Mathematical and numerical modeling of wave propagation in viscoelastic and poroelastic media

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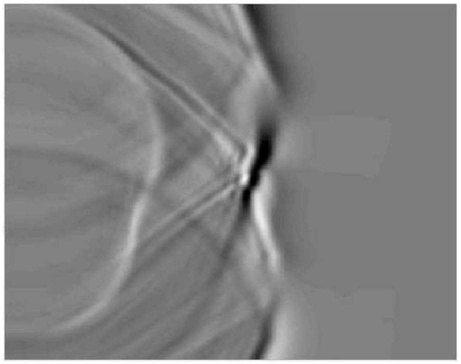


- Applications : Seismic, Geophysic, Undersea and underground, Trabecular bone propagation, Modeling of polymers ...
- Objective : Powerful numerical methods
  - fast.
  - precise.
  - robust.
  - Realistic.

- I. Viscoelastic waves
- II. Poroelastic waves
  - Wave propagation model
  - Mathematical analysis
  - Numerical analysis
  - Implementation

I-Viscoelastic waves

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Viscoelastic model (Solids with memory) :

R :Tensorial Relaxation function $\quad R_{ijkl} = R_{jikl} = R_{klij}$

$$\begin{aligned}
\sigma_{ij}(x,t) &= \int_{-\infty}^{t} R_{ijkl}(x,t-\tau)\frac{\partial}{\partial \tau}\varepsilon_{kl}(x,\tau)\,d\tau \\
&= \int_{-\infty}^{+\infty} R_{ijkl}(x,t-\tau)\frac{\partial}{\partial \tau}\varepsilon_{kl}(x,\tau)\,d\tau \\
&= R * \frac{\partial \varepsilon}{\partial t} = m * \varepsilon \quad \left(m = \frac{\partial R}{\partial t}\right)
\end{aligned}$$

- Hooke's law (Elastic model)

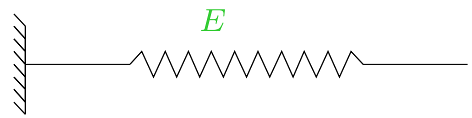
$$R = C(x)H(t) \;\Leftrightarrow\; m = C(x)\delta \;\Leftrightarrow\; \sigma = C\varepsilon$$

- 1D model: $P(D)\sigma = T(D)\varepsilon(u)$

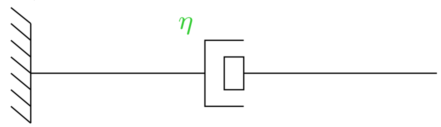
$$P(D) = \sum_{k=0}^{k=n} a_k \frac{\partial^k}{\partial t^k};\ T(D) = \sum_{k=0}^{k=m} b_k \frac{\partial^k}{\partial t^k}$$

- Elementary models :

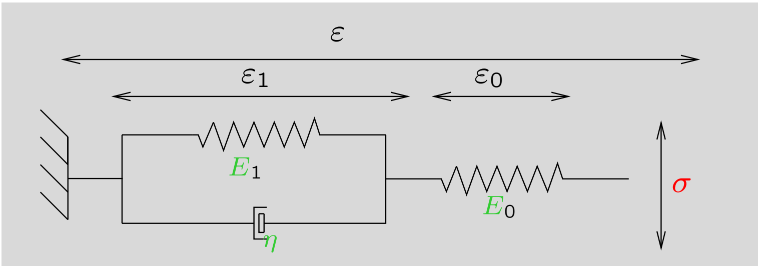
(a) : $\sigma = E\varepsilon$ Spring



(b) : $\sigma = \eta\dot{\varepsilon}$ Dahpot



- Combination of elementary models :



$$\sigma + \tau_0 \, \partial_t \sigma = \mu \Big[\varepsilon(u) + \tau_1 \, \varepsilon(\partial_t u) \Big]$$

Absorption condition?

$\tau_1 - \tau_0 > 0.$

Relaxation function : $R(x,t) = \mu \Big(1 + \frac{\tau_1 - \tau_0}{\tau_0} \, e^{-\frac{t}{\tau_0}} \Big) H(t)$

- General case :

$$\sigma + \tau_0 \, \partial_t \sigma = C \varepsilon(u) + \tau_0 D \varepsilon(\partial_t u)$$

C, D are symmetric > 0

Absorption condition ?

$(D - C) > 0$

Tensorial Relaxation :

$$R(x,t) = \Big[C(x) + \Big(D(x) - C(x) \Big) e^{-\frac{t}{\tau_0(x)}} \Big] H(t)$$

$$
\begin{cases}
\rho \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot \sigma = f, & \mathbb{R}^d \times]0, T], \\[2ex]
\sigma + \tau_0 \dfrac{\partial \sigma}{\partial t} = \boldsymbol{C}\, \varepsilon(u) + \tau_0 \boldsymbol{D}\, \varepsilon(\dfrac{\partial u}{\partial t}), & \mathbb{R}^d \times]0.T], \\[2ex]
u(x,0) = u_0,\ \partial_t u(x,0) = u_1,\ \sigma(x,0) = \sigma_0, & \mathbb{R}^d.
\end{cases}
$$

- $d = 1, 2, 3.$
- $(\nabla \cdot \sigma)_i = \displaystyle\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}.$
- $\rho(x)$, $\tau_0(x)$, $C(x)$ and $D(x)$ are measurable.
- $0 < \rho_-,\ \tau_{0-} \leq \rho(x),\ \tau_0(x) \leq \rho_+,\ \tau_{0+} < +\infty \quad a.e.\ x \in \mathbb{R}^d.$

$$Z(x) = (D(x) - C(x)) > 0 \;\; (\Leftrightarrow \tau_1 > \tau_0 \text{ in 1D})$$

- $L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d)) = \left\{\sigma : \mathbb{R}^d \to \mathcal{L}^{sym}(\mathbb{R}^d) \;/\; \int_{\mathbb{R}^d} |\sigma|^2 \, dx < \infty\right\}$
- $H^{sym}(div, \mathbb{R}^d) = \left\{\sigma \in L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d)) \;/\; \nabla \cdot \sigma \in (L^2(\mathbb{R}^n))^n\right\}$

Theorem 1

- $(u_0, u_1, \sigma_0) \in \left([H^1(\mathbb{R}^d)]^d\right)^2 \times H^{sym}(div, \mathbb{R}^d)$,
- $f \in C^1(0, T, [L^2(\mathbb{R}^d)]^d)$,
- $u \in C^1\left(0, T; [H^1(\mathbb{R}^d)]^d\right) \cap C^2\left(0, T; [L^2(\mathbb{R}^d)]^d\right)$,
- $\sigma \in C^0(0, T; H^{sym}(div, \mathbb{R}^d)) \cap C^1\left(0, T; L^2(\mathbb{R}^d, \mathcal{L}^{sym}(\mathbb{R}^d))\right)$

- $\boldsymbol{Z}_\tau = \tau_0 \boldsymbol{Z} = \tau_0 (D - C)$,
- $\|\sigma\|_G^2 = \int_{\mathbb{R}^d} \left[G(x)\sigma(x) : \sigma(x) \right] dx.$

Energy

The energy of (u, σ) at time t is given by :

$$E_c(u, \sigma, t) = \frac{1}{2} \|\frac{\partial u}{\partial t}\|_\rho^2 + \frac{1}{2} \|\varepsilon(u)\|_C^2 + \frac{1}{2} \|\sigma - \boldsymbol{C}\varepsilon(u)\|_{\boldsymbol{Z}^{-1}}^2$$

Theorem 2

$$\frac{dE_c}{dt} = -\|\sigma - C\varepsilon(u)\|_{\boldsymbol{Z}_\tau^{-1}}^2 + \int_{\mathbb{R}^d} f \frac{\partial u}{\partial t} dx.$$

$$\begin{cases} f = 0 \\ \boldsymbol{Z} > 0 \end{cases} \Rightarrow \frac{dE_c}{dt} \leq 0.$$

I.2 Mathematical analysis

Plane wave analysis

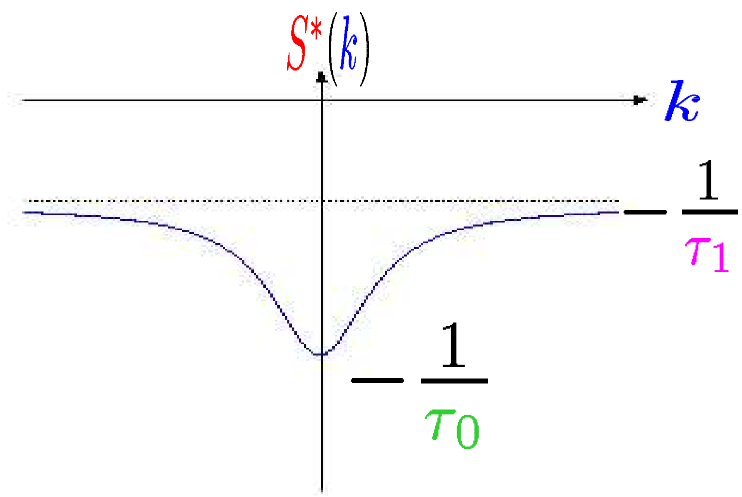
- Plane wave solutions of the form :

$$u(x,t) = \bar{u}\, e^{i(\omega t - kx)}, \qquad \sigma(x,t) = \bar{\sigma}\, e^{i(\omega t - kx)}$$

- Dispersion equation :

$$\omega^2 = c^2 k^2 \left(\frac{1 + i\omega\tau_1}{1 + i\omega\tau_0}\right), \quad c = \sqrt{\frac{\mu}{\rho}}$$

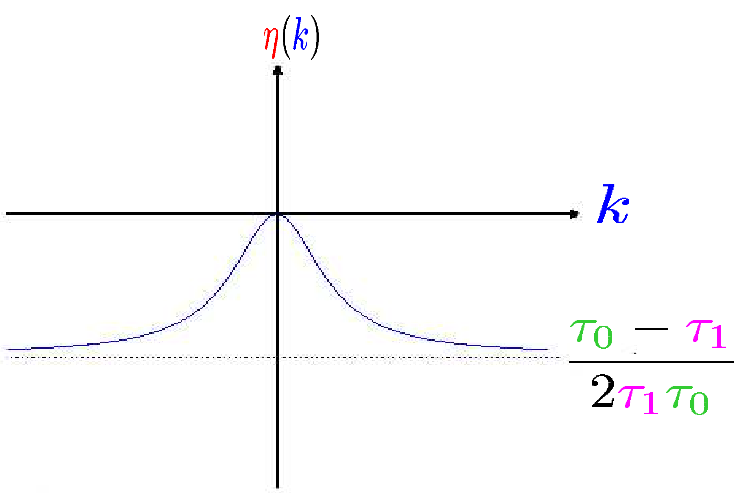
- Three solutions:
  - a purely damped mode : $\omega(k) = -iS^*(k) \in \mathbb{C}/\{\mathbb{R}\}$

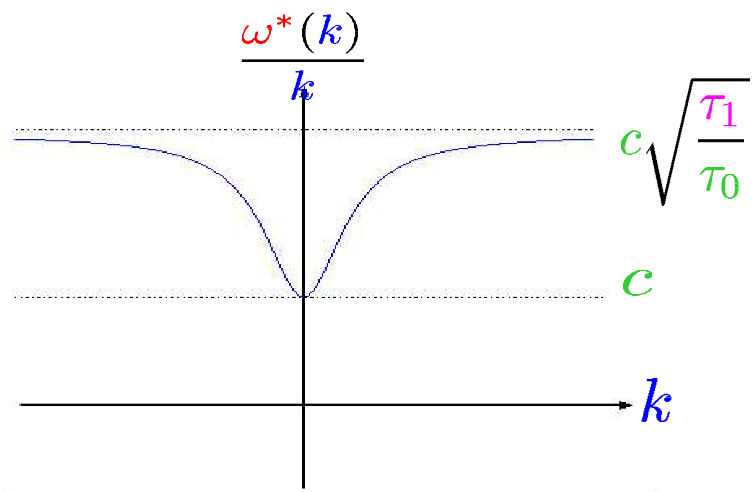


I.2 Mathematical analysis — Plane wave analysis

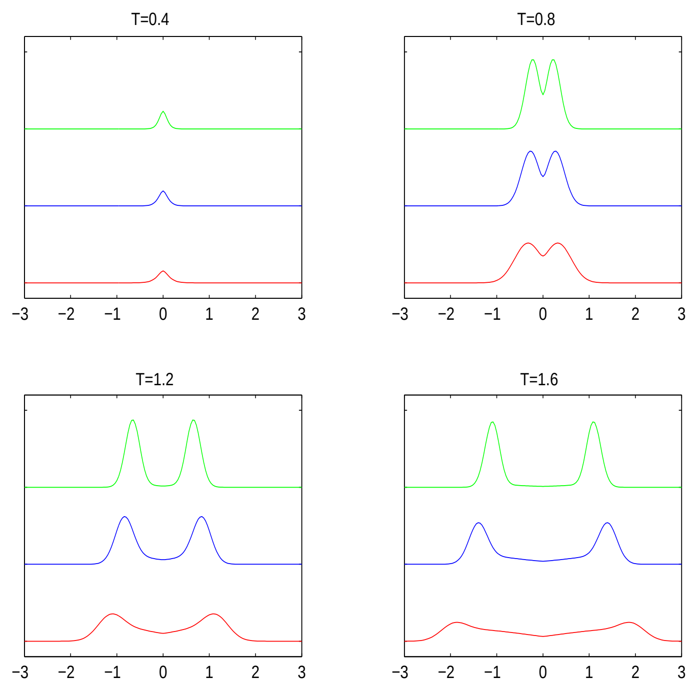
- two propagative damped modes: $\omega(k) = \pm\omega^*(k) - i\eta(k) \in \mathbb{C}$

$$\left|\begin{array}{lll} \eta < 0 & \longrightarrow & \text{dissipation} \\ \omega^* & \longrightarrow & \text{propagation} \end{array}\right.$$





Numerical illustration in 1D case



$$\frac{\tau_1}{\tau_0} = 4, \quad \frac{\tau_1}{\tau_0} = 2, \quad \frac{\tau_1}{\tau_0} = 1.2$$

3D isotropic case

We consider the problem

$$\begin{cases} \rho\, \ddot{u} - \nabla \cdot \sigma = 0 \\ \sigma + \tau_0 \dot{\sigma} = \lambda \Big[\mathbf{Tr}(\varepsilon(u)) + \gamma_\lambda \mathbf{Tr}(\varepsilon(\dot{u})) \Big] \boldsymbol{I} + 2\tau_0 \mu \Big[\varepsilon(u) + \gamma_\mu \varepsilon(\dot{u}) \Big] \end{cases}$$

- 2 cases :
  - $\vec{k} \parallel \vec{d}$ **P** wave : $\omega^2(1 + i\omega\tau_0) = {v_p}^2 |k|^2 (1 + i\omega\tau_p)$
  - $\vec{k} \perp \vec{d}$ **S** wave : $\omega^2(1 + i\omega\tau_0) = {v_s}^2 |k|^2 (1 + i\omega\tau_s)$

$$\omega^2(1 + i\omega\tau_0) = c^2 k^2 (1 + i\omega\tau_1), \quad c = \sqrt{\frac{\mu}{\rho}}$$

$$(MP) \begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot \sigma = f \\ \sigma + \tau_0 \dfrac{\partial \sigma}{\partial t} = C\, \varepsilon(u) + \tau_0 D\, \varepsilon(\dfrac{\partial u}{\partial t}) \end{cases}$$

We introduce : $s = \sigma - \boldsymbol{C}\, \varepsilon(u)$

$$(MP) \quad \Longleftrightarrow \quad \begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot \sigma = f \\ \mathcal{M}_\tau\, s + \mathcal{M} \dfrac{\partial s}{\partial t} = \varepsilon(\dfrac{\partial u}{\partial t}) \\ \mathcal{A}\, \sigma - \mathcal{A}\, s = \varepsilon(u) \end{cases}$$

$$\mathcal{M} = (\boldsymbol{D} - \boldsymbol{C})^{-1},\ \mathcal{M}_\tau = \tau_0^{-1} \mathcal{M},\ \mathcal{A} = \boldsymbol{C}^{-1}$$

Find $(u_h(t), \sigma_h(t), s_h(t)) \in \underline{M}_h \times (\underline{\underline{H}}_h^{sym})^2$ such that :

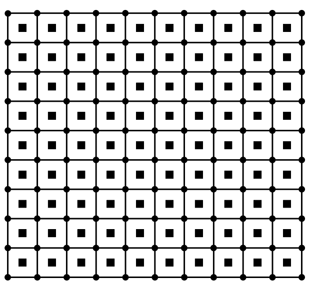
$$\begin{cases} \int_\Omega \rho\, \ddot{u}_h \cdot \tilde{u}_h \, dx - \int_\Omega \mathbf{div}\, \sigma_h \cdot \tilde{u}_h \, dx - \int_\Omega f \cdot \tilde{u}_h \, dx = 0 & \forall\, \tilde{u}_h \in \underline{M}_h \\ \int_\Omega \mathcal{M}_\tau\, s_h : \tilde{s}_h \, dx + \int_\Omega \mathcal{M}\, \dot{s}_h : \tilde{s}_h \, dx + \int_\Omega \dot{u}_h \cdot \mathbf{div}\, \tilde{s}_h \, dx = 0 & \forall\, \tilde{s}_h \in \underline{\underline{H}}_h^{sym} \\ \int_\Omega \mathcal{A}\, \sigma_h : \tilde{\sigma}_h \, dx - \int_\Omega \mathcal{A}\, s_h : \tilde{\sigma}_h \, dx + \int_\Omega u_h \cdot \mathbf{div}\, \tilde{\sigma}_h \, dx = 0 & \forall\, \tilde{\sigma}_h \in \underline{\underline{H}}_h^{sym} \end{cases}$$

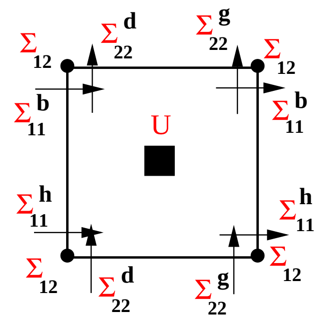
$$\underline{M} = \left[L^2(\Omega)\right]^2,$$

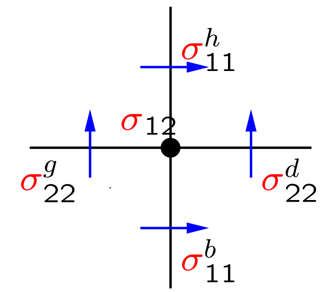
$$\underline{\underline{H}}^{sym} = \left\{\sigma \in L^2(\Omega, \mathcal{L}^{sym}(\mathbb{R}^2)) \,/\, \mathbf{div}\, \sigma \in \underline{M}\right\}.$$

$$\underline{M}_h = \{u_h \in \underline{M} \,/\, \forall K \in \mathcal{T}_h,\, u_h/K \in Q_0\},$$

$$\underline{\underline{H}}_h^{sym} = \{\sigma_h \in \underline{\underline{H}}^{sym} \,/\, \forall K \in \mathcal{T}_h,\, \sigma_h/K \in Q_1\}.$$







$$\begin{cases} M_u \dfrac{d^2U}{dt^2} - B\,\Sigma = F \\ M_\tau\, S + M_s \dfrac{dS}{dt} + B^\star \dfrac{dU}{dt} = 0 \\ A\,\Sigma - A\,S + B^\star\, U = 0 \end{cases}$$

Bécache et al., *A new family of mixed finite elements for the linear elastodynamic problem, 2002.*

$$\left\{\begin{array}{l} M_u \dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} - B\,\Sigma^n = F^n \\ M_\tau \dfrac{S^{n+1} + S^n}{2} + M_s \dfrac{S^{n+1} - S^n}{\Delta t} + B^\star \dfrac{U^{n+1} - U^n}{\Delta t} = 0 \\ A\,\Sigma^{n+1} - A\,S^{n+1} + B^\star\,U^{n+1} = 0 \end{array}\right.$$

$$\Updownarrow$$

$$\left\{\begin{array}{l} M_u \dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} - B\,S^n + B\,A^{-1}B^\star\,U^n = F^n \\ M_\tau \dfrac{S^{n+1} + S^n}{2} + M_s \dfrac{S^{n+1} - S^n}{\Delta t} + B^\star \dfrac{U^{n+1} - U^n}{\Delta t} = 0 \end{array}\right.$$

- Result of discrete energy decay

$$E_d^{n+\frac{1}{2}} = \frac{1}{2}\|\frac{U^{n+1}-U^n}{\Delta t}\|^2_{M_u} + \frac{1}{2}\Big(A^{-1}B^\star U^{n+1}, B^\star U^n\Big)$$
$$+\frac{1}{4}(\|S^{n+1}\|^2_{M_s} + \|S^n\|^2_{M_s}) - \frac{\Delta t^2}{4}\Big(\frac{U^{n+1}-U^n}{\Delta t}, B\,\frac{S^{n+1}-S^n}{\Delta t}\Big)$$

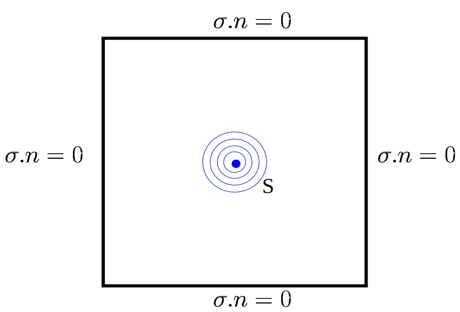
$$\frac{\Delta t}{2}\|B\| \leq 1$$ Sufficient stability condition

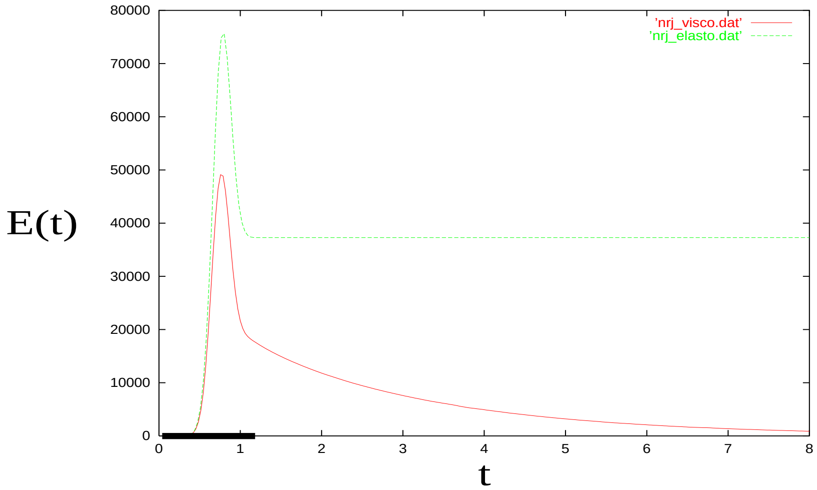
$$\|B\| = \sup_{U,\Sigma\neq 0} \frac{(B\Sigma, U)}{\|\Sigma\|_{\mathbb{K}}\|U\|_{M_u}}, \quad \mathbb{K} = (A^{-1} + M_s^{-1})^{-1}$$

Isotropic homogeneous case :
$$\begin{aligned} \Delta t &\leq h \max\left\{(v_p\sqrt{\tfrac{\tau_p}{\tau_0}})^{-1}, (v_s\sqrt{\tfrac{\tau_s}{\tau_0}})^{-1}\right\} \\ &\leq h\, v_p^{-1} \end{aligned}$$

$$\rho = 1. \quad ; \quad v_p = 2.74 \quad ; \quad v_s = 1.43$$

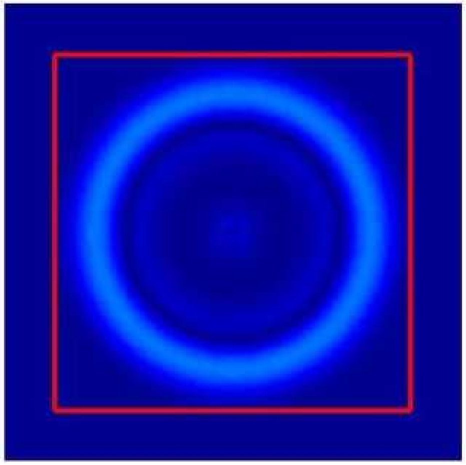
$$\tau_0 = 0.7 \quad ; \quad \tau_p = 1.013 \quad ; \quad \tau_s = 1.015$$

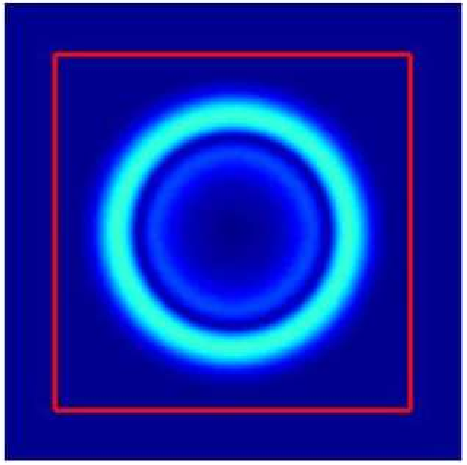


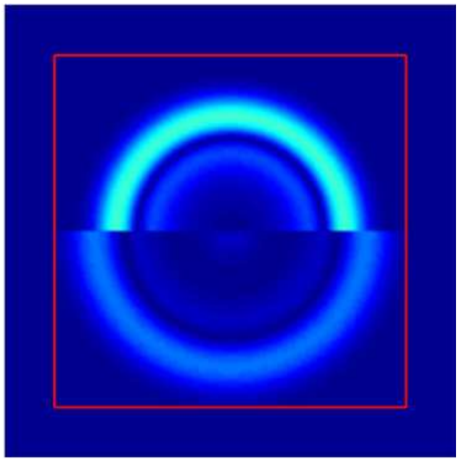


I.4 Numerical results

Elastic and viscoelastic waves in homogeneous medium







- 1D model: $\rho\, \partial^2_{tt} u = \partial_x \sigma, \;\; \sigma = m * \partial_x u.$
- Solutions of the form: $u = \bar{u}\, e^{i(\omega t - kx)}, \;\; \sigma = \bar{\sigma}\, e^{i(\omega t - kx)}$

$$-M(\omega) k^2 + \rho \omega^2 = 0, \;\; \omega = \omega_* + i\eta, \;\; u = \bar{u}\, e^{-\eta t} e^{i(\omega_* t - kx)}$$

$$M(\omega) = \int_{-\infty}^{+\infty} m(t)\, e^{-i\omega t}\, dt = \Re e\, M(\omega) + i\, \Im m\, M(\omega)$$

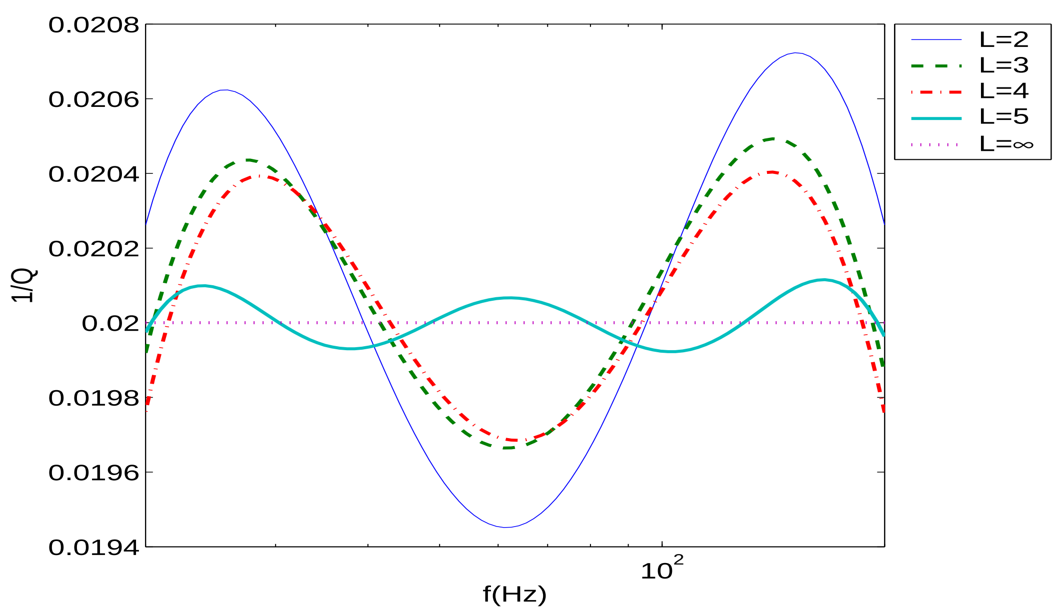
- $\eta(k)^2 = \frac{|M| - \Re e\, M}{2\rho} k^2 \quad = \Re e\, M \frac{\sqrt{1 + Q^{-2}} - 1}{2\rho} k^2$
- $\omega_*(k)^2 = \frac{|M| + \Re e\, M}{2\rho} k^2 \quad = \Re e\, M \frac{\sqrt{1 + Q^{-2}} + 1}{2\rho} k^2$
- Quality factor : $Q(\omega) = \frac{\Re e\, M}{\Im m\, M} \quad Q << \Rightarrow$ dissipation and dispersion

- Zener model $\sigma + \tau_0 \dot{\sigma} = \mu(\varepsilon(u) + \tau_1 \varepsilon(\dot{u})) \Rightarrow Q(\omega) = \dfrac{1 + \omega^2 \tau_1 \tau_0}{\omega(\tau_1 - \tau_0)}$

- Generalized Zener model

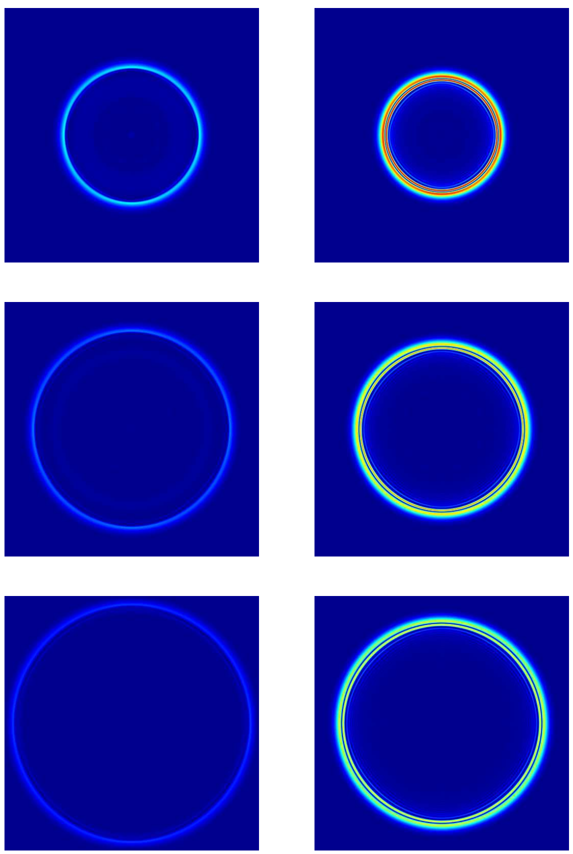
$$\sigma = \sum_{l=1}^{L} \sigma_l, \; \sigma_l + \tau_{0l} \dot{\sigma}_l = \mu(\varepsilon(u) + \tau_{1l} \varepsilon(\dot{u})) \quad l = 1, \cdots, L$$

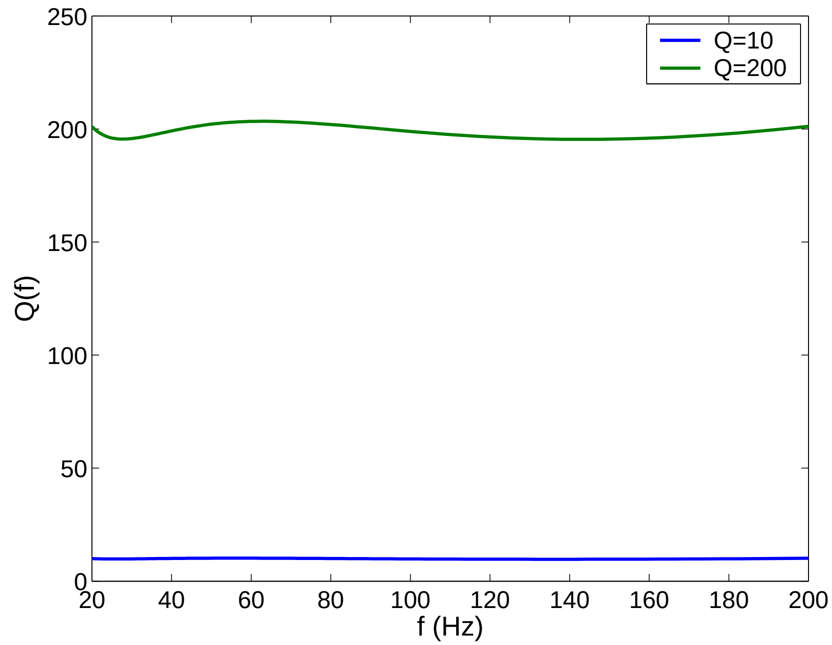
- $\min\limits_{\tau_{0l}, \tau_{1l}} F(\tau_{01}, \cdots, \tau_{0L}, \tau_{11}, \cdots, \tau_{1L}) = \min\limits_{\tau_{0l}, \tau_{1l}} \int_{\omega_a}^{\omega_b} (Q(\omega) - Q_0)^2 \, d\omega$



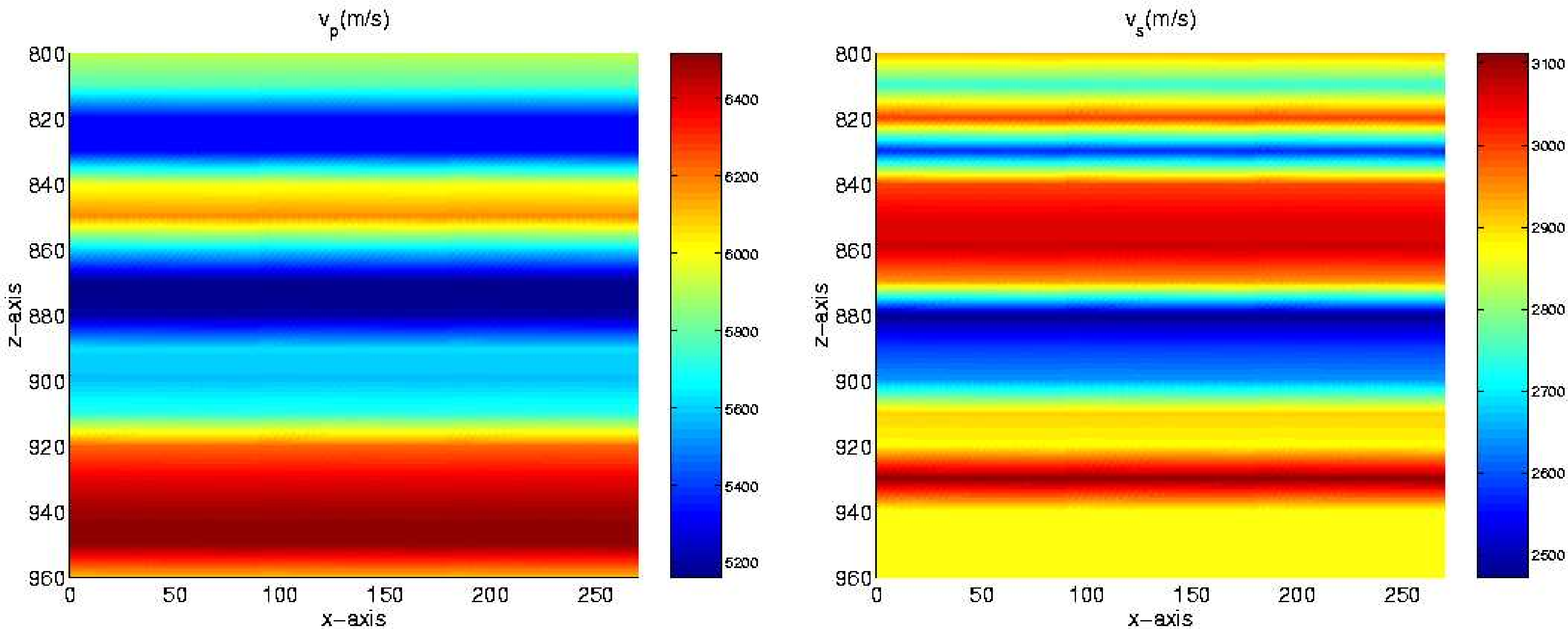
Generalized Model in 3D

$$\sigma = \sum_{l=1}^{L} \sigma_l, \quad \sigma_l + \tau_{0l}\dot{\sigma}_l = \boldsymbol{C}\varepsilon(u) + \tau_{0l}\boldsymbol{D}_l\varepsilon(\dot{u}) \quad \forall l = 1, \cdots, L$$



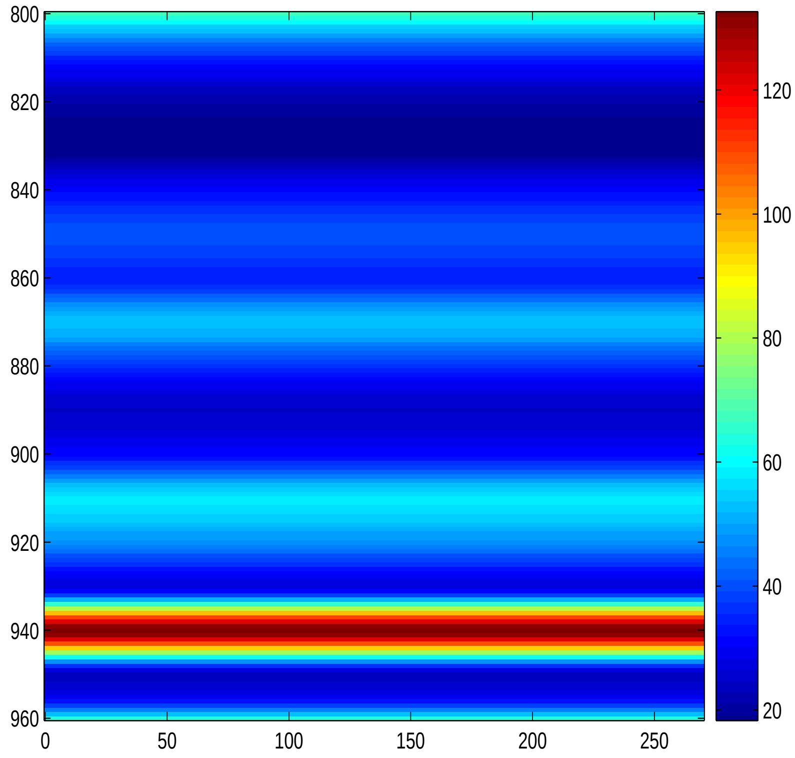


I.5 Realistic models Viscoelastic waves in heteregeneous medium

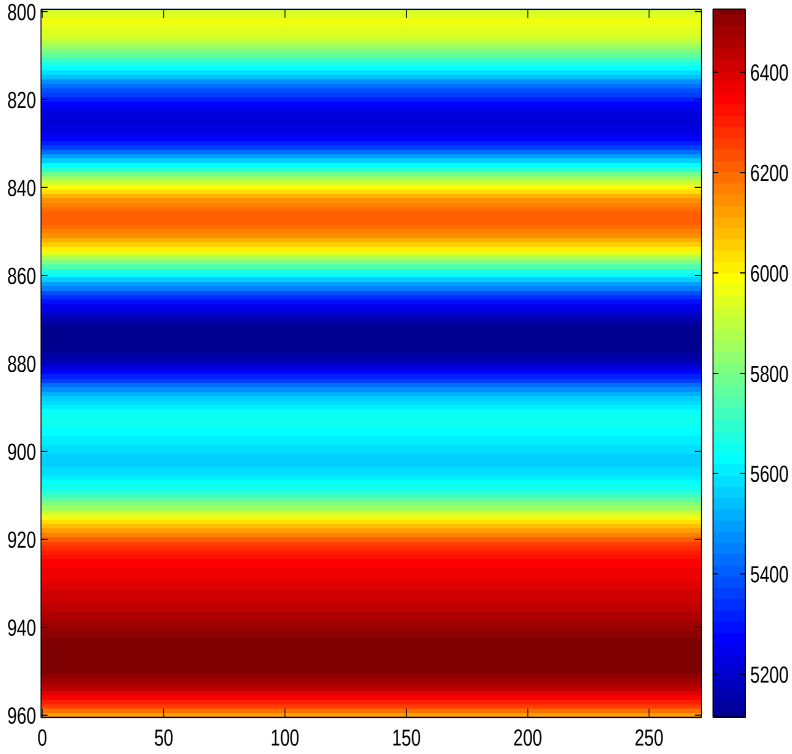


v_p v_s

I.5 Realistic models   Viscoelastic waves in heteregeneous medium



I.5 Realistic models Viscoelastic waves in heteregeneous medium



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II. Poroelastic waves

II.1 Biot's model

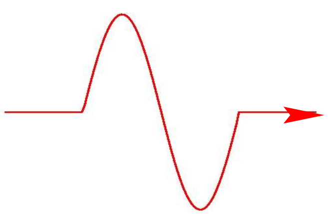
Assumptions

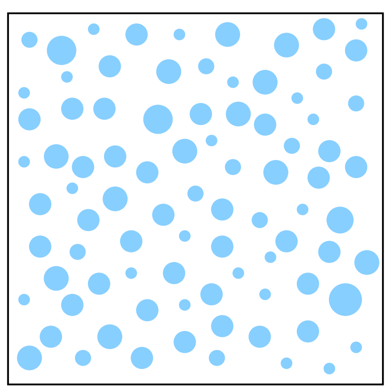
- The wavelength is large in comparison with the size of the macroscopic elementary volume of the material.

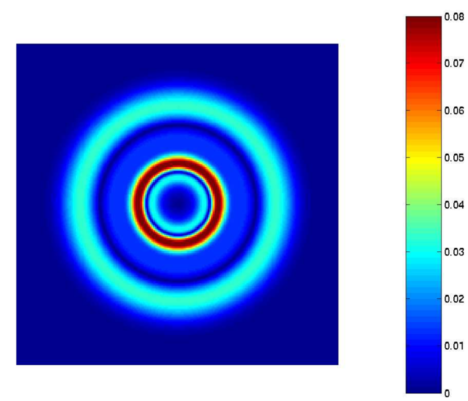
- The displacement is small (for solid and fluid phase)

- The fluid phase is continuous (saturated medium)

- No thermomechanical coupling







II.1 Biot's model

$$\boldsymbol{w} = \phi(\boldsymbol{u}_f - \boldsymbol{u}_s)$$

- Biot's law*:

$$\begin{cases} \rho\,\ddot{\boldsymbol{u}}_s + \rho_f\,\ddot{\boldsymbol{w}} - \boldsymbol{\nabla}\cdot\sigma = f_u & \text{in } \mathbb{R}^d \times]0,T], \\ \rho_f\,\ddot{\boldsymbol{u}}_s + \rho_w\,\ddot{\boldsymbol{w}} + \mathcal{K}\dot{\boldsymbol{w}} + \nabla p = f_w & \text{in } \mathbb{R}^d \times]0,T], \\ \sigma = C\varepsilon(\boldsymbol{u}_s) - \beta\, p \boldsymbol{I_d} & \text{in } \mathbb{R}^d \times]0,T], \\ -p = m(\beta\nabla\cdot\boldsymbol{u}_s + \nabla\cdot\boldsymbol{w}) & \text{in } \mathbb{R}^d \times]0,T], \\ \boldsymbol{u}_s(x,0) = \boldsymbol{u}_0(x),\ \partial_t\boldsymbol{u}_s(x,0) = \boldsymbol{u}_1(x) & \text{in } \mathbb{R}^d, \\ \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x),\ \partial_t\boldsymbol{w}(x,0) = \boldsymbol{w}_1(x) & \text{in } \mathbb{R}^d. \end{cases}$$

*M. A. Biot. Theory of propagation of elatic waves in a fluid-saturated porous solid. I. low-frequency range. JASA, 1956

$$H = [H^1(\mathbb{R}^d)]^d \times [L^2(\mathbb{R}^d)]^d \times \Big([L^2(\mathbb{R}^d)]^d\Big)^2 \times L^2(\mathbb{R}^d)$$

$$D = \left\{ \begin{array}{l} (\boldsymbol{u}, \boldsymbol{w}, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}, p) \in H \;/\; \tilde{\boldsymbol{u}} \in [H^1(\mathbb{R}^d)]^d,\ p \in H^1(\mathbb{R}^d),\ \tilde{\boldsymbol{w}} \in H(div, \mathbb{R}^d), \\ \boldsymbol{\nabla} \cdot \Big(C\varepsilon(\boldsymbol{u})\Big) - \nabla(\beta\, p) \in [L^2(\mathbb{R}^d)]^d \end{array} \right\}$$

Theorem 4

- $(\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{u}_1, \boldsymbol{w}_1, p_0) \in D$,
- $(f_u,\ f_w) \in C^1(0, T, \big[L^2(\mathbb{R}^d)\big]^d) \times C^1(0, T, \big[L^2(\mathbb{R}^d)\big]^d)$,
- $\boldsymbol{u}_s \in C^1(0, T; [H^1(\mathbb{R}^d)]^d) \cap C^2(0, T; [L^2(\mathbb{R}^d)]^d)$,
- $\boldsymbol{w} \in C^1(0, T; H(div, \mathbb{R}^d)) \cap C^2(0, T; [L^2(\mathbb{R}^d)]^d)$,
- $p \in C^0(0, T; H^1(\mathbb{R}^d)) \cap C^1(0, T; L^2(\mathbb{R}^d))$.

Energy

The energy of $(\boldsymbol{u}_s, \boldsymbol{w}, p)$ at time t is given by :

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} \left[\rho\,|\dot{\boldsymbol{u}}_s|^2 + C\,\varepsilon(\boldsymbol{u}_s) : \varepsilon(\boldsymbol{u}_s)\right] dx$$

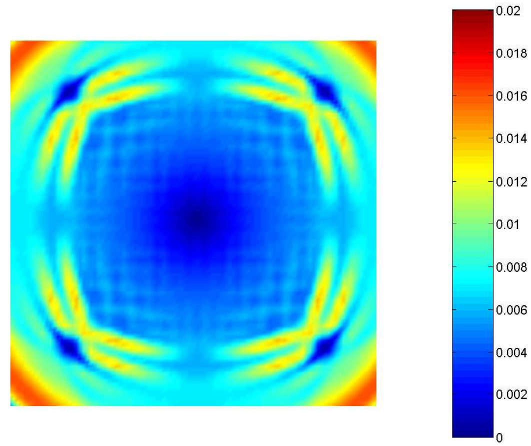
$$+ \frac{1}{2}\int_{\mathbb{R}^d} \left[\rho_w\,|\dot{\boldsymbol{w}}|^2 + \frac{1}{m}|p|^2\right] dx + \int_{\mathbb{R}^d} \rho_f\,\dot{\boldsymbol{u}}_s \cdot \dot{\boldsymbol{w}}\,dx$$

$$\rho\rho_w - \rho_f^2 \geq 0 \Rightarrow E(t) \geq 0$$

Theorem 5

$$\frac{dE}{dt} = -\int_{\mathbb{R}^d} \mathcal{K}|\dot{\boldsymbol{w}}|^2\,dx$$

- Numerical dispersion of slow wave



- Higher-order method
- Mixed finite element*
  - No classical formulation $H^1 - L^2$
  - Problem with ∇ and $\nabla\cdot$ ($\equiv \nabla^*$)

*S. Fauqueux. Eléments finis mixtes spectraux et PML pour la propagation d'ondes élastique en regime transitoire. PhD thesis, Université Paris 9, 2003

$$(A_{ij})_{kl} = C_{ikjl}$$

$$\begin{cases} \rho\,\dot{u}_s + \rho_f\,\dot{w} - \boldsymbol{\nabla}.\sigma = f_u & \Omega\times]0,T], \\ \rho_f\,\dot{u}_s + \rho_w\,\dot{w} + \mathcal{K}\,w + \nabla p = f_w & \Omega\times]0,T], \\ \dot{\gamma}_i = \nabla u_i \;\; i = 1, d & \Omega\times]0,T], \\ \sigma_i = \sum_{j=1}^{d} A_{ij}\gamma_j - \beta\,p\,\vec{e_i} \;\; i = 1, d & \Omega\times]0,T], \\ \frac{1}{m}\dot{p} + \beta \sum_{i=1}^{d} \dot{\gamma}_i \cdot \vec{e_i} + \nabla \cdot w = 0 & \Omega\times]0,T], \\ + I.C \text{ and } B.C. \end{cases}$$

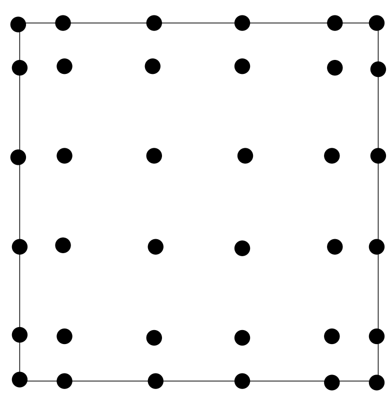
$$
\left\{
\begin{aligned}
&\int_\Omega \rho\, \dot{u}_h \cdot \tilde{u}_h + \int_\Omega \rho_f\, \dot{w}_h \cdot \tilde{u}_h + \sum_{i=1}^{d} \int_\Omega \sigma_{i\,h} \cdot \nabla \tilde{u}_{i\,h} - \int_\Omega f_u \cdot \tilde{u}_h = 0 \\
&\int_\Omega \rho_f\, \dot{u}_h \cdot \tilde{w}_h + \int_\Omega \rho_w\, \dot{w}_h \cdot \tilde{w}_h + \int_\Omega \mathcal{K}\, w_h \cdot \tilde{w}_h + \int_\Omega \nabla p_h \cdot \tilde{w}_h - \int_\Omega f_w \cdot \tilde{w}_h = 0 \\
&\int_\Omega \dot{\gamma}_{i\,h} \cdot \tilde{\gamma}_h - \int_\Omega \nabla u_{i\,h} \cdot \tilde{\gamma}_h = 0 \quad i = 1, d \\
&\int_\Omega \sigma_{i\,h} \cdot \tilde{\sigma}_h - \sum_{j=1}^{d} \int_\Omega A_{ij} \gamma_{j\,h} \cdot \tilde{\sigma}_h + \int_\Omega \beta\, p_h \vec{e}_i \cdot \tilde{\sigma}_h = 0 \quad i = 1, d \\
&\int_\Omega \frac{1}{m} \dot{p}_h \tilde{p}_h + \sum_{i=1}^{d} \int_\Omega \beta\, \dot{\gamma}_{i\,h} \cdot e_i \tilde{p}_h - \int_\Omega w_h \cdot \nabla \tilde{p}_h - \int_\Omega f_p \tilde{p}_h = 0 \\
&\forall\, (\tilde{u}_h, \tilde{w}_h, \tilde{\gamma}_h, \tilde{\sigma}_h, \tilde{p}_h) \in \mathbb{U}_h \times \mathbb{W}_h \times \mathbb{W}_h \times \mathbb{W}_h \times \mathbb{P}_h
\end{aligned}
\right.
$$

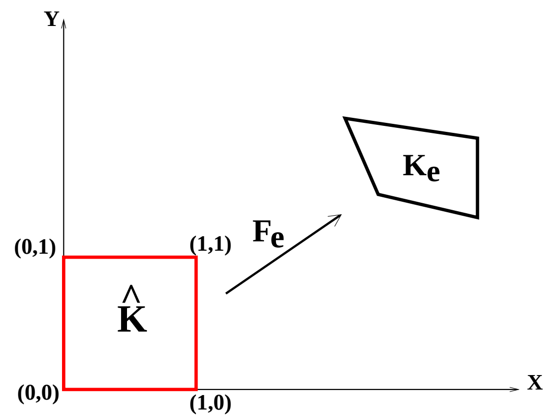
$$
\mathbb{U}_h \subset \mathbb{U} = [H^1(\Omega)]^d, \; \mathbb{W}_h \subset \mathbb{W} = [L^2(\Omega)]^d, \; \mathbb{P}_h \subset \mathbb{P} = H^1_0(\Omega)
$$

$$\mathbb{P}_h = \{\varphi \in C^0(\Omega) \,/\, \varphi|_{K_e} \circ F_e \in Q_r(\hat{K}) \text{ and } \varphi|_{\partial\Omega} = 0\} \subset H_0^1(\mathbb{R}^d),$$

$$\mathbb{U}_h = \{\phi \in C^0(\Omega) \,/\, \phi|_{K_e} \circ F_e \in [Q_r(\hat{K})]^d\} \subset [H^1(\mathbb{R}^d)]^d,$$

$$\mathbb{W}_h = \{\psi \in [L^2(\Omega)]^d \,/\, |J_e| DF_e^{-1}\psi|_{K_e} \circ F_e \in [Q_r(\hat{K})]^d\} \subset [L^2(\mathbb{R}^d)]^d,$$





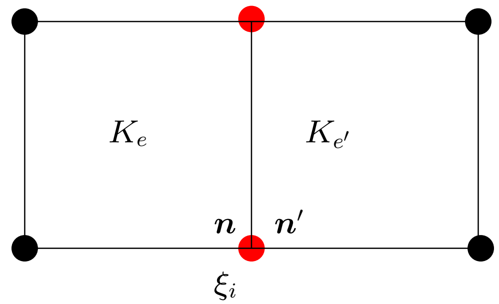
$$\left\{\begin{array}{l}
D_u \dfrac{U_h^{n+1} - U_h^n}{\Delta t} + D_{uw} \dfrac{W_h^{n+1} - W_h^n}{\Delta t} + \displaystyle\sum_{i=1}^{d} R \Sigma_{i\,h}^{n+\frac{1}{2}} = F_u^{n+\frac{1}{2}} \\
D_{uw}^* \dfrac{U_h^{n+1} - U_h^n}{\Delta t} + D_w \dfrac{W_h^{n+1} - W_h^n}{\Delta t} + K \dfrac{W_h^{n+1} + W_h^n}{2} + R^* P_h^{n+\frac{1}{2}} = F_w^{n+\frac{1}{2}} \\
D_\gamma \dfrac{\Gamma_{i\,h}^{n+\frac{1}{2}} - \Gamma_{i\,h}^{n-\frac{1}{2}}}{\Delta t} - R^* U_{i\,h}^n = 0 \quad i = 1, d \\
D_\gamma \Sigma_{i\,h}^{n+\frac{1}{2}} - \displaystyle\sum_{j=1}^{d} T_{ij} \Gamma_{j\,h}^{n+\frac{1}{2}} + G_i P_h^{n+\frac{1}{2}} = 0 \quad i = 1, d \\
D_p \dfrac{P_h^{n+\frac{1}{2}} - P_h^{n-\frac{1}{2}}}{\Delta t} + \displaystyle\sum_{i=1}^{d} G_i^* \dfrac{\Gamma_{i\,h}^{n+\frac{1}{2}} - \Gamma_{i\,h}^{n-\frac{1}{2}}}{\Delta t} - R W_h^n = F_p^n
\end{array}\right.$$

$$
\begin{cases}
HU_h^{n+1} = F, \\
H = D_u - D_{uw}(D_w + \frac{\Delta t}{2}K)^{-1} D_{uw}^*, \\
F = F_1 - D_{uw}(D_w + \frac{\Delta t}{2}K)^{-1} F_2.
\end{cases}
$$

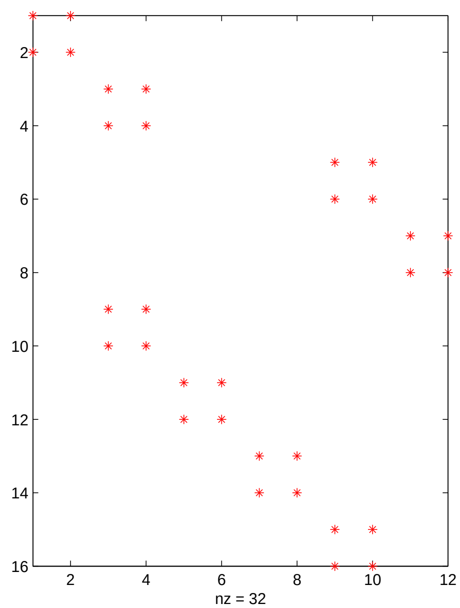
$$
\left[\begin{array}{c|c} D_u & D_{uw} \\ \hline D_{uw}^* & D_w + \frac{\Delta t}{2}K \end{array}\right]
\left[\begin{array}{c} U_h^{n+1} \\ \hline W_h^{n+1} \end{array}\right]
=
\left[\begin{array}{c} F_1 \\ \hline F_2 \end{array}\right]
$$

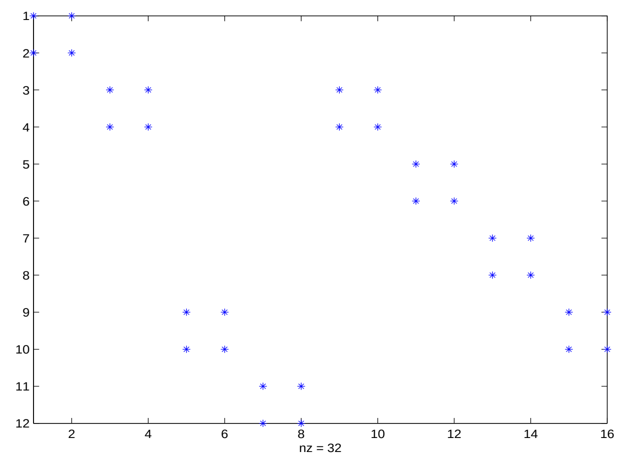
H est diagonale par bloc, avec des blocs de taille $d \times d$.

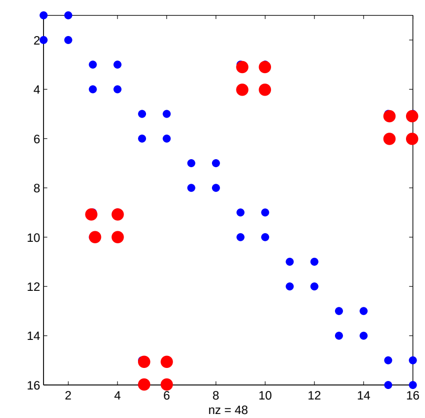
- Exemple :

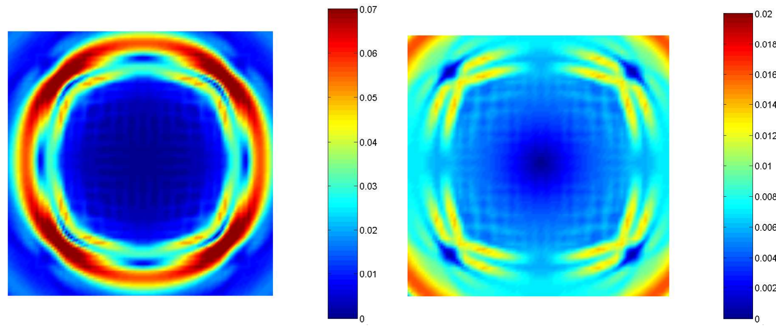


$$K_e \cap K_e' \neq \emptyset \text{ et } glob_e(\boldsymbol{n}) = glob_{e'}(\boldsymbol{n}')$$

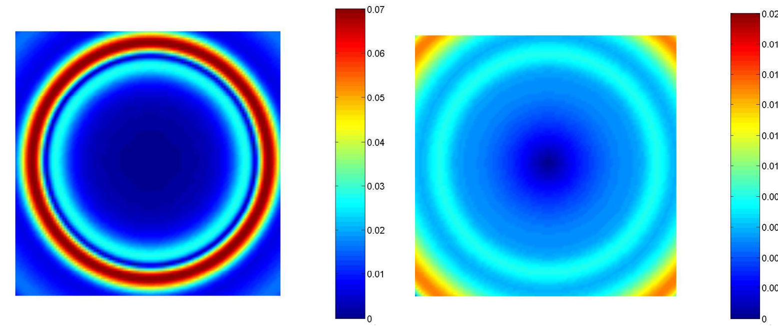




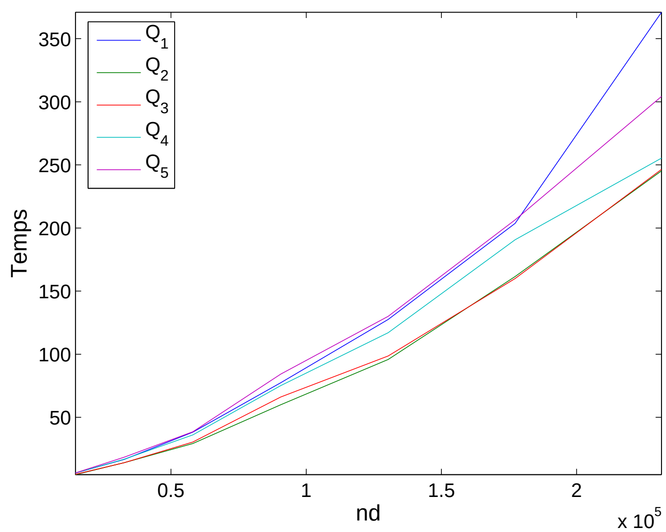


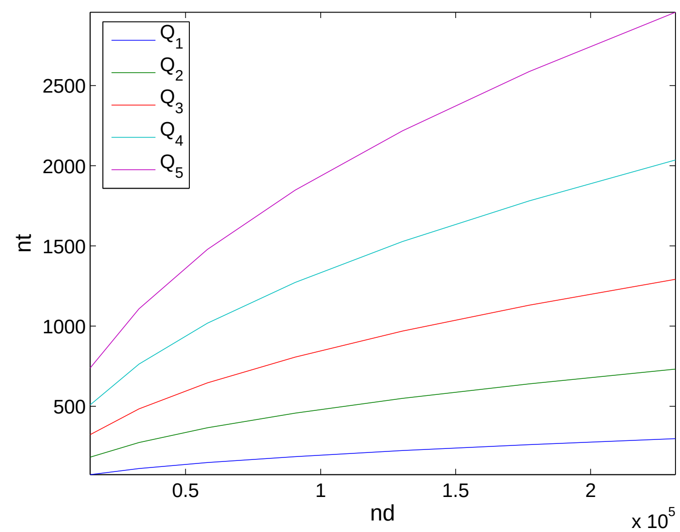


Q_1-57600 elements



Q_5-2304 elements

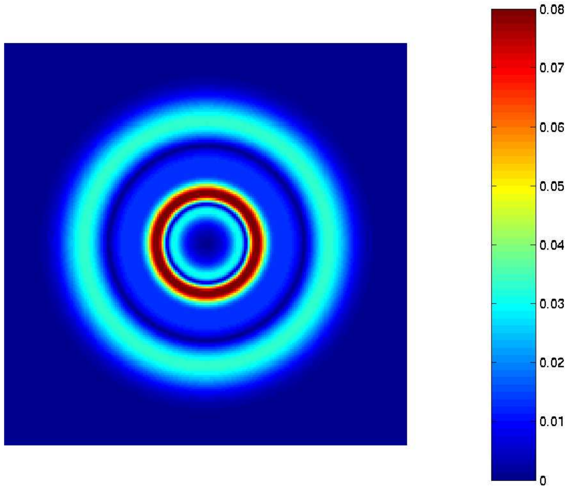




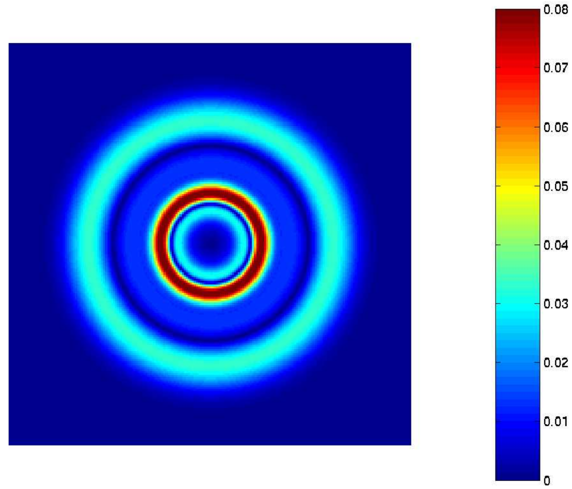
II.4 Numerical results

- infinite homogenuous medium:

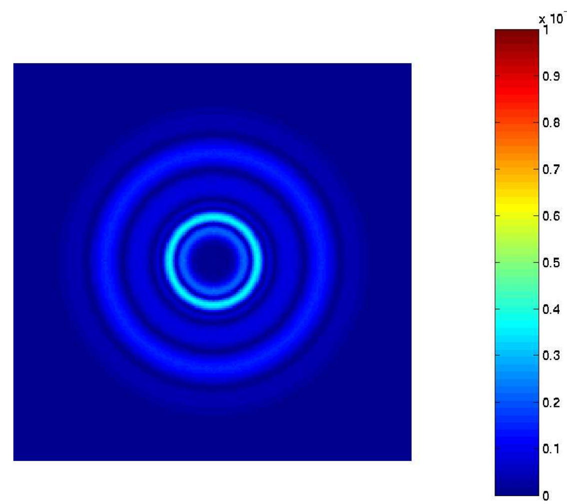
  $V_{pf} = 2.927 m/s$, $V_{ps} = 1.195 m/s$, $V_s = 1.549 m/s$
- gaussian source: $F(x, y, t) = \delta(x)\delta(y)f(t)$
- $\Omega = [0, 6] \times [0, 6]$, $t = 1s$, $h = 0.1$, $\Delta t = 1.\, 10^{-3}$
- Q_5-3600 elements



Analytical solution



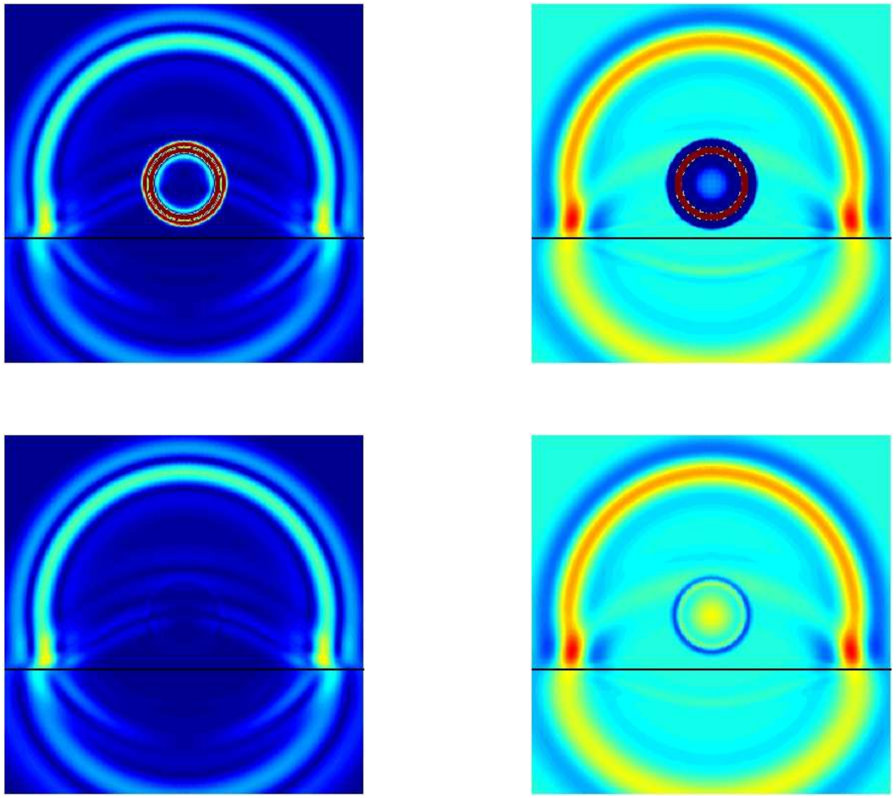
Numerical solution



Relative error $\leq 0.43\%$

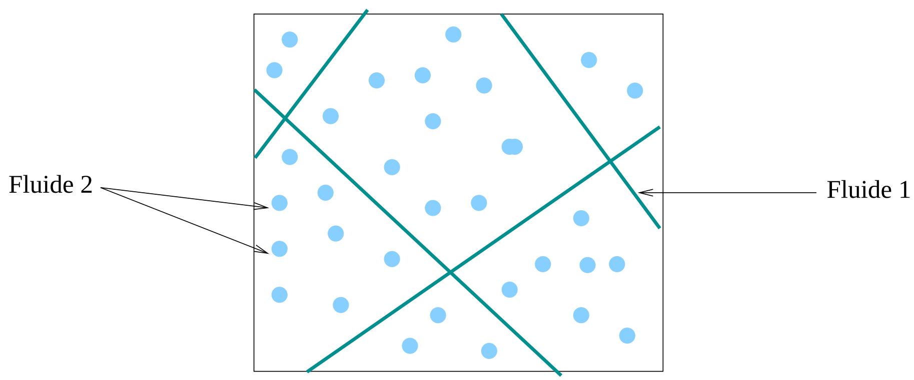
II.4 Numerical results

heteregeneous medium



Prospects

- 3D implementation and validation
- Poroviscoelastic model
- Multiphasic poroelastic model



- Asymptotic method ($V_{ps} << V_{pf}$)