Theoretical and numerical studies of the primitive equations of the ocean without viscosity

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A mes parents.

あいしてる

Theoretical and Numerical Studies of the Primitive Equations of the Ocean Without Viscosity

Abstract

This work is dedicated to the Primitive Equations (PEs) of the ocean, both from the theoretical and numerical viewpoints. The PEs are fundamental equations of geophysical fluid dynamics, based on the hydrostatic and Boussinesq approximations. Here we consider them without viscosity, in a bounded domain, which actually makes the problem ill-posed with any set of local boundary conditions, as shown in the introduction.

In the first part (chapters 1 to 4), we study a slight modification of the hydrostatic equation, adding a friction term of size δ, which is a small parameter. We establish existence, uniqueness and regularity results, and study the asymptotic behaviour of the solutions as δ goes to 0. The numerical simulations evidence some boundary layers and reflexion phenomena at the boundary of the domain. We then confirm the numerical observations with a rigorous proof, obtained with the help of the corrector theory.

In the second part (chapters 5 and 6), we go back to the original hydrostatic formulation of the PEs, and propose a set of transparent boundary conditions for the linearized equations. We prove the well-posedness of the corresponding boundary value problem, and perform numerical simulations in which we implement the boundary conditions that we introduced, both in the linear and nonlinear cases. As expected, the boundary layers and reflexion phenomena encountered before disappear.

Key words: Primitive Equations, boundary conditions, semigroup theory, finite elements, finite differences, boundary layers, transparent boundary conditions.

AMS Classification (2000): 35L50, 35Q35, 47D03, 65Mxx, 65Nxx, 76Bxx, 76Mxx, 86-08, 86A05.

Etudes théoriques et numériques des équations primitives de l'océan sans viscosité

Résumé

Cette thèse regroupe un ensemble d'analyses mathématiques et de simulations numériques relatives aux Equations Primitives de l'océan (EPs) sans viscosité, en domaine borné. Les EPs sont des équations bien connues de la mécanique des fluides, qui s'appuient sur les approximations hydrostatique et de Boussinesq. On rappelle en introduction pourquoi ces équations, considérées avec des conditions aux limites de type local, sont mal posées.

Dans une première partie (chapitres 1 à 4), on s'intéresse à une modification de l'équation hydrostatique au moyen d'un terme de friction proportionnel à un petit paramètre δ. On démontre des résultats d'existence, d'unicité et de régularité des solutions avant d'étudier le comportement de ces solutions lorsque δ tends vers 0. Des résultats numériques montrent que des couches limites et des réflexions se produisent aux frontières du domaine. Les phénomènes observés numériquement sont alors confirmés par une preuve rigoureuse effectuée grâce à la théorie des correcteurs.

Dans une seconde partie (chapitres 5 et 6), on revient à la formulation hydrostatique d'origine des EPs, et l'on propose un jeu de conditions aux limites transparentes pour le système linéarisé. Une preuve du caractère bien posé du problème aux limites ainsi obtenu justifie l'introduction de telles conditions aux limites, qui sont ensuite implémentées dans une simulation numérique confirmant que les phénomènes de couches limites et de réflexions aux frontières sont ainsi évités, aussi bien sur les équations non linéaires que sur le linéarisé.

Mots clés : Equations Primitives, conditions aux limites, théorie des semi-groupes, éléments finis, différences finies, couches limites, conditions aux limites transparentes.

Classification AMS (2000) : 35L50, 35Q35, 47D03, 65Mxx, 65Nxx, 76Bxx, 76Mxx, 86-08, 86A05.

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Introduction

Il nous parait utile pour présenter cette thèse d'en donner les motivations et le contexte. L'essentiel de ce travail porte sur des problèmes linéaires (d'un type tout à fait nouveau) mais, bien sûr, notre objectif est l'étude de problèmes non linéaires, et comme nous l'expliquons ci-après, les problèmes linéaires sous-jacents contiennent l'essentiel de la difficulté et sont, de toute façon, une étape nécessaire.

Ce travail voudrait apporter quelques contributions et remarques à ce que l'on considère comme un problème numérique majeur pour les écoulements de fluides géophysiques dans les années à venir, à savoir les conditions aux limites pour des simulations en **domaine limité** avec une frontière qui, en partie ou en totalité, n'est pas physique. Les simulations en domaine limité, dans l'océan et l'atmosphère, font à présent l'objet de demandes multiples, conduisant même à des logiciels de type commercial en vue de la prévision, pour des besoins aussi variés que la pisciculture, la conservation des fonds marins, l'agriculture, la navigation aérienne, le tourisme, etc., voir e.g.. [WPT97].

Les équations utilisées dans ce type de prévision sont bien connues, ce sont les équations primitives (EP) que nous rappelons ci-dessous et les équations de Saint-Venant (*Shallow Water Equations*) que nous ne considérons pas dans ce travail. Plus précisément les équations primitives considérées sont **sans viscosité** car l'effet des termes de viscosité ne se manifeste en principe qu'au-delà de quelques jours, donc au-delà de la durée de la prévision.

Il y a, en fait, abondance de données numériques que l'on pourrait utiliser pour les conditions aux limites, provenant de mesures ou de simulations faites à grande échelle avec un pas de discrétisation plus grand. Dans ces conditions, le problème numérique devient, de fait, un problème mathématique théorique : déterminer les conditions aux limites que l'on doit adjoindre aux EP sans viscosité pour obtenir un problème bien posé. Dans le cas des EP avec viscosité, des progrès importants pour les questions d'existence, unicité et régularité de solutions ont été obtenus dans les travaux pionniers de Lions, Temam et Wang [LTW92a, LTW92b], et ceux de nombreux auteurs à la suite ; voir [TZ04] pour une synthèse de ces résultats que l'on peut résumer ainsi : bien que les équations primitives avec viscosité soient un peu moins régulières que les équations de Navier-Stokes des fluides incompressibles, on a à présent en dimension deux et trois d'espace les mêmes types de résultats d'existence et d'unicité de solutions, à savoir existence de solutions faibles pour tout temps en dimension 2 et 3 [LTW92a, LTW92b] ; existence et unicité pour tout temps de solutions fortes en dimension 2 et existence et unicité de solutions fortes pour un temps limite en dimension d'espace 3. Conformément à la terminologie utilisée

en mécanique des fluides, les solutions faibles (resp. fortes) sont celles dont la norme L^2 (resp. H^1) est bornée pour tout temps et la norme H^1 (resp. H^2) est L^2 pour des temps finis.
En revanche l'étude du cas des EP sans viscosité (en domaine limité) n'a fait, à notre connaissance, aucun progrès depuis le résultat négatif de Oliger et Sundström [OS78] qui ont montré que les EP sans viscosité, et d'autres équations d'écoulements géophysiques, ne sont bien posées pour aucun ensemble de conditions aux limites de type local. Tandis que les EP avec viscosité sont comparables en bien des points aux équations de Navier-Stokes incompressibles comme nous l'avons rappelé ci-dessus, le résultat d'Oliger et Sundström [OS78] contredit l'idée intuitive selon laquelle les EP sans viscosité seraient alors comparables aux équations d'Euler.
En fait pour l'analyse mathématique des équations primitives sans viscosité, nous entrons dans le domaine mathématique des problèmes aux limites dans un ouvert borné pour des systèmes hyperboliques (linéaires ou non), et l'étude bibliographique que nous avons menée nous a montré que ce domaine était encore très peu exploré, tandis que le sujet dans son ensemble des équations et systèmes hyperboliques était en plein essor.
Une partie de notre effort a de fait consisté à étudier l'existence et l'unicité de solutions pour les équations primitives sans viscosité en dimension deux, linéarisées autour d'un écoulement dominant. Cette étude est présentée au chapitre 5. Bien que l'étude repose sur l'utilisation classique du théorème de Hille-Yosida, la vérification des hypothèses n'en est pas élémentaire, et le problème aux limites obtenu est, à notre connaissance, d'un type tout à fait nouveau, faisant apparaitre, pour les conditions aux limites, une infinité d'équations intégrales.
Ayant clarifié et résolu le problème théorique, nous sommes en mesure au chapitre 6 de revenir sur le problème numérique initial, les simulations en domaine limité (en dimension deux d'espace), et obtenons des résultats qui nous semblent d'une bonne précision pour un tel problème, avec des courbes simulées parfois indifférentiables des courbes témoin. L'extension des résultats théoriques à la dimension trois est en cours, ainsi que leur extension aux problèmes non linéaires.
Le chapitre 6 présente les résultats des simulations numériques en domaine limité dans le cas **non linéaire**, en se basant sur des formulations heuristiques en l'absence de résultats rigoureux, avec l'idée, communément admise, que les conditions aux limites devraient être en général (c'est-à-dire en-dehors des cas limites) celles du problème linéarisé.
Pour compléter la description des motivations numériques de ce travail, il est utile de mentionner que diverses méthodes empiriques sont utilisées depuis longtemps (premier travaux de von Neumann et Charney dans les années cinquante), pour simuler les EP sans viscosité dans un domaine limité, introduisant en particulier diverses couches limites artificielles à la frontière du domaine, mais les experts considèrent que ces méthodes génèreront des modes parasites ("*spurious modes*") tout à fait inacceptables lorsque, dans les années à venir, des méthodes à haute résolution seront utilisées. Signalons aussi que des questions et discussions qui ont suivi la présentation des travaux qui font l'objet de ce mémoire, ont montré que le problème du choix des conditions aux limites était présent dans un grand nombre de phénomènes en physique mathématique.

Comme nous venons de l'expliquer, ce travail de thèse s'articule autour des équations primitives (EP) de l'océan, obtenues à partir des équations fondamentales de conservation de la physique moyennant bien sûr certaines hypothèses simplificatrices. Nous terminons cette introduction par une brève description des équations primitive et un résumé plus technique de notre travail.

1 Les Equations Primitives de l'Océan

Les lois fondamentales exprimant la conservation d'un certain nombre de quantités physiques (moments, masse, énergie, salinité de l'eau, humidité de l'air, etc.) sont centrales dans la modélisation des phénomènes océaniques et atmosphériques. Toutefois, si l'on considère la forme brute de ces équations, il paraît sans espoir, dans l'état actuel de nos connaissances, d'en extraire quelque résultat que ce soit.
C'est pourquoi les physiciens et les mathématiciens qui s'intéressent à ces problèmes effectuent un certain nombre d'hypothèses simplificatrices raisonnables, construisant ainsi une hiérarchie de modèles océaniques, atmosphériques, ou bien couplant ces deux domaines. Bien entendu, à mesure que l'on simplifie les modèles, les résultats mathématiques associés deviennent plus précis. Le jeu - passionnant ! - qui consiste a retirer une hypothèse simplificatrice et à essayer de transposer à un modèle plus complexe des résultats bien connus donne aujourd'hui matière à réflexion à une importante communauté de scientifiques à travers le monde.

1.1 Les lois fondamentales

Nous présentons ici brièvement les lois fondamentales de la physique dont découle le modéle étudié dans cette thèse. Commençons par la loi de conservation des moments, qui s'exprime comme suit :

$$\begin{aligned} \rho \frac{d\mathbf{V}}{dt} &= \text{gradient de pression + force de Coriolis + gravité} \\ &= -\nabla p - f\,\rho\,\mathbf{V} \times \mathbf{e_z} - \rho\, g\, \mathbf{e_z}, \end{aligned} \tag{1.1}$$

où $\mathbf{V}$ représente la vitesse tridimensionnelle d'une particule de fluide, ρ sa densité, et p la pression qui s'exerce sur celle-ci. La fonction scalaire f (non nécéssairement constante) représente la rotation de la Terre, qui génère la force d'inertie de Coriolis, et g est la constante universelle de gravitation. Enfin le symbole ∇ représente l'opérateur gradient tridimensionnel, et

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla).$$

Il est important de remarquer dans (1.1) que nous considérons un modèle *non visqueux.* Ce choix de modélisation est crucial car c'est l'absence de terme de diffusion qui rend l'étude plus difficile, et qui justifie donc en partie notre travail.

L'équation de conservation de la masse, encore appelée équation de continuité, s'écrit :

$$\frac{d\rho}{dt} + \rho\, div\mathbf{V} = 0. \tag{1.2}$$

L'équation de conservation d'énergie, représentant le transport de la chaleur par les particules de fluides, s'écrit simplement (là encore, on ignore les termes de diffusion) :

$$\frac{dT}{dt} = Q_T, \tag{1.3}$$

où Q_T représente une source de chaleur (le soleil par exemple).

Enfin, l'équation d'état, qui donne la loi de dépendance de la densité en fonction de la température pour le cas de l'océan, s'énonce ainsi :

$$\rho = \rho_0(1 - \beta(T - T_0)), \tag{1.4}$$

où ρ_0 et T_0 sont des valeurs de référence (valeurs moyennes) de la densité et de la température.

1.2 Approximations essentielles à notre étude

Comme nous l'avons précisé ci-dessus, il est à l'heure actuelle encore utopique de travailler avec de telles équations. Nous allons donc formuler ici quelques hypothèses naturelles qui permettent de simplifier notre étude sans toutefois dénaturer tragiquement les phénomènes physiques.
Nous commençons par la célèbre approximation de l'hydrodynamicien Joseph Valentin Boussinesq (1842-1929), qui énonçait en 1903 dans son livre **Théorie analytique de la Chaleur** :

"Il faut savoir que dans la plupart des mouvements provoqués par la chaleur sur nos fluides pesants, les volumes ou les densités se conservent à très peu près, quoique la variation correspondante du poids de l'unité de volume soit justement la cause des phénomènes qu'il s'agit d'analyser. De là résulte la possibilité de négliger les variations de la densité, là où elles ne sont pas multipliées par la gravité g, tout en conservant, dans les calculs, leur produit par celle-ci."

Grâce à cette approximation, les équations (1.1) et (1.2) deviennent :

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho_0}\nabla p - f\,\mathbf{V}\times\mathbf{e_z} - \frac{\rho}{\rho_0}\,g\,\mathbf{e_z}, \tag{1.5}$$

$$div\mathbf{V} = 0. \tag{1.6}$$

Vient ensuite l'approximation hydrostatique. Elle consiste à négliger l'accélération verticale du fluide. La troisième équation de mouvement (1.1) devient alors :

$$\frac{\partial p}{\partial z} = -\rho\,g. \tag{1.7}$$

Remarquons que c'est cette approximation qui différencie les EP de celles, plus célèbres, de Navier-Stokes. De ce fait, la vitesse verticale w, contrairement au cas de NS, est pour nous une variable diagnostique, c'est à dire qu'elle est calculée, ainsi d'ailleurs que la pression p, en fonction des variables dites prognostiques que sont la vitesse horizontale $\mathbf{v} = (u, v)$ et la température T.

Finalement, les équations primitives de l'océan, telles que nous les considérerons tout au long de ce travail, s'écrivent comme suit, en notant $\phi = p/\rho_0$:

$$\begin{aligned}
\frac{d\mathbf{v}}{dt} &= -\nabla\phi + f\,\mathbf{v}\times\mathbf{e_z}, && (1.8a)\\
\frac{\partial\phi}{\partial z} &= -\frac{\rho}{\rho_0}\,g, && (1.8b)\\
div\mathbf{V} &= 0, && (1.8c)\\
\frac{dT}{dt} &= Q_T. && (1.8d)
\end{aligned}$$

1.3 Le cas de l'atmosphère

Le cadre dans lequel nous nous situons pour cette thèse est celui des équations primitives de l'océan. Toutefois, il est bon de dire un mot sur le cas de l'atmosphère, à la fois comparable et très différent du cas qui nous préoccupe.
En effet, si l'approximation hydrostatique est toujours valable dans le cas de l'atmosphère, la densité du fluide (l'air) y est très variable, de sorte que l'approximation de Boussinesq n'est plus valable. En particulier, l'équation de continuité ne peut donc se réduire à (1.6). Cependant, en changeant la variable verticale z en p (ce qui est admissible d'après (1.7)), on retrouve une forme très proche des équations décrites plus haut.
Les nouvelles coordonnées sont donc x, y et p, et les inconnues sont $\mathbf{v} = (u, v)$ la vitesse horizontale, ω la vitesse verticale (dérivée de la pression par rapport au temps, $\omega \neq w$), ρ et θ la densité et la température potentielle de l'air, ainsi que le géopotentiel $\phi = gz$. Les équations primitives de l'atmosphère, finalement tout à fait semblables à celles décrites plus haut, s'écrivent alors[1] :

$$\begin{aligned}
\frac{D\mathbf{v}}{dt} &= -\frac{1}{\rho}\nabla p + \rho f\,\mathbf{v}\times\mathbf{e_z}, && (1.9a)\\
\frac{\partial\phi}{\partial p} &= \frac{R\,\theta}{p_0}\Big(\frac{p_0}{p}\Big)^{1/\gamma}, && (1.9b)\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial \omega}{\partial p} &= 0, && (1.9c)\\
\frac{D\theta}{dt} &= Q_\theta, && (1.9d)
\end{aligned}$$

où $\gamma \simeq 1.4$ dans le cas de l'air sec, et où l'on note

$$\frac{D}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}.$$

[1] Voir e.g. [Sal98] pour plus de détails.

2 Position du problème. Caractère mal posé des équations

Rappelons ici deux choses essentielles : les EP que nous considérerons ici sont **sans viscosité**, et dans un **domaine borné**. Dans le cas des EP en domaine borné avec viscosité, un travail conséquent a déjà été produit dans les années 90, avec les travaux fondateurs de Lions, Temam et Wang (voir [LTW92a, LTW92b]). Concernant les EPs sans viscosité dans tout l'espace, leur caractère bien posé a déjà été établi dans [Ift99].
Si l'on considère donc les EP sans viscosité et en domaine borné, et que l'on leur adjoint des conditions aux limites de type local, ce qui est classique en dynamique des fluides, le problème aux limites associé est malheureusement mal posé, et c'est précisement ce a motivé ce travail de thèse. Nous rappelons ici les arguments de [TT03], explicitant un raisonnement de [OS78], et qui prouvent le caractère mal posé de ce problème aux limites.

Commençant par des problèmes linéaires, on considère alors les équations primitives linéarisées autour du flot $\overline{U}_0\,\mathbf{e_x}$, et sans dépendance en y. Le domaine considéré sera donc bidimensionnel, par exemple de la forme $\mathcal{M} = (0, L_1) \times (-L_3, 0)$. Après avoir retiré la stratification (voir Section 1.1 du chapitre 5 pour plus de détails), on obtient les équations suivantes, où $\psi = -\rho g/\rho_0$:

$$\frac{\partial u}{\partial t} + \overline{U}_0 \frac{\partial u}{\partial x} - f\,v + \frac{\partial \phi}{\partial x} = F_u, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + \overline{U}_0 \frac{\partial v}{\partial x} + f\,u = F_v - f\,\overline{U}_0, \tag{2.1b}$$

$$\frac{\partial \psi}{\partial t} + \overline{U}_0 \frac{\partial \psi}{\partial x} + N^2\,w = F_\psi, \tag{2.1c}$$

$$\frac{\partial \phi}{\partial z} = -\frac{\rho}{\rho_0} g = \psi, \tag{2.1d}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{2.1e}$$

On procède alors par séparation des variables, en cherchant une solution de (2.1) sous la forme[2]

$$\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = U(z) \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\phi} \end{pmatrix}, w = \mathcal{W}(z)\hat{w}, \tag{2.2}$$

où les quantités $\hat{u}, \hat{v}, \hat{w}$, et $\hat{\phi}$ ne dépendent que de x et t. En remplaçant dans (2.1c) et (2.1e) il vient :

$$\frac{\mathcal{U}'}{N^2\mathcal{W}} = -\frac{\hat{w}}{\hat{\phi}_t + U_0\hat{\phi}_x} \quad (= c_1),$$

$$\frac{\mathcal{U}}{\mathcal{W}'} = -\frac{\hat{w}}{\hat{u}_x} \quad (= c_2).$$

[2] A priori on cherche la solution sous la forme plus générale $u = \mathcal{U}\hat{u}$, $v = \mathcal{V}\hat{v}$, $\phi = \varphi\hat{\phi}$, mais (1.8a) implique alors que $\mathcal{U}, \mathcal{V}, \varphi$ sont des quantités proportionnelles, donc identiques.

On remarque alors que les membres de gauche des équations ci-dessus ne dépendent que de z alors que ceux de droite ne dépendent que de x et t. Finalement les quantités ci-dessus sont donc constantes ($= c_1, c_2$).

En combinant ces équations on obtient :

$$\left(\frac{\mathcal{U}_z}{N^2}\right)_z + \lambda^2 \mathcal{U} = 0, \qquad \mathcal{W}_{zz} + \lambda^2 N^2 \mathcal{W} = 0, \tag{2.3}$$

où $\lambda^2 = -c_1/c_2$, et on résoud maintenant le problème (2.3) comme un problème aux valeurs propres dans lequel doivent intervenir les conditions aux limites. Classiquement, si l'on cherche à avoir $w = 0$ au fond et à la surface de l'océan, on a nécessairement :

$$\mathcal{W}(0) = \mathcal{W}(-L_3) = \mathcal{U}'(0) = \mathcal{U}'(-L_3) = 0. \tag{2.4}$$

On résoud (2.3) avec les conditions aux limites (2.4). On note λ_n^2 les valeurs propres associées, $\mathcal{U}_n, \mathcal{W}_n$ les modes propres correspondants (dits "modes normaux") et $\hat{u}_n, \hat{v}_n, \hat{\phi}_n, \hat{w}_n$ les solutions correspondantes. On élimine alors $\hat{w}_n$ et $\hat{\phi}_n$ pour obtenir (on omet pour l'instant les indices n) :

$$\left\{\begin{array}{l} \hat{u}_t - f\hat{v} - \dfrac{1}{N\lambda_n}\hat{\psi}_x + U_0\hat{u}_x = 0, \\ \hat{v}_t + f\hat{u} + U_0\hat{v}_x = 0, \\ \hat{\psi}_t + U_0\hat{\psi}_x - \dfrac{N}{\lambda_n}\hat{u}_x = 0. \end{array}\right. \tag{2.5}$$

Posons $\xi = u - \phi/N, \eta = u + \psi/N$ et regardons les nouvelles équations en (ξ, v, η). On obtient

$$\left\{\begin{array}{l} \hat{\xi}_t + (U_0 + \frac{1}{\lambda})\hat{\xi}_x - f\hat{v} = 0, \\ \hat{v}_t + U_0\hat{v}_x + \frac{f}{2}(\hat{\xi} + \hat{\eta}) = 0, \\ \hat{\eta}_t + (U_0 - \frac{1}{\lambda})\hat{\eta}_x - f\hat{v} = 0. \end{array}\right. \tag{2.6}$$

Les valeurs caractéristiques, qui dépendent en fait de l'indice n, et que l'on détermine par la partie du premier ordre (en faisant $f = 0$ dans (2.6)), sont $\overline{U}_0 + 1/\lambda_n$, $\overline{U}_0$ et $\overline{U}_0 - 1/\lambda_n$. Puisque les quantités $\overline{U}_0$ et λ_n sont toujours positives, on est assurés que $\overline{U}_0 + 1/\lambda_n$ le soit, et donc les quantités ξ_n et v_n requièrent une condition aux limites en $x = 0$, et ce quel que soit le mode considéré. En revanche, la quantité $\overline{U}_0 - 1/\lambda_n$ peut être positive ou négative selon le mode, si bien que la condition aux limites en η_n peut être requise en $x = 0$ ou bien en $x = L_1$. C'est ainsi que les auteurs de [OS78], repris dans [TT03], démontrent qu'il n'existe pas de conditions aux limites de type local qui rende le problème (2.1) (et *a fortiori* (1.8)) bien posé.

3 Première Partie : Analyse des Equations δ-Primitives

Dans leur premier article sur le problème des conditions aux limites pour les équations primitives sans viscosité, Temam et Tribbia [TT03] ont proposé d'introduire une viscosité "très faible" en remplaçant l'équation (1.7) par

$$\delta\, w + \frac{\partial \phi}{\partial z} = -\frac{\rho}{\rho_0} g, \tag{3.1}$$

où $\delta > 0$ est un "petit" paramètre. Les auteurs ont montré dans [TT03] que le système correspondant que nous appelons ici équations δ-primitives (δ-EP) était bien posé avec des conditions aux limites de type locales (semblables aux conditions aux limites pour les équations d'Euler : unicité de solutions, dépendance continue des données). Ils ont aussi montré qu'avec un choix convenable - empirique - de δ, le terme $\delta\, w$ produisait un bon filtrage des modes parasites dans des simulations numériques réalistes.
Notre étude présentée dans la Première Partie (chapitres 1 à 4) a consisté à

(*i*) Avancer l'étude du caractère bien posé des δ-EP (Chapitre 1). On montre des résultats d'existence, d'unicité et de régularité de solutions des δ-EP avec des conditions aux limites périodiques.

(*ii*) Etudier le rôle du terme visqueux $\delta\, w$. L'étude faite sur un seul mode normal a suffit à montrer l'existence d'ondes réfléchies indésirables à la frontière du domaine (Chapitres 2 et 3). Ces réflexions et couches limites (lorsque δ tend vers 0) étant présentes sur un seul mode du problème linéarisé, il est raisonnable de penser que ces propriétés non physiques persisteront - voire s'aggraveront - lors de l'étude de modèles plus complets (par exemple non linéaires et tridimensionnel).

De plus le Chapitre 3, préparatoire à la seconde partie, explore le choix de conditions aux limites non réfléchissantes que l'on peut adjoindre à chaque mode. Enfin le chapitre 4, lié aux deux chapitres qui le précèdent, présente une analyse de stabilité des schémas numériques qui y sont présentés.

4 Seconde Partie : Retour aux Equations Primitives

A la suite des phénomènes observés et étudiés sur les δ- EP, nous revenons aux EP d'origine (toujours sans viscosité) et au problème posé par les géophysiciens avec l'espoir de trouver un ensemble de conditions aux limites qui satisfasse les points suivants :

(*i*) Assurer le caractère bien posé du problème aux limites ainsi obtenu.

(*ii*) Faire disparaître les phénomènes de couches limites et les réflexions aux frontières du domaine, phénomènes qui ne sont pas souhaitables car la frontière n'est pas physique.

Etant donnée l'étude déjà mentionnée de [OS78], nous sommes conscients du fait que le point (*i*) ci-dessus ne pourra être assuré qu'avec des conditions aux limites non locales. Dans le Chapitre 5, qui entame cette seconde partie, nous proposons pour les EP sans viscosité bidimensionnelles linéarisées des conditions aux limites sous forme intégrale qui s'inspirent de l'étude de la première partie de cette thèse. Nous obtenons ainsi un problème aux limites tout à fait nouveau à notre connaissance (conditions aux limites consistant en une infinité de conditions intégrales). Ensuite nous montrons dans ce chapitre le caractère bien posé du problème d'évolution ainsi obtenu.

Enfin, le Chapitre 6 présente des simulations numériques des EP bidimensionnelles, considérées à la fois sous leurs formes linéarisée et *non linéaire*. La partie qui concerne le modèle linéaire illustre les résultats théoriques obtenu dans le Chapitre 5 et valide les propriétés non-réfléchissantes des conditions aux limites introduites. Quant aux calculs effectués sur le modèle non linéaire, ils permettent d'envisager avec optimisme la preuve de résultats théoriques comparables à ceux du Chapitre 5 pour le problème non linéaire. Cette étude constituera - et constitue déjà - une grande partie de mes efforts de recherche présents et à venir.

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Première Partie

Sur les équations δ-primitives de l'océan.

Chapitre 1

Sur l'existence et l'unicité des solutions des Équations δ-Primitives

On the δ-Primitive Equations

Ce chapitre est constitué de l'article **On the δ-Primitive and Boussinesq type Equations** écrit en collaboration avec M. Petcu, et qui est à paraître en 2005 dans *Advanced in Differential Equations*. Il décrit des résultats d'existence, unicité et régularité de solutions pour les Equations δ-Primitives nonlinéaires tridimensionnelles, dans le cadre de conditions aux limites périodiques. Ces équations, proposées dans [TT03], sont constituées des Equations Primitives habituelles, sans viscosité, dans lesquelles on substitue à l'équation hydrostatique l'équation suivante, dans laquelle δ est un réel positif :

$$\delta w + \phi_z = -\frac{\rho g}{\phi_0}.$$

Dans les chapitres 2 et 3 nous étudierons le rôle du terme δw et le comportement des solutions lorsque le petit paramètre δ tend vers 0.

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On the δ-Primitive and Boussinesq type Equations.

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Abstract

In this article we consider the Primitive Equations without horizontal viscosity but with a mild vertical viscosity added in the hydrostatic equation, as in [Sal98] and [TT03], which are the so-called $\delta-$Primitive Equations. We prove that the problem is well posed in the sense of Hadamard in a certain type of spaces. This means that we prove the finite in time existence, uniqueness and continuous dependence on data for appropriate solutions. The results given in the 3D periodic space, easily extend to dimension 2.
We also consider a Boussinesq type of equations, meaning that the mild vertical viscosity present in the hydrostatic equation, is replaced by the time derivative of the vertical velocity. We prove the same type of results as for the $\delta-$Primitive Equations; periodic boundary conditions are similarly considered.

1 Introduction

One of the major challenges in the mathematical and physical sciences is to study and improve the long-term weather prediction and to understand the climate changes. This consists in studying the mathematical equations and the models governing the motion of the atmosphere and the oceans, and advancing the techniques for their numerical simulations. The general equations describing these motions are derived from the basic conservation laws. The resulting equations are very complex and unfortunately too complicated to be analyzed but using some scale analysis methods and meteorological observations, these equations are well approximated by a somehow simplier system called the Primitive Equations.
In this article we are interested in deriving various results regarding the so-called δ-Primitive Equations and a Boussinesq type equation. By δ-Primitive Equations we understand the equations governing the movement of the geophysical fluids (atmosphere, oceans) as follows; we consider the laws of conservation of horizontal momentum with some minor geometrical approximations and we add a dissipation term in the hydrostatic equation (namely the term δw in (1.1c) as in [Sal98] and [TT03]). The Boussinesq type equations considered in this article are the same as the δ-Primitive Equations, the only difference being the hydrostatic equation, where the dissipative term δw is substituted by the time derivative of the vertical velocity multiplied by $\delta > 0$, $\delta \partial w/\partial t$, see e.g. [MHPA97].
We prove in this article that the equations obtained in this manner lead to the well-posedness of the problem, meaning that in a certain class of functions and in limited time, the equations

have solutions which are unique and depend continuously in the initial data (for a similar result in the context of the Euler equation, see e.g. Kato [Kat72], [Kat67] or Temam [Tem75b]).
The δ-Primitive Equations for the ocean read[1]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = F_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial y} = F_v, \tag{1.1b}$$

$$\delta w + \frac{\partial p}{\partial z} = -\rho g, \tag{1.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.1d}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = F_T. \tag{1.1e}$$

In the system above, $(u,\, v,\, w)$ are the three components of the velocity vector, p, ρ and T are respectively the perturbations of the pressure, of the density and of the temperature from a reference (average) state p_0, ρ_0 and T_0. The relation between the temperature and the density is given by the equation of state and we consider here a version of this equation linearized around the reference state ρ_0, T_0:

$$\rho = \rho_0(-\beta_T(T - T_0)). \tag{1.2}$$

In the system (1.1), f is the Coriolis parameter, $(F_u,\, F_v)$ represent the body forces per unit of mass and F_T represents a heating source. In applications, F_u, F_v and F_T vanish for the ocean (but we consider here the nonzero forcing for mathematical generality).
The Boussinesq type equations are the same as (1.1), with the only difference that the hydrostatic equation (1.1c) is substituted by:

$$\delta\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = -\rho g, \quad \delta > 0. \tag{1.3}$$

We recall here that the usual Primitive Equations correspond to $\delta = 0$ in (1.1c). For $\delta > 0$, the term δw is a friction (vertical viscosity) term. The term δw, called a drag term, is on one hand a mathematical remedy to ensure the well-posedness of the problem. On the other hand, as shown in [TT03], it has a smoothing effect, numerically filtering some undesirable oscillations. An interesting problem raised by these equations is the asymptotic study when $\delta \to 0$ (see e.g. [RTT04], [RTT05]). For more details regarding the motivations of this term see Temam and Tribbia [TT03] and also Salmon [Sal98]. A coherent model can be also obtained substituting the term δw with $\delta\, \partial w/\partial t$; as we announced we also intend to study this model from the mathematical point of view.
The article is organized as follows: in Section 2 we recall the δ-Primitive Equations, then we prove the existence of solutions in some class of functions, and we finally derive some a priori estimates, showing the continuous dependence on the data and the uniqueness of the regular

[1] Some slight modifications are necessary for the atmosphere, we refer the interested reader to Salmon [Sal98], where it is shown that working with the potential temperature instead of the temperature and changing the coordinates, the equations for the lower atmosphere will have the same form as the equations (1.1) for the ocean.

solutions and a regularity in time. In Section 3 we consider the Boussinesq type of equations and we prove the existence and uniqueness of regular solutions and also the regularity in time of the solutions.
For the interested reader, we mention that much work is available on the mathematical theory of the Primitive Equations in different contexts: the well-posedness of the Primitive Equations in the presence of viscosity has been established by Lions, Temam and Wang (see [LTW92a], [LTW92b]) for both the ocean and the atmosphere; improved results based on an anisotropic treatment of the vertical direction can be found in Petcu, Temam and Wirosoetisno, see [PTW04]. The same problem, of the well-posedness of the Primitive Equations, has been considered in a thin domain by Hu, Temam and Ziane [HTZ02]. High regularity results for the Primitive Equations in 2D periodic space were derived in [PTW04]. A review of numerous results available in the mathematical theory of geophysical fluid dynamics (as far as existence, uniqueness and regularity of solutions are concerned) can be found in [TZ04].
For details regarding the derivation of these models (PEs, δ-PEs, Boussinesq type model) from the physical laws, we refer the reader to classical references in Geophysical Fluid Dynamics, e.g. Haltiner and Williams [HW80], Gill [Gil82], Pedlosky [Ped87], Washington and Parkinson [WP86], and the references therein, as well as the references already quoted of Salmon, and Temam and Tribbia.

2 The δ-Primitive Equations

In this section we consider the δ-Primitive Equations as described in the Introduction and we prove that the problem is well-posed in the sense of Hadamard in a certain type of spaces.

2.1 The main result: existence and uniqueness of solutions

In this article we work in a limited domain $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ and we assume space periodicity with period Ω, meaning that all functions are taken to satisfy:

$$f(x_1 + L_1, x_2, x_3, t) = f(x_1, x_2, x_3, t) = f(x_1, x_2 + L_2, x_3, t) = f(x_1, x_2, x_3 + L_3, t),$$

when extended to $\mathbb{R}^3$. All the functions being periodic, they admit Fourier series expansions, hence we can write:

$$f = \sum_{k \in \mathbb{Z}^3} f_k e^{i(k_1' x_1 + k_2' x_2 + k_3' x_3)}, \tag{2.1}$$

where $k_j' = 2\pi k_j / L_j$ for $j = 1, 2, 3$.
Our aim is to study the existence and regularity of the solutions of problem (1.1) with some initial data. In system (1.1), the prognostic variables are u, v and T, whereas ρ, w and p are the diagnostic variables. Indeed, the density is already expressed in terms of the temperature T by the state equation (1.2), hence taking into account that in (1.1) ρ and T are respectively the perturbations of the density and of the temperature from an average value, we have:

$$\rho = -\rho_0 \beta_T T. \tag{2.2}$$

In order to determine the vertical velocity in terms of the prognostic variables, we write the equations (1.1c) and (1.1d) in Fourier modes and we obtain:

$$\delta w_k + i k_3' p_k = -g\rho_k, \tag{2.3}$$

and

$$k_1' u_k + k_2' v_k + k_3' w_k = 0. \tag{2.4}$$

From equation (2.3) we find:

$$w_k = -\frac{g\rho_k}{\delta}, \quad \text{for} \quad k_3' = 0, \tag{2.5}$$

and from equation (2.4) we find:

$$w_k = -\frac{k_1' u_k + k_2' v_k}{k_3'}, \quad \text{for} \quad k_3' \neq 0. \tag{2.6}$$

So, for each $U = (u,\, v,\, T)$ we can define $w = w(U)$ by its Fourier series, namely:

$$w(U)_k = \begin{cases} -\dfrac{k_1' u_k + k_2' v_k}{k_3'}, & \text{for } k_3' \neq 0, \\ \dfrac{g}{\delta}\rho_0 \beta_T T_k, & \text{for } k_3 = 0. \end{cases} \tag{2.7}$$

From (2.3) we then determine the pressure p in terms of the diagnostic variables, up to its vertical average. This means that we can fully determine the Fourier coefficients p_k of the pressure p for $k_3 \neq 0$ but not for $k_3 = 0$. The part of the pressure which can not be expressed in terms of the prognostic variables is the average of the pressure in the vertical direction:

$$\frac{1}{L_3}\int_0^{L_3} p(x_1,\, x_2,\, x_3)\,\mathrm{d}x_3 = \sum_{k,\, k_3=0} p_k(t) e^{i(k_1' x_1 + k_2' x_2)}. \tag{2.8}$$

Some natural function spaces for this problem are as follows:

$$\mathbf{V} = \{(u,\, v,\, T) \in (\dot{H}^1_{\text{per}}(\Omega))^3;\ \int_0^{L_3} (u_x + v_y)\,\mathrm{d}z = 0\}, \tag{2.9}$$

and

$$\mathbf{H} = \text{ the closure of } \mathbf{V} \text{ in } (\dot{L}^2(\Omega))^3. \tag{2.10}$$

Here the dot above $\dot{H}^1_{\text{per}}$ and $\dot{L}^2$ denotes the functions with average in Ω equal to zero. These spaces are endowed with the usual scalar products, meaning that on $\mathbf{H}$ we take the scalar product from $L^2(\Omega)$ and on $\mathbf{V}$ we work with the following scalar product:

$$((\phi,\, \tilde{\phi}))_{\mathbf{V}} = \int_\Omega \Big(\frac{\partial\phi}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\tilde{\phi}}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\tilde{\phi}}{\partial z}\Big)\,\mathrm{d}\Omega. \tag{2.11}$$

Note that because of the assumption that all the functions have zero average, the Poincaré inequality holds, meaning:

$$|U|_{L^2} \leq c_0 \|U\|, \quad \forall\, U \in \mathbf{V}, \tag{2.12}$$

which indeed guarantees that $\|\cdot\|$ is a norm on $\mathbf{V}$ equivalent to the usual norm on H^1.
In order to obtain the variational formulation of this problem, we consider a test function $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{T}) \in \mathbf{V}$, multiply (1.1a) by $\tilde{u}$, (1.1b) by $\tilde{v}$, and (1.1e) by $\tilde{T}$, and integrate over Ω. Using the integration by parts and the space periodicity we find that system (1.1) is formally equivalent to the following problem:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_{L^2} + b(U, U, \tilde{U}) + e(U, \tilde{U}) = (F,\tilde{U})_{L^2}, \quad \forall \tilde{U} \in \mathbf{V}, \\ &U(0) = U_0. \end{aligned} \tag{2.13}$$

In (2.13) we have defined the bilinear form e as being:

$$e(U,\tilde{U}) = f\int_\Omega u\tilde{v}\,\mathrm{d}\Omega - f\int_\Omega v\tilde{u}\,\mathrm{d}\Omega - \beta_T g\int_\Omega T\tilde{w}\,\mathrm{d}\Omega + \frac{\delta}{\rho_0}\int_\Omega w\tilde{w}\,\mathrm{d}\Omega, \tag{2.14}$$

and the trilinear form b as:

$$\begin{aligned} b(U,U^\sharp,\tilde{U}) = &\int_\Omega (u\frac{\partial u^\sharp}{\partial x}\tilde{u} + v\frac{\partial u^\sharp}{\partial y}\tilde{u} + w(U)\frac{\partial u^\sharp}{\partial z}\tilde{u})\,\mathrm{d}\Omega \\ &+ \int_\Omega (u\frac{\partial v^\sharp}{\partial x}\tilde{v} + v\frac{\partial v^\sharp}{\partial y}\tilde{v} + w(U)\frac{\partial v^\sharp}{\partial z}\tilde{v})\,\mathrm{d}\Omega \\ &+ \int_\Omega (u\frac{\partial T^\sharp}{\partial x}\tilde{T} + v\frac{\partial T^\sharp}{\partial y}\tilde{T} + w(U)\frac{\partial T^\sharp}{\partial z}\tilde{T})\,\mathrm{d}\Omega. \end{aligned} \tag{2.15}$$

We also introduce the following notation: we denote by $(f,g)_m$ and $|f|_m$ the scalar product and the norm in $\dot{H}^m_{\mathrm{per}}(\Omega)$,

$$(f,g)_m = \sum_{|\alpha|=m} (D^\alpha f, D^\alpha g)_{L^2}, \tag{2.16}$$

where D^α is a multi-index derivation; $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.
In all that follows in this article, we are interested in proving the existence of solutions for this problem on a certain interval of time, the uniqueness and the continuous dependence on the data for a certain class of solutions. The main result is the existence and uniqueness theorem, stated here below:

Theorem 2.1. *Let there be given $m \geq 3$, and $L_1, L_2, L_3 > 0$, $\Omega = (0,L_1) \times (0,L_2) \times (0,L_3)$ as above. Then for each U_0 given in $\mathbf{V} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^3$ and F given in $L^\infty(0,t_1;\, (\dot{H}^m_{\mathrm{per}}(\Omega))^3)$, there exists a $t_\star \leq t_1$, depending on the data (L_1, L_2, L_3, U_0, F) but independent of m, and a unique solution U of problem (2.13) defined on the interval $(0,t_\star)$, with*

$$U \in L^\infty(0,t_\star; \mathbf{V} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^3).$$

Proof : The proof of the existence of solutions is based on the Galerkin-Fourier method, using the a priori estimates obtained in the subsection below (for more details see e.g., in the context of the Euler equations, [Kat67], [Tem75a], [Tem75b]). The uniqueness of solution will be proved in the next subsection. □

Remark 2.1. *The same result of existence can be obtained in any dimension d, the proof is identical; because of the dimension of the space in the Sobolov imbedding theorems, we then require $m > 1 + d/2$.*

2.2 Existence of regular solutions

In this section we are interested in obtaining some estimates on the high order derivatives, from which we will then derive the existence of solutions for the δ-Primitive Equations in $(\dot{H}^m_{\text{per}}(\Omega))^3$, for m specified later on, and sufficiently large.
In all that follows, we assume that $m > 5/2$ so that $H^{m-1}(\Omega)$ is a multiplicative algebra.
We start by deriving the a priori estimates necessary to prove the existence results. Let α be a multi-index, $|\alpha| = m$. We apply the operator D^α to equations (1.1a), (1.1b) and (1.1e), then multiply the equations respectively by $D^\alpha u$, $D^\alpha v$ and $D^\alpha T$, integrate over Ω and add all these equations for $|\alpha| = m$. In this way we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|_m^2 &+ (u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w(U)\frac{\partial u}{\partial z}, u)_m + (u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w(U)\frac{\partial v}{\partial z}, v)_m \\ &+ (u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w(U)\frac{\partial T}{\partial z}, T)_m + \frac{1}{\rho_0}(\frac{\partial p}{\partial x}, u)_m + \frac{1}{\rho_0}(\frac{\partial p}{\partial y}, v)_m \\ &= (F, U)_m.\end{aligned} \tag{2.17}$$

Using periodicity and integrating by parts, we obtain:

$$\frac{1}{\rho_0}(\frac{\partial p}{\partial x}, u)_m + \frac{1}{\rho_0}(\frac{\partial p}{\partial y}, v)_m = -\frac{1}{\rho_0}(p, u_x + v_y)_m. \tag{2.18}$$

Using (1.1c), (1.1d) and integrating by parts we find:

$$-\frac{1}{\rho_0}(p, u_x + v_y)_m = -\frac{1}{\rho_0}(p_z, w)_m = \frac{\delta}{\rho_0}|w|_m^2 - \beta_T\frac{g}{\rho_0}(T, w)_m. \tag{2.19}$$

It now remains to estimate the nonlinear terms:

$$\begin{aligned}I_1 &= (u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w(U)\frac{\partial u}{\partial z}, u)_m, \\ I_2 &= (u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w(U)\frac{\partial v}{\partial z}, v)_m, \\ I_3 &= (u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w(U)\frac{\partial T}{\partial z}, T)_m.\end{aligned}$$

Since the three terms I_1, I_2 and I_3 have a similar structure, it suffices to estimate I_1 which we write in the form:

$$I_1 = \sum_{|\alpha|=m} (D^\alpha\psi, D^\alpha u)_{L^2}, \tag{2.20}$$

where $\psi = u\,\partial u/\partial x + v\,\partial u/\partial y + w(U)\,\partial u/\partial z$.
Using the Leibnitz rule we find:

$$\begin{aligned} D^{\alpha}\psi =& u\frac{\partial D^{\alpha}u}{\partial x} + v\frac{\partial D^{\alpha}u}{\partial y} + w(U)\frac{\partial D^{\alpha}u}{\partial z} \\ &+ \sum_{0<\beta\leq\alpha} c_{\alpha,\beta}(D^{\beta}u\frac{\partial D^{\alpha-\beta}u}{\partial x} + D^{\beta}v\frac{\partial D^{\alpha-\beta}u}{\partial y} + D^{\beta}w(U)\frac{\partial D^{\alpha-\beta}u}{\partial z}), \end{aligned} \tag{2.21}$$

where $c_{\alpha,\beta}$ are some suitable coefficients.
The contribution in (2.20) of the first three terms from (2.21) is zero, because of the conservation of mass law (1.1d). In this way, I_1 becomes:

$$I_1 = \sum_{\substack{|\alpha|=m\\0<\beta\leq\alpha}} c_{\alpha,\beta}(D^{\beta}u\frac{\partial D^{\alpha-\beta}u}{\partial x} + D^{\beta}v\frac{\partial D^{\alpha-\beta}u}{\partial y} + D^{\beta}w(U)\frac{\partial D^{\alpha-\beta}u}{\partial z},\ D^{\alpha}u)_{L^2}. \tag{2.22}$$

We bound I_1 as follows, c denoting an absolute constant which may be different at different places:

$$\begin{aligned} |I_1| \leq& c\sum_{\substack{|\alpha|=m\\0<\beta\leq\alpha}} \big[\big|D^{\beta}u\frac{\partial D^{\alpha-\beta}u}{\partial x}\big|_{L^2} + \big|D^{\beta}v\frac{\partial D^{\alpha-\beta}u}{\partial y}\big|_{L^2} \\ &+ \big|D^{\beta}w(U)\frac{\partial D^{\alpha-\beta}u}{\partial z}\big|_{L^2}\big]|D^{\alpha}u|_{L^2}. \end{aligned} \tag{2.23}$$

We know that:

$$|D^{\alpha}u|_{L^2} \leq |u|_m \leq |U|_m, \quad \forall\, \alpha \,\text{with}\, |\alpha| = m. \tag{2.24}$$

The problem now reduces to finding a good way to estimate the terms:

$$\big|D^{\beta}u\frac{\partial D^{\alpha-\beta}u}{\partial x}\big|_{L^2(\Omega)}, \tag{2.25}$$

and

$$\big|D^{\beta}w(U)\frac{\partial D^{\alpha-\beta}u}{\partial z}\big|_{L^2(\Omega)}, \tag{2.26}$$

for α with $|\alpha| = m$ and $0 < \beta \leq \alpha$.
In order to bound these terms, we use the following inequalities:

$$|\xi\eta|_{L^2(\Omega)} \leq c_1'|\xi|_{H^2(\Omega)}|\eta|_{L^2(\Omega)}, \tag{2.27}$$

and

$$|\xi\eta|_{L^2(\Omega)} \leq c_2'|\xi|_{H^1(\Omega)}|\eta|_{H^1(\Omega)}, \tag{2.28}$$

where c_1' and c_2' are constants depending only on Ω.
We obtain:

$$\sum_{\substack{|\alpha|=m\\0<\beta\leq\alpha}} \big|D^{\beta}u\frac{\partial D^{\alpha-\beta}u}{\partial x}\big|_{L^2} \leq c_1(|U|_m|U|_3 + |U|^2_{m-1}). \tag{2.29}$$

For the sum from I_1 containing the terms of the form (2.26), we obtain:

$$\sum_{\substack{|\alpha|=m\\0<\beta\leq\alpha}} |D^\beta w(U)\frac{\partial D^{\alpha-\beta}u}{\partial z}|_{L^2} \leq c_2(|w(U)|_m|U|_3 + |w(U)|_{m-1}|U|_{m-1} + |w(U)|_3|U|_m). \tag{2.30}$$

Taking into account these estimates and using Young's inequality, we find the following energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_m^2 + \frac{\delta}{\rho_0}|w(U)|_m^2 \leq \eta(t)|U|_m^2 + \xi(t)|U|_m, \tag{2.31}$$

where

$$\eta(t) = c_1 + c_2|U|_3^2 + |w(U)|_3,$$

and

$$\xi(t) = 2|F|_m + c_3(|U|_{m-1}^2 + |w(U)|_{m-1}|U|_{m-1}).$$

For $m = 3$ the differential inequality writes as:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_3^2 + \frac{\delta}{\rho_0}|w(U)|_3^2 \leq (c_1 + c_2|U|_3^2)|U|_3^2 + 2|U|_3|F|_3. \tag{2.32}$$

We find that there exists a t_1 depending only on the initial data, such that:

$$|U(t)|_3 \leq 1 + 2|U_0|_3,\ \forall\, 0 \leq t \leq t_1, \tag{2.33}$$

which leads us to:

$$U \in L^\infty(0, t_1; (\dot{H}^3_{\text{per}}(\Omega))^3),\ w(U) \in L^2(0, t_1; \dot{H}^3_{\text{per}}(\Omega)). \tag{2.34}$$

Recursively we find that, for $m \geq 3$, $|U(t)|_m$ remains bounded on $(0, t_1)$, where t_1 is exactly the time determined for $m = 3$.
Gathering all these estimates and using classical methods (the Galerkin-Fourier method), we obtain the existence of the solutions as enounced above. The Galerkin-Fourier method consists in constructing approximate solutions by the Galerkin approximation. The approximate solutions are the solutions of a finite-dimensional equation with bilinear nonlinearity. The a priori estimates work for each approximate solution. Since the bound is independent of the solution, we can pass to the limit finding the solution of the problem.

2.3 Uniqueness and continuous dependence on the data

In this section we prove the continuous dependence on data of the solutions. Hence, with the existence result proved above, we obtain that problem (2.13) is well posed in the sense of Hadamard, in suitable spaces.
Let us consider two solutions for the problem (2.13), namely $U' = (u', v', T')$ and $U'' =$

(u'', v'', T''), which respectively correspond to the initial data $U_0' = (u_0', v_0', T_0')$ and $U_0'' = (u_0'', v_0'', T_0'')$ and to the forcing $F' = (F_u', F_v', F_T')$ and $F'' = (F_u'', F_v'', F_T'')$. We set:

$$
\begin{aligned}
&u = u' - u'', && v = v' - v'',\\
&w(U) = w'(U') - w''(U''), && T = T' - T'',\\
&u_0 = u_0' - u_0'', && v_0 = v_0' - v_0'',\\
&T_0 = T_0' - T_0'', && F = F' - F''.
\end{aligned}
$$

Then, $U = (u, v, T)$ obeys the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde U)_{L^2} + b(U',U,\tilde U) + b(U,U'',\tilde U) + e(U,\tilde U) = (F,\tilde U)_{L^2}, \quad \forall \tilde U \in \mathbf{V}, \tag{2.35a}$$

$$U(0) = U_0. \tag{2.35b}$$

We set in all that follows $\mathbf{v} = (u, v)$.
In equation (2.35) we take formally $\tilde U = U(t)$ for t fixed but arbitrary. We notice that:

$$\int_\Omega [(\mathbf{v}' \cdot \nabla)u\,u + w'\frac{\partial u}{\partial z}u]\,\mathrm{d}\Omega = 0, \tag{2.36}$$

because $(\mathbf{v}', w')$ obeys the conservation of mass equation, namely (1.1d) (the same argument is used for the terms similar with (2.36)).
We obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[|u|^2_{L^2} + |v|^2_{L^2} + |T|^2_{L^2}] + \frac{1}{\rho_0}\int_\Omega \frac{\partial p}{\partial x}u\,\mathrm{d}\Omega + \frac{1}{\rho_0}\int_\Omega \frac{\partial p}{\partial y}v\,\mathrm{d}\Omega = (F,U)_{L^2} + \eta_1 + \eta_2 + \eta_3, \tag{2.37}$$

where:

$$
\begin{aligned}
\eta_1 &= \int_\Omega [(\mathbf{v}\cdot\nabla)u'' + w\frac{\partial u''}{\partial z}]u\,\mathrm{d}\Omega,\\
\eta_2 &= \int_\Omega [(\mathbf{v}\cdot\nabla)v'' + w\frac{\partial v''}{\partial z}]v\,\mathrm{d}\Omega,\\
\eta_3 &= \int_\Omega [(\mathbf{v}\cdot\nabla)T'' + w\frac{\partial T''}{\partial z}]T\,\mathrm{d}\Omega.
\end{aligned}
$$

Using integration by parts we notice that:

$$
\begin{aligned}
\frac{1}{\rho_0}\int_\Omega \frac{\partial p}{\partial x}u\,\mathrm{d}\Omega + \frac{1}{\rho_0}\int_\Omega \frac{\partial p}{\partial y}v\,\mathrm{d}\Omega &= \frac{1}{\rho_0}\int_\Omega p w_z\,\mathrm{d}\Omega = -\frac{1}{\rho_0}\int_\Omega p w_z\,\mathrm{d}\Omega\\
&= \frac{\delta}{\rho_0}|w|^2_{L^2} + \frac{g}{\rho_0}\int_\Omega \rho w\,\mathrm{d}\Omega.
\end{aligned} \tag{2.38}
$$

We need to estimate η_1, η_2 and η_3. Supposing that the solutions are functions having their gradient in L^∞, i.e. their first order spatial derivatives are bounded, we find:

$$
\begin{aligned}
|\eta_j| &\le c_1'|\nabla U''|_{L^\infty}|U|^2_{L^2} + c_2'|w(U)|_{L^2}|U|_{L^2}\left|\frac{\partial U''}{\partial z}\right|_{L^\infty}\\
&\le k_1|U|^2_{L^2} + k_2, \ \text{ for } j = 1,2,3,
\end{aligned} \tag{2.39}
$$

where k_1 and k_2 are constants depending on $U'' = (u'', v'', T'')$ through the L^∞ norm of its spatial gradient.
Going back to (2.37), we find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|_{L^2}^2 + \frac{\delta}{\rho_0}|w(U)|_{L^2}^2 \leq \Big|\frac{g}{\rho_0}\int_\Omega \rho w(U)\,\mathrm{d}\Omega\Big| + |F|_{L^2}|U|_{L^2} + k_1|U|_{L^2}^2 + k_2|w(U)|_{L^2}|U|_{L^2}. \tag{2.40}$$

Using Young's inequality, (2.40) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_{L^2}^2 + \frac{\delta}{\rho_0}|w(U)|_{L^2}^2 \leq k_1|U|_{L^2}^2 + |F|_{L^2}^2. \tag{2.41}$$

By the Gronwall lemma, we find:

$$|U(t)|_{L^2}^2 \leq |U_0|_{L^2}^2 e^{k_1 t} + e^t\int_0^t |F(s)|_{L^2}^2 \mathrm{d}s. \tag{2.42}$$

From the estimate (2.42) we deduce immediately that the solutions having their first derivatives uniformly bounded, depend continuously on the data in the root-mean-sense.
Uniqueness of the solutions belonging to the class mentioned above can be deduced from (2.42), taking the same initial data $U_0' = U_0''$ and the same forcing $F' = F''$, which leads to $U_0 = 0$ and $F = 0$, so

$$|U(t)|_{L^2}^2 \leq 0, \quad \forall\, t > 0.$$

We now define the following function spaces:

$$Y = \{U \in L^\infty(0, t_\star; (\dot{H}^1_{\mathrm{per}}(\Omega))^3); D_j U \in (L^\infty(\Omega \times (0, t_1)))^3,\ j = 1,\ 2,\ 3\},$$
$$X = [(\dot{H}^m_{\mathrm{per}}(\Omega))^3 \cap \mathbf{V}] \times [L^\infty(0, t_1; (\dot{H}^m_{\mathrm{per}}(\Omega))^3)],$$

where $D_j = \partial/\partial x_j$, $m \geq 3$, $t_1 > 0$ arbitrarily chosen and $t_\star$ defined in Theorem 2.1. Both spaces are equipped with their natural norms which make them Banach spaces. For Theorem 2.2 we also consider these spaces equipped with the L^2 norm for X and with the norm $|U| = |\nabla U|_{L^\infty(\Omega\times(0,t_1))}$ and then call $\tilde{X}$ and $\tilde{Y}$ these spaces (which are normed, non-complete spaces).
We conclude this section by the following theorem:

Theorem 2.2. *For $(U_0, F) \in X$ and $m \geq 3$, the system (2.13) has a unique solution U in $L^\infty(0, t_\star; (\dot{H}^m_{\mathrm{per}}(\Omega))^3 \cap \mathbf{V}) \subset Y$. Furthermore, the mapping $(U_0, F) \to U$ is bounded from X into Y and continuous from the bounded sets of X into Y for the norms of $\tilde{X}$ and $\tilde{Y}$.*

2.4 Time regularity

In this subsection we are interested in deriving some regularity results in time. We first derive the necessary a priori estimates. We differentiate the equations (1.1) l times in t and then take the scalar product in H^m of the equations resulting from (1.1a), (1.1b) and (1.1e) respectively with $u^{(l)}$, $v^{(l)}$ and $T^{(l)}$. Here and in all that follows we set $u^{(l)} = \partial^l u/\partial t^l$. We find:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U^{(l)}|_m^2 + &\sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial u^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial u^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial u^{(l-k)}}{\partial z}, u^{(l)})_m \\
&+ \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial v^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial v^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial v^{(l-k)}}{\partial z}, v^{(l)})_m \\
&+ \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial T^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial T^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial T^{(l-k)}}{\partial z}, T^{(l)})_m \\
&+ \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, u^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial y}, v^{(l)})_m \\
&= (F^{(l)}, U^{(l)})_m.
\end{aligned} \tag{2.43}$$

Using periodicity, integrating by parts and taking into account (1.1c), (1.1d), we obtain:

$$\begin{aligned}
\frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, u^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial y}, v^{(l)})_m &= -\frac{1}{\rho_0}(p^{(l)}, u_x^{(l)} + v_y^{(l)})_m \\
&= \frac{\delta}{\rho_0}|w(U)^{(l)}|_m^2 - \beta_T\frac{g}{\rho_0}(T^{(l)}, w(U)^{(l)})_m.
\end{aligned} \tag{2.44}$$

We now need to estimate the terms:

$$\begin{aligned}
J_1 &= \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial u^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial u^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial u^{(l-k)}}{\partial z}, u^{(l)})_m \\
J_2 &= \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial v^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial v^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial v^{(l-k)}}{\partial z}, v^{(l)})_m \\
J_3 &= \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial T^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial T^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial T^{(l-k)}}{\partial z}, T^{(l)})_m,
\end{aligned} \tag{2.45}$$

where the C_l^k are the binomial coefficients.
Since the terms J_1, J_2 and J_3 are similar, we concentrate our attention only on J_1. We notice that:

$$J_1 = \sum_{|\alpha|=m} (D^\alpha \eta, D^\alpha u^{(l)})_{L^2}, \tag{2.46}$$

where

$$\eta = \sum_{k=0}^{l} C_l^k \big(u^{(k)}\frac{\partial u^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial u^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial u^{(l-k)}}{\partial z}\big). \tag{2.47}$$

Computing $D^\alpha \eta$ we find:

$$\begin{aligned}
D^\alpha \eta = &\sum_{k=0}^{l} C_l^k \big(u^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial x} + v^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial y} + w(U)^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial z}\big) \\
&+ \sum_{0<\beta\leq\alpha} c_{\alpha,\beta} \sum_{k=0}^{l} C_l^k \big(D^\beta u^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial x} + D^\beta v^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial y} \\
&\quad + D^\beta w(U)^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial z}\big).
\end{aligned} \tag{2.48}$$

For $k = 0$, the corresponding terms from the first sum will have the scalar product with $D^\alpha u^{(l)}$ equal to zero, because of the conservation of mass law (1.1d). Taking into account this simplification, J_1 writes as:

$$\begin{aligned}
J_1 = &\sum_{k=0}^{l} C_l^k \big(u^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial x} + v^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial y} + w(U)^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial z}, D^\alpha u^{(l)}\big)_{L^2} \\
&+ \sum_{0<\beta\leq\alpha} c_{\alpha,\beta} \sum_{k=0}^{l} C_l^k \big(D^\beta u^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial x} + D^\beta v^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial y} \\
&\quad + D^\beta w(U)^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial z}, D^\alpha u^{(l)}\big)_{L^2}.
\end{aligned} \tag{2.49}$$

Using the inequalities (2.27) and (2.28), we estimate J_1 as follows:

$$\begin{aligned}
|J_1| \leq &c_1 |U|_{m+1} |U^{(l)}|_m^2 + c_2 |w(U)^{(l)}|_m |U|_{m+1} |U^{(l)}|_m \\
&+ c_3 \sum_{k=1}^{l-1} |U^{(l-k)}|_{m+1} (|U^{(k)}|_m + |w(U)^{(k)}|_m) |U^{(l)}|_m.
\end{aligned} \tag{2.50}$$

Gathering the estimates obtained above and using also Young's inequality, we arrive at the following differential inequality:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} |U^{(l)}|_m^2 + \frac{\delta}{\rho_0} |w(U)^{(l)}|_m^2 \leq &(c_1 + c_2 |U|_{m+1}^2) |U^{(l)}|_m^2 + |F^{(l)}|_m |U^{(l)}|_m \\
&+ c_3 |U^{(l)}|_m \sum_{k=1}^{l-1} |U^{(l-k)}|_{m+1} (|U^{(k)}|_m + |w(U)^{(k)}|_m).
\end{aligned} \tag{2.51}$$

By classical methods (meaning we use the Fourier-Galerkin method, constructing approximating solutions for which all the a priori estimates deduced above hold, and then passing to the limit) and using an induction argument, we find that exists a time $t_\star$ depending only on the initial data, such that:

$$\frac{\partial^l U}{\partial t^l} \in L^\infty(0, t_\star, H^m_{\text{per}}(\Omega)^3), \tag{2.52}$$

for all $l \geq 1$ and all $m \geq 3$. This way we prove the following result:

Theorem 2.3. *Let there be given* $m \geq 3$, $l \geq 1$ *and* $L_1, L_2, L_3 > 0$, $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ *as above. Then for each* U_0 *given in* $\mathbf{V} \cap (\dot{H}^m_{\text{per}}(\Omega))^3$ *and* F *given such that* $F^{(l)}$ *is in* $L^\infty(0, t_1, \dot{H}^m_{\text{per}}(\Omega)^3)$, *there exists a time* $t_\star$ *depending on the initial data and not on* m *nor on* l, *and a unique solution* U *of problem (2.13) defined on the interval* $(0, t_\star)$, *with*

$$U^{(l)} \in L^\infty(0, t_\star, \dot{H}^m_{\text{per}}(\Omega)^3).$$

We conclude this section by the following theorem:

Theorem 2.4. *Let there be given* $m \geq 3$, *and* $L_1, L_2, L_3 > 0$, $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ *as above. Then for each* U_0 *given in* $\mathbf{V} \cap (\mathcal{C}^\infty_{\text{per}}(\bar{\Omega}))^3$ *and* F *given in* $\mathcal{C}^\infty(0, t_1; \mathcal{C}^\infty_{\text{per}}(\bar{\Omega}))^3$, *there exists a unique solution* U *of problem (2.13) defined on the interval* $(0, t_\star)$, *with*

$$U \in \mathcal{C}^\infty(0, t_\star; \mathcal{C}^\infty_{\text{per}}(\bar{\Omega})^3),$$

where $t_\star = \min(t_1, t_2)$, t_2 *given by (2.33).*

Proof : To prove Theorem 2.4 we apply Theorem 2.3 for each $m \geq 3$, remembering that:

$$\mathcal{C}^\infty_{\text{per}}(\bar{\Omega}) = \cap_{m \geq 3} H^m_{\text{per}}(\Omega);$$

of importance here is the fact that $t_\star$ in Theorem 2.3 is independent of m. We denote by $\mathcal{C}^\infty_{\text{per}}(\bar{\Omega})$ the set of functions in $\mathcal{C}^\infty(\bar{\Omega})$ whose periodic extension beyond Ω is $\mathcal{C}^\infty$ on $\mathbb{R}^3$ (smooth matching at the boundary of Ω). □

3 A Boussinesq type equation

As we announced in the Introduction of this article, we are also interested in considering a Boussinesq type of equations, given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = F_u, \tag{3.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial y} = F_v, \tag{3.1b}$$

$$\delta\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = -\rho g, \tag{3.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.1d}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = F_T, \tag{3.1e}$$

where $\delta > 0$ is given.

3.1 Existence and uniqueness of regular solutions

Our aim is to study the existence and regularity of solutions of problem (3.1) on a periodic domain with suitable initial data.
The natural function spaces for this problem are:

$$\tilde{\mathbf{V}} = \{(u,\, v,\, w,\, T) \in (\dot{H}^1_{\mathrm{per}}(\Omega))^4;\; u_x + v_y + w_z = 0\}, \tag{3.2}$$

and

$$\tilde{\mathbf{H}} = \text{ the closure of } \tilde{\mathbf{V}} \text{ in } (\dot{L}^2(\Omega))^4. \tag{3.3}$$

As before, the dot above the spaces H^1_{per} and L^2 denotes the functions with average zero; for these functions the Poincaré inequality holds. The spaces are endowed with the usual scalar products.
We first derive the variational formulation of the problem. We consider a test function $\tilde{U} = (u,\, v,\, w,\, T) \in \tilde{\mathbf{V}}$, multiply (3.1a) by $\tilde{u}$, (3.1b) by $\tilde{v}$, (3.1c) by $\tilde{w}$ and (3.1e) by $\tilde{T}$ and integrate over Ω. Using the integration by parts and the space periodicity, we find that system (3.1) is formally equivalent to the variational problem:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_{L^2} + b(U,U,\tilde{U}) + e(U,\tilde{U}) = (F,(\tilde{u},\tilde{v},\tilde{T}))_{L^2}, \quad \tilde{U} \in \tilde{\mathbf{V}},\\ &U(0) = U_0. \end{aligned} \tag{3.4}$$

In relation (3.4) we defined the bilinear form e as:

$$e(U,\tilde{U}) = f\int_\Omega u\tilde{v}\,\mathrm{d}\Omega - f\int_\Omega v\tilde{u}\,\mathrm{d}\Omega - g\beta_T\int_\Omega T\tilde{w}\,\mathrm{d}\Omega, \tag{3.5}$$

and the trilinear form b as:

$$\begin{aligned} b(U,U^\sharp,\tilde{U}) = &\int_\Omega (u\frac{\partial u^\sharp}{\partial x}\tilde{u} + v\frac{\partial u^\sharp}{\partial y}\tilde{u} + w(U)\frac{\partial u^\sharp}{\partial z}\tilde{u})\,\mathrm{d}\Omega\\ &+ \int_\Omega (u\frac{\partial v^\sharp}{\partial x}\tilde{v} + v\frac{\partial v^\sharp}{\partial y}\tilde{v} + w(U)\frac{\partial v^\sharp}{\partial z}\tilde{v})\,\mathrm{d}\Omega\\ &+ \int_\Omega (u\frac{\partial T^\sharp}{\partial x}\tilde{T} + v\frac{\partial T^\sharp}{\partial y}\tilde{T} + w(U)\frac{\partial T^\sharp}{\partial z}\tilde{T})\,\mathrm{d}\Omega; \end{aligned} \tag{3.6}$$

by F we understand $F = (F_u,\, F_v,\, F_T)$.
In this section we prove the existence and uniqueness of solutions locally in time for this problem. We state here the main result for these equations:

Theorem 3.1. *Let there be given $m \geq 3$, and $L_1, L_2, L_3 > 0$, $\Omega = (0,L_1)\times(0,L_2)\times(0,L_3)$ as above. Then for each U_0 given in $\tilde{\mathbf{V}} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^4$ and F given in $L^\infty(0,t_1;\, (\dot{H}^m_{\mathrm{per}}(\Omega))^3)$, there exists a $0 < t_\star \leq t_1$, independent of m, and a unique solution U of problem (2.13) defined on the interval $(0,t_\star)$, with*

$$U \in L^\infty(0,t_\star;\tilde{\mathbf{V}} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^4).$$

Proof : The proof of the existence of solutions is based on the Galerkin-Fourier method. The method is based on the priori estimates obtained below.
The estimates we deduce here are on the high order derivatives; these estimates lead us to conclude the existence of solutions for the Boussinesq type equations in $(\dot{H}^m_{\rm per}(\Omega))^4$, with $m > 5/2$.
Let α be a multi-index, $|\alpha| = m$. We take the operator D^α and apply it to equations (3.1a), (3.1b), (3.1d) and (3.1e), then we multiply these equations respectively by $D^\alpha u$, $D^\alpha v$, $D^\alpha w/\rho_0$ and $D^\alpha T$, integrate over Ω and add all these equations for $|\alpha| = m$.
The terms containing the Coriolis parameter obviously disappear. Integrating by parts and using the periodicity and the conservation of mass law we also have:

$$(p_x, u)_m + (p_y, v)_m + (p_z, w)_m = -(p, u_x + v_y + w_z)_m = 0. \tag{3.7}$$

We then have:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{|u|_m^2 + |v|_m^2 + \frac{\delta}{\rho_0}|w|_m^2 + |T|_m^2\} - g\beta_T(T, w)_m \\ &\quad + (([(u, v, w)\cdot \mathrm{grad}]u, u))_m + (([(u, v, w)\cdot \mathrm{grad}]v, v))_m \\ &\quad + (([(u, v, w)\cdot \mathrm{grad}]T, T))_m = (F, (u, v, T))_m. \end{aligned} \tag{3.8}$$

We easily estimate:

$$|g\beta_T(T, w)_m| \leq g\beta_T|T|_m|w|_m, \tag{3.9}$$

and

$$|(F, (u, v, T))_m| \leq |F|_m(|u|_m + |v|_m + |T|_m). \tag{3.10}$$

It remains to estimate the terms:

$$\begin{aligned} &I_1 = ([(u, v, w)\cdot \mathrm{grad}]u, u)_m, \quad I_2 = ([(u, v, w)\cdot \mathrm{grad}]v, v)_m, \\ &I_3 = ([(u, v, w)\cdot \mathrm{grad}]T, T)_m. \end{aligned} \tag{3.11}$$

Since the terms from (3.11) are the same as the terms considered in the previous section for the Primitive Equations, we can use the inequalities (2.27) and (2.28) and apply the same kind of reasonings. We do not repeat here the details of the computations.
We also introduce the following definition:

$$\|U\|_m^2 = |u|_m^2 + |v|_m^2 + \frac{\delta}{\rho_0}|w|_m^2 + |T|_m^2; \tag{3.12}$$

$\|\cdot\|_m$ is a norm on $(\dot{H}^m_{\rm per}(\Omega))^4$ equivalent to the usual norm.
Returning to (3.8), we find the following estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|_m^2 \leq g\beta_T\rho_0\|U\|_m^2 + |F|_m\|U\|_m + c_1\|U\|_m^2\|U\|_3 + c_2\|U\|_{m-1}^2\|U\|_m, \tag{3.13}$$

where c_1 and c_2 are some constants independent of the initial data, which may vary at different appearances.
For $m = 3$ we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|_3^2 \leq 2g\beta_T\rho_0\|U\|_3^2 + 2|F|_3\|U\|_3 + c_1\|U\|_3^3 + c_2\|U\|_2^2\|U\|_3. \tag{3.14}$$

Applying the Gronwall lemma to the estimate (3.14), we find that there exists a time $t_\star$ depending on the initial data such that the following estimate in $L^\infty(0,t_\star;(\dot{H}^3(\Omega))^4)$ holds:

$$\|U(t)\|_3 \leq 2\|U_0\|_3, \quad \forall\, 0 \leq t \leq t_\star. \tag{3.15}$$

Recursively we find that, for $m > 3$, $|U(t)|_m$ remains bounded in $(0,t_\star)$, where $t_\star$ is the time determined for $m = 3$. Gathering these estimates and using the Galerkin–Fourier method, we obtain the existence of the solutions.

In order to prove the uniqueness of the solutions, we consider two solutions of the problem (3.4), namely $U' = (u', v', w', T')$ and $U'' = (u'', v'', w'', T'')$. We set $U = U' - U''$. Substituting the corresponding equation (3.4) for U'' from the equation for U' we find that U satisfies the following equation:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_{\tilde{\mathbf{H}}} + b(U',U,\tilde{U}) + b(U,U'',\tilde{U}) + e(U,\tilde{U}) = 0, \quad \forall\, \tilde{U} \in \mathbf{V},\\
&U(0) = 0,
\end{aligned} \tag{3.16}$$

where the scalar product on $\tilde{\mathbf{H}}$ is:

$$(U,\tilde{U})_{\tilde{\mathbf{H}}} = (u,\tilde{u})_{L^2} + (v,\tilde{v})_{L^2} + \frac{\delta}{\rho_0}(w,\tilde{w})_{L^2} + (T,\tilde{T})_{L^2}. \tag{3.17}$$

In equation (3.16), we take $\tilde{U} = U(t)$ for an arbitrary but fixed instant of time t. Applying the conservation of mass equation (1.1d) we find $b(U',U,U) = 0$. From the definition of the form e we also find $e(U,U) = 0$. We then obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_{\tilde{\mathbf{H}}} + b(U,U'',U) = 0, \tag{3.18}$$

which leads us to the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_{\tilde{\mathbf{H}}} \leq c|DU'|_{L^\infty}|U|^2_{\tilde{\mathbf{H}}}. \tag{3.19}$$

Since the solutions are in $(\dot{H}^3_{\mathrm{per}}(\Omega))^4$, we find $U = 0$, so the solution is unique. □

Similar to the case of δ–Primitive Equations, we can prove the continuous dependence on the data of the solutions. We define the following spaces:

$$\begin{aligned}
&X_1 = [(\dot{H}^m_{\mathrm{per}}(\Omega))^4 \cap \tilde{\mathbf{V}}] \times [L^\infty(0,t_1;(\dot{H}^m_{\mathrm{per}}(\Omega))^4)],\\
&Y_1 = \{U \in L^\infty(0,t_\star;(\dot{H}^1_{\mathrm{per}}(\Omega))^4);\ D_jU \in (L^\infty(\Omega \times (0,t_\star)))^4,\ j = 1,2,3,4.\},
\end{aligned}$$

where $t_\star$ is defined in Theorem 3.1.

We equip these spaces with their natural norms, which make them Banach spaces. We also consider X_1 equipped with the L^2–norm and Y_1 equipped with the L^∞–norm of the spatial gradient ($|U| = |\nabla U|_{L^\infty(\Omega\times(0,t_\star))}$) and we call these spaces respectively $\tilde{X}_1$ and $\tilde{Y}_1$. Then we can prove that:

Remark 3.1. *For the spaces defined above, the analogue of Theorem 2.2 holds.*

3.2 Time regularity

In this section we prove that a time regularity result, similar to the result obtained for the δ-Primitive Equations, is available for the Boussinesq type of equations considered in this section. As before, we are interested in obtaining some a priori estimates. In order to derive the necessary a priori estimates, we differentiate l times in time the equations (3.1a), (3.1b), (3.1c) and (3.1e), then we take the scalar product in H^m of the resulting equations respectively with $u^{(l)}$, $v^{(l)}$, $w^{(l)}/\rho_0$ and $T^{(l)}$. We find:

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U^{(l)}\|_m^2 + \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial u^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial u^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial u^{(l-k)}}{\partial z}, u^{(l)})_m \\ + \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial v^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial v^{(l-k)}}{\partial y} + w^{(k)}\frac{\partial v^{(l-k)}}{\partial z}, v^{(l)})_m \\ + \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial T^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial T^{(l-k)}}{\partial y} + w^{(k)}\frac{\partial T^{(l-k)}}{\partial z}, T^{(l)})_m \\ + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, u^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial y}, v^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, w^{(l)})_m \\ = (F^{(l)}, (u,v,T)^{(l)})_m. \end{aligned} \tag{3.20}$$

Using periodicity, the conservation of mass (3.1d) and integrating by parts, we have:

$$\frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, u^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial y}, v^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial z}, w^{(l)})_m = 0. \tag{3.21}$$

The terms that remain to be estimated are the same as the terms obtained for the δ-Primitive Equations, so the computations are identical and we skip them here. We obtain the following a priori estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U^{(l)}\|_m^2 \le c_1\|U\|_{m+1}\|U^{(l)}\|_m^2 + c_2\sum_{k=1}^{l-1}\|U^{(l-k)}\|_{m+1}\|U^{(k)}\|_m\|U^{(l)}\|_m + 2|F^{(l)}|_m\|U^{(l)}\|_m. \tag{3.22}$$

Using the same arguments as before we find that there exists a time $t_\star$, depending on the initial data, such that:

$$\frac{\partial^l U}{\partial t^l} \in L^\infty(0, t_\star, (\dot{H}^m_{\mathrm{per}}(\Omega))^4), \tag{3.23}$$

for all $l \ge 0$ and all $m \ge 3$.

We can now state the following results, similar to the results obtained for the δ-Primitive Equations:

Theorem 3.2. *Let there be given $m \ge 3$, $l \ge 0$ and $L_1, L_2, L_3 > 0$, $\Omega = (0,L_1)\times(0,L_2)\times(0,L_3)$ as above. Then for each U_0 given in $\tilde{\mathbf{V}} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^4$ and F given such that $F^{(l)}$ is in $L^\infty(0,t_1,\dot{H}^m_{\mathrm{per}}(\Omega)^3)$, there exists a time $t_\star$ depending only on the initial data and a unique solution U of problem (2.13) defined on the interval $(0,t_\star)$, with*

$$U^{(l)} \in L^\infty(0, t_\star, \dot{H}^m_{\mathrm{per}}(\Omega)^4).$$

Using the a priori estimates above, we can also find:

Theorem 3.3. *Let there be given* $m \geq 3$, *and* $L_1, L_2, L_3 > 0$, $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ *as above. Then for each* U_0 *given in* $\tilde{\mathbf{V}} \cap (\mathcal{C}^\infty_{\text{per}}(\bar{\Omega}))^4$ *and* F *given in* $\mathcal{C}^\infty(0, t_0; \mathcal{C}^\infty_{\text{per}}(\bar{\Omega})^3)$, *there exists a time* $t_\star$, $0 < t_\star \leq t_0$ *and a unique solution* U *of problem (2.13) defined on the interval* $(0, t_\star)$, *with*

$$U \in \mathcal{C}^\infty(0, t_\star; \mathcal{C}^\infty_{\text{per}}(\bar{\Omega})^4).$$

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Chapitre 2

Etude numérique de l'influence du petit paramètre dans les δ-EPs : couches limites

Numerical study of the boundary layers generated by the small parameter in the δ-PEs

Ce chapitre correspond à l'article **Boundary Layers in an Ocean Related System**, J. Sci. Comput., **21** (2004), pp.405-432. Ce travail en collaboration avec R. Temam et J.Tribbia consiste à comprendre le comportement numérique des solutions des Equations δ-Primitives ; les conditions aux limites considérées ici sont les conditions aux limites de Dirichlet, qui génèrent des phénomènes de couches limites et de réflexions à la frontière.
On regarde en fait un modèle constitué de l'un des modes (sous-critique) des δ-EPs bidimensionnelles linéarisées autour de $\overline{U}_0\mathbf{e_x}$, dans le cadre d'une décomposition en modes dits "normaux". On observe alors, pour les modes fondamentaux et lorsque δ tend vers 0, une couche limite au voisinage de la frontière $x = 0$, et des réflexions d'ondes en $x = 0$ et $x = L$.

Dans le chapitre suivant, on démontrera rigoureusement les résultats de convergence conjecturés ici.

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Boundary Layers in an Ocean Related System.

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Abstract

In this article two regularizations of an hyperbolic model system derived from the primitive equations of the ocean (or the atmosphere) are presented. The two regularized systems converge to different limits as the regularization parameter converges to 0. Numerical approximations of these equations and numerical simulations are also presented.

1 Introduction

The Primitive Equations of the ocean and the atmosphere are fundamental equations of geophysical fluid mechanics ([Ped87],[WP86]). In the presence of viscosity, it has been shown - in various contexts - that these equations are well-posed (see e.g. [LTW92a], [LTW92b], and the review article [TZ04]).
In the absence of viscosity, it is known that the PEs are not well-posed for any set of boundary conditions of local type. This difficulty is analyzed in [OS78] using a modal analysis in the vertical direction (see also [TT03]). To overcome this difficulty a modification of the PEs was proposed in [TT03], for which a set of local boundary conditions also given in [TT03] produces a decay of energy (in the absence of forcing), an important positive step in proving well-posedness. The modified PEs introduced in [TT03] contain an added friction term δw in the hydrostatic equation, so that the model is actually nonhydrostatic ; we will call this system the δ-PEs equations.
A first objective of this work was to study the δ-PEs equations linearized around a flow of constant velocity U_0 in direction (Ox). For that purpose we considered an expansion of the solutions using suitable normal modes in the vertical direction, thus leading to a system of three equations (see below for more details) :

$$\begin{cases} u_t - f\,v + \phi_x - \dfrac{\delta\, g}{N^2\,\lambda^2}\,u_{xx} + U_0\,u_x = 0, \\ v_t + f\,u + U_0\,v_x = 0, \\ \phi_t + U_0\,\phi_x + \dfrac{1}{\lambda^2}\,u_x = 0. \end{cases} \tag{1.1}$$

This system is supplemented with suitable boundary conditions described below (see (2.3)) and we were interested in finding the limit of the solutions of this system as $\delta \to 0$ (or $\varepsilon \to 0$, $\varepsilon = \delta\, g/2\, N^2\, \lambda^2$). Actually, we were surprised to learn that the limit system was not what one could believe at first. Furthermore, we found the boundary conditions for the limit system thanks to numerical simulations, by inspection of the solutions to system (1.1) (see also (1.8)) for small ε. Subsequent theorical studies, to be reported in [RTT05], confirmed that the boundary conditions of the limit system were those hinted at by numerical simulations. The latter were done using a numerical procedure described in Section 2.3 and Remark 2.3.
Depending on the problem that we study, the limit system for (1.1) (or (1.8)) provided in Section 2 may or may not be suitable. In Section 3 we propose another version of system (1.8) (equations and boundary conditions), inspired by the concept of transparent boundary conditions. In that case, the limit system is the same as before (one mode of the nonviscous PEs) as $\varepsilon \to 0$, but the boundary conditions are different, somehow more natural. We realize of course that deciding what are the "good" boundary conditions for the limit system is a matter of physical intuition and may depend on the physical problem considered. For the problem of the PEs studied in [OS78] and [TT03], the δ-PEs are supplemented with boundary conditions of local type, but this modal analysis suggests that the model produces boundary layers and some undesirable reflection of waves at the boundary. These difficulties disappear in the TBC (transparent boundary condition) model studied in Section 3. However, passing from one mode to the whole PEs, this study suggests that nonlocal (mode dependant) boundary conditions and implicit or semi-implicit procedures are required for the TBC approach, whereas easier local boundary conditions and fully explicit schemes are available for the δ-PEs. Finally, the choice will depend on the objectives, that is the desired precision versus the amount of computational effort that one wishes to invest. In the case of the PEs this study will be conducted elsewhere.
We conclude this section by recalling the derivation of system (1.1)-(1.8) studied in Section 2.

We start from the primitive equations of the ocean without viscosity :

$$\left\{\begin{array}{c} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + w\,\dfrac{\partial \mathbf{v}}{\partial z} + f\,\mathbf{k}\times\mathbf{v} + \nabla\phi = 0,\\ \delta w + \dfrac{\partial p}{\partial z} = -\rho\, g,\\ \nabla\cdot\mathbf{v} + \dfrac{\partial w}{\partial z} = 0,\\ \dfrac{\partial \theta}{\partial t} + (\mathbf{v}\cdot\nabla)\theta + w\dfrac{\partial \theta}{\partial z} = 0,\\ \rho = \rho(\theta). \end{array}\right. \tag{1.2}$$

In these equations $\mathbf{v} = (u, v)$ is the horizontal velocity, w the vertical velocity, ϕ the pressure, ρ the density, and θ the temperature. g is the gravitational acceleration, and f the Coriolis parameter. The horizontal gradient is denoted by ∇.
Here $\delta \geq 0$; in the original version of the PEs, $\delta = 0$, and for $\delta > 0$ we obtain the δ-PEs considered in [TT03] (see also [Sal98] in a different context).

A linearization of (1.2) around a constant flow velocity U_0 in the x direction ($U_0 > 0$), with no dependance in y reads :

$$\begin{cases} u_t - f\,v + \phi_x + U_0\,u_x = 0, \\ v_t + f\,u + U_0\,v_x = 0, \\ \theta_t + N^2 \dfrac{\theta_0}{g}\,w + U_0\,\theta_x = 0, \\ u_x + w_z = 0, \\ \delta\,w + \phi_z = \dfrac{g\theta}{\theta_0}, \end{cases} \tag{1.3}$$

where N is the Brunt-Väisälä (or buoyancy) frequency, supposed constant.

The boundary conditions proposed in [TT03] for the δ-system (1.2) induce the following boundary conditions for (1.3) (valid under the hypothesis that u is small compared with U_0) :

$$\begin{cases} u(0,z,t) = u_{gl}(z,t), & u(L,z,t) = u_{gr}(z,t), & z \in [-H,0], \quad t > 0, \\ v(0,z,t) = v_{gl}(z,t), & \theta(0,z,t) = \theta_{gl}(z,t), & z \in [-H,0], \quad t > 0, \\ w(x,-H,t) = 0, & w(x,0,t) = 0, & x \in [0,L], \quad t > 0, \end{cases} \tag{1.4}$$

with $\int_{-H}^{0} u_{gl}(z,t)\,dz = \int_{-H}^{0} u_{gr}(z,t)\,dz$, so that we can integrate the 4th equation of (1.3) and guarantee that $w = 0$ at $z = -H$ and 0.

The method used in [TT03] to show the decay of energy for the δ-PEs extends to system (1.3)-(1.4) in the following way. We assume here for simplicity that u_{gl}, u_{gr}, v_{gl} and θ_{gl} vanish ; the treatment of the nonhomogeneous boundary conditions will be explained below for our actual equations (see Remark 2.2). For an arbitrary $\kappa > 0$, we multiply (1.3a) by u, (1.3b) by v, (1.3c) by $\kappa\,\theta$ and (1.3e) by w, integrate over $\mathcal{M} = (0,L) \times (-H,0)$ and add these relations. We find

$$\frac{d}{dt} \int_{\mathcal{M}} (u^2 + v^2 + \kappa\,\theta^2)\,d\mathcal{M} + 2\,\delta \int_{\mathcal{M}} w^2\,d\mathcal{M} + I_1 + I_2 + I_3 = I_4, \tag{1.5}$$

where,

$$\begin{cases} I_1 = 2\int_{\mathcal{M}} (\phi_x\,u + \phi_z\,w)\,d\mathcal{M}, \\ I_2 = U_0 \int_{-H}^{0} \{ \left[u^2(x,z) + v^2(x,z) + \kappa\,\theta^2(x,z) \right]_{x=0}^{x=L} \}\,dz, \\ I_3 = 2\,\kappa\,N^2 \int_{\mathcal{M}} \dfrac{\theta_0\,\theta}{g}\,w\,d\mathcal{M}, \\ I_4 = 2\,\int_{\mathcal{M}} \dfrac{g\,\theta}{\theta_0}\,w\,d\mathcal{M}. \end{cases} \tag{1.6}$$

Setting $\kappa = (g/N\,\theta_0)^2$, we find $I_3 = I_4$, and these terms cancel each other in (1.5) ; it is then easy to check with (1.3d) and (1.4) that $I_1 = 0$, $I_2 \geq 0$. That is, (1.5) ensures that the energy $E_\kappa(t) = \int_{\mathcal{M}} (u^2 + v^2 + \kappa\,\theta^2)\,d\mathcal{M}$ is time-decreasing (for $\kappa = (g/N\,\theta_0)^2$), which is an essential step in the proof of well-posedness of the linear system (1.3)-(1.4) (see [Lio65] or [Yos80] for more details).

In [OS78] and [TT03], a mode analysis of system (1.3) in the z direction was implemented ; the authors proceed by separations of variables when looking for a solution of (1.3). They search u

and the other variables of the form $u(x,z,t) = U(z)\,\hat{u}(x,t), v(x,z,t) = U(z)\,\hat{v}(x,t), \phi(x,z,t) = U(z)\,\hat{\phi}(x,t)$. Substituting these expressions into the equations, and combining them together we obtain the following modal equation (again, see [OS78] or [TT03] for the definition of the function $U(z)$, and more details) :

$$\begin{cases} \hat{u}_{n\,t} - f\,\hat{v_n} + \hat{\phi}_{n\,x} + U_0\,\hat{u}_{n\,x} = 0, \\ \hat{v}_{n\,t} + f\,\hat{u}_n + U_0\,\hat{v}_{n\,x} = 0, \\ \hat{\phi}_{n\,t} + U_0\,\hat{\phi}_{n\,x} + \dfrac{1}{(\lambda_n)^2}\,\hat{u}_{n\,x} = 0, \end{cases} \quad \text{for the } n\text{th mode.} \tag{1.7}$$

The characteristic values of (1.7) - determined by the first order parts - are U_0, $U_0 + \lambda^{-1}$ and $U_0 - \lambda^{-1}$ ($\lambda = \lambda_n$). Hence, since the domain is $(0, L)$ in the x direction, if $\lambda^{-1} < U_0$ three boundary conditions are needed at $x = 0$. Conversely, if $\lambda^{-1} > U_0$, only two boundary conditions are needed at $x = 0$, and one is needed at $x = L$. This is why the global problem is ill-posed for a local set of boundary conditions. The study below focuses on the most problematical case, $\lambda^{-1} > U_0$, corresponding to the so called subcritical modes.

For notational convenience, we now write u, v, ϕ and λ instead of $\tilde{u}_n$,$\tilde{v}_n$, $\tilde{\phi}_n$ and λ_n. With $\delta > 0$, the same study leads to the following system, which naturally reduces to (1.7) if $\delta = 0$:

$$\begin{cases} u_t - f\,v + \phi_x - \dfrac{\delta\,g}{N^2\,\lambda^2}\,u_{xx} + U_0\,u_x = 0, \\ v_t + f\,u + U_0\,v_x = 0, \\ \phi_t + U_0\,\phi_x + \dfrac{1}{\lambda^2}\,u_x = 0. \end{cases} \tag{1.8}$$

From (1.4), one can derive, after some approximations, the following boundary conditions for system (1.8) :

$$\begin{cases} u(0,t) = u_{gl}(t), \quad u(L,t) = u_{gr}(t), \\ v(0,t) = v_{gl}(t), \quad \phi(0,t) = \phi_{gl}(t), \end{cases} \quad t > 0. \tag{1.9}$$

From now on, we set $f = 0$ since the Coriolis effect is not an essential element in the well-posedness issue. With this simplification, the second equation (1.8) can be solved independently of the other equations, and we can study the system (1.8),(1.9), in which the v variable is now omitted.

2 One mode analysis of the δ-PEs

2.1 The equations and boundary condition

With the simplification $f = 0$, we can now focus on the equations coupling u and ϕ :

$$\begin{cases} u_t + \phi_x - 2\,\varepsilon\,u_{xx} + U_0\,u_x = F_u, \\ \phi_t + U_0\,\phi_x + \dfrac{1}{\lambda^2}\,u_x = F_\phi. \end{cases} \tag{2.1}$$

$$\begin{cases} u(0,t) = u_{gl}(t), \ u(L,t) = u_{gr}(t), \\ \phi(0,t) = \phi_{gl}(t), \end{cases} \qquad t > 0. \tag{2.2}$$

In (2.1) we have introduced for mathematical generality the source terms F_u and F_ϕ, which do not exist in the original problem ; we also set $\varepsilon = \delta\, g/2N^2\lambda^2$, which thus depends on $\lambda = \lambda_n$. Using a change of variable suggested by the diagonalization of the limit system ((2.1) with $\varepsilon = 0$, see below), we set $\xi = u + \lambda\,\phi$ and $\eta = u - \lambda\,\phi$. The system (2.1) becomes :

$$\begin{cases} \xi_t + \alpha\,\xi_x - \varepsilon(\xi_{xx} + \eta_{xx}) = f_\xi, \\ \eta_t - \beta\,\eta_x - \varepsilon(\xi_{xx} + \eta_{xx}) = f_\eta. \end{cases} \tag{2.3}$$

where

$$\begin{cases} \alpha = U_0 + \dfrac{1}{\lambda} > 0, \\ \beta = -U_0 + \dfrac{1}{\lambda}. \end{cases} \tag{2.4}$$

In the two following sections, we will study the system (2.3) with two different sets of boundary conditions. From now on we will also suppose $\beta > 0$, i.e. $U_0 < \lambda^{-1}$ which, as we said, is the most problematic case in the physical problem (subcritical modes).

Boundary conditions

We are interested in studying the system (2.3), with the Dirichlet boundary conditions proposed in [TT03].
For system (2.1) the natural boundary conditions are as follows :

$$\begin{cases} u(0,t) = u_{gl}(t), \ u(L,t) = u_{gr}(t), \\ \phi(0,t) = \phi_{gl}(t), \end{cases} \qquad t > 0. \tag{2.5}$$

In the (ξ, η) variables, the boundary conditions (2.2) yield the following boundary conditions for the system (2.3) :

$$\begin{cases} \xi(0,t) = \xi_{gl}(t), \ \eta(0,t) = \eta_{gl}(t), \\ (\xi + \eta)(L,t) = 2\,u_{gr}(t), \end{cases} \qquad t > 0, \tag{2.6}$$

where we remember that $\xi + \eta = 2\,u$, and set $\xi_{gl}(t) = u_{gl}(t) + \lambda\,\phi_{gl}(t), \eta_{gl}(t) = u_{gl}(t) - \lambda\,\phi_{gl}(t)$.

The limit system

We discuss the (formal) passage to the limit $\varepsilon \to 0$ in the homogeneous boundary condition case, where u_{gl}, u_{gr} and ϕ_{gl} vanish. Setting formally $\varepsilon = 0$ in (2.1) and (2.3), we obtain the limit systems

$$\begin{cases} u_t^0 + \phi_x^0 + U_0\, u_x^0 = 0, \\ \phi_t^0 + U_0\, \phi_x^0 + \dfrac{1}{\lambda^2}\, u_x^0 = 0. \end{cases} \tag{2.7}$$

and

$$\begin{cases} \xi^0_t + \alpha\, \xi^0_x = 0, \\ \eta^0_t - \beta\, \eta^0_x = 0. \end{cases} \tag{2.8}$$

It is not clear to which solutions of (2.7) the solutions of (2.1),(2.2) converge as $\varepsilon \to 0$. If we consider the system (2.8), then the most natural set of boundary conditions for ξ^0, η^0 is the following :

$$\begin{cases} \xi^0(0,t) = 0, \\ \eta^0(L,t) = 0, \end{cases} \qquad t > 0. \tag{2.9}$$

In the (u, ϕ) variables, these conditions read :

$$\begin{cases} u^0(0,t) + \lambda\, \phi^0(0,t) = 0, \\ u^0(L,t) - \lambda\, \phi^0(L,t) = 0, \end{cases} \qquad t > 0. \tag{2.10}$$

Indeed it is well-known and obvious that (2.8),(2.9) - or equivalently (2.7),(2.10) - is a well-posed system. We define an energy for the system by setting :

$$E^0(t) = \int_0^L \{(\xi^0(x,t))^2 + (\eta^0(x,t))^2\}\, dx \tag{2.11}$$

Evaluating the derivative of E^0 we find :

$$\begin{aligned} \frac{d}{dt} E^0(t) &= 2 \int_0^L \{\xi^0_t(x,t)\, \xi^0(x,t) + \eta^0_t(x,t)\, \eta^0(x,t)\}\, dx \\ &= 2 \int_0^L \{-\alpha\, \xi^0_x(x,t)\, \xi^0(x,t) + \beta\, \eta^0_x(x,t)\, \eta^0(x,t)\}\, dx \\ &= -\alpha\, (\xi^0(L,t))^2 - \beta\, (\eta^0(0,t))^2, \end{aligned}$$

and therefore

$$\frac{d}{dt} E^0(t) \leq 0. \tag{2.12}$$

That is, the energy decreases in time. In a fully rigorous mathematical setting, (2.12) can be used to show that the system is well-posed, using the theory of linear contraction semi-groups (see eg. [Lio65],[Yos80] for the general context, and [RTT05] for this specific system).
Actually, as we said, one of the most surprising aspects of this study was to find out that, when ε goes to zero, the solutions of (2.3),(2.6) converge to the system (2.8) supplemented with a different set of boundary conditions, namely :

$$\begin{cases} \xi^0(0,t) + \dfrac{\beta}{\alpha}\, \eta^0(0,t) = 0, \\ \xi^0(L,t) + \eta^0(L,t) = 0, \end{cases} \qquad t > 0. \tag{2.13}$$

Equivalently, in the (u, ϕ) variables, the solutions of (2.1),(2.2) converge, as ε goes to zero, to the system (2.7) supplemented with the following boundary conditions :

$$\begin{cases} u(0,t) + \lambda^2 U_0 \, \phi(0,t) = 0, \\ u(L,t) = 0, \end{cases} \qquad t > 0. \tag{2.14}$$

A full mathematical proof of this result is beyond the scope of this article ; it will be presented in [RTT05]. In the next subsection we want to prove the decay of energy for (2.3),(2.6) and (2.8),(2.13).

2.2 Non-increasing energy for the systems

As we mentioned previously, the system (2.8) with the natural boundary condition (2.9) is well-posed. In the following, we want to give an indication of well-posedness for both (2.3),(2.6) and the new limit system (2.8),(2.13) by showing that some energy function is non-increasing.

The system (2.8), with the boundary conditions (2.13), has a time-decreasing energy.

As in (2.11), let us define an energy for the system (2.8) :

$$E^0(t) = \int_0^L \{(\xi^0(x,t))^2 + (\eta^0(x,t))^2\} \, dx.$$

Then,

$$\begin{aligned} \frac{d}{dt} E^0(t) &= 2 \int_0^L \{\xi_t^0(x,t) \, \xi^0(x,t) + \eta_t^0(x,t) \, \eta^0(x,t)\} \, dx \\ &= -\alpha \left((\xi^0(L,t))^2 - (\xi^0(0,t))^2\right) + \beta \left((\eta^0(L,t))^2 - (\eta^0(0,t))^2\right). \end{aligned}$$

We now use the boundary conditions (2.13) to conclude that :

$$\frac{d}{dt} E^0(t) = (\beta - \alpha) \, (\eta^0(L,t))^2 + \frac{\beta}{\alpha} \, (\beta - \alpha) \, ((\eta^0(0,t))^2.$$

Taking into account that $\alpha > \beta$ (see (2.4)), we find that indeed $dE^0(t)/dt \leq 0$.

In the same way, let us prove the following :

The system (2.1), with the boundary conditions (2.2), or equivalently (2.3) with (2.6), has a time-decreasing energy.

$$E(t) = \int_0^L \{u^2(x,t) + \lambda^2 \phi^2(x,t)\} \, dx \tag{2.15}$$

Indeed, we have

$$\begin{aligned}\frac{d}{dt}\,E(t) &= 2\int_0^L \{u_t(x,t)\,u(x,t) + \lambda^2\phi_t(x,t)\,\phi(x,t)\}\,dx \\ &= 2\int_0^L \{u(x,t)\,(-\phi_x + 2\varepsilon\,u_{xx} + U_0\,u_x)(x,t) \\ &\quad +\phi(x,t)\,(-\lambda^2 U_0\,\phi_x - u_x)(x,t)\}\,dx.\end{aligned}$$

Using the boundary conditions, we obtain :

$$\begin{aligned}\frac{d}{dt}\,E(t) &= -2\big[u(x,t)\,\phi(x,t)\big]_{x=0}^{x=L} + 4\varepsilon\int_0^L u(x,t)u_{xx}(x,t)\,dx - \lambda^2\,U_0\,\phi^2(L,t) \\ &= -4\varepsilon\int_0^L u_x^2\,dx - \lambda^2\,U_0\,\phi^2(L,t).\end{aligned}$$

Finally,

$$\frac{d}{dt}\,E(t) \leq 0. \tag{2.16}$$

This achieves the proof of non-increasing energy for the system (2.1)-(2.3).

Remark 2.1. *:* In the same way, we could have obtained the same result in the (ξ,η) variables, defining $E(t) = \int_0^L \{\xi^2(x,t) + \eta^2(x,t)\}\,dx$, which is in fact identical to (2.15).

Remark 2.2. *:* In the nonhomogeneous case and also if F_u and F_ϕ do not vanish, the similar result can be proved. However we do not obtain then a decay of energy but a bound on the energy involving the forcing terms F_u, F_ϕ, and the boundary values $u_{gl}, u_{gr}, \phi_{gl}$. We first extend the boundary data in the form $u_g(x,t), \phi_g(x,t)$ with :

$$\begin{cases} u_g(0,t) = u_{gl}(t),\ u_g(L,t) = u_{gr}(t), \\ \phi_g(L,t) = \phi_{gl}(t), \end{cases} \qquad t>0. \tag{2.17}$$

Then, as in [TT03], we set $u = u_g + u'$, $\phi = \phi_g + \phi'$, and we proceed as above, obtaining :

$$\frac{d}{dt}\,E'(t) \leq E'(t) + \int_0^L (F_u'^2(x,t) + \lambda^2\,F_\phi'^2(x,t))\,dx, \tag{2.18}$$

with :

$$\begin{cases} E'(t) = \int_0^L (u'^2(x,t) + \lambda\,\phi'^2(x,t))\,dx, \\ F'_u = F_u - u_{gt} - \phi_{gx} + 2\,\varepsilon\,u_{gxx} - U_0\,u_{gx}, \\ F'_\phi = F_\phi - \phi_{gt} - U_0\,\phi_{gx} - \dfrac{1}{\lambda^2}\,u_{gx}. \end{cases} \tag{2.19}$$

The bound on E' follows from (2.18) and Gronwall's lemma. Note that the bound is not uniform in time in this case, unlike (2.16).

2.3 Numerical scheme

Another task of this article is to propose an efficient numerical scheme for computing the solutions of system (2.3),(2.6). Actually one can see, comparing the boundary conditions (2.6) with (2.14), that when ε goes to zero a boundary layer shall appear at $x = 0$. The approximation of such a problem by a classical discrete method (finite elements or finite differences) would require the use of a fine mesh, at least of the order of the size of the boundary layer, that is here $\mathcal{O}(\varepsilon)$. To avoid using such a small and expensive mesh, it was proposed in [CT02], [CTW00] for a similar problem (with one equation only), to add to the Galerkin basis a function representing the boundary layer corrector in the context of a finite element discretization[1].
In spite of the time dependence of the equations, we want to focus in this article on the space discretization, taking into account that the time does not change the existence of the boundary layers.

In a classical finite element method we consider the standard piecewise linear base functions

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{for } x_{j-1} \le x \le x_j, \\ \dfrac{x_{j+1} - x}{h} & \text{for } x_j \le x \le x_{j+1}, \\ 0 & \text{elsewhere,} \end{cases} \qquad 1 \le j \le M-1. \tag{2.20}$$

$$\varphi_M(x) = \begin{cases} \dfrac{x - x_{M-1}}{h} & \text{for } x_{M-1} \le x \le x_M, \\ 0 & \text{elsewhere,} \end{cases} \tag{2.21}$$

where $h = L/M$, $x_j = j\,h$, and we look for an approximate solution

$$\begin{pmatrix} \xi_M \\ \eta_M \end{pmatrix} \in \begin{pmatrix} \xi_g \\ \eta_g \end{pmatrix} + V_M, \tag{2.22}$$

where $\xi_g = u_g + \lambda\,\phi_g, \eta_g = u_g - \lambda\,\phi_g$, and

$$V_M = Span\{\begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix}, \cdots, \begin{pmatrix} \varphi_{M-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_{M-1} \end{pmatrix}, \begin{pmatrix} \varphi_M \\ -\varphi_M \end{pmatrix}\}.$$

The approximate solution satisfies :

$$\begin{cases} \dfrac{d}{dt}(\xi_M, \chi_1) + \alpha\,(\xi_{M_x}, \chi_1) + \varepsilon\,(\xi_{M_x} + \eta_{M_x}, \chi_{1_x}) &= 0, \\ \dfrac{d}{dt}(\eta_M, \chi_2) - \beta\,(\eta_{M_x}, \chi_2) + \varepsilon\,(\xi_{M_x} + \eta_{M_x}, \chi_{2_x}) &= 0, \end{cases} \qquad \forall \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in V_M, \tag{2.23}$$

[1] This assumes of course that the shape of the corrector is known, see Remark 2.3. The case of finite differences will be discussed in a forthcoming article.

where $(f,g) = \int_0^L f(x)\,g(x)\,dx$ is the usual $\mathbb{L}^2$ inner product. A similar weak formulation of (2.3) is also valid.

Unfortunately, the numerical results are not satisfying as soon as $h \gtrsim \varepsilon$. To improve the numerical resolution, we add the following corrector :

$$\tilde{\varphi_0}(x) = e^{-r\,x/\varepsilon}, \text{ where } r = \frac{\alpha\,\beta}{\alpha-\beta} > 0.$$

To insert this corrector in the basis functions, and make it consistent with the boundary conditions, we slightly modify $\tilde{\varphi_0}$ and replace it with :

$$\varphi_0(x) = e^{-\frac{r\,x}{\varepsilon}} + (1 - e^{-\frac{r\,L}{\varepsilon}})\,\frac{x}{L} - 1, \tag{2.24}$$

The expression of φ_0 is very similar to what appears in [CT02] and [CTW00], but since we have two *coupled* equations the corrector here is more difficult to determine.

The space V_M is replaced by :

$$\tilde{V_M} = Span\{\begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}, \ldots, \begin{pmatrix} \varphi_{M-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_{M-1} \end{pmatrix}, \begin{pmatrix} \varphi_M \\ -\varphi_M \end{pmatrix}\}. \tag{2.25}$$

The new formulation reads :

$$\begin{pmatrix} \xi_M \\ \eta_M \end{pmatrix} \in \begin{pmatrix} \xi_g \\ \eta_g \end{pmatrix} + \tilde{V_M}, \text{ and :}$$

$$\left\{ \begin{array}{lcl} \frac{d}{dt}\,(\xi_M, \chi_1) + \alpha\,(\xi_{M_x}, \chi_1) + \varepsilon\,(\xi_{M_x} + \eta_{M_x}, \chi_{1_x}) & = & 0, \\ \frac{d}{dt}\,(\eta_M, \chi_2) - \beta\,(\eta_{M_x}, \chi_2) + \varepsilon\,(\xi_{M_x} + \eta_{M_x}, \chi_{2_x}) & = & 0, \end{array} \right. \quad \forall \xi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in \tilde{V_M}. \tag{2.26}$$

One can easily observe that (2.26) is a linear differential system for the coefficients a_j, b_j defining ξ_M and η_M :

$$\left\{ \begin{array}{lcl} \xi_M(x,t) & = & \xi_g(x,t) + a_0(t)\,\varphi_0(x) + \sum_{j=1}^M a_j(t)\,\varphi_j(x), \\ \eta_M(x,t) & = & \eta_g(x,t) + b_0(t)\,\varphi_0(x) + \sum_{j=1}^M b_j(t)\,\varphi_j(x). \end{array} \right. \tag{2.27}$$

Remark 2.3. *:* The corrector φ_0 was not known at the beginning of our work. To perform the numerical simulations which led us to the boundary conditions (2.13), we used an empirical parabolic corrector φ_0^* which was used in the algorithm above instead of φ_0. Further applications of this empirical corrector will be discussed elsewhere.

2.4 Numerical results

We emphasize the case where u_g, v_g, ϕ_g vanish which leads to more intuitive figures. The case where these data do not vanish do not raise any additional numerical difficulty : an example is given at the end of the section.
The results are presented below in the (ξ, η) variables, because the plots are more readable. We will use the same initial data for ξ and η, corresponding to the bump plotted in Figure 1. For the computations shown hereafter, we use these values of the parameters :

$$\left\{\begin{array}{lcr} U_0 & = & 1, \\ \lambda & = & 1/2, \end{array}\right. \qquad \Leftrightarrow \qquad \left\{\begin{array}{lcl} \alpha & = & 3, \\ \beta & = & 1. \end{array}\right. \tag{2.28}$$

$$\left\{\begin{array}{lcr} \varepsilon & = & 10^{-3}, \\ M & = & 200. \end{array}\right. \tag{2.29}$$

For $t = 0.15$, for either the classical finite element method or the new algorithm proposed, we have the same numerical results (see Figure 2) : ξ is travelling to the right with a speed α, whereas η is moving to the left with a speed β. On Figure 2, ξ is going out of the frame at the boundary $x = L$, and η is coming in. This illustrates the boundary condition $\xi + \eta = 0$. We can also notice a decrease of the $\mathbb{L}^\infty$-norm, due to the diffusion ε.

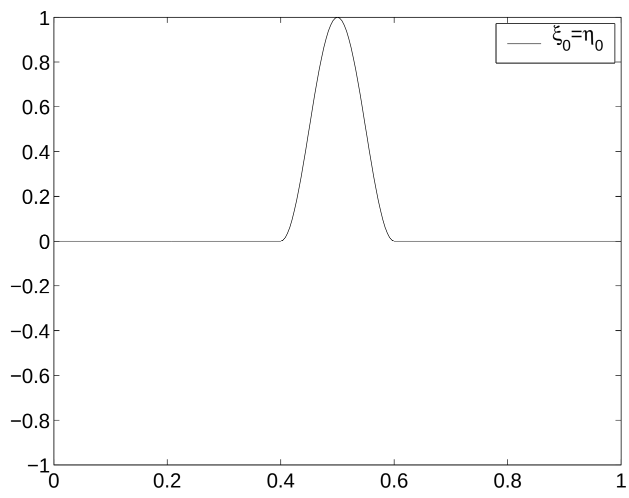

Figure 2.1: Dirichlet Boundary Condition. Initial data ξ_0, η_0.

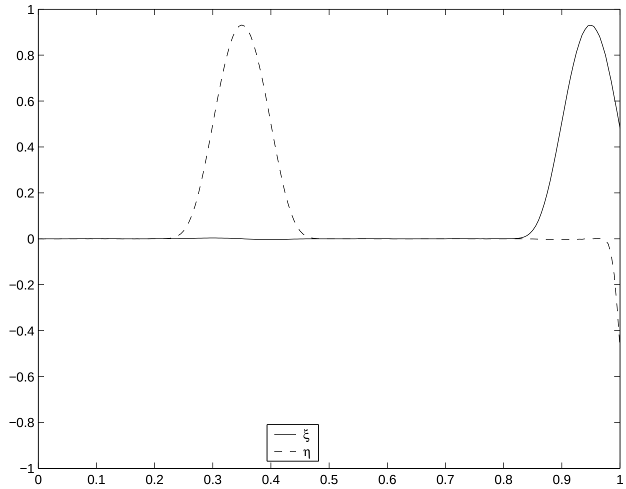

Figure 2.2: Values of ξ and η at $t = 0.15$.

As soon as the boundary $x = 0$ is reached (here at $t = 0.45$), the classical method is no longer satisfying and the boundary layer can not be computed with such a small number of points ($\varepsilon = 10^{-3}$, $M = 200$) : see Figure 3, displaying a zoom at $x = 0$. On the contrary, with the same values of the parameters, the method using the new function φ_0 gives very good results, even if $h > \varepsilon$ (Figure 4).

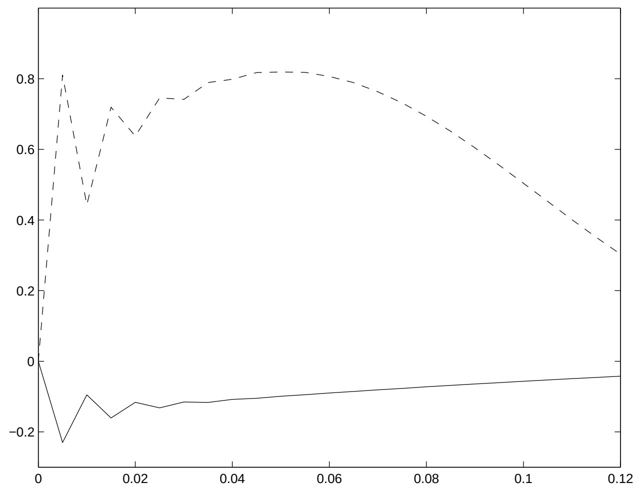

Figure 2.3: Oscillating solution without numerical corrector, with $\varepsilon = 10^{-3}$, $t = 0.45$.

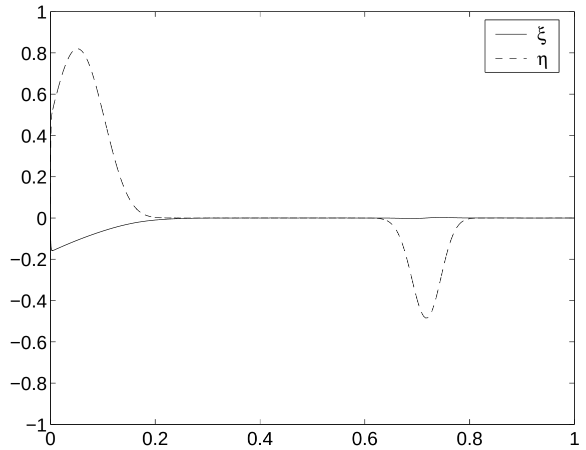

Figure 2.4: Nonoscillating Solution with numerical corrector (same ε and t as in Fig 2.3).

We now want to show the plots that confirm the convergence of (2.3),(2.6) to (2.8),(2.13). To this end we compare the two systems at $t = 0.5$ for different values of the parameter ε (α and β are unchanged). Figure 5 presents the numerical solution of the limit system ($\varepsilon = 0$). On can notice that the values of ξ^0 and η^0 satisfy the boundary conditions (2.13). Figures 6 and 7 respectively display the numerical solutions of (2.3) with $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-5}$. They illustrate the convergence indicated. Figure 8 displays a zoom of (ξ, η) at the boundary $x = 0$ for $\varepsilon = 10^{-5}$, in order to see the boundary layer that has just appeared.

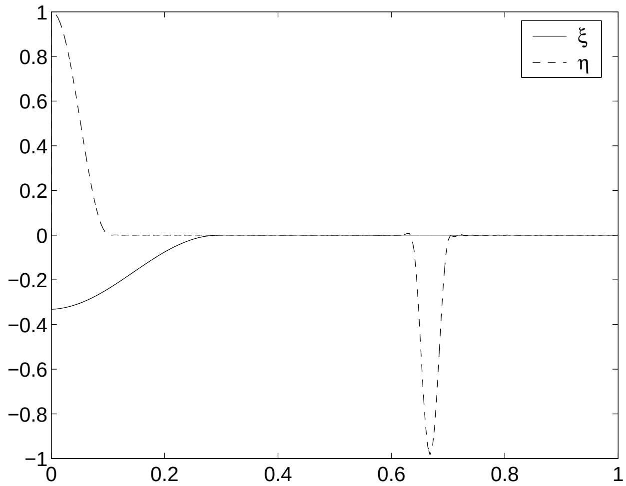

Figure 2.5: Limit Solution, $\varepsilon = 0$, $t = 0.5$.

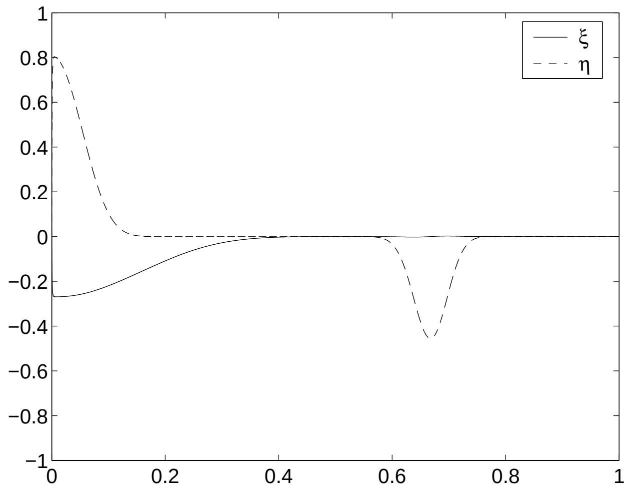

Figure 2.6: Solution with corrector, with $\varepsilon = 10^{-2}$, $t = 0.5$.

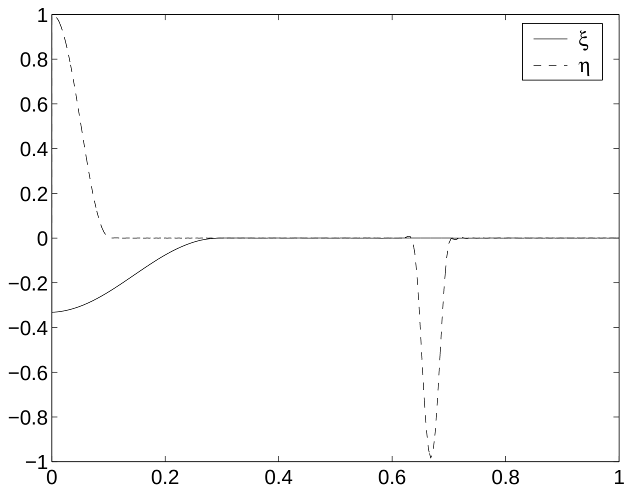

Figure 2.7: Solution with corrector, with $\varepsilon = 10^{-5}$, $t = 0.5$.

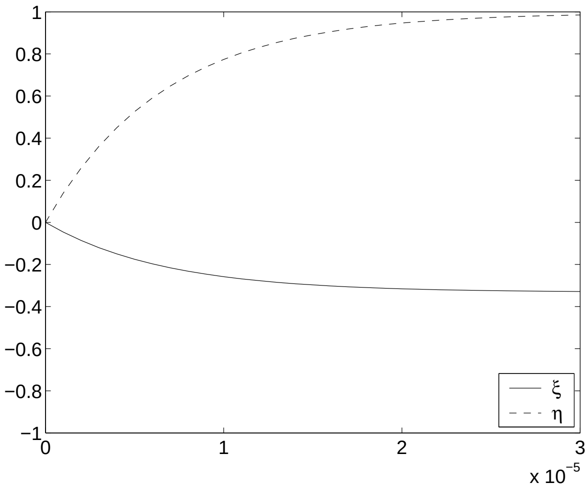

Figure 2.8: Zoom at $x = 0$ of Fig. 2.7.

To finish with this section, we show two plots with nonhomogeneous boundary conditions. The two following figures show the numerical solutions of the system for $\xi_{gl} = -0.05, \eta_{gl} = 0.1$, and $u_{gr} = 0.1$. Figure 9 is the initial data, and Figure 10 plots the solutions at $t = 0.65$, with $\varepsilon = 10^{-6}$. Some other tests, with boundary conditions depending on time have also been done, but since the difficulty is not here we chose not to show them.

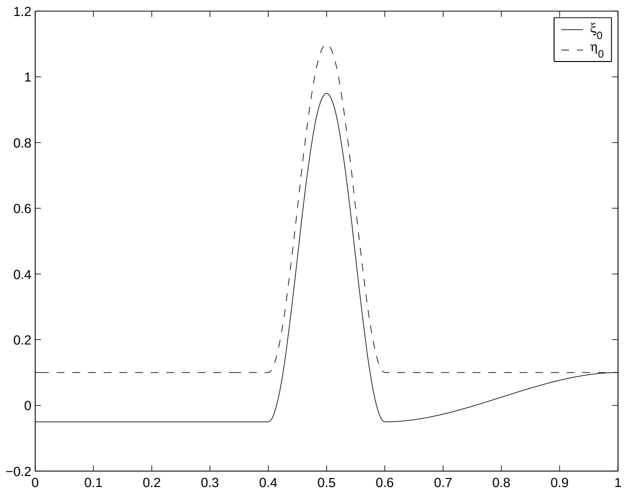

Figure 2.9: Initial data for Nonhomogeneous Dirichlet Boundary Conditions.

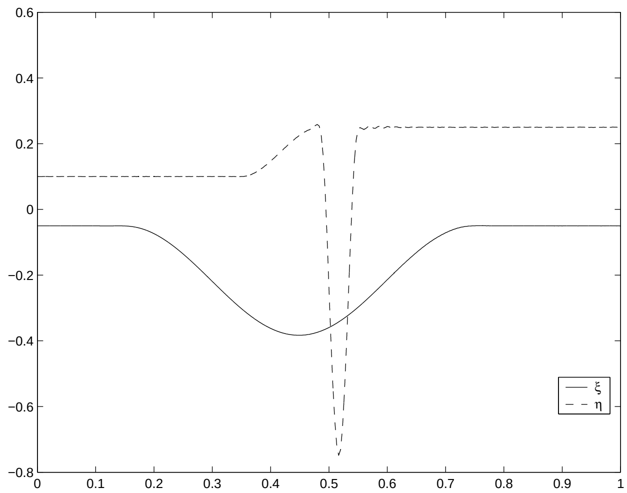

Figure 2.10: Solution with corrector, $\varepsilon = 10^{-6}$, $t = 0.65$ (nonhomogeneous boundary conditions).

3 Transparent boundary conditions

The previous numerical results show that the limit system is not what one could believe at first. Despite the mathematical interest of this study, the reflections that appeared at the boundaries may not be physically relevant, and this is why we consider hereafter a set of transparent boundary conditions (TBC) removing these reflections. They enjoy the following two important properties :

- for $\varepsilon > 0$ there is no reflection at the boundaries.
- the boundary conditions for the limit system ($\varepsilon \to 0$) are the natural conditions (2.10), identical to (2.9).

The aim of the following is to provide numerical evidence of these properties of the TBC. The reader is referred to [RTT05] for an alternate set of transparent conditions with full mathematical proofs of existence and uniqueness of solutions and convergence as $\varepsilon \to 0$.

3.1 The boundary conditions

We are now interested in studying the same system (2.3), but with a set of boundary conditions that would avoid the boundary layers and reflections which appeared in the previous section. For system (2.3), we will use the following transparent boundary conditions which are consistant with (2.8) :

$$\begin{cases} \xi(0,t) = \xi_{gl}(t), \\ \eta(L,t) = \eta_{gr}(t), \\ \eta_t(0,t) - \beta\,(\xi_x + \eta_x)(0,t) = f_\eta(0,t) - \dfrac{\beta}{\alpha} f_\xi(0,t), \\ \xi_t(L,t) + \alpha\,(\xi_x + \eta_x)(L,t) = f_\xi(L,t) - \dfrac{\alpha}{\beta} f_\eta(L,t), \end{cases} \qquad t > 0, \tag{3.1}$$

where ξ_{gl} and η_{gr} are given.

The system consisting of (2.3) and (3.1) is obviously equivalent to (2.1) supplemented with the following boundary conditions :

$$\begin{cases} u(0,t) + \lambda\,\phi(0,t) = \xi_{gl}(t), \\ u(L,t) - \lambda\,\phi(L,t) = \eta_{gr}(t), \\ u_t(0,t) - \lambda\,\phi_t(0,t) - 2\beta\,u_x(0,t) = f_\eta(0,t) - \dfrac{\beta}{\alpha} f_\xi(0,t), \\ u_t(L,t) + \lambda\,\phi_t(L,t) + 2\alpha\,u_x(L,t) = f_\xi(L,t) - \dfrac{\alpha}{\beta} f_\eta(L,t), \end{cases} \qquad t > 0. \tag{3.2}$$

Here, even when the parameter ε tends to zero, there is no boundary layer for the functions themselves, neither at $x = 0$ nor at $x = L$. On the contrary to the previous section, we do not need to improve the numerical algorithm because of the absence of boundary layer. Therefore we can choose h greater than ε with a classical finite element method, and the motion looks very regular, even close to the boundary.

We first check that the proposed TBC lead to a decreasing energy for the system, and then we show numerically the convergence of (2.1),(3.2) to (2.7),(2.10) - or equivalently of (2.3),(3.1) to (2.8),(2.9) - when ε goes to zero.

3.2 Non-increasing energy for the system

As we already did in Section 2.2 for system (2.3) with boundary conditions (2.6), we want to prove that the energy of the system (2.1),(3.2) does not increase with time in the absence of forcing, that is :

$$\xi_{gl} = \eta_{gr} = 0. \tag{3.3}$$

The system (2.1), with the boundary conditions (3.2), or equivalently (2.3) with (3.1), and (3.3), has a time-decreasing energy.

As in (2.11), let us define an energy for the system (2.1) :

$$E(t) \quad = \quad \int_0^L \{u^2(x,t) + \lambda^2 \phi^2(x,t)\}\, dx, \tag{3.4}$$

and let us evaluate the derivative :

$$\begin{aligned}
\frac{d}{dt}E(t) &= 2\int_0^L \{u_t(x,t)\,u(x,t) + \lambda^2\phi_t(x,t)\,\phi(x,t)\}\,dx \\
&= 2\int_0^L \{u(x,t)\,(-\phi_x + 2\varepsilon\,u_{xx} - U_0\,u_x)(x,t) + \phi(x,t)\,(-\lambda^2 U_0\,\phi_x - u_x)(x,t)\}\,dx \\
&= -2\big[u(x,t)\,\phi(x,t)\big]_{x=0}^{x=L} + 4\varepsilon\int_0^L u(x,t)u_{xx}(x,t)\,dx \\
&\quad + U_0(u^2(0,t) + \lambda\,\phi^2(0,t) - u^2(L,t) - \lambda\,\phi^2(L,t))
\end{aligned}$$

Using the boundary conditions (3.2a),(3.2c), and recalling that $\alpha = U_0 + 1/\lambda$, $\beta = -U_0 + 1/\lambda$, we obtain :

$$\begin{aligned}
\frac{d}{dt}E(t) &= -2\,\alpha\,u^2(L,t) - 2\,\beta\,u^2(0,T) - 4\varepsilon\int_0^L u_x^2(x,t)\,dx \\
&\quad + 4\varepsilon\,\big[u(x,t)u_x(x,t)\big]_{x=0}^{x=L}
\end{aligned} \tag{3.5}$$

Taking into account (3.2b),(3.2d), we evaluate the following term :

$$\begin{aligned}
4\varepsilon\,\big[u(x,t)u_x(x,t)\big]_{x=0}^{x=L} &= -\frac{2\varepsilon}{\alpha}\,u(L,t)\,(u_t(L,t) + \lambda\,\phi_t(L,t)) \\
&\quad -\frac{2\varepsilon}{\beta}\,u(0,t)\,(u_t(0,t) - \lambda\,\phi_t(L,t)) \\
&= -2\varepsilon\frac{d}{dt}\Big(\frac{u^2(L,t)}{\alpha} + \frac{u^2(0,t)}{\beta}\Big)
\end{aligned} \tag{3.6}$$

From (3.5),(3.6), and since $\alpha, \beta > 0$, we find :

$$\frac{d}{dt}\{E(t) + \frac{2\varepsilon}{\alpha}\,u^2(L,t) + \frac{2\varepsilon}{\beta}\,u^2(0,t)\} \leq 0. \tag{3.7}$$

Hence,

$$\begin{aligned}
E(t) + \frac{2\varepsilon}{\alpha}\,u^2(L,t) + \frac{2\varepsilon}{\beta}\,u^2(0,t) &\leq E(0) + \frac{2\varepsilon}{\alpha}\,u^2(L,0) + \frac{2\varepsilon}{\beta}\,u^2(0,0) \\
&= E(0)
\end{aligned}$$

Finally :

$$E(t) \leq E(0).$$

This achieves the proof of non-increasing energy for the system (2.1) with boundary conditions (3.2).

Remark 3.1. *:* We could have obtained a similar result in the (ξ, η) variables, defining $E(t) = \int_0^L \{\xi^2(x,t) + \eta^2(x,t)\}\,dx$, which is identical to (3.4).

Remark 3.2. *:* In the presence of forcing, i.e. if $F_u, F_\phi, \xi_{gl}, \eta_{gr}$ do not vanish, we proceed as in Remark 2.2.

3.3 Numerical Scheme

As there is no boundary layer to compute, we can implement the classical finite element method. We do not need to use φ_0 anymore, but we will replace it with the classical φ_0 (see below) as the functions do not verify $\xi(0,t) = \eta(0,t) = 0$ anymore. Then, as we already did in Section 2, we extend the boundary data inside the interval $(0, L)$ in the form $\xi_g(x,t), \eta_g(x,t)$ with :

$$\left\{ \begin{array}{l} \xi_g(0,t) = \xi_{gl}(t), \\ \eta_g(L,t) = \eta_{gr}(t), \end{array} \right. \qquad t > 0. \tag{3.8}$$

Hence we will look for an approximate function :

$$\begin{pmatrix} \xi_M \\ \eta_M \end{pmatrix} \in \begin{pmatrix} \xi_g \\ \eta_g \end{pmatrix} + V_M, \tag{3.9}$$

with :

$$V_M = Span\{ \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}, \cdots, \begin{pmatrix} \varphi_M \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_M \end{pmatrix} \}, \tag{3.10}$$

where φ_j, $1 \le j \le M$ are as in (2.20),(2.21), and :

$$\varphi_0(x) = \left\{ \begin{array}{ll} \dfrac{x_1 - x}{h} & \text{for } 0 \le x \le x_1, \\ 0 & \text{elsewhere.} \end{array} \right. \tag{3.11}$$

The approximate solution (ξ_M, η_M) satisfies :

$$\left\{ \begin{array}{lll} \xi_M(x,t) & = & \xi_g(t) + \sum_{j=0}^{M} a_j(t)\, \varphi_j(x), \\ \eta_M(x,t) & = & \eta_g(t) + \sum_{j=0}^{M} b_j(t)\, \varphi_j(x). \end{array} \right. \tag{3.12}$$

One more time, there is a linear differential system to solve to determine the coefficients (a_j, b_j), which is then solved by time discretization. This is all classical and no more details are needed.

3.4 Numerical results

Again, we will show some plots with the variables (ξ, η). Also, as we did in Section 2, the results are mainly presented with homogeneous boundary conditions. An example of nonhomogeneous case is given in the last figures.

The aim of the following is to show that the system (2.3), with the boundary conditions (3.1), converges to the equations (2.8),(2.9) when ε goes to zero.

The decoupled system (2.8),(2.9) consists of two transport equations and the exact solutions are the following functions :

$$\begin{cases} \xi^0(x,t) = \xi_0(x - \alpha\, t), \\ \eta^0(x,t) = \eta_0(x + \beta\, t). \end{cases} \tag{3.13}$$

The following plots illustrate the convergence announced. Figure 11 represents the initial functions $\xi_0 = \eta_0$, same as those in Figure 1. Figures 12, 13 and 14 represent ξ and η at time $t = 0.15$, for different values of ε (respectively 10^{-3}, 10^{-4} and 10^{-5}). For these 3 plots, we can notice that there is no reflection at $x = L$. Figure 14 really looks like the exact solution (3.13), at the precision of the figure.

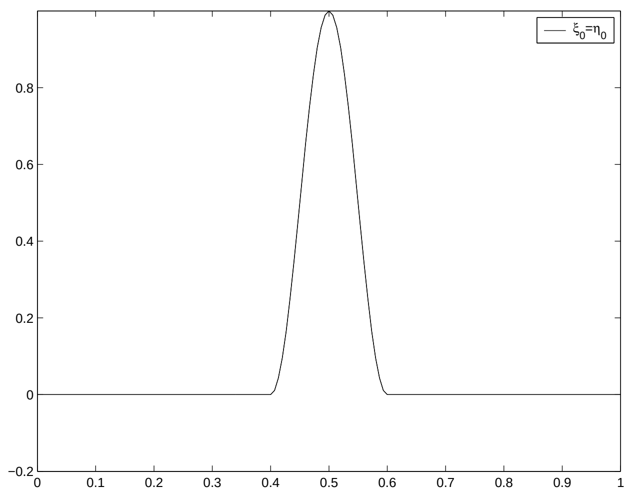

Figure 2.11: Transparent Boundary Condition. Initial data ξ_0, η_0.

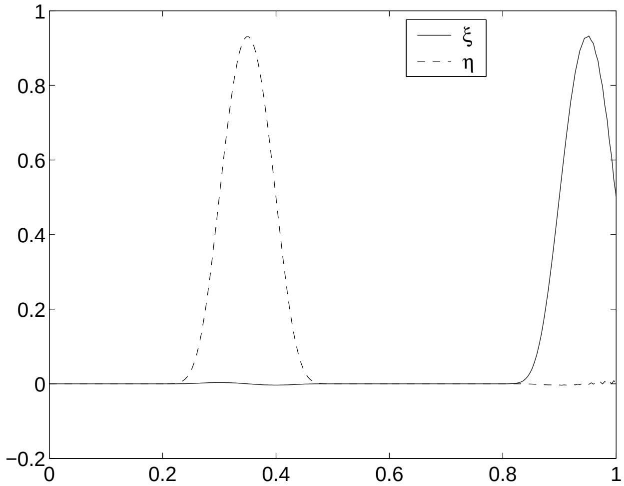

Figure 2.12: Solution with $\varepsilon = 10^{-3}$, $t = 0.15$.

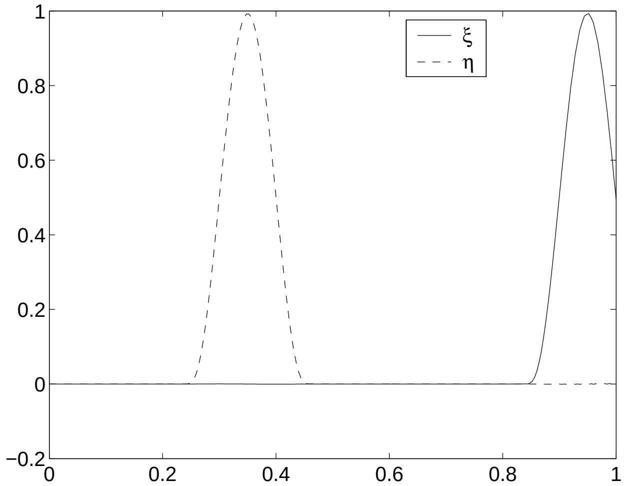

Figure 2.13: Solution with $\varepsilon = 10^{-4}$, $t = 0.15$.

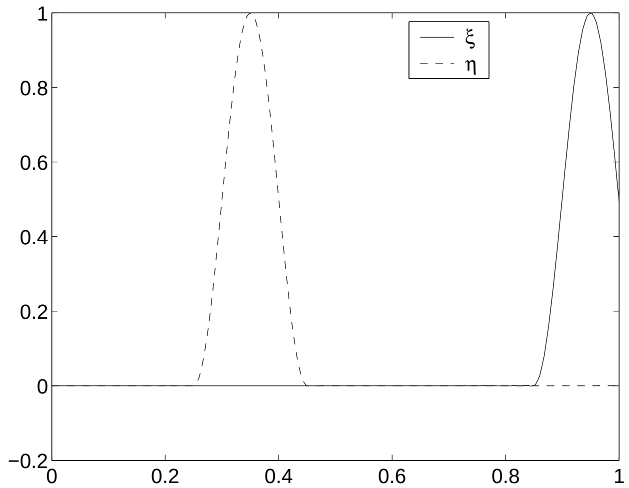

Figure 2.14: Solution with $\varepsilon = 10^{-5}$, $t = 0.15$.

In Figure 15, we also set $\varepsilon = 10^{-5}$, but $t = 0.45$; ξ now equals to zero everywhere, and η is getting out through the boundary $x = 0$. We can notice that there is naturally no boundary layer here. Finally, Figure 16 shows the numerical solutions after a time (here $t = 1$) such that both of them equal zero.

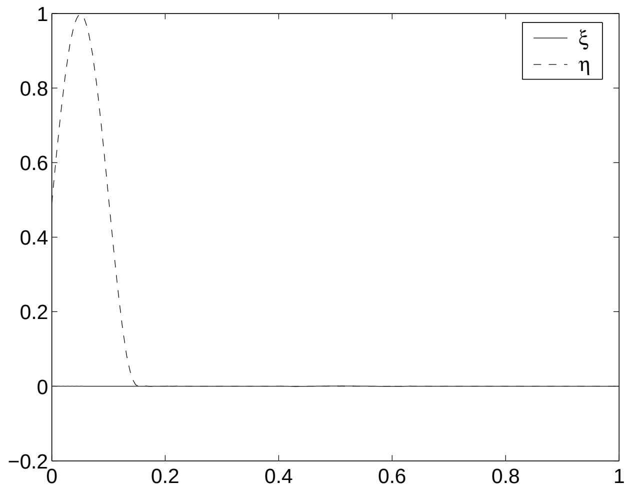

Figure 2.15: Solution with $\varepsilon = 10^{-5}$, $t = 0.45$. The right wave has crossed the right boundary.

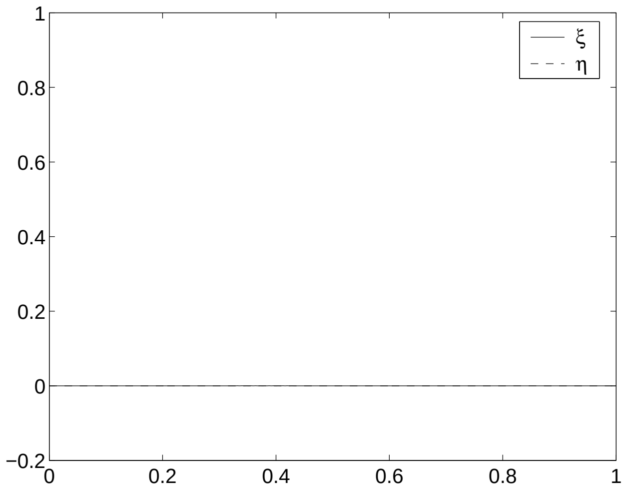

Figure 2.16: Solution with $\varepsilon = 10^{-5}$, $t = 1.0$. Both waves have left the domain.

As we did for the Dirichlet boundary conditions, we end this section with a nonhomogeneous case, namely with $\xi_{gl} = 0.05, \eta_{gr} = -.1$. Figure 17 shows the initial condition, whereas Figure 18 gives the solution at time $t = 0.3$.

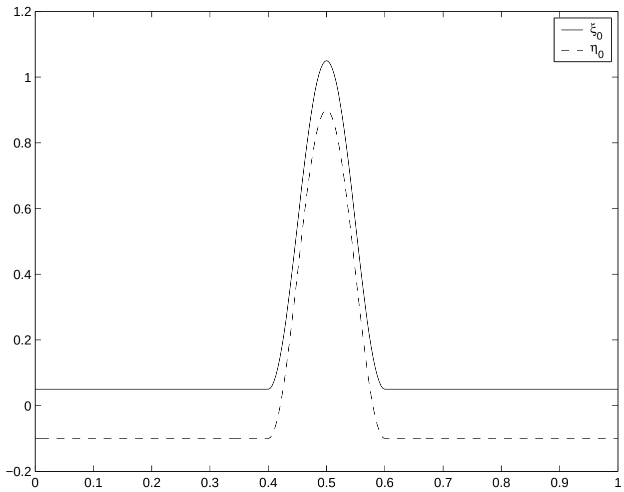

Figure 2.17: Initial data for Nonhomogeneous Transparent Boundary Conditions.

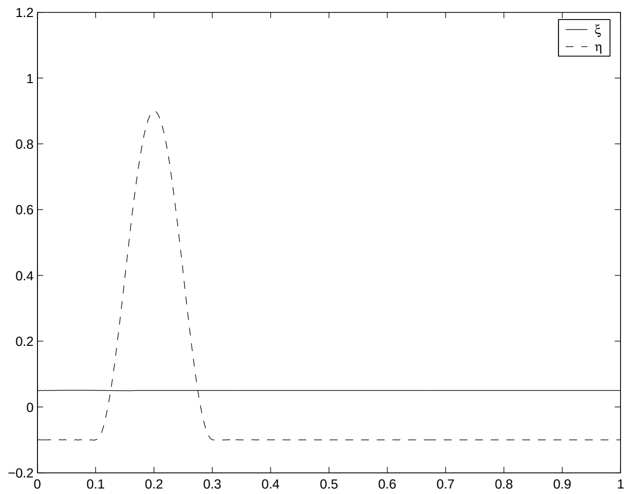

Figure 2.18: Solution with $\varepsilon = 10^{-5}$, $t = 0.3$ (nonhomogeneous boundary conditions).

4 Conclusion

In this article, a linear differential system consisting of two coupled scalar evolution equations in space dimension one was considered ; this system was derived from a modal analysis of the Primitive Equations of the ocean. We have shown numerically that, by adjunction of a small viscosity, the system converges to some unusual, unexpected limit system thus producing boundary layers and reflections of waves at the boundary. We have also proposed an alternate set of boundary conditions of transparent type for the viscous systems and, in this case, the viscous system does not produce boundary layers nor reflections of waves at the boundary.
For the Primitive Equations of the atmosphere we learned from this study, that a mild friction term can produce unexpected spurious waves in the case of a limited domain. We leave to subsequent studies the choice between accepting such undesirable waves and coping with them, versus the implementation of more involved schemes.

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Chapitre 3

Etude analytique de l'influence du petit paramètre dans les δ-EPs : conditions aux limites

Boundary conditions for the δ-PEs

Ce chapitre est constitué de l'article **Boundary Conditions for an Ocean Related System with a Small Parameter**, à paraître en 2005 dans *"Nonlinear PDEs and Related Analysis", Contemporary Mathematics, AMS.* Réalisé avec les mêmes auteurs que ceux du chapitre précédent, il est constitué de deux parties. La première partie confirme par des arguments analytiques rigoureux l'existence de solutions pour les équations considérées dans le chapitre 2, ainsi que la convergence lorsque δ tend vers 0 de la solution vers une solution limite dont on donne la forme explicite.
Dans la seconde partie de ce chapitre, un nouveau jeu de conditions aux limites, de type non réfléchissantes, est étudié. Là encore, des preuves d'existence et d'unicité de solutions à $\delta > 0$ fixé et de convergence lorsque δ tend vers 0 sont données.

"Nonlinear PDEs and Related Analysis", Contemporary Mathematics, AMS, 2005, to appear.

Boundary Conditions for an Ocean Related System with a Small Parameter.

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Abstract

A linear system derived from the Primitive Equations (PEs) of the atmosphere and the ocean is considered. Existence and uniqueness of solutions, behavior as a small viscosity parameter tends to zero are studied for different boundary conditions ; one of the boundary condition is of Dirichlet type and it produces reflections of waves at the boundary, the other one is of transparent type. Computational issues are addressed in a companion paper [RTT04], which also contains more details on the derivation of the system that we study here.

1 Introduction

The Primitive Equations of the ocean and the atmosphere are fundamental equations of geophysical fluid mechanics ([Ped87],[WP86],[Sal98]). In the presence of viscosity, it has been shown, in various contexts, that these equations are well-posed (see e.g. [LTW92a],[LTW92b], and the review article [TZ04]).
In the absence of viscosity, it is known that the PEs are not well-posed for any set of boundary conditions of local type (see [OS78] and also [TT03]). This difficulty is analyzed in [TT03] using a modal analysis in the vertical direction. To overcome this difficulty a modification of the PEs was proposed in [TT03], for which a set of local boundary conditions also given in this reference produces a decay of energy (in the absence of forcing).
The modified PEs introduced in [TT03] contain an added friction term δw in the hydrostatic equation, so that the model is actually nonhydrostatic; here we call this system the δ-PEs equations. Partial results of well-posedness of the δ-PEs were proven and full well-posedness is conjectured at least in dimension two; these questions will be addressed elsewhere.
In a companion article [RTT04], we showed how to derive the system (1.1) below from the δ-PEs after linearization of the δ-PEs equations around a stratified flow with a constant velocity $\overline{U}_0$

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in the (west-east) direction Ox and after performing a modal decomposition of the dependant unknowns in the vertical direction. Dropping the indices n, we obtain for the n-th mode u_n, ψ_n,

$$\begin{cases} u_t + \overline{U}_0 u_x + \psi_x - \dfrac{\delta g}{N^2 \lambda^2} u_{xx} = 0, \\ \psi_t + \overline{U}_0 \psi_x + \dfrac{1}{\lambda^2} u_x = 0, \end{cases} \tag{1.1}$$

where (u, v) is the horizontal velocity, and ψ is proportional to the temperature. The equation for v is omitted here because it does not raise any additional difficulty; the coupling between u and v has been suppressed by setting the Coriolis parameter equal to 0.

Here $\lambda = \lambda_n > 0$ is the n-th eigenvalue in the vertical decomposition. We only consider the subcritical case where $\overline{U}_0 < 1/\lambda$, the supercritical case being easier to solve; $g > 0$ is the gravity constant and $N > 0$ is a parameter related to the background stratified state. Solutions independent of the y (south-north) variable are considered, and the Coriolis term which is not significant here is neglected so that the equation for the second component v of the velocity is decoupled from the system (1.1). The system (1.1) is supplemented with the physically natural boundary conditions described below in (2.2) which are of Dirichlet type.

The primary motivation of the article [RTT04] and the present one is to study the effect on such a mode of the added viscosity $\delta > 0$. Although the limit of system (1.1) is very simple as $\delta \longrightarrow 0$ (in particular after diagonalization), the numerical simulations reported in [RTT04] surprisingly showed that the boundary conditions of the limit system are not those expected. Our aim in Section 2 is to study the well-posedness of the perturbed problem and of the limit problem as δ (or $\varepsilon = \delta g / 2 N^2 \lambda^2$) goes to zero; and the convergence of the solution with an explicit derivation of the *correctors* for the corresponding boundary layer.

A drawback of the system (1.1) in the physical context it was introduced, is that it produces a non physical reflection of waves at the boundary (see below and the discussion in [RTT04]). Hence in Section 3 we propose another set of boundary conditions for (1.1), inspired by the concept of transparent boundary conditions (TBC) and which does not produce reflection at the boundary, see (3.9)-(3.10). We prove that the problem is also well-posed, we provide a proof of convergence as δ (or $\varepsilon = \delta g / 2 N^2 \lambda^2$) goes to zero to a natural limit system made up with uncoupled scalar transport equations (see (3.3)).

Finally in Section 4 we propose another set of transparent boundary conditions for (1.1), less satisfactory from the theorical viewpoint, but numerically more efficient, in fact the system used in the numerical simulations of [RTT04] (see the details in Section 4).

From the mathematical viewpoint the system that we consider here is a linear incompletely parabolic perturbation of a hyperbolic system. The problem of finding an appropriate set of boundary conditions for such systems has been addressed by a number of authors, with the purpose of obtaining a well-posed initial boundary value problem, or with the purpose of finding nonreflecting boundary conditions or both. Early papers devoted in general to the (nonlinear) compressible Navier-Stokes equations or the Shallow Water equations are those of Lions and Raviart [LR66], Strikwerda [Str77], Gustafsson and Kreiss [GK79], Rudy and Strikwerda [RS80, RS81], Michelson [Mic85]. More recent articles are e.g. those of Halpern and Schatzman [HS89], Halpern [Hal91] and Tourette [Tou97]. The problem of finding nonreflective boundary conditions in a general context is addressed in the classical articles of Engquist and Majda [EM77, EM79]. Section 2 of our article differs from the references above in that the boundary conditions for the incompletely parabolic system are known and derived from physical considerations and from the study in [TT03]. On the contrary in Sections 3 and 4, motivated by computational preoccupations, we look for nonreflective boundary conditions. To the best of our knowledge the issue of reflective and nonreflective boundary conditions for the Primitive Equations of the ocean and the atmosphere have not been addressed in the past.

2 The systems with the Dirichlet boundary conditions

In this section we consider the system (1.1) and the limit system obtained by setting formally $\varepsilon = 0$. Both systems are supplemented with suitable boundary conditions. We show the well-posedness for each system and we then prove in Section 2.3 that when ε goes to zero, the solution of (2.1) converges to the solution of (2.27) in relevant spaces, and with the boundary conditions presented below.

We start by presenting the two systems and studying their well-posedness.

2.1 The ε-system

We are interested in studying the following system:

$$\begin{cases} u_t^\varepsilon + \overline{U}_0\, u_x^\varepsilon + \psi_x^\varepsilon - 2\,\varepsilon\, u_{xx}^\varepsilon = f, \\ \psi_t^\varepsilon + \overline{U}_0\, \psi_x^\varepsilon + \dfrac{1}{\lambda^2}\, u_x^\varepsilon = g. \end{cases} \quad 0 < x < L, \quad t > 0, \tag{2.1}$$

with the following boundary conditions and initial data:

$$\begin{cases} u^\varepsilon(0,t) = 0, \\ \psi^\varepsilon(0,t) = 0, \\ u^\varepsilon(L,t) = 0, \end{cases} \qquad t > 0. \tag{2.2}$$

$$\begin{cases} u^\varepsilon(x,0) = u_0(x), \\ \psi^\varepsilon(x,0) = \psi_0(x), \end{cases} \qquad 0 < x < L. \tag{2.3}$$

Remark 2.1. *:* For $\varepsilon > 0$, the boundary conditions (2.2) are derived from the boundary conditions for a limited domain as proposed in [TT03]; namely: prescription of the normal velocity everywhere on the boundary and of the horizontal velocity and temperature on the parts of the boundary where the flow is incoming. Taking into account that the flow under consideration is a perturbation of the flow $\overline{U}_0\, e_x$, we find the boundary conditions (2.2): indeed $u^\varepsilon(0,t)$ and $u^\varepsilon(L,t)$ are the normal components of the velocity at $x=0$ and L, and $\psi^\varepsilon(0,t)$ corresponds to prescribing the temperature at $x=0$ where the flow is incoming.

For more convenience we will now write (u,ψ) instead of $(u^\varepsilon,\psi^\varepsilon)$, so that (2.1) is the same as (1.1) with $\varepsilon = \delta\ g/2\,N^2\,\lambda^2$, the right-hand sides f,g have been added for mathematical generality.

We now want to set the functional framework for (2.1)-(2.3). In the context of the linear semi-group theory ([Yos80], [Bré73], [Hen81], [Lio65], [Paz83]) the natural spaces and operators are the following:

$$H = L^2(0,L)\times L^2(0,L),$$
$$D(A) = \{U=(u,\psi)\in H,\ u_x,\psi_x,\ \text{and}\ u_{xx}\in L^2(0,L),$$
$$u(0)=\psi(0)=u(L)=0\},$$
$$A\begin{pmatrix} u\\ \psi\end{pmatrix} = \begin{pmatrix}\overline{U}_0\,u_x+\psi_x-2\,\varepsilon\,u_{xx}\\ \overline{U}_0\,\psi_x+\dfrac{1}{\lambda^2}\,u_x\end{pmatrix},\quad \forall\begin{pmatrix} u\\ \psi\end{pmatrix}\in D(A).$$

Then (2.1)-(2.3) can be written as follows:

$$\frac{dU}{dt}+A\,U=F,\ \ U(0)=U_0, \tag{2.4}$$

where $U_0=(u_0,\psi_0)^T\in D(A)$, $U(t)=(u(t),\psi(t))^T\in D(A)$ for all time and $F=(f,g)^T\in H$.

To prove the well-posedness of this problem, we will use the Hille-Yosida theorem (see the references above):

Theorem 2.1. *Let H be a Hilbert space and let $A: D(A)\longrightarrow H$ be a linear unbounded operator, with domain $D(A)\subset H$. Assume the following:*
(i) $D(A)$ is dense in H and A is closed,
(ii) A is ≥ 0, i.e. $(AU,U)_H\geq 0,\quad \forall U\in D(A)$,
(iii) $\forall\mu>0$, $A+\mu I$ is onto.
Then $-A$ is infinitesimal generator of a strongly continuous semigroup of contractions $\{S(t)\}_{t\geq 0}$, and for every $U_0\in H$ and $F\in L^1(0,T;H)$, there exists a unique solution $U\in\mathcal{C}([0,T];H)$ of

$$U(t)=S(t)\,U_0+\int_0^t S(t-s)\,F(s)\,ds, \tag{2.5}$$

that is a weak form of (2.4).

If furthermore $\overline{U}_0 \in D(A)$ *and* $F' = dF/dt \in L^1(0,T;H)$ *then* U *satisfies* (2.4)

$$U \in \mathcal{C}([0,T];H) \cap L^\infty(0,T;D(A)), \ \frac{dU}{dt} \in L^\infty(0,T;H). \tag{2.6}$$

The spaces H, $D(A)$, and the operator A have been defined above. It is clear that H, supplemented with the inner product $(U,V)_H = \int_0^L (U_1\,V_1 + \lambda^2\,U_2\,V_2)\,dx$, is a Hilbert space. We will achieve the proof of well-posedness for the problem (2.4) within three steps, corresponding to the three main hypotheses of Theorem 2.1.

We start with (i):

Lemma 2.1. *$D(A)$ is dense in H and A is a closed operator.*

Proof : We write $D(A) = \{(u,\psi) \in (H^2 \cap H_0^1) \times H_l^1\}$, where $H_l^1 = \{\psi \in H^1(0,L), \psi(0) = 0\}$.

To prove that $D(A)$ is dense in H, it is sufficient to observe that $\mathcal{C}_0^\infty(0,L)$ is dense in $L^2(0,L)$ since $\mathcal{C}_0^\infty(0,L) \times \mathcal{C}_0^\infty(0,L) \subset D(A)$.

To show that A is closed, we consider a sequence $(u_n,\psi_n) = U_n$ of $D(A)$, such that:

$$U_n \longrightarrow U \text{ in } H, \tag{2.7}$$

$$A\,U_n = F_n \longrightarrow F \text{ in } H, \tag{2.8}$$

and we want to verify that $U = (u,\psi) \in D(A)$ and $F = A\,U$, so that the graph of A is closed.

By the definition of A

$$\left\{ \begin{array}{llllll} \overline{U}_0 \dfrac{\partial u_n}{\partial x} & + & \dfrac{\partial \psi_n}{\partial x} & - & 2\,\varepsilon\,\dfrac{\partial^2 u_n}{\partial x^2} & = f_n, \\ \overline{U}_0 \dfrac{\partial \psi_n}{\partial x} & + & \dfrac{1}{\lambda^2}\dfrac{\partial u_n}{\partial x} & & & = g_n. \end{array} \right. \tag{2.9}$$

Hence, passing to the limit in the distribution sense we find

$$\left\{ \begin{array}{llllll} \overline{U}_0 \dfrac{\partial u}{\partial x} & + & \dfrac{\partial \psi}{\partial x} & - & 2\,\varepsilon\,\dfrac{\partial^2 u}{\partial x^2} & = f, \\ \overline{U}_0 \dfrac{\partial \psi}{\partial x} & + & \dfrac{1}{\lambda^2}\dfrac{\partial u}{\partial x} & & & = g. \end{array} \right. \tag{2.10}$$

By combination of (2.9a) and (2.9b), we find:

$$(\overline{U}_0^2 - \frac{1}{\lambda^2})\,\frac{\partial u_n}{\partial x} - 2\,\varepsilon\,\overline{U}_0\,\frac{\partial^2 u_n}{\partial x^2} = \overline{U}_0\,f_n - g_n. \tag{2.11}$$

We integrate on $(0,x)$, and obtain:

$$\frac{\partial u_n}{\partial x}(x) = \frac{\partial u_n}{\partial x}(0) + \Phi(g_n - \overline{U}_0\, f_n)(x), \tag{2.12}$$

where:

$$\Phi(\varphi)(x) = \frac{1}{2\,\varepsilon\,\overline{U}_0}\int_0^x \varphi(x')\, e^{-(1/2\,\varepsilon\,\overline{U}_0\,\lambda^2 - \overline{U}_0/\,2\,\varepsilon)(x-x')}\, dx'. \tag{2.13}$$

From the latter expression, we find that $\Phi : L^2 \longrightarrow H^1$ is a linear continuous mapping.
Thanks to (2.8), it is clear that $\Phi(g_n - \overline{U}_0\, f_n)$ is bounded in $H^1(0,L)$. Differentiating (2.12), we then conclude that $\partial^2 u_n/\partial x^2$ is bounded in L^2. Hence u_n is bounded in $H^2 \cap H^1_0$, and by (2.7), u is in $H^2(0,L) \cap H^1_0(0,L)$.
Similarly (2.9b) shows that the sequence ψ_n is bounded in H^1_l and then by (2.7) ψ is also in H^1_l.
In conclusion $U = (u,\psi) \in D(A)$ and equations (2.10) show that $A\,U = F$, thus concluding the proof of Lemma 2.1. □
To continue with the proof of well-posedness, we want to show that the operator A is positive, that is:

Lemma 2.2. *For every $U \in D(A), (A\,U,U)_H \geq 0$.*

Proof : Let us compute the inner product, with any $U \in D(A)$:

$$\begin{aligned}(A\,U,U)_H &= \int_0^L \{(\overline{U}_0\, u_x + \psi_x - 2\varepsilon\, u_{xx})\, u + (\overline{U}_0\, \lambda^2\, \psi_x + u_x)\, \psi\}\, dx \\ &= \frac{1}{2}\overline{U}_0\, (u^2(L) - u^2(0) + \lambda^2\, \psi^2(L) - \lambda^2 \psi^2(0)) \\ &\quad -2\,\varepsilon\, [u_x\, u]_{x=0}^{x=L} + 2\,\varepsilon \int_0^L u_x^2\, dx.\end{aligned}$$

Thanks to the boundary conditions (2.2), we find:

$$(A\,U,U)_H = \frac{1}{2}\overline{U}_0\, \lambda^2\, \psi^2(L) + 2\,\varepsilon \int_0^L u_x^2\, dx \geq 0. \tag{2.14}$$

This shows that A is a positive operator, and ends the proof of Lemma 2.2 □

Remark 2.2. *:* One can derive from Lemma 2.2 a result of decrease of the energy of the system. If one defines :

$$E(t) = \int_0^L (u^2(x,t) + \lambda^2 \psi^2(x,t))\, dx, \tag{2.15}$$

then it is easy to show, following the proof of Lemma 2.2, that if $F = 0$, any solution of (2.4) in $L^1(0,T;D(A))$ satisfies:

$$\frac{d}{dt} E(t) \leq 0, \quad 0 \leq t \leq T. \tag{2.16}$$

We achieve the proof of well-posedness for (2.4) with the following lemma:

Lemma 2.3. *The operator $A + \mu I$ is onto, $\forall \mu > 0$.*

Proof : Let μ be a positive real number, let $(f, g) \in H$; we look for $(u, \psi) \in D(A)$ such that:

$$\left\{\begin{array}{lllllll} \overline{U}_0\, u_x & + & \psi_x & - & 2\,\varepsilon\, u_{xx} & + & \mu\, u = f, \\ \overline{U}_0\, \psi_x & + & \frac{1}{\lambda^2}\, u_x & & & + & \mu\, \psi = g. \end{array}\right. \tag{2.17}$$

with:

$$\left\{\begin{array}{l} u(0) = u(L) = 0, \\ \psi(0) = 0. \end{array}\right. \tag{2.18}$$

From (2.17b), we find

$$\psi_x + \frac{\mu}{\overline{U}_0}\, \psi = \frac{g}{\overline{U}_0} - \frac{1}{\overline{U}_0\, \lambda^2}\, u_x,$$

and integrating this equation we obtain:

$$\psi = \Psi(g) - \frac{1}{\lambda^2}\, \Psi(u_x), \tag{2.19}$$

where

$$\Psi(\varphi) = \frac{1}{\overline{U}_0} \int_0^x \varphi(x')\, e^{-\mu\,(x-x')/\overline{U}_0}\, dx'. \tag{2.20}$$

The mapping Ψ is similar to the mapping Φ defined in (2.13), and only some constants differ. In particular $u \in H^1 \mapsto \Psi(u_x) \in H^1$ is continuous, and $\psi(g)$ is in H^1.

Thanks to (2.19) and (2.20), equations (2.17) and (2.18) are equivalent to finding $u \in H_0^1(0, L)$ such that

$$-2\,\varepsilon\, u_{xx} + \overline{U}_0\, u_x - \frac{1}{\lambda^2}\, \frac{d}{dx}\, \Psi(u_x) + \mu\, u = f - \frac{d}{dx} \Psi(g). \tag{2.21}$$

The variational formulation of (2.21) reads:
To find $u \in H_0^1(0, L)$, such that

$$\begin{aligned} 2\,\varepsilon\,(u_x, \tilde{u}_x) + \overline{U}_0\,(u_x, \tilde{u}) + \frac{1}{\lambda^2}\,(\Psi(u_x), \tilde{u}_x) + \mu\,(u, \tilde{u}) \\ = (f - \frac{d}{dx}\, \Psi(g), \tilde{u}), \quad \forall \tilde{u} \in H_0^1(0, L). \end{aligned}$$

We solve this problem using the Lax-Milgram theorem; the main point is to prove the coercivity on $H_0^1(0, L)$ of the bilinear form

$$a(u, \tilde{u}) = 2\,\varepsilon\,(u_x, \tilde{u}_x) + \overline{U}_0\,(u_x, \tilde{u}) + \frac{1}{\lambda^2}\,(\Psi(u_x), \tilde{u}_x) + \mu\,(u, \tilde{u}), \tag{2.22}$$

and the continuity of

$$l(\tilde{u}) = (f - \frac{d}{dx}\, \Psi(g), \tilde{u}). \tag{2.23}$$

The latter is easy, observing that $f - d\Psi(g)/dx \in L^2$. Regarding the coercivity of a, it is sufficient to prove it for $u \in \mathcal{C}_0^\infty(0,L)$ since this space is dense in $H_0^1(0,L)$. For such a u:

$$a(u,u) = 2\,\varepsilon\,|u_x|_{L^2}^2 + \frac{1}{\lambda^2}\,(\Psi(u_x), u_x) + \mu\,|u|_{L^2}^2. \tag{2.24}$$

From (2.20), we know that $\Psi(u_x) = \Psi_0$ is solution of:

$$\begin{cases} \overline{U}_0\,\dfrac{\partial \Psi_0}{\partial x} + \mu\,\Psi_0 = u_x, \\ \Psi_0(0) = 0 \end{cases} \tag{2.25}$$

Hence:

$$\begin{aligned} (\Psi_0, u_x) &= (\Psi_0, \overline{U}_0\,\frac{\partial \Psi_0}{\partial x} + \mu\,\Psi_0) \\ &= \mu\,|\Psi_0|_{L^2}^2 + \frac{1}{2}\,\overline{U}_0\,(\Psi_0^2(L) - \Psi_0^2(0)) \\ &= \mu\,|\Psi_0|_{L^2}^2 + \frac{1}{2}\,\overline{U}_0\,\Psi_0^2(L) \end{aligned}$$

Finally, inserting this in (2.24), we find:

$$a(u,u) \geq 2\,\varepsilon\,|u_x|_{L^2}^2 + \mu\,|u|_{L^2}^2, \tag{2.26}$$

and the coercivity of a is proven. Hence, thanks to Lax-Milgram theorem, there exists a unique solution $u \in H_0^1$ of (2.21). Thanks to (2.19), $\psi \in H^1$, and $\psi(0) = 0$. Lastly, we obtain from (2.17a) that $u \in H^2 \cap H_0^1$. The lemma is proven. □
All hypotheses have been proved and we thus infer from Theorem 2.1 the following result:

Theorem 2.2. *Problem (2.1)-(2.3) is well-posed, that is for every* $(f,g) \in L^1(0,T;H)$, *and every* $(u_0,\psi_0) \in D(A)$ [1], *there exists a unique solution* $U = (u,\psi)$ *of (2.1)-(2.3) satisfying* (2.6).

Remark 2.3. *:* We have proven that $-A$ is generator of a strongly continuous semigroup, which is sufficient for Theorem 2.2 (and Theorem 2.3 below). However $-A$ is not generator of an analytic semi-group and this would produce additional difficulties in the nonlinear case.

[1] And e.g. (f,g) is continuous in H at $t = 0$; see footnote 1

2.2 The limit system ($\varepsilon = 0$)

If we formally set $\varepsilon = 0$ in (2.1), we obtain the following limit system:

$$\begin{cases} u_t^0 + \overline{U}_0 \, u_x^0 + \psi_x^0 = f, \\ \psi_t^0 + \overline{U}_0 \, \psi_x^0 + \dfrac{1}{\lambda^2} u_x^0 = g, \end{cases} \quad 0 < x < L, \quad t > 0. \tag{2.27}$$

What was conjectured in [RTT04], based on the results of the numerical simulations, is that the boundary conditions for the limit system should be

$$\begin{cases} u^0(0,t) + \overline{U}_0 \, \lambda^2 \, \psi^0(0,t) = 0, \\ u^0(L,t) = 0. \end{cases} \quad t > 0. \tag{2.28}$$

Of course, we impose the same initial data to (u^0, ψ^0):

$$\begin{cases} u^0(x,0) = u_0(x), \\ \psi^0(x,0) = \psi_0(x). \end{cases} \quad 0 < x < L. \tag{2.29}$$

We consider the change of variables which diagonalize (2.27), and we set

$$\xi^0 = u^0 + \lambda \, \psi^0, \eta^0 = u^0 - \lambda \, \psi^0. \tag{2.30}$$

In terms of ξ^0, η^0, equations (2.27)-(2.29) become

$$\begin{cases} \xi_t^0 + \alpha \, \xi_x^0 = f + \lambda \, g, \\ \eta_t^0 - \beta \, \eta_x^0 = f - \lambda \, g, \end{cases} \quad 0 < x < L, \;\; t > 0, \tag{2.31}$$

$$\begin{cases} (\xi^0 + \dfrac{\beta}{\alpha} \eta^0)(0,t) = 0, \\ (\xi^0 + \eta^0)(L,t) = 0, \end{cases} \quad t > 0, \tag{2.32}$$

$$\begin{cases} \xi^0(x,0) = \xi_0(x) = u_0(x) + \lambda \, \psi_0(x), \\ \eta^0(x,0) = \eta_0(x) = u_0(x) - \lambda \, \psi_0(x), \end{cases} \quad 0 < x < L, \tag{2.33}$$

where $\alpha = \overline{U}_0 + 1/\lambda$, and $\beta = -\overline{U}_0 + 1/\lambda$. We recall that we consider the subcritical case where $\overline{U}_0 < 1/\lambda$, so that $\alpha > \beta > 0$.

In the rest of this section we show the well-posedness of problem (2.27)-(2.29) (or equivalently (2.30)-(2.33)). Then, in Section 2.3, we do prove that $(u^\varepsilon, \psi^\varepsilon)$ converges to (u^0, ψ^0) as $\varepsilon \longrightarrow 0$.

For (2.27)-(2.29), the natural functional framework is as follows:

$$H = L^2(0,L) \times L^2(0,L),$$

$$D(A^0) = \{U^0 = (u^0, \psi^0) \in H, u_x^0, \psi_x^0 \in L^2(0,L), \\ u^0(0) + \overline{U}_0 \, \lambda^2 \, \psi^0(0) = u^0(L) = 0\},$$

$$A^0 \begin{pmatrix} u^0 \\ \psi^0 \end{pmatrix} = \begin{pmatrix} \overline{U}_0 \, u_x^0 + \psi_x^0 \\ \overline{U}_0 \, \psi_x^0 + \dfrac{1}{\lambda^2} u_x^0 \end{pmatrix}, \quad \forall \begin{pmatrix} u^0 \\ \psi^0 \end{pmatrix} \in D(A).$$

Equation (2.27) can then be written as follows:

$$\frac{dU^0}{dt} + A^0\, U^0 = F,\;\; U^0(0) = U_0, \tag{2.34}$$

where $U_0 = (u_0, \psi_0)^T \in D(A^0)$, $U^0(t) = (u^0(t), \psi^0(t))^T \in D(A^0)$ for all time, and $F = (f, g)^T \in H$.

Here again, we use Theorem 2.1, and we have to check (i), (ii), and (iii).

The proof of (i) is similar to what was done in Section 2.1, and we do not repeat it.
Let us now proceed with the proof of (ii):

Lemma 2.4. *For every $U \in D(A^0), (A^0\,U, U)_H \geq 0$.*

Proof : For $U \in D(A^0)$ we write:

$$\begin{aligned}
(A^0\,U, U)_H &= \int_0^L \left[(\overline{U}_0\, u_x + \psi_x)\, u + (\overline{U}_0\, \lambda^2\, \psi_x + \, u_x)\, \psi\right] dx \\
&= \frac{1}{2}\,\overline{U}_0 \left[u^2(L) - u^2(0) + \lambda^2\, \psi^2(L) - \lambda^2 \psi^2(0)\right] + [u\, \psi]_{x=0}^{x=L}
\end{aligned}$$

Thanks to the boundary conditions (2.28), we find:

$$(A^0\,U, U)_H = \frac{1}{2}\,\overline{U}_0\, \lambda^2\, \psi^2(L) + \frac{1}{2}\,\overline{U}_0\, \lambda^2\,(1 - \overline{U}_0^2\, \lambda^2)\, \psi^2(0). \tag{2.35}$$

Since $1/\lambda > \overline{U}_0 > 0$, this last equation shows that A is a positive operator, and ends the proof of Lemma 2.4. □

To apply Theorem 2.1, we need one more result, proved in the following lemma:

Lemma 2.5. *The operator $A^0 + \mu I$ is onto, $\forall\, \mu > 0$.*

Proof :
Let μ be a positive real, and let $(f, g) \in H$; we look for $(u^0, \psi^0) \in D(A^0)$ such that:

$$\left\{\begin{array}{ccccccc}
\overline{U}_0\, u_x^0 & + & \psi_x^0 & + & \mu\, u^0 & = & f, \\
\overline{U}_0\, \psi_x^0 & + & \frac{1}{\lambda^2}\, u_x^0 & + & \mu\, \psi^0 & = & g.
\end{array}\right. \tag{2.36}$$

with:

$$\left\{\begin{array}{l}
u^0(0) + \overline{U}_0\, \lambda^2\, \psi^0(0) = 0, \\
u^0(L) = 0.
\end{array}\right. \tag{2.37}$$

We need to integrate the system by elementary calculations. Classically, we first look for the general solution of the homogeneous system:

$$\left\{\begin{array}{lclclcl} \overline{U}_0\, u^0_x & + & \psi^0_x & + & \mu\, u^0 & = & 0, \\ \overline{U}_0\, \psi^0_x & + & \dfrac{1}{\lambda^2}\, u^0_x & + & \mu\, \psi^0 & = & 0. \end{array}\right. \tag{2.38}$$

Looking for (u^0, ψ^0) of the form $(A^0, B^0)\, e^{R^0\, x}$ we obtain:

$$\left\{\begin{array}{lclcl} A^0\,(\overline{U}_0\, R^0 + \mu) & + & R^0\, B^0 & = & 0, \\ B^0\,(\overline{U}_0\, R^0 + \mu) & + & \dfrac{1}{\lambda^2}\, R^0\, A^0 & = & 0, \end{array}\right. \tag{2.39}$$

from which we infer the characteristic equation (for R^0):

$$P^0(R^0) = (\overline{U}_0\, R^0 + \mu)^2 - \frac{R^{0\,2}}{\lambda^2} = 0 \tag{2.40}$$

The roots of the polynomial P^0 are:

$$R^0_1 = -\frac{\mu}{1/\lambda + \overline{U}_0} < 0, \text{ and } R^0_2 = \frac{\mu}{1/\lambda - \overline{U}_0} > 0. \tag{2.41}$$

Thus the general solution of problem (2.38) is of the form:

$$\left(\begin{array}{c} u^0 \\ \psi^0 \end{array}\right) = \sum_{i=1}^{2} \left(\begin{array}{c} A^0_i \\ B^0_i \end{array}\right) e^{R^0_i\, x} \tag{2.42}$$

In (2.38), A^0_i is related to B^0_i by (2.39), that is

$$B^0_i = b^0_i\, A^0_i \tag{2.43}$$

where $b^0_i = \mu/R^0_i + \overline{U}_0$. Actually, $(b^0_1, b^0_2) = (-1/\lambda, 1/\lambda)$.
Hence, the general solution of (2.38) is:

$$\left(\begin{array}{c} u^0 \\ \psi^0 \end{array}\right) = \sum_{i=1}^{2} \left(\begin{array}{c} 1 \\ b^0_i \end{array}\right) A^0_i\, e^{R^0_i\, x} \tag{2.44}$$

In view of using the Duhamel formula, we rewrite (2.36) in the form

$$M\, Y'(x) + \mu\, Y(x) = B(x) \tag{2.45}$$

where:

$$Y(x) = \left(\begin{array}{c} u^0 \\ \psi^0 \end{array}\right), M = \left(\begin{array}{cc} \overline{U}_0 & 1 \\ \dfrac{1}{\lambda^2} & \overline{U}_0 \end{array}\right), B(x) = \left(\begin{array}{c} f(x) \\ g(x) \end{array}\right). \tag{2.46}$$

The general solution of (2.38) is:

$$Y = \sum_{i=1}^{2} \begin{pmatrix} 1 \\ b_i^0 \end{pmatrix} A_i^0 \, e^{R_i^0 \, x}, \tag{2.47}$$

and we look for a solution of (2.36) of the form

$$Y(x) = \sum_{i=1}^{2} \begin{pmatrix} 1 \\ b_i^0 \end{pmatrix} A_i^0(x) \, e^{R_i^0 \, x}. \tag{2.48}$$

The matrix M is invertible, since $\overline{U}_0^2 - 1/\lambda^2 < 0$, thus, we should have:

$$\sum_{i=1}^{2} \begin{pmatrix} 1 \\ b_i^0 \end{pmatrix} A_i^{0\,\prime}(x) \, e^{R_i^0 \, x} = M^{-1} \, B(x) =: \tilde{B}(x). \tag{2.49}$$

Hence:

$$\left\{ \begin{array}{rcrcl} A_1^{0\,\prime} \, e^{R_1^0 \, x} & + & A_2^{0\,\prime} \, e^{R_2^0 \, x} & = & \tilde{B}_1(x), \\ b_1^0 \, A_1^{0\,\prime} \, e^{R_1^0 \, x} & + & b_2^0 \, A_2^{0\,\prime} \, e^{R_2^0 \, x} & = & \tilde{B}_2(x). \end{array} \right. \tag{2.50}$$

System (2.50) is clearly regular since $b_1^0 \neq b_2^0$, and from it we compute the $A_i^{0\,\prime}$ using the Cramer's formula, and we then write:

$$A_i^0(x) = \int_0^x A_i^{0\,\prime}(u) \, du + A_i^0(0), \quad i = 1..2. \tag{2.51}$$

Since the $A_i^{0\,\prime}$ were derived from (2.50), we only need to determine the coefficients $A_i^0(0)$, which we do using the boundary conditions (2.28). Hence we obtain the following system:

$$\left\{ \begin{array}{rcrcl} A_1^0(0) \, (1 + \lambda \, \overline{U}_0) & + & A_2^0(0) \, (1 - \lambda \, \overline{U}_0) & = & 0, \\ A_1^0(0) \, e^{R_1^0 \, L} & + & A_2^0(0) \, e^{R_2^0 \, L} & = & \lambda^0, \end{array} \right. \tag{2.52}$$

where $\lambda^0 = - \sum_{i=1}^{2} \{ e^{R_i^0 \, L} \int_0^L A_i^{0\,\prime}(t) \, dt \}$.

This system is regular, we can uniquely determine the coefficients $A_i^0(0)$, and then obtain the exact expressions of the A_i^0, thanks to (2.51). We then have the solution of system (2.36) with (2.48); this shows that the operator $A^0 + \mu I$ is onto. □

We are now able to apply theorem (2.1) and we find:

Theorem 2.3. *The problem (2.1)-(2.3) is well-posed, that is for every* $(f, g) \in L^1(0, T; H)$, *and every* $(u_0, \psi_0) \in D(A^0)$, *there exists a unique solution* $U^0 = (u^0, \psi^0)$ *of (2.27)-(2.29), with* $U^0 \in \mathcal{C}([0, T]; H), dU^0/dt \in L^\infty(0, T; H)$.

2.3 Convergence as ε goes to 0

We want to prove that, when ε goes to zero, the solutions of (2.1)-(2.3) do converge to those of (2.27)-(2.29). As numerically shown in [RTT04], some boundary layers appear at the boundary $x = 0$. Actually we will prove this convergence by using and explicitly computing the *correctors* corresponding to these boundary layers..

The *ansatz* for the correctors (θ_u, θ_ψ) is the following:

$$\begin{pmatrix} \theta_u \\ \theta_\psi \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{-r\,x/\varepsilon} \tag{2.53}$$

where θ_u, θ_ψ satisfy

$$\left\{ \begin{array}{lllllll} \overline{U}_0\,\theta_{u\,x} & + & \theta_{\psi\,x} & - & 2\,\varepsilon\,\theta_{u\,xx} & = & 0, \\ \overline{U}_0\,\theta_{\psi\,x} & + & \frac{1}{\lambda^2}\,\theta_{u\,x} & & & = & 0. \end{array} \right. \tag{2.54}$$

and the boundary conditions

$$\left\{ \begin{array}{l} \theta_u(0) = -u^0(0), \\ \theta_\psi(0) = -\psi^0(0), \\ \theta_u(L) = 0. \end{array} \right. \tag{2.55}$$

Hence we should have:

$$\left\{ \begin{array}{lllllll} -\overline{U}_0\,\frac{r}{\varepsilon}\,A & - & \frac{r}{\varepsilon}\,B & - & 2\,\varepsilon\,\frac{r^2}{\varepsilon^2}\,A & = & 0, \\ -\overline{U}_0\,\frac{r}{\varepsilon}\,B & - & \frac{1}{\lambda^2}\,\frac{r}{\varepsilon}\,A & & & = & 0. \end{array} \right. \tag{2.56}$$

The characteristic equation of system (2.54) reads

$$r \begin{vmatrix} \overline{U}_0 + 2\,r & 1 \\ \frac{1}{\lambda^2} & \overline{U}_0 \end{vmatrix} = 0, \tag{2.57}$$

producing the solution $r = 0$ and the solution

$$r = \frac{1}{2\,\overline{U}_0\,\lambda^2} - \frac{\overline{U}_0}{2} > 0. \tag{2.58}$$

We can thus write θ_u and θ_ψ as follows:

$$\left\{ \begin{array}{lllll} \theta_u & = & A\,e^{-r\,x/\varepsilon} & + & C, \\ \theta_\psi & = & B\,e^{-r\,x/\varepsilon} & + & D, \end{array} \right. \tag{2.59}$$

where A and B are linked by the second equation (2.54), so that:

$$B = -\frac{1}{\overline{U}_0\,\lambda^2}\,A \tag{2.60}$$

We then determine the coefficients A, C, D, using the boundary conditions (2.55) and we find

$$\left\{\begin{array}{rcrcl} A & + & C & = & -u^0(0), \\ B & + & D & = & -\psi^0(0), \\ A\,e^{-r\,L/\varepsilon} & + & C & = & 0. \end{array}\right. \tag{2.61}$$

After computation, we obtain:

$$\left\{\begin{array}{l} A = -\dfrac{1}{1-e^{-r\,L/\varepsilon}}\,u^0(0), \\ B = \dfrac{1}{\overline{U}_0\,\lambda^2\,(1-e^{-r\,L/\varepsilon})}\,u^0(0), \\ C = \dfrac{e^{-r\,L/\varepsilon}}{1-e^{-r\,L/\varepsilon}}\,u^0(0), \\ D = -\dfrac{e^{-r\,L/\varepsilon}}{\overline{U}_0\,\lambda^2\,(1-e^{-r\,L/\varepsilon})}\,u^0(0). \end{array}\right. \tag{2.62}$$

Thus:

$$\left\{\begin{array}{rcrcl} \theta_u(x) & = & -u^0(0)\,e^{-r\,x/\varepsilon} & + & e_u, \\ \theta_\psi(x) & = & -\psi^0(0)\,e^{-r\,x/\varepsilon} & + & e_\psi. \end{array}\right. \tag{2.63}$$

where e_u and e_ψ are some terms exponentially small in norm in all classical spaces.

Having determined the correctors θ_u and θ_ψ we set $w_u = u - u^0 - \theta_u$, $w_\psi = \psi - \psi^0 - \theta_\psi$, where $(u^\varepsilon, \psi^\varepsilon)$ is solution of (2.1)-(2.3), and (u^0, ψ^0) is solution of (2.27)-(2.29). We intend to prove the following result which concludes this study:

Theorem 2.4. *We assume that* $u^0_{xx} \in L^\infty(0,T;L^2(0,L))^2$. *Then, as* $\varepsilon \longrightarrow 0$, w_u *and* w_ψ *are* $O(\varepsilon^{1/2})$ *in* $L^\infty(0,T;L^2(0,L))$.
Since θ_u *and* θ_ψ *are* $O(\varepsilon^{1/2})$ *in* $L^\infty(L^2)$, *we infer from the previous assumption that, as* $\varepsilon \longrightarrow 0$,

$$u^\varepsilon \longrightarrow u^0, \psi^\varepsilon \longrightarrow \psi^0 \textit{ in } L^\infty(0,T;L^2(0,L)). \tag{2.64}$$

Proof : It is clear that w_u and w_ψ verify:

$$\left\{\begin{array}{lclclcl} w_{u\,t} & + & \overline{U}_0\,w_{u\,x} & + & w_{\psi\,x} & - & 2\,\varepsilon\,w_{u\,xx} = 2\varepsilon\,u^0_{xx} - \theta_{u\,t}, \\ w_{\psi\,t} & + & \overline{U}_0\,w_{\psi\,x} & + & \dfrac{1}{\lambda^2}\,w_{u\,x} & & = -\theta_{\psi\,t}. \end{array}\right. \tag{2.65}$$

with

$$w_u(0,t) = 0,\; w_\psi(0,t) = 0,\; w_u(L,t) = 0, \tag{2.66}$$

and

$$w_u(x,0) = 0,\; w_\psi(x,0) = 0. \tag{2.67}$$

But

$$\left\{\begin{array}{rcrcrcl} \theta_u(x) & = & -u^0(0)\,e^{-r\,x/\varepsilon} & + & e_u & = & O(\varepsilon^{1/2}) \text{ in } L^\infty(L^2), \\ \theta_\psi(x) & = & -\psi^0(0)\,e^{-r\,x/\varepsilon} & + & e_\psi & = & O(\varepsilon^{1/2}) \text{ in } L^\infty(L^2). \end{array}\right. \tag{2.68}$$

[2] See Remark 2.4 hereafter.

Hence we have:

$$\left\{\begin{array}{lllllll} w_{u\,t} & + & \overline{U}_0\, w_{u\,x} & + & w_{\psi\,x} & - & 2\,\varepsilon\, w_{u\,xx} & = & h_u, \\ w_{\psi\,t} & + & \overline{U}_0\, w_{\psi\,x} & + & \dfrac{1}{\lambda^2}\, w_{u\,x} & & & = & h_\psi, \end{array}\right. \tag{2.69}$$

where h_u and h_ψ are the right-hand sides of the two equations (2.69), so that $\|h_u\|$, $\|h_\psi\| = O(\varepsilon^{1/2})$ in $L^\infty(0,T;L^2(0,L))$.

Multiplying the first equation of (2.65) by $2\,w_u$, and the second equation by $2\,\lambda^2\,w_\psi$, summing and integrating, we obtain:

$$\begin{aligned} \frac{d}{dt}\int_0^L (w_u^2(x,t) + \lambda^2\, w_\psi^2(x,t))\,dx &= -\overline{U}_0\, w_\psi^2(L) - 4\,\varepsilon \int_0^L w_{u\,x}^2(x,t)\,dx \\ &- \int_0^L (w_u(x,t)\, w_{\psi\,x}(x,t) + w_{u\,x}(x,t)\, w_\psi(x,t))\,dx \\ &+2 \int_0^L (h_u(x,t)\, w_u(x,t) + \lambda^2\, h_\psi(x,t)\, w_\psi(x,t))\,dx \end{aligned}$$

Dropping the negative terms, and using the inequality $2\,a\,b \le a^2 + b^2$, we finally find:

$$\frac{d}{dt}\int_0^L (w_u^2(x,t) + \lambda^2\, w_\psi^2(x,t))\,dx \le O(\varepsilon) + \int_0^L (w_u^2 + \lambda^2\, w_\psi^2)\,dx, \tag{2.70}$$

from which we derive, thanks to the Gronwall lemma, that w_u and w_ψ are $O(\varepsilon^{1/2})$ in $L^\infty(0,T;L^2(0,L))$.

Finally, we use (2.68) and the definitions of (w_u, w_ψ) to achieve the proof of (2.64). □

Remark 2.4. : Using the so called compatibility conditions (as in e.g [Tem82]), we can derive conditions on the initial data u_0, ψ_0 (or ξ_0, η_0) which guarantee this regularity property on u^0_{xx}. For instance, differentiating equations (2.27) in time, we obtain for $U^0_t = (u^0_t, \psi^0_t)$ an equation very similar to (2.27) for which a result similar to Theorem 2.3 applies. For that purpose we need to assume that $(f_t, g_t) \in L^1(0,T;H)$ and that $U^0_t|_t = 0$ is in $D(A^0)$. In view of (2.27), (u^0_t, ψ^0_t) belongs to $D(A^0)$ if, firstly, $(u^0_{t\,x}, \psi^0_{t\,x})|_{t=0}$ belongs to $L^2(0,L)^2$; differenciating equations (2.27) in x, we see that this is true if $u_{0\,xx}, \psi_{0\,xx}, f_x|_{t=0}, g_x|_{t=0}$ are in $L^2(0,L)$. Furthermore we need the boundary conditions

$$(u^0_t + \overline{U}_0\,\lambda^2\,\psi^0_t)(x=0,t=0) = u^0_t(x=L,t=0) = 0, \tag{2.71}$$

to be satisfied. In view of (2.27) written at $t = 0$, we obtain the *compatibility conditions* for u_0, ψ_0, f, g:

$$\left\{\begin{array}{r} 2\,\overline{U}_0\, u_{0\,x}(0) + (1 + \overline{U}_0^2\,\lambda^2)\,\psi_{0\,x}(0) = f(0,0) + \overline{U}_0\,\lambda^2\, g(0,0), \\ \overline{U}_0\, u_{0\,x}(L) + \psi_{0\,x}(L) = f(L,0). \end{array}\right. \tag{2.72}$$

Hence, if conditions (2.72) are satisfied, as well as the regularity conditions above and below ($f_t, g_t \in L^1(0,T;L^2(0,L))$, $u_{0\,xx}, \psi_{0\,xx}, f_x|_{t=0}, g_x|_{t=0}$ are in $L^2(0,L)$), then, as for Theorem 2.3, the Hille-Yosida theorem gives the existence and uniqueness of $U^0_t = (u^0_t, \psi^0_t)$ in $\mathcal{C}([0,T];H)$, with $U^0_{tt} = (u^0_{tt}, \psi^0_{tt})$ in $L^\infty(0,T;H)$. Therefore $u_{0\,xt}$, $\psi_{0\,xt}$ are in $L^\infty(0,T;L^2(0,L))$. Differenciating equations (2.27) in x and assuming also that f_x, g_x are in $L^\infty(0,T;L^2(0,L))$, we conclude that u^0_{xx} and ψ^0_{xx} are in $L^\infty(0,T;L^2(0,L))$.

Remark 2.5. *:* This article is devoted to the subcritical case i.e when $\lambda^{-1} > \overline{U}_0 > 0$ and $\beta > 0$ in (2.31). This case is more difficult than the supercritical case corresponding to $\overline{U}_0 > \lambda^{-1} > 0$ so that $\beta < 0$ in (2.31). In the supercritical case both characteristics enter the domain on the left-hand side, and the natural boundary conditions for (2.31) read

$$\xi^0(0,t) = \eta^0(0,t) = 0, \tag{2.73}$$

or in the u^0, ψ^0 variables,

$$u^0(0,t) = \psi^0(0,t) = 0. \tag{2.74}$$

With exactly the same methods as for Theorem 2.4, one can prove that $(u^\varepsilon, \psi^\varepsilon)$ converges to (u^0, ψ^0), where u^0, ψ^0, are now solutions of (2.27), (2.74) and (2.34) (and a similar result for $\xi^\varepsilon, \eta^\varepsilon$). In that case there is no reflection of wave at the right boundary $x = L$: the natural boundary conditions for the system (ξ^0, η^0) or (u^0, ψ^0) are already "transparent", and there is no need at all for the developments to come in Section 3.
For further explanations the reader is refered to classical references on hyperbolic systems, such as e.g. [Lax86, Ser96].

3 Transparent boundary conditions

As shown in [RTT04] and as evidenced by the first condition of (2.28), the system (2.1) supplemented with the boundary conditions (2.2) produces, in the limit $\varepsilon \longrightarrow 0$, some reflection phenomena at the boundary. These reflections are not desirable in the context in which these equations were introduced (see the discussion in [RTT04]). For that reason, in this section and in the next one, we propose to supplement the equations (2.1) with a different set of boundary conditions of transparent type (see e.g. [EM77],[TH86]). With such conditions, the solution of (2.1) converge - when ε goes to zero - to a solution of the limit system (2.27), supplemented with boundary conditions which do not produce reflections at the boundary and which are in some sense, the "natural" boundary conditions for (2.27).

3.1 The limit system with natural boundary conditions

In order to better understand the limit system (2.27), we use its diagonal form (2.31). We recall that $1/\lambda > \overline{U}_0$ so that $\alpha > \beta > 0$. In the ξ, η variables, the system (2.1) becomes:

$$\begin{cases} \xi^\varepsilon_t + \alpha\,\xi^\varepsilon_x - \varepsilon\,(\xi^\varepsilon + \eta^\varepsilon)_{xx} = f + \lambda\,g, \\ \eta^\varepsilon_t - \beta\,\eta^\varepsilon_x - \varepsilon\,(\xi^\varepsilon + \eta^\varepsilon)_{xx} = f - \lambda\,g, \end{cases} \quad 0 < x < L,\ \ t > 0. \tag{3.1}$$

Setting formally $\varepsilon = 0$ in (3.1), we obtain the above equations (2.31). These two equations are uncoupled, they are simple transport equations, with waves propagating at speed α for ξ and $-\beta$ for η. The most "natural" boundary conditions for (2.31) are conditions of the type

$$\xi^0(0,t) = \xi_l(t),\ \eta^0(L,t) = \eta_r(t),\ \ t > 0. \tag{3.2}$$

Of course, the problem (2.31)-(3.2) with initial data (2.33) is well-posed, and its well-known solution, in the homogeneous case $f = g = 0$, $\xi_l = \eta_r = 0$, is the pair of travelling waves[3]

$$\begin{cases} \xi^0(x,t) = \xi_0(x - \alpha\, t), \\ \eta^0(x,t) = \eta_0(x + \beta\, t). \end{cases} \tag{3.3}$$

Back to the (u, ψ) variables, the homogeneous problem is:

$$\begin{cases} u_t^0 + \overline{U}_0\, u_x^0 + \psi_x^0 = 0, \\ \psi_t^0 + \overline{U}_0\, \psi_x^0 + \dfrac{1}{\lambda^2}\, u_x^0 = 0, \end{cases} \qquad 0 < x < L, \qquad t > 0, \tag{3.4}$$

with the boundary conditions:

$$\begin{cases} u^0(0,t) + \lambda\, \psi^0(0,t) = 0, \\ u^0(L,t) - \lambda\, \psi^0(L,t) = 0, \end{cases} \qquad t > 0. \tag{3.5}$$

Its solution reads, thanks to (3.3):

$$\begin{aligned} u^0(x,t) &= \frac{1}{2}\Big[(u_0 + \lambda\,\psi_0)(x - \alpha\, t) + (u_0 - \lambda\,\psi_0)(x + \beta\, t)\Big], \\ \psi^0(x,t) &= \frac{1}{2\,\lambda}\Big[(u_0 + \lambda\,\psi_0)(x - \alpha\, t) - (u_0 - \lambda\,\psi_0)(x + \beta\, t)\Big]. \end{aligned} \tag{3.6}$$

Naturally, in the nonhomogeneous case, (u^0, ψ^0) are also well-defined, but their expression are slightly more complicated.

From now on, we restrict ourselves to the homogeneous boundary conditions $\xi_l = \eta_r = 0$, and we assume that (u^0, ψ^0) are given, computed by (3.3). The aim of the following section is to propose and study a set of boundary conditions supplementing (2.1) and such that its solution converges, as ε goes to zero, to the "natural" solution given in (3.3). These boundary conditions are called transparent boundary conditions (TBC)[4].

3.2 The transparent boundary conditions

We consider the following equations, in the (u, ψ) variables:

$$\begin{cases} u_t^\varepsilon + \overline{U}_0\, u_x^\varepsilon + \psi_x^\varepsilon - 2\,\varepsilon\, u_{xx}^\varepsilon = f, \\ \psi_t^\varepsilon + \overline{U}_0\, \psi_x^\varepsilon + \dfrac{1}{\lambda^2}\, u_x^\varepsilon = g, \end{cases} \qquad 0 < x < L,\ \ t > 0, \tag{3.7}$$

[3]Here we extend the definition of the initial data to all of $\mathbb{R}$ by setting: $\xi_0(x), \eta_0(x) = 0$ if $x \notin [0, L]$.

[4]For the nonspecialist we recall that this expression comes from the fact that there is no reflection of waves at the boundary so that waves go through the boundary, as if it were not there. This is an essential feature for the problem which motivated this study, namely numerical predictions (or simulations) in a limited domain with artificial (nonnatural) boundaries. The expressions "transparent" and "nonreflecting" boundary conditions are both used in the literature [Str77, RS80, RS81, Hal91, Tou97].

with initial data

$$\begin{cases} u^\varepsilon(x,0) = u_0(x), \\ \psi^\varepsilon(x,0) = \psi_0(x), \end{cases} \qquad 0 < x < L. \tag{3.8}$$

We supplement these equations with the following boundary conditions:

$$\begin{cases} \frac{d}{dt}\, u^\varepsilon(0) & + & \frac{1}{2}\,\overline{U}_0\, u^\varepsilon(0) & - & 2\,\varepsilon\, u^\varepsilon_x(0) & = & h, \\ \lambda^2\, \frac{d}{dt}\, \psi^\varepsilon(0) & + & \frac{1}{2}\,\overline{U}_0\, \lambda^2\, \psi^\varepsilon(0) & + & u^\varepsilon(0) & = & i, \\ \frac{d}{dt}\, u^\varepsilon(L) & + & \frac{1}{2\,\overline{U}_0\,\lambda^2}\, u^\varepsilon(L) & + & 2\,\varepsilon\, u^\varepsilon_x(L) & = & j, \end{cases} \tag{3.9}$$

where h, i, j are the following functions of t (through u^0 and ψ^0):

$$\begin{cases} h & = & \frac{d}{dt}\, u^0(0) & + & \frac{1}{2}\,\overline{U}_0\, u^0(0), \\ i & = & \lambda^2\, \frac{d}{dt}\, \psi^0(0) & + & \frac{1}{2}\,\overline{U}_0\, \lambda^2\, \psi^0(0) & + & u^0(0), \\ j & = & \frac{d}{dt}\, u^0(L) & + & \frac{1}{2\,\overline{U}_0\,\lambda^2}\, u^0(L). \end{cases} \tag{3.10}$$

Remark 3.1. *:* Since $(u,\psi) \in D(A)$, $(u,\psi) \in H^2 \times H^1 \subset \mathcal{C}^1 \times \mathcal{C}^0$, and the quantities in (3.9) are well defined.

Remark 3.2. *:* As in Section 2, from now on we drop the superscript ε and write (u,ψ) instead of $(u^\varepsilon, \psi^\varepsilon)$.

We want to prove the well-posedness of problem (3.7)-(3.9), and the convergence of its solutions to the solution (u^0, ψ^0) of (3.6) as ε goes to zero. For the well-posedness we will use the Hille-Yosida theorem again, in the following functional framework:

$$H = L^2(0,L) \times L^2(0,L) \times \mathbb{R}^3,$$

$$D(A) = \{U = (U_j)_{1 \le j \le 5} \in H,\ (U_1, U_2) = (u,\psi) \in H^2 \times H^1, \\ U_3 = u(0), U_4 = \psi(0), U_5 = u(L)\},$$

$$A \begin{pmatrix} u \\ \psi \\ u(0) \\ \psi(0) \\ u(L) \end{pmatrix} = \begin{pmatrix} \overline{U}_0\, u_x + \psi_x - 2\,\varepsilon\, u_{xx} \\ \overline{U}_0\, \psi_x + \frac{1}{\lambda^2}\, u_x \\ \frac{1}{2}\,\overline{U}_0\, u(0) - 2\,\varepsilon\, u_x(0) \\ \frac{1}{2}\,\overline{U}_0\, \lambda^2\, \psi(0) + u(0) \\ \frac{1}{2\,\overline{U}_0\,\lambda^2}\, u(L) + 2\,\varepsilon\, u_x(L) \end{pmatrix}, \quad \forall \begin{pmatrix} u \\ \psi \\ u(0) \\ \psi(0) \\ u(L) \end{pmatrix} \in D(A).$$

We endow H with the following inner product:

$$(U,V)_H = \int_0^L \{U_1\, V_1) + \lambda^2\, U_2\, V_2\}\, dx + U_3\, V_3 + U_4\, V_4 + U_5\, V_5.$$

Equations (3.7)-(3.9) can now be written in the functional form

$$\frac{dU}{dt} + A\,U = F,\; U(0) = U_0, \tag{3.11}$$

where

$$\begin{aligned} & U_0 = (u_0, \psi_0, u_0(0), \psi_0(0), u_0(L))^T \in D(A), \\ & U(t) = (u(t), \psi(t), u(0,t), \psi(0,t), u(L,t))^T \in D(A) \text{ for all time}, \\ \text{and} \quad & F = (f, g, h, i, j)^T \in L^1(0,T;H). \end{aligned}$$

In order to apply Theorem 2.1 we have to check its hypotheses (i), (ii), and (iii). We skip the technical proof of (i), done as in Lemma 2.1, and in the following Lemmas 3.1 and 3.2, we check the two other hypotheses of the Hille-Yosida theorem.

Lemma 3.1. *For every $U \in D(A)$, $(A\,U, U)_H \geq 0$.*

Proof : For $U \in D(A)$, we compute the inner product

$$\begin{aligned} (A\,U, U)_H \;=\; & \frac{1}{2}\overline{U}_0\, u^2(L) - \frac{1}{2}\overline{U}_0\, u^2(0) + u(L)\,\psi(L) - u(0)\,\psi(0) \\ & + \frac{1}{2}\overline{U}_0\,\lambda^2\,\psi^2(L) - \frac{1}{2}\overline{U}_0\,\lambda^2\,\psi^2(0) - 2\,\varepsilon\left[u_x\,u\right]_0^L \\ & + 2\,\varepsilon\,|u_x|^2_{L^2} \\ & + \frac{1}{2}\overline{U}_0\, u^2(0) - 2\,\varepsilon\, u_x(0)\, u_(0) \\ & + \frac{1}{2}\overline{U}_0\,\lambda^2\,\psi^2(0) + u(0)\,\psi(0) \\ & + \frac{1}{2\,\overline{U}_0\,\lambda^2}\, u^2(L) + 2\,\varepsilon\, u_x(L)\, u_(L) \end{aligned}$$

Dropping terms that cancel each other, we find:

$$(A\,U, U)_H = 2\,\varepsilon\,|u_x|^2_{L^2} + \frac{1}{2}\overline{U}_0\, u^2(L) + \frac{1}{2\,\overline{U}_0\,\lambda^2}\,(u(L) + \overline{U}_0\,\lambda^2\,\psi(L))^2.$$

This quantity is nonnegative, and the lemma is proven. □

The last step is the following:

Lemma 3.2. *The operator $A + \mu I$ is onto, $\forall\,\mu > 0$.*

Proof : Let μ be a positive real, and let $F = (f, g, h, j, k) \in H$.

We look for $(u, \psi, u(0), \psi(0), u(L)) \in D(A)$ such that:

$$\left\{\begin{array}{rcrcl} \overline{U}_0\, u_x + \psi_x - 2\,\varepsilon\, u_{xx} & + & \mu\, u & = & f, \\ \overline{U}_0\, \psi_x + \dfrac{1}{\lambda^2}\, u_x & + & \mu\, \psi & = & g, \\ \frac{1}{2}\,\overline{U}_0\, u^\varepsilon(0) - 2\,\varepsilon\, u^\varepsilon_x(0) & + & \mu\, u(0) & = & h, \\ \frac{1}{2}\,\overline{U}_0\, \lambda^2\, \psi^\varepsilon(0) + u^\varepsilon(0) & + & \mu\, \psi(0) & = & j, \\ \dfrac{1}{2\,\overline{U}_0\, \lambda^2}\, u^\varepsilon(L) + 2\,\varepsilon\, u^\varepsilon_x(L) & + & \mu\, u(L) & = & k. \end{array}\right. \tag{3.12}$$

We intend to reduce this system to a variational equation for u and use Lax-Milgram theorem for its solution. Before that, we express ψ as a function of u.

Thanks to (3.12b), we have:

$$\psi_x + \frac{\mu}{\overline{U}_0}\, \psi = \frac{g}{\overline{U}_0} - \frac{1}{\overline{U}_0\, \lambda^2}\, u_x. \tag{3.13}$$

We integrate this equation and find:

$$\psi = \Psi_0 + \Psi_1(u), \tag{3.14}$$

where

$$\Psi_0 = \frac{2}{2\,\mu + \overline{U}_0\, \lambda^2}\, j\, e^{-\mu\, x/\overline{U}_0} + \frac{1}{\overline{U}_0} \int_0^x g(x')\, e^{-\mu\,(x-x')/\overline{U}_0}\, dx', \tag{3.15}$$

$$\Psi_1(u) = \frac{-2\, u(0)\, e^{-\mu\, x/\overline{U}_0}}{2\,\mu + \overline{U}_0\, \lambda^2} - \frac{1}{\overline{U}_0\, \lambda^2} \int_0^x u_x(x')\, e^{-\mu\,(x-x')/\overline{U}_0}\, dx'. \tag{3.16}$$

From (3.15) and (3.16), we see that Ψ_1 is a linear mapping from H^1 into itself, and that $\Psi_0 \in H^1$. We thus have the following problem in u:

$$\left\{\begin{array}{rcl} \overline{U}_0\, u_x + \dfrac{d}{dx}\, \Psi_1(u) - 2\,\varepsilon\, u_{xx} + \mu\, u & = & f - \dfrac{d}{dx}\, \Psi_0, \\ \frac{1}{2}\,\overline{U}_0\, u^\varepsilon(0) - 2\,\varepsilon\, u^\varepsilon_x(0) + \mu\, u(0) & = & h, \\ \dfrac{1}{2\,\overline{U}_0\, \lambda^2}\, u^\varepsilon(L) + 2\,\varepsilon\, u^\varepsilon_x(L) + \mu\, u(L) & = & k. \end{array}\right. \tag{3.17}$$

The variational formulation of (3.17) reads: To find $u \in H^1(0, L)$, such that

$$a(u, \tilde{u}) = l(\tilde{u}),\ \forall \tilde{u} \in H^1(0, L), \tag{3.18}$$

where

$$\begin{array}{rcl} a(u, \tilde{u}) & = & \overline{U}_0\, (u_x, \tilde{u})_{L^2} + (\dfrac{d}{dx}\, \Psi_1(u), \tilde{u})_{L^2} + 2\,\varepsilon\, (u_x, \tilde{u}_x)_{L^2} + \mu\, (u, \tilde{u})_{L^2} \\ & & +(\dfrac{1}{2}\, \overline{U}_0 + \mu)\, u(0)\, \tilde{u}(0) + (\dfrac{1}{2\,\overline{U}_0\, \lambda^2} + \mu)\, u(L)\, \tilde{u}(L), \\ l(\tilde{u}) & = & (f - \dfrac{d}{dx}\, \Psi_0, \tilde{u}) + h\, \tilde{u}(0) + k\, \tilde{u}(L). \end{array}$$

It is clear that a is defined and bilinear continuous on $H^1(0,L)$ and that l is defined and linear continuous on that space. We ought to prove the coercivity of a on $H^1(0,L)$, and it is sufficient to establish it for u belonging to a dense subspace of $H^1(0,L)$, say $\mathcal{C}^2([0,L])$. For such a u we compute $a(u,u)$:

$$\begin{aligned} a(u,u) &= \frac{1}{2}\overline{U}_0\, u^2(L) - \frac{1}{2}\overline{U}_0\, u^2(0) + (\frac{d}{dx}\,\Psi_1(u), u) + 2\,\varepsilon\,|u_x|^2_{L^2} \\ &+ (\frac{1}{2}\,\overline{U}_0 + \mu)\,u^2(0) + (\frac{1}{2\,\overline{U}_0\,\lambda^2} + \mu)\,u^2(L) + \mu\,|u|^2_{L^2}. \end{aligned} \tag{3.19}$$

We know that $\Psi_1 = \Psi_1(u)$ is solution of:

$$\begin{cases} \dfrac{\partial\Psi_1}{\partial x} + \dfrac{\mu}{\overline{U}_0}\,\Psi_1 = \dfrac{-1}{\overline{U}_0\,\lambda^2}\,u_x, \\ \Psi_1(0) = \dfrac{-2}{2\mu + \overline{U}_0\,\lambda^2}\,u(0). \end{cases} \tag{3.20}$$

Hence we find:

$$\begin{aligned} a(u,u) &= \mu\,|u|^2_{L^2} + 2\,\varepsilon\,|u_x|^2_{L^2} + \frac{1}{2}\,\overline{U}_0\,u^2(L) + \mu\,(u^2(0) + u^2(L)) \\ &+ \Big[\Psi_1\,u\Big]_0^L - (\Psi_1, \overline{U}_0\,\lambda^2\,\frac{d}{dx}\,\Psi_1 + \mu\,\lambda^2\,\Psi_1) + \frac{1}{2\,\overline{U}_0\,\lambda^2}\,u^2(L). \end{aligned}$$

Finally, using the boundary condition (3.20b)¡ we obtain the coercivity:

$$a(u,u) \geq 2\,\varepsilon\,|u_x|^2_{L^2} + \mu\,|u|^2_{L^2}. \tag{3.21}$$

Hence we can use the Lax-Milgram theorem and we obtain the existence and uniqueness of a solution u for (3.17). From (3.14), we see that $\psi \in H^1(0,L)$, and finally $u \in H^2(0,L)$, thanks to (3.12a). Hence $U = (u, \psi, u(0), \psi(0), u(L)) \in D(A)$, and U is solution of $A\,U + \mu\,U = F$. The lemma is proven. □

To conclude this section, we can now write the following result:

Theorem 3.1. *The problem (3.7)-(3.9) is well-posed, that is for every $(f,g) \in H$, and every $(u_0, \psi_0) \in D(A)$, there exists a unique solution $U = (u, \psi)$ of (3.7)-(3.9), with $U \in \mathcal{C}([0,T]; H)$, $dU/dt \in L^\infty(0,T;H)$.*

We next prove the convergence, when ε goes to zero, of the solution of (3.7)-(3.9) to the solution (3.6) of the limit problem.

3.3 Convergence for the transparent boundary conditions

We are here interested in two systems of equations. The first system is (2.27), supplemented with the boundary condition (3.5). The solutions of this system is given by (3.6). The second perturbed system of equations is (2.1) with boundary conditions (3.9).

Thanks to the previous results, we know that these two problems are well-posed. Our aim is now to prove that $(u^\varepsilon, \psi^\varepsilon)$, solution of (2.1),(3.9), converges to (u^0, ψ^0), solution of (2.27),(3.5), as $\varepsilon \longrightarrow 0$.
We set

$$\tilde{u} = u^\varepsilon - u^0,\ \tilde{\psi} = \psi^\varepsilon - \psi^0, \tag{3.22}$$

and prove the following:

Theorem 3.2. *As $\varepsilon \longrightarrow 0$, $\tilde{u}$ and $\tilde{\psi}$ are $O(\varepsilon^{1/2})$ in $L^\infty(0,T;L^2(0,L))$.*

Proof : The equations for $(\tilde{u}, \tilde{\psi})$ read:

$$\left\{ \begin{array}{rcll} \tilde{u}_t & + & \overline{U}_0\,\tilde{u}_x + \tilde{\psi}_x - 2\,\varepsilon\,\tilde{u}_{xx} & = \; 2\,\varepsilon\,u^0_{xx}, \\ \tilde{\psi}_t & + & \overline{U}_0\,\tilde{\psi}_x + \dfrac{1}{\lambda^2}\,\tilde{u}_x & = \; 0, \\ \dfrac{d}{dt}\,\tilde{u}(0) & + & \frac{1}{2}\overline{U}_0\,\tilde{u}(0) - 2\,\varepsilon\,\tilde{u}_x(0) & = \; 2\,\varepsilon\,u^0_x(0), \\ \lambda^2\,\dfrac{d}{dt}\,\tilde{\psi}(0) & + & \frac{1}{2}\overline{U}_0\,\lambda^2\,\tilde{\psi}(0) + \tilde{u}(0) & = \; 0, \\ \dfrac{d}{dt}\,\tilde{u}(L) & + & \dfrac{1}{2\,\overline{U}_0\,\lambda^2}\,\tilde{u}(L) + 2\,\varepsilon\,\tilde{u}_x(L) & = \; -2\,\varepsilon\,u^0_x(L). \end{array} \right. \tag{3.23}$$

Taking the scalar product of each side of equation (3.23) with the corresponding component of $\tilde{U}$, we find after simplification[5]:

$$\begin{aligned} &\frac{1}{2}\,\frac{d}{dt}\,E(t) + 2\,\varepsilon\,|\tilde{u}_x|^2_{L^2} + \frac{1}{2}\,\overline{U}_0\,\lambda^2\,(\tilde{\psi}(L) + \frac{1}{\overline{U}_0\,\lambda^2}\,\tilde{u}(L))^2 + \frac{1}{2}\,\overline{U}_0\,\tilde{u}^2(L) \\ &= 2\,\varepsilon\,\Big\{ \int_0^L u^0_{xx}\,\tilde{u}\,dx + u^0_x(0)\,\tilde{u}(0) - u^0_x(L)\,\tilde{u}(L) \Big\}, \end{aligned}$$

where $E(t) = \int_0^L (\tilde{u}^2 + \lambda^2 \tilde{\psi}^2)\,dx + \tilde{u}^2(0) + \tilde{\psi}^2(0) + \tilde{u}^2(L)$.

Hence, using the Cauchy-Schwarz inequality, we finally obtain:

$$\frac{1}{2}\,\frac{d}{dt}\,E(t) + 2\,\varepsilon\,|\tilde{u}_x|^2_{L^2} \le E(t) + K_1\,\varepsilon^2, \tag{3.24}$$

where K_1 depends only on (u^0, ψ^0).

Using the Gronwall lemma, we conclude the proof of Theorem 3.2. □

[5] Numerous terms cancel each other, which is one of the reasons for the choice of equations (3.9)

Remark 3.3. *:* We notice that for the purpose of this proof, we did not explicitly use the boundary conditions (3.5), and only used that (u^0, ψ^0) are defined and satisfy (2.27). Hence we could also use (u^0, ψ^0) from the boundary conditions (2.28) from Section 2, and still obtain convergence. Doing so, we would find a numerical solution, with $\varepsilon > 0$ and these TBC, that produces some reflections. In other words, the boundary conditions are actually transparent only if (u^0, ψ^0) are well chosen !

4 Another set of TBC

The solution of the perturbed system (2.1),(3.9), necessitates the knowledge of the solution (u^0, ψ^0) of the limit system (2.27), which is easy to find in this case. In cases where the limit system is not easily accessible, the procedure above may not be practical. In fact in our computations described in [RTT04], we used a different set of perturbed equations which does not necessitate the knowledge of the limit solution. We now describe this system in *its discrete form*, in the case of homogeneous boundary conditions. We have not been able to show that, in the limit $\Delta x \longrightarrow 0$, this system is well-posed and, in fact, a naive count of the number of equations indicates that the continuous system may be overdetermined if no precautions are taken. However, as we said, this perturbed system has some computational advantages, and for $\Delta x > 0$ fixed it is well-posed, the limit system is well-posed (but this is standard), and the perturbed system does converge to the expected limit when $\varepsilon \longrightarrow 0$ (Δx fixed).

We now give, for the sake of completeness, a brief description of these results which do not raise any essential difficulty, the systems being finite dimensional.

4.1 Existence of a discrete solution

Let N be a given integer. We set $h = L/(N+1)$. We want to compute the step functions

$$\begin{cases} \xi_h(x,t) = \displaystyle\sum_{j=0}^{N} \xi_j(t)\, w_j^h(x), \\ \eta_h(x,t) = \displaystyle\sum_{j=0}^{N} \eta_j(t)\, w_j^h(x), \end{cases} \qquad \forall t > 0, \tag{4.1}$$

where $(\xi_j, \eta_j)(t)$ are unknown, and where $w_j^h = \chi_{[jh,(j+1)h)}$.

Similarly, we introduce f_h and g_h of the form

$$\begin{cases} f_h(x,t) = \displaystyle\sum_{j=0}^{N} f_j(t)\, w_j^h(x), \\ g_h(x,t) = \displaystyle\sum_{j=0}^{N} g_j(t)\, w_j^h(x), \end{cases} \qquad \forall t > 0, \tag{4.2}$$

where $f_j(t) = f(j\,h, t)$ and $g_j(t) = g(j\,h, t)$.
We define the following discrete operators ∇_h and $\overline{\nabla}_h$:

$$\begin{cases} (\nabla_h.\varphi)_j = \dfrac{\varphi_{j+1} - \varphi_j}{h}, \ \forall j = 0..N-1, \\ (\overline{\nabla}_h.\varphi)_j = \dfrac{\varphi_j - \varphi_{j-1}}{h}, \ \forall j = 1..N. \end{cases} \tag{4.3}$$

To be consistent with (3.2) in the homogeneous case, we set

$$\xi_0(t) = 0,\ \eta_N(t) = 0,\ \ \forall t > 0. \tag{4.4}$$

The discrete equations, derived from (4.1) for all $t > 0$, read for $j = 1, ..., N-1$:

$$\begin{cases} \dfrac{d\xi_j}{dt}(t) + \alpha\,(\overline{\nabla}\,\xi)_j(t) - \varepsilon\,[\nabla\,\overline{\nabla}\,(\xi+\eta)]_j(t) = f_j(t), \\ \dfrac{d\eta_j}{dt}(t) - \beta\,(\nabla\,\eta)_j(t) - \varepsilon\,[\nabla\,\overline{\nabla}\,(\xi+\eta)]_j(t) = g_j(t). \end{cases} \tag{4.5}$$

We choose to supplement this system with the discrete boundary conditions

$$\begin{cases} \dfrac{d\xi_N}{dt}(t) + \dfrac{\alpha}{h}\,(\xi_N - \xi_{N-1} - \eta_{N-1})(t) = f_N(t) - \dfrac{\alpha}{\beta}\,g_N(t), \\ \dfrac{d\eta_0}{dt}(t) - \dfrac{\beta}{h}\,(\xi_1 + \eta_1 - \eta_0)(t) = g_0(t) - \dfrac{\beta}{\alpha}\,f_0(t). \end{cases} \tag{4.6}$$

We consider the space $H = \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R} \simeq \mathbb{R}^{2N}$. The domain of the operator, usually denoted $D(A_h)$, is all of H since our operator A_h is bounded. Each element U_h in H is of the form $U_h = (\xi_1, ..., \xi_{N-1}, \eta_1, ..., \eta_{N-1}, \xi_N, \eta_0)$.

For every U_h in H, we set

$$A_h\,U_h = \begin{pmatrix} \alpha\,(\overline{\nabla}\,\xi)_j - \varepsilon\,[\nabla\overline{\nabla}(\xi+\eta)]_j, & j = 1..N-1 \\ -\beta\,(\nabla\,\eta)_j - \varepsilon\,[\nabla\overline{\nabla}(\xi+\eta)]_j, & j = 1..N-1 \\ \dfrac{\alpha}{h}\,(\xi_N - \xi_{N-1} - \eta_{N-1}) & \\ -\dfrac{\beta}{h}\,(\xi_1 + \eta_1 - \eta_0) & \end{pmatrix}. \tag{4.7}$$

Hence we can write (4.5) in the form:

$$\frac{dU_h}{dt} + A_h\,U_h = F_h,\ \ U_h(0) = U_{0,h}, \tag{4.8}$$

where:

$$U_h = (\xi_1, ..., \xi_{N-1}, \eta_1, ..., \eta_{N-1}, \xi_N, \eta_0)^T,$$
$$F_h = (f_j, g_j, f_N - \frac{\alpha}{\beta}\, g_N, g_0 - \frac{\beta}{\alpha}\, f_0)^T, \tag{4.9}$$

and $U_{0,h}$ is a spatial discretization of the initial data $U_0 = (u_0, \psi_0)$. We notice the following theorem which is easy since (4.8) is a linear ordinary differential system with as many equations as unknowns.

Theorem 4.1. *The problem (4.4)-(4.6) is well-posed, that is for every $(f,g) \in L^1(0,T;H)$, and every $U_{0,h} \in H$, there exists a unique solution U_h of (4.4)-(4.6), with $U_h \in \mathcal{C}([0,T];H)$, $dU_h/dt \in L^1(0,T;H)$.*

We then consider the limit system obtained by setting formally $\varepsilon = 0$ in (4.5), that is

$$\left\{ \begin{array}{lcllll} \dfrac{d\xi_j^0}{dt}(t) & + & \alpha\,(\overline{\nabla}\,\xi^0)_j(t) & = & f_j(t), & 1 \le j \le N,\ t > 0, \\ \dfrac{d\eta_j^0}{dt}(t) & - & \beta\,(\nabla\,\xi^0)_j(t) & = & g_j(t), & 0 \le j \le N-1,\ t > 0. \end{array} \right. \tag{4.10}$$

We define ξ_h^0 and η_h^0 as in (4.1), and impose the boundary conditions

$$\xi_0^0 = 0,\ \eta_N^0 = 0, \tag{4.11}$$

For the functional setting, the Hilbert space H^0 is $\mathbb{R}^{2N}$, and for every $U = (\xi_1, ..., \xi_{N-1}, \eta_1, ..., \eta_{N-1}, \xi_N, \eta_0) \in H^0$ we write

$$A_h^0\, U = \left(\begin{array}{ll} \alpha\,(\overline{\nabla}\,\xi)_j, & j = 1..N \\ -\beta\,(\nabla\,\eta)_j, & j = 0..N-1 \end{array} \right). \tag{4.12}$$

The initial value problem is written as

$$\frac{dU_h^0}{dt} + A_h^0\, U_h^0 = F_h,\ \ U_h^0(0) = U_{0,h}. \tag{4.13}$$

This is again a linear differential system with as many equations as unknowns, so that the analogue of Theorem 4.1 is also easy:

Theorem 4.2. *The problem (4.10)-(4.11) is well-posed, that is for every $(f,g) \in L^1(0,T;H^0)$, and every $U_{0,h} \in H^0$, there exists a unique solution U_h^0 of (4.10)-(4.11), with $U_h^0 \in \mathcal{C}([0,T];H^0)$, $dU_h/dt \in L^1(0,T;H^0)$.*

4.2 Convergence as ε goes to 0

To conclude this section we prove a convergence result as ε goes to zero of the discrete solution of (4.4)-(4.6) to that of (4.10)-(4.11), $h = \Delta x$ being fixed.

We first endow H with the inner product:

$$(U,V)_H = h \sum_{i=1}^{2N-2} U_j V_j + \frac{\varepsilon}{\alpha} U_{2N-1} V_{2N-1} + \frac{\varepsilon}{\beta} U_{2N} V_{2N}. \tag{4.14}$$

We have the following discrete integration by parts formula, whose proof is elementary:

Lemma 4.1. *Let* $\varphi_h = \sum_{j=0}^{N} \varphi_j w_j^h$, *and* $\psi_h = \sum_{j=0}^{N} \psi_j w_j^h$.

Then:

$$\int_h^{L-h} \overline{\nabla} \varphi_h . \psi_h \, dx = \varphi_{N-1} \psi_N - \varphi_0 \psi_1 - \int_h^{L-h} \varphi_h . \nabla \psi_h \, dx$$

We then observe that the operator A_h is semi-definite positive:

Lemma 4.2. *For every* $U \in H$, $(A_h U, U)_H \geq 0$.

Proof : Let $U = (\xi_1, ..., \xi_{N-1}, \eta_1, ..., \eta_{N-1}, \xi_N, \eta_0)$ be an element of $D(A_h) = H$. From the definition of the inner product (4.14) on H, we find

$$\begin{aligned} (A_h U, U)_H &= \alpha h \sum_{j=1}^{N-1} (\overline{\nabla} \xi)_j \xi_j - \beta h \sum_{j=1}^{N-1} (\nabla \eta)_j \eta_j \\ &\quad - \varepsilon h \sum_{j=1}^{N-1} [\nabla \overline{\nabla} (\xi + \eta)]_j (\xi + \eta)_j \\ &\quad + \frac{\varepsilon}{h} (\xi_N - \xi_{N-1} - \eta_{N-1}) \xi_N - \frac{\varepsilon}{h} (\xi_1 + \eta_1 - \eta_0) \eta_0 \end{aligned} \tag{4.15}$$

Thanks to Lemma 4.1, and using the boundary conditions (4.4) we find

$$\begin{aligned} (A_h U, U)_H &= \frac{\alpha}{2} \xi_{N-1}^2 + \frac{\alpha}{2} \sum_{j=1}^{N-1} |\xi_j - \xi_{j-1}|^2 \\ &\quad + \frac{\beta}{2} \eta_1^2 + \frac{\beta}{2} \sum_{j=1}^{N-1} |\eta_{j+1} - \eta_j|^2 \\ &\quad + \varepsilon h \sum_{j=1}^{N} [\frac{(\xi+\eta)_j - (\xi+\eta)_{j-1}}{h}]^2 \end{aligned} \tag{4.16}$$

Finally, we have:

$$(A_h U, U)_H \geq 0, \tag{4.17}$$

and the lemma is proven. □

We now confirm the numerical simulations of [RTT04] with the following result:

Theorem 4.3. *The solutions* U_h *of* (4.4),(4.6) *converge in* $L^\infty(0,T;H)$ *to the solution* U_h^0 *of* (4.10),(4.11), *when* ε *goes to zero.*

Proof : We proceed as in Section 3 and set $\tilde{\xi}_h = \xi_h^\varepsilon - \xi_h^0$, $\tilde{\eta}_h = \eta_h^\varepsilon - \eta_h^0$. From equations (4.5),(4.10) and boundary conditions (4.4),(4.6),(4.11) we obtain:

$$\left\{\begin{array}{lllll} \frac{d\tilde{\xi}_j}{dt} & + & \alpha\,(\overline{\nabla}\,\tilde{\xi})_j - \varepsilon\,[\nabla\,\overline{\nabla}\,(\tilde{\xi}+\tilde{\eta})]_j & = & \varepsilon\,[\nabla\,\overline{\nabla}\,(\xi^0+\eta^0)]_j, \\ \frac{d\tilde{\eta}_j}{dt} & - & \beta\,(\nabla\,\tilde{\eta})_j(t) - \varepsilon\,[\nabla\,\overline{\nabla}\,(\tilde{\xi}+\tilde{\eta})]_j & = & \varepsilon\,[\nabla\,\overline{\nabla}\,(\xi^0+\eta^0)]_j, \\ \frac{d\tilde{\xi}_N}{dt} & + & \frac{\alpha}{h}\,(\xi_N - \xi_{N-1} - \eta_{N-1}) & = & 0, \\ \frac{d\tilde{\eta}_0}{dt} & - & \frac{\beta}{h}\,(\xi_1 + \eta_1 - \eta_0) & = & 0. \end{array}\right. \tag{4.18}$$

Taking the scalar product of each side of (4.18) with $\overline{U}$ we find after simplification:

$$\frac{1}{2}\,\frac{d}{dt}\sum_{j=0}^{N}(\tilde{\xi}_j^2+\tilde{\eta}_j^2) \leq \varepsilon\sum_{j=1}^{N-1}\left|[\nabla\,\overline{\nabla}\,(\xi^0+\eta^0)]_j\right|\left|\xi_j^0+\eta_j^0\right|. \tag{4.19}$$

Thanks to the Cauchy-Schwarz inequality, we infer that

$$\frac{1}{2}\,\frac{d}{dt}E_N(t) \leq E_N(t) + K_2\,\varepsilon^2, \tag{4.20}$$

where $E_N(t) = \sum_{j=0}^{N}(\tilde{\xi}_j^2(t)+\tilde{\eta}_j^2(t))$, and K_2 depends only on ξ^0, η^0.

Then we obtain the theorem thanks to the discrete Gronwall lemma. □

Conclusion

In this article we have considered a system of linear PDEs in one space dimension with a small viscosity parameter. This system was derived from the Primitive Equations of the ocean with mild viscosity thanks to a modal decomposition in the vertical direction (see [TT03]). This simple system serves as a model to the study of the important issue of the boundary conditions for the atmosphere and the ocean, in particular for simulations in limited domains. We supplemented the equations with two different sets of boundary conditions, and proved their well-posedness and the convergence of their solutions $(u^\varepsilon, \psi^\varepsilon)$ to (u^0, ψ^0). The latter is a solution of the limit system obtained by setting formally $\varepsilon = 0$ in the initial perturbed system.
The first set of boundary conditions is of Dirichlet type and produces some boundary layers and reflections at the boundary. We have also proposed an alternate set of transparent boundary conditions that avoid these reflections. These results are confirmed in the numerical simulations performed in [RTT04].

For the full Primitive Equations, this study suggests that the subcritical and supercritical parts of the solutions need to be considered carefully, in particular regarding the boundary conditions. We leave this to subsequent studies.

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Chapitre 4

Schémas numériques pour un système d'EDP issu de l'océanographie

Numerical time-schemes for an ocean related system of PDEs

Ce chapitre est constitué de l'article rédigé avec Madalina Petcu, **Numerical time-schemes for an ocean related system of PDEs**, qui est à paraître dans *Numerical Methods for Partial Differential Equations.* Nous y considérons un système d'équations introduites dans le chapitre 2, que nous munissons de conditions aux limites de type transparent (voir aussi les sections 3 et 4 du chapitre 3).
Nous étudions ici la stabilité de différents schémas numériques dont certains ont été implémentés dans le chapitre 2. Lorsque le petit paramètre δ est pris égal à zéro, les conditions de stabilités établies dans ce chapitre sont conformes à la condition de stabilité habituelle des équations de transport (CFL).

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Numerical time-schemes for an ocean related system of PDEs

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Abstract

In this article we consider a system of equations related to the δ–Primitive Equations of the ocean and the atmosphere, linearized around a stratified flow, and we supplement the equations with transparent boundary conditions. We study the stability of different numerical schemes and we show that for each case, taking the vertical viscosity $\delta = 0$, the stability conditions are the same as the classical CFL conditions for the transport equation.

1 Introduction

The issue of open boundary conditions for the Primitive Equations (PEs) of the ocean and the atmosphere is fundamental in the field of computational fluid dynamics (see e.g. [Ped87, TZ04, TT03]). The PEs, supplemented with any set of local boundary conditions, were shown to be ill-posed (see [OS78], [TT03]). To overcome this difficulty, the so-called δ-PEs were introduced with different motivations in [TT03] and [Sal98]. This new model consists in the addition of a friction term of the form δw in the hydrostatic equation, which is sufficient to ensure well-posedness (see [PR05]).

In a recent article [RTT04], the authors make a modal analysis of the δ-PEs linearized around a stratified flow, and perform numerical simulations of the so-called subcritical modes, that are the most challenging ones (see [RTT04]). In the case of classical Dirichlet boundary conditions, some reflexions of waves and boundary layers occur, and thus the authors consider another set of boundary conditions, of transparent type, in order to avoid these boundary layers as δ goes to zero. For these models, some energy estimates are given, and a full proof of well-posedness and convergence as δ goes to zero are given in [RTT05]. In the present article we intend to study the stability of the schemes considered in these articles. Hereafter we consider different discretizations of the equations and boundary conditions that have been proposed in [RTT04, RTT05], and present the stability results. In the case when a stability condition occurs (e.g. in Section 4), we notice that if δ is taken equal to zero, the condition matches with the classical CFL condition for the transport equation.

The article is organized as follows. In Section 2, we recall the equations and boundary conditions introduced in [RTT05], and set the functionnal framework of our study. We then start the stability studies in Section 3 with an implicit Euler scheme, which is proved to be unconditionally stable. For the explicit scheme, we derive in Section 4 a stability condition involving $\Delta t, \Delta x$, and δ. We then prove the stability of the Crank-Nicholson scheme, with no condition on the parameters, and end this article with a study in Section 6 of the so-called fractional scheme method, which is shown to be easier to implement in the numerical computations, while it remains consistent and stable without any additional stability condition on the parameters $\Delta t, h$, and ε). The approach for the study of stability is the classical one, based on energy estimates which is more appropriate than the Von Neumann spectral method for nonperiodic boundary value problems.

2 Equations and functional framework

The δ-PEs of the ocean with no Coriolis force, in a 2D domain $\mathcal{M} = (-H, 0) \times (0, L)$, and linearized around a constant stratified flow $\overline{U}_0 e_x$ with $\overline{U}_0 > 0$, read:

$$\frac{\partial u}{\partial t} + \overline{U}_0 \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial x} = F_u, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + \overline{U}_0 \frac{\partial v}{\partial x} = F_v, \tag{2.2}$$

$$\frac{\partial \psi}{\partial t} + \overline{U}_0 \frac{\partial \psi}{\partial x} + N^2 w = F_\psi, \tag{2.3}$$

$$\delta w + \frac{\partial \phi}{\partial z} = \psi, \tag{2.4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{2.5}$$

Here u, v, w, ϕ and ρ are all perturbed quantities; (u, v) is the horizontal velocity, w the vertical velocity, ϕ the pressure, ρ the density, and ψ the temperature. The constant g is the gravitational acceleration.

We perform the so-called normal mode decomposition, that is we look for some solutions of the form[1]:

$$(u, v, \phi) = \sum_{n \geq 0} \cos(N\lambda_n z)\,(u_n, v_n, \phi_n)\,(x, t), \tag{2.6}$$

$$(w, \psi) = \sum_{n \geq 1} \sin(N\lambda_n z)\,(w_n, \psi_n)\,(x, t). \tag{2.7}$$

Here $N\lambda_n = n\pi/H$, where N is the constant Brunt-Väisälä (or buoyancy) frequency, and n is the number of the considered mode.

[1] See [RTT04, RTT05] for more details.

We obtain for each mode $n \geq 1$ the following system of equations:

$$\left\{\begin{array}{lcl}\dfrac{\partial u_n}{\partial t} + \overline{U}_0 \dfrac{\partial u_n}{\partial x} + \dfrac{\partial \phi_n}{\partial x} & = & F_{u,n}, \\ \dfrac{\partial v_n}{\partial t} + \overline{U}_0 \dfrac{\partial v_n}{\partial x} & = & F_{v,n}, \\ \dfrac{\partial \psi_n}{\partial t} + \overline{U}_0 \dfrac{\partial \psi_n}{\partial x} + N^2 w_n & = & F_{\psi,n}, \\ \phi_n = -\dfrac{1}{N\lambda_n}(\psi_n - \delta w_n), & & \\ w_n = -\dfrac{1}{N\lambda_n}\dfrac{\partial u_n}{\partial x}. & & \end{array}\right. \tag{2.8}$$

Dropping the equation on v_n that can be solved independently (in the absence of Coriolis force), and replacing ϕ_n and w_n by their expression in the equations for u_n and ψ_n, we obtain:

$$\left\{\begin{array}{lcl}\dfrac{\partial u_n}{\partial t} + \overline{U}_0 \dfrac{\partial u_n}{\partial x} - \dfrac{1}{N\lambda_n}\dfrac{\partial \psi_n}{\partial x} - \dfrac{\delta}{N^2\lambda_n^2}\dfrac{\partial^2 u_n}{\partial x^2} & = & F_{u,n}, \\ \dfrac{\partial \psi_n}{\partial t} + \overline{U}_0 \dfrac{\partial \psi_n}{\partial x} - \dfrac{N}{\lambda_n}\dfrac{\partial u_n}{\partial x} & = & F_{\psi,n}.\end{array}\right. \tag{2.9}$$

Finally, we set $\xi = u_n - \psi_n/N$, $\eta = u_n + \psi_n/N$, and we find for every $(x,t) \in (0,L) \times (0,T)$:

$$\left\{\begin{array}{lcl}\dfrac{\partial \xi}{\partial t}(x,t) + \alpha\dfrac{\partial \xi}{\partial x}(x,t) - \varepsilon(\dfrac{\partial^2 \xi}{\partial x^2} + \dfrac{\partial^2 \eta}{\partial x^2})(x,t) & = & f(x,t), \\ \dfrac{\partial \eta}{\partial t}(x,t) - \beta\dfrac{\partial \eta}{\partial x}(x,t) - \varepsilon(\dfrac{\partial^2 \xi}{\partial x^2} + \dfrac{\partial^2 \eta}{\partial x^2})(x,t) & = & g(x,t),\end{array}\right. \tag{2.10}$$

where $\alpha = \overline{U}_0 + \lambda_n^{-1}$, $\beta = -\overline{U}_0 + \lambda_n^{-1}$ are some constants depending on the mode that we consider. We restrict ourselves to the subcritical modes that are the most important and the most challenging and, in that case n is such that $\beta > 0$. The reader is referred to the articles quoted before for more discussion about these modes. The parameter $\varepsilon = \delta/2N^2\lambda_n^2$ is proportional to δ, hence is devoted to tend to zero. We supplement these equations with the following (nonreflecting) boundary conditions:

$$\left\{\begin{array}{rcl}\xi(0,t) & = & 0, \\ \eta(L,t) & = & 0, \\ \dfrac{\partial \xi}{\partial t}(L,t) + \alpha(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial x})(L,t) & = & f(L,t) - \dfrac{\alpha}{\beta}g(L,t), \\ \dfrac{\partial \eta}{\partial t}(0,t) - \beta(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial x})(0,t) & = & g(0,t) - \dfrac{\beta}{\alpha}f(0,t),\end{array}\right. \tag{2.11}$$

for every $t > 0$.

Working with finite differences in space, for $0 \leq j \leq N$ we set:

$$\left\{\begin{array}{l}\xi_j(t) = \xi(jh,t), \\ \eta_j(t) = \eta(jh,t),\end{array}\right. \tag{2.12}$$

where $h = \Delta x = L/N$ is the mesh size. The discretization in space given in [RTT05] reads, for every $1 \leq j \leq N-1$, $t > 0$:

$$\begin{cases} \frac{d\xi_j}{dt}(t) + \alpha\,(\overline{\nabla}_h \xi)_j(t) - \varepsilon\,[\nabla_h \overline{\nabla}_h (\xi+\eta)]_j(t) = f_j(t), \\ \frac{d\eta_j}{dt}(t) - \beta\,(\nabla_h \eta)_j(t) - \varepsilon\,[\nabla_h \overline{\nabla}_h (\xi+\eta)]_j(t) = g_j(t), \end{cases} \tag{2.13}$$

where ∇_h and $\overline{\nabla}_h$ are the following discrete operators:

$$\begin{aligned} (\nabla_h \varphi)_j &= \frac{\varphi_{j+1} - \varphi_j}{h},\ \forall j = 0..N-1, \\ (\overline{\nabla}_h \varphi)_j &= \frac{\varphi_j - \varphi_{j-1}}{h},\ \forall j = 1..N. \end{aligned}$$

Finally, we have the following discrete boundary conditions:

$$\begin{cases} \xi_0(t) = 0, \\ \eta_N(t) = 0, \end{cases} \quad \forall t > 0, \tag{2.14}$$

and for every $t > 0$,

$$\begin{cases} \frac{d\xi_N}{dt}(t) + \frac{\alpha}{h}\,(\xi_N - \xi_{N-1} - \eta_{N-1})(t) = f_N(t) - \frac{\alpha}{\beta}\,g_N(t), \\ \frac{d\eta_0}{dt}(t) - \frac{\beta}{h}\,(\xi_1 + \eta_1 - \eta_0)(t) = g_0(t) - \frac{\beta}{\alpha}\,f_0(t). \end{cases} \tag{2.15}$$

Let us now set the functional framework of the problem.

For $U = (\xi_1, ..., \xi_{N-1}, \eta_1, ..., \eta_{N-1}, \xi_N, \eta_0) \in H = \mathbb{R}^{2N}$, we define the following scalar product :

$$(U, \tilde{U})_H = \sum_{j=1}^{N-1} h\,\xi_j \tilde{\xi}_j + \sum_{j=1}^{N-1} h\,\eta_j \tilde{\eta}_j + \frac{\varepsilon}{\alpha}\,\xi_N\,\tilde{\xi}_N + \frac{\varepsilon}{\beta}\,\eta_0\,\tilde{\eta}_0. \tag{2.16}$$

Given some continuous functions (f, g), we set, using the same notation as in (2.12):

$$F = (f_1, ..., f_{N-1}, g_1, ..., g_{N-1}, f_N - \frac{\alpha}{\beta}\,g_N, g_0 - \frac{\beta}{\alpha}\,f_0). \tag{2.17}$$

In the sequel, we will prove the stability of classical time discretisation schemes applied to equations (2.13)-(2.15), the consistency with the continuous equations (2.10)-(2.11) being easy. However, we have not been able to show that, in the limit $\Delta x \longrightarrow 0$, the boundary value problem (2.10)-(2.11) is well-posed and, in fact, a naive count of the number of equations indicates that the continuous system may be overdetermined if no precautions are taken. However, as we said in [RTT04, RTT05], this perturbed system has some computational advantages, and for $\Delta x > 0$ fixed it is well-posed, the limit system is also well-posed (but this is standard), and the perturbed system does converge to the expected limit when $\varepsilon \longrightarrow 0$ (Δx fixed).

3 The Implicit Euler time scheme

3.1 Discretization of the equations and boundary conditions

We now give the time discretization for (2.13) and (2.15) based on the implicit Euler scheme. For each $m \leq M$, we denote by u^m the quantity $u(m\Delta t)$ where $\Delta t = T/M$ is the time meshsize. Inside the domain, we have, $\forall 1 \leq j \leq N-1$, $\forall 1 \leq m \leq M$:

$$\left\{ \begin{array}{lcl} \dfrac{\xi_j^m - \xi_j^{m-1}}{\Delta t} + \alpha\, \dfrac{\xi_j^m - \xi_{j-1}^m}{h} - \varepsilon\, \dfrac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{h^2} & = & f_j^m, \\ \dfrac{\eta_j^m - \eta_j^{m-1}}{\Delta t} - \beta\, \dfrac{\eta_{j+1}^m - \eta_j^m}{h} - \varepsilon\, \dfrac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{h^2} & = & g_j^m, \end{array} \right. \tag{3.1}$$

For the sake of simplicity, in the relation (3.3) above and in the sequel, we denote by v_j^m the quantity $(\xi_j^m + \eta_j^m)$. On the boundary, equation (2.15) gives, $\forall 1 \leq m \leq M$:

$$\left\{ \begin{array}{lclclcl} \dfrac{\xi_N^m - \xi_N^{m-1}}{\Delta t} & + & \alpha\, \dfrac{\xi_N^m - \xi_{N-1}^m}{h} & - & \dfrac{\alpha}{h}\, \eta_{N-1}^m & = & f_N^m - \dfrac{\alpha}{\beta}\, g_N^m, \\ \dfrac{\eta_0^m - \eta_0^{m-1}}{\Delta t} & - & \beta\, \dfrac{\eta_1^m - \eta_0^m}{h} & - & \dfrac{\beta}{h}\, \xi_1^m & = & g_0^m - \dfrac{\beta}{\alpha}\, f_0^m. \end{array} \right. \tag{3.2}$$

3.2 Proof of stability

Let us prove that the implicit scheme (3.1)-(3.2) is stable, with no condition on Δt, Δx or ε. In order to recover the scalar product in H, we multiply $(3.1)_1$ by $h\,\xi_j^m$ for $1 \leq j \leq N-1$, $(3.1)_2$ by $h\,\eta_j^m$ for $1 \leq j \leq N-1$, $(3.2)_1$ by $\varepsilon\,\xi_N^m/\alpha$, and $(3.2)_2$ by $\varepsilon\,\eta_0^m/\beta$ and sum all these equations, and obtain:

$$
\begin{aligned}
&\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^m|^2 + \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^m|^2 + \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m|^2 \\
&-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m-1}|^2 - \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m-1}|^2 - \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m-1}|^2 - \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m-1}|^2 \\
&+\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^m - \xi_j^{m-1}|^2 + \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^m - \eta_j^{m-1}|^2 \\
&+\frac{\alpha}{2}\sum_{j=1}^{N-1} |\xi_j^m - \xi_{j-1}^m|^2 + \frac{\beta}{2}\sum_{j=1}^{N-1} |\eta_{j+1}^m - \eta_j^m|^2 + \frac{\alpha}{2}|\xi_{N-1}^m|^2 + \frac{\beta}{2}|\eta_1^m|^2 \\
&+\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m - \xi_N^{m-1}|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m - \eta_0^{m-1}|^2 \\
&-\frac{\varepsilon}{h}\sum_{j=1}^{N-1} (v_{j+1}^m - v_j^m)\,v_j^m - \frac{\varepsilon}{h}\sum_{j=1}^{N-1} (v_{j-1}^m - v_j^m)\,v_j^m \\
&+\frac{\varepsilon}{h}\,(\xi_N^m - \xi_{N-1}^m - \eta_{N-1}^m)\,\xi_N^m - \frac{\varepsilon}{h}\,(\eta_1^m - \eta_0^m + \xi_1^m)\,\eta_0^m \\
&= \sum_{j=1}^{N-1} h\, f_j^m\,\xi_j^m + \sum_{j=1}^{N-1} h\, g_j^m\,\eta_j^m + \frac{\varepsilon}{\alpha}\,(f_N^m - \frac{\alpha}{\beta}\,g_N^m)\,\xi_N^m + \frac{\varepsilon}{\beta}\,(g_0^m - \frac{\beta}{\alpha}\,f_0^m)\,\eta_0^m.
\end{aligned}
\tag{3.3}
$$

Thanks to some easy computations, we find:

$$
\begin{aligned}
&-\frac{\varepsilon}{h}\sum_{j=1}^{N-1} (v_{j+1}^m - v_j^m)\,v_j^m - \frac{\varepsilon}{h}\sum_{j=1}^{N-1} (v_{j-1}^m - v_j^m)\,v_j^m \\
&= \frac{\varepsilon}{h}\sum_{j=0}^{N-1} (v_{j+1}^m - v_j^m)^2 \\
&-\frac{\varepsilon}{h}\,(\eta_0^m - \xi_1^m - \eta_1^m)\,\eta_0^m + \frac{\varepsilon}{h}\,(\xi_{N-1}^m + \eta_{N-1}^m - \xi_N^m)\,\xi_N^m
\end{aligned}
\tag{3.4}
$$

Using (3.4), equation (3.3) becomes:

$$
\begin{aligned}
&\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^m|^2 + \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^m|^2 + \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m|^2 \\
&-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m-1}|^2 - \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m-1}|^2 - \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m-1}|^2 - \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m-1}|^2 \\
&+\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^m - \xi_j^{m-1}|^2 + \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^m - \eta_j^{m-1}|^2 \\
&+\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m - \xi_N^{m-1}|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m - \eta_0^{m-1}|^2 \\
&+\frac{\alpha}{2}\sum_{j=1}^{N-1} |\xi_j^m - \xi_{j-1}^m|^2 + \frac{\beta}{2}\sum_{j=1}^{N-1} |\eta_{j+1}^m - \eta_j^m|^2 + \frac{\alpha}{2}|\xi_{N-1}^m|^2 + \frac{\beta}{2}|\eta_1^m|^2 \\
&+\frac{\varepsilon}{h}\sum_{j=0}^{N-1} (v_{j+1}^m - v_j^m)^2
\end{aligned}
\tag{3.5}
$$

$$
= \sum_{j=1}^{N-1} h\, f_j^m\, \xi_j^m + \sum_{j=1}^{N-1} h\, g_j^m\, \eta_j^m + \frac{\varepsilon}{\alpha}\,(f_N^m - \frac{\alpha}{\beta}\, g_N^m)\, \xi_N^m + \frac{\varepsilon}{\beta}\,(g_0^m - \frac{\beta}{\alpha}\, f_0^m)\, \eta_0^m.
$$

Using the functional framework defined in Section 2, equation (3.5) can be rewritten as:

$$
\begin{aligned}
&\frac{1}{2\,\Delta t}\,|U^m|_H^2 - \frac{1}{2\,\Delta t}\,|U^{m-1}|_H^2 + \frac{1}{2\,\Delta t}\,|U^m - U^{m-1}|_H^2 \\
&+\frac{\alpha}{2\,h}\sum_{j=1}^{N-1} h\,|\xi_j^m - \xi_{j-1}^m|^2 + \frac{\beta}{2\,h}\sum_{j=1}^{N-1} h\,|\eta_{j+1}^m - \eta_j^m|^2 \\
&+\frac{\alpha}{2}\,|\xi_{N-1}^m|^2 + \frac{\beta}{2}\,|\eta_1^m|^2 + \frac{\varepsilon}{h}\sum_{j=0}^{N-1} (v_{j+1}^m - v_j^m)^2 \\
&= (F, U^m)_H.
\end{aligned}
\tag{3.6}
$$

After dropping some positive terms[2], and using the Cauchy-Schwarz inequality we find, with $F \in L^\infty(0,T;H)$:

$$
\frac{1}{2\,\Delta t}\,|U^m|_H^2 \le \frac{1}{2\,\Delta t}\,|U^{m-1}|_H^2 + |F|_\infty\,|U^m|_H. \tag{3.7}
$$

Thanks to some easy computations, (3.7) implies:

$$
|U^m|_H^2 \le \frac{1}{1-\Delta t}\,|U^{m-1}|_H^2 + \frac{\Delta t}{1-\Delta t}\,|F|_\infty^2. \tag{3.8}
$$

Writing (3.8) for $m = 1, ..., m_0$, $m_0 \le M$, we find recursively:

$$
|U^{m_0}|_H^2 \le \frac{1}{(1-\Delta t)^{m_0}}\,|U^0|_H^2 + \frac{\Delta t}{1-\Delta t}\,|F|_\infty^2\,(1 + \frac{1}{1-\Delta t} + ... + \frac{1}{(1-\Delta t)^{m_0-1}}). \tag{3.9}
$$

[2]We recall that the modes we consider are such that $\alpha > \beta > 0$.

Assuming that $\Delta t < 1$, we obtain:

$$|U^{m_0}|_H^2 \leq \frac{1}{(1-\Delta t)^{m_0}}\,|U^0|_H^2 + \frac{1}{(1-\Delta t)^{m_0}}\,|F|_\infty^2. \tag{3.10}$$

Finally, we use the classical inequality $e^{-2x} \geq 1 - x$ valid for every $x \in [0, x^*]$ where $x^* =?$. Assuming then that $0 < \Delta t < min(1, x^*)$, we find for every $m \leq M$:

$$|U^m|_H^2 \leq e^{2m\Delta t}\,(|U^0|_H^2 + |F|_\infty^2) \leq e^{2T}\,(|U^0|_H^2 + |F|_\infty^2), \tag{3.11}$$

which guarantees the stability of our scheme.

4 The explicit Euler time scheme

4.1 Discretization of the equations and boundary conditions

We now give the time discretization of (2.13) and (2.15) using the explicit Euler scheme. Inside the domain, we have, $\forall 1 \leq j \leq N-1, 1 \leq m \leq M$:

$$\left\{\begin{array}{lllllll} \dfrac{\xi_j^{m+1}-\xi_j^m}{\Delta t} & + & \alpha\,\dfrac{\xi_j^m-\xi_{j-1}^m}{h} & - & \varepsilon\,\dfrac{v_{j+1}^m-2\,v_j^m+v_{j-1}^m}{h^2} & = & f_j^m, \\ \dfrac{\eta_j^{m+1}-\eta_j^m}{\Delta t} & - & \beta\,\dfrac{\eta_{j+1}^m-\eta_j^m}{h} & - & \varepsilon\,\dfrac{v_{j+1}^m-2\,v_j^m+v_{j-1}^m}{h^2} & = & g_j^m. \end{array}\right. \tag{4.1}$$

On the boundary, equation (2.15) gives, $\forall 1 \leq m \leq M$:

$$\left\{\begin{array}{lllllll} \frac{\xi_N^{m+1}-\xi_N^m}{\Delta t} & + & \alpha\,\frac{\xi_N^m-\xi_{N-1}^m}{h} & - & \frac{\alpha}{h}\,\eta_{N-1}^m & = & f_N^m - \frac{\alpha}{\beta}\,g_N^m, \\ \frac{\eta_0^{m+1}-\eta_0^m}{\Delta t} & - & \beta\,\frac{\eta_1^m-\xi_0^m}{h} & - & \frac{\beta}{h}\,\xi_1^m & = & g_0^m - \frac{\beta}{\alpha}\,f_0^m. \end{array}\right. \tag{4.2}$$

4.2 Proof of stability

Here we expect a condition on $(\Delta t, h, \varepsilon)$ for the scheme to be stable. Proceeding like in Section 3.2, we multiply $(4.1)_1$ by $h\,\xi_j^m$ for $1 \leq j \leq N-1$, $(4.1)_2$ by $h\,\eta_j^m$ for $1 \leq j \leq N-1$, $(4.2)_1$ by

$\varepsilon\,\xi_N^m/\alpha$, and $(4.2)_2$ by $\varepsilon\,\eta_0^m/\beta$. We sum all these equations, and obtain:

$$
\begin{aligned}
&\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m+1}|^2 + \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m+1}|^2 + \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m+1}|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m+1}|^2 \\
&-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m}|^2 - \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m}|^2 - \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m}|^2 - \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m}|^2 \\
&-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m+1}-\xi_j^m|^2 - \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m+1}-\eta_j^m|^2 \\
&-\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m+1}-\xi_N^m|^2 - \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m+1}-\eta_0^m|^2 \\
&+\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)^2 + \frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^m-\eta_j^m)^2 + \frac{\alpha}{2}\,|\xi_{N-1}^m|^2 + \frac{\beta}{2}\,|\eta_1^m|^2 \\
&-\frac{\varepsilon}{h}\sum_{j=1}^{N-1}(v_{j+1}^m - v_j^m)\,v_j^m - \frac{\varepsilon}{h}\sum_{j=1}^{N-1}(v_{j-1}^m - v_j^m)\,v_j^m \\
&+\frac{\varepsilon}{h}\,(\xi_N^m - \xi_{N-1}^m - \eta_{N-1}^m)\,\xi_N^m - \frac{\varepsilon}{h}\,(\eta_1^m - \eta_0^m + \xi_1^m)\,\eta_0^m \\[1em]
&= \sum_{j=1}^{N-1} h\,f_j^m\,\xi_j^m + \sum_{j=1}^{N-1} h\,g_j^m\,\eta_j^m + \frac{\varepsilon}{\alpha}\,(f_N^m - \frac{\alpha}{\beta}\,g_N^m)\,\xi_N^m + \frac{\varepsilon}{\beta}\,(g_0^m - \frac{\beta}{\alpha}\,f_0^m)\,\eta_0^m.
\end{aligned}
\tag{4.3}
$$

where v_j^m has been defined above.

Using again (3.4) and the notations defined above, we obtain:

$$
\begin{aligned}
&\frac{1}{2\,\Delta t}\,|U^{m+1}|_H^2 - \frac{1}{2\,\Delta t}\,|U^m|_H^2 - \frac{1}{2\,\Delta t}\,|U^{m+1}-U^m|_H^2 \\
&+\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)^2 + \frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^m-\eta_j^m)^2 \\
&+\frac{\alpha}{2}\,|\xi_{N-1}^m|^2 + \frac{\beta}{2}\,|\eta_1^m|^2 + \frac{\varepsilon}{h}\sum_{j=0}^{N-1}(v_{j+1}^m - v_j^m)^2 \\[1em]
&= (F, U^m)_H.
\end{aligned}
\tag{4.4}
$$

We now need to estimate the quantity $|U^{m+1}-U^m|_H^2$. To this aim, we multiply $(4.1)_1$ by $h\,(\xi_j^{m+1}-\xi_j^m)/2$ for $1 \le j \le N-1$, $(4.1)_2$ by $h\,(\eta_j^{m+1}-\eta_j^m)/2$ for $1 \le j \le N-1$, $(4.2)_1$ by $\varepsilon\,(\xi_N^{m+1}-\xi_N^m)/2\,\alpha$, and $(4.2)_2$ by $\varepsilon\,(\eta_0^{m+1}-\eta_0^m)/2\,\beta$. We sum all these equations, and obtain:

$$\begin{aligned}
&\frac{1}{2\,\Delta t}\,|U^{m+1}-U^m|^2 = \\
&-\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)\,(\xi_j^{m+1}-\xi_j^m)+\frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^m-\eta_j^m)\,(\eta_j^{m+1}-\eta_j^m) \\
&+\frac{\varepsilon}{2\,h}\sum_{j=1}^{N-1}(v_{j+1}^m-2\,v_j^m+v_{j-1}^m)\,(v_j^{m+1}-v_j^m) \\
&-\frac{\varepsilon}{2\,h}\,(\xi_N^m-\xi_{N-1}^m-\eta_{N-1}^m)\,(\xi_N^{m+1}-\xi_N^m)+\frac{\varepsilon}{2\,h}(\eta_1^m-\eta_0^m+\xi_1^m)\,(\eta_0^{m+1}-\eta_0^m) \\
&+\frac{1}{2}\,(F,U^{m+1}-U^m)_H.
\end{aligned} \tag{4.5}$$

Let us now bound terms that appear in the right hand side of equation. Firstly, we have:

$$\begin{aligned}
&-\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)\,(\xi_j^{m+1}-\xi_j^m)+\frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^m-\eta_j^m)\,(\eta_j^{m+1}-\eta_j^m) \\
&\le \frac{\alpha}{4}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)^2+\frac{\alpha}{4\,h}\sum_{j=1}^{N-1}h\,|\xi_j^{m+1}-\xi_j^m|^2 \\
&+\frac{\beta}{4}\sum_{j=1}^{N-1}(\eta_{j+1}^m j-\xi_j^m)^2+\frac{\beta}{4\,h}\sum_{j=1}^{N-1}h\,|\eta_j^{m+1}-\eta_j^m|^2
\end{aligned} \tag{4.6}$$

For the terms with ε, the main part can be bounded this way

$$\begin{aligned}
&\frac{\varepsilon}{2\,h}\sum_{j=1}^{N-1}(v_{j+1}^m-v_j^m)\,(v_j^{m+1}-v_j^m) \\
&\le \frac{\varepsilon}{4\,h}\sum_{j=1}^{N-1}[(\xi_{j+1}^m-\xi_j^m)+(\eta_{j+1}^m-\eta_j^m)]^2 \\
&+\frac{\varepsilon}{2\,h^2}\sum_{j=1}^{N-1}h\,|\xi_j^{m+1}-\xi_j^m|^2+\frac{\varepsilon}{2\,h^2}\sum_{j=1}^{N-1}h\,|\eta_j^{m+1}-\eta_j^m|^2
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\frac{\varepsilon}{2\,h}\sum_{j=1}^{N-1}(v_{j-1}^m-v_j^m)\,(v_j^{m+1}-v_j^m) \\
&\le \frac{\varepsilon}{4\,h}\sum_{j=1}^{N-1}[(\xi_j^m-\xi_{j-1}^m)+(\eta_j^m-\eta_{j-1}^m)]^2 \\
&+\frac{\varepsilon}{2\,h^2}\sum_{j=1}^{N-1}h\,|\xi_j^{m+1}-\xi_j^m|^2+\frac{\varepsilon}{2\,h^2}\sum_{j=1}^{N-1}h\,|\eta_j^{m+1}-\eta_j^m|^2
\end{aligned} \tag{4.8}$$

Similarly, for the boundary terms:

$$\left\{\begin{array}{l} -\frac{\varepsilon}{2h}(\xi_N^m-\xi_{N-1}^m-\xi_{N-1}^m)(\xi_N^{m+1}-\xi_N^m)\leq\frac{\varepsilon}{4h}(\xi_N^m-\xi_{N-1}^m-\xi_{N-1}^m)^2+\frac{\varepsilon}{4h}(\xi_N^{m+1}-\xi_N^m)^2,\\ \frac{\varepsilon}{2h}(\eta_1^m-\eta_0^m+\xi_1^m)(\eta_0^{m+1}-\eta_0^m)\leq\frac{\varepsilon}{4h}(\eta_1^m-\eta_0^m+\xi_1^m)^2+\frac{\varepsilon}{4h}(\eta_0^{m+1}-\eta_0^m)^2. \end{array}\right. \tag{4.9}$$

Finally, using inequalities (4.6)-(4.9) and the former equation (3.4), equation (4.5) becomes:

$$\begin{array}{l} \frac{1}{2\,\Delta t}\,|U^{m+1}-U^m|^2\leq\frac{1}{2}\,|F|_H\,|U^{m+1}-U^m|_H\\ +\frac{\alpha}{4}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)^2+(\frac{\alpha}{4\,h}+\frac{\varepsilon}{2\,h^2})\sum_{j=1}^{N-1}h\,|\xi_j^{m+1}-\xi_j^m|^2\\ +\frac{\beta}{4}\sum_{j=1}^{N-1}(\eta_{j+1}^m-\eta_j^m)^2+(\frac{\beta}{4\,h}+\frac{\varepsilon}{2\,h^2})\sum_{j=1}^{N-1}h\,|\eta_j^{m+1}-\eta_j^m|^2\\ +\frac{\varepsilon}{2\,h}\sum_{j=0}^{N-1}(v_{j+1}^m-v_j^m)^2+\frac{\varepsilon}{4\,h}(\xi_N^{m+1}-\xi_N^m)^2+\frac{\varepsilon}{4\,h}(\eta_0^{m+1}-\eta_0^m)^2 \end{array} \tag{4.10}$$

Now, since $a\,b\leq a^2/2\,\mu+\mu\,b^2/2$ for every $(a,b)\in\mathbb{R}^2$ and $\mu>0$, we find, thanks to the fact that $0<\beta<\alpha$:

$$\begin{array}{l} (\frac{1}{2\,\Delta t}-\frac{1}{4\,\mu\,\Delta t}-\frac{\alpha}{4\,h}-\frac{\varepsilon}{h^2})\,|U^{m+1}-U^m|^2\\ \leq\frac{\alpha}{4}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)^2+\frac{\beta}{4}\sum_{j=1}^{N-1}(\eta_{j+1}^m-\eta_j^m)^2\\ +\frac{\varepsilon}{2\,h}\sum_{j=0}^{N-1}(v_{j+1}^m-v_j^m)^2\\ +\frac{\mu\,\Delta t}{4}\,|F|_\infty^2 \end{array} \tag{4.11}$$

Let us assume that $R=1-\frac{1}{2\,\mu}-\frac{\alpha\,\Delta t}{2\,h}-\frac{2\,\varepsilon\,\Delta t}{h^2}$ is positive. Returning to (4.4), we obtain:

$$\begin{array}{l} \frac{1}{2\,\Delta t}\,|U^{m+1}|_H^2-\frac{1}{2\,\Delta t}\,|U^m|_H^2+\frac{\varepsilon}{h}\sum_{j=0}^{N-1}(v_{j+1}^m-v_j^m)^2\\ +\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)^2+\frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^m-\eta_j^m)^2+\frac{\alpha}{2}\,|\xi_{N-1}^m|^2+\frac{\beta}{2}\,|\eta_1^m|^2\\ \leq|F|_\infty\,|U^m|_H+\frac{\varepsilon}{2\,h\,R}\sum_{j=0}^{N-1}(v_{j+1}^m-v_j^m)^2\\ +\frac{\alpha}{4\,R}\sum_{j=1}^{N-1}(\xi_j^m-\xi_{j-1}^m)^2+\frac{\beta}{4\,R}\sum_{j=1}^{N-1}(\eta_{j+1}^m-\eta_j^m)^2 \end{array} \tag{4.12}$$

We obtain the stability result if $R > 1/2$, that is[3]:

$$0 < \frac{\alpha\,\Delta t}{h} + \frac{4\,\Delta t\,\varepsilon}{h^2} < 1 \tag{4.13}$$

Remark 4.1. *:* We note that the condition (4.13) matches the classical CFL condition if ε equals to zero.

5 Crank Nicholson scheme

5.1 Discretization of the equations and boundary conditions

The discretization in time of equations (2.10)-(2.11) using the C-N scheme reads:

$$\left\{\begin{array}{lcl} \dfrac{\xi_j^{m+1}-\xi_j^m}{\Delta t} + \alpha\,\dfrac{\xi_j^{m+1/2}-\xi_{j-1}^{m+1/2}}{h} - \varepsilon\,\dfrac{v_{j+1}^{m+1/2}-2\,v_j^{m+1/2}+v_{j-1}^{m+1/2}}{h^2} & = & f_j^{m+1/2}, \\ \dfrac{\eta_j^{m+1}-\eta_j^m}{\Delta t} - \beta\,\dfrac{\eta_{j+1}^{m+1/2}-\eta_j^{m+1/2}}{h} - \varepsilon\,\dfrac{v_{j+1}^{m+1/2}-2\,v_j^{m+1/2}+v_{j-1}^{m+1/2}}{h^2} & = & g_j^{m+1/2}, \end{array}\right. \tag{5.1}$$

where $u^{m+1/2}$ naturally denotes the quantity $(u^{m+1}+u^m)/2$.

The boundary conditions read:

$$\left\{\begin{array}{lcl} \dfrac{\xi_N^{m+1}-\xi_N^m}{\Delta t} + \alpha\,\dfrac{\xi_N^{m+1/2}-\xi_{N-1}^{m+1/2}}{h} - \dfrac{\alpha}{h}\,\eta_{N-1}^{m+1/2} & = & f_N^{m+1/2} - \dfrac{\alpha}{\beta}\,g_N^{m+1/2}, \\ \dfrac{\eta_0^{m+1}-\eta_0^m}{\Delta t} - \beta\,\dfrac{\eta_1^{m+1/2}-\xi_0^{m+1/2}}{h} + \dfrac{\beta}{h}\,\xi_1^{m+1/2} & = & g_0^{m+1/2} - \dfrac{\beta}{\alpha}\,f_0^{m+1/2}. \end{array}\right. \tag{5.2}$$

5.2 Proof of stability

We claim that the stability holds for every set of parameters $(\Delta t, h, \varepsilon)$ (unconditional stability). This time we multiply $(5.1)_1$ by $h\,\xi_j^{m+1/2}$ for $1 \le j \le N-1$, $(5.1)_2$ by $h\,\eta_j^{m+1/2}$ for $1 \le j \le N-1$,

[3]If the condition (4.13) is satisfied, one can easily find $\mu > 0$ such that $R > 1/2$.

$(5.2)_1$ by $\varepsilon\,\xi_N^{m+1/2}/\alpha$, and $(5.2)_2$ by $\varepsilon\,\eta_0^{m+1/2}/\beta$. We sum all these equations, and obtain:

$$\begin{aligned}
&\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m+1}|^2 + \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m+1}|^2 + \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m+1}|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m+1}|^2\\
&-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m}|^2 - \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m}|^2 - \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m}|^2 - \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m}|^2\\
&+\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^{m+1/2}-\xi_{j-1}^{m+1/2})^2 + \frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^{m+1/2}-\eta_j^{m+1/2})^2 + \frac{\alpha}{2}\,|\xi_{N-1}^{m+1/2}|^2 + \frac{\beta}{2}\,|\eta_1^{m+1/2}|^2\\
&-\frac{\varepsilon}{h^2}\sum_{j=1}^{N-1}[v_{j+1}^{m+1/2}-2v_j^{m+1/2}+v_{j-1}^{m+1/2}]v_j^{m+1/2}\\
&+\frac{\varepsilon}{h}\,(\xi_N^{m+1/2}-\xi_{N-1}^{m+1/2}-\eta_{N-1}^{m+1/2})\,\xi_N^{m+1/2} - \frac{\varepsilon}{h}\,(\eta_1^{m+1/2}-\eta_0^{m+1/2}+\xi_1^{m+1/2})\,\eta_0^{m+1/2}\\
&= (F, U^{m+1/2})_H
\end{aligned} \tag{5.3}$$

Again, we use (3.4) and obtain, for the Crank Nicholson scheme:

$$\begin{aligned}
&\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m+1}|^2 + \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m+1}|^2 + \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m+1}|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m+1}|^2\\
&-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m}|^2 - \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m}|^2 - \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m}|^2 - \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m}|^2\\
&+\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^{m+1/2}-\xi_{j-1}^{m+1/2})^2 + \frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^{m+1/2}-\eta_j^{m+1/2})^2 + \frac{\alpha}{2}\,|\xi_{N-1}^{m+1/2}|^2 + \frac{\beta}{2}\,|\eta_1^{m+1/2}|^2\\
&+\frac{\varepsilon}{h}\sum_{j=0}^{N-1}(v_{j+1}^{m}-v_j^{m})^2 = (F, U^{m+1/2})_H
\end{aligned} \tag{5.4}$$

Hence, we find:

$$\frac{1}{\Delta t}\,|U^{m+1}|_H^2 \le \frac{1}{\Delta t}\,|U^m|_H^2 + |F|_H\,|U^{m+1}+U^m|_H. \tag{5.5}$$

After some computations, we obtain:

$$|U^{m+1}|_H^2 \le \frac{2+\Delta t}{2-\Delta t}\,|U^m|_H^2 + \frac{2\,\Delta t}{2-\Delta t}\,|F|_\infty^2 \tag{5.6}$$

Similarly to Section 4, we write these inequalities for every m, and finally obtain:

$$\begin{aligned}
|U^{m+1}|_H^2 &\le \left(\frac{2+\Delta t}{2-\Delta t}\right)^{m+1}|U^0|_H^2 + \left(\frac{2+\Delta t}{2-\Delta t}\right)^{m}|F|_\infty^2\\
&\le e^{2T}\,(|U^0|_H^2 + |F|_\infty^2),
\end{aligned} \tag{5.7}$$

which guarantees the finite time stability of the scheme with no additional condition.

6 Fractional scheme

6.1 Discretization of the equations and boundary conditions

In this section we use the fractional scheme method, (see [Mar71] and, for the Navier-Stokes equations, [Tem69]), which consists in splitting each time step into several (here two) intermediate steps. The advantage is that the numerical computations for each intermediate step are easier, while the stability result does not require any condition on the parameters $(\Delta t, h, \varepsilon)$.
Let us now describe the two intermediate steps, and give the semi-discretized (in time) schemes. We consider the previous system (2.9) with the subscripts n dropped, and reintroduce the parameter ε:

$$\left\{ \begin{array}{lllllllll} \dfrac{\partial u}{\partial t} & + & \overline{U}_0 \dfrac{\partial u}{\partial x} & - & \dfrac{1}{N\lambda}\dfrac{\partial \psi}{\partial x} & - & 2\varepsilon \dfrac{\partial^2 u}{\partial x^2} & = & F_u, \\ \dfrac{\partial \psi}{\partial t} & + & \overline{U}_0 \dfrac{\partial \psi}{\partial x} & - & \dfrac{N}{\lambda}\dfrac{\partial u}{\partial x} & & & = & F_\psi. \end{array} \right. \tag{6.1}$$

Since the second space derivative does not occur in the second equation (6.1), we choose the two intermediate steps as follows. The first intermediate step $m+1/2$ (between m and $m+1$) reads:

$$\left\{ \begin{array}{lll} \dfrac{u^{m+1/2} - u^m}{\Delta t} + \overline{U}_0\, u_x^{m+1/2} - \dfrac{1}{N\lambda}\psi_x^{m+1/2} & = & F_u^m, \\ \dfrac{\psi^{m+1/2} - \psi^m}{\Delta t} + \overline{U}_0\, \psi_x^{m+1/2} - \dfrac{N}{\lambda} u_x^{m+1/2} & = & F_\psi^m. \end{array} \right. \tag{6.2}$$

For the second intermediate step, we set:

$$\left\{ \begin{array}{lll} \dfrac{u^{m+1} - u^{m+1/2}}{\Delta t} - 2\,\varepsilon\, u_{xx}^{m+1} & = & 0, \\ \dfrac{\psi^{m+1} - \psi^{m+1/2}}{\Delta t} & = & 0. \end{array} \right. \tag{6.3}$$

Let us now go back to the notation (ξ, η), and rewrite (6.2) and (6.3), discretized in space. For every $1 \leq j \leq N-1$:

$$\frac{\xi_j^{m+1/2} - \xi_j^m}{\Delta t} + \alpha\, \frac{\xi_j^{m+1/2} - \xi_{j-1}^{m+1/2}}{h} = f_j^m, \tag{6.4a}$$

$$\frac{\eta_j^{m+1/2} - \eta_j^m}{\Delta t} - \beta\, \frac{\eta_{j+1}^{m+1/2} - \eta_j^{m+1/2}}{h} = g_j^m, \tag{6.4b}$$

We supplement equations (6.4) with the following natural boundary conditions:

$$\xi_0^{m+1/2} = 0, \tag{6.5a}$$

$$\eta_N^{m+1/2} = 0, \tag{6.5b}$$

$$\frac{\xi_N^{m+1/2} - \xi_N^m}{\Delta t} = f_N^m - \frac{\alpha}{\beta} g_N^m, \tag{6.5c}$$

$$\frac{\eta_0^{m+1/2} - \eta_0^m}{\Delta t} = g_0^m - \frac{\beta}{\alpha} f_0^m. \tag{6.5d}$$

Given (ξ_j^m, η_j^m) with $0 \le j \le N$, one can easily compute from (6.4) and (6.5) the intermediate solution $(\xi_j^{m+1/2}, \eta_j^{m+1/2})$ with $0 \le j \le N$. For the second intermediate step $m+1$, we have for $1 \le j \le N-1$:

$$\frac{\xi_j^{m+1} - \xi_j^{m+1/2}}{\Delta t} - \varepsilon \frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{h^2} = 0, \tag{6.6a}$$

$$\frac{\eta_j^{m+1} - \eta_j^{m+1/2}}{\Delta t} - \varepsilon \frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{h^2} = 0, \tag{6.6b}$$

We supplement equations (6.6) with the following boundary conditions:

$$\xi_0^{m+1} = 0, \tag{6.7a}$$

$$\eta_N^{m+1} = 0, \tag{6.7b}$$

$$\frac{\xi_N^{m+1} - \xi_N^{m+1/2}}{\Delta t} + \alpha \frac{\xi_N^{m+1} - \xi_{N-1}^{m+1} - \eta_{N-1}^{m+1}}{h} = 0, \tag{6.7c}$$

$$\frac{\eta_0^{m+1} - \eta_0^{m+1/2}}{\Delta t} - \beta \frac{\eta_1^{m+1} - \eta_0^{m+1} + \xi_1^{m+1}}{h} = 0. \tag{6.7d}$$

Knowing $(\xi_j^{m+1/2}, \eta_j^{m+1/2})$ with $0 \le j \le N$, we can finally compute $(\xi_j^{m+1}, \eta_j^{m+1})$ with $0 \le j \le N$, using the relations (6.6) and (6.7).
Before going into the proof of stability, let us observe that this two-steps scheme is consistent. To this aim, we express $(\xi^{m+1/2}, \eta^{m+1/2})$ from (6.6) and substitute it in (6.4). Eventually we recover the usual consistency, with the help of Taylor expansions.

6.2 Proof of stability

We start by multiplying (6.4a) by $h\,\xi_j^{m+1/2}$ and (6.4b) by $h\,\eta_j^{m+1/2}$ for $1 \le j \le N-1$. We sum and obtain, thanks to (6.5a) and (6.5b):

$$\begin{cases} \frac{1}{2\Delta t}\sum_{j=1}^{N-1} h|\xi_j^{m+1/2}|^2 \le \frac{1}{2\Delta t}\sum_{j=1}^{N-1} h|\xi_j^m|^2 + \sum_{j=1}^{N-1} hf_j^m\xi_j^{m+1/2}, \\ \frac{1}{2\Delta t}\sum_{j=1}^{N-1} h|\eta_j^{m+1/2}|^2 \le \frac{1}{2\Delta t}\sum_{j=1}^{N-1} h|\eta_j^m|^2 + \sum_{j=1}^{N-1} hg_j^m\eta_j^{m+1/2}. \end{cases} \tag{6.8}$$

In order to recover the scalar product of H, we also multiply (6.5c) by $\varepsilon\xi_N^{m+1/2}/\alpha$, (6.5d) by $\varepsilon\eta_0^{m+1/2}/\beta$ and obtain:

$$\begin{cases} \frac{1}{2\Delta t}\frac{\varepsilon}{\alpha}|\xi_N^{m+1/2}|^2 \le \frac{1}{2\Delta t}\frac{\varepsilon}{\alpha}|\xi_N^m|^2 + \frac{\varepsilon}{\alpha}\xi_N^{m+1/2}(f_N^m - \frac{\alpha}{\beta}g_N^m), \\ \frac{1}{2\Delta t}\frac{\varepsilon}{\beta}|\eta_0^{m+1/2}|^2 \le \frac{1}{2\Delta t}\frac{\varepsilon}{\beta}|\eta_0^m|^2 + \frac{\varepsilon}{\beta}\eta_0^{m+1/2}(g_0^m - \frac{\beta}{\alpha}f_0^m). \end{cases} \tag{6.9}$$

We now use (6.8) together with (6.9), and find:

$$\frac{1}{2\Delta t}|U^{m+1/2}|_H^2 \le \frac{1}{2\Delta t}|U^m|_H^2 + |F|_\infty\,|U^{m+1/2}|_H. \tag{6.10}$$

For the second intermediate step, we multiply (6.6a) by $h\,\xi_j^{m+1}$ and (6.4b) by $h\,\eta_j^{m+1/2}$, for $1 \le j \le N-1$. We also multiply (6.7c) by $\varepsilon\,\xi_N^{m+1}/\alpha$, (6.7d) by $\varepsilon\,\eta_0^{m+1}/\beta$ and we add the resulting equations. Using (3.4) again, we finally have:

$$|U^{m+1}|_H^2 \le |U^{m+1/2}|_H^2. \tag{6.11}$$

From inequalities (6.10) and (6.11), we easily obtain:

$$|U^{m+1}|_H^2 \le \frac{\Delta t}{1-\Delta t}\,|F|_\infty^2 + \frac{1}{1-\Delta t}\,|U^m|_H^2, \tag{6.12}$$

and we are thus back to (3.8) of Section 3, which guarantees the stability of the scheme, with no condition on $(\Delta t, h, \varepsilon)$.

To conclude this section, we emphasize the fact that from the numerical point of view, the above fractional scheme method is more convenient; the first part (6.4)-(6.5) is quite easy to implement while the second part (6.6)-(6.7) can be written as follows, with $v = \xi + \eta$ $(= 2u)$ and $w = \xi - \eta$ $(= -2\psi/N)$:

$$\frac{v_j^{m+1} - v_j^{m+1/2}}{\Delta t} - 2\varepsilon\frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{\Delta x^2} = 0, \tag{6.13}$$

with the boundary conditions:

$$\begin{cases} \frac{v_N^{m+1} - v_N^{m+1/2}}{\Delta t} + \alpha\frac{v_N^{m+1} - v_{N-1}^{m+1}}{h} = 0, \\ \frac{v_0^{m+1} - v_0^{m+1/2}}{\Delta t} + \beta\frac{v_1^{m+1} - v_0^{m+1}}{h} = 0. \end{cases} \tag{6.14}$$

These equations on v are decoupled from those on w which are:

$$\frac{w_j^{m+1} - w_j^{m+1/2}}{\Delta t} = 0, \quad 1 \leq j \leq N - 1. \tag{6.15}$$

The advantage for these equations in w is that they do not depend on the space discretization, so that there is no linear system to solve. Also, although we did not perform error analysese in this article, we conjecture that, alternating the steps $(m + 1/2,\ m + 1)$, using the classical procedure of Strang [Str68], we would obtain here a scheme of second order in time; these questions will be addressed elewhere.

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Seconde Partie

Sur les équations primitives de l'océan sans viscosité.

Chapitre 5

Conditions aux limites pour les EPs de l'océan 2D linéarisées

Boundary conditions for the 2D linearized PEs of the Ocean

Ce chapitre est constitué de l'article **Boundary conditions for the 2D linearized PEs of the Ocean in the absence of viscosity**, à paraître en 2005 dans *Discrete and Continuous Dynamical Systems.* Réalisé avec les mêmes auteurs que ceux des chapitres 2 et 3, ce travail met en évidence le caractère bien posé des EPs sans viscosité munies de conditions aux limites (nonlocales) adaptées. L'outil principal est la théorie des semi-groupes linéaires, grâce à laquelle on démontre des résultats d'existence et d'unicité de solutions pour les équations primitives sans viscosité bidimensionnelles linéarisées.

Discrete and Continuous Dynamical Systems, 2005, to appear.

Boundary conditions for the 2D linearized PEs of the Ocean in the absence of viscosity

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Abstract

The linearized Primitive Equations with vanishing viscosity are considered. Some new boundary conditions (of transparent type) are introduced in the context of a modal expansion of the solution which consist of an infinite sequence of integral equations. Applying the linear semi-group theory, existence and uniqueness of solutions is established. The case with nonhomogeneous boundary values, encountered in numerical simulations in limited domains, is also discussed.

Introduction

The Primitive Equations of the ocean and the atmosphere are fundamental equations of geophysical fluid mechanics ([Ped87],[WP86],[Sal98]). In the presence of viscosity, it has been shown, in various contexts, that these equations are well-posed (see e.g. [LTW92a],[LTW92b], and the review article [TZ04]). The viscosity appearing in [LTW92a] is the usual second order dissipation term. Other viscosity terms have also been considered as in the so-called δ-PEs proposed with different motivations in [TT03] and [Sal98]. It has been shown in [PR05] and [TT03] that the mild vertical viscosity appearing in the δ-PEs is sufficient to guarantee well-posedness.
It is generally accepted that the viscosity terms do not affect numerical simulations (predictions) in a limited domain, over a period of a few days, and these viscosities are generally not used, see [WPT97].
Now, for the PEs without viscosity, and to the best of our knowledge, no result of well-posedness has ever been proven, since the negative result of Oliger and Sundström [OS78] showing that

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these equations are ill-posed for any set of local boundary conditions (see also the analysis in [TT03]).
Whereas the analysis of the PEs with viscosity bears some similarity with that of the incompressible Navier Stokes equations (see [LTW92a, LTW92b, TZ04]), it is noteworthy that the result of [OS78] shows that the PEs without viscosity are definitely different from the Euler equations of fluid dynamics, and it is expected that totally different boundary conditions of nonlocal type will be required.
In this article the full 2D-PEs, without viscosity, and linearized around a stratified state with constant velocity are considered. The proposed boundary conditions are of a totally new type; they consist of nonlocal boundary conditions, defined mode by mode. The well-posedness of the corresponding linearized PEs is established using the linear semi-group theory. Although the use of the Hille-Yosida theorem in this context is classical, the verification of its hypotheses is not straightforward.
Results concerning the linearized 3D-PEs will appear elsewhere. The additional difficulty in dimension three is that the verification of the hypotheses of the Hille-Yosida theorem necessitates the solution of partial differential equations, whereas in space dimension two it involves the resolution of ordinary differential equations.
A few words are in order about the nonlinear case which is our ultimate goal. Concerning well-posedness, we are faced with boundary value problems for nonlinear hyperbolic systems of equations in a limited domain, a subject not yet extensively studied (see however the important results of [Maj84, Guè90]). We believe and intend to prove that the appropriate boundary conditions for the nonlinear PEs correspond in general to those of the corresponding linearized equations. In any case the study of the well-posedness of the linear primitive equations is a necessary and important step for the problem of well-posedness of the nonlinear PEs, and this fully justifies the attention devoted here to the linearized PEs.

This article is organised as follows. In Section 1 we recall the PEs, describe the equations linearized around the stratified flow, and perform the normal modes expansion, which evidences as in [OS78, TT03] the can not be well-posed for any problem when supplemented with a set of local boundary conditions.
In Section 2 we introduce the boundary conditions which are distinct for the first set of modes (called subcritical modes) and the remaining ones (called supercritical modes). The proposed boundary conditions are furthermore of nonreflective (transparent) type (see e.g. [EM77, HR95]), making them appropriate for computations. This initial boundary value problem is then set as an abstract linear evolution equation in a suitable Hilbert space (Section 2.1). The result of existence and uniqueness of the solution is stated in Section 2.2, and Section 2.3 is devoted to the proof of the hypotheses of the Hille-Yosida theorem. To conclude, we study in Section 2.4 the case, actually encountered in numerical simulations, of nonhomogeneous boundary conditions.

1 Ill-posedness of the classical PEs

The Primitive Equations of the Ocean read :

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + (\tilde{\mathbf{v}}.\nabla)\,\tilde{\mathbf{v}} + \tilde{w}\,\frac{\partial \tilde{\mathbf{v}}}{\partial z} + f\,k \times \tilde{\mathbf{v}} + \nabla \tilde{p} = F, \tag{1.1}$$

$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho}\,g, \tag{1.2}$$

$$\nabla\,\tilde{\mathbf{v}} + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{1.3}$$

$$\frac{\partial \tilde{T}}{\partial t} + (\tilde{\mathbf{v}}.\nabla)\,\tilde{T} + \tilde{w}\,\frac{\partial \tilde{T}}{\partial z} = Q_T, \tag{1.4}$$

$$\tilde{\rho} = \rho_0\,(1 - \alpha\,(\tilde{T} - T_0)). \tag{1.5}$$

In these equations $\tilde{\mathbf{v}} = (\tilde{u}, \tilde{v})$ is the horizontal velocity, $\tilde{w}$ the vertical velocity, $\tilde{p}$ the pressure, $\tilde{\rho}$ the density, and $\tilde{T}$ the temperature; g is the gravitational acceleration, and f the Coriolis parameter. The horizontal gradient is denoted by ∇. Equation (1.5) is the equation of state of the fluid, ρ_0 and T_0 are constant reference values of $\tilde{\rho}$ and $\tilde{T}$, and $\alpha > 0$; this equation of state is linear.
Equation (1.2) is the so-called hydrostatic equation. The other equations correspond to the Boussinesq approximation (see e.g. [Ped87] and [Sal98] for more details).

1.1 Reference flow and stratification

We now consider a reference stratified flow with constant velocity $\overline{\mathbf{v}}_0 = (\overline{U_0}, 0) = \overline{U_0}\,e_x$, and density, temperature and pressure of the form $\rho_0 + \overline{\rho}$, $T_0 + \overline{T}$, $p_0 + \overline{p}$ with $d\overline{p}/dz$ constant and thus

$$\begin{aligned}
\overline{T}(z) &= \frac{N^2}{\alpha\,g}\,z, && (1.6)\\
\overline{\rho}(z) &= -\rho_0\,\alpha\,\overline{T}(z) = -\frac{\rho_0\,N^2}{g}\,z, && (1.7)\\
\frac{d\overline{T}}{dz}(z) &= \frac{N^2}{\alpha\,g}, && (1.8)\\
\frac{d\overline{\rho}}{dz}(z) &= -\frac{\rho_0}{g}\,N^2, && (1.9)\\
\frac{d\overline{p}}{dz}(z) &= -(\rho_0 + \overline{\rho})\,g. && (1.10)
\end{aligned}$$

Here N is the buoyancy frequency, assumed to be constant.

We then decompose the unknown functions $\tilde{\mathbf{v}}, \tilde{\rho}, \tilde{T}, \tilde{p}$ in the following way:

$$\left\{\begin{array}{lllllll} \tilde{\mathbf{v}} & = & \overline{U_0}\, e_x & + & \mathbf{v}(x,y,z,t), & & \\ \tilde{\rho} & = & \rho_0 & + & \overline{\rho}(z) & + & \rho(x,y,z,t), \\ \tilde{T} & = & T_0 & + & \overline{T}(z) & + & T(x,y,z,t), \\ \tilde{p} & = & p_0 & + & \overline{p}(z) & + & p(x,y,z,t). \end{array}\right. \tag{1.11}$$

Equations (1.2), (1.4) and (1.5) become

$$\frac{\partial p}{\partial z} = -\rho\, g, \tag{1.12}$$

$$\rho = -\rho_0\, \alpha\, T, \tag{1.13}$$

$$\frac{\partial T}{\partial t} + (\tilde{\mathbf{v}}.\nabla)\, T + w\, \frac{\partial T}{\partial z} + \frac{N^2}{\alpha\, g}\, w = F_T. \tag{1.14}$$

Restricting now to a 2D problem, we assume that all variables in (1.11) are independent of y and we infer from (1.1)-(1.5) and (1.12),(1.13) the following equations for u, v, w, $\phi = p/\rho_0$ and $\psi = \phi_z = \alpha\, g\, T$:

$$\frac{\partial u}{\partial t} + \overline{U_0}\, \frac{\partial u}{\partial x} + u\, \frac{\partial u}{\partial x} + w\, \frac{\partial u}{\partial z} - f\, v + \frac{\partial \phi}{\partial x} = F_u, \tag{1.15}$$

$$\frac{\partial v}{\partial t} + \overline{U_0}\, \frac{\partial v}{\partial x} + u\, \frac{\partial v}{\partial x} + w\, \frac{\partial v}{\partial z} + f\, u = F_v - f\, \overline{U_0}, \tag{1.16}$$

$$\frac{\partial \psi}{\partial t} + \overline{U_0}\, \frac{\partial \psi}{\partial x} + u\, \frac{\partial \psi}{\partial x} + w\, \frac{\partial \psi}{\partial z} + N^2\, w = F_\psi, \tag{1.17}$$

$$\frac{\partial \phi}{\partial z} = -\frac{\rho}{\rho_0}\, g = \psi, \tag{1.18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{1.19}$$

From equations (1.18) and (1.19)we find:

$$\frac{\partial \phi}{\partial x}(x,z) = \phi_s'(x) - \int_z^0 \frac{\partial \psi}{\partial x}(x,z')\, dz', \tag{1.20}$$

$$w(x,z) = \int_z^0 \frac{\partial u}{\partial x}(x,z')\, dz', \tag{1.21}$$

where $\phi_s(x,t) = \phi(x, z=0, t)$ is the surface pressure, divided by ρ_0, and ϕ_s' its derivative with respect to x.

The PEs (1.15)-(1.19), linearized around the stratified flow $\overline{\mathbf{v}}_0 = \overline{U_0}\, e_x, \overline{\rho}, \overline{T}, \overline{p}$, read:

$$\begin{aligned}
\frac{\partial u}{\partial t} + \overline{U_0}\,\frac{\partial u}{\partial x} - f\,v + \frac{\partial \phi}{\partial x} &= F_u, && (1.22)\\
\frac{\partial v}{\partial t} + \overline{U_0}\,\frac{\partial v}{\partial x} + f\,u &= F_v - f\,\overline{U_0}, && (1.23)\\
\frac{\partial \psi}{\partial t} + \overline{U_0}\,\frac{\partial \psi}{\partial x} + N^2\,w &= F_\psi, && (1.24)\\
\frac{\partial \phi}{\partial z} = -\frac{\rho}{\rho_0}\,g &= \psi, && (1.25)\\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. && (1.26)
\end{aligned}$$

We will consider the flow in the 2D domain $\mathcal{M} = (0, L_1) \times (-L_3, 0)$. Naturally, we supplement equations (1.22)-(1.26) with the following top and bottom boundary conditions (just imposed by kinematics):

$$w(x, z = -L_3, t) = w(x, z = 0, t) = 0, \;\; \forall x \in (0, L_1), t > 0. \qquad (1.27)$$

The aim of this article is to consider some lateral boundary conditions at $x = 0$ and $x = L_1$ that are both physically reasonable and computationnally satisfying[1] , and that lead to the well-posedness of the problem (1.22)-(1.26).

1.2 Normal modes

We consider a normal mode decomposition of the solution of the following form (see [TT03] for the details and the justifications):

$$(u, v, \phi) = \sum_{n \geq 0} \mathcal{U}_n(z)\, (\hat{u}_n, \hat{v}_n, \hat{\phi}_n)\, (x, t), \qquad (1.28)$$

$$(w, \psi) = \sum_{n \geq 1} \mathcal{W}_n(z)\, (\hat{w}_n, \hat{\psi}_n)\, (x, t). \qquad (1.29)$$

As explained in [TT03], for every $n \geq 1$ the functions $\mathcal{U}_n$ and $\mathcal{W}_n$ are solutions of the following eigenvalue problem:

$$\begin{aligned}
\Big(\frac{\mathcal{U}_n}{N^2}\Big)_{zz} + \lambda_n^2\, \mathcal{U}_n &= 0, && (1.30)\\
(\mathcal{W}_n)_{zz} + \lambda_n^2\, N^2\, \mathcal{W}_n &= 0, && (1.31)\\
\mathcal{U}_n' &= N^2\, \mathcal{W}_n\, c_{1,n}, && (1.32)\\
\mathcal{U}_n &= c_{2,n}\, \mathcal{W}_n', && (1.33)\\
\lambda_n^2 = \frac{1}{g\, H_n} &= -\frac{c_{1,n}}{c_{2,n}}. && (1.34)
\end{aligned}$$

[1] Assuming that we are willing to pay the price of a nonlocal (mode by mode) boundary condition, for increased accuracy; see [RTT04] for an alternate solution and a discussion on this issue.

where $c_{1,n}, c_{2,n}$ are appropriate constants and the λ_n the eigenvalues of these two-point boundary value problems. By (1.27) we should have:

$$\begin{cases} \mathcal{W}_n(0) &= \mathcal{W}_n(-L_3) &= 0, \\ \mathcal{U}'_n(0) &= \mathcal{U}'_n(-L_3) &= 0. \end{cases} \tag{1.35}$$

In a standard manner, we infer from (1.31) and the boundary conditions (1.35), that $\mathcal{W}_n(z) = C_n\, sin(\lambda_n\, N\, z)$ with

$$\lambda_n = \frac{n\,\pi}{N\,L_3} \text{ and thus } H_n = \frac{N^2\,L_3^2}{g\,n^2\,\pi^2}. \tag{1.36}$$

One of the constants $c_{1,n}$, $c_{2,n}$ has not yet been imposed; we choose it by orthonormalization of $\mathcal{W}_n$, that is we set $\|\mathcal{W}_n\|_{L^2(-L_3,0)} = 1$, and we find

$$C_n = \sqrt{\frac{2}{L_3}}, \tag{1.37}$$

so that C_n is in fact independent of n. The discussion above refers to the modes $n \geq 1$. For $n = 0$, $\lambda_0 = 0$, so that $\mathcal{W}_0$ vanishes identically, whereas $\mathcal{U}_0$ is constant. Finally we find:

$$\mathcal{U}_0(z) = \frac{1}{\sqrt{L_3}}, \tag{1.38}$$

and for $n \geq 1$:

$$\begin{cases} \mathcal{U}_n(z) &= \sqrt{\frac{2}{L_3}}\, cos(\lambda_n\, N\, z) &= \sqrt{\frac{2}{L_3}}\, cos(\frac{n\,\pi\,z}{L_3}), \\ \mathcal{W}_n(z) &= \sqrt{\frac{2}{L_3}}\, sin(\lambda_n\, N\, z) &= \sqrt{\frac{2}{L_3}}\, sin(\frac{n\,\pi\,z}{L_3}). \end{cases} \tag{1.39}$$

We notice that $\forall n \geq 1, m \geq 0$, we have, as usual:

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \mathcal{U}_n(z)\,\mathcal{U}_m(z)\,dz &= \delta_{n,m}, \\ \displaystyle\int_{-L_3}^{0} \mathcal{U}_n(z)\,\mathcal{W}_m(z)\,dz &= 0, \\ \mathcal{U}'_n(z) &= -N\,\lambda_n\,\mathcal{W}_n(z), \\ \mathcal{W}'_n(z) &= N\,\lambda_n\,\mathcal{U}_n(z). \end{cases} \tag{1.40}$$

Remark 1.1. If we look for a solution "more general" than (1.28), (1.29), that is $u = \sum_{n\geq 1} \mathcal{U}_n \hat{u}_n$, $v = \sum_{n\geq 1} \mathcal{V}_n \hat{v}_n$, $\phi = \sum_{n\geq 1} \Phi_n \hat{\phi}_n$, then (1.22) and (1.23) imply that $\mathcal{U}_n, \mathcal{V}_n, \Phi_n$ are proportional, hence they can be taken equal.

1.3 The modal equations for $(\hat{u}_n, \hat{v}_n, \hat{w}_n, \hat{\psi}_n, \hat{\phi}_n)$

From now on and when no confusion can occur, we drop the hats and write $(u_n, v_n, w_n, \psi_n, \phi_n)$ instead of $(\hat{u}_n, \hat{v}_n, \hat{w}_n, \hat{\psi}_n, \hat{\phi}_n)$. The constant mode in z (n=0) is different (simpler), and we postpone its study to Section 2.5 below. For every $n \geq 1$, since $\psi(x,z,t) = \phi_z(x,z,t)$ we have:

$$\psi_n(x,t) = -N\,\lambda_n\,\phi_n(x,t). \tag{1.41}$$

We now introduce the expansion (1.28)-(1.29) into equations (1.22)-(1.26). We multiply (1.22), (1.23) and (1.26) by $\mathcal{U}_n$, (1.24) and (1.25) by $\mathcal{W}_n$ and integrate on $(-L_3, 0)$, and we find:

$$\left\{\begin{array}{lclclclcl}
\dfrac{\partial u_n}{\partial t} & + & \overline{U_0}\,\dfrac{\partial u_n}{\partial x} & - & f\,v_n & + & \dfrac{\partial \phi_n}{\partial x} & = & F_{u,n}, \\
\dfrac{\partial v_n}{\partial t} & + & \overline{U_0}\,\dfrac{\partial v_n}{\partial x} & + & f\,u_n & & & = & F_{v,n}, \\
\dfrac{\partial \psi_n}{\partial t} & + & \overline{U_0}\,\dfrac{\partial \psi_n}{\partial x} & + & N^2\,w_n & & & = & F_{\psi,n}, \\
\phi_n & = & -\dfrac{1}{N\,\lambda_n}\psi_n, & & & & & & \\
w_n & = & -\dfrac{1}{N\,\lambda_n}\,\dfrac{\partial u_n}{\partial x}. & & & & & &
\end{array}\right. \tag{1.42}$$

Taking into account the last two equations of (1.42) the first three become:

$$\left\{\begin{array}{lclclclcl}
\dfrac{\partial u_n}{\partial t} & + & \overline{U_0}\,\dfrac{\partial u_n}{\partial x} & - & f\,v_n & - & \dfrac{1}{N\,\lambda_n}\,\dfrac{\partial \psi_n}{\partial x} & = & F_{u,n}, \\
\dfrac{\partial v_n}{\partial t} & + & \overline{U_0}\,\dfrac{\partial v_n}{\partial x} & + & f\,u_n & & & = & F_{v,n}, \\
\dfrac{\partial \psi_n}{\partial t} & + & \overline{U_0}\,\dfrac{\partial \psi_n}{\partial x} & - & \dfrac{N}{\lambda_n}\,\dfrac{\partial u_n}{\partial x} & & & = & F_{\psi,n}.
\end{array}\right. \tag{1.43}$$

Let us now introduce the lateral boundary conditions which, for each $n \geq 1$, will supplement this system.

1.4 Boundary conditions at $x = 0$ and $x = L_1$

Looking at (1.43), we find that the characteristic values of this first order system are $\overline{U_0} - 1/\lambda_n$, $\overline{U_0}$ and $\overline{U_0} + 1/\lambda_n$; they are the eigenvalues of the matrix:

$$A_n = \begin{pmatrix} \overline{U_0} & 0 & -\dfrac{1}{N\,\lambda_n} \\ 0 & \overline{U_0} & 0 \\ -\dfrac{N}{\lambda_n} & 0 & \overline{U_0} \end{pmatrix}.$$

Since $\overline{U_0} > 0$, $\lambda_n > 0$, we always have at least two positive eigenvalues. But $\overline{U_0} - 1/\lambda_n$ can either be positive or negative. We say that the corresponding mode is supercritical in the first case and

subcritical in the second case, it appears then that the subcritical modes require two boundary values on the left of the domain ($x = 0$) and one boundary value on the right ($x = L_1$), whereas the supercritical modes require three boundary values at $x = 0$. Based on this remark, Oliger and Sundström concluded in [OS78] that the boundary value problem associated with (1.42)-(1.43) is ill-posed for any set of local boundary conditions (see also [TT03]). Instead different boundary conditions for the two types of modes must be provided and one of our aims in this article is to show the well-posedness of the system consisting of (1.22)-(1.26) supplemented with an appropriate set of boundary conditions.
Since $\lambda_n = n\,\pi/N\,L_3 \longrightarrow \infty$ as $n \longrightarrow \infty$, there is only a finite number of subcritical modes, let us say n_c:

Definition 1.1. *We denote by n_c the number of subcritical modes, defined by:*

$$\frac{n_c\,\pi}{N\,L_3} = \lambda_{n_c} \leq \frac{1}{\overline{U_0}} < \lambda_{n_c+1} = \frac{(n_c+1)\,\pi}{N\,L_3}.$$

In physical applications most of the modes are supercritical, but the few subcritical modes carrying most of the energy are particularly important.

The boundary conditions for the subcritical modes were discussed in [RTT05a, RTT04], they are recalled below. The boundary conditions for the supercritical modes are less problematic, we now present them. For $n > n_c$, a set of natural boundary conditions for system (1.43) is:

$$\begin{cases} u_n(0,t) = 0, \\ v_n(0,t) = 0, \\ \psi_n(0,t) = 0. \end{cases} \tag{1.44}$$

In (1.44) and (1.46) we chose, for simplicity, homogeneous boundary conditions, but we discuss in Section 2.4 below the case of nonzero boundary values.
For $1 \leq n \leq n_c$, $\overline{U_0} - 1/\lambda_n < 0$, and the corresponding eigenvector is $\eta_n = u_n + \psi_n/N$. The eigenvectors related to $\overline{U_0}$ and $\overline{U_0} + 1/\lambda_n$ are respectively v_n and $\xi_n = u_n - \psi_n/N$. Thanks to (1.41), we have, for $n \geq 1$, $(\xi_n, \eta_n) = (u_n + \lambda_n\,\phi_n, u_n - \lambda_n\,\phi_n)$.

Using the variables ξ_n, v_n, η_n we rewrite (1.43) as follows:

$$\left\{\begin{array}{llllll} \dfrac{\partial \xi_n}{\partial t} & + & (\overline{U_0} + \dfrac{1}{\lambda_n})\,\dfrac{\partial \xi_n}{\partial x} & - & f\,v_n & = & F_{\xi,n}, \\ \dfrac{\partial v_n}{\partial t} & + & \overline{U_0}\,\dfrac{\partial v_n}{\partial x} & + & \frac{1}{2}\,f\,(\xi_n + \eta_n) & = & F_{v,n}, \\ \dfrac{\partial \eta_n}{\partial t} & + & (\overline{U_0} - \dfrac{1}{\lambda_n})\,\dfrac{\partial \eta_n}{\partial x} & & & = & F_{\eta,n}. \end{array}\right. \tag{1.45}$$

Hence, for these subcritical modes ($n \leq n_c$), a set of natural and nonreflective boundary conditions is the following

$$\begin{cases} \xi_n(0,t) = 0, \\ v_n(0,t) = 0, \\ \eta_n(L_1,t) = 0. \end{cases} \tag{1.46}$$

In Section 2 of this article, we will prove the well-posedness of the linear Primitive Equations (1.22)-(1.26) (equivalent mode by mode to (1.42)) with the modal boundary conditions (1.44) and (1.46).

2 Well-posedness of the linear PEs with modal boundary conditions

We aim to implement (1.44) and (1.46), and we first set the functional framework appropriate to these boundary conditions.

2.1 Preliminary settings

We aim to write the initial value problem under consideration as a functional evolution in an appropriate Hilbert space H:

$$\begin{cases} \frac{dU}{dt} + A\,U = F, \\ U(0) = U_0. \end{cases} \tag{2.1}$$

Here A is an unbounded operator with domain $D(A) \subset H$, the forcing $F \in H$ and the initial data $U_0 \in D(A)$ are given.
We define H by setting

$$H = H_u \times H_v \times H_\psi, \tag{2.2}$$

$$H_u = \left\{ u \in L^2(\mathcal{M}) \ / \ \int_{-L_3}^{0} u(x,z)\,dz = 0 \text{ a.e. in } (0,L_1) \right\},$$

$$H_v = H_\psi = L^2(\mathcal{M}),$$

where $\mathcal{M}$ is the 2D domain $(0,L_1) \times (-L_3,0)$. We endow H with the scalar product[2]

$$(U,\tilde{U})_H = \int_{\mathcal{M}} (u\,\tilde{u} + v\,\tilde{v} + \frac{1}{N^2}\,\psi\,\tilde{\psi})\,d\mathcal{M}, \qquad \forall (U,\tilde{U}) \in H^2. \tag{2.3}$$

The space H_u is clearly closed in $L^2(\mathcal{M})$, and $H = H_u \times H_v \times H_\psi$ is a closed subspace of $(L^2(\mathcal{M}))^3$, which we endow with the scalar product and norm derived from (2.3) and equivalent

[2]It is not surprising to have $1/N^2$ as a multiplicative coefficient in front of the last term of $(U,\tilde{U})_H$, since $\int_{\mathcal{M}} u^2 + v^2 d\mathcal{M}$ represents the kinetic energy whereas $1/N^2 \int_{\mathcal{M}} \psi^2\,d\mathcal{M}$ is the available potential energy.

to those of $(L^2(\mathcal{M}))^3$. We denote by P the orthogonal projector from $L^2(\mathcal{M})$ onto H_u. For every $g \in L^2(\mathcal{M})$,

$$P(g)(x,z) = g(x,z) - \frac{1}{L_3}\int_{-L_3}^{0} g(x,z')\,dz', \tag{2.4}$$

$$(I-P)(g)(x,z) = \frac{1}{L_3}\int_{-L_3}^{0} g(x,z')\,dz'. \tag{2.5}$$

It is easily checked that $Pg \in H_u$ and $(I-P)g \perp Pg$. Finally $H_u^{\perp}$ is identical to $L_x^2(0,L_1)$. Indeed for $g \in H_u^{\perp}$, $(I-P)\,g = g$, so that g does not depend on z and belongs to $L_x^2(0,L_1)$. Conversely if $h \in L_x^2(0,L_1)$, then for every $u \in H_u$, $(u,h)_{L^2(\mathcal{M})} = \int_0^{L_1} h(x)\int_{-L_3}^0 u(x,z)dz\,dx = 0$ and $h \in H_u^{\perp}$.

We are now in position to define the operator A; its domain $D(A)$ is defined by

$$D(A) = \left\{ U = (u,v,\psi) \in H \;/\; \left| \begin{array}{l} (u_x,v_x,\psi_x) \in L^2(\mathcal{M}) \\ (u,v,\psi) \text{ verify (2.7) and (2.8)} \end{array} \right. \right\}. \tag{2.6}$$

Here and in the sequel u_x, u_z denote the partial derivatives $\partial u/\partial x$, $\partial u/\partial z$ of a function u.

The boundary conditions (2.7) and (2.8), identical to (1.44) and (1.46), are written in the following form[3]:

$$\left\{ \begin{array}{l} \displaystyle\int_{-L_3}^{0} u(0,z)\,\mathcal{U}_n(z)\,dz - \frac{1}{N}\int_{-L_3}^{0} \psi(0,z)\,\mathcal{W}_n(z)\,dz = 0, \\ \displaystyle\int_{-L_3}^{0} v(0,z)\,\mathcal{U}_n(z)\,dz = 0, \\ \displaystyle\int_{-L_3}^{0} u(L_1,z)\,\mathcal{U}_n(z)\,dz + \frac{1}{N}\int_{-L_3}^{0} \psi(L_1,z)\,\mathcal{W}_n(z)\,dz = 0, \end{array} \right. \qquad \forall\, 1 \le n \le n_c, \tag{2.7}$$

and

$$\left\{ \begin{array}{rcl} \displaystyle\int_{-L_3}^{0} u(0,z)\,\mathcal{U}_n(z)\,dz &=& 0, \\ \displaystyle\int_{-L_3}^{0} v(0,z)\,\mathcal{U}_n(z)\,dz &=& 0, \\ \displaystyle\int_{-L_3}^{0} \psi(0,z)\,\mathcal{W}_n(z)\,dz &=& 0, \end{array} \right. \qquad \forall n > n_c. \tag{2.8}$$

For every $U = (u,v,\psi) \in D(A)$, AU is given by:

$$AU = \begin{pmatrix} P\big[\,\overline{U_0}\,u_x - f\,v - \displaystyle\int_z^0 \psi_x(x,z')\,dz'\big] \\ \overline{U_0}\,v_x + f\,u \\ \overline{U_0}\,\psi_x + N^2\,w \end{pmatrix} \tag{2.9}$$

[3]We note that the boundary conditions on v do not depend on the modes (see also the boundary condition on the constant mode v_0 in Section 2.5 below), hence they could be written in the form $v(0,z) = 0,\ \forall z \in (-L_3, 0)$. However we keep the modal notation by analogy with the other functions u and ψ, and because this is the way this boundary condition is actually implemented in numerical simulations [RTT05b].

where $w = w(u)$ is given by (1.21).

We now intend to prove, in the context of the linear semi-group theory ([Yos80], [Bré73], [BG], [Hen81], [Lio65], [Paz83]), the well-posedness for equation (2.1), corresponding to the linearized PEs supplemented with the boundary conditions (2.7) and (2.8).

2.2 The main result

To prove the well-posedness of the initial value problem (2.1), we will use the following version of the Hille-Yosida theorem borrowed from [BG] (see also [Bré73], [Hen81],[Lio65],[Paz83],[Yos80]):

Theorem 2.1. (Hille-Yosida Theorem) *Let H be a Hilbert space and let $A : D(A) \longrightarrow H$ be a linear unbounded operator, with domain $D(A) \subset H$. Assume the following :*
(i) $D(A)$ is dense in H and A is closed,
(ii) A is ≥ 0, i.e. $(AU, U)_H \geq 0, \quad \forall U \in D(A)$,
(iii) $\exists \mu_0 > 0$, such that $A + \mu_0 I$ is onto.

Then $-A$ is infinitesimal generator of a semigroup of contractions $\{S(t)\}_{t\geq 0}$ in H, and for every $U_0 \in H$ and $F \in L^1(0,T;H)$, there exists a unique solution $U \in \mathcal{C}([0,T];H)$ of (2.1),

$$U(t) = S(t)\, U_0 + \int_0^t S(t-s)\, F(s)\, ds. \tag{2.10}$$

If furthermore $\overline{U_0} \in D(A)$ and $F' = dF/dt \in L^1(0,T;H)$ then U satisfies (2.1) *and*

$$U \in \mathcal{C}([0,T];H) \cap L^\infty(0,T;D(A)), \ \frac{dU}{dt} \in L^\infty(0,T;H). \tag{2.11}$$

The hypotheses of Theorem 2.1 being proved in Section 2.3, Theorem 2.1 readily implies our main result for the homogeneous boundary conditions:

Theorem 2.2. *Let H be the Hilbert space defined in* (2.2) *and A be the linear operator defined in* (2.9) *corresponding to the linearized Primitive Equations with vanishing viscosity and homogeneous modal boundary conditions.*
Then the initial value problem (2.1)*, corresponding to equations* (1.22)-(1.26) *supplemented with the boundary conditions* (2.7) *and* (2.8) *is well-posed, that is for every initial data $U_0 \in D(A)$ and forcing $F \in L^1(0,T;H)$, there exists a unique solution $U \in \mathcal{C}([0,T];H)$ of* (2.1).

2.3 Proof of Theorem 2.2

We now want to apply Theorem 2.1 to equation (2.1). To this aim we verify the hypotheses (i), (ii) and (iii) of the Hille-Yosida theorem (Theorem 2.1); we start with (ii) and (iii), and postpone the proof of (i) to Lemma 2.3 below. We start with the proof of (ii):

Lemma 2.1. *For every $U \in D(A)$, $(A\,U, U)_H \geq 0$.*

Proof : For any $U \in H$, let us compute the scalar product $(A\,U, U)_H$:

$$\begin{aligned}(A\,U, U)_H &= \int_{\mathcal{M}} P(\overline{U_0}\,u_x - f\,v - \int_z^0 \psi_x(x, z')\,dz')\,u\,d\mathcal{M} \\ &\quad + \int_{\mathcal{M}} (\overline{U_0}\,v_x + f\,u)\,v\,d\mathcal{M} + \int_{\mathcal{M}} (\overline{U_0}\,\psi_x + N^2\,w)\,\frac{\psi}{N^2}\,d\mathcal{M}.\end{aligned}$$

Since $u \in H_u$, we have, using (1.21):

$$\begin{aligned}(A\,U, U)_H &= \int_{\mathcal{M}} (\overline{U_0}\,u_x - f\,v - \int_z^0 \psi_x(x, z')\,dz')\,u\,d\mathcal{M} \\ &\quad + \int_{\mathcal{M}} (\overline{U_0}\,v_x + f\,u)\,v\,d\mathcal{M} + \int_{\mathcal{M}} (\overline{U_0}\,\psi_x + N^2\,w)\,\frac{\psi}{N^2}\,d\mathcal{M} \\ &= \int_{-L_3}^0 \frac{\overline{U_0}}{2}\Big(u^2(L_1) - u^2(0) + v^2(L_1) - v^2(0) + \frac{1}{N^2}\,\psi^2(L_1) - \frac{1}{N^2}\,\psi^2(0)\Big)\,dz \\ &\quad - \int_{\mathcal{M}} \Big\{u(x,z)\int_z^0 \psi_x(x, z')\,dz' - \psi(x,z)\int_z^0 u_x(x, z')\,dz'\Big\}\,dx\,dz.\end{aligned}$$

Here $u(L_1)$, $u(0)$ stands for $u(L_1, z)$, $u(0, z)$, etc. Using the expansion (1.28), (1.29) with (1.39), it is easy to check that:

$$\left\{\begin{aligned} -\int_z^0 \psi_x(x, z')\,dz' &= \sum_{n\geq 1} \frac{\psi_{n\,x}(x)}{N\,\lambda_n}\,(1 - \mathcal{U}_n(z)) \\ &= \theta(x) - \sum_{n\geq 1} \frac{\psi_{n\,x}(x)}{N\,\lambda_n}\,\mathcal{U}_n(z), \\ \int_z^0 u_x(x, z')\,dz' &= -\sum_{n\geq 1} \frac{u_{n\,x}(x)}{N\,\lambda_n}\,\mathcal{W}_n(z). \end{aligned}\right. \tag{2.12}$$

where $\theta = \theta(x)$ is an L^2-function depending only on x.

Using again the expansion (1.28), (1.29), and remembering that $u \in H_u$, the integral $\int_{\mathcal{M}} u\,\theta\,d\mathcal{M}$ vanishes and we find:

$$\begin{aligned}(A\,U, U)_H &= \sum_{n\geq 1} \frac{\overline{U_0}}{2}\Big(u_n^2(L_1) - u_n^2(0) + v_n^2(L_1) - v_n^2(0) + \frac{1}{N^2}\,\psi_n^2(L_1) - \frac{1}{N^2}\,\psi_n^2(0)\Big) \\ &\quad + \frac{\overline{U_0}}{2}\Big(v_0^2(L_1) - v_0^2(0)\Big) - \sum_{n\geq 1} \frac{1}{N\,\lambda_n} \int_0^{L_1} (\psi_{n\,x}\,u_n + \psi_n\,u_{n\,x})\,dx.\end{aligned}$$

Using the boundary conditions (2.7) for the subcritical modes and (2.8) for the supercritical ones, we find:

$$\begin{aligned}
(AU,U)_H \; = \; & \sum_{1\le n\le n_c} \frac{\overline{U_0}}{2} \Big(u_n^2(L_1) - u_n^2(0) + v_n^2(L_1) + u_n^2(L_1) - u_n^2(0) \Big) \\
& + \frac{\overline{U_0}}{2} v_0^2(L_1) + \sum_{1\le n\le n_c} \frac{1}{\lambda_n} \Big(u_n^2(L_1) + u_n^2(0) \Big) \\
& + \sum_{n>n_c} \frac{\overline{U_0}}{2} \Big(u_n^2(L_1) + v_n^2(L_1) + \frac{1}{N^2} \psi_n^2(L_1) \Big) \\
& - \sum_{n>n_c} \frac{1}{N\,\lambda_n} u_n(L_1)\, \psi_n(L_1).
\end{aligned}$$

For every subcritical mode (when $n \le n_c$):

$$\begin{aligned}
& \overline{U_0} \Big(u_n^2(L_1) - u_n^2(0) + \frac{1}{2} v_n^2(L_1) \Big) + \frac{1}{\lambda_n} \Big(u_n^2(L_1) + u_n^2(0) \Big) \\
= \; & (\overline{U_0} + \frac{1}{\lambda_n})\, u_n^2(L_1) + \frac{\overline{U_0}}{2} v_n^2(L_1) + (\frac{1}{\lambda_n} - \overline{U_0})\, u_n^2(0) \ge 0;
\end{aligned}$$

the latter quantity is nonnegative, thanks to the definition of n_c. For every supercritical mode (when $n > n_c$):

$$\begin{aligned}
& \frac{\overline{U_0}}{2} \Big(u_n^2(L_1) + v_n^2(L_1) + \frac{1}{N^2} \psi_n^2(L_1) \Big) - \frac{1}{N\,\lambda_n} u_n(L_1)\, \psi_n(L_1) \\
= \; & \frac{\overline{U_0}}{2} v_n^2(L_1) + \frac{\overline{U_0}}{2} \Big(u_n(L_1) - \frac{1}{\overline{U_0}\, N\, \lambda_n} \psi_n(L_1) \Big)^2 + \frac{\overline{U_0}}{2\, N^2} \Big(1 - \frac{1}{\overline{U_0}^2\, \lambda_n^2} \Big) \psi_n^2(L_1) \ge 0.
\end{aligned}$$

This quantity is also nonnegative, which achieves the proof of Lemma 2.1. □

In order to simplify the following study, we now assume that $\overline{U_0}$ is not a critical value, that is

$$\overline{U_0} \notin \Big\{ \frac{1}{\lambda_n},\ n \ge 1 \Big\}, \text{ or equivalently } \frac{N\, L_3\, \overline{U_0}}{\pi} \notin \mathbb{N}. \tag{2.13}$$

The case where (2.13) is not satisfied ($\overline{U_0} = \lambda_n^{-1}$) is actually simpler and will be discussed in Remark 2.1 below. Assuming (2.13), we choose μ_0 such that:

$$\mu_0 \notin \Big\{ f^2\, (1 - \overline{U_0}^2\, \lambda_n^2),\ n \ge 1 \Big\}, \tag{2.14}$$

$$\mu_0 \notin \Big\{ f^2\, \overline{U_0}^2\, \lambda_n^2,\ n \ge 1 \Big\}. \tag{2.15}$$

With this choice of μ_0, we can prove the following lemma:

Lemma 2.2. *The operator $A + \mu_0 I$ is onto from $D(A)$ onto H, where μ_o satisfies* (2.14) *and* (2.15).

Proof : For μ_0 as indicated, we are given $F = (F_u, F_v, F_\psi)$ in H, and we look for $U = (u, v, \psi)$ in $D(A)$ such that $(A + \mu_0\, I)\, U = F$. Writing this equation componentwise, we find:

$$\left\{\begin{aligned} \overline{U_0}\, u_x(x,z) - f\, v(x,z) + \mu_0\, u(x,z) & \\ - \int_z^0 \psi_x(x,z')\, dz' + \phi_s'(x) &= F_u(x,z), \\ \overline{U_0}\, v_x(x,z) + f\, u(x,z) + \mu_0\, v(x,z) &= F_v(x,z), \\ \overline{U_0}\, \psi_x(x,z) + N^2\, w(x,z) + \mu_0\, \psi(x,z) &= F_\psi(x,z). \end{aligned}\right. \tag{2.16}$$

To obtain the modal equations corresponding to (2.16), we multiply the three equations by $\mathcal{U}_n$, $\mathcal{U}_n$ and $\mathcal{W}_n$ respectively, and integrate on $(-L_3, 0)$.

Of course, since $F = (F_u, F_v, F_\psi) \in H$, we also have the following modal decompositions:

$$\left\{\begin{aligned} F_u(x,z) &= \sum_{n\geq 1} \mathcal{U}_n(z)\, \hat{F}_{u,n}(x), \\ F_v(x,z) &= \sum_{n\geq 0} \mathcal{U}_n(z)\, \hat{F}_{v,n}(x), \\ F_\psi(x,z) &= \sum_{n\geq 1} \mathcal{W}_n(z)\, \hat{F}_{\psi,n}(x). \end{aligned}\right. \tag{2.17}$$

Note that for F as for U, since $F_u \in H_u \subset L^2(\mathcal{M})$, $\hat{F}_{u,0} = 0$ and the decomposition of F_u starts from $n = 1$.

For the mode $n = 0$ (constant in the variable z), we only consider the first two equations, since multiplying the third one by $\mathcal{W}_0 = 0$ would be useless. Integrating the equation for v and reporting in the equation for u (in which $\hat{u}_0 = 0$, see above), we find v_0 (formerly denoted $\hat{v}_0$) and the surface pressure ϕ_s, up to an additive constant $\phi_s(0)$:

$$\left\{\begin{aligned} v_0(x) &= \frac{1}{\overline{U_0}} \int_0^x F_{v\,0}(x')\, e^{(x'-x)\,\mu_0/\overline{U_0}}\, dx', \\ \phi_s(x) &= \phi_s(0) + \int_0^x \Big(f\, v_0(x') - \frac{L_3^2}{\pi} \sum_{n\geq 1} \psi_{n\,x}(x') \Big)\, dx'. \end{aligned}\right. \tag{2.18}$$

We recall that the nth mode is now denoted by $(u_n, v_n, w_n, \psi_n, \phi_n)$ instead of ($\hat{u}_n$, $\hat{v}_n$, $\hat{w}_n$, $\hat{\psi}_n$, $\hat{\phi}_n$). Naturally, the above expression of ϕ_s depends on the other modes ($n \geq 1$). We now write the corresponding equations, derived from (2.16) mode by mode:

$$\left\{\begin{aligned} \overline{U_0}\, u_{n\,x} - f\, v_n + \mu_0\, u_n - \frac{1}{N\,\lambda_n}\, \psi_{n\,x} &= F_{u,n}, \\ \overline{U_0}\, v_{n\,x} + f\, u_n + \mu_0\, v_n &= F_{v,n}, \\ \overline{U_0}\, \psi_{n\,x} - \frac{N}{\lambda_n}\, u_{n\,x} + \mu_0\, \psi_n &= F_{\psi,n}. \end{aligned}\right. \tag{2.19}$$

We recall that the functions (u_n, v_n, ψ_n) only depend on the x variable. Hence (2.19) is just a linear system of ordinary differential equations for u_n, v_n, ψ_n.

As usual, to solve (2.19), we first consider the corresponding homogeneous system. Dropping the subscripts n for the moment, we write:

$$\left\{\begin{array}{lllllllll} \overline{U_0}\,\dfrac{du}{dx} & - & f\,v & - & \dfrac{1}{N\,\lambda_n}\,\dfrac{d\psi}{dx} & + & \mu_0\,u & = & 0, \\ \overline{U_0}\,\dfrac{dv}{dx} & + & f\,u & & & + & \mu_0\,v & = & 0, \\ \overline{U_0}\,\dfrac{d\psi}{dx} & & & + & \dfrac{N}{\lambda_n}\,\dfrac{du}{dx} & + & \mu_0\,\psi & = & 0. \end{array}\right. \tag{2.20}$$

The general solution of this linear system is of the form

$$(u, v, \psi) = \sum_{i=1}^{3} (A_i, B_i, C_i)\, e^{R_i\, x} \tag{2.21}$$

where the coefficients R_i are as follows:

$$\left\{\begin{array}{lll} R_1 & = & -\dfrac{\mu_0}{\overline{U_0}}, \\ R_2 & = & \dfrac{-\mu_0\,\overline{U_0} + \dfrac{1}{\lambda}\Big(\mu_0^2 - f^2\,(\overline{U_0}^2\,\lambda^2 - 1)\Big)^{1/2}}{\overline{U_0}^2 - \dfrac{1}{\lambda^2}}, \\ R_3 & = & \dfrac{-\mu_0\,\overline{U_0} - \dfrac{1}{\lambda}\Big(\mu_0^2 - f^2\,(\overline{U_0}^2\,\lambda^2 - 1)\Big)^{1/2}}{\overline{U_0}^2 - \dfrac{1}{\lambda^2}}. \end{array}\right. \tag{2.22}$$

The $(A_i, B_i, C_i)_{1 \le i \le 3}$ satisfy the equations:

$$\left\{\begin{array}{lll} A_i & = & a_i\, B_i, \\ C_i & = & c_i\, B_i, \end{array}\right. \tag{2.23}$$

with

$$\left\{\begin{array}{lll} a_1 & = & 0, \\ c_1 & = & -\dfrac{f\,N\,\lambda}{R_1}, \end{array}\right. \tag{2.24}$$

and, for $i = 2, 3$:

$$\left\{\begin{array}{lll} a_i & = & -\dfrac{\overline{U_0}\,R_i + \mu_0}{f}, \\ c_i & = & \dfrac{N\,R_i}{\lambda\,(\overline{U_0}\,R_i + \mu_0)}. \end{array}\right. \tag{2.25}$$

Now, returning to the nonhomogeneous system (2.19), we look for a solution $(u_n, v_n, \psi_n) = (u, v, \psi)$ of the form:

$$Y = (u, v, \psi)^t = \sum_{i=1}^{3} (a_i, 1, c_i,)^t\, B_i(x)\, e^{R_i\, x}, \tag{2.26}$$

where the (a_i, c_i) and R_i have been defined above. Equation (2.19) reads then:

$$M\,Y' + N\,Y = F, \tag{2.27}$$

where $\lambda = \lambda_n$ and

$$M = \begin{pmatrix} \overline{U_0} & 0 & -\frac{1}{N\,\lambda} \\ 0 & \overline{U_0} & 0 \\ -\frac{N}{\lambda} & 0 & \overline{U_0} \end{pmatrix} \quad , \quad N = \begin{pmatrix} \mu_0 & -f & 0 \\ f & \mu_0 & 0 \\ 0 & 0 & \mu_0 \end{pmatrix}, \tag{2.28}$$

$$F = (F_u, F_v, F_\psi)^t. \tag{2.29}$$

Thanks to assumption (2.13), $\overline{U_0} \neq 1/\lambda_n$, the matrix M is regular and it can be inverted. Equation (2.27) then implies:

$$\sum_{i=1}^{3} (a_i, 1, c_i)^t\, B_i'(x)\, e^{R_i\,x} = M^{-1} F =: \tilde{F}. \tag{2.30}$$

We now write the latter equation component by component. We find:

$$\Lambda(x).(B_1'(x), B_2'(x), B_3'(x))^t = (\tilde{F}_1(x), \tilde{F}_2(x), \tilde{F}_3(x))^t, \tag{2.31}$$

with

$$\Lambda(x) = \begin{pmatrix} 0 & a_2\, e^{R_2\,x} & a_3\, e^{R_3\,x} \\ e^{R_1\,x} & e^{R_2\,x} & e^{R_3\,x} \\ c_1\, e^{R_1\,x} & c_2\, e^{R_2\,x} & c_3\, e^{R_3\,x} \end{pmatrix}. \tag{2.32}$$

Let us check that the matrix $\Lambda(x)$ is regular for every $x \in \mathbb{R}$; it is clearly sufficient to do so for $x = 0$, for which

$$\Lambda(0) = \begin{pmatrix} 0 & a_2 & a_3 \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

We call $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ the lines of $\Lambda(0)$. It is clear that $\mathcal{L}_1$ and $\mathcal{L}_2$ are linearly independent vectors. Then if $\Lambda(0)$ were not regular there would exist $(\alpha, \beta) \in \mathbb{R}^2$ such that $\mathcal{L}_3 = \alpha\, \mathcal{L}_1 + \beta\, \mathcal{L}_2$. After some easy computations we would find that necessarily:

$$a_3\,(c_2 - c_1) = a_2\,(c_3 - c_1), \tag{2.33}$$

which leads (see (2.24) and (2.25)) to:

$$\overline{U_0}\,(R_3 - R_2)\, f^2\, \lambda^2 = -\mu_0\, R_1\,(R_3 - R_2). \tag{2.34}$$

From (2.14) we find that $R_2 \neq R_3$, and thanks to the definition of R_1 equation (2.34) becomes:

$$\overline{U_0}^2\, f^2\, \lambda^2 = \mu_0^2, \tag{2.35}$$

which contradicts (2.15). Thus the matrix $\Lambda(x)$ is regular for every $x \in \mathbb{R}$.

Back to equation (2.31), and thanks to the latter result, the functions $B_i'(x)$ are uniquely determined for $i = 1, 2, 3$. It remains to use the modal boundary conditions in order to determine the constants $B_i(0)$ and thus the functions $B_i(x)$.

At this point, it is desirable to reintroduce the indices n i.e. to return to the notation (u_n, v_n, ψ_n), since the boundary conditions depend on the mode considered. For the supercritical modes $(n > n_c)$, the modal boundary condition is the one of (1.44). We thus look for the $B_i(0)$ satisfying:

$$\left\{ \begin{array}{rcrcrcl} & & a_2\, B_2(0) & + & a_3\, B_3(0) & = & 0, \\ B_1(0) & + & B_2(0) & + & B_3(0) & = & 0, \\ c_1\, B_1(0) & + & c_2\, B_2(0) & + & c_3\, B_3(0) & = & 0. \end{array} \right. \tag{2.36}$$

The matrix of this system is again $\Lambda(0)$ which was shown to be regular (see above). We conclude that the constants $B_i(0)$ are uniquely determined by (2.36) and equal to zero. The functions $B_i(x)$ for the supercritical modes $(n > n_c)$ are now fully determined.

If $n \leq n_c$, the mode is subcritical and we consider the boundary condition (1.46). We thus want to solve the following system:

$$\left\{ \begin{array}{rcrcrcl} -N\, c_1\, B_1(0) & + & (a_2 - N\, c_2)\, B_2(0) & + & (a_3 - N\, c_3)\, B_3(0) & = & 0, \\ B_1(0) & + & B_2(0) & + & B_3(0) & = & 0, \\ N\, c_1\, B_1(0) & + & (a_2 + N\, c_2)\, B_2(0) & + & (a_3 + N\, c_3)\, B_3(0) & = & \Gamma, \end{array} \right. \tag{2.37}$$

where

$$\Gamma = -\sum_{i=1}^{3} \int_0^{L_1} (a_i + N\, c_i)\, B_i'(x)\, dx. \tag{2.38}$$

The quantity Γ depends only on the data and on the B_i', hence it is known at this stage. After some computations and using hypotheses (2.14) and (2.15), we check that the matrix of the linear system (2.37) is regular (same proof exactly as for $\Lambda(0)$). This achieves the determination of the B_i in the subcritical case, and the lemma is proved. □

Remark 2.1. The case when there exists $n \geq 1$ such that $\overline{U_0} = 1/\lambda_n$ is slightly different and actually simpler since the third equation (1.45) becomes $\partial\eta_n(x,t)/\partial t = F_{\eta,n}(t)$, which can be integrated directly. We note that no boundary condition (neither in the subcritical case nor in the supercritical one) would then be required for η_n so that (2.7),(2.8) would have to be modified. But we do not want to go into the details since this nongeneric situation seldom occurs in numerical simulations.

To conclude there remains to prove the hypothesis (i) of the Hille-Yosida theorem, that is:

Lemma 2.3. *The domain $D(A)$ of A is dense in H, and the operator A is closed.*

Proof : We first verify that the orthogonal in H of $D(A)$, $D(A)^\perp$, is reduced to $\{0\}$.

Let v be an element of $D(A)^\perp$. Since $A+\mu_0 I$ is onto, there exists $u \in D(A)$ such that $(A+\mu_0 I)\,u = v$. Then:

$$0 = (v,u)_H = \Big((A+\mu_0 I)\,u, u\Big)_H \geq \mu_0 \|u\|_H^2;$$

hence $u = v = 0$, which implies that $D(A)^\perp = \{0\}$, and $D(A)$ is dense in H.

To show that A is closed, we consider a sequence $(u_j, v_j, \psi_j) = U_j$ of $D(A)$, such that :

$$U_j \longrightarrow U \text{ in } H, \tag{2.39}$$
$$A\,U_j = F_j \longrightarrow F \text{ in } H, \tag{2.40}$$

and we want to verify that $U = (u, v, \psi) \in D(A)$ and $F = A\,U$, so that the graph of A is closed.

Thanks to (2.39), we know that

$$u_j \longrightarrow u \text{ in } H_u \subset L^2(\mathcal{M}), \tag{2.41}$$
$$v_j \longrightarrow v \text{ in } L^2(\mathcal{M}). \tag{2.42}$$

We also find from (2.9) and (2.40) that

$$\overline{U_0}\,\frac{dv_j}{dx} + f\,u_j \longrightarrow F_2 \text{ in } L^2(\mathcal{M}). \tag{2.43}$$

Hence the sequence $(dv_j/dx)_{j\in\mathbb{N}}$ is bounded in $L^2(\mathcal{M})$, and thanks to (2.42) we obtain that $v_x \in L^2(\mathcal{M})$.
In view of proving that $(u_x, \psi_x) \in L^2(\mathcal{M})$, we consider the decomposition in normal modes, introduced in Section 1.2. Thanks to (2.39), we have for every $n \geq 1$:

$$\hat{u}_{j,n} \longrightarrow \hat{u}_n \text{ in } L^2(0, L_1), \tag{2.44}$$
$$\hat{v}_{j,n} \longrightarrow \hat{v}_n \text{ in } L^2(0, L_1), \tag{2.45}$$
$$\hat{\psi}_{j,n} \longrightarrow \hat{\psi}_n \text{ in } L^2(0, L_1), \tag{2.46}$$

and the quantities $\sum_{n\geq 1} |\hat{u}_{j\,n}|^2$, $\sum_{n\geq 1} |\hat{v}_{j\,n}|^2$ and $\sum_{n\geq 1} |\hat{\psi}_{j\,n}|^2$ are bounded uniformly in j.

Similarly, we infer from (2.40) that for every $n \geq 1$:

$$\overline{U_0}\,\frac{d\hat{u}_{j,n}}{dx} - f\,v_n - \frac{1}{N\,\lambda_n}\,\frac{d\hat{\psi}_{j,n}}{dx} = F^j_{u,n} \longrightarrow F_{u,n} \text{ in } L^2(0, L_1), \tag{2.47}$$

$$\overline{U_0}\,\frac{d\hat{v}_{j,n}}{dx} + f\,\hat{u}_n = F^j_{v,n} \longrightarrow F_{v,n} \text{ in } L^2(0, L_1), \tag{2.48}$$

$$\overline{U_0}\,\frac{d\hat{\psi}_{j,n}}{dx} - \frac{N}{\lambda_n}\,\frac{d\hat{u}_{j,n}}{dx} = F^j_{\psi,n} \longrightarrow F_{\psi,n} \text{ in } L^2(0, L_1), \tag{2.49}$$

and the quantities $\sum_{n\geq 1} |F^j_{u,n}|^2$, $\sum_{n\geq 1} |F^j_{v,n}|^2$ and $\sum_{n\geq 1} |F^j_{\psi,n}|^2$ are bounded uniformly in j.

Combining (2.47) and (2.49), we find that:

$$\frac{d\hat{u}_{j,n}}{dx} = \frac{1}{\overline{U_0}^2 - 1/\lambda_n^2}\,(\overline{U_0}\,F^j_{u,n} + f\,\overline{U_0}\,\hat{v}_{j,n} + \frac{F^j_{\psi,n}}{N\,\lambda_n}), \tag{2.50}$$

hence the $(d\hat{u}_{j,n}/dx)_{j\geq 1}$ are bounded in $L^2(0, L_1)$ and $(d\hat{u}_n/dx) \in L^2(0, L_1)$. Moreover, we find that[4]

$$\sum_{n\geq 1}\Big|\frac{d\hat{u}_{j,n}}{dx}\Big|^2 \leq \frac{4}{\min\limits_{n\geq 1}|\overline{U_0}^2 - 1/\lambda_n^2|^2}\sum_{n\geq 1}(\overline{U_0}^2|F^j_{u,n}|^2 + f^2\,\overline{U_0}^2\,|\hat{v}_{j,n}|^2 + \Big|\frac{F^j_{\psi,n}}{N\,\lambda_n}\Big|^2), \tag{2.51}$$

so that the latter quantity is bounded uniformly in j. This guarantees that $u_x \in L^2(\mathcal{M})$. Following the same idea, and using either (2.47) or (2.49), we also prove that $\psi_x \in L^2(\mathcal{M})$.

To insure that $U \in D(A)$, we need to verify that the modal boundary conditions (1.44) and (1.46) are satisfied by U. This is clear since the convergence of $(\hat{u}_{j,n}, \hat{v}_{j,n}, \hat{\psi}_{j,n})$ to $(\hat{u}_n, \hat{v}_n, \hat{\psi}_n)$ is in fact in $H^1(0, L_1)$, so that the boundary conditions pass to the limit.

Finally, let us show that $A\,U = F$. Thanks to (2.39), we find that $A\,U_j \to A\,U$ in $\mathcal{D}'(\mathcal{M})$, hence $A\,U = F$ in $\mathcal{D}'(\mathcal{M})$. We infer from $U \in D(A)$ that $A\,U \in L^2(\mathcal{M})$, and conclude that $A\,U = F$ in $L^2(\mathcal{M})$, which ends the proof of Lemma 2.3. □

2.4 The case of nonhomogeneous boundary conditions

In practical simulations, we want to solve the PEs with nonhomogeneous boundary conditions on U at $x = 0$ and $x = L_1$, that is $U^{g,l}$ and $U^{g,r}$. The latter are boundary values derived from a solution $\tilde{U}$ computed on a domain $\tilde{\mathcal{M}}$ larger than $\mathcal{M}$[5] .
We discussed in Section 2.3 above the case when $U^{g,l} = U^{g,r} = 0$. The issue is now to determine which components of $U^{g,l}$ and $U^{g,r}$ are needed to obtain a well-posed problem. In this context all components of $U^{g,l}$ and $U^{g,r}$ are available but we know (or surmise at this point) that they will not be all used, those used depending on the mode that we consider.
Based on the data $U^{g,l}$,$U^{g,r}$, let us now construct the following $U^g = (u^g, v^g, \psi^g)$:

$$(u^g, v^g, \psi^g)(z,t) = \sum_{n\geq 1}\Big(u^g_n(t)\,\mathcal{U}_n(z), v^g_n(t)\mathcal{U}_n(z), \psi^g_n(t)\mathcal{W}_n(z)\Big), \tag{2.52}$$

[4]Thanks to (2.13), we know that $\min\limits_{n\geq 1}|\overline{U_0}^2 - 1/\lambda_n^2| > 0$.

[5]Assuming e.g. periodical boundary conditions for $\tilde{\mathcal{M}}$.

where (u_n^g, v_n^g, ψ_n^g) are found using the boundary values $U^{g,l}$ and $U^{g,r}$ by:

$$\left\{\begin{array}{rcl} u_n^g(t) - \frac{1}{N}\,\psi_n^g(t) &=& u_n^{g,l}(t) - \frac{1}{N}\,\psi_n^{g,l}(t), \\ v_n^g(t) &=& v_n^{g,l}(t), \qquad\qquad 1 \leq n \leq n_c, \\ u_n^g(t) + \frac{1}{N}\,\psi_n^g(t) &=& u_n^{g,r}(t) + \frac{1}{N}\,\psi_n^{g,r}(t), \end{array}\right. \tag{2.53}$$

$$\left\{\begin{array}{rcl} u_n^g(t) &=& u_n^{g,l}(t), \\ v_n^g(t) &=& v_n^{g,l}(t), \qquad n > n_c. \\ \psi_n^g(t) &=& \psi_n^{g,l}(t), \end{array}\right. \tag{2.54}$$

We note that U^g is a function of z and t, and hence it does not depend on the horizontal coordinate x. Setting $F^{\#} = F - dU^g/dt$ and $U_0^{\#} = U_0 - U_0^g$ where $U_0^g = U^g(t=0)$, we will look for $U^{\#}$ solution of

$$\left\{\begin{array}{l} \frac{dU^{\#}}{dt} + A\,U^{\#} = F^{\#}, \\ U^{\#}(t=0) = U_0^{\#}. \end{array}\right. \tag{2.55}$$

Like (2.1) this equation corresponds to the case with homogeneous boundary conditions, and Theorem 2.2 applies[6]. Writting $U = U^{\#} + U^g$, we find that U is solution of (1.22) -(1.27), and the boundary conditions of U at $x = 0$ and $x = L_1$ are those of U^g, that is for the subcritical modes ($1 \leq n \leq n_c$):

$$\left\{\begin{array}{l} \int_{-L_3}^{0} u(0,z,t)\,\mathcal{U}_n(z)\,dz - \frac{1}{N}\int_{-L_3}^{0} \psi(0,z,t)\,\mathcal{W}_n(z)\,dz \\ \quad = \int_{-L_3}^{0} u^{g,l}(z,t)\,\mathcal{U}_n(z)\,dz - \frac{1}{N}\int_{-L_3}^{0} \psi^{g,l}(z,t)\,\mathcal{W}_n(z)\,dz, \\ \int_{-L_3}^{0} v(0,z,t)\,\mathcal{U}_n(z)\,dz = \int_{-L_3}^{0} v^{g,l}(z,t)\,\mathcal{U}_n(z)\,dz, \\ \int_{-L_3}^{0} u(L_1,z,t)\,\mathcal{U}_n(z)\,dz + \frac{1}{N}\int_{-L_3}^{0} \psi(L_1,z,t)\,\mathcal{W}_n(z)\,dz \\ \quad = \int_{-L_3}^{0} u^{g,r}(z,t)\,\mathcal{U}_n(z)\,dz + \frac{1}{N}\int_{-L_3}^{0} \psi^{g,r}(z,t)\,\mathcal{W}_n(z)\,dz, \end{array}\right. \tag{2.56}$$

and for the supercritical ones $(n > n_c)$:

$$\left\{\begin{array}{rcl} \int_{-L_3}^{0} u(0,z,t)\,\mathcal{U}_n(z)\,dz &=& \int_{-L_3}^{0} u^{g,l}(z,t)\,\mathcal{U}_n(z)\,dz, \\ \int_{-L_3}^{0} v(0,z,t)\,\mathcal{U}_n(z)\,dz &=& \int_{-L_3}^{0} v^{g,l}(z,t)\,\mathcal{U}_n(z)\,dz, \\ \int_{-L_3}^{0} \psi(0,z,t)\,\mathcal{W}_n(z)\,dz &=& \int_{-L_3}^{0} \psi^{g,l}(z,t)\,\mathcal{W}_n(z)\,dz. \end{array}\right. \tag{2.57}$$

Thus we have established the following result:

[6]We will state in Theorem 2.3 below some assumptions on $U^{g,l}$ and $U^{g,r}$ so that $U_0^{\#}$ and $f^{\#}$ are as in Theorem 2.2.

Theorem 2.3. *Let H be the Hilbert space defined in (2.2) and A be the linear operator defined in (2.9) corresponding to the linearized Primitive Equations with vanishing viscosity. We are given the boundary values $U^{g,l}$ and $U^{g,r}$ which are in $L^1(0,T;L^2(-L_3,0)^3)$, together with their first time derivative, F and $F' = dF/dt \in L^1(0,T;H)$.*
Then the initial value problem corresponding to equations (1.22)-(1.27), supplemented with the boundary conditions (2.56) and (2.57) is well-posed, that is for every initial data $U_0 \in U_0^g + D(A)$[7], there exists a unique solution $U \in \mathcal{C}([0,T];H)$ of (1.22)-(1.27) verifying (2.56) and (2.57).

2.5 The mode constant in z.

We now return to the mode constant in z, when $n = 0$. This mode does not raise any mathematical difficulty, but it is fundamental in the numerical simulations, since it carries much energy. Integrating (1.22),(1.23), and (1.26) on $(-L_3, 0)$ we find:

$$\frac{\partial u_0}{\partial t} + \overline{U_0}\,\frac{\partial u_0}{\partial x} - f\,v_0 + \frac{\partial \phi_0}{\partial x} = F_{u,0}, \tag{2.58}$$

$$\frac{\partial v_0}{\partial t} + \overline{U_0}\,\frac{\partial v_0}{\partial x} + f\,u_0 = F_{v,0}, \tag{2.59}$$

$$\frac{\partial u_0}{\partial x} = 0. \tag{2.60}$$

We propose to supplement this system with the following boundary conditions:

$$u_0(0,t) = u_l(t), \tag{2.61}$$

$$v_0(0,t) = v_l(t), \tag{2.62}$$

with u_l, v_l given (not necessarily zero, as in Section 2.4).

Then, since $\partial u_0/\partial x = 0$, u_0 does not depend on x, and it is thus equal to $u_l(t)$ everywhere, so that (2.61) means in fact that

$$u_0(x,t) = u_l(t), \quad \forall (x,t) \in (0, L_1) \times \mathbb{R}_+^*. \tag{2.63}$$

Introducing (2.63) in (2.59), we find that:

$$\frac{\partial v_0}{\partial t} + \overline{U_0}\,\frac{\partial v_0}{\partial x} = F_{v,0} - f\,(\overline{U_0} + u_l). \tag{2.64}$$

When we supplement (2.64) with the boundary condition (2.62), we have a simple well-posed problem and v_0 is given in terms of the data by integration along the characteristics.

[7]This means that U_0 has the same smoothness as a function of $D(A)$ and (2.56),(2.57) are satisfied at $t = 0$.

Finally, since both u_0 and v_0 are known, equation (2.58) gives ϕ_0, up to an additive constant (as expected):

$$\begin{aligned} \phi_0(x,t) &= \phi_0(0,t) + \int_0^x \{f\, v_0(x',t) - \frac{\partial u_0}{\partial t}(x',t)\}dx' \\ &= \phi_0(0,t) - x\, u_l'(t) + f \int_0^x v_0(x',t)dx'. \end{aligned} \tag{2.65}$$

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Chapitre 6

Simulations numériques des équations primitives de l'océan

Numerical simulations of the primitive equations of the ocean

Ce chapitre est consacré aux simulations numériques des équations primitives (EP) de l'océan sans viscosité, ce qui correspond à la motivation générale de ce travail ; les cas linéaire et non linéaire sont considérés en dimension deux d'espace. Dans le cas non linéaire on résoud tout d'abord les EP de référence dans un domaine Ω_0 avec conditions aux limites périodiques. Ensuite on se restreint à un domaine Ω_1 plus petit, dans lequel on effectue un second calcul, cette fois avec des conditions aux limites transparentes ; aucune explosion numérique ne se produit ce qui conforte la conjecture que les conditions aux limites proposées conduisent à un problème non linéaire bien posé. Enfin on compare les deux solutions numériques dans le sous-domaine Ω_1, et la bonne coincidence des courbes (à quelques pourcents près) confirme d'autre part la bonne validité numérique des conditions aux limites que nous avons proposées.
On termine ce chapitre par des simulations numériques comparables pour le cas des équations linéarisées, qui prolongent ainsi les résultats théoriques nouveaux établis dans le chapitre précédent ; le caractère transparent des conditions aux limites utilisées est là aussi mis en évidence.

Numerical simulations of the PEs of the ocean

Abstract

This chapter is dedicated to the numerical computations of the Primitive Equations (PEs) of the ocean without viscosity with the boundary conditions introduced in Chapter 5. We consider the 2D nonlinear PEs, and firstly proceed to computations in a large domain Ω_0 with periodic boundary conditions. Then we consider a subdomain Ω_1, in which we compute a second numerical solution with transparent boundary conditions, and evidence the accuracy of these boundary conditions thanks to a comparison in the subdomain Ω_1 of the two numerical solutions. We end this chapter with some numerical simulations of the linearized primitive equations, which extend the theoretical results established in Chapter 5, and evidence the transparent properties of the boundary conditions.

1 Introduction and objectives

In this chapter, we intend to present our numerical simulations of the 2D Primitive Equations (PEs). We want to emphasize on the transparent boundary conditions introduced in Chapter 5. To this aim, we first compute the PEs in a large (x, z) domain $\Omega_0 = (0, L) \times (-H, 0)$, with no flux boundary conditions at top and bottom, and periodic boundary conditions in the horizontal direction:

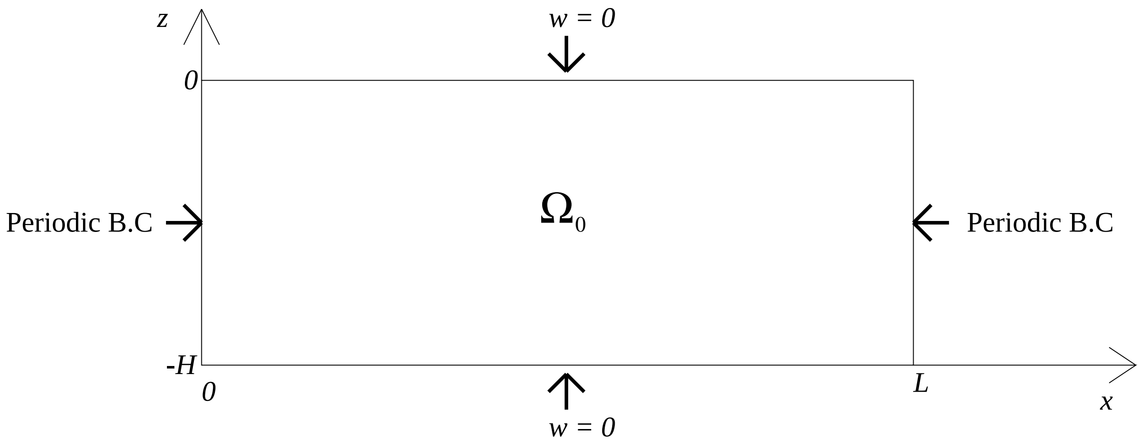


Figure 6.1: Domain Ω_0

We then consider a sub-domain $\Omega_1 = (a, b) \times (-H, 0)$, where $0 < a < b < L$, so that $\Omega_1 \subset \Omega_0$. We perform numerical simulations of the PEs on Ω_1, with the no flux boundary condition at top and bottom, but we set the transparent boundary conditions at $x = a$ and $x = b$ (the actual values are taken taken from the calculations in Ω_0). The initial condition is the same as on Ω_0

(restriction to Ω_1):

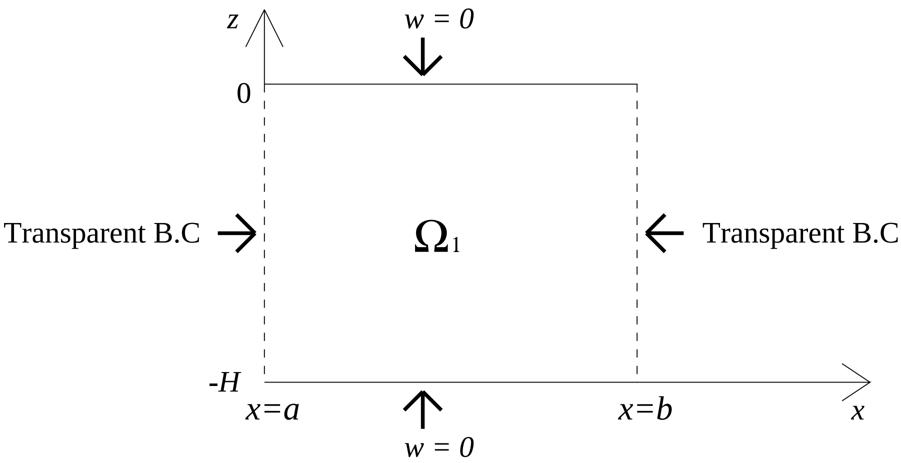


Figure 6.2: Subdomain Ω_1

Our objectives are twofold: firstly to show that these boundary conditions are well adapted to the PEs without viscosity, since no numerical blow-up occurs, and secondly to show that they are well-suited for the problem of numerical simulations in a limited domain. This is shown by observation that the solutions computed on Ω_1 only (with the with the nonreflecting boundary conditions) matche well on Ω_1 with the solutions computed in the whole domain Ω_0.

1.1 Discretization of the equations

We start from the Primitive Equations of the Ocean, independant of y, as in (1.15)-(1.19) of Chapter 5:

$$\frac{\partial u}{\partial t} + (\overline{U_0} + u)\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} - f\,v + \frac{\partial \phi}{\partial x} = F_u, \quad (1.1a)$$

$$\frac{\partial v}{\partial t} + (\overline{U_0} + u)\,\frac{\partial v}{\partial x} + w\,\frac{\partial v}{\partial z} + f\,u = F_v - f\,\overline{U_0}, \quad (1.1b)$$

$$\frac{\partial \psi}{\partial t} + (\overline{U_0} + u)\,\frac{\partial \psi}{\partial x} + (N^2 + \frac{\partial \psi}{\partial z})\,w = F_\psi, \quad (1.1c)$$

$$\frac{\partial \phi}{\partial z} = -\frac{\rho}{\rho_0}\,g = \psi, \quad (1.1d)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (1.1e)$$

We supplement these equations with an initial data u_0, v_0, ψ_0, and consider the equations in the bidimensional domain $\Omega_0 = (0, L) \times (-H, 0)$. The boundary condition taken at $z = -H$ and $z = 0$ is classically $w = 0$, and we will consider in this study two different sets of boundary conditions in the horizontal direction: the periodic ones and the non-reflective ones.

Classically we proceed by separation of variables and actually look for the unknown functions (u, v, w, ϕ, ψ) under the form (see Chapter 5 for more details):

$$(u, v, \phi) = \mathcal{U}(z)\,(\hat{u}, \hat{v}, \hat{\phi})\,(x, t), \quad (w, \psi) = \mathcal{W}(z)\,(\hat{w}, \hat{\psi})\,(x, t), \tag{1.2}$$

where the functions $\hat{u}, \hat{v}, \hat{w}$, and $\hat{\phi}$ only depend on x and t. Introducing the decomposition (1.2) into equations (1.1) shows that $\mathcal{W}$ (and then $\mathcal{U}$) solves a two-point eigenvalue problem with the boundary condition $w = 0$ at top and bottom ($z = 0, -H$). We thus obtain the normal modes $\mathcal{W}_m(z)$ and $\mathcal{U}_m(z)$ and then look for the general solution in the form a series

$$(u, v, \phi) = \sum_{m \geq 0} \mathcal{U}_m(z)\,(\hat{u}_m, \hat{v}_m, \hat{\phi}_m)\,(x, t), \tag{1.3}$$

$$(w, \psi) = \sum_{m \geq 1} \mathcal{W}_m(z)\,(\hat{w}_m, \hat{\psi}_m)\,(x, t), \tag{1.4}$$

where:

$$\mathcal{U}_m(z) = \sqrt{\frac{2}{H}}\,cos(N\,\lambda_m\,z), \quad \mathcal{U}_0(z) = \frac{1}{\sqrt{H}}, \tag{1.5}$$

$$\mathcal{W}_m(z) = \sqrt{\frac{2}{H}}\,sin(N\,\lambda_m\,z), \tag{1.6}$$

$$\lambda_m = \frac{m\,\pi}{N\,H}. \tag{1.7}$$

We notice that $\forall m' \geq 0, m \geq 1$, we have the orthogonality properties:

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \mathcal{U}_{m'}(z)\,\mathcal{U}_m(z)\,dz = \delta_{m',m}, \\ \displaystyle\int_{-L_3}^{0} \mathcal{U}_{m'}(z)\,\mathcal{W}_m(z)\,dz = 0, \\ \mathcal{U}'_m(z) = -N\,\lambda_m\,\mathcal{W}_m(z), \\ \mathcal{W}'_m(z) = N\,\lambda_m\,\mathcal{U}_m(z). \end{cases} \tag{1.8}$$

In the numerical simulations, we will truncate the series after M terms. Naturally, the larger M is, the more accurate the method is expected to be, but the heavier the computations are. Typically, $M = 10$ is satisfying from the physical point of view.
The case of the steady mode $m = 0$ is very simple, and is explained in Section 2.5 of Chapter 5. From now on we only consider the modes $m \geq 1$.

Writing the PEs mode by mode, and writing (u_m, v_m, w_m, ψ_m, ϕ_m) instead of ($\hat{u}_m$, $\hat{v}_m$, $\hat{w}_m$,

$\hat{\psi}_m$, $\hat{\phi}_m$), we obtain the following system of integrodifferential equations ($1 \leq m \leq M$):

$$\frac{\partial u_m}{\partial t} + \overline{U_0}\,\frac{\partial u_m}{\partial x} - f\,v_m + \frac{\partial \phi_m}{\partial x} + B_{u,m}(U) = F_{u,m}, \tag{1.9a}$$

$$\frac{\partial v_m}{\partial t} + \overline{U_0}\,\frac{\partial v_m}{\partial x} + f\,u_m + B_{v,m}(U) = F_{v,m}, \tag{1.9b}$$

$$\frac{\partial \psi_m}{\partial t} + \overline{U_0}\,\frac{\partial \psi_n}{\partial x} + N^2\,w_n + B_{\psi,m}(U) = F_{\psi,m}, \tag{1.9c}$$

$$\phi_m = -\frac{\psi_m}{N\,\lambda_m}, \tag{1.9d}$$

$$w_m = -\frac{1}{N\,\lambda_m}\,\frac{\partial u_m}{\partial x}; \tag{1.9e}$$

here $B_{u,m}$, $B_{v,m}$ and $B_{\psi,m}$ are the following modal parts of the nonlinearities:

$$B_{u,m} = \int_{-H}^{0} (u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z})\,\mathcal{U}_m\,dz, \tag{1.10a}$$

$$B_{v,m} = \int_{-H}^{0} (u\,\frac{\partial v}{\partial x} + w\,\frac{\partial v}{\partial z})\,\mathcal{U}_m\,dz, \tag{1.10b}$$

$$B_{\psi,m} = \int_{-H}^{0} (u\,\frac{\partial \psi}{\partial x} + w\,\frac{\partial \psi}{\partial z})\,\mathcal{W}_m\,dz. \tag{1.10c}$$

with u, v, ψ, w truncated to M modes. Taking equations (1.9d) and (1.9e) into account, equations (1.9a)-(1.9c) become:

$$\frac{\partial u_m}{\partial t} + \overline{U_0}\,\frac{\partial u_m}{\partial x} - f\,v_m - \frac{1}{N\,\lambda_m}\,\frac{\partial \psi_m}{\partial x} + B_{u,m}(U) = F_{u,m}, \tag{1.11a}$$

$$\frac{\partial v_m}{\partial t} + \overline{U_0}\,\frac{\partial v_m}{\partial x} + f\,u_m + B_{v,m}(U) = F_{v,m}, \tag{1.11b}$$

$$\frac{\partial \psi_m}{\partial t} + \overline{U_0}\,\frac{\partial \psi_m}{\partial x} - \frac{N}{\lambda_m}\,\frac{\partial u_m}{\partial x} + B_{\psi,m}(U) = F_{\psi,m}. \tag{1.11c}$$

We have now M systems of three coupled integro-differential equations (time dependant with one space variable). We will discretize this system in Section 2 below.

For every $m \leq M$, we introduce $\xi_m = u_m - \psi_m/N, \eta_m = u_m + \psi_m/N$, and let v_m unchanged. In terms of these variables, the system (1.11) becomes:

$$\frac{\partial \xi_m}{\partial t} + (\overline{U_0} + \frac{1}{\lambda_m})\,\frac{\partial \xi_m}{\partial x} - f\,v_m + B_{\xi,m}(U) = F_{\xi,m}, \tag{1.12a}$$

$$\frac{\partial v_m}{\partial t} + \overline{U_0}\,\frac{\partial v_m}{\partial x} + f\,\frac{\xi_m + \eta_m}{2} + B_{v,m}(U) = F_{v,m}, \tag{1.12b}$$

$$\frac{\partial \eta_m}{\partial t} + (\overline{U_0} - \frac{1}{\lambda_m})\,\frac{\partial \eta_m}{\partial x} - f\,v_m + B_{\eta,m}(U) = F_{\eta,m}. \tag{1.12c}$$

where $B_{\xi,m} = B_{u,m} - B_{\psi,m}/N$ and $B_{\eta,m} = B_{u,m} + B_{\psi,m}/N$. The physical quantities can be obtained from ξ_m, η_m and v_m with:

$$u_m(x,t) = \frac{\xi_m + \eta_m}{2}(x,t), \tag{1.13a}$$

$$w_m(x,t) = -\frac{u_{m_x}}{N\,\lambda_m}(x,t), \tag{1.13b}$$

$$\psi_m(x,t) = \frac{N\,(\eta_m - \xi_m)}{2}(x,t), \tag{1.13c}$$

$$\phi_m(x,t) = -\frac{\psi_m}{N\,\lambda_m}(x,t). \tag{1.13d}$$

It is crucial to notice that $\overline{U_0}$, $\overline{U_0} + 1/\lambda_m$ are always positive, whereas $\overline{U_0} - 1/\lambda_m$ can either be positive or negative[1], depending on the value of $m \le M$; actually, the sign of these three characteristic values will determine the way we discretize the equations (1.12) in the horizontal direction.
Thanks to (1.7), there exists a critical value m_c such that $\overline{U_0} - 1/\lambda_m$ is negative (resp. positive) if $m \le m_c$ (resp. $m > m_c$). The corresponding modes are then called subcritical (resp. supercritical).

As in [TT03] we assume in the following that the initial data is such that the nonlinear part is small compared to the stratified flow $(\overline{U_0}, 0, 0)$, so that the characteristic values do not change sign, at least during a certain period of time. Assuming so, we conjecture that the boundary conditions provided for the linearized system will give a well-posed problem for the nonlinear equations. We leave the theoretical analysis to subsequent studies, and perform here the corresponding numerical simulations based on this hypothesis, which is conforted by the lack of numerical blow-up.

This chapter is organized as follows. In Section 2, we present the numerical schemes that we use in order to solve the primitive equations (1.1). The spectral method is employed for the vertical direction, whereas the x and t derivatives are discretized using finite differences. In Section 3, we present the numerical computations of the nonlinear PEs. The first results are dedicated to the periodic boundary conditions; then we implement the nonreflective boundary conditions introduced in Chapter 5, and we end this section with a comparison between the two different numerical solutions. The same type of numerical simulations is implemented in Section 4, but for the case of the linearized equations; this is of course a natural application of the theoretical results presented in Chapter 5.

[1] This is actually why the PEs in a bounded domain are ill-posed with any set of local boundary conditions, see [OS78, TT03, PR05, RTT05a, RTT05b].

2 Numerical scheme

2.1 Vertical discretization by spectral method

In the vertical direction, we proceed by normal modes decomposition as in (1.3), (1.4). From the numerical point of view, we will need to transform some grid-data into modal coefficients in the $\mathcal{U}_m$ or $\mathcal{W}_m$ bases of $L^2(-H,0)$, and *vice versa.*
Given a function f represented by its values f_l on a grid $z_l = -H + l\Delta z$, $0 \le l \le l_{max}$, $\Delta z = H\,/\,l_{max}$, we want to transform it into coefficients f_m, $0 \le m \le M$. To this aim we use the second order central point integration method, with the z_l as collocation points. For the functions u, v and ϕ, we decompose them in the $\mathcal{U}_m$ basis of $L^2(-H,0)$. For $0 \le m \le M$:

$$\{u_m, v_m, \phi_m\} = \Delta z \sum_{l=0}^{l_{max}-1} \frac{\mathcal{U}_m(z_l)\cdot\{u,v,\phi\}(z_l) + \mathcal{U}_m(z_{l+1})\cdot\{u,v,\phi\}(z_{l+1})}{2}, \tag{2.1}$$

and for w and ψ, $1 \le m \le M$:

$$\{w_m, \psi_m\} = \Delta z \sum_{l=0}^{l_{max}-1} \frac{\mathcal{W}_m(z_l)\cdot\{w,\psi\}(z_l) + \mathcal{W}_m(z_{l+1})\cdot\{w,\psi\}(z_{l+1})}{2}. \tag{2.2}$$

This approach which is that proposed by the physicists is different from the more mathematical approach to spectral and pseudo spectral methods (as in e.g. [BM97, GH01]). The advantage of such a choice is that the orthogonality relations (1.8) are satisfied from the numerical point of view. Further studies and comparisons of approaches will be needed in the future.

On the contrary, if the function is given by its modal coefficients, the values on the z-grid z_l, $0 \le l \le l_{max}$ is simply given by:

$$(u,v,\phi)(z_l) = \sum_{m=0}^{M} (u_m, v_m, \phi_m)\,\mathcal{U}_m(z_l), \tag{2.3}$$

$$(w,\psi)(z_l) = \sum_{m=0}^{M} (w_m, \psi_m)\,\mathcal{W}_m(z_l). \tag{2.4}$$

In the numerical simulations, we are given some initial data on the physical grid $(z_l)_{0\le l\le l_max}$. We transform them into modal coefficients thanks to formulas (2.1) or (2.2), and if the problem is linear, we keep them all along the computations, except for graphic purposes, for which we use inverse formulas (2.3)-(2.4) to get back to physical space. Naturally, in the nonlinear case, we need to operate (2.1)-(2.4) once at every time step, in order to avoid the computation of a convolution product, that would cost too much in term of CPU time. We compute the nonlinear terms of the equations in the physical space (x,z) thanks to Fourier and inverse Fourier transforms.

2.2 Finite differences

Looking at the form of (1.12), we choose to discretize these equations with the finite differences method. Naturally, care has to be taken to the sign of the characteristic values, in order to take an upwind (hence stable) spatial discretization of the x-derivative. Whereas $\overline{U_0}$ and $\overline{U_0}+1/\lambda_m$ are always positive, the third characteristic value of the mth mode - in the linear case - is $\overline{U_0}-1/\lambda_m$. and can either be positive or negative. The corresponding mode is then called supercritical (resp. subcritical). We define by m_c the number of subcritical modes, that is such that:

$$\frac{m_c\,\pi}{N\,L_3}=\lambda_{m_c}\leq\frac{1}{\overline{U_0}}<\lambda_{m_c+1}=\frac{(m_c+1)\,\pi}{N\,L_3}. \tag{2.5}$$

In the non linear case, since the initial data is small compared to $\overline{U_0}$ ([TT03]), we implicitly assume that m_c remains unchanged for a certain period of time. Until a full nonlinear theory is performed, a first step in the verification of this hypothesis would be to linearize Equation (1.9) (or (1.11)-(1.12)) around the current state which may amount to replacing $\overline{U_0}$ by $\overline{U_0}+u$, but may also involve a more complex analysis already in the linearized case. These involved issues are left to subsequent work.

That is, for every subcritical mode $m\leq m_c$, we discretize (1.12) as follows:

$$\frac{\xi_{m,j}^{n+1}-\xi_{m,j}^{n}}{\Delta t^n}+(\overline{U_0}+\frac{1}{\lambda_m})\,\frac{\xi_{m,j}^{n}-\xi_{m,j-1}^{n}}{\Delta x}-f\,v_{m,j}^{n}=F_{\xi,m,j}^{n}-B_{\xi,m,j}^{n},\quad\forall 1\leq j\leq J, \tag{2.6a}$$

$$\frac{v_{m,j}^{n+1}-v_{m,j}^{n}}{\Delta t^n}+\overline{U_0}\,\frac{v_{m,j}^{n}-v_{m,j-1}^{n}}{\Delta x}+f\,\frac{\xi_{m,j}^{n}+\eta_{m,j}^{n}}{2}=F_{v,m,j}^{n}-B_{v,m,j}^{n},\quad\forall 1\leq j\leq J, \tag{2.6b}$$

$$\frac{\eta_{m,j}^{n+1}-\eta_{m,j}^{n}}{\Delta t^n}+(\overline{U_0}-\frac{1}{\lambda_m})\,\frac{\eta_{m,j+1}^{n}-\eta_{m,j}^{n}}{\Delta x}-f\,v_{m,j}^{n}=F_{\eta,m,j}^{n}-B_{\eta,m,j}^{n},\quad\forall 0\leq j\leq J-1, \tag{2.6c}$$

where the right-hand-side contain the nonlinear terms, computed explicitely thanks to an Adams-Bashforth scheme.

There are no equations for $\xi_{m,0}^{n+1}$, $v_{m,0}^{n+1}$ and $\eta_{m,J}^{n+1}$, these quantities being given by the boundary conditions as required by (1.46) of Chapter 5. On the contrary, if $m>m_c$ (supercritical case), we choose:

$$\frac{\xi_{m,j}^{n+1}-\xi_{m,j}^{n}}{\Delta t^n}+(\overline{U_0}+\frac{1}{\lambda_m})\,\frac{\xi_{m,j}^{n}-\xi_{m,j-1}^{n}}{\Delta x}-f\,v_{m,j}^{n}=F_{\xi,m,j}^{n}-B_{\xi,m,j}^{n},\quad\forall 1\leq j\leq J, \tag{2.7a}$$

$$\frac{v_{m,j}^{n+1}-v_{m,j}^{n}}{\Delta t^n}+\overline{U_0}\,\frac{v_{m,j}^{n}-v_{m,j-1}^{n}}{\Delta x}+f\,\frac{\xi_{m,j}^{n}+\eta_{m,j}^{n}}{2}=F_{v,m,j}^{n}-B_{v,m,j}^{n},\quad\forall 1\leq j\leq J, \tag{2.7b}$$

$$\frac{\eta_{m,j}^{n+1}-\eta_{m,j}^{n}}{\Delta t^n}+(\overline{U_0}-\frac{1}{\lambda_m})\,\frac{\eta_{m,j}^{n}-\eta_{m,j-1}^{n}}{\Delta x}-f\,v_{m,j}^{n}=F_{\eta,m,j}^{n}-B_{\eta,m,j}^{n},\quad\forall 1\leq j\leq J. \tag{2.7c}$$

Either $\xi_{m,0}^{n+1}$, $v_{m,0}^{n+1}$ and $\eta_{m,0}^{n+1}$ are given by the boundary conditions defined as in (1.44) of Chapter 5 (transparent boundary conditions case), or they satisfy the periodicity conditions (3.1) below (periodical case).
For every function $f(x,z,t)$, $f_{m,j}^n$ represents $f_m(x_j,t_n)$ for $0 \leq j \leq J$, $0 \leq n \leq n_{max}$, with

$$0 = x_0 < x_1 < ... < x_j < ... < x_J = L, \tag{2.8}$$

$$0 = t_0 < t_1 < ... < t_n < ... < t_{n_{max}} = T, \tag{2.9}$$

$$\Delta x = x_{j+1} - x_j = \frac{L}{J}, \tag{2.10}$$

$$\Delta t^n = t_{n+1} - t_n. \tag{2.11}$$

In the numerical experiments, we choose an homogeneous space discretization ($\Delta x = const = L/J$). For the sake of simplicity, we choose an explicit time-scheme, with a constant time-step Δt, which will be restricted by the well-known CFL condition to guarantee stability in the linear case:

$$\Delta t \leq \frac{\Delta x}{\max\limits_{1 \leq m \leq M}(\overline{U_0}, \overline{U_0} + \frac{1}{\lambda_m}, |\overline{U_0} - \frac{1}{\lambda_m}|)} = \frac{\Delta x}{\overline{U_0} + \frac{1}{\lambda_1}}. \tag{2.12}$$

Naturally, when the equations are nonlinear, the CFL condition is not enough to guarantee stability. Actually, the characteristic values depend on time since $\overline{U_0}$ has to be replaced by $u + \overline{U_0}$, but we assume that the initial data is such that $|u_0| << \overline{U_0}$, which is physically relevant, [TT03]. We actually base our computations on those of the quoted article [TT03]: the initial data is such that the ratio between the perturbation and the reference flow $\overline{U_0}\,\mathbf{e_x}$ is less than 10%, which is - according to Joseph Tribbia [TT03] - physically relevant. In the case of numerical simulations with periodic boundary conditions, we multiply the initial data of [TT03] by $sin(\pi\, x/L)$ to make it periodic and avoid any boundary layer at $t = 0$.

3 Numerical simulations of the nonlinear primitive equations

The computations are done as follows. We fix M (the number of modes) and compute (u_m^0, v_m^0, $\psi_m^0)_{0<m\leq M}$ from the given data u^0, v^0, ψ^0 thanks to (2.1)-(2.2).
Then, for every mode $m \leq M$, we consider the modal equations (1.12) and their discretization (2.6)-(2.7), and supplement them with the appropriate boundary conditions, either (3.2) for the periodical case or (3.6)-(3.7) for the case of transparent boundary conditions. We recall here that for every m, (ξ_m, η_m)=$(u_m - \psi_m/\lambda_m, u_m + \psi_m/\lambda_m)$ will be the numerical unknowns to be computed.

3.1 Periodic boundary conditions for the large domain Ω_0.

In the periodical case, we consider the following modal boundary conditions:

$$\xi_m(0,t) = \xi_m(L,t), \tag{3.1a}$$

$$v_m(0,t) = v_m(L,t), \tag{3.1b}$$

$$\eta_m(0,t) = \eta_m(L,t). \tag{3.1c}$$

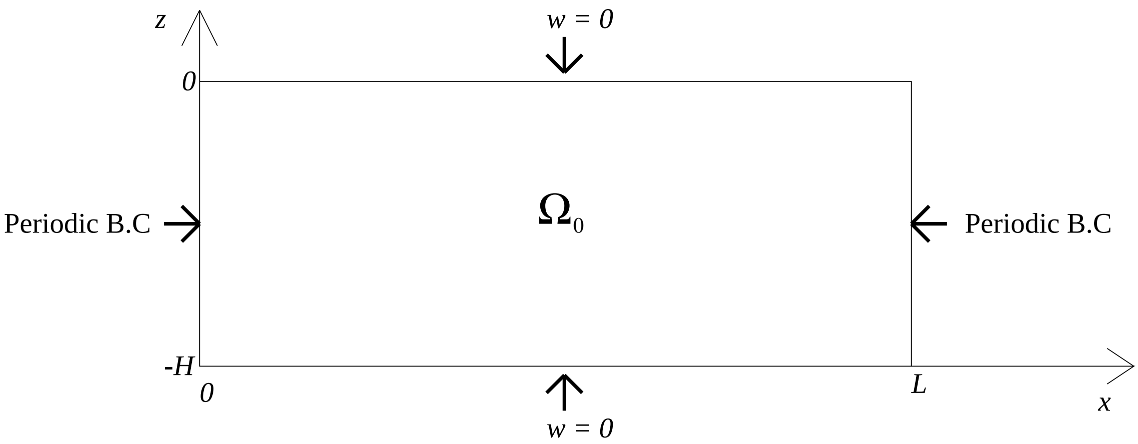


Figure 6.3: Domain Ω_0

For each time step $\Delta t^n = \Delta t$ satisfying (2.12) we compute the unknown functions (ξ_m^{n+1}, v_m^{n+1}, η_m^{n+1}) thanks to (2.6) and (2.7), with the numerical boundary conditions:

$$\xi_{m,0}^{n+1} = \xi_{m,J}^{n+1}, \tag{3.2a}$$

$$v_{m,0}^{n+1} = v_{m,J}^{n+1}, \tag{3.2b}$$

$$\eta_{m,0}^{n+1} = \eta_{m,J}^{n+1}. \tag{3.2c}$$

The following figures plot u, v and ψ in the domain Ω_0 at two different times. Figures 4, 5 and 6 represent the initial data ($t = 0$) for these three quantities, whereas Figures 7, 8 and 9 represent u, v and ψ at $t = t_1 > 0$.

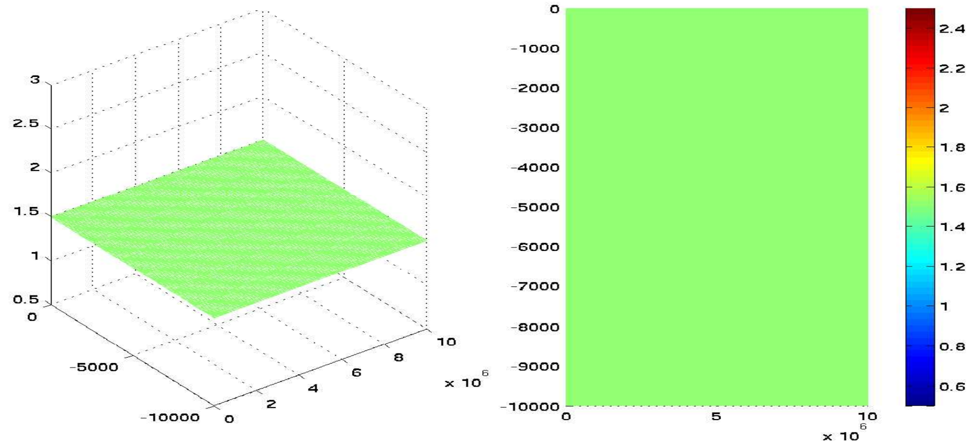


Figure 6.4: Periodic Boundary Condition. Initial data u_0.

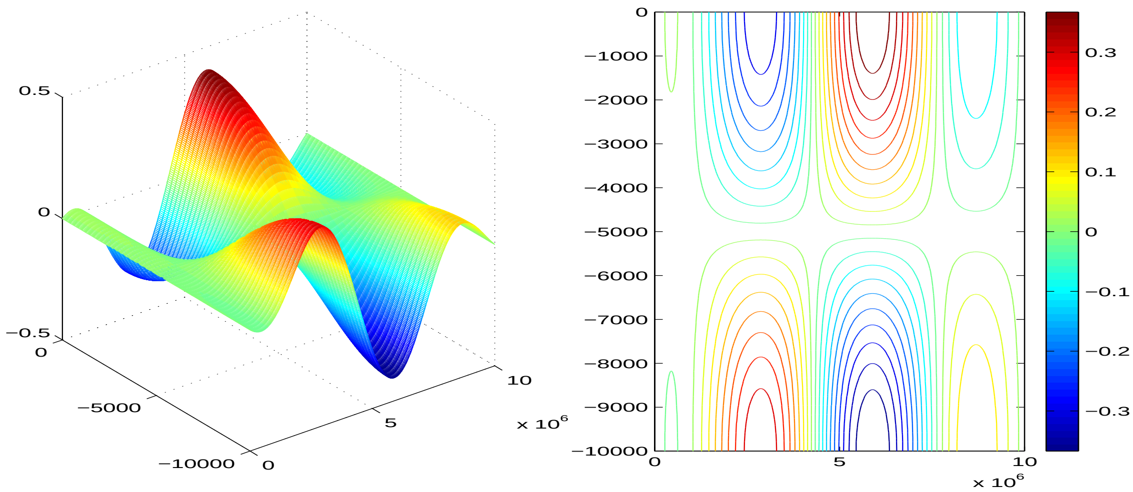


Figure 6.5: Periodic Boundary Condition. Initial data v_0.

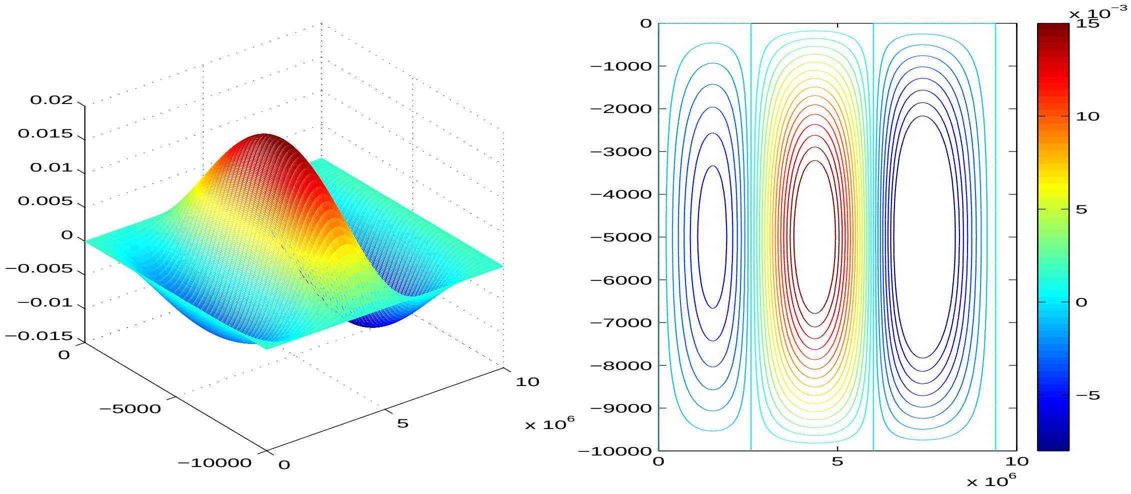


Figure 6.6: Periodic Boundary Condition. Initial data ψ_0.

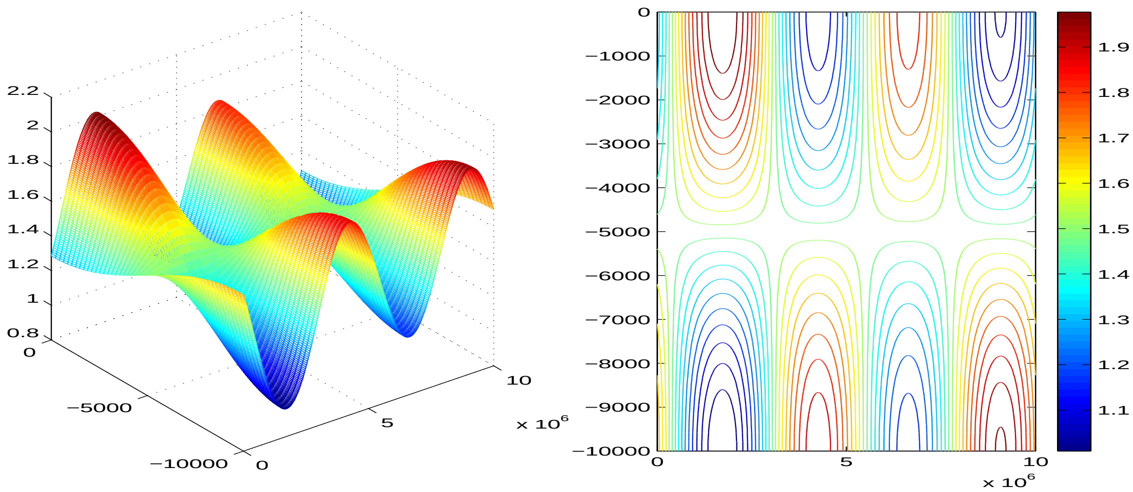


Figure 6.7: Periodic Boundary Condition. Values of u at $t = t_1$.

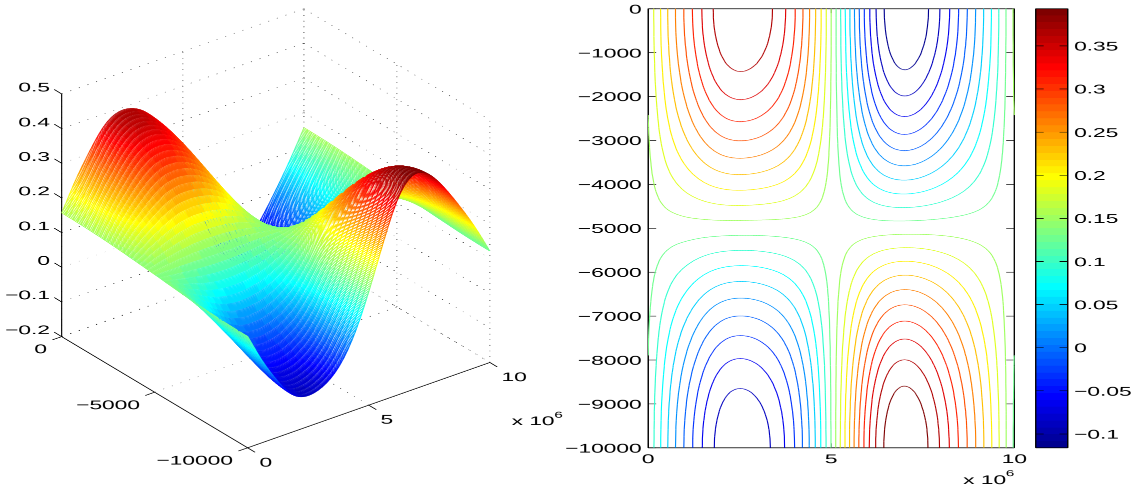


Figure 6.8: Periodic Boundary Condition. Values of v at $t = t_1$.

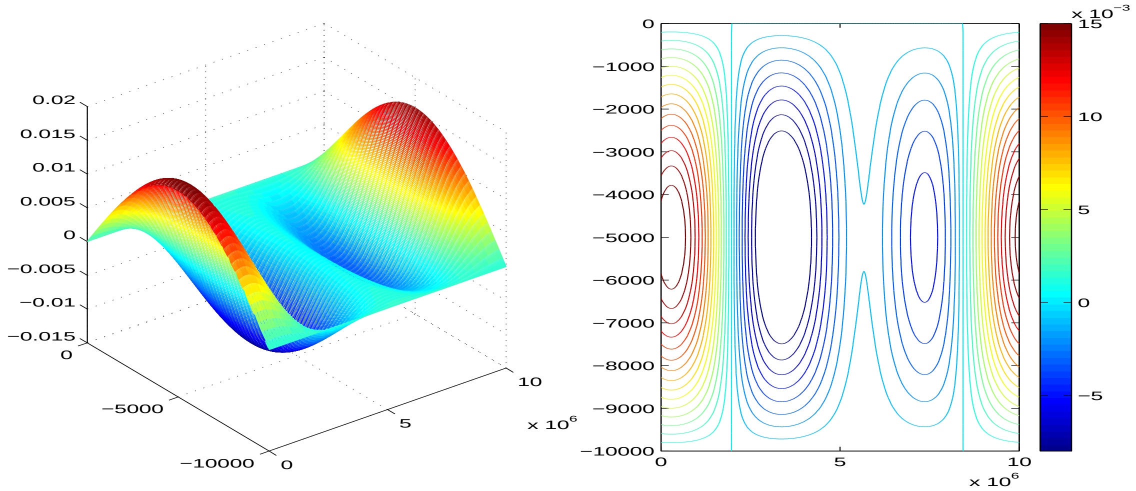


Figure 6.9: Periodic Boundary Condition. Values of ψ at $t = t_1$.

3.2 Transparent boundary conditions for the subdomain $\Omega_1 \subset \Omega_0$

We now intend to simulate the PEs in the subdomain $\Omega_1 = (a, b) \times (-H, 0)$ and the boundary conditions are the nonlinear version of those introduced in Chapter 5. In the numerical simulations below we will consider a domain $\Omega_1 = (a, b) \times (-H, 0)$ such that $0 < a < b < L$, so that Ω_1 is actually imbedded in $\Omega_0 = (0, L) \times (-H, 0)$. The space discretization is now changed to $x_j = a + j(b - a)/J$, $0 \leq j \leq J$.
At the boundaries $x = a$ and $x = b$, we will consider the nonhomogeneous form of the transparent boundary conditions of Chapter 5. We use the computations of Section 3.1 above to provide

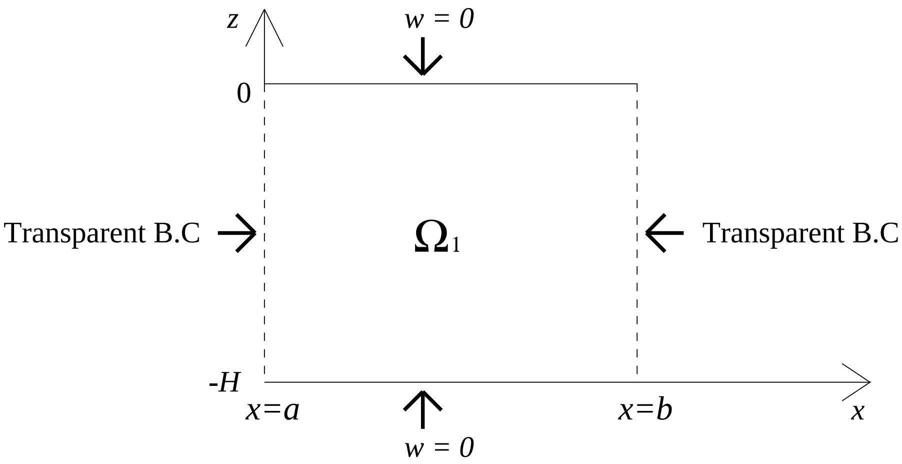


Figure 6.10: Subdomain Ω_1

the right-hand-side of the boundary conditions (3.4) and (3.5) below, and afterwards use them for comparison in the subdomain Ω_1. These boundary conditions, expressed in a general way, are equations (2.57) of Chapter 5, *that is an infinite set of integral boundary conditions.* For example:

$$\int_{-H}^{0} v(a,z,t)\,\mathcal{U}_m(z)\,dz = \int_{-L_3}^{0} \tilde{v}(a,z,t)\,\mathcal{U}_m(z)\,dz, \quad \forall m \leq M, \tag{3.3}$$

where $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\psi}, \tilde{\phi})$ are some known functions, computed in the domain Ω_0 with some periodic boundary conditions (see Section 3.1 above).
Hence, for every subcritical mode ($m \leq m_c$) and every time $t > 0$, we have:

$$\xi_m(a,t) = \tilde{\xi}_m(a,t), \tag{3.4a}$$
$$v_m(a,t) = \tilde{v}_m(a,t), \tag{3.4b}$$
$$\eta_m(b,t) = \tilde{\eta}_m(b,t), \tag{3.4c}$$

where $\tilde{\xi}_m$ and $\tilde{\eta}_m$ are defined as usual.
For the supercritical modes, we set for every $t > 0$:

$$\xi_m(a,t) = \tilde{\xi}_m(a,t), \tag{3.5a}$$
$$v_m(a,t) = \tilde{v}_m(a,t), \tag{3.5b}$$
$$\eta_m(a,t) = \tilde{\eta}_m(a,t). \tag{3.5c}$$

To implement these boundary conditions, we discretize equations (1.12) with the finite differences method, taking into account the sign of $\overline{U_0} - 1/\lambda_m$ for the discretization of the first x-derivative of η_m in equation (1.12c) (see equations (2.6) and (2.7) of Section 3 above).
For each time step $\Delta t^n = \Delta t$ satisfying (2.12) we compute the unknown functions (ξ_m^{n+1}, v_m^{n+1},

η_m^{n+1}) thanks to (2.6) and (2.7), with the numerical boundary conditions:

$$\xi_{m,0}^{n+1} = \tilde{\xi}_m(a, t_{n+1}), \tag{3.6a}$$
$$v_{m,0}^{n+1} = \tilde{v}_m(a, t_{n+1}), \tag{3.6b}$$
$$\eta_{m,J}^{n+1} = \tilde{\eta}_m(b, t_{n+1}), \tag{3.6c}$$

if m is subcritical ($m \leq m_c$). If m is supercritical ($m > m_c$), we set

$$\xi_{m,0}^{n+1} = \tilde{\xi}_m(a, t_{n+1}), \tag{3.7a}$$
$$v_{m,0}^{n+1} = \tilde{v}_m(a, t_{n+1}), \tag{3.7b}$$
$$\eta_{m,0}^{n+1} = \tilde{\eta}_m(a, t_{n+1}). \tag{3.7c}$$

The following figures plot u, v and ψ in the domain Ω_1 at two different times. Figures 11, 12 and 13 represent the initial data ($t = 0$) for these three quantities, whereas Figures 14, 15 and 16 represent u, v and ψ at $t = t_1 > 0$.

Here, one can see that Figures 14, 15 and 16 respectively match with Figures 7, 8 and 9 in the domain Ω_1.

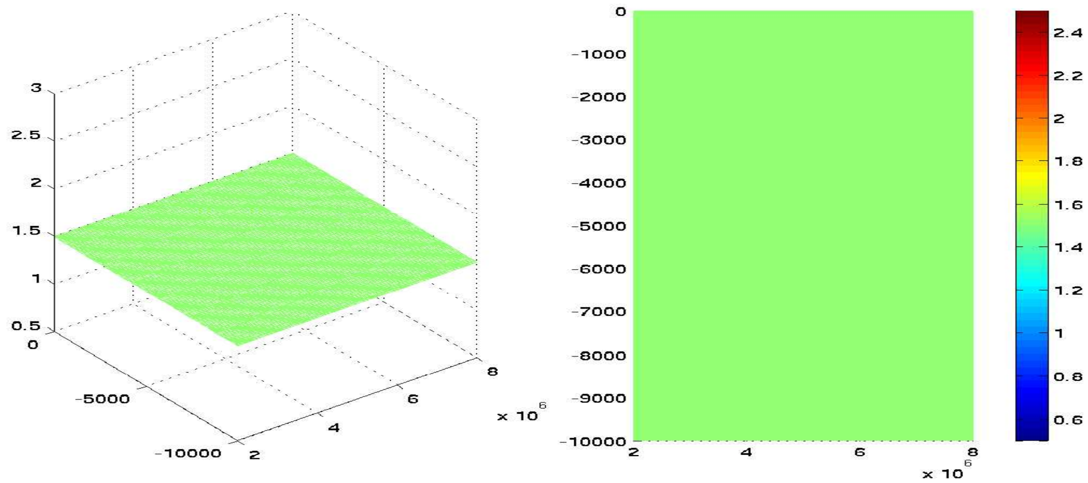


Figure 6.11: Transparent Boundary Condition. Initial data u_0.

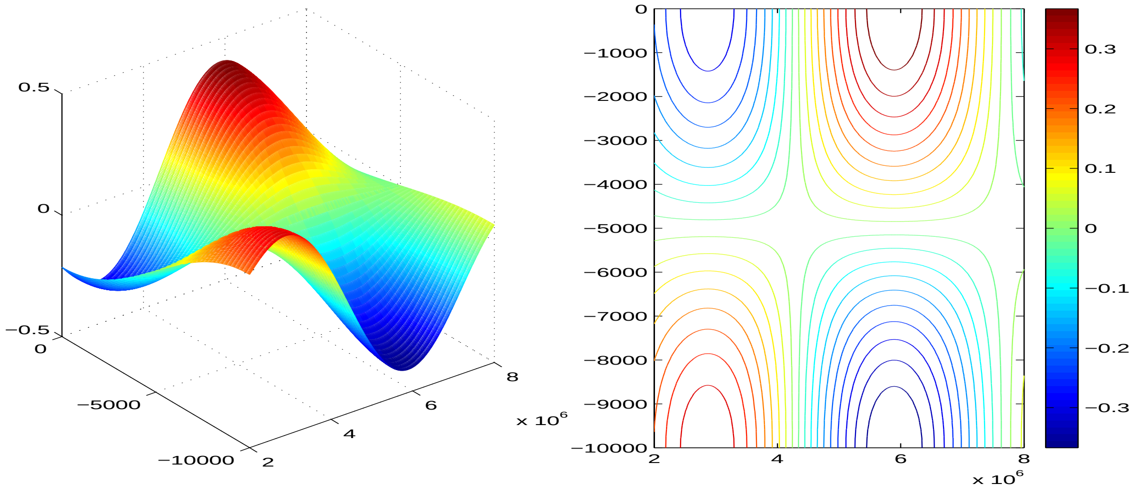


Figure 6.12: Transparent Boundary Condition. Initial data v_0.

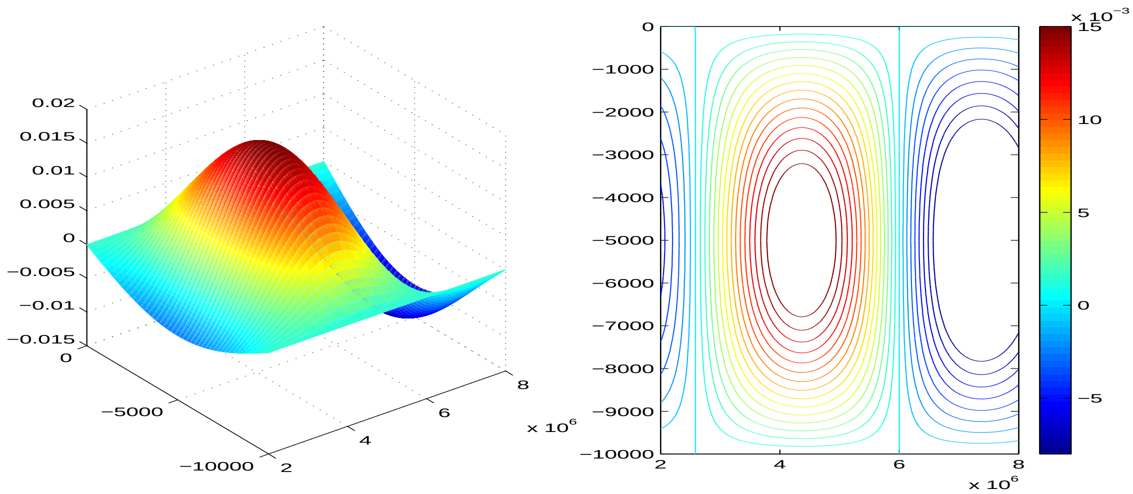


Figure 6.13: Transparent Boundary Condition. Initial data ψ_0.

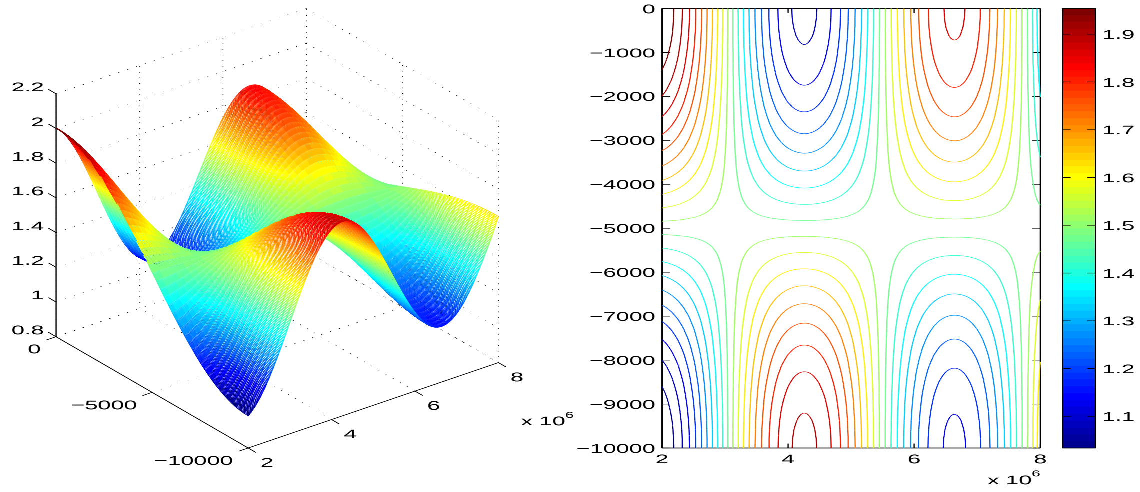


Figure 6.14: Transparent Boundary Condition. Values of u at $t = t_1$.

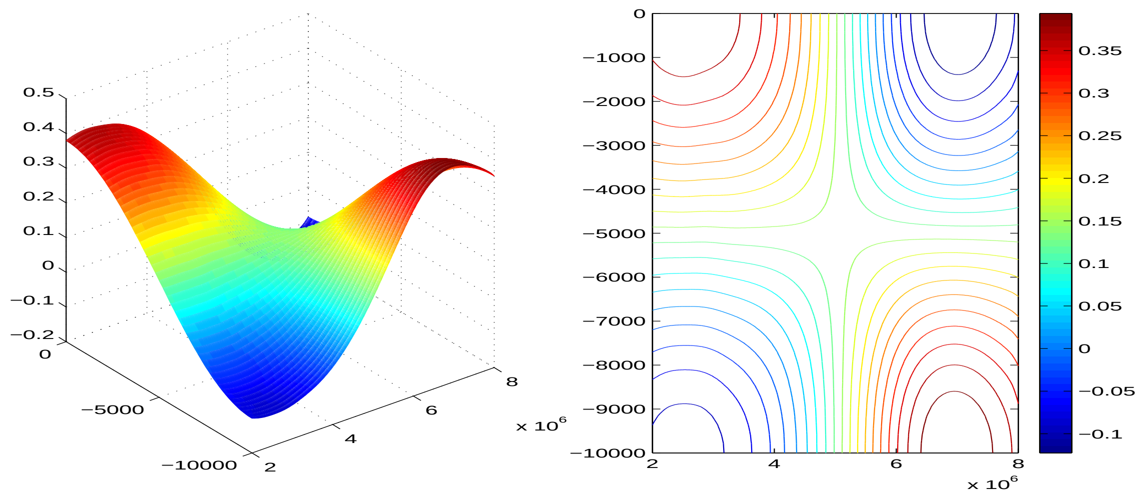


Figure 6.15: Transparent Boundary Condition. Values of v at $t = t_1$.

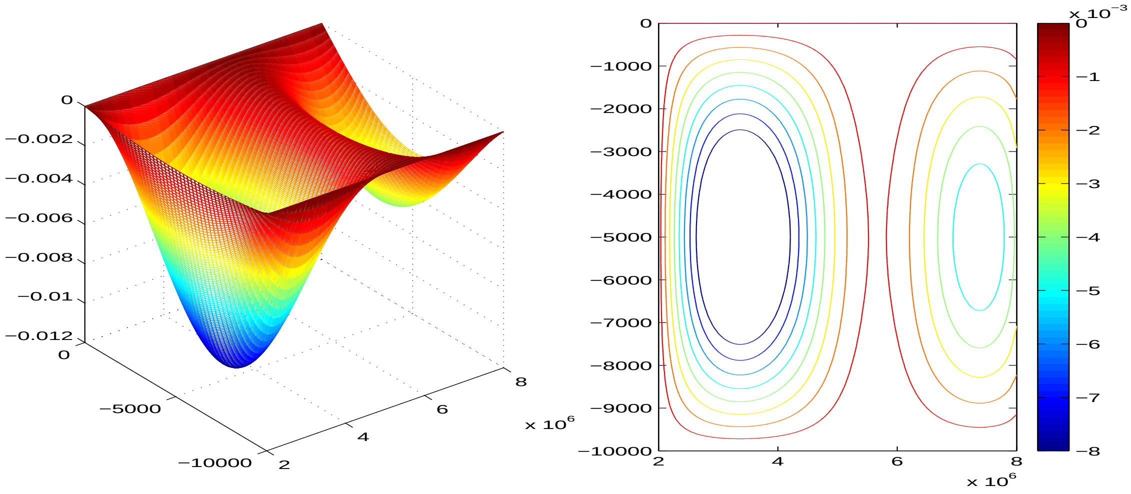


Figure 6.16: Transparent Boundary Condition. Values of ψ at $t = t_1$.

3.3 Comparisons

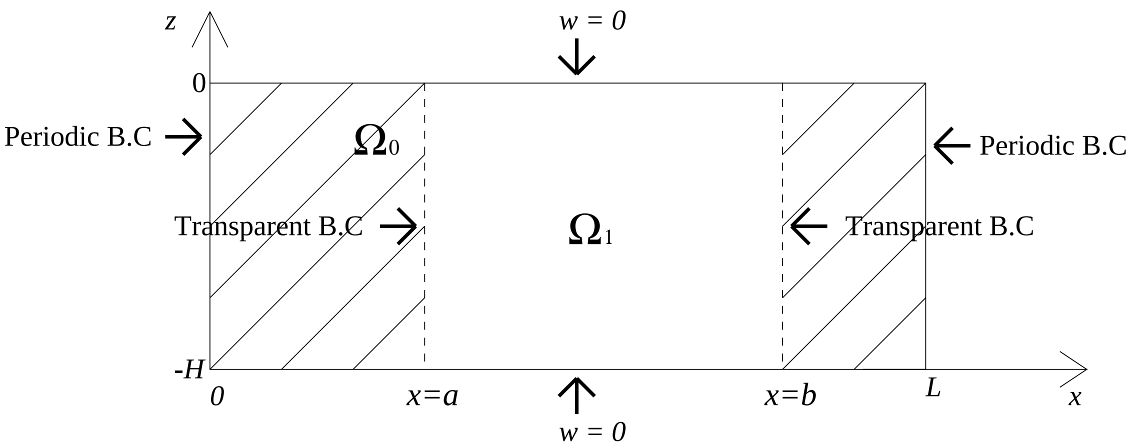


Figure 6.17: Subdomains Ω_0 and Ω_1

In order to confirm what can be observed, we finally choose an interior point $(x_0, z_0) = (5.8 \times 10^6, -4.0 \times 10^3) \in \Omega_1$, and plot in Figure 18 the values of $(u, v, \psi)(x_0, z_0, t)$ computed in Ω_1 with transparent boundary conditions, compared to $(u, v, \psi)(x_0, z_0, t)$ computed in Ω_0 with periodic boundary conditions. The results are similar if one considers another choice of (x_0, z_0); this shows the transparency property of the boundary conditions (3.4)-(3.5).
On the left part, we plot the different quantities $(u, v, \psi)(x_0, z_0, t)$ computed with the two types of boundary conditions. On the right part, we plot the corresponding relative errors $|f_{\Omega_0} - f_{\Omega_1}|/|f_{\Omega_0}|$ where f is successively u, v and ψ. The reader might think that the relative error

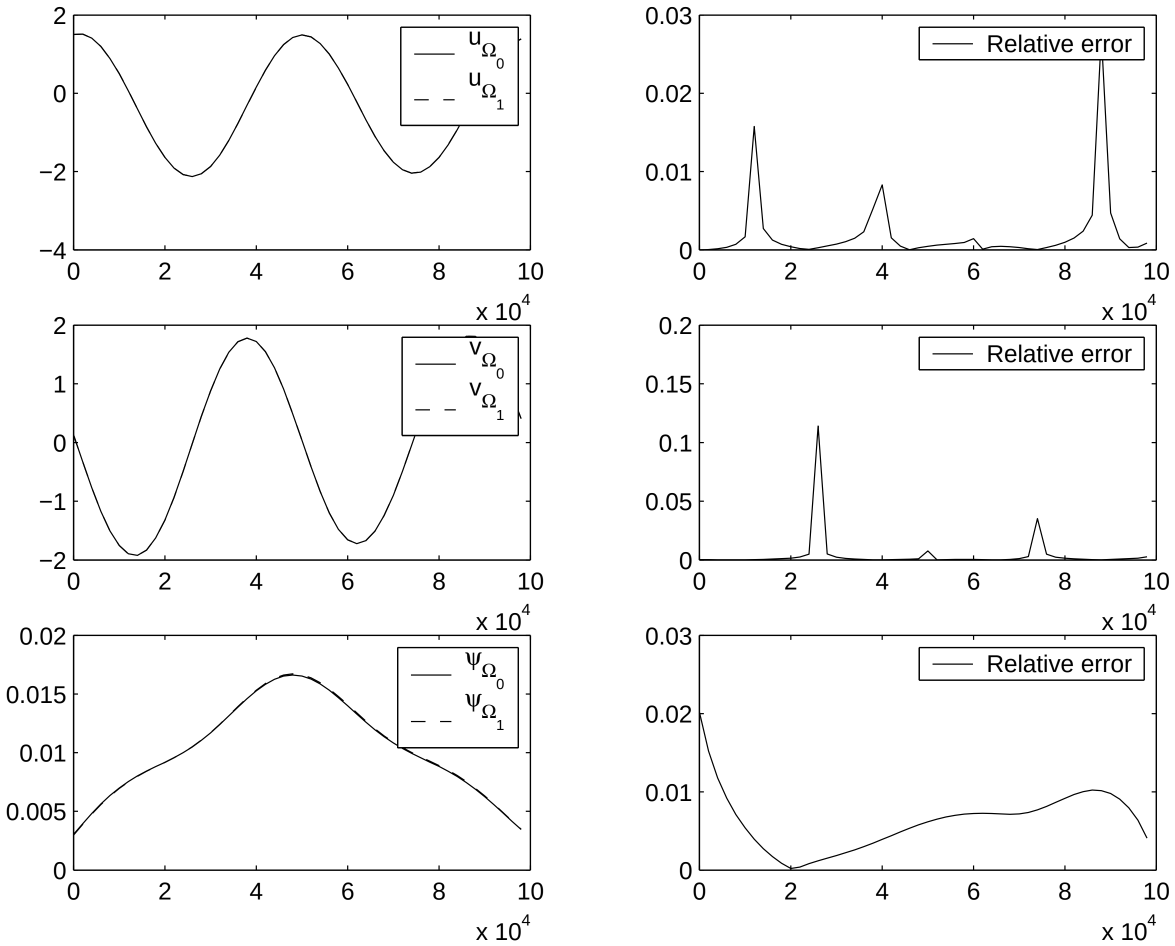


Figure 6.18: Two different computations of $\psi(x_0, z_0, t)$ (left). Relative error (right).

reaches some local high values, but this is actually due to the fact that the quantity u_{Ω_0} (or v_{Ω_0}, ψ_{Ω_0}) vanishes; these local maximum are not meaningful.

4 Numerical simulations of the linearized primitive equations

We present hereafter some results in the linear case. Although the numerical issues are less challenging than for the the linear case, this section is a natural extension to the theoretical results established in Chapter 5.

We proceed as we did in the previous section: periodic boundary conditions are implemented in the domain Ω_0 whereas we use the transparent boundary conditions in the subdomain Ω_1. Finally, we compare the numerical values obtained with these two sets of numerical computations.

4.1 Periodic boundary conditions for the large domain Ω_0.

As for the linear case, we set:

$$\xi_m(0,t) = \xi_m(L,t), \tag{4.1a}$$
$$v_m(0,t) = v_m(L,t), \tag{4.1b}$$
$$\eta_m(0,t) = \eta_m(L,t). \tag{4.1c}$$

From the numerical point of view, equation (4.1) can be written as:

$$\xi_{m,0}^{n+1} = \xi_{m,J}^{n+1}, \tag{4.2a}$$
$$v_{m,0}^{n+1} = v_{m,J}^{n+1}, \tag{4.2b}$$
$$\eta_{m,0}^{n+1} = \eta_{m,J}^{n+1}. \tag{4.2c}$$

The following figures plot u, v and ψ in the domain Ω_0 at two different times. Figures 19, 20 and 21 represent the initial data ($t = 0$) for these three quantities, whereas Figures 22, 23 and 24 represent u, v and ψ at $t = t_1 > 0$.

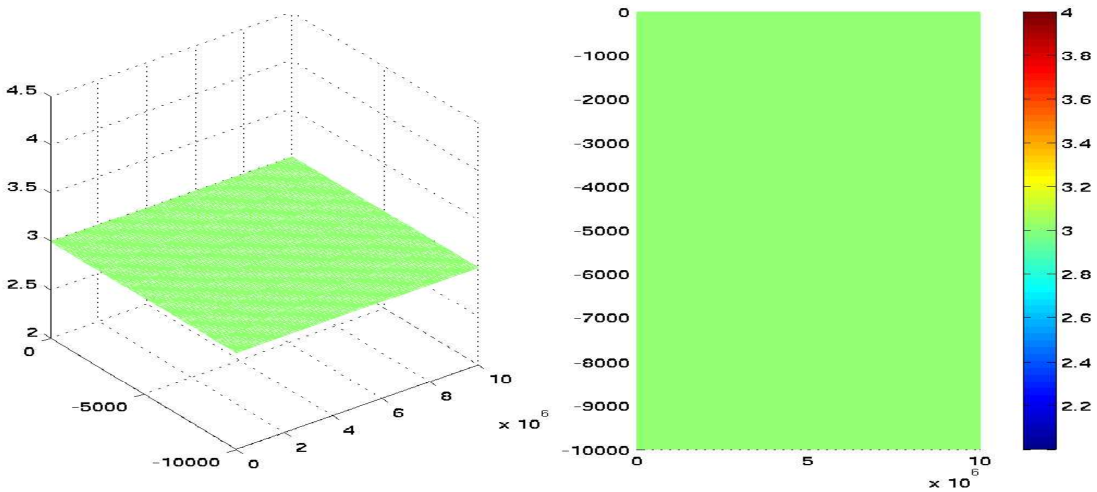


Figure 6.19: Periodic Boundary Condition. Initial data u_0.

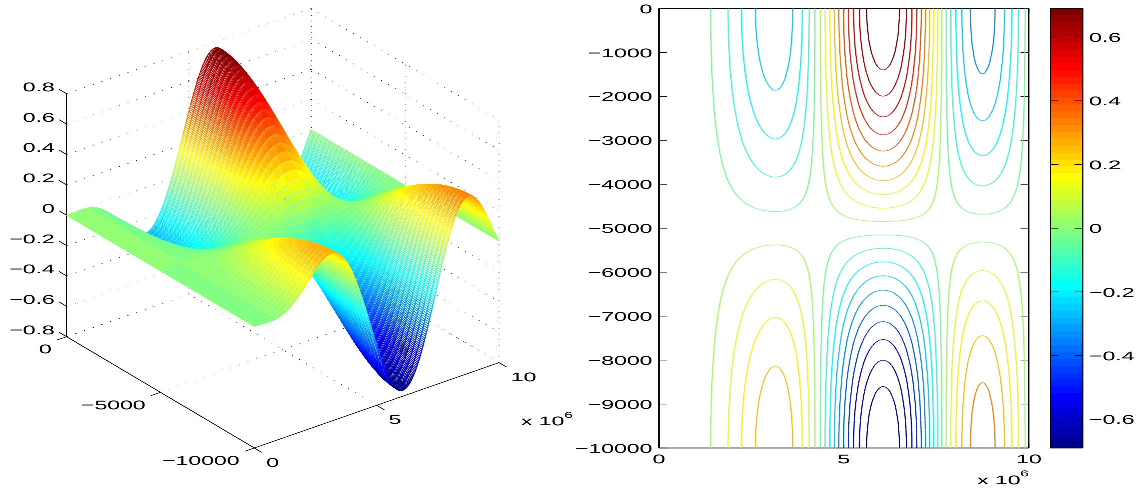


Figure 6.20: Periodic Boundary Condition. Initial data v_0.

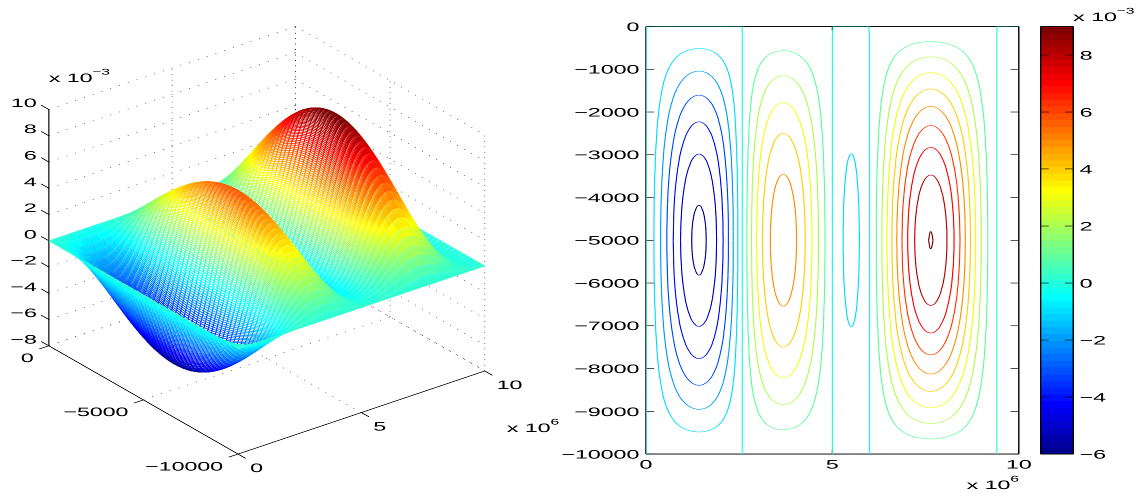


Figure 6.21: Periodic Boundary Condition. Initial data ψ_0.

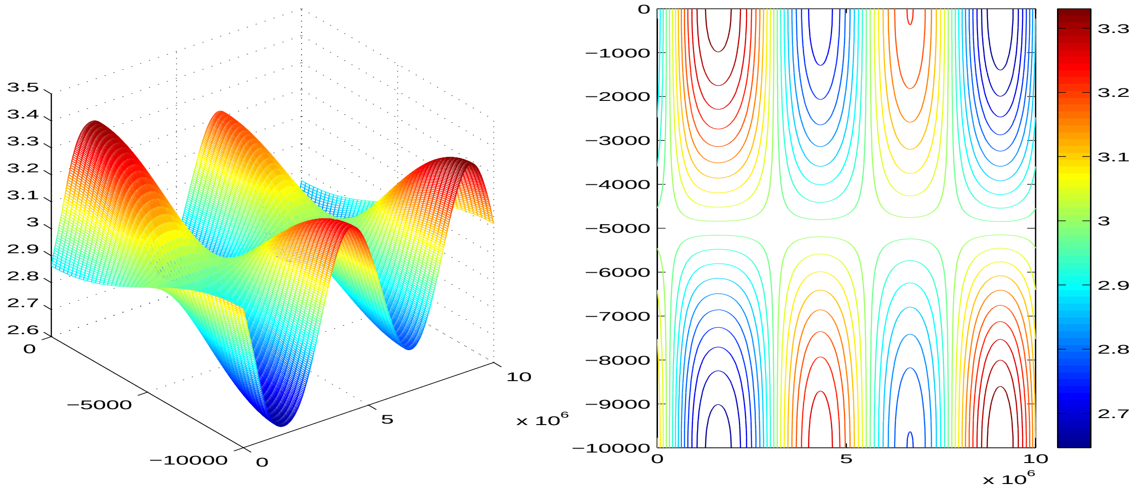


Figure 6.22: Periodic Boundary Condition. Values of u at $t = t_1$.

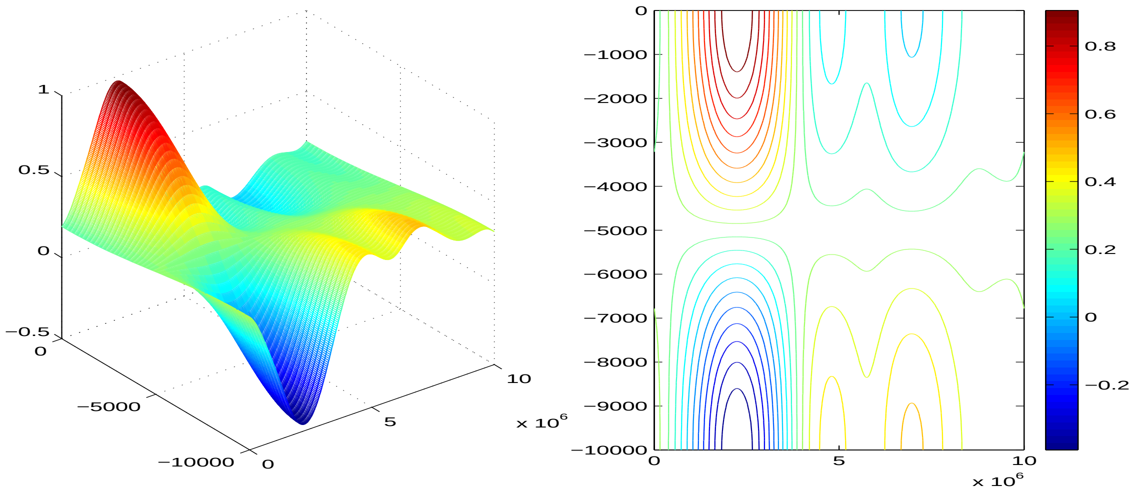
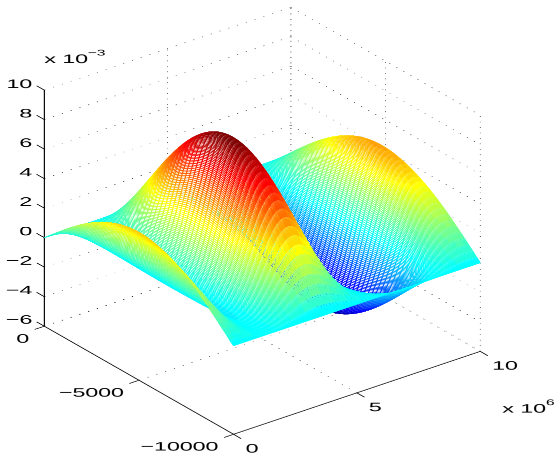


Figure 6.23: Periodic Boundary Condition. Values of v at $t = t_1$.



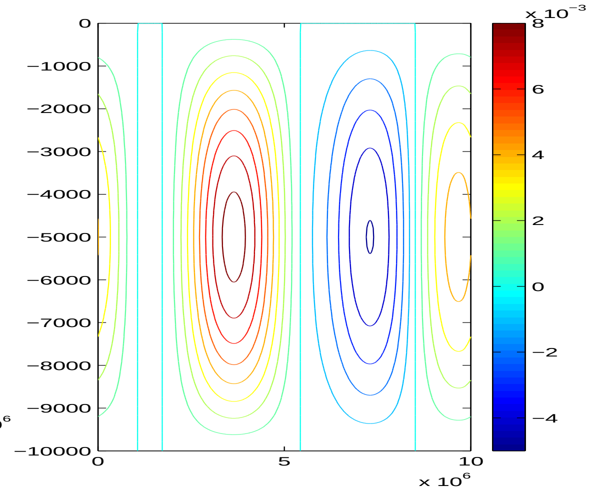


Figure 6.24: Periodic Boundary Condition. Values of ψ at $t = t_1$.

4.2 Transparent boundary conditions for the subdomain $\Omega_1 \subset \Omega_0$

Here the boundary conditions depend on the normal mode that is concerned. For ervery subcritical $m \leq m_c$, we have:

$$\xi_m(a,t) = \tilde{\xi}_m(a,t), \tag{4.3a}$$
$$v_m(a,t) = \tilde{v}_m(a,t), \tag{4.3b}$$
$$\eta_m(b,t) = \tilde{\eta}_m(b,t), \tag{4.3c}$$

where $\tilde{U}$ is a known data (e.g. provided by the computations of Section 4.1 above).
For the supercritical modes, all the informations are required at $x = a$:

$$\xi_m(a,t) = \tilde{\xi}_m(a,t), \tag{4.4a}$$
$$v_m(a,t) = \tilde{v}_m(a,t), \tag{4.4b}$$
$$\eta_m(a,t) = \tilde{\eta}_m(a,t), \tag{4.4c}$$

The following figures plot u, v and ψ in the domain Ω_1 at two different times, computed with the transparent boundary conditions. Figures 25, 26 and 27 represent the initial data ($t = 0$) for these three quantities, whereas Figures 28, 29 and 30 represent u, v and ψ at $t = t_1 > 0$.

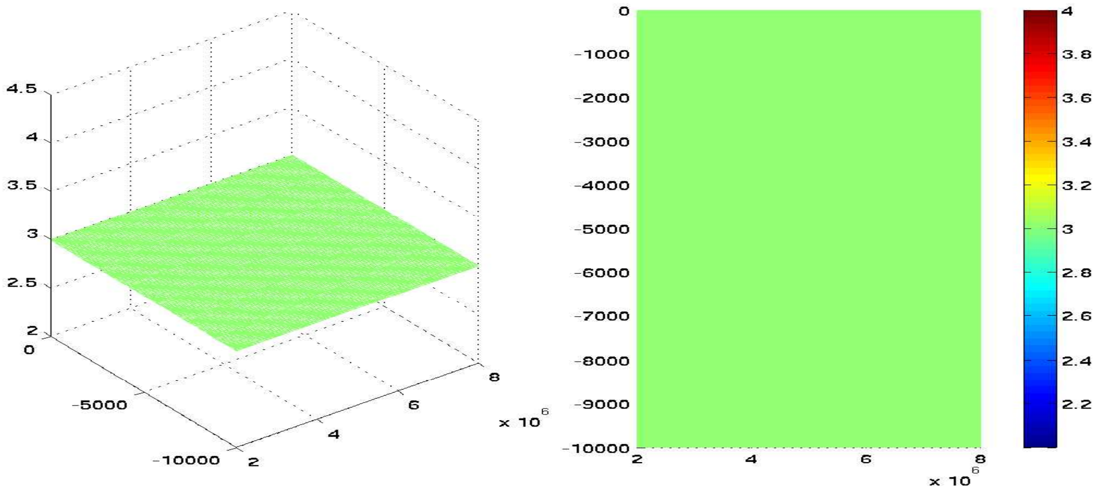


Figure 6.25: Transparent Boundary Condition. Initial data u_0.

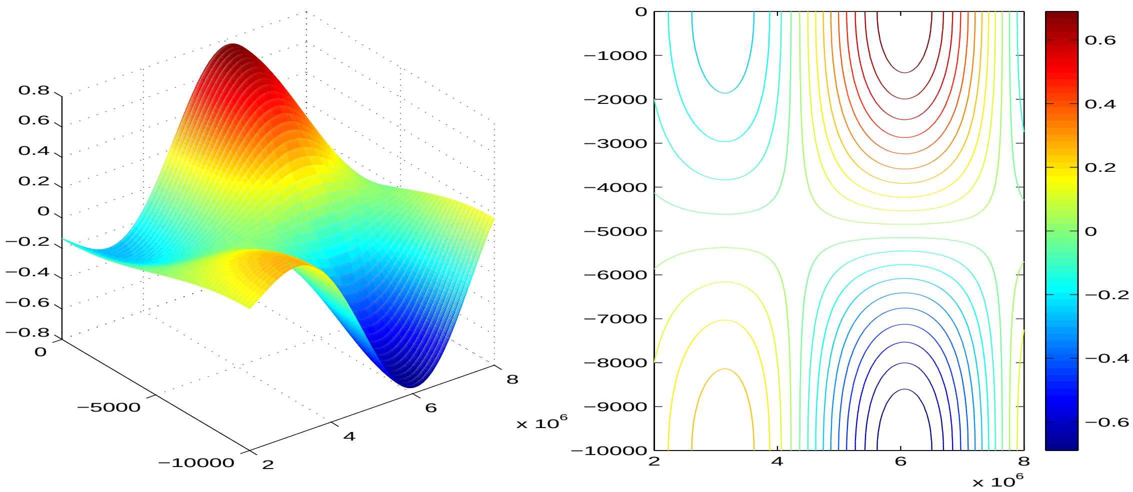


Figure 6.26: Transparent Boundary Condition. Initial data v_0.

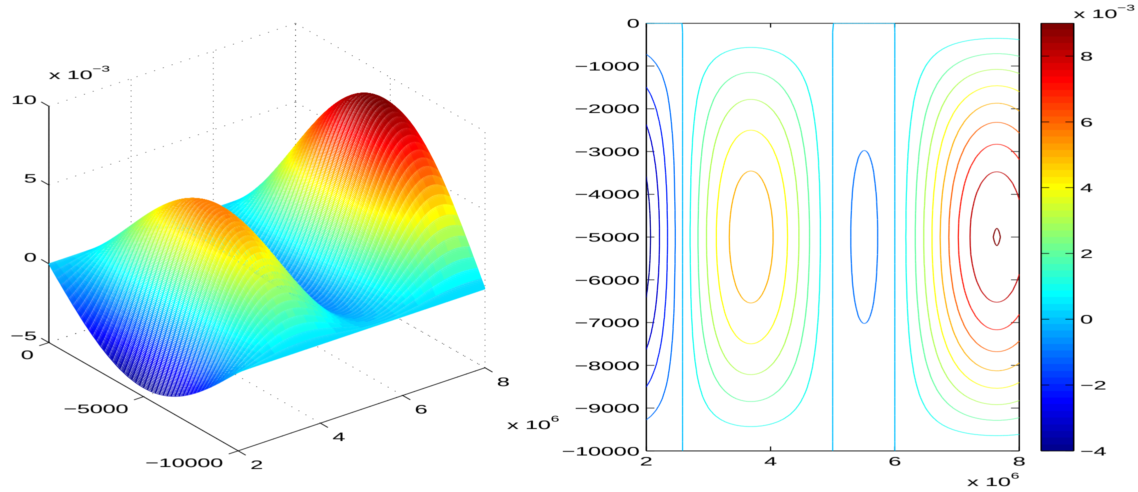


Figure 6.27: Transparent Boundary Condition. Initial data ψ_0.

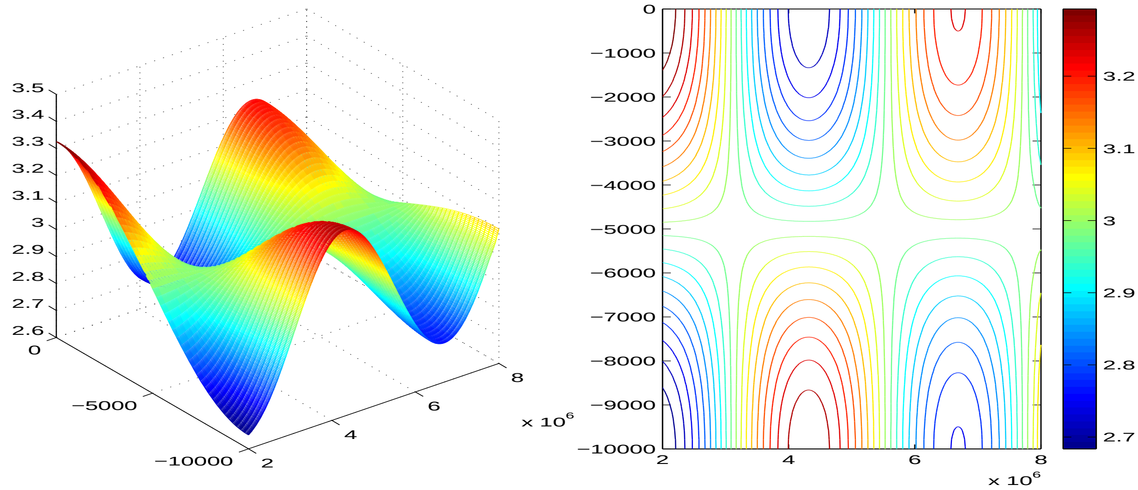


Figure 6.28: Transparent Boundary Condition. Values of u at $t = t_1$.

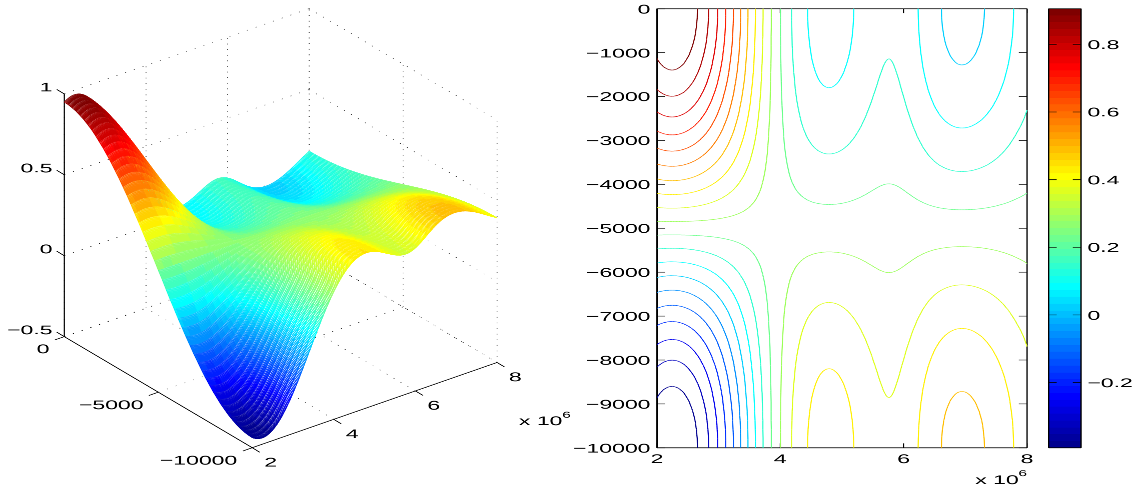


Figure 6.29: Transparent Boundary Condition. Values of v at $t = t_1$.

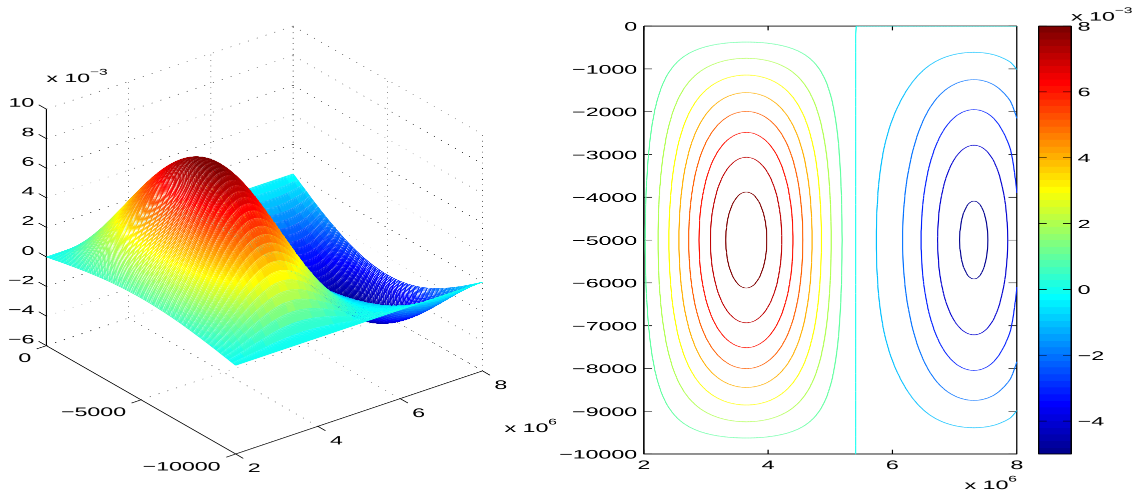


Figure 6.30: Transparent Boundary Condition. Values of ψ at $t = t_1$.

4.3 Comparisons

As we did in the nonliear case we choose an interior point $(x_0, z_0) = (5.8 \times 10^6, -4.0 \times 10^3) \in \Omega_1$, and plot in Figure 31 the values of $(u, v, \psi)(x_0, z_0, t)$ computed in Ω_1 with transparent boundary conditions, compared to $(u, v, \psi)(x_0, z_0, t)$ computed in Ω_0 with periodic boundary conditions. The results are similar if one considers another choice of (x_0, z_0); this shows the transparency property of the boundary conditions (3.4)-(3.5).

On the left part, we plot the different quantities $(u, v, \psi)(x_0, z_0, t)$ computed with the two types of boundary conditions. On the right part, we plot the corresponding relative errors $|f_{\Omega_0} -$

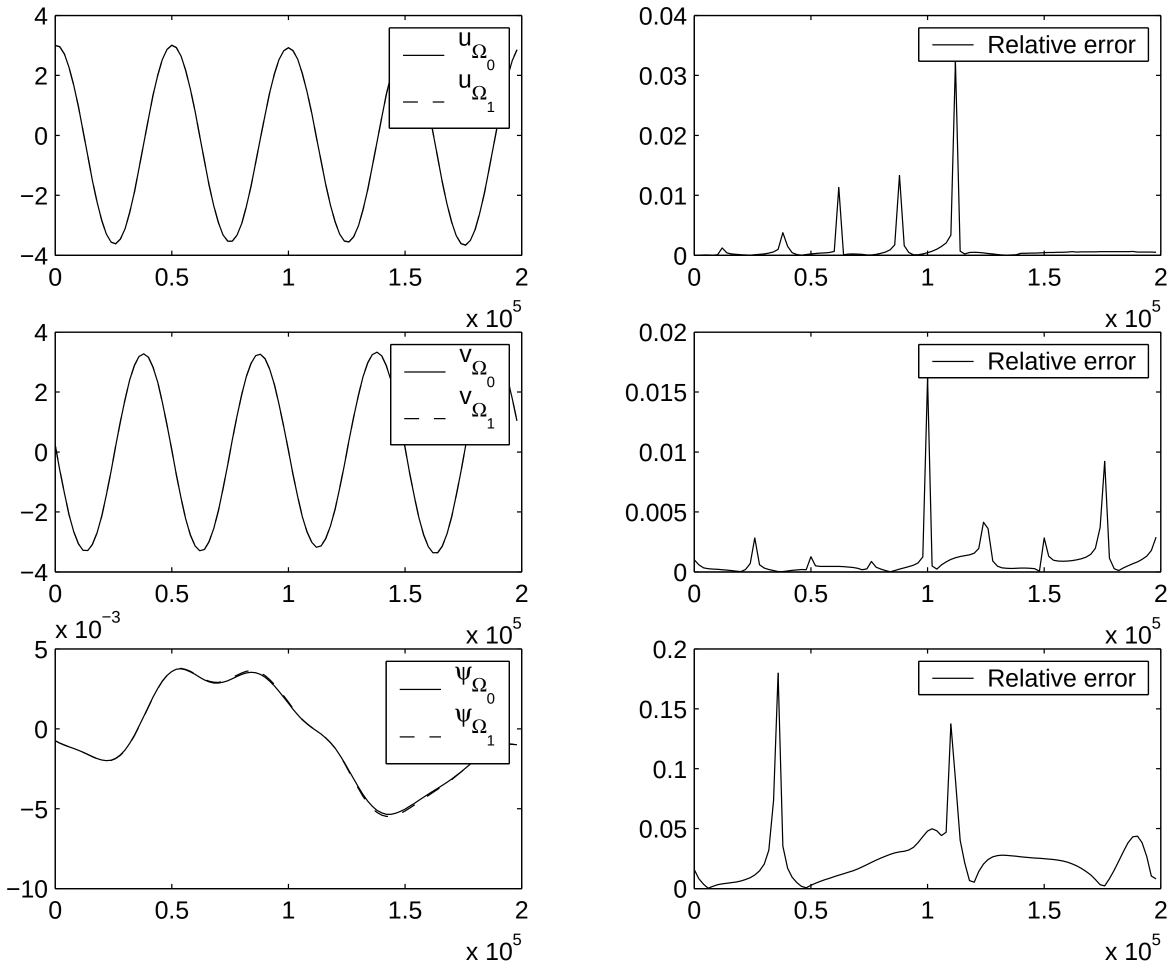


Figure 6.31: Two different computations of $\psi(x_0, z_0, t)$ (left). Relative error (right).

$f_{\Omega_1}|/|f_{\Omega_0}|$ where f is successively u, v and ψ. The reader might think that the relative error reaches some local high values, but this is actually due to the fact that the quantity u_{Ω_0} (or v_{Ω_0}, ψ_{Ω_0}) vanishes; these local maximum are not meaningful.

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Conclusion

Nous avons introduit les équations primitives de l'océan sans viscosité, dont la caractéristique principale, si on les compare aux équations de Navier-Stokes ou d'Euler, est l'approximation hydrostatique :

$$\phi_z = -\rho\, g.$$

Après avoir montré le caractère mal posé de ces équations, considérées en domaine borné et munies de conditions aux limites locales, nous nous sommes intéressés aux équations dites δ-primitives, pour lesquelles l'équation hydrostatique a été remplacée par :

$$\delta w + \phi_z = -\rho\, g.$$

Dans le premier chapitre, on a montré des résultats d'existence et d'unicité de solutions pour ce problème, que l'on a ensuite étudié numériquement (sous sa forme linéarisée et bidimensionnelle) dans le chapitre 2, en particulier lorsque le petit paramètre δ tend vers 0. Dans ce chapitre, une partie est consacrée aux conditions aux limites de Dirichlet, pour lesquelles on fait appel à un correcteur numérique afin de capter les phénomènes de couche limite. Le second chapitre a consisté à implémenter des conditions aux limites permettant d'éviter les réflexions à la frontière, qui peuvent ne pas être souhaitables si l'on se place dans le contexte de frontières ouvertes. La première partie se poursuit avec le chapitre 3 qui a permis de confirmer les comportements asymptotiques (lorsque $\delta \longrightarrow 0$) conjecturés dans le chapitre 2, mettant ainsi en évidence des couches limites et des réflexions aux frontières du domaine. On propose d'ailleurs, dans ce même chapitre, un jeu de conditions aux limites transparentes, qui conduisent également à un problème bien posé, mais qui sont non locales. Enfin, on propose d'autres conditions aux limites, qui sont locales et donc faciles a implémenter numériquement, mais qui ne sont bien posées que pour $\Delta x > 0$. La stabilité des schémas relatifs à ces conditions aux limites est étudiée dans le chapitre 4, qui clôt cette première partie.

Dans la seconde partie (chapitres 5 et 6), on revient à la formulation hydrostatique $\delta = 0$. On propose dans le chapitre 5 un jeu de conditions aux limites (non locales) qui rendent le problème aux limites linéaire bien posé. La démonstration de ce résultat utilise des techniques de semi-groupes. En attendant de parvenir à une preuve similaire dans le cas non linéaire (localement en temps), on propose dans le chapitre 6 une série de simulations numériques utilisant les conditions aux limites nouvellement introduites pour le problème bidimensionnel. Ces simulations confirment en évidence le caractère transparent des conditions aux limites considérées, ce qui est essentiel pour les simulations numériques.

En résumé nos principales contributions ont consisté du point de vue qualitatif à mettre en évidence les réflexions d'ondes à la frontière pour les équations δ-primitives[1] ; du point de vue théorique à étudier les équations primitives sans viscosité linéarisées, en dimension deux d'espace, ce qui, à notre connaissance, est le premier élément nouveau dans cette direction depuis le résultat négatif d'Oliger et Sundström en 1978 [OS78] (Chapitre 5) ; enfin à simuler les équations primitives non linéaires sans viscosité en dimension deux d'espace avec des conditions aux limites appropriées (Chapitre 6), ce qui constitue une toute petite étape dans un problème que va étudier la communauté océan-atmosphère dans les prochaines années [WPT97].

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[1] Ce qui ne condamne pas ces équations qui ont montré leur intérêt dans [TT03] (première partie), mais cet inconvénient est à comparer à la simplicité du modèle.

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Numéro d'impression ? ? ? ?,

2ème trimestre 2005.

Etudes théoriques et numériques des équations primitives de l'océan sans viscosité

Résumé

Cette thèse regroupe un ensemble d'analyses mathématiques et de simulations numériques relatives aux Equations Primitives de l'océan (EPs) sans viscosité, en domaine borné. Les EPs sont des équations bien connues de la mécanique des fluides, qui s'appuient sur les approximations hydrostatique et de Boussinesq. On rappelle en introduction pourquoi ces équations, considérées avec des conditions aux limites de type local, sont mal posées.

Dans une première partie (chapitres 1 à 4), on s'intéresse à une modification de l'équation hydrostatique au moyen d'un terme de friction proportionnel à un petit paramètre δ. On démontre des résultats d'existence, d'unicité et de régularité des solutions avant d'étudier le comportement de ces solutions lorsque δ tends vers 0. Des résultats numériques montrent que des couches limites et des réflexions se produisent aux frontières du domaine. Les phénomènes observés numériquement sont alors confirmés par une preuve rigoureuse effectuée grâce à la théorie des correcteurs.

Dans une seconde partie (chapitres 5 et 6), on revient à la formulation hydrostatique d'origine des EPs, et l'on propose un jeu de conditions aux limites transparentes pour le système linéarisé. Une preuve du caractère bien posé du problème aux limites ainsi obtenu justifie l'introduction de telles conditions aux limites, qui sont ensuite implémentées dans une simulation numérique confirmant que les phénomènes de couches limites et de réflexions aux frontières sont ainsi évités, aussi bien sur les équations non linéaires que sur le linéarisé.

Mots clés : Equations Primitives, conditions aux limites, théorie des semi-groupes, éléments finis, différences finies, couches limites, conditions aux limites transparentes.

Classification AMS (2000) : 35L50, 35Q35, 47D03, 65Mxx, 65Nxx, 76Bxx, 76Mxx, 86-08, 86A05.