}(y) \ \forall y \in \omega, \ \forall \alpha, \beta \in \{1, \dots, p\}, \ \forall i \in \{1, \dots, q\},$

(3.32) $\quad (iii) \ \big\langle \partial_\alpha N^i(y), N^j(y) \big\rangle = T^{ij}_\alpha(y) \ \forall y \in \omega, \ \forall \alpha \in \{1, \dots, p\}, \ \forall i, j \in \{1, \dots, q\}.$

Moreover, if a mapping $\tilde{\theta}$ and the vector fields $\tilde{N}^1, \dots, \tilde{N}^q$ normal to $\tilde{\theta}(\omega)$ satisfy the same relations on ω, then there exist a vector $a \in \mathbb{R}^{p+q}$ and a matrix $Q \in \mathbb{O}^{p+q}$ such that

$$\tilde{\theta}(y) = a + Q\theta(y) \text{ for all } y \in \omega$$
$$\tilde{N}^i(y) = QN^i(y) \text{ for all } y \in \omega, \text{ and for all } i \in \{1, \dots, q\}.$$

Proof: The existence of the mapping $\theta \in \mathcal{C}^3(\omega; \mathbb{R}^{p+q})$ and of the normal vector fields $N^1, \dots, N^q \in \mathcal{C}^2(\omega; \mathbb{R}^{p+q})$ such that the relations (3.30), (3.31) and (3.32) are satisfied is obtained directly from Theorem 3.1.

The second part is a consequence of the fact that any two solutions which differ by an isometry at the fixed point y_0, differ by the same isometry at all points of ω. □

3.2. Reconstruction of a submanifold with boundary.

The aim of this section is to establish that, under *ad hoc* assumptions, the reconstruction of a submanifold of $\mathbb{R}^{p+q}$ can be done "up to the boundary". More specifically, the existence and uniqueness of the isometric immersion $\theta \colon \omega \subset \mathbb{R}^p \to \mathbb{R}^{p+q}$ established in Theorem 3.1

can be extended as follows to the closure $\overline{\omega}$ of the subset $\omega \subset \mathbb{R}^p$. This theorem extends previous results obtained in [**8**] and [**9**].

THEOREM 3.2. *Let ω be a simply connected open subset of $\mathbb{R}^p$ that satisfies the geodesic property (Definition 2.1). Consider a positive definite symmetric matrix field $A = (a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>)$ (Definition 2.2), q symmetric matrix fields $B^i = (b^i_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)$, $i \in \{1,\dots,q\}$, and p anti-symmetric matrix fields $T_\alpha = (T^{ij}_\alpha) \in \mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)$, $\alpha \in \{1,\dots,p\}$, such that the relations* (3.15), (3.16) *and* (3.17) *are satisfied. Then there exist a mapping $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$ and an orthonormal family of q vector fields $N^1,\dots, N^q \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ normal to $\theta(\overline{\omega})$ satisfying*

$$(3.33) \qquad (i)\ \left\langle \overline{\partial_\alpha\theta}(y), \overline{\partial_\beta\theta}(y) \right\rangle = \overline{a_{\alpha\beta}}(y),\ \forall y \in \overline{\omega},\ \forall \alpha,\beta \in \{1,\dots,p\},$$

$$(3.34) \qquad (ii)\ \left\langle \overline{\partial_{\alpha\beta}\theta}(y), \overline{N^i}(y) \right\rangle = -\,\overline{b^i_{\alpha\beta}}(y)\ \forall y \in \overline{\omega},\ \forall \alpha,\beta \in \{1,\dots,p\},\ \forall i \in \{1,\dots,q\},$$

$$(3.35) \qquad (iii)\ \left\langle \overline{\partial_\alpha N^i}(y), \overline{N^j}(y) \right\rangle = \overline{T^{ij}_\alpha}(y)\ \forall y \in \overline{\omega},\ \forall \alpha \in \{1,\dots,p\},\ \forall i,j \in \{1,\dots,q\},$$

such that $\theta(y_0) = \mathbf{0}$ and the α-th column of the matrix $G_0^{1/2}$ is $\partial_\alpha\theta(y_0)$ for all $\alpha \in \{1,\dots,p\}$, where the matrix G_0 is given by relation (3.18).

Proof: The proof is broken into four steps, in order to provide a clearer presentation.

(i) *Preliminaries.* Since the open subset ω is connected (it has the geodesic property) and simply-connected, and the components of the matrix fields $A = (a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>)$, $B^i = (b^i_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)$, $T_\alpha = (T^{ij}_\alpha) \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^p)$ satisfy together relations (3.15),(3.16) and (3.17), we deduce from Theorem 3.1 that there exist a mapping $\theta \in \mathcal{C}^3(\omega; \mathbb{R}^{p+q})$ and an orthonormal family of q vector fields $N^1,\dots,N^q \in \mathcal{C}^2(\omega; \mathbb{R}^{p+q})$ normal to $\theta(\omega)$ such that relations (3.19), (3.20) and (3.21) are satisfied. Our objective is to prove that we can extend these applications by continuity to $\overline{\omega}$.

We introduce the following notations: $\mathbf{f}_\alpha(y) := \partial_\alpha\theta(y)$ for $1 \le \alpha \le p$, $\mathbf{f}_{p+i}(y) := N^i(y)$ for $1 \le i \le q$, and $F(y) \in \mathbb{M}^{p+q}$ is the matrix whose m-th column is $\mathbf{f}_m(y)$, $m \in \{1,\dots,p+q\}$, for each $y \in \omega$. We also recall that the matrix $\Gamma_\alpha(y)$ is defined in Step 1 of the proof of Theorem 3.1.

Then an immediate computation shows that the matrix fields $F \in \mathcal{C}^2(\omega; \mathbb{M}^{p+q})$ and $\Gamma_\alpha \in \mathcal{C}^1(\omega; \mathbb{M}^{p+q})$ satisfy the following equality

$$(3.36) \qquad \partial_\alpha F(y) = F(y)\Gamma_\alpha(y) \text{ for all } y \in \omega \text{ and for all } \alpha \in \{1,\dots,p\}.$$

Let us also point out that the matrix fields Γ_α are in fact of class $\mathcal{C}^1(\overline{\omega}; \mathbb{M}^{p+q})$. Indeed, since $(a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>)$, we have in particular that $\det(\overline{a_{\alpha\beta}}(y)) > 0$ for all $y \in \overline{\omega}$. This implies that $(a^{\sigma\tau}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>)$, because the functions $a^{\sigma\tau}$ and $\partial^{\boldsymbol{\alpha}} a^{\sigma\tau}$, $1 \le |\boldsymbol{\alpha}| \le 2$, are obtained as rational fractions of the functions $a_{\alpha\beta}$ and their derivatives. Consequently, the definition of the Christoffel symbols $\Gamma^\tau_{\alpha\beta}$ and the definition of the mixed components of the second fundamental form $b^i_{\alpha\beta}$ imply that they belong to $\mathcal{C}^1(\overline{\omega})$. From these relations and from the fact that each function T_α^{ij} is of class $\mathcal{C}^1(\overline{\omega})$, we finally infer that the matrix fields Γ_α belong to $\mathcal{C}^1(\overline{\omega}; \mathbb{M}^{p+q})$, as claimed.

(ii) *Let K be a compact subset of $\mathbb{R}^p$. Then* $\sup_{y\in K\cap\omega} |F(y)| < +\infty$.

For any $y \in \omega$,

$$\begin{aligned} |F(y)|^2 &\le \sum_{m=1}^{p+q} |\mathbf{f}_m(y)|^2 \\ &= \sum_{\alpha=1}^{p} \langle \mathbf{f}_\alpha(y), \mathbf{f}_\alpha(y)\rangle + \sum_{i=1}^{q} \langle \mathbf{f}_{p+i}(y), \mathbf{f}_{p+i}(y)\rangle \\ &= \sum_{\alpha=1}^{p} a_{\alpha\alpha}(y) + q, \end{aligned}$$

hence

$$\sup_{y\in K\cap\omega} |F(y)|^2 \le \sup_{y\in K\cap\overline{\omega}} \big(\sum_{\alpha=1}^{p} \overline{a_{\alpha\alpha}}(y) + q\big) < +\infty,$$

since the functions $\overline{a_{\alpha\alpha}}$ belong to the space $\mathcal{C}^0(\overline{\omega})$ by assumption.

(iii) *The matrix field $F \in \mathcal{C}^2(\omega; \mathbb{M}^{p+q})$ defined in (i) belongs to the space $\mathcal{C}^2(\overline{\omega}; \mathbb{M}^{p+q})$.*

To begin with, let $y_0 \in \partial\omega$ be an arbitrary, but fixed point of the boundary of ω, and let $K_0 := B(y_0; 1)$. Let $\varepsilon > 0$ be given. Because ω satisfies the geodesic property, there exists $\delta(\varepsilon) > 0$ such that, given any two points $x, y \in B(y_0; \delta(\varepsilon)) \cap \omega$, there exists a path $\gamma = (\gamma_\alpha)$ joining x to y in ω whose length satisfies $L(\gamma) \le \frac{\varepsilon}{\max\{c_0, 2\}}$, where

$$c_0 := \Big(\sup_{y\in K_0\cap\omega} |F(y)|\Big)\Big(\sup_{y\in K_0\cap\omega} \big(\sum_{\alpha=1}^{p} |\Gamma_\alpha(y)|^2\big)^{1/2}\Big).$$

Note that the constant c_0 is finite, as a consequence of the properties established in steps (i) and (ii). Moreover, to ensure that that set $\gamma([0,1])$ is contained in the set K_0, we can assume, without loss of generality, that $\varepsilon \le 1$ and $\delta(\varepsilon) \le \frac{1}{2}$.

We consider the matrix field $Y := F \circ \gamma \in \mathcal{C}^1([0,1]; \mathbb{M}^{p+q})$ associated with any such path γ. The relations (3.36) then imply that the matrix field Y satisfies

$$Y'(t) = \gamma_\alpha'(t) Y(t) \Gamma_\alpha(\gamma(t)) \text{ for all } 0 \le t \le 1.$$

Since $Y(1) = Y(0) + \int_0^1 Y'(t)\mathrm{d}t$, we have, for any two points $x, y \in B(y_0; \delta(\varepsilon)) \cap \omega$,

$$\begin{aligned}|F(y) - F(x)| =& |Y(1) - Y(0)| \\ \leq& \Big(\sup_{0\leq t\leq 1} |F(\gamma(t))|\Big) \int_0^1 |\gamma'_\alpha(t)||\Gamma_\alpha(\gamma(t))|\mathrm{d}t \\ \leq& \Big(\sup_{y\in K_0\cap\omega} |F(y)|\Big)\Big(\sup_{y\in K_0\cap\omega} \big(\sum_{\alpha=1}^{p} |\Gamma_\alpha(y)|^2\big)^{1/2}\Big) L(\gamma).\end{aligned}$$

We infer from this inequality and from the particular choice of $\delta(\varepsilon)$ that

$$|F(y) - F(x)| \leq \varepsilon \text{ for all } x, y \in B(y_0; \delta(\varepsilon)) \cap \omega.$$

Let $(y_m)_{m\geq 1}$ be any sequence of points in ω such that $\lim_{m\to\infty} y_m = y_0$, hence for any $\varepsilon > 0$, there exists $m_0(\varepsilon)$ such that $y_m \in B(y_0; \delta(\varepsilon))$ for all $m \geq m_0(\varepsilon)$. The last inequality then implies that the sequence $(F(y_m))_{m\geq 1}$ is a Cauchy sequence, hence $\lim_{m\to\infty} F(y_m)$ exists and it is clearly independent of the sequence $(y_m)_{m\geq 1}$. We define $F(y_0) := \lim_{m\to\infty} F(y_m)$, which shows that the field $F \in \mathcal{C}^2(\omega; \mathbb{M}^{p+q})$ can be extended to a field that is continuous on $\overline{\omega}$. Next, from relations (3.36) and from the fact that $A_\alpha \in \mathcal{C}^1(\overline{\omega}; \mathbb{M}^{p+q})$, we deduce that each field $\partial_\alpha F$ can be extended to a field that is continuous on $\overline{\omega}$, hence $F \in \mathcal{C}^1(\overline{\omega}; \mathbb{M}^{p+q})$. Finally, by differentiating the relations (3.36), we deduce that $F \in \mathcal{C}^2(\overline{\omega}; \mathbb{M}^{p+q})$.

(iv) *The mapping $\theta \in \mathcal{C}^3(\omega; \mathbb{R}^{p+q})$ belongs to the space $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$ and the vector fields $N^1, \ldots, N^q \in \mathcal{C}^2(\omega; \mathbb{R}^{p+q})$ belong to the space $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$.*

First, the vector fields $N^1, \ldots, N^q$ belong to the space $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$, as a direct consequence of the definition of the matrix field F and of the result established in part (iii).

In order to prove the second statement, we proceed as in part (iii): given any $y_0 \in \partial\omega$ and any $\varepsilon > 0$, the number $\delta(\varepsilon) > 0$ is being now chosen such that $L(\gamma) \leq \frac{\varepsilon}{\max\{c_1, 2\}}$, where the constant c_1 is given by

$$c_1 := \frac{1}{\sqrt{p}}\Big(\sup_{y\in K_0\cap\omega} |F(y)|\Big)^{-1} < +\infty.$$

Again without loss of generality, we assume that $\varepsilon \leq 1$ and $\delta(\varepsilon) \leq \frac{1}{2}$.

From the definition of the vector field F made in part (i), we have in particular that $\partial_\alpha\theta(y) = \mathbf{f}_\alpha(y)$ for all $y \in \omega$, which implies that the vector field $\varphi := \theta \circ \gamma \in \mathcal{C}^1([0,1]; \mathbb{R}^{p+q})$ associated with any such path γ joining x to y in ω satisfies

$$\varphi'(t) = \gamma'_\alpha(t)\mathbf{f}_\alpha(\gamma(t)) \text{ for all } 0 \leq t \leq 1.$$

Consequently, for any two points $x, y \in B(y_0; \delta(\varepsilon)) \cap \omega)$, we have

$$\begin{aligned}|\theta(y)-\theta(x)| =&|\varphi(1)-\varphi(0)|\\ \leq& \int_0^1 |\gamma'_\alpha(t)\mathbf{f}_\alpha(\gamma(t))|\mathrm{d}t\\ \leq& L(\gamma)\sup_{y\in K_0\cap\omega}\big(\sum_{\alpha=1}^p|\mathbf{f}_\alpha(y)|^2\big)^{1/2}\\ \leq&\sqrt{p}L(\gamma)\sup_{y\in K_0\cap\omega}|F(y)|\leq\varepsilon,\end{aligned}$$

the last inequality being satisfied thanks to the particular choice of $\delta(\varepsilon)$. Arguing as in (iii), we thus conclude that the mapping $\theta \in \mathcal{C}^3(\omega; \mathbb{R}^{p+q})$ can be extended to a field that is continuous on $\overline{\omega}$.

Differentiating the relations $\partial_\alpha\theta(y) = \mathbf{f}_\alpha(y)$ in ω and noting that $\mathbf{f}_\alpha \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ by (iii) finally imply that $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$. □

The result obtained in Corollary 3.1 can be also extended "up to the boundary" of the subset $\omega \subset \mathbb{R}^p$, as follows:

COROLLARY 3.2. *Let the assumptions on the set ω, on the symmetric matrix fields $A = (a_{\alpha\beta})$, $B^i = (b^i_{\alpha\beta})$, for $i \in \{1, \ldots, q\}$ and on the anti-symmetric matrix fields $T_\alpha = (T^{ij}_\alpha)$ be as in Theorem 3.2. Then there exist an immersion $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$ and an orthonormal family of q vector fields $N^1, \ldots, N^q \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ normal to $\theta(\overline{\omega})$ satisfying*

$$(3.37)\qquad (i)\ \langle\overline{\partial_\alpha\theta}(y), \overline{\partial_\beta\theta}(y)\rangle = \overline{a_{\alpha\beta}}(y),\ \forall y \in \overline{\omega},\ \forall\alpha,\beta\in\{1,\ldots,p\},$$

$$(3.38)\qquad (ii)\ \Big\langle\overline{\partial_{\alpha\beta}\theta}(y), \overline{N^i}(y)\Big\rangle = -\,\overline{b^i_{\alpha\beta}}(y)\ \forall y \in \overline{\omega},\ \forall\alpha,\beta\in\{1,\ldots,p\},\ \forall i\in\{1,\ldots,q\},$$

$$(3.39)\qquad (iii)\ \Big\langle\overline{\partial_\alpha N^i}(y), \overline{N^j}(y)\Big\rangle = \overline{T^{ij}_\alpha}(y)\ \forall y \in \overline{\omega},\ \forall\alpha\in\{1,\ldots,p\},\ \forall i,j\in\{1,\ldots,q\}.$$

Moreover, if a mapping $\tilde{\theta}$ and the vector fields $\tilde{N}^1, \ldots, \tilde{N}^q$ normal to $\tilde{\theta}(\overline{\omega})$ satisfy the same relations on $\overline{\omega}$, then there exist a vector $a \in \mathbb{R}^{p+q}$ and a matrix $Q \in \mathbb{O}^{p+q}$ such that

$$\begin{aligned}\overline{\tilde{\theta}}(y) =& a + Q\overline{\theta}(y)\ \textit{for all}\ y \in \overline{\omega},\\ \overline{\tilde{N}^i}(y) =& Q\overline{N^i}(y),\ i \in \{1,\ldots,q\},\ \textit{for all}\ y \in \overline{\omega}.\end{aligned}$$

Proof: Since the open subset ω, the matrix fields A, (B^i), (T_α), $i \in \{1, \ldots, q\}$ and $\alpha \in \{1, \ldots, p\}$, satisfy the hypotheses of Theorem 3.2, there exists a mapping $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$ and an orthonormal family of q vector fields $N^1, \ldots, N^q \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ normal to $\theta(\overline{\omega})$ such that conditions (3.37), (3.38) and (3.39) are satisfied.

If a mapping $\tilde{\theta}$ and the vector fields $\tilde{N}^1, \ldots, \tilde{N}^q$ normal to $\tilde{\theta}(\overline{\omega})$ also satisfy the same relations on $\overline{\omega}$, then by Corollary 3.1, there exist a vector $a \in \mathbb{R}^{p+q}$ and a matrix $Q \in \mathbb{O}^{p+q}$

such that

$$\begin{aligned}\tilde{\theta}(y) =& a + Q\theta(y) \text{ for all } y \in \omega \\ \tilde{N}^i(y) =& QN^i(y),\ i \in \{1, \dots, q\} \text{ for all } y \in \omega.\end{aligned}$$

Consequently, the continuous extensions $\overline{\theta}$, $\overline{N^1}, \dots, \overline{N^q}$ and $\overline{\tilde{\theta}}$, $\overline{\tilde{N}^1}, \dots, \overline{\tilde{N}^q}$ satisfy

$$\begin{aligned}\overline{\tilde{\theta}}(y) =& a + Q\overline{\theta}(y) \text{ for all } y \in \overline{\omega} \\ \overline{\tilde{N}^i}(y) =& Q\overline{N^i}(y),\ i \in \{1, \dots, q\} \text{ for all } y \in \overline{\omega}.\end{aligned}$$

□

4. Continuity of a submanifold with boundary

We establish in this section a continuity result for the mapping defined by the existence and uniqueness result of Theorem 3.2.

More specifically, let ω be a simply-connected open subset of $\mathbb{R}^p$ that satisfies the geodesic property (Definition 2.1). We introduce the following notation:

$$\begin{aligned}X(\overline{\omega}) :=& \{\big(A, (B^i), (T_\alpha)\big) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)\big)^p \\ & \text{such that relations (3.15), (3.16) and (3.17) are satisfied}\},\end{aligned} \tag{4.1}$$

where (3.15), (3.16) and (3.17) are the Gauss-Ricci-Codazzi equations written in local coordinates. Fix a point $y_0 \in \omega$ and consider the matrix G_0 given by relation (3.18). By Theorem 3.2, there exists a well-defined mapping

$$\mathcal{F}\colon X(\overline{\omega}) \to \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q$$

that associates with any point $\big(A, (B^i), (T_\alpha)\big) \in X(\overline{\omega})$ the uniquely determined element $\big(\theta, (N^i)\big) \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q$ satisfying

$$\begin{aligned}\left\langle \overline{\partial_\alpha \theta}(y), \overline{N^i}(y) \right\rangle =& 0\ \forall y \in \overline{\omega},\ \forall \alpha \in \{1, \dots, p\},\ \forall i \in \{1, \dots, q\}, \\ \langle \overline{\partial_\alpha \theta}(y), \overline{\partial_\beta \theta}(y) \rangle =& \overline{a_{\alpha\beta}}(y),\ \forall y \in \overline{\omega},\ \forall \alpha, \beta \in \{1, \dots, p\}, \\ \left\langle \overline{\partial_{\alpha\beta} \theta}(y), \overline{N^i}(y) \right\rangle =& -\overline{b^i_{\alpha\beta}}(y)\ \forall y \in \overline{\omega},\ \forall \alpha, \beta \in \{1, \dots, p\},\ \forall i \in \{1, \dots, q\}, \\ \left\langle \overline{\partial_\alpha N^i}(y), \overline{N^j}(y) \right\rangle =& \overline{T^{ij}_\alpha}(y)\ \forall y \in \overline{\omega},\ \forall \alpha \in \{1, \dots, p\},\ \forall i, j \in \{1, \dots, q\},\end{aligned}$$

such that $\theta(y_0) = \mathbf{0}$ and the α-th column of the matrix $G_0^{1/2}$ is $\partial_\alpha \theta(y_0)$ for all $\alpha \in \{1, \dots, p\}$.

The second objective of this paper is to study the properties of this mapping.

We first note that, if in addition the set ω is bounded, the spaces $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p)$, $\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)$, $\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)$, $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ and $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$, endowed with their natural norms, become Banach spaces. The product space $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)\big)^p$, endowed with the norm

$$\|\big(A, (B^i), (T_\alpha)\big)\|_{2,1,1,\overline{\omega}} = \|A\|_{2,\overline{\omega}} + \max_i \|B^i\|_{1,\overline{\omega}} + \max_\alpha \|T_\alpha\|_{1,\overline{\omega}}$$

becomes a Banach space and thus in this case the set $X(\overline{\omega})$ becomes a metric space when it is equipped with the induced topology. The product space $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q$, endowed with the norm

$$\|\big(\theta, (N^i)\big)\|_{3,2,\overline{\omega}} = \|\theta\|_{3,\overline{\omega}} + \max_i \|N^i\|_{2,\overline{\omega}}$$

is also a Banach space.

We are now in a position to state the main result of this section, which generalizes the results obtained in [**8**] and [**9**].

THEOREM 4.1. *Let ω be a simply-connected and bounded open subset of $\mathbb{R}^p$ that satisfies the geodesic property. Let the spaces $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)\big)^p$ and $\mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q$ be endowed with the norms $\|\cdot\|_{2,1,1,\overline{\omega}}$ and $\|\cdot\|_{3,2,\overline{\omega}}$ respectively, and let the set $X(\overline{\omega})$ be equipped with the metric induced by the norm $\|\cdot\|_{2,1,1,\overline{\omega}}$. Then the mapping*

$$\mathcal{F}\colon \big(A, (B^i), (T_\alpha)\big) \in X(\overline{\omega}) \to \big(\theta, (N^i)\big) \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q}) \times \big(\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})\big)^q,$$

is continuous. It is even locally Lipschitz-continuous over the set $X(\overline{\omega})$, in the sense of Definition 2.3.

Proof: For clarity, the theorem is established is a series of four steps, numbered (i) to (iv).

(i) *Preliminaries.* We recall that the image

$$\big(\theta, (N^i)\big) = \mathcal{F}\big(A, (B^i), (T_\alpha)\big)$$

of an arbitrary element of the space $X(\overline{\omega})$ is defined as follows (see the proof of Theorem 3.2):

First, we construct the matrix fields $\Gamma_\alpha \in \mathcal{C}^1(\overline{\omega}; \mathbb{M}^{p+q})$, for $\alpha \in \{1, \dots, p\}$.

Second, the matrix field $F \in \mathcal{C}^2(\overline{\omega}; \mathbb{M}^{p+q})$ is defined as being the unique one that satisfies

$$\begin{aligned} \partial_\alpha F &= F\Gamma_\alpha \text{ in } \omega, \\ F(y_0) &= G_0^{1/2}. \end{aligned}$$

Third, the mapping $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^{p+q})$ is defined as being the unique one that satisfies

$$\begin{aligned} \partial_\alpha \theta &= \mathbf{f}_\alpha \text{ in } \omega,\ \alpha \in \{1, \dots, p\}, \\ \theta(y_0) &= \mathbf{0}, \end{aligned}$$

and the vector fields $N^i \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^{p+q})$ are defined by the relations

$$N^i := \mathbf{f}_{p+i}, \text{ in } \omega, \ i \in \{1, \dots, q\},$$

where $\mathbf{f}_m(y)$ is the m-th column of the matrix $F(y)$, $m \in \{1, \dots, p+q\}$.

Accordingly, our strategy will consist in establishing that each of the above mappings is locally Lipschitz-continuous, since a composite mapping is locally Lipschitz-continuous if all its component mappings share this property.

(ii) *In order to simplify the notation, a generic element of the space* $X(\overline{\omega})$ *is denoted* $X := \big(A, (B^i), (T_\alpha)\big)$. *Let* $\hat{X} := \big(\hat{A}, (\hat{B}^i), (\hat{T}_\alpha)\big)$ *be an arbitrary point of the space* $X(\overline{\omega})$. *Then there exist constants* $c_1(\hat{X}) > 0$ *and* $\delta(\hat{X}) > 0$ *such that*

$$|G_0^{1/2} - \tilde{G}_0^{1/2}| + \max_\alpha \|\Gamma_\alpha - \tilde{\Gamma}_\alpha\|_{1,\overline{\omega}} \leq c_1(\hat{X}) \|X - \tilde{X}\|_{2,1,1,\overline{\omega}},$$

for all $X, \tilde{X} \in B(\hat{X}; \delta(\hat{X})) \cap X(\overline{\omega})$, *where the matrix fields* Γ_α, $\tilde{\Gamma}_\alpha$ *and the matrices* $G_0^{1/2}$, $\tilde{G}_0^{1/2}$ *are constructed in terms of the elements* $X, \tilde{X} \in X(\overline{\omega})$ *as in (i).*

Our strategy consists in establishing this inequality as a consequence of the local Lipschitz-continuity of the mapping

$$\text{(4.2)} \quad \begin{aligned} \psi\colon \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{R}^p)\big)^{q^2} &\to \big(\mathcal{C}^1(\overline{\omega}; \mathbb{M}^{p+q})\big)^p \times \mathbb{S}^{p+q}_>, \\ \big(A, (B^i), (T_\sigma)\big) &\mapsto \big(\Gamma_\alpha, G_0^{1/2}\big). \end{aligned}$$

To begin with, notice that it makes sense to study the differentiability of mappings defined over the set $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>)$, because this set in open in the Banach space $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p)$.

First, the mapping $A \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to G_0^{1/2}$ can be written as a composite mapping $Q \circ P$, where

$$P\colon A \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to G_0 \in \mathbb{S}^p_>$$

is linear and continuous, hence of class $\mathcal{C}^\infty$, and

$$Q\colon G \in \mathbb{S}^p_> \to G^{1/2} \in \mathbb{S}^p_>$$

is of class $\mathcal{C}^\infty$ by Lemma 2.2. Consequently, the mapping $A \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to G_0^{1/2}$ is also of class $\mathcal{C}^\infty$. Hence, it is locally Lipschitz-continuous, since the space $\mathbb{S}^p_>$ is finite-dimensional.

Consider next each mapping

$$\text{(4.3)} \quad \chi_\alpha\colon \big(A, (B^i), (T_\sigma)\big) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)\big)^p \to \Gamma_\alpha \in \mathcal{C}^1(\overline{\omega}; \mathbb{M}^{p+q}).$$

First, we claim that the mapping

$$\chi\colon A \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to \Gamma^\tau_{\alpha\beta} \in \mathcal{C}^1(\overline{\omega})$$

is locally Lipschitz-continuous (for brevity, the dependence with respect to the indices α, β, τ is dropped in the notation used for such a mapping).

On one hand, each linear mapping

$$A \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to (\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\alpha\sigma} - \partial_\sigma a_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega})$$

is clearly continuous, hence of class $\mathcal{C}^\infty$. On the other hand, each mapping

$$A = (a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to a^{\sigma\tau} \in \mathcal{C}^2(\overline{\omega}),$$

that associates to the matrix field A a function $a^{\sigma\tau}$, obtained as a quotient by $\det A$ of a homogeneous polynomial $c^{\sigma\tau}(A)$ of degree $(p-1)$ in terms of the functions $a_{\alpha\beta}$, is also of class $\mathcal{C}^\infty$. Indeed, we first note that each mapping

$$A \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to \big(\big(c^{\sigma\tau}(A)\big), \det A\big) \in \big(\mathcal{C}^2(\overline{\omega})\big)^{p^2+1}$$

is of class $\mathcal{C}^\infty$, because each one of its components is a sum of continuous multilinear mappings. Therefore, since

$$\det A \in U := \{f \in \mathcal{C}^2(\overline{\omega}); \overline{f}(y) > 0 \text{ for all } y \in \overline{\omega}\},$$

it suffices to establish that the mapping

$$\varphi\colon\, f \in U \subset \mathcal{C}^2(\overline{\omega}) \to \frac{1}{f} \in \mathcal{C}^2(\overline{\omega})$$

is of class $\mathcal{C}^\infty$ (note that this question makes sense, since the set U is open in $\mathcal{C}^2(\overline{\omega})$). For a proof, see [**8**, Theorem 3.1].

We proved until now that the mapping

$$\chi\colon\, A \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to \Gamma^\tau_{\alpha\beta} \in \mathcal{C}^1(\overline{\omega})$$

is of class $\mathcal{C}^\infty$, hence in particular it is Fréchet differentiable.

The next step consists in showing that its Fréchet derivative is locally bounded. To this end, we compute the difference $\chi(A+\Delta A)-\chi(A)$ for a variation $\Delta A = (\Delta a_{\sigma\tau}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p)$ at $A = (a_{\alpha\beta})$ and then we take the linear part with respect to ΔA. It is found in this fashion that $D\chi(A)\Delta A$ it is a sum of polynomials of degree $(p-1)$ in terms of the functions $a_{\alpha\beta}$ and of degree one in terms of the functions $\partial_\gamma a_{\nu\mu}$, times some components $\Delta a_{\sigma\tau}$, and divided by $\det(a_{\alpha\beta})$ or $(\det(a_{\alpha\beta}))^2$. Hence, given any two constants $M > 0$ and $d > 0$, there exists a constant $c(M, d) > 0$ such that

$$\|D\chi(A)\|_{\mathcal{L}(\mathcal{C}^2(\overline{\omega};\mathbb{S}^p);\mathcal{C}^1(\overline{\omega}))} \leq c(M, d)$$

for any matrix field $A \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>)$ that satisfies

$$\|A\|_{2,\overline{\omega}} \leq M \text{ and } \det \overline{A}(y) \geq d \text{ for all } y \in \overline{\omega}.$$

We thus conclude that the mapping χ is locally Lipschitz-continuous.

We consider next any one of the mappings

$$\Phi\colon\, (A, B^i) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^p) \to b^{i\tau}_\sigma \in \mathcal{C}^1(\overline{\omega}).$$

The mapping

$$B^i \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^p) \to b^i_{\sigma\tau} \in \mathcal{C}^1(\overline{\omega})$$

is linear and continuous, hence of class $\mathcal{C}^\infty$, and we have already proved that the mapping

$$A = (a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega}; \mathbb{S}^p_>) \to a^{\nu\tau} \in \mathcal{C}^2(\overline{\omega})$$

is of class $\mathcal{C}^\infty$. Consequently, the mapping Φ is also of class $\mathcal{C}^\infty$. Then, by applying the same method as that used in the proof for the mapping $\mathcal{X}$, we establish that the Fréchet derivative of the mapping Φ is locally bounded, allowing us to conclude that the mapping Φ is also locally Lipschitz-continuous.

The linear mappings

$$B^i \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^p) \to b^i_{\sigma\tau} \in \mathcal{C}^1(\overline{\omega}), \tag{4.4}$$

$$T_\alpha \in \mathcal{C}^1(\overline{\omega}; \mathbb{A}^q) \to T^{ij}_\alpha \in \mathcal{C}^1(\overline{\omega}) \tag{4.5}$$

are clearly continuous; hence they are of class $\mathcal{C}^\infty$ and their Fréchet derivatives are bounded over $B(\widehat{B^i}; 1)$ and $B(\widehat{T_\alpha}; 1)$. Hence, we deduce that the mappings defined by (4.4) and (4.5) are locally Lipschitz-continuous.

Finally, we conclude that each mapping $\mathcal{X}_\alpha$ defined in relation (4.3) is locally Lipschitz-continuous and consequently, that the mapping ψ given by the relation (4.2) is also locally Lipschitz-continuous.

(iii) *Define the matrix fields* $\Gamma_\alpha, \tilde{\Gamma}_\alpha \in \mathcal{C}^1(\overline{\omega}; \mathbb{M}^{p+q})$ *as in (i), the matrices* $G_0, \tilde{G}_0$ *as in relation* (3.18) *and let the matrix fields* $F, \tilde{F} \in \mathcal{C}^2(\overline{\omega}; \mathbb{M}^{p+q})$ *satisfy*

$$\partial_\alpha F(y) = F(y)\Gamma_\alpha(y) \text{ for all } y \in \omega \text{ and } F(y_0) = G_0^{1/2}, \tag{4.6}$$

$$\partial_\alpha \tilde{F}(y) = \tilde{F}(y)\tilde{\Gamma}_\alpha(y) \text{ for all } y \in \omega \text{ and } \tilde{F}(y_0) = \tilde{G}_0^{1/2}. \tag{4.7}$$

Then, for any given element $\hat{X} \in X(\overline{\omega})$, *there exists a constant* $c_2(\hat{X}) > 0$ *such that*

$$\|F - \tilde{F}\|_{2,\overline{\omega}} \leq c_2(\hat{X})\big(|G_0^{1/2} - \tilde{G}_0^{1/2}| + \max_\alpha \|\Gamma_\alpha - \tilde{\Gamma}_\alpha\|_{1,\overline{\omega}}\big)$$

for all $X, \tilde{X} \in B(\hat{X}; \delta(\hat{X})) \cap X(\overline{\omega})$, *where* $\delta(\hat{X}) > 0$ *is the constant found in (ii).*

Since the open set ω satisfies the geodesic property and is bounded, Lemma 2.1 implies that its geodesic diameter is bounded. By definition of D_ω, there thus exists a constant $\Lambda > 0$ such that, given any $y \in \omega$, there exists a path γ joining y_0 to y whose length satisfies $L(\gamma) \leq \Lambda$. Fix $y \in \omega$, consider such a path $\gamma = (\gamma_\alpha)$ and a matrix field $Z := (F - \tilde{F}) \circ \gamma \in \mathcal{C}^1([0,1]; \mathbb{M}^{p+q})$ associated with the path γ. The relations (4.6) and (4.7) then imply that the matrix field Z satisfies

$$Z'(t) = \gamma'_\alpha(t)Z(t)\Gamma_\alpha(\gamma(t)) + \gamma'_\alpha(t)\tilde{F}(\gamma(t))\big(\Gamma_\alpha(\gamma(t)) - \tilde{\Gamma}_\alpha(\gamma(t))\big) \text{ for all } 0 \leq t \leq 1,$$

so that, by Lemma 2.3,

$$\begin{aligned} |Z(1)| \leq & |Z(0)| \exp\Big(\int_0^1 \gamma'_\alpha(\tau)\Gamma_\alpha(\gamma(\tau))d\tau\Big) \\ & + \int_0^1 |\gamma'_\alpha(s)\tilde{F}(\gamma(s))\big(\Gamma_\alpha(\gamma(s)) - \tilde{\Gamma}_\alpha(\gamma(s))\big)| \exp\Big(\int_s^1 \gamma'_\alpha(\tau)\Gamma_\alpha(\gamma(\tau))d\tau\Big) ds. \end{aligned}$$

First, we infer from Cauchy-Schwarz inequality that

$$\begin{aligned} \int_s^1 \gamma'_\alpha(\tau)\Gamma_\alpha(\gamma(\tau))d\tau \leq & \int_0^1 \big(\sum_\alpha |\gamma'_\alpha(\tau)|^2\big)^{1/2}\big(\sum_\alpha |\Gamma_\alpha(\gamma(\tau))|^2\big)^{1/2} d\tau \\ \leq & \Lambda\sqrt{p} \max_\alpha \|\Gamma_\alpha\|_{1,\overline{\omega}} \end{aligned}$$

for all $0 \leq s \leq 1$. Then, by using the inequality found in part (ii) for any $X \in B(\hat{X}; \delta(\hat{X})) \cap X(\overline{\omega})$, we deduce that the associated matrix fields $\Gamma_\alpha \in \mathcal{C}^1([0,1]; \mathbb{M}^{p+q})$ are such that

$$\max_\alpha \|\Gamma_\alpha\|_{1,\overline{\omega}} \leq \max_\alpha \|\hat{\Gamma}_\alpha\|_{1,\overline{\omega}} + c_1(\hat{X})\delta(\hat{X}) =: a_1(\hat{X}).$$

Consequently,

$$\int_s^1 \gamma'_\alpha(\tau)\Gamma_\alpha(\gamma(\tau))d\tau \leq \Lambda\sqrt{p}a_1(\hat{X}),$$

for all $0 \leq s \leq 1$, and likewise,

$$\int_0^1 |\gamma'_\alpha(s)\tilde{F}(\gamma(s))\big(\Gamma_\alpha(\gamma(s)) - \tilde{\Gamma}_\alpha(\gamma(s))\big)| ds \leq \sqrt{p}\Lambda\big(\sup_{y\in\omega} |\tilde{F}(y)|\big) \max_\alpha \sup_{y\in\omega} |\Gamma_\alpha(y) - \tilde{\Gamma}_\alpha(y)|.$$

Since $Z(1) = (F - \tilde{F})(y)$ and $Z(0) = G_0^{1/2} - \tilde{G}_0^{1/2}$, we have thus shown until now that

$$\begin{aligned} \sup_{y\in\omega} |(F - \tilde{F})(y)| \leq & \exp(\sqrt{p}\Lambda a_1(\hat{X}))\big(|G_0^{1/2} - \tilde{G}_0^{1/2}| + \\ & + \sqrt{p}\Lambda \sup_{y\in\omega} |\tilde{F}(y)| \max_\alpha \|\Gamma_\alpha(y) - \tilde{\Gamma}_\alpha(y)\|_{1,\overline{\omega}}\big). \end{aligned} \tag{4.8}$$

In particular, the last inequality implies for the matrix fields $\tilde{F}$ and $\hat{F}$ that

$$\begin{aligned} \sup_{y\in\omega} |\tilde{F}(y)| \leq & \sup_{y\in\omega} |\hat{F}(y)| + \exp(\sqrt{p}\Lambda a_1(\hat{X}))\big(|\tilde{G}_0^{1/2} - \hat{G}_0^{1/2}| + \\ & + \sqrt{p}\Lambda \sup_{y\in\omega} |\hat{F}(y)| \max_\alpha \|\tilde{\Gamma}_\alpha(y) - \hat{\Gamma}_\alpha(y)\|_{1,\overline{\omega}}\big) \\ \leq & \sup_{y\in\omega} |\hat{F}(y)| + \exp(\sqrt{p}\Lambda a_1(\hat{X}))\big(c_1(\hat{X})\delta(\hat{X}) + \\ & + \sqrt{p}\Lambda \sup_{y\in\omega} |\hat{F}(y)| c_1(\hat{X})\delta(\hat{X})\big) =: a_2(\hat{X}). \end{aligned}$$

If we combine this last result with relation (4.8), we eventually obtain

$$\begin{aligned}\sup_{y\in\omega}|(F-\tilde F)(y)| \le & \exp(\sqrt{p}\Lambda a_1(\hat X))\big(|G_0^{1/2}-\tilde G_0^{1/2}|+\\ &+\sqrt{p}\Lambda a_2(\hat X)\max_\alpha\|\Gamma_\alpha(y)-\tilde\Gamma_\alpha(y)\|_{1,\overline\omega}\big).\end{aligned}\tag{4.9}$$

Next, the relations

$$\partial_\alpha(F-\tilde F)=(F-\tilde F)\Gamma_\alpha+\tilde F(\Gamma_\alpha-\tilde\Gamma_\alpha) \text{ in } \omega$$

imply that

$$\sup_{y\in\omega}|\partial_\alpha(F-\tilde F)(y)|\le a_1(\hat X)\sup_{y\in\omega}|(F-\tilde F)(y)|+a_2(\hat X)\|\Gamma_\alpha(y)-\tilde\Gamma_\alpha(y)\|_{1,\overline\omega}\tag{4.10}$$

and the relations

$$\partial_{\alpha\beta}(F-\tilde F)=\partial_\beta(F-\tilde F)\Gamma_\alpha+(F-\tilde F)\partial_\beta\Gamma_\alpha+\partial_\beta\tilde F(\Gamma_\alpha-\tilde\Gamma_\alpha)+\tilde F\partial_\beta(\Gamma_\alpha-\tilde\Gamma_\alpha) \text{ in } \omega$$

similarly imply that

$$\begin{aligned}\sup_{y\in\omega}|\partial_{\alpha\beta}(F-\tilde F)(y)| \le & a_1(\hat X)\big(\sup_{y\in\omega}|\partial_\alpha(F-\tilde F)(y)|+\sup_{y\in\omega}|(F-\tilde F)(y)|\big)+\\ &+a_2(\hat X)(1+a_2(\hat X))\|\Gamma_\alpha(y)-\tilde\Gamma_\alpha(y)\|_{1,\overline\omega}.\end{aligned}\tag{4.11}$$

Finally, from inequalities (4.9), (4.10) and (4.11), we deduce the announced upper bound for the norm $\|F-\tilde F\|_{2,\overline\omega}$.

(iv) *The matrix fields $F,\tilde F\in\mathcal C^2(\overline\omega;\mathbb M^{p+q})$ being defined as in (iii), let the mappings $\theta,\tilde\theta\in\mathcal C^3(\overline\omega;\mathbb R^{p+q})$ satisfy*

$$\partial_\alpha\theta(y)=\mathbf f_\alpha(y) \text{ for all } y\in\omega \text{ and } \theta(y_0)=\mathbf 0,$$
$$\partial_\alpha\tilde\theta(y)=\tilde{\mathbf f}_\alpha(y) \text{ for all } y\in\omega \text{ and } \tilde\theta(y_0)=\mathbf 0,$$

where $\mathbf f_\alpha(y)$ and $\tilde{\mathbf f}_\alpha(y)$ respectively denote the α-th column of $F(y)$ and $\tilde F(y)$. Let the vector fields $N^1,\dots,N^q\in\mathcal C^2(\overline\omega;\mathbb R^{p+q})$ and $\tilde N^1,\dots,\tilde N^q\in\mathcal C^2(\overline\omega;\mathbb R^{p+q})$ be given by

$$N^i(y)=\mathbf f_{p+i}(y) \text{ for all } y\in\omega \text{ and for all } i\in\{1,\dots,q\},$$
$$\tilde N^i(y)=\tilde{\mathbf f}_{p+i}(y) \text{ for all } y\in\omega \text{ and for all } i\in\{1,\dots,q\},$$

where $\mathbf f_{p+i}(y)$ and $\tilde{\mathbf f}_{p+i}(y)$ respectively, denote the $(p+i)$-th column of $F(y)$ and $\tilde F(y)$. Then there exists a constant $c_3>0$ independent of these fields such that

$$\|\big(\theta,(N^i)\big)-\big(\tilde\theta,(\tilde N^i)\big)\|_{3,2,\overline\omega}\le c_3\|F-\tilde F\|_{2,\overline\omega}.$$

To prove this inequality, we proceed as in part (iii): for any fixed point $y\in\omega$, we can find a path $\gamma=(\gamma_\alpha)$ joining y_0 to y, such that $L(\gamma)\le\Lambda$. Let the vector field $z\in\mathcal C^1([0,1];\mathbb R^{p+q})$ be defined by $z:=(\theta-\tilde\theta)\circ\gamma$. Then, from the relations satisfied par θ and $\tilde\theta$, we infer that

$$z'(t)=\gamma'_\alpha(t)\big(\mathbf f_\alpha(\gamma(t))-\tilde{\mathbf f}_\alpha(\gamma(t))\big)$$

for all $0 \leq t \leq 1$. Noting that

$$\theta(y) - \tilde{\theta}(y) = z(1) - z(0) = \int_0^1 z'(t)dt,$$

we have that

$$\begin{aligned}|\theta(y) - \tilde{\theta}(y)| \leq & \int_0^1 |\gamma'_\alpha(t)\big(\mathbf{f}_\alpha(\gamma(t)) - \tilde{\mathbf{f}}_\alpha(\gamma(t))\big)|dt \\ \leq & L(\gamma) \sup_{y\in\omega} \big(\sum_\alpha |(\mathbf{f}_\alpha - \tilde{\mathbf{f}}_\alpha)(y)|^2\big)^{1/2} \\ \leq & \Lambda\sqrt{p} \sup_{y\in\omega} |(F - \tilde{F})(y)| \\ \leq & \Lambda\sqrt{p}\|F - \tilde{F}\|_{2,\overline{\omega}}.\end{aligned}$$

From the last inequality and from the fact that

$$\|\partial_\alpha(\theta - \tilde{\theta})\|_{2,\overline{\omega}} = \|\mathbf{f}_\alpha - \tilde{\mathbf{f}}_\alpha\|_{2,\overline{\omega}} \leq \|F - \tilde{F}\|_{2,\overline{\omega}}$$

we obtain on the one hand that

$$\|\theta - \tilde{\theta}\|_{3,\overline{\omega}} \leq c\|F - \tilde{F}\|_{2,\overline{\omega}},$$

where $c > 0$ is a real constant. On the other hand, we have that

$$\|N^i - \tilde{N}^i\|_{2,\overline{\omega}} = \|\mathbf{f}_{p+i} - \tilde{\mathbf{f}}_{p+i}\|_{2,\overline{\omega}} \leq \|F - \tilde{F}\|_{2,\overline{\omega}}$$

for all $i \in \{1, \dots, q\}$.

The announced upper bound on the norm $\|\big(\theta, (N^i)\big) - \big(\tilde{\theta}, (\tilde{N}^i)\big)\|_{3,2,\overline{\omega}}$ thus follows from the last two inequalities. □

Remarks. 1. Since $X(\overline{\omega})$ is not an open subset of the space $\mathcal{C}^2(\overline{\omega}; \mathbb{S}^p) \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{S}^p)\big)^q \times \big(\mathcal{C}^1(\overline{\omega}; \mathbb{A}^q)\big)^p$, the Fréchet differentiability of the mapping $\mathcal{F}$ cannot be defined in the usual manner. Otherwise this would have been a convenient way to establish that the mapping $\mathcal{F}$ is pointwise Lipschitz-continuous.

2. We emphasize the important role played by the hypothesis that ω satisfies the *geodesic property*, which appears both in Theorems 3.2 and 4.1: it provides estimates on the solutions of ordinary differential equations along a path joining two points in ω that eventually depend only on the length of the path, but not on the path itself. □

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CHAPITRE 3

Sur des inégalités de Korn linéaires et non linéaires pour les courbes

Linear and nonlinear Korn inequalities on curves in $\mathbb{R}^3$

Joint work with Sorin Mardare

ABSTRACT. We establish several linear and nonlinear inequalities of Korn's type for curves in the three-dimensional euclidean space. These inequalities are obtained under weak regularity assumptions on the curve of reference. We also establish an infinitesimal rigid displacement lemma for curves.

1. Introduction

Inequalities of Korn's type play an important rôle in the mathematical analysis of linearized elasticity, since they provide a basic tool for the existence of solutions of linearized displacement-traction equations (see Section 1.7 and Chapter 2 of Ciarlet [**4**], and the references therein, for the three-dimensional Korn inequality and Korn's inequality on a surface, or the article of Chen and Jost [**2**] for a Riemannian version of these results).

The three-dimensional linear Korn inequality involves the *linearized change of metric tensor* and the linear Korn inequality for surfaces involves the *linearized change of metric* and the *linearized change of curvature tensors.* Therefore, it is natural to expect that the *linearized change of metric, change of curvature, and change of torsion, tensors* will appear in linear Korn inequalities for curves.

The *first* objective of this paper is to establish, by analogy to the three-dimensional and bi-dimensional cases, *a linear Korn inequality* for curves in a three-dimensional euclidean space. Our strategy follows the "classical" steps: *first*, we establish a Korn inequality "without boundary conditions", *second*, we prove a linearized rigid displacement lemma and *third*, the Korn inequality is obtained as a consequence of the above Korn inequality "without boundary conditions" and of the rigid displacement lemma.

The *second* objective of this paper is to provide several *nonlinear inequalities of Korn's type* for curves in the three-dimensional euclidean space. To this end, we fix a curve $\boldsymbol{\gamma}$, which is seen as the *reference* curve, and we derive some estimates of the norm of the difference $\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$, where $\tilde{\boldsymbol{\gamma}}$ is a *deformed* curve, *i.e.*, the curve defined by $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} + \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is a displacement vector field. As corollaries, we also obtain some inequalities of Korn's type for isometric deformations.

We emphasize that the linear Korn inequality is obtained for curves *"with little regularity"*, in the sense that the regularity assumptions on the reference curve involve the derivatives in the distributional sense, instead of classical derivatives (for a similar approach in the case of surfaces, see Blouza and Le Dret [**1**]).

2. Notations

The three dimensional euclidean space is identified with $\mathbb{R}^3$ and it is endowed with the Euclidean norm defined by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$, where $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the Euclidean inner product of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. The Euclidean vector product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ is denoted $\boldsymbol{a} \times \boldsymbol{b}$. The space of all real matrices with n rows and q columns is denoted $\mathbb{M}^{n \times q}$, or $\mathbb{M}^n$ if $n = q$, and is endowed with the operator norm $|\cdot|$, defined by

$$|\boldsymbol{A}| := \sup_{\boldsymbol{v} \in \mathbb{R}^q \setminus \{\boldsymbol{0}\}} \frac{|\boldsymbol{A}\boldsymbol{v}|}{|\boldsymbol{v}|},$$

for all $\boldsymbol{A} \in \mathbb{M}^{n \times q}$. The identity matrix in $\mathbb{M}^n$ is denoted $\boldsymbol{I}_n$ and the set of all ortogonal matrices $\boldsymbol{Q} \in \mathbb{M}^n$ with $\det \boldsymbol{Q} = 1$ is denoted $\mathbb{O}^n_+$.

Let $I = (0, L)$ be a bounded interval of $\mathbb{R}$. The derivatives of a function f with respect to $t \in I$ are denoted f', f'', etc. The usual Lebesgue, resp. Sobolev, spaces are denoted $L^p(I; \mathbb{M}^{n \times q})$, resp. $W^{m,p}(I; \mathbb{M}^{n \times q})$. For real-valued function spaces we shall use the notation $L^p(I)$ instead of $L^p(I; \mathbb{R})$, $W^{m,p}(I)$ instead of $W^{m,p}(I; \mathbb{R})$, etc. The space of all continuous functions on I is denoted $\mathcal{C}(I)$.

The geometrical framework of our problem is the following: let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3$ be a *regular curve*, in the sense that the vectors $\boldsymbol{\gamma}'(t)$ and $\boldsymbol{\gamma}''(t)$ are linearly independent for all $t \in I$. One can then show (see, e.g., [**8**]) that there exists a unique Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ associated with this curve, where $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B} : I \to \mathbb{R}^3$ are respectively the tangent, normal, and binormal vector fields.

The *Frenet equations* associated with $\boldsymbol{\gamma}$ read, in matrix form,

$$\begin{pmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{pmatrix} = |\gamma'| \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix},$$

where k and τ denote respectively the curvature and the torsion of the curve $\boldsymbol{\gamma}$, $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$ are row vector fields and $\boldsymbol{T}'$, $\boldsymbol{N}'$, and $\boldsymbol{B}'$ are their derivatives.

The curvature and the torsion of the curve $\boldsymbol{\gamma}$ are given by

$$k(t) = \frac{|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)|}{|\boldsymbol{\gamma}'(t)|^3} \tag{2.1}$$

and

$$\tau(t) = \frac{\det(\boldsymbol{\gamma}'(t), \boldsymbol{\gamma}''(t), \boldsymbol{\gamma}'''(t))}{|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)|^2}, \tag{2.2}$$

where $\boldsymbol{\gamma}'(t)$, $\boldsymbol{\gamma}''(t)$, and $\boldsymbol{\gamma}'''(t)$ are column vector fields. The metric of the curve $\boldsymbol{\gamma}$ is a^2, where $a := |\boldsymbol{\gamma}'|$ is the velocity.

In this section, we gather various preliminary results that will be subsequently needed. The first lemma establishes the existence and uniqueness of the solution to a system of differential equations under weak regularity assumptions on the initial data. In what follows, the derivatives are to be understood in the distributional sense and classes of functions in the Sobolev spaces $W^{1,1}((0,L);\mathbb{M}^{n\times q})$ are identified with their continuous representatives, in view of the Sobolev embedding theorem. In particular, the relation (2.4) below means that the continuous representative of $\boldsymbol{Y} \in W^{1,1}((0,L);\mathbb{M}^{n\times q})$ takes the value $\overline{\boldsymbol{Y}}$ at $t = 0$.

LEMMA 2.1. *Consider the system of differential equations:*

$$\boldsymbol{Y}'(t) = \boldsymbol{A}(t)\boldsymbol{Y}(t) + \boldsymbol{B}(t) \text{ for almost all } t \text{ in } (0,L), \tag{2.3}$$

$$\boldsymbol{Y}(0) = \overline{\boldsymbol{Y}}, \tag{2.4}$$

where $\boldsymbol{A} \in L^1((0,L);\mathbb{M}^n)$, $\boldsymbol{B} \in L^1((0,L);\mathbb{M}^{n\times q})$ and $\overline{\boldsymbol{Y}}$ is a matrix of $\mathbb{M}^{n\times q}$.

Then there exists a unique solution $\boldsymbol{Y} \in W^{1,1}((0,L);\mathbb{M}^{n\times q})$ to this system.

PROOF. Although the proof of this lemma is elementary, we nevertheless record it here for completeness. It consists in three steps.

(i) *If $\boldsymbol{A} \in L^1((a,b);\mathbb{M}^n)$, $\boldsymbol{X}_0 \in W^{1,1}((a,b);\mathbb{M}^{n\times q})$ and, for each integer $m \geq 1$, $\boldsymbol{X}_m(t) := \int_a^t \boldsymbol{A}(s)\boldsymbol{X}_{m-1}(s)\,ds$ for all $t \in (a,b)$, then $\boldsymbol{X}_m \in W^{1,1}((a,b);\mathbb{M}^{n\times q})$ and*

$$\|\boldsymbol{X}_m\|_{L^\infty(a,b)} \leq \|\boldsymbol{A}\|^m_{L^1(a,b)} \|\boldsymbol{X}_0\|_{L^\infty(a,b)}\,.$$

Since

$$|\boldsymbol{X}_m(t)| \leq \int_a^t |\boldsymbol{A}(s)||\boldsymbol{X}_{m-1}(s)|\,ds \leq \int_a^b |\boldsymbol{A}(s)||\boldsymbol{X}_{m-1}(s)|\,ds \leq \|\boldsymbol{A}\|_{L^1(a,b)}\|\boldsymbol{X}_{m-1}\|_{L^\infty(a,b)}$$

for all $t \in (a,b)$, we deduce that $\|\boldsymbol{X}_m\|_{L^\infty(a,b)} \leq \|\boldsymbol{A}\|_{L^1(a,b)}\|\boldsymbol{X}_{m-1}\|_{L^\infty(a,b)}$. The desired inequality follows immediately by a recursion argument.

(ii) *Existence of a solution to the system* (2.3)-(2.4).

We use the classical method of approximating solutions. Since $\boldsymbol{A} \in L^1((0,L);\mathbb{M}^n)$, there exists $\varepsilon > 0$ such that $\|\boldsymbol{A}\|_{L^1(a,b)} \leq \frac{1}{2}$ for all $a, b \in (0,L)$ with $|a-b| \leq \varepsilon$. For some integer $r \in \mathbb{N}$, we have $(0,L) = (0,\varepsilon] \cup (\varepsilon, 2\varepsilon] \cup \cdots \cup (r\varepsilon, L)$. We will first solve the system on $(0,\varepsilon)$, then on $(\varepsilon, 2\varepsilon)$, and so on.

In order to construct the solution on $(0,\varepsilon)$, we define the following sequence of approximate solutions:

$$\boldsymbol{Y}_0(t) = \boldsymbol{0} \text{ for all } t \in (0,\varepsilon),$$

$$\boldsymbol{Y}_m(t) := \overline{\boldsymbol{Y}} + \int_0^t \big(\boldsymbol{A}(s)\boldsymbol{Y}_{m-1}(s) + \boldsymbol{B}(s)\big)\,ds,\ m \geq 1,\ \text{for all } t \in (0,\varepsilon). \tag{2.5}$$

By substracting $\boldsymbol{Y}_{m-1}$ from $\boldsymbol{Y}_m$, we obtain $(\boldsymbol{Y}_m - \boldsymbol{Y}_{m-1})(t) = \int_0^t \boldsymbol{A}(s)(\boldsymbol{Y}_{m-1} - \boldsymbol{Y}_{m-2})(s)\,ds$ for all $m \geq 2$. Note that $(\boldsymbol{Y}_1 - \boldsymbol{Y}_0)(t) = \boldsymbol{Y}_1(t) = \overline{\boldsymbol{Y}} + \int_0^t \boldsymbol{B}(s)\,ds$, so that $\boldsymbol{Y}_1 - \boldsymbol{Y}_0 = \boldsymbol{Y}_1 \in W^{1,1}((0,\varepsilon);\mathbb{M}^{n\times q})$. By applying the result of step (i) with $\boldsymbol{X}_m := \boldsymbol{Y}_{m+1} - \boldsymbol{Y}_m$, $m \geq 0$, we next obtain

$$\|\boldsymbol{Y}_m - \boldsymbol{Y}_{m-1}\|_{L^\infty(0,\varepsilon)} \leq \|\boldsymbol{A}\|^{m-1}_{L^1(0,\varepsilon)} \|\boldsymbol{Y}_1\|_{L^\infty(0,\varepsilon)} \leq \frac{1}{2^{m-1}} \|\boldsymbol{Y}_1\|_{L^\infty(0,\varepsilon)}\,.$$

This shows that $(\boldsymbol{Y}_m)$ is a Cauchy sequence in the space $L^\infty((0,\varepsilon);\mathbb{M}^{n\times q})$. Consequently, there exists $\boldsymbol{Y}[0] \in L^\infty((0,\varepsilon);\mathbb{M}^{n\times q})$ such that $\boldsymbol{Y}_m \to \boldsymbol{Y}[0]$ as $m \to \infty$ in $L^\infty((0,\varepsilon);\mathbb{M}^{n\times q})$. By passing to limit in (2.5), we deduce that

$$\boldsymbol{Y}[0](t) = \overline{\boldsymbol{Y}} + \int_0^t \big(\boldsymbol{A}(s)\boldsymbol{Y}[0](s) + \boldsymbol{B}(s)\big)\,ds \text{ for all } t \in (0,\varepsilon),$$

hence $\boldsymbol{Y}[0] \in W^{1,1}((0,\varepsilon);\mathbb{M}^{n\times q})$. The continuous representative of $\boldsymbol{Y}[0]$, for simplicity still denoted $\boldsymbol{Y}[0]$, satisfies in addition the initial condition $\boldsymbol{Y}[0](0) = \overline{\boldsymbol{Y}}$.

Now we construct the solution $\boldsymbol{Y}[1]$ on the interval $(\varepsilon, 2\varepsilon)$ as the limit of the following sequence:

$$\begin{aligned}
&\boldsymbol{Y}_0(t) = \boldsymbol{0} \text{ for all } t \in (\varepsilon, 2\varepsilon),\\
&\boldsymbol{Y}_m(t) = \boldsymbol{Y}[0](\varepsilon) + \int_\varepsilon^t \big(\boldsymbol{A}(s)\boldsymbol{Y}_{m-1}(s) + \boldsymbol{B}(s)\big)\,ds,\ m \geq 1, \text{ for all } t \in (\varepsilon, 2\varepsilon)
\end{aligned}$$

As before, the sequence $(\boldsymbol{Y}_m)$ converges in $L^\infty((\varepsilon,2\varepsilon);\mathbb{M}^{n\times q})$ to a limit $\boldsymbol{Y}[1]$ that satisfies

$$\boldsymbol{Y}[1](t) = \boldsymbol{Y}[0](\varepsilon) + \int_\varepsilon^t \big(\boldsymbol{A}(s)\boldsymbol{Y}[1](s) + \boldsymbol{B}(s)\big)\,ds \text{ for all } t \in (\varepsilon, 2\varepsilon).$$

We continue this algorithm until the last interval $(r\varepsilon, L)$. In this fashion, we define $(r+1)$ matrix fields $\boldsymbol{Y}[i]$ that satisfy, for all $i \in \{0,1,\ldots,r\}$,

$$\boldsymbol{Y}[i](t) = \boldsymbol{Y}[i-1](i\varepsilon) + \int_{i\varepsilon}^t \big(\boldsymbol{A}(s)\boldsymbol{Y}[i](s) + \boldsymbol{B}(s)\big)\,ds$$

for all $t \in \big(i\varepsilon, \min\{(i+1)\varepsilon, L\}\big)$, where $\boldsymbol{Y}[-1](0) := \overline{\boldsymbol{Y}}$.

Let $\boldsymbol{Y} = \boldsymbol{Y}[0]\mathbf{1}_{[0,\varepsilon)} + \boldsymbol{Y}[1]\mathbf{1}_{[\varepsilon,2\varepsilon)} + \cdots + \boldsymbol{Y}[r]\mathbf{1}_{[r\varepsilon,L]}$, where $\mathbf{1}_J$ is the characteristic function of the interval J. Then it is easy to verify that the matrix field $\boldsymbol{Y}$ belongs to the space $W^{1,1}((0,L);\mathbb{M}^{n\times q})$ and satisfies

$$\boldsymbol{Y}(t) = \overline{\boldsymbol{Y}} + \int_0^t \big(\boldsymbol{A}(s)\boldsymbol{Y}(s) + \boldsymbol{B}(s)\big)\,ds \text{ for all } t \in (0,L).$$

This implies that $\boldsymbol{Y}$ is a solution of the system (2.3).

(iii) *Uniqueness of the solution to the system* (2.3)-(2.4).

It suffices to prove that $\boldsymbol{X} = \boldsymbol{0}$ is the unique solution in $W^{1,1}((0,L);\mathbb{M}^{n\times q})$ to the system

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} \text{ a.e. in } (0,L), \tag{2.6}$$

$$\boldsymbol{X}(0) = \boldsymbol{0}. \tag{2.7}$$

Define the set $\mathcal{A} := \{t \in [0,L]\ ;\ \boldsymbol{X}(t) = \boldsymbol{0}\}$. Since $0 \in \mathcal{A}$, $\mathcal{A}$ is non empty. It thus suffices to prove that $\mathcal{A}$ is open and closed in $[0,L]$. First, the set $\mathcal{A}$ is closed in $[0,L]$, since $\boldsymbol{X} \in W^{1,1}((0,L);\mathbb{M}^{n\times q}) \subset \mathcal{C}([0,L];\mathbb{M}^{n\times q})$. It remains to prove that $\mathcal{A}$ is also open in $[0,L]$. Let $t_0 \in \mathcal{A}$. Since $\boldsymbol{X}$ satisfy relation (2.6), we have

$$\boldsymbol{X}(t) = \boldsymbol{X}(t_0) + \int_{t_0}^{t} \boldsymbol{A}(s)\boldsymbol{X}(s)\,ds = \int_{t_0}^{t} \boldsymbol{A}(s)\boldsymbol{X}(s)\,ds$$

for all $t \in [0,L]$. Let $J = (t_0 - \varepsilon, t_0 + \varepsilon) \cap [0,L]$, where ε is defined in step (ii). Then, for all $t \in J$, we have

$$|\boldsymbol{X}(t)| \leq \int_{t_0}^{t} |\boldsymbol{A}(s)||\boldsymbol{X}(s)|\,ds \leq \|\boldsymbol{A}\|_{L^1(t_0,t)}\|\boldsymbol{X}\|_{L^\infty(t_0,t)} \leq \frac{1}{2}\|\boldsymbol{X}\|_{L^\infty(J)},$$

and therefore $\|\boldsymbol{X}\|_{L^\infty(J)} \leq \frac{1}{2}\|\boldsymbol{X}\|_{L^\infty(J)}$. Consequently, $\|\boldsymbol{X}\|_{L^\infty(J)} = 0$, so that $\boldsymbol{X}(t) = \boldsymbol{0}$ for all $t \in J$. This proves that $J \subset \mathcal{A}$, which in turn implies that $\mathcal{A}$ is open in $[0,L]$. □

REMARKS 1. (a) If $\boldsymbol{A} \in L^p((0,L);\mathbb{M}^n)$, $\boldsymbol{B} \in L^p((0,L);\mathbb{M}^{n\times q})$, $1 \leq p \leq +\infty$, then the solution to the system (2.3)-(2.4) belongs to the space $W^{1,p}((0,L);\mathbb{M}^{n\times q})$, since $\boldsymbol{Y} \in W^{1,1}((0,L);\mathbb{M}^{n\times q}) \subset L^\infty((0,L);\mathbb{M}^{n\times q})$ and $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{B} \in L^p((0,L);\mathbb{M}^{n\times q})$.

(b) If $\boldsymbol{A} \in L^2((0,L);\mathbb{M}^n)$ and $\boldsymbol{B} \in L^2((0,L);\mathbb{M}^{n\times q})$, Lemma 2.1 is a consequence of Theorem 4.1 and Remark 4.3 of Lions and Magenes [**9**]. □

The following result is an immediate consequence of Lemma 2.1:

COROLLARY 2.1. *Let $\boldsymbol{A} \in L^1((0,L);\mathbb{M}^n)$ and consider the homogeneous linear system*

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y} \quad \textit{a.e. in} \quad (0,L).$$

Then the set of all solutions of class $W^{1,1}((0,L);\mathbb{M}^{n\times q})$ of this system is a subspace of dimension nq of the space $W^{1,1}((0,L);\mathbb{M}^{n\times q})$.

We conclude this section with the following inequality of Gronwall's type.

LEMMA 2.2. *Let there be given matrix fields $\boldsymbol{A} \in L^1((0,L);\mathbb{M}^n)$, $\boldsymbol{B} \in L^1((0,L);\mathbb{M}^{n\times q})$ and $\boldsymbol{Z} \in W^{1,1}((0,L);\mathbb{M}^{n\times q})$ that satisfy*

$$\boldsymbol{Z}'(t) = \boldsymbol{A}(t)\boldsymbol{Z}(t) + \boldsymbol{B}(t),$$

for almost all $t \in (0,L)$. Then

$$|\boldsymbol{Z}(t)| \leq |\boldsymbol{Z}(0)| \exp\Big(\int_0^t |\boldsymbol{A}(\tau)|d\tau\Big) + \int_0^t |\boldsymbol{B}(s)| \exp\Big(\int_s^t |\boldsymbol{A}(\tau)|d\tau\Big)ds \textit{ for all } t \in (0,L).$$

PROOF. The proof is similar to that of the usual Gronwall inequality. It suffices to note that, given a function $g \in L^1_{loc}((0,L))$ and a number $t_0 \in (0,L)$, the function $f:(0,L) \to \mathbb{R}$ defined by $f(t) = \int_{t_0}^t g(s)ds$ belongs to the space $\mathcal{C}((0,L))$ and is differentiable with $f' = g$ almost everywhere in $(0,L)$. □

REMARK 2. The previous Gronwall inequality provides another proof of the uniqueness of the solution to the system (2.3)-(2.4). □

3. Linear Korn inequalities for curves

Throughout this section, in order to simplify the notations, the Lebesgue and the Sobolev norms are respectively denoted $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$, where $m \in \{1,2,3\}$ and $p \geq 1$, irrespective of whether the functions are scalar or vector valued. For exemple, if $f \in L^p(I;\mathbb{R})$ and $\boldsymbol{\eta} \in W^{1,p}(I;\mathbb{R}^3)$, we will write $\|f\|_{L^p}$ instead of $\|f\|_{L^p(I;\mathbb{R})}$ and $\|\boldsymbol{\eta}\|_{W^{1,p}}$ instead of $\|\boldsymbol{\eta}\|_{W^{1,p}(I;\mathbb{R}^3)}$.

In this section, we establish several *linear Korn inequalities for curves* in the three-dimensional euclidean space $\mathbb{R}^3$. As expected, the linearized change of metric, change of curvature, and change of torsion, tensors appear in the right term of the inequality. Accordingly, we begin by explicitly computing these tensors.

Let $\boldsymbol{\gamma}: I \to \mathbb{R}^3$ be a curve parametrized by its arc length s, and let $\boldsymbol{\eta}: I \to \mathbb{R}^3$ be an arbitrary displacement field of this curve. Assume that the curve $\boldsymbol{\gamma}$ is regular so that there exists a unique Frenet frame $(\boldsymbol{T},\boldsymbol{N},\boldsymbol{B})$ associated with this curve (see Section 2). In what follows, whenever no confusion should arise, the dependence with respect to $s \in I$ is omitted.

LEMMA 3.1. *The linearized change of metric tensor associated with the vector field* $\boldsymbol{\eta}$ *is given by*

$$a_{lin}(\boldsymbol{\eta}) = \boldsymbol{\eta}' \cdot \boldsymbol{T}, \tag{3.1}$$

the linearized change of curvature tensor associated with the vector field $\boldsymbol{\eta}$ *is given by*

$$\begin{aligned} k_{lin}(\boldsymbol{\eta}) &= \boldsymbol{\eta}'' \cdot \boldsymbol{N} - 2k\boldsymbol{\eta}' \cdot \boldsymbol{T} \\ &= (\boldsymbol{\eta}' \cdot \boldsymbol{N})' - k\boldsymbol{\eta}' \cdot \boldsymbol{T} - \tau\boldsymbol{\eta}' \cdot \boldsymbol{B}, \end{aligned} \tag{3.2}$$

and the linearized change of torsion tensor associated with the vector field $\boldsymbol{\eta}$ *is given by*

$$\begin{aligned} \tau_{lin}(\boldsymbol{\eta}) &= -\tau\boldsymbol{\eta}' \cdot \boldsymbol{T} - \frac{\tau}{k}\boldsymbol{\eta}'' \cdot \boldsymbol{N} + (\frac{1}{k}\boldsymbol{\eta}''' - \frac{k'}{k^2}\boldsymbol{\eta}'' + k\boldsymbol{\eta}') \cdot \boldsymbol{B} \\ &= -\tau\boldsymbol{\eta}' \cdot \boldsymbol{T} + (k\boldsymbol{\eta}' - \frac{k'}{k^2}\boldsymbol{\eta}'') \cdot \boldsymbol{B} + \frac{1}{k}(\boldsymbol{\eta}'' \cdot \boldsymbol{B})'. \end{aligned} \tag{3.3}$$

PROOF. Throughout the proof, the metric, curvature, and torsion, tensors associated with a given curve $\boldsymbol{\gamma}$ are respectively denoted $a^2(\boldsymbol{\gamma})$, $k(\boldsymbol{\gamma})$, and $\tau(\boldsymbol{\gamma})$.

The linearized change of metric tensor is defined by

$$a_{lin}(\boldsymbol{\eta}) = \frac{1}{2}[a^2(\boldsymbol{\gamma}+\boldsymbol{\eta}) - a^2(\boldsymbol{\gamma})]^{lin},$$

where $[\ldots]^{lin}$ denotes the linear part with respect to $\boldsymbol{\eta}$ in the expression $[\ldots]$. In order to compute this linear part, we consider a one-parameter family of deformed curves $\{(\boldsymbol{\gamma}+t\boldsymbol{\eta})(I), t\in\mathbb{R}\}$, for t small enough, and we differentiate the metric tensor associated with the curves of this family with respect to t at $t=0$. Therefore, we have

$$\begin{aligned} a_{lin}(\boldsymbol{\eta}) &= \frac{1}{2}\frac{d}{dt}\Big[(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\cdot(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\Big]_{t=0} \\ &= \frac{d}{dt}\Big[\boldsymbol{\gamma}'+t\boldsymbol{\eta}'\Big]_{t=0}\cdot\Big[\boldsymbol{\gamma}'+t\boldsymbol{\eta}'\Big]_{t=0} \\ &= \boldsymbol{\eta}'\cdot\boldsymbol{\gamma}', \end{aligned}$$

which establishes formula (3.1), since $\boldsymbol{\gamma}'=\boldsymbol{T}$.

In order to obtain the linearized change of curvature tensor, we now compute

$$\begin{aligned} k_{lin}(\boldsymbol{\eta}) &= [k(\boldsymbol{\gamma}+\boldsymbol{\eta})-k(\boldsymbol{\gamma})]^{lin} \\ &= \frac{d}{dt}\Big[\frac{|(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\times(\boldsymbol{\gamma}''+t\boldsymbol{\eta}'')|}{|(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')|^3}\Big]_{t=0} \\ &= \frac{|\boldsymbol{\gamma}'|^3\frac{d}{dt}\Big[|(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\times(\boldsymbol{\gamma}''+t\boldsymbol{\eta}'')|\Big]_{t=0}-|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''|\frac{d}{dt}\Big[|\boldsymbol{\gamma}'+t\boldsymbol{\eta}'|^3\Big]_{t=0}}{|\boldsymbol{\gamma}'|^6}. \end{aligned} \tag{3.4}$$

First, we note that

$$\begin{aligned} &\frac{d}{dt}\Big[|(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\times(\boldsymbol{\gamma}''+t\boldsymbol{\eta}'')|\Big]_{t=0} \\ &\quad= \Big[|(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\times(\boldsymbol{\gamma}''+t\boldsymbol{\eta}'')|^{-1}\Big]_{t=0}\frac{d}{dt}\Big[(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\times(\boldsymbol{\gamma}''+t\boldsymbol{\eta}'')\Big]_{t=0}\cdot(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}'') \\ &\quad= \frac{1}{|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''|}(\boldsymbol{\eta}'\times\boldsymbol{\gamma}''+\boldsymbol{\gamma}'\times\boldsymbol{\eta}'')\cdot(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''). \end{aligned}$$

Next, since $\boldsymbol{\gamma}'=\boldsymbol{T}$, $\boldsymbol{\gamma}''=k\boldsymbol{N}$ and $\frac{\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''}{|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''|}=\boldsymbol{B}$, we obtain

$$\begin{aligned} \frac{d}{dt}\Big[|(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\times(\boldsymbol{\gamma}''+t\boldsymbol{\eta}'')|\Big]_{t=0} &= (\boldsymbol{\eta}'\times k\boldsymbol{N})\cdot\boldsymbol{B}+(\boldsymbol{T}\times\boldsymbol{\eta}'')\cdot\boldsymbol{B} \\ &= k\boldsymbol{\eta}'\cdot\boldsymbol{T}+\boldsymbol{\eta}''\cdot\boldsymbol{N}. \end{aligned}$$

By combining this relation with relations (3.1) and (3.4), we deduce that

$$k_{lin}(\boldsymbol{\eta})=\boldsymbol{\eta}''\cdot\boldsymbol{N}-2k\boldsymbol{\eta}'\cdot\boldsymbol{T},$$

which is the desired result.

Finally, we compute the linearized change of torsion tensor. We have:

$$
\begin{aligned}
\tau_{lin}(\boldsymbol{\eta}) &= [\tau(\boldsymbol{\gamma}+\boldsymbol{\eta}) - \tau(\boldsymbol{\gamma})]^{lin} \\
&= \frac{d}{dt}\Big[\frac{\det\big(\boldsymbol{\gamma}'+t\boldsymbol{\eta}', \boldsymbol{\gamma}''+t\boldsymbol{\eta}'', \boldsymbol{\gamma}'''+t\boldsymbol{\eta}'''\big)}{|(\boldsymbol{\gamma}'+t\boldsymbol{\eta}')\times(\boldsymbol{\gamma}''+t\boldsymbol{\eta}'')|^2}\Big]_{t=0} \\
&= \frac{\big[\det\big(\boldsymbol{\eta}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}'''\big) + \det\big(\boldsymbol{\gamma}', \boldsymbol{\eta}'', \boldsymbol{\gamma}'''\big) + \det\big(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\eta}'''\big)\big]}{|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''|^2} \\
&\quad -2\frac{\det\big(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}'''\big)(\boldsymbol{\eta}'\times\boldsymbol{\gamma}''+\boldsymbol{\gamma}'\times\boldsymbol{\eta}'')\cdot(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}'')}{|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''|^4}.
\end{aligned}
\tag{3.5}
$$

Since the curve $\boldsymbol{\gamma}$ is parametrized by its arc length, the Frenet equations imply that

$$
\begin{aligned}
\boldsymbol{\gamma}' &= \boldsymbol{T}, \\
\boldsymbol{\gamma}'' &= k\boldsymbol{N}, \\
\boldsymbol{\gamma}''' &= -k^2\boldsymbol{T} + k'\boldsymbol{N} + k\tau\boldsymbol{B}.
\end{aligned}
$$

By using these formulas and elementary properties of determinants, we obtain:

$$
\begin{aligned}
\det\big(\boldsymbol{\eta}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}'''\big) &= \det\big(\boldsymbol{\eta}', k\boldsymbol{N}, -k^2\boldsymbol{T} + k'\boldsymbol{N} + k\tau\boldsymbol{B}\big) \\
&= k\boldsymbol{\eta}'\cdot\big(\boldsymbol{N}\times(-k^2\boldsymbol{T}+k\tau\boldsymbol{B})\big) \\
&= k^3\boldsymbol{\eta}'\cdot\boldsymbol{B} + k^2\tau\boldsymbol{\eta}'\cdot\boldsymbol{T},
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
\det\big(\boldsymbol{\gamma}', \boldsymbol{\eta}'', \boldsymbol{\gamma}'''\big) &= -k'\boldsymbol{\eta}''\cdot\boldsymbol{B} + k\tau\boldsymbol{\eta}''\cdot\boldsymbol{N}, \\
\det\big(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\eta}'''\big) &= k\boldsymbol{\eta}'''\cdot\boldsymbol{B}, \\
\det\big(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}'''\big) &= k^2\tau.
\end{aligned}
$$

By using these expressions in (3.5), we finally deduce that:

$$
\begin{aligned}
\tau_{lin}(\boldsymbol{\eta}) &= \frac{1}{k^2}\big(k^3\boldsymbol{\eta}'\cdot\boldsymbol{B} + k^2\tau\boldsymbol{\eta}'\cdot\boldsymbol{T} - k'\boldsymbol{\eta}''\cdot\boldsymbol{B} + k\tau\boldsymbol{\eta}''\cdot\boldsymbol{N} + k\boldsymbol{\eta}'''\cdot\boldsymbol{B}\big) \\
&\quad -2k^2\tau\frac{\big(\boldsymbol{\eta}'\times(k\boldsymbol{N}) + \boldsymbol{T}\times\boldsymbol{\eta}''\big)\cdot\boldsymbol{B}}{|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''|^3} \\
&= \tau\boldsymbol{\eta}'\cdot\boldsymbol{T} + (\frac{1}{k}\boldsymbol{\eta}''' - \frac{k'}{k^2}\boldsymbol{\eta}'' + k\boldsymbol{\eta}')\cdot\boldsymbol{B} + \frac{\tau}{k}\boldsymbol{\eta}''\cdot\boldsymbol{N} - 2\tau\boldsymbol{\eta}'\cdot\boldsymbol{T} - 2\frac{\tau}{k}\boldsymbol{\eta}''\cdot\boldsymbol{N} \\
&= -\tau\boldsymbol{\eta}'\cdot\boldsymbol{T} - \frac{\tau}{k}\boldsymbol{\eta}''\cdot\boldsymbol{N} + (\frac{1}{k}\boldsymbol{\eta}''' - \frac{k'}{k^2}\boldsymbol{\eta}'' + k\boldsymbol{\eta}')\cdot\boldsymbol{B},
\end{aligned}
$$

which is exactly the announced expression for $\tau_{lin}(\boldsymbol{\eta})$. □

REMARK 3. The linearized change of metric tensor $a_{lin}(\boldsymbol{\eta})$ is also the linear part of the change of velocity tensor, i.e., $a_{lin}(\boldsymbol{\eta}) = [a(\boldsymbol{\gamma}+\boldsymbol{\eta}) - a(\boldsymbol{\gamma})]^{lin}$. □

In what follows, we establish Korn's inequalities for curves in a framework similar to that of Blouza-Le Dret [**1**], in the sense that we consider curves with as little regularity as possible. More specifically, we assume that the reference curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, parametrized by its arc length, is of class $W^{3,\infty}$ over I and that its Frenet frame is defined at each point of the curve. This means that $\boldsymbol{\gamma}'(s) \times \boldsymbol{\gamma}''(s) \neq \mathbf{0}$ for all $s \in I$, i.e., that the curve $\boldsymbol{\gamma}$ is *regular* according to the definition given in Section 2. These assumptions imply that $\boldsymbol{T} \in W^{2,\infty}(I;\mathbb{R}^3)$, $\boldsymbol{N}, \boldsymbol{B} \in W^{1,\infty}(I;\mathbb{R}^3)$, $k \in W^{1,\infty}(I)$ and $\tau \in L^\infty(I)$. Moreover, since $W^{1,\infty}(I) \subset \mathcal{C}(\overline{I})$ and $k(s) > 0$ for all $s \in \overline{I}$, they also imply that $(1/k) \in W^{1,\infty}(I)$.

Note also that, if $\boldsymbol{\eta} \in W^{1,p}(I;\mathbb{R}^3)$, $1 \leq p \leq +\infty$, satisfies $\boldsymbol{\eta}' \cdot \boldsymbol{N},\ \boldsymbol{\eta}'' \cdot \boldsymbol{B} \in W^{1,p}(I)$, then the linearized change of metric, change of curvature, and change of torsion, tensors $a_{lin}(\boldsymbol{\eta}), k_{lin}(\boldsymbol{\eta})$ and $\tau_{lin}(\boldsymbol{\eta})$ associated with the displacement field $\boldsymbol{\eta}$ all belong to $L^p(I)$.

We begin by establishing a *Korn inequality without boundary conditions*:

THEOREM 3.1. *Let there be given a curve $\boldsymbol{\gamma} \in W^{3,\infty}(I;\mathbb{R}^3)$ that is regular and let*

$$V := \{\boldsymbol{\eta} \in W^{1,p}(I;\mathbb{R}^3)\ ;\ \boldsymbol{\eta}' \cdot \boldsymbol{N} \in W^{1,p}(I) \text{ and } \boldsymbol{\eta}'' \cdot \boldsymbol{B} \in W^{1,p}(I)\,\}.$$

Then there exists a constant $C > 0$ depending only on the curve $\boldsymbol{\gamma}$ such that

$$\text{(3.6)}\quad \|\boldsymbol{\eta}\|_{W^{1,p}} + \|\boldsymbol{\eta}' \cdot \boldsymbol{N}\|_{W^{1,p}} + \|\boldsymbol{\eta}'' \cdot \boldsymbol{B}\|_{W^{1,p}} \leq C\Big(\|\boldsymbol{\eta}\|_{L^p} + \|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|k_{lin}(\boldsymbol{\eta})\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta})\|_{L^p}\Big)$$

for all $\boldsymbol{\eta} \in V$.

PROOF. We denote the left hand side of the inequality (3.6) by $\|\boldsymbol{\eta}\|_V$, since $\|\cdot\|_V$ is indeed the natural norm of the space V. Note that the space V endowed with this norm is a Banach space.

First we will prove the following inequality:

$$\text{(3.7)}\quad \|\boldsymbol{\eta}\|_{W^{1,p}} + \|\boldsymbol{\eta}' \cdot \boldsymbol{N}\|_{W^{1,p}} + \|\boldsymbol{\eta}'' \cdot \boldsymbol{B}\|_{W^{1,p}} \leq C\Big(\|\boldsymbol{\eta}\|_{L^p} + \|\boldsymbol{\eta}' \cdot \boldsymbol{N}\|_{L^p} + \|\boldsymbol{\eta}' \cdot \boldsymbol{B}\|_{L^p} + \|\boldsymbol{\eta}'' \cdot \boldsymbol{B}\|_{L^p} + \|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|k_{lin}(\boldsymbol{\eta})\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta})\|_{L^p}\Big).$$

Note that, thanks to the regularity of the components of the Frenet frame, the norms $\boldsymbol{\eta} \mapsto \|\boldsymbol{\eta}\|_{W^{1,p}}$ and $\boldsymbol{\eta} \mapsto \|\boldsymbol{\eta} \cdot \boldsymbol{T}\|_{W^{1,p}} + \|\boldsymbol{\eta} \cdot \boldsymbol{N}\|_{W^{1,p}} + \|\boldsymbol{\eta} \cdot \boldsymbol{B}\|_{W^{1,p}}$ are equivalent. Likewise, the norms $\boldsymbol{\eta} \mapsto \|\boldsymbol{\eta}\|_{L^p}$ and $\boldsymbol{\eta} \mapsto \|\boldsymbol{\eta} \cdot \boldsymbol{T}\|_{L^p} + \|\boldsymbol{\eta} \cdot \boldsymbol{N}\|_{L^p} + \|\boldsymbol{\eta} \cdot \boldsymbol{B}\|_{L^p}$ are equivalent. To begin with, we use the second expressions, those which use the Frenet frame.

For proving inequality (3.7), we use the Frenet equations, the Hölder inequality, and the regularity of k, τ and $(1/k)$. More specifically, since (cf. Lemma 3.1)

$$\begin{aligned}
(\boldsymbol{\eta}\cdot\boldsymbol{T})' &= \boldsymbol{\eta}'\cdot\boldsymbol{T} + k\boldsymbol{\eta}\cdot\boldsymbol{N} = a_{lin}(\boldsymbol{\eta}) + k\boldsymbol{\eta}\cdot\boldsymbol{N},\\
(\boldsymbol{\eta}\cdot\boldsymbol{N})' &= \boldsymbol{\eta}'\cdot\boldsymbol{N} + \boldsymbol{\eta}\cdot(-k\boldsymbol{T}+\tau\boldsymbol{B}),\\
(\boldsymbol{\eta}\cdot\boldsymbol{B})' &= \boldsymbol{\eta}'\cdot\boldsymbol{B} - \tau\boldsymbol{\eta}\cdot\boldsymbol{N},\\
(\boldsymbol{\eta}'\cdot\boldsymbol{N})' &= \boldsymbol{\eta}''\cdot\boldsymbol{N} + \boldsymbol{\eta}'\cdot(-k\boldsymbol{T}+\tau\boldsymbol{B}) = k_{lin}(\boldsymbol{\eta}) + k a_{lin}(\boldsymbol{\eta}) + \tau\boldsymbol{\eta}'\cdot\boldsymbol{B},\\
(\boldsymbol{\eta}''\cdot\boldsymbol{B})' &= k\tau_{lin}(\boldsymbol{\eta}) + k\tau a_{lin}(\boldsymbol{\eta}) - k^2\boldsymbol{\eta}'\cdot\boldsymbol{B} + \frac{k'}{k}\boldsymbol{\eta}''\cdot\boldsymbol{B},
\end{aligned}$$

it follows that

$$\begin{aligned}
\|\boldsymbol{\eta}\cdot\boldsymbol{T}\|_{W^{1,p}} &\le C(\|\boldsymbol{\eta}\cdot\boldsymbol{T}\|_{L^p} + \|\boldsymbol{\eta}\cdot\boldsymbol{N}\|_{L^p} + \|a_{lin}(\boldsymbol{\eta})\|_{L^p}),\\
\|\boldsymbol{\eta}\cdot\boldsymbol{N}\|_{W^{1,p}} &\le C(\|\boldsymbol{\eta}\cdot\boldsymbol{T}\|_{L^p} + \|\boldsymbol{\eta}\cdot\boldsymbol{N}\|_{L^p} + \|\boldsymbol{\eta}\cdot\boldsymbol{B}\|_{L^p} + \|\boldsymbol{\eta}'\cdot\boldsymbol{N}\|_{L^p}),\\
\|\boldsymbol{\eta}\cdot\boldsymbol{B}\|_{W^{1,p}} &\le C(\|\boldsymbol{\eta}\cdot\boldsymbol{N}\|_{L^p} + \|\boldsymbol{\eta}\cdot\boldsymbol{B}\|_{L^p} + \|\boldsymbol{\eta}'\cdot\boldsymbol{B}\|_{L^p}),\\
\|\boldsymbol{\eta}'\cdot\boldsymbol{N}\|_{W^{1,p}} &\le C(\|\boldsymbol{\eta}'\cdot\boldsymbol{N}\|_{L^p} + \|\boldsymbol{\eta}'\cdot\boldsymbol{B}\|_{L^p} + \|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|k_{lin}(\boldsymbol{\eta})\|_{L^p}),\\
\|\boldsymbol{\eta}''\cdot\boldsymbol{B}\|_{W^{1,p}} &\le C(\|\boldsymbol{\eta}'\cdot\boldsymbol{B}\|_{L^p} + \|\boldsymbol{\eta}''\cdot\boldsymbol{B}\|_{L^p} + \|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta})\|_{L^p}).
\end{aligned}$$

Adding these inequalities gives inequality (3.7).

We now establish the inequality (3.6) by contradiction. Assume that inequality (3.6) does not hold. Then there exists a sequence $(\boldsymbol{\eta}_n)$ in V such that

$$\|\boldsymbol{\eta}_n\|_V = \|\boldsymbol{\eta}_n\|_{W^{1,p}} + \|\boldsymbol{\eta}_n'\cdot\boldsymbol{N}\|_{W^{1,p}} + \|\boldsymbol{\eta}_n''\cdot\boldsymbol{B}\|_{W^{1,p}} = 1 \tag{3.8}$$

and

$$\|\boldsymbol{\eta}_n\|_{L^p} + \|a_{lin}(\boldsymbol{\eta}_n)\|_{L^p} + \|k_{lin}(\boldsymbol{\eta}_n)\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta}_n)\|_{L^p} \to 0 \text{ as } n\to+\infty. \tag{3.9}$$

Note that for any $\boldsymbol{\eta}\in V$, we have $(\boldsymbol{\eta}'\cdot\boldsymbol{B})' = \boldsymbol{\eta}''\cdot\boldsymbol{B} - \tau\boldsymbol{\eta}'\cdot\boldsymbol{N}$; hence $\boldsymbol{\eta}'\cdot\boldsymbol{B}\in W^{1,p}(I)$. Moreover, there exists a constant $C>0$ depending only on the curve $\boldsymbol{\gamma}$ such that

$$\|\boldsymbol{\eta}'\cdot\boldsymbol{B}\|_{W^{1,p}} \le C(\|\boldsymbol{\eta}'\|_{L^p} + \|\boldsymbol{\eta}''\cdot\boldsymbol{B}\|_{L^p}) \text{ for all } \boldsymbol{\eta}\in V.$$

By combining this inequality with relation (3.8), we deduce that the sequences $(\boldsymbol{\eta}_n)$, $(\boldsymbol{\eta}_n'\cdot\boldsymbol{N})$, $(\boldsymbol{\eta}_n'\cdot\boldsymbol{B})$, and $(\boldsymbol{\eta}_n''\cdot\boldsymbol{B})$ are bounded in $W^{1,p}(I)$. Therefore, since the inclusion $W^{1,p}(I)\subset L^p(I)$ is compact, there exists a subsequence $(\boldsymbol{\eta}_j)$ such that the sequences $(\boldsymbol{\eta}_j)$, $(\boldsymbol{\eta}_j'\cdot\boldsymbol{N})$, $(\boldsymbol{\eta}_j'\cdot\boldsymbol{B})$ and $(\boldsymbol{\eta}_j''\cdot\boldsymbol{B})$ converge in $L^p(I)$.

In addition, the sequences $(a_{lin}(\boldsymbol{\eta}_j))$, $(k_{lin}(\boldsymbol{\eta}_j))$, and $(\tau_{lin}(\boldsymbol{\eta}_j))$ converge to zero in $L^p(I)$ as $j\to\infty$ by (3.9). Consequently, $(\boldsymbol{\eta}_j)$, $(\boldsymbol{\eta}_j'\cdot\boldsymbol{N})$, $(\boldsymbol{\eta}_j'\cdot\boldsymbol{B})$, $(\boldsymbol{\eta}_j''\cdot\boldsymbol{B})$, $(a_{lin}(\boldsymbol{\eta}_j))$, $(k_{lin}(\boldsymbol{\eta}_j))$, and $(\tau_{lin}(\boldsymbol{\eta}_j))$ are Cauchy sequences in $L^p(I)$. By inequality (3.7), this implies that $(\boldsymbol{\eta}_j)$ is a Cauchy sequence for the norm of V. Hence there exists $\boldsymbol{\eta}\in V$ such that $\boldsymbol{\eta}_j\to\boldsymbol{\eta}$ in V as $j\to\infty$. In particular, $\|\boldsymbol{\eta}_j\|_{L^p}\to\|\boldsymbol{\eta}\|_{L^p}$ as $j\to\infty$. But we already know by (3.9) that $\|\boldsymbol{\eta}_j\|_{L^p}\to 0$. Hence $\boldsymbol{\eta}=\boldsymbol{0}$. But this implies that $\|\boldsymbol{\eta}_j\|_V\to\|\boldsymbol{\eta}\|_V = 0$ as $j\to\infty$, which contradicts relation (3.8).

□

If we assume in addition that the curve $\boldsymbol{\gamma}$ belongs to the space $W^{5,\infty}(I;\mathbb{R}^3)$, the Korn inequality established in Theorem 3.1 can be written in a simpler form, which is similar to the Korn inequality for surfaces immersed in $\mathbb{R}^3$ (see Ciarlet [**4**, Theorem 2.6-1]). More specifically, we have the following *Korn inequality without boundary conditions for smooth curves*:

COROLLARY 3.1. *If $\boldsymbol{\gamma} \in W^{5,\infty}(I;\mathbb{R}^3)$, the space V of Theorem 3.1 can be also defined as*

$$V = \{\boldsymbol{\eta} = \eta_1 \boldsymbol{T} + \eta_2 \boldsymbol{N} + \eta_3 \boldsymbol{B} \ ; \ \eta_1 \in W^{1,p}(I), \eta_2 \in W^{2,p}(I), \eta_3 \in W^{3,p}(I)\}, \tag{3.10}$$

and there exists a constant $C > 0$ depending only on $\boldsymbol{\gamma}$ such that

$$\|\eta_1\|_{W^{1,p}} + \|\eta_2\|_{W^{2,p}} + \|\eta_3\|_{W^{3,p}} \leq C\Big(\|\boldsymbol{\eta}\|_{L^p} + \|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|k_{lin}(\boldsymbol{\eta})\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta})\|_{L^p}\Big)$$

for all $\boldsymbol{\eta} \in V$.

PROOF. If $\boldsymbol{\gamma} \in W^{5,\infty}(I;\mathbb{R}^3)$, then $\boldsymbol{T} \in W^{4,\infty}(I;\mathbb{R}^3)$, $\boldsymbol{N}, \boldsymbol{B} \in W^{3,\infty}(I;\mathbb{R}^3)$, $k \in W^{3,\infty}(I)$, $\tau \in W^{2,\infty}(I)$, and the components of the vector field $\boldsymbol{\eta}$ over the basis $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ satisfy the relations

$$\begin{aligned}
&\eta_1 = \boldsymbol{\eta} \cdot \boldsymbol{T},\\
&\eta_2' = (\boldsymbol{\eta} \cdot \boldsymbol{N})' = \boldsymbol{\eta}' \cdot \boldsymbol{N} - k\boldsymbol{\eta} \cdot \boldsymbol{T} + \tau\boldsymbol{\eta} \cdot \boldsymbol{B},\\
&\eta_3'' = (\boldsymbol{\eta} \cdot \boldsymbol{B})'' = \boldsymbol{\eta}'' \cdot \boldsymbol{B} - 2\tau\boldsymbol{\eta}' \cdot \boldsymbol{N} + k\tau\boldsymbol{\eta} \cdot \boldsymbol{T} - \tau'\boldsymbol{\eta} \cdot \boldsymbol{N} - \tau^2\boldsymbol{\eta} \cdot \boldsymbol{B}.
\end{aligned}$$

These relations show that the space V defined in Theorem 3.1 is also given by (3.10) and that the norm $\boldsymbol{\eta} \in V \mapsto \|\boldsymbol{\eta}\|_V$ is equivalent to the norm $\boldsymbol{\eta} \in V \mapsto \|\eta_1\|_{W^{1,p}} + \|\eta_2\|_{W^{2,p}} + \|\eta_3\|_{W^{3,p}}$. The desired inequality then follows from (3.6). □

If the displacement field $\boldsymbol{\eta}$ satisfies in addition some appropriate *boundary conditions*, the term $\|\boldsymbol{\eta}\|_{L^p}$ appearing in the right hand side term of the Korn inequality (3.6) becomes irrelevant. The inequality obtained in this fashion (see Theorem 3.2) is called *Korn's inequality for curves.* For proving it, we proceed as in Ciarlet [**3, 4**] where Korn inequalities were established for open subsets of $\mathbb{R}^n$ and for surfaces immersed in $\mathbb{R}^3$. Accordingly, we first establish an *infinitesimal rigid displacement lemma for curves*:

LEMMA 3.2. *Let $\boldsymbol{\eta} \in V$ satisfy $a_{lin}(\boldsymbol{\eta}) = k_{lin}(\boldsymbol{\eta}) = \tau_{lin}(\boldsymbol{\eta}) = 0$. Then there exists two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ such that*

$$\boldsymbol{\eta}(s) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{\gamma}(s) \text{ for all } s \in I\,.$$

PROOF. Since $a_{lin}(\boldsymbol{\eta}) = k_{lin}(\boldsymbol{\eta}) = \tau_{lin}(\boldsymbol{\eta}) = 0$, the following differential system is satisfied:

$$\begin{aligned}
&(\boldsymbol{\eta}' \cdot \boldsymbol{N})' = \tau\boldsymbol{\eta}' \cdot \boldsymbol{B},\\
&(\boldsymbol{\eta}' \cdot \boldsymbol{B})' = -\tau\boldsymbol{\eta}' \cdot \boldsymbol{N} + \boldsymbol{\eta}'' \cdot \boldsymbol{B},\\
&(\boldsymbol{\eta}'' \cdot \boldsymbol{B})' = -k^2\boldsymbol{\eta}' \cdot \boldsymbol{B} + \frac{k'}{k}\boldsymbol{\eta}'' \cdot \boldsymbol{B}.
\end{aligned}$$

This system can be written as a matrix equation of the form $\boldsymbol{Y}' = \boldsymbol{M}\boldsymbol{Y}$, where

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{\eta}' \cdot \boldsymbol{N} \\ \boldsymbol{\eta}' \cdot \boldsymbol{B} \\ \boldsymbol{\eta}'' \cdot \boldsymbol{B} \end{pmatrix} \in W^{1,p}(I;\mathbb{R}^3)$$

and

$$\boldsymbol{M} = \begin{pmatrix} 0 & \tau & 0 \\ -\tau & 0 & 1 \\ 0 & -k^2 & k'/k \end{pmatrix} \in L^\infty(I;\mathbb{M}^3).$$

Since this system is linear and homogeneous, Corollary 2.1 shows that the set of all its solutions is a three-dimensional subspace of $W^{1,p}(I;\mathbb{R}^3)$.

For any $\boldsymbol{b} \in \mathbb{R}^3$, define the vector field $\boldsymbol{Y}_{\boldsymbol{b}} : I \to \mathbb{R}^3$ by

$$\boldsymbol{Y}_{\boldsymbol{b}} = \begin{pmatrix} \boldsymbol{b} \cdot \boldsymbol{B} \\ -\boldsymbol{b} \cdot \boldsymbol{N} \\ k\boldsymbol{b} \cdot \boldsymbol{T} \end{pmatrix},$$

and note that $\boldsymbol{Y}_{\boldsymbol{b}}$ satisfies the system $\boldsymbol{Y}'_{\boldsymbol{b}} = \boldsymbol{M}\boldsymbol{Y}_{\boldsymbol{b}}$ (these solutions are obtained by replacing $\boldsymbol{\eta}$ with $\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{\gamma}$ in the expression of $\boldsymbol{Y}$). Let $(\boldsymbol{e}_i)$ be the canonical basis of $\mathbb{R}^3$. Since $\{\boldsymbol{Y}_{\boldsymbol{e}_i}; i \in \{1,2,3\}\}$ is a basis of the space $\{\boldsymbol{Y}_{\boldsymbol{b}} \in \mathbb{R}^3 \ ; \ \boldsymbol{b} \in \mathbb{R}^3\}$, the latter is a three-dimensional subspace of $W^{1,p}(I;\mathbb{R}^3)$. Therefore all the solutions to the system $\boldsymbol{Y}' = \boldsymbol{M}\boldsymbol{Y}$ belong to this space.

This implies that there exists a vector $\boldsymbol{b} \in \mathbb{R}^3$ such that $\boldsymbol{Y} = \boldsymbol{Y}_{\boldsymbol{b}}$, i.e., such that

$$\begin{aligned} \boldsymbol{\eta}' \cdot \boldsymbol{N} &= \boldsymbol{b} \cdot \boldsymbol{B}, \\ \boldsymbol{\eta}' \cdot \boldsymbol{B} &= -\boldsymbol{b} \cdot \boldsymbol{N}, \\ \boldsymbol{\eta}'' \cdot \boldsymbol{B} &= k\boldsymbol{b} \cdot \boldsymbol{T}. \end{aligned}$$

In addition, $\boldsymbol{\eta}' \cdot \boldsymbol{T} = 0$ since $a_{lin}(\boldsymbol{\eta}) = 0$ by assumption. Hence $\boldsymbol{\eta}' = (\boldsymbol{b} \cdot \boldsymbol{B})\boldsymbol{N} - (\boldsymbol{b} \cdot \boldsymbol{N})\boldsymbol{B} = \boldsymbol{b} \times \boldsymbol{T}$. Since $\boldsymbol{\gamma}' = \boldsymbol{T}$, we eventually obtain that $\boldsymbol{\eta} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{\gamma}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in $\mathbb{R}^3$.

□

REMARK 4. The infinitesimal rigid displacement lemma for curves (Lemma 3.2) can also be proved by using a strategy similar to that of Ciarlet and S. Mardare [**6**], in the sense that it can be obtained as a consequence of the three-dimensional infinitesimal rigid displacement lemma in curvilinear coordinates. More specifically, let $\Omega = (0, L) \times (-\varepsilon, \varepsilon) \times (-\delta, \delta)$ for some numbers $\varepsilon, \delta > 0$ and let $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\Theta}(s, s_2, s_3) = \boldsymbol{\gamma}(s) + s_2\boldsymbol{N}(s) + s_3\boldsymbol{B}(s)$$

for all $(s, s_2, s_3) \in \Omega$. With any displacement field $\boldsymbol{\eta} \in V$, let there be associated the three-dimensional displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ of the set $\boldsymbol{\Theta}(\Omega)$ defined by

$$\boldsymbol{u}(s, s_2, s_3) = \boldsymbol{\eta}(s) + s_2\boldsymbol{\zeta}(s) + s_3\boldsymbol{\xi}(s)$$

for all $(s, s_2, s_3) \in \Omega$, where the vector fields $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ are defined by

$$\boldsymbol{\zeta}(s) = -\boldsymbol{\eta}'(s) \cdot \boldsymbol{N}(s)\boldsymbol{T}(s) + \frac{1}{k}\boldsymbol{\eta}''(s) \cdot \boldsymbol{B}(s)\boldsymbol{B}(s)$$

and

$$\boldsymbol{\xi}(s) = -\boldsymbol{\eta}'(s) \cdot \boldsymbol{B}(s)\boldsymbol{T}(s) - \frac{1}{k}\boldsymbol{\eta}''(s) \cdot \boldsymbol{B}(s)\boldsymbol{N}(s)$$

for all $s \in (0, L)$. If $\boldsymbol{\eta}$ satisfies the assumptions of Lemma 3.2, then one can see that $\boldsymbol{u}$ is an infinitesimal rigid displacement field of the set $\boldsymbol{\Theta}(\Omega)$. Consequently, the three-dimensional infinitesimal rigid displacement lemma in curvilinear coordinates (see Ciarlet [**4**]) shows that there exist two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ such that $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{\Theta}$ over Ω. In particular, this implies that $\boldsymbol{\eta} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{\gamma}$ over $(0, L)$.

Note that this latter approach is close to that used by Griso [**7**], where an "elementary displacement" of a rod is defined by

$$\boldsymbol{\eta}(s, s_2, s_3) = \mathcal{A}(s) + \mathcal{B}(s) \times (s_2\boldsymbol{N}(s) + s_3\boldsymbol{B}(s))$$

for all $(s, s_2, s_3) \in (0, L) \times D(0; \delta)$, where $D(0; \delta)$ denotes the open disk of center 0 and radius δ, and $\mathcal{A}$ and $\mathcal{B}$ are vector fields in $W^{1,p}((0, L); \mathbb{R}^3)$. □

We are now in position to establish the main result of this section, namely the *Korn inequality for a curve*:

THEOREM 3.2. *Let there be given a curve* $\boldsymbol{\gamma} \in W^{3,\infty}(I; \mathbb{R}^3)$ *that is regular and let*

$$V_0 := \Big\{\boldsymbol{\eta} \in W^{1,p}(I; \mathbb{R}^3)\ ;\ \boldsymbol{\eta}' \cdot \boldsymbol{N},\ \boldsymbol{\eta}'' \cdot \boldsymbol{B} \in W^{1,p}(I) \text{ and}$$
$$\boldsymbol{\eta}(0) = \boldsymbol{0},\ (\boldsymbol{\eta}' \cdot \boldsymbol{N})(0) = (\boldsymbol{\eta}' \cdot \boldsymbol{B})(0) = (\boldsymbol{\eta}'' \cdot \boldsymbol{B})(0) = 0\Big\}.$$

Then there exists a constant $C > 0$ *depending only on the curve* $\boldsymbol{\gamma}$ *such that*

$$\|\boldsymbol{\eta}\|_{W^{1,p}} + \|\boldsymbol{\eta}' \cdot \boldsymbol{N}\|_{W^{1,p}} + \|\boldsymbol{\eta}'' \cdot \boldsymbol{B}\|_{W^{1,p}} \leq C\Big(\|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|k_{lin}(\boldsymbol{\eta})\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta})\|_{L^p}\Big) \quad (3.11)$$

for all $\boldsymbol{\eta} \in V_0$.

REMARK 5. The set V_0 endowed with the norm $\|\cdot\|_V$ is a closed subspace of V. To see this, it suffices to use the relation $(\boldsymbol{\eta}' \cdot \boldsymbol{B})' = \boldsymbol{\eta}'' \cdot \boldsymbol{B} - \tau\boldsymbol{\eta}' \cdot \boldsymbol{N}$ (established in the proof of Theorem 3.1) and the continuous inclusion $W^{1,p}(I) \subset L^\infty(I)$, which shows that the boundary conditions are preserved when passing to limit with respect to the norm of V. □

PROOF. We proceed by contradiction. Assume that inequality (3.11) does not hold. Then there exists a sequence $(\boldsymbol{\eta}_n)$ in V_0 such that

$$\|\boldsymbol{\eta}_n\|_V = \|\boldsymbol{\eta}_n\|_{W^{1,p}} + \|\boldsymbol{\eta}_n' \cdot \boldsymbol{N}\|_{W^{1,p}} + \|\boldsymbol{\eta}_n'' \cdot \boldsymbol{B}\|_{W^{1,p}} = 1 \quad (3.12)$$

and

$$\|a_{lin}(\boldsymbol{\eta}_n)\|_{L^p} + \|k_{lin}(\boldsymbol{\eta}_n)\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta}_n)\|_{L^p} \to 0 \text{ as } n \to +\infty. \quad (3.13)$$

By the compact inclusion $W^{1,p}(I) \subset L^p(I)$, there exists a subsequence, still denoted $(\boldsymbol{\eta}_n)$, that converges in $L^p(I)$. On the other hand, relation (3.13) shows that the sequences $(a_{lin}(\boldsymbol{\eta}_n))$, $(k_{lin}(\boldsymbol{\eta}_n))$ and $(\tau_{lin}(\boldsymbol{\eta}_n))$ converge to zero in $L^p(I)$.

Therefore $(\boldsymbol{\eta}_n)$, $(a_{lin}(\boldsymbol{\eta}_n))$, $(k_{lin}(\boldsymbol{\eta}_n))$ and $(\tau_{lin}(\boldsymbol{\eta}_n))$ are Cauchy sequences with respect to the $L^p(I)$-norm. The Korn inequality of Theorem 3.1 next shows that $(\boldsymbol{\eta}_n)$ is a Cauchy sequence for the norm of V. Since V is complete, there exists $\boldsymbol{\eta} \in V$ such that $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$ in V.

Using now the expressions of $a_{lin}(\boldsymbol{\eta}_n)$, $k_{lin}(\boldsymbol{\eta}_n)$ and $\tau_{lin}(\boldsymbol{\eta}_n)$, we obtain on the one hand that $a_{lin}(\boldsymbol{\eta}_n) \to a_{lin}(\boldsymbol{\eta})$, $k_{lin}(\boldsymbol{\eta}_n) \to k_{lin}(\boldsymbol{\eta})$ and $\tau_{lin}(\boldsymbol{\eta}_n) \to \tau_{lin}(\boldsymbol{\eta})$ in $L^p(I)$. On the other hand, we know by (3.13) that $\|a_{lin}(\boldsymbol{\eta}_n)\|_{L^p} \to 0$, $\|k_{lin}(\boldsymbol{\eta}_n)\|_{L^p} \to 0$ and $\|\tau_{lin}(\boldsymbol{\eta}_n)\|_{L^p} \to 0$. Hence $a_{lin}(\boldsymbol{\eta}) = k_{lin}(\boldsymbol{\eta}) = \tau_{lin}(\boldsymbol{\eta}) = 0$. Then Lemma 3.2 shows that $\boldsymbol{\eta} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{\gamma}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in $\mathbb{R}^3$. But $\boldsymbol{\eta} \in V_0$, since $\boldsymbol{\eta}_n \in V_0$, $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$ in V, and V_0 is a closed subspace of V. The boundary conditions satisfied by $\boldsymbol{\eta}$ then imply that $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$, so that $\boldsymbol{\eta} = \boldsymbol{0}$.

Therefore the sequence $(\boldsymbol{\eta}_n)$ converges to zero in V. This implies that $\|\boldsymbol{\eta}_n\|_V \to \|\boldsymbol{\eta}\|_V = 0$ as $n \to \infty$, which contradicts (3.12). □

REMARK 6. Korn's inequality for a curve remains valid for displacement fields satisfying other sets of boundary conditions. For instance, we can replace the space V_0 of Theorem 3.2 by the space

$$\Big\{\boldsymbol{\eta} \in V;\ \boldsymbol{\eta}(L) = \boldsymbol{0},\, (\boldsymbol{\eta}' \cdot \boldsymbol{N})(L) = (\boldsymbol{\eta}' \cdot \boldsymbol{B})(L) = (\boldsymbol{\eta}'' \cdot \boldsymbol{B})(L) = 0\Big\},$$

or by the space

$$\Big\{\boldsymbol{\eta} \in V;\ \boldsymbol{\eta}(0) = \boldsymbol{0},\, (\boldsymbol{\eta}' \cdot \boldsymbol{N})(L) = (\boldsymbol{\eta}' \cdot \boldsymbol{B})(L) = (\boldsymbol{\eta}'' \cdot \boldsymbol{B})(L) = 0\Big\}.$$

□

If we assume that the curve is smooth enough, then the Korn inequality for a curve takes a simpler form, similar to the Korn inequality on a surface (see Ciarlet [**4**, Theorem 2.6-4]). More specifically, we have the following *Korn inequality for smooth curves* as a corollary to Theorem 3.2:

COROLLARY 3.2. *If $\boldsymbol{\gamma} \in W^{5,\infty}(I;\mathbb{R}^3)$, the space V_0 defined as in Theorem 3.2 can be also defined as*

$$V_0 = \{\boldsymbol{\eta} = \eta_1 \boldsymbol{T} + \eta_2 \boldsymbol{N} + \eta_3 \boldsymbol{B}\ ;\ \eta_1 \in W^{1,p}(I), \eta_2 \in W^{2,p}(I), \eta_3 \in W^{3,p}(I)$$
$$\eta_1(0) = \eta_2(0) = \eta_3(0) = \eta_2'(0) = \eta_3'(0) = \eta_3''(0) = 0\},$$

and there exists a constant $C > 0$ depending only on $\boldsymbol{\gamma}$ such that

$$\|\eta_1\|_{W^{1,p}} + \|\eta_2\|_{W^{2,p}} + \|\eta_3\|_{W^{3,p}} \leq C\Big(\|a_{lin}(\boldsymbol{\eta})\|_{L^p} + \|k_{lin}(\boldsymbol{\eta})\|_{L^p} + \|\tau_{lin}(\boldsymbol{\eta})\|_{L^p}\Big)$$

for all $\boldsymbol{\eta} \in V_0$.

4. Nonlinear Korn inequalities for curves

We now establish several *nonlinear Korn inequalities for curves*, i.e., inequalities showing that $W^{m,p}$-norms of the displacement field of a curve are bounded above by appropriate Sobolev norms of expressions depending only on the "full" change of metric, change of curvature, and change of torsion, tensors of the curve. Such inequalities are thus natural extensions to curves in space of the nonlinear Korn inequality for open subsets of $\mathbb{R}^n$ established in Ciarlet and C. Mardare [**5**].

Let $I = (0, L) \subset \mathbb{R}$ and let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the canonical basis of $\mathbb{R}^3$. Let $\boldsymbol{\gamma} \in W^{3,p}(I; \mathbb{R}^3)$, $1 \le p \le +\infty$, be a regular curve (in the sense given in Section 2) in $\mathbb{R}^3$, let $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$ denote the Frenet frame associated with $\boldsymbol{\gamma}$, and let (a^2, k, τ) denote the metric, curvature, and torsion, tensors of $\boldsymbol{\gamma}$. We recall that $a(t) = |\boldsymbol{\gamma}'(t)|$ for all $t \in I$ and that k and τ are defined by formulas (2.1) and (2.2), respectively. With the notations

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \tag{4.1}$$

the Frenet equations associated with $\boldsymbol{\gamma}$ read

$$\boldsymbol{X}' = \big(a\boldsymbol{A}\big)\boldsymbol{X} \text{ a.e. in } I. \tag{4.2}$$

In what follows, $\boldsymbol{\gamma}$ is viewed as a *reference curve*, i.e., fixed once and for all. A generic regular curve $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I; \mathbb{R}^3)$ will be viewed as a *deformed curve*, i.e., a curve obtained by a deformation of $\boldsymbol{\gamma}$. Let $(\tilde{\boldsymbol{T}}, \tilde{\boldsymbol{N}}, \tilde{\boldsymbol{B}})$ denote the Frenet frame associated with $\tilde{\boldsymbol{\gamma}}$ and let $(\tilde{a}^2, \tilde{k}, \tilde{\tau})$ respectively denote the metric, curvature, and torsion tensors of $\tilde{\boldsymbol{\gamma}}$.

The following theorem establishes a *first nonlinear Korn inequality for curves.*

THEOREM 4.1. *Let $p \in [1, +\infty]$ and let $\boldsymbol{\gamma} \in W^{3,p}(I; \mathbb{R}^3)$ be a regular curve that satisfies $\boldsymbol{\gamma}(0) = \boldsymbol{0}$ and $\boldsymbol{X}(0) = \boldsymbol{I}_3$. Then there exists a constant $C_1(\boldsymbol{\gamma}) > 0$ depending only on the curve $\boldsymbol{\gamma}$ such that*

$$\|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{1,p}(I;\mathbb{R}^3)} \le C_1(\boldsymbol{\gamma})\big\{\|\tilde{a} - a\|_{L^p(I)} + \|\tilde{a}\tilde{k} - ak\|_{L^p(I)} + \|\tilde{a}\tilde{\tau} - a\tau\|_{L^p(I)}\big\} \tag{4.3}$$

for any regular curve $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I; \mathbb{R}^3)$ that satisfies $\tilde{\boldsymbol{\gamma}}(0) = \boldsymbol{0}$ and $\tilde{\boldsymbol{X}}(0) = \boldsymbol{I}_3$.

PROOF. Let the matrix fields $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{A}}$ associated with the curve $\tilde{\boldsymbol{\gamma}}$ be defined as above. Then the field $\tilde{\boldsymbol{X}}$ belongs to the space $W^{1,p}(I; \mathbb{M}^3)$ and satisfies the differential system

$$\tilde{\boldsymbol{X}}' = \big(\tilde{a}\tilde{\boldsymbol{A}}\big)\tilde{\boldsymbol{X}} \text{ a.e. in } I.$$

We first show that there exists a constant $C(a\boldsymbol{A}) > 0$ (depending only on $a\boldsymbol{A}$, hence only on the curve $\boldsymbol{\gamma}$) such that

$$\|\tilde{\boldsymbol{X}} - \boldsymbol{X}\|_{L^p(I;\mathbb{M}^3)} \le C(a\boldsymbol{A})\|\tilde{a}\tilde{\boldsymbol{A}} - a\boldsymbol{A}\|_{L^p(I;\mathbb{M}^3)}. \tag{4.4}$$

In view of the differential systems satisfied by $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$, we deduce that the field $(\tilde{\boldsymbol{X}} - \boldsymbol{X})$ is a solution to the system:

$$\begin{aligned} (\tilde{\boldsymbol{X}} - \boldsymbol{X})' &= a\boldsymbol{A}(\tilde{\boldsymbol{X}} - \boldsymbol{X}) + (\tilde{a}\tilde{\boldsymbol{A}} - a\boldsymbol{A})\tilde{\boldsymbol{X}} \text{ a.e. in } I, \\ (\tilde{\boldsymbol{X}} - \boldsymbol{X})(0) &= \boldsymbol{0}. \end{aligned} \tag{4.5}$$

Then we infer from Gronwall's inequality of Lemma 2.2 that, for all $t \in I$,

$$\begin{aligned} |(\tilde{\boldsymbol{X}} - \boldsymbol{X})(t)| &\leq \int_0^t |(\tilde{a}\tilde{\boldsymbol{A}} - a\boldsymbol{A})(s)\tilde{\boldsymbol{X}}(s)| \exp\Big(\int_s^t |a(r)\boldsymbol{A}(r)|dr\Big)ds \\ &\leq \int_0^t |(\tilde{a}\tilde{\boldsymbol{A}} - a\boldsymbol{A})(s)\tilde{\boldsymbol{X}}(s)| \exp\big(\|a\boldsymbol{A}\|_{L^1(I;\mathbb{M}^3)}\big)ds \\ &\leq C(a\boldsymbol{A})\int_0^t |(\tilde{a}\boldsymbol{A} - a\boldsymbol{A})(s)|ds. \end{aligned} \tag{4.6}$$

Note that in the last inequality we have used the fact that $|\tilde{\boldsymbol{X}}| \leq 1$ on I, an inequality itself a consequence of the relations $|\tilde{\boldsymbol{T}}| = |\tilde{\boldsymbol{N}}| = |\tilde{\boldsymbol{B}}| = 1$ and $\tilde{\boldsymbol{T}} \cdot \tilde{\boldsymbol{N}} = \tilde{\boldsymbol{T}} \cdot \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{N}} \cdot \tilde{\boldsymbol{B}} = 0$. From relation (4.6), we then immediately deduce inequality (4.4).

Using the Poincaré inequality and the fact that the field $(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})$ satisfies the system

$$\begin{aligned} (\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})' &= (\tilde{a} - a)\tilde{\boldsymbol{T}} + a(\tilde{\boldsymbol{T}} - \boldsymbol{T}) \text{ a.e. in } I, \\ (\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})(0) &= \boldsymbol{0}, \end{aligned} \tag{4.7}$$

we next deduce that:

$$\begin{aligned} \|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{1,p}(I;\mathbb{R}^3)} &\leq C\|(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})'\|_{L^p(I;\mathbb{R}^3)} \\ &= C\|(\tilde{a} - a)\tilde{\boldsymbol{T}} + a(\tilde{\boldsymbol{T}} - \boldsymbol{T})\|_{L^p(I;\mathbb{R}^3)} \\ &\leq C(\|\tilde{a} - a\|_{L^p(I)}\|\tilde{\boldsymbol{T}}\|_{L^\infty(I;\mathbb{R}^3)} + \|a\|_{L^\infty(I)}\|\tilde{\boldsymbol{T}} - \boldsymbol{T}\|_{L^p(I;\mathbb{R}^3)}) \\ &\leq C(\|\tilde{a} - a\|_{L^p(I)} + \|a\|_{L^\infty(I)}\|\tilde{\boldsymbol{X}} - \boldsymbol{X}\|_{L^p(I;\mathbb{M}^3)}). \end{aligned} \tag{4.8}$$

Then, by using the estimate (4.4) of the L^p-norm of $(\tilde{\boldsymbol{X}} - \boldsymbol{X})$, we obtain that

$$\begin{aligned} \|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{1,p}(I;\mathbb{R}^3)} &\leq C\big(\|\tilde{a} - a\|_{L^p(I)} + \|a\|_{L^\infty(I)}C(a\boldsymbol{A})\|\tilde{a}\tilde{\boldsymbol{A}} - a\boldsymbol{A}\|_{L^p(I;\mathbb{M}^3)}\big) \\ &\leq C_1(\boldsymbol{\gamma})\big(\|\tilde{a} - a\|_{L^p(I)} + \|\tilde{a}\tilde{k} - ak\|_{L^p(I)} + \|\tilde{a}\tilde{\tau} - a\tau\|_{L^p(I)}\big), \end{aligned} \tag{4.9}$$

which is exactly the announced inequality. □

REMARK 7. For curves not necessarily parametrized by their arc length, the "right" geometric quantities to consider in the Korn inequality are $(a, ak, a\tau)$ instead of (a^2, k, τ). □

The next theorem provides a *second nonlinear inequality of Korn's type* that involves the L^p-norm of the second-order generalized derivatives of the difference $(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})$:

THEOREM 4.2. *Let $p \in [1, +\infty]$ and let $\boldsymbol{\gamma} \in W^{3,p}(I;\mathbb{R}^3)$ be a regular curve that satisfies $\boldsymbol{\gamma}(0) = \boldsymbol{0}$ and $\boldsymbol{X}(0) = \boldsymbol{I}_3$. Then there exists a constant $C_2(\boldsymbol{\gamma}) > 0$ depending only on the* *curve* $\boldsymbol{\gamma}$ *such that*

$$\begin{aligned}\|\tilde{\boldsymbol{\gamma}}-\boldsymbol{\gamma}\|_{W^{2,p}(I;\mathbb{R}^3)} &\leq C_2(\boldsymbol{\gamma})\big\{\|\tilde{a}-a\|_{W^{1,p}(I)}+\|\tilde{a}^2\tilde{k}-a^2k\|_{L^p(I)}\\ &\quad+\|\tilde{a}\tilde{k}-ak\|_{L^p(I)}+\|\tilde{a}\tilde{\tau}-a\tau\|_{L^p(I)}\big\}\end{aligned}\tag{4.10}$$

for any regular curve $\tilde{\boldsymbol{\gamma}}\in W^{3,p}(I;\mathbb{R}^3)$ *that satisfies* $\tilde{\boldsymbol{\gamma}}(0)=\mathbf{0}$ *and* $\tilde{\boldsymbol{X}}(0)=\boldsymbol{I}_3$.

PROOF. We use the same notations as in the proof of Theorem 4.1. We deduce from the differential systems satisfied by the fields $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ that

$$\|(\tilde{\boldsymbol{X}}-\boldsymbol{X})'\|_{L^p(I;\mathbb{M}^3)}\leq\|a\boldsymbol{A}\|_{L^p(I;\mathbb{M}^3)}\|\tilde{\boldsymbol{X}}-\boldsymbol{X}\|_{L^\infty(I;\mathbb{M}^3)}+\|\tilde{a}\tilde{\boldsymbol{A}}-a\boldsymbol{A}\|_{L^p(I;\mathbb{M}^3)}\|\tilde{\boldsymbol{X}}\|_{L^\infty(I;\mathbb{M}^3)}.$$

By combining this inequality with the estimate of the L^∞-norm of the difference $(\tilde{\boldsymbol{X}}-\boldsymbol{X})$ furnished by relation (4.6), we then obtain the estimate

$$\|\tilde{\boldsymbol{X}}-\boldsymbol{X}\|_{W^{1,p}(I;\mathbb{M}^3)}\leq C(a\boldsymbol{A})\|\tilde{a}\tilde{\boldsymbol{A}}-a\boldsymbol{A}\|_{L^p(I;\mathbb{M}^3)},\tag{4.11}$$

where $C(a\boldsymbol{A})>0$ is a constant.

We also need to estimate the L^p-norm of the difference between the second derivatives of $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$. Since

$$(\tilde{\boldsymbol{\gamma}}-\boldsymbol{\gamma})''=(\tilde{a}\tilde{\boldsymbol{T}})'-(a\boldsymbol{T})'=\tilde{a}'\tilde{\boldsymbol{T}}+\tilde{a}\tilde{\boldsymbol{T}}'-a'\boldsymbol{T}-a\boldsymbol{T}'$$

and since $\boldsymbol{T}'=ak\boldsymbol{N}$ and $\tilde{\boldsymbol{T}}'=\tilde{a}\tilde{k}\tilde{\boldsymbol{N}}$, we infer that

$$\begin{aligned}(\tilde{\boldsymbol{\gamma}}-\boldsymbol{\gamma})'' &= (\tilde{a}'-a')\tilde{\boldsymbol{T}}+a'(\tilde{\boldsymbol{T}}-\boldsymbol{T})+\tilde{a}(\tilde{a}\tilde{k})\tilde{\boldsymbol{N}}-a(ak)\boldsymbol{N}\\ &= (\tilde{a}'-a')\tilde{\boldsymbol{T}}+a'(\tilde{\boldsymbol{T}}-\boldsymbol{T})+(\tilde{a}^2\tilde{k}-a^2k)\tilde{\boldsymbol{N}}+a^2k(\tilde{\boldsymbol{N}}-\boldsymbol{N}).\end{aligned}\tag{4.12}$$

Consequently,

$$\begin{aligned}\|(\tilde{\boldsymbol{\gamma}}-\boldsymbol{\gamma})''\|_{L^p(I;\mathbb{R}^3)} &\leq \|\tilde{a}'-a'\|_{L^p(I)}\|\tilde{\boldsymbol{T}}\|_{L^\infty(I;\mathbb{R}^3)}+\|a'\|_{L^p(I)}\|\tilde{\boldsymbol{T}}-\boldsymbol{T}\|_{L^\infty(I;\mathbb{R}^3)}\\ &\quad+\|\tilde{a}^2\tilde{k}-a^2k\|_{L^p(I)}\|\tilde{\boldsymbol{N}}\|_{L^\infty(I;\mathbb{R}^3)}+\|a^2k\|_{L^\infty(I)}\|\tilde{\boldsymbol{N}}-\boldsymbol{N}\|_{L^p(I;\mathbb{R}^3)}\\ &\leq \|\tilde{a}'-a'\|_{L^p(I)}+\|\tilde{a}^2\tilde{k}-a^2k\|_{L^p(I)}\\ &\quad+(\|a'\|_{L^p(I)}+\|a^2k\|_{L^\infty(I)})\|\tilde{\boldsymbol{X}}-\boldsymbol{X}\|_{W^{1,p}(I;\mathbb{M}^3)}.\end{aligned}\tag{4.13}$$

By combining this inequality with inequality (4.11), we obtain, on the one hand,

$$\begin{aligned}\|(\tilde{\boldsymbol{\gamma}}-\boldsymbol{\gamma})''\|_{L^p(I;\mathbb{R}^3)} &\leq \|\tilde{a}'-a'\|_{L^p(I)}+\|\tilde{a}^2\tilde{k}-a^2k\|_{L^p(I)}\\ &\quad+C(\boldsymbol{\gamma})(\|\tilde{a}\tilde{k}-ak\|_{L^p(I)}+\|\tilde{a}\tilde{\tau}-a\tau\|_{L^p(I)}).\end{aligned}\tag{4.14}$$

On the other hand, we know from Theorem 4.1 that

$$\|\tilde{\boldsymbol{\gamma}}-\boldsymbol{\gamma}\|_{W^{1,p}(I;\mathbb{R}^3)}\leq C_1(\boldsymbol{\gamma})(\|\tilde{a}-a\|_{L^p(I)}+\|\tilde{a}\tilde{k}-ak\|_{L^p(I)}+\|\tilde{a}\tilde{\tau}-a\tau\|_{L^p(I)}).\tag{4.15}$$

From the last two inequalities, we conclude that there exists a constant $C_2(\boldsymbol{\gamma})>0$ such that

$$\begin{aligned}\|\tilde{\boldsymbol{\gamma}}-\boldsymbol{\gamma}\|_{W^{2,p}(I;\mathbb{R}^3)} &\leq C_2(\boldsymbol{\gamma})\big(\|\tilde{a}-a\|_{W^{1,p}(I)}+\|\tilde{a}^2\tilde{k}-a^2k\|_{L^p(I)}\\ &\quad+\|\tilde{a}\tilde{k}-ak\|_{L^p(I)}+\|\tilde{a}\tilde{\tau}-a\tau\|_{L^p(I)}\big),\end{aligned}\tag{4.16}$$

which is the desired inequality. ☐

We can generalize the previous inequalities one step further, by establishing a *third nonlinear Korn inequality* that takes into account the third order derivatives of the curves, according to the following theorem:

THEOREM 4.3. *Let $p \in [1, +\infty]$ and let $\boldsymbol{\gamma} \in W^{3,p}(I; \mathbb{R}^3)$ be a regular curve that satisfies $\boldsymbol{\gamma}(0) = \mathbf{0}$ and $\boldsymbol{X}(0) = \boldsymbol{I}_3$. Then there exists a constant $C_3(\boldsymbol{\gamma}) > 0$ depending only on the curve $\boldsymbol{\gamma}$ such that*

$$\begin{aligned} \|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{3,p}(I;\mathbb{R}^3)} \leq\ & C_3(\boldsymbol{\gamma}) \big\{ \|\tilde{a} - a\|_{W^{2,p}(I)} + \|\tilde{a}^2\tilde{k} - a^2 k\|_{W^{1,p}(I)} \\ & + \|\tilde{a}\tilde{k} - ak\|_{L^p(I)} + \|(\tilde{a}^2)'\tilde{k} - (a^2)'k\|_{L^p(I)} + \|\tilde{a}\tilde{\tau} - a\tau\|_{L^p(I)} \\ & + \|\tilde{a}^3\tilde{k}^2 - a^3k^2\|_{L^p(I)} + \|\tilde{a}^3\tilde{k}\tilde{\tau} - a^3k\tau\|_{L^p(I)} \big\} \end{aligned} \tag{4.17}$$

for any regular curve $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I; \mathbb{R}^3)$ that satisfies $\tilde{\boldsymbol{\gamma}}(0) = \mathbf{0}$ and $\tilde{\boldsymbol{X}}(0) = \boldsymbol{I}_3$.

PROOF. We first notice that the Frenet equations for the curve $\boldsymbol{\gamma}$ imply that

$$\begin{aligned} \boldsymbol{\gamma}' &= a\boldsymbol{T}, \\ \boldsymbol{\gamma}'' &= a'\boldsymbol{T} + a^2k\boldsymbol{N}, \\ \boldsymbol{\gamma}''' &= (a'' - a^3k^2)\boldsymbol{T} + \big((a^2k)' + \frac{(a^2)'k}{2}\big)\boldsymbol{N} + a^3k\tau\boldsymbol{B}. \end{aligned} \tag{4.18}$$

Similar equations are obtained from the Frenet equations for the curve $\tilde{\boldsymbol{\gamma}}$. By substracting these relations, we get

$$\begin{aligned} (\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})''' =\ & (\tilde{a}'' - a'')\tilde{\boldsymbol{T}} - (\tilde{a}^3\tilde{k}^2 - a^3k^2)\tilde{\boldsymbol{T}} + (a'' - a^3k^2)(\tilde{\boldsymbol{T}} - \boldsymbol{T}) \\ & + (\tilde{a}^2\tilde{k} - a^2k)'\tilde{\boldsymbol{N}} + \big(\frac{(\tilde{a}^2)'\tilde{k}}{2} - \frac{(a^2)'k}{2}\big)\tilde{\boldsymbol{N}} + \big((a^2k)' + \frac{(a^2)'k}{2}\big)(\tilde{\boldsymbol{N}} - \boldsymbol{N}) \\ & + (\tilde{a}^3\tilde{k}\tilde{\tau} - a^3k\tau)\tilde{\boldsymbol{B}} + a^3k\tau(\tilde{\boldsymbol{B}} - \boldsymbol{B}). \end{aligned} \tag{4.19}$$

Consequently,

$$\begin{aligned} \|(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})'''\|_{L^p(I;\mathbb{R}^3)} \leq\ & \Big(\|a'' - a^3k^2\|_{L^p(I)} + \big\|(a^2k)' + \frac{(a^2)'k}{2}\big\|_{L^p(I)} + \|a^3k\tau\|_{L^p(I)}\Big)\|\tilde{\boldsymbol{X}} - \boldsymbol{X}\|_{L^\infty(I;\mathbb{M}^3)} \\ & + \|(\tilde{a} - a)''\|_{L^p(I)} + \|\tilde{a}^3\tilde{k}^2 - a^3k^2\|_{L^p(I)} + \|(\tilde{a}^2\tilde{k} - a^2k)'\|_{L^p(I)} \\ & + \frac{1}{2}\|(\tilde{a}^2)'\tilde{k} - (a^2)'k\|_{L^p(I)} + \|\tilde{a}^3\tilde{k}\tilde{\tau} - a^3k\tau\|_{L^p(I)}. \end{aligned} \tag{4.20}$$

Using inequality (4.11) and the Sobolev embedding theorem to estimate the L^∞-norm of $(\tilde{\boldsymbol{X}} - \boldsymbol{X})$ appearing in the previous inequality, then combining with the inequality obtained in Theorem 4.2, we obtain

$$\begin{aligned} \|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{3,p}(I;\mathbb{R}^3)} \leq\ & C_3(\boldsymbol{\gamma}) \big(\|\tilde{a} - a\|_{W^{2,p}(I)} + \|\tilde{a}^2\tilde{k} - a^2 k\|_{W^{1,p}(I)} \\ & + \|\tilde{a}\tilde{k} - ak\|_{L^p(I)} + \|(\tilde{a}^2)'\tilde{k} - (a^2)'k\|_{L^p(I)} + \|\tilde{a}\tilde{\tau} - a\tau\|_{L^p(I)} \\ & + \|\tilde{a}^3\tilde{k}^2 - a^3k^2\|_{L^p(I)} + \|\tilde{a}^3\tilde{k}\tilde{\tau} - a^3k\tau\|_{L^p(I)} \big), \end{aligned} \tag{4.21}$$

which is the desired inequality. Note that all the terms appearing in the inequality (4.21) make sense, since the curves $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are regular and belong to the space $W^{3,p}(I; \mathbb{R}^3)$. □

REMARK 8. Note that inequalities (4.3), (4.10) and (4.17) can be interpreted as non-linear versions of the linear Korn inequality for curves established in Section 3, in the sense that the latter can be obtained by formal linearisation with respect to the displacement field $\boldsymbol{\eta} = \tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$ in the former. □

In the particular case of *isometric curves*, i.e., curves parametrized by their arc length, the previous inequalities of Korn's type take the following simpler forms:

COROLLARY 4.1. *Let $p \in [1, +\infty]$ and let $\boldsymbol{\gamma} \in W^{3,p}(I; \mathbb{R}^3)$ be a regular curve parametrized by its arc length that satisfies $\boldsymbol{\gamma}(0) = \mathbf{0}$ and $\boldsymbol{X}(0) = \boldsymbol{I}_3$ (the matrix field X being related to the curve $\boldsymbol{\gamma}$ by relation (4.1)). Then there exist constants $c_2(\boldsymbol{\gamma}) > 0$ and $c_3(\boldsymbol{\gamma}) > 0$ depending only on the curve $\boldsymbol{\gamma}$ such that*

$$\|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{2,p}(I;\mathbb{R}^3)} \leq c_2(\boldsymbol{\gamma})(\|\tilde{k} - k\|_{L^p(I)} + \|\tilde{\tau} - \tau\|_{L^p(I)}), \tag{4.22}$$

and

$$\begin{aligned} \|\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_{W^{3,p}(I;\mathbb{R}^3)} \leq\ & c_3(\boldsymbol{\gamma})(\|\tilde{k} - k\|_{W^{1,p}(I)} + \|\tilde{\tau} - \tau\|_{L^p(I)} \\ & + \|\tilde{k}^2 - k^2\|_{L^p(I)} + \|\tilde{k}\tilde{\tau} - k\tau\|_{L^p(I)}), \end{aligned} \tag{4.23}$$

for any regular curve $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I; \mathbb{R}^3)$ parametrized by its arc length that satisfies $\tilde{\boldsymbol{\gamma}}(0) = \mathbf{0}$ and $\tilde{\boldsymbol{X}}(0) = \boldsymbol{I}_3$.

PROOF. Inequalities (4.22) and (4.23) are obtained by using the fact that $\tilde{a} = a = 1$ in relations (4.10) and (4.17). Note that the inequality obtained by taking $\tilde{a} = a = 1$ in relation (4.3) is a particular case of inequality (4.22). □

The nonlinear Korn inequalities of Theorems 4.1, 4.2 and 4.3 provide estimates in appropriate Sobolev norms of the difference between the deformed configuration and the reference configuration of a curve that satisfies specific boundary conditions, such as $\tilde{\boldsymbol{\gamma}}(0) = \boldsymbol{\gamma}(0) = \mathbf{0}$ and $\tilde{\boldsymbol{X}}(0) = \boldsymbol{X}(0) = \boldsymbol{I}_3$. The consideration of such boundary conditions can be altogether eliminated if the difference between the deformed and the reference configurations is measured in an appropriate *quotient* space. More specifically, we have the following corollary to Theorem 4.1 (similar corollaries can be derived from Theorems 4.2 and 4.3):

COROLLARY 4.2. *Let $p \in [1, +\infty]$ and let $\boldsymbol{\gamma} \in W^{3,p}(I; \mathbb{R}^3)$ be a regular curve in $\mathbb{R}^3$. Then there exists a constant $C_1(\boldsymbol{\gamma}) > 0$ depending only on $\boldsymbol{\gamma}$ with the following property: Given any regular curve $\tilde{\boldsymbol{\gamma}} \in W^{3,p}(I; \mathbb{R}^3)$, there exist a vector $\boldsymbol{a} \in \mathbb{R}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that*

$$\|\tilde{\boldsymbol{\gamma}} - (\boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\gamma})\|_{W^{1,p}(I;\mathbb{R}^3)} \leq C_1(\boldsymbol{\gamma})\big\{\|\tilde{a} - a\|_{L^p(I)} + \|\tilde{a}\tilde{k} - ak\|_{L^p(I)} + \|\tilde{a}\tilde{\tau} - a\tau\|_{L^p(I)}\big\}.$$

PROOF. It suffices to apply Theorem 4.1 to the curves $\boldsymbol{X}(0)^{-1}(\boldsymbol{\gamma} - \boldsymbol{\gamma}(0))$ and $\tilde{\boldsymbol{X}}(0)^{-1}(\tilde{\boldsymbol{\gamma}} - \tilde{\boldsymbol{\gamma}}(0))$, where the matrix field $\boldsymbol{X}$, resp. $\tilde{\boldsymbol{X}}$, associated with $\boldsymbol{\gamma}$, resp. $\tilde{\boldsymbol{\gamma}}$, are defined at the beginning of Section 4. This gives the desired inequality with $\boldsymbol{Q} = \tilde{\boldsymbol{X}}(0)\boldsymbol{X}(0)^{-1}$ and $\boldsymbol{a} = \tilde{\boldsymbol{\gamma}}(0) - \boldsymbol{Q}\boldsymbol{\gamma}(0)$. □

An immediate consequence of the above nonlinear Korn inequality is the following *rigid displacement lemma for curves* (see Lemma 3.2 for its linear counterpart):

LEMMA 4.1. *Let two curves* $\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}} \in W^{3,p}(I; \mathbb{R}^3)$ *that are regular satisfy* $a = \tilde{a}$, $k = \tilde{k}$, *and* $\tau = \tilde{\tau}$ *a.e. in* I. *Then there exist a vector* $\boldsymbol{a} \in \mathbb{R}^3$ *and a matrix* $\boldsymbol{Q} \in \mathbb{O}^3_+$ *such that*

$$\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{a} + \boldsymbol{Q}\boldsymbol{\gamma}(t) \text{ for all } t \in I\,.$$

REMARK 9. The previous rigid displacement lemma for curves can also be obtained as a direct consequence of the uniqueness part of Corollary 3.1 of Szopos [**10**], where the author establishes the existence and uniqueness up to rigid motions of a curve isometrically immersed in the three-dimensional euclidean space, with prescribed curvature and torsion functions. □

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Annexe

Dans cette partie, on développe un outil permettant de reconstruire des courbes en dimension 2 ou 3 dont les fonctions de courbure et de torsion sont dérivables au sens de Sobolev, mais pas au sens classique.

Pour cela, on a fait des simulations numériques à l'aide du Scilab, qui est un logiciel libre développé par l'INRIA (http ://www.scilab.org) et on a utilisé les résultats théoriques obtenus dans la première partie.

1. Reconstruction des courbes planes

Dans cette section, on considère le cas d'une courbe plane $\alpha\colon (0,T) \to \mathbb{R}^2$, paramétrée par la longueur d'arc. Ce cas ne rentre pas dans le cadre général décrit dans le Chapitre 1, où sont étudiées les courbes immergées dans un espace euclidien de dimension au moins égale à 3, mais il permet d'obtenir une première interprétation des notions qui vont apparaître dans la suite.

Dans le cas particulier des courbes planes, la courbure est définie de la manière suivante : soit $t(s) := \frac{d\alpha}{ds}(s)$ le vecteur tangent à la courbe α au point $\alpha(s)$, où $s \in (0,T)$ et soit $n(s)$ le vecteur normal unitaire orienté tel que la base $\{t(s), n(s)\}$ ait la même orientation que la base canonique $\{(1,0),(0,1)\}$ de $\mathbb{R}^2$. La courbure $k(s)$ est alors donnée par la relation

$$t'(s) = k(s)n(s), \tag{1.1}$$

pour tout $s \in (0,T)$. Il est à remarquer que dans le cas des courbes planes la courbure peut être ou bien positive ou bien négative (contrairement au cas des courbes dans l'espace, où elle est toujours positive).

Les exemples les plus simples sont les droites, qui ont une courbure nulle, ou les cercles, qui ont une courbure constante, égale à l'inverse du rayon. Intuitivement, la courbure d'une courbe mesure l'écart, au point considéré, entre la courbe et la droite tangente.

Réciproquement, on peut se poser la question suivante : si on se donne une fonction $k\colon (0,T) \to \mathbb{R}$, peut-on trouver une courbe $\alpha\colon (0,T) \to \mathbb{R}^2$ telle que sa courbure soit exactement k ? La réponse est donnée par le théorème fondamental de la théorie des courbes planes, qui affirme qu'une telle courbe existe, qu'elle est unique à une isométrie de $\mathbb{R}^2$ près, et qu'elle s'obtient en intégrant les équations suivantes :

$$\theta'(s) = k(s) \tag{1.2}$$

$$\alpha'(s) = \big(\cos\theta(s),\ \sin\theta(s)\big). \tag{1.3}$$

De même que dans le Chapitre 1, on peut considérer que ces équations sont vérifiées au sens des distributions et trouver des solutions faibles du problème. On peut ainsi reconstruire des courbes planes dont la fonction de courbure n'est pas nécessairement dérivable au sens classique.

Nous présentons dans la suite un exemple de courbe obtenue par cette méthode : il s'agit d'une courbe plane dont la courbure est donnée par une fonction du type "dents de scie", dont le graphe est le suivant :

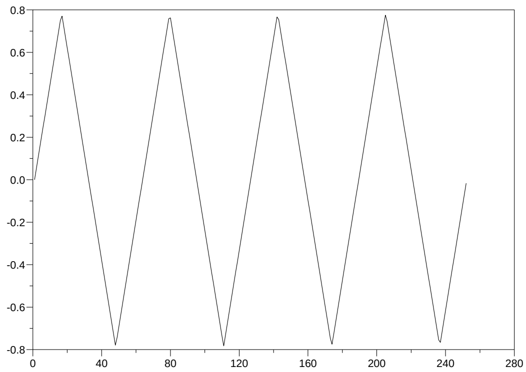

FIG. 1 – Graphe de la fonction "dents de scie"

et qui est donnée par la formule :

$$k(s) = \arctan\Big(|\tan(\frac{s}{2} + \frac{\pi}{4})| \Big) - \frac{\pi}{4}. \tag{1.4}$$

L'intégration numérique des équations (1.2) et (1.3) permet d'obtenir la courbe plane de courbure k suivante :

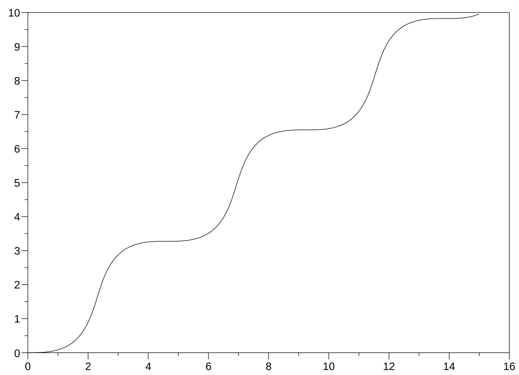

FIG. 2 – Courbe plane dont la courbure est une fonction du type "dents de scie"

Il est à remarquer que la périodicité de la fonction de courbure a comme conséquence le comportement périodique de la courbe ainsi reconstruite.

Cet exemple est à comparer avec le cas d'une fonction de courbure qui est, cette fois-ci, dérivable au sens classique, et qui a le graphe suivant :

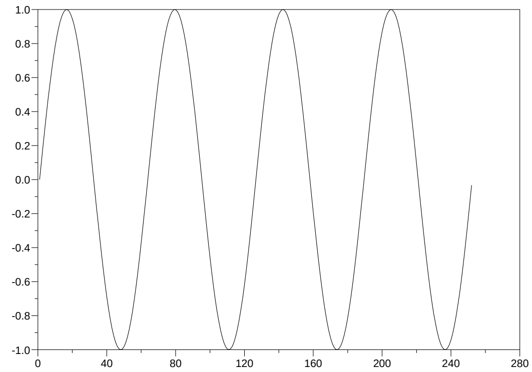

FIG. 3 – Graphe de la fonction sinus

Il s'agit de la fonction $k(s) = \sin s$, qui a un comportement très proche de la fonction "dents de scie", dans le sens où on retrouve la périodicité et l'alternence du signe, mais qui n'a plus de points angulaires. Dans ce cas, on obtient par simulation numérique la courbe suivante :

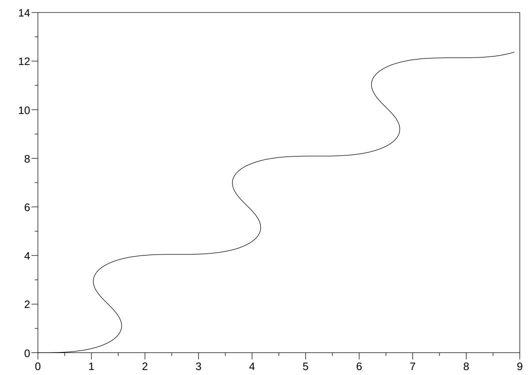

FIG. 4 – Courbe plane de courbure une fonction du type "sinusoïde"

A nouveau, la périodicité de la fonction de courbure se retrouve dans la périodicité de la courbe. De plus, en faisant une comparaison entre les deux courbes présentées auparavant, on retrouve le même comportement, ce qui s'explique par le fait que leurs fonctions de courbure avaient la même allure et par la propriété de stabilité de la solution du système (1.2)-(1.3).

2. Reconstruction des courbes gauches

Soit $\beta\colon (0,T) \to \mathbb{R}^3$, $\beta(s) = \big(x_1(s), x_2(s), x_3(s)\big)$, une courbe gauche (i.e. immergée en $\mathbb{R}^3$) paramétrée par la longueur d'arc et telle que sa courbure et sa torsion soient données par $k\colon (0,T) \to \mathbb{R}^*_+$ et $\tau\colon (0,T) \to \mathbb{R}$, respectivement. On note par $\{t(s), n(s), b(s)\}$ son repère de Frenet en tout point $s \in (0,T)$.

D'un point de vue intuitif, la courbure d'une telle courbe peut être interprétée de la même manière que dans le cas d'une courbe plane ; quant à la torsion, elle donne une indication sur la façon dont la courbe étudiée s'écarte d'une courbe plane (dans le sens où une courbe a la torsion nulle si, et seulement si, elle se trouve dans un plan).

Le résultat de reconstruction d'une courbe décrit dans le Chapitre 1 (voir Corollaire 3.1) nous permet de reconstruire des courbes qui ont une courbure et une torsion données par des fonctions qui ne sont pas nécessairement dérivables au sens classique. Grâce à ce résultat, il est possible de considérer une fonction de courbure positive dans $H^1(I)$, une fonction de torsion dans $L^2(I)$, et d'obtenir une courbe c dans $H^3(I;\mathbb{R}^3)$ dont les courbures sont exactement les fonctions données.

La stratégie consiste à intégrer les équations suivantes :

$$x_1'(s) = t_1(s), \tag{2.1}$$
$$x_2'(s) = t_2(s), \tag{2.2}$$
$$x_3'(s) = t_3(s), \tag{2.3}$$

où les fonctions t_1, t_2, t_3 sont obtenues comme solutions des équations de Frenet :

$$t_1'(s) = k(s)n_1(s), \tag{2.4}$$
$$t_2'(s) = k(s)n_2(s), \tag{2.5}$$
$$t_3'(s) = k(s)n_3(s), \tag{2.6}$$
$$n_1'(s) = -k(s)t_1(s) + \tau(s)b_1(s), \tag{2.7}$$
$$n_2'(s) = -k(s)t_2(s) + \tau(s)b_2(s), \tag{2.8}$$
$$n_3'(s) = -k(s)t_3(s) + \tau(s)b_3(s), \tag{2.9}$$
$$b_1'(s) = -\tau(s)n_1(s), \tag{2.10}$$
$$b_2'(s) = -\tau(s)n_2(s), \tag{2.11}$$
$$b_3'(s) = -\tau(s)n_3(s). \tag{2.12}$$

Pour illustrer ce résultat théorique, considérons l'exemple suivant : soient $k(s) = |s|$ une fonction qui n'est pas dérivable au sens classique en 0 et $\tau(s) = 0.3$ une fonction constante. Si on résout ce système numériquement et on représente graphiquement le résultat, on obtient la courbe suivante :

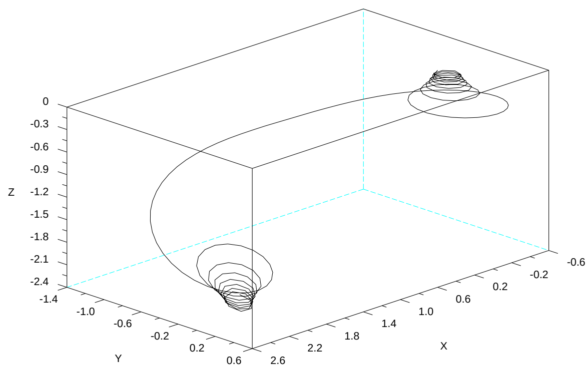

FIG. 5 – Courbe gauche de courbure $|s|$ et torsion 0.3

Cet exemple peut être comparé avec le cas de l'hélice représentée ci-dessous, qui a la même torsion $\tau(s) = 0.3$, mais qui a la courbure constante, $k(s) = 1$:

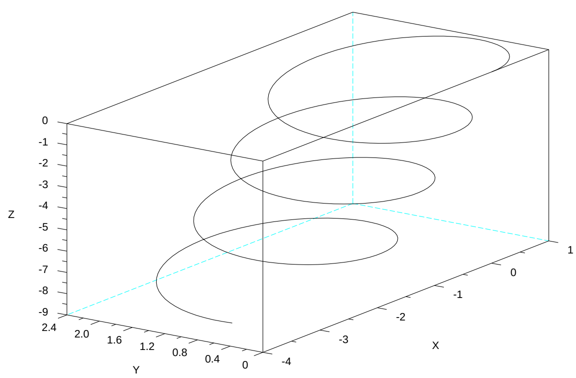

FIG. 6 – Hélice de courbure 1 et torsion 0.3

Dans les deux cas, le "pas" (c'est-à-dire la distance entre deux spires consécutives) est constant, ce qui est une conséquence du fait que la torsion est constante. Par contre, si pour l'hélice la courbure est constante, la première courbe a l'allure d'une droite au voisinage du point obtenu pour $s = 0$, car la courbure y est très petite, et a le comportement d'une spirale pour s grand, car k tend vers l'infini lorsque s tend vers infini.