On the modeling of thin plates in nonlinear elasticity

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Sur la modélisation des plaques minces en élasticité non linéaire

THÈSE

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Résumé

Cette thèse est consacrée à la modélisation mathématique des plaques minces en élasticité non linéaire. Plus précisément, il s'agit d'obtenir des modèles non linéaires bidimensionnels de plaques à partir de l'élasticité non linéaire tridimensionnelle en employant essentiellement deux méthodes : le développement asymptotique formel et la Γ-convergence. Deux classes de matériaux hyperélastiques réalistes à densités d'énergie singulières sont étudiées. Pour la première classe, l'énergie tend vers l'infini lorsque le déterminant du gradient de la déformation tend vers zéro i.e. l'on ne peut comprimer un volume en un point. Pour ce type de plaques, on obtient, en employant la première méthode, un nouveau modèle membranaire non linéaire qui empêche la formation de *plis* et qui approche le modèle classique pour les petites déformations. On retrouve aussi le modèle inextensionnel non linéaire classique. Ensuite, on considère les matériaux incompressibles i.e. la densité d'énergie est infinie pour les déformations dont le déterminant du gradient est différent de un. On produit grâce à la deuxième méthode un modèle membranaire non linéaire. Enfin, accessoirement on montre un résultat de non existence de minimiseurs pour le modèle membranaire non linéaire classique comprimé et quelques remarques générales sont faites à ce sujet.

Mots-clés: Plaques élastiques non linéaires, dérivation de modèles bidimensionnels, non existence pour membrane comprimée, matériaux de type Ogden, matériaux incompressibles, développement asymptotique formel, Γ-convergence.

Abstract

This PhD thesis is dedicated to the mathematical modelling of nonlinearly elastic thin plates. Namely, nonlinear two-dimensional models of plates are derived from three-dimensional elasticity via two methods : the formal asymptotic expansions and Γ-convergence. Two classes of realistic hyperelastic materials with singular stored energy function are considered. For the first class of materials, the energy tends to infinity as the determinant of the deformation gradient tends to zero hence preventing the annihilation of volumes. Here, using the first method, we draw a new nonlinear membrane model that precludes the formation of *folds* and that approximates the classical model for small deformations. The classical nonlinear inextensional model is also retrieved. Then incompressible materials are considered i.e. stored energy functions are infinite whenever the determinant of the deformation gradient is different from one. A nonlinear membrane model is produced using the second method. Finally, we show the non-existence of minimizers for membranes under compression and make some general remarks on the topic.

Keywords: Nonlinearly elastic plates, derivation of two-dimensional models, non-existence for membranes under compression, Ogden materials, incompressible materials, formal asymptotic expansions, Γ-convergence.

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Introduction générale

Introduction générale

Cette thèse est consacrée à la modélisation mathématique des plaques minces en élasticité non linéaire. Plus précisément, il s'agit d'obtenir des modèles nonlinéaires bidimensionnels de plaques à partir de l'élasticité nonlinéaire tridimensionnelle en employant essentiellement deux méthodes mathématiques : le développement asymptotique formel et la Γ-convergence . Tout d'abord, qu'entend-t-on par plaque ? Une plaque, dans sa configuration de référence (entendre au repos), est un objet cylindrique de la forme

$$\Omega = \omega \times (-\varepsilon, \varepsilon) \, ,$$

où $\omega \subset \mathbb{R}^2$ est un ouvert borné représentant la surface moyenne de la plaque et $\varepsilon > 0$ en est l'épaisseur. Une plaque est dite mince lorsque son épaisseur est très petite comparée aux dimensions de sa surface moyenne

$$\varepsilon << |\omega| \, ,$$

d'où l'idée de définir cet objet comme la déformation de sa surface moyenne ω en une surface de $\mathbb{R}^3$. Pourquoi ? Parce que du point de vue de la physique une telle modélisation semble naturelle et devrait pouvoir préserver les propriétés constitutives et qualitatives de l'objet tridimensionnel du départ et puis parce que du point de vue de l'analyse numérique l'implémentation de tels problèmes paraît difficile du fait de la disparité entre les dimensions.

Pratiquement, il s'agira de considérer des problèmes de minimisation de l'énergie totale d'une plaque mince et d'en faire l'étude asymptotique avec ε comme petit paramètre (comprendre $\varepsilon \longrightarrow 0$) afin d'obtenir des problèmes de minimisation posés sur la surface moyenne ω.

Après une brève introduction à la théorie mathématique de l'élasticité non linéaire tridimensionnelle et ses principaux concepts culminant avec la notion d'hyperélasticité, on se propose de rappeler différents résultats du calcul des variations qui nous permetteront de comprendre certains choix de modélisation et d'en analyser les résultats. Ensuite, on fournira un aperçu de l'état de l'art intimement lié à nos problèmes en se limitant aux modèles *classiques*. Au passage, on décrira les méthodes employées pour leurs justifications. Notre objectif est d'une part de rendre la lecture de cette thèse aussi aisée que possible et d'autre part de mettre en évidence les différentes sources qui l'ont motivé. En dernier lieu, on décrit les résultats obtenus.

1 L'élasticité mathématique tridimensionnelle

Dans ce qui va suivre, nous allons essentiellement nous intéresser à la description de la position d'équilibre d'un corps élastique qui occupe, au repos, une configuration de référence $\bar{\Omega} \subset \mathbb{R}^3$ où Ω est une ouvert borné connexe à frontière lipschitzienne. Lorsqu'il est soumis à des forces, ce corps occupe une configuration déformée $\varphi(\bar{\Omega})$ définie par une application $\varphi : \bar{\Omega} \longrightarrow \mathbb{R}^3$ dite déformation qui doit en particulier être localement injective sur Ω et conserver l'orientation dans $\bar{\Omega}$ pour être physiquement admissible. Ces deux contraintes se traduisent pratiquement par la condition

$$\det \nabla\varphi(x) > 0$$

pour tout $x \in \Omega$.

L'état d'équilibre statique d'une configuration déformée $\varphi(\bar{\Omega})$ est décrit par un champ de tenseurs symétriques applé *tenseur des contraintes de Cauchy*

$$\mathcal{T}^{\varphi} : \varphi(\bar{\Omega}) \longrightarrow \mathcal{S}^3$$

où $\mathcal{S}^3$ est l'ensemble des matrices symétriques de $\mathbb{R}^{3\times 3}$. L'existence de ce tenseur est donnée par le théorème de Cauchy qui est lui-même une conséquence de l'axiome connu sous le nom de *principe des contraintes d'Euler-Cauchy*. Ce théorème nous dit que la frontière d'un sous domaine de $\varphi(\bar{\Omega})$ est soumise à une force de surface exercée par le reste du corps dont la densité vectorielle dépend linéairement de l'orientation de cette même frontière par l'intermédiaire du tenseur de Cauchy. Ainsi, sous l'action de la force interne $f^{\varphi} : \varphi(\bar{\Omega}) \longrightarrow \mathbb{R}^3$ et de la force surfacique $g^{\varphi} : \Gamma_1^{\varphi} \longrightarrow \mathbb{R}^3$, où Γ_1^{φ} est une portion de la frontière Γ^{φ} de $\varphi(\bar{\Omega})$, l'état d'équilibre statique de la configuration déformée est traduit par le système

$$\left\{ \begin{array}{rcll} -\operatorname{div}^{\varphi} \mathcal{T}^{\varphi} &=& f^{\varphi} & \text{dans} \quad \varphi(\Omega), \\ \mathcal{T}^{\varphi} &=& \mathcal{T}^{\varphi t} & \text{dans} \quad \varphi(\Omega), \\ \mathcal{T}^{\varphi} n^{\varphi} &=& g^{\varphi} & \text{sur} \quad \Gamma_1^{\varphi}, \end{array} \right. \tag{1}$$

où n^{φ} est la normale sortante unitaire à Γ_1^{φ}. Cependant, ce tenseur est défini sur la configuration déformée $\varphi(\bar{\Omega})$ qui est une inconnue du problème, on préférera donc reformuler les équations d'équilibre en fonction de la configuration de référence $\bar{\Omega}$. Ceci est rendu possible par l'introduction du premier tenseur de contraintes de Piola-Kirchhoff $\mathcal{T} : \bar{\Omega} \longrightarrow \mathbb{R}^{3\times 3}$, qu'on définit de la manière suivante

$$\mathcal{T}(x) = \mathcal{T}^{\varphi}(y) \operatorname{Cof} \nabla\varphi(x), \quad y = \varphi(x),$$

où $\operatorname{Cof} \nabla\varphi \in \mathbb{R}^{3\times 3}$ est la matrice des cofacteurs de $\nabla\varphi \in \mathbb{R}^{3\times 3}$. Notons que ce tenseur n'est pas nécessairement symétrique. Maintenant on peut exprimer les équations d'équilibre dans la configuration de référence

$$\left\{ \begin{array}{rcll} -\operatorname{div} \mathcal{T} &=& f & \text{dans} \quad \Omega, \\ \nabla\varphi \mathcal{T}^t &=& \mathcal{T} \nabla\varphi^t & \text{dans} \quad \Omega, \\ \mathcal{T} n &=& g & \text{sur} \quad \Gamma_1, \end{array} \right. \tag{2}$$

où l'équation sur Γ_1 traduit une condition aux limites de traction que l'on peut compléter par une condition de déplacement en préscrivant la déformation sur une partie ou tout le reste du bord $\varphi = \varphi_0$ sur $\Gamma_0 = \Gamma \backslash \Gamma_1$. Dans la suite, on va noter $u \cdot v = u_i v_i$ le produit intérieur vectoriel Euclidien et $A : B = A_{ij} B_{ij} = \text{tr } A^t B$ le produit intérieur matriciel. On est à présent en mesure de donner une forme variationnelle au système ci-dessus comme suit

$$\int_\Omega \mathcal{T} : \nabla \psi \, dx = \int_\Omega f \cdot \psi + \int_{\Gamma_1} g \cdot \psi \, da,$$

pour tout champs de vecteur assez régulier $\psi : \Omega \longrightarrow \mathbb{R}^3$ tel que $\psi_{|\Gamma \backslash \Gamma_1} = 0$. La formulation variationnelle associée aux équation d'équilibre est connue sous le nom de *principe du travail virtuel*. L'adjectif "virtuel" empreinté à la mécanique des milieux continus traduit le fait que les champs de vecteurs ψ sont des quantités mathématiques exemptes d'interprétation physique. Remarquons que par souci de clarté et pour alléger les notations, on suppose que toutes les forces appliquées sont mortes c.à.d. qu'elles ne dépendent pas de la configuration déformée.

Il est clair que le modèle mathématique développé ci-dessus est incomplet tant d'un point de vue mathématique puisque nos 3 équations font intervenir 9 inconnues (les 3 composantes de la déformation φ plus les 6 composantes du tenseur associé $\mathcal{T}$), que d'un point de vue physique car ces mêmes équations ne tiennent pas compte de la nature du matériau (solide, liquide, gaz, etc.). A cet effet, nous allons idéaliser les matériaux élastiques en supposant que le tenseur de Cauchy $\mathcal{T}^\varphi$ ne dépend que du gradient de la déformation $\nabla \varphi$ en tout point de la configuration déformée $\varphi(\bar{\Omega})$. Par conséquent, on dira d'un matériau qu'il est élastique s'il existe une application $\hat{\mathcal{T}} : \bar{\Omega} \times \mathbb{R}^{3\times 3}_+ \longrightarrow \mathbb{R}^{3\times 3}$ telle que

$$\mathcal{T}(x) = \hat{\mathcal{T}}(x, \nabla \varphi(x)), \quad x \in \bar{\Omega}, \tag{3}$$

où $\mathbb{R}^{3\times 3}_+$ est l'ensemble des matrices $\mathbb{R}^{3\times 3}$ à déterminant strictement positif. La fonction $\hat{\mathcal{T}}$ est dite *loi de comportement* du premier tenseur de Piola-Kirchhoff. Il est aussi utile de définir un tenseur symétrique Σ qu'on appellera *second tenseur de contraintes de Piola-Kirchhoff* et sa loi de comportement comme suit

$$\Sigma(x) = \hat{\Sigma}(x, \nabla \varphi(x)) = \nabla \varphi(x)^{-1} \hat{\mathcal{T}}(x, \nabla \varphi(x)), \quad x \in \bar{\Omega}. \tag{4}$$

Les identités (3) et (4) sont les *équations constitutives* du matériau qu'elles décrivent.

Le matériau dans sa configuration de référence Ω est dit *homogène* si sa loi de comportement est indépendente du point $x \in \bar{\Omega}$ considéré. Le plus souvent, cette propriété concerne une certaine configuration de référence et se perd si on changeait cette configuration. Nous allons dès à présent considérer que des matériaux homogènes, ainsi

$$\mathcal{T}(x) = \hat{\mathcal{T}}(\nabla \varphi(x)) \quad \text{et} \quad \Sigma(x) = \hat{\Sigma}(\nabla \varphi(x)), \quad x \in \bar{\Omega}.$$

Il va de soi que ces lois de comportement ne peuvent être quelconques. Elles doivent en effet satisfaire un axiome fondamental de la mécanique qui est le *principe d'indifférence matérielle*. Cet axiome exprime le fait qu'une déformation ne dépend pas du repère orthogonal dans laquelle elle est calculée. En d'autres termes, si l'on effectue une déformation arbitraire suivie d'une déformation *rigide* (i.e. une rotation suivie d'une translation), cette deuxième déformation ne devrait pas altérer les contraintes subies par le corps. C'est le caractère intrinsèque des contraintes que cet axiome aussi connu sous le nom d'*axiome d'objectivité* signifie. Cette restriction va réduire la classe des lois de comportement, seules seront admissibles les applications $\hat{\mathcal{T}}$ qui satisfont

$$\hat{\mathcal{T}}(RF) = R\hat{\mathcal{T}}(F), \quad \varphi \in \mathbb{R}^{3\times 3}_{+},\ R \in SO(3), \tag{5}$$

ou encore

$$\hat{\Sigma}(RF) = \hat{\Sigma}(F), \quad \varphi \in \mathbb{R}^{3\times 3}_{+},\ R \in SO(3). \tag{6}$$

Cette dernière identité dit qu'une équation constitutive est une relation fonctionnelle entre une mesure de la déformation, le *tenseur des contraintes de Cauchy-Green* $C = \nabla\varphi^t\nabla\varphi$, et une mesure des contraintes, le tenseur Σ.

Pour réduire d'avantage la classe des lois de comportement, on peut supposer qu'un matériau possède la propriété d'*isotropie*. C'est une propriété de symétrie matérielle, qui exprime le fait qu'en un point donné le comportement du matériel est le même dans toutes les directions, dont l'énoncé mathématique est le suivant

$$\hat{\mathcal{T}}(FR) = \hat{\mathcal{T}}(F)R, \quad \varphi \in \mathbb{R}^{3\times 3}_{+},\ R \in SO(3), \tag{7}$$

$$\hat{\Sigma}(FR) = R^t\,\hat{\Sigma}(F)R, \quad \varphi \in \mathbb{R}^{3\times 3}_{+},\ R \in SO(3). \tag{8}$$

On peut caractériser un matériau isotrope grâce au théorème de représentation Rivlin-Ericksen qui stipule que la loi de comportement du second tenseur de Piola-Kirchhoff d'un matériau isotrope est nécessairement de la forme suivante

$$\hat{\Sigma}(F) = \tilde{\Sigma}(C) = \gamma_0(\imath_C)I + \gamma_1(\imath_C)C + \gamma_2(\imath_C)C^2, \quad C = F^tF,\ F \in \mathbb{R}^{3\times 3}_{+},$$

où $\imath_C = (\operatorname{tr} C, \det \operatorname{Cof} C, \det C)$ et les γ_i sont des fonctions à valeurs réelles.

Nous allons maintenant introduire le *tenseur de contraintes de Green-Saint Venant*

$$E = \frac{1}{2}(C - I), \quad C = \nabla\varphi^t\nabla\varphi$$

qui mesure dans un certain sens la différence entre une déformation donnée φ et une déformation rigide $R \in SO(3)$ pour laquelle $C = I$. Il est donc tout à fait légitime de calculer, jusqu'à un certain ordre de puissances de $|E|$, la différence $\tilde{\Sigma}(I+2E)-\tilde{\Sigma}(I)$. On montre que pour un matériau élastique homogène et isotrope dont la configuration de référence est un état naturel et dont les fonctions γ_i sont différentiables au point $(3, 3, 1)$, il existe deux constantes (appelées *constantes de Lamé*) λ et μ telles que

$$\hat{\Sigma}(F) = \tilde{\Sigma}(C) = \bar{\Sigma}(E) = \lambda(\operatorname{tr} E)I + 2\mu E + o(E), \quad C = F^tF = I + 2E,\ F \in \mathbb{R}^{3\times 3}_{+}.$$

On obtient de la sorte notre premier candidat pour une loi de comportement

$$\bar{\Sigma}(E) = \tilde{\Sigma}(I + 2E) = \lambda(\operatorname{tr} E)I + 2\mu E, \quad I + 2E \in S^3_+,$$

qui définit le matériau de *Saint Venant-Kirchhoff*.

Une classe importante de matériaux élastiques est celle des matériaux *hyperélastiques*. Par définition, un tel matériau admet une loi de comportement qui dérive d'un potentiel. Plus précisément, il existe une densité d'énergie interne $\hat{W} : \mathbb{R}^{3\times 3}_+ \longrightarrow \mathbb{R}$ telle que

$$\hat{\mathcal{T}}(F) = \frac{\partial \hat{W}}{\partial F}(F), \quad F \in \mathbb{R}^{3\times 3}_+.$$

L'intérêt de recourir à un matériau hyperélastique est que son comportement peut être décrit simplement par une formulation variationnelle. A cet effet, combinons dans un premier temps les équations d'équilibre du corps avec la loi de comportement du matériau

$$\begin{cases} -\operatorname{div} \hat{\mathcal{T}}(\nabla\varphi(x)) &= \hat{f}(x), \quad x \in \Omega, \\ \hat{\mathcal{T}}(\nabla\varphi(x))n &= \hat{g}(x), \quad x \in \Gamma_1, \\ \varphi(x) &= \varphi_0(x), \quad x \in \Gamma_0 \end{cases} \tag{9}$$

Nous rappelons que nous faisons l'hypothèse simplificatrice de considérer des corps élastiques fait de matériaux homogènes et de ne les soumettre qu'à des forces mortes. Le système ci-dessus est un système quasilinéaire du second ordre et on ne dispose pas de théorie d'existence générale pour ce type de système. Cependant, dans le cas hyperélastique, une autre approche est de chercher la déformation comme minimum de l'énergie

$$J(\psi) = \int_\Omega \hat{W}(\nabla\varphi)\, dx - \int_\Omega \hat{f} \cdot \psi \, dx - \int_{\Gamma_1} \hat{g} \cdot \psi \, da, \tag{10}$$

sur un ensemble de déformations admissibles ad hoc. Un minimiseur assez régulier de l'énergie I vérifiera le problème aux limites (9) qui n'est autre que les équations d'Euler-Lagrange associée au problème de minimisation sus-cité. L'analyse mathématique de ce problème sera abordée dans la section suivante.

Par analogie avec la loi de comportement du matériau, les notions d'indifférence matérielle et d'isotropie s'étendent naturellement à la densité d'énergie interne $\hat{W}$. En effet, le matériau vérifie le principe d'indifférence matérielle si

$$\hat{W}(RF) = \hat{W}(F), \quad F \in \mathbb{R}^{3\times 3}, \ R \in SO(3),$$

ou de manière équivalente s'il existe $\tilde{W} : S^3_+ \longrightarrow \mathbb{R}$ telle que

$$\hat{W}(F) = \tilde{W}(F^t F), \quad F \in \mathbb{R}^{3\times 3};$$

et il est isotrope si

$$\hat{W}(FR) = \hat{W}(F), \quad F \in \mathbb{R}^{3\times 3},\ R \in SO(3).$$

Maintenant pour un matériau hyperélastique vérifiant l'axiome d'objectivité et isotrope, on obtient un résultat comparable au théorème de Rivlin-Ericksen qui est une caractérisation de la densité d'énergie

$$\hat{W}(F) = \check{W}(\imath_{F^tF}) = \check{W}(\imath_{FF^t}), \quad f \in \mathbb{R}^{3\times 3},$$

où $\check{W} : i(S^3_+) \longrightarrow \mathbb{R}$ et $\imath(S^3_+) = \{\imath_A \in \mathbb{R}^3 : A \in S^3_+\}$. Enfin, si on écrit $\hat{W}(F) = \check{W}(\imath_{F^tF}) = \bar{W}(E)$ et que la fonction $\check{W}$ est deux fois différentiable au point $\imath_I$ alors

$$\bar{W}(E) = \frac{\lambda}{2}(\operatorname{tr} E)^2 + \mu \operatorname{tr} E^2 + o(|E|^2). \tag{11}$$

Remarquons au passage que le matériau hyperélastique le plus simple est celui de Saint Venant-Kirchhoff et que sa densité d'énergie n'est autre que

$$W(F) = \frac{\lambda}{8}(\operatorname{tr}(F^tF - I))^2 + \frac{\mu}{4}\operatorname{tr}(F^tF - I)^2 + o(|E|^2).$$

On conclut cette introduction par le comportement de la densité d'énergie pour les grandes contraintes. Il est assez raisonnable d'exiger de cette fonction de vérifier les deux conditions suivantes

$$\hat{W}(F) \longrightarrow +\infty \quad \text{lorsque} \quad \det F \longrightarrow 0^+, \quad F \in \mathbb{R}^{3\times 3}_+,$$

$$\hat{W}(F) \longrightarrow +\infty \quad \text{lorsque} \quad (|F| + |\operatorname{Cof} F| + |\det F|) \longrightarrow +\infty.$$

La première condition garantie que l'on ne peut comprimer un volume en un point et la deuxième est tout simplement une conséquence de l'*inégalité de coércivité* qui semble mesurer la capacité d'un matériau à résister aux grands déplacements. Il est clair que les hypothèses concernant les grandes contraintes sont essentiellement mathématiques puisqu'elles ne sont pas complètement vérifiables.

Nous concluons ce paragraphe par une liste non exhaustive d'ouvrages spécialisés dans la matière. Tout d'abord, Ciarlet [11] dont ce paragraphe s'inspire et qui consacre une grande partie à la modélisation de l'élasticité tridimensionnelle et l'exposé de certains résultats d'existence. Ensuite Ogden [64], Truesdell & Noll [74], Mardsen & Hughes [56].

2 Eléments de convexité vectorielle et de calcul variationnel

Dans cette partie, nous survolons les concepts mathématiques qui permettent l'application de la méthode directe du calcul des variations en vue de la minimisation de la fonctionnelle (10). Une condition nécessaire au succés de cette démarche est la semicontinuité inférieur faible séquentielle de la fonctionnelle sur l'espace des déformations admissibles ; le cadre fonnctionnel naturel pour ce type de problème étant l'espace de Sobolev $W^{1,p}(\Omega;\mathbb{R}^3)$ pour un certain $p \in [1,\infty]$.

Signalons tout d'abord que la manière la plus simple de régler ce problème est de supposer que $\hat{W}$ est convexe ; voir Ekeland & Temam [38]. Or, cette propriété est incompatible avec le comportement $\hat{W}(F) \longrightarrow +\infty$ lorsque $\det F \longrightarrow 0^+$ (cf. Antman [3]) d'une part et implique des restrictions inconcevables sur les valeurs propres du tenseur de Cauchy (cf. Coleman & Noll [26]) d'autre part. Ainsi, la convexité est donc bannie une fois pour toutes.

Une première alternative proposée par Morrey [61, 62] est une condition nécessaire et suffisante de semicontinuité inférieure faible-$\star$ dans $W^{1,\infty}(\Omega;\mathbb{R}^3)$. La fonction W est *quasiconvexe* si pour tout $F \in \mathbb{R}^{3\times3}$

$$|\Omega|\, W(F) \leq \int_\Omega W(F+\nabla\varphi)\, dx$$

pour tout $\varphi \in C_0^1(\Omega;\mathbb{R}^3)$. La quasiconvexité stipule que parmi les déformations qui valent $\varphi_0(x) = F_0 x$ sur le bord $\partial\Omega$, φ_0 réalise le minimum de l'énergie. Cette condition étant globale, elle n'est pas vérifiable en général sauf dans des cas très particuliers. On définit l'enveloppe quasiconvexe de W comme étant la plus grande fonction quasiconvexe qui la minore et on la note $\mathbf{Q}W$.

Une condition nécessaire mais pas suffisante de semicontinuité inférieure faible dans $W^{1,\infty}(\Omega;\mathbb{R}^3)$ (cf. Tartar [70]) est la *rang-1-convexité* :

$$W(\lambda A + (1-\lambda)B) \leq \lambda W(A) + (1-\lambda)W(B)$$

pour tout $\lambda \in [0,1]$, et pour tout $A, B \in \mathbb{R}^{3\times3}$ tels que $\operatorname{rang}(A-B) \leq 1$. C'est une condition qui est impliquée par la quasiconvexité dans le cas fini. Dans la cas où W n'est pas partout finie Ball & Murat [5] ont montré qu'il est nécessaire que W soit en plus continue pour que cela reste vrai. Enfin, lorsque W est de clase C^2, cette condition est équivalente à la condition d'ellipticité de *Legendre-Hadamard* :

$$\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}} \lambda^i \lambda^k \mu^j \mu^l \geq 0,$$

pour tout $\lambda, \mu \in \mathbb{R}^3$, $F = (F_{ij})_{1\leq i,j\leq 3} \in \mathbb{R}^{3\times3}$. On définit de manière analogue au cas quasiconvexe l'enveloppe rang-1-convexe $\mathbf{R}W$. Enfin, Kohn & Strang [48] ont mis en évidence un algorithme pour le calcul de la dite enveloppe.

Proposition 0.2.1. (Kohn & Strang [48]) *Soit f une fonction Borel mesurable et minorée. On définit la suite $(\mathrm{R}_k f)_{k\in\mathbb{N}}$ par $\mathrm{R}_0 f = f$ et*

$$\begin{cases} \mathrm{R}_0 f = f, \\ \mathrm{R}_{k+1} f(A) = \inf\limits_{(a,b)\in\mathbb{R}^{n\times m}} \{(1-\lambda)\mathrm{R}_k f(A - \lambda\, a\otimes b) + \lambda\, \mathrm{R}_k f(A + (1-\lambda) a\otimes b)\} \end{cases}$$

pour tout entier $k \geq 1$ et toute matrice $A \in \mathbb{R}^{n\times N}$. Alors $(\mathrm{R}_k f)_{k\in\mathbb{N}}$ décroît vers $\mathbf{R}f$.

Malheureusement, rien ne nous garantit que l'enveloppe est atteinte au bout d'un nombre fini d'itérations.

Ball a proposé une classe plus restreinte de fonctions quasiconvexes pour subvenir aux besoins de l'élasticité. Une fonction W est *polyconvexe* s'il existe une fonction convexe $\dot{W} : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \longrightarrow \bar{\mathbb{R}}$ telle que

$$W(F) = \dot{W}(F, \operatorname{Cof} F, \det F).$$

La polyconvexité implique toujours la quasiconvexité ainsi que la rang-1-convexité. De plus, elle a la bon goût de respecter tous les prérequis physiques y compris la conservation de l'orientation d'où la légitimité de cette modélisation. En vertue de la convexité de $\dot{W}$, la semicontinuité inférieure faible se ramène à la continuité faible des opérateurs $\varphi \mapsto \operatorname{Cof} \nabla\varphi$ et $\varphi \mapsto \det \nabla\varphi$ qui est vérifiée dans un sens raisonnable et on arrive moyennant des conditions ad-hoc de coercivité et de croissance à montrer l'existence de minimiseurs pour l'énergie d'un corps hyperélastique. L'exemple par excellence de densité d'énergie polyconvexe sont les matériaux de type *Ogden* :

$$\hat{W}(F) = \sum_{i=1}^{M} a_i \left[\operatorname{tr} C^{\frac{\gamma_i}{2}} - 3\right] + \sum_{j=1}^{N} b_j \left[\operatorname{tr} (\operatorname{Cof} C)^{\frac{\delta_j}{2}} - 3\right] + \Gamma(\det F), \quad C = F^t F, \tag{12}$$

où $a_i,\ b_j > 0,\ \gamma_i\ , \delta_j \geq 1$ et $\Gamma : (0,\infty) \longrightarrow \mathbb{R}$ est une fonction convexe telle que $\Gamma(\delta) \longrightarrow +\infty$ lorsque $\delta \longrightarrow 0^+$.

Naturellement, il existe une vaste collection de résultats d'existence autour de ces notions de convexité vectorielle dont certains seront rappelés plus loin dans la thèse lorsque le besoin s'en fera ressentir. Nous renvoyons le lecteur intéressé par ces questions vers les ouvrages de Dacorogna [33] et Buttazzo [10] ou encore les articles de Ball & Murat [5] et Fonseca [39] et aux références qui y figurent. Parmi ces résultats, il existe des résultats de *relaxation*. Relaxer un problème de minimisation veut dire identifier l'enveloppe semicontinue inférieure faible de sa fonctionnelle par rapport à une certaine topologie et la remplacer par celle-ci. Les deux résultats de relaxation avec représentation intégrale de l'énergie relaxée auxquels on s'intéresse particulièrement sont les suivants :

Théorème 0.2.2. (Acerbi & Fusco [2]) *Soient* $1 \leq p \leq \infty$ *et* $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ *une fonction Borel mesurable satisfaisant*

$$\begin{aligned} 0 \leq f(x,u,F) &\leq C(1+|F|^p), && \textit{si } p < \infty; \\ 0 \leq f(x,u,F) &\leq b(F), && \textit{si } p = \infty; \end{aligned}$$

où $C \geq 0$ *et* b *est une fonction positive et localement bornée sur* $\times\mathbb{R}^{n\times m}$. *Alors si on pose* $I(\varphi) = \int_D f(\nabla\varphi)$ *pour tout* $\varphi \in W^{1,p}(D;\mathbb{R}^m)$, *sa fonctionnelle relaxée pour la topologie faible de* $W^{1,p}(D;\mathbb{R}^m)$ *(respectivement faible-*$\star$ $W^{1,\infty}(D;\mathbb{R}^m)$*) est* $\bar{I}(\varphi) = \int_D \mathbf{Q}f(\nabla\varphi)$.

Soulignons au passage que $\inf I = \inf \bar{I}$ sur $W^{1,p}(D;\mathbb{R}^m)$ et qu'il existe une suite de quasi-minimiseurs de I qui converge faiblement vers un minimiseur de $\bar{I}$. Ce premier théorème concerne uniquement les fonctionnelles finies et ne peut à priori s'appliquer à une énergie W telle que $W(F) \longrightarrow +\infty$ lorsque $\det F \longrightarrow 0^+$. Nous renvoyons le lecteur vers Ball & Murat [5] et Fonseca [39] pour des discussions et des contre-exemples. Remarquons que l'énoncé ci-dessus n'est pas le plus général qui soit, c'est une version qui convient aux énergies homogènes qui ne dépendent que du gradient de la déformation. Le deuxième théorème concerne des fonctionnelles singulières et nécessite l'introduction de certaines notations.

Définition 0.2.3. Soit D un ouvert borné de $\mathbb{R}^n$. Une fonction $\phi \in W^{1,\infty}(D;\mathbb{R}^m)$ est *Vitali affine par morceaux* sur D si
- $\operatorname{card}\{\nabla\phi(x);\ x \in D\}$ est fini ,
- il existe une famille au plus dénombrable d'ouverts disjoints $(D_k)_{k\in K}$ tels que $|\partial D_k| = 0$ pour tout $k \in K$ et $|D \setminus \cup_{k\in K} D_k| = 0$ tels que $\phi_{|_{D_k}}$ est affine.

On note l'espace des fonctions Vitali affine par morceaux et nulles sur le bord ∂D $\mathrm{Aff}_0^{\mathrm{V}}(D;\mathbb{R}^m)$.

Pour une fonction $f : \mathbb{R}^{n\times m} \longrightarrow \bar{\mathbb{R}}$, on définit son domaine effectif $\mathcal{D}_e(f) := \{F \in \mathbb{R}^{n\times m} :\ f(F) < \infty\}$. Enfin, on définit l'application

$$\bar{\mathrm{Z}}_D f(\xi) := \inf\left\{\frac{1}{|D|}\int_D f(\xi + \nabla\varphi)\,dx :\ \varphi \in \mathrm{Aff}_0^{\mathrm{V}}(D;\mathbb{R}^m)\right\}. \tag{13}$$

Notons que $\bar{\mathrm{Z}}_D f = \bar{\mathrm{Z}} f$ (c.à d. que la quantité est indépendante de l'ouvert) si $|\partial D| = 0$. De plus, $\bar{\mathrm{Z}} f = \mathbf{Q} f$ lorsque f est partout finie.

Théorème 0.2.4. (Ben Belgacem [8]) *Soit $f : \mathbb{R}^{n\times m} \longrightarrow \bar{\mathbb{R}}$ une fonction Borel mesurable satisfaisant*

(i) $\mathcal{O}_f := \operatorname{int}\{F \in \mathbb{R}^{n\times m} : \bar{\mathrm{Z}}\mathrm{R}_k f(F) \le \mathrm{R}_{k+1}(F),\ \forall k \in \mathbb{N}\}$ *est dense dans* $\mathbb{R}^{n\times m}$ *;*

(ii) Pour tout $F \in \mathbb{R}^{n\times m}$ et tout $k \in \mathbb{N}^*$, si $(F_\varepsilon)_{\varepsilon>0} \subset \mathcal{O}_f$ vérifie $F_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} F$, alors

$$\limsup_{\varepsilon \longrightarrow 0^+} \mathrm{R}_k f(F_\varepsilon) \le \mathrm{R}_k f(F);$$

(iii) $\mathbf{R} f$ est partout finie et pour un certain $p \ge 1$ il existe $c > 0$ telle que pour tout $F \in \mathbb{R}^{n\times m}$, on a

$$\mathbf{R} f(F) \le c\,(1 + |F|^p)\,;$$

Alors si on pose $I(\varphi) = \int_D f(\nabla\varphi)$ pour tout $\varphi \in W^{1,p}(D;\mathbb{R}^m)$, sa fonctionnelle relaxée pour la topologie faible de $W^{1,p}(D;\mathbb{R}^m)$ (respectivement faible-$\star$ $W^{1,\infty}(D;\mathbb{R}^m)$) est $\bar{I}(\varphi) = \int_D \mathbf{QR} f(\nabla\varphi)$.

Ce résultat semble convenir aux besoins des fonctionnelles d'énergies élastiques. En effet, malgré l'aspect "décourageant" de l'hypothèse technique (i) elle semble s'accomoder sans problèmes avec la condition $W(F) \longrightarrow +\infty$ lorsque $\det F \longrightarrow 0^+$; dans la pratique il arrive souvent que $\mathcal{O}_f = \mathcal{D}_e(f)$. Remarquez qu'il généralise le théorème précédent tel qu'il est énoncé.

3 Sur la théorie des plaques minces non linéaires

La théorie des plaques minces consiste en la recherche de modèles bidimensionnels qui approximent dans un sens qui reste à définir le modèle tridimensionnel tout en conservant les propriétés qualitatives de la plaque. Notre propos n'étant pas de faire un inventaire des modèles existants, nous allons rappeler les modèles classiques ainsi que les méthodes asymptotiques mises en œuvre pour leurs justifications. Rappelons toutefois que les tout premiers modèles ont été proposés par des physiciens à partir de considérations mécaniques sur les contraintes ; voir par exemple Naghdi [63], Morgenstern [60] ou encore Koiter [49] pour le cas des coques. Parmi, ces modèles certains ont été justifiés mathématiquement. Pour fixer les idées nous allons noter

$$(P^\varepsilon) \qquad J_\varepsilon(\varphi^\varepsilon) = \inf_{\psi \in \mathcal{F}(\Omega^\varepsilon)} J_\varepsilon(\psi)$$

le problème de minimisation de l'énergie tridimensionnelle totale du corps élastique posé sur toute la plaque : $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ et

$$(P) \qquad J(\varphi) = \inf_{\psi \in \mathcal{F}(\omega)} J(\psi)$$

le problème de minimisation de l'énergie bidimensionnelle posé sur la surface moyenne de la plaque. Il s'agit donc de montrer que la suite de solutions $(\varphi^\varepsilon)_{\varepsilon>0}$ converge vers une solution du deuxième problème dans un sens qui reste à préciser. Dans le cas linéaire, des résultats de convergence forte souvent dans des espaces de Sobolev ont été établis légitimant de la sorte les équations classiques. On citera Ciarlet & Destuynder [16, 17] pour la théorie de *Kirchhoff-Love*, Miara [58, 59] pour les modèles membranaires et inextensionnels et Ciarlet & Lods [21, 22, 25] et Ciarlet, Lods & Miara [23] pour les coques. Pour le cas non linéaire, le cadre mathématiqe qui s'impose dès que l'on cherche à décrire de grands déplacements, les résultats sont plus faibles.

Rappelons enfin qu'il est préférable que les modèles bidimensionnels conservent les propriétés de base d'un matériau élastique telle que l'indifférence matérielle ou la conservation de l'orientation ou encore l'isotropie lorsque cette hypothèse est faite. Une première approche est celle dite du *développement asymptotique formel* et qui a été ajustée par Ciarlet & Destuynder [16] au cadre mathématique moderne. Cette méthode n'a cessé de subir des raffinements culminant avec la version de Fox, Raoult & Simo [40] où des hypothèses minimales sont faites outre que les données du problème admettent des développement en puissances de ε. Cette méthode consiste à résoudre une série d'équations d'Euler-Lagrange pour chaque ordre de grandeur en $\varepsilon^p, p \in \mathbb{Z}$ après avoir écrit le problème sous sa forme variationnelle. Procédant de la sorte, Fox & al. [40] obtiennent une hiérarchie de modèles bidimensionnels pour le matériau de Saint Venant-Kirchhoff :

le modèle membranaire (p=0)

$$W_{FRS}(\nabla\varphi) = \frac{2\lambda\mu}{\lambda+2\mu}\left(\mathrm{tr}\left(\nabla\varphi^t\nabla\varphi - I\right)\right)^2 + \mu\,\mathrm{tr}\left(\nabla\varphi^t\nabla\varphi - I\right)^2; \tag{14}$$

le modèle inextensionnel (p=2)

$$W_b(F) = \frac{2\lambda\mu}{3(\lambda+2\mu)}\, b_{\sigma\sigma}(\varphi) b_{\tau\tau}(\varphi) + \frac{2\mu}{3}\, b_{\alpha\beta}(\varphi) b_{\alpha\beta}(\varphi) \tag{15}$$

$$b_{\alpha\beta}(\varphi) = -\, n\cdot\varphi_{,\alpha\beta}, \quad n = \varphi_1 \wedge \varphi_2, \quad \varphi_{,1}\cdot\varphi_{,2} = \delta^\alpha_\beta; \tag{16}$$

et le modèle de Von Kármán (p=4) qui fait partie de la théorie de Kirchhoff-Love qui stipule que les déplacements tridmensionnels $u = \varphi - I$ sont de la forme $u^\alpha = \zeta^\alpha - x_3\zeta^3_{,\alpha}$ et $u^3 = \zeta^3$ avec ζ une fonction définie sur la surface moyenne. Notons que ce dernier modèle a été justifié par Ciarlet [12] avec les mêmes hypothèses sur les forces extérieures ; voir aussi Ciarlet & Gratie [19] and Ciarlet, Gratie & Sabu [20] pour *les équations de Von Kármán généralisées*. Néanmoins il présente un inconvénient de taille : celui de ne pas satisfaire le principe d'indifférence matérielle. Les deux premiers modèles sont *classiques* (cf. Green & Zerna [47]) et décomposent l'énergie élastique en deux énergies correspondant aux deux effets qui régissent la déformation d'un corps élastique et par conséquent aux deux quantités qui les caractérisent. En effet, la première dépend uniquement de la différence entre le tenseur métrique de la déformation et celui de la configuration de référence $[\nabla\varphi^t\nabla\varphi - I]$ et la deuxième n'est valable que pour les déformations inextensionnelles et dépend uniquement de la courbure de la déformation. Malheureusement, l'énergie membranaire n'est pas semicontinue inférieurement faible et n'a été justifiée que pour des petits déplacements

$$\mathrm{tr}\left(\nabla\varphi^t\nabla\varphi - I\right) < \frac{\lambda+2\mu}{2\lambda} \quad \text{in } \omega. \tag{17}$$

De plus, si on considère la densité d'énergie qui couple les deux effets *à la Koiter* :

$$W^\varepsilon_K = W_{FRS} + \varepsilon^2 W_b,$$

elle n'est pas coercive par rapport à l'espace naturel de minimisation $W^{2,2}$. Pour une couverture assez large des modèles de plaques obtenus par ce type de méthode nous renvoyons le lecteur vers Ciarlet [13] ; voir Ciarlet [14] pour le cas des coques.

Une nouvelle étape a été franchie lorsque Le Dret & Raoult [50, 53] ont obtenu un modèle de membrane pour une famille de matériaux comprenant celui de Saint Venant-Kirchhoff par des arguments de Γ-convergence. En effet, leur méthode, inspirée par le travail de Acerbi, Buttazzo & Percivale [1] qui proposent un modèle unidimensionnel de fils élastiques, fournit un résultat de convergence faible d'une suite de quasi-minimiseurs de l'énergie vers une solution du problème bidimensionnel dont la densité d'énergie interne est la suivante

$$\hat{W}_m = \mathbf{Q}\hat{W}_0, \quad \hat{W}_0(F) = \inf_{z\in\mathbb{R}^3} \hat{W}((F|z)), \quad F \in \mathbb{R}^{3\times 2}.$$

Dans le cas particulier du matériau de Saint Venant-Kirchhoff on a

$$W_0(\nabla\varphi) = W_{FRS}(\nabla\varphi) + \frac{1}{4(\lambda+2\mu)}[2\lambda \mathrm{tr}\left(\nabla\varphi^t\nabla\varphi - I\right) - (\lambda+2\mu)]_+^2, \qquad (18)$$

A première vue, il semblerait que l'énergie de Le Dret & Raoult soit différente de celle de Fox & al. La première n'est que la relaxée de la deuxième à un détail près : l'hypothèse restrictive (17) est due à la condition de conservation de l'orientation $\det\nabla\varphi > 0$ dont ne tient pas compte le modèle relaxé. D'ailleurs, Pantz [65, 66] l'a montré en menant une étude asymptotique équivalente à celle de Fox & al sans imposer l'injectivité locale. Globalement, il propose de résoudre la série de problèmes de minimisations associées aux équations d'Euler-Lagrange qu'ils considèrent. Cette approche semble plus naturelle et plus performante. En effet, elle se résume à des calculs de minima de fonctions numériques à opposer à des choix particuliers de fonctions test. Cela permet aussi de travailler sur des énergies tridimensionnelles plus complexes donc plus réalistes ; voir Chapitre 2.

Signalons que Friesecke, James & Müller [44] ont obtenu un résultat de convergence similaire pour le cas inextensionnel juste après une tentative de justification partielle de Pantz [67, 68].

Pour conclure, citons le modèle qui nous a inspiré le Chapitre 3 et établi dans Ben Belgacem [6, 7]. C'est une modélisation semblable à celle de Le Dret & Raoult à ceci près que la densité d'énergie est telle que $\hat{W}(F) \longrightarrow +\infty$ lorsque $\det F \longrightarrow 0^+$ garantissant automatiquement de la sorte la conservation de l'orientation. A cet effet, Ben Belgacem commence par établir un théorème de relaxation (cf. [8] et Théorème 0.2.4) pour aboutir au final à

$$\ddot{W}_m = \mathbf{QR}\ddot{W}_0, \quad \ddot{W}_0(F) = \inf_{z\in\mathbb{R}^3} \ddot{W}((F|z)).$$

Son modèle présente toutes les propriétés physiques *standard* requises.

4 Présentation des résultats de la thèse

4.1 Chapitre 1. Sur les plaques membranaires comprimées

Dans cette partie de la thèse, on établit un résultat de non existence de minimiseurs pour le modèle classique (14) de plaque mince membranaire. La densité d'énergie membranaire non linéaire classique est la suivante :

$$\forall\, \xi \in \mathbb{R}^{3\times 2},\; W(\xi) = \frac{E\nu}{2(1-\nu^2)}\left(\mathrm{tr}\left(\xi^t\xi - I\right)\right)^2 + \frac{E}{2(1+\nu)}\,\mathrm{tr}\left(\xi^t\xi - I\right)^2, \qquad (19)$$

où ξ est le gradient de la déformation bidimensionnelle, $E > 0$ est le module de Young et $0 \leq \nu < \frac{1}{2}$ est le quotient de Poisson. On rappelle que

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \quad \text{et} \quad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}.$$

On rappelle aussi que la fonction ci-dessus exprime la différence exacte entre le tenseur métrique de la surface inconnue et celui de la configuration de référence. On s'intéresse au problème de minimisation associé. Soit $\omega \subset \mathbb{R}^2$ un ouvert borné et on note $\varphi : \omega \longrightarrow \mathbb{R}^3$ la déformation. Enfin, $f \in L^1(\omega; \mathbb{R}^3)$ désigne une force volumique morte et soit $\xi_0 \in \mathbb{R}^{3\times 2}$. Le problème en question est alors

$$(P) \qquad \inf\left\{ I(\varphi) = \int_\omega W(\nabla\varphi)\, dx : \ \varphi \in \varphi_0 + W_0^{1,4}(\omega; \mathbb{R}^3) \right\},$$

où $\varphi_0 = \xi_0 x, x \in \overline{\omega}$. Malheureusement, la fonctionnelle d'énergie associée I n'est pas semicontinue inférieurement pour la topologie faible-$\star$ de $W^{1,\infty}(\omega; \mathbb{R}^3)$ qui est dans ce cas une condition nécessaire de semicontinuité inférieure faible sur $W^{1,4}(\omega; \mathbb{R}^3)$. En effet, Morrey [61, 62] a montré l'équivalence entre semicontinuité inférieure faible et quasiconvexité pour des conditions de croissance et de coércivité appropriées pour toute une classe de fonctionnelles dépendant d'un gradient, voir aussi Meyers [57] et Acerbi & Fusco [2] pour des versions améliorées de ce résultat. Pour le problème (P), Coutand [28] a mis en évidence un contre-exemple à la semicontinuité inférieure lorsque la membrane est soumise à des forces planes. En fait, cette densité d'énergie n'est pas quasiconvexe puisqu'elle n'est même pas rang-1-convexe comme l'a remarqué Raoult [69] qui a montré qu'une fonction de la forme

$$\hat{W}(F) = a \operatorname{tr} C + b \operatorname{tr} C^2 + c \operatorname{tr} \operatorname{Cof} C, \quad C = F^t F,$$

avec $a < 0$ et $b, c > 0$ ne peut être rang-1-convexe ; voir aussi Ciarlet [11]. Or, il se trouve que

$$W(F) = -3E\,\frac{2\nu+1}{1-\nu^2} \operatorname{tr} C + \frac{E}{2(1-\nu^2)} \operatorname{tr} C^2 + \frac{E}{1-\nu^2} \operatorname{tr} \operatorname{Cof} C + \frac{E}{2}\frac{8\nu+1}{1-\nu^2}.$$

De plus Le Dret & Raoult [53] ont calculé l'enveloppe quasiconvexe de W prouvant encore une fois indirectement que cette fonction ne peut être quasiconvexe. Si pour $F \in \mathbb{R}^{3\times 2}$, on note $0 \le s_1(F) \le s_2(F)$ ses valeurs singulières par ordre croissant alors

$$\mathbf{Q}W(F) = W^{**}(F) = \begin{cases} 0 & \text{si } F \in D_1, \\ \dfrac{E}{2}(s_2^2 - 1)^2 & \text{si } F \in D_2, \\ W(F) & \text{si } F \notin D_1 \cup D_2, \end{cases} \qquad \forall\, F \in \mathbb{R}^{3\times 2}, \tag{20}$$

où

$$\begin{aligned} &0 \le s_1(F) \le s_2(F), \\ &D_1 = \{F \in \mathbb{R}^{3\times 2} : s_2 \le 1 \quad \text{et} \quad s_1^2 + \nu s_2^2 < 1 + \nu\}, \\ &D_2 = \{F \in \mathbb{R}^{3\times 2} : s_2 > 1 \quad \text{et} \quad s_1^2 + \nu s_2^2 < 1 + \nu\}. \end{aligned} \tag{21}$$

Notons que $\mathbf{Q}W = \mathbf{C}W$ l'enveloppe convexe de W, ce qui semble avoir rendu ce calcul possible. Remarquons qu'alors que la fonction W est strictement positive pour les états compressifs (gradient de déformation à valeurs singulières inférieures à 1), $\mathbf{Q}W$ est nulle ce qui semble être plus réaliste puisqu'une membrane est plutôt sensée céder à la compression sans opposer la moindre résistance comme cela a déjà été démontré dans Le Dret & Raoult [52, 53]. En effet, la membrane se met dans un état de flexion sans changer la valeur de son tenseur métrique. De façon générale, on a

Proposition 0.4.1. *Soit* $W : \mathbb{R}^{3\times 2} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ *une fonction rang-1-convexe, isotrope et vérifiant le principe d'indiffrence matérielle. Supposons en plus que* $W(\eta) = 0$ *où* $\eta = (\delta^i_j)_{i\leq 3, j\leq 2}$, *alors*

$$W(\xi) = 0, \quad \forall\, \xi \in \mathbb{R}^{3\times 2} \text{ telle que } s_2(\xi) \in [0,1].$$

La rang-1-convexité étant une condition nécessaire de semicontinuité inférieure faible-$\star$ d'après Tartar [70], on ne peut concevoir une énergie membranaire non linéaire qui soit non nulle pour les états compressifs. Il est alors naturel de considérer le problème relaxé au sens précisé plus haut (cf. Théorème 0.2.2) :

$$(QP) \qquad \inf\left\{ \bar{I}(\varphi) = \int_\omega \mathbf{Q}W(\nabla\varphi)\,dx : \ \varphi \in \varphi_0 + W_0^{1,4}(\omega;\mathbb{R}^3) \right\}.$$

Rappelons que l'infimum dans (P) et le minimum dans (QP) coïncident ; voir Dacorogna [33] pour de plus amples détails en la matière. Rappelons que le calcul de $\mathbf{Q}W$ a été motivé par l'obtention du problème (QP) comme approximation bidimensionnelle de l'élasticité tridimensionnelle pour des forces extérieures d'ordre 0 en fonction de l'épaisseur pour toute une classe de matériaux.

Cependant Coutand [27, 30, 29, 28] a obtenu des résultats d'existence de minimiseurs pour l'énergie membranaire pour des petites forces grâce au théorème d'inversion locale dans certains cas particuliers où la membrane est étirée. D'ailleurs, dans [28], il est montré que pour une membrane compressée sur son bord, les points critiques obtenus ne peuvent être des minimiseurs de l'énergie en aucun sens raisonnable. Dacorogna & Marcellini [35] donnent des conditions nécessaires et suffisantes d'existence pour l'énergie de Saint Venant-Kirchhoff plane qui d'un point de vue strictement mathématique est identique à l'énergie membranaire dans le plan. Ce résultat est une application d'un résultat plus général. Avant de l'énoncer nous rappelons une définition.

Definition 0.4.1. Une fonction convexe $H : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ est dite strictement convexe en $\xi_0 = (\xi_0^\alpha)_{1\leq\alpha\leq N} \in \mathbb{R}^{n\times N}$ dans au moins N directions s'il existe $\lambda = (\lambda^\alpha)_{1\leq\alpha\leq N} \in \mathbb{R}^{n\times N}$ tel que

$$\lambda^\alpha \neq 0 \quad \text{et} \quad \langle \lambda^\alpha ; \xi^\alpha - \xi_0^\alpha \rangle_{\mathbb{R}^n} = 0 \quad \forall\, \alpha = 1, 2, ..., N,$$

lorsque $\xi = (\xi^\alpha)_{1\leq\alpha\leq N}$ vérifie la condition

$$\frac{H(\xi) + H(\xi_0)}{2} = H\left(\frac{\xi + \xi_0}{2}\right). \tag{22}$$

Théorème 0.4.2. (Dacorogna & Marcellini [35]) *Soit* $g : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ *une fonction semicontinue inférieurement et soit* $\xi_0 \in \mathbb{R}^{n\times N}$ *tels que*
(i) $G^{**}(\xi_0) = QG(\xi_0) < g(\xi_0)$,
(ii) G^{**} *est strictement convexe en au moins* N *directions au point* ξ_0 *;*
alors (P) n'a pas de solution.

Nous utilisons le résultat ci-dessus pour obtenir

Théorème 0.4.3. *Le problème* (P) *admet des solutions si et seulement si*

$$s_1^2(\xi_0) + \nu s_2^2(\xi_0) \geq 1 + \nu.$$

Ce résultat a été annoncé dans Trabelsi [72]. La preuve dans notre cas est plus difficile techniquement du fait que le gradient de la déformation est une matrice rectangulaire compliquant ainsi la manipulation des valeurs singulières. La démonstration fait appel également à la formule de l'enveloppe quasiconvexe de l'énergie (1.5)-(1.6). Pour les questions d'unicité liées à ce type de problème, nous renvoyons le lecteur vers Knops & Stuart [48].

Enfin, notons que la faiblesse de ce résultat réside dans le fait qu'il ne prend pas en compte une énergie externe. On s'est également intéressé à un résultat semblable dû à Ball & Murat [5] pour un travail des forces extérieures strictement positif. Leur démonstration ne fonctionne pas pour un travail nul. Néanmoins, on a obtenu un borne supérieure pour l'infimum de l'énergie dans le cas général en fonction des données du bord

Proposition 0.4.4. *Soit* $f \in L^2(\omega; \mathbb{R}^3)$ *et supposons que* $|\partial\omega| = 0$*, alors*

$$\inf_{\bar\varphi \in M_{\xi_0}} J(\bar\varphi) \leq \inf_{\bar\varphi \in M_{\xi_0}} I(\bar\varphi) - \int_\omega f \cdot \xi_0 x \, dx,$$

où

$$J(\bar\varphi) = I(\bar\varphi) - \int_\omega f \cdot \bar\varphi \, dx \quad \text{et} \quad M_{\xi_0} = \xi_0 x + W^{1,4}(\omega; \mathbb{R}^3).$$

De plus, si $s_1^2(\xi_0) + \nu s_2^2(\xi_0) \geq 1 + \nu$*, alors*

$$\inf_{\bar\varphi \in M_{\xi_0}} J(\bar\varphi) \leq |\omega| \, W(\xi_0) - \int_\omega f \cdot \xi_0 x \, dx.$$

4.2 Chapitre 2. Plaques minces non linéaires pour des matériaux de type Ogden

Notre motivation dans cette partie a été de produire un modèle de plaques minces bidimensionnel pour un matériau hyperélastique plus réaliste que celui de Saint Venant-Kirchhoff. En d'autres termes, un matériau qui ait un comportement raisonnable pour les grandes et petites déformations tout en exhibant des propriétés mathématiques satisfaisantes telles que la coercivité de sa densité d'énergie et, pourquoi pas, l'existence de minimiseurs sous des conditions justifiées.

Soit $(e_i)_{i=1,2,3}$ une base orthonormale de $\mathbb{R}^3$. La dérivée partielle d'une fonction ou d'un champ de vecteurs ψ par rapport au vecteur e_i est notée $\psi_{,i}$ et son gradient est noté $\nabla\psi$. Dans sa configuration de référence, la plaque occupe le domaine $\Omega^\varepsilon = \omega\times]-\varepsilon,\ \varepsilon[$, où l'intérieur de de sa surface moyenne ω est un ouvert de l'espace engendré par e_1 et e_2, et ε est le petit paramètre. La frontière de Ω^ε est partitionnée de la sorte $\partial\Omega^\varepsilon = \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \cup \Gamma^\varepsilon$, où $\Gamma_\pm^\varepsilon = \omega\times\{\pm\varepsilon\}$ et $\Gamma^\varepsilon = \partial\omega\times[-\varepsilon,\ \varepsilon]$. La frontière de ω est composée de deux parties $\partial\omega = \partial\omega_\sigma \cup \partial\omega$, et on définit une partition de $\Gamma^\varepsilon = \Gamma_\sigma^\varepsilon \cup \Gamma_\phi^\varepsilon$, où $\Gamma_\sigma^\varepsilon = \partial\omega_\sigma\times]-\varepsilon,\ \varepsilon[$ et $\Gamma_\phi^\varepsilon = \partial\omega_\phi\times]-\varepsilon,\ \varepsilon[$. La plaque est soumise à des forces volumiques b^ε, et à des forces surfaciques $g_\pm^\varepsilon$ sur les faces supérieure et inférieure $\Gamma_\pm^\varepsilon$ respectivement et h^ε sur la portion $\Gamma_\sigma^\varepsilon$ de sa face latérale, tandis que la déformation est imposée par $\bar\phi^\varepsilon$ sur Γ_ϕ^ε. On suppose que la plaque est constituée d'un matériau isotropique, homogène et hyperélastique. Dans cette configuration la plaque subit une déformation $\varphi^\varepsilon : \bar\Omega^\varepsilon \to \mathbb{R}^3$ qui devrait être un minimiseur de l'énergie $J_\varepsilon(\varphi^\varepsilon) = I_\varepsilon(\varphi^\varepsilon) - \ell_\varepsilon(\varphi^\varepsilon)$, où

$$\ell^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} b^\varepsilon\cdot\varphi^\varepsilon\,dx + \int_{\Gamma_+^\varepsilon\cup\Gamma^\varepsilon+} g^\varepsilon\cdot\varphi^\varepsilon\,dx_H + \int_{\Gamma_\sigma^\varepsilon} h^\varepsilon\cdot\varphi^\varepsilon\,ds,$$

$$I^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} W(\nabla\varphi^\varepsilon)\,dx.$$

sur l'espace des déformations admissibles

$$\mathcal{M}^\varepsilon = \left\{\psi^\varepsilon \in \mathcal{F}(\bar\Omega^\varepsilon;\mathbb{R}^3) : \det\nabla\psi^\varepsilon > 0 \quad \text{et} \quad \psi^\varepsilon_{|\Gamma_\phi^\varepsilon} = \bar\varphi^\varepsilon_{|\Gamma_\phi^\varepsilon}\right\},$$

avec $\mathcal{F}(\overline{\Omega}^\varepsilon;\mathbb{R}^3)$ un espace de fonctions assez régulières pour que toutes les expressions où elles interviennent aient un sens. On note ce problème de minimisation (P^ε). Enfin, on choisit une famille de matériaux de type Ogden (cf. (12)) proposée par Ciarlet & Geymonat [18] et dont l'expression de leurs densités d'énergie est donnée par

$$W(F) = a\,|F|^2 + b\,|\mathrm{adj}_3\,F|^2 + c\,\det F^tF - d\,\ln\det F^tF + e \quad \forall F\in\mathbb{R}^{3\times3}, \quad (23)$$

L'intérêt d'un tel matériau est multiple. D'abord, il est plus réaliste que le matériau de Saint Venant-Kirchhoff en ce sens qu'il ne permet pas la compression d'un volume en un point grâce au terme logarithmique ; condition qui soulignons-le intégre naturellement la condition de conservation de l'orientation locale

$$\det\nabla\psi^\varepsilon(x) > 0 \quad \text{a.e. } x\in\Omega^\varepsilon$$

qui est retenue ici. Ensuite, la fonction W s'accorde avec le développement (11) pour les petits déplacements. Enfin, sa densité d'énergie est polyconvexe et coercive par rapport aux invariants du gradient de la déformation ce qui fournit une théorie d'existence puissante pour le problème (P^ε) ; voir Ball [4] ou encore Ciarlet [11] pour un exposé de ce type de résultat.

Afin de mettre en œuvre une analyse asymptotique, il est utile de se ramener à un domaine fixe. A cet effet, on opère le changement d'échelle π_ε defini par $(\pi_\varepsilon f)(x^1, x^2, x^3) = f(x^1, x^2, \varepsilon x^3)$ et pour toute fonction f^ε et toute fonctionelle G^ε, on définit $f(\varepsilon) = \pi_\varepsilon^{-1} f^\varepsilon$ et $G(\varepsilon)(\psi) = G_\varepsilon(\pi_\varepsilon^{-1}\psi)$. De plus on note $\Omega = \Omega^1$, $\Gamma_\sigma = \Gamma_\sigma^1$, $\Gamma_\phi = \Gamma_\phi^1$, $\Gamma_\pm = \Gamma_\pm^1$ et $x = (x_H, \xi) \in \omega \times [-1, 1]$. Puis, on suppose que les déformations ainsi que toutes les données du problème admettent des développements en puissances positives de ε. Pour une fonction $f(\varepsilon)$, on écrit $f(\varepsilon) = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \cdots$. Cela induit un développement en puissances de ε sur la fonctionnelle $J(\varepsilon)$ qu'on note en conséquence. Un dernier prérequis mathématique est de supposer que $\bar{\phi}(\varepsilon)(x_H, \xi) = \bar{\varphi}(x_H)$ pour éviter le phénomène de couche-limite qui n'est pas pris en compte par cette analyse ; voir Friedrichs & Dressler [41].

On est maintenant en mesure de reformuler la nouvelle famille de problèmes $(P(\varepsilon))_{\varepsilon>0}$ en une suite récursive de problèmes de minimisation indépendants de ε grâce à la proposition suivante

Proposition 0.4.5. (Pantz [65, 66]) *La solution $\varphi(\varepsilon) = \varphi^0 + \varphi^1\varepsilon + \cdots$ du problème $P(\varepsilon)$ est telle que*

$$\varphi \in \bigcap_{n=-1}^{\infty} \mathcal{M}_n$$

où

$$\mathcal{M}_{n+1} = \left\{ \psi \in \mathcal{M}_n \;:\; J^n(\psi) = \inf_{\breve{\psi} \in \mathcal{M}_n} J^n(\breve{\psi}) \right\},$$

$$\mathcal{M}_{-1} = \left\{ \psi \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^3)^{\mathbb{N}} \;:\; \sum_n \psi^n \varepsilon^n \in \mathcal{M}(\varepsilon) \right\},$$

$$\varphi = \left(\varphi^0, \varphi^1, \varphi^2, \cdots\right) \quad et \quad J(\varepsilon) = \varepsilon^{-1} J^{-1} + J^0 + \varepsilon J^1 + \varepsilon J^2 + \cdots .$$

En d'autres termes, si l'on note P_n le problème qui consiste à minimiser la fonctionnelle J^n sur l'ensemble $\mathcal{M}_n$, la résolution des problèmes $(P(\varepsilon))_{\varepsilon>0}$ équivaut à la résolution de la suite de problèmes $(P_n)_{n\geq -1}$. Cette procédure asymptotique mise au point par Pantz [66] est équivalente à celle formulée par Fox, Raoult & Simo [40] où les auteurs résolvent une suite de problèmes variationnels.

La résolution des problèmes P_{-1}, P_0 et P_1 nous permet d'obtenir un modèle membranaire en ce sens qu'il ne dépend que de la première forme fondamentale de la déformation de la surface moyenne de la plaque.

Théorème 0.4.6. *Si $(\varphi^0, \psi^1, \psi^2, \ldots)$ est solution du problème P_1, alors le terme dominant φ^0 dans le développement asymptotique minimise l'énergie*

$$J_m^1(\phi^0) = \int_\omega W_m(\nabla \phi^0)\, dx_H - \int_\omega f^0 \cdot \phi^0\, dx_H - \int_{\omega_\sigma} \overline{f^0} \cdot \phi^0\, ds,$$

sur l'ensemble des déformations $\phi^0 : \omega \to \mathbb{R}^3$ telles que $\phi^0_{|\partial\omega_\phi} = \overline{\phi}_{|\partial\omega_\phi}$, où

$$W_m(F) = 2a\,|F|^2 + 2b\,\det F^t F + 2d \ln\left\{\frac{a + b\,|F|^2}{\det F^t F} + c\right\} + e',$$

$$f^0 = \int_{-1}^{1} b^0 \, d\xi + g_+^1 + g_-^1, \quad \overline{f^0} = \int_{-1}^{1} h^0 \, d\xi \quad et \quad e' = 2e + 2d\left(1 + \ln\frac{c}{d}\right).$$

Ce modèle généralise le modèle membranaire nonlinéaire classique pour un matériau de Saint Venant-Kirchhoff. En effet, pour toutes les valeurs des constantes de la densité d'énergie (23) les termes dominants du développement de celle-ci près d'un état naturel sont identiques à ceux du modèle classique.

Proposition 0.4.7. *Pour toute matrice $F \in \mathbb{R}^{3\times 2}$, soit $E = \frac{1}{2}\left(F^tF - I\right)$, alors*

$$W_m(F) = \frac{2\lambda\mu}{\lambda + 2\mu}\,(\operatorname{tr} E)^2 + 2\mu \operatorname{tr} E^2 + O\left(|E|^3\right). \tag{24}$$

De plus, l'énergie obtenue possède la propriété de devenir infinie lorsque la normale s'annule :

$$W_m(F) \longrightarrow +\infty \quad \text{lorsque} \quad \det F^tF = |F_1 \wedge F_2|^2 \longrightarrow 0$$

Ceci est une conséquence directe du comportement du matériau de Ciarlet-Geymonat qui interdit la compression d'un volume en un point. Ce comportement empêche la formation de plis mais la fonctionnelle obtenue n'est pas pour autant semicontinue inférieure faible puisque W n'est pas rang-1-convexe. Notons qu'aucune restriction n'est imposée sur les déplacements confirmant ainsi le commentaire de Fox & al. qui expliquait les contraintes injustifiées sur leur modèle par la mauvaise approximation du matériau de Saint Venant-Kirchhoff. On montre que la relaxation du problème de minimisation associé entre dans le cadre du résultat de Ben Belgacem [8], voir Théorème 0.2.4. Enfin, le modèle hérite de l'indifférence matérielle et de l'isotropie du modèle tridimensionnel.

Le résolution du problème P_2 nous amène à imposer certaines conditions sur les données. Ces conditions s'accordent parfaitement avec notre objectif d'obtenir un modèle de flexion pure d'où leur légitimité. Pratiquement, on impose que les forces membranaires s'annulent et on restreint en conséquence les déformations admissibles à celles qui annulent cette énergie soient celles dont le premier terme du développement est une isométrie. De plus, on admet qu'il existe de telles déformations non-triviales. Enfin, on impose aux forces extérieures qu'elles soient de type flexion c.à d. à résultante nulle, afin d'éliminer tout effet membranaire. La résolution du problème P_3 nous permet d'obtenir un modèle nonlinéaire de flexion pure en ce sens que sa densité d'énergie ne dépend que de la deuxième forme fondamentale de la déformation de la surface moyenne de la plaque.

Théorème 0.4.8. *Pour des forces extérieures telles que $b^0 = 0$, $g^1 = 0$, $h^0 = 0$, $f^1 = 0$ et $\bar{f}^1 = 0$, si $\psi = (\varphi^0, \psi^1, \psi^2, \ldots)$ est une solution du problème P_3^{iso}, alors le terme dominant φ^0 dans le développement asymptotique minimise l'énergie*

$$J_b^3(\phi^0) = \int_\omega W_b(\phi^0)\, dx_H - \int_\omega \left(f^2 \cdot \phi^0 + p^1 \cdot n\right) dx_H - \int_{\omega_\sigma} \left(\overline{f}^2 \cdot \phi^0 + \overline{p}^1 \cdot n\right) ds$$

sur l'ensemble des déformations $\phi^0 : \omega \to \mathbb{R}^3$ *telles que* $\phi^0_{,\alpha} \cdot \phi^0_{,\beta} = \delta^\alpha_\beta$ *et* $\phi^0_{|\partial\omega_\phi} = \overline{\phi}_{|\partial\omega_\phi}$, *où*

$$W_b(\phi^0) = \frac{4}{3}(a+b)\left(b^2_{11}(\phi^0) + b^2_{22}(\phi^0) + 2\,b^2_{12}(\phi^0)\right) + \frac{4}{3}\frac{(a+b)(b+c)}{d}\,b^2_{\alpha\alpha}(\phi^0), \quad (25)$$

with

$$b_{\alpha\beta}(\phi^0) = -\,n(\phi^0)\cdot\phi^0_{,\alpha\beta}, \quad n(\phi^0) = \varphi^0_{,1}\wedge\varphi^0_{,2},$$

$$f^2 = \int_{-1}^{1} b^2\,d\xi + g^3_+ + g^3_-, \quad \overline{f^2} = \int_{-1}^{1} h^2\,d\xi,$$

$$p^1 = \int_{-1}^{1} \xi b^1\,d\xi + g^2_+ + g^2_-, \quad et \quad \overline{p^1} = \int_{-1}^{1} \xi h^1\,d\xi.$$

On remarquera qu'une fois de plus l'énergie (25) généralise l'énergie de flexion classique pour un matériau de Saint Venant-Kirchhoff. En effet, pour toutes les valeurs des constantes admissibles de (23), on obtient exactement l'énergie classique (15).

Pour ce qui est de la théorie d'existence relative au problème de minimisation soulevé par le théorème 0.4.8, elle se déduit de celle menée par Ciarlet & Coutand [15] dans le cas des coques en flexion ; voir aussi Ciarlet [14] et Coutand [31].

4.3 Chapitre 3. Plaques minces membranaires non linéaires incompressibles

L'objectif de ce chapitre a été de produire un modèle de plaques minces bidimensionnel pour un matériau incompressible par une méthode de convergence. A notre connaissance, c'est la première fois qu'un tel modèle est justifié de la sorte.

Soit $(e_i)_{i=1,2,3}$ une base orthonormale de $\mathbb{R}^3$. La dérivée partielle d'une fonction ou d'un champ de vecteurs ψ par rapport au vecteur e_i est notée $\psi_{,i}$ et son gradient est noté $\nabla\psi$. Dans sa configuration de référence, la plaque occupe le domaine $\Omega^\varepsilon = \omega\times]-\varepsilon,\ \varepsilon[$, où l'intérieur de de sa surface moyenne ω est un ouvert de l'espace engendré par e_1 et e_2, et ε est le petit paramètre. La frontière de Ω^ε est partitionnée de la sorte $\partial\Omega^\varepsilon = \Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_- \cup \Gamma^\varepsilon$, où $\Gamma^\varepsilon_\pm = \omega \times \{\pm\varepsilon\}$ et $\Gamma^\varepsilon = \partial\omega \times [-\varepsilon,\ \varepsilon]$. La plaque est soumise à des forces volumiques b^ε, et à des forces surfaciques $g^\varepsilon_\pm$ sur les faces supérieure et inférieure $\Gamma^\varepsilon_\pm$ respectivement. On suppose que la plaque est constituée du même matériau isotropique, homogène et hyperélastique. Dans cette configuration la plaque subit une déformation $\varphi^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ qui devrait être un minimiseur de l'énergie $J_\varepsilon(\varphi^\varepsilon) = I_\varepsilon(\varphi^\varepsilon) - \ell_\varepsilon(\varphi^\varepsilon)$, où

$$\ell^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} b^\varepsilon\cdot\varphi^\varepsilon\,dx + \int_{\Gamma^\varepsilon_+\cup\Gamma^\varepsilon +} g^\varepsilon\cdot\varphi^\varepsilon\,dx_H,$$

$$I^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} W(\nabla\varphi^\varepsilon)\,dx,$$

sur l'espace des déformations admissibles

$$\mathcal{V}^\varepsilon = \left\{ \psi^\varepsilon \in W^{1,p}(\Omega^\varepsilon; \mathbb{R}^3) : \det \nabla \psi^\varepsilon = 1 \text{ p.p.} \quad \text{et} \quad \psi^\varepsilon_{|\Gamma^\varepsilon_\phi} = id \right\},$$

où la contrainte $\det \nabla \psi^\varepsilon = 1$ traduit l'incompressibilité du matériau. En d'autres termes tout volume élémentaire du matériau est conservé durant la déformation. La densité d'énergie interne du matériau est définie de la manière suivante :

$$W(F) = \begin{cases} W^*(F) & \text{si } \det F = 1 \\ +\infty & \text{sinon,} \end{cases}$$

où $F \in \mathbb{R}^{3\times 2}$ représente le gradient de la déformation. De plus on fait les hypothèses suivantes :

(i) $W^* : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}$ est continue,
(ii) (croissance) $\exists C > 0$ telle que $W^*(F) \leq C(1 + |F|^p)$, (H)
(iii) (coercivité) $\exists \alpha > 0$ et $\beta \in \mathbb{R}$ telles que $W^*(F) \geq \alpha |F|^p + \beta$.

Comme densité d'énergie on peut considérer celle du matériau de *Mooney-Rivlin*

$$W^*(F) = a\,|F|^2 + b\,|\text{Cof}\, F|^2 + c.$$

On peut aussi prendre l'énergie de Saint Venant-Kirchhoff ou un matériau de type Ogden (12) quelconque.

Afin de mettre en œuvre une analyse asymptotique, il est utile de se ramener à un domaine fixe. A cet effet, on opère le changement d'échelle π_ε defini par $(\pi_\varepsilon f)(x^1, x^2, x^3) = f(x^1, x^2, \varepsilon x^3)$ et pour toute fonction f^ε et toute fonctionelle G^ε, on définit $f(\varepsilon) = \pi_\varepsilon^{-1} f^\varepsilon$ et $G(\varepsilon)(\psi) = G_\varepsilon(\pi_\varepsilon^{-1}\psi)$. De plus on note $\Omega = \Omega^1$, $\Gamma = \Gamma^1$, $\Gamma_\pm = \Gamma^1_\pm$ et $x = (x_H, \xi) \in \omega \times [-1, 1]$. Ensuite on pose

$$\bar{J}(\varepsilon)(\psi(\varepsilon)) = \begin{cases} \tilde{J}(\varepsilon)(\psi(\varepsilon)) & \text{si } \psi(\varepsilon) \in \mathcal{V}(\varepsilon) \\ +\infty & \text{si } \psi \in L^p(\Omega; \mathbb{R}^3) \text{ et } \psi \notin \mathcal{V}(\varepsilon), \end{cases}$$

où

$$\mathcal{V}(\varepsilon) = \left\{ \psi(\varepsilon) \in W^{1,p}(\Omega; \mathbb{R}^3) : \det \nabla \psi(\varepsilon) = \varepsilon \text{ p.p.} \quad \text{et} \quad \psi(\varepsilon)_{|\Gamma} = (x_H, \varepsilon\xi) \right\},$$

pour éviter à avoir à utiliser la topologie faible de $W^{1,p}(\Omega; \mathbb{R}^3)$ qui est non-métrisable sur les ensembles non-bornés. Enfin, si on considère une suite de quasi-minimiseurs $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ c. à d.

$$\bar{J}(\varepsilon)(\psi(\varepsilon)) \leq \inf_{\phi \in id + W_0^{1,p}(\Omega; \mathbb{R}^3)} \bar{J}(\varepsilon)(\phi) + \varepsilon h(\varepsilon) \quad \forall \varepsilon > 0,$$

son comportement est bien décrit par la Γ-limite de la suite de fonctionnelles $(\bar{J}(\varepsilon))_{\varepsilon>0}$ par rapport à la topologie faible de $W^{1,p}(\Omega; \mathbb{R}^3)$. Plus précisément, on a le résultat suivant

Théorème 0.4.9. *$\bar{J}(\varepsilon)$ is Γ-converge vers $\bar{J}(0)$ par rapport à la topologie faible de l'espace $W^{1,p}(\Omega;\mathbb{R}^3)$ et on a*

$$\bar{J}(0)(\varphi) = \begin{cases} 2\displaystyle\int_\omega \mathbf{QR}W_0(\nabla\varphi)\,dx_H - \ell(0)(\varphi) & \text{si } \varphi \in id + W_0^{1,p}(\omega;\mathbb{R}^3) \\ +\infty & \text{sinon,} \end{cases} \tag{26}$$

où

$$W_0(F) = \inf_{z\in\mathbb{R}^3} W((F|z)), \quad F \in \mathbb{R}^{3\times 2}, \tag{27}$$

et

$$\ell(0)(\varphi) = \int_\omega \left\{ \int_{-1}^{1} b(x_H,\xi)\,d\xi + g(x_H,-1) + g(x_H,1) \right\} \cdot \varphi\,dx_H.$$

Il est clair que l'énergie définie ci-dessus est membranaire puisqu'elle ne dépend que de la première forme fondamentale de la déformation ou plus précisément du tenseur métrique $\nabla\varphi^t\nabla\varphi$. De plus, cette énergie admet des minimiseurs dans $id+W_0^{1,p}(\omega;\mathbb{R}^3)$ d'après le Théorème 0.2.4. Ceci est aussi justifié par la nature même de la fonctionnelle en tant que Γ-limite.

Le plan de la démonstration est le suivant : on procède par double inégalité. L'obtention de la borne inférieure est aisée ; c'est une conséquence des propriétés des enveloppes rang-1-convexe et quasiconvexe. La borne supérieure est beaucoup plus délicate à justifier. On commence par obtenir une borne vérifiée par les isométries.

Proposition 0.4.10. *If $\psi \in C^1(\overline{\omega};\mathbb{R}^3)$ est une immersion telle que $\psi_{|\partial\omega} = id$ et $\theta \in C(\overline{\omega};\mathbb{R}^3)$ est une fonction telle que $\theta_{|\partial\omega} = e_3$ et $\det(\psi_{,1}|\psi_{,2}|\theta) = 1$, alors on a*

$$\bar{J}(0)(\psi) \leq 2\int_\omega W\left((\psi_{,1}|\psi_{,2}|\theta)\right)dx_H - \ell(0)(\psi). \tag{28}$$

L'étape suivante consiste en la dérivation d'une autre borne supérieure obtenue pour les applications affines par morceaux et localement injectives $\psi \in W^{1,\infty}(\omega;\mathbb{R}^3)$,

$$\bar{J}(0)(\psi) \leq 2\int_\omega W_0\left(\nabla\psi\right)dx_H - \ell(0)(\psi). \tag{29}$$

La borne n'est vérifiée que par des fonctions localement injectives parce que $W_0(F) \longrightarrow 0$ lorsque $\det F \longrightarrow 0^+$ et si on perdait l'injectivité locale on ne pourrait plus utiliser l'argument de la convergence dominée. Enfin, on relaxe cette borne grâce à l'algorithme de Kohn & Strang [48] et au Théorème 0.2.2 après avoir montré que $\mathbf{R}W_0$ était partout finie, et on conclut par une série d'arguments de densité et le Théorème 0.2.2. Deux approches sont proposées, l'une faisant intervenir l'enveloppe Z ; voir (3.3).

Par ailleurs, on examine le comportement des suites quasi-minimisantes :

Proposition 0.4.11. *Soit $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ une suite diagonale de quasi-minimiseurs c. à d. qu'il existe une fonction $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ telle que $h(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$ et*

$$\bar{J}(\varepsilon)(\psi(\varepsilon)) \leq \inf_{\phi\in id+W_0^{1,p}(\Omega;\mathbb{R}^3)} \bar{J}(\varepsilon)(\phi) + h(\varepsilon) \quad \forall\, \varepsilon > 0,$$

alors ses points d'adhérence sont dans $id + W_0^{1,p}(\omega;\mathbb{R}^3)$ et sont solutions du problème de minimisation suivant

$$\bar{J}(0)(\varphi) = \inf_{\psi\in\mathcal{M}(\omega)} \bar{J}(0)(\psi).$$

Enfin, on vérifie que les propriétés constitutives du matériau sont bien conservées.

Proposition 0.4.12. *Si la fonction W^* satisfait le principe d'indifférence matérielle alors $\mathbf{QR}W_0$ aussi satisfait le principe*

$$\mathbf{QR}W_0(F) = \mathbf{QR}W_0(RF) \quad \forall R \in SO(3),\ F \in \mathbb{R}^{3\times 2}.$$

Si la fonction W^* est isotrope, alors $\mathbf{QR}W_0$ l'est aussi

$$\mathbf{QR}W_0(F) = \mathbf{QR}W_0(FR) \quad \forall R \in SO(2),\ F \in \mathbb{R}^{3\times 2}.$$

Si la fonction W^* est isotrope et vérifie le principe d'indifférence matérielle alors il existe une fonction symétrique $\phi : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ telle que

$$\mathbf{QR}W_0(F) = \phi\left(s_1(F), s_2(F)\right)$$

où $s_1(F)$ and $s_2(F)$ sont les valeurs singulières de la matrice $F \in \mathbb{R}^{3\times 2}$. Si de plus W^* est positive et vérifie $W^*(id) = 0$, alors $\phi(x,y) = 0$ pour tout $(x,y) \in [0,1]^2$.

Les résultats obtenus dans ce chapitre sont annoncés dans Trabelsi [71]

Chapitre 1
On nonlinear membrane thin plates

Chapitre 1

On nonlinear membrane thin plates

Abstract

The classical equations of a nonlinearly membrane plate made of Saint Venant-Kirchhoff material have been justified by Fox, Raoult & Simo [40] and Pantz [65, 66]. We show that, under compression, the associated minimization problem admits no solution. The proof is based on a result of non-existence of minimizers of non-convex functionals due to Dacorogna & Marcellini [35]. We generalize the application of their result from plane elasticity to membrane plates.

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1.1 Introduction

Let $\mathbb{R}^{3\times 2}$ be the space of real 3×2 matrices endowed with the usual Euclidian norm $|F| = (\mathrm{tr}(F^T F))^{1/2}$. The classical two-dimensional stored energy function for a nonlinearly elastic plane membrane is the following :

$$\forall \xi \in \mathbb{R}^{3\times 2},\ W(\xi) = \frac{E\nu}{2(1-\nu^2)} \left(\mathrm{tr}\left(\xi^t \xi - I\right)\right)^2 + \frac{E}{2(1+\nu)} \mathrm{tr}\left(\xi^t \xi - I\right)^2, \tag{1.1}$$

where ξ stands for the two-dimensional deformation gradient, $E > 0$ is the Young modulus and $0 \leq \nu < \frac{1}{2}$ is Poisson's ratio. The above function expresses the exact difference between the metric tensor of the unknown surface and that of the reference configuration. We are interested in the associated minimization problem. Let $\omega \subset \mathbb{R}^2$ be a bounded open set. Let $\varphi : \omega \longrightarrow \mathbb{R}^3$ denote the deformation. Finally, let $f \in L^{12}(\omega;\mathbb{R}^3)$ and $\xi_0 \in \mathbb{R}^{3\times 2}$. Then the considered minimization problem is

$$(P) \qquad \inf\left\{ I(\varphi) = \int_\omega W(\nabla\varphi)\,dx : \ \varphi \in \varphi_0 + W_0^{1,4}(\omega;\mathbb{R}^3) \right\},$$

where $\varphi_0 = \xi_0 x, x \in \overline{\omega}$.

The classical two-dimensional equations of a nonlinearly elastic membrane plate, as found in the mechanical literature (see Green and Zerna [47], for instance), have been justified by Fox, Raoult and Simo [40] by means of the method of formal asymptotic expansions, introduced by Ciarlet & Destuynder [16, 17] (see also Ciarlet [13] for an extensive presentation of the different nonlinear plate theories) applied to the three-dimensional equations of the nonlinear elasticity for a Saint Venant-Kirchhoff material. A remarkable property of this nonlinear plate theory is the material frame indifference of its stored energy function (2.4.1) which it inherited from the original model.

Unfortunately, the associated energy functional $J(\varphi) = \int_\omega W(\nabla\varphi)dx$ is not weakly lower semicontinuous on $W^{1,\infty}(\omega;\mathbb{R}^3)$ which is a necessary condition for the lower semicontinuity with respect to the topology of $W^{1,4}(\omega;\mathbb{R}^3)$. Actually, Morrey [61, 62] has shown the equivalence between weakly lower semicontinuity and quasiconvexity, under some ad hoc growth and coercivity conditions, for a whole class of finite functionals depending on a gradient. See also Meyers [57] and Acerbi & Fusco [2] for refined versions of this result. In the case of problem (P), Coutand [28] gave a counter-example to the weak lower semicontinuity of its functional when the membrane is subjected to plane forces. In fact, the above stored energy function (2.4.1) is neither polyconvex nor quasiconvex since it is not rank-one-convex, as it is shown in Genevey [45] (see also Raoult [69] and Ciarlet [11] for the case of the Saint Venant-Kirchhoff stored energy function), which is a necesary condition for weak-$\star$ lower semicontinuity by Tartar [70].

Le Dret & Raoult [52] have computed the quasiconvex envelope of the three-dimensional Saint Venant-Kirchhoff stored energy function. It turns out that the latter function is not quasiconvex and consequently not weakly lower semicontinuous in light of Morrey's result as well.

For the reasons stated above, it is natural to consider the relaxed minimization problem ; see for example Dacorogna [33], in which the stored energy function is substituted by its quasiconvex envelope. It is well-known that the infimum of the associated relaxed problem

$$(QP) \qquad \inf\left\{ \bar{I}(\varphi) = \int_\omega QW(\nabla\varphi)\,dx : \ \varphi \in \varphi_0 + W_0^{1,4}(\omega;\mathbb{R}^3) \right\}$$

coincides with that of the original problem (P) and that any minimizing sequence of problem (P) contains a subsequence which weakly converges in $W^{1,4}(\omega;\mathbb{R}^3)$ towards a minimizer of problem (QP) ; see Dacorogna [33] for a complete survey of the subject.

The motivation for the computation of the quasiconvex envelope carried out by Le Dret & Raoult [52] was their justification of another nonlinear plane membrane model by means of Γ-convergence theory ; see Le Dret & Raoult [50, 53]. Their approach gives a convergence result, as the thickness of the plate tends to zero, of a diagonal infimizing sequence of deformations of the original three-dimensional energy towards a minimizer of the two-dimensional membrane energy. The latter is proved to be equal to the quasiconvex envelope of the functional obtained by minimizing the Saint Venant-Kirchhoff Stored energy function with respect to the third column vector. Besides giving the first rigorous result in the derivation of nonlinear membrane theories, the existence of a minimizer to the energy they obtain is established through the convergence. An important feature of their membrane model is that it cannot sustain any compression. That is, it can be compressed with nil stored energy which is not, intuitively, surprising that much since a membrane energy only measures the changes in the surface metric.

Coutand [27, 30, 29, 28] has managed to show some existence results of minimizers for the classical model albeit in particular instances. He proves the existence of a solution to the equations by means of the inverse function theorem before showing it actually is a local or global minimizer depending on the case. In one of the considered cases, he proves that the found solution (critical point) does not minimize the energy in any reasonable space (cf. Coutand [28]). In fact, it turns out to be the only case where the membrane is under slight compression on its boundary whereas all the other results deal with membranes extended or clamped at their boundary. In a similar vein, Dacorogna & Marcellini [35] give necessary and sufficient conditions to the existence of minimizers to the two-dimensional Saint Venant-Kirchhoff stored energy functional. These conditions draw a direct link between the existence of minimizers and the prescribed condition on the boundary whether it is a compression or an extension in absence of body loads. Their application is based on a general non-existence theorem that they establish for non-convex functionals with a linear application as the prescribed deformation on the boundary and on the computation of the quasiconvex envelope of the Saint Venant-Kirchhoff stored energy function performed by Le Dret & Raoult [52, 54]. For issues dealing with the uniqueness of the solution, we send the reader to Knops & Stuart [48].

These existence results were the incentive behind this chapter. Whereas Coutand [27, 30, 29, 28] uses an implicit method which does not bring out the link between the convexity properties of the functional and the prescribed conditions on the boundary, Dacorogna & Marcellini apply a direct method but to a less satisfying instance because of an algebraic restriction namely the lack of convenient expressions for the singular values of the deformation gradient. Here, we use the non-existence result obtained by Dacorogna & Marcellini to prove the non-existence of minimizers to the classical nonlinear membrane plate energy model under compression conditions on the boundary.

In the next section, we recall the notion of quasiconvexity and give the quasi-convex envelope of the nonlinear plate membrane stored energy function. We also recall the non-existence theory established by Dacorogna & Marcellini which yields the announced result. In section 2, we give and prove an algebraic lemma, that allows the application of this theory. We terminate the chapter with a compilation of remarks on nonlinear membrane plates in the last section. Namely we derive an upper bound for the minimum of the elastic membrane energy with dead body loads and remark that membranes do not resist compression.

1.2 Preliminaries

We recall that a Borel measurable and locally integrable function $G : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is said to be quasiconvex if

$$G(x,u,A) \leq \frac{1}{\operatorname{meas} D} \int_D G(x,u,A+\nabla\varphi)\,dx$$

for every bounded domain $D \subset \mathbb{R}^n$, for every $A \in \mathbb{R}^{n\times N}$ and for every $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^N)$. The notion of quasiconvexity was first introduced by Morrey [61]. Since, several authors have worked on the relationship between quasiconvexity and lower semicontinuity. Here, we give the result as stated in Butazzo [10] and we send the reader, for instance, to Morrey [62], Dacorogna, Acerbi & Fusco [2] and Marcellini for proofs of this result and other variations on the data.

Theorem 1.2.1. *Let $1 \leq p \leq \infty$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $G : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be a Cathéodory integrand satisfying the following estimate*

$$\begin{aligned} &0 \leq G(x,u,A) \leq a(x,|u|)\,(1+|A|^p) && \text{if} \quad p<\infty, \\ &0 \leq G(x,u,A) \leq \alpha(x,|u|,|A|) && \text{if} \quad p=\infty, \end{aligned}$$

where $a(x,s)$ and $\alpha(x,s,t)$ are summable in x and increasing in s and t. Then the following statements are equivalent :

1. *for a.e. $x \in \Omega$ and every $u \in \mathbb{R}^N$, the function $G(x,u,.)$ is quasiconvex ;*
2. *the functional $F : u \in \mathbb{R}^N \to \int_\Omega G(x,u,\nabla u)\,dx$ is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega;\mathbb{R}^N)$ (sequentially weakly-$\star$ lower semicontinuous on $W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ if $p=\infty$).*

Remark 1.2.2. Ball & Murat [5] introduced the notion of $W^{1,p}$-quasiconvexity in an attempt to weaken the notion of quasiconvexity of Morrey (which corresponds in their terminology corresponds to the $W^{1,\infty}$-quasiconvexity) and thus draw a direct link between the $W^{1,p}$-quasiconvexity of a stored energy function and the sequential weak lower semicontinuity with respect to the topology of

$W^{1,p}$. In particular, they show that in certain cases $W^{1,p}$-quasiconvexity and $W^{1,\infty}$-quasiconvexity are equivalent. However, the above theorem clearly shows that for finite functionals quasiconvexity lies behind sequential weak lower semicontinuity in Sobolev spaces.

Next we give the definitions of the convex and quasiconvex envelopes respectively of a function $G : \mathbb{R}^{n\times N} \to \mathbb{R}^N$

$$G^{**} = \sup\{Z : \mathbb{R}^{n\times N} \to \mathbb{R}^N;\ Z \text{ convex},\ Z \leq g\},$$

$$\mathbf{Q}G = \sup\{Z : \mathbb{R}^{n\times N} \to \mathbb{R}^N;\ Z \text{ quasiconvex},\ Z \leq g\},$$

Furthermore if G is locally bounded and Borel measurable then a characterization of the quasiconvex envelope, due to Dacorogna [33], is given by

$$\mathbf{Q}G(\xi) = \inf_{\varphi \in W_0^{1,\infty}(D;\mathbb{R}^N)} I_\xi(\varphi) \tag{1.2}$$

where

$$I_\xi(\varphi) = \frac{1}{\operatorname{meas} D} \int_\omega G(\xi + \nabla\varphi)\, dx, \tag{1.3}$$

for every $\xi \in \mathbb{R}^{n\times N}$, and $D \subset \mathbb{R}^n$ a bounded domain. In particular, the infinum in identity (1.2) is independent of the choice of D.

All of the above notions as well as their properties are fully exposed in Dacorogna [33] ; see also Butazzo [10] and Giusti [46]. So we refer to these books for details, proofs and references. In the next section, we will explain how we intend to use the above theory. The quasiconvex envelope of the stored energy function (1.1) is easily deduced from the expression of the quasiconvex envelope of the Saint-Venant Kirchhoff stored energy function carried out by Le Dret & Raoult [52, 54], or can be computed in the same fashion as they have also done. We give its expression in the next proposition, but first we recall the membrane stored energy function (1.1) expressed through the singular values s_1 and s_2 (ordered such that $s_1 \leq s_2$), also known as the principal stretches, of the matrix ξ (i.e. the eigenvalues of $(\xi^t\xi)^{\frac{1}{2}}$)

$$W(\xi) = \frac{E\nu}{2(1-\nu^2)}(s_1^2 + s_2^2 - 2)^2 + \frac{E}{2(1+\nu)}\left((s_1^2 - 1)^2 + (s_1^2 - 1)^2\right). \tag{1.4}$$

Proposition 1.2.3. *The quasiconvex envelope of the nonlinear plane membrane is*

$$\mathbf{Q}W(\xi) = W^{**}(\xi) = \begin{cases} 0 & \text{if } \xi \in D_1, \\ \dfrac{E}{2}(s_2^2 - 1)^2 & \text{if } \xi \in D_2, \\ W(\xi) & \text{if } \xi \notin D_1 \cup D_2, \end{cases} \qquad \forall \xi \in \mathbb{R}^{3\times 2}, \tag{1.5}$$

where

$$\begin{aligned} &0 \leq s_1(\xi) \leq s_2(\xi), \\ &D_1 = \{\xi \in \mathbb{R}^{3\times 2} : s_2 \leq 1 \text{ and } s_1^2 + \nu s_2^2 < 1 + \nu\}, \\ &D_2 = \{\xi \in \mathbb{R}^{3\times 2} : s_2 > 1 \text{ and } s_1^2 + \nu s_2^2 < 1 + \nu\}. \end{aligned} \tag{1.6}$$

Remark 1.2.4. Inspecting the Saint Venant-Kirchhoff stored energy function expressed in terms of the singular values $0 \leq s_1 \leq s_2 \leq s_3$ of the deformation gradient $F \in \mathbb{R}^{3\times 2}$

$$\bar{W}(F) = \frac{E\nu}{8(1+\nu)(1-2\nu)}\left(\text{tr}\left(F^tF - I\right)\right)^2 + \frac{E}{8(1+\nu)}\,\text{tr}\left(F^tF - I\right)^2, \qquad (1.7)$$

it is clear that, from a mathematical viewpoint, it is not very different from the energy (1.4) we are concerned with. However Dacorogna & Marcellini don't consider the right quasiconvex envelope of the two-dimensional stored energy function for an elastic body made of Saint Venant-Kirchhoff material in their paper [35]. Indeed, they seem to have recovered the result by merely taking the greatest singular value equal to zero in the three-dimensional case. Whereas, the right guess would have been to make the smallest one vanish.

We now present the non-existence theorem we shall use to achieve our goal. This result is due to Dacorogna & Marcellini [35]. First of all, we need to introduce a notion of strict convexity that is central to the proof.

Definition 1.2.5. A convex function $H : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ is said to be strictly convex at $\xi_0 = (\xi_0^\alpha)_{1\leq\alpha\leq N} \in \mathbb{R}^{n\times N}$ in at least N directions if there exists $\lambda = (\lambda^\alpha)_{1\leq\alpha\leq N} \in \mathbb{R}^{n\times N}$ such that

$$\lambda^\alpha \neq 0 \quad and \quad \langle\lambda^\alpha;\xi^\alpha - \xi_0^\alpha\rangle_{\mathbb{R}^n} = 0 \quad \forall\, \alpha = 1,2,...,N,$$

whenever $\xi = (\xi^\alpha)_{1\leq\alpha\leq N}$ satisfies the condition

$$\frac{H(\xi) + H(\xi_0)}{2} = H\left(\frac{\xi + \xi_0}{2}\right). \qquad (1.8)$$

Theorem 1.2.6. *Let $g : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ be a lower semicontinuous function and let $\xi_0 \in \mathbb{R}^{n\times N}$ be such that*
(i) $G^{**}(\xi_0) = QG(\xi_0) < g(\xi_0)$,
(ii) G^{**} is strictly convex in at least N directions ;
then (P) has no solution.

Remark 1.2.7. As pointed out by the authors, the weakness of this theorem is that it is expressed in terms of G^{**} and not of QG. Fortunately, as already observed above, in our case the two envelopes coincide and the same goes for the three-dimensional Saint Venant-Kirchhoff stored energy function, see Le Dret & Raoult [52, 54].

1.3 Main Result

As already mentioned in the Introduction, the purpose of this paper is to draw a link between weak lower semicontinuity of the minimization functional and the boundary condition in the case without exterior energy.

Theorem 1.3.1. *Problem (P) has solutions if and only if*

$$s_1^2(\xi_0) + \nu s_2^2(\xi_0) \geq 1 + \nu.$$

Remark 1.3.2. The above result is still valid if we replace W_m in problem (P) by the following stored energy function

$$W_0(\xi) = W(\xi) + \frac{1}{4}\left[\frac{\nu}{1-\nu}\operatorname{tr}\left(\xi^t\xi - I\right) - 2\right]_+^2 \quad \forall \xi \in \mathbb{R}^{3\times 2},$$

which was obtained by Pantz in [65, 66] for a membrane plate made of Saint Venant-Kirchhoff material without imposing the orientation-preserving condition on the set of admissible three-dimensional deformations. Note that this function can also be obtained by minimizing the functional $\bar{W}$ along the third column vector as in Le Dret & Raoult [53].

In practice, the above result means that a membrane plate *behaves badly* under compression. This behaviour was already observed by Le Dret & Raoult [50, 53] in their asymptotic analysis. As a matter of fact, their two-dimensional limit model comes as a relaxed minimization problem. In another context, Coutand [7] showed that the solution he found to the local boundary-value problem of the membrane under slight compression via the implicit function theorem is not even a local minimizer. This property is further investigated in the next section.

We can now announce the algebraic lemma that allows us to generalize the result obtained by Dacorogna & Marcellini, who applied their Theorem 1.2.6 to the two-dimensional Saint Venant-Kirchhoff material, to the nonlinear plane membrane made of Saint Venant-Kirchhoff material. We'll give the proof later after proving theorem 1.3.1.

Lemma 1.3.3. *Let $\zeta_0 \in \mathbb{R}^{3\times 2}$ such that $s_1(\zeta_0) < s_2(\zeta_0)$ then there exists $\lambda = (\lambda^\alpha)_{1\leq\alpha\leq N} \in \mathbb{R}^{3\times 2}$ such that*

$$\lambda^\alpha \neq 0 \quad \textit{and} \quad \langle \lambda^\alpha, \zeta^\alpha - \zeta_0^\alpha \rangle_{\mathbb{R}^3} = 0, \quad \forall\, \alpha = 1, 2,$$

whenever $\zeta \in E = \left\{ \xi \in \mathbb{R}^{3\times 2} : \dfrac{s_2(\xi) + s_2(\xi_0)}{2} = s_2\left(\dfrac{\xi + \xi_0}{2}\right) \right\}.$

Remark 1.3.4. Note that the statement above does not say that function s_2 is strictly convex in at least two directions at ζ_0 since function s_2 is not even convex.

Proof of theorem 1.3.1. Here we follow Dacorogna & Marcellini [35]. First we assume that $s_1^2(\xi_0) + \nu s_2^2(\xi_0) \geq 1 + \nu$. By definition of quasiconvexity we write

$$|\omega|\, \mathbf{Q}W(\xi_0) \leq \int_\omega \mathbf{Q}W(\xi_0 + \nabla\varphi(x))\, dx,$$

for all $\varphi \in W^{1,\infty}(\omega; \mathbb{R}^3)$ and by a result in Ball & Murat [5], the above still holds for all $\varphi \in W^{1,4}(\omega; \mathbb{R}^3)$ by the lower semicontinuity of W. Since by defintion $\mathbf{Q}W \leq W$ on $\mathbb{R}^{3\times 2}$ and by assumption $\mathbf{Q}W(\xi_0) = W(\xi_0)$ (cf. Proposition 1.2.3), we infer

$$|\omega|\, W(\xi_0) \leq \int_\omega W(\xi_0 + \nabla\varphi(x))\, dx,$$

for all $\varphi \in W^{1,4}(\omega;\mathbb{R}^3)$ which means that the function defined by $\varphi_0(x) = \xi_0 x$ on $\bar{\omega}$ is a solution of problem (P).
Next, we assume that $s_1^2(\xi_0) + \nu s_2^2(\xi_0) < 1 + \nu$ then ξ_0 is either in D_1 or in D_2 (see (1.5) and (1.6)). We study these two cases separately.

Step 1. Assume that $\xi_0 \in D_2$. The non-existence of solutions in this case will follow from Theorem 1.2.6. Therefore, we have to prove that W^{**} is strictly convex at ξ_0 in at least two directions. Consider $\xi_0 \in D_2$ (this is possible since D_2 is open) satisfying condition (1.8) i.e.

$$\frac{W^{**}(\xi) + W^{**}(\xi_0)}{2} = W^{**}\left(\frac{\xi + \xi_0}{2}\right). \tag{1.9}$$

Consider now the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \frac{E}{2}(x^2 - 1)^2.$$

h is strictly convex as long as $x > 1$. And we can write

$$W^{**}(\xi) = h(s_2(\xi)). \tag{1.10}$$

Since $s_2(\xi) > 1$, we deduce from (1.9), (1.10) and the strict convexity of h that

$$\frac{s_2(\xi) + s_2(\xi_0)}{2} = s_2\left(\frac{\xi + \xi_0}{2}\right).$$

In other words, we have shown that

$$\xi \in E = \left\{\zeta \in \mathbb{R}^{3\times 2} :\ \frac{s_2(\zeta) + s_2(\xi_0)}{2} = s_2\left(\frac{\zeta + \xi_0}{2}\right)\right\}.$$

Now $\xi_0 \in D_2$ entails that $s_1(\xi_0) \leq 1 < s_2(\xi_0)$. Hence, we can apply Lemma 1.3.3 to infer that W^{**} is strictly convex at ξ_0 in at least two directions. Then we conclude by Theorem 1.2.6 that (P) has no solution if $\xi_0 \in D_2$.

Step 2. Assume that $\xi_0 \in D_1$. According to proposition 1.2.3, we have

$$\mathbf{Q}W(\xi_0) = W^{**}(\xi_0) = 0.$$

Assume now that (P) has a solution $\varphi \in \varphi_0 + W_0^{1,\infty}(\omega;\mathbb{R}^3)$. Then, recalling that the infima of problems (P) and (QP) coincide (cf. preliminaries), we necessarily have

$$W(\nabla\varphi) = 0 \quad \text{a.e. in } \omega.$$

From (1.4), we deduce that

$$s_1(\nabla\varphi) = s_2(\nabla\varphi) = 1 \quad \text{and} \quad \det \nabla\varphi^t\nabla\varphi = 1,$$

almost everywhere in ω. On the one hand, this implies that

$$\int_\omega \det \nabla\varphi^t\nabla\varphi \, dx = \operatorname{meas}\omega,$$

and on the other hand, the boundary data and the fact that $\varphi \in W^{1,\infty}(\omega;\mathbb{R}^3)$ yield

$$\int_\omega \det \nabla\varphi^t \nabla\varphi \, dx = \frac{1}{2}\int_{\partial\omega} \left\{(\mathrm{Cof}\, \nabla{\varphi_0}^t\nabla\varphi_0) n(x)\right\} \cdot \left\{(\nabla{\varphi_0}^t\nabla\varphi_0)x\right\} \, da. \qquad (1.11)$$

For a proof of (1.11), we refer to Ciarlet [11] (Theorem 2.7-1). Next, observing that

$$(\mathrm{Cof}\, \nabla{\varphi_0}^t\nabla\varphi_0) = \det \nabla{\varphi_0}^t\nabla\varphi_0 (\nabla{\varphi_0}^t\nabla\varphi_0)^{-1},$$

identity (1.11) entails

$$\mathrm{meas}\,\omega = \frac{1}{2} \, \det \nabla{\varphi_0}^t\nabla\varphi_0 \int_{\partial\omega} \left\{(\nabla{\varphi_0}^t\nabla\varphi_0)^{-1} n(x)\right\} \cdot \left\{(\nabla{\varphi_0}^t\nabla\varphi_0 x)\right\} da.$$

Then, by the symmetry of $\nabla{\varphi_0}^t\nabla\varphi_0$, we have

$$\mathrm{meas}\,\omega = \frac{1}{2} \, \det \nabla{\varphi_0}^t\nabla\varphi_0 \int_{\partial\omega} n(x)\cdot x \, da. \qquad (1.12)$$

Besides, Stokes formula applied to the identity vector field gives

$$\int_{\partial\omega} n(x)\cdot x\, da = \int_\omega \mathrm{div}\, x \, dx = 2\,\mathrm{meas}\,\omega,$$

and injecting the above in (1.12) we get

$$\det \nabla{\varphi_0}^t\nabla\varphi_0 = \det \xi_0^t\xi_0 = s_1(\xi_0)s_2(\xi_0) = 1.$$

And as $0 \le s_1(\xi_0) \le s_2(\xi_0) \le 1$ since $\xi_0 \in D_1$ we necessarily have

$$s_1(\xi_0) = s_2(\xi_0) = 1.$$

However this contradicts the data since $s_1(\xi_0)^2 + \nu s_2(\xi_0)^2 < 1+\nu$, when $\xi_0 \in D_1$ and the proof is complete □

Proof of lemma 1.3.3. We divide the proof in two steps.

Step1. For all matrices $\zeta \in \mathbb{R}^{3\times 2}$, let us note $v_1(\zeta)$ and $v_2(\zeta)$ the two positive eigenvalues of the symmetric matrix $\zeta^t\zeta$, $s_\alpha(\zeta) = v_\alpha^{\frac{1}{2}}(\zeta)$ for $\alpha = 1,2$ its singular values and $\{e_1, e_2\}$ an orthonormal basis of eigenvectors of $\zeta_0^t\zeta_0$ so that $\zeta_0^t\zeta_0 \; e_\alpha = v_\alpha(\zeta_0)e_\alpha$. First of all, we remark that

$$\begin{aligned} v_2(\zeta_0) &= \sup_{a_1^2+a_2^2=1} \left\{a_1^2 v_1(\zeta_0) + a_2^2 v_2(\zeta_0)\right\} \\ &= \sup_{a_1^2+a_2^2=1} \left\langle a_1e_1 + a_2e_2, (\zeta_0^t\zeta_0)(a_1e_1+a_2e_2)\right\rangle \\ &= \sup_{|u|=1} \left\langle u, (\zeta_0^t\zeta_0)u\right\rangle \\ &= \sup_{|u|=1} |\zeta_0 u|^2. \end{aligned}$$

In the same manner as above, we prove that

$$\sup_{|u|=1} |\zeta u| = s_2(\zeta) \quad \forall \zeta \in M_{3\times 2}. \tag{1.13}$$

Let us show now that

$$\zeta e_2 = \lambda \zeta_0 e_2 \quad \text{where} \quad \lambda \in \mathbb{R}. \tag{1.14}$$

Let $w \in \mathbb{R}^2$ be such that

$$|w| = 1 \quad \text{and} \quad |(\zeta + \zeta_0)w| = \sup_{|u|=1} |(\zeta + \zeta_0)u|.$$

Then

$$s_2(\zeta + \zeta_0) = \sup_{|u|=1} |(\zeta + \zeta_0)u| \leq |\zeta w| + |\zeta_0 w| \leq \sup_{|u|=1} |\zeta u| + \sup_{|u|=1} |\zeta_0 u| = s_2(\zeta) + s_2(\zeta_0).$$

Since $\zeta \in E$, the above necessarily implies that

$$|\zeta w| + |\zeta_0 w| = \sup_{|u|=1} |\zeta u| + \sup_{|u|=1} |\zeta_0 u|.$$

Next, as

$$0 \geq |\zeta w| - \sup_{|u|=1} |\zeta u| = \sup_{|u|=1} |\zeta_0 u| - |\zeta_0 w| \leq 0,$$

we draw

$$|\zeta w| = \sup_{|u|=1} |\zeta u| = s_2(\zeta) \quad \text{and} \quad |\zeta_0 w| = \sup_{|u|=1} |\zeta_0 u| = s_2(\zeta_0).$$

Recalling that e_2 is the unique vector satisfying $|e_2| = 1$ and $\zeta_0 e_2 = s_2(\zeta_0) e_2$, we deduce that $w = e_2$. We can now write that

$$s_2(\zeta_0 + \zeta) = |(\zeta + \zeta_0)e_2| \leq |\zeta e_2| + |\zeta_0 e_2| = s_2(\zeta) + s_2(\zeta_0).$$

As $\zeta \in E$, we necessarily have

$$|(\zeta + \zeta_0)e_2| = |\zeta e_2| + |\zeta_0 e_2|,$$

and it follows that

$$\exists\, \lambda \in \mathbb{R}_+ \quad \text{such that} \quad \zeta e_2 = \lambda \zeta_0 e_2.$$

We have so far shown that

$$E = \left\{ \zeta \in M_{3\times 2} : \sup_{|u|=1} |\zeta u| = |\zeta e_2| \text{ and } \exists\, \lambda \in \mathbb{R}_+ : \ \zeta e_2 = \lambda \zeta_0 e_2 \right\}.$$

Step2. First of all we remark that $\zeta_0 e_1$ and $\zeta_0 e_2$ are orthogonal since

$$\langle \zeta_0 e_1, \zeta_0 e_2 \rangle = \left\langle e_1, \zeta_0^t \zeta_0 e_2 \right\rangle = v_2(\zeta_0) \langle e_1, e_2 \rangle = 0.$$

Therefore we can set $f_\alpha = \dfrac{\zeta e_\alpha}{|\zeta e_\alpha|}$ and let $\{f_1, f_2, f_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Now recall that $|\zeta e_2| = s_2(\zeta)$ and $\zeta e_2 = \lambda\zeta_0 e_2$, so we can write

$$\zeta_0 = \begin{pmatrix} b_0 & 0 \\ 0 & s_2(\zeta_0) \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \zeta = \begin{pmatrix} p & 0 \\ 0 & s_2(\zeta) \\ q & 0 \end{pmatrix},$$

where

$$b_0 = |\zeta_0 e_1|, \quad p = \langle \zeta e_1, f_1\rangle \quad \text{and} \quad q = \langle \zeta e_1, f_3\rangle .$$

Finally,

$$\zeta - \zeta_0 = \begin{pmatrix} p - b_0 & 0 \\ 0 & s_2(\zeta) - s_2(\zeta_0) \\ q & 0 \end{pmatrix}$$

thereby choosing, for instance,

$$\lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

completes the argument. □

Remark 1.3.5. The result presented in this section was announced in Trabelsi [72].

1.4 Some remarks on nonlinear membrane plates

For completeness, in this section we compile some general remarks on nonlinear membrane models.

1.4.1 An upper bound for the minimization problem with non-vanishing external forces

In [5], Ball & Murat obtain a non-existence result similar to Theorem 1.2.6 albeit in the particular instance of a strictly positive work of external forces. Their method does not apply to the case of vanishing external loads, nor does the latter theorem apply in their instance. Moreover, neither method applies to the case of ad hoc body forces. Nevertheless, the method of Ball & Murat still gives an upper bound for the infimum of the total energy in the general case. More precisely, if we define $M_{\xi_0} = \xi_0 x + W_0^{1,4}(\omega; \mathbb{R}^3)$, we have

Proposition 1.4.1. *Let $f \in L^2(\omega; \mathbb{R}^3)$ and suppose $\partial\omega = 0$, then*

$$\inf_{\bar\varphi \in M_{\xi_0}} J(\bar\varphi) \le \inf_{\bar\varphi \in M_{\xi_0}} I(\bar\varphi) - \int_\omega f \cdot \xi_0 x \, dx,$$

where

$$J(\bar\varphi) = I(\bar\varphi) - \int_\omega f \cdot \bar\varphi \, dx.$$

Proof. Let $\varepsilon > 0$ and $\varphi = \xi_0 x + \psi \in M_{\xi_0}$ satisfy the following inequality

$$I(\varphi) \leq \inf_{\bar{\varphi} \in M_{\xi_0}} I(\bar{\varphi}) + \varepsilon.$$

By the Vitali covering theorem, given $n \in \mathbb{N}$ there exists a finite or countable disjoint sequence $(a_p + \varepsilon_p \overline{\omega})_{p \in A_n} \subset \omega$ where $a_p \in \mathbb{R}^2$ and $0 < \varepsilon_p \leq \dfrac{1}{n}$ such that $\text{meas}(\omega \backslash \cup_{p \in A_n} (a_p + \varepsilon_p \overline{\omega})) = 0$. Since $\partial\omega = 0$, we also have that $\sum_{p \in A_n} \varepsilon_p^2 = 1$. Now define

$$\varphi_n(x) = \begin{cases} \xi_0 x + \varepsilon_p \psi \left(\dfrac{x - a_p}{\varepsilon_p} \right) & \text{if } x \in a_p + \varepsilon_p \overline{\omega}, \\ \xi_0 x & \text{otherwise}. \end{cases}$$

Then $\varphi_n \in M_{\xi_0}$ verifies

$$J(\varphi_n) = \sum_{p \in A_n} \int_{a_p + \varepsilon_p \omega} W \left(\xi_0 + \nabla \psi \left(\frac{x - a_p}{\varepsilon_p} \right) \right) dx - \int_\omega f \cdot \varphi_n \, dx.$$

Next we perform in each of the integrals above the corresponding change of variable $y = \dfrac{x - a_p}{\varepsilon_p}$ to bring up

$$\begin{aligned} J(\varphi_n) &= \sum_{p \in A_n} \varepsilon_p^2 \int_\omega W \left(\xi_0 + \nabla \psi(y) \right) dy - \int_\omega f \cdot \xi_0 x \, dx \\ &\qquad - \sum_{p \in A_n} \varepsilon_p^3 \int_{a_p + \varepsilon_p \omega} f \cdot \psi(y) \, dy, \\ &= I(\varphi) - \int_\omega f \cdot \xi_0 x \, dx - \sum_{p \in A_n} \varepsilon_p^3 \int_{a_p + \varepsilon_p \omega} f(a_p + \varepsilon_p y) \cdot \psi(y) \, dy \end{aligned}$$

Now recalling the properties of $(\varepsilon_p)_{p \in A_n}$, the following holds

$$\left| \sum_{p \in A_n} \varepsilon_p^3 \int_{a_p + \varepsilon_p \omega} f(a_p + \varepsilon_p y) \cdot \psi(y) \, dy \right| \leq \frac{1}{n} \sum_{p \in A_n} \varepsilon_p^2 \, |f|_{L^2(\omega;\mathbb{R}^3)} \, |\psi|_{L^2(\omega;\mathbb{R}^3)}$$

Hence, by Lebesgue's dominated convergence theorem we deduce that

$$\inf_{\bar{\varphi} \in M_{\xi_0}} J(\bar{\varphi}) \leq J(\varphi_n) \leq I(\varphi) - \int_\omega f \cdot \xi_0 x \, dx \leq \inf_{\bar{\varphi} \in M_{\xi_0}} I(\bar{\varphi}) - \int_\omega f \cdot \xi_0 x \, dx + \varepsilon,$$

for all $\varepsilon > 0$ which yields the aforementioned bound. □

1.4.2 Membranes do not resist compression on the boundary

This fact was observed by Le Dret & Raoult [53] who show that the stored energy function of the membrane model they obtain by Γ-convergence arguments from finite nonlinear elasticity vanishes for deformations whose singular values are less than one - which correspond to compressive states. Here we show that any sensible membrane stored energy function verifies the latter property. First, let us recall that for all matrix $\xi \in \mathbb{R}^{3\times 2}$, we note $0 \le s_1(\xi) \le s_2(\xi)$ its singular values, that is the eigenvalues of the matrix $(\xi^t\xi)^{\frac{1}{2}}$. We also recall that a function $F : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ is rank-one-convex if

$$F(\lambda A + (1-\lambda)B) \le \lambda F(A) + (1-\lambda)F(B)$$

for all $\lambda \in [0,1]$ and all $A, B \in \mathbb{R}^{n\times m}$ such that $\mathrm{rank}\,(A-B) \le 1$.

Proposition 1.4.2. *Let $W : \mathbb{R}^{3\times 2} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ be a rank-one-convex isotropic and frame-indifferent function i.e.*

$$W(\xi) = W(\rho\xi), \quad \forall \xi \in \mathbb{R}^{3\times 2},\ \forall \rho \in SO(3).$$

Suppose furthermore that $W(\eta) = 0$ where $\eta = (\delta^i_j)_{i\le 3, j\le 2}$, then

$$W(\xi) = 0, \quad \forall \xi \in \mathbb{R}^{3\times 2} \text{ such that } s_2(\xi) \in [0,1].$$

Proof. We first show that

$$W(\xi) = 0, \quad \forall \xi \in \mathbb{R}^{3\times 2} \text{ such that } s_1(\xi)s_2(\xi) = 1. \tag{1.15}$$

Indeed, as pointed out by Le Dret & Raoult [53], any matrix $\xi \in \mathbb{R}^{3\times 2}$ admits the following *polar factorization*

$$\xi = \rho\eta(\xi^t\xi)^{\frac{1}{2}}, \tag{1.16}$$

where $\rho \in SO(3)$. Next, let $\zeta \in SO(2)$ be such that

$$\xi^t\xi = \zeta^t \mathrm{diag}\,(s^2_\alpha(\xi))\zeta.$$

Then identity (1.16) gives

$$\xi\zeta^t = \rho\eta\left(\zeta(\xi^t\xi)\zeta^t\right)^{\frac{1}{2}} = \rho\eta\,\mathrm{diag}\,(s_\alpha(\xi)).$$

Hence isotropy and frame-indifference of the membrane stored energy function W imply

$$W(\xi) = W(\xi\zeta^t) = W(\rho\eta\,\mathrm{diag}\,(s_\alpha(\xi))) = W(\eta\,\mathrm{diag}\,(s_\alpha(\xi))),$$

so that if we suppose furthermore that $s_1(\xi)s_2(\xi) = 1$ then $\mathrm{diag}\,(s_\alpha(\xi)) \in SO(2)$. This together with isotropy bring about

$$W(\xi) = W(\eta\,\mathrm{diag}\,(s_\alpha(\xi))) = W(\eta) = 0,$$

by definition of the function W and property (1.15) is proved. Now we set out to prove the announced result. For simplicity, let $\xi \in \mathbb{R}^{3\times 2}$ be such that $s_2(\xi) \le 1$

and $\xi = \mathrm{diag}\,(s_\alpha(\xi))$. Let $r_\alpha \in [0,1]$ be such that $s_\alpha(\xi) = -r_\alpha + (1 - r_\alpha)$ and note $\zeta^\alpha_\beta = \big((-1)^\alpha e_1|(-1)^\beta e_2\big) \in \mathbb{R}^{3\times 2}$. Accordingly writing ξ in this fashion

$$\xi = r_1 \left[r_2\zeta^1_1 + (1-r_2)\zeta^1_2\right] + (1-r_1)\left[r_2\zeta^2_1 + (1-r_2)\zeta^2_2\right]$$

yields

$$W(\xi) \le r_1 W\left(r_2\zeta^1_1 + (1-r_2)\zeta^1_2\right) + (1-r_1)W\left(r_2\zeta^2_1 + (1-r_2)\zeta^2_2\right),$$

using the rank-one-convexity of W. Again the rank-one-convexity of W raises

$$W(\xi) \le r_1 r_2 W(\zeta^1_1) + r_1(1-r_2)W(\zeta^1_2) + (1-r_1)r_2 W(\zeta^2_1) + (1-r_1)(1-r_2)W(\zeta^2_2).$$

Lastly, since $s_1(\zeta^\alpha_\beta)s_2(\zeta^\alpha_\beta) = 1$ property (1.15) entails that $W(\zeta^\alpha_\beta) = 0$ and the result is fully justified. □

Remark 1.4.3. (i) Note that in the particular case of a finite membrane energy W, a corollory of the above is the result obtained by Le Dret & Raoult [52, 53] who show that quasiconvex finite energy membranes behave likewise. Indeed, in the finite case quasiconvexity implies rank-one-convexity and accordingly $\mathbf{QR}W = \mathbf{Q}W$. Thus, to conclude it suffices to mention that 0 is quasiconvex. This argument is false in the general case since rank-one-convexity does not imply quasiconvexity ; we send the reader to Ball & Murat [5] for a counterexample when the stored energy function is not continuous. What is more, the issue of weak lower semicontinuity related to vectorial convexity in the general case is far from being fully understood ; see Ben Belgacem [8] for a partial answer.
(ii) As was already remarked in [53], the observed phenomenon is a consequence of the reference configuration being a natural state. In fact, if the reference configuration was an extended state the membrane would tend to shrink back to a natural position and compressive deformations would accordingly be expected.

In light of the above observation, we deduce that compressive states do not realize a finite minimum of the membrane energy if an external load is applied whether the stored energy function is rank-one-convex or even polyconvex since polyconvexity implies rank-one-convexity. This phenomenon agrees with Tartar [70] who showed that for such a functional to be $W^{1,\infty}$ sequentially weakly lower semicontinuous, it has to be rank-one-convex.

Chapitre 2
Nonlinear thin plates for a family of Ogden materials

Chapitre 2

Nonlinear thin plates for a family of Ogden materials

Abstract

A new two-dimensional nonlinear membrane plate theory is derived via a formal asymptotic procedure a family of hyperelastic nonlinear materials proposed by Ciarlet & Geymonat [18] whose stored energy function is polyconvex and becomes infinite when the determinant of the deformation gradient tends to zero, and can be adjusted to arbitrary Lamé constants. The classical nonlinear bending plate model is also retrieved.

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Introduction

The problem of deriving nonlinear two-dimensional plate models from nonlinear three-dimensional elasticity meeting the fundamental requirement of continuum mechanics frame-indifference has been extensively studied since the pioneering papers of Ciarlet & Destuynder [16, 17], where the method of formal asymptotic expansions in powers of the thickness was cast in a modern functional framework. Since, the method has been widely used to justify a multitude of plate models such as those of Koiter and Naghdi, as well as a variety of shell models. Meanwhile, the procedure itself has undergone numerous developments culminating in the most refined framework set up by Fox, Raoult & Simo in [40], where minimal assumptions are made on the data to produce a whole hierarchy of nonlinear two-dimensional models depending on the order of magnitude of the loads for a Saint Venant-Kirchhoff material. The first two models obtained are what is referred to in the mechanical litterature, see for instance Green & Zerna [47], as the classical nonlinear membrane plate model and the classical nonlinear inextensional bending plate model. Both models are frame-indifferent which is not true for the models raised for smaller loads such as the von Kármán plate model previously justified by Ciarlet [12]. For an almost exhaustive survey of these models and related litterature, we send the reader to Ciarlet [13] for the plates and to Ciarlet [14] for the shells.

A rigorous proof of convergence to a nonlinear membrane model was provided in Le Dret & Raoult [50, 53] where plate-like bodies were considered with the same external loads assumptions as in Fox, Raoult & Simo [40]. Futhermore, their result applies to a general hyperelastic material. Their method relies on Γ-convergence arguments and was inspired by the paper of Acerbi, Buttazzo & Percivale [1] dealing with nonlinearly elastic strings. Let us emphasize the fact that in the latter case, the limit model being one-dimensional, convexity arguments could have been used whereas in the two-dimensional case, convexity is not sufficient. Therefore, the asymptotic analysis performed by Le Dret & Raoult is far from being a mere direct generalization of the one-dimensional case. Another advantage of this asymptotic justification besides being a rigorous convergence result is that the minimization problem associated to the limit two-dimensional problem is relaxed thus providing solutions which is not the case for the classical nonlinear membrane plate model. As a matter of fact, local minimizers are obtained in various instances of such membranes under tension by Coutand [30] and a non-existence result for the nonlinear membrane under compression is provided in Trabelsi [73]. Note that, Friesecke, James & Müller [44] obtained a similar convergence result producing the classical nonlinear inextensional bending plate model. Prior to this paper, an attempt by Pantz [67, 68] to produce this result made a partial justification of the convergence.

Another difference between the two approaches described above is that Fox, Raoult & Simo [40] impose the orientation-preserving condition which is expressed by the strict positivity of the determinant of the deformation gradient. Due to technical reasons, this physical requirement was dropped by Le Dret & Raoult [**?**]. In the former case, this condition resulted in a restriction on the set of admissible two-dimensional deformations ; see section 4. Hence, the attempt of Pantz [65, 66], to show that if the orientation-preserving condition is dropped, one can retrieve the limit model in Le Dret & Raoult [50, 53] by means of a purely formal asymptotic procedure, albeit before relaxation. What is more, the author converts the classical method from solving a sequence of local boundary-value problems to solving the associated sequence of minimization problems which proves to be, if not more efficient, at least more agreeable since the expressions dealt with tend to be simpler as will be illustrated below. In addition, instead of choosing particular test functions to solve the problems in the classical method, we minimize numerical functions defined on $\mathbb{R}^3$ which is obviously more natural and therefore more legitimate.

Our motivation for this work was to produce two-dimensional nonlinear plate models for a more realistic hyperelastic material that agrees with all the fundamental physical requirements such as frame-indifference, the behaviour for large deformations and the behaviour for small deformations, while exhibiting good mathematical properties, mainly ensuring the existence of solutions to the minimization problem describing the state of equilibrium of an elastic body made of such a material. It turns out that a certain family of Ogden materials proposed by Ciarlet & Geymonat [18] and defined through its stored energy function, referred to here as the Ciarlet-Geymonat stored energy function, fulfills all of the above prerequisites. Our initial hope was that the eventual membrane model would present good mathematical properties that would allow for the existence of minimizers as the classical nonlinear membrane plate model inherited the bad behaviour of the Saint Venant-Kirchhoff material. Unfortunately, the associated minimization problem has to be relaxed yet.

The outline of of this paper reads as follows. Section 2 introduces the three-dimensional problem. We consider a family of plates of thickness 2ε and midsurface a bounded open subset of $\mathbb{R}^2$. All of these plates are assumed to be made of the same homogenous and isotropic hyperelastic material whose stored energy function was given by Ciarlet & Geymonat. The state of equilibrium of the family of plates is defined as a family of energy minimization problems over a set of admissible deformations smooth enough for all expressions involving them to make sense and satisfying the orientation-preserving condition. In Section 3, we transpose the problems to a fixed domain no longer dependent of ε to be able to carry out an asymptotic analysis and we duly renote the data of the problem. Then, we make the Ansatz that the deformations as well as the other data of our problem admit asymptotic expansions in powers of ε which will in turn induce expansions on the energy terms resulting in the generation of a sequence of energy minimization problems associated to the different powers of ε. Finally, we expose the asymptotic procedure set up by Pantz. Section 4 is devoted to solving the problems of negative order. We show that the leading term of the expansion of the deformation does not depend on the third variable that is the thickness.

In Section 5, we obtain a two-dimensional energy minimization problem that models the state of equilibrium of a nonlinear membrane plate and whose solution is the leading term in the expansion of the deformation. The order of this energy with respect to ε is one. We compare the result to those obtained for a Saint-Venant Kirchhoff material and show our stored energy function behaves exactly as the classical membrane energy for small deformations. Section 6 is dedicated to deriving a nonlinear inextensional bending plate model which turns out to be the classical model itself so the existence of minimizers is proved *de facto* by Coutand [27].

2.1 Formulation of the three-dimensional problem

A plate is an elastic body whose natural reference configuration is a cylinder whose height, also called thickness, is small compared to the other two dimensions. We denote by ω the interior mid-surface (here, a plane) parallel to the base of the cylinder. Let $(e_i)_{i=1,2,3}$ be an orthonormal basis of $\mathbb{R}^3$ so that ω lies in the plane spanned by e_1 and e_2. Greek indices take on values 1 or 2, while Latin indices take on values 1, 2 or 3. Partial differentiation of a function or a vector field ψ with respect to the vector e_i is denoted $\psi_{,i}$ and its gradient (matrix) is denoted $\nabla\psi$. We adopt the classical convention of summation on repeated indices.

2.1.1 Reference configuration, loading and boundary conditions

In its reference configuration, the plate occupies the domain

$$\Omega^\varepsilon = \omega \times]-\varepsilon,\ \varepsilon[,$$

where ω is an open subset of $\mathbb{R}^2$ and ε is the small parameter (i.e. small compared to the dimensions of ω). The boundary $\partial\Omega^\varepsilon$ of Ω^ε is partitioned as follows :

$$\Gamma_+^\varepsilon = \omega \times \{\varepsilon\}, \quad \text{the top surface,}$$
$$\Gamma_-^\varepsilon = \omega \times \{-\varepsilon\}, \quad \text{the bottom surface,}$$
$$\Gamma^\varepsilon = \partial\omega \times [-\varepsilon,\ \varepsilon], \quad \text{the lateral surface.}$$

The boundary of the set ω is in turn partitioned as $\partial\omega = \partial\omega_\sigma \cup \partial\omega_\phi$, which induces a corresponding partition of Γ^ε as

$$\Gamma^\varepsilon = \Gamma_\sigma^\varepsilon \cup \Gamma_\phi^\varepsilon, \quad \text{where} \quad \Gamma_\sigma^\varepsilon = \partial\omega_\sigma \times]-\varepsilon,\ \varepsilon[\quad \text{and} \quad \Gamma_\phi^\varepsilon = \partial\omega_\phi \times]-\varepsilon,\ \varepsilon[.$$

The coordinates of a point x in the reference configuration $\overline{\Omega^\varepsilon}$ are

$$x = (x^1, x^2, x^3), \quad \text{where} \quad (x^1, x^2) = x_H \in \overline{\omega} \quad \text{and} \quad x^3 \in [-\varepsilon,\ \varepsilon].$$

The volume element is denoted $dx = dx^1dx^2dx^3$, while the area element is $dx_H = dx^1dx^2$. The plate is subjected to a dead body force $b^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$, to dead surface tractions $g^\varepsilon_\pm : \Gamma^\varepsilon_\pm \to \mathbb{R}^3$ on the top and bottom surfaces and to $h^\varepsilon : \Gamma^\varepsilon_\sigma \to \mathbb{R}^3$ on a portion of the lateral surface, while the deformation is prescribed as $\overline{\varphi^\varepsilon}$ on the remainder of the lateral surface that is Γ^ε_ϕ.
The manifold of admissible deformations is

$$\mathcal{M}^\varepsilon = \left\{\psi^\varepsilon \in \mathcal{F}(\overline{\Omega}^\varepsilon; \mathbb{R}^3) : \det \nabla\psi^\varepsilon > 0 \quad \text{and} \quad \psi^\varepsilon_{|\Gamma^\varepsilon_\phi} = \overline{\varphi^\varepsilon}_{|\Gamma^\varepsilon_\phi}\right\},$$

where $\mathcal{F}(\overline{\Omega}^\varepsilon; \mathbb{R}^3)$ denotes a space of functions smooth enough for all expressions involving elements of $\mathcal{M}^\varepsilon$ to make sense and $\det \nabla\psi^\varepsilon > 0$ merely expresses the local orientation-preserving character of the deformation.

All the plates introduced above are assumed to be made of the same homogeneous and isotropic hyperelastic material. Such a material, when subjected to the loads and boundary conditions described above, undergoes a deformation $\varphi^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ which should be a stationary point of the energy defined by

$$J_\varepsilon(\varphi^\varepsilon) = I_\varepsilon(\varphi^\varepsilon) - \ell_\varepsilon(\varphi^\varepsilon),$$

where the linear form ℓ^ε is the work of the external forces

$$\ell^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} b^\varepsilon \cdot \varphi^\varepsilon \, dx + \int_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon +} g^\varepsilon \cdot \varphi^\varepsilon \, dx_H + \int_{\Gamma^\varepsilon_\sigma} h^\varepsilon \cdot \varphi^\varepsilon \, ds,$$

and the functional I^ε measures the internal energy of the plate

$$I^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} W(\nabla\varphi^\varepsilon) \, dx.$$

W is the stored energy function of our hyperelastic material proposed by Ciarlet & Geymonat and presented in the next subsection. Hence, the state of equilibrium of the family of plates of thickness 2ε is defined as being the solution to the minimization problem

$$(P^\varepsilon) \qquad \varphi^\varepsilon \in \mathcal{M}^\varepsilon \quad \text{and} \quad J_\varepsilon(\varphi^\varepsilon) = \inf_{\psi^\varepsilon \in \mathcal{M}^\varepsilon} J_\varepsilon(\psi^\varepsilon).$$

2.1.2 The Ciarlet-Geymonat polyconvex stored energy function

We introduce $|\,.\,|$ the euclidien norm on $\mathbb{R}^{n\times m}$ and $\mathrm{adj}_{n_\wedge m} F$ the matrix of all $k \times k$ minors of the matrix $F \in \mathbb{R}^{n\times m}$, where $2 \le k \le n_\wedge m = \min(n, m)$. The definition of the Ciarlet-Geymonat polyconvex stored energy function (see Ciarlet & Geymonat [18], see also Ciarlet [11]) is given in the theorem below

Theorem 2.1.1. *Let $\lambda > 0$ and $\mu > 0$ be two given Lamé constants. The stored energy functions defined by*

$$W(F) = a\,|F|^2 + b\,|\mathrm{adj}_3\, F|^2 + \Gamma(\det F^t F) + e \quad \forall F \in \mathbb{R}^{3\times 3}, \tag{2.1}$$

where

$$a = s + \mu, \quad b = -\left(s + \frac{\mu}{2}\right),$$

$$\Gamma(\delta) = c\,\delta - d\ln\delta, \quad c = s + \frac{1}{2}\left(\frac{\lambda}{2} + \mu\right), \quad d = \frac{1}{2}\left(\frac{\lambda}{2} + \mu\right),$$

$$e = -\left(\frac{3}{2}\mu + s\right) \quad \textit{and} \quad s \in \left(-\mu, -\frac{\mu}{2}\right)$$

are polyconvex and satisfy the following

$$W(F) = \frac{\lambda}{2}\,(\operatorname{tr} E)^2 + \,\mu \operatorname{tr} E^2 + \,O(E^3) \quad \textit{where} \quad E(F) = \frac{1}{2}\,(F^tF - I), \tag{2.2}$$

$$W(F) \to +\infty \quad \textit{as} \quad \det F \to 0^+, \tag{2.3}$$

and

$$W(F) \geq A\left(|F|^2 + |\mathrm{Cof}\,F|^2 + (\det F)^2\right) + B, \quad A > 0. \tag{2.4}$$

Remark 2.1.2. (i) In the above, F stands for the deformation gradient.
(ii) By polyconvexity, we mean that W is convex as a function of the invariants of the matrix F, that is F, $\mathrm{adj}_3\, F$ and $\det F$. The class of polyconvex functions was proposed by Ball [4] as presenting good mathematical properties permitting the existence of minimizers to the associated equilibrium problem while modelling realistic materials namely Ogden materials and Mooney-Rivlin materials (in the incompressible case).
(iii) The tensor E defined in (2.2) is called the Lagrangian strain tensor or deformation tensor.
(iv) Note that relation (2.2) is the beginning of the expansion of the stored energy function of a homogenous, isotropic, hyperelastic material near a natural state. The Saint Venant-Kirchhoff matrial is the simplest one that agrees with this expansion.
(v) The requirement (2.3) imposes that an infinite amount of energy is necessary to annihilate a volume. In other words, it excludes the possibility of squashing a volume into a point. This natural physical prerequisite is not met by the Saint Vernant-Kirchhoff material nor by the family of materials studied by Le Dret & Raoult [50, 53].
(vi) Inequality (2.4) merely expresses the coercivenes of the function.

Proof of Theorem 2.1.1. The following identities hold

$$
\begin{aligned}
|F|^2 &= \operatorname{tr} F^tF = \operatorname{tr}(I+2E) = 3 + 2\operatorname{tr} E,\\
|\operatorname{adj}_3 F|^2 &= |\operatorname{Cof} F|^2 = \operatorname{tr}\operatorname{Cof} F^tF\\
&= \frac{1}{2}(\operatorname{tr} F^tF)^2 - \frac{1}{2}\operatorname{tr}(F^tF)^2\\
&= \frac{1}{2}\{\operatorname{tr}(I+2E)\}^2 - \frac{1}{2}\operatorname{tr}(I+2E)^2\\
&= 3 + 4\operatorname{tr} E + 2(\operatorname{tr} E)^2 - 2\operatorname{tr} E^2,\\
\det F^tF &= \frac{1}{6}(\operatorname{tr} F^tF)^3 - \frac{1}{2}\operatorname{tr} F^tF\operatorname{tr}(F^tF)^2 + \frac{1}{3}\operatorname{tr}(F^tF)^3\\
&= 1 + 2\operatorname{tr} E + 2(\operatorname{tr} E)^2 - 2\operatorname{tr} E^2 + O(|E|^3),\\
\Gamma(\det F^tF) &= \Gamma(1 + 2\operatorname{tr} E + 2(\operatorname{tr} E)^2 - 2\operatorname{tr} E^2 + O(|E|^3))\\
&= \Gamma(1) + \Gamma'(1)\{2\operatorname{tr} E + 2(\operatorname{tr} E)^2 - 2\operatorname{tr} E^2\}\\
&\qquad + \frac{1}{2}\Gamma''(1)(2\operatorname{tr} E)^2 + O(|E|^3)\\
&= \Gamma(1) + 2\Gamma'(1)\operatorname{tr} E + 2\{\Gamma'(1) + \Gamma''(1)\}(\operatorname{tr} E)^2\\
&\qquad - 2\Gamma'(1)\operatorname{tr} E^2 + O(|E|^3).
\end{aligned}
\tag{2.5}
$$

In order that

$$
a|F|^2 + b\,|\operatorname{adj}_3 F|^2 + \Gamma(\det F^tF) + e = \frac{\lambda}{2}(\operatorname{tr} E)^2 + \mu\operatorname{tr} E^2 + O(|E|^3),
$$

we must have

$$
\begin{cases}
3a + 3b + \Gamma(1) + e = 0,\\
2a + 4b + 2\Gamma'(1) = 0,\\
2b + 2\Gamma'(1) + 2\Gamma''(1) = \dfrac{\lambda}{2},\\
-2b - 2\Gamma'(1) = \mu.
\end{cases}
$$

where the first equation expresses the fact that $W(I) = 0$. Moreover, these equations have to be solved in such a way that

$$
a > 0, \quad b > 0, \quad \text{and} \quad \Gamma''(1) \geq 0,
$$

to verify the polyconvexity condition. Note that the strict positivity of d is necessary to ensure the requested behaviour as the determinant tends to 0. Therefore, as $\Gamma''(1) = d$, the last condition above is *de facto* verified. We rewrite the above system as follows

$$\begin{cases} 3a + 3b + c - d + e &= 0, \\ a + 2b + \Gamma'(1) &= 0, \\ b + \Gamma'(1) + d &= \dfrac{\lambda}{4}, \\ -b - \Gamma'(1) &= \dfrac{\mu}{2}. \end{cases}$$

Adding the last two equations above yields

$$d = \frac{1}{2}\left(\frac{\lambda}{2} + \mu\right).$$

Now, let $s = \Gamma'(1)$, then the last equation in the system gives

$$b = -\left(s + \frac{\mu}{2}\right),$$

and as $b > 0$, we infer that necessarily

$$s < -\frac{\mu}{2}.$$

Hence, equation two implies

$$a = s + \mu,$$

and as $a > 0$, we deduce that

$$s > -\mu.$$

Then $s = \Gamma'(1) = c - d$ entail

$$c = s + \frac{1}{2}\left(\frac{\lambda}{2} + \mu\right).$$

Finally,

$$e = d - (3a + 3b + c) = -\left(\frac{3}{2}\mu + s\right).$$

The family of stored energy functions defined by

$$W(F) = (s+\mu)|F|^2 - \left(s + \frac{\mu}{2}\right)\,|\mathrm{adj}_3\,F|^2 + \left[s + \frac{1}{2}\left(\frac{\lambda}{2} + \mu\right)\right]\det F^tF - \frac{1}{2}\left(\frac{\lambda}{2} + \mu\right)\ln\det F^tF - \left(\frac{3}{2}\mu + s\right),$$

for all $F \in \mathbb{R}^{3\times 3}$ and $s \in (-\mu, -\frac{\mu}{2})$, satisfy all the requirements stated in the theorem ; the coerciveness inequality clearly holds with $A = \min\{a, b, c\}$ and $B = e$. □

Remark 2.1.3. (i) Note that the stored energy function (2.1) is slightly different from the one proposed by Ciarlet & Geymonat in [18] :

$$\hat{W}(F) = a\,|F|^2 + b\,|\mathrm{adj}_3\,F|^2 + c\,(\det F)^2 - d\ln(\det F)^2 + e.$$

(ii) We draw the attention of the reader to a slip in the proof of the theorem in Ciarlet & Geymonat [18] where the authors claim that they find strictly positive constants a, b, c, and d such that $\hat{W}$ is polyconvex and satisfies the requirements (2.2)-(2.4). In fact, they propose $a \in \left(-\frac{\lambda}{4} + \frac{\mu}{2}, \frac{\mu}{2}\right)$ where clearly a can take negative values (for instance, for lead, aluminium and rubber $-\frac{\lambda}{4} + \frac{\mu}{2} < 0$; cf. Ciarlet [11]). However, this can be rectified as it seems that in their proof condition $a > 0$ has simply not been taken into consideration ; we send the reader to Appendix B for an alternative proof. Indeed, the result as stated in their paper remains nevertheless valid.

Remark 2.1.4. For further use, note that in the above theorem, constants a, b, c and d verify the following relations

$$a + b = \frac{\mu}{2}, \quad b + c = \frac{\lambda}{4}, \quad \text{and} \quad a + 2b + c = d. \tag{2.6}$$

2.1.3 Practical notations

To carry out the computations involved in the asymptotic analysis, we need to introduce two notations to slightly condense the expressions involved. Practically, these notations serve to make up for the lack of the explicit expression of the Lagrange strain tensor in the definition of the Ciarlet-Geymonat stored energy function. Unfortunately, given the nature of the expression of the latter energy function, no satisfying mechanically meaningful notations were found. Hence we define the 3×3 tensors $\mathcal{C}^\varepsilon(\psi^\varepsilon)$ and $\mathcal{E}^\varepsilon(\psi^\varepsilon)$ whose components are given below :

$$\mathcal{C}^\varepsilon_{ij}(\psi^\varepsilon) = \psi^\varepsilon_{,i} \cdot \psi^\varepsilon_{,j}, \tag{2.7}$$

$$\mathcal{E}^\varepsilon_{ij}(\psi^\varepsilon) = \mathcal{C}^\varepsilon_{ii}(\psi^\varepsilon)\mathcal{C}^\varepsilon_{jj}(\psi^\varepsilon) - (\mathcal{C}^\varepsilon_{ij}(\psi^\varepsilon))^2, \tag{2.8}$$

so that we can express the Ciarlet-Geymonat stored energy function (2.1) as

$$\begin{aligned} W^\epsilon(\nabla\psi^\varepsilon) = a\,\mathcal{C}^\varepsilon_{ii}(\psi^\varepsilon) + b\,(\mathcal{E}^\varepsilon_{12}(\psi^\varepsilon) + \mathcal{E}^\varepsilon_{23}(\psi^\varepsilon) + \mathcal{E}^\varepsilon_{31}(\psi^\varepsilon)) \\ + \Gamma^\epsilon(\det \nabla\psi^{\varepsilon t}\nabla\psi^\varepsilon) + e \end{aligned}$$

where

$$\begin{aligned} \det\nabla\psi^{\varepsilon t}\nabla\psi^\varepsilon = \mathcal{C}^\varepsilon_{11}(\psi^\varepsilon)\mathcal{E}^\varepsilon_{23}(\psi^\varepsilon) + 2\mathcal{C}^\varepsilon_{12}(\psi^\varepsilon)\mathcal{C}^\varepsilon_{23}(\psi^\varepsilon)\mathcal{C}^\varepsilon_{31}(\psi^\varepsilon) \\ - \mathcal{C}^\varepsilon_{33}(\psi^\varepsilon)(\mathcal{C}^\varepsilon_{12}(\psi^\varepsilon))^2 - \mathcal{C}^\varepsilon_{22}(\psi^\varepsilon)(\mathcal{C}^\varepsilon_{31}(\psi^\varepsilon))^2, \end{aligned}$$

for $\psi^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$.

Note that the newly introduced tensors are symmetrical. Also recall that we use the summation convention on repeated indices and $i, j \in \{1, 2, 3\}$.

2.2 Setting of the asymptotic procedure

2.2.1 Scaling and change of coordinates

The solutions φ^ε to the above three-dimensional problem are defined on the set $\overline{\Omega^\varepsilon}$ which varies when ε does. In order to carry out an asymptotic analysis, it is useful to operate a scaling that transposes the problem onto a fixed domain. Therefore, we define the operator π_ε by

$$(\pi_\varepsilon f)(x^1, x^2, x^3) = f(x^1, x^2, \varepsilon x^3).$$

For every function f^ε and every functional G^ε, we define

$$f(\varepsilon) = \pi_\varepsilon^{-1} f^\varepsilon,$$
$$G(\varepsilon)(\psi) = G_\varepsilon(\pi_\varepsilon^{-1}\psi).$$

We also introduce the following notations

$$\Omega = \Omega^1 = \omega \times]-1, 1[,$$

$$\begin{array}{lllll} \Gamma_\sigma & = & \Gamma_\sigma^1 & = & \partial\omega_\sigma \times]-1,1[, \\ \Gamma_+ & = & \Gamma_+^1 & = & \omega \times \{+1\}, \end{array} \qquad \begin{array}{lllll} \Gamma_\phi & = & \Gamma_\phi^1 & = & \partial\omega_\phi \times]-1,1[, \\ \Gamma_- & = & \Gamma_-^1 & = & \omega \times \{-1\}. \end{array}$$

An element in Ω has coordinates (x^1, x^2, ξ) ; differentiation with respect to ξ is denoted by the subscript $_{,\xi}$. The minimization problem equivalent to P^ε and associated with the ε-independent domain Ω^ε consists in finding solutions to the problem

$$\varphi(\varepsilon) \in \mathcal{M}(\varepsilon) \quad \text{and} \quad J(\varepsilon)(\varphi(\varepsilon)) = \inf_{\psi(\varepsilon) \in \bar{\mathcal{M}}(\varepsilon)} J(\varepsilon)(\psi(\varepsilon)),$$

where the manifold of admissible deformations is now

$$\bar{\mathcal{M}}(\varepsilon) = \left\{\psi(\varepsilon) \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^3) : \det \nabla\psi(\varepsilon) > 0 \quad \text{and} \quad \psi(\varepsilon)_{|\Gamma_\phi} = \bar{\varphi}(\varepsilon)_{|\Gamma_\phi}\right\}.$$

2.2.2 Asymptotic expansion of the configuration

Before exposing the asymptotic procedure that will be applied to the above configuration, we need to make two assumptions that will ensure the consistency of our analysis. The first one is necessary because of the presence of boundary-layer regions around the neighborhoods of the clamped parts of the plates, which are not encompassed by the formal procedure below. Therefore we modify the boundary datum by defining $\bar{\phi} \in \mathcal{F}(\bar{\omega}; \mathbb{R}^3)$ as follows

$$\bar{\phi}(\varepsilon)(x^1, x^2) = \frac{1}{2}\int_{-1}^{1} \bar{\varphi}(\varepsilon)(x^1, x^2, \xi)\, d\xi,$$

and we consider the following minimization problem

$$(P(\varepsilon)) \qquad \varphi(\varepsilon) \in \mathcal{M}(\varepsilon) \quad \text{and} \quad J(\varepsilon)(\varphi(\varepsilon)) = \inf_{\psi(\varepsilon)\in\mathcal{M}(\varepsilon)} J(\varepsilon)(\psi(\varepsilon)),$$

where the manifold of admissible deformations is

$$\mathcal{M}(\varepsilon) = \left\{ \psi(\varepsilon) \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^3) : \det \nabla\psi(\varepsilon) > 0 \quad \text{and} \quad \frac{1}{2}\int_{-1}^{1} \psi(\varepsilon)_{|\Gamma_\phi}\, d\xi = \bar{\phi}(\varepsilon)_{|\Gamma_\phi} \right\}.$$

Moreover , we assume that $\bar{\phi}(\varepsilon)$ is independent of ε :

Ansatz 1. There exists $\bar{\phi} \in \mathcal{F}(\bar{\omega}; \mathbb{R}^3)$ such that for all $\varepsilon > 0$, $\bar{\phi}(\varepsilon) = \bar{\phi}$.

Furthermore, as a solution $\varphi(\varepsilon) \in \mathcal{M}(\varepsilon)$ depends on the parameter ε, we assume that all the other data of the problem $P(\varepsilon)$ as well as the solutions $\varphi(\varepsilon)$ admit asymptotic expansions. More precisely,

Ansatz 2. The data in problem $P(\varepsilon)$ admits the following asymptotic expansion in powers of ε :

$$\begin{aligned} b(\varepsilon) &= b^0 + \varepsilon b^1 + \varepsilon^2 b^2 + ... && \text{in } \Omega, \\ g(\varepsilon) &= g^0 + \varepsilon g^1 + \varepsilon^2 g^2 + ... && \text{on } \Gamma_+ \cup \Gamma_-, \\ h(\varepsilon) &= h^0 + \varepsilon h^1 + \varepsilon^2 h^2 + ... && \text{on } \Gamma_\sigma. \end{aligned}$$

The above function $g : \Gamma_+ \cup \Gamma_- \to \mathbb{R}^3$ is obviously defined in this fashion :

$$g = \begin{cases} g_+ & \text{on } \Gamma_+, \\ g_- & \text{on } \Gamma_-. \end{cases}$$

Ansatz 3. Deformations $\varphi(\varepsilon)$ can be likewise expanded in powers of ε :

$$\varphi(\varepsilon) = \varphi^0 + \varepsilon\varphi^1 + \varepsilon^2\varphi^2 + ... \quad \text{in } \Omega.$$

For any functional $F(\varepsilon)$, we associate the following functional

$$\hat{F}(\varepsilon)(\psi^0, \psi^1, \psi^2, ...) = F(\varepsilon)(\psi^0 + \varepsilon\psi^1 + \varepsilon^2\psi^2 + ...).$$

And from now on, φ and ψ will stand for the sequences $(\varphi^i)_{i\in\mathbb{N}}$ and $(\psi^i)_{i\in\mathbb{N}}$ respectively.

2.2.3 Asymptotic expansion of the energy terms

We recall that $\mathcal{C}_{ij}(\varepsilon)$ and $\mathcal{E}_{ij}(\varepsilon)$ are defined by

$$\mathcal{C}_{ij}(\varepsilon)(\psi(\varepsilon)) = \mathcal{C}_{ij}^\varepsilon(\psi^\varepsilon) \quad \text{and} \quad \mathcal{E}_{ij}(\varepsilon)(\psi(\varepsilon)) = \mathcal{E}_{ij}^\varepsilon(\psi^\varepsilon).$$

We also recall that

$$\begin{aligned} \hat{\mathcal{C}}_{ij}(\varepsilon)(\psi) &= \mathcal{C}_{ij}(\varepsilon)(\psi^0 + \varepsilon\psi^1 + \varepsilon^2\psi^2 + ...), \\ \hat{\mathcal{E}}_{ij}(\varepsilon)(\psi) &= \mathcal{E}_{ij}(\varepsilon)(\psi^0 + \varepsilon\psi^1 + \varepsilon^2\psi^2 + ...). \end{aligned}$$

In the same obvious manner we define the 3×3 tensors $\mathcal{C}(\varepsilon)$, $\mathcal{E}(\varepsilon)$, $\hat{\mathcal{C}}(\varepsilon)$ and $\hat{\mathcal{E}}(\varepsilon)$, all of which are symmetrical.

In order to compute the asymptotic expansions of the energy terms, we need to begin by computing the asymptotic expansion of $\nabla\psi^\varepsilon$. Recall that

$$\psi(\varepsilon)(x^1, x^2, \xi) = \psi^\varepsilon(x^1, x^2, \varepsilon\xi).$$

This implies that

$$\nabla\psi^\varepsilon = \left(\psi(\varepsilon)_{,1} \mid \psi(\varepsilon)_{,2} \mid \frac{1}{\varepsilon}\,\psi(\varepsilon)_{,\xi}\right).$$

As $\psi(\varepsilon) = \sum_{n=0}^{\infty} \psi^n \varepsilon^n$, we get

$$\nabla\psi^\varepsilon = (0 \mid 0 \mid \psi^0_{,\xi})\varepsilon^{-1} + \sum_{n=0}^{\infty} (\psi^n_{,1} \mid \psi^n_{,2} \mid \psi^{n+1}_{,\xi})\varepsilon^n. \tag{2.9}$$

Lemma 2.2.1. *The tensor $\hat{\mathcal{C}}(\varepsilon)$ admits the following asymptotic expansion in powers of ε*

$$\hat{\mathcal{C}}(\varepsilon)(\psi) = \sum_{n=-2}^{\infty} \mathcal{C}^n(\psi)\varepsilon^n,$$

where the components of the tensors $\mathcal{C}^n(\psi)$ have the following expressions

$$\mathcal{C}^n_{\alpha\beta}(\psi) = \sum_{p=0}^{n} \psi^p_{,\alpha} \cdot \psi^{n-p}_{,\beta} \quad \forall n \geq 0, \qquad \mathcal{C}^n_{\alpha 3}(\psi) = \sum_{p=0}^{n+1} \psi^p_{,\alpha} \cdot \psi^{n+1-p}_{,\xi} \quad \forall n \geq -1,$$

$$\textit{and} \quad \mathcal{C}^n_{33}(\psi) = \sum_{p=0}^{n+2} \psi^p_{,\xi} \cdot \psi^{n+2-p}_{,\xi} \quad \forall n \geq -2.$$

Proof. From the expansion of the deformation gradient (2.9), we recover the following identities

$$\psi_{,\alpha}(\varepsilon) = \sum_{n=0}^{\infty} \psi^n_{,\alpha}\varepsilon^n \quad \text{and} \quad \psi_{,\xi}(\varepsilon) = \sum_{n=-1}^{\infty} \psi^{n+1}_{,\xi}\varepsilon^n.$$

Then, it suffices to factorize with respect to the powers of ε in the following products

$$\hat{\mathcal{C}}_{\alpha\beta}(\varepsilon) = \left(\sum_{n=0}^{\infty} \psi^n_{,\alpha}\varepsilon^n\right)\left(\sum_{n=0}^{\infty} \psi^n_{,\beta}\varepsilon^n\right), \quad \hat{\mathcal{C}}_{\alpha 3}(\varepsilon) = \left(\sum_{n=0}^{\infty} \psi^n_{,\alpha}\varepsilon^n\right)\left(\sum_{n=-1}^{\infty} \psi^{n+1}_{,\xi}\varepsilon^n\right)$$

and $\hat{\mathcal{C}}_{33}(\varepsilon) = \left(\sum_{n=-1}^{\infty} \psi^{n+1}_{,\xi}\varepsilon^n\right)^2$ □

Lemma 2.2.2. *The tensor $\hat{\mathcal{E}}(\varepsilon)$ admits the following asymptotic expansion in powers of ε*

$$\hat{\mathcal{E}}(\varepsilon)(\psi) = \sum_{n=-2}^{\infty} \mathcal{E}^n(\psi)\varepsilon^n,$$

where the components of the tensors $\mathcal{E}^n(\varepsilon)$ have the following expressions

$$\mathcal{E}^n_{\alpha\beta}(\psi) = \sum_{k+m+p+q=n} (\psi^k_{,\alpha} \cdot \psi^m_{,\alpha})(\psi^p_{,\beta} \cdot \psi^q_{,\beta}) - (\psi^k_{,\alpha} \cdot \psi^m_{,\beta})(\psi^p_{,\alpha} \cdot \psi^q_{,\beta}) \quad \forall n \geq 0,$$

$$\mathcal{E}^n_{\alpha 3}(\psi) = \sum_{k+m+p+q=n+2} (\psi^k_{,\alpha} \cdot \psi^m_{,\alpha})(\psi^p_{,\xi} \cdot \psi^q_{,\xi}) - (\psi^k_{,\alpha} \cdot \psi^m_{,\xi})(\psi^p_{,\alpha} \cdot \psi^q_{,\xi}) \quad \forall n \geq -2.$$

Proof. As in the precedent lemma, we write

$$\mathcal{E}^n_{\alpha\beta}(\psi) = \left|\sum_{n=0}^{\infty} \psi^n_{,\alpha}\varepsilon^n\right|^2 \left|\sum_{n=0}^{\infty} \psi^n_{,\beta}\varepsilon^n\right|^2 - \left(\sum_{n=0}^{\infty} \psi^n_{,\alpha}\varepsilon^n \cdot \sum_{n=0}^{\infty} \psi^n_{,\beta}\varepsilon^n\right)^2,$$

$$\mathcal{E}^n_{\alpha 3}(\psi) = \left|\sum_{n=0}^{\infty} \psi^n_{,\alpha}\varepsilon^n\right|^2 \left|\sum_{n=0}^{\infty} \psi^n_{,\xi}\varepsilon^n\right|^2 - \left(\sum_{n=0}^{\infty} \psi^n_{,\alpha}\varepsilon^n \cdot \sum_{n=0}^{\infty} \psi^n_{,\xi}\varepsilon^n\right)^2$$

and factorize with respect to the powers of ε to obtain the announced expressions. □

Let us introduce the following notation for the logarithm term in the expression of the Ciarlet-Geymonat stored energy function

$$\mathcal{L}(\varepsilon)(\psi(\varepsilon)) = \ln \mathcal{D}(\varepsilon)(\psi(\varepsilon)) \quad \text{where} \quad \mathcal{D}(\varepsilon)(\psi(\varepsilon)) = \det \nabla\psi^t(\varepsilon)\nabla\psi(\varepsilon).$$

Accordingly, we have

$$\hat{\mathcal{L}}(\varepsilon)(\psi) = \mathcal{L}(\varepsilon)(\psi^0 + \varepsilon\psi^1 + \varepsilon^2\psi^2 + ...) \quad \text{and} \quad \hat{\mathcal{D}}(\varepsilon)(\psi) = \mathcal{D}(\varepsilon)(\psi^0 + \varepsilon\psi^1 + \varepsilon^2\psi^2 + ...).$$

Lemma 2.2.3. *The function $\hat{\mathcal{D}}(\varepsilon)$ admits the following asymptotic expansion in powers of ε*

$$\hat{\mathcal{D}}(\varepsilon)(\psi) = \sum_{n=-2}^{\infty} \mathcal{D}^n(\psi)\varepsilon^n.$$

where the components have the following expression

$$\mathcal{D}^n(\psi) = \sum_{p+q=n} \mathcal{C}^p_{11}\mathcal{E}^q_{23} + \sum_{p+q+k=n} \left(2\mathcal{C}^p_{12}\mathcal{C}^q_{23}\mathcal{C}^k_{31} - \mathcal{C}^p_{33}\mathcal{C}^q_{12}\mathcal{C}^k_{12} - \mathcal{C}^p_{22}\mathcal{C}^q_{31}\mathcal{C}^k_{31}\right).$$

In the above formula, we do not expand the general expression of $\mathcal{D}^n(\psi)$ in terms of the derivatives of the expansion of the deformation which is obtained by injecting the expressions of the components of $\hat{\mathcal{C}}(\varepsilon)(\psi)$ and $\hat{\mathcal{E}}(\varepsilon)(\psi)$, given in Lemmas 2.2.1 and 2.2.2 respectively. In fact, the computations are very tedious and, as will be seen later, only the first few terms will be required.

Lemma 2.2.4. *The function $\hat{\mathcal{L}}(\varepsilon)$ admits the following asymptotic expansion in powers of ε*

$$\hat{\mathcal{L}}(\varepsilon)(\psi) = \sum_{n=0}^{\infty} \mathcal{L}^n(\psi)\varepsilon^n,$$

where

$$\begin{cases} \mathcal{L}^0(\psi) &= \ln\left(\varepsilon^{-2}\mathcal{D}^{-2}(\psi) + \varepsilon^{-1}\mathcal{D}^{-1}(\psi) + \mathcal{D}^0(\psi)\right), \\ \mathcal{L}^n(\psi) &= -\sum_{p=1}^{n} \frac{1}{p}\left(\frac{-1}{\sum_{q=-2}^{0}\varepsilon^q\mathcal{D}^q(\psi)}\right)^p \sum_{\sum_1^p k_i = n} \prod_1^p \mathcal{D}^{k_i}(\psi), \forall n \geq 1. \end{cases}$$

Proof. We have

$$\begin{aligned} \hat{\mathcal{L}}(\varepsilon)(\psi) &= \ln \hat{\mathcal{D}}(\varepsilon)(\psi) = \ln \sum_{n=0}^{\infty} \mathcal{D}^n(\psi)\varepsilon^n, \\ &= \ln\left\{\varepsilon^{-2}\mathcal{D}^{-2}(\psi) + \varepsilon^{-1}\mathcal{D}^{-1}(\psi) + \mathcal{D}^0(\psi) + \varepsilon\sum_{n=0}^{\infty}\mathcal{D}^{n+1}(\psi)\varepsilon^n\right\}, \\ &= \mathcal{L}^0(\psi) + \ln\left\{1 + \frac{\varepsilon}{\sum_{q=-2}^{0}\varepsilon^q\mathcal{D}^q(\psi)}\sum_{n=0}^{\infty}\mathcal{D}^{n+1}(\psi)\varepsilon^n\right\}, \\ &= \mathcal{L}^0(\psi) + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{\varepsilon}{\sum_{q=-2}^{0}\varepsilon^q\mathcal{D}^q(\psi)}\right)^n\left(\sum_{k=0}^{\infty}\mathcal{D}^{k+1}(\psi)\varepsilon^k\right)^n. \end{aligned}$$

Then

$$\hat{\mathcal{L}}(\varepsilon)(\psi) = \mathcal{L}^0(\psi) - \sum_{n=1}^{\infty}\varepsilon^n\left\{\sum_{p=1}^{n}\frac{1}{p}\left(\frac{-1}{\sum_{q=-2}^{0}\varepsilon^q\mathcal{D}^q(\psi)}\right)^p\sum_{\sum_1^p k_i=n}\prod_1^p\mathcal{D}^{k_i}(\psi)\right\}. \qquad \square$$

Lemma 2.2.5. *The functional $\hat{I}(\varepsilon)$ admits the following asymptotic expansion in powers of ε*

$$\hat{I}(\varepsilon)(\psi) = \sum_{n=-1}^{\infty} I^n(\psi)\varepsilon^n$$

Proof. The above expansion is obtained by replacing the terms $\mathcal{C}_{ij}(\psi)$ and $\mathcal{E}_{\alpha i}(\psi)$ by their respective asymptotic expansions given in Lemmas 2.2.1 and 2.2.2 and operating the change of variable $x^3 = \varepsilon\xi$ in the integral. The lowest-order terms (i.e. $\mathcal{C}_{33}(\psi)$, $\mathcal{E}_{\alpha 3}(\psi)$, $\ldots$) are of order -2 in ε. However, the change of variable multiplies all the integrals by ε, so that the lowest-order term in the expansion of the internal energy is of order -1. The general expression of the terms $I^n(\psi)$ is too long to be given here and pointless as only the first terms will be needed in what follows. We will compute these terms whenever they will be needed while solving problems P_n. $\square$

Lemma 2.2.6. *The linear form $\hat{\ell}(\varepsilon)$ admits the following asymptotic expansion in powers of ε.*

$$\hat{\ell}(\varepsilon)(\psi) = \sum_{n=0}^{\infty} \ell^n(\psi)\varepsilon^n,$$

where

$$\ell^n(\psi) = \int_\Omega \sum_{p=0}^{n-1} b^p \cdot \psi^{n-1-p}\, dx + \int_{\Gamma_+ \cup \Gamma_-} \sum_{p=0}^{n} g^p \cdot \psi^{n-p}\, dx_H + \int_{\Gamma_\sigma} \sum_{p=0}^{n-1} h^p \cdot \psi^{n-1-p}\, ds d\xi.$$

Proof. Again, this is obtained by replacing the different terms in the expression of $\hat{\ell}(\varepsilon)(\psi)$ by their respective asymptotic expansions and making the change of variable $x^3 = \varepsilon\xi$ in the integral. □

Proposition 2.2.7. *Functional $\hat{J}(\varepsilon)$ admits the following asymptotic expansion in powers of ε*

$$\hat{J}(\varepsilon)(\psi) = \sum_{n=-1}^{\infty} J^n(\psi)\varepsilon^n, \quad \textit{where} \quad J^n = I^n - \ell^n.$$

Proof. The above result is a direct consequence of the equality $\hat{J}(\varepsilon) = \hat{I}(\varepsilon) - \hat{\ell}(\varepsilon)$ and Lemmas 2.2.5 and 2.2.6. □

Finally, the local orientation-preserving condition $\det \nabla\psi(\varepsilon) > 0$ translates as follows

$$(\psi^0 + \varepsilon\psi^1 + \cdots)_{,1} \wedge (\psi^0 + \varepsilon\psi^1 + \cdots)_{,2} \cdot (\psi^0 + \varepsilon\psi^1 + \cdots)_{,\xi} > 0, \tag{2.10}$$

and should be understood for ε small enough so that the positivity of the above expression is equivalent to the positivity of the leading term in the expansion in powers of ε.

2.2.4 The asymptotic procedure

We can now break problem $P(\varepsilon)$ into a sequence of ε-independent minimization problems as follows :

Proposition 2.2.8. *The solution $\varphi(\varepsilon) = \varphi^0 + \varphi^1\varepsilon + \cdots$ to problem $P(\varepsilon)$ satisfies*

$$\varphi = \left(\varphi^0, \varphi^1, \varphi^2, \cdots\right) \in \bigcap_{n=-1}^{\infty} \mathcal{M}_n$$

where

$$\mathcal{M}_{n+1} = \left\{\psi \in \mathcal{M}_n \ : \ J^n(\psi) = \inf_{\breve{\psi} \in \mathcal{M}_n} J^n(\breve{\psi})\right\},$$

$$\mathcal{M}_{-1} = \left\{\psi \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^3)^{\mathbb{N}} : \sum_n \psi^n \varepsilon^n \in \mathcal{M}(\varepsilon)\right\}.$$

In other words, if we call P_n the problem consisting of minimizing functional J^n over the set $\mathcal{M}_n$, then solving problem $P(\varepsilon)$ is equivalent to solving the sequence of minimization problems $(P_n)_{n\in\mathbb{N}}$. The above proposition can be proved by induction ; see Appendix A, see also Pantz [65, 66].

2.3 Solving problems (P_n) for $n \leq 0$

In this section, we solve problems $(P_n)_{n\leq 0}$. These problems have the property that of being independent of the applied forces. In particular, we show that the solutions to these problems are the deformations $\psi = (\psi^0, \psi^1, \cdots)$ that satisfy

$$\psi^0_{,\xi} = 0.$$

2.3.1 Problems without exterior forces

Proposition 2.3.1. *The solutions to the lowest order problem P_{-1} are the deformations ψ in $\mathcal{M}_{-1}$ such that*

$$\psi^0_{,\xi} = 0. \tag{2.11}$$

Proof. We consider the first problem (P_{-1}) which consists of finding ψ in $\mathcal{M}_{-1}$ such that

$$J^{-1}(\psi) = \inf_{\breve{\psi}\in\mathcal{M}_{-1}} J^{-1}(\breve{\psi}).$$

where

$$J^{-1}(\psi) = I^{-1}(\psi),$$

by Proposition 2.2.7 and Lemma 2.2.6. Next,

$$I^{-1}(\psi) = \int_\Omega \Big\{ a\,\mathcal{C}_{33}^{-2} + b(\mathcal{E}_{23}^{-2} + \mathcal{E}_{31}^{-2}) + \cdots$$
$$\cdots + c\left(\mathcal{C}_{11}^{0}\mathcal{E}_{23}^{-2} + 2\,\mathcal{C}_{12}^{0}\mathcal{C}_{23}^{-1}\mathcal{C}_{31}^{-1} - \mathcal{C}_{33}^{-2}\mathcal{C}_{12}^{0\,2} - \mathcal{C}_{22}^{0}\mathcal{C}_{31}^{-1\,2}\right)\Big\}\, dx_H d\xi,$$

according to Lemma 2.2.5. Note that

$$\mathcal{C}_{11}^{0}\mathcal{E}_{23}^{-2} + 2\mathcal{C}_{12}^{0}\mathcal{C}_{23}^{-1}\mathcal{C}_{31}^{-1} - \mathcal{C}_{33}^{-2}\mathcal{C}_{12}^{0\,2} - \mathcal{C}_{22}^{0}\mathcal{C}_{31}^{-1\,2} = \det \nabla\psi^{0^t}\nabla\psi^0 \geq 0,$$

$$\mathcal{C}_{33}^{-2} = |\psi^0_\xi|^2 \geq 0, \quad \text{and} \quad \mathcal{E}_{\alpha\xi}^{-2} = |\psi^0_\alpha|^2|\psi^0_\xi|^2 - \ \psi^0_\alpha\cdot\ \psi^0_\xi \geq 0.$$

Then we necessarily have

$$\inf_{\psi\in\mathcal{M}_{-1}} J^{-1}(\psi) = 0,$$

since $J^{-1}(0) = 0$. Hence the elements ψ in $\mathcal{M}_0$ that minimize J^{-1} satisfy $J^{-1}(\psi) = 0$. In other words, they verify

$$\mathcal{C}_{33}^{-2} = \left(\mathcal{E}_{23}^{-2} + \mathcal{E}_{31}^{-2}\right) = \det \nabla\psi^{0^t}\nabla\psi^0 = 0$$

which implies the aforementioned result $\psi^0_{,\xi} = 0$. □

Now, as $\mathcal{M}_{-1} = \mathcal{M}(\varepsilon)$, we have

$$\mathcal{M}_{-1} = \left\{\psi \in \mathcal{F}(\bar{\Omega};\mathbb{R}^3)^{\mathbb{N}} : \frac{1}{2}\int_{-1}^{1} \psi^0_{|\omega_\phi}\, d\xi = \bar{\phi}_{|\omega_\phi} \quad \text{and} \quad \int_{-1}^{1} \psi^p_{|\omega_\phi}\, d\xi = 0\ \forall p \geq 1\right\}.$$

We deduce that

$$\mathcal{M}_0 = \Big\{ \psi \in \mathcal{F}(\bar{\Omega};\mathbb{R}^3)^{\mathbb{N}} : \psi^0_{,\xi} = 0, \quad \frac{1}{2}\int_{-1}^{1} \psi^0_{|\omega_\phi}\, d\xi = \bar{\phi}_{|\omega_\phi} \\ \text{and} \quad \int_{-1}^{1} \psi^p_{|\omega_\phi}\, d\xi = 0\ \forall p \geq 1 \Big\}, \\ = \Big\{ \psi \in \mathcal{F}(\bar{\Omega};\mathbb{R}^3)^{\mathbb{N}} : \exists\, \varphi^0 \in \mathcal{F}(\bar{\omega};\mathbb{R}^3) \quad \text{such that} \quad \psi^0(x_H,\xi) = \varphi^0(x_H), \\ \varphi^0_{|\omega_\phi} = \bar{\phi}_{|\omega_\phi} \quad \text{and} \quad \int_{-1}^{1} \psi^p_{|\omega_\phi}\, d\xi = 0\ \forall p \geq 1 \Big\}.$$

2.3.2 Solving problem P_0

A direct consequence of the above result is

Corollary 2.3.2. *$\mathcal{C}^{-2}_{i3} = \mathcal{C}^{-1}_{i3} = \mathcal{E}^{-2}_{\alpha 3} = \mathcal{E}^{-1}_{\alpha 3} = 0$ for all $\psi \in \mathcal{M}_0$, and the asymptotic expansion of the logarithm term reduces to*

$$\hat{\mathcal{L}}(\varepsilon)(\psi) = \sum_{n=0}^{\infty} \mathcal{L}^n(\psi)\varepsilon^n,$$

where

$$\begin{cases} \mathcal{L}^0(\psi) &=& \ln\left(\mathcal{D}^0(\psi)\right), \\ \mathcal{L}^n(\psi) &=& -\displaystyle\sum_{p=1}^{n} \frac{1}{p}\left(\frac{-1}{\mathcal{D}^0(\psi)}\right)^p \sum_{\sum_1^p k_i = n} \prod_1^p \mathcal{D}^{k_i}(\psi), \forall n \geq 1. \end{cases}$$

Proof. All of the above quantities have $\psi^0_{,\xi} = 0$ in factor, cf. Lemmas 2.2.1-2.2.2, therefore they trivially vanish. For the logarithm term expansion, it suffices to notice that as all negative-order terms vanish, we necessarily have $\mathcal{D}^{-2}(\psi) = \mathcal{D}^{-1}(\psi) = 0$ for all $\psi \in \mathcal{M}_0$, and the result is deduced from Lemma 2.2.4. ☐

The above corollary entails that

$$I^0(\psi) = 0 \quad \forall \psi \in \mathcal{M}_0,$$

so that

$$J^0(\psi) = -\ell^0(\psi) \quad \forall \psi \in \mathcal{M}_0.$$

Thus problem P_0 consists of finding configurations ψ in $\mathcal{M}_0$ that minimize

$$J^0(\psi) = -\int_{\Gamma_+\cup\Gamma_-} g^0 \cdot \varphi^0\, dx_H = -\int_{\Gamma_+\cup\Gamma_-} (g^0_+ + g^0_-) \cdot \varphi^0\, dx_H.$$

However, problem P_0 admits minimizers if and only if

$$g^0_+ + g^0_- = 0.$$

Henceforward, we make the stronger assumption that

$$g_+ = 0 \text{ on } \Gamma_+ \quad \text{and} \quad g_- = 0 \text{ on } \Gamma_-,$$

which implies that the leading-order terms in the loading on the top and bottom surfaces are of order 1. This assumption is physically legitimate as it means that the plate of thickness 2ε cannot endure a non-vanishing resultant surface load as the thickness ε tends to zero. Finally, once this condition is satisfied, problem P_0 becomes trivial and consequently $\mathcal{M}_1 = \mathcal{M}_0$.

Remark 2.3.3. Now that $\psi^0_{,\xi} = 0$, inequality (2.10) implies that the orientation-preserving condition $\det \nabla\psi(\varepsilon) > 0$ becomes

$$\left(\varphi^0_{,1} \wedge \varphi^0_{,2}\right) \cdot \psi^1_{,\xi} + \varepsilon \left[\left(\varphi^0_{,1} \wedge \varphi^0_{,2}\right) \cdot \psi^2_{,\xi} + \left(\varphi^0_{,1} \wedge \psi^1_{,2} + \psi^1_{,1} \wedge \varphi^0_{,2}\right) \cdot \psi^1_{,\xi}\right] + \cdots > 0.$$

In other words, we have

$$\det \nabla\psi(\varepsilon) = \left(\varphi^0_{,1} \wedge \varphi^0_{,2}\right) \cdot \psi^1_{,\xi} + O(\varepsilon).$$

Then, for ε small enough, we are left with the following requirement

$$\left(\varphi^0_{,1} \wedge \varphi^0_{,2}\right) \cdot \psi^1_{,\xi} > 0 \quad \text{in } \omega. \tag{2.12}$$

2.4 A nonlinear membrane theory

In this section we show that problem P_1 gives rise to a nonlinear membrane model. In other words, the two-dimensional stored energy function obtained below depends only on the first fundamental form of the deformed plate's midsurface.

2.4.1 A nonlinear membrane plate model

Theorem 2.4.1. *If $(\varphi^0, \psi^1, \psi^2, \ldots)$ is a solution of problem P_1, then the leading term φ^0 in the asymptotic expansion minimizes the energy defined by*

$$J^1_m(\phi^0) = \int_\omega W_m(\nabla\phi^0)\, dx_H - \int_\omega f^0 \cdot \phi^0\, dx_H - \int_{\omega_\sigma} \bar{f}^0 \cdot \phi^0\, ds,$$

on the set of deformations $\phi^0 : \omega \to \mathbb{R}^3$ such that $\phi^0_{|\partial\omega_\phi} = \bar{\phi}_{|\partial\omega_\phi}$, where

$$W_m(F) = 2a\,|F|^2 + 2b\,\det F^tF + 2d\ln\left\{\frac{a + b\,|F|^2}{c\,\det F^tF} + 1\right\} + e',$$

$$f^0 = \int_{-1}^{1} b^0\, d\xi + g^1_+ + g^1_-, \quad \bar{f}^0 = \int_{-1}^{1} h^0\, d\xi \quad \textit{and} \quad e' = 2e + 2d\left(1 + \ln\frac{c}{d}\right).$$

For clarity, we break the proof of Theorem 2.4.1 into a series of lemmas.

Lemma 2.4.2. *If $(\phi^0, \psi^1, \psi^2, \ldots)$ belongs to $\mathcal{M}^1$, then the following holds*

$$\mathcal{C}^0_{\alpha\beta}(\psi) = \phi^0_{,\alpha} \cdot \phi^0_{,\beta}, \quad \mathcal{C}^0_{\alpha 3}(\psi) = \phi^0_{,\alpha} \cdot \psi^1_{,\xi}, \quad \mathcal{C}^0_{33}(\psi) = |\psi^1_{,\xi}|^2,$$

$$\mathcal{E}^0_{12}(\psi) = |\phi^0_{,1}|^2|\phi^0_{,2}|^2 - \mathcal{C}^0_{\alpha\beta}(\psi)^2,$$
$$\mathcal{E}^0_{\alpha 3}(\psi) = |\phi^0_{,\alpha}|^2|\psi^1_{,\xi}|^2 - \mathcal{C}^0_{\alpha 3}(\psi)^2.$$

and

$$\mathcal{D}^0(\psi) = \mathcal{C}^0_{11}\mathcal{E}^0_{23} + 2\,\mathcal{C}^0_{12}\mathcal{C}^0_{23}\mathcal{C}^0_{31} - \mathcal{C}^0_{33}{\mathcal{C}^0_{12}}^2 - \mathcal{C}^0_{22}{\mathcal{C}^0_{31}}^2,$$

Proof. The above is a direct consequence of Lemmas 2.2.1-2.2.4, bearing in mind that $\phi^0_{,\xi} = 0$. □

Lemma 2.4.3. *If $(\varphi^0, \psi^1, \psi^2, \ldots)$ is a solution of problem P_1, then the leading term φ^0 in the asymptotic expansion minimizes the energy defined by*

$$J^1_m(\phi^0) = J^1_0(\phi^0) + \inf_{\psi \in \mathcal{M}^1_1} J^1_1(\phi^0, \psi^1),$$

on the set of deformations $\phi^0 : \omega \to \mathbb{R}^3$ such that $\phi^0_{|\partial\omega_\phi} = \bar{\phi}_{|\partial\omega_\phi}$, where

$$J^1_0(\psi) = 2\int_\omega \left\{a\,\mathcal{C}^0_{\alpha\alpha} + b\,\mathcal{E}^0_{12}\right\} dx_H - \int_\omega f_0 \cdot \phi^0\, dx_H - \int_{\partial\omega_\sigma} \bar{f}^0 \cdot \phi^0\, ds + e\,|\Omega|, \quad (2.13)$$

$$J^1_1(\psi) = 2\int_\Omega \left\{a\,\mathcal{C}^0_{33} + b\left(\mathcal{E}^0_{23} + \mathcal{E}^0_{31}\right) + c\,\mathcal{D}^0 - d\ln\mathcal{D}^0\right\} dx_H, \quad (2.14)$$

and

$$\mathcal{M}^1_1 = \left\{\psi^1 \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^3) : \int_{-1}^{1} \psi^1_{|\partial\omega_\phi}\, d\xi = 0\right\}.$$

Proof. We consider problem P_1 i.e. finding a deformation ψ such that

$$J^1(\psi) = \inf_{\breve{\psi} \in \mathcal{M}_1} J^1(\breve{\psi}).$$

First of all, we sum the total energy along the thickness. The contribution of the external forces to the energy is

$$\begin{aligned} \ell^1(\psi) &= \int_\Omega b^0 \cdot \phi^0\, dx_H d\xi + \int_{\Gamma_+ \cup \Gamma_-} g^1 \cdot \phi^0\, dx_H + \int_{\Gamma_\sigma} h^0 \cdot \phi^0\, ds d\xi, \\ &= \int_\omega f_0 \cdot \phi^0\, dx_H + \int_{\partial\omega_\sigma} \bar{f}^0 \cdot \phi^0\, ds; \end{aligned}$$

while the contribution of the internal energy is

$$I^1(\psi) = \int_\Omega \left\{a\,\mathcal{C}^0_{ii} + b\left(\mathcal{E}^0_{12} + \mathcal{E}^0_{23} + \mathcal{E}^0_{31}\right) + c\,\mathcal{D}^0 - d\ln\mathcal{D}^0\right\} dx_H d\xi + e\,|\Omega|.$$

Now, we can write

$$J^1(\psi) = I^1(\psi) + \ell^1(\psi) = J_0^1(\psi) + J_1^1(\psi),$$

where J_0^1 and J_1^1 are defined in the lemma. Observe that J_0^1 is a functional that depends only on ψ^0 and that J_1^1 depends only on ψ^0 and ψ^1. Moreover, if we let

$$J_m^1(\phi^0) = \inf_{\{(\psi^1,\psi^2,\dots):\ (\phi^0,\psi^1,\dots)\in\mathcal{M}_1\}} J^1(\phi^0,\psi^1,\dots),$$

then considering $(\varphi^0,\varphi^1,\dots)$ a solution of problem P_1, we get

$$\begin{aligned} J_m^1(\varphi^0) &= \inf_{\psi\in\mathcal{M}_1} J^1(\psi) \\ &= \inf_{\{\phi^0:\ \phi^0_{|\partial\omega_\phi}=\bar{\phi}^0\}} \left\{ \inf_{\{(\psi^1,\psi^2,\dots):\ (\phi^0,\psi^1,\dots)\in\mathcal{M}_1\}} J^1(\phi_0,\psi^1,\dots) \right\} \\ &= \inf_{\{\phi^0:\ \phi^0_{|\partial\omega_\phi}=\bar{\phi}^0\}} J_m^1(\phi^0) \end{aligned}$$

which in turn implies the announced result since

$$\begin{aligned} \mathcal{M}_1^1 &= \left\{\psi^1 \in \mathcal{F}(\bar{\Omega};\mathbb{R}^3) :\ \exists(\psi^2,\psi^3,\dots) \text{ such that } (\phi^0,\psi^1,\psi^2,\dots)\in\mathcal{M}_1\right\} \\ &= \left\{\psi^1 \in \mathcal{F}(\bar{\Omega};\mathbb{R}^3) :\ \int_{-1}^{1} \psi^1_{|\partial\omega_\phi}\, d\xi = 0\right\}. \end{aligned}$$ □

The following result is a technical lemma that will be used to compute

$$\inf_{\psi\in\mathcal{M}_1^1} J_1^1(\phi^0,\psi^1).$$

Lemma 2.4.4. *Let b, d, K, Q, $K_{\alpha\alpha}$ be strictly positive constants and let K_{12} be real. Furthermore, suppose that*

$$\frac{K K_{\alpha\alpha}}{Q} - b > 0. \tag{2.15}$$

Consider the function

$$\begin{aligned} F(t,X,Y) = 2\,(K+Q)\,t^2 + 4\,K_{12}XY - 2\,(b+K_{11})Y^2 - 2\,(b+K_{22})X^2 \\ - 2\,d\ln\left\{Q\,t^2 + 2\,K_{12}XY - K_{11}Y^2 - K_{22}X^2\right\}, \end{aligned}$$

$\forall\,(t,X,Y) \in \left\{(x,y,z)\in\mathbb{R}^+\backslash\{0\}\times\mathbb{R}^2 :\ Q\,x^2 + 2\,K_{12}yz - K_{11}z^2 - K_{22}y^2 > 0\right\}$. *Then, the unique extremum of F is the minimum $m = \left(\sqrt{d/(K+Q)}, 0, 0\right)$. Moreover, we have*

$$F(m) = 2d\left(1 - \ln\frac{d\,Q}{K+Q}\right).$$

Proof. We investigate the critical points of F that is the triplets $(t, X, Y) \in \mathbb{R}^+\backslash\{0\} \times \mathbb{R}^2$ satisfying

$$F_{,t}(t, X, Y) = F_{,X}(t, X, Y) = F_{,Y}(t, X, Y) = 0,$$

i.e.

$$\begin{cases} (K+Q)\,t - \dfrac{d\,Q\,t}{Q\,t^2 + 2\,K_{12}XY - K_{11}Y^2 - K_{22}X^2} &= 0, \\[2ex] K_{12}Y - (b+K_{22})X - \dfrac{d\,K_{12}Y - d\,K_{22}X}{Q\,t^2 + 2\,K_{12}XY - K_{11}Y^2 - K_{22}X^2} &= 0, \\[2ex] K_{12}X - (b+K_{11})Y - \dfrac{d\,K_{12}X - d\,K_{11}Y}{Q\,t^2 + 2\,K_{12}XY - K_{11}Y^2 - K_{22}X^2} &= 0. \end{cases} \tag{2.16}$$

First of all, we remark that the first equation in the system above is equivalent to

$$Q\,t^2 + 2\,K_{12}XY - K_{11}Y^2 - K_{22}X^2 = \frac{d\,Q}{K+Q}. \tag{2.17}$$

Hence, the couple (X, Y) is the solution to the following system

$$\begin{cases} K_{12}Y - (b+K_{22})X - \left(\dfrac{K}{Q}+1\right)(K_{12}Y - K_{22}X) &= 0, \\[2ex] K_{12}X - (b+K_{11})Y - \left(\dfrac{K}{Q}+1\right)(K_{12}X - K_{11}Y) &= 0, \end{cases}$$

which in turn reduces to the Kramer system

$$\begin{cases} (K K_{22} - b\,Q)\,X - K K_{12}\,Y &= 0, \\[1ex] K K_{12}\,X - (K K_{11} - b\,Q)\,Y &= 0, \end{cases} \tag{2.18}$$

whose determinant is

$$\Delta = (K K_{22} - b\,Q)\,(K K_{11} - b\,Q) - (K K_{12})^2\,.$$

Now either $\Delta = 0$ and the solution is a line whose equation is given by either of the equations (2.18) above or $\Delta \neq 0$ and the unique solution to the above system is $(X, Y) = (0, 0)$. We investigate both cases separately.

Case 1 : $\Delta = 0$. Here, two events are possible. Either K_{12} vanishes or not. If $K_{12} = 0$, system (2.18) reduces to

$$(K K_{22} - b\,Q)\,X = (K K_{11} - b\,Q)\,Y = 0,$$

which implies by condition (2.15) that $(X,Y) = (0,0)$. Hence, equation (2.17) gives

$$t^2 = \frac{d}{K+Q}.$$

Now, if $K_{12} \neq 0$, from the first equation of system (2.18), we draw

$$Y = \frac{KK_{22} - b\,Q}{KK_{12}}\,X. \tag{2.19}$$

Then, from identity (2.17), we deduce that

$$(K+Q)t^2 = d + \left(1 + \frac{K}{Q}\right)\left(K_{11}Y^2 + K_{22}X^2 - 2\,K_{12}XY\right),$$

Finally, injecting the two above identities in F, we obtain

$$\inf_E F = \inf_{\mathbb{R}} \bar{F}(X),$$

where

$$\bar{F}(X) = \left(\frac{KK_{22}}{Q} - b\right)\left[\frac{(KK_{11} - b\,Q)(KK_{22} - b\,Q)}{(KK_{12})^2} + 1\right]X^2 + 2\,d\left[1 - \ln\frac{d\,Q}{K+Q}\right].$$

To compute the minimum of $\bar{F}$, we need to recall the condition (2.15) so that the factor of X^2 is strictly positive and $\inf_{\mathbb{R}} \bar{F}(X) = \bar{F}(0) = F(m)$. Furthermore, from identities (2.19) and (2.17), we retrieve $m = \left(\sqrt{d/(K+Q)}, 0, 0\right)$.

Case 2 : $\Delta \neq 0$. In this case, the unique solution to the Kramer system (2.18) is $(X,Y) = (0,0)$, then we conclude as above. This terminates the proof of the lemma. □

Lemma 2.4.5. $\displaystyle\inf_{\psi \in \mathcal{M}_1^1} J_1^1(\phi^0,\psi^1) = 2\,d - 2\,d\ln\frac{dc\,\mathcal{E}_{12}^0}{a + b\,|\nabla\phi^0|^2 + c\,\mathcal{E}_{12}^0}.$

Proof. We substitute $\mathcal{E}_{\alpha 3}^0$ and $\mathcal{C}_{\alpha 3}^0$ by their expressions from Lemma 3.10 in identity (2.14)

$$\begin{aligned} J_1^1(\phi^0,\psi^1) = {} & 2\int_\omega \left\{\left(a + b\,\mathcal{C}_{\alpha\alpha}^0 + c\,\mathcal{E}_{12}^0\right)|\psi^1_{,\xi}|^2 + 2\,c\,\mathcal{C}_{12}^0\left(\phi^0_{,1}\cdot\psi^1_{,\xi}\right)\left(\phi^0_{,2}\cdot\psi^1_{,\xi}\right)\right\}dx \\ & - 2\int_\omega \left\{\left(b + c\,\mathcal{C}_{11}^0\right)\left(\phi^0_{,2}\cdot\psi^1_{,\xi}\right)^2 + \left(b + c\,\mathcal{C}_{22}^0\right)\left(\phi^0_{,1}\cdot\psi^1_{,\xi}\right)^2\right\}dx \\ & + 2d\int_\omega \ln\left\{\mathcal{E}_{12}^0|\psi^1_{,\xi}|^2 + 2\,\mathcal{C}_{12}^0\left(\phi^0_{,1}\cdot\psi^1_{,\xi}\right)\left(\phi^0_{,2}\cdot\psi^1_{,\xi}\right) - \dots\right. \\ & \qquad \left.\dots - \mathcal{C}_{22}^0\left(\phi^0_{,1}\cdot\psi^1_{,\xi}\right)^2 - \mathcal{C}_{11}^0\left(\phi^0_{,2}\cdot\psi^1_{,\xi}\right)^2\right\}dx. \end{aligned}$$

Then, we remark that

$$J_1^1(\phi^0, \psi^1) = \int_\omega F\left(|\psi^1_{,\xi}|, \phi^0_{,1} \cdot \psi^1_{,\xi}, \phi^0_{,2} \cdot \psi^1_{,\xi}\right) dx_H,$$

with

$$K = a + b\,\mathcal{C}^0_{\alpha\alpha}, \quad Q = c\,\mathcal{E}^0_{12}, \quad \text{and} \quad K_{\alpha\beta} = c\,\mathcal{C}^0_{\alpha\beta}, \tag{2.20}$$

provided that condition (2.15) is satisfied and $|\psi^1_{,\xi}| > 0$. Indeed,

$$KK_{11} - b\,Q = c\,\mathcal{C}^0_{11}(a + b\,\mathcal{C}^0_{\alpha\alpha}) - bc\,\mathcal{E}^0_{12} = c\,\mathcal{C}^0_{11}(a + b\,\mathcal{C}^0_{11}) + bc\left(\mathcal{C}^0_{12}\right)^2 > 0.$$

If $|\psi^1_{,\xi}| = 0$, then $\psi^1_{,\xi} = 0$ and consequently $\phi^0_{,1} \cdot \psi^1_{,\xi} = \phi^0_{,2} \cdot \psi^1_{,\xi} = 0$. However $J_1^1(\phi^0, 0) = +\infty$, therefore this cannot be a minimum of the functional J_1^1. Now, we can use the result of Lemma 2.4.4 to compute $\inf\limits_{\psi \in \mathcal{M}_1^1} J_1^1(\phi^0, \psi^1)$. More precisely,

$$\inf_{\psi \in \mathcal{M}_1^1} J_1^1(\phi^0, \psi^1) = F\left(\sqrt{d/\left(a + b\,\mathcal{C}^0_{\alpha\alpha} + c\,\mathcal{E}^0_{12}\right)}, 0, 0\right),$$

which yields the announced result and the deformations ψ that minimize the functional J_1^1 are such that

$$\begin{cases} \phi^0_{,\alpha} \cdot \psi^1_{,\xi} &= 0, \\ |\psi^1_{,\xi}| &= \eta, \end{cases} \quad \text{where} \quad \eta^2 = \frac{d}{a + b\,|\nabla\phi^0|^2 + c\,\mathcal{E}^0_{12}}. \tag{2.21}$$

Proof of Theorem 2.4.1. It suffices to use Lemma 3.11 in light of Lemma 2.4.4 to obtain the announced expression of J_1^m. □

The result of the next proposition has partially been shown in the above proof of Lemma 2.4.5.

Proposition 2.4.6. *The solutions ψ of problem P_1 are the elements of the set*

$$\begin{aligned} \mathcal{M}_2 = \Big\{ & \psi \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^3) \; : \exists\, \varphi^0, \; u \text{ and } v \in \mathcal{F}(\bar{\omega}; \mathbb{R}^3) \quad \text{such that} \\ & \psi^0(x^1, x^2, \xi) = \varphi^0(x^1, x^2) \quad \text{and} \quad J_m^1(\varphi^0) = \inf_{\phi^0 \in \mathcal{M}_1^1} J_m^1(\phi^0), \\ & \psi^1(x^1, x^2, \xi) = u(x^1, x^2) + \xi\, \eta(\nabla\varphi^0)\, n(\nabla\varphi^0), \\ & u_{|\partial\omega_\phi} = 0, \quad \varphi^0_{|\partial\omega_\phi} = \bar{\phi}^0{}_{|\partial\omega_\phi} \quad \text{and} \quad \frac{1}{2}\int_{-1}^1 \psi^p_{|\partial\omega_\phi}\, d\xi = 0 \;\forall p \geq 2 \Big\}, \end{aligned}$$

where $n(F) = \dfrac{F_1 \wedge F_2}{|F_1 \wedge F_2|}$, $\eta^2(F) = \dfrac{d}{a + b\,|F|^2 + c\,|F_1 \wedge F_2|^2}$, $F = (F_1, F_2) \in \mathbb{R}^{3\times 2}$.

Proof. Now, to be consistent, it remains to show that there exists such $\psi \in \mathcal{M}_1^1$ verifying the above conditions (2.21). Here we recall the orientation-preserving condition and more precisely inequality (2.22)

$$\left(\varphi^0_{,1} \wedge \varphi^0_{,2}\right) \cdot \psi^1_{,\xi} > 0 \quad \text{in } \omega, \tag{2.22}$$

which together with the system (2.21) fully determine ψ^1 as

$$\psi^1(x^1, x^2, \xi) = u(x^1, x^2) + \xi\, \eta\, n(x^1, x^2)$$

where

$$n(x^1, x^2) = \frac{\phi^0_{,1} \wedge \phi^0_{,2}}{|\phi^0_{,1} \wedge \phi^0_{,2}|},$$

and the proof is finished. □

Remark 2.4.7. (i) For notational brevity, the dependance on ϕ^0, whenever $\psi = (\phi^0, \psi^1, \psi^2, \cdots)$ is a deformation, will be dropped without notice.
(ii) Note that the following identities hold

$$\mathcal{E}^0_{12} = |\phi^0_{,1} \wedge \phi^0_{,2}|^2 = |\mathrm{adj}_2 \nabla\phi^0|^2 = \det \nabla\phi^{0^t} \nabla\phi^0.$$

(ii) Another convenient notation is the expression of $\mathcal{D}^0(\psi) = \mathcal{D}^0(\phi^0) = \mathcal{D}^0$, which can be retrieved from the computations involved in the proof of Theorem 2.4.1. In fact, from equation (2.17), we deduce that

$$\mathcal{D}^0 = \frac{d\, \mathcal{E}^0_{12}}{a + b\, |\nabla\phi^0|^2 + c\, \mathcal{E}^0_{12}} \tag{2.23}$$

2.4.2 The local boundary-value problem

In this section, we give the Euler-Lagrange equations satisfied by a nonlinear membrane plate in a state of equilibrium described by the total energy given in theorem 2.4.1. This is made possible by the fact that we have formally assumed the deformations to be smooth enough to carry out all the required computations.

Theorem 2.4.8. *Under the same data as in theorem 2.4.1, if $\psi = (\varphi^0, \psi^1, \psi^2, \cdots) \in \mathcal{M}_2$ is such that φ^0 is a deformation that minimizes the nonlinear membrane plate energy J^1_m, then φ^0 satisfies the following boundary-value problem*

$$\begin{cases} (M^\alpha_0)_{,\alpha} + f^0 = 0 & \textit{in } \omega, \\ M^\alpha_0 \cdot \nu_\alpha = \bar{f}^0 & \textit{on } \omega_\sigma, \end{cases} \tag{2.24}$$

where

$$M^\alpha_0 = 4(a + b\,\eta^2)\, \varphi^0_{,\alpha} + 2\left(b - \frac{a + b\,|\nabla\varphi^0|^2}{\mathcal{E}^0_{12}}\,\eta^2\right)\left\{|\nabla\varphi^0|^2 \varphi^0_{,\alpha} - (\varphi^0_{,\beta} \cdot \varphi^0_{,\alpha})\varphi^0_{,\beta}\right\}.$$

Proof. By definition, the deformation φ^0 verifies

$$J^1_m(\varphi^0 + t\varphi^*) \geq J^1_m(\varphi^0),$$

for all $t \in \mathbb{R}$ and for all $\varphi^* \in \mathcal{F}(\bar{\omega}; \mathbb{R}^3)$ such that $\varphi^*_{|\partial\omega_\phi} = 0$. However, as J^1_m is differentiable with respect to φ^0, the above inequality is equivalent to the simpler identity $DJ_m^1(\varphi^0).\varphi^* = 0$, where D denotes the differential with respect to φ^0. We start by computing the differential of the nonlinear membrane stored energy function

$$DW_m(\nabla\varphi^0).\varphi^* = 4\,a\,\varphi^0_{,\alpha}\cdot\varphi^*_{,\alpha} + 2\,b\,D\mathcal{E}^0_{12}(\varphi^0).\varphi^* - 2\,d\,\frac{D\mathcal{D}^0(\varphi^0).\varphi^*}{\mathcal{D}^0(\varphi^0)}. \tag{2.25}$$

It is obvious that in the above formula, W_m is regarded as a function of φ^0. Note that the expression of $\mathcal{D}^0$ is given in the remark 2.4.7. More precisely we have

$$\begin{aligned}
D\mathcal{E}^0_{12}(\varphi^0).\varphi^* &= (\varphi^0_{,\alpha}\cdot\varphi^0_{,\alpha})(\varphi^0_{,\beta}\cdot\varphi^*_{,\beta}) - (\varphi^0_{,\alpha}\cdot\varphi^0_{,\beta})(\varphi^0_{,\alpha}\cdot\varphi^*_{,\beta})\\
&= |\nabla\varphi^0|^2(\varphi^0_{,\beta}\cdot\varphi^*_{,\beta}) - (\varphi^0_{,\alpha}\cdot\varphi^0_{,\beta})(\varphi^0_{,\alpha}\cdot\varphi^*_{,\beta}),\\
D\mathcal{D}^0(\varphi^0).\varphi^* &= \frac{d\,D\mathcal{E}^0_{12}(\varphi^0).\varphi^*}{(a+b\,|\nabla\varphi^0|^2 + c\,\mathcal{E}^0_{12})} - d\,\mathcal{E}^0_{12}\frac{2b\,\varphi^0_{,\alpha}\cdot\varphi^*_{,\alpha} + c\,D\mathcal{E}^0_{12}(\varphi^0).\varphi^*}{(a+b\,|\nabla\varphi^0|^2 + c\,\mathcal{E}^0_{12})^2}\\
&= \left\{\frac{a+b\,|\nabla\varphi^0|^2}{\mathcal{E}^0_{12}}\,D\mathcal{E}^0_{12}(\varphi^0).\varphi^* - 2\,b\,\varphi^0_{,\alpha}\cdot\varphi^*_{,\alpha}\right\}\frac{1}{d}\,\eta^2\mathcal{D}^0(\varphi^0),
\end{aligned}$$

so that expression (2.25) becomes

$$DW_m(\nabla\varphi^0).\varphi^* = 4\,(a+b\,\eta^2)\,\varphi^0_{,\alpha}\cdot\varphi^*_{,\alpha} + 2\left(b - \frac{a+b\,|\nabla\varphi^0|^2}{\mathcal{E}^0_{12}}\,\eta^2\right)D\mathcal{E}^0_{12}(\varphi^0).\varphi^*.$$

Now integrating by parts $DJ_m^1(\varphi^0).\varphi^* = 0$ rises the following equality

$$\int_\omega \left\{(M_0^\alpha)_{,\alpha} + f^0\right\}\cdot\varphi^*\,dx_H + \int_{\omega_\sigma}\left\{\bar{f}^0 - M_0^\alpha\cdot\nu_\alpha\right\}\cdot\varphi^*\,ds = 0, \tag{2.26}$$

where

$$M_0^\alpha = 4\,(a+b\,\eta^2)\,\varphi^0_{,\alpha} + 2\left(b - \frac{a+b\,|\nabla\varphi^0|^2}{\mathcal{E}^0_{12}}\,\eta^2\right)\left\{|\nabla\varphi^0|^2\varphi^0_{,\alpha} - (\varphi^0_{,\beta}\cdot\varphi^0_{,\alpha})\varphi^0_{,\beta}\right\}.$$

The above integration by parts is licit only if φ^0 is smooth enough for $M_0^{\alpha\beta}$ to be, for instance say in $W^{1,\infty}(\omega;\mathbb{R}^3)$. Then we can deduce from (2.26) the announced result. □

2.4.3 Commentaries on the nonlinear membrane plate model

Comparison with nonlinear membrane models derived for a Saint Venant-Kirchhoff material

Before proceeding with a closer inspection of the membrane plate model obtained in this section, we recall the nonlinear membrane models derived for a Saint Venant-Kirchhoff material. In Fox, Raoult & Simo [40], the authors use a reformulation of the classical asymptotic procedure introduced in Ciarlet & Destuynder[16] to obtain a model which consists of minimizing the following total energy

$$J_{FRS}(\phi^0) = \int_\omega W_{FRS}(\nabla\phi^0)\,dx_H - \int_\omega f^0\cdot\phi^0\,dx_H - \int_{\omega_\sigma}\bar{f}^0\cdot\phi^0\,ds, \tag{2.27}$$

where

$$W_{_{FRS}}(F) = \frac{2\lambda\mu}{\lambda+2\mu}\left(\text{tr}\left(F^tF - I\right)\right)^2 + \mu\,\text{tr}\left(F^tF - I\right)^2, \tag{2.28}$$

for all matrices $F \in \mathbb{R}^{3\times 2}$, on the set of deformations $\phi^0 : \omega \to \mathbb{R}^3$ that satisfy

$$\text{tr}\left(\nabla\phi^{0t}\nabla\phi^0 - I\right) < \frac{\lambda+2\mu}{2\lambda} \quad \text{in } \omega. \tag{2.29}$$

The main difference between their asymptotic procedure and our approach is that instead of solving the sequence of minimization problems they solve the sequence of associated local boundary-value problems. As in our work they require the orientation-preserving condition $\det\nabla\psi(\varepsilon) > 0$, which is a legitimate requirement all the more considering that a Saint Venant-Kirchhoff material allows for annihilating volumes. However, apparently this latter requirement is to blame for the restrictive condition (2.29) on the set of admissible deformations as it does not allow the normal to the surface to vanish. Actually, Pantz [66, 67] uses the asymptotic procedure we use here, to derive a nonlinear membrane model for a Saint Venant-Kirchhoff material without imposing that the deformation satisfy $\det\nabla\psi(\varepsilon) > 0$ and consequently obtains a model that makes no restriction on the deformations. More precisely, the latter model consists of minimizing the following total energy

$$J^1_*(\phi^0) = \int_\omega W_0(\nabla\phi^0)\,dx_H - \int_\omega f^0\cdot\phi^0\,dx_H - \int_{\omega_\sigma}\bar{f}^0\cdot\phi^0\,ds, \tag{2.30}$$

where

$$W_0(F) = W_{_{FRS}}(F) + \frac{1}{4(\lambda+2\mu)}[2\lambda\text{tr}\left(F^tF - I\right) - (\lambda+2\mu)]^2_+, \tag{2.31}$$

for all matrices $F \in \mathbb{R}^{3\times 2}$, on the set of deformations $\phi^0 : \omega \to \mathbb{R}^3$ such that $\phi^0_{|\partial\omega_\phi} = \bar{\phi}_{|\partial\omega_\phi}$.

Le Dret and Raoult [50, 53] obtain the first nonlinear membrane plate model to be derived via a rigorous asymptotic procedure which uses Γ- convergence arguments. Actually, their result applies to a whole family of elastic materials whose respective stored energy functions satisfy some ad hoc growth and coercivity conditions. However, these materials bear the drawback of allowing for annihilating volumes. The simplest instance of such a material is the Saint Venant-Kirchhoff. In this particular case, and without making the orientation-preserving requirement, the derived model consists of minimizing the following functional

$$J_\Gamma(\phi^0) = \int_\omega \mathbf{Q}W_0(\nabla\phi^0)\,dx_H - \int_\omega f^0\cdot\phi^0\,dx_H - \int_{\omega_\sigma}\bar{f}^0\cdot\phi^0\,ds, \tag{2.32}$$

on the set of deformations $\phi^0 : \omega \to \mathbb{R}^3$ such that $\phi^0_{|\partial\omega_\phi} = \bar{\phi}_{|\partial\omega_\phi}$, where $\mathbf{Q}W_0$ is the quasiconvex envelope of W_0. Note that they derive the exact relaxation of the problem (2.30)-(2.31) obtained by Pantz [66, 67] *à posteriori*. Hence as a consequence of their convergence result, they also prove the existence of minimizers to the functional (2.32). Moreover their result gives more evidence to the fact that membrane plates do not resist compression as $QW_0(F) = 0$ if $\det \nabla F \leq 1$; see Trabelsi [72] for non-existence of minimizers to the functionals (2.27) and (2.30) above; see also Chapter 1.

Constitutive properties of W_m

In our case, the model is derived for an elastic material which is physically more realistic as it obeys the additional property that

$$W(F) \longrightarrow +\infty \quad \text{as} \quad \det \nabla F \longrightarrow 0;$$

in this fashion, the orientation-preserving condition is naturally imposed. A first consequence is that the normal to the plate in the two-dimensional model is prevented from vanishing i.e.

$$W_m(F) \to +\infty \quad \text{as} \quad |F_1 \wedge F_2| \to 0,$$

and no further restriction is made on the admissible deformations. Nevertheless, this is not enough to make for a lack of convexity that would ensure the existence of minimizers to the associated minimization problem stated in Theorem 2.4.1; this issue is discussed in the next paragraph. Besides, it inherits the material frame-indifference property satisfied by the Ciarlet-Geymonat model which states that the energy should be independent of the cartesian frame in which it is computed, namely we have

$$W_m(RF) = W_m(F) \quad \forall\, F \in \left\{\mathbb{R}^{3\times 2} : F_1 \wedge F_2 \neq 0\right\},\ \forall\, R \in SO(3). \tag{2.33}$$

The above property is usually restricted to matrices with a strictly positive determinant in the three-dimensional case; see for instance Ciarlet [11]. Here, we restrict it to what *remains* of that initial property after the asymptotic procedure was carried out. Similarly, we mention that W_m is isotropic meaning that

$$W_m(FR) = W_m(F) \quad \forall\, F \in \left\{\mathbb{R}^{3\times 2} : F_1 \wedge F_2 \neq 0\right\},\ \forall\, R \in SO(2). \tag{2.34}$$

Finally, near a natural state, the function W_m behaves like the classical stored energy function for a membrane plate. More precisely, we have

Proposition 2.4.9. *For all matrices $F \in \mathbb{R}^{3\times 2}$, let $E = \frac{1}{2}\left(F^t F - I\right)$, then*

$$W_m(F) = \frac{2\lambda\mu}{\lambda + 2\mu}\left(\operatorname{tr} E\right)^2 + 2\mu \operatorname{tr} E^2 + O\left(|E|^3\right). \tag{2.35}$$

Proof. The following identities hold

$$
\begin{aligned}
|F|^2 &= \text{tr}\, F^t F = \text{tr}(I + 2E) = 2 + 2\,\text{tr}\, E,\\
\det F^t F &= \frac{1}{6}(\text{tr}\, F^t F)^3 - \frac{1}{2}\text{tr}\, F^t F\, \text{tr}\, (F^t F)^2 + \frac{1}{3}\text{tr}\, (F^t F)^3\\
&= 1 + 2\,\text{tr}\, E + 2\,(\text{tr}\, E)^2 - 2\,\text{tr}\, E^2 + O(|E|^3),\\
\ln\det F^t F &= \ln\left(1 + 2\,\text{tr}\, E + 2\,(\text{tr}\, E)^2 - 2\,\text{tr}\, E^2 + O(|E|^3)\right),\\
&= 2\,\text{tr}\, E - 2\,\text{tr}\, E^2 + O(|E|^3),\\
\ln\left(a + b\,|F|^2 + c\,\det F^t F\right) &= \ln\big[d + 2(b+c)\text{tr}\, E + 2c\,(\text{tr}\, E)^2\\
&\qquad\qquad - 2c\,\text{tr}\, E^2 + O(|E|^3)\big],\\
&= \ln d + \frac{2(b+c)}{d}\,\text{tr}\, E + 2\left[\frac{c}{d} - \frac{(b+c)^2}{d^2}\right](\text{tr}\, E)^2\\
&\qquad\qquad - 2\,\frac{c}{d}\,\text{tr}\, E^2 + O(|E|^3).
\end{aligned}
$$

Then using the relation $d = a + 2b + c$ (cf. *Remark 2.1.4*), we obtain

$$
\begin{aligned}
W_m(F) &= a\,|F|^2 + b\,\det F^t F + 2d\,\ln\left(a + b\,|F|^2 + c\,\det F^t F\right)\\
&\qquad\qquad - 2d\,\ln\det F^t F + e'\\
&= 4a + 2b + 2d\ln d + e' + [4a + 4b + 4(b+c) - 4d]\,\text{tr}\, E^2\\
&\quad + 4(b+c)\left[1 - \frac{b+c}{d}\right](\text{tr}\, E)^2 + [-4b - 4c + 4d]\,\text{tr}\, E^2 + O(|E|^3)\\
&= 4\frac{(a+b)(b+c)}{d}(\text{tr}\, E)^2 + 4(a+b)\text{tr}\, E^2 + O(|E|^3).
\end{aligned}
$$

Finally, recalling *Remark 2.1.4*, we get the aforementioned expansion (2.35). □

Mathematical properties of W_m

A first observation is that W_m is not rank-one-convex. For instance, consider vectors e_α of the canonical base of $\mathbb{R}^3$, then we have

$$
\frac{1}{2}\,(e_1|-e_2) + \frac{1}{2}\,(e_1|\,e_2) = (e_1|\,0) \quad \text{and} \quad \text{rank}\,[(e_1|\,e_2) - (e_1|-e_2)] = 1.
$$

However,

$$
W_m(e_1|\,0) = +\infty \quad \text{and} \quad W_m(e_1|-e_2) + W_m(e_1|\,e_2) < \infty.
$$

Another way of verifying this is submitting W_m to one of the criterions of rank-one-convexity in Dacorogna, Douchet, Gangbo & Rappaz [34]. Tartar proves in [70] that the rank-one-convexity of W_m is necessary for the associated functional J^1_m to be sequentially weakly-$\star$ lower semicontinous on $W^{1,\infty}(\omega;\mathbb{R}^3)$. In fact, by the relaxation theorem provided by Ben Belgacem [8] and dealing with singular functionals defined on Sobolev spaces, we claim the following

Proposition 2.4.10. *The integral representation of the relaxed minimization problem associated to the stored energy function W_m is the following*

$$\min_{\varphi \in W^{1,p}(\omega;\mathbb{R}^3)} \bar{J}_m(\varphi) = \inf_{\psi \in W^{1,p}(\omega;\mathbb{R}^3)} J_m(\psi)$$

where

$$J_m(\varphi) = \int_\omega W_m(\nabla\varphi)\, dx - L.\varphi \quad \textit{and} \quad \bar{J}_m(\varphi) = \int_\omega \mathbf{QR}W_m(\nabla\varphi)\, dx - L.\varphi$$

and L is a linear functional.

Proof. All we have to do is check that function W_m satisfies the requirements of the relaxation result that is Theorem 0.2.4. Indeed, function $W_m : \mathbb{R}^{3\times 2} \longrightarrow \bar{\mathbb{R}}$ is continuous and satisfies $W_m(F) \longrightarrow +\infty$ if $\det F^tF \longrightarrow 0^+$. Moreover, for all $F \in \mathbb{R}^{3\times 2}$ such that $\det F^tF \geq \delta$, we have

$$\begin{aligned} W_m(F) &\leq 2a\,|F|^2 + 2b\det F^tF + 2d\left\{1 + \frac{1}{c\delta}\left(a + b\,|F|^2\right)\right\} + e', \\ &\leq C(\delta)\left(1 + |F|^2 + \det F^tF\right). \end{aligned}$$

Besides, for all $F \in \mathbb{R}^{3\times 2}$, we have

$$W_m(F) \geq e' + 2a\,|F|^2 + 2b\det F^tF.$$

Moreover, in this instance we have

$$\mathcal{D}_e(W_m) = \mathcal{D}_e(\mathrm{R}_kW_m) = \mathcal{O}_{W_m} = \left\{F \in \mathbb{R}^{3\times 2} \,:\, \operatorname{rank} F = 2\right\}, \quad k \in \mathbb{N}^*.$$

Lastly, it is easy to see that the Kohn & Strang sequence $(\mathrm{R}_kW_m)_{k\in\mathbb{N}}$ (cf. Proposition 0.2.1) is upper semicontinuous as an infimum of a family of upper semicontinuous functions ; cf. Proposition 3.1.1. □

Remark 2.4.11. For further properties of the relaxed stored energy function for a nonlinear membrane plate $\mathbf{QR}W_m$ derived in this fashion, we send the reader to Proposition 1.4.2.

2.5 Continuation of the asymptotic procedure

2.5.1 Configuration and loads for the inextensional theory

In this section, we solve problem P_2. We obtain a model without internal constraints. More precisely, we get the following result :

Theorem 2.5.1. *If $\psi = (\varphi^0, \psi^1, \psi^2, \ldots)$ is a solution of problem P_2, then it minimizes the energy J^2_* on $\mathcal{M}_2$ where*

$$J^2_*(\psi) = -\int_\omega (p^0 \cdot \eta n + f^1 \cdot \phi^0)\, dx_H - \int_{\omega_\sigma} (\bar{p^0} \cdot \eta n + \bar{f^1} \cdot \phi^0)\, ds,$$

on the set of deformations $\phi^0 : \omega \to \mathbb{R}^3$ *such that* $\phi^0_{|\partial\omega_\phi} = \bar{\phi}_{|\partial\omega_\phi}$, *and*

$$
\begin{aligned}
f^k &= \int_{-1}^{1} b^k \, d\xi + g_+^{k+1} + g_-^{k+1}, \quad \bar{f}^k = \int_{-1}^{1} h^k \, d\xi \quad \textit{for } k = 0, 1\,, \\
p^0 &= \int_{-1}^{1} \xi b^0 \, d\xi + g_+^1 + g_-^1, \quad \textit{and} \quad \bar{p^0} = \int_{-1}^{1} \xi h^0 \, d\xi.
\end{aligned}
\tag{2.36}
$$

Proof. We consider problem P_2 i.e. finding a deformation ψ such that

$$J^2(\psi) = \inf_{\breve{\psi} \in \mathcal{M}_2} J^2(\breve{\psi}).$$

Let $\psi = (\phi^0, \psi^1, \psi^2, \ldots) \in \mathcal{M}_2$ be such that $\psi^1(x^1, x^2, \xi) = u + \xi\eta n$. The contribution of the internal energy to the total energy is the following

$$
\begin{aligned}
I^2(\psi) = \int_\Omega \Big\{ a\, \mathcal{C}^1_{ii}(\psi) + b(\mathcal{E}^1_{12}(\psi) + \mathcal{E}^1_{23}(\psi) + \mathcal{E}^1_{31}(\psi)) + \cdots \\
\cdots + \left(c - \frac{d}{\mathcal{D}^0(\psi)} \right) \mathcal{D}^1(\psi) \Big\} \, dx,
\end{aligned}
$$

with

$$\mathcal{D}^1(\psi) = \sum_{p+q=1} \mathcal{C}^p_{11} \mathcal{E}^q_{23} + \sum_{p+q+k=1} \left(2\mathcal{C}^p_{12}\mathcal{C}^q_{23}\mathcal{C}^k_{31} - \mathcal{C}^p_{33}\mathcal{C}^q_{12}\mathcal{C}^k_{12} - \mathcal{C}^p_{22}\mathcal{C}^q_{31}\mathcal{C}^k_{31} \right),$$

where

$$
\begin{aligned}
\mathcal{C}^0_{\alpha\beta}(\psi) &= \phi^0_{,\alpha} \cdot \phi^0_{,\beta}, \\
\mathcal{C}^0_{\alpha 3}(\psi) &= 0, \quad \mathcal{C}^0_{33}(\psi) = \eta^2, \\
\mathcal{C}^1_{\alpha\beta}(\psi) &= \phi^0_{,\alpha} \cdot \psi^1_{,\beta} + \psi^1_{,\alpha} \cdot \phi^0_{,\beta}, \\
\mathcal{C}^1_{\alpha 3}(\psi) &= \phi^0_{,\alpha} \cdot \psi^2_{,\xi} + \phi^1_{,\alpha} \cdot \psi^1_{,\xi}, \\
\mathcal{C}^1_{33}(\psi) &= 2\, \psi^1_{,\xi} \cdot \psi^2_{,\xi},
\end{aligned}
$$

and also

$$
\begin{aligned}
\mathcal{E}^0_{12}(\psi) &= |\phi^0_{,1}|^2 |\phi^0_{,2}|^2 - \mathcal{C}^0_{12}(\psi)^2, \quad \mathcal{E}^0_{\alpha 3}(\psi) = |\phi^0_{,\alpha}|^2 \eta^2, \\
\mathcal{E}^1_{12}(\psi) &= 2|\phi^0_{,1}|^2 (\phi^0_{,2} \cdot \psi^1_{,2}) + 2(\phi^0_{,1} \cdot \psi^1_{,1}) |\phi^0_{,2}|^2 + 2\, \mathcal{C}^0_{12}(\psi) \mathcal{C}^1_{12}(\psi), \\
\mathcal{E}^1_{\alpha 3}(\psi) &= 2|\phi^0_{,\alpha}|^2 (\psi^1_{,\xi} \cdot \psi^2_{,\xi}) + 2(\phi^0_{,\alpha} \cdot \psi^1_{,\alpha}) \eta^2.
\end{aligned}
$$

are given by Lemmas 2.2.1-2.2.4 and the fact that $\phi^0_{,\alpha} \cdot \psi^1_{,\xi} = 0$. A close inspection of the above terms yields the following identities and simplifications

$$
\begin{aligned}
\mathcal{E}^1_{12}(\psi) &= D\mathcal{E}^0_{12}(\phi^0).\psi^1, \\
&= 2(\phi^0_{,\alpha}\cdot\phi^0_{,\alpha})(\phi^0_{,\beta}\cdot\psi^1_{,\beta}) - 2(\phi^0_{,\alpha}\cdot\phi^0_{,\beta})(\phi^0_{,\alpha}\cdot\psi^1_{,\beta}) \\
\mathcal{D}^1(\psi) &= |\phi^0_{,1}|^2\mathcal{E}^1_{23} + 2\,\eta^2|\phi^0_{,2}|^2(\phi^0_{,1}\cdot\psi^1_{,1}) - \sum_{p+q+k=1}\mathcal{C}^p_{33}\mathcal{C}^q_{12}\mathcal{C}^k_{12} \qquad (2.37)\\
&= 2\,\mathcal{E}^0_{12}(\psi^1_{,\xi}\cdot\psi^2_{,\xi}) + \eta^2 D\mathcal{E}^0_{12}(\phi^0).\psi^1, \\
c - \frac{d}{\mathcal{D}^0} &= -\frac{a+b\,|\nabla\phi^0|^2}{\mathcal{E}^0_{12}}, \\
a\,\mathcal{C}^1_{ii} + b\left(\mathcal{E}^1_{23}+\mathcal{E}^1_{31}\right) &= 2\,a\,\phi^0_{,\alpha}\cdot\psi^1_{,\alpha} + 2(a+b\,|\nabla\phi^0|^2)(\psi^1_{,\xi}\cdot\psi^2_{,\xi}) + 2b\,\eta^2\phi^0_{,\alpha}\cdot\psi^1_{,\alpha}.
\end{aligned}
$$

Injecting the above in the expression of the internal energy gives

$$
\begin{aligned}
I^2(\psi) &= \int_\Omega \left\{2(a+b\,\eta^2)\,\phi^0_{,\alpha}\cdot\psi^1_{,\alpha} + \left(b - \frac{a+b\,|\nabla\phi^0|^2}{\mathcal{E}^0_{12}}\,\eta^2\right) D\mathcal{E}^0_{12}(\phi^0).\psi^1\right\} dx \\
&= \frac{1}{2}\int_\Omega M^\alpha_0\cdot\psi^1_{,\alpha}\,dx_H d\xi \\
&= \frac{1}{2}\int_\omega\int_{-1}^1 M^\alpha_0\cdot\{u(x_H)+\xi\eta(x_H)n(x_H)\}_{,\alpha}\,d\xi dx_H \\
&= \frac{1}{2}\int_{-1}^1 d\xi\int_\omega M^\alpha_0\cdot u_{,\alpha}\,dx_H + \frac{1}{2}\int_{-1}^1\xi\,d\xi\int_\omega \eta n\,dx_H \\
&= \int_\omega M^\alpha_0\cdot u_{,\alpha}\,dx_H
\end{aligned}
$$

As $u_{|\partial\omega_\phi} = 0$, we conclude that

$$
I^2(\psi) = \int_\omega f^0\cdot u\,dx_H + \int_{\omega_\sigma}\bar{f}^0\cdot u\,ds,
$$

using Theorem 2.4.8.
Now let us compute the contribution of the external energy

$$
\begin{aligned}
\ell^2(\psi) &= \int_\Omega\sum_{p=0}^1 b^p\cdot\psi^{1-p}\,dx_H d\xi + \int_{\Gamma_+\cup\Gamma_-}\sum_{p=0}^2 g^p\cdot\psi^{2-p}\,dx_H \\
&\qquad + \int_{\Gamma_\sigma}\sum_{p=0}^1 h^p\cdot\psi^{1-p}\,ds d\xi \\
&= \int_\omega f^1\cdot\phi^0\,dx_H + \int_{\omega_\sigma}\bar{f}^1\cdot\phi^0\,ds + \int_\Omega(b^0+g^1)\cdot\psi^1\,dx_H d\xi \\
&\qquad + \int_{\Gamma_\sigma} h^0\cdot\psi^1\,ds d\xi \\
&= \int_\omega(f^1\cdot\phi^0 + f^0\cdot u + p^0\cdot\eta n)\,dx_H + \int_{\omega_\sigma}(\bar{f}^1\cdot\phi^0 + \bar{f}^0\cdot u + \bar{p}^0\cdot\eta n)\,ds,
\end{aligned}
$$

where f^0, $\bar{f}^0$, f^1, $\bar{f}^1$, p^0 and $\bar{p}^0$ are defined in (2.36). Finally the stated total energy $J_*^2(\psi) = J_*^2(\phi^0, u) = I^2(\psi) + \ell^2(\psi)$ is recovered thus completing the proof. □

The above result brings up several considerations induced by the nature of the membrane theory and other mathematical requirements that have been omitted. Actually, we are interested in obtaining a pure bending model and in order to exclude membrane effects, the choice of a vanishing membrane loading, that is $f^0 = 0$ and $\bar{f}^0 = 0$, appears to be legitimate. This implies that the deformation φ^0 verifies $W_m(\varphi^0) = 0$. As a consequence and since $W_m(\mathrm{id}) = 0$, any surface isometric to the plane which satisfies the Dirichlet boundary conditions will minimize the membrane stored energy function (cf. paragraph 4.3). Moreover, in order to be consistent, we need to assume that the Dirichlet boundary conditions are such that non-trivial isometric deformations are possible solutions. More precisely, we replace the set $\mathcal{M}_2$ by the set

$$
\begin{aligned}
\mathcal{M}_2^{iso} = \Big\{ & \psi \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^3) : \exists \varphi^0,\ u \in \mathcal{F}(\bar{\omega}; \mathbb{R}^3) \quad \text{such that} \\
& \psi^0(x^1, x^2, \xi) = \varphi^0(x^1, x^2) \quad \text{and} \quad \mathcal{C}^0_{\alpha\beta}(\psi) = \delta^\alpha_\beta, \\
& \psi^1(x^1, x^2, \xi) = u(x^1, x^2) + \xi n(\nabla\varphi^0) \quad \text{where} \quad n(\nabla\varphi^0) = \varphi^0_{,1} \wedge \varphi^0_{,2}, \\
& \qquad u_{|\partial\omega_\phi} = 0, \quad \varphi^0_{|\partial\omega_\phi} = \bar{\phi}^0{}_{|\partial\omega_\phi} \quad \text{and} \quad \frac{1}{2}\int_{-1}^{1} \psi^p_{|\partial\omega_\phi}\, d\xi = 0\ \forall p \geq 2 \Big\},
\end{aligned}
$$

and we assume that $\mathcal{F}_0^{iso} = \{\varphi \in \mathcal{F}(\bar{\omega}; \mathbb{R}^3) :\ \mathcal{C}_{\alpha\beta}(\varphi^0) = \delta^\alpha_\beta\} \neq \{\mathrm{id}\}$. We accordingly rename problem P_2, P_2^{iso}.

Furthermore, another sensible requirement is for the leading-order external loading to be of pure bending type, that is a vanishing resultant force and a non-vanishing resultant torque. This leads us to choose $f^1 = 0$ and $\bar{f}^1 = 0$. Taking account of all these requirements and substituting $\mathcal{M}_2$ by $\mathcal{M}_2^{iso}$ reduces problem P_2^{iso} to a trivial problem and therefore $\mathcal{M}_3^{iso} = \mathcal{M}_2^{iso}$.

All the above requirements are thouroughly justified in Fox, Raoult and Simo [40].

2.5.2 A nonlinear bending plate model

In this section, we justify a bending plate model while solving problem P_3^{iso}. Let us first remark that some recurrent quantities have noticeably simplified, namely we have

Lemma 2.5.2. *Let $\psi = (\phi^0, \psi^1, \psi^2, \ldots) \in \mathcal{M}_3^{iso}$ then the following identities hold*

$$
\mathcal{E}^0_{12}(\psi) = \mathcal{E}^0_{\alpha 3}(\psi) = \eta = \mathcal{D}^0(\psi) = \mathcal{C}^0_{33} = 1 \quad \textit{and} \quad \mathcal{C}^0_{\alpha 3} = 0. \tag{2.38}
$$

Proof. Let $\psi = (\phi^0, \psi^1, \psi^2, \ldots) \in \mathcal{M}_3^{iso}$, then

$$
\mathcal{C}^0_{\alpha\beta}(\psi) = \delta^\alpha_\beta \quad \text{and} \quad \mathcal{E}^0_{12}(\psi) = \mathcal{C}^0_{11}\mathcal{C}^0_{22} - (\mathcal{C}^0_{12})^2 = 1.
$$

This together with (2.21) implies that

$$\mathcal{C}^0_{33} = |\psi^1_{,\xi}|^2 = \eta^2 = \frac{d}{a + b\,|\nabla\phi^0|^2 + c\,\mathcal{E}^0_{12}} = \frac{d}{a+2b+c} = 1,$$

by definition of the Ciarlet-Geymonat stored energy function. Next, we easily deduce the remaining identites. First $\mathcal{D}^0(\psi) = \eta^2\,\mathcal{E}^0_{12}(\psi) = 1$ and $\mathcal{C}^0_{\alpha 3} = \phi^0_{,\alpha}\cdot(\phi^0_{,1}\wedge\phi^0_{,2}) = 0$. Lastly, $\mathcal{E}^0_{\alpha 3}(\psi) = \mathcal{C}^0_{\alpha\alpha}\eta^2 - (\mathcal{C}^0_{\alpha 3})^2 = 1$ □

As $\eta = 1$, from now on we will adopt the notation $\psi^1_{,\xi} = n$.

Theorem 2.5.3. *For external loadings satisfying $b^0 = 0$, $g^1 = 0$, $h^0 = 0$, $f^1 = 0$ and $\bar{f}^1 = 0$, if $\psi = (\varphi^0, \psi^1, \psi^2, \ldots)$ is a solution of problem P_3^{iso}, then the leading term φ^0 in the asymptotic expansion minimizes the energy defined by*

$$J^3_b(\phi^0) = \int_\omega W_b(\phi^0)\,dx_H - \int_\omega \left(f^2\cdot\phi^0 + p^1\cdot n\right)dx_H - \int_{\omega_\sigma}\left(\bar{f}^2\cdot\phi^0 + \bar{p}^1\cdot n\right)ds$$

on the set of deformations $\phi^0 : \omega \to \mathbb{R}^3$ such that $\phi^0_{,\alpha}\cdot\phi^0_{,\beta} = \delta^\alpha_\beta$ and $\phi^0_{|\partial\omega_\phi} = \bar{\phi}_{|\partial\omega_\phi}$, where

$$W_b(\phi^0) = \frac{4}{3}(a+b)\left(b^2_{11}(\phi^0) + b^2_{22}(\phi^0) + 2\,b^2_{12}(\phi^0)\right) + \frac{4}{3}\frac{(a+b)(b+c)}{d}\,b^2_{\alpha\alpha}(\phi^0),$$

with

$$b_{\alpha\beta}(\phi^0) = -\,n(\phi^0)\cdot\phi^0_{,\alpha\beta},\quad n(\phi^0) = \varphi^0_{,1}\wedge\varphi^0_{,2},$$
$$f^2 = \int_{-1}^1 b^2\,d\xi + g^3_+ + g^3_-,\quad \bar{f}^2 = \int_{-1}^1 h^2\,d\xi,$$
$$p^1 = \int_{-1}^1 \xi b^1\,d\xi + g^2_+ + g^2_-,\quad \text{and}\quad \bar{p}^1 = \int_{-1}^1 \xi h^1\,d\xi.$$

For clarity, the proof of the above theorem is broken into a series of lemmas.

Lemma 2.5.4. *If $(\phi^0, \psi^1, \psi^2, \ldots)$ belongs to $\mathcal{M}^{iso}_2$, then the following holds*

$$\mathcal{C}^1_{\alpha\beta}(\psi) = \phi^0_{,\alpha}\cdot\psi^1_{,\beta} + \psi^1_{,\alpha}\cdot\phi^0_{,\beta},\ \mathcal{C}^1_{\alpha 3}(\psi) = \phi^0_{,\alpha}\cdot\psi^2_{,\xi} + n\cdot\psi^1_{,\alpha},$$
$$\mathcal{C}^1_{33}(\psi) = 2\,n\cdot\psi^2_{,\xi},\quad \mathcal{C}^2_{\alpha\beta}(\psi) = \phi^0_{,\alpha}\cdot\psi^2_{,\beta} + \psi^1_{,\alpha}\cdot\psi^1_{,\beta} + \psi^2_{,\alpha}\cdot\phi^0_{,\beta},$$
$$\mathcal{C}^2_{\alpha 3}(\psi) = \phi^0_{,\alpha}\cdot\psi^3_{,\xi} + \psi^1_{,\alpha}\cdot\psi^2_{,\xi} + n\cdot\psi^2_{,\alpha},\quad \mathcal{C}^2_{33}(\psi) = |\psi^2_{,\xi}|^2 + 2\,n\cdot\psi^3_{,\xi},$$
$$\mathcal{E}^1_{12}(\psi) = 2\,\phi^0_{,\alpha}\cdot\psi^1_{,\alpha},\quad \mathcal{E}^1_{\alpha 3}(\psi) = 2\,n\cdot\psi^2_{,\xi} + 2\,\phi^0_{,\alpha}\cdot\psi^1_{,\alpha},$$
$$\mathcal{E}^2_{12}(\psi) = \sum_{k+m+p+q=2}(\psi^k_{,1}\cdot\psi^m_{,1})(\psi^p_{,2}\cdot\psi^q_{,2}) - (\psi^k_{,1}\cdot\psi^m_{,2})(\psi^p_{,1}\cdot\psi^q_{,2})$$
$$= \psi^1_{,\alpha}\cdot\psi^1_{,\alpha} + 2\phi^0_{,\alpha}\cdot\psi^2_{,\alpha} + 4(\phi^0_{,1}\cdot\psi^1_{,1})(\phi^0_{,2}\cdot\psi^1_{,2}) - \left(\phi^0_{,1}\cdot\psi^1_{,2} + \psi^1_{,1}\cdot\phi^0_{,2}\right)^2,$$
$$\mathcal{E}^2_{\alpha 3}(\psi) = \sum_{k+m+p+q=4}(\psi^k_{,\alpha}\cdot\psi^m_{,\alpha})(\psi^p_{,\xi}\cdot\psi^q_{,\xi}) - (\psi^k_{,\alpha}\cdot\psi^m_{,\xi})(\psi^p_{,\alpha}\cdot\psi^q_{,\xi})$$
$$= |\psi^1_{,\alpha}|^2 + 2\,\phi^0_{,\alpha}\cdot\psi^2_{,\alpha} + |\psi^2_{,\xi}|^2 + 2\,n\cdot\psi^3_{,\xi} + 4(\phi^0_{,\alpha}\cdot\psi^1_{,\alpha})(n\cdot\psi^2_{,\xi})$$
$$-\left(\phi^0_{,\alpha}\cdot\psi^2_{,\xi} + n\cdot\psi^1_{,\alpha}\right)^2.$$

Proof. We get the general expressions of the above components from Lemmas 2.2.1 and 2.2.2, then we simplify them using Lemma 2.5.2 and the fact that $(\phi^0, \psi^1, \psi^2, \dots)$ belongs to $\mathcal{M}_2^{iso}$. □

Lemma 2.5.5. *If $(\phi^0, \psi^1, \psi^2, \dots)$ belongs to $\mathcal{M}_2^{iso}$, then the following holds*

$$\begin{aligned}
\mathcal{D}^1(\psi) &= 2\, n \cdot \psi^2_{,\xi} + 2\, \phi^0_{,\alpha} \cdot \psi^1_{,\alpha},\\
\mathcal{D}^2(\psi) &= \psi^1_{,\alpha} \cdot \psi^1_{,\alpha} + 2\, \phi^0_{,\alpha} \cdot \psi^2_{,\alpha} + 4(\phi^0_{,1} \cdot \psi^1_{,1})(\phi^0_{,2} \cdot \psi^1_{,2}) + 4(\phi^0_{,\alpha} \cdot \psi^1_{,\alpha})(n \cdot \psi^2_{,\xi})\\
&\quad - \left(\phi^0_{,\alpha} \cdot \psi^2_{,\xi} + n \cdot \psi^1_{,\alpha}\right)\left(\phi^0_{,\alpha} \cdot \psi^2_{,\xi} + n \cdot \psi^1_{,\alpha}\right) - \left(\phi^0_{,1} \cdot \psi^1_{,2} + \phi^0_{,2} \cdot \psi^1_{,1}\right)^2.
\end{aligned}$$

Proof. Lemma 2.2.3 gives $\mathcal{D}^1(\psi) = 2\,\mathcal{E}^0_{12}(n \cdot \psi^2_{,\xi}) + \eta^2\,\mathcal{E}^1_{12}(\psi)$ and

$$\begin{aligned}
\mathcal{D}^2(\psi) &= \sum_{p+q=2} \mathcal{C}^p_{11}\mathcal{E}^q_{23} + \sum_{p+q+k=2} \left(2\,\mathcal{C}^p_{12}\mathcal{C}^q_{23}\mathcal{C}^k_{31} - \mathcal{C}^p_{33}\mathcal{C}^q_{12}\mathcal{C}^k_{12} - \mathcal{C}^p_{22}\mathcal{C}^q_{31}\mathcal{C}^k_{31}\right)\\
&= \mathcal{E}^2_{23} + \mathcal{C}^1_{11}\mathcal{E}^1_{23} + \mathcal{C}^2_{11} - \left(\mathcal{C}^1_{12}\right)^2 - \left(\mathcal{C}^1_{13}\right)^2.
\end{aligned}$$

Then we conclude by Lemma 2.5.4. □

Lemma 2.5.6. *For external loadings satisfying $b^0 = 0$, $g^1 = 0$, $h^0 = 0$, $f^1 = 0$ and $\bar{f}^1 = 0$, if $\psi = (\varphi^0, \psi^1, \psi^2, \dots)$ is a solution of problem P_3^{iso}, then the leading terms φ^0 and $u = \psi^1_{,\xi}$ in the asymptotic expansion minimize the energy defined by*

$$J^3_{iso}(\phi^0, u) = J^3_1(\phi^0, u) + \inf_{\left\{\psi^2 : \int_{-1}^1 \psi^2_{|\partial\omega_\sigma}\, d\xi = 0\right\}} J^3_2(\phi^0, u, \psi^2). \tag{2.39}$$

on the set of functions $\phi^0 : \omega \to \mathbb{R}^3$ such that $\phi^0_{,\alpha} \cdot \phi^0_{,\beta} = \delta^\alpha_\beta$, $\phi^0_{|\partial\omega_\phi} = \bar{\phi}_{|\partial\omega_\phi}$, and $u : \omega \to \mathbb{R}^3$ such that $u_{|\partial\omega_\phi} = 0$, where

$$\begin{aligned}
J^3_1(\psi) &= (a+b)\int_\Omega \left\{\left(\phi^0_{,1} \cdot \psi^1_{,2} + \phi^0_{,2} \cdot \psi^1_{,1}\right)^2 - 4(\phi^0_{,1} \cdot \psi^1_{,1})(\phi^0_{,2} \cdot \psi^1_{,2})\right\} dx - \ell^3(\phi^0),\\
J^3_2(\psi) &= (a+b)\int_\Omega \left(\phi^0_{,\alpha} \cdot \psi^2_{,\xi} + n \cdot \psi^1_{,\alpha}\right)\left(\phi^0_{,\alpha} \cdot \psi^2_{,\xi} + n \cdot \psi^1_{,\alpha}\right) dx\\
&\quad + 2d\int_\Omega \left(\phi^0_{,\alpha} \cdot \psi^1_{,\alpha} + n \cdot \psi^2_{,\xi}\right)^2 dx - 4(a+b)\int_\Omega (\phi^0_{,\alpha} \cdot \psi^1_{,\alpha})(n \cdot \psi^2_{,\xi})\, dx,
\end{aligned}$$

and $\psi^1 = u + \xi\, \phi^0_{,1} \wedge \phi^0_{,2}$.

Proof. We consider problem P_3^{iso} i.e. finding a deformation ψ such that

$$J^3(\psi) = \inf_{\breve{\psi} \in \mathcal{M}_3^{iso}} J^3(\breve{\psi}).$$

We start by computing the external energy

$$\begin{aligned}
\ell^3(\psi) = \int_\Omega \sum_{p=0}^2 b^p \cdot \psi^{2-p}\, dx_H d\xi &+ \int_{\Gamma_+ \cup \Gamma_-} \sum_{p=0}^3 g^p \cdot \psi^{3-p}\, dx_H\\
&+ \int_{\Gamma_\sigma} \sum_{p=0}^2 h^p \cdot \psi^{2-p}\, ds d\xi,
\end{aligned}$$

which is given by Lemma 2.2.6. After taking into consideration the requirements stated in section 5.1 and integrating along the thickness, we obtain the following

$$\ell^3(\psi) = \ell^3(\phi^0) = \int_\omega \left(f^2 \cdot \phi^0 + p^1 \cdot n\right) dx_H + \int_{\omega_\sigma} \left(\bar{f}^2 \cdot \phi^0 + \bar{p}^1 \cdot n\right) ds,$$

where f^2, p^1, $\bar{f}^2$, $\bar{p}^1$ are defined in Theorem 2.5.3. The contribution of the internal energy to the total energy is

$$I^3(\psi) = \int_\Omega \Big\{ a\, \mathcal{C}^2_{ii}(\psi) + b\left(\mathcal{E}^2_{12}(\psi) + \mathcal{E}^2_{23}(\psi) + \mathcal{E}^2_{31}(\psi)\right) + \cdots \\ \cdots + \frac{d}{2}\mathcal{D}^1(\psi)^2 + (c-d)\mathcal{D}^2(\psi) \Big\}\, dx \tag{2.40}$$

where we have used Lemma 2.2.4 for the expansion of the logarithm and the fact that $\mathcal{D}^0(\psi) = 1$. First of all, we remark that in the expression of the internal energy (2.40), we have

$$(a + 2b + c - d) \int_\Omega \left\{ \psi^1_{,\alpha} \cdot \psi^1_{,\alpha} + |\psi^2_{,\xi}|^2 + 2\left(\phi^0_{,\alpha} \cdot \psi^2_{,\alpha} + n \cdot \psi^3_{,\xi}\right)\right\} dx = 0,$$

since, by definition of the Ciarlet-Geymonat polyconvex stored energy function (see Theorem 2.1.1), the above factor vanishes. Hence we are left with

$$J^3(\psi) = J^3_1(\psi) + J^3_2(\psi) \quad \text{where} \quad \begin{cases} J^3_1(\psi) = J^3_1(\phi^0, u), \\ J^3_2(\psi) = J^3_2(\phi^0, u, \psi^2). \end{cases} \tag{2.41}$$

where J^3_1 and J^3_2 are defined in the lemma. Now, let us set

$$J^3_{iso}(\phi^0, u) = \inf_{\left\{(\psi^2, \psi^3, \ldots) : (\phi^0, u + \xi n, \psi^2, \ldots) \in \mathcal{M}^{iso}_3\right\}} J^3(\psi),$$

then (2.41) implies the announced result (2.39). □

Remark 2.5.7. Note that we also have

$$\inf_{\psi \in \mathcal{M}^{iso}_3} J^3_{iso}(\phi^0, u) = \inf_{\psi \in \mathcal{M}^{iso}_3} J^3(\psi).$$

Lemma 2.5.8.

$$\inf_{\left\{\psi^2 : \int_{-1}^{1} \psi^2_{|\partial\omega_\sigma}\, d\xi = 0\right\}} J^3_2(\phi^0, u, \psi^2) = \\ 2(a+b)\left(1 + \frac{b+c}{d}\right) \int_\Omega \left(\phi^0_{,\alpha} \cdot \psi^1_{,\alpha}\right)^2 dx. \tag{2.42}$$

Proof. To compute the above infimum, let us first introduce the function

$$G(X_1, X_2, Z) = (a+b)(X_\alpha + B_\alpha)(X_\alpha + B_\alpha) + 2\,d(B_3 + Z)^2 - 4(a+b)B_3\, Z$$

where

$$X_\alpha = \phi^0_{,\alpha} \cdot \psi^2_{,\xi}, \quad Z = n \cdot \psi^2_{,\xi}, \quad B_\alpha = n \cdot \psi^1_{,\alpha} \quad \text{and} \quad B_3 = 4(b+c)(\phi^0_{,\alpha} \cdot \psi^1_{,\alpha}).$$

Then $J^3_2(\phi^0, u, \psi^2) = \int_\Omega G(X_1, X_2, Z)\, dx$. Thereby, it suffices to compute the minimum of G now. If (X_1, X_2, Z) is a critical point of G, it verifies

$$G_{,X_1}(X_1, X_2, Z) = G_{,X_2}(X_1, X_2, Z) = G_{,Z}(X_1, X_2, Z) = 0,$$

i.e.

$$\begin{cases} X_\alpha + B_\alpha &= 0, \\ 4d(B_3 + Z) - 4(a+b)Z &= 0. \end{cases}$$

In other words, the functions ψ^2 that minimize the functional J^3_2 must verify

$$\begin{cases} \phi^0_{,\alpha} \cdot \psi^2_{,\xi} &= -\, n \cdot \psi^1_{,\alpha}, \\ n \cdot \psi^2_{,\xi} &= -\dfrac{b+c}{d}(\phi^0_{,\alpha} \cdot \psi^1_{,\alpha}). \end{cases}$$

Replacing these terms by their respective values in the expression of J^3_2 yields its minimum on $\left\{\psi^2 : \int_{-1}^1 \psi^2_{|\partial\omega_\sigma}\, d\xi = 0\right\}$ which only depends on ϕ^0 and u as it appears in (2.42). □

Proof of Theorem 2.5.3. To finalize the proof of the theorem, we carry on with the resolution of problem P^{iso}_3. Indeed, Lemmas 2.5.6 and 2.5.8 entail that if $\psi = (\varphi^0, \psi^1, \psi^2, \ldots)$ is a solution of problem P^{iso}_3, then the leading terms φ^0 and ψ^1 in the asymptotic expansion minimize the energy below

$$\begin{aligned} J^3_{iso}(\phi^0, u) = 2(a+b)\left(1 + \frac{b+c}{d}\right) &\int_\Omega (\phi^0_{,\alpha} \cdot \psi^1_{,\alpha})^2\, dx \\ - 4(a+b) &\int_\Omega (\phi^0_{,1} \cdot \psi^1_{,1})(\phi^0_{,2} \cdot \psi^1_{,2})\, dx \\ + (a+b) &\int_\Omega \left(\phi^0_{,1} \cdot \psi^1_{,2} + \phi^0_{,2} \cdot \psi^1_{,1}\right)^2 dx - \ell^3(\phi^0). \end{aligned}$$

Now recall that $\psi^1(x) = u(x_H) + \xi\, n(x_H)$ and that $d = a + 2b + c$, then after integrating along ξ the above expression becomes

$$J^3_{iso}(\phi^0, u) = 3\int_\omega W_{iso}(\nabla\phi^0, \nabla u)\, dx_H + \int_\omega W =_{iso} (\nabla\phi^0, \nabla n)\, dx_H - \ell^3(\phi^0), \quad (2.43)$$

where

$$\begin{aligned} W_{iso}(F, G) = \frac{4}{3}(a+b)\left(1 + \frac{b+c}{d}\right) &(F_\alpha \cdot G_\alpha)^2 \\ -\frac{8}{3}&(a+b)(F_1 \cdot G_1)(F_2 \cdot G_2) \\ +\frac{2}{3}&(a+b)(F_1 \cdot G_2 + F_2 \cdot G_1)^2, \end{aligned}$$

for all matrices $F = (F_1|F_2),\ G = (G_1|\,G_2) \in \mathbb{R}^{3\times 2}$.
We first observe that the linear term in the total energy (2.43) does not depend on u and $W_{iso}(F,G) \geq W_{iso}(F,0) \geq 0$, so a solution $\psi \in \mathcal{M}_3^{iso}$ of problem P_3^{iso} is necessarily such that $u = 0$. Hence we have

$$\inf_{\psi\in\mathcal{M}_3^{iso}} J_b^3(\phi^0) = \inf_{\psi\in\mathcal{M}_3^{iso}} J^3(\psi),$$

where

$$J_b^3(\phi^0) = \int_\omega W_{iso}(\nabla\phi^0, \nabla n)\, dx_H - \ell^3(\phi^0). \tag{2.44}$$

Finally, to *get rid* of the derivatives of the normal n, we mention that $\phi^0_{,\alpha} \cdot n = 0$ and thereby

$$\phi^0_{,\alpha} \cdot n_{,\beta} = -\,\phi^0_{,\alpha\beta} \cdot n,$$

Next, we substitute $\phi^0_{,\alpha} \cdot n_{,\beta}$ in identity (2.44) and infer that

$$J_b^3(\phi^0) = \int_\omega W_b(\phi^0)\, dx_H - \ell^3(\phi^0), \tag{2.45}$$

where

$$W_b(\phi^0) = \frac{4}{3}(a+b)\left(1 + \frac{b+c}{d}\right) b^2_{\alpha\alpha}(\phi^0) - \frac{8}{3}(a+b)\, b_{11}(\phi^0)\, b_{22}(\phi^0) + \frac{8}{3}(a+b)\, b^2_{12}(\phi^0),$$

and $b_{\alpha\beta}(\phi^0) = -\,n\cdot\phi^0_{,\alpha\beta}$ is the second fundamental form. Ultimately, it remains to rearrange the terms in the stored energy function W_b in a form more reminiscent of the classical nonlinear bending plate model to yield the announced formula. □

2.5.3 Commentaries on the nonlinear bending plate model

The stored energy function W_b obtained in Theorem 2.5.3 is of bending type as it only depends on the second fundamental form of the deformation ϕ^0 i.e. $b_{\alpha\beta}(\phi^0) = -\,n(\phi^0)\cdot\phi^0_{,\alpha\beta}$. Moreover, if we recall remark 2.1.4, we notice that the function W_b is actually exactly the classical stored energy function of a bending plate as stated in mechanical litterature and as justified by Fox, Raoult & Simo [40], by means of formal asymptotic expansions, and by Friesecke, James & Müller [44] via Γ-convergence arguments. Namely, we have

$$W_b(F) = \frac{2\lambda\mu}{3(\lambda+2\mu)} b_{\sigma\sigma}(F) b_{\tau\tau}(F) + \frac{2\mu}{3} b_{\alpha\beta}(F) b_{\alpha\beta}(F)$$

for all $F = (F_1|F_2) \in \mathbb{R}^{3\times 2}$ such that $F_\alpha \cdot F_\beta = \delta^\alpha_\beta$. What is more, we derived this limit model for an infinity of Ogden materials thereby validating in a certain empirical sense the legitimacy of the original family of materials as well as bringing strong evidence to the accuracy of the approximation of the two-dimensional model.

Coutand provided an existence theorem for the minimization problem associated to the above stored energy function in [27]. We recall the result below

Theorem 2.5.9. *There exists $\varphi \in \mathcal{M}_b$ such that :*

$$J_b^3(\varphi) = \inf_{\psi \in \mathcal{M}_b} J_b^3(\psi),$$

where

$$\mathcal{M}_b = \left\{ \psi \in H^2(\omega) : \ \psi_\alpha \cdot \psi_\beta = \delta_\beta^\alpha \quad \textit{and} \quad \psi_{|\partial\omega_\phi} = \bar{\phi}_{|\partial\omega_\phi} \right\}.$$

Note that the result above is also recovered by Friesecke, James & Müller [44] as an *upshot* of their asymptotic analysis since the model as a Γ-limit functional which is by definition a relaxed energy ; see also Ciarlet & Coutand [15] and Friesecke, James, Mora & Müller [42] for a generalization to the case of flexural shells.

The results presented and proved in this chapter were announced in Trabelsi [73].

Annexe A

The recursive asymptotic procedure

In this appendix, we give a proof of the asymptotic procedure proposed by Pantz [65, 66]. We recall the result, see Proposition 2.2.4

Proposition A.0.10. *The solution* $\varphi(\varepsilon) = \varphi^0 + \varphi^1\varepsilon + \cdots$ *to problem* $P(\varepsilon)$ *is such that*

$$\varphi \in \bigcap_{n=-1}^{\infty} \mathcal{M}_n$$

where

$$\mathcal{M}_{n+1} = \left\{ \psi \in \mathcal{M}_n \; : \; J^n(\psi) = \inf_{\breve{\psi} \in \mathcal{M}_n} J^n(\breve{\psi}) \right\},$$

$$\mathcal{M}_{-1} = \left\{ \psi \in \mathcal{F}(\overline{\Omega}; \mathbb{R}^3)^{\mathbb{N}} : \sum_n \psi^n \varepsilon^n \in \mathcal{M}(\varepsilon) \right\}.$$

Proof. We prove the result by induction. To this intent, we define the following property

$$(\mathcal{P}_n) \qquad \varphi(\varepsilon) \in \bigcap_{p=-1}^{n} \mathcal{M}_p = \mathcal{M}_n$$

As $\varphi(\varepsilon)$ is by definition a solution to problem $P(\varepsilon)$, property $(\mathcal{P}_{-1})$ is *de facto* verified. Thus, it remains to show that if $\varphi(\varepsilon)$ verifies $\mathcal{P}_n$ then it verifies $\mathcal{P}_{n+1}$.
Let $\varphi \in \mathcal{M}_n$, then according to Proposition 2.2.7, we have

$$\hat{J}(\varepsilon)(\varphi) = \sum_{p=-1}^{n-1} J^p(\varphi)\varepsilon^p + \varepsilon^n \left\{ J^n(\varphi) + \varepsilon \sum_{p=0}^{\infty} J^{n+1+p}(\varphi)\varepsilon^p \right\}. \tag{A.1}$$

Furthermore, for all $p < n$, $\varphi \in \mathcal{M}_{p+1}$, hence

$$J^p(\varphi) = \inf_{\bar{\varphi} \in \mathcal{M}_p} J^p(\bar{\varphi}). \tag{A.2}$$

Now let us introduce the following notation

$$S^n(\varepsilon) = \sum_{p=-1}^{n-1} J^p(\varphi)\varepsilon^p,$$

then equations (A.1) and (A.2) imply that for all $\varphi(\varepsilon) \in \mathcal{M}_n$ we have

$$\hat{J}(\varepsilon)(\varphi) = S^n(\varepsilon) + \varepsilon^n \left\{ J^n(\varphi) + \varepsilon \sum_{p=0}^{\infty} J^{n+1+p}(\varphi)\varepsilon^p \right\}. \tag{A.3}$$

Let us recall that $\varphi(\varepsilon)$ is the solution to problem $P(\varepsilon)$, that is

$$\hat{J}(\varepsilon)(\varphi) = \inf_{\psi \in \mathcal{M}(\varepsilon)} J^p(\varepsilon)(\psi).$$

Therefore, the following inequality is trivially verified for all $\varepsilon > 0$

$$\hat{J}(\varepsilon)(\varphi) \leq \inf_{\psi \in \mathcal{M}_n} J^p(\varepsilon)(\psi).$$

Next, from identity (A.3), we deduce that

$$J^n(\varphi) + \sum_{p=0}^{\infty} J^{n+1+p}(\varphi)\varepsilon^p \leq \inf_{\psi \in \mathcal{M}_n} \left\{ J^n(\psi) + \varepsilon \sum_{p=0}^{\infty} J^{n+1+p}(\psi)\varepsilon^p \right\}. \tag{A.4}$$

Next, for all $\epsilon > 0$, there exists $\psi_\epsilon \in \mathcal{M}_n$ such that

$$J^n(\psi_\epsilon) \leq \inf_{\psi \in \mathcal{M}_n} J^p(\varepsilon)(\psi) + \epsilon.$$

The above inequality combined with inequality (A.4) yield

$$\begin{aligned} J^n(\varphi) + \sum_{p=0}^{\infty} J^{n+1+p}(\varphi)\varepsilon^p &\leq J^n(\psi_\epsilon) + \varepsilon \sum_{p=0}^{\infty} J^{n+1+p}(\psi_\epsilon)\varepsilon^p \\ &\leq \inf_{\psi \in \mathcal{M}_n} J^p(\varepsilon)(\psi) + \epsilon + \varepsilon \sum_{p=0}^{\infty} J^{n+1+p}(\psi_\epsilon)\varepsilon^p, \end{aligned}$$

for all $\varepsilon > 0$. Thus, as ε tends to zero, we get

$$J^n(\varphi) \leq \inf_{\psi \in \mathcal{M}_n} J^p(\varepsilon)(\psi) + \epsilon,$$

for all $\epsilon > 0$. In other words, we have

$$J^n(\varphi) \leq \inf_{\psi \in \mathcal{M}_n} J^p(\varepsilon)(\psi),$$

and since $\varphi \in \mathcal{M}_n$, we deduce that $\varphi \in \mathcal{M}_{n+1}$. We have therefore shown that φ satisfies property $\mathcal{P}_{n+1}$ which concludes the proof. □

Annexe B

The Ciarlet-Geymonat stored energy function

In this appendix we give a revised proof to the result of Ciarlet & Geymonat [18] with the same notations as in their paper ; see also Ciarlet [11]. We first recall the result in its original form.

Theorem B.0.11. *Let $\lambda > 0$ and $\mu > 0$ be two given Lamé constants. There exist polyconvex stored energy functions of the form*

$$F \in M^3_> \to \hat{W}(F) = a\,|F|^2 + \,b\,|\mathrm{adj}_3\,F|^2 + \,\Gamma(F) + \,e,$$

with

$$a > 0,\ b > 0,\ \Gamma(\delta) = c\delta^2 - d\ln\delta,\ c > 0,\ d > 0,\ e \in \mathbb{R},$$

that satisfy

$$\hat{W}(F) = \check{W}(E) = \frac{\lambda}{2}\,(\mathrm{tr}\,E)^2 + \,\mu\,\mathrm{tr}\,E^2 + \,O(E^2) \quad where \quad E(F) = \frac{1}{2}\,(F^tF - I).$$

A stored energy function of this form verifies the coercivenes inequality

$$W(F) \geq \alpha\left(|F|^2 + |\mathrm{Cof}\,F|^2 + (\det F)^2\right) + \beta, \quad \alpha > 0.$$

Proof. The beginning of the proof is identical to the one given in the proof of

Theorem 2.1.1. Recall the The following identities hold

$$\begin{aligned}
|F|^2 &= \operatorname{tr} F^tF = \operatorname{tr}(I+2E) = 3 + 2\operatorname{tr} E,\\
|\operatorname{adj}_3 F|^2 &= |\operatorname{Cof} F|^2 = \operatorname{tr}\operatorname{Cof} F^tF\\
&= \frac{1}{2}(\operatorname{tr} F^tF)^2 - \frac{1}{2}\operatorname{tr}(F^tF)^2\\
&= \frac{1}{2}\{\operatorname{tr}(I+2E)\}^2 - \frac{1}{2}\operatorname{tr}(I+2E)^2\\
&= 3 + 4\operatorname{tr} E + 2(\operatorname{tr} E)^2 - 2\operatorname{tr} E^2,\\
\det F^tF &= \frac{1}{6}(\operatorname{tr} F^tF)^3 - \frac{1}{2}\operatorname{tr} F^tF\operatorname{tr}(F^tF)^2 + \frac{1}{3}\operatorname{tr}(F^tF)^3\\
&= 1 + 2\operatorname{tr} E + 2(\operatorname{tr} E)^2 - 2\operatorname{tr} E^2 + O(|E|^3),\\
\Gamma(\det F) &= \Gamma(\{\det F^tF\}^{\frac{1}{2}})\\
&= \Gamma(1 + \operatorname{tr} E + \frac{1}{2}(\operatorname{tr} E)^2 - \operatorname{tr} E^2 + O(|E|^3))\\
&= \Gamma(1) + \Gamma'(1)\left\{\operatorname{tr} E + \frac{1}{2}(\operatorname{tr} E)^2 - \operatorname{tr} E^2\right\}\\
&\qquad + \frac{1}{2}\Gamma''(1)(\operatorname{tr} E)^2 + O(|E|^3)\\
&= \Gamma(1) + \Gamma'(1)\operatorname{tr} E + \frac{1}{2}\{\Gamma'(1) + \Gamma''(1)\}(\operatorname{tr} E)^2\\
&\qquad - \Gamma'(1)\operatorname{tr} E^2 + O(|E|^3).
\end{aligned} \tag{B.1}$$

In order to have $\check{W}$ agree with the expansion of a stored energy function of a homogenous, isotropic, hyperelastic material near a natural state, we impose that

$$a|F|^2 + b\,|\operatorname{adj}_3 F|^2 + \Gamma(\det F^tF) + e = \frac{\lambda}{2}(\operatorname{tr} E)^2 + \mu\operatorname{tr} E^2 + O(|E|^3).$$

In other words, using the above expansions B.1, we have to solve the following system

$$\begin{cases}
3a + 3b + \Gamma(1) + e &= 0,\\
2a + 4b + \Gamma'(1) &= 0,\\
4b + \Gamma'(1) + \Gamma''(1) &= \lambda,\\
-2b - \Gamma'(1) &= \mu.
\end{cases}$$

where the first equation expresses the fact that $\hat{W}(I) = 0$ i.e. the reference configuration Ω is a natural state. Moreover, these equations have to be solved in such a way that

$$a > 0, \quad b > 0, \quad c > 0 \quad \text{and} \quad \Gamma''(1) = 2c + d \geq 0,$$

to verify the polyconvexity condition. Let us first introduce the following notation :$s = \Gamma'(1) = 2c - d$. From the first equation above we draw the value of e

$$e = -(3a + 3b + c).$$

Then we rearrange the remaining equations in order to express all constants in function of s as follows

$$\begin{cases} a &= \mu + \frac{1}{2}s, \\ b &= -\frac{1}{2}(\mu + s), \\ c &= \frac{\mu}{2} + \frac{\lambda}{4} + \frac{1}{2}s \\ d &= 2c - s. \end{cases}$$

Finally the imposed positivity on constantsc$a$, b and c implies the following restrictions on s :

$$s \in \left(- \min \left\{ \frac{\lambda}{2}, \mu \right\} - \mu, -\mu \right).$$

Hence, the theorem is proved by choosing, for instance,

$$\alpha = \min\{a, b, c\} \quad \text{and} \quad \beta = e.$$

□

Chapitre 3
Incompressible nonlinear membrane thin plates

Chapitre 3

Incompressible nonlinear membrane thin plates

Abstract

In this paper, we derive nonlinear membrane plate models for hyperelastic incompressible materials using Γ-convergence arguments. We obtain an integral representation of the limit two-dimensional internal energy owing to a result of singular functionals relaxation due to Ben Belgacem [8].

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The theory of plates and the derivation of nonlinear two-dimensional thin plate models from three-dimensional elasticity has received extensive attention over the last two decades. Starting with the seminal papers of Ciarlet & Destuynder [16, 17] where the method of formal asymptotic expansions was cast in a modern mathematical framework for the very first time, it has undergone several refinements while producing justifications for a variety of models such as the Von

Kàrmàn models by Ciarlet [12] and the membrane and inextensional models by Miara [59]. This approach consisted of solving a series of Euler-Lagrange equations where the limit model was progressively drawn through judicious choices of test functions after having supposed that the deformations (or displacements or strains) as well as the data of the problem admitted expansions in powers of the small parameter that is the thickness of the plate. Furthermore, formal assumptions on the scaling of the deformation were made. In this fashion, Fox, Raoult & Simo [40] achieved an important result using what stands out as the most refined version of the method of formal asymptotic expansions. Indeed, they rejustify a whole hierarchy of nonlinear thin plate models comprising the celebrated membrane, inextensional and Von Kàrmàn models while making minimal assumptions on the data. In this vein, Pantz [65, 66] has recently set this method to an even more agreeable and natural format by transposing the approach to solving a series of minimization problems hence putting an end to the guessing of the *right* test functions. In this manner, he justified once more the classical membrane and inextensional models. All of these attempts at deriving two-dimensional models have been carried out for the Saint Venant-Kirchhoff material. The latter method yielding simpler computations, it permitted to tackle a somewhat more realistic material proposed by Ciarlet & Geymonat [18] to quite successful results. Actually, in Trabelsi [73] (see Chapter 2) a new membrane model is obtained and the classical inextensional model is derived once more. In particular, the new model behaves like the classical membrane model for small displacements and its stored energy function becomes infinite where the normal to the middle surface is not defined.

Another crucial step was achieved by Le Dret & Raoult [50, 51] who derived a membrane model via Γ-convergence arguments thereby providing the first *rigorous* justification of a two-dimensional plate model. The limit stored energy function is relaxed as a Γ-limit consequently obtaining minimizers of the elastic energy which was not the case with the membrane energies obtained through the formal approach. As a matter of fact, Le Dret & Raoult dropped the orientation-preserving condition i.e. the strict positivity of the determinant of the deformation gradient for technical reasons. Ben Belgacem [6, 7] managed to respect this condition by imposing a more restrictive requirement on the admissible three-dimensional deformations which is that of having the energy to become infinite as the determinant of the deformation gradient tend to zero. To suit the needs of the asymptotic analysis, the author relaxes the associated minimization problem by identifying the weakly lower semicontinous envelope of the energy functional for the topology of the considered Sobolev space ; see also [8]. However, his asymptotic analysis did not benefit from Fonseca [39] and the ensuing precision in the approximation of the relawed energy. Note that more recently Friesecke, James & Müller [44, 43] likewise justified the inextensional model after a partially successful attempt by Pantz [66, 67, 68].

Here we derive an incompressible nonlinear membrane model. This is made possible owing to the aforementioned relaxation result which supplies an integral representation of the limit elastic energy and to density and approximation results due to Bennequin [6] and based on works by Whitney [75, 76, 77]. More precisely, we consider a family of thin plates of thickness 2ε and midsurface a bounded open domain of $\mathbb{R}^2$. We assume that all of them are made of the same homogenous isotropic and frame-indifferent incompressible material whose stored energy function is finite for deformations preserving the volume and infinite otherwise. The incompressibility condition imposes on the determinant of the gradient of the deformation to be equal to one hence it is yet more restrictive than the strict positivity of the determinant. This three-dimensional energy is neither relaxed nor continuous. Then through an asymptotic analysis based on Γ-convergence, we show that a sequence of quasi-minimizers of this energy converges towards a minimizer of a limit two-dimensional energy for the weak topology of the appropriate Sobolev space. To this intent, we obtain the model for deformations that are immersions or locally injective piecewise linear mappings. The latter energy is relaxed as in the non-singular case. These results are to be compared with those obtained by Ben Belgacem [6, 7].

The outline of this chapter reads as follows. Section 1 is devoted to the introduction of the different variational notions and tools to be used in the subsequent sections most crucial of which are an approximation result due to Bennequin [6] and the relaxation of singular functionals results of Fonseca [39]. Section 2 introduces the three-dimensional problem and prepares the grounds for the asymptotic analysis to follow. In section 3, we present the main result with some commentaries, comparaisons and straightforward properties. In Section 4, we prove some preliminary results to be used henceforth. Finally, Sections 5 and 6 are dedicated to the identification of the Γ-limit energy by deriving a lower and then an upper bound which turn out to be equal thus proving the purpose.

3.1 Notations, definitions and variational tools

In this section, we recall some elementary notions of vectorial convexity and results dealing with semicontinuity in the calculus of variations. For a thorough introduction to the subject, see Buttazzo [10] and Dacorogna [33] ; see also Giusti [46].

3.1.1 Vectorial convexity in the calculus of variations

A function $f : \mathbb{R}^{n\times N} \longrightarrow \bar{\mathbb{R}}$ is said to be ***rank-one-convex*** if

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B),$$

for all $\lambda \in [0,1],\ A,\ B \ \in \mathbb{R}^{n\times N}$ avec $\mathrm{rank}(A-B) \le 1$. The rank-one-convex envelope of f is

$$\mathbf{R}f = \sup\{h \le f :\ h \text{ rank-one convex}\}.$$

A Borel measurable and locally integrable function $f : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ is said to be ***quasiconvex*** if

$$|D|\, f(A) \leq \int_D f(A + \nabla\varphi)\, dx,$$

for every bounded domain $D \subset \mathbb{R}^n$, $A \in \mathbb{R}^{n\times N}$ and for every $\varphi \in W_0^{1,\infty}(D;\mathbb{R})$. The quasiconvex envelope of f is

$$\mathbf{Q}f = \sup\{h \leq f : \ h \text{ quasiconvex}\}.$$

A characterization of the above envelope was obtained by Dacorogna [32, 33]

$$\mathbf{Q}f(A) = \inf\left\{\frac{1}{|D|}\int_D f(A+\nabla\varphi)\, dx : \ \varphi \in W_0^{1,\infty}(D;\mathbb{R})\right\}, \tag{3.1}$$

for all $A \in \mathbb{R}^{n\times N}$ and all bounded open set $D \subset \mathbb{R}^n$.

The proposition below is an iterative algorithm proposed by Kohn & Strang for the computation of rank-one-convex envelopes.

Proposition 3.1.1. (Kohn & Strang [48]) *Let f be a bounded below Borel measurable function. Define the sequence $(\mathrm{R}_k f)_{k\in\mathbb{N}}$ by $\mathrm{R}_0 f = f$ and*

$$\begin{cases} \mathrm{R}_0 f = f, \\ \mathrm{R}_{k+1} f(A) = \inf\limits_{(a,b)\in\mathbb{R}^{n\times m}} \{(1-\lambda)\mathrm{R}_k f(A - \lambda\, a\otimes b) + \lambda\, \mathrm{R}_k f(A + (1-\lambda) a\otimes b)\} \end{cases}$$

for all $k \geq 1$ and for all $A \in \mathbb{R}^{n\times N}$. Then $(\mathrm{R}_k f)_{k\in\mathbb{N}}$ decreases to $\mathbf{R}f$.

The above algorithm is an extension of the algorithm that computes the convex envelope. Indeed, if $n = 1$, $\mathrm{R}_1(f)$ is merely the convex envelope of f. Next we give a result that will be useful to derive the upper bound in Proposition 3.7.2.

Proposition 3.1.2. *Let $f : \mathbb{R}^{n\times N} \longrightarrow \bar{\mathbb{R}}$ be a continuous function satisfying the bound*

$$f(A) \geq \alpha\, |A|^p + \beta, \quad A \in \mathbb{R}^{n\times N}, \tag{3.2}$$

where $(\alpha,\beta) \in \mathbb{R}_+ \times \mathbb{R}$ and $p > 1$. Let $k \in \mathbb{N}^*$, then for all $A \in \mathbb{R}^{n\times N}$ there exists $(a,b) \in \mathbb{R}^n \times \mathbb{R}^N$ and $\lambda \in [0,1]$ such that

$$(P_k) \qquad \mathrm{R}_k f(A) = (1-\lambda)\mathrm{R}_{k-1} f(A - \lambda\, a\otimes b) + \lambda\, \mathrm{R}_{k-1} f(A + (1-\lambda) a\otimes b).$$

Proof. We show that for all $k \in \mathbb{N}^*$, $\mathrm{R}_k f$ is lower semicontinuous and satisfies the bound 3.2. We proceed by induction on $k \in \mathbb{N}^*$.

Claim 3.1.1. $\mathrm{R}_1 f$ satisfies property (P_1).

Let $A \in \mathbb{R}^{n\times N}$. Without loss of generality, we assume that $R_1 f(A) < \infty$ (note that this is anyway verified by W_0, the function to which this result will be applied later on). Let $(a_m, b_m, \lambda_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^N \times [0,1]$ be a minimizing sequence of $R_1 f(A)$ i.e.

$$(1-\lambda_m) f(A - \lambda_m\, a_m \otimes b_m) + \lambda_m\, f(A + (1-\lambda_m) a_m \otimes b_m) \leq R_1 f(A) + \frac{1}{m}.$$

We first observe that $\lambda_m \in [0,1]$ for all $m \in \mathbb{N}$ and that either

$$\liminf_{n \longrightarrow \infty} \lambda_m > 0 \quad \text{or} \quad \liminf_{n \longrightarrow \infty} (1-\lambda_m) > 0.$$

Besides, the coercivity inequality implies

$$\alpha(1-\lambda_m)\,|A - \lambda_m\, a_m \otimes b_m|^p + \alpha\lambda_m\,|A + (1-\lambda_m) a_m \otimes b_m|^p \leq 1 - 2\beta + R_1 f(A)$$

for all $m \in \mathbb{N}$. As $p > 1$, the last inequality ensues that $(a_m \otimes b_m, \lambda_m)_{m\in\mathbb{N}}$ is bounded, therefore should we extract a subsequence,

$$\lambda_m \underset{m\to\infty}{\longrightarrow} \lambda \in [0,1] \quad \text{and} \quad (a_m \otimes b_m) \underset{m\to\infty}{\longrightarrow} (a, b) \in \mathbb{R}^n \times \mathbb{R}^N,$$

since the set of rank-one-connected matrices is closed by Proposition D.0.6. Hence by the lower semicontinuity of f, we infer

$$\begin{aligned} &(1-\lambda) f(A - \lambda\, a \otimes b) + \lambda\, f(A + (1-\lambda) a \otimes b) \leq \\ &\quad \liminf_{m \longrightarrow \infty} \{(1-\lambda_m) f(A - \lambda_m\, a_m \otimes b_m) + \lambda_m\, f(A + (1-\lambda_m) a_m \otimes b_m)\} \leq \\ &\qquad \mathrm{R}_1 f(A), \end{aligned}$$

In other words,

$$(1-\lambda) f(A - \lambda\, a \otimes b) + \lambda\, f(A + (1-\lambda) a \otimes b) = \mathrm{R}_1 f(A),$$

and the claim is justified.
Claim 3.1.2. $\mathrm{R}_1 f$ is lower semicontinuous.
In other words, we have to show that if $A \in \mathbb{R}^{n\times N}$ and $c > 0$ are such that $\mathrm{R}_1 f(A) > c$ then there exists $\epsilon > 0$ satisfying

$$B \in \mathbb{R}^{n\times N} : \ |A - B| \leq \epsilon \implies \mathrm{R}_1 f(B) > c.$$

Again, by contraposition, we assume that $\mathrm{R}_1 f$ is not lower semicontinuous at $A \in \mathbb{R}^{n\times N}$ then for all $m \in \mathbb{N}$ there exists $A_m \in \mathbb{R}^{n\times N}$ such that

$$|A_m - A| \leq \frac{1}{m} \quad \text{and} \quad \mathrm{R}_1 f(A_m) \leq c.$$

By *Claim* 3.1.1, for all $m \in \mathbb{N}$ there exists $(a_m, b_m, \lambda_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^N \times [0,1]$ such that

$$\mathrm{R}_1 f(A_m) = (1-\lambda_m) f(A_m - \lambda_m\, a_m \otimes b_m) + \lambda_m\, f(A_m + (1-\lambda_m) a_m \otimes b_m).$$

Now the coercivity condition imposes

$$\alpha(1-\lambda_m)\,|A_m - \lambda_m\, a_m \otimes b_m|^p + \alpha\lambda_m\,|A_m + (1-\lambda_m) a_m \otimes b_m|^p \leq c - 2\beta,$$

therefore $(a_m \otimes b_m)_{m\in\mathbb{N}}$ is necessarily bounded since so are $(A_m)_{m\in\mathbb{N}}$ and $(\lambda_m)_{m\in\mathbb{N}}$. As a consequence, should we extract a subsequence, we have

$$\lambda_m \underset{m\to\infty}{\longrightarrow} \lambda \in [0,1] \quad \text{and} \quad (a_m \otimes b_m) \underset{m\to\infty}{\longrightarrow} (a \otimes b) \in \mathbb{R}^n \times \mathbb{R}^N,$$

since the set of rank-one-connected matrices is closed; see Proposition D.0.6. Next, by the lower semicontinuity of f we infer

$$c < \mathrm{R}_1 f(A) \leq (1-\lambda) f(A - \lambda\, a \otimes b) + \lambda\, f(A + (1-\lambda) a \otimes b) \leq \liminf_{m \longrightarrow \infty} \{(1-\lambda_m) f(A_m - \lambda_m\, a_m \otimes b_m) + \lambda_m\, f(A_m + (1-\lambda_m) a_m \otimes b_m)\} \leq \liminf_{m \longrightarrow \infty} \mathrm{R}_1 f(A_m) \leq c,$$

hence leading to a contradiction.

Claim 3.1.3. $\mathrm{R}_1 f$ satisfies the bound 3.2.

Setting $L : A \in \mathbb{R}^{n\times N} \longrightarrow L(A) = \alpha\,|A|^p + \beta$, it suffices to observe that $\mathrm{R}_1 L = L$ to conclude.

Claim 3.1.4. $(P_k) \implies (P_{k+1})$.

Suppose $\mathrm{R}_k f$ is lower semicontinuous, verifies $L \leq \mathrm{R}_k f$ and satisfies property (P_k). Note that

$$\mathrm{R}_1[\mathrm{R}_k f] = \mathrm{R}_{k+1} f,$$

and therefore claims 3.1.1-3.1.3 entail *Claim* 3.1.4. □

We conclude this paragraph with a result that is a core ingredient to obtain the Γ-limit. Indeed, a crucial step in the derivation of an upper bound for the said limit is provided by a relaxation result due to Ben Belgacem [8]; see Theorem 0.2.4. The latter identifies the integral representation of relaxed singular functionals defined on ad hoc Sobolev spaces and whose rank-one-convex envelopes are everywhere finite. His proof is largely founded on partial relaxation results due to Fonseca [39] which we recall below. The notations involved below were defined in the Introduction before Theorem 0.2.4.

Proposition 3.1.3. (Fonseca [39]) *Let $D \subset \mathbb{R}^n$ be an open bounded set with $|\partial D| = 0$. The function*

$$\mathrm{Z}_D f(\xi) := \inf\left\{\frac{1}{|D|}\int_D f(\xi + \nabla\varphi)\,dx : \ \varphi \in \mathrm{Aff}_0(D;\mathbb{R}^m)\right\} \tag{3.3}$$

satisfies the following

(i) $\mathrm{Z}_D f(A) = \mathrm{Z} f(A)$, *for all* $A \in \mathbb{R}^{n\times m}$.

(ii) If $\phi \in \mathrm{Aff}_0^{\mathrm{V}}(D;\mathbb{R}^m)$, *then for all* $A \in \mathbb{R}^{n\times m}$

$$\mathrm{Z} f(A) \leq \frac{1}{|D|}\int_D \mathrm{Z} f(A + \nabla\phi(x))\,dx.$$

(iii) $\mathrm{Z} f$ *is rank-one-convex on the interior of its effective domain* $\mathcal{D}_e(\mathrm{Z} f)$.

In the above proposition, statement (i) means that the expression of $\mathrm{Z} does not depend on set \mathrm{D}$.

3.1.2 Approximation and density results

This paragraph is dedicated to the statement of density results that will serve in the different approximations we will have to go through in Section 5. In the following $\omega \subset \mathbb{R}^2$ is an open bounded set.

Definition 3.1.4. A mapping $\psi \in C^1(\omega;\mathbb{R}^3)$ is said to be an immersion at $x \in \omega$ if $Df(x)$ is injective. Otherwise, f is said to be singular at x.

Lemma 3.1.5. (Bennequin [6]) *Let $\psi \in \mathrm{Aff}^{\mathrm{V}}(\omega;\mathbb{R}^3)$ be a locally injective function such that $\psi_{|_V} = id$ where V is a neighbourhood of $\partial\omega$. Then there exists a sequence of immersions $(\psi^n)_{n\in\mathbb{N}} \subset C^1(\omega;\mathbb{R}^3)$ that satisfies the following*

- $\|\psi^n - \psi\|_{W^{1,p}(\omega;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0$;
- $\psi^n{}_{|_V} = id$, *for all* $n \in \mathbb{N}$;
- *there exists* $\delta > 0$ *such that* $\left|\psi^n_{,1} \wedge \psi^n_{,2}\right| \geq \delta$, *for all* $n \in \mathbb{N}$.

The above was proved in Ben Belgacem [6] for locally injective piecewise affine functions, the proof is reproduced in the appendix. The extension to $\mathrm{Aff}^{\mathrm{V}}(\omega;\mathbb{R}^3)$ is merely a technical matter. Next, we state and prove a density result useful to the derivation of the upper bound in Section 6, Proposition 3.7.2.

Proposition 3.1.6. *The class of functions*

$$E = \{\varphi \in \mathrm{Aff}(D;\mathbb{R}^m) : \mathrm{rank}\, \nabla\varphi = n \text{ a.e.}\}$$

is dense in $\mathrm{Aff}(D;\mathbb{R}^m)$ *endowed with its strong topology.*

Proof. It suffices to show that E is dense in $\mathrm{Aff}(D;\mathbb{R}^m)$ for the strong topology of $W^{1,\infty}(D;\mathbb{R}^m)$ since D is bounded and the density of $\mathrm{Aff}(D;\mathbb{R}^m)$ in $W^{1,p}(D;\mathbb{R}^m)$ endowed with its strong topology is a well-known result ; for instance see Ciarlet & Raviart [24] ; see also Ciarlet [12].

Now, let $\varphi \in \mathrm{Aff}(D;\mathbb{R}^m)$ then, by definition, there exists a finite family of subsets $(D_j)_{j\in J} \subset D$ such that $|D\backslash \cup_{j\in J} D_j| = 0$ and $\nabla\varphi_{|_{D_j}} = A_j \in \mathbb{R}^{n\times m}$. Let $B \in \mathbb{R}^{n\times m}$ such that $\mathrm{rank}\, B = n$. Next, for $\varepsilon > 0$ set $\varphi^\varepsilon = \varphi + \varepsilon B \in \mathrm{Aff}(D;\mathbb{R}^m)$. We claim that, in fact we have $\varphi^\varepsilon \in E$ for ε small enough. Indeed, let $(e_k)_{1\leq k\leq n}$ be the canonical basis of $\mathbb{R}^n$, then as B is injective $(Be_k)_{1\leq k\leq n}$ is a linearly independent family of $\mathbb{R}^m$. Hence, for $j \in J$, we can write

$$A_j e_k = \sum_{i=1}^{n} \lambda^j_{k,i} B e_i + b^j_k$$

for all $1 \leq k \leq n$, where b^j_k is orthogonal to vect $(Be_k)_{1\leq k\leq n}$. To check, that φ^ε is of maximal rank, let $(\mu^\varepsilon_k)_{1\leq k\leq n} \subset \mathbb{R}$ be such that

$$\sum_{k=1}^{n} \mu^\varepsilon_k (\nabla\varphi^\varepsilon)_k = \sum_{k=1}^{n} \mu^\varepsilon_k (A_j e_k + \varepsilon B e_k) = 0.$$

In other words, for all $j \in J$,

$$\sum_{k=1}^{n} \mu^\varepsilon_k \left(\sum_{i=1}^{n} \lambda^j_{k,i} B e_i + b^j_k + \varepsilon B e_k\right) = \sum_{k=1}^{n} \mu^\varepsilon_k b^j_k + \sum_{k=1}^{n} \left(\varepsilon\mu^\varepsilon_k + \sum_{i=1}^{n} \mu^\varepsilon_i \lambda^j_{i,k}\right) B e_k = 0.$$

Thus, for all $j \in J$,

$$\begin{cases} \displaystyle\sum_{k=1}^{n} \mu^\varepsilon_k b^j_k = 0 \\ \displaystyle\varepsilon\mu^\varepsilon_k + \sum_{i=1}^{n} \mu^\varepsilon_i \lambda^j_{i,k} = 0 \quad \forall\, 1 \leq k \leq n \end{cases}$$

Therefore, if we set $\mu^\varepsilon = (\mu_k^\varepsilon)_{1\le k\le n} \in \mathbb{R}^n$ and $\Lambda^j = (\lambda_{i,k}^j)_{1\le i,k\le n} \subset \mathbb{R}^{n\times m}$ we can write for all $j \in J$,

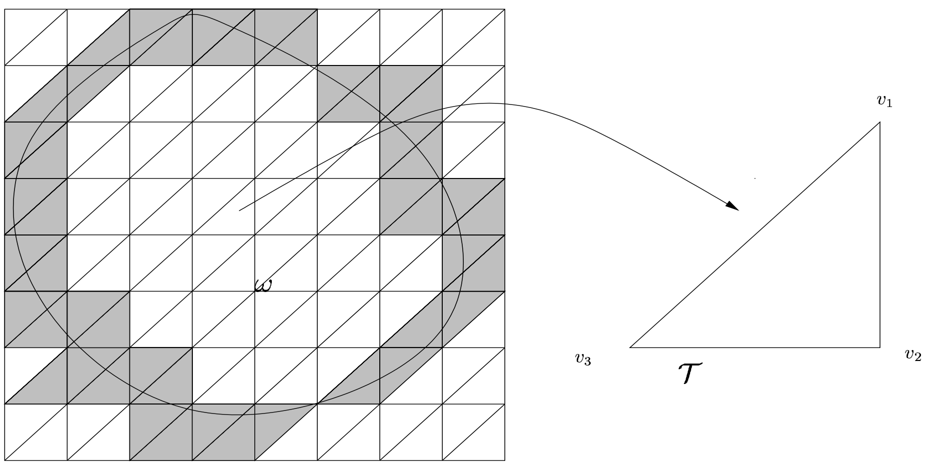
$$\left(\Lambda^j + \varepsilon\, id\right) \mu^\varepsilon = 0.$$

The discriminant of the above homogenous linear system $\det\left(\Lambda^j + \varepsilon\, id\right)$ is the characteristic polynomial associated to matrix Λ^j which is not identically equal to zero and has at most n solutions. It suffices to fix $\varepsilon_0 > 0$ smaller than the smallest positive solution of the polynome so that for all $\varepsilon < \varepsilon_0$, $\mu^\varepsilon = 0$ and consequently rank $\nabla\varphi^\varepsilon = n$. □

The next proposition is about locally injective piecewise-linear interpolation of immersions.

Proposition 3.1.7. *If $\psi \in C^1(\omega;\mathbb{R}^3)$ is an immersion such that $\psi_{|\omega} = id$, then there exists a sequence of locally injective functions $(\psi^n)_{n\in\mathbb{N}} \subset id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$ such that $\|\psi^n - \psi\|_{W^{1,p}(\omega;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0$.*

Proof. Without loss of generality, let $\psi \in C^2(\omega;\mathbb{R}^3)$ be an immersion such that $\psi_{|V} = id$ where V is a neighbourhood of $\partial\omega$. We consider a regular mesh of ω as in the figure below that is a lattice of squares of size $h > 0$ and each square is divided into two triangles. Then we interpolate ψ by an affine function on each triangle through the values at the vertices and note the interpolation $\psi_h \in \mathrm{Aff}(\omega;\mathbb{R}^3)$; in doing so, we prescribe $\psi_h = id$ on the triangles that are intersected by the boundary $\partial\omega$ (these correspond to the shaded areas below).



Claim 3.1.5. There exists $C(1) > 0$ such that

$$\|\psi - \psi_1\|_{W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)} \le C(1)\, |\psi|_{W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)}\,.$$

To prove the claim, we proceed by contraposition. Assume the above is false, then there exists $(\psi^n)_{n\in\mathbb{N}} \subset W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)$ such that

$$\|\phi^n\|_{W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)} = 1 \quad \text{and} \quad |\phi^n|_{W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0,$$

where $\phi^n = \psi - \psi_1$. We first observe that by the compact inclusion of $W^{2,\infty}$ into $W^{1,\infty}$ and the boundedness of $(\phi^n)_{n\in\mathbb{N}}$ in $W^{2,\infty}$, there exists $\phi \in W^{1,\infty}(\mathcal{T}_1;\mathbb{R}^3)$ such that $\phi^n \underset{n\to\infty}{\longrightarrow} \phi$ in $W^{1,\infty}$ after the extraction of a subsequence. Hence if we write,

$$\left\|\phi^n - \phi^{n+p}\right\|_{W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)} = \left\|\phi^n - \phi^{n+p}\right\|_{W^{1,\infty}(\mathcal{T}_1;\mathbb{R}^3)} + \left|\phi^n - \phi^{n+p}\right|_{W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)},$$

we infer that $\|\phi^n - \phi^{n+p}\|_{W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0$ by definition of $(\phi^n)_{n\in\mathbb{N}}$ so that the completeness of $W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)$ ensures the convergence of the Cauchy sequence $(\phi^n)_{n\in\mathbb{N}}$ to ϕ in $W^{2,\infty}$. As a consequence, on the one hand

$$|\phi|_{W^{2,\infty}(\mathcal{T}_1;\mathbb{R}^3)} = 0 \qquad \Longrightarrow \qquad \phi \in \mathrm{Aff}(\mathcal{T}_1;\mathbb{R}^{\mathbb{R}^3}),$$

and on the other hand the pointwise convergence $\phi^n(x) \underset{n\to\infty}{\longrightarrow} \phi(x)$ for all $x \in \mathcal{T}_1$ ensues that ϕ vanishes at the vertices of $\mathcal{T}_1$. As a conclusion, $\phi = 0$ on $\mathcal{T}_1$ which contradicts $\|\phi^n\|_{W^{1,\infty}(\mathcal{T}_1;\mathbb{R}^3)} = 1$ thus completeing the justification of the claim.

Claim 3.1.6. For $h > 0$ small enough function ψ_h is locally injective.

We first remark that by dilatation we derive the following bound from the previous claim :

$$\|\psi - \psi_h\|_{W^{1,\infty}(\omega;\mathbb{R}^3)} \le C(1)h^2\,|\psi|_{W^{2,\infty}(\omega;\mathbb{R}^3)}\,.$$

Next, by the local injectivity of the immersion ψ, for all $x \in \omega$ there exists $\delta,\ c > 0$ such that

$$y, z \in B(x,\delta) \qquad \Longrightarrow \qquad |\psi(y) - \psi(z)| \ge c\,|y - z|\,.$$

Therefore setting $E_h = \psi - \psi_h$, we can write

$$\begin{aligned} |\psi_h(y) - \psi_h(z)| &\ge |\psi(y) - \psi(z)| - |E_h(y) - E_h(z)|\,,\\ &\ge c\,|y - z| - |\nabla E_h|_{L^\infty(\omega;\mathbb{R}^3)}\,|y - z|\,,\\ &\ge (c - C(1)h)\,|y - z|\,, \end{aligned}$$

which entails the local injectivity of ψ_h for h small enough and the proof is complete.□

The proof above is inspired by the finite element method ; for a broad introduction to this kind of interpolation methods and related bibliography we refer the reader to Ciarlet [12]. We conclude this section with a density result that we conjecture because we could not provide a rigorous proof.

Conjecture 3.1.8. Let $\psi \in W^{1,p}(\omega;\mathbb{R}^3)$, then there exists a sequence of immersions $(\psi^n)_{n\in\mathbb{N}} \subset C^1(\omega;\mathbb{R}^3)$ such that $\|\psi^n - \psi\|_{W^{1,p}(\omega;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0$.

A sketch of an attempt at proving the above result was given in Ben Belgacem [6] and is due to Bennequin. It mainly adapts Whitney's result dealing with the density of the class of semi-regular functions in the class C^1 functions with respect to the uniform convergence norm ; see [75, 76, 77]. The conjecture seems to be quite sensible and the main obstacle to give a neat proof of this result is the seeming difficulty to fit the geometrical arguments to a Sobolev space framework.

3.2 Formulation of the three-dimensional problem

A plate is an elastic body whose natural reference configuration is a cylinder whose height, also called thickness, is small compared to the other two dimensions. We denote by ω the interior mid-surface (here, a plane) parallel to the base of the cylinder. Let $(e_i)_{i=1,2,3}$ be an orthonormal basis of $\mathbb{R}^3$ so that ω lies in the plane spanned by e_1 and e_2. Greek indices take on values 1 or 2, while Latin indices take on values 1, 2 or 3. Partial differentiation of a function or a vector field ψ with respect to the vector e_i is denoted $\psi_{,i}$ and its gradient (matrix) is denoted $\nabla\psi$. We adopt the classical convention of summation on repeated indices.

3.2.1 Reference configuration, loading and boundary conditions

In its reference configuration, the plate occupies the domain

$$\Omega^\varepsilon = \omega \times]-\varepsilon,\ \varepsilon[,$$

where ω is an open subset of $\mathbb{R}^2$ and ε is the small parameter (i.e. small compared to the dimensions of ω). The boundary $\partial\Omega^\varepsilon$ of Ω^ε is partitioned as follows :

$$\begin{aligned} \Gamma_+^\varepsilon &= \omega \times \{\varepsilon\}, \quad \text{the top surface,} \\ \Gamma_-^\varepsilon &= \omega \times \{-\varepsilon\}, \quad \text{the bottom surface,} \\ \Gamma^\varepsilon &= \partial\omega \times [-\varepsilon,\ \varepsilon], \quad \text{the lateral surface.} \end{aligned}$$

The coordinates of a point x in the reference configuration $\overline{\Omega^\varepsilon}$ are

$$x = (x^1, x^2, x^3), \quad \text{where} \quad (x^1, x^2) = x_H \in \overline{\omega} \quad \text{and} \quad x^3 \in [-\varepsilon,\ \varepsilon].$$

The volume element is denoted $dx = dx^1 dx^2 dx^3$, while the area element is $dx_H = dx^1 dx^2$. The plate is subjected to a dead body force $b^\varepsilon \in L^q(\Omega^\varepsilon; \mathbb{R}^3)$ and to dead surface tractions $g_\pm^\varepsilon \in L^q(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \cup; \mathbb{R}^3)$ on the top and bottom surfaces, while it is clamped on the whole lateral surface that is Γ^ε. The manifold of admissible deformations is

$$\mathcal{V}^\varepsilon = \left\{ \psi^\varepsilon \in W^{1,p}(\Omega^\varepsilon; \mathbb{R}^3) : \det \nabla\psi^\varepsilon = 1 \text{ a.e.} \quad \text{and} \quad \psi^\varepsilon_{|\Gamma^\varepsilon_\phi} = id \right\},$$

where the constraint $\det \nabla\psi^\varepsilon = 1$ translates the incompressibility of the material, in other words any elementary volume of an incompressible material is preserved in the course of deformation. Furthermore, we assume that $\frac{1}{p} + \frac{1}{q} = 1$ so that the energy is clearly well-defined. Indeed, other choices are possible.

All the plates introduced above are assumed to be made of the same homogeneous and isotropic hyperelastic material. Such a material, when subjected to the loads and boundary conditions described above, undergoes a deformation $\varphi^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ which should be a stationary point of the energy defined by

$$J_\varepsilon(\varphi^\varepsilon) = I_\varepsilon(\varphi^\varepsilon) - \ell_\varepsilon(\varphi^\varepsilon),$$

where the linear form ℓ_ε is the work of the external forces

$$\ell_\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} b^\varepsilon \cdot \varphi^\varepsilon \, dx + \int_{\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon} g^\varepsilon \cdot \varphi^\varepsilon \, dx_H,$$

and the functional I^ε measures the internal energy of the plate

$$I^\varepsilon(\varphi^\varepsilon) = \int_{\Omega^\varepsilon} W(\nabla \varphi^\varepsilon) \, dx.$$

The stored energy function of the material W is defined in the following fashion

$$W(\xi) = \begin{cases} W^*(F) & \text{if } \det F = 1 \\ +\infty & \text{otherwise,} \end{cases}$$

where $F \in \mathbb{R}^{3\times 3}$ stands for the deformation gradient. We make the following ad hoc hypotheses on the energy, for all fixed $p > 1$,

(i) $W^* : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}$ is continuous,
(ii) (growth) $\exists C > 0$ such that $W^*(F) \leq C(1 + |F|^p)$, (H)
(iii) (coercivity) $\exists \alpha > 0$ and $\beta \in \mathbb{R}$ such that $W^*(F) \geq \alpha |F|^p + \beta$.

The simplest instance of such an energy is that of the *Mooney-Rivlin* material :

$$W^*(F) = a\,|F|^2 + b\,|\mathrm{Cof}\, F|^2 + c.$$

More generally, Ogden materials can be considered. Finally, the state of equilibrium of the family of plates of thickness 2ε is defined as being the solution to the minimization problem

$$(P^\varepsilon) \qquad \varphi^\varepsilon \in \mathcal{V}^\varepsilon \quad \text{and} \quad J_\varepsilon(\varphi^\varepsilon) = \inf_{\psi^\varepsilon \in \mathcal{V}^\varepsilon} J_\varepsilon(\psi^\varepsilon).$$

Remark 3.2.1. Note that we make no convexity assumption on function W^*. Ball [4] showed that the polyconvexity condition is sufficient for minimizing singular functionals. More recently, Ben Belgacem [8] identified the integral representation of a whole family of singular functionals as was exposed in the previous section.

Our goal is to study the asymptotic behaviour of the family of plates as the thickness 2ε tends to zero. To do so, the order of magnitude of the applied loads with respect to ε must be specified. More accurately, as we wish to obtain a membrane model, it turns out that the right order of magnitude is

$$|b^\varepsilon|_{L^p(\Omega^\varepsilon;\mathbb{R}^3)} \leq C\varepsilon^{\frac{1}{q}} \quad \text{and} \quad |g^\varepsilon|_{L^p(\Omega^\varepsilon;\mathbb{R}^3)}\,.$$

This fact was already observed by Fox, Raoult & Simo [40] and Le Dret & Raoult [53]. Practically, we let the parameter ε tend to zero and we consider a sequence of minimizing (or almost minimizing as will be remarked later) deformations $(\psi^\varepsilon)_{\varepsilon>0} \subset \mathcal{V}^\varepsilon$ such that

$$J_\varepsilon(\psi^\varepsilon) \leq \inf_{\phi \in id + W_0^{1,p}(\Omega;\mathbb{R}^3)} J_\varepsilon(\phi) + \varepsilon h(\varepsilon) \quad \forall \varepsilon > 0,$$

where function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is such that $h(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

3.2.2 Scaling and change of coordinates

The solutions φ^ε to the above three-dimensional problem are defined on the set Ω^ε which varies when ε does. In order to carry out an asymptotic analysis, it is useful to operate a scaling that transposes the problem onto a fixed domain. Therefore, we define the operator π_ε by

$$(\pi_\varepsilon f)(x^1, x^2, x^3) = f(x^1, x^2, \varepsilon x^3).$$

For every function f^ε and every functional G^ε, we define

$$\begin{aligned} f(\varepsilon) &= \pi_\varepsilon^{-1} f^\varepsilon, \\ G(\varepsilon)(\psi) &= G_\varepsilon(\pi_\varepsilon^{-1}\psi). \end{aligned}$$

We also introduce the following notations

$$\begin{aligned} \Omega &= \Omega^1 = \omega\times]-1,1[, \\ \Gamma &= \partial\omega\times]-1,1[, \\ \Gamma_\pm &= \Gamma^1_\pm = \omega\times\{\pm 1\}. \end{aligned}$$

An element in Ω has coordinates (x^1, x^2, ξ); differentiation with respect to ξ is denoted by the subscript $_{,\xi}$. The minimization problem equivalent to P^ε and associated with the ε-independent domain Ω^ε consists in finding solutions to the problem

$$\varphi(\varepsilon) \in \mathcal{V}(\varepsilon) \quad \text{and} \quad J(\varepsilon)(\varphi(\varepsilon)) = \inf_{\psi(\varepsilon)\in\mathcal{V}(\varepsilon)} J(\varepsilon)(\psi(\varepsilon)),$$

where the total energy $J(\varepsilon) = I(\varepsilon) - \ell(\varepsilon)$ is such that

$$I(\varepsilon)(\psi(\varepsilon)) = \varepsilon\tilde{I}(\varepsilon) = \varepsilon\int_\Omega W\left(\left(\psi_{,1}(\varepsilon)\middle|\psi_{,2}(\varepsilon)\middle|\frac{1}{\varepsilon}\psi_{,\xi}(\varepsilon)\right)\right) dx,$$

$$\ell(\varepsilon)(\psi(\varepsilon)) = \varepsilon\tilde{\ell}(\varepsilon)(\psi) = \varepsilon\int_\Omega b(\varepsilon)\cdot\psi(\varepsilon)\, dx + \int_S g(\varepsilon)\cdot\psi(\varepsilon)\, dx_H,$$

where we have noted $\Gamma_+ \cup \Gamma_- = S$. The manifold of admissible deformations is now

$$\mathcal{V}(\varepsilon) = \left\{\psi(\varepsilon) \in W^{1,p}(\Omega;\mathbb{R}^3) : \det\nabla\psi(\varepsilon) = \varepsilon \text{ a.e.} \quad \text{and} \quad \psi(\varepsilon)_{|\Gamma} = (x_H, \varepsilon\xi)\right\}.$$

Observe that in the above we have used that

$$\nabla\psi^\varepsilon = \left(\psi_{,1}(\varepsilon)\middle|\psi_{,2}(\varepsilon)\middle|\frac{1}{\varepsilon}\psi_{,\xi}(\varepsilon)\right).$$

For simplicity, we will suppose that $b(\varepsilon) = b$ and $g(\varepsilon) = \varepsilon g$ where f and g are independent of ε.

3.2.3 Setting of the asymptotic procedure and Γ-convergence

We recall that a sequence of functionals $(G_\varepsilon)_{\varepsilon>0}$ defined on a topological space X and taking its values in $\bar{\mathbb{R}}$ is said to Γ-***converge*** towards G for the topology of X if for all $x \in X$, the two conditions below are satisfied

– $\forall (x_\varepsilon)_{\varepsilon>0} \subset X, \quad x_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} x \quad \Longrightarrow \quad G(x) \leq \liminf_{\varepsilon\to 0} G_\varepsilon(x_\varepsilon),$

– $\exists (\bar{x}_\varepsilon)_{\varepsilon>0} \subset X$ such that $\bar{x}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} x$ and $G(x) = \liminf_{\varepsilon\to 0} G_\varepsilon(\bar{x}_\varepsilon).$

Moreover the limit is also defined by

$$G(x) = \min\left\{ \liminf_{\varepsilon\to 0} G_\varepsilon(x_\varepsilon) : \ x_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} x \right\}.$$

If in addition, X is separable and locally metrisable, then functions defined on X and taking values in $\bar{\mathbb{R}}$ possess a property of sequential compacity for Γ-convergence. Actually, any sequence $G_\varepsilon : X \longrightarrow \bar{\mathbb{R}}$ is relatively compact with respect to the topology of Γ-convergence i.e. we can extract a subsequence $(G_{\varepsilon'})_{\varepsilon'>0} \subset (G_\varepsilon)_{\varepsilon>0}$ that is Γ-convergent. The primary relevance of Γ-convergence is that if minimizers of $(G_\varepsilon)_{\varepsilon>0}$ remain in a compact set of X then their cluster points are minimizers of the Γ-limit G. In fact, if $(x_\varepsilon)_{\varepsilon>0} \subset X$ is such that

– $\left| G_\varepsilon(x_\varepsilon) - \inf_{x\in X} G_\varepsilon(x) \right| \longrightarrow 0,$

– $(x_\varepsilon)_{\varepsilon>0}$ is in a compact set of X

then cluster points of $(x_\varepsilon)_{\varepsilon>0}$ are minimizers of G. Hence, the validity of this property does not require the attainment of the infimum of G_ε. For a broad introduction to this subject, we refer to De Giorgi & Franzoni [37] and Dal Maso [36].

Now, to avoid the use of the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$ which is not metrizable on unbounded sets, a classical trick is to extend the the energies to $L^p(\Omega;\mathbb{R}^3)$ by setting

$$\bar{J}(\varepsilon)(\psi(\varepsilon)) = \begin{cases} \tilde{J}(\varepsilon)(\psi(\varepsilon)) & \text{if } \psi(\varepsilon) \in \mathcal{V}(\varepsilon) \\ +\infty & \text{if } \psi \in L^p(\Omega;\mathbb{R}^3) \text{ and } \psi \notin \mathcal{V}(\varepsilon). \end{cases}$$

As we consider a diagonal sequence of quasi-minimizers $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ i.e.

$$\bar{J}(\varepsilon)(\psi(\varepsilon)) \leq \inf_{\phi\in id+W_0^{1,p}(\Omega;\mathbb{R}^3)} \bar{J}(\varepsilon)(\phi) + h(\varepsilon) \quad \forall \varepsilon > 0,$$

the characterization of the asymptotic behaviour of such a sequence is best described by the Γ-limit of the sequence of functionals $(\bar{J}(\varepsilon))_{\varepsilon>0}$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$.

3.3 Main result

The main result of this chapter is the obtention of a nonlinear membrane plate model made of incompressible material through the identification of the Γ-limit of the sequence of functionals $(\bar{J}(\varepsilon))_{\varepsilon>0}$. Let us first introduce the space of membrane deformations

$$\begin{aligned}\mathcal{M}(\omega) &= \left\{\varphi \in W^{1,p}(\omega;\mathbb{R}^3) : \ \varphi_{|\partial\omega}(x_H) = (x_H, 0)\right\}, \\ &\approx \left\{\bar{\varphi} \in W^{1,p}(\Omega;\mathbb{R}^3) : \ \bar{\varphi}_{,\xi} = 0 \ \text{ and } \ \bar{\varphi}_{|\Gamma}(x_H,\xi) = (x_H, 0)\right\}.\end{aligned}$$

As expected, membrane deformations are no more dependent on the third variable. Now we can enunciate the precise statement of our result.

Theorem 3.3.1. *$\bar{J}(\varepsilon)$ is Γ-convergent towards to $\bar{J}(0)$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$ and we have*

$$\bar{J}(0)(\varphi) = \begin{cases} 2\displaystyle\int_\omega \mathbf{QR}W_0(\nabla\varphi)\,dx_H - \ell(0) & \text{if } \varphi \in \mathcal{M}(\omega) \\ +\infty & \text{otherwise,}\end{cases} \tag{3.4}$$

where

$$W_0(F) = \inf_{z\in\mathbb{R}^3} W((F|z)) \quad \textit{for all } F \in \mathbb{R}^{3\times 2}, \tag{3.5}$$

and

$$\ell(0)(\varphi) = \int_\omega \left\{\int_{-1}^{1} b(x_H,\xi)\,d\xi + g(x_H,-1) + g(x_H,1)\right\}\cdot\varphi\,dx_H.$$

The proof of the above theorem is the object of sections 5 and 6. Here, we point out the constitutive and mathematical properties of the derived membrane energy and the consequences for the minimizing sequences. First, we make a few remarks on the spot.

Remark 3.3.2. (i) At first sight, the internal energy associated with the limit model (3.4) is bidimensional, it no longer depends on the thickness of the plate meaning that the admissible deformations are now deformations of the middle surface ω into a surface of $\mathbb{R}^3$. Furthermore, the elastic energy depends solely on the gradient of the deformation i.e. the first fundamental form, or more precisely on the metric tensor $\nabla\varphi^t\nabla\varphi$. Hence, bending effects associated to the curvature are dismissed and only stretching of the surface is accounted for in the energy which describes in this fashion a membrane elastic body.
(ii) The above functional (3.4) admits minimizers in $\mathcal{M}(\omega)$. This claim is vindicated either by the functional being a Γ-limit of coercive functionals (cf. [36]) or by the $W^{1,p}(\omega;\mathbb{R}^3)$ weak lower semicontinuity of the limit functional and its coercivity and growth conditions according to Theorem 0.2.2 ; see also [8]. Let us mention that in our case all required assumptions on W_0 are satisfied as will be seen. In particular, $\mathcal{D}_e(W_0) = \mathcal{O}_{W_0} = \{F \in \mathbb{R}^{3\times 2} : \operatorname{rank} F = 2\}$.
(iii) More general loadings may be considered.
(iv) In light of recent papers, first by Pantz [66, 67] then by Friesecke, James & Müller [43, 44] justifying nonlinear bending plate models roughly through identifying the Γ-limit of $\frac{1}{\varepsilon^2}\bar{J}(\varepsilon)(\psi(\varepsilon))$ as ε tends to zero, it would be interesting to investigate the incompressible case. As a matter of fact, this problem will be addressed elsewhere.

For a starter, we examine the behaviour of the minimizing sequences.

Proposition 3.3.3. *Let $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ be a diagonal sequence of quasi-minimizers i.e. there exists a function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $h(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$ and*

$$\bar{J}(\varepsilon)(\psi(\varepsilon)) \leq \inf_{\phi\in id+W_0^{1,p}(\Omega;\mathbb{R}^3)} \bar{J}(\varepsilon)(\phi) + h(\varepsilon) \quad \forall\, \varepsilon > 0,$$

then its cluster points belong to $\mathcal{M}(\omega)$ and are solutions to the minimization problem

$$\bar{J}(0)(\varphi) = \inf_{\psi\in\mathcal{M}(\omega)} \bar{J}(0)(\psi).$$

Proof. Let $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ be a minimizing sequence of $\left(\bar{J}(\varepsilon)\right)_{\varepsilon>0}$ then, as $\left(\bar{J}(\varepsilon)(\psi(\varepsilon))\right)_{\varepsilon>0}$ is uniformly bounded, by Lemma 3.5.1, we infer that $(\psi(\varepsilon))_{\varepsilon>0}$ is relatively weakly compact in $W^{1,p}(\Omega;\mathbb{R}^3)$ and its cluster points belong to $\mathcal{M}(\omega)$. Let ψ be such a cluster point, by definition we have

$$\begin{cases} \psi(\varepsilon') \rightharpoonup \psi \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^3), \\ \bar{J}(0)(\psi) = \lim\limits_{\varepsilon'\to 0} \bar{J}(\varepsilon')(\psi(\varepsilon)) \end{cases}$$

Let $\phi \in \mathcal{M}(\omega)$. By definition of the Γ-limit, there exists $(\phi(\varepsilon))_{\varepsilon>0} \subset W^{1,p}(\Omega;\mathbb{R}^3)$

$$\begin{cases} \phi(\varepsilon') \rightharpoonup \phi \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^3), \\ \bar{J}(0)(\phi) \leq \liminf\limits_{\varepsilon'\to 0} \bar{J}(\varepsilon')(\phi(\varepsilon')) \end{cases}$$

Consequently, we can write

$$\bar{J}(0)(\phi) \leq \liminf_{\varepsilon'\to 0} \bar{J}(\varepsilon')(\phi(\varepsilon')) \leq \liminf_{\varepsilon'\to 0} \left\{\bar{J}(\varepsilon')(\psi(\varepsilon')) + h(\varepsilon')\right\} = \bar{J}(0)(\psi),$$

establishing that ϕ is minimizer of $\bar{J}(0)$ on $\mathcal{M}(\omega)$. □

In the next proposition, we make sure that the basic properties we usually require the 3-d energy to meet, such as frame-indifference and isotropy, are transmitted to the membrane energy in so far as it is possible.

Proposition 3.3.4. *If the function W^* satisfies the principle of material frame-indifference then $\mathbf{QR}W_0$ is frame-indifferent*

$$\mathbf{QR}W_0(F) = \mathbf{QR}W_0(RF) \quad \forall R \in SO(3),\ F \in \mathbb{R}^{3\times 2}.$$

If the function W^* is isotropic, then function $\mathbf{QR}W_0$ is isotropic

$$\mathbf{QR}W_0(F) = \mathbf{QR}W_0(FR) \quad \forall R \in SO(2),\ F \in \mathbb{R}^{3\times 2}.$$

If the function W^* is isotropic and frame-indifferent, then there exists a symmetric function $\phi : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ such that

$$\mathbf{QR}W_0(F) = \phi\left(s_1(F), s_2(F)\right)$$

where $s_1(F)$ and $s_2(F)$ are the singular values of matrix $F \in \mathbb{R}^{3\times 2}$. If in addition the function W^* is positive and verifies

$$W^*(id) = \min\left\{W^*(\bar{F}) : \bar{F} \in \mathbb{R}^{3\times 3} \textit{ such that } \det \bar{F} = 1\right\} = 0,$$

then $\phi(x, y) = 0$ for all $(x, y) \in [0, 1]^2$.

Proof. We first show that the function W_0 satisfies the principle of material frame-indifference. By defintion, we have

$$W_0(F) = \inf_{z\in\mathbb{R}^3} W((F|z)) \quad \text{for all } F \in \mathbb{R}^{3\times 2},$$

therefore for all $F \in \mathbb{R}^{3\times 2},\ z \in \mathbb{R}^3 \in$ and $R \in SO(3)$,

$$W_0(F) \le W(F|z) \le W(R(F|z)) = W((RF|Rz)),$$

where we have used the frame-indifference of function W. Hence, minimizing the right-hand side with respect to the third column-vector yields

$$W_0(F) \le W_0(RF), \quad \text{for all } F \in \mathbb{R}^{3\times 2} \text{ and all } R \in SO(3).$$

Rewriting the above inequality for $R^{-1} \in SO(3)$ and applying it to $RF \in \mathbb{R}^{3\times 2}$, we bring out the reverse inequality and conclude that

$$W_0(F) = W_0(RF), \quad \text{for all } F \in \mathbb{R}^{3\times 2} \text{ and all } R \in SO(3).$$

Next, we remark that for all $R \in SO(3)$, function $\mathrm{R}W_0^R : F \in \mathbb{R}^{3\times 2} \longrightarrow \mathbf{R}W_0(RF)$ is rank-one convex and smaller than W_0. Therefore, as by definition the rank-one convex envelope $\mathbf{R}W_0$ is the greatest rank-one convex function smaller than W_0, we necessarily have $\mathbf{R}W_0^R \le \mathbf{R}W_0$. In particular, we also have $\mathbf{R}W_0^{R^{-1}} \le \mathbf{R}W_0$ which at point $RF \in \mathbb{R}^{3\times 2}$ raises the reverse inequality $\mathbf{R}W_0 \le \mathbf{R}W_0^R$ thus completing the proof of the frame-indifference of $\mathbf{R}W_0$. We follow the same argumentation to show the frame-indifference of $\mathbf{QR}W_0$. To this end, we use Dacorogna's representation formula (3.1)

$$\mathbf{QR}W_0(F) = \inf\left\{\frac{1}{|D|}\int_D \mathbf{R}W_0(F + \nabla\varphi)\, dx :\ \varphi \in W_0^{1,\infty}(D;\mathbb{R})\right\}, \tag{3.6}$$

for all $F \in \mathbb{R}^{3\times 2}$ and any bounded domain $D \subset \mathbb{R}^2$. In our case, the right choice for the domain would be a disk so that for all $\varphi \in W_0^{1,\infty}(D;\mathbb{R})$ and for all $R \in SO(3)$, we have $R\varphi \in W_0^{1,\infty}(D;\mathbb{R})$ as well as $\nabla R\varphi = R\nabla\varphi$. The rest of the proof reads exactly as above. The isotropy property is verified likewise. The existence of function ϕ is a straightforward consequence of the frame-indifference. Indeed, by a sort of *polar factorization*, we have that for all $F \in \mathbb{R}^{3\times 2}$, $F = Rj(F^tF)^{\frac{1}{2}}$ where $j : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{3\times 2}$, $A \longmapsto A$. Hence function $\phi : (\mathbb{R}_+)^2 \longrightarrow \mathbb{R}$, $(y_1, y_2) \longmapsto \phi(y_1, y_2) = \mathbf{QR}W_0(j\,\mathrm{diag}\, y_\alpha)$ is symmetric and answers the question. Finally, to show that $\phi \equiv 0$ on $[0, 1]^2$, it suffices to recall a property proved in Section 3 of Chapter 1 that states that $\mathbf{R}W_0(F) = 0$ for matrices F whose singular values are less than one and then to conclude by remarking that 0 is quasiconvex. □

Remark 3.3.5. (i) A natural question arises : could we not avoid the rank-one-convexification and quasiconvexification steps ? Well, the answer seems to be no in the general case according to the previous proposition since for compressive states i.e. deformations with singular values less than 1, we clearly have $\mathbf{QR}W_0 < W_0$ (we refer to the next section for more evidence on the subject). One of the instances, as was observed in Le Dret & Raoult [53], where the rank-one-convexity (the same holds for quasiconvexity and polyconvexity) of the original three-dimensional energy W is transmitted to W_0 is when the minimum in 3.4.1 does not depend on the deformation i.e. there exists $z_0 \in \mathbb{R}^3$ such that $W_0(F) = W((F|z_0))$ for all $F \in \mathbb{R}^{3\times 2}$. However, as the minimum can only be attained for deformations locally preserving the volume, it requires that $\det((F|z_0)) = 1$ for all $F \in \mathbb{R}^{3\times 2}$. As a conclusion, it is obvious that this assumption is incompatible with the incompressibility of the material.
(ii) Another natural question is : couldn't we take $(\mathbf{QR}W)_0$ instead ? The answer is no for multiple reasons. The first one is that $\mathbf{R}W$ is not finite, since there are no rank-one-connected matrices with determinants equal to one, and therefore we would not know how to relax the functional. Secondly, nothing guarantees that $(\mathbf{QR}W)_0$ is relaxed.
Remark 3.3.6. In the case of thin plates made of Saint-Venant Kirchhoff material that is

$$\begin{aligned}
W^*(F) = W^*\left((F_1|F_2|F_3)\right) &= \frac{\mu}{4}\left(\sum_{i,j=1}^{3}(F_i \cdot F_j - \delta_j^i)^2\right) + \frac{\lambda}{8}\left(\sum_{i=1}^{3}(|F_i|^2 - 1)\right)^2, \\
&= \frac{\mu}{4}\left(\sum_{\alpha,\beta=1}^{2}(F_\alpha \cdot F_\beta - \delta_\beta^\alpha)^2\right) + \frac{\lambda}{8}\left(\sum_{\alpha=1}^{2}(|F_\alpha|^2 - 1)\right)^2 \\
&\quad + \frac{\mu}{2}\left(\sum_{\alpha=1}^{2}(F_\alpha \cdot F_3)^2\right) + \frac{2\mu+\lambda}{8}(|F_3|^2 - 1)^2 \\
&\quad + \frac{\lambda}{4}(|F_3|^2 - 1)\left(\sum_{\alpha=1}^{2}(|F_\alpha|^2 - 1)\right),
\end{aligned}$$

we see that minimizers $F = (F_1|F_2|F_3) \in \mathbb{R}^{3\times 3}$of W^* with respect to the third column are such that F_3 is orthogonal to $\mathrm{vect}(F_1, F_2)$. Moreover as $\det(F_1|F_2|F_3) = (F_1 \wedge F_2) \cdot F_3 = 1$, necessarily

$$|F_3| = \frac{1}{|F_1 \wedge F_2|},$$

so accordingly for $\bar{F} = (F_1|F_2) \in \mathbb{R}^{3\times 2}$

$$\begin{aligned}
W_0(\bar{F}) &= \frac{\mu}{4}\left(\sum_{\alpha,\beta=1}^{2}(F_\alpha \cdot F_\beta - \delta_\beta^\alpha)^2\right) + \frac{\lambda}{8}\left(\sum_{\alpha=1}^{2}(|F_\alpha|^2 - 1)\right)^2 \\
&\quad + \frac{2\mu+\lambda}{8}\left(\frac{1}{|F_1 \wedge F_2|^2} - 1\right)^2 + \frac{\lambda}{4}\left(\frac{1}{|F_1 \wedge F_2|^2} - 1\right)\left(\sum_{\alpha=1}^{2}(|F_\alpha|^2 - 1)\right)
\end{aligned}$$

It is a challenging problem to compute the envelope $\mathbf{QR}W_0$ as no general methods apply as is the case for the membrane stored energy function W_m of the Ciarlet-Geymonat-Ogden material in Chapter 2.

3.4 Preliminaries

This section is dedicated to the properties of function W_0 defined in (3.4.1). Namely, the first proposition deals with the properties it inherits from the 3-d stored energy function W.

Proposition 3.4.1. *For all $F \in \mathbb{R}^{3\times 2}$, we have*

$$W_0(F) = \inf\left\{W^*((F|z)) :\ z \in \mathbb{R}^3 \text{ such that } \det(F|z) = 1\right\}. \tag{3.7}$$

Moreover, the following properties are satisfied

(i) W_0 is continuous,

(ii) $W_0(F) \longrightarrow +\infty$ as $\det F^tF \longrightarrow 0$,

(iii) $\forall\delta > 0$, $\exists C(\delta) > 0$ such that $\det F^tF \geq \delta \implies W_0(F) \leq C(\delta)(1 + |F|^p)$,

(iv) $\exists\alpha > 0$ and $\beta \in \mathbb{R}$ such that $W_0(F) \geq \alpha|F|^p + \beta$.

Proof. First of all, let us recall that for all $F \in \mathbb{R}^{3\times 2}$,

$$W_0(F) = \inf\left\{W((F|z)) :\ z \in \mathbb{R}^3 \text{ and } \det(F|z) = 1\right\}.$$

If matrix F is such that $\det F^tF = 0$, then for all $z \in \mathbb{R}^3$, we have $\det(F|z) = 0$ and $W((F|z)) = +\infty$ i.e. $W_0(\xi) = +\infty$. Otherwise $\det F^tF > 0$ so there exists $z \in \mathbb{R}^3$ such that $\det(F|z) = 1$. Hence $W((F|z)) = W^*((F|z))$, and $W_0(F) \leq W^*((F|z))$ by definition which leads to $W_0(F) \leq W_0^*(F)$. Conversely, by definition $W((F|z)) \geq W^*((F|z))$ so that trivially $W_0(F) \geq W_0^*(F)$. Thus we have proved that

$$W_0(F) = \begin{cases} W_0^*(F) & \text{if } \det F^tF > 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{3.8}$$

where

$$W_0^*(F) = \inf\left\{W^*((F|z)) :\ z \in \mathbb{R}^3 \text{ such that } \det(F|z) = 1\right\}.$$

(i) Since W_0^* is the infimum of a family of continuous functions, it is sequentially upper semicontinuous. For the lower semicontinuity, let $F \in \mathbb{R}^{3\times 2}$ be such that $\det F^tF > 0$ and let us consider a sequence $(F^k)_{k\in\mathbb{N}} \subset \mathbb{R}^{3\times 2}$ such that $\det {F^k}^tF^k > 0$ for all $k \in \mathbb{N}$ and that converges towards F. By contradiction, we assume that

$$\liminf_{k\longrightarrow\infty} W_0(F^k) < W_0(F).$$

Without loss of generality, we suppose that $(W_0(F^k))_{k\in\mathbb{N}}$ is convergent. The above strict inequality entails the existence of $c \in \mathbb{R}$ such that

$$\liminf_{k\longrightarrow\infty} W_0(F^k) < c < W_0(F).$$

Then there exists $K \in \mathbb{N}$ such that for all $k \geq K,\ W_0(F^k) < c$. Now, the coercivity together with the continuity of W^* imply that for all $k \geq K$, there exists $z^k \in \mathbb{R}^3$ such that $W_0(F^k) = W^*((F^k|z^k))$. Again, the coercivity of W^* (cf. $(H)-(iii)$) yields that

$$\alpha \left|z^k\right|^p + \beta \leq c.$$

Hence as $(z^k)_{k\in\mathbb{N}}$ is bounded, up to the extraction of a subsequence it is convergent. Let $z \in \mathbb{R}^3$ be its limit. The function W^* being continuous, we deduce that

$$\lim_{k\to\infty} W^*((F^k|z^k)) = W^*((F|z)).$$

Recalling that $W^*((F^k|z^k)) = W_0(F^k) < c$ for all $k \geq K$, we infer that $W^*((F|z)) < c$ on the one hand. On the other hand, by defintion of W_0, we have that $W_0(\xi) \leq W^*((F|z))$. Consequently, we have showed the following contradiction

$$c < W_0(F) \leq c$$

and therefore function W_0 is lower semicontinuous on $\{F \in \mathbb{R}^{3\times 2} :\ \det F^tF > 0\}$. The continuity on $\mathbb{R}^{3\times 2}$ is completed by the next point.
(ii) Let $(F^k)_{k\in\mathbb{N}} \subset \mathbb{R}^{3\times 2}$ be such that $F_k \underset{k\to\infty}{\longrightarrow} F \in \mathbb{R}^{3\times 2}$ and $\det {F^k}^t F^k > 0$ with $\det F^tF = 0$. The coercivity together with the continuity of W^* yield the existence of a sequence $(z^k)_{k\in\mathbb{N}} \subset \mathbb{R}^3$ such that $W_0(F^k) = W^*((F^k|z^k))$ for all $k \in \mathbb{N}$. Next, consider an orthonormal basis (τ_1^k, τ_2^k) of the plane defined by $F^k = (F_1^k|F_2^k)$. Then we can write

$$z^k = \alpha_1^k \tau_1^n + \alpha_2^k \tau_2^n + \beta^k \frac{F_1^k \wedge F_2^k}{\left|F_1^k \wedge F_2^k\right|},$$

where $(\alpha_1^k, \alpha_2^k, \beta^k) \in \mathbb{R}^3$. Now condition $\det(F^k|z^k) = 1$ translates as

$$(F_1^k \wedge F_2^k) \cdot z^k = \beta^k \left|F_1^k \wedge F_2^k\right| = 1.$$

From the coercivity of W^*, we infer that

$$\frac{\alpha}{\left|F_1^k \wedge F_2^k\right|^p} + \beta \leq W^*((F^k|z^k)),$$

which ensues

$$W_0(F^k) = W^*((F^k|z^k)) \underset{k\to\infty}{\longrightarrow} +\infty = W_0(F),$$

since $\left|F_1^k \wedge F_2^k\right|^2 = \det {F^k}^t F^k \underset{k\to\infty}{\longrightarrow} \det F^tF = 0$.
(iii) Let $\delta > 0$ and $F \in \mathbb{R}^{3\times 2}$ such that $\det F^tF \geq \delta$. We choose $z = \dfrac{F_1 \wedge F_2}{\det F^tF} \in \mathbb{R}^3$

so that $\det(F|z) = 1$. Then we write the growth condition on W at the point $(F|z)$

$$
\begin{aligned}
W_0(F) \leq W((F|z)) &\leq C(1 + |(F|z)|^p) \\
&\leq C(1 + (|F|^2 + |z|^2)^{\frac{p}{2}}) \\
&\leq C\left(1 + \left(|F|^2 + \frac{1}{\det F^tF}\right)^{\frac{p}{2}}\right) \\
&\leq C'\left(1 + \left(|F|^2 + \frac{1}{\delta^2}\right)^{\frac{p}{2}}\right) \\
&\leq C(\delta)(1 + |F|^p),
\end{aligned}
$$

where we have used the fact that $|F_1 \wedge F_2|^2 = \det F^tF$.
(iv) The coercivity inequality is a trivial consequence of the coercivity of function W (cf. $(H) - (iii)$). □

Remark 3.4.2. In the growth condition (iii) above, the constant we obtain depends on the determinant of the deformation tensor. In fact, the constant explodes as the determinant approaches zero.
Remark 3.4.3. We remark that the procedure of minimization with respect to the third column vector has smoothened the energy. Although the 3-d function W is discontinuous, the bidimensional stored energy function W_0 is continuous.

We now show another interesting and important property of W_0 which is that its rank-one-convex envelope is everywhere finite.

Proposition 3.4.4. *There exists $C > 0$ such that for all $F \in \mathbb{R}^{3\times 2}$,*

$$\alpha |F|^p + \beta \leq \mathbf{R}W_0(F) \leq C\left(1 + |F|^p\right).$$

Proof. The left inequality is a straightforward consequence of the coerciveness inequality of W_0 which is inherited by $\mathrm{R}W_0$ by virtue of the rank-one-convexity of the bound below. For the upper bound, we first notice that by proposition 3.4.1(ii), there exists $C(1) > 0$ such that for all $F \in \mathbb{R}^{3\times 2}$ verifying $\det F^tF \geq 1$ we have

$$\mathbf{R}W_0(F) \leq W_0(F) \leq C(1)\left(1 + |F|^p\right). \tag{3.9}$$

Moreover, we have already put forward, in section 3 of Chapter 1, that a positive frame-indifferent isotropic rank-one-convex membrane energy (i.e. it only depends on the gradient of the deformation) whose minimum is attained for the identity vanishes for gradients whose singular values are less than one. If we remove the positivity assumption in there, we end up with the following

$$\mathbf{R}W_0(F) \leq 0,$$

for all $F \in \mathbb{R}^{3\times 2}$ such that its singular values $s_\alpha(F) \leq 1$. Consequently, it remains to investigate the case where $\det F^tF \leq 1$ and $s_1(F) \leq 1 < s_2(F)$. Indeed, as $\dfrac{1}{s_2(F)} < 1$, there exists $t \in [0,1]$ such that $s_1(F) = 1 - 2\dfrac{t}{s_2(F)}$. Now let us write the polar factorization of $F \in \mathbb{R}^{3\times 2}$: there exists $R \in SO(3)$ such that $F = RJ(F^tF)^{\frac{1}{2}}$ where $J = (\delta^i_j)_{1\le i\le 3, 1\le j\le 2}$. Furthermore, there exists $Q \in SO(2)$ such that $(F^tF)^{\frac{1}{2}} = Q\,\mathrm{diag}(s_1(F), s_2(F))Q^t$. Then setting $G^{\pm} = RJQ\,\mathrm{diag}\big(\pm\frac{1}{s_2(F)}, s_2(F)\big)Q^t$, we can write

$$F = RJQ\,\mathrm{diag}(s_1(F), s_2(F))Q^t = RJQ\,\mathrm{diag}\left(1 - 2\frac{t}{s_2(F)}, s_2(F)\right)Q^t$$
$$= (1-t)G^+Q^t + RJQG^-Q^t.$$

As $\mathrm{rank}(G^+ - G^-) \le 1$, the rank-one-convexity of RW_0 entails

$$\mathbf{R}W_0(F) \le (1-t)\mathbf{R}W_0(G^+) + t\mathbf{R}W_0(G^-).$$

Lastly, we remark that

$$\det {G^{\pm}}^t G^{\pm} = 1,$$

so that by (3.9) we get

$$\begin{aligned}
\mathbf{R}W_0(G^{\pm}) &\le C(1)\left(1 + |G^{\pm}|^p\right)\\
&\le C(1)\left(1 + \left|RJQ\,\mathrm{diag}\left(\pm\frac{1}{s_2(F)}, s_2(F)\right)Q^t\right|^p\right)\\
&\le C(1)\left(1 + \left(\frac{1}{s_2^2(F)} + s_2^2(F)\right)^{\frac{p}{2}}\right)\\
&\le C(1)\left(1 + \left(1 + s_2^2(F)\right)^{\frac{p}{2}}\right)\\
&\le 2C(1)\left(1 + s_2^p(F)\right)\\
&\le 2C(1)\left(1 + |F|^p\right),
\end{aligned}$$

which finishes the proof. □

3.5 A lower bound for the Γ-limit

Before proceeding with the derivation of a lower bound for the Γ-limit $\bar{J}(0)$, we show some preliminary results that will be useful henceforth. Namely, we inspect the sequences that keep the energy bounded.

Lemma 3.5.1. *Let $(\psi(\varepsilon))_{\varepsilon>0} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ be a sequence of deformations such that the sequence $\big(\bar{J}(\varepsilon)(\psi(\varepsilon))\big)_{\varepsilon>0} \subset \mathbb{R}$ is bounded. Then the sequence $(\psi(\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$ and its cluster points for the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$ are in $\mathcal{M}(\omega)$.*

Proof. Let $(\psi(\varepsilon))_{\varepsilon>0} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ be such that there exists a finite constant C satisfying

$$\bar{J}(\varepsilon)(\psi(\varepsilon)) \le C.$$

We first deduce that necessarily $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ since the energies remain bounded. Next, the coercivity condition $(H)-(iii)$ ensues

$$\alpha \int_\Omega \left| \left(\psi(\varepsilon)_{,1} | \psi(\varepsilon)_{,2} | \frac{1}{\varepsilon} \psi(\varepsilon)_{,\xi} \right) \right|^p dx - \int_\Omega b \cdot \psi(\varepsilon)\, dx - \int_S g \cdot \psi(\varepsilon)\, dx_H \leq C + \beta$$

for all $\varepsilon > 0$, then by Hölder's inequality we get

$$\begin{aligned} \alpha \left| \left(\psi(\varepsilon)_{,1} | \psi(\varepsilon)_{,2} | \frac{1}{\varepsilon} \psi(\varepsilon)_{,\xi} \right) \right|^p_{L^p(\Omega;\mathbb{R}^3)} &- |b|_{L^p(\Omega;\mathbb{R}^3)} |\psi(\varepsilon)|_{L^p(\Omega;\mathbb{R}^3)} \\ &- |g|_{L^p(S;\mathbb{R}^3)} |\psi(\varepsilon)|_{L^p(S;\mathbb{R}^3)} \leq C + \beta. \end{aligned} \tag{3.10}$$

The above being valid for all $\varepsilon > 0$, it still holds for $\varepsilon \leq 1$ and by Poincaré's inequality applied to $\psi(\varepsilon)$, we infer that

$$\alpha \|\psi(\varepsilon)\|^p_{W^{1,p}(\Omega;\mathbb{R}^3)} - C' \|\psi(\varepsilon)\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leq C + \beta. \tag{3.11}$$

As $p > 1$, we deduce that the sequence $(\psi(\varepsilon))_{\varepsilon>0}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$ and therefore it is weakly relatively compact in $W^{1,p}(\Omega;\mathbb{R}^3)$. Furthermore, from inequalitites (3.10) and (3.11), we yield

$$\left|\psi(\varepsilon)_{,\xi}\right|_{L^p(\Omega;\mathbb{R}^3)} \leq C''\varepsilon \quad \text{for all } \varepsilon > 0.$$

As a consequence, $\psi(\varepsilon)_{,\xi} \longrightarrow 0$ in $L^p(\Omega;\mathbb{R}^3)$. Moreover, as $\psi(\varepsilon)_{|\Gamma}(x_H,\xi) = (x_H, \varepsilon\xi)$ for all $\varepsilon > 0$, sequence $(\psi(\varepsilon))_{\varepsilon>0}$ converges towards function $id_2 : (x_H,\xi) \longrightarrow (x_H, 0)$ on Γ. Finally, we invoke the continuity of the trace operator defined between $W^{1,p}(\Omega;\mathbb{R}^3)$ and $L^p(S;\mathbb{R}^3)$ to bring out the relative compactness of $(\psi(\varepsilon)_{|\Gamma})_{\varepsilon>0}$ in $L^p(S;\mathbb{R}^3)$. Hence, all the cluster points of sequence $(\psi(\varepsilon))_{\varepsilon>0}$ for the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$ belong to $\mathcal{M}(\omega)$. □

Corollary 3.5.2. *If $\psi \in L^p(\Omega;\mathbb{R}^3)$ and $\psi \notin \mathcal{M}(\omega)$, then $\bar{J}(0)(\psi) = +\infty$.*

Proof. Conversely, if $\psi \in L^p(\Omega;\mathbb{R}^3)$ is such that $\bar{J}(0)(\psi) < +\infty$, then by definition of Γ-convergence there exists $(\psi(\varepsilon))_{\varepsilon>0} \subset L^p(\Omega;\mathbb{R}^3)$ that converges strongly towards ψ and such that $\bar{J}(\varepsilon)(\psi(\varepsilon)) \underset{\varepsilon\to 0}{\longrightarrow} \bar{J}(0)(\psi)$. The latter convergence entails that sequence $\left(\bar{J}(\varepsilon)(\psi(\varepsilon))\right)_{\varepsilon>0}$ is bounded and by Lemma 3.5.1, we conclude that $\psi \in \mathcal{M}(\omega)$. □

We are now in a position to give a lower bound for the Γ-limit energy $\bar{J}(0)$.

Proposition 3.5.3. *For all $\varphi \in \mathcal{M}(\omega)$, we have*

$$\bar{J}(0)(\varphi) \geq 2 \int_\omega \mathrm{QR}W_0(\nabla\varphi)\, dx_H - \ell(0),$$

where

$$\ell(0)(\varphi) = \int_\omega \left\{ \int_{-1}^{1} b(x_H,\xi)\, d\xi + g(x_H,-1) + g(x_H,1) \right\} \cdot \varphi\, dx_H.$$

Proof. Consider $\varphi \in \mathcal{M}(\omega)$. As $\big(\bar{J}(\varepsilon)(\varphi)\big)_{\varepsilon>0}$ is uniformly upper bounded, by definition of Γ-convergence we infer that $\bar{J}(0)(\varphi) < +\infty$. Again by definition, there exists $(\varphi(\varepsilon))_{\varepsilon>0}$ such that

$$\begin{cases} \varphi(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \varphi & \text{in } L^p(S;\mathbb{R}^3) \\ \bar{J}(\varepsilon)(\varphi(\varepsilon)) \underset{\varepsilon\to 0}{\longrightarrow} J(0)(\psi). \end{cases}$$

The latter convergence implicates that sequence $\big(\bar{J}(\varepsilon)(\varphi(\varepsilon))\big)_{\varepsilon>0}$ is bounded and by Lemma 3.5.1, we deduce that $\varphi(\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \varphi$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$.
First, it is straightforward that $\ell(\varepsilon)\varphi(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \ell(0)(\varphi)$ where

$$\ell(\varepsilon)(\varphi) = \int_\Omega b\cdot\varphi(\varepsilon)\,dx + \int_S g\cdot\varphi(\varepsilon)\,dx_H.$$

Next, for the internal energy we write for all $\varepsilon > 0$

$$\begin{aligned}\int_\Omega W\left(\left(\varphi(\varepsilon)_{,1}|\varphi(\varepsilon)_{,2}|\frac{1}{\varepsilon}\varphi(\varepsilon)_{,\xi}\right)\right)dx &\geq \int_\Omega W_0\left((\varphi(\varepsilon)_{,1}|\varphi(\varepsilon)_{,2})\right)dx \\ &\geq \int_\Omega \mathbf{QR}W_0\left((\varphi(\varepsilon)_{,1}|\varphi(\varepsilon)_{,2})\right)dx,\end{aligned}$$

by definition of function W_0 and the rank-one-convex and quasiconvex envelopes. Now, the right-hand side being sequentially lower semicontinuous for the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$, we infer that

$$\begin{aligned}2\int_\omega \mathbf{QR}W_0\left((\varphi_{,1}|\varphi_{,2})\right)dx_H &= \int_\Omega \mathbf{QR}W_0\left((\varphi_{,1}|\varphi_{,2})\right)dx \\ &\leq \liminf_{\varepsilon\longrightarrow 0}\int_\Omega \mathbf{QR}W_0\left((\varphi(\varepsilon)_{,1}|\varphi(\varepsilon)_{,2})\right)dx.\end{aligned}$$

Ultimately, we get

$$J(0)(\varphi) = \liminf_{\varepsilon\longrightarrow 0}\bar{J}(\varepsilon)(\varphi(\varepsilon)) \geq 2\int_\omega \mathbf{QR}W_0\left((\varphi_{,1}|\varphi_{,2})\right)dx_H - \ell(0)(\varphi),$$

as was stated in the proposition. □

3.6 An upper bound for the Γ-limit satisfied by immersions

This section is the first step in the derivation of an upper bound for the Γ-limit. The first upper bound we draw concerns C^1 bidimensional immersions, the original third derivative being replaced by an ad hoc function. Namely, we prove

Proposition 3.6.1. *If* $\psi \in C^1(\bar{\omega};\mathbb{R}^3)$ *is an immersion such that* $\psi_{|\partial\omega} = id$ *and* $\theta \in C(\bar{\omega};\mathbb{R}^3)$ *a function such that* $\theta_{|\partial\omega} = e_3$ *and* $\det(\psi_{,1}|\psi_{,2}|\theta) = 1$*, then the following inequality holds*

$$\bar{J}(0)(\psi) \leq 2\int_\omega W\left((\psi_{,1}|\psi_{,2}|\theta)\right) dx_H - \ell(0)(\psi), \tag{3.12}$$

where

$$\ell(0)(\psi) = \int_\omega \left\{ \int_{-1}^{1} b(x_H,\xi)\, d\xi + g(x_H,-1) + g(x_H,1) \right\} \cdot \psi \, dx_H.$$

Before proving the above proposition, we show the following lemma.

Lemma 3.6.2. *For all immersion* $\psi \in C^2(\bar{\omega};\mathbb{R}^3)$ *such that* $\psi_{|\partial\omega} = id$ *and all function* $\theta \in C^1(\bar{\omega};\mathbb{R}^3)$ *such that* $\theta_{|\partial\omega} = e_3$ *and* $\det(\psi_{,1}|\psi_{,2}|\theta) = 1$*, there exists a sequence* $(L(\varepsilon))_{\varepsilon>0} \subset C^1(\bar{\Omega};\mathbb{R})$ *such that the sequence defined by* $(\psi(\varepsilon))_{\varepsilon>0} = (\psi + L(\varepsilon)\theta)_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$ *and converges to* ψ *in* $W^{1,p}(\omega;\mathbb{R}^3)$*. Moreover,* $\dfrac{L(\varepsilon)_{,\xi}}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 1$ *uniformly.*

Remark 3.6.3. We would like to insist on the fact that the construction of such a (minimizing) sequence is almost straightforward in the compressible case where the sequence would be at most required to be orientation-preserving. Therefore, the natural candidate

$$\psi(\varepsilon)(x_H,\xi) = \psi(x_H) + \varepsilon\,\xi\theta(x_H),$$

would do. However, in the incompressible case, intuitively such a sequence cannot be expected to be an affine function of the thickness. As a matter of fact, it is not as shown below.

***Proof of Lemma* 3.6.2.** Define for all $\varepsilon > 0$,

$$\psi(\varepsilon)(x_H,\xi) = \psi(x_H) + L(\varepsilon)(x_H,\xi)\theta(x_H).$$

Then, we have

$$\begin{aligned}\nabla\psi(\varepsilon) &= \left(\psi(\varepsilon)_{,1}|\psi(\varepsilon)_{,2}|\psi(\varepsilon)_{,\xi}\right)\\ &= \left(\psi_{,1} + L(\varepsilon)_{,1}\theta + L(\varepsilon)\theta_{,1}|\psi_{,2} + L(\varepsilon)_{,2}\theta + L(\varepsilon)\theta_{,2}|L(\varepsilon)_{,\xi}\theta\right),\end{aligned}$$

and the incompressibility condition $\det\nabla\psi(\varepsilon) = \varepsilon$ translates as

$$\begin{aligned}L(\varepsilon)_{,\xi}L^2(\varepsilon)\det(\theta_{,1}|\theta_{,2}|\theta) + L(\varepsilon)_{,\xi}L(\varepsilon)\left[\det(\psi_{,1}|\theta_{,2}|\theta) + \det(\theta_{,1}|\psi_{,2}|\theta)\right]&\\ +L(\varepsilon)_{,\xi}\det(\psi_{,1}|\psi_{,2}|\theta) = \varepsilon.&\end{aligned}$$

We integrate the above expression with respect to ξ,

$$\begin{aligned}\frac{L^3(\varepsilon)}{3}\det(\theta_{,1}|\theta_{,2}|\theta) + \frac{L^2(\varepsilon)}{2}\left[\det(\psi_{,1}|\theta_{,2}|\theta) + \det(\theta_{,1}|\psi_{,2}|\theta)\right]&\\ + L(\varepsilon) = \varepsilon\,\xi + c,&\end{aligned} \tag{3.13}$$

where $c : \omega \longrightarrow \mathbb{R}$ and deduce that for all $(x_H, \xi) \in \Omega$, $L(\varepsilon)(x_H, \xi)$ can be chosen as a real solution of the above third degree equation. Furthermore, the continuity of the coefficients implies that $L(\varepsilon)$ is continuous on $\bar{\Omega}$ and the function is of the form $L(\varepsilon)(x_H, \xi) = L(x_H, \varepsilon\,\xi)$. Moreover, for all $\varepsilon > 0$ it is required that $\psi(\varepsilon)_{|\Gamma} = (x_H, \varepsilon\,\xi)$. We infer from the above identity that for all $(x_H, \xi) \in \Gamma$, $L(\varepsilon) = \varepsilon\,\xi + c$ which leads to $\psi(\varepsilon)_{|\Gamma} = (x_H, \varepsilon\,\xi + c)$ and accordingly $c(x_H) = 0$, for all $x_H \in \partial\omega$. For simplicity, we choose $c \equiv 0$. This ensures that $(\psi(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\varepsilon)$. Now, we have to check that $(L(\varepsilon))_{\varepsilon>0}$ can be chosen so that sequence $(\psi(\varepsilon))_{\varepsilon>0}$ converges to ψ in $W^{1,p}(\omega; \mathbb{R}^3)$. Actually, inspecting equation (3.13), we notice that when ε vanishes 0 is a solution. Knowing that the said equation always admits one real solution, we deduce that

$$L(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} L(0) = 0.$$

As a direct consequence, we raise the convergence of $(\psi(\varepsilon))_{\varepsilon>0}$ to ψ in $W^{1,p}(\omega; \mathbb{R}^3)$. Next, for $\varepsilon > 0$ small enough, we deduce from equation (3.13) that

$$L(\varepsilon) = \varepsilon\,\xi + o(\varepsilon),$$

and ultimately

$$\frac{L(\varepsilon)_{,\xi}}{\varepsilon} = 1 + o(1).$$

□

Proof of Proposition 3.6.1. Without loss of generality let $\phi \in C^2(\bar{\omega}; \mathbb{R}^3)$ be an immersion such that $\phi_{|V} = id$ and let $\Theta \in C^1(\bar{\omega}; \mathbb{R}^3)$ be a function such that $\Theta_{|\partial\omega} = e_3$ and $\det(\phi_{,1}|\phi_{,2}|\Theta) = 1$. Lemma (3.6.2) above guarantees the existence of a sequence $(\phi(\varepsilon))_{\varepsilon>0}$ that converges to ϕ in $W^{1,p}(\omega; \mathbb{R}^3)$. Hence, it is trivial to see that

$$\ell(\varepsilon)(\phi(\varepsilon)) = \int_\Omega b \cdot \phi(\varepsilon)\, dx + \int_S g \cdot \phi(\varepsilon)\, dx_H \underset{\varepsilon \to 0}{\longrightarrow} \int_\Omega b \cdot \phi\, dx + \int_S g \cdot \phi\, dx_H = \ell(0)(\phi).$$

As $\det \nabla\phi(\varepsilon) = \varepsilon$, $W\left(\left(\phi(\varepsilon)_{,1}|\phi(\varepsilon)_{,2}|\frac{1}{\varepsilon}\phi(\varepsilon)_{,\xi}\right)\right) = W^*\left(\left(\phi(\varepsilon)_{,1}|\phi(\varepsilon)_{,2}|\frac{1}{\varepsilon}\phi(\varepsilon)_{,\xi}\right)\right)$ and $(H) - (ii - iii)$ imply

$$\beta \leq W\left(\left(\phi(\varepsilon)_{,1}|\phi(\varepsilon)_{,2}|\frac{1}{\varepsilon}\phi(\varepsilon)_{,\xi}\right)\right) \leq C\left(1 + \left|\left(\phi(\varepsilon)_{,1}|\phi(\varepsilon)_{,2}|\frac{1}{\varepsilon}\phi(\varepsilon)_{,\xi}\right)\right|^p\right).$$

Hence, the continuity of W and Lebesgue's theorem allow us to conclude that

$$\lim_{\varepsilon \longrightarrow 0} \bar{I}(\phi(\varepsilon)) = \int_\Omega W\left((\phi_{,1}|\phi_{,2}|\Theta)\right)\, dx = 2\int_\omega W\left((\phi_{,1}|\phi_{,2}|\Theta)\right)\, dx_H,$$

where we have used that $\left(\phi(\varepsilon)_{,1}|\phi(\varepsilon)_{,2}|\frac{1}{\varepsilon}\phi(\varepsilon)_{,\xi}\right)$ converges to $(\phi_{,1}|\phi_{,2}|\Theta)$ in $L^p(\Omega; \mathbb{R}^3)$. Now recollecting the above convergences together with the first property of Γ-convergence yields

$$\begin{aligned} J(0)(\phi) &\leq \liminf_{\varepsilon \longrightarrow 0} J(0)(\phi(\varepsilon)) \\ &\leq \liminf_{\varepsilon \longrightarrow 0} \bar{J}(\varepsilon)(\phi(\varepsilon)) = \lim_{\varepsilon \longrightarrow 0} \left[\bar{I}(\phi(\varepsilon)) - \bar{\ell}(\phi(\varepsilon))\right], \end{aligned}$$

which brings up inequality (3.12) albeit for functions ϕ and Θ smoother than announced. Now let ψ and θ be as in the statetment of the proposition. As in Proposition 3.4.1, we express θ in the local coordinate system defined by the tangent plane at each point of the plate. In other words, we write

$$\theta = \mu_\alpha \psi_{,\alpha} + \lambda \left(\psi_{,1} \wedge \psi_{,2}\right)$$

and injecting it in $\det\left(\psi_{,1}|\psi_{,2}|\theta\right) = \left(\psi_{,1} \wedge \psi_{,2}\right) \cdot \theta = 1$ brings $\lambda = \dfrac{1}{\left|\psi_{,1} \wedge \psi_{,2}\right|^2}$. Now let us consider functions $\rho_\alpha \in C_c^\infty(\mathbb{R}^2; \mathbb{R})$ such that

$$\left\{ \begin{array}{l} \operatorname{supp} \rho_\alpha \subset B(0,1), \quad \rho_\alpha \geq 0 \quad \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} \rho_\alpha(x)\, dx > 0 \quad \text{and} \quad \int_{\mathbb{R}^2} x_\alpha \rho_\alpha(x)\, dx = 0. \end{array} \right.$$

Then take the associated sequence of mollifiers $(\rho^n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^3; \mathbb{R})$ defined by $\rho_\alpha^n(x) = C_\alpha n^2 \rho_\alpha(nx)$ where $C_\alpha = \left(\displaystyle\int \rho_\alpha\right)^{-1}$ and set

$$\bar{\psi}(x) = \left\{ \begin{array}{ll} \psi(x) & \text{if } x \in \omega \\ (x,0) & \text{if } \mathbb{R}^2 \backslash \omega \end{array} \right. \quad \text{and} \quad \bar{\mu}_\alpha(x) = \left\{ \begin{array}{ll} \mu_\alpha & \text{if } x \in \omega \\ 0 & \text{if } \mathbb{R}^2 \backslash \omega, \end{array} \right.$$

so that $\bar{\psi} \in C^1(\mathbb{R}^2; \mathbb{R})$ and $\bar{\mu}_\alpha \in C(\mathbb{R}^2; \mathbb{R})$. Next, we define the following mollified sequences

$$\left\{ \begin{array}{lll} \bar{\psi}_\alpha^n & = & \rho_\alpha^n * \bar{\psi}_\alpha, \\ \bar{\mu}_\alpha^n & = & \rho_\alpha^n * \bar{\mu}_\alpha, \\ \bar{\theta}^n & = & \bar{\mu}_\alpha^n \bar{\psi}_{,\alpha}^n + \dfrac{\bar{\psi}_{,1}^n \wedge \bar{\psi}_{,2}^n}{\left|\bar{\psi}_{,1}^n \wedge \bar{\psi}_{,2}^n\right|^2}, \end{array} \right.$$

where the convolution is made with respect to each component of the deformation $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)$. We have constructed the above sequences making sure that for all $n \in \mathbb{N}$,

$$\det\left(\nabla \bar{\psi}^n | \bar{\theta}^n\right) = 1.$$

It remains to verify that the boundary conditions are satisfied. We first observe that since V is a neighbourhood of $\partial\omega$, if $x \in V$ then for n large enough $B(x, \frac{1}{n}) \subset V$ and consequently the following identities hold

$$\begin{aligned} \bar{\psi}_\alpha^n(x) = \rho_\alpha^n * \bar{\psi}_\alpha(x) &= \int_{\mathbb{R}^2} \psi_\alpha(x-y) \rho_\alpha^n(y)\, dy \\ &= \int_{B(0,\frac{1}{n})} \psi_\alpha(x-y) \rho_\alpha^n(y)\, dy = \int_{B(0,\frac{1}{n})} (x_\alpha - y_\alpha) \rho_\alpha^n(y)\, dy \\ &= x_\alpha - \frac{1}{n} \int_{B(0,1)} z_\alpha \rho_\alpha(z)\, dz \\ &= x_\alpha. \end{aligned}$$

Hence, $\bar{\psi}^n{}_{|V} = id$ and likewise we check that $\bar{\mu}^n_{\alpha|V} = 0$ (let us remark that as $\theta_{|V} = e_3$, $\mu_{\alpha|V} = 0$) so that $\bar{\theta}^n{}_{|V} = e_3$. Again, for n large enough, we know that

$$\operatorname{supp}\bar{\psi}^n,\ \operatorname{supp}\bar{\theta}^n \subset \omega.$$

Therefore, it suffices to set $\psi^n = \bar{\psi}^n{}_{|\omega} \in C^2(\omega;\mathbb{R}^3)$ and $\theta^n = \bar{\theta}^n{}_{|\omega} \in C^1(\omega;\mathbb{R}^3)$, then a classical argument (cf. Brézis [9]) ensures that

$$\|\psi^n - \psi\|_{W^{1,\infty}(\omega;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \|\theta^n - \theta\|_{L^\infty(\omega;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.14}$$

Lastly, we infer from the above that there exists $N \in \mathbb{N}$ such that functions $(\psi^n)_{n\geq N}$ and $(\theta^n)_{n\geq N}$ satisfy inequality (3.12). Moreover, as $\det(\nabla\psi^n|\theta^n) = 1$ in ω, by assumptions $(H)-(ii)$ and (iii) there exists $\eta > 0$, such that for n large enough

$$\begin{aligned}\beta \leq W\left((\nabla\psi^n|\theta^n)\right) &\leq C\left(1 + |(\nabla\psi^n|\theta^n)|^p\right)\\ &\leq C\left(1 + \left(\|\nabla\psi\|^2_{L^\infty(\omega;\mathbb{R}^3)} + \|\theta\|^2_{L^\infty(\omega;\mathbb{R}^3)} + \eta^2\right)^{\frac{p}{2}}\right),\end{aligned}$$

where we have used the triangular inequality and the convergences (3.14). We finally terminate the proof by invoking Lebesgue's dominated convergence theorem and concluding as in the smoother case above. □

3.7 Relaxation of the upper bound

In order to perform the relaxation of the upper bound, we need to have it verified by locally injective piecewise affine functions.

Proposition 3.7.1. *For all locally injective function $\psi \in id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$, we have*

$$\bar{J}(0)(\psi) \leq 2\int_\omega W_0(\nabla\psi)\,dx_H - \ell(0)(\psi), \tag{3.15}$$

where

$$W_0(F) = \inf\left\{W((F|z)) :\ z \in \mathbb{R}^3 \text{ and } \det(F|z) = 1\right\}.$$

and

$$\ell(0)(\psi) = \int_\omega \left\{\int_{-1}^1 b(x_H,\xi)\,d\xi + g(x_H,-1) + g(x_H,1)\right\}\cdot\psi\,dx_H.$$

Proof. Without loss of generality let $\psi \in id + \mathrm{Aff}_0(\omega;\mathbb{R}^m)$ be a locally injective function such that $\psi_{|V} = id$ then there exists $\delta > 0$ such that $|\psi_{,1} \wedge \psi_{,2}| \geq \delta$ almost everywhere in ω. Hence, by Proposition 3.4.1-(iii) there exists $C > 0$ such that

$$W_0(\nabla\psi) \leq C\left(1 + |\nabla\psi|^p\right) \quad \text{a.e. in } \omega.$$

Adding the fact that W is bounded from below (cf. coercivity $(H)-(iii)$), elicits the existence for almost every $x \in \omega$ of a vector $\theta(x) \in \mathbb{R}^3$ such that

$$W_0(\nabla\psi) = W\left((\nabla\psi|\theta(x))\right).$$

and $\det(\nabla\psi|\theta(x)) = 1$. Moreover, as ψ is piecewise affine, $\nabla\psi$ is piecewise constant and accordingly θ can be chosen piecewise constant a.e. in ω. By Lemma 3.1.5, there exists a sequence of immersions $(\psi^n)_{n\in\mathbb{N}} \subset C^1(\omega;\mathbb{R}^3)$ that converges to ψ in $W^{1,p}(\omega;\mathbb{R}^3)$ and such that $\left|\psi^n_{,1} \wedge \psi^n_{,2}\right| \geq \delta > 0$ for all $n \in \mathbb{N}$. We consider a sequence of mollifiers as in the proof of Proposition 3.6.1 :

$$(\rho^n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^2;\mathbb{R}), \quad \operatorname{supp}\rho^n \subset B(0,\frac{1}{n}), \quad \int_{\mathbb{R}^2}\rho^n = 1, \quad \text{and} \quad \rho^n \geq 0 \text{ in } \mathbb{R}^2.$$

Next we define the following sequence

$$\mu^n_\alpha = (\rho^n * \bar{\mu}_\alpha)_{|V}$$

where we have expressed θ in the local coordinate system defined by the tangent plane

$$\theta = \mu_\alpha \psi_{,\alpha} + \frac{\psi_{,1}\wedge\psi_{,2}}{\left|\psi_{,1}\wedge\psi_{,2}\right|^2}$$

before extending functions μ_α to $\mathbb{R}^2$ by 0 outside ω. Let us note that functions μ_α are piecewise constant and therefore $(\mu^n_\alpha)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\omega;\mathbb{R}^3)$. We can now set the continuous approximation of function θ as follows

$$\theta^n = \mu^n_\alpha \psi^n_{,\alpha} + \frac{\psi^n_{,1}\wedge\psi^n_{,2}}{\left|\psi^n_{,1}\wedge\psi^n_{,2}\right|^2},$$

which clearly converges almost everywhere towards θ. Furthermore, if $x \in V$, then

$$\begin{aligned}\mu^n_\alpha(x) = \rho^n_\alpha * \mu_\alpha(x) &= \int_{\mathbb{R}^2}\mu_\alpha(x-y)\rho^n_\alpha(y)\,dy \\ &= \int_{B(0,\frac{1}{n})}\mu_\alpha(x-y)\rho^n_\alpha(y)\,dy = 0,\end{aligned}$$

for n large enough to have $x - B(0,\frac{1}{n}) \subset V$. We have thus constructed $(\theta^n)_{n\in\mathbb{N}}$ so that $\det((\nabla\psi^n|\theta^n) = 1$ and by assumptions $(H)-(ii)$ and (iii), we get

$$\begin{aligned}\beta \leq W\left((\nabla\psi^n|\theta^n)\right) &\leq C\left(1 + |(\nabla\psi^n|\theta^n)|^p\right) \\ &\leq C\left(1 + \left(|\nabla\psi^n|^2 + \left|\mu^n_\alpha\psi^n_{,\alpha}\right|^2 + \frac{1}{\left|\psi^n_{,1}\wedge\psi^n_{,2}\right|^2}\right)^{\frac{p}{2}}\right), \\ &\leq C'\left(1 + \left(\left[1 + \|\theta\|_{L^\infty(\omega;\mathbb{R}^3)}\right]|\nabla\psi^n|^2 + \frac{1}{\delta^2}\right)^{\frac{p}{2}}\right), \\ &\leq C(\delta, \|\theta\|_{L^\infty(\omega;\mathbb{R}^3)})\left(1 + |\nabla\psi^n|^p\right).\end{aligned}$$

Hence, the above bounds together with the convergence

$$\|\psi^n - \psi\|_{W^{1,p}(\omega;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0$$

and the pointwise convergence

$$W\left((\nabla\psi^n|\theta^n)\right) \underset{n\to\infty}{\longrightarrow} = W\left((\nabla\psi|\theta)\right) = W_0\left(\nabla\psi\right)$$

entail that

$$\int_\omega W\left((\nabla\psi^n(x)|\theta^n(x))\right) dx \underset{n\to\infty}{\longrightarrow} \int_\omega W_0\left(\nabla\psi(x)\right) dx,$$

owing to Lebesgue's dominated convergence theorem. Besides, we obviously have $\ell(0)(\psi^n) \underset{n\to\infty}{\longrightarrow} \ell(0)(\psi)$, therefore

$$\bar{J}(0)(\psi) \leq \liminf_{n\longrightarrow\infty} \bar{J}(0)(\psi^n) \leq 2\int_\omega W_0\left(\nabla\psi(x)\right) dx - \ell(0)(\psi),$$

since the Γ-limit is sequentially weakly lower semicontinuous and the proof is complete. □

The last step in the indentification of the Γ-limit $\bar{J}(0)$ is the relaxation of the upper bound drawn so far. The next result is the most delicate move towards our objective. We follow the argument in Ben Belgacem [6] albeit with the envelope Z defined on piecewise affine functions like in Fonseca [39] and Proposition 3.1.3. Moreover, boundary conditions are taken into account.

Proposition 3.7.2. *For all locally injective function $\psi \in id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$, we have*

$$\bar{J}(0)(\psi) \leq 2\int_\omega \mathrm{R}_k W_0\left(\nabla\psi\right) dx_H - \ell(0)(\psi), \quad k \in \mathbb{N}. \tag{3.16}$$

Proof. The proof is by induction. For $k = 0$, $\mathrm{R}_0 W_0 = W_0$ which is true by virtue of Proposition 3.7.1. Next, we assume that the above inequality holds for some $k \in \mathbb{N}$ and we aim to prove it for $k+1$. Let $\psi \in id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$ be a locally injective function ; we name $(F_j)_{j\in J}$ the set of values its gradient takes and $(O_j)_{j\in J}$ the associated partition of ω. Then by definition of $\mathrm{ZR}_k W_0$ and Proposition 3.1.3(i), there exists $(\varphi_j)_{j\in J} \subset \mathrm{Aff}_0(S;\mathbb{R}^3)$ where $S \subset \mathbb{R}^2$ is the unit square such that for all $j \in J$,

$$\int_S \mathrm{R}_k W_0(F_j + \nabla\varphi_j(x))\, dx \leq |S|\, \mathrm{ZR}_k W_0(F_j) + \epsilon\, |S|\,. \tag{3.17}$$

We extend φ_j to $\mathbb{R}^2$ by periodicity for all $j \in J$ and consider $(\varphi_j^n)_{n\in\mathbb{N}^*}$ defined by

$$\varphi_j^n(x) = \frac{1}{n}\varphi_j(nx), \quad \forall\, x \in \mathbb{R}^2$$

Now, for all $j \in J$, there exists $j \in J$ such that $\nabla\psi = F_j$ a.e. in O_j. As by Vitali's covering theorem, there exists a countable disjoint family $(a_j^i + \alpha_j^i S)_{i\in I} \subset O_j$ with $(a_j^i)_{i\in I} \subset \mathbb{R}^2$ and $(\alpha_j^i)_{i\in I} \subset (0,1)$ such that

$$\left|O_j \backslash \cup_{i\in I}\left(a_j^i + \alpha_j^i \bar{S}\right)\right| = 0,$$

we set for all $(n, N) \in \mathbb{N}^* \times \mathbb{N}$ (we can identify I with $\mathbb{N}$),

$$\varphi(n,N)(x) = \begin{cases} \alpha_j^i \varphi_j^n \left(\dfrac{x - a_j^i}{\alpha_j^i} \right) & \text{if } x \in a_j^i + \alpha_j^i S, \quad j \in J, \quad i = 1, \dots, N, \\ 0 & \text{otherwise,} \end{cases}$$

so that $(\varphi(n, N))_{(n,N)\in\mathbb{N}^*\times\mathbb{N}} \subset \mathrm{Aff}_0(\omega; \mathbb{R}^3)$. We first notice that

$$(\varphi(n, N)) \underset{N\to\infty}{\longrightarrow} \varphi(n), \quad \text{in } W^{1,\infty}(\omega; \mathbb{R}^3),$$

where

$$\varphi(n)(x) = \begin{cases} \alpha_j^i \varphi_j^n \left(\dfrac{x - a_j^i}{\alpha_j^i} \right) & \text{if } x \in a_j^i + \alpha_j^i S, \quad j \in J, \quad i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, observing that $\nabla\varphi(n) = \nabla\varphi(1)$ for all $n \in \mathbb{N}^*$, we can write

$$\begin{aligned} \int_\omega \mathrm{R}_k W_0(\nabla\psi(x) + \nabla\varphi(x))\, dx &= \int_\omega \mathrm{R}_k W_0(\nabla\psi(x) + \nabla\varphi(1)(x))\, dx, \\ &= \sum_{i\in I, j\in J} \int_{a_j^i + \alpha_j^i S} \mathrm{R}_k W_0 \left(F_j + \nabla\varphi_j \left(\frac{x - a_j^i}{\alpha_j^i} \right) \right) dx, \\ &= \sum_{i\in I, j\in J} (\alpha_j^i)^2 \int_S \mathrm{R}_k W_0 \left(F_j + \nabla\varphi_j(x) \right) dx. \end{aligned}$$

Thus using (3.17), $\left| \sum_{i\in I, j\in J} \left(\alpha_j^i S \right) \right| = |\omega|$ and $0 < \alpha_j^i < 1$, we infer

$$\begin{aligned} \int_\omega \mathrm{R}_k W_0(\nabla\psi(x) + \nabla\varphi(1)(x))\, dx &\le |\omega| \sum_{i\in I} \mathrm{ZR}_k W_0(F_j) + \epsilon\, |\omega|\,, \\ &= \int_\omega \mathrm{ZR}_k W_0(\nabla\psi(x))\, dx + \epsilon\, |\omega|\,. \end{aligned} \qquad (3.18)$$

Now, we remark that there is no argument to impose that $\mathrm{rank}\, \nabla\psi + \nabla\varphi(1) = 2$. However by virtue of Proposition 3.1.6, there exists $(A_\eta)_{\eta>0} \subset \mathbb{R}^{3\times 2}$ such that $|A_\eta| < \eta$ and

$$\mathrm{rank}\, \nabla\psi + \nabla\varphi(1) + A_\eta = 2, \quad \eta > 0, \quad \text{a.e. in } \omega,$$

since $\mathrm{card}\, \nabla\psi + \nabla\varphi; x \in \omega$ is finite. The trouble now is that such a gross transformation while ensuring the non-degeneracy of the gradients inside the squares $a_j^i + \alpha_j^i S$ that cover ω, does not restore the local injectivity that was lost through the perturbation φ^1 of the original deformation ψ nor does it preserve the boundary condition. To make for this discrepancy, we need to introduce the following *cut-off*-like function

$$\zeta_\delta(x) = \begin{cases} \dfrac{1}{\delta}\, \mathrm{dist}(x, \partial S) & \text{if } \mathrm{dist}(x, \partial S) \le \delta, \\ 1 & \text{otherwise,} \end{cases}$$

where $\delta << 1$ and define

$$\chi_N^\delta(x) = \begin{cases} \alpha_j^i \zeta_\delta \left(\dfrac{x - a_j^i}{\alpha_j^i} \right) & \text{if } x \in a_j^i + \alpha_j^i S, \quad j \in J, \quad i = 1, \ldots, N, \\ 0 & \text{otherwise.} \end{cases}$$

Let us mention that

$$\chi_N^\delta \underset{N \to \infty}{\longrightarrow} \chi^\delta, \quad \text{in } W^{1,\infty}(\omega; \mathbb{R}^3),$$

where

$$\chi^\delta(x) = \begin{cases} \alpha_j^i \zeta_\delta \left(\dfrac{x - a_j^i}{\alpha_j^i} \right) & \text{if } x \in a_j^i + \alpha_j^i S, \quad j \in J, \quad i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we set

$$\psi_\eta(n, N) = \psi + \varphi(n, N) + \phi_\eta^N, \quad \phi_\eta^N(x) = \chi_N^\delta(x) A_\eta x, \quad x \in \omega, \quad n \in \mathbb{N}^*.$$

We claim that for $\eta,\ \delta > 0$ small enough, $\psi_\eta(n, N)$ is locally injective and belongs to $id{+}\mathrm{Aff}_0(\omega; \mathbb{R}^3)$ for all n and N because $\nabla\psi_\eta(n, N) = \nabla\psi$ on $\omega \backslash \cup_{i \in I, j \in J}(a_j^i + \alpha_j^i \bar{S})$ so the local injectivity is preserved because A_η ensures the local injectivity on $a_j^i + \alpha_j^i \bar{S}$. Hence, by the induction argument we can write

$$\bar{J}(0)(\psi_\eta(n, N)) \leq 2 \int_\omega \mathrm{R}_k W_0 \left(\nabla\psi_\eta(n, N) \right) dx_H - \ell(0)(\psi_\eta(n, N)).$$

Next, noting $\phi_\eta(x) = \chi^\delta(x) A_\eta x$ and taking $N \longrightarrow \infty$ yields

$$\bar{J}(0)(\psi + \varphi(n) + \phi_\eta) \leq 2 \int_\omega \mathrm{R}_k W_0 \left(\nabla\psi + \nabla\varphi(1) + \nabla\phi_\eta \right) dx_H - \ell(0)(\psi + \varphi(n) + \phi_\eta),$$

where $\phi_\eta(x) = \chi^\delta(x) A_\eta x$. Hence taking $n \longrightarrow \infty$ ensues

$$\bar{J}(0)(\psi + \phi_\eta) \leq 2 \int_\omega \mathrm{R}_k W_0 \left(\nabla\psi + \nabla\varphi(1) + \nabla\phi_\eta \right) dx_H - \ell(0)(\psi + \phi_\eta).$$

for all $\eta > 0$ since by definition of Γ-convergence

$$\bar{J}(0)(\psi + \phi_\eta) \leq \liminf_{n \longrightarrow \infty} \bar{J}(0)(\psi + \varphi(n) + \phi_\eta)$$

and obviously $\ell(0)(\psi + \varphi(n) + \phi_\eta) \underset{n \to \infty}{\longrightarrow} \ell(0)(\psi + \phi_\eta)$. We turn to taking the limit $\eta \longrightarrow 0$: first of all observe that W_0 being continous (cf. Proposition 3.4.1-(i)), by induction on $k \in \mathbb{N},\ \mathrm{R}_k W_0$ is sequentially upper semicontinuous as an infimum of a family of sequentially upper semicontinuous functions (recall the definition in Proposition 3.1.1). Moreover, we have that $\|\phi_\eta\|_{W^{1,\infty}(\omega;\mathbb{R}^3)} \underset{\eta\to 0}{\longrightarrow} 0$ since $|A_\eta| < \eta$. Consequently, we infer

$$\begin{aligned}
\bar{J}(0)(\psi) &\le \liminf_{\eta\longrightarrow 0^+} \bar{J}(0)(\psi+\phi_\eta) \le \limsup_{\eta\longrightarrow 0^+} \bar{J}(0)(\psi+\phi_\eta) \\
&\le \limsup_{\eta\longrightarrow 0^+}\ 2\int_\omega \mathrm{R}_k W_0\left(\nabla\psi + \nabla\varphi^1 + \nabla\phi_\eta\right) dx_H - \lim_{\eta\to 0^+} \ell(0)(\psi+\phi_\eta) \\
&\le \int_\omega \mathrm{R}_k W_0(\nabla\psi(x) + \nabla\varphi^1(x))\, dx - \ell(0)(\psi)
\end{aligned}$$

which entails

$$\bar{J}(0)(\psi) \le \int_\omega \mathrm{ZR}_k W_0(\nabla\psi(x))\, dx + \epsilon\, |\omega|$$

by (3.18). Here we invoke Proposition 3.1.3-(iii) to assert that $\mathrm{ZR}_k W_0$ is rank-one-convex on the interior of the effective domain of $\mathrm{R}_k W_0$, $\mathcal{D}_e(\mathrm{R}_k W_0) = \mathcal{D}_e(W_0) = \{F \in \mathbb{R}^{3\times 2} : \operatorname{rank} F = 2\}$, but $\psi \in id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$ and is locally injective so $\nabla\psi(x) \in \mathcal{D}_e(\mathrm{R}_k W_0)$ a.e. in ω and accordingly we deduce

$$\mathrm{ZR}_k W_0(\nabla\psi(x)) \le \mathbf{R} W_0(\nabla\psi(x)) \le \mathrm{R}_{k+1} W_0(\nabla\psi(x)), \quad \text{a.e. in } \omega,$$

which brings up the induction argument for $k+1$ thereby ending the proof. □

Next we sketch a different proof of the above proposition which does not require the introduction of function Z and Proposition 3.1.3, but makes use of property (P_k) in Proposition 3.1.2.

Alternative proof of Proposition 3.7.2. We proceed by induction on $k \in \mathbb{N}$. Indeed, we suppose that (3.16) is satisfied and go on proving it for $k+1$. Let $\psi \in id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$ and name $(F_j)_{j\in J}$ the finite set of values its gradient takes and $(O_j)_{j\in J}$ the associated partition of ω. According to Proposition 3.1.2, for all $j \in J$, there exists $(a_j, b_j, \lambda_j) \in \mathbb{R}^2 \times \mathbb{R}^3 \times [0,1]$ such that

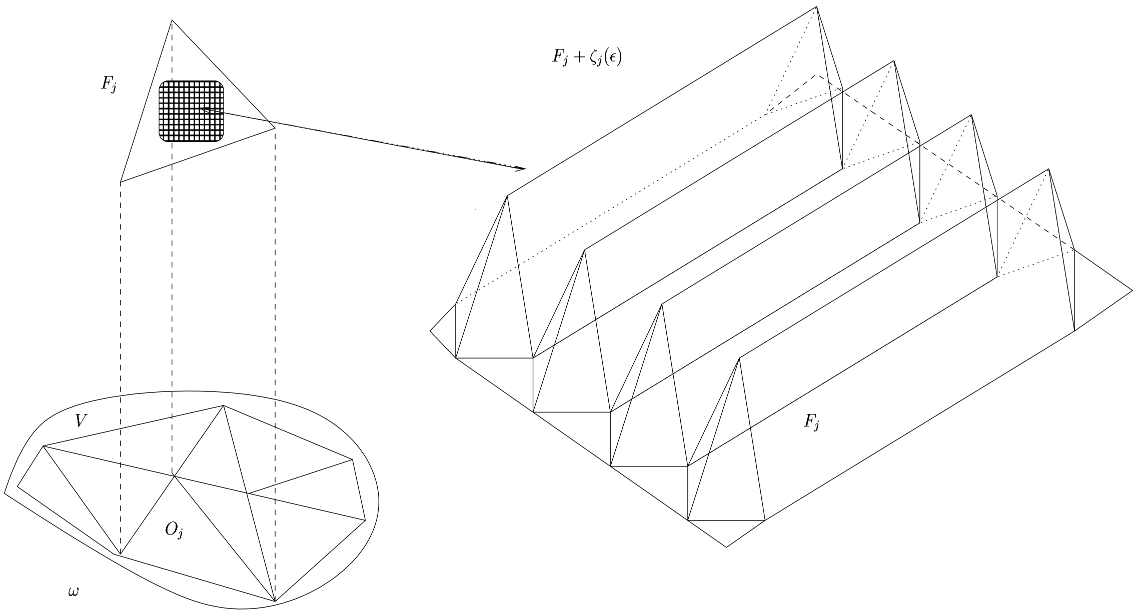
$$\mathrm{R}_{k+1} W_0(F_j) = (1-\lambda_j)\mathrm{R}_k W_0(F_j - \lambda_j\, a_j \otimes b_j) + \lambda_j\, \mathrm{R}_k W_0(F_j + (1-\lambda_j) a_j \otimes b_j).$$

Then, we introduce functions $\zeta_j \in \mathrm{Aff}(S;\mathbb{R}^3)$ on $S = [0,1]^2$ defined as follows

$$\nabla\zeta_j(x) = \begin{cases} -\lambda_j(a_j\otimes b_j) & \text{if } 0 \le x_1 \le 1-\lambda_j, \\ (1-\lambda_j)(a_j\otimes b_j) & \text{if } 1-\lambda_j \le x_1 \le 1, \end{cases} \qquad j \in J,$$

so that we have

$$\begin{aligned}
\int_S \mathrm{R}_k W_0(F_j + \nabla\zeta_j)\, dx &= \int_{\{x\in S: 0\le x_1\le 1-\lambda_j\}} \mathrm{R}_k W_0(F_j - \lambda_j(a_j\otimes b_j))\, dx \\
&\quad + \int_{\{x\in S: 1\ge x_1\ge 1-\lambda_j\}} \mathrm{R}_k W_0(F_j + (1-\lambda_j)(a_j\otimes b_j))\, dx, \\
&= |S|\,[(1-\lambda_j)\mathrm{R}_k W_0(F_j - \lambda_j\, a_j\otimes b_j) \\
&\quad + \lambda_j\, \mathrm{R}_k W_0(F_j + (1-\lambda_j) a_j\otimes b_j)] \\
&= |S|\, \mathrm{R}_{k+1} W_0(F_j) = \int_S \mathrm{R}_{k+1} W_0(F_j)\, dx.
\end{aligned}$$



It suffices then to construct a sequence in the same fashion as in the previous proof i.e. a sequence $(\zeta_j(\epsilon))_{\epsilon>0} \subset \mathrm{Aff}_0(\omega;\mathbb{R}^3)$ such that

- $\zeta_j(\epsilon)_{|S} = 0$,
- $\mathrm{rank}(F_j + \nabla\zeta_j(\epsilon)) = 2$,
- $\lim_{\epsilon\to 0} \int_S \mathrm{R}_k W_0(F_j + \nabla\zeta_j(\epsilon))\, dx = |S|\, \mathrm{R}_{k+1} W_0(F_j)$.

The next step is to extend by periodicity $(\zeta_j(\epsilon))_{j\in J}$ to $\mathbb{R}^2$ as in the previous proof and conclude likewise by the Vitali covering and the definition of piecewise affine sequence that converges to the Vitali piecewise affine perturbation of the initial deformation making sure that the local injectivity is preserved as well as boundary conditions. For an illustration of the construction see the figure above. □

What follows is deduced from the previous proposition by classical arguments.

Corollary 3.7.3. *For all immersions $\psi \in \mathcal{C}^1(\omega;\mathbb{R}^3)$ such that $\psi_{|\partial\omega} = id$, we have*

$$\bar{J}(0)(\psi) \leq 2\int_\omega \mathbf{QR}W_0\,(\nabla\psi)\, dx_H - \ell(0)(\psi). \tag{3.19}$$

Proof. First let $\psi \in id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$ be a locally injective function. By Proposition 3.1.1, $\mathrm{R}_k W_0(\nabla\psi) \underset{k\to\infty}{\longrightarrow} \mathbf{R}W_0(\nabla\psi)$. Therefore, since the set of values $\nabla\psi$ takes is finite, there exists $\epsilon > 0$ and $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\mathrm{R}_k W_0(\nabla\psi(x)) \leq \mathbf{R}W_0(\nabla\psi(x)) + \epsilon \quad \text{a.e. in } \omega.$$

According to Proposition 3.4.4, $\mathbf{R}W_0$ is everywhere finite so we can reason by applying Lebesgue's monotone convergence theorem to (3.16) since the sequence $(\mathrm{R}_k W_0)_{k\in\mathbb{N}}$ is decreasing to ensue that

$$\bar{J}(0)(\psi) \leq 2\int_\omega \mathbf{R}W_0\,(\nabla\psi)\,dx_H - \ell(0)(\psi), \tag{3.20}$$

for all locally injective $\psi \in id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$. Now assume $\psi \in \mathcal{C}^1(\omega;\mathbb{R}^3)$ is an immersion such that $\psi_{|\partial\omega} = id$, by Proposition 3.1.7 there exists a sequence of locally injective functions $(\psi^n)_{n>0} \subset id + \mathrm{Aff}_0(\omega;\mathbb{R}^3)$ such that $\|\psi - \psi^n\|_{W^{1,\infty}(\omega;\mathbb{R}^3)} \underset{n\to\infty}{\longrightarrow} 0$. Moreover, we can write

$$\bar{J}(0)(\psi^n) \leq 2\int_\omega \mathbf{R}W_0\,(\nabla\psi^n)\,dx_H - \ell(0)(\psi^n), \quad n \in \mathbb{N}.$$

Next, we assert that $\mathrm{R}W_0$ is continuous owing to its boundedness and convexity with respect to each variable (this is a consequence of rank-one-convexity) ; it is actually locally lipschitz, see Lemma D.0.5 for more details. In this fashion, the sequential weak lower semicontinuity of the Γ-limit $\bar{J}(0)$ and the dominated convergence theorem raise

$$\bar{J}(0)(\psi) \leq 2\int_\omega \mathbf{R}W_0\,(\nabla\psi)\,dx_H - \ell(0)(\psi).$$

Lastly, we invoke Acerbi & Fusco's relaxation result [2] (cf. Theorem 0.2.2) for finite functionals defined on Sobolev spaces (cf. Introduction) which together with the dominated convergence theorem yields the announced inequality. □

To conclude this section, we mention that to get the actual Γ-limit for all functions in $W^{1,p}(\omega;\mathbb{R}^3)$ as stated in Theorem 3.3.1, we use Conjecture 3.1.8 for lack of a better argument at this moment.

The results proved in this chapter are announced in Trabelsi [71].

Annexe C

Approximation of piecewise affine functions by immersions

Here we prove Lemma 3.1.5. For convenience, we recall its precise statement.

Proposition C.0.4. *(*Bennequin [8]*) If $\psi \in \mathrm{Aff}(\omega;\mathbb{R}^3)$ is a locally injective function such that $\psi_{|_V} = id$ where V is a neighbourhood of $\partial\omega$, then there exists a sequence of immersions $(\psi^n)_{n\in\mathbb{N}} \subset C^1(\bar{\omega};\mathbb{R}^3)$ such that*

- *$\psi^n \underset{n\to\infty}{\longrightarrow} \psi$ in $W^{1,p}(\omega;\mathbb{R}^3)$,*
- *there exists $\delta > 0$ such that $\left|\psi^n_{,1} \wedge \psi^n_{,2}\right| > \delta$ for all $n \in \mathbb{N}$,*
- *$\psi^n{}_{|_V} = id$ for all $n \in \mathbb{N}$.*

Proof. Let D be the unit disk of $\mathbb{R}^2$. We first prove the lemma for a mapping $\varphi : D \longrightarrow \mathbb{R}^3$ such that

- $\varphi(D)$ is a cone of $\mathbb{R}^3$,
- $\varphi \in C^1(\bar{D};\mathbb{R}^3)$ is locally injective except at $O = (0,0)$,
- there exists $\rho > 0$ such that $|\varphi_{,1} \wedge \varphi_{,2}| > \rho$.

Now assume without loss of generality that $\varphi(O) = \bar{O} = (0,0,0)$. By the local injectivity of φ, we infer that there exists $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$, $\mathcal{S}(O,\frac{1}{n}) \cap \varphi(D)$ is a simple closed curve γ^n. The curve γ^n divides $\mathcal{S}(O,\frac{1}{n})$ into two connected sets $\mathcal{S}_n^{\pm}$. Likewise, $\varphi(D)$ is divided into two connected sets $\mathcal{C}_n^{\pm}$ where $\varphi(O) = \bar{O} \in \mathcal{C}_n^{+}$. Next, we define the set $\mathcal{U}_n^{+} = \varphi^{-1}(\mathcal{C}_n^{+})$. Due to the local injectivity, there exists a diffeomorphism $\eta_{n_0}^{+}$ between $\mathcal{U}_{n_0}^{+}$ and $\mathcal{S}_{n_0}^{+}$. We introduce the function

$$\bar{\varphi}_{n_0} = \begin{cases} \eta_{n_0}^{+}, & \text{if } x \in \mathcal{U}_{n_0}^{+}, \\ \varphi & \text{otherwise.} \end{cases}$$

The mapping $\bar{\varphi}_{n_0}$ is locally injective and differentiable on $\mathcal{D}\backslash\varphi^{-1}(\gamma_{n_0})$. Hence after mollifying the function $\bar{\varphi}_{n_0}$ around $\varphi^{-1}(\gamma_{n_0})$ we yield an immersion φ_{n_0}. Lastly, for all $n \geq n_0$ we consider the dilation Λ_n of $\mathbb{R}^3$ of center the origin $\bar{O}$ and ratio $\frac{n}{n_0}$ and set

$$\bar{\varphi}_n = \begin{cases} \Lambda_n^{-1}\eta_{n_0}^+\Lambda_n, & \text{if } x \in \mathcal{U}_n^+, \\ \varphi & \text{otherwise.} \end{cases}$$

We remark that the construction obtained in this fashion is selfsimilar thereby ensuring that

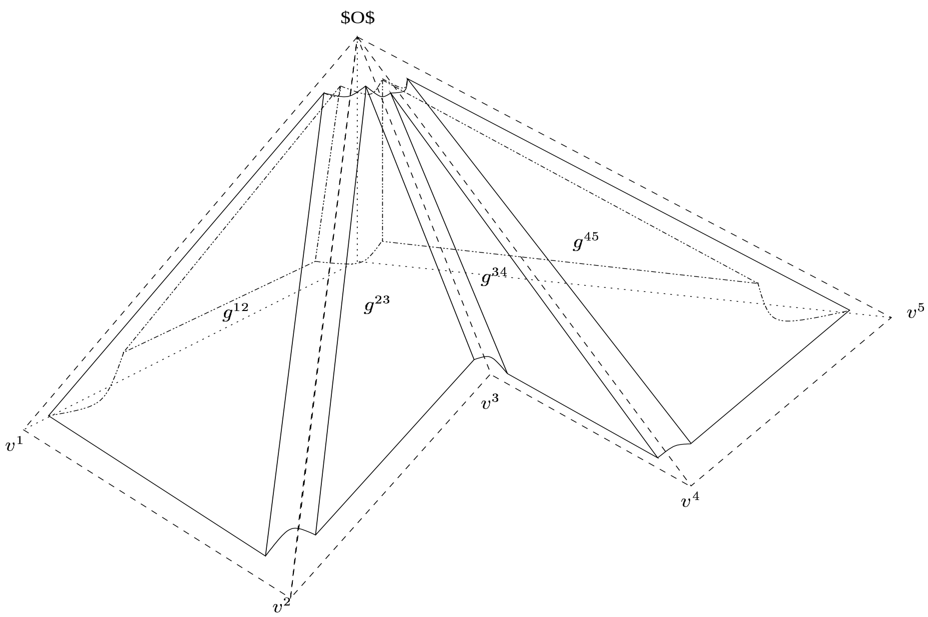
$$\left|\varphi^n_{,1} \wedge \varphi^n_{,2}\right| > \delta, \quad n \geq n_0,$$

where $\delta > 0$ is does not depend on n. Furthermore, it is clear that $\varphi^n \underset{n\to\infty}{\longrightarrow} \varphi$ in $W^{1,p}(\mathcal{D};\mathbb{R}^{\ni})$.

Now we consider a piecewise linear mapping ψ satisfying the data of the lemma. We assume, without loss of generality, that ψ is piecewise affine on a regular triangular mesh of $\omega \setminus V$ with 6 equilateral trinangles around each vertice. Let O be such a vertice and let $(v^k)_{1\leq k\leq 6}$ be the associated adjacent vertices. We call $(g^{k,k+1})_{1\leq k\leq 5}$ and $g^{6,1}$ the centres of gravity of triangles $(O, v^k, v^{k+1})_{1\leq k\leq 5}$ and (O, v^6, v^1) respectively. Next, we define the dilation $\Lambda_{n_0}^{k,l}$ of centre $g^{k,l}$ and ratio $1 - \frac{1}{n_0}$, where n_0 is a positive integer to be chosen later, for each $(k,l) \in \{(1,2), \ldots, (6,1)\}$. Accordingly, we set the notations

$$O_{n_0}^{k,l} = \Lambda_{n_0}^{k,l}(O), \quad v^{k,k+1} = \Lambda_{n_0}^{k,k+1}(v^k), \quad v^{k+1,k} = \Lambda_{n_0}^{k,k+1}(v^{k+1}),$$

for $1 \leq k \leq 5$ and $v^{6,1} = \Lambda_{n_0}^{6,1}(v^6)$, $v^{1,6} = \Lambda_{n_0}^{6,1}(v^1)$. Hence we mollify ψ around the sides of the triangles thereby obtaining the following function $\bar{\psi}^{n_0}$ as it is depicted in the figure below.



Thence, we extend $\bar{\psi}_{n_0}$ around the vertice O to obtain an application whose image is a cone.

This construction is to be repeated around each vertice of the triangulation providing a mapping $\bar{\psi}^{n_0}$ diffrerentiable on ω minus its vertices. Its image around each vertice is a cone by construction, $\bar{\psi}^{n_0}{}_{|V} = id$ and it is locally injective for n_0 large enough. Moreover, as $\mathrm{card}\{\nabla\psi(x) : \ x \in \omega\}$ is finite, there exists $\rho > 0$ such that

$$\left|\bar{\psi}^{n_0}_{,1} \wedge \bar{\psi}^{n_0}_{,2}\right| \geq \rho, \quad \text{in } \omega.$$

Hence, $\bar{\psi}^{n_0}$ verifies the assumptions made on the function considered at the beginning of the proof. As a result, there exists an immersion $\psi^{n_0} \in C^1(\bar{\omega}; \mathbb{R}^3)$ and $\delta > 0$ such that

$$\psi^{n_0}{}_{|V} = id, \quad \left|\psi^{n_0}_{,1} \wedge \psi^{n_0}_{,2}\right| \geq \delta, \quad \text{and} \quad \left\|\psi^{n_0} - \bar{\psi}^{n_0}\right\|_{W^{1,p}(\omega;\mathbb{R}^3)} \leq \frac{1}{n_0}.$$

We repeat the same argument for all $n \geq n_0$, exhibiting in this fashion a sequence of immersions $(\psi^n)_{n \geq n_0} \subset C^1(\bar{\omega}; \mathbb{R}^3)$ satisfying the requirements of the proposition. □

All in all, the above proof is merely a procedure of *rounding the corners*.

Annexe D

Two elementary results

We devote the appendix to the full statement and proof of two standard results that have been used in this chapter. We start by a lemma dealing with the continuity of rank-one-convex functions satisfying an ad hoc bound. This was put forward in the proof of Proposition 3.7.2. The proof we give is due to Giusti [46].

Lemma D.0.5. (Marcellini [55]) *Let $W : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ be a rank-one-convex function satisfying*

$$|W(A)| \leq c(\lambda + |A|^p), \quad \lambda \geq 0,$$

with $p \geq 1$. Then

$$|W(A) - W(B)| \leq c(\lambda + |A| + |B|)^{p-1} |A - B| .$$

Proof. Let matrix $O \in \mathbb{R}^{n\times N}$ be such that only one component is different from zero. Then $\operatorname{rank} O = 1$. Set $w(t) = W(U+tO)$ where $t \in \mathbb{R}$ and $U \in \mathbb{R}^{n\times N}$. Function w is convex since W is rank-one-convex. Hence $\mathcal{W} : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$\mathcal{W}(t) = \frac{w(t) - w(0)}{t},$$

is increasing. Next, we consider a matrix $A \in \mathbb{R}^{n\times N}$ as a vector with components A_k, $k = 1, 2, \ldots, nN$. Let $B \in \mathbb{R}^{n\times N}$ and define

$$A^{(k)} = (B_1, \ldots, B_k, A_{k+1}, \ldots, A_{nN}), \quad k = 0, 1, 2, \ldots, nN,$$

so that $A^{(0)} = A$ and $A^{(nN)} = B$. Furthermore, if we take $U = A^{(k)}$, $O = A^{(k+1)} - A^{(k)}$ and $t = \frac{\lambda+|A|+|B|}{|A-B|} > 1$, we can write

$$\begin{aligned} W(A^{(k+1)}) - W(A^{(k)}) &= \mathcal{W}(1) \leq \mathcal{W}(t) \\ &= \frac{W(A^{(k)} + t(A^{(k+1)} - A^{(k)})) - W(A^{(k)})}{\lambda + |A| + |B|} |A - B| . \end{aligned}$$

Thus remarking that $\left|A^{(k)}\right| \leq |A| + |B|$ and $\left|A^{(k)} + t(A^{(k+1)} - A^{(k)})\right| \leq c(\lambda + |A| + |B|)$, by the growth condition satisfied by W, we infer

$$W(A^{(k+1)}) - W(A^{(k)}) \leq c(\lambda + |A| + |B|)^{p-1} |A - B| .$$

Ultimately, summing over $k = 0, 1, 2, \ldots, nN$ and exchanging the roles of A and B the desired inequality is established. □

The above lemma entails that such functions are locally Lipschitz.

The following was invoked in the proof of Proposition 3.1.2 and states that the set of rank-one-connected matrices is closed.

Proposition D.0.6. *The set of matrices*

$$E = \left\{(A, B) \in (\mathbb{R}^{n\times m})^2 : \text{rank}\,(A - B) \leq 1\right\}$$

is closed.

Proof. Let (A, B) belong to the cluster set of E i.e. there exists $(A^k, B^k)_{k\in\mathbb{N}} \subset E$ such that $(A^k, B^k) \underset{k\to\infty}{\longrightarrow} (A, B)$. As for all $n \in \mathbb{N}$ we have $\text{rank}\,(A_k - B_k) \leq 1$, there exists $(a_k, b_k)_{k\in\mathbb{N}} \subset \mathbb{R}^{n\times m}$ such that $A^k = B^k + {a_k}^t b_k$. Moreover, we can choose $|a_k| = 1$. Now, S^{n-1} is compact, therefore we can extract a subsequence $a_k \underset{k\to\infty}{\longrightarrow} a \in S^{n-1}$. As a consequence, for all $k \in \mathbb{N}$

$$b_k = \frac{1}{\langle a_k, a\rangle}(A_k a - B_k a) \underset{k\to\infty}{\longrightarrow} b \in \mathbb{R}^m.$$

Thus $A - B = a^t b$ and $(A, B) \in E$ □

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